SIMPLE GROUPS WITH BRAUER TREES OF PRINCIPAL BLOCKS
IN THE SHAPE OF A STAR

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Abstract. We have found a list of finite simple groups with cyclic Sylow $p$-subgroup whose principal $p$-blocks have Brauer trees in the shape of a star, that is a tree of diameter at most 2. Moreover, for an arbitrary finite group $G$ with cyclic Sylow $p$-subgroup, we have obtained a necessary condition when the Brauer tree of the principal $p$-block of $G$ is a star.

Introduction

Let $G$ be a finite group, $p$ be a prime number dividing the order of $G$. Suppose that a Sylow $p$-subgroup of $G$ is cyclic. Then the Brauer graph of a $p$-block of $G$ is uniquely defined. Moreover, this graph is a tree.

Denote by $\mathcal{X}_p$ the class of finite groups with non-trivial cyclic Sylow $p$-subgroup such that the Brauer tree of the principal $p$-block is a star, that is a tree of diameter at most 2. Note that the Brauer graph of a $p$-block with cyclic defect group is a star if and only if every $p$-modular irreducible character of this block lifts to an ordinary irreducible character (see [37, Lemma 3.1]).

We are interested in the description of the class $\mathcal{X}_p$. The problem arises from the work [4], where the author studied the properties of such groups. For instance, he has shown that if $G \in \mathcal{X}_p$ and $G$ is not a $p$-solvable group, then the star has an even number of edges. A similar problem was studied in [21], where the author considered the class $\mathcal{L}_p$ of groups all whose absolutely irreducible $p$-modular characters are liftable.

It is known that if $G$ is a $p$-solvable group with cyclic Sylow $p$-subgroup, then Brauer trees of all $p$-blocks are starts, in particularly $G \in \mathcal{X}_p$. But there exist also nonsolvable groups having this property. For instance, $A_5 \in \mathcal{X}_3$.

The main goal of this work is to find all simple finite groups with the property $\mathcal{X}_p$. But some non-simple groups (namely, symmetric and classical groups) also will be covered during our study.

For simple groups, we have obtained the following result.

Theorem 1. Let $G$ be a finite simple group, and let $p$ be a primer dividing the order of $G$. Then $G \in \mathcal{X}_p$ if and only if one of the following statements holds.

1) $G = C_p$;
2) $G = \text{PSL}_2(q)$, $p \neq 2$ and $p$ divides $q \pm 1$;
3) $G = \text{PSL}_3(q)$, $p \neq 2$ and $p$ divides $q + 1$;
4) $G = \text{PSU}_3(q^2)$, $p \neq 2$ and $p$ divides $q - 1$;
5) $G = A_5$, $p \in \{3, 5\}$;
6) $G = A_6$ and $p = 5$;
7) $G = \text{Sz}(q^2)$, $q^2 = 2^{2n+1}$ ($n \geq 1$), where $p \neq 2$ divides $q - 1$ or $q + r + 1$;
8) $G = \text{Sz}(q^2)$, $q^2 = 3^{2n+1}$ ($n \geq 1$), where $p \neq 2$ divides $q^2 - 1$ or $q^2 + \sqrt{3}q + 1$;

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6) \( G \in \{ M_{11}, M_{12}, J_3 \} \) and \( p = 5 \);
7) \( G = J_1 \) and \( p \in \{3,5 \} \).

Moreover, for an arbitrary finite group \( G \), the following theorem gives a necessary condition when \( G \in \mathcal{X}_p \).

**Theorem 2.** Let \( G \) be a non-\( p \)-solvable group with a non-trivial cyclic Sylow \( p \)-subgroup \( P \). Then there exists the smallest normal subgroup \( K \) in \( G \) properly containing \( O_{p'}(G) \), and the quotient group \( L = K/O_{p'}(G) \) is simple non-abelian. Moreover, if \( G \in \mathcal{X}_p \), then \( L \in \mathcal{X}_p \).

1. Preliminaries

Recall some basic facts about Brauer trees of finite groups. We refer the reader to [3, 12] for details.

Let \( p \) be a primer dividing the order of a finite group \( G \). We will denote by \( \text{Irr}(G) \) the set of irreducible ordinary characters of \( G \) and by \( \text{IBr}_p(G) \) the set of irreducible Brauer (\( p \)-modular) characters of \( G \). The restriction \( \chi^o \) of \( \chi \in \text{Irr}(G) \) to the set on \( p \)-regular elements of \( G \) can be decomposed as

\[
\chi^o = \sum_{\varphi \in \text{IBr}_p(G)} d_{\chi \varphi} \varphi.
\]

The coefficients \( d_{\chi \varphi} \) are called the decomposition numbers. They form a decomposition matrix.

The Brauer graph of \( G \) is an undirected graph, the vertices of which are labelled by elements of \( \text{Irr}(G) \), and the edges are labelled by elements of \( \text{IBr}(G) \). Some vertices may be labelled by a few irreducible ordinary characters \( \chi_1, \ldots, \chi_m \) if they have the same restriction to the \( p \)-regular conjugacy classes. We call a such vertex exceptional with multiplicity \( m \). Two vertices labelled by \( \chi, \psi \in \text{Irr}(G) \) are adjacent if there exists \( \varphi \in \text{IBr}_p(G) \) such that \( d_{\chi \varphi} \neq 0 \) and \( d_{\psi \varphi} \neq 0 \). The connected components of the Brauer graph are called \( p \)-blocks of \( G \).

Let \( B \) be a \( p \)-block of \( G \) with cyclic defect group. Then the Brauer graph corresponding to \( B \) is a tree, which we will denote by \( \tau(B) \). If \( e \) is the number of edges in \( \tau(B) \), then \( \tau(B) \) has \( e + 1 \) vertices. A \( p \)-block may have not more then one exceptional vertex (with multiplicity \( m > 1 \)).

A vertex of a Brauer graph is called non-real if its character has a non-real value for some \( p \)-regular element. The real stem of a Brauer tree is a subtree obtained by removing all non-real vertices. The real stem always has the shape of a straight line (see [22, p. 3]).

**Lemma 1.** [3, p. 212] If a group \( G \) has a cyclic Sylow \( p \)-subgroup \( P \), then the number \( e \) of edges of Brauer tree of the principal \( p \)-block of \( G \) is equal to \( |N_G(P)|/C_G(P)| \), and the multiplicity of the exceptional vertex is \( m = (|P| - 1)/e \).

**Lemma 2.** [4, Theorem 1, Corollary 1] Let \( G \) be a simple non-abelian group with a non-trivial cyclic Sylow \( p \)-subgroup \( P \). Suppose that the Brauer tree of the principal \( p \)-block of \( G \) is a star with \( e \) edges. Then 1) \( e \) is even, 2) if the number \( |C_G(P)| \) is odd, then all involutions of \( G \) form a unique conjugacy class.

**Lemma 3.** Suppose that \( G \) is a group with cyclic Sylow \( p \)-subgroup \( P \). Suppose that \( H \) is a normal subgroup of \( G \) such that \(|G/H|, p \) = 1. Let \( e_G, e_H \) be the numbers of edges of Brauer tree of the principal \( p \)-block of the groups \( G \) and \( H \), correspondently. Then \( e_H | e_G \).
Proof. Since \( P \in \text{Syl}_p(H) \), applying Frattini’s argument, we obtain

\[
e_H = \frac{|N_H(P)|}{|C_H(P)|} = \frac{|C_G(P)H|}{|G|} \cdot e_G,
\]

where \( C_G(P)H \) is a subgroup of \( G \), because \( H \) is normal in \( G \). \( \square \)

Let \((\tau, Q)\) be the Brauer tree of a block \( B \) with the exceptional vertex \( Q \). Then the tree \((\tau, Q)^n\) is obtained by winding up \( \tau \) around \( Q \) which created \( n \) branches, where the original tree is considered as one of these branches.

We say that the Brauer trees of blocks \( B_1 \) and \( B_2 \) are similar if there is a tree \((\tau, Q)\) such that \( \tau(B_1) = (\tau, Q)^m \) and \( \tau(B_2) = (\tau, Q)^n \) for some \( m, n \in \mathbb{N} \).

