AN IRRATIONAL-SLOPE THOMPSON’S GROUP
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Abstract. The purpose of this paper is to study the properties of the irrational-
slope Thompson’s group \( F_\tau \) introduced by Cleary in [10]. We construct presenta-
tions, both finite and infinite and we describe its combinatorial structure using
binary trees. We show that its commutator group is simple. Finally, inspired
by the case of Thompson’s group \( F \), we define a unique normal form for the
elements of the group and study the metric properties for the elements based
on this normal form. As a corollary, we see that several embeddings of \( F \) in \( F_\tau \)
are undistorted.

Introduction

Thompson’s groups were introduced in the 1960s and soon captured the interest
of group theorists for their interesting properties. They have spawned a family of
groups that have properties similar to Thompson’s groups, but each of which has
its own interesting particularities. The purpose of this paper is to study one of
these groups, namely, the group \( F_\tau \) of piecewise linear homeomorphisms of \([0, 1] \)
having breakpoints in \( \mathbb{Z}[\tau] \) and slopes that are powers of \( \tau \), where \( \tau \) is the golden
number \((\sqrt{5} - 1)/2\). Hence breakpoints of the elements in \( F_\tau \) are irrational numbers
in the unit interval. This group will present a structure which is similar to that of
Thompson’s group \( F \), and will also share many of its properties.

The group \( F_\tau \) was introduced by Sean Cleary in [10], where it is first described and
proved to be of type \( F_\infty \). The group \( F_\tau = G([0, 1]; \mathbb{Z}[\tau], \langle \tau \rangle) \) is also mentioned in
the Bieri-Strebel notes [1], although finite presentations there are only considered
for groups with rational slopes [11, D.15.10]

The structure of this paper is as follows. After general sections on the group, an
introductory one and another one which specifies the multiplication algorithm, we
introduce its presentations (both infinite and finite, as is common in Thompson’s
groups), we compute its abelianisation and its commutator subgroup, and in sim-
ilar fashion to \( F \), we prove that the commutator subgroup is simple. We show
that the abelianisation of \( F_\tau \) is isomorphic to \( \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \), giving an example of
a torsion-free Thompson-like group with torsion in its abelianisation. We thank

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**Question 1.** Does every finitely presented subgroup of Thompson’s group \( F \) have torsion-free abelianisation?

Note that there are finitely generated subgroups of \( F \) with torsion in their abelianisations [5].

We describe a normal form for the elements of \( F_\tau \), and use it to find estimates for the word metric. Finally we show that several copies of \( F \) inside \( F_\tau \) are undistorted. It is interesting to remark that this group is somewhat similar to other Thompson’s groups (such as \( F(2, 3) \) and \( 2V \)) which require two types of carets to describe their elements. In these groups, subgroups of elements using only one type of carets are usually exponentially distorted (see [13] and [6]). However, we show that here, the analogous subgroups of \( F_\tau \) using only carets of a single type give undistorted copies of \( F \).

The reader is assumed to have some familiarity with Thompson’s group \( F \). Many of the arguments for \( F_\tau \) will be similar to those for \( F \). In order to avoid repetitions and making this paper unnecessarily long, we will refer to the corresponding constructions and results for \( F \) when necessary. A good introduction for \( F \) which contains many results which apply here can be found in [9].

1. **Definition and first properties**

Let \( \tau \) be the small golden ratio \( \frac{\sqrt{5} - 1}{2} = 0.6180339887... \), which is a zero of the polynomial \( X^2 + X - 1 \). We will consider the ring \( \mathbb{Z}[\tau] \) of elements of the type \( a + b\tau \), where \( a \) and \( b \) are integers. Observe that \( \tau = (1 + \tau)^{-1} \) is a unit of this ring, so the multiplicative cyclic group of powers of \( \tau \) is a group of units and then we can consider the group \( G([0, 1]; \mathbb{Z}[\tau], \langle \tau \rangle) \), according to the notation in the Bieri-Strebel notes [1]. This group was introduced by Cleary in [10], where it is proven that the group is of type \( F_\infty \), so in particular, it is finitely presented. In that paper, Cleary also describes a combinatorial structure for \( F_\tau \), which we are going to develop here, since it will be used extensively throughout this paper.

Observe that the equality \( 1 = \tau + \tau^2 \) can be used to give a subdivision of the unit interval in two intervals in two ways:

\[
[0, 1] = [0, \tau] \cup [\tau, 1], \quad \text{and} \quad [0, 1] = [0, \tau^2] \cup [\tau^2, 1].
\]

The unit interval is subdivided in two intervals of lengths \( \tau \) and \( \tau^2 \). Since we also have that \( \tau^k = \tau^{k+1} + \tau^{k+2} \) (for all \( k = 0, 1, 2, \ldots \)), each subinterval can be subdivided further, and the smaller intervals have length equal to a power of \( \tau \). Hence, the interval can be subdivided several times into a subdivision with \( n \)
intervals. If we have two such subdivisions into \( n \) intervals, and we map linearly the intervals in order-preserving fashion, we obtain an element of \( F_\tau \). In \[10\], Cleary proves that every element of \( F_\tau \) can be obtained in this way.

This opens the door to a combinatorial approach to \( F_\tau \) using binary trees, with a caret representing a subdivision, in a very similar fashion as is usually done for Thompson’s group \( F \). The only particularity here is that we will need two types of carets, since the same interval can be subdivided in two ways. As is common, a subdivision will be represented by a caret, however the carets here will have two edges of different lengths corresponding to the fact that the two intervals also have different lengths.

Subdivisions are represented by carets the following way. In Figure 1 we have the two subdivisions of the unit interval given above, represented by their carets.

![Figure 1. The two subdivisions of the unit interval and their carets.](image)

And an example of an iterated subdivision with its corresponding tree is given in Figure 2.

It may seem surprising that the longer subinterval is represented by a short edge in the caret. The reason this is done this way is that then the nodes in the tree are organized by levels according to the lengths of their corresponding subintervals. A node in level \( k \) will correspond to an interval of length \( \tau^k \). Hence, doing it this way, the tree carries more information than just the combinatorial structure of the intervals. See Figure 2 again for the levels of each node, including the leaves, and verify that the level corresponds to the length of the interval.

**Definition 1.1.** A caret with a long left edge and a short right edge is called an \( x \)-caret or a caret of \( x \)-type, whereas the other type is called a \( y \)-caret.

The reason for this nomenclature will be apparent later. And, as stated above, recall that a long edge corresponds to a shorter subinterval and a short edge to a long subinterval.

An interesting feature of this group is that there are subdivisions which correspond to more than one tree. In particular, the easiest example of this phenomenon is
the subdivision of the unit interval into

\[ [0, \tau^2] \quad [\tau^2, \tau] \quad [\tau, 1] \]

and recall that \( \tau^2 = 1 - \tau \), so the subdivision is symmetric, with intervals of length \( \tau^2, \tau^3, \tau^2 \). Hence, it can be obtained by two subdivisions, one with two \( x \)-carets and one with two \( y \)-carets, as seen in Figure 3. This difficulty for the diagrams in \( F_{\tau} \) will be fixed with the definition of normal form in Section 6, and a particular type of diagram which will be unique for each element.

