Abstract

Recently it was suggested that the problem of species doubling with Kogut-Susskind lattice fermions entails, at finite chemical potential, a confusion of particles with antiparticles. What happens instead is that the familiar correspondence of positive-energy spinors to particles, and of negative-energy spinors to antiparticles, ceases to hold for the Kogut-Susskind time derivative. To show this we highlight the role of the spinorial “energy” in the Osterwalder-Schrader reconstruction of the Fock space of non-interacting lattice fermions at zero temperature and nonzero chemical potential. We consider Kogut-Susskind fermions and, for comparison, fermions with an asymmetric one-step time derivative.
1 Introduction

The implementation of finite baryon density on the lattice \([1, 2]\) has proven to be a hard task. Several proposals have been made to explain the unexpectedly low chemical potential \(\mu\) at which the baryon density and other quantities start to grow. The onset at a \(\mu\) much lower than the value \(m_N/3\), as expected for a free nucleon gas, could be due to the enhanced number of flavours (when using Kogut-Susskind fermions) producing a much stronger nuclear binding than in nature \([3, 4]\).

As the onset is so early that its energy can be almost confounded with the existence of a ‘baryonic’ Goldstone mode, it was also interesting to search for possible enhancement of mesonic modes due to some kind of lattice artifact. In \([5]\) it was pointed out that by selecting only the positive-energy poles of the free Kogut-Susskind propagator one still obtains energies and baryon number densities composed of particles as well as antiparticles. This is in contrast to Dirac fermions in space-time continuum where positive-energy spinors always correspond to particles and negative-energy spinors to antiparticles. As observed in \([5]\), that feature of continuous fermions is reproduced on the lattice if an asymmetric, one-step time derivative is used. (In fact, such fermions have the same Hilbert space as Susskind’s fermions \([6]\) on the Hamiltonian lattice.) It was then suggested in \([5]\) that the Kogut-Susskind action does not allow a clear distinction of particles and antiparticles. That conclusion, however, is not justified. In fact, particles and antiparticles can be reliably identified only from the Hamiltonian and baryon number operators in Fock space which we therefore reconstruct below. Apart from the usual doubling of flavours, nothing peculiar will be seen here to occur with Kogut-Susskind fermions. Rather, it will be the correspondence of positive/negative-energy spinors to particles/antiparticles which is destroyed by the Kogut-Susskind two-step time derivative.

Intuitively, in this respect, the essence of the technical, field-theoretic arguments presented in the next sections is as follows. In Euclidean time \(\tau\), “particle” solutions of the Dirac equation are those which remain bounded for \(\tau \to +\infty\) while “antiparticles” remain bounded for \(\tau \to -\infty\); this corresponds to the Minkowskian notion of particles moving forward and antiparticles backward in time. The rule is implicit in the Osterwalder-Schrader reconstruction algorithm (cf. remark following eqn. (16)). For the Dirac equation in continuous Euclidean time, \(\dot{\psi} = -H\psi\), this implies that the positive eigenvalues of \(H\) are associated with particles and the negative eigenvalues with antiparticles. Now, if time is discretized, the Dirac equation is converted into a recursion relation. The most straightforward, asymmetric discretization would give (with \(a_T\) being the lattice spacing and the spinorial Hamiltonian diagonalized)

\[
\psi_{\tau+1} - \psi_{\tau} = -a_T E \psi_{\tau}
\]

Depending on the size of \(a_T E\) cutoff this will distort the continuous-time dependencies somewhat but it will preserve the qualitative features. In contrast, the symmetric discretization would give the recursion relation

\[
\psi_{\tau+1} - \psi_{\tau-1} = -2a_T E \psi_{\tau}
\]

which is for any \(a_T\) a qualitatively different type of equation. In fact, looking for “particle” solutions with an exponential decay \(e^{-\omega \tau}\) in the positive time direction, we
now find two of them:

\[
\begin{align*}
&\{ e^{-\omega \tau} & (-1)^\tau e^{-\omega \tau} \} \text{ emerging from } \\
&\{ a_T E = + \sinh \omega > 0 & a_T E = - \sinh \omega < 0 \}
\end{align*}
\]

A similar degeneracy, involving both signs of spinorial energy, is obtained with solutions bounded for \( \tau \to -\infty \). Hence, positive spinorial energies are no longer reserved for particles, but occur as antiparticle solutions as well. Of course, those arguments require formal substantiation. In the field theory that we consider here, particles and antiparticles will be clearly identified by the baryon number.

The Hilbert spaces corresponding to space-time lattice fermions have all been reconstructed long ago as part of the issue of the positivity of the transfer matrices. This was done in \([7, 8, 9]\) for Wilson fermions, and in \([10]\) for staggered fermions. In all those references, a direct comparison was made between matrix elements in a standard fermion Fock space and the Grassmann functional integral. While this method enables one to deal with interacting fermions from the outset, it does not assign a Fock-space interpretation to the path-integral variables themselves. For Kogut-Susskind fermions, in particular, auxiliary Grassmann integrations had to be introduced, preventing the Fock space interpretation of the original variables. In the present context, where we want to identify the role of positive- and negative-energy spinors (the Fourier transforms of the path-integral variables) in the fermion Fock space, we therefore prefer to work our way through the Osterwalder-Schrader reconstruction scheme \([11, 12]\) which does not anticipate any knowledge of the Hilbert space. For a detailed introduction to this formalism we refer the reader to \([13]\); we shall only need the simplest of the formal concepts, and the mathematical tools required in the following will not go beyond some suitably organized algebra of Grassmann derivatives.

Fortunately, the present issue allows us to restrict ourselves to non-interacting fermions at zero temperature, but at finite chemical potential \(\mu\).

Furthermore, we will restrict our examples to cases where the inverse fermion matrix can be given explicitly, in terms of spinorial energy eigenstates and eigenvalues. Thus we consider fermion actions of the general form

\[
S = \bar{\psi}_n K_{nn'} \psi_{n'} \quad n = (n_1, n_2, n_3, n_4) = (\vec{n}, n_4)
\]

such that the time-changing (derivative) part \(D\) commutes with the equal-time, spinorial “energy” part \(E\). This form can be achieved both for Kogut-Susskind and for time-asymmetric \([5]\) fermions. For Wilson fermions, such a simplification does not seem to be possible.

