The Wave-Mechanical Propagation Way

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Abstract. An exact and intuitive picture of point-particle dynamics is allowed by the Hamiltonian ray-tracing system associated with the time-independent Schrödinger equation, starting from any wave-front assignment. Matter waves may be viewed as a general property of moving particles.

1 - Introduction

Let us begin the present considerations by quoting the starting lines of a paper written by Louis de Broglie in 1959 [1], 20 years after the Nobel Prize rewarding his foundation of Wave Mechanics:

In my first works on Wave Mechanics, dating back to 1923, I had clearly perceived that it was necessary, in a general way, to associate with the movement of any corpuscle the propagation of a wave. But the homogeneous wave that I had been led to consider, and that became the wave \( \psi \) of the usual wave mechanics, did not seem to me to describe the physical reality (...). Giving no particular prerogative to any point in space, it was not capable of representing the position of the corpuscle: we could suppose at most, as was very shortly done, that it gave, by its square, the "probability of presence" of the corpuscle in each point.

2 – De Broglie’s Ansatz

In de Broglie’s Wave Mechanics [2] (including Schrödinger’s contribution [3]), any point-particle of mass \( m \), launched with momentum \( p_0 \) and total energy \( E = p_0^2 / 2m \) into a force field deriving from a stationary potential energy \( V(r) \), is assumed to be associated with a stationary “matter wave”

\[
\psi(r,E) = R(r,E) e^{i \varphi(r,E)},
\]

with real amplitude \( R(r,E) \), phase \( \varphi(r,E) \) and wave-vector

\[
\mathbf{k} = \nabla \varphi.
\]

The wave-like behavior of matter is viewed, in this perspective, as a general property of any moving particle, according to de Broglie’s founding Ansatz

\[
p = \hbar \mathbf{k},
\]

having the threefold role of:

1) addressing the particle momentum along the field-lines of the wave-vector \( \mathbf{k} \) of the matter wave,

2) specifying the value of the relevant wavelength \( \lambda = 2\pi / k = 2\pi \hbar / p \) : a value which was immediately verified by the Davisson-Germer experiments on electron diffraction by a crystal of Nickel [4, 5], and

3) establishing that particle trajectories (coinciding with the ray-trajectories of the relevant matter waves) do exist, contrary to widespread different opinions.
Recalling that any Helmholtz-like equation is associated with an exact Hamiltonian system of ray-trajectories [6-11], the behavior of matter waves may be described, thanks to eq. (3), in terms of a suitable Helmholtz-like equation, obtained under the condition that the Classical Dynamics of the particles correspond to the Geometrical Optics of their matter waves. The function \( u(r,E) \) turns out to solve, within this scheme, the Helmholtz-like time-independent Schrödinger equation

\[
\nabla^2 u(r,E) + \frac{2m}{\hbar^2} [E-V(r)] u(r,E) = 0. \tag{4}
\]

By introducing the function (1) into eq. (4) and separating real from imaginary parts, the associated Hamiltonian system of ray-trajectories (along which the particle motion is driven) is obtained in the form

\[
\begin{align*}
\frac{dr}{dt} &= \frac{\partial H}{\partial p} = \frac{p}{m} \\
\frac{dp}{dt} &= -\frac{\partial H}{\partial r} = -\nabla [V(r) + W(r,E)] \\
\nabla \cdot (R^2 p) &= 0 \\
|p(t=0)| &\equiv p_0 = \sqrt{2mE}
\end{align*}
\]

[\text{where}]

\[
W(r,E) = -\frac{\hbar^2}{2m} \frac{V^2(R,E)}{R(r,E)} \tag{6}
\]

[\text{and}]

\[
H(r,p,E) = \frac{p^2}{2m} + W(r,E) + V(r). \tag{7}
\]

The system (5) is seen to reduce to standard Classical Dynamics in the absence of the trajectory-coupling “Wave Potential” term \( W(r,E) \). Another peculiar property of the function \( W(r,E) \) is to act perpendicularly to the particle momentum, leaving therefore its energy unchanged. Notice that the easily tractable, energy-dependent Wave Potential \( W(r,E) \) has little to share with Bohm’s almost intractable Quantum Potential \( Q(r,t) \) [12], mixing together the whole set of energy eigen-values of eq. (4).

