A Noncommutative Deformation of Topological Field Theory∗

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(Dated: March 27, 2022)

Abstract

Cohomological Yang-Mills theory is formulated on a noncommutative differentiable four manifold through the $\theta$-deformation of its corresponding BRST algebra. The resulting noncommutative field theory is a natural setting to define the $\theta$-deformation of Donaldson invariants and they are interpreted as a mapping between the Chevalley-Eilenberg homology of noncommutative spacetime and the Chevalley-Eilenberg cohomology of noncommutative moduli of instantons. In the process we find that in the weak coupling limit the quantum theory is localized at the moduli space of noncommutative instantons.

∗ This work is dedicated to Professor Alberto García on the occasion of his 60th birthday.
1. INTRODUCTION

Quantum field theories on noncommutative spaces are very interesting theories that have recently motivated a great deal of work (for a recent review see, [1]). Quantum field theories of topological nature describing the cohomology of the moduli space of Yang-Mills instantons (with certain gauge group and instanton number $k$) constitute a class of interesting theories whose correlation functions of BRST-invariant field functionals $\mathcal{O}$ give rise to topological invariants known as Donaldson polynomial invariants [2]. A gravitational analog of Donaldson theory has been, by the first time, constructed in Ref. [3]. These kind of theories are known generically as Cohomological Field Theories and it is natural to ask about the formulation of a possible noncommutative deformation of these theories. A proposal for a noncommutative deformation of cohomological field theory was given in Ref. [4], where some relations to noncommutative solitons was found. However, in the present paper, we follow a different approach making emphasis in the definition of topological invariants. Formulating a noncommutative topological theory is another way of finding noncommutative topological invariants. Noncommutative classical invariants (as the Euler number or signature) are known in the literature from some years ago in the context of the standard formulation of spectral triples from noncommutative geometry [5] and recently has been also pursued by using the method of Moyal product and the Seiberg-Witten map [6].

Noncommutative gauge theories are gauge theories on noncommutative spaces which can be understood in terms of algebras of operators on certain Hilbert space. The Weyl-Wigner-Moyal correspondence establishes an isomorphic relation between it and the algebra of functions on $\mathbb{R}^n$, but instead of having the usual commutative product of functions we have a noncommutative and associative star product $\star$ called the Moyal product. For instance a noncommutative deformation of space $\mathbb{R}^n$ where the usual coordinates satisfy the commutation relations: $[x^\alpha, x^\beta] = i\theta^{\alpha\beta}$ with $x^\alpha \in \text{End}(L^2(\mathbb{R}^n))$, is given by a noncommutative and associative algebra $\mathcal{A}_\star \equiv \mathbb{R}_\star^n$ with the usual Moyal bracket of functions.

Usually gauge fields are defined as connections on $G$ gauge bundles $E$ over spacetime manifold $X$. In noncommutative geometry one substitutes vector bundles $E$ by the noncommutative Moyal deformation of the right (or left) $\mathcal{A}$-module denoted by $\mathcal{E}$. For instance if $S$ is a submanifold of $X$ and denoting $\mathcal{A}(S)$ as the set of sections of the gauge bundle $E$ over $S$. $\mathcal{A}(S)$ can be regarded as an $\mathcal{A}_\star(X)$-module that we denote $\mathcal{E}$, where $\mathcal{A}_\star(X)$ denotes
the Moyal algebra of functions on $X$. Alternatively the algebra $\mathcal{A}_\theta(X)$ will be denoted simply as $X_\theta$. Thus in noncommutative geometry the gauge bundle $E$ can be replaced by $\mathcal{E}$. A noncommutative gauge field $A^I_\alpha$ can be regarded as a connection in the $X_\theta$-module $\mathcal{E}$.

In the present paper we formulate a noncommutative deformation of Witten’s cohomological field theory by replacing its underlying Lie algebra by its enveloping algebra following [7]. We will show that this procedure leads to define a cohomology theory of the moduli space of noncommutative Yang-Mills instantons [8]. However a rather different approach proposed in Ref. [9] will be more useful for our purposes. Noncommutative instantons are instantons on a noncommutative deformation of euclidean spacetime and in [8] a noncommutative of ADHM description of instantons was proposed. This description gives rise to a definition of the moduli space of noncommutative instantons as a smooth manifold free from small instanton singularities, because these singularities are blow up by a smooth deformation of ADHM equations. In Ref. [10] the smooth structure of the moduli space of noncommutative instantons was corroborated and reinterpreted in terms of D-brane processes. The description of the ADHM constructions and its relation to noncommutative twistor transform was given in Ref. [11].

