Geometric extensions of many-particle Hardy inequalities

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Abstract

Certain many-particle Hardy inequalities are derived in a simple and systematic way using the so-called ground state representation for the Laplacian on a subdomain of $\mathbb{R}^n$. This includes geometric extensions of the standard Hardy inequalities to involve volumes of simplices spanned by a subset of points. Clifford/multilinear algebra is employed to simplify geometric computations. These results and the techniques involved are relevant for classes of exactly solvable quantum systems such as the Calogero–Sutherland models and their higher-dimensional generalizations, as well as for membrane matrix models, and models of more complicated particle interactions of geometric character.

Keywords: uncertainty principle, many-body interactions, Hardy inequality, Calogero–Sutherland models, Clifford algebra

1. Introduction

During the past century, Hardy inequalities have appeared in a variety of different forms in the literature and have played an important role in analysis and mathematical physics (see e.g. the books [1, 2] and the reviews in [3, 4]). The standard Hardy inequality associated to the Laplacian in $\mathbb{R}^d$, $d \geq 3$, is given by

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \geq C_d \left( \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \, dx \right), \quad (1)$$

with the sharp constant $C_d = \left( \frac{d-2}{2} \right)^2$, where $u$ is any function in the Sobolev space $H^1(\mathbb{R}^d)$, i.e. the square-integrable functions for which the lhs of (1) is finite. It states explicitly that the Laplace operator $-\Delta$ on $\mathbb{R}^d$ (defined via its quadratic form) is not only non-negative, but in a sense strictly positive, in that it is bounded below by a potential which even increases in

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \geq C_d \left( \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \, dx \right), \quad (1)$$
strength unboundedly as the distance to the origin tends to zero. In quantum mechanics, this is a concrete manifestation of the uncertainty principle, and inequalities of this form have been crucial for e.g. rigorous proofs for the stability of matter (see e.g. [5]). In such many-particle contexts it becomes relevant to also consider extensions of (1) involving mutual distances between a (possibly large) number, say $N$, of particles. Also the sharp values of the corresponding constants as well as their dependence on $N$ are relevant for physical applications.

An area where the importance of the Hardy inequality and its many-particle generalizations becomes particularly transparent is in the context of exactly solvable quantum systems. For example, in the Calogero–Sutherland model [6, 7] for $N$ particles on the real line $\mathbb{R}$ with inverse-square interactions, the corresponding many-particle Hardy inequality\(^1\)

\[
\int_{\mathbb{R}^N} \sum_{j=1}^{N} |\partial_j u|^2 \, dx \geq \frac{1}{2} \int_{\mathbb{R}^N} \sum_{j \neq k} \frac{|u|^2}{(x_j - x_k)^2} \, dx, \quad u \in H_0^1(\mathbb{R}^N \cap \{ x_j \neq x_k \}_{j \neq k}),
\]

(2)
guarantees the stability of the model, i.e. the boundedness of the energy from below, for a certain range of coupling parameters for which the interaction may be attractive and for a wide class of external potentials. Although first arising as mathematical toy models, the Calogero–Sutherland models have proved to be very useful in the study of a variety of physical phenomena, such as soliton wave propagation [8], quantum spin chains [9, 10], random matrices [11], as well as for anyons in the lowest Landau level [12, 13]. Various generalizations of the particle interactions in the Calogero–Sutherland model, including to higher dimensions and more complicated geometric potentials, have also been considered; see e.g. [14–18]. Considering the success of the original Calogero–Sutherland models, some of these generalizations are likewise also expected to be important, for example in the study of strongly correlated systems (further physical motivations for studying such models are discussed in [14–17, 19]).

Hoffmann-Ostenhof et al have in [20] studied many-particle generalizations of the standard Hardy inequality (1) of the conventional type (2) in arbitrary dimensions, both for bosonic and fermionic particles (i.e. completely symmetric resp. antisymmetric wave functions), as well as for magnetically interacting particles in two dimensions, and determined the optimal behavior for the associated constants in these inequalities in the large-$N$ limit for the bosonic or distinguishable case (hence with implications for the coupling parameters in associated exactly solvable models). In the fermionic case, the optimal large-$N$ behavior of the corresponding constants was studied in [21]. The magnetic case, relevant for models of anyons in two dimensions, was reconsidered and improved in [22], leading to rigorous bounds for the energy of the anyon gas (see also [23, 24] for recent applications).

In this note we will focus on the case of bosons or distinguishable particles and use the so-called ground state representation (GSR) for the Laplacian on a subdomain of $\mathbb{R}^n$ to derive the conventional (bosonic) many-particle Hardy inequalities in a simple and systematic way. Using multilinear and Clifford algebra, the approach we take straightforwardly generalizes to other types of many-particle Hardy inequalities, involving geometric relatives and higher-dimensional analogs of distances between particles. Some of these generalizations coincide with the interaction potentials of generalized Calogero–Sutherland models such as those studied in [14, 15, 17, 18]. Also the case of critical dimension, which for the standard bosonic case is equal to two, is covered, and corresponding inequalities involving logarithms found.

\(^1\) We denote by $H_0^1(\Omega)$ the space of functions in $H^1(\Omega)$ vanishing on the boundary $\partial\Omega$, or more precisely the closure of the smooth functions of compact support $C_0^\infty(\Omega) \subseteq H^1(\Omega)$.
We point out that some of the generalizations presented have also been considered in the context of Hardy inequalities by Laptev et al.\cite{20,25}. For purposes of illustrating the method and techniques involved in a more accessible way, we choose to start from a simple setting and build up a more general framework rather than to start with the most general (but technically complicated) theorem and then specialize from that.

The main purpose of this work is twofold. First, the new bounds which are derived in theorems\ref{thm:10}–\ref{thm:14} prove that the classes of models considered in\cite{15,17} are, or can be made, well-defined independently of the choice of external potentials for a corresponding range of coupling parameters, in that the corresponding quadratic forms are bounded from below. Furthermore, the GSRs given here will be of use in the spectral analysis of the associated operators (compare e.g. the analysis of natural self-adjoint extensions for the Calogero–Sutherland operators in\cite{23}, for which knowledge of the GSR was essential). However, the study of optimal constants in the large-$N$ limit for some of the models is unfortunately very complicated due to the geometrically complicated nature of the potentials and will have to be addressed in future work, but we do derive the sharp constants for the constituent potentials with a fixed number of particles in the interaction (see appendix\ref{appendix:B}).

Second, although the ground state approach employed here is well known in the literature on Hardy inequalities (see e.g.\cite{26–28} and references therein), it is hoped that the structure of the paper and its systematic study of such inequalities will help bring out the usefulness of the technique to an even wider community of mathematicians and physicists. In particular, in combination with the new approach employed here using geometric algebra (whose usefulness is also well known within a certain community, although the intersection between these two communities seems unfortunately to be rather small\cite{29}) the emphasis is on the fact that both technically and geometrically complicated results can be obtained in an efficient way. The geometrically most general form, the corollary to theorem\ref{thm:14} which shows that a class of generalized particle interactions involving volumes can be considered as a small perturbation of the free kinetic energy operator, would probably not have been manageable without these tools and the systematic exposition.

The paper is organized as follows. In section 2 we state the preliminary setup which allows for a straightforward and systematic derivation of the results. The main results are given in section 3 as GSRs for the conventional many-particle Hardy inequalities in all dimensions (theorems\ref{thm:4}–\ref{thm:7}, with an extension in theorem\ref{thm:8}), for some alternative geometric cases where the origin is singled out as a special point (theorems\ref{thm:9}–\ref{thm:11}), and finally for inequalities involving volumes of simplices of points (theorems\ref{thm:12}–\ref{thm:14}). Conclusions are given in section 4. Some computations involving multilinear algebra, and a brief note on the sharpness of the derived constants, have been placed in an appendix.

2. Preliminaries: single-particle Hardy inequalities

As a preparation, we start by recalling the GSR for the Laplacian in a form which is well suited for our applications, and use it to derive the standard single-particle Hardy inequalities w.r.t. a point and a higher-dimensional subspace in $\mathbb{R}^d$.

2.1. The GSR

We have the following simple but general GSR for the (Dirichlet) Laplacian on a domain in $\mathbb{R}^n$.
Proposition 1 (GSR). Let \( \Omega \) be an open set in \( \mathbb{R}^n \) and let \( f : \Omega \to \mathbb{R}_+ := (0, \infty) \) be twice differentiable. Then, for any \( u \in C^0_0(\Omega) \) and \( \alpha \in \mathbb{R} \),
\[
\int_{\Omega} |Vu|^2 \, dx = \int_{\Omega} \left( \alpha (1 - \alpha) \frac{|Vf|^2}{f^2} + \alpha - \frac{\Delta f}{f} \right) |u|^2 \, dx + \int_{\Omega} |Vv|^2 f^{2\alpha} \, dx,
\]
where \( v := f^{-\alpha} u \).

**Proof.** We have for \( u = f^\alpha v \) that \( Vu = af^{2\alpha-1}(Vf)v + f^\alpha Vv \), hence
\[
|Vu|^2 = a^2 f^{2(\alpha-1)} |Vf|^2 |v|^2 + af^{2\alpha-1}(Vf) \cdot V |v|^2 + f^{2\alpha} |Vv|^2.
\]
Integrating this expression over \( \Omega \), we find that the middle term on the rhs produces after partial integration
\[
-\alpha \int_{\Omega} V \cdot \left( f^{2\alpha-1} Vf \right) |v|^2 \, dx.
\]
Now, using that
\[
V \cdot \left( f^{2\alpha-1} Vf \right) = (2\alpha - 1)f^{2\alpha-2} |Vf|^2 + f^{2\alpha-1} \Delta f,
\]
and collecting the terms we arrive at (3). \( \square \)

It will in the sequel be very convenient to introduce some terminology related to the GSR. We refer to \( f \) as the (exact or approximate) ground state and to \( \alpha \) as the GSR weight, while the potential term arising in (3), i.e.
\[
\alpha (1 - \alpha) \frac{|Vf|^2}{f^2} + \alpha - \frac{\Delta f}{f},
\]
will be called the GSR potential. Note that the choice \( \alpha = \frac{1}{2} \) maximizes the first term in the GSR potential, which would be the relevant term if \( \Delta f = 0 \) on \( \Omega \), i.e. if \( f \) is a generalized zero eigenfunction for the Laplacian on \( \Omega \). In this case the resulting GSR (3) will usually be called a Hardy GSR, in anticipation of Hardy-type inequalities. A general idea of this approach, however, is to try to find as good an inequality as possible by considering also approximate ground states, say of a particular form convenient for computations, with the possibility to optimize over the GSR weight \( \alpha \). We further emphasize that an important advantage of having the GSR for a Hardy inequality is that the integral term involving \( v \) in (3) provides a guide for proving sharpness (cp. appendix B), and also opens up for further improvements of the inequality.

