MEAN-FIELD LIMIT AND NUMERICAL ANALYSIS FOR ENSEMBLE KALMAN INVERSION: LINEAR SETTING

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Abstract. Ensemble Kalman inversion (EKI) is a method introduced in [14] to find samples from the targeted posterior distribution in the Bayesian formulation. As a deviation from Ensemble Kalman filter [6], it introduces a pseudo-time along which the particles sampled from the prior distribution are pushed to fit the profile of the posterior distribution. To today, however, the thorough analysis on EKI is still unavailable. In this article, we analyze the continuous version of EKI, a coupled SDE system, and prove the solution to this SDE system convergences, as the number of particles goes to infinity, to the target posterior distribution in Wasserstein distance in finite time.

1. INTRODUCTION

Bayes’ law plays a rather important role in inverse problems. It provides a way to blend one’s prior knowledge, and collected data, to produce a so-called posterior distribution that characterizes the probability distribution of the to-be-reconstructed parameter. Bayesian inference is used in almost every aspect of inverse problems. Its generality and stability largely explains its popularity.

A big challenge in the Bayesian formulation, however, comes from sampling, especially when the to-be-reconstructed parameter is of high dimensional. Suppose there are 1000 parameters to be reconstructed, and we have a budget of making 10,000 samples, then how do we design algorithms so that these 10,000 samples look like that they are i.i.d. drawn from the posterior distribution?

There are abundant studies in this direction, and a lot of algorithms have been proposed. Traditional methods such as Markov chain Monte Carlo (MCMC) like Metropolis Hastings type algorithm, and sequential Monte Carlo (SMC) have garnered a large amount of investigations both on the theoretical and numerical sides [5] [17] [4], and newer methods such as stein variational gradient descent (SVGD) based on Kernelized Stein Discrepancy [15] and the ensemble Kalman inversion (EKI) quickly drew attention from many related areas. All the methods have certain advantages and disadvantages on one or another aspect.

In this paper, we will study in depth of Ensemble Kalman Inversion (EKI) method [8] [14]. The method can be viewed as a variation of Ensemble Kalman filter (EnKF). EnKF was introduced initially for dynamical systems in [6] [11] [7] [13]: one sequentially mixes in newly available data and evolve the probability distribution of the to-be-reconstructed parameters along the evolution of the dynamical system. In EKI, the problems are typically static: one is given a static problem with unknown parameters, and independent-on-time measurements are taken to infer these parameters. In EKI [14], the authors introduced a pseudo-time: one i.i.d. samples a fixed number of particles according to the prior distribution and call them the initial data at \( t = 0 \), and in (pseudo-)time moves the particles around according to certain dynamics, hoping at \( t = 1 \) the particles look like they are i.i.d. sampled from the posterior distribution. There are a number of theoretical studies about this method [18] [19] [1], but the thorough understanding of the convergence is far from being complete. It is unknown, for example, with what rate in what sense, the ensemble distribution provided by the algorithm approximates the target posterior distribution. In this paper we will give a theorem that states such convergence in Wasserstein 2-metric with a precise rate in \( J \), the number of particles. This is by far the only theoretical result on this matter known to us. We would like to mention three related works: in [16] the authors proved the continuous version of SVGD is the weak solution to a transport type equation whose equilibrium state at infinite time is the target posterior distribution. They did not obtain convergence for the infinite time but it is the only theoretical work that rigorously justified the mean-field limit using Dobrushin’s argument known to us; in [10] the authors proposed a new algorithm based on a Fokker-Planck
type equation that requires convergence in infinite time; in [12] the authors investigated the convergence of the moments with kinetic tools.

In Section 2, we will give a quick overview of the method, and collect the theoretical results obtained in literature. We will also state the main result that we obtain, with a layout of the strategy of the proof. The proof is divided into two steps, studied in Section 3 and 4 respectively. Some calculations are rather technical and we leave them in appendix.

2. Ensemble Kalman Inversion setup and statement of our result

The aim of ensemble Kalman inversion is to find samples that look like they are drawn i.i.d. from the target posterior distribution.

Suppose \( u \in \mathcal{X} \) is the to-be-reconstructed parameter and it could be a long vector, and let \( \mathcal{G} : \mathcal{X} \to \mathcal{Y} \) be the parameter-to-observable map, namely:

\[
y = \mathcal{G}(u) + \eta,
\]

where \( y \in \mathcal{Y} \) collects the observed data with \( \eta \) denoting the noise in the measurement-taking. The inverse problem amounts to reconstructing \( u \) from \( y \). Without loss of generality, we assume \( \mathcal{X} = \mathbb{R}^L \), \( \mathcal{Y} = \mathbb{R}^K \) and \( \eta \sim \mathcal{N}(0, \Gamma) \) is a Gaussian noise independent of \( u \).

Denoting the loss functional \( \Phi(\cdot; y) : \mathbb{R}^N \to \mathbb{R} \) by

\[
\Phi(u; y) = \frac{1}{2} |y - \mathcal{G}(u)|^2_{\Gamma}, \quad \text{where} \quad | \cdot |_{\Gamma} := \left| \Gamma^{-\frac{1}{2}} \cdot \right|
\]

then the Bayes’ theorem, derived simply from the equivalence of the joint probability, states that the posterior distribution is the (normalized) product of the prior distribution and the likelihood function:

\[
\mu_{\text{pos}}(u) du = \frac{1}{Z} \exp \left( -\Phi(u; y) \right) \mu_0(u) du, \quad \text{with} \quad Z := \int_{\mathcal{X}} \exp \left( -\Phi(u; y) \right) \mu_0(u) du.
\] (1)

Here \( Z \) serves as the normalization factor, \( \exp \left( -\Phi(u; y) \right) \) is the likelihood function and \( \mu_0 \) is the prior distribution that collects people’s prior knowledge about the distribution of \( u \). This so-called posterior distribution represents the probability measure of the to-be-reconstructed parameter \( u \), blending the prior knowledge and the collected data \( y \), taking \( \eta \), the measurement error into account.

More explicitly, it is a classical derivation that assuming

- \( \mathcal{G} \) is linear in the sense that there exists a matrix \( A \) so that

\[
\mathcal{G} (\cdot) = A \cdot, \quad \text{with} \quad A \in \mathcal{L}(\mathbb{R}^L, \mathbb{R}^K),
\] (2)

- and \( \mu_0 \), the prior distribution, is a Gaussian distribution with mean \( \mu_0 \) and covariance \( \Gamma_0 \):

\[
\mu_0(u) = \frac{1}{Z(0)} \exp \left( -\frac{1}{2} (u - u_0)^\top \Gamma_0^{-1} (u - u_0) \right),
\] (3)

then under the assumption that

\[
\det (tA^\top \Gamma_0^{-1} A + \Gamma_0^{-1}) > C > 0, \quad \forall t \in [0, 1],
\] (4)

the posterior distribution is also a Gaussian distribution with explicit expressible mean and covariance

\[
m = (A^\top \Gamma_0^{-1} A + \Gamma_0^{-1})^{-1} (A^\top \Gamma_0^{-1} A u_0^\top + \Gamma_0^{-1} u_0), \quad \Gamma_{\text{pos}} = (A^\top \Gamma_0^{-1} A + \Gamma_0^{-1})^{-1}.
\]

More details on Bayesian inversion can be found in [3] [20]. For later use, we denote the “closest” solution \( u^l \) with noise \( r \) such that:

\[
y = Au^l + r, \quad \text{with} \quad r \perp \text{range}\{A\}.
\] (5)

2.1. Ensemble Kalman Inverse. Mathematically, with [11] explicitly written down, the inverse problem is complete. In practice, however, one still needs to keep looking for a good representative of \( u \). One usually uses either the mean of the posterior distribution or MAP (maximum a posteriori). Finding this mean or MAP point, however, is genuinely challenging: since it is unlikely to plot out the whole distribution function (especially in high dimension space), one typically samples a large number of particles according to the target distribution. How to generate a fixed number of samples that look like i.i.d. sampled from a usually arbitrarily looking distribution?

There are a large number of algorithms developed towards this end, including the classical MCMC (Markov chain Monte Carlo) method and the newly developed SVGD (Stein variational Gradient Descent) method.
It is not our intention to compare these different methods. In this paper, we would like to focus on Ensemble Kalman Inversion and give a relatively sharp estimate to the convergence rate of the method.

EKI is a variation of EnKF tailored to fit static problem setups. It samples a fixed number of particles according to the prior distribution first, call them \( \{ u_0^j \} \) (with 0 in the subscript standing for initial time), and introduces the pseudo-time along which particles are propagated, according to a certain flow defined by the ensemble mean and covariance, hoping in finite time, the ensemble of the particles represents the posterior distribution. The algorithm is summarized in Algorithm 1.

Algorithm 1 Ensemble Kalman Inverse

**Preparation:**
1. Input: \( J \gg 1; h \ll 1 \) (time step); \( N = 1/h \) (stopping index); \( \Gamma \); and \( y \) (data).
2. Initial: \( \{ u_0^j \} \) sampled from initial distribution \( \mu_{\text{prior}} \).

**Run:** Set time step \( n = 0 \);

While \( n < N \): 1. Define empirical means and covariance:
\[
\bar{\mu}_n = \frac{1}{J} \sum_{j=1}^{J} u_n^j, \quad \text{and} \quad \bar{\Sigma}_n = \frac{1}{J} \sum_{j=1}^{J} \mathcal{G}(u_n^j),
\]
\[
C_{pp}^{\mu}(u) = \frac{1}{J} \sum_{j=1}^{J} (\mathcal{G}(u_n^j) - \bar{\Sigma}_n) \otimes (\mathcal{G}(u_n^j) - \bar{\Sigma}_n), \quad \text{and} \quad C_{pp}^{\mu}(u) = \frac{1}{J} \sum_{j=1}^{J} (u_n^j - \bar{\mu}_n) \otimes (u_n^j - \bar{\mu}_n). \tag{6}
\]

2. Artificially perturb data (with \( \xi_{n+1}^j \) drawn i.i.d. from \( \mathcal{N}(0, h^{-1}\Gamma) \)):
\[
y_{n+1}^j = y + \xi_{n+1}^j, \quad j = 1, \ldots, J.
\]
3. Update (set \( n \rightarrow n + 1 \))
\[
u_{n+1}^j = u_{n}^j + C_{pp}^{\mu}(u_n) \left( C_{pp}^{\mu}(u_n) + h^{-1}\Gamma \right)^{-1} \left( y_{n+1}^j - \mathcal{G}(u_n^j) \right), \quad \forall 1 \leq j \leq J. \tag{7}
\]

end

Output: \( \{ u_{N}^j \} \).

Prior to running the algorithm, one first specifies the number of samples needed (denote by \( J \)), and the number of steps one can take (denote by \( N \)). The time-step size, then is simply \( h = 1/N \). This is to ensure \( t = 1 \) is the final time. So in total, there are two parameters in the algorithm:

1. The pseudo-time-step \( h \).
2. The number of particles \( J \).

Along the evolution, at each time step, one computes the sample mean and covariance in (6), and uses them to move the samples around according to (7). If the system is linear (2), the update formula could be further simplified to
\[
u_{n+1}^j = u_{n}^j + C^{uu} A^{\ast} (AC^{uu} A^{\ast} + h^{-1}\Gamma)^{-1} (y_{n+1}^j - Au_{n}^j),
\]
with \( C^{uu} \) being the covariance matrix of \( \{ u_0^j \} \):
\[
C^{uu} = \frac{1}{J} \sum_{j=1}^{J} (u_n^j - \bar{u}) \otimes (u_n^j - \bar{u}).
\]

Upon finishing the algorithm in \( N \) steps, one obtains a list of particles \( \{ u_N^j \} \) and defines the ensemble distribution:
\[
M_{\mu}(u)du = \frac{1}{J} \sum_{j=1}^{J} \delta^{(j)}(u)du, \tag{8}
\]
hoping this ensemble distribution, in some sense, is close to the target posterior distribution \( \mu_{\text{post}}du \).

