DUALITY, BRIDGELAND WALL-CROSSING AND FLIPS OF SECANT VARIETIES

CRISTIAN MARTINEZ

Abstract. We prove that for any smooth projective surface $X$, the functor $R\text{Hom}(\cdot, \omega_X)[1]$ induces an isomorphism between suitable Bridgeland moduli spaces. In particular, when $X = \mathbb{P}^2$ this reproves a result due to Maican [Mai10b] for Gieseker moduli spaces of sheaves supported on curves. We use the duality result to describe a sequence of flips of secant varieties of Veronese surfaces.

CONTENTS

1. Introduction. 1
2. Preliminaries 3
   2.1. Stability conditions on $\mathbb{P}^2$ 5
   2.2. Wall and chamber structure 7
3. Duality 8
4. Wall-crossing 11
5. Bridgeland walls for 1-dimensional plane sheaves 17
   5.1. Rank one walls 19
6. The embedded problem: flips of secant varieties. 20
   6.1. The divisorial contraction 28
   6.2. The last birational model 29
References 31

1. Introduction.

Stability conditions on triangulated categories were introduced by Bridgeland ([Bri07]) who also constructed the first family of nontrivial examples for $K3$ surfaces ([Bri08]). In [AB13] Arcara and Bertram extended these examples for an arbitrary smooth projective surface. [AB13] also provides us with a new perspective, by studying the variation of a particular family of stability conditions on a simple $K3$ surface $(X, H)$ for a particular topological type the authors construct a sequence of birational transformations for the blow up of the complete linear series $|H|$ along $X$ flipping the “secant” varieties of $X$. 

1
The wall-crossing phenomena was studied in [ABCH13] for the Hilbert scheme of points on $\mathbb{P}^2$ where it was indicated that varying the family of stability conditions introduced in [AB13] for the topological type $(1, 0, -n)$ corresponds to running a directed MMP on $\text{Hilb}^n(\mathbb{P}^2)$. This was proven in [BMW13] for any del Pezzo surface and a primitive topological type.

The aim of this paper is to interpret another “classical” problem in algebraic geometry in terms of Bridgeland stability conditions. In [Mai10b] M. Maican proves that the map $F \mapsto \text{Ext}^{n-1}(F, \omega_{\mathbb{P}^n})$ induces an isomorphism between the moduli spaces $N_{\text{Gieseker}}(r, \chi)$ and $N_{\text{Gieseker}}(r, -\chi)$ of Gieseker semistable sheaves with Hilbert polynomials $P = rm + \chi$ and $P_D = rm - \chi$ respectively. The moduli spaces $N_X(r, \chi)$ were constructed by C. Simpson [Sim94] for any smooth projective surface via invariant theory and they were proved to be projective so one could ask whether Maican’s result extends to any surface. We prove this by identifying $N_X(r, \chi)$ with a moduli space of Bridgeland semistable objects on $X$. The main result is

**Theorem (Theorem 3.1).** The functor $R\text{Hom}(-, \omega_X)[1]$ induces an isomorphism between the Bridgeland moduli spaces $M_{D,tH}(v)$ and $M_{-D+K_X,tH}(v)$ provided these moduli spaces exist and $Z_{D,tH}(v)$ belongs to the open upper half plane.

and its corollary

**Corollary (Corollary 3.3).** There is an isomorphism $N_X([C], \chi) \cong N_X([C], -\chi)$ mapping the $S$-equivalence class of a sheaf $F$ to the $S$-equivalence class of $\text{Ext}^1(F, \omega_X)$.

In a latter paper [Mai12], Maican constructs cohomological stratifications of the Gieseker moduli $N_{\mathbb{P}^2}(6, \chi)$. Using those strata we can get exceptional loci for birational transformations of $N_{\mathbb{P}^2}(6, \chi)$ as it was done in [BMW13] for $N(4, 2)$ and $N(5, 0)$. However, there is no bijective correspondence between the cohomological strata and the Bridgeland walls, as shown in [CCT13] for the case of $N(6, 1)$ a cohomological strata may be object of several contractions when running the MMP, giving rise to several Bridgeland walls. Nevertheless, when $\chi = 0$ we can identify all rank-1 walls even when Maican-type stratifications are unknown. In this case, by restricting the Bridgeland wall-crossing on a suitable subvariety of a model of $N(d, 0)$ ($d$ odd), and following the spirit of [AB13], we construct a sequence of flips for the blow-up of the linear series $|\mathcal{O}(d - 3)|$ along the Veronese surface, the first of these flips coincide with the one constructed by Vermeire in [Ver01].

**Notation.** Other than specified we will use the following standard notation:

- $\Re(z)$, $\Im(z)$ denote the real and imaginary parts of the complex number $z$.
- $D^b(X)$ is the bounded derived category of $X$. 
• We use $H^i(\cdot)$ to denote the cohomology sheaves of an object in the derived category and $H^i(\cdot)$ for the cohomology groups of a sheaf.

• For a smooth projective surface $X$, the topological type $v \in \mathbb{Z} \oplus NS(X) \oplus \frac{1}{2} \mathbb{Z}$ of an object $E \in D^b(X)$ is its Chern character vector.

• $M_H(v)$ denotes the moduli space of Gieseker semistable sheaves of topological type $v$ with respect to the polarization $H \in \text{Pic}X$.

• $M_{s,t}(v)$ denotes the Bridgeland moduli space of $\mu_{s,t}$-semistable objects of topological type $v$.

• We refer to an object $F$ fitting into an exact sequence $A \hookrightarrow F \twoheadrightarrow B$ as an extension.

Acknowledgments. I would like to thank my advisor, Professor Aaron Bertram, for his constant encouragement and patience, and for numerous discussions which were crucial in the preparation of the present work. Also, I thank to Kiryong Chung for pointing out a misplaced reference and for making me aware of his joint work with Jinwon Choi on one-dimensional plane sheaves of Euler-Poincaré characteristic $1$ [CCT13].

2. Preliminary

We assume familiarity with the concept of stability conditions introduced by Bridgeland [Bri07]. We quote here the relevant theorems and definitions but for a more detailed presentation the unfamiliar reader is encouraged to consult Bridgeland’s original papers [Bri07, Bri08], or the introduction to the topic by D. Huybrechts [Huy11].

Let $X$ be a smooth projective variety.

Definition 2.1. A pre-stability condition on $X$ is a pair $(Z, A)$ consisting of a linear function $Z : K(X) \to \mathbb{C}$ called the charge and the heart $A$ of a $t$-structure on $D^b(X)$, such that:

(a) $\mathcal{J}(Z(E)) \geq 0$ for all $E \in A$ and

(b) If $\mathcal{J}(Z(E)) = 0$ and $E \neq 0$ then $\Re(Z(E)) < 0$.

For every pre-stability condition one can define a slope function

$$\mu_Z = \frac{-\Re(Z)}{\Im(Z)}$$

which gives us a notion of (semi)stability: an object $E \in A$ is said to be $Z$-(semi)stable if for any inclusion $A \hookrightarrow E$ of objects in $A$ one has

$$\mu_Z(A)(\leq) < \mu_Z(E).$$

Definition 2.2. A pre-stability condition $(Z, A)$ is a stability condition if it has the Harder-Narasimhan property:
Every nonzero object \( E \in \mathcal{A} \) admits a finite filtration in \( \mathcal{A} \)

\[
0 \subset E_0 \subset E_1 \subset \cdots \subset E_n = E
\]

uniquely determined by the property that each quotient \( F_i := E_i/E_{i-1} \) is \( \mathbb{Z} \)-semi-stable and \( \mu_\mathbb{Z}(F_1) > \mu_\mathbb{Z}(F_2) > \cdots > \mu_\mathbb{Z}(F_n-1) \).

**Example.** If \( X = C \) is a smooth projective curve then ordinary degree and rank of coherent sheaves give a stability condition on \( \mathcal{A} = D^b(\text{CohC}) \):

\[
Z(F) = -\deg(F) + \sqrt{-1}\text{rk}(F).
\]

However when \( X \) is a surface this is not the case. One can still define a Mumford slope (with respect to some polarization \( H \)):

\[
\mu_H(E) = \frac{c_1(E) \cdot H}{\text{rk}(E)}
\]

but this does not come from any stability condition on \( \text{CohX} \) since \( c_1(C_p) = \text{rk}(C_p) = 0 \). However, it is true that every coherent sheaf \( E \) has a filtration

\[
E_0 \subset \cdots \subset E_n = E
\]

such that \( E_0 \) is the torsion subsheaf of \( E \) and for every \( i > 0 \), the factors \( E_i/E_{i-1} \) are semistable of degreasing slopes. We set \( \mu_{\text{max}} = \mu_H(E_1/E_0) \) and \( \mu_{\text{min}} = \mu_H(E/E_{n-1}) \).

Let \( \sigma = (Z, \mathcal{A}) \) be a stability condition on \( X \). For any nonzero object \( E \in \mathcal{A} \) one can write \( Z(E) = |Z(E)|e^{\pi \sqrt{-1}\phi} \) for a unique \( \phi \in (0,1] \). We say that \( E \) has phase \( \phi \). For every \( \phi \in (0,1] \) we denote by \( \mathcal{P}(\phi) \) the subcategory consisting of \( \sigma \)-semistable objects of phase \( \phi \). Inductively one can define \( \mathcal{P}(\phi + 1) := \mathcal{P}(\phi)[1] \).

For a bounded interval \( I \subset \mathbb{R} \) we denote \( \mathcal{P}(I) \) the subcategory extension-generated by \( \sigma \)-semistable objects of phase in the interval \( I \). For instance, \( \mathcal{P}(0,1] = \mathcal{A} \).

One can define (semi)stability in terms of phase just by declaring an object \( E \) to be (semi)stable if every subobject has (smaller)strictly smaller phase. This is equivalent to the definition using slopes since for an object \( E \in \mathcal{A} \) of phase \( \phi \) one has

\[
\mu_\mathbb{Z}(E) = -\cot(\pi \phi).
\]

An easy but important consequence of the definition of stability is

**Proposition 2.3** (Schur’s lemma). Let \( \sigma = (Z, \mathcal{A}) \) be a stability condition.

(a) If \( E \) is \( \sigma \)-stable then \( \text{Hom}(E,E) = \mathbb{C} \).

(b) If \( A, B \) are different \( \sigma \)-stable objects of the same phase then \( \text{Hom}(A,B) = 0 \).

(c) If \( A \in \mathcal{P}(\phi_1), B \in \mathcal{P}(\phi_2) \) with \( \phi_1 > \phi_2 \) then \( \text{Hom}(A,B) = 0 \).
Let $E \in \mathcal{P}(\phi)$. A finite Jordan-Holder filtration of $E$ is a chain

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that the factors $E_i/E_{i-1}$ are stable of phase $\phi$. Jordan-Holder filtrations always exist but what it is not clear is if they are finite. Even if they are finite they are not unique, what it is true is that the stable factors are unique up to a permutation.

