DIAMOND-SHAPED REGIONS AS MICROCOSMOI

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Abstract. We give a geometrically intrinsic construction of a global time function for relatively compact diamond-shaped regions in arbitrary spacetimes. In the case of Minkowski spacetime, the flow of diffeomorphisms associated to a suitably normalized gradient of this time function becomes the conformal isotropy subgroup of the diamond. In full generality, this time function is elegantly expressed in terms of the Lorentzian distance function, and it has an asymptotic behavior at large absolute times similar to the one in Minkowski spacetime.

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1. Introduction

1.1. A somewhat revised (hi)story. A great deal of all physics is the study of time evolution of a physical system, with the objective to be able to predict (or at least estimate) its future behavior from the knowledge of the past. However, the choice of what we mean by “time” has always been considered a more philosophical issue from a physical viewpoint, whereas from a mathematical viewpoint – that is, (typically) the study of differential equations –, it has been nothing more than a choice of a convenient “flow parameter”. With the advent of General Relativity, such a choice was forcefully brought to the conceptual forefront. Namely, the choice of “time” must be ultimately devoid of any operational meaning, in the sense that it cannot fixed by any concrete physical procedure, but the laws ruling such procedures must, however, be local and independent of such a choice – such was the lesson put forward by Einstein. This automatically leads to a more sensible question: what choices make manifest the predictability of physical laws? Or, from a more mathematical standpoint: which choices render the initial value problem for the dynamics of a physical system (at least locally) well posed?

If we impose as well the requirement that physical effects propagate locally with speed less than or equal to the speed of light (microcausality), we are immediately led to the study of hyperbolic partial differential equations in general spacetimes, started by Hadamard [24] and Riesz [35] in a local setting. A more detailed study of the (Lorentzian) geometry of spacetimes reveals that the regions where global well-posedness for this class of equations holds must have the property that, roughly, effects that can causally reach an event \( p \) from the past and another event \( q \) from the future should be spread over a finitely extended subregion. This was proven

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\[1\]Unless one makes some additional assumptions on “test devices”, which amount to an approximate idealization which is not intrinsic to the theory [39].
by Leray in his Princeton lectures [29] – more precisely, the Cauchy problem for hyperbolic partial differential equations in a Lorentzian manifold \((\mathcal{M}, g)\) is well posed within the domain of dependence \(D(\Sigma)\) of the initial data hypersurface \(\Sigma\) (for these and all other notions of Lorentzian geometry employed in this work, we refer the reader to Subsection 1.3).

\[
D(\Sigma) \doteq \{ p \in \mathcal{M} : \forall \text{ inextendible causal curve } \gamma \text{ s.t. } \gamma(\lambda_0) = p \exists! \lambda_1 \text{ s.t. } \gamma(\lambda_1) \in \Sigma \}.
\]

For \((\mathcal{M}, g)\) causal [9], we have that \(p \leq q \in D(\Sigma)\) iff

\[
J^+(p) \cap J^-(q), J^+(p) \cap J^-(\Sigma), J^+(\Sigma) \cap J^-(q) \text{ are compact},
\]

that is, any dynamical information that propagates from/to \(\Sigma\) with speed up to that of light cannot leak to infinity. Such a property is tantamount to well-posedness of Cauchy problems of the aforementioned kind, and due to Leray is the christening “global hyperbolicity” for this property.

**Definition 1.1.** A spacetime \((\mathcal{M}, g)\) such that \(\mathcal{M} = \text{int}D(\Sigma)\) for some achronal \(C^0\) hypersurface \(\Sigma\) is called globally hyperbolic, and \(\Sigma\) is then said to be a Cauchy hypersurface for \((\mathcal{M}, g)\).

The equivalence of Definition 1.1 with the validity of (2) for all \(p \leq q \in \mathcal{M}\) was proven by Geroch [22] (with some gaps filled later by Dieckmann [18]); moreover, in this case he constructed a continuous, surjective function \(t : \mathcal{M} \to \mathbb{R}\) which is strictly increasing along any future-directed causal curve (i.e. a global time function) such that \(t^{-1}(\lambda)\) satisfies (1) for all \(\lambda \in \mathbb{R}\) (i.e. \(t\) is a Cauchy time function). A question that immediately arose, and became one of the main “folk theorems” in general relativity, was: can \(t\) be chosen smooth? This is a highly desirable property – for instance, it implies that \(dt\) is then a future directed timelike covector. A first partial answer was proposed by Seifert [37], but his smoothing procedure is difficult to understand. A satisfactory answer to this question ended up coming much later, with the seminal work of Bernal and Sánchez [6, 7, 8]. These papers actually work in the more general case of stably causal spacetimes, that is, the ones which admit a (continuous) global time function \(t\), and one of their main results is that one can always regularize \(t\) so as to make it smooth and still retain the characteristic traits of a global time function. Moreover, if \(t\) has Cauchy level surfaces, the regularization procedure proposed by Bernal and Sánchez also retains this property.

Of course, the procedure discussed above still retains a lot of freedom. One may try to go further and ask whether there are choices of Cauchy time functions which are geometrically natural, that is, depend only on the intrinsic geometry of spacetime. This question is interesting not only in itself, but also may lead to relevant tools for addressing the dynamics of hyperbolic partial differential equations in the large by purely geometrical means. This is absolutely essential for the analysis of Einstein’s equations, as dramatically manifested in the seminal work of Christodoulou and Klainerman on the global nonlinear stability of Minkowski spacetime [12] and the developments originated from their ideas. For static spacetimes,

\footnote{Actually, Leray’s definition of global hyperbolicity demands that the space of piecewise smooth causal functions linking \(p\) and \(q\) is either empty or compact w.r.t. the compact-open topology. It turns out, however, that this property is equivalent to [2]. See [35] and [9] for a detailed discussion on this matter.}
the answer to this question is an obvious yes – just pick a time function \( \tau \) whose flow induced by the foliation of \( \mathcal{M} \) by the level sets of \( \tau \) is generated by a Killing vector field. In a cosmological context, this question was addressed in considerable generality by Andersson, Galloway and Howard [1]. Roughly, the Lorentzian distance to the initial cosmological singularity (which generalizes the conformal time of Friedmann-Robertson-Walker spacetimes) provides us such a choice. This time function is generally not smooth, but twice differentiable almost everywhere.

1.2. Aims of the present work. Our objective is to show that, for certain globally hyperbolic regions of general spacetimes, the question posed at the last paragraph of the Introduction always has a positive answer. To wit, we give a procedure to construct global time functions from \( g \) alone for regions of the form \( \mathcal{O}_{p,q} = \mathcal{I}^+(p) \cap \mathcal{I}^-(q), \ p \ll q \in \mathcal{M} \). Such regions are hereby called diamonds. If \( (\mathcal{M}, g) \) is causally simple (that is, \( J^\pm(p) \) are closed for any \( p \in \mathcal{M} \)) and \( J^+(p) \cap J^-(q) = \mathcal{O}_{p,q} \) is compact, then such regions are automatically globally hyperbolic (see next Section), and can be considered as “dynamically closed” spacetimes in their own right. This viewpoint pervades the whole of the paper, and is expounded in Section 2.

An important aspect of our construction – which, to begin with, was its original motivation – is that it generalizes the flow parameter of the one-parameter subgroup of the conformal group of the Minkowski spacetime \( \mathbb{R}^{1,d-1} \) that preserves a particular diamond and whose orbits within the latter are timelike. This result is discussed in detail in Subsection 2.3. Indeed, for \( p \) and \( q \) in general spacetimes, the diffeomorphism flow generated by our time function approaches this flow parameter in a rather precise fashion. This is shown in Section 3, which forms the core of the paper. The general construction is performed in Subsection 3.1, making substantial use of tools from global Lorentzian geometry, namely the Lorentzian distance function [2] and its fine differentiability properties, as discussed in [1] and also in a different context by Moretti [31]. More detailed properties of our global time function for diamonds (particularly, the aforementioned asymptotic behavior) can be obtained when the latter are contained in a geodesically convex neighborhood, as done in Subsection 3.2. A slight generalization of our construction for half-diamonds, i.e., regions of the form \( \mathcal{I}^+(q) \cap \Sigma \) for an acausal hypersurface \( \Sigma \), is presented in Section 4. This paper concludes with some comments on possible future uses of our framework.

1.3. Notation and nomenclature. Here we collect all basic notions of Lorentzian geometry and fix the notation we will use throughout the paper – the reader can safely skip this Subsection at a first reading and return to it when necessary. All our manifolds \( \mathcal{M} \) are smooth, paracompact, second countable and oriented. Recall that a \((d\)-dimensional\) pseudo-Riemannian manifold is a pair \((\mathcal{M}, g)\), where the smooth section \( g \in \Gamma^\infty(\mathcal{M}, \bigwedge^2 T^* \mathcal{M}) \) is at each \( p \in \mathcal{M} \) a non-degenerate symmetric bilinear form on \( T_p \mathcal{M} \) with \( r \) negative and \( d-r \) positive eigenvalues (by Sylvester’s law of inertia, \( r \) is independent of the local trivialization). We say then that \( g \) has index \( r \), or signature \((r,d-r)\). We also say that \((\mathcal{M}, g)\) is Lorentzian if \( r = 1 \), and a spacetime if, in addition, there is a vector field \( T \in \Gamma^\infty(\mathcal{M}, T \mathcal{M}) \) such that \( g(T,T) < 0 \) (that is, \((\mathcal{M}, g)\) is time oriented by \( T \)). We denote the Levi-Civita
connection associated to \( g \) simply by \( \nabla \), since no confusion will arise from this.

Given a spacetime \((\mathcal{M}, g)\), \( 0 \neq X(p) \in T_p\mathcal{M} \) is said to be causal or non-spacelike (resp. timelike, spacelike, null) if \( g(X(p), X(p))(p) \leq 0 \) (resp. \( < 0 \), \( > 0 \), \( = 0 \)) – these properties are said to define the causal character of \( X(p) \). A vector field \( X \) is then said to have a certain causal character if \( X(p) \) has the same causal character for each \( p \in \mathcal{M} \). For example, \( T \) as in the previous paragraph is timelike, and so on; one defines analogously the causal character of covectors and 1-forms. Likewise, a piecewise smooth curve \( \gamma : (a, b) \to \mathcal{M} \) is said to be causal or non-spacelike (resp. timelike, spacelike, null) if \( g(\dot{\gamma}(\lambda), \dot{\gamma}(\lambda))(\gamma(\lambda)) \leq 0 \) (resp. \( < 0 \), \( > 0 \), \( = 0 \)) for all \( \lambda \in (a, b) \). \( X(p) \in T_p\mathcal{M}, \omega(p) \in T_p^\ast \mathcal{M} \) are said to be future (resp. past) directed if \( g(T(p), X(p))(p), \omega(T)(p) > 0 \) (resp. \( < 0 \)) – this implies that \( X(p) \) and \( \omega(p) \) are necessarily causal. \( X \in \Gamma^\infty(\mathcal{M}, T\mathcal{M}), \omega \in \Gamma^\infty(\mathcal{M}, T^\ast\mathcal{M}) \), \( \gamma : (a, b) \to \mathcal{M} \) are likewise said to be future (resp. past) directed if \( g(T, X), \omega(T) > 0 \) (resp. \( < 0 \)) everywhere on \( \mathcal{M} \) and \( g(T, (\gamma)(t)) > 0 \) (resp. \( < 0 \)) for all \( t \in (a, b) \).

