The $L^p$ boundedness of wave operators for Schrödinger operators with threshold singularities II. Even dimensional case

Domenico Finco* and Kenji Yajima†

1 Introduction

Let $H = -\Delta + V(x)$ be a Schrödinger operator on $\mathbb{R}^m$, $m \geq 1$, with real potential $V(x)$ such that $|V(x)| \leq C\langle x \rangle^{-\delta}$, $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$, for some $\delta > 2$. Then, $H$ with domain $D(H) = H^2(\mathbb{R}^m)$, the Sobolev space of order 2, is selfadjoint in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^m)$ and $C^\infty_0(\mathbb{R}^m)$ is a core. The spectrum $\sigma(H)$ of $H$ consists of absolutely continuous part $[0, \infty)$ and a finite number of non-positive eigenvalues $\{\lambda_j\}$ of finite multiplicities. The singular continuous spectrum and positive eigenvalues are absent from $H$. We denote the point, the continuous and the absolutely continuous subspaces for $H$ by $\mathcal{H}_p$, $\mathcal{H}_c$ and $\mathcal{H}_{ac}$ respectively, and the orthogonal projections onto the respective subspaces by $P_p$, $P_c$ and $P_{ac}$. We have $\mathcal{H}_{ac} = \mathcal{H}_c$ and $P_{ac} = P_c$; $H_0 = -\Delta$ is the free Schrödinger operator.

The wave operators $W_\pm$ are defined by the following strong limits

$$W_\pm = \lim_{t \to \pm \infty} e^{itH}e^{-itH_0}$$

in $\mathcal{H} \equiv L^2(\mathbb{R}^m)$. It is well known that the limits exist and are complete in the sense that $\text{Image } W_\pm = \mathcal{H}_{ac}$. The wave operators satisfy the so called intertwining property and the continuous part of $H$ is unitarily equivalent to $H_0$ via $W_\pm$: For any Borel functions on $\mathbb{R}$, we have

$$f(H)P_{ac}(H) = W_\pm f(H_0)W_\pm^*.$$  

(1.1)

*Department of Mathematics, Gakushuin University, 1-5-1 Mejiro, Toshima-ku, Tokyo 171-8588, Japan. Present address: Institute für Angewandte Mathematik

†Department of Mathematics, Gakushuin University, 1-5-1 Mejiro, Toshima-ku, Tokyo 171-8588, Japan. Supported by JSPS grant in aid for scientific research No. 14340039
It follows that the mapping properties of \( f(H)P_{ac}(H) \) may be deduced from those of \( f(H_0) \) once corresponding properties of \( W_\pm \) are known. In this paper we shall prove the following theorem. We say that \( H \) is of exceptional type if there exist no non-trivial solutions of \(-\Delta \phi + V(x)\phi = 0\) which satisfies \(|\phi(x)| \leq C|x|^{2-m}; H \) is of generic type otherwise (see Definition 3.3 for an equivalent definition). We write \( \mathcal{F} \) for the Fourier transform. Throughout this paper, we assume that \( V \) satisfies the following condition:

\[
\mathcal{F}(\langle x \rangle^{2\sigma} V) \in L^{m_*} \quad \text{for} \quad \sigma > \frac{1}{m_*} = \frac{m-2}{m-1}.
\]

(1.2)

For integers \( k \geq 0, W^{k,p}(\mathbb{R}^m) \) is the Sobolev space of order \( k \).

**Theorem 1.1.** Let \( m \geq 6 \) be even and \( V \) satisfy (1.2).

1. Suppose, in addition, that \(|V(x)| \leq C|x|^{-(m+2+\varepsilon)}\) for some \( C > 0 \) and \( \varepsilon > 0 \) and that \( H \) is of generic type. Then, for all \( 1 \leq p \leq \infty, W_\pm \) extend to bounded operators in \( L^p(\mathbb{R}^m) \):

\[
\|W_\pm u\|_{L^p} \leq C_p\|u\|_{L^p}, \quad u \in L^p(\mathbb{R}^m) \cap L^2(\mathbb{R}^m).
\]

(1.3)

For \( 1 < p < \infty, W_\pm \) actually are bounded in \( W^{k,p}(\mathbb{R}^m) \) for \( 0 \leq k \leq 2 \). If derivatives \( \partial^\alpha V(x) \) are bounded for \(|\alpha| \leq \ell \) in addition, then \( W_\pm \) are bounded in \( W^{k,p}(\mathbb{R}^m) \) for all \( 0 \leq k \leq \ell + 2 \) and \( 1 < p < \infty \). For \( p = 1, \infty \) the same holds if \( \partial^\alpha V(x), |\alpha| \leq \ell, \) satisfy (1.2) and \(|\partial^\alpha V(x)| \leq C|x|^{-(m+2+\varepsilon)}\) for some \( C > 0 \) and \( \varepsilon > 0 \).

2. Suppose, in addition, that \(|V(x)| \leq C|x|^{-(m+4+\varepsilon)}\) if \( m = 6 \), and \(|V(x)| \leq C|x|^{-(m+3+\varepsilon)}\) if \( m \geq 8 \) for some \( C > 0 \) and \( \varepsilon > 0 \), and that \( H \) is of exceptional type. Then, for \( m/(m-2) < p < m/2 \) and \( 0 \leq k \leq 2 \), \( W_\pm \) extend to bounded operators in \( W^{k,p}(\mathbb{R}^m) \):

\[
\|W_\pm u\|_{W^{k,p}} \leq C_p\|u\|_{W^{k,p}}, \quad u \in W^{k,p}(\mathbb{R}^m) \cap L^2(\mathbb{R}^m).
\]

(1.4)

If \( \partial^\alpha V(x) \) are bounded for \(|\alpha| \leq \ell \) in addition, then (1.4) holds for \( 0 \leq k \leq \ell + 2 \) and \( W_\pm \) are bounded in \( W^{k,p}(\mathbb{R}^m) \) for all \( 0 \leq k \leq \ell + 2 \).

Some remarks are in order.

**Remark 1.2.** Some condition like (1.2) is necessary for the theorem by virtue of the counter example due to [10] to the dispersive estimates for the corresponding time dependent Schrödinger equation, see below.

**Remark 1.3.** When \( m \geq 3 \) is odd, it is proved in a recent paper [25] that \( W_\pm \) are bounded in \( L^p(\mathbb{R}^m) \) for all \( 1 \leq p \leq \infty \) if \( V \) satisfies (1.2) and
\[ |V(x)| \leq C\langle x\rangle^{-(m+2+\varepsilon)} \] and \( H \) is of generic type; for \( p \) between \( \frac{m}{m-2} \) and \( \frac{m}{2} \) if \( V \) satisfies (1.2) and \( |V(x)| \leq C\langle x\rangle^{-(m+3+\varepsilon)} \) and \( H \) is of exceptional type. The argument in Section 7 below implies that this result extends to the continuity of \( W_\pm \) in \( W^{k,p}(\mathbb{R}^m) \) as in Theorem 1.1. The paper [25] will be referred to as [I] in what follows. In [4] an extension for some non-selfadjoint cases in \( m = 3 \) and its application to nonlinear equations is presented. When \( m = 1 \), it is recently shown ([6]) that \( W_\pm \) are bounded in \( L^p \) for \( 1 < p < \infty \) (but not for \( p = 1 \) or \( p = \infty \)) if \( \int_\mathbb{R} \langle x\rangle |V(x)| dx < \infty \) and \( H \) is of generic type, or if \( \int_\mathbb{R} \langle x\rangle^2 |V(x)| dx < \infty \) and \( H \) is of exceptional type (see [20], [2] for earlier results).

**Remark 1.4.** When \( m \geq 4 \) is even, it is long known ([22]) that (1.3) is satisfied for all \( 1 \leq p \leq \infty \) if \( V \) satisfies

\[
\sum_{\alpha \leq k+(m-2)/2} \left( \int_{|x-y| \leq 1} |\partial_\alpha^p V(y)|^{p_0} dy \right)^{\frac{1}{p_0}} \leq C\langle x\rangle^{-(\frac{3m+1+\varepsilon}{2})} \tag{1.5}
\]

for some \( p_0 > \frac{m}{2} \) and \( \varepsilon > 0 \) and if \( H \) is of generic type. If \( m \geq 6 \), condition (1.5) implies that \( \partial^\alpha V, |\alpha| \leq k \), satisfy both (1.2) and \( |\partial_\alpha^p V(x)| \leq C\langle x\rangle^{-(m+2+\varepsilon)} \) and Theorem 1.1 (1) improves the result of [22] for \( m \geq 6 \). When \( m = 2 \), it is known ([23], [12]) that \( W_\pm \) is bounded in \( L^p(\mathbb{R}^2) \) for \( 1 < p < \infty \) if \( V \) satisfies \( |V(x)| \leq C\langle x\rangle^{-6-\varepsilon} \) and if \( H \) is of generic type.

**Remark 1.5.** If \( m \geq 4 \) and if \( H \) is of exceptional type, \( W_\pm \) is not bounded in \( L^p(\mathbb{R}^m) \) when \( 1 \leq p < \frac{m}{m-2} \) because this would contradict Murata’s result ([16]) on the decay in time in weighted \( L^2 \) spaces of solutions \( e^{-itH}u \) of the corresponding time dependent Schrödinger equation. We strongly believe the same is true for \( \frac{m}{2} < p \leq \infty \) though the proof is missing. Notice that when \( m = 4, \frac{m}{m-2} = \frac{m}{2} = 2 \).

**Remark 1.6.** By interpolating (1.3) for different \( k' \)'s by the real interpolation method ([3]), estimates of Theorem 1.1 can be extended to the ones between Besov spaces.

When \( f(\lambda) = e^{-i\lambda}, \) (1.1) and (1.3) implies the so called \( L^p-L^q \) estimates for the propagator of the corresponding time dependent Schrödinger equation

\[
\|e^{-itH}P_c u\|_p \leq C|t|^{-m\left(\frac{1}{2} - \frac{1}{p}\right)} \|u\|_q \tag{1.6}
\]

where \( p, q \) are dual exponents, viz. \( 1/p + 1/q = 1 \), and \( 2 \leq p \leq \infty \) if \( H \) is of generic type and \( 2 \leq p < m/2 \) if \( H \) is of generic case. When \( 1 \leq m \leq 3 \) and if \( H \) is of generic type, the \( L^p-L^q \) estimate has been proven for \( 2 \leq p \leq \infty \) for
much wider class of potentials by more direct methods ([9], [17], [8]); when

\( m = 3 \) and \( H \) is of exceptional type it is proved that (1.6) holds for \( 2 \leq p < 3 \) and

\[
\|e^{-itH}P_c u\|_{L^3,\infty} \leq C_p t^{-\frac{1}{2}}\|u\|_{L^{2,1}}, \tag{1.7}
\]

replaces (1.6) at the end point where \( L^{p,q} \) are Lorentz spaces ([7], [24]). However, when \( m \geq 4 \), the result obtained by using wave operators via Theorem 1.1 (1) or [22] gives the best estimates so far as far as the decay and smoothness assumption on the potentials is concerned. We should also emphasize that the \( L^p-L^q \) estimate (1.6) is proven for the first time when \( m \geq 6 \) and \( H \) is exceptional type.

The intertwining property and the boundedness results, (1.1) and (1.3), may be applied for various other functions \( f(H)P_c \) and can provide useful estimates. We refer the readers to [I] as well as [21] and [22] for some more applications, and we shall be devoted to the proof of Theorem 1.1 in the rest of the paper.

We prove Theorem 1.1 only for \( W_- \), which we denote by \( W \) for brevity. We shall mainly discuss the \( L^p \) boundedness, as the extension to Sobolev spaces is immediate as will be shown in Section 7. We write \( R(z) = (H-z)^{-1} \) and \( R_0(z) = (H_0-z)^{-1} \) for resolvents. We parametrize \( z \in \mathbb{C} \setminus \{0, \infty\} \) by \( z = \lambda^2 \) by \( \lambda \in \mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\} \) and define \( G(\lambda) = R(\lambda^2) \) and \( G_0(\lambda) = R_0(\lambda^2) \) for \( \lambda \in \mathbb{C}^+ \). They are \( \mathcal{B}(\mathcal{H}) \)-valued meromorphic functions of \( \lambda \in \mathbb{C}^+ \) and the limiting absorption principle (LAP for short) asserts that \( G(\lambda) \) and \( G_0(\lambda) \) when considered as \( \mathcal{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma}) \)-valued functions have continuous extensions to \( \overline{\mathbb{C}}^+ = \{z : \Im z \geq 0\} \), the closure of \( \mathbb{C}^+ \), where \( \mathcal{H}_\gamma = L^2(\mathbb{R}^m, \langle x \rangle^{2\gamma}dx) \) is the weighted \( L^2 \) space and \( \sigma > \frac{1}{2} \). Our theory is based on the stationary representation of wave operators which expresses \( W \) via boundary values of the resolvents (cf. [14], [15]):

\[
W u = u - \frac{1}{\pi i} \int_0^\infty G(\lambda)V(G_0(\lambda) - G_0(-\lambda))u\lambda d\lambda. \tag{1.8}
\]

As in odd dimensional cases, we decompose \( W \) into the high and the low energy parts \( W = W_\geq + W_\leq \equiv W \Psi(H_0)^2 + W \Phi(H_0)^2 \), by using cut off functions \( \Phi(\lambda) \) and \( \Psi(\lambda) \) such that \( \Phi(\lambda)^2 + \Psi(\lambda)^2 \equiv 1 \), \( \Phi(\lambda) = 1 \) near \( \lambda = 0 \) and \( \Phi(\lambda) = 0 \) for \( |\lambda| > \lambda_0^2 \) for a small constant \( \lambda_0 > 0 \) to be specified below. The operators \( \Phi(H) \) and \( \Phi(H_0) \) have continuous integral kernels bounded by \( C_N(\lambda + 1 - \lambda_{\geq})^{-N} \) for any \( N \) and they are are bounded in \( L^p(\mathbb{R}^m) \) for any \( 1 \leq p \leq \infty \). By virtue of the intertwining property we have \( W_\geq = \Phi(H)W \Psi(H_0) \) and \( W_\leq = \Phi(H)W \Phi(H_0) \) and, combining this with (1.8)

\[
W_\leq = \Phi(H)\Phi(H_0) - \int_0^\infty \Phi(H)G(\lambda)V(G_0(\lambda) - G_0(-\lambda))\Phi(H_0)\lambda \frac{d\lambda}{\pi i}. \tag{1.9}
\]
\[ W_\geq = \Psi(H)\Psi(H_0) - \int_0^\infty \Psi(H)G(\lambda)V(G_0(\lambda) - G_0(-\lambda))\Psi(H_0)\frac{d\lambda}{\pi i}. \]  

(1.10)

We study the operators defined by the integrals in (1.9) and (1.10) separately. We use the following terminology.

**Definition 1.7.** We say that the integral kernel \( K(x, y) \) is admissible if

\[
\sup_x \int_{\mathbb{R}^m} |K(x, y)| \, dy + \sup_y \int_{\mathbb{R}^m} |K(x, y)| \, dx < \infty. 
\]  

(1.11)

It is well known that integral operators with admissible integral kernels are bounded in \( L^p \) for any \( 1 \leq p \leq \infty \).

**Definition 1.8.** The operator valued function \( K(\lambda) \) of \( \lambda \in (-\lambda_0, \lambda_0) \) which acts on functions on \( \mathbb{R}^m \) is said to satisfy property \((K)_\rho, \rho > 0\), if it satisfies the following two conditions:

1. For \( 0 \leq \gamma \leq m - 2 \), \( \lambda \mapsto \langle x \rangle^{\rho - \gamma} K(\lambda) \langle x \rangle^{\rho - \gamma} \in B(\mathcal{H}) \) is of class \( C^\gamma \).
2. For \( m - 2 \leq \gamma \leq m + 2 \), it is of class \( C^\gamma \) for \( \lambda \neq 0 \) and, for some \( C > 0 \) and \( N > 0 \),

\[
\|\langle x \rangle^{\rho - \frac{m-2}{2}} K^{(\frac{m}{2})} K(\lambda) \langle x \rangle^{\rho - \frac{m-2}{2}} \|_{B(\mathcal{H})} \leq C \langle \log \lambda \rangle^N, 
\]

\[
\|\langle x \rangle^{\rho - \frac{m+2}{2}} K^{(\frac{m}{2}+2)} K(\lambda) \langle x \rangle^{\rho - \frac{m+2}{2}} \|_{B(\mathcal{H})} \leq C |\lambda|^{-1} \langle \log \lambda \rangle^N. 
\]

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1. For \( 0 \leq \gamma \leq m - 2 \), \( \lambda \mapsto \langle x \rangle^{\rho - \gamma} K(\lambda) \langle x \rangle^{\rho - \gamma} \in B(\mathcal{H}) \) is of class \( C^\gamma \).
2. For \( m - 2 \leq \gamma \leq m + 2 \), it is of class \( C^\gamma \) for \( \lambda \neq 0 \) and, for some \( C > 0 \) and \( N > 0 \),

\[
\|\langle x \rangle^{\rho - \frac{m-2}{2}} K^{(\frac{m}{2})} K(\lambda) \langle x \rangle^{\rho - \frac{m-2}{2}} \|_{B(\mathcal{H})} \leq C \langle \log \lambda \rangle^N, 
\]

\[
\|\langle x \rangle^{\rho - \frac{m+2}{2}} K^{(\frac{m}{2}+2)} K(\lambda) \langle x \rangle^{\rho - \frac{m+2}{2}} \|_{B(\mathcal{H})} \leq C |\lambda|^{-1} \langle \log \lambda \rangle^N. 
\]

(1.12)

The plan of the paper is as follows. Section 2 is preparatory in nature. In subsection 2.1, we collect some results on the behavior of the free resolvent \( G_0(\lambda) \) on the reals, near the threshold \( \lambda = 0 \) in particular. When \( m \) is even, \( G_0(\lambda) \) is convolution operator with the kernel

\[
G_0(\lambda, x) = C_m e^{i\lambda|x|} \int_0^\infty e^{-t\frac{m-2}{2}} \left(\frac{t}{2} - i\lambda|x|\right)^\frac{m-3}{2} \, dt 
\]

(1.14)

where \( C_m = \frac{ie^{-i(2\nu+1)\pi/4}}{2(2\pi)^{\nu+1}\Gamma(\nu + \frac{1}{2})} \) and \( G_0(\lambda) \) contains a term whose \( (m - 2) \)-nd derivative becomes logarithmically singular at \( \lambda = 0 \). We study mapping properties of the derivatives of such operators in detail (see Proposition 2.6).

In subsection 2.2, we recall from [21] the result on the \( L^p \) boundedness of Born approximations of wave operators. Using these results, we study the behavior of \( (1 + G_0(\lambda)V)^{-1} \) in Section 3 and show that \( V(1 + G_0(\lambda)V)^{-1} \) satisfies the property \((K)_\rho\) for any \( \rho < \delta - 1 \) if \( H \) is of generic type; and
when $H$ is of exceptional type that $(1 + G_0(\lambda)V)^{-1}$ has the expansion near $\lambda = 0$ of the following form

$$(1 + G_0(\lambda)V)^{-1} = \frac{P_0 V}{\lambda^2} + \sum_{j=0}^{2} \sum_{k=1}^{2} \lambda^j \log^k \lambda D_{jk} + I + R_\tau(\lambda),$$

(1.15)

where $P_0$ is the orthogonal projection onto the 0 eigenspace of $H$, $VD_{jk}$ are finite rank operators from $\mathcal{H}_{-(\delta-3-\varepsilon)}$ to $\mathcal{H}_{\delta-3-\varepsilon}$ for any $\varepsilon > 0$, and $V R_\tau(\lambda)$ satisfies the condition $(K)_\rho$ for any $\rho < \delta - 3$ if $m = 6$ and $\rho < \delta - 2$ if $m \geq 8$. When the dimension $m$ becomes the larger the formula (1.15) for $(1 + G_0(\lambda)V)^{-1}$ becomes the less complex thanks to the fact that $\lambda^{m-2} \log \lambda$ becomes less singular as $m$ increases.

We show in Section 4 and Section 5 that the low energy part $W_<$ is bounded in $L^p$ for all $1 \leq p \leq \infty$ when $H$ is of generic type, and for $\frac{m-2}{2} < p < \frac{m}{2}$ when $H$ is of exceptional type, respectively. For proving this we substitute $G_0(\lambda)V(1 + G_0(\lambda)V)^{-1}$ for $G(\lambda)V$ on the right of (1.9). In view of the fact that $V(1 + G_0(\lambda)V)^{-1}$ in generic case and $VR_\tau(\lambda)$ in exceptional case satisfy property $(K)_\rho$ for some $\rho > m + 1$, we show in Section 4 that, if $K(\lambda)$ satisfies this property, then the operator $\Omega$ defined by

$$\Omega = \int_{0}^{\infty} \Phi(H)G_0(\lambda)K(\lambda)(G_0(\lambda) - G_0(-\lambda))\Phi(H_0)\lambda \Phi(\lambda) d\lambda$$

(1.16)

with an additional cut-off function $\Phi \in C_0^\infty(\mathbb{R})$ such that $\Phi(\lambda)\Phi(\lambda) = \Phi(\lambda)$ is an integral operator with an admissible integral kernel (see Proposition 4.2). The basic idea of the proof is similar to the one used for similar purpose in odd dimensions, however, because of the more complex structure of $G_0(\lambda,x)$, the argument becomes a bit more involved and a rather unexpected cancellation between $G_0(\lambda)$ and $G_0(-\lambda)$ plays a crucial role.

For studying $W_<$ when $H$ is of exceptional type, we substitute (1.15) for $(1 + G_0(\lambda)V)^{-1}$. Then, the identity $I$ produces the first Born approximation, which is bounded in $L^p$ for all $1 \leq p \leq \infty$ (see Lemma 2.7); Proposition 4.2 implies that $R_\tau(\lambda)$ produces an integral operator with admissible integral kernel; and we study operators produced by the singular terms $\lambda^{-2}P_0 V + \sum_{j=0}^{2} \sum_{k=1}^{2} \lambda^j \log^k \lambda D_{jk}$ in Section 5. We shall deal with the one produced by the most singular term $\lambda^{-2}P_0 V$ in Subsection 5.1. Here again the basic idea is similar to the odd dimensional case: If $P_0 = \sum \phi_j \otimes \phi_j$, the operator under investigation $\int_{0}^{\infty} G_0(\lambda)V P_0 V(G_0(\lambda) - G_0(-\lambda))\lambda^{-2} \Phi(\lambda) d\lambda$ is a linear combination of

$$Z_j u(x) = \int_{\mathbb{R}^m} \frac{(V \phi_j)(x) F_j u(|x - y|)}{|x - y|^{m-2}} dy$$
where, with sperical average $M_j u(r) = |\Sigma|^{-1} \int_{\Sigma} (V \phi_j * \hat{u})(r \omega) d\omega$, $\hat{u}(x) = u(-x)$ and $\Sigma = S^{m-1}$ being the unit sphere of $\mathbb{R}^m$, $F_j u(\rho)$ is given by

$$F_j u(\rho) = \int_0^\infty \int_0^\infty e^{-(t+s)}(ts)^{-\frac{m-1}{2}} dt ds \times \left\{ \int_0^\infty e^{-i\rho(s + 2i\lambda \rho)} \left( \int_{\mathbb{R}} e^{i\lambda r} (t - 2i\lambda r)^{-\frac{m-3}{2}} r^j M_j u(r) dr \right) d\lambda \right\}.$$

Observing that $F_j u(\rho)$ and $M_j u(r)$ are one dimensional objects, we apply some one dimensional harmonic analysis machineries, the weighted inequality for the Hilbert transform $\mathcal{H}$ and the Hardy-Littlewood maximal operator $\mathcal{M}$. However, as the comparison of formulae above with those in the odd dimensional case suggests, the analysis in even dimensions becomes much more intricate. In Subsection 5.2 we shall indicate how to modify the argument in subsection 5.1 for dealing with the operators produced by $\lambda^j \log^k \lambda D_{jk}$.

In Section 6, we prove that the high energy part $W_\gamma$ is bounded in $L^p$ for any $1 \leq p \leq \infty$. As the high energy part is insensitive to the low energy singularities and as the argument used for the same purpose in [25] for odd dimensions applies, we shall only very briefly sketch the proof. In Section 7, we show the continuity of $W$ in Sobolev spaces and complete the proof of Theorem 1.1. For $1 < p < \infty$, this follows from the intertwining property $W = (H - z)^{-j} W (H_0 - z)^j$ and the well known mapping property of the resolvent. For $p = 1$ and $p = \infty$, we may adopt the commutator argument as in [21] and we omit the discussion.

We use the same notation and conventions as in [1]: For $u \in \mathcal{H}_{-\gamma}$ and $v \in \mathcal{H}_\gamma$, $\langle u, v \rangle = \int_{\mathbb{R}^n} \overline{u}(x) v(x) dx$ is the standard coupling of functions; $\langle u \rangle = u \otimes v$ will be interchangeably used to denote the rank 1 operator $\phi \mapsto \langle v, \phi \rangle u$. For Banach spaces $X$ and $Y$, $B(X,Y)$ (resp. $B_\infty(X,Y)$) is the Banach space of bounded (resp. compact) operators from $X$ to $Y$, $B(X) = B(X,X)$ (resp. $B_\infty(X) = B_\infty(X,X)$). The identity operator is denoted by $1$. The norm of $L^p$-spaces, $1 \leq p \leq \infty$, is denoted by $\| u \|_p = \| u \|_{L^p}$. We write $\mathcal{S}(\mathbb{R}^m)$ for the space of rapidly decreasing functions. The Fourier transform is defined by

$$\hat{u}(\xi) = \mathcal{F} u(\xi) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-ix\xi} u(x) dx$$

and $\mathcal{F}^* u(\xi) = \mathcal{F} u(-\xi)$ is the conjugate Fourier transform. For functions $f$ on the line $f^{(j)}$ is the $j$-th derivative of $f$, $j = 1, 2, \ldots$. For $a \in \mathbb{R}$, $a_+$ or $a_-$ is an arbitrary number larger or smaller than $a$ respectively; $[a]$ is the largest integer not larger than $a$. When $I$ is an open or closed interval which contains $0$, $C_0^s(I)$ is the set of functions of order $C^s$ on $I$ which vanishes at $\lambda = 0$ along with the derivatives up to the order $[s]$. We sometimes say that $u$ is of order $C_0^s$ when $u \in C_0^s(I)$. 

7
2 Preliminaries

2.1 Free resolvent

Recall that \( H_\gamma = L^2_\gamma(\mathbb{R}^m, \langle x \rangle^{2\gamma} dx) \) and \( \Sigma \) is the unit sphere of \( \mathbb{R}^m \). As is well known the Fourier transform \( \mathcal{F} \) is an isomorphism from \( H_\gamma \) to the Sobolev space \( H^\gamma(\mathbb{R}^m) \) and the mapping

\[ \tilde{\Gamma}: H_\gamma(\mathbb{R}^m) \ni u \mapsto \lambda^{m-1/2} \hat{u}(\lambda \cdot) \in H^\gamma_0((0, \infty), L^2(\Sigma)) \tag{2.1} \]

is bounded if \( 0 \leq \gamma < \frac{m}{2} \). The upper bound for \( \gamma \), however, is relevant only at \( \lambda = 0 \) and, for any \( \varepsilon > 0 \), the map (2.1) is bounded for any \( 0 \leq \gamma \) if \( H^\gamma_0((0, \infty), L^2(\Sigma)) \) is replaced by \( H^\gamma((\varepsilon, \infty), L^2(\Sigma)) \). It follows by the Sobolev embedding theorem that the \( B(H_\gamma, L^2(\Sigma))-\)valued function defined by

\[ \Gamma(\lambda): H_\gamma \ni u \mapsto \lambda^{(m-1)/2} \hat{u}(\lambda \cdot) \in L^2(\Sigma), \]

is of class \( C^{\gamma-1/2} \) over \([0, \infty)\) and vanishes at \( \lambda = 0 \) along with the derivatives up to the order \([\gamma - 1/2]\) if \( \frac{1}{2} < \gamma < \frac{m}{2} \); and it is of class \( C^{\gamma-1/2} \) over \((\varepsilon, \infty)\) for any \( \frac{1}{2} < \gamma \) and \( \varepsilon > 0 \). We shall use the following well known lemma on the division in Sobolev spaces. The lemma is a result of repeated application of Hardy’s inequality when \( s \) is an integer and from the complex interpolation theory when \( s \) is not an integer.