**Definition 2.4.** A stability condition is called locally finite if there is some $\delta > 0$ such that each quasi-abelian category $\mathcal{P}(\phi - \delta, \phi + \delta)$ is of finite length. For a locally finite stability condition the categories $\mathcal{P}(\phi)$ have finite length. In particular, every semistable object has a finite Jordan-Holder filtration.

**Definition 2.5.** Let $\sigma$ be a locally finite stability condition. Two objects $A, B \in \mathcal{P}(\phi)$ are called $S$-equivalent if they have isomorphic stable factors.

**Definition 2.6.** [AP06]. Let $S$ be a scheme of finite type over $\mathbb{C}$. A family of objects in $\mathcal{A}$ parametrized by $S$ is an object $E \in D^b(X \times S)$ such that for every closed point $s \in S$ we have

$$L_i^s(E) \in \mathcal{A}.$$

2.1. **Stability conditions on $\mathbb{P}^2$.** As we explained before, the standard rank and degree of a coherent sheaf do not define a stability condition on any surface. A large class of examples of stability conditions on surfaces were constructed by Bridgeland [Bri08] in the case of $K3$ surfaces and generalized by Arcara-Bertram [AB13] for any smooth projective surface. The idea is to define a nice subcategory of $D^b(X)$ where some generalized rank and degree functions form good stability functions giving actual stability conditions. Fix a very ample line bundle $\omega \in \text{Pic}(X)$. One defines, for every $s \in \mathbb{R}$, the full subcategories:

- $Q_s = \{ E \in \text{Coh}(X) : E \text{ is torsion or } \mu_{\text{min}} > s \}$,
- $F_s = \{ E \in \text{Coh}(X) : E \text{ torsion-free and } \mu_{\text{max}} \leq s \}$.

The subcategories $Q_s, F_s$ are full and $(Q_s, F_s)$ is a torsion pair, i.e.,

- $\text{Hom}(Q, F) = 0$ for all $Q \in Q_s, F \in F_s$.
- Every coherent sheaf $E$ fits into an exact sequence

$$0 \to Q \to E \to F \to 0$$

for some $Q \in Q_s, F \in F_s$. This short exact sequence is unique up to isomorphisms of extensions.

By general theory of torsion pairs we know that the extension closure of $(Q_s, F_s[1])$ is the heart of a $t$-structure, more precisely it is the full subcategory

$$\mathcal{A}_s = \{ E \in D^b(X) : H^{-1}(E) \in F_s, \ H^0(E) \in Q_s \text{ and } H^i(E) = 0 \text{ if } i \neq -1, 0 \}.$$
Theorem 2.7 ([AB13, Bri08]). Let $\beta, \omega \in \text{Num}_R(X)$, $\omega$ ample. Then for any $t > 0$

$$Z_{\beta, \omega}(E) = -\int_X e^{-\beta - \sqrt{-1}t\omega} \text{ch}(E)$$

is the charge of a locally finite stability condition on $A_s$.

We now concentrate in the case $X = \mathbb{P}^2$, in this case (Picard number 1) one can treat $\text{ch}(E)$ as a vector with numerical entries. Choosing $\omega = H$ the hyperplane class, and $\beta = sH$ the central charge takes the form

$$Z_{s,t}(ch_0, ch_1, ch_2) = (-ch_2 + ch_1s - \frac{ch_0}{2}(s^2 - t^2)) + \sqrt{-1}t(ch_1 - ch_0s).$$

One of the most important results in [ABCH13] is the following:

Theorem 2.8 ([ABCH13]). There are projective coarse moduli spaces $M_{s,t}(v)$ classifying $S$-equivalence classes of families of $Z_{s,t}$-semistable objects in $A_s$ of topological type $v$.

The idea is to identify $(s,t)$-stability with quiver stability. Let $k \in \mathbb{Z}$ and consider the extension closure

$$A(k) = \langle \mathcal{O}(k-2)[2], \mathcal{O}(k-1)[1], \mathcal{O}(k) \rangle.$$

An element of $A(k)$ is a complex

$$\mathbb{C}^{n_0} \otimes \mathcal{O}(k-2) \to \mathbb{C}^{n_1} \otimes \mathcal{O}(k-1) \to \mathbb{C}^{n_2} \otimes \mathcal{O}(k)$$

the vector $n = (n_0, n_1, n_2)$ is its dimension vector. Let $a$ be a vector orthogonal to $n$. An object of dimension vector $n$ is said to be quiver (semi)stable with respect to $a$ if for any subcomplex in $A(k)$ of dimension vector $n'$ one has $n' \cdot m(\geq) > 0$.

Moduli spaces of quiver semistable complexes of fixed dimension vector with respect to a fixed polarization have a construction via GIT given in [Kim94]. [ABCH13, Proposition 7.3] shows that for every $(s,t)$ in the region

$$(s - (k-1))^2 + t^2 < 1$$

there exists a choice of a polarization $a_{s,t}$ such that moduli spaces of $(s,t)$-stable objects are isomorphic to moduli spaces of stable objects in $A(k)$ with respect to $a_{s,t}$. The potential walls foliate the $(s,t)$ plane and every potential wall intersects one of the quiver regions above, then since the moduli of stable objects remains unchanged along every potential wall one has that for every $(s,t)$ and every choice of invariants the moduli spaces $M_{s,t}(v)$ are projective and may be constructed by GIT.

The change from Chern classes to dimension vectors is given by the matrix

$$
\begin{bmatrix}
\frac{k(k-1)}{2} & -\frac{(2k-1)}{2} & 1 \\
k(k-2) & -(2k-2) & 2 \\
\frac{(k-1)(k-2)}{2} & -\frac{(2k-3)}{2} & 1
\end{bmatrix}
$$
2.2. Wall and chamber structure. It was proven by Bridgeland \cite{Bri07} that the set of locally finite stability conditions has the structure of a complex manifold, the local charts given by sending a stability condition to its central charge in $\text{Hom}(K(X), \mathbb{C})$. Moreover, for each chern character vector $ch$ there is a locally finite collection of (real) codimension 1 subvarieties of the stability manifold called walls. Every connected component of the complement of the walls is called a chamber. Roughly speaking, a stability condition $\sigma = (Z, A)$ is on a wall if there is an object $E \in A$ that is semistable for $\sigma$, it is stable for stability conditions in some chamber and unstable for stability conditions on another.

In the case of $\mathbb{P}^2$ and the stability conditions $(Z_{s,t}, A_s)$, a wall for a chern character $ch$ is produced when there is an object $E$ with $ch(E) = ch$ and an inclusion $A \hookrightarrow E$ in some $A_{s_0}$ such that

$$\mu_{s_0,t}(A) = \mu_{s_0,t}(E).$$

using the explicit formula for $\mu_{s,t}$ it is proven in \cite{ABCH13} that the walls are nested semicircles in the $(s, t)$-upper half plane with center on the real axis. Denote by $W_{ch(A), ch(E)}$ the wall corresponding to the inclusion $A \hookrightarrow E$.

Lemma 2.9. \cite{ABCH13} Lemma 6.3] Let $E$ be a coherent sheaf on $\mathbb{P}^2$ which is either a torsion sheaf supported in codimension 1 or a torsion-free sheaf (not necessarily Mumford-semistable) satisfying the Bogomolov inequality:

$$ch_2(E) < \frac{ch_1(E)^2}{2r(E)}$$

and suppose $A \rightarrow E$ is a map of coherent sheaves which is an inclusion of $\mu_{s_0,t_0}$-semi-stable objects of $A_{s_0}$ of the same slope for some

$$(s_0, t_0) \in W := W_{ch(A), ch(E)}.$$

Then $A \rightarrow E$ is an inclusion of $\mu_{s,t}$-semi-stable objects of $A_s$ of the same slope for every point $(s, t) \in W$.

The lemma above was used in \cite{ABCH13} to provided specific bounds on the radius of the walls and via an identification of $(s, t)$-stability with quiver stability, it is shown that if $E$ is a Mumford stable torsion-free sheaf of primitive chern vector $ch(E)$ then there are finitely many isomorphism types of moduli spaces of $(s, t)$-stable objects with invariants $ch(E)$ i.e., finitely many walls intersecting the $(s, t)$-slide. In the next section we will see that similar results are obtained for 1-dimensional sheaves.

We finish this section by recalling that for a primitive chern vector $v$ the moduli space $M_H(v)$ of semistable torsion-free sheaves of type $v$ is smooth and a Mori dream space (see \cite{HL10}, or \cite{BMW13} for a detailed explanation). It is shown in
that above the outermost wall $M_{s,t}(v) \cong M_H(v)$ and by the argument given in [BMW13] that decreasing $t$ corresponds to run a directed minimal model program for $M_H(v)$ so that the main component (the one whose generic element is a sheaf) of each $M_{s,t}(v)$ is a birational model.

Things are slightly different when studying the Gieseker moduli of 1-dimensional sheaves although most of the arguments are the same. By the work of Le Potier [LP93] we know that these moduli spaces are irreducible, locally factorial and their Picard group is free abelian of rank 2, moreover a specific set of generators is given, namely: the determinant line bundle and the line bundle giving the support map. It is not hard then to prove that the Gieseker moduli of 1-dimensional plane sheaves of fixed invariants is also a Mori dream space, an argument can be found in [Woo13].

We notice that in general the Gieseker moduli of 1-dimensional plane sheaves is not smooth (although its singular locus has high codimension). We can fix this by making the following

**Definition 2.10.** An object $F \in A_s$ of chern character $\text{ch}(F) = (0, ch_1, ch_2)$ is said to be $(s, t)$-pseudo stable if for any inclusion $A \hookrightarrow F$ in $A_s$ one has

$$\mu_{s,t}(A) \leq \mu_{s,t}(F) \quad \text{and} \quad \mu_{s,t}(A) = \mu_{s,t}(F) \Rightarrow \text{ch}_0(A) = 0.$$  

**Remark.** With this new terminology it is easy to see that on any chamber one has $(s, t)$-semistable $= (s, t)$ pseudo-stable. Also for $t >> 0$ a sheaf is $(s, t)$-pseudo-stable if and only if it is Gieseker semistable and these are all the $(s, t)$-semistable objects (see proof of Corollary [3.3]). Thus, Theorem 1.1 in [BMW13] holds for 1-dimensional plane sheaves, with no restrictions on the topological type, when replacing stable by pseudo-stable. □

3. **Duality**

Let $X$ be a smooth projective surface, $D, H \in \text{Num} \mathbb{R}(X)$ with $H$ ample. Let $\sigma_{D,H} = (Z_{D,H}, A_{D,H})$ be the stability condition of Theorem [2.7] and assume that projective coarse moduli spaces for $\sigma_{D,H}$ and $\sigma_{-D+K_X,H}$ are known to exist. For example, $X$ can be $\mathbb{P}^2$ [ABCH13] or a K3 surface [BM12]. This section is devoted to prove

**Theorem 3.1.** The functor $F \mapsto F^D := R\text{Hom}(F, \omega_X)[1]$ induces an isomorphism between the Bridgeland moduli $M_{D,H}(\text{ch}(F)) \cong M_{-D+K_X,H}(\text{ch}(F^D))$ provided that $\text{ch}(F)$ is the chern character of an object in $A_{D,H}$ of phase in $(0, 1)$. 

