Unified-entropy trade-off relations for a single quantum channel

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Abstract
Many important properties of quantum channels are quantified by means of entropic functionals. Characteristics of such a kind are closely related to different representations of a quantum channel. In the Jamiołkowski–Choi representation, the given quantum channel is described by the so-called dynamical matrix. Entropies of the rescaled dynamical matrix known as map entropies describe a degree of introduced decoherence. Within the so-called natural representation, the quantum channel is formally posed by another matrix, obtained as the reshuffling of the dynamical matrix. The corresponding entropies characterize an amount, with which the receiver \textit{a priori} knows the channel output. As was previously shown, the map and receiver entropies are mutually complementary characteristics. Indeed, there exists a non-trivial lower bound on their sum. First, we extend the concept of receiver entropy to the family of unified entropies. Developing the previous results, we further derive non-trivial lower bounds on the sum of the map and receiver \((q, s)\)-entropies. The derivation is essentially based on some inequalities with the Schatten norms and anti-norms.

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1. Introduction

One of the fundamental distinctions of the quantum world from the classical world is expressed by the uncertainty principle [1]. We mention here only several developments concerning this issue. More extensive discussion with the corresponding references can be found in the reviews [2–4]. Recent studies of entropic relations are often connected with advances in quantum information science. It treats quantum states and effects as informational resources [5, 6]. Due to the influential works [7, 8], entropic functionals are now typically used in formulating the uncertainty principle. Entropic relations are essential in analyzing the security of quantum cryptographic schemes [9, 10]. Reformulations in the presence of quantum memory are discussed in [11–13]. With respect to quantum operations, an entropic approach has been
developed in [14]. In expressing entropic relations, various kinds of entropic functionals have found to be useful. Such entropic bounds were utilized in studying several specific topics such as quantifying number-phase uncertainties [15, 16], incompatibilities of anti-commuting observables [17] and reformulations in quasi-Hermitian models [18]. The Rényi [19] and Tsallis’ entropies [20] are both especially important. They are particular cases of unified \((q, s)\)-entropies introduced in [21] and further studied in [22]. Formally, entropic uncertainty relations are posed as lower bounds on the sum of two or more entropies. Each of these entropies characterizes an uncertainty in measurement statistics with respect to the corresponding basis.

In paper [14], the authors derived a lower bound on the sum of two entropies, corresponding to complementary characteristics of the same quantum operation. Inequalities of such a kind are usually referred to as trade-off relations. The formalism of quantum operations provides a general tool for the description of open quantum systems. Deterministic protocols are represented by trace-preserving operations also known as quantum channels. In quantum information theory, several channel representations are significant. Both the Jamiołkowski–Choi and natural representations are utilized in entropy characterizing a given quantum channel. The entropy exchange [23, 24] and the map entropy [25] are respectively used to describe entanglement transmission and decoherence induced by a quantum channel. A family of map entropies is defined in terms of the dynamical matrix. The writers of [14] have defined the receiver entropies related to the natural representation of a channel. They further showed that the sum of the map and receiver entropies is bounded from below. This bound is written in terms of system dimensionality and compares two complementary characteristics of a single quantum channel. Thus, it differs in character from uncertainty relations posed in [26] for extremal unraveling of two channels. The writers of [14] utilized Rényi’s entropic functionals. Other entropic functionals have been found to be useful in studying various quantum properties [27–31].

The aim of this work is to formulate trade-off relations for a single quantum channel in terms of the map and receiver \((q, s)\)-entropies. This paper is organized as follows. In section 2, the required material is reviewed. First, definitions of the Schatten norms and anti-norms are discussed. Second, we briefly recall the concept of map entropies studied in [25, 32–34]. Then, the family of receiver entropies is extended in terms of quantum \((q, s)\)-entropies. Section 3 is devoted to formulating the main results. Before their derivation, we collect several auxiliary statements. In particular, some relations with the Schatten norms and anti-norms are obtained. These inequalities could be used in other questions, in which we deal with norms and anti-norms of matrices related by reordering the same entries. Non-trivial lower bounds on the sum of the map and receiver \((q, s)\)-entropies are obtained for all considered values of the parameters. Here, we establish lower bounds for arbitrary quantum channels as well as stronger bounds for unital channels. In section 4, we conclude the paper with a summary of results.

