1 Introduction

It is well-known that two fundamental constants — light velocity $c$ and Planck constant $\hbar$ — are introduced respectively by relativistic kinematics (Einstein’s special relativity) and quantum mechanics. In quantum mechanics the noncommutativity of the position and momentum observables

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad (1.1)$$

implies the uncertainty relation

$$\Delta_\phi \hat{x}_i \Delta_\phi \hat{p}_i \overset{df}{=} \Delta x \Delta p \geq \frac{\hbar}{2}, \quad (1.2)$$

where $x_i^\phi = \langle \phi | \hat{x}_i | \phi \rangle$ is a mean position and $\Delta_\phi \hat{x}_i = \left( \langle \phi | (\hat{x}_i - x_i^\phi)^2 | \phi \rangle \right)^{1/2}$. In standard quantum mechanics the commutativity of the position operators $\hat{x}_i$ implies the possibility to measure the position of quantum particle with arbitrary accuracy. Due to this property the Schrödinger wave function $\psi(\vec{x}, t)$ is a classical field, with the arguments described by commuting space-time coordinates.

Recently there has been a considerable progress in the description of noncommutative or “quantum” geometry, which deals with algebra of functions on a “noncommutative manifold”. The simplest example is provided by quantum phase space (1.1) and the algebra of functions $f(\vec{x}, \vec{p})$. However, one can consider the noncommutative structure also in space-time — by assuming that $[\hat{x}_\mu, \hat{x}_\nu] \neq 0 \ (\mu, \nu = 0, 1, 2, 3)$ (see e.g. [1]). Physically, nonvanishing commutation relations of space-time coordinates could be the effects caused by quantum gravity (see e.g. [2-5]) or quantum string theory (see e.g. [6-9]). Below we shall outline some of the arguments.

1.1 Elementary Planck length and quantum gravity

It is known that quantum mechanics (Heisenberg uncertainty relation (1.2)) and relativistic kinematics put together allows to consider the concept of particle only in the space intervals larger that the Compton wave length (for simplicity we drop the three-space vector indices):

$$\Delta x > \frac{\hbar}{m_0 c}. \quad (1.3)$$

Indeed, because for relativistic particles energy $E = c(p^2 + m_0^2 c^2)^{1/2}$ we have $\Delta E = c \Delta p (p^2 + m_0^2 c^2)^{-1/2}$ and for $p \gg m_0^2$ one can write $\Delta x \Delta E \sim c \Delta x \Delta p \geq \hbar c$. If $\Delta E$ is larger or equal to the rest energy $m_0 c^2$ the concept of mass looses its meaning [10]. We see therefore that if we put $\Delta E_m < m_0 c^2$, one gets (1.3) from (1.2).

The uncertainty relation (1.3) leads effectively to the existence of fundamental length, where $m_0$ is the rest mass of the stable particle, if the creation and annihilation processes would not take place. Because for $E >> m_0 c^2$ this is not the case, therefore one should look for the universal limitations on $\Delta x$ from below in another place e.g. in gravity theory, describing the space-time manifold as a dynamical system. The advantage of gravity is its universal nature, its coupling to any matter in the universe.

Let us consider the measurement process of the length in general relativity. Let us observe that the Einstein equations for the metric

$$\partial^2 g \sim \frac{1}{\kappa^2 \rho}, \quad (1.4)$$

imply the following relation between the fluctuations of the metric $\Delta g$ and the fluctuation of the
energy density $\rho = \frac{\Delta E}{(\Delta x)^3}$,

$$\frac{\Delta g}{(\Delta x)^2} \sim \frac{1}{\kappa^2} \frac{\Delta E}{(\Delta x)^3}. \quad (1.5)$$

Because photon localizing with accuracy $\Delta x$ should have energy larger than $E = h\nu = \frac{\hbar}{\Delta x}$, one gets

$$(\Delta g)(\Delta x)^2 > \frac{\hbar}{\kappa^2} = \lambda_p^2. \quad (1.6)$$

Writing $(\Delta s)^2 = g(\Delta x)^2 > (\Delta g)(\Delta x)^2$ one gets

$$\Delta s > \lambda_p$$

is the Planck length.

The impossibility of localizing in quantized general relativity an event with the accuracy below the Planck length follows from the creation of gravitational field by the energy necessary for the measurement process.

It should be mentioned that similar conclusion can be reached in the framework of the lattice quantum gravity and functional interaction approach to quantum gravity [11,12].