This result was proven by Maican [Mai10a] for moduli spaces of Gieseker semistable sheaves on $\mathbb{P}^n$ supported on curves. The theorem above recovers Maican’s for $X = \mathbb{P}^2$ and $t >> 0$. The proof in this context is identical to Maican’s original proof modulo the following
Lemma 3.2. Let $E$ be a $\sigma_{D,H}$-(semi)stable object in $\mathcal{P}(0,1)$. Then

(a) If $E$ is stable then it is quasi-isomorphic to a two-term complex of vector bundles $E^{-1} \to E^0$.

(b) $\mathcal{H}^{-1}(E)$ is torsion-free with semistable factors of slope $< DH$.

(c) If $A \in \mathcal{A}_{DH}$ is an object all of whose semistable factors belong to $\mathcal{P}(0,1)$ then $A^D \in \mathcal{A}_{(-D+K_X)H}$.

(d) $E^D \in \mathcal{A}_{(-D+K_X)H}$ is $\sigma_{-D+K_X,H}$-(semi)stable.

(e) If $E, F \in \mathcal{A}_{DH}$ are $S$-equivalent then so are $E^D, F^D \in \mathcal{A}_{(-D+K_X)H}$.

(f) For any flat family $F \in D^b(S \times X)$ with fibers of invariants $\text{ch}(F_s)$ there is a flat family $F^D \in D^b(S \times X)$ with fibers of invariants $\text{ch}(F_s^D)$ such that $L^*_s(F^D) \cong (L^*_sF)^D$.

Proof. Part (a) is the content of [Bri8, 10.1(a)]. For (b) notice that if $\mu_{\text{max}}(\mathcal{H}^{-1}(E)) = DH$ then $\mathcal{H}^{-1}(E)[1]$ will have a subobject of phase 1 destabilizing $E$. Assume that $E$ is stable, to prove that $E^D \in \mathcal{A}_{(-D+K_X)H}$ note that for any coherent sheaf $F$ with semistable factors of slope $< DH$ (resp. $> DH$) we have

$$\mathcal{H}^i(F^D) = \begin{cases} 
\text{torsion free sheaf with } \mu_{\text{min}} > -DH + KH & \text{if } i = -1 \\
\text{torsion or 0} & \text{if } i = 0 \\
\text{0-dim torsion sheaf or 0} & \text{if } i = 1 \\
0 & \text{otherwise.}
\end{cases}$$

(which can be proven by taking a minimal free resolution for $F$). Taking cohomology on the short exact sequence

$$0 \to \mathcal{H}^0(E)^D \to E^D \to \mathcal{H}^{-1}(E)[1]^D \to 0$$

we get the long exact sequence of sheaves

$$0 \to \mathcal{H}^{-1}(E)^D \to \mathcal{H}^{-1}(E^D) \to \mathcal{H}^{-1}(\mathcal{H}^{-1}(E)[1]^D) \to \mathcal{H}^0(\mathcal{H}^0(E)^D) \to \mathcal{H}^0(E^D) \to \mathcal{H}^0(\mathcal{H}^{-1}(E)[1]^D) \to \mathcal{H}^1(\mathcal{H}^0(E)^D) \to 0$$

since by (a) $E^D$ is a two-term complex of vector bundles. But $\mathcal{H}^{-1}(E)[1]^D = \mathcal{H}^{-1}(E)^D[-1]$ and so

$$\mathcal{H}^{-1}(\mathcal{H}^{-1}(E)[1]^D) = 0.$$ 

This implies that $\mathcal{H}^{-1}(E)^D \in \mathcal{F}_{(-D+K_X)H}$, $\mathcal{H}^0(\mathcal{H}^0(E)^D)$ is the torsion subsheaf of $\mathcal{H}^0(E^D)$ and because $\mathcal{H}^1(\mathcal{H}^0(E)^D)$ is zero-dimensional we have $\mathcal{H}^0(E^D) \in \mathcal{Q}_{(-D+K_X)H}$.

Moreover, this proves that if $A \in \mathcal{P}(0,1)$ is stable then $A^D$ is an element of $\mathcal{A}_{(-D+K_X)H}$. For arbitrary $A \in \mathcal{P}(0,1)$, $A$ is in the extension closure of some stable objects $A_1, \ldots, A_k \in \mathcal{A}_s$ of the same phase and so

$$A^D \in \langle A_1^D, \ldots, A_k^D \rangle \subset \mathcal{A}_{-s-3}.$$
By the same argument we get (c).

Assume for the moment that $E$ is stable then there is no injective map

$$0 \to K \to E^D$$

in $\mathcal{A}_{(-D+K)H}$ with $K$ having at least one of its semistable factors of phase 1. If so, there would be an inclusion

$$0 \to A \to E^D$$

with $A \in \mathcal{P}(1)$ stable, i.e., $A = \mathbb{C}_x$ or $A = F[1]$ for some locally free sheaf $F$ with $\mu_H$-semistable factors of slope $(-D+K)H$ ([Br10, 10.1(b)]). But if $E$ is stable then $E^D$ is derived equivalent to a two-term complex of vector bundles implying $\text{Hom}(\mathbb{C}_x, E^D) = 0$. Also

$$\text{Hom}(F[1], E^D) = \text{Hom}(F, \mathcal{H}^{-1}(E^D)) = 0$$

in virtue of Schur’s lemma.

This allows to conclude that $E^D$ is stable, indeed, if there is a destabilizing sequence

$$0 \to A \to E^D \to B \to 0$$

we can choose $B$ to be stable and by the argument above we know that all semistable factors of $A$ have phase in $(0, 1)$, then by dualizing this sequence we get a destabilizing sequence for $E$ in $\mathcal{A}_{DH}$ since

$$\mu_{D,tH}(\cdot) = -\mu_{-D+K,tH}(\cdot)^D.$$ 

We conclude that $E^D$ is semistable for all semistable objects $E$ of phase in $(0, 1)$ just by dualizing the Jordan-Holder filtration of $E$.

Let $0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E$ be a Jordan-Holder filtration for $E$ in $\mathcal{A}_{DH}$ then $(E/F_n)^D \subset (E/F_{n-1})^D \subset \cdots \subset E^D$ is a Jordan-Holder filtration for $E^D$ in $\mathcal{A}_{(-D+K)H}$ with stable factors $(F_i/F_{i-1})^D$. This also gives part (d).

For the last part let $F^D := R\text{Hom}(F, \omega_{S\times X/S})$ then

$$\text{Li}_s^*(R\text{Hom}(F, \omega_{S\times X/S})) \cong R\text{Hom}(\text{Li}_s^*F, \omega_X) \in \mathcal{A}_{(-D+K)H}.$$ 

\[ \square \]

Proof. (of Theorem 3.1, [Mai10b, Theorem 13]) Every flat family $F \in D^b(S \times X)$ gives a morphism $\pi : S \to M_{D,tH}(v)$ and a morphism $\pi^D : S \to M_{-D+K,tH}(v^D)$ corresponding to the family $F^D$ of the lemma. Since $\pi^D$ is constant on the fibers of $\pi$ by part (c) then $\pi^D$ factors through a morphism $M_{D,tH}(v) \to M_{-D+K,tH}(v^D)$ which sends the closed point representing $E$ to the closed point representing $E^D$. 

\[ \square \]
The symmetry of the situation and the fact that $(\triangle^D)^D = Id$ prove that such morphism is an isomorphism.

**Remark.** In the special case when $X = \mathbb{P}^2$ and $v = (0, d, -3d/2)$ duality gives an automorphism $(\triangle^D) : M_{-3/2,t}(v) \cong M_{-3/2,t}(v)$ for all $t > 0$.

**Corollary 3.3.** Let $N([C], \chi)$ denote the moduli space of Gieseker semistable sheaves of Euler-Poincare characteristic $\chi$ supported on a curve of class $[C]$. The functor $\mathcal{F} \mapsto \mathcal{E xt}^1(\mathcal{F}, \omega_X)$ induces an isomorphism between $N([C], \chi)$ and $N([C], -\chi)$.

**Proof.** Take $D = K/2$ in the duality theorem. If $r(E) = 0$ and $c_1(E) = [C]$ then

$$\mu_{K/2, tH}(E) = \frac{\chi(E)}{tC : H}$$

and therefore a sheaf of those invariants that is $\sigma_{K/2, tH}$ semistable is also Gieseker semistable. By [LQ11] we know that the values of $t$ for which there is an inclusion of objects $A \hookrightarrow E$ with $\mu_{K/2, tH}(A) = \mu_{K/2, tH}(E)$ is bounded above (this also follows by a result of Maciocia [Mac12, Theorem 3.11] when considering the family of stability conditions $\sigma_{K/2+tH}$. If $E$ is an object that is $\sigma_{K/2+tH}$-semistable for all $t > 0$ then $E$ must be a sheaf since otherwise $H^{-1}(E)[1]$ would destabilize $E$. If $A \rightarrow E$ is an inclusion in $A_{KH/2}$ then $A$ must be a sheaf and if it destabilizes $E$, it must be a sheaf of positive rank. But a simple computation shows that for $t >> 0$ one has

$$\mu_{K/2, tH}(A) < \mu_{K/2, tH}(E)$$

and so the inclusion $A \rightarrow E$ must produce a wall, since the walls are bounded above we conclude that above all walls $E$ is $\sigma_{K/2, tH}$-semistable. The coarse moduli spaces $N([C], \chi)$ were constructed by C. Simpson [Sim94] via invariant theory, then the conclusion follows from the duality theorem.

**4. Wall-crossing**

The results in this section seem to be known to the experts but we decided to include here some proofs for the seek of completeness.

In [ABCH13] the authors describe what new objects become stable after crossing a wall. The idea is the following: assume that a wall for the family of stability conditions $\sigma_{D, tH}$ is produced by a destabilizing sequence

$$0 \rightarrow A \rightarrow E^+ \rightarrow B \rightarrow 0$$

and assume furthermore that $A$ and $B$ are stable at the wall and $E^+$ is stable above the wall. Then the destabilizing sequence is a Jordan-Holder filtration for the semistable object $E^+$ at the wall. Crossing the wall will produce semistable
objects that are $S$-equivalent to $E^+$ at the wall, i.e., the new objects must have $A$ and $B$ as stable factors and so they must be extensions of the form

$$0 \to B \to E^- \to A \to 0.$$  

But even more is true,

**Proposition 4.1.** Assume that $\mu_{D,t_0,H}(A) = \mu_{D,t_0,H}(B)$ for some objects $A, B \in \mathcal{A}_{DH}$ and that there is $\epsilon > 0$ such that $A$ and $B$ are $\mu_{D,t_1}$-stable with $\mu_{D,t_1}(A) < \mu_{D,t_1}(B)$ for $t_0 \leq t < t_0 + \epsilon$. If $\text{Ext}^1(B, A) \neq 0$ in $\mathcal{A}_{DH}$ then there exists $\delta > 0$ such that every non-split extension

$$0 \to A \to E \to B \to 0$$

is $\mu_{D,t_1}$-stable (or $\mu_{D,t_1}$-pseudo-stable when $r(E) = 0$) for all $t_0 < t < t_0 + \delta$.