2. Definitions and notation

In this section, we introduce terms and conventions that will be used throughout the text. First, required material on operator norms and anti-norms is presented. Second, we discuss the map \((q, s)\)-entropy. Then the notion of the receiver \((q, s)\)-entropy is introduced.

2.1. Operators, norms and anti-norm

Let \(\mathcal{L}(\mathcal{H})\) be the space of linear operators on \(d\)-dimensional Hilbert space \(\mathcal{H}\). By \(\mathcal{L}_+ (\mathcal{H})\) and \(\mathcal{L}_{++} (\mathcal{H})\), we respectively denote the set of positive semidefinite operators on \(\mathcal{H}\) and the set
of strictly positive ones. A density operator $\rho \in \mathcal{L}_+(\mathcal{H})$ has a unit trace, i.e. $\text{Tr}(\rho) = 1$. For $X, Y \in \mathcal{L}(\mathcal{H})$, we define the Hilbert–Schmidt inner product by \cite{6}

$$\langle X, Y \rangle_{\text{hs}} := \text{Tr}(X^*Y).$$

(1)

The Schatten norms, which form an important class of unitarily invariant norms, are defined in terms of singular values. The singular values $\sigma_j(X)$ of operator $X$ are put as eigenvalues of $|X| = \sqrt{XX^*} \in \mathcal{L}_+(\mathcal{H})$. For $q \geq 1$, the Schatten $q$-norm of $X \in \mathcal{L}(\mathcal{H})$ is then defined by \cite{6, 35}

$$\|X\|_q := \left( \sum_{j=1}^d \sigma_j(X)^q \right)^{1/q} = \text{Tr}((X^*X)^{q/2})^{1/q}.$$  

(2)

Here, the singular values should be inserted according to their multiplicities. The family (2) gives the trace norm $\|X\|_1$ for $q = 1$, the Frobenius norm $\|X\|_2$ for $q = 2$ and the spectral norm $\|X\|_{\infty} = \max(\sigma_j(X) : 1 \leq j \leq d)$ for $q = \infty$. Note that the Frobenius norm can be rewritten as

$$\|X\|_2 = \langle X, X \rangle_{\text{hs}}^{1/2}.$$  

(3)

For this reason, it is often called the Hilbert–Schmidt norm. This norm induces the Hilbert–Schmidt distance, which is very useful to analyze a geometry of quantum states \cite{36}. Formula (3) shows that the squared 2-norm is expressed as the sum of squared modulus of all matrix entries. In the following, this fact will be essential.

In the papers \cite{37, 38}, Bourin and Hiai examined symmetric anti-norms of positive operators. They form a class of functionals containing the right-hand side of (2) for $q \in (0; 1)$ and, with strictly positive matrices, for $q < 0$. For arbitrary $X \in \mathcal{L}_+(\mathcal{H})$, we consider a functional $X \mapsto \|X\|_q$, taking values on $[0; \infty)$. If this functional enjoys the homogeneity, the symmetry $\|X\|_q = \|VXV^*\|_q$ for all unitary $V$ and the superadditivity

$$\|X + Y\|_q \geq \|X\|_q + \|Y\|_q,$$  

(4)

it is said to be a symmetric anti-norm \cite{37}. Let $\lambda_j(X)$ denote the $j$th eigenvalue of operator $X$. For $X \in \mathcal{L}_{++}(\mathcal{H})$ and $q \leq 1 \neq 0$, we put the Schatten $q$-anti-norm as

$$\|X\|_q = \left( \sum_{j=1}^d \lambda_j(X)^q \right)^{1/q}.$$  

(5)

Using the given anti-norm $\|X\|_q$, for any $q \in (0; 1)$, we can build another anti-norm (see proposition 3.7 of \cite{37})

$$X \mapsto \|X^*\|_q^{1/q}.$$  

(6)

Note that the trace norm is actually an anti-norm on positive matrices. Indeed, the superadditivity inequality is satisfied here with equality. Combining this with (6), the right-hand side of (5) is an anti-norm on positive operators for $q \in (0; 1)$. With $X \in \mathcal{L}_{++}(\mathcal{H})$ and arbitrary non-zero $q \leq 1$, the inequality (4) for the Schatten $q$-anti-norm was proved in \cite{39}. Note that the restriction to positive matrices is essential in the sense of the superadditivity condition (4). Following \cite{37, 38}, the Schatten $q$-anti-norm of $X$ will also be denoted as $\|X\|_q$.