We denote by \( B_0(G) \) the principal \( p \)-block of \( G \), by \( \tau_0(G) \) the Brauer tree of \( B_0(G) \), and by \( e_0(G) \) or \( e_G \) the number of edges in \( \tau_0(G) \).

**Lemma 4.** Let \( G \) be a group with a cyclic Sylow \( p \)-subgroup \( P \). Let \( H \) be a normal subgroup of \( G \) of index coprime to \( p \). Then:

1) \( \tau_0(G) \) is similar to \( \tau_0(H) \);

2) If \( \tau_0(G) \) is a line, then the same holds true for \( \tau_0(H) \).

Proof. The first part of the lemma follows from [12, Lemma 4.2].

Suppose that \( \tau_0(G) \) is a line with \( e_G \) edges. Since \( \tau_0(G) \) is similar to \( \tau_0(H) \), we have that \( \tau_0(G) = \tau^m \) and \( \tau_0(H) = \tau^n \) for some tree \( \tau \). If \( e_G \) is odd, then \( \tau \) coincides with \( \tau_0(G) \). It follows from \( e_H \leq e_G \) that \( \tau_0(H) = \tau \).

If \( e_G \) is even, then either \( \tau_0(G) = \tau \) or \( \tau_0(G) = \tau^2 \). In both cases, we obtain that \( \tau_0(H) \) is a line. \( \square \)

The following Lemma describes some simple properties of the class \( \mathfrak{X}_p \).

**Proposition 1.** Let \( G \) be a finite group, and \( P \) be a Sylow \( p \)-subgroup of \( G \).

1) If \( G \) is \( p \)-solvable and \( P \) is cyclic, then \( G \in \mathfrak{X}_p \).

2) \( G \in \mathfrak{X}_p \) if and only if \( G/O_{p'}(G) \in \mathfrak{X}_p \).

3) Suppose that \( H \) is a normal subgroup of \( G \) such that \( |G/H| \) is coprime to \( p \). If \( G \in \mathfrak{X}_p \), then \( H \in \mathfrak{X}_p \).

Proof. 1) If \( G \) is a \( p \)-solvable group, then according to [12, Lemma X.4.2], the Brauer tree of any \( p \)-block of \( G \) with a cyclic defect group is a star.

2) The second statement holds because the kernel of the principal \( p \)-block of \( G \) is equal to \( O_{p'}(G) \).

3) The third statement follows from Lemmas 1 and 3. \( \square \)

**Proof of Theorem 2.** Let \( G \) be a non-\( p \)-solvable group with a non-trivial cyclic Sylow \( p \)-subgroup \( P \). Let \( H = G/O_{p'}(G) \). Then, by [31, Lemma 6.1], there is a unique minimal normal subgroup \( L \) in \( H \), and \( L \) is simple.

Denote by \( K \) a subgroup of \( G \) such that \( L \cong K/O_{p'}(G) \). Then \( K \) is normal in \( G \), and \( K \) containing \( O_{p'}(G) \) properly, because \( L \neq 1 \).
Since $G$ is not $p$-solvable, the subgroup $H$ can not be a Frobenius group with kernel $P$. Thus, according to [4, Lemma 5.1], $L$ contains $P$, hence $p$ doesn’t divide $|G/K|$. It gives that $K$ is not abelian.

If $G \in \mathfrak{X}_p$, then using Proposition 1 we obtain that $K \in \mathfrak{X}_p$ and $L \in \mathfrak{X}_p$. □

The proof of Theorem 1 is based on the classification of simple finite groups [8]. We will consequentially consider cyclic groups, alternating groups, classical groups, exceptional groups of Lie type and sporadic groups.

2. Cyclic, symmetric and alternating groups

For cyclic groups, the result is simple.

**Proposition 2.** $C_p \in \mathfrak{X}_p$ for any prime $p$.

*Proof.* The statement holds because each abelian group is solvable, i.e. it is $p$-solvable for any $p$ dividing the order of the group. □

Note that by definition the trivial group is not in $\mathfrak{X}_p$.

For symmetric and alternating groups, we have the following result.

**Proposition 3.**

1) $S_n \in \mathfrak{X}_p$ if and only if $p = 2$ and $n \in \{2, 3\}$, or $p = 3$ and $n \in \{3, 4, 5\}$.

2) $A_n \in \mathfrak{X}_p$ if and only if $p = 3$ and $n \in \{3, 4, 5\}$, or $p = 5$ and $n \in \{5, 6\}$.

*Proof.* 1) Let $G = S_n$ $(n \geq 2)$. A sylow $p$-subgroup $P$ of $G$ is cyclic if and only if $p \leq n < 2p$ (or $n/2 < p \leq n$). Suppose that $P$ is cyclic. Since all ordinary irreducible characters of $S_n$ are real, they lay on the real steam which is a line, and there are no exceptional characters (i.e. multiplicity $m = 1$). Thus, the number of edges in $\tau_0(G)$ is $e = |P| - 1$. This tree is a star if and only if $e \leq 2$. This holds if and only if $|P| = 3$ and $n \in \{3, 4, 5\}$, or $|P| = 2$ and $n \in \{2, 3\}$.

2) For $G = A_n$ $(n \geq 3)$, the number of edges in $\tau_0(G)$ is $e = p - 1$ if $n/2 < p < n - 1$, and $e = (p - 1)/2$ if $p \in \{n - 1, n\}$ (see [21, p.282]). It gives desired. □

3. Sporadic groups

**Proposition 4.** Let $G$ be one of the sporadic groups. Then $G \in \mathfrak{X}_p$ if and only if one of the following statements holds.

1) $G = M_{11}$ and $p = 5$;

2) $G = M_{23}$ and $p = 5$;

3) $G = J_1$ and $p \in \{3, 5\}$;

4) $G = J_3$ and $p = 5$.

*Proof.* Brauer trees of sporadic groups for most values of $p$ can be found in [22]. Some trees can be also easily constructed from decomposition matrices [2]. Also, using information about the orders of the centralizer $C_G(P)$ and normalizer $N_G(P)$ of a Sylow $p$-subgroup $P$ of $G$, it is easy to show with Lemma 2 that $\tau_0(G)$ is not a star for most sporadic groups.

For instance, consider the Mathieu group $G = M_{23}$ of the order $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. For $p \in \{2, 3\}$, a Sylow $p$-subgroup $P$ of $G$ is not cyclic. The Brauer tree of the principal 5-block is
a star, according to [22]. For \( p \in \{7, 23\} \), the order \( |N_G(P)/C_G(P)| \) is odd, hence \( G \notin X_p \). For \( p = 11 \), the order of \( C_G(P) \) is odd and \( G \) has more than one conjugacy class of involutions, and \( G \notin X_p \) by Lemma 2. Other groups are analysed the same way. Only two groups, Baby Monster \( BM \) for \( p \in \{13, 19\} \) and Fisher group \( F_{24}^0 \) for \( p \in \{11, 13\} \), are required special consideration. In [29, Lemma 5.1], it has been showed that in these cases the graph \( \tau_0(G) \) contains a path of length at least 3.

\[ \square \]

4. Classical groups

In this section, we will consider Brauer trees of finite classical groups: linear, symplectic, unitary, and orthogonal.

**Lemma 5.** Let \( G \) be a finite group which is not solvable. Then a Sylow 2-subgroup of \( G \) is not cyclic.

**Proof.** Let \( G \) is a non-solvable group. Then by the Feit-Thompson theorem, 2 divides \(|G|\).

Suppose that a Sylow 2-subgroup \( P \) of \( G \) is cyclic. Then according to [26, Satz IV.2.8], \( G \) is 2-nilpotent, i.e. \( G \) has a normal subgroup \( H \) such that \( G/H \cong P \). It gives a contradiction.

\[ \square \]

**Lemma 6.** [33, Proposition 5.1] Let \( G \) be a simple finite group of Lie type over \( \mathbb{F}_q \), which is not isomorphic \( \text{PSL}_2(q) \). If \( p \mid q \), then Sylow \( p \)-subgroup of \( G \) is not abelian.

