Abstract

We study the lightlike foliations that appear on Lorentzian manifolds with weakly irreducible not irreducible holonomy algebra. We give global structure equations for the foliation that generalize the Gauss and Weingarten equations for one lightlike hypersurface. This gives us some global operators on the manifold. Using these operators, we decompose the curvature tensor of the manifold into several components. We give a criteria how to find the type of the holonomy algebras (there are 4 possible types) in terms of the global operators.

Keywords: Lorentzian manifold, holonomy algebra, lightlike foliation

Introduction

Weakly irreducible not irreducible holonomy algebras of Lorentzian manifolds were classified recently. In [4] L. Berard Bergery and A. Ikemakhen divided these holonomy algebras into 4 types and associated to each such algebra a subalgebra of $\mathfrak{so}(n)$ that is called the orthogonal part (the dimension of the manifold is $n+2$). In [9] T. Leistner proved that the orthogonal part of the holonomy algebra of a Lorentzian manifold is the holonomy algebra of a Riemannian manifold. We recall this classification in section 3.

If the holonomy algebra of a connected Lorentzian manifold $\tilde{M}$ is weakly irreducible and not irreducible (i.e. it preserves an isotropic line and does not preserve any nondegenerate vector subspace of the tangent space), then we obtain on $\tilde{M}$ a parallel distribution $D$ of isotropic lines and the distribution $D^\perp$ of degenerate hypersubspaces of the tangent spaces. The distribution $D^\perp$ is parallel and, in particular, it is involutive. This gives us a foliation in lightlike hypersurfaces.
Our purpose is to describe the geometry of Lorentzian manifolds with weakly irreducible not irreducible holonomy algebras of each type and with each orthogonal part in terms of the foliation in lightlike hypersurfaces. There were some approaches to study geometry of these manifolds in local coordinates, see \[12, 11, 8, 9, 5\]. We hope that our approach can give some global description.

The case of one lightlike hypersurface of a Lorentzian manifold was studied, for example, in \[6\]. In section \[1\] we recall some results from this book. Note that in order to obtain a connection on a lightlike hypersurface \( M \) of a Lorentzian manifold \( \bar{M} \) we must choose a distribution (a screen distribution) \( S(TM) \subset TM \) that is any complement distribution to \( TM^\perp \subset TM \) or, equivalently, choose a complementary isotropic vector bundle (a transversal vector bundle) \( tr(TM) \subset TM \subset T\bar{M}|_M \). Such choice is not always canonical. Some approaches to a canonical choice of \( S(TM) \) (equivalently of \( tr(TM) \)) can be found in \[3\].

In the beginning of section \[2\] we rewrite the structure equations from \[6\] for one hypersurface of the foliation. Since we have not one lightlike hypersurface but a foliation in lightlike hypersurfaces, it is natural to have global structure equations. Such equations can be obtained by choosing a global screen distribution (or transversal bundle), the existence of the last is guaranteed by theorem \[1\]. From the global equations we obtain some operators on \( \bar{M} \) (which depend on the choice of the screen distribution). These operators generalize the operators for one hypersurface from \[6\].

A global screen distribution gives us a decomposition of the tangent space of \( \bar{M} \) at each point. The same decomposition was used in \[7\] in order to decompose the curvature tensor at a point into several components, see section \[4\]. Knowing these components at each point of the manifold, we know the type of the holonomy algebra. In section \[4\] we consider these components for the curvature tensor of the manifold \( \bar{M} \) and express them in terms of the global operators that we have defined.

In section \[6\] we give a criterion how to find the type of the holonomy algebra in terms of our global operators. First we distinguish holonomy algebras of type 2 and 4 from the holonomy algebras of type 1 and 3. Then we give criterions for holonomy algebras of type 3 and 4.

Finally we consider a local example, where the holonomy algebra is Abelian.

Another open problem is to find a canonical way of choosing the screen distribution.

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1 Case of one lightlike hypersurface of a Lorentzian manifold

In this section we recall some results from [6].

