On Some Numerical Integration Formulas on the $d$-Dimensional Simplex

Filomena Di Tommaso and Benaissa Zerroudi

Abstract. In this paper, we consider the problem of the approximation of the integral of a function $f$ over a $d$-dimensional simplex $S$ of $\mathbb{R}^d$ by some quadrature formulas which use only the functional and derivative values of $f$ on the boundary of the simplex $S$ or function data at the vertices of $S$, at points on its facets and at its center of gravity. The quadrature formulas are computed by integrating over $S$ a polynomial approximant of $f$ which uses functional and derivative values at the vertices of $S$.

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1. Introduction

The problem of the determination of quadrature rules for triangles, tetrahedra and, in general, for $d$-dimensional simplicial domains has reached the attention of a number of scholars starting from the middle of the nineteenth century up today (see [13] and the references therein). Although many papers focus on quadrature rules for triangles [3,10,12,16], only a limited literature is available on the integration in three or higher dimensions [4,14,17]. In this paper, we approach the problem of integration over general $d$-dimensional simplices by special type integration formulas which use functional and derivative values of the integrand function $f$ mainly on points on the boundary of the $d$-dimensional simplex $S$. When the nodes lie only on the boundary of $S$ these formulas are called boundary type quadrature formulas and are used when the values of $f$ and its derivatives inside the simplex are not given or are not easily determinable. Applications of these formulas can be realized

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in the framework of the numerical solution of boundary value problems of partial differential equations (see [9] and the references therein).

To reach our goal, we follow the approach proposed in Refs. [1,2]. More precisely, we approximate the integrand function \( f \) with a polynomial interpolant \( L^S_r[f](\mathbf{x}) \) which uses functional and derivative data values up to a fixed order \( r \in \mathbb{N} \) at the vertices of \( S \), i.e.

\[
f(\mathbf{x}) = L^S_r[f](\mathbf{x}) + R^S_r[f](\mathbf{x}), \quad \mathbf{x} \in S,
\]

and then, we integrate both sides of (1.1) over the \( d \)-dimensional simplex \( S \) to get the quadrature formula

\[
\int_S f(\mathbf{x})d\mathbf{x} = Q^S_r[f] + E^S_r[f]
\]

where

\[
Q^S_r[f] = \int_S L^S_r[f](\mathbf{x})d\mathbf{x} \quad \text{and} \quad E^S_r[f] = \int_S R^S_r[f](\mathbf{x})d\mathbf{x}.
\]

The obtained quadrature formula (1.2) uses function and derivative data up to the order \( r \) at the vertices of \( S \) and has degree of exactness \( 1 + r \), i.e. \( E^S_r[f] = 0 \) whenever \( f \) is a polynomial in \( d \) variables of total degree at most \( 1 + r \). The main feature of the quadrature formula \( Q^S_r[f] \) is that it relies only on function and derivative data up to the order \( r \) at the vertices of \( S \). This motivates us to look for approximations of those derivatives to obtain quadrature formulas which do not use any derivative data. To this end, we restrict to the case \( r = 1 \) and, by following the technique described in Ref. [6], we approximate the first order derivative data by three-point finite differences approximation. According to the choice of the approximation of the derivative data, we get different quadrature formulas with degree of exactness 2 and, to increase the degree of exactness of such formulas, we consider the convex combination of two of them to get a quadrature formula with degree of exactness 3 (see Sect. 3). Finally, we restrict to the two-dimensional case (see Sect. 4) and we provide numerical results to test the approximation accuracies of the proposed formulas (see Sect. 5).

2. A Quadrature Formula on the \( d \)-Dimensional Simplex

2.1. Preliminaries and Notations

Let \( S \subset \mathbb{R}^d \) be a not degenerated \( d \)-dimensional simplex with vertices \( \mathbf{v}_0, \ldots, \mathbf{v}_d \in \mathbb{R}^d \) and

\[
A(S) = \begin{vmatrix}
1 & \mathbf{v}_0 \\
1 & \mathbf{v}_1 \\
1 & \vdots \\
1 & \mathbf{v}_d
\end{vmatrix}
\]

the signed volume of the hyperparallelepiped with vertices \( \mathbf{v}_0, \ldots, \mathbf{v}_d \). For a point \( \mathbf{x} \in S \) and for each \( l = 0, 1, \ldots, d \) we denote by \( S_l(\mathbf{x}) \) the \( d \)-dimensional
simplex of vertices $v_0, \ldots, v_{l-1}, x, v_{l+1}, \ldots, v_d$. The barycentric coordinates of $x$ with respect to the simplex $S$ are then defined by

$$\lambda_l(x) = \frac{A(S_l)}{A(S)}, \quad l = 0, 1, \ldots, d. \quad (2.1)$$

For each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, as usual, we denote by

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d, \quad \alpha! = \alpha_1! \cdots \alpha_d!, \quad x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}. \quad (2.2)$$

We also set $\lambda = (\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_d)$ and $\lambda_l = (\lambda_0, \ldots, \lambda_{l-1}, \lambda_{l+1}, \ldots, \lambda_d)$ for each $l = 0, \ldots, d$. Moreover, we denote by $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}$ and by

$$D^k_{x-v} f = \sum_{|\alpha|=k, \alpha \in \mathbb{N}^d} \frac{k!}{\alpha!} (x-v)^\alpha D^\alpha f \quad (2.2)$$

the $k$-th order directional derivative of $f$ along the line segment between $x$ and $v$. Finally, we use the notations

$$D_{ij} f = (v_i - v_j) \cdot \nabla f, \quad i, j = 0, 1, \ldots, d \quad (2.3)$$

for the derivative of $f$ along the directed line segment from $x_j$ to $x_i$ (as usual, $\cdot$ denotes the dot product) and

$$D^\alpha_l = D^\alpha_{0,l} D^\alpha_{1,l} \cdots D^\alpha_{l-1,l} D^\alpha_{l+1,l} \cdots D^\alpha_{d,l}, \quad l = 0, 1, \ldots, d, \quad (2.4)$$

for the composition of derivatives along the directed sides of the simplex. Under these assumptions, we get the following result.

**Lemma 2.1.** Let $f \in C^r(S)$, then

$$D^k_{x-v} f(v_l) = \sum_{|\alpha|=k, \alpha \in \mathbb{N}^d} \frac{k!}{\alpha!} D^\alpha_l f(v_l) \lambda_l^\alpha(x), \quad l = 0, 1, \ldots, d, \quad (2.5)$$

for any $k \in \mathbb{N}, k \leq r$ and $x \in \mathbb{R}^d$.