Thus, it is sufficient to consider only cases when \( p \neq 2 \), and when \( p \) does not divide \( q \) (except linear groups). We will use Stather’s result [36, Table 1] about sizes of Sylow \( p \)-subgroups for such case. Also, the following Lemma shows that, in this case, the number of edges in a \( p \)-block depends only on \( \text{ord}_p(q) \), the multiplicative order of \( q \) modulo \( p \).

**Lemma 7.** [18] Let \( G \) be a finite group of Lie type over a field of \( q \) elements with cyclic (non-trivial) Sylow \( p \)-subgroup. Suppose that \( p \nmid q \), and let \( d = \text{ord}_p(q) \). Then \( p \)-modular decomposition matrix directly depends only on \( d \), and does not depend on the particular choice of \( q \) and \( p \).

**Lemma 8.** Suppose that \( G \in \{ \text{GL}_n(q), \text{Sp}_{2n}(q) \} \), where \( q \geq 2 \), or \( G \in \{ \text{GU}_n(q^2), \text{SO}_{2n+1}(q), \text{CSO}^\pm_{2n}(q) \} \), where \( q \) is odd, or \( G \in \{ \text{SU}_n(q^2), \text{GO}^\pm_{2n}(q) \} \), where \( q \) is even. Suppose also that a Sylow \( p \)-subgroup of \( G \) is cyclic, \( p \nmid q \) and \( p \neq 2 \). Then the Brauer tree of the principal block of \( G \) is a line.

**Proof.** The result has been proven in [13] for \( \text{GL}_n(q) \) with any \( q \), and in [14] for the groups \( \text{Sp}_{2n}(q), \text{GU}_n(q^2), \text{SO}_{2n+1}(q) \), and \( \text{CSO}_{2n}^\pm(q) \), when \( q \) is odd.

Assume that \( G \) is one of the groups \( \text{Sp}_{2n}(q) \) or \( \text{GO}^\pm_{2n}(q) \), where \( q \) is even. Then, by Gow [20], each element of the group \( G \) is a product of two involutions. It follows that each element of \( G \) is conjugated to its inverse. Therefore, each ordinary character of \( G \) is real-valued. Thus, the Brauer tree of each block with cyclic defect group coincides with its real stem, i.e. it has the shape of a line.

Assume that \( G = \text{SU}_n(q^2), q \) is even. Since \( G \) is quasi-simple, it follows from [32, Section 6] that each non-exceptional character in \( B_0(G) \) is rational-valued, and hence it is located on the real stem. Thus, \( \tau_0(G) \) is a line.

\[ \square \]
Thus, for these groups, we need just to find the number $e_0(G)$ of edges in the principal block of $G$, i.e. the order of $N_G(P)/C_G(P)$. As we will see soon, for general linear groups, this number is equal to the multiplicative order of $q$ modulo $p$. This is not always true for other classical groups, but the approach is similar.

4.1. Linear groups. Recall that the order of the general linear group $\text{GL}(n, q)$ over the finite field $\mathbb{F}_q$ with $q$ elements is equal to

$$|\text{GL}_n(q)| = q^{n(n-1)/2} \cdot (q-1) \cdot \ldots \cdot (q^n - 1).$$

The special linear group $\text{SL}(n, q)$ is a normal subgroup of $\text{GL}(n, q)$ of index $q-1$. The projective special linear group $\text{PSL}(n, q)$ is obtained from $\text{SL}(n, q)$ by factoring out its center $Z$ whose order equals $(n, q-1)$.

Lemma 9. [28, Lemma 2] Let $G = \text{GL}_n(q)$, $p \nmid q$ and $d = \text{ord}_p(q)$. Then

1) A Sylow $p$-subgroup $P$ of $G$ is cyclic and non-trivial if and only if $d \leq n < 2d$.

2) If $d \leq n < 2d$, then $|N_G(P)/C_G(P)| = d$.

Proposition 5. Let $G = \text{GL}_n(q)$, $n \geq 2$. Then $G \in \mathcal{X}_p$ if and only if one of the following statements holds.

1) $n = 2$ and $p = q \in \{2, 3\}$;

2) $n \in \{2, 3\}$, $p \neq 2$ and $p \mid q + 1$.

Proof. Assume that $p \mid q$. In this case, $P$ is cyclic if and only if $n = 2$ and $q = p$. Let $G = \text{GL}_2(p)$. Then $|N_G(P)| = p(p-1)^2$ and $|C_G(P)| = p(p-1)$. Hence, $\tau_0(G)$ is a line (by Lemma 8) with $|N_G(P)/C_G(P)| = p-1$ edges. Therefore, $\text{GL}_2(p) \in \mathcal{X}_p$ if and only if $p \in \{2, 3\}$.

Assume now that $p \nmid q$, and $d = \text{ord}_p(q)$. Since $P$ is cyclic, by Lemma 9 we have that $p \neq 2$, $n < 2d$ and $|N_G(P)/C_G(P)| = d$. In particular, $d \geq 2$. Thus, $\tau_0(G)$ is a star if and only if $d = 2$ and $n = 2, 3$.

\[ \square \]

Lemma 10. Let $G = \text{GL}_n(q)$, $H = \text{SL}_n(q)$. Suppose that a Sylow $p$-subgroup $P$ of $H$ is cyclic (and non-trivial).

1) If $p \nmid q$, then $\tau_0(G)$ and $\tau_0(H)$ contain the same number of edges.

2) If $p \nmid q$ and $p \nmid q - 1$, then $\tau_0(G) = \tau_0(H)$.

Proof. 1) According to Frattini’s argument, $G = N_G(P)H$. Therefore,

$$G/N_G(P) = H/(H \cap N_G(P)) = H/N_H(P),$$

and $|N_G(P)|/|N_H(P)| = |G|/|H| = q - 1$.

Since $p \nmid q$, the centralizers $C_G(P)$ and $C_H(P)$ coincide with Singer cycles in $G$ and $H$. They have orders $q^n - 1$ and $(q^n - 1)/(q-1)$, correspondingly. It gives that $|N_H(P)/C_H(P)| = |N_G(P)/C_G(P)|$, as desired.

2) If also $(|G/H|, p) = 1$, then by Lemma 1, Brauer trees $\tau_0(G)$ and $\tau_0(H)$ are similar. And since they contain the same number of edges, they have the same shape.

\[ \square \]
Proposition 6. Let \( H = \text{PSL}_n(q) \) and \( G = \text{SL}_n(q) \), where \( n \geq 2 \) and \( q \geq 5 \). Then \( H \in \mathcal{X}_p \) \((G \in \mathcal{X}_p)\) if and only if one of the following statements holds.

1) \( n = 2 \) and \( p = q = 5 \);
2) \( n = 2 \), \( p \neq 2 \) and \( p | q - 1 \);
3) \( n \in \{2, 3\} \), \( p \neq 2 \) and \( p | q + 1 \);

Proof. We may assume that \( p > 2 \), otherwise Sylow \( p \)-subgroups of both \( \text{SL}_n(q) \) and \( \text{PSL}_n(q) \) for \( q \geq 5 \) are not cyclic.

The center \( Z \) of \( \text{SL}_n(q) \) has the order \((n, q - 1)\). Therefore, if \( p \nmid q - 1 \), then \( Z \) is in the kernel of the block \( B_0(\text{SL}_n(q)) \); hence, this block consists with \( B_0(\text{PSL}_n(q)) \).

First, assume that \( n = 2 \). We will consider three cases: 1) \( p | q \), 2) \( p | q - 1 \), and 3) \( p \) doesn’t divide \( q \) neither \( q - 1 \).

1) Let \( p | q \). Then a Sylow \( p \)-subgroup \( P \) of \( \text{SL}_2(q) \) is cyclic if and only if \( p = q \). According to [11, Theorem 71.3], the Brauer tree of the principal \( p \)-block of \( \text{SL}_2(p) \) is a line with \((p - 1)/2\) edges. Thus, \( \text{SL}_2(p) \in \mathcal{X}_p \) if and only if \( p \leq 5 \).

