Boundary $K$-matrix for the quantum Mikhailov-Shabat model

J. D. Kim

Department of Mathematical Sciences,
University of Durham, Durham DH1 3LE, U.K.

ABSTRACT

We present complete solutions of $K$-matrix for the quantum Mikhailov-Shabat model. It has been known that there are three diagonal solutions with no free parameters, one being trivial identity solution, the others non-trivial. The most general solutions which we found consist of three families corresponding to each diagonal solutions. One family of solutions depends on two arbitrary parameters. If one of the parameters vanishes, the other must also vanish so that the solutions reduces to trivial identity solution. The other two families for each non-trivial diagonal solutions have only one arbitrary parameter.

*jideog.kim@durham.ac.uk
†On leave of absence from Korea Advanced Institute of Science and Technology
I. Introduction

The Yang-Baxter equation whose solution is called $R$-matrix appears frequently in physics, *eg.*, factorizable scattering theory on a line and solvable lattice models. The actual adaptation of $R$-matrix to each of these theories requires some more ingredients.

In the factorizable scattering theory, the solutions of the Yang-Baxter equation can be used to compute the $S$-matrix, which is given by $R$-matrix multiplied by overall factor chosen to satisfy crossing relation and unitarity, *etc.* Here the spectral parameter is related with coupling constant as well as the rapidity parameter of the particles and the deformation parameter is related with coupling constant. In the solvable lattice theory, the $L$-operator is given by $R$-matrix and the bulk Hamiltonian can be computed from transfer matrix which is given by the trace of products of $L$-operators.

Generalization of factorizable scattering theory to a half-line lead to the discovery of the reflection equation whose solution is called $K$-matrix[*1*]. This equation describes interactions related with boundary. By similar way as in the bulk theory, boundary $S$-matrix can be computed from $K$-matrix[*2*, *3*]. This idea was also used to prove the quantum integrability of open spin chains in the framework of quantum inverse scattering theory[*4*]. This gives us the Hamiltonian including boundary terms.

At the early stage of development, the solutions of the Yang-Baxter equation were computed in a rather direct way, *ie.*, by solving the functional equations component-wise. The substantial number of solutions were compiled in Ref.[*5*] before the general algebraic solutions were incidentally obtained[*6*, *7*, *8*].

The most general non-diagonal solutions of reflection equation for XXX, XXZ and XYZ type $R$-matrix which are $4 \times 4$ matrices were solved in Ref.[*1*, *4*, *9*, *10*]. The diagonal solutions for bigger size $R$-matrices were found in Ref.[*11*] for the Zamolodchikov-Fateev model[*12*], in Ref.[*13*] for the Mikhailov-Shabat model and in Ref.[*9*] for $A_n^{(1)}$ model. Spectral parameter independent solutions for various models are considered in Ref.[*14*].
We consider the most general non-diagonal solutions of reflection equation for the $9 \times 9$ $R$-matrix, especially corresponding to the quantum Mikhailov-Shabat model\cite{15}. This model is interesting because it is one of the relativistic quantum field theory with single scalar field having non-trivial families of boundary interaction\cite{16}.

This model was known to have two non-trivial diagonal solutions of $K$-matrix as well as trivial identity solution\cite{13}. These solutions have no free parameters. The most general solutions which we found consist of three families corresponding to each diagonal solutions. One family of solutions depends on two arbitrary parameters. If one of the parameters vanishes, the other must also vanish so that the solutions reduces to trivial identity solution. The other two families for each non-trivial diagonal solutions have only one arbitrary parameter and have upper-lower triangular structures.

The plan of this paper is as follows. This introduction is the section I. In section II, we consider the whole $9 \times 9$ equations resulting from matrix reflection equation and describe their general features. In section III, we solve the equations to obtain solutions. Finally, we make some discussions in section IV.
II. Reflection Equation

The bulk S-matrix of integrable quantum field theories can be described by the R-matrix which is the solution of the Yang-Baxter equation. The R-matrix acts on $V \otimes V$, where $V$ is $\mathbb{C}^3$ for the Mikhailov-Shabat model, since this model has $3 \times 3$ Lax-pair representation even though it involves only single scalar field.

$$R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u),$$

where $R_{12}(u)$, $R_{13}(u)$ and $R_{23}(u)$ act on $V \otimes V \otimes V$, with $R_{12}(u) = R(u) \otimes 1$, $R_{23}(u) = 1 \otimes R(u)$ etc.

The R-matrix for the Mikhailov-Shabat or $A_2^{(2)}$ model was obtained in Ref.[15] and it was used to compute S-matrix in Ref.[17].

$$R(u) = \begin{pmatrix}
  c & b & e \\
  d & g & f \\
  \bar{e} & \bar{g} & a \\
  \bar{f} & \bar{g} & d \\
  \bar{e} & b & c
\end{pmatrix},$$

where

$$a(u) = -e^{u-3\eta} + e^{-u+3\eta} + e^{-\eta} - e^{\eta} + e^{-3\eta} - e^{3\eta} - e^{-5\eta} + e^{5\eta},$$

$$b(u) = -e^{u-3\eta} + e^{-u+3\eta} + e^{-3\eta} - e^{3\eta},$$

$$c(u) = -e^{u-5\eta} + e^{-u+5\eta} + e^{-\eta} - e^{\eta},$$

$$d(u) = -e^{u-\eta} + e^{-u+\eta} + e^{-\eta} - e^{\eta},$$

$$e(u) = e^{-\eta} - e^{-u+\eta} - e^{-5\eta} + e^{-u+5\eta},$$

$$\bar{e}(u) = e^{u-\eta} - e^{\eta} - e^{u-5\eta} + e^{5\eta},$$

$$f(u) = -e^{-u+\eta} + e^{-u-\eta} - e^{-u+3\eta} + e^{3\eta} + e^{-u+5\eta} - e^{-5\eta},$$

