Fractality of Massive Graphs: Scalable Analysis with Sketch-Based Box-Covering Algorithm

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Abstract—Analysis and modeling of networked objects are fundamental pieces of modern data mining. Most real-world networks, from biological to social ones, are known to have common structural properties. These properties allow us to model the growth processes of networks and to develop useful algorithms. One remarkable example is the fractality of networks, which suggests the self-similar organization of global network structure. To determine the fractality of a network, we need to solve the so-called box-covering problem, where preceding algorithms are not feasible for large-scale networks. The lack of an efficient algorithm prevents us from investigating the fractal nature of large-scale networks. To overcome this issue, we propose a new box-covering algorithm based on recently emerging sketching techniques. We theoretically show that it works in near-linear time with a guarantee of solution accuracy. In experiments, we have confirmed that the algorithm enables us to study the fractality of million-scale networks for the first time. We have observed that its outputs are sufficiently accurate and that its time and space requirements are orders of magnitude smaller than those of previous algorithms.

I. INTRODUCTION

Graph representation of real-world systems, such as social relationship, biological reactions, and hyperlink structure, gives us a strong tool to analyze and control these complex objects [21]. For the last two decades, we have witnessed the spark of network science that unveils common structural properties across a variety of real networks. We can exploit these frequently observed properties to model the generation processes of real networked systems [20] and to develop graph algorithms that are applicable to various objects [19].

A notable example of such properties is the scale-free property [3, 7], which manifests a power-law scaling in the vertex degree distribution and existence of well-connected vertices (often called hubs). The scale-free property, existence of hubs especially, underlies efficient performance of practical graph algorithms on realistic networks [1, 17].

Although the scale-free property inspires us to design better network models and algorithms, it is purely based on the local property of networks, i.e., the vertex degree. Real-world networks should possess other common properties beyond the local level. As a remarkable example of such non-local properties, the fractality of complex networks was found in network science [27, 14]. The fractality of a network suggests that the network shows a self-similar structure; if we replace groups of adjacent vertices with supervertices, the resultant network holds a similar structure to the original network (see Section II-C for its formal definition). The fractality of networks gives us unique insights into modeling of growth processes of real-world networks [28]. In addition, fractal and non-fractal networks, even with the same degree distribution, indicate striking differences in facility of spreading [25] and vulnerability against failure [16]. Aside from theoretical studies, the fractality provides us with useful information about network topology. Examples include the backbone structure of networks [15] and the hierarchical organization of functional modules in the Internet [8], metabolic [28] and brain [13] networks, to name a few.

Determination of the fractality of a network is based on the so-called box-covering problem [27] (also see Section II-C). We locally cover a group of adjacent vertices with a box such that all vertices in a box are within a given distance from each other, and then we count the number of boxes we use to cover the whole network. In principle, we have to minimize the number of boxes that cover the network, which is known to be an NP-hard problem (see [26] and references therein). Although different heuristic algorithms are proposed in the previous work (e.g., [26, 24]), they are still not so efficient as to be able to process networks with millions of vertices. This limitation leaves the fractal nature of large-scale networks far from our understanding.

Contributions: The main contribution of the present study is to propose a new type of box-covering algorithm that is much more scalable than previous algorithms. In general, previous algorithms first explicitly instantiate all boxes and then reduce the box cover problem to the famous set cover problem. This approach requires quadratic $Θ(n^2)$ space for representing neighbor sets and is obviously infeasible for large-scale networks with millions of vertices. In contrast, the central idea underlying the proposed method is to solve the problem in the sketch space. That is, we do not explicitly instantiate neighbor sets; instead, we construct and use the bottom-k min-hash sketch representation [9, 10] of boxes.

Technically, we introduce several new concepts and algorithms. First, to make the sketch-based approach feasible, we introduce a slightly relaxed problem called $(1 - ϵ)$-BoxCover. We also define a key subproblem called the
(1 − ε)-SetCover problem. The proposed box-cover algorithm consists of two parts. First, we generate min-hash sketches of all boxes to reduce the (1 − ε)-BoxCover problem to the (1 − ε)-SetCover problem. Our sketch generation algorithm does not require explicit instantiation of actual boxes and is efficient in terms of both time and space. Second, we apply our efficient sketch-space set-cover algorithm to obtain the final result. Our sketch-space set-cover algorithm is based on a greedy approach, but is carefully designed with event-driven data structure operations to achieve near-linear time complexity.

We theoretically guarantee both the scalability and the solution quality of the proposed box-cover algorithm. Specifically, for a given trade-off parameter k and radius parameter r, it works in O((n + m)k log k min{r, log n}) time and O(nk + m) space. The produced result is a solution of (1 − ε)-BoxCover within a factor 1 + 2 ln n of the optimum for BoxCover for ε ≥ 2√5(ln n)/k, with a high probability that asymptotically approaches 1.

In experiments, we have confirmed the practicability of the proposed method. First, we observed that its outputs are quite close to those of previous algorithms and are sufficiently accurate to recognize networks with ground-truth fractality. Second, the time and space requirements are orders of magnitude smaller than those of previous algorithms, resulting in the capability of handling large-scale networks with tens of millions of vertices and edges. Finally, we applied our algorithm to a real-world million-scale network and accomplish its fractality analysis for the first time.

Organization: The remainder of this paper is organized as follows. We describe the definitions and notations in Section II. In Section III, we present our algorithm for sketch-space SetCover. We explain our sketch construction algorithm to complete the proposed method for BoxCover in Section IV. In Section V, we present a few empirical techniques to further improve the proposed method. We explain the experimental evaluation of the proposed method in Section VI. We conclude in Section VII.

II. Preliminaries

A. Notations

We focus on networks that are modeled as undirected unweighted graphs. Let G = (V, E) be a graph, where V and E are the vertex set and edge set, respectively. We use n and m to denote |V| and |E|, respectively. For d ≥ 0 and v ∈ V, we define Nd(v) as the set of vertices with distance at most d from v. We call Nd(v) the d-neighbor. When d = 1, we sometimes omit the subscript, i.e., N(v) = N1(v). We also define Nd(S) for a set S ⊆ V as Nd(S) = ∪v∈SNd(v). In other words, N0(S) represents the set of vertices with distance at most 0 from at least one vertex in S. The notations we will frequently use hereafter are summarized in Table I.

| Notation | Description |
|----------|-------------|
| G = (V, E) | The graph. |
| n, m | The numbers of vertices and edges in G. |
| Nd(v) | The vertices with distance at most d from v. |

B. Bottom-k Min-Hash Sketch

In this subsection, we review the bottom-k min-hash sketch and its cardinality estimator [9], [10]. Let X denote the ground set of items. We first assign a random rank value ri ∼ U(0, 1) to each item i ∈ X, where U(0, 1) is the uniform distribution on (0, 1). Let S be a subset of X. For an integer k ≥ 1, the bottom-k min-hash sketch of S is defined as ˜S, where ˜S = S if |S| < k.