These two trees are quite important in the development of this group. Since they represent the same subdivision, they are completely interchangeable. A basic move is a process of replacing one of these two configurations inside a tree by the other one. Performing a basic move on a tree yields the same subdivision. See Figure 4 where the tree in Figure 2 has had a basic move performed on it on the two carets marked with thicker lines.
Figure 3. A subdivision of the unit interval and two trees that represent it.

Figure 4. Performing a basic move on the tree in Figure 2.

We will make use of this basic move extensively in this paper.

Hence, an element in $F_\tau$ will be given by two trees with the same number of leaves. Clearly, an element can have more than one tree pair diagram representing it. Besides the usual phenomenon of nonreduced diagrams giving the same element by reducing a diagram (deleting meaningless subdivisions), here we can have two reduced diagrams representing the same element. An example is given in Figure 5. A basic move on the right-hand-side diagram will allow one caret to be erased and we obtain the diagram on the left.

2. Multiplication

Multiplication in the group $F_\tau$ is performed by composition of the two maps. When the two elements are given by tree pairs, to be able to multiply we find a common subdivision for the target tree of the first element, and the source tree of the second element. Take two elements given by tree pairs $(T_1, T_2)$ and $(S_1, S_2)$. If it happens
that \( T_2 = S_1 \), then the composed element will be \((T_1, S_2)\). Hence, if those trees are not the same, we find larger tree pairs for each element in such a way that those two trees are the same. This can be done using the following Lemma.

**Proposition 2.1.** Given two trees \( T \) and \( T' \) (with \( x \)- and \( y \)-carets), there exists a tree \( T'' \) which is a finer subdivision to both.

**Proof.** In cases when the carets are all the same type, as it happens in \( F \), this can be done by just adding a few carets to construct the least common refinement. In our case, carets are of two types, so some carets need to be switched to the other type. Use Figure 6 as a reference to see how carets can be made to switch types. The following Lemma will also be useful later.

**Lemma 2.2.** Any caret in a tree \( T \) can always be switched from \( x \)-type to \( y \)-type or vice versa, by adding at most one caret to \( T \).

**Proof of Lemma.** If the caret to be switched can have a basic move performed to it, then that switches the type. If the child in the short edge is not there, add one of the same type to perform the basic move (see picture 2 of Figure 6). And if the child caret on the short side is the opposite type, keep going down short sides until a basic move can be done. Should this process not result in a basic move, add a caret to the bottom one and perform several basic moves going back up to the caret to be switched. See pictures 3, 4 and 5 on Figure 6.

By changing the types of some carets (maybe with the aid of some added carets), we can find a subdivision of \( T \) which is also a subdivision of \( T' \). Add the necessary carets to make them equal and construct the common subdivision \( T'' \) (see picture 6 in Figure 6). This ends the proof of Proposition 2.1.

This procedure finishes the construction of the algorithm to perform the multiplication of two elements given by two tree pairs \((T_1, T_2)\) and \((S_1, S_2)\). Find the tree \( T_3 \) which is the common subdivision for \( T_2 \) and \( S_1 \), and find two tree pairs which represent the same elements which look like \((T'_1, T_3)\) and \((T_3, S'_2)\). This is done adding to \( T_1 \) the carets corresponding to those we have added to \( T_2 \), and the same for the other pair. Finally, the multiplication element is given by \((T'_1, S'_2)\). See a simple example in Figure 7.
3. Presentation

To find generators for $F_\tau$ we can follow the model for $F$. The infinite generating set for $F$ has generators which extend down the right hand side of the tree, on a row of carets which are all right children. In the case of $F_\tau$, we have two types of carets, so it would appear that we need generators where the row of carets could be of both types. But in view of the Lemma 2.2, a caret can always switch types. This
Figure 7. An example of how to multiply two elements when the corresponding carets do not coincide. In dashed lines the carets which are being added to be able to perform the multiplication.

Figure 8. A spine.

fact suggests that only one type of carets is needed for the row of right-side carets. We begin with an auxiliary definition which will save notation in the future.

Definition 3.1. A tree which has only right-side carets of $x$-type is called a spine.
Generators for $F_\tau$ will have a spine to which an extra caret is added at the end, as a left child on the source tree, and as a right child on the target one.

**Definition 3.2.** We define the elements $x_n$ in $F_\tau$, for $n \geq 0$, by a tree-pair diagram $(T_1, T_2)$, where $T_2$ is a spine with $n + 2$ carets, and where $T_1$ is a spine with $n + 1$ carets together with an extra $x$-caret on the last left leg. Note that all carets in $x_n$ are $x$-carets, see Figure 9. The same way, we define the generators $y_n$ by having the same spine, but then the caret added in the source tree is of $y$-type.

Observe from Figure 9 that the key caret is of type $x$ for the generators $x_n$ and of type $y$ for $y_n$, and this is the reason the carets are called this way. However, the spines have only $x$-carets in both cases.

Our goal is to prove that the set of generators $x_n$ and $y_n$, for $n \geq 0$, is a set of generators for $F_\tau$. In a similar fashion to that for $F$, if the target tree of an element
is a spine, this element is the product of generators (without taking inverses). In Figure 10 we can see an example of an element which is the product of three generators, obtaining a pair made of a tree and a spine. This is all quite similar to the situation in Thompson’s group $F$.

In Figure 11 we see why any element given by a tree and a spine can be written as a product of generators $x_n$ and $y_n$. If we multiply an element with a spine as a target tree by the generator $x_i$ or $y_i$, the result is to attach a caret of the corresponding type to the $i$-th leaf. In this way we can construct a tree paired with a spine. Observe though that the tree constructed this way will always have in the right-hand side all carets of the $x$-type. This is because all generators have only $x$-carets in the right-hand sides.

Using this construction we can prove our first result.

**Proposition 3.3.** The set of elements $x_n$ and $y_n$, for $n \geq 0$, is a set of generators for $F_r$.

**Proof.** Take any element of $F_r$ given as a pair $(T_1, T_2)$ of trees. Using Lemma 2.2 we can assume that all carets of the right-hand side of each tree are $x$-carets. If the trees have an $y$-caret in the right-hand side, use the Lemma to change the type of these carets, at the price of adding a few carets to the trees. But the result will be a pair of trees whose right-hand sides have only $x$-carets.

Now, put a spine $S$ in between the two trees. The first tree pair $(T_1, S)$ gives an element which, by the construction specified above, is the product of generators $x_n$ or $y_n$. The second pair $(S, T_2)$ is the inverse of an element also of this type. Hence, any element is product of generators $x_n$ or $y_n$. $\square$
It is not hard to see that there are some relations which are satisfied by these generators. The combinatorics of the carets, similar to those of $F$, give the following four sets of relators:

1. $x_jx_i = x_ix_{j+1}$
2. $x_jy_i = y_ix_{j+1}$
3. $y_jx_i = x_iy_{j+1}$
4. $y_jy_i = y_iy_{j+1}$

where in all cases we have $i < j$. Another clear set of relators given by the subdivision which admits two expressions as carets, since both expressions can be written in terms of the generators. These relations are $y_n^2 = x_nx_{n+1}$. The goal of the next theorem is to show that these are all relations needed to have a presentation for $F_\tau$.