**Proof.** Let $0 < \delta \leq \epsilon$ such that there are no walls for $E$ between $t_0$ and $t_0 + \delta$ (this is possible because the walls are locally finite). It is enough to prove that there is no stable subobject $E' \hookrightarrow E$ destabilizing $E$. If there were such $E'$ then at the wall $W := W_{ch(A),ch(B)}$ $E'$ is semistable and $\mu_{D,t_0,H}(E') = \mu_{D,t_0,H}(E)$ otherwise it would destabilize $E$. The map $E' \to B$ must be surjective otherwise it would be the zero map and therefore we would get an inclusion $E' \hookrightarrow A$ in which case $\mu(E') < \mu(A) < \mu(E)$ above the wall. Let $K$ be its kernel, then there is an inclusion $K \hookrightarrow A$, since the slopes of $K$ and $A$ are equal at $W$ then either $K = 0$ in which case the sequence $A \to E \to B$ splits or $K = A$ and therefore $E' = E$. \qed

Moreover, the more general result holds

**Proposition 4.2.** Let $E$ be an object in $\mathcal{A}_{DH}$ which is strictly semistable for some $t_0 > 0$ and assume that $E$ has a Jordan-Holder filtration at the wall determined by $t_0$ that becomes the Harder-Narasimhan filtration of $E$ on one of the chambers determined by $t_0$, then $E$ is stable (or pseudo-stable when $r(E) = 0$) on the other chamber.

**Proof.** Let us assume that the length of a Jordan-Holder filtration for $E$ (and so of any) is 2, and that $E$ fits into a diagram

$$
\begin{array}{ccc}
B' & \text{B} & \\
\downarrow & & \downarrow \\
A' & \text{E} & \text{B} \\
\downarrow & & \downarrow \\
B'' & & \\
\end{array}
$$
where $A, B'$ and $B''$ are the stable factors of $E$. We can assume that there is $\epsilon > 0$ such that for all $t \in (t_0, t_0 + \epsilon)$ $A, B', B''$ are $\sigma_{D,1H}$-stable and $\mu_{D,1H}(A) < \mu_{D,1H}(B') < \mu_{D,1H}(B'')$. By Proposition 4.1 $B$ is also stable for all such $t$. Assume that there is a destabilizing sequence $0 \to S_1 \to E \to S_2 \to 0$ for some $t$. If $S_2$ is stable at the wall then a simple comparison of the slopes shows that it can not destabilize $E$ above the wall. Thus we can assume that $S_2$ is strictly semistable at the wall, then there is a commutative diagram

$$\begin{array}{cc}
S_1 & B' \\
\downarrow & \downarrow \\
A & E & B \\
\downarrow & \downarrow & \downarrow \\
S_2 & B'' \\
\downarrow \\
S'_2
\end{array}$$

for any stable quotient $S'_2$ of $S_2$ at the wall. Since $B$ is stable above the wall and $S'_2$ is a stable factor of both $E$ and $B$ at the wall then $S'_2 = B_2$. Let $K$ be the kernel of the map $S_2 \to S'_2 = B''$, since the length of any Jordan-Holder filtration for $E$ is two, we must have $S_1$ stable at the wall. The conclusion follows by considering the surjection $E \to B''$ instead of $E \to B$.

We proceed by induction on the length of a Jordan-Holder filtration for $E$. The argument given above is exactly how the proof goes in general. Assume that $E$ fits into a diagram

$$\begin{array}{cc}
 & B_1 \\
& \downarrow \\
& A \leftarrow E \rightarrow B \\
& \downarrow \\
& \tilde{B}
\end{array}$$

where $A$ and $B_1$ are stable at the wall, and moreover we can assume that $B$ has a Jordan-Holder filtration $0 \subset B_1 \subset B_2 \subset \cdots \subset B_n \subset B$ such that

$$\mu_{D,1H}(B_i/B_{i-1}) < \mu_{D,1H}(B_{i+1}/B_i)$$
for all $i$ and for any $t$ sufficiently near and above the wall determined by $t_0$. As before for any destabilizing sequence $0 \to S_1 \to E \to S_2 \to 0$ above the wall we get a diagram

\[ \begin{array}{cc}
S_1 & \quad B_1 \\
\downarrow & \downarrow \\
A & E \\
\downarrow & \downarrow \\
S_2 & \quad B \\
\downarrow & \downarrow \\
S'_2 & \quad \tilde{B} \\
\end{array} \]

so that $S'_2$ is a stable factor of $B$, but since $B$ is stable above the wall by induction then $S'_2 \neq B_1$. Consider the diagram

\[ \begin{array}{cc}
& B_2/B_1 \\
A_1 & \quad E \\
\downarrow & \downarrow \\
-0 & \quad B/B_1 \\
\downarrow & \downarrow \\
S'_2 & \quad B/B_2 \\
\end{array} \]

The map $A_1 \to S'_2$ must be the zero map since otherwise $S'_2$ would be a stable factor of $A_1$, but the stable factors of $A_1$ are $A$ and $B_1$. Since the hypothesis on the Jordan-Holder filtration for $B/B_1$ are also satisfied then $B/B_1$ is stable above the wall and therefore $S'_2 \neq B_2/B_1$. Since there are only finitely many stable factors
we will end up with a diagram

```
\begin{array}{c}
\text{\(S_1\)} & \text{\(B_n/B_{n-1}\)} \\
\downarrow & \downarrow \\
A_{n-1} & \text{\(E \rightarrow B/B_{n-1}\)} \\
\downarrow & \downarrow \\
K & \text{\(S_2 \rightarrow B/B_n\)} \\
\downarrow & \downarrow \\
T
\end{array}
```

If \(\ell(\cdot)\) denotes the number of Jordan-Holder factors then we have

\[
\ell(E) = \ell(A_{n-1}) + 2 = \ell(S_1) + \ell(S_2) = \ell(S_1) + 1 + \ell(K) = \ell(S_1) + 1 + \ell(A_{n-1}) + \ell(T)
\]

and therefore \(\ell(S_1) + \ell(T) = 1\) which is only possible if \(T = 0\) and \(S_1\) is stable. The rest of the argument is exactly as in the case where \(\ell(E) = 3\) discussed above. □

**Proposition 4.3.** Let \(\delta, A, B', B''\) as in the first part of the proof of Proposition 4.2. Then objects \(\tilde{E}\) that are extensions of the form

```
\begin{array}{c}
0 \\
\uparrow \\
B' \\
\uparrow \\
A \\
\uparrow \\
\tilde{E} \\
\uparrow \\
\hat{B} \\
\uparrow \\
B'' \\
\uparrow \\
0
\end{array}
```

can not be stable for any \(t_0 < t < t_0 + \delta\).
Proof. Fix \( t_0 < t < t_0 + \delta \). By proposition 4.1 all extensions \( 0 \to A \to E \to B \to 0 \) are \( \mu_{D,1H} \)-stable. If \( \tilde{E} \) is \( \mu_{D,1H} \)-stable then \( \mu_{D,1H}(A) < \mu_{D,1H}(\tilde{E}) < \mu_{D,1H}(B') \) and so \( \text{Hom}(B', A) = 0 \) and so we get an inclusion

\[
\text{Ext}^1(B'', A) \hookrightarrow \text{Ext}^1(B, A).
\]

The image of every nonzero element corresponds to a non split extension which is stable by Proposition 4.2. Such extensions admit an injective morphism \( B' \hookrightarrow E \) that can be visualized in the diagram

\[
\begin{array}{ccc}
0 & \downarrow & \\
& B' & \\
& \downarrow & \\
0 & \xrightarrow{3} & A \xrightarrow{E} B \xrightarrow{0} \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \xrightarrow{0} & A \xrightarrow{G} B'' \xrightarrow{0} \\
& \downarrow & \downarrow & \downarrow & \\
& 0 & & & \\
\end{array}
\]

Stability of \( E \) implies \( \mu_{D,1H}(B') < \mu_{D,1H}(E) = \mu_{D,1H}(\tilde{E}) \) and so \( B' \) destabilizes \( \tilde{E} \). Thus the only possibility is \( \text{Ext}^1(B'', A) = 0 \) which gives a surjective map

\[
\text{Ext}^1(B', A) \twoheadrightarrow \text{Ext}^1(\tilde{B}, A)
\]

implying that \( \tilde{E} \) is a pullback of an extension of \( B' \) by \( A \). As before, there is an injective map \( B'' \hookrightarrow \tilde{E} \)

\[
\begin{array}{ccc}
0 & \downarrow & \\
& B'' & \\
& \downarrow & \\
0 & \xrightarrow{3} & A \xrightarrow{\tilde{E}} \tilde{B} \xrightarrow{0} \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \xrightarrow{0} & A \xrightarrow{\tilde{G}} B' \xrightarrow{0} \\
& \downarrow & \downarrow & \downarrow & \\
& 0 & & & \\
\end{array}
\]

Again the stability of \( E \) implies \( \mu_{D,1H}(B'') > \mu_{D,1H}(E) = \mu_{D,1H}(\tilde{E}) \) destabilizing \( \tilde{E} \).
\[\square\]
For the following Corollary assume that there is a small open interval $J \subset (0, +\infty)$ such that the family stability conditions $\{\sigma_{D,t,H}\}_{t \in J}$ have coarse moduli spaces.

**Corollary 4.4** (Set-theoretic wall-crossing). Let $0 \to A \to E \to B \to 0$ be an exact sequence in $A_{DH}$ producing a wall $W := W_{\text{ch}(A),\text{ch}(E)}$ at $t_0 \in J$. Then

(a) There exists $\delta > 0$ such that $A$ and $B$ and so $E$ are $\mu_{D,H}$-stable (or pseudo-stable when $r(E) = 0$) for all $t_0 < t < t_0 + \delta$, and the Bridgeland moduli spaces for the invariants $\text{ch}(A)$ and $\text{ch}(B)$ are constant for all such $t$.

(b) Denote by $M_{D,t+H}(\text{ch}(A))$ and $M_{D,t+H}(\text{ch}(B))$ the Bridgeland moduli spaces above and sufficiently near $W$. Crossing $W$ interchanges extensions

\[
0 \to A^+ \to E \to B^+ \to 0 \quad A^+ \in M_{D,t+H}(\text{ch}(A)), B^+ \in M_{D,t+H}(\text{ch}(B))
\]

by extensions

\[
0 \to B^- \to F \to A^- \to 0 \quad B^- \in M_{D,t-H}(\text{ch}(B)), A^- \in M_{D,t-H}(\text{ch}(B))
\]

where $M_{D,t-H}(\text{ch}(B))$ and $M_{D,t-H}(\text{ch}(A))$ denote the Bridgeland moduli spaces below and sufficiently near $W$.