For a set S ⊆ X, the threshold rank τ(S) is defined as follows. If |S| ≥ k, τ(S) = k-th{rj | j ∈ S}. Otherwise, τ(S) = (k − 1)/|S|. Note that τ(S) = τ( ˜S). Using sketch ˜S, we estimate the cardinality |S| as ˜C(S) = (k − 1)/τ(S). Its relative error is theoretically bounded as follows.

Lemma 1 (Bottom-k cardinality estimator [9], [10]): The cardinality estimation ˜C(S) is an unbiased estimator of |S|, and has a coefficient of variation (CV)) of at most 1/√k − 2.

The following corollary can be obtained by applying Chernoff bounds [11].

Corollary 2: For ε > 0 and c > 1, by setting k ≥ (2 + c)ε−2 ln |X|, the probability of the estimation having a relative error larger than ε is at most 1/|X|c.

In addition, our algorithms heavily rely on the mergeability of min-hash sketches. Suppose S1, S2 ⊆ X and S3 = S1 ∪ S2. Then, since S3 ⊆ S1 ∪ S2, S3 can be obtained only from S1 and S2. We denote this procedure as Merge-and-Purify (e.g., S3 = Merge-and-Purify(S1, S2)).

For simplicity, we assume that r1 is unique for i ∈ X, and sometimes identify i with ri. In particular, we use the comparison between elements such as i < j for i, j ∈ X, where we actually compare ri and rj. We also define k-th(S) as the element with the k-th smallest rank in S ⊆ X.

C. Problem Definition

1) Graph Fractality: The fractality of a network [27] is a generalization of the fractality of a geometric object in Euclidean space [12]. A standard way to determine the fractality of a geometric object is to use the so-called box-counting

1The CV is the ratio of the standard deviation to the mean.
method; we tile the object with cubes of a fixed length and
count the number of cubes needed. If the number of cubes
follows a power-law function of the cube length, the object
is said to be fractal. A fractal object holds a self-similar property
so that we observe similar structure in it when we zoom in
and out to it.

The idea of the box counting method is generalized to
analyze the fractality of networks [27]. The box-covering
method for a network works by covering the network with
boxes of finite length $\ell$, which refers to a subset of vertices in
which all vertices are within distance $\ell$. For example, a box
with $\ell = 1$ is a set of nodes all adjacent to each other. If
the number of boxes of length $\ell$ needed to cover the whole
network, denoted by $b(\ell)$, follows a power-law function of $\ell$: $b(\ell) \propto \ell^{-d}$, the network is said to be fractal. The exponent $d$ is
called the fractal dimension. As can be noticed, $b(\ell)$ crucially
depends on how we put the boxes. Theoretically, we have to
put boxes such that $b(\ell)$ is minimized to assess its precise
scaling. However, this box-covering problem is NP-hard and
that is why we propose our new approximation algorithm of
this problem in the rest of this paper.

After computing $b(\ell)$ for a network, we want to decide
whether the network is fractal or not. A typical indicator of
a non-fractal network is an exponential form: $b(\ell) \propto \exp(c\ell)$,
where $c$ is a constant factor [29]. Therefore, comparison
of the fitting of the obtained $b(\ell)$ to power-law and
exponential functions enables us to determine the fractality of
the network. Figure 1 illustrates the comparison for $(3, 3, 7)$-
flower, a network model with ground-truth fractality [23] (see Section VI-A). Since $b(\ell)$ is closer to the power-law than to the
exponential function in this case, the fitting procedure correctly
indicates the fractality of this network model.

It should be noted that the fractality of a network suggests
its self-similarity. Let us aggregate the vertices in a box into a
supervertex and then aggregate the edges spanning two boxes
into a superedge. Then we obtain a coarse-grained version of
the original network. If the original network is fractal, the vertex degree distributions of the original and coarse-
grained networks are (statistically) the same [27]. Note that
the fractality and self-similarity of a scale-free network are
not equivalent, and a non-fractal scale-free network can be
self-similar under certain conditions [18].

2) Box Cover: As we described in the previous section, the
fractality of graphs is analyzed by solving the box covering
problem. The problem has two slightly different versions: the
diameter version [27] and the radius version [26]. It has been
empirically shown that these two versions yield negligible
difference in the results. In this study, we focus on the radius
version, which is defined as follows.

**Problem 1 (BoxCover):** In the BoxCover problem, given
a graph $G$ and a radius limit $\ell > 0$, the objective is to find a
set $S \subseteq V$ of the minimum size such that $N_\ell(S) = V$.

The size of set $S$ is equal to $b(\ell)$ discussed in Section II-C1.
In this study, we consider a slightly relaxed variant of the
BoxCover problem, named $(1 - \epsilon)$-BoxCover. The $(1 - \epsilon)$-
BoxCover problem is defined as follows.

**Problem 2 ((1 - \epsilon)-BoxCover):** In the $(1 - \epsilon)$-BoxCover problem, we are given a graph $G$, a radius limit $\ell > 0$
and an error tolerance parameter $\epsilon > 0$. The objective is to find a
set $S \subseteq V$ of the minimum size such that $|N_\ell(S)| \geq (1 - \epsilon)n$.

3) Set Cover: The BoxCover problem is a special case of
the SetCover problem, which is defined as follows.

**Problem 3 (SetCover):** In the SetCover problem, we are
given a set family $\{S_p\}_{p \in P}$. The objective is to find a set
$R \subseteq P$ of the minimum size such that $\bigcup_{p \in R} S_p = \bigcup_{p \in P} S_p$.

The proposed box-covering algorithm deals with a slightly
different version of SetCover, named $(1 - \epsilon)$-SetCover
with sketched input as a key subproblem, which is defined as
follows.

**Problem 4 ((1 - \epsilon)-SetCover with sketched input):** In the
sketched input version of the $(1 - \epsilon)$-SetCover problem, we
are given the min-hash sketches of a set family $\{S_p\}_{p \in P}$
and an error tolerance parameter $\epsilon > 0$. The objective is to find
a set $R \subseteq P$ of the minimum size such that $|\bigcup_{p \in R} S_p| \geq (1 - \epsilon)|\bigcup_{p \in P} S_p|.$

We first design an efficient approximation algorithm for
$(1 - \epsilon)$-SetCover (Section III). We then propose a new box-
covering algorithm using it (Section IV).