Let $\bar{M}, \bar{g}$ be a connected Lorentzian manifold of dimension $n+2$ and let $M \subset \bar{M}$ be a lightlike hypersurface of $\bar{M}$, i.e. the restriction of $\bar{g}_x$ to $T_xM$ is degenerate for all $x \in M$. Since $T_xM$ is degenerate, the perpendicular $T_xM^\perp$ to $T_xM$ in $T_x\bar{M}$ is an isotropic line, which is contained in $T_xM$. We get on $M$ the vector subbundle $TM^\perp = \bigcup_{x \in M} T_xM^\perp$ of $TM$.

In order to obtain a connection on $M$ we fix a complementary vector bundle $S(TM)$ of $TM^\perp$ in $TM$. We have

$$TM = S(TM) \oplus TM^\perp.$$  \hfill (1)

If $M$ is paracompact, $S(TM)$ always exists. The distribution $S(TM) \subset TM$ is called a screen distribution.

Since any maximal isotropic subspace of $T_x\bar{M}$ is 1-dimensional, we see that the distribution $S(TM)$ is not degenerate. Hence we obtain the decomposition

$$T\bar{M}|_M = S(TM) \oplus S(TM)^\perp,$$  \hfill (2)

where $S(TM)^\perp$ is the orthogonal complementary vector bundle to $S(TM)$ in $T\bar{M}|_M$.

In [6] it was proved that for a given screen distribution $S(TM)$ there exists a unique vector bundle $\text{tr}(TM)$ of rank 1 over $M$, such that for any non-zero section $\xi$ of $TM^\perp$ on a coordinate neighborhood $U \subset M$, there exists a unique section $N$ of $\text{tr}(TM)$ on $U$ such that

$$\bar{g}(\xi, N) = 1 \text{ and } \bar{g}(N, N) = \bar{g}(N, W) = 0 \text{ for all } W \in \Gamma(S(TM)|_U).$$

The vector bundle $\text{tr}(TM)$ is called the lightlike transversal vector bundle of $M$ with respect to $S(TM)$. We have the following decompositions of $T\bar{M}|_M$:

$$T\bar{M}|_M = S(TM) \oplus (TM^\perp \oplus \text{tr}(TM)) = TM \oplus \text{tr}(TM).$$  \hfill (3)

Suppose that we have a screen distribution on a lightlike hypersurface $M$ of a Lorentzian manifold $(\bar{M}, \bar{g})$. Using the second form of the decomposition (3), we obtain

$$\bar{\nabla}_X Y = \underbrace{\nabla_X Y}_{\in \Gamma(TM)} + \underbrace{h(X, Y)}_{\in \Gamma(\text{tr}(TM))}$$  \hfill (4)

and

$$\bar{\nabla}_X V = -A_V X + \underbrace{\nabla^i_X V}_{\in \Gamma(\text{tr}(TM))}$$  \hfill (5)
for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(\text{tr}(TM))$. It is known that $\nabla$ is a torsion-free linear connection on $M$, $h$ is a $\Gamma(\text{tr}(TM))$-valued symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$, $A_V : \Gamma(TM) \to \Gamma(TM)$ is a $\mathcal{F}(M)$-linear operator and $\nabla^t$ is a linear connection on the vector bundle $\text{tr}(TM)$.

The connections $\nabla$ and $\nabla^t$ are called the induced connections on $M$ and $\text{tr}(TM)$ respectively. The bilinear form $h$ and the operator $A_V$ are called the second fundamental form and the shape operator respectively. The formulas (4) and (5) are called the Gauss and Weingarten formulas respectively.

Using decomposition (1), we obtain

$$\nabla_X Y = \sum_{\gamma \in \Gamma(S(TM))} \*^{\gamma_X Y} + \sum_{\gamma \in \Gamma(TM^\perp)} \*^{h(X,Y)}$$  \quad (6)

and

$$\nabla_X U = -\sum_{\gamma \in \Gamma(S(TM))} \*^{A_U X} + \sum_{\gamma \in \Gamma(TM^\perp)} \*^{\nabla_X U}$$  \quad (7)

for any $X \in \Gamma(TM)$, $Y \in \Gamma(S(TM))$ and $U \in \Gamma(TM^\perp)$. We have that $\nabla$ and $\nabla^t$ are linear connections on vector bundles $S(TM)$ and $TM^\perp$ respectively, $\* : \Gamma(TM) \times \Gamma(S(TM)) \to \Gamma(TM^\perp)$ is a $\mathcal{F}(M)$-bilinear form and $\*^{A_U} : \Gamma(TM) \to \Gamma(S(TM))$ is a $\mathcal{F}(M)$-linear operator.