For the asymmetric fermions it will be advantageous to allow for an anisotropy of the space-time lattice; hence the appearance in \([4]\) of the spatial and temporal lattice spacings, \(a_S\) and \(a_T\).

In section 2 we recall the basic steps in the procedure of Osterwalder-Schrader reconstruction, and the standard tools we have at hand for non-interacting fermions. In section 3 we apply this, in some detail, to fermions with a one-step time derivative. We thus recover the familiar zero-temperature scenario at finite chemical potential. In
section 4 we provide the analogous expressions for Kogut-Susskind fermions. A comparison of the operator expressions for the energy and for the baryon number in Fock space (as opposed to the linear space spanned by the Grassmann variables) will show that a chemical potential enhancing the baryons will deplete the antibaryons. In the concluding section 5 we discuss relations with the positivity of the transfer matrices and with finite temperature.

2 Basic ideas of reconstructing $\hat{H}$ and $\hat{B}$

From the path-integral expectation values and correlations we wish to recover the multi-fermion Hilbert space, the scalar product of the Hilbert space vectors, and the operators of time evolution and baryon number. The basic ideas are as follows [13]. Both bra and ket vectors contribute to $\langle \cdots \rangle_{\text{path integral}}$. The ket vectors are the functionals of Grassmann variables $\psi_\tau$ and $\bar{\psi}_\tau$ with Euclidean times $\tau > 0$ (more precisely, they are equivalence classes of such functionals; cf. below). The bra vectors are the same with $\tau < 0$. Thus bra and ket functionals are related through a time-reversal operation $\Theta$. On Grassmann variables it acts as

$$\Theta \psi(\tau) = \bar{\psi}(-\tau) \quad \Theta \bar{\psi}(\tau) = \psi(-\tau)$$

and is extended as an antilinear operation on functionals of $\psi$ and $\bar{\psi}$. We shall adopt the rule that complex conjugation of a product of Grassmann variables implies reversed ordering of factors.

The scalar product of a bra and ket vector is given by a path-integral correlation: if $F$ and $G$ are ket functionals, and $|F\rangle$ and $|G\rangle$ their interpretations as Hilbert space vectors, their scalar product is

$$\langle F | G \rangle_{\text{path integral}} = \frac{1}{Z} \int [d\psi][d\bar{\psi}] \, e^S \Theta F[\bar{\psi}, \psi]_{\tau < 0} G[\bar{\psi}, \psi]_{\tau > 0}$$

For a quantum-statistical interpretation the scalar product must satisfy $\langle F | F \rangle \geq 0$ which it does in all the cases we consider. Null-norm functionals (which have $\langle F | F \rangle = 0$ although $F \neq 0$) can then be factored away. This will, in fact, simplify the structure of the Hilbert space.

We will extract the transfer matrix operator $\hat{T}$ from the path integral correlation $\langle F | \hat{T} | G \rangle$ in such a way that it will reproduce these matrix elements for arbitrary $F$ and $G$. The transfer matrix, which defines the true energy (as opposed to spinorial “energy”) by $\hat{T} = e^{-\hat{H}}$, derives from a unit shift of the time index on all the Grassmann variables in a ket functional

$$\langle F[\bar{\psi}_\tau, \psi_\tau] | \hat{T} | G[\bar{\psi}_\tau, \psi_\tau] \rangle = \langle F[\bar{\psi}_\tau, \psi_\tau] | G[\bar{\psi}_{\tau+1}, \psi_{\tau+1}] \rangle$$

The baryon number operator will likewise be identified from its matrix elements $\langle F | \hat{B} | G \rangle$, i.e. from the path integral correlation $\langle \Theta F B | G \rangle$ in which $B[\bar{\psi}, \psi] = \partial S/\partial \mu$ is the usual observable.
It will be essential in the following to have an explicit representation of Grassmann-integral expectation values and correlations at hand. In the most general finite-dimensional case, if $\psi_n$ and $\bar{\psi}_n$ are the integration variables and if $O[\psi, \bar{\psi}]$ is any observable or correlation in terms of them, then

$$
\langle O \rangle_{\text{path integral}} = \frac{1}{\det K} \prod_n d\psi_n d\bar{\psi}_n \exp \left( \sum_{n,m} \bar{\psi}_n K_{nm} \psi_m \right) O[\psi, \bar{\psi}]
$$

$$
= \exp \left( \sum_{n,m} K_{nm}^{-1} \frac{\partial}{\partial \psi_n} \frac{\partial}{\partial \bar{\psi}_m} \right) O[\psi, \bar{\psi}] \bigg|_{\psi=\bar{\psi}=0}
$$

To keep the notation as simple as possible, we shall refer only to Grassmann derivatives “from left to right” ($\partial/\partial \psi = \rightarrow \partial$). The reader who wishes to check the crucial signs for himself will find it convenient to use Grassmann derivatives “from right to left” as well, noting that $\rightarrow \partial \rightarrow \partial F \rightarrow G = - \bigg( \tilde{\partial} \tilde{\partial} \tilde{F} \tilde{G} \bigg)$.

We shall be only concerned here with the Euclidean time structure of the inverse action kernel $K_{nm}^{-1}$, where $n = (n_1, n_2, n_3, n_4)$ etc. We shall use two different labelings of time slices: $n_4 = 0, \pm 1, \pm 2, \ldots$ as appropriate for the space-time functional-integral approach, and $\tau = \pm 1, \pm 2, \ldots$ which will emphasize later on the correspondence between bra and ket functionals. For later convenience we note that

$$
\tau = n_4 \quad \text{for} \quad n_4 > 0 \quad \tau = n_4 - 1 \quad \text{for} \quad n_4 \leq 0
$$

With respect to the spatial indices, we transform into a basis in which the equal-time part of the action is diagonal. For Kogut-Susskind fermions this involves spatial Fourier transformation and linear combination of the momenta at the edges of the Brillouin zone. Let $p$ denote a suitable (multi)index for the basis vectors. For example, $p$ can be chosen as

$$
p = (p_1, p_2, p_3, \sigma, s, f) \quad p_k \in \left[ -\frac{\pi}{2a_S}, \frac{\pi}{2a_S} \right], \quad \sigma = \pm 1, \quad s = \pm \frac{1}{2}, \quad f = u, d
$$

where $p_k$ are the components of spatial momentum. The rest of the Brillouin zone, reachable by addition of “edge” momenta ($\pm \pi/a, \pm \pi/a, \pm \pi/a$) corresponds to the degrees of freedom of spinorial “energy” sign $\sigma$, spin $s$, and flavour $f$. In the following, $\sigma(p)$ will be of particular interest to us. Using

$$
E(p) = \sqrt{\sum \sin^2 p_k + m^2}
$$

the spinorial “energy” is $\sigma(p)E(p)$.