III. SET COVER IN SKETCH SPACE

In this section, we design an efficient approximation algo-
rithm for the sketched-input version of $(1 - \epsilon)$-SetCover
(Problem 4). We call each $p \in P$ a collection and $i \in S_p$ an element. Because of the connection to the BoxCover
problems, we assume that the numbers of collections and
elements are equal. We denote them by $n$, that is, $|P| = |igcup_{p \in P} S_p| = n$. For $R \subseteq P$, we define $S_R = \bigcup_{p \in R} S_p$.
Moreover, for simplicity, we denote $\tilde{C}(S_R)$ by $\tilde{C}(R)$, which
can be calculated from merged min-hash sketch $\tilde{S}_R$.

We first explain the basic greedy algorithm that runs in
$O(n^2 k)$ time, and then present its theoretical solution guar-
antee. Finally, we propose an efficient greedy algorithm, which
runs in $O(nk \log n)$ time and produces the exact same solution
as the basic algorithm.
Let us assume that we are in the main approaches $1$, described as Algorithm 1. We start with an empty set $\mathcal{R}$ and $\mathcal{C}$.

B. Theoretical Solution Guarantee

The complexity analysis of the algorithm is as follows.

Lemma 3: Algorithm Select-Greedily-Naive runs in $O(n^2 k)$ time and $O(nk)$ space.

Proof Sketch. This algorithm always terminates as, even in the worst case, after the $n$-th iteration, $R$ gets $P$. Each iteration takes $O(n k)$ time, and the number of iterations is at most $n$. Therefore, the time complexity is $O(n^2 k)$ time.

B. Theoretical Solution Guarantee

We can guarantee the quality of the solution produced by the above algorithm as follows.

Lemma 4: For $\epsilon \geq 2\sqrt{5(\ln n)/k}$, algorithm Select-Greedily-Naive produces a solution of $(1-\epsilon)$-SetCover within a factor $1 + 2\ln n$ of the optimum for SetCover with a probability of at least $1 - 1/n$.

In other words, with a high probability that asymptotically approaches $1$, $R$ is the solution of $(1-\epsilon)$-SetCover and $|R| \leq (1 + 2\ln n)|R^*|$, where $R$ is the output of algorithm Select-Greedily-Naive and $R^*$ is the optimum solution of SetCover (with the same set family as the input).

Proof. Let $R$ be the output of algorithm Select-Greedily-Naive, and $R^*$ be the optimum solution to SetCover. Let $R_0 = \emptyset$ and $R_i \subseteq P$ be the currently selected sets after the $i$-th iteration of algorithm Select-Greedily-Naive. Let $C(R) = \bigcup_{p \in R} \mathcal{S}_p$. From Corollary 2, for set $R \subseteq P$, the probability of $C(R)$ having a relative error larger than $\epsilon/2$ is at most $1/n^3$.

At the $i$-th iteration, there is some collection $p$ such that

$$n - C(R_i \cup \{p\}) \leq \left(1 - \frac{1}{|R^*|}\right) \left(n - C(R_i)\right),$$

since otherwise there would not be any solution $R^*$ of that size to SetCover. During the $i$-th iteration, there are at most $n$ new sets to be examined, and thus the union bound implies that the relative error between $C$ and $\tilde{C}$ is at most $\epsilon/2$ with a probability of at least $1 - 1/n^2$. Therefore, with that probability,

$$\tilde{C}(R_{i+1}) \geq \tilde{C}(R_i \cup \{p\}) \geq \left(1 - \frac{\epsilon}{2}\right) C(R_i \cup \{p\}).$$

From Inequalities 1 and 2 with some calculation, we have

$$(1 - \frac{\epsilon}{2}) n - \tilde{C}(R_{i+1}) \leq \left(1 - \frac{1}{|R^*|}\right) \left((1 - \frac{\epsilon}{2}) n - C(R_i)\right),$$

with a probability of at least $1 - 1/n^2$. As the number of iterations is at most $n$, by applying the union bound over all iterations, we obtain

$$(1 - \frac{\epsilon}{2}) n - C(R_i) \leq \left(1 - \frac{1}{|R^*|}\right)^i n < e^{-\frac{\epsilon \ln n}{2}},$$

with a probability of at least $1 - 1/n$. If $i$ is at least $2|R^*|\ln n$, it becomes strictly less than $1/n$, which is smaller than the resolution of $C$. Therefore, the number of iterations is at most $|2|R^*|\ln n|$, and thus $|R| \leq |2|R^*|\ln n| \leq (1 + 2\ln n)|R^*|$. Moreover, $C(R) \geq (1 - \epsilon/2)C(R^*) \geq (1 - \epsilon)n$, and thus $R$ is the solution to the $(1-\epsilon)$-SetCover problem.

C. Near-Linear Time Greedy Algorithm

Algorithm Select-Greedily-Naive takes quadratic time, which is unacceptable for large-scale set families. Therefore, we then design an efficient greedy algorithm Select-Greedily-Fast, which produces the exact same output as algorithm Select-Greedily-Naive but runs in $O(nk \log n)$ time. As the input size is $O(nk)$, this algorithm is near-linear time.

The behavior of Select-Greedily-Fast at a high level is the same as that of Select-Greedily-Naive. That is, we start with an empty set $R = \{\}$. and, at each iteration, it adds $p \in P \setminus R$ with the maximum gain on $C$ to $R$. The central idea underlying the speed-up is to classify the state of each $p \in P$ at each iteration into two types and manage differently to reduce the reevaluation of the gain. To this end, we closely look at the relation between sketches $\mathcal{S}_p$ and $\mathcal{S}_R$.

Types and Variables: Let us assume that we are in the main loop of the greedy algorithm. Here, we have a currently incomplete solution $R \subseteq P$. Let $p \in P \setminus R$. We define that $p$ belongs to type $A$ if the $k$-th element of $\mathcal{S}_{R,p}$ is in $\mathcal{S}_p$, i.e., $k$-th$(\mathcal{S}_{R,p}) \in \mathcal{S}_p$. Otherwise, $p$ is type $B$. Note that \(\mathcal{S}_{R,p} = \text{Merge-and-Purify}(\mathcal{S}_R, \mathcal{S}_p)\).

We define $a_p = |\mathcal{S}_{R,p} \cap \mathcal{S}_p|$, $b_p = |\mathcal{S}_{R,p} \cap \mathcal{S}_R|$, and $c_p = |\mathcal{S}_R \cap \mathcal{S}_p|$. Please note that, if $p$ is a type-$A$ collection, $\tilde{C}(R \cup \{p\})$ is determined by $k$-th$(\mathcal{S}_{R,p}) = a_p$-th$(\mathcal{S}_p)$.