There are two parameters in the algorithm, and thus the convergence result of the algorithm to the posterior distribution should be established in the \( h \rightarrow 0 \) and \( J \rightarrow \infty \) limit.

**Remark 2.1.** Two comments are in order:

1. **We emphasize that** \( N \) and \( h \) **satisfy a certain relation:** \( Nh = 1 \), **and thus** \( N \) **is not a free parameter.**
   This fact is easily overlooked. In fact, in all the previous theoretical studies that we found [15, 4],...
people have been looking for convergence result where $h \to 0$ first and $N \to \infty$ afterwards. Namely it is
\[
\lim_{N \to \infty} \lim_{h \to 0} \quad \text{instead of} \quad \lim_{N h = 1, h \to 0}
\]
that has been studied. We would like to emphasize, however, that the two limits do not commute. It is rather dangerous to investigate $h \to 0$, the continuum limit, before passing $N \to \infty$, long time limit. This leads to an artificial “collapsing” phenomenon. In this article, we stick to what the algorithm requires, and we look at finite time $t = Nh = 1$ dynamics of the system.

2. Although we do not aim at comparing different methods, but one immediate advantage of this method over MCMC is that the number of samples are fixed, and the number of steps are also fixed. So instead of tracing the error in time and terminating the process on-the-fly whenever tolerance is met, the number of particles is pre-set, and thus the numerical cost is known ahead of the computation. Indeed, exactly because of this, the error analysis is rather crucial: based on the error analysis, one can pre-determine the size of $J$ and $h$.

2.2. Strategy of our proof. The crucial difference between our approach and the previous ones, as discussed in the remark above, is that we look at finite time convergence and the convergence is taken on $J \to \infty$. We view $N$ as a fixed number once $h$ is set, exactly as what the algorithm requires us to do.

We do emphasize that we build our analysis on the results obtained in [18, 2]. It was argued in that paper that in the continuum limit ($h \to 0$), the algorithm formally becomes the Euler-Maruyama discretization to the following SDE:
\[
du_j^t = C^{ap}(u)\Gamma^{-1} \left(y - G(u_j^t)\right) dt + C^{ap}(u)\Gamma^{-1/2}dW_j^t.
\]
This suggests that SDE system can be viewed as the continuous version of the algorithm, and the solutions are close. Furthermore, with the linear assumption \([2]\), the equation is reduced to:
\[
du_j^t = \text{Cov}_u(t)A^\top\Gamma^{-1}A \left(\dot{u}^t - u_j^t\right) dt + \text{Cov}_u(t)A^\top\Gamma^{-1/2}dW_j^t, \tag{9}
\]
where we use $y = Au^\top + r$ and $\text{Cov}_u(t)$ is the empirical covariance:
\[
\text{Cov}_u(t) = \frac{1}{J} \sum_{j=1}^{J} \left( u_j^t - \bar{u}_t \right) \otimes \left( u_j^t - \bar{u}_t \right), \quad \text{with} \quad \bar{u}_t = \frac{1}{J} \sum_{j=1}^{J} u_j^t.
\]

The SDE \([3]\) is well-defined with the following space: let $\Omega$ be the sample space and $\mathcal{F}_0$ being the $\sigma$-algebra: $\sigma(\{w_j(t) = 0, 1 \leq j \leq J\})$, then the filtration is introduced by the dynamics:
\[
\mathcal{F}_t = \sigma(\{w_j(t = 0), W_j^s, 1 \leq j \leq J, s \leq t\}). \tag{10}
\]

We do not intend to rigorously prove the “equivalence” between the SDE system and the algorithm (which corresponds to showing the $h \to 0$ limit). Rather we will focus on the $J \to \infty$ limit, assuming this SDE system indeed is a close approximation to the algorithm. In particular, we will show that as the number of particles going to infinity, the solution to the SDE system indeed converges, in the Wasserstein’s sense, to the posterior distribution.

The rigorous statement of the main result is the following:

**Theorem 2.1.** (Main result: linking SDE with $\mu_{pos}$) Under assumptions \([2]-[4]\), let $w_0^j$ i.i.d. sampled from $\mu_0(du)$, then for any $\epsilon > 0$, there exists $J_\epsilon > 0$, such that for any $J > J_\epsilon$
\[
\mathbb{E}(W_J(\mu_{pos}(du), M_0(u)du)) \leq \epsilon,
\]
where $M_0(du)$ is the ensemble distribution of $w_{i=1}^J$, defined in \([S]\), with $\{w_j\}$ solving the SDE \([S]\).

We will also give the decay rate in the later sections.

To show this theorem, we take two steps. We will first show that $\mu_{pos}$ is the solution to a Fokker-Planck-like PDE at $t = 1$ if the initial data is given as the prior distribution. We then will connect the PDE with the SDE using the standard Dobrushin’s argument. Before laying out the strategy, we first unify the notations
in the following: we always denote $\mathbb{E}$ the expectation in the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$. For any vectors $\{m^j\}_{j=1}^J$ and $\{n^j\}_{j=1}^J$, we denote

$$\text{Cov}_{m,n} = \frac{1}{J} \sum_{j=1}^J \left( m^j - \overline{m}_t \right) \otimes \left( n^j - \overline{n}_t \right),$$

and denote $\text{Cov}_m = \text{Cov}_{m,m}$. The two main steps towards showing the convergence are as followed:

**Step 1:** Find the underlying PDE for the posterior distribution. To that end we first define

$$\mu(t,u)du = \frac{1}{Z(t)} \exp(-t\Phi(u,y)) \mu_0(u)du, \quad \text{with} \quad Z(t) := \int_X \exp(-t\Phi(u,y)) \mu_0(u)du. \tag{11}$$

Then it is clear that

$$\mu(t=0,u)du = \mu_0 du, \quad \text{and} \quad \mu(t=1,u)du = \mu_{\text{pos}} du,$$

which means this new definition (11) finds a smooth transition that moves the prior distribution to the posterior, our target distribution. This transition can be further characterized by the following Fokker-Planck-like PDE:

$$\partial_t \rho(t,u) + \nabla \cdot \left( (u - u)^* A^* \Gamma^{-1} A \text{Cov}_\mu(t) \rho \right) = \frac{1}{2} \text{Tr} \left( \text{Cov}_\mu(t) A^* \Gamma^{-1} A \text{Cov}_\mu(t) \mathcal{H}_u(\rho) \right). \tag{12}$$

In particular, we will show:

**Theorem 2.2** (Linking $\mu_{\text{pos}}$ with the PDE). Under assumptions (2)-(11), the Fokker-Planck-like equation (12) characterizes the transition from the prior distribution $\mu_0$ to the target posterior distribution $\mu_{\text{pos}}$. Namely, $\mu(u,t)$, defined in (11), is a unique solution to the Fokker-Planck-like PDE (12). In particular, with initial condition set to be $\rho(t=0,u) = \mu_0$, we have

$$\rho(t=1,u) = \mu_{\text{pos}}.$$

Like every other Fokker-Planck equation, this PDE is associated with a particle system that follows its flow in probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$. Realizing that the first order terms represent the velocity of particle and the second order term introduces the Brownian motion, the particle system then writes:

$$dv^j_t = \text{Cov}_\mu(t) A^* \Gamma^{-1} A \left( u^j - v^j \right) dt + \text{Cov}_\mu(t) A^* \Gamma^{-1/2} dW^j_t. \tag{13}$$

This particle system $\{v^j\}$ looks rather similar to the underlying SDE of the algorithm (9). However, instead of having Cov, to determine the flow, which makes (9) nonlinear, this $\{v^j\}$ system is linear with the speed and the strength of the Brownian motion preset by $\mu$. Since the flow of $\{v^j\}$ is determined by the PDE (12), it is rather intuitive that the ensemble distribution of $\{v_j\}$ should be similar to the solution to the PDE in some sense.

**Step 2:** We show the “equivalence” between the SDE (9) and the PDE (12) in Step 2, through linking $\{u^j\}$ and $\{v^j\}$ system by comparing (9) and (13).

We will show they are equivalent in the sense that with $J \to \infty$, the distance between the ensemble distribution generated by $\{u^j\}$ and the solution to the PDE converges. Here the ensemble distribution is the normalized summation of many delta-functions, defined in (5) and $\{u^j\}$ is the solution to the coupled SDE (9). To measure the distance we use the Wasserstein distance:

**Definition 1.** Let $\nu_1, \nu_2$ be two probability measures in $(\mathbb{R}^L, \mathcal{B}_{\mathbb{R}^L})$, then the $W_2$-Wasserstein distance between $\nu_1, \nu_2$ is defined as

$$W_2(\nu_1, \nu_2) := \left( \inf_{\gamma \in \Gamma(\nu_1, \nu_2)} \int_{\mathbb{R}^L \times \mathbb{R}^L} |x - y|^2 d\gamma(x, y) \right)^{1/2},$$

where $\Gamma(\nu_1, \nu_2)$ denotes the collection of all measures on $\mathbb{R}^L \times \mathbb{R}^L$ with marginals $\nu_1$ and $\nu_2$.

The precise statement of the result states as the following:
Theorem 2.3 (Linking the PDE with the SDE). Under assumptions [2]-[4], for any \( \epsilon > 0 \), there exists \( J_\epsilon > 0 \), such that for any \( J > J_\epsilon \),

\[
\mathbb{E}(W_2(\rho(t = 1, u)du, M_\alpha(u)du)) \leq \epsilon,
\]

where \( \rho(t = 1, u) \) is the solution to (11) and \( M_\alpha \) is the ensemble distribution of SDE system (9) at \( t = 1 \), defined in (8), and \( \{u_0^j\} \) are drawn i.i.d. from \( \rho(t = 0, u)du \).

As argued above, this amounts to showing both

\[
\mathbb{E}(W_2(\rho(t = 1, u)du, M_\alpha(u)du)) \rightarrow 0, \quad \text{and} \quad \mathbb{E}(W_2(M_\alpha(u)du, M_\alpha(u)du)) \rightarrow 0.
\]

where \( M_\alpha(u)du \) is the ensemble distribution of \( \{v^j\} \) particles. These two smallness will be stated in Theorem 3.1 and Theorem 4.1 below respectively. Then Theorem 2.3 would be a direct consequence.

The main result, Theorem 2.1 is a direct consequence from Theorem 2.2 and Theorem 2.3. Below we designate Section 3 and Section 4 to show the proof for Theorem 2.2 and Theorem 2.3 respectively.

Remark 2.2. Two remarks are needed:

- Note that we do not aim at making the derivation of SDE rigorous in this paper. Rigorously speaking, we are still one step away from showing

\[
\lim_{J \to \infty} \lim_{h \to 0} M_\alpha(u) \sim \mu_{\text{pos}}.
\]

This step requires rigorous justification of the Euler-Maruyama method applied on the SDE (9), and is not pursued in the current paper.

- All previous results arrive at the continuum in time limit (11) and continued in discussing the validity of the system, its simplification and its long time behavior. We regard them as important stepping-stone in the sense that they provide some crucial estimates on the bounds, but we believe the long time behavior of the SDE has limited connection to the method EKI, and that the “collapsing” phenomenon is artificial. What we are interested in is what happens to (9) exactly at \( t = 1 \) given the initial \( u_0^j \) are i.i.d. samples from the prior distribution.