**Lemma 2.1.** For any \( s > 0 \), the operator \( f(x) \mapsto x^{-s} f(x) \) is bounded from \( H^0_0(\mathbb{R}^+, L^2(\Sigma)) \) to \( H^{-s}_0(\mathbb{R}^+, L^2(\Sigma)) \).

We define operator valued function \( A(\lambda) \) for \( \lambda \in \mathbb{R} \) by

\[ A(\lambda)u(x) = \frac{1}{(2\pi)^m} \int_{\Sigma} \int_{\mathbb{R}^m} e^{i\omega(x-y)} u(y)dyd\omega, \quad x \in \mathbb{R}^m. \tag{2.2} \]

It is clear that \( A(\lambda) \) is even with respect to \( \lambda \in \mathbb{R} \) and \( \mu^{m-1} A(\mu) = \Gamma(\mu)^* \Gamma(\mu) \). We shall use the following expressions for \( G_0(\lambda), \lambda \in \mathbb{C}^+ \).

\[ G_0(\lambda) = \frac{1}{2\lambda} \left( \int_{0}^{\infty} \frac{\Gamma(\mu)^* \Gamma(\mu)}{\mu - \lambda} d\mu - \int_{0}^{\infty} \frac{\Gamma(\mu)^* \Gamma(\mu)}{\mu + \lambda} d\mu \right) \tag{2.3} \]

\[ = \int_{0}^{\infty} \frac{\mu^{m-1} A(\mu)}{\mu^2 - \lambda^2} d\mu = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\mu^{m-2} \text{sign} \mu A(\mu)}{\mu - \lambda} d\mu. \tag{2.4} \]

It can be see from the last expression, \( G_0(\lambda) \) become logarithmically singular when it is differentiated by \( \lambda \) more than \( m - 3 \) times. The following lemmas
are basic to the following analysis. We let \( D_1, D_2 \) and \( D_3 \) be the closed domains of the first quadrant of \((k, \ell)\) plane defined by
\[
D_1 = \{(k, \ell) : k, \ell \geq 0, k + \ell \leq m - 1, \ell \leq k\},
D_2 = \{(k, \ell) : k, \ell \geq 0, k \leq \frac{m-1}{2}, \ell \geq k\},
D_3 = \{(k, \ell) : k, \ell \geq 0, k + \ell \leq m - 1, \frac{m-1}{2} \leq k \leq m - 1\}.
\]
They have disjoint interiors and \( D_1 \cup D_2 \cup D_3 = \{(k, \ell) : 0 \leq k \leq m-1, 0 \leq \ell\} \).

Define the function \( \sigma_0(k, \ell) \) for \( 0 \leq k \leq m - 1 \) and \( 0 \leq \ell \) by
\[
\sigma_0(k, \ell) = \begin{cases} 
\frac{k+\ell+1}{2}, & (k, \ell) \in D_1 \\
\ell + \frac{1}{2}, & (k, \ell) \in D_2, \\
\frac{m-2}{2} & (k, \ell) \in D_3.
\end{cases} \tag{2.5}
\]

The function \( \sigma_0(k, \ell) \) is continuous, separately increasing with respect to \( k \) and \( \ell \) and, on lines \( k + \ell = c \) with fixed \( c \), decreases with \( k \).

**Lemma 2.2.** Let \( \ell \geq 0 \) be an integer and let \( 0 \leq k \leq m - 1 \). Let \( \sigma_0 = \sigma_0(k, \ell) \) as above and \( \sigma > \sigma_0 \). Then, \( \lambda^{m-1-k}A^{(\ell)}(\lambda) \) is a \( B(H_\sigma, H_{-\sigma}) \) valued function of \( \lambda \in \mathbb{R} \) of class \( C^{\sigma-\sigma_0} \).

**Proof.** Define \( \rho(\lambda)u(\omega) = \tilde{u}(\lambda \omega) \) for \( \lambda \in \mathbb{R} \) and \( \omega \in \Sigma \) and write \( \Gamma_\lambda = \Gamma(\lambda) \) and \( \rho_\lambda = \rho(\lambda) \) for shortening formulae. We also write \( X_\sigma \equiv B(H_\sigma, L^2(\Sigma)) \).

We have \( (A(\lambda)u, v) = (\rho_\lambda u, \rho_\lambda v) \). By differentiation,
\[
\rho_\lambda^{(k)}u(\omega) = \frac{1}{(2\pi)^{\frac{m-2}{2}}} \int_{\mathbb{R}^m} (i\omega x)^k e^{i\lambda \omega x} u(x) dx = \sum_{|\alpha|=k} C_\alpha \omega^\alpha \rho_\lambda(x^\alpha u)(\omega).
\]

It follows by Leibniz’ rule that
\[
(A^{(\ell)}(\lambda)u, v) = \sum_{|\alpha|+|\beta|=\ell} C_{\alpha\beta} (\omega^\alpha \rho_\lambda(x^\alpha u), \omega^\beta \rho_\lambda(x^\beta u)).
\]

In terms of \( \Gamma_\lambda \) we may write this in the form
\[
\lambda^{m-1-k}A^{(\ell)}(\lambda)u, v) = \lambda^{-k} \sum_{|\alpha|+|\beta|=\ell} C_{\alpha\beta} (\omega^\alpha \Gamma_\lambda(x^\alpha u), \omega^\beta \Gamma_\lambda(x^\beta u))
\]

It is an elementary to check that \( \sigma_0 = \sigma_0(k, \ell) \) is equal to
\[
\max_{|\alpha|+|\beta| = \ell} \min\{\max(a + |\alpha| + \frac{1}{2}, b + |\beta| + \frac{1}{2}) : 0 \leq a, b \leq \frac{m-1}{2}, a + b = k\}.
\]

It follows that, if \( \sigma > \sigma_0 \) then for any \( \alpha, \beta \) such that \( |\alpha| + |\beta| = \ell \) we can find \( 0 \leq a, b \leq \frac{m-1}{2} \) such that
\[
a + b = k, \quad a < \sigma - |\alpha| - \frac{1}{2} \text{ and } b < \sigma - |\beta| - \frac{1}{2}. \tag{2.6}
\]
For these \( a, b, \lambda^{-a} \Gamma_{\lambda}(x)^{[a]} \) and \( \lambda^{-b} \Gamma_{\lambda}(x)^{[b]} \) are \( \mathcal{X}_{\sigma} \)-valued continuous. Indeed, if \( a = \frac{m-1}{2} \), then, \( \lambda^{-a} \Gamma_{\lambda}(x)^{[a]} = \rho_{\lambda}(x)^{[a]} \) is a \( \mathcal{X}_{\sigma} \)-valued function of class \( C^{\sigma-|a| - \frac{m}{2}} \) by virtue of Sobolev embedding theorem because \( \sigma - |a| > \frac{m}{2} \); if \( a < \frac{m-1}{2} \), then, \( \Gamma_{\lambda}(x)^{[a]} \) is \( \mathcal{X}_{\sigma} \)-valued function of class \( C^{\gamma} \), \( \gamma = \min(\frac{m}{2}, \sigma - |a| - \frac{1}{2}) \) on \( \lambda \in [0, \infty) \) which vanishes at \( \lambda = 0 \) along with the derivatives of order up to \( [\gamma] \), and \( \lambda^{-a} \Gamma_{\lambda}(x)^{[a]} \) is of class \( C^{\gamma-a} \) as a \( \mathcal{X}_{\sigma} \)-valued function. A similar proof applies to \( \lambda^{-b} \Gamma_{\lambda}(x)^{[b]} \). This and the identity

\[
\lambda^{-k} \langle \omega^{\alpha} \Gamma_{\lambda}(x^{a}u), \omega^{\beta} \Gamma_{\lambda}(x^{b}u) \rangle = \langle \omega^{\alpha} \lambda^{-a} \Gamma_{\lambda}(x^{a}u), \omega^{\beta} \lambda^{-b} \Gamma_{\lambda}(x^{b}u) \rangle
\]

imply that \( \lambda \mapsto \lambda^{m-1-k} A^{(\ell)}(\lambda) \) is \( \mathcal{B}(\mathcal{H}_{\sigma}, \mathcal{H}_{-\sigma}) \) valued continuous on \( \mathbb{R} \). To conclude the proof, it suffices to show that it is of class \( C^{(\sigma-\sigma_{0})-} \) when \( 0 \leq \sigma - \sigma_{0} \leq 1 \). However, if \( \sigma > \sigma_{0} + 1 \), a differentiation implies that \( \langle x \rangle^{-\sigma} \lambda^{m-1-k} A^{(\ell)}(\lambda) \) is of class \( C^{1} \). The lemma then follows by interpolation.

\textbf{Corollary 2.3.} \textit{Let} \( 0 \leq a \leq m-1 \) \textit{and} \( b \geq 0 \). \textit{Let} \( j \geq 0 \) \textit{and} \( \sigma > \sigma_{0}(a, b+j) \). \textit{Then,} \( \lambda^{m-1-a} A^{(b)}(\lambda) \) \textit{is} \( \mathcal{B}(\mathcal{H}_{\sigma}, \mathcal{H}_{-\sigma}) \)-valued continuous function of \( \lambda \in \mathbb{R} \) \textit{of class} \( C^{j+(\sigma-\sigma_{0})-} \).

\textbf{Proof.} It suffices to show that \( \lambda \mapsto \lambda^{m-1-a-a'} A^{(b+b')}(\lambda) \in \mathcal{B}(\mathcal{H}_{\sigma}, \mathcal{H}_{-\sigma}) \) are continuous if \( a' + b' \leq j \) and \( a + a' \leq m - 1 \). This follows from Lemma 2.2 since, on the segment \( \{(k, \ell) : k + \ell = a + b + j', 0 \leq a \leq k \leq m - 1\} \), \( \sigma_{0} \) attains its maximum at \( (a, b + j') \) and \( \sigma_{0}(a, b + j') \) increases with \( 0 \leq j' \).

The following Lemma 2.4 is a slight improvement of Corollary 2.3 for small \( \sigma \). We omit the proof as it is identical with that of Lemma 2.1 of [I] for the odd dimensional case.

\textbf{Lemma 2.4.} \textit{Let} \( \frac{1}{2} \leq \sigma, \tau < \frac{3}{2} \) \textit{be such that} \( \sigma + \tau > 2 \) \textit{and define} \( \rho_{0} = \tau + \sigma - 2 \). \textit{Then,} \( \lambda \mapsto \lambda^{m-2} A^{(\ell)}(\lambda) \) \textit{is of class} \( C^{\rho} \) \textit{for any} \( \rho < \rho_{0} \) \textit{in} \( \mathbb{R} \) \textit{and of class} \( C^{(\min(\sigma, \frac{1}{2}), \sigma-\frac{1}{2})} \) \textit{in} \( \mathbb{R} \setminus \{0\} \).

\textbf{Lemma 2.5.} \textit{(1)} \textit{Let} \( 1/2 < \sigma \). \textit{Then,} \( G_{0}(\lambda) \) \textit{is a} \( \mathcal{B}_{\infty}(\mathcal{H}_{\sigma}, \mathcal{H}_{-\sigma}) \)-valued function of \( \lambda \in \mathbb{C}^{+} \setminus \{0\} \) \textit{of class} \( C^{(\sigma-\frac{1}{2})-} \). \textit{For non-negative integers} \( j < \sigma - \frac{1}{2} \),

\[
\|G_{0}^{(j)}(\lambda)\|_{\mathcal{B}(\mathcal{H}_{\sigma}, \mathcal{H}_{-\sigma})} \leq C_{j} \omega|\lambda|^{-1}, \quad |\lambda| \geq 1.
\]

\textit{(2)} \textit{Let} \( \frac{1}{2} < \sigma, \tau < m - \frac{3}{2} \) \textit{satisfy} \( \sigma + \tau > 2 \). \textit{Then,} \( G_{0}(\lambda) \) \textit{is a} \( \mathcal{B}_{\infty}(\mathcal{H}_{\sigma}, \mathcal{H}_{-\tau}) \)-valued function of \( \lambda \in \mathbb{C}^{+} \) \textit{of class} \( C^{\rho-} \), \( \rho_{*} = \min(\tau + \sigma - 2, \tau - 1/2, \sigma - 1/2) \).
Proof. The first statement is well known and follows immediately from (2.3) and the property of $\Gamma(\lambda)$ stated at the beginning of this subsection. By virtue of Corollary 2.3 and Lemma 2.4, $\text{sign} \mu \mu^{m-2}A(\mu)$ is a $B_\infty(\mathcal{H}_\sigma, \mathcal{H}_{-\tau})$-valued function of $\mu \in \mathbb{R}$ since $\rho_* < m - 2$. We apply the Privaloff theorem to the last expression of (2.4). The second statement follows.

The $(m - 2)$-th derivative of $\text{sign} \mu \mu^{m-2}A(\mu)$ contains Heaviside type singularity at $\mu = 0$ and, for any large $\sigma$, $\lambda \mapsto \langle x \rangle^{-\sigma}G_0^{(m-2)}(\lambda)\langle x \rangle^{-\sigma} \in B(\mathcal{H})$ is not continuous at $\lambda = 0$. We now examine this singularity. Let

$$J_k(\lambda) = \frac{1}{\lambda^k} \left( G_0(\lambda) - G_0(0) - \cdots - G_0^{(k-1)}(0) \frac{\lambda^{k-1}}{(k-1)!} \right).$$

**Proposition 2.6.** Let $m \geq 4$ be even. Then:

1. Let $k = 0, 1, \ldots, m - 3$ and $0 < \rho < m - 2 - k$. Let $\sigma > \sigma_0(k + 1, \rho)$. Then $J_k(\lambda)$ is a $B(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$-valued function of $\lambda \in \mathbb{R}$ of class $C^{\rho}$.

2. For $0 < \pm \lambda < \frac{1}{\xi}$, $G_0(\lambda)$ has the following expression:

$$\sum_{j=0}^{m-4} \lambda^{2j}(\Delta)^{-j-1} + \lambda^{m-2}\left( \pm \frac{i\pi}{2}A(\lambda) - \log |\lambda| A(\lambda) \right) + \lambda^{m-2}F(\lambda),$$

where $\lambda \mapsto F(\lambda)$ is even and, for $k = 0, \ldots, m - 1$, $\lambda^{m-1-k}F(\lambda)$ satisfies the same smoothness property as $\lambda^{m-1-k}A(\lambda)$ as stated in Corollary 2.3 and Lemma 2.4.

Remark that proof of Lemma 2.2 implies that for $\lambda \mapsto \sum_{j=0}^{m-4} \lambda^{2j}(\Delta)^{-j-1}$ is a $B(\mathcal{H}_{(m-1)_+}, \mathcal{H}_{-(m-1)})$ valued polynomial and hence is analytic.

Proof. If $k = 0$, statement (1) is contained in Lemma 2.5 (2). Let $k > 0$. Substituting $\sum_{j=0}^{k-1} \lambda^j \mu^{-j-1} + (\mu - \lambda)^{-1} \lambda^k \mu^{-k}$ for $(\mu - \lambda)^{-1}$ in the second equation of (2.4), we have for $\lambda \in C^+$ that

$$G_0(\lambda) = \sum_{j=0}^{k-1} \frac{\lambda^j}{2} \int_{-\infty}^{\infty} \mu^{m-j-3}\text{sign} \mu A(\mu)d\mu + \lambda^k \int_{-\infty}^{\infty} \frac{\mu^{m-1}\text{sign} \mu A(\mu)d\mu}{2\mu^{k+1}(\mu - \lambda)}.$$

Since $A(\mu)$ is even, the integrals in the sum vanish for odd $j$ and for even $j$

$$\frac{1}{2} \int_{-\infty}^{\infty} \mu^{m-j-3}\text{sign} \mu A(\mu)d\mu = \int_{0}^{\infty} \frac{\mu^{m-1}A(\mu)}{\mu^{j+2}}d\mu = (\Delta)^{-\frac{j}{2}}\lambda^j.$$
Thus, we have for \( \lambda \in \mathbb{C}^+ \)

\[
J_k(\lambda) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\mu^{m-2-k} \text{sign} \mu \ A(\mu)}{\mu - \lambda} \, d\mu. \tag{2.8}
\]

If \( \sigma > \sigma_0(k+1, \rho) \) and \( \rho < m-2-k, \mu \mapsto \mu^{m-2-k} \text{sign} \mu \ A(\mu) \in \mathcal{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma}) \) is of class \( C^\rho \) by virtue of Corollary 2.3 and (1) follows by Privaloff’s theorem.

(2) We substitute \( \mu^{m-2} = (\mu^2 - \lambda^2)(\mu^{m-4} + \lambda^2 \mu^{m-6} + \cdots + \lambda^{m-4}) + \lambda^{m-2} \) in the first of (2.4). The result is:

\[
G_0(\lambda) = \sum_{j=0}^{m-4} \lambda^{2j} (-\Delta)^{-j} + \lambda^{m-2} \int_0^{\infty} \frac{\mu A(\mu)}{\mu^2 - \lambda^2} \, d\mu, \quad \lambda \in \mathbb{C}^+. \tag{2.9}
\]

Take an even function \( \chi \in C^\infty_0(\mathbb{R}) \) such that \( \chi(\mu) = 1 \) for \( |\mu| \leq 1/4 \) and \( \chi(\mu) = 0 \) for \( |\mu| \geq 1/2 \), and split the last integral:

\[
\int_0^{\infty} \frac{\mu A(\mu)}{\mu^2 - \lambda^2} \, d\mu = \int_0^{\infty} \frac{\mu \chi(\mu) A(\mu)}{\mu^2 - \lambda^2} \, d\mu + \int_0^{\infty} \frac{\mu (1 - \chi(\mu)) A(\mu)}{\mu^2 - \lambda^2} \, d\mu \tag{2.10}
\]

The second integral yields a \( \mathcal{B}(\mathcal{H}) \)-valued analytic function of \( \lambda \) in a neighbourhood of the interval \((-\frac{1}{8}, \frac{1}{8})\) and we include it in \( F(\lambda) \). We denote \( B(\mu) = \chi(\mu) A(\mu) \), write the first integral in the form

\[
\frac{1}{2} \left( \int_0^{\infty} \frac{B(\mu)}{\mu + \lambda} \, d\mu + \int_0^{\infty} \frac{B(\mu)}{\mu - \lambda} \, d\mu \right) \tag{2.11}
\]

and take the boundary values at \(-\frac{1}{8} < \lambda < \frac{1}{8}\):

\[
\frac{1}{2} \left( \int_0^{1} \frac{B(\mu)}{\mu + |\lambda|} \, d\mu \pm i \pi B(\lambda) + \text{p.v.} \int_0^{\infty} \frac{B(\mu)}{\mu - |\lambda|} \, d\mu \right), \quad 0 < |\lambda| < \frac{1}{8}. \tag{2.12}
\]

To fix the idea we let \( 0 < \lambda < \frac{1}{8} \). We split the domain of integral of the second integral \([0, \infty) = [0, 2\lambda] \cup [2\lambda, \infty)\). The integral over \([2\lambda, \infty)\) is equal to \( \int_{2\lambda}^{1} B(\mu - \lambda) \mu^{-1} \, d\mu \), and we add it to the first integral which is equal to \( \int_{2\lambda}^{1} B(\lambda + \mu) \mu^{-1} \, d\mu \). We write the sum in the form

\[
\left( \int_0^1 - \int_0^\lambda \right) \frac{B(\mu + \lambda) + B(\mu - \lambda) - 2B(\lambda)}{2\mu} \, d\mu - (\log \lambda) A(\lambda)
\]

and add this to

\[
\text{p.v.} \int_0^{2\lambda} \frac{B(\mu)}{\mu - \lambda} \, d\mu = \int_0^\lambda \frac{(B(\lambda + \mu) - B(\lambda - \mu))}{2\mu} \, d\mu.
\]

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We arrive at the desired expression with

\[
F(\lambda) = \int_0^\infty \frac{\mu(1 - \chi(\mu))A(\mu)}{\mu^2 - \lambda^2} d\mu - \int_0^\lambda \frac{(B(\lambda - \mu) - B(\lambda))}{\mu} d\mu
+ \int_0^1 \frac{B(\mu + \lambda) + B(\mu - \lambda) - 2B(\lambda)}{2\mu} d\mu.
\] (2.13)

It is immediate to check that \(F(\lambda) = F(-\lambda)\). We prove that \(\lambda^{m-1-k}F(\lambda)\) satisfies the desired smoothness properties on \((-\frac{1}{8}, \frac{1}{8})\). The first integral defines a \(\mathbf{B}(\mathcal{H})\)-valued analytic as remark above and we ignore it. Let \(k = m-1\) first. We have \(\sigma(m-1, j) = j + \frac{m}{2}\). Hence, if \(\sigma > \frac{m}{2}\) and \(0 \leq t < \sigma - \frac{m}{2}\), \(B(\mu)\) is a \(\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})\) valued function of class \(C^t\) and the last two integrals satisfy the same property. Indeed, if \(0 < t < 1\). Then

\[
\left\| \int_\lambda^{\lambda+h} \frac{B(\lambda + h - \mu) - B(\lambda + h)}{\mu} d\mu \right\| \leq \int_\lambda^{\lambda+h} \mu^{-t} d\mu \leq Ch^t.
\]

Since \(\|(B(\lambda + h - \mu) - B(\lambda + h)) - (B(\lambda - \mu) - B(\lambda))\| \leq C\min(\mu, h^t)\),

\[
\left\| \int_0^\lambda ((B(\lambda + h - \mu) - B(\lambda + h)) - (B(\lambda - \mu) - B(\lambda))) \frac{d\mu}{\mu} \right\| \leq \int_0^h C\mu^{-t} d\mu + Ch^t \int_h^\lambda \frac{d\mu}{\mu} \leq Ch^t (1 + |\log h|)h^t.
\]

This prove that the first integral on the right of (2.13) is of class \(C^{t-}\). When \(t \geq 1\), we differentiate it and apply similar estimates. In this way we prove that it is of class \(C^{(\sigma - \frac{m}{2})^t}\). The proof for the second integral is simpler. We next let \(0 \leq k \leq m - 2\). Write \(m - 1 - k = t\),

\[
\lambda^t \int_0^\lambda \frac{B(\lambda - \mu) - B(\lambda)}{\mu} d\mu = \int_0^\lambda \frac{(\lambda - \mu)^t B(\lambda - \mu) - \lambda^t B(\lambda)}{\mu} d\mu
+ \sum_{\ell=1}^t \binom{t}{\ell} \int_0^\lambda \mu^{t-1}(\lambda - \mu)^{t-\ell} B(\lambda - \mu) d\mu \tag{2.14}
\]

The argument for the case \(k = 0\) implies that the first integral on the right satisfies the same smoothness property as \(\lambda^{m-1-k}A(\lambda)\) as stated in Corollary 2.3. We write the second integral in the form

\[
\int_0^\lambda \mu^{t-\ell}(\lambda - \mu)^{t-1} B(\mu) d\mu = (\ell - 1)! \int_0^\lambda \cdots \int_0^{s_2} \int_0^{s_1} \mu^{t-\ell} B(\mu) d\mu ds_1 \cdots ds_{\ell-1}
\]
Since \( \sigma(k,j) \geq \sigma(k - \ell, j - \ell) \) for any \( k, \ell \geq 0 \), the operator valued function defined by the right hand side enjoys the desired smoothness property. To prove the same for the last integral in (2.13):

\[
\int_0^1 \frac{B(\mu + \lambda) - B(\lambda)}{2\mu} d\mu + \int_0^1 \frac{B(\mu - \lambda) - B(\lambda)}{2\mu} d\mu, \tag{2.15}
\]

we multiply it by \( \lambda^t \), \( t = m - 1 - k \) and write the resulting function as in (2.14). For the first integral, as previously, it suffices to show

\[
\int_0^1 (\lambda + \mu)^{t-\ell} B(\lambda + \mu) \mu^{\ell-1} d\mu = \int_0^{\lambda+1} \mu^{t-\ell} B(\mu - \lambda)^{\ell-1} d\mu \tag{2.16}
\]

satisfies the desired property. However, the derivative of the right side is

\[
(\lambda + 1)^{t-\ell} B(\lambda + 1) + (\ell - 1) \int_0^{\lambda+1} \mu^{t-\ell} B(\mu - \lambda)^{\ell-2} d\mu
\]

and the first term is as smooth as \( \lambda^{m-1} A(\lambda) \) as \( \frac{7}{8} < 1 + \lambda \) when \( -\frac{1}{8} < \lambda < \frac{1}{8} \). Thus, by induction, (2.16) satisfies the desired property. The same holds for the second integral of (2.15). We omit the details. \( \square \)

### 2.2 Born Terms

If we formally expand the right of \( G(\lambda)V = (1 + G_0(\lambda)V)^{-1}G_0(\lambda)V \) into the series \( \sum_{n=1}^{\infty} (-1)^{n-1}(G_0(\lambda)V)^n \) and substitute it for \( G(\lambda)V \) in the stationary formula (1.8), then we have \( W = 1 - \Omega_1 + \Omega_2 - \cdots \) where

\[
\Omega_n u = \frac{1}{\pi i} \int_0^\infty (G_0(\lambda)V)^n(G_0(\lambda) - G_0(-\lambda))u\lambda d\lambda, \ n = 1, 2, \ldots
\]

The sum \( I - \Omega_1 + \cdots + (-1)^n\Omega_n \) is called the \( n \)-th Born approximation of \( W \) and individual \( \Omega_n \) is called the \( n \)-Born term. The following lemma is proved in any dimension \( m \geq 3 \) ([21]) and it will be used for studying both the low and the high energy parts of \( W \).

**Lemma 2.7.** Let \( \sigma > 1/m^* \). Then there exists a constant \( C > 0 \) such that

\[
\|\Omega_1 u\|_{W^{k,p}} \leq C \sum_{|\alpha| \leq k} \|F(x)^\sigma (\partial^\alpha V)\|_{L^{m*}(R^n)} \|u\|_{W^{k,p}}, \tag{2.17}
\]

\[
\|\Omega_n u\|_{W^{k,p}} \leq C^n \left( \sum_{|\alpha| \leq k} \|F(x)^{2\sigma} (\partial^\alpha V)\|_{L^{m*}(R^n)} \right)^n \|u\|_{W^{k,p}}, \quad n = 2, \ldots \tag{2.18}
\]

for any \( 1 \leq p \leq \infty \).
3 Threshold singularities

The resolvent $G(\lambda) = (H - \lambda^2)^{-1}$ of $H = -\Delta + V$ is a $\mathcal{B}(\mathcal{H})$-valued meromorphic function of $\lambda \in \mathbb{C}^+$ with possible poles $i\kappa_1, \ldots, i\kappa_n$ on $i\mathbb{R}^+$ such that $-\kappa_1^2, \ldots, -\kappa_n^2$ are eigenvalues of $H$ and outside the poles we have

$$G(\lambda) = (1 + G_0(\lambda)V)^{-1}G_0(\lambda), \quad \lambda \in \mathbb{C}^+. \quad (3.1)$$

For $\lambda \in \mathbb{R}$, $G_0(\lambda)V \in \mathcal{B}_\infty(\mathcal{H}_-, \mathcal{H}_-)$ for all $\frac{1}{2} < \gamma < \delta - \frac{1}{2}$ and $1 + G_0(\lambda)V$, $\lambda \neq 0$, is invertible if and only if $\lambda$ is an eigenvalue of $H$ ([1]). Since positive eigenvalues are absent from $H$ ([13]), (3.1) is satisfied for all $\lambda \in \mathbb{C}^+ \setminus \{0\}$ and $G(\lambda)$ considered as a $\mathcal{B}(\mathcal{H}_-, \mathcal{H}_-)$ valued function satisfies the same regularity properties as $G_0(\lambda)$ as stated in Lemma 2.5 except possibly at $\lambda = 0$. We omit the proof of the following well known lemma (see [1], [15]).

