Optimizing expected utility of dividend payments for a Erlang risk process

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Abstract

We consider the problem of maximizing the expected utility of discounted dividend payments of an insurance company whose reserves are modeled as a Cramér risk process with Erlang claims. We focus on the exponential claims and power and logarithmic utility functions. Finally we also analyze asymptotic behaviour of the value function and identify the asymptotic optimal strategy. We also give the numerical procedure of finding considered value function.

1 Introduction

The problem of finding optimal dividend strategies for an insurance company have been studied many times. A mathematical formalization of this problem was proposed by Gerber [3]. He assumed that the reserve process \((R_t)_{t \geq 0}\) of an insurance company is a classical Cramér-Lundberg risk process given by

\[
R_t = x + \mu t - \sum_{i=1}^{N_t} Y_i,
\]

\[1\]

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where $Y_1, Y_2, \ldots$ are i.i.d positive random variables representing the claims, $N = (N_t)_{t \geq 0}$ is an independent Poisson process with intensity $\lambda > 0$ modeling the times at which the claims occur, $x > 0$ denotes the initial surplus and $\mu$ is a premium intensity. Additionally we will assume that generic claim $Y$ has Erlang distribution $\text{Erlang}(k, \xi)$ which is probably the most common distribution in actuarial science.

Apart of the reserve process (1) we can consider the dividend payments. Let $(C_t)_{t \geq 0}$ be an adapted and nondecreasing process representing all accumulated dividend payments up to time $t$. Then the regulated process $X = (X_t)_{t \geq 0}$ is given by:

$$X_t = R_t - C_t. \quad (2)$$

We observe the regulated process $X_t$ until the time of ruin:

$$\tau = \inf\{t \geq 0: X_t \leq 0\}.$$

Obviously the time of the ruin of an insurance company depend on dividend strategy and after ruin occurs no dividends are paid.

Following Hubalek and Schachermayer [5] who considered the Brownian risk process with drift we assume that $(C_t)_{t \geq 0}$ is absolutely continuous with respect to Lebesgue measure which means that for each $t \geq 0$:

$$C_t = \int_0^t c_s ds \text{ a.s.}$$

and we define the target value function as

$$v(x) = \sup_{c_t} E_x \left( \int_0^\tau e^{-\beta t} U(c_t) dt \right),$$

where $U$ is some fixed utility function and $E_x$ means expectation with respect of $P_x(\cdot) = P(\cdot | X_0 = x)$. We maximize the value function $v(x)$ over all admissible dividend strategies $(c_t)_{t \geq 0}$. We assume that dividend density process $(c_t)_{t \geq 0}$ is admissible, if it is a nonnegative, adapted process and there is no dividend after ruin occurs: $c_t = 0$ for all $t \geq \tau$.

For above dividend problem we will prove the verification Theorem [1] producing the Hamilton-Jacobi-Bellman (HJB) equation for optimal value function. Like in many other cases the HJB equation cannot be solved explicitly. Therefore we analyze the asymptotic values of the value function focusing on the exponential claim sizes and the power and logarithmic utility function (see Section [3]).

In Section [4] we give new algorithm of identifying the value function.
2 Hamilton-Jacobi-Bellman equation

From now on we will assume that $U \in C^\infty(\mathbb{R}_{\geq 0})$ is concave such that $U(0) = 0$ and the Inada conditions are satisfied i.e. $\lim_{x \to 0} U''(x) = \infty$ and $\lim_{x \to \infty} U'(x) = 0$. We also assume that:

$$\lim_{x \to \infty} U(x) = \infty.$$  \hspace{1cm} (3)

Then if we consider strategies approximating lump sum payment at the beginning (i.e. $c_t \to \infty$ as $t \downarrow 0$) condition (3) implies that also:

$$\lim_{x \to \infty} v(x) = \infty.$$  \hspace{1cm} (4)

We will use the variational approach based on the verification theorem and Hamiltonian-Jacobi-Bellman equation.

