Some remarks on recent developments in micropolar continuum theory

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Abstract. This paper considers micropolar media that can undergo structural changes and do not a priori consist of indestructible material particles. Initially the pertinent literature is reviewed. Then the necessary theoretical framework for a continuum of that type is presented. The standard macroscopic equations for mass, linear and angular momentum are complemented by a recently proposed balance for the moment of inertia tensor, which contains a production term. Two examples illustrate the effect of the production. In the first example, we study a continuous stream of matter on a conveyor belt going through a crusher so that the total number of particles will change. In context with this example, it is also clear that the traditional Lagrangian way of describing the motion of solids is no longer adequate and must be replaced by the Eulerian point of view known from fluid mechanics. The second example deals with hollow particles which rotate because of the presence of body couples. Now a transient temperature field is superimposed such that the moment of inertia field changes due to the thermal expansion of particles. This in turn results in rotational motion that is no longer constant but varies in space and time.

1. Introduction: Short review of the state-of-the-art

Generalized Continuum Theories (GCTs) have always been interesting to the materials science community in order to model and quantitatively describe high performance materials with inner degrees of freedom. Such materials can be used in large and small scale applications ranging from light-weight aerospace and automotive panels to micromechanics and microelectronic gadgets. A special GCT is the so-called theory of micropolar materials. It emphasizes the aspect of inner rotational degrees of freedom in a material. The introduction of such a feature is particularly useful for applications to soils, polycrystalline and composite matter, granular and powder-like materials, and even to porous media and foams.

It should be noted that the tensor field of the moment of inertia $J$ traditionally plays an important role in context with rotational motion, specifically in combination with the angular velocity vector $\omega$ describing, as a product, the local angular momentum in a continuum. Like the mass it is traditionally assigned to a material element. As such it is conserved and known a priori, see [1], pg. 14; [2], pg. 966; [3], Section 13; and [4], pg. 33. A different approach was suggested in [5], where it was assumed that the inertia of polar particles may change as the continuum deforms. This idea was further elaborated by [6] who clearly stated that the moment...
of inertia tensor should be treated as an independent field, just like the inertia associated with the linear momentum, namely the field of mass density, $\rho$. In fact, they proposed a balance equation for $\mathbf{J}$, which contains a production term $\chi J$ of moment of inertia due to “structural transformations” as they called it. This means that the moment of inertia will change due to combination or fragmentation of particles during mechanical crushing or chemical reactions. In addition, a production may arise because of phase transitions or physical property changes, such as electric magnetization or polarization.

However, the departure from the idea of an indestructible material particle also necessitates us to abandon the Lagrangian description and turn to the Eulerian perspective (a.k.a. spatial description) instead. Originally, the Eulerian description stems from fluid mechanics. It does not impose strict constraints on the motion of “points.” Rather it follows the idea of an open system a priori allowing for the exchange of mass, momentum, energy, moment of inertia, etc. between and within the cells of the Eulerian grid.

We therefore need to extend the original goals of micropolar theory. In fact, in what follows, we initially formulate the corresponding equations in a rather general manner, such that they are ready for future investigations of static or dynamic problems which involve linear and angular momentum simultaneously. However, we do not allow for a coupling between the two. The theoretical parts of this aspect are addressed (for example) in [4], Section 4.3; [6], or [7]. However, concrete examples were yet not provided. It is to be expected that the coupling will open up another degree of freedom in describing complex motion as it was shown for a point particle in [8], pp. 193. We now proceed to explain the relevant theory for structurally transforming micropolar materials.

2. Outline of the theory
In what follows, we consider the theoretical aspects of micropolar theory, first, from the macroscopic, i.e., the continuum point of view, which results in a complete set of balances including one for the rotational inertia. Because the balance for the inertia tensor field is extended by the production term, we, second, present mesoscopic aspects of relevant fields in terms of microproperties of rigid particles. The situation is illustrated in figure 1.