**Proof.** Due to the properties satisfied by the barycentric coordinates, for any $l = 0, 1, \ldots, d$, we have

$$x - v_l = v_l \lambda_l(x) - v_l + \sum_{j=0}^d v_j \lambda_j(x)$$

$$= v_l (\lambda_l(x) - 1) + \sum_{j=0}^d v_j \lambda_j(x)$$

$$= -v_l \left( \sum_{j=0}^d \lambda_j(x) \right) + \sum_{j=0}^d v_j \lambda_j(x)$$

$$= \sum_{j=0}^d \lambda_j(x) v_j - \sum_{j \neq l} \lambda_j(x) v_j = \lambda_l(x) v_l.$$
\[ = \sum_{j=0}^{d} (v_j - v_l) \lambda_j(x) \]
\[ = (v_0 - v_l, \ldots, v_{l-1} - v_l, v_{l+1} - v_l, \ldots, v_d - v_l) \cdot \lambda_l(x). \quad (2.6) \]

By substituting (2.6) in (2.2) and by definition (2.4), we have
\[
D^k \frac{x - v_l}{f(v_l)} = \sum_{|\alpha| = k} \frac{k!}{\alpha!} \left( (v_0 - v_l, \ldots, v_{l-1} - v_l, v_{l+1} - v_l, \ldots, v_d - v_l) \cdot \lambda_l(x) \right) \lambda_l(x)^{\alpha}. \quad (2.6)
\]

Proposition 2.2. Let \( S \subset \mathbb{R}^d \) be a not degenerated \( d \)-dimensional simplex with vertices \( v_0, \ldots, v_d \) then
\[
\int_S \lambda^\alpha(x) \, dx = \frac{A(S) \alpha!}{(d + |\alpha|)!}, \quad \alpha \in \mathbb{N}^{d+1}. \quad (2.7)
\]
Proof. See [15, Theorem 2.2]. \( \square \)

Proposition 2.3. Let \( S \subset \mathbb{R}^d \) be a not degenerated \( d \)-dimensional simplex with vertices \( v_0, v_1, \ldots, v_d \). For any \( x \in S \) and \( r \in \mathbb{N} \), we have
\[
\int_S \sum_{i=0}^{d} \|x - v_i\|^{r+2} \lambda_i(x) \, dx \leq \frac{A(S)}{(d + 2)!} \sum_{i=0}^{d} \sum_{j=0}^{d} \|v_i - v_j\|^{r+2}. \quad (2.8)
\]
Proof. By the equality (2.6) and by recalling that \( 0 \leq \lambda_j(x) \leq 1 \) for each \( x \in S \), we easily obtain
\[
\|x - v_l\|^{r+2} \leq \left( \sum_{j=0}^{d} \|v_j - v_l\| \lambda_j(x) \right)^{r+2} \leq \sum_{j=0}^{d} \|v_j - v_l\|^{r+2} \lambda_j(x)
\]
for any \( l = 0, \ldots, d \). Consequently,
\[
\int_S \sum_{i=0}^{d} \|x - v_i\|^{r+2} \lambda_i(x) \, dx \leq \int_S \sum_{i=0}^{d} \sum_{j=0}^{d} \|v_j - v_i\|^{r+2} \lambda_j(x) \lambda_i(x) \, dx
\]
\[
\leq \sum_{i=0}^{d} \sum_{j=0}^{d} \|v_j - v_i\|^{r+2} \lambda_j(x) \lambda_i(x) \, dx
\]
and, by Proposition 2.2, we easily get the inequality (2.8). \( \square \)
2.2. Construction of the Quadrature Formula

The multivariate Lagrange interpolation polynomial on the simplex $S$ in barycentric coordinates is

$$L^S[f](x) = \sum_{l=0}^{d} \lambda_l(x) f(v_l). \quad (2.9)$$

The operator $L^S$ reproduces polynomials up to the degree 1 and interpolates the values of $f$ at the vertices $v_l$ of the simplex $S$. If the function $f$ belongs to $C^r(S)$, we can replace the values $f(v_l)$ by the modified Taylor polynomial of degree $r$ at $v_\ell$ proposed in [6], the resulting polynomial operator is

$$L^S_r[f](x) = \sum_{l=0}^{d} \left( \sum_{k=0}^{r} \frac{a_{r,k}}{k!} D^k_{x-v_l} f(v_l) \right) \lambda_l(x), \quad x \in \mathbb{R}^d, \quad (2.10)$$

where $a_{r,k} = \frac{(1+r-k)!}{(1+r)!(r-k)!}$. As specified in [6], the operator $L^S_r[f](x)$ reproduces polynomials up to the degree max $\{1, 1+r\} = 1+r$. Moreover, for each $x \in S$ and $f \in C^{r+2}(S)$, its remainder term $R^S_r[f](x) = f(x) - L^S_r[f](x)$ can be explicitly represented as

$$R^S_r[f](x) = \sum_{l=0}^{d} \lambda_l(x) \int_0^1 \frac{t(1-t)^r}{(1+r)!} D^{r+2}_{x-v_l} f(v_l + t(x - v_l)) \, dt. \quad (2.11)$$

**Remark 2.4.** Since $S$ is a compact convex domain and $L^S$ is a linear bounded operator, in line with [8], $L^S_r$ can be interpreted as

$$L^S_r[f](x) = L^S \left[ \sum_{k=0}^{r} \frac{a_{r,k}}{k!} D^k_{x}. f \right](x)$$

and, from [8, Proposition 3.4], it follows that $L^S_r$ inherits the interpolation properties of the Lagrange operator (2.9).

To obtain the desired quadrature formula, we rearrange polynomial (2.10) by taking into account Lemma 2.1. More precisely,

$$L^S_r[f](x) = \sum_{l=0}^{d} \left( \sum_{k=0}^{r} \frac{a_{r,k}}{k!} \sum_{|\alpha|=k}^{1+r} \frac{1}{\alpha!} D^\alpha_{x} f(v_l) \lambda^\alpha_l(x) \right) \lambda_l(x)$$

$$= \sum_{l=0}^{d} \left( \sum_{k=0}^{r} \frac{(1+r-k)!}{(1+r)!(r-k)!} \sum_{|\alpha|=k}^{1+r} \frac{1}{\alpha!} D^\alpha_{x} f(v_l) \lambda^\alpha_l(x) \right) \lambda_l(x)$$

$$= \sum_{l=0}^{d} \left( \sum_{k=0}^{r} \frac{1+r-k}{1+r} \sum_{|\alpha|=k}^{1+r} \frac{1}{\alpha!} D^\alpha_{x} f(v_l) \lambda^\alpha_l(x) \right) \lambda_l(x) \quad (2.12)$$
and, by the change of dummy index, we get

\[ L^S_r[f](x) = \sum_{k=0}^{r} \frac{1 + r - k}{1 + r} \left( \sum_{l=0}^{d} \sum_{|\alpha| = k \atop \alpha \in \mathbb{N}^d} \frac{1}{\alpha!} D^\alpha_l f(v_l) \lambda_l^\alpha(x) \lambda_l(x) \right). \]  

(2.13)

The quadrature formula is then computed by integrating the right hand side of (2.13) on the simplex \( S \).