**Theorem 1.** The value function uniquely solves HJB equation:

$$-\beta v + \sup_{c \geq 0} \left\{ (\mu - c) v_x + \frac{\lambda^k}{(k-1)!} \int_0^\infty [v(x - y) - v(x)] y^{k-1} e^{-\xi y} dy + U(c) \right\} = 0.$$  \hspace{1cm} (5)

**Proof.** If $v$ is absolutely continuous then using classical arguments we know that $v$ solves uniquely \cite{5} (see e.g. \cite{8} Sec. 2.1) which follows from that fact that infinitesimal generator of the risk process $R$ has the following form:

$$\mathcal{A}f = \mu f_x + \frac{\lambda^k}{(k-1)!} \int_0^\infty [f(x - y) - f(x)] y^{k-1} e^{-\xi y} dy$$

and $v$ is in its domain; see e.g. \cite{7} p. 460.

For any $x$ we can write

$$x + \frac{U(\mu)}{\lambda + \beta} \leq v(x) \leq x + \frac{U(\mu)}{\beta}.$$  \hspace{1cm} (6)

The lower bound of $v(x)$ is obtained by paying $x$ out immediately and then to pay out the premia at rate $\mu$ until the first claim occurs. The upper bound comes from the pseudo strategy paying out $x$ immediately and thereafter to pay out a dividend at rate $\mu$, not stopping at ruin. Inequality above leads to
conclusion that $v(x)$ is locally bounded. Considering for $X_0 = x + \mu h \ (h > 0)$ and for any strategy $c_t$ the new strategy:

$$\tilde{c}_t = \begin{cases} 
0 & t \leq h \ \text{or} \ T \leq h \\
\tilde{c}_t - h & T \land t > h
\end{cases}$$

one can derive two-sided bounds:

$$v(x) \geq e^{-(\lambda + \beta)h}v(x + \mu h) \geq e^{-(\lambda + \beta)h}v(x),$$

where $T$ is a moment when first claim arrives. By (6) it means that $v$ is locally Lipschitz continuous and using [2, p. 164] one can conclude that $v$ is absolutely continuous. Hence equation (5) holds true if one interpret $v_x$ as a right-derivative. The optimal strategy equals $c_t = c^*(X_t)$ for some measurable function $c^*$. Obviously supremum in (6) is attained at

$$c^*(x) = (U')^{-1}(v_x) \quad (7)$$

Putting (7) to equation (5) we obtain

$$\mu v_x - c^* v_x - \beta v + U(c^*) + \frac{\lambda \xi \xi}{(k - 1)!} \int_0^\infty [v(x - y) - v(x)]y^{k-1}e^{-\xi y}dy = 0. \quad (8)$$

Since $v$ is continuous it follows that $v_x$ is well-defined. This completes the proof. \hfill \square

The equation (8) can be rewritten as:

$$\mu v_x - c^* v_x - (\beta + \lambda) v + U(c^*) + \frac{\lambda \xi \xi}{(k - 1)!} \int_0^x v(x - y)y^{k-1}e^{-\xi y}dy = 0$$

Integration by substitution leads to:

$$\mu v_x - c^* v_x - (\beta + \lambda) v + U(c^*) + \frac{\lambda \xi \xi}{(k - 1)!} \int_0^x v(z)(x - z)^{k-1}e^{\xi z}dz = 0 \quad (9)$$

From the equation above follows that $v \in C^2(R_{\geq 0})$. If we differentiate this equation by $x$ we get

$$\mu v_{xx} - c^*_x v_{xx} - c^* v_{xx} - (\beta + \lambda) v_x + U_x(c^*) - \xi \frac{\lambda \xi \xi}{(k - 1)!} \int_0^x v(z)(x - z)^{k-1}e^{\xi z}dz + \frac{\lambda \xi \xi}{(k - 1)!} (k - 1) \int_0^x v(z)(x - z)^{k-2}e^{\xi z}dz = 0. \quad (10)$$
Then by calculating the integral from equation (9) and putting it back to the equation (10) we derive:

\[
\begin{align*}
\mu v_{xx} - c_2^* v_x - c^* v_{xx} - (\beta + \lambda) v_x + U_x(c^*) + \xi(\mu v_x - c^* v_x - (\beta + \lambda) v + U(c^*)) \\
+ \frac{\lambda \xi^k e^{-\xi x}}{(k-2)!} \int_0^x v(z)(x-z)^{k-2} e^{\xi z} dz = 0. 
\end{align*}
\]  

