ON THE RELATION BETWEEN DENJOY–KHINTchine AND HK_r-INTEGRALS

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Abstract. We locate Musial & Sagher’s concept of HK_r-integration within the approximate Henstock–Kurzweil integral theory. If to restrict HK_r-integral by requirement that the indefinite HK_r-integral is continuous, then it is included even in the classical Denjoy–Khintchine integral. We provide a direct argument demonstrating that this inclusion is proper.

Keywords: L^r-derivative, HK_r-integral, variational measure, Denjoy–Khintchine integral

1. Introduction

The concept of derivation defined with L_r integral means has been applied to build a counterpart of Perron integration theory by Louis Gordon in [1, 2]. An analogous approach to Riemann integration was introduced and investigated much later in a work by Musial and Sagher [3], related to the first author’s PhD dissertation [4]. Despite a connection between these Perron and Riemann (Henstock–Kurzweil) type integrals has been established in [3, 4] (L^r-Perron integrability implies HK_r-integrability), a true impetus the topic has received only very recently, with a surprising result (co-authored by the first author of the present note) that the HK_r-integral is strictly more general than the L^r-Perron integral, contradicting an intuition from classical theory; see also [5, 6, 7, 8, 9].

In the present note we locate Musial & Sagher’s concept of HK_r-integration within another counterpart of the Henstock–Kurzweil integral, the so-called approximate Henstock–Kurzweil integral (AH-integral). Having shown here that HK_r-integral is included into AH-integral (Corollary 12), we obtain that it is covered by the integrals of a numerous family of approximately continuous integrals (see [10]), in particular, it is included in the Kubota integral (see [11, 12]). Moreover, if to restrict HK_r-integral by requirement that the indefinite HK_r-integral is continuous, then it is included even in the classical Denjoy–Khintchine (aka wide Denjoy or D-) integral. We provide a direct argument demonstrating that this inclusion is proper. In fact our construction shows that the HK_r-integral does not cover even the Khintchine integral which is more narrow than the D-integral.

2. Terminology and notation

2.1. Riemann approach terminology. We work in a fixed segment (compact interval) [a, b] ⊂ R (in some cases we shall use an alternative notation (a, b) for a subsegment of R meaning that any of the points a and b can be on the left to the other one). A tagged interval is a pair (I, x) where I ⊂ [a, b] is a segment and x ∈ I (its tag). We say that a tagged interval (I, x) is tagged in a set E ⊂ [a, b]
exists a gauge \( \delta \)

A definition some related notions from [1].

Consider the respective Lebesgue spaces [2].

In common. A division is said to be tagged in \((I, x)\) if there exists a real number \(\alpha\) \((x - \delta(x), x + \delta(x))\).

A division is a finite collection \(\pi\) of tagged intervals, where distinct elements \((I', x')\) and \((I'', x'')\) in \(\pi\) have \(I'\) and \(I''\) non-overlapping, i.e., without inner points in common. A division is said to be tagged in a set/\(\delta\)-fine if all its elements have the respective property.

2.2. \(L^r\) related notions. Throughout this paper we assume that \(r \geq 1\) and consider the respective Lebesgue spaces \(L^r[a, b]\). \(\mu\) denotes the Lebesgue measure on \(a, b\). We recall the definition of the \(L^r\)-Henstock–Kurzweil integral given in [3] and some related notions from [1].

Definition 1. A function \(f: [a, b] \rightarrow \mathbb{R}\) is said to be \(L^r\)-Henstock–Kurzweil integrable (HK\(_r\)-integrable) if there exists a function \(F \in L^r[a, b]\) such that for any \(\varepsilon > 0\) there exists a gauge \(\delta\) with the property that for any \(\delta\)-fine division \(\pi = \{(c_i, d_i, x_i)\}_{i=1}^{q}\),

\[
\sum_{i=1}^{q} \left( \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} < \varepsilon.
\]

By [3, Theorem 5], the function \(F\) in Definition 1 is unique up to an additive constant, so, putting \(F(a) = 0\), we can consider the indefinite HK\(_r\)-integral

\[
F(x) = (\text{HK}_r) \int_a^x f, \quad x \in (a, b].
\]

It can be easily checked that the value of HK\(_r\)-integral does not depend on the values of the function \(f\) on a set of measure zero (provided only finite values of \(f\) are considered).