We can now rewrite the action kernel of equation (1) as

$$
K_{nn'} = \delta_{p,p'} \left( D(\mu)_{n_4,n'_4} + \delta_{n_4,n'_4} \frac{\alpha_T}{a_S} \sigma(p) E(p) \right)
$$

In the following we reconstruct the fermion Fock space, and the operators of energy and baryon number in particular, for two versions of the time derivative.
3 Asymmetric time derivative

We first analyse fermions with a one-step, asymmetric time derivative. This will lead to expectation values (and operators) for the energy and baryon number resembling those familiar from space-time continuum [3].

The time-asymmetric action kernel, corresponding to equation (2.15) of ref. [3], is

\[ K_{nn'} = \frac{aT}{aS} m \delta_{nn'} + \frac{1}{2} \frac{aT}{aS} \sum_{k=1}^{3} \gamma_k (\delta_{n,n'+\hat{k}} - \delta_{n,n'-\hat{k}}) + \gamma_4 (e^{\mu} \delta_{n,n'+\hat{4}} - \delta_{n,n'}) \]

where the Euclidean gamma matrices are characterized by \{\gamma_{\nu}, \gamma_{\nu}'\} = \delta_{\nu\nu'} and \gamma_4 = \gamma_\nu. To reduce the fermion degeneracy, the 4 x 4 matrix structure can be eliminated by first absorbing a \gamma_4 factor into the \psi integration variables and then diagonalizing in spin and restricting to one of the components by analogy to Kawamoto-Smit. Hence let us define \( \alpha_k = i \gamma_4 \gamma_k \) and

\[ \bar{\chi}(n) = \bar{\psi}(n) \gamma_4 \alpha_3 \alpha_2 \alpha_1^{n_1} \bar{\alpha} \quad \chi(n) = \alpha_1^{n_1} \alpha_2^{n_2} \alpha_3^{n_3} \psi(n) \]

With respect to the \( \chi \) and \( \bar{\chi} \) spinors the action kernel is

\[ K_{nn'} = \frac{aT}{aS} m \delta_{nn'} \gamma_4 \Gamma_4(n) + \frac{1}{2} \frac{aT}{aS} \sum_{k=1}^{3} \Gamma_k(n) (\delta_{n,n'+\hat{k}} - \delta_{n,n'-\hat{k}}) + e^{\mu} \delta_{n,n'+\hat{4}} - \delta_{n,n'} \]

where \( \Gamma_\nu(n) = (-1)^{n_1 + \cdots + n_\nu - 1} \). A \( \gamma_4 \) matrix has remained in the mass term but, assuming it to be diagonalized, the spinor components are now decoupled so that we can build a flavour-reduced fermionic system by projecting onto the first component. We note that the equal-time part of this action coincides with Susskind's spatial fermions [3]. After spatial Fourier transformation the asymmetric kernel becomes

\[ K_{nn'} = \delta_{pp'} \left( e^{\mu} \delta_{n_4,n'_4} - \delta_{n_4,n'_4} \right) + \sigma(p) E(p) \frac{aT}{aS} \delta_{n_4,n'_4} \]

To evaluate any scalar product or matrix element, such as \( \langle (\Theta F) B G \rangle \), using (2), we need to know the inverse of the action kernel. Using

\[ \epsilon \equiv \ln \left( 1 - \frac{aT}{aS} \sigma(p) E(p) \right) \]

we obtain

\[ K_{nn'}^{-1} = \begin{cases} -e^{-\epsilon} e^{(\epsilon-\mu)(n_4-n'_4)} & \text{if } n_4 \leq n'_4 \\ 0 & \text{if } n_4 > n'_4 \end{cases} \delta_{pp'} \text{ for } \epsilon > \mu \]  

(4)

\[ K_{nn'}^{-1} = \begin{cases} e^{-\epsilon} e^{(\epsilon-\mu)(n_4-n'_4)} & \text{if } n_4 \leq n'_4 \\ 0 & \text{if } n_4 > n'_4 \end{cases} \delta_{pp'} \text{ for } \epsilon < \mu \]  

(5)

It should be mentioned here that the zeros above will produce zero-norm states later on. Right at \( \epsilon = \mu \) the inverse kernel is unambiguously determined by the antiperiodic boundary condition with respect to time. The appropriate limit is

\[ K^{-1} = \frac{1}{2} \left( K_{\epsilon\rightarrow\mu-0}^{-1} + K_{\epsilon\rightarrow\mu+0}^{-1} \right) \]
It is convenient to adapt operation (2) to the special case of \( O = \bar{F} \cdot G \) with \( \bar{F} \) a negative-time and \( G \) a positive-time functional. (This also covers the case of a matrix element like \( \langle \Theta F BG \rangle \).) Sorting out the derivatives acting on \( \bar{F} \) or \( G \) exclusively, we can write

\[
\exp \sum_{nn'} K_{nn'}^{-1} \frac{\partial}{\partial \bar{\psi}_n} \frac{\partial}{\partial \psi_{n'}} \bar{F} G = C_0 ((C_- \bar{F})(C_+ G))
\]

where the \( C \) stand for “contractions” and

\[
C_\pm = \exp \sum_p \sum_{\tau, \tau' > 0} K_{\tau \tau'}^{-1}(p) \frac{\partial}{\partial \psi_{p \tau}} \frac{\partial}{\partial \bar{\psi}_{p \tau'}}
\]