(11)

If we repeat this procedure \(k - 1\) times we will derive the following equation for the value function:

\[
\sum_{i=0}^{k-1} \binom{k}{i} \xi^i \left( \mu v^{(k+1-i)} - (\beta + \lambda) v^{(k-i)} - (c^* v_x)^{(k-i)} + U^{(k-i)}(c^*) \right) + \lambda \xi^k v = 0,
\]

where \(f^{(k)}\) means \(\frac{d^k f}{dx^k}\). By some simple calculation, changing the limits of summation and using the general Leibniz rule we can convert this equation into:

\[
\begin{align*}
\mu v^{(k+1)} + \sum_{j=0}^{k-1} \xi^j \left( \xi \mu \binom{k}{j+1} - \binom{k}{j}(\beta + \lambda) \right) v^{(k-j)} + \sum_{i=0}^{k} \binom{k}{i} \xi^i U^{(k-i)}(c^*) \\
- \sum_{i=0}^{k} \binom{k}{i} \xi^i \sum_{l=0}^{k-i} \binom{k-i}{l} c^{(i+l)} v^{(k-i-l+1)} - \xi^k \beta v = 0. 
\end{align*}
\]  

(12)

This is nonlinear \((k+1)\)th order nonlinear differential equation. Let notice that from equality \(7\) it follows that for \(k = 1\) equation \(12\) simplifies to the following one:

\[
\mu v_{xx} + (\xi \mu - \beta - \lambda) v_x - \xi \beta v - c^* v_{xx} - c^* v_x + \xi U(c^*) = 0.
\]  

(13)

It is still difficult to solve above equations. Therefore we will analyze it asymptotically for large reserves \(x \to \infty\) and give numerical procedure of solving them.

3 Asymptotic solution

We start from asymptotic analysis and the following lemma.

**Lemma 2.** \(\lim_{x \to \infty} v_x(x) = 0.\)
Proof. At the beginning we show that \( \lim_{x \to \infty} c(x) = \infty \). For the proof by contradiction let assume that there exist \( K > 0 \) such that for all \( x \geq 0 \) we have \( c(x) < K \). Then

\[
v(x) = \sup \mathbb{E}_x \left( \int_0^\tau e^{-\beta t} U(c_t) dt \right) \leq \mathbb{E}_x \left( \int_0^\tau e^{-\beta t} U(K) dt \right)
\]

\[
\leq \mathbb{E}_x \left( \int_0^\infty e^{-\beta t} U(K) dt \right) \leq \frac{U(K)}{\beta} < \infty.
\]

This means that \( v(x) \) is bounded which is a contradiction with (4). Thus indeed \( c(x) \to \infty \) as \( x \to \infty \). Then from equality (7) it follows that

\[
\lim_{x \to \infty} v_x(x) = \lim_{x \to \infty} U'(c^*(x)) = 0,
\]

where the last equality in this equation comes from Inada condition \( \lim_{x \to \infty} U'(x) = 0 \), which we required from the utility function. \( \square \)

From now on we will assume that claims have exponential distribution with parameter \( \xi \), that is \( Y_i \overset{D}{=} \text{Exp}(\xi) \).

### 3.1 Power utility function

In this section we will consider the power utility function:

\[
U(x) = \frac{x^\alpha}{\alpha}, \quad x \geq 0, \quad \alpha \in (0, 1).
\]  

(14)

Then the supremum in (5) is attained at

\[
c^* = (U')^{-1}(v_x) = v_x^{\frac{1}{1-\alpha}}
\]

and the equation (13) simplifies to:

\[
\mu v_{xx} + (\xi \mu - \beta - \lambda)v_x - \xi \beta v + \frac{1 - \alpha}{\alpha} v_x^{\frac{\alpha}{1-\alpha}} - v_x^{\frac{1}{1-\alpha}} v_{xx} = 0.
\]

(16)