2) Let \( p | q - 1 \). For \( G = \text{SL}_2(q) \), we have that \( P = Syl_p(G) \) is cyclic, the centralizer \( C_G(P) = \langle \alpha \rangle \) and the normalizer \( N_G(y) = \langle \alpha, \beta \rangle \), where \( \alpha = \left( \begin{smallmatrix} 0 & \nu^{-1} \\ \nu & 0 \end{smallmatrix} \right) \) and \( \beta = \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \). Here \( \nu \) is a generator of the multiplicative group \( \mathbb{F}_q^{*} \) (see [11, p. 230]). Therefore, \(|N_G(P)/C_G(P)| = 2 \), and \( G \in \mathcal{X}_p \).

Now consider \( H = \text{PSL}_2(q) \), where \( p | q - 1 \). A Sylow \( p \)-subgroup of \( H \) is also cyclic, and it follows from [5] that the tree \( \tau_0(H) \) is a line with two edges.

3) Let \( p \nmid q \) and \( p \nmid q - 1 \). In this case, the tree \( \tau_0(H) \) consists with \( \tau_0(G) \). But \( \tau_0(G) \) consists also with \( \tau_0(\text{GL}_n(q)) \) by Lemma 10. Thus, the trees \( \tau_0(H) \) and \( \tau_0(G) \) are both stars if \( p | q + 1 \) (see Proposition 5).

Now suppose that \( n \geq 3 \). In this case, a Sylow \( p \)-subgroup of \( \text{SL}_n(q) \) is cyclic if and only if \( p | q \) and \( p \nmid q - 1 \). As well as above, the trees of all three groups \( \text{PSL}_n(q) \), \( \text{SL}_n(q) \) and \( \text{GL}_n(q) \) are the same, and we can apply Proposition 5. 

\[ \Box \]

4.2. Symplectic groups. Let \( V \) be a vector space of even dimension \( n = 2m \) over the field \( \mathbb{F}_q \). Let \( f : V \times V \to V \) be a skew-symmetric form on \( V \), i.e. a bilinear form such that \( f(u, v) = -f(v, u) \) for all \( u, v \in V \). If \( f \) is given by a matrix \( W \), then \( W = -W^t \), where \( t \) means transpose.

A symplectic group \( \text{Sp}_{2m}(q) \) consists of invertible matrices \( A \) of order \( 2m \) which preserve \( f \), i.e. \( AWA^t = W \). It has the order \[ |\text{Sp}_{2m}(q)| = q^{m^2} \cdot (q^2 - 1) \cdot (q^4 - 1) \cdot \ldots \cdot (q^{2m} - 1). \]

We may take \( W = \left( \begin{smallmatrix} 0 & I_m \\ -I_m & 0 \end{smallmatrix} \right) \). Then the rule
\[
\varphi : A \mapsto \left( \begin{smallmatrix} A & 0 \\ 0 & A^{-t} \end{smallmatrix} \right)
\]
defines an embedding of \( \text{GL}_m(q) \) into \( \text{Sp}_{2m}(q) \).

The projective symplectic group is \( \text{PSp}_{2m}(q) = \text{Sp}_{2m}(q)/Z \), where \( Z \) is the center of \( \text{Sp}_{2m}(q) \).

The groups \( \text{Sp}_2(q) \cong \text{SL}_2(q) \) and \( \text{PSp}_2(q) \cong \text{PSL}_2(q) \) have been already considered in the previous section. For \( m \geq 2 \), the groups \( \text{PSp}_{2m}(q) \) are simple, except \( \text{PSp}_4(2) \cong \text{Sp}_4(2) \cong S_6 \).

If \( p \neq 2 \), then the principal \( p \)-blocks of \( \text{Sp}_n(q) \) and \( \text{PSp}_n(q) \) are the same, because \( |Z| = (2, q - 1) \). Moreover, \( \text{Sp}_{2m}(q) \cong \text{PSp}_{2m}(q) \) for even \( q \).
Proposition 7. Let $G$ be one of the groups $PSp_{2m}(q)$ or $Sp_{2m}(q)$, where $m \geq 2$, and let $p$ be a prime dividing $|G|$. Then $G \not\in \mathcal{X}_p$.

Proof. Using Lemmas 5 and 6, we may assume that $p > 2$ and $p \nmid q$. In this case, Sylow subgroups of $PSp_{2m}(q)$ and $Sp_{2m}(q)$ are isomorphic, and the principal blocks of this groups are the same, because $p$ doesn’t divide $|Z(\text{Sp}_{2m}(q))|$. Thus, it suffices to consider only $Sp_{2m}(q)$.

Assume that $G = \text{Sp}_{2m}(q) \in \mathcal{X}_p$ for some $m, q$ (where $m \geq 2$).

Let $d = \text{ord}_d(q)$. If $d = 1$, then, by Lemma 9, a Sylow $p$-subgroup of $\text{GL}_m(q)$ is not cyclic for $m \geq 2$. The diagonal embedding 2 shows that the same holds true for a Sylow subgroup $P$ of $\text{Sp}_{2m}(q)$.

Therefore, we can assume that $d \neq 1$. By Lemma 1, the number $e$ of edges in the Brauer tree $\tau_0(G)$ is equal to $|N_G(P)/C_G(P)|$.

We will show that $e \geq d \geq 3$, that gives a contradiction with the assumption, because $\tau_0(G)$ is a line by Lemma 8.

The structure of $P$ depends on $d$.

(1) Even $d$. According to [36, Table 1], $P$ coincides with a Sylow $p$-subgroup of the ambient group $\text{GL}_{2m}(q)$. Since $P$ is non-trivial and cyclic, we conclude that $m < d \leq 2m$. Further, $m \geq 2$ yields $d \geq 4$.

Under a suitable choice of the matrix $W$, the group $\text{Sp}_d(q) \times \text{Sp}_{2m-d}(q)$ is embedded into $\text{Sp}_{2m}(q)$ as a block diagonal (see [1, Remark 3.2] for details); hence, $P$ can be chosen in the left upper corner $G_d = \text{Sp}_d(q)$. Now taking into account Lemma 3, it suffices to show that $e_d = |N_{G_d}(P)/C_{G_d}(P)| \geq d$.

Let $\alpha$ be a generator of $P$. According to [36, Lemma 4.6], $N_{G_d}(P)/C_{G_d}(P)$ is a cyclic group generated by an element $\gamma$ acting by conjugation as $\alpha^\gamma = \alpha^d$. Since this action has order $d$, we obtain $e_d \geq d$. Thus, $e \geq d \geq 4$ as desired.

(2) Odd $d$. According to [36], the order of $P$ is equal to the order of a Sylow $p$-subgroup $P'$ of $\text{GL}_m(q)$. We may assume that $P$ is the image of $P'$ under embedding 2. Since $P'$ is cyclic, we have $m/2 < d \leq m$. Thus, $d \geq 3$.

It remains to show that $e \geq d$. As above, we may assume that $m = d$. Consider an element $y' \in \text{GL}_d(q)$ which acts by conjugation on $P'$ as an automorphism of order $d$. The diagonal image of this element belongs to $N_G(P)$ and acts as an automorphism of order $d$ on this subgroup. \hfill \Box

4.3. Unitary groups. Let us denote by $\bar{a}$ the involution $a \mapsto a^q$ of the field $\mathbb{F}_q$, and let $V$ be an $n$-dimensional vector space over this field. Then there exists an unique non-singular conjugate-sesquilinear form $f : V \times V \rightarrow \mathbb{F}_q$, i.e. $f(u, v) = \overline{f(v, u)}$ for all $u, v \in V$. If $f$ is given by a matrix $W$, then $W = \overline{W}$. For instance, $W$ can be the identity matrix.