We will for the following also find it useful to note that a simple modification of proposition 1 to involve a product ground state ansatz \( gh^\beta \) (i.e. \( u = g^\alpha h^\beta v \)) produces the GSR potential
\[
\alpha (1 - \alpha) \frac{|Vg|^2}{g^2} + \alpha \frac{-\Delta g}{g} + \beta (1 - \beta) \frac{|Vh|^2}{h^2} + \beta \frac{-\Delta h}{h} - 2\alpha \beta \frac{Vg \cdot Vh}{gh}.
\]
2.2. The standard Hardy inequalities in \( \mathbb{R}^d \)

The obvious ground states \( f \) for the Laplacian in \( \mathbb{R}^d \) are the fundamental solutions,

\[
\begin{align*}
\hat{f}_{d \neq 2}(x) &= |x|^{-(d-2)}, \\
f_2(x) &= \ln |x|,
\end{align*}
\]

where \( \delta_0 \) are Dirac delta distributions supported at the origin and \( c_d \) some irrelevant constants.

Hence, for \( d \neq 2 \) we can consider the domain \( \Omega := \mathbb{R}^d \setminus \{0\} \) on which \( f = f_2 > 0 \) and \( \Delta f = 0 \). (3) is therefore optimal for \( \alpha = \frac{1}{2} \), and yields the GSR associated to the standard Hardy inequality (1) in \( \mathbb{R}^d \):

\[
\int_\Omega |\nabla u|^2 \, dx = \frac{(d-2)^2}{4} \int_\Omega \frac{|u|^2}{|x|^2} \, dx = \int_\Omega |\nabla f_2|^2 |x|^{-(d-2)} \, dx \geq 0. \tag{6}
\]

The inequality (6) holds for all \( u \in C^\infty_0(\Omega) \), and hence the lhs is non-negative on the Sobolev space \( H^1(\Omega) \) for \( d \geq 2 \) by closure.

For \( d = 2 \) we can take the domain \( \Omega := \mathbb{R}^2 \setminus (\{0\} \cup S^1) \) and ground state \( f = |f_2| \), so that \( f > 0 \) and \( \Delta f = 0 \) on \( \Omega \). (3) then produces the corresponding two-dimensional Hardy GSR

\[
\int_\Omega |\nabla u|^2 \, dx - \frac{1}{4} \int_\Omega \frac{|u|^2}{|x|^2 (\ln |x|)^2} \, dx = \int_\Omega |\nabla f_2| |\ln |x|| \, dx \geq 0. \tag{7}
\]

By closure, the lhs is non-negative for \( u \in H^1_0(\Omega) = H^1_0(\mathbb{R}^2 \setminus S^1) \).

In the above we used the standard fact that \( C^\infty_0(\mathbb{R}^n \setminus \{0\}) \) is dense in \( H^1(\mathbb{R}^n) \) (with the Sobolev norm) for \( n \geq 2 \), while the closure of \( C^\infty_0(\mathbb{R} \setminus \{0\}) \) is \( H^1(\mathbb{R} \setminus \{0\}) \subseteq H^1(\mathbb{R}) \). See e.g. lemma 3 in [23] for an explicit proof in the case of critical dimension \( n = 2 \). Similar density arguments are valid for \( C^\infty_0(\mathbb{R}^n \setminus K) \) for the codimension \( k \geq 2 \) subsets \( K \) we consider in the following, typically being finite unions of closed smooth or cone-like submanifolds of dimension \( n-k \) (see e.g. section 9 in [1]).

2.3. Hardy inequalities outside subspaces

Let us also briefly recall the generalizations of the standard Hardy inequalities (6)–(7) w.r.t. the point \( \{0\} \), to corresponding inequalities w.r.t. any linear subspace in \( \mathbb{R}^d \). For later purposes, it is most convenient to state and derive these GSR in the language of geometric algebra (involving the Clifford algebra over \( \mathbb{R}^d \); see [30], or the brief introduction in appendix A).

Let \( A := a_1 \wedge \ldots \wedge a_p \neq 0 \) be a \( p \)-blade, \( 0 \leq p < d \), i.e. an exterior product of \( p \) vectors \( a_j \in \mathbb{R}^d \), representing the oriented \( p \)-dimensional linear subspace \( \mathbb{A} \subseteq \mathbb{R}^d \) spanned by \( \{a_j\} \), with magnitude \( |A| \). Note that the \( p+1 \)-blade \( x \wedge A = 0 \) if and only if \( x \in \mathbb{A} \). Further, \( \delta(x) := |x \wedge A| |A|^{-1} \) is the minimal distance from \( x \) to \( \mathbb{A} \), while the Clifford product \( (x \wedge A)A^{-1} = (1 - P_A)x \), where \( P_A \) is the orthogonal projection on \( \mathbb{A} \). We then have the following simple Hardy GSRs, which reduce to (6) resp. (7) for \( p=0 \), and which will shortly be generalized to many-particle versions:

**Theorem 2** \( (d-p \neq 2) \). Let \( \Omega := \{x \in \mathbb{R}^d; |x \wedge A| > 0\} \). Taking the ground state \( f(x) := |x \wedge A|^{-(d-p-2)} \propto \delta(x)^{-1} \) we obtain
\[ \int_{\Omega} |V u|^2 \, dx - \frac{(d - p - 2)^2}{4} \int_{\Omega} \frac{|A|^2}{|x \wedge A|^2} |u|^2 \, dx = \int_{\Omega} |V f^{\frac{2}{2}} u|^2 f \, dx \geq 0, \quad (8) \]

for \( u \in C_0^\infty(\Omega) \). The corresponding Hardy inequality for the lhs holds for \( u \in H^1_0(\Omega) \) (\( = H^1(\mathbb{R}^d) \) for \( d - p \geq 2 \)).

**Proof.** By choosing a basis and coordinate system appropriately, one easily computes that \( \Delta_\delta - (d - p - 2) = 0 \) on \( \Omega \) (\( f \) is a fundamental solution to the Laplacian w.r.t. the subspace \( \bar{A} \)). Furthermore, \( \nabla f = -(d - p - 2) V \delta / \delta \), and it follows that the optimal weight is the standard Hardy \( \alpha = \frac{1}{2} \), with \( |V f|^2 / f^2 = (d - p - 2)^2 / \delta^2 \).

Alternatively, by employing geometric algebra we can avoid introducing coordinates and directly obtain \( \nabla f = -(d - p - 2)(x \wedge A)^{-1} A f \) and \( \Delta f = 0 \) (see appendix A). \( \Box \)

**Theorem 3** \((d - p = 2)\). Fix a length scale \( R > 0 \) and consider \( \Omega := \{ x \in \mathbb{R}^d \colon 0 < |x \wedge A| / R \neq 1 \} \). Taking \( f(x) := \left| \ln \frac{1}{R} |x \wedge A| \right| \) we obtain

\[ \int_{\Omega} |V u|^2 \, dx - \frac{1}{4} \int_{\Omega} \frac{|A|^2}{|x \wedge A|^2 \left( \ln \frac{1}{R} |x \wedge A| \right)^2} |u|^2 \, dx = \int_{\Omega} |V f^{\frac{2}{2}} u|^2 f \, dx \geq 0, \quad (9) \]

for \( u \in C_0^\infty(\Omega) \).

**Proof.** Here we have \( V f = \pm(x \wedge A)^{-1} A \) (with a sign depending on \( p \) and which part of \( \Omega \) we consider) and \( \Delta f = \pm(d - p - 2)|A|^2 / |x \wedge A|^2 = 0 \) (see appendix A). \( \Box \)

Due to the invariance of the Laplacian under translations, corresponding GSR of course hold also for affine subspaces \( a + \bar{A} , a \in \mathbb{R}^d \), simply by translation \( x \mapsto x + a \) and considering \( \Omega := \{|(x - a) \wedge A| > 0\} \) (analogously for \( d - p = 2 \)). Furthermore, we note that the constants in (8) are sharp, just as for \( p = 0 \) (cp. appendix B).

### 3. Many-particle Hardy inequalities

We now turn to a systematic application of the GSR with approximate ground states to the setting of many-particle Hardy inequalities.

#### 3.1. Conventional many-particle inequalities

Consider a tuple \((x_1, \ldots, x_N)\) of \( N \) points, or particles, in \( \mathbb{R}^d \). We define the distance \( e_{ij} := |x_i - x_j| \) between two particles, and the circumradius \( R_{ijk} \) associated to three non-coincident particles,

\[ \frac{1}{2R_{ijk}^2} := \sum_{\text{cyclic in } i,j,k} (x_i - x_j)^{-1} \cdot (x_i - x_k)^{-1}, \]

i.e. the radius of the circle that the particles \( x_i, x_j, x_k \) inscribe \( (R_{ijk} := \infty \) for collinear particles, for which the rhs is zero; cp. lemma 3.2 in [20]). Let us first consider the total separation measured by the distance-squared between all pairs of particles:
Theorem 4 (Total separation of \( N \geq 2 \) particles). Let
\[
\Omega := \mathbb{R}^{dN} \setminus \left\{ x_1 = x_2 = \ldots = x_N \right\}.
\]
Taking the ground state \( \rho(x)^2 := \sum_{i<j} |x_i - x_j|^2 \) we obtain
\[
\int_{\Omega} |\nabla u|^2 \, dx - N \left( \frac{N-1}{2} d - 1 \right)^2 \int_{\Omega} \frac{|u|^2}{\rho^2} \, dx = \int_{\Omega} |\nabla \rho^{2/4} |^2 \rho^{4/4} \, dx \geq 0, \tag{10}
\]
for all \( u \in C_0^\infty(\Omega) \), with the optimal weight \( \alpha := -\frac{(N-1)d-2}{4} \).