In the relativistic case Eq. (4) is replaced [8-11] by a suitable (Helmholtz-like) Klein-Gordon equation, associated with an analogous trajectory system, holding even in case of vanishing rest mass.

On the wave-like side of the wave-mechanical context, the launching wave amplitude \( R(r,E,t=0) \) coincides with the particle distribution over an assigned surface, representing the wave-front at \( t=0 \).

\text{The particle side of the context will require, in turn, the assignment of the particle mass} \( m \), \text{energy} \( E \) \text{and momentum amplitude} \( p(E,t=0) = \sqrt{2mE} \), \text{together with the momentum distribution} \( p(E,t=0) \) \text{normal to the launching front. The trajectory system will be basically determined by the form and extension of the launching wave-front. The third equation of the system (5) (expressing the constancy of the flux of the vector field} \( R^2 p \) \text{along any tube formed by the trajectories) shall be employed in the time integration of the system (5) in order to obtain, at each time-step, the relevant wave amplitude} \( R(r,E,t) \) \text{and Wave Potential function} \( W(r,E,t) \) \text{encountered by the moving particle on the next wave-front.}
Contrary to the popular narrative, but in agreement both with the Davisson-Germer experiment [4] and with the “ontologic parsimony” expressed by Ockham’s Razor [13], the wave-mechanical “duality” does not predict a wave travelling with the particle. It predicts a stationary wave (solving eq.(4) in the wave-mechanical context) normally to whose wave-fronts classical-looking point-particles are addressed, without any energy exchange. Once the launching context is chosen, the time-integration of the wave-mechanical system (5) provides a simple solution of de Broglie’s problem, without resorting to any kind of probabilistic or hydrodynamic interpretation. We shall obtain, indeed, a stationary pattern of virtual trajectories stemming (each one with its time-table) from the assigned launching front, and mutually coupled, at each point of space, by the local value of the Wave Potential. Both the trajectory and the time-table of the considered particle shall be simply picked up from the overall display of numerical results by selecting its launching position and time. In the absence of wave-mechanical coupling the trajectory pattern and motion law would clearly reduce to Classical Mechanics.

Let us expound the relevant physics in detail, by means of a couple of simple examples of wave-mechanical contexts. Assuming, for simplicity sake, a geometry limited to the $(x, z)$-plane, we consider, to begin with, the diffraction of a single particle launched along the z-axis, in the absence of any external field $V(\mathbf{r})$, through a slit of half-width $w_0$ (centered at $x=0, z=0$) practiced on a screen placed along the x-axis. A particle launched with momentum components $p_x(t=0)=0$ and $p_z(t=0)=p_0=\sqrt{2mE}$ shall be guided by a matter wave with launching wavelength $\lambda_0=2\pi \hbar / \sqrt{2mE}$. In the absence of any external field, and under the rule of the energy-preserving Wave Potential alone, we’ll have $\lambda = \lambda_0$ along the entire particle motion; and let us assume, say, a ratio $\lambda_0 / w_0 = 10^{-4}$. We shall approximate the launching amplitude distribution of the matter wave by a Gaussian function of the form $R(x;z=0) \propto \exp(-x^2 / w_0^2)$, somewhat exceeding the width of the slit, in order to avoid most numerical difficulties, while respecting the physical content of our “experiment”. The numerical solution of the Hamiltonian system (5) provides both a stationary virtual pattern of coupled trajectories starting from the positions with co-ordinates $-w_0 \leq x \leq w_0; \ z = 0$, and the particle time-table along them.
The full virtual pattern is plotted in Fig.1 in terms of the dimensionless co-ordinates \( x/w_0 \) and \( z/w_0 \); and the two trajectories starting from \( x/w_0 = \pm 1 \) turn out to be in excellent numerical agreement with their well-known "paraxial" [14] analytical expression

\[
\frac{x}{w_0} = \pm \sqrt{1 + \left( \frac{\lambda_0 z}{\pi w_0^2} \right)^2}.
\] (8)

The analogous case of two neighbouring (Gaussian) slits is plotted in Fig.2, where a virtual pattern of coupled trajectories is seen to be launched from the \( x \)-axis. In any case, there is no “Feynman mystery” [15] in the particle diffractive behaviour, which is simply due to the choice of a bounded launching wave-front.