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\[ f(u) = e^{u-\eta} - e^{u+\eta} + e^{u-3\eta} - e^{-3\eta} - e^{u-5\eta} + e^{5\eta}, \]
\[ g(u) = -e^{-u} + 1 + e^{-u+4\eta} - e^{4\eta}, \]
\[ \bar{g}(u) = -1 + e^u + e^{-4\eta} - e^{-u-4\eta}. \]

This \( R \)-matrix is a meromorphic function of \( e^u \). It has only combined \( PT \) symmetry.

\[ \mathcal{P}_{12} R_{12}(u) \mathcal{P}_{12} = R_{12}(u)^{t_1 t_2}, \tag{4} \]

where \( \mathcal{P}_{ij} \) is the permutation operator on \( V_i \otimes V_j \) defined by \( \mathcal{P}(x \otimes y) = (y \otimes x) \) and \( t_i \) denotes transposition in the \( i \)-th space.

The boundary versions of the Yang-Baxter equation so-called reflection equations for \( PT \) symmetric \( R \)-matrix are given by [13]

\[ R_{12}(u-v) \frac{1}{2} (K_- (u) R_{12}^{t_1 t_2} (u+v) K_- (v)) = K_- (v) R_{12}(u+v) \frac{1}{2} (K_- (u) R_{12}^{t_1 t_2} (u-v)), \tag{5} \]

\[ R_{12}(-u+v) \frac{1}{2} (K_+^{t_2} (u) M^{-1} R_{12}^{t_1 t_2} (-u-v-2\rho) M^{-1} (K_+^{t_2}) (v)) = (K_+^{t_2} (v) M^{-1} R_{12}(-u-v-2\rho) M^{-1} (K_+^{t_2}) (u) R_{12}^{t_1 t_2} (-u+v)), \tag{6} \]

where \( K_- (u) = K_- (u) \otimes 1, K_+ (u) = 1 \otimes K_-(u), \) etc. and \( M \) is given by \( V \) matrix defined by the crossing unitarity relation.

\[ R_{12}(u) = V^t R_{12}^{t_2} (-u-\rho) V^{-1}, \quad M = V^t V = M^t. \tag{7} \]

For \( A_{2}^{(2)} \) model, \( \rho = -6\eta - i\pi \) and the matrix \( V \) is given by

\[ V = \begin{pmatrix} -e^{-\eta} \\ 1 \\ -e^\eta \end{pmatrix}. \tag{8} \]

In practice, if \( K_- (u) \) is a solution of (4) then

\[ K_+ (u) = K_-^t (-u-\rho) M \tag{9} \]

is a solution of (3).
Now we write down the $9 \times 9$ component equations in the following parametrization of $K$-matrix.

$$K(u, \eta) = \begin{pmatrix} \beta & x & z \\ \bar{x} & \alpha & y \\ \bar{z} & \bar{y} & \gamma \end{pmatrix}. \quad (10)$$

2 Unknowns

1133 : $zzcd = zzcd$

1232 : $xxbg + yxed = yybg + xyed$

3212 : $\bar{y}ybg + \bar{x}yed = \bar{x}xbg + \bar{y}xed$

3311 : $\bar{z}zcd = \bar{z}zcd$

3 Unknowns

1123 : $xzc b + zycg = yzeg + xzeb + xzbd$

1132 : $xzcd = yzbg + zzbb + xz\bar{e}d$

1233 : $xzbg + yzbb + yzed = yzcd$

2133 : $xz\bar{e}g + zyeb + yzbd = yzcb + xzc\bar{g}$

2311 : $\bar{x}zcb + \bar{z}ycg = \bar{y}zeg + \bar{z}xeb + \bar{x}zbd$

3211 : $\bar{x}zbg + \bar{z}ybb + \bar{y}zed = \bar{z}ycd$

3321 : $\bar{y}zcb + \bar{z}xcb = \bar{x}z\bar{e}g + \bar{z}yeb + \bar{y}zbd$

4 Unknowns

1111 : $x\bar{e}ce + z\bar{z}cf = x\bar{e}ce + \bar{z}zcf$

1212 : $xybg + \bar{x}xec + y\bar{y}ef = \bar{x}ybg + x\bar{e}ce + \bar{y}yef$

2222 : $xygb + \bar{x}ygb + \bar{x}xae + y\bar{y}ae = y\bar{x}gb + \bar{x}ygb + x\bar{x}ae + \bar{y}yae$

3232 : $\bar{y}xbg + \bar{x}x\bar{e}f + y\bar{y}ec = \bar{x}ybg + x\bar{x}\bar{e}f + \bar{y}yec$

3333 : $\bar{y}yce + \bar{z}zcf = \bar{y}yce + \bar{z}zcf$

5 Unknowns

2123 : $xy\bar{e}a + \beta z\bar{e}e + z\gamma \bar{e}e + yybg = yxe a + z\beta e e + xx\bar{b}g + \gamma zee$