The bilinear form $\*^{h}$ and the operator $\*^{A_U}$ are called the second fundamental form and the shape operator of the screen distribution $S(TM)$ respectively. The equations (6) and (7) are called the Gauss and Weingarten equations for the screen distribution respectively.

Let $\bar{R}$ and $R$ be the curvature tensors for the connections $\bar{\nabla}$ and $\nabla$ respectively. Then for $X, Y, Z \in \Gamma(TM)$ we have

$$\bar{R}(X, Y)Z = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z),$$  \quad (8)

where

$$(\nabla_X h)(Y, Z) = \nabla^t_X (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The equation (8) is called the Gauss-Codazzi equation.

## 2 Global structure equations

Now consider a case when we obtain a foliation of lightlike hypersurfaces on a Lorentzian manifold.

Suppose that the holonomy group of a connected $n+2$-dimensional Lorentzian manifold $(\tilde{M}, \tilde{g})$ is weakly irreducible and not irreducible, i.e. for any $x \in \tilde{M}$ the holonomy group $\text{Hol}_x \subset O(T_x \tilde{M}, \tilde{g}_x)$ preserves an isotropic line of $T_x \tilde{M}$ and does not preserve any nondegenerate vector subspace of $T_x \tilde{M}$. Then we obtain on $\tilde{M}$ a parallel distribution $D$ of isotropic lines. Since
Hol_x \subset O(T_x \hat{M}, \bar{g}_x), we see that the perpendicular $D_x^\perp \subset T_x \hat{M}$ is also preserved. Obviously, $D_x \subset D_x^\perp$, $\dim D^\perp = n+1$ and the subspace $D^\perp$ is degenerate. We obtain a parallel distribution $D^\perp$ on $\hat{M}$. It is known that the distributions $D$ and $D^\perp$ are smooth. Denote $D^\perp$ by $T$, then $D = T^\perp$. Since the torsion of the Levi-Civita connection is zero, for any $X, Y \in \Gamma(T)$ we have $[X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(T)$, hence the distribution $T$ is involutive. By the Frobenius theorem, for any point $x \in \hat{M}$ we have a maximal integral manifold $M_x$ of the distribution $T$. Obviously, $M_x \subset \hat{M}$ is a lightlike hypersurface. Thus we obtain a foliations of lightlike hypersurfaces on $\hat{M}$.

Let $x \in \hat{M}$. We will rewrite the above formulas for the lightlike hypersurface $M_x \subset \hat{M}$. Suppose that we have some screen distribution $S(TM_x)$. Since the distribution $T$ is parallel, from (4) we obtain

$$\bar{\nabla}_X Y = \nabla_X Y \in \Gamma(TM_x)$$

for all $X, Y \in \Gamma(TM_x)$, i.e. $h = 0$.

The equations (5) and (6) remain without change. From (7) we obtain

$$\nabla_X U = \bar{\nabla}_X U \in \Gamma(TM_x^\perp)$$

for all $X \in \Gamma(TM_x)$, $U \in \Gamma(TM_x^\perp)$, i.e. $\tilde{A} = 0$.

Since $h = 0$, from (8) it follows that for $X, Y, Z \in \Gamma(TM_x)$ we have

$$\bar{R}(X, Y)Z = R(X, Y)Z.$$

Since we have a foliation of lightlike hypersurfaces, it is natural to construct a global screen distribution and the corresponding transversal bundle.

**Theorem 1** For the distribution $T^\perp$ there exists on $\hat{M}$ a (not unique) distribution $Tr$ of rank 1 that satisfies the condition: for any $\xi \in \Gamma(T^\perp)$ nonzero at all points of some open set $U \subset \hat{M}$, there exists a unique section $N \in \Gamma(Tr|U)$ satisfying $\bar{g}(N, N) = 0$ and $\bar{g}(\xi, N) = 1$ on $U$.