3. DERIVATION OF THE FOKKER-PланCK EQUATION

In this section we justify Step 1. In particular, we will show Theorem 2.2. Furthermore we will show the ensemble distribution of \( \{v^j\} \), the particle system that follows the flow of the PDE, is indeed a good representation of \( \rho(t, u) \). We further give estimates of the boundedness of the moments.

3.1. \( \mu_{\text{pos}} \) AND THE FOKKER-PланCK EQUATION. Theorem 2.2 provides a smooth transition that transforms the prior distribution to the posterior distribution in pseudo-time \( t \), changing from 0 to 1.

To show Theorem 2.2 amounts to direct deriving the derivatives and compare terms. Before we start the proof, note that according to the definition (11), and the linear assumption (2), we can explicitly express, for all \( t \geq 0 \):

\[
\mathbb{E}_\mu(t) = (t A^* \Gamma^{-1} A + \Gamma_0^{-1})^{-1} (t A^* \Gamma^{-1} A u^1 + \Gamma_0^{-1} u_0) \quad \text{and} \quad \text{Cov}_\mu(t) = (t A^* \Gamma^{-1} A + \Gamma_0^{-1})^{-1}.
\]

Proof. To show \( \mu(t, u) \) is the solution to the PDE, we simply plug it in the equation and check if the two sides balance. Without loss of generality, we assume \( y = au^1 \) in (13) (one arrives at the same derivation with when \( r \neq 0 \)). As a preparation we first calculate the derivatives of \( \mu \). The time derivative is:

\[
\partial_t \mu(u, t) = -\Phi (u; y) \mu(u, t) - \frac{\partial_t Z(t)}{Z(t)} \mu(u, t),
\]

in which:

\[
\Phi (u; y) = (u^1 - u)^* A^* \Gamma^{-1} A (u^1 - u)/2;
\]

and

\[
\frac{\partial_t Z}{Z} = \int -((u - \mathbb{E}_\mu + \mathbb{E}_\mu - u)^* A^* \Gamma^{-1} A(u - \mathbb{E}_\mu + \mathbb{E}_\mu - u^1)^*/2 \mu du = -\text{Tr} [\text{Cov}_\mu A^* \Gamma^{-1} A] \} / 2 - (u^1 - \mathbb{E}_\mu)^* A^* \Gamma^{-1} A (u^1 - \mathbb{E}_\mu) / 2.
\]

Similarly the gradient in \( u \) is:

\[
\nabla_u \mu(u, t) = t A^* \Gamma^{-1} A (u^1 - u) \mu(u, t) + \Gamma_0^{-1} (u_0 - u) \mu(u, t).
\]
The Hessian in \(u\) can also be computed:
\[
\mathcal{H}_u(\mu(u, t)) = (-\text{Cov}_\mu)^{-1} + (\text{Cov}_\mu)^{-1}(u - \mathbb{E}_\mu)(u - \mathbb{E}_\mu)^* (\text{Cov}_\mu)^{-1}) \mu.
\] (18)

Putting them back into the equation, one has
\[
\partial_t \mu + \nabla_u \cdot \left( (u^\dagger - u)^* A^* \Gamma^{-1} A \text{Cov}_\mu \mu \right) = \frac{1}{2} \text{Tr} \left( \text{Cov}_\mu A^* \Gamma^{-1} A \text{Cov}_\mu \mathcal{H}_u(\mu) \right)
\]
\[
= \partial_t \mu + (u^\dagger - u)^* A^* \Gamma^{-1} A \text{Cov}_\mu \nabla_u \mu + \nabla_u \cdot \left( (u^\dagger - u)^* A^* \Gamma^{-1} A \text{Cov}_\mu \right) \mu
\]
\[
- \frac{1}{2} \text{Tr} \left( \text{Cov}_\mu A^* \Gamma^{-1} A \text{Cov}_\mu \mathcal{H}_u(\mu) \right) = \text{term I} + \text{term II} + \text{term III} + \text{term IV}.
\]

Term I is computed in (15). To handle term II, we plug in (17) for:
\[
(u^\dagger - u)^* A^* \Gamma^{-1} A \text{Cov}_\mu \nabla_u \mu = t (u^\dagger - u)^* A^* \Gamma^{-1} A \text{Cov}_\mu A^* \Gamma^{-1} A (u^\dagger - u) \mu
\]
\[
+ (u^\dagger - u)^* A^* \Gamma^{-1} A \text{Cov}_\mu \Gamma_0^{-1} (u_0 - u) \mu
\]
\[
= t (u^\dagger - u)^* A^* \Gamma^{-1} A (u^\dagger - u) \mu - (u^\dagger - u)^* A^* \Gamma^{-1} A \text{Cov}_\mu \Gamma_0^{-1} (u^\dagger - u_0) \mu
\]
\[
= t (u^\dagger - u)^* A^* \Gamma^{-1} A (u^\dagger - u) \mu - (u^\dagger - u)^* A^* \Gamma^{-1} A (u^\dagger - \mathbb{E}_\mu) \mu
\]
where we have used (15) and
\[
y - AE_\mu = A (u^\dagger - E_\mu) = A (tA^* \Gamma^{-1} A + \Gamma_0^{-1})^{-1} \Gamma_0^{-1} (u^\dagger - u_0) = A \text{Cov}_\mu \Gamma_0^{-1} (u^\dagger - u_0).
\]

Term III becomes:
\[
\nabla_u \cdot \left( (u^\dagger - u)^* A^* \Gamma^{-1} A \text{Cov}_\mu \right) \mu = -\text{Tr} \left[ \text{Cov}_\mu A^* \Gamma^{-1} A \right] \mu.
\]

By (18), Term IV turns to:
\[
- \frac{1}{2} \text{Tr} \left( \text{Cov}_\mu A^* \Gamma^{-1} A \text{Cov}_\mu \mathcal{H}_u(\mu) \right) = \frac{1}{2} \text{Tr} \left( \text{Cov}_\mu A^* \Gamma^{-1} A \right) + \frac{1}{2}(u - \mathbb{E}_\mu)^* A^* \Gamma^{-1} A (u - \mathbb{E}_\mu) \mu.
\]

We conclude simply by adding up all the terms. \(\square\)

A direct consequence of the theorem above is the following boundedness in moments. We collect it here for later use.

**Proposition 3.1.** For any \(1 \leq p < \infty\), and \(0 \leq t \leq 1\), there exits a constant \(C_p\) independent of \(t\) such that:
\[
\int_{\mathbb{R}^L} |u - \mathbb{E}_\mu(t)|^p \mu(u, t) du < C_p, \quad \text{and} \quad \int_{\mathbb{R}^L} |u - u^\dagger|^p \mu(u, t) du < C_p.
\] (19)

**Proof.** Because \(\mu(u, t) du\) is a Gaussian distribution, we have
\[
\int_{\mathbb{R}^L} |u - \mathbb{E}_\mu(t)|^p \mu(u, t) du \\
= \frac{1}{\sqrt{(2\pi)^L \det(\text{Cov}_\mu^{-1}(t))}} \int_{\mathbb{R}^L} \exp \left( -\frac{1}{2} u^* \text{Cov}_\mu^{-1}(t) u \right) du,
\]
\[
= \frac{1}{\det(\text{Cov}_\mu^{-1}(t))^{(L+p+1)/2}} \int_{\mathbb{R}^L} |v|^p \exp \left( -\frac{1}{2} v^* \left[ \text{Cov}_\mu^{-1}(t) / \det(\text{Cov}_\mu^{-1}(t)) \right] v \right) dv,
\]
\[
\leq \frac{1}{\det(\text{Cov}_\mu^{-1}(t))^{(L+p+1)/2}} \leq C,
\]
where the second equation comes from changing variable \(v = u\det(\text{Cov}_\mu^{-1}(t))^{1/2}\) and the last inequality comes from (15).

To estimate the second inequality, we notice, according to (15) that
\[
\mathbb{E}_\mu(t) - u^\dagger = (tA^* \Gamma^{-1} A + \Gamma_0^{-1})^{-1} \Gamma_0^{-1} (u_0 - u^\dagger),
\]
and the inequality follows directly from the first inequality. \(\square\)
3.2. \{v_j\} and the Fokker-Planck-like equation. We now investigate the particle system \{v_j\} that is designed to follow the flow of the PDE. Reformulating (13), we have:

\[ dv_j^i = \text{Cov}_{\mu}(t) A \Gamma^{-1} A \left( v_j^i - v_j^t \right) dt + \text{Cov}_{\mu}(t) A \Gamma^{-1/2} dW_j. \]

We show below that if the initial condition for this SDE system is consistent with \( \rho(t = 0, u) \), meaning \( \{v_j\} \) are drawn i.i.d. from \( \rho(t = 0, u) \), then the ensemble distribution of \( \{v_j\} \) is equivalent to \( \rho \) for all finite time.

**Theorem 3.1** (Linking \( \{v_j\} \) with Fokker-Planck-like PDE). Under assumptions (2) - (4), let \( \{v_j^i=0\} \) drawn i.i.d. from \( \rho(t = 0, u) \), then at \( t = 1 \), there exists a constant \( C \) independent on \( J \) such that, for all \( J \geq 1 \)

\[
\mathbb{E}(W_2(M_v(u)du, \rho(t = 1, u)du)) \leq C \begin{cases} J^{-1/2}, & L < 4 \\ J^{-1/2}\log(1 + J), & L = 4 \\ J^{-2/L}, & L > 4 \end{cases}.
\]

Here \( W_2 \) stands for the \( W_2 \)-Wasserstein distance between two measures, \( \rho \) is the solution to the Fokker-Planck equation, and \( M_v \) is the ensemble distribution of \( v_{i=1}^J \):

\[ M_v(u)du = \frac{1}{J} \sum_{j=1}^{J} \delta_{v_j^i} du. \]

We note that \( t = 1 \) can be replace by any finite time, with the constant \( C < \infty \) (for all \( t < \infty \)) adjusted accordingly.

This is a rather standard result that SDE generated from the underlying PDE has its dynamics following that of the PDE. There are many famous results related to it. For the completeness of the paper we here simply cite one from [3].

**Theorem 3.2** (Theorem 1 in [3]). Let \( \mu(u)du \) be a probability measure on \( \mathbb{R}^L \) and let \( p > 0 \). Assume that

\[ M_q(\mu) := \int_{\mathbb{R}^d} |x|^q \mu(dx) < \infty \]

for some \( q > p \). Consider an i.i.d sequence \( (X_k)_{k \geq 1} \) of \( \mu du \)-distributed random variables and, for \( N \geq 1 \), define the empirical measure

\[ \mu_N := \frac{1}{N} \sum_{k=1}^{N} \delta_{X_k}. \]

There exists a constant \( C \) depending only on \( p, L, q \) such that, for all \( N \geq 1 \),

\[
\mathbb{E}(W_p(\mu_N du, \mu du)) \leq C M_q^p(\mu) \begin{cases} N^{-1/2} + N^{-(q-p)/q}, & \text{if } p > L/2 \text{ and } q \neq 2p \\ N^{-1/2}\log(1 + N) + N^{-(q-p)/q}, & \text{if } p = L/2 \text{ and } q \neq 2p \\ N^{-p/L} + N^{-(q-p)/q}, & \text{if } p \in (0, L/2), \text{ if } p \in (0, L/2) \text{ and } q \neq L/(L-p) \end{cases}.
\]

Our result, Theorem 3.1 is a straightforward consequence. Considering we are in a linear setup, \( \rho \) keeps having a Gaussian profile and thus all moments are bounded. So one can simply choose a large enough \( q \) to have the first terms in Theorem 3.2 being the dominant term, our Theorem 3.1 then directly holds true, noting that \( \{v_j^i\} \) are i.i.d. samples of \( \mu \).