**Lemma 3.1.** Let $\frac{1}{2} < \gamma < \delta - \frac{1}{2}$. Then, $G(\lambda)$ is a $\mathcal{B}_\infty(\mathcal{H}_+, \mathcal{H}_+)$ valued function of $\lambda \in \mathbb{C}^+ \setminus \{0\}$ of class $C(\gamma - \frac{1}{2})$ for all $0 \leq j < \gamma - \frac{1}{2}$.

$$\|G^{(j)}(\lambda)\|_{\mathcal{B}(\mathcal{H}_+, \mathcal{H}_+)} \leq C_{j\gamma}|\lambda|^{-1}, \quad |\lambda| \geq 1. \quad (3.2)$$

Following [11], with $D_0 = G_0(0)$ as in Proposition 2.6 we define:

$$\mathcal{N} = \{\phi \in \mathcal{H}_- : (1 + D_0V)\phi = 0\}. \quad (3.3)$$

It is well known ([11], [24]) that $\mathcal{N}$ is finite dimensional and it is independent of $1/2 < \gamma < \delta - 1/2$; $(Vu, u)$ defines an inner product of $\mathcal{N}$; and if $\{\phi_1, \ldots, \phi_d\}$ is an orthonormal basis of $\mathcal{N}$, $\{-V\phi_1, \ldots, -V\phi_d\}$ is the dual basis of the dual space $\mathcal{N}^* = \{\psi \in \mathcal{H}_+ : (1 + VG_0(0))\psi = 0\}$. It follows that the spectral projection $Q$ in $\mathcal{H}_-$ for the eigenvalue $-1$ of $G_0(0)V$ is given by $Q = -\sum_{j=1}^d \phi_j \otimes (V\phi_j)$. We set $\overline{Q} = 1 - Q$.

**Lemma 3.2.** Let $D_2$ be as in Proposition 2.6 and let $\phi \in \mathcal{N}$. Then:

$$V\phi \in H_{(\delta + \frac{m}{2})_+} : \|(D_2V\phi)(x)\| \leq C\langle x\rangle^{4-m} \quad \text{and} \quad D_2V\phi \in H_{(\frac{m}{2})_+}. \quad (3.4)$$

**Proof.** The lemma follows since $\phi \in \mathcal{N}$ satisfy $|\phi(x)| \leq C\langle x\rangle^{-(m-2)}$ and $D_2$ has the integral kernel $C|x - y|^{4-m}$. \hfill \Box

By virtue of (3.4), $\mathcal{N}$ coincides with the eigenspace $\mathcal{E}$ of $H$ with eigenvalue 0 if $m \geq 6$ and the following definition is consistent with the one given in the introduction.

**Definition 3.3.** We say that the operator $H$ is of generic type if $\mathcal{N} = \{0\}$ and that $H$ is of exceptional type if otherwise.
3.1 Generic Case

When $H$ is of generic type, $G(\lambda)$ as a $B(H_{\gamma}, H_{-\gamma})$ valued function, $\frac{1}{2} < \gamma < \delta - \frac{1}{2}$, satisfies the same regularity properties as $G_0(\lambda)$ as stated in Lemma 2.5 on $\mathbb{R}$. We write $M(\lambda) = I + G_0(\lambda)V$ in what follows.

**Definition 3.4.** For an integer $\rho > 0$ and a $C^{\rho-1}$ function $f(\lambda)$ defined on an open interval $I$ containing 0 we say $f$ is of class $C^\rho$ on $I$ if $f \in C^{\rho}(I \setminus \{0\})$ and it satisfies $\|f^{(\rho)}(\lambda)\| \leq C(\log \lambda)^N$ for constants $C > 0$ and $N > 0$, $\lambda \neq 0$.

**Lemma 3.5.** Let $\frac{1}{2} < \gamma, \tau < \delta - \frac{1}{2}$ be such that $\gamma + \tau > 2$. Let $\rho_0 = \min(\gamma - 1/2, \delta - \gamma - 1/2)$ and $\rho_* = \min(\gamma - 1/2, \tau - 1/2, \tau + \gamma - 2)$. Suppose $H$ is of generic type. Then:

1. If $\rho_0 \leq m - 2$, $M^{-1}(\lambda)$ is a $B(H_{\gamma})$ valued function of $\lambda$ of class $C^{(\rho_0)}$. If $\rho_0 > m - 2$, it is of class $C^{(\rho_0)}$ for $\lambda \neq 0$ and of class $C^{(\rho_0)}$ on $\mathbb{R}$.

2. For any $\lambda \in \mathbb{R}$, $M(\lambda)^{-1} - 1$ may be extended to a bounded operator from $\mathcal{H}_{-\delta + \gamma}$ to $\mathcal{H}_{-\tau}$. If $\rho_* \leq m - 2$, it is a $B(\mathcal{H}_{-\delta + \gamma}, \mathcal{H}_{-\tau})$ valued function of class $C^{(\rho_*)}$. If $\rho_* > m - 2$, it is of class $C^{(\rho_*)}$ for $\lambda \neq 0$ and of class $C^{(\rho_*)}$ on $\mathbb{R}$. If $m = 4$ and $\rho_* > 3$, $\lambda(M(\lambda)^{-1} - 1)$ is of class $C^{(\rho_*)}$ for $\lambda \neq 0$ and of class $C^3$ on $\mathbb{R}$.

**Proof.** We prove the estimates $\|\partial^m \lambda M^{-1}(\lambda)\|_{B(H_{-\delta + \gamma}, H_{-\tau})} \leq C(\log \lambda)$ only, assuming $\rho_* > m - 2$, as the rest may be proved, by virtue of Lemma 2.5, by an almost word by word repetition of the proof of Lemma 2.7 of [I]. By using the identity $\partial^m \lambda M^{-1}(\lambda) = -M^{-1}(\lambda)G_0^{(\rho_*)}(\lambda)V M^{-1}(\lambda)$ we $m - 2$ times formally differentiate $M^{-1}(\lambda) - I$. This produces a linear combination over $j_1 + \cdots + j_k = m - 2$, $j_1, \ldots, j_k \geq 1$ of

$$M^{-1}(\lambda)G_0^{(j_1)}(\lambda)V M^{-1}(\lambda) \cdots M^{-1}(\lambda)G_0^{(j_k)}(\lambda)V M^{-1}(\lambda).$$

If $k \geq 2$, this is bounded in $B(H_{-\delta + \gamma}, H_{-\tau})$ near $\lambda = 0$ by the proof of Lemma 2.7 of [I]; and if $k = 1$, this is bounded by $C(\log \lambda)$ by virtue of Proposition 2.6 (2) and of the estimate

$$\|\partial^m \lambda (\lambda^{m-2} \log \lambda A(\lambda))\|_{B(H_{\gamma}, H_{-\tau})} \leq C(\log \lambda), \quad |\lambda| < 1$$

obtained via Corollary 2.3 and Lemma 2.2. The desired estimate follows.

3.2 Exceptional Case

In this subsection we assume $H$ is of exceptional type. Then $(1 + G_0(\lambda)V)^{-1}$ is singular at $\lambda = 0$. When $m$ is even, the logarithmic singularities appear in
addition to those due to the 0 eigenspace of $H$ and the analysis becomes more complex than in odd dimensions. In this subsection we prove the following expansion formulae for $(1 + G_0(\lambda)V)^{-1}$. Recall that $P_0$ is the orthogonal projection onto the zero eigenspace of $H$.

**Proposition 3.6.** (1) Let $m = 6$ and $|V(x)| \leq C|x|^\delta$ with $\delta > 10$. Then, with $E(\lambda)$ such that $VE(\lambda)$ satisfies the condition $(K)_\rho$ with $\rho > m + 1$,

$$(1 + G_0(\lambda)V)^{-1} - 1 = \frac{P_0 V}{\lambda^2} + \sum_{j=0}^{m} \sum_{k=1}^{2} D_{jk} \lambda^j \log k \lambda + E(\lambda). \quad (3.5)$$

Here $D_{jk}$ are finite rank operators of the form

$$D_{jk} = P_0 V D_{jk}^{(1)} P_0 V + D_{jk}^{(2)} P_0 V + P_0 V D_{jk}^{(3)}, \quad (3.6)$$

where $D_{jk}^{(1)} \in B(\mathcal{N})$, $D_{jk}^{(2)} \in B(\mathcal{N}, \mathcal{H}_{-\delta})$, $D_{jk}^{(3)} \in B(\mathcal{H}_{-\delta+3}, \mathcal{N})$.

(2) Let $m \geq 8$ and $|V(x)| \leq C|x|^\delta$ with $\delta > m + 3$. Then, with $E(\lambda)$ such that $VE(\lambda)$ satisfies the condition $(K)_\rho$ with $\rho > m + 1$,

$$(1 + G_0(\lambda)V)^{-1} - 1 = \frac{P_0 V}{\lambda^2} + c_m \varphi \otimes (V \varphi)\lambda^{m-6} \log \lambda + E(\lambda). \quad (3.7)$$

Here $\varphi = P_0 V$ with $V$ being considered as a function. If $m \geq 12$, then $c_m \varphi \otimes (V \varphi)\lambda^{m-6} \log \lambda$ may be included in $E(\lambda)$.

The rest of this subsection is devoted to the proof of Proposition 3.6. We use the following lemma as in the odd dimensional case.

**Lemma 3.7.** Let $\mathcal{X} = \mathcal{X}_0 + \mathcal{X}_1$ be a direct sum decomposition of a vector space $\mathcal{X}$. Suppose that a linear operator $L$ in $\mathcal{X}$ is written in the form

$$L = \begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix}$$

in this decomposition and that $L_{00}^{-1}$ exists. Set $C = L_{11} - L_{10} L_{00}^{-1} L_{01}$. Then, $L^{-1}$ exists if and only if $C^{-1}$ exists. In this case

$$L^{-1} = \begin{pmatrix} L_{00}^{-1} + L_{00}^{-1} L_{01} C^{-1} L_{10} L_{00}^{-1} & -L_{00}^{-1} L_{01} C^{-1} \\ -C^{-1} L_{10} L_{00}^{-1} & C^{-1} \end{pmatrix}. \quad (3.8)$$

Using the spectral projections $Q$ and $\overline{Q} = 1 - Q$, we decompose $\mathcal{H}_{-\gamma} = \overline{Q} \mathcal{H}_{-\gamma} + \mathcal{N}$ as a direct sum. With respect to this decomposition, we write

$$M(\lambda) = \begin{pmatrix} \overline{Q} M(\lambda) \overline{Q} \\ Q M(\lambda) \overline{Q} \end{pmatrix} \equiv \begin{pmatrix} L_{00}(\lambda) & L_{01}(\lambda) \\ L_{10}(\lambda) & L_{11}(\lambda) \end{pmatrix}. \quad (3.9)$$
where the right side is the definition. We begin by studying \( L_{00}^{-1}(\lambda) \). Since \( L_{00}(0) \in B(\overline{QH}_{-\gamma}) \) is invertible by the separation of spectrum theorem for compact operators, \( L_{00}(\lambda) \) is also invertible for small \( |\lambda| < \lambda_0 \). We omit the proof of the following lemma which is similar to that of Lemma 3.5.

**Lemma 3.8.** Let \( \frac{1}{2} < \gamma, \tau < \delta - \frac{1}{2} \) and \( \gamma + \tau > 2 \). Let \( \rho_0 = \min(\gamma - 1/2, \delta - \gamma - 1/2) \) and \( \rho_* = \min(\gamma - 1/2, \tau - 1/2, \tau + \gamma - 2) \). Then:

1. If \( \rho_0 \leq m - 2 \), \( L_{00}^{-1}(\lambda) \) is a \( B(\overline{QH}_{-\gamma}) \) valued function of \( \lambda \in (-\lambda_0, \lambda_0) \) of class \( C^{(\rho_0)}_- \). If \( \rho_0 > m - 2 \), it is of class \( C^{(\rho_0)}_- \) for \( \lambda \neq 0 \) and of class \( C^{m-2}_\gamma \) on \((-\lambda_0, \lambda_0)\).

2. For any \( \lambda \in \mathbb{R} \), \( L_{00}^{-1}(\lambda) - \overline{Q} \) may be extended to a bounded operator from \( \overline{QH}_{-\delta+\gamma} \) to \( \overline{QH}_{-\tau} \). If \( \rho_* \leq m - 2 \), it is of class \( C^{(\rho_*)-} \) as a \( B(\overline{QH}_{-\delta+\gamma}, \overline{QH}_{-\tau}) \)-valued function. If \( \rho_* > m - 2 \), then it is of class \( C^{(\rho_*)-} \) for \( \lambda \neq 0 \) and of class \( C^{m-2}_\gamma \) on \((-\lambda_0, \lambda_0)\).

Removing the singular part \(-\lambda^{m-2}\log \lambda A(\lambda)\) from \( G_0(\lambda) \), we define

\[
G_{0\text{reg}}(\lambda) = G_0(\lambda) + \lambda^{m-2} \log \lambda A(\lambda), \quad N(\lambda) = \overline{Q}(1 + G_{0\text{reg}}(\lambda)V)\overline{Q}.
\]

If \( \gamma \) and \( \rho_0 \) is as in Lemma 3.8, Proposition 2.6 implies that \( N(\lambda) \) is a \( B(\overline{QH}_{-\gamma}) \) valued function of \( \lambda \in \mathbb{R} \) of class \( C^{(\rho_0)}_- \) and \( N(\lambda) \) is invertible in \( \overline{QH}_{-\gamma} \) for \( \lambda \in (-\lambda_0, \lambda_0) \) if \( \lambda_0 > 0 \) is chosen small enough. We write

\[
\tilde{L}(\lambda) = L_{00}^{-1}(\lambda) - \overline{Q}, \quad X(\lambda) = N^{-1}(\lambda), \quad \tilde{X}(\lambda) = X(\lambda) - \overline{Q}. \tag{3.10}
\]

We omit the proof of the following lemma which is also similar to that of Lemma 3.5.

**Lemma 3.9.** Let \( \gamma, \tau \) and \( \rho_0, \rho_* \) be as in Lemma 3.8. Then:

1. \( X(\lambda) \) is a \( B(\overline{QH}_{-\gamma}) \) functions of \( \lambda \in (-\lambda_0, \lambda_0) \) of class \( C^{(\rho_0)}_- \).

2. For \( \lambda \in (-\lambda_0, \lambda_0) \), \( \tilde{X}(\lambda) \) extends to a bounded operator from \( \overline{QH}_{-\delta+\gamma} \) to \( \overline{QH}_{-\tau} \) and it is \( B(\overline{QH}_{-\delta+\gamma}, \overline{QH}_{-\tau}) \)-valued function of class \( C^{(\rho_*)-} \).

We define \( Y(\lambda) = L_{00}^{-1}(\lambda) - X(\lambda) \). In what follows we shall often use the arguments similar to the ones which will be used in (i) to (iv) of the proof of the following corollary. We use the following elementary lemma which follows by a direct estimate:

**Lemma 3.10.** Suppose \( f(x) \) is of class \( C^{s\lambda}_0(\mathbb{R}), 0 < s \leq 1 \), then \( \log x f(x) \) is of class \( C^{s\lambda}_0(\mathbb{R}) \).
Corollary 3.11. Let \( \gamma, \tau \) and \( \rho_* \) be as in Lemma 3.8 and \( j, k = 0, 1, \ldots \). Then, \( Y_{jk}(\lambda) \equiv \lambda^j (\log \lambda)^k Y(\lambda) \), \( \lambda \neq 0 \), may be extended to a bounded operator from \( \overline{Q} \mathcal{H}_{\delta + \gamma} \) to \( \overline{Q} \mathcal{H}_{\tau} \). Define \( Y_{jk}(0) = 0 \). If \( \rho_* \leq m - 2 \), then \( Y_{jk}(\lambda) \) is of class \( C^{(\rho_*)} \) - as a \( \mathcal{B}(\overline{Q} \mathcal{H}_{\delta + \gamma}, \overline{Q} \mathcal{H}_{\tau}) \) - valued function of \( \lambda \in (-\lambda_0, \lambda_0) \). If \( \rho_* > m - 2 \), then it is of class \( C^{(\rho_*)} \) for \( \lambda \neq 0 \) and of class \( C^{m-2} \) on \( (-\lambda_0, \lambda_0) \).

Proof. We insert \( \lambda^{-1} Y_{jk}(\lambda) = \overline{Q} + \hat{\lambda}(\lambda) \) and \( X(\lambda) = \overline{Q} + \hat{X}(\lambda) \) into

\[
Y(\lambda) = \lambda^{-1} Y_{jk}(\lambda)(\lambda^{m-2} \log \lambda \overline{Q} A(\lambda) V \overline{Q}) X(\lambda) \tag{3.11}
\]

This produces four terms and the smoothness property of each operator outside \( \lambda = 0 \) is easy to check. We treat them near \( \lambda = 0 \).

(i) \( \lambda^{m-2} \log \lambda \overline{Q} A(\lambda) V \overline{Q} \) enjoys the desired property by virtue of Lemma 2.4;

(ii) To see the same for \( \tilde{\lambda} \) in (ii), we compute the \( \ell \)-th derivative via Leibniz' formula. If \( \ell = \alpha + \beta < \rho_* \), we choose \( \kappa \) such that \( b + \frac{1}{2} < \kappa < \delta - a - \frac{1}{2}, \tau + (\delta - \kappa) > 2 \) and \( \kappa + \gamma > 2 \). This is possible since \( \delta - a - \frac{1}{2} > b + \frac{1}{2}, b + \frac{1}{2} < \delta + \tau - 2 \) and \( 2 - \gamma < \delta - a - \frac{1}{2} \). Then, as a \( \mathcal{B}(\overline{Q} \mathcal{H}) \)-valued function,

\[
\langle x \rangle^{-\tau} \tilde{\lambda}^{(\alpha)}(\lambda) \langle x \rangle^{\kappa} \cdot \langle x \rangle^{-\tau} (\lambda^{m-2} \log \lambda \overline{Q} A(\lambda) V \overline{Q})^{(\beta)} \langle x \rangle^{\delta - \gamma}
\]

is continuous with respect to \( \lambda \in (-\lambda_0, \lambda_0) \setminus \{0\} \). If \( b < m - 2 \), this is continuous also at \( \lambda = 0 \), and is bounded by \( \langle \log \lambda \rangle \) if \( b = m - 2 \).

(iii) The argument which is entirely similar to the one used in (ii) proves that \( \overline{Q} (\lambda^{m-2} \log \lambda A(\lambda)) V \overline{Q} \hat{\lambda}(\lambda) \) satisfies the corollary.

(iv) In view of the result in (ii), we estimate for \( \ell = \alpha + \beta < \rho_* \)

\[
\langle x \rangle^{-\tau} \{ \tilde{\lambda}(\lambda) \lambda^{m-2} \log \lambda \overline{Q} A(\lambda) V \overline{Q} \}^{(\alpha)} \langle x \rangle^{\kappa} \cdot \langle x \rangle^{-\tau} \hat{\lambda}(\lambda) \hat{\lambda}(\lambda) \langle x \rangle^{\delta - \gamma}
\]

by choosing \( \kappa \) such that \( b + \frac{1}{2} < \kappa < \delta - a - \frac{1}{2}, \tau + (\delta - \kappa) > 2 \) and \( \kappa + \gamma > 2 \). Such \( \kappa \) exists by the same reason as in (ii). Then, as a \( \mathcal{B}(\overline{Q} \mathcal{H}) \)-valued function, this is continuous with respect to \( \lambda \in (-\lambda_0, \lambda_0) \setminus \{0\} \). If \( a < m - 2 \) it is continuous also at \( \lambda = 0 \), and is bounded by \( \langle \log \lambda \rangle \) if \( a = m - 2 \). Hence \( \tilde{\lambda}(\lambda) \lambda^{m-2} \log \lambda \overline{Q} A(\lambda) V \overline{Q} \hat{\lambda}(\lambda) \) has the desired property. \( \Box \)

Since the logarithmic singularity appears in the form \( \lambda^{m-2} \log \lambda A(\lambda) \) in \( G_0(\lambda) \) as in Proposition 2.6 and \( \lambda^{m-2} \log \lambda \) is less singular in higher dimensions, the proof of the proposition becomes easier as the spatial dimension \( m \) increases. Thus, we study the case \( m = 6 \) first and then discuss the case \( m \geq 8 \) only briefly. In what follows we shall indiscriminately write \( E_0(\lambda) \) for \( \mathcal{B} \mathcal{N} \) valued functions which satisfy the following property for some \( N \geq 0 \):

\[
E_0(\lambda) \text{ is of class } C^{m-2}(\lambda^{m-2} \log \lambda A(\lambda) \setminus \{0\}) \text{ and } C^{m-2}(\lambda^{m-2} \log \lambda A(\lambda) \setminus \{0\}) \text{ and }
\]

\[
\| E_{\|}(\lambda) \| \leq C \langle \log \lambda \rangle^N, \| \lambda E_{\|}(\lambda) \| \leq C \langle \log \lambda \rangle^N. \tag{3.12}
\]
Note that the function of class $C^{\frac{m+2}{2}}$ on $(-\lambda_0, \lambda_0)$ clearly satisfies the condition (3.12). We often omit the variable $\lambda$ of operator valued functions. Note that $m \geq 2 \geq \frac{m+2}{2}$ when $m \geq 6$ with strict inequality when $m > 6$.

### 3.2.1 Proof of Proposition 3.6 for $m = 6$

In view of Lemma 3.7, we first study $C(\lambda) = L_{11} - L_{10}^{-1}L_{01}$. We have

$$G_0(\lambda) = D_0 + \lambda^2D_2 - \lambda^4(\log \lambda)A(\lambda) + \lambda^4F(\lambda)$$

by virtue of Proposition 2.6. Since $(1 + D_0V)Q = Q(1 + VD_0) = 0$,

$$L_{11}(\lambda) = \lambda^2Q(D_2 - \lambda^2\log \lambda A(\lambda) + \lambda^2F(\lambda))VQ,$$

$$L_{01}(\lambda) = \lambda^2Q(D_2 - \lambda^2\log \lambda A(\lambda) + \lambda^2F(\lambda))VQ.$$ (3.13)

We know that $QD_2VQ$ is invertible in $\mathcal{N}$ and $(QD_2VQ)^{-1} = P_0V$, $P_0VQ = P_0V$ and $VQp_0 = VP_0$. Then, $C(\lambda)$ may be written in the form

$$C(\lambda) = \lambda^2(QD_2VQ)(1 - P_0V E_{2}^*(\lambda)),$$

$$E_{2}^*(\lambda) = \lambda^2Q_{00} + \lambda^2\log \lambda F_{01} + \lambda^4F_{20} + \lambda^4\log \lambda F_{21} + \lambda^2E_{0}(\lambda),$$

where $F_{00}(\lambda), F_{01}(\lambda), F_{20}(\lambda)$ and $F_{21}(\lambda)$ are defined by

$$F_{00}(\lambda) = -Q(F(\lambda) - D_2VQ^{-1}QD_2)VQ, \quad F_{01} = QA(\lambda)VQ,$$

$$F_{20}(\lambda) = Q(F(\lambda)VQ^{-1}QD_2 + D_2VQ^{-1}QF(\lambda))VQ,$$

$$F_{21}(\lambda) = -Q(A(\lambda)VQ^{-1}QD_2 + D_2VQ^{-1}QF(\lambda))VQ.$$ (3.15)

and $E_{0}(\lambda) = \lambda^4Q(\log \lambda A(\lambda) - F(\lambda))VQ^{-1}Q(\log \lambda A(\lambda) - F(\lambda))VQ$ is of class $C^4$ thanks to (3.4) (recall $\delta > 10$). We write $\tilde{F}_{jk}(\lambda)$ for the operator obtained from $F_{jk}(\lambda)$ of (3.15) by replacing $L_{00}^{-1}(\lambda)$ by $X(\lambda) = N^{-1}(\lambda)$.

**Lemma 3.12.** As $\textbf{B}(\mathcal{N})$-valued functions of $\lambda \in (-\lambda_0, \lambda_0)$,

1. $F_{00}(\lambda), F_{20}(\lambda)$ and $F_{21}(\lambda)$ are of class $C^4$;
2. $\tilde{F}_{00}(\lambda), F_{01}(\lambda), \tilde{F}_{20}(\lambda)$ and $\tilde{F}_{21}(\lambda)$ are of class $C^{\delta-4}$;
3. $QA(\lambda)VQ^{-1}QL^{-1}(\lambda)VQ^{-1}QF(\lambda)VQ$ is of class $C^4$;
4. $QA(\lambda)VQ^{-1}QL^{-1}(\lambda)VQ^{-1}QF(\lambda)VQ$ is of class $C^{\delta-4}$.

The same holds for the operators which are obtained by replacing one or both of $A(\lambda)$ and $F(\lambda)$ by the other operator.
Proof. We prove statements (2). The proof for others is similar.

(i) Proposition 2.6 and properties (3.4) of φ ∈ N imply that QF(λ)VQ is of class C^{(δ-2)-}; and the operators QF(λ)VQD_2VQ and QD_2VQF(λ)VQ are of class C^{(δ-4)-}. The same holds when F(λ) is replaced by A(λ).

(ii) By virtue of Lemma 3.9 and (3.4), QD_2VX(λ)D_2VQ implies that the same holds for Lemma 3.13.

(iii) QF(λ)VX(λ)D_2VQ is of class C^{(δ-2)-}. To see this we differentiate it by λ by using Leibniz’ formula. Then, by virtue of Lemma 3.9 and (3.4), for some ε > 0, ⟨x⟩^{−δ−1+ε}F(k_1)(λ)VX(k_2)(λ)⟨x⟩^{1+ε} is B(H)-valued continuous as long as

\[ k_1 + 3 + k_2 + \frac{1}{2} < \delta, \quad k_1 + 3 < \delta + 1, \quad k_2 + \frac{1}{2} < \delta − 1 \]

and the latter inequalities trivially hold if k_1 + k_2 < δ − \frac{7}{2}. Similar argument implies that the same holds for QD_2VQX(λ)QF(λ)VQ and for the operators obtained by replacing F(λ) by A(λ).

Combining (i), (ii) and (iii) we obtain statement (2).

Lemma 3.13. There exist λ_0 > 0 such that for λ ∈ (−λ_0, λ_0) \ {0} C(λ) is invertible in N and C(λ)^{-1} may be written in the form

\[ \lambda^{-2}P_0V + P_0V \left( \log \lambda D_{10} + \sum_{1 \leq k \leq j \leq 2} \lambda^k (\log \lambda)^k D_{jk} + E_0(\lambda) \right) P_0V \]  

(3.16)

with D_{jk} ∈ B(N) and a B(N)-valued C^4 function E_0(λ).

Proof. By virtue of Lemma 3.12, \|P_0VE^*_2(λ)\| → 0 as λ → 0. It follows that C(λ) is invertible for small λ ≠ 0 and

\[ C^{-1}(λ) = \lambda^{-2} \sum_{n=0}^{\infty} (P_0VE^*_2(λ))^n P_0V \]

Here \lambda^{-2} \sum_{n=0}^{\infty} (P_0VE^*_2(λ))^n P_0V is of the form P_0VE_0(λ)P_0V with a C^4 function E_0(λ) again by Lemma 3.12. We also put all terms into E_0(λ) which is produced by \lambda^{-2} \sum_{n=1}^{2} (P_0VE^*_2(λ))^n P_0V and which do not contain any log λ factors or contain factors λ^j with j ≥ 3. In this way we obtain

\[ C^{-1}(λ) = \lambda^{-2}P_0V + \log λP_0VF_{01}P_0V + \lambda^2 \log λP_0V(F_{21} + F_{00}P_0VF_{01} + F_{01}P_0VF_{00})P_0V \]

(3.17)

+ \lambda^2 \log^2 λ(P_0VF_{01})^2 P_0V + E_0(λ).