**Proof.** In (11) it was proved that locally there exist coordinates $x^0, x^1, \ldots, x^n, x^{n+1}$ such that the metric $\bar{g}$ has the form

$$\bar{g} = 2dx^0dx^{n+1} + \sum_{i,j=1}^ng_{ij}dx^idx^j + f \cdot (dx^{n+1})^2,$$

where $g_{ij}$ are functions of $x^1, \ldots, x^n, x^{n+1}$ and $f$ is a function of $x^0, \ldots, x^n, x^{n+1}$. For all $x$ from this coordinate neighborhood we have $T^\perp_x = \mathbb{R}(\frac{\partial}{\partial x^0})_x$ and we choose $Tr_x = \mathbb{R}(-\frac{1}{2}f(\frac{\partial}{\partial x^0})_x + (\frac{\partial}{\partial x^{n+1}})_x)$. Thus on some neighborhood of each point $x \in \hat{M}$ we obtain a distribution that satisfy the theorem. We assume that $\hat{M}$ is paracompact, then we have a locally finite open covering $(U_i)_{i \in I}$ of $\hat{M}$, where $U_i$ are coordinate neighborhoods as above. On each $U_i$ we have a distribution
Using decomposition (12), we obtain

$$\bar{\nabla}_W Y = \nabla_W Y \in \Gamma(T)$$

(14)

Using the second form of the decomposition \[13\], we obtain

$$\bar{\nabla}_W V = -A_V W + \underbrace{\nabla^t_W V}_{\in \Gamma(T^\perp)}$$

(15)

for any $W \in \Gamma(TM)$, $Y \in \Gamma(T)$ and $V \in \Gamma(Tr)$. It can be proved that $\nabla$ is a linear connection on the bundle $T$, $A_V : \Gamma(TM) \to \Gamma(T)$ is a $\mathcal{F}(\tilde{M})$-linear operator, and $\nabla^t$ is a linear connection on the vector bundle $Tr$. We call $\nabla$ and $\nabla^t$ the induced connections on $T$ and $Tr$ respectively. We call $A_V$ the shape operator of $\tilde{M}$ with respect to the screen distribution $S$.

Using decomposition \[12\], we obtain

$$\nabla_W Y = \underbrace{\nabla^*_W Y}_{\in \Gamma(S)} + \underbrace{h(W,Y)}_{\in \Gamma(T)}$$

(16)

and

$$\nabla_W U = \underbrace{\nabla^t_W U}_{\in \Gamma(T^\perp)}$$

(17)
for any $W \in \Gamma(TM)$, $Y \in \Gamma(S)$ and $U \in \Gamma(T^\perp)$. It can be proved that $\hat{\nabla}$ and $\hat{\nabla}^t$ are linear connections on vector bundles $S$ and $T^\perp$ respectively, $\hat{h} : \Gamma(TM) \times \Gamma(S) \to \Gamma(T^\perp)$ is a $\mathcal{F}(M)$-bilinear form. We call $\hat{h}$ the second fundamental form of the screen distribution $S$.

For a hypersurface $M_x$ the operators $\nabla$, $A_V$, $\nabla^t$, $\hat{\nabla}$ and $\hat{\nabla}^t$ can be obtained from the defined above by taking the obvious restrictions. Restricting to $M_x$ (14), (15), (16) and (17) we obtain (9), (5), (6) and (10) respectively.

Let $V \in \Gamma(Tr)$. We have $\bar{g}(V,V) = 0$, hence for any $W \in \Gamma(TM)$ holds $\bar{g}(\bar{\nabla}_W V, V) + \bar{g}(V, \bar{\nabla}_W V) = 0$ and

$$\bar{g}(\bar{\nabla}_W V, V) = 0.$$  \hspace{1cm} (18)

From (15) we obtain $\bar{g}(-A_V W + \nabla^t_W V, V) = 0$. Thus,

$$\bar{g}(A_V W, V) = 0.$$ \hspace{1cm} (19)

This means, that $A_V$ takes values in $\Gamma(S)$. Thus we can consider $A_V$ as

$$A_V : \Gamma(TM) \to \Gamma(S).$$ \hspace{1cm} (20)

By analogy, for any $Y \in \Gamma(S)$ and $V \in \Gamma(Tr)$ we have $\bar{g}(Y, V) = 0$, hence for any $W \in \Gamma(TM)$ holds $\bar{g}(\bar{\nabla}_W Y, V) + \bar{g}(Y, \bar{\nabla}_W V) = 0$. Using (15) and (16), we obtain

$$\bar{g}(h(W,Y), V) = \bar{g}(A_V W, Y)$$ \hspace{1cm} (21)

for all $W \in \Gamma(TM)$, $Y \in \Gamma(S)$ and $V \in \Gamma(Tr)$. Hence $A_V$ and $\hat{h}$ can be found from each other.

