Orbital Entanglement Entropy of Short Range Correlated Pairs in Nuclear Structure

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We calculate the nuclear structure entanglement entropy of short range correlations (SRC) based on the nuclear scale separation. Using the decoupling of low and high-momenta which was established by the similarity normalization group we obtain a simple general expression of the entanglement entropy in terms of SRC by employing the generalized contact formalism. Comparing our analytical calculation for the SRC entanglement entropy for \(^4\)He to previously obtained numerical calculations for the single orbital entanglement entropy we demonstrate that most of entanglement entropy can possibly be attributed to the SRC entanglement entropy. Obtaining a general formula for the SRC entanglement entropy of a nuclear structure in terms of the nuclear contact, allows us to obtain the scaling of the entropy in terms the mass number, \(A\). We find that the entanglement entropy associated with the SRC in large nuclei is linearly dependent on \(A\), thus the SRC nuclear entanglement entropy is extensive.

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1. INTRODUCTION

Entanglement entropy, an entropy arising not from a lack of knowledge, due to thermal fluctuations, of the micro-state the system is in, but is rather an entropy arising due to the entanglement between sub-parts of the system. It arises when one chooses to integrate over part of the system. As such, this entropy also exists for the ground state. The entropy of the reduced state of a sub-region tends to grow like the boundary area of that sub-region, often with a small, typically logarithmic correction, rather than extensively with the volume [1]. The scaling of the entanglement with the area of a region is referred to as an area law and holds insights with regard to the distribution of quantum correlations in many-body systems and thus offers a measure of the complexity of the system. Since entanglement is an important resource for: quantum computation, sensing, as well as for communication, there is a growing interest in calculating the entanglement entropy of different many-body systems. One obtains area laws for one and higher-dimensional lattices, for fermionic or spin degrees of freedom [2]. The entanglement entropy has been calculated for electronic orbitals [3] and area laws have been obtained for condensed matter systems [4]. The entanglement entropy is also a good measure for identifying correlations in many-body systems and as such it was also used as a tool for identifying topological order [5].

In addition to the theoretical interest in trying to understand the quantum structure of the nucleus, calculating the entanglement entropy of nuclear structure has practical implications because it can play a role as an organizational principle. Entanglement entropy measures the distribution of wave-function coefficients, and thus it acts as a measure for the viability of truncation of a computational model space. As such it has many implications regarding the reduction of the exponentially large computational basis [6] where knowledge of the entanglement entropy allows one to require less computational resources while retaining similar accuracy. More specifically, Ref. [7] demonstrates how employing a quantum entropy based optimization procedure can effectively reduce the number of block states needed to provide an accurate calculation for the ground and first excited state of $^{28}$Si. Recently it has also been shown how nuclear structure can be investigated on a quantum computer where the evaluation of the entanglement entropy plays an important role in assessing the required quantum resources in the number of qubits and circuit depths [8].

The calculation of the entanglement entropy for nuclear structures has commenced with an initial study of the single-orbital entropy and two-orbital mutual information performed for $^{28}$Si, $^{56}$Ni and $^{64}$Ge employing the density matrix renormalization group (DMRG) [7]. It continued, rather recently, with the calculation for the state of two interacting particles [9] and shortly after with a detailed numerical calculation for the entanglement entropy of $^{4}$He and $^{6}$He [10]. Mode entanglement has also been investigated under the framework of the Lipkin model [11]. In this work instead of performing a calculation of the entanglement entropy of a specific nucleus, as was previously done, we obtain a general formula for the entanglement entropy in terms of the high momentum states. Whereas typically entanglement is used as a diagnostic tool for defining quantum correlations [12], here we reverse this role and use correlations as a basis to evaluate entropy entanglement. This sort of approach allows us also to look at the spatial scaling of the entanglement entropy identifying possible area laws.