The equation (3.17) remains valid if F_{00}, F_{01} and F_{21} are replaced by \tilde{F}_{00}, \tilde{F}_{01} and \tilde{F}_{21} respectively because the difference is of class C^4 by virtue of Corollary 3.11. We then expand various operators in the resulting equation in powers of λ by using Taylor’s formula. More specifically, we expand
Lemma 3.14. There exist $\lambda_0 > 0$ such that for $\lambda \in (-\lambda_0, \lambda_0) \setminus \{0\}$

$$L_{00}^{-1}(\lambda)L_{01}(\lambda)C_r(\lambda) = \lambda^2 \log \lambda D_{21}^{(1)} P_0 V + R_{01}(\lambda) P_0 V. \quad (3.18)$$

Here $D_{21}^{(1)} \in \mathcal{B}(\mathcal{N}, \mathcal{H}_{(-1)\cdot})$ and $R_{01}(\lambda)$ is such that as $\mathcal{B}(\mathcal{N}, \mathcal{H})$ valued functions, $\langle x \rangle^{-(\sigma+1)\cdot} R_{01}(\lambda)$ is of class $C^\sigma$ for $0 \leq \sigma \leq 2$; for $\sigma = 3$ and $4$, $\langle x \rangle^{-(\sigma+\frac{1}{2})\cdot} R_{01}(\lambda)$ is of class $C^\sigma$ for $\lambda \neq 0$ and for some $N > 0$

$$\|\langle x \rangle^{-(\frac{5}{2})\cdot} R_{01}^{(3)}(\lambda)\|_{\mathcal{B}(\mathcal{N}, \mathcal{H})} + \|\langle x \rangle^{-(\frac{3}{2})\cdot} \lambda R_{01}^{(4)}(\lambda)\|_{\mathcal{B}(\mathcal{N}, \mathcal{H})} \leq C \langle \log \lambda \rangle^N. \quad (3.19)$$

Proof. Since $L_{01} = \overline{Q}(1 + G_0 V)Q$ is a $\mathcal{B}(\mathcal{N}, \mathcal{H}_{-(\gamma + \frac{1}{2})\cdot})$-valued function of $\lambda$ of class $C^\gamma$ for $\gamma < 4$ and of class $C^4_\ast$ if $\gamma \geq 4$, Lemma 3.8 implies that

$$L_{00}^{-1} L_{01}(\lambda) E_0(\lambda) P_0 V = (\overline{Q} L_{01}(\lambda) E_0(\lambda) + \tilde{L}(\lambda) L_{01}(\lambda) E_0(\lambda)) P_0 V$$

satisfies the same property. We have only to consider $\lambda^j (\log \lambda)^k L_{00}^{-1} L_{01} D_{jk}$. (In this proof we ignore the factor $P_0 V$ for shortening formulas). As a $\mathcal{B}(\mathcal{N}, \mathcal{H}_{-(\gamma + \frac{1}{2})\cdot})$-valued function $\lambda^{4+j}(\log \lambda)^k \overline{Q}(\log \lambda A(\lambda) + F(\lambda))VQ$ is of class $C^\gamma$ if $\gamma < 4$ and of class $C^4_\ast$ if $\gamma \geq 4$ by virtue of Corollary 2.3, Lemma 2.4 and Proposition 2.6. It follows by writing again $L_{00}^{-1} = \overline{Q} + \tilde{L}(\lambda)$ and applying Lemma 3.8 that

$$L_{00}^{-1}(\lambda) (\lambda^{m-2+j}(\log \lambda)^k \overline{Q}(\log \lambda A(\lambda) + F(\lambda))VQ) D_{jk}$$

shares this property. Writing $L_{00}^{-1} = \overline{Q} + Y + X$, we are left with

$$\lambda^{2+j}(\log \lambda)^k L_{00}^{-1}(\lambda)\overline{Q} D_2 V Q D_{jk} = \lambda^{2+j}(\log \lambda)^k \overline{Q} D_2 V Q D_{jk}$$
\[ + \lambda^{2+j}(\log \lambda)^k Y(\lambda) \overline{Q} D_2 V Q D_{jk} + \lambda^{2+j}(\log \lambda)^k \bar{X}(\lambda) \overline{Q} D_2 V Q D_{jk} \]  

(3.20)

Recalling that \( \overline{Q} D_2 V \in B(\mathcal{N}, \mathcal{H}_{\mathcal{H}}) \) for any \( \varepsilon > 0 \), we put the first term on the right to \( \lambda^2 \log \lambda D_{21}^{(1)} \) if \( j = 0 \) and into \( R_{01}(\lambda) \) otherwise. Note that if \( j = 0 \), we have only \( k = 1 \) term, see (3.16). Corollary 3.11 with \( \gamma = \delta - 1 - \varepsilon \) and \( \tau = \sigma + 1 + \varepsilon \) implies that the second term may also be put into \( R_{01}(\lambda) \). To deal with the last term, we write \( \bar{X}(\lambda) = \bar{X}(0) + (\bar{X}_1(\lambda) - \bar{X}(0)) \). Remarking that \( \bar{X}(0) \overline{Q} D_2 V Q \in B(\mathcal{N}, \mathcal{H}_{\mathcal{H}}) \) for any \( \varepsilon > 0 \) we put \( \lambda^{2+j}(\log \lambda)^k \bar{X}(0) \overline{Q} D_2 V Q D_{jk} \) into \( \lambda^2 \log \lambda D_{21}^{(1)} \) if \( j = 0 \) and into \( R_{01}(\lambda) \) otherwise. Finally, \( \lambda^{2+j}(\log \lambda)^k \bar{X}(\lambda) - \bar{X}(0)) \overline{Q} D_2 V Q D_{jk} \) may also be put into \( R_{01}(\lambda) \). This can be seen by differentiating via Leibniz’ rule and by applying Lemma 3.9. This completes the proof.

Recall \( \sigma_0(3, \sigma) = \frac{\sigma}{2} + 2 \) for \( \sigma \leq 3 \). In the following two lemmas, we set

\[
\gamma(\sigma) = \begin{cases} 
\max(\sigma_0(3, \sigma), 3), & \text{if } \sigma < 2, \\
\sigma + 1, & \text{if } 2 \leq \sigma \leq 4.
\end{cases}
\]

(3.21)

We remark that the condition \( \delta > 10 = m + 4 \) when \( m = 6 \) is originated from the fact that \( \gamma(0) = 3 \) and that, for \( VR_{02}(\lambda)P_0V \) and \( VP_0VR_{30}(\lambda) \) to satisfy the property \( (K)_\rho \) with \( \rho > m + 1 \), we need \( \delta - 3 > m + 1 \).

**Lemma 3.15.** With \( D_{2,1}^{(2)} \in B(\mathcal{N}, \mathcal{H}_{\mathcal{H}}) \) and \( R_{02}(\lambda) \), we have

\[ L_{00}^{-1}(\lambda)L_{01}(\lambda)C^{-1}(\lambda) = \lambda^2 \log \lambda D_{2,1}^{(2)} P_0 V + R_{02}(\lambda)P_0 V. \]  

(3.22)

where, as a \( B(\mathcal{N}, \mathcal{H}) \) valued functions, \( \langle x \rangle^{-\sigma} R_{02}(\lambda) \) is of class \( C^\sigma \) on \( (-\lambda_0, \lambda_0) \) for \( 0 \leq \sigma \leq 2 \), for \( \lambda \neq 0 \) for \( \sigma = 3, 4 \) and

\[ \| \langle x \rangle^{-4} R_{02}^{(3)}(\lambda) \|_{B(\mathcal{N}, \mathcal{H})} + \| \langle x \rangle^{-5} R_{02}^{(4)}(\lambda) \|_{B(\mathcal{N}, \mathcal{H})} \leq C(\log \lambda). \]  

(3.23)

**Proof.** In view of Lemma 3.14 it suffices to prove the lemma with \( \lambda^{-2}P_0 V \) in place of \( C^{-1}(\lambda) \). We multiply the following by \( L_{00}^{-1}(\lambda) \) from the left:

\[ L_{01}(\lambda)\lambda^{-2}P_0 V = \overline{Q} D_2 V P_0 V - \lambda^2 \log \lambda \overline{Q} A(\lambda) V P_0 V + \lambda^2 \overline{Q} F(\lambda) V P_0 V. \]

(i) We may include \( L_{00}^{-1}(\lambda) \overline{Q} D_2 V P_0 V = (\overline{Q} + \tilde{L}(\lambda)) \overline{Q} D_2 V P_0 V \) into \( R_{02}(\lambda) \) by virtue of Lemma 3.8.
(ii) In \( L_{00}^{-1}(\lambda) \overline{Q} \lambda^2 \log \lambda A(\lambda) V P_0 V \) we substitute \( Y(\lambda) + \bar{X}(\lambda) + \overline{Q} \) for \( L_{00}^{-1} \). We may put \( Y(\lambda) \overline{Q} \lambda^2 \log \lambda A(\lambda) V P_0 V = \lambda^2 \log \lambda Y(\lambda) \cdot \overline{Q} A(\lambda) V P_0 V \) into \( R_{02}(\lambda) \) by virtue of Corollary 3.11. We write in the form

\[
\lambda^2 \log \lambda \bar{X}(\lambda) \overline{Q} A(\lambda) V = \lambda^2 \log \lambda \bar{X}(0) \overline{Q} A(0) V + \lambda^2 \log \lambda \bar{X}(\lambda)(A(\lambda) - A(0)) V.
\]

(3.24)

Then, \( \bar{X}(0) \overline{Q} A(0) V \in B(\mathcal{N}, \mathcal{H}_{(-\frac{1}{2}, -)}^{+}) \) and we put the first term on the right into \( \lambda^2 \log \lambda D_{21}^{(2)} \); it is easy to check by using Lemma 3.9 (2) that the last two terms satisfy the properties of \( R_{02}(\lambda) \). We write \( \overline{Q} \lambda^2 \log \lambda A(\lambda) V P_0 V \) as

\[
\overline{Q} \lambda^2 \log \lambda A(0) V P_0 V + \overline{Q} \lambda^2 \log \lambda (A(\lambda) - A(0)) V P_0 V.
\]

Since \( \overline{Q} A(0) V P_0 V \in B(\mathcal{N}, \mathcal{H}_{-3, +}) \), we put the first term into \( \lambda^2 \log \lambda D_{21}^{(2)} \). It can be checked that the second term satisfies the properties of \( R_{02} \) by differentiating it by \( \lambda \) and by applying Lemma 2.2 and Corollary 2.3.

(iii) Since \( \lambda^2 F(\lambda) V \) is also of class \( C^\sigma \) as a \( B(\mathcal{N}, \mathcal{H}_{-\gamma(\sigma), +}) \)-valued function by virtue of Proposition 2.6, we may put \( L_{00}^{-1}(\lambda) \overline{Q} \lambda^2 F(\lambda) V P_0 V \) into \( R_{02}(\lambda) \). This completes the proof.

\( \square \)

We omit the proof of the following lemma which goes entirely in parallel with that of the previous Lemma 3.15.

**Lemma 3.16.** There exists an operator \( D_{12}^{(3)} \in B(\mathcal{H}_{(-\delta, 3, +)} \setminus \mathcal{N}) \) and an operator valued function \( R_{03}(\lambda) \) which satisfies the property below such that

\[
C^{-1} L_{10}^{-1}(\lambda) L_{00}(\lambda) = \lambda^2 \log \lambda P_0 V D_k^{(3)} + P_0 V R_{30}(\lambda).
\]

Here, as a \( B(\mathcal{H}, \mathcal{N}) \) valued function, \( R_{03}(\lambda) \langle x \rangle^{(\delta - \gamma(\sigma)) -} \) is of class \( C^\sigma \) on \((-\lambda_0, \lambda_0)\) if \( 0 \leq \sigma \leq 2 \), for \( \lambda \neq 0 \) if \( \sigma = 3, 4 \), and with some \( N > 0 \)

\[
\| R_{03}^{(3)}(\lambda) \langle x \rangle^{\delta - \gamma(3)} - \| \|_{B(\mathcal{H}, \mathcal{N})} + \| R_{03}^{(4)}(\lambda) \langle x \rangle^{\delta - \gamma(4)} - \| \|_{B(\mathcal{H}, \mathcal{N})} \leq C(\log \lambda)^N.
\]

Since \( V P_0 V E_0(\lambda) R_0 V, V R_{01}(\lambda) P_0 V, V R_{02}(\lambda) P_0 V \) and \( V P_0 V R_{03}(\lambda) \) satisfy property \((K)_\rho \) with \( \rho > m + 1 \), the following lemma completes the proof of Proposition 3.6 for \( m = 6 \).

**Lemma 3.17.** The operator valued function \( V L_{00}^{-1} L_{01} C^{-1} L_{10} L_{00}^{-1} \) satisfies the condition \((K)_\rho \) with \( \rho > m + 1 \).

**Proof.** We substitute \( \lambda^{-2} P_0 V + C_r(\lambda) \) for \( C(\lambda) \) and write \( L_{01} \lambda^{-2} P_0 V L_{10} \) in the form

\[
\overline{Q}(\lambda D_2 - \lambda^3 \log \lambda A(\lambda) + \lambda^3 F(\lambda)) V P_0 V (\lambda D_2 - \lambda^3 \log \lambda A(\lambda) + \lambda^3 F(\lambda)) \overline{Q}.
\]

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It follows by virtue of Corollary 2.3 that this is a \( B(\mathcal{H}_{-\delta+\sigma_0(2j)}, \mathcal{H}_{-\sigma_0(2j)}) \)-valued function of class \( C^j \) on \((-\lambda_0, \lambda_0)\) for \(0 \leq j \leq 2\), on \((-\lambda_0, \lambda_0)\) \(\{0\}\) for \(j = 3\) and \(j = 4\), and the third and fourth derivatives are bounded by \( C(\log \lambda) \) and \( C|\lambda|^{-1}(\log \lambda) \) in respective norms. It follows, by writing \( L^{-1}_{00}(\lambda) = \overline{Q} + (L^{-1}_{00} - \overline{Q}) \) as usual, that \( VL^{-1}_{00}L_{01}(\lambda^{-2}P_0V)L_{10}L^{-1}_{00} \) satisfies the condition \((K)_\rho\) with \( \rho > m + 1 \). It is then obvious that so does \( VL^{-1}_{00}L_{01}C_r(\lambda)L_{10}L^{-1}_{00} \).

The lemma follows. \(\Box\)

### 3.2.2 The case \( m \geq 8 \) is even

Let now \( m \geq 8 \). Define \( F_0(\lambda), F_1(\lambda) \) and \( F_2(\lambda) \) by

\[
F_0(\lambda) = D_0 + \lambda^2D_2 + \cdots + \lambda^{m-4}D_{m-4} + \lambda^{m-2}F(\lambda) \\
F_1(\lambda) = D_2 + \cdots + \lambda^{m-6}D_{m-4} + \lambda^{m-4}F(\lambda), \\
F_2(\lambda) = D_4 + \cdots + \lambda^{m-8}D_{m-4} + \lambda^{m-6}F(\lambda)
\]

so that \( G_0(\lambda) = F_0(\lambda) - \lambda^{m-2}\log \lambda A(\lambda), \) \( F_0 = D_0 + \lambda^2F_2(\lambda) \) and \( F_2(\lambda) = D_2 + \lambda^2F_4(\lambda) \). Since \((1 + D_0V)Q = 0\), we then have

\[
L_{11}(\lambda) = \lambda^2Q(D_2 + \lambda^2F_4(\lambda) - \lambda^{-m-4}\log \lambda A(\lambda))VQ; \quad (3.26)
\]

and \( L_{10}(\lambda) \) and \( L_{01}(\lambda) \) are obtained from (3.26) by replacing one of \( Q \) by \( \overline{Q} \) as in (3.13). Recall that \((QD_2VQ)^{-1} = P_0V, VQP_0 = VP_0 \) and \( P_0VQ = P_0V \).

**Lemma 3.18.** Let \( \varphi = P_0V \) where \( V \) is considered as a function. Then there exists \( \lambda_0 \) such that

\[
C^{-1}(\lambda) = \lambda^{-2}P_0V + c_m\lambda^{m-6}\log \lambda \varphi \otimes (V\varphi) + P_0VE_0(\lambda)P_0V, \quad (3.27)
\]

where \( c_m = (2\pi)^{-\frac{m}{2}}(m!!)^{-1} \) and \( E_0(\lambda) \) is a \( B(\mathcal{N}) \)-valued function of \( \lambda \in (-\lambda_0, \lambda_0) \) which satisfies the property (3.12).

**Proof.** In this proof the smoothness of operator valued functions will be referred to as \( B(\mathcal{N}) \) valued functions. By virtue of Proposition 2.6, \( E_0(\lambda) \equiv QF_2(\lambda)VQ \) is of class \( C^{\frac{m+2}{2}} \). Likewise Lemma 3.8, Proposition 2.6, property (3.4) of \( \phi \in \mathcal{N} \) and that \( 2(m - 4) > \frac{m+2}{2} \) imply that

\[
E_{02}(\lambda) \equiv \lambda^{2(m-4)}(\log \lambda)^2QA(\lambda)V\overline{Q}L^{-1}_{00}(\lambda)\overline{Q}A(\lambda)VQ, \\
E_{03}(\lambda) \equiv QF_1V\overline{Q}L^{-1}_{00}(\lambda)\overline{Q}F_1VQ, \\
F_{20}(\lambda) = Q(F_1V\overline{Q}L^{-1}_{00}(\lambda)A + AV\overline{Q}L^{-1}_{00}(\lambda)F_1)VQ
\]

are all of class \( C^{\frac{m+2}{2}} \). With these definitions, we may write

\[
L_{10}(\lambda)L_{00}(\lambda)^{-1}L_{01}(\lambda) = \lambda^4(E_{02} + E_{03} - \lambda^{m-4}\log \lambda F_{20}(\lambda)), \quad (3.28)
\]
Thus, defining
\[
\tilde{E}_0(\lambda) = P_0 V (E_{01}(\lambda) - E_{02}(\lambda) - E_{03}(\lambda)),
\]
\[
\tilde{F}_{10}(\lambda) = P_0 V A(\lambda) V Q, \quad \tilde{F}_{20}(\lambda) = P_0 V F_{20}(\lambda),
\]
subtracting (3.28) from (3.26) and factoring out $\lambda^2 Q D_2 V Q$, we obtain
\[
C(\lambda) = \lambda^2 Q D_2 V Q (1 - \lambda^{-m-4} \log \lambda \tilde{F}_{10}(\lambda) + \lambda^{-m-2} \log \lambda \tilde{F}_{20}(\lambda) + \tilde{\lambda}^2 \tilde{E}_0(\lambda)).
\]
It follows that $C(\lambda)$ is invertible in $\mathcal{N}$ for $0 < |\lambda| < \lambda_0$ for small enough $\lambda_0$ and
\[
C^{-1}(\lambda) = \lambda^{-2} \sum_{n=0}^{\infty} (\lambda^{-m-4} \log \lambda \tilde{F}_{10}(\lambda) - \lambda^{-m-2} \log \lambda \tilde{F}_{20}(\lambda) - \tilde{\lambda}^2 \tilde{E}_0(\lambda))^n P_0 V.
\]
It is easy to see by counting the powers of $\lambda$ in front of powers of $\log \lambda$ that the series over $2 \leq n < \infty$ produces a function of class $C^{\frac{m+2}{2}}$. Thus, writing $E_0(\lambda)$ for $C^{\frac{m+2}{2}}$ functions indiscriminately, we have
\[
C^{-1}(\lambda) = \lambda^{-2} P_0 V + \lambda^{m-6} \log \lambda \tilde{F}_{10}(\lambda) P_0 V - \lambda^{m-4} \log \lambda \tilde{F}_{20}(\lambda) P_0 V + E_0(\lambda).
\]
Since $F_{20}(\lambda)$ is of class $C^{\frac{m+2}{2}}$ as mentioned above and $m - 4 \geq \frac{m}{2}$ if $m \geq 8$, $\lambda^{m-4} \log \lambda \tilde{F}_{20}(\lambda) P_0 V$ satisfies the property (3.12). If we expand $A(\lambda) = A(0) + \lambda A'(0) + A_2(\lambda)$ with $A_2(\lambda) = A(\lambda) - A(0) - \lambda A'(0)$ in
\[
\lambda^{m-6} \log \lambda \tilde{F}_{10}(\lambda) P_0 V = \lambda^{m-6} \log \lambda P_0 V A(\lambda) V P_0 V,
\]
then, $\lambda^{m-6} \log \lambda P_0 V A_2(\lambda) V P_0 V$ satisfies the property (3.12). Since $A(0) = c_m 1 \otimes 1$ and $A'(0) = 0$, the lemma follows. 

Lemma 3.18 and the following lemma complete the proof of Proposition 3.6 for $m \geq 8$. We use the following short hand notation.
\[
R_i(\lambda) = C^{-1}(\lambda) L_{10}(\lambda) L_{00}^{-1}(\lambda), \quad R_r(\lambda) = L_{00}^{-1}(\lambda) L_{01}(\lambda) C^{-1}(\lambda)
\]
\[
R_c(\lambda) = L_{00}^{-1}(\lambda) L_{01}(\lambda) C^{-1}(\lambda) L_{10}(\lambda) L_{00}^{-1}(\lambda)
\]

**Lemma 3.19.** For sufficiently small $\lambda_0 > 0$ the following properties are satisfied:

(1) For $\sigma \leq \frac{m-2}{2}$, $R_i(\lambda)$ is a $\mathcal{X}_\sigma \equiv \mathcal{B}(\mathcal{H}_{(\sigma+2-\delta)+}, \mathcal{N})$-valued function of class $C^\sigma$ on $(-\lambda_0, \lambda_0)$; it is of class $C^\sigma$ for $\lambda \neq 0$ for $\frac{m}{2} \leq \sigma \leq \frac{m+4}{2}$ and
\[
\|R_i^{(m-2)}(\lambda)\|_{\mathcal{X}_\infty} + |\lambda||R_i^{(m-2)}(\lambda)\|_{\mathcal{X}_{m+2}} \leq C(\log \lambda).
\]
(2) For \( \sigma \leq \frac{m-2}{2} \), \( R_\sigma(\lambda) \) is a \( \mathcal{Y}_\sigma \equiv B(N, \mathcal{H}_{-(\sigma+2),+}) \)-valued function of class \( C^\sigma \); it is of class \( C^\sigma \) for \( \lambda \neq 0 \) for \( \frac{m}{2} \leq \sigma \leq \frac{m+2}{2} \) and
\[
\| R_\sigma^{(m)}(\lambda) \|_{\mathcal{Y}_{\frac{m}{2}}} + |\lambda| \| R_\sigma^{(m+\delta)}(\lambda) \|_{\mathcal{Y}_{\frac{m+\delta}{2}}} \leq C(\log \lambda).
\]

(3) For \( \sigma \leq \frac{m-2}{2} \), \( R_\sigma(\lambda) \) is a \( \mathcal{Z}_\sigma \equiv B(\mathcal{H}_{(\sigma+2)-\delta,+}, \mathcal{H}_{-(\sigma+2),+}) \)-valued function of class \( C^\sigma \). Moreover it is of class \( C^\sigma \) for \( \lambda \neq 0 \) for \( \frac{m}{2} \leq \sigma \leq \frac{m+2}{2} \) and
\[
\| R_\sigma^{(m)}(\lambda) \|_{\mathcal{Z}_{\frac{m}{2}}} + |\lambda| \| R_\sigma^{(m+\delta)}(\lambda) \|_{\mathcal{Z}_{\frac{m+\delta}{2}}} \leq C(\log \lambda).
\]

Proof. We have \( \delta - \frac{m}{2} > \frac{m+2}{2} \) and
\[
P_0 V L_{01}(\lambda) V Q L_{00}^{-1}(\lambda) = \lambda^2 P_0 V J_2(\lambda) V Q (Q + \tilde{L}(\lambda)).
\]

Proposition 2.6 and Corollary 2.3 imply that \( T(\lambda) = P_0 V J_2(\lambda) V Q \) satisfies the property of \( R_t(\lambda) \) of the lemma (recall that \( J_2(\lambda) \) contains \( \lambda^{m-4} \log \lambda A(\lambda) \) and \( m - 4 \geq \frac{m}{2} \)). Since
\[
\lambda^2 C^{-1}(\lambda) = P_0 V + c_m \lambda^{m-4} \log \lambda \phi \otimes V \phi + \lambda^2 P_0 V E_0(\lambda) P_0 V
\]
satisfies property (3.12), statement (1) follows. We likewise see that
\[
\tilde{T}(\lambda) = (\overline{Q} + \tilde{L}(\lambda)) \overline{Q} J_2(\lambda) V Q
\]
satisfies the property of \( R_r(\lambda) \) of the lemma. Then statement (2) follows since \( \lambda^2 C^{-1}(\lambda) \) satisfies the property (3.12). Statement (3) is obvious since \( C_\sigma(\lambda) = \tilde{T}(\lambda) C_\sigma(\lambda), \ C_\sigma(\lambda) \) satisfies (1) and \( \tilde{T}(\lambda) \) satisfies the property of \( R_r(\lambda) \).

\[\square\]

4 Low energy estimate I, generic case

In the following two sections, we study the low energy part \( W_< \) of the wave operator \( W_- \). We take and fix \( \lambda_0 > 0 \) arbitrarily if \( H \) is generic type, otherwise small enough so that Proposition 3.6 is satisfied. We take cut-off functions \( \Phi \) and \( \Psi \) as in the introduction and define \( W_< \) as in (1.9):
\[
W_< = \Phi(H) \Phi(H_0) - \int_0^\infty \Phi(H) G(\lambda) V(G_0(\lambda) - G_0(-\lambda)) \Phi(H_0) \lambda d\lambda / \pi i.
\]

In this section we study \( W_< \) in the case that \( H \) is of generic type and prove the following proposition. We assume that \( V \) satisfies the condition
\[
\mathcal{F}(|x|^{2\sigma} V) \in L^{m+1}(\mathbb{R}^m) \quad \text{and} \quad |V(x)| \leq C|x|^{-\delta} \quad \text{for some} \ \delta > m+2. \quad (4.1)
\]
Proposition 4.1. Let $m \geq 6$ be even and let $V$ satisfy (4.1). Suppose that $H$ is of generic type. Then $W_\varphi$ is bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$.

The integral kernels $\Phi_0(x, y)$ and $\Phi(x, y)$ of $\Phi(H_0)$ and $\Phi(H)$ respectively are continuous and bounded by $C_N \langle x - y \rangle^{-N}$ for any $N$ ([22]) and a fortiori $\Phi(H)$ and $\Phi(H_0)$ are bounded in $L^p$ for all $1 \leq p \leq \infty$. Hence, we have only to discuss the operator defined by the integral

$$
\int_0^\infty \Phi(H_0)G_0(\lambda)V(G_0(\lambda) - G_0(-\lambda))\lambda\Phi(H_0)d\lambda. \quad (4.2)
$$

By iterating the resolvent equation, we have, with $L(\lambda) = (1 + G_0(\lambda)V)^{-1} - 1$ as in Lemma 3.5,

$$
G(\lambda) = G_0(\lambda) - G_0(\lambda)V G_0(\lambda) - G_0(\lambda) V L(\lambda) G_0(\lambda).
$$

We substitute this for $G(\lambda)$ in (4.2). The first two terms produce the Born approximation $\Phi(H)(\Omega_1 - \Omega_2)\Phi(H_0)$, which is bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$ by virtue of Lemma 2.7. The last term produces

$$
\int_0^\infty \Phi(H)G_0(\lambda)V L(\lambda)G_0(\lambda)V(G_0(\lambda) - G_0(-\lambda))\lambda\Phi(H_0)\tilde{\Phi}(\lambda)d\lambda. \quad (4.3)
$$

where we have introduced another cut off function $\tilde{\Phi}(\lambda) \in C_0^\infty(\mathbb{R})$ which satisfies

$$
\tilde{\Phi}(\lambda)\Phi(\lambda^2) = \Phi(\lambda^2), \quad \text{and} \quad \tilde{\Phi}(\lambda) = 0 \text{ for } |\lambda| \geq \lambda_0^2.
$$

We prove that (4.3) is bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$ in the following slightly more general setting for a later purpose. Note that, by virtue of Lemma 3.5, $VL(\lambda)G_0(\lambda)V$ satisfies property the $(K)_\rho$ with $\rho = \delta - 1 - \varepsilon$ for any $\varepsilon > 0$ and, by choosing $\varepsilon > 0$ small enough, $\rho$ can be taken larger than $m - 1$ as $\delta > m + 2$.