This is nonlinear second order ODE. Obviously we need some initial conditions. Putting \( x = 0 \) into equation (9) we get:

\[
v(0) = \frac{\mu}{\beta + \lambda} v_x(0) + \frac{1 - \alpha}{\alpha(\beta + \lambda)} v_x(0)^{\frac{\alpha}{1-\alpha}}
\]

(17)
By Riccati’s substitution we transform this problem to nonlinear first order ODE. Let \( v_x(x) = y(v) \) then \( v_{xx} = y_x(v) = y_v x = v_x y \). Including this into equation (16) produces the following equation

\[
\mu y_v y + (\xi \mu - \beta - \lambda) y - \xi \beta v + \xi \frac{1 - \alpha}{\alpha} y^{\frac{\alpha}{1 - \alpha}} - y^{\frac{\alpha}{1 - \alpha}} y_v = 0 \tag{18}
\]

which is equivalent with

\[
y_v = \frac{(\xi \mu - \beta - \lambda) y - \xi \beta v + \xi \frac{1 - \alpha}{\alpha} y^{\frac{\alpha}{1 - \alpha}}}{\mu y - y^{\frac{\alpha}{1 - \alpha}}} \tag{19}
\]

This is nonlinear first order ODE without known explicit solution. Therefore we will focus on its asymptotical behaviour producing also the asymptotic optimal strategy \( c_t = c^*(X_t) \) of paying dividends being the asymptotical function \( c^*(x) \) depending on the present amount of reserves. We write \( f(x) \sim g(x) \) iff \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \).

**Theorem 3.** Let \( \alpha = \frac{p}{q} \in (0, 1) \), where \( p, q \in \mathbb{N}, p < q \). Then as \( x \to \infty \),

\[
v(x) \sim \left( \frac{1 - \alpha}{\beta} \right)^{1 - \alpha} \frac{x^\alpha}{\alpha}, \quad \tag{20}
\]

\[
v_x(x) \sim \left( \frac{1 - \alpha}{\beta} \right)^{1 - \alpha} x^{\alpha - 1}, \quad \tag{21}
\]

\[
c^*(x) \sim \frac{\beta}{1 - \alpha} x. \quad \tag{22}
\]

**Remark 4.** Note that assumption that \( \alpha \) is rational is not very restrictive because the set of rational numbers is sufficient in modelling all kind of shapes of the power utility function.

**Proof.** When \( \alpha = \frac{p}{q} \) then equation (18) has the following form:

\[
\mu y_v y + (\xi \mu - \beta - \lambda) y - \xi \beta v + \xi \frac{q - p}{p} y^{\frac{p}{p-q}} - y^{\frac{p}{p-q}} y_v = 0.
\]

If we make substitution \( z = y^{\frac{1}{p-q}} \), then \( z_v = \frac{1}{p-q} y_v z z^{q-p} \). Putting this into equation above we get:

\[
(\xi \mu - \beta - \lambda) z - \xi \beta v z^{q-p+1} + \xi \frac{q - p}{p} z^{q+1} - z_v (q - p) \left( \mu z^{p-q} - z^p \right) = 0.
\]
If we multiple both sides of equation above by $z^{q-p}$ we obtain

$$(\xi \mu - \beta - \lambda) z^{q-p+1} - \xi \beta v z^{2q-2p+1} + \xi \frac{q-p}{p} z^{2q-p+1} - (q-p) \mu z_v - (p-q) z^q z_v = 0.$$  

(23)

This is equation of the form

$$P(v, z) - z_v Q(v, z) = 0,$$  

(24)

where $P, Q$ are polynomials in $v$ and $z$. Any term of the left side of (24) is of one of the forms $z^m a_m(v)$ or $z_v z^n a_n(v)$. Vojislav Marić in [6] proved that if for two functions $a_m, a_n \in H$ (where $H$ denote the class of Hardy functions) then the set of all terms of the left side of equation (24) is totally ordered with respect to the relation $\succeq$. Where $a \succeq b$, for $v \to \infty$ means that either $\frac{a}{b} \to \infty$ or $\frac{a}{b} \to l(\neq 0)$ as $v \to \infty$. Furthermore in [6, p. 195] it was proved that in this set exist two terms of the same order i.e. whose quotient tends to a finite limit $l \neq 0$ for $v \to \infty$. Using this we can derive asymptotic of solution of equation $P(v, z) - z_v Q(v, z) = 0$, for $v \to \infty$.