Some relations for anti-norms and their applications to entropic functionals are examined in \cite{40}. We will further assume that the use of anti-norms tacitly implies the corresponding restriction on matrices. For all $0 < p < q$, we have

$$\|X\|_q \leq \|X\|_p.$$  

(7)

This relation is actually no more than theorem 19 of \cite{41}. In the following, we will use the Schatten $q$-norms for $q \in [1; \infty]$ and the Schatten $q$-anti-norms of positive operators for $q \in (0; 1)$.
The completely mixed input \([32, 48]\). Properties of the map entropy coincides with the entropy exchange calculated for a given channel with (10) is a two-parametric extension of the standard map entropy introduced in [25]. Note that we will usually omit the summation range, when it is clear from the context. Expression

\[
M_q^0(\Phi) = \frac{q}{1 - q} \ln \|\Phi\|_q - \ln d = \frac{1}{1 - q} \left\{ \ln \left( \sum_{j=1}^{d^2} \lambda_j(\Phi)^q \right) - q \ln d \right\}.
\]

In the limit \(s \to 0\), one leads to the map Rényi entropy defined as

\[
M_q^0(\Phi) = \frac{q}{1 - q} \ln \|\Phi\|_q - \ln d = \frac{1}{1 - q} \left\{ \ln \left( \sum_{j=1}^{d^2} \lambda_j(\Phi)^q \right) - q \ln d \right\}.
\]

Taking \(s = 1\), we have the case of the Tsallis entropy

\[
M_q^1(\Phi) = \frac{d^{-q}}{1 - q} \ln \|\Phi\|_q - \ln d = \sum_{j=1}^{d^2} \left( \frac{\lambda_j(\Phi)}{d} \right)^q \ln \left( \frac{\lambda_j(\Phi)}{d} \right),
\]

where the \(q\)-logarithm \(\ln_q(x) = (x^{1-q} - 1)/(1 - q)\) is defined for \(q > 0 \neq 1\) and \(x > 0\). When \(q \to 1\), we obtain the usual logarithm and the map von Neumann entropy. For brevity, we will usually omit the summation range, when it is clear from the context. Expression (10) is a two-parametric extension of the standard map entropy introduced in [25]. Note that the map entropy coincides with the entropy exchange calculated for a given channel with the completely mixed input \([32, 48]\). Properties of the map \((q, s)\)-entropies are examined in

2.2. Two channel representations and the corresponding \((q, s)\)-entropies

By \(\mathcal{H}\), we mean the \(d\)-dimensional space of the principal quantum system \(Q\). Consider a linear map \(\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})\), which also satisfies the condition of complete positivity. Let \(\text{id}^\Phi\) be the identity map on \(L(\mathcal{H}_R)\), where the space \(\mathcal{H}_R\) is assigned to an auxiliary reference system. The complete positivity implies that \(\Phi \otimes \text{id}^\Phi\) transforms a positive operator into a positive operator again for all extensions. When the identity operator is mapped into itself, i.e. \(\Phi(\mid) = 1\), the map is unital. Unital quantum channels are often called bistochastic [42].

There are many interesting classes of quantum channels such as unistochastic channels and degradable channels. The former implies that the principal system unitarily interacts with the \(d\)-dimensional environment, which was initially completely mixed [42]. The notion of degradable channels is related to studying capacities for various processing scenarios [43].

Linear maps can formally be described in several ways. We will use two representations, when the map is represented by a single matrix of size \(d^2 \times d^2\). The family of map entropies is introduced within the Jamiołkowski–Choi representation [44, 45], for which we take \(\mathcal{H}_R = \mathcal{H}\). To the fixed orthonormal basis \(\{\mid i\rangle\}\) in \(\mathcal{H}\), we assign the normalized pure state

\[
\mid \phi_+ \rangle := \frac{1}{\sqrt{d}} \sum_{v=1}^{d} \mid v\rangle \otimes \mid v\rangle.
\]