This gives a generalization to \( N \neq 3 \) of (3.5) in [20]. Note that codim \( \Omega^c = dN - d = d(N - 1) \) and hence that the corresponding Hardy inequality on \( H^1(\Omega^c) \) holds on \( H^1(\mathbb{R}^{dN}) \) unless \( d = 1 \) and \( N = 2 \). The constant is sharp, as shown explicitly in the appendix (proposition 18).

Proof. One computes
\[
V_k \rho^2 = 2 \sum_{j \neq k} (x_k - x_j),
\]
hence
\[
\Delta \rho^2 = 2 \sum_k \sum_{j \neq k} V_k \cdot (x_k - x_j) = 2N(N - 1)d,
\]
and
\[
|\nabla \rho^2|^2 = 8 \sum_{i<j} r_{ij}^2 + 8 \sum_k \sum_{i<j} (x_k - x_i) \cdot (x_k - x_j) = 4N \rho^2,
\]
where in the last step we used the identity
\[
\sum_{i<j} (x_k - x_i) \cdot (x_k - x_j) = \frac{N - 2}{2} \sum_{i<j} |x_i - x_j|^2. \tag{11}
\]
It follows that we have a GSR potential
\[
\alpha (1 - \alpha) \frac{|\nabla \rho^2|^2}{\rho^4} - \alpha \frac{\Delta \rho^2}{\rho^2} = 4N \alpha \left( \frac{2 - (N - 1)d}{2} - \alpha \right) \frac{1}{\rho^2},
\]
which by optimization proves the theorem. \( \square \)

Next, we have as a special case of the following, the so-called ‘standard’ many-particle Hardy inequality:

Theorem 5 (Separation and circumradii of pairs and triples of particles in \( d \geq 3 \)). Let
\[
\Omega := \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^{dN} : x_i \neq x_j, \forall i \neq j \right\}. \tag{12}
\]
Taking the ground state \( f(x) := \prod_{j<k} |x_j - x_k|^{-(d-2)} \) we obtain
\[
\int_{\Omega} |Vu|^2 \, dx - (d-2)^2 \int_{\Omega} \left( 2\alpha(1 - \alpha) \sum_{i<j} \frac{1}{r_{ij}^2} - \alpha^2 \sum_{i<j<k} \frac{1}{R_{ijk}^2} \right) |u|^2 \, dx
= \int_{\Omega} |Vf^{-\alpha}u|^2 f^{2\alpha} \, dx \geq 0, \tag{13}
\]
for all \( u \in C^\infty_0(\Omega) \) and \( \alpha \in \mathbb{R} \). In particular, defining
\[
K_{d,N} := \sup_{x \in \Omega} \frac{\sum_{i<j<k} 1/R_{ijk}^2}{\sum_{i<j} 1/r_{ij}^2} \leq N - 2 < \infty, \tag{14}
\]
(the upper bound following from geometric relations; cp. lemma 3.3 in [20]), and taking the then optimal weight \( \alpha := 1/(2 + K_{d,N}) \) we have
\[
\int_{\Omega} |Vu|^2 \, dx - \frac{(d-2)^2}{2 + K_{d,N}} \sum_{i<j<k} \int_{\Omega} \frac{|u|^2}{R_{ijk}^2} \, dx \geq \int_{\Omega} |Vf^{-\alpha}u|^2 f^{2\alpha} \, dx > 0, \tag{15}
\]
On the other hand, assuming \( \alpha(1 - \alpha) \geq 0 \) in (13) and now using the bound (14) on the term involving \( r_{ij} \) we obtain
\[
\int_{\Omega} |Vu|^2 \, dx - (d-2)^2 \frac{1}{K_{d,N}} \sum_{i<j<k} \int_{\Omega} \frac{|u|^2}{R_{ijk}^2} \, dx \geq \int_{\Omega} |Vf^{-\alpha}u|^2 f^{2\alpha} \, dx > 0, \tag{16}
\]
again with the optimal weight \( \alpha := \frac{1}{2 + K_{d,N}} \).

As pointed out in [20], using the geometric relations between separation and circumradii, the \( N = 3 \) case of theorem 4 also implies
\[
\int_{\Omega} |Vu|^2 \, dx \geq \frac{(d-1)^2}{3} \left( \frac{N}{3} \right)^{-1} \sum_{i<j<k} \int_{\Omega} \frac{|u|^2}{R_{ijk}^2} \, dx. \tag{17}
\]
Combining this with (13), one is led to maximize \( \frac{\alpha(1 - \alpha)}{1 + ca^2} \) with
\[
c := \frac{3}{2} \frac{(d-2)^2}{(d-1)^2} (N - 1)(N - 2).
\]
This results in
\[
\int_{\Omega} |Vu|^2 \, dx - \alpha(d-2)^2 \sum_{i<j} \int_{\Omega} \frac{|u|^2}{r_{ij}^2} \, dx \geq \frac{1}{1 + ca^2} \int_{\Omega} |Vf^{-\alpha}u|^2 f^{2\alpha} \, dx \geq 0, \tag{18}
\]
with the optimal \( \alpha := (1 + \sqrt{1 + ca})^{-1} \). The Hardy inequalities corresponding to (15) and (18) were given in the form of theorems 2.1, 4.9, 4.11 in [20].

We also note that all corresponding Hardy inequalities from (15)–(18) hold on the full space of functions \( H^1(\mathbb{R}^d) \) since codim \( \Omega^c = dN - d - (N - 2)d = d \geq 3 \). For \( N = 2 \) we simply have \( \alpha = \frac{1}{2} \) and (15) and (18) reduce to (10) with the sharp constant \( (d - 2)^2/2 \). It was also noted in [20] that the large-\( N \) behavior of the constant in (15)/(18) with \( K_{d,N} \sim N \) cannot be improved.
Proof. One computes
\[ V_i f = -(d - 2)f \sum_{j \neq k} (x_k - x_j)^{-1}, \]
hence
\[ |V f|^2 = (d - 2)^2 f^2 \left( 2 \sum_{i<j} \frac{1}{t_{ij}} + \sum_{i<j<k} \frac{1}{R^2_{ijk}} \right), \]
and, due to \( |x_k - x_j|^{(d-2)} = 0 \) on \( \Omega \),
\[ \Delta f = (d - 2)^2 \sum_{i<j<k} \frac{1}{R^2_{ijk}}. \]
This gives the GSR (13) in the theorem. Bounding the (in total positive) term involving \( R_{ijk} \) in that equation by the term involving \( r_{ij} \) by means of \( K_{d,N} \) in (14), we obtain the total constant in (15)
\[ (d - 2)^2 (2(1 + \alpha) - aK_{d,N} ) = (d - 2)^2 (2 + K_{d,N}) \alpha \left( \frac{2}{2 + K_{d,N}} - \alpha \right), \]
which is optimal for \( \alpha = (2 + K_{d,N})^{-1} \). The bound (16) follows similarly.

The one-dimensional case, on the other hand, is much simpler due to collinearity of the particles:

**Theorem 6** (Separation of pairs of particles in \( d = 1 \)). Let
\[ \Omega := \{ (x_1, \ldots, x_N) \in \mathbb{R}^N; x_i \neq x_j \; \forall \; i \neq j \}. \]
Taking the ground state \( f(x) := \prod_{i<j} |x_j - x_i| \) we obtain
\[ \int_{\Omega} |V u|^2 \; dx - \frac{1}{2} \int_{\Omega} \left( \sum_{i<j} \frac{1}{t_{ij}} \right) |u|^2 \; dx = \int_{\Omega} |V f|^{-2} u^2 \prod_{i<j} t_{ij} \; dx \geq 0, \quad (19) \]
for all \( u \in C^\infty_0(\Omega) \).

**Proof.** In this case \( R_{ijk} = \infty \) and \( \Delta f = 0 \) on \( \Omega \). Hence \( \alpha = \frac{1}{2} \) optimizes the GSR.

This is a GSR version of (2) and theorem 2.5 in [20]. The corresponding inequality holds for all \( u \in H^2_0(\Omega) \) (note that codim \( \mathcal{Q}^* = 1 \) in this case) and is sharp. This lower bound (19) (as well as a corresponding identity for \( \alpha > 1/2 \)) plays an important role for operators appearing in the Calogero–Sutherland models [6, 7, 31], and for a model of identical particles in one dimension with generalized statistics [23, 32, 33].

For the two-dimensional case we fix a length scale \( R > 0 \) and define
\[ \tilde{t}_{ij} := \left| x_i - x_j \right| \ln \left| \frac{1}{R} \left| x_i - x_j \right| \right|. \]
and \( \hat{R}_{ijk} \) by

\[
\frac{1}{2\hat{R}_{ijk}^2} = \sum_{\text{cycl}} \left( \frac{1}{R} \ln \frac{1}{R} |x_i - x_j| \right) - \left( \frac{1}{R} \ln \frac{1}{R} |x_i - x_k| \right)
\]  \hspace{1cm} (20)

**Theorem 7** (Separation of pairs of particles in \( d = 2 \)). Let \( \Omega = \left\{ \left( x_1, \ldots, x_N \right) \in \mathbb{R}^{2N} : x_i \neq x_j \quad \forall \; i \neq j \right\} \cap (B_{R/2}(0))^N \).