Given \( p, q \in \mathcal{M} \), we say that \( p \) chronologically (resp. causally) precedes \( q \), written as \( p \ll q \) (resp. \( p \leq q \)) if there is a future directed timelike (resp. causal) curve segment \( \gamma : [a, b] \to \mathcal{M} \) such that \( \gamma(a) = p, \gamma(b) = q \) – we write the reverse relations as \( q \gg p \) (resp. \( q \geq p \)), and say that \( q \) chronologically (resp. causally) succeeds \( p \).

We now define the chronological past (resp. future) \( I^\mp(p) \) of \( p \in \mathcal{M} \) and the causal past (resp. future) \( J^\mp(p) \) of \( p \in \mathcal{M} \) as

\[
I^{-/+}(p) = \{ q \in \mathcal{M} : q \ll / \gg p \}, J^{-/+}(p) = \{ q \in \mathcal{M} : q = p \text{ or } \leq / \geq p \}.
\]

We say that \( \mathcal{W} \subset \mathcal{M} \) is achronal (resp. acausal) if there are no \( p, q \in \mathcal{W} \) such that \( p \ll q \) (resp. \( p \leq q \)). Given an achronal set \( \mathcal{W} \subset \mathcal{M} \), the edge \( \mathcal{W} \) of \( \mathcal{W} \) is defined as the set of the \( p \in \mathcal{WM} \) such that there is an open neighborhood \( \mathcal{V} \ni p, \mathcal{V} \ni \gamma^{-}(p) \), \( r \in \mathcal{V} \cap J^+(p) \) and a future directed causal curve segment \( \gamma : [a, b] \to \mathcal{M} \) such that \( \gamma(a) = q, \gamma(b) = r \) and \( (a, b) \cap \mathcal{W} = \emptyset \). One can show \( \mathcal{W} \) that any achronal set \( \mathcal{U} \) such that \( \mathcal{U} \cap \mathcal{W} = \emptyset \) is an embedded, locally Lipschitz submanifold of \( \mathcal{M} \) with codimension one. Recalling from the Introduction the notion of Cauchy development \( D(\mathcal{W}) \) of \( \mathcal{W} \), one can further show \( \mathcal{W} \) that, if in addition \( \mathcal{W} \) is acausal, then \( D(\mathcal{W}) \) is open and hence globally hyperbolic with Cauchy hypersurface \( \mathcal{W} \).

We say that a spacetime \((\mathcal{M}, g)\) is chronological (resp. causal) if \( p \ll p \) (resp. \( p \leq p \)) for all \( p \in \mathcal{M} \), i.e. there are no closed timelike (resp. causal) curves in \( \mathcal{M} \), and strongly causal if, for all \( p \in \mathcal{M} \), there is an open neighborhood \( \mathcal{U} \ni p \) such that \( \gamma((a, b)) \cap \mathcal{U} \) is connected for all causal curves \( \gamma : (a, b) \to \mathcal{M} \) – this property is equivalent to \( q \leq r \in \mathcal{U} \) implying that all causal curve segments from \( q \) to \( r \) are contained in \( \mathcal{U} \), we say that such a \( \mathcal{U} \) is causally convex. \((\mathcal{M}, g)\) is stably causal if there is a continuous, non-vanishing, future directed timelike vector field \( S \) such that \((\mathcal{M}, g - S \otimes S)\) is a chronological spacetime, or, equivalently \( \mathcal{M}, g \) if there is \( t \in C^\infty(\mathcal{M}) \) such that \( t \cdot \gamma \) is strictly increasing for all future directed causal curves \( \gamma : (a, b) \to \mathcal{M} \) (and, hence, \( dt \) is a future directed timelike covector).

We then say that such a \( t \) is a global time function on \((\mathcal{M}, g)\). The following chain of implications is immediate:

\[
(\mathcal{M}, g) \text{ strongly causal } \Rightarrow (\mathcal{M}, g) \text{ causal } \Rightarrow (\mathcal{M}, g) \text{ chronological}.
\]
Let now \( t \in C^\infty(\mathcal{M}) \) be a global time function, and \( p \in t^{-1}(0) \). Then there is \( \epsilon > 0 \) and a coordinate chart \( (\mathcal{V}, x = (x^0, x)) \) around \( p \) such that \( x(p) = 0, t|_\mathcal{V} = x^0 \) and \( \mathcal{U} = x^{-1}(\{(x^0, x) : |x^0| < \epsilon^2 - |x|^2\}) \subseteq \mathcal{V} \subseteq \mathcal{V}^c \). By taking \( \epsilon \) sufficiently small, we can make the two “halves” \( \partial^\pm \mathcal{U} = x^{-1}(\{(x^0, x) : x^0 = \pm(\epsilon^2 - |x|^2)\}) \) of the boundary \( \partial \mathcal{U} \) of the “lens-shaped” open neighborhood \( \mathcal{U} \) acausal, so that any future directed causal curve \( \gamma : (a, b) \to \mathcal{M} \) necessarily, if ever, enters (resp. leaves) \( \mathcal{U} \) by first crossing \( \partial^- \mathcal{U} \) (resp. \( \partial^+ \mathcal{U} \)), which implies that \( \gamma((a, b)) \cap \mathcal{U} \) is always connected since \( t \) is a global time function. Therefore,

\[(\mathcal{M}, g) \text{ stably causal } \Rightarrow (\mathcal{M}, g) \text{ strongly causal.}\]

It can, last but not least, be shown [4] that

\[(\mathcal{M}, g) \text{ globally hyperbolic } \Rightarrow (\mathcal{M}, g) \text{ causally simple } \Rightarrow (\mathcal{M}, g) \text{ stably causal.}\]

2. Structure of relatively compact diamonds

2.1. Generalities. For any spacetime, \( J^+(p) \cap J^-(q) \) is the maximal region \textit{causally accessible} to any observer world-line between the events \( p \) and \( q \). Whenever \( J^+(p) \cap J^-(q) \) is compact, we have that \( J^+(p') \cap J^-(q') \) is also compact for all \( p \leq p' \ll q' \leq q \), if the latter set is \textit{closed}. The latter fact is always true for \textit{causally simple} spacetime, thus any relatively compact diamond

\[(3) \quad \partial_{p, q} = \mathcal{I}^+(p) \cap \mathcal{I}^-(q) = \text{int}(\mathcal{I}^+(p) \cap \mathcal{I}^-(q)), \quad p \ll q \in \mathcal{M}\]

in a causally simple spacetime \((\mathcal{M}, g)\) is \textit{globally hyperbolic}. It is easy to see that the Cauchy surfaces of \( \partial_{p, q} \) have edge \( \partial \mathcal{I}^+(p) \cap \partial \mathcal{I}^-(q) \). Notice that the edge may be empty, which is precisely the case when the Cauchy surfaces of \( \partial_{p, q} \) are closed with respect to \( \mathcal{M} \) and hence compact. From now on, \( \partial_{p, q} \) will always be \textit{relatively compact}, unless otherwise stated.

Out of technical convenience, we may (and will) make instead the stronger assumption that \( \overline{\partial_{p, q}} \) is also contained in an open, globally hyperbolic region of \( \mathcal{M} \), which particularly on its turn allows us to drop the assumption of causal simplicity. It is appealing to point out that the former can always be achieved in the causally simple case (which may be taken by the reader as a model), for by taking \( p \ll p' \ll q' \ll q \) with \( \partial_{p, q} \) relatively compact, \( p' \) (resp. \( q' \)) arbitrarily close to \( p \) (resp. \( q \)), and considering \( \partial_{p', q'} \) instead of \( \partial_{p, q} \).

2.2. Manifold structure of the boundary. The boundary of a diamond \( \partial_{p, q} \) is made of three disjoint pieces:

\[(4) \quad \partial \partial_{p, q} = \partial_+ \partial_{p, q} \cup \partial_- \partial_{p, q} \cup \mathcal{E}_{p, q},\]

where

\[(5) \quad \partial_+ \partial_{p, q} = \partial \mathcal{I}^-(q) \cap \mathcal{I}^+(p), \quad \partial_- \partial_{p, q} = \partial \mathcal{I}^+(p) \cap \mathcal{I}^-(q), \quad \mathcal{E}_{p, q} = \partial \mathcal{I}^+(p) \cap \partial \mathcal{I}^-(q)\]

are respectively called the \textit{future horizon}, the \textit{past horizon} and the \textit{edge} of \( \partial_{p, q} \).

Recall that \( \partial \mathcal{I}^-(q) \cap \partial \mathcal{I}^+(p) \) being a future (resp. past) achronal boundary, is a locally Lipschitz hypersurface ruled by future (resp. past) directed null geodesics which, due to causal simplicity, have a future (resp. past) endpoint
q (resp. p). These null geodesics are called the null generators of $\partial I^-(q)$ (resp. $\partial I^+(p)$), and are supposed to be extended towards the past (resp. future) as long as they remain achronal. Therefore the null generators of $\partial I^-(q)$ (resp. $\partial I^+(p)$) are either past (resp. future) inextendible, hence past (resp. future) complete by causal simplicity, or have a past (resp. future) endpoint beyond which any extension ceases to be achronal. Such a point is called simply a past (resp. future) endpoint of $\partial I^-(q)$ (resp. $\partial I^+(p)$). Accordingly, the restriction of the null generators of $\partial I^-(q)$ (resp. $\partial I^+(p)$) to $\partial_+ \mathcal{E}_{p,q}$ (resp. $\partial_- \mathcal{E}_{p,q}$) are called the null generators of $\partial_+ \mathcal{E}_{p,q}$ (resp. $\partial_- \mathcal{E}_{p,q}$).