## 3 Weakly irreducible not irreducible holonomy algebras of Lorentzian manifolds

Let $(\mathbb{R}^{1,n+1}, \eta)$ be a Minkowski space of dimension $n+2$, where $\eta$ is a metric on $\mathbb{R}^{n+2}$ of signature $(1,n+1)$. We fix a basis $U, X_1, \ldots, X_n, V$ of $\mathbb{R}^{1,n+1}$ with respect to which the Gram matrix of $\eta$ has the form

$$\begin{pmatrix}
0 & 0 & 1 \\
0 & E_n & 0 \\
1 & 0 & 0
\end{pmatrix},$$

where $E_n$ is the $n$-dimensional identity matrix.

A subalgebra $\mathfrak{g} \subset \mathfrak{so}(1,n+1)$ is called irreducible if it does not preserve any proper subspace of $\mathbb{R}^{1,n+1}$; $\mathfrak{g}$ is called weakly irreducible if it does not preserve any nondegenerate proper subspace of $\mathbb{R}^{1,n+1}$. Obviously, if $\mathfrak{g} \subset \mathfrak{so}(1,n+1)$ is irreducible, then it is weakly irreducible. If $\mathfrak{g} \subset \mathfrak{so}(1,n+1)$ preserves a degenerate proper subspace $W \subset \mathbb{R}^{1,n+1}$, then it preserves the isotropic line $W \cap W^\perp$. 

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From the classification of M. Berger (see [1]) it follows that the only irreducible holonomy algebra of Lorentzian manifolds is isomorphic to \( \mathfrak{so}(1, n+1) \), so we consider only the case of weakly irreducible not irreducible holonomy algebra.

Let \((\bar{M}, \bar{g})\) be an \( n + 2 \)-dimensional connected Lorentzian manifold with weakly irreducible not irreducible holonomy algebra. Let \( x \in \bar{M} \). We identify \((T_x \bar{M}, \bar{g}_x)\) with \((\mathbb{R}^{1,n+1}, \eta)\). We assume that the subalgebra \( \mathfrak{hol} \subset \mathfrak{so}(1, n+1) \) corresponding to the holonomy algebra \( \mathfrak{hol}_x \subset \mathfrak{so}(T_x \bar{M}, \bar{g}_x) \) preserves the isotropic line \( \mathbb{R}U \), i.e. \( \mathfrak{hol} \) is contained in the subalgebra \( \mathfrak{so}(1, n+1)_{\mathbb{R}U} \) of \( \mathfrak{so}(1, n+1) \) that preserves the line \( \mathbb{R}U \). Above we had a decomposition \( T_x \bar{M} = T_x^+ \oplus S_x \oplus T_{rx} \).

We assume that \( X_1, ..., X_n \) correspond to a basis of \( S_x \) and \( V \) corresponds to a vector of \( T_{rx} \).

The Lie algebra \( \mathfrak{so}(1, n+1)_{\mathbb{R}U} \) can be identified with the following algebra of matrices:

\[
\mathfrak{so}(1, n+1)_{\mathbb{R}U} = \left\{ \begin{pmatrix} a & X & 0 \\ 0 & A & -X^t \\ 0 & 0 & -a \end{pmatrix} : a \in \mathbb{R}, X \in \mathbb{R}^n, A \in \mathfrak{so}(n) \right\}.
\]

Let \( \mathfrak{h} \subset \mathfrak{so}(n) \) be a subalgebra. Recall that \( \mathfrak{h} \) is a compact Lie algebra and we have the decomposition \( \mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}(\mathfrak{h}) \), where \( \mathfrak{h}' \) is the commutant of \( \mathfrak{h} \) and \( \mathfrak{z}(\mathfrak{h}) \) is the center of \( \mathfrak{h} \).

