ZETA CORRESPONDENCES IN RANK-

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1. INTRODUCTION.

The purpose of this work is to study in the case of function fields and rank-\( n \) certain correspondences associated with Jacobi sums [C]. These correspondences are called zeta correspondences and have been widely studied [An2], [An3], [An4], [AnDP], [Si]. Their importance is the relationship with Stickelberger’s Theorem and \( L \)-series evaluators, [An2], [An3], [An4], [AnDP]. In work [C] they are used to prove that the Frobenius map is essentially a Gauss or Jacobi sum for Fermat and Artin-Schreier curves.

In rank 1, these zeta correspondences are closely related to the theta divisor defined over the Jacobian, [An2]. The strategy in works [An2], [An3] is to study this subject in the setting of Drinfeld modules. Here we shall follow the same strategy but considering \( \mathbb{F}_q[t] \)-Drinfeld modules (elliptic sheaves) with level structures over an effective divisor \( D \) on \( Spec(\mathbb{F}_q[t]) \). In this way, the moduli of Drinfeld modules (elliptic sheaves) with level structures are given by the coefficients of a polynomial. These coefficients are torsion elements for a universal Drinfeld module (c.f [Al]). In this setting, zeta correspondences are given explicitly in terms of these elements. To study rank-\( n \) zeta correspondences, we have taken into account the generalized theta function considered in [An3] and the morphism given in 4. [An2], but considering

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the generalized genus \[\text{Se}\] pg.86. Firstly, we shall introduce zeta subschemes over the moduli of vector bundles with level structures as a generalization of the \(n^2\)-generalized theta divisor. We recall that the \(n^2\)-generalized theta divisor over a curve \(C\) of genus \(g\) is formed by the vector bundles of rank \(n^2\) and degree \(n^2(g-1)\) without global sections. The restriction via a certain morphism, similar to that of 4. \[\text{An2}\], of these zeta subschemes to the moduli of rank-\(n\) Drinfeld modules (elliptic sheaves) will be the zeta correspondences that we wish to study.

In this way we shall explicitly obtain, for \(P_1\), the zeta correspondences as the zero locus of certain functions. In the rank-\(n\) case, they are the zero locus of \(n^2\) functions given by a \(n \times n\)-matrix. These functions involve the torsion elements of a rank-\(n\) universal \(\mathbb{F}_q[t]\)-Drinfeld module. To obtain these results we shall prove a similar Lemma to \(^\chi = 0 \Rightarrow h^0 = h^1 = 0\" Lemma 3.3.1, \[\text{An2}\] for level structures in the rank-\(n\) case and \(P_1\). We can prove this Lemma in our case because for semistable bundles over \(P_1\) the generalized theta divisor is the empty subset. For rank-1, as in \[\text{An2}\], we shall see a relationship of these zeta correspondences with the theta divisor for level structures. One observes that instead of the zero locus of the \(n \times n\)-matrix considered one could take the zero locus of its determinant and would thus obtain a 1-correspondence that involves the zeta correspondence considered before. The examples that we provide in the rank-2 case give us zeta correspondences whose irreducible components are essentially the same as in the rank-1 case. They are given by graphs of automorphisms given by central matrices and the Frobenius morphism.

Above we have considered Drinfeld modules with level structures over \(\infty + D\), \(\infty\) being the pole of the Drinfeld modules. However if we consider another rational point \(p\) different from \(\infty\), we can extend the above results to an arbitrary proper and smooth curve \(C\) over a finite field. We shall obtain zeta subschemes for rank-\(n\) shtukas; these zeta subschemes would be formed by pairs of isogenous shtukas although to complete the result it would be necessary to prove the \(^\chi = 0 \Rightarrow h^0 = h^1 = 0\"-Lemma for rank \(n\). This result tells us, via the immersion of \(A\)-Drinfeld modules in rank-\(n\) shtukas, that for \(P_1\) zeta correspondences \((p\) instead of \(\infty\)) are given by pairs of isogenous \(A\)-Drinfeld modules.

In the case of a general curve, these zeta subschemes are again given, now locally, by the zero locus of \(n^2\)-functions. We shall check that these functions are the restriction of hyperplane sections via a Plucker morphism. In this way, these zeta subschemes are the intersection of \(n^2\)-generalized theta divisors.

I believe that it would be very interesting to complete section 4.1 in the setting of the iterated shtukas and Lafforgue’s compactification for the shtuka variety \[\text{An1}\], \[\text{La2}\] and the arithmetic counterpart.

The paper is organized in the following way:

Section 2 is devoted to recalling some definitions and results of rank-\(n\) vector bundles with level structures. In subsection 2.2, we introduce zeta correspondences for the moduli of semistable vector bundles with level structures over \(P_1\), which are analogous to the \(n^2\)-generalized theta divisor. Moreover, zeta correspondences are given by the zero locus of \(n^2\)-functions, obtained from a matrix.

In section 3, we recall some definitions and results for rank-\(n\) Drinfeld modules and elliptic sheaves with level structures. Via the immersion of elliptic sheaves in the moduli of semistable vector bundles, we gain a notion of zeta correspondences for rank-\(n\) Drinfeld modules. By means of a similar Lemma to Lemma 3.3.1 \[\text{An2}\],
we can give some information about the irreducible components of these zeta correspondences. We provide rank-2 examples and the known rank-1 examples, [An3], [AnDF], [Si].

In section 4, by taking a rational point $p$ instead of $\infty$, we can obtain in a more precise way the results of the last sections for a general curve $C$ and rank-$n$ shtukas. We state a similar Lemma to Lemma 3.3.1 [An2] for rank-$n$ shtukas, although in this case we should establish this Lemma in rank $n$ without level structures to complete this Lemma.

**List of notations and previous definitions**

$\mathbb{F}_q$ is a finite field with $q$-elements, ($q = p^m$)

$\otimes$ denotes $\otimes_{\mathbb{F}_q}$

$R$ is a $\mathbb{F}_q$-algebra

$C$ is a smooth, proper and geometrically irreducible curve over $\mathbb{F}_q$

$g$ is the genus of $C$

$\infty$ and $p$ are rational points in $C$

$A = H^0(C \setminus \{\infty\}, \mathcal{O}_C)$

$D$ is an effective rational divisor over $Spec(A)$

If $M$ is a vector bundle over $C$, $M(k)$ denotes $M \otimes_{\mathcal{O}_C} \mathcal{O}_C(k\infty)$, $k \in \mathbb{Z}$.

$R^\times$ denotes the group of units in a ring $R$

If $s \in Spec(R)$, we denote by $m_s$ the prime ideal associated with $s$.

If $M$ is a vector bundle over $C \times Spec(R)$, and $s \in Spec(R)$, $deg(M_s)$ denotes the degree of the vector bundle, $M_s$, over $C \times k(s)$. In this paper we shall consider vector bundles of constant degree, $deg(M)$, for each $s \in Spec(R)$.

Let $M$ be a vector bundle over $\mathcal{O}_C$; $M^\vee$ denotes the dual vector bundle

$(\mathbb{G}_a)_R$ is the additive line group over a ring $R$

$F$ denotes the Frobenius morphism over a scheme $Spec(R)$

For a vector bundle, $M$, over $C \times Spec(R)$, $F^#(M)$ denotes the pull-back $(Id \times F)^*M$.

If $T : M \to M'$ is a morphism of vector bundles over a scheme, $H^0(T)$ denotes the morphism induced among the global sections.
2. Vector bundles with level structures and zeta correspondences over \( \mathbb{P}_1 \)

2.1. Vector bundles with level structures.

**Definition 2.1.** A \( \infty D \)-level structure is a pair \((M, f_{\infty D})\), where \( M \) is a rank-\( n \) vector bundle over \( C \times \text{Spec}(R) \) and

\[
\phi_{\infty D} : M \to (\mathcal{O}_C/\mathcal{O}_C(-\infty - D) \otimes R)^n
\]

is a surjective morphism of \( \mathcal{O}_C \otimes R \)-modules.

By \( \mathcal{O}_C/\mathcal{O}_C(-\infty - D) \) and \( \mathcal{O}_{\infty D} \) we denote \( (\mathcal{O}_{\infty D})^n \) and \( H^0(C, (\mathcal{O}_C/\mathcal{O}_C(-\infty - D))) \) respectively.

A morphism of two pairs, \((M, f_{\infty D}), (M', f'_{\infty D})\), is a morphism of \( \mathcal{O}_C \otimes R \)-modules, \( \phi : M \to M' \), such that the diagram

\[
\begin{array}{c}
M \xrightarrow{\phi} M' \\
\downarrow f_{\infty D} \quad \downarrow f'_{\infty D} \\
(\mathcal{O}_{-\infty - D})^n \otimes R
\end{array}
\]

is commutative. Two pairs \((M, f_{\infty D}), (M', f'_{\infty D})\) are said to be equivalent if there exists an isomorphism \( \phi \).

**Proposition 2.2.** If \( \phi : M \to M' \) is a morphism between two pairs \((M, f_{\infty D}), (M', f'_{\infty D})\) defined over a field \( K \), then \( \phi \) is injective.

**Proof.** Since \( M/Ker(\phi) \rightarrow M', M/Ker(\phi) \) is a free torsion module, and since

\[
M/Ker(\phi) \otimes \mathcal{O}_{\infty D} \simeq (\mathcal{O}_{\infty D})^n \otimes K,
\]

\( M/Ker(\phi) \) is locally free of rank \( n \), and hence \( Ker(\phi) = 0 \) \( \square \)

We recall that \( M \) is said to be semistable if for each geometric point \( s \in \text{Spec}(R) \), \( M_s := M \otimes_R k(s) \) is semistable; i.e, for each \( F \subset M_s \)

\[
\mu(F) := deg(F)/rank(F) \leq \mu(M_s)
\]

Let denote us by \( \mathcal{M}^D_C(n, h, \infty D) \) the moduli scheme of pairs \((M, f_{\infty D})\) of semistable vector bundles, \( M \), of rank \( n \), degree \( h \), with a \( \infty D \)-level structure, over a curve \( C \) (c.f. [24]). Then,

\[
\text{Hom}_{\text{Schem}}(\text{Spec}(R), \mathcal{M}^D_C(n, h, \infty D)) = \{ \text{pairs over } R, (M, f_{\infty D}) \}/\text{Up to isomor.}
\]

In the same way, we shall denote \( \mathcal{M}_C(n, h, \infty D) \), the moduli stack, \( \mathcal{M}_C(n, h, \infty D) \), of pairs \((M, f_{\infty D})\), of vector bundles, \( M \), of rank \( n \), degree \( h \), with a \( \infty D \)-level structure.

In all this section, \( C = \mathbb{P}_1 \). By choosing a local parameter \( 1/t \) in \( \infty \), \( A = \mathbb{F}_q[t] \). We consider an isomorphism \( H^0(\mathcal{O}_{\infty D}, \mathcal{O}_{\infty D}/\mathcal{O}_{\infty D}(-\infty - D)) = \mathbb{F}_q[t]/p(t) \times \mathbb{F}_q \), with \( p(t) = (t-\alpha_1)^{r_1}(t-\alpha_2)^{r_2} \cdots (t-\alpha_l)^{r_l} \) and \( \alpha_i = r_1x_1 + \cdots + r_1x_i \), where \( x_i \) is the point associated with the maximal \( (t-\alpha_i)\mathbb{F}_q[t] \). Recall that \( d = \text{deg}(D) \).
Proposition 2.3. Let $M$ be a semistable vector bundle over $\mathbb{P}_1 \times Spec(R)$ of rank $n$ and degree 0 with a $\infty D$-level structure $f_{\infty D}$. Then:

1) $H^0(\mathbb{P}_1 \times Spec(R), M)$ is a free $R$-module of rank $n$, and $M = H^0(\mathbb{P}_1 \times Spec(R), M) \otimes O_C$.

2) Two pairs $(M, f_{\infty D}), (\tilde{M}, \tilde{f}_{\infty D})$, $deg(M) = deg(\tilde{M}) = 0$, are equivalent if and only if by choosing bases $\{s_1, \cdots, s_n\}$ and $\{\tilde{s}_1, \cdots, \tilde{s}_n\}$ for $H^0(\mathbb{P}_1 \times Spec(R), M)$ and $H^0(\mathbb{P}_1 \times Spec(R), \tilde{M})$ respectively, there exists $g \in Gl_n(R)$, with

$$f_{\infty D}(s_i) = \tilde{f}_{\infty D}(g(s_i))$$

Proof. 1) By taking global sections in the exact sequence of $O_{\mathbb{P}_1} \otimes R$-modules

$$0 \to M(-\infty) \to M \to M/M(-\infty) \to 0,$$

and bearing in mind the morphism given by the $D$-level structure $f_{\infty} : M \to (O_\infty \otimes R)^n$, we obtain an isomorphism

$$M/M(-\infty) \cong (O_\infty \otimes R)^n$$

and we conclude because, as $M$ is semistable, we have

$$H^0(\mathbb{P}_1 \times Spec(R), M(-\infty)) = H^1(\mathbb{P}_1 \times Spec(R), M(-\infty)) = 0$$

and therefore $H^0(f_{\infty}) : H^0(\mathbb{P}_1 \times Spec(R), M) \to (O_\infty \otimes R)^n$ is an isomorphism. The second assertion is easily deduced because $H^0(\mathbb{P}_1 \times Spec(R), M) \otimes O_C \subseteq M$ and $deg(M) = 0$.

2) If $(M, f_{\infty D}), (\tilde{M}, \tilde{f}_{\infty D})$ are equivalent, there exists an isomorphism of modules $\phi : M \to \tilde{M}$ such that $\tilde{f}_{\infty D} \phi = f_{\infty D}$. In this case, $g \in Gl_n(R)$ is given by considering $H^0(\phi)$ and bases for $H^0(\mathbb{P}_1 \times Spec(R), M)$ and $H^0(\mathbb{P}_1 \times Spec(R), \tilde{M})$.

Conversely, if $\{s_1, \cdots, s_n\}$ and $\{\tilde{s}_1, \cdots, \tilde{s}_n\}$ are bases for $H^0(\mathbb{P}_1 \times Spec(R), M)$ and $H^0(\mathbb{P}_1 \times Spec(R), \tilde{M})$ respectively, satisfying

$$f_{\infty D}(s_i) = \tilde{f}_{\infty D}(g(s_i))$$

for all $i$, $\phi$ is obtained from $g$, since $g$ can be interpreted as an isomorphism:

$$M \cong (O_{\mathbb{P}_1} \otimes R).s_1, \cdots, (O_{\mathbb{P}_1} \otimes R).s_n \to \tilde{M} \cong (O_{\mathbb{P}_1} \otimes R).\tilde{s}_1, \cdots, (O_{\mathbb{P}_1} \otimes R).\tilde{s}_n$$

□

Remark 1. By considering a basis $\{s_1, \cdots, s_n\}$, by the last Lemma $(M, f_{\infty D})$ has associated, in a bijective way, a matrix:

$$(\Delta_0 + \Delta_1 t + \cdots + \Delta_{d-1} t^{d-1}) \times \Delta_\infty$$

where $\Delta_k, \Delta_\infty$ are $n \times n$-matrices with entry elements in $R$. These matrices are given by $\{f_{\infty D}(s_1), \cdots, f_{\infty D}(s_n)\} \subset (O_\infty)^n \otimes R$.

One can consider the vector bundle (scheme) over $\mathbb{F}_q$, $V(\text{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^n, (O_\infty)^n))$ associated with the $\mathbb{F}_q$-vector space $\text{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^n, (O_\infty)^n)$. Then, for this vector bundle there exists a universal object

$$(D_0 + D_1 t + \cdots + D_{d-1} t^{d-1}) \times D_\infty$$

$D_k$ and $D_\infty$ being matrices with the independent variables $d_{i,j,k}$ and $d_{i,j,\infty}$ as entries, respectively.
Lemma 2.4. There exists an injective immersion (of schemes)

\[ \Psi : \mathcal{M}_{\mathbb{P}^1}(n, 0, \infty D) \to V(\text{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^n, (O_{\infty D})^n))/Gl_n \]

where $Gl_n$ denotes the $n$-linear algebraic group $(\mathbb{F}_q)$ acting on

\[ V(\text{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^n, (O_{\infty D})^n)) \]

over the term $\mathbb{F}_q^n$.