Proposition 4.2. Let $m \geq 6$. Suppose $K(\lambda)$ satisfies property $(K)_\rho$ for some $\rho > m + 1$. Let $\Phi, \tilde{\Phi} \in C_0^\infty(\mathbb{R})$ be as above and $\Omega$ be defined by

$$
\Omega = \int_0^\infty \Phi(H)G_0(\lambda)K(\lambda)(G_0(\lambda) - G_0(-\lambda))\lambda\Phi(H_0)\tilde{\Phi}(\lambda)d\lambda. \quad (4.4)
$$

Then, $\Omega$ is an integral operator with admissible integral kernel.

We prove Proposition 4.2 by using a series of lemma. We first remark that (4.4) may be considered as Riemann integral of $B(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})$ valued continuous function and that $\Omega$ may be extended to a bounded operator in $\mathcal{H}$. Indeed,
since the multiplication by \( \langle x \rangle^{-\gamma}, \gamma > 1 \) is \( H_0 \)-smooth in the sense of Kato ([14]), we have

\[
\| \Omega f, g \| \leq \sup_{\lambda \in \mathbb{R}} \| \Phi(\lambda) \langle x \rangle^{-\gamma} K(\lambda) \langle x \rangle^{-\gamma} \|_{\mathcal{B}(\mathcal{H})} \| \langle x \rangle^{-\gamma} G_0(\lambda) \Phi(H_0) f \|_{L^2(\mathbb{R}; \mathcal{H}, \lambda d\lambda)} \times \| \langle x \rangle^{-\gamma} G_0(\lambda) \Phi(H) g \|_{L^2(\mathbb{R}; \mathcal{H}, \lambda d\lambda)} \leq C \| f \| \| g \| .
\]

Define \( \Omega(x, y) = \Omega_+(x, y) - \Omega_-(x, y) \), where

\[
\Omega_\pm(x, y) = \int_0^\infty \langle K(\lambda) G_0(\pm \lambda) \Phi_0(\cdot, y), G_0(\mp \lambda) \Phi(\cdot, x) \rangle \lambda \Phi(\lambda) d\lambda. \tag{4.5}
\]

**Lemma 4.3.** The function \( \Omega(x, y) \) is continuous and \( \Omega \) is an integral operator with the integral kernel \( \Omega(x, y) \).

**Proof.** For \( \gamma > 1 \), \( x \mapsto \Phi_0(\cdot, x) \) and \( y \mapsto \Phi_0(\cdot, y) \) are \( \mathcal{H}_\gamma \)-valued continuous, and \( \Omega_\pm(x, y) \) are continuous functions of \((x, y)\). For \( f, g \in C_0^\infty(\mathbb{R}_m) \), \( \Phi(H_0) f(\cdot) = \int \Phi(\cdot, y) f(y) dx \) and \( \Phi(H) g(\cdot) = \int \Phi(\cdot, x) g(x) dx \) converge as Riemann integrals in \( \mathcal{H}_\gamma \). It follows by Fubini’s theorem that

\[
\langle \Omega f, g \rangle = \sum_\pm \int \langle K(\lambda) G_0(\pm \lambda) \Phi(H_0) f, G_0(\mp \lambda) \Phi(H) g \rangle \Phi(\lambda) d\lambda
\]

is equal to \( \int \Omega(x, y) f(y) g(x) dy dx \). The lemma follows. \( \square \)

Introducing the notation

\[
G_0(\lambda, \cdot, y) = e^{-i\lambda y} G_0(\lambda) \Phi_0(\cdot, y), \quad G_0(\lambda, \cdot, x) = e^{-i\lambda x} G_0(\lambda) \Phi(\cdot, x), \quad F_\pm(\lambda, x, y) = \langle K(\lambda) G_0(\pm \lambda, \cdot, y), G_0(\lambda, \cdot, x) \rangle \Phi(\lambda), \tag{4.6}
\]

we write (4.5) in the form

\[
\Omega_\pm(x, y) = \int_0^\infty e^{i\lambda (|x| + |y|)} F_\pm(\lambda, x, y) \lambda d\lambda. \tag{4.8}
\]

**Lemma 4.4.** Let \( \gamma > \frac{1}{2} \) and \( \beta \geq 0 \) be an integer and let \( x, y \in \mathbb{R}^m \). Then:

1. As \( \mathcal{H} \)-valued functions of \( \lambda \), \( \langle \cdot \rangle^{-\beta-\gamma} G_0(\lambda, \cdot, y) \) and \( \langle \cdot \rangle^{-\beta-\gamma} G_0(\lambda, \cdot, x) \) are of class \( C^{\beta}_{\mathcal{H}}(\mathbb{R}) \) for \( 0 \leq \beta \leq \frac{m+2}{2} \) if \( m \geq 8 \); they are of class \( C^{\beta}_{\mathcal{R}}(\mathbb{R}) \) for \( \beta \leq \frac{m}{2} \) and of class \( C^{\beta}_{\mathcal{R}}(\mathbb{R}) \) for \( \beta = \frac{m+2}{2} \) if \( m = 6 \).
2. For \( 0 \leq \beta \leq \frac{m+2}{2} \) and \( \epsilon > 0 \), we have

\[
\| \langle \cdot \rangle^{-\beta-\epsilon} G_0^{(\beta)}(\lambda, \cdot, y) \| \leq C \lambda e^{\min(0, \frac{m-4}{2} - \beta)} (y)^{-\frac{m-1}{2}}, \quad 0 < |\lambda| < \lambda_0. \tag{4.9}
\]
(3) For $0 \leq \beta \leq \frac{m-2}{2}$, we have at $\lambda = 0$:

$$|G_{0l}(0, z, y)| \leq C \sum_{\beta_1+\beta_2 = \beta} \langle z \rangle^{\beta_1} |z-y|^{m-2-\beta_2}.$$  \hspace{1cm} (4.10)

(4) With obvious modifications $G_{0r}(\lambda, z, x)$ satisfies (4.9) and (4.10).

Proof. Statement (1) follows from the LAP, viz. from Lemma 2.5 and Proposition 2.6. The following proof will be a bit more than necessary for the lemma for a later purpose. By Leibniz’ rule $C^{(\beta)}(\lambda, z, y)$ is a linear combination of

$$K_{\beta_1, \beta_2}(\lambda, z, y) = \int_{\mathbb{R}^m} \frac{(i\psi(w, z, y))^{\beta_1} e^{i\lambda \psi(w, y)}}{|z-w|^{m-2-\beta_2}} H_{\beta_2}(\lambda |w-z|) \Phi_0(w, y) dw,$$

$$H_{\beta}(s) = \int_0^{\infty} e^{-t \frac{m-2}{\beta}} (s + \frac{it}{2})^{-\frac{m-2}{\beta}} dt \hspace{1cm} (4.11)$$

over the indices $\beta_1, \beta_2$ such that $\beta_1 + \beta_2 = \beta$. Here $\psi(w, z, y) \equiv |w-z| - |y|$ satisfies $|\psi(w, z, y)| \leq |w-y| + |z|$. It follows when $\beta \leq \frac{m-2}{2}$ that

$$|K_{\beta_1, \beta_2}(0, z, y)| \leq C_{\beta_1, \beta_2} \int_{\mathbb{R}^m} \frac{\langle z \rangle^{\beta_1} (w-y)^{-N}}{|z-w|^{m-2-\beta_2}} dw \leq C_{\beta_1, \beta_2} \frac{\langle z \rangle^{\beta_1}}{|z-y|^{m-2-\beta_2}}$$

for any $N$ and we obtain (4.10).

If $\beta_2 \leq \frac{m-3}{2}$, we have $|s + \frac{it}{2}|^{\frac{m-3}{2} - \beta_2} \leq C(s^{\frac{m-3}{2} - \beta_2} + |t|^{\frac{m-3}{2} - \beta_2})$ and $|H_{\beta_2}(s)| \leq C(s + 1)^{\frac{m-3}{2} - \beta_2}$. It follows that

$$|K_{\beta_1, \beta_2}(\lambda, z, y)| \leq C \int_{\mathbb{R}^m} \frac{|z|^{\beta_1} + |w-y|^{\beta_1}}{|z-w|^{m-2-\beta_2}} (\lambda |z-w| + 1)^{\frac{m-3}{2} - \beta_2} |\Phi_0(w, y)| dw$$

$$\leq C(\langle z \rangle^{\beta_1} (z-y)^{-\frac{m-1}{2}}) \text{ for all } \lambda \leq \lambda_0. \hspace{1cm} (4.12)$$

Hence, if $\beta \leq \frac{m-3}{2}$, $\langle \cdot \rangle^{\beta - \gamma} G_0(\lambda, \cdot, y)$ is $\mathcal{H}$ valued function of class $C^\beta$ and the estimate (4.9) follows.

If $\frac{m-3}{2} \leq \beta_2 \leq \frac{m+2}{2}$, $|s + \frac{it}{2}|^{\frac{m-3}{2} - \beta_2} \leq C \min\{s^{\frac{m-3}{2} - \beta_2}, |t|^{\frac{m-3}{2} - \beta_2}\}$ and

$$|H_{\beta_2}(s)| \leq C \left( s^{\frac{m-3}{2} - \beta_2} \int_0^s e^{-t \frac{m-3}{2}} dt + \int_s^{\infty} e^{-t \frac{m-3}{2}} dt \right) \leq C \left\{ \begin{array}{ll} \min\{s^{\frac{m-3}{2} - \beta_2}, \log(1/s)\}, & \text{if } \beta_2 = m-2, \\ \min\{s^{\frac{m-3}{2} - \beta_2}, 1\}, & \text{if otherwise.} \end{array} \right.$$ 

The smoothness property of $\langle \cdot \rangle^{\beta - \gamma} G_0(\lambda, \cdot, y)$ and $\langle \cdot \rangle^{\beta - \gamma} G_{0r}(\lambda, \cdot, x)$ for $\frac{m-2}{2} \leq \beta \leq \frac{m+2}{2}$ follows from this estimate and Lebesgue’s dominated convergence theorem. We have $\beta_2 = m-2$ if and only if $(m, \beta_2) = (6, 4)$. It follows
that, if \((m, \beta_2) = (6, 4)\),
\[
|K_{\beta_1, \beta_2}(\lambda, z, y)| \leq C\langle z \rangle^{\beta_1} \min \left( \frac{\lambda^{m-3}_m - \beta_2}{(z-y)^{m-2}_m}, \langle \log \lambda \rangle + \log \langle z - y \rangle \right)
\]  
(4.13)

and if otherwise
\[
|K_{\beta_1, \beta_2}(\lambda, z, y)| \leq C\langle z \rangle^{\beta_1} \min \left( \frac{\lambda^{m-3}_m - \beta_2}{(z-y)^{m-2}_m}, \frac{1}{(z-y)^{m-2-\beta_2}_m} \right).
\]  
(4.14)

Estimate (4.9) follows from the first estimates of (4.13) and (4.14). (We shall use the second estimates shortly.) The proof for \(G_{0r}(\lambda, \cdot, x)\) is similar and we omit it.

By virtue of Lemma 4.4, \(F_\pm(\lambda, x, y)\) is of class \(C_{m+2}^{m+2}\) on \(\mathbb{R}\) with respect to \(\lambda\) for every fixed \(x, y \in \mathbb{R}\) and it satisfies
\[
|\Omega(x, y)| \leq C\langle x \rangle^{-m-1}_m \langle y \rangle^{-m-1}_m.
\]
It is then easy to check that
\[
\sup_{x \in \mathbb{R}^m} \int_{||x|-|y|| < 1} |\Omega(x, y)| dy + \sup_{y \in \mathbb{R}^m} \int_{||x|-|y|| < 1} |\Omega(x, y)| dx < \infty.
\]  
(4.15)

Thus, we hereafter consider \(\Omega(x, y)\) only on the domain \(||x|-|y|| > 1\). We apply integration by parts \(k = (m+2)/2\) times to
\[
\Omega_\pm(x, y) = \frac{1}{(i(||x|+|y||))^k} \int_0^\infty \left( \frac{\partial}{\partial \lambda} \right)^k e^{i\lambda(||x|+|y||)} \cdot F_\pm(\lambda, x, y) \lambda d\lambda.
\]  
(4.16)

The result is that \(\Omega(x, y)\) is the sum of
\[
I_1(x, y) = \sum_{\pm} \frac{\pm i^{m+2}}{2(||x|+|y||)^{m+2}} \int_0^\infty e^{i\lambda(||x|+|y||)} F_\pm^{(m+2)}(\lambda, x, y) \lambda d\lambda,
\]  
(4.17)
\[
I_2(x, y) = \sum_{\pm} \frac{\pm (m+2) i^{m+2}}{2(||x|+|y||)^{m+2}} \int_0^\infty e^{i\lambda(||x|+|y||)} F_\pm^{(m+2)}(\lambda, x, y) d\lambda,
\]  
(4.18)

and the boundary terms:
\[
B(x, y) = \sum_{j=0}^{m-2} i^j (j+1) \left( \frac{F_\pm^{(j)}(0, x, y)}{||x|+|y||^{j+2}} - \frac{F_\pm^{(j)}(0, x, y)}{||x|-|y||^{j+2}} \right).
\]  
(4.19)

**Lemma 4.5.** The function \(B(x, y)\) of (4.19) is an admissible integral kernel.
Proof. Derivatives \( F^{(j)}_\pm(0, x, y) \) are linear combinations over \( \alpha + \beta + \gamma = j \) of

\[
(\pm 1)^\beta \langle K^{(\alpha)}(0) G^{(\beta)}_{0l}(0, \cdot, y), G^{(\gamma)}_{0r}(0, \cdot, x) \rangle
\]

(4.20)

with coefficients \((-1)^\gamma j! / \alpha! \beta! \gamma! \). In (4.20), we have for arbitrarily small \( \varepsilon > 0 \)

\[
\| \langle z \rangle^{1+j-m-\varepsilon-\beta} G^{(\beta)}_{0l}(0, z, y) \| \leq C \begin{cases} 
\langle y \rangle^{j+2-m} & \text{if } \beta = j, \\
\langle y \rangle^{j+1-m} & \text{if otherwise.}
\end{cases}
\]

\[
\| \langle z \rangle^{1+j-m-\varepsilon-\gamma} G^{(\gamma)}_{0r}(0, z, x) \| \leq C \begin{cases} 
\langle x \rangle^{j+2-m} & \text{if } \gamma = j, \\
\langle x \rangle^{j+1-m} & \text{if otherwise.}
\end{cases}
\]

(4.21)

This can be seen as follows. By virtue of (4.10) we have

\[
\left| \frac{G^{(\beta)}_{0l}(0, z, y)}{\langle z \rangle^{m-j-1+\beta+\varepsilon}} \right| \leq \sum_{\beta_2=0}^{\beta} \frac{C}{\langle z \rangle^{m-\beta_1-1+\beta_2+\varepsilon}} \frac{C}{\langle y \rangle^{m-2-\beta_2}}
\]

and the like for \( G^{(\gamma)}_{0r}(0, \cdot, x) \). Since \((m - j - 1 + \beta_2 + \varepsilon) + (m - 2 - \beta_2) > m\), we have either \( m - j - 1 + \beta_2 + \varepsilon > \frac{m}{2} \) or \( m - 2 - \beta_2 > \frac{m}{2} \). Hence

\[
\| \langle z \rangle^{j+1-m-\beta_2-\varepsilon} (z-y)^{2+j-m} \| \leq C \begin{cases} 
\langle y \rangle^{2+j-m}, & \text{if } \beta = j, \\
\langle y \rangle^{1+j-m}, & \text{if otherwise.}
\end{cases}
\]

(4.22)

Since \( \rho > m + 1 \), we have for \( 0 < \varepsilon \leq 1 \) that

\[
\max(m - 1 - (j - \beta) + \varepsilon, m - 1 - (j - \gamma) + \varepsilon) < \rho - \alpha
\]

and \( \langle \cdot \rangle^{m-1-(j-\gamma)+\varepsilon} K^{(\alpha)}(0) \langle \cdot \rangle^{m-1-(j-\beta)+\varepsilon} \in B(H) \) by property \((K)_\rho\). Thus, (4.20) is bounded in modulus by a constant times

\[
Y_{\alpha\beta\gamma}(x, y) = \begin{cases} 
\langle x \rangle^{1+j-m} \langle y \rangle^{1+j-m} & \text{if } \beta \neq 0, j, \\
\langle x \rangle^{1+j-m} \langle y \rangle^{2+j-m} & \text{if } \beta = j, \\
\langle x \rangle^{2+j-m} \langle y \rangle^{1+j-m} & \text{if } \beta = 0.
\end{cases}
\]

(4.23)

It follows that the \( j \)-th summand of (4.19) is bounded by

\[
C \sum_{\alpha+\beta+\gamma=j} \frac{1}{|x|+|y|^{j+2}} - \frac{(-1)^\beta}{(|x| - |y|)^{j+2}} Y_{\alpha\beta\gamma}(x, y)
\]

(4.24)

and it is an easy exercise to prove that this is an admissible integral kernel. (Indeed, summands with \( \beta \neq 0, j \) are admissible by virtue of Lemma 3.6 of [I]; those with \( \beta = 0 \) or \( \beta = j \) are the same as (3.21) of [I] and the argument in [I] following (3.21) applies also for \( \beta \leq \frac{m-2}{2} \) or \( \gamma \leq \frac{m-2}{2} \) if \( m \geq 4 \).) This completes the proof. \( \square \)
**Lemma 4.6.** The integral kernel $I_2(x, y)$ defined by (4.18) is admissible.

**Proof.** By Leibniz’ rule $F_{±}^{(\alpha)}(\lambda, x, y)$ is a linear combination of

$$X_{ξ,±}(λ, x, y) = (±1)^{β}⟨K^{(α)}(λ) G^{(β)}_0(±λ, ·, y), G^{(γ)}_0(λ, ·, x)⟩\tilde{Φ}(η)(λ)$$  \hspace{1cm} (4.25)

with ± independent coefficients $(-1)^{γ}(\frac{m}{2})!|λ|!|β|!|γ|!$ over multi-indices $ξ = (\alpha, β, γ, η)$ of length $|ξ| = \frac{m}{2}$. Thus, if we define

$$Ω_{ξ,±}^{(1)}(x, y) = \int_{0}^{∞} e^{iλ(|x|±|y|)} X_{ξ,±}(λ, x, y)dλ,$$  \hspace{1cm} (4.26)

then $I_2(x, y)$ is a linear combination over $ξ = (α, β, γ, η)$ with $|ξ| = \frac{m}{2}$ of

$$I_{2,ξ}(x, y) = \left( \frac{Ω_{ξ,+}^{(1)}(x, y)}{(|x| + |y|)^{\frac{m+2}{2}}} - \frac{Ω_{ξ,-}^{(1)}(x, y)}{(|x| - |y|)^{\frac{m+2}{2}}} \right).$$  \hspace{1cm} (4.27)

We estimate $I_{2,ξ}(x, y)$ for various cases of $ξ$ separately.

(1) **The case** $ξ \neq (0, \frac{m}{2}, 0, 0), (0, 0, \frac{m}{2}, 0)$. In view of property $(K)_ρ$, we estimate

$$|X_{ξ}(λ, x, y)| \leq C\|⟨x⟩^{ρ-α} K^{(α)}(λ)⟨x⟩^{ρ-α}\|_{B(H)}$$

$$×\|⟨·⟩^{-(ρ-α)} C^{(β)}_0(±λ, ·, y)\|\|⟨·⟩^{-(ρ-α)} C^{(γ)}_0(±λ, ·, x)\|.$$  \hspace{1cm} (4.28)

Since $ρ > m + 1$ and $α + β + γ ≤ \frac{m}{2}$, we have for $0 < ε < 1$

$$\max(β + \frac{m}{2} + ε, γ + \frac{m}{2} + ε) < ρ - α.$$  \hspace{1cm} (4.29)

Hence, by virtue of (4.9), we have that

$$|X_{ξ}(λ, x, y)| ≤ C \begin{cases} ⟨x⟩^{-\frac{m+1}{2}} ⟨y⟩^{-\frac{m+1}{2}}, & \text{if both } β, γ ≤ \frac{m-3}{2}, \\ λ^{-\frac{1}{2}}⟨x⟩^{-\frac{m+1}{2}} ⟨y⟩^{-\frac{m+1}{2}}, & \text{if one of } β, γ = \frac{m-2}{2}, \end{cases}$$  \hspace{1cm} (4.30)

where we have to modify the first line on the right by multiplying by $⟨\log λ⟩^N$ when $ξ = (\frac{m}{2}, 0, 0, 0)$. Thus after integrating with respect to $λ$ we obtain for $||x| - |y|| > 1$ that

$$\left| \frac{Ω_{ξ,±}^{(1)}(x, y)}{(|x| ± |y|)^{\frac{m+2}{2}}} \right| ≤ \frac{C}{(|x| ± |y|)^{\frac{m+1}{2}}⟨x⟩^{\frac{m+1}{2}} ⟨y⟩^{\frac{m+1}{2}}}. \hspace{1cm} (4.31)$$

It follows that $I_{2,ξ}$ are admissible for these $ξ$’s.
(2) The case $\xi = (0, \frac{m}{2}, 0, 0)$. Recall the definition (4.11) of $K_{\beta_1 \beta_2}(\lambda, z, y)$.

We substitute $\sum_{\beta_1 + \beta_2 = \frac{m}{2}} C_{\beta_1 \beta_2} K_{\beta_1 \beta_2}(\lambda, z, y)$ for $G_{0r}^{(2)}(\lambda, z, y)$ in

$$X_{\xi, \pm}(\lambda, x, y) = (\pm 1)^{\frac{m}{2}} \langle K(\lambda) G_{0r}^{(2)}(\pm \lambda, \cdot, y), G_{0r}(-\lambda, \cdot, x) \rangle \tilde{\Phi}(\lambda).$$

If $(\beta_1, \beta_2) \neq (0, \frac{m}{2})$, we have $\beta_2 \leq \frac{m-2}{2}$ and the first estimate of (4.14) implies

$$\| \langle \cdot \rangle^{-(\beta + \frac{m}{2} + \varepsilon)} K_{\beta_1 \beta_2}(\lambda, \cdot, y) \| \leq C \lambda^{-\frac{1}{2}} \langle y \rangle^{-\frac{m-1}{2}}. \quad (4.32)$$

Since $\beta + \frac{m}{2} + \varepsilon < \rho$ for $0 < \varepsilon \leq 1$, it follows via the argument similar to the one used for (4.28) that, for $(\beta_1, \beta_2) \neq (0, \frac{m}{2})$,

$$|\langle K(\lambda) K_{\beta_1 \beta_2}(\pm \lambda, \cdot, y), G_{0r}(-\lambda, \cdot, x) \rangle| \leq C \lambda^{-\frac{1}{2}} \langle x \rangle^{-\frac{m-1}{2}} \langle y \rangle^{-\frac{m-1}{2}}.$$

This implies that all members under the summation sign of

$$\sum_{\beta_1 + \beta_2 = \frac{m}{2}} \frac{\pm (1)^{\frac{m}{2}} C_{\beta_1 \beta_2}}{(|x| \pm |y|)^{n+2}} \int_0^\infty e^{i\lambda(|x| \pm |y|)} \langle K(\lambda) K_{0r}^{(2)}(\pm \lambda, \cdot, y), G_{0r}(-\lambda, \cdot, x) \rangle d\lambda,$$

are admissible except those with $(\beta_1, \beta_2) = (0, \frac{m}{2})$. We are thus left with

$$I_{2r} = \sum_{\pm} \frac{\pm (1)^{\frac{m}{2}}}{(|x| \pm |y|)^{n+2}} \int_0^\infty e^{i\lambda(|x| \pm |y|)} \langle K(\lambda) K_{0r}^{(2)}(\pm \lambda, \cdot, y), G_{0r}(-\lambda, \cdot, x) \rangle d\lambda.$$

For proving that $I_{2r}(x, y)$ is admissible, we restore the factors $e^{i\lambda(|x| \pm |y|)}$ to the original position. Thus defining $G_{0r}(\lambda, \cdot, x)$ and $G_{0r}^{(2)}(\lambda, z, y)$ by

$$\tilde{G}_{0r}(\lambda, \cdot, x) = G_0(\lambda) \Phi(\cdot, x) = e^{-i\lambda|x|} G_0(\lambda, \cdot, x)$$

and $G_{0r}^{(2)}(\lambda, z, y) = e^{i\lambda|y|} K_{0r}^{(2)}(\lambda, z, y)$ respectively, we rewrite, ignoring the unimportant constant, $I_{2r}$ in the form

$$\sum_{\pm} \frac{\pm (1)^{n+2}}{(|x| \pm |y|)^{n+2}} \int_0^\infty \langle K(\lambda) G_{0r}^{(2)}(\pm \lambda, \cdot, y), \tilde{G}_{0r}(-\lambda, \cdot, x) \rangle \tilde{\Phi}(\lambda) d\lambda. \quad (4.33)$$

More explicitly $G_{0r}^{(2)}(\lambda, z, y)$ is given by

$$\int_{R^m} e^{i\lambda|z-w|} \left( \int_0^\infty e^{-t \frac{m-2}{2}} \left( \lambda|z-w| + \frac{it}{2} \right)^{-\frac{4}{2}} dt \right) \Phi_0(w, y) dw. \quad (4.34)$$
Lemma 4.7.  (1) There exists a constant $C > 0$ such that

$$|\tilde{G}_0(\lambda, z, x) - \tilde{G}_0(0, z, x)| \leq C|\lambda| \langle z - x \rangle^{-\frac{m-1}{2}},$$  

$$(4.35)$$

$$|G_m^{\lambda}(\lambda, z, y)| \leq C \min \left( \lambda^{-\frac{3}{2}} \langle z - y \rangle^{-\frac{m-1}{2}}, \langle z - y \rangle^{-\frac{m-3}{2}} \right).$$  

$$(4.36)$$

(2) For a fixed $(z, y)$, $G_m^\lambda(\lambda, z, y)$ is continuous with respect to $\lambda \in \mathbb{R}$; for a fixed $y$ and, for $\gamma > \frac{3}{2}$, $G_m^\lambda(\lambda, :, y)$ is $\mathcal{H}_{-\gamma}$ valued integrable on $\mathbb{R}$ and is continuous for $\lambda \neq 0$.

(3) The integrand of (4.34) is integrable with respect to $(t, \lambda, w)$.

Proof.  (1) Write the convolution kernel of $G(\lambda) - G(0)$ in the form

$$C^m e^{i|\lambda|x} \int_0^\infty e^{-t\frac{m-3}{2}} \left\{ \left( \frac{t}{2} - i|\lambda||x| \right)^{-\frac{m-3}{2}} - \left( \frac{t}{2} \right)^{-\frac{m-3}{2}} \right\} dt + \frac{C^m (e^{i|\lambda|x} - 1)}{|x|^{m-2}}$$

and estimate it by $C|\lambda|(|x|^{3-m} + |x|^{3-m})$ for $|\lambda| \leq \lambda_0$. This yields (4.35) since $\frac{m-1}{2} \leq m - 3$ for $m \geq 6$. Estimate (4.36) is contained in (4.14).