2321 : $\bar{x}y\bar{e}a + \beta \bar{z}\bar{e}e + \bar{z}\gamma \bar{e}e + \bar{y}ybg = \bar{y}xe a + \bar{z}\beta e e + \bar{x}x\bar{b}g + \gamma \bar{z}ee$

6 Unknowns

1112 : $\bar{x}zbg + \alpha xce + \beta xxb + x\beta ec + \bar{y}ze f = x\alpha e + \beta xcc + z\bar{y}ef$

1113 : $\alpha z\bar{g} + yxfe + xx\bar{g}b + z\beta fc + \beta zdd + \gamma zf f = xyce + \beta zcc + z\gamma cf$

1121 : $\bar{x}z\bar{e}g + \alpha xbe + \beta x\bar{e}b + x\beta bc + \bar{y}zbf = x\beta cb + \bar{x}zcg$
\[\begin{align*}
1122 & : \alpha z a g + y x g e + x x a b + z \beta g c + \beta z g d + \gamma z g f = x x c b + z a c g \\
1131 & : \alpha z g g + y x d e + x x g b + z \beta d c + \beta z f d + \gamma z d f = z \beta c d \\
1211 & : x z b g + \alpha x e e + \beta x b b + \bar{x} \beta e c + y z f f = \bar{x} \alpha c e + \beta x c c + \bar{z} y c f \\
1223 & : x y b a + \alpha z e b + \beta z b e + \gamma z b e = z a e b + x y b d \\
1311 & : \alpha z g g + \bar{y} x f e + \bar{x} x g b + \bar{z} \beta f c + \beta \bar{z} d d + \bar{z} y f f = \bar{x} y c e + \beta z c c + \bar{z} y c f \\
1313 & : \alpha z g g + y x d e + \beta z d f + y y g b + \gamma z f d + z \gamma d c = z \gamma c d \\
2111 & : x z \bar{e} g + \alpha x b e + \beta x \bar{e} b + \bar{x} \beta b f = \bar{x} \beta c b + \bar{x} x c g \\
2121 & : x x e a + z \bar{z} e e + \bar{y} x b g = \bar{x} x e a + \bar{z} z e e + \bar{y} x b g \\
2313 & : \alpha z g g + x y f e + \beta \bar{z} f f + y y g b + \gamma z d d + \bar{z} y f c = y x c e + \beta z b f + \gamma z c c \\
3221 & : \bar{y} x b a + \alpha z e b + \bar{z} \beta b e + \gamma \bar{z} b e = z a e b + y x b d \\
3233 & : y z b g + \alpha y e e + \bar{x} z e \bar{f} + \bar{y} y b b + y y e c = y a c e + z \bar{x} c f + \gamma y c c \\
3313 & : \alpha z g g + \bar{x} y d e + \bar{y} y g b + \beta \bar{z} d f + \gamma \bar{z} f d + \bar{z} \gamma d c = \bar{z} \gamma c d \\
3322 & : \alpha z a g + \bar{x} y g g + \bar{y} y a b + \beta \bar{z} g f + \bar{y} \gamma g d + \bar{z} \bar{y} g c = \bar{y} e y c + \bar{z} a c g \\
3323 & : y z e g + \alpha y b e + x z b \bar{f} + \gamma y e b + y \gamma b c = \bar{y} \gamma c b + \bar{y} y c g \\
3331 & : \alpha z g g + \bar{x} y f e + \beta \bar{z} f f + \bar{y} y g b + \gamma z d d + \bar{z} y f c = \bar{y} x c e + \beta \beta c f + \gamma z c c \\
3332 & : y z b g + \alpha y e e + x z \bar{e} \bar{f} + \gamma y b b + \bar{y} y e c = \bar{y} a c e + \bar{z} x c f + \gamma y c c \\
7 & \text{Unknowns} \\
1213 & : \alpha y g g + y y f e + x a g b + z \bar{x} f c + \beta y d d + \gamma y f f \\
& \quad = x \gamma b g + \alpha y e e + \beta y b b + \bar{x} z e c + y y e f \\
1231 & : \alpha y g g + y a d e + x a g b + z \bar{x} d c + \beta y f d + \gamma y d f \\
& \quad = x \beta b g + z \bar{x} b b + y \beta e d \\
1312 & : \alpha y g g + \bar{y} \alpha f e + x a g b + z x f c + \beta y d d + \gamma y f f \\
& \quad = \bar{x} \gamma b g + \alpha y e e + \beta y b b + x \bar{z} e c + \bar{y} y e f
\end{align*}\]
1332: \( \alpha xg\bar{g} + x\alpha d\bar{e} + \beta x\bar{d}f + y\alpha gb + \gamma xf d + z\bar{y}dc \)
\[ = y\gamma bg + z\bar{y}bb + x\gamma \bar{e}d \]
2212: \( \alpha yag + \bar{y}oge + \bar{x}oab + \bar{z}xgc + \beta y\bar{g}d + \gamma \bar{y}gf \)
\[ = \bar{x}oba + \alpha \bar{x}eb + \beta \bar{x}b\bar{e} + \bar{z}ybe + \bar{y}\alpha eg \]
3112: \( \alpha y\bar{g}g + \bar{y}ode + \bar{x}o\bar{g}b + \bar{z}xdc + \beta \bar{y}fd + \gamma \bar{y}df \)
\[ = \bar{x}\beta \bar{b}g + \bar{z}xbb + \bar{y}\beta ed \]
3132: \( \alpha x\bar{g}g + x\bar{a}f\bar{e} + \beta x\bar{f}f + y\alpha gb + \gamma xdd + z\bar{y}fc \)
\[ = y\beta bg + ax\bar{e}e + x\beta \bar{e}f + \gamma xbd + \gamma \bar{z}ec \]
3213: \( \alpha x\bar{g}g + x\bar{a}fe + \beta x\bar{f}f + y\alpha gb + \gamma x\bar{d}d + z\bar{y}fc \)
\[ = y\beta bg + \alpha x\bar{e}e + \bar{x}\beta \bar{e}f + \gamma xbd + \bar{y}\z\bar{e}c \]
8 Unknowns
1221: \( \bar{x}y\bar{e}g + a\alpha be + \beta a\bar{e}b + x\bar{x}bc + \bar{y}ybf \)
\[ = \bar{x}eba + \alpha \beta eb + \beta \beta be + z\bar{z}be + \bar{y}\bar{x}eg \]
1222: \( \alpha yag + yoge + xoab + z\bar{x}gc + \beta y\bar{g}d + \gamma ygf \)
\[ = \bar{x}oba + \alpha xeb + \beta xbe + \bar{y}ge + \alpha eg \]
1321: \( \alpha \bar{x}ga + \bar{y}\beta fb + x\bar{z}db + \bar{x}\beta g\bar{e} + \beta x\bar{d}g + \bar{y}\bar{z}ge + \beta x\bar{f}g \)
\[ = \bar{x}\gamma eg + \alpha ybe + \beta \bar{y}eb + \bar{x}zbc + \bar{y}\gamma bf \]
1323: \( \alpha yga + \bar{y}zfb + x\gamma \bar{d}b + \bar{x}z\bar{g}e + \beta \bar{y}dg + \bar{y}\gamma ge + \gamma yfg \)
\[ = y\gamma eg + \bar{z}yeb + x\gamma bd \]
2112: \( x\bar{y}eg + a\alpha be + \beta a\bar{e}b + \bar{x}zbc + \bar{y}ybf \)
\[ = \bar{x}eba + \alpha \beta eb + \beta \beta be + \bar{z}zbe + \bar{y}\bar{x}eg \]
2113: \( \alpha xga + y\beta fb + \bar{x}z\bar{d}b + x\beta ge + \beta x\bar{d}g + \bar{y}zge + \gamma xfg \)
\[ = x\gamma eg + \alpha ybe + \beta \bar{y}eb + \bar{x}zbc + \gamma ybf \]
2122: \( \alpha xaa + y\beta gb + \bar{x}zgb + x\beta a\bar{e} + \beta x\bar{g}g + \bar{y}zae + \gamma xgg \)
\[ = xa\alpha a + \alpha xbb + \beta x\bar{e}e + \bar{z}y\bar{e}e + \alpha ybg \]
2131: \( \alpha x\bar{g}a + y\beta db + \bar{x}z\bar{f}b + x\beta ge + \beta x\bar{fg} + \bar{y}z\bar{ge} + \gamma x\bar{dg} \)
\[ = \beta \bar{e}g + \bar{z}x\bar{e}b + y\beta bd \]
2221: \( \alpha \bar{x}aa + y\beta gb + x\bar{z}gb + \bar{x}\beta a\bar{e} + \beta \bar{x}gg + \bar{y}zae + \gamma xgg \)
\[ = \bar{x}a\alpha a + \alpha x\bar{bb} + \beta \bar{x}\bar{e}e + \bar{z}y\bar{ee} + \alpha ybg \]
2223: \( \alpha yaa + \bar{y}zgb + x\gamma \bar{g}b + \bar{x}z\bar{ae} + \beta y\bar{g}g + \gamma y\bar{ae} + \gamma ygg \)
\[ = y\alpha a + \alpha ybb + \bar{z}x\bar{e}e + x\alpha bg + \gamma yee \]
respectively the arguments