1.2 Elementary length and string theory

The string theories (or rather superstrings theories which do not have tachyons and have consistent string loop expansions — see e.g. [13]) introduce the fundamental string length $\lambda_s$ by the dimensionfull string tension $T$ as follows

$$\lambda_s^2 = \frac{\hbar}{\pi T} = \frac{\hbar^2}{M_s^2}, \quad (1.8)$$

where $M_s$ denotes the fundamental string mass [6]. The Regge slope $\alpha'$ of the string trajectories is given by the formula

$$\alpha' = \frac{1}{2\pi T}. \quad (1.9)$$

The relation between the string mass and the Planck mass $M_p$ is obtained from the description of the graviton-graviton scattering by the string tree amplitude, with dimensionless string coupling constant $g$. One obtains that the Newton constant $G = \lambda_p^2$ is given by

$$G = g^2 \frac{1}{M_s^2} \quad (1.10)$$

i.e one obtains $M_s = gM_p$. Indeed, the quantum gravity perturbative series in the coupling constant $\frac{1}{M_p^2}$ correspond to the string perturbative expansion with the coupling constant $\frac{\alpha'}{\lambda_p^2}$. The quantum mechanical uncertainty relation (1.2) is modified for the fundamental strings. The uncertainty in the position $\Delta x$ is the sum of two terms:

a) standard term is due to Heisenberg uncertainty relation for point-like canonically quantized objects,

b) new term is related with the size of the string increasing linearly with the energy.

One obtains (see e.g. [8,9])

$$\Delta x \geq \frac{\hbar}{\Delta p} + k\lambda_p^2 \Delta p. \quad (1.11)$$

The minimal value of $\Delta x$ is obtained for $(\Delta p)^2 \simeq \frac{\hbar}{kl_p^2}$ i.e.

$$\Delta x \geq 2\sqrt{k}\lambda_p. \quad (1.12)$$

We see therefore that again it follows from the fundamental string theory that the Planck fundamental length $l_p$ describes the accuracy of the measurements of space-time distances.

2 Quantum $\kappa$-deformations and noncommutative space-time

In order to introduce the noncommutative space-time coordinates one could follow two approaches:

i) One can consider the noncommutative Minkowski space only as the representation space of the Poincaré symmetries, without identification of the space-time coordinates with the translation sector of the Poincaré group. In such a case the algebraic properties of the space-time coordinates and the translation generators belonging to the Poincaré group might be different; in particular one can introduce the noncommutative Minkowski space coordinates and classical Poincaré symmetries with commuting Poincaré group parameters. Such an approach, outside of the framework of quantum groups was recently proposed by Doplicher, Fredenagen and Roberts [14].
ii) In another approach we introduce the noncommutative Minkowski space described by the translation sector of the quantum Poincaré group. In such an approach the quantum Poincaré group as well as the quantum Poincaré algebra are the examples of noncommutative and noncocommutative Hopf algebras, which provide the algebraic generalization of the notions of the Lie group as well as the Lie algebra. First quantum deformation $U_h(\mathcal{P}_4)$ of $D = 4$ Poincaré algebra has been proposed in [15], with mass-like quantum deformation parameter $\kappa$. The introduction of fundamental mass parameter $\kappa$ does not modify the nonrelativistic $O(3)$ symmetries and one gets the following noncommuting $\kappa$-deformed Minkowski space coordinates (see e.g. [16-18])

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_i, \hat{x}_0] = \frac{i}{\kappa} \hat{x}_i,$$  \hspace{1cm} (2.1)

It is easy to see from (2.1) that for standard $\kappa$-deformed Poincaré symmetries the space directions are classical and the quantum deformation affects the time direction. In particular the mass shell condition is modified as follows:

$$p_0^2 - \vec{p}^2 = m^2 \rightarrow (2\kappa \sinh \frac{p_0}{2\kappa})^2 - \vec{p}^2 = m^2,$$  \hspace{1cm} (2.2)

where the fourmomentum coordinates $p_\mu$ are the commuting variables. One can introduce corresponding $\kappa$-deformed free KG fields and the spacetime picture in two different ways:

i) One can introduce the commuting spacetime $x_\mu$ coordinates by standard Fourier transform. In such a case the deformed free K-G equation takes the form

$$[\Delta - (2\kappa \sinh \frac{\partial}{2\kappa})^2 - m^2] \phi(\hat{x}) = 0.$$  \hspace{1cm} (2.3)

Such a deformation of scalar field theory was firstly considered in [19].

ii) Recently there were introduced fields $\varphi(\hat{x})$ depending on the noncommutative Minkowski space coordinates [20-22]. In such a case after introducing the noncommutative differential calculus on $\kappa$-deformed Minkowski space (2.3) the $\kappa$-deformed Klein-Gordon equation takes the classical form:

$$[(\frac{\partial}{\partial \hat{x}_i})^2 - (\frac{\partial}{\partial \hat{x}_0})^2 - m^2] \phi(\hat{x}) = 0$$  \hspace{1cm} (2.4)