**Theorem 2.5.** Let \( f \in C^{r+2}(S) \). Then

\[ \int_S f(x) \, dx = Q^S_r[f] + E^S_r[f], \]

where

\[ Q^S_r[f] = \frac{A(S)}{(1 + r)} \sum_{k=0}^{r} \frac{1 + r - k}{(d + 1 + k)!} \left( \sum_{l=0}^{d} \sum_{|\alpha| = k \atop \alpha \in \mathbb{N}^d} D^\alpha_l f(v_l) \right) \]  

(2.14)

and

\[ E^S_r[f] = \int_S R^S_d[f](x) \]

\[ = \int_S \sum_{l=0}^{d} \left( \int_0^1 \frac{1 - t(1 - t)^r}{(1 + r)!} D^r_{x-v_l} f(v_l + t(x-v_l)) \, dt \right) \lambda_l(x) \, dx. \]  

(2.15)

Moreover, the quadrature formula \( Q^S_r[f] \) has degree of exactness \( 1 + r \).

**Proof.** By integrating the right hand side of equality (2.13), we get

\[ Q^S_r[f](x) = \int_S L^S_d[f](x) \, dx \]

\[ = \sum_{k=0}^{r} \frac{1 + r - k}{(1 + r)} \left( \sum_{l=0}^{d} \sum_{|\alpha| = k \atop \alpha \in \mathbb{N}^d} \frac{1}{\alpha!} D^\alpha_l f(v_l) \int_S \lambda_l^\alpha(x) \lambda_l(x) \, dx \right). \]  

(2.16)

By Proposition (2.2)

\[ \int_S \lambda_l^\alpha(x) \lambda_l(x) \, dx = \frac{A(S)\alpha!}{(d + 1 + |\alpha|)!} \]

and then (2.16) becomes

\[ Q^S_r[f] = \sum_{k=0}^{r} \frac{1 + r - k}{(1 + r)} \left( \sum_{l=0}^{d} \sum_{|\alpha| = k \atop \alpha \in \mathbb{N}^d} \frac{1}{\alpha!} D^\alpha_l f(v_l) \frac{A(S)\alpha!}{(d + 1 + |\alpha|)!} \right). \]  

(2.17)
The expression of $E_r^S[f]$ is obtained by integrating on the simplex $S$ the remainder term $R_r^S[f](x)$ in formula (2.11). Since $R_r^S[f](x)$ vanishes whenever $f$ is a polynomial in $d$ variables of total degree at most $1 + r$, $E_r^S[f]$ inherits this property and the quadrature formula has degree of exactness $1 + r$. □

2.3. Error Bounds

To give a bound for the remainder term $E_r^S[f]$ of the quadrature formula in Theorem 2.5, we need some additional notations. More precisely, for a $k$-times continuous differentiable function $f : S \rightarrow \mathbb{R}$, we introduce the norm

$$|D^k f|_S := \sup_{x \in S} \{ |D^k f(x)| : y \in \mathbb{R}^d, \|y\| = 1 \}. \tag{2.18}$$

where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^d$, and $y$ is assumed to be a column vector. Consequently, for any $x \in S$ and $y \in \mathbb{R}^d$, we have

$$|D^k y f(x)| \leq |D^k f|_S \| y \|^{r+2}. \tag{2.19}$$

Proposition 2.6. Let $S \subset \mathbb{R}^d$ be a not degenerated $d$-dimensional simplex with vertices $v_0, \ldots, v_d$ and $f \in C^{r+2}(S)$. Then

$$|E_r^S[f]| \leq \frac{|D^{r+2} f|_S}{(r+2)! (r+1)(d+2)!} A(S) \sum_{l=0}^{d} \sum_{j=0}^{d} \| v_l - v_j \|^{r+2}. \tag{2.20}$$

Proof. By taking the modulus of both sides of equality (2.15), by applying the triangular inequality and by bounding the directional derivative of $f$ of order $r + 2$ by (2.19), we have

$$|E_r^S[f]| \leq \left| \int_S \sum_{l=0}^{d} \left( \int_0^1 \frac{t(1-t)^r}{(1+r)!} D^{r+2} f_{v_l, v_l}(x) \cdot (x - v_l) \, dx \right) \lambda_l(x) \, dx \right|$$

$$\leq |D^{r+2} f|_S \int_S \sum_{l=0}^{d} \left( \int_0^1 \frac{t(1-t)^r}{(1+r)!} \| x - v_l \|^{r+2} \, dt \right) \lambda_l(x) \, dx$$

$$\leq \frac{|D^{r+2} f|_S}{(1+r)!} \int_S \sum_{l=0}^{d} \| x - v_l \|^{r+2} \left( \int_0^1 t(1-t)^r \, dt \right) \lambda_l(x) \, dx. \tag{2.21}$$

Using the inequality in Proposition (2.3), and by the fact that

$$\int_0^1 t(1-t)^r \, dt = \frac{1}{(r+2)(r+1)},$$

we have

$$|E_r^S[f]| \leq \frac{|D^{r+2} f|_S}{(1+r)! (r+1)(r+2)} A(S) \sum_{l=0}^{d} \sum_{j=0}^{d} \| v_l - v_j \|^{r+2}$$

and then (2.20). □

Remark 2.7. It is worth noting that Theorem 2.5 gives a quadrature formula obtained by integrating both sides of the expression in [6, Theorem 1] and the bound in Proposition 2.6 is nothing but the integral of the bound given in [6, Theorem 2], where $\Omega = S$, $m = 1$ and $\phi_i(x) = \lambda_i(x)$. Consequently, the bound (2.20) is the best possible estimation for each $r \in \mathbb{N}_0$. 