As a direct consequence of Proposition 3.2, we can also bound the high moments of \( \{v_j\} \). We collect the results below for later use.

**Proposition 3.2.** With the basic setting (2) - (4), for any fixed even number \( 2 \leq p < \infty \) and large enough \( J \), there exists a constant \( C_p \) independent of \( J \) such that for all \( 0 \leq t \leq 1 \):

\[
\mathbb{E}|v_{i}^{j}|^p \leq C_p, \quad \mathbb{E}|v_{i}^{j} - u_{i}^{j}|^p \leq C_p, \quad \mathbb{E}|v_{i}^{j} - \overline{v}_i|^p \leq C_p, \quad \forall 1 \leq j \leq J,
\]

and

\[
(\mathbb{E} \|\text{Cov}_v(t) - \text{Cov}_{\mu}(t)\|_2^p)^{1/p} \lesssim J^{-1/2}.
\]
Proof. Since \( \{v^k_t\} \) are i.i.d sampled from Gaussian distribution \( \mu(u, t)du \), (21) is a direct result from (19). To show (22), without loss of generality, we first assume \( \mathbb{E}(v^k_t) = 0 \), then we write \( \text{Cov}_v(t) \) as

\[
\text{Cov}_v(t) = \frac{J - 1}{J^2} \left( \sum_{j=1}^J v^j_t \otimes v^j_t \right) - \frac{1}{J^3} \sum_{j \neq k} v^j_t \otimes v^k_t.
\]

Now we divide (22) into three parts

\[
(\mathbb{E} \|\text{Cov}_v(t) - \text{Cov}_\mu(t)\|_2^p)^{1/p} \leq \left( \mathbb{E} \left\| \frac{J - 1}{J^2} \left( \sum_{j=1}^J v^j_t \otimes v^j_t \right) - \frac{J - 1}{J^2} \left( \sum_{j=1}^J \text{Cov}_\mu \right) \right\|_2^p \right)^{1/p} + \left( \mathbb{E} \left\| \frac{1}{J^3} \sum_{j \neq k} v^j_t \otimes v^k_t \right\|_2^p \right)^{1/p} + \left( \mathbb{E} \left\| \frac{1}{J^2} \text{Cov}_\mu \right\|_2^p \right)^{1/p}.
\]

The latter two terms are bounded by \( J^{-1} \) and \( J^{-2} \) respectively using (21) and (19) respectively. To control the first term, we have

\[
\left( \mathbb{E} \left\| \frac{J - 1}{J^2} \left( \sum_{j=1}^J v^j_t \otimes v^j_t \right) - \frac{J - 1}{J^2} \left( \sum_{j=1}^J \text{Cov}_\mu \right) \right\|_2^p \right)^{1/p} \leq C_p \left( \mathbb{E} \left\| \frac{1}{J} \left( \sum_{j=1}^J v^j_t \otimes v^j_t \right) - \text{Cov}_\mu \right\|_F^p \right)^{1/p} \leq C_{p, L} \sum_{m,n=1}^L \left( \mathbb{E} \left( \frac{1}{J} \sum_{j=1}^J v^j_t \otimes v^j_t - \text{Cov}_\mu \right)_{m,n}^p \right)^{1/p} = C_{p, L} \sum_{m,n=1}^L \left( \mathbb{E} \left( \frac{1}{J} \sum_{j=1}^J v^j_t \otimes v^j_t - \text{Cov}_\mu \right)_{m,n} \right)^p \quad \frac{1}{p},
\]

where \( \left( \frac{1}{J} \sum_{j=1}^J v^j_t \otimes v^j_t - \text{Cov}_\mu \right)_{m,n} \) means the \((m, n)\)th entry of matrix. Using the central limit theorem, for any \( 1 \leq m, n \leq L \), we have

\[
\frac{\sum_{j=1}^J v^j_t \otimes v^j_t - \text{Cov}_\mu}{\sqrt{J}} \rightarrow N(0, V_{m,n}),
\]

where \( V_{m,n} \) is determined by \( \mathbb{E}_\mu, \text{Cov}_\mu \). This implies

\[
\mathbb{E} \left[ \frac{\sum_{j=1}^J v^j_t \otimes v^j_t - \text{Cov}_\mu}{\sqrt{J}} \right]^p \sim O(1).
\]

In conclusion, we finally obtain

\[
\left( \mathbb{E} \left\| \frac{J - 1}{J^2} \left( \sum_{j=1}^J v^j_t \otimes v^j_t \right) - \frac{J - 1}{J^2} \left( \sum_{j=1}^J \text{Cov}_\mu \right) \right\|_2^p \right)^{1/p} \approx J^{-1/2}, \quad (\mathbb{E} \|\text{Cov}_v(t) - \text{Cov}_\mu(t)\|_2^p)^{1/p} \lesssim J^{-1/2}.
\]
4. Equivalence of SDE and PDE, mean field limit

As argued before, to link \{\{u^j\}\} system, governed by the SDE \([9]\), with \(\mu(t, u)\), the solution to the PDE \([12]\), we will route through the connection to \{\{v^j\}\}, governed by the corresponding SDE of the Fokker-Planck equation \([13]\). As already shown in Theorem \([5.1]\) in W-2 distance, the ensemble distribution of \{\{v^j\}\} indeed well presents the solution the PDE, and thus in this section we only need to compare the two SDE systems \([9]\) and \([13]\).

The precise statement is the following:

**Theorem 4.1.** (Linking \{\{u^j\}\} with \{\{v^j\}\}) Under assumptions \([2]-[4]\), let \{\{u^j\}\} solve \([9]\) and \{\{v^j\}\} solve \([13]\) with the same initial data in probability space \((\Omega, \mathcal{F}_1, \P)\). At time \(t = 1\), the two SDE systems are close in the following sense: for any \(\epsilon > 0\) there is a constant \(0 < C_\epsilon < \infty\) so that

\[
\frac{1}{J} \sum_{j=1}^J \E |u^j_{t=1} - v^j_{t=1}|^2 \leq C_\epsilon J^{-1+\epsilon}.
\]

Furthermore, denote \(M_o\) and \(M_u\) the ensemble distributions of \{\{v^j\}\} and \{\{u^j\}\} at \(t = 1\) respectively, then

\[
\E(W_2(M_o(\mu)du, M_u(\mu)du)) \leq \left( \frac{1}{J} \sum_{j=1}^J \E |u^j_t - v^j_t|^2 \right)^{1/2} \leq C_\epsilon J^{-1/2+\epsilon}.
\]

This theorem states that the two particle systems are almost identical in the \(J \to \infty\) limit. Combined with Theorem \([3.1]\) it is straightforward to show Theorem \([2.3]\).

**Proof of Theorem 2.3.** Considering \([20]\) and \([24]\), by triangle inequality, one has:

\[
\E(W_2(M_u du, \rho(t = 1, u)du)) \leq \E(W_2(M_u du, M_u du)) + \E(W_2(M_u du, \rho(t = 1, u)du)) \leq C \begin{cases} J^{-1/2+\epsilon}, & L \leq 4 \\ J^{-2/L}, & L > 4 \end{cases},
\]

which finishes the proof. \(\square\)

To show Theorem 4.1, we first unify the notations. Without loss of generality, we let \(\bar{u}^t = \bar{0}\). We further use the following notations for conciseness. Let

\[
x^t_i = u^t_i - \bar{u}^t, \quad p^t_i = x^t_i - \bar{x}^t, \quad q^t_i = v^t_i - \bar{v}^t,
\]

and denote (call them observables)

\[
x^t_i = \Gamma^{-1/2} A x^t_i, \quad u^t_i = \Gamma^{-1/2} A u^t_i, \quad v^t_i = \Gamma^{-1/2} A v^t_i, \quad p^t_i = \Gamma^{-1/2} A (x^t_i - \bar{x}^t), \quad q^t_i = \Gamma^{-1/2} A (v^t_i - \bar{v}^t).
\]

To prove the theorem amounts to trace the evolution of \(\E|x^t_i|^2\) as a function of time and \(J\). For that we use the bootstrapping argument, namely, we assume \(\E|x^t_i|^2\) decays in \(J\) with certain rate (could be 0), then by following the flow of the SDE we can show the rate can be tightened till a threshold is achieved. This threshold is exactly the rate one needs to prove in Theorem 4.1.

Below we first demonstrate some basic a-priori estimates of \{\{u^j\}\} in Proposition \([4.1]\) and Corollary \([4.3]\) before showing the lemma that states the tightening procedure, namely Lemma \([4.1]\) and Lemma \([4.2]\). The proof of the theorem is an immediate consequence.

In the proofs we will constantly use the fact that

\[
\E|p^t_i|^2 = \E|p^t_i|^2, \quad \E|x^t_i|^2 = \E|x^t_i|^2, \quad \forall 1 \leq j \leq J, 0 \leq t \leq 1.
\]

When the context is clear, we also omit subscript \(t\) for the simplicity of the notation.

**Proposition 4.1.** Under assumptions \([2]-[4]\), let \(p \geq 2\), then for \(J\) large enough, \(p\)-th moment of particles are uniformly bounded for finite time, namely there is \(C_p > 0\) depending only on \(p\) so that for all \(0 \leq t \leq 1\)

\[
\E|u^t_i|^p \leq C_p \quad \text{and} \quad \E \left| \text{Cov}_u(t) - \text{Cov}_u(t) \right|^p_2 \leq C_p, \quad \forall 1 \leq j \leq J.
\]

Furthermore,

\[
\E \left| u^t_i - \bar{u}^t \right|^p \leq C_p, \quad \text{and} \quad \E \left| u^t_i - v^t_i \right|^p \leq C_p, \quad \forall 1 \leq j \leq J.
\]

\[
(23) \quad C_\epsilon J^{-1+\epsilon}.
\]
We note that the case of \( p = 2 \) was studied in \[1\] (Proposition 4.11 and 5.1). For later use, we need to extend the proof for arbitrary \( p \geq 2 \). The proof is shown in Appendix A. Combining Proposition 3.2 and Proposition 4.1 using triangle inequality we have:

**Corollary 4.1.** Under assumptions \([2]-[4]\), for all \( 2 \leq p < \infty \) and large enough \( J \), we have a constant \( C_p \) independent of \( J \) such that for all \( 0 \leq t \leq 1 \)

\[
\mathbb{E}|u^j_t - v^j_t|^p = \mathbb{E}|u^j_0 - v^j_t|^p \leq C_p, \quad \forall 1 \leq j \leq J.
\]

We will first show if we already have an a-priori estimate for \( \{x^j_t\} \), we can have a better boundedness for \( \{x^j_t\} \).