This paper is organized as follows: in section 2 we propose a noncommutative deformation of the BRST invariant gauge theory. We will mainly follow the notation of Ref. [2]. Observables of this deformed theory are discussed in section 3. Section 4 is devoted to provide a deformation of the Donaldson invariants and therefore to define noncommutative topological invariants and to give an expression for the noncommutative Donaldson invariants in terms of an integration of differential forms on the moduli space of noncommutative instantons. Throughout all the paper we have assumed that the metric field on the manifold $X$ is not a dynamical field (spectator field) and it is not noncommutatively coupled to noncommutative matter fields. Finally in section 5 we give our concluding remarks.

2. THE NONCOMMUTATIVE LAGRANGIAN

We start from a noncommutative deformation of the BRST-like invariant Lagrangian of the usual topological Yang-Mills theory given by

\[ \mathcal{L} = \int d^4x \sqrt{|g|} \left( \mathcal{F}^2 - \mathcal{D} \mathcal{F} \right) \]
The Yang-Mills field can be expanded in a basis \{t^I\} of \mathcal{U}(su(2), \text{ad}) as \(A_\alpha(x) = A^I_\alpha(x)t^I\) and therefore

\[ [A^I_\alpha ; A_J^J] = \frac{1}{2} \left\{ A^I_\alpha ; A_J^J \right\} [t_I, t_J] + \frac{1}{2} \left[ A^I_\alpha ; A^J_\beta \right] \{t_I, t_J\}, \tag{2} \]

where \{A^I ; B_J\} \equiv A \ast B \ast A. Here indices \(I, J\) runs over all possible number of independent generators of enveloping algebra \mathcal{U}(su(2), \text{ad}).

Then all the products of the generators \(t_I\) will be needed in order to close the algebra \mathcal{U}(su(2), \text{ad}). Its structure can be obtained by successively computing the commutators and anticommutators \([t^I, t^J] = f^{IJK}t^K\) and \\{t^I, t^J\} = d^{IJK}t^K.\n
Thus under the assumption \(h^{IJK}t^I = t^J \cdot t^K\), the field strength \(F_{\alpha\beta}\) can be written as

\[ F^I_{\alpha\beta} = \partial_\alpha A^I_\beta - \partial_\beta A^I_\alpha + h^{IJK} A^J_\alpha \ast A^K_\beta - h^{IJK} A^K_\beta \ast A^J_\alpha \]

\[ = \partial_\alpha A^I_\beta - \partial_\beta A^I_\alpha + \frac{1}{2} f^{IJK} \left\{ A^J_\alpha ; A^K_\beta \right\} + \frac{1}{2} d^{IJK} \left[ A^I_\alpha ; A^K_\beta \right], \tag{3} \]

where for definitiveness all fields and transformation parameters will be \mathcal{U}(su(2), \text{ad})-valued fields in the adjoint representation \text{ad} of \(su(2)\). Here \mathcal{U}(su(2), \text{ad}) is the universal enveloping algebra of the Lie algebra \(su(2)\) of \(SU(2)\). The field \(A_\mu(x)\) is a Yang-Mills gauge field with field strength \(F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha \ast A_\beta]\) and \(\tilde{F}^{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta}\). The commutator is defined as follows: \([A^I ; B_J] \equiv A \ast B \ast B \ast A\), and it satisfies the Leibnitz rule when acting on products of noncommutative fields. The \* product is the usual Moyal product \(f(x) \ast g(x) \equiv \exp \left(\frac{i}{2} \theta^{uv} \frac{\partial}{\partial x^u} \frac{\partial}{\partial y^v}\right) f(y)g(z)\bigg|_{y=z=x}\) together with the matrix multiplication. The field contents in \(L\) are: the gauge field \(A_\alpha\), the ghost fields \(\psi_\alpha\) and \(\phi\), the anti-ghost fields \(\lambda, \eta, \chi_{\alpha\beta}\) and an auxiliary field \(H_{\alpha\beta}\). All these fields \((A_\alpha, \psi_\alpha, \phi, \lambda, \eta, \chi_{\alpha\beta}, H_{\alpha\beta})\) are \mathcal{U}(su(2), \text{ad})-valued fields. Lagrangian \(L\) has a global symmetry identified with the ghost number \(U\). The assignation of \(U\) for all fields is respectively: \((0, 1, 2, -2, -1, -1, 0)\). In the Lagrangian \(L\) the covariant derivative \(D_\alpha\) is defined by \(D_\alpha \Theta = \partial_\alpha \Theta + [A_\alpha \ast \Theta]\).
where \( h^{IJK} = \frac{1}{2}(f^{IJK} + d^{IJK}) \).