(2) The continuity of $\lambda \mapsto G_m^\lambda(\lambda, z, y)$ is obvious by Lebesgue’s dominated convergence theorem. Then the second statement follow from the estimate (4.36) which also implies $|G_m^\lambda(\lambda, z, y)| \leq C \lambda^{-\frac{1}{2}} \langle z - y \rangle^{-\frac{m-3}{2}}$ by interpolation.

(3) Integrating with respect to $\lambda$ first, we have

$$\int_{\mathbb{R}^m} \int_0^\infty \int_{\mathbb{R}} \frac{e^{-t\frac{m-3}{2}}}{|z - w|^{\frac{m-2}{2}} \langle |\lambda||z - w| + t \rangle^\frac{1}{2}} |\Phi_0(w, y)| dw dt d\lambda$$

$$= C \int_{\mathbb{R}^m} \frac{|\Phi_0(w, y)|}{|z - w|^{\frac{m-2}{2}}} dw \cdot \int_0^\infty e^{-t\frac{m-6}{2}} dt < \infty.$$

We first show that it is sufficient to show that (4.33) is admissible if $K(\lambda)$, $\tilde{G}_0(-\lambda, :, x)$ and $\tilde{H}(\lambda)$ are replaced by $K(0)$, $\tilde{G}_0(0, :, x)$ and the constant function $1$ respectively.

(i) Let $I_{2r}^{(1)}(x, y)$ be defined by (4.33) with $K(0)$ in place of $K(\lambda)$. Via Taylor’s formula, $K(\lambda) - K(0) = \lambda \int_0^1 K'(\theta \lambda) d\theta$ and property $(K)_\rho$ implies that

$$\left\| \langle x \rangle^{\rho-1} \left( \int_0^1 K'(\theta \lambda) d\theta \right) \langle x \rangle^{\rho-1} \right\|_{B(H)} \leq C.$$

Hence, using (4.9) for $\tilde{G}_0(-\lambda, :, x)$ and (4.36), we obtain
Thus the problem is reduced to proving that first via Fubini’s theorem. For \( \lambda > 0 \), then, by virtue of Lemma 4.7 (3), we may integrate (4.34) with respect to \( \lambda \):

\[
\left\| \left\langle \cdot \right\rangle^{-\frac{3}{2}} \lambda G_{\Phi}^m(\lambda, \cdot, y) \right\| \left\| \left\langle \cdot \right\rangle^{-\frac{3}{2}} \tilde{G}_{0r}(\lambda, \cdot, x) \right\| \leq C \lambda^{-\frac{1}{2}} \langle x \rangle^{-\frac{m+1}{2}} \langle y \rangle^{-\frac{m+1}{2}}.
\]

It follows after integration with respect to \( \lambda \) that

\[
|I_{2r}(x, y) - I_{2r}^{(1)}(x, y)| \leq \frac{C}{\langle x \rangle^{\frac{m+2}{2}} \langle y \rangle^{\frac{m+2}{2}} \langle |x| - |y| \rangle^{\frac{m+2}{2}}} \quad (4.37)
\]

and \( I_{2r}(x, y) - I_{2r}^{(1)}(x, y) \) is an admissible kernel.

(ii) We then let \( I_{2r}^{(2)}(x, y) \) be defined by (4.33) with \( K(0) \) and \( \tilde{G}_{0r}(\cdot, x) \) in places of \( K(\lambda) \) and \( \tilde{G}_{0r}(\lambda, \cdot, x) \) respectively. Then, trading the factor \( \lambda \) of (4.35) for estimating \( G_{\Phi}^m(\lambda, z, y) \) as above, we obtain

\[
|I_{2r}^{(1)}(x, y) - I_{2r}^{(2)}(x, y)| \leq \frac{C}{\langle x \rangle^{\frac{m+1}{2}} \langle y \rangle^{\frac{m+1}{2}} \langle |x| - |y| \rangle^{\frac{m+1}{2}}} \quad (4.38)
\]

and \( I_{2r}^{(1)}(x, y) - I_{2r}^{(2)}(x, y) \) is also admissible.

(iii) In view of arguments in (i) and (ii), to see that we may further replace \( \tilde{\Phi}(\lambda) \) by \( \tilde{\Phi}(0) = 1 \) it is sufficient to notice that \( |G_{\Phi}^m(\pm\lambda, z, y)| \leq C \lambda^{-\frac{1}{2}} \langle z - y \rangle^{-\frac{m+1}{2}} \) on the support of \( 1 - \Phi(\lambda) \), which is obvious from (4.36).

Thus the problem is reduced to proving that

\[
\tilde{I}_2(x, y) = \sum_{\pm} \frac{(\pm 1)^{\frac{m+1}{2}}}{\langle |x| \pm |y| \rangle^{\frac{m+2}{2}}} \int_0^\infty \langle K(0)G_{\Phi}^m(\pm\lambda, \cdot, y), \tilde{G}_{0r}(0, \cdot, x) \rangle d\lambda \quad (4.39)
\]

is an admissible kernel. Since \( G_{\Phi}^m(\lambda, \cdot, y) \) satisfies the continuity property of Lemma 4.7, \( \langle x \rangle^\gamma K(0) \langle x \rangle^\gamma \in B(\mathcal{H}) \) and \( \tilde{G}_{0r}(0, \cdot, x) \in \mathcal{H}_{-\gamma} \) for some \( \gamma > \frac{3}{2} \), we may perform the integration in (4.39) before taking the inner product and write the integral in the form

\[
\left\langle K(0) \int_0^\infty G_{\Phi}^m(\pm\lambda, \cdot, y) d\lambda, \tilde{G}_{0r}(\cdot, x) \right\rangle.
\]

Here the integral on the right is the Riemann integral of an \( \mathcal{H}_{-\gamma} \) valued function, however, Lemma 4.7 (2) implies that we may replace it by the standard Riemann integral of the scalar continuous function \( G_{\Phi}^m(\pm\lambda, z, y) \). Then, by virtue of Lemma 4.7 (3), we may integrate (4.34) with respect to \( \lambda \) first via Fubini’s theorem. For \( a > 0 \) and \( t > 0 \) we have by residue theorem that

\[
\int_0^\infty e^{i\lambda a} \left( \lambda a + \frac{it}{2} \right)^{-\frac{3}{2}} d\lambda = -\int_0^\infty e^{-i\lambda a} \left( -\lambda a + \frac{it}{2} \right)^{-\frac{3}{2}} d\lambda.
\]
and both sides are bounded in modulus by $Cat^{-\frac{1}{2}}$. It follows that

$$\int_0^\infty G_m(\lambda, z, y)d\lambda = -\int_0^\infty G_m(-\lambda, z, y)d\lambda \equiv J(z, y),$$

(4.40)

and $|J(z, y)| \leq \int_{\mathbb{R}^m} C|\Phi_0(w, y)|dw \leq \frac{C}{(z - y)^\frac{m-2}{2}}$. (4.41)

Thus, we have $|\langle K(0)J(\cdot, y), G_{00}(\cdot, x)\rangle| \leq C\langle x \rangle^{-(m-2)}\langle y \rangle^{-\frac{m-2}{2}}$ and

$$|\tilde{I}_2(x, y)| \leq C\left|\frac{1}{(|x| + |y|)^\frac{m+2}{2}} - \frac{1}{(|y| - |x|)^\frac{m+2}{2}}\right| \langle x \rangle^{-(m-2)}\langle y \rangle^{-\frac{m-2}{2}}. (4.42)$$

The right side is the same as the summand in (4.24) with $j = \beta = \frac{m-2}{2}$ and $\alpha = \gamma = 0$ and, hence, $\tilde{I}_2(x, y)$ is admissible.

(3) The case $\xi = (0, 0, \frac{m}{2}, 0)$. Define $\tilde{G}_m(\lambda, z, x)$ by (4.34) with $\Phi(w, x)$ in place of $\Phi_0(w, y)$ and

$$\tilde{J}(z, x) = \int_0^\infty \tilde{G}_m(-\lambda, z, x)d\lambda.$$

Proceeding virtually in the same way as in the case $\xi = (0, \frac{m}{2}, 0, 0)$, we see that it suffices to show that

$$I_3(x, y) = \left(\frac{1}{(|x| + |y|)^\frac{m+2}{2}} - \frac{1}{(|y| - |x|)^\frac{m+2}{2}}\right) \langle K(0)\tilde{G}_{00}(0, \cdot, y), \tilde{J}(\cdot, x)\rangle$$

is admissible. It is obvious from the argument which lead to (4.41) that $|\tilde{J}(z, x)| \leq C\langle z - x \rangle^{-\frac{m-2}{2}}$ and have

$$|\langle K(0)\tilde{G}_{00}(0, \cdot, y), \tilde{J}(\cdot, x)\rangle| \leq C\langle x \rangle^{-\frac{m-2}{2}}\langle y \rangle^{-(m-2)}.$$

Thus, $I_3(x, y)$ is bounded by the right of (4.42) with $x$ and $y$ interchanged and is therefore admissible. This completes the proof of Lemma 4.6.

**Lemma 4.8.** The integral kernel $I_4(x, y)$ defined by the integral (4.17):

$$I_4(x, y) = \sum_{\pm} \frac{\pm i^{\frac{m+2}{2}}}{(|x| + |y|)^\frac{m+2}{2}} \int_0^\infty e^{i\lambda(|x|+|y|)} F_\pm^{(\frac{m+2}{2})}(\lambda, x, y)\lambda d\lambda$$

is admissible.
Proof. We proceed as in the proof of Lemma 4.6. Let as in (4.25):

\[
X_{\xi, \pm}(\lambda, x, y) = (\pm 1)^{\beta}(K^{(\alpha)}(\lambda)G^{(\beta)}_{0_\ell}(\pm\lambda, \cdot, y), G^{(\gamma)}_{0r}(\mp\lambda, \cdot, x))\tilde{\Phi}(\eta)(\lambda)
\]

for \(\xi = (\alpha, \beta, \gamma, \eta)\) and define

\[
\Omega^{(2)}_{\xi, \pm}(x, y) = \int_{0}^{\infty} e^{i\lambda(|x|+|y|)}X_{\xi, \pm}(\lambda, x, y)\lambda d\lambda. \tag{4.43}
\]

By Leibniz’ formula we have

\[
I_4(x, y) = \sum_{|\xi| = \frac{m+2}{2}} C_\xi \left( \frac{\Omega^{(2)}_{\xi, +}(x, y)}{(|x| + |y|)^{\frac{m+2}{2}}} - \frac{\Omega^{(2)}_{\xi, -}(x, y)}{(|x| - |y|)^{\frac{m+2}{2}}} \right). \tag{4.44}
\]

Let first \(\xi \neq (0, \frac{m+2}{2}, 0, 0), (0, 0, \frac{m+2}{2}, 0)\). Since \(\alpha + \max(\beta + \frac{m}{2}, \gamma + \frac{m}{2}) \leq m+1\) and \(\rho > m+1\), there exists \(\varepsilon > 0\) such that \(\max(\beta + \frac{m}{2}, \gamma + \frac{m}{2}) + \varepsilon < \rho\). By virtue of (4.9) and the property \((K)_{\rho}\), we have with this \(\varepsilon > 0\) that

\[
|\lambda||X_{\xi, \pm}(\lambda, x, y)| \leq |\lambda||\langle x \rangle^{\rho-\alpha}K^{(\alpha)}(\lambda)\langle x \rangle^{\rho-\alpha}||\langle x \rangle^{-(\rho-\alpha)}G^{(\beta)}_{0_\ell}(\lambda, \cdot, y)||_{\mathcal{H}}
\times||\langle x \rangle^{-(\rho-\alpha)}G^{(\gamma)}_{0r}(\lambda, \cdot, x)||_{\mathcal{H}} \leq C|\lambda|^{-\frac{\rho}{2}}(|\lambda|^{N}y^{-\frac{\alpha}{2}}\langle x \rangle^{-\frac{m-1}{2}}. \tag{4.45}
\]

This implies that for the summands in (4.44) with these \(\xi\) we have

\[
\left| \frac{\Omega^{(2)}_{\xi, \pm}(x, y)}{(|x| + |y|)^{\frac{m+2}{2}}} \right| \leq \frac{C}{(|x| + |y|)^{\frac{m+2}{2}}} \cdot \frac{1}{(|x|^{\frac{m-1}{2}}/\langle x \rangle)^{\frac{m-1}{2}}}, \tag{4.46}
\]

and these are therefore admissible. We are left with those either with \(\xi = (0, \frac{m+2}{2}, 0, 0)\) or \(\xi = (0, 0, \frac{m+2}{2}, 0)\) and we shall deal with the former case only as the other case may be treated similarly. So let \(\xi = (0, 0, \frac{m+2}{2}, 0)\) in what follows. We substitute \(\sum_{\beta_1 + \beta_2 = \frac{m+2}{2}} C_{\beta_1, \beta_2} K_{\beta_1, \beta_2}(\lambda, z, y)\) for \(G^{(\frac{m+2}{2})}_{0_\ell}(\lambda, z, y)\) in (4.25) and plug this into (4.43). This produces several functions indexed by \(\beta_1\) and \(\beta_2\) in the obvious manner and, by virtue of (4.14) and estimates corresponding to (4.45), they are all admissible except the one with index \((\beta_1, \beta_2) = (0, \frac{m+2}{2})\) which is written in the form as follows as in (4.33) after restoring the exponents \(e^{i\lambda(|x|+|y|)}\) to the original position:

\[
\sum_{\pm} \pm(\pm 1)^{\frac{m+2}{2}} \int_{0}^{\infty} \langle K(\lambda)G^{(\frac{m+2}{2})}_{0_\ell}(\pm\lambda, \cdot, y), G^{(0r)}_{0r}(\mp\lambda, \cdot, x)\rangle\tilde{\Phi}(\lambda)\lambda d\lambda. \tag{4.47}
\]

The same argument as in the proof of Lemma 4.6 shows that it suffices to show that (4.47) is admissible after replacing \(G^{(0r)}_{0r}(\lambda, z, x)\) by \(G_{0r}(0, z, x),\)
K(λ) by K(0) and Φ(λ) by the constant function 1. In this case the residue theorem implies, for a>0 and t>0, that
\[ \int_0^\infty e^{i\lambda a} \left( \lambda a + \frac{it}{2} \right)^{-\frac{5}{2}} \lambda d\lambda = \int_0^\infty e^{-i\lambda a} \left( -\lambda + \frac{it}{2} \right)^{-\frac{5}{2}} \lambda d\lambda \]
and the both sides are bounded in modulus by Ca^{-2}t^{-\frac{1}{2}}. Thus, we have
\[ \int_0^\infty G_{\frac{m}{m+2}}(\lambda, \cdot, y) \lambda d\lambda \]
and both sides are bounded in modulus by C(1-|x|)^{m-\frac{2}{2}}. It follows that
\[ (4.47) \]
with this change is bounded in modulus by
\[ C \frac{1}{(|y|-|x|)^{m+2}} - \frac{1}{(|y|+|x|)^{m+2}} \frac{1}{(y)^{m-2}} \frac{1}{(x)^{m-2}} \]
and is admissible. This completes the proof of Lemma 4.8 and therefore that of Proposition 4.2.

5 Low energy estimate II, Exceptional case

In this section we discuss the low energy part \( W_\langle \) in the case when \( H \) is of exceptional type, assuming
\[ |V(x)| \leq C \langle x \rangle^{-\delta} \]
with \( \delta > m + 3 \) if \( m \geq 8 \) and \( \delta > 10 \) if \( m = 6 \). (5.1)
so that the results of Proposition 3.6 apply. We substitute (3.5) when \( m = 6 \)
or (3.7) when \( m = 8 \) for \( L(\lambda) = (1 + G_0(\lambda) V)^{-1} - I \) in formula (4.3).
As \( \text{VE}_0(\lambda) \) satisfies property \( (K)_\rho \) with \( \rho > m + 1 \), Proposition 4.2 implies that \( E_0(\lambda) \) produces an operator with admissible integral kernel. Thus, we have only to discuss operators produced by singular parts \( \lambda^{-2}P_0V \) and \( \sum_{ab} \lambda^a (\log \lambda)^b D_{ab} \) (note that we have changed indices \( j, k \) to \( a, b \)). In Subsection 5.1 we prove that the operator produced by \( \lambda^{-2}P_0V \),
\[ W_{s,m} = \int_0^\infty G_0(\lambda) VP_0V(G_0(\lambda) - G_0(-\lambda)) \Phi(\lambda) \lambda^{-1} d\lambda, \]
is bounded in \( L^p \) for \( \frac{m}{m-2} < p < \frac{m}{2} \) and in Subsection 5.2 we indicate how the argument in Subsection 5.1 can be modified to prove the same for \( \Phi(H)W_{s,ab}\Phi(H_0) \) where
\[ W_{s,ab} = \int_0^\infty G_0(\lambda) V D_{ab}(G_0(\lambda) - G_0(-\lambda)) \Phi(\lambda) \lambda^{a+1}(\log \lambda)^b d\lambda. \]
5.1 Estimate for $W_{s,m}$

In this subsection, we prove the following proposition. We shall often write $\nu = (m - 3)/2$.

Proposition 5.1. Let $V$ satisfy (5.1). Then, for any $\frac{m}{m-2} < p < \frac{m}{2}$, there exists a constant $C_p$ such that

$$
\|W_{s,m}u\|_p \leq C_p\|u\|_p, \quad u \in C_0^\infty(\mathbb{R}^m).
$$

(5.4)

We first state two lemmas which will be used in what follows for proving the proposition. The first one can be found in [19].

Lemma 5.2. The function $|r|^a$ on $\mathbb{R}$ is a one dimensional $(A)_p$ weight if and only if $-1 < a < p - 1$. The Hilbert transform $\tilde{H}$ and the Hardy-Littlewood maximal operator $M$ are bounded operators in $L^p(\mathbb{R}, w(r)dr)$ for $(A)_p$ weights $w(r)$.

For a function $f$ on $\mathbb{R}^m$, $M(r, u)$ is the spherical average of $f$:

$$
M(r, u) = \frac{1}{|\Sigma|} \int_{\Sigma} u(r\omega) d\omega, \quad r \in \mathbb{R}.
$$

Lemma 5.3. Let $m \geq 3$. Let $\psi \in L^1(\mathbb{R}^m)$ and $u \in S(\mathbb{R}^m)$. Then

$$
F_{\psi, u}(\lambda) = \langle \psi, (G_0(\lambda) - G_0(-\lambda))u \rangle
$$

$$
= C_m \int_0^\infty e^{-t^\frac{m-3}{2}} \left( \int_{\mathbb{R}} e^{i\lambda r (t + 2i\lambda r)^\frac{m-3}{2}} r M(r, \psi \ast \bar{u}) dr \right) dt
$$

(5.5)

where $\bar{u}(x) = u(-x)$ and $C_m = -|\Sigma|/(4\pi)^{\frac{m-2}{2}} \Gamma(m-2)/2! = -2^{\frac{m-1}{2}}/(m-3)!$.

Proof. Recall (1.14). By Fubini’s theorem and by using polar coordinates,

$$
\langle \psi, G_0(\lambda)u \rangle = \int_{\mathbb{R}^m} G_0(\lambda, y)(\psi \ast \bar{u})(y) dy
$$

$$
= -C_m \int_0^\infty e^{-t^\nu \frac{1}{2}} \left( \int_{\mathbb{R}} e^{i\lambda r (t - 2i\lambda r)^\nu \frac{1}{2}} r M(r, \psi \ast \bar{u}) dr \right) dt.
$$

Since $M(r) = M(-r)$, It follows that $-\langle \psi, G_0(-\lambda)u \rangle$ is given by

$$
C_m \int_0^\infty e^{-t^\nu \frac{1}{2}} \left( \int_{0}^{\infty} e^{i\lambda r (t + 2i\lambda r)^\nu \frac{1}{2}} r M(r, \psi \ast \bar{u}) dr \right) dt
$$

$$
= -C_m \int_0^\infty e^{-t^\nu \frac{1}{2}} \left( \int_{0}^{-\infty} e^{i\lambda r (t - 2i\lambda r)^\nu \frac{1}{2}} r M(r, \psi \ast \bar{u}) dr \right) dt.
$$

Adding the two equations and changing $r \to -r$, we obtain the lemma. □
For fixed $f, g \in L^1(\mathbb{R}^m)$, we define the operator $Z = Z(f \otimes g)$ by

$$Z(f \otimes g)u = \int_0^\infty G_0(\lambda)(f \otimes g)(G_0(\lambda) - G_0(-\lambda)) \tilde{\Phi}(\lambda) \lambda^{-1} d\lambda \quad (5.6)$$

If we write $P_0 = \sum_{j=1}^d \phi_j \otimes \phi_j$ in terms of an orthonormal basis of $P_0\mathcal{H}$, we have $W_{s,m} = \sum_{j=1}^d Z((V\phi_j) \otimes (V\phi_j))$.

**Lemma 5.4.** With suitable constants $C_{jk}$ we have

$$Zu(x) = \sum_{j,k=0}^{m-4} C_{jk} \int_{\mathbb{R}^m} \frac{f(y)K_{jk}u(|x-y|)}{|x-y|^{m-2}} dy \quad (5.7)$$

where with $M(r, \overline{g} \ast \hat{u}) = M(r)$, $K_{jk}u(|x-y|)$, $0 \leq j, k \leq \frac{m-4}{2}$, are defined by

$$K_{jk}u(\rho) = \rho^j \int_0^\infty e^{i\lambda\rho} \lambda^{j+k-1} \tilde{\Phi}(\lambda) \left\{ \int_0^\infty \int_0^\infty e^{-(t+s)} \times \right.$$

$$t^{2\nu-\frac{1}{2}-k} s^{2\nu-\frac{3}{2}-j} (s - 2i\lambda \rho)^{\frac{1}{2}} \left( \int_{\mathbb{R}} e^{i\lambda(t+2i\lambda r)\frac{1}{2} - k+1} M(r) dr ds \right) d\lambda.$$  

(5.8)

**Proof.** We remark that, for $u \in C_0^\infty(\mathbb{R}^m)$, (5.8) is well defined for all $j, k$ because $M(r)$ is smooth, $\langle r \rangle^{m-1} M^{(\ell)}(r)$ is integrable and $\int_{\mathbb{R}} r M(r) dr = 0$ because $M(r)$ is even. By virtue of (5.5), we have

$$Zu(x) = \int_0^\infty G_0(\lambda)f(x) \cdot F_{g,u}(\lambda) d\lambda$$

We substitute (5.5) for $F_{g,u}(\lambda)$ and the expression

$$\frac{1}{(4\pi)^{m/2} \Gamma(m+2/2)} \int_{\mathbb{R}^m} e^{i|x-y|\lambda} f(y) \left( \int_0^\infty e^{-s} s^{m-3} (s - 2i\lambda|x-y|)^{m-3} ds \right) dy$$

for $G_0(\lambda)f(x)$. We then change the order of integrations with respect to $d\lambda$ and $dy$ and, using the binomial formula, write

$$(t + 2i\lambda r)^{m/2} = \sum_{j=0}^{m-4} \binom{m-4}{j} t^{m/2-j} (2i\lambda r)^j (t + 2i\lambda r)^{\frac{1}{2}}$$

and similarly for $(s - 2i\lambda \rho)^{m/2}$. The lemma follows. \qed
In what follows we fix \( f, g \) which satisfy for some \( \varepsilon > 0 \) and \( C > 0 \)

\[
|f(x)| \leq C|x|^{-m-\varepsilon}, \quad |g(x)| \leq C|x|^{-m-\varepsilon}
\]

(5.9)

and define operator \( W_{jk} \) for \( 0 \leq j, k \leq \frac{m-4}{2} \) by

\[
W_{jk}u(x) = \int_{\mathbb{R}^m} \frac{f(y) K_{jk} u(|x-y|)}{|x-y|^{m-2}} dy
\]

(5.10)

so that \( Z = \sum C_{jk} W_{jk} \). We use the following lemma.

**Lemma 5.5.** Let \( M(r) = M(r, \vec{g} \ast u) \). Then, for \( \frac{m}{m-2} < p < \frac{m}{2} \), we have

\[
\int_{0}^{\infty} \langle r \rangle |M(r)|dr \leq C\|u\|_p.
\]

**Proof.** Let \( q = \frac{p}{p-1} \) be the conjugate exponent of \( p \). We have

\[
\int_{\mathbb{R}} \langle r \rangle |M(r)|dr \leq C \int_{\mathbb{R}^m} \frac{\langle x \rangle |(\vec{g} \ast \hat{u})(x)|}{|x|^{m-1}} dx
\]

\[
\leq C \int_{|x|<1} \frac{\|\vec{g} \ast \hat{u}\|_\infty}{|x|^{m-1}} dx + C \left( \int_{|x|>1} \frac{\|\vec{g} \ast \hat{u}\|_p}{|x|^q} dx \right)^{\frac{1}{q}} \leq C(\|\vec{g}\|_q + \|\vec{g}\|_1)\|u\|_p
\]

since \( q(m-2) > m \) for \( \frac{m}{m-2} < p < \frac{m}{2} \).

It is clear that \( V_{\phi_j}, j = 1, \ldots, d \), satisfy the condition (5.9) and Proposition 5.1 follows from the following proposition.

**Proposition 5.6.** Let \( f, g \) satisfy (5.9). Then, \( W_{jk}, 0 \leq j, k \leq \frac{m-4}{2} \), are bounded in \( L^p(\mathbb{R}^m) \) for \( \frac{m}{m-2} < p < \frac{m}{2} \).

We prove Proposition 5.6 for various cases of \( j, k \) separately. By interpolation, we have only to show Proposition 5.6 for \( p = \frac{m}{m-2-\varepsilon} \) and \( p = \frac{m}{2+\varepsilon} \) with arbitrary small \( \varepsilon > 0 \). We denote the Hilbert transform by \( \mathcal{H} \) and \( \mathcal{H} = (1 + \mathcal{H})/2 \). By lemma Lemma 5.2 \( |r|^{m-1-p\theta} \) is a one dimensional \((A)_{p}\) weight if and only if \( 0 < \frac{m}{p} - \theta < 1 \), viz.

\[
\begin{align*}
  m - 3 - \varepsilon < \theta < m - 2 - \varepsilon & \quad \text{if } p = \frac{m}{(m - 2 - \varepsilon)}, \\
  1 + \varepsilon < \theta < 2 + \varepsilon & \quad \text{if } p = \frac{m}{(2 + \varepsilon)}.
\end{align*}
\]

(5.11)

**1) The case \( j, k \geq 1 \).** If \( 1 \leq j, k \leq \frac{m-4}{2} \) the integrand of (5.8) is integrable with respect to \( dt ds dr d\lambda \) and we are free to change the order of integration. Thus, we may write

\[
K_{j,k}u(\rho) = \int_{\mathbb{R}} M(r) T_{jk}(\rho, r) dr,
\]

(5.12)
\[
T_{jk}(\rho, r) = \int_0^\infty \int_0^\infty e^{-(t+s)}t^{2\nu-\frac{3}{4}-k}s^{2\nu-\frac{3}{4}-j}J_{jk}(s, t, \rho, r)dt
ds,
\]
(5.13)

\[
J_{jk} = \rho^j r^{k+1} \int_0^\infty e^{i\lambda(\rho-r)}\lambda^{j+k-1}\tilde{\Phi}(\lambda)(s - 2i\lambda \rho)^{\frac{3}{2}}(t + 2i\lambda r)^{\frac{3}{2}}d\lambda.
\]
(5.14)

**Lemma 5.7.** Let \(j, k \geq 1\). Then, with a constant \(C = C_{jk}\), we have

\[
|T_{jk}(\rho, r)| \leq C \left| \frac{\langle \rho \rangle^{j+1/2}\rho^{k+1}\langle r \rangle^{1/2}}{(r - \rho)^{j+k}} \right|.
\]
(5.15)

Estimate (5.15) holds when \(\tilde{\Phi}\) is replaced by any smooth function with compact support and \(t^{2\nu-\frac{3}{4}-k}\) and/or \(s^{2\nu-\frac{3}{4}-j}\) by \(t^a\) and/or \(s^b\) with \(a, b \geq 0\).

