CONSTRUCTION OF THE LINDSTRÖM VALUATION OF AN ALGEBRAIC EXTENSION

DUSTIN CARTWRIGHT

Abstract. Recently, Bollen, Draisma, and Pendavingh have introduced the Lindström valuation on the algebraic matroid of a field extension of characteristic $p$. Their construction passes through what they call a matroid flock and builds on some of the associated theory of matroid flocks which they develop. In this paper, we give a direct construction of the Lindström valuated matroid using the theory of inseparable field extensions. In particular, we give a description of the valuation, the valuated circuits, and the valuated cocircuits.

The algebraic matroid of a field extension records which subsets of a fixed set of elements of the field are algebraically independent. In characteristic 0, the algebraic matroid coincides with the linear matroid of the vector configuration of differentials, and, as a consequence the class of matroids with algebraic realizations over a field of characteristic 0 is exactly equivalent to the class of matroids with linear realizations in characteristic 0 [Ing71]. However, in positive characteristic, there are strictly more algebraic matroids than linear matroids, and without an equivalence to linear matroids, the class of algebraic matroids is not well understood.

Pioneering work of Lindström has shown the power of first applying well-chosen powers of the Frobenius morphism to the field elements, before taking differentials. In particular, he constructed an infinite family of matroids (the Fano matroid among them) for which any algebraic realization over a field of finite characteristic, after applying appropriate powers of Frobenius and taking differentials, yields a linear representation of the same matroid [Lin85].

In general, no single choice of powers of Frobenius may capture the full algebraic matroid, and so Bollen, Draisma, and Pendavingh went one step further by looking at the matroids of differentials after all possible powers of Frobenius applied to the chosen field elements [BDP18]. These matroids fit together to form what they call a matroid flock, and they show that a matroid flock is equivalent to a valuated matroid [BDP18 Thm. 7]. Therefore, the matroid flock of differentials defines a valuation on the algebraic matroid of the field extension, called the Lindström valuation of the algebraic matroid. In this paper we give a direct construction of this valuation, without reference to matroid flocks.

We now explain the construction of the Lindström valuation of an algebraic matroid. Throughout this paper, we will work with an extension of fields $L \supset K$ of characteristic $p > 0$ as well as fixed elements $x_1, \ldots, x_n \in L$. We
also assume that \( L \) is a finite extension of \( K(x_1, \ldots, x_n) \), for example, by replacing \( L \) with \( K(x_1, \ldots, x_n) \). The algebraic matroid of this extension can be described in terms of its bases, which are subsets \( B \subset E = \{1, \ldots, n\} \) such that the extension of \( L \) over \( K(x_B) = K(x_i : i \in B) \) is algebraic. We recall from \([\text{Lan02, Sec. V.6}]\) that if \( K(x_B)^{\text{sep}} \) denotes the set of elements of \( L \) which are separable over \( K(x_B) \), then \( L \) is a purely inseparable extension of \( K(x_B)^{\text{sep}} \), and the degree of this extension, \( [L : K(x_B)]_{i}^{\text{sep}} \) is called the \textit{inseparable degree} and denoted by adding a subscript: \( [L : K(x_B)]_{i} \).

Now, we define a valuation on the algebraic matroid of \( L \) as the following function \( \nu \) from the set of bases to \( \mathbb{Z} \):

\[
\nu(B) = \log_p [L : K(x_B)]_{i}.
\]

Note that \( \nu(B) \) is finite because we assumed that \( L \) was a finitely generated algebraic extension of \( K(x_B) \) and it is an integer because \( [L : K(x_B)]_{i} \) is the degree of a purely inseparable extension, which is always a power of \( p \) \([\text{Lan02, Cor. V.6.2}]\).

\textbf{Theorem 1.} The function \( \nu \) in (1) defines a valuation on the algebraic matroid of \( L \supset K \), such that the associated matroid flock is the matroid flock of the extension.

In addition to the valuation given in (1), we give descriptions of the valuated circuits of the Lindström valuated matroid in the beginning of Section 1 and of the valuated cocircuits and minors in Section 3. The description of the circuits gives an algorithm for computing the Lindström valuated matroid using Gröbner bases, assuming that \( L \) is finitely generated over a prime field (see Remark 6 for details).

\textbf{Remark 2.} There are two different sign conventions used in the literature on valuated matroids. We use the convention which is compatible with the “min-plus” convention in tropical geometry, which is the opposite of what was used in the original paper of Dress and Wenzel \([\text{DW92}]\), but is consistent with \([\text{BDP18}]\).

\textbf{Acknowledgments.} I’d like to thank Jan Draisma for useful discussion about the results in \([\text{BDP18}]\), which prompted this paper, Rudi Pendavingh for suggesting the results appearing in Section 3 and Felipe Rincón for helpful feedback. The author was supported by NSA Young Investigator grant H98230-16-1-0019.

