Unitarity constraints on trimaximal mixing

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Abstract

When the neutrino mass eigenstate $\nu_2$ is trimaximally mixed, the mixing matrix is called trimaximal. The middle column of trimaximal mixing matrix is identical to tri-bimaximal mixing and the other two columns are subject to unitarity constraints. This corresponds to a mixing matrix with four independent parameters in the most general case. Apart from the two Majorana phases, the mixing matrix has only one free parameter in the CP conserving limit. Trimaximality results into interesting interplay between mixing angles and CP violation. A notion of maximal CP violation naturally emerges here: CP violation is maximal for maximal 2-3 mixing. Similarly, there is a natural constraint on the deviation from maximal 2-3 mixing which takes its maximal value in the CP conserving limit.

1 Introduction

The historical discovery of neutrino oscillations constitutes the only indication of physics beyond the standard model. In the absence of a basic understanding of fermion masses and mixings, it is natural to seek simplified patterns in the mixing matrix which are consistent with the present experimental knowledge and provide guidance for the future experimental searches. Such phenomenological patterns are, at best, first approximations to the experimental data and the next step is to study deviations from these patterns.

A successful phenomenological ansatz for neutrino mixing matrix which is consistent with the present neutrino data [1] was proposed by Harrison, Perkins and Scott [2] and is given by

$$U = \begin{pmatrix}
\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}. \quad (1)
$$

In this mixing scheme, the second neutrino mass eigenstates $\nu_2$ has trimaximal character as it arises from maximal mixing of the three flavor eigenstates ($\nu_e$, $\nu_\mu$ and $\nu_\tau$). Moreover, the
third mass eigenstate $\nu_3$ has bimaximal character as it results from maximal mixing of two flavor eigenstates viz. $\nu_\mu$ and $\nu_\tau$. Hence, this mixing scheme is called tri-bimaximal (TBM) mixing and gives vanishing $\theta_{13}$ and maximal $\theta_{23}$. The mixing angle $\theta_{13}$ vanishes in TBM mixing because of bimaximal character of $\nu_3$, a feature needs to be abandoned in order to allow for a non-vanishing $\theta_{13}$ and, hence, the possibility of CP violation. However, there is no need to abandon the trimaximal character of the $\nu_2$ in this generalization of TBM mixing. Such a mixing matrix, called trimaximal (TM) [3], will be of the form

$$U = \begin{pmatrix} U_{11} & \frac{1}{\sqrt{3}} & U_{13} \\ U_{21} & \frac{1}{\sqrt{3}} & U_{23} \\ U_{31} & \frac{1}{\sqrt{3}} & U_{33} \end{pmatrix}$$

(2)

where the first and third columns are subject solely to unitarity constraints and the middle column has trimaximal character. The definition of trimaximality adopted here requires that magnitudes and phases of all the three elements in the middle column of the mixing matrix are equal. The common phase of the middle column can be rephased away by phase-redefinition of the lepton fields.

The relationship between TBM mixing and TM mixing needs also to be understood from the standpoint of the symmetry groups associated with these mixing schemes. The TM mixing follows from a neutrino mass matrix which can be parametrized as

$$M = \begin{pmatrix} a + 2d & c - d & b - d \\ c - d & b + 2d & a - d \\ b - d & a - d & c + 2d \end{pmatrix}$$

(3)

in terms of four complex parameters $a$, $b$, $c$ and $d$ having $A_4$ symmetry in the flavor basis where the charged lepton mass matrix is diagonal [4]. The matrix is called magic mass matrix since its rows and columns all add up to $a + b + c$ and hence one of its eigenvectors is $(1/\sqrt{3} \ 1/\sqrt{3} \ 1/\sqrt{3})^T$ within an overall multiplicative phase [3]. The above mass matrix becomes $\mu$-$\tau$ symmetric as well if the number of free parameters are reduced to three by demanding $b = c$. The mixing matrix become TBM and the symmetry group enlarges to $S_4$ after making the above fine tuning. Thus, the neutrino mass matrix having $A_4$ symmetry and bringing about TM mixing contains one more complex parameter than the mass matrix having $S_4$ symmetry and producing TBM mixing which can be used to accommodate a non-vanishing $\theta_{13}$ and a Dirac-type CP-violating phase $\delta$. This additional parameter needs to be fine tuned to recover the TBM mixing in a mass model based upon $A_4$ symmetry. The neutrino mixing matrix will be TM in absence of any such fine tuning of parameters [4]. It becomes imperative to analyse the phenomenology of TM mixing scheme in this context.