Definition 2. A function \(F \in L^r[a, b]\) is said to be \(L^r\)-continuous at \(x \in [a, b]\) if

\[
\lim_{h \to 0} \frac{1}{h} \int_{-h}^{h} |F(x + t) - F(x)|^r \, dt = 0.
\]

If \(F\) is \(L^r\)-continuous at all \(x \in E\), we say that \(F\) is \(L^r\)-continuous on \(E\).

Definition 3. A function \(F \in L^r[a, b]\) is said to be \(L^r\)-differentiable at \(x\), if there exists a real number \(\alpha\) such that

\[
\left( \frac{1}{h} \int_{-h}^{h} |F(x + t) - F(x) - \alpha t|^r \, dt \right)^{1/r} = o(h).
\]

In this case we say that \(\alpha\) is the \(L^r\)-derivative of \(F\) at \(x\) and write \(F'_r(x) = \alpha\).

It was proved in [3] that if \(F\) is the indefinite HK\(_r\)-integral of \(f\) then \(F\) is \(L^r\)-continuous on \([a, b]\), and \(F'_r\) exists and is equal to \(f\) a.e. on \([a, b]\). For \(F \in L^r[a, b]\), \(x \in [a, b]\), and a tagged interval \((I, x)\) we denote

\[
\Delta_r F(I, x) = \left( \frac{1}{|I|} \int_I |F(y) - F(x)|^r \, dy \right)^{1/r}.
\]

Now let us recall the notion of \(L^r\)-variational measure generated by a function \(F \in L^r[a, b]\); it has been defined in [3].
Definition 4. For $F \in L^r[a, b]$, a set $E \subset [a, b]$, and a fixed gauge $\delta$ on $E$, we define the $(\delta, r)$-variation of $F$ on $E$ by
\[
\operatorname{Var}(E, F, \delta, r) = \sup \sum_{i=1}^{q} \Delta_r F(I_i, x_i)
\]
where the sup is taken over all $\delta$-fine divisions $\{(I_i, x_i)\}_{i=1}^{q}$ in $[a, b]$ that are tagged in $E$. The $L^r$-variational measure of $E \subset [a, b]$ generated by $F \in L^r[a, b]$ is defined by
\[
\nu^r_F(E) = \inf_{\delta} \operatorname{Var}(E, F, \delta, r)
\]
where the inf is taken over all gauges on $E$. A variational measure $\nu^r_F$ is said to be absolutely continuous on $E$ if for every $N \subset E$ such that $\mu(N) = 0$, Note that, in particular, this condition applied to $N = \{x\}$ says $\Delta_r F(I, x) \to 0$ as $|I| \to 0$, i.e., that $F$ is $L^r$-continuous at $x$ (and so on $E$).

The following descriptive characterization of the indefinite HK$_r$-integral in terms of $L^r$-variational measure was given in [3]:

Theorem 5. A function $f : [a, b] \to \mathbb{R}$ is HK$_r$-integrable if and only if there exists a function $F$ on $[a, b]$ with absolutely continuous $\nu^r_F$ and such that $F'_r = f$ a.e.; the function $x \mapsto F(x) - F(a)$ is the indefinite HK$_r$-integral of $f$.

2.3. Approximate Kurzweil–Henstock integral. The HK$_r$-integral, as we are going to demonstrate, can be related to the so-called approximate Kurzweil–Henstock integral (AH-integral). We avoid here its Riemann-type definition and introduce it via a characterization in terms of corresponding variational measure (similar to Theorem 5), which is sufficient for our needs here. More information on this theory can be found in [13, Chapter 16].

Definition 6. A measurable set $E \subset \mathbb{R}$ is said to have a density point at $x \in \mathbb{R}$, if
\[
\frac{\mu(E \cap [x-h, x+h])}{2h} \to 1, \quad h \searrow 0.
\]
Analogously lefthand and righthand density points are defined. A function $F : [a, b] \to \mathbb{R}$ is said to be approximately continuous at $x \in (a, b)$ if for each $\varepsilon > 0$, the set $\{y \in [a, b] : |F(y) - F(x)| < \varepsilon\}$ has 0 as a density point. Analogously unilateral modes of this notion are defined. The corresponding notion of approximate differentiability and approximate derivative $F'_r$ one defines in the same fashion.

An approximate full cover (AFC) on $E \subset \mathbb{R}$ is a collection $\mathcal{S} = \{S_x\}_{x \in E}$ of measurable sets $S_x \ni x$ having a density point at $x$, for all $x \in E$.