Similarly, if $p$ is a type-$B$ collection, $\tilde{C}(R \cup \{p\})$ is determined by $k$-th$(\mathcal{S}_{R,p}) = b_p$-th$(\mathcal{S}_R)$.

\begin{algorithm}
\caption{Select-Greedily-Naive(\(\{\mathcal{S}_p\}_{p \in P}\))}
\begin{algorithmic}
\STATE $R \leftarrow \{\}$, $\mathcal{S}_R \leftarrow \{\}$
\WHILE{$R \neq P$ and $\tilde{C}(R) < (1 - \epsilon/2)n$}
\STATE $p \leftarrow \arg\max\{\tilde{C}(R \cup \{p\}) | p \in P \setminus R\}$
\STATE $R \leftarrow R \cup \{p\}$, $\mathcal{S}_R \leftarrow \text{Merge-and-Purify}(\mathcal{S}_R, \mathcal{S}_p)$
\STATE $\mathcal{S}_R \leftarrow \text{Merge-and-Purify}(\mathcal{S}_R, \mathcal{S}_p)$
\ENDWHILE
\RETURN $R$
\end{algorithmic}
\end{algorithm}
Algorithm 2 Select-Greedily-Fast(\{S_p\}_{p \in P})

1: // Initialization
2: R ← \{\}; S_R ← \{\}
3: Q_A ← an empty min-queue (key: ranks, value: collections)
4: Q_B ← an empty min-queue (key: integers, value: collections)
5: for j = 1, 2, ..., k do
6: T(j) ← a binary search tree (key: ranks, value: collections)
7: I_* ← \{p \in P | i \in S_p\}
8: for all p \in P do
9: Insert (p, k-th(r_i | i \in \bar{S}_p)) to Q_A and T(1)
10: a_p ← b_p ← 0
11: // Main loop
12: while R ≠ \{\} and \bar{C}(R) < (1 - \epsilon/2)n do
13: // Selection
14: p ← argmax\{\bar{C}(R \cup \{p\}) | p is at the top of Q_A or Q_B\}
15: R ← R \cup \{p\}
16: Remove p from Q_A, Q_B and T
17: \bar{S}_R ← Merge-and-Purify(\bar{S}_R, \bar{S}_p)
18: \Delta ← \bar{S}_R \setminus \bar{S}_i, \bar{S}_i ← \bar{S}_R
19: // Notifying events of type 3
20: for all i \in \Delta do
21: for all p \in I_i do
22: if p \in Q_A then
23: Move p from T(b_p) to T(b_p + 1)
24: else if a_p-th(\bar{S}_p) \equiv i then
25: while a_p-th(\bar{S}_p) = i do a_p ← a_p - 1
26: Update p’s key in T(b_p + 1) to a_p-th(\bar{S}_p)
27: Remove p from Q_A and Insert (p, b_p + 1) to Q_B
28: else
29: Move p from T(b_p + 1) to T(b_p + 2)
30: b_p ← b_p + 1
31: // Notifying events of types 1 and 2-1
32: for j = 1, 2, ..., k do
33: P* ← Retrieve those with keys \geq j-th(\bar{S}_R) from T(j)
34: for all p \in P* do
35: r ← j-th(\bar{S}_R)
36: Remove p from Q_A, Q_B and T(j)
37: if p \in Q_A then
38: a_p ← a_p - 1, b_p ← b_p + 1
39: if a_p-th(\bar{S}_p) \in \bar{S}_R then
40: b_p ← b_p - 1, r ← (j-1)-th(\bar{S}_R)
41: while a_p-th(\bar{S}_p) \in \bar{S}_R do a_p ← a_p - 1
42: if a_p-th(\bar{S}_p) > r then
43: Insert (p, a_p-th(\bar{S}_p)) to Q_A and T(b_p + 1)
44: else
45: Insert (p, b_p) to Q_B
46: Insert (p, a_p-th(\bar{S}_p)) to T(b_p)
47: return R

Events to be Captured: Suppose that we have decided to adopt a new collection and R is about to be updated to R’ (i.e., R’ = R \cup \{p\} for some collection p’ \in P). Let us first assume that a single element appeared in the merged sketch, i.e., \bar{S}_R \setminus \bar{S}_p = \{i\}. Let p \in P \setminus R’. In the following, we examine and classify the events where the evaluation of p is updated, i.e., \bar{C}(R \cup \{p\}) ≠ \bar{C}(R \cup \{p\}) (types 1 and 2), or c_p is updated (type 3).

Type 1: We assume that i \not\in \bar{S}_p and p is type A. From the definition, \tau(\bar{S}_R \cup p) = a_p-th(\bar{S}_p), and, b_p-th(\bar{S}_R) \leq a_p-th(\bar{S}_p) < (b_p + 1)-th(\bar{S}_p). Therefore, \tau(\bar{S}_R \cup p) if and only if (b_p+1)-th(\bar{S}_R) \neq (b_p+1)-th(\bar{S}_R) and (b_p+1)-th(\bar{S}_R) < a_p-th(\bar{S}_p). We define that a type-1 event happens to p when this condition holds.

Type 2: Similarly, we assume that i \not\in \bar{S}_p and p is type B. From the definition, \tau(\bar{S}_R \cup p) = b_p-th(\bar{S}_R) and a_p-th(\bar{S}_p) < b_p-th(\bar{S}_R). Thus, \tau(\bar{S}_R \cup p) ≠ \tau(\bar{S}_R \cup p), if and only if b_p-th(\bar{S}_R) ≠ b_p-th(\bar{S}_p). There are two cases: b_p-th(\bar{S}_R) ≤ a_p-th(\bar{S}_p) (type 2-1), after which p becomes type A, or b_p-th(\bar{S}_R) > a_p-th(\bar{S}_p) (type 2-2), after which p still belongs to type B.

Type 3: If i \in \bar{S}_p, then c_p will be incremented.

The following lemma is the key to the efficiency of our algorithm.

Lemma 5: For each p \in P, throughout the algorithm execution, events of type 1, type 2-1, or type 3 occur at most 3k times in total.

Proof Sketch. We use the progress indicator \Phi = k - a_p + b_p + c_p. Initially, a_p = k and b_p = c_p = 0; hence \Phi = 0. For each event occurrence, \Phi increases by at least one. As a_p ≥ 0 and b_p, c_p ≤ k, \Phi ≤ 3k.

Please note that events of type 2-2 are not considered in the above lemma, and, indeed, they happen \Theta(n) times in the worst case for each collection. Therefore, we design the algorithm so that we do not need to capture type-2-2 events.