One defines the operator \(\omega(\Phi) := \Phi \otimes \text{id} \mid \phi_+ \rangle \langle \phi_+ \mid\), acting on the space \(\mathcal{H} \otimes \mathcal{H}\). The dynamical matrix is then written as \(D(\Phi) = d\omega(\Phi)\) [25]. In the shortened notation, we write the dynamical matrix and the rescaled one as \(D_\Phi\) and \(\omega_\Phi\), respectively. For all \(X \in \mathcal{L}(\mathcal{H})\), the action of \(\Phi\) can be expressed as [6]

\[
\Phi(X) = \text{Tr}_R(D_\Phi (1 \otimes X^T)),
\]

where \(X^T\) denotes the transpose operator to \(X\). The map \(\Phi\) is completely positive, if and only if the matrix \(D(\Phi)\) is positive. Furthermore, the preservation of the trace is expressed as \(\text{Tr}_Q(D_\Phi) = 1\), whence \(\text{Tr}(D_\Phi) = d\) and \(\text{Tr}(\omega_\Phi) = 1\).

The concept of entropy is one of the fundamentals in both statistical physics and information theory [46, 47]. We focus our attention on the notions of the map and receiver entropies. The former is put through the Jamiołkowski–Choi representation. For any quantum channel, the dynamical matrix is the positive one with the trace

\[
\operatorname{Tr}(\Phi) = \sum_{j=1}^{d^2} \lambda_j(\Phi)^q = 1,
\]

where \(\lambda_j(\Phi)^q = |\langle j|\Phi\mid \rangle|^2\) is the \(j\)-th eigenvalue of the channel \(\Phi\). When the map is represented by a single matrix of size \(d^2 \times d^2\), the former is put through the Jamiołkowski–Choi representation. For any quantum channel, the dynamical matrix is the positive one with the trace

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\[
\operatorname{Tr}(\Phi) = \sum_{j=1}^{d^2} \lambda_j(\Phi)^q = 1,
\]
between the above matrices is essential. The standard basis in which the receiver
∥ where
\text{R}^\text{entropy} \rightarrow \mathcal{H} \otimes \mathcal{H}. With respect to the standard basis \{\ket{\nu}\}, one defines the mapping [6]
\begin{equation}
\vec{\langle m|\nu\rangle} = \nu \otimes |\nu\rangle.
\end{equation}

By linearity, this mapping is determined for an arbitrary input \(X \in \mathcal{L}(\mathcal{H})\). It is an isometry in the sense that \(\langle X, Y \rangle_\text{HS} = \langle \vec{\text{vec}(X)} | \vec{\text{vec}(Y)} \rangle\) [6]. We now define the matrix \(\mathbf{K}(\Phi) \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})\) such that, for all \(X \in \mathcal{L}(\mathcal{H})\),
\begin{equation}
\vec{\text{vec}(\Phi(X))} = \mathbf{K}(\Phi) |\text{vec}(X)\rangle.
\end{equation}

Following [14, 48], this matrix will be referred to as the superoperator matrix. Some properties of superoperator matrices are indicated in section 5.2.1 of [6]. We now define the matrix \(\mathbf{K}(\Phi) \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})\) such that, for all \(X \in \mathcal{L}(\mathcal{H})\),
\begin{equation}
\vec{\text{vec}(\Phi(X))} = \mathbf{K}(\Phi) |\text{vec}(X)\rangle.
\end{equation}

Follow ing [14, 48], this matrix will be referred to as the superoperator matrix. Some properties of superoperator matrices are indicated in section 5.2.1 of [6]. Note that the superoperator matrix is not generally Hermitian. In characterizations of quantum channels, the natural representation is not so frequently used as other representations [6]. Nevertheless, the authors of [14] have shown that entropic characteristics of \(\mathbf{K}_\Phi\) are demonstrative as well. Namely, the family of receiver entropies is defined in terms of the superoperator matrix [14].