Proof. Noticing that $V(\text{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^n, (O_{\infty D})^n))/Gl_n$ is the Grassmannian of $n$-planes in $(O_{\infty D})^n$, then $\Psi(M, f_{\infty D})$ is defined as the rank-$n$ subbundle of $(O_{\infty D})^n$ generated by \{f_{\infty D}(s_1), \ldots, f_{\infty D}(s_n)\}. If

\[ (M, f_{\infty D}) \in \mathcal{M}_{\mathbb{P}^1}(n, 0, \infty D)(R), \]

then $< f_{\infty D}(s_1), \ldots, f_{\infty D}(s_n) >$ is a rank-$n$ subbundle of $(O_{\infty D})^n \otimes R$, because taking sections in the exact sequence

\[ 0 \to M(-\infty - D) \to M \xrightarrow{f_{\infty D}} (O_{\infty D})^n \otimes R \to 0 \]

one obtains the exact sequence

\[ < f_{\infty D}(s_1), \ldots, f_{\infty D}(s_n) > \to (O_{\infty D})^n \otimes R \to H^1(\mathbb{P}_1 \times \text{Spec}(R), M(-\infty - D)) \to 0 \]

To obtain this last exact sequence we have used the fact that $M$ is semistable and hence $H^1(\mathbb{P}_1 \times \text{Spec}(R), M) = 0$ and $H^0(\mathbb{P}_1 \times \text{Spec}(R), M(-\infty - D)) = 0$. Moreover, by Grauert’s theorem $H^1(\mathbb{P}_1 \times \text{Spec}(R), M(-\infty - D))$ is a locally free $R$-module. \[ \square \]

Remark 2. $\Psi$ takes values in the open subscheme, $U$, of the vector bundle of

\[ \text{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^n, (O_{\infty D})^n) \]

where $U$ are the morphisms such that their composition with the natural projection $(O_{\infty D})^n \to (O_{\infty})^n$ is an isomorphism. By fixing an isomorphism $\mathbb{F}_q^n \to (O_{\infty})^n$, $U$ can be identified with the vector bundle

\[ V(\text{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^n, (O_{\infty D})^n)). \]

Thus, by $\Psi$, $\mathcal{M}_{\mathbb{P}^1}(n, 0, \infty D)$ is isomorphic to $V(\text{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^n, (O_{\infty D})^n))$

2.2. Zeta correspondences for vector bundles over $\mathbb{P}_1$ with level structures. In this section we shall construct a zeta correspondence for $\mathcal{M}_{\mathbb{P}^1}(n, 0, \infty D)$. To do so, we take into account the generalized theta function \[ \text{Kn} \] 6.1.1.

Over $\mathcal{O}_{\mathbb{P}_1}(k, \infty)$ one can consider the following $\infty D$-level structure, $\pi_{\infty D}^k$:

\[ \pi_{\infty D}^k := \pi_{\infty}^k \times \pi_D : \mathcal{O}_{\mathbb{P}_1}(k) \to \mathcal{O}_{\mathbb{P}_1}/\mathcal{O}_{\mathbb{P}_1}(-1) \times \mathcal{O}_{\mathbb{P}_1}/\mathcal{O}_{\mathbb{P}_1}(-D) \]

$\pi_D$ being induced by the natural epimorphism $\mathcal{O}_{\mathbb{P}_1} \to \mathcal{O}_{\mathbb{P}_1}/\mathcal{O}_{\mathbb{P}_1}(-D)$ and the natural inclusion $\mathcal{O}_{\mathbb{P}_1} \hookrightarrow \mathcal{O}_{\mathbb{P}_1}(k)$. $\pi_{\infty}^k$ is obtained from the epimorphism

\[ \mathcal{O}_{\mathbb{P}_1}(k) \to \mathcal{O}_{\mathbb{P}_1}(k)/\mathcal{O}_{\mathbb{P}_1}(k - 1) \]

and the isomorphism induced by the multiplication by $1/t^{k - 1}$

\[ \mathcal{O}_{\mathbb{P}_1}(k)/\mathcal{O}_{\mathbb{P}_1}(k - 1) \cong \mathcal{O}_{\mathbb{P}_1}/\mathcal{O}_{\mathbb{P}_1}(-1). \]

In this way if $\lambda_0 + \lambda_1 t + \cdots + \lambda_{k-1} t^{k-1} \in H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(k))$, $H^0(\pi_{\infty D}^k)(\lambda_0 + \lambda_1 t + \cdots + \lambda_{k-1} t^{k-1}) = (\lambda_0 + \lambda_1 t + \cdots + \lambda_{k-1} t^{k-1}, \lambda_{k-1}) \in O_D \times O_{\infty}$
**Definition 2.5.** We define the $\infty D$-zeta correspondence, $Z^\infty_n$, as the subscheme of $\mathcal{M}^*_{\infty}(n, 0, \infty D) \times \mathcal{M}^*_{\infty}(n, 0, \infty D)$ defined by

$$[(\tilde{M}, \tilde{f}_\infty D), (M, f_\infty D)] \in \text{Hom}_{\text{ schemes}}(\text{Spec}(R), Z^\infty_n) \text{ if and only if there exists a morphism of modules } T : \tilde{M} \to M(d - 1) \text{ (} d = \deg(D)\text{), where the diagram :}$$

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{T} & M(d - 1) \\
\tilde{f}_\infty D & \downarrow & \downarrow \pi_{\infty D} \\
(\mathcal{O}_\infty D)^n \otimes R & \end{array}$$

is commutative.

Therefore, $[(\tilde{M}, \tilde{f}_\infty D), (M, f_\infty D)] \in Z^\infty_n$ if and only if

$$\text{Hom}_{\text{pairs}}((\tilde{M}, \tilde{f}_\infty D), (M(d - 1), f_\infty D \otimes \pi_{\infty D}^{-1})) \neq 0.$$  

From Lemma 2.8 it can be deduced that this subscheme is closed.

**Lemma 2.6.** Given level structures $(\tilde{M}, \tilde{f}_\infty D), (M, f_\infty D) \in \mathcal{M}^*_{\infty}(n, 0, \infty D)$ over $R$, there exists a unique morphism of vector bundles $T : \tilde{M} \to M(d - 1)$ with the diagram

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{T} & M(d - 1) \\
\tilde{f}_D & \downarrow & \downarrow \pi_D \\
(\mathcal{O}_D)^n \otimes R & \end{array}$$

commutative.

**Proof.** By choosing bases $\{s_1, \ldots, s_n\}$ and $\{\tilde{s}_1, \ldots, \tilde{s}_n\}$ for $H^0(\mathbb{P}_1 \times \text{Spec}(R), M)$ and $H^0(\mathbb{P}_1 \times \text{Spec}(R), \tilde{M})$ respectively.

By Remark 2.8, $(\tilde{M}, \tilde{f}_\infty D), (M, f_\infty D)$ has associated

$$(\Delta_0 + \Delta_1 t + \cdots + \Delta_{d-1} t^{d-1}) \times \Delta_{\infty}$$

and

$$(\bar{\Delta}_0 + \bar{\Delta}_1 t + \cdots + \bar{\Delta}_{d-1} t^{d-1}) \times \bar{\Delta}_{\infty},$$

respectively.

Since

$$H^0(\mathbb{P}_1 \times \text{Spec}(R), M(d - 1)) = \bigoplus_{i=0}^{d-1} H^0(\mathbb{P}_1 \times \text{Spec}(R), M) t^i$$

by considering the above basis, $T$ is given by

$$A_0 + A_1 t + \cdots + A_{d-1} t^{d-1},$$

where $A_i$ are $n \times n$-matrices with entries in $R$.

These matrices can be obtained from the relation, $\tilde{f}_D = f_D . T|_D$, where $T|_D$ is the isomorphism

$$T|_D : \tilde{M}/\tilde{M}(-D) \to M(d - 1)/M(d - 1)(-D).$$

Thus, $T$ is given by

$$A_0 + A_1 t + \cdots + A_{d-1} t^{d-1} = (\Delta_0 + \Delta_1 t + \cdots + \Delta_{d-1} t^{d-1})^{-1} . (\Delta_0 + \Delta_1 t + \cdots + \Delta_{d-1} t^{d-1}).$$
In this Lemma we have considered an isomorphism \( O_D \cong \mathbb{F}_q[t]/p(t) \) and \( A_i \) matrices for linear morphisms between \( H^0(\mathbb{P}_1 \times \text{Spec}(R), \tilde{M}) \) and \( H^0(\mathbb{P}_1 \times \text{Spec}(R), M) \) expressed in the bases \( \{ s_1, \cdots, s_n \} \) and \( \{ \tilde{s}_1, \cdots, \tilde{s}_n \} \).

**Lemma 2.7.** With the above notation, \( T : \tilde{M} \to M(d - 1) \) makes the diagram

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{T} & M(d - 1) \\
\downarrow{\tilde{f}^{\infty}_D} & & \downarrow{f^{\infty}_D \circ \pi^{d-1}_D} \\
(O^{\infty}_D)^n \otimes R & \rightarrow & (O^{\infty}_D)^n \otimes R
\end{array}
\]

commutative if and only if \( \Delta_{\infty} = \Delta_{\infty} \cdot A_{d-1} \).

**Proof.** By following the same notation as in the latter lemma, let us consider \( T = A_0 + A_1 t + \cdots + A_{d-1} t^{d-1} \). Then, the only condition that \( T \) must satisfy for the diagram

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{T} & M(d - 1) \\
\downarrow{\tilde{f}^{\infty}_D} & & \downarrow{f^{\infty}_D \circ \pi^{d-1}_D} \\
(O^{\infty}_D)^n \otimes R & \rightarrow & (O^{\infty}_D)^n \otimes R
\end{array}
\]

to be commutative is that

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{T} & M(d - 1) \\
\downarrow{\tilde{f}} & & \downarrow{f^{\infty}_D \circ \pi^{d-1}_D} \\
(O^{\infty}_D)^n \otimes R & \rightarrow & (O^{\infty}_D)^n \otimes R
\end{array}
\]

must be commutative. But from the definition of \( \pi^{d-1}_D \), if \( i < d - 1 \) then \( \pi^{d-1}_D(t) = 0 \) and hence the condition that \( A_0 + A_1 t + \cdots + A_{d-1} t^{d-1} \) must be satisfied to make the above diagram commutative is \( \Delta_{\infty} = \Delta_{\infty} \cdot A_{d-1} \). \( \square \)

**Remark 3.** From Remark \( [3] \)

\[
V(\text{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^n, (O^{\infty}_D)^n)) = \text{Spec}(\mathbb{F}_q[\{d_{i,j,r}\}_{i,j,r}, \{w_{i,\infty}\}_{i,\infty}])
\]

"\( d_{i,j,r} \)" and "\( d_{i,j,\infty} \)" being the matrix entries for \( D_r \) and \( D_{\infty} \), respectively, with

\[
(D_0 + D_1 t + \cdots + D_{d-1} t^{d-1}) \times D_{\infty}
\]

a universal object for \( V := V(\text{Hom}_{\mathbb{F}_q}(\mathbb{F}_q^n, (O^{\infty}_D)^n)) \). Let us denote \( p_1, p_2 : V \times V \to V \) the natural projections and \( A_0 + A_1 t + \cdots + A_{d-1} t^{d-1} \) the matrix composition

\[
(p^*_2 D_0 + p^*_2 D_1 t + \cdots + p^*_2 D_{d-1} t^{d-1})^{-1} \cdot (p^*_1 D_0 + \cdots + p^*_1 D_{d-1} t^{d-1}).
\]

We now consider \( I^n_{\infty D} \), the subscheme of \( V \times V \), given by the relation

\[
p^*_1 D_{\infty} = p^*_2 D_{\infty} \cdot A_{d-1}.
\]

Since \( I^n_{\infty D} \) is invariant, by the action of \( \text{Gl}_n \times \text{Gl}_n \), we obtain a subscheme

\[
I^n_{\infty D}/\text{Gl}_n \times \text{Gl}_n \subset V/\text{Gl}_n \times V/\text{Gl}_n.
\]

Moreover, by the very definition of \( I^n_{\infty D} \) it is given by the zero locus of \( n^2 \)-regular functions of

\[
\mathbb{F}_q[\{d_{i,j,r}\}_{i,j,r}, \{d_{i,j,\infty}\}_{i,j}, \{d_{i,\infty}\}_{i,j}] \otimes \mathbb{F}_q[\{d_{i,j,r}\}_{i,j,r}, \{d_{i,\infty}\}_{i,j}]
\]
Lemma 2.8. Bearing in mind the morphism $\Psi$ of [2.4], we have

$$(\Psi \times \Psi)^{-1}(I_{\infty D}^n / Gl_n \times Gl_n) = Z^\infty_{n D}$$

and hence $Z^\infty_{n D}$ is the zero locus of $n^2$-regular functions of $\mathcal{M}^a_{P_1}(n, 0, \infty D) \times \mathcal{M}^a_{P_1}(n, 0, \infty D)$. Recall that the subscheme $Z^\infty_{n D}$ is defined in [2.3].

**Proof.** This Lemma is deduced from Lemma 2.7 and Remark 3.

$Z^\infty_{n D}$ is the zero locus of $n^2$-regular functions of $\mathcal{M}^a_{P_1}(n, 0, \infty D) \times \mathcal{M}^a_{P_1}(n, 0, \infty D)$ because of Remark 3. Moreover, one can calculate in an explicit way these $n^2$-regular functions from the relation

$$0 = (p^*_D \mathcal{D})^* - (p^*_D \mathcal{D})_\ast.$$  

$\square$

For rank 1, we have:

**Definition 2.9.** We define the $\infty D$-generalized theta divisor, $\Theta_{\infty D}$, as the subscheme of $\mathcal{J}_{\infty D} \mathcal{P}_1$, 0 defined by:

$$(L, f_{\infty D}) \in \text{Hom}_{\text{schemes}}(\text{Spec}(R), \Theta_{\infty D})$$

if and only if there exists a morphism of modules $T : \mathcal{O}_{P_1} \otimes R \to L(d-1)$ ($d = \text{deg}(D)$), where the diagram:

$$\begin{array}{ccc}
\mathcal{O}_{P_1} \otimes R & \xrightarrow{T} & L(d-1) \\
\downarrow{\pi_D} & & \downarrow{f_{\infty D} \otimes \pi_D} \\
\mathcal{O}_{\infty D} \otimes R & \xleftarrow{f_{\infty D} \otimes \pi_D} & (\mathcal{O}_{\infty D})^n \otimes R
\end{array}$$

is commutative.

Here, $J_{\infty D}$ denotes the $\infty D$-generalized Jacobian of line bundles of degree 0, $\mathcal{S}_{\infty D}$.

Now, we recall the definition of the tensor product and dual level structures for vector bundles.

**Remark 4.** The tensor product of two $\infty D$-level structures $(M, f_{\infty D})$ and $(\tilde{M}, \tilde{f}_{\infty D})$ is the $\infty D$-level structure

$$(M \otimes \tilde{M}, f_{\infty D} \otimes \tilde{f}_{\infty D})$$

where $f_{\infty D} \otimes \tilde{f}_{\infty D}$ denotes the morphism

$$M \otimes \tilde{M} \to (\mathcal{O}_{\infty D})^n \otimes (\mathcal{O}_{\infty D})^n \otimes R \simeq (\mathcal{O}_{\infty D})^{n^2} \otimes R.$$  

The dual $\infty D$-level structure $(M^\vee, f^\vee_{\infty D})$ for $(M, f_{\infty D})$, is given by a level structure, $f^\vee_{\infty D}$, for the dual vector bundle $M^\vee$ i.e., $f_{\infty D}$ induces an isomorphism

$$M/M(-\infty - D) \simeq (\mathcal{O}_{\infty D})^n \otimes R$$

from which we deduce an isomorphism

$$M^\vee \otimes \mathcal{O}_{\infty D} \otimes R \to (\mathcal{O}_{\infty D})^n \otimes R.$$  

$f^\vee_{\infty D}$ is given by this isomorphism.
Let us now consider the analogous morphism to the morphism \[^4.1\] but for line bundles with level structures. Notice that the genus considered in the case of \(\infty\)-level structures is \(g(P_1) + \deg(\infty D) - 1 = d\). [Se] pg.86.