**Theorem 3.4.** A presentation for $F_\tau$ is given by the generators $x_i, y_i$, with the relations

1. $x_jx_i = x_ix_{j+1}$
2. $x_jy_i = y_ix_{j+1}$
3. $y_jx_i = x_iy_{j+1}$
4. $y_jy_i = y_iy_{j+1}$
5. $y_i^2 = x_ix_{i+1}$

for $0 \leq i < j$. 
The proof of this theorem will be based on the following lemma.

**Lemma 3.5.** Given two trees representing the same subdivision of the unit interval, we can always go from one to the other by a sequence of basic moves.

**Proof of the lemma.** Let $T_1$ and $T_2$ be the two trees which represent the same subdivision, and assume that their root caret is different. Assume that $T_1$ has an $x$-type root caret and $T_2$ has a $y$-type root caret. Looking at the tree $T_2$, since it has a $y$-caret at the root, this forces the common subdivision to have a break at the point $\tau$ in the interval. This means that at $T_1$, the right edge (which is short) needs to be subdivided further, because we need the break at $\tau$ on $T_1$ too (recall that the two subdivisions are the same). We are going to show that in order for the break at $\tau$ to show up at $T_1$, there has to be two consecutive carets of the same type somewhere on $T_1$, so that a basic move can be performed.

If the right edge of the root caret in $T_1$ is subdivided further and this subdivision is of $x$-type, then we have two consecutive carets of the same type and a basic move could be performed at the root, so the root caret would become a $y$-caret. Hence, assume that the short edge of the $x$-type root has a $y$-caret as child. But then the breaks are at the points $\tau^2$ and $2\tau^2 = 1 - \tau^3$, and observe that we have

$$\tau^2 < \tau < 1 - \tau^3,$$

so the desired breakpoint still has not been produced. See Figure 13.

So the tree $T_1$ needs to be subdivided further. The point of the proof is that it is strictly necessary to have two consecutive subdivisions of the same type ($x$- or $y$-depending on the parity) to have the break at $\tau$. This is because of the following sequence (for even $n$, the odd case is similar):

$$\tau = \tau^2 + \tau^3$$
$$= \tau^2 + \tau^4 + \tau^5$$
$$= \tau^2 + \tau^4 + \tau^6 + \tau^7$$
$$\ldots$$
$$= \sum_{k=1}^{n} \tau^{2k} + \tau^{2n+1}$$
$$\ldots$$

and the odd power can only be produced with two consecutive carets of the same type, see Figure 12.

If below the short edge of the root caret the subsequent carets are alternating the type on their short edges, we never reach the value $\tau$ because of the strict inequalities

$$\sum_{k=1}^{n} \tau^{2k} < \tau < 1 - \sum_{k=1}^{n} \tau^{2k+1}$$
caused by the fact that we need infinite series to get $\tau$, i.e.

$$\sum_{k=1}^{\infty} \tau^{2k} = \tau = 1 - \sum_{k=1}^{\infty} \tau^{2k+1}$$

and an infinite tree would be required. Again, Figure 12 should give an idea of why this is true.

The conclusion of this argument is that since the tree is finite, to be able to agree the breaks among the two trees, in one of the trees there has to be two consecutive carets of the same type and a basic move can be performed on them. Once this is done, the root caret is of the same type on both trees, and observe that no caret needs to be added at any moment, unlike in the multiplication algorithm. Hence, we can keep going down the tree switching types of all the carets of different type, adding nothing, until the two trees are exactly the same. \hfill \Box

Once the lemma has been established, we can prove the theorem.

**Proof of Theorem 3.4.** Given a word in the generators $x_i, y_i$ which is the identity, when we construct its corresponding tree-pair diagram, the two trees have to give the same subdivision. Also, the two trees will have a spine (all $x$-carets) in their right hand sides. According to Lemma 3.5, we can go from one to the other by applying basic moves to one of them, and in this case, the basic moves are never performed on a vertex on the right hand side of the tree. Observe that each basic move corresponds exactly to multiplying our word by a conjugate of relation (5), noting that all instances of relation (5) have spines and hence are precisely those that we need. Hence, using relation (5) we can obtain a word which yields a diagram where the two trees are the same. The exact same way as it is done in Thompson’s group $F$, this diagram is consequence of relations of the type (1) to (4), using the appropriate relation to the type of caret we have at each step. Hence, our identity word is a consequence of the relations (1) to (5). \hfill \Box
This presentation allows us to establish a correspondence between tree-pair diagrams and a particular type of words. This correspondence is completely analogous to that in Thompson’s group $F$, based on leaf exponents. See [9 Theorem 2.5] or
Observe that relations (1)-(4) allow for the ordering of the generators in a word by index, increasingly for positives and decreasingly for negative elements. Namely, we have the following result.

**Proposition 3.6.** Any element of $F_{\tau}$ admits an expression of the type

$$a_{i_1}a_{i_2}\ldots a_{i_n}b_{j_m}^{-1}\ldots b_{j_2}^{-1}b_{j_1}^{-1}$$

where:

1. The letters $a$ and $b$ represent either $x$ or $y$.
2. $i_1 \leq i_2 \leq \ldots \leq i_n$ and $j_1 \leq j_2 \leq \ldots \leq j_m$.

This is analogous to the normal form for Thompson’s group $F$. This expression for an element corresponds to its tree pair diagram using leaf exponents. The only difference between $F_{\tau}$ and $F$ is that since here we have two types of carets corresponding to two different generators, we can alternate generators $x_i$ and $y_i$ within the same index, as in the example at Figure 10, where we have the element $y_0x_1y_1$.

It is not difficult to deduce a finite presentation from the infinite one. From the relations (1)-(4) it is easily seen that the generators with index 2 or higher are conjugates to those with index 1. Hence, the only generators needed are $x_0, x_1, y_0, y_1$. The same way, for each family of relators (1) to (4), only two are needed in the exact same fashion as it happens in Thompson’s group $F$, see, for instance, [9]. For the family (5), observe that if $i ≥ 2$, the relation $y_i^2 = x_i y_{i+1}$ is a conjugate (by the appropriate power of $x_0$) of $y_1^2 = x_1 x_2$. Hence, all relations we need are:

$$
x_2x_1 = x_1x_3 \quad x_3x_1 = x_1x_4
\quad x_2y_1 = y_1x_3 \quad x_3y_1 = y_1x_4
\quad y_2x_1 = x_1y_3 \quad y_3x_1 = x_1y_4
\quad y_2y_1 = y_1y_3 \quad y_3y_1 = y_1y_4
\quad y_2^2 = x_0x_1 \quad y_4^2 = x_1x_2.
$$

We do not claim that this presentation is optimal, and it is possible that there is a presentation with fewer generators or with fewer relations.