Firstly, we have to notice that $z \to \infty$ since $y \to 0$ for $v \to \infty$ and $p < q$. Because of that we note that in the equation (23) the element $(q-p) \mu z_v$ is of smaller order than the element $y_v$. Similarly, the element $(\xi \mu - \beta - \lambda) z^{q-p+1}$ has smaller order than other elements of the equation (23) which not contain $y_v$. Because we know that there exist two of the terms of the equation (23) of the same order we have three possibilities which can produce the asymptotics of the solution $v$ of the equation (16):

a) $\xi \beta v z^{2q-2p+1}$ and $\xi \frac{q-p}{p} z^{2q-p+1}$;

b) $\xi \frac{q-p}{p} z^{2q-p+1}$ and $(p-q) z^q z_v$;

c) $\xi \beta v z^{2q-2p+1}$ and $(q-p) z^q z_v$.

In the case a) both of these terms are of the same order. Hence

$$\lim_{v \to \infty} \frac{\xi \beta v z^{2q-2p+1}}{\xi \frac{q-p}{p} z^{2q-p+1}} = l(\neq 0),$$

which gives

$$z(v) \sim \left(\frac{\beta p}{l(q-p)}\right)^{\frac{1}{p}} v^\frac{1}{p}.$$
Putting above asymptotics into the equation (23) and dividing by \( v^{2q-p+1} \) gives \( l = 1 \). Finally we get the following asymptotics of \( z(v) \):

\[
z(v) \sim \left( \frac{\beta p}{q-p} \right)^{\frac{1}{p}} v^{\frac{1}{p}}. \tag{25}
\]

Obviously in this case \( z \to \infty \) for \( v \to \infty \) as we required from \( z \).

Similarly, in the case b) we have:

\[
\lim_{v \to \infty} \left( \frac{p-q}{p} \right) z^q z_v v^\xi = l(\neq 0),
\]

L'Hopital rule guarantees that \( z \) has the same asymptotics as the solution \( a \) of the differential equation of separated variables:

\[
-a^{p-q-1} a_v = \frac{l \xi}{p},
\]

which gives

\[
z(v) \sim \left( \frac{l \xi (q-p)}{p} \right)^{\frac{1}{p-q}} (v + c)^{\frac{1}{p-q}}.
\]

But for \( p - q < 0 \) we have \( z \to 0 \) as \( v \to \infty \) which contradicts the assumption that \( z \to \infty \) for \( v \to \infty \).

In the case c) we have:

\[
\lim_{v \to \infty} \left( \frac{q-p}{p} \right) z^q z_v v^\xi = l(\neq 0),
\]

which simplifies into considering the following equation:

\[
z^{2p-q-1} z_v = \frac{l \xi \beta}{q-p} v.
\]

We will distinguish two cases.

I. If \( q \neq 2p \), then

\[
z(v) \sim \left( \frac{l \xi \beta (2p-q)}{q-p} \right)^{\frac{1}{2p-q}} \left( v^2 + c \right)^{\frac{1}{2p-q}}.
\]
II. If \( q = 2p \), then
\[
z(v) \sim e^{\frac{\beta p}{v^{p}} \left( \frac{v^2}{v + c} \right)}.
\]
Note that in the case I the asymptotics of \( z \) have sense only if \( q < 2p \) because otherwise \( z \to 0 \) for \( v \to \infty \) and we get again the contradiction. After substitution above asymptotics into the equation (23) the increment coming from \( \xi \frac{q-p}{p} z^{2q-p+1} \) dominates any other increment. Dividing both sides of the equation (23) by this asymptotically "largest" element leads to false identity \( 1 = 0 \).