Taking the ground state \( f(x) = \prod_{i<j} \left| \ln \frac{1}{R} |x_i - x_j| \right| \) we obtain

\[
\int_{\Omega} |Vu|^2 \, dx = \int_{\Omega} \left( 2\alpha (1 - \alpha) \sum_{i<j} \frac{1}{R_{ij}} - \alpha^2 \sum_{i<j<k} \frac{1}{R_{ijk}} \right) |u|^2 \, dx = \int_{\Omega} |Vf^{-\alpha} u|^2 f^{2\alpha} \, dx \geq 0,
\]

for all \( u \in C_0^\infty(\Omega) \). Hence, if \( K_{2,N} := \sup_{\Omega} \sum_{i<j} R_{ij}^{-2} / R_{ij}^2 \), then

\[
\int_{\Omega} |Vu|^2 \, dx - \frac{1}{2 + K_{2,N}} \sum_{i<j<k} \int_{\Omega} \frac{|u|^2}{R_{ijk}^2} \, dx \geq \int_{\Omega} |Vf^{-\alpha} u|^2 f^{2\alpha} \, dx \geq 0, \hspace{1cm} (21)
\]

and

\[
\int_{\Omega} |Vu|^2 \, dx - \frac{1}{2 + K_{2,N}} \sum_{i<j<k} \int_{\Omega} \frac{|u|^2}{R_{ijk}^2} \, dx \geq \int_{\Omega} |Vf^{-\alpha} u|^2 f^{2\alpha} \, dx \geq 0, \hspace{1cm} (22)
\]

with \( \alpha := \frac{1}{2 + K_{2,N}} \).

**Proof.** This theorem follows just as in the proof of theorem 5, using that \( \nabla_k f = f \sum_{i \neq k} \left| \ln \frac{1}{R} |x_k - x_i| \right|^{-1} (x_k - x_i)^{-1} \) and \( \Delta f = f \sum_{i \neq j} \hat{R}_{ijk}^{-2} \).

Here we have a rough bound \( K_{2,N} \leq 2(N - 2) \), which simply follows by applying the Cauchy–Schwarz inequality in \( \mathbb{R}^2 \) to each term in (20).

We have from (21) a non-trivial two-dimensional many-particle Hardy inequality on \( H^1(\Omega) = H^1(\mathbb{R}^2(0)^N) \). This corresponds to the physical situation where we consider the particles to be confined to a finite area. Also note that, despite the logarithms, this inequality gives a rough lower bound

\[
\inf_{u \in H^1(\Omega)} \left\{ \int_{\Omega} |Vu|^2 \, dx \right\} \geq \frac{\text{const} \cdot \left( \frac{N}{2} \right)}{(2 + K_{2,N})R^2}
\]

for the ground state energy of a confined gas of two-dimensional non-interacting bosonic particles, which hence is of the same form as for \( d \geq 3 \).

As an illustration of the freedom for improvement left in the remainder terms in the above GSRs, we show that it is e.g. possible to combine and improve theorem 4 with theorem 5:
Theorem 8 (Combined pairwise separation and total separation). Taking $\Omega$ as in (12) and the ground state $\rho := \prod_{i<j} e^{-\beta_{ij}(d-2)}$ with $\alpha = (2 + K_{d,N})^{-1}$ and $\beta = \frac{1}{4}(\alpha N(d-2) - d)(N-1) + 2$, we obtain

$$\int_{\mathbb{R}^d} |V u|^2 \, dx \geq \int_{\mathbb{R}^d} \left( \frac{(d-2)^2}{2 + K_{d,N}} \sum_{i<j} \frac{1}{r_{ij}} \right) \, dx + \frac{N}{4} \left( \frac{N(d-2)}{2 + K_{d,N}} - d \right)(N-1) + 2 \left( \frac{1}{\rho^2} \right)^2 |u|^2 \, dx$$

for all $u \in H^1(\mathbb{R}^d)$, $d \geq 3$.

Proof. Taking $g(x) := \prod_{i<j} e^{-\beta_{ij}(d-2)}$ and $h(x) := \rho^2$ in the product GSR potential (5), we find, together with the above computations,

$$V g \cdot V h = -\beta_{ij}(d-2)N^2(N-1)g,$$

which follows from an identity related to (11),

$$\sum_{k \neq j} \sum_{j \neq k} (x_k - x_j) \cdot (x_k - x_j)^{-1} = \frac{1}{2} N^2(N-1). \quad (24)$$

This identity is most easily proved (cp., e.g., equation (3.9b) in [34]) by introducing the center-of-mass $X := \frac{1}{N} \sum_j x_j$ and writing $\sum_{j \neq k} (x_k - x_j) = N(x_k - X)$, so that the lhs of (24) becomes

$$N \sum_{k \neq j} (x_k - X) \cdot (x_k - x_j)^{-1}$$

$$= N \left( \sum_{k \neq j} (x_k - X) \cdot (x_k - x_j)^{-1} - \sum_{j \neq k} (x_j - X) \cdot (x_k - x_j)^{-1} \right)$$

$$= N \frac{1}{2} \sum_{j \neq k} 1.$$

Now, collecting the constants in front of the terms in (5) involving $1/\rho^2$ gives

$$4N\beta ((\alpha (d-2)N(N-1) - d(N-1) + 2)\beta),$$

which with the earlier bound (14) on the circumradius terms and the corresponding choice of $\alpha$, and again by optimizing in the weight $\beta$, produces the statement of the theorem. \qed

3.2. Some other inequalities of many-particle type

Associated to the original one-dimensional Hardy inequality away from the origin is also the following ‘many-particle’ version, which can be viewed as a certain limiting case of theorem 8 for $d = 1$:
Theorem 9. Let $\Omega \equiv \{ (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 \ldots x_N \neq 0 \}$. Taking the ground state $f(x) \equiv |x|^{1-N} \prod_{k=1}^N |x_k|$ we obtain

$$\int_{\Omega} |\nabla f|^2 \, dx - \int_{\Omega} \left( \frac{1}{4} \sum_{k=1}^N \frac{1}{x_k^2} + (N-1)^2 \frac{1}{|x|^2} \right) |f|^2 \, dx$$

$$= \int_{\Omega} |\nabla f|^{-2} \, \prod_{k=1}^N |x_k| \, |x|^{2(1-N)} \, dx \geq 0,$$

for all $u \in C_0^\infty(\Omega)$. Hence, the lhs is non-negative on $H_0^1(\Omega)$.

The corresponding Hardy inequality was proved and applied in [4].

Proof. With $g \equiv \prod_k |x_k|$ and $h \equiv |x|^2$ in (5), and using that $\nabla_k g = g/x_k$, $\Delta g = 0$, $\nabla_k h = 2x_k$, and $\Delta h = 2N$, we find the potential

$$-\alpha (1 - \alpha) \sum_k \frac{1}{x_k^2} + 4\beta \left( \frac{2 - (1 + 2\alpha)N}{2} - \beta \right) \frac{1}{|x|^2},$$

which with the condition that the first term be optimal yields $\alpha = \frac{1}{2}$ and $\beta = \frac{1-N}{2}$. We also note that codim $\Omega^N = 1$. \qed

The following application of our systematic approach arises in the context of bosonic and supersymmetric matrix models (see e.g. [35]), and operators of the corresponding form also appear in a class of solvable models [15, 17]. Consider an $N$-tuple $(x_1, \ldots, x_N)$ of vectors in $\mathbb{R}^d$ and define the bivectors (two-blades) $B_{ij} \equiv x_i \wedge x_j, i < j$, with magnitudes

$$|B_{ij}| = \sqrt{|x_i|^2 |x_j|^2 - (x_i \cdot x_j)^2} = |x_i| |x_j| \sin \theta_{ij},$$

and inverses $B_{ij}^{-1} = (x_i \wedge x_j)^{-1} = -(x_i \wedge x_j)/|B_{ij}|^2$ (again, cp. [30] or appendix A). Note that $B_{ij} = 0$ if and only if $x_i$ and $x_j$ are parallel. We define the corresponding geometric quantities

$$\Sigma_1(x) = \sum_{j \neq k} \left| x_j \wedge (x_j \wedge x_k)^{-1} \right|^2 = \sum_{j < k} \frac{|x_j|^2 + |x_k|^2}{|x_j \wedge x_k|^2},$$

and

$$\Sigma_2(x) = \sum_{i \neq j \neq k} \left( x_i \wedge (x_i \wedge x_k)^{-1} \right) : \left( x_j \wedge (x_j \wedge x_k)^{-1} \right),$$

where we note that

$$-x_i \wedge (x_i \wedge x_k)^{-1} = x_i (x_i \wedge x_k) |x_i \wedge x_k|^{-2}$$

is the vector in the (oriented) plane $x_i \wedge x_k$ obtained by rotating $x_i$ by $90^\circ$ toward $x_k$ and rescaling by the inverse area $|x_i \wedge x_k|^{-1}$. We then have the following result, which one could think of as a higher-dimensional combination of theorem 2 with theorem 5, involving one-dimensional subspaces ($A = x_j$) instead of zero-dimensional ($A = 1$):

\[ J. Phys. A: Math. Theor. 48 (2015) 175203 \]
Theorem 10 (Parallelity of pairs of vectors in $d > 3$ or $d = 2$). Let

$$\Omega := \{ (x_1, \ldots, x_N) \in \mathbb{R}^{dN} : x_i \wedge x_j \neq 0 \ \forall \ i \neq j \}. $$

Taking the ground state

$$f(x) := \prod_{j<k} |B_{jk}|^{-(d-3)}$$

one obtains

$$\int_\Omega |\nabla u|^2 \, dx - (d-3)^2 \int_\Omega \left( \alpha (1 - \alpha) \Sigma_1 - \alpha^2 \Sigma_2 \right) |u|^2 \, dx = \int_\Omega |\nabla f|^2 u^{2\alpha} \, dx \geq 0,$$

for all $u \in C^\infty_0(\Omega)$. Hence, if $C_{d,N} = \sup_{x \in \Omega} \Sigma_2(x)/\Sigma_1(x)$, then

$$\int_\Omega |\nabla u|^2 \, dx - \frac{(d-3)^2}{4(1 + C_{d,N})} \int_\Omega \Sigma_1 |u|^2 \, dx \geq \int_\Omega |\nabla f|^2 u^{2\alpha} \, dx \geq 0,$$

and

$$\int_\Omega |\nabla u|^2 \, dx - \frac{(d-3)^2}{4C_{d,N} (1 + C_{d,N})} \int_\Omega \Sigma_2 |u|^2 \, dx \geq \int_\Omega |\nabla f|^2 u^{2\alpha} \, dx \geq 0,$$

with $\alpha := \frac{1}{2(1 + C_{d,N})}$ (in both inequalities).