The dramatic context-dependence of the particle trajectories is clearly shown in Fig.3, where an external potential field \( V(x,z) \) representing a lens-like focalizing structure [16] is taken into account in eqs.(4) and (5). The point-like focus obtained in the Geometrical Optics limit (Black trajectories) by dropping the Wave Potential term \( W(r,E) \) from the equation system (5), is replaced by a finite focal waist when the diffractive role of the Wave Potential is duly taken into account (red trajectories).

Fig.3. Lens-like potential

Black trajectories: point-like focusing when the Wave Potential term is neglected.
Red trajectories: finite focal waist when the Wave Potential term is taken into account.

3 – Discussion

The present paper stresses the peculiar role of the time-independent Schrödinger equation (4) in the matter wave propagation mode.
1) Besides providing the main items of the whole quantum theory, such as energy eigen-values and harmonic oscillators [5],
2) this equation lends itself to describe, *thanks to its Helmholtz-like nature*, the wave-like particle motion along a *virtual* pattern of coupled trajectories, determined by the launching conditions, under the diffractive and energy preserving action of Wave Potential. Moreover,
3) *thanks to its energy-dependence*, it presents the **wave-mechanical dynamics** as a direct extension of **Classical Dynamics**, including a peculiar force term, \( \nabla W(r,E) \), of wave-mechanical origin.

And what about the **time-dependent** Schrödinger equation? Under Planck's condition

\[ E = \hbar \omega \quad (9) \]

let us refer to a discrete set of eigen-energies \( E_n \), eigen-frequencies \( \omega_n = E_n / \hbar \) and eigen-solutions \( u_n(r) \) of eq.(4), and “complete” the matter wave (1) in the form

\[ \psi_n(r, t) = u_n(r) e^{-i \omega_n t} \equiv u_n(r) e^{-i E_n t / \hbar} \quad (10) \]

Any linear superposition, now, of the form

\[ \psi(r,t) = \sum_n c_n \psi_n(r,t) \quad (11) \]

with arbitrary constant coefficients \( c_n \), turns out to be a solution of the (so-called) **time-dependent Schrödinger equation** [5], *directly obtained from eqs. (4) and (10)*:

\[ i \hbar \frac{\partial \psi}{\partial t} = - \frac{\hbar^2}{2m} \nabla^2 \psi + V(r) \psi \quad (12) \]

in itself, NOT a standard-looking wave equation. The insertion of the function (11) into eq.(12) puts it in the form

\[ \sum_n c_n e^{-i E_n t / \hbar} \left\{ - \frac{\hbar^2}{2m} \nabla^2 u_n(r) + \frac{2m}{\hbar^2} (E_n - V(r)) u_n(r) \right\} = 0 \quad (13) \]

where each term of the sum is of the form (4). Eq. (13) provides the closest approach between the two Schrödinger equations (4) and (12). At this point, two different routes are followed in the way of dealing with quantum problems:

1) **Standard Quantum Mechanics** (SQM) is centered on eq.(12), which is taken, in spite of its counter-intuitive structure, as a **Primary Assumption** [17]. The function (11) is conjectured to carry the most complete physical information, ranging over the full set of particle eigen-energies, and collapsing into one of them in case of observation. A “probability current density” is obtained in the form

\[ J = \frac{\hbar}{2mi} (\psi^* \nabla \psi - \psi \nabla \psi^*) \quad (14) \]

where \( \psi \psi^* \equiv |\psi|^2 \equiv R^2 \), and no definite particle trajectory is admitted, in an odd-looking “scrambling of ontic and epistemic contextualities” [18, 19].
2) The Bohmian approach [12, 20] conjectures, in turn, a “guidance velocity” provided by a mixture of dynamics and hydro-dynamics of the form

$$\frac{d r(t)}{dt} = J / R^2 = \frac{\hbar}{2 m i} \frac{\psi^* \nabla \psi - \psi \nabla \psi^*}{\psi^* \psi}, \tag{15}$$

to be evaluated in parallel with the numerical solution of the time-dependent eq.(12), and addressing the particle motion along the flux lines "along which the probability density is transported" [20].