The CP violating structure in trimaximal mixing is an interesting feature to study because of the mathematical restriction that one of the eigenvector of the mass matrix be $(1 \ 1 \ 1)^T$ which severely restricts the elements of the matrix to coordinate in a way so that they produce a real eigenvector. This coordination results into a unique correlation between CP violation and the mixing angles.

A notion of maximal CP violation automatically emerges when unitarity constraints are combined with trimaximality. The Jarlskog rephasing invariant $J$ [5] takes its maximal
value \( \frac{1}{2} \sqrt{3} |U_{11}| |U_{13}| \) when \( \theta_{23} \) is maximal. Similarly, the deviation from maximal 2-3 mixing is proportional to \( \theta_{13} \) and is naturally restricted to lie within the present experimental range by unitarity for values of \( \theta_{13} \) within its experimental bounds \(^1\). This deviation is zero when CP violation is maximal and is maximal when there is no CP violation.

2 Mixing angles and CP violating phases

Trimaximality, as defined in Eq. (2), equates magnitudes as well as phases of the three elements of the middle column of \( U \). Although, the equality of the magnitudes is preserved after the phase redefinitions of the lepton fields, same is not true for the equality of the phases. Hence, the definition of trimaximal mixing adopted here is not rephasing invariant. This is quite natural since a symmetry may prefer a basis, called symmetry basis, for its definition. So, it becomes necessary to find the consequences of TM mixing in terms of rephasing invariant quantities. The mixing angles and CP violating phases must be deduced from these rephasing invariants only.

There are four independent CP-even quadratic invariants \(^6\) which can conveniently be chosen as \( U_{11}^* U_{11}, U_{13}^* U_{13}, U_{21}^* U_{21} \) and \( U_{23}^* U_{23} \) and three independent CP-odd quartic invariants \(^6\)

\[
J = Im(U_{11}^* U_{12}^* U_{21}^* U_{22})
\]

(4)

\[
I_1 = Im[(U_{11}^* U_{12})^2]
\]

(5)

and

\[
I_2 = Im[(U_{11}^* U_{13})^2].
\]

(6)

The Jarlskog rephasing invariant \( J \) \(^5\) is relevant for CP violation in lepton number conserving process like neutrino oscillations and \( I_1 \) and \( I_2 \) are relevant for CP violation in lepton number violating processes like neutrinoless double beta decay.

The consequences of unitarity for TM mixing will be worked out in terms of the above rephasing invariant quantities \( J, I_1, I_2 \). However, it will be rewarding to express the phenomenological implications of trimaximality in term of the usual parameters \( \theta_{12}, \theta_{23}, \theta_{13}, \alpha, \beta \) and \( \delta \) of a widely used parametrisation \(^1\) \( 7\)

\[
U = \begin{pmatrix}
   c_{13} c_{12} & c_{13} s_{12} & s_{13} e^{-i\delta} \\
   -c_{23} s_{12} - c_{12} s_{23} s_{13} e^{i\delta} & -c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & c_{13} s_{23} \\
   s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - c_{23} s_{12} s_{13} e^{i\delta} & c_{13} c_{23}
\end{pmatrix}
\]

(7)

where \( s_{ij} = \sin \theta_{ij} \) and \( c_{ij} = \cos \theta_{ij} \). This will be achieved through the relations \( |U_{11}| = c_{13} c_{12}, |U_{13}| = s_{13}, |U_{23}| = c_{13} s_{23}, \)

\[
I_1 = c_{13}^4 c_{12}^2 s_{12}^2 \sin 2\alpha,
\]

(8)

\[
I_2 = c_{13}^2 c_{12}^2 s_{13}^2 \sin 2\beta
\]

(9)

and

\[
J = c_{13}^2 c_{23} c_{12} s_{12} s_{23} s_{13} \sin \delta.
\]

(10)
However, some of the relationships will be more apparent when written in terms of rephasing invariant quantities themselves.