3. Integration Formulas on the Simplex with Only Function Data

The main feature of the quadrature formula (2.14) is that it uses only derivatives of \( f \) along the edges of \( S \); this motivates to consider approximations of those derivatives to obtain quadrature formulas which use the function data at the vertices of the simplex \( S \), at points on its facets or at its center of gravity. To this aim, we focus on the case \( r = 1 \) and we consider different kinds of approximation of the derivatives in (2.14). For \( r = 1 \), the quadrature formula (2.14) in Theorem 2.5 becomes

\[
Q^S_1[f] = \frac{A(S)}{(d+1)!} \sum_{l=0}^{d} d f(v_l) + \frac{A(S)}{2(d+2)!} \sum_{l=0}^{d} \sum_{|\alpha|=1}^{d} D_{l}^{\alpha} f(v_l). 
\]  

(3.1)

Proposition 3.1. Let \( f : S \rightarrow \mathbb{R} \) be a 3-times continuous differentiable function on \( S \), then

\[
\tilde{Q}^S_1[f] = \frac{A(S)}{(d+2)!} \left( (3d+2) \sum_{l=0}^{d} f(v_l) - 4 \sum_{l=0}^{d-1} \sum_{r=l+1}^{d} f \left( \frac{v_r + v_l}{2} \right) \right) 
+ \frac{A(S)}{2(d+2)!} \sum_{l=0}^{d-1} \sum_{r=l+1}^{d} \varepsilon_{l,r}. 
\]  

(3.2)

where

\[
\varepsilon_{l,r}[f] = \quad D_{v_l - v_r}^3 f(v_r + \xi_1(v_l - v_r)) 
\quad - D_{v_l - v_r}^3 f(v_l + \xi_2(v_l - v_r)), \quad \xi_1, \xi_2 \in [0, 1].
\]

Moreover, the quadrature formula (3.2) has degree of exactness 2.

Proof. By definition (2.4), the sum of first-order derivatives in the second term of \( Q^S_1[f] \) can be rewritten as

\[
\sum_{l=0}^{d} \sum_{\alpha \in \mathbb{N}^d} D_{l}^{\alpha} f(v_l) = \sum_{l=0}^{d} \sum_{r=0}^{d} D_{l}^{\alpha} f(v_l) = \sum_{l=0}^{d-1} \sum_{r=l+1}^{d} (D_{l}^{\alpha} f(v_r) - D_{l}^{\alpha} f(v_l)), 
\]  

(3.3)

where the differences of directional derivatives along the edges of the simplex \( S \) can be replaced by a three-point finite difference approximation. To do this, let us recall that for a univariate function \( g \), it is possible to consider the derivation formula

\[
g'(a - h) = \frac{1}{h} \left( -\frac{1}{2} g(a + h) + 2g(a) - \frac{3}{2} g(a - h) \right) + \frac{h^2}{3} g'''(\xi) 
\]  

(3.4)

for some \( \xi \in [a - h, a + h] \). Using this formula with \( h = \pm 1/2 \) and \( a = 1/2 \) we get a three-point approximation for \( g'(0) - g'(1) \) with a remainder term
which is expressed in terms of the modulus of continuity of $g''''$ [6, Section 5.1]. By applying the formula (3.4) along the edges of $S$ we get

$$D_{lr} f(v_r) - D_{lr} f(v_l) = 4 \left( f(v_r) - 2 f \left( \frac{v_r + v_l}{2} \right) + f(v_l) \right) + \varepsilon_{l,r} \quad (3.5)$$

with

$$|\varepsilon_{l,r}| \leq \frac{1}{12} \omega \left( D^3 f \left( (1 - t) v_l + t v_r \right), 1 \right),$$

where $\omega$ denotes the modulus of continuity with respect to $t \in [0,1]$. By substituting expression (3.5) in (3.3) and by rearranging, we get

$$\sum_{l=0}^{d} \sum_{|\alpha|=1}^{d} D^{|\alpha|}_l f(v_l) = 4d \sum_{l=0}^{d} f(v_l) - 8 \sum_{l=0}^{d-1} \sum_{r=l+1}^{d} f \left( \frac{v_r + v_l}{2} \right) + \sum_{l=0}^{d-1} \sum_{r=l+1}^{d} \varepsilon_{l,r}. \quad (3.6)$$

Finally, by substituting (3.6) in (3.1), we get

$$\tilde{Q}^S_1[f] = \frac{A(S)}{(d+2)!} \left( (3d + 2) \sum_{l=0}^{d} f(v_l) - 4 \sum_{l=0}^{d-1} \sum_{r=l+1}^{d} f \left( \frac{v_r + v_l}{2} \right) \right)$$

$$+ \frac{A(S)}{2(d+2)!} \sum_{l=0}^{d-1} \sum_{r=l+1}^{d} \varepsilon_{l,r}.$$ \hfill \Box

**Proposition 3.2.** Let $S \subset \mathbb{R}^d$ be a not degenerated $d$-dimensional simplex with vertices $(v_i)_{i=0,1,\ldots,d}$. Let us denote by $s_{l}^{d-1}$, $l = 0, 1, \ldots, d$ the facet opposite to the vertex $v_l$ and by $g_l$ the barycenter of $s_{l}^{d-1}$. For all $\alpha \in (0,1)$ we have

$$\int_S f(x) dx = \frac{A(S)}{2(d+2)!} \left( \frac{\alpha (d+4) - d}{\alpha} \sum_{l=0}^{d} f(v_l) + \frac{d}{\alpha - \alpha^2} \sum_{l=0}^{d} f(y_l(\alpha)) \right)$$

$$+ \frac{\alpha d}{\alpha - 1} \sum_{l=0}^{d} f(g_l) + R(d, \alpha)[f], \quad (3.7)$$

with $y_l(\alpha) = v_l + \alpha (g_l - v_l)$ and

$$R(d, \alpha)[f] = \frac{dA(S)}{4(d+2)!} \sum_{l=0}^{d} \left( \frac{1}{\alpha - \alpha^2} \int_0^1 (1 - t)^2 D^3 g_{l}^{d-1} v, f(v_l + t(g_l - v_l)) dt \right.$$

$$\left. - \int_0^\alpha (\alpha - t)^2 D^3 g_{l}^{d-1} v, f(v_l + t(g_l - v_l)) dt \right). \quad (3.8)$$

The quadrature formula (3.7) has degree of exactness 2.
Proof. Let $g_l$ be the barycenter of $s_l^{d-1}$, by Lemma (2.1) the sum of first-order derivatives along the edges of $S$ in (3.1) can be rewritten as

$$
\sum_{l=0}^{d} \sum_{\alpha \in \mathbb{N}^d, |\alpha|=1} D_l^\alpha f(v_l) \lambda^\alpha(g_l) = \sum_{l=0}^{d} D_{g_l-v_l} f(v_l)
$$

and, since $\lambda_k(g_l) = \frac{1}{d}$, for each $l, k = 0, \ldots, d$, then

$$
\sum_{l=0}^{d} \sum_{\alpha \in \mathbb{N}^d, |\alpha|=1} D_l^\alpha f(v_l) = d \sum_{l=0}^{d} D_{g_l-v_l} f(v_l).
$$

