LETTER TO THE EDITOR:
A SHORT COMPLEX-VARIABLE PROOF OF THE
TITCHMARSH CONVOLUTION THEOREM

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Abstract. The Titchmarsh convolution theorem is a celebrated result about the support of the convolution of two functions. We present a simple proof based on the canonical factorization theorem for bounded holomorphic functions on the unit disk.

1. Introduction

Let $f, g : \mathbb{R} \to \mathbb{C}$ be integrable functions, and let $f * g$ be their convolution product, namely

$$(f * g)(t) := \int_{-\infty}^{\infty} f(t - s)g(s) \, ds \quad (t \in \mathbb{R}).$$

It is clear that if $f = 0$ a.e. on $(-\infty, a)$ and if $g = 0$ a.e. on $(-\infty, b)$, then $f * g = 0$ a.e. on $(-\infty, a + b)$. Hence, defining

$$\alpha(f) := \sup\{a \in \mathbb{R} : f = 0 \text{ a.e. on } (-\infty, a)\},$$

and likewise for $g$ and $f * g$, we have $\alpha(f * g) \geq \alpha(f) + \alpha(g)$. Much less obvious is the following theorem.

**Theorem 1.1.** If $\alpha(f) > -\infty$ and $\alpha(g) > -\infty$, then

$$\alpha(f * g) = \alpha(f) + \alpha(g).$$

Theorem 1.1 is a version of the Titchmarsh convolution theorem [13]. This celebrated result has applications in a number of fields, including harmonic analysis, Banach algebras, operator theory and partial differential equations.

There are several proofs of the Titchmarsh convolution theorem, none of them easy. Broadly speaking, they divide into two groups: those using real-variable methods (see e.g. [4, 7, 11, 12]), and those based on complex-variable techniques (Titchmarsh’s original proof was of this kind, see also [11, 2, 5, 8, 9]). The complex-variable proofs are maybe more natural, but tend to require more advanced results from function theory. In this paper we present a simple proof that uses nothing beyond the well-known canonical factorization theorem for bounded holomorphic functions on the unit disk.

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2. LAPLACE TRANSFORMS

Complex-variable proofs of Titchmarsh’s theorem usually proceed via the Laplace transform, so we begin by briefly reviewing this notion. In what follows, we write \( \mathbb{H} \) for the right half-plane, namely \( \mathbb{H} := \{ z \in \mathbb{C} : \text{Re} z > 0 \} \).

Let \( f : \mathbb{R} \to \mathbb{C} \) be an integrable function with \( \alpha(f) > -\infty \). Its Laplace transform \( \mathcal{L}f : \mathbb{H} \to \mathbb{C} \) is defined by

\[
\mathcal{L}f(z) := \int_{-\infty}^{\infty} f(t)e^{-zt} \, dt \quad (z \in \mathbb{H}).
\]

It is easy to see that \( \mathcal{L}f \) is holomorphic on \( \mathbb{H} \). Furthermore, a straightforward estimate shows that

\[
|\mathcal{L}f(z)| \leq Ae^{-\alpha(f)\text{Re} z} \quad (z \in \mathbb{H}),
\]

where \( A := \int_{-\infty}^{\infty} |f(t)| \, dt \). In particular, if \( \alpha(f) \geq 0 \), then \( \mathcal{L}f \) is bounded on \( \mathbb{H} \). The converse is also true:

**Lemma 2.1.** If \( \mathcal{L}f \) is bounded on \( \mathbb{H} \), then \( \alpha(f) \geq 0 \).

**Proof.** Set \( f_1 := f1_{(-\infty,0]} \) and \( f_2 := f1_{(0,\infty)} \). As \( f_1 \) is supported on a compact subset of \( (-\infty,0] \), its Laplace transform \( \mathcal{L}f_1 \) is an entire function which is bounded on the left half-plane \( \text{Re} z \leq 0 \). On the right half-plane \( \text{Re} z > 0 \), we have \( \mathcal{L}f_1 = \mathcal{L}f - \mathcal{L}f_2 \), where \( \mathcal{L}f_2 \) is clearly bounded, and \( \mathcal{L}f \) is bounded by assumption. So in fact \( \mathcal{L}f_1 \) is bounded on the whole complex plane. By Liouville’s theorem \( \mathcal{L}f_1 \) is constant. It is easily seen that \( \lim_{x \to -\infty} \mathcal{L}f_1(x) = 0 \), so the constant must be zero, i.e. \( \mathcal{L}f_1 \equiv 0 \). By the uniqueness theorem for Laplace transforms, \( f_1 = 0 \) a.e. Hence \( f = f_2 \) a.e., and so, finally, \( \alpha(f) = \alpha(f_2) \geq 0 \). \( \square \)

Lastly in this section, we remark that, if \( f, g \) are integrable functions such that \( \alpha(f), \alpha(g) > -\infty \), then \( \mathcal{L}(f \ast g) = \mathcal{L}f \mathcal{L}g \), the pointwise product on \( \mathbb{H} \). The proof is a routine argument using Fubini’s theorem.