Summarizing, the asymptotic solution of \( z \) is given by (25). We made substitution \( y = z^{p-q} \) and the asymptotics of \( y(v) \) is given by:
\[
y(v) \sim \left( \frac{\beta p}{q-p} \right)^{\frac{p-q}{p}} v^{\frac{p-q}{p}},
\]
which is equivalent with:
\[
y(v) \sim \left( \frac{1 - \alpha}{\alpha \beta} \right)^{\frac{1-\alpha}{\alpha}} v^{\frac{(1-\alpha)}{\alpha}}.
\]
Recall that \( y(v) = v_x(x) \). Using L'Hopital rule and solving equation:
\[
v_x(x) = \left( \frac{1 - \alpha}{\alpha \beta} \right)^{\frac{1-\alpha}{\alpha}} v(x)^{(1-\alpha)}.
\]
give (20). This completes the proof by (15).

3.2 Asymptotic solution of the HJB equation for logarithmic utility function

In this section we will consider the logarithmic utility function:
\[
U(x) = \ln(x + 1), \quad x \geq 0.
\]
Then supremum in the equation (5) is attained for:
\[
c^* = (U')^{-1}(v_x) = \frac{1}{v_x} - 1
\]
and the equation (13) simplifies to:

\[(\mu + 1)v_{xx} + (\xi\mu + \xi - \beta - \lambda)v_x - \xi\beta v - \xi \ln v_x - \frac{1}{v_x}v_{xx} - \xi = 0.\]  \hspace{1cm} (28)

This is a nonlinear second order ODE. By the Riccati’s substitution \(v_x(x) = y(v)\) we can transform this problem into the nonlinear first order ODE:

\[(\xi\mu + \xi - \beta - \lambda)y - \xi \ln y - \xi\beta v - \xi + (\mu + 1)yy_v - y_v = 0\]  \hspace{1cm} (29)

**Theorem 5.** As \(x \to \infty\) we have,

\[v(x) \sim \frac{1}{\beta} (\ln(\beta(x+1)) - 1); \hspace{1cm} (30)\]

\[v_x(x) \sim \frac{1}{\beta(x+1)}; \hspace{1cm} (31)\]

\[c^*(x) \sim \beta x + \beta - 1. \hspace{1cm} (32)\]

**Proof.** We will use similar arguments like in the proof of Theorem 3. In fact, one can derive the same expression like (24) with this difference that now in \(P\) and \(Q\) expressions of the form \(v^m \ln v\) and \(v^n\) will appear. What is important that both these expressions will appear there (to satisfy eliminating procedure giving [6, Eq. (3.3)]). Then mimic all arguments of [6] one can conclude that also in the case of the logarithmic utility function there exist two of the terms of the equation (29) of the same order.

Note now that in the equation (29) the increment \((\mu + 1)yy_v\) is of smaller order than \(y_v\). Similarly, the increment \((\xi\mu + \xi - \beta - \lambda)y\) is of smaller order than all other elements which not contain \(y_v\). We have then three possibilities:

a) \(y_v\) and \(\xi\beta v + \xi\);

b) \(y_v\) and \(\xi \ln y\);

c) \(\xi\beta v + \xi\) and \(\xi \ln y\).

In the case a):

\[
\lim_{v \to \infty} \frac{y_v}{\xi\beta v + \xi} = l(\neq 0),
\]

which gives

\[y(v) \sim l\xi\beta \frac{v^2}{2} + l\xi v + c.\]
When \( v \to \infty \) then \( y \to \pm \infty \) and we get contradiction since by Lemma 2 \( y \to 0 \) when \( v \to \infty \).

In the case b):

\[
\lim_{v \to \infty} \frac{yv}{\xi \ln y} = l(\neq 0),
\]

and by L’Hopital rule \( v \sim a \) for \( a \) satisfying

\[
\frac{1}{y} \int_0^y \frac{1}{\ln s} ds = \frac{l\xi a + C}{y}. \tag{33}
\]

Then the right side of equation (33) goes to \( \pm \infty \) depending on value of \( l \) when \( v \to \infty \). The left side of this equation goes to \( \lim_{y \to 0} \frac{1}{\ln y} = 0 \) and we get contradiction.

