Analysis of level-dependent subdivision schemes near extraordinary vertices and faces

Costanza Conti · Marco Donatelli · Paola Novara · Lucia Romani

Received: date / Accepted: date

Abstract Convergence and smoothness analysis of a bivariate level-dependent (non-stationary) subdivision scheme for 2-manifold meshes with arbitrary topology is still an open issue. In this paper we focus on the problem of analyzing convergence and tangent plane continuity - also known as $G^1$-continuity - of non-stationary subdivision schemes. Exploiting ideas from the theory of asymptotically equivalent subdivision schemes, we derive new sufficient conditions for establishing $G^1$-continuity of any non-stationary subdivision surface at the limit points of extraordinary vertices and/or extraordinary faces.

Keywords Non-stationary subdivision · Extraordinary vertex/face · Convergence · Tangent plane ($G^1$) continuity

Mathematics Subject Classification (2010) 26A15 · 68U07

1 Introduction

This paper provides a general procedure to check convergence of level-dependent subdivision schemes in the neighborhood of an extraordinary vertex/face. It also gives sufficient conditions for the limit surface to be tangent plane continuous at the limit point of an extraordinary vertex/face. To the best of our knowledge the only contribution in this domain...
is the work of Jena et al. in [21] where the behaviour of the limit surfaces generated by a specific non-stationary subdivision scheme in the neighborhood of an extraordinary face is studied. The specific scheme is a generalization of the celebrated Doo-Sabin’s proposal [13].

The sufficient conditions we propose are used for the analysis of the family of approximating non-stationary subdivision schemes presented in [17]. The members of the latter family are a generalization of exponential spline surfaces to quadrilateral meshes of arbitrary topology whose tangent plane continuity is conjectured and shown only by numerical evidence in [17, Section 5].

Due to the lack of existing theoretical results for the analysis of level-dependent subdivision schemes, we believe that our contribution could mark a first step forward towards a deeper understanding of non-stationary subdivision with a consequent increase of its use in different fields of application.

1.1 Motivation

Non-stationary subdivision schemes were introduced in the last 10 years with the aim of enriching the class of limit functions of stationary schemes and have very different and distinguished properties. Indeed, it is well-known that stationary subdivision schemes are not capable of generating circles, ellipses, or to deal with level-dependent tension parameters that allow to arbitrarily modify the shape of a subdivision limit. Non-stationary schemes generate function spaces that are much richer. For example, in the univariate case, they include exponential B-splines or \( C^\infty \) limits with bounded support as the Rvachev-type function (see, e.g., [16]). Since the support size of the subdivision limit is dominated by the effect of the first few refinement steps while the smoothness by the last ones, a narrow support and a high level of derivative continuity can be achieved by using non-stationary schemes. The generation capability of level-dependent schemes (especially the capability of generating exponential-polynomials) is important in several applications, e.g., in biological imaging [1, 8, 10, 12, 31], geometric design-approximation [11, 23, 24, 29, 34] and in isogeometric analysis [20]. Level-dependent subdivision schemes include Hermite schemes that do not only model curves and surfaces, but also their gradient fields (such schemes are used in geometric modelling and biological imaging, see e.g. [5, 6, 8, 22, 28]). Additionally, non-stationary wavelet and frame constructions are level adapted and more flexible [9, 14, 19, 33]. Unfortunately, in practice, the use of subdivision is mostly restricted to the class of stationary subdivision schemes even though the non-stationary ones are equally relatively simple to implement and highly intuitive in use: from an implementation point of view changing coefficients with the levels is not a crucial matter also in consideration of the fact that, in practice, only few subdivision iterations are performed. On the contrary, a crucial limitation to the spread of level-dependent schemes, is a lack of general analysis methods, especially methods for their convergence and regularity analysis. This motivates our study.

1.2 Subdivision framework

Subdivision schemes are efficient iterative algorithms to produce smooth surfaces as the limit of a recursive process starting from a given coarse 2-manifold polygon mesh (a polygon mesh is considered to be 2-manifold if it does neither contain non-manifold edges,
non-manifold vertices, nor self-intersections, see [18]). Each step of the recursive process produces a finer 2-manifold polygon mesh than the original one, containing many more vertices and polygonal faces. Vertices and faces of a polygon mesh are identified by the so-called vertex valence and face valence, respectively. While the valence of a vertex is the number of edges incident to it, the valence of a face counts the number of edges that delimit it. For a quadrilateral mesh, vertices and faces of valence 4 are called regular. Differently, for a triangular mesh regular vertices are the ones with valence 6 while regular faces have valence 3. A regular mesh is a mesh that contains regular vertices and regular faces only. Non regular vertices and faces are called extraordinary vertices/faces (see Figure 1 for a graphical illustration of these two cases) and, whenever they appear, the mesh is said to be of arbitrary topology. It is evident that to ensure the convergence of a subdivision scheme to a (tangent plane) continuous limit surface, the refinement rules to be applied in the neighborhood of extraordinary vertices/faces have to be different from the ones used in the regular regions, since they must strictly depend on the vertex/face valence (see, e.g., [26]). The refinement rules to be used in the regular regions and in the neighbourhood of extraordinary vertices/faces may change with the refinement level or not. In the latter case the subdivision scheme is called stationary, non-stationary or level-dependent otherwise. A known analysis tool to investigate convergence and smoothness properties of stationary subdivision schemes for regular meshes is the one proposed by Dyn and Levin in [16]. To study convergence and smoothness of a non-stationary subdivision scheme for regular meshes, Dyn and Levin [15] proposed a method based on the comparison with a stationary scheme whose regularity properties are known.

In the case of meshes with arbitrary topology, we are currently able to study only convergence and regularity of stationary subdivision schemes near extraordinary vertices/faces, thanks to the results in [26, 27, 32, 35, 36]. However, in literature we can find no general results to analyze level-dependent subdivision schemes at extraordinary vertices/faces. To the best of our knowledge, the only contribution in this domain is the work of Jena et al. in [21] where a specific scheme is considered.

Therefore, the goal of our paper is to propose a general procedure to check if a non-stationary subdivision scheme is convergent in the neighborhood of an extraordinary vertex/face. Moreover, it also aims at giving sufficient conditions for the limit surface to be tangent plane continuous at the limit point of an extraordinary vertex/face.

The paper is organized as follows. In Section 2 we provide preliminaries on bivariate subdivision schemes. Then, in Section 3, we prove new results concerning non-stationary subdivision schemes. Next, in Section 4 (specifically, Subsection 4.2) sufficient conditions for proving convergence of a non-stationary subdivision scheme in correspondence to extraordinary vertices/faces are considered. Finally, in Subsection 4.3 we give sufficient conditions to verify if the limit surfaces generated by an arbitrary convergent, non-stationary subdivision scheme are tangent plane continuous at the limit points of extraordinary elements. Some application examples of the derived conditions are shown in Section 5.

2 Preliminaries on bivariate subdivision schemes

A bivariate subdivision scheme is an iterative method capable of producing a smooth surface starting from a given coarse polygonal mesh. Unless explicitly specified, we consider level-dependent subdivision schemes, non-stationary subdivision schemes, alternatively.
In the following we denote by $S$ a subdivision scheme for meshes of arbitrary topology. For any given initial mesh $\mathcal{M}^{(1)}$ of arbitrary topology, we denote by $\mathcal{R}^{(1)}$ and $\mathcal{E}^{(1)}$ the submeshes of $\mathcal{M}^{(1)}$ that determine the behaviour of the limit surface on the one-ring of a regular vertex and an extraordinary element (vertex or face), respectively. The submesh $\mathcal{R}^{(1)}$ is also called the neighborhood of a regular vertex. Similarly, the submesh $\mathcal{E}^{(1)}$ is also called the neighborhood of an extraordinary element.

The action of $S$ on $\mathcal{R}^{(1)}$, defined by 3D control points having components $\{f^{(k)}_{\alpha} \in \mathbb{R}, \alpha \in \mathbb{Z}^2\}, k \in \mathbb{N}$, can be described either by the componentwise application of the refinement rules

$$f^{(k+1)}_{\alpha} = \sum_{\beta \in \mathbb{Z}^2} c^{(k)}_{\alpha-2\beta} f^{(k)}_{\beta}, \quad k \in \mathbb{N}, \alpha \in \mathbb{Z}^2,$$

or, equivalently, by the application of the sequence of subdivision operators $\{\mathcal{S}^{(k)}_{c}, k \in \mathbb{N}\}$, mapping componentwise the vector $f^{(1)} = \{f^{(1)}_{\alpha}, \alpha \in \mathbb{Z}^2\} \in \ell(\mathbb{Z}^2)$ with the initial control points into the corresponding vector of level $k+1$, i.e.,

$$f^{(k+1)} = \mathcal{S}^{(k)}_{c} \mathcal{S}^{(k-1)}_c \ldots \mathcal{S}^{(1)}_c f^{(1)}.$$

(In the sequel we use $k \geq 1$ instead of $k \in \mathbb{N}$, omitting the trivial information that the refinement level is always assumed to be an integer).

The coefficients in (1) can be conveniently collected in the so-called $k$-th level subdivision mask

$$c^{(k)} = \{c^{(k)}_{\alpha}, \alpha \in \mathbb{Z}^2\},$$

or incorporated in the $k$-th level subdivision symbol

$$c^{(k)}(z) = \sum_{\alpha \in \mathbb{Z}^2} c^{(k)}_{\alpha} z^\alpha, \quad z \in (\mathbb{C}\setminus\{0\})^2.$$

The notation $\|\mathcal{S}^{(k)}_{c}\|_\infty$ is for the norm of the operator $\mathcal{S}^{(k)}_{c}$, i.e.,

$$\|\mathcal{S}^{(k)}_{c}\|_\infty := \max \left\{ \sum_{\beta \in \mathbb{Z}^2} |c^{(k)}_{\alpha-2\beta}| : \alpha \in \{(0,0),(0,1),(1,0),(1,1)\} \right\}.$$ 

In conclusion, when applied on $\mathcal{R}^{(1)}$, a subdivision scheme $S$ can be equivalently identified by the sequence of subdivision operators $\{\mathcal{S}^{(k)}_{c}, k \geq 1\}$, by the sequence of subdivision masks $\{c^{(k)}, k \geq 1\}$ or by the sequence of associated subdivision symbols $\{c^{(k)}(z), k \geq 1\}$. 

---

Fig. 1 Example of quadrilateral mesh containing an extraordinary face (left) and of triangular mesh containing an extraordinary vertex (right).
Instead, in \( E^{(1)} \) the subdivision rules relating the vertices of the \( k \)-th level mesh with those of the next level \( k + 1 \) are encoded in the rows of a local subdivision matrix \( S_k \). Thus, in the neighborhood of an extraordinary element the action of the subdivision scheme \( S \) is described by a sequence of local subdivision matrices \( \{S_k, k \geq 1\} \).

**Remark 2.1.** When the valence of the extraordinary element reduces to the regular value, the local subdivision matrix \( S_k \) provides another alternative way to represent a subdivision step in the regular case.

In the stationary setting we will use the notation \( \bar{S} \) to refer to a subdivision scheme that is not level-dependent. Hence, it will be identified by

- a subdivision operator, say \( \mathcal{S}_c \), a subdivision mask \( c \) or an associated subdivision symbol \( c(z) \), when applied in \( \mathcal{R}^{(1)} \);
- a local subdivision matrix \( S \), when applied in \( E^{(1)} \).

**2.1 Preliminaries for studying convergence of subdivision schemes in regular submeshes**

In the following, after recalling some well-known definitions, we present several useful results dealing with the convergence of a non-stationary subdivision scheme in \( \mathcal{R}^{(1)} \) (see [15] for further details).

**Definition 2.1.** The non-stationary subdivision scheme \( S \) applied to the initial data \( f^{(1)} \in \ell(\mathbb{Z}^2) \) is called convergent if there exists a limit function \( g_{f^{(1)}} \in C(\mathbb{R}^2) \) (which is nonzero for at least one initial nonzero sequence \( f^{(1)} \)) such that

\[
\lim_{\ell \to +\infty} \sup_{\alpha \in \mathbb{Z}^2} |g_{f^{(1)}}(2^{-\ell} \alpha) - f^{(1)}_{\alpha}| = 0.
\]

We call the subdivision scheme \( S \) \( C^r \)-convergent, \( r \in \mathbb{N}_0 \), if \( g_{f^{(1)}} \in C^r(\mathbb{R}^2) \). The limit function is often denoted as \( g_{f^{(1)}} = \lim_{\ell \to +\infty} (\mathcal{S}_c)^{\ell} f^{(1)} \).

**Definition 2.2.** Let \( \delta = \{\delta_0, \alpha \in \mathbb{Z}^2\} \). For a convergent, stationary subdivision scheme \( \bar{S} := \{\mathcal{S}_c\} \) the function

\[
\bar{\phi} := \lim_{\ell \to +\infty} (\mathcal{S}_c)^{\ell} \delta,
\]

is the basic limit function of the subdivision scheme.

**Definition 2.3.** Let \( \delta = \{\delta_0, \alpha \in \mathbb{Z}^2\} \). For a convergent, non-stationary subdivision scheme \( S := \{\mathcal{S}_c^{(k)}, k \geq 1\} \) the function

\[
\phi_k := \lim_{\ell \to +\infty} \mathcal{S}_c^{(k+\ell)} \mathcal{S}_c^{(k+\ell-1)} \ldots \mathcal{S}_c^{(k)} \delta, \quad k \geq 1,
\]

is the \( k \)-th member of the family of basic limit functions \( \{\phi_k, k \geq 1\} \).

**Definition 2.4.** Let \( S \) and \( \bar{S} \) be subdivision schemes defined in \( \mathcal{R}^{(1)} \) by the subdivision masks \( \{c^{(k)} \in \ell(\mathbb{Z}^2), k \geq 1\} \) and \( c \in \ell(\mathbb{Z}^2) \), respectively. If

\[
\sum_{k=1}^{+\infty} \|\mathcal{S}_c^{(k)} - \mathcal{S}_c\|_\infty < +\infty,
\]

then \( S \) and \( \bar{S} \) are said to be asymptotically equivalent schemes.
Remark 2.2. As observed in [7, page 2], (5) holds if and only if
\[ \sum_{k=1}^{+\infty} \|c^{(k)} - c\|_\infty < +\infty \ \text{where} \ \|c\|_\infty = \sup_{\alpha \in \mathbb{Z}^2} |c_\alpha|. \]

Theorem 2.1. [15, Theorems 7-8 and Lemma 15] Let \( S \) and \( \tilde{S} \) be asymptotically equivalent subdivision schemes defined in \( \mathbb{R}^{(1)} \) by the subdivision masks \( \{c^{(k)} \} \in \ell(\mathbb{Z}^2), k \geq 1 \) and \( c \in \ell(\mathbb{Z}^2) \), respectively. If \( \tilde{S} \) is convergent, then \( S \) is also convergent and
\[ \lim_{k \to +\infty} \sup_{(u,v) \in \mathbb{R}^2} |\phi_k(u,v) - \tilde{\phi}(u,v)| = 0, \]
where \( \tilde{\phi} \) is the basic limit function of \( \tilde{S} \) defined in (3) and \( \{\phi_k, k \geq 1\} \) the family of basic limit functions of \( S \) defined in (4).