At this point, a question that naturally arises is whether the closed, achronal set $\mathcal{E}_{p,q}$ also has a manifold structure. For $\mathcal{E}_{p,q}$ contained in a geodesically convex open neighborhood, it turns out that $\partial I^+(p)$ and $\partial I^-(q)$ are transverse at $\mathcal{E}_{p,q}$, so the latter is indeed a submanifold. In the general case, there are two potential sources of problems:

1. $\mathcal{E}_{p,q}$ is generally just the intersection of two locally Lipschitz submanifolds, so it is not clear what it means for these submanifolds to be transverse;

2. There may be a (non achronal) null geodesic segment $\gamma : [a, b] \to \mathcal{M}$ from $p$ to $q$ which belongs to $\partial \mathcal{E}_{p,q}$, in which case $\partial I^+(p)$ and $\partial I^-(q)$ cannot be transverse at $\gamma([a, b]) \cap \mathcal{E}_{p,q}$ in any reasonable sense.

Recall that $\partial I^+(p) \setminus \{p\}$ and $\partial I^-(q) \setminus \{q\}$ can be written in a neighborhood of each of their points as the graph of a Lipschitz function in suitable coordinate charts. Since transversality is a local property, we can work in an open domain $V \subset \mathbb{R}^d$ which is the range of a coordinate chart $x : U \ni r \to V = x(U)$ around a point $r \in \mathcal{E}_{p,q}$. Suppose without loss of generality that $p, q \notin U$; writing the components of $x$ as $x^0, \ldots, x^{d-1}$, where the coordinate vector field $\partial_{x^0}$ is supposed to be timelike and the coordinate vector fields $\partial_{x^1}, \ldots, \partial_{x^{d-1}}$ are supposed to be spacelike, we have that

$$x(\partial I^+(p) \cap U) = \{(x^0 = f_+(x), x) : x = (x^1, \ldots, x^{d-1}) \in \bar{V}\},$$

$$x(\partial I^-(q) \cap U) = \{(x^0 = f_-(x), x) : x \in \bar{V}\},$$

where $f_+, f_- : \bar{V} \to \mathbb{R}$ are Lipschitz on

$$\bar{V} = \{x \in \mathbb{R}^{d-1} : (x^0(r), x) \in V\}.$$

If $\partial I^+(p)$ and $\partial I^-(q)$ are $\mathcal{C}^1$ at $r$, then they are transverse at $r$ if and only if $df_+ \circ x(r)$ and $df_- \circ x(r)$ are linearly independent. At points where either $f_+$ or $f_-$ is not differentiable, we can use the fact that any Lipschitz function $f$ on $\bar{V}$ is differentiable almost everywhere (Rademacher’s theorem) and define the generalized differential of $f$ at $r \in \mathbb{R}^{d-1}$ as the set

$$\partial f(r) \doteq \text{co}\{\lim df(x_i) : x_i \to x, x_i \notin \Omega f\}$$

where co denotes “convex hull” and $\Omega f$ denotes the (null) set of points of $\bar{V}$ where $f$ is not partially differentiable with respect to all coordinates. If $f$ is $\mathcal{C}^1$ at $r$, then one can show that $\partial f(r) = \{df(r)\}$. Using this definition, we say that $\partial I^+(p)$ and $\partial I^-(q)$ are transverse at $r$ if any element of $\partial f_+(r)$ is linearly independent of any element of $\partial f_-(r)$. This definition is independent of the choice of coordinates and functions $f_+, f_-$ used to represent $\partial I^+(p) \cap U$ and $\partial I^-(q) \cap U$. If $\partial I^+(p)$ and $\partial I^-(q)$ are transverse in an open neighborhood of $r$ in $\mathcal{E}_{p,q}$, then we can invoke
the same argument used in the $\mathcal{C}^1$ case to reduce the problem of proving that the transverse intersection of two submanifolds is also a submanifold to an application of the inverse function theorem \cite{23}. At this point, we employ instead of the following

**Theorem 2.1 (Clarke’s Inverse Function Theorem \cite{13}).** Let $U \subset \mathbb{R}^n$ open, $F : U \rightarrow \mathbb{R}^n$ Lipschitz, $x_0 \in U$. Define the generalized Jacobian of $F$ at $x_0$ as the set of $n \times n$ matrices

$$\partial F(x_0) = \co\{\lim dF(x_i) : x_i \rightarrow x, x_i \notin \Omega F\},$$

where $\Omega F$ denotes the (null) set of points of $\bar{V}$ where the components of $F$ are not all partially differentiable with respect to all coordinates. If all elements of $\partial F(x_0)$ are nonsingular, then there are open neighborhoods $V \subset U, W$ of $x_0$ and a Lipschitz function $G : W \rightarrow \mathbb{R}^n$ such that $G \circ F(v) = v$ for all $v \in V$ and $F \circ G(w) = w$ for all $w \in W$. \hfill $\Box$

To summarize, we have proven the

**Corollary 2.2.** Let $f_+, f_-$ be transverse at $x_0$. Then there is a neighborhood $\bar{W} \subset V$ of $x_0$ such that the set

$$x(\mathcal{E}_{p,q}) \cap ((\mathbb{R} \times \bar{W}) \cap V) = \{(x^0 = f_+(x) = f_-(x), x) : x \in \bar{W}\}$$

is the graph of a Lipschitz function. \hfill $\Box$

Using the local graph description of Lipschitz submanifolds outlined above, we can define the conormal cone at a point $r$ of $\partial I^+(p) \setminus \{p\}$ or $\partial I^-(q) \setminus \{q\}$ as the convex hull of the limits of conormal directions along sequences of differentiable points converging to $r$. Since any point of an achronal boundary which is not an endpoint of more than one geodesic is differentiable, we see that the conormal cone of $\partial I^+(p) \setminus \{p\}$ (resp. $\partial I^-(q) \setminus \{q\}$) at each point is given by the convex hull of covectors of the form $g(\gamma, \cdot)$, where $\gamma$ is a future (resp. past) directed null generator of $\partial I^+(p) \setminus \{p\}$ (resp. $\partial I^-(q) \setminus \{q\}$) with respect to some affine parametrization.

Now we are in a position to be more precise about the second problem, for it describes exactly what happens at points where the intersection of $\partial I^+(p)$ with $\partial I^-(q)$ is not transverse even in the above generalized sense. It is clear that, if $\gamma : [a_-, b_+] \rightarrow M$ is any null geodesic segment connecting $p \leq q$, there must be $a_- < a_+ < b_- < b_+$ such that $p_+ = \gamma(a_+)$ is a future endpoint of $\partial I^+(p)$ and $q_- = \gamma(b_-)$ is a past endpoint of $\partial I^-(q)$.

**Lemma 2.3.** Let $\gamma$ be as above and suppose that it is completely contained in $\partial \mathcal{O}_{p,q}$. Then $p_+$ and $q_-$ must belong to $\mathcal{E}_{p,q}$, in which case we must have $b_- \leq a_+$. Moreover, if $q_- \neq p_+$, then the whole null geodesic segment $\gamma|_{[a_+, b_-]}$ must belong to $\mathcal{E}_{p,q}$.

**Proof.** Indeed, if either $p_+$ or $q_-$ do not belong to $\mathcal{E}_{p,q}$, then $\gamma$ must leave $\partial \mathcal{O}_{p,q}$ at one of these points, in contradiction with the hypothesis. If $a_+ < b_- < a_+, b_+$ (resp. $\gamma|_{[a_+, b_+]}$) would be a non achronal null geodesic segment connecting $p$ (resp. $p_+ \in \partial I^-(q)$) to $q_- \in \partial I^+(p)$ (resp. $q$), violating the achronality of both $\partial I^+(p)$ and $\partial I^-(q)$. Suppose now that there are $a_+ < c < d < b_+$ such that $\gamma(c), \gamma(d) \in \mathcal{E}_{p,q}$ and $\gamma((c, d)) \cap \mathcal{E}_{p,q} = \emptyset$. Then by hypothesis either $\gamma((c, d)) \subset \partial_+ \mathcal{O}_{p,q}$ or $\gamma((c, d)) \subset \partial_- \mathcal{O}_{p,q}$. The first (resp. second) case can take place only if $\gamma$ leaves $\partial_+ \mathcal{O}_{p,q}$ (resp. $\partial_- \mathcal{O}_{p,q}$) at $\gamma(c)$ and returns at $\gamma(d)$, which implies that $\gamma(d)$ is a past
(resp. future) endpoint of $\partial_+ \mathcal{O}_{p,q}$ (resp. $\partial_- \mathcal{O}_{p,q}$). However, this implies that $\gamma|_{[a,b]}$ (resp. $\gamma|_{[a,m]}$) is a non achronal null geodesic connecting $\gamma(c) \in \partial I^-(q)$ (resp. $p$) to $q$ (resp. $\gamma(d) \in \partial I^+(p)$), in contradiction with the achronality of $\partial I^-(q)$ (resp. $\partial I^+(p)$).

Conversely, we have the following:

**Lemma 2.4.** Let $p \ll q$ belong to a causally simple spacetime $(\mathcal{M}, g)$. If $\mathcal{E}_{p,q}$ contains a null geodesic segment $\gamma : [a,b] \to \mathcal{M}$, then $\gamma$ can be extended to a (non achronal) null geodesic segment from $p$ to $q$ which is completely contained in $\partial \mathcal{O}_{p,q}$ and whose intersection with $\mathcal{E}_{p,q}$ is a null geodesic segment whose past (resp. future) endpoint is a future (resp. past) endpoint of $\partial I^+(p)$ (resp. $\partial I^-(q)$) and whose interior intersects no null generator of either $\partial I^-(q)$ (resp. $\partial I^+(p)$).

**Proof.** If $\gamma$ is as in the hypothesis, then it can be past (resp. future) extended beyond $\gamma(a)$ (resp. $\gamma(b)$) up to $p$ (resp. $q$). Let $a_{\pm}$ and $b_+$, the values of the affine parameter of $\gamma$ such that $p = \gamma(a_{\pm})$ and $q = \gamma(b_+)$. It follows that $\gamma|_{[a_{\pm},b]}$ (resp. $\gamma|_{[a_{\pm},b_{\pm}]}$) is a null generator of $\partial_- \mathcal{O}_{p,q}$ (resp. $\partial_+ \mathcal{O}_{p,q}$). However, the fully extended $\gamma$ ranging from $a_{\pm}$ to $p_+$ cannot be achronal, for $p \ll q$. This implies that there must be $a_- < a_+ \leq a$, $b \leq b_- < b_+$ such that $\gamma(b_-)$ (resp. $\gamma(a_+)$) is a future (resp. past) endpoint of the null generator $\gamma|_{[a_-,b_-]}$ (resp. $\gamma|_{[a_+,b_+]}$) of $\partial_- \mathcal{O}_{p,q}$ (resp. $\partial_+ \mathcal{O}_{p,q}$), and $\gamma|_{[a_{\pm},b_{\pm}]}$ is the maximal extension of $\gamma$ that still belongs to $\mathcal{E}_{p,q}$. The very existence of such a $\gamma$ entails, in addition, that $\gamma(a_+)$ (resp. $\gamma(b_-)$) is not a future (resp. past) endpoint of $\partial_- \mathcal{O}_{p,q}$ (resp. $\partial_+ \mathcal{O}_{p,q}$). This implies that no other generator of either $\partial_- \mathcal{O}_{p,q}$ or $\partial_+ \mathcal{O}_{p,q}$ can cross $\gamma|_{[a_{\pm},b_{\pm}]}$.