2.1. The macroscopic aspect
Mathematically speaking, in a very general form, the objective of micropolar theory is to determine the following primary fields: (a) the scalar field of mass density $\rho(x,t)$; (b) the vector field of linear velocity $\mathbf{v}(x,t)$; (c) the symmetric, second rank, and positive definite specific moment of inertia tensor field $\mathbf{J}(x,t)$ in units of $m^2$; and (d) the spin (a.k.a. angular velocity)
field $\omega(x, t)$ at all points $x$ and at all times $t$ within a region of space $B$, which can be either a material volume, i.e., it consists of the same matter at all times, or a region through which the matter is flowing (see the left inset of figure 1).

The determination of these fields relies on field equations for the primary fields. The field equations are based on balance laws and need to be complemented by suitable constitutive relations. At regular points, these macroscopic balances read as follows:

- balance of mass:
  \[ \frac{\delta \rho}{\delta t} + \rho \nabla \cdot v = 0, \]  
  \[ (1) \]

- balance of momentum:
  \[ \frac{\delta v}{\delta t} = \nabla \cdot \sigma + \rho f, \]  
  \[ (2) \]

- balance of the moment of inertia tensor:
  \[ \frac{\delta J}{\delta t} + J \times \omega - \omega \times J = \chi_J, \]  
  \[ (3) \]

- balance of spin:
  \[ \frac{\delta J \cdot \omega}{\delta t} = \nabla \cdot \mu + \sigma \times + \rho \gamma + \rho \chi_J \cdot \omega. \]  
  \[ (4) \]

We denote by
\[ \frac{\delta (\cdot)}{\delta t} = \frac{d (\cdot)}{dt} + (v - w) \cdot \nabla (\cdot) \]  
the substantial (a.k.a. material) derivative of a field quantity. Here $d(\cdot)/dt$ is the total derivative and $w$ the mapping velocity of the observation point (see [9]). Moreover, $\sigma$ is the (nonsymmetric) Cauchy stress tensor, $f$ is the specific body force, $\chi_J$ (a second-rank symmetric tensor) is the production related to the moment of inertia tensor $J$; $\mu$ is the couple stress tensor, $\sigma \times := \epsilon \cdot \sigma$ is the Gibbsian cross applied to the (nonsymmetric) Cauchy stress tensor, $\epsilon$ being the Levi-Civita tensor, and $m$ are specific volume couples.

Equation (3) is worthy of a detailed discussion. It has already been mentioned that no need for the conservation of microinertia was seen in [5]. However, in the presented form, this equation can only be found in a recent paper by one of the authors [6]. There is a precedent to the balance for the inertia tensor $J$, namely what is called “conservation of microinertia” in [1], pg. 15. However, no production term $\chi_J$ can be found in there, because the particle $X$ is material, i.e., indestructible. Following [4], pg. 33 we may write if the translational and rotational inertia is conserved:

\[ \rho = \frac{\rho_0}{\det \hat{F}}, \]  
\[ \hat{F} := \nabla \hat{x}(X, t) \]  
\[ \text{for } J = JI : \ J = J_0, \]  
\[ (6) \]

where all functions of the current configuration depend on $(X, t)$ and the ones in the reference placement (identified by the subscript 0) on $(X, t_0)$. Obviously the Lagrangian viewpoint is applicable with no problem whatsoever.

However, as indicated before, there is a catch. Consider our first example to be studied in the next section: A granular medium is milled. This affects the material particle, because its subunits will be crushed. They will change their individual masses and moments of inertia and, what is more, during the milling process there might even be an exchange of crushed subunits between neighboring material particles, which are then no longer material in the Lagrangian sense. Consequently, on a macroscopic scale the moments of inertia will change as well and all of this gives rise to the production term $\chi_J$. On the macroscopic continuum level, this new term could be interpreted as a constitutive quantity and its concrete form depending on the primary fields should be found following the established principle of continuum theory. However, the mesoscopic point of view may give us some guidance as to how the production term should look like. We proceed to discuss this issue.
2.2. The mesoscopic aspect

In what follows, we summarize some essential relations on the mesoscopic level from paper [6]. In there and in [10], more information can be found.