Let \(m_{\infty D}\) be the morphism
\[
m_{\infty D} : J_{\infty D}^{P_1,0} \times J_{\infty D}^{P_1,0} \to J_{\infty D}^{P_1,0},
\]
defined by
\[
m_{\infty D}[(L, f_{\infty D}), (\bar{L}, \bar{f}_{\infty D})] = (\bar{L}, \bar{f}_{\infty D}) \otimes (L', f'_{\infty D}).
\]
We have

**Lemma 2.10.** For \(n = 1\), \(m_{\infty D}^{-1}(\Theta_{\infty D}^1) = Z_{\infty D}^1\) and \(Z_{\infty D}^1\) is a principal Weil divisor on \(J_{\infty D}^{P_1,0} \times J_{\infty D}^{P_1,0}\).

**Proof.** This is deduced from the latter definition, from the definition of \(Z_{\infty D}^n\), and from Lemma 2.8. \(\square\)

### 3. DRINFELD MODULES AND ZETA CORRESPONDENCES

#### 3.1. A-Drinfeld modules, elliptic sheaves and their antiequivalence

In this section we recall the definition of Drinfeld modules, elliptic sheaves, level structures and their antiequivalence [BlSt], [Dr1], [Dr2], [LRS], [L], [Mu].

**Definition 3.1.** A Drinfeld module, \(\phi\), of rank \(n\) over \(R\) is a ring morphism:
\[
\phi : A \to \text{End}_R((\mathbb{G}_a)_R)
\]
with \(\phi_a = \sum_{i=0}^{n v_\infty(a)} a_i \sigma^i\), \(a \in A\), \(v_\infty\) is the valuation for \(\infty\), \(a_i \in R\), \(a_m\) is a unit in \(R\) and \((\mathbb{G}_a)_R\) is the additive line group over \(R\), \(\text{End}_R((\mathbb{G}_a)_R)\) are its endomorphisms, and \(\sigma\) is the endomorphism on \((\mathbb{G}_a)_R\), \(\sigma(\gamma) = \gamma^q\). In this way, \(\text{End}_R((\mathbb{G}_a)_R) = R\{\sigma\}\), where \(R\{\sigma\}\) is the ring of polynomials in \(\sigma\), with the twisted rule of multiplication \(\sigma \cdot b = b^q \sigma\).

From this definition, one deduces a morphism of rings
\[
c : A \to R\{\sigma\} \xrightarrow{\sigma = 0} R
\]
called the characteristic morphism of \(\phi\).

**Definition 3.2.** An elliptic sheaf of rank \(n\) over \(R\), \((E_j, i_j, \tau)\), is a commutative diagram of vector bundles of rank \(n\) over \(C \times \text{Spec}(R)\), and injective morphisms of modules \(\{i_h\}_{h \in \mathbb{N}}\), \(\tau\):

\[
\begin{array}{cccccccc}
E_1 & \xrightarrow{i_1} & E_2 & \xrightarrow{i_2} & \cdots & \xrightarrow{i_{(n-1)}} & E_n & \xrightarrow{i_n} & \cdots \\
\tau | \downarrow & & \tau | \downarrow & & \cdots & \tau | \downarrow & & \cdots \\
F\# E_0 & \xrightarrow{F\# i_0} & F\# E_1 & \xrightarrow{F\# i_1} & \cdots & \xrightarrow{F\# i_n} & F\# E_n & \cdots \\
\end{array}
\]
satisfying:

- a) For any \(z \in \text{Spec}(R)\), \(\deg((E_h)_z) = ng + h\).

- b) For all \(i \in \mathbb{Z}\), \(E_{i+n} = E_i(1)\).
c) $E_i + \tau(F^#E_i) = E_{i+1}$.

d) $j_*(E_i/E_{i-1})$ is a rank-one free module over $R$. $j$ is the inclusion $\times \times \text{Spec}(R) \hookrightarrow C \times \text{Spec}(R)$.

Recall that $F^#$ denotes $(\text{Id} \times F)^*$.

Remark 5. From these properties, it is deduced that $h^0(E_h) = h$ and $h^1(E_h) = 0$, $h \geq 0$ c.f. [Dr2]. In these conditions it is not hard to prove that the "$E_{in}$" are semistable.

Moreover, it is proved that there exists a basis $\{s, \sigma.s, \ldots, \sigma^{n+i-1}.s\}$ for $H^0(C \times \text{Spec}(R), E_i)$ ($i > 0$), with $\sigma.s := \tau(F^#s)$ and $\sigma^i.s := \tau(F^#\sigma^i).s$.

In [Dr2], the anti-equivalence between rank-$n$-Drinfeld modules and rank-$n$-elliptic sheaves is settled in the following way:

Remark 6. Given a rank-$n$-Drinfeld module over $R$

$$\phi : A \to \text{End}_R((\mathbb{G}_a)R) = R\{\sigma\}$$

$R\{\sigma\}$ has a structure of an $A \otimes R$-module by means of $\phi$ ($R$-module on the left and $A$-module on the right). Moreover, $R\{\sigma\}$ has a graduation: $\text{deg}(\sum_j b_j.\sigma^j) = m$.

Bearing in mind this graduation, the $A \otimes R$-module $R\{\sigma\}$ has associated a coherent sheaf $E_0$ over $C \times \text{Spec}(R)$. The "$E_i$" are obtained by translating the degree over $R\{\sigma\}$. $\tau$ is obtained from the multiplication on the left over $R\{\sigma\}$ by $\sigma$. It is not hard to prove that $E_i$ are vector bundles of rank $n$.

Conversely, given an elliptic sheaf $(E_j, i_j, \tau)$ over $C \times \text{Spec}(R)$, by regarding the diagram associated with this elliptic sheaf, for each $h \in \mathbb{N}$ one deduces injective morphisms, which we denote by $\tau^h$

$$\tau^h : F^#^hE_i \to E_{i+h}$$

and a chain of modules

$$\tau^h(F^#^hE_i) \subset \tau^{h-1}(F^#^{h-1}E_{i+1}) \subset \cdots \subset F^{#h-k}E_{h+k}$$

($h \geq k$).

If $<s> = H^0(C \times \text{Spec}(R), E_{i-n})$ by c) is

$$\bigcup_i H^0(C \times \text{Spec}(R), E_i) = R\{\sigma\}.s$$

with $\sigma^r.s = \tau^r.((F^#)^r(s))$, then $a.s = \sum_j b_j.\sigma^j.s$ ($a \in A$) and the Drinfeld module associated with $(E_j, i_j, \tau)$ is $\phi_a = \sum_j b_j.\sigma^j$.

Since rank$(E_i) = \text{rank}(F^#E_{i-1})$ and deg$(E_i) = \text{deg}(F^#E_{i-1})+1$, $E_i/\tau(F^#E_{i-1})$ is a coherent sheaf over $C \times \text{Spec}(R)$ such that for each $s \in \text{Spec}(R)$

$$E_i/\tau(F^#E_{i-1}) \otimes_R k(s)$$

is concentrated on $p_s \in \text{Spec}(A)$ and hence one obtains a morphism $c^* : \text{Spec}(R) \to \text{Spec}(A)$. This morphism is the characteristic morphism associated with $\phi$. We say that $(E_j, i_j, \tau)$ has the generic characteristic if the characteristic morphism $c : A \to R$ is injective.
Remark 7. Let $(E_j, i_j, \tau)$ be an elliptic sheaf defined over a field $K$ and with generic characteristic $c : A \to K$. One can see that $F^h E_{i_j+1}/\tau(F^h+1 E_j)$ is concentrated on $y_h \in \text{Spec}(A) \times \text{Spec}(K)$, with $y_0 := (Id \times F^h)(y_h)$ the characteristic of $(E_j, i_j, \tau)$ and $E_n/\tau^n(F^h E_0)$ is concentrated on $y_0, \ldots, y_{n-1} \in \text{Spec}(A) \times \text{Spec}(K)$. Moreover, by considering the points "$y^n_i$" as morphisms

$$y_i : \text{Spec}(K) \to \text{Spec}(A) \times \text{Spec}(K)$$

they are given by the morphism $c^i : A \to K$ on the first entry and the identity morphism on the second entry. Therefore, $y_i \neq y_j$ ($i \neq j$) because $c^i \neq c^j$ since $c$ is injective.

Proposition 3.3. Let consider us two elliptic sheaves with generic characteristic $(E_j, i_j, \tau)$ and $(\bar{E}_j, i_j, \bar{\tau})$ defined over a field $K$. If $T : E_0 \to \bar{E}_r$ is a morphism of vector bundles with the diagram

$$
\begin{array}{ccc}
E_0 & \xrightarrow{T} & \bar{E}_r \\
\downarrow{\bar{\tau}^n} & & \downarrow{id} \\
F^h E_{-n} & \xrightarrow{F^n T(-1)} & F^h \bar{E}_{r-n}
\end{array}
$$

commutative, there exists a maximum $i \in \mathbb{N}$ such that $T(E_0) \subseteq \bar{\tau}^i(F^h \bar{E}_{r-i}) \subset \bar{E}_r$. In this case, $(Id \times F^i)y_0 = z_0$, where $y_0$ and $z_0$ denote the characteristic of $(E_j, i_j, \tau)$ and $(\bar{E}_j, i_j, \bar{\tau})$ respectively. We denote by $T(-1)$ the morphism induced by $T$ over $E_{-n} := E_0(-1)$.

Proof. If $i$ is the maximum $i \in \mathbb{N}$ such that $T(E_0) \subseteq \bar{\tau}^i(F^h \bar{E}_{r-i})$, then we have a non-trivial morphism

$$T : E_0/F^h E_{-n} \to \bar{\tau}^i(F^h \bar{E}_{r-i})/\bar{\tau}^{i+1}(F^{h+1} \bar{E}_{r-i-1})$$

where $\bar{\tau}^i(F^h \bar{E}_{r-i})/\bar{\tau}^{i+1}(F^{h+1} \bar{E}_{r-i-1})$ is concentrated on $z_i$ and $E_0/F^h E_{-n}$ is concentrated on $y_0, \ldots, y_{n-1}$, $(Id \times F^i)z_i = z_0$ and $(Id \times F^j)y_j = y_0$. We thus have $(Id \times F^j)z_0 = (Id \times F^i)y_0$. To conclude it suffices to prove that $j = 0$.

We shall assume that $j \geq 1$. However, in doing so we shall arrive at a contradiction. It is not difficult to see that each case is reduced to studying $j = 1$ and $i = 0$. In this case, $z_0 = y_1$ because $(Id \times F)z_0 = y_0 = (Id \times F)y_1$. Thus,

$$T|_{F^h E_{-1}} : F^h E_{-1} \to \bar{E}_r/\bar{\tau}(F^h \bar{E}_{r-1})$$

gives an isomorphism

$$F^h E_{-1}/\tau(F^{h+1} E_{-2}) \cong \bar{E}_r/\bar{\tau}(F^{h+1} \bar{E}_{r-1})$$
(recall that $F^\#E_{-1}/\tau(F^\#E_{-2})$ is concentrated on $y_1$). We therefore have a commutative diagram

\[
\begin{array}{ccc}
E_0 \xrightarrow{T} & E_r \\
\downarrow \tau & \downarrow \tau \\
F^\#E_{-1} & \xrightarrow{\hat{\tau}} & F^\#E_{r-1}
\end{array}
\]

where $F^\#T_{-1}$ is the restriction of $T$ to $F^\#E_{-1}$. Since $z_0 = y_1$, we have $z_{n-1} = y_n$ because $(Id \times F^{n-1})z_{n-1} = z_0 = y_1 = (Id \times F^{n-1})y_n$ and $z_{n-1}$ and $y_n$ are given by injective morphisms $A \hookrightarrow K$. Thus, in the same way as before we also have a commutative diagram

\[
\begin{array}{ccc}
F^\#_{n-1}E_{r+1} \xrightarrow{\bar{\tau}} & F^\#_{n}E_{r-n} \\
\downarrow \bar{\tau} & \downarrow \bar{\tau} \\
F^\#_{n-1}E_{r-1} & \xrightarrow{\bar{\tau}} & F^\#_{n}E_{r-n}
\end{array}
\]

$F^\#_{n-1}T_{-n}$ being the restriction of $T$ to $F^\#_{n-1}E_{-n}$. Thus, this morphism gives an isomorphism

\[
F^\#_{n}E_{-n}/\tau(F^\#_{n+1}E_{-n-1}) \simeq F^\#_{n-1}E_{r-n+1}/\bar{\tau}(F^\#_{n}E_{r-n}).
\]

However, this contradicts the inclusion

\[
F^\#_{n}T(-1)(F^\#_{n}E_{-n}) \subseteq F^\#_{n}E_{r-n}.
\]

3.2. A-Drinfeld modules and elliptic sheaves with level structures. Let us now briefly recall the definitions of level structures for elliptic sheaves and Drinfeld modules: $(E_j, i_j, \tau)$ and $\phi$ respectively.

Let $E_I$ be the subgroup scheme of $I$ division points of $(\mathbb{G}_a)_R$ as an $A$-module (via $\phi$). $(I^{-1}/A)^n$ will denote the constant sheaf of stalk $(I^{-1}/A)^n$. $I$ is a proper ideal in $\text{Spec}(A)$.

**Definition 3.4.** An I-level structure for $\phi$ is a pair $(\phi, \iota_I)$. $\iota_I$ is an isomorphism of $A$-modules

\[
\iota_I : E_I(R) \rightarrow (I^{-1}/A)^n(R).
\]

**Definition 3.5.** A D-level structure for the elliptic sheaf $(E_j, i_j, \tau)$, is a D-level structure, $f^D_j$, for each vector bundle $E_j$ compatible with the morphisms $\{i_j, \tau\}$. i.e., $f^D_{i_j} = f^D_j$ and $f^D_{i_j+1, \tau} = (Id \times F)^* f^D_j$. 

In [An1], the anti-equivalence between Drinfeld modules and elliptic sheaves with level structures is established. There exists an affine scheme, $\mathcal{E}_n^D = \text{Spec}(\mathcal{R}_n^D)$, that represents rank-$n$ elliptic sheaves with $D$-level structures (equivalently, rank-$n$ Drinfeld modules with $I$-level structures).

One can calculate in an explicit way the sections $\sigma_i$. These computations are done in [An1] (Theorem 5), [Al] (Remark 3.1).

Let $I$ be the ideal associated with the effective divisor $D \subset \text{Spec}(A)$, $D = r_1 \cdot x_1 + \cdots + r_l \cdot x_l$, $x_i$ being the point in $\text{Spec}(A)$ associated with the maximal ideal $m_{x_i}$, and $t_{x_i}$ a local uniformizer for $m_{x_i}$.