Remark 2.3. The condition of asymptotical equivalence in (5), that guarantees convergence, could be relaxed by considering the fulfillment of the weaker condition of asymptotical similarity together with approximate sum rules of order 1, as recently shown in [4].

2.2 Preliminaries for studying convergence of subdivision schemes in irregular submeshes

In the neighborhood of an extraordinary vertex/face, each step of a subdivision algorithm can be conveniently encoded in the rows of a local subdivision matrix \( S_k \) relating the vertices of the \( k \)-th level mesh with those of the next level. The matrix \( S_k \) has a different structure depending on the kind of extraordinary element (face or vertex) appearing in the \( k \)-th level mesh. Precisely, if the mesh contains an extraordinary face of valence \( n \), in view of the fact that the valence-\( n \) extraordinary face is surrounded by \( n \) sectors, each composed by \( p \) vertices, the local subdivision matrix \( S_k \) is of the form
\[ S_k = \begin{pmatrix}
B_{0,k} & B_{1,k} & \cdots & B_{n-1,k} \\
B_{n-1,k} & B_{0,k} & \cdots & B_{n-2,k} \\
\vdots & \vdots & \ddots & \vdots \\
B_{1,k} & \cdots & B_{n-1,k} & B_{0,k}
\end{pmatrix}, \tag{6}
\]
where \( B_{i,k} \in \mathbb{R}^{p \times p}, i = 0, \ldots, n-1 \). Thus \( S_k \in \mathbb{R}^{N \times N} \) with \( N = pn \) has a block-circulant structure. For short we write \( S_k := \text{circ}(B_{0,k}, \ldots, B_{n-1,k}) \).

Remark 2.4. Due to the structure of \( S_k \), it is not difficult to prove that
\[ \|S_k\|_\infty \leq \sum_{i=0}^{n-1} \|B_{i,k}\|_\infty. \]

If the \( k \)-th level mesh contains an extraordinary vertex of valence \( n \), the refinement rules in its neighborhood involve \( pn + 1 \) points instead of \( pn \): \( p \) points in each of the \( n \) sectors plus
the extraordinary vertex. Thus, to construct the local subdivision matrix \( S_k \) we first build the matrix

\[
\tilde{S}_k = \begin{pmatrix}
\alpha_k & \beta_k^T & \beta_k^T & \cdots & \beta_k^T \\
\gamma_k & \tilde{B}_{0,k} & \tilde{B}_{1,k} & \cdots & \tilde{B}_{n-1,k} \\
\gamma_k & \tilde{B}_{n-1,k} & \tilde{B}_{0,k} & \cdots & \tilde{B}_{n-2,k} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\gamma_k & \tilde{B}_{1,k} & \cdots & \tilde{B}_{n-1,k} & \tilde{B}_{0,k}
\end{pmatrix},
\]

(7)

where \( \alpha_k \in \mathbb{R} \), \( \beta_k \in \mathbb{R}^p \) and \( \tilde{B}_{i,k} \in \mathbb{R}^{p \times p} \), \( i = 0, \ldots, n-1 \). Then, following the method shown in [26, Example 5.14], we transform the matrix \( \tilde{S}_k \) in a block-circulant matrix \( S_k \) of the form

\[
S_k := \text{circ}(B_{0,k}, \ldots, B_{n-1,k}) \quad \text{with} \quad B_{j,k} = \left( \frac{\alpha_k}{n} \beta_k^T \right), \quad j = 0, \ldots, n-1.
\]

(8)

It follows that \( S_k \in \mathbb{R}^{N \times N} \), with \( N = n(p+1) \), has a block-circulant structure. Hence, without loss of generality, we can always assume that the local subdivision matrix \( S_k \) has a block-circulant structure with blocks of dimension \( m \times m \), where \( m = p \) if the \( k \)-th level mesh contains an extraordinary face and \( m = p + 1 \) if it contains an extraordinary vertex.

We continue by introducing some important notation from [26, 27, 32]. We start by assuming that near an isolated extraordinary vertex or face of valence \( n \) the regular subdivision surface \( r \) is defined on the local domain \( D_n := \Omega \times \mathbb{Z}_n \) (consisting of \( n \) copies of \( \Omega \)) with

\[
\Omega := \begin{cases}
[0,2] \times [0,2] & \text{in case of quadrilateral mesh}, \\
\{(u,v) \in \mathbb{R}^2 \mid u,v \geq 0 \text{ and } 0 \leq u+v \leq 2\} & \text{in case of triangular mesh},
\end{cases}
\]

and \( \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z} \). In the case of triangular and quadrilateral meshes, if we apply one step of refinement to the local domain \( D_n \), we obtain a new domain with \( 4n \) cells: \( 3n \) outer ordinary cells and \( n \) inner cells that contain the extraordinary element. The restriction \( r_1 \) of \( r \) to the outer cells is called ring. Denoting by \( \bar{r} \) the inner part of \( r \), that is \( \bar{r} := r \setminus r_1 \), we can repeat the refinement process only for \( \bar{r} \) to obtain a second ring \( r_2 \) and an even smaller inner part. Hence, iterated refinement generates a sequence of rings \( \{r_k, k \geq 1\} \) which covers all of the surface except for the central point (limit of the extraordinary vertex or face), that hereinafter we denote by \( r_e \). Precisely, assuming the central point to be placed at \( 0 \) and introducing the notation

\[
\bar{\Omega} := \begin{cases}
[0,1] \times [0,1] & \text{in case of quadrilateral mesh}, \\
\{(u,v) \in \mathbb{R}^2 \mid u,v \geq 0 \text{ and } 0 \leq u+v \leq 1\} & \text{in case of triangular mesh},
\end{cases}
\]

and

\[
\Omega_k := 2^{1-k}(\Omega \setminus \bar{\Omega}), \quad D_{n,k} := \Omega_k \times \mathbb{Z}_n, \quad k \geq 1,
\]

we see the ring \( r_k \) as the restriction of the subdivision surface \( r : D_n \rightarrow \mathbb{R}^3 \) to the set \( D_{n,k} \), i.e.,

\[
r_k := r|_{D_{n,k}},
\]

and the subdivision surface \( r \) as (see Figures 2 and 3)

\[
r = \bigcup_{k \geq 1} r_k \cup \{r_e\}.
\]
Specifically, in the case of quadrilateral meshes, the set $\Omega_k$ is explicitly given by
\[ \Omega_k = \{(u,v) \in \mathbb{R}^2 | u,v \geq 0 \text{ and } 2^{1-k} \leq \max\{u,v\} \leq 2^{2-k}\}, \]
while in the case of triangular meshes
\[ \Omega_k = \{(u,v) \in \mathbb{R}^2 | u,v \geq 0 \text{ and } 2^{1-k} \leq u+v \leq 2^{2-k}\}, \]
(see Figure 4). As a consequence, both in the case of triangular and quadrilateral meshes, the set $\Omega_k$ is constituted by the union of 3 cells, say $\omega_k^1$, $\omega_k^2$, and $\omega_k^3$, implying that the domain $D_{n,k}$ is indeed made of $3n$ cells. There follows that the entire surface ring $r_k$ is the union of $3n$ patches, each one denoted by $r_k^{[j]}$ and corresponding to the restriction of the regular subdivision surface $r$ to the single cell $\omega_k^{[j]}$.

![Fig. 2 Domains $\Omega_1, \Omega_2, \Omega_3$ corresponding to three subdivision steps in the case of a quadrilateral mesh containing an extraordinary vertex.](image)

![Fig. 3 Ring $r_k$ in the case of a quadrilateral mesh with an extraordinary vertex (figure taken from [26]).](image)

In the following, we denote by $d_k^{[j]} \in \mathbb{R}^P$, $P < N$, the vector of control points of each patch $r_k^{[j]}$, and with $\Phi_k^{[j]} \in \mathbb{R}^P$ the function vector containing all $P$ shifts of the basic limit function $\phi_k$, whose support intersects $\omega_k^{[j]}$. We assume that the functions in $\Phi_k^{[j]}$ are ordered as the points in the vector $d_k^{[j]}$, and thus we call them the associated basic limit functions.
Thus, for each \( j \in \mathbb{Z}_{3n} \), we have

\[
\mathbf{r}_k^{[j]} : \omega_k^{[j]} \rightarrow \mathbb{R}^3 \\
(u, v) \mapsto \mathbf{r}_k^{[j]} (u, v) = (\mathbf{d}_k^{[j]})^T \tilde{\Phi}_k^{[j]}(u, v).
\] (9)

Denoting by \( \tilde{\Phi}_k^{[j]} \in \mathbb{R}^P \) the vector containing all shifts of the basic limit function \( \tilde{\phi} \), whose support intersects \( \omega_k^{[j]} \), if assumptions of Theorem 2.1 are satisfied, we have that

\[
\lim_{k \rightarrow +\infty} \sup_{(u, v) \in \omega_k^{[j]}} \| \tilde{\Phi}_k^{[j]}(u, v) - \tilde{\Phi}_k^{[j]}(u, v) \|_\infty = 0, \quad \forall j \in \mathbb{Z}_{3n}.
\]

**Remark 2.5.** Let \( \mathbf{x}_0 := (1, 1, \ldots, 1)^T \in \mathbb{R}^P \). We observe that, for a convergent stationary subdivision scheme \( \mathcal{S} \), we have \( \tilde{\Phi}_k^{[j]}(u, v)^T \mathbf{x}_0 = 1 \) for all \( (u, v) \in \mathbb{R}^2, \ j \in \mathbb{Z}_{3n} \). Instead, for a non-stationary subdivision scheme \( \mathcal{S} \), \( \tilde{\Phi}_k^{[j]}(u, v)^T \mathbf{x}_0 = 1 \) for all \( k \geq 1 \) and for all \( (u, v) \in \mathbb{R}^2, \ j \in \mathbb{Z}_{3n} \) if and only if \( \mathcal{S} \) has the property of stepwise reproduction of constants (see, e.g., [3] for more details). In general, \( \tilde{\Phi}_k^{[j]}(u, v)^T \mathbf{x}_0 = \alpha_k \) with \( \alpha_k \in \mathbb{R} \), for all \( j \in \mathbb{Z}_{3n} \).

Now, let \( \mathbf{d}_1 \in \mathbb{R}^{N \times 3} \) be the collection of the vectors of control points \( \mathbf{d}_k^{[j]} \) of all patches \( \mathbf{r}_k^{[j]} \), \( j \in \mathbb{Z}_{3n} \). Denoted by \( \{ S_k \in \mathbb{R}^{N \times N}, k \geq 1 \} \) the matrix sequence that defines a non-stationary subdivision scheme \( \mathcal{S} \) in \( \mathcal{E}^{(1)} \), we can obtain the entire set of the \( (k+1) \)-th level control points representing the whole ring \( \mathbf{r}_{k+1} \) by the matrix multiplication

\[
\mathbf{d}_{k+1} = S_k \mathbf{d}_k = S_k S_{k-1} \mathbf{d}_{k-1} = \ldots = S^{(k)} \mathbf{d}_1 \quad \text{with} \quad S^{(k)} := \begin{cases} S_k S_{k-1} \cdots S_1, & k \geq 1, \\ I, & k = 0. \end{cases}
\] (10)

Moreover, denoting by \( \Phi_{k+1} \) the function vector with blocks \( \tilde{\Phi}_k^{[j]}, j \in \mathbb{Z}_{3n} \), we can rewrite each patch \( \mathbf{r}_k^{[j]}(u, v) = (\mathbf{d}_k^{[j]})^T \tilde{\Phi}_k^{[j]}(u, v), \ (u, v) \in \omega_k^{[j]} \), of the surface ring \( \mathbf{r}_{k+1} \) as

\[
\mathbf{r}_k^{[j]}(u, v) = (\mathbf{d}_k^{[j]})^T \tilde{\Phi}_k^{[j]}(u, v),
\]

(i.e. independently of \( j \)) since the function vector \( \Phi_{k+1} \in \mathbb{R}^N \) indeed contains only \( P \) functions that are non-zero on \( \omega_k^{[j]} \).

Exploiting the given definition of \( \mathbf{r}_{k+1} \), we can now provide the following notion of convergence of a non-stationary subdivision scheme \( \mathcal{S} \) in \( \mathcal{E}^{(1)} \).
Definition 2.5. Let $S$ be a (non-stationary) subdivision scheme whose action in $\mathcal{E}^{(1)}$ is described by the matrix sequence $\{S_k \in \mathbb{R}^{N \times N}, k \geq 1\}$. Moreover, let $d_1 \in \mathbb{R}^{N \times 3}$ be the vertices of $\mathcal{E}^{(1)}$. $S$ is said to be convergent in $\mathcal{E}^{(1)}$ (i.e., in the neighborhood of an extraordinary vertex/face of valence $n$) if, for all bounded initial data $d_1$, there exists a limit point $r_c \in \mathbb{R}^3$ such that

$$\lim_{k \to +\infty} \sup_{(u,v) \in D_{n,k}} ||r_k(u,v) - r_c||_\infty = 0.$$ 

We conclude by observing that, if the subdivision scheme $S$ converges, then $r = \bigcup_{k \geq 1} r_k \cup \{r_c\}$ is a surface without gap, i.e. $r$ is a surface which is continuous at all points including $r_c$ (which is in fact $r(0,0)$). In the following we call $r$ the limit surface of the subdivision scheme $S$.

3 New results related to the $C^1$-continuity analysis of subdivision schemes in regular submeshes

The goal of this section is to prove new preliminary results that allow us to derive a general criterion for verifying if the limit surface $r$ generated by an arbitrary non-stationary subdivision scheme is tangent plane continuous at the limit points of extraordinary elements. Tangent plane continuity of $r$ at the limit point $r_c$ is a weaker definition of $C^1$-continuity at the limit point $r_c$, named $G^1$-continuity. It reads as follows.

Definition 3.1. Let $S$ be a convergent (non-stationary) subdivision scheme and let $n(r_c)$ denote the normal vector at the limit point $r_c$ of an extraordinary vertex/face of valence $n$. The surface $r$, limit of $S$, is tangent plane continuous at $r_c$ if there exists a unique vector $n(r_c)$ such that, for all sequences of normal vectors $\{n_k(u,v) := \frac{\partial r_k(u,v) \wedge \partial r_k(u,v)}{||\partial r_k(u,v) \wedge \partial r_k(u,v)||_2}, (u,v) \in D_{n,k}, k \geq 1\}$,

$$\lim_{k \to +\infty} \sup_{(u,v) \in D_{n,k}} ||n_k(u,v) - n(r_c)||_\infty = 0.$$ 

We remark that a scheme $S$ which is tangent plane continuous at each limit point is named $G^1$-continuous.

The preliminary results required in Subsection 4.3 to give sufficient conditions for verifying tangent plane continuity at the limit points of extraordinary elements, deal with new results connected with the $C^1$-continuity analysis of subdivision schemes in regular submeshes $\mathcal{B}^{(1)}$. For them we recall the well-known notions of asymptotical equivalence of order 1 and divided-difference scheme, plus related results proven in [15].