3. BEGINNING OF THE PROOF OF THEOREM 1.1

We argue by contradiction. Suppose, if possible, that there are integrable functions \( f, g \) with \( \alpha(f) > -\infty \) and \( \alpha(g) > -\infty \) such that \( \alpha(f \ast g) > \alpha(f) + \alpha(g) \). Replacing \( f \) and \( g \) by appropriate translates, we may further assume that \( \alpha(f) < 0 \) and \( \alpha(g) < 0 \) and \( \alpha(f \ast g) > 0 \). By Lemma 2.1, the Laplace transforms \( \mathcal{L}f, \mathcal{L}g \) are unbounded on \( \mathbb{H} \), whereas their product \( \mathcal{L}f \mathcal{L}g \) is bounded on \( \mathbb{H} \). So, to obtain a contradiction, and thereby complete the proof of Theorem 1.1, it suffices to establish the following general result about holomorphic functions.

**Proposition 3.1.** Let \( F \) and \( G \) be holomorphic functions on \( \mathbb{H} \) satisfying

\[
|F(z)| \leq Ae^{\alpha(z)\text{Re} z} \quad \text{and} \quad |G(z)| \leq Be^{\beta(z)\text{Re} z}
\]

for some positive constants \( A, a, B, b \). If both \( F \) and \( G \) are unbounded on \( \mathbb{H} \), then so is their product \( FG \).
Up till now, we have followed a fairly standard route. The novelty in our approach lies in our proof Proposition 3.1, for which we shall use the canonical factorization theorem for bounded holomorphic functions on the unit disk. We pause briefly to review this theorem.

4. Canonical factorization theorem

Let $\mathbb{D}$ denote the open unit disk and $\mathbb{T}$ denote the unit circle.

A Blaschke product on $\mathbb{D}$ is a function of the form

$$B(z) := cz^m \prod \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z} \quad (z \in \mathbb{D}),$$

where $c$ is a unimodular constant, $m$ is a non-negative integer, and $(a_n)$ is a (finite or infinite) sequence of points in $\mathbb{D}$ such that $\sum_n (1 - |a_n|) < \infty$.

A bounded outer function on $\mathbb{D}$ is a function of the form

$$O(z) := \exp \left( \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \omega(\theta) \frac{d\theta}{2\pi} \right) \quad (z \in \mathbb{D}),$$

where $\omega : \mathbb{T} \to (0, \infty)$ is a bounded positive function such that $\log \omega \in L^1(\mathbb{T})$.

A singular inner function on $\mathbb{D}$ is a function of the form

$$S(z) := \exp \left( - \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(e^{i\theta}) \right) \quad (z \in \mathbb{D}),$$

where $\sigma$ is finite positive Borel measure on $\mathbb{T}$ that is singular with respect to Lebesgue measure. The measure $\sigma$ is uniquely determined by $S$. We shall often write $S = S_\sigma$ to indicate the dependence of $S$ upon $\sigma$.

The following result is known as the canonical factorization theorem. A proof can be found for example in [10, Theorem 2.8].

**Theorem 4.1.** Every bounded holomorphic function $H$ on $\mathbb{D}$ with $H \not\equiv 0$ has a unique factorization of the form $H = BOS$, where $B$ is a Blaschke product, $O$ is a bounded outer function, and $S$ is a singular inner function.

The following corollary will prove crucial in what follows.

**Corollary 4.2.** Let $B$ be a Blaschke product, let $O$ be a bounded outer function, and let $S_\sigma_1$ and $S_\sigma_2$ be singular inner functions. Then $BOS_{\sigma_1}/S_{\sigma_2}$ is bounded on $\mathbb{D}$ if and only if $\sigma_1 - \sigma_2$ is a positive measure.

**Proof.** The ‘if’ is obvious, since $BOS_{\sigma_1}/S_{\sigma_2} = BOS_{\sigma_1 - \sigma_2}$. For the ‘only if’, suppose that $BOS_{\sigma_1}/S_{\sigma_2}$ is bounded. Then, by Theorem 4.1, we can write it as $\tilde{B}OS$, where $\tilde{B}$ is a Blaschke product, $\tilde{O}$ is a bounded outer function, and $\tilde{S}$ is a singular inner function, say $\tilde{S} = S_\sigma$. Multiplying up by $S_{\sigma_2}$, we then have $BOS_{\sigma_1} = \tilde{B}OS_{\sigma_1 + \sigma}$. By the uniqueness part of Theorem 4.1 it follows that $\sigma_1 = \sigma_2 + \sigma$. Thus $\sigma_1 - \sigma_2$ is a positive measure. □
5. Completion of the proof of Theorem 1.1

Proof of Proposition 3.1. By assumption, the function \( F(z)e^{-az} \) is bounded on \( \mathbb{H} \). Composing with \( \phi(z) := (1 + z)/(1 - z) \), which maps \( \mathbb{D} \) conformally onto \( \mathbb{H} \), we obtain a bounded holomorphic function on \( \mathbb{D} \). Hence, by Theorem 4.1,

\[
F(\phi(z))e^{-a\phi(z)} = B(z)O(z)S_\sigma(z) \quad (z \in \mathbb{D}),
\]

where \( B \) is a Blaschke product, \( O \) is a bounded outer function, and \( S_\sigma \) is a singular inner function. Notice also that \( e^{-a\phi(z)} = S_{a\delta_1}(z) \), where \( \delta_1 \) denotes the Dirac mass at the point 1. Thus we have

\[
F \circ \phi = BOS_\sigma/S_{a\delta_1}.
\]

If \( F \) is unbounded on \( \mathbb{H} \), then \( F \circ \phi \) is unbounded on \( \mathbb{D} \), so by Corollary 4.2 \( \sigma - a\delta_1 \) is not a positive measure, in other words \( \sigma(\{1\}) < a \).