**Proof.** By induction on the number of stable factors at the wall and by Proposition 4.3 we have that semistable objects at the wall satisfy the hypothesis of Proposition 4.2. \qed

5. **Bridgeland walls for 1-dimensional plane sheaves**

As mentioned in the preliminaries, moduli spaces of Gieseker semistable plane sheaves of Hilbert polynomial $P(t) = ct + \chi$ were initially studied by J. Le Potier in [LP93] where it is shown that these moduli spaces are projective, irreducible, locally factorial and smooth at the stable points. For small values of $c$ it is possible to find nice stratifications of these moduli spaces by studying their resolutions, see [DM11] for $c = 4$ and [Mai10a] and [Mai12] for $c = 5$ and 6. Studying a sheaf by studying its possible resolutions is same as replacing such sheaf for an equivalent element in the derived category, indeed each strata in the stratifications given by Drezet and Maican in [DM11] and by Maican in [Mai10a] and [Mai12] can be interpreted as a set of extensions in a tilted category [BMW13].

But the story goes on, as in [BMW13], each set of extensions produces a Bridgeland wall and these are all the walls for the directed MMP. The following numerical bound coming from Lemma 2.9 produces some of these sets of extensions for arbitrary value of $c$ even when Maican-type stratifications are unknown.
Let $E$ be a sheaf of topological type $(0, c, d)$ with $c > 0$ and let $F$ be a destabilizing object (which is necessarily a sheaf) then $E$ and $F$ fit into an exact sequence

$$0 \to K \to F \to E.$$ 

By Lemma 2.9 we must have $K \in \mathcal{F}_s$ and $F \in \mathcal{Q}_s$ for all $s$ along the wall. If $ch(F) = (r', c', d')$ then in our case where the wall is a semicircle with center $(d/c, 0)$ and radius $R$ this says

$$\frac{ch_1(K)}{r(K)} \leq \frac{d}{c} - R \quad \text{and} \quad \frac{c'}{r'} \geq \frac{d}{c} + R.$$ 

Since $r(K) = r'$ and $ch_1(K) - c' + c \geq 0$ then combining the inequalities above we get

$$(5.1) \quad R + \frac{d}{c} \leq \frac{c'}{r'} \leq \frac{d}{c} + \frac{c}{r'} - R$$

which immediately produces

$$(5.2) \quad R \leq \frac{c}{2r'}.$$ 

Fix a numerical class $v = (0, d, -\frac{3}{2}d)$ with $d$ odd. One of the key ingredients in the computation that follows is the existence of a collapsing wall. The generic element of $M_H(v)$ corresponds to a sheaf $\mathcal{F}$ satisfying $H^0(\mathcal{F}) = 0$ (see [Mai12, Proposition 6.1.1]), by using the Beilinson spectral sequence one can conclude that the general element of $M_H(v)$ has a resolution of the form

$$0 \to d\mathcal{O}(-2) \to d\mathcal{O}(-1) \to \mathcal{F} \to 0.$$ 

In particular $\mathcal{O}(-1) \in \mathcal{A}_{-3/2}$ produces a wall contracting an open set. The corresponding wall $W_C$ has center $(-3/2, 0)$ and radius

$$R = \sqrt{\frac{1}{4} + \frac{2\chi}{r}} = \frac{1}{2}.$$ 

The complement of such open set is the theta divisor ([LP93]) and is the set of semistable sheaves that have at least one section, i.e., those that have $\mathcal{O}$ as a subobject. The corresponding wall $W_\Theta$ has radius $R = \frac{3}{2}$. Crossing $W_\Theta$ corresponds to a divisorial contraction and since $M_H(v)$ has Picard number 2 then there are no walls between $W_C$ and $W_\Theta$. This improves our bound for the walls corresponding to flips:

**Proposition 5.1.** Let $A$ be a coherent sheaf of rank $r > 0$ and Euler characteristic $\chi$ and let $E$ be a coherent sheaf with $ch(E) = (0, d, -\frac{3}{2}d)$. A morphism of sheaves $A \to E$ which is an inclusion of objects in the category $\mathcal{A}_{-3/2}$ produces a wall corresponding to a flip if and only if

$$(5.3) \quad \frac{3}{2} < \sqrt{\frac{1}{4} + \frac{2\chi}{r}} \leq \frac{d}{2r}.$$
It is useful to know whether the new objects we get after crossing a wall are stable or pseudo-stable, we can answer this in a very special case:

**Proposition 5.2.** Let \( v = (0, d, -3d/2) \) and assume that \( E \in A_{-3/2} \) is an object with \( \text{ch}(E) = v \) that has a Jordan-Holder filtration of length 1 at a wall \( W \). Then \( E \) is stable on one of the chambers determined by \( W \).

**Proof.** Assume that the Jordan-Holder filtration for \( E \) at \( W \) is
\[
0 \to A \to E \to B \to 0
\]
and that \( \mu_{-3/2,t}(A) > \mu_{-3/2,t}(B) \) above \( W \), then \( E \) is pseudo-stable below \( W \) by Proposition 4.1. Assume that \( E \) is not stable then there should be a subobject \( E' \twoheadrightarrow E \) such that \( r(E') = r(E) = 0 \), \( c_1(E') < c_1(E) = d \) and \( \chi(E') = \chi(E) = 0 \). Let \( K = \ker(E' \to B) \) and \( c_1(E') = d - s \), then we have a diagram
\[
\begin{array}{ccc}
K & \longrightarrow & E' \\
\downarrow & & \downarrow \\
A & \longrightarrow & E & \longrightarrow & B
\end{array}
\]
If \( r(B) = r', c_1(B) = c' \) and \( \chi(B) = \chi' \) then
\[
(r(K), c_1(K), \chi(K)) = (-r', d - s - c', -\chi')
\]
\[
(r(A), c_1(A), \chi(A)) = (-r', d - c', -\chi').
\]
Note that in this case \( d_{-3/2,t}(K) = d_{-3/2,t}(A) \) and \( r_{-3/2,t}(K) = r_{-3/2,t}(A) - st \). But \( A \) is stable at \( W \) and so it is stable for \( t \) sufficiently near \( W \) therefore
\[
\mu_{-3/2,t}(K) < \mu_{-3/2,t}(A)
\]
for \( t \) above and below \( W \). This implies that \( sd_{-3/2,t}(A) < 0 \) above and below \( W \) and so \( s = 0 \). Thus \( E \) is stable. \( \square \)

5.1. **Rank one walls.** Setting \( r = 1 \) in [5.3] one finds the set of admissible values for the Euler characteristic of a destabilizing object producing a wall corresponding to a flip:
\[
\chi = 2, \ldots, \frac{d^2 - 1}{8}.
\]
The possible values for the first Chern class come from solving the inequality [5.1]. The first Chern class \( c \) of a rank 1 destabilizing object with Euler characteristic \( \chi = \frac{d^2 - 1}{8} - \ell \) must satisfy
\[
\sqrt{\frac{d^2}{4} - 2\ell - \frac{3}{2}} \leq c \leq -\frac{3}{2} + d - \sqrt{\frac{d^2}{4} - 2\ell}.
\]
(5.4)
It is easy to check that \( c = \frac{d - 3}{2} \) is always a solution. These are the invariants of a twisted ideal sheaf of a zero-subscheme \( Z \) of length \( \ell \). Moreover, since for generic
zero-subschemes $Z$ and $W$ of length $\ell$ we have $\text{Hom}(\mathcal{O}, \mathcal{I}_W \otimes_D \mathcal{I}_Z(d)) \neq 0$ then there are nontrivial extensions

$$0 \to \mathcal{I}_Z((d-3)/2) \to E \to \mathcal{I}_W((d-3)/2)[1] \to 0$$

producing sheaves $E \in A_{-3/2}$ which are Bridgeland stable for $t > \sqrt{\frac{d^2}{4} - 2\ell}$. This proves that the number of actual rank 1 walls corresponding to flips is $\frac{d^2 - 9}{8}$.

Notice that the exceptional loci for a rank 1 flip is not irreducible in general. Indeed, the inequality (5.4) has unique solution only when $\ell < \frac{d-1}{2}$. However, setting

$$G_{\ell,i}^W := \mathcal{I}_W \left(\frac{d-3}{2} + i\right), \quad \text{length}(W) = \ell + \frac{i(d+i)}{2}, \quad i \in \mathbb{Z}$$

we have

**Proposition 5.3.** If $c = \frac{d-3}{2} + i$ is solution for (5.4) then so is $c = \frac{d-3}{2} - i$. Generically, the corresponding destabilizing objects are of the form $G_{\ell,i}^W$ and $G_{\ell,-i}^Y$ respectively. Moreover, if $E_{\ell,k}$ denotes the component containing the locus of sheaves destabilized by an object of the form $G_{\ell,k}^W$ then $E_{\ell,-k}$ is the image of $E_{\ell,k}$ by the duality automorphism.

**Proof.** First part is a trivial computation. For the second one only has to notice that a generic destabilizing sequence is of the form

$$0 \to G_{\ell,i}^W \to E \to (G_{\ell,-i}^Y)^D \to 0$$

which is again a trivial computation of the invariants. \qed

6. The embedded problem: flips of secant varieties.

In [Ver01] and [Ver02] P. Vermeire describes a sequence of flips for the secant varieties of an embedding $X \hookrightarrow \mathbb{P}^N$ of an algebraic surface. This sequence of flips is constructed in similar fashion to the flips obtained by Thaddeus [Tha94] when studying variation of GIT for moduli spaces of stable pairs on curves. The first of these flips is easy to describe and it is the content of [Ver01, 4.13]. Roughly speaking, if the embedding of $X$ is sufficiently ample such that it can be generated by quadrics with only linear syzygies then there is a flip diagram

```
\begin{tikzcd}
\hat{M} \\
| \downarrow \varphi^+ | \\
bl_X(\mathbb{P}^N) \\
| \downarrow \varphi^- | \\
\mathbb{P}^N \\
\end{tikzcd}
```
where $\varphi : \mathbb{P}^N \to \mathbb{P}^s$ is the rational map given by the forms defining $X$ and $\tilde{M}$ is the blow-up of $\text{bl}_X(\mathbb{P}^N)$ along the strict transform of the secant variety $\widetilde{\text{Sec}}X$. The diagram restricts to

$$
\begin{array}{ccc}
\pi & \sim & h \\
\mathbb{P}(\mathcal{E}) & \xrightarrow{\varphi^+} & \mathbb{P}(\mathcal{F}) \\
\varphi^- & \xrightarrow{\sim} & \text{Hilb}^2(X)
\end{array}
$$

where $\mathbb{P}(\mathcal{E}) \cong \widetilde{\text{Sec}}X$ and $\mathcal{F} = \varphi^+(N^*_\mathcal{E}/\text{bl}_X(\mathbb{P}^N) \otimes \mathcal{O}(\mathcal{E})(-1))$.