3 Generalized $\kappa$-deformations of $D = 4$ relativistic symmetries

The most general class of noncommutative spacetime coordinates described by the translation sector of quantum Poincaré group was considered in [23]. One gets the following algebraic relations:

$$(R-1)_{\mu\nu}^{\rho\sigma}(\hat{x}_\rho \hat{x}_\sigma + \frac{1}{\kappa} T_{\rho\sigma} \lambda \hat{x}_\lambda + \frac{1}{\kappa^2} C_{\rho\sigma}) = 0,$$  \hspace{1cm} (3.1)

where the matrix $R$ describes the quantum $R$-matrix for the Lorentz group satisfying the condition $R^2 = 1$, $\kappa$ is a masslike deformation parameter and $T_{\mu\nu}$, $C_{\mu\nu}$ are the numerical coefficients (for details see [23]) which are dimensionless. The condition $R^2 = 1$ can be removed if we consider quantum Poincaré groups belonging to larger class of so called braided Hopf algebras (see e.g. [24]).

The relations (3.1) follows as a special case of the relation $[\hat{x}_1, \hat{x}_2] = \frac{i}{\kappa} \hat{x}_1$, with $R = \tau = (\tau a \otimes \bar{b}) = b \otimes a$ describing classical Lorentz symmetry, $C_{\mu\nu} = 0$ and a particular choice of $T_{\mu\nu}$. Recently there were also considered the $\kappa$-deformations along one of the space axes, for example $x^3$ (see [25]; this is so called tachyonic $\kappa$-deformation with $O(2,1)$ classical subalgebra). Other interesting $\kappa$-deformation is the null-plane quantum Poincaré algebra [26] with the “quantized” light cone coordinate $x_+ = x_0 + x_3$ and classical $E(2)$ subalgebra. The generalized $\kappa$-deformations of $D = 4$ Poincaré symmetries were recently proposed in [27] and describe the $\kappa$-deformation in any direction $y_0 = R_{\mu\nu} x_\mu$ in Minkowski space. Because the change of the linear basis in standard Minkowski space $x_\mu \rightarrow y_\mu = R_{\mu\nu} x_\nu$ implies the replacement of Minkowski metric $g^{\mu\nu} \rightarrow g'^{\mu\nu} = R^\rho_{\mu\nu} g^{\rho\sigma} R_{\sigma\nu}$, $(R^\rho_{\mu\nu} = (R^{\mu\nu})^T)$ generalized $\kappa$-deformations are obtained by deforming the classical Poincaré algebra with arbitrary symmetric metric $g^{\mu\nu}$. From the formulae presented in [27] follows distinction between the $\kappa$-deformations in the case $g_{00} \neq 0$ and $g_{00} = 0$. It appears that

i) If $g_{00} \neq 0$ the $\kappa$-deformation is described by the classical $r$-matrix satisfying modified Y-B equation. Further, it can be shown [28] that the minimal dimension of the bicovariant differential calculus on $\kappa$-deformed Minkowski space is five, i.e. in the equation
one obtains
\[ d\phi(\hat{x}) = d\hat{x}_\mu \frac{\partial}{\partial \hat{x}_\mu} \phi + \omega \Omega \phi \] (3.2)

where e.g. for standard \( \kappa \)-deformation described by (2.1) the additional one-form \( \omega = d(\hat{x}^2 + \frac{n}{\kappa} x^0) - 2\hat{x}_\mu d\hat{x}^\mu \) and \( \Omega \) is the nonpolynomial vector field described by \( \kappa \)-deformed mass Casimir. When \( g_{00} \neq 0 \) the basic relations of \( \kappa \)-deformed differential calculus are the following [20]
\[ [d\hat{x}_\mu, \hat{x}_\nu] = i g^{00}_\mu d\hat{x}_\nu + \frac{i}{\kappa} g^{\mu\nu} d\hat{x}_0 + \frac{1}{4} g_{\mu\nu} \omega, \] (3.3a)
\[ [\omega, \hat{x}_\mu] = - \frac{4}{\kappa^2} d\hat{x}_\mu. \] (3.3b)

ii) If \( g_{00} = 0 \) the \( \kappa \)-deformation is described by the classical \( r \)-matrix satisfying classical YB equation, which permits the extension of such \( \kappa \)-deformation to the conformal algebra [29]. Further, it can be shown that if \( g_{00} = 0 \) the differential calculus is fourdimensional, with standard basis of the one-forms described by \( d\hat{x}_\mu \). The formula (3.2) has only first term on the rhs, and the commutator \([d\hat{x}_\mu, \hat{x}_\nu]\) can be obtained from (3.3a) by neglecting the last term on the rhs.

Having the differential calculi on \( \kappa \)-deformed Minkowski spaces one can consider the corresponding \( \kappa \)-deformed field theory for both cases \( g_{00} \neq 0 \) and \( g_{00} = 0 \). This programme is now under consideration [22].

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