From AdS to Schrödinger/Lifshitz dual space-times without or with hyperscaling violation

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Abstract

It is observed that the (intersecting) branes of M/string theory, which are known to give AdS geometry (directly or upto a conformal transformation) in the near horizon limit, do also lead to Schrödinger/Lifshitz dual space-times (without or with hyperscaling violation) upon using appropriate solution generating transformation and dimensional reduction. We show that the dynamical exponents of the Schrödinger and the Lifshitz space-times obtained in this way always add upto 2. We illustrate this by several examples, including M2-, M5-branes of M-theory and D(p + 1)-branes (p \neq 4, since in this case the near horizon limit does not give AdS geometry) of string theory as well as many of their intersecting solutions. The Schrödinger space-time can be obtained by the standard wave generating technique along one of the brane directions (for single brane) or one of the common brane directions (for intersecting branes) and then interchanging the light-cone coordinates by double Wick rotations, whereas, the Lifshitz space-time can be obtained by dimensionally reducing (for M-theory) along the wave direction or taking T-duality (for string theory) along the same direction. We thus obtain Schrödinger/Lifshitz dual space-times without or with hyperscaling violation from the same M/string theory solutions and they preserve some fraction of the supersymmetry.

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1 Introduction

Both Schrödinger and Lifshitz symmetries are non-relativistic symmetries as the space and
the time scale differently: \( t \rightarrow \lambda t, \ x^i \rightarrow \lambda x^i \), \( (z \neq 1) \), where \( i = 1, 2, \ldots, d \), \( d \) being the
number of spatial dimensions and \( z \), the dynamical critical exponent, unlike the relativistic
symmetry where the space-time scale as \( t \rightarrow \lambda t, \ x^i \rightarrow \lambda x^i \). However, the Schrödinger
symmetry is much larger than the Lifshitz symmetry. While the Schrödinger symmetry \([1, 2, 3, 4]\) consists of time and space translations, spatial rotations, Galilean boosts, dilatation
or a scaling symmetry (mentioned above), a particle number symmet ry and in addition a
special conformal transformation which appears only for \( z = 2 \), the Lifshitz symmetry \([5, 6]\)
consists of only space-time translations, spatial rotations and a scaling symmetry. Such non-
relativistic symmetries arise in some strongly coupled condensed matter systems near their
quantum critical point \([7, 8]\). In particular, the Schrödinger symmetry arises in fermionic
cold atom system at unitarity \([9, 10, 11]\) and the Lifshitz symmetry arises in some strongly
correlated electron system such as certain dimer model \([12]\) and also in some lattice models
\([13, 14, 15]\). The gravity models which realize these symmetries as isometries \([16, 17, 18]\) are
of interest as they provide the calculational tool to study these strongly coupled condensed
matter systems \([19, 20, 21, 22]\) in the spirit of AdS/CFT correspondence \([23, 24, 25]\).

The metric having Schrödinger symmetry as an isometry has the form,

\[
ds^2 = -\frac{2dt^2}{u^{2z}} + \frac{-2d\xi dt + \sum_{i=1}^{d}(dx^i)^2 + du^2}{u^2},
\]

(1)

whereas, the metric having Lifshitz symmetry as an isometry has the form,

\[
ds^2 = -\frac{dt^2}{u^{2z}} + \frac{\sum_{i=1}^{d}(dx^i)^2 + du^2}{u^2}
\]

(2)

In (1) \( \xi \) is a space-like coordinate whose conjugate \( i\partial/\partial \xi \) is an operator associated w ith
the conserved particle number and so, \( \xi \) is compact. \( z \) is the dynamical critical exponent as
mentioned above. \( u \) in (1) and (2) is the radial coordinate, which in the boundary theory
is related to the energy parameter giving rise to the RG flow. It is clear from (1) that the
metric has a scaling symmetry \( t \rightarrow \lambda^z t, \ \xi \rightarrow \lambda^{2-z} \xi, \ x^i \rightarrow \lambda x^i \) and \( u \rightarrow \lambda u \). On the other
hand the metric in (2) has a scaling symmetry \( t \rightarrow \lambda^z t, \ x^i \rightarrow \lambda x^i \) and \( u \rightarrow \lambda u \). Apart
from the above scaling symmetry the metric in (1) also has space-time translation, spatial
rotation, particle number (\( \xi \) translation) and boost (\( x^i \rightarrow x^i - v^i t, \ \xi \rightarrow \xi + v^i x^i - (1/2)v^2 t \))
symmetry. Furthermore, for \( z = 2 \), the metric has in addition a special conformal symmetry.
Thus the metric (1) has Schrödinger symmetry as an isometry. Whereas the Lifshitz metric
in (2) has apart from the above mentioned scaling symmetry, space-time translation and
spatial rotation symmetry and thus has Lifshitz symmetry as an isometry.

It is well-known in a ‘top-down’ approach that Schrödinger metric (1) can be obtained
either by the so-called ‘Null Melvin Twist’ \([26, 27, 28, 29, 16]\) or by the TsT transformation
\([30]\) on the standard \( p \)-brane/M-brane solutions of string/M theory and then taking
dimensional reduction. However, the non-relativistic branes or the Schrödinger metric obtained this way are not supersymmetric. Supersymmetric non-relativistic branes have been obtained in \[31\, 32\]. Lifshitz metric \[2\], on the other hand, is not easy to obtain directly from string/M theory. The gravity dual of Lifshitz metric was first obtained in \[18\]. Lifshitz solution with a specific dynamical exponent, namely, \(z = 2\), has been obtained by AdS null deformations of known string theory solution in \[33\]. It has been either generalized \[34\] or embedded in a ‘bottom-up’ approach in various gauged supergravities as well as in string/M theory in \[35\]. However, a direct ‘top-down’ approach of obtaining Lifshitz metric was still missing. In \[36\, 37\, 38\], it was shown how to obtain Lifshitz metric directly from certain intersecting brane solutions of string/M theory by taking the near horizon limit and then reducing them to lower dimensions. It should be remarked here that the metrics obtained by these methods are not invariant under the Schrödinger or Lifshitz symmetries, rather, are conformal to the Schrödinger or Lifshitz metrics. The conformal factor is related to the so-called hyperscaling violation exponent of the boundary theory \[39\] and so the above mentioned metrics are called the Schrödinger or Lifshitz metrics with hyperscaling violation. Such space-times and their properties have been studied in \[40\, 41\, 42\, 43\, 44\, 45\, 46\].

In this paper we argue that the single brane or the intersecting brane solutions of string/M theory which are known to lead to AdS geometry (directly or upto a conformal transformation) in the near horizon limit also give Schrödinger/Lifshitz dual space-times (without or with hyperscaling violation) upon using appropriate solution generating technique and dimensional reduction. In order to have this, the dimensions of the AdS space must be greater than 2. Also, both the Schrödinger and Lifshitz space-times obtained this way are supersymmetric as they are obtained from BPS solutions of string/M theory using solution generating transformation (without breaking supersymmetry). We illustrate this with several examples. Among the single branes M2- and M5-branes of M-theory lead directly to AdS4 and AdS7 geometry respectively in the near horizon limit. Similarly, D3-brane of string theory leads to AdS5 geometry directly. Other D(\(p + 1\))-branes of string theory lead to AdS\(_{p+3}\) geometry upto a conformal transformation except for \(p = 4\). We will see that in all these cases the solutions lead to Schrödinger/Lifshitz dual space-times. The intersecting branes of string/M theory which give AdS geometry have been given in \[47\, 48\]. Among them the ones which yield AdS3 (AdS2 is excluded as we mentioned) geometry are M2 \(\perp\) M5 and M5 \(\perp\) M5 \(\perp\) M5 \[49\] of M-theory and D1 \(\perp\) D5, D2 \(\perp\) D4, D3 \(\perp\) D3 and F \(\perp\) NS5 \[36\, 37\] of type II string theories (the solutions involving triple intersections of string theory branes are not known explicitly and even if they give AdS geometry, they will either give AdS3 or AdS2 similar to the solutions involving double intersections and therefore we will not consider them here to avoid repetitions). We will consider some of these cases to show how they lead to Schrödinger/Lifshitz dual space-times (except F \(\perp\) NS5). Now starting from the AdS geometry we first generate a pp-wave along one of the original brane directions for single
brane or one of the original common brane directions for the intersecting branes by standard method [30]. We then express the resulting solution in the light-cone coordinates and by further taking double Wick rotations involving the light cone coordinates we generate the Schrödinger space-time (upon dimensional reductions). On the other hand to obtain Lifshitz space-time, we either dimensionally reduce the solution (for M-theory branes) in Poincare coordinates along the wave direction or take T-duality (for string theory branes) along the same direction (and making further dimensional reductions). This way we generate Schrödinger/Lifshitz dual space-times starting from the same solution of string/M theory. The Schrödinger or Lifshitz space-times obtained this way can have hyperscaling violations in some cases. We will also show that the dynamical critical exponents of these two space-times always add up to 2. As the Schrödinger/Lifshitz space-times are obtained from BPS solutions of string/M theory, they preserve some fraction of space-time supersymmetry. Hyperscaling violating Lifshitz solutions have also been obtained from Schrödinger space-time and branes with waves in [46].

This paper is organized as follows. In section 2, we give the general argument of obtaining Schrödinger/Lifshitz dual space-times from the AdS geometry. The examples illustrating this are given in the next two sections. In section 3, we discuss the cases of single branes i.e., M2-, M5- and D(p + 1)-branes. In section 4, we discuss the cases of intersecting branes including M2 ⊥ M5, M5 ⊥ M5 ⊥ M5 of M theory and D1 ⊥ D5, D2 ⊥ D4 and F ⊥ NS5 of string theory. Finally, we conclude in section 5.