1. \textbf{The Lindström valuated matroid}

In this section, we verify that the function (1) from the introduction is a valuation on the algebraic matroid of the extension \( L \supset K \) and the elements \( x_1, \ldots, x_n \). We do this by first constructing the valuated matroid in terms of its valuated circuits, and then showing that the corresponding valuation agrees with the function (1). Throughout the rest of the paper, we will use \( F \) to denote the Frobenius morphism \( x \mapsto x^p \).
Recall that a (non-valuated) circuit of the algebraic matroid of the elements \(x_1, \ldots, x_n\) in the extension \(L \supset K\) is an inclusion-wise minimal set \(C \subset E\) such that \(K(x_C)\) has transcendence degree \(|C| - 1\) over \(K\). Therefore, there is a unique (up to scaling) polynomial relation among the \(x_i\), which we call the circuit polynomial, following [KRT13]. More precisely, we let \(K[X_C]\) be the polynomial ring whose variables are denoted \(X_i\) for \(i \in C\). The aforementioned circuit polynomial is a (unique up to scaling) generator \(f_C\) of the kernel of the homomorphism \(K[X_C] \to K(x_C)\) which sends \(X_i\) to \(x_i\). We write this polynomial:

\[
f_C = \sum_{u \in J} c_u X^u \in K[X_C] \subset K[X_E]
\]

where \(J \subset \mathbb{Z}_{\geq 0}^n\) is a finite set of exponents and \(c_u \neq 0\) for all \(u \in J\). Then, we define \(C(f_C)\) to be the vector in \((\mathbb{Z} \cup \{\infty\})^n\) with components:

\[
(C(f_C))_i = \min\{\text{val}_p u_i \mid u \in J, u_i \neq 0\},
\]

where \(\text{val}_p u_i\) denotes the \(p\)-adic valuation, which is defined to be the power of \(p\) in the prime factorization of the positive integer \(u_i\). If \(u_i = 0\) for all \(u \in J\), then we take \(C(f_C)_i\) to be \(\infty\). For any vector \(C \in (\mathbb{Z} \cup \{\infty\})^n\), the support of \(C\), denoted \(\text{supp} \ C\), is the set \(\{i \in E \mid C_i < \infty\}\). Since \(f_C\) is a polynomial in the variables \(X_i\) for \(i \in C\), but not in any proper subset of them, the support of \(C(f_C)\) is exactly the circuit \(C\).

We will take the valuated circuits of the Lindström valuation to be the set of vectors:

\[
C = \{C(f_C) + \lambda \mathbf{1} \mid C \text{ is a circuit of } L \supset K, \lambda \in \mathbb{Z} \subset (\mathbb{Z} \cup \{\infty\})^n\},
\]

where \(\mathbf{1}\) denotes the vector \((1, \ldots, 1)\). Before verifying that this collection of vectors satisfies the axioms, we prove the following preliminary lemma relating the definition in (2) to the inseparable degree:

**Lemma 3.** Let \(S \subset E\) be a set of rank \(|S| - 1\), and let \(C\) be the unique circuit contained in \(S\). If we abbreviate the vector \(C(f_C)\) as \(C\), then

\[
[K(x_S) : K(x_{S \setminus \{i\}})]_i = p^{C(f_C)_i}
\]

for any \(i \in C\). In particular, \(K(x_S)\) is a separable extension of \(K(x_{S \setminus \{i\}})\) if and only if \(C_i = 0\).

**Proof.** For \(i \in C\), we let \(Y_i\) denote the monomial \(X_i^{pC_i}\) in \(K[X_S]\). Then, the polynomial \(f_C\) lies in the polynomial subring \(K[X_S \setminus \{i\}, Y_i]\), by the definition of \(C_i\). Similarly, we let \(y_i\) denote the element \(x_i^{pC_i} = F^{C_i}x_i\) in \(K(x_S)\). Then, \(f_C\), as a polynomial in \(K[X_S \setminus \{i\}, Y_i]\), is the minimal defining relation for \(K(x_{S \setminus \{i\}}, y_i)\) as an extension of \(K(x_{S \setminus \{i\}})\). By the definition of \(C_i\), some term of \(f_C\) is of the form \(X_i^a Y_i^a\), where \(a\) is not divisible by \(p\), and so \(\partial f_C/\partial Y_i\) is a non-zero polynomial. Therefore, \(f_C\) is a separable polynomial of \(Y_i\), and so \(K(x_{S \setminus \{i\}}, y_i)\) is a separable extension of \(K(x_{S \setminus \{i\}})\).
On the other hand, \( K(x_S) \) is a purely inseparable extension of \( K(x_{S\setminus \{i\}}, y_i) \), defined by the minimal relation

\[
x_i^{p_{C_i}} - y_i = 0.
\]

Therefore, this extension has degree \( p_{C_i} \), which is thus the inseparable degree \( [K(x_S) : K(x_{S\setminus \{i\}})] \), as desired.