The general unitary group $\text{GU}_n(q^2)$ consists of matrices $A \in \text{GL}_n(q^2)$ preserving $f$, i.e. $AWA^t = W$. The order is

$$|\text{GU}_n(q^2)| = q^{n(n-1)/2} \cdot (q + 1) \cdot (q^2 - 1) \cdots \cdot (q^n - (-1)^n).$$
The unitary matrices of determinant 1 form a normal subgroup in $\text{GU}_n(q^2)$ of index $q + 1$, called the special unitary group $\text{SU}_n(q^2)$. The center $Z$ of $\text{SU}_n(q^2)$ consists of scalar matrices, and $|Z| = (n, q + 1)$. The quotient group $\text{PSU}_n(q^2) = \text{SU}_n(q^2)/Z$ is called the projective special unitary group. If $n \geq 3$, then this group is simple, except $\text{PSU}_3(2^2)$.

Note that $\text{PSU}_2(q^2) \cong \text{PSL}_2(q)$. Thus, it suffices to consider $n \geq 3$. 

**Proposition 8.** Let $G$ be one of the groups $\text{PSU}_n(q^2)$ or $\text{SU}_n(q^2)$, where $n \geq 3$, and let $p$ be a prime dividing $|G|$. Then $G \in \mathcal{X}_p$ if and only if $n = 3$, $p > 2$ and $p$ divides $q - 1$.

**Proof.** As before, we assume that $p \neq 2$ and $p \nmid q$. Write $d = \text{ord}_p(q)$, and $s = \text{ord}_p(q^2)$.

We first consider the case $n = 3$. If $p = 3$ and $p \mid q + 1$, then Sylow subgroups of $\text{PSU}_3(q^2)$ and $\text{SU}_3(q^2)$ are not cyclic. Otherwise, the Sylow subgroups are cyclic, and the principal $p$-block of $\text{PSU}_3(q^2)$ coincides with the principal $p$-block $B_0$ of $\text{SU}_3(q^2)$, because $B_0$ is annihilated by the center of the group $\text{SU}_3(q^2)$. It follows from [16] that the Brauer tree of $B_0$ is a start if and only if $p \neq 2$ and $p \mid q - 1$.

Now set $n \geq 4$. In this case, if $p$ divides $q \pm 1$, Sylow $p$-subgroups of $\text{PSU}_n(q^2)$ and $\text{SU}_n(q^2)$ are not cyclic.

Assume that $p \nmid q \pm 1$. Then $2 < d \leq 2n$. Further, the principal block of $\text{PSU}_n(q^2)$ coincides with the principal block $B_0$ of the group $\text{SU}_n(q^2)$.

A Sylow $p$-subgroup $P$ of $H = \text{SU}_n(q^2)$ coincides with a Sylow $p$-subgroup of $G = \text{GU}_n(q^2)$. Moreover, $|N_G(P)/C_G(P)| = |N_H(P)/C_H(P)|$. Therefore, $\tau_0(H) = \tau_0(G)$. This tree is a line by Lemma 8 (for odd and even $q$). Thus, it suffices to show that the number of edges in $\tau_0(G)$ is greater than 2.

There are two possibilities for $P$ in depending on $d$.

1. $d \equiv 2 \pmod{4}$. According to [36], $P$ is also a Sylow $p$-subgroup of the ambient group $\text{GL}_n(q^2)$. Because $P$ is cyclic, we conclude that $n/2 < s = d/2 \leq n$. In particular, $s \geq 3$.

The group $\text{GU}_s(q^2) \times \text{GU}_{n-s}(q^2)$ is embedded into $\text{GU}_n(q^2)$, and $P$ can be chosen in the left upper corner $G_s = \text{GU}_s(q^2)$. As well as for symplectic groups, it is easy to show that the number of edges $e_0(G) \geq e_0(G_s) = |N_{G_s}(P)/C_{G_s}(P)| \geq s$, as desired.

2. $d \equiv 0, 1, 3 \pmod{4}$. Let $n = 2m + \varepsilon$, where $\varepsilon = 0, 1$. According to [36], the order of $P$ equals the order of a Sylow $p$-subgroup $P'$ of $\text{GL}_m(q^2)$. If $\varepsilon = 0$, then choosing $W = \left( \begin{smallmatrix} I_m & 0_m \\ 0_m & I_m \end{smallmatrix} \right)$, we obtain the embedding from $\text{GL}_m(q^2)$ into $\text{GU}_n(q^2)$, which sends $A$ to $\left( \begin{smallmatrix} A & 0_m \\ 0_m & A^{-1} \end{smallmatrix} \right)$. And a similar embedding takes place for $\varepsilon = 1$ if we add 1 to the lower right corner of $W$.

We first consider the case when $d$ is odd; hence, $s = d > 2$. Because $P'$ is cyclic and non-trivial, we conclude that $m/2 < s \leq m$. As above, to estimate $e_0(G)$, we may assume that $n = 2d$. Choose an element in $\text{GL}_d(q^2)$ which acts by conjugation on $P'$ as an automorphism of order $s$. By multiplying by a constant we may assume that this element has determinant 1. Expanding diagonally, we conclude that $e \geq s > 2$, as desired.

It remains to consider the last case when $d$ is divisible by 4. In this case, $d = 2s$. If $d > 4$, then, using the diagonal embedding, we conclude that $e \geq s > 2$.

Finally, if $d = 4$, i.e. $G = \text{GU}_4(q^2)$ and $p \mid q^2 + 1$, then the generator $A = \alpha'$ for $P'$ can be chosen such that $A \cdot \alpha'^{t} = I_2$. As above, using the diagonal embedding we obtain that $e \geq 2$. But also the matrix $W$ normalizers $P$, hence $e \geq 4$, as desired. \qed
4.4. Orthogonal groups in odd dimension. Let $n = 2m + 1$, where $m \geq 1$, and let $V$ be an $n$-dimensional vector space over $\mathbb{F}_q$. The general orthogonal group $GO_n(q)$ consists of invertible matrices of order $n$ which preserve a scalar product given by a matrix $W$, i.e. $A \in GO_n(q)$ if and only if $AWA^t = W$. For instance, we may take $W = \begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It gives the embedding

$$\text{GL}_m(q) \to GO_{2m+1}(q), \ A \mapsto \begin{pmatrix} A & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & I \end{pmatrix}.$$ 

The group $GO_n(q)$ contains a special orthogonal group $SO_n(q)$, consisting of matrices with determinant 1. The derived subgroup of $SO_n(q)$ is denoted by $\Omega_n(q)$.

If $q$ is even, then $\Omega_{2m+1}(q) \cong \text{PSp}_{2m}(q)$ for any $m \geq 1$. If $q$ is odd, then $\Omega_3(q) \cong \text{PSL}_2(q)$ and $\Omega_5(q) \cong \text{PSp}_4(q)$, according to [27, p. 43].

Thus, it suffices to consider the groups $\Omega_{2m+1}(q)$ for odd $q$ and $m \geq 3$. They are all simple, and the order is

$$|\Omega_{2m+1}(q)| = \frac{1}{2} \cdot q^{m^2} \cdot (q^2 - 1) \cdot (q^4 - 1) \cdots (q^{2m} - 1).$$

**Lemma 11.** Suppose that $G = GO_{2m+1}(q)$ and $H = SO_{2m+1}(q)$, where $q$ is odd. Suppose that Sylow $p$-subgroup $P$ of $H$ is cyclic and non-trivial, and that $p$ does not divide $q$. Then the Brauer trees of the principal blocks of $G$ and $H$ are the same.

**Proof.** Since $\tau_0(H)$ is similar to $\tau_0(G)$ by Lemma 1, it is sufficient to show that $|N_G(P)/C_G(P)| = |N_H(P)/C_H(P)|$.

The equality $|G| = |H|/2$ yields $|N_H(P)| = |N_G(P)|/2$. To deduce the same equality for centralizers of $P$, it suffices to show that there exists a matrix $Z \in G$ such that $\det(Z) = -1$ and $Z$ centralizes $P$. One may take $Z = \text{diag}(1,1,\ldots,1,-1)$. 

**Theorem 3.** Let $G$ be one of the groups $\Omega_{2m+1}(q)$, $SO_{2m+1}(q)$ or $GO_{2m+1}(q)$, where $m \geq 3$ and $q$ is odd. Let $p$ be a prime dividing the order of $G$. Then $G \notin \mathfrak{X}_p$.