The Lagrangian (1) is invariant under a noncommutative BRST-like transformation \([2]\). Thus the generic noncommutative functionals \( O \) satisfying the relation \( \delta_\varepsilon O = -i\varepsilon(x) \star \{Q \star O, \} \), where and \( Q \) is the BRST-like charge which it can be easily showed, with the help of \([12]\), that it satisfies \( Q^2 = 0 \) up to gauge transformations. In general \( O \) is a functional of the fields of the theory.

In Refs. \([13, 14]\) it was showed that fixing the topological symmetry given by the shift \( A_\alpha \rightarrow A_\alpha + \varepsilon_\alpha + D_\alpha \varepsilon \), there exists a genuine BRST symmetry which appears after BRST quantize the classical action \( \int_X \text{Tr} F_{\alpha \beta} \tilde{F}^{\alpha \beta} \). In the procedure in order to enforce the gauge constraint \( \partial \cdot A = 0 \) it is necessary to introduce a Yang-Mills triplet \((c, \bar{c}, b)\) with corresponding ghost number \( U \) to be \((1, -1, 0)\). In this noncommutative case the full BRST symmetry is described by the transformations:

\[
\begin{align*}
\delta A^I_\alpha &= D_{\alpha}^{\text{ad}} c^I + h^{IJK} [A^J_\alpha \star c^K], \\
\delta \psi^I_\alpha &= -D_{\alpha}^{\text{ad}} \phi^I - h^{IJK} [A^J_\alpha \star \phi^K] - h^{IJK} [c^J \star \psi^K], \\
\delta \phi^I &= -h^{IJK} [c^J \star \phi^K], \\
\delta \chi_{\alpha \beta} &= H_{\alpha \beta}, \\
\delta B_{\alpha \beta} &= 0, \\
\delta \lambda &= \eta, \\
\delta \eta &= 0,
\end{align*}
\]  

(4)

\[
\begin{align*}
\delta c^I &= \phi + \frac{1}{2} h^{IJK} [c^J \star c^K], \\
\delta \bar{c} &= b, \\
\delta b &= 0,
\end{align*}
\]  

(6)

where \( D_{\alpha}^{\text{ad}} \) is the usual covariant derivative in the \( \text{ad} \) representation of the gauge group given by \( D_{\alpha}^{\text{ad}} \Theta = \delta^{IJK} \partial_\alpha + f^{IJK} A^K_\alpha \Theta \). It can be easily showed, following a similar procedure than \([12]\) that the BRST charge \( Q \) is nilpotent \( i.e. \) \( Q^2 = 0 \).

Action (1) is BRST-invariant and anomaly free \( (\Delta U = 0) \). If gravity is not dynamical and only enters as an spectator field and its is coupled to matter only in the usual commutative way then action can be rewritten as a exact-BRST action \( i.e. \) as a BRST commutator plus a topological term

\[
L' = -\frac{1}{4} \int d^4 x \sqrt{g} \text{Tr} F_{\alpha \beta} \tilde{F}^{\alpha \beta} + \frac{1}{e^2} \int \{Q \star V\},
\]  

(7)

where

\[
V = \int_X d^4 \sqrt{g} \text{Tr} \left[ \lambda^* \left( D_\alpha \psi^\alpha - \frac{1}{2} \beta \eta \right) - \chi_{\alpha \beta} \star \left( \frac{1}{2} \alpha H^{\alpha \beta} - \frac{1}{2} (F^{\alpha \beta} - \tilde{F}^{\alpha \beta}) \right) + \bar{c}^* \left( \partial \cdot A - \frac{1}{2} \gamma b \right) \right].
\]  

(8)
Lagrangian \( \Pi \) can be obtained by taking \( \alpha = 1 \) and \( \beta = \gamma = 0 \). Substituting \( V \) into Eq. \( (7) \) and integrating out the auxiliary field \( H_{\alpha\beta} \) we can recover Lagrangian \( \Pi \).