In line with the right-hand sides of (10) and (11), we define the receiver \((q, s)\)-entropy by
\begin{align}
R_q^{(s)}(\Phi) &:= \frac{1}{1- q} \left\{ \frac{1}{q} \sum_{j=1}^d \sigma_j(\mathbf{K}_\Phi)^q \right\} \right\} (s \neq 0), \\
R_q^{(0)}(\Phi) &:= \frac{1}{1- q} \left\{ \ln \sum_{j=1}^d \sigma_j(\mathbf{K}_\Phi)^q \right\} \right\} (16)
\end{align}
where \(|\mathbf{K}_\Phi| = \sum_j \sigma_j(\mathbf{K}_\Phi)\). This definition can be rewritten in terms of the Schatten \(q\)-norm of \(\mathbf{K}_\Phi\) for \(q \geq 1\) and \(q\)-anti-norm of positive \(|\mathbf{K}_\Phi|\) for \(q \in (0, 1)\). The right-hand side of (16), i.e. R\(\text{eny’s variety}, was previously introduced in [14]. It characterizes an amount with which the receiver \textit{a priori} knows the channel output. For our purposes, the following relation between the above matrices is essential. The standard basis in \(\mathcal{H} \otimes \mathcal{H}\) is formed by the vectors \(|\mu\nu\rangle \equiv |\mu\rangle \otimes |\nu\rangle\). For arbitrary vectors of this basis, we have
\begin{equation}
\langle \alpha \beta | \mathbf{K}_\Phi | \mu \nu \rangle = \langle \alpha | \mathbf{D}_\Phi | \beta \nu \rangle.
\end{equation}

In other words, the superoperator matrix is obtained from the dynamical one by the reshuffling operation [14]. It is for this reason that the matrices \(\mathbf{D}_\Phi\) and \(\mathbf{K}_\Phi\) instead of rescaled ones are to be used in our definitions. Matrix operations of such a kind and their essential role in quantum information are reviewed in [49]. The map end receiver entropies provide two mutually complementary characteristics of a given quantum channel. This issue is considered in the following section.

3. Trade-off relations in terms of the \((q, s)\)-entropies

In this section, we formulate trade-off relations for a single quantum channel in terms of the map and receiver \((q, s)\)-entropies. To avoid too long a derivation, we first present several auxiliary results. In particular, some relations between Schatten norms and anti-norms are given. Then, lower bounds on the \((q, s)\)-entropic sum are derived for all values of the parameters.
3.1. Preliminaries

As already mentioned, the dynamical and superoperator matrices are $d^2 \times d^2$-matrices related by the reshuffling operation. Since the operation does not change the matrix entries, these matrices have the same Schatten 2-norm [14]. Hence, relations between 2-norm and other norms or anti-norms are essential for our purposes.

**Proposition 1.** For all $X \in \mathcal{L}(\mathcal{H})$ and $q \geq 1$, the Schatten $q$-norm satisfies

$$
\|X\|_q^q \leq \|X\|_2^{2(q-1)} \|X\|_1^{2-q} \quad (1 \leq q \leq 2),
$$

(18)

$$
\|X\|_q^q \geq \|X\|_2^{2(q-1)} \|X\|_1^{2-q} \quad (2 \leq q < \infty).
$$

(19)

For $X \in \mathcal{L}_+(\mathcal{H})$ and $0 < q < 1$, the Schatten $q$-anti-norm satisfies (19). When $X \in \mathcal{L}_+(\mathcal{H})$, relation (19) holds for $q < 0$ as well.

**Proof.** Let us introduce positive numbers $x_j = \sigma_j(X) \|X\|_1^{-1}$, which obey $\sum_j x_j = 1$. Assuming $q \geq 1$, we apply Jensen’s inequality to the function $x \mapsto x^{q-1}$. As this function is concave for $q \in [1; 2]$ and convex for $q \in [2; \infty)$, we have

$$
\sum_j x_j^q = \sum_j x_j x_j^{q-1} \left\{\begin{array}{ll}
\leq & 1 \leq q \leq 2 \\
\geq & 2 \leq q < \infty
\end{array}\right\} \left(\sum_j x_j^2\right)^{q-1}.
$$

(20)

The left-hand side of (20) is equal to $\|X\|_2^q \|X\|_1^{-q}$, while the right-hand side is equal to $(\|X\|_2^2 \|X\|_1^{-2})^{q-1}$. After substituting these expressions, we respectively obtain (18) and (19).

Restricting a consideration to the strictly positive $X$, we take $x_j = \lambda_j(X) \|X\|_1^{-1} > 0$. Since the function $x \mapsto x^{q-1}$ is convex for $q < 1$, the above relation provides the claim (19) for the Schatten $q$-anti-norm.