Definition 3.2. Let $S$ and $\tilde{S}$ be subdivision schemes defined in $\mathcal{B}^{(1)}$ by the subdivision masks $\{c^{(k)} \in \ell(\mathbb{Z}^2), k \geq 1\}$ and $c \in \ell(\mathbb{Z}^2)$, respectively. If

$$\sum_{k=1}^{+\infty} 2^k ||S^{(k)} - \tilde{S}||_\infty < +\infty,$$

then $S$ and $\tilde{S}$ are said to be asymptotically equivalent schemes of order 1.

Remark 3.1. Asymptotical equivalence of order 1 implies asymptotical equivalence.

Theorem 3.1. [15, Theorem 8] Let $S$ and $\tilde{S}$ be subdivision schemes defined in $\mathcal{B}^{(1)}$ by the subdivision masks $\{c^{(k)} \in \ell(\mathbb{Z}^2), k \geq 1\}$ and $c \in \ell(\mathbb{Z}^2)$, respectively. If $S$ and $\tilde{S}$ are asymptotically equivalent of order 1, then $C^1$-convergence of $\tilde{S}$ implies $C^1$-convergence of $S$. 
Definition 3.3. For the two perpendicular directions \( e_1 = (1,0)^T, e_2 = (0,1)^T \), we define as
\[
(\Delta e_j^{(l)} f^{(l)})_{\alpha} := \frac{f^{(l)}_{\alpha} - f^{(l)}_{\alpha - e_j}}{2^{-l}}, \quad \alpha \in \mathbb{Z}^2, \ j \in \{1,2\}, \ l \geq 1,
\]
the \( e_j \)-directional divided-difference operator.

Lemma 3.1. Let \( j \in \{1,2\} \). If \( c^{(l)}(z) = \frac{1}{2}(1 + z_j) b^{(l)}_{e_j}(z) \), then
\[
\Delta e_j^{(l+1)} f^{(l+1)}(z) = b^{(l)}_{e_j}(z) \Delta e_j^{(l)} f^{(l)}(z^2),
\]
and \( \mathcal{S} b^{(l)}_{e_j}, \ l \geq 1 \) is called the \( e_j \)-directional divided difference scheme of \( \mathcal{S} f^{(l)}_{e_i}, \ l \geq 1 \).

Proof. Introducing the notation \( f^{(l)}(z) = \sum_{\alpha \in \mathbb{Z}^2} f^{(l)}_{\alpha} z^\alpha \), where the power is intended componentwise, we can write
\[
\Delta e_j^{(l+1)} f^{(l+1)}(z) = \frac{1}{2^{-(l+1)}} (1 - z_j) f^{(l+1)}(z).
\]
Since \( f^{(l+1)}(z) = c^{(l)}(z) f^{(l)}(z^2) \), in view of the factorized form of \( c^{(l)}(z) \), we obtain
\[
\Delta e_j^{(l+1)} f^{(l+1)}(z) = \frac{1}{2^{-l}} (1 - z_j^2) b^{(l)}_{e_j}(z) f^{(l)}(z^2).
\]
Now, taking into account equation (11), the claim follows. \( \square \)

From Lemma 3.1 we have that
\[
\Delta e_j^{(l+1)} f^{(l+1)} = \mathcal{S} b^{(l)}_{e_j} \Delta e_j^{(l)} f^{(l)} \iff \Delta e_j^{(l+1)} f^{(l+1)} = \mathcal{S} b^{(l)}_{e_j} \Delta e_j^{(l)} f^{(l)}.
\]

Lemma 3.2. Let \( \mathcal{S} \) and \( \bar{\mathcal{S}} \) be subdivision schemes defined in \( \mathcal{S}^{(1)} \) by the subdivision symbols \( \{c^{(k)}(z), k \geq 1\} \) and \( c(z) \), respectively. Assume that:

i) \( \mathcal{S} \) and \( \bar{\mathcal{S}} \) are asymptotically equivalent of order 1;

ii) the factor \( (1 + z_1)(1 + z_2) \) is contained in the symbols \( c(z) \) and \( c^{(k)}(z) \), for all \( k \geq 1 \).

Then, the divided difference schemes with symbols \( b_{e_j}(z) := \frac{2c(z)}{1+z_j}, j \in \{1,2\} \) and \( b^{(k)}_{e_j}(z) := \frac{2(c^{(k)}(z))}{1+z_j}, j \in \{1,2\} \), are asymptotically equivalent of order 1.

Proof. We only consider the case corresponding to \( j = 1 \), since the case \( j = 2 \) can be treated analogously. To simplify the notation we denote \( b_{e_1}(z) \) and \( b^{(k)}_{e_1}(z) \) by \( b(z) \) and \( b^{(k)}(z) \), respectively. We start by considering the relation
\[
2c(z) = (1 + z_1)b(z)
\]
with
\[
c(z) := \sum_{\alpha \in [0,N_1] \times [0,N_2]} c_\alpha z^\alpha \quad \text{and} \quad b(z) := \sum_{\alpha \in [0,N_1-1] \times [0,N_2]} b_\alpha z^\alpha.
\]
Comparing the same power of \( z \) we easily see that,
\[
c_0. = \frac{1}{2} b_{0.}, \ c_{N_1.} = \frac{1}{2} b_{N_1-1.}, \ c_{\alpha} = \frac{1}{2} (b_\alpha + b_{\alpha - e_1}), \ \alpha \in [1,N_1-1] \times [0,N_2],
\]
which means
\[ b_{0, \alpha_2} = 2c_{0, \alpha_2}, \; b_{N_1 - 1, \alpha_2} = 2c_{N_1, \alpha_2}, \; b_{\alpha} = 2 \sum_{\beta_1 = 0}^{\alpha} (-1)^{\alpha_1 - \beta_1} c_{\beta_1, \alpha_2}, \; \alpha \in [1, N_1 - 2] \times [0, N_2]. \]

Analogously, working with the relation \( 2c^{(k)}(z) = (1 + z_1) b^{(k)}(z) \) we get
\[ b_{0, \alpha_2} = 2c_{0, \alpha_2}, \; b_{N_1 - 1, \alpha_2} = 2c_{N_1, \alpha_2}, \; b_{\alpha} = 2 \sum_{\beta_1 = 0}^{\alpha} (-1)^{\alpha_1 - \beta_1} c_{\beta_1, \alpha_2}, \; \alpha \in [1, N_1 - 2] \times [0, N_2]. \]

Therefore,
\[ \|b^{(k)} - b\|_\infty \leq 2N_1 \|c^{(k)} - c\|_\infty \quad \text{and} \quad \sum_{k=1}^{\infty} 2^k \|b^{(k)} - b\|_\infty \leq 2N_1 \sum_{k=1}^{\infty} 2^k \|c^{(k)} - c\|_\infty < \infty. \]

Thus, in light of Remark 2.2, the result is proven.

The previous Lemma is useful for the next result.

**Proposition 3.1.** Let \( S \) and \( \tilde{S} \) be subdivision schemes defined in \( \mathcal{R}^{(1)} \). Assume that:

i) \( S \) and \( \tilde{S} \) are asymptotically equivalent of order 1.

ii) \( \tilde{S} \) is \( C^1 \)-convergent in \( \mathcal{R}^{(1)} \), with symbol \( c(z) \) that contains the factor \( (1 + z_1)(1 + z_2) \);

iii) \( S \) is defined in \( \mathcal{R}^{(1)} \) by the subdivision symbols \( \{c^{(\ell)}(z), \ell \geq 1\} \) all containing the factor \( (1 + z_1)(1 + z_2) \).

Then, the associated divided difference schemes with symbols \( b_{e_j}(z) := \frac{2c(z)}{1 + z_j} \) and \( b_{e_j}^{(\ell)}(z) := \frac{2c^{(\ell)}(z)}{1 + z_j}, \; j \in \{1, 2\} \), satisfy the following properties:

a) the sequence of basic limit functions of \( \{\mathcal{S}_{b_{e_j}}, \ell \geq 1\} \) converges uniformly to the basic limit function of \( \{\mathcal{S}_{b_{e_j}}\} \);  
b) \[ \lim_{\ell \to +\infty} \mathcal{S}_{b_{e_j}}^{(k+\ell)} \mathcal{S}_{b_{e_j}}^{(k+\ell-1)} \cdots \mathcal{S}_{b_{e_j}}^{(k)} \Delta_{e_j}^{(k)} \delta = \partial_{e_j} \phi_k \] and \[ \lim_{\ell \to +\infty} \mathcal{S}_{b_{e_j}}^{(k+1)} \Delta_{e_j}^{(1)} \delta = \partial_{e_j} \tilde{\phi}, \]

for \( \delta = \{\delta_0, \alpha \in \mathbb{Z}^2\} \) and with \( \phi_k \) defined as in (4) and \( \tilde{\phi} \) as in (3).

**Proof.** The result in (a) is a direct consequence of Lemma 3.2 and Theorem 2.1 (see also [15, Lemma 15]).

To show (b) we proceed as follows. In view of the factorization properties of \( c^{(\ell)}(z) \), we can apply Lemma 3.1 to conclude the existence of the \( e_j \)-directional divided difference scheme of order 1 of \( \{\mathcal{S}_{c^{(\ell)}}, \ell \geq 1\} \). Then, to show convergence of the \( e_j \)-directional divided difference scheme of order 1, we just recall the result in (a). Next, we exploit (12) and write
\[ \mathcal{S}_{b_{e_j}}^{(k)} \Delta_{e_j}^{(k)} \delta = \Delta_{e_j}^{(k+1)} \mathcal{S}_{c^{(k)}} \delta, \]

so that
\[ \mathcal{S}_{b_{e_j}}^{(k+\ell)} \mathcal{S}_{b_{e_j}}^{(k+\ell-1)} \cdots \mathcal{S}_{b_{e_j}}^{(k)} \Delta_{e_j}^{(k)} \delta = \Delta_{e_j}^{(k+\ell+1)} \mathcal{S}_{c^{(k+\ell)}} \mathcal{S}_{c^{(k+\ell-1)}} \cdots \mathcal{S}_{c^{(\ell)}} \delta. \]
Moreover, introducing the notation \( \delta^{(\ell+1)} := \mathcal{J}_{(k+\ell)} \mathcal{J}_{(k+\ell-1)} \ldots \mathcal{J}_{(k)} \delta \), we have that
\[
\Delta_{e_j}^{(k+\ell+1)} \delta^{(\ell+1)} = \frac{\delta^{(\ell+1)} - (\delta^{(\ell+1)})_{-e_j}}{2^{-k-\ell-1}}, \quad j \in \{1, 2\}.
\]
Thus
\[
\lim_{\ell \to +\infty} \mathcal{J}_{(k+\ell)} \mathcal{J}_{(k+\ell-1)} \ldots \mathcal{J}_{(k)} \delta = \lim_{\ell \to +\infty} \Delta_{e_j}^{(k+\ell+1)} \mathcal{J}_{(k+\ell)} \mathcal{J}_{(k+\ell-1)} \ldots \mathcal{J}_{(k)} \delta = \lim_{\ell \to +\infty} \mathcal{J}_{(k) \ldots \mathcal{J}_{(k)}} \delta - (\mathcal{J}_{(k+\ell)} \mathcal{J}_{(k+\ell-1)} \ldots \mathcal{J}_{(k)})_{-e_j} = \frac{\delta^{(\ell+1)} - (\delta^{(\ell+1)})_{-e_j}}{2^{-k-\ell-1}}.
\]
in view of the fact that \( \lim_{\ell \to +\infty} \mathcal{J}_{(k+\ell)} \mathcal{J}_{(k+\ell-1)} \ldots \mathcal{J}_{(k)} \delta = \phi_k \) and \( \phi_k \) is \( C^1 \).
The result for the stationary scheme follows by taking \( \mathcal{J}_{(\ell)} = \mathcal{J} \) for all \( \ell \geq 1 \) and using Theorem 2.1.

As a consequence of the previous Proposition we have

**Corollary 3.1.** Under the assumptions of Proposition 3.1
\[
\lim_{k \to +\infty} \sup_{(u,v) \in \mathbb{R}^2} |\partial_{e_j} \phi_k(u,v) - \partial_{e_j} \phi(u,v)| = 0, \quad j \in \{1, 2\}.
\]

### 4 Analysis of (non-stationary) subdivision schemes in irregular submeshes

Before focusing on the sufficient conditions that guarantee the convergence of a non-stationary subdivision scheme in \( \mathcal{E}^{(1)} \) (Theorem 4.1), we present a few linear algebra results to be used for the subdivision analysis.

#### 4.1 Auxiliary linear algebra results

Let \( M \in \mathbb{R}^{N \times N} \). In the following, two simple results based on the Jordan decomposition of \( M \) are proven. In the first one we assume \( d \in \mathbb{R}^{N \times 3} \) and consider the sequence \( \{M^k d, k \geq 0\} \). Then, under suitable assumptions on the matrix \( M \), we show its convergence. In the second one (which is a well known result so that we omit its proof) we study the properties of \( M^k \), \( k \geq 0 \), again with the help of its Jordan decomposition.

**Proposition 4.1.** Let \( XJX^{-1} \) be the Jordan decomposition of \( M \in \mathbb{R}^{N \times N} \). Assume that the unique dominant eigenvalue of \( J \) is 1 and the associated eigenvector is \( \mathbf{x} = (1,1,\ldots,1)^T \). Then, for all bounded \( d \in \mathbb{R}^{N \times 3} \),
\[
\lim_{k \to +\infty} M^k d = \mathbf{x} \mathbf{q}^T,
\]
with \( \mathbf{q}^T = \mathbf{\bar{x}}^T d \in \mathbb{R}^{1 \times 3}, \ \mathbf{\bar{x}}^T = e_1^T X^{-1} \in \mathbb{R}^{1 \times N} \) and \( e_1^T = (1,0,\ldots,0) \in \mathbb{R}^{1 \times N} \). Moreover, \( \mathbf{\bar{x}}^T M = \mathbf{\bar{x}}^T \).
Proof. Using the Jordan decomposition of $M$ and introducing the notation $w := X^{-1}d$, we can write $M^k d = X J^k w$. Hence, recalling that 1 is the unique dominant eigenvalue of $J$ and the associated eigenvector is $x = (1, 1, ..., 1)^T$, we have

\[
\lim_{k \to +\infty} M^k d = \lim_{k \to +\infty} X J^k w = X \left( \begin{array}{cccc}
1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0
\end{array} \right) w = x q^T,
\]

with $q^T = \tilde{x}^T d$ and $\tilde{x}^T = e_1^T X^{-1}$. To conclude we observe that $\tilde{x}^TM = \tilde{x}^T JX^{-1} = e_1^T X^{-1} = \tilde{x}^T$, where the last but one equality is due to the structure of $J$. \qed

In Propositions 4.2 and 4.3, $\| \cdot \|$ refers to any vector norm and its associated induced matrix norm.