**Proof.** Since (5.15) is obvious for \(|\rho - r| \leq 1\), we prove it only for \(|\rho - r| \geq 1\). By integrating by parts \(j + k\) times with respect to \(\lambda\), we have

\[
J_{jk}(s, t, \rho, r) = \frac{i^{j+k}\sqrt{st}\rho^j r^{k+1}}{(\rho - r)^{j+k}}(j + k)! +
\]
\[
- \frac{i^{j+k}\rho^j r^{k+1}}{(\rho - r)^{j+k}} \int_0^\infty e^{i\lambda(\rho-r)} \{\lambda^{j+k-1}\tilde{\Phi}(\lambda)(s - 2i\lambda \rho)^{1/2}(t + 2i\lambda r)^{1/2}\}^{(j+k)} d\lambda.
\]

We insert this into (5.13). The boundary term produces

\[
T_{jk,b}(\rho, r) = C\frac{\rho^j r^{k+1}}{(r - \rho)^{j+k}},
\]
(5.16)

and this clearly satisfies (5.15). We compute the derivative via Leibniz’ rule:

\[
\left( \frac{d}{d\lambda} \right)^{j+k} \left\{ \lambda^{j+k-1}\tilde{\Phi}(\lambda)(s - 2i\lambda \rho)^{1/2}(t + 2i\lambda r)^{1/2} \right\}
\]
\[
= \sum_{a+b+c=j+k} C_{abc} \Psi_a(\lambda)(s - 2i\lambda \rho)^{1/2 - b}(2i\rho)^b(t + 2i\lambda r)^{1/2 - c}(-2ir)^c,
\]

where \(\Psi_a(\lambda) = \{\lambda^{j+k-1}\tilde{\Phi}(\lambda)\}^{(a)}\). Denoting summands on the right by \(E_{abc} = E_{abc}(\lambda, s, t, \rho, r)\), we define

\[
J_{abc}(s, t, \rho, r) \equiv \frac{i^{j+k}\rho^j r^{k+1}}{(\rho - r)^{j+k}} \int_0^\infty e^{i\lambda(\rho-r)}E_{abc}(\lambda, s, t, \rho, r)d\lambda,
\]
(5.17)

and \(T_{abc}(\rho, r)\) by the right of (5.13) with \(J_{jk}\) replacing \(J_{jk}\). By using obvious estimates \(|(s - 2i\lambda \rho)^{-1}(2i\lambda \rho)| \leq 1\) and \(|(t + 2i\lambda r)^{-1}(-2i\lambda r)| \leq 1\), we obtain

\[
|E_{abc}| \leq C|\Psi_a(\lambda)| \times \begin{cases} 
(s^{\frac{1}{2}} + |\lambda \rho|^{\frac{1}{2}})(t^{\frac{1}{2}} + |\lambda r|^{\frac{1}{2}}), & \text{if } b = c = 0; \\
(\rho r)^{\frac{1}{2}}|\lambda|^{1-(b+c)}, & \text{if } b \neq 0, c = 0; \\
(r^{\frac{1}{2}}(s^{\frac{1}{2}} + |\lambda \rho|^{\frac{1}{2}})|\lambda|^{\frac{1}{2}-c}, & \text{if } b = 0, c \neq 0; \\
(t^{\frac{1}{2}} + |\lambda \rho|^{\frac{1}{2}})|\lambda|^{\frac{1}{2}-b}, & \text{if } b \neq 0, c = 0.
\end{cases}
\]
Note that \( \lambda^{1-(b+c)} \Psi_a(\lambda), \lambda^{\frac{1}{2}-b} \Psi_a(\lambda) \) and \( \lambda^{\frac{1}{2}-c} \Psi_a(\lambda) \) are integrable functions with compact supports in respective cases. It immediately follows that

\[
|T_{abc}(\rho, r)| \leq C \frac{\rho^j(\rho)\frac{1}{2}r^{k+1}(r)^{\frac{1}{2}}}{r^{j+k}}
\]  

(5.18)

and, by summing up, we obtain (5.15). The only property of \( \tilde{\Phi} \) which is used in the argument above is that it is smooth and compactly supported; all estimate above go through if \( e^{-t(t^{2\nu} - \frac{3}{4})} \) or \( e^{-s^{2\nu} - \frac{3}{4}} \) are replaced by \( e^{-t^\alpha} \) or \( e^{-s^b} \), \( a, b \geq 0 \), and the last statement of the lemma follows.

**Lemma 5.8.** Let \( j, k \geq 1 \) and let \( T_{jk}(\rho, r) \) satisfy (5.15). Let \( W_{jk} \) be defined by (5.10) with \( K_{jk} \) given by (5.12). Then \( W_{jk} \) satisfies Proposition 5.6.

**Proof.** By splitting the domain of integration, we estimate

\[
|W_{jk}u(x)| \leq \left( \int_{|x-y|<1} + \int_{|x-y|\geq1} \right) \frac{|f(y)K_{jk}(|x-y|)|}{|x-y|m-2} dy
\]

\[
= I_1(x) + I_2(x)
\]

(5.19)

By Young’s inequality, (5.15) and Lemma 5.5

\[
\|I_1\|_p \leq C\|f\|_p \left( \int_{|x|<1} \frac{|K_{jk}(x)|}{|x|^{m-2}} dx \right) \leq C \sup_{|\rho|<1} |K_{jk}(\rho)|
\]

\[
\leq C \sup_{0 \leq \rho \leq 1} \int_{\mathbb{R}} \frac{|r|^{k+1}(r)^{\frac{1}{2}}}{(r-\rho)^{j+k}} |M(r)| dr \leq C \int_{\mathbb{R}} |M(r)| dr \leq C\|u\|_p.
\]

Let \( p = \frac{m}{2+\varepsilon} \), \( 0 < \varepsilon < 1 \) and choose \( \theta = 2 \), see (5.11). By Young’s inequality we have by using polar coordinates that

\[
\|I_2\|_p \leq \|f\|_1 \left( \int_1^\infty \rho^{m-1} \left( \frac{1}{\rho^{m-2}} \int_{\mathbb{R}} T_{jk}(\rho, r)M(r) dr \right)^p d\rho \right)^{\frac{1}{p}}
\]

\[
\leq C \int_1^\infty \rho^{m-1-p\theta} \left( \frac{\rho^{j+\frac{1}{2}}}{\rho^{m-4}} \int_{\mathbb{R}} \frac{|r|^{k+1}(r)^{\frac{1}{2}}}{(r-\rho)^{j+k}} |M(r)| dr \right)^p d\rho.
\]

(5.20)

Since \( |r|^{k-1}(r)^{\frac{1}{2}} \leq C(r-\rho)^{k-\frac{1}{2}}p^{k-\frac{1}{2}} \) (recall \( k \geq 1 \)) and \( m - 4 \geq j + k \),

\[
\frac{\rho^{j+\frac{1}{2}}|r|^{k+1}(r)^{\frac{1}{2}}}{\rho^{m-4}(r-\rho)^{j+k}} \leq C \frac{\rho^{j+k}|r|^2}{\rho^{m-4}(r-\rho)^{j+\frac{1}{2}}} \leq C \frac{|r|^2}{(r-\rho)^{j+\frac{1}{2}}}, \quad \rho > 1.
\]

Hence, the right of (5.20) is bounded by

\[
C \int_1^\infty \rho^{m-1-p\theta} M(|r|^2M)(\rho)^p d\rho \leq C \left( \int_0^\infty r^{m-1} |M(r)|^p dr \right)^{\frac{1}{p}}
\]

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by virtue of the weighted inequality for the maximal functions. By Hölder’s inequality the right side is bounded by $C\|g \ast \tilde{u}\|_p \leq C\|u\|_p$.

When $p = \frac{m}{m-2} - \varepsilon$, $0 < \varepsilon < 1$, we choose $\theta = m - 3$. Again by using Young’s inequality and (5.15)

$$\|I_2\|_p^p \leq C \int_1^\infty \rho^{m-1-p\theta} \left( \int \frac{|r|^{k+1}(r)^j}{(r-\rho)^{j+k}} |M(r)|dr \right)^p d\rho \quad (5.21)$$

Since $\rho^{j-\frac{1}{\theta}} \leq \langle r \rangle^{j-\frac{1}{\theta}}(r-\rho)^{-\frac{1}{\theta}}$ and $m - 1 - p\theta$ is an $(A)_p$ weight, the right hand side is further estimated by

$$C \int_1^\infty \rho^{m-1-p\theta} \left( \int \frac{|r|^{k+1}(r)^j}{(r-\rho)^{j+k}} |M(r)|dr \right)^p d\rho \leq C \int_1^\infty \rho^{m-1-p\theta} M(|r|^{k+1}(r)^j M(r)) (\rho)^p d\rho \leq C \int_0^\infty r^{m-1-p\theta} (|r|^{k+1}(r)^j M(r))^p dr$$

Since $k + j + 1 \leq m - 3 = \theta$, the last integral is bounded by a constant times

$$\int_0^1 r^{m-1-p(\theta-k-1)} M(r)^p dr + \int_1^\infty r^{m-1} M(r)^p dr$$

$$\leq C \int_{|x| < 1} \frac{\|g \ast \tilde{u}\|^p}{|x|^{p(q-k-1)}} dx + \|g \ast \tilde{u}\|^p \leq C(\|g\|_q + \|g\|_1)^p \|u\|_p^p,$$

where $q$ is the conjugate exponent of $p$. This completes the proof. \[\square\]

(2) The case $j = 0$, $k \geq 1$. We now prove Proposition 5.6 for $j = 0$ and $1 \leq k \leq \nu - 1 = \frac{m-4}{2}$ by induction on $k$, using also the already proven result for the case $j, k \geq 1$. For this and for the purpose for the next case (3) we define as follows.

**Definition 5.9.** (1) We say $J_{0k}(s, t, \rho, r)$ is $\ell$-admissible if operators $W_{0k,ln}$, $0 \leq l, n \leq \nu - \ell$, defined by (5.10) with $K_{0k,ln}(\rho) = \int T_{0k,ln}(\rho, r) M(r) dr$ in place of $K_{jk}(\rho)$ are bounded in $L^p(\mathbb{R}^m)$ for $\frac{m}{m-2} < p < \frac{m}{2}$ where

$$T_{0k,ln}(\rho, r) = \int_0^\infty \int_0^\infty e^{-(t+s)\frac{1}{2}+n} s^{2\nu-n} \frac{1}{2} \int J_{0k}(s, t, \rho, r) dt ds. \quad (5.22)$$

(2) We say $J_{0,ln}(s, t, \rho, r)$ is $\ell$-admissible if operators $W_{0,ln}$, $0 \leq l, n \leq \nu - \ell$, defined by (5.10) with $K_{0,ln}(\rho) = \int T_{0,ln}(\rho, r) M(r) dr$ in place of $K_{jk}(\rho)$ are bounded in $L^p(\mathbb{R}^m)$ for $\frac{m}{m-2} < p < \frac{m}{2}$ where

$$T_{0,ln}(\rho, r) = \int_0^\infty \int_0^\infty e^{-(t+s)\frac{1}{2}+n} s^{2\nu-n} \frac{1}{2} \int J_{0}(s, t, \rho, r) dt ds. \quad (5.23)$$
Lemma 5.10. Let $J_{0k}(s, t, \rho, r)$, $1 \leq k \leq \nu - 1$, be defined by (5.14) with $\Phi \in C_0^\infty(\mathbb{R})$ which is a constant near $\lambda = 0$. Then, $J_{0k}(s, t, \rho, r)$ are $k$-admissible.

Proof. We prove the lemma by induction on $k$. We begin with the case $k = 1$.

Lemma 5.11. For all $0 \leq l, n \leq \nu - 1$, $T_{01,ln}(\rho, r)$ satisfies the estimate

$$|T_{01,ln}(\rho, r)| \leq C \frac{|r|^2(\langle \rho \rangle + \langle r \rangle)}{\langle r - \rho \rangle^2}$$  \hspace{1cm} (5.24)

Proof. This is obvious when $|\rho - r| \leq 1$ and we assume $|\rho - r| > 1$ in what follows. Integrating by parts twice with respect to $\lambda$, we have

$$J_{01}(s, t, \rho, r) = \frac{ir^2}{(r - \rho)^2} \sqrt{st} + \frac{ir^2}{(r - \rho)^2} (\rho(t/s)^{1/2} - r(s/t)^{1/2}) + \frac{r^2}{(r - \rho)^2} \int_0^\infty e^{i\lambda(r - \rho)} \left( \frac{\partial}{\partial \lambda} \right)^2 \left( \Phi(\lambda)(s - 2i\lambda\rho)^{1/2}(t + 2i\lambda r)^{1/2} \right) d\lambda \hspace{1cm} (5.25)$$

We substitute this for $J_{01}$ in (5.22). Then the functions produced by the boundary terms are bounded by

$$C \left( \frac{r^2}{\langle r - \rho \rangle} + \frac{r^2(\langle \rho \rangle + \langle r \rangle)}{\langle r - \rho \rangle^2} \right) \leq C \frac{|r|^2(\langle \rho \rangle + \langle r \rangle)}{\langle r - \rho \rangle^2}.$$

Denoting by $'$ the derivative with respect to the variable $\lambda$, we compute:

$$\left( \Phi(\lambda)(s - 2i\lambda\rho)^{1/2}(t + 2i\lambda r)^{1/2} \right)'' = \Phi''(\lambda)(s - 2i\lambda\rho)^{1/2}(t + 2i\lambda r)^{1/2} + 2\Phi'(\lambda) \left( (s - 2i\lambda\rho)^{1/2}(t + 2i\lambda r)^{1/2} \right)' + \Phi(\lambda) \left( (s - 2i\lambda\rho)^{1/2}(t + 2i\lambda r)^{1/2} \right)$$.  \hspace{1cm} (5.26)

Since $\Phi'(\lambda) = 0$ near $\lambda = 0$ and $|(s - 2i\lambda\rho)^{-\frac{3}{2}} 2\rho| \leq C(\rho/s)^{\frac{3}{2}}$ for $|\lambda| \geq C > 0$, this is bounded in modulus by a constant times

$$|\Phi''(\lambda)| \left( \frac{s}{\sqrt{t}} + |\lambda\rho|^{\frac{3}{2}} \right) \left( t^{\frac{3}{2}} + |\lambda r|^{\frac{3}{2}} \right) + |\Phi'(\lambda)| \left( \frac{\rho}{\sqrt{s}} + |\lambda|^{\frac{3}{2}} \right) \left( t^{\frac{3}{2}} + |\lambda r|^{\frac{3}{2}} \right) + |\Phi(\lambda)| \left( t - s \rho r \right)^{\frac{5}{2}} \left( s + |\lambda r| \right)^{\frac{3}{2}}$$

Hence integrating by $dt$ first and using also elementary estimates

$$\int_0^\infty \frac{e^{-t^a}}{(t + |\lambda r|^b)^{\frac{5}{2}}} dt \leq C(\lambda r)^{-b} \hspace{1cm} 0 < b < a + 1,$$  \hspace{1cm} (5.26)
\[
\left| \int_0^\infty \frac{\Phi(\lambda)}{\langle \lambda r \rangle^a \langle \lambda \rho \rangle^b} \right| \leq C \frac{1}{\langle r \rangle + \langle \rho \rangle}, \quad a, b > 0, a + b > 1, \tag{5.27}
\]
we obtain estimate (5.24). \qed

**Lemma 5.12.** Let \( J_{01}(s, t, \rho, r) \) be defined by (5.14) with \( \Phi \in C_0^\infty(\mathbb{R}) \) which is a constant near \( \lambda = 0 \). Then, \( J_{01}(s, t, \rho, r) \) is 1-admissible.

**Proof.** We have
\[
\frac{|r|^2 \langle \rho \rangle + \langle r \rangle)}{\langle r - \rho \rangle^2} \leq \frac{2|\rho|^2}{\langle r - \rho \rangle^2} + \frac{|r|^2}{\langle r - \rho \rangle^2}.
\]
The first term on the right satisfies (5.15) with \( j = k = 1 \) and, by virtue of Lemma 5.8, it suffices to show that \( W_{01} \) is bounded in \( L^p(\mathbb{R}^m) \) for \( \frac{m}{m-2} < p < \frac{m}{2} \) if
\[
|T_{01}(\rho, r)| \leq C \frac{|r|^2}{\langle r - \rho \rangle} \leq C \left( |r| + \frac{|r\rho|}{\langle \rho - r \rangle} \right). \tag{5.28}
\]
We estimate \( |W_{01}u(x)| \leq I_1(x) + I_2(x) \) as in (5.19). For \( I_1(x) \) we use the first of (5.28) and proceed as in the proof of Lemma 5.8. We have
\[
\|I_1\|_p \leq C\|f\|_p \sup_{|\rho| \leq 1} \int_{\mathbb{R}} \frac{|r|^2 |M(r)|}{\langle r - \rho \rangle} \leq C\|f\|_p \int_{\mathbb{R}} |r| |M(r)| \, dr \leq C\|u\|_p.
\]
For \( I_2(x) \) we use Young’s inequality and the second of (5.28) to obtain:
\[
\|I_2\|_p \leq \|f\|_1 \left( \int_1^\infty \rho^{m-1} \left( \frac{1}{\rho^{m-2}} \int_{\mathbb{R}} \left| \frac{|r|^2 |M(r)|}{\langle r - \rho \rangle} \right|^p \, d\rho \right)^{\frac{1}{p}} \right.
\]
\[
+ \|f\|_1 \left( \int_1^\infty \rho^{m-1} \left( \frac{1}{\rho^{m-2}} \int_{\mathbb{R}} \left| \frac{|r|^2 |M(r)|}{\langle r - \rho \rangle} \right|^p \, d\rho \right)^{\frac{1}{p}} \right)
\tag{5.29}
\]
The first term is bounded by \( C \int_{\mathbb{R}} |r| |M(r)| \, dr \leq C\|u\|_p \) since \( p(m-2) > m \) for \( \frac{m}{m-2} < p < \frac{m}{2} \). For estimating the second, take \( \varepsilon > 0 \) arbitrarily small and fix \( p \in (\frac{m}{m-2}, \frac{m}{2}, \frac{m}{m-1}, \frac{m}{2} + \varepsilon) \). Take \( 0 < \varepsilon' < \varepsilon \) and choose \( \frac{m}{p} - 1 < \theta < \frac{m}{p} \) sufficiently close to \( \frac{m}{p} - 1 \) so that \( m - 1 - p\theta \) is an \( (A)_p \) weight and so that \( 1 + \varepsilon' < \theta \leq m - 3 - \varepsilon' \). Then, using \( \langle \rho - r \rangle^{-1} \leq C_{\varepsilon'} \langle \rho \rangle^\varepsilon' \langle r \rangle'^\varepsilon' \langle \rho - r \rangle^{-(1+\varepsilon')} \), we estimate the second integral by a constant times
\[
\left( \int_1^\infty \rho^{m-1} \left( \frac{1}{\rho^{m-2-\varepsilon}} \int_{\mathbb{R}} \left| \frac{|r|^2 |M(r)|}{\langle r - \rho \rangle^{1+\varepsilon'}} \right|^p \, d\rho \right)^{\frac{1}{p}} \right)
\]
\[
\leq C \left( \int_1^\infty \rho^{m-1-p\theta} \left( \int_{\mathbb{R}} \left| \frac{|r|^2 |M(r)|}{\langle r - \rho \rangle^{1+\varepsilon'}} \, d\rho \right|^p \right)^{\frac{1}{p}} \right)
\]
\[
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\]
\[ \leq C \left( \int_{\mathbb{R}} r^{m-1-p(\theta-1)} \langle r \rangle^{p'} |M(r)|^{p} \, dr \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}} \frac{\langle x \rangle^{p'} |g \ast u(x)|}{|x|^{p(\theta-1)}} \, dx \right)^{\frac{1}{p}}. \]

Since \( p* < p(\theta-1) < m \), the right hand side is bounded by \( C\|u\|_p \). This proves that \( \|F_2\|_p \leq C\|u\|_p \). This completes the proof. \( \square \)

**Completion of the proof** of Lemma 5.10. The lemma is satisfied when \( k = 1 \) by virtue of Lemma 5.11. We let \( k \geq 2 \) and assume that the lemma is already proved for smaller values of \( k \). We write \( r^{k+1} = r^k \rho - r^k (\rho - r) \) in the definition (5.14) for \( J_{0k}(s, t, \rho, r) \) and apply integration by part to the integral containing \( r^k (\rho - r) \). We obtain

\[ J_{0k}(s, t, \rho, r) = J_{1(k-1)}(s, t, \rho, r) - ir^k \int_{0}^{\infty} e^{i(t-\rho)} \left( \frac{\partial}{\partial \lambda} \right) \left( \lambda^{k-1} \Phi(\lambda)(s - 2i\lambda \rho)^{\frac{1}{2}}(t + 2i\lambda r)^{\frac{1}{2}} \right) \, d\lambda. \]  

(5.30)

Thanks to results in case (1), \( J_{1(k-1)}(s, t, \rho, r) \) is \( (k-1) \)-admissible and it may be ignored. We insert the following for the derivative in the integrand:

\[ (k-1)\lambda^{k-2} \Phi(s - 2i\lambda \rho)^{\frac{1}{2}}(t + 2i\lambda r)^{\frac{1}{2}} + \lambda^{k-1} \Phi'(s - 2i\lambda \rho)^{\frac{1}{2}}(t + 2i\lambda r)^{\frac{1}{2}} \]

\[ - 2\lambda^k \Phi \left( \frac{\partial}{\partial \rho} \right) \left( s - 2i\lambda \rho \right)^{\frac{1}{2}}(t + 2i\lambda r)^{\frac{1}{2}} + \lambda^{k-1} \Phi(s - 2i\lambda \rho)^{\frac{1}{2}}i\lambda r(t + 2i\lambda r)^{-\frac{1}{2}} \]

The first term produces \((k-1)J_{0(k-1)}(s, t, \rho, r)\), which is \((k-1)\)-admissible by induction hypothesis; the second does \( J_{0(k-1)}(s, t, \rho, r) \) with \( \lambda \Phi'(\lambda) \) replacing \( \Phi \), which is also \((k-1)\)-admissible since \( \lambda \Phi'(\lambda) = 0 \) near \( \lambda = 0 \). Define

\[ J_{0(k-1)} = 2r^k \int_{0}^{\infty} e^{i(t-\rho)} \lambda^{k-1} \Phi(\lambda) \left( \frac{\partial}{\partial \rho} \right) \left( s - 2i\lambda \rho \right)^{\frac{1}{2}}(t + 2i\lambda r)^{\frac{1}{2}} \, d\lambda \]

and substitute this for \( J_{jk}(s, t, \rho, r) \) in (5.22). This yields after integration by parts with respect to the \( s \)-integral

\[ -2T_{1(k-1), ln}(\rho, r) + 2(2\nu - \frac{3}{2} - l)T_{1(k-1), (l+1)n}(\rho, r) \]

and the result of case (1) implies \( J_{0k(3)}(s, t, \rho, r) \) is \( k \)-admissible. We rewrite the last term \( \lambda^{k-1} \Phi(s - 2i\lambda \rho)^{\frac{1}{2}}i\lambda r(t + 2i\lambda r)^{-\frac{1}{2}} \) in the form

\[ \frac{1}{2} \lambda^{k-2} \Phi(\lambda)(s - 2i\lambda \rho)^{\frac{1}{2}}(t + 2i\lambda r)^{\frac{1}{2}} - \lambda^{k-2} \Phi(\lambda)(s - 2i\lambda \rho)^{\frac{1}{2}} t \left( \frac{\partial}{\partial t} \right) (t + 2i\lambda r)^{\frac{1}{2}} \]

The first term again produces \( \frac{1}{2} J_{0(k-1)}(s, t, \rho, r) \). Define

\[ J_{0k(4)}(s, t, \rho, r) = r^k \int_{0}^{\infty} e^{i(t-\rho)} \lambda^{k-2} \Phi(\lambda)(s - 2i\lambda \rho)^{\frac{1}{2}} \left( \frac{\partial}{\partial t} \right) (t + 2i\lambda r)^{\frac{1}{2}} \, d\lambda \]
and substitute $J_{0k(4)}(s, t, \rho, r)$ for $J_{jk}(s, t, \rho, r)$ in (5.22). This yields, after integration by parts with respect to the $t$-integral,

$$-T_{0(k-1),l(n+1)}(\rho, r) + (2\nu - \frac{1}{2} - k + n)T_{0(k-1),ln}(\rho, r)$$

It follows by induction hypothesis that $J_{0k(4)}(s, t, \rho, r)$ is also $k$-admissible. This completes the proof.

(3) The case $j \geq 1$ and $k = 0$. We next prove Proposition 5.6 for $j \geq 1$ and $k = 0$. It suffices to prove the following lemma.

**Lemma 5.13.** Let $J_{j0}(s, t, \rho, r)$, $j = 1, \ldots, \nu - 1$, be defined by (5.14) with $\tilde{\Phi} \in C_0^\infty(\mathbb{R})$ which is a constant near $\lambda = 0$. Then, $J_{j0}(s, t, \rho, r)$ are $j$-admissible.

**Proof.** We prove the lemma by induction on $j$. Thus, we let $j = 1$ first. Comparing definitions of $J_{01}$ and $J_{10}$ and (5.24), it is obvious that

$$|T_{10,(ln)}(\rho, r)| \leq C\frac{|r| \rho (\langle \rho \rangle + \langle r \rangle)}{\langle \rho - r \rangle^2}$$

Since $\langle \rho \rangle + \langle r \rangle \leq 4(\langle \rho - r \rangle + |r|)$ and $|r|^2 \rho (\rho - r)^{-2}$ satisfies (5.15) with $j = k = 1$, it suffices to show that $W_{10}$ has the desired property when

$$|T_{10}(\rho, r)| \leq C\frac{|r| \rho}{\langle \rho \rangle - r}.$$ 

However, the right hand side is the same as the second term on the right of (5.28) and the proof of Lemma 5.12 shows that Lemma 5.13 is satisfied when $j = 1$. We then let $j \geq 2$ and assume that the lemma is already proved for smaller values of $j$. We write $\rho^j r = r^{j-1} r^2 + r^{j-1} (\rho - r)$ in the definition (5.14) of $J_{j0}(s, t, \rho, r)$. The first term $r^{j-1} r^2$ produces $J_{(j-1)l}(t, s, \rho, r)$ and we may ignore it by virtue of results in case (1). We need study the operators corresponding to $W_{jk}$ produced by functions

$$\rho^{j-1} r \int_0^\infty e^{i\lambda(\rho - r)} \left( \frac{\partial}{\partial \lambda} \right) \left( \lambda^{k-1} \tilde{\Phi}(\lambda)(s - 2i\lambda \rho)^{\frac{1}{2}} (t + 2i\lambda r)^{\frac{1}{2}} \right) d\lambda.$$ 

However, after this point the argument completely in parallel with that of Lemma 5.10 after (5.30) and we omit the repetitious details.