In both cases, the exact dynamics allowed by eqs.(3) - (5) is replaced by a probabilistic description based on eq.(12), and the (experimentally well established) role of de Broglie’s monochromatic matter waves [1-4] is, to say the least, overshadowed. SQM and Bohmian Mechanics, in conclusion, stemming from the time-dependent eq.(12), appear to be based on heavily conjectural foundations, and to tell us, with respect to the Wave Mechanics stemming from the time-independent eq.(4), a quite different story.

Let us notice that both the assumption of de Broglie’s Ansatz in the (popular but incomplete) scalar form $\lambda = 2\pi \hbar / p$ (rather than in the complete vectorial form (3)) and the assumption of eq.(12) as a (strongly counter-intuitive) starting stage have contributed to the widespread reputation of Quantum Mechanics (QM) as a scientific mystery.

Indeed, “there remain several features of QM that resist an intuitive explanation (…). It is an open question whether unexpected future developments will produce an explanation of these features, or they will just become familiar by getting used to it (…). After all, “understanding” is a human mental state conditioned by our brain that has reached its state after (...) a few million years of evolution, but might not necessarily be adequate for the microscopic world.”[21].

References

[1] de Broglie, L. 1959 L’interprétation de la Mécanique Ondulatoire, Le Journal de Physique et le Radium, 20, 963-979. Mc Graw-Hill Company, Inc.
[2] de Broglie, L. 1924 Recherches sur la théorie des quanta. Thèse de doctorat.
[3] Schrödinger, E. 1926 Quantisierung als Eigenwertproblem I and II, Annalen der Physik, 79, 361 and 489
[4] Davisson, C. J., and Germer, L. H. 1927 The Scattering of Electrons by a Single Crystal of Nickel, Nature 119, 558.
[5] Messiah, A. 1959 Mécanique Quantique, Dunod
[6] Orefice, A., Giovanelli, R., Ditto, D. 2009 Complete Hamiltonian Description of Wave-Like Features in Classical and Quantum Physics, Foundations of Physics 39, 256.
[7] Orefice, A., Giovanelli, R., Ditto, D. 2013 A non-probabilistic insight into Wave Mechanics, Annales de la Fondation Louis de Broglie 38, 7.
[8] Orefice, A., Giovanelli, R., Ditto, D. 2015 From Classical to Wave-Mechanical Dynamics, Annales de la Fondation Louis de Broglie 40, 95.
[9] Orefice, A., Giovanelli, R., Ditto, D. 2015 Is wave mechanics consistent with classical logic? Physics Essays 28, 515.
[10] Orefice, A., Giovanelli, R., Ditto, D. 2018 The Dynamics of Wave-Particle Duality, Journal of Applied Mathematics and Physics, 6, 1840.
[11] Orefice, A., Giovanelli, R., Ditto, D. 2018 Primary Assumptions and Guidance Laws in Wave Mechanics, Journal of Applied Mathematics and Physics, 6, 2621
[12] Bohm, D. J., 1952 A Suggested Interpretation of the Quantum Theory in Terms of Hidden Particles. Phys. Rev. 85, 166 and 180.
[13] Baker, A. (2004) Simplicity, Stanford Encyclopedia of Philosophy.
[14] Kogelnik, H., and Li, T. 1966 Laser Beams and Resonators, Applied Optics 5, 1550
[15] Feynman, R. P. 1965 The Character of Physical Law, BBC Publications.
[16] Nowak, S., and Orefice, A. 1993 Quasi-optical treatment of e.m. Gaussian beams in inhomogeneous and isotropic plasmas, Phys.Fluids B 5 (7), 1945
[17] Pauling, L. and Wilson Jr., E.B. 1935 Introduction to Quantum Mechanics, McGraw-Hill Company, Inc., New York, London
[18] Jaynes, E.T. 1990 Complexity, Entropy and the Physics of Information, ed. by Zurecks, W., H., Addison-Wesley.
[19] de Ronde, C. 2017, Unscrambling the Omelette of Quantum Contextuality (Part I): Pre-Existent Properties or Measurement Outcomes?, arXiv:1606.03967v4
[20] Dürr, D., and Teufel, S., 2009 Bohmian Mechanics, Springer-Verlag
[21] de la Torre, A. C. 2016, Do we finally understand Quantum Mechanics?, arXiv:1605.00672 v2