**Lemma 4.1.** For any \( \alpha < 1 \), and \( 0 \leq t \leq 1 \), if one has:

\[
\mathbb{E}|x^j_t|^2 \leq O(J^{-\alpha}),
\]

then, for any \( \epsilon > 0 \), there is \( C_\epsilon < \infty \) so that

\[
\mathbb{E}|p^j_t|^2 = \mathbb{E}\left| x^j_t - \frac{1}{J} \sum_{k \neq j} x^j_t \right|^2 \leq C_\epsilon J^{-1/2-\alpha/2+\epsilon}, \quad \text{and} \quad \mathbb{E}|x^j_t|^2 \leq C_\epsilon J^{-1/2-\alpha/2+\epsilon}.
\]

**Proof.** Firstly, due to \([20]\), we have a rough estimate for \( x^j \)

\[
\mathbb{E}|x^j_0|^2 \leq O(J^{-\alpha}), \quad \forall 1 \leq j \leq J,
\]

and it also leads to

\[
(\mathbb{E}|p^j|^2)^{1/2} = \left( \mathbb{E}\left| \frac{1}{J} \sum_{k \neq j} x^j \right|^2 \right)^{1/2} \leq 2 \left( \mathbb{E}|x^j_0|^2 \right)^{1/2} \leq O(J^{-\alpha/2}), \quad \forall 1 \leq j \leq J.
\]

Apply \( \Gamma^{-1/2}A \) on both sides of \([9]\) and \([13]\), we find the evolution of the observables:

\[
du^j = -\text{Cov}_u(t)u^j dt + \text{Cov}_u(t)dW^j_t, \\
d\tilde{\nu}^j = -\Gamma^{-1/2}A\text{Cov}_\mu(t)A^*\Gamma^{-1/2}\tilde{\nu}^j dt + \Gamma^{-1/2}A\text{Cov}_\mu(t)A^*\Gamma^{-1/2}dW^j_t.
\]

Subtracting the two equations we can derive the evolution of \( x^j \). With some calculation (shown in Appendix D equation \([29]\)), for any \( \epsilon > 0 \), we have:

\[
\frac{1}{J} \sum_{j=1}^J \frac{d\mathbb{E}|x^j|^2}{dt} \leq -2 \sum_{j=1}^J \mathbb{E} \left\{ \langle p^k + q^k, x^j \rangle \langle x^j, p^k + q^k \rangle + \langle p^k + q^k, x^j \rangle \langle p^k, \tilde{\nu}^j \rangle + \langle p^k, x^j \rangle \langle q^k, \tilde{\nu}^j \rangle \right\} + O(J^{-1/2-\alpha/2+\epsilon}).
\]

Here the constant in the \( O \) notation depends on \( \epsilon \). Similar to \([24]\), we take the average of \([28]\) and subtract the two equations to have:

\[
d \langle \overline{u} - \overline{\nu} \rangle = \left[ -\text{Cov}_u(t)\overline{u} + \Gamma^{-1/2}A\text{Cov}_\mu(t)A^*\Gamma^{-1/2}\overline{\nu} \right] dt + [\text{Cov}_u(t) - \text{Cov}_\mu(t)] d\overline{W}_t,
\]

which leads to

\[
\frac{d\mathbb{E}|\overline{x}|^2}{dt} = 2\mathbb{E} \langle \overline{u} - \overline{\nu}, -\text{Cov}_u(t)\overline{u} + \text{Cov}_\nu(t)\overline{\nu} \rangle dt + O(J^{-1/2})
\]

\[
= -2 \sum_{k=1}^J \mathbb{E} \left\{ \langle p^k + q^k, \overline{x} \rangle \langle \overline{x}, p^k + q^k \rangle + \langle p^k + q^k, \overline{x} \rangle \langle p^k, \overline{\nu} \rangle + \langle p^k, \overline{x} \rangle \langle q^k, \overline{\nu} \rangle \right\} + O(J^{-1}).
\]

Noticing

\[
\mathbb{E}|\overline{x}(0)|^2 = \frac{1}{J} \sum_{j=1}^J \mathbb{E}|x^j(0)|^2 = 0, \quad \text{and} \quad \mathbb{E}|\overline{x}|^2 = \mathbb{E} \left( \frac{1}{J} \sum_{j=1}^J x^j \right)^2 \leq \frac{1}{J} \sum_{j=1}^J \mathbb{E}|x^j|^2,
\]
by comparing (29) and (30), we get

\[ \frac{d}{dt} \left( \frac{1}{2} \sum_{j=1}^{J} E|x_j|^2 - E|x|^2 \right) \leq O(J^{-1/2-\alpha/2+\epsilon}) \Rightarrow \left| \frac{1}{J} \sum_{j=1}^{J} E|x_j|^2 - E|x|^2 \right| \leq O(J^{-1/2-\alpha/2+\epsilon}). \]

Noticing:

\[ E|x|^2 = \frac{1}{J^2} \sum_{j=1}^{J} E|x_j|^2 + \frac{1}{J^2} \sum_{j \neq k} E\langle x_j, x_k \rangle, \]

then for all \( 1 \leq j, k, l \leq J \) one has:

\[ |E|x_j|^2 - E\langle x_k, x_l \rangle| \leq O(J^{-1/2-\alpha/2+\epsilon}). \]

Considering the definition of \( p \):

\[ E|p|^2 = E\left| x_j - \frac{1}{J} \sum_{k} x_k \right|^2 = \frac{J-1}{J} E|x_j|^2 - \frac{J-1}{J} E\langle x_j, x \rangle \leq O \left( J^{-1/2-\alpha/2+\epsilon} \right). \]

To have the bound for \( x_j \) in (24), we notice that

\[
\left( E|p|^2 + q^k |\mathbf{x}|^2 \right)^{1/2} \leq \left( E|p|^2 + q^k |\mathbf{x}|^2 \right)^{1/2} \leq \left( E|p|^2 + q^k |\mathbf{x}|^2 \right)^{1/2} \leq C \left( E|\mathbf{x}|^2 \right)^{1/4} \leq C \left( E|\mathbf{x}|^2 \right)^{1/4},
\]

where the second inequality comes from Hölder’s inequality, and Proposition 3.2 is used in the third inequality. The forth inequality comes from Proposition 3.1 and Cauchy Schwartz inequality. Lastly we have:

\[ \left( E|x_j|^2 \right)^{1/2} \leq \frac{1}{J} \sum_{j=1}^{J} \left( E|x_j|^2 \right)^{1/2} = \left( E|x_1|^2 \right)^{1/2}. \]

Similarly we can apply this to \( E\|q^k, \mathbf{x}\|_2^2 \) and obtain

\[ \left( E\|q^k, \mathbf{x}\|_2^2 \right)^{1/2} \leq C \left( E|x|^2 \right)^{1/4}. \]

Inserting these two back into (29) again, and noticing the first term on the right of (29) is positive, we can obtain

\[
\frac{dE|x_j|^2}{dt} = \frac{1}{J} \sum_{j=1}^{J} \frac{dE|x_j|^2}{dt} \leq -C \frac{1}{J} \sum_{k=1}^{J} \left( E|p|^2 \right)^{1/2} \left( E|x_j|^2 \right)^{1/2} \leq C J^{-1/4-\alpha/4+\epsilon/2} \left( E|x_j|^2 \right)^{1/2} + O(J^{-1/2-\alpha/2+\epsilon}),
\]

which implies

\[ E|x_j|^2 = E|x|^2 \leq O \left( J^{-1/2-\alpha/2+\epsilon} \right). \]

This allows us to give a tighter bound for \( E|x_j|^2 \):

**Lemma 4.2.** For any \( 0 < \alpha < 1 \), \( 0 \leq t \leq 1 \), if we have an estimate of:

\[ E|x_j|^2 \leq O \left( J^{-\alpha} \right), \tag{31} \]

then one can tighten it to: for any \( \epsilon > 0 \), there is a constant \( C_\epsilon \) so that

\[ E|p|^2 \leq C_\epsilon J^{-1/2-\alpha/2+\epsilon} \quad \text{and} \quad E|x|^2 \leq C_\epsilon J^{-1/2-\alpha/2+\epsilon}. \tag{32} \]
Proof. First, from (31), it is immediate that for all $1 \leq j \leq J$:

$$
(\mathbb{E}|p_j|^2)^{1/2} = \left( \mathbb{E} \left[ \frac{J-1}{J}x_j^2 - \frac{1}{J} \sum_{k \neq j} x_k^2 \right] \right)^{1/2} \leq 2(\mathbb{E}|x|^2)^{1/2} \lesssim O\left( J^{-\alpha/2} \right).
$$

Similar to deriving (29), we subtract the two particle systems (2) and (13). With some calculation (seen in Appendix B (47)) and Lemma 3.4 for any $\epsilon > 0$, we have:

$$
\frac{1}{J} \sum_{j=1}^{J} \frac{d\mathbb{E}|x_j|^2}{dt} \leq -\frac{2}{J} \mathbb{E} \left\{ \sum_{k=1}^{J} (p_k + q_k, x) \langle \mathbf{x}, p_k + q_k \rangle + (p_k + q_k, x) \langle p_k, \mathbf{v} \rangle + (p_k, x) \langle q_k, \mathbf{v} \rangle \right\} + C_{\epsilon} \left( (\mathbb{E}|p|^2)^{1-\epsilon/4} + (\mathbb{E}|p|^2)^{1/2} (\mathbb{E}|p|^2)^{(2-\epsilon)/4} \right) + C_{\epsilon} J^{-1/2} \left[ (\mathbb{E}|x|^2)^{1/2} + (\mathbb{E}|p|^2)^{(2-\epsilon)/4} \right] + C_{\epsilon} J^{-1} \leq -\frac{2}{J} \mathbb{E} \left\{ \sum_{k=1}^{J} (p_k + q_k, x) \langle \mathbf{x}, p_k + q_k \rangle + (p_k + q_k, x) \langle p_k, \mathbf{v} \rangle + (p_k, x) \langle q_k, \mathbf{v} \rangle \right\} + C_{\epsilon} J^{-1/2} (\mathbb{E}|x|^2)^{1/2} + C_{\epsilon} J^{-1/4-\alpha/4+\epsilon/2} (\mathbb{E}|p|^2)^{(2-\epsilon)/4} + C_{\epsilon} J^{-1}
$$

$$
\leq -\frac{2}{J} \mathbb{E} \left\{ \sum_{k=1}^{J} (p_k + q_k, x) \langle \mathbf{x}, p_k + q_k \rangle + (p_k + q_k, x) \langle p_k, \mathbf{v} \rangle + (p_k, x) \langle q_k, \mathbf{v} \rangle \right\} + C_{\epsilon} J^{-1/4-\alpha/4+\epsilon/2} (\mathbb{E}|p|^2)^{(2-\epsilon)/4} + C_{\epsilon} J^{-1/2-\alpha/2},
$$

where we directly plug (31) into $(\mathbb{E}|x|^2)^{1/2}$ to obtain last inequality. Similar to deriving (30), we also have:

$$
\frac{d\mathbb{E}|\mathbf{p}|^2}{dt} = -\frac{2}{J} \sum_{k=1}^{J} \langle (p_k + q_k, x) \langle \mathbf{x}, p_k + q_k \rangle + (p_k + q_k, x) \langle p_k, \mathbf{v} \rangle + (p_k, x) \langle q_k, \mathbf{v} \rangle \rangle + O(J^{-1}).
$$

(34)

Subtracting (33) and (34) for:

$$
\frac{d(\mathbb{E}|p|^2)}{dt} = \frac{d}{dt} \left( \frac{1}{J} \sum_{j=1}^{J} \mathbb{E}|x_j|^2 - \mathbb{E}|\mathbf{p}|^2 \right) \leq C_{\epsilon} J^{-1/4-\alpha/4+\epsilon/2} (\mathbb{E}|p|^2)^{(2-\epsilon)/4} + C_{\epsilon} J^{-1/2-\alpha/2},
$$

which implies

$$
\mathbb{E}|p|^2 = \mathbb{E}|p|^2 \lesssim O\left( J^{(1-\alpha-2\epsilon)}/(2-\epsilon) \right).
$$