Finding the Maximum Gain: To adopt a new collection in each iteration, we need to efficiently find the collection that gives the maximum gain. We clarify the ordering relation in each type.

Type A: Let p, q \in P \setminus R be type-A collections, then \bar{C}(R \cup \{p\}) \geq \bar{C}(R \cup \{q\}) if and only if a_p-th(\bar{S}_p) ≤ a_q-th(\bar{S}_q).

Type B: Let p, q \in P \setminus R be type-B collections, then \bar{C}(R \cup \{p\}) \geq \bar{C}(R \cup \{q\}) if and only if b_p-th(\bar{S}_R) ≤ b_q-th(\bar{S}_R), which is equivalent to b_p ≤ b_q.

Data Structures: We use the following data structures to notify the collections about an event occurrence.

Type 1: For each type-A collection p, as we observed above, p wants to be notified about a type-1 event when (b_p + 1)-th(\bar{S}_R) becomes smaller than or equal to a_p-th(\bar{S}_p). Therefore, for j = 1, 2, ..., k, we prepare a binary search tree T(j), where values are collections and keys are ranks (i.e., collections are managed in the ascending order of ranks in each tree). For each type-A collection p, we put p in T(b_p + 1) with key a_p-th(\bar{S}_p). Then, when j-th(\bar{S}_R) is updated to a new value, from T(j), we retrieve collections with keys larger than or equal to the new value and notify them about an event.

Type 2-1: Similarly, for each type-B collection p, p wants to be notified about a type-2-1 event when b_p-th(\bar{S}_R) becomes smaller than or equal to a_p-th(\bar{S}_p). Thus, we store p in T(b_p) and set its key to a_p-th(\bar{S}_p). Then, when j-th(\bar{S}_R) is updated to a new value, we retrieve those in T(j) with keys larger than or equal to the new value and notify them about an event.
To capture type-3 events, the use of an inverted index suffices. That is, for each \( i \in X \), we precompute \( I_i = \{ p \in P \mid i \in S_p \} \). When \( i \) comes to \( S_R \), we notify the collections in \( I_i \).

Moreover, we also need data structures to find the collections with the maximum gain as follows.

**Type A:** Type-A collections are managed in a minimum-oriented priority queue, where the key of a collection \( p \) is \( a_p \cdot \theta(S_p) \).

**Type B:** Type-B collections are managed in another minimum-oriented priority queue, where the key of a collection \( p \) is \( b_p \).

**Overall Set-Cover Algorithm:** The overall algorithm of Select-Greedily-Fast is described as Algorithm 2. In each iteration, we adopt the new collection with maximum gain, which can be identified by comparing the top elements of the two priority queues. Then, we process events to update variables and data structures. At the beginning of Section III-C, we assumed that a single element appears in the new sketch. When more than one elements come to the new sketch, we basically process each of them separately. See Algorithm 2 for the details of the update procedure. The algorithm complexity and solution quality are guaranteed as follows.

**Lemma 6:** Algorithm Select-Greedily-Fast runs in \( O(nk \log n) \) time and \( O(nk) \) space.

**Proof Sketch.** Each data structure operation takes \( O(\log n) \) time, which, from Lemma 5, happens at most 3k times for each collection.

**Lemma 7:** Algorithm Select-Greedily-Fast produces the same solution as algorithm Select-Greedily-Naive.

**Proof Sketch.** Both algorithms choose the collection with the maximum gain in each iteration.

IV. Sketch-Based Box Covering

In this section, we complete our sketch-based box-covering algorithm for the \((1 - \epsilon)\)-BoxCover problem (Problem 2). We first propose an efficient algorithm to construct min-hash sketches representing the \( \ell \)-neighbors, and then present and analyze the overall box-covering algorithm.

**A. Sketch Generation**

For \( v \in V \), we denote the min-hash sketch of \( N_{\ell}(v) \) as \( \tilde{N}_{\ell}(v) \). Here, we construct \( \tilde{N}_{\ell}(v) \) for all vertices \( v \in V \) to reduce the \((1 - \epsilon)\)-BoxCover problem to the \((1 - \epsilon)\)-SetCover problem (Problem 3). Our sketch construction algorithm Build-Sketches is described as Algorithm 3.

It receives a graph \( G \) and a radius parameter \( \ell \). Each vertex \( v \) manages a tentative min-hash sketch \( X_v \). Initially, \( X_v \) only includes the vertex itself, i.e., \( X_v = \{ v \} \), which corresponds to \( \tilde{N}_0(v) \). Then, we repeat the following procedure for \( \ell \) times so that, after the \( \ell \)-th iteration, \( X_v = N_{\ell}(v) \). This algorithm has a similar flavor to algorithms for approximated neighborhood functions and all-distances sketches [22], [5], [10].

**Algorithm 3 Build-Sketches(\( G, \ell \))**

1. \( X_v \leftarrow \{ v \} \) for all \( v \in V \).
2. for \( \ell \) times do
3. for all \( v \in V \) in the increasing order of \( r_v \) do
4. \( A_v \leftarrow \{ u \in V \mid v \) is added to \( X_u \) in the last iteration \}
5. for all \( w \in N(A_v) \) do
6. \( X_w \leftarrow \text{Merge-and-Purify}(X_w, \{ v \}) \)
7. if \( X_v \) was not modified for any \( v \in V \) then
8. break
9. return \( \{ X_v \}_{v \in V} \)

In each iteration, for each vertex, we essentially merge the sketches of its neighbors into its sketch in a message-passing-like manner. Two speed-up techniques are employed here to avoid an unnecessary insertion check. For \( v \in V \), let \( A_v \) be the vertices in whose sketches \( v \) is added to in the last iteration. First, for each \( v \in V \), we try to insert \( v \) only into the sketches of the vertices that are neighbors of \( A_v \), as \( v \) cannot be inserted into other vertices. Second, we conduct the procedure above in the increasing order of ranks, since this decreases the unnecessary insertion. We prove its correctness and complexity as follows.

**Lemma 8:** In algorithm Build-Sketches, after the \( i \)-th iteration, \( X_v = \tilde{N}_{i}(v) \) for all \( v \in V \).

**Proof Sketch.** We prove the lemma by mathematical induction on \( i \). Since \( \{ v \} = \tilde{N}_0(v) \), it is true for \( i = 0 \). Now we assume it holds for \( i \) and prove it also holds for \( i + 1 \). Let \( B = \{ u \in V \mid (v, u) \in E \} \). Since \( N_{i+1}(v) = \{ v \} \cup \bigcup_{u \in B} N_i(u) \), and \( \{ v \} \subseteq N_{i}(v) \subseteq N_{i+1}(v) \), \( \tilde{N}_{i+1}(v) \) can be obtained by merging \( N_i(u) \) for all \( u \in B \cup \{ v \} \).