while the remaining “mixing” term is

\[
C_0 = \exp \sum_{p: \epsilon > \mu} (-e^{-\mu}) \left( \sum_{\tau > 0} e^{(\mu-\epsilon)\tau} \frac{\partial}{\partial \psi_{p,-\tau}} \right) \left( \sum_{\tau' > 0} e^{(\mu-\epsilon)\tau'} \frac{\partial}{\partial \bar{\psi}_{p,-\tau'}} \right) \times
\]

\[
\times \exp \sum_{p: \epsilon < \mu} e^{\mu-2\epsilon} \left( \sum_{\tau > 0} e^{(\epsilon-\mu)\tau} \frac{\partial}{\partial \bar{\psi}_{p,-\tau}} \right) \left( \sum_{\tau' > 0} e^{(\epsilon-\mu)\tau'} \frac{\partial}{\partial \psi_{p,-\tau'}} \right)
\]

The operations \( C_+ \) and \( C_- \) only modify the ket and bra functionals, respectively. The operation intertwining kets and bras is \( C_0 \), and the last step is to discard all remaining \( \psi \)s and \( \bar{\psi} \)s. It will be most convenient to formulate the fermion Hilbert space in terms of \( \bar{F}' = C_- \bar{F} \) and \( G' = C_+ G \), i.e., at the \( C_0 \) stage after the \( C_+ \) and \( C_- \) operations.

We are aiming at operator expressions for the energy (transfer matrix \( \hat{T} \) or Hamiltonian \( \hat{H} \)) and for the baryon number. As \( C_+ \) and \( C_- \) depend on \( \tau - \tau' \) only, they commute with the time evolution. Thus \( \hat{T} \) can be determined at the \( C_0 \) stage immediately; see below. The baryon number operator is most easily identified from the path-integral observable “at \( \tau = 0 \)” (i.e., involving \( \tau = -1 \) and \( \tau = 1 \)) which is

\[
B = \frac{\partial S}{\partial \mu} = \sum_{\vec{n}} e^\mu \bar{\psi}_{\vec{n},-1} \psi_{\vec{n},1} = \sum_{p} e^\mu \bar{\psi}_{p,-1} \psi_{p,1}
\]

In the second equality we changed to the eigenbasis of the spinorial energy, and arranged Grassmann variables in time order. \( \bar{\psi}_{p,-1} \) and \( \psi_{p,1} \) can now be interpreted as multiplication operators acting on bra and ket functionals, respectively. Thus the matrix elements of the baryon number operator are

\[
\langle F|\bar{B}|G \rangle = e^\mu \langle \Theta \bar{\psi}_{p,1} F \rangle \psi_{p,1} G \rangle_{\text{path integral}} = e^\mu C_0 \{C_- (\Theta \bar{\psi}_{p,1} F) C_+ \bar{\psi}_{p,1} G \}
\]

To bring the field operators (multiplication operators) of the above equation to the \( C_0 \) stage, we apply the Baker-Hausdorff formula, obtaining for example the following primed form (with any ket functional \( G \))

\[
C_+ \bar{\psi}_{p,1} G = \left( \psi_{p,1} + \left\{ \begin{array}{ll} 1 & \epsilon > \mu \\ 0 & \epsilon < \mu \end{array} \right\} \right) \times e^{-\mu} \sum_{\tau > 0} e^{(\mu-\epsilon)\tau} \frac{\partial}{\partial \psi_{p,\tau}} \right) G'
\]

\[
(9)
\]
On the bra side, we obtain the \( \Theta \)-reflected expression. In a similar way, by commuting \( \psi_s \) and \( \bar{\psi}_s \) through \( C_0 \) one obtains the adjoints of field operators. For example, we have

\[
C_0 \bar{\psi}_{p,-1} = \left( \bar{\psi}_{p,-1} + \begin{array}{c} 0 \\ 1 \end{array} \begin{array}{c} \epsilon > \mu \\ \epsilon < \mu \end{array} \right) \times e^{- \epsilon} \sum_{\tau > 0} e^{(\epsilon - \mu)\tau} \frac{\partial}{\partial \psi_{p,\tau}} C_0
\]

\[
= \bar{\psi}_{p,-1} C_0 + C_0 \left( \begin{array}{c} 0 \\ 1 \end{array} \begin{array}{c} \epsilon > \mu \\ \epsilon < \mu \end{array} \right) \times e^{- \epsilon} \sum_{\tau > 0} e^{(\epsilon - \mu)\tau} \frac{\partial}{\partial \psi_{p,\tau}}
\]

The \( \bar{\psi}_{p,-1} \) on the RHS is discarded when all \( \psi \) and \( \bar{\psi} \) are set to zero in the final step of the scalar product. The bracketed expression on the RHS is the ket-adjoint to the bra-multiplication by \( \bar{\psi}_{p,-1} \). To further develop our example, this can be used to rewrite \( C_0 \left\{ (\Theta \psi_{p,1} F') \psi_{p,1} G' \right\} \) as

\[
C_0 \left\{ (\Theta F') \psi_{p,1} \psi_{p,1} G' \right\}
\]

which is part of the baryon number operator we are interested in. From an expression such as the last one, in which all operator action is thrown on the kets, one finally abstracts away the functionals and the scalar product operation.