We mentioned above that $\|D_\Phi\| = \text{Tr}(D_\Phi) = d$. Deriving entropic relations, we will also use an upper bound on $\|K_\Phi\|_{\infty}$. The statement of theorem 1 in [14] says that

$$
\|K_\Phi\|_{\infty} \leq d^{1/2} \|\Phi(\rho_\Phi)\|_1^{1/2},
$$

(21)

where $\rho_\Phi = \mathbb{1}/d$ is the completely mixed state. Clearly, one has $\|\Phi(\rho_\Phi)\|_{\infty} \leq \|\Phi(\rho_\Phi)\|_1 = \text{Tr}(\Phi(\rho_\Phi)) = 1$ for any quantum channel, whence $\|K_\Phi\|_{\infty} \leq d^{1/2}$. For unital channels, we merely obtain $\|\Phi(\rho_\Phi)\|_{\infty} = \|\rho_\Phi\|_{\infty} = 1/d$ and $\|K_\Phi\|_{\infty} \leq 1$. It follows from lemma 3 of [14] that, for arbitrary $X \in \mathcal{L}(\mathcal{H})$,

$$
\|X\|_2 \leq \|X\|_1 / \|X\|_{\infty} \leq \|X\|_1^{1/2}.
$$

(22)

Combining the above facts with the equality $\|D_\Phi\|_2 = \|K_\Phi\|_2$, we then write

$$
\|D_\Phi\|_1 / \|K_\Phi\|_1 \geq \|D_\Phi\|_1 / \|K_\Phi\|_2 \geq \|D_\Phi\|_1 / \|K_\Phi\|_{\infty} \geq \left\{\begin{array}{ll}
d^{1/2}, & \text{for all channels,} \\
d, & \text{for unital ones.}
\end{array}\right\}
$$

(23)

Thus, the left-hand side of (23) is bounded from below. To obtain trade-off relations from bounds on norms and anti-norms, some analytical results are required as well. Let $a$ and $b$ be strictly positive numbers such that $a < 1$ and $1 < b$. We consider the following two-dimensional domains:

$$
D_a := \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, xy \leq a\},
$$

(24)

$$
D_b := \{(x, y) : 1 \leq x < \infty, 1 \leq y < \infty, b \leq xy\}.
$$

(25)

The minimum of the function $(x, y) \mapsto 2 - x - y$ in the domain (24) is equal to

$$
\min\{2 - x - y : (x, y) \in D_a\} = 1 - a.
$$

(26)
Indeed, this function decreases with each of $x$ and $y$. Hence, the minimum is reached on the part of the curve $xy = a$ between the points $(a, 1)$ and $(1, a)$. As the function $x \mapsto 2 - x - a/x$ is concave and takes the same value (26) at the end points, this value is actually the minimum. Furthermore, the minimum of $(x, y) \mapsto x + y - 2$ in the domain (25) is equal to

$$
\min \{x + y - 2 : (x, y) \in D_0\} = 2(\sqrt{b} - 1).
$$

Since this function increases with each of $x$ and $y$, the minimum is reached on the part of the curve $xy = b$ between $(b, 1)$ and $(1, b)$. Then the claim is merely obtained by applying the ordinary inequality between the arithmetic and geometric mean. Results (26) and (27) will be used in obtaining desired unified-entropy relations.

### 3.2. Lower bounds on the sum of map and receiver entropies

Using the above preliminaries, we now derive lower bounds on the sum of the map and receiver $(q, s)$-entropies for all values of the parameters.

**Proposition 2.** For $q > 0$ and all real $s$, the sum of the map and receiver $(q, s)$-entropies is bounded from below. For $q \in (0; 1) \cup (1; \infty)$ and $s \neq 0$, the lower bound is written as

$$
M_q^{(s)}(\Phi) + R_q^{(s)}(\Phi) \geq \frac{s}{q} \ln_q(d^{2s/q}) \quad \text{for all channels},
$$

$$
M_q^{(s)}(\Phi) + R_q^{(s)}(\Phi) \geq \frac{s}{q} \ln_q(d^{2s/q}) \quad \text{for unital ones}.
$$

Here, the parameters $\gamma$ and $\alpha$ are defined as

$$
\gamma := \begin{cases} 
1, & (1 - q)s < 0, \\
2, & (1 - q)s > 0,
\end{cases} \quad \alpha := \begin{cases} 
1, & (q(2(q - 1))^{-1}, q \in (0; 2], \\
q \in [2; \infty).
\end{cases}
$$