In the standard commutative case we have that the energy-momentum tensor \( T_{\alpha\beta} \) of matter fields is conserved with the usual noncommutative covariant derivative \( D_\alpha T^{\alpha\beta} = 0 \). This energy-momentum tensor is determined through its BRST algebra \( [2] \) and it has the form of a BRST commutator. As far as we are considering a noncommutative deformation of the BRST algebra the corresponding noncommutative version will have the form of a deformed BRST commutator in the form

\[
T_{\alpha \beta} = \{ Q^* ; \lambda_{\alpha \beta} \},
\]

where

\[
\lambda_{\alpha \beta} = \frac{1}{2} \text{Tr} \left[ F_{\alpha \sigma} * \chi_{\beta}^{\sigma} + F_{\beta \sigma} * \chi_{\alpha}^{\sigma} - \frac{1}{2} g_{\alpha \beta} F_{\mu \nu} * \chi^{\mu \nu} \right] + \frac{1}{2} \text{Tr} \left( \psi_\alpha * D_\beta \lambda + \psi_\beta * D_\alpha \lambda - g_{\alpha \beta} \psi_\sigma * D^\sigma \lambda \right) + \frac{1}{4} g_{\alpha \beta} \text{Tr} \left( \eta * [\phi^*, \lambda] \right).
\]

Of course the energy-momentum tensor of the noncommutative matter is conserved with the noncommutative covariant derivative \( D_\alpha \Theta = \partial_\alpha \Theta + [A_\alpha^* \Theta] \).

3. OBSERVABLES

Physical observables of the topological theory are BRST cohomology classes of \( Q \). From the previous section we have that the scalar field \( \phi^I(x) \) is a BRST invariant since \( [Q^* ; \phi^I(x)] = 0 \). Any gauge invariant polynomial of this field is a physical observable, for instance take

\[
\mathcal{O}_{k,0}(x) = \text{Tr} \phi^k_* \equiv \text{Tr} \left( \phi(x) * \phi(x) * \cdots * \phi(x) \right), \quad k \text{ times}
\]

with ghost number \( U = 2k \).

Using the properties of \( X_\theta \), the property of the differential operator on \( X_\theta \) i.e. \( d(f_1 * f_2) = df_1 * f_2 + f_1 * df_2 \) (with \( f_1, f_2 \in X_\theta \)) and the BRST algebra \( (4) \) we get

\[
\frac{\partial \mathcal{O}_{k,0}}{\partial x^\alpha} = k \text{Tr} \phi^{k-1}_* * D_\alpha \phi(x) = \{ Q^* - k \text{Tr} \phi^{k-1}_* \psi_\alpha(x) \}.
\]
We will use the standard definition of noncommutative wedge product \( \wedge \) 

\[
F \wedge G \equiv F_{\alpha_1...\alpha_p}(x) \ast G_{\beta_1...\beta_q} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p} \wedge dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_q}.
\]

(13)

where \( F \in \Lambda^p(T^\ast X_\theta) \) and \( G \in \Lambda^q(T^\ast X_\theta) \).