Likewise, we can write

\[
G \circ \phi = \tilde{B}\tilde{O}S_\tau/S_{b\delta_1},
\]

where \( \tilde{B} \) is a Blaschke product, \( \tilde{O} \) is a bounded outer function, \( S_\tau \) is a singular inner function, and, if \( G \) is unbounded, then \( \tau(\{1\}) < b \).

Multiplying these equations together, we obtain

\[
(FG) \circ \phi = (BB)(O\tilde{O})S_{\sigma+\tau}/S_{(a+b)\delta_1}.
\]

If \( F \) and \( G \) are both unbounded, then \( (\sigma + \tau)(\{1\}) < a + b \), so the measure \( (\sigma + \tau) - (a+b)\delta_1 \) is not positive. Corollary 4.2 then shows that \( (FG) \circ \phi \) is unbounded on \( \mathbb{D} \), whence \( FG \) is unbounded on \( \mathbb{H} \). This completes the proof of Proposition 3.1 and therefore also that of Theorem 1.1. \( \square \)

6. Concluding remarks

The Titchmarsh convolution theorem can be expressed in terms of supports. The support an integrable function \( f : \mathbb{R} \rightarrow \mathbb{C} \) is the smallest closed subset of \( \mathbb{R} \) such that \( f = 0 \) a.e. on the complement. Writing \( \text{supp}(f) \) for this support, it is easy to see that, for all integrable functions \( f, g \), we have

\[
\text{supp}(f * g) \subset \text{supp}(f) + \text{supp}(g).
\]

If further \( \text{supp}(f) \) and \( \text{supp}(g) \) are both compact, then Theorem 1.1 tells us that the two sides of (6.1) have the same minimum and (after reflection) the same maximum. Thus, writing \( \text{conv}(\cdot) \) for the convex hull, we have

\[
\text{conv}(\text{supp}(f * g)) = \text{conv}(\text{supp}(f)) + \text{conv}(\text{supp}(g)).
\]

In this form, the result generalizes to higher dimensions. If \( f, g : \mathbb{R}^n \rightarrow \mathbb{C} \) are integrable functions with compact supports, then (6.2) holds. More generally still, (6.2) holds whenever \( f \) and \( g \) are distributions on \( \mathbb{R}^n \) with compact supports. This generalization is due to Lions [10]. It can be deduced quite easily from the basic version, Theorem 1.1 (see e.g. [3, §45]).
TITCHMARSH CONVOLUTION THEOREM

REFERENCES

1. R. P. Boas, Jr., *Entire functions*, Academic Press Inc., New York, 1954. MR 0068627
2. M. M. Crum, *On the resultant of two functions*, Quart. J. Math., Oxford Ser. 12 (1941), 108–111. MR 0004650
3. W. F. Donoghue, Jr., *Distributions and Fourier transforms*, Pure and Applied Mathematics, vol. 32, Academic Press, New York, 1969. MR 3363413
4. R. Doss, *An elementary proof of Titchmarsh’s convolution theorem*, Proc. Amer. Math. Soc. 104 (1988), no. 1, 181–184. MR 958063
5. J. Dufresnoy, *Sur le produit de composition de deux fonctions*, C. R. Acad. Sci. Paris 225 (1947), 857–859. MR 0022631
6. P. L. Duren, *Theory of $H^p$ spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London, 1970. MR 0268655
7. G. K. Kalisch, *A functional analysis proof of Titchmarsh’s theorem on convolution*, J. Math. Anal. Appl. 5 (1962), 176–183. MR 0140893
8. P. Koosis, *On functions which are mean periodic on a half-line*, Comm. Pure Appl. Math. 10 (1957), 133–149. MR 0089297
9. P. D. Lax, *Translation invariant spaces*, Acta Math. 101 (1959), 163–178. MR 0105620
10. J. L. Lions, *Supports dans la transformation de Laplace*, J. Analyse Math. 2 (1953), 369–380. MR 0058013
11. J. G. Mikusiński, *A new proof of Titchmarsh’s theorem on convolution*, Studia Math. 13 (1953), 56–58. MR 0058668
12. ———, *Operational calculus*, International Series of Monographs on Pure and Applied Mathematics, Vol. 8, Pergamon Press, New York-London-Paris-Los Angeles; Państwowe Wydawnictwo Naukowe, Warsaw, 1959. MR 0105594
13. E. C. Titchmarsh, *The Zeros of Certain Integral Functions*, Proc. London Math. Soc. (2) 25 (1926), 283–302. MR 1575285

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