For an AFC $\mathcal{S} = \{S_x\}_{x \in E}$, we say a tagged interval $([y, z], x)$ is $\mathcal{S}$-fine if $y, z \in S_x$. Analogously we understand $\mathcal{S}$-fine divisions. The following definition is analogous to Definition 6. Let $F : \mathbb{R} \to \mathbb{R}$, $E \subset \mathbb{R}$, and $\mathcal{S}$ be an AFC on $E$. Set
\[
\operatorname{Var}_{ap}(E, F, \mathcal{S}) = \sup \sum_{i=1}^{q} |F(z_i) - F(y_i)|,
\]
where sup ranges over all $\mathcal{S}$-fine divisions $\{(y_i, z_i, x_i)\}_{i=1}^{q}$ tagged in $E$, and
\[
\nu_{ap}^r(F, E) = \inf_{\mathcal{S}} \operatorname{Var}_{ap}(E, F, \mathcal{S}).
\]
The latter is called the \emph{approximate Kurzweil–Henstock integral} of \(E\) generated by \(F\). The approximate Kurzweil–Henstock integral can be characterized in terms of \(V^\text{ap}_F\) (like the HK\(_r\)-integral in Theorem 5).

\textbf{Definition 7.} We say a function \(f: [a, b] \to \mathbb{R}\) is AH-integrable if there exists a function \(F\) on \([a, b]\) with absolutely continuous \(V^\text{ap}_F\) such that the approximate derivative \(F'_\text{ap} = f\) a.e.; the function \(x \mapsto F(x) - F(a)\) is the indefinite AH-integral of \(f\).

Note that the following result is true: every \(F\) with absolutely continuous \(V^\text{ap}_F\) is almost everywhere approximately differentiable, so that the class of indefinite AH-integrals coincides with that of functions \(F, F(a) = 0\), generating \(\text{absolutely continuous} V^\text{ap}_F\); see [14, 15].

\section{2.4. Khintchine and Denjoy–Khintchine integrals.}

\textbf{Definition 8.} An \(F: [a, b] \to \mathbb{R}\) is said to be an ACG-function if \([a, b] = \bigcup_{n=1}^{\infty} D_n\), where \(F \upharpoonright D_n\) is absolutely continuous for all \(n\). If, moreover, all \(D_n\) can be chosen closed, we say \(F\) is an \(\text{ACG}\)-function.

\textbf{Definition 9.} A function \(f: [a, b] \to \mathbb{R}\) is said to be Khintchine/Denjoy–Khintchine integrable if there exists a continuous ACG-function \(F: [a, b] \to \mathbb{R}\) such that \(F'(x) = f(x)\) (resp. \(F'_\text{ap}(x) = f(x)\)) at almost all \(x \in [a, b]\). One then defines \(\int_a^b f = F(b) - F(a)\).

\section{3. Results}

\textbf{Lemma 10.} Every function \(F: [a, b] \to \mathbb{R}, F \in L^r,\) with absolutely continuous \(V^r_F\) generates also absolutely continuous \(V^\text{ap}_F\).

\textbf{Proof.} Given \(x \in [a, b]\), denote for brevity

\[
\omega_x(h) = \Delta_x F(x, x + h, x), \quad h \neq 0, \ (x, x + h) \subset [a, b].
\]

We prove that if \(x \in [a, b]\), \(h > 0\), the set

\[
S_x(h) = \{t \in (0, h) : |F(x + t) - F(x)| \leq \omega_x(h)\}
\]

has 0 as a righthand density point. If this were not true, for some \(\eta > 0\) arbitrarily small \(k > 0\) with \(|C_k| > \eta k\), where

\[
C_k = \{t \in [0, k] : |F(x + t) - F(x)| > \omega_x(h)\},
\]

should be found. For such a \(k > 0\), integrating \(|F(x + t) - F(x)|^r\) over \(C_k\) produces \(k \cdot \omega_x(k)^r \geq \eta k \cdot \omega_x(h)^r\), i.e., \(\omega_x(k)^r \geq \eta \cdot \omega_x(h)^r\), which can’t hold (due to \(L^r\)-continuity of \(F\)) for arbitrarily small \(k < h\) as long as \(F\) isn’t a.e. constant in some righthand neighborhood of \(x\) (since only then one could have \(\omega_x(h) = 0\)). Similarly we prove that for \(x \in (a, b)\) and \(h < 0\) the set (2) has 0 as a lefthand density point.