We find that we can rewrite the Eq. \((\text{12})\) as

\[
d\mathcal{O}_{k,0} = \{ Q \ast \mathcal{O}_{k,1} \},
\]

(14)

where \( \mathcal{O}_{k,1} = -k \text{Tr} \phi_s^{k-1} \ast \psi_\alpha(x) dx^\alpha \). In this noncommutative extension of the BRST symmetry the observables \( \mathcal{O}_{k,a} \) with \( a = 0, 1, 2, 3, 4 \) also generate noncommutative BRST-invariant operators \( W_k(S_a) \) where \( S_a \) is a homology cycle of \( X \) of dimension \( a \). These operators are defined as

\[
W_k(S_a) = \int_{S_a} \mathcal{O}_{k,a}.
\]

(15)

It is easy to show that in the noncommutative BRST algebra we have

\[
\{ Q \ast W_k(S_a) \} = \int_{S_a} \{ Q \ast \mathcal{O}_{k,a} \} = \int_{S_a} d\mathcal{O}_{k,a} = 0.
\]

(16)

Thus \( W_k(S_a) \) are BRST-invariant operators and they are organized in BRST cohomology classes of the BRST charge \( Q \).

The rest of the operators \( \mathcal{O}_{k,a} \) with \( a > 1 \) can be computed easily. Thus we have

\[
\mathcal{O}_{k,2} = \text{Tr} \left( i\psi \wedge F + \frac{1}{2} \psi \wedge \psi \right),
\]

(17)

\[
\mathcal{O}_{k,3} = i \text{Tr} \left( \psi \wedge F \right),
\]

(18)

\[
\mathcal{O}_{k,4} = -\frac{1}{2} \text{Tr} \left( F \wedge F \right).
\]

(19)

In the commutative case \([\mathcal{O}_{k,a}]\) are organized in cohomology classes of de Rham cohomology. \( W_k(S_a) \) gives a pairing relating \( \mathcal{O}_{k,a} \) to homology cycles \([S_a]\) by Poincaré duality. In the noncommutative case \([\mathcal{O}_{k,a}]\) defines cohomology classes in the Chevalley-Eilenberg cohomology \( H^\ast(X_\theta) \) of the Moyal algebra \( X_\theta \). By Poincaré duality classes \([\mathcal{O}_{k,a}]\) can be coupled to Chevalley-Eilenberg homology cycles \([S_a] \in H_\ast(X_\theta)\).
4. NONCOMMUTATIVE DONALDSON INVARIANTS

4.1. Moduli Space of Noncommutative Instantons

As in the usual commutative case, in the noncommutative theory the partition function $Z$ is given by

$$Z = \int (\mathcal{D}X) \exp \left( - \frac{L'}{e^2} \right),$$

(20)

where the measure $\mathcal{D}X$ is a notation to abbreviate $\mathcal{D}A_\alpha^I \cdot \mathcal{D}\psi^I_\alpha \cdot \mathcal{D}\phi^I \cdot \ldots \cdot \mathcal{D}\chi$.

In the case when the gravity is a spectator and it is not coupled noncommutatively to the matter fields, partition function $Z$ is a topological invariant of the noncommutative manifold $X_\theta$. To see that we can use the fact that $L'$ is a $\theta$-deformed BRST commutator (see Eq. (7)). These invariants are independent on the metric but they may depend on the differentiable structure of the noncommutative manifold $X_\theta$. In this section we will argued that these topological invariants would be considered as the noncommutative deformation of Donaldson polynomial invariants of $X_\theta$.

For the moduli space $\hat{\mathcal{M}}$ of zero dimension, the only topological invariant is the partition function $Z$. For positive dimension the non-vanishing topological invariants are the correlation functions given by

$$Z(\mathcal{O}) = \langle \mathcal{O} \rangle = \int (\mathcal{D}X) \exp \left( - \frac{L'}{e^2} \right) \cdot \mathcal{O},$$

(21)

where $\mathcal{O}$ is a functional of the involved noncommutative fields.

The statement of that $\langle \{Q, \mathcal{O}\} \rangle = 0$ (for any $\mathcal{O}$) valid in the commutative theory is still valid in the noncommutative case since the necessary and sufficient condition is that our noncommutative Lagrangian [7] be a noncommutative BRST invariant and therefore $\{Q, L'\} = 0$ and therefore $\langle \{Q, \mathcal{O}\} \rangle = 0$. For similar reasons the correlation function $\langle A * \{Q, B\} \rangle = 0$ for any functional $B$ if $\{Q, A\} = 0$ for any functional $A$.