**Proposition 4.2.** If the dominant eigenvalue of $M$ is 1 and its algebraic multiplicity is 1, then there exists a finite positive constant $C$ (independent of $k$) such that

\[
\|M^k\| \leq C, \quad \forall k \geq 0.
\]

**Remark 4.1.** It is important to remark that the subdivision matrix $S$ defining a convergent, stationary subdivision scheme $\bar{S}$ in irregular submeshes $\mathcal{E}^{(1)}$, always verifies the assumptions of Propositions 4.1 and 4.2 since

a.1) the unique dominant eigenvalue of $S$ is $\lambda_0 = 1$,
a.2) the algebraic multiplicity of $\lambda_0$ is 1,
a.3) the eigenvector associated with $\lambda_0$ is $x_0 = (1, 1, ..., 1)^T$,

see, e.g., [26, 27, 35].

In the next Proposition we replace the $k$-th power of the matrix $M$ with the product of $k$ different matrices $M_k M_{k-1} \cdots M_1$ and we successively consider hybrid combinations of the two.

**Proposition 4.3.** Let $M^{(0)} := I \in \mathbb{R}^{N \times N}$ and for all $k \geq 1$ let $M^{(k)} := M_k M_{k-1} \cdots M_1$ with $M_j \in \mathbb{R}^{N \times N}$, for all $j = 1, ..., k$. Let $M \in \mathbb{R}^{N \times N}$ be a nonsingular matrix having 1 as dominant eigenvalue with algebraic multiplicity 1. If, for all $k \geq 1$, $\|M_k - M\| \leq \frac{C}{\sigma^k}$ with $\sigma > 1$ and some finite positive constant $C$ (independent of $k$), then

\[
\|M^{(k)}\| \leq \hat{C}, \quad \forall k \geq 1
\]

with $\hat{C}$ a finite positive constant (independent of $k$).

**Proof.** The proof takes inspiration from [15, Theorem 5]. Let $y \in \mathbb{R}^N$. The claim is proven by introducing a new vector norm

\[
\|y\|_M := \sup_{k \geq 1} \|M^{k+1} y\|
\]

associated to the given nonsingular matrix $M \in \mathbb{R}^{N \times N}$. In view of Proposition 4.2, our assumption on $M$ implies the existence of a finite positive constant $\hat{C}$ such that $\|M^k\| \leq \hat{C}$.
for all $k \geq 0$. Moreover, $\|y\| \leq \|y\|_M$ since $\|M^{k+1}y\| = \|y\|$ when $k = -1$. There follows that

$$\|y\| \leq \|y\|_M \leq \tilde{C} \|y\|,$$

with $\tilde{C}$ a finite positive constant, i.e. any standard vector norm and the $\| \cdot \|_M$ norm are uniformly equivalent. We now consider the induced norm for the matrix $M$ itself, and denote it as $\| \cdot \|_M$. Then

$$\|M\|_M := \sup_{\|y\|_M = 1} \|My\| = \sup_{\|y\|_M = 1} \|M^{k+1}(My)\| = \sup_{\|y\|_M = 1} \sup_{k \geq -1} \|M^{k+1}y\| \leq \sup_{\|y\|_M = 1} \|M\| = 1.$$

We continue by exploiting the uniform equivalence of norms to bound $\|M^{(k)}\|$, $k \geq 1$. From the definition of induced norm, it holds the submultiplicative property

$$\|M_kM_{k-1} \cdots M_1\|_M \leq \|M_k\|_M \|M_{k-1}\|_M \cdots \|M_1\|_M.$$

Moreover, in view of the fact that

$$\|M_j - M\|_M \leq C \|M_j - M\| \leq \tilde{C} \frac{C}{\sigma^j},$$

where $C$ is the finite positive constant appearing in the assumption, we can finally arrive at the following result for any arbitrary $k \geq 1$:

$$\|M^{(k)}\| \leq \prod_{j=1}^k \|M_j\|_M \leq \prod_{j=1}^k (\|M\|_M + \|M_j - M\|_M) \leq \prod_{j=1}^k \left(1 + \tilde{C} \frac{C}{\sigma^j}\right) = e^{\log_e \left(\prod_{j=1}^k \left(1 + \frac{\tilde{C} C}{\sigma^j}\right)\right)} = e^{\sum_{j=1}^k \log_e \left(1 + \frac{\tilde{C} C}{\sigma^j}\right)} \leq e^{\sum_{j=1}^k \frac{\tilde{C} C}{\sigma^j}} \leq e^{\sum_{j=1}^{+\infty} \frac{1}{\sigma^j}},$$

where the last but one inequality follows from the fact that $\log_e (1 + x) \leq x$ for all $x \geq 0$. Since $\sum_{j=1}^{+\infty} \frac{1}{\sigma^j} < \infty$ the claim follows.

We conclude this section with another useful intermediate result relating $M^{(k)}$ with $M^k$.

**Proposition 4.4.** Let $M \in \mathbb{R}^{N \times N}$, $M^{(0)} := I \in \mathbb{R}^{N \times N}$ and for all $k \geq 1$ let $M^{(k)} := M_k \cdots M_1 \in \mathbb{R}^{N \times N}$. Then,

$$M^{(k)} = M^k + \sum_{j=1}^k M^{k-j}(M_j - M)M^{(j-1)}, \quad \text{for all} \quad k \geq 1.$$
Proof. Assuming \( \sum_{j=1}^{k-1} M^{k-j} (M_j - M) M^{(j-1)} \) to be 0 when \( k = 1 \), we can write

\[
M^k + \sum_{j=1}^{k-1} M^{k-j} (M_j - M) M^{(j-1)} = \]

\[
M^k + \sum_{j=1}^{k-1} M^{k-j} (M_j - M) M^{(j-1)} + (M_k - M) M^{(k-1)} = \]

\[
M^k + M^{(k)} + \sum_{j=1}^{k-1} M^{k-j} M^{(j)} - MM^{(k-1)} - \sum_{j=1}^{k-1} M^{k-j} M^{(j-1)} = \]

\[
M^k + M^{(k)} + \sum_{j=1}^{k-2} M^{k-j} M^{(j)} - \sum_{j=0}^{k-2} M^{k-j} M^{(j)} = \]

\[
M^k + M^{(k)} - M^k = M^{(k)}, \]

so concluding the proof. \( \square \)

4.2 Convergence analysis in irregular submeshes

In this section we make use of the previous linear algebra results to provide sufficient conditions for establishing the convergence of a non-stationary subdivision scheme \( S \) defined in irregular submeshes \( E^{(1)} \) by the matrix sequence \( \{S_k \in \mathbb{R}^{N \times N}, k \geq 1 \} \). With the notation previously introduced, let \( d_1 \in \mathbb{R}^{N \times 3} \) be the collection of the vectors of control points \( d[i,j] \) of all patches \( r[i,j] \), \( j \in \mathbb{Z}_{3n} \). According to Definition 2.5, our goal is to study the convergence of the sequence of regular rings \( \{r[k+1], k \geq 0 \} \) whose patches \( r[i,j] \) are described by the equation

\[
r[i,j+1] = (d[i,j])^T \Phi[i,j] = d[i,j+1] \Phi[i,j], \quad j \in \mathbb{Z}_{3n}. \]

According to (10), the entire set of the \( (k+1) \)-th level control points \( d[k+1] \) representing the whole ring \( r[k+1] \) is given by the matrix multiplication

\[
d[k+1] = S[k](d), \quad \text{with} \quad S[k] := \begin{cases} S[k-1] \cdots S_1, & k \geq 1, \\ I, & k = 0. \end{cases} \]

The key idea to prove convergence of \( S \) is to write the product matrix \( S[k] \) in terms of the stationary matrix \( S[k] \). Indeed, from Proposition 4.4 we write

\[
d[k+1] = S[k]d + y[k] \quad \text{with} \quad y[k] := \sum_{j=1}^{k} S[k-j] (S_j - S) S^{(j-1)} d, \]

and then show our first main result.

**Theorem 4.1.** Let \( S \) be a non-stationary subdivision scheme whose action in \( E^{(1)} \) is described by the matrix sequence \( \{S_k, k \geq 1 \} \). Moreover, let \( \tilde{S} \) be a stationary subdivision scheme that in \( E^{(1)} \) is identified by \( \{S\} \). Assume that:

(i) \( \tilde{S} \) is convergent both in \( E^{(1)} \) and in \( E^{(1)} \),

(ii) \( S \) is asymptotically equivalent to \( \tilde{S} \) in \( E^{(1)} \),

(iii) in \( E^{(1)} \) the matrices \( S_k \) and \( S \) satisfy, for all \( k \geq 1 \), \( \|S_k - S\|_\infty \leq \frac{C}{\sigma} \) with \( C \) some finite positive constant and \( \sigma > 1 \).
Then, for all bounded initial data $d_1 \in \mathbb{R}^{N \times 3}$, the non-stationary subdivision scheme $S$ is convergent also in $\mathcal{E}^{(1)}$. In particular,

$$
\lim_{k \to +\infty} \sup_{(u,v) \in \Omega_{k+1}} \|r_{k+1}(u,v) - (q_0 + \beta_0)\|_\infty = 0,
$$

where

- $q_0 = d_1^T \bar{x}_0 \in \mathbb{R}^3$ with $\bar{x}_0$ such that $S^T \bar{x}_0 = \bar{x}_0$,

- $\beta_0 = (\lim_{k \to +\infty} y_k)^T \frac{x_0}{x_0^T x_0} \in \mathbb{R}^3$ for $y_k = \sum_{j=1}^k S^{k-j} (S_j - S) S^{(j-1)} d_1$ and $x_0$ such that $S x_0 = x_0$.

**Proof.** The proof follows the line of reasoning of the proof of [15, Theorem 6]. For $d_1 \in \mathbb{R}^{N \times 3}$ we define

$$
u_{k+1,\ell} := S^\ell S^{(k)} d_1, \quad \ell \geq 0, \quad k \geq 0.
$$

From assumption $(i)$ we know that

$$
\lim_{\ell \to +\infty} \nu_{k+1,\ell} =: \nu_{k+1}, \quad k \geq 0
$$

namely $\forall \varepsilon > 0$ there exists $L \in \mathbb{N}$ such that $\|\nu_{k+1,\ell} - \nu_{k+1}\|_\infty < \varepsilon$ for all $\ell > L$. We continue by proving that the sequence $\{\nu_{k+1,\ell}, \ k \geq 0\}$ is a Cauchy sequence. Indeed, in view of Proposition 4.2, Proposition 4.3 and assumption $(iii)$ we have

$$
\|\nu_{k+1} - \nu_k\|_\infty = \left\| \lim_{\ell \to +\infty} S^\ell (S_k - S) S^{(k-1)} d_1 \right\|_\infty \leq \bar{C} \|S_k - S\|_\infty \|d_1\|_\infty \leq \frac{\bar{C}}{\sigma^k},
$$

and thus, for $s \geq 1$,

$$
\|\nu_{k+s} - \nu_k\|_\infty \leq \sum_{j=1}^s \|\nu_{k+j} - \nu_{k+j-1}\|_\infty \leq \bar{C} \frac{1}{\sigma^s} \sum_{j=0}^{s-1} \frac{1}{\sigma^j}
$$

(15)

with $\bar{C}$ and $\hat{C}$ finite positive constants. From above we can conclude the existence of the vector $u := \lim_{k \to +\infty} \nu_k$ and write that $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$
\|\nu_{k+1,\ell} - \nu_k\|_\infty \leq \|\nu_{k+1,\ell} - \nu_{k+1}\|_\infty + \|\nu_{k+1} - \nu_k\|_\infty < \varepsilon, \quad \forall k, \ell > N.
$$

Now, using the notation $d_{k,\ell} := S^{(k+\ell)} d_1 = S_{k+\ell} \cdots S_1 d_1$ we estimate $\|d_{k,\ell} - u\|_\infty$ as

$$
\|d_{k,\ell} - u\|_\infty \leq \|d_{k,\ell} - \nu_{k+1,\ell}\|_\infty + \|\nu_{k+1,\ell} - u\|_\infty.
$$

We then write

$$
\|d_{k,\ell} - \nu_{k+1,\ell}\|_\infty = \left\| \left( S_{k+\ell} \cdots S_{k+1} - S^\ell \right) S_{k} \cdots S_1 d_1 \right\|_\infty \leq C \|S_{k+\ell} \cdots S_{k+1} - S^\ell\|_\infty
$$

with $C$ a finite positive constant. In view of Proposition 4.4 (with $S_{k+1}$ playing the role of $M_1$), using again $(iii)$ we arrive at

$$
\|d_{k,\ell} - \nu_{k+1,\ell}\|_\infty \leq \bar{C} \frac{1}{\sigma^s} \sum_{j=k+1}^{k+\ell} \|S^{k+j} (S_j - S) S^{(j-1)} d_1\|_\infty \leq \hat{C} \frac{1}{\sigma^s} \sum_{j=k+1}^{k+\ell} \frac{1}{\sigma^j},
$$

(16)
where again \( \tilde{c}, \tilde{c} \) are finite positive constants. Since the last sum in (16) can be made arbitrarily small, we can conclude that, \( \forall \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for all \( k, \ell > N \)
\[
\| d_{k, \ell} - u \|_\infty < \varepsilon.
\]
As a consequence, \( \forall \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for all \( m = k + \ell > 2N + 1 \)
\[
\| S^{(m)} d_1 - u \|_\infty < \varepsilon,
\]
that is
\[
\lim_{m \to +\infty} S^{(m)} d_1 = u. \tag{17}
\]
We continue by showing that the vector \( u \) is an eigenvector of \( S \) associated with the eigenvalue \( \lambda \) (i.e., \( S u = u \)). Indeed, observing that \( S u_{k+1, \ell} = u_{k+1, \ell + 1} \) we write
\[
\| S u - u \|_\infty \leq \| S u - S u_{k+1, \ell} \|_\infty + \| u_{k+1, \ell + 1} - d_{k, \ell + 1} \|_\infty + \| d_{k, \ell + 1} - u \|_\infty,
\]
with the right hand side that tends to 0 for \( k \) and \( \ell \) going to \(+\infty\). In view of assumption \((i)\) and \((17)\), we can thus conclude convergence of the sequence
\[
\{ y_k, k \geq 0 \}, \text{ with } y_k := S^{(k)} d_1 - S^k d_1 = d_{k+1} - S^k d_1 = \sum_{j=1}^{k} S^{k-j} (S_j - S) S^{(j-1)} d_1.
\]
Moreover, denoting \( y := \lim_{k \to +\infty} y_k \), from the fact that \( S u = u \) we can also conclude that \( S y = y \), which means that \( y \) lies in the eigenspace corresponding to the right eigenvector of \( S \) associated to the eigenvalue \( \lambda_0 = 1 \). There follows that \( y \) is of the form \( y = x_0 \beta_0^T \) with \( x_0 = (1, 1, ..., 1)^T \), which implies that \( \beta_0 \) can be written as \( \beta_0 = y^T \frac{x_0}{\|x_0\|} \).