Calculating the nuclear structure is highly complex. Due to the repulsive-core of nucleon-nucleon interactions, combined with the large numbers of many-body states, it was initially considered to be unachievable in a reliable way. It should be noted that it is primarily the entanglement of the many-body nuclear wave function that complicates the calculation since the system can not be well approximated by separable states. The status of nuclear structure calculations has greatly advanced since the 1970’s through the use of modern field theoretic techniques. A major tool in enabling the study of nuclear structure is the Similarity Renormalization Group (SRG) which helps resolve the following apparent contradiction [13, 14]. On the one hand, short range correlations (SRC) have been experimentally observed [15–22] in which one observes components in the high resolution nuclear wave function with relative pair momenta greater than the Fermi momentum [23–29]. On the other hand, the shell model for the nucleus, which has been highly successful for calculating nuclear properties [30] seems to include no such short range structure thus seemingly contradicting the SRC description. Through SRG it was recently demonstrated [13, 14] that these two viable seemingly contradicting descriptions for the nucleus, the Fermi sea (FS) like independent particle model and SRC can in fact be connected. This sort of connection has also been previously studied and demonstrated for the $^{4}$He case [31]. Employing a SRG transformation specifically enables one to obtain a factorization of the long from short-distance physics, providing a clean scale separation. The SRC are manifested when considering high RG resolution. Whereas in the low-RG resolution the features of SRC phenomenology are clearly identified using simple two-body operators and local-density approximations with uncorrelated wave functions reconciling the contrasting pictures. One can thus naively view SRC as the two-body extension to the single independent particle form of the shell model. In this naive view the physics of the nucleus is encapsulated in one plus correlated two-body model which are scale separated. This separation of scales is extremely effective for the calculation of the entanglement entropy of the nucleus. In the present work we exemplify how the entanglement entropy is mainly given in terms of SRC. To do so, we consider only the SRC composed of neutron-proton pairs which are in a $s$-wave configuration. It is these sorts of pairs which have been found experimentally to constitute the vast majority of SRC. In only considering such pairs we in fact neglect the contribution of neutron-neutron and
proton-proton pairs to the entanglement entropy. However this contribution has been shown to be small. In fact in the electron scattering experiments kinematically designed to measure SRC a typical ratio of proton-neutron SRC pairs to proton-proton SRC pairs was found to be around 20 [17, 19, 22–27].

We start by a brief reminder of the orbital entanglement entropy given in Sec. II. In Sec. III the generalized contact formalism (GCF) is succinctly introduced and is used for calculating the nuclear SRC orbital entanglement in terms of the nuclear contact. In Sec. IV we compare our results with previously obtained results from the literature for the entanglement entropy of $^4\text{He}$ [10]. In Sec. V we approximate the entanglement entropy of the low-momentum states in terms of a general expression for the entanglement entropy of a FS which is shown to be an insufficient approximation. The results for the entanglement entropy for the SRC of the nuclear structure are summarized and future research directions are discussed in Sec. VI.

II. ORBITAL ENTANGLEMENT ENTROPY

The entanglement entropy measures the correlation between two parts of a bipartite system [32, 33]. In the case of distinguishable particles, the notion of the entanglement is based on the structure of the tensor product for the Hilbert spaces of the subsystems. However, trying to define a similar concept for identical particles is not straightforward, since the decomposition into particle subsystems does not correspond to a tensor product structure of the Fock space. In order to address this issue, the notion of the mode (orbital) entanglement was introduced [34–36].

One can define the eigenstates of the nucleus, $|\Psi\rangle$, by choosing a single particle basis defined by the quantum numbers $\{i\} = \{n_i, J_i, l_i, m_i, s_i, \tau_i\}$, which correspond to the principal quantum number, the total angular momentum, the orbital angular momentum, and the spin, isospin projections. The nucleus eigenstates can then be written as a linear combination of Slater determinants $|\phi_\alpha\rangle$ for the nucleon wave functions

$$|\Psi\rangle = \sum_\eta A_\eta |\phi_\eta\rangle,$$

where the Slater determinant is given in terms of applying creation operators on the true particle vacuum $|0\rangle$

$$|\phi_\eta\rangle = \prod_{i\in\eta} a_i|0\rangle,$$

where $A$ is the number of nucleons in the nucleus.

Employing this notation, the single orbital entanglement entropy $S_\gamma(1)$ is given in terms of the one-orbital density matrix [10]

$$\rho^{(i)} = \begin{pmatrix} 1 - \gamma_{ii} & 0 \\ 0 & \gamma_{ii} \end{pmatrix},$$

where the occupation of orbital, $i$, is given by $\gamma_{ii} = \langle \Psi | a_i^\dagger a_i | \Psi \rangle$.