**Proof.** In the first place, we consider the case $q \in (0; 1)$. The sum of the map and receiver $(q, s)$-entropies is represented as the function

$$
f(x, y) := \frac{x + y - 2}{(1 - q)s}, \quad x := \frac{\|D_{\Phi}q\|^q_{q}}{\|D_{\Phi}\|_1^q}, \quad y := \frac{\|Y_{\Phi}\|^q_{q}}{\|Y_{\Phi}\|_2^q},
$$

where $Y_{\Phi} = |K_{\Phi}| \in L_+ (\mathcal{H} \otimes \mathcal{H})$. Suppose also that $s > 0$; then $x \geq 1$ and $y \geq 1$. Due to (19) and (23), the anti-norms of the positive matrices $D_{\Phi}$ and $Y_{\Phi}$ satisfy

$$
\left(\frac{\|D_{\Phi}\|_1}{\|D_{\Phi}\|_1} \cdot \frac{\|Y_{\Phi}\|_1}{\|Y_{\Phi}\|_1}\right)^{\alpha} \geq \left(\frac{\|D_{\Phi}\|_2}{\|D_{\Phi}\|_2} \cdot \frac{\|Y_{\Phi}\|_2}{\|Y_{\Phi}\|_2}\right)^{\alpha} \geq d^{1-q}, \quad \text{for all channels},
$$

$$
d^{2(1-q)}, \quad \text{for unital ones}.
$$

Therefore, the function $f(x, y)$ should be minimized in the domain (25) with $b = d^{1-q}$ for all quantum channels and $b = d^{2(1-q)}$ for unital channels. Using (27), we obtain the lower bound

$$
\min \{f(x, y) : (x, y) \in D_0\} = \frac{2}{s} \ln_q(d^{q/2}), \quad \text{for all channels},
$$

$$
\ln_q(d^q), \quad \text{for unital ones}.
$$

Since $\gamma = 2$ for $q \in (0; 1)$ and $s > 0$, the bound (33) respectively concurs with (28) and (29).

Defining $x$ and $y$ by (31), we have $x \leq 1$ and $y \leq 1$ in the case $q \in (0; 1)$ and $s < 0$. Since $(1 - q)s < 0$, the sum of the map and receiver $(q, s)$-entropies is rewritten as the function

$$
g(x, y) := \frac{2 - x - y}{|(1 - q)s|}.
$$
Raising (32) to the power $s < 0$, we have arrived at the following task. The function $g(x, y)$ should be minimized in the domain (24) with $a = d^{x(1 - q)}$ for all quantum channels and $a = d^{2x(1 - q)}$ for unital channels. Using (26), we obtain the lower bound

$$
\min\{g(x, y) : (x, y) \in D_α\} = \frac{1}{s} \ln q(d^s), \quad \text{for all channels},
$$

$$
\frac{1}{s} \ln q(d^s), \quad \text{for unital ones.}
$$

(35)

Since $γ = 1$ for $q \in (0, 1)$ and $s < 0$, the bound (35) concurs with (28) and (29) as well.

In the second place, we consider the case $q \in (1, \infty)$. For $s > 0$, the entropic sum is also taken as (34) due to $(1 - q)s < 0$. Taking (18) for $1 \leq q \leq 2$ and (7) for $2 \leq q < \infty$, one gives

$$
\left(\frac{∥D_φ∥_q ∥K_φ∥_s}{∥D_φ∥_1 ∥K_φ∥_1}\right)^q \leq \left(\frac{∥D_φ∥_2 ∥K_φ∥_2}{∥D_φ∥_1 ∥K_φ∥_1}\right)^{2s(q - 1)} \leq \left\{\begin{array}{ll}
d^{x(1 - q)}, & \text{for all channels}, \\
d^{2x(1 - q)}, & \text{for unital ones.}
\end{array}\right.
$$

(36)