2 From AdS to Schrödinger/Lifshitz dual space-times

It is well-known that the non-dilatonic branes of string/M theory lead to AdS geometry in the near horizon limit. So, for example, M2- and M5-branes of M theory give AdS\(_4\) and AdS\(_7\) respectively, whereas D3-brane of string theory gives AdS\(_5\) space-times in the near horizon limit apart from some spherical part which we will not need in our discussion here. However, the other D(p + 1) branes (p ≠ 2) of string theory do not give AdS geometry in the near horizon limit directly, but they give geometries which are conformal to AdS\(_{p+3}\) except for p = 4. Other BPS branes like F-string or NS5-branes do not give AdS geometry and so, the only single branes that are relevant for our purpose here are the M2-, M5- and D(p + 1)-branes (for p ≠ 4). Intersecting branes can also lead to AdS geometries, the first example being M2-brane intersecting with M5-brane on a common string, M2 ⊥ M5, which gives AdS\(_3\) geometry in the near horizon limit. This solution preserves 1/4 supersymmetry. Another example of intersecting M-branes which gives AdS\(_3\) geometry is triple intersections, namely, three M5-branes intersecting on a string and pairwise intersecting on 3-branes, M5 ⊥ M5 ⊥ M5 [49]. This solution preserves 1/8 supersymmetry. There are other intersecting solutions of M-branes, for example, M2 ⊥ M2 ⊥ M2, M2 ⊥ M2 ⊥ M5 ⊥ M5, which preserve
some supersymmetries and give AdS geometries \[48\], but these latter solutions give AdS which does not contain any brane direction and therefore are not suitable for generating Schrödinger/Lifshitz dual space-times. Intersecting solutions of string theory can also lead to AdS geometry in the near horizon limit. Some of the double intersecting solutions are discussed in \[36, 37\]. Among them those which give AdS$_3$ are D$_1 \perp$ D$_5$, D$_2 \perp$ D$_4$, D$_3 \perp$ D$_3$ and F $\perp$ NS5. Note that although F-string and NS5-brane individually does not give AdS geometry their intersection gives AdS$_3$. We will discuss D$_1 \perp$ D$_5$, D$_2 \perp$ D$_4$ to show how they give rise to Schrödinger/Lifshitz dual space-times. We also discuss the case F $\perp$ NS5 as an exception. We point out that even if it gives AdS geometry, it does not yield Schrödinger/Lifshitz dual space-times. We do not consider triple intersections in string theory for the reasons mentioned earlier.

In this section we will give the general arguments to show how starting from AdS solutions (obtained from various single brane or double/triple intersecting solutions of string/M theory) we can generate Schrödinger/Lifshitz dual space-times by using some solution generating technique. Suppose we start from any such solution of string/M theory described in the first paragraph whose near horizon limit gives AdS geometry either directly or upto a conformal transformation. Then the near horizon metric can be written as,

$$ds^2 = u^\beta \left[ -dt^2 + (dx^1)^2 + \frac{\sum_{i=2}^{d}(dx^i)^2 + du^2}{u^2} \right]$$  \hspace{1cm} (3)

where we have put all the charges associated with the branes to unity for convenience. In the above $u$ is a radial coordinate which is related to the original radial coordinate $r$, transverse to the branes, by some coordinate transformation. The overall $u^\beta$, where $\beta$ is a constant, indicates that the metric is conformal to AdS$_{d+2}$ space. Note that we have isolated one of the brane directions $x^1$ along which waves will be generated. Also, we have ignored the spherical part and some Euclidean part as they will not play any role in our discussion here.

In order to obtain Schrödinger space-time from here we will generate waves along one of the brane directions $x^1$ (say). The standard way to generate waves along a brane direction is to first write the black solution and then boost the solution along that direction and then take double scaling limit \[50\]. The resultant solution is an extremal brane solution with waves along $x^1$. Applying this method, the above metric (3) reduces to,

$$ds^2 = u^\beta \left[ -dt^2 + (dx^1)^2 + (H - 1)(dt - dx^1)^2 + \frac{\sum_{i=2}^{d}(dx^i)^2 + du^2}{u^2} \right]$$ \hspace{1cm} (4)

where $H = (1 + u^\alpha)$ is a harmonic function and $\alpha$ is another constant. Now defining the light-cone coordinates by

$$t_{\text{new}} = \frac{1}{\sqrt{2}}(t + x^1), \quad \xi = \frac{1}{\sqrt{2}}(t - x^1)$$ \hspace{1cm} (5)
we rewrite (4) as,

$$ds^2 = u^\beta \left[ -2 \frac{dt d\xi}{u^2} + 2u^\alpha d\xi^2 + \sum_{i=2}^d (dx^i)^2 + du^2 \right]$$  \hspace{1cm} (6)

Note that in writing (6) we have replaced $t_{\text{new}} \rightarrow t$. Now if we make a double Wick rotation $t \rightarrow i\xi$ and $\xi \rightarrow -it$, the metric (6) takes the form

$$ds^2 = u^\beta \left[ -\frac{2dt^2}{u^{2-\alpha}} + \frac{-2d\xi dt + \sum_{i=2}^d (dx^i)^2 + du^2}{u^2} \right]$$  \hspace{1cm} (7)

Now comparing (7) with the Schrödinger metric given in (1) we find that apart from overall $u^\beta$ factor the metric has exactly the same form with the dynamical critical exponent $z = 1 - \alpha/2$. The overall factor represents that the whole metric actually transforms under the scaling and the parameter $\beta$ is related to the so-called hyperscaling violation exponent. This therefore shows that how starting from AdS geometry (obtained from the near horizon limit of brane solutions of string/M theory) we can get Schrödinger space-times with (for $\beta \neq 0$) or without (for $\beta = 0$) hyperscaling violation.

Now in order to get Lifshitz space-times we note that we can rewrite the metric in (4) as,

$$ds^2 = u^\beta \left[ -H^{-1}dt^2 + H \left( (1 - H^{-1})dt - dx^1 \right)^2 + \sum_{i=2}^d (dx^i)^2 + du^2 \right]$$  \hspace{1cm} (8)

Now if the above metric is obtained from M-theory solution we can dimensionally reduce it along $x^1$ to go to a string theory solution which can be written as,

$$ds^2 = u^{\beta+\gamma} \left[ -H^{-1}dt^2 + \sum_{i=2}^d (dx^i)^2 + du^2 \right]$$  \hspace{1cm} (9)

where $\gamma$ is another constant. Note that the second term in (8) is absent in the reduced solution and a gauge field $A_0$ will be generated in string theory solution. Thus here the dimensionality of the solution or the boundary theory is reduced by one. The solution (9) in the near horizon limit takes the form,

$$ds^2 = u^{\beta+\gamma} \left[ -\frac{dt^2}{u^{\alpha+2}} + \frac{\sum_{i=2}^d (dx^i)^2 + du^2}{u^2} \right]$$  \hspace{1cm} (10)

In the above we have used the fact that in the near horizon limit $H = 1 + u^\alpha \approx u^\alpha$. Actually, as will see in the specific examples in the next sections that the near horizon limit implies $u \rightarrow \infty$ ($u \rightarrow 0$) and in that case $\alpha$ is positive (negative) and so, we always have $u^\alpha \gg 1$. On the other hand if the metric (8) is obtained from a string theory solution, we can take T-duality along $x^1$ and the resulting solution in that case can be written as,

$$ds^2 = u^\beta \left[ -\frac{dt^2}{u^{\alpha+2}} + \frac{\sum_{i=1}^d (dx^i)^2 + du^2}{u^2} \right]$$  \hspace{1cm} (11)
In this case if we start from a type IIA solution we will end up with a type IIB solution and vice-versa. However, the dimensionality of the solution or the boundary theory does not change. Thus comparing (10) and (11) with (2), we find that in either case we get Lifshitz solution without or with hyperscaling violation with the dynamical critical exponent \( z = 1 + \alpha/2 \). Now since for the Schrödinger solution we get \( z = 1 - \alpha/2 \) we thus find that the sum of the dynamical critical exponents of the Schrödinger/Lifshitz dual space-times obtained this way add up to 2.

Before we close this section, we would like to emphasize that the reason we were able to obtain Schrödinger/Lifshitz dual space-times from a given AdS solution of string/M theory lies in our ability to write the AdS + pp-wave solution in two different ways as given in eqs. (4) and (8) above. In the first case we obtain Schrödinger metric (upto a conformal factor) by going to the light-cone coordinates and then making a double Wick rotation, whereas, in the second case we obtain Lifshitz metric (upto a conformal factor) by either dimensionally reducing the solution (for M-theory) or taking T-duality transformation (for string theory). It is well known that AdS solution of string/M theory can be deformed more generally by some non-normalizable terms which include the matter sector of the theory and in particular, certain AdS null deformations can lead to Lifshitz solutions [33]. Although these deformations are more general than the pp-wave deformations we consider in this paper, they may not lead to Schrödinger/Lifshitz dual space-times in general. In other words, for the occurrence of the dual space-times, we need to consider a special deformation of the AdS solution, namely, the deformation by pp-wave. Also note that since \( \alpha \) as given after eq. (4) is a positive number, the dynamical critical exponent \( z = 1 - \alpha/2 \) for the Schrödinger case can become negative. In fact \( z \) can become negative even for some Lifshitz (with hyperscaling violation) solutions as given in section 4.2 below. This may seem surprising but in the boundary theory this simply means that the dynamical system near the critical point has smaller relaxation time (instead of longer relaxation time for \( z > 0 \)) or in this case there will be a critical speeding up (instead of slowing down) of the system. This is known to occur for some random-cluster model [51] and also in 3-dimensional single cluster O(4) sigma model [52]. Note that negative \( z \) implies that the time and the space coordinates scale in the opposite way, however, this does not pose any problem to consider the theory at finite temperature for Lifshitz case. In order to add a temperature we need to put a suitable Schwarzschild factor, as usual, in \( dt^2 \) and \( dr^2 \) terms since the Lifshitz metric is diagonal, but we do not consider this case here. The finite temperature solution in this case can be shown to be stable even for \( z < 0 \) as long as the hyperscaling parameter satisfies \( \theta > d \). Indeed this happens for some cases considered later in section 4. However, how to add temperature for the Schrödinger case is not obvious at all because of the presence of a cross-term \( dt d\xi \) in the metric, but this problem is in no way related to the negative value of the dynamical exponent. Since the particular Schrödinger space-time considered here is obtained by generating a wave
along one of the brane directions and that involves taking a zero temperature and infinite boost limit (such that their product remains constant) of the boosted black brane solutions, it is not clear how to further add temperature to these solutions.