To obtain the entanglement entropy essentially one needs to perform a calculation of the von Neumann entropy of the partial density matrix, Eq. (3)

$$S_\gamma^{(1)} = -Tr[\rho^{(i)} \ln \rho^{(i)}] = -\sum_{k=1}^2 \omega_k^{(i)} \ln \omega_k^{(i)},$$

where $\omega_k$ are the eigenvalues of $\rho^{(i)}$

For the sake of completeness we also present the two-orbital entanglement entropy which is similarly given by

$$S_{\gamma ij}^{(2)} = -Tr[\rho^{(ij)} \ln \rho^{(ij)}] = -\sum_{k=1}^4 \eta_k^{(ij)} \ln \eta_k^{(ij)},$$

where $\rho^{(ij)}$ is the following $4 \times 4$ matrix with eigenvalues $\eta_k^{(ij)}$ [10]

$$\rho^{(ij)} = \begin{pmatrix} 1 - \gamma_{ii} - \gamma_{jj} - \gamma_{ijij} & 0 & 0 & 0 \\ 0 & \gamma_{jj} - \gamma_{ijij} & \gamma_{ij} & 0 \\ 0 & \gamma_{ij} & \gamma_{jj} - \gamma_{ijij} & 0 \\ 0 & 0 & 0 & \gamma_{ijij} \end{pmatrix},$$

defined in terms of $\gamma_{ij} = \langle \Psi | a_i^\dagger a_i | \Psi \rangle$ and $\gamma_{ijij} = \langle \Psi | a_i^\dagger a_j^\dagger a_j a_i | \Psi \rangle$. 
III. ENTANGLEMENT ENTROPY IN TERMS OF THE NUCLEAR CONTACT

Having introduced the orbital entanglement entropy, we proceed to demonstrate how the entanglement entropy for the high-momentum states part of the nuclear wave function is defined by the entanglement entropy of the SRC. We show specifically that the splitting of the wave function into two parts, which is the basis of GCF, provides a very convenient framework for this calculation.

The GCF fundamentally complements the shell model enabling one to incorporate SRC into the model. Though the shell model is essentially a mean field theory, it provides detailed information about the nuclear shell structure. One of the major success of the shell model is explaining the stability of a certain number of nucleons in nuclei called the magic numbers. However, it does not capture the complete picture of the nucleus structure. The shell model fails to predict the occupancy of the shells since in particular it does not describe the strong short range nucleon-nucleon part of the potential which is responsible for the SRC. The SRC are manifested in a high-momentum tail of the nucleon momentum distribution with momenta exceeding the Fermi momentum. Experimental evidence for the existence of SRC was obtained by performing electron scattering experiments. SRC related physics was observed by choosing the right kinematics in these experiments, specifically inclusive scattering at Bjorken \( x_B > 1 \) [15, 16, 18] and also by performing exclusive experiments in which the two-body currents could be identified and distinguished from final state interactions [22–24].

Starting off with the ground state of the nucleus with hard interactions, i.e., low-RG resolution, there are both high-momentum and low-momentum contributions to the wave function. By employing a SRG transformation to high-RG resolution it was demonstrated that operator expectation values exhibit factorization in the two-nucleon approximation is viable when the wave function is viewed on a short length scale, important for calculating short range neutron-proton pairs dominate \((pp, pn)\) in terms of isospin, \(\pi\) of the pair, and projection respectively. This state has been shown to match experimental data to a very high precision \([7]\). Through this factorization a model can be obtained in which SRC, which are identified as components of the nuclear wave function in which nucleon pairs with relative momenta well above the Fermi momentum, naturally emerge. This factorization is the basis of the GCF \([37–40]\), which is a model introduced to provide a simple framework for describing the SRC. The GCF is a very convenient theoretical tool for analyzing experiments designed to probe SRC in the nucleus \([24, 41–43]\). It has been benchmarked against ab-initio many-body calculations, proving its validity, and was successfully applied to a wide range of topics, most notably to the analysis of electron scattering measurements \([27, 44–47]\). The contact which is at the heart of the GCF will be shown to be an instrumental tool for expressing the SRC entanglement entropy. Currently the GCF does not treat three-body effects. However, the three-body contributions are expected to be less important than those of the leading two-body.