Partition function and correlation functions are also independent of the gauge coupling constant $e$. Hence we can evaluate the path integral for small values of $e^2$ i.e. $e^2 \to 0$. The application of the stationary phase approximation leads to the fact that the path integral is dominated by the classical minima and thus it is concentrated precisely at the
noncommutative instanton field configuration\[8, 9\]

\[ F^I_{\alpha\beta} = -\tilde{F}^I_{\alpha\beta}. \]  \(22\)

Solutions to this equation define the moduli space of noncommutative instantons \(\widetilde{M}_{\text{inst}}^\theta\) which is a smooth manifold already without small instanton singularities \(8, 9, 10\). In this case the self-duality condition is deformed by the last term of the Eq. \(3\) given by \(\frac{1}{2} d^{IJK} [A'^I_\alpha, A^K_\beta]\). As parameter \(\theta \rightarrow 0\) this term goes also to zero. This is precisely the term that deforms the ADHM equations of \(8, 9\).

4.2. Noncommutative Deformation of Moduli Space

In the commutative case the moduli space of instantons \(M_{\text{inst}}\) has singularities where the size of instantons goes to zero, they are called small instanton singularities which are related to the existence of zero modes for ghost fields. Then when one integrate out on this space it is usually assumed that they are no present. However in the noncommutative case these singularities are absent because of the existence of a length scale provided by the noncommutative parameter \(\theta\) \(8, 9\). Thus it is not necessary of dropping out the zero modes of other fields than gauge field \(A'^I_\alpha\) and \(\psi'^I_\alpha\). After evaluating Feynman integral in the weak coupling limit \((e \rightarrow 0)\) and integrating out the non-zero bosonic and fermionic modes we get from the stationary phase approximation that

\[ Z(\mathcal{O}) = \int_{\widetilde{M}_{\text{inst}}^\theta} da_1 \ldots da_n d\psi_1 \ldots d\psi_n \mathcal{O}, \]  \(23\)

where \((a_1, a_2, \ldots a_n)\) are the coordinates of the moduli space of noncommutative instantons \(\widetilde{M}_{\text{inst}}^\theta\) and \(\mathcal{O}\) being any BRST-invariant functional. The part of \(\mathcal{O}\) depending only on its zero modes is given by

\[ \mathcal{O}' = \Phi_{i_1 \ldots i_n}(a^k) \cdot \psi'^{i_1} \ast \psi'^{i_2} \ast \ldots \ast \psi'^{i_n}. \]  \(24\)

Then correlation function of the functional \(\mathcal{O}\) leads to an integration of the a differentiable closed \(n\)-form \(\Phi\) on the moduli space \(\widetilde{M}_{\text{inst}}^\theta\)

\[ Z(\mathcal{O}) = \int_{\widetilde{M}_{\text{inst}}^\theta} da_1 \ldots da_n d\psi_1 \ldots d\psi_n \Phi_{i_1 \ldots i_n}(a^k) \cdot \psi'^{i_1} \ast \ldots \ast \psi'^{i_n} \]
If we take the Moyal product of $k$ functionals of fields we have $\mathcal{O} = \mathcal{O}_1 \ast \mathcal{O}_2 \ast \cdots \ast \mathcal{O}_k$, where each $\mathcal{O}_r$ has $U = n_r$ such that $n = \sum_r n_r$. After integrating out the zero modes we get

$$
\mathcal{O}'_r = \Phi^{(r)}_{i_1 \cdots i_{n_r}} \psi^{i_1} \ast \cdots \ast \psi^{i_{n_r}}.
$$

(26)

This leads to reexpress the product as $\mathcal{O}' = \mathcal{O}'_1 \ast \mathcal{O}'_2 \ast \cdots \ast \mathcal{O}'_k$. Then correlation functions are given by

$$
Z(\mathcal{O}_{\alpha_1} \ast \mathcal{O}_{\alpha_2} \ast \cdots \ast \mathcal{O}_{\alpha_s}) = \int_{\hat{M}_{\theta}^{\text{inst}}} \hat{\Phi}^{(\alpha_1)} \ast \hat{\Phi}^{(\alpha_2)} \ast \cdots \ast \hat{\Phi}^{(\alpha_n)},
$$

(28)

where now the wedge product $\ast$ is defined on $\hat{M}_{\theta}^{\text{inst}}$.