**Proof.** Assume that $G \in \mathfrak{X}_p$. Then a Sylow $p$-subgroup $P$ of $G$ is cyclic, and $p \nmid q$.

Since all indices in the normal series $\Omega_n(q) \subset SO_n(q) \subset GO_n(q)$ are equal to 2, the Brauer trees of the principal blocks of these groups are similar to each other. Moreover, by Lemma 11, the principal blocks of $SO_n(q)$ and $GO_n(q)$ have the same Brauer trees. Further, since this tree is a line (by Lemma 8), it suffices to show that the number $e$ of edges in $\tau_0(GO_n(q))$ is larger than 4, or equals 4 but the exception vertex is not in the center.

Write $d = \text{ord}_p(q)$.

(1) Even $d$. In this case, $P$ can be chosen as a subgroup of the ambient group $\text{GL}_{2m}(q)$. Because $P$ is cyclic and non-trivial, we obtain $m < d \leq 2m$; hence, $d > 3$. Using [36, Lemma 4.5], we derive that $e \geq d$. Thus, if $m \geq 4$, then $d \geq 6$ implies $e \geq 6$, a contradiction.

Consider the remaining case $m = 3$ and $d = 4$, i.e. $G = \text{SO}_7(q)$, where $p \mid q^2 + 1$. Using [7, p. 466–467], we can calculate the Brauer tree of the principal block. It has the following shape, where the numbers near vertices show the degrees and parameterization symbols of characters.

\[
\begin{array}{cccccc}
1 & q^2(q^4+q^2+1) & q^4(q^3+1)(q+1)/2 & (q^6-1)(q^2-1) & q^4(q^3-1)(q-1)/2 \\
\left(\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 \\ 1 \\ 3 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 \\ 1 \\ 2 \end{smallmatrix}\right)
\end{array}
\]
The character degrees are comparable with $1, -1, 1, 4, 1$ modulo $p$; therefore, the second on the right character is exceptional (see [12, Theorem 7.2.16]).

(2) Odd $d$. In this case, the order of $P$ equals the order of a Sylow $p$-subgroup $P'$ of $GL_m(q)$. Thus, $P$ is the image of $P'$ via the embedding (3).

Since $P$ is cyclic, we conclude that $m/2 < d \leq m$; hence, $d \geq 3$. Calculating the normalizer of $P'$ in $GL_d(q)$, we get $e \geq d$. If $m > 5$, then $d > 3$ that gives the desired.

Now consider the remaining cases with $3 \leq m \leq 5$ and $d = 3$ (i.e. $p \mid q^2 + q + 1$). It follows from Lemma 7 that the number of edges $e_0(G)$ in the principal block of $G = SO_{2m+1}(q)$ depends on $d$ only. For instance, we may take $q = 3$ and $p = 13$. A calculation in GAP [15] gives $e_0(G) = |N_G(P)/C_G(P)| = 6$ for $m = 3, 4, 5$.

\[\square\]

4.5. Orthogonal groups in even dimension. Write $n = 2m$, where $m \geq 1$. A general orthogonal group $GO_n^\pm(q)$ consists of invertible matrices of order $n$, preserving the quadratic form correspondingly $Q^{(+)} = (0 \quad I_m)$ and $Q^{(-)} = (0 \quad I_{m-1} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad a \quad b)$, where $a = \gamma + \gamma^\alpha$, $b = \gamma^{q + 1}$, and $\gamma$ is a primitive element of $\mathbb{F}_q^2$. The corresponding bilinear form for $Q^{(\pm)}$ is $W = Q + Q^T$.

The conformal orthogonal group $CO_n^\pm(q)$ consists of invertible matrices which preserve this form up to a multiplicative constant. The special orthogonal group $SO_n^\pm(q)$ is a subgroup of $GO_n^\pm(q)$ of matrices with determinant 1. The conformal special orthogonal group $CSO_{2m}^\pm(q)$ consists of matrices $A$ such that the multiplicative constant $\lambda = \lambda(A)$ satisfies $\lambda^m = \det A$ (see [30, p. 13]).

For odd $q$, we have the following diagram of normal subgroups of $CO_n^\pm(q)$.

\[\begin{array}{c}
CO_n^\pm(q) \\
CSO_n^\pm(q) \\
GO_n^\pm(q) \\
SO_n^\pm(q) \\
\Omega_n^\pm(q)
\end{array}\]

For even $q$, we have $GO_n^\pm(q) \cong SO_n^\pm(q)$ and $CO_n^\pm(q) \cong CSO_n^\pm(q)$.

The centre $Z$ of $\Omega_n^\pm(q)$ consists of scalar matrices, and $|Z| = (4, q^m - 1)/2$. The factor group $P\Omega_{n}^\pm(q) = \Omega^\pm(q)/Z$ is called a projective orthogonal group. It has the following order.

$$|P\Omega_{2m}^\pm(q)| = \frac{1}{(4, q^m - 1)} \cdot q^{m(m - 1)/2} \cdot (q^m + 1) \cdot \prod_{i=1}^{m-1}(q^{2i} - 1).$$

Note (see [27, p. 43]) that the group $\Omega_2^\pm(q)$ is isomorphic to the dihedral group $D_{2(q-1)}$, which is solvable.

Further, the group $P\Omega_{2m}^\pm(q) \cong 2.(PSL_2(q) \times PSL_2(q))$ is not simple. The simple groups $P\Omega_4^\pm(q) \cong PSL_2(q^2)$ have been already considered.

If $m \geq 3$, all groups $P\Omega_{2m}^\pm(q)$ are simple. We already know the answer for $P\Omega_6^+(q) \cong PSL_4(q)$ and $P\Omega_6^-(q) \cong PSU_4(q^2)$. Thus, it is remaining to consider the groups with $m \geq 4$.
**Theorem 4.** Let $G$ be one of groups $\mathrm{PO}_m^\pm(q)$, $\Omega_m^\pm(q)$, $\mathrm{SO}_m^\pm(q)$, $\mathrm{GO}_m^\pm(q)$, where $m \geq 4$, and let $p$ be a prime dividing the order of $G$. Then $G \notin \mathcal{X}_p$.

**Proof.** As usual, we exclude the case $p = 2$, and also when $p$ divides $q$ or $q - 1$. Thus, we may assume that $d \geq 2$, and a Sylow $p$-subgroup $P$ of $G$ is cyclic and non-trivial.

First, assume that $q$ is odd. By Fact 8, $\tau_0(\mathrm{CSO}_n^+(q))$ is a line. By Lemma 1, the Brauer trees of the principal blocks of $\mathrm{SO}_n^+(q)$ and $\Omega_n^+(q)$ are similar to $\tau_0(\mathrm{CSO}_n^+(q))$, hence they are lines, too. Moreover, $\tau_0(\mathrm{GO}_n^+(q)) = \tau_0(\mathrm{SO}_n^+(q))$, because they have the same number of edges. Also $\tau_0(\mathrm{PO}_n^+(q))$ coincide with $\tau_0(\Omega_n^+(q))$ because $Z \subseteq O_p(\Omega_n^+(q))$. Therefore, $e_0(\mathrm{GO}_n^+(q)) > 4$ implies $e_0(\mathrm{PO}_n^+(q)) > 2$.

Now assume that $q$ is even. Then $\mathrm{PO}_n^+(q) \cong \Omega_n^+(q)$. By Fact 8, the Brauer tree of the principal block of $\mathrm{GO}_n^+(q) \cong \mathrm{SO}_n^+(q)$ is a line. And, by Fact 1, the tree $\tau_0(\Omega_n^+(q))$ is similar to $\tau_0(\mathrm{GO}_n^+(q))$.

Thus, for both odd and even $q$, it suffices to show that the number of edges in $\tau_0(\mathrm{GO}_n^+(q))$ exceeds 4.