**Corollary 9:** Algorithm Build-Sketches computes \( \tilde{N}_{i}(v) \) for all \( v \in V \).

**Lemma 10:** Algorithm Build-Sketches runs in \( O((n + m)k \log k \min \{ \ell, \log n \}) \) expected time and \( O(nk + m) \) space.

**Proof Sketch.** In addition to the graph, the algorithm stores a sketch of size \( k \) for each vertex, and hence it works in \( O(nk + m) \) space. Each insertion trial takes \( O(\log k) \) time (Line 6). Therefore, it suffices to prove that the number of traversed edges is \( O((n + m)k \ell) \) and \( O((n + m)k \log n) \). The former bound is easier, since, in each iteration, the number of last inserted elements in each sketch is at most \( k \), and thus we traverse each edge at most \( k \) times.

For the latter bound, we count the expected number of vertices that are inserted once into \( X_v \) for a vertex \( v \in V \). The vertex that is \( i \)-th to arrive at \( v \) is inserted into \( X_v \) with a probability of \( \min \{ 1, k/i \} \), and thus it is at most

\[
\sum_{i=1}^{n} \min \left\{ \frac{k}{i}, 1 \right\} = k + k(H(n) - H(k)) = O(k \log n),
\]

where \( H(i) \) is the \( i \)-th Harmonic number. Therefore, each edge is traversed at most \( O(k \log n) \) times in total.
B. Overall Box-Cover Algorithm

The overall box-covering algorithm Sketch-Box-Cover is as follows. We first construct the min-hash sketches using algorithm Build-Sketches and then solve the set cover problem in the sketch space using algorithm Select-Greedily-Fast. The guarantees on performance and accuracy are immediate from the previous lemmas and corollaries as below.

Theorem 11 (Scalability guarantee): Algorithm Sketch-Box-Cover works in $O((n + m)k \log k \min \{\ell, \log n\})$ time and $O(nk\ell) + m$ space.

Theorem 12 (Solution accuracy guarantee): With a probability of at least $1 - 1/n$, for $\epsilon \geq 2\sqrt{5(\ln n)/k}$, algorithm Sketch-Box-Cover produces a solution to the $(1 - \epsilon)$-BoxCover problem within a factor $1 + 2\ln n$ of the optimum for the BoxCover problem.

Assuming $k$ is a constant, the time and space complexities are near-linear. Similarly, given a constant $\epsilon$, the time and space complexities are still near-linear, since it suffices to set $k = \lceil 20\epsilon^{-2}\ln n \rceil$. In practice, as seen in our experiments, the algorithm produces solutions that are much closer to the optimum than what is expected from the above approximation ratio with much smaller $k$.

V. PRACTICAL IMPROVEMENT

In this section, we propose techniques to improve the practicality of the proposed method.

Exact Coverage Management: For the termination condition in the greedy selection algorithm (i.e., Line 12 in Algorithm 2), when applied to the box cover problem, we propose to use the exact coverage $C(R)$ instead of the estimated coverage $\hat{C}(R)$. This technique makes the results more stable. We can efficiently manage the exact coverage as follows.

First, we prepare an array $\delta$, and initialize it as $\delta[v] = \infty$ for all $v \in V$. After selecting a vertex $v$ in each iteration, we conduct a pruned breadth-first search (BFS) from $v$. Suppose we are visiting vertex $u$ with distance $d$ in this BFS. If $\delta[u] \leq d$, then we prune this BFS, i.e., we do not traverse the edges from $u$. Otherwise, we set $\delta[u] = d$ and continue the search. We do not visit vertices with a distance larger than $\ell$.

The number of covered vertices is the number of non-infinity values in array $\delta$. Since the value of $\delta[u]$ changes at most $\ell + 1$ times, each vertex or edge is visited $O(\ell)$ times. Therefore, the total time consumption of this process throughout all iterations is $O((n + m)\ell)$.

Multi-Pass Execution: On the basis of the above exact coverage management technique, we sometimes detect that, even while the estimated coverage is saturated (i.e., $\hat{C}(R) = \hat{C}(P)$), the actual coverage is below the specified threshold. In that case, to choose more vertices, we propose to repeat the algorithm from sketch construction until the actual coverage becomes higher than the threshold.

In the $i$-th pass, we only care for vertices that are not covered by the previous passes. This can be easily realized by modifying the algorithm Build-Sketches so that, at Line 1, we set $X_v = \emptyset$ for already covered vertices. For accurate results, node ranks should be reassigned for each pass.

Exact Neighborhood: To further improve the accuracy, we propose to combine our sketch-based algorithm with a non-sketch-based algorithm. For a very small radius parameter $\ell$, neighborhood $N_{\ell}(v)$ is sometimes much smaller than $k$. Moreover, even for a larger $\ell$, when using the above multi-pass execution technique, the remaining neighbors may become small in later passes. In these cases, the sketching approach has little advantage. Therefore, we detect such circumstances and switch to a non-sketch-based greedy algorithm. Interestingly, this switching can be done seamlessly. If $|N_{\ell}(v)| \leq k$, then $N_{\ell}(v) = N_i(v)$. Therefore, under such circumstances, the output of algorithm Build-Sketches can be immediately given to the non-sketch-based greedy algorithm.

The proposed overall procedure is as follows. We specify a parameter $\alpha$. We start by constructing the “sketches” with algorithm Build-Sketches, but, at first, we apply the algorithm as if $k = \infty$, i.e., we do not conduct purification on the min-hash sketches. During the construction, if the total number of elements in all min-hash sketches exceeds $\alpha nk$ at some point, then we conduct purification on all the min-hash sketches, continue the construction with the actual $k$ value, and pass the resulting sketches to the sketch-based greedy algorithm. Otherwise, we apply the non-sketch-based greedy algorithm to the resulting “sketches,” which are actually exact neighborhood sets. Assuming parameter $\alpha$ is a constant, the total time and space complexity remain the same.

Exact Box Covering: Together with the preceding three techniques, to further make the results reliable, we propose to use our algorithm for solving the original BoxCover problem (Problem 1) rather than $(1 - \epsilon)$-BoxCover problem (Problem 2). In other words, we recommend setting $\epsilon = 0$ to ensure that all vertices are completely covered. As we will see in the experiments, even with this seemingly extreme threshold, thanks to the above techniques, both the running time and the solution quality are reasonable.

VI. EXPERIMENTS

In this section, we present our experimental results to verify the performance of the algorithm. Specifically, we compare our algorithm with other preceding algorithms in terms of accuracy and computation time.