**Remark 8.** if $(\Phi, t_I)$ is a universal Drinfeld module with an $I$-level structure over $\mathcal{R}_n^D$. Therefore, $t_I$ is an isomorphism

$$t_I : \mathcal{E}_I(\mathcal{R}_n^D) \to \left( \prod_{i \in \{1, \ldots, l\}} t_{x_i}^{r_i} \mathcal{F}_q[[t_{x_i}]]/\mathcal{F}_q[[t_{x_i}]] \right)^n$$

with $t_I(\alpha_{j,h}^{x_i}) = (0, \ldots, v_j, \ldots, 0)$ and

$$v_j = (0, \ldots, t_{x_i}^{-1-h}, \ldots, 0) \in \prod_{i \in \{1, \ldots, l\}} t_{x_i}^{r_i} \mathcal{F}_q[[t_{x_i}]]/\mathcal{F}_q[[t_{x_i}]].$$

We denote by $(\mathcal{E}_j, i_j, \tau, f_{I_j})$ the corresponding universal elliptic sheaf of rank $n$ with a $D$-level structure associated with $(\Phi, t_I)$. Thus, the value of the sections $\sigma_i$ is $H^0(C \otimes \mathcal{R}_n^D, \mathcal{E}_m)$ via the level structure $f_{I_j}^m : \mathcal{E}_m \to \mathcal{O}_D^m \otimes \mathcal{R}_n^D$ is:

$$H^0(f_{I_j}^m)(\sigma)(s) = (s_{1}^{k}, \ldots, s_{n}^{k}) \in (\mathcal{O}_D \otimes \mathcal{R}_n^D)^n$$

By using the isomorphism

$$(\mathcal{O}_D \otimes \mathcal{R}_n^D)^n = \left( \prod_{i \in \{1, \ldots, l\}} \mathcal{R}_n^D[[t_{x_i}]]/t_{x_i}^{r_i} \mathcal{R}_n^D[[t_{x_i}]] \right)^n$$

we have:

$$s_i^{k} = \sum_{h=0}^{r_i-1} (\alpha_{j,h}^{x_i})^{k_{h}} t_{x_i}^{h}, \ldots, \sum_{h=0}^{r_i-1} (\alpha_{j,h}^{x_i})^{k_{h}} t_{x_i}^{h}.$$ 

**Definition 3.6.** A $\infty$-level structure for a rank-$n$ elliptic sheaf $(\mathcal{E}_j, i_j, \tau)$ over $R$ is a $\infty$-level structure $(E_0, f_{\infty})$ such that the diagram

$$\begin{array}{ccc}
F^# & \xrightarrow{\tau} & E_0(1) \\
\downarrow{f_{\infty}} & & \downarrow{f_{\infty}} \\
(O_{\infty})^n \otimes R & & (O_{\infty})^n \otimes R
\end{array}$$

is commutative. $\pi_{\infty}$ is defined in [2].

To give a $\infty$-$D$-level structure for an elliptic sheaf is to give an $\infty$-level structure together with a $D$-level structure for the elliptic sheaf.
The idea of considering ∞-level structures in this setting has been suggested to me by G.W. Anderson.

As for D-level structures, there exists an affine scheme, \( E_n^\infty = \text{Spec}(\mathcal{R}_n^\infty) \), that represents rank-n Elliptic sheaves with ∞D-level structures.

**Remark 9.** Let \((E_j, i_j, \tau)\) be a universal rank-n elliptic sheaf with an ∞D-level structure \( f_{\infty D} = f_\infty \times f_D \) over \( \text{Spec}(\mathcal{R}_n^\infty) \). By choosing the basis \( \{ s, \sigma, s, \cdots, \sigma^{n-1}.s \} \)

in \( H^0(C \otimes \mathcal{R}_n^\infty, E_0) \), Remark 9 associates a matrix with \((E_0, f_{\infty D})\). By using the results and notation of the latter Remark, this matrix is

\[
\begin{pmatrix}
\alpha_{1, h}^{x_1} & (\alpha_{1, h}^{x_1})^q & \cdots & (\alpha_{1, h}^{x_1})^{q^{n-1}} \\
\alpha_{2, h}^{x_2} & (\alpha_{2, h}^{x_2})^q & \cdots & (\alpha_{2, h}^{x_2})^{q^{n-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n, h}^{x_n} & (\alpha_{n, h}^{x_n})^q & \cdots & (\alpha_{n, h}^{x_n})^{q^{n-1}}
\end{pmatrix}.
\]

\( A_\infty \) is a \( n \times n \)-matrix with coefficients in \( \mathcal{R}_n^\infty \) that we shall calculate in the case \( C = \mathbb{F}_1 \) in the following subsection.

### 3.3. \( \mathbb{F}_q[t] \)-Drinfeld modules and zeta correspondences.

In the rest of this section \( C = \mathbb{F}_1 \). We denote by \( E_n^\infty \) the moduli (scheme) of rank-n elliptic sheaves with ∞D-level structures for \( A = \mathbb{F}_q[t] \). We denote an object of this moduli by \((E_j, i_j, \tau, f_{\infty D})\).

In a direct way, one can obtain a morphism

\[
\Phi : E_n^\infty \to \mathcal{M}_{\mathbb{F}_1}^\infty(n, 0, \infty D)
\]

given by \( \Phi(E_j, i_j, \tau, f_{\infty D}) = (E_0, f_\infty \times f_D^0) \). Recall that by Remark 5, \( E_0 \) are semistable.

The next three subsections are devoted to studying the zeta correspondences

\[
(\Phi \times \Phi)^{-1}(Z_n^\infty)
\]

in \( E_n^\infty \times E_n^\infty \), where \( Z_n^\infty \) is defined as in Definition 5.

\( \Phi \times \Phi \) is the analogous morphism to that defined in [An2] 4.1.

**Lemma 3.7.** If

\[
[(E_j, i_j, \tau, f_{\infty D}), (\bar{E}_j, \bar{i}_j, \bar{\tau}, \bar{f}_{\infty D})] \in (\Phi \times \Phi)^{-1}(Z_n^\infty),
\]

then there exists \( h \in \mathbb{N} \) with \( h \leq n(d-1) \) such that \( (Id \times F^h)y_0 = z_0 \), \( y_0 \) and \( z_0 \) being the characteristics of \((E_j, i_j, \tau)\) and \((\bar{E}_j, \bar{i}_j, \bar{\tau})\) respectively.
Proof. We assume the elliptic sheaves defined over a field $R$. By Definition 2.5, there exists a diagram (not necessarily commutative)

$$\begin{array}{ccc}
E_0 & \xrightarrow{T} & \tilde{E}_{n(d-1)} \\
\tau^n & \xrightarrow{(O_{\infty D})^{\otimes} R} & \tilde{\tau}^n \\
F^s E_{-n} & \xrightarrow{F^s T(-1)} & F^s \tilde{E}_{n(d-2)}
\end{array}$$

(oblique arrows are given by level structures). This diagram is commutative after tensoring by $O_{\infty D}$ and we therefore have that the morphism of vector bundles

$$T, \tau^n - \tilde{\tau}^n, F^s T(-1) : F^s E_{-n} \to \tilde{E}_{n(d-1)}$$

takes values in $\tilde{E}_{n(d-1)}(-\infty - D)$. However, we are in $\mathbb{P}_1$ and $\tilde{E}_{n(d-1)}$ and $F^s E_{-n}$ are semistable. Therefore,

$$\tilde{E}_{n(d-1)} \otimes (F^s E_{-n})^\vee (-\infty - D)$$

is also semistable of degree $-n$. In this way,

$$H^0(\mathbb{P}_1 \otimes R, \tilde{E}_{n(d-1)} \otimes (F^s E_{-n})^\vee (-\infty - D)) = 0.$$ 

Hence $T, \tau^n - \tilde{\tau}^n, F^s T(-1) = 0$. We therefore conclude because we are in the conditions of Proposition 3.3.

Let $K$ be the $F_q(t)$-algebra, $(t \to x)$, $K := F_q(x)[a_1, \ldots, a_{n-1}]$. Let $\phi$ be the rank-$n$ Drinfeld module with generic characteristic, defined over $K$:

$$\phi_t = \sigma^n + a_{n-1} \sigma^{n-1} + \cdots + a_1 \sigma + x.$$ 

Let $(E_j, i_j, \tau)$ be an elliptic sheaf associated with $\phi$ and $s \in H^0(\mathbb{P}_1 \otimes K, E_{1-n})$, with $t, s \equiv x, s + a_1, \sigma, s + \cdots + a_{n-1}, \sigma^{n-1}, s + \sigma^n, s$.

Bearing in mind Definition 3.4, we study when an $\infty$-level structure $f_{\infty}$ for $E_0$ is an $\infty$-level structure for the elliptic sheaf $(E_j, i_j, \tau)$.

Since $H^0(\mathbb{P}_1 \otimes K, E_0(1)) = H^0(\mathbb{P}_1 \otimes K, E_0) \oplus t H^0(\mathbb{P}_1 \otimes K, E_0)$ if we consider the bases $\{s, \sigma, s, \ldots, \sigma^{n-1}, s\}$ and $\{F^s(s), \sigma F^s(s), \ldots, \sigma^{n-1}, F^s(s)\}$ for $H^0(\mathbb{P}_1 \otimes K, E_0)$ and $H^0(\mathbb{P}_1 \otimes K, F^s E_0)$ respectively, the morphism $\tau^n : F^s E_0 \to E_0(1)$ is given in these basis by $A.t + B$, $A, B$ being matrices with entries in $K$.

Lemma 3.8. We have that

$$A.t + B = F^{s-1} C, F^{s-2} C \ldots F^{s} C, C$$

with

$$C = \begin{pmatrix}
0 & 0 & \cdots & 0 & t - x \\
1 & 0 & \cdots & 0 & -a_{n-1} \\
0 & 1 & \cdots & 0 & -a_{n-2} \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & -a_1
\end{pmatrix}$$
Proof. Bearing in mind Remark 3, \( \tau^n \) is given by

\[
\tau^n : F^{\#n} K\{\sigma\} \xrightarrow{\sigma^{\otimes n}} K\{\sigma\},
\]

where \( \sigma \) is the multiplication on the left over \( K\{\sigma\} \). In this setting, the bases \( \{s, \sigma.s, \ldots, \sigma^{n-1}.s\} \) and \( \{F^{\#n}(s), F^{\#n}(\sigma.s), \ldots, F^{\#n}(\sigma^{n-1}.s)\} \) are \( \{1, \sigma, \ldots, \sigma^{n-1}\} \) and \( \{F^{\#n}(1), F^{\#n}(\sigma), \ldots, F^{\#n}(\sigma^{n-1})\} \) respectively. We conclude bearing in mind that \( t.s = x.s + a_1.\sigma.s + \cdots + a_{n-1}.\sigma^{n-1}.s + \sigma^n.s \).

Now we shall calculate \( A_\infty \) (Remark 3).

Lemma 3.9. Let \( (E_j, i_j, \tau) \) be a rank-\( n \) elliptic sheaf, \( f_\infty \) an \( \infty \)-level structure over \( E_0 \), and \( (\nu_{ij})_{i,j} \) the matrix associated with

\[
f_\infty : H^0(\mathbb{P}_1 \otimes K, E_0) \to K^{\infty n}
\]

for the basis \( \{s, \sigma.s, \ldots, \sigma^{n-1}.s\} \) and the standard basis in \( K^n \). Then \( (E_0, f_\infty) \) is an \( \infty \)-level structure for the elliptic sheaf \( (E_j, i_j, \tau) \) if and only if \( (\nu_{ij}^0)_{i,j} = (\nu_{ij})_{i,j}.A \), where \( A \) is defined as in the above Lemma.

Proof. Because of the definition of \( \infty \)-level structures for elliptic sheaves, the diagram

\[
\begin{array}{ccc}
F^{\#n}E_0 & \xrightarrow{\tau^n} & E_0(1) \\
\downarrow F^{\#nf_\infty} & & \downarrow (f_\infty \otimes \pi_\infty) \\
\mathcal{O}_\infty & \to & (\mathcal{O}_\infty)^n \otimes R
\end{array}
\]

must be commutative.

Bearing in mind 1) of Proposition 2.3 and the above Lemma, we conclude by taking the bases \( \{F^{\#n}(s), F^{\#n}(\sigma.s), \ldots, F^{\#n}(\sigma^{n-1}.s)\} \) in \( H^0(\mathbb{P}_1 \otimes K, F^{\#n}E_0) \) and

\[
\{s, \sigma.s, \ldots, \sigma^{n-1}.s\} \cup t.\{s, \sigma.s, \ldots, \sigma^{n-1}.s\}
\]

for \( H^0(\mathbb{P}_1 \otimes K, E_0(1)) \).

\( A_\infty \) (Remark 3) is \( (\nu_{ij})_{i,j} \).

We shall now consider \( K_n^{\infty D} := K[\{\alpha_{ij}^n\}_{i,j,h}, \{\nu_{ij}\}_{i,j}] \) as a subring of an algebraic closed field containing \( K \), where the \( \alpha_{ij}^n \) are defined as in Remark 3 and "\( \nu_{ij} \)" are solutions for the equation \( (\nu_{ij}^n)_{i,j} = (\nu_{ij})_{i,j}.A \).

It is not hard to prove that there exists a morphism \( \text{Spec}(K_n^{\infty D}) \to \mathcal{E}_n^{\infty D} \).

Theorem 3.10. If we denote by \( \beta \) the morphism composition

\[
\text{Spec}(K_{n,\infty D}) \times \text{Spec}(K_{n,\infty D}) \to \mathcal{E}_n^{\infty D} \times \mathcal{E}_n^{\infty D} \xrightarrow{\Phi \times \Phi} \mathcal{M}_{\mathbb{P}_1}(n, 0, \infty D) \times \mathcal{M}_{\mathbb{P}_1}(n, 0, \infty D),
\]

then

\[
\beta^{-1}(Z_n^{\infty D})
\]

is the zero locus of the \( n^2 \)-functions deduced from an equation

\[
\Gamma_\infty \tilde{\Gamma}_{d-1} - Id_n = 0,
\]

\( \tilde{\Gamma}_{d-1}, \Gamma_\infty \) being \( n \times n \)-matrices with entries in \( K_n^{\infty D} \otimes K_n^{\infty D} \).
Proof. This Theorem is a direct consequence of Lemma 2.8 and one can make an explicit calculation of $\Gamma_{d-1, 1, \infty}$ bearing in mind Remark 1 for parameters $t_{x_i} = t - \alpha_i$, Lemma 3.9, and the Chinese remainder theorem: i.e., there exists a ring isomorphism

$$\delta : \prod_{i \in \{1, \cdots, t\}} \mathbb{F}_q[t]/(t - \alpha_i) \rightarrow \mathcal{O}_D := \mathbb{F}_q[t]/(p(t))$$

given by

$$\delta(h_1(t), \cdots, h_t(t)) = p_1(t - \alpha_1), h_1(t).p(t)/(t - \alpha_1)^{r_1} + \cdots + p_t(t - \alpha_t), h_t(t).p(t)/(t - \alpha_t)^{r_t}$$

with $\deg(p_i(t - \alpha_i)) < r_i$ and

$$1/p(t) = \sum_{i \in \{1, \cdots, t\}} p_i(t - \alpha_i)/(t - \alpha_i)^{r_i}.$$ 

Recall that $D = (p(t))_0$, with $p(t) = \prod_{i \in \{1, \cdots, t\}} (t - \alpha_i)^{r_i}$. 

\[\Box\]

3.4. Some explicit irreducible components for zeta correspondences. The finite group $GL_{O_D}(O_{\infty D}^n) = GL_n(\mathbb{F}_q) \times GL_D(O_D^n)$ acts on $K_{\infty D}^n$ by acting on the level structures. By considering the restrictions over $K_{\infty D}^n$

$$a_1 = q_1(x), \cdots, a_{n-1} = q_{n-1}(x)$$

with $q_1(x), \cdots, q_{n-1}(x) \in \mathbb{F}_q[x]$, if we denote

$$\tilde{K}_{\infty D}^n := K_{\infty D}^n/(a_1 - q_1(x), \cdots, a_{n-1} - q_{n-1}(x))$$

we have

$$\text{Spec}(\tilde{K}_{\infty D}^n)/GL_n(\mathbb{F}_q) \times GL_D(O_D^n) = \text{Spec}(\mathbb{F}_q(x)).$$

Lemma 3.11. Let $\tilde{\beta}$ be the restriction of $\beta$ to $\text{Spec}(\tilde{K}_{\infty D}^n) \times \text{Spec}(\tilde{K}_{\infty D}^n)$:

$$\tilde{\beta} : \text{Spec}(\tilde{K}_{\infty D}^n) \times \text{Spec}(\tilde{K}_{\infty D}^n) \rightarrow \mathcal{M}_{\psi, 1}^n(n, 0, \infty D) \times \mathcal{M}_{\psi, 1}^n(n, 0, \infty D).$$

Then,

$$(\tilde{\beta})^{-1}(\mathcal{Z}_{\infty D}^n) = \bigcup_{\text{For certain } g \in GL_{O_{\infty D}^n}(O_{\infty D}^n)} \Gamma_{g, F^i}.$$ 

$F$ is the Frobenius morphism and $\Gamma_{g, F^i}$ is the graph for the morphism $g.F^i$. $g$ is considered as an automorphism over $K_{\infty D}^n$.