A. The nuclear contact

The GCF is based on the factorization ansatz of the many-body wave function into a two-body problem of a nuclei pair close in space (correlated), which is a universal part, common to all nuclei, multiplied by a particular part, that depends on the specific state. The specific state part describes the remaining, \(A - 2\), nuclei in the nucleus \([40]\) and also depends on the center of mass coordinate for the nuclei pair

\[
\Psi \rightarrow \sum_{\alpha} \varphi_{ab}^\alpha (r_{ab}) A_{ab}^\alpha (R_{ab}, \{r_c\}_{c \neq a,b}). \tag{6}
\]

Here \(\varphi_{ab}^\alpha\) are the two-body universal functions defining the SRC, \(A_{ab}^\alpha\) are the so called regular part of the many-body wave function, the index \(\alpha\) defines the quantum numbers for the two-body states and the indices \(ab\) define the SRC in terms of isospin, \((pp, pn)\) and \((nn)\) pairs as well as the specific particle indices in \(r_{ab}\) and \(R_{ab}\). This sort of approximation is viable when the wave function is viewed on a short length scale, important for calculating short range observables. For simplicity and since most experimental data on SRC is in the kinematical region where neutron-proton pairs dominate \([17, 19, 22–27]\). In this work we consider neutron-proton pairs in the deuteron like quantum state (quasi-deuteron), defined as \(a_1 \equiv (S = 1, \pi = +1, J = 1, M = 0 \pm 1)\), where \(S\) is the total spin of the pair, \(\pi\) is their parity, and \(J\) and \(M\) are the quantum numbers defining the SRC, which are the total angular momentum of the pair and its projection respectively. This state has been shown to match experimental data to a high precision \([17, 19, 22–27]\). Under this approximation the nuclear contact in GCF is defined as

\[
C_{pm} = N(A, Z) \langle A_{pm}^{\alpha_1} | A_{pm}^{\alpha_1} \rangle, \tag{7}
\]

where \(N(A, Z)\) is the number of pairs one can produce from \(Z\) protons and, \(A - Z\), neutrons. In this work we will only consider symmetric nuclei such that \(Z = A/2\).

The function \(\varphi_{ab}^\alpha (r_{ab})\) in Eq. (6) is a function of solely the distance between the SRC proton and neutron, \(r_{ab}\), and not their center of mass coordinate \(R_{ab}\) which appears in \(A_{ab}^\alpha (R_{ab}, \{r_c\}_{c \neq a,b})\). This function is obtained by solving the two-body, zero energy, Schrödinger equation with the full many-body potential.
Since the contact is obtained by tracing out the degrees of freedom exterior to the SRC pair, it is clear why it plays an important part in determining the entanglement entropy of a SRC pair with respect to the rest of the nucleus structure.

Based on the GCF ansatz, Eq. (6), which can be shown to be a result of high-RG resolution [14] the Hilbert space for the nucleus can now be written as a product \( \mathcal{H} = \mathcal{H}_{A-2} \otimes \mathcal{H}_2 \) and as such the many-body nuclear wave function can be decomposed into a sum of tensor products

\[
|\Psi\rangle = \sum_{q'q} \mathcal{B}_{q'q} |\phi_{q'}\rangle \otimes |\varphi_{q}^{\alpha_1}\rangle, \tag{8}
\]

whereas \( |\phi_{q'}\rangle \) are single particle momentum states below and around \( k \lesssim k_F \) is a Slater determinant for a FS composed of \( A - 2 \) nuclei, and \( |\varphi_{\alpha_1}\rangle \) are two particle states composed of a proton-neutron pair with quantum numbers \( \alpha_1 \) and relative high-momentum, i.e. larger than \( q > k_F \).

\[
|\phi_{q'}\rangle = \prod_{k \lesssim k_F} a^\dagger_k |0\rangle \quad |\varphi_{q}^{\alpha_1}\rangle = b^\dagger_q |0\rangle, \tag{9}
\]

where the creation operator \( a^\dagger_k \) creates a nucleon with momentum \( k \lesssim k_F \) and the the creation operator \( b^\dagger_q \) creates a pair of back to back neutron and proton pair in a SRC with relative momentum \( |\pm q| > k_F \). We have specifically used different creation operators for the neutron proton pair in the SRC \( b^\dagger_q \) than for the low-momentum nuclei \( a^\dagger_k \) since the SRC composed of a neutron proton SRC pair above the Fermi energy with \( S = 1, J = 1 \) can not be viewed simply as a double particle hole excitation of the FS. The coefficients \( B_{q'q} \) appearing in Eq. (8) signify that the nuclear wave function can not be expressed as a single Slater determinant or even as a single Slater determinant for \( A - 2 \) nuclei times a SRC pair. The distribution of the coefficients \( B_{q'q} \) results from a spread of momentum of the single particles composing the FS. The spread is due to long range correlations which constitute an intermediate momentum range resulting in a smearing of the sharp edge of the Fermi momentum distribution. The other source for the distribution of \( B_{q'q} \) is from the undefined center of mass coordinates corresponding to the SRC.