We mainly focus on model networks, instead of empirical ones, in order to validate the results of our algorithm with ground-truth theoretical solutions and to investigate the scalability of the algorithm for various sizes of networks. However, we also demonstrate the practicalness of our algorithm by applying it to a real million-scale web graph. On the basis of the result, we reveal the fractality of such large-scale real graph for the first time.

A. Setup

Environment: Experiments were conducted on a Linux server with Intel Xeon X5650 (2.67 GHz) and 96GB of main
memory. Algorithms were implemented in C++ and compiled by gcc 4.8.4 with -O3 option.

**Algorithms:** For comparison, we used a naive algorithm named greedy coloring (GC) and three advanced and popular algorithms, named maximum excluded mass burning (MEMB), minimal value burning (MVb), and compact box burning (CBB). GC, MEMB, and CBB were introduced in [26] and MVb was in [24].

**Network Models:** We used two network models with ground-truth fractality: the $(u,v,g)$-flower [23] and the Song-Havlin-Makse (SHM) [28] model. These models have power-law degree distributions, the representative characteristic of complex networks. Both models can be either fractal or non-fractal, depending on the structural parameter values. We refer to them as the $(u,v,g)$-flower and $(c,e,g)$-SHM model to indicate the parameter settings. The common parameter $g = (1, 2, 3, \ldots)$ determines the network size $n$: $n = (w − 2/w − 1)^w + w/w − 1$, where $w \equiv u + v$ for the $(u,v,g)$-flower, and $n = (2c + 1)^g n_0$ for the $(c,e,g)$-SHM model. In addition to the flower and SHM models, we considered the Barabási-Albert (BA) network model [3] as one of the most famous models of complex networks. The BA model is not fractal [27]. We refer to this model as $(c,t)$-BA, where $c$ is the number of edges that a new node has and $t$ sets the network size as $n = 125 \times 2^t$.

**Fractality Decision Procedure:** After the computation of the box-covering algorithms, we determined whether the obtained $b(\ell)$ indicates the fractality or not. This task was done by fitting the $b(\ell)$ curve with a power-law function (i.e., fractal) and an exponential function (i.e., non-fractal) by using optimize.leastsq function in SciPy package of Python. We used the parameters estimated by fitting the curves to linearized models as the initial values for the nonlinear fitting. The key quantity was the ratio between the residual error of fitting to a power-law function and that to an exponential function, denoted by $r_{fit}$. If $-\log_{10} r_{fit}$ is positive (i.e., $r_{fit} < 1$), the network was supposed to be fractal. Otherwise, it was supposed to be non-fractal. This procedure of fitting and comparison follows that used in [29].

**B. Parameter Settings**

First of all, we have to decide the parameter values of our algorithm: $\epsilon$ (error tolerance), $k$ (sketch size), and $\alpha$ (exact neighborhood switch threshold). In principle, the accuracy of results as well as running time increases with $k$ and $\alpha$, and it decreases with $\epsilon$. As we discussed in Section V we fixed $\epsilon = 0$. To choose $k$ and $\alpha$, we plotted the average approximation ratio of our results to the theoretical solutions for several fractal network models as a function of $k$ and $\alpha$ in Figure 2. The average approximation ratio is defined by $\rho \equiv \langle b_{\text{sketch}}(\ell) / b_{\text{theory}}(\ell) \rangle_t$, where $\langle \cdot \rangle_t$ is the average over $\ell$. To compute $b_{\text{sketch}}(\ell)$, we executed the algorithm for ten times and took the average of the resulting $b(\ell)$ over the ten runs.

In the left panel of Figure 2 we varied $2^4 \leq k \leq 2^{10}$ while fixing $\alpha = 1$. The $\rho$ values were affected slightly by $k$ for the SHM models and tended to decrease with $k$ for the flower network. On the basis of the results, we decided to use $k = 2^7$ throughout the following experiments. In the right panel of Figure 2 we varied $2^{-3} \leq \alpha \leq 2^3$ while fixing $k = 2^7$. The $\rho$ values were almost constant regardless of the $\alpha$ values for all of the three networks considered. Therefore, taking into account the running time, we decided to use $\alpha = 1$ throughout the following experiments. It is worth noting that Figure 2 also demonstrates the high accuracy and robustness of our algorithm over a broad range of parameter values.

**C. Accuracy and Scalability**

Table II summarizes the main results of this paper and shows the comparison of our algorithm (Sketch) with other preceding algorithms for fractal and non-fractal network models with various sizes. We evaluated the performance of algorithms by two measures. The first was the accuracy given by $-\log_{10} r_{fit}$ (Section VI-A). If this measure took a positive (negative) value for a fractal (non-fractal) network, the algorithm correctly distinguished the fractality of the network. The second was computation time in seconds.

**Discrimination Ability:** As we can see in Table II, the sketch algorithm perfectly distinguishes between the fractal and non-fractal networks as the other algorithms do (except for CBB for $(3,4,5)$-flower). The proposed algorithm shows its advantage in computation time: the algorithm is generally faster than both the previous algorithms and is able to handle large networks that other algorithms do not terminate. Although MEMB is faster than Sketch for some relatively small network models, this result is expected because actual neighborhood sets are not significantly larger than sketch sizes in these networks. As a summary, (i) the sketch algorithm correctly detected the fractality of network models with around ten times smaller computation time than the fastest previous algorithm. In addition, (ii) the algorithm was able to deal with networks with millions of nodes with acceptable computation time (within 1 day), whereas other algorithms could not in our machine environment.

**Time and Memory Consumption:** The proposed algorithm is scalable for not only for computation time but also for memory usage. In Figure 3 computation time (seconds) and memory usage (KB) of the five algorithms were plotted as a function of the number of vertices. We use $(2,2,g)$-flower $(3 \leq g \leq 11)$ and $(2,t)$-BA $(0 \leq t \leq 15)$ networks as the example of a fractal and a non-fractal network, respectively.
### Table II
Running time in seconds (Time) and the relative error ratio of a power-law function, $-\log_{10} \alpha_{fit}$ (Fit). DNF means that it did not finish in one day or ran out of memory.