Note that here $C_{d,N} \leq N - 2$ by the Cauchy–Schwarz inequality in $\mathbb{R}^d$. Furthermore, $\text{codim}(\Omega^c) = dN - (N-2)d - d - 1 = d - 1$, so the corresponding inequalities hold on $H^1(\mathbb{R}^{dN})$ for $d > 2$ and $H^1_0(\Omega)$ for $d = 2$. For $N = 2$ we have $\Sigma_2 = 0$ and hence the optimal and sharp two-particle Hardy inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \geq \frac{(d-3)^2}{4} \int_{\mathbb{R}^d} \frac{|x_1|^2 + |x_2|^2}{|x_1 \wedge x_2|^2} |u|^2 \, dx,$$

(25)

for $u \in H^1(\mathbb{R}^d)$, $d > 3$, or $u \in H^1_0(\Omega)$, $d = 2$ (cp. theorem 2).

Proof. One computes (see appendix A)

$$\nabla_i f = -\frac{(d-3)}{2} \sum_{j \neq k} B_{jk} \nabla_i B_{jk} \left( \begin{array}{c} x_j \wedge (x_j \wedge x_k) \end{array} \right)^{-1},$$

implying

$$|\nabla f|^2 = (d-3)^2 f^2 \left( \Sigma_1 + \Sigma_2 \right),$$

as well as $\Delta f = (d-3)^2 \Sigma_2$, due to $\Delta_k |B_{kj}|^{-(d-3)} = 0$ on $\Omega \forall j$. \hfill \Box

For the critical case $d = 3$ we once again need a length scale. One natural way to obtain this in the context of matrix models is to introduce the matrix model potential $W := \sum_{j<k} |B_{jk}|^2$ and consider e.g. $\Omega_0 := \Omega \cap \{ W < R \}$ for which $H^1_0(\Omega_0) = H^1_0(\{ W < R \})$. Defining

2 The case with $d = 3$ and $N$ particles corresponds in the matrix models to dimensionally reduced SU(2) Yang–Mills theory from $N + 1$ spacetime dimensions.
\[ \bar{\Sigma}_1(x) := \sum_{j<k} \left| x_j \right|^2 + \left| x_k \right|^2, \]

and

\[ \bar{\Sigma}_2(x) := \sum_{i\neq j\neq k \neq \ell} \frac{x_i \perp B_{jk}^{-1}}{\ln \frac{1}{R_i^2}} \cdot \frac{x_j \perp B_{jk}^{-1}}{\ln \frac{1}{R_j^2}}. \]

we then have the following analog of theorem 10.

**Theorem 11 (Parallelity of pairs of vectors in \(d = 3\)).** Taking the ground state \( f(x) := \prod_{j<k} \left| \ln \frac{1}{R_j^2} \right| \) one obtains

\[
\int_{\Omega_0} |Vu|^2 \, dx = \int_{\Omega_0} \left( \alpha(1-\alpha)\bar{\Sigma}_1 - \alpha^2\bar{\Sigma}_2 \right) |u|^2 \, dx
\]

\[= \int_{\Omega_0} |Vf^{\alpha-\alpha}u|^2 f^{2\alpha} \, dx \geq 0, \]

for all \( u \in C_0^\infty(\Omega_0) \). Hence, if \( C_{3,N} := \sup_{x \in \Omega_0} \bar{\Sigma}_2(x)/\bar{\Sigma}_1(x) (\leq N - 2) \), then

\[\int_{\Omega_n} |Vu|^2 \, dx - \frac{1}{4(1 + C_{3,N})} \int_{\Omega_0} \bar{\Sigma}_1 |u|^2 \, dx \geq \int_{\Omega_n} |Vf^{\alpha-\alpha}u|^2 f^{2\alpha} \, dx \geq 0, \]

and

\[\int_{\Omega_0} |Vu|^2 \, dx - \frac{1}{4C_{d,N}(1 + C_{3,N})} \int_{\Omega_0} \bar{\Sigma}_2 |u|^2 \, dx \geq \int_{\Omega_0} |Vf^{\alpha-\alpha}u|^2 f^{2\alpha} \, dx \geq 0, \]

with \( \alpha := \frac{1}{2(1 + C_{3,N})} \) (in both inequalities). In particular, with \( N = 2 \),

\[\int_{\mathbb{R}^3} |Vu|^2 \, dx \geq \frac{1}{4} \int_{\mathbb{R}^3} \left[ \frac{|x_1|^2 + |x_2|^2}{|x_1 \wedge x_2|^2} \left( \ln \frac{1}{R_1^2} \right) \right] |u|^2 \, dx, \quad (26)\]

for all \( u \in H^1_0(\{W < R\}) \).

The proof is completely analogous to the proof of theorem 10, with similar modifications as in theorem 7.

### 3.3. Inequalities involving volumes of simplices of points

Consider again a tuple of \( N \) vectors in \( \mathbb{R}^d \), but now think of them as points. These points span a (possibly degenerate) \( N - 1 \)-simplex with volume given by

\[ V(x_1, \ldots, x_N) := \frac{1}{(N-1)!} \left| (x_1 - x_N) \wedge \ldots \wedge (x_{N-1} - x_N) \right| \]

\[= \frac{1}{(N-1)!} \left| \det \left[ (x_j - x_N) \cdot (x_k - x_N) \right] \right|_{j,k \neq N}. \]

Note that this expression is invariant under any permutation of the points (this is also shown explicitly in appendix A). We have then the following geometric generalization of the \( N = 2 \) case in theorem 4 (or 5):
Theorem 12 (Volume of the \( N - 1 \)-simplex of \( N \) points in \( \mathbb{R}^d \), \( d > N \) or \( d = N - 1 \)). Consider \( \Omega := \{ (x_1, \ldots, x_N) \in \mathbb{R}^{dN} : V(x) > 0 \} \) and the ground state \( f(x) := (N - 1)! V(x)^{-(d-N)} \). Denoting
\[
\Sigma^{(N)}(x) := \frac{\sum_{j=1}^{N} V(x_1, \ldots, x_j, \ldots x_N)^2}{(N - 1)^2 V(x_1, \ldots, x_N)^2},
\]
(27)
(where * means deletion), we then have
\[
\int_{\Omega} |V u|^2 \ dx - \frac{(d - N)^2}{4} \int_{\Omega} \Sigma^{(N)} |u|^2 \ dx = \int_{\Omega} |V f^{-\frac{1}{2}} u|^2 f \ dx \geq 0,
\]
(28)
for all \( u \in C_0^\infty(\Omega) \). The corresponding sharp Hardy inequalities hold on \( H^1(\mathbb{R}^{dN}) \) for \( d > N \) and \( H^1_0(\Omega) \) for \( d = N - 1 \).

Before proving this theorem, it is convenient to introduce the following notation:
\[
A_k(x) := (-1)^{k-1} \bigwedge_{1 \leq j \neq k < N} (x_j - x_N), \quad k = 1, \ldots, N - 1,
\]
\[
A_{k=N}(x) := (-1)^{N-1} \bigwedge_{1 \leq j < N-1} (x_j - x_{N-1}), \quad \text{and}
\]
\[
A(x) := \bigwedge_{1 \leq j < N} (x_j - x_N),
\]
so that
\[
\Sigma^{(N)}(x) = \sum_{k=1}^{N} |A_k \wedge A^{-1}|^2 = \sum_{k=1}^{N} |A_k|^2 |A|^2.
\]
Note that \( \Sigma^{(N)} \) describes a ratio of a mean of squares of volumes of all \( N - 2 \)-dimensional subsimplices \( A_k \) to the square of the volume of the full simplex \( A \). In particular, for \( N = 3 \),
\[
\Sigma^{(3)} = \frac{r_2^2 + r_3^2 + r_1^2}{4V(x_1, x_2, x_3)^2} \geq \frac{4}{27} \left( \frac{\rho^2}{(r_{123})^d} \right)^d,
\]
(29)
where we used that the simplex area \( V \) is bounded by \( \frac{3\sqrt{3}}{4\pi} \) times the area of the circumcircle.

**Proof.** (Proof of theorem 12) Note that we have defined \( A_{k < N} \) and \( A_N \) s.t. (cp. appendix A)
\[
A = (x_N - x_N) \wedge A_k = (x_N - x_{N-1}) \wedge A_N.
\]
(30)
For each fixed \( k < N \) we then have \( f(x) = |(x_k - x_N) \wedge A_k|^{-(d-N)} \) and just as in theorem 2 that \( V_{A_k} f = (1)_{N-1}^{(N-1)} (d - N)(A_k \wedge A^{-1}) f \) and \( A_k f = 0 \), and hence the following optimal single-particle Hardy GSR:
\[
\int_{\Omega} |V_k u|^2 \ dx - \frac{(d - N)^2}{4} \int_{\Omega} |A_k \wedge A^{-1}|^2 |u|^2 \ dx = \int_{\Omega} |V_k f^{-\frac{1}{2}} u|^2 f \ dx.
\]
(31)
For \( k = N \) we use the invariance of \( V \) under permutations, i.e. (30), and write instead \( f(x) = |(x_N - x_{N-1}) \wedge A_N|^{-(d-N)} \), with analogous conclusions. Hence, \( |V|^2 = \sum_k |V_k|^2 = (d - N)^2 \Sigma^{(N)} f^2 \), \( \Delta f = \sum_k A_k f = 0 \), and the GSR (28) follows. Finally, note that in this case \( \text{codim} \Omega^d = dN - (N - 1)d - (N - 2) = d - N + 2 \) and hence \( H^1_0(\Omega) = H^1(\mathbb{R}^{dN}) \) for \( d > N - 1 \). Sharpness is proven in appendix B. \( \square \)
Again, for the codimension-critical case $d = N$ we fix a length scale $R > 0$ and restrict to,
e.g., $\Omega_R := \{(x_1, \ldots, x_N) \in \mathbb{R}^d : V(x) < R\}$.