We will see that in the case $X = \mathbb{P}^2$ such flips appear naturally when running the MMP for the Gieseker moduli $M(0, d, -\frac{3d}{2})$ for $d$ odd (or $M(0, d, -d)$ for $d$ even).

Consider the exceptional loci for the first flip of $M(0, d, -\frac{3d}{2})$ ($d \geq 5$ odd):

$$
E_0^+ : 0 \to \mathcal{O}((d-3)/2) \to F \to \mathcal{O}((-d-3)/2) [1] \to 0 \\
E_0^- : 0 \to \mathcal{O}((-d-3)/2) [1] \to \mathcal{G} \to \mathcal{O}((d-3)/2) \to 0
$$

these are obtained from the set-theoretic wall-crossing since the objects $\mathcal{O}((d-3)/2)$ and $\mathcal{O}((-d-3)/2) [1]$ are stable for every value of $t$. $E_0^+$ and $E_0^-$ are projective spaces, indeed:

$$
E_0^+ \cong \mathbb{P}(\text{Ext}^1(\mathcal{O}((-d-3)/2) [1], \mathcal{O}((d-3)/2))), \\
E_0^- \cong \mathbb{P}(\text{Ext}^1(\mathcal{O}((d-3)/2), \mathcal{O}((-d-3)/2) [1])) \\
= \mathbb{P}(H^2(\mathbb{P}^2, \mathcal{O}(-d))) \\
= \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(d-3))^\vee).
$$

There is a natural $\mathbb{P}^2$ embedded in $E_0^-$ by the complete linear series $|\mathcal{O}(d-3)|$. This Veronese surface can be described in terms of extensions, it is the set of complexes
Note that $G$ unique (up to scalars) since $\text{ext}^1(C_p, \mathcal{O}((-d-3)/2)[1]) = 1$. Thus $G^\bullet_p$ is the image under the pullback homomorphism

$$\text{Ext}^1(C_p, \mathcal{O}((-d-3)/2)[1]) \twoheadrightarrow \text{Ext}^1(\mathcal{O}((-d-3)/2), \mathcal{O}((-d-3)/2)[1]).$$

But we know that $\text{Ext}^1(C_p, \mathcal{O}((-d-3)/2)[1]) \cong \text{Ext}^1(\mathcal{O}((-d-3)/2)[1]^{P}, (\mathcal{C}_p)^P) = \text{Ext}^1(\mathcal{O}((-d-3)/2), (\mathcal{C}_p)^P)$, so $G^\bullet_p$ is also the image under the push-forward map

$$\text{Ext}^1(\mathcal{O}((-d-3)/2), (\mathcal{C}_p)^P) \twoheadrightarrow \text{Ext}^1(\mathcal{O}((-d-3)/2), \mathcal{O}((-d-3)/2)[1]).$$

Applying the functor $(\_)^D$ to the pullback diagram above gives us the push-forward diagram

$$
\begin{array}{c}
\mathbb{C}^D_p \searrow \nearrow \mathcal{G}^D \nearrow \mathcal{O}((d-3)/2) \\
\downarrow \downarrow \downarrow \\
\mathcal{O}((-d-3)/2)[1] \searrow \nearrow (G^\bullet_p)^D \nearrow \mathcal{O}((d-3)/2) \\
\downarrow \downarrow \downarrow \\
(\mathcal{I}_p((d-3)/2))^D
\end{array}
$$

**Proposition 6.1.** The elements of $E_0^-$ are fixed by the duality automorphism.

*Proof.* From the discussion above we know that $G^\bullet_p = (G^\bullet_p)^D$ and so the duality automorphism which restricts to an automorphism $(\_)^D|_{E_0^-} : E_0^- \rightarrow E_0^-$ fixes the $(d-3)$-uple embedding of $\mathbb{P}^2$, since every automorphism of $E_0^- \cong \mathbb{P}^N$ is linear then $(\_)^D|_{E_0^-}$ must be the identity. \hfill $\Box$

From now on we denote by $X$ the $(d-3)$-uple embedding of $\mathbb{P}^2$ inside $E_0^-$. The exceptional loci for the second flip are

$$
\begin{align*}
E_1^+ & : 0 \rightarrow \mathcal{I}_p((d-3)/2) \rightarrow F \rightarrow \mathcal{I}_q^*((-d-3)/2)[1] \rightarrow 0; \quad p, q \in \mathbb{P}^2 \\
E_1^- & : 0 \rightarrow \mathcal{I}_q^*((-d-3)/2)[1] \rightarrow G \rightarrow \mathcal{I}_p((d-3)/2) \rightarrow 0; \quad p, q \in \mathbb{P}^2.
\end{align*}
$$
Since the Bridgeland moduli for the Hilbert scheme of 1 point is constant (equal to \( \mathbb{P}^2 \)) then the description of \( E_1^- \) above is given by Corollary 4.4.

**Proposition 6.2.** (a) \( E_1^+ \) and \( E_1^- \) are both projective bundles over \( \mathbb{P}^2 \times \mathbb{P}^2 \).
(b) \( E_1^+ \cap E_0^- = X \).
(c) The closure of \( E_0^- \setminus X \) in \( M_1^- \) is isomorphic to the blow-up of \( E_0^- \) along \( X \).

**Proof.** For part (a) one only needs to verify that \( \text{ext}^1(\mathcal{I}_q^{((d-3)/2)}|_1, \mathcal{I}_p((d-3)/2)) \) and \( \text{ext}^1(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^{((d-3)/2)}|_1) \) are constant, the rest of the argument follows as in [AB13, Proposition 4.2]. We have

\[
\text{Ext}^1(\mathcal{I}_q^{((d-3)/2)}|_1, \mathcal{I}_p((d-3)/2)) = \text{Hom}(\mathcal{O}, \mathcal{I}_p \otimes \mathcal{I}_q(d)),
\]

\[
\text{Ext}^1(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^{((d-3)/2)}|_1) = \text{Ext}^2(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^{((d-3)/2)}|_1) \cong \text{Hom}(\mathcal{O}, \mathcal{I}_p \otimes \mathcal{I}_q(d-3))^{-1}.
\]

Note that we can use ordinary tensor instead of derived tensor, this is because ideal sheaves have a two-term resolution by locally free sheaves. For \( p \neq q \) there is no problem. For \( p = q \) one gets constant dimension because

\[
H^1(\mathbb{P}^2, \mathcal{I}_p^2(k)) = 0 \quad \text{for } k > 0
\]

which follows for example by Bertram-Ein-Lazarsfeld vanishing:

**Theorem 6.3** (Bertram-Ein-Lazarsfeld, [BEL91]). Assume that \( X \subset \mathbb{P}^r \) is (scheme-theoretically) cut out by hypersurfaces of degrees \( d_1 \geq d_2 \geq \cdots \geq d_m \). Then

\[
H^i(\mathbb{P}^r, \mathcal{I}_X^b(k)) = 0 \quad \text{for all } i \geq 1
\]

provided that \( k \geq ad_1 + d_2 + \cdots + d_e - r \), where \( e = \text{codim}(X, \mathbb{P}^r) \).

For part (b) diagram (6.1) already shows that \( X \subset E_1^+ \cap E_0^- \). For the other inclusion one notices that \( \text{hom}(\mathcal{I}_p((d-3)/2), \mathcal{O}((d-3)/2)) = 1 \) and \( \text{hom}(\mathcal{I}_p((d-3)/2), \mathcal{O}((d-3)/2)|_1) = 0 \).

More can be said, since \( E_0^- \) is fixed by the duality automorphism then \( E_1^+ \) intersects \( E_0^- \) along a section over the diagonal \( \Delta \subset \mathbb{P}^2 \times \mathbb{P}^2 \). Since the morphism \( \pi_1^+: M_1^+ \to M_1 \) collapses the fibers of \( E_1^+ \) then \( \pi_1|_{E_0^-}: E_0^- \to M_1 \) is a closed
immersion. By Lemma 6.4 we have a diagram

\[
\begin{array}{c}
\text{bl}_{E_1^+} M_1^+ \\
\downarrow \\
\text{bl}_X E_0^- \\
\downarrow \\
M_1^+ \\
\downarrow \\
M_1^- \\
\downarrow \\
E_0^- \quad \pi_1^*|_{E_0^-} \\
\downarrow \\
M_1 \\
\end{array}
\]

which proves \( \text{bl}_X E_0^- \subset M_1^- \) completing the proof of part (c). \( \square \)

**Lemma 6.4.** The fiber product \( M_1^+ \times_{M_1} M_1^- \) is isomorphic to the common blow-up \( \text{bl}_{E_1^+} M_1^+ = \text{bl}_{E_1^-} M_1^- \).

**Proof.** A proof of this statement was already given in [BMW13] for the case \( d = 5 \), it generalizes for all \( d \) (odd) without change. One notices the following vanishing

\[
\begin{align*}
\text{Hom}(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^\vee((-d-3)/2)[1]) &= 0 \\
\text{Ext}^2(\mathcal{I}_q^\vee((-d-3)/2)[1], \mathcal{I}_q^\vee((-d-3)/2)[1]) &= 0 \\
\text{Ext}^2(\mathcal{I}_p((d-3)/2), \mathcal{I}_p((d-3)/2)) &= 0 \\
\text{Ext}^2(\mathcal{I}_q^\vee((-d-3)/2)[1], F) &= 0 \\
\text{Ext}^1(\mathcal{I}_q^\vee((-d-3)/2)[1], \mathcal{I}_q^\vee((-d-3)/2)[1]) \\
\text{Ext}^1(F, \mathcal{I}_p((d-3)/2)) \\
\text{Ext}^1(F, F) \\
\text{Ext}^1(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^\vee((-d-3)/2)[1]) \\
\text{Ext}^1(\mathcal{I}_q^\vee((-d-3)/2)[1], F) \\
0
\end{align*}
\]

for every \( p, q \in \mathbb{P}^2 \) and \( F \in E_1^+ \). The first is obvious when \( p \neq q \), for \( p = q \) one uses Serre duality and Bertram-Ein-Lazarsfeld vanishing. The last two are obtained by using Serre duality and the fact that \( E \) is Bridgeland stable. This allows us to get diagrams

\[
\begin{array}{c}
\text{Ext}^1(\mathcal{I}_q^\vee((-d-3)/2)[1], \mathcal{I}_q^\vee((-d-3)/2)[1]) \downarrow \\
\text{Ext}^1(F, \mathcal{I}_p((d-3)/2)) \rightarrow \text{Ext}^1(F, F) \rightarrow \text{Ext}^1(F, \mathcal{I}_q^\vee((-d-3)/2)[1]) \downarrow \\
\text{Ext}^1(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^\vee((-d-3)/2)[1]) \downarrow \\
0
\end{array}
\]

and
Then we get an exact sequence

\[ 0 \rightarrow \ker f \rightarrow \text{Ext}^1(F, F) \rightarrow \text{Ext}^1(I_p((d-3)/2), I_q((-d-3)/2)[1]) \rightarrow 0 \]

where \( \ker f \) fits into an exact sequence

\[ 0 \rightarrow \mathbb{C} \rightarrow \text{Ext}^1(I_q((-d-3)/2)[1], I_p((d-3)/2)) \rightarrow \ker f \rightarrow \text{Ext}^1(I_q((-d-3)/2)[1], I_q((-d-3)/2)[1]) \oplus \text{Ext}^1(I_p((d-3)/2), I_p((d-3)/2)) \rightarrow 0 \]