In the case c) we have

\[
\lim_{v \to \infty} \frac{\xi \beta v + \xi}{\xi \ln y} = l(\neq 0),
\]

which gives:

\[
y(v) \sim e^{\frac{\beta}{1}}. \tag{34}
\]

Asymptotics above has sense only if \( l < 0 \), because otherwise \( y \to \infty \) when \( v \to \infty \). Putting the asymptotics (34) into (29) gives \( l = -1 \). Hence:

\[
y(v) \sim e^{-\beta v - 1}.
\]

Recall that \( y(v) = v_x(x) \). Thus \( v \sim a \) with \( a \) solving:

\[
a_x(x) = e^{-\beta a(x) - 1}.
\]

This gives

\[
v(x) \sim \frac{1}{\beta} (\ln (\beta (x + C)) - 1).
\]

Deriving (31) and (32) is straightforward. \( \square \)

4 Numerical analysis for exponentially distributed claims

In this section we give numerical algorithm of finding the value function for exponentially distributed claims and the power utility function (14). To do this we will find \( v_x(0) \), then (based on the boundary condition (17)) find \( v(0) \) and numerically solve the equation (16).
Remark 6. The choice of \( v_x(0) \) is crucial in the context of optimality of solution of HJB equation. Indeed, if we choose \( v_x(0) \) too big (see Figure 1), then \( v(x) \) and \( v_x(x) \) go to infinity as \( x \to \infty \). In fact, by (15) the discounted cumulative dividends goes to 0 (see Table 1). This situation corresponds to a bubble, i.e. value of the company is not materialized by the dividend payments and we can not discuss the optimal solution.

![Figure 1: Functions \( v(x) \) and \( v_x(x) \) for \( \alpha = 0.5, \beta = 0.05, \mu = 0.26, \xi = 0.4, \lambda = 0.1 \) and \( v_x(0) = 2, v(0) = 6.8 \).](image)

(Figure 1: Functions \( v(x) \) and \( v_x(x) \) for \( \alpha = 0.5, \beta = 0.05, \mu = 0.26, \xi = 0.4, \lambda = 0.1 \) and \( v_x(0) = 2, v(0) = 6.8 \).)

When \( v_x(0) \) is sufficiently large like Figure 2 shows, then the function \( v(x) \) is concave and \( v_x(x) \) tends to 0 as \( x \to \infty \) allowing the cumulative discounted dividend payments to increase (see Table 2).
Table 1: Functions $v(x)$ and $v_x(x)$ for $\alpha = 0.5$, $\beta = 0.05$, $\mu = 0.26$, $\xi = 0.4$, $\lambda = 0.1$ and $v_x(0) = 2$, $v(0) = 6.8$.

| $x$ | $v(x)$ | $v_x(x)$ | $c(x)$ |
|-----|--------|----------|--------|
| 0   | 6.8000 | 2.0000   | 0.2500 |
| 1   | 9.4022 | 3.1941   | 0.0980 |
| 2   | 13.3275| 4.7502   | 0.0443 |
| 3   | 19.1343| 7.0039   | 0.0204 |
| 4   | 27.6771| 10.2878  | 0.0094 |
| 5   | 40.2103| 15.0801  | 0.0044 |
| 6   | 58.5692| 22.0787  | 0.0021 |
| 7   | 85.4378| 32.3029  | 0.0010 |
| 8   | 124.7394| 47.2425 | 0.0004 |
| 9   | 182.2094| 69.0750 | 0.0002 |
| 10  | 266.2320|100.9833 | 0.0001 |

Figure 2: Functions $v(x)$ and $v_x(x)$ for $\alpha = 0.5$, $\beta = 0.05$, $\mu = 0.26$, $\xi = 0.4$, $\lambda = 0.1$ and $v_x(0) = 1.9$, $v(0) = 6.8021$. 
Table 2: Functions $v(x)$ and $v_x(x)$ for $\alpha = 0.5$, $\beta = 0.05$, $\mu = 0.26$, $\xi = 0.4$, $\lambda = 0.1$ and $v_x(0) = 1.9$, $v(0) = 6.8021$. 

| $x$ | $v(x)$ | $v_x(x)$ | $c(x)$ |
|-----|--------|----------|--------|
| 0   | 6.8021 | 1.9000   | 0.2770 |
| 1   | 8.5790 | 1.6929   | 0.3489 |
| 2   | 10.2022| 1.5575   | 0.4122 |
| 3   | 11.7010| 1.4431   | 0.4802 |
| 4   | 13.0940| 1.3454   | 0.5525 |
| 5   | 14.3963| 1.2613   | 0.6286 |
| 6   | 15.6203| 1.1884   | 0.7081 |
| 7   | 16.7762| 1.1247   | 0.7905 |
| 8   | 17.8723| 1.0687   | 0.8755 |
| 9   | 18.9158| 1.0192   | 0.9626 |
| 10  | 19.9126| 0.9752   | 1.0515 |
To find \( v_x(0) \) we propose the following algorithm.