Assume now \(V^r_F\) is absolutely continuous and consider a nullset \(N \subset [a, b]\). Then, given \(\varepsilon > 0\), there is a gauge \(\delta\) such that if a division \(\{(x_i, y_i), r_i\}_{i=1}^k\) is \(\delta\)-fine, tagged in \(N\), then \(\sum_{i=1}^k \omega_r(y_i - x_i) < \varepsilon\). Let the sets \(S_x(h), S_x(-h)\) be defined at \(x\) with respect to \(h = \delta(x)\). We set \(S_x = x + (S_x(h) \cup S_x(-h)), x \in N\). From what we have proved above, \(S_x\) has \(x\) as a density point, i.e., \(S = \{S_x\}_{x \in N}\) is an AFC
on $N$. If $\{(y_j, z_j, x_j)\}_{j=1}^l$ is an $\mathcal{S}$-fine division tagged in $N$, from the definition of $S_x(h)$ and $S_x(-h)$, it follows

$$\sum_{j=1}^l |F(y_j) - F(z_j)| \leq \sum_{j=1}^l |F(y_j) - F(x_j)| + \sum_{j=1}^l |F(x_j) - F(z_j)|$$

$$\leq \sum_{j=1}^l \omega_{x_j}(x_j - y_j) + \sum_{j=1}^l \omega_{x_j}(z_j - x_j) < \varepsilon + \varepsilon.$$  

That is, $\text{Var}_{ap}(N, F, \mathcal{S}) \leq 2\varepsilon$ and so, since $\varepsilon > 0$ was arbitrary, $\text{Var}_{ap}^p(N) = 0$. \qed

A straightforward consequence of the above argument is \cite{3} Theorem 6.

Remark 11. If $F: [a, b] \to \mathbb{R}$, $F \in L^r$, is $L^r$-continuous at $x \in [a, b]$, then it is approximately continuous at $x$.

Corollary 12. Every HK$_r$-integrable function is AH-integrable and both integrals coincide.

Proof. Let $f: [a, b] \to \mathbb{R}$ be HK$_r$-integrable, $F = \int f$. By Theorem\cite{5} $F' = f$ almost everywhere in $[a, b]$. At every point where $F$ is $L^r$-differentiable, it is approximately differentiable with the same derivative (see \cite{1} Theorem 2), that is, $F'_{ap}(x) = f(x)$ at almost every $x \in [a, b]$. By Theorem \cite{5} and Lemma \cite{10} $V^p_F$ is absolutely continuous, so $f$ is AH-integrable with $\int f = F$. \qed

In consequence, in view of \cite{12} Theorem 2.1, we have a slight improvement of \cite{3} Corollary 1.

Corollary 13. Each indefinite HK$_r$-integral is an [ACG]-function.

We show next that the HK$_r$-integral does not cover the Denjoy–Khintchine integral. Namely, we prove the following

Theorem 14. There exists a function which is Khintchine (and so Denjoy–Khintchine) integrable on $[0, 1]$ but which is HK$_r$-integrable on $[0, 1]$ for no $r$.

Let us remark that, thanks to Corollary\cite{12} the theorem can be deduced from the theory of AH-integration; see e.g. \cite{10}. However, the argument we provide here has the advantage of being direct.

Proof. Let $C \subset [0, 1]$ be the classical Cantor ternary set (of measure zero) with contiguous intervals $u_n$ of rank $n = 1, 2, \ldots$ having length $u_n = 3^{-n}$\footnote{Here and in what follows, in a slight abuse of notation, we identify intervals $u_n, v_n, r_n$ and their lengths, so that there is no distinction made between different intervals of the same kind and the same rank.}. The set that is left after removing all contiguous intervals up to rank $n$ from $[0, 1]$, is constituted by $2^n$ segments $r_n$ of length $3^{-n}$ which are called segments of rank $n$. Note that each $u_n$ is an interval concentric with some segment $r_{n-1}$ (we put $r_0 = [0, 1]$). So each segment $r_n$, for each nonnegative integer $l$, can be represented as the union of $2^l$ segments $r_{n+l}$ and $2^l - 1$ intervals $u_{n+l}$, all of them being of length $3^{-(n+l)}$.