Thus we have shown how starting from AdS geometry obtained from some string/M theory solutions we can generate both the Schrödinger and the Lifshitz space-times without and with hyperscaling violations. Here we have argued in generality in a schematic fashion and the details will be given as we discuss various specific cases in the next two sections.

3 Schrödinger/Lifshitz from single brane solutions

In this section we will consider single brane solutions, namely, M2-, M5-branes of M-theory and D\((p + 1)\)-branes of string theory which are known to lead to AdS geometry in the near horizon limit either directly or upto a conformal transformation. We will show how they lead to Schrödinger/Lifshitz dual space-times upon using some solution generating techniques.

3.1 M2-brane

M2-brane solution has the form,

\[
\begin{align*}
 ds^2 & = H_2^{-\frac{5}{2}} (-dt^2 + (dx^1)^2 + (dx^2)^2) + H_2^{\frac{1}{2}} (dr^2 + r^2 d\Omega_7^2) \\
 A_{[3]} & = H_2^{-1} dt \wedge dx^1 \wedge dx^2
\end{align*}
\]

(12)

Here \(H_2\) is a harmonic function given as \(H_2(r) = 1 + Q_2/r^6\), where \(Q_2\) is the charge associated with the M2-brane and \(A_{[3]}\) is the 3-form gauge field which couples to M2-brane. In the near horizon limit, \(r \rightarrow 0\), the above metric reduces to

\[
ds^2 = \frac{r^4}{Q_2^3} (-dt^2 + (dx^1)^2 + (dx^2)^2) + \frac{Q_2^\frac{5}{2}}{r^2} \left( dr^2 + r^2 d\Omega_7^2 \right)
\]

(13)

Now taking \(r \rightarrow 1/r\) and defining a new variable by the relation \(u^2 = r^4\), we can rewrite (13) and the gauge field in (12) as,

\[
\begin{align*}
 ds^2 & = Q_2^{\frac{5}{2}} \left[ \frac{-dt^2 + (dx^1)^2 + (dx^2)^2 + \frac{1}{4} du^2}{Q_2 u^2} + d\Omega_7^2 \right] \\
 A_{[3]} & = \frac{1}{Q_2 u^3} dt \wedge dx^1 \wedge dx^2
\end{align*}
\]

(14)

This is the standard AdS\(_4 \times S^7\) metric which comes from M2-brane in the near horizon limit. Now as mentioned in the previous section, to obtain the Schrödinger metric, we generate a pp-wave along \(x^1\) direction by the standard technique. The resultant metric then takes the form,

\[
 ds^2 = Q_2^{\frac{7}{2}} \left[ \frac{-dt^2 + (dx^1)^2 + (H_1 - 1)(dt - dx^1)^2 + (dx^2)^2 + \frac{1}{4} du^2}{Q_2 u^2} + d\Omega_7^2 \right] \]

(15)
where $H_1 = 1 + Q_1/r^6$ is another harmonic function and $Q_1$ is the asymptotic momentum carried by the wave. As before taking the coordinate transformation $r \to 1/r$ and using the new variable $u$, the harmonic function takes the form $H_1 = 1 + Q_1 u^3$. Note here that going to the near horizon means $u \to \infty$. Writing the metric in (15) and the form-field in (14) in the light-cone coordinates as defined in the previous section we have,

$$d s^2 = Q_2^{\frac{1}{2}} \left[ -2 d t d \xi + 2 Q_1 u^3 d \xi^2 + (d x^2)^2 + \frac{1}{4} Q_2 d u^2 \right] + d \Omega_7^2$$

$$A_{[3]} = - \frac{1}{Q_2 u^3} d t \wedge d \xi \wedge d x^2$$

Now dimensionally reducing the solution on $S^7$ and taking the double Wick rotation $t \to i \xi$, $\xi \to -i t$, the above solution takes the form,

$$d s^2 = Q_2^{\frac{3}{2}} \left[ -2 Q_1 u d t^2 + \frac{-2 d \xi d t + (d x^2)^2 + \frac{1}{4} Q_2 d u^2}{Q_2 u^2} \right]$$

$$A_{[3]} = \frac{1}{Q_2 u^3} d t \wedge d \xi \wedge d x^2$$

Comparing the metric in (17) with (1), we find that the metric has a Schrödinger symmetry with dynamical critical exponent $z = -1/2$, spatial dimension of the boundary theory $d = 1$ and no hyperscaling violation.

Now to obtain Lifshitz metric, we rewrite the metric in (15), as mentioned in section 2, as

$$d s^2 = Q_2^{\frac{1}{2}} \left[ -H_1^{-1} d t^2 + H_1 \left( (1 - H_1^{-1}) d t - d x^1 \right)^2 + (d x^2)^2 + \frac{1}{4} Q_2 d u^2 \right] + d \Omega_7^2$$

Reducing the above solution along the wave direction, i.e. along $x^1$, we obtain a string theory solution which has the form,

$$d s^2 = Q_1^{\frac{1}{2}} u^{\frac{3}{2}} \left[ -\frac{d t^2}{Q_1 Q_2 u^5} + \frac{(d x^2)^2 + \frac{1}{4} Q_2 d u^2}{Q_2 u^2} + d \Omega_7^2 \right]$$

$$e^{2\phi} = \frac{Q_1^2}{Q_2} u^{\frac{3}{2}}$$

$$B_{[2]} = \frac{1}{Q_2 u^3} d t \wedge d x^2, \quad A_{[1]} = \frac{1}{Q_1 u^3} d t$$

where we have written the metric in the string frame and used $H_1 \approx Q_1 u^3$. It can be easily seen that indeed the above metric has Lifshitz symmetry with hyperscaling violation. Actually we recognize the solution (19) to be the near horizon limit of F-D0 solution discussed in [36]. This solution has, as discussed in [36], dynamical critical exponent $z = 5/2$, the spatial dimension of boundary theory $d = 1$ and the hyperscaling violation exponent $\theta = 1/2$. 
We thus obtain both Schrödinger and Lifshitz space-times starting from the AdS\(_4\) solution which is the near horizon limit of M2-brane solution. The dynamical critical exponents of these two space-times are \(z = -1/2\) and \(z = 5/2\), which add up to 2. Note that since M2-brane is a BPS solution both the Schrödinger and the Lifshitz solution obtained this way are supersymmetric.

### 3.2 M5-brane

Here we proceed exactly as in M2-brane case discussed in the previous subsection. The M5-brane solution has the form,

\[
ds^2 = H_2^{-3/2} \left( -dt^2 + (dx^1)^2 + \sum_{i=2}^{5} (dx^i)^2 \right) + H_2^2 \left( dr^2 + r^2 d\Omega_4^2 \right) \]

\[
A_6 = H_2^{-1} dt \wedge dx^1 \ldots \wedge dx^5 \tag{20}
\]

In this case the harmonic function \(H_2\) is given as \(H_2(r) = 1 + Q_2/r^3\), where \(Q_2\) is the charge associated with M5-brane. Also M5-brane couples to a 6-form gauge field given in \(20\). In the near horizon limit \(r \to 0\), M5-brane solution takes the form,

\[
ds^2 = \frac{r}{Q_2^3} \left( -dt^2 + (dx^1)^2 + \sum_{i=2}^{5} (dx^i)^2 \right) + \frac{Q_2^3}{r^2} \left( dr^2 + r^2 d\Omega_4^2 \right) \]

\[
A_6 = \frac{r^3}{Q_2} dt \wedge dx^1 \ldots \wedge dx^5 \tag{21}
\]

Now taking the coordinate transformation \(r \to 1/r\) and defining a new coordinate by \(u^2 = r\), we can rewrite \(21\) as,

\[
ds^2 = Q_2^3 \left[ \frac{-dt^2 + (dx^1)^2 + \sum_{i=2}^{5} (dx^i)^2 + 4Q_2 du^2}{Q_2 u^2} + d\Omega_4^2 \right] \]

\[
A_6 = \frac{1}{Q_2 u^6} dt \wedge dx^1 \ldots \wedge dx^5 \tag{22}
\]

The metric in \(22\) has the standard AdS\(_7\) \(\times \) S\(_4\) structure obtained from the near horizon limit of M5-brane. Now in order to obtain Schrödinger solution we generate pp-waves along \(x^1\) direction by the standard procedure and then the metric will be given by,

\[
ds^2 = Q_2^3 \left[ \frac{-dt^2 + (dx^1)^2 + (H_1 - 1)(dt - dx^1)^2 + \sum_{i=2}^{5} (dx^i)^2 + 4Q_2 du^2}{Q_2 u^2} + d\Omega_4^2 \right] \tag{23}
\]

where \(H_1 = 1 + Q_1/r^3\) is another harmonic function, with \(Q_1\), the asymptotic momentum carried by the wave. Writing in terms of variable \(u\), the harmonic function has the form...
$H_1 = 1 + Q_1 u^6$. As before here also going to the near horizon means $u \to \infty$. In light cone coordinates the metric (23) and the form field in (22) take the forms,

$$
\begin{align*}
ds^2 &= Q_2^2 \left[ -2dt \, d\xi + 2Q_1 u^6 dt \, d\xi^2 + \frac{\sum_{i=2}^5 (dx^i)^2 + 4Q_2 du^2}{Q_2 u^2} \right] + d\Omega_4^2 \\
A_{[6]} &= -\frac{1}{Q_2 u^6} dt \wedge d\xi \wedge dx^2 \ldots \wedge dx^5
\end{align*}
$$

(24)

Dimensionally reducing the solution on $S^4$ and taking the double Wick rotation as before $t \to i\xi$ and $\xi \to -it$, we get,

$$
\begin{align*}
ns^2 &= Q_2^2 \left[ -2Q_1 u^4 dt^2 + \frac{-2d\xi dt + \sum_{i=2}^5 (dx^i)^2 + 4Q_2 du^2}{Q_2 u^2} \right] \\
A_{[6]} &= \frac{1}{Q_2 u^6} dt \wedge d\xi \wedge dx^2 \ldots \wedge dx^5
\end{align*}
$$

(25)

Comparing the above metric with the Schrödinger metric given in (11), we find that this metric has a Schrödinger symmetry with $z = -2$, $d = 4$ and no hypserscaling violation.