Recall the (representative) volume element $\Delta V$ at a fixed position $x$ in space (see figure 1, center) containing matter carrying inertial characteristics $\rho (x,t), J(x,t)$ (in units of m$^2$) moving with linear and angular velocities $v(x,t)$ and $\omega (x,t)$, respectively, all of which are fields. Note that $\Delta V$ does not refer to the volume occupied by a material point. Rather imagine it to be a small cell in a spatial Eulerian grid.

Within the volume element $\Delta V$, there are sufficiently many “rigid particles,” $i = 1, \ldots, N(x,t)$. The presence of a very large number of particles is required since establishing a continuous field theory would not be possible otherwise, and fluctuations would become dominant. Furthermore, note that the total particle number $N$ can depend on space $x$ and on time $t$.

One of the rigid particles, $i$, is depicted on the very right in figure 1. It carries known time-independent inertial characteristics $m_i$ and $\hat{J}_i$ (in units of kg $\cdot$ m$^2$, w.r.t. the center of $\Delta V$). Moreover, the particle moves with individual independent linear and angular velocities $v_i(t)$ and $\omega_i(t)$, respectively, both of which can be time-dependent. Both are no fields but discrete quantities. We are now going to relate the macroscopic and particle worlds by forming mesoscopic averages. The idea is to replace the particles within an elementary volume by an ensemble of identical particles each having an average mass and an average tensor of inertia. To this end, the inertial characteristics of the elementary volume are assumed to coincide with those of the average particle. Moreover, the linear and angular momenta of the elementary volume consisting of original particles are required to equal those of the elementary volume consisting of average particles.

We start by defining average inertial characteristics within $\Delta V$ by

$$m(x,t) := \frac{\sum_{i=1}^{N(x,t)} m_i}{N(x,t)}, \quad J(x,t) := \frac{\sum_{i=1}^{N(x,t)} \hat{J}_i}{N(x,t)},$$

followed by the fields of particle and mass densities,

$$n(x,t) := \frac{N(x,t)}{\Delta V}, \quad \rho(x,t) := \frac{\sum_{i=1}^{N(x,t)} m_i}{\Delta V},$$

which are then related by

$$\rho(x,t) = m(x,t)n(x,t).$$

For the rotational inertial characteristic, we have by definition:

$$\rho(x,t)J(x,t) := \frac{\sum_{i=1}^{N(x,t)} \hat{J}_i}{\Delta V},$$

so that

$$\rho(x,t)J(x,t) = n(x,t)\hat{J}(x,t).$$

We shall now connect the linear and angular momenta of the macroscopic world with those of the particles. If we ignore coupling (see [4], Section 4.3; or [11], Sections 3.2.3ff for a discussion of coupling effects in general; and [7] and [10] in context with mesoscopic considerations of micropolar media in particular), then the linear and the angular momenta of one particle simply read

$$\vec{K}_{1,i}(t) = m_i v_i(t), \quad \vec{K}_{2,i}(t) = \hat{J}_i \cdot \omega_i(t).$$
We define average linear and angular momenta fields within $\Delta V$ by

$$K_1(x,t) := \frac{\sum_{i=1}^{N(x,t)} K_{1,i}(t)}{N(x,t)}, \quad K_2(x,t) := \frac{\sum_{i=1}^{N(x,t)} K_{2,i}(t)}{N(x,t)},$$

and postulate the following connection between the macroscopic linear momentum and the spin with the particle averages from equation (13):

$$\rho \mathbf{v} = n \mathbf{K}_1, \quad \rho \mathbf{J} \cdot \mathbf{\omega} = n \mathbf{K}_2,$$

where all quantities are fields depending on $x$ and $t$. This in turn means that there are no simple relations between the particle velocities $v_i, \omega_i$ and the macroscopic fields $v, \omega$ and, in particular, we must conclude

$$v(x,t) \neq \frac{\sum_{i=1}^{N(x,t)} v_{1,i}(t)}{N(x,t)}, \quad \omega(x,t) \neq \frac{\sum_{i=1}^{N(x,t)} \omega_{2,i}(t)}{N(x,t)}.$$