From (14) we then write
\[
\lim_{k \to +\infty} d_{k+1} = \lim_{k \to +\infty} S^k d_1 + x_0 \beta_0^T, \tag{18}
\]
and in view of Proposition 4.1, after replacing (13) in equation (18), we arrive at
\[
\lim_{k \to +\infty} d_{k+1} = x_0 (q_0 + \beta_0)^T, \quad \text{with} \quad q_0 = d_1^T x_0. \tag{19}
\]
Then, taking into consideration assumption \((ii)\) and Theorem 2.1, we have that
\[
\lim_{k \to +\infty} \sup_{(u,v) \in \Omega_{k+1}} \| \Phi_{k+1}(u,v) - \overline{\Phi}(u,v) \|_\infty = 0. \tag{20}
\]
After recalling that \( \overline{\Phi}(u,v)^T x_0 = 1 \) for all \( (u,v) \in \Omega_{k+1} \) (in light of the arguments in Remark 2.5), we continue by writing, for all \( j \in \mathbb{Z}_{3n} \),
\[
\sup_{(u,v) \in \Omega_{k+1}} \| r_{k+1}^{[j]} (u,v)^T - (q_0 + \beta_0)^T \|_\infty = \sup_{(u,v) \in \Omega_{k+1}} \| \Phi_{k+1}(u,v)^T d_{k+1} - \overline{\Phi}(u,v)^T x_0 (q_0 + \beta_0)^T \|_\infty = \sup_{(u,v) \in \Omega_{k+1}} \| \Phi_{k+1}(u,v)^T d_{k+1} - \Phi_{k+1}(u,v)^T x_0 (q_0 + \beta_0)^T + \sup_{(u,v) \in \Omega_{k+1}} \| \Phi_{k+1}(u,v)^T d_{k+1} - \Phi_{k+1}(u,v)^T x_0 (q_0 + \beta_0)^T \|_\infty + \sup_{(u,v) \in \Omega_{k+1}} \| \Phi_{k+1}(u,v)^T x_0 (q_0 + \beta_0)^T - \overline{\Phi}(u,v)^T x_0 (q_0 + \beta_0)^T \|_\infty \leq \sup_{(u,v) \in \Omega_{k+1}} \| \Phi_{k+1}(u,v)^T \|_\infty \| d_{k+1} - x_0 (q_0 + \beta_0)^T \|_\infty + \sup_{(u,v) \in \Omega_{k+1}} \| \Phi_{k+1}(u,v)^T - \overline{\Phi}(u,v)^T \|_\infty \| x_0 (q_0 + \beta_0)^T \|_\infty. \]
Since $\lim_{k \to +\infty} ||d_{k+1} - x_0(q_0 + \beta_0)^T||_\infty = 0$ (in light of (19)), $\lim_{k \to +\infty} \sup_{(u,v) \in \Omega_{k+1}} ||\mathbf{P}_{k+1}(u,v)^T - \mathbf{F}(u,v)^T||_\infty = 0$ (in light of (20)), $\sup_{(u,v) \in \Omega_{k+1}} ||\mathbf{P}_{k+1}(u,v)^T||_\infty$ is uniformly bounded and $||x_0(q_0 + \beta_0)^T||_\infty$ is bounded, we finally obtain

$$\lim_{k \to +\infty} \sup_{(u,v) \in \alpha_{j+1}} ||x_j||_{k+1}(u,v)^T - (q_0 + \beta_0)^T||_\infty = 0, \quad \forall \ j \in \mathbb{Z}_n,$$

which concludes the proof. 

Now, following the notation in [26], we denote with $\lambda_r$, $r = 0, \ldots, r$, $0 \leq r \leq N - 1$, the $r + 1$ different eigenvalues of $S \in \mathbb{R}^{N \times N}$ sorted in decreasing order according to their magnitude, i.e. $|\lambda_0| \geq |\lambda_1| \geq \ldots \geq |\lambda_r|$. Moreover, for $\ell \geq 0$ we denote by $\ell + 1$, the algebraic multiplicity of $\lambda_1$. As emphasized in Remark 4.1, it is a known fact that, for a convergent stationary scheme $\bar{S}$ identified by $\{S\}$, all $r + 1$ eigenvalues have magnitude less than 1, except $\lambda_0$ which is required to be exactly 1 and with algebraic and geometric multiplicity 1. It means that $1 = \lambda_0 > |\lambda_1| \geq \ldots \geq |\lambda_r|$ and $\ell_0 = 0$. Moreover, the eigenvector associated to the unique dominant eigenvalue $\lambda_0 = 1$ is required to be $x_0 = (1, 1, \ldots, 1)^T \in \mathbb{R}^N$ (see, e.g., [26, 27, 35]). Thus, exploiting the Jordan decomposition of $S^k$ and the equality $S^k d_1 = X J^k w$ with $w = X^{-1} d_1$, for $k$ sufficiently large we can write

$$S^k d_1 = x_0 q_0^T + O(|\lambda_1|^k),$$

Equation (21) implies the following convergence rate result for the sequence $\{y_k, k \geq 0\}$.

**Corollary 4.1.** Let $\bar{S}$ be a convergent, stationary scheme identified by $\{S\}$ and denote by $\lambda_1$ the subdominant eigenvalue of $S$. Under the assumptions of Theorem 4.1 with the additional requirement that $\sigma > \frac{1}{|\lambda_1|} > 1$, for $u = \lim_{k \to +\infty} S^{(k)} d_1$ we have

$$S^{(k)} d_1 = u + O\left(\frac{1}{\sigma^k}\right).$$

Thus

$$y_k = x_0 \beta_0^T + O\left(\frac{1}{\sigma^k}\right).$$

**Proof.** We use the notation of the proof of Theorem 4.1 and start proving (22). First we write

$$u - S^{(k+\ell)} d_1 = (u - u_{k+1}) + (u_{k+1} - u_{k+1,\ell}) + (u_{k+1,\ell} - S^{(k+\ell)} d_1).$$

Then, by (15), (16) and (21) we obtain

$$||u - S^{(k+\ell)} d_1||_\infty \leq ||u - u_{k+1}||_\infty + ||u_{k+1} - u_{k+1,\ell}||_\infty + ||u_{k+1,\ell} - S^{(k+\ell)} d_1||_\infty \leq C_1 |\lambda_1|^\ell + \frac{C_2}{\sigma^k},$$

with $C_1, C_2$ finite positive constants. Since $|\lambda_1| < 1$, we can find $\bar{L}$ such that $|\lambda_1|^\ell < \frac{1}{\sigma^\ell}$ for all $\ell > \bar{L}$. Therefore,

$$||u - S^{(k+L+1)} d_1||_\infty \leq \frac{C_3}{\sigma^\ell},$$
with $\zeta_3$ a finite positive constant. Hence, taking the limit to $+\infty$ with respect to $k$, (22) follows.

Similarly, to prove the result in (23) we write
\[ y_{k+\ell} - x_0 \beta_0^T = (S^{k+\ell} d_1 - u) - (S^{k+\ell} d_1 - x_0 q_0^T) \]
and consider the triangular inequality
\[
\|y_{k+\ell} - x_0 \beta_0^T\|_\infty \leq \|S^{k+\ell} d_1 - u_{k+1,\ell}\|_\infty + \|u_{k+1,\ell} - u_{k+1}\|_\infty \\
+ \|u_{k+1} - u\|_\infty + \|S^{k+\ell} d_1 - x_0 q_0^T\|_\infty.
\]
Using again (15), (16) and (21), we obtain the upper bound
\[
\|y_{k+\ell} - x_0 \beta_0^T\|_\infty \leq \tilde{c}_1|\lambda_1|^\ell + \frac{\tilde{c}_2}{\sigma^\ell} + \tilde{c}_3|\lambda_1|^{k+\ell}.
\]
Then, applying the same reasoning as before, (23) is proven. \qed

4.3 Tangent plane continuity analysis at the limit point of an extraordinary element

Aim of this section is to provide sufficient conditions to show that a convergent, non-stationary subdivision scheme $S$ produces tangent plane continuous limit surfaces at the limit point.

We recall that, if a symmetric stationary subdivision scheme $\bar{S}$ is tangent plane continuous, the ordered eigenvalues of the associated local subdivision matrix $S$ satisfy
\[ 1 = \lambda_0 > \lambda_1 > |\lambda_2| \quad \text{with} \quad \lambda_1 \in \mathbb{R}^+, \ell_1 = 1, \]
i.e. the sub-dominant eigenvalue $\lambda_1$ is real, double and with geometric multiplicity equal to algebraic multiplicity. In this case, the eigenvectors associated to $\lambda_1$ are linearly independent. In the following we denote by $x_0 = (1, 1, \ldots, 1)^T \in \mathbb{R}^N$ the eigenvector associated to $\lambda_0 = 1$, and by $x_1^0, x_1^1 \in \mathbb{R}^N$ the two linearly independent eigenvectors associated to $\lambda_1$.

It is a well-known fact that, if the stationary scheme $\bar{S}$ produces tangent plane continuous limit surfaces, then its characteristic map is regular (see [26] for details), i.e. the planar ring defined by $\bar{V}(u, v)^T = \bar{F}(u, v)^T (x_1^0, x_1^1) \in \mathbb{R}^{1\times 2}$, where $\bar{F}(u, v) \in \mathbb{R}^N$ is the corresponding basic limit function vector, is such that $\det(\partial \bar{V}(u, v)^T)$ is non-zero and of constant sign with
\[
\partial \bar{V}(u, v)^T := \left( \partial_u \bar{V}(u, v)^T \right) \in \mathbb{R}^{2\times 2}, \quad (u, v) \in \Omega_1.
\]
In the following we provide sufficient conditions to show that a non-stationary subdivision scheme $\tilde{S}$ produces tangent plane continuous limit surfaces at the limit point $r_c = q_0 + \beta_0$ (see Definition 3.1).

**Theorem 4.2.** Let $S$ be a non-stationary subdivision scheme whose action in $\mathcal{E}^{(1)}$ is described by the matrix sequence $\{S_k, k \geq 1\}$. Moreover, let $\bar{S}$ be a stationary subdivision scheme that in $\mathcal{E}^{(1)}$ is identified by $\{S\}$. Assume that:

1. $\bar{S}$ is $C^1$-convergent in $\mathcal{R}^{(1)}$ with symbol $c(z)$ containing the factor $(1 + z_1)(1 + z_2)$, and $G^1$-convergent in $\mathcal{E}^{(1)}$.
(ii) $S$ is defined in $\mathcal{R}^{(1)}$ by the symbols $\{c^{(k)}(z), k \geq 1\}$ where each $c^{(k)}(z)$ contains the factor $(1 + z_i)(1 + z_j)$;

(iii) $S$ is asymptotically equivalent of order 1 to $\overline{S}$ in $\mathcal{R}^{(1)}$;

(iv) in $\mathcal{E}^{(1)}$ the matrices $S_k$ and $S$ satisfy, for all $k \geq 1$, $\|S_k - S\|_\infty \leq \frac{C}{r^k}$ with $C$ some finite positive constant, $\sigma > \frac{1}{r} > 1$ with $\lambda_1 \in \mathbb{R}_+$, the subdominant eigenvalue of $S$ which is double and non-defective.

Then the subdivision surface generated by $S$ is tangent plane continuous at the limit point $\mathbf{r}_c$.

Proof. First we observe that from (i) and Remark 4.1 the matrix $S$ has a simple dominant eigenvalue $\lambda_0 = 1$. Also, from Theorem 3.1 we know that $S$ is $C^1$-convergent in $\mathcal{R}^{(1)}$ and from Theorem 4.1 we also know that $S$ is convergent in $\mathcal{E}^{(1)}$. From Definition 2.5 it follows that, if $\mathbf{r}_{k+1}$ is the regular patches ring near an extraordinary vertex/face and if \{d_{k+1}, k \geq 0\} is the sequence of control points related to $\mathbf{r}_{k+1}$, then the two sequences \{d_{k+1}, k \geq 0\} and \{\mathbf{r}_{k+1}, k \geq 0\} converge. In particular, in view of Proposition 4.4, we know that

$$d_{k+1} = S^k d_1 + y_k \quad \text{with} \quad y_k = \sum_{j=1}^k S^{k-j}(S_j - S)S^{j-1}d_1.$$ 

To simplify the following analysis, we do not consider the full expression of a sequence of rings, but only the asymptotic behavior of the dominant terms as $k$ tends to infinity. Since the subdivision surface generated by the stationary scheme $\bar{S}$ is tangent plane continuous at its limit point (assumption (i)), the eigenvalues of $S$ satisfy $1 = \lambda_0 > \lambda_1 > |\lambda_i|, \; i = 2, \ldots, r$ and the subdominant eigenvalue $\lambda_1$ has geometric multiplicity and algebraic multiplicity two [26]. Thus, exploiting the Jordan decomposition of $S^k$ and the equality $S^k d_1 = X J^k w$ with $w = X^{-1} d_1$, for $k$ sufficiently large we can write that

$$S^k d_1 = x_0 q_0^T + \lambda_1^k (x_1^0(q_0^0)^T + x_1^1(q_1^1)^T) + o(\lambda_1^k)$$

with $x_1^0$ and $x_1^1$ denoting the two linearly independent eigenvectors associated to $\lambda_1$ and $q_0^0, q_1^1$ two vectors in $\mathbb{R}^3$. Since in view of Corollary 4.1 we also have that

$$y_k = x_0 \beta_0^T + O\left(\frac{1}{\sigma^k}\right),$$

we arrive at

$$d_{k+1} = x_0 (q_0^T + \beta_0^T) + \lambda_1^k (x_1^0(q_0^0)^T + x_1^1(q_1^1)^T) + \theta_k,$$

where $\theta_k$ denotes a vector in $\mathbb{R}^{N \times 3}$ with all its entries behaving as $o(\lambda_1^k) + O\left(\frac{1}{\sigma^k}\right)$. Parameterizing the regular patches ring $\mathbf{r}_{k+1}$ using the basic limit function vector $\Phi_{k+1}$, we can write $(\mathbf{r}_{k+1}^j)^T$, for each $j \in \mathbb{Z}_{3n}$, as (cf. equation (9))

$$(\mathbf{r}_{k+1}^j(u, v))^T = \Phi_{k+1}^T(u, v)d_{k+1}, \quad (u, v) \in \omega_{k+1}^j, \quad j \in \mathbb{Z}_{3n}.$$ 