(3.9)

By substituting (3.9) in (3.1), we get

$$
Q^S_l[f] = \frac{A(S)}{(d+1)!} \sum_{l=0}^{d} f(v_l) + \frac{dA(S)}{2(d+2)!} \sum_{l=0}^{d} D_{g_l-v_l} f(v_l).
$$

(3.10)

To have a three point finite difference approximation of the directional derivatives in (3.10), for each $l = 0, \ldots, d$, let us introduce the univariate function

$$
h_l : [0, 1] \to \mathbb{R},
$$

$$
t \mapsto h_l(t) = f(v_l + t(g_l - v_l)).
$$

(3.11)

For $t = 1$ and $t = \alpha \in (0, 1)$, the second-order Taylor expansion of $h_l(1)$ and $h_l(\alpha)$ centered at 0 with integral remainder are

$$
h_l(1) = h_l(0) + h'_l(0) + \frac{1}{2} h''_l(0) + \frac{1}{2} \int_0^1 (1 - t)^2 h'''_l(t) dt
$$

(3.12)

and

$$
h_l(\alpha) = h_l(0) + \alpha h'_l(0) + \frac{1}{2} \alpha^2 h''_l(0) + \frac{1}{2} \int_0^\alpha (\alpha - t)^2 h'''_l(t) dt.
$$

(3.13)

Then, by (3.12) and (3.13), we get

$$
h_l(\alpha) - \alpha^2 h_l(1) = (1 - \alpha^2) h_l(0) + (\alpha - \alpha^2) h'_l(0) + R_l[f](\alpha)
$$

(3.14)

where

$$
R_l[f](\alpha) = \frac{1}{2} \int_0^\alpha (\alpha - t)^2 h'''_l(t) dt - \alpha^2 \frac{1}{2} \int_0^1 (1 - t)^2 h'''_l(t) dt.
$$

Therefore, by (3.14) it follows that

$$
h'_l(0) = \frac{1}{\alpha - \alpha^2} h_l(\alpha) - \frac{\alpha}{1 - \alpha} h_l(1) - \frac{1 + \alpha}{\alpha} h'_l(0) + \frac{1}{\alpha - \alpha^2} R_l(\alpha)[f].
$$

(3.15)

By rewriting equality (3.15) in terms of $f$ we get

$$
D_{g_l-v_l} f(v_l) = \frac{1}{\alpha - \alpha^2} f(\alpha g_l + (1-\alpha)v_l) - \frac{\alpha}{1-\alpha} f(g_l) - \frac{1 + \alpha}{\alpha} f(v_l)
$$

$$
+ \frac{1}{\alpha - \alpha^2} R_l(\alpha)[f],
$$

(3.16)
where
\begin{align*}
R_l(\alpha)[f] &= \frac{1}{2} \left( \alpha^2 \int_0^1 (1 - t)^2 D^3_{g_l - v_l} f(v_l + t(g_l - v_l)) dt \\
&\quad - \int_0^\alpha (\alpha - t)^2 D^3_{g_l - v_l} f(v_l + t(g_l - v_l)) dt \right). \quad (3.17)
\end{align*}

Finally, by substituting (3.16) in (3.1), we get (3.7). $R(\alpha)[f] = 0$ whenever $f$ is a polynomial in $d$ variables of total degree 2 and, therefore, the quadrature formula (3.7) has degree of exactness 2.

\textbf{Remark 3.3.} 1. For $\alpha = \frac{d}{d + 4}$ the formula (3.7) becomes
\begin{align*}
\int_S f(x) dx &= \frac{A(S)}{2(d + 2)!} \left( \frac{(d + 4)^2}{4} \sum_{l=0}^d f \left( \frac{4g_l + 4v_l}{d + 4} \right) - \frac{d^2}{4} \sum_{l=0}^d f(g_l) \right) \\
&\quad + R \left( d, \frac{d}{d + 4} \right)[f], \quad (3.18)
\end{align*}

that is a quadrature formula which uses only the function data at the points $g_l$ and $\frac{4g_l + 4v_l}{d + 4}$, $l = 0, \ldots, d$, and is exact for all polynomial of degree less than or equal to 2.

2. For $\alpha = \frac{d}{d + 1}$, the quadrature formula (3.7) becomes,
\begin{align*}
\int_S f(x) dx &= \frac{A(S)}{2(d + 2)!} \left( 3 \sum_{l=0}^d f(v_l) + (d + 1)^3 f(x^*) - d^2 \sum_{l=0}^d f(g_l) \right) \\
&\quad + R \left( d, \frac{d}{d + 1} \right)[f] \quad (3.19)
\end{align*}

where $x^*$ is the center of gravity of $S$.

To improve the approximation accuracy of the quadrature formula (3.19), let us consider a convex combination of this formula with a multivariate Simpson rule for a simplex proposed in \cite[Theorem 5.1]{7}. For a particular value of the parameter of the linear combination, we are able to get a quadrature formula with an higher degree of exactness.

\textbf{Corollary 3.4.} Let $f : S \to \mathbb{R}$ be a 3-times continuous differentiable function on $S$. Let us denote by $x^*$ the center of gravity of $S$, by $s_{d-1}^l$, $l = 0, 1, \ldots, d$ the facets of $S$ opposite to the vertex $v_l$ and by $g_l$ the barycenter of $s_{d-1}^l$. Then,
\begin{align*}
\int_S f(x) dx &= \tau F_1[f] + (1 - \tau) F_2[f] + \tilde{R}(\tau)[f], \quad \tau \in \mathbb{R}, \quad (3.20)
\end{align*}

where
\begin{align*}
F_1[f] &= \frac{A(S)}{d!} \left( \frac{d + 1}{d + 2} f(x^*) + \frac{1}{(d + 1)(d + 2)} \sum_{l=0}^d f(v_l) \right)
\end{align*}
is the multivariate Simpson rule for a simplex [7, Theorem 5.1],

\[ F_2[f] = \frac{A(S)}{2(d + 2)!} \left( 3 \sum_{l=0}^{d} f(v_l) + (d + 1)^3 f(x^*) - d^2 \sum_{l=0}^{d} f(g_l) \right) \]

is given by (3.7) for \( \alpha = \frac{d}{d+1} \) and

\[ \tilde{R}(\tau)[f] = \tau R^S_{d}[f] + (1 - \tau) R \left( d, \frac{d}{d+1} \right)[f] \]  

(3.21)

with \( R^S_{d}[f] \) denoting the remainder term in the multivariate Simpson rule for a simplex. For all \( \tau \in \mathbb{R} \) we have \( \tilde{R}(\tau)[f] = 0 \), whenever \( f \) is a polynomial in \( d \) variables of total degree at most 2.