On the \( C_0 \) stage, as we see from (7) and (9), Grassmann derivatives occur only in those linear combinations of time indices which, for each momentum \( p \), correspond to “physical” time dependency. Linear combinations of \( \psi_\tau \) and \( \bar{\psi}_\tau \) orthogonal to it drop out from all matrix elements (they create the null-norm functionals). The only relevant variables are \( \Psi(p) \propto \sum_{\tau > 0} e^{(\epsilon - \mu)\tau} \partial \bar{\psi}_{p,\tau} \) and \( \bar{\Psi}(p) \propto \sum_{\tau > 0} e^{(\mu - \epsilon)\tau} \partial \psi_{p,\tau} \). They are conveniently defined by

\[
\sum_{\tau > 0} e^{(\mu - \epsilon)\tau} \frac{\partial}{\partial \bar{\psi}_{p,\tau}} = \frac{\partial}{\partial \bar{\Psi}(p)} \quad \epsilon > \mu
\]

\[
\sum_{\tau > 0} e^{(\epsilon - \mu)\tau} \frac{\partial}{\partial \psi_{p,\tau}} = \frac{\partial}{\partial \Psi(p)} \quad \epsilon < \mu
\]

so that, in the ket functionals, we have

\[
\bar{\psi}_{p,\tau} = e^{(\mu - \epsilon)\tau} \bar{\Psi}(p) + \text{orth.} \quad \epsilon > \mu
\]

\[
\psi_{p,\tau} = e^{(\epsilon - \mu)\tau} \Psi(p) + \text{orth.} \quad \epsilon < \mu
\]

(The \( \Psi(p) \) and \( \bar{\Psi}(p) \) actually represent equivalence classes.) Omitting the irrelevant “orthogonal” parts we obtain functionals of \( \Psi(p) \) and \( \bar{\Psi}(p) \) only. We note that the \( \Psi \) are non-null only for \( \epsilon < \mu \) (in ket functionals) and the \( \bar{\Psi} \) only for \( \epsilon > \mu \). This will be different for Kogut-Susskind fermions, leading to an additional doubling of species there.

In terms of the new variables, the baryon number (8) for time-asymmetric fermions takes the simple operator form

\[
\hat{B} = \sum_{p: \epsilon > \mu} \bar{\Psi}(p) \frac{\partial}{\partial \bar{\Psi}(p)} + \sum_{p: \epsilon < \mu} \left( 1 - \Psi(p) \frac{\partial}{\partial \Psi(p)} \right)
\]

(11)

At the \( C_0 \) stage, the transfer matrix is also easily identified. If all Grassmann variables in a ket functional are shifted by one time step, equation (11) shows that \( \bar{\Psi}(p) \) picks up a factor of \( e^{\mu - \epsilon} \), and \( \Psi(p) \) a factor of \( e^{\epsilon - \mu} \). The generator of such a transformation is

\[
\hat{H} - \mu \hat{B} = -\log \hat{T} = \sum_{p: \epsilon > \mu} (\epsilon - \mu) \bar{\Psi}(p) \frac{\partial}{\partial \bar{\Psi}(p)} + \sum_{p: \epsilon < \mu} (-\epsilon + \mu) \Psi(p) \frac{\partial}{\partial \Psi(p)}
\]

(12)
We can rewrite this in the standard Fock-space form, using $\epsilon_m = \log(1 + \frac{a_T}{a_S}m)$:

$$\hat{H} - \mu \hat{B} = \sum_{p: \epsilon > \epsilon_m} (\epsilon - \mu) b^*(p)b(p) + \sum_{p: \epsilon > \epsilon_m} (\epsilon + \mu) \bar{b}^*(p)\bar{b}(p) + \sum_{p: \epsilon > \epsilon_m} (\epsilon - \mu) \quad (13)$$

$$\hat{B} = \sum_{p: \epsilon > \epsilon_m} (b^*(p)b(p) - \bar{b}^*(p)\bar{b}(p)) + \sum_{p: \epsilon < -1} 1 \quad (14)$$

where we have used $\epsilon_m = \log(1 + \frac{a_T}{a_S}m)$ and defined creation and annihilation operators for antibaryons (denoting by $\bar{p}$ the multiindex $p$ as in (3) with a reversed sign of spinorial energy) by

$$\bar{b}^*(p) = \Psi(\bar{p}) \quad \bar{b}(p) = \frac{\partial}{\partial \Psi(\bar{p})} \quad \epsilon(p) > \epsilon_m$$

and for baryons by

$$b^*(p) = \left\{ \begin{array}{ll} \Psi(p) & \epsilon > \mu \\ \frac{\partial}{\partial \Psi(p)} & \mu > \epsilon > \epsilon_m \end{array} \right.$$  

$$b(p) = \left\{ \begin{array}{ll} \frac{\partial}{\partial \Psi(p)} & \epsilon > \mu \\ \Psi(p) & \mu > \epsilon > \epsilon_m \end{array} \right.$$  

The $b^*$ and $\bar{b}^*$ satisfy canonical anticommutation relations with $b$ and $\bar{b}$, implying standard identification for number operators. However, for $\epsilon(p)$ or $\mu$ large (comparable to the UV cutoff) they deviate from the hermitian adjoints of $b$ and $\bar{b}$ by a real factor

$$\bar{b}^\dagger(p) = e^{-2\epsilon - \mu} \bar{b}^*(p) \quad \text{for all } \epsilon > \epsilon_m$$  

$$b^\dagger(p) = e^{\mu} b^*(p) \quad \epsilon > \mu \quad \bar{b}^\dagger(p) = e^{\mu - 2\epsilon} b^*(p) \quad \mu > \epsilon > \epsilon_m$$

As we see from (12) the functional representing the ground state $|\mu\rangle$ at chemical potential $\mu$ is $G_0 = 1$. In terms of Fock space operators it is characterized by

$$\bar{b}(p)|\mu\rangle = 0 \quad \text{all } p$$  

$$b(p)|\mu\rangle = 0 \quad \text{for } \epsilon > \mu$$  

$$\bar{b}^\dagger(p)|\mu\rangle = 0 \quad \text{for } \mu > \epsilon > \epsilon_m$$  

Thus the ground state is filled with baryons of all momenta, spins and flavours with energies in the range $\mu > \epsilon(p) > \epsilon_m$.

The zero-point energy in (13) is due to the above definition of $\hat{H} - \mu \hat{B}$ as a plain generator of a temporal translation. Had we defined the baryon number operator as a plain generator of phase rotations, we would have avoided the divergent but $\mu$-independent contribution in (14) which we must now subtract by hand.