Now again in order to obtain Lifshitz metric we rewrite (23) as,

$$
\begin{align*}
ds^2 &= Q_2^2 \left[ -H_1^1 dt^2 + H_1 \left( (1 - H_1^{-1}) dt - dx^1 \right)^2 + \sum_{i=2}^5 (dx^i)^2 + 4Q_2 du^2 \right] + d\Omega_4^2
\end{align*}
$$

(26)

Dimensionally reducing the solution along the wave direction, i.e., along $x^1$, we obtain the following string theory solution,

$$
\begin{align*}
ds^2 &= Q_1^{1/2} Q_2^{1/2} u^2 \left[ -\frac{dt^2}{Q_1 Q_2 u^8} + \frac{\sum_{i=2}^5 (dx^i)^2 + 4Q_2 du^2}{Q_2 u^2} \right] + d\Omega_4^2 \\
e^{2\phi} &= \frac{Q_2^2}{Q_1^2} u^6 \\
A_{[1]} &= \frac{1}{Q_1 u^6} dt, \quad A_{[5]} = \frac{1}{Q_2 u^6} dt \wedge dx^2 \ldots \wedge dx^5
\end{align*}
$$

(27)

where we have used $H_1 \approx Q_1 u^6$ and the metric is written in the string frame. By comparing with the Lifshitz metric (2), we immediately recognize that the metric in (27) has a Lifshitz symmetry with hypserscaling violation. In fact we notice that the above solution is nothing but the near horizon limit of D0-D4 solution discussed in [37]. As found there D0-D4 solution in the near horizon limit indeed has hyperscaling violating Lifshitz symmetry with dynamical critical exponent $z = 4$, spatial dimension of the boundary theory $d = 4$ and the hypserscaling violation exponent $\theta = 2$.

Thus we again found Schrödinger/Lifshitz space-times from AdS$_7$ solution which is the near horizon geometry of M5-brane. Here we note that the dynamical critical exponents of Schrödinger/Lifshitz space-times are given as $z = -2$ and $z = 4$ respectively and so they again add upto 2 as expected.
3.3 D(p+1)-branes

The D(p+1)-brane solution of type II string theory has the form,

\[ ds^2 = H_2^{-\frac{1}{2}} \left( -dt^2 + (dx^1)^2 + \sum_{i=2}^{p+1} (dx^i)^2 \right) + H_2^{\frac{1}{2}} \left( dr^2 + r^2 d\Omega_{7-p}^2 \right) \]

\[ e^{2\phi} = H_2^2 \]

\[ A_{[p+2]} = H_2^{-1} dt \wedge dx^1 \ldots \wedge dx^{p+1} \]  

(28)

Here the metric is written in the string frame. The harmonic function \( H_2 \) has the form \( H_2 = 1 + Q_2/r^{6-p} \). Note that we have isolated one of the brane directions \( x^1 \) along which pp-waves will be generated. This is the reason \( p \neq 0 \) and in fact we have \( 1 \leq p \leq 5 \). \( Q_2 \) is the charge associated with D(p+1)-brane. \( \phi \) is the dilaton and \( A_{[p+2]} \) is a \((p+2)\)-form field which couples to D(p+1)-brane. In the near horizon limit the above metric takes the form,

\[ ds^2 = Q_2^2 r^\frac{p-2}{p} \left[ -\frac{r^{4-p}}{Q_2} \left( -dt^2 + (dx^1)^2 + \sum_{i=2}^{p+1} (dx^i)^2 \right) + \frac{dr^2}{r^2} + d\Omega_{7-p}^2 \right] \]  

(29)

Now going to a coordinate \( r \to 1/r \) and introducing a new variable by the relation \( u^2 = r^{4-p} \) (for \( p \neq 4 \)), we can rewrite the metric (29) along with the other fields given in (28) as,

\[ ds^2 = Q_2^2 u^\frac{2-p}{4-p} \left[ -dt^2 + (dx^1)^2 + \sum_{i=2}^{p+1} (dx^i)^2 + \frac{4}{(4-p)p} Q_2 du^2 \right] + d\Omega_{7-p}^2 \]

\[ e^{2\phi} = Q_2^2 u^\frac{2-p}{4-p} \]

\[ A_{[p+2]} = \frac{1}{Q_2 u^\frac{2-p}{4-p}} dt \wedge dx^1 \ldots \wedge dx^{p+1} \]  

(30)

Here the metric has AdS_{p+3} \times S^{7-p} structure for \( 1 \leq p \leq 5 \) except for \( p = 4 \) (or D5-brane) up to a conformal factor \((u^{2-p})/(4-p)\). Note that the conformal factor actually vanishes for \( p = 2 \) or D3-brane as is well-known. For \( p = 4 \) or D5-brane, we do not get AdS in the near horizon limit, rather we get \( \mathcal{M}_7 \times S^3 \) up to a conformal factor, where \( \mathcal{M}_7 \) represents the seven dimensional Minkowski space. Now we will show how starting from this AdS_{p+3} geometry we can generate Schrödinger/Lifshitz space-times by some solution generating transformation as was done for M2- and M5-brane cases.

In order to get Schrödinger space-times we generate pp-waves, by standard technique [50], along \( x^1 \) direction as before and thus we obtain the metric

\[ ds^2 = Q_2^2 u^\frac{2-p}{4-p} \left[ -dt^2 + (dx^1)^2 + \left( H_1 - 1 \right) (dt - dx^1)^2 + \sum_{i=2}^{p+1} (dx^i)^2 + \frac{4}{(4-p)p} Q_2 du^2 \right] + d\Omega_{7-p}^2 \]  

(31)
where \( H_1 = 1 + Q_1/r^{6-p} \) is a harmonic function with \( Q_1 \), the asymptotic momentum carried by the wave. By first changing \( r \to 1/r \) and then defining \( u^2 = r^{4-p} \), the harmonic function can be written as \( H_1 = 1 + Q_1 u^{2(6-p)/(4-p)} \). Note that for \( p < 4 \), the near horizon limit \( r \to \infty \) implies \( u \to \infty \), but for \( p > 4 \), \( r \to \infty \) implies \( u \to 0 \). However, in both cases, in the near horizon limit \( H_1 \approx Q_1 u^{2(6-p)/(4-p)} \). Now in the light-cone coordinates defined earlier, the above metric (32) can be written as,

\[
    ds^2 = Q_2^2 u^{\frac{2-p}{4}} \left[ -2dt d\xi + 2Q_1 u^{\frac{2(6-p)}{4-p}} d\xi^2 + \sum_{i=2}^{p+1} (dx^i)^2 + \frac{4}{(4-p)^2} Q_2 du^2 + d\Omega_{7-p}^2 \right]
\]

Dimensions reducing the solution on \( S^7 \) and expressing the resultant metric in the Einstein frame and then further taking the double Wick rotation \( (t \to i \xi \text{ and } \xi \to -it) \), the metric (32) and the other fields in (31) take the forms,

\[
    ds^2 = \frac{2^{\frac{p+2}{2}}}{Q_2^2} \frac{u^{2(6-p)/(4-p)}}{u^{(4-p)}} \left[ -2Q_1 u^{\frac{4}{4-p}} dt^2 + \frac{-2dt d\xi + \sum_{i=2}^{p+1} (dx^i)^2 + \frac{4}{(4-p)^2} Q_2 du^2}{Q_2 u^2} \right]
\]

\[
    e^{2\phi} = \frac{2^{\frac{2-p}{2}}}{Q_2^2} \frac{u^{(2-p)(6-p)}}{(4-p)} dt \wedge d\xi \wedge dx^2 \ldots \wedge dx^{p+1}
\]

Comparing the metric in (33) with the Schrödinger metric given in (11) we immediately notice that this metric has a Schrödinger symmetry with hyperscaling violation. Under the scaling symmetry \( t \to \lambda^{-\frac{2}{4-p}} t, \xi \to \lambda^{\frac{2}{4-p}} \xi, x^i \to \lambda x^i \) (for \( i = 2, \ldots, (p+1) \)), \( u \to \lambda u \), the metric in the square bracket remains invariant. However, as there is a hyperscaling violation the full metric is not invariant under the scaling, but changes as \( ds \to \lambda^{(p-2)^2/((p-4)(p+1))} ds \equiv \lambda^{\theta/D} ds \), where \( D \) is kept arbitrary as in (33). We thus find that the metric in (33) has a hyperscaling violating Schrödinger symmetry with the dynamical critical exponent \( z = -2/(4-p) \), hyperscaling violation exponent \( \theta/D = (p-2)^2/((p-4)(p+1)) \) and the spatial dimension of the boundary theory \( d = p \). We note that for \( p = 2 \), i.e. for D3-brane \( \phi = \) constant and \( \theta/D = 0 \) and for \( p < 4 \), \( \theta/D < 0 \). Therefore, there is no hyperscaling violation for \( p = 2 \) and hyperscaling violating exponent is negative for \( p < 4 \). Similar observations were made [9] for the non-relativistic \( D_p \)-branes obtained by the Null Melvin Twist [53].