### B. Entanglement entropy of SRC in terms of the contact

Since each of the SRC is given by a universal wave function \( \varphi_{\alpha_1}^{pn}(r_{ab}) \) in the limit where \( r_{ab} \to 0 \), we consider it as a single orbital. In calculating the entanglement entropy we consider SRC as independent thus the entanglement entropy for the total SRC is given as a sum over the individual entanglement entropy for each of the single SRC

\[
S^{SRC}_A = \sum_i^N S^{SRC}_i. \tag{10}
\]

The entanglement entropy then just depends on the occupation of the SRC ”orbitals” which are the high-momentum states \( q > k_F \). In analogy to Eq. (3) the single SRC density matrix is thus given as

\[
\rho^{(\alpha_1)} = \begin{pmatrix} 1 - \gamma_{SRC} & 0 \\ 0 & \gamma_{SRC} \end{pmatrix}. \tag{11}
\]

Where \( \gamma_{SRC} = \langle \Psi | b^\dagger_q b_q | \Psi \rangle \) is the probability of finding a single proton-neutron SRC with quantum numbers \( \alpha_1 \) in the the many-body wave function. From the one SRC density matrix one obtains from Eq. (4) the entanglement entropy for a SRC

\[
S_{\alpha_1}^{SRC} = -Tr[\rho^{(\alpha_1)} \ln \rho^{(\alpha_1)}] = -\sum_{k=1}^{2} \omega_{k}^{(\alpha_1)} \ln \omega_{k}^{(\alpha_1)} \tag{12}
\]

It should be noted that \( \gamma_{SRC} \) is essentially the probability for obtaining a single SRC whereas the contact is the probability for finding an SRC in general. Specifically the expression for the contact Eq. (7) has a \( N(A, Z) \) part which is the number of pairs one can produce from \( Z \) protons and, \( A - Z \), neutrons. Were since we are considering symmetric nuclei, \( N = Z \). So that the number of SRC pairs in Eq. (7) is simply

\[
N(A, Z = A/2) = \frac{A}{2}. \tag{13}
\]
Due to this dependence, Eq. (13) we introduce the reduced contact which gives the probability for obtaining a single SRC

\[ c_{pn} = \frac{C_{pn}}{(A/2)}. \]  

(14)

Thus \( \gamma_{SRC} \) is simply given by the reduced contact Eq. (14)

\[ \gamma_{SRC} = c_{pn} \]  

(15)

From Eqs. (11,15) we directly obtain the following eigenvalues: \( \omega_{k}^{(pn)} = \{1 - c_{pn}, c_{pn}\} \), which denote the probabilities of observing a single nucleon in a proton-neutron SRC or observing it in the rest of the system. The entanglement entropy for a single SRC, is directly obtained through Eq. (12)

\[ S^{SRC}_{pn} = - \left[ c_{pn} \ln \left( \frac{c_{pn}}{1 - c_{pn}} \right) + \ln (1 - c_{pn}) \right]. \]  

(16)

To obtain the total SRC entanglement entropy one has to multiply the single SRC entanglement entropy by \( N(A, Z) \) as expressed in Eq. (16) plugging the expression for the number of neutron proton pairs, Eq. (13) the total SRC entanglement entropy is obtained,

\[ S^{SRC}_{A} = - \frac{A}{2} \left[ c_{pn} \ln \left( \frac{c_{pn}}{1 - c_{pn}} \right) + \ln (1 - c_{pn}) \right]. \]  

(17)

We have thus obtained a simple general expression viable to any nucleus of the entanglement entropy in terms of the reduced contact.

C. Scaling of the SRC entanglement entropy

The above result, Eq. (17), shows an explicit linear dependence of the SRC entanglement entropy on \( A \). This is a rather surprising result since the SRC entanglement primarily scales like the nuclei volume \( A \), making the entanglement entropy extensive which means that it does not obey an area law as could be expected. This extensive scaling of the entanglement entropy can be traced back to the number of neutron proton pairs, \( N(A, Z) \propto A \), which can be combined to form a SRC. However there might be a further dependence on \( A \) through the reduced contact, \( c_{pn}(A) \).