The following result is due to L. Berard Bergery and A. Ikemakhen.

**Theorem** Suppose \( \mathfrak{hol} \subset \mathfrak{so}(1, n+1)_{\mathbb{R}U} \) is a weakly irreducible holonomy algebra. Then \( \mathfrak{hol} \) belongs to one of the following types

**Type 1.** \( \mathfrak{hol}^{1,h} = \left\{ \begin{pmatrix} a & X & 0 \\ 0 & A & -X^t \\ 0 & 0 & -a \end{pmatrix} : a \in \mathbb{R}, X \in \mathbb{R}^n, A \in \mathfrak{h} \right\} \), where \( \mathfrak{h} \subset \mathfrak{so}(n) \) is a subalgebra;

**Type 2.** \( \mathfrak{hol}^{2,h} = \left\{ \begin{pmatrix} 0 & X & 0 \\ 0 & A & -X^t \\ 0 & 0 & 0 \end{pmatrix} : X \in \mathbb{R}^n, A \in \mathfrak{h} \right\} \);

**Type 3.** \( \mathfrak{hol}^{3,h,\varphi} = \left\{ \begin{pmatrix} \varphi(B) & X & 0 \\ 0 & A + B & -X^t \\ 0 & 0 & -\varphi(B) \end{pmatrix} : X \in \mathbb{R}^n, A \in \mathfrak{h}', B \in \mathfrak{z}(\mathfrak{h}) \right\} \), where \( \varphi : \mathfrak{h} \to \mathbb{R} \) is a non-zero linear map with \( \varphi|_{\mathfrak{h}'} = 0 \);

**Type 4.** \( \mathfrak{hol}^{4,h,\psi} = \left\{ \begin{pmatrix} 0 & X & \psi(B) & 0 \\ 0 & A + B & 0 & -X^t \\ 0 & 0 & 0 & -\psi(B)^t \\ 0 & 0 & 0 & 0 \end{pmatrix} : X \in \mathbb{R}^{n_1}, A \in \mathfrak{h}', B \in \mathfrak{z}(\mathfrak{h}) \right\} \), where we have a non-trivial decomposition \( n = n_1 + n_2 \) such that \( \mathfrak{h} \subset \mathfrak{so}(n_1) \); and \( \psi : \mathfrak{h} \to \mathbb{R}^{n_2} \) is a surjective linear map with \( \psi|_{\mathfrak{h}'} = 0 \).
The subalgebra $\mathfrak{h} \subseteq \mathfrak{so}(n)$ associated to a holonomy algebra in the above theorem is called the orthogonal part of the holonomy algebra. In [10, 9] T. Leistner proved the following theorem:

**Theorem** The orthogonal part of the weakly irreducible not irreducible holonomy algebra of a Lorentzian manifold is the holonomy algebra of a Riemannian manifold.

### 4 Decomposition of an abstract curvature tensor

In this section we use the notations of section [3](#). We recall the decomposition of a curvature tensor for the holonomy algebras given in [7].

Let $W$ be a vector space and $\mathfrak{f} \subseteq \mathfrak{gl}(W)$ a subalgebra. The space of curvature tensors for the Lie algebra $\mathfrak{f}$ is defined as follows

$$\mathcal{R}(\mathfrak{f}) = \{ R \in \text{Hom}(W \otimes W, \mathfrak{f}) : R(u, v) = -R(v, u), R(u, v)w + R(v, w)u + R(w, u)v = 0 \text{ for all } u, v, w \in W \}.$$  

Suppose that the holonomy algebra of $\bar{M}$ is of type 1, i.e. $\mathfrak{hol} = \mathfrak{hol}^{1, h}$ for some $\mathfrak{h} \subseteq \mathfrak{so}(n)$. Let $R_x$ be the curvature tensor of the manifold $\bar{M}$ at a point $x \in \bar{M}$, then $R_x \in \mathcal{R}(\mathfrak{hol}^{1, h})$, where $x \in \bar{M}$ (we use the identifications as in section [3](#)). Having the decomposition $\mathbb{R}^{1,n+1} = \mathbb{R}U \oplus \mathbb{R}^n \oplus \mathbb{R}V$, $R_x$ can be decomposed into 5 components, $R_x = \bar{R}_{x1} + \bar{R}_{x2} + \bar{R}_{x3} + \bar{R}_{x4} + \bar{R}_{x5}$. These components are given by elements