It is understood that the first, second, third and fourth factors in each terms have respectively the arguments \(u, v, u-v\) and \(u+v\). Eqs. (1133, 3311) are automatically
satisfied. We numbered equations by $(ijkl)$, where $ij$ denotes the rows and $kl$ represents the columns of the matrix reflection equation in tensor notation. Equations are sorted out according to the number of $\alpha, \beta, \gamma, x, y, z, \bar{x}, \bar{y},$ and $\bar{z}$ variables.

Diagonal solutions for $K_-(u)$ have been obtained in [13]. In this case, it is sufficient to solve Eqs.(1221,1331,1132) only. It turns out that there are three solutions, being $K_-(u) = 1$, $K_-(u) = \Gamma^+(u)$ and $K_-(u) = \Gamma^-(u)$, with

$$
\Gamma^\pm(u) = \begin{pmatrix} B_\pm(u) & 1 \\ G_\pm(u) & \end{pmatrix},
$$

(11)

where

$$
B_\pm = \frac{2 + b^\pm_1 (1-e^{-u})}{2 + b^\pm_1 (1-e^u)}, \quad b^\pm_1 = \frac{2 \left(-1 \pm i e^{-3\eta}\right)}{1 + e^{-6\eta}},
$$

(12)

$$
G_\pm = \frac{2 - g^\pm_1 (1-e^u)}{2 - g^\pm_1 (1-e^{-u})}, \quad g^\pm_1 = \frac{2 \left(1 \pm i e^{3\eta}\right)}{1 + e^{6\eta}}.
$$

These solutions have no free parameters. By the automorphism (9), three solutions for $K_+(u)$ follow.
III. Solutions

Now we solve the $9 \times 9$ equations listed in the previous section. The basic method of solving which we will use is the same as that used when solving the Yang-Baxter equation in a direct way\cite{5}. The known solutions of $K$-matrix up to now are all obtained by this method. The way is, first divide equations by $\alpha(u)$, then take the derivative with respect to the variable $v$ and finally set $v$ to zero. This gives functional equations involving the variable $u$ only among the elements of $K$-matrix, which we need to solve.