3. Two examples

The effects and consequences of a balance for the moment of inertia with a production term are analyzed by two examples. In the first one, the crusher, an expression for the production is postulated phenomenologically, which allows us to mimic the milling of particles. In the second one, referred to as contracting and expanding hollow spheres, the production term is postulated based on a mesoscopic model of thermally sensitive hollow spheres in a thermal field varying in space and time. Here their state of rotation couples to the change of moment of inertia.

3.1. Analysis of a one-dimensional crusher

Consider the following problem in infinite one-dimensional space $-\infty < x < +\infty$ depicted in figure 2. A continuous flow of spherical particles with initial density $\rho(x,t=0)$ is coming in from the left and keeps moving to the right at a constant prescribed speed $v_0$. On its way it enters a region $-\delta \leq x \leq +\delta$ symmetrically arranged around the position $x = 0$, where it is continuously crushed to form smaller and smaller particles. Hence we assume that the momentum balance shown in (2) is identically satisfied and the balance of mass (1) and of moment of inertia (3) read

$$\frac{\partial \rho}{\partial t} + v_0 \frac{\partial \rho}{\partial x} = 0, \quad \frac{\partial J}{\partial t} + v_0 \frac{\partial J}{\partial x} = \chi J.$$  

For the production of moment of inertia we postulate the following relationship:

$$\chi J(x,t) = \begin{cases} 0, & \text{if } -\infty < x < -\delta, \\ -\alpha \langle \rho(x,t) \rangle^0 \langle J(x,t) - J_0 \rangle, & \text{if } -\delta \leq x \leq \delta, \\ 0, & \text{if } \delta < x < +\infty, \end{cases}$$  

Here $\langle \cdot \rangle$ denotes the microscopic average over the particles.
where $\alpha$ is a positive rate constant, $J_s$ indicates the moment of inertia pertinent to the minimum crushing size of the particles, and $(\cdot)$ denotes the Macaulay bracket\(^1\). The density field is incorporated through a Heaviside function in order to ensure that there is no production if the density is zero.

A few more remarks regarding the constitutive character of equation (17) are in order. We may want to interpret the quantity $J_s$ as a rotational inertial characteristic of the size of “glued” grains constituting the incoming ore. As such it is a characteristic of the material and not of the sieve size of the crusher (say), which has no inertial characteristics. Similarly the coefficient $\alpha$ could be interpreted as a measure of the dynamic resistance of the glue, i.e., also as a constitutive property. On the other hand, it might also depend on the state of stress or on the stress rate in terms of an equivalent measure, just like yield stress, hence “converting” the action of the crusher blades to a material response. Thereby it is also related to the effectiveness or to the rotational speed of the crusher blades and transmission of its exerted forces. Hence, at least in the presented case the production term shows both, material as well as process, characteristics.

The field of the mass density can be obtained in closed form from equation (16) in combination with the piecewise constant initial condition

$$\rho(x, t = 0) =: \rho_0(x) = \begin{cases} \rho_0 = \text{const}, & \text{if } -\infty < x < -\delta, \\ 0, & \text{if } -\delta \leq x \leq +\delta, \\ 0, & \text{if } +\delta < x < +\infty, \end{cases}$$

and by using the method of characteristics for initial-value problems on an infinite domain (see [7] for details):