**Proof.** Since \( R^S_{d}[f] = 0 \) and \( R \left( d, \frac{d}{d+1} \right)[f] = 0 \) whenever \( f \) is a polynomial in \( d \) variables of total degree less than or equal to 2, it easily follows that \( \tilde{R}(\tau)[f] \) vanishes for each polynomial of degree at most 2. \( \square \)

For \( \tau = \frac{3(d+1)}{d+3} \), the family of quadrature formulas (3.20) yields to a formula which has degree of exactness 3.

**Theorem 3.5.** Let \( f : S \to \mathbb{R} \) be a 3-times continuous differentiable function on \( S \) and let us denote by \( x^* \) the center of gravity of \( S \), by \( s_l^{d-1}, l = 0, 1, \ldots, d \) the facet opposite to the vertex \( v_l \) and by \( g_l \) the barycenter of \( s_l^{d-1} \). The quadrature formula

\[ \int_S f(x) \, dx = \frac{A(S)}{(d + 3)!} \left( 3 \sum_{l=0}^{d} f(v_l) + d^3 \sum_{l=0}^{d} f(g_l) + (d + 1)^3(3 - d)f(x^*) \right) \]

\[ + \tilde{R} \left( \frac{3(d+1)}{d+3} \right)[f], \]  

(3.22)

with \( \tilde{R} \left( \frac{3(d+1)}{d+3} \right)[f] \) defined in (3.21), has degree of exactness 3.

**Proof.** For \( \tau = \frac{3(d+1)}{d+3} \), the quadrature formula (3.20) reduces to

\[ \int_S f(x) \, dx \approx F_3[f] \]

\[ = \frac{A(S)}{(d + 3)!} \left( 3 \sum_{l=0}^{d} f(v_l) + d^3 \sum_{l=0}^{d} f(g_l) + (d + 1)^3(3 - d)f(x^*) \right) \]  

(3.23)

and, by Corollary 3.4, it follows that \( F_3[f] \) has degree of exactness 2. Let \( P_3(x) \) be a polynomial in \( d \) variables of degree 3; we can write \( P_3(x) \) as

\[ P_3(x) = P_2(x) + \sum_{i=0}^{d} c_i x_i^3 + \sum_{i=0}^{d} \sum_{j \neq i} b_{ij} x_i^2 x_j + \sum_{i=0}^{d} \sum_{j \neq i} \sum_{k \neq i,j} d_{ijk} x_i x_j x_k \]
where $P_2(x)$ is a polynomial of degree 2. Therefore, it is sufficient to prove that $F_3[M]$ is exact for the monomials

\[ M_1 = x_i^3; \ M_2 = x_i^2 x_j \quad (j \neq i); \ M_3 = x_i x_j x_k \quad (k \neq i, j) \quad i, j, k = 0, \ldots, d. \]

Thanks to the linear isomorphism which maps the standard simplex $\Delta_d$ of $\mathbb{R}^d$ to a generic simplex $S$, without loss of generality, we can restrict to the case of the simplex $\Delta_d$ of vertices $v_0 = (0, \ldots, 0); v_1 = (1, 0, \ldots, 0); v_l = (0, 0, \ldots, 1, \ldots, 0); v_d = (0, \ldots, 0, 1).$ The center of gravity of $\Delta_d$ is $x^* = \left( \frac{1}{d+1}, \ldots, \frac{1}{d+1} \right)$ and the barycenter of the facets are

\[ g_0 = \left( \frac{1}{d}, \frac{1}{d}, \ldots, \frac{1}{d} \right), g_1 = \left( 0, \frac{1}{d}, \ldots, \frac{1}{d} \right), \ldots, g_d = \left( \frac{1}{d}, \frac{1}{d}, \ldots, 0 \right). \]

Let us now consider $M_1 = x_i^3$, $i = 0, \ldots, d$. The exact integral of $M_1$ over the simplex $\Delta_d$ is \[ \int_{\Delta_d} M_1 \, dx = \frac{6}{(d+3)!}, \]
and, in addition,

\begin{align*}
\sum_{l=0}^{d} f(v_l) &= 1, \\
\sum_{l=0}^{d} f(g_l) &= \frac{d}{d^3}, \\
f(x^*) &= \frac{1}{(d+1)^3}.
\end{align*}

By substituting equalities (3.24) in the quadrature formula (3.23) we have

\[ F_3[M_1] = \frac{A(\Delta_d)}{(d+3)!} \left( 3 + d^3 \frac{d}{d^3} + (d+1)^3(3-d) \frac{1}{(d+1)^3} \right) = \frac{6}{(d+3)!} = \int_{\Delta_d} M_1 \, dx. \]

Let us consider $M_2 = x_i^3 x_j$, $i, j = 0, \ldots, d$ and $j \neq i$. The exact integral of $M_2$ over the simplex $\Delta_d$ is \[ \int_{\Delta_d} M_2 \, dx = \frac{2}{(d+3)!}, \]
and, in addition,

\begin{align*}
\sum_{l=0}^{d} f(v_l) &= 0, \\
\sum_{l=0}^{d} f(g_l) &= \frac{d-1}{d^3}, \\
f(x^*) &= \frac{1}{(d+1)^3}.
\end{align*}

By substituting equalities (3.25) in the quadrature formula (3.23) we have

\[ F_3[M_2] = \frac{A(\Delta_d)}{(d+3)!} \left( 0 + d^3 \frac{d-1}{d^3} + (d+1)^3(3-d) \frac{1}{(d+1)^3} \right). \]
Finally, let us consider $M_3 = x_ix_jx_k$, $i, j, k = 0, \ldots, d$ and $k \neq i, j$ for which the exact integral over the simplex $\Delta_d$ is $\frac{1}{(d+3)!}$

and

$$\sum_{l=0}^{d-2} f(g_l) = \frac{d-2}{d^3}, \quad f(x^*) = \frac{1}{(d+1)!}.$$  \hfill (3.26)

By substituting equalities (3.26) in the quadrature formula (3.23), we have

$$F_3[M_3] = \frac{A(S)}{(d+3)!} \left( 0 + d^2 \frac{d-2}{d^3} + (d+1)^2 (3-d) \frac{1}{(d+1)!} \right)$$