We suppose that the solutions to be proportional to identity when the spectral parameter $u$ is zero.

$$X(0) = \tilde{X}(0) = Y(0) = \tilde{Y}(0) = Z(0) = \tilde{Z}(0) = 0, \quad B(0) = G(0) = 1.$$ (13)

We introduce the following functions normalized by $\alpha(u)$

$$X(u) = \frac{x(u)}{\alpha(u)}, \quad Y(u) = \frac{y(u)}{\alpha(u)}, \quad Z(u) = \frac{z(u)}{\alpha(u)}, \quad B(u) = \frac{\beta(u)}{\alpha(u)}, \quad G(u) = \frac{\gamma(u)}{\alpha(u)},$$ (14)

and define their first derivatives at $u = 0$ as follows.

$$x_1 = X'(u)|_{u=0}, \quad y_1 = Y'(u)|_{u=0}, \quad z_1 = Z'(u)|_{u=0},$$ (15)

To begin with, let us suppose that all the elements of $K$-matrix in Eq.(10) are non-zero and see what happens. Following the procedure described above, we get relations between $Y(u), X(u)$ from Eq.(1232) and $\tilde{Y}(u), \tilde{X}(u)$ from Eq.(3212) for the 2 unknowns.

$$Y(u) = \frac{-x_1bg + y_1ed}{x_1ed - y_1bg}X(u),$$ (16)

$$= \frac{e^{-\eta + u}(x_1 - e^u x_1 + e^{2\eta}y_1 + e^{\eta + u}y_1)}{e^{2\eta}x_1 + e^u x_1 - e^{3\eta}y_1 + e^{3\eta + u}y_1}X(u),$$
\[
\tilde{Y}(u) = \frac{-x_1 b \tilde{g} + y_1 \tilde{e} d}{\bar{x}_1 e d - \bar{y}_1 b g} \tilde{X}(u),
\]
\[
= e^{-\eta+u}(\bar{x}_1 - e^u \bar{x}_1 + e^{3\eta} \bar{y}_1 + e^{\eta+u} \bar{y}_1) \bar{X}(u).
\] (17)

Next we go to the equations involving three unknowns. These equations can be used to express \(Z(u)\) and \(\bar{Z}(u)\) in terms of \(X(u)\) and \(\bar{X}(u)\) respectively. Eqs.(1123, 1132, 1233, 2133) give the same \(Z(u)\) and Eqs.(2311, 3211, 3312, 3321) give the same \(\bar{Z}(u)\).

\[
Z(u) = \frac{(e^{2\eta} + e^{2u}) \bar{z}_1}{e^{2\eta} \bar{x}_1 + e^{u} \bar{x}_1 - e^{3\eta} \bar{y}_1 + e^{3\eta+u} \bar{y}_1} \bar{X}(u),
\]
\[
\bar{Z}(u) = \frac{(e^{2\eta} + e^{2u}) \bar{z}_1}{e^{2\eta} \bar{x}_1 + e^{u} \bar{x}_1 - e^{3\eta} \bar{y}_1 + e^{3\eta+u} \bar{y}_1} \bar{X}(u).
\] (18) (19)

Until now, the equations involved \(X, Y, Z\) and their barred partners separately. However, the equations for 4 unknowns begin to connect the unbarred and barred variables. We use these equations to determine \(\bar{X}(u)\) from \(X(u)\). It turns out that five equations give the same simple relations between \(\bar{X}(u)\) and \(X(u)\)

\[
\bar{X}(u) = \frac{\bar{x}_1}{x_1} X(u),
\] (20)
if the following relations holds.

\[
\bar{y}_1 x_1 = \bar{x}_1 y_1.
\] (21)

Eq.(21) also give some simplifications to already obtained relations.

Now we have relations among off-diagonal elements of \(K\)-matrix which can be completely determined once we know \(X(u)\). To determine \(X(u)\) we turn to equations involving 6 unknowns, skipping equations involving 5 unknowns. There are sufficient number of equations for 6 unknowns to determine \(\bar{X}(u)\). However, the answer is rather unexpected in the sense that there is no solutions with every elements of \(K\)-matrix non-vanishing. Let us see how this happens.