Now in order to get Lifshitz space-time we rewrite the metric with waves in (31) as follows,

\[
    ds^2 = \frac{1}{2} \frac{2^{\frac{2-p}{2}}}{Q_2^2} \frac{u^{\frac{2-p}{4}}}{u^{(4-p)}} \left[ \frac{-H_1^{-1} dt^2 + H_1 \left( (1 - H_1^{-1}) dt - dx^1 \right)^2 + \sum_{i=2}^{p+1} (dx^i)^2 + \frac{4}{(4-p)^2} Q_2 du^2}{Q_2 u^2} + d\Omega_{7-p}^2 \right]
\]

Since here we are dealing with string theory solutions, we will take T-duality along the wave direction, i.e., \( x^1 \) to obtain Lifshitz metric. After T-duality the only metric components that
will change are $g_{tt}$, $g_{tx^1}$ and $g_{x^1 x^1}$. Denoting the new metric components by a ‘tilde’, we have

\[
\begin{align*}
\tilde{g}_{tt} & = g_{tt} - \frac{g_{x^1 x^1}^2}{g_{x^1 x^1}} = -Q_2 u^{2-p} H_1^{-1} = -Q_2 u^{\frac{2-p}{4-p}} H_1^{-1}, \\
\tilde{g}_{tx^1} & = \frac{B_{tx^1}}{g_{x^1 x^1}} = 0, \\
\tilde{g}_{x^1 x^1} & = \frac{1}{g_{x^1 x^1}} = \frac{Q_2 u^{\frac{2-p}{4-p}}}{Q_1 u^2}
\end{align*}
\]

where we have used the metric (34). We have also used $H_1 \approx Q_1 u^{2(6-p)/(4-p)}$ in the near horizon limit. $\tilde{g}_{tx^1}$ vanishes because $D(p+1)$-brane solution with wave does not have an NSNS $B$-field. However, in the T-dual solution a $B$-field will be generated because of a non-vanishing $g_{tx^1}$ component in (34). Using (35) and the other field configuration given in (30) the complete T-dual solution can be written as,

\[
\begin{align*}
\quad 
\quad ds^2 &= Q_2^\frac{1}{2} u^{\frac{2-p}{4-p}} \left[-\frac{dt^2}{Q_1 Q_2 u^{4(5-p)/(4-p)}} + \frac{Q_2 (dx^1)^2}{Q_2 u^2} + \sum_{i=2}^{p+1} (dx^i)^2 + \frac{4}{(4-p)^2} Q_2 du^2 u^{-2(6-p)/(4-p)} + d\Omega^2_{7-p} \right] \\
\quad e^{2\phi} &= \frac{Q_2^{\frac{1}{2-p}}}{Q_1} u^{(6-p)(1-p)/(4-p)} \\
\quad B_{[2]} &= \frac{1}{Q_1 u^{2(6-p)/(4-p)}} dt \wedge dx^1, \\
\quad A_{[p+1]} &= \frac{1}{Q_2 u^{2(6-p)/(4-p)}} dt \wedge dx^2 \wedge \ldots \wedge dx^{p+1},
\end{align*}
\]

By comparing with the metric (2), we find that the metric in (36) indeed has a Lifshitz symmetry with hyperscaling violation. In fact we recognize this solution as the near horizon limit of the 1/4 BPS, F-Dp solution found in [36]. Here F-string lies along $x^1$ and Dp-brane lies along $x^2, \ldots, x^{p+1}$. Note that the gauge fields differ by a sign from those in [36] due to a slightly different convention we use here. As discussed in [36], the metric in (36) has a hyperscaling violating Lifshitz symmetry with the dynamical critical exponent $z = 2(5-p)/(4-p)$, hyperscaling violation exponent $\theta = p - (p-2)/(4-p)$ and the spatial dimension of the boundary theory $d = p+1$. Interestingly, as noted also in [36], for $p = 2$, or D3-brane we have $\theta = p = d - 1$. The entanglement entropy of the boundary theory in that case is well-known to show a logarithmic violation of the area law and therefore represents compressible metallic state with hidden fermi surface [40, 41]. The entanglement entropy for the F-Dp system has been calculated in [54].

Thus we have shown that starting from the AdS$_{p+3}$ geometry (upto a conformal transformation) obtained from the near horizon limit of D($p+1$)-brane solutions of type II string theory, we can generate both the Schrödinger as well as Lifshitz space-times by making use of some solution generating techniques. Note that since the critical dynamical exponent for the Schrödinger symmetry has the value $z = -2/(4-p)$ and that of the Lifshitz symmetry has the value $z = 2(5-p)/(4-p)$, they indeed add upto 2 as we argued in section 2.
Before we conclude this section, we would like to remark that, for the case of Dp-branes, Schrödinger/Lifshitz space-times have also been obtained in [45]. However, their connections with AdS geometry was not clear there. Lifshitz geometry was obtained in [45] by first taking a double scaling limit of the boosted black Dp-brane (this is a standard procedure for generating pp-waves [50]), then going to the light-cone coordinates and finally compactifying the solution along the space-like light-cone direction. Note that in this procedure the solution becomes nine-dimensional and also the dimensionality of the boundary theory gets reduced by one. Since the light-cone coordinates involve one of the brane directions, it is not clear why one should compactify that direction. However, we think that the proper way to identify the Lifshitz geometry from Dp-branes, as is done in this paper, is to remain in Poincare coordinates and take a T-duality. This way there is no need to compactify one of the brane directions and the dimensionality of the boundary theory does not get reduced. Also our method clarifies the connection of Lifshitz space-times with AdS geometry as a deformation of the latter by the pp-waves and a performance by T-duality. Note, however, that for M-theory solutions we performed a dimensional reduction along the wave direction in order to obtain Lifshitz symmetry (there is no T-duality here). But since this extra dimension of M-theory is related to the string coupling constant, this means that Lifshitz symmetry is manifest only at small string coupling. When the coupling is large we have to lift the solution to M-theory and in that case we get an asymmetric Lifshitz scaling of this extra dimension. The Schrödinger space-time, on the other hand, was obtained in [45], by first going to the corresponding bubble solutions with a double Wick rotation, then taking a double scaling limit on the boosted bubble solutions and finally going to the light-cone coordinates. However, in this paper, we started from AdS solution and then deformed it by a pp-wave. We further introduced the light-cone coordinates and then took the double Wick rotation. This way we obtained Schrödinger space-times with hyperscaling violation. We also identified the hyperscaling violation exponent in this case, a feature never mentioned in [45]. We have seen that the dynamical critical exponents of the Schrödinger/Lifshitz dual space-times add up to 2. This was also observed in [45] in various cases of Dp-branes. We have given a general argument in section 2 explaining why this is so. Finally, in [45], it was shown how Lifshitz and Schrödinger space-times arise from Dp-brane solutions of string theory. Their origin was never really understood. In this paper we have shown that whenever we get an AdS solution (directly or up to a conformal transformation) in string or M-theory, not necessarily only Dp-branes, we can deform it (and apply other solution generating techniques) to generate Schrödinger/Lifshitz dual space-times.
4 Schrödinger/Lifshitz from intersecting solutions

In this section we will show how one can get Schrödinger/Lifshitz dual space-times from some intersecting solutions of M/string theory. The intersecting solutions we will consider give AdS geometry in the near horizon limit. For M-theory it is known that there are two intersecting solutions which lead to AdS$_3$ geometry and they are M2 $\perp$ M5 and M5 $\perp$ M5 [48]. There are others which lead to AdS$_2$ geometry, but we will not consider them for the reason mentioned earlier. For string theory solutions there are double and triple intersections which give AdS geometry, but we will consider only double intersections since the explicit triple intersecting solutions of string theory are not known and they will mostly give lower dimensional AdS space like the double intersecting solutions, therefore, Schrödinger/Lifshitz space-times will be quite similar as the double intersecting solutions. So, for string theory, we will consider D1 $\perp$ D5 (type IIB), D2 $\perp$ D4 (type IIA) and F $\perp$ NS5 (type IIA or IIB) solution all of which are known to lead to AdS$_3$ geometry in the near horizon limit. We will discuss the string theory solutions in brief as they give AdS$_3$ and the Schrödinger/Lifshitz space-time they lead to are very similar to intersecting M-brane solutions. The low dimensional AdS space, namely, AdS$_3$ along with waves and the resulting Schrödinger space have also been considered in [55].

4.1 M2 $\perp$ M5 solution

M2-brane intersecting with M5-brane on a string has a solution of the form,

$$ds^2 = H_2^\frac{2}{3}H_3^\frac{1}{3} \left[ H_2^{-1}H_1^{-1} (-dt^2 + (dx_1)^2) + H_2^{-1} \sum_{i=2}^{5} (dx_i)^2 + H_3^{-1}(dx_6)^2 + dr^2 + r^2 d\Omega_3^2 \right]$$

$$A[3] = H_3^{-1} dt \wedge dx_1 \wedge dx_6, \quad A[6] = H_2^{-1} dt \wedge dx_1 \wedge \ldots \wedge dx_5$$

(37)

Here the harmonic functions are given as $H_{2,3} = 1 + Q_{2,3}/r^2$ and $Q_{2,3}$ are the charges associated with M5-brane and M2-brane respectively. Note that M2-brane lies along $x^1, x^6$ and M5-brane lies along $x^1, x^2, \ldots, x^5$. In the near horizon limit $H_{2,3} \approx Q_{2,3}/r^2$ and the metric in (37) takes the form,

$$ds^2 = Q_2^\frac{2}{3}Q_3^\frac{1}{3} \left[ \frac{r^2}{Q_2Q_3} (-dt^2 + (dx_1)^2) + \sum_{i=2}^{5} \frac{(dx_i)^2}{Q_2} + \frac{(dx_6)^2}{Q_3} + \frac{dr^2}{r^2} + d\Omega_3^2 \right]$$

(38)

Now defining a new coordinate by $u = 1/r$, we can write the full solution (37) in the near horizon limit as,

$$ds^2 = Q_2^\frac{2}{3}Q_3^\frac{1}{3} \left[ -dt^2 + (dx_1)^2 + Q_2Q_3du^2 \frac{u^2}{Q_2Q_3} + \sum_{i=2}^{5} \frac{(dx_i)^2}{Q_2} + \frac{(dx_6)^2}{Q_3} + d\Omega_3^2 \right]$$

$$A[3] = \frac{1}{Q_3u^2} dt \wedge dx_1 \wedge dx_6, \quad A[6] = \frac{1}{Q_2u^2} dt \wedge dx_1 \wedge \ldots \wedge dx_5$$

(39)
It is clear that the above metric has AdS$_3 \times$ E$^5 \times$ S$^3$ structure.