Then

$$\int_{\Delta_d} P_3(x) dx = F_3[P_3]$$

and this shows that the degree of exactness of the quadrature formula (3.22) is 3. \hfill \Box

4. The 2-Simplex Case

In this section, we restrict to the case $d = 2$ in which $S$ is a triangle of vertices $v_0, v_1, v_2$. In this particular case, the quadrature formula (2.14) reduces to

$$Q_r^S[f] = \frac{1}{(1+r)} \sum_{k=0}^{r} \frac{1+r-k}{(3+k)!} \sum_{j=0}^{k} \left( D_{v_1-v_0}^j D_{v_2-v_0}^{k-j} f(v_0) + D_{v_1-v_0}^j D_{v_2-v_1}^{k-j} f(v_1) + D_{v_0-v_2}^j D_{v_1-v_2}^{k-j} f(v_2) \right)$$

and, by easy computations, the bound for the approximation error becomes

$$|E_r^{\Delta_2}[f]| \leq \frac{|D^{r+2}f| S}{12(r+2)!(1+r)} \left( 1 + \sqrt{2r} \right).$$  \hfill (4.2)

The quadrature formula (3.19), which has degree of exactness 2 and uses only function data at the vertices of $S$, at the midpoints of its sides and at its center of gravity, becomes

$$\int_S f(x) dx = \tilde{Q}_1^S[f] + R \left( 2, \frac{2}{3} \right) [f]$$
where
\[
\hat{Q}^S_2[f] = \frac{A(S)}{16} \left( \sum_{l=0}^{2} f(v_l) - \frac{4}{3} \sum_{l=0}^{1} \sum_{r=l+1}^{2} f(g_l) + 9f(x^*) \right) \quad (4.3)
\]
and
\[
R\left(2, \frac{2}{3}\right)[f] = \frac{3A(S)}{32} \sum_{l=0}^{2} \left( \frac{4}{9} \int_{0}^{1} (1-t)^2 D_{g_l-v_l} f(v_l + t(g_l - v_l)) dt \right.
\]
\[
- \int_{0}^{\frac{2}{3}} \left( \frac{2}{3} - t \right)^2 D_{g_l-v_l} f(v_l + t(g_l - v_l)) dt \right) .
\]

The quadrature formula (3.22), which has degree of exactness 3 and uses only function data at the vertices of \(S\), at the midpoints of its sides and at its center of gravity, becomes
\[
\hat{Q}^S_3[f] = \frac{A(S)}{40} \left( \sum_{l=0}^{d} f(v_l) + \frac{8}{3} \sum_{l=0}^{d} f(g_l) + 9f(x^*) \right) , \quad (4.4)
\]
where \(g_l = \frac{v_l + v_r}{2}\), i.e. \(g_l\) is nothing but the midpoint of the side of the triangle \(S\) opposite to \(v_l\). For \(d = 2\) and \(\alpha = 1/3\) the formula (3.7), which has degree of exactness 2 and uses function data at the midpoints of the sides of \(S\) and at the points \(\frac{g_l + 2v_l}{3}\), \(l = 0, \ldots, d\), becomes
\[
\int_{S} f(x) dx = \hat{Q}^S_2[f] + R\left(2, \frac{1}{3}\right)[f] \quad (4.5)
\]
with
\[
\hat{Q}^S_2[f] = \frac{A(S)}{48} \left( 9 \sum_{l=0}^{2} f\left(\frac{g_l + 2v_l}{3}\right) - \sum_{l=0}^{2} f(g_l) \right) \quad (4.6)
\]
and \(R\left(2, \frac{1}{3}\right)[f]\) defined in (3.8).

To enhance the degree of exactness of the quadrature formula (4.6), let us consider the midpoint formula for the 2-dimensional simplex [5]
\[
\int_{S} f(x) dx = \frac{A(S)}{6} \sum_{r=0}^{2} f(g_r) + E[f]
\]
where \(E[f] = 0\) whenever \(f\) is a polynomial in 2 variables of total degree at most 2. We set
\[
Q^{mid}_2[f] = \frac{A(S)}{6} \sum_{l=0}^{2} f(g_l)
\]
and define the convex combination
\[
\int_{S} f(x) dx = \alpha\hat{Q}^S_2[f] + (1 - \alpha)Q^{mid}_2[f] + E_\alpha[f], \quad \alpha \in \mathbb{R} . \quad (4.7)
\]

where
\[
E_\alpha[f] = \alpha R\left(2, \frac{1}{3}\right)[f] + (1 - \alpha)E[f] . \quad (4.8)
\]
Since, for all $\alpha \in \mathbb{R}$, $E_\alpha[f] = 0$ whenever $f$ is a polynomial in 2 variables of total degree at least 2, the quadrature formula $\text{(4.7)}$ has at least degree of exactness 2.

**Theorem 4.1.** Let $f : S \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a 3-times continuous differentiable function on $S$. Then, the quadrature formula

$$\int_S f(x)dx = MQ_3^S[f] + E_{\frac{4}{5}}[f]$$

(4.9)

with

$$MQ_3^S[f] = \frac{A(S)}{20} \left( \sum_{l=0}^{2} f\left(\frac{g_l + 2v_l}{3}\right) + \frac{1}{3} \sum_{l=1}^{3} f(g_l) \right)$$

(4.10)

has degree of exactness 3.

**Proof.** Equality (4.10) follows by setting $\alpha = \frac{4}{5}$ in (4.7) and by rearranging. To prove the degree of exactness of the formula (4.9), it is sufficient to follow the same arguments used in the proof of Theorem 3.5 for $d = 2$. □

Finally, to obtain a quadrature formula over $S$ with degree of exactness 4, we consider the convex combination of the quadrature formulas (4.4) and (4.10)

$$\int_S f(x)dx = \alpha \tilde{Q}_3^S[f] + (1 - \alpha)MQ_3^S[f] + E'_\alpha[f], \quad \alpha \in \mathbb{R},$$

(4.11)

where

$$E'_\alpha[f] = \alpha \tilde{R}\left(\frac{9}{5}\right)[f] + (1 - \alpha)E_{\frac{4}{5}}[f],$$

with $\tilde{R}\left(\frac{9}{5}\right)[f]$ given by the Eq. (3.21) and $E_{\frac{4}{5}}[f]$ by the equation (4.8). Since, for all $\alpha \in \mathbb{R}$, $E'_\alpha[f] = 0$ whenever $f$ is a polynomial in 2 variables of total degree at least 3, the quadrature formula (4.11) has at least degree of exactness 3.