Like $\mu - \tau$ symmetry and TBM mixing, the TM mixing is mass-independent texture putting no constraints on neutrino masses and Majorana phases $\alpha$ and $\beta$ \cite{8}. Hence, trimaximality will have no physical consequences for the Majorana phases.

3 Strong and weak trimaximality

A naive way to work out the consequences of unitarity for trimaximal mixing would have been to compare the magnitudes of the elements of trimaximal mixing matrix of Eq. (2) with the unitary mixing matrix of Eq. (7). This would mean that $|U_{21}|$, $|U_{22}|$ and $|U_{23}|$ in Eq. (7) are all equal to $\frac{1}{\sqrt{3}}$ with no restriction on their phases. In this version of TM mixing, trimaximality is imposed only in a weak sense as the elements of the middle column of $U$ in Eq. (7) have unequal phases for any non zero value of $\delta$ and $\theta_{13}$. This weak form of trimaximality has been already analysed in the literature \cite{9} for its phenomenological consequences. If a similar analysis to is to be performed for the trimaximality defined by Eq. (2), called strong trimaximality to distinguish it from weak trimaximality, which demands that magnitudes as well as the phases of $U_{21}$, $U_{22}$ and $U_{23}$ are equal, one is required to compare the trimaximal $U$ of Eq. (2) with the unitary $U$ of Eq. (7) only after making its middle column completely real by making a phase rotation on the lepton fields. This process of making a basis transformation on a mixing matrix to the symmetry basis where the TM mixing is defined is quite natural if the definition of symmetry is not rephasing invariant and prefers some basis. As stated already, the strong trimaximality is not a rephasing invariant concept unlike weak trimaximality which is a rephasing invariant notion. However, instead of adapting a unitary matrix by phase rotation to match it with trimaximal mixing as outlined above, the present work attempts to obtain an unitary parametrisation which is trimaximal in the strong sense from the very outset.

The strong trimaximility implies that the mass eigenstate $\nu_2$ is a democratic and coherent combination of the three flavor eigenstates whereas the weak trimaximality relaxes the condition of coherence allowing different flavor eigenstates to mix with equal amplitudes but mismatching phases. However, the distinction between strong and weak trimaximality is not merely conceptual, it is also phenomenological in nature having distinct predictions for CP violation. The weak trimaximality is less interesting as it doesn’t fully utilise the phase restrictions on the mass matrix coming from $A_4$ symmetry.

A mixing matrix which is trimaximal in the strong sense was first proposed and studied phenomenologically by Bjorken, Harrison and Scott (BHS) \cite{10} in which the only free complex parameter was $U_{13}$. A neutrino mass matrix having magic symmetry in the basis of a diagonal charged lepton mass matrix was prescribed by Lam \cite{11} which gives rise to TM mixing in BHS parametrisation. In another attempt for the minimal modification to the TBM mixing, He and Zee \cite{12} obtained a mixing matrix, which is also trimaximal in strong sense, containing only one free complex parameter. The neutrino mass matrix in a model proposed by Grimus and Lavoura \cite{13} is magic and hence gives rise to TM mixing in the strong sense. However,
the mass matrix contains three complex parameters and therefore is more restrictive than a most general magic mass matrix [Eq. (2)] containing four complex parameters. Similarly, the resulting TM mixing matrix also contains lesser number of parameters than the most general case. All the above forms of TM mixing studied phenomenologically in the literature are more constrained than the most general form given by Eq. (2) and can be expressed as the product of the TBM mixing matrix of Eq. (1) by a generalized 1-3 rotation of the type

$$\begin{pmatrix}
\cos \theta & 0 & \sin \theta e^{-i\phi} \\
0 & 1 & 0 \\
-\sin \theta e^{i\phi} & 0 & \cos \theta
\end{pmatrix}$$

from the right. However, the resulting mixing matrix is not the most general TM mixing matrix.