We construct a continuous ACG-function $F$ on $[0, 1]$, differentiable everywhere outside of $C$, which is the indefinite Khintchine integral of its derivative $f = F'$ existing a.e. We then show that $F$ is not HK$_r$-integrable. For each interval $u_n$, let $v_n \subset u_n$ be the segment concentric with $u_n$ and of length $u_n/2$. We put $F = 0$
on $C$ and $F = 1/n$ on every $v_n$, and then extend $F$ smoothly over both intervals $u_n \setminus v_n$ (in all $u_n$) so that the resulting function $F \geq 0$ is continuous on $[0,1]$ and differentiable everywhere outside of $C$. So we have for each contiguous interval $u_n$, 

$$\int_{u_n} F^r > \int_{v_n} F^r = \frac{v_n}{n^r} = \frac{1}{2n^r \cdot 3^u}.$$ 

We show that the $L'$-variational measure $V^r_F$ is not absolutely continuous. Fix arbitrary gauge $\delta$ on $C$ and for $m \in \mathbb{N}$ denote $C^m = \{x \in C : \delta(x) > 1/m\}$, $C = \bigcup_{m=1}^{\infty} C^m$. By the Baire category theorem there exists $m_0$ such that $C_{m_0}$ is metrically dense in some nonempty portion of $C$ defined by an interval $(c,d)$, i.e., $C \cap (c,d)$. We can assume that $d - c < 1/m_0$. There exists a segment $r_n$ with $r_n \subset (c,d)$. As we have already noted, for each $l$ there are $2^l - 1$ intervals $u_{n+l}$ of rank $n + l$ lying within the considered segment $r_n$. Denote them $(\alpha_k, \beta_k)$, $1 \leq k \leq 2^l - 1$. Choose for each of them a point $x_k \in C_{m_0}$ which belongs to the segment $r_{n+l}$ adjoining $(\alpha_k, \beta_k)$ on the left. Then 

$$\beta_k - x_k \leq \frac{2}{3^{n+l}} < \frac{1}{m_0} < \delta(x_k)$$

and so pairs $([x_k, \beta_k], x_k)$ form a $\delta$-fine division tagged in $C$. Having in mind (3) we get

$$\Delta_r F([x_k, \beta_k], x_k) = \left(\frac{1}{\beta_k - x_k} \int_{x_k}^{\beta_k} |F(y) - F(x_k)|^r \, dy \right)^{1/r} \geq \left(\frac{3^{n+l}}{2} \cdot \frac{1}{2(n + l)^r \cdot 3^{n+l}}\right)^{1/r} = \frac{1}{4^1/(n + l)}.$$ 

Summing up over all $k$ we obtain

$$\text{Var}(C, F, \delta, r) \geq \sum_{k=1}^{2^l - 1} \Delta_r F([x_k, \beta_k], x_k) > \frac{2^l - 1}{4^1/(n + l)}.$$ 

As $n$ is fixed and $l$ is arbitrary we obtain that $\text{Var}(C, F, \delta, r) = \infty$ for any gauge $\delta$. Hence $V^r_F(C) = \infty$ and so $V^r_F$ is not absolutely continuous. This proves that $F$ is not the indefinite HK$_r$-integral for its derivative.

To see that $F'$ is not HK$_r$-integrable it is enough to apply Remark 11 and Corollary 13. If this were not true, $G = (\text{HK}_r) \int F'$ would be an approximately continuous [ACG]-function with $F' = G'_{\text{ap}}$ a.e. in $[a, b]$. Thus, from the monotonicity property of [ACG]-functions [11], $F - G = \text{const}$, and so $F = G$, a contradiction. □

Remark 15. An example in the opposite direction, i.e., of an HK$_r$-integrand which is not a Denjoy–Khintchine integrand, can be provided via a discontinuous and everywhere $L'$-differentiable function (which can be constructed like $F$ in the proof of Theorem 13 with $v_n/u_n \to 0$ suitably quickly and $F = 1$ over each $v_n$).

Remark 16. Note that an HK$_r$-integrable function whose indefinite integral is continuous is necessarily Denjoy–Khintchine integrable, while not necessarily Khintchine integrable. An example here is more delicate and can be constructed similarly as a continuous ACG-function which is not a.e. differentiable [17, p. 224]: take a Cantor-like set $H \subset [0,1]$ of positive measure with contiguous intervals $I_1, I_2, \ldots$; set $F = |I_n| + \rho_n$ on a segment $J_n \subset I_n$, where $\rho_n$ is the maximal length of a subinterval of $[0,1]$ disjoint to all $I_1, \ldots, I_n$, $F = 0$ elsewhere. It can be shown as in
that $F$ is not differentiable at any point of $H$, while if $J_n$ are thin enough in $I_n$ it is $L'$-differentiable everywhere in $H$. Then one can modify $F$ smoothly around endpoints of all $J_n$ so that the resulting function is continuous, while nondifferentiability and $L'$-differentiability on $H$ are not affected.

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