First we notice that from the paired equations like (1121, 2111) we get the following relations.

\[
\bar{z}_1 = \frac{\bar{x}_1 \bar{x}_1}{x_1 x_1} \bar{z}_1.
\] (22)

Among 28 equations, we use Eqs.(1211, 2111) to express \(B(u)\) in terms of \(X(u)\). Two equations give two different expressions for the same \(B(u)\) function.

\[
B^1(u) = \frac{1}{x_1 e b} \left(-X \bar{z}_1 \bar{e} g - \bar{x}_1 b e - Y \bar{z}_1 b f + \bar{Z} x_1 c g + 2 \bar{X}((b, c))\right),
\] (23)
\[ B2(u) = \frac{1}{\bar{x}_1(bb - cc)} \left( -X \bar{z}_1 bg - \bar{x}_1 ee - \bar{X} b_1 ec - Y \bar{z}_1 ef + \bar{Z} y_1 cf + 2 \bar{X}(\langle e, c \rangle) \right), \]  

(24)

where we introduced Wronskian of two functions as follows.

\[ ((f, g)) = f'g - fg'. \]  

(25)

Equating these two different expressions, we can get an expression for \( X(u) \). Similarly we use the equations (2333,3233) to get another expression for \( X(u) \) through \( G(u) \) function.

\[ G1(u) = \frac{1}{y_1 \bar{e}b} \left( -\bar{Y} z_1 \bar{e}g - y_1 \bar{b} \bar{e} - \bar{X} z_1 b \bar{f} + Z y_1 c \bar{g} + 2 Y'((b, c)) \right), \]  

(26)

\[ G2(u) = \frac{1}{y_1(bb - cc)} \left( -\bar{Y} z_1 \bar{b} \bar{g} - y_1 \bar{e} \bar{e} - Y g_1 \bar{e} c - \bar{X} z_1 \bar{e} \bar{f} + Z \bar{x}_1 c \bar{f} + 2 Y'((\bar{e}, c)) \right). \]  

(27)

So we have two different expressions for the same \( X(u) \) function. Requiring these to be identical, we get important information about the solutions. There are two possibilities.

\[ \bar{x}_1 z_1 = 0, \quad y_1 = e^{-\eta} x_1, \quad b_1 \ x_1 = 0, \quad g_1 \ x_1 = 0. \]  

(28)

\[ \bar{x}_1 z_1 = 0, \quad y_1 = -e^{-\eta} x_1, \quad b_1 = -\frac{4}{e^{2\eta} + 1}, \quad g_1 = \frac{4e^{2\eta}}{e^{2\eta} + 1}. \]  

(29)

It should be noted that the first derivatives of each elements of \( K \)-matrix were assumed as non-zeros. So the conclusion is, no solution of \( K \)-matrix with every components non-vanishing.

Now we consider the cases when some of the elements of \( K \)-matrix vanish. When \( \bar{x}_1 \) is non-zero, above two conditions are still valid so \( z_1 \) must be zero. When \( z_1 = 0 \), the four equations(1131, 1223, 1333, 2132) imply \( X(u) = Y(u) = 0 \). So in this case, only \( \bar{X}(u), \bar{Y}(u) \) and \( \bar{Z}(u) \) can be non-zero. The remaining possibility is only the case when \( \bar{x}_1 \) itself vanishes. Eq.(3212) for 2 unknowns means \( \bar{Y} \) must also vanish. Moreover, we can easily see that Eqs.(2212,1312,3213,3231,3112) for 7 unknowns imply either \( \bar{Z} = 0 \) or that \( X(u) = Y(x) = 0 \). So in this case, there are two possibilities, non-zero \( X(u), Y(u), Z(u) \) or non-zero \( \bar{Z}, Z(u) \).
In summary, there are three possibilities. Case (I): non-zero \( Z(u) \), \( \bar{Z}(u) \), Case (II): non-zero \( X(u) \), \( Y(u) \), \( Z(u) \), Case (III): non-zero \( \bar{X}(u) \), \( \bar{Y}(u) \), \( \bar{Z}(u) \). For any cases, \( B(u) \), \( G(u) \) are non-zero. Let me first consider the case (I).

Case (I): It is easy to see that \( \bar{Z}(u) \) should be proportional to \( Z(u) \) eg., from Eq.(1111).

\[
\bar{Z}(u) = \frac{\bar{z}_1}{z_1} Z(u). \tag{30}
\]

It is useful to start from the Eqs.(2112,2332) to express the \( B(u) \) and \( G(u) \) in terms of \( Z(u) \). They give

\[
B(u) = \frac{2((\bar{e}, b)) - \bar{z}_1 Z(u) b\bar{e} - b_1 \bar{e} b}{2((\bar{e}, b)) + b_1 \bar{b}\bar{e}} \tag{31}
\]

\[
= \frac{2 - (b_1 + \bar{z}_1 Z(u))(-1 + e^{-u})}{2 + b_1 (1 - e^u)},
\]

\[
G(u) = \frac{2((\bar{e}, b)) - \bar{z}_1 Z(u) b\bar{e} - g_1 \bar{e} b}{2((\bar{e}, b)) + g_1 \bar{b}\bar{e}} \tag{32}
\]

\[
= \frac{2 - (g_1 + \bar{z}_1 Z(u))(1 - e^u)}{2 + g_1 (e^{-u} - 1)}.
\]

From Eq.(2123) for 5 unknowns, we get the following relation.

\[
B(u)z_1 \bar{e}\bar{e} - G(u)z_1 ee = Z(u)(b_1 - g_1) e\bar{e} + 2Z(u)((\bar{e}, e)). \tag{33}
\]

Inserting the expressions for \( B(u) \) and \( G(u) \) in Eqs.(31,32) into Eq.(33), we can get an expression for \( Z(u) \).

\[
Z(u) = \frac{2 (-1 + e^u)(1 + e^u) \bar{z}_1}{ZF}, \tag{34}
\]

where

\[
ZF = 4e^u + 2b_1(e^u - 2e^{2u}) + 2g_1(1 - e^u) + b_1g_1(1 - 2e^u + e^{2u}) + \bar{z}_1 z_1(-1 + 2e^u - e^{2u}). \tag{35}
\]

Inserting back this into Eqs.(31,32), we obtain \( B(u) \) and \( G(u) \) free of \( Z(u) \). We determined the essential form of the solutions.