$$\rho(x, t) = \rho_0(x - v_0 t).$$

In other words, the solution is a step function of height $\rho_0$ moving steadily and uniformly from the left to the right. As expected, the solution for the rotational inertia shows a more complex behavior. By using the method of characteristics for equation (16) in combination with the initial condition

$$J(x, t = 0) =: J_0(x) = \begin{cases} J_0 = \text{const}, & \text{if } -\infty < x < -\delta, \\ 0, & \text{if } -\delta \leq x \leq +\delta, \\ 0, & \text{if } +\delta < x < +\infty, \end{cases}$$

we obtain

- to the left of the crusher region, i.e., at positions $-\infty < x < -\delta$ and at times $0 \leq t < \infty$:

$$J(x, t) = J_0;$$

- the crusher region at times $0 \leq t < 2\delta/v_0$ for $-\delta \leq x < x_s$:

$$J(x, t) = J_s + (J_0 - J_s) \exp \left[ -\frac{\alpha}{v_0}(x + \delta) \right];$$

and for $x_s \leq x < +\delta$:

$$J(x, t) = 0,$$

and at times $2\delta/v_0 \leq t < \infty$ for $-\delta \leq x < +\delta$:

$$J(x, t) = J_s + (J_0 - J_s) \exp \left[ -\frac{\alpha}{v_0}(x + \delta) \right],$$

whilst the shock front moves at a constant speed $dx_s/dt = v_0$;\(^1\) The Macaulay bracket is defined by $\langle x \rangle = \frac{1}{2}(|x| + x)$.
Figure 3. Evolving moment of inertia field for matter going through crusher region.

- to the right of the crusher region at times $0 \leq t < 2\delta/v_0$ for $\delta \leq x < \infty$:
  \[ J(x, t) = 0; \]
  (25)

and at times $2\delta/v_0 \leq t < \infty$ for $\delta \leq x < v_0 t - \delta$:

  \[ J(x, t) = J_* + (J_0 - J_*) \exp \left( -\frac{2\alpha \delta}{v_0} \right), \]

  (26)

and for $v_0 t - \delta \leq x < \infty$:

  \[ J(x, t) = 0, \]

  (27)

and for $-\delta \leq x < \infty$:

  \[ J(x, t) = 0. \]

  (28)

This result is graphically depicted in figure 3. The exponential decay in the crusher related to the milling and refinement of particles is visible. In fact, parameters were chosen such that the time the matter spends within the crusher was too short for milling down to the finest size $J_*$. Moreover, various finite volume schemes were used to solve the problem numerically, the FiPy method due to van Leer and the Kurganov–Tadmor scheme. The problem is to capture the abrupt decline of shock fronts accurately. This problem was analyzed in more detail in [12].

3.2. Analysis of rotating contracting and expanding hollow spheres

Consider a medium consisting of empty hollow elastic spheres homogeneously distributed within a one-dimensional region $x \in [0, l]$. Their initial inner and outer radii are $R_i$ and $R_o$, respectively.
By positioning the medium in between two reservoirs kept at temperatures $T_0$ and $T_l$ and attached at positions $x = 0$ and $x = l$ of the region, respectively, the temperature of this medium will gradually change from an initially constant value $T_{ini}$. The development of temperature $T$ is governed by Fourier heat conduction and the solution to the temperature problem is given by ([13], Section 3.4):

\[ \bar{T}(\bar{x}, \bar{t}) = \bar{T}_0 + (\bar{T}_l - \bar{T}_0) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\bar{T}_l \cos(n\pi) - \bar{T}_0}{n} \sin(n\pi \bar{x}) \exp(-n^2 \pi^2 \bar{t}) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos(n\pi)}{n} \sin(n\pi \bar{x}) \exp(-n^2 \pi^2 \bar{t}). \] (29)