We first consider the group $G = \mathrm{GO}_n^+(q)$. Since a Sylow $p$-subgroup $P$ of $G$ is cyclic, it gives that $d \leq 2m - 2$. According to [36, Table 1], we have the following possibilities for $P$.

(1.1) *Odd $d$. Then $P$ can be chosen in $\GL_m(q)$. Since $P$ is non-trivial and cyclic, we obtain $m/2 < d \leq m$. Using the diagonal embedding $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ from $\GL_m(q)$ into $\GL_2^m(q)$, we obtain $e \geq d$.

If $m \geq 8$, then $e > 4$, as desired. Since $e \geq d$, it remains to consider only cases when $d = 3$ and $m \in \{4, 5\}$. By Lemma 7, the shape of $\tau_0(\mathrm{GO}_n^+(q))$ depends only on $d$ and $m$. Taking $p = 7$ and $q = 2$, we have found with GAP that $e = 6$ for both $m \in \{4, 5\}$.

(1.2) *Even $d$ is even*. The inequality $d \leq 2m - 2$ implies that the integer part of the fraction $d/2m$ equals zero; hence, $P$ can be chosen as a subgroup of $\GL_2^m(q)$. Since $P$ is cyclic and non-trivial, we obtain $m < d$. Now, using [36, Lemma 4.6], we get $e \geq d$. Then $d > 4$ yields $e > 4$, as desired.

Now consider $G = \mathrm{GO}_n^-(q)$. Since a Sylow $p$-subgroup $P$ of $G$ is cyclic, we have $2 \leq d \leq 2m$. Notice that $d = 2m$ may occur, when $p \mid q^n + 1$. We will consider possibilities for $P$, according to [36, Table 1].

(2.1) *Odd $d$. Then $P$ can be chosen as a subgroup of $\GL_{m-1}(q)$. Since $P$ is cyclic, we conclude that $(m - 1)/2 < d$. By [36, Lemma 4.6], we have $e \geq d$.

If $m \geq 7$, then $e \geq d \geq 5$, as desired. We should consider only $m \in \{4, 5, 6\}$ and $d = 3$. In all these cases, we have found with GAP that $e = 6$.

(2.2) *Even $d$. This case splits into two subcases.*

If the integer part of $d/2m$ is odd, then $d \leq 2m$ yields $d = 2m$. From [36, Table 1] we see, that $P$ can be chosen as a subgroup of $\GL_{2m}(q)$. Since $P$ is cyclic, we conclude that $m < d$ yields $d \geq 5$. Using [36, Lemma 4.6], we conclude that $e \geq d \geq 5$, as desired.

Otherwise, the integer part of $d/2m$ is zero. Hence, $P$ can be chosen as a subgroup of $\GL_{2m-2}(q)$. Since $P$ is cyclic, we get $m - 1 < d$. If $m > 4$, it follows from [36, Lemma 4.6] that $e \geq d > 4$.

For the remaining case when $m = d = 4$, we have found with GAP that the number of edges in the principal blocks of both groups $\mathrm{GO}_8^-(q)$ and $\mathrm{PO}_8^-(q)$ are equal 4. This completes the proof. \qed
5. Exceptional groups of Lie type

In this section, we consider the finite exceptional groups of Lie type, namely $E_6$, $E_7$, $E_8$, $F_4$, $G_2$ and twisted groups $^2B_2$, $^3D_4$, $^2E_6$, $^2F_4$, $^2G_2$. The last two types are called Ree groups. The groups of the type $^2B_2$ are known as Suzuki groups.

All these groups are simple except $G_2(2)$ and $^2F_4(2)$. The group $G_2(2)$ has order $2^6 \cdot 3^3 \cdot 7$, and its derived subgroup $G_2(2)' \cong \text{PSU}(3, 3)$ is simple of order $2^2 \cdot 3^3 \cdot 7$.

The group $^2F_4(2)$ has the derived subgroup $^2F_4(2)'$ which is simple and called the Tits group.

**Proposition 9.** Let $G = ^2F_4(2)'$ and $p$ divides $|G|$. Then $G \not\in \mathcal{X}_p$.

**Proof.** The order of $G$ is $2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$. Since $G$ contains maximal subgroups $\text{PSL}(3, 3).C_2$ and $\text{PSL}(2, 25)$ (see [7, p. 74]), we conclude that a Sylow $p$-subgroup of $G$ is not cyclic for $p = 2, 3, 5$.

If $p = 13$, then it follows from decomposition matrices [2] that $\tau_0(G)$ is not a star. □

**Proposition 10.** Let $G$ be a simple group of any type $^3D_4$, $E_6$, $^2E_6$, $E_7$, $E_8$, $F_4$, $^2F_4$ or $G_2$. Then $G \not\in \mathcal{X}_p$ for any $p$ dividing the order of $G$.

**Proof.** Assume that $G \in \mathcal{X}_p$ for some $p$ dividing $|G|$. Then Sylow $p$-subgroup $P$ of $G$ is cyclic and $p \nmid q$. Let $d = \text{ord}_p(q)$. In particular, $d$ divides $p - 1$. We will consider groups from the list one by one.

1. Let $G = ^3D_4(q)$, where $q$ is a prime power. It has the order

$$|^3D_4(q)| = q^{12}(q^2 - 1)^2(q^4 - q^2 + 1)(q^4 + q^2 + 1)^2.$$

Since $P$ is cyclic, it follows from [10, Prop. 5.6c and Table 1.1] that $P$ is a subgroup of the maximal torus $T_5 = C_{q^4 - q^2 + 1}$, i.e. $p$ divides $q^4 - q^2 + 1$. In this case according to [17, p. 3265], the Brauer tree of $B_0$ is a line with 4 edges; hence, $\tau_0(G)$ is not a star, a contradiction.

2. Let $G = E_6(q)$. Looking at the order

$$|E_6(q)| = \frac{1}{(3, q - 1)} \cdot q^{36}(q^2 - 1)(q^5 - 1)(q^6 - 1)(q^8 - 1)(q^9 - 1)(q^{12} - 1),$$

we see that the possible values for $d$ are $1, \ldots, 6, 8, 9, 12$.

If $p = 3$, then $p$ divides either $q + 1$ or $q - 1$. According to [9, p. 897], $G$ contains the maximal torus $C_{q + 1} \times C_{q - 1}$, so Sylow 3-subgroups of $G$ are not cyclic.

Assume now that $p > 3$. If $d \in \{1, 6\}$, then by [24, Theorem 3.1] there are no unipotent blocks with cyclic defect group. Note that the principal $p$-block is always unipotent, and its cyclic defect group is a Sylow $p$-subgroup of $G$.

For $d \in \{3, 4, 5, 8, 9, 12\}$, according to [24, Theorem 3.1], there are no unipotent blocks whose Brauer tree is a star.

If $d = 2$, then $p \mid q + 1$. In this case, $P$ is not cyclic because $G$ contains a subgroup which is isomorphic to $C_{q + 1} \times C_{q + 1}$.

3. Let $G = ^2E_6(q)$. Since

$$|^2E_6(q)| = \frac{1}{(3, q + 1)} \cdot q^{36}(q^2 - 1)(q^5 + 1)(q^6 - 1)(q^8 - 1)(q^9 + 1)(q^{12} - 1),$$

the possible values for $d$ are $1, 2, 3, 4, 6, 8, 10, 12, 18$. 

13
Again we first consider the case \( p > 3 \). If \( d \in \{2, 3\} \), by [25, Theorem 2.2] the group \( G \) has no unipotent blocks with cyclic defect group. If \( d \in \{4, 6, 8, 10, 12, 18\} \), then there are no unipotent blocks whose Brauer tree is a star and whose defect group is cyclic.

Finally, assume that \( d = 1 \), i.e. \( p | q - 1 \). According to [9, p. 903], the maximal tori of \( G \) can be obtained from the corresponding list for \( E_6(q) \) by a formal substitution \( q \mapsto -q \).

It follows from [9, p. 897] that \( G \) contains the maximal torus \( T_{11} = C_{q^2-1} \times C_{q^4-1} \). Therefore, \( P \) is not cyclic.