The above results give us a picture about the most pathological points of $\mathcal{E}_{p,q}$. Now we show that, for “most” choices of pairs of points $p \ll q$, these points are rather rare.

**Proposition 2.5.** Let $(\mathcal{M}, g)$ be globally hyperbolic, $q \in \mathcal{M}$. Then the set of points $p \ll q$ such that there are only a finite number of future directed nonspacelike geodesics from $p$ to $q$ is residual in $I^-(q)$.

**Sketch.** By Sard’s theorem, the set of points $p \ll q$ which are not conjugate along any geodesic are residual in $\mathcal{M}$, so let us concentrate only on these. The argument is completed along the lines of Proposition 10.42 of [1], pp. 364.

The results of this Subsection can be summarized by the following

**Theorem 2.6.** Let $(\mathcal{M}, g)$ be globally hyperbolic, $q \in \mathcal{M}$. Then, for a residual set of points $p \in I^-(q)$, the edge $\mathcal{E}_{p,q}$ of $\mathcal{O}_{p,q}$ is a compact Lipschitz submanifold of $\mathcal{M}$, up to a finite set of points and a finite set of null geodesic segments.

It is easy to see from [1] that any Cauchy surface $\Sigma$ for $\mathcal{O}_{p,q}$ has $\mathcal{E}_{p,q}$ as its boundary but, although $\mathcal{O}_{p,q}$ is of the form $D(\Sigma)$ as a spacetime by itself, it is not of this form as a proper region of $\mathcal{M}$, for generally $\partial_+ \mathcal{O}_{p,q}$ (resp. $\partial_- \mathcal{O}_{p,q}$) possesses achronal geodesic segments with future (resp. past) endpoints, that is, there are points which belong to null generators which cease to be achronal before reaching $\mathcal{E}_{p,q}$. Hence, we have that $\text{int}(D(\Sigma) \cap \mathcal{O}_{p,q}) = \mathcal{E}_{p,q}$ — equality occurs if and only if the null geodesic segments which generate $\partial_+ \mathcal{O}_{p,q}$ and $\partial_- \mathcal{O}_{p,q}$ are achronal, as for example in the case that $\mathcal{E}_{p,q}$ is contained in a geodesically convex neighborhood.
2.3. A remark on conformal isotropy groups of diamonds in Minkowski spacetime. Let us consider now the simplest case, that of diamonds in Minkowski spacetime $\mathbb{R}^{1,d-1}$. Each $O_{p,q}$ possesses a one-parameter isotropy subgroup of the conformal group $SO(2,d)$ of $\mathbb{R}^{1,d-1}$, given in standard Cartesian coordinates $x = (x^0 = t, \mathbf{x})$, $\mathbf{x} = (x^1, \ldots, x^{d-1})$, $|\mathbf{x}| = r$ by

$$x \mapsto u_{p,q}^\lambda(x) = K_{p,q}^{-1}(e^\lambda K_{p,q}(x)),$$

where $K_{p,q}$ is the Lorentz boost around the origin making the direction $x(p)x(q)$ parallel to the $x^0$ axis, $K(x) = I(x - (1,0)) - (\frac{1}{2},0)$ and $I(x) = -\frac{x}{\eta(x,x)}$ is the relativistic ray inversion map. That is, the discrete conformal transformation $K_{p,q}$ is a diffeomorphism of $O_{p,q}$ onto the forward light cone $I^+(0)$ which maps the (multiplicative) subgroup of dilations (written additively by putting the dilation in the exponential form $e^\lambda$) onto $\lambda \mapsto u_{p,q}^\lambda$ by conjugation. Since conformal transformations preserve the causal structure of strongly causal spacetimes and the orbits of $\lambda \mapsto e^\lambda$ in $I^+(0)$ are everywhere timelike, so are the orbits of $\lambda \mapsto u_{p,q}^\lambda$ in $O_{p,q}$.

Another description of $\lambda \mapsto u_{p,q}^\lambda$ can be given as follows. Consider the radial null coordinates $x^- = t - r$ (retarded time) and $x^- = t + r$ (advanced time), whose level sets are respectively the forward and backward light cones $\partial I^+((t,0))$ and $\partial I^-((t,0))$. In the case that $p = p_0 = (-1,0)$ and $q = q_0 = (1,0)$, $u_{p_0,q_0}^\lambda$ acts only on the coordinates $x^\pm$, and can be simplified to

$$x^\pm(u_{p_0,q_0}^\lambda(p)) = \frac{(1 + x^\pm) - e^{-\lambda}(1 - x^\pm)}{(1 + x^\pm) + e^{-\lambda}(1 - x^\pm)}, \lambda \in \mathbb{R}.$$

This leads to a quite appealing, intrinsic geometric characterization of the parameter $\lambda$, which we state already in coordinate-independent form:

**Theorem 2.7.** If $\lambda$ is the parameter of $u_{p,q}^\lambda$, seen as a global time function on $O_{p,q}$, then

$$\lambda(r) = \frac{1}{2} \log \left[ \frac{d_\eta(p,r)^2}{d_\eta(r,q)^2} \right] = \frac{1}{d} \log \left[ \frac{\text{Vol}O_{p,r}}{\text{Vol}O_{r,q}} \right],$$

where $d_\eta^2(r,q)$ is the square of the Lorentzian distance associated to the Minkowski spacetime metric $\eta$.

**Proof.** Due to the rotational symmetry of $u_{p_0,q_0}^\lambda$, we can restrict our considerations to the $x^0 - x^1$ plane. In this case, let us write $x^0(p) = (0, x^1, 0, \ldots, 0)$, $x^1 \in [-1,1]$, whence it follows that

$$x^\pm(u_{p_0,q_0}^\lambda(p)) = \frac{(1 \pm x^1) - e^{-\lambda}(1 \mp x^1)}{(1 \pm x^1) + e^{-\lambda}(1 \mp x^1)},$$

and thus

$$x^0(u_{p_0,q_0}^\lambda(p)) = \frac{1}{2}(x^+(u_{p_0,q_0}^\lambda(p)) + x^-(u_{p_0,q_0}^\lambda(p)))$$

$$= \frac{(1 - (x^1)^2)(1 - e^{-2\lambda})}{(1 - (x^1)^2)(1 + e^{-2\lambda}) + 2e^{-\lambda}(1 + (x^1)^2)}.$$
and
\[
x^1(u_{p_0,q_0}^\lambda(p)) = \frac{1}{2}(x^+(u_{p_0,q_0}^\lambda(p)) - x^-(u_{p_0,q_0}^\lambda(p))) = 4x^1e^{-\lambda} \left(1 - (x^1)^2\right) \left(1 + e^{-2\lambda}\right) + 2e^{-\lambda}(1 + (x^1)^2).
\]

Let \( q = (t,0) \), \( t \in (-1,1) \). The diamond \( \partial_{p_0,q} \) is a translation of \( \frac{1+t}{2} \partial_{p_0,q_0} \), and the diamond \( \partial_{q_0} \), a translation of \( \frac{1-t}{2} \partial_{p_0,q_0} \) - hence, we have \( |\partial_{p_0,q}| = \left(\frac{1+t}{2}\right)^{d-1} |\partial_{p_0,q_0}| \text{ and } |\partial_{q_0}| = \left(\frac{1-t}{2}\right)^{d-1} |\partial_{p_0,q_0}| \). More generally, if \( q = (t,r,0,\ldots,0) \), there exists a Lorentz boost in the \( x^0 - x^1 \) plane around \( p_0 \) which takes \( \partial_{p_0,q} \) to \( \partial_{p_0,q^+} \), where \( q^+ = ((1+t)^2 - r^2)^{\frac{1}{2}} - 1,0) \), and a Lorentz boost in the \( x^0 - x^1 \) plane around \( q_0 \) which takes \( \partial_{p_0,q_0} \) to \( \partial_{q_0^-} \), where \( q^- = (1 - (1-t)^2 - r^2)^{\frac{1}{2}},0) \). As Lorentz transformations preserve volume, we have
\[
|\partial_{p_0,q}| = \left(\frac{((1+t)^2 - r^2)^{\frac{1}{2}}}{2}\right)^d |\partial_{p_0,q_0}|
\]
and
\[
|\partial_{q_0}| = \left(\frac{((1-t)^2 - r^2)^{\frac{1}{2}}}{2}\right)^d |\partial_{p_0,q_0}|.
\]

Finally, taking \( q = q(\lambda) = u_{p_0,q_0}^\lambda(p) \), it follows that
\[
(1 + x^0(q(\lambda)))^2 - x^1(q(\lambda))^2 = \frac{4(1 - (x^1)^2)}{(1 - (x^1)^2)(1 + e^{-2\lambda}) + 2e^{-\lambda}(1 + (x^1)^2)}
\]
and
\[
(1 - x^0(q(\lambda)))^2 - x^1(q(\lambda))^2 = \frac{4e^{-2\lambda}(1 - (x^1)^2)}{(1 - (x^1)^2)(1 + e^{-2\lambda}) + 2e^{-\lambda}(1 + (x^1)^2)},
\]
and hence formula (7). The second identity follows from the fact that \( |\partial_{p,q}| = \frac{\text{Vol}S^{d-2}d_y(p,q)^d}{x - \frac{1}{d_x(d_y - 1)}d_y(p,q)^d} \). □

3. Cosmological time functions in general diamonds

3.1. The structure in the large. Let us denote by
\[
\log \frac{d_y(p,-)}{d_y(-,q)} \doteq \lambda_y^{q,p} : \partial_{p,q} \to \mathbb{R}
\]
the function given by the first identity in (7) with the Lorentzian distance \( d_y \) associated to the Minkowski metric \( \eta \) replaced by the one associated to \( g \), henceforth called \( \lambda_y \) (see below).