The bar on symbols refers to dimensionless quantities, namely,

\[ \bar{x} := \frac{x}{l}, \quad \bar{t} := \frac{Dt}{l^2}, \quad \bar{T} := \frac{T}{T_{ini}}, \] (30)

where $D$ is the thermal diffusivity. According to this temperature field, the hollow spheres do now contract and expand their inner and outer radii. Their moment of inertia will change accordingly. For simplicity, we assume that all of this happens instantaneously:

\[ J = JI, \quad J = J_0[1 + \alpha T_{ini}(\bar{T} - 1)]^2, \quad J_0 := \frac{2}{5} R_o^2 \frac{1 - \beta^5}{1 - \beta^3}, \quad \beta := \frac{R_i}{R_o}. \] (31)

The production can now be found on the basis of equation (3):

\[ \chi_J = \chi_J I, \quad \chi_J I \equiv \frac{\partial J}{\partial \bar{t}} = \chi_0[1 + \alpha T_{ini}(\bar{T} - 1)] \frac{\partial \bar{T}}{\partial \bar{t}}, \quad \chi_0 := 2\alpha T_{ini}J_0 \frac{D}{l^2}. \] (32)

Figure 4 shows the development of the (normalized) moment of inertia on the left ($\alpha T_{ini}=0.5$). The blue and red curves depict the situation shortly after the heat conduction process has started. The green curve represents the situation corresponding to the less stationary temperature distribution (linear profile in $\bar{T}$). The picture on the right shows the temporal evolution of the production at various points, namely, at $\bar{x} = 0.1$ (blue), $\bar{x} = 0.5$ (red), $\bar{x} = 1/3$ (magenta, where $\bar{T} = 1$ for the choice of parameters), and $\bar{x} = 0.9$ (green). The production vanishes for long times when we move toward the stationary temperature profile.

We now turn to the problem of how the angular velocity of the hollow spheres will be influenced when subjected to the nonstationary temperature profile. To this end, we need to make some assumptions regarding the body couple $\mathbf{m}$. We consider the micro-model shown on the left in figure 5. Following the ideas presented in [14] we imagine the hollow spheres to be electrically polarized dipoles with zero net charge, $q^+ = -q^- = q$. Now we apply a constant external electric field $E_0$ in the negative $x$-direction. The total Coulomb force, and therefore (after homogenization) the body force (in the $x$-direction), will then vanish. However, the moment couple acting on the sphere will not:

\[ \mathbf{M} = (q^+ - q^-)R_o \times E_0 = 2qR_oE_0\cos \varphi(t)e_z \neq \mathbf{0}. \] (33)

Then the body moment couple becomes:

\[ \mathbf{m} = m_0 \cos \left[ \int_{t=0}^{\bar{t}} \omega(t)dt \right] e_z, \quad m_0 := \frac{q}{m_p}R_oE_0 \] (34)
for $\varphi(0) = 0$ and $m_p$ being the mass of one particle. We assume that the stress tensor and the couple stress tensor vanish (dust) and rewrite the spin balance (4) as

$$\frac{\partial}{\partial t} \left( J \frac{\partial \omega}{\partial t} \right) = -\omega \sqrt{1 - \left( J \frac{\partial \omega}{\partial t} \right)^2}. \quad (35)$$

A numerical solution of this ODE in time for each fixed position with $\omega(x, t = 0) = 0.4$, $\omega(x, t = 0) = 0$ leads to oscillating behavior shown in figure 5 on the right. The different colors refer to different (normalized) positions ($\bar{x} = 0.1$ (blue), $\bar{x} = 1/3$ (magenta, where $\bar{T} = 1$). Noticeably the different speeds of temperature change influence the oscillation frequencies. More details are presented in [15].

**Summary**

The intention of this paper was to draw attention on some recent activities in the field of micropolar media capable of structural change. An extended set of balance equations was presented, where the extension consisted of an independent balance for the field of moment of inertia and production fields for hollow spheres (see text).
inertia with production. Two examples were presented showing the effect of this new term. More research in this field is planned for the future in order to study fully coupled problems for micropolar media.

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