We now verify that the collection \((\text{3})\) satisfies the axioms of valuated circuits. Several equivalent characterizations of valuated circuits are given in [MT01], and we will use the characterization in the following proposition:

**Proposition 4** (Thm. 3.2 in [MT01]). A set of vectors \( C \subset (\mathbb{Z} \cup \{\infty\})^n \) is the set of valuated circuits of a valuated matroid if and only if it satisfies the following properties:

1. The collection of sets \( \{\text{supp} \ C \mid C \in C\} \) satisfies the axioms of the circuits of a non-valuated matroid.
2. If \( C \) is a valuated circuit, then \( C + \lambda \mathbb{1} \) is a valuated circuit for all \( \lambda \in \mathbb{Z} \).
3. Conversely, if \( C \) and \( C' \) are valuated circuits with \( \text{supp} \ C = \text{supp} \ C' \), then \( C = C' + \lambda \mathbb{1} \) for some integer \( \lambda \).
4. Suppose \( C \) and \( C' \) are in \( C \) such that
   \[ \text{rank}(\text{supp} \ C \cup \text{supp} \ C') = |\text{supp} \ C \cup \text{supp} \ C'| - 2, \]
   and \( u, v \in E \) are elements such that \( C_u = C'_u \) and \( C_v < C'_v = \infty \). Then there exists a vector \( C'' \in C \) such that \( C''_u = \infty \), \( C''_v = C_v \), and \( C''_i \geq \min\{C_i, C'_i\} \) for all \( i \in E \).

The first property from Proposition 4 is equivalent to axioms VC1, VC2, and MCE from [MT01] and the three after that are denoted VC3, VC3e, VC\text{loc1}, respectively.

**Proposition 5.** The collection \( C \) of vectors given in \((\text{3})\) defines the valuated circuits of a valuated matroid.

**Proof.** The first axiom from Proposition 4 follows because each valuated circuit is constructed to have support equal to a non-valuated circuit. The second axiom follows immediately from the construction, and the third follows from the uniqueness of circuit polynomials.

Thus, it remains only to check (4) from Proposition 4. Suppose that \( C \) and \( C' \) are valuated circuits and \( u, v \in E \) are elements satisfying the hypotheses of condition (4). We can write \( C = C(f) + \lambda \mathbb{1} \) and \( C' = C(f') + \lambda' \mathbb{1} \) for circuit polynomials \( f \) and \( f' \) in \( K[X_1, \ldots, X_n] \). Note that \( C(F^m f) = C(f) + m \mathbb{1} \), and so by either replacing \( f \) with \( F^m f \) or replacing \( f' \) with \( F^m f' \), for some integer \( m \), we can assume that \( \lambda = \lambda' \). Moreover, since the fourth axiom only depends on the relative values of the entries of \( C \) and \( C' \), it is sufficient to check the axiom for \( C \) and \( C' \) replaced by \( C(f) = C - \lambda \mathbb{1} \) and \( C(f') = C' - \lambda \mathbb{1} \), respectively.
We now define an injective homomorphism $\psi$ from the polynomial ring $K[Y_1, \ldots, Y_n]$ to $K[X_1, \ldots, X_n]$ by

$$\psi(Y_i) = F_{\min\{C_i, C'_i\}} X_i$$

Thus, there exist polynomials $g$ and $g'$ in $K[Y_1, \ldots, Y_n]$ such that $f = \psi(g)$ and $f' = \psi(g')$. In particular, since $C_i(g) = C_i - \min\{C_i, C'_i\}$ and $C_i(g') = C'_i - \min\{C_i, C'_i\}$, then our assumptions on $u$ and $v$ imply that $C_i(g)_{u} = C_i(g')_{u} = 0$.

Likewise, we define $y_i = F_{\min\{C_i, C'_i\}} x_i$ so that the elements $y_i \in L$ satisfy the polynomials $g$ and $g'$. Thus, Lemma 3 shows that $g$ is separable in the variable $Y_i$, and so if $S$ denotes the set $\{C \cup \text{supp} C'\}$, then $K(y_S)$ is a separable extension of $K(y_{S \setminus \{v\}})$. Likewise, $g'$ is separable in the variable $Y_u$ and doesn’t use the variable $Y_v$, and so $K(y_{S \setminus \{v\}})$ is a separable extension of $K(y_{S \setminus \{u\}})$. Since the composition of separable extensions is separable, $y_v$ is separable over $K(y_{S \setminus \{u\}})$, $\psi$ is given as the fraction $y_v$.

Since algebraic extensions have transcendence degree 0, then the field $K(y_{S \setminus \{u\}})$ has the same transcendence degree over $K$ as $K(y_S)$ does, and that transcendence degree is $|S| - 2$, because we assumed that rank$(S) = |S| - 2$. In addition, we have containments $K(y_{S \setminus \{u\}}) \subset K(y_{S \setminus \{u\}}) \subset K(y_S)$, so that $K(y_{S \setminus \{u\}})$ also has transcendence degree $|S| - 2$, and therefore there exists a unique (up to scaling) polynomial relation $g'' \in K[Y_{S \setminus \{u\}}]$ among the elements $y_i$ for $i \in S \setminus \{u\}$. Since $y_v$ is finite and separable over $K(y_{S \setminus \{u\}})$, $C_i(g'')_{v} = 0$ by Lemma 3.