### 4. Abelianisation and the commutator subgroup

Once we have a presentation, it is easy to abelianise the group. The abelianised group has four generators $\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1$, and observe that the relations (1)-(4) abelianise trivially. Hence the quotient abelian group has two relations, namely

$$2\bar{y}_0 = \bar{x}_0 + \bar{x}_1 \quad 2\bar{y}_1 = 2\bar{x}_1,$$

changing to additive notation for the abelian group. From the first relation we can eliminate the generator $\bar{x}_0$, and we can also consider the element $\bar{z} = \bar{x}_1 - \bar{y}_1$,
and the abelianisation can be considered as generated by \( \bar{x}_1, \bar{y}_0, \bar{z} \) with the relation \( 2\bar{z} = 0 \). The abelianisation is hence isomorphic to \( \mathbb{Z}^2 \oplus \mathbb{Z}_2 \).

The commutator subgroup, i.e. the kernel of the abelianisation map, can also be completely understood. Looking at the generators \( \bar{x}_1 \) and \( \bar{y}_0 \), we see that they represent the slopes at 0 and at 1, in a much similar way as it is done for Thompson’s group \( F \). The map from \( F_\tau \) to \( \mathbb{Z}^2 \) given by the two components of the abelianisation map generated by \( \bar{x}_1 \) and \( \bar{y}_0 \) coincides (up to a change of basis in \( \mathbb{Z}^2 \)) with the map that sends every element to the two slopes at 0 and 1. The situation is quite similar to that of \( F \).

**Definition 4.1.** We say that an element \( f \in F_\tau \) has support bounded away from 0 and 1, or just bounded support for short, if there exists \( \varepsilon > 0 \) such that \( f \) is the identity in the intervals \([0, \varepsilon]\) and \([1 - \varepsilon, 1]\). We will denote the subgroup of elements with bounded support by \( F_\tau^c \).

Observe then that the commutator subgroup is contained in \( F_\tau^c \). But the \( \mathbb{Z}_2 \) component means that it is not equal to it. To describe it clearly, let \( z = x_1 y_1^{-1} \), and observe that \( z \) maps to the \( \bar{z} \) mentioned above by the abelianisation map.

**Proposition 4.2.** The commutator subgroup for \( F_\tau \) is formed exactly by those elements in \( F_\tau^c \) such that the total exponents in \( \bar{x}_1 \) and \( \bar{y}_1 \) are both even. Equivalently, they are the elements in \( F_\tau^c \) which have even exponent for \( z \), i.e., which abelianise to zero on the \( \mathbb{Z}_2 \) component.

The proof is elementary by looking at the interpretations of the abelianisation given above.

According to Proposition 3.6 and the corresponding word-diagram, the extra condition for an element to be in \( F_\tau^c \) (the total exponents in \( \bar{x}_1 \) and \( \bar{y}_1 \) in the abelianisation are both even) can be read off the diagram. Recall that a binary tree has left, right and interior carets according to their location in the tree. Left carets are on the left side of the tree, each of them connected to the root by a chain of left children. Right carets are connected to the root by right children, and interior carets are those carets that are neither left or right, see, for instance, [11] or [4].

Let a diagram have the trees \( T_1 \) and \( T_2 \). We can identify the total exponent for \( \bar{x}_1 \) and \( \bar{y}_1 \) according to the number of carets in the diagram. Define the following numbers, for \( i = 1, 2 \):

- Let \( n_i \) be the number of interior \( x \)-carets in \( T_i \).
- Let \( m_i \) be the number of interior \( y \)-carets in \( T_i \).
- Let \( r_i \) be the number of left \( x \)-carets in \( T_i \).
- Let \( s_i \) be the number of left \( y \)-carets in \( T_i \).

Then we have the following situation:
Proposition 4.3. When the element given by the diagram \((T_1, T_2)\) is abelianised, the component for \(\overline{x}_1\) is \(n_1 - r_1 - n_2 + r_2\). Also, the total number of \(\overline{y}_1\) is \(m_1 - m_2\).

Furthermore, the number \(s_1 - s_2\) gives the component for \(\overline{y}_0\), but this is irrelevant for the extra condition.

Observe that to obtain the total exponent for \(\overline{x}_1\) one has to take the \(n_i\), which correspond to the generators \(x_i, i \geq 1\), but also the \(r_i\), which correspond to the generators \(x_0\) in the word. This is because in the abelianisation, each \(\overline{x}_0\) is replaced by \(-\overline{x}_1 + 2\overline{y}_0\). Hence, each \(\overline{x}_0\) contributes with a unit to the exponent for \(\overline{x}_1\). Observe that we will only be interested in parity, so we can discard all negative signs.

Hence, just for looking at the tree-pair diagram we can know if an element is in the commutator subgroup or not.

Theorem 4.4. An element given by a diagram \((T_1, T_2)\) is in \(F'_{\tau}\) if and only if the following conditions are all satisfied:

1. The level of the leftmost leaf is the same for \(T_1\) and \(T_2\), that is, \(2r_1 + s_1 = 2r_2 + s_2\). This corresponds to the fact that the element must be the identity in a neighbourhood of 0.
2. The level of the rightmost leaf is the same for \(T_1\) and \(T_2\). This corresponds to the fact that the element must be the identity in a neighbourhood of 1.
3. The total exponents for \(\overline{x}_1\) and \(\overline{y}_1\) are both even, i.e. \(n_1 + r_1 + n_2 + r_2\) and \(m_1 + m_2\) are both even.

This interpretation will be useful in the next section.

5. Simplicity

The goal of this section is to prove that the commutator subgroup \(F'_{\tau}\) is a simple group. This proof will follow several steps.

Definition 5.1. Let \(a, b \in \mathbb{Z}[\tau]\), with \(0 < a < b < 1\). Then we denote by \(F_{\tau}[a, b]\) the subgroup of \(F_{\tau}\) of those elements whose support is included in \([a, b]\). Within \(F_{\tau}[a, b]\), we denote by \(F'_{\tau}[a, b]\) its commutator subgroup and also \(F_{\tau}^c[a, b]\) its subgroup of elements with bounded support (i.e. they are the identity in a neighbourhood of \(a\) and in one of \(b\)). For clarity, observe that the support of an element in \(F_{\tau}^c[a, b]\) is included in \([a + \varepsilon, b - \varepsilon]\) for some \(\varepsilon > 0\).

We have that for any \(a, b\) the subgroup \(F_{\tau}[a, b]\) is isomorphic to \(F_{\tau}\).

Proposition 5.2. \(F_{\tau}[a, b] \cong F_{\tau}\).

Proof. According to [10, Corollary 1], there exists an element \(f \in F_{\tau}\) such that \(f(\tau^2) = a\) and \(f(\tau) = b\). Conjugating by \(f\), we see that \(F_{\tau}[a, b] \cong F_{\tau}[\tau^2, \tau]\). To see
that $F_τ[τ^2, τ]$ is isomorphic to $F_τ$, we only need to scale the maps by a factor of $τ^3$, which is the length of $[τ^2, τ]$, and observe that $τ$ is a unit of the ring $ℤ[τ]$. □

Since the support of the elements of $F_τ[τ^2, τ]$ is contained in $[τ^2, τ]$ we can identify these by a tree-pair diagram given by the 2-caret spine which appears at the root, and the rest of the diagram hanging only from the middle leaf of this 2-caret spine, see Figure 14.