Inserting this back into (33) to replace term $(\mathbb{E}|p|^2)^{(2-\epsilon)/4}$, we can obtain

$$
\frac{1}{J} \sum_{j=1}^{J} \frac{d\mathbb{E}|x_j|^2}{dt} \leq -\frac{2}{J} \sum_{k=1}^{J} \mathbb{E} \left\{ (p_k + q_k, x) \langle \mathbf{x}, p_k + q_k \rangle + (p_k + q_k, x) \langle p_k, \mathbf{v} \rangle + (p_k, x) \langle q_k, \mathbf{v} \rangle \right\} + O\left( J^{-1/2-\alpha/2} \right)
$$

$$
\leq C_{\epsilon} \sum_{k=1}^{J} (\mathbb{E}|p|^2)^{1/2} \left\{ (\mathbb{E}|p_k + q_k, x| |\mathbf{x}|^2)^{1/2} + (\mathbb{E}|q_k, \mathbf{v}| |\mathbf{p}|^2)^{1/2} \right\} + O\left( J^{-1/2-\alpha/2} \right)
$$

$$
\leq C(\mathbb{E}|p|^2)^{1/2} \left\{ (\mathbb{E}|p^1 + q^1, x| |\mathbf{x}|^2)^{1/2} + (\mathbb{E}|q^1, \mathbf{v}| |\mathbf{p}|^2)^{1/2} \right\} + O\left( J^{-1/2-\alpha/2} \right)
$$

$$
\leq C(\mathbb{E}|p|^2)^{1/2} \left\{ \left( \mathbb{E}|p^1|^2 + q^1 |\mathbf{x}|^2 |\mathbf{v}|^2 |\mathbf{p}|^2 \right)^{1/2} + (\mathbb{E}|q^1|^2 |\mathbf{p}|^2 |\mathbf{p}|^2)^{1/2} \right\} + O\left( J^{-1/2-\alpha/2} \right)
$$

$$
\leq C_{\epsilon} J^{-1/4-\alpha/4+\epsilon/2} (\mathbb{E}|x|^2)^{(2-\epsilon)/4} \left\{ \left( \mathbb{E}|p^1|^2 + q^1 |\mathbf{x}|^2 |\mathbf{v}|^2 |\mathbf{p}|^2 \right)^{1/2} + (\mathbb{E}|q^1|^2 |\mathbf{p}|^2 |\mathbf{p}|^2)^{1/2} \right\} + O\left( J^{-1/2-\alpha/2} \right)
$$

$$
\leq C_{\epsilon} J^{-1/4-\alpha/4+\epsilon/2} (\mathbb{E}|x|^2)^{(2-\epsilon)/4} + O\left( J^{-1/2-\alpha/2} \right).
$$

(35)
In the second inequality, we used the positivity of \( \sum_{k=1}^{J}(p^{k} + q^{k}, x, p^{k} + q^{k}) \) and last inequality comes from boundedness of high moments. Now, expand \( |\pi|^{2} \) we can obtain

\[
\left( \mathbb{E} |\pi|^{2} \right)^{1/2} \leq \frac{1}{J} \sum_{j=1}^{J} \left( \mathbb{E} |x_{j}^{1}|^{2} \right)^{1/2} = \left( \mathbb{E} |x^{1}|^{2} \right)^{1/2}.
\]

Therefore, \( \frac{d\mathbb{E}|x_{j}^{1}|^{2}}{dt} \) finally implies

\[
\frac{d\mathbb{E}|x_{j}^{1}|^{2}}{dt} = \frac{1}{J} \sum_{j=1}^{J} \frac{d\mathbb{E}|x_{j}^{1}|^{2}}{dt} \leq C_{\epsilon}J^{-1/4 - \alpha/4 + \epsilon/2} \left( \mathbb{E} |x_{j}^{1}|^{2} \right)^{(2-\epsilon)/4} + O \left( J^{-1/2 - \alpha/2} \right),
\]

from which we obtain

\[
\mathbb{E}|x_{j}^{1}|^{2} = \mathbb{E}|x_{j}^{1}|^{2} \lesssim O \left( J^{-1 - \alpha - 2\epsilon}/(2-\epsilon) \right).
\]

Finally, we are ready to prove Theorem \( \text{[4.1]} \).

**Proof.** We first note that by the definition of \( L^{2} \)-Wasserstein distance,

\[
\mathbb{E}(W_{2}^{2}(M_{n}(u)du, M_{n}(u)du)) \leq \left( \frac{1}{J} \sum_{j=1}^{J} \mathbb{E}|u_{j}^{1} - v_{j}^{1}|^{2} \right)^{1/2},
\]

and thus the estimate \( \text{[23]} \) holds true once \( \text{[23]} \) is shown. For that we directly apply Lemma \( \text{[4.2]} \). Starting with \( \alpha_{0} = 0 \) we recursively use the lemma, equation \( \text{[32]} \) in particular, for

\[
\alpha_{n} = 1/2 + \alpha_{n-1}/2 - \epsilon
\]

till the rate saturates to \( \lim_{n \to \infty} \alpha_{n} = 1 - 2\epsilon \). Since \( \epsilon \) is an arbitrary small number, we conclude the proof. \( \Box \)

**Appendix A. Bound of high moments of \{u^j\}**

First, we present a lemma similar to \( \text{[1]} \) Theorem 4.5. For convenience, denote

\[
\epsilon^{j}(t) = u^{j}(t) - \pi(t), \quad \epsilon^{j}(t) = \Gamma^{-1/2}Ae^{j}(t).
\]

**Lemma A.1.** Let \( p > 2 \) and \( u^{j}_{0} \) i.i.d. sampled from \( \mu_{0}(u)du \), then for \( J \) large enough, we have

\[
V_{p}(e(t)) := \mathbb{E} \left( \sum_{m=1}^{K} \left( \frac{1}{J} \sum_{j=1}^{J} |\epsilon_{m}^{j}(t)|^{2} \right)^{p/2} \right) < \infty,
\]

is monotonically decreasing in \( t \), meaning

\[
V_{p}(e(t)) \leq V_{p}(e(0)) < C_{p},
\]

where constant \( C_{p} \) only depends on \( p \).

**Proof.** Without loss of generality, assume \( u^{\dagger} = \tilde{0} \) and let

\[
V_{p}(e(t)) = \mathbb{E} \left( \sum_{m=1}^{K} \left( \frac{1}{J} \sum_{j=1}^{J} |\epsilon_{m}^{j}(t)|^{2} \right)^{p/2} \right),
\]

then we have

\[
d\epsilon_{m}^{j} = -\frac{1}{J} \sum_{k=1}^{J} \epsilon_{n}^{k} \langle \epsilon^{k}, \epsilon^{j} \rangle dt + \frac{1}{J} \sum_{k=1}^{J} \epsilon_{n}^{k} \langle \epsilon^{k}, d(W^{J} - \pi) \rangle,
\]

and

\[
dV_{p}(e) = \sum_{m=1}^{K} \sum_{j=1}^{J} \frac{\partial V_{p}}{\partial e_{m}^{j}} de_{m}^{j} + \frac{1}{2} \sum_{m',m=1}^{K} \sum_{j=1}^{J} de_{m}^{j} \frac{\partial^{2} V_{p}}{\partial e_{m}^{j} \partial e_{m'}^{j}} de_{m'}^{j}.
\]
Proof. Using Ito’s formula, for fix \( 1 \leq j \leq J \) and \( p \geq 1 \), we obtain

\[
\frac{dE[u_j|^2p]}{dt} = -2pE\left(|u_j|^2(p-1) \langle u_j, \text{Cov}_u A^* \Gamma^{-1} Au_j \rangle \right) + E\left(|u_j|^2(p-1) \sum_{j,k=1}^{J} \langle e_j, e_k \rangle \langle e_j, e_k \rangle \right).
\]

Since the first term is negative, one has

\[
E|u_j|^2p \leq CE\left(|u_j|^2(p-1) \left( \frac{1}{J^2} \sum_{j,k=1}^{J} \langle e_j, e_k \rangle \langle e_j, e_k \rangle \right) \right) \leq \left( \sum_{j=1}^{J} \langle e_j, e_j \rangle \right)^{2p} \frac{1}{p} \]

Then, we can prove the boundedness of high moments:

The expectation is given by

\[
CE\left(|u_j|^2(p-1) \sum_{m=1}^{K} \left( \sum_{j=1}^{J} |e_m|^2 \right) \sum_{n=1}^{K} e_n^k e_n \right) \leq \frac{p}{J^2} \sum_{j,k=1}^{J} \langle e_j, e_k \rangle \langle e_j, e_k \rangle \]

By Lemma [1\textsuperscript{41}] and [2\textsuperscript{41}], we obtain the boundedness for \( E \|u_j\|_2^2 \). Then to prove the second inequality of \( 2\textsuperscript{41} \), it suffices to prove

\[
(E \|\text{Cov}_u(t)\|_2^p)^{1/p} \leq C_p,
\]

which is a direct result by expansion of \( \text{Cov}_u(t) \) and triangle inequality:

\[
(E \|\text{Cov}_u(t)\|_2^p)^{1/p} \leq \frac{1}{J} \sum_{j=1}^{J} \left( E \|u_j - \bar{u}\|_2^p \right)^{1/p} \leq \frac{1}{J} \sum_{j=1}^{J} \left( E \|u_j - \bar{u}\|_2^{2p} \right)^{1/p} \leq C.
\]

Here the last inequality comes from each term of the sum has a bound

\[
\left( E \|u_j - \bar{u}\|_2^{2p} \right)^{1/p} \leq \left( E \|u_j - \bar{u}\|_2^{2p} \right)^{1/2p} \leq \left( E \|u_j - \bar{u}\|_2^{2p} \right)^{1/2p} \leq C.
\]

\( \square \)
Consider $x^j = u^j - v^j$, Corollary 4.1 and particle symmetry. We also have, by Proposition 3.2 and Proposition 4.1:

\[ v^j \]

where the boundedness on $v$'s moments are given by Proposition 3.2 and the last inequality comes from Corollary 4.1 and particle symmetry. We also have, by Proposition 3.2 and Proposition 4.1:

\[ \mathbb{E} \left( \| u^j - v^j, [\text{Cov}_v(t) - \text{Cov}_\mu(t)] A^\ast \Gamma^{-1} A v^j \right) \]

\[ \leq C \left( \mathbb{E} \| u^j - v^j \|^2 \right)^{1/2} \left( \mathbb{E} \| \text{Cov}_v(t) - \text{Cov}_\mu(t) \|_{4}^2 \right)^{1/4} \left( \mathbb{E} \| A^\ast \Gamma^{-1} A v^j \|_4^4 \right)^{1/4}, \]

where the boundedness on $v$'s moments are given by Proposition 3.2 and the last inequality comes from Corollary 4.1 and particle symmetry. We also have, by Proposition 3.2 and Proposition 4.1:

\[ \mathbb{E} \left( \| u^j - v^j, [\text{Cov}_v(t) - \text{Cov}_\mu(t)] A^\ast \Gamma^{-1} A v^j \right) \]

\[ \leq \frac{C}{J^{1/2}} \left( \mathbb{E} \| u^j - v^j \|^2 \right)^{1/2} \left( \mathbb{E} \| \text{Cov}_v(t) - \text{Cov}_\mu(t) \|_{4}^2 \right)^{1/4} \left( \mathbb{E} \left( 2 \sum_{j=1}^{J} | \Gamma^{-1/2} \left[ \text{Cov}_v(t) - \text{Cov}_\mu(t) \right] \right) \left( \text{Cov}_v(t) - \text{Cov}_\mu(t) \right) \right) \]

\[ \leq C \left( \mathbb{E} \| \text{Cov}_v(t) - \text{Cov}_\mu(t) \|_{2}^2 \right)^{1/2} J^{-1/2}. \]