We claim that the $C'' = C(\psi(g''))$ satisfies the desired conclusions of the axiom. First,

$$C''_{v} = C(g''_{v})_{v} + \min\{C_v, C'_v\} = 0 + \min\{C_v, \infty\} = C_v,$$

as desired. Similarly,

$$C''_{i} = C(g''_{v})_{i} + \min\{C_i, C'_i\} \geq \min\{C_i, C'_i\},$$

and, finally, $C''_{u} = \infty$ because $g''$ was chosen to be a polynomial in the variables $Y_{S \setminus \{u\}}$. \hfill $\Box$

**Remark 6.** The valued circuits defined in Proposition 5 are effectively computable from a suitable description of $L$ and the $x_i$. More precisely, suppose $K$ is a finitely generated extension of $\mathbb{F}_p$ and $L$ is given as the fraction field of $K[x_1, \ldots, x_n]/I$ for a prime ideal $I$. Then $I$ can be represented in computer algebra software, and the elimination ideals $I \cap K[x_S]$ can be computed for any subset $S \subset E$ using Gröbner basis methods. The circuits of the algebraic matroid are the minimal subsets $C$ for which $I \cap K[x_C]$ is not the zero ideal, in which case the elimination ideal will be principal, generated by the circuit polynomial $f_C$. By computing all of these elimination ideals, we can determine the circuits of the algebraic matroid, and from the corresponding generators, we get the valued circuits by the formula 3.
Example 7. One case where the connection between the Lindström valuated matroid and linear algebraic valuated matroids is most transparent is when the variables $x_i$ are monomials. This example is given in [BDP18, Thm. 45], but we discuss it here in terms of our description of the valuated circuits.

We let $A$ be any $d \times n$ integer matrix, and then we take $L = K(t_1, \ldots, t_d)$ for any field $K$ of characteristic $p$, and we let $x_i$ be the monomial $t_1^{A_{i1}} \cdots t_d^{A_{id}}$, whose exponents are the $i$th column of $A$. Then the algebraic matroid of $x_1, \ldots, x_n$ is the same as the linear matroid of the vector configuration formed by taking the columns of $A$. Moreover, we claim that the Lindström valuated matroid is the same as the valuated matroid of the same vector configuration with respect to the $p$-adic valuation on $\mathbb{Q}$.

To see this, we look at the valuated circuits of both valuated matroids. A circuit of the linear matroid is determined by an $n \times 1$ vector $u$ with minimal support such that $Au = 0$. The circuit is the support of the vector $u$, and the valuated circuit is the entry-wise $p$-adic valuation of $u$. The support of $u$ is also a circuit of $x_1, \ldots, x_n$ with circuit polynomial

$$f = X_1^{u_1^+} \cdots X_n^{u_n^+} - X_1^{u_1^-} \cdots X_n^{u_n^-}$$

where

$$u_i^+ = \min\{0, u_i\} \quad u_i^- = -\max\{0, u_i\}$$

so that $u = u^+ - u^-$. Then, since one of $\text{val}_p(u_i^-)$ and $\text{val}_p(u_i^+)$ equals $\text{val}_p(u_i)$ and the other is infinite, $C(f)$ is the same as the entry-wise $p$-adic valuation of $u$, which is the valuated circuit of the linear matroid. Thus, the valuated circuits of the linear and algebraic matroids are the same.

Proposition 8. The Lindström valuated matroid given by the circuits in (3) agrees with the valuation (1) given in the introduction.

Proof. The essential relation between the valuation and the valuated circuits is that if $B$ is a basis, $u \in B$, $v \in E \setminus B$, and $C$ is a valuated circuit whose support is contained in $B \cup \{v\}$, then:

$$\nu(B) + C_u = \nu(B \setminus \{u\} \cup \{v\}) + C_v$$

This relation is used at the beginning of [MT01, Sec. 3.1] to define the valuated circuits in terms of the valuation, and in the other direction with (10) from [MT01]. In (4), we adopt the convention that $\nu(B \setminus \{u\} \cup \{v\})$ is $\infty$ if $B \setminus \{u\} \cup \{v\}$ is not a basis.

The only quantities in (4) which can be infinite are $C_u$ and $\nu(B \setminus \{u\} \cup \{v\})$, because if $C_v$ were infinite, then supp$C$ would be contained in $B$, which contradicts $B$ being a basis. However $B \setminus \{u\} \cup \{v\}$ is not a basis if and only the support of $C$ is contained in $B \setminus \{u\} \cup \{v\}$, which is true if and only $C_u = \infty$. Therefore, the left hand side of (4) is infinite if and only if the right hand side is, so for the rest of the proof, we can assume that all of the terms of (4) are finite.
By the multiplicativity of inseparable degrees \[\text{[Lan02 Cor. V.6.4]},\] we have
\[
\nu(B) = \log_p[L : K(x_B)]_i
\]
\[
= \log_p[L : K(x_{B \cup \{v\}})]_i + \log_p[K(x_{B \cup \{v\}}) : K(x_B)]_i
\]
\[
= \log_p[L : K(x_{B \cup \{v\}})]_i + C_v,
\]
by Lemma \[3\] Similarly, we also have
\[
\nu(B \setminus \{u\} \cup \{v\}) = \log_p[L : K(x_{B \setminus \{u\} \cup \{v\}})]_i
\]
\[
= \log_p[L : K(x_{B \setminus \{u\}})]_i + \log_p[K(x_{B \setminus \{u\}}) : K(x_{B \setminus \{u\} \cup \{v\}})]_i
\]
\[
= \log_p[L : K(x_{B \setminus \{u\}})]_i + C_u,
\]
again, using Lemma \[3\] for the last step. Therefore,
\[
\nu(B) - C_v = \log_p[L : K(x_{B \setminus \{v\}})]_i = \nu(B \setminus \{u\} \cup \{v\}) - C_u,
\]
which is just a rearrangement of the desired equation \[4\].