- Calculate initial value \( v_x(0) =: b \),
- From the equality (17) derive initial value \( v(0) =: a \);
- Solve numerically the differential equation (16) with the initial condition \( v(0) =: a \);
- Calculate \( c(x) \) using \( c(x) = v_x(x)^{-\frac{1}{\alpha}} \);
- Using the least squares method match to \( c(x) \) with the linear function \( \hat{c}(x) = a_1 x + b_1 \);
- Let \( x(t) \) be a trajectory of the regulated process starting from 0 until the first claim arrival \( T \). Hence
  \[
  \mu - \hat{c}(x(t)) = x'(t), \quad x(0) = 0,
  \]
  i.e.
  \[
  x(t) = \frac{\mu - b_1}{a_1} - \frac{\mu - b_1}{a_1} e^{-a_1 t};
  \]
- Using the least squares method match to \( v(x) \) the function of the form \( \hat{v}(x) = a_2 x^\alpha + b_2 \);
- Calculate
  \[
  A = \mathbb{E} \left[ e^{-\beta S} \hat{v}(x(S) - T) \right] + \mathbb{E} \left[ \int_0^S e^{-\beta S} U(\hat{c}(x(S))) ds \right],
  \]
  where \( S \overset{D}{=} \text{Exp}(\lambda), \; T \overset{D}{=} \text{Exp}(\xi) \).
- Calculate value \( a - A \);
- Repeat until \( |a - A| < \epsilon \) for fixed \( \epsilon > 0 \).

If we choose \( v_x(0) \) hence also \( v(0) = a \) correctly then observing the regulated process just right after the first jump occurs the left hand side \( A \) of (36) gives true estimator of \( v(0) \). Hence \( A \) will approximate \( a \). In practice we should look for "correct" \( a \) changing \( v_x(0) \) by some small fixed value \( d > 0 \) until \( |a - A| < \epsilon \) for the prescribed precision \( \epsilon \).

We applied above procedure with ten points least square algorithm to the data given below the Figure 2. The results are described in the Figures 3, 4 and the Table 3.
Figure 3: Functions $c(x)$, $\hat{c}(x)$ and trajectory $x(t)$ for $\alpha = 0.5$, $\beta = 0.05$, $\mu = 0.26$, $\xi = 0.4$, $\lambda = 0.1$ and $v_x(0) = 1.9$, $v(0) = 6.8021$.

Figure 4: Functions $v(x)$ and $\hat{v}(x)$ for $\alpha = 0.5$, $\beta = 0.05$, $\mu = 0.26$, $\xi = 0.4$, $\lambda = 0.1$ and $v_x(0) = 1.9$, $v(0) = 6.8021$. 
Table 3: The values of initial conditions obtained from procedure of finding $v_x(0)$.

| $b$      | $a$           | $A$           | $a - A$         |
|----------|---------------|---------------|-----------------|
| 1,9      | 6,802105263   | 6,794392618   | 0,007712645     |
| 1,89     | 6,803336861   | 6,796662198   | 0,006674663     |
| 1,882    | 6,804464186   | 6,798652236   | 0,005811950     |
| 1,8819   | 6,804479085   | 6,798679195   | 0,005799890     |
| 1,88186  | 6,804485051   | 6,798690050   | 0,005795001     |
| 1,881851 | 6,804486392   | 6,798692504   | 0,005793888     |
| 1,8818504| 6,804486482   | 6,798692667   | 0,005793815     |
| 1,88185035| 6,804486489  | 6,798692681   | 0,005793808     |
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