Thus \( \ker f \) can be identified with the tangent space of \( E_1^+ \) at the point \([F]\) and we get an exact sequence

\[ 0 \rightarrow (TE_1^+)[F] \rightarrow (TM_1^+|_{E_1^+})[F] \rightarrow \text{Ext}^1(I_p((d-3)/2), I_q((-d-3)/2)[1]) \rightarrow 0. \]

and therefore an exact sequence of sheaves

\[ 0 \rightarrow TE_1^+ \rightarrow TM_1^+|_{E_1^+} \rightarrow (\pi_1^+|_{E_1^+})^*(((\pi_1^-|_{E_1^-})_*O_{E_1^-}(1)) \rightarrow 0. \]

Similarly one gets

\[ 0 \rightarrow TE_1^- \rightarrow TM_1^-|_{E_1^-} \rightarrow (\pi_1^-|_{E_1^-})^*(((\pi_1^+|_{E_1^+})_*O_{E_1^+}(1)) \rightarrow 0. \]
This proves that we have a fiber square

\[
\begin{array}{ccc}
P(N_{E_i^+} / M_i^+) & \cong & P(N_{E_i^-} / M_i^-) \\ & \downarrow & \downarrow \\ E_i^+ & \xrightarrow{\pi_i | E_i^+} & \mathbb{P}^2 \times \mathbb{P}^2
\end{array}
\]

which completes the proof. □

We now study the third flip for \(d \geq 7\) odd. Since \(d \geq 7\) then ideal sheaves of length two zero-subschemes are Bridgeland stable at the wall and since \(2 < \frac{d - 1}{2}\) then there is unique solution for the inequality 5.4 and so the exceptional loci are:

\[
\begin{align*}
E_2^+ : & \quad 0 \to \mathcal{I}_Z((d - 3)/2) \to F \to \mathcal{I}_W((-d - 3)/2)[1] \to 0 \quad |Z| = |W| = 2 \\
E_2^- : & \quad 0 \to \mathcal{I}_W((-d - 3)/2)[1] \to G \to \mathcal{I}_Z((d - 3)/2) \to 0 \quad |Z| = |W| = 2.
\end{align*}
\]

Again, Bertram-Ein-Lazarsfeld vanishing exposes \(E_2^+\) and \(E_2^-\) as projective bundles over \(\text{Hilb}^2(\mathbb{P}^2) \times \text{Hilb}^2(\mathbb{P}^2)\).

Our plan is to study the restriction of the directed MMP for \(M(0, d, -3d/2)\) to \(E_0^-\). It is convenient to fix some notation. Inductively define \(Y_1^+ := E_0^-\), \(Y_i^-\) is the closure of the image of \(Y_i^+\) by the rational map \(M_i^+ \dashrightarrow M_i^-\) and \(Y_{i+1}^+ := Y_i^-\). Then, for instance, \(Y_1^- = Y_2^+ = \text{bl}_X E_0^-\).

**Proposition 6.5.** \(E_2^+\) intersects \(Y_2^+\) along the strict transform of the secant variety \(\widehat{\text{Sec}X}\) which is a projective bundle over \(\text{Hilb}^2(\mathbb{P}^2)\).

**Proof.** The computation is very similar to the one we did when computing \(E_1^+ \cap E_0^-\).

Let \(Z = p + q\) where \(p, q \in \mathbb{P}^2\) and \(p \neq q\). We have a pullback diagram

\[
\begin{array}{cccc}
\mathcal{I}_Z((d - 3)/2) & \xleftarrow{\mathcal{O}((-d - 3)/2)[1]} & G^* & \xrightarrow{\mathcal{O}((d - 3)/2)} \\
\downarrow & & \downarrow & \\
\mathcal{O}((-d - 3)/2)[1] & \xleftarrow{G} & \mathcal{O}((d - 3)/2) & \xrightarrow{\mathcal{C}_Z} \\
\end{array}
\]

(6.2)

The difference here is that \(\text{ext}^1(\mathcal{C}_Z, \mathcal{O}((-d - 3)/2)[1]) = 2\) which corresponds to the line passing through \(p\) and \(q\) removing \(p\) and \(q\). Thus intersection of \(E_2^+ \setminus E_1^-\) with \(E_0^- \setminus X\) is \(\widehat{\text{Sec}X} \setminus X\) which proves the first claim.
The problem arises when considering $Z = 2p$, in this case $\text{ext}_1^1(C_Z, O((-d - 3)/2)[1]) = 3$. But since in $M_2^+$ we already flipped $E_1^+$ then not all the complexes $G_Z$ obtained this way are Bridgeland stable. Instead, the complexes $G_{2p}$ fitting into a commutative diagram

\[
\begin{array}{c}
\mathcal{I}_p((-d - 3)/2)[1] \rightarrow G_{2p} \rightarrow \mathcal{I}_p((d - 3)/2)
\end{array}
\]

are Bridgeland stable. The objects $G_{2p}$ form the fiber of $\widetilde{\text{Sec}}X$ over $Z = 2p$. □

**Lemma 6.6.** The fiber product $M_2^+ \times_{M_2} M_2^-$ is isomorphic to the common blow-up $\text{bl}_{E_2^+} M_2^+ = \text{bl}_{E_2^-} M_2^-$.  

**Proof.** The proof is similar to the proof of Lemma 6.4. The right vanishing is again a consequence of Bertram-Ein-Lazarsfeld vanishing. □

This completes Vermeire’s first flip since by restricting the fiber diagram of Lemma 6.6 one gets

\[
\begin{array}{c}
E \leftarrow \text{bl}_{\widetilde{\text{Sec}}X}(\text{bl}_X E_0^-) \rightarrow M_2^+ \times_{M_2} M_2^- \rightarrow M_2^-
\end{array}
\]

\[
\text{bl}_X(E_0^-) \rightarrow M_2^+ \rightarrow M_2
\]

**Remark.** In [Ver01] it is mentioned that flips of secant varieties are closely related to the geometry of $\text{Hilb}^n(X)$. By Proposition 5.3 and Corollary 4.4 and the results of this section, one sees that indeed flips of secant varieties of Veronese surfaces are related to the geometry of $\text{Hilb}^n(\mathbb{P}^2)$, and more precisely to its birational geometry. □

By using diagrams similar to 6.1 and 6.2 one sees that every rank-1 wall produces a
birational transformation of $E_0^-$ whose exceptional locus contains the strict transform of some higher secant variety of $X$. Indeed, for $\ell < (d - 1)/2$ the exceptional locus for the induced birational transformation of $E_0^-$, corresponding to crossing the wall $W_\ell$, is the strict transform of $\widetilde{\text{Sec}}^{\ell-1} X$. For $\ell \geq (d - 1)/2$ the exceptional locus is reducible and the middle component $E_{\ell,0}$ intersects $E_0^-$ along the strict transform of $\text{Sec}^{\ell-1} X$. For $\ell = (d - 1)/2$ the exceptional locus for the induced birational transformation of $E_0^-$, corresponding to crossing the wall $W_\ell$, is the strict transform of $\text{Sec}^{\ell-1} X$. For $\ell \geq (d - 1)/2$ the exceptional locus is reducible and the middle component $E_{\ell,0}$ intersects $E_0^-$ along the strict transform of $\text{Sec}^{\ell-1} X$. For $\ell \geq (d - 1)/2$ the exceptional locus is reducible and the middle component $E_{\ell,0}$ intersects $E_0^-$ along the strict transform of $\text{Sec}^{\ell-1} X$. For $\ell \geq (d - 1)/2$ the exceptional locus is reducible and the middle component $E_{\ell,0}$ intersects $E_0^-$ along the strict transform of $\text{Sec}^{\ell-1} X$. For $\ell \geq (d - 1)/2$ the exceptional locus is reducible and the middle component $E_{\ell,0}$ intersects $E_0^-$ along the strict transform of $\text{Sec}^{\ell-1} X$. For $\ell \geq (d - 1)/2$ the exceptional locus is reducible and the middle component $E_{\ell,0}$ intersects $E_0^-$ along the strict transform of $\text{Sec}^{\ell-1} X$.

6.1. The divisorial contraction. We want to study what happens to our restricted MMP when crossing the wall $W_0$ corresponding to the theta divisor (i.e., the closure of the set of those sheaves that admit at least one nonzero section). The theta divisor is fixed by the duality automorphism and therefore it corresponds to extensions of the form

$$0 \to N \to F \to \mathcal{O}(-3)[1] \to 0$$

where $N$ is an element in the corresponding model $N$ of $\text{Hilb}^n(\mathbb{P}^2) \otimes \mathcal{O}(d - 3)$ of $n = d(d - 3)/2$ points on the plane.

**Remark.** One can originally think of the dual extensions but this version allows us to compute the intersection with the first flipped locus more effectively. □

The intersection of the divisor $\Theta$ with $E_0^-$ corresponds to the extensions $F$ fitting into the push-forward diagrams

$$\begin{array}{ccccccc}
\mathcal{O}_C(-3) & \xleftarrow{\phi} & G & \xrightarrow{\psi} & \mathcal{O}((d - 3)/2) \\
\mathcal{O}((d - 3)/2)[1] & \xleftarrow{\phi} & F & \xrightarrow{\psi} & \mathcal{O}((d - 3)/2) \\
\mathcal{O}(-3)[1] & & & & & & \\
\end{array}$$

where $C \subset \mathbb{P}^2$ is a curve of degree $(d - 3)/2$. Indeed, the middle vertical sequence of arrows is exact and $G$ corresponds to those complexes produced when flipping the locus in $\text{Hilb}^n(\mathbb{P}^2)$ of $n$ points on a curve of degree $(d - 3)/2$. This intersection is therefore a projective bundle over the Hilbert scheme of plane curves of degree $(d - 3)/2$. An example of this situation was already observed in [BMW13] for the case $d = 5$ where the intersection of $\Theta$ with $E_0^-$ was exactly the strict transform of...
the secant variety of the Veronese surface in \( \mathbb{P}^5 \).