Proof. Let $(E_j, i_j, \tau, f_{\infty D})$ and $(\tilde{E}_j, \tilde{i}_j, \tilde{\tau}, \tilde{f}_{\infty D})$ be two rank-$n$ elliptic sheaves with an $\infty D$-level structure defined over $K_{\infty D}^n$. For Proposition 3.8 if

$$[(E_0, f_{\infty D}), (\tilde{E}_0, \tilde{f}_{\infty D})] \in (\tilde{\beta})^{-1}(\mathcal{Z}_{\infty D}^n),$$

then $F^j$ (Characteristic($E_j, i_j, \tau, f_{\infty D}$)) = Characteristic($\tilde{E}_j, \tilde{i}_j, \tilde{\tau}, \tilde{f}_{\infty D}$) for some $j \in \mathbb{N}$. Thus, for some $g \in GL_{O_{\infty D}^n}(O_{\infty D}^n)$ is

$$g.F^j(E_j, i_j, \tau, f_{\infty D}) = (\tilde{E}_j, \tilde{i}_j, \tilde{\tau}, \tilde{f}_{\infty D})$$

because

$$\text{Spec}(\tilde{K}_{\infty D}^n) \rightarrow \text{Spec}(\tilde{K}_{\infty D}^n)/GL_{O_{\infty D}^n}(O_{\infty D}^n) = \text{Spec}(\mathbb{F}_q(x))$$

is the characteristic morphism. 

\[\Box\]
Lemma 3.12. Let \( s(t) \) be a monic polynomial with \( d - 1 - i = \deg(s(t)) \leq d - 1 \). We denote by \( h_{s(t)} \) the central matrix \( \text{Id}_n \times \text{diag}(s(t), \cdots, s(t)) \in \text{GL}_n(F_q) \times \text{GL}_D(O_D) \). Then,

\[
\Gamma_{h_{s(t)}.F^n} \subset (\tilde{\beta})^{-1}(Z_n^D).
\]

Moreover, if

\[
\Gamma_{g.F^{n-1}} \subset (\tilde{\beta})^{-1}(Z_n^D),
\]

then \( g = \text{Id} \).

Proof. It suffices to find \( T : F^{n,i}(E_0) \to E_0(d-1) \) such that the diagram

\[
\begin{array}{ccc}
F^{n,i}(E_0) & \xrightarrow{T} & E_0(d-1) \\
\downarrow h_{s(t).F^{n,i}} & & \downarrow f_{\infty D} \otimes \pi_{\infty D}^{-1} \\
(O_{\infty D})^n \otimes \bar{K}_n^D & & \\
\end{array}
\]

is commutative. However, since

\[
\begin{array}{ccc}
F^{n,i}(E_0) & \xrightarrow{\tau^{n,i}} & E_0(i) \\
\downarrow f_{\infty D} \otimes \pi_{\infty D}^{-1} & & \downarrow f_{\infty D} \otimes \pi_{\infty D}^{-1} \\
(O_{\infty D})^n \otimes \bar{K}_n^D & & \\
\end{array}
\]

is commutative because \( f_{\infty D} \) is an \( \infty D \)-level structure for the elliptic sheaf \( (E_j, \iota_j, \tau) \), it suffices to find \( T' : E_0(i) \to E_0(d-1) \) with

\[
\begin{array}{ccc}
E_0(i) & \xrightarrow{T'} & E_0(d-1) \\
\downarrow h_{s(t).F^{d-1}} & & \downarrow f_{\infty D} \otimes \pi_{\infty D}^{-1} \\
(O_{\infty D})^n \otimes \bar{K}_n^D & & \\
\end{array}
\]

commutative. We shall take \( T' \) defined as the homothety by \( s(t) \). It is clear that it is defined from \( E_0(i) \) to \( E_0(d-1) \), because \( \deg(s(t)) = d - 1 - i \). Moreover, the above diagram is commutative because

\[
\begin{array}{ccc}
E_0(i) & \xrightarrow{T'} & E_0(d-1) \\
\downarrow h_{s(t).F_{\infty D}} & & \downarrow f_{\infty D} \otimes \pi_{D} \\
(O_{D})^n \otimes \bar{K}_n^D & & \\
\end{array}
\]

is commutative by the definition of \( T' \). And

\[
\begin{array}{ccc}
E_0(i) & \xrightarrow{T'} & E_0(d-1) \\
\downarrow h_{s(t).F_{\infty D}} & & \downarrow f_{\infty D} \otimes \pi_{\infty D}^{-1} \\
(O_{\infty D})^n \otimes \bar{K}_n^D & & \\
\end{array}
\]

is commutative because \( s(t) \) is monic.

The second assertion of the Lemma is deduced because \( T \) takes values in

\[
\tau^{n(d-1)}(E_0^{n(d-1)}) \subset E_{n(d-1)}
\]
because of Proposition 3.3. We thus have $\tau^{n(d-1)} = T$, (up to isomorphisms). In this case, if 

$$(f_{\infty D} \otimes \sigma_{\infty D}^{d-1})^{n(d-1)} = g F^\# n^{(d-1)} f_{\infty D},$$

this implies that $g = 1d$ because of the definition of $\infty D$-level structure for an elliptic sheaf is 

$$\left( f_{\infty D} \otimes \sigma_{\infty D}^{d-1} \right)^{n(d-1)} = F^\# n^{(d-1)} f_{\infty D}. $$

From this Lemma we deduce:

**Theorem 3.13.** (c.f [An3], [AnDP], [Si]) For $n = 1$

$$(\tilde{\beta}_1)^{-1}(\Theta_{1,a \times 1}^\infty D) = \bigcup_{s(t) \in (P_q[t]/p(t))^c} \Gamma_{s(t), F^i}.$$

Here, $\tilde{\beta}_1$ denotes $m_{\infty D, \tilde{\beta}}$ and we follow the notation of Lemma 2.10.

We can now obtain the same result as in [An3] merely by considering all possibilities over the $\infty$-level structures: From Lemma 3.9 for rank 1, two $\infty$-level structures for a rank-1 elliptic sheaf differ in a $a \in F_q^\times$. Thus, 

$$\bigcup_{a \in F_q^\times} (\tilde{\beta}_1)^{-1}(\Theta_{1,a \times 1}^\infty D) = \bigcup_{s(t) \in (P_q[t]/p(t))^c} \Gamma_{s(t), F^i}.$$

Here, $a \times 1 \in F_q^\times \otimes O_D^\times$, $a \times 1$ acts on $\mathcal{M}^{\ast s}(1, 0, \infty D)$ (this scheme is the generalized Jacobian $J_{\tilde{F}_1}^{\infty D}$), and $\Theta_{1,a \times 1}^\infty D$ denotes the transformation of $\Theta_{1,a \times 1}^\infty D$ by $a \times 1$.

3.5. **Examples.** As above, $\mathbb{C}$ will be $\mathbb{F}_1$. 

**Example.** 1 ([An3], [Q]) We shall study $n = 1$, $p(t) = t(t-1)$. We thus denote $D$ by $0 + 1$, and $K_1^{\infty D} = F_q[x][\alpha_0, \alpha_1, \nu]$, where 

$$\alpha_0^2 + \alpha_0 x = 0, \alpha_1^2 + \alpha_1 (x-1) = 0,$$

and $\nu \in F_q^\times$. We can set $\nu = 1$. With the variable changes $u = \alpha_0$ and $v = \alpha_1$, we have 

$$F_q(x)[\alpha_0, \alpha_1, \nu] = F_q(u)[v, 1/v]/u^{q-1} - v^{q-1} + 1.$$

Now bearing in mind the Chinese remainder theorem, there exists a ring isomorphism 

$$\delta : F_q[t]/(t) \times F_q[t]/(t-1) = F_q[t]/(t(t-1)),$$

given by 

$$\delta(c, d) = ct + d(1-t).$$

In this way, $ct + d(1-t)$ is monic $\in (F_q[t]/(t(t-1)))^\times$ if and only if $c - d = 1$ and $c \neq 0, 1$.

For Theorem 3.13, 

$$(\tilde{\beta}_1)^{-1}(\Theta_{1,0+1}^{\infty D}) = \bigcup_{c \in F_q^\times} \Gamma_{[c, c-1]} \cup \Gamma_F,$$
Lemma 3.9, \( \tilde{\beta} \) defined in 3.13. By Theorem 3.10 and by the explicit calculations in Remark 9 and Lemma 3.9, \( \tilde{\beta}_1 \) has associated the 1 \times 1" matrix" 

\[
(1 \otimes u \times 1 \otimes v \times 1 \otimes 1)^{-1} \cdot (u \otimes 1 \otimes v \otimes 1 \otimes 1)
\]

and 

\[
(\tilde{\beta}_1)^{-1}(\Theta_{1_{\infty+0+1}}) = \left( \frac{u \otimes 1}{1 \otimes u} - \frac{v \otimes 1}{1 \otimes v} - 1 \otimes 1 \right) 0.
\]

If we consider \( \nu = a \in \mathbb{F}_q \), we obtain 

\[
(\tilde{\beta}_1)^{-1}(\Theta_{1_{\infty+0+1}}) = \left( \frac{u \otimes 1}{1 \otimes u} - \frac{v \otimes 1}{1 \otimes v} - a \otimes 1 \right) 0
\]

and 

\[
(\tilde{\beta}_1)^{-1}(\Theta_{1_{\infty+0+1}}) = \bigcup_{\mathbb{F}_q^\times} \Gamma_{[c,c-a]} \cup \Gamma_F
\]

**Example. 2** In this example we shall study the last example in a more general way. \( n = 1, \ p(t) = \prod_{i \in \{1, \ldots, l\}} (t - \alpha_i) \). Then, \( D = \alpha_1 + \cdots + \alpha_l \) and 

\[
\tilde{K}_{1+D} = \mathbb{F}_q(x)[\alpha_1, \ldots, \alpha_l, \nu],
\]

where \( \alpha_i + (x - \alpha_i) = 0 \) with \( 1 \leq i \leq l \) and \( \nu \in \mathbb{F}_q^\times \). We can set \( \nu = 1 \) as in the above example.

Let us consider the decomposition in simple fractions

\[
1/p(t) = \sum_{i \in \{1, \ldots, l\}} m_i/t - \alpha_i,
\]

\( m_i \in \mathbb{F}_q^\times \). The Chinese remainder theorem is then settled in the following way: There exists a ring isomorphism

\[
\delta : \prod_{i \in \{1, \ldots, l\}} \mathbb{F}_q[t]/(t - \alpha_i) \rightarrow \mathbb{F}_q[t]/(p(t))
\]

given by

\[
\delta(h_1, \cdots, h_l) = m_1.h_1.p(t)/t - \alpha_1 + \cdots + m_l.h_l.p(t)/t - \alpha_l.
\]

In this way, \( \delta(h_1, \cdots, h_l) \in (\mathbb{F}_q[t]/(p(t)))^\times \) if \( h_i \neq 0 \) for all \( i \).

For Theorem 3.13 \( (\tilde{\beta}_1)^{-1}(\Theta_{1_{\infty+0+1}}) \) is

\[
\bigcup_{0 \leq i \leq \deg(p(t))} \bigcup_{\deg(\delta(h_1, \cdots, h_l)) = \deg(p(t)) - i} \Gamma_{\delta(h_1, \cdots, h_l). F^i}
\]

\( \Gamma_{\delta(h_1, \cdots, h_l). F^i} \) is the graph of the morphism in \( \tilde{K}_{1+0+1} \) defined by \( \alpha_r \rightarrow h_r \alpha_r^{q}\), for each \( i \).

By Theorem 3.10 and by the explicit calculations in Remark 3 and Lemma 3.9, \( \tilde{\beta}_1 \) has associated the 1 \times 1" matrix" 

\[
(1 \otimes \alpha_1 \times \cdots \times 1 \otimes \alpha_l \times 1 \otimes 1)^{-1} \cdot (\alpha_1 \otimes 1 \times \cdots \times \alpha_l \otimes 1 \otimes 1)
\]
and 
\[(\hat{\beta}_1)^{-1}\Theta_1^{\infty+\alpha_1+\cdots+\alpha_l} = (m_1, \frac{\alpha_1 \otimes 1}{1 \otimes \alpha_1} + \cdots + m_l, \frac{\alpha_l \otimes 1}{1 \otimes \alpha_l} - 1 \otimes 1)_0.\]

**Example 3** We now consider \(n = 2\), \(p(t) = t\). Then, \(D = 0\) and we follow the notation of Remark 3:

\[K_2^{\infty+0} = \mathbb{F}_2(x)[\alpha^0_{1,0}, \alpha^0_{2,0}, \nu_1, \nu_2, \nu_2] \]

where 
\[(\alpha^0_{i,0})^q + q_1(x). (\alpha^0_{i,0})^q + \alpha^0_{i,0}. x = 0\]

for \(i = 1, 2\) and \((\nu_{ij})_i\) satisfies \((\nu^q_{ij})_i = (\nu_{ij})_i A\), with \(A\) defined as in Lemma 3.3.

Thus, 
\[A = \begin{pmatrix} 1 & -q_1(x) \\ 0 & 1 \end{pmatrix},\]

and hence \(\nu^q_{21} = \nu_{21}, \nu^q_{22} = \nu_{22}\) and \(\nu^q_{12} = \nu_{12} + q_1(x). \nu_{12} = 0, \nu^q_{22} - \nu_{22} + q_1(x). \nu_{22} = 0\).

We can set \(\nu_{21} = \nu_{22} = 1\) and \(\nu_{11} - \nu_{12} = 1\).

In this case, 
\[(\hat{\beta})^{-1}(Z_2^{\infty+0}) = \Delta\]

(\(\Delta\) denotes the diagonal correspondence) because there exists a morphism of level structures between two vector bundles with the same degree and with \(\infty D\)-level structures. This morphism must be an isomorphism (Proposition 2.4).

As before, by Theorem 3.10 and by the explicity calculations in Remark 3 and Lemma 3.3, \(\hat{\beta}\) has associated the 2 \(\times\) 2-matrix:

\[
\begin{pmatrix}
1 \otimes a^0_{1,0} & 1 \otimes (a^0_{1,0})^q \\
1 \otimes a^0_{2,0} & 1 \otimes (a^0_{2,0})^q
\end{pmatrix}^{-1}
= \begin{pmatrix}
\nu_{11} & \nu_{12} \\
\nu_{11} & \nu_{12}
\end{pmatrix}^{-1}
\times
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\]

If we denote by
\[A^0_{0,1} := \begin{pmatrix}
\nu_{11} & \nu_{12} \\
\nu_{11} & \nu_{12}
\end{pmatrix}, A^0_{1,0} := \begin{pmatrix}
\nu_{11} & \nu_{12} \\
\nu_{11} & \nu_{12}
\end{pmatrix}, A^1_{0,1} := \begin{pmatrix}
\nu_{11} & \nu_{12} \\
\nu_{11} & \nu_{12}
\end{pmatrix}, A^1_{1,0} := \begin{pmatrix}
\nu_{11} & \nu_{12} \\
\nu_{11} & \nu_{12}
\end{pmatrix}\]

then by Theorem 3.10,
\[\hat{\beta}^{-1}(Z_2^{\infty+0}) = (A^1_{\infty}(A^1_{0,1})^{-1} A^0_{0,1}(A^0_{\infty})^{-1} - \text{Id}) = 0\]

**Example 4** Now, \(n = 2\), \(p(t) = t(t - 1)\). Thus, \(D = 0 + 1\) and as in the above example we follow the notation of Remark 3:

\[K_2^{\infty+0} = \mathbb{F}_2(x)[\alpha^0_{1,0}, \alpha^0_{2,0}, \beta^0_{1,0}, \beta^0_{2,0}, \nu_1, \nu_2, \nu_2]\]

where \(\alpha^0_{i,0}\) is as in the above example and
\[(\beta^0_{1,0})^q + q_1(x). (\beta^0_{1,0})^q + \beta^0_{1,0}. (x - 1) = 0\]
for $i = 1, 2$ and $(\nu_i)_{ij}$ as before.