(4) The case $j = k = 0$. Finally we prove Proposition 5.6 for $j = k = 0$. Recall the definition (5.8) and (5.13). In (5.8) we substitute

$$(s - 2i\lambda \rho)^{\frac{1}{2}} ((t + 2i\lambda r)^{\frac{1}{2}} - t^{\frac{1}{2}}) + ((s - 2i\lambda \rho)^{\frac{1}{2}} - s^{\frac{1}{2}}) t^{\frac{1}{2}} + s^{\frac{1}{2}} t^{\frac{1}{2}}.$$
for \((s - 2i\lambda \rho) \frac{1}{2}(t + 2i\lambda r) \frac{1}{2}\) and denote by \(K_j\) the operator produced by \(j\)-th summand, \(j = 1, 2, 3\), so that \(K_{00} = K_1 + K_2 + K_3\). Define \(W_j\) by (5.10) with \(K_j\) in place of \(K_{jk}\) so that \(W_{00} = W_1 + W_2 + W_3\). For \(j = 1\) and \(j = 2\), we may change the order of integrations and write \(K_ju(\rho)\) in the following form:

\[
K_ju(\rho) = \int_{\mathbb{R}} T_j(\rho, r) r M(r) dr, \quad j = 1, 2
\]

where \(T_1(\rho, r)\) and \(T_2(\rho, r)\) are given by constants times

\[
T_1(\rho, r) = r \int_0^\infty \int_0^\infty e^{-(t+s)i2\nu - \frac{r}{2}s^2} \times
\]

\[
\left( \int_0^\infty e^{i\lambda(\rho-r)} \Phi(\lambda) \left\{ \frac{2ir(s - 2i\lambda \rho)^{\frac{1}{2}}}{(t + 2i\lambda r)^{\frac{1}{2}} + t^{\frac{1}{2}}} \right\} d\lambda \right) dsdt,
\]

\[
T_2(\rho, r) = r \int_0^\infty e^{-s} s^{2\nu - \frac{3}{2}} \left( \int_0^\infty e^{i\lambda(\rho-r)} \Phi(\lambda) \left\{ \frac{2i\rho}{(s - 2i\lambda \rho)^{\frac{1}{2}} + s^{\frac{1}{2}}} \right\} d\lambda \right) ds.
\]

**Lemma 5.14.** We have estimate

\[
|T_1(\rho, r)| \leq C \left( |r| + \frac{|r|\rho}{|r - \rho|} + \frac{|r|\rho}{(r - \rho)^2} + \frac{|r|^2\langle \rho \rangle^{\frac{1}{2}}}{(r - \rho)^2} \right), \quad (5.31)
\]

\[
|T_2(\rho, r)| \leq C \frac{|\rho|\langle r \rangle}{(r - \rho)}. \quad (5.32)
\]

**Proof.** The estimates are trivial for \(|r - \rho| \leq 1\) and we suppose \(|\rho - r| > 1\). We first prove (5.32). Integrating by parts, we estimate the inner integral by the boundary contribution \(\rho|\rho - r|^{-1} s^{-\frac{1}{2}}\) plus

\[
\left| \frac{1}{\rho - r} \left( \int_0^\infty \frac{2i\rho e^{i\lambda(\rho-r)}\Phi(\lambda)}{(s - 2i\lambda \rho)^{\frac{1}{2}} + s^{\frac{1}{2}}} d\lambda \right) \right| \leq \frac{C\rho}{(|r - \rho|)} \left( \frac{1}{\sqrt{s}} + \int_0^\infty \frac{\rho|\Phi(\lambda)|}{(|s| + |\lambda \rho|)^{\frac{1}{2}}} d\lambda \right).
\]

The desired estimate follows since

\[
\int_0^\infty \left( \int_0^\infty e^{-s} s^{2\nu - \frac{3}{2}} \frac{|\Phi(\lambda)| ds}{(|s| + |\lambda \rho|)^{\frac{1}{2}}} \right) d\lambda \leq \int_0^\infty \frac{C d\lambda}{\langle \lambda \rho \rangle^{\frac{1}{2}}} \leq C \frac{\rho}{\rho}. \quad (5.33)
\]

For proving (5.31) for \(T_1(\rho, r)\) we apply integration by parts twice to the inner integral. The result is
\[
\frac{2r}{\rho - r} \sqrt{\frac{s}{2t}} - \frac{r}{(\rho - r)^2} \left( \frac{\rho}{\sqrt{ts}} + \frac{\sqrt{sr}}{t^{3/2}} \right) \\
- \frac{ir}{(\rho - r)^2} \int_0^\infty e^{i\lambda(\rho - r)} \left( \frac{\partial}{\partial \lambda} \right)^2 \left( \Phi(\lambda) - \frac{(s - 2i\lambda\rho)^{1/2}}{(t + 2i\lambda r)^{3/2}} + t^{1/2} \right) d\lambda. \tag{5.34}
\]

We estimate the second derivative by a constant times

\[
\left| \Phi''(\lambda) \right| \left( s + |\lambda\rho| \right)^{1/2} + \frac{\rho|\Phi'(\lambda)|}{\sqrt{st}} + \frac{\Phi'(\lambda)|r(\lambda\rho)^{1/2}}{(t + |\lambda r|)^{3/2}} \\
+ \frac{\Phi'(\lambda)|\rho^2}{\sqrt{t}(s + |\lambda\rho|)^{1/2}} + \frac{|\Phi(\lambda)|r \rho}{\sqrt{s}(t + |\lambda r|)^{3/2}} + \frac{|\Phi(\lambda)|r^2(s + |\lambda\rho|)^{1/2}}{(t + |\lambda r|)^{3/2}}.
\]

Desired estimate follows after integration via estimates similar to (5.33).  

**Lemma 5.15.** For \( \frac{m}{m-2} < p < \frac{m}{2} \), \( W_1 \) and \( W_2 \) are bounded in \( L^p(\mathbb{R}^m) \)

**Proof.** \( T_1(\rho, r) \) is bounded by the right of (5.15) with \( j = k = 1 \); \( T_2(\rho, r) \) by the right of (5.28). The lemma follows from Lemma 5.8 and Lemma 5.12.

Finally we deal with \( W_3 \). Recall that \( K_3u(\rho) \) is defined by (5.8) with \( (st)^{1/2} \) replacing \( (s - 2i\lambda\rho)^{1/2}(t + 2i\lambda r)^{1/2} \). Then, the inner most integral becomes \( t \) independent and we may integrate out the \( (t, s) \) integral. Result is

\[
K_3u(\rho) = C \int_0^\infty \Phi(\lambda) e^{-i\lambda\rho} \lambda^{-1} \left( \int_{\mathbb{R}} e^{i\lambda v} M(r) dr \right) d\lambda \tag{5.35}
\]

with a suitable constant \( C \). Since \( M(r) \) is even, we may write

\[
\frac{1}{\lambda} \int_{\mathbb{R}} e^{i\lambda r} M(r) dr = \int_{\mathbb{R}} \left( e^{i\lambda r} - 1 \right) \lambda r M(r) dr = i \int_{\mathbb{R}} r M(r) \left( \int_0^r e^{i\lambda v} dv \right) dr.
\]

Thus, if we define \( F(v) \) by

\[
F(v) = \pm v \int_v^{\pm \infty} r M(r) dr, \quad \text{for } \pm v > 0
\]

and change the order of integration, we have

\[
K_3u(\rho) = C \int_0^\infty \Phi(\lambda) e^{-i\lambda\rho} \left( \int_{\mathbb{R}} e^{i\lambda v} F(v) dv \right) d\lambda = C \left[ \mathcal{H}(\hat{\Phi} \ast F) \right](\rho). \tag{5.36}
\]

We estimate \( |W_3u(x)| \leq I_1(x) + I_2(x) \) as in (5.19). Recall that for \( \frac{m}{m-2} < p < \frac{m}{2} \) and \( \frac{m}{p} - 1 < \theta < \frac{m}{p} \) we have \( m - 2 - \theta > 0 \). Let \( p = \frac{m}{2+\varepsilon} \) and \( \theta = 2 \)
first for an arbitrarily small \( \varepsilon > 0 \). Then, applying Lemma 5.2 twice, once for \( \mathcal{H} \) and once for \( \mathcal{M} \), we obtain

\[
\|I_2\|_p^p \leq C\|f\|_1^p \int_1^\infty \rho^{m-1} \left( \frac{|K_3(\rho)|}{\rho^{m-2}} \right)^p d\rho \\
\leq C \int_0^\infty \rho^{m-1-\theta} |\mathcal{H}(\Phi \ast F)(\rho)|^p d\rho \leq C \int_0^\infty \rho^{m-1-\theta} |(\Phi \ast F)(\rho)|^p d\rho \\
\leq C \int_0^\infty \rho^{m-1-\theta} |\mathcal{M}(F)(\rho)|^p d\rho \leq C \int_0^\infty \rho^{m-1-2\theta} |F(\rho)|^p d\rho.
\] (5.37)

We then apply Hardy’s then Hölder’s inequalities and estimate the right by

\[
C \int_0^\infty r^{m-1} |M(r)|^p dr \leq \int_{\mathbb{R}^m} |(\overline{g} \ast \overline{u})(x)|^p \leq C\|u\|^p_p. \tag{5.38}
\]

Let \( q = \frac{m}{m-2-\varepsilon} \) be the dual exponent of \( p = \frac{m}{2+\varepsilon} \). By Hölder’s inequality

\[
|I_1(x)| \leq C \left( \int_{|y|<1} \left| \frac{K_3 u(|y|)}{|y|^2} \right|^p \right)^{\frac{1}{p}} \left( \int_{|x-y|<1} \left| \frac{|f(y)|}{|x-y|^{m-4}} \right|^q \right)^{\frac{1}{q}}.
\]

The second factor on the right is an \( L^p \) function of \( x \in \mathbb{R}^m \) since \( |f(x)| \leq C(x)^{-m-\varepsilon} \). Then estimates (5.37) and (5.38) implies

\[
\|I_1\|^p_p \leq C \int_{|y|<1} \left| \frac{K_3 u(|y|)}{|y|^2} \right|^p \right)^{\frac{1}{p}} \int_0^1 \rho^{m-1-2\theta} |\mathcal{H}(\Phi \ast F)(\rho)|^p d\rho \leq C\|u\|^p_p.
\]

Let \( p = \frac{m}{m-2-\varepsilon} \) and \( \theta = m - 3 \) next. Then, using Lemma 5.2 twice as in (5.37), we obtain

\[
\|I_2\|_p \leq C\|f\|_1 \left( \int_1^\infty \rho^{m-1} \left( \frac{|K_3 u(\rho)|}{\rho^{m-2}} \right)^p d\rho \right)^{\frac{1}{p}} \\
\leq C \left( \int_0^\infty \rho^{m-1-\theta} |F(\rho)|^p d\rho \right)^{\frac{1}{p}}.
\] (5.39)

Then, Hardy’s inequality implies that the right side is bounded by

\[
C \left( \int_0^\infty \rho^{m-1-p(\theta-2)} |M(\rho)|^p d\rho \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}^m} \frac{|(\overline{g} \ast u)(x)|^p}{|x|^{p(\theta-2)}} dx \right)^{\frac{1}{p}}. \tag{5.40}
\]

Since \( p(\theta-2) < m \), the right side is bounded by \( C\|u\|_p \). For \( I_1(x) \) we proceed as previously. By Hölder’s inequality

\[
|I_1(x)| \leq \left( \int_{|y|<1} \left| \frac{K_3 u(|y|)}{|y|^{m-3}} \right|^p \right)^{\frac{1}{p}} \left( \int_{|x-y|<1} \left| \frac{f(y)}{|x-y|} \right|^q \right)^{\frac{1}{q}}.
\]
The second factor on the right is an \( L^p \) function of \( x \in \mathbb{R}^m \) as previously. Then (5.39) and (5.40) imply that the right hand side of

\[
\|I_1\|_p \leq C \left( \int_{|y|<1} \frac{|K_3(|y|)|}{|y|^{m-3}} dy \right)^{\frac{1}{p}} \leq C \left( \int_0^1 \rho^{m-1-p\theta} |\mathcal{H}(\tilde{\Phi} * F)(\rho)|^p d\rho \right)^{\frac{1}{p}}
\]

is bounded by \( C\|u\|_p \). This proves \( W_3 \) is bounded in \( L^p \) for \( \frac{m}{m-2} < p < \frac{m}{2} \) and completes the proof of Proposition 5.1.

5.2 Estimate for \( W_{s,ab} \)

In this subsection, we indicate how the discussion in the previous subsection may be modified for proving the following proposition.

**Proposition 5.16.** Let \( V \) satisfy (5.1) and let \( W_{ab} \) be defined by (5.3). Then, for any \( \frac{m}{m-2} < p < \frac{m}{2} \),

\[
\|\Phi(H)W_{s,ab}\Phi(H_0)u\|_p \leq C_p\|u\|_p, \quad u \in C_0^\infty(\mathbb{R}^m). \quad (5.41)
\]

**Proof.** By virtue of Proposition 3.6, \( VD_{ab} \) are finite rank operators from \( \mathcal{H}_{-(\delta-3)} \) to \( \mathcal{H}_{(\delta-3)} \). Hence, they are finite linear combinations of rank one operators \( f \otimes g \) with \( f, g \in \mathcal{H}_{m+\varepsilon} \) for some \( \varepsilon > 0 \), and it suffices to prove (5.41) for \( \Phi(H)\tilde{Z}\Phi(H_0) \), where \( \tilde{Z} \) is the operator defined by

\[
\tilde{Z} = \int_0^\infty G_0(\lambda)(f \otimes g)(G_0(\lambda) - G_0(-\lambda)) \tilde{\Phi}(\lambda)\lambda^{a+1} \log b \lambda d\lambda \quad (5.42)
\]

with such \( f, g \), which is the same as (5.6) if \( \lambda^{-1} \) replaces \( \lambda^{a+1} \log b \lambda \). Notice that \( \tilde{Z}\Phi(H_0)u \) is given by the same formula (5.42) with \( \Phi(H_0)g \) in place of \( g \) and \( |\Phi(H_0)g(x)| \leq C\langle x \rangle^{-m-\varepsilon} \). Thus, we may (and do) assume that \( g \) satisfies the condition (5.9), \( |g(x)| \leq C\langle x \rangle^{-m-\varepsilon} \), and ignore \( \Phi(H_0) \). After this, we proceed as in the previous section: Define \( K_{jk}^{ab}u(\rho) \) by the right side of (5.8) with \( \lambda^{j+k+1+a}(\log \lambda)^b \) in place of \( \lambda^{j+k-1} \) and

\[
W_{jk}^{ab}u(x) = \int_{\mathbb{R}^m} \frac{f(y)K_{jk}^{ab}(|x-y|)}{|x-y|^{m-2}} dy \quad (5.43)
\]

so that \( \tilde{Z} \) is a linear combination of \( K_{jk}^{ab} \):

\[
\tilde{Z}u(x) = \sum_{j,k=0}^{\frac{m-4}{2}} C_{jk}W_{jk}^{ab}u(x), \quad (5.44)
\]
see Lemma 5.4. We then follow the argument in the proof of Proposition 5.6 for the case \(j, k \geq 1\). The function \(\lambda^{j+k+1+a}(\log \lambda)^b\) is certainly less singular than \(\lambda^{j+k-1}\) at \(\lambda = 0\) and the proof of Lemma 5.7 implies, as previously,

\[
K_{jk}^{ab}(\rho) = \int_{\mathbb{R}} M(\mathcal{F} \ast u, r)T_{jk}^{ab}(r)dr
\]

with \(T_{jk}^{ab}(r)\) which satisfies estimate (5.15):

\[
|T_{jk}^{ab}(\rho, r)| \leq C \left| \frac{\langle \rho \rangle^{j+1/2}r^{k+1/2}}{\langle r - \rho \rangle^{j+k}} \right|.
\]

We then want to apply the argument in the proof of Lemma 5.8. Here it is important to observe that we may pretend that \(f\) satisfies (5.9) as well:

\[
|f(x)| \leq C \langle x \rangle^{m-\delta}. \quad \text{Indeed, we have}
\]

\[
|\Phi(H)W_{jk}^{ab}u(x)| = \left| \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} \Phi(x, z)f(z - y)dz \right) \frac{K_{jk}^{ab}u(|y|)}{|y|^{m-2}}dy \right|
\]

\[
\leq C \int_{\mathbb{R}^m} \langle y \rangle^{-(m+\delta)}\frac{|K_{jk}^{ab}u(|x - y|)|}{|x - y|^{m-2}}dy
\]

since \(|\Phi(x, z)| \leq C_N \langle x - z \rangle^{-N}\). Then, the proof of Lemma 5.8 applies without any change and we obtain \(\|\Phi(H)W_{jk}^{ab}u\|_p \leq C\|u\|_p\) for any \(\frac{m}{m-2} < p < \frac{m}{2}\).

6 High energy estimate

In this section we prove the following proposition. Recall that \(m_* = \frac{m-1}{m-2}\).

**Proposition 6.1.** Let \(V\) satisfy (1.2) and, in addition, \(|V(x)| \leq C\langle x \rangle^{-\delta}\) for some \(\delta > m + 2\). Let \(\Psi(\lambda) \in C^\infty(\mathbb{R})\) be such that \(\Psi(\lambda) = 0\) for \(|\lambda| < \lambda_0\) for some \(\lambda_0\). Then \(W_>\) is bounded in \(L^p(\mathbb{R}^m)\) for all \(1 \leq p \leq \infty\).

Since the proof is entirely similar to the corresponding one in [I], we shall only sketch it very briefly pointing out what modifications are necessary for even dimensions. Iterating the resolvent equation, we have \(G(\lambda)V = \sum_{1}^{2n}(-1)^{j-1}(G_0(\lambda)V)^j + G_0(\lambda)N_n(\lambda)\), where

\[
N_n(\lambda) = (VG_0(\lambda))^{n-1}V^{\lambda}G(\lambda)V(G_0(\lambda)V)^{\lambda}.\]

If we substitute this for \(G(\lambda)V\) in the right of (1.10), we have

\[
W_> = \Psi(H_0)^2 + \sum_{j=1}^{2n}(-1)^j\Omega_j\Psi(H_0)^2 - \tilde{\Omega}_{2n+1}, \quad (6.1)
\]

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\[ \tilde{\Omega}_{2n+1} = \frac{1}{i\pi} \int_0^\infty G_0(\lambda)N_n(G_0(\lambda) - G_0(-\lambda))\tilde{\Psi}(\lambda)d\lambda, \]  

(6.2)

where \( \tilde{\Phi}(\lambda) = \lambda^2 \Psi(\lambda)^2 \). The operators \( \Psi(H_0) \) and \( \Omega_1, \ldots, \Omega_{2n} \) are bounded in \( L^p \) for any \( 1 \leq p \leq \infty \) by virtue of Lemma 2.7. We show that, if \( n \) is large enough, the integral kernel

\[ \tilde{\Omega}_{2n+1}(x, y) = \int_0^\infty (N_n(\lambda)(G_0(\lambda) - G_0(-\lambda))\delta_y, G_0(\lambda)\delta_x)\lambda^2 \Psi(\lambda^2)d\lambda, \]

of \( \tilde{\Omega}_{2n+1} \) is admissible. We define \( \tilde{G}_0(\lambda, z, x) = e^{-i\lambda|x|}G_0(\lambda, x - z) \) and \( \psi(z, x) = |x - z| - |x| \) as previously.

**Lemma 6.2.** Let \( j = 0, 1, 2, \ldots \). We have for \( |\lambda| \geq 1 \) that

\[ \left| \left( \frac{\partial}{\partial \lambda} \right)^j \tilde{G}_0(\lambda, z, x) \right| \leq C_j \left( \frac{\langle z \rangle^j}{|x - z|^{m-2}} + \frac{\lambda^{m-2} \langle z \rangle^j}{|x - z|^{m-1}} \right). \]  

(6.3)

**Proof.** Differentiate \( \tilde{G}_0^{(j)}(\lambda, z, x) \) by using Leibniz’s formula. The result is a linear combination over \( (\alpha, \beta) \) such that \( \alpha + \beta = j \) of

\[ \frac{e^{i\lambda\psi(z, x)}\psi(z, x)^\alpha}{|x - z|^{m-2-\beta}} \int_0^\infty e^{-t}t^{\nu-\frac{1}{2}} \left( \frac{t}{2} - i\lambda|x - z| \right)^{\nu-\frac{1}{2}-\beta} dt. \]

Since \( |\psi(z, x)|^\alpha \leq \langle z \rangle^j \) for \( 0 \leq \alpha \leq j \) and \( |z - x| \leq |\frac{t}{2} - i\lambda|z - x| \leq (t + \lambda|z - x|) \) when \( |\lambda| \geq 1 \), (6.3) follows.

Define \( T_\pm(\lambda, x, y) = \langle N_n(\lambda)\tilde{G}_0(\pm \lambda, \cdot, y), \tilde{G}_0(-\lambda, \cdot, x) \rangle \) so that

\[ \tilde{\Omega}_{2n+1}(x, y) = \frac{1}{\pi i} \int_0^\infty (e^{i\lambda(|x| + |y|)}T_+(\lambda, x, y) - e^{i\lambda(|x| - |y|)}T_-(\lambda, x, y))\tilde{\Psi}(\lambda)d\lambda. \]

The following lemma may be proved by repeating line by line the proof of Lemma 3.14 of [I] by using (6.3) and Lemma 2.5.

**Lemma 6.3.** Let \( 0 \leq s \leq \frac{m+2}{2} \). For sufficiently large \( n \), we have

\[ \left| \left( \frac{\partial}{\partial \lambda} \right)^s T_\pm(\lambda, x, y) \right| \leq C_{ns}\lambda^{-3}\langle x \rangle^{-\frac{m+1}{2}}\langle y \rangle^{-\frac{m-1}{2}}. \]  

(6.4)

We then integrate by parts \( 0 \leq s \leq (m + 2)/2 \) times to obtain

\[ \int_0^\infty e^{i\lambda(|x| + |y|)}T_\pm(\lambda, x, y)\tilde{\Psi}(\lambda)d\lambda \]

\[ = \frac{1}{(|x| + |y|)^s} \int_0^\infty e^{i\lambda(|x| + |y|)} \left( \frac{\partial}{\partial \lambda} \right)^s \left( T_\pm(\lambda, x, y)\tilde{\Psi}(\lambda) \right) d\lambda \]

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and estimate the right hand side by using (6.4). We obtain

$$|\tilde{\Omega}_{n+1}(x, y)| \leq C \sum_{\pm} (|x| \pm |y|)^{-\frac{m+2}{2}} \langle x \rangle^{-\frac{m-1}{2}} \langle y \rangle^{-\frac{m-1}{2}}$$

and $\tilde{\Omega}_{n+1}(x, y)$ is admissible. Proposition 6.1 follows.

## 7 Completion of proof of Theorem

To complete the proof of Theorem 1.1 we have only to prove the continuity of $W$ in Sobolev spaces. We prove this for the case $1 < p < \infty$ only. For the cases $p = 1$ and $p = \infty$, we may apply without any change the proof presented in Section 4 of [21] for odd dimensional cases where we estimated the multiple commutators $[p_{i_1}, [p_{i_2}, \cdots, [p_{i_r}, W_{\pm} \cdots ]]]$. We use the following two lemmas.

**Lemma 7.1.** Let $1 < p < \infty$ and $|V(x)| \leq C < \infty$. Then for large negative $\lambda$, $R(\lambda) \in \mathcal{B}(L^p(\mathbb{R}^m), W^{2,p}(\mathbb{R}^m))$ and $R(\lambda)^{\frac{1}{2}} \in \mathcal{B}(L^p(\mathbb{R}^m), W^{1,p}(\mathbb{R}^m)).$

**Proof.** We first remark that $H$ is bounded from below and $R(\lambda)^{\frac{1}{2}}$ is well defined bounded operator in $\mathcal{H}$ for large negative $\lambda$ and $R(\lambda)^{\frac{1}{2}} \in \mathcal{B}(L^p, W^{1,p})$ means that $R(\lambda)^{\frac{1}{2}}$ defined on $L^2 \cap L^p$ can be extended to such an operator. For $\lambda < 0$, we have $\|R_0(\lambda)\|_{p,p} \leq C|\lambda|^{-1}$ and $\|\nabla R_0(\lambda)\|_{p,p} \leq C_p|\lambda|^{-\frac{1}{2}}$. It follows that, for large negative $\lambda$, $1+R_0(\lambda)V$ is an isomorphism of $L^p$, $R(\lambda) = (1+R_0(\lambda)V)^{-1}R_0(\lambda)$ also in $L^p$ and $\|R(\lambda)\|_{p,p} \leq C|\lambda|^{-1}$. Hence, the resolvent equation is also valid in $L^p$,

$$R(\lambda) = R_0(\lambda) - R_0(\lambda)V R(\lambda), \quad (7.1)$$

and this implies $R(\lambda) \in \mathcal{B}(L^p, W^{2,p})$. It also follows that the integral in

$$\nabla R(\lambda)^{\frac{1}{2}} = \nabla R_0(\lambda)^{\frac{1}{2}} - C \int_{0}^{\infty} \mu^{-\frac{1}{2}} \nabla R_0(\lambda - \mu)^{-1} V R(\lambda - \mu) d\mu$$

converges in the norm of $\mathcal{B}(L^p)$ and $\nabla R(\lambda)^{\frac{1}{2}}$ is bounded in $L^p(\mathbb{R}^m)$. \hfill \Box

**Lemma 7.2.** Let $1 < p < \infty$ and $n = 1, 2, \ldots$. Then, for large negative $\lambda$ the following statements are satisfied:

1. Let $|\partial^\alpha V(x)| \leq C_\alpha$ for $|\alpha| \leq 2(n-1)$. Then, $R(\lambda)^n \in \mathcal{B}(L^p, W^{2n,p}).$
2. Let $|\partial^\alpha V(x)| \leq C_\alpha$ for $|\alpha| \leq 2n-1$. Then, $R(\lambda)^n \in \mathcal{B}(W^{1,p}, W^{2n+1,p}).$
Proof. We first prove (1) by induction on \( n \). If \( n = 1 \), (1) is contained in Lemma 7.1. Let \( n \geq 2 \) and suppose that (1) is already proved for smaller values of \( n \). By virtue of (7.1),

\[
R(\lambda)^n = R_0(\lambda)R(\lambda)^{n-1} - R_0(\lambda)VR(\lambda)^{n-1}R(\lambda).
\]

(7.2)

By the assumption on \( V \) and the induction hypothesis \( R(\lambda)^{n-1}, VR(\lambda)^{n-1} \in B(L^p, W^{2n,p}) \) and (1) follows since \( R_0(\lambda) \) maps \( W^{2n,p} \) to \( W^{2n+2,p} \) boundedly.

We next prove (2). Let \( n = 1 \) first. Then, in (7.1), \( R_0(\lambda) \in B(W^{1,p}, W^{3,p}) \) and \( VR(\lambda) \in B(W^{1,p}) \) by (1) for \( n = 1 \) and the assumption on \( V \). Hence (1) holds for \( n = 1 \). Let \( n \geq 2 \) and suppose that (2) is already proved for smaller values of \( n \). Then in (7.2), \( R(\lambda)^{n-1} \in B(W^{1,p}, W^{2n-1,p}) \) by the induction hypothesis, and \( VR(\lambda)^{n-1}R(\lambda) \in B(W^{1,p}, W^{2n-1,p}) \) also by the assumption on \( V \) and Lemma 7.1. Since \( R_0(\lambda) \in B(W^{2n-1,p}, W^{2n+1,p}) \), (2) follows.

By intertwing property we have for sufficient large negative \( \lambda \)

\[
R(\lambda)^nW_\pm = W_\pm R_0(\lambda)^n, \quad R(\lambda)^{n+\frac{1}{2}}W_\pm = W_\pm R_0(\lambda)^{n+\frac{1}{2}}
\]

From the first equation and Lemma 7.2 (1) we see that, if \( |\partial^\alpha V(x)| \leq C_\alpha \) for \( |\alpha| \leq 2(n-1) \), \( W_\pm \in B(W^{2n,p}, W^{2n,p}) \). Likewise from Lemma 7.1 and Lemma 7.2 (2), we have \( W_\pm \in B(W^{2n+1,p}, W^{2n+1,p}) \) if \( |\partial^\alpha V(x)| \leq C_\alpha \) for \( |\alpha| \leq 2n-1 \). This completes the proof of Theorem 1.1.

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