Choose any $\epsilon > 0$ and small enough, we further estimate difference of covariance by:

\[ \left( \mathbb{E} \| \text{Cov}_v(t) - \text{Cov}_\mu(t) \|_{2}^2 \right)^{1/2} \]

\[ \leq \frac{1}{J} \sum_{j=1}^{J} \left( \mathbb{E} \| p^j \|^2 \right)^{1/2} \left( \mathbb{E} \| q^j \|^2 \right)^{1/2} \]

\[ \leq \frac{1}{J} \sum_{j=1}^{J} \left( \mathbb{E} \| p^j \|^{2-\epsilon} \right)^{1/2} \left( \mathbb{E} \| q^j \|^{2-\epsilon} \right)^{1/2} \]

\[ \leq C \left( \mathbb{E} \| p^j \|^{2-\epsilon} \right)^{1/2} \left( \mathbb{E} \| q^j \|^{2-\epsilon} \right)^{1/2} \]

\[ \leq C_{\epsilon} \left( \mathbb{E} \| p^j \|^2 \right)^{(2-\epsilon)/4} \left( \mathbb{E} \| q^j \|^2 \right)^{(2-\epsilon)/4} \]

where the third inequality comes from Hölder’s inequality and the last one comes from particle symmetry and terms with power $\epsilon/4$ are bounded by Proposition 4.1 and Cauchy Schwartz inequality. Therefore,
we have a bound for second term \([10]\) as
\[
\mathbb{E} \text{Tr} \left( \left( \text{Cov}_{u}(t) - \text{Cov}_{v}(t) \right) A^{*} \Gamma^{-1} A \left( \text{Cov}_{v}(t) - \text{Cov}_{\mu}(t) \right) \right) \leq C_{c} J^{-1/2} \left( \mathbb{E} \left| p_{j}^{1} \right|^{2} \right)^{(2-\epsilon)/4},
\]
for any \(\epsilon > 0\) and small enough. In the end, by Proposition \([8,2]\)
\[
\mathbb{E} \text{Tr} \left( \left( \text{Cov}_{v}(t) - \text{Cov}_{\mu}(t) \right) A^{*} \Gamma^{-1} A \left( \text{Cov}_{v}(t) - \text{Cov}_{\mu}(t) \right) \right) \leq C \mathbb{E} \| \text{Cov}_{v}(t) - \text{Cov}_{\mu}(t) \|_{2}^{2} \leq CJ^{-1}.
\]
These lead to the fact that
\[
R_{j} \leq C_{e} \left[ J^{-1/2} \left( \mathbb{E} \left| x_{j}^{1} \right|^{2} \right)^{1/2} + J^{-1/2} \left( \mathbb{E} \left| p_{j}^{1} \right|^{2} \right)^{(2-\epsilon)/4} \right] + CJ^{-1}. \tag{42}
\]
Besides, we need to mention \([12]\) at least implies
\[
R_{j} \lesssim O(J^{-1/2})
\]
by Proposition \([5,2]\) and Proposition \([1,1]\).

To control \(L_{j}\) we first write it to:
\[
L_{j} = 2\mathbb{E} \left( x_{j}^{1}, -\text{Cov}_{x,x}(x_{j} + v_{j}) - \left( \text{Cov}_{x,v} + \text{Cov}_{v,x} \right) (x_{j} + v_{j}) - \text{Cov}_{v,v}(x_{j} + v_{j}) \right) + \mathbb{E} \text{Tr} \left[ \left( \text{Cov}_{x,x} + \text{Cov}_{x,v} + \text{Cov}_{v,x} \right) \left( \text{Cov}_{x,x} + \text{Cov}_{x,v} + \text{Cov}_{v,x} \right)^{*} \right] .
\]
\[
= \mathbb{E} \text{Term1}_{j} + \mathbb{E} \text{Term2}_{j}
\]
Expand Term1 and sum up with \(j\), we obtain
\[
\frac{1}{J} \sum_{j=1}^{J} \text{Term1}_{j} = -\frac{2}{J^{2}} \sum_{j,k=1}^{J} \left\{ \left( p^{k}_{j}, x_{j}^{1} \right) \left( p^{k}_{j}, x_{j}^{1} \right) + \left( q^{k}_{j}, x_{j}^{1} \right) \left( q^{k}_{j}, x_{j}^{1} \right) + \left( p^{k}_{j}, p^{1}_{j} \right) \left( p^{k}_{j}, p^{1}_{j} \right) + \left( q^{k}_{j}, p^{1}_{j} \right) \left( q^{k}_{j}, p^{1}_{j} \right) + \left( p^{k}_{j}, p^{1}_{j} \right) \left( q^{k}_{j}, p^{1}_{j} \right) \right. \\
+ \left( p^{k}_{j}, q^{1}_{j} \right) \left( p^{k}_{j}, q^{1}_{j} \right) + \left( q^{k}_{j}, q^{1}_{j} \right) \left( q^{k}_{j}, q^{1}_{j} \right) + \left( p^{k}_{j}, q^{1}_{j} \right) \left( q^{k}_{j}, q^{1}_{j} \right) + \left( q^{k}_{j}, q^{1}_{j} \right) \left( p^{k}_{j}, q^{1}_{j} \right) \right\} \tag{43}
\]
Now insert \(\mathbf{x}, \mathbf{v}\), we can further write
\[
\frac{1}{J} \sum_{j=1}^{J} \text{Term1}_{j} = I + II,
\]
where
\[
I = -\frac{2}{J^{2}} \sum_{j,k=1}^{J} \left\{ \left( p^{k}_{j}, p^{1}_{j} \right) \left( p^{k}_{j}, p^{1}_{j} \right) + \left( q^{k}_{j}, p^{1}_{j} \right) \left( q^{k}_{j}, p^{1}_{j} \right) + \left( p^{k}_{j}, p^{1}_{j} \right) \left( p^{k}_{j}, p^{1}_{j} \right) + \left( q^{k}_{j}, p^{1}_{j} \right) \left( q^{k}_{j}, p^{1}_{j} \right) + \left( p^{k}_{j}, p^{1}_{j} \right) \left( q^{k}_{j}, p^{1}_{j} \right) \right. \\
+ \left( p^{k}_{j}, q^{1}_{j} \right) \left( p^{k}_{j}, q^{1}_{j} \right) + \left( q^{k}_{j}, q^{1}_{j} \right) \left( q^{k}_{j}, q^{1}_{j} \right) + \left( p^{k}_{j}, q^{1}_{j} \right) \left( q^{k}_{j}, q^{1}_{j} \right) + \left( q^{k}_{j}, q^{1}_{j} \right) \left( p^{k}_{j}, q^{1}_{j} \right) \right\}
\]
and
\[
II = -\frac{2}{J} \sum_{k=1}^{J} \left\{ \left( p^{k}_{j}, \mathbf{x} \right) \left( p^{k}_{j}, \mathbf{x} \right) + \left( q^{k}_{j}, \mathbf{x} \right) \left( q^{k}_{j}, \mathbf{x} \right) + \left( p^{k}_{j}, \mathbf{x} \right) \left( q^{k}_{j}, \mathbf{x} \right) + \left( q^{k}_{j}, \mathbf{x} \right) \left( q^{k}_{j}, \mathbf{x} \right) \right. \\
+ \left( p^{k}_{j}, \mathbf{v} \right) \left( p^{k}_{j}, \mathbf{v} \right) + \left( q^{k}_{j}, \mathbf{v} \right) \left( q^{k}_{j}, \mathbf{v} \right) + \left( p^{k}_{j}, \mathbf{v} \right) \left( q^{k}_{j}, \mathbf{v} \right) + \left( q^{k}_{j}, \mathbf{v} \right) \left( q^{k}_{j}, \mathbf{v} \right) \right\} \tag{44}
\]
Similarly we expand Term2 to obtain
\[
\frac{1}{J} \sum_{j=1}^{J} \text{Term2}_{j} = \frac{1}{J^{2}} \sum_{j,k=1}^{J} \left\{ \left( p^{k}_{j}, p^{1}_{j} \right) \left( p^{k}_{j}, p^{1}_{j} \right) + \left( p^{k}_{j}, p^{1}_{j} \right) \left( q^{k}_{j}, q^{1}_{j} \right) + \left( q^{k}_{j}, q^{1}_{j} \right) \left( p^{k}_{j}, p^{1}_{j} \right) \right. \\
+ 2 \left( q^{k}_{j}, q^{1}_{j} \right) \left( p^{k}_{j}, p^{1}_{j} \right) + 2 \left( p^{k}_{j}, p^{1}_{j} \right) \left( p^{k}_{j}, q^{1}_{j} \right) + 2 \left( q^{k}_{j}, q^{1}_{j} \right) \left( p^{k}_{j}, q^{1}_{j} \right) \right\} .
\]
Combine this with $I$, we have

$$I + \frac{1}{J} \sum_{j=1}^{J} \text{Term2}_j = -\frac{1}{J^2} \sum_{j,k=1}^{J} \left\{ \langle p^k, p^j \rangle \langle p^k, p^j \rangle + 2 \langle p^k, p^j \rangle \langle q^k, p^j \rangle + 2 \langle q^k, p^j \rangle \langle q^k, p^j \rangle \right\}$$

$$- \frac{1}{J^2} \sum_{j,k=1}^{J} \left\{ \langle p^k, p^j \rangle \langle q^k, q^j \rangle - \langle q^k, q^j \rangle \langle p^k, p^j \rangle \right\} .$$

(45)

Noticing

$$- \langle q^k, p^j \rangle \langle q^k, p^j \rangle = (q^k)^* (p^j \otimes p^j) A^* \Gamma^{-1} A q^k \leq 0$$

and

$$- \frac{1}{J^2} \left\{ \sum_{j,k=1}^{J} \langle p^k, p^j \rangle \langle q^k, q^j \rangle \right\} = -\frac{1}{J^2} \left\{ \text{Tr} \left[ \left( \sum_{j=1}^{J} p^k \otimes q^j \right) \left( \sum_{j=1}^{J} p^k \otimes q^j \right)^* \right] \right\} \leq 0 .$$

Term 1, 3 and 4 can be eliminated from (45). Then for any $\epsilon > 0$ small enough, we use Hölder’s inequality similar as (43):

$$\mathbb{E} \left( I + \frac{1}{J} \sum_{j=1}^{J} \text{Term2}_j \right) \leq \frac{1}{J^2} \sum_{j,k=1}^{J} \mathbb{E} \left( \langle q^k, q^j \rangle \langle p^k, p^j \rangle + \langle p^k, p^j \rangle \langle q^k, p^j \rangle \right)$$

$$\leq \frac{1}{J} \sum_{j=1}^{J} \left\{ \frac{1}{J} \sum_{k=1}^{J} \left( \mathbb{E} \left| \langle q^k, q^j \rangle \langle p^k, p^j \rangle \right| \right)^{1/2} + \left( \mathbb{E} \left| \langle p^k, p^j \rangle \langle q^k, p^j \rangle \right| \right)^{1/2} \right\}$$

$$\leq \frac{1}{J} \sum_{j=1}^{J} \left\{ \mathbb{E} \left| p^j \right|^{2} \left( \mathbb{E} \left| q^j \right|^{2} + \mathbb{E} \left| q^j \right|^{2} \right)^{1/2} + \left( \mathbb{E} \left| p^j \right|^{2} \left| q^j \right|^{2} \right)^{1/2} \right\}$$

$$\leq \frac{1}{J} \sum_{j=1}^{J} \left\{ \mathbb{E} \left| p^j \right|^{2} \left( \mathbb{E} \left| q^j \right|^{2} + \mathbb{E} \left| q^j \right|^{2} \right)^{1/2} + \left( \mathbb{E} \left| p^j \right|^{2} \left| q^j \right|^{2} \right)^{1/2} \right\}$$

$$\leq C_\epsilon \left( \mathbb{E} \left| p^j \right|^{2} \right)^{2-\epsilon/4} + \left( \mathbb{E} \left| p^j \right|^{2} \right)^{2-\epsilon/4} = C_\epsilon \left( \mathbb{E} \left| p^j \right|^{2} \right)^{1-\epsilon/4} + \left( \mathbb{E} \left| p^j \right|^{2} \right)^{2-\epsilon/4} ,$$