$A_{\infty}^0$, $A_{\infty}^1$, $A_{0,1}^0$, and $A_{0,1}^1$ follow the same notation as in the last example and $B_{0,1}^0$, and $B_{0,1}^1$ denote

$$
\begin{pmatrix}
\beta_{1,0}^0 \otimes 1 & (\beta_{1,0}^0)^\varphi \otimes 1 \\
\beta_{2,0}^0 \otimes 1 & (\beta_{2,0}^0)^\varphi \otimes 1 
\end{pmatrix},
\begin{pmatrix}
1 \otimes \beta_{1,0}^0 & 1 \otimes (\beta_{1,0}^0)^\varphi \\
1 \otimes \beta_{2,0}^0 & 1 \otimes (\beta_{2,0}^0)^\varphi 
\end{pmatrix}
$$

respectively.

Again by Theorem 3.10 and by the explicit calculations in Remark 9 and Lemma 3.9 $\tilde{\beta}$ has associated the $2 \times 2$-matrix:

$$
(A_{\infty}^1)^{-1} \cdot A_{0,1}^0 \times (B_{\infty}^1)^{-1} \cdot B_{0,1}^0 \times (A_{\infty}^1)^{-1} \cdot A_{0,1}^0
$$

and $(\tilde{\beta})^{-1}(Z_2^{\infty+0+1})$ is

$$(A_{\infty}^1(A_{0,1}^1)^{-1}, A_{0,1}^0(A_{\infty}^1)^{-1} - A_{\infty}^1(B_{0,1}^1)^{-1} \cdot B_{0,1}^0 \cdot (A_{\infty}^1)^{-1} - Id_2 = 0).$$

The coefficients $+$ and $-$ are deduced bearing in mind the Chinese remainder theorem. We shall now attempt to calculate the irreducible components of $(\tilde{\beta})^{-1}(Z_2^{\infty+0+1})$.

From Lemma 3.11, we know that these components are graphs $\Gamma_{g,F_i}$, where $i \leq 2$ and $g \in GL_2(F_q^\infty) \times GL_2(F_q) \times GL_2(F_q^1)$. We try to prove that $\Gamma_h \subset \Gamma_{F^2}$ and $\Gamma_{h_{d+i}}$, are contained in $(\tilde{\beta})^{-1}(Z_2^{\infty+0+1})$.

We know from Lemma 3.12 that the graphs $\Gamma_{F^2}$ and $\Gamma_{h_{d+i}}$, are contained in $(\tilde{\beta})^{-1}(Z_2^{\infty+0+1})$.

We try to prove that

$$(\tilde{\beta})^{-1}(Z_2^{\infty+0+1})^{-1} = \bigcup_{d \neq 0} \Gamma_{h_{d+i}} \cup \Gamma_{F^2}.$$ 

Let $g = g_{\infty} \times g_{0+1}$ be $B \times C \times D \in GL_2(F_q^\infty) \times GL_2(F_q) \times GL_2(F_q^1)$. If $\Gamma_g \subset (\tilde{\beta})^{-1}(Z_2^{\infty+0+1})$, we have

$$(A_{0,1})^{-1} \cdot C \cdot A_{0,1} - (B_{0,1})^{-1} \cdot D \cdot B_{0,1} = (A_{\infty})^{-1} \cdot B \cdot A_{\infty}$$

where

$$B_{0,1} := \begin{pmatrix}
\beta_{1,0}^0 & (\beta_{1,0}^0)^\varphi \\
\beta_{2,0}^0 & (\beta_{2,0}^0)^\varphi 
\end{pmatrix},
A_{0,1} := \begin{pmatrix}
\alpha_{1,0}^0 & (\alpha_{1,0}^0)^\varphi \\
\alpha_{2,0}^0 & (\alpha_{2,0}^0)^\varphi 
\end{pmatrix}
$$

and

$$A_{\infty} := \begin{pmatrix}
\nu_{1,1} & \nu_{1,2} \\
1 & 1
\end{pmatrix}.$$

By acting on this equality by $Id \times \Omega \times Id \in GL_2(F_q^\infty) \times GL_2(F_q) \times GL_2(F_q^1)$, we obtain the expression

$$(A_{0,1})^{-1} \cdot C \cdot A_{0,1} - (B_{0,1})^{-1} \cdot D \cdot B_{0,1} = (A_{\infty})^{-1} \cdot B \cdot A_{\infty}.$$

Thus, for each $\Omega \in GL_2(F_q)$ we have that $\Omega^{-1} \cdot C \cdot \Omega = C$, ans hence $C$ is a central matrix. In an analogous way, we can check that $B$ and $D$ must be central matrices. As $F_q$ acts on $K_2^{\infty+0+1}$ by the identity, we can assume that $B = Id$. Then, $C - D = Id$, and hence $g = h_{d+i}$.

In a similar way, one can check that there does not exist $g$ with

$$
\Gamma_{g,F} \subset (\tilde{\beta})^{-1}(Z_2^{\infty+0+1}).
$$
We thus obtain what were looking for:

\[(\tilde{\beta})^{-1}(Z_2^{\infty+0+1}) = \bigcup_{d \neq 0} \Gamma_{h_{d+1}} \cup \Gamma_{f^2}.
\]

4. Case of a general curve

In this section we shall study zeta subschemes for a general curve \(C\). We shall translate the results of the above sections to this case. Now, instead of \(\infty\) we shall consider a rational point \(p \in \text{Spec} (A)\) such that \(p \notin \text{Supp}(D)\). In this case, we shall state in a more general way the result of Lemma 3.11, and the relation between these zeta subschemes and the general theta divisor.

Let us recall some notation and results: \(\mathcal{M}_C(n, h)\) denotes the moduli stack, \([LM]\), of vector bundles of rank \(n\) and degree \(h\). In the same way, \(\mathcal{M}_C(n, h, p + D)\) denotes the moduli stack of pairs, \((M, f_{p + D})\), of vector bundles of rank \(n\) and degree \(h\) with a \(p + D\)-level structure. There exists a natural epimorphism of forgetting the level structures:

\[L : \mathcal{M}_C(n, h, p + D) \to \mathcal{M}_C(n, h)\]

Let us denote by \(\Theta_n \subset \mathcal{M}_C(n, n(g - 1))\) the general theta divisor \([DN]\): \(\Theta_n\) is the closed substack of \(\mathcal{M}_C(n, n(g - 1))\) of vector bundles \(M\) such that \(H^0(C, M) = 0\). We set \(\Theta_n^{\infty, D} := L^{-1}(\Theta_n)\).

By \(\mathcal{E}_n^{p+D}\) we denote the moduli (scheme for \(\text{deg}(D) > 0\)) of rank \(n\)-A-elliptic sheaves, \((E_j, i_j, \tau)\), with \(p + D\)-level structures. As in the preceding sections, we have a natural morphism

\[\Phi : \mathcal{E}_n^{p+D} \to \mathcal{M}_C(n, ng, p + D),\]

given by \(\Phi(E_j, i_j, \tau, f_{p + D}) = (E_0, f_{0}^{p+D})\).

In this section, \(\pi\) denotes the natural morphism \(\mathcal{O}_C(g + d - 1) \to \mathcal{O}_C/\mathcal{O}_C(-p - D)\).

As in the above sections

\[\mathcal{O}_{pD} := \mathcal{O}_C/\mathcal{O}_C(-p - D)\text{ and }\mathcal{O}_{pD} := H^0(C, \mathcal{O}_C/\mathcal{O}_C(-p - D)).\]

Remark 10. We consider the open substack

\[U_d \subseteq \mathcal{M}_C(n, h, p + D) \times \mathcal{M}_C(n, h, p + D)\]

of pairs \([(\tilde{M}, \tilde{f}_{p + D}), (M, f_{p + D})]\) satisfying \(H^1(C, \tilde{M}^\vee \otimes M(g + d - 1)) = 0\). In this case, if \([(\tilde{M}, \tilde{f}_{p + D}), (M, f_{p + D})]\) is a universal object for \(U_d\) we have

\[h^0(\tilde{M}^\vee \otimes M(g + d - 1)) = n^2d.\]

Let \(\{s_1, \ldots, s_{n^2d}\}\) be a basis for the space global sections of

\[\tilde{M}^\vee \otimes M(g + d - 1),\]

and we set

\[o_i = H^0(\tilde{f}_{p + D}^\vee \otimes f_{p + D} \otimes \pi)(s_i)\].

In this last Remark and in the following Lemmas we assume, for easy notation, that the global sections of \(\tilde{M}^\vee \otimes M(g + d - 1)\) are a free module of rank \(n^2d\). By
Grauert’s Theorem, it occurs locally, so instead of $U_d$ we should write $U_d^\alpha$, where $U_d = \cup U_d^\alpha$. The localization by $U_d^\alpha$ of the space of global sections of

$$M^\vee \otimes_{O_C \otimes O_{U_d}} M(g + d - 1)$$

is a $O_{U_d^\alpha}$-free module.

We shall now define an analogous subscheme to $Z_n^D$.

**Definition 4.1.** Let $S$ be a scheme. We define the $p + D$-generalized zeta substack, $Z_n^{p+D}$, as the closed substack of $U_d$ defined by: $[(\bar{M}, f_{p+D}), (M, f_{p+D})] \in \text{Hom}_{\text{stacks}} (S, Z_n^{p+D})$ if over the open subscheme, $\bar{U}$, of $S$ formed by the $s \in S$ such that

$$H^0(C \otimes k(s), \bar{M}_s^\vee \otimes_{O_{C \otimes k(s)}} M_s((g + d - 1)\infty - p - D)) = 0$$

there exists a morphism of modules $T : \bar{M}_\bar{U} \to M(g + d - 1)_{\bar{U}}$ ($d = \deg(D)$), where the diagram:

$$\begin{array}{ccc}
\bar{M}_{\bar{U}} & \xrightarrow{T} & M(g + d - 1)_{\bar{U}} \\
\downarrow f_{p+D} & & \downarrow f_{p+D} \otimes \pi \\
(O_{pD})^n \otimes O_{\bar{U}} & &
\end{array}$$

is commutative.

**Remark 11.** From the last Lemma, if

$$[(\bar{M}, f_{p+D}), (M, f_{p+D})] \in \text{Hom}_{\text{stacks}} (S, Z_n^{p+D})$$

then there exists an open subscheme $\bar{U} \subset S$ such that over this open subscheme there exists a non-trivial morphism $T : \bar{M} \to M(g + d - 1)$. And if $s \not\in \bar{U}$, there exists a non-trivial morphism $T_s : \bar{M}_s \to M_s((g + d - 1)\infty - p - D)) \subset M(g + d - 1)_s$.

Hence $Z_n^{p+D} = V^{p+D}_n \cup (V^{p+D}_n)^\circ$, with $V^{p+D}_n$ an open substack of $Z_n^{p+D}$.

**Remark 12.** A pair $[(\bar{M}, f_{p+D}), (M, f_{p+D})] \in \text{Hom}_{\text{stacks}} (\text{Spec}(R), Z_n^{p+D})$ gives a morphism

$$\bar{f}_{p+D}^\vee \otimes f_{p+D} \otimes \pi : \bar{M}_{\bar{U}}^\vee \otimes_{RC} M(g + d - 1) \to (O_{pD})^n \otimes R.$$ 

Therefore, by taking global sections we obtain a morphism

$$H^0(\bar{f}_{p+D}^\vee \otimes f_{p+D} \otimes \pi) : H^0(O_{C} \otimes R, \bar{M}_{\bar{U}}^\vee \otimes_{RC} M(g + d - 1)) \to (O_{pD})^n \otimes R.$$ 

We shall denote by $\Xi$ the natural morphism

$$\Xi : O_C \otimes R \to \bar{M}_{\bar{U}}^\vee \otimes_{RC} \bar{M}_{\bar{U}}^\vee \otimes_{f_{p+D} \otimes \pi} \otimes (\bar{O}_{pD})^n \otimes R$$

where the first morphism in the composition is the diagonal morphism. Therefore, by taking global sections we obtain a morphism

$$H^0(\Xi) : R \to (O_{pD})^n \otimes R$$

given by the diagonal matrix with entries over $O_{pD}$.

Let us now consider the morphism

$$\Sigma := H^0(\Xi) + H^0(\bar{f}_{p+D}^\vee \otimes f_{p+D} \otimes \pi)$$
from $R \oplus H^0(\mathcal{O}_C \otimes R, \bar{M}^\vee_{\mathcal{O}_C \otimes R} \otimes M(g + d - 1))$ to $(O_{pD})^n \otimes R$.

Lemma 4.2. A pair $[(\bar{M}, \bar{f}_{p+D}), (M, f_{p+D})] \in \text{Hom}_{\text{stacks}}(\text{Spec}(R), U_d)$ belongs to $\text{Hom}_{\text{stacks}}(\text{Spec}(R), Z_n^{p+D})$

if and only if $\Sigma$ is not injective.

Proof. Since if $s \in \text{Spec}(R)$ satisfies

$$H^0(C \otimes k(s), M_s^{\vee} \otimes M_s((g + d - 1)\infty - p - D)) \neq 0$$

we have $\text{Ker}(H^0(f_{p+D}^\vee \otimes f_{p+D} \otimes \pi)_s) \neq 0$. Thus, $\Sigma_s$ is not injective and we can assume that for all $s \in \text{Spec}(R)$ we have:

$$H^0(C \otimes k(s), M_s^{\vee} \otimes M_s((g + d - 1)\infty - p - D)) = 0.$$ 

If there exists a morphism $T$ between the level structures considered

$$\bar{M} \xrightarrow{f_{p+D}} M(g + d - 1)$$

by tensoring by $\bar{M}^\vee$ and considering $\bar{f}_{p+D}$, we obtain a section

$$w \in H^0(\mathcal{O}_C \otimes R, \bar{M}^\vee_{\mathcal{O}_C \otimes R} \otimes M(g + d - 1))$$

such that

$$H^0(\Xi)(1) = H^0(\bar{f}_{p+D}^\vee \otimes f_{p+D} \otimes \pi)(w).$$

Thus, in this case $1 \oplus -w \in \text{Ker}(\Sigma)$.