Thus, we’ve proved the first part of Theorem \[1\] namely that the function \(\nu\) given in \[1\] defines a valuation on the algebraic matroid \(M\). In the next section, we turn to the second part of Theorem \[1\] and show that this valuation is compatible with the matroid flock studied in \[BDP18\].

2. Matroid Flocks

We now show that the matroid flock defined by the valuated matroid from the previous section is the same as the matroid flock defined from the extension \(L \supset K\) in \[BDP18\]. A matroid flock is a function \(M\) which maps each vector \(\alpha \in \mathbb{Z}^n\) to a matroid \(M_\alpha\) on the set \(E\), such that:

(1) \(M_\alpha/i = M_{\alpha + e_i}\) \(\setminus i\) for all \(\alpha \in \mathbb{Z}^n\) and \(i \in E\),
(2) \(M_\alpha = M_{\alpha + e_i}\) for all \(\alpha \in \mathbb{Z}^n\).

In the first axiom, the matroids \(M_\alpha/i\) and \(M_{\alpha + e_i} \setminus i\) are the contraction and deletion of the respective matroids with respect to the single element \(i\).

To any valuated matroid \(M\), the associated matroid flock, which we also denote by \(M\), is defined by letting \(M_\alpha\) be the matroid whose bases consist of those bases of \(M\) such that \(e_B \cdot \alpha - \nu(B) = g(\alpha)\), where \(e_B\) is the indicator vector with entry \((e_B)_i = 1\) for \(i \in B\) and \((e_B)_i = 0\) otherwise, and where

\[
g(\alpha) = \max\{e_B \cdot \alpha - \nu(B) \mid B\text{ is a basis of } M\}.
\]

Moreover, any matroid flock comes from a valuated matroid in this way by Theorem 7 in \[BDP18\].

On the other hand, \[BDP18\] also associates a matroid flock directly to the extension \(L \supset K\) and the elements \(x_1, \ldots, x_n\). Their construction is in terms of algebraic varieties and the tangent spaces at sufficiently general points. Here, we recast their definition using the language of field theory and derivations. Define \(\hat{L}\) to be the perfect closure of \(L\), which is equal to the union \(\bigcup_{\nu \geq 0} L(x_1^{1/p^\nu}, \ldots, x_n^{1/p^\nu})\) of the infinite tower of purely inseparable extensions of \(L\). For a vector \(\alpha \in \mathbb{Z}^n\), we define \(F^{-\alpha}x_E\) to be the vector...
in \( \tilde{L}^n \) with \( (F^{-\alpha}x_E)_i = F^{-\alpha}x_i \), and \( K(F^{-\alpha}x_E) \) to be the field generated by these elements. Recall from field theory, e.g. [Lan02, Sec. XIX.3], that the vector space of differentials \( \Omega_{K(F^{-\alpha}x_E)/K} \) is defined algebraically over \( K(F^{-\alpha}x_E) \), generated by the differentials \( d(F^{-\alpha}x_i) \) as \( i \) ranges over the set \( E \). We define \( \nu \) to be the matroid on \( E \) of the configuration of these vectors \( d(F^{-\alpha}x_i) \) in \( \Omega_{K(F^{-\alpha}x_E)/K} \), and then the function \( \nu \) which sends \( \alpha \), to \( \nu \), is a matroid flock [BDP18, Thm. 34].

**Proof of Theorem 4.** The function \( \nu \) is a valuation on \( M \) by Propositions 5 and 8 so it only remains to show that the matroid flock associated to this valuation coincides with the matroid flock \( N \) defined above. Let \( \alpha \) be a vector in \( \mathbb{Z}^n \). Since both \( M \) and \( N \) are matroid flocks, they are invariant under shifting \( \alpha \) by the vector \( \mathbb{1} \), as in the second axiom of a matroid flock. Therefore, we can shift \( \alpha \) by a multiple of \( \mathbb{1} \) such that all entries of \( \alpha \) are non-negative and it suffices to show that \( M_\alpha = N_\alpha \) in this case.