4 Unitary constraints

The TM mixing matrix of Eq. (2) can be written as

$$U = \begin{pmatrix}
U_{11} & \frac{1}{\sqrt{3}} U_{13} & U_{13} \\
U_{21} & \frac{1}{\sqrt{3}} U_{23} & U_{23} \\
-U_{11} \sqrt{\frac{2}{3}} & -U_{21} \sqrt{\frac{2}{3}} & -U_{13} - U_{23}
\end{pmatrix} \quad (11)$$

by straightforwardly eliminating two elements \(U_{31}\) and \(U_{33}\) using the two orthogonality conditions for the three columns. The remaining 4 elements are further constrained by the following seven constraints coming from the unitarity condition \(UU^\dagger = U^\dagger U = 1\):

$$|U_{11}|^2 + |U_{13}|^2 = \frac{2}{3},$$

$$|U_{21}|^2 + |U_{23}|^2 = \frac{2}{3},$$

$$|U_{11}|^2 + |U_{21}|^2 + Re(U_{11}^* U_{21}) = \frac{1}{2},$$

$$|U_{13}|^2 + |U_{23}|^2 + Re(U_{13}^* U_{23}) = \frac{1}{2},$$

$$Im(U_{11}^* U_{21}) + Im(U_{13}^* U_{23}) = 0,$$

$$2Re(U_{11} U_{13}^*) + Re(U_{21} U_{13}^*) + Re(U_{11} U_{23}^*) + 2Re(U_{21} U_{23}^*) = 0 \quad (17)$$

and

$$2Im(U_{11} U_{13}^*) + Im(U_{21} U_{13}^*) + Im(U_{11} U_{23}^*) + 2Im(U_{21} U_{23}^*) = 0 \quad (18)$$

Eq. (11) is a unitary parametrisation of TM mixing subject to the above seven unitarity constraints. All the parametrizations of TM mixing proposed in the literature \cite{10, 12, 13} are special cases of the above parametrization and satisfy these seven constraints. However, they are more constrained than the general parametrization obtained here. A special property of
the TM mixing matrix of Eq. (11), also shared by its earlier parametrizations \[10, 12, 13\], is that its first and third columns add up to zero.

The first four of the seven unitarity constraints can be seen as normalization conditions and the last three as orthogonalization conditions. The three orthogonalization relations can be solved to get three real variables, say, \(ReU_{23}, ImU_{23}\) and \(ImU_{21}\) which can be substituted back into the four normalization relations. This eliminates these three parameters. Two classes of solutions are obtained on solving the four normalisation relations in the remaining five parameters: one is CP conserving while the other is CP violating.

5 The CP conserving solution

The first solution has three independent parameters which can be chosen as \(ReU_{11}, ReU_{13}\) and \(ImU_{13}\). The imaginary part of \(U_{11}\) can be calculated from the equation

\[ |U_{11}| = \sqrt{2/3} - |U_{13}|^2. \]  

The elements \(U_{21}\) and \(U_{23}\) are given by

\[ U_{21} = \frac{U_{11}}{2} \left( -1 \pm \frac{\sqrt{3}|U_{13}|}{|U_{11}|} \right) \]  

and

\[ U_{23} = \frac{U_{13}}{2} \left( -1 \mp \frac{\sqrt{3}|U_{11}|}{|U_{13}|} \right). \]

The resulting mixing matrix has no Dirac type CP violation since first two elements in either first or third column have identical phases which results in a vanishing \(J\). There are no constraints on the phases of \(U_{11}\) and \(U_{13}\) and they are not necessary to describe the phenomenon of neutrino oscillations. However, the CP-odd rephasing invariants \(I_1\) and \(I_2\) do depend upon these phases and are given by

\[ I_1 = -\frac{2}{3} ReU_{11} ImU_{11} \]  

and

\[ I_2 = Re(U_{11}^2 Im(U_{13}^2) - Im(U_{11}^2) Re(U_{13}^2). \]