**Theorem 4.2.** Let $f : S \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a 3-times continuous differentiable function on $S$ and $x^*$ the center of gravity of $S$. Then, the quadrature formula

$$\int_S f(x)dx = MQ_4^S[f] + E'_{\frac{5}{3}}[f]$$

(4.12)

where

$$MQ_4^S[f] = \frac{A(S)}{120} \left( \sum_{l=0}^{2} f(v_l) + 4 \sum_{l=0}^{2} f(g_l) + 12 f\left(\frac{g_l + 2v_l}{3}\right) + 9f(x^*) \right)$$

(4.13)

and

$$E'_{\frac{5}{3}}[f] = \frac{1}{3} \tilde{R}\left(\frac{9}{5}\right)[f] + \left(1 - \frac{1}{3}\right) E_{\frac{4}{5}}[f],$$

has degree of exactness 4.
Table 1. Absolute value of the remainder terms $E_{r}^{\Delta_{2}}[f_{i}] = Q_{r}^{\Delta_{2}}[f_{i}] - \int_{\Delta_{2}} f_{i}(x)dx$, $i = 1,\ldots,4$; $r = 1,\ldots,10$

| $r$ | $|E_{r}^{\Delta_{2}}[f_{1}]|$ | $|E_{r}^{\Delta_{2}}[f_{2}]|$ | $|E_{r}^{\Delta_{2}}[f_{3}]|$ | $|E_{r}^{\Delta_{2}}[f_{4}]|$ |
|-----|-------------------------|-------------------------|-------------------------|-------------------------|
| 1   | 2.67e−3                 | 4.39e−2                 | 3.02e−4                 | 1.91e−4                 |
| 2   | 1.06e−3                 | 1.03e−2                 | 3.54e−5                 | 3.87e−5                 |
| 3   | 7.03e−5                 | 1.86e−2                 | 2.55e−6                 | 1.68e−6                 |
| 4   | 1.73e−5                 | 4.35e−3                 | 1.75e−7                 | 1.93e−7                 |
| 5   | 1.18e−6                 | 5.51e−3                 | 1.36e−8                 | 8.93e−9                 |
| 6   | 2.15e−7                 | 4.01e−3                 | 6.76e−10                | 7.55e−10                |
| 7   | 1.35e−8                 | 6.67e−4                 | 5.11e−11                | 3.30e−11                |
| 8   | 2.60e−9                 | 1.85e−3                 | 2.02e−12                | 2.28e−12                |
| 9   | 6.48e−10                | 3.82e−4                 | 1.43e−13                | 9.02e−14                |
| 10  | 5.07e−10                | 5.86e−4                 | 5.16e−15                | 4.94e−15                |

Table 2. Absolute value of the remainder terms $\tilde{E}_{r}^{\Delta_{2}}[f_{i}]$ , $\tilde{E}_{3}^{\Delta_{2}}[f_{i}]$, $\tilde{E}_{MQ_{3}}^{\Delta_{2}}[f_{i}]$, $\tilde{E}_{MQ_{4}}^{\Delta_{2}}[f_{i}]$, $i = 1,\ldots,4$

| $f_{i}$ | $|\tilde{E}_{2}^{\Delta_{2}}[f_{i}]|$ | $|\tilde{E}_{3}^{\Delta_{2}}[f_{i}]|$ | $|\tilde{E}_{MQ_{3}}^{\Delta_{2}}[f_{i}]|$ | $|\tilde{E}_{MQ_{4}}^{\Delta_{2}}[f_{i}]|$ |
|---------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $f_{1}$ | 1.55e−3                         | 2.86e−4                         | 1.46e−4                         | 1.12e−6                         |
| $f_{2}$ | 3.85e−3                         | 8.64e−3                         | 4.59e−3                         | 1.83e−4                         |
| $f_{3}$ | 3.71e−4                         | 1.01e−5                         | 5.17e−6                         | 7.89e−8                         |
| $f_{4}$ | 3.76e−4                         | 1.08e−5                         | 5.51e−6                         | 7.91e−8                         |

Proof. Equality (4.13) follows by setting $\alpha = \frac{1}{3}$ in (4.11). To prove the degree of exactness of the formula (4.12), we proceed by verifying the exactness of the quadrature formula for the monomials $x^{4}$, $x^{3}y$, $xy^{3}$, $x^{2}y^{2}$, $y^{4}$, similarly to the proof of Theorem 3.5 for $d = 2$. □

5. Numerical Results in $d = 2$

To test the approximation accuracies of the proposed formulas, we consider the case $d = 2$ and the standard triangle $S = \Delta_{2}$ of vertices $v_{0} = (0,0)$, $v_{1} = (1,0)$, $v_{2} = (0,1)$. The numerical experiments are conducted by considering the following set of test functions [1]

\[
\begin{align*}
 f_{1}(x, y) &= \cos \left( \sqrt{1 + x^{2} + y^{2}} \right), \\
 f_{2}(x, y) &= \exp \left( - \frac{(3x - 2)^{2} + (3y - 2)^{2}}{4} \right), \\
 f_{3}(x, y) &= \sin \left( \frac{\pi}{4} x + \frac{\pi}{6} y \right), \\
 f_{4}(x, y) &= \sinh \left( \frac{\pi}{4} x + \frac{\pi}{6} y \right). 
\end{align*}
\]
In all the experiments, the exact value of the integrals for functions $f_1$ and $f_2$ are computed by assuming as exact the numerical integration performed by Mathematica. In Table 1, we report the absolute value of the remainder terms $E_{\Delta^2}[f_i] = Q_{\Delta^2}[f_i] - \int_{\Delta^2} f_i(x)dx$, $i = 1, \ldots, 4$, $r = 1, \ldots, 10$, and, in Table 2, we display the absolute value of the remainder terms

$$
\tilde{E}_{\Delta^2}[f_i] = \tilde{Q}_{\Delta^2}[f_i] - \int_{\Delta^2} f_i(x)dx, \quad i = 1, \ldots, 4,
$$

$$
\tilde{E}_{\Delta^2}[f_i] = \tilde{Q}_{\Delta^2}[f_i] - \int_{\Delta^2} f_i(x)dx, \quad i = 1, \ldots, 4,
$$

$$
\tilde{E}_{MQ_{\Delta^2}}[f_i] = MQ_{\Delta^2}[f_i] - \int_{\Delta^2} f_i(x)dx, \quad i = 1, \ldots, 4
$$

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Filomena Di Tommaso
Department of Mathematics and Computer Science
University of Calabria
Via P. Bucci, cubo 30A
87036 Rende
Italy
e-mail: filomena.ditommaso@unical.it

Benaissa Zerroudi
University Ibn Tofail
Kenitra
Morocco
e-mail: zerroudi@gmail.com

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