It should be noted that definition (13), based on a single time step, is valid only if $\epsilon$ in the above definition $\epsilon = \ln(1 - \frac{a_T}{a_S}\sigma(p)E(p))$ is a real number for all energy-momentum indices $p$. Since the maximal eigenvalue of the free staggered Hamiltonian is $\sqrt{3 + m^2}$, $m$ being the fermion mass, the condition is $a_T < a_S/\sqrt{3 + m^2}$. If the temporal spacing is too coarse to satisfy this, then one can still define a two-step Hamiltonian, $\hat{H} - \mu \hat{B} = -\frac{1}{2} \log \hat{T}^2$ as customary for staggered fermions.
4 Kogut-Susskind time derivative

We now consider the two-step time derivative as it is used with Kogut-Susskind fermions [1, 2]. The action kernel, allowing for a space-time anisotropy, is

\[ K_{nn'} = \frac{aT}{aS} m \delta_{nn'} + \frac{1}{2} \frac{aT}{aS} \sum_{k=1}^{3} \Gamma_k(n)(\delta_{n+k,n'} - \delta_{n-k,n'}) + \Gamma_4(n)(e^\mu \delta_{n+4,n'} - e^{-\mu} \delta_{n-4,n'}) \]

Absorbing a factor of \( \Gamma_4 \) into the \( \bar{\psi} \) integration and Fourier transforming with respect to \( \bar{n} \) we arrive at

\[ K_{nn'} = \delta_{pp'} K_{n_4 n'_4}(p) \]

where \( K_{n_4 n'_4}(p) = \frac{1}{2} \left( e^\mu \delta_{n_4+1,n'_4} - e^{-\mu} \delta_{n_4-1,n'_4} \right) + \sigma(p) E(p) n_4 n'_4 \delta_{n_4 n'_4} \]

Let us restrict to \( \mu \geq 0 \) and put

\[ \epsilon \equiv \text{arsinh} \left( \frac{aT}{aS} \sigma E \right) \]

The inverse kernel \( K^{-1}_{n_4 n'_4}(p) \) then reads

\[ e^{-\mu(n_4-n'_4)} e^{-\epsilon|n_4-n'_4|} \cosh \epsilon \times \left\{ \begin{array}{c} 1 \quad \text{for } n_4 \geq n'_4 \\ (-1)^{n_4-n'_4} \quad \text{for } n_4 \leq n'_4 \end{array} \right\} \epsilon > \mu \]

\[ e^{-\mu(n_4-n'_4)} e^{\epsilon|n_4-n'_4|} \cosh \epsilon \times \left\{ \begin{array}{c} 1 \quad \text{for } n_4 \geq n'_4 \\ (-1)^{n_4-n'_4} \quad \text{for } n_4 \leq n'_4 \end{array} \right\} \epsilon < -\mu \]

\[ e^{-\mu(n_4-n'_4)} e^{-\epsilon|n_4-n'_4|} \cosh \epsilon \times \left\{ \begin{array}{c} 1 \quad \text{for } n_4 > n'_4 \\ 0 \quad \text{for } n_4 \leq n'_4 \end{array} \right\} |\epsilon| < \mu \]

A general remark may be in order here. The inverse kernel consists of positive-time and negative-time solutions of the discretized Dirac equation, suitably patched together at \( n_4 = n'_4 \). In eqn. [2] the index \( n_4 \) always goes with a \( \partial / \partial \tau \). If for a particular solution \( n_4 \) runs over the positive half-line, then \( \psi \) are detected in the ket functional; if \( n_4 \) runs over the negative half-line, the \( \psi \)'s are in the bra functional where they are interpreted as complex-conjugates. Whatever corresponds to a “particle” in one case thus corresponds to an “antiparticle” in the other.

To carry on with the technicalities, we change to time labels \( \tau, \tau' \) and split the summation over time according to [3]. In particular, the intertwining operation analogous to [3] now reads

\[ C_0 = \exp \sum_{p: \epsilon > -\mu} \frac{e^{\epsilon + \mu}}{\cosh \epsilon} \left( \sum_{\tau > 0} e^{-(\epsilon + \mu)\tau} \frac{\partial}{\partial \psi_{p,\tau}} \right) \left( \sum_{\tau' > 0} e^{-(\epsilon + \mu)\tau'} \frac{\partial}{\partial \psi_{p,-\tau'}} \right) \]

\[ \times \exp \sum_{p: \epsilon < -\mu} \frac{e^{\epsilon - \mu}}{\cosh \epsilon} \left( \sum_{\tau > 0} (-1)^\tau e^{(\epsilon - \mu)\tau} \frac{\partial}{\partial \psi_{p,\tau}} \right) \left( \sum_{\tau' > 0} (-1)^{\tau'} e^{(\epsilon - \mu)\tau'} \frac{\partial}{\partial \psi_{p,-\tau'}} \right) \]

\[ \times \exp \sum_{p: \epsilon > \mu} \frac{-e^{-\epsilon - \mu}}{\cosh \epsilon} \left( \sum_{\tau > 0} (-1)^\tau e^{(\epsilon - \mu)\tau} \frac{\partial}{\partial \psi_{p,\tau}} \right) \left( \sum_{\tau' > 0} (-1)^{\tau'} e^{(\epsilon - \mu)\tau'} \frac{\partial}{\partial \psi_{p,-\tau'}} \right) \]
It is interesting to note that the asymptotic form of the calculations, the alternating modes of free fermions do not allow the continuum limit to be taken only at the very end their directions giving degree of freedom. The same kind of combinations have to be considered in the spatial directions giving difficulties. From this we can identify the relevant Grassmann variables at the $C_0$ stage of the scalar product. The procedure is largely the same as with asymmetric fermions. Again, the new variables can be conveniently defined by

$$
\sum_{\tau > 0} e^{-|\epsilon + \mu| \tau} \frac{\partial}{\partial \psi_{p, \tau}} =: \frac{\partial}{\partial \Psi(p, +)} \quad \text{and} \quad \sum_{\tau > 0} e^{-|\epsilon - \mu| \tau} \frac{\partial}{\partial \psi_{p, \tau}} =: \frac{\partial}{\partial \Psi(p, -)}
$$

so that, in the ket functionalss, the Grassmann variables decompose as follows:

- $\epsilon > \mu$ \quad $\tilde{\psi}_{p, \tau} = (-1)^\tau e^{(\mu - \epsilon)\tau} \Psi(p, -) + \text{orth.}$ \quad $\psi_{p, \tau} = e^{-(\epsilon + \mu)\tau} \Psi(p, +) + \text{orth.}$
- $|\epsilon| < \mu$ \quad $\tilde{\psi}_{p, \tau} = 0 + \text{orth.}$ \quad $\psi_{p, \tau} = e^{-(\epsilon - \mu)\tau} \Psi(p, +) + (-1)^\tau e^{(\epsilon - \mu)\tau} \Psi(p, -) + \text{orth.}$
- $\epsilon < -\mu$ \quad $\tilde{\psi}_{p, \tau} = e^{(\epsilon + \mu)\tau} \tilde{\Psi}(p, +) + \text{orth.}$ \quad $\psi_{p, \tau} = (-1)^\tau e^{(\epsilon - \mu)\tau} \tilde{\Psi}(p, -) + \text{orth.}$

It is interesting to note that the limit $a_T \to 0$ can be taken only after these transformations when the $(-1)^\tau$ time-dependency mode has “transmuted” into an internal degree of freedom. The same kind of combinations have to be considered in the spatial directions giving their flavour degeneracy. However, as one is ultimately interested in interacting fermions which allow the continuum limit to be taken only at the very end of the calculations, the alternating modes of free fermions do not a priori signal any difficulties.

The baryon number as a path-integral observable \[1, 2\] on the zero-time slice is

$$
B = \frac{\partial S}{\partial \mu} = \sum_p \left( e^\mu \tilde{\psi}_{p, -1} \psi_{p, 1} + e^{-\mu} \tilde{\psi}_{p, 1} \psi_{p, -1} \right)
$$

An intermediate result for the baryon number operator at $C_0$ level is

$$
\begin{align*}
\hat{B} &= \frac{1}{2} e^\mu \sum_{p: \epsilon > \mu} \left( \frac{1}{\cosh \epsilon} \frac{\partial}{\partial \Psi(p, +)} + \tilde{\Psi}(p, -) \right) \left( e^{-\epsilon - \mu} \Psi(p, +) + \frac{e^{\epsilon - \mu}}{\cosh \epsilon} \frac{\partial}{\partial \Psi(p, -)} \right) + \\
&- \frac{1}{2} e^{-\mu} \sum_{p: \epsilon > \mu} \left( \frac{1}{\cosh \epsilon} \frac{\partial}{\partial \Psi(p, +)} + \Psi(p, +) \right) \left( -e^{\epsilon - \mu} \Psi(p, -) + \frac{e^{\epsilon + \mu}}{\cosh \epsilon} \frac{\partial}{\partial \Psi(p, +)} \right) + \\
&+ \frac{1}{2} e^\mu \sum_{p: |\epsilon| < \mu} \left( \frac{1}{\cosh \epsilon} \frac{\partial}{\partial \Psi(p, +)} - \frac{1}{\cosh \epsilon} \frac{\partial}{\partial \Psi(p, -)} \right) \left( e^{-\epsilon - \mu} \Psi(p, +) - e^{-\epsilon \mu} \Psi(p, -) \right) - \\
&- \frac{1}{2} e^{-\mu} \sum_{p: |\epsilon| < \mu} \left( \Psi(p, +) + \Psi(p, -) \right) \left( \frac{e^{\epsilon + \mu}}{\cosh \epsilon} \frac{\partial}{\partial \Psi(p, +)} + \frac{e^{\epsilon - \mu}}{\cosh \epsilon} \frac{\partial}{\partial \Psi(p, -)} \right) + \\
&+ \frac{1}{2} e^\mu \sum_{p: \epsilon < -\mu} \left( -\frac{1}{\cosh \epsilon} \frac{\partial}{\partial \Psi(p, -)} + \tilde{\Psi}(p, +) \right) \left( -e^{\epsilon - \mu} \Psi(p, -) + \frac{e^{\epsilon - \mu}}{\cosh \epsilon} \frac{\partial}{\partial \Psi(p, +)} \right)
\end{align*}
$$
The transfer matrix of Kogut-Susskind fermions is known to be

\[ B = e^{-\mu} \sum_{p: \epsilon < -\mu} \left( \frac{1}{\cosh \epsilon} \frac{\partial}{\partial \Psi(p, +)} - \Psi(p, -) \right) \left( e^{\mu + \epsilon} \Psi(p, +) + e^{\mu - \epsilon} \frac{\partial}{\partial \Psi(p, -)} \right) \]

This actually simplifies to

\[
\hat{B} = \sum_{p: \epsilon > \mu} \left( \Psi(p, -) \frac{\partial}{\partial \Psi(p, -)} - \Psi(p, +) \frac{\partial}{\partial \Psi(p, +)} \right) + \sum_{p: \epsilon < -\mu} \left( \Psi(p, +) \frac{\partial}{\partial \Psi(p, +)} - \Psi(p, -) \frac{\partial}{\partial \Psi(p, -)} \right) + \sum_{p: |\epsilon| < \mu} \left( 1 - \Psi(p, -) \frac{\partial}{\partial \Psi(p, -)} - \Psi(p, +) \frac{\partial}{\partial \Psi(p, +)} \right)
\]

The transfer matrix of Kogut-Susskind fermions is known to be not of the preferable form \( e^{-\hat{H}} \) with \( \hat{H} \) hermitian \([10]\). Nevertheless one can define a quantum field theory using the two-step Hamiltonian \( \tilde{H} = -\frac{1}{2} \log T^2 \). This is hermitian whenever \( \tilde{T} \) is. Thus we obtain, generalizing to \( \mu > 0 \),

\[
\hat{H} - \mu \hat{B} = \sum_{p: \epsilon > \mu} \left( (\epsilon + \mu) \Psi(p, +) \frac{\partial}{\partial \Psi(p, +)} + (\epsilon - \mu) \Psi(p, -) \frac{\partial}{\partial \Psi(p, -)} \right)
\]

The corresponding baryon and antibaryon operators are defined as follows:

\[
\bar{b}(p, \eta) = \partial/\partial \Psi(\eta p, \eta) \quad \bar{b}^*(p, \eta) = \Psi(\eta p, \eta) \quad \epsilon(p) > \epsilon_m
\]