Using Remark 2.5 and introducing the notation $\alpha_{k+1}$ for the value $\Phi_{k+1}^T x_0 \in \mathbb{R}$, thanks to (24), we have

$$(\mathbf{r}_{k+1}^j(u, v))^T = \Phi_{k+1}^T(u, v)^T \left(x_0 (q_0^T + \beta_0^T) + \lambda_1^k (x_1^0(q_0^0)^T + x_1^1(q_1^1)^T) + \theta_k\right)$$

$$= \alpha_{k+1} (q_0^T + \beta_0^T) + \lambda_1^k \Phi_{k+1}^T(u, v)^T (x_1^0(q_0^0)^T + x_1^1(q_1^1)^T)$$
To verify the tangent plane continuity of the limit surface at the limit point \( \mathbf{r}_c = \mathbf{q}_0 + \beta_0 \), we first observe that
\[
\partial_u \alpha_{k+1} = \partial_v \alpha_{k+1} = 0,
\]
and then write
\[
\partial_u (\mathbf{r}_{k+1}^{[j]} (u,v))^T = \partial_u \left( \alpha_{k+1} (\mathbf{q}_0^T + \beta_0^T) + \lambda_i^k \mathbf{F}_{k+1} (u,v)^T (\mathbf{x}_1^0 (\mathbf{q}_1^0)^T + \mathbf{x}_1^1 (\mathbf{q}_1^1)^T) \right.
+ \Phi_{k+1} (u,v)^T \theta_k
= \lambda_i^k \partial_u \mathbf{F}_{k+1} (u,v)^T (\mathbf{x}_1^0 (\mathbf{q}_1^0)^T + \mathbf{x}_1^1 (\mathbf{q}_1^1)^T) + \partial_u \Phi_{k+1} (u,v)^T \theta_k
= \lambda_i^k \partial_u \mathbf{F}_{k+1} (u,v)^T \left( (\mathbf{x}_1^0 (\mathbf{q}_1^0)^T + \mathbf{x}_1^1 (\mathbf{q}_1^1)^T) + \frac{\partial_x \lambda_i}{\lambda_i^k} \right).
\]

and
\[
\partial_v (\mathbf{r}_{k+1}^{[j]} (u,v))^T = \partial_v \left( \alpha_{k+1} (\mathbf{q}_0^T + \beta_0^T) + \lambda_i^k \mathbf{F}_{k+1} (u,v)^T (\mathbf{x}_1^0 (\mathbf{q}_1^0)^T + \mathbf{x}_1^1 (\mathbf{q}_1^1)^T) \right.
+ \Phi_{k+1} (u,v)^T \theta_k
= \lambda_i^k \partial_v \mathbf{F}_{k+1} (u,v)^T (\mathbf{x}_1^0 (\mathbf{q}_1^0)^T + \mathbf{x}_1^1 (\mathbf{q}_1^1)^T) + \partial_v \Phi_{k+1} (u,v)^T \theta_k
= \lambda_i^k \partial_v \mathbf{F}_{k+1} (u,v)^T \left( (\mathbf{x}_1^0 (\mathbf{q}_1^0)^T + \mathbf{x}_1^1 (\mathbf{q}_1^1)^T) + \frac{\partial_x \lambda_i}{\lambda_i^k} \right).
\]

Therefore, in order to study the evolution of the direction of the normal vectors to the \( j \)-th surface patch we write
\[
\partial_u (\mathbf{r}_{k+1}^{[j]} (u,v))^T \land \partial_v (\mathbf{r}_{k+1}^{[j]} (u,v))^T = \lambda_i^{2k} \left( \partial_u \mathbf{F}_{k+1} (u,v)^T \left( (\mathbf{x}_1^0 (\mathbf{q}_1^0)^T + \mathbf{x}_1^1 (\mathbf{q}_1^1)^T) + \frac{\partial_x \lambda_i}{\lambda_i^k} \right) \right.
\land \partial_v \mathbf{F}_{k+1} (u,v)^T \left( (\mathbf{x}_1^0 (\mathbf{q}_1^0)^T + \mathbf{x}_1^1 (\mathbf{q}_1^1)^T) + \frac{\partial_x \lambda_i}{\lambda_i^k} \right),
\]
and, with the help of the formula
\[
(k_1 \mathbf{a} + k_2 \mathbf{b}) \land (h_1 \mathbf{a} + h_2 \mathbf{b}) = \det \begin{pmatrix} k_1 & k_2 \\ h_1 & h_2 \end{pmatrix} (\mathbf{a} \land \mathbf{b}), \text{ for } k_1, k_2, h_1, h_2 \in \mathbb{R} \text{ and } \mathbf{a}, \mathbf{b} \in \mathbb{R}^3,
\]
we arrive at
\[
\partial_u (\mathbf{r}_{k+1}^{[j]} (u,v))^T \land \partial_v (\mathbf{r}_{k+1}^{[j]} (u,v))^T = \lambda_i^{2k} \left( \det( \mathbf{J} \Phi_{k+1} (u,v)^T ((\mathbf{q}_1^0)^T \land (\mathbf{q}_1^1)^T) \right.
\land \partial_v \mathbf{F}_{k+1} (u,v)^T (\mathbf{x}_1^0 (\mathbf{q}_1^0)^T + \mathbf{x}_1^1 (\mathbf{q}_1^1)^T)
+ \partial_u \Phi_{k+1} (u,v)^T (\mathbf{x}_1^0 (\mathbf{q}_1^0)^T + \mathbf{x}_1^1 (\mathbf{q}_1^1)^T) \land (\partial_v \mathbf{F}_{k+1} (u,v)^T \frac{\partial_x \lambda_i}{\lambda_i^k})
+ \partial_u \mathbf{F}_{k+1} (u,v)^T (\mathbf{x}_1^0 (\mathbf{q}_1^0)^T + \mathbf{x}_1^1 (\mathbf{q}_1^1)^T) \land \partial_v \mathbf{F}_{k+1} (u,v)^T \frac{\partial_x \lambda_i}{\lambda_i^k}
\land \partial_v \mathbf{F}_{k+1} (u,v)^T \frac{\partial_x \lambda_i}{\lambda_i^k} \land (\partial_v \mathbf{F}_{k+1} (u,v)^T \frac{\partial_x \lambda_i}{\lambda_i^k}).
\]
where $\Psi_{k+1}(u,v)^T := \Phi_{k+1}(u,v)^T (x_0^j, x_1^j)$ and

$$J \Phi_{k+1}(u,v)^T := \left( \frac{\partial u \Psi_{k+1}(u,v)^T}{\partial u \Psi_{k+1}(u,v)^T} x_0^j, \frac{\partial v \Phi_{k+1}(u,v)^T}{\partial v \Phi_{k+1}(u,v)^T} x_1^j \right).$$

Since in our notation the partial derivatives $\partial_a$ and $\partial_i$ are directional derivatives with respect to the two perpendicular axes directions $e_1$ and $e_2$, using assumptions (i), (ii), (iii) we have that $\partial_a(\Phi_{k+1}(u,v))^T$ and $\partial_i(\Phi_{k+1}(u,v))^T$ are uniformly bounded in view of Corollary 3.1. Recalling also that $\theta_k$ denotes a vector in $\mathbb{R}^{N \times 3}$ with all its entries behaving as $o(\lambda_k^4) + O \left( \frac{1}{\sigma} \right)$ and $\lambda_3 \sigma > 1$ (in light of assumption (iv)), this allows us to obtain

$$\partial_a(r_{k+1}^j(u,v))^T \land \partial_i(r_{k+1}^j(u,v))^T = \lambda_k^{2k} \left( \det(J \Psi_{k+1}(u,v))^T ((q_0^j)^T \land (q_1^j)^T) + o(1) \right),$$

$$(u,v) \in \omega_{k+1}^j, \ j \in \mathbb{Z}_{3n}.$$

Therefore, when computing the sequence $\{n_{k+1}^j(u,v), (u,v) \in \omega_{k+1}^j, k \geq 0\}$ of normal vectors to the $j$-th surface patch we obtain

$$n_{k+1}^j(u,v) = \frac{\partial_a(r_{k+1}^j(u,v))^T \land \partial_i(r_{k+1}^j(u,v))^T}{\| \partial_a(r_{k+1}^j(u,v))^T \land \partial_i(r_{k+1}^j(u,v))^T \|_2} \left( \det(J \Psi_{k+1}(u,v))^T ((q_0^j)^T \land (q_1^j)^T) + o(1) \right) \left( \| ((q_0^j)^T \land (q_1^j)^T) + o(1) \|_2 \right),$$

$$(u,v) \in \omega_{k+1}^j, \ j \in \mathbb{Z}_{3n}.$$

Again in view of Proposition 3.1 and Corollary 3.1 we can write

$$\lim_{k \to +\infty} \sup_{(u,v) \in \Omega_{k+1}} \| \partial_u \Phi_{k+1}(u,v)^T - \partial_u \Phi(u,v)^T \|_\infty = 0,$$

and

$$\lim_{k \to +\infty} \sup_{(u,v) \in \Omega_{k+1}} \| \partial_v \Phi_{k+1}(u,v)^T - \partial_v \Phi(u,v)^T \|_\infty = 0.$$

From the latter we obtain

$$\lim_{k \to +\infty} \sup_{(u,v) \in \Omega_{k+1}} \| J \Psi_{k+1}(u,v)^T - \Phi(u,v)^T \|_\infty = 0$$

and also

$$\lim_{k \to +\infty} \sup_{(u,v) \in \Omega_{k+1}} \| \partial_u \Phi_{k+1}(u,v)^T - \partial_u \Phi(u,v)^T \|_\infty = 0.$$

Therefore, taking into account that

$$\lim_{k \to +\infty} \sup_{(u,v) \in \Omega_{k+1}} \| r_{k+1}(u,v) - r_c \|_\infty = 0$$

and

$$n(r_c) := \text{sign}(\det(J \Phi(0,0)^T) (q_0^0)^T \land (q_1^0)^T \| ((q_0^0)^T \land (q_1^0)^T \|_2),$$
it has been proven that, for all \( j \in \mathbb{Z} \),
\[
\lim_{k \to +\infty} \sup_{(u, v) \in \omega_{k+1}} \| \frac{n}{k+1}(u, v) - n(r) \|_\infty = 0.
\]

Hence the limit surface \( r \) obtained by the non-stationary subdivision scheme \( S \) is tangent plane continuous at the limit point \( r_c = r(0,0) \). \( \square \)

5 Application examples

In this section we use Theorem 4.1 to study the convergence of two non-stationary subdivision schemes in the neighborhood of extraordinary elements. Also, we use Theorem 4.2 to prove that the limit surfaces obtained by such schemes are tangent plane continuous at the limit points of the corresponding extraordinary elements. This proves a conjecture given in [17, Section 5] where only numerical evidence was shown.

Both examples deal with approximating subdivision schemes defined on quadrilateral meshes of arbitrary topology. As a matter of fact, although our theorems apply also to non-stationary interpolatory schemes and to schemes defined on triangular meshes, we are only aware of the above examples of non-stationary subdivision schemes.

5.1 Generalized trigonometric spline surfaces of order 3

In [21], the authors presented a non-stationary subdivision scheme which produces tensor-product trigonometric spline surfaces of order 3 except in the neighborhood of extraordinary faces. This non-stationary scheme can be seen as a generalization of the well-known stationary Doo-Sabin’s scheme [13] yielding polynomial spline surfaces of order 3 except in the neighborhood of extraordinary faces. In \( \mathcal{E}^{(1)} \), Doo-Sabin’s scheme is described by the subdivision mask
\[
c = \begin{pmatrix}
\frac{1}{16} & \frac{3}{16} & \frac{3}{16} & \frac{1}{16} \\
\frac{3}{16} & \frac{9}{16} & \frac{9}{16} & \frac{3}{16} \\
\frac{3}{16} & \frac{9}{16} & \frac{9}{16} & \frac{3}{16} \\
\frac{1}{16} & \frac{3}{16} & \frac{3}{16} & \frac{1}{16}
\end{pmatrix},
\]
while in \( \mathcal{E}^{(1)} \) the refinement rules are written in terms of a subdivision matrix \( S \) having the structure in (6) with blocks
\[
B_0 = \begin{pmatrix}
\frac{1}{4n} + \frac{1}{8} & 0 & 0 & 0 \\
\frac{3}{16} & 0 & 0 & 0 \\
\frac{3}{16} & 0 & 0 & 0 \\
\frac{3}{16} & 0 & 0 & 0
\end{pmatrix}, \\
B_1 = \begin{pmatrix}
\frac{1}{4n} + \frac{1}{8} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
B_i = \begin{pmatrix}
\frac{1}{4n} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
B_{n-1} = \begin{pmatrix}
\frac{3}{16} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
It is a well-known fact that Doo-Sabin’s scheme is convergent both in $R^{(1)}$ and in $E^{(1)}$, and the limit surface is $C^1$-continuous in correspondence to regular regions of the mesh and tangent plane continuous at the limit points of extraordinary faces. Moreover, in $R^{(1)}$ the associated subdivision symbol is
\[
c(z_1, z_2) = \frac{(z_1 + 1)^3(z_2 + 1)^3}{16},
\]
namely it contains the factor $(1 + z_1)(1 + z_2)$. Thus it verifies assumption $(i)$ of Theorem 4.1 and assumption $(i)$ of Theorem 4.2. The non-stationary scheme in $[21]$, is described in $R^{(1)}$ by the $k$-level mask $(k \geq 1)$
\[
e(k) = \begin{pmatrix}
c_{4,k} & b_k + c_{4,k} & b_k + c_{4,k} & c_{4,k} \\
b_k + c_{4,k} & a_k + c_{4,k} & a_k + c_{4,k} & b_k + c_{4,k} \\
b_k + c_{4,k} & a_k + c_{4,k} & a_k + c_{4,k} & b_k + c_{4,k} \\
c_{4,k} & b_k + c_{4,k} & b_k + c_{4,k} & c_{4,k}
\end{pmatrix},
\]
where
\[
c_{n,k} = \frac{1}{4n \cos^2 \left( \frac{h}{2} \right) \cos^2 \left( \frac{h}{2^{k-1}} \right)}, \quad n \in \mathbb{N}, \quad n \geq 4, \quad h \in \left[ 0, \frac{\pi}{3} \right), \quad k \geq 1,
\]
and
\[
a_k = \frac{1}{4 \cos^2 \left( \frac{h}{2} \right) \cos \left( \frac{h}{2^{k-1}} \right)} + \frac{1}{4 \cos^2 \left( \frac{h}{2^{k-1}} \right)}, \quad b_k = \frac{1}{8 \cos^2 \left( \frac{h}{2} \right) \cos \left( \frac{h}{2^{k-1}} \right)},
\]
h $\in \left[ 0, \frac{\pi}{3} \right)$, $k \geq 1$.