Now in order to get Schrödinger space-time we generate pp-waves along the common brane direction $x^1$ by the standard technique. The metric in (39) then takes the form,

$$ds^2 = Q_2^2 \frac{1}{Q_3^3} \left[ -\frac{dt^2 + (dx^1)^2 + (H_1 - 1)(dt - dx^1)^2 + Q_2 Q_3 du^2}{Q_2 Q_3 u^2} + \sum_{i=2}^{5}(dx^i)^2 \frac{Q_3^2}{Q_2} + (dx^6)^2 \frac{Q_3}{Q_2} + d\Omega_3^2 \right]$$

(40)

where $H_1 = 1 + Q_1/r^2$ is another harmonic function and $Q_1$ is the asymptotic momentum of the wave. Going to the light cone coordinates and further taking double Wick rotation as before we arrive at the solution,

$$ds^2 = Q_2^2 \frac{1}{Q_3^3} \left[ -\frac{2Q_1}{Q_2 Q_3}dt^2 + \frac{2d\xi dt + Q_2 Q_3 du^2}{Q_2 Q_3 u^2} + \sum_{i=2}^{5}(dx^i)^2 \frac{Q_3^2}{Q_2} + (dx^6)^2 \frac{Q_3}{Q_2} + d\Omega_3^2 \right]$$

$$A_{[5]} = \frac{1}{Q_3 u^2} dt \wedge d\xi \wedge dx^6, \quad A_{[6]} = \frac{1}{Q_2 u^2} dt \wedge d\xi \wedge dx \wedge \ldots \wedge dx^5$$

(41)

The metric in (41) has a Schrödinger symmetry under the scaling $t \to \lambda t, \xi \to \lambda^2 \xi$ and $u \to \lambda u$. Note that the coordinates $x^2, \ldots, x^6$ do not scale and we can compactify the solution on S$^3$ as well as E$^5$ to get a zero dimensional Schrödinger metric. This Schrödinger space-time has $d = 0$, the dynamical critical exponent $z = 0$ and no hyperscaling violation, i.e., $\theta = 0$.

Lifshitz space-time can be obtained by rewriting the metric (40) as follows,

$$ds^2 = Q_2^2 \frac{1}{Q_3^3} \left[ -\frac{H_1^{-1}dt^2 + H_1 ((1 - H_1^{-1})dt - dx^1)^2 + Q_2 Q_3 du^2}{Q_2 Q_3 u^2} + \sum_{i=2}^{5}(dx^i)^2 \frac{Q_3^2}{Q_2} + (dx^6)^2 \frac{Q_3}{Q_2} + d\Omega_3^2 \right]$$

(42)

Now compactifying along $x^1$, the wave direction, we get the ten dimensional string theory solution as,

$$ds^2 = Q_1^2 \frac{1}{Q_2^2} \left[ -\frac{dt^2}{Q_1 Q_2 Q_3 u^4} + \frac{du^2}{u^2} + \sum_{i=2}^{5}(dx^i)^2 \frac{Q_3^2}{Q_2} + (dx^6)^2 \frac{Q_3}{Q_2} + d\Omega_3^2 \right]$$

$$e^{2\phi} = Q_1^3 \frac{1}{Q_2^2 Q_3}$$

$$B_{[2]} = \frac{1}{Q_3 u^2} dt \wedge dx^6, \quad A_{[5]} = \frac{1}{Q_2 u^2} dt \wedge dx^2 \wedge \ldots \wedge dx^5, \quad A_{[1]} = \frac{1}{Q_1 u^2} dt$$

(43)

Note that the metric has a scaling symmetry $t \to \lambda^2 t, u \to \lambda u$ and since the other spatial coordinates do not scale one can compactify the solution on S$^3 \times$ E$^5$ to get a zero dimensional Lifshitz metric. The dilaton is constant. The dynamical critical exponent here is $z = 2$ and hyperscaling violation exponent $\theta = 0$. Now since the corresponding Schrödinger metric has dynamical critical exponent $z = 0$ they indeed add up to 2. However, note that since the...
other spatial dimensions do not scale, the zero dimensional Lifshitz metric (consisting of
the first two terms of the metric in \(43\)) can be cast into an \(\text{AdS}_2\) form by defining a new
variable \(\tilde{u} = u^2\). In this sense the zero dimensional Lifshitz is kind of trivial.

### 4.2 More Lifshitz-like space-times

As the solution \(43\) is obtained from the dimensional reduction (along the wave direction)
of the near horizon limit of ‘\(M2 \perp M5 + \text{wave}\)’ solution of M-theory, it must correspond to
the near horizon limit of \(F \perp D4 \perp D0\) solution of string theory. Indeed one can check that
this is the case by explicitly constructing this solution (from the dimensional reduction of
the complete ‘\(M2 \perp M5 + \text{wave}\)’ solution) and taking the near horizon limit. This way the
complete \(F \perp D4 \perp D0\) solution takes the form,

\[
\begin{align*}
    ds^2 &= \frac{1}{H_1^3} \left( H_3^{-1} \left( -H_1^{-1} H_2^{-1} dt^2 + (dx^6)^2 \right) + H_2^{-1} \sum_{i=2}^{5} (dx^i)^2 + dr^2 + r^2 d\Omega_2^2 \right) \\
    e^{2\phi} &= \frac{H_1^3}{H_2^3 H_3}, \quad B_2 = H_3^{-1} dt \wedge dx^6 \\
    A[1] &= H_1^{-1} dt, \quad A[5] = H_2^{-1} dt \wedge dx^2 \wedge \ldots \wedge dx^5 \\
\end{align*}
\]

(44)

Here F-string is along \(x^6\) and D4-brane is along \(x^2, x^3, x^4, x^5\). It can be easily checked
that the near horizon limit of \(44\) gives \(43\). One can get more Lifshitz-like solution with
hyperscaling violation by taking T-dualities on the above \(F \perp D4 \perp D0\) solution. So, for
example, by taking T-duality along one of the common transverse directions, say \(x^7\), we can
generate \(F \perp D5 \perp D1\) solution given as,

\[
\begin{align*}
    ds^2 &= \frac{1}{H_1^3} \left( H_3^{-1} \left( -H_1^{-1} H_2^{-1} dt^2 + (dx^6)^2 \right) + H_2^{-1} \sum_{i=2}^{5} (dx^i)^2 + H_1^{-1} H_2^{-1} (dx^7)^2 + dr^2 + r^2 d\Omega_2^2 \right) \\
    e^{2\phi} &= \frac{H_1}{H_3}, \quad B_2 = H_3^{-1} dt \wedge dx^6 \\
    A[2] &= H_1^{-1} dt \wedge dx^7, \quad A[6] = H_2^{-1} dt \wedge dx^2 \wedge \ldots \wedge dx^5 \wedge dx^7 \\
\end{align*}
\]

(45)

Here F-string is along \(x^6\), D5-brane is along \(x^2, \ldots, x^5, x^7\) and D1-brane is along \(x^7\). The
harmonic functions are given as \(H_{1,2,3} = 1 + Q_{1,2,3}/r\), where \(Q_{1,2,3}\) are the charges associated
with D1-brane, D5-brane and F-string respectively. In the near horizon limit \(r \to 0\) and
further defining the coordinate \(u^2 = r\), we can write the solution \(45\) as,

\[
\begin{align*}
    ds^2 &= Q_1^{\frac{1}{2}} Q_2^{\frac{1}{2}} u^2 \left[ -\frac{u^2}{Q_1 Q_2 Q_3} dt^2 + \frac{(dx^6)^2}{u^2 Q_3} + \sum_{i=2}^{5} \frac{(dx^i)^2}{u^2 Q_2} + \frac{(dx^7)^2}{Q_1 Q_2} + \frac{4 du^2}{u^2} + d\Omega_2^2 \right] \\
    e^{2\phi} &= \frac{Q_1 u^2}{Q_2 Q_3}, \quad B_2 = \frac{u^2}{Q_3} dt \wedge dx^6 \\
    A[2] &= \frac{u^2}{Q_1} dt \wedge dx^7, \quad A[6] = \frac{u^2}{Q_2} dt \wedge dx^2 \wedge \ldots \wedge dx^5 \wedge dx^7 \\
\end{align*}
\]

(46)
It is clear from (46) that the part of the metric in square bracket is invariant under the scaling $t \to \lambda^{-1}t$, $x^1, \ldots, 5 \to \lambda x^1, \ldots, 5$, $u \to \lambda u$. However the full metric is not scale invariant. This tells us that the metric has Lifshitz scaling (with $z = -1$) with hyperscaling violation. To obtain the hyperscaling violation exponent we have to compactify the metric on $S^2 \times \mathbb{R}$ and express the resultant metric in the Einstein frame. The compactified metric has the form,

$$ds^2 = Q_1^\frac{1}{5}Q_2Q_3^{\frac{2}{5}}u^\frac{4}{5} \left[ -\frac{u^2}{Q_1Q_2Q_3}dt^2 + \frac{(dx^6)^2}{u^2Q_2} + \sum_{i=2}^{5} \frac{(dx^i)^2}{w^2Q_2} + 4\frac{du^2}{u^2} \right]$$

(47)

Since under the scaling mentioned above this metric changes as $ds \to \lambda^\frac{6}{5}ds \equiv \lambda^\frac{6}{5}ds$, where $d$ is the spatial dimension of the boundary theory (which is 5 in this case), we have the hyperscaling violation exponent $\theta = 6$. Thus $F \perp D5 \perp D1$ solution in the near horizon limit gives a Lifshitz-like metric with $z = -1$, $\theta = 6$ and $d = 5$.