The reduced contact is expected to be \( A \) independent for nuclei with a large \( A \) for which the Fermi gas approximation is expected to be valid. The linear dependence of the contact on \( A \) and the independence of the reduced contact in the Fermi gas approximation can be inferred from Ref. [48]. There the contact was calculated by taking the limit when interparticle distance \( r \) goes to zero of the two-body Fermi gas correlation function \( g(r) \). The obtained result for the Fermi gas model was shown to be linearly dependent on the number of nuclei in the nucleus and the slope of the linear dependence on \( A \) was then estimated by comparison to experimental results as well as to calculations based on the AV18 potential [48]. The obtained result for the reduced contact for a symmetric nuclei was given as \( c^{FG}_{pn} = 0.146 \pm 0.002 \). Thus under the Fermi gas approximation the SRC entanglement entropy, Eq. (17), is simply linear in \( A \).

The reduced contact was also calculated using a variational Monte Carlo (VMC) method and calculations were compared to experimental results and found to be in agreement. Through the VMC results one can also justify numerically the neglect of other channels differing from the proton-neutron channel (\( S = 1, \pi = +1, J = 1, M = 0 \pm 1 \)). The proton-proton and neutron-neutron calculated contacts were shown to be an order of magnitude smaller then the proton-neutron channel and the \( S = 0 \) proton-neutron channel is even smaller. The VMC calculated values in momentum space, for the reduced contact in Ref. [44], range from \( c_{pn} \approx 0.125 \) for \( ^4\text{He}, ^8\text{Be}, ^9\text{Be}, ^{10}\text{B} \) to \( c_{pn} = 0.106 \) for \( ^6\text{Li}, ^7\text{Li} \) and a higher value \( c_{pn} = 0.168 \) for \( ^{12}\text{C} \). The experimental values obtained for \( ^4\text{He} \) were slightly higher than the VMC calculation \( c^{exp}_{pn} \approx 0.149 \) as was the experimental evaluated contact for \( ^{12}\text{C} \) which was \( c^{exp}_{pn} \approx 0.168 \). It is difficult to determine if there is a slight mass number dependence of the reduced contact due to the limited range of nuclei calculated. A first rough approximation can be to take the contact as \( c_{pn} \approx 0.15 \) and independent of \( A \). However, it seems that there could be some corrections to the simple linear (extensive) dependence of the entanglement entropy on \( A \).

A word of caution should be given when estimating the value for the reduced contact by calibrating the result through experimental data. Whereas the relevant scale for momentum separation is well defined theoretically by SRG it is not always clear that this was the relevant scale in the experiments. The obtained experimental results depends on physical scale through the specific kinematics. These two scales, the theoretical and experimental, need not necessarily coincide. This issue is elaborated in detail in [14]. The scale mismatch if occurring can cause errors in estimating the reduced contact and thus also effect the entanglement entropy estimation through Eq. (17).
IV. COMPARISON TO PREVIOUS RESULTS

Calculating the entanglement entropy for a nuclei can involve considerable computational efforts. The sole source for an entanglement entropy calculation which we found easy to compare to quantitatively was the calculation performed for $^4\text{He}$ and $^3\text{He}$ [10]. The calculation of the entanglement entropy was performed by a full diagonalization on a 7 shell model space. It should be noted that the calculation that was performed with only 6 major active shells, containing the first 114 single-particle states, was not large enough to give satisfactory results. On the other side it seems rather simple using our SRC entanglement expression Eq. (17) to calculate the SRC entanglement entropy once the relevant reduced contact is known. It is thus of interest to try to evaluate how much of the entanglement entropy can be related to SRC. In this section we compare the results for the entanglement entropy of $^4\text{He}$ from Ref. [10] to the SRC entanglement entropy from our analytical expression with the appropriate reduced contact and mass number for $^4\text{He}$.

In Ref. [10] a comparison was made between different basis states to find the appropriate base in which to work in. It was found that the two most accurate basis sets were NAT, a term for natural, and VNAT, a term for, for variational natural which gave the most accurate results. The total sum of the single orbital entanglement entropy for the calculations in these basis gave a result of $S_{\text{tot}}^{1}(^4\text{He}) = 1.006$. In calculating the single-orbital entanglement entropy for $^4\text{He}$ solely from Eq. (17) we obtain a value of $S_{\text{SRC}}^{1}(^4\text{He}) = 0.74$ for $c_{ pn} = 0.12$ and $S_{\text{SRC}}^{0}(^4\text{He}) = 0.86$ for the value for the reduced contact obtained experimentally $c_{ pn}^{\text{exp}} = 0.15$ [44]. The results are relatively similar to those of Ref. [10] especially taking in mind that contributions from proton-proton and neutron-neutron SRC pairs was not considered as well as the contribution for single low-momentum nucleons. Based on this comparison it can be stated that most of the nuclear entanglement is due to SRC.