The associated subdivision symbol is therefore
\[
c^{(k)}(z_1, z_2) = \frac{i^\frac{h}{2^{k-1}}(z_1+1)(z_2+1)(z_1+i^\frac{h}{2^{k-1}})(z_1+i^\frac{h}{2^{k-1}}+1)(z_2+i^\frac{h}{2^{k-1}}+1)}{(e^{i\frac{h}{2^{k-2}}+1})^2(e^{i\frac{h}{2^{k-1}}+1})^2},
\]
which contains the factor $(1 + z_1)(1 + z_2)$, thus satisfying assumption $(ii)$ of Theorem 4.2. Differently, in $E^{(1)}$ the refinement rules are given in terms of the $k$-level matrix $S_k$ having the structure in $(6)$ with blocks
\[
B_{0,k} = \begin{pmatrix}
a_k + c_{n,k} & 0 & 0 & 0 \\
a_k + c_{4,k} & b_k + c_{4,k} & 0 & 0 \\
a_k + c_{4,k} & b_k + c_{4,k} & c_{4,k} & b_k + c_{4,k} \\
a_k + c_{4,k} & 0 & 0 & b_k + c_{4,k}
\end{pmatrix}, \quad B_{1,k} = \begin{pmatrix}
b_k + c_{n,k} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
b_k + c_{n,k} & c_{4,k} & 0 & 0
\end{pmatrix},
\]

\[
B_{i,k} = \begin{pmatrix}
c_{n,k} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad i = 2, \ldots, n-2, \quad B_{n-1,k} = \begin{pmatrix}
b_k + c_{n,k} & 0 & 0 & 0 \\
b_k + c_{4,k} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Using $(25)$ and $(27)$, we verify that the stationary and non-stationary subdivision schemes are asymptotically equivalent of order 1. To see it, we use the Lagrange form of the remainder of the Taylor expansion to write
\[
\cos(2^{-k}h) = 1 - \frac{h^2}{2}2^{-2k} + \frac{h^4}{24}2^{-4k} \cos(\xi), \quad \xi \in (0, 2^{-k}h),
\]
and
\[
\cos^2(2^{-kh}) = 1 - h^2 2^{-2k} + \frac{h^4}{3} 2^{-4k} \cos(2\xi), \quad \xi \in (0, 2^{-kh}).
\]
The previous expression allows us to get the bounds
\[
|a_k - \frac{1}{2}| \leq \frac{3}{4k}, \quad |b_k - \frac{1}{8}| \leq \frac{9}{32k}, \quad |c_{n,k} - \frac{1}{4n}| \leq \frac{n^{-1}C}{4k}, \quad \forall n \geq 4
\]
with $A$, $B$, $C$ finite positive constants independent of $n$ and $k$.

The latter bounds can then be used to show that
\[
\| S_{e(k)} - S \|_\infty \leq \frac{A + 2B + C}{2k} \sum_{k=1}^{\infty} \frac{1}{2k^2} < +\infty.
\]
Summarizing, assumptions $(i)$ -- $(iii)$ of Theorem 4.2 and assumption $(ii)$ of Theorem 4.1 are satisfied. Next, we show that $\| S_k - S \|_\infty \leq \frac{M_0}{k}$ for all $k \geq 1$, $n \geq 5$ and $h \in [0, \frac{1}{2}]$, with $M$ a finite positive constant. Indeed, by (26) and (28) we have
\[
\| S_k - S \|_\infty \leq \| B_{0,k} - B_0 \|_\infty + \| B_{1,k} - B_1 \|_\infty + \sum_{i=2}^{n-2} \| B_{i,k} - B_i \|_\infty + \| B_{n-1,k} - B_{n-1} \|_\infty.
\]
Since
\[
\| B_{0,k} - B_0 \|_\infty = \max \left\{ |a_k + c_{n,k} - \frac{1}{4n} + \frac{1}{2}|, |a_k + c_{n,k} - \frac{9}{32} + \frac{1}{8}|, |b_k + c_{n,k} - \frac{3}{16}|, |a_k + c_{n,k} - \frac{3}{16} + 2|b_k + c_{n,k} - \frac{3}{16}|, |a_k + c_{n,k} - \frac{1}{8}| + |c_{n,k} - \frac{1}{16}| \right\},
\]
\[
= \max \left\{ |a_k + c_{n,k} - \frac{1}{4n} + \frac{1}{2}|, |a_k + c_{n,k} - \frac{9}{32} + \frac{1}{8}|, |b_k + c_{n,k} - \frac{3}{16}|, |c_{n,k} - \frac{1}{16}| \right\},
\]
\[
\leq \max \left\{ |a_k - \frac{1}{2}| + |c_{n,k} - \frac{1}{4n}|, |a_k - \frac{1}{2}| + 2|b_k - \frac{1}{8}| + |c_{n,k} - \frac{1}{16}| \right\},
\]
\[
\leq \frac{1}{2} \max \left\{ A + n^{-1}C, A + 2B + C \right\} = \frac{M_0}{2k},
\]
\[
\| B_{1,k} - B_1 \|_\infty = \| B_{n-1,k} - B_{n-1} \|_\infty
\]
\[
= \max \left\{ |b_k + c_{n,k} - \frac{1}{4n} + \frac{1}{2}|, |b_k + c_{n,k} - \frac{3}{16}|, |c_{n,k} - \frac{1}{16}| \right\},
\]
\[
\leq \max \left\{ |b_k - \frac{1}{2}| + |c_{n,k} - \frac{1}{4n}|, |b_k - \frac{1}{2}| + 2|c_{n,k} - \frac{1}{16}| \right\},
\]
\[
\leq \frac{1}{2} \max \left\{ B + n^{-1}C, B + 2C \right\} = \frac{M_1}{2k},
\]
\[
\| B_{i,k} - B_i \|_\infty = \| c_{n,k} - \frac{1}{4n} \| \leq \frac{n^{-1}C}{2k}, \quad i = 2, \ldots, n-2,
\]
for $n \geq 5$ we finally obtain the bound
\[
\| S_k - S \|_\infty \leq \frac{M_0 + M_1 + (1 - \frac{3}{n})C}{4k} \leq \frac{M}{4k},
\]
with $M := M_0 + M_1 + C$ a finite positive constant independent of $n$ and $k$. In other words assumption $(iii)$ of Theorem 4.1 and assumption $(iv)$ of Theorem 4.2 are satisfied. Since $S$ has a dominant single eigenvalue $\lambda_0 = 1$ and a subdominant eigenvalue $0.5 < \lambda_1 < 1$ with algebraic and geometric multiplicity 2 (i.e. is a double non-defective eigenvalue), all the assumptions of Theorem 4.1 and Theorem 4.2 are verified with $\sigma = 4$. Hence, the non-stationary version of Doo-Sabin’s scheme is convergent at extraordinary faces and the limit surfaces obtained by such a scheme are tangent plane continuous at the limit points of extraordinary faces.
5.2 Generalized exponential spline surfaces of order $\geq 3$

In [30] a generalization of order-$d$ polynomial spline surfaces to quadrilateral meshes of arbitrary topology has been proposed. For $d = 4$, the refinement rules of the corresponding scheme are the rules of Catmull-Clark’s subdivision scheme [2] which, in the regular regions of the mesh, can be given in terms of the subdivision mask

$$c = \begin{pmatrix}
\frac{1}{64} & \frac{1}{16} & \frac{3}{32} & \frac{1}{16} & \frac{1}{64} \\
\frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} \\
\frac{3}{32} & \frac{3}{8} & \frac{9}{16} & \frac{3}{8} & \frac{3}{32} \\
\frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} \\
\frac{1}{64} & \frac{1}{16} & \frac{3}{32} & \frac{1}{16} & \frac{1}{64}
\end{pmatrix}.
$$

(D29)

Differently, in the neighborhood of an extraordinary vertex of valence $n \geq 5$, the subdivision matrix $S_k$ of the order-4 scheme is as in (7) with $\bar{\alpha} = 1 - \frac{7}{4n}$, $\bar{\beta} = \left(\frac{3}{2n^2}, \frac{1}{4n^2}, 0, 0, 0, 0\right)^T$, $\bar{\gamma} = \left(\frac{3}{8}, \frac{1}{4}, \frac{3}{32}, \frac{1}{16}, \frac{1}{64}, \frac{1}{16}\right)^T$ and $6 \times 6$ blocks

$$\bar{B}_0 = \begin{pmatrix}
\frac{3}{8} & \frac{1}{16} & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\
\frac{9}{16} & \frac{3}{32} & \frac{3}{16} & 0 & 0 \\
\frac{3}{8} & \frac{1}{16} & 0 & 0 & 0 \\
\frac{3}{32} & \frac{9}{16} & \frac{1}{64} & \frac{3}{32} & \frac{1}{16} \\
\frac{1}{16} & \frac{3}{8} & 0 & 0 & 0
\end{pmatrix}, \quad \bar{B}_1 = \begin{pmatrix}
\frac{1}{64} & 0 & 0 & 0 & 0 \\
\frac{1}{4} & 0 & 0 & 0 & 0 \\
\frac{1}{16} & 0 & 0 & 0 & 0 \\
\frac{3}{32} & 0 & 0 & 0 & 0 \\
\frac{3}{16} & 0 & 0 & 0 & 0 \\
\frac{1}{16} & \frac{3}{16} & 0 & 0 & 0
\end{pmatrix},
$$

$$\bar{B}_i = 0_{6 \times 6}, \quad i = 2, \ldots, n - 2, \quad \bar{B}_{n-1} = \begin{pmatrix}
\frac{1}{64} & \frac{1}{16} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\frac{1}{64} & \frac{3}{32} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

It is a well-known fact that Catmull-Clark’s scheme is convergent both in $R^{(1)}$ and in $E^{(1)}$, and the limit surface is $C^2$ continuous in correspondence to regular regions of the mesh and tangent plane continuous at the limit points of extraordinary vertices. The subdivision symbol associated to the scheme is

$$c(z_1, z_2) = \frac{(z_1 + 1)^4(z_2 + 1)^4}{64},$$

containing the factor $(1 + z_1)(1 + z_2)$. Thus it verifies assumption $(i)$ of Theorem 4.1 and assumption $(i)$ of Theorem 4.2.

The family of approximating subdivision schemes discussed in [17] is a non-stationary extension of the family in [30], and provides a generalization of order-$d$ exponential spline surfaces to quadrilateral meshes of arbitrary topology. The refinement rules defining this family depend on the subdivision level $k$ and, in particular, they are chosen in such a way that the schemes could reproduce particular shapes such as spheres, tori or conical shapes.
when the initial meshes are suitably selected. In addition, when the initial mesh is regular, these schemes are tensor-product exponential splines (namely tensor-product polynomial splines but also tensor-product trigonometric and hyperbolic splines) [25]. More precisely, the $k$-level ($k \geq 1$) refinement rules characterizing this family of non-stationary subdivision schemes depend on a $k$-level parameter $v_k$ defined as

$$v_k = \frac{1}{2} \left( e^{\frac{i \theta}{2}} + e^{-\frac{i \theta}{2}} \right), \quad \theta \in [0, \pi) \cup i(0, 2acosh(500)), \quad k \geq 1$$

which satisfies

$$\begin{align*}
(a) & \quad v_{k+1} = \sqrt{v_k + 1}, \\
(b) & \quad \lim_{k \to +\infty} v_k = 1.
\end{align*}$$

Note that

$$v_1 = \frac{1}{2} \left( e^{\frac{i \theta}{2}} + e^{-\frac{i \theta}{2}} \right) = \begin{cases} \cos \left( \frac{\theta}{2} \right) & \text{if } \theta \in (0, \pi), \\
\cosh \left( \frac{\Im(\theta)}{2} \right) & \text{if } \theta \in i(0, 2acosh(500)).
\end{cases}$$

For the non-stationary approximating scheme of order $d = 4$ (non-stationary version of Catmull-Clark’s scheme), the $k$-level subdivision mask to be used in the regular regions of the mesh is

$$c^{(k)} = \begin{pmatrix} c_k & e_k & b_k & e_k & c_k \\
e_k & \frac{1}{4} & d_k & \frac{1}{4} & e_k \\
b_k & d_k & a_k & d_k & b_k \\
e_k & \frac{1}{4} & d_k & \frac{1}{4} & e_k \\
c_k & e_k & b_k & e_k & c_k \end{pmatrix}, \quad (30)$$

where

$$a_k = \frac{(2v_k + 1)^2}{4(v_k + 1)^2}, \quad b_k = \frac{4v_k + 2}{16(v_k + 1)^2}, \quad c_k = \frac{1}{16(v_k + 1)^2},$$

$$d_k = \frac{2v_k + 1}{4(v_k + 1)}, \quad e_k = \frac{1}{8(v_k + 1)}.$$

Hence, the associated symbol reads as

$$c^{(k)}(z_1, z_2) = \frac{(z_1 + 1)^2(z_2 + 1)^2(z_1 e^{i \frac{\theta}{2}} + 1)(z_1 + e^{i \frac{\theta}{2}})(z_2 e^{i \frac{\theta}{2}} + 1)(z_2 + e^{i \frac{\theta}{2}})}{4(e^{i \frac{\theta}{2}} + 1)^4},$$

and contains the factor $(1 + z_1)(1 + z_2)$.

Differently, the $k$-level subdivision matrix $\tilde{S}_k$, defined near an extraordinary vertex of valence $n \geq 5$, is of the form (7) with

$$\tilde{S}_k = 1 - \frac{4v_k + 2}{n(v_k + 1)^2}, \quad \tilde{S}_k = \begin{pmatrix} 2(2v_k + 1) \\ n^2(v_k + 1)^2 \\ n^2(v_k + 1)^2 \end{pmatrix}, \quad 0, 0, 0, 0^T,$$

$$\tilde{S}_k = \begin{pmatrix} \frac{1}{4} & b_k, e_k, c_k, e_k \end{pmatrix}^T.$$
and $6 \times 6$ blocks

$$
\tilde{B}_{0,k} = \begin{pmatrix}
   d_k & e_k & 0 & 0 & 0 & 0 \\
   \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
   a_k & b_k & b_k & c_k & 0 & 0 \\
   d_k & d_k & e_k & e_k & 0 & 0 \\
   b_k & a_k & c_k & b_k & c_k & b_k \\
   e_k & d_k & 0 & 0 & 0 & e_k
\end{pmatrix}, \quad \tilde{B}_{1,k} = \begin{pmatrix}
   e_k & 0 & 0 & 0 & 0 & 0 \\
   \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
   c_k & 0 & 0 & 0 & 0 & 0 \\
   e_k & 0 & 0 & 0 & 0 & 0 \\
   b_k & 0 & c_k & 0 & 0 & 0 \\
   d_k & 0 & e_k & 0 & 0 & 0
\end{pmatrix},
$$

$$
\tilde{B}_{i,k} = \mathbf{0}_{6 \times 6}, i = 2, \ldots, n - 2, \quad \tilde{B}_{n-1,k} = \begin{pmatrix}
   c_k & b_k & 0 & 0 & 0 & c_k \\
   0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

The choice of $v_k$ specifies the kind of spline surface we get in the limit, in correspondence to the regular regions of the mesh. In fact, if $v_k < 1$ the schemes yield trigonometric splines, if $v_k = 1$ polynomial splines and if $v_k > 1$ hyperbolic splines.