Conversely, if $1 \oplus -s \in \text{Ker}(\Sigma)$ we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_C \otimes R & \xrightarrow{\Xi} & \bar{M}^\vee_{\mathcal{O}_C \otimes R} \otimes M(g + d - 1) \\
\downarrow f_{p+D}^\vee \otimes f_{p+D} \otimes \pi & & \\
(O_{pD})^n \otimes R
\end{array}$$

Therefore, we conclude by tensoring by $\bar{M}$ and bearing in mind $\bar{f}_{p+D}$.

\[ \square \]

Remark 13. If $Z_n^{p+D} = V_n^{p+D} \cup (V_n^{p+D})^c$, we deduce that that:

$$(V_n^{p+D})^c = \{[(\bar{M}, \bar{f}_{p+D}), (M, f_{p+D})] \in Z_n^{p+D}, \text{ with } \text{Ker}(H^0(\bar{f}_{p+D}^\vee \otimes f_{p+D} \otimes \pi)) \neq 0\}$$

and $[(\bar{M}, \bar{f}_{p+D}), (M, f_{p+D})] \in V_n^{p+D}$ if

$$H^0(\Xi)(1) \in H^0(\bar{f}_{p+D}^\vee \otimes f_{p+D} \otimes \pi).$$

Remark 14. From $H^0(\Xi)(1)$ and $\{o_1, \cdots, o_{n^2d}\}$, we obtain a $(n^2d + 1)(\text{columns}) \times (n^2d + 1)(\text{rows})$-matrix $(H^0(\Xi)(1), o_1, \cdots, o_{n^2d})$ with entries over $H^0(U_d, \mathcal{O}_{U_d})$. Recall that $\{H^0(\Xi)(1), o_1, \cdots, o_{n^2d}\} \subset (O_{pD})^n \otimes H^0(U_d, \mathcal{O}_{U_d})$ and $\text{dim}_{\mathbb{Q}} O_{pD} = d + 1$. 

Let \( \tilde{U}_d \) be the open substack of \( U_d \) such that \( [(\tilde{M}, \tilde{f}_{p+D}), (M, f_{p+D})] \in \tilde{U}_d \) if and only if \( h^0(\tilde{M}^\vee \otimes M(g + d - 1)(-p - D)) = 0 \). In this case, \( \text{rank}(o_1, \cdots, o_{n^2d}) = n^2d \).

**Lemma 4.3.** \( Z_n^{p+D} \) is a closed substack of \( U_d \) and is formed by the pairs
\[
[(\tilde{M}, \tilde{f}_{p+D}), (M, f_{p+D})] \in U_d,
\]
such that
\[
\text{rank}((O_{p+D})^{\otimes n^2} < H^0(\Xi)(1), o_1, \cdots, o_{n^2d}) \geq n^2.
\]
Moreover, \( V_n^{p+D} = \tilde{U}_d \cap Z_n^{p+D} \) is locally the zero locus of \( n^2 \)-regular functions.

**Proof.** The first assertion is easily deduced from Lemma 4.2.
For the second assertion, we denote by
\[
N_i^{(\text{H}^0(\Xi)(1), o_1, \cdots, o_{n^2d})}(1 \leq i_1 < \cdots < i_{n^2d+1} \leq n^2(d + 1)) \text{ the } n^2d + 1 \text{-minor of the matrix } (H^0(\Xi)(1), o_1, \cdots, o_{n^2d}),
\]
where the \( k \)-th row is the \( i_k \)-th row of \( (H^0(\Xi)(1), o_1, \cdots, o_{n^2d}) \).

Analogously, we set
\[
N_i^{(o_1, \cdots, o_{n^2d})}(1 \leq i_1, \cdots, i_{n^2d} \leq n^2) \text{ for the } (n^2d)(\text{columns}) \times (n^2)(\text{rows})\text{-matrix } (o_1, \cdots, o_{n^2d}).
\]
Let us take the open covering of \( \tilde{U}_d \)
\[
\tilde{U}_d = \bigcup_{1 \leq i_1, \cdots, i_{n^2d} \leq n^2(d + 1)} \tilde{U}_d^{i_1, \cdots, i_{n^2d}}
\]
\( \tilde{U}_d^{i_1, \cdots, i_{n^2d}} \) being the open substack of \( \tilde{U}_d \) of pairs \( [(\tilde{M}, \tilde{f}_{p+D}), (M, f_{p+D})] \in U_d \), such that
\[
\text{rank}(N_i^{(o_1, \cdots, o_{n^2d})}) = \text{rank}(o_1, \cdots, o_{n^2d}) = n^2d.
\]
Thus, we have that \( \tilde{U}_d^{i_1, \cdots, i_{n^2d}} \cap Z_n^{p+D} \) is the zero locus of the functions
\[
\{ \det(N_i^{(\text{H}^0(\Xi)(1), o_1, \cdots, o_{n^2d})}), \cdots, \det(N_i^{(\text{H}^0(\Xi)(1), o_1, \cdots, o_{n^2d})}) \}
\]
with \( \{1, 2, \cdots, n^2(d + 1)\} \setminus \{i_1, \cdots, i_{n^2d}\} = \{j_1, j_2, \cdots, j_{n^2d + n^2}\} \).

If
\[
[(\tilde{M}, \tilde{f}_{p+D}), (M, f_{p+D})] \in \text{Hom}_{\text{stacks}}(\text{Spec}(R), \tilde{U}_d),
\]
we have
\[
H^0(C \otimes R, \tilde{M}^\vee \otimes M(g + d - 1)(-p - D)) = 0
\]
and
\[
H^1(C \otimes R, \tilde{M}^\vee \otimes M(g + d - 1)) = 0.
\]
In this way,
\[
H^0(f^\vee_{p+D} \otimes f_{p+D} \otimes \pi)(H^0(C \otimes R, \tilde{M}^\vee \otimes M(g + d - 1)))
\]
is a rank \( n^2d \)-sub-vector bundle of \( (O_{pD})^{n^2} \otimes R \). Thus,
\[
\bigwedge (H^0(f^\vee_{p+D} \otimes f_{p+D} \otimes \pi)(H^0(M^\vee \otimes M(g + d - 1)))) \in \mathbb{P} \bigwedge \bigwedge (O_{pD})^{n^2} \otimes R
\]
and we obtain a morphism
\[ \tilde{U}_d \rightarrow \mathbb{P}(\bigwedge^n (O_{P,D}))^{n^2 d}. \]

It is not hard to prove that
\[ \det(N_{i_1,\cdots,i_{n^2 d},j_{n^2 d+k}}(H^0(\Xi)(1),o_1,\cdots,o_{n^2 d})) \]
can be considered as the restriction to \( \tilde{U}_d \) of a certain hyperplane section of \( \mathbb{P}(\bigwedge^n (O_{P,D}))^{n^2 d} \).

In this way, \( U_d^{i_1,\cdots,i_{n^2 d}} \cap Z^{p+D}_1 \) is obtained as the intersection of \( n^2 d \) hyperplane sections of \( \mathbb{P}(\bigwedge^n (O_{P,D}))^{n^2 d} \).

For \( n = 1 \), we have a direct relation of \( Z^{p+D}_1 \) with a theta divisor. We take \( d > 2g - 2 \).

**Definition 4.4.** We define the \( p+D \)-generalized theta divisor, \( \Theta^{p+D} \), as the closed subscheme of \( J^{p+D}_{C,0} \) defined by:
\[ \Theta^{p+D} = \text{Closed subscheme of } J^{p+D}_{C,0} \text{ such that } (L, f_{p+D}) \in \text{Hom}_{\text{schemes}}(\text{Spec}(k), \Theta^{p+D}) (k \text{ a field}) \text{ if and only if there exists a morphism of modules } \]
\[ T: O_C \otimes R \rightarrow L(g + d - 1) \]
\( (d = \deg(D)) \), where the diagram:
\[ \begin{array}{ccc}
O_C \otimes K & \xrightarrow{T} & L(g + d - 1) \\
\downarrow & & \downarrow \\
O_{p+D} \otimes K & \xrightarrow{f_{p+D} \otimes \pi} & (L, f_{p+D})
\end{array} \]

is commutative. The vertical arrow is the natural epimorphism.

Let us consider \( J^{p+D}_{C,g} \), where \( J^{p+D}_{C,g} \) is the \( p+D \)-generalized Rosenlich’s Jacobian (c.f. [Se]).

**Remark 15.** By regarding the morphism
\[ m^{p+D}_{C,g} : J^{p+D}_{C,g} \times J^{\infty}_{C,g} \rightarrow J^{p+D}_{C,0} \]
defined by
\[ m^{p+D}_{C,g}([L, f_{p+D}],[L, f_{p+D}]) = (L', f'_{p+D}) \otimes (L, f_{p+D}), \]
we have that \( m_{\infty}^{-1}(\Theta^{\infty}_D) = Z^{\infty}_1 \) for \( d > 2g - 2 \), because of the definition of \( Z^{\infty}_1 \) and \( U_d = J^{p+D}_{C,g} \times J^{\infty}_{C,g} \).

Moreover, from the last Lemma we can say that \( U_d^{i_1,\cdots,i_{n^2 d}} \cap Z^{p+D}_1 \) is the zero locus of the function
\[ \det(N_{i_1,\cdots,i_{n^2 d}}) \]
over \( U_d^{i_1,\cdots,i_{n^2 d}} \) (for \( n = 1 \) \( H^0(\Xi)(1) = 1 \in O_{P,D} \)). For different \( \{i_1, \cdots, i_d\} \), these determinants are the determinant \( \det(1, o_1, \cdots, o_d) \) up to a sign. Then,
\[ \tilde{U}_d \cap Z^{p+D}_1 = \tilde{U}_d \cap (\det(1, o_1, \cdots, o_d))_0. \]

Recall that
\[ \tilde{U}_d = \bigcup_{1 \leq i_1 < \cdots < i_{n^2 d} \leq d+1} \tilde{U}_d^{i_1,\cdots,i_{n^2 d}}. \]
On the other hand, as $(\bar{U}_d)^c$ is formed by the pairs where \(\{o_1, \cdots, o_d\}\) are linearly dependent, then
\[
(\bar{U}_d)^c \subseteq (\text{det}(1, o_1, \cdots, o_d))_0,
\]
and we conclude that \(Z_1^{p+D} = (\text{det}(1, o_1, \cdots, o_d))_0\).

Recall that actually this result is settled locally over \(J_{C,g}^{p+D} \times J_{C,g}^{p+D}\). To be precise, one must take a universal object \([(\bar{L}, \bar{f}), (L, f)]\) and an open covering for \(J_{C,g}^{p+D} \times J_{C,g}^{p+D}\) such that
\[
p_*(\bar{L}^\vee \otimes_{\mathcal{O}_C \otimes O_J} L)
\]
is trivialized by this covering. \(p\) denotes the natural projection \(C \times J_{C,g}^{p+D} \rightarrow J_{C,g}^{p+D}\).

4.1. **Zeta subschemes for rank-\(n\)-Shtukas.** In this section we shall study zeta subschemes in the case of rank-\(n\)-Shtukas and \(A\)-Drinfeld modules defined over an arbitrary curve \(C\). Professor G.W. Anderson has suggested that I should study this topic.

First, we shall state some previous definitions.

**Definition 4.5.** A rank-\(n\) shtuka over \(C \times \text{Spec}(R)\) is a diagram of rank-\(n\)-vector bundles over \(C \times \text{Spec}(R)\)
\[
L_r \mapsto L_{r+1} \leftarrow F^\# L_r,
\]
such that \(L_{r+1}/j(L_r)\) and \(L_{r+1}/t(F^\# L_r)\) are line bundles as \(R\)-modules and they are supported by two morphisms \(\Gamma_\alpha\) and \(\Gamma_\gamma\) \(\text{Spec}(R) \rightarrow C\); the pole and zero, respectively. We denote the line bundles associated with \(\Gamma_\alpha\) and \(\Gamma_\gamma\) by \(O_{C, \otimes \mathcal{O}_J}^\alpha\) and \(O_{C, \otimes \mathcal{O}_J}^\gamma\) respectively. We assume \(j(L_r) + t(F^\# L_r) = L_{r+1}\). We set \(\text{deg}(L_r) = ng + r\).

**Definition 4.6.** Similar to the above sections, to give a \(p + D\)-level structure over
\[
L_r \mapsto L_{r+1} \leftarrow F^\# L_r,
\]
is to give level structures \((L_r, f_{p+D}^r)\) and \((L_{r+1}, f_{p+D}^{r+1})\) with commutative diagrams
\[
\begin{array}{ccc}
L_r & \xrightarrow{j} & L_{r+1} \\
\downarrow f_{p+D}^r & & \downarrow f_{p+D}^{r+1} \\
(O_{p+D}^n) \otimes R & & (O_{p+D}^n) \otimes R
\end{array}
\]
and
\[
\begin{array}{ccc}
F^\# L_r & \xrightarrow{t} & L_{r+1} \\
\downarrow F^\# f_{p+D}^r & & \downarrow F^\# f_{p+D}^{r+1} \\
(O_{p+D}^n) \otimes R & & (O_{p+D}^n) \otimes R
\end{array}
\]
p \notin \{\alpha, \gamma, \text{sup}(D)\}.

**Definition 4.7.** An isogeny between two rank-\(n\) Shtukas
\[
L_r \xrightarrow{j} L_{r+1} \leftarrow F^\# L_r
\]
and
\[
N_h \xrightarrow{j} N_{h+1} \leftarrow F^\# N_h
\]
is defined by two morphisms of $O_C \otimes R$-modules $T : \mathcal{L}_r \to \mathcal{N}_h$ and $H : \mathcal{L}_{r+1} \to \mathcal{N}_{h+1}$ such that $j'.T = H.j$ and $t'.F^\#T = F^\#H.t$.

Analogously, one can define a $p + D$-isogeny between shtukas with $p + D$-level structures, except that one must take $T$ and $H$ compatible with the level structures.

We now consider the stack, $\text{Cht}^{n,ng+r}$, of shtukas of rank $n$ and $\deg(\mathcal{L}_r) = ng+r$. We follow the same notation for $\text{Cht}^{n,ng+r}$ but considering $p + D$-level structures. Let $P^{n,ng+r}$ be the closed substack of $\text{Cht}^{n,ng+r} \times \text{Cht}^{n,ng+r}$ of shtukas with the same pole. Analogously, we define $P^{n,ng+r}$.

**Proposition 4.8.** Let $[(\mathcal{L}_r, \mathcal{L}_{r+1}, j, t), (\mathcal{N}_h, \mathcal{N}_{h+1}, j', t')]$ be an element of $P^{n,ng+r}$ with pole $\alpha$ and defined over a $F_q$-reduced algebra $R$. Then, there exists an isogeny between these shtukas if and only if there exists a morphism of modules $T : \mathcal{L}_r \to \mathcal{N}_h$, together with the commutative diagram

$$
\begin{array}{ccc}
\mathcal{L}_r(\alpha) & \xrightarrow{T(\alpha)} & \mathcal{N}_h(\alpha) \\
\kappa \downarrow & & \kappa' \downarrow \\
\mathcal{L}_{r+1} & \xrightarrow{\kappa} & \mathcal{N}_{h+1} \\
\downarrow j & & \downarrow j' \\
F^\# \mathcal{L}_r & \xrightarrow{T} & F^\# \mathcal{N}_h
\end{array}
$$

$T(\alpha)$ denotes the morphism induced by $T$ from $\mathcal{L}_r(\alpha)$ to $\mathcal{N}_h(\alpha)$. The injective morphisms $\kappa : \mathcal{L}_{r+1} \to \mathcal{L}_r(\alpha)$ and $\kappa' : \mathcal{N}_{h+1} \to \mathcal{N}_h(\alpha)$ are morphisms such that the composition with $j$ and $j'$ are the morphisms $\mathcal{L}_r \hookrightarrow \mathcal{L}_r(\alpha)$ and $\mathcal{N}_h \hookrightarrow \mathcal{N}_h(\alpha)$ induced by the natural inclusion $O_C \otimes R \subset O_C \otimes R(\alpha)$. Recall that the cokernels of $\mathcal{L}_r \to \mathcal{L}_{r+1}$ and $\mathcal{N}_h \to \mathcal{N}_{h+1}$ are supported over $\alpha$.