**Theorem 13** (Volume of the $d - 1$-simplex of $d$ points in $\mathbb{R}^d$). Consider $\Omega := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 < V(x) < R\}$ and the ground state $f(x) := \left[\ln \frac{1}{R} V(x)\right]$. Denoting

$$\Sigma^{(d)}(x) := \frac{\sum_{k=1}^{d} V(x_1, \ldots, \hat{x}_k, \ldots, x_d)^2}{(d - 1)^2 V(x_1, \ldots, x_d)^2 \left(\ln \frac{1}{R} V(x_1, \ldots, x_d)\right)^2},$$

we then have

$$\int_{\Omega} |V u|^2 \, dx - \frac{1}{4} \int_{\Omega} \Sigma^{(d)} |u|^2 \, dx = \int_{\Omega} |\nabla \cdot u|^2 f \, dx \geq 0,$$

for all $u \in C^\infty_0(\Omega)$. The corresponding Hardy inequality holds on $H^1_0(\Omega_R)$.

**Proof.** This is completely analogous to the proof of theorem 12, using that

$${\nabla \cdot f} = \left(\frac{d - 1}{2}\right) (\Lambda_R \wedge A^{-1})$$

and $\Delta_R f = 0$. □

Now, whereas theorem 10 involved all possible quantities $|R_j| = |(x_i - 0) \wedge (x_j - 0)|$ among $N$ points, i.e. (twice) the volumes of all $2$-simplices spanned by selections of points of the form $(x_i, x_j, 0)$, this can naturally be generalized to higher dimensions as follows. This is both a geometrically and combinatorially more complete generalization of theorem 5, which corresponds to $p = 2$.

**Theorem 14** (Volumes of all simplices of $p$ points among $N$ points in $d \neq p$). Consider $\Omega := \{(x_1, \ldots, x_N) \in \mathbb{R}^d : V(x) > 0\}$, where

$$\Psi(x) := \frac{1}{((p - 1)!)(\frac{N}{p})} \prod_{1 \leq i < j \leq N} \left| (x_i - x_j) \wedge \ldots \wedge (x_{p-1} - x_p) \right|$$

is the product of the volumes of all $p - 1$-simplices in $\mathbb{R}^d$ spanned by $p$ of the points \{x_{i=1,\ldots,N}\}, and take the ground state $f := \left((p - 1)!\right)^{(\frac{N}{p})} \Psi^{(d-p)}$.

Denote by $A = A(p, N)$ the set of ordered subsets $\lambda = (\lambda_1, \ldots, \lambda_p) \subseteq \{1, \ldots, N\}$ of $p$ elements out of $N$. For $\lambda \in A$ define the $p - 1$-blade $A_{\lambda} := A(x_1, \ldots, x_N)$, and for each $k \in \lambda$ let $A_{\lambda,k}$ denote a $p - 2$-blade s.t. $A_{\lambda,k} = (x_k - x_{\lambda_k}) \wedge A_{\lambda,k}$ for some $\lambda_k \in \lambda \setminus k$. With

$$\Sigma^{(p,N)}_1(x) := \sum_{k=1}^{N} \sum_{\lambda \in A \setminus k} \left| A_{\lambda,k} \wedge A_{\lambda}^{-1} \right|^2 = \sum_{\lambda \in A} \sum_{k \in \lambda} \left| A_{\lambda,k} \right|^2 / |A_{\lambda}^2|,$$

and

$$\Sigma^{(p,N)}_2(x) := \sum_{k=1}^{N} \sum_{\lambda_1, \mu \in A \setminus k} \left( A_{\lambda_1,k} \wedge A_{\lambda_1}^{-1} \right) \cdot \left( A_{\mu,k} \wedge A_{\mu}^{-1} \right),$$
we then have
\[ \int_\Omega |Vu|^2 \, dx - (d-p)^2 \int_\Omega \left( \alpha (1 - \alpha) \Sigma_{(p,N)}^{(p,N)} - \alpha^2 \Sigma_{(p,N)}^{(p,N)} \right) |u|^2 \, dx \\
\geq \int_\Omega \left| V f^\alpha u \right|^2 f^{2\alpha} \, dx \geq 0, \tag{34} \]
for all \( u \in C_0^\infty (\Omega) \). Hence, if \( C_{d,N}^{(p)} = \sup_{x \in \Omega} \Sigma_{(p,N)}^{(p,N)}(x)/\Sigma_{(p,N)}^{(p,N)}(x) \), then
\[ \int_\Omega |Vu|^2 \, dx - \frac{(d-p)^2}{4(1 + C_{d,N}^{(p)})} \int_\Omega \Sigma_{(p,N)}^{(p,N)} |u|^2 \, dx \geq \int_\Omega \left| V f^\alpha u \right|^2 f^{2\alpha} \, dx \geq 0, \tag{35} \]
and
\[ \int_\Omega |Vu|^2 \, dx - \frac{(d-p)^2}{4C_{d,N}^{(p)}(1 + C_{d,N}^{(p)})} \int_\Omega \Sigma_{(p,N)}^{(p,N)} |u|^2 \, dx \geq \int_\Omega \left| V f^\alpha u \right|^2 f^{2\alpha} \, dx \geq 0, \tag{36} \]
with \( \alpha := \frac{1}{2(1 + C_{d,N}^{(p)})} \).

Note that by Cauchy–Schwarz in \( \mathbb{R}^d \),
\[ \Sigma_{(p,N)}^{(p,N)} \leq \sum_k \sum_{\lambda \neq \mu} \sum_{\rho \neq \lambda} |A_{\lambda,k} \ll A_{\lambda}^{-1}| \left| A_{\mu,k} \ll A_{\mu}^{-1} \right| \\
\leq \sum_k \sum_{\lambda \neq \mu} \frac{1}{2} \left( |A_{\lambda,k} \ll A_{\lambda}^{-1}|^2 + |A_{\mu,k} \ll A_{\mu}^{-1}|^2 \right) \\
= \sum_k \sum_{\lambda \neq \mu} |A_{\lambda,k} \ll A_{\lambda}^{-1}|^2 \sum_{\rho \neq \lambda} \left( \left( \frac{N-1}{p-1} \right) - 1 \right) \Sigma_{(p,N)}^{(p,N)}, \]
hence \( C_{d,N}^{(p)} \leq \left( \frac{N-1}{p-1} \right) - 1 \). Furthermore, as in the single-volume case one finds that \( \text{codim} \Omega^p = d - p + 2 \). We therefore have the following bound for the Laplacian in terms of a potential which can be interpreted as a geometrically generalized many-particle interaction:

**Corollary 1.** For \( d > p \) we have the generalized many-particle Hardy inequality
\[ \int_{\mathbb{R}^n} |Vu|^2 \, dx \geq \frac{(d-p)^2}{4(1 + C_{d,N}^{(p)})} \int_{\mathbb{R}^n} \sum_{\lambda \in \Lambda_{(p,N)}} V\left(x_{\lambda_1}, \ldots, x_{\lambda_k}, \ldots, x_{\lambda_p}\right) \left( x_{\lambda_1}, \ldots, x_{\lambda_k}, \ldots, x_{\lambda_p}\right)^2 |u|^2 \, dx, \tag{37} \]
for all \( u \in H^1(\mathbb{R}^{dn}) \). The inequality holds for \( u \in H^1_0(\Omega) \) when \( d = p - 1 \).
Proof. We have
\[ f = \prod_{i \in A(p,N)} |A_i|^{(d-p)} \] with \( A_i \prod_{\mu \in A(p,N) : \mu \neq \lambda} |A_{\mu}|^{-(d-p)} = 0 \) on \( \Omega \forall k, \lambda \), and hence
\[ v_k f = \sum_{i, k \in A(p,N) : k \neq \lambda} v_k |A_i|^{(d-p)} \prod_{\mu \in A(p,N) : \mu \neq \lambda} |A_{\mu}|^{-(d-p)} = \frac{(d-p)^2}{4} \sum_{i, k \neq \lambda} |A_i|^{2} v_k |A_{\lambda}|^{2} |A_{\mu}|^{2} v_k |A_{\mu}|^{2}. \]

and
\[ A_k f = \sum_{i \neq k, \mu \neq k} v_k |A_i|^{(d-p)} \prod_{\mu \neq i, \mu \neq k} |A_{\mu}|^{-(d-p)} = \frac{(d-p)^2}{4} \sum_{i \neq k, \mu \neq k} v_k |A_i|^{2} |A_{\lambda}|^{2} |A_{\mu}|^{2} v_k |A_{\mu}|^{2}. \]

Then, again using (A.1) from appendix A,
\[ v_k |A_i|^2 = v_k \left( x_k - x_{\lambda} \right) \wedge A_{\lambda,k} \right|^2 = (-1)^{\left( \frac{p-1}{2} \right)} 2 A_{\lambda,k} \perp A_{\lambda}, \]
we therefore obtain
\[ |V f|^2 = \sum_k |V_k f|^2 = (d-p)^2 f^2 \left( \Sigma_1^{(p,N)} + \Sigma_2^{(p,N)} \right) \]
and
\[ \Delta f = \sum_k A_k f = (d-p)^2 f \Sigma_2^{(p,N)}. \]