Remark 3.1. In Minkowski spacetime, the second identity in (7) equates \( \lambda_y^{q,p} \) up to a constant factor, to the global time function originally built in [22]. For diamonds in general, causally simple spacetimes, the second identity no longer holds, due to curvature effects - the intuitive reason is that the “packing” number of small diamonds inside a larger one need not grow linearly with the volume of the latter. This heuristic argument can be made rigorous by employing semi-Riemannian volume comparison estimates [19].
Although $\lambda^g_{p,q}$ suggests itself as a natural choice of global time function for $\mathcal{O}_{p,q}$ for general spacetimes, the remark at the beginning of Section 2 raises concerns about whether the level sets of $\lambda^g_{p,q}$ are Cauchy surfaces or not. Before proceeding any further, let us recall the general definition of $d_g$:

\begin{equation}
\begin{aligned}
d_g(p, q) \doteq & \begin{cases} 
\sup_{\gamma \in \Omega_{p,q}} \sum_{i=1}^{k} \int_{\lambda_{i-1}}^{\lambda_i} \sqrt{-g(\dot{\gamma}(\lambda), \dot{\gamma}(\lambda))} d\lambda & \text{if } p \leq q \\
0 & \text{otherwise }
\end{cases},
\end{aligned}
\end{equation}

where $\Omega_{p,q}$ is the set of future directed, piecewise $C^\infty$ causal curves from $p$ to $q$. $d_g$ as defined in (9) enjoys the fundamental reverse triangular inequality

\begin{equation}
\begin{aligned}
d_g(p, q) & \geq d_g(p, r) + d_g(r, q).
\end{aligned}
\end{equation}

With (9) at our disposal, we can prove the following

**Proposition 3.1.** The level sets of

$$\lambda^g_{p,q} = \frac{1}{2} \log \frac{d_g(p, \cdot)^2}{d_g(\cdot, q)^2}$$

are acausal.

**Proof.** Since $\frac{1}{2} \log$ is injective and strictly monotonically increasing, it suffices to establish the claim for the ratio $\frac{d_g(p, \cdot)}{d_g(\cdot, q)}$. It follows from (10) that, if $\gamma : [0, 1] \to \mathcal{M}$ is a future directed causal curve segment in $\mathcal{O}_{p,q}$, then $\lambda \mapsto d_g(\gamma(\lambda), q)$ (resp. $\lambda \mapsto d_g(p, \gamma(\lambda))$) is a function bounded by $d_g(p, q)$ ($< +\infty$ by virtue of the compactness of $\mathcal{O}_{p,q}$ and the definition of Lorentzian distance) and strictly decreasing (resp. increasing) in $\lambda$. To see the latter fact, recall that any maximal causal curve (i.e. whose arc length between any two of its points is equal to the Lorentzian distance) is necessarily a smooth geodesic, up to reparametrization (8). Therefore, even if $\gamma$ is an achronal null geodesic segment (and thus $d_g(\gamma(0), \gamma(1)) = 0$), the fact that $\gamma(0), \gamma(1) \in \mathcal{O}_{p,q}$ implies that any future directed causal curve segment linking either $p$ to $\gamma(1)$ or $\gamma(0)$ to $q$, and containing $\gamma$, will either explicitly have a larger Lorentzian arc length or be a broken causal curve segment, which for no reparametrization can be made a smooth geodesic.

The above argument shows particularly that the level sets of $d_g(\cdot, q)$ and $d_g(p, \cdot)$ are acausal at nonzero values. To reach the conclusion of the theorem, we argue by *reductio ad absurdum* and assume that

$$\frac{d_g(p, \gamma(0))}{d_g(\gamma(0), q)} = \frac{d_g(p, \gamma(1))}{d_g(\gamma(1), q)}.$$ 

This implies that

$$d_g(p, \gamma(0)) = d_g(\gamma(0), q) \cdot \frac{d_g(p, \gamma(1))}{d_g(\gamma(1), q)}.$$ 

However, we have seen that $d_g(p, \gamma(0)) < d_g(p, \gamma(1))$ and $d_g(\gamma(0), q) > d_g(\gamma(1), q)$ – the second inequality implies that $d_g(p, \gamma(0)) > d_g(p, \gamma(1))$, in contradiction with the first inequality.

As for Cauchy property of the level sets proper, notice that, due to strong causality, any inextendable causal curve $\gamma : (0, 1) \to \mathcal{O}_{p,q}$ has a past endpoint $\gamma_-$ in $I^-(q) \cap \partial I^+(p)$ and a future endpoint $\gamma_+$ in $I^+(p) \cap \partial I^-(q)$. In this case, we have

$$d_g(p, \gamma_-) = d_g(\gamma_+, q) = 0,$$

and

$$d_g(\gamma_-), d_g(p, \gamma_+) \neq 0.$$
Continuity of $\gamma$ will then assure that $\frac{d_{\gamma}(p, \gamma(t))}{d_{\gamma}(\gamma(t), q)}$ assume all possible values in $(0, +\infty)$, provided that the latter ratio is continuous as well, which on its turn is a consequence of the fact that $C_{p, q}$ is assumed to be contained in a globally hyperbolic region of $(M, g)$, in which $d_{\gamma}$ is thus jointly continuous and finite valued (Lemma 4.5, page 140 in [4]). This yields

**Corollary 3.2.** $\lambda_{p, q}$ is continuous and the level sets of $\lambda_{p, q}: C_{p, q} \rightarrow \mathbb{R}$ are acausal Cauchy surfaces.

Actually, one can say even more. Under our hypotheses on $p, q$, we have that the part of the future non-spacelike cut locus of $p$ and the part of the past non-spacelike cut locus of $q$ within $C_{p, q}$ are closed [4]. Moreover, by definition any open neighborhood of a future non-spacelike cut point of $p$ contains a point which can be connected to $p$ by a unique maximal past directed causal geodesic segment; likewise, any open neighborhood of a past non-spacelike cut point of $q$ contains a point which can be connected to $q$ by a unique maximal future directed causal geodesic segment. In other words,

**Lemma 3.3.** Given $p \ll q$ contained in a globally hyperbolic region of $(M, g)$, the complement of the union of the future non-spacelike cut locus of $p$ and the past non-spacelike cut locus of $q$ in $C_{p, q}$ is open and dense in the relative topology.

This also implies that $\lambda_{p, q}$ is smooth almost everywhere, for the set of non-spacelike cut points of $p$ and $q$ within $C_{p, q}$ is then compact and nowhere dense.

**Proposition 3.4.** $d_{g}(p, \cdot)$ and $d_{g}(\cdot, q)$ are semi-convex, i.e., for any $r \in C_{p, q}$ there exists a neighborhood $\mathcal{V} \ni r$, a local chart $x: \mathcal{V} \rightarrow \mathbb{R}^{d}$ and $f \in \mathcal{C}^{\infty}(\mathcal{V})$ such that $(d_{g}(p, \cdot)|_{\mathcal{V}} + f) \circ x^{-1}$ and $(d_{g}(\cdot, q)|_{\mathcal{V}} + f) \circ x^{-1}$ are convex in $x(\mathcal{V})$.

**Proof.** (Sketch; for more details, see [1] and references therein) We will present the argument only for $d_{g}(p, \cdot)$, for the argument for $d_{g}(\cdot, q)$ is analogous. There is a future-directed minimizing timelike geodesic segment, say $\gamma_{p}: [0, d_{g}(p, r)] \rightarrow M$, from $p$ to $r \in C_{p, q}$ (that is, $\gamma_{p}(0) = p$, $\gamma_{p}(d_{g}(p, r)) = r$ and $d_{g}(\gamma_{p}(\lambda), \gamma_{p}(\lambda')) = \lambda' - \lambda$ for all $\lambda' > \lambda$). Since $\gamma_{p}$ is maximizing, the segment $\gamma_{p}(\epsilon, d_{g}(p, r))$ is free of cut points for any $0 < \epsilon < d_{g}(p, r)$ (from now on fixed) and hence there is an open neighborhood $\mathcal{U}_{\epsilon}$ of $\gamma_{p}(\epsilon, d_{g}(p, r))$ in $C_{p, q}$ where $d_{g}(\gamma_{p}(\cdot, \epsilon))$ is smooth. We use $\gamma_{p}(\epsilon)$ instead of $p$ in the first entry of $d_{g}$ because, whereas $\gamma_{p}[\epsilon, d_{g}(p, r)]$ is open and nowhere dense in $\mathcal{U}_{\epsilon}$, the unique maximizing timelike geodesic segment linking $\gamma_{p}(\epsilon)$ to $r$, it is not necessarily true that $\gamma_{p}$ is the unique maximizing timelike geodesic segment linking $p$ to $r$. Nevertheless, we still have $d_{g}(p, r) = d_{g}(p, \gamma_{p}(\epsilon)) + d_{g}(\gamma_{p}(\epsilon), r)$, of course but not necessarily $d_{g}(p, \gamma_{p}(d_{g}(p, r) + \delta)) = d_{g}(p, \gamma_{p}(\epsilon)) + d_{g}(\gamma_{p}(\epsilon), \gamma_{p}(d_{g}(p, r) + \delta))$, no matter how small $0 < \delta$ is.

Particularly, we have $g^{-1}(dd_{g}(\gamma_{p}(\epsilon), \cdot), dd_{g}(\gamma_{p}(\epsilon), \cdot)) = -1$ everywhere in $\mathcal{U}_{\epsilon}$ and the covariant Hessian $\text{Hess}_{g}(p, \cdot)(X, Y) = \nabla_{X} \nabla_{Y} d_{g}(\gamma_{p}(\epsilon), \cdot)$ exists everywhere in $\mathcal{U}_{\epsilon}$ and defines the second fundamental form (whose associated linear operator is the Weingarten map) of the level hypersurfaces of $d_{g}(\gamma_{p}(\epsilon), \cdot)$ in $\mathcal{U}_{\epsilon}$. Consider particularly an open normal coordinate neighborhood $\mathcal{V} \subset \mathcal{U}_{\epsilon}$ of $r$. Then $\text{Hess}_{g}(p, \cdot)$ as a quadratic form satisfies two-sided bounds in $\mathcal{V}$ in terms of existing two-sided bounds on the sectional curvature of 2-planes containing $\gamma_{p}$ [2].