Thus, it remains to consider the case \( p = 3 \). Since \( q \equiv \pm 1 \ (\text{mod } 3) \), we conclude that \( P \) is not cyclic by considering the same torus as above.

(4) Let \( G = E_7 \). Since

\[
|E_7(q)| = \frac{1}{(2, q - 1)} \cdot q^{63}(q^2 - 1)(q^6 - 1)(q^8 - 1)(q^{10} - 1)(q^{12} - 1)(q^{14} - 1)(q^{18} - 1),
\]

the possible values for \( d \) are \( 1, \ldots, 10, 12, 14, 18 \).

If \( d \in \{5, 7, 8, 9, 10, 12, 14, 18\} \), then it follows from [18, Theorem 12.6, and remark on p. 2970] that all unipotent blocks with cyclic defect group have Brauer tree in the shape of a line of length \( e \geq 4 \).

For remaining values of \( d \), considering the maximal tori of \( G \), it is easy to see that a Sylow \( p \)-subgroup of \( G \) is not cyclic. Indeed, we can take the torus \( T_{10} = C_{q-1} \times C_{q^2-1} \) (see [9, p. 898]) for \( d \in \{1, 3\} \), the torus \( T_6 = C_{q-1} \times C_{q+1} \times C_{q^2-1} \) for \( d = 2 \), and \( T_{28} = C_{(q-1)(q^2+1)} \times C_{q^2-1} \times C_{q^2+1} \) for \( d = 4 \). Finally, for \( d = 6 \) replacing \( q \) by \(-q\) in \( T_{10} \), we see that the group contains the torus \( C_{q+1} \times C_{q^2+1} \).

Note that the case \( p = 3 \), when either \( p | q + 1 \) or \( p | q - 1 \), is already included into consideration above.

(5) Let \( G = E_8 \). The order is

\[
|E_8(q)| = q^{120}(q^2 - 1)(q^8 - 1)(q^{12} - 1)(q^{14} - 1)(q^{18} - 1)(q^{20} - 1)(q^{24} - 1)(q^{30} - 1).
\]

Hence the possible values for \( d \) are \( 1, \ldots, 10, 12, 14, 15, 18, 20, 24, 30 \).

If \( d \in \{7, 9, 14, 15, 18, 20, 24, 30\} \), then it follows from [18, Thm. 12.7] that the Brauer trees of all unipotent blocks with cyclic defect group are not stars.

For remaining values of \( d \), a Sylow \( p \)-subgroup of \( G \) is not cyclic because \( G \) contains the following tori (see [9, p. 899–901]): \( T_{34} = C_{(q+1)(q^2-1)} \) for \( d = 1, 2, 3 \); \( T_{36} = C_{q^2-1} \) for \( d = 4 \); \( T_{57} = C_{q^4+q^3+q^2+q+1} \) for \( d = 5 \); \( T_{61} = C_{q^4+1} \) for \( d = 8 \); and \( T_{67} = C_{q^3-q^2+1} \) for \( d = 12 \). For \( d = 6 \), we can take the torus \( C_{(q^4-q^2+1)(q^2-q-1)} \times C_{q^2-q+1} \) that obtained from \( T_{62} \) by plugging \( q \mapsto -q \). And for \( d = 10 \), we obtain the torus \( C_{q^4-q^3+q^2-q+1} \) from \( T_{57} \) by the same substitution.

(6) Let \( G = F_4(q) \). Then

\[
|F_4(q)| = q^{24}(q - 1)^4(q + 1)^4(q^2 - q + 1)(q^2 + 1)^2(q^2 + q + 1)(q^4 - q^2 + 1)(q^4 + 1)(q^4 + q^2 + 1),
\]

the possible values for \( d \) are \( 1, 2, 3, 4, 6, 8, 12 \).

First assume that \( p > 3 \). If \( d \in \{1, 2, 3, 6\} \), then it follows from [25, Theorem 2.1] that there are no unipotent blocks with cyclic defect group. If \( d \in \{4, 8, 12\} \), then by the same reference the Brauer tree of unipotent blocks is not a star.
Now suppose that \( p = 3 \). It follows from [19, Table 4.7.3] that \( G \) has three conjugacy classes of 3-elements. Hence, Sylow 3-subgroups of \( G \) are not cyclic.

(7) Let \( G = 2F_4(q^2) \), where \( q^2 = 2^{2n+1} \), \( n > 0 \). Then \( G \) is a simple group of the order
\[
q^{24}(q^2 - 1)^2(q^2 + 1)^2(q^4 - q^2 + 1)(q^4 + 1)^2(q^4 \pm \sqrt{2}q^3 + q^2 \pm \sqrt{2}q + 1).
\]

If \( P \) is cyclic, then it follows from [23, Section 4.2] that \( p > 3 \) and \( p \) divides \( q^4 - q^2 + 1 \) or \( q^4 \pm \sqrt{2}q^3 + q^2 \pm \sqrt{2}q + 1 \). According to [23, Theorems 4.5–4.7], in each of these cases \( \tau_0(G) \) is not a star.

(8) Let \( G = G_2(q) \), where \( q \geq 3 \). This group is simple, and its order is
\[
|G_2(q)| = q^6(q^2 - 1)(q^2 - q + 1)(q^2 + q + 1).
\]

If \( p \) divides \( q \pm 1 \), then the Sylow \( p \)-subgroup of \( G \) is not cyclic, because \( G \) contains the tori \( C_2^2 \) (see [34, p. 1902]).

Suppose that \( p \) divides \( q^2 \pm q + 1 \). If \( p \neq 3 \), then by [35, p. 380–381] \( \tau_0(G) \) is not a star.

Now assume that \( p = 3 \). If \( 3 \) divides \( q^2 + q + 1 \), then \( q \equiv 1 \pmod{3} \), and this case has been already considered above. Similarly, \( 3 | q^2 - q + 1 \) leads to \( 3 | q + 1 \).

\[ \Box \]

Now consider the series of groups \( 2G_2(q^2) \), where \( q^2 = 3^{2n+1} \). They are simple if and only if \( n \geq 1 \). The order is
\[
|2G_2(q^2)| = q^6(q^2 - 1)(q^2 + 1) = q^6(q^2 - 1)(q^2 + 3q + 1)(q^2 - \sqrt{3}q + 1).
\]

**Proposition 11.** Let \( G = 2G_2(q^2) \), where \( q^2 = 3^{2n+1} \), \( n \geq 1 \). Then \( G \in \mathcal{X}_p \) if and only if \( p > 2 \) and \( p \) divides either \( q^2 - 1 \) or \( q^2 + \sqrt{3}q + 1 \).

**Proof.** Again we may assume that \( p \nmid q \) and \( p > 2 \). According to [23, Theorems 4.1-4.4], if \( p \) divides \( q^2 + 1 \) or \( q^2 - \sqrt{3}q + 1 \), then \( P \) is cyclic and \( \tau_0(G) \) is a star. If \( p \) divides \( q^2 + \sqrt{3}q + 1 \) or \( q^2 - 1 \), then \( P \) is cyclic, but \( \tau_0(G) \) is not a star.

\[ \Box \]

Finally, we will consider the Suzuki groups \( 2B_2(q) \).

**Proposition 12.** Let \( G = 2B_2(q) \), where \( q = 2^{2n+1} \), \( n \geq 1 \), and let \( r = 2^{n+1} \). Then \( G \in \mathcal{X}_p \) if and only if \( p \) divides either \( q - 1 \) or \( q + r + 1 \).

**Proof.** Since \( r^2 = 2q \), the order of \( G \) can be written as
\[
|G| = q^2(q^2 + 1)(q - 1) = q^2(q + r + 1)(q - r + 1)(q - 1).
\]

It is known that \( |G| \) is not divisible by 3, and that a Sylow \( p \)-subgroup of \( G \) is cyclic for \( p > 3 \). The tree \( \tau_0(G) \) is described in [6]. It is a star when \( p \) divides either \( q - 1 \) or \( q + r + 1 \), and it is not a star when \( p \) divides \( q - r + 1 \).

\[ \Box \]

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