Now let \( B \) be a basis of \( M \) and we want to show that the differentials \( d(F^{-\alpha}x_i) \), for \( i \in B \), form a basis for \( \Omega_{K(F^{-\alpha}x_E)/K} \) if and only if \( e_B \cdot \alpha - \nu(B) \) equals \( g(\alpha) \), as defined in [5]. Since the field \( K(F^{-\alpha}x_B) \) is generated by the algebraically independent elements \( F^{-\alpha}x_i \) as \( i \) ranges over the elements of \( B \), the differentials \( d(F^{-\alpha}x_i) \) do form a basis for \( \Omega_{K(F^{-\alpha}x_B)/K} \). Moreover, the natural map \( \Omega_{K(F^{-\alpha}x_B)/K} \to \Omega_{K(F^{-\alpha}x_E)/K} \) is an isomorphism if and only if the \( K(F^{-\alpha}x_E) \) is a separable extension of \( K(F^{-\alpha}x_B) \) [Lan02 Prop. VIII.5.2], i.e. if and only if its inseparable degree is 1. Therefore, \( B \) is a basis for \( N_\alpha \) if and only if \( [K(F^{-\alpha}x_E): K(F^{-\alpha}x_B)]_i = 1 \).

We list the inseparable degrees:

\[
\begin{align*}
[L : K(x_B)]_i &= p^{\nu(B)} \\
[K(F^{-\alpha}x_B) : K(x_B)]_i &= p^{e_B \cdot \alpha} \\
[K(F^{-\alpha}x_E) : K(x_E)]_i &= p^{m(\alpha)} \\
[L : K(x_E)]_i &= p^\ell
\end{align*}
\]

The first of these equalities is by definition, the second is because \( K(F^{-\alpha}x_B) \) is the purely inseparable extension of \( K(x_B) \) defined by adjoining a \( p^{\alpha_i} \)-root of \( x_i \) for each \( i \), and the third and fourth we take to be the definitions of the integers \( m(\alpha) \) and \( \ell \), respectively. By the multiplicativity of inseparable degrees, and taking logarithms, we have:

\[
\log_p [K(F^{-\alpha}x_E) : K(F^{-\alpha}x_B)]_i = \log_p [K(F^{-\alpha}x_E) : K(x_B)]_i - e_B \cdot \alpha = m(\alpha) + [K(x_E) : K(x_B)]_i - e_B \cdot \alpha = m(\alpha) - \ell + \nu(B) - e_B \cdot \alpha
\]

(6)

As noted above, \( B \) is a basis of \( N_\alpha \) if and only if the left hand side of (6) is zero, and \( B \) is a basis of \( M_\alpha \) if and only if \( e_B \cdot \alpha - \nu(B) = g(\alpha) \). Thus, it suffices to show that \( m(\alpha) - \ell \) equals \( g(\alpha) \).
These vectors are projectively equivalent to the Fano configuration, and, therefore, the matroid flock is the Fano matroid. In particular, we have the linear relation $t_3 dx_4 + t_2 dx_5 + t_1 dx_6 = 0$, among the differentials, even though $\{4, 5, 6\}$ is a basis of the algebraic matroid.

On the other hand, if we let $\alpha = (-1, -1, -1, 0, 0, 0, -1)$, then $K(F^{-\alpha}x_E)$ is the subfield $K(x_4, x_5, x_6) \subset L$, because

$$Fx_1 = x_1^2 = x_4 x_5 x_6^{-1} \quad Fx_3 = x_3^2 = x_4^{-1} x_5 x_6$$

$$Fx_2 = x_2^2 = x_4 x_5^{-1} x_6 \quad Fx_7 = x_7^2 = x_4 x_5 x_6$$

Since (6) is always non-negative, we have the inequality

$$m(\alpha) - \ell \geq e_B \cdot \alpha - \nu(B)$$

for all bases $B$, and thus $m(\alpha) - \ell \geq g(\alpha)$. On the other hand, if $m(\alpha) - \ell > g(\alpha)$, then (6) will always be positive, so no subset of the differentials $d(F^{\alpha_i} x_i)$ will form a basis for $\Omega_{K(F^{-\alpha}x_E)/K}$. However, this would contradict the fact that the complete set of differentials $d(F^{-\alpha_i} x_i)$ for all $i \in E$ forms a generating set for $\Omega_{K(F^{-\alpha}x_E)/K}$, and therefore, some subset forms a basis. Thus, $m(\alpha)$ must equal $g(\alpha) + l$, which completes the proof that the two matroid flocks coincide. \qed

**Remark 9.** By [BDP18, Thm. 7], any matroid flock, such as that of an algebraic extension, comes from a valued matroid, but the valuation is not unique. In particular, two valuations $\nu$ and $\nu'$ are called equivalent if they differ by a shift $\nu'(B) = \nu(B) + \lambda$ for some constant $\lambda$ [DW92, Def. 1.1], and equivalent valuations define the same matroid flock. However, among all equivalent valuations giving the matroid flock of an algebraic extension, the formula (1) nevertheless gives a distinguished valuation. For example, if $L = K(x_E)$, then this distinguished valuation $\nu$ is the unique representative such that the minimum $\min_B \nu(B)$ over all bases $B$ is 0. If $L$ is a proper extension of $K(x_E)$, then the valuation $\nu$ records the inseparable degree $[L : K(x_E)]$, which was denoted $p^\ell$ in the proof of Theorem 1.