This solution is called CP conserving as there is no Dirac type CP violation here. Tri-maximality is a mass-independent texture and it puts no constraints on neutrino masses and Majorana phases \[8\]. Hence, a special case with vanishing Majorana phases will have real \(U_{11}\) and \(U_{13}\) parametrizable by a single parameter, say \(\theta\). The mixing matrix will be completely real in this special case and is given by

\[ U = \begin{pmatrix} \sqrt{\frac{2}{3}} c & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} s \\ -\frac{c}{\sqrt{6}} \pm \frac{s}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{s}{\sqrt{6}} \pm \frac{c}{\sqrt{2}} \\ -\frac{c}{\sqrt{6}} \pm \frac{s}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{s}{\sqrt{6}} \pm \frac{c}{\sqrt{2}} \end{pmatrix} \]
where \( c = \cos \theta \), \( s = \sin \theta \) and the parameter \( \theta \) can be related to the mixing angle \( \theta_{13} \) through the relation \( \sin \theta_{13} = \sqrt{\frac{2}{3}} \sin \theta \). This mixing matrix coincides with the mixing matrix obtained in a lepton mass model proposed by Grimus, Lavoura and Singraber \cite{14} for the first sign choice.

6 The CP violating solution

The second solution has four independent parameters which can be chosen as \( \text{Re}U_{11}, \text{Re}U_{13}, \text{Re}U_{21} \) and \( \text{Im}U_{13} \). The imaginary parts of \( U_{11} \) and \( U_{21} \) are calculated from the equations

\[
|U_{11}| = \sqrt{\frac{2}{3} - |U_{13}|^2} \tag{25}
\]

and

\[
\text{Im}U_{21} = -\frac{1}{2} (\text{Im}U_{11} + D) \tag{26}
\]

where the real parameter \( D \) is given by

\[
D^2 = 3|U_{13}|^2 - [\text{Re}(U_{11} + 2U_{21})]^2 \tag{27}
\]

which can either be positive or negative. The element \( U_{23} \) is given by

\[
U_{23} = -\frac{1}{2} \left[ U_{13} + \frac{U_{11}^*}{U_{13}^*} \left( \text{Re}(U_{11} + 2U_{21}) - iD \right) \right]. \tag{28}
\]

The Jarlskog invariant \( J \) is given by

\[
J = \frac{1}{6} (\text{Re}U_{11}\text{Im}U_{11} + 2\text{Re}U_{21}\text{Im}U_{11} + D\text{Re}U_{11}). \tag{29}
\]

The solution is called CP violating solution since \( J \) is non-zero here. The CP conserving solution can be recovered from this solution in the limiting case where \( J = 0 \). \( J \) vanishes for the two values of \( U_{23} \) given by Eq. (21) which are also the two limiting values within which \( U_{23} \) varies in the CP violating solution [Eq. (28)]. Thus, the CP conserving solution for \( U_{23} \) defines the limiting region in which \( U_{23} \) varies in the CP violating solution.

A notion of maximal CP violation naturally emerges here. It can be seen from the Eqs. (28) and (29) that \( J \) is maximal when \( \theta_{23} \) is maximal and the maximal value is given by

\[
J^2 = \frac{1}{12} |U_{11}|^2 |U_{13}|^2. \tag{30}
\]

It should be noted that the maximal CP violation does not mean that the CP violating phase \( \delta \) will also be maximum when \( J \) and \( \theta_{23} \) are maximal. The maximal value of \( J \) obtained here coincides with the intrinsic CP violation in the mass model by He and Zee \cite{12}. This is because the mixing matrix in the model is a special case of the most general unitary TM mixing matrix considered here.
7 Numerical analysis and results

The relationship between $\theta_{12}$ and $\theta_{13}$ given by Eq. (19) or (25) has been depicted in Fig. 1 and is identical for both the solutions. The relationship is well known and has been reproduced here to show the values of $\theta_{12}$ and $\theta_{13}$ which will enter into the numerical calculations through $U_{11}$ and $U_{13}$. The value of $\theta_{12}$ is constrained automatically to be well below its 3$\sigma$ range when $\theta_{13}$ is varied upto its 3$\sigma$ upper bound [1].

Both the CP conserving and CP violating solutions have identical predictions for the $ee$ element of the neutrino mass matrix

$$M_{ee} = m_1 U_{11}^2 + \frac{m_2}{3} + m_3 U_{13}^2$$

(31)

whose magnitude, also called effective Majorana mass, will be observable in neutrinoless double beta day experiments. Varying the lowest neutrino mass over a reasonable range ($10^{-3}$ eV-1 eV), calculating other mass eigenvalues from the squared-mass differences $\Delta m^2_{12}$ and $\Delta m^2_{23}$ (Table 1) sampled using normal distribution and using the values of $\theta_{12}$ and $\theta_{13}$ sampled in Fig. 1, the predictions for $|M_{ee}|$ have been shown in Fig. 2 for normal and inverted orderings of neutrino masses. This graph matches with usual plots of $|M_{ee}|$ as the function of lowest neutrino mass and has been reproduced here merely to check correctness of the numerical analysis.