More precisely, there exist $\phi_{p, r, \epsilon} \in C^{\infty}(\mathcal{V})$ such that $d_{g}(p, r) = \phi_{p, r}(r)$ and $d_{g}(r, q) = \phi_{r, q}(r)$, $d_{g}(p, \cdot) \geq \phi_{p, r}$ and $d_{g}(\cdot, q) \geq \phi_{r, q}$ in $\mathcal{V}$ and the Hessians $D^{2}\phi_{p, r}$
and $D^2\phi_{c,q}$ are such that $D^2\phi_{c,p}(r) - c_p \cdot 1$ and $D^2\phi_{c,q}(r) - c_q \cdot 2$ are positive semi-definite matrices for $c_p, c_q \in \mathbb{R}$, which not only implies semi-convexity in the sense of Proposition 3.3 4, but also guarantees that the given definition is independent of coordinates. In these circumstances, we can invoke the classical result of Aleksandrov 20, which tells us that a convex function is not only locally Lipschitz, but is also twice differentiable almost everywhere with respect to Lebesgue measure. Such a result obviously extends to semi-convex functions. As the restriction of $\mu_g$ to normal neighborhoods is absolutely continuous with respect to Lebesgue measure, we thus obtain the following

**Proposition 3.5.** $d_g(p,.)^2$ and $d_g(.,q)^2$ are locally Lipschitz and twice differentiable almost everywhere in $\mathcal{C}_{p,q}$. □

Now we want to prove a simple but important approximation property for $\lambda^g_{p,q}$, already suggested by the proof of Proposition 3.4.

**Proposition 3.6.** Let $(p_n)_{n \in \mathbb{N}}, (q_n)_{n \in \mathbb{N}}$ be two sequences of points in $\mathbb{M}$ such that $p \leq p_n \ll q_n \leq q$, $p_n \xrightarrow{n \to \infty} p$, $q_n \xrightarrow{n \to \infty} q$. Then, for any $K \subset \mathcal{C}_{p,q}$ compact, we have (possibly after passing to subsequences so as to guarantee that $K \subset \mathcal{C}_{p_n,q_n}$ for all $p_n, q_n$ in the respective subsequences) that $\sup_K \left| \lambda^g_{p_n,q_n} - \lambda^g_{p,q} \right| \xrightarrow{n \to \infty} 0$.

**Proof.** Obviously, we have $\mathcal{C}_{p_n,q_n} \subset \mathcal{C}_{p,q}$ for all $n$. From now on, we tacitly assume without loss of generality that $K \subset \mathcal{C}_{p_n,q_n}$ for all $n$. The assertion then follows from the continuity of $d_g$. □

An important peculiarity of $\lambda^g_{p,q}$ in Minkowski spacetime which also survives to a great extent in the general case is that a timelike geodesic segment linking $p$ to $q$ which maximizes the Lorentzian distance between these two points will be, up to (an inevitable) reparametrization, a complete orbit of the diffeomorphism flow associated to the foliation induced by $\lambda^g_{p,q}$.

**Proposition 3.7.** Any maximal, unit-speed future directed timelike geodesic segment $\gamma : [0, d_g(p,q)]$ between $p \ll q$ realizes the Lorentzian distance to the level sets of $\lambda^g_{p,q}$. That is, for each $\lambda \in \mathbb{R}$, if $t_\lambda \in (0, d_g(p,q))$ is such that $\gamma((0,d_g(p,q))) \cap (\lambda^g_{p,q})^{-1}(\lambda) = \{\gamma(t_\lambda)\}$, then

$$d_g(\gamma(t'), (\lambda^g_{p,q})^{-1}(\lambda)) \leq \sup_{r'' \in \Sigma_\lambda} d_g(\gamma(t'), r'') = d_g(\gamma(t'), \gamma(t_\lambda)), \forall t' \leq t_\lambda,$$

$$d_g((\lambda^g_{p,q})^{-1}(\lambda), \gamma(t')) \leq \sup_{r'' \in \Sigma_\lambda} d_g(r'', \gamma(t')) = d_g(\gamma(t_\lambda), \gamma(t'')), \forall t' \geq t_\lambda.$$

**Proof.** We will just prove the claim for $t' \geq t_\lambda$, the other case being analogous. Let $r \in \Sigma_\lambda$ realize the Lorentzian distance between $\Sigma_\lambda \simeq (\lambda^g_{p,q})^{-1}(\lambda)$ and $r' \simeq \gamma(t')$, i.e. $d_g(r, r') = d_g(\Sigma_\lambda, r')$, and $r_\lambda = \gamma(t_\lambda)$. The existence of $r$ follows from the continuity of $d_g$ within $\mathcal{C}_{p,q}$ and the fact that all past directed, past inextendible causal curves in $\mathbb{M}$ issuing from $r'$ must cross $\Sigma_\lambda$ in a compact subset thereof before leaving $\mathcal{C}_{p,q}$ since the former is a Cauchy hypersurface w.r.t. the latter. We have the following six facts: from (a) the maximality of $\gamma$, it follows that

- $d_g(p, r_\lambda) + d_g(r_\lambda, r') = d_g(p, r')$;
- $d_g(r_\lambda, r') + d_g(r', q) = d_g(r', q)$;
- from (b) the definition of $\Sigma_\lambda$;

$b_1$ $d_g(p, r) = e^\lambda d_g(r, q)$;
by maximality and the unit parametrization of \( \gamma \) first part of the claim. Since \( d \) is given by \( \gamma \) from [21] the following properties of \( \Gamma \) with \( g \) with \( \gamma \) 

We have then the following implications:

\[
\begin{align*}
(b1) & \quad d_g(p, r \lambda) = e^\lambda d_g(r \lambda, q); \\
\text{(c1)} & \quad d_g(p, r) + d_g(r, r') \leq d_g(p, r'); \\
\text{(c2)} & \quad d_g(r, r') + d_g(r', q) \leq d_g(r, q).
\end{align*}
\]

Finally, from (c) the reverse triangular inequality for \( d_g \),

\[
\begin{align*}
(b2) & \quad d_g(p, r) + d_g(r, r') \leq d_g(p, r'); \\
(b2) & \quad d_g(r, r') + d_g(r', q) \leq d_g(r, q).
\end{align*}
\]

We have then the following implications:

\[
\begin{align*}
(b1) + (c1) + (c2) & \quad (d1) \quad d_g(p, r') \geq e^\lambda d_g(r', q) + (1 + e^\lambda) d_g(r, r'); \\
(b2) + (a1) + (a2) & \quad (d2) \quad d_g(p, r') = e^\lambda d_g(r', q) + (1 + e^\lambda) d_g(r \lambda, r'); \\
(d1) + (d2) & \quad (d3) \quad d_g(r \lambda, r') \geq d_g(r, r').
\end{align*}
\]

However, from the definition of \( r \) we must have \( d_g(r \lambda, r') \leq d_g(r, r') \), hence the claim follows.

Corollary 3.8. Any maximal, unit-speed future directed timelike geodesic segment \( \gamma : [0, d_g(p, q)] \) between \( p \ll q \) is an orbit of \( \lambda^g_{p,q} \), up to reparametrization.

Proof. Notice that, due to the maximality of \( \gamma \), \( \gamma(t) \) doesn’t belong to the timelike cut locus of neither \( p \) nor \( q \), hence \( d_g(p, \cdot) \) and \( d_g(\cdot, q) \) are smooth in an open neighborhood of \( \gamma(t) \) for any \( t \in (0, d_g(p, q)) \), and thus are the level sets of \( \lambda^g_{p,q} \). Moreover, maximality of \( \gamma \) together with Proposition 3.7 also imply that the direction of the tangent vector \( \dot{\gamma}(t) \) coincides with the direction of greatest variation of the ratio \( \frac{d_g(p, \cdot)}{d_g(\cdot, q)} \). Since \( \log \) is strictly increasing, this shows that the covector \( g(\dot{\gamma}(t), \cdot)(\gamma(t)) \) points in the same direction as \( d\lambda^g_{p,q}(\gamma(t)) \), thus establishing the first part of the claim. Since \( d_g(\gamma(t), q) = d_g(p, q) - d_g(p, \gamma(t)) \) and \( t = d_g(p, \gamma(t)) \) by maximality and the unit parametrization of \( \gamma \), the desired reparametrization of \( \gamma \) is given by

\[
t \mapsto \lambda^g_{p,q}(\gamma(t)) = \log \left( \frac{t}{d_g(p, q) - t} \right).
\]

Proposition 3.7 together with formula (d2) in its proof gives two interesting alternative expressions for \( \lambda^g_{p,q} \):

\[
\lambda^g_{p,q}(r) = \log \left( \frac{d_g(p, r)}{d_g(r, q)} \right) = \begin{cases} 
- \log \left( e^\lambda + (1 + e^\lambda) \frac{d_g(r, (\lambda^g_{p,q})^{-1}(r))}{d_g(p, r)} \right) & (\lambda^g_{p,q}(r) \leq \lambda) \\
\log \left( e^\lambda + (1 + e^\lambda) \frac{d_g((\lambda^g_{p,q})^{-1}(r), r)}{d_g(r, q)} \right) & (\lambda^g_{p,q}(r) \geq \lambda)
\end{cases}
\]

We will use this formula in Section 4 to give a generalization of \( \lambda^g_{p,q} \) which is defined when \( p \) (resp. \( q \)) is not given, but instead an acausal hypersurface to the past of \( q \) (resp. future of \( p \)) is.

3.2. Finer details in the small. If \( \mathcal{O}_{p,q} \) is suitably small, then much more information about \( \lambda^g_{p,q} \) can be obtained. To this avail, recall now that, for \( (\mathcal{M}, g) \) strongly causal and \( p \ll q \) belonging to geodesically convex regions, \( \frac{1}{2}d^2_g \) coincides within \( \{(r, s) : p \ll r \leq s \ll q\} \) with Synge’s world-function, given by

\[
\Gamma_g(p, q) = -\frac{1}{2} \int_0^1 \tilde{g}(\gamma_{p,q}(s), \dot{\gamma}_{p,q}(s)) ds,
\]

with \( \gamma_{p,q} \) the (only) geodesic segment from \( p = \gamma_{p,q}(0) \) to \( q = \gamma_{p,q}(1) \). We recall from [21] the following properties of \( \Gamma_g(p, q) \):
\( \Gamma_g \in C^\infty \left( \bigcup_{p \in \mathcal{M}} \{ p \} \times \mathcal{U}_p \right) \), where \( \mathcal{U}_p \) is an open, geodesically convex neighborhood of \( p \);

- \( \Gamma_g(p, q) = \Gamma_g(q, p) \);

- \( \nabla_a \Gamma_g(p, \cdot) = -\gamma^a_{p, \cdot} \) and \( \nabla^a \Gamma_g(\cdot, q) = -\gamma^a_{\cdot, q} \), where \( \cdot \) denotes the variable on which \( \nabla \) acts. We immediately have the fundamental Gauss’s Lemma

\[
(12) \quad g^{-1}(d_p \Gamma_g(p, q), d_p \Gamma_g(p, q)) = g^{-1}(d_q \Gamma_g(p, q), d_q \Gamma_g(p, q)) = -2 \Gamma_g(p, q),
\]

where \( d_p \) and \( d_q \) denote respectively the differential with respect to the first and second variables.