Notice that this intersection is not exactly what gets contracted when crossing \( W_\Theta \) since after several flips we may have replaced some of these objects by new ones. What we know is that the object \( G \) above must have \( \mathcal{O} \) as a subobject and therefore crossing \( W_\Theta \) must introduce objects \( E \) fitting into an exact sequence

\[
0 \to \mathcal{O}(-3)[1] \to E \to \mathcal{O} \oplus \mathcal{G} \to 0
\]

where \( \text{ch}(\mathcal{G}) = (0, d - 3, -3(d - 3)/2) \). Thus these new objects \( E \) are all strictly semistable, in fact pseudo-stable. Further analysis tell us that if \( \mathcal{G} \) is a sheaf then it must fit into an exact sequence

\[
0 \to \mathcal{O}_C(-3) \to \mathcal{G} \to \mathcal{O}_C((d - 3)/2) \to 0
\]

Semistability of \( \mathcal{G} \) at \( W_\Theta \) says that if \( \mathcal{G} \) is a complex then at least it has to fit into an exact sequence of the form

\[
0 \to A \to \mathcal{G} \to A^D \to 0
\]

where \( A \) is a semistable object of invariants \( \text{ch}(A) = \left(0, \frac{d - 3}{2}, -\frac{(d - 3)(d + 9)}{8}\right) \).

6.2. The last birational model. One could ask what is the moduli space we obtain after the divisorial contraction and what is the strict transform of \( E_0^- \). The answer to the first question comes from the identification with the quiver moduli. Values of \( (-3/2, t) \) near \( W_C \) are all inside the quiver region corresponding to \( k = -1 \), in this case we can compute the dimension invariants

\[
\begin{bmatrix}
\frac{k(k-1)}{2} & \frac{-(2k-1)}{2} & 1 \\
k(k-2) & \frac{-(2k-2)}{2} & 2 \\
\frac{(k-1)(k-2)}{2} & \frac{-(2k-3)}{2} & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
d \\
-3d/2
\end{bmatrix}
= \begin{bmatrix}
0 \\
d \\
d
\end{bmatrix}
\]

A more detailed analysis shows that above \( W_C \) the polarization \((-\theta, \theta)\) satisfies \( \theta > 0 \), at \( W_C \) we have \( \theta = 0 \), and below \( W_C \) it satisfies \( \theta < 0 \). Therefore the last model corresponds to the moduli space \( N(3, d, d) \) studied in [DM11] of semistable morphisms

\[
W\mathcal{O}(-2) \to W^*\mathcal{O}(-1),
\]

where \( \dim W = d \). The moduli space at \( W_C \) is just a point and below \( W_C \) is empty, which proves our assertion that \( W_C \) was the collapsing wall.

In order to understand what is going to be the last birational model of \( E_0^- \), let us take a look at the simplest but yet interesting \( M_H(0, 3, -9/2) \) studied by Le Potier. Le Potier showed that \( M_H(0, 3, -9/2) \) is the blow up of \( N(3, 3, 3) \) at the
complement of the dense open subset of injective morphisms \(3\mathcal{O}(-2) \hookrightarrow 3\mathcal{O}(-1)\), this complement consists of a single orbit which is the skew orbit

\[
\begin{array}{c}
\mathcal{O}(-3) \\
3\mathcal{O}(-2) \\
\Omega^1
\end{array} \xrightarrow{egin{pmatrix}
0 & -x & x \\
\frac{z}{y} & 0 & -y \\
-x & y & 0
\end{pmatrix}} \begin{array}{c}
\mathcal{O}(-1) \\
3\mathcal{O}(-1)
\end{array}
\]

As in the example above the general skew-map \(W\mathcal{O}(-2) \to W^*\mathcal{O}(-1)\) will drop rank by 1 everywhere and therefore it must have a kernel and a cokernel that are line bundles, indeed, as a complex it should fit into an exact sequence

\[
0 \to \mathcal{O}((-d - 3)/2)[1] \to E \to \mathcal{O}((d - 3)/2) \to 0.
\]

On the other hand, by Proposition 5.2 all the complexes in \(E_0^-\) are stable rather than pseudo-stable. Any stable complex in the last model for \(E_0^-\) must be, as \(E_0^-\) itself, fixed by the duality automorphism and therefore it must correspond to the orbit of a skew-map \(W\mathcal{O}(-2) \to W^*\mathcal{O}(-1)\).

For \(d = 5\) we have four walls (see [BMW13] for details): The walls produced by the destabilizing objects \(\mathcal{O}(1)\) and \(I_p(1)\), \(W_\varnothing\) and \(W_C\). At the first two walls the Jordan-Holder filtrations have length 1 and so the strict transform of \(E_0^-\) before the divisorial contraction consists only of stable objects. As above, the divisorial contraction produces objects that are \(S\)-equivalent to complexes fitting into an exact sequence

\[
0 \to \mathcal{O}(-3)[1] \to E \to \mathcal{O} \oplus \mathcal{O}_p(-3) \oplus \mathcal{O}_p(1) \to 0.
\]

In \(N(3, 5, 5)\) these correspond to the \(GL(W) \times GL(W^*)\)-orbits of matrices

\[
\begin{pmatrix}
0 & -z & x & 0 & 0 \\
z & 0 & -y & 0 & 0 \\
-x & y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & L \\
0 & 0 & 0 & -L & 0
\end{pmatrix}
\]

where \(L\) is a linear equation defining \(\ell\). The \(GL(W) \times GL(W^*)\)-orbits of these matrices are strictly semistable in \(N(3, 5, 5)\).

For \(d > 5\) the orbits generated after the divisorial contraction are the orbits of
elements of the form
\[
\begin{pmatrix}
0 & -z & x & 0 & 0 \\
z & 0 & -y & 0 & 0 \\
-x & y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A \\
0 & 0 & 0 & -A^t & 0
\end{pmatrix}
\]
where \( A \) is a square matrix of order \((d-3)/2\) but we can not say much about it, only that when \( G \) is a sheaf then \( \det A \) must give an equation for the curve \( C \).

Now assume that \( B \) is a skew-symmetric matrix giving a \( GL(W) \times GL(W^*) \)-stable orbit. If there are invertible matrices \( T, S \in GL(W) \) such that \( TBS^t \) is again skew-symmetric then
\[
B = (S^{-1}T)B(T^{-1}S)^t
\]
and therefore \( S = \lambda T \) for some \( \lambda \in \mathbb{C}^* \) since \( B \) is stable and so \( \text{Hom}(B, B) = \mathbb{C} \).

Since \( GL(W) \) can be embedded via the diagonal \( T \mapsto (T, T^t) \) into \( GL(W) \times GL(W^*) \) then a skew-symmetric matrix that is \( GL(W) \times GL(W^*) \)-stable is also \( GL(W) \)-stable for the diagonal action. Thus we can define an injective map
\[
\{ \text{Stable Skew } GL(W) \times GL(W^*) \text{-orbits} \} \longrightarrow \wedge^2 W \otimes V//GL(W)
\]
where \( V = \text{Hom}(O(-2), O(-1)) \). In the examples above, this map can be extended to the semistable orbits that have a skew representative. In fact, in a personal communication to the author, Aaron Bertram has made the following

**Conjecture 6.7.** The last birational model of \( \text{bl}_X E_0^- \) is isomorphic to the GIT quotient \( \wedge^2 W \otimes V//GL(W) \).

**References**

[AB13] Daniele Arcara and Aaron Bertram. Bridgeland-stable moduli spaces for \( K \)-trivial surfaces. *J. Eur. Math. Soc. (JEMS)*, 15(1):1–38, 2013. With an appendix by Max Lieblich.

[ABCH13] Daniele Arcara, Aaron Bertram, Izzet Coskun, and Jack Huizenga. The minimal model program for the Hilbert scheme of points on \( P^2 \) and Bridgeland stability. *Adv. Math.*, 235:580–626, 2013.

[AP06] Dan Abramovich and Alexander Polishchuk. Sheaves of \( t \)-structures and valuative criteria for stable complexes. *J. Reine Angew. Math.*, 590:89–130, 2006.

[BEL91] Aaron Bertram, Lawrence Ein, and Robert Lazarsfeld. Vanishing theorems, a theorem of Severi, and the equations defining projective varieties. *J. Amer. Math. Soc.*, 4(3):587–602, 1991.

[BM12] A. Bayer and E. Macri. Projectivity and Birational Geometry of Bridgeland moduli spaces. *ArXiv e-prints*, (arXiv:1203.4613), March 2012.

[BMW13] A. Bertram, C. Martinez, and J. Wang. The birational geometry of moduli space of sheaves on the projective plane. *ArXiv e-prints*, (arXiv:1301.2011), January 2013.

[Bri07] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007.
Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008.

J. Choi and K. Chung. On the geometry of the moduli space of one-dimensional sheaves. *ArXiv e-prints*, (arXiv:1311.0134), November 2013.

Jean-Marc Drézet and Mario Maican. On the geometry of the moduli spaces of semi-stable sheaves supported on plane quartics. *Geom. Dedicata*, 152:17–49, 2011.

Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

D. Huybrechts. Introduction to stability conditions. *ArXiv e-prints*, (arXiv:1111.1745), November 2011.

A. D. King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, 45(180):515–530, 1994.

J. Le Potier. Faisceaux semi-stables de dimension 1 sur le plan projectif. *Rev. Roumaine Math. Pures Appl.*, 38(7-8):635–678, 1993.

J. Lo and Z. Qin. Mini-walls for Bridgeland stability conditions on the derived category of sheaves over surfaces. *ArXiv e-prints*, (arXiv:1103.4352), March 2011.

A. Maciocia. Computing the Walls Associated to Bridgeland Stability Conditions on Projective Surfaces. *ArXiv e-prints*, (arXiv:1202.4587), February 2012.

M. Maican. On the moduli spaces of semi-stable plane sheaves of dimension one and multiplicity five. *ArXiv e-prints*, (arXiv:1007.1815), July 2010.

Mario Maican. A duality result for moduli spaces of semistable sheaves supported on projective curves. *Rend. Semin. Mat. Univ. Padova*, 123:55–68, 2010.

M. Maican. The classification of semi-stable plane sheaves supported on sextic curves. *ArXiv e-prints*, (arXiv:1205.0278), May 2012.

Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. I. *Inst. Hautes Études Sci. Publ. Math.*, (79):47–129, 1994.

Michael Thaddeus. Stable pairs, linear systems and the Verlinde formula. *Invent. Math.*, 117(2):317–353, 1994.

Peter Vermeire. Some results on secant varieties leading to a geometric flip construction. *Compositio Math.*, 125(3):263–282, 2001.

Peter Vermeire. Secant varieties and birational geometry. *Math. Z.*, 242(1):75–95, 2002.

M. Woolf. Nef and Effective Cones on the Moduli Space of Torsion Sheaves on the Projective Plane. *ArXiv e-prints*, (arXiv:1305.1465), May 2013.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, 155 S 1300 E, SALT LAKE CITY, UT 84112, martinez@math.utah.edu.