So we know now what $F_τ[a, b]$ looks like. Now we will look at its commutator subgroup.

**Lemma 5.3.** Let $f ∈ F'_τ$, and choose $a, b ∈ ℤ[τ]$ such that $f$ is actually in $F'_τ[a, b]$ (i.e. its support is strictly included in $[a, b]$, see above). Then, $f ∈ F'_τ[a, b]$.

Observe that from the fact that $f ∈ F'_τ$ we cannot conclude that $f$ is in $F'_τ[a, b]$, because the extra condition, see Theorem [14] for $x_1$ and $y_1$ refers to the generators of $F_τ$ and not to those of $F_τ[a, b]$. We need to relate the generators of both groups to be able to establish the result, and this is what is done in the proof.

**Proof.** As in the previous proof, we can assume that $a = τ^2$ and $b = τ$. The element $f$ has bounded support in $F_τ[τ^2, τ]$, but in order for it to be in $F'_τ[τ^2, τ]$ then it must satisfy the extra condition with respect to the generators of $F_τ[τ^2, τ]$.

Let $φ : F_τ → F'_τ[τ^2, τ]$ be the isomorphism described above, i.e., by hanging trees to the middle leaf of a two-caret spine. Then, $F'_τ[τ^2, τ]$ is generated by $φ(x_0), φ(x_1), φ(y_0), φ(y_1)$. See Figure 14 to clarify this situation.

Consider the two trees $(T_1, T_2)$ such that the diagram for $f$ is obtained by attaching $T_1$ and $T_2$ to the middle leaf of a 2-caret spine, as we did above. Let $n_i, m_i, r_i, s_i$ be the numbers of right and interior $x$-type and $y$-type carets as defined in the previous section. Then, observe that we know $f$ is in $F'_τ$ and we want to see that $f ∈ F'_τ[τ^2, τ]$. But the diagram for $f$ when considered in $F'_τ[τ^2, τ]$ would be $(T_1, T_2)$, whereas the diagram for $f$ when considered in $F_τ$ has the trees $T_1$ and $T_2$ attached to a two-caret spine.

Hence, the number of $x_1$ and $y_1$ for $f$ in $F_τ$ has to consider all the carets in $T_1$ and $T_2$ as interior carets, since they hang from the middle leaf in a two-caret spine. This means that by being in $F'_τ$ we know that the numbers $n_1 + r_1 + n_2 + r_2$ and $m_1 + s_1 + m_2 + s_2$ are even (observe that the right carets in $T_1$ and $T_2$, which are now interior, are the same number in both trees so their number is always even). And to see that $f ∈ F'_τ[τ^2, τ]$ we need that the numbers which have to be even are now $n_1 + r_1 + n_2 + r_2$ and $m_1 + m_2$. The first of these numbers is the same in both cases, and for $m_1 + m_2$ we only need to see that since $f$ is the identity in a neighbourhood of $τ^2$, we have that $2r_1 + s_1 = 2r_2 + s_2$ and then $s_1 - s_2$, and hence $s_1 + s_2$, is even. So from this and from $m_1 + s_1 + m_2 + s_2$ being even, we conclude that $m_1 + m_2$ is even and hence $f ∈ F'_τ[τ^2, τ]$. □

We are now in situation of stating and proving the main theorem.
Theorem 5.4. The group $F'_\tau$ is a simple group.

The proof will be spelled out in the remaining part of this section. It will be based in the following theorem, due to Higman. Let $\Gamma$ be a group of bijections of some set $E$. For $g \in \Gamma$ define its moved set $D(g)$ as the set of points $x \in E$ such that
\(g(x) \neq x\). This is analogous to the support, but since \textit{a priori} there is no topology on \(E\), we do not take the closure.

**Theorem 5.5** (Higman). Suppose that for all \(\alpha, \beta, \gamma \in \Gamma \setminus \{1\}\), there is a \(\rho \in \Gamma\) such that the following holds: \(\gamma(\rho(S)) \cap \rho(S) = \emptyset\) where \(S = D(\alpha) \cup D(\beta)\). Then the commutator subgroup \(\Gamma'\) is simple.

The proof can be seen in \[12\].

The idea of using this theorem is to take advantage of the high transitivity of Thompson-like groups to see that they easily fulfill the conditions of Higman’s theorem. As we have already used before, and \[10\] Corollary 1 implies that given two closed intervals \(A\) and \(B\), such that \(0, 1 \notin A\), there exists an element of \(\Gamma'\) such that \(f(A) \subset B\). Hence, the conditions of Higman’s theorem are easily satisfied: Since \(\gamma \neq 1\) it is easy to find an interval \(C\) such that \(\gamma(C) \cap C = \emptyset\). Also, use transitivity to find \(\rho\) to send \(S\) inside \(C\).

The only thing is that the condition \(0, 1 \notin A\) means that Higman’s theorem cannot be applied to \(F_{\tau}\), because there are many elements whose support is the whole unit interval. Hence, we can only apply Higman’s theorem to the commutator \(F_{\tau}'\), because all its elements have bounded support. The conclusion of the application of Higman’s theorem is then that the \textit{second} commutator \(F_{\tau}''\) is simple. The proof of Theorem 5.4 will be finished when we prove the following lemma.

**Lemma 5.6.** We have that \(F_{\tau}'' = F_{\tau}'\).

\textit{Proof of Lemma 5.6.} Clearly we have that \(F_{\tau}'' \subset F_{\tau}'\). For the reverse inclusion, take \(f \in F_{\tau}'\). Since \(f \in F_{\tau}'\), choose \(a, b \in \mathbb{Z}[\tau]\) such that \(f \in F_{\tau}'[a, b]\), namely, if the support of \(f\) is included in the interval \([c, d]\), choose \(a, b\) satisfying \(0 < a < c < d < b < 1\). According to Lemma 5.3 we have that \(f \in F_{\tau}'[a, b]\). Hence, we have that \(f = [p_1, q_1][p_2, q_2] \ldots [p_k, q_k]\), where \(p_i, q_i \in F_{\tau}[a, b] \subset F_{\tau}'\). To finish the proof and see that \(f \in F_{\tau}'\), it would be enough to prove that \(p_i, q_i\) are in \(F_{\tau}'\), but this need not be true. We will modify these elements to get the desired result.