- \( (\nabla_a \nabla_b \Gamma_g(p, \cdot))(p) = -\gamma_{ab}(p) \).

Recalling that any causally simple spacetime is strongly causal, we write \( \lambda^a_{p,q}(r) = \frac{1}{2} (\log(\Gamma_g(p, r)) - \log(\Gamma_g(r, q))) \), whence it follows that

\[
(13) \quad T^a = \frac{\nabla^a \lambda^2_{p,q}}{g^{-1}(d^a \lambda^2_{p,q}, d^2 \lambda^2_{p,q})} = \frac{\Gamma_g(p, r) \nabla^a \Gamma_g(r, q) - \Gamma_g(r, q) \nabla^a \Gamma_g(p, r)}{\Gamma_g(p, r) + \Gamma_g(r, q) + g^{-1}(d \Gamma_g(p, r), d \Gamma_g(r, q))}
\]

generates the flux of diffeomorphisms \( \lambda \mapsto u^a_{p,q} \), associated to the foliation induced by \( \lambda^a_{p,q} \) that is, \( T \) is the unique vector field satisfying the following properties:

- \( d \lambda^a_{p,q}(r)(T) = 1 \) for all \( r \in \mathcal{O}_{p,q} \);
- \( g(T, X) = 0 \) for all \( X^a \) tangent to \( (\lambda^a_{p,q})^{-1}(t), t \in \mathbb{R} \).

To obtain formula (13), notice that the above properties imply that

\[
(14) \quad g(T, T) = \frac{1}{g^{-1}(d \lambda^a_{p,q}, d \lambda^a_{p,q})},
\]

where

\[
(15) \quad g^{-1}(d \lambda^a_{p,q}, d \lambda^a_{p,q}) = -\frac{1}{2} \left( \frac{1}{\Gamma_g(p, \cdot)} + \frac{1}{\Gamma_g(\cdot, q)} + \frac{g^{-1}(d \Gamma_g(p, \cdot), d \Gamma_g(\cdot, q))}{\Gamma_g(p, \cdot) \Gamma_g(\cdot, q)} \right),
\]

of which (13) is an immediate consequence. Let us now pay attention to the limiting form of \( T \) in the future (resp. past) horizon \( \partial_+ \mathcal{O}_{p,q} \) (resp. \( \partial_- \mathcal{O}_{p,q} \)). We obtain these limits by sending \( \Gamma_g(r, q) \to 0 \) (resp. \( \Gamma_g(p, r) \to 0 \)) in (11) while keeping \( \Gamma_g(p, r) \) (resp. \( \Gamma_g(r, q) \)) constant (notation: \( \lim_{\partial_+ \mathcal{O}_{p,q}} \)), yielding

\[
(16) \quad \lim_{\partial_+ \mathcal{O}_{p,q}} T^a = \frac{-\Gamma_g(\cdot, q) \nabla^a \Gamma_g(p, \cdot)}{\Gamma_g(p, \cdot) \nabla^a \Gamma_g(p, q) + g^{-1}(d \Gamma_g(p, \cdot), d \Gamma_g(\cdot, q))},
\]

\[
\lim_{\partial_- \mathcal{O}_{p,q}} T^a = \frac{-\Gamma_g(\cdot, \cdot) \nabla^a \Gamma_g(p, \cdot)}{\Gamma_g(p, \cdot) \nabla^a \Gamma_g(p, q) + g^{-1}(d \Gamma_g(p, \cdot), d \Gamma_g(\cdot, q))},
\]

That is, \( T \) extends continuously to a null vector field over \( \partial \mathcal{O}_{p,q} \) which is tangent and normal to \( \partial_+ \mathcal{O}_{p,q} \) and \( \partial_- \mathcal{O}_{p,q} \), and vanishing at \( p \) and \( q \). Particularly, \( T \) is tangent to the null generators of \( \partial_+ \mathcal{O}_{p,q} \) and \( \partial_- \mathcal{O}_{p,q} \), which entails that \( \nabla_T T = \kappa^a_{p,q} T \), where \( \kappa^a_{p,q} \) (resp. \( \kappa_{p,q}^- \)) is a scalar function on \( \partial_+ \mathcal{O}_{p,q} \) (resp. \( \partial_- \mathcal{O}_{p,q} \)) which measures the failure of any extension of \( \lambda^2_{p,q} \) to \( \partial_+ \mathcal{O}_{p,q} \) (resp. \( \partial_- \mathcal{O}_{p,q} \)) in being an affine parameter for the latter’s null generators. Another way of seeing \( \kappa^a_{p,q} \) is as the magnitude (up to a sign) of the “near-horizon” acceleration \( g(T, T)^{-1} \nabla_T T \) of the orbits of \( T \) multiplied by the redshift factor \( (-\gamma(T, T))^{\frac{1}{2}} \). As such, it is fair to

---

3We remark, however, that \( T \) determines this extension up to a smooth choice of an additive constant on each null generator, in the same way as the extension of isometries in Minkowski spacetime to null infinity in the sense of Penrose suffers from the so-called supertranslation ambiguity [89].
call \( \kappa_{p,q}^\pm \) (resp. \( \kappa_{p,q}^- \)) the *future* (resp. *past*) surface gravity of \( \mathcal{O}_{p,q} \), in the spirit of the zeroth law of black hole dynamics [33].

To compute the tangential acceleration of \( T \) and, therefore, \( \kappa_{p,q}^\pm \) we define the auxiliary functions

\[
\begin{align*}
 h_{p,q}(r) & \doteq \Gamma_g(p, r) + \Gamma_g(r, q) + g^{-1}(d\Gamma_g(p, r), d\Gamma_g(r, q)), \\
f_p(r) & \doteq \frac{\Gamma_g(p, r)}{h_{p,q}(r)}, \\
f_q(r) & \doteq \frac{\Gamma_g(r, q)}{h_{p,q}(r)},
\end{align*}
\]

which allows us to write

\[
\begin{align*}
 T^a &= \frac{1}{2} (f_q \nabla^a \Gamma_g(p, r) - f_p \nabla^a \Gamma_g(r, q)), \\
g(T, T) &= -2 \frac{\Gamma_g(p, \cdot) \Gamma_g(\cdot, q)}{h_{p,q}} \\
 &= -2 f_p \Gamma_g(\cdot, q) \\
 &= -2 f_q \Gamma_g(p, \cdot) \\
 &= -2 h_{p,q} f_p f_q.
\end{align*}
\]

This gives us

\[
\begin{align*}
\nabla_a T_b &= \frac{1}{2} (f_q \nabla_a \nabla_b \Gamma_g(p, \cdot) - f_p \nabla_a \nabla_b \Gamma_g(\cdot, q)) \\
&\quad \quad + \frac{1}{2} (\nabla_b \Gamma_g(p, \cdot) \nabla_a f_q - \nabla_b \Gamma_g(\cdot, q) \nabla_a f_p) \\
&\quad \quad = \frac{1}{2} (f_q \nabla_a \nabla_b \Gamma_g(p, \cdot) - f_p \nabla_a \nabla_b \Gamma_g(\cdot, q)) \\
&\quad \quad \quad \quad + \frac{1}{2 h_{p,q}} (\nabla_a \Gamma_g(\cdot, q) \nabla_b \Gamma_g(p, \cdot) - \nabla_b \Gamma_g(\cdot, q) \nabla_a \Gamma_g(p, \cdot)) \\
&\quad \quad \quad \quad - \frac{1}{h_{p,q}} T_b \nabla_a h_{p,q}.
\end{align*}
\]

The first term is manifestly symmetric, whereas the second is manifestly antisymmetric. Hence

\[
\begin{align*}
\nabla_a T_b &= \frac{1}{4} (f_q \nabla^a \Gamma_g(p, r) - f_p \nabla^a \Gamma_g(r, q)) (f_q \nabla_a \nabla_b \Gamma_g(p, \cdot) - f_p \nabla_a \nabla_b \Gamma_g(\cdot, q)) \\
&\quad \quad + \frac{1}{4 h_{p,q}} (f_q \nabla^a \Gamma_g(p, r) - f_p \nabla^a \Gamma_g(r, q)) (\nabla_a \Gamma_g(\cdot, q) \nabla_b \Gamma_g(p, \cdot) \\
&\quad \quad \quad \quad - \nabla_b \Gamma_g(\cdot, q) \nabla_a \Gamma_g(p, \cdot)) + \frac{1}{h_{p,q}} T^b T^a \nabla_a h_{p,q} \\
&= - \frac{1}{2} (f_q \nabla^a \Gamma_g(p, \cdot) + f_p \nabla^a \Gamma_g(\cdot, q) + f_p f_q \nabla^a g^{-1}(d\Gamma_g(p, \cdot), d\Gamma_g(\cdot, q)) \\
&\quad \quad + \frac{1}{4 h_{p,q}} [(f_q \nabla^a \Gamma_g(p, \cdot) + f_p \nabla^a \Gamma_g(\cdot, q)) g^{-1}(d\Gamma_g(p, \cdot), d\Gamma_g(\cdot, q)) \\
&\quad \quad \quad \quad + 2 (f_q \Gamma_g(p, \cdot) \nabla_b \Gamma_g(\cdot, q) - f_p \Gamma_g(\cdot, q) \nabla_b \Gamma_g(p, \cdot))] + \frac{1}{h_{p,q}} T^b T^a \nabla_a h_{p,q},
\end{align*}
\]
where we have used formula (12) in the last passage, and, recalling that the second term is antisymmetric,

\begin{equation}
T^a T^b \nabla_a T_b = \frac{1}{2} T^a \nabla_a g(T, T)
\end{equation}

\begin{align*}
&= - \frac{1}{4} \left( f_q \nabla^b \Gamma_g(p, r) - f_p \nabla^b \Gamma_g(r, q) \right) \left( f_q^2 \nabla_b \Gamma_g(p, \cdot) + f_p^2 \nabla_b \Gamma_g(\cdot, q) \right) \\
&\quad + f_p f_q \nabla_b g^{-1}(d\Gamma_g(p, \cdot), d\Gamma_g(\cdot, q)) + \frac{g(T, T)}{h_{p, q}} T^a \nabla_a h_{p, q} \\
&= - \frac{g(T, T)}{4} \left( f_q^2 + f_p^2 - (f_p - f_q) \frac{1}{2h_{p, q}} g^{-1}(d\Gamma_g(p, \cdot), d\Gamma_g(\cdot, q)) \right) \\
&\quad - \frac{1}{2h_{p, q}} T^b \nabla_b g^{-1}(d\Gamma_g(p, \cdot), d\Gamma_g(\cdot, q)) + \frac{g(T, T)}{h_{p, q}} T^a \nabla_a h_{p, q}.
\end{align*}

To compute

\[ \kappa_{p, q}^\pm = \lim_{\to \pm \partial \kappa_{p, q}} \frac{T^a T^b \nabla_a T_b}{g(T, T)}, \]

we need to know the limiting form of \( h_{p, q}(r) \), \( f_p(r) \) and \( f_q(r) \) as \( r \) approaches \( \partial \kappa_{p, q} \).