Equation (30) provides a map from Chevalley-Eilenberg homology cycles $[S_a]$ of $X_\theta$ to the Chevalley-Eilenberg cohomology cycles of the moduli space seen as a noncommutative smooth space $\hat{M}_{\theta}^{\text{inst}}$. In this case we have that the Donaldson map is given by $Z : H_a(X_\theta) \to \cdots$
$H^{4-a}(\widehat{\mathcal{M}}^\theta_{\text{inst}})$ and they represent a noncommutative deformation of Donaldson invariants given by integral (30), which can be interpreted as the intersection form of $\widehat{\mathcal{M}}^\theta_{\text{inst}}$.

5. CONCLUDING REMARKS

In this paper we have proposed a noncommutative deformation of Witten’s cohomological field theory for Yang-Mills gauge theory [2]. This deformation is performed by deforming the underlying Lie algebra and changing it by the enveloping algebra following [7]. The application of these ideas leads us to a consistent noncommutative deformation of the full BRST fermionic symmetry of this topological theory given by Eqs. (4), (5) and (6). The results of noncommutative deformations of a standard BRST quantization of Ref. [12] help us to show that in our case BRST charge is nilpotent i.e. $Q^2 = 0$.

Observables of this deformed theory are computed and showed to be BRST and gauge invariant in the noncommutative sense. Partition function and correlation functions of these operators are proved to be topological invariants and independent of the gauge coupling $e$ as in the standard commutative case. Weak coupling limit ($e \to 0$) and the stationary phase approximation are used to show that the main contribution of Feynman integrals gives precisely the condition that satisfies the noncommutative Yang-Mills instantons [8, 9].

Gathering all that we are able to define deformed Donaldson invariants as integrals on the noncommutative moduli space of noncommutative differential forms. This form can be interpreted as the deformed Donaldson map $Z : H_a(X_\theta) \to H^{4-a}(\widehat{\mathcal{M}}^\theta_{\text{inst}})$, given by a mapping between the Chevalley-Eilenberg homology of noncommutative spacetime $X_\theta$ and the Chevalley-Eilenberg cohomology of the noncommutative moduli space of instantons $\widehat{\mathcal{M}}^\theta_{\text{inst}}$.

Acknowledgments

This work was supported in part by CONACyT México Grant 33951E. P.P. is supported by a CONACyT graduate fellowship. It is a pleasure to thank A. Dymarsky, O. Obregón and C. Ramírez for very useful discussions.
[1] M.R. Douglas and N.A. Nekrasov, Rev. Mod. Phys. 73 (2002), 977.
[2] E. Witten, “Topological Quantum Field Theories”, Commun. Math. Phys. 117 (1988) 117.
[3] E. Witten, “Topological Gravity”, Phys. Lett. B 206 (1988) 601.
[4] T. Ishikawa, S-I. Kuroki and A. Sako, “Noncommutative Cohomological Field Theory and GMS Soliton”, J. Math. Phys. 43 (2002) 872, [hep-th/0107033]
[5] A. Connes, Noncommutative Geometry, Academic Press (1994).
[6] H. García-Compeán, O. Obregón, C. Ramírez, and M. Sabido, Phys. Rev. D. 68 (2003) 045010, [hep-th/0210203]
[7] X. Calmet, B. Jurco, P. Schupp, J. Wess and M. Wohlgenannt, Eur.Phys. J. C 23 (2002) 363.
[8] N. Nekrasov and A. Schwarz, Commun. Math. Phys. 198 (1998) 689.
[9] A. Schwarz, “Noncommutative Instantons: A New Approach”, Commun. Math. Phys. 221 (2001) 433.
[10] N. Seiberg and E. Witten, JHEP 9909:032 (1999).
[11] A. Kapustin, A. Kuznetsov and D. Orlov, “Noncommutative Instantons and Twistor Space”, Commun. Math. Phys. 220 (2001) 385.
[12] M. Sorosh, “BRST Quantization of Noncommutative Gauge Theories”, Phys. Rev. D 67 (2003) 105005-1.
[13] L. Baulieu and I.M. Singer, “Topological Yang-Mills Symmetry”, Nucl. Phys. B (Proc. Suppl.) 5 (1988) 12.
[14] R. Brooks, D. Montano and J. Sonnenschein, Phys. Lett. B 214 (1988) 91.