Observe that \(p_i, q_i\) have support in \([a, b]\), but we have no information on whether they have an even or odd number of generators \(z\) when abelianised. For \(p_i, q_i\) to be in \(F_{\tau}'\) we would need each of them to have an even number of generators \(z\), and this is not necessarily the case. To solve this, we will do the following. Observe that the element \(z = x_1y_1^{-1}\) has bounded support, namely, it is in \(F_{\tau}'\). Choose now a tiny interval \([u, v]\) such that \([a, b] \cap [u, v] = \emptyset\). This means that either \(0 < u < v < a\) or \(b < u < v < 1\), either one works. As we have done before, and according to \[10\] Corollary, choose an element \(g \in F_{\tau}\) which maps the support of \(z\) inside \([u, v]\). Let \(z'\) be the conjugate of \(z\) by \(g\) in such a way that the support of \(z'\) is now inside \([u, v]\). Finally, since \([u, v]\) is disjoint with \([a, b]\), we have that \(z'\) commutes with each of the \(p_i, q_i\), for all \(i = 1, \ldots, k\). Hence, because of this
commuting, we have that for each \( i = 1, \ldots, k \),
\[
[p_i, q_i] = [p_i z', q_i] = [p_i z' z'_i] = [p_i z', q_i z'_i]
\]
and since \( z' \) is a conjugate of \( z \), it contributes exactly with one generator \( \bar{z} \) to the abelianisation. Hence, for each \( i \), exactly one of these four commutators has both terms with an even number of \( \bar{z} \). For instance, if \( p_1 \) has an odd number of generators \( \bar{z} \) and \( q_1 \) has an even number, the commutator we choose to have two elements with even \( \bar{z} \) will be \([p_1 z', q_1]\).

By choosing the appropriate commutator for each \( i \), we can get all \( k \) commutators to have two terms with even number of \( \bar{z} \), and hence we conclude that all terms involved in all commutators are in \( F' \), and from this, finally, that \( f \in F'' \). \( \square \)

This lemma, together with Higman’s theorem applied to \( F' \), implies that \( F' \) is simple.

6. Normal Form

In this section we describe a normal form (with uniqueness) for \( F_\tau \) that is very similar to that for \( F \). A word over the \( x_i \) and \( y_i \) will be said to be in seminormal form if it has the following form:
\[
x_0^{a_0} y_0^{\epsilon_0} x_1^{a_1} y_1^{\epsilon_1} \cdots x_n^{a_n} y_n^{\epsilon_n} x_{-b_m}^{-b_m} x_{-b_{m-1}}^{-b_{m-1}} \cdots x_1^{-b_1} x_0^{-b_0}
\]
where \( a_i, b_i \geq 0 \) and \( \epsilon_i \in \{0, 1\} \). Observe that \( y \) generators only appear in the positive part of the word, and that they are only allowed to have exponents zero or one. From the correspondence between diagrams and words on the generators via leaf exponents described in Section 3, the existence of a seminormal form for each element of \( F_\tau \) follows from the following result.

**Lemma 6.1.** Let \((S_1, S_2)\) be a tree-pair with \( S_i \) having only \( x \)-carets down the right spine. There exists a tree-pair \((T_1, T_2)\) representing the same element of \( F_\tau \) that satisfies the following:

1. \( T_2 \) has no \( y \)-carets,
2. \( y \)-carets in \( T_1 \) have no left children.

Moreover, the number of carets in \( T_i \) is bounded above by three times the number of carets in \( S_i \).

**Proof.** As noted previously all elements of \( F_\tau \) have tree pairs in which \( T_1 \) and \( T_2 \) have only \( x \)-carets on their right side. Starting with such a pair, we first modify \( T_2 \) so that all \( y \)-carets in \( T_2 \) have no left children. This is done by working from right to left as follows. Suppose a \( y \)-caret is such that it has left children, but that all \( y \)-carets of higher leaf index do not have left children. We want to swap the type of the caret. If the immediate left child is also a \( y \)-caret, then we perform a basic move. Suppose then that we have an \( y \)-caret whose left child is an \( x \)-caret.
There are three possibilities determined by the right child of the $x$-caret. These are illustrated in Figure 15. Note that in the third case in the picture, the bottom $y$-caret has a higher leaf index than the top one, so it must have no left child. In each case, after adding at most one caret, the original $y$-caret can be moved down closer to the leaf. The new tree $T_2$ now has the property that each $y$-caret has no left child. Now for each $y$-caret in $T_2$ add another $y$-caret as the left child and perform a basic move. The resulting tree $T_2$ now has no $y$-carets.

Following the same process as above, we can move $y$-carets in $T_1$ down the tree to ensure that $T_1$ satisfies (2). We need to be careful not to add any $y$-carets to $T_2$. To that end we modify the third case to be that shown in Figure 16, adding two $x$-carets instead of a $y$-caret. Notice that given a tree-pair $(S_1, S_2)$ the above proof

![Figure 15. Changing the type of a $y$-caret. The $y$-carets are bold. In the first case, the indicated leaf is subdivided by the addition of an $x$-caret and then two basic moves are carried out. In the second case no subdivision is needed. In the third situation a $y$-caret is added and then a sequence of three basic moves applied. Note that in each case the original (topmost) $y$-caret has been moved down the tree.](image-url)
produces a tree-pair \((T_1, T_2)\) satisfying the conclusion of the lemma and such that the number of carets added is at most twice the original number of carets. \(\square\)

Two different words, each in seminormal form, can represent the same element of \(F_T\). This can happen in two ways. First, we can have a reduction similar to the one in \(F\), where Thompson relators can be applied to reduce the subscripts of many generators in the word. This corresponds to a diagram being nonreduced and the erasing of exposed matching carets. The second way this can happen is more subtle and corresponds to an example such as \(x_0y_0x_2x_1^{-1}x_0^{-1} = y_0\). Both these words are in seminormal form and both are reduced, but after performing a couple of basic moves, two carets become exposed and they can be cancelled. See Figure 17. This will be called a hidden cancellation. Fortunately, the only possible hidden cancellations will be exactly of this type. A hidden cancellation shows up every time we have a subword of the form \(x_iy_ix_{i+2}ux_{i+1}^{-1}x_i^{-1}\) where \(u\) is a word involving generators of index at least \(i+3\). If that happens, we can do the following sequence of equalities using relators:

\[
\begin{align*}
x_iy_ix_{i+2}ux_{i+1}^{-1}x_i^{-1} &= x_ix_{i+1}y_iux_{i+1}^{-1}x_i^{-1} = y_i^3uux_{i+1}^{-1}x_i^{-1} = y_i^3ux_{i+1}^{-1}x_i^{-1} = y_i^3ux_{i+1}^{-1}x_i^{-1} = y_iu'
\end{align*}
\]

where \(u'\) is the same word as \(u\) but with all subscripts lowered by 2.

These two types of reductions are the only possible obstructions for the uniqueness of the seminormal form, as we will show next. Hence, we define a normal form as a word which is not allowed to have any of these possible reductions.

**Definition 6.2.** A word \(w\) is said to be in normal form if it is in seminormal form and, in addition, for all \(i\) we have:

1. If \(a_i\) and \(b_i\) are both nonzero, then at least one of \(a_{i+1}, b_{i+1}, \epsilon_i, \epsilon_{i+1}\) is nonzero.
2. If \(w\) contains a subword of the form \(x_iy_ix_{i+2}ux_{i+1}^{-1}x_i^{-1}\), then \(u\) contains a generator with index either \(i+1\) or \(i+2\).
As said before, these conditions are best understood in terms of tree-pair diagrams. The first condition, as for $F$, corresponds to matching exposed carets that can be eliminated. The second condition corresponds to a situation in which two basic moves result in matching exposed carets. This is illustrated in Figure 17.