The GSR (34) and inequalities (35)–(36) now follow as in the earlier theorems. \( \square \)

For the optimal large-\( N \) dependence of the constants in these many-particle inequalities, it becomes relevant to study the ratio of the geometric quantities w.r.t. the optimal asymptotic probability distribution \( \rho \) of points in \( \mathbb{R}^d \);
\[ C_d^{(p,q)} := \sup_{\rho \geq 0} \int_{\mathbb{R}^d} \left( \Sigma_{\rho}(x_1, \ldots, x_q) \lambda_{\rho}(x_{p+1}, \ldots, x_{2p-1}) \right) \prod_{i=1}^{p-1} \mathbb{R}^d \mathbb{R}^d \prod_{i=1}^{q} \mathbb{R}^d \mathbb{R}^d \mathbb{R}^d, \]
where \( q = p + 1, \ldots, 2p - 1 \), \( \Sigma^{(p)} \) was defined in (27), and
\[ \Sigma^{(p,q)} : = \sum_{1 \leq i, j \leq A(p,q) : x \in S} \left( A_{\pi(i), \pi(j)} \perp A_{\pi(j), \pi(i)} \right) \cdot \left( A_{\pi(\rho), \pi(j)} \perp A_{\pi(j), \pi(\rho)} \right) \]
are higher-dimensional generalizations of a single circumradius contribution \( R_1^{(p,q)} \). Related optimizations involving lower-dimensional geometric quantities have been discussed in remarks 2.2 (iv) in [20] (see also [4]), section 3.1 in [21], and also e.g. equation (2) in [36].

It is of course possible to generalize theorem 14 even further and consider the volumes of all simplices among \( N \) points (i.e. all simplex dimensions simultaneously), including the case of critical codimension. We will not state the corresponding theorem explicitly here.
4. Conclusions

We have studied operator inequalities for the Laplacian, i.e. uncertainty principles, involving many-particle interaction potentials of increasingly generalized geometric form. Such interactions appear e.g. in membrane matrix models [35] and in higher-dimensional generalized Calogero–Sutherland models such as those studied in [14, 15, 17, 18], and the bounds provided here address the issue of whether the spectrum of the corresponding model is bounded from below independently of the choice of external potentials, i.e. whether the uncertainty principle is strong enough to prevent many-body collapse in these cases. Furthermore, the explicit GSRs provided in these bounds will be of use for the detailed spectral analysis of the associated operators. We have furthermore illustrated the novel use of techniques from geometric algebra to conveniently facilitate these otherwise technically and geometrically complicated computations.

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Appendix A. Some applications of geometric algebra

The geometric algebra over \( \mathbb{R}^d \) is the exterior algebra \( \wedge \mathbb{R}^d \) together with the left- and right-interior products \( (A, B) \mapsto AB \), \( (A, B) \mapsto A \ll B \) and the associative Clifford product \( (A, B) \mapsto AB \), which are all inherited from the usual Euclidean scalar product \( (x, y) \mapsto x \cdot y \). A \( p \)-blade \( A \) is an exterior product of \( p \) vectors and is uniquely determined by its corresponding \( p \)-dimensional subspace \( \hat{A} \subseteq \mathbb{R}^d \), orientation (sign), and magnitude \( |A| = \sqrt{A^\dagger A} \), where \( A' := (-1)^{\frac{p(p-1)}{2}} A \) denotes the reverse of \( A \). Note that if \( A, B \) are blades then their exterior and interior products \( A \ll B \) resp. \( A \ll B \) are blades as well, and that \( A \ll B = AB \) if \( \hat{A} \subseteq \hat{B} \) (see e.g. section 3 in [30]).

If \( A \) is a \( p \)-blade then we have for the gradient

\[
V_x | x \wedge A |^2 = 2(x \wedge A)A' = 2A'(A \wedge x) = (-1)^{\frac{p+1}{2}} 2A \ll (x \wedge A).
\]

(A.1)

One way to see this is to note that it is trivially true for \( A = 0 \), and that for \( A \neq 0 \) we can write any point \( x \) in \( \mathbb{R}^d \) uniquely as

\[
x = xA A^{-1} = (x \ll A) A^{-1} + (x \wedge A) A^{-1} = x_\parallel + x_\perp,
\]

where \( A^{-1} = A' / |A|^2 \), \( x_\parallel := P_A x \) is the orthogonal projection on \( \hat{A} \) and \( x_\perp := (1 - P_A) x \) (the so-called rejection on \( \hat{A} \); see e.g. section 3.3 in [30]). Hence
The remaining equalities in (A.1) follow by taking the reverse.

From (A.1) it also follows that

\[
\left| V_x |x \wedge A|^2 \right|^2 = 4 \left( (x \wedge A)A^\dagger \right) \left( (x \wedge A)A^\dagger \right)^\dagger = 4 |A|^2 |x \wedge A|^2,
\]

\[
V_x |x \wedge A|^2 \cdot V_y |y \wedge B|^2 = 4 \left( (x \wedge A)A^\dagger \right) \cdot \left( (y \wedge B)B^\dagger \right)
= 4 \left( A^\dagger \right) \left( B^\dagger \right) \left( (x \wedge A) \cdot (y \wedge B) \right)
= (-1)^{p+1} \left( \frac{q+1}{2} \right) 4 (A \wedge (x \wedge A)) \cdot (B \wedge (y \wedge B)),
\]

for \( A, B \) \( p \)- resp. \( q \)-blades, and, by e.g. exercise 3.6 in [30] (with \( \{e_k\}_k \) denoting an orthonormal basis in \( \mathbb{R}^d \)),

\[
\Delta_x |x \wedge A|^2 = 2 V_x (x \wedge A)A^\dagger = \sum_{k=1}^d (e_k \wedge A)A^\dagger = 2(d - p) |A|^2,
\]

as well as (for \( x \) outside the singular set)

\[
\Delta_x |x \wedge A|^{2\beta} = (\beta - 1) |x \wedge A|^{2\beta-4} \frac{1}{R^2} V_x |x \wedge A|^2
+ \beta |x \wedge A|^{2\beta-2} \Delta_x |x \wedge A|^2
= 2\beta (2\beta + d - p - 2) |A|^2 |x \wedge A|^{2\beta-2},
\]

which is zero for \( \beta = -d + p - 2)/2. Furthermore, (again on a suitable domain)

\[
\Delta_x \ln \frac{1}{R} |x \wedge A| = \frac{1}{2} |x \wedge A|^{-2} V_x |x \wedge A|^2 = (x \wedge A)^{-1} A^\dagger,
\]

\[
\Delta_x \ln \frac{1}{R} |x \wedge A| = \frac{1}{2} V_x \cdot \left( |x \wedge A|^{-2} V_x |x \wedge A|^2 \right) = (d - p - 2) |A|^2 |x \wedge A|^{-2}.
\]

Lastly, we have the following explicit invariance of the simplex volume under permutations of the points:

**Proposition 15.** Let \( A := \bigwedge_{1 \leq j \leq N} (x_j - x_N) \) be the \( N - 1 \)-blade associated to the points \( (x_1, \ldots, x_N) \). Then \( A \) is invariant, up to a sign, under any permutation of the points.

**Proof.** \( A \) is due to the total antisymmetry of the exterior product clearly invariant under any permutation \( \sigma \) of \( (x_1, \ldots, x_{N-1}) \), up to a sign (\( \text{sgn} \sigma \)). Furthermore, for any \( 1 \leq k < N \),

\[
A_{k} \cdot (x_{k} - x_N) = \text{sgn} \sigma \cdot \left| A_{k} \right|^2 |x_{k} - x_N|^2.
\]
\[ A = (x_1 - x_N) \wedge \ldots \wedge (x_k - x_N) \wedge \ldots \wedge (x_{k-1} - x_N) \]
\[ = \bigwedge_{j=1}^{k-1} (x_j - x_k + x_N) \wedge (x_k - x_N) \wedge \bigwedge_{j=k+1}^{N-1} (x_j - x_k) \]
\[ = (-1)^{N-k} \bigwedge_{j=1}^{N-1} (x_j - x_k) \]

by multilinearity and antisymmetry. The proposition then follows by composition of permutations. \[ \square \]

Appendix B. Sharpness of derived constants

For completeness, we briefly note in this appendix how the explicit ground states \( f \) in the derived Hardy GSRs also can be used to determine sharpness of the constants in the corresponding Hardy inequalities.

Lemma 16. Suppose that \( \Omega \) and \( f \) in proposition 1 are such that \( \Omega = \mathbb{R}^d \) and

(i) \( \int_{\mathbb{R}^d} f^{1-\delta} e^{-|x|} \, dx \) is uniformly bounded for small \( \delta > 0 \),

(ii) \( \int_{\mathbb{R}^d} \frac{|V f|^2}{f^2} f^{1-\delta} e^{-|x|} \, dx \) is finite for small \( \delta > 0 \), but \( \rightarrow \infty \) as \( \delta \rightarrow 0 \).

Then, for every \( \epsilon > 0 \) there exists \( u_\delta := f^2 f^{-2} e^{-2|x|} \in H^1(\mathbb{R}^d) \), with \( \delta > 0 \), s.t.

\[ \int_{\mathbb{R}^d} |V u_\delta|^2 \, dx \leq \left( \frac{1}{4} + \epsilon \right) \int_{\mathbb{R}^d} \frac{|V f|^2}{f^2} |u_\delta|^2 \, dx. \]  

(B.1)

If \( f(x) \rightarrow 0 \) as \( x \rightarrow \partial \Omega \), then we reverse the sign on \( \delta \) above and note that \( u_\delta \in H^1_0(\Omega) \) for small \( \delta \).