The variation of $|U_{23}|$ with $|U_{13}|$ for the CP conserving solution can be calculated directly from Eq. (21) as

$$|U_{23}| = \frac{1}{2} \left( \sqrt{3} |U_{11}| \pm |U_{13}| \right)$$

(32)

and has been plotted in Fig. 3. It is noted that $|U_{23}|$ depends only on $|U_{13}|$ for the CP conserving solution. However, $U_{23}$ depends on $ReU_{11}$, $ReU_{13}$, $ImU_{13}$ and $ReU_{21}$ in the CP violating solution. Varying these free parameters randomly in their allowed ranges, the variation of $|U_{23}|$ with $|U_{13}|$ for the CP violating solution can be calculated from Eq. (28) and has been shown in Fig. 4. One can easily see that the CP conserving solution depicted in Fig. 3 sets the upper and lower limits within which the CP violating solution shown in Fig. 4 can vary. The same fact is true for the variation between $\theta_{23}$ and $\theta_{13}$ depicted in Fig. 5 for CP conserving solution and in Fig. 6 for CP violating solution. The two branches of CP conserving solution for $\theta_{23}$ can be viewed as the bounds within which $\theta_{23}$ varies in the CP violating solution. It can be easily seen that $\theta_{23}$ can be maximal even for non-zero $\theta_{13}$ in trimaximal mixing. The allowed span of $\theta_{23}$ around the maximal value increases with $\theta_{13}$. When $\theta_{13}$ reaches its 3$\sigma$ upper bound, $\theta_{23}$ also spans its 3$\sigma$ allowed range [1]. The span within which $\theta_{23}$ is allowed to vary increases almost linearly with $\theta_{13}$ showing a non-linear behaviour only at very large values of $\theta_{13}$ which has not been depicted in Figures 5 and 6. The deviation from maximal 2-3 mixing is zero when CP violation is maximal and it increases as CP violation decreases and, finally, there is no CP violation at the point where the deviation from maximality takes its maximal value allowed in TM mixing.

For the CP violating solution, $J$, $I_1$ and $I_2$ are calculated from Eqs. (22), (23) and (29) by varying the four free parameters $ReU_{11}$, $ReU_{13}$, $ImU_{13}$ and $ReU_{21}$ in their full ranges allowed by experimental as well as the unitarity constraints. The relationship between $J$
with $|U_{23}|$ for $|U_{13}| = 0, 0.1$ and $0.2$ has been depicted in Fig. 7. The central point is the TBM solution for which both $J$ and $U_{13}$ are zero and $|U_{23}| = \frac{1}{\sqrt{2}}$. However, there are values of $|U_{23}|$ other than the TBM value for which $J$ is zero [Eq. (32)] and there are values of $J$ other than zero for which $|U_{23}|$ takes its TBM value [Eq. (30)]. The relationship between $U_{13}$, $U_{23}$ and $J$ shown in Fig. (7) is unique to trimaximal mixing not found in other mixing schemes deviating from TBM mixing [9] (and references therein) and subject to verification in future neutrino oscillation experiments.

The relationship between $\delta$ and $\theta_{23}$ for $\theta_{13} = 4, 8$ and $12$ degrees has been depicted in Fig. 8. It is apparent from this figure that $\delta$ does not attain its maximum value at $\theta_{23} = \frac{\pi}{4}$. So, the point of maximal CP violation is not the point at which $\delta$ is maximum. The maximum of $\delta$ occurs when $\theta_{23}$ is slightly below the maximal value. Furthermore, the span in which $\theta_{23}$ is allowed to vary scales up with $\theta_{13}$ (see Fig. 6 as well) but the allowed range of $\delta$ is almost unaffected by variation in $\theta_{13}$ showing a marginal increase of about 10 degrees with increase in $\theta_{13}$. The relationship between CP violation and deviation from maximality can be visualized in the most apparent way in Fig. 9 which plots $s_{23}^2$ as a function of $J$ for $\theta_{13} = 4, 8$ and $12$ degrees. For a given $\theta_{13}$, the allowed values of $J$ and $s_{23}^2$ form an ellipse around the TBM point ($J = 0, s_{23}^2 = \frac{1}{2}$). The semi-major axis of the ellipse is given by the maximal value of $J$ [Eq. (30)] and the semi-minor axis of the ellipse is defined by the CP conserving values of $s_{23}^2$ [Eq. (32)].