**Proof.** This can be deduced because

$$
\begin{array}{ccc}
\mathcal{L}_r(\alpha) & \xrightarrow{T(\alpha)} & \mathcal{N}_h(\alpha) \\
\kappa \downarrow & & \kappa' \downarrow \\
\mathcal{L}_{r+1} & \xrightarrow{\kappa} & \mathcal{N}_{h+1} \\
\downarrow j & & \downarrow j' \\
\mathcal{L}_r & \xrightarrow{T} & \mathcal{N}_h
\end{array}
$$

is commutative, and

$$j(\mathcal{L}_r) + t(F^\# \mathcal{L}_r) = \mathcal{L}_{r+1}, \quad j'(\mathcal{N}_h) + t'(F^\# \mathcal{N}_h) = \mathcal{N}_{h+1}.
$$

\[\square\]

Let consider us the natural morphism $\tilde{\Phi} : \text{Cht}^{n,ng+r}_{p+D} \to \mathcal{M}_C(n-ng+r,p+D)$ given by $\tilde{\Phi}((\mathcal{L}_r, f_{r+D}^r), (\mathcal{L}_{r+1}, f_{r+1}^{r+1}, j, t)) = (\mathcal{L}_r, f_{r+D}^r)$. Hence, we deduce a morphism

$$\tilde{\beta} : P^{n,ng+r}_{p+D} \to \text{Cht}^{n,ng+r}_{p+D} \times \text{Cht}^{n,ng+r}_{p+D} \xrightarrow{\tilde{\Phi} \times \tilde{\Phi}} \mathcal{M}_C(n-ng+r,p+D) \times \mathcal{M}_C(n-ng+r,p+D).
$$

For easy notation, we denote by $(\mathcal{L}_r, \mathcal{L}_{r+1}, f_{r+D})$ an element of $\text{Cht}^{n,ng+r}_{p+D}$. Because $p \neq \alpha$ in the following Lemma, we are be able to set, for shtukas, the results of
Lemma 3.7 in a more precise way. This Lemma attempts to be an approximation for shtukas with level structures of Lemma 3.3 [An2].

**Lemma 4.9.** Let \([(L_0, L_1, f_{p+D}), (\mathcal{N}_0, \mathcal{N}_1, g_{p+D})] \in \bar{\beta}^{-1}(\mathbb{Z}_n^{p+D})\) be defined over a field \(K\). Then:

1) There exists a \(p + D\)-isogeny between the shtukas with \(p + D\)-level structures \([(L_0, L_1, f_{p+D})\) and \((\mathcal{N}_0, \mathcal{N}_1, g_{p+D} \otimes \pi)\].

2) There exists an isogeny between the shtuka \((L_0, L_1, j, t)\) and the twisted shtuka \((\mathcal{N}_0, \mathcal{N}_1, j', t') \otimes \mathcal{O}_C((g + d - 1)\infty - p - D)\).

3) \((F^\# L_0)^\vee \otimes \mathcal{O}_C(\mathcal{N}_0((g + d - 1)\infty - p - D + \alpha) \in \Theta_n^2\).

**Proof.** Let us consider the diagram (not necessarily commutative)

\[
\begin{array}{ccc}
L_0(\alpha) & \xrightarrow{T(\alpha)} & N_0(g + d - 1)(\alpha) \\
\downarrow{\kappa} & & \downarrow{\kappa'} \\
L_1 & \xrightarrow{(\mathcal{O}_D^p)^n \otimes R} & N_1(g + d - 1) \\
\uptau \downarrow & & \uptau' \downarrow \\
F^\# L_0 & \xrightarrow{F^\# T} & F^\# N_0(g + d - 1)
\end{array}
\]

\(T\) is a morphism given by \([(L_0, f_{p+D}^0), (\mathcal{N}_0, g_{p+D}^0)] \in \mathbb{Z}_n^{p+D}\). The diagonal morphisms are the level structures and the vertical morphisms \(\kappa\) and \(\kappa'\) are induced by \(j\) and \(j'\) since these two shtukas have the same pole, \(\alpha\). From Definition 3.3 we have two possibilities: either \(T : L_0 \rightarrow N_0(g + d - 1)\) takes values in \(N_0((g + d - 1)(-p - D))\) or the latter diagram is commutative after tensoring by \(\mathcal{O}_D\). In both cases, one obtains a morphism:

\[T(\alpha), \kappa, t - \kappa', t'. F^\# T : F^\# L_0 \rightarrow N_0(g + d - 1)\]

When \(T(\alpha), \kappa, t - \kappa', t'. F^\# T = 0\), we obtain cases 1) and 2) and when it is \(\neq 0\) it tell us that \((F^\# L_0)^\vee \otimes \mathcal{O}_C(\mathcal{N}_0((g + d - 1)\infty - p - D + \alpha) \in \Theta_n^2\).

4.2. Zeta subschemes and \(A\)-Drinfeld modules. In this subsection, by means of the natural immersion of elliptic sheaves into shtukas (c.f. [Dr3]), we shall be able to translate the results of the above section to the case of Drinfeld modules. We use the notation and definitions of section 3.

**Proposition 4.10.** There exists an immersion of stacks

\[
\Upsilon : E_{n}^{p+D} \hookrightarrow Ch_{n}^{p+D}
\]

**Proof.** This is defined by

\[
\Upsilon(E_i, i_j, \tau, f_{p+D}) = ((E_0, f_{p+D}^0), (E_1, f_{p+D}^1), i_0, \tau)
\]

This immersion is given in [Dr3] (1.3). □

We recall the definition of isogeny for \(A\)-Drinfeld modules
Definition 4.11. An isogeny of degree \( r \) between two Drinfeld modules \( \phi \) and \( \tilde{\phi} \) is a polynomial of degree \( r \) in \( \sigma, q(\sigma) \), such that the endomorphism \( q(\sigma) : (\mathbb{G}_a)_R \to (\mathbb{G}_a)_R \) satisfies
\[
q(\sigma)(\phi_a(\lambda)) = \tilde{\phi}_a(q(\sigma)(\lambda))
\]
for all \( a \in A \).

Definition 4.12. Two elliptic sheaves \( (\tilde{E}_j, \tilde{i}_j, \tilde{\tau}) \) and \( (E_j, i_j, \tau) \) are said to be isogenous by an isogeny of degree \( \leq r \) if and only if there exist morphisms of modules \( T_j : \tilde{E}_j \to E_{j+r} \), for each \( j \), compatible with the diagrams that define the elliptic sheaves.

Analogously, one can define a \( p + D \)-isogeny between elliptic sheaves with \( p + D \)-level structures, except that one must take “\( T_j \)” compatible with the level structures.

There exists an equivalence between the notion of isogenous elliptic sheaves and isogenous Drinfeld modules:

Proposition 4.13. Two Drinfeld modules \( \phi \) and \( \tilde{\phi} \) associated with elliptic sheaves \( (\tilde{E}_j, \tilde{i}_j, \tilde{\tau}) \) and \( (E_j, i_j, \tau) \), respectively, are isogenous by an isogeny of degree \( \leq r \) if and only if there exists an isogeny of degree \( \leq r \) between the elliptic sheaves \( (\tilde{E}_j, \tilde{i}_j, \tilde{\tau}) \) and \( (E_{j+r}, i_{j+r}, \tau) \).

The following Proposition together with Proposition 4.8 tell us that two elliptic sheaves are isogenous as shtukas if and only if they are isogenous as elliptic sheaves. Recall that the elliptic sheaves considered have \( \infty \) as their pole.

Proposition 4.14. Two elliptic sheaves \( (\tilde{E}_j, \tilde{i}_j, \tilde{\tau}) \) and \( (E_j, i_j, \tau) \) defined over a \( \mathbb{F}_q \)-reduced algebra, \( R \), are isogenous by an isogeny of degree \( \leq r \) if and only if there exists a morphism of vector bundles
\[
T : \tilde{E}_0 \to E_r
\]
with the diagram
\[
\begin{array}{c}
\tilde{E}_0 \\
\uparrow \tilde{\tau} \\
F^# \tilde{E}_{-n} \\
\downarrow \tau \\
E_r \\
\end{array}
\begin{array}{c}
\downarrow T \\
F^# \tau \\
\uparrow T(-1) \\
F^# E_{r-n} \\
\end{array}
\]
commutative. Here, \( T(-1) \) denotes the morphism induced by \( T \) over
\[
\tilde{E}_{-n} \to E_{r-n},
\]
bearing in mind that \( E_{k-n} := E_k(-\infty) \).

Proof. Since \( \tilde{E}_{1-n} = \tilde{\tau}(F^# \tilde{E}_{-n}) + \tilde{E}_{-n} \subset \tilde{E}_0 \) and \( E_{r+1-n} = \tau(F^# E_{r-n}) + E_{r-n} \subset E_r \), by observing the latter diagram we have \( T(\tilde{E}_{1-n}) \subset E_{r+1-n} \). Hence, we obtain the commutative diagram
\[
\begin{array}{ccc}
\tilde{E}_0 & \xrightarrow{T} & E_r \\
\uparrow & & \uparrow \\
\tilde{E}_{1-n} & \xrightarrow{T_{1-n}} & E_{r+1-n} \\
\downarrow & & \downarrow \\
\tilde{E}_{-n} & \xrightarrow{T(-1)} & E_{r-n}
\end{array}
\]
where $T_{1-n}$ denotes the restriction of $T$ to $\tilde{E}_{1-n}$ and the injective morphisms are the "$i$" and "$\tau$" inclusions given in the elliptic sheaves.

From the last commutative diagram we also obtain a commutative diagram

\[
\begin{array}{ccc}
F^# \tilde{E}_{1-n} & \xrightarrow{F^# T_{1-n}} & F^# E_{r+1-n} \\
\downarrow & & \downarrow \\
F^# E_{r-n} & \xrightarrow{F^# T(-1)} & F^# E_{r-n}
\end{array}
\]

and in the case of $R$ being a field,

\[
\begin{array}{ccc}
\tilde{E}_0 & \xrightarrow{T} & E_r \\
\downarrow & & \downarrow \\
\tilde{E}_{1-n} & \xrightarrow{F^# T_{1-n}} & F^# E_{r+1-n}
\end{array}
\]

($\tau$ and $\tau$ are morphisms induced by the definition of elliptic sheaves) is also commutative because $(T, \tau - \tau. F^# T_{1-n})_{F^# E_{1-n}} = 0$. Then,

\[F^# \tilde{E}_{1-n} \subset Ker(T, \tau - \tau. F^# T_{1-n})\]

and hence $\text{rank}(Ker(T, \tilde{E}_{1-n} - \tau. F^# T_{1-n})) = n$. Because $T, \tilde{E}_{1-n} - \tau. F^# T_{1-n}$ defines an injective morphism

\[F^# \tilde{E}_{1-n} / Ker(T, \tau - \tau. F^# T_{1-n}) \hookrightarrow E_{r-n},\]

we conclude that $T, \tilde{E}_{1-n} - \tau. F^# T_{1-n} = 0$. We finish the proof because for each $s \in \text{Spec}(R)$ we have $\text{Im}(T, \tau - \tau. F^# T_{1-n}) \subset m_s.E_r$ for all $s \in \text{Spec}(R)$ then, as $E_r$ is locally free and $R$ is reduced, we have $T, \tilde{E}_{1-n} - \tau. F^# T_{1-n} = 0$.

We repeat the same argument to obtain $T_{1-n} : \tilde{E}_{1-n} \to E_{r+l-n}$ with $0 \leq l \leq n$.

By looking at $\Upsilon$ and at the morphism defined in the last subsection,

$\tilde{\beta} : P_{p+D}^{n,ng+r} \to CH_{p+D}^{n,ng+r} \times CH_{p+D}^{n,ng+r} \xrightarrow{\Phi \times \Phi} \mathcal{M}_C(n,ng+r,p+D) \times \mathcal{M}_C(n,ng+r,p+D)$,

we obtain, in the setting of elliptic sheaves, the morphism $\tilde{\beta}(\Upsilon \times \Upsilon)$, which is the analogous morphism to $\Phi \times \Phi$ defined in subsection 3.3. We again denote this morphism by $\tilde{\beta}$.

From the last Lemma and Lemma 4.14 we obtain:

**Lemma 4.15.** If

\[ [(\tilde{E}_j, \tilde{i}_j, \tau, g_{p+D}), (E_j, i_j, \tau, f_{p+D})] \in \tilde{\beta}^{-1}(Z_{p+D}^{n,D}) \]

defined over a field $K$, then:

1) There exists a $p + D$-isogeny between the elliptic sheaves with $p + D$-level structures $(\tilde{E}_j, \tilde{i}_j, \tau, g_{p+D})$ and $(E_j, i_j, \tau, f_{p+D})$.

or

2) There exists an isogeny between the elliptic sheaf $(\tilde{E}_j, \tilde{i}_j, \tau)$ and the twisted elliptic sheaf $(E_j, i_j, \tau) \otimes \mathcal{O}_C(-p-D)$.

or

3) $(F^# \tilde{E}_0) \otimes_{\mathcal{O}_C \otimes K} E_0((g + d)\infty - p - D) \in \Theta_{n^2}$. 

Proof. Let consider us the diagram (not necessarily commutative)

\[
\begin{array}{c}
\hat{E}_n \\
\uparrow l \\
\hat{E}_1 \\
\uparrow \tau \\
F^#E_0 \\
\downarrow F^#T
\end{array} \rightarrow 
\begin{array}{c}
T(1) \\
E_n(g + d - 1) \\
E_1(g + d - 1) \\
F^#E_0(g + d - 1)
\end{array}
\]

\(T\) is a morphism given by \([[(\hat{E}_j, \hat{t}_j, \hat{\tau}, g_{p+D})], (E_j, i_j, \tau, f_{p+D})] \in \tilde{\beta}^{-1}(Z_{p+D})\). The diagonal morphisms are the level structures, and the vertical morphisms \(l\) and \(l\) are induced by the “\(i_j\)” and “\(\hat{i}_j\).” From Definition 4.1, we have two possibilities: either \(T: \hat{E}_n \rightarrow E_0(g + d - 1)\) takes values in \(E_0(g + d - 1)(-p - D)\) or the latter diagram is commutative after tensoring by \(O_{pD}\). In both cases, one obtains a morphism:

\[T(1), l, \tau, l.\tau.F^#T : F^#\hat{E}_n \rightarrow E_0((g + d)\infty - p - D).\]

When \(T, l, \tau, l.\tau.F^#T = 0\), we obtain cases 1) and 2) and when it is \(\neq 0\) it tells us that \((F^#\hat{E}_n)^\vee \otimes_{O_C \otimes K} E_0((g + d)\infty - p - D) \notin \Theta_{n^2}.\)

Thanks to [An2] Lemma 3.3, for \(n = 1\) it is possible to complete 3) in terms of Drinfeld modules. It would be very interesting to study this Lemma for rank-\(n\)-shtukas and rank-\(n\)-Drinfeld modules. For \(C = \mathbb{F}_1\), because \(E_0\) and \(\hat{E}_0\) are semi-stable of 0 degree we have:

\[
(F^#\hat{E}_0)^\vee \otimes_{O_C \otimes K} E_0((g + d)\infty - p - D) \notin \Theta_{n^2}.
\]

One can study explicit examples, as in section 3, merely by changing monic polynomials “\(q(t)\)” for polynomials \(q(t)\) with \(q(\alpha_p) = 1\), where \(m_p = (t - \alpha_p)F_q[t].\)

Remark 16. Given a rank-\(n\)-elliptic sheaf \((E_j, i_j, \tau)\), we have that

\[h^0(E_{-n}) = h^1(E_{-n}) = 0.\]

Thus, \(E_{-n}\) is a semistable vector bundle. In this way, there exists a \(d >> 0\) such that for every pair of elliptic sheaves \([(\hat{E}_j, \hat{i}_j, \hat{\tau}), (E_j, i_j, \tau)]\) defined over a ring \(R\) is

\[H^1(C \otimes R, \hat{E}_j^\vee \otimes_{O_C \otimes R} E_0((g + d - 1)) = 0.\]

Therefore, \([(\hat{E}_j, \hat{i}_j, \hat{\tau}), (E_j, i_j, \tau)] \in \mathcal{U}_d.\)

In the case of shtukas, this result is not true unless one considers shtukas with truncation \([La1]\) and \([La2]\). In a following work, it would be interesting to study this issue in the setting of Lafforgue’s shtuka compactifications \([La1]\).

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