$R_0 \in \mathcal{R}(\mathfrak{h})$, $P \in \{ P \in \text{Hom}(\mathbb{R}^n, \mathfrak{h}) : \eta(P(u)v, w) + \eta(P(v)w, u) + \eta(P(w)u, v) = 0 \text{ for all } u, v, w \in \mathbb{R}^n \}$,

$T \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ with $T^* = T$, $L \in \text{Hom}(\mathbb{R}^n, \mathbb{R})$ and $\lambda \in \mathbb{R}$,

and can be found from the following conditions

$$\bar{R}_{x1}(X, Y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & R_h(X, Y) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{R}_{x2}(V, X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P(X) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{R}_{x2}(X, Y) = \begin{pmatrix} 0 & P(X,Y) & 0 \\ 0 & 0 & -P^*(X,Y)^t \\ 0 & 0 & 0 \end{pmatrix},$$

$$\bar{R}_{x3}(V, X) = \begin{pmatrix} 0 & T(X) & 0 \\ 0 & 0 & -T(X)^t \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{R}_{x4}(U, V) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\lambda \end{pmatrix},$$

$$\bar{R}_{x5}(V, X) = \begin{pmatrix} L(X) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -L(X) \end{pmatrix}, \quad \bar{R}_{x5}(U, V) = \begin{pmatrix} 0 & L^*(1) & 0 \\ 0 & 0 & -L^*(1)^t \\ 0 & 0 & 0 \end{pmatrix},$$
where all $X, Y \in \mathbb{R}^n$. We assume that each $\mathring{R}_{xi}$ is zero on vectors for which $\mathring{R}_{xi}$ was not defined.

Since any holonomy algebra $\mathfrak{hol}^h$ of any other type with the orthogonal part $\mathfrak{h}$ is contained in $\mathfrak{hol}^{1,h}$, for the curvature tensor $\mathring{R}_x \in \mathcal{R}(\mathfrak{hol}^h)$ we have $\mathring{R}_x \in \mathcal{R}(\mathfrak{hol}^{1,h})$, and the decomposition for $\mathring{R}_x$ can be obtained from the above decomposition and the condition that $\mathring{R}_x$ takes values in $\mathfrak{hol}^h$.

For $R_x \in \mathcal{R}(\mathfrak{hol}^{2,h})$ we have

$$\mathring{R}_{x4} = \mathring{R}_{x5} = 0 \text{ and } \mathring{R}_x = \mathring{R}_{x1} + \mathring{R}_{x2} + \mathring{R}_{x3}. \quad (22)$$

For $\mathring{R}_x \in \mathcal{R}(\mathfrak{hol}^{3,h,\varphi})$ we have

$$\mathring{R}_{x4} = 0, \mathring{R}_{x1}(X,Y) \in \ker \varphi \text{ and } \mathring{R}_{x5}(V,X)U = \varphi(\mathring{R}_{x2}(V,X))U \text{ for all } X,Y \in \mathbb{R}^n. \quad (23)$$

For $\mathring{R}_x \in \mathcal{R}(\mathfrak{hol}^{4,h,\psi})$ we have

$$R_{x4} = R_{x5} = 0, R_{x1}(Z_1,Z_2) \in \ker \psi, R_{x3}(V,Y)|_{\mathbb{R}^n_2} = 0, \text{ and } R_{x3}(V,X)Y = \eta(\psi(R_{x2}(V,X)),Y)U$$

for all $Z_1, Z_2 \in \mathbb{R}^n$ and $X + Y \in \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} = \mathbb{R}^n$.