We can generate more such intersecting solutions from $F \perp D4 \perp D0$ by applying T-duality along $x^2$ direction. The resultant solution is $F \perp D3 \perp D1$ and then by further taking T-duality along $x^7$ direction we get $F \perp D4 \perp D2$ solution. One can easily check that both these solutions yield Lifshitz-like metric in the near horizon limit. The former one, $F \perp D3 \perp D1$, gives zero dimensional Lifshitz metric (which can be recast into $AdS_2$ form) in the near horizon limit very similar to $F \perp D4 \perp D0$ and the latter one, $F \perp D4 \perp D2$ gives five dimensional Lifshitz metric in the near horizon limit (and dimensional reduction) with $z = -1$ and $\theta = 6$ very similar to $F \perp D5 \perp D1$ solution. By applying T-duality on $F \perp D3 \perp D1$ along $x^3$ direction we can generate intersecting solution $F \perp D2 \perp D2$ and by further applying T-duality along $x^7$, we can generate $F \perp D3 \perp D3$ solutions. Again we find that both these solutions yield Lifshitz-like metric in the near horizon limit. $F \perp D2 \perp D2$ gives zero dimensional Lifshitz metric very similar to $F \perp D4 \perp D0$ and $F \perp D3 \perp D3$ gives five dimensional Lifshitz metric with $z = -1$ and $\theta = 6$ very similar to $F \perp D5 \perp D1$. Therefore, we do not give any further details of these solutions. S-dual of the type IIB solutions can also give more intersecting solutions of the type discussed here, but they also yield very similar Lifshitz space-time as their original counterpart and therefore we do not discuss them here.
4.3 $M_5 \perp M_5 \perp M_5$ solution

Apart from $M_2 \perp M_5$ solution the other solution which yields AdS geometry in the near horizon limit is $M_5 \perp M_5 \perp M_5$. The solution has the form \cite{49},

$$
\begin{align*}
    ds^2 &= (H_2 H_3 H_4)^{\frac{1}{3}} \left[ (H_2 H_3 H_4)^{-1}(- dt^2 + (dx^1)^2) + (H_2 H_3)^{-1}((dx^2)^2 + (dx^3)^2) \\
    &\quad + (H_2 H_4)^{-1}((dx^4)^2 + (dx^5)^2) + (H_3 H_4)^{-1}((dx^6)^2 + (dx^7)^2) + dt^2 + r^2 d\Omega_2^2 \right] \\
    A_{[6]} &= H_2^{-1} dt \wedge dx^1 \wedge \ldots \wedge dx^5, \quad A'_{[6]} = H_2^{-1} dt \wedge dx^1 \wedge dx^4 \wedge \ldots \wedge dx^7 \\
    A''_{[6]} &= H_4^{-1} dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^6 \wedge dx^7
\end{align*}
$$

(48)

Note that here the three $M_5$-branes intersect on a string along $x^1$ and they intersect pairwise on 3-branes along $x^1$, $x^2$, $x^3$, along $x^1$, $x^4$, $x^5$ and along $x^1$, $x^6$, $x^7$. The three harmonic functions are given as $H_{2,3,4} = 1 + Q_{2,3,4}/r$, where $Q_{2,3,4}$ are the charges associated with the three $M_5$-branes. The $M_5$-branes are electric and so, $A_{[6]}$, $A'_{[6]}$ and $A''_{[6]}$ are the three 6-form gauge fields to which they couple. In the near horizon limit $H_{2,3,4} \approx Q_{2,3,4}/r$ and further defining a new coordinate by $u^2 = 1/r$, we can rewrite the metric in (48) as

$$
\begin{align*}
    ds^2 &= (Q_2 Q_3 Q_4)^{\frac{1}{3}} \left[ \frac{-dt^2 + (dx^1)^2 + 4Q_2Q_3Q_4 du^2}{Q_2 Q_3} + \frac{(dx^2)^2 + (dx^3)^2}{Q_2 Q_3 Q_4} \\
    &\quad + \frac{(dx^4)^2 + (dx^5)^2}{Q_2 Q_4} + \frac{(dx^6)^2 + (dx^7)^2}{Q_3 Q_4} + d\Omega_2^2 \right]
\end{align*}
$$

(49)

It is clear from the metric (49) that it has the structure $AdS_3 \times E^6 \times S^2$. As discussed in section 2, we will show how starting from this AdS geometry we can generate Schrödinger/Lifshitz dual space-times by solution generating techniques.

To obtain Schrödinger space-times we generate pp-waves along $x^1$ by standard technique. The above metric in that case takes the form

$$
\begin{align*}
    ds^2 &= (Q_2 Q_3 Q_4)^{\frac{1}{3}} \left[ \frac{-dt^2 + (dx^1)^2 + (H_1 - 1)(dt - dx^1)^2 + 4Q_2 Q_3 Q_4 du^2}{Q_2 Q_3 Q_4 u^2} \\
    &\quad + \frac{(dx^2)^2 + (dx^3)^2}{Q_2 Q_3} + \frac{(dx^4)^2 + (dx^5)^2}{Q_2 Q_4} + \frac{(dx^6)^2 + (dx^7)^2}{Q_3 Q_4} + d\Omega_2^2 \right]
\end{align*}
$$

(50)

where $H_1 = 1 + Q_1/r$ is another harmonic function and $Q_1$ is the asymptotic charge carried by the wave. In terms of $u$, harmonic function is given as $H_1 = 1 + Q_1 u^2$. Substituting this in the metric \cite{50}, going to the light cone coordinates defined earlier and further taking the double Wick rotation ($t \to i\xi$ and $\xi \to -it$), the metric as well as the gauge fields take the
forms,
\[ ds^2 = (Q_2Q_3Q_4)^2 \left[ -\frac{2Q_1}{Q_2Q_3Q_4}dt^2 + \frac{-2d\xi dt + 4Q_2Q_3Q_4du^2}{Q_2Q_3Q_4u^2} \right. \\
+ \left. \frac{(dx)^2}{Q_2Q_3} + \frac{(dx^3)^2}{Q_2Q_4} + \frac{(dx^4)^2 + (dx^5)^2}{Q_3Q_4} + \frac{(dx^6)^2 + (dx^7)^2}{Q_3Q_4} + d\Omega_2^2 \right] \]
\[ A_{[6]} = \frac{1}{Q_2u^2} dt \wedge d\xi \wedge \ldots \wedge dx^5, \quad A'_{[6]} = \frac{1}{Q_3u^2} dt \wedge d\xi \wedge dx^4 \wedge \ldots \wedge dx^7 \]
\[ A''_{[6]} = \frac{1}{Q_4u^2} dt \wedge dx^2 \wedge dx^3 \wedge dx^6 \wedge dx^7 \]  

Under the scaling symmetry \( t \rightarrow \lambda^6 t, \xi \rightarrow \lambda^2 \xi \) and \( u \rightarrow \lambda u \), the metric in (51) remains invariant. Note that the other coordinates do not scale and therefore we can compactify the metric on \( E^6 \times S^2 \), to get a zero dimensional Schrödinger metric with \( z = 0 \) and \( \theta = 0 \), very much like the case we discussed for \( M2 \perp M5 \) solution in subsection 4.1.

To obtain Lifshitz space-time we rewrite the metric (51) as follows,
\[ ds^2 = (Q_2Q_3Q_4)^2 \left[ -\frac{H^{-1} - 1}{Q_2Q_3Q_4}dt + \frac{H_1 - 1}{Q_2Q_3Q_4}dx^1 \right. \\
+ \left. \frac{(dx)^2}{Q_2Q_3} + \frac{(dx^3)^2}{Q_2Q_4} + \frac{(dx^4)^2 + (dx^5)^2}{Q_3Q_4} + \frac{(dx^6)^2 + (dx^7)^2}{Q_3Q_4} + d\Omega_2^2 \right] \]  

Compactifying along \( x^1 \), the wave direction, we obtain the string theory solution as,
\[ ds^2 = (Q_1Q_2Q_3Q_4)^2 \left[ -\frac{dt^2}{Q_1Q_2Q_3Q_4u^4} + \frac{4du^2}{u^2} + \frac{(dx)^2}{Q_2Q_3} \right. \\
+ \left. \frac{(dx^4)^2 + (dx^5)^2}{Q_2Q_4} + \frac{(dx^6)^2 + (dx^7)^2}{Q_3Q_4} + d\Omega_2^2 \right] \]
\[ e^{2\phi} = \frac{Q_1^2}{(Q_2Q_3Q_4)^2} \]
\[ A_{[5]} = \frac{1}{Q_2u^2} dt \wedge dx^2 \wedge \ldots \wedge dx^5, \quad A'_{[5]} = \frac{1}{Q_3u^2} dt \wedge dx^4 \wedge \ldots \wedge dx^7 \]
\[ A''_{[5]} = \frac{1}{Q_4u^2} dt \wedge dx^2 \wedge dx^3 \wedge dx^6 \wedge dx^7, \quad A_{[1]} = \frac{1}{Q_1u^2} dt \]  

The metric above has a scaling symmetry \( t \rightarrow \lambda^2 t, u \rightarrow \lambda u \). Since the other spatial coordinates do not scale we can compactify them to obtain a zero dimensional Lifshitz metric. However, we note that by defining a new parameter \( \bar{u} = u^2 \) we can recast the metric into \( AdS_2 \) form. The dilaton in this case is constant very much like \( M2 \perp M5 \) case we discussed before. By looking at the solution (53) we recognize this to be the near horizon limit of the intersecting 1/16 BPS type IIA string theory solution \( D4 \perp D4 \perp D4 \perp D0 \). By applying T- and S-dualities to this solution one can construct many such intersecting 1/16 BPS solutions. These solutions in the near horizon limit will not give AdS geometry, but it will be
worthwhile to see whether some of these solutions can give rise to interesting Lifshitz-like space-time. We leave this for a future investigation.