**Example 10.** We look at the matroid flock and Lindström valuation of an algebraic realization of the non-Fano matroid $M$ over $K = \mathbb{F}_2$, which is a special case of the construction in Example 7. The realization is given by the elements

\[
\begin{align*}
  x_1 &= t_1 & x_3 &= t_3 & x_5 &= t_1 t_3 & x_7 &= t_1 t_2 t_3 \\
  x_2 &= t_2 & x_4 &= t_1 t_2 & x_6 &= t_2 t_3
\end{align*}
\]

in the field $L = K(t_1, t_2, t_3)$. The differentials of these elements in $\Omega_{L/K}$ are:

\[
\begin{align*}
  dx_1 &= dt_1 & dx_4 &= t_2 dt_1 + t_1 dt_2 \\
  dx_2 &= dt_2 & dx_5 &= t_3 dt_1 + t_1 dt_3 \\
  dx_3 &= dt_3 & dx_6 &= t_3 dt_2 + t_2 dt_3 \\
  & & dx_7 &= t_2 t_3 dt_1 + t_1 t_3 dt_2 + t_1 t_2 dt_3.
\end{align*}
\]

These vectors are projectively equivalent to the Fano configuration, and, therefore, the matroid $M_{(0,0,0,0,0,0,0,0)}$ of the matroid flock is the Fano matroid.
Therefore, $\{4, 5, 6\}$ is a basis for the matroid $M_\alpha$. Using the basis $dx_4, dx_5, dx_6$ for $\Omega_{K(x-E)/K}$, one can check that the vectors $d(Fx_i)$, for $i = 1, 2, 3, 7$ are all parallel to each other, and thus the bases of $M$ which contain at least two of these indices is not a basis for $M_\alpha$.

We claim that the Lindström valuation $\nu$ of the field extension $L$ of $K$ takes the value 0 for every basis of $M$ except that $\nu(\{4, 5, 6\}) = 1$. This can be seen directly from the definition (1) because one can check that every basis other than $\{4, 5, 6\}$ generates the field $L$, and $L \supset K(x_4, x_5, x_6)$ is an index 2, purely inseparable extension.

Alternatively, the fact that the vector configuration of the differentials $dx_i$ in $\Omega_{L/K}$ is the Fano matroid means that its bases consist of all bases of $M$ except for $\{4, 5, 6\}$, and so the bases of the Fano matroid have the same valuation, except for $\{4, 5, 6\}$, which has larger valuation. As in Remark 9, the matroid flock only determines the valuation up to equivalence, so we can take $\nu(B) = 0$ for $B$ a basis of the Fano matroid. Then, the computation of $M_\alpha$ above shows that both $\{4, 5, 6\}$ and $\{3, 5, 6\}$ are bases, and thus,

$$e_{\{4, 5, 6\}} \cdot \alpha - \nu(\{4, 5, 6\}) = e_{\{3, 5, 6\}} \cdot \alpha - \nu(\{3, 5, 6\}) = -1 - 0 = -1$$

and so we can solve for $\nu(\{4, 5, 6\}) = 1$.

Finally, a third way of computing the Lindström valuation is to use Example 7, which shows that the valuation is the same as that of the vector configuration given by the columns of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

over the field of rational numbers $\mathbb{Q}$ with the 2-adic valuation. The valuation of a vector configuration is given by the 2-adic valuation of the determinant of the submatrices. The submatrices of $A$ corresponding to bases of $M$ all have determinant $\pm 1$ except for the one with columns $\{4, 5, 6\}$, whose determinant is $-2$, which has 2-adic valuation equal to 1.

3. Cocircuits and minors

In this section, we consider further properties of the Lindström valuated matroid which can be understood in terms of the field theory of the extension. In particular, we give constructions of the valuated cocircuits and minors of the Lindström valuated matroid.

First, a hyperplane of the algebraic matroid of $L$ is a maximal subset $H$ of $E$ such that $L$ has transcendence degree 1 over $K(x_E)$. For any hyperplane $H$, we define a vector in $(\mathbb{Z} \cup \{\infty\})^n$:

$$C^{\infty}(H)_i = \begin{cases} \infty & \text{if } i \in H \\ \log_p[L : K(x_{H \cup \{i\}})] & \text{if } i \notin H \end{cases}$$
The expression in the second case is an integer by [Lan02, Cor. V.6.2] and finite because, by the assumption that \( H \) is a hyperplane, \( L \) must be an algebraic extension of \( K(x_{H_\cup(i)}) \), for \( i \notin H \).

**Proposition 11.** The collection of vectors:

\[ \{ C^{co}(H) + \lambda \mathbb{1} \mid H \text{ is a hyperplane of the algebraic matroid of } L, \lambda \in \mathbb{Z} \} \]

define the cocircuits of the Lindström valuation of the field \( L \) and the elements \( x_1, \ldots, x_n \).