Now we have to see what constraints must be made on the parameters. Only one equation is sufficient to determine it. Let us use Eq.(2211) for 6 unknowns.

\[
\bar{z}_1 ag + \bar{Z}b_1 gc - 2\bar{Z}((g, c)) + B\bar{z}_1 \bar{g}d + G\bar{z}_1 gf = 0. \tag{36}
\]
From above equation, we can get
\[ g_1 = e^{2\eta}b_1, \quad \bar{z}_1 = e^{2\eta}b_1^2 z_1. \] (37)

In all, the solutions for the case (I) is the following.
\begin{align}
Z(u) &= \frac{(-1 + e^{2u}) z_1}{2e^u + b_1(e^{2\eta} + e^u - e^{2\eta} - e^{2u})}, \quad \text{(38)} \\
\bar{Z}(u) &= \frac{e^{2\eta}(-1 + e^{2u})b_1^2}{(2e^u + b_1(e^{2\eta} + e^u - e^{2\eta} - e^{2u})) z_1}, \\
B(u) &= \frac{2e^u + b_1(-1 + e^{2\eta} + e^u - e^{2\eta} + u)}{2e^u + b_1(e^{2\eta} + e^u - e^{2\eta} - e^{2u})}, \\
G(u) &= \frac{e^u(2 + b_1(1 - e^{2\eta} - e^u + e^{2\eta} + u))}{2e^u + b_1(e^{2\eta} + e^u - e^{2\eta} - e^{2u})}. \end{align}

\( b_1 \) and \( z_1 \) are the two free parameters. If \( b_1 \) is zero, \( g_1, z_1, \bar{z}_1 \) also vanish so that the solution reduces to identity. On the other hand, if \( z_1 \) is zero, so is \( \bar{z}_1 \). Then this becomes diagonal situation.

Case (II): In this case, it is quite useful to notice that the Eqs.(1221-2112, 2332-3223, 2231-3122, 1322-2213,1331-3113) which are relevant for the diagonal solutions remain intact. So the diagonal part for this case is the same as the diagonal solutions. Now it is sufficient to find out the \( X(u), Y(u) \) and \( Z(u) \). We use Eqs.(1112, 1121) for 6 unknowns to determine \( X(u) \). Eq.(1121) leads to
\[ X(u) = \frac{x_1}{2((b, c))}(be + B(u)eb). \] (39)

Eq.(1112) leads to
\[ X(u) = \frac{x_1}{b_1 ec + 2((c, e))}(B(u)(cc - bb) - ee). \] (40)

To determine \( Y(u) \), Eqs.(2333 or 3233) can be used. Eq.(2333) leads to
\[ Y(u) = \frac{y_1}{2((b, c))}(b\bar{e} + G(u)eb). \] (41)

Eq.(3233) leads to
\[ Y(u) = \frac{y_1}{g_1 \bar{e}c + 2((c, \bar{e}))}(G(u)(cc - bb) - \bar{e}\bar{e}). \] (42)
Above two different expressions for $X(u), Y(u)$ give the same result.

For $b_1 = g_1 = 0$, $X(u), Y(u)$ reduce to the following.

$$X(u) = \frac{(-1 + e^u) (1 + e^u) x_1}{2 e^u}, \quad (43)$$

$$Y(u) = \frac{(-1 + e^u) (1 + e^u) y_1}{2 e^u}. \quad (44)$$

For non-trivial diagonal solutions, $X(u), Y(u)$ become

$$X(u) = \frac{(\pm i + e^{3\eta}) (-1 + e^u) (1 + e^u) x_1}{2e^u (\pm i + e^{3\eta+u})}, \quad (45)$$

where $\pm$ corresponds to two non-trivial diagonal solutions in Eq.(11).

$$Y(u) = \frac{(\pm i + e^{3\eta}) (-1 + e^u) (1 + e^u) y_1}{2 (\pm i + e^{3\eta+u})}. \quad (46)$$

To determine the possible conditions on $x_1$ and $y_1$, we use Eq.(1232) for the 2 unknowns. It gives the following constraints.

$$y_1 = e^{-\eta} x_1, \quad (47)$$

for Eqs.(13, 14) and

$$y_1 = \pm i e^{-2\eta} x_1, \quad (48)$$

for Eqs.(15, 16) irrespective of the chosen $b_1, g_1$ in Eq.(12). So there are four combinations to consider, which we denote $(++)$, $(+−)$, $(-+)$, $(-−)$, where the first sign refers to $b_1, g_1$ and the second sign refers to $y_1$.