4.4 D1 \perp D5 solution

This type IIB string theory solution is known to lead to $\text{AdS}_3 \times \mathbb{E}^4 \times S^3$ metric in the near horizon limit. So, we can obtain Schrödinger/Lifshitz dual space-times starting from this metric. In order to see this we first write the solution,

$$
\begin{align*}
 ds^2 &= H_2^2 H_3^2 \left[ H_2^{-1} H_3^{-1} (-dt^2 + (dx^1)^2) + H_3^{-1} \sum_{i=2}^{5} (dx^i)^2 + dr^2 + r^2 d\Omega_3^2 \right] \\
 e^{2\phi} &= \frac{H_2}{H_3} \\
 A_{[2]} &= H_2^{-1} dt \wedge dx^1, \quad A_{[6]} = H_3^{-1} dt \wedge dx^1 \wedge \ldots \wedge dx^5
\end{align*}
$$

(54)

Here $H_{2,3} = 1 + Q_{2,3}/r^2$ and $Q_{2,3}$ are the charges associated with D1-brane and D5-brane respectively. D1-brane lies along $x^1$ and D5-brane lies along $x^1, \ldots, x^5$. In the near horizon limit $H_{2,3} \approx Q_{2,3}/r^2$ and then defining a new coordinate $u = 1/r$, we write the solution as,

$$
\begin{align*}
 ds^2 &= Q_2^2 Q_3^2 \left[ \frac{-dt^2 + (dx^1)^2 + Q_2 Q_3 du^2}{Q_2 Q_3 u^2} + \frac{1}{Q_3} \sum_{i=2}^{5} (dx^i)^2 + d\Omega_3^2 \right] \\
 e^{2\phi} &= \frac{Q_2}{Q_3} \\
 A_{[2]} &= \frac{1}{Q_2 u^2} dt \wedge dx^1, \quad A_{[6]} = \frac{1}{Q_3 u^2} dt \wedge dx^1 \wedge \ldots \wedge dx^5
\end{align*}
$$

(55)

The metric above is $\text{AdS}_3 \times \mathbb{E}^4 \times S^3$. Once we have $\text{AdS}_3$, we can get Schrödinger space-time as before by first generating a pp-wave along the common brane direction $x^1$, going to the light-cone coordinates defined before and then taking a double Wick rotation. The resultant Schrödinger metric takes the form,

$$
\begin{align*}
 ds^2 &= Q_2^2 Q_3^2 \left[ \frac{-2Q_1}{Q_2 Q_3} dt^2 + \frac{-2d\xi dt + Q_2 Q_3 du^2}{Q_2 Q_3 u^2} + \frac{1}{Q_3} \sum_{i=2}^{5} (dx^i)^2 + d\Omega_3^2 \right] 
\end{align*}
$$

(56)

where $Q_1$ is the asymptotic momentum carried by the wave. The metric is invariant under the scaling $t \to \lambda t$, $\xi \to \lambda^2 \xi$ and $u \to \lambda u$. Also, since the other spatial coordinates do not scale we can compactify the solution on $T^4 \times S^3$ to obtain a zero dimensional Schrödinger space-time very similar to M2 $\perp$ M5 case.

Lifshitz space-time can also be obtained as before by rewriting the metric with waves in a suitable form as described in section 2 and then taking T-duality along the wave direction $x^1$. The resultant metric will be the near horizon limit of the T-dual solution of ‘D1 $\perp$ D5 + wave’ solution we just described. This T-dual solution is nothing but the F $\perp$ D4 $\perp$
D0 solution (44) we described in subsection 4.2. In the near horizon limit it gives a zero dimensional Lifshitz metric which can be written in AdS$_2$ form by a suitable coordinate transformation as shown in (43).

4.5 D2 $\perp$ D4 solution

This is a type IIA string theory solution which can be obtained from D1 $\perp$ D5 solution by applying T-duality along one of the D5-brane direction transverse to D1-brane. Here D2 and D4 intersect on a string and is 1/4 BPS, unlike 1/2 BPS D2-D4 solution where D2-brane is completely inside the D4-brane. The solution has been discussed in [37], where we found that in the near horizon limit it gives AdS$_3 \times $ E$^4 \times$ S$^3$ as in D1 $\perp$ D5 system. We can generate Schr"odinger/Lifshitz dual space-times starting from this AdS$_3$ geometry. As in D1 $\perp$ D5 case here also we get zero dimensional Schr"odinger metric with dynamical critical exponent $z = 0$ and with no hyperscaling violation. The corresponding Lifshitz is also zero dimensional with $z = 2$ and $\theta = 0$, which can also be recast into AdS$_2$ form.

D3 $\perp$ D3 solution of type IIB string theory [37] is also known to give AdS$_3$ in the near horizon limit. The Schr"odinger/Lifshitz dual space-times obtained from this solution also have very similar forms as those of D2 $\perp$ D4 or D1 $\perp$ D5 solutions. So, we do not give any further details for this solution.

4.6 F $\perp$ NS5 solution: an exception

F $\perp$ NS5 solution can be obtained from D1 $\perp$ D5 solution by applying S-duality. Like D1 $\perp$ D5, this solution also leads to AdS metric in the near horizon limit. F $\perp$ NS5 solution of either type IIA or IIB string theory has the form,

$$ds^2 = H_3 \left[ H_2^{-1}H_3^{-1}(-dt^2 + (dx^1)^2) + H_3^{-1} \sum_{i=2}^{5} (dx^i)^2 + dr^2 + r^2 d\Omega_3^2 \right]$$

$$e^{2\phi} = \frac{H_3}{H_2}$$

$$B_{[2]} = H_2^{-1}dt \wedge dx^1, \quad B_{[6]} = Q_3 \text{Vol}(\Omega_3)$$

(57)

where $H_{2,3} = 1 + Q_{2,3}/r^2$ and $Q_{2,3}$ are the charges associated with the F-string and the NS5-brane respectively. In the near horizon limit $H_{2,3} \approx Q_{2,3}/r^2$ and then defining a new coordinate by $u = 1/r$ we can write the metric in (57) as,

$$ds^2 = Q_3 \left[ \frac{-dt^2 + (dx^1)^2 + Q_2Q_3du^2}{Q_2Q_3u^2} + \frac{1}{Q_3} \sum_{i=2}^{5} (dx^i)^2 + d\Omega_3^2 \right]$$

(58)

It is clear that the metric (58) has AdS$_3 \times $ E$^4 \times$ S$^3$ structure. As we argued in section 2, we might expect to get Schr"odinger/Lifshitz dual space-times starting from this AdS geometry.
However, we will argue that this is not the case and $F \perp NS5$ is an exception. Schrödinger space-time can be obtained by generating a wave along the common brane direction $x^1$, then rewriting the resulting solution in the light-cone coordinates and finally taking the double Wick rotation. As in $D1 \perp D5$ case, here also we get a zero dimensional Schrödinger metric with $z = 0$ and $\theta = 0$. We can then try to get Lifshitz space-time by rewriting the metric as given in (5) and then taking T-duality along the wave direction $x^1$. However, since the solution ‘$F \perp NS5 + \text{wave}$’ is invariant under T-duality, we cannot generate Lifshitz metric in this case and the resultant metric will still exhibit Schrödinger symmetry. So, this is the only case where our algorithm of generating Schrödinger/Lifshitz dual space-times starting from AdS geometry does not work.

5 Conclusion

To conclude, in this paper we have shown how starting from AdS geometry (obtained from various string/M theory solutions in the near horizon limit either directly or up to a conformal transformation), one can generate Schrödinger/Lifshitz dual space times (without or with hyperscaling violation) by some solution generating techniques known for string/M theory. Schrödinger/Lifshitz space-times obtained in this way are dual in the sense that their dynamical critical exponents add up to 2. We have studied various examples including simple branes and double and triple intersecting branes. Among simple brane solutions we have studied M2-, M5- and D3-branes, which are known to give AdS$_4$, AdS$_7$ and AdS$_5$ geometries respectively, directly in the near horizon limit. We have also studied D$(p + 1)$-branes ($p \neq 3, 4$) which are known to give AdS$_{p+3}$ geometry up to a conformal transformation. Then we have studied double and triple intersecting M-brane solutions $M2 \perp M5$ and $M5 \perp M5 \perp M5$ which are known to give AdS$_3$ geometry and some double intersecting string solutions $D1 \perp D5$ (of type IIB), $D2 \perp D4$ (of type IIA) and $F \perp NS5$ (of type IIA or IIB) which are also known to give AdS$_3$ geometries. In all these cases we have obtained Schrödinger/Lifshitz dual space-times except $F \perp NS5$ case. For obtaining Schrödinger space-times we had to deform the AdS geometry by introducing pp-waves along one of the brane directions (for single brane) and one of the common brane directions (for the intersecting branes) and then wrote the solution in the light-cone coordinate. By further taking a double Wick rotation we obtained metric having Schrödinger symmetry without or with hyperscaling violation. On the other hand to obtain Lifshitz space-times we took the deformed solution in Poincaré coordinates and either dimensionally reduced it along the wave direction (for M-theory solutions) or took T-duality (for string theory solutions) along the same direction. These reduced or T-dual solutions then exhibited Lifshitz scaling symmetry without or with hyperscaling violation. More Lifshitz (but not Schrödinger) space-times were obtained by taking further T-duality along other directions as shown for specific case in subsection 4.2. Thus start-
ing from the same string or M-theory solutions, which are known to give AdS geometry in the near horizon decoupling limit, we obtained both Schrödinger and Lifshitz space-times by using solution generating techniques. Since these solution generating techniques do not break supersymmetry and the string/M-theory solutions are BPS, we expect the Schrödinger and the Lifshitz space-times we obtained this way also preserve some fraction of space-time supersymmetries. Intersecting branes give low dimensional or AdS$_3$ space and consequently the Schrödinger/Lifshitz dual space-times obtained in these cases are zero dimensional. Zero dimensional Lifshitz can also be cast into AdS$_2$ form by some coordinate transformation. However, we found that we can not obtain Schrödinger/Lifshitz dual space-times for $F \perp$ NS5 intersecting solution even if it gives AdS$_3$ geometry (in the near horizon limit). We can get Schrödinger metric, but no Lifshitz metric can be obtained by T-duality as this solution with pp-wave (along the common brane direction) is T-duality invariant. Our results can be interpreted from the dual field theory point of view as obtaining certain strongly correlated condensed matter system (having Schrödinger or Lifshitz scaling symmetry) by some sort of a deformation of a relativistic system. The precise form of the deformation in field theory is not clear to us.

**Note added:**

After submitting this paper to the archive, we were informed by the authors of [46] that there is quite a bit of overlap of this paper with sections 5 and 6 of [46]. In section 2 we have shown that the dynamical critical exponents of Schrödinger/Lifshitz dual space-times add upto 2 and this was also observed in section 5 of their paper. Also it can be checked that their general formula of $z$ and $\theta$ given in eqs.(6.43) and (6.44) for Lifshitz space-times in [46] match with the results given in sections 3 and 4 of our paper for different values of the parameters defined in [46]. We are grateful to Jakob Gath, Jelle Hartong, Ricardo Monteiro and Niels Obers for informing us about this.

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