Here, we also used inequality (23) and definition (30). Raising (36) to the power $s > 0$, the variables $x$ and $y$ range in the domain (24) with $a = d^{x(1 - q)}$ for all quantum channels and $a = d^{2x(1 - q)}$ for the unital channels. By (26), the minimum is formally posed just as (35), but with the term $sxy$ instead of $s$ in powers of $d$ in the $q$-logarithm. As $γ = 1$ for $(1 - q)s < 0$, this minimum concurs with (28) and (29). For $s < 0$, the entropic sum is taken as (31) due to $(1 - q)s > 0$. Raising (36) to the power $s < 0$, the variables $x$ and $y$ range in the domain (25) with $b = d^{x(1 - q)}$ for all quantum channels and $b = d^{2x(1 - q)}$ for the unital channels. Using (27), the minimum is formally posed as (33), but again with the term $sxy$ instead of $s$ in powers of $d$ in the $q$-logarithm. As $γ = 2$ for $(1 - q)s > 0$, this minimum concurs with (28) and (29) as well.

The inequalities (28) and (29) provide unified-entropy trade-off relations for a given quantum channel $Φ$. They are an extension of the results of the paper [14] with the use of the family of $(q, s)$-entropies. In more details, the physical sense of such bounds was considered in [14]. When $s = 1$, formulas (28) and (29) give trade-off relations in terms of Tsallis’ entropies, namely

$$
M_1^{(1)}(Φ) + R_1^{(1)}(Φ) \geq \left\{\begin{array}{ll}
γ \ln q(d^{x/y}), & \text{for all channels}, \\
γ \ln q(d^{2x/y}), & \text{for unital ones.}
\end{array}\right.
$$

(37)

Taking the limit $s \to 0$, we recover the Rényi formulation, which is written as

$$
M_1^{(0)}(Φ) + R_1^{(0)}(Φ) \geq \left\{\begin{array}{ll}
x \ln d, & \text{for all channels}, \\
2x \ln d, & \text{for unital ones.}
\end{array}\right.
$$

(38)

These trade-off relations for a single quantum channel were originally formulated in [14]. Of course, we can obtain (38) directly by taking the logarithm of both (32) and (36). Since the Rényi case was explicitly examined in [14], we refrain from presenting the details here.

As is shown in [21], values of the $(q, s)$-entropy of finite-dimensional density matrix $ρ$ are bounded from above. Adapting this result for the considered case, we obtain

$$
M_q^{(1)}(Φ) \leq \frac{1}{s} \ln q(\text{rank}(D_φ))^s \leq \frac{1}{s} \ln q(d^s) \quad (s \neq 0),
$$

(39)

$$
M_q^{(0)}(Φ) \leq \ln(\text{rank}(D_φ)) \leq 2 \ln d,
$$

(40)

since $\text{rank}(D_φ) \leq d^2$. The right-hand sides of (39) and (40) give the upper bound on the receiver $(q, s)$-entropy as well. The value (39) is also obtained from (29) for $q \in (0; 1)$ and $s < 0$ as well for $q \in (1; 2)$ and $s > 0$. Here, the sum of the map and receiver $(q, s)$-entropies is not less than the maximal possible value for each of them. Thus, they may be regarded as mutually complementary characteristics of a single quantum channel.
4. Conclusions

We have obtained a two-parametric family of entropic bounds for a single quantum channel. These entropic relations are the unified-entropy extension of the relations recently formulated in [14]. In general, the existence of such lower bounds has a physical relevance as some trade-off property in a single measuring process [14]. For a given quantum channel, two entropic characteristics are expressed in terms of singular values of the dynamical and superoperator matrices, respectively. These matrices are related to the Jamiołkowski–Choi representation and the natural representation of the channel. It is significant that the two matrices are connected by the reshuffling operation. Hence, they have the same Schatten 2-norm. Entropic trade-off relations for a single quantum channel have been formulated in terms of the map and receiver \((q, s)\)-entropies for all considered values of the parameters. The derivation is essentially based on some inequalities for the Schatten norms and anti-norms. Such inequalities could be interesting in other contexts, in which we deal with matrices related by reordering the same matrix entries. Both the cases of arbitrary and unital quantum channels were examined. In the latter case, when bistochastic maps are treated, we see the following. For a wide range of the parameters \(q \) and \(s \), the lower bound on the sum of two entropies coincides with the upper bound on each of the summands. This result gives an additional reason for the fact that the map and receiver entropies characterize mutually complementary properties of a given quantum channel. Hence, it is interesting to study entropic bounds for more specialized types of quantum channels, including unistochastic maps and degradable channels. This issue could be the subject of a separate research.

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