Figure 17. A hidden cancellation. The tree-pair on the left is reduced and corresponds to $x_0y_0x_2x_1^{-1}x_0^{-1}$. After performing two basic moves, the two carets in bold can be cancelled. The diagram we obtain is $y_0$.

**Theorem 6.3.** Each element of $F_\tau$ has a unique normal form representative.

**Proof.** The proof is similar to that for $F$. Familiarity with the proof of uniqueness of the normal form for $F$ (as shown, for instance, in [2]) will be of great help understanding this proof.

That each element of $F_\tau$ has a representative word in normal form is straightforward with what we have already shown. The first four relations as listed in Theorem 3.4 can be used, as with $F$, to have the indices in increasing order in the positive part and decreasing in the negative one. Then use Lemma 6.1 to transform this word into seminormal form. If this word then fails either of the conditions in the definition of the normal form, then there is a strictly shorter representative in seminormal form. Keep reducing the word until both conditions are satisfied. Equivalently, keep reducing the diagram both for exposed matching carets and for hidden cancellations.

For uniqueness, consider two normal form words $u$ and $v$ that represent the same element of $F_\tau$ and have minimum total length among all such pairs in the whole group. Let the words be given by

$$u \equiv x_0^{a_0} y_0^{\epsilon_0} u_1 x_0^{-b_0} \quad v \equiv x_0^{c_0} y_0^{\zeta_0} v_1 x_0^{-d_0}$$

where $a_0, b_0, c_0, d_0 \geq 0$, $\epsilon_0, \zeta_0 \in \{0, 1\}$ and $u_1$ and $v_1$ are normal form words in which all subscripts are at least 1. The symbol $\equiv$ is being used to denote equality as words. We will assume that not all $a_0, b_0, c_0, d_0, \epsilon_0, \zeta_0$ are zero. If this were not the case then the following argument can be readily modified by moving to the least subscript for which this is true, but we keep the case of zero for simplicity and clarity.
Equating the slopes at 0 for the piecewise linear maps determined by \( u \) and \( v \) we obtain \( 2a_0 + \epsilon_0 - 2b_0 = 2c_0 + \zeta_0 - 2d_0 \), from which it follows that

\[(*) \quad a_0 - b_0 = c_0 - d_0 \quad \text{and} \quad \epsilon_0 = \zeta_0.\]

Since \( u \) and \( v \) were chosen to minimise the total length we have that one of \( a_0 \) and \( c_0 \) must be zero, or else an \( x_0 \) could be cancelled to obtain shorter words. Similarly, one of \( b_0 \) and \( d_0 \) must be zero. We can assume that \( c_0 = 0 \). It then follows from (\( \Xi \)) that \( d_0 = 0 \) and \( a_0 = b_0 \neq 0 \). We deal separately with the two possible cases: \( \epsilon_0 = 0 \) and \( \epsilon_0 = 1 \), which will correspond to conditions (1) and (2) of the normal form, respectively.

In the case in which \( \epsilon_0 = 0 \) we move the generators \( x_0 \) from \( u \) to \( v \) so we have \( u_1 = x_0^{-a_0} v_1 x_0^{-a_0} = v_2 \) where \( v_2 \) is the word obtained from \( v_1 \) after increasing all subscripts by \( a_0 \). The word \( v_2 \) is in normal form and all subscripts appearing in it are 2 or more. Since \( u_1 = v_2 \), both words are in normal form and the total length is strictly less than that for the original pair, we conclude that \( u_1 \equiv v_2 \). But then the original word \( u \equiv x_0^{-a_0} v_2 x_0^{-a_0} \) would have violated condition (1) in Definition 6.2.

Suppose now that \( \epsilon_0 = 1 \). Our words are now \( u \equiv x_0^{-a_0} y_0 u_1 x_0^{-a_0} \) and \( v \equiv y_0 v_1 \). We move one generator \( x_0 \) from each side of \( u \) to \( v \), so we have

\[
x_0^{-a_0} y_0 u_1 x_0^{-(a_0-1)} = x_0^{-1} y_0 v_1 x_0 = x_0^{-1} y_0 x_0 x_0^{-1} v_2 = x_0^{-1} y_0 x_0 x_1 x_1^{-1} v_2 = x_0^{-1} y_0 x_1 x_1^{-1} v_2 = x_1 y_0 v_3 x_1^{-1} = y_0 x_2 v_3 x_1^{-1}
\]

where \( v_2 \) is the word obtained by increasing all subscripts in \( v_1 \) by 1 (by moving the \( x_0 \) across it), and then subsequently \( v_3 \) also obtained from \( v_2 \) increasing the subscripts while moving \( x_1^{-1} \) across. Notice that all indices for \( v_3 \) are at least 3 and hence the final word is still in normal form. Observe too that the total length for these two words is exactly the same as the original pair, because we have added two generators and later eliminated two more. Now repeating the above for all pairs of \( x_0 \) until they are exhausted, we get

\[
y_0 u_1 = y_0 x_2 v' x_1^{-1}
\]

where \( v' \) is a normal form word in which all subscripts are at least 3, and the length is still the same as the original one. But cancelling the \( y_0 \), and since \( u_1 \) and \( x_2 v' x_1^{-1} \) are in normal form, we conclude, by the minimality of the original pair, that \( u_1 \equiv x_2 v' x_1^{-1} \). But then the original word \( u \equiv x_0^{-a_0} y_0 x_2 v' x_1^{-1} x_0^{-a_0} \) would not have satisfied part (2) of the definition of the normal form, having a forbidden subword with all subscripts for \( v' \) being at least 3. \( \square \)

7. Metric Properties

Once we have a unique normal form for the elements of \( F_r \), we can compute some estimates for the word metric of elements, based on the normal form and the
unique reduced diagram that relates to it. The idea and the procedures are very similar to those for $F$, see [3] and [7].

Given an element $g \in F_r$, take its normal form

$$g = x_0^{a_0} y_0^{a_1} x_1^{a_1} y_1^{a_1} \cdots x_n^{a_n} y_n^{a_n} x_m^{-b_m} \cdots x_1^{-b_1} x_0^{-b_0},$$

where both $a_n + \epsilon_n$ and $b_n$ are nonzero (i.e. we have a positive generator of index $n$ and a negative one of index $m$), and there are no cancellations between $x_n$ and $x_m^{-1}$ (i.e. either $\epsilon_n = 1$ or else $n \neq m$).

**Definition 7.1.** We define the number

$$D(g) = a_0 + a_1 + \cdots + a_m + \epsilon_0 + \epsilon_1 + \cdots + \epsilon_n + b_0 + b_1 + \cdots + b_m + n + m$$

and we denote by $N(g)$ the number of carets of either tree of the unique diagram which corresponds to the normal form, that is, a reduced diagram with no hidden cancellations.

These two quantities are good estimates for the word metric.