Let us turn to $Z(u)$. Eqs.(1123, 1132, 1233, 2133) give the same result for each different choices of $y_1$ and $b_1, g_1$. For the trivial diagonal solutions, $Z(u)$ becomes

$$Z(u) = \frac{(-1 + e^u) (1 + e^u) (e^{2\eta} + e^{2u}) z_1}{2 e^{2u} (1 + e^{2\eta})}, \quad (49)$$

for the $y_1$ in Eq.(17). For the non-trivial diagonal solutions, $Z(u)$ becomes

$$Z(u) = \frac{(-1 \pm i e^{\eta} + e^{2\eta}) (e^\eta \mp i e^u) (-1 + e^u) (1 + e^u) z_1}{2 e^u (\pm i + e^{3\eta+u})}, \quad (50)$$
for the 

\[(++)\], 

\[(-+)\] choices and 

\[Z(u) = \frac{(\mp i + e^\eta)(-1 \pm ie^\eta + e^{2\eta})(e^\eta \pm i e^u)(-1 + e^u)(1 + e^u)z_1}{2e^{\eta}(\pm i + e^\eta)(\pm i + e^{3\eta+u})}\]  \hfill (51)

for the 

\[(-+)\], 

\[(++)\] choices.

We need to determine the possible conditions on \(z_1\). We use Eqs.(1223 or 2132). For \(Z(u)\) in Eq.(49), there does not exist consistent solution for \(z_1\), \(ie.\) \(z_1, x_1\) must be zero. For the \((++)\), 

\[(-+)\] choices,

\[z_1 = \frac{(\mp i + e^\eta)(1 \mp ie^\eta - e^{2\eta})x_1^2}{2e^{\eta}(\pm i + e^\eta)}\]  \hfill (52)

For the \((+-)\), \((--\) choices, it turns out there is no non-trivial solution for \(z_1\), \(ie.\) \(z_1, x_1\) must vanish. This is the end of the Case(II). The result is the following for the sign choices \((++)\), 

\[(-+)\].

\[X(u) = \frac{(\mp i + e^\eta)(-1 \pm ie^\eta + e^{2\eta})(-1 + e^u)(1 + e^u)x_1}{2e^{\eta}(\pm i + e^{3\eta+u})},\]

\[Y(u) = \frac{(1 \pm ie^\eta)(-1 \pm ie^\eta + e^{2\eta})(-1 + e^u)(1 + e^u)x_1}{2e^{2\eta}(\pm i + e^{3\eta+u})},\]

\[Z(u) = \frac{e^{-3\eta-u}(\mp i + e^\eta)(-1 \pm ie^\eta + e^{2\eta})^2(1 - e^u)(e^\eta \mp ie^u)(1 + e^u)x_1^2}{4(\pm i + e^\eta)(\pm i + e^{3\eta+u})},\]  \hfill (53)

\[B(u) = \frac{(e^{3\eta} \pm ie^u)}{e^{\eta}(\pm i + e^{3\eta+u})},\]

\[G(u) = \frac{e^{\eta}(e^{3\eta} \pm ie^u)}{(\pm i + e^{3\eta+u})},\]

The solutions which we found satisfies all \(9 \times 9\) equations.

Case(III): This case can be solved in the same way as in the case (II) since the equations for this case is exactly the same. We have only to replace every unbarked variables with barred ones.

\[\bar{X}(u) = \frac{(\mp i + e^\eta)(-1 \pm ie^\eta + e^{2\eta})(-1 + e^u)(1 + e^u)x_1}{2e^{\eta}(\pm i + e^{3\eta+u})},\]

\[\bar{Y}(u) = \frac{(1 \pm ie^\eta)(-1 \pm ie^\eta + e^{2\eta})(-1 + e^u)(1 + e^u)x_1}{2e^{2\eta}(\pm i + e^{3\eta+u})},\]

\[\bar{Z}(u) = \frac{e^{-3\eta-u}(\mp i + e^\eta)(-1 \pm ie^\eta + e^{2\eta})^2(1 - e^u)(e^\eta \mp ie^u)(1 + e^u)x_1^2}{4(\pm i + e^\eta)(\pm i + e^{3\eta+u})},\]  \hfill (54)
\[ B(u) = \frac{e^{3\eta} \pm ie^u}{e^u(\pm i + e^{3\eta + u})}, \]
\[ G(u) = \frac{e^u(e^{3\eta} \pm ie^u)}{(\pm i + e^{3\eta + u})}. \]
IV. Discussions

The most general solutions of $K$-matrix to be determined from reflection equation for known $R$-matrix are highly desired. Obviously it would be best to obtain them in algebraic way. But until now, only small number of solutions are known for small size $R$-matrices in a direct way as in the present paper.

We solved the reflection equation for the quantum Mikhailov-Shabat model. This model is known to have two non-trivial diagonal solutions as well as trivial diagonal solution. The most general solutions which we found consist of three families corresponding to each diagonal solutions. One family of solutions depends on two arbitrary parameters. If one of the parameters vanishes, the other must also vanish so that the solutions reduces to trivial identity solution. The other two families for each non-trivial diagonal solutions have only one arbitrary parameter.

This solution can be used to compute the Hamiltonian including the boundary terms for the corresponding open spin chains along the lines in Ref.[4]. It can also be used to calculate the boundary $S$-matrix of the model along the lines in Ref.[2] using the $K$-matrix and the bulk $S$-matrix. The bulk $S$-matrix for this model was obtained in Ref.[17] from the $R$-matrix.

On the other hand, the quantum Mikhailov-Shabat model has Lagrangian description in terms of single scalar field. The classical boundary $S$-matrix can also be computed in the way pursued in Ref.[18]. It would be nice if we can see whether they give the consistent result.

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