| Model | Graph | $|V|$ | $|E|$ | Sketch | MEMB | GC | MVB | CBB |
|-------|-------|------|------|--------|------|----|-----|-----|
|       |       |      |      | Time | Time | Time | Time | Time |
| (2, 2, 4)-flower | 172 256 | 0.8 | 0.8 | 1.0 | 28.7 | 199 1.0 | 0 28.0 |
| (2, 2, 7)-flower | 10,924 16,384 | 15 2.5 | 10 3.4 | 228 27.7 | DNF | DNF | 122 27.4 |
| (2, 10)-flower | 699,052 1,048,576 | 8,628 3.5 | DNF | DNF | DNF | DNF | — |
| (2, 11)-flower | 2,796,204 4,194,304 | 62,138 4.0 | DNF | DNF | DNF | DNF | — |
| (2, 3, 6)-flower | 11,720 15,625 | 2 1.2 | 14 1.1 | 146 0.1 | DNF | DNF | 5,593 0.5 |
| (2, 3, 7)-flower | 58,595 78,125 | 286 1.1 | 377 1.0 | 8,538 0.1 | DNF | DNF | — |
| (2, 3, 8)-flower | 292,970 390,625 | 2,913 1.0 | DNF | DNF | DNF | DNF | — |
| (2, 4, 6)-flower | 37,326 46,666 | 138 1.1 | 121 1.0 | 2,422 1.7 | DNF | 2,481 0.8 |
| (2, 4, 7)-flower | 223,950 279,936 | 1,526 1.0 | DNF | DNF | DNF | DNF | — |
| (3, 3, 6)-flower | 37,326 46,666 | 148 1.1 | 101 1.3 | 10,751 2.1 | DNF | DNF | 1,284 1.4 |
| (3, 3, 7)-flower | 223,950 279,936 | 1,779 1.2 | DNF | DNF | DNF | DNF | 61,562 1.5 |
| (3, 4, 5)-flower | 14,007 16,807 | 34 0.7 | 16 0.9 | 560 0.1 | DNF | DNF | 3,380 0.4 |
| (3, 4, 7)-flower | 686,287 823,543 | 8,873 0.8 | DNF | DNF | DNF | DNF | — |
| (2, 0, 6)-SHM | 12,501 12,500 | 33 1.2 | 8 1.1 | 872 1.1 | 32 1.1 | 325 0.7 |
| (2, 0, 7)-SHM | 62,501 62,500 | 224 1.2 | 206 1.1 | 48,116 1.1 | 1,126 1.1 | 6,579 0.9 |
| (2, 0, 8)-SHM | 312,501 312,500 | 2,728 1.1 | DNF | DNF | DNF | DNF | — |
| (3, 0, 6)-SHM | 67,229 67,228 | 207 1.0 | 190 0.9 | 21,726 0.9 | 628 0.9 | 4,623 0.9 |

### Networks with ground-truth fractality ("Fit" values are expected to be positive.)

### Networks with ground-truth non-fractality ("Fit" values are expected to be negative.)

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The symbols corresponding to an algorithm were not shown if the algorithm did not stop within 24 hours or could not execute owing to memory shortage. The performance of the proposed algorithm is comparable to or worse than some other algorithms when the network is relatively small (i.e., $n < 10^4$). However, the algorithm is orders of magnitude faster than other algorithms for large networks. Also, it achieves such high a speed with incomparably smaller memory usage than MEMB, the second fastest algorithm.

**Robustness over Randomness:** The sketch algorithm accurately recovers $b(\ell)$ of theoretical prediction for fractal network graphs.

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Fig. 3. Scalability of computation time (top) and memory usage (bottom) for (2,2,4)-flower (left) and (2,7)-BA (right) networks.

Fig. 4. Results of different runs for (3,3,7)-flower. (Left) $b(\ell)$ and (Right) $CV$ of them as a function of $\ell$. 
models, and the results are robust over different execution runs. The left panel of Figure 4 shows \(b(\ell)\) of ten different runs of the proposed algorithm on \((3,3,7)-\text{flower}\). The \(b(\ell)\) values follow well the theoretical solution, which is indicated by the solid line. As we can clearly observe, the fluctuation in the \(b(\ell)\) values due to the randomness is very small. The consistency over different runs is captured by the CV of \(b(\ell)\) (i.e., the ratio of the standard deviation of \(b(\ell)\) to its average over ten runs) as a function of \(\ell\) (right panel of Figure 4). The CV values tend to increase with \(\ell\). This tendency can be explained by the following two factors. First, the \(b(\ell)\) value takes a positive integer value and monotonically decreases with \(\ell\) by definition. Thus, even a change of \(\pm 1\) in \(b(\ell)\) might cause a large CV value if \(\ell\) is large. Second, our algorithm intrinsically fluctuates more when \(\ell\) is larger. This could be because the sizes of the solutions become smaller for larger \(\ell\), and hence the algorithm gets a little more sensitive to estimation errors. Nevertheless, it should be noted that the variance of our algorithm was considerably small even for large \(\ell\) (i.e., \(CV \sim 0.19\) at most). This magnitude of variance would have little impact on the estimation of fractality.

D. Application to Real Large Network

In closing this section, we applied the sketch algorithm to a large-scale real graph to show the scalability of the proposed algorithm with an empirical instance. The results also gave us some insight on the fractality of large-scale real-world networks, which is beyond the reach of previous algorithms. As a representative instance of a real-world large graph, we considered the in-2004 network [6], [4], which is a crawled web graph of 1,382,908 vertices and 16,917,053 edges. We discarded the direction of the edges (i.e., hyperlinks) to make the network undirected. The algorithm took 11.7 hours in total.

The resulting \(b(\ell)\) of the sketch algorithm and the fitting curves are shown in Figure 5. We omitted the three points with the smallest \(\ell\) values from the fitting because empirical networks would not show a perfect fractality, contrary to well-designed network models. A large part of the points fall on the line of the fitted power-law function and, indeed, our fractality decision procedure yielded \(-\log_{10} P_{\text{fit}} = 0.79\), which suggests the fractality of the in-2004 network. It is worth mentioning that the fractality of this network was unveiled for the first time for the sake of our algorithm.

VII. CONCLUSIONS

Fractality is an interesting property that appears in some classes of real networks. In the present study, we designed a new box-covering algorithm, which is useful for analyzing the fractality of large-scale networks. In theory, we have shown desirable guarantees on scalability and solution quality. In the experiments, we confirmed that the algorithm’s outputs are sufficiently accurate and that it can handle large networks with millions of vertices and edges. We hope that our method enables further exploration of graph fractality and its applications such as graph coarsening.

Repeatability: Our implementation of the proposed and previous box-cover algorithms is available at http://git.io/fractality. It also contains the generators of the synthetic network models, and thus the results in this paper can be perfectly replicated. We hope that our public software will enable further exploration of graph fractality and its applications.

Acknowledgment: This work was supported by JSPS KAKENHI (No. 15H06828), JST, ERATO, Kawarabayashi Large Graph Project, and JST, PRESTO. Web graph data was downloaded from http://law.di.unimi.it/datasets.php T.T. thanks to K. Takemoto for valuable discussions.

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