For completeness, the variation of $J$ and $I_2$ with $U_{13}$ has been depicted in Fig. 10. There are no constraints on $I_1$. Since the phases of $U_{11}$ and $U_{13}$ are not constrained in both the solutions for TM mixing, there are no constraints on Majorana-type CP violating phases $\alpha$ and $\beta$.

8 Conclusion

In conclusion, TM matrix has been defined as a unitary matrix in which the three elements of the middle column are equal and their phases can be rephased away by phase redefinition of lepton fields. This definition of strong trimaximality is not rephasing invariant and differs from the weak trimaximality which demands only the equality of magnitude of the elements of the middle column without putting any constraints on the phases and, consequently, is less restrictive. The unitarity constraints on the TM mixing matrix have been solved and two solutions have been obtained. The first solution is CP conserving with vanishing Jarlskog rephasing invariant $J$ and has three free parameters viz. $ReU_{11}$, $ReU_{13}$ and $ImU_{13}$ which completely determine the mixing matrix. The deviation of $\theta_{23}$ from maximality scales up with $\theta_{13}$ and is within experimental constraints. The second solution, called CP violating solution, has $ReU_{21}$ as one more free parameter and reduces to the CP conserving solution in the limiting case of vanishing $J$. The CP conserving solution defines the upper and lower bounds within which $\theta_{23}$ varies with respect to $\theta_{13}$ in the CP violating solution. So, $J$ vanishes whenever $\theta_{13}$ is zero or $\theta_{23}$ equals its upper or lower bound for the given value of $\theta_{13}$. The Jarlskog invariant $J$ takes its maximal value $\frac{1}{2\sqrt{3}}|U_{11}||U_{13}|$ when $\theta_{23}$ is maximal. So, a notion of maximal CP violation naturally arises here: CP violation is maximal for maximal
2-3 mixing. However, the Dirac-type CP violating phase $\delta$ does not take its maximum value at the point of the maximal CP violation. The maximal value of $\delta$ occurs when $\theta_{23}$ is slightly less than its maximal value. The allowed range within which $\delta$ can vary shows a marginal variation of about 10 degrees with $\theta_{13}$. For a given value of $\theta_{13}$, $s_{23}^2$ and $J$ form an ellipse whose center is defined by the TBM mixing and whose axes are governed by the maximal values of CP violation and deviation from maximal 2-3 mixing. Trimaximality puts no constraints on the Majorana-type CP violating phases $\alpha$ and $\beta$ since it is a mass-independent texture.

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Figure 1: Relationship between $\theta_{12}$ and $\theta_{13}$.

Figure 2: The effective Majorana mass $|M_{ee}|$ as the function of lightest neutrino mass $m_0$ for normal hierarchy (lower branch) and inverted hierarchy (upper branch).
Figure 3: $|U_{23}|$ as a function of $|U_{13}|$ for CP conserving solution.

Figure 4: $|U_{23}|$ as a function of $|U_{13}|$ for CP violating solution.
Figure 5: Relationship of $\theta_{23}$ with $\theta_{13}$ for CP conserving solution.

Figure 6: Relationship of $\theta_{23}$ with $\theta_{13}$ for CP violating solution.
Figure 7: Relationship between $J$ and $|U_{23}|$ for $|U_{13}| = $ 0, 0.1 and 0.2.

Figure 8: Relationship between $\delta$ and $\theta_{23}$ for $\theta_{13} =$ 4, 8 and 12 degrees.
Figure 9: The variation of $J$ with $s_{23}^2$ for $\theta_{13} = 4, 8$ and 12 degrees.

Figure 10: The variation of $J$ and $I_2$ with $|U_{13}|$. 