In fact, we have that

\[ h_{p, q} \left\{ \begin{array}{c}
\to_+ \partial \kappa_{p, q} \quad \Gamma_g(p, \cdot) + g^{-1}(d\Gamma_g(p, \cdot), d\Gamma_g(\cdot, q)) \doteq h_{p, q}^+
\\
\to_- \partial \kappa_{p, q} \quad \Gamma_g(\cdot, q) + g^{-1}(d\Gamma_g(p, \cdot), d\Gamma_g(\cdot, q)) \doteq h_{p, q}^-
\end{array} \right. \]

Therefore

\[ f_p \left\{ \begin{array}{c}
\to_+ \partial \kappa_{p, q} \quad \Gamma_g(p, \cdot) \doteq f_p^+
\\
\to_- \partial \kappa_{p, q} \quad 0
\end{array} \right. \]

\[ f_q \left\{ \begin{array}{c}
\to_+ \partial \kappa_{p, q} \quad 0
\\
\to_- \partial \kappa_{p, q} \quad \Gamma_g(\cdot, q) \doteq f_q^-
\end{array} \right. \]

whence we conclude that

\[ \kappa_{p, q}^+ = - \frac{1}{4} \left[ f_p^+ \frac{2}{h_{p, q}} \left( \Gamma_g(p, \cdot) - \frac{1}{2} g^{-1}(d\Gamma_g(p, \cdot), d\Gamma_g(\cdot, q)) \right) \\
\quad - \frac{1}{2h_{p, q}} T^b \nabla_b g^{-1}(d\Gamma_g(p, \cdot), d\Gamma_g(\cdot, q)) \right] + \frac{1}{h_{p, q}} T^a \nabla_a h_{p, q}^+, \]

\[ \kappa_{p, q}^- = - \frac{1}{4} \left[ f_q^- \frac{2}{h_{p, q}} \left( \Gamma_g(\cdot, q) + \frac{1}{2} g^{-1}(d\Gamma_g(p, \cdot), d\Gamma_g(\cdot, q)) \right) \\
\quad - \frac{1}{2h_{p, q}} T^b \nabla_b g^{-1}(d\Gamma_g(p, \cdot), d\Gamma_g(\cdot, q)) \right] + \frac{1}{h_{p, q}} T^a \nabla_a h_{p, q}^-. \]

**Remark 3.2.** There is a hypothesis about the normalization of \( T^a \) implicit in the definition of \( \kappa_{p, q}^\pm \). More precisely, our definition is conditioned to the following fact: if \( r \) is the middle point of the maximal timelike geodesic in \( \mathcal{M} \) linking \( p \) to \( q \) (and hence \( \Gamma_g(p, r) = \Gamma_g(q, r) \)), clearly implying that \( \nabla^a \Gamma_g(p, r) = -\nabla^a \Gamma_g(r, q) \), then \( g(T, T)(r) = -\frac{d_g(p, q)^2}{16} \). If we rescale \( T^a \) by a factor \( R \in \mathbb{R} \), \( \kappa_{p, q}^\pm \) is rescaled by the same factor, by definition. In Minkowski spacetime, for instance, this has
the consequence that, if we take \( p = p_0 \) and \( q = q_0 \) as in Subsection 2.3 and rescale the \( T^a \) associated to the diamond \( \mathcal{O}_{R_{p_0,q_0}} \) by a factor \( \frac{\tau}{\tau_0} \), so as to maintain 
g(T, T)(r) = \eta(T, T)(0) = -1, \) it follows that \( \kappa_{\pm}^{\pm} \rightarrow^{R \rightarrow +\infty} 0, \) in a way consistent
with the fact that \( T^a R^{a \rightarrow +\infty} (\partial_0)^a \) (physically, the “Unruh temperature” associated to time translations in Minkowski spacetime is zero).

4. A variation over the theme: half-diamonds

In this section, we will apply our strategy to a slightly different kind of region. Consider a causally simple spacetime \((\mathcal{M}, g)\) endowed with a smooth global time function \( \tau \), and denote the latter’s level hypersurfaces by \( \Sigma_t \equiv \tau^{-1}(t) \).

Definition 4.1. Let \( t \in \mathbb{R} \), \( p \in \Gamma^+ \Sigma_t \), \( q \in \Gamma^+ \Sigma_t \) such that \( J^+(p) \cap J^-(\Sigma_t) \) and \( J^-(q) \cap J^+(\Sigma_t) \) are compact. The (relatively compact) regions of the form \( \mathcal{O}^{-}(t) \equiv I^+(p) \cap I^-(\Sigma_t) \) and \( \mathcal{O}^{+}(t) \equiv I^-(q) \cap I^+(\Sigma_t) \) are respectively called past and future half-diamonds at time \( t \).

Half-diamonds also enjoy the property of being globally hyperbolic, but, similarly to the case of diamonds, generally are not of the form \( \text{int} D^{-}(J^+(p) \cap \Sigma_t) \) or \( \text{int} D^{+}(J^-(q) \cap \Sigma_t) \) with respect to the ambient spacetime, unless they are contained in a geodesically convex neighborhood.

The alternative formulae for \( \lambda_{p,q} \) derived in Section 3 suggest the following intrinsic global time functions for \( \mathcal{O}^{-}(t) \) and \( \mathcal{O}^{+}(t) \):

\[
(21) \quad \mathcal{O}^{-}(t) \ni r \mapsto \lambda_{p,q}^{\pm}(r) \equiv - \log \left( e^t + (1 + e^t) \frac{d_g(r, \Sigma_t)}{d_g(p, r)} \right) \in (-\infty, t];
\]

\[
(22) \quad \mathcal{O}^{+}(t) \ni r \mapsto \lambda_{p,q}^{\pm}(r) \equiv \log \left( e^t + (1 + e^t) \frac{d_g(\Sigma_t, r)}{d_g(r, q)} \right) \in [t, +\infty).
\]

Notice that the definition of \( \lambda_{p,q}^{\pm}(r) \) doesn’t use any information whatsoever about \( \Sigma_t \setminus J^+(p) \) (resp. \( \Sigma_t \setminus J^-(q) \)) — particularly, if \( p \ll q \) and \( \Sigma_t \cap I^+(p) \) (resp. \( \Sigma_t \cap I^-(q) \)) equals \( \lambda_{p,q}^{-1}(0) \), then \( \lambda_{p,q}^{\pm}(r) \equiv \lambda_{p,q}^{\pm}(0) \) (resp. \( \lambda_{p,q}^{\pm}(r) \equiv \lambda_{p,q}^{\pm}(t) \)) (resp. \( \mathcal{O}^{+}(t) \)).

Employing the same methods used in our study of \( \lambda_{p,q}^{\pm}(r) \), one concludes that \( \lambda_{p,q}^{\pm}(r) \) and \( \lambda_{q,p}^{\pm}(r) \) are locally Lipschitz and twice differentiable almost everywhere, and foliate respectively \( \mathcal{O}^{-}(t) \) and \( \mathcal{O}^{+}(t) \) by acausal Cauchy surfaces. Moreover, for \( \mathcal{O}^{-}(t) \), \( \mathcal{O}^{+}(t) \) contained in a geodesic neighborhood, one can derive analogous formulae for the vector fields generating the diffeomorphism flows associated to these foliations — particularly, though it is not immediately obvious from the formulae above, the vector fields generating the flows do admit smooth extensions respectively to the past horizon \( \partial^- \mathcal{O}^{-}(t) \equiv \partial I^+(p) \cap J^-(\Sigma_t) \) of \( \mathcal{O}^{-}(t) \) and the future horizon \( \partial^+ \mathcal{O}^{+}(t) \equiv \partial I^-(q) \cap J^+(\Sigma_t) \) of \( \mathcal{O}^{+}(t) \), being there once more tangent to the null geodesic generators. Moreover, the asymptotic surface gravities near \( p \) and \( q \) are the same as in the case of diamonds.

5. Conclusions and remarks (or: a quantum coda)

We have presented a rather general procedure of building global time functions for diamonds in general spacetimes. Our assumptions seem to be optimal. The method is sufficiently robust to be adapted so as to satisfy boundary conditions at
conformal infinity, as for instance in the case of wedges in spacetimes endowed with past and future null infinity, in the spirit of [33, 34] (to be addressed in a future publication).

One must mention some potential applications for quantum field theory in curved spacetimes, which to a great extent inspired the developments presented above: it is known [26, 10] that the vacuum state $\omega_0$ of a conformally invariant quantum field theory in Minkowski spacetime is a KMS (i.e., finite-temperature) state w.r.t. the $W^*$-dynamical system $(\mathfrak{A}(\Theta_{p,q}), \alpha_\lambda)$, where $\mathfrak{A}(\Theta_{p,q})$ is the local von Neumann algebra of observables associated to $\Theta_{p,q}$ and $\alpha_\lambda$ is the group of $*$-automorphisms induced by $u_\lambda^{p,q}$, and hence $-2\pi \lambda$ is precisely the flow parameter of the Tomita-Takesaki modular group intrinsically associated to the standard pair $(\mathfrak{A}(\Theta_{p,q}), \omega_0)$. This result led Martinetti and Rovelli [30] to propose that $\lambda$ should then be realized as a “thermal time” for finite lifetime observers in curved spacetime. Heuristically, in the general situation we have discussed, an asymptotic form of the KMS periodicity condition as $\lambda \to \pm \infty$ (i.e. a kind of “return to equilibrium”) is strongly suggested by leading term of the asymptotic short-distance expansion of the two-point function in powers of Synge’s world function [35, 21, 3]. An honest proof of this fact, however, requires obtaining certain decay estimates for solutions of the wave and Klein-Gordon equations in diamonds at large “cosmological times” $\lambda_{g_{p,q}}^\theta \to \pm \infty$. Work on such estimates is in progress. Having asymptotic freedom in mind, one is hence led to the possibility that the “thermal time hypothesis” is only realized asymptotically as a return to equilibrium for more realistic QFT’s, since this becomes essentially (at least, on a geometrical level) a scaling limit.

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