**Proof.** By definition, the cocircuits of a valuated matroid \( M \) are the circuits of the dual \( M^* \), and the dual valuation is defined by \( \nu(B^*) = \nu(E \setminus B^*) \) for any subset \( B^* \subseteq E \) such that \( E \setminus B^* \) is a basis of \( M \). Suppose \( B^* \) and \( B^* \setminus \{u\} \cup \{v\} \) are bases of \( M^* \), and \( C^{co}(H) \) is a cocircuit contained in \( B^* \cup \{v\} \). Then, as in the proof of Proposition 8, we have to show the relation:

\[
\nu^*(B^*) + C^{co}(H)_u = \nu^*(B^* \setminus \{u\} \cup \{v\}) + C^{co}(H)_v
\]

We write \( B \) for the complement \( E \setminus B^* \), which is a basis of \( M \). We can then expand these expressions using their definitions and multiplicativity of the inseparable degree:

\[
\nu^*(B^*) = \log_p [L : K(x_{H \cup \{v\}})]_i + \log_p [K(x_{H \cup \{v\}}) : K(x_{B})]_i \\
C^{co}(H)_u = \log_p [L : K(x_{H \cup \{u\}})]_i \\
\nu^*(B^* \setminus \{u\} \cup \{v\}) = \log_p [L : K(x_{H \cup \{u\}})]_i \\
+ \log_p [K(x_{H \cup \{u\}}) : K(x_{B \setminus \{v\} \cup \{u\}})]_i \\
C^{co}(H)_v = \log_p [L : K(x_{H \cup \{v\}})]_i
\]

Therefore, to show the relation (7), it is sufficient to show that

\[
[K(x_{H \cup \{v\}}) : K(x_{B})]_i = [K(x_{H \cup \{u\}}) : K(x_{B \setminus \{v\} \cup \{u\}})]_i
\]

We claim that (8) is true because both sides are equal to the inseparable degree \([K(x_H) : K(x_{B \setminus \{v\}})]_i \). Indeed, the extensions on either side of (8) are given adjoining to the extension \( K(x_H) \supset K(X_{B \setminus \{v\}}) \) a single transcendental element, namely, \( x_v \) on the left, and \( x_u \) on the right. Such a transcendental element has no relations with the other elements of \( x_H \) and so doesn’t affect the inseparable degree.

Minors of a valuated matroid are defined in [DW92, Prop. 1.2 and 1.3]. Note that the definition of the valuation on the minor depends on an auxiliary choice of a set of vectors, and the valuation is only defined up to equivalence.

**Proposition 12.** Let \( F \) and \( G \) be disjoint subsets of \( E \). Then the minor \( M \setminus G/F \), denoting the deletion of \( G \) and the contraction of \( F \), is equivalent to the Lindström valuation of the extension \( K(x_{E \setminus G}) \supset K(x_F) \) with the elements \( x_i \) for \( i \in E \setminus (F \cup G) \).
Proof. The valuated circuits of the deletion $M \setminus G$ are equal to the restriction of the valuated circuits $C$ such that $\text{supp } C \cap G = \emptyset$ to the indices $E \setminus G$. Likewise, the circuits and circuit polynomials of the algebraic extension $K(x_{E'}) \supset K$ are those of $L \supset K$ such that the support and variable indices, respectively, are disjoint from $G$. Therefore, the valuated circuits of the Lindström matroid of $K(x_{E \setminus G})$ as an extension of $K$ are the same as those of the deletion $M \setminus G$.

Dually, the valuated cocircuits of the contraction $M \setminus G/F$ are the restrictions of the cocircuits $C^{\text{co}}$ of $M \setminus G$ such that $\text{supp } C^{\text{co}} \cap F = \emptyset$ to the indices in $E \setminus (F \cup G)$. The hyperplanes of the extension $K(x_{E \setminus G}) \supset K(x_F)$ are the hyperplanes of $K(x_{E \setminus G}) \supset K$ which contain $F$ and so the valuated cocircuits are the valuated cocircuits which are disjoint from $F$ and with indices restricted to the indices $E \setminus (F \cup G)$. Therefore, the Lindström valuated matroid of $K(x_{E \setminus G}) \supset K(x_F)$ is the same as the minor $M \setminus G/F$. □

References

[BDP18] Guus P. Bollen, Jan Draisma, and Rudi Pendavingh, Algebraic matroids and Forbenius flocks, Adv. Math. 323 (2018), 688–719.

[DW92] Andreas W. M. Dress and Walter Wenzel, Valuated matroids, Adv. Math. 93 (1992), no. 2, 214–250.

[Ing71] A. W. Ingleton, Representation of matroids, Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969), Academic Press, London, 1971, pp. 149–167.

[KRT13] Franz J. Király, Zvi Rosen, and Louis Theran, Algebraic matroids with graph symmetry, 2013. preprint, [arXiv:1312.3777]

[Lan02] Serge Lang, Algebra, Graduate Texts in Mathematics, vol. 211, Springer, 2002.

[Lin85] Bernt Lindström, On the algebraic characteristic set for a class of matroids, Proc. AMS 95 (1985), no. 1, 147–151.

[MT01] Kazuo Murota and Akihisa Tamura, On circuit valuations of matroids, Adv. Appl. Math. 26 (2001), 192–225.

Department of Mathematics, University of Tennessee, 227 Ayres Hall, Knoxville, TN 37996

E-mail address: cartwright@utk.edu