(46)

where the third inequality comes from Hölder’s inequality and the last inequality comes from particle symmetry (we write all expectation w.r.t one particle for convenience) while other terms are all bounded by Proposition 5.2, 4.4 and Cauchy Schwartz inequality. Inserting (42), (44), and (46) back into (45), we obtain

$$\frac{1}{J} \sum_{j=1}^{J} \frac{d}{dt} \mathbb{E} \left| x^j \right|^{2} \leq \text{II} + \mathbb{E} \left( I + \frac{1}{J} \sum_{j=1}^{J} \text{Term2}_j \right) + \frac{1}{J} \sum_{j=1}^{J} R^j$$

$$\leq -\frac{2}{J} \mathbb{E} \left\{ \sum_{k=1}^{J} \langle p^k + q^k, \mathbf{x} \rangle \langle \mathbf{x}, p^k + q^k \rangle + \langle p^k + q^k, \mathbf{x} \rangle \langle p^k, \mathbf{v} \rangle + \langle p^k, \mathbf{x} \rangle \langle q^k, \mathbf{v} \rangle \right\} ,$$

$$+ C_\epsilon \left( \mathbb{E} \left| p^j \right|^{2} \right)^{1-\epsilon/4} + \left( \mathbb{E} \left| p^j \right|^{2} \right)^{2-\epsilon/4}$$

$$+ C_\epsilon J^{-1/2} \left[ \left( \mathbb{E} \left| x^j \right|^{2} \right)^{1/2} + \left( \mathbb{E} \left| p^j \right|^{2} \right)^{2-\epsilon/4} \right] + C J^{-1}$$

(47)

for any $\epsilon > 0$ small enough.

**APPENDIX C. EXPANSION OF $\frac{1}{J} \sum_{j=1}^{J} \mathbb{E} \left| x^j \right|^{2}$**

By Ito’s formula, one has:

$$d \left| u^j - v^j \right|^{2} = 2 \langle du^j - dv^j, u^j - v^j \rangle + \langle du^j - dv^j, du^j - dv^j \rangle ,$$

and plugging in (29) and (13), we have

$$\langle du^j - dv^j, du^j - dv^j \rangle = \text{Tr} \left[ \left( \text{Cov}_u(t) - \text{Cov}_v(t) \right) A^* \Gamma^{-1} A \left[ \text{Cov}_u(t) - \text{Cov}_v(t) \right] \right] dt ,$$

(48)
and thus one has the following ODE the error:

\[
\frac{dE[u^j - v^j]^2}{dt} = 2E\left(du^j - dv^j, u^j - v^j\right) + E\left(du^j - dv^j, du^j - dv^j\right) / dt
\]

\[
= 2E\left(u^j - v^j, -Cov_u(t)u^j + \Gamma^{-1/2}A Cov_\mu(t)A^{*}\Gamma^{-1/2}v^j\right)
\]

\[
+ E Tr\left(\Gamma^{-1/2}A [Cov_u(t) - Cov_\mu(t)] A^{*}\Gamma^{-1/2}A [Cov_u(t) - Cov_\mu(t)] A^{*}\Gamma^{-1/2}\right),
\]

\[
= 2E\left(u^j - v^j, -Cov_u(t)u^j + Cov_v(t)v^j\right)
\]

\[
+ E Tr\left([Cov_u(t) - Cov_\mu(t)]^2\right) + R_j,
\]

\[
= L_j(t) + R_j(t)
\]

where \(R_j\) is the remaining term comes from replacing \(Cov_\mu\) by \(Cov_v\):

\[
R_j(t) = 2E\left(u^j - v^j, \Gamma^{-1/2}A [Cov_v(t) - Cov_\mu(t)] A^{*}\Gamma^{-1/2}v^j\right)
\]

\[
- 2E Tr\left(\Gamma^{-1/2}A [Cov_u(t) - Cov_\mu(t)] A^{*}\Gamma^{-1/2}A [Cov_u(t) - Cov_\mu(t)] A^{*}\Gamma^{-1/2}\right)
\]

\[
+ E Tr\left(\Gamma^{-1/2}A [Cov_v(t) - Cov_\mu(t)] A^{*}\Gamma^{-1/2}A [Cov_v(t) - Cov_\mu(t)] A^{*}\Gamma^{-1/2}\right).
\]

The three terms in \(R_j\) all decay in \(J\). With Cauchy-Schwarz inequality:

\[
E\left(u^j - v^j, -\Gamma^{-1/2}A [Cov_v(t) - Cov_\mu(t)] A^{*}\Gamma^{-1/2}v^j\right)
\]

\[
\leq C \left(E\left|u^j - v^j\right|^2\right)^{1/2} \left(E\|Cov_v(t) - Cov_\mu(t)\|_2^{1/2} \left(E\|v^j\|^4\right)^{1/4}\right).
\]

\[
\leq \frac{C}{J^{1/2}} \left(E\left|u^j - v^j\right|^2\right)^{1/2} \leq CJ^{-1/2} \left(E\|u^j - v^j\|^2\right)^{1/2}
\]

where the boundedness of \(v\)'s moment comes from Proposition \([3.2]\) and the boundedness of \(u - v\) comes from Corollary \([4.1]\). Similarly to \([5.1]\),

\[
E Tr\left([Cov_u(t) - Cov_\mu(t)] (A^{*}\Gamma^{-1/2}\Gamma^{-1/2}A [Cov_v(t) - Cov_\mu(t)])\right)
\]

\[
\leq C \left(E\|Cov_u(t) - Cov_v(t)\|_2\right)^{1/2} \left(E\|Cov_\mu(t)\|_2\right)^{1/2} \left(E\|Cov_v(t) - Cov_\mu(t)\|_2\right)^{1/2},
\]

\[
\leq C\epsilon J^{-1/2} \left(E\left|p^j\right|^2\right)^{(2-\epsilon)/4}
\]

for any \(\epsilon > 0\) and small enough. Similarly, we also have

\[
E Tr\left([Cov_v(t) - Cov_\mu(t)] (A^{*}\Gamma^{-1/2}\Gamma^{-1/2}A [Cov_v(t) - Cov_\mu(t)])\right) \leq C\epsilon \left(E\|Cov_v(t) - Cov_\mu(t)\|_2^2\right) \leq CJ^{-1}.
\]

These altogether give

\[
R_j(t) \leq C\epsilon \left[J^{-1/2} \left(E\|x^j\|^2\right)^{1/2} + J^{-1/2} \left(E\|p^j\|^2\right)^{(2-\epsilon)/4}\right] + CJ^{-1}.
\]

To deal with \(L_j(t)\) in \([5.2]\) we first rewrite \(L_j\) as (eliminating subscript \(t\)):

\[
L_j = 2E\left(x^j, -Cov_{x,x}(x^j + v^j) - (Cov_{x,v} + Cov_{v,v})(x^j + v^j) - Cov_v x^j\right)
\]

\[
+ E Tr\left([Cov_{x,x} + Cov_{x,v} + Cov_{v,v})(Cov_{x,x} + Cov_{x,v} + Cov_{v,v})\right) - E \text{Term1}_j + E \text{Term2}_j.
\]
Expand Term1 and sum up with $j$, we obtain
\[
\frac{1}{J} \sum_{j=1}^{J} \text{Term1}_j = -\frac{2}{J} \left\{ \sum_{j,k=1}^{J} \langle p^k, x^j \rangle \langle p^k, x^j \rangle + \sum_{j,k=1}^{J} \langle q^k, x^j \rangle \langle p^k, x^j \rangle \\
+ \sum_{j,k=1}^{J} \langle p^k, x^j \rangle \langle p^k, v^j \rangle + \sum_{j,k=1}^{J} \langle q^k, x^j \rangle \langle p^k, v^j \rangle \\
+ \sum_{j,k=1}^{J} \langle p^k, x^j \rangle \langle q^k, x^j \rangle + \sum_{j,k=1}^{J} \langle q^k, x^j \rangle \langle q^k, x^j \rangle + \sum_{j,k=1}^{J} \langle p^k, x^j \rangle \langle q^k, v^j \rangle \right\}
= I + \Pi,
\]
where we use the same technique as in (43) for:
\[
I = -\frac{2}{J^2} \sum_{j,k=1}^{J} \left\{ \langle p^k, x^j \rangle \langle p^k, x^j \rangle + \langle q^k, x^j \rangle \langle p^k, x^j \rangle + \langle p^k, x^j \rangle \langle p^k, q^j \rangle + \langle q^k, x^j \rangle \langle p^k, q^j \rangle + \langle q^k, p^j \rangle \langle q^k, q^j \rangle \right\}
\]
and
\[
\Pi = -\frac{2}{J^2} \sum_{j,k=1}^{J} \left\{ \langle p^k, x^j \rangle \langle p^k, x^j \rangle + \langle q^k, x^j \rangle \langle p^k, x^j \rangle + \langle p^k, x^j \rangle \langle q^k, q^j \rangle + \langle q^k, x^j \rangle \langle q^k, q^j \rangle \right\}.
\]

Similarly we expand Term2:
\[
\frac{1}{J} \sum_{j=1}^{J} \text{Term2}_j = \frac{1}{J^2} \sum_{j,k=1}^{J} \left\{ \langle p^k, p^j \rangle \langle p^k, p^j \rangle + \langle q^k, p^j \rangle \langle q^k, p^j \rangle + \langle q^k, q^j \rangle \langle p^k, p^j \rangle + 2 \langle q^k, p^j \rangle \langle p^k, q^j \rangle + 2 \langle p^k, p^j \rangle \langle p^k, q^j \rangle \right\}.
\]

Combine this with $I$, we have
\[
I + \frac{1}{J} \sum_{j=1}^{J} \text{Term2}_j = -\frac{1}{J^2} \left\{ \sum_{j,k=1}^{J} \left( \langle p^k, p^j \rangle + \langle q^k, p^j \rangle \right)^2 + \left( \langle q^k, p^j \rangle \right)^2 \right\} \leq 0.
\]

Further combine with (51) and (52) to plug in (49):
\[
\frac{1}{J} \sum_{j=1}^{J} \frac{dE|\mathbf{x}|^2}{dt} \leq -\frac{2}{J} E \left\{ \sum_{k=1}^{J} \left( \langle p^k + q^k, \mathbf{x} \rangle \langle \mathbf{x}, p^k + q^k \rangle + \langle p^k + q^k, \mathbf{x} \rangle \langle \mathbf{x}, p^k \rangle + \langle p^k, \mathbf{x} \rangle \langle q^k, \mathbf{x} \rangle \right) \right\}
+ C_r J^{-1/2} \left[ (E|\mathbf{x}|^2)^{1/2} + (E|\mathbf{p}|^2)^{(2-\epsilon)/4} \right] + CJ^{-1},
\]
as desired.

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