**Theorem 7.2.** There exists a constant $C > 0$ such that we have

$$\frac{D(g)}{C} \leq \|g\| \leq C D(g) \quad \text{and} \quad \frac{N(g)}{C} \leq \|g\| \leq C N(g)$$

where $\|g\|$ represents the word metric with respect to the generating set $x_0, x_1, y_0, y_1$.

**Proof.** Since each $x$ or $y$ generator is represented by a caret, we have the obvious inequalities:

$$N(g) \geq a_0 + a_1 + \cdots + a_n + \epsilon_0 + \epsilon_1 + \cdots + \epsilon_n$$

$$N(g) \geq b_0 + b_1 + \cdots + b_m$$

$$N(g) \geq n \quad N(g) \geq m$$

which yield the inequality $D(g) \leq 4 N(g)$.

For the upper bounds, take each generator $x_i$ and $y_i$ with $i \geq 2$ and rewrite it in terms of $x_0, x_1, y_0, y_1$ to obtain the desired bound. The positive part of the word can be written as

$$x_0^{a_0} y_0^{a_1} x_1^{a_1} y_1^{a_1} x_0^{-1} y_1^{-1} x_1^{-1} y_1^{-1} \cdots x_0^{a_n} y_n^{a_n} x_0^{-1} x_1^{-1} x_1^{-1}$$

because observe that a sequence $\ldots x_i^{a_i} y_i^{a_i} x_{i+1}^{a_{i+1}} \ldots$ will have a large number of generators $x_0$ cancelled in between:

$$\ldots x_i^{a_i} y_i^{a_i} x_{i+1}^{a_{i+1}} \ldots = \ldots (x_0^{-i+1} x_1^{a_i} x_0^{a_i-1}) (x_0^{-i+1} y_1^{a_i} x_0^{a_i-1}) (x_0^{-i} x_1^{a_i} x_0^{a_i} x_1^{a_i+1} \ldots$$

and hence for the word we only have one generator $x_0^{-1}$ every time the index grows by 1. Similarly, we do the same for the negative part. Clearly then, we have that the length of this word in $x_0, x_1, y_0, y_1$ is, for instance, at most $2 D(g)$. 

For the lower bound, we use the number of carets. Start with a shortest word for an element \( g \), with length \( L = \|g\| \). When multiplying by a generator, observe that since a generator has at most three carets, the number of carets of the diagram can increase by at most three carets, plus possibly added carets needed to perform the multiplication. But a generator has only one caret which is not on the spine, and since the spine has only \( x \)-carets all the time, only one caret may need to be changed to multiply and then only one caret may have to be added. Hence, when we multiply by a generator the number of carets can grow by at most four. From the shortest word we can obtain then a diagram which has at most \( 4L \) carets.

This diagram will have \( x \) and \( y \) generators mixed in each index (see Proposition 3.6), so it has to be modified so that only one \( y \)-caret appears for each index and with no left children, according to Lemma 6.1. We observe carefully the process described in that proof, and as it is indicated there, the number of carets can at most triple, because we may need to add two carets for each original one. Hence, the total number of carets of the diagram corresponding to the seminormal form is at most \( 12L \). Reducing and eliminating hidden cancellations can only decrease the number of carets. From here we have that \( \|g\| \geq N(g)/12 \). Summarizing all inequalities, we have

\[
\frac{D(g)}{48} \leq \frac{N(g)}{12} \leq \|g\| \leq 2D(g) \leq 8N(g)
\]

and this finishes the proof.

8. Distortion

The similarities between the metric properties of \( F_\tau \) and \( F \) allow us to state some distortion results of subgroups in \( F_\tau \) which are isomorphic to \( F \).

Diagrams in \( F_\tau \) may have two types of carets. We can consider the subgroup of \( F_\tau \) of those elements which can be written with only one type. But if only one type (say \( x \)) of carets is used, then the combinatorics are exactly those of \( F \), and the subgroup is obviously isomorphic to \( F \). We will call \( F_x \) the copy of \( F \) inside \( F_\tau \) given by elements with a diagram with only \( x \)-carets. Observe that this subgroup is the subgroup generated by all the \( x_i \) generators, or, if we prefer, generated by \( x_0 \) and \( x_1 \), clearly yielding a copy of \( F \) inside \( F_\tau \). We have the following result:

**Proposition 8.1.** The inclusion of \( F_x \) inside \( F_\tau \) is undistorted.

**Proof.** Observe that if an element of \( F \) has a diagram (with regular equally-sided carets), then the same diagram but now with \( x \)-carets will give the reduced diagram for \( F_\tau \). Observe also that the normal form in \( F \) is also a normal form in \( F_\tau \). Hence the number of carets is the same for both groups. Since in both cases the number of carets is equivalent to the word metric, we obtain the desired result. 

\( \square \)
The $y$-sided counterpart of this result is a bit more complicated. We can clearly consider the subgroup of $F_\tau$ generated by $y_0$ and $y_1$. This subgroup is also clearly isomorphic to $F$, for instance observe that the combinatorics of the diagrams are exactly the same, but diagrams here have $x$-carets on the spine and $y$-carets everywhere else. Hence, due to the bias we have chosen for the generators (and hence the normal forms) having $x$-careted spines, this subgroup is not the same as the subgroup of $F_\tau$ with all carets of $y$-type. This latter subgroup will be called $F_z$ and it is generated by the following two elements:

\[ z_0 = y_0y_2 \quad \text{and} \quad z_1 = y_2y_4, \]

see Figure 18. This is a proper subgroup of $F_y$, and is also isomorphic to $F$. But both these subgroups behave well.

**Figure 18.** The generators of $F_z$ transformed into elements with $x$-careted spines. Actually these are their normal forms. Originally they only have $y$-carets, but their expressions in the $y$ generators (and hence their normal forms) need to have $x$-carets on the spine.

**Proposition 8.2.** The inclusions of $F_y$ and $F_z$ inside $F_\tau$ are both undistorted.

**Proof.** The case of $F_z$ of elements with only $y$-carets is actually symmetric to $F_x$. We chose to have generators biased towards the right-hand side of the tree, and the spine to have $x$-carets and the normal form to have a majority of $x$ generators, and we deduced from here that $F_x$ is undistorted. But we could have chosen the opposite direction, with preference for $y$-carets, spines on the left-hand side, and
so on, and then we would have obtained that what we call now $F_2$ is undistorted. There is no reason for the copy of $F$ of elements with $x$-carets to be undistorted and its symmetric image to be distorted.

For the subgroup $F_y$, elements here have $x$-carets in the spine and $y$-carets in the interior and left side of the trees. To compute the number of carets of their normal forms most of the $y$-carets have to be transformed into $x$-carets (except a few at the bottom with no left children), but as we have seen in Lemma 6.1 the number of carets can at most triple in this process. Hence the number of carets in $F$ and in $F_\tau$ differ by a multiplicative constant, so the distances do too, and the inclusion is undistorted.

It is interesting to remark that in previous examples of groups of the Thompson family where two different types of carets appear, copies of $F$ inside which use only one type of caret were always distorted. See [13] and [6]. Hence, $F_\tau$ is the first known example of a group of the Thompson family whose elements have two types of carets but whose $F$ subgroups of a single type of caret are undistorted.

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