Proof. Using that \( u_\delta \in C^1(\Omega \setminus \{0\}) \) and

\[ |V u_\delta| = \left| \frac{V f}{2f} u_\delta - \frac{\delta V f}{2f} u_\delta - \frac{V |x|}{2} u_\delta \right| \leq \frac{1}{2} |\delta| \left| \frac{V f}{f} \right| |u_\delta| + \frac{1}{2} |u_\delta| \]

on \( \Omega \setminus \{0\} \), we find by (i) and (ii) that \( u_\delta \in H^1(\mathbb{R}^d) \) for sufficiently small \( \delta > 0 \), and that

\[ \|V u_\delta\|_{L^2} \leq \left( \frac{1}{2} + \frac{|\delta|}{2} \right) \left\| \frac{V f}{f} u_\delta \right\|_{L^2} + \frac{1}{2} \|u_\delta\|_{L^2} \]

\[ \leq \left( \frac{1}{2} + \frac{|\delta|}{2} + \frac{\sqrt{C}}{2} \right) \left\| \frac{V f}{f} u_\delta \right\|_{L^2}^{1/2} \left\| \frac{V f}{f} u_\delta \right\|_{L^2}^{1/2}, \]

where \( C < \infty \) denotes a bound for (i). The result follows by taking \( \delta \rightarrow 0 \). \[ \square \]
The conditions (i) and (ii) above typically hold when \( f \) is of the form \( \delta_\Omega^{(k-2)} \) near \( \partial \Omega \), where \( \delta_\Omega \) is the distance to \( \Omega \) and \( k \) the codimension of \( \Omega \). Let us prove this explicitly for some of the Hardy GSRs considered in this paper.

**Proposition 17.** The constant \((d - p - 2)^2/4\) in (8) is sharp.

**Proof.** As in appendix A, we split the space \( \mathbb{R}^d \) into variables \( x_\perp \in \bar{A} \), and \( x_\parallel \) orthogonal to \( A \). Then, since \( f(x) \propto |x_\perp|^{-(d-p-2)} \),

\[
\int_{\mathbb{R}^d} f^{1-\delta} e^{-|x|^2} \, dx \propto \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-p}} |x_\perp|^{-(d-p-2)(1-\delta)} e^{-|x_\perp|^2} \, dx_\perp \, dx_\parallel \leq |\mathbb{S}^{d-p-1}| \int_{\mathbb{R}^p} e^{-\frac{1}{4}|x_\parallel|^2} \, dx_\parallel \int_{r=0}^{\infty} r^{-(d-p-2)(1-\delta)} e^{-L_p r} \, dr,
\]

which is uniformly bounded for \( 0 < |\delta| < \delta_0 \), and

\[
\int_{\mathbb{R}^d} \frac{|\nabla f|^2}{f^2} f^{1-\delta} e^{-|x|^2} \, dx \propto \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-p}} |x_\perp|^{-(d-p-2)(1-\delta)} e^{-|x_\perp|^2} \, dx_\perp \, dx_\parallel \propto \int_{r=0}^{\infty} \int_{\mathbb{R}^p} e^{-\sqrt{|x_\parallel|^2 + r^2}} \, dx_\parallel \, r^{-(d-p-2)\delta} \, dr,
\]

which is finite for \( 0 < +\delta < \delta_0 \), but tends to infinity when \( \delta \to 0 \). Hence, by lemma 16, the constant in the GSR potential \( \frac{|\nabla f|^2}{4f^2} = \frac{(d-p-2)^2}{4} \frac{1}{|x_\perp|^2} \) is sharp for \( d \neq p + 2 \). \( \square \)

**Proposition 18.** The constant \( N ((N - 1)d - 2)^2/4 \) in (10) is sharp for \((N - 1)d > 2\).

**Proof.** The set \( \Omega \) in theorem 4 is a linear subspace of \( \mathbb{R}^{dN} \) which can be parameterized by, say, \( x_N \in \mathbb{R}^d \). We then have the ground state

\[
f(x) := \rho^{4N} \left( \sum_{i \in N} |x_i|^2 + \sum_{i < j \in N} |x_i - x_j|^2 \right)^{-\frac{(N-1)d-2}{2}},
\]

where we for fixed \( x_N \) define \( \bar{y}_i := x_i - x_N \). Hence, using that

\[
|y|^2 = \sum_{i \in N} |y_i|^2 \leq \rho^2 \leq \sum_{i \in N} |y_i|^2 + \sum_{i < j \in N} \left( |y_i|^2 + |y_j|^2 \right) \leq C |y|^2
\]

(here and in the following, \( C \) will denote some unspecified positive constants), we find

\[
\int_{\mathbb{R}^d} f^{1-\delta} e^{-|x|^2} \, dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1d}} \rho^{-(N-1)(d-2)(1-\delta)} e^{-|x_N|^2} \, dy \, dx_N \leq C \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1d}} e^{-\sqrt{|y|^2 + |x_N|^2}} \left| y \right|^{-(N-1)(d-2)(1-\delta)} \, dy \, dx_N
\]
which is uniformly bounded for \( 0 < \delta < \delta_0 \), and
\[
\int_{\mathbb{R}^d} \frac{|V|^2}{f^2} f^{1-\delta} e^{-1\|x\|} \, dx \\
\sim C \int_{\mathbb{R}^d} \int_{\mathbb{R}^{N-d}} e^{-\sqrt{|V|^2 + F_0}} \, |x|^{2 - ((N-1)d - 2)(1 - \delta)} \, dy \, dx_N,
\]
which is finite for \( 0 < \delta < \delta_0 \), but tends to infinity as \( \delta \to 0 \). Hence, the GSR constant in (10) is sharp by lemma 16. □

**Proposition 19.** The constant \((d-3)^2/4\) in (25) is sharp.

**Proof.** Here \( f(x) = |x_1 \wedge x_2|^{(d-3)} \) and \( \Omega' = \{ (x_1, x_2) \in \mathbb{R}^{2d} : x_1 \wedge x_2 = 0 \} \) is a cone-like set which can be parameterized by \( x_1 \in \mathbb{R}^d \setminus \{0\} \), \( x_2 \in \mathbb{R}^d \), and \( x_1 = 0, x_2 \in \mathbb{R}^d \). Hence, for each fixed \( x_1 \neq 0 \) we split the second variable into \( x_2 || \) along, and \( x_2 \perp \) orthogonal to, the line \( \mathbb{R} x_1 \), write \( |x_1 \wedge x_2| = |x_1||x_2||, \) and deduce (with \( \delta' := (d-3)\delta \))
\[
\int_{\mathbb{R}^{2d}} f^{1-\delta} e^{-1\|x\|} \, dx \\
= \int_{\mathbb{R}^{d} \setminus \{0\}} \int_{\mathbb{R}^{d-1}} e^{-1\|x_2||} \ |x_2||^{(d-3)+\delta} \, dx_2 \ |x_1|^{(d-3)+\delta} \, dx_1,
\]
and
\[
\int_{\mathbb{R}^{2d}} \frac{|V|^2}{f^2} f^{1-\delta} e^{-1\|x\|} \, dx \\
\sim \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1}} e^{-\sqrt{|V|^2 + F_0}} \, |x_2||^{2 - ((d-3)d - 2)(1 - \delta)} \, dx_2 \ |x_1|^{2 - ((d-3)d - 2)(1 - \delta)} \, dx_1,
\]
with similar conclusions as in our earlier examples.

**Proposition 20.** The constant \((d - N)^2/4\) in (28) is sharp.

**Proof.** Here \( f(x) = |A|^{(d-N)} \), with \( A = \bigwedge_{j=1}^{N-1} (x_j - x_N) \), and \( \Omega' = \{ x \in \mathbb{R}^{dN} : A = 0 \} \) can be parameterized by \( x_N \in \mathbb{R}^d \) and, defining \( y_j := x_j - x_N \) for each fixed \( x_N \), by
\[
y_1 \in \mathbb{R}^d \setminus \{0\}, \quad y_2 \in \mathbb{R}^d \setminus y_1, \quad y_3 \in \mathbb{R}^d \setminus y_1 \wedge y_2, \quad \ldots, \quad y_{N-1} \in B, \quad (B.2)
\]
with \( B := y_1 \wedge \ldots \wedge y_{N-2} \), plus, \( y_1 = 0, y_{j=2, \ldots, N-1} \in \mathbb{R}^d \), and so forth.

Hence, for each fixed \( x_N \in \mathbb{R}^d \) and \( y_{j=1, \ldots, N-2} \) in general position as in (B.2) we split the last variable \( y_{N-1} \) into \( y_{N-1} \in B \) and \( y_{N-1} \perp B \) orthogonal to \( B \), write \( |A| = |B \wedge y_{N-1}| = |B||y_{N-1}| \), and deduce (with \( \delta' := (d - N)\delta \))
\[ \int_{\mathbb{R}^m} f^{1-\delta} e^{-|x|^2} dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} \cdots \int_{\mathbb{R}^d \setminus \{0\} \cdots \{0\} \cdots} \int_{\mathbb{R}^{d-n+2}} \int_{\mathbb{R}^{d-n+2}} e^{-|y|_2} dy = \cdot \left[y_1\right]^{-(d-N)+\delta'} dy_2 \left[B\right]^{-(d-N)+\delta'} dy_{n-2} \cdots dy_1 dy_N, \]

and

\[ \int_{\mathbb{R}^m} \frac{|V|^2}{2} f^{1-\delta} e^{-|x|^2} dx \propto \int_{\mathbb{R}^d} \sum(N) f^{1-\delta} e^{-|x|^2} dx = N \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} \cdots \int_{\mathbb{R}^d \setminus \{0\} \cdots \{0\} \cdots} \int_{\mathbb{R}^{d-n+2}} \int_{\mathbb{R}^{d-n+2}} e^{-|y|_2} dy = \cdot \left[y_1\right]^{-(d-N)+\delta'} dy_2 \left[B\right]^{-(d-N)+\delta'} dy_{n-2} \cdots dy_1 dy_N, \]

with similar conclusions as in our earlier examples.

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