## 5 Decomposition of the curvature tensor on a manifold

In this section we use the notations of section 2. Let $x \in \mathring{M}$. For $T_x \mathring{M}$ we have the decomposition

$$T_x \mathring{M} = T_x^\perp \oplus S_x \oplus Tr_x.$$

In section we saw that having such decomposition of $T_x \mathring{M}$, we can decompose the curvature tensor of $\mathring{M}$ at the point $x$ into 5 components, $\mathring{R}_x = \mathring{R}_{x1} + \mathring{R}_{x2} + \mathring{R}_{x3} + \mathring{R}_{x4} + \mathring{R}_{x5}$. Thus we have

$$\mathring{R} = \mathring{R}_1 + \mathring{R}_2 + \mathring{R}_3 + \mathring{R}_4 + \mathring{R}_5. \quad (25)$$
From the results of section it follows that for all $U, U_1 \in \Gamma(T^\perp)$, $X, Y, Z \in \Gamma(S)$ and $V, V_1 \in \Gamma(Tr)$ we have

$$R(X, Y)Z = R_1(X, Y)Z + R_2(X, Y)Z, \quad \text{in } \Gamma(S) \quad \text{and } \Gamma(T^\perp)$$

$$R(X, Y)U = 0, \quad \text{in } \Gamma(S) \quad \text{and } \Gamma(T^\perp)$$

$$R(X, V)Y = R_2(X, V)Y + R_3(X, V)Y, \quad \text{in } \Gamma(S) \quad \text{and } \Gamma(T^\perp)$$

$$R(X, V)U = R_5(X, V)U \in \Gamma(T^\perp), \quad \text{in } \Gamma(S) \quad \text{and } \Gamma(T^\perp)$$

$$R(U, V)U_1 = R_4(U, V)U_1 \in \Gamma(T^\perp), \quad \text{in } \Gamma(S) \quad \text{and } \Gamma(T^\perp)$$

$$R(U, V)V_1 = R_5(U, V)V_1 + R_4(U, V)V_1, \quad \text{in } \Gamma(S) \quad \text{and } \Gamma(T^\perp)$$

Using this we will express $\bar{R}_1, \bar{R}_2, \bar{R}_3, \bar{R}_4$ and $\bar{R}_5$ in terms of the operators that we defined above.

Let $X, Y, Z \in \Gamma(S)$, then

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z \quad \text{in } \Gamma(S)$$

$$\bar{R}(X, Y)U = 0, \quad \text{in } \Gamma(S)$$

$$\bar{R}(X, V)Y = \bar{R}_2(X, V)Y + \bar{R}_3(X, V)Y, \quad \text{in } \Gamma(S)$$

$$\bar{R}(X, V)U = \bar{R}_5(X, V)U \in \Gamma(T^\perp), \quad \text{in } \Gamma(S)$$

$$\bar{R}(U, V)U_1 = \bar{R}_4(U, V)U_1 \in \Gamma(T^\perp), \quad \text{in } \Gamma(S)$$

$$\bar{R}(U, V)V_1 = \bar{R}_5(U, V)V_1 + \bar{R}_4(U, V)V_1, \quad \text{in } \Gamma(S)$$

We have

$$[X, Y] = \bar{\nabla}_X Y - \bar{\nabla}_Y X = \nabla_X Y - \nabla_Y X = \bar{\nabla}_X Y + h(X, Y) - \bar{\nabla}_Y X - h(Y, X). \quad (32)$$

Thus,

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{h}(Y, Z) - \bar{\nabla}_Y \bar{h}(X, Z) - \bar{h}([X,Y], Z) \quad \text{in } \Gamma(S)$$

$$\bar{R}(X, Y)U = 0, \quad \text{in } \Gamma(S)$$

$$\bar{R}(X, V)Y = \bar{\nabla}_X \bar{h}(Y, Z) + \bar{h}(X, \bar{\nabla}_Y Z) - \bar{\nabla}_Y \bar{h}(X, Z) - h([X,Y], Z)$$

From (26) it follows that

$$\bar{R}_1(X, Y)Z = \bar{R}(X, Y)Z \quad (35)$$

and

$$\bar{R}_2(X, Y)Z = (\bar{\nabla}_X \bar{h})(Y, Z) - (\bar{\nabla}_Y \bar{h})(X, Z) - \bar{h}([X,Y], Z) \quad (36)$$
for all $X, Y, Z \in \Gamma(S)$.

Note that from (32) it follows that the distribution $S$ is involutive if and only if
\[
\hat{h}(X, Y) = \hat{h}(Y, X) \text{ for all } X, Y \in \Gamma(S).
\]