In comparing our results with those presented for $^4\text{He}$ it might not be clear why we should compare the SRC entanglement entropy to that of the single orbital entropy calculated in [10]. Naively one can expect that the SRC entanglement entropy should be compared to the the two-orbital entanglement entropy, Eq. (5). However, it should also be kept in mind that the calculation of the entropy entanglement in [10] was performed for the low-RG case, whereas the results obtained in this work, Eqs. (17,19) are for the high-RG case. In Ref. [14] a SRG connection was established showing that calculations performed in the high-RG resolution involving SRC pairs in the GCF can be mapped via SRG to the low-RG resolution were he nucleons are best described by a FS. Thus we claim that actually the single orbital entanglement entropy obtained in [10] is transformed by the SRG to SRC entanglement entropy. It still needs to be shown how the SRG connects these two methods for calculating the entanglement entropy and how it continuously transform from one to the other. An indication for this connection can be found in the claim made in Ref. [10] that the most important couplings are 1s-1s, 1s-1p$_{1/2}$ and 1p$_{1/2}$-1p$_{1/2}$ since they are related to deuteron-type correlations. A further qualitative comparison can also be made to Ref. [7] in which the single orbital and two-orbital mutual information was calculated employing the DMRG for $^{28}\text{Si}$, $^{56}\text{Ni}$ and $^{64}\text{Ge}$. The total entanglement entropy for these calculations was not presented thus allowing only a qualitative comparison. The authors of of Ref. [7] state that they could see significant entanglement between proton-neutron maximally aligned states for the $p3/2$ and $f5/2$ orbits. However, it is also claimed that the fact that the two orbital mutual information is approximately equal for the $p1/2$, $p3/2$, and $f5/2$ orbits, and independent of their $M$ projections, is an indication for the presence of a strong $T = 1$ proton-proton and neutron-neutron pairing coherence. Whereas in our calculations we have neglected such SRC pairs. Future work should establish how the entanglement entropy is effected by the SRG.

V. CALCULATING THE FERMI SEA ENTANGLEMENT ENTROPY

The wave function separation expressed in Eq. (8) separates the nuclear wave function into SRC and a part well approximated as a FS [14]. Under the assumption that the whole nuclear structure is composed of SRC and a FS the change of the entanglement entropy of the two sub systems due to the coupling between them should be proportional $\Delta S_{A}^{FS} \propto S_{A}^{SRC}$ where $\Delta S_{A}^{FS}$ is the change in the FS entanglement entropy due to the fact pairs can now occupy another effective orbital, the SRC.

If we naively assume that the low-momentum states are a FS we can attempt to calculate their entanglement entropy through a known formula [49] devised for the general calculation of the entanglement entropy of a FS

\[
S_{FS} = \frac{L^{d-1}}{(2\pi)^{d-1}} \log \frac{L}{12} \int \int |n_{x} \cdot n_{k}| dA_{x} dA_{k}
\]

(18)

where, $L$ is the region in space in which the fermions are located which in the nuclear case corresponds to the nuclear radius, $n_{x}$ and $n_{k}$ are unit normals for the real space boundary and the Fermi surface, respectively. The integration is over the Fermi surface and a scaled version of the real space boundary (hence the $L^{d-1}$ factor). It should be noted that the logarithm in the above Eq. (18) results from counting the number of chiral one-dimensional
excitations at low-energy. These chiral excitations are the low-energy modes of the FS. The entanglement entropy considered here is therefore a single orbital entanglement entropy.