The corresponding baryon operators are

\[
b(p, \eta) = \partial/\partial \Psi(-\eta p, \eta) \quad b^*(p, \eta) = \Psi(-\eta p, \eta) \quad \epsilon(p) > \mu
\]

We then have

\[
\hat{B} = \sum_{p: \epsilon > \epsilon_m} \sum_{\eta} (b^*(p, \eta) b(p, \eta) - \bar{b}^*(p, \eta) \bar{b}(p, \eta))
\]

and

\[
\hat{H} - \mu \hat{B} = \sum_{p: \epsilon > \epsilon_m} \sum_{\eta} ((\epsilon - \mu) b^*(p, \eta) b(p, \eta) + (\epsilon + \mu) \bar{b}^*(p, \eta) \bar{b}(p, \eta)) + \sum_{p: \mu > \epsilon > \epsilon_m} 2(\epsilon - \mu)
\]

We note that creation and annihilation operators are only defined for multi-indices \( p \) with a positive energy. In terms of these operators the Hamiltonian and baryon number take the standard form of baryons and antibaryons with an extra two-component flavour.
The chemical potential $\mu > 0$ decreases the energy of baryons and increases it for antibaryons, as expected.

As in the asymmetric case, the $b^*$ and $\bar{b}^*$ at large $\epsilon$ or $\mu$ (comparable to the UV cutoff) deviate from the hermitian adjoints of $b$ and $b$:

$$b(p, \eta) = e^{-\epsilon - \mu} \cosh \epsilon \bar{b}^*(p, \eta) \quad \epsilon > \epsilon_m$$

$$b^T(p, \eta) = e^{-\epsilon + \mu} \cosh \epsilon b^*(p, \eta) \quad \epsilon > \mu \quad b^T(p, \eta) = \frac{e^{-\epsilon + \mu}}{\cosh \epsilon} b^*(p, \eta) \quad \mu > \epsilon > \epsilon_m$$

where $\epsilon = \epsilon(p)$. The hermiticity of $\hat{H}$ is not affected by those relations.

As with asymmetric fermions, the ground state functional is $G_0[\Psi, \bar{\Psi}] = 1$. In terms of creation and annihilation operators this implies the same relations as in (15) for both of the time-like flavours. The ground state has a nonzero baryon density if $\sinh \mu > a^T a^S m$. Thus, again, $\mu$ and $m$ are related in a nonlinear way, reminding us of a degree of ambiguity in the quantification of chemical potentials on the lattice [16, 17].

5 Conclusions

We have elaborated on how the sign of the “energy” of Grassmann functional integral variables is associated with the Fock-space degrees of freedom in the two cases of Kogut-Susskind and time-asymmetric lattice fermions; the central equations in this respect are (10) and (17). Our main conclusion is that the usual implementation of a chemical potential for the baryon number, namely through real-valued propagation factors in the discretized time derivative for any kind of fermions, will indeed enhance or deplete the baryons and antibaryons just as it is supposed to do.

It is quite plausible that time evolution with Kogut-Susskind fermions should involve some peculiarities. In continuous Euclidean time, the positivity of the transfer matrix is guaranteed already by the basic requirement $\langle F|F \rangle \geq 0$ [11, 8]. By contrast, the Kogut-Susskind one-step transfer matrix has both positive and negative eigenvalues at any time-like lattice spacing [10] and is therefore usually regarded as involving both time-evolution and flavour rotation [13]. In fact, the hermiticity of the energy in this case is accomplished by simply discarding the sign of the negative eigenvalues of $\hat{T}$, namely in defining $\hat{H} = -\frac{1}{2} \log \hat{T}^2$ instead of $\hat{H} = -\log \hat{T}$. The fact that this leads to a sound quantum field theory in principle has been well-known [10]. In the practice of Monte Carlo simulations, when dealing with propagators of particles composed of Kogut-Susskind fermions, a contribution of “odd parity” states (involving $(-1)^\tau$) is therefore fitted to a free Kogut-Susskind particle with the same flavour rotation and time evolution.

Complications arising from the nonpositivity of the transfer matrix are avoided with Wilson fermions [3], and also with the asymmetric fermions of section 3, provided that the temporal lattice spacing is sufficiently smaller than the spatial one. For non-interacting fermions the condition is $a_T < a_S/\sqrt{3 + m^2}$; similar bounds exist for fermions interacting with a compact gauge field.

In computational practice, and related analytical study such as [2, 16, 5], the energy expectation value $\epsilon$ of lattice fermions at finite temperature is not derived from
the logarithm of the transfer matrix, $\varepsilon = \text{tr} \hat{H} e^{-\hat{H} N_{\beta}}$, but by differentiating the action with respect to the temporal lattice spacing. In terms of Fock space operators this can be obtained from $\varepsilon = \text{tr} \hat{E} T^{N_{\beta}} = \text{tr} \hat{E} e^{-\hat{H} N_{\beta}}$ where $\hat{E}$ is the operator reconstructed from the path-integral observable $\sum_p \bar{\psi}_{p,1} E(p) \psi_{p,1}$ for one-step fermions, or $\frac{1}{2} \sum_p (\bar{\psi}_{p,-1} E(p) \psi_{p,-1} + \bar{\psi}_{p,1} E(p) \psi_{p,1})$ for Kogut-Susskind fermions. In fact, $\hat{E}$ and $\hat{H}$ are different only for momenta comparable to the uv cutoff so that the use of $\hat{E}$ to obtain an energy density is certainly justified by its computational advantages.

It is reassuring to find that particles and antiparticles are safely recognized by their internal transformation properties (of which we have considered the baryon number) and not by any kind of spinorial energy. The latter would be an ill-defined concept, indeed, in the presence of dynamical gauge fields. The Hamiltonian of interacting fermions can only be reconstructed as part of the complete, translation-invariant fermion-boson field theory. Naturally, such a reconstruction had to emphasize local terms (as it did in [7, 8, 10]) rather than global diagonalizations.

As the time-asymmetric action could provide an exact algorithm with two flavours, it will be interesting to study it further in the SU(3) interactive case, in concordance with our view that the early onset problem could be due to the excessive proliferation of degenerate flavours.

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