In [17], the authors prove that the limit surface obtained by applying the generalized spline schemes of order $d$ to a regular mesh is $C^{d-2}$-continuous, while in the neighborhood of extraordinary elements the tangent plane continuity of the limit surface is shown only by numerical evidence. Here we use Theorem 4.1 and Theorem 4.2 to prove convergence and tangent plane continuity of the limit surfaces at the limit points of extraordinary elements. To prove that the non-stationary version of Catmull-Clark’s scheme is convergent and produces tangent plane continuous surfaces at the limit points of extraordinary vertices, we first show that the subdivision masks $c$ and $e_k$ in (29) and (30) are asymptotically equivalent of order 1. To this purpose we again write

$$
\cos(2^{-k} \theta) = 1 - \frac{\theta^2}{2} 2^{-2k} + \frac{\theta^4}{24} 2^{-4k} \cos(\xi), \quad \xi \in (0, 2^{-k} \theta),
$$

$$
\cos^2(2^{-k} \theta) = 1 - \theta^2 2^{-2k} + \frac{\theta^4}{24} 2^{-4k} \cos(2\xi), \quad \xi \in (0, 2^{-k} \theta),
$$

and

$$
\cosh(2^{-k} \theta) = 1 + \theta^2 2^{-2k} + \frac{\theta^4}{24} 2^{-4k} \cosh(\eta), \quad \eta \in (0, 2^{-k} \theta),
$$

$$
\cosh^2(2^{-k} \theta) = 1 + 2 \theta^2 2^{-2k} + \frac{\theta^4}{24} 2^{-4k} \cosh(2\eta), \quad \eta \in (0, 2^{-k} \theta),
$$

from which we obtain

$$
|d_k - \frac{9}{10}| \leq \frac{\vartheta}{2^k}, \quad |b_k - \frac{3}{32}| \leq \frac{\vartheta}{2^k}, \quad |c_k - \frac{1}{16}| \leq \frac{\vartheta}{2^k}, \quad |d_k - \frac{3}{8}| \leq \frac{\vartheta}{2^k}, \quad |e_k - \frac{1}{16}| \leq \frac{\vartheta}{2^k}
$$

with $A, B, C, D, E$ finite positive constants independent of $n$ and $k$. Thus, we get

$$
\|S_c^{(k)} - S_c\|_{\infty} = \max \left\{ \left| a_k - \frac{9}{10} \right| + 4 \left| b_k - \frac{3}{32} \right| + 4 \left| c_k - \frac{1}{16} \right|, \right.

\left. 2 \left| d_k - \frac{3}{8} \right| + 4 \left| e_k - \frac{1}{16} \right| \right\}

\leq \frac{1}{4} \max \{ A + 4B + 4C, 2D + 4E \},
$$

so that

$$
\sum_{k=1}^{+\infty} 2^k \|S_c^{(k)} - S_c\|_{\infty} \leq \max \{ A + 4B + 4C, 2D + 4E \} \sum_{k=1}^{+\infty} \frac{1}{2^k} < +\infty.
$$
As a consequence, assumptions (i)-(iii) of Theorem 4.2 and assumption (ii) of Theorem 4.1 are satisfied.

Next, we use formula (8) to transform the matrices $\tilde{S}$ and $\tilde{S}_k$ in the block-circulant matrices denoted by $S$ and $S_k$, and verify the existence of a finite positive constant $M$ independent of $n$ and $k$ such that $\|S_k - S\|_\infty \leq \frac{M}{k^2}$ for all $k \geq 1$, $n \geq 5$ and $\theta \in [0, \pi) \cup i(0, 2\text{acosh}(500))$.

As before, we first write

$$\|S_k - S\|_\infty \leq \|B_{0,k} - B_0\|_\infty + \|B_{1,k} - B_1\|_\infty + \sum_{i=2}^{n-2} \|B_{i,k} - B_i\|_\infty + \|B_{n-1,k} - B_{n-1}\|_\infty,$$

and explicitly compute each norm on the right hand side as

$$\|B_{0,k} - B_0\|_\infty = \left\| \frac{\alpha_n - \alpha}{n} \tilde{\beta}_{k} - \tilde{\beta}_0 \right\|_\infty$$

$$= \max \left\{ \frac{1}{n} |d_k - \frac{3}{\alpha} | + |e_k - \frac{3}{\alpha n} |, \right.$$  

$$\left. \frac{1}{n} |d_k - \frac{3}{\alpha} | + |e_k - \frac{3}{\alpha n} |, \right.$$  

$$\left. \left( \frac{1}{2} + 1 \right) |d_k - \frac{3}{\alpha} | + |e_k - \frac{3}{\alpha n} |, \right.$$  

$$\left. \left( \frac{1}{2} + 1 \right) |d_k - \frac{3}{\alpha} | + |e_k - \frac{3}{\alpha n} |. \right.$$  

$$\|B_{1,k} - B_1\|_\infty = \left\| \frac{\alpha_n - \alpha}{n} \tilde{\beta}_{k} - \tilde{\beta}_1 \right\|_\infty$$

$$= \max \left\{ \frac{1}{n} |d_k - \frac{3}{\alpha} | + |e_k - \frac{3}{\alpha n} |, \right.$$  

$$\left. \frac{1}{n} |d_k - \frac{3}{\alpha} | + |e_k - \frac{3}{\alpha n} |, \right.$$  

$$\left. \left( \frac{1}{2} + 1 \right) |d_k - \frac{3}{\alpha} | + |e_k - \frac{3}{\alpha n} |, \right.$$  

$$\left. \left( \frac{1}{2} + 1 \right) |d_k - \frac{3}{\alpha} | + |e_k - \frac{3}{\alpha n} |. \right.$$  

$$\|B_{i,k} - B_i\|_\infty = \left\| \frac{\alpha_n - \alpha}{n} \tilde{\beta}_{k} - \tilde{\beta}_i \right\|_\infty$$

$$= \max \left\{ \frac{1}{n} |d_k - \frac{3}{\alpha} | + |e_k - \frac{3}{\alpha n} |, \right.$$  

$$\left. \frac{1}{n} |d_k - \frac{3}{\alpha} | + |e_k - \frac{3}{\alpha n} |, \right.$$  

$$\left. \left( \frac{1}{2} + 1 \right) |d_k - \frac{3}{\alpha} | + |e_k - \frac{3}{\alpha n} |, \right.$$  

$$\left. \left( \frac{1}{2} + 1 \right) |d_k - \frac{3}{\alpha} | + |e_k - \frac{3}{\alpha n} |. \right.$$  

Moreover, in view of the bounds

$$\left| \frac{3}{2} - \frac{4y + 3}{(y + 1)^2} \right| = 16 \left| \frac{3}{2} - \frac{4y + 3}{16(y + 1)^2} \right|$$

$$\leq \left| \frac{3}{2} - \frac{4y + 3}{16(y + 1)^2} \right| + \left| \frac{3}{2} - \frac{4y + 3}{16(y + 1)^2} \right|$$

$$\leq \frac{16C}{4k^2},$$

$$\frac{2(2v + 1)}{(v + 1)^2} - \frac{3}{2} = 16 \left| \frac{2(2v + 1)}{16(v + 1)^2} - \frac{3}{32} \right| \leq \frac{16C}{4k^2}.$$
we are finally able to bound the norms of the blocks as

\[ \|B_{0,k} - B_0\|_\infty \leq \max \left\{ \frac{32n^{-2}(b+c)}{d^3}, \frac{(n^{-1}+1)d+e}{d^3}, \frac{(n^{-1}+2)b+a+c}{d^3}, \frac{(n^{-1}+2)c+2d}{d^3}, \right\} \]

\[ \leq \max \left\{ \frac{32(b+c)}{d^3}, \frac{2d+e}{d^3}, \frac{3a+4c+2d}{d^3} \right\} =: \frac{M_0}{d^3}, \]

\[ \|B_{1,k} - B_1\|_\infty \leq \max \left\{ \frac{32n^{-2}(b+c)}{d^3}, \frac{n^{-1}d+e}{d^3}, \frac{n^{-1}a+c}{d^3} \right\} \]

\[ \leq \max \left\{ \frac{32(b+c)}{d^3}, \frac{d+e}{d^3}, \frac{c+2d}{d^3} \right\} =: \frac{M_1}{d^3}, \]

\[ \|B_{n,k} - B_{n-1}\|_\infty \leq \max \left\{ \frac{32n^{-2}(b+c)}{d^3}, \frac{n^{-1}d+2e}{d^3}, \frac{n^{-1}a+2c}{d^3} \right\} \]

\[ \leq \max \left\{ \frac{32(b+c)}{d^3}, \frac{2d+2e}{d^3}, \frac{2c}{d^3} \right\} =: \frac{M_3}{d^3}, \]

Hence, for all \( n \geq 5, \)

\[ \|S_k - S\|_\infty \leq \frac{M_0 + M_1 + (1 - \frac{3}{n})M_2 + M_3}{4^k} \leq \frac{M}{4^k}, \]

with \( M := M_0 + M_1 + M_2 + M_3 \) a finite positive constant independent of \( n \) and \( k \).

The above proves that (iii) of Theorem 4.1 is satisfied. Moreover, since \( S \) has a dominant single eigenvalue \( \lambda_0 = 1 \) and a subdominant eigenvalue \( 0.5 < \lambda_1 < 1 \) with algebraic and geometric multiplicity 2 (double non defective eigenvalue), (iv) of Theorem 4.2 is also satisfied with \( \sigma = 4 \). It follows that all the assumptions of Theorem 4.1 and Theorem 4.2 are verified. Thus, this non-stationary version of Catmull-Clark’s scheme is convergent at extraordinary vertices and the limit surfaces obtained by such a scheme are tangent plane continuous at the limit points of extraordinary vertices.

Acknowledgments

This research has been accomplished within RITA (Research ITalian network on Approximation).

References

1. Badoual, A., Novara, P., Romani, L., Schmitter, D., Unser, M.: A non-stationary subdivision scheme for the construction of deformable models with sphere-like topology. Graphical Models 94, 38–51 (2017)
2. Catmull, E., Clark, J.: Recursively generated B-splines surfaces on arbitrary topological meshes. Comput. Aided Design 10(6), 350–355 (1978)
3. Charina, M., Conti, C., Romani, L.: Reproduction of exponential polynomials by multivariate non-stationary subdivision schemes with a general dilation matrix. Numer. Math. 127(2), 223–254 (2014)
4. Charina, M., Conti, C., Guglielmi, N., Protasov, V.: Regularity of non-stationary subdivision: a matrix approach. Numer. Math. 135(3), 639–678 (2017)
5. Conti, C., Cotronei, M., Sauer, T.: Factorization of Hermite subdivision operators preserving exponentials and polynomials. Adv. Comput. Math. 42, 1055–1079 (2016)
6. Conti, C., Cotronei, M., Sauer, T.: Convergence of level-dependent Hermite subdivision schemes. Appl. Numer. Math. 116, 119–128 (2017)
7. Conti, C., Dyn, N., Manni, C., Mazure, M.-L.: Convergence of univariate non-stationary subdivision schemes via asymptotic similarity. Comput. Aided Geom. Design 37, 1–8 (2015)
8. Conti, C., Romani, L., Unser, M.: Ellipse-preserving Hermite interpolation and subdivision. J. Math. Anal. Appl. 426, 211–227 (2015)
9. Cotronei, M., Sissouno, N.: A note on Hermite multiwavelets with polynomial and exponential vanishing moments. Appl. Numer. Math. 120, 21–34 (2017)
10. Delgado-Gonzalo, R., Thevenaz, P., Seelamantula, C.S., Unser, M.: Snakes with an ellipse-reproducing property. IEEE Trans. Image Process. 21, 1258–1271 (2012)
11. Delgado-Gonzalo, R., Thevenaz, P., Unser, M.: Exponential splines and minimal-support bases for curve representation. Comput. Aided Geom. Des. 29, 109–128 (2012)
12. Delgado-Gonzalo, R., Unser, M.: Spline-based framework for interactive segmentation in biomedical imaging. IRBM Ingenierie et Recherche Biomedicale/BioMed. Eng. Res. 34, 235–243 (2013)
13. Doo, D., Sabin, M.: Behavior of recursive division surfaces near extraordinary points. Comput. Aided Design 10(6), 356–360 (1978)
14. Dyn, N., Kounchev, O., Levin, D., Render, H.: Regularity of generalized Daubechies waveletets reproducing exponential polynomials with real-valued parameters. Appl. Comput. Harmonic Anal. 37, 288–306 (2014)
15. Dyn, N., Levin, D.: Analysis of asymptotically equivalent binary subdivision schemes. J. Math. Anal. Appl. 193, 594–621 (1995)
16. Fang, M., Ma, W., Wang, G.: A generalized surface subdivision scheme of arbitrary order with a tension parameter. Comput. Aided Design 49, 8–17 (2014)
17. Gotsman, C., Gunhold, S., Kobbelt, L.: Simplification and compression of 3D meshes. Tutorials on Multiresolution in Geometric Modelling, Part of the series Mathematics and Visualization, A. Iske, E. Quak and M.S. Floater eds., 319–361 (2002)
18. Han, B.: Non homogeneous weavelet systems in high dimensions. Appl.Comput.Harmonic Anal.32, 169–196 (2012)
19. Hughes, T.J.R., Cottrell, J.A., Bazilevs, Y.: Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. Computer Methods in Applied Mechanics and Engineering, 194 (39-41), 4135–4195 (2005)
20. Jeong, B., Yoon, J.: Construction of Hermite subdivision schemes reproducing polynomials. J. Math. Anal. Appl. 451(1), 565–582 (2017)
21. Jia, R.Q., Lei, J.J.: Approximation by piecewise exponentials. SIAM J. Math. Anal. 22, 1776–1789 (1991)
22. Jena, M.K., Shunmugaraj, P., Das, P.C.: A non-stationary subdivision scheme for generalizing trigonometric spline surfaces to arbitrary meshes. Comput. Aided Geom. Design 20, 61–77 (2003)
23. Lee, Y.-J., Yoon, J.: Non-stationary subdivision schemes for surface interpolation based on exponential polynomials. Appl. Numer. Math. 60, 130–141 (2010)
24. Peters, J., Reif, U.: A unified approach to subdivision algorithms near extraordinary vertices. Comput. Aided Geom. Design 12, 153–174 (1995)
25. Romani, L.: A circle-preserving $C^2$ Hermite interpolatory subdivision scheme with tension control. Comput. Aided Geom. Design 27(1), 36–47 (2010)
26. Stam, J., Mederos, V.H., Sarlabous, J.E.: Exact evaluation of a class of nonstationary approximating subdivision algorithms and related applications. IMA J. Numer. Anal. 36(1), 380–399 (2016)
27. Uhlmann, V., Delgado-Gonzalo, R., Conti, C., Romani, L., Unser, M.: Exponential Hermite splines for the analysis of biomedical images. ICASSP, IEEE International Conference on Acoustics, Speech and Signal Processing - Proceedings 6853874, pp. 1631-1634 (2014)
28. Umlauf, G.: Analyzing the characteristic map of triangular subdivision schemes. Constr. Approx. 16, 145–155 (2000)
29. Vonesch, C., Blu, T., Unser, M.: Generalized Daubechies wavelet families. IEEE Trans.Signal Process. 55, 4415–4429 (2007)
30. Warren, J., Weimer, H.: Subdivision methods for geometric design. Morgan-Kaufmann, New York (2002)
31. Zorin, D.: A method for analysis of $C^1$-continuity of subdivision surfaces. SIAM J. Numer. Anal. 35(5), 1677–1708 (2000)
32. Zorin, D., Schröder, P., Sweldens, W.: Interpolating subdivision for meshes with arbitrary topology. In Proceedings of 23rd International Conference on Computer Graphics and Interactive Techniques (SIGGRAPH'96), 189–192 (1996)