In applying Eq. (18) to the nuclear case as a good approximation one can consider $L \sim A^{1/3}$, where it is measured in units of $r_0 = 1.2 \, fm$ which correspond to $\rho^{-1/3}$, where $\rho$ is the nuclear density which is roughly constant, respectively the momentum is measured in units of $(1/r_0)$. The integration over the Fermi momentum over the surface $dA_L \sim dE_F$ corresponds to an averaged nuclear surface energy and can be approximated by the liquid drop model $\int |n_L \cdot n_k|dA_LdA_k \sim \tilde{u}_s A^{2/3}$, where $\tilde{u}_s$ is some constant. Under a naive underlying assumption that nuclei have a probability of being in the SRC or the bulk low-momentum states one should multiply the result by the probability for a nucleon to be in these low-momentum states, which is given by $1 - c_{pn}$ for each single SRC. From this one obtains under this approximation the change in entanglement entropy of the FS is give by

$$\Delta S_{FS} = c_{pn} \log L \frac{L}{12} \tilde{u}_s A^{2/3}$$

$$= c_{pn} \log A \frac{A}{36} \tilde{u}_s A^{2/3}$$

(19)

From Eq. (19) it can be seen that the naive change in the entanglement entropy for the low-momentum part of the nuclear structure grows like a modified fermion area law $[49]$ multiplied by the the volume $A$, i.e, scaling as $A^{2/3} \ln A$. In counting the number of modes one gets a different scaling then that for the SRC which is extensive. Thus it seems that the naive assumption that creating a SRC simply reduces the number of modes in the FS and does not effect the other modes is virtually too simplistic. It is pretty clear that coupling to the SRC level introduces two-orbital entanglement entropy in FS. Moreover, there are some major reservations with regard to the estimate introduced in Eq. (19). First, the evaluation of the entanglement entropy was based on Eq. (18) which was devised for a system with a sharp Fermi surface and we have also assumed $|n_x \cdot n_k| \sim 1$. Whereas it should be noted that the SRC modify the Fermi surface inducing a long tail in momentum space. Thus there should be a correction to the estimation based on Eq. (18), since less modes impinge the Fermi surface. Though not directly related to the difference in the scaling we should also note that we have neglected the difference between the proton Fermi energy and that of the neutrons in obtaining Eq. (19). We estimate that this latter effect is of higher order and should be addressed in future research.

VI. SUMMARY AND DISCUSSION

Basing our work on previously obtained theoretical results demonstrating that through applying an SRG transformation on the nuclear structure one obtains scale separation $[14]$, we have calculated the nuclear structure entanglement entropy. We have demonstrated how such scale separation greatly simplifies the computation of the nuclear structure’s entanglement entropy. In employing it we have been able to obtain a general formula for the SRC entanglement entropy, Eq. (17), applicable to all nuclei which is given in terms of the probability for obtaining a SRC in that specific nuclei given in terms of the reduced nuclear contact. Through this general formula we have been able to demonstrate that entanglement entropy for the SRC part of the nuclear structure scales extensively and not according to an area law.

More specifically, we utilized in our calculations the fact that in the high-RG resolution the nuclear structure can be viewed as a bipartite system where the high-momentum short length scale properties of the system are given in terms of SRC pairs and the low-momenta degrees of freedom are well described by a FS. We have performed our calculation for the SRC entanglement entropy in the framework of the single orbital entanglement entropy. In this calculation we considered the universal two-body SRC wave function, which is part of the many-body wave function, as an orbital which can be occupied by many SRC. The reduced nuclear contact for a neutron-proton SRC pair which is obtained by integrating out the long length scales in the nucleus corresponding to the low-momentum degrees of freedom and then normalizing the result, naturally appears as the measure for the orbital entanglement entropy related to a single SRC. Using this connection, expressing the matrix elements of the single orbital density matrix in terms of the reduced nuclear contact, Eq. (15). Summing over the entanglement entropy for all SRC pairs we obtained that the nuclear entanglement entropy obeys an extensive entropy law since it scales linearly with the mass number $A$ and since the nuclear structure density is essentially constant this means it scales like the volume of the nucleus.

Even though in obtaining our results quite a few contributions were neglected, e.g., neutron-neutron and proton-proton SRC still we find them to be relatively close to the more elaborate work performed in calculating the entanglement entropy for $^4\text{He}$ $[10]$. Considering this we argue that most of the nuclear entanglement entropy is related to SRC and thus our general Eq. (17) is a good measure for the entanglement entropy of all nuclei.

The work presented here is essentially preliminary. It introduced the idea that the SRG scale separation is an invaluable tool for performing the calculations for the nuclear structure entanglement entropy and demonstrating
the scaling of the entanglement entropy. Further research has yet to incorporate proton-proton and neutron-neutron SRC and establish how the entanglement entropy is modified by the SRG transformation. It is of importance to obtain a better estimation method for calculating the entanglement entropy of the low-momentum states. An extension of this work will allow to make more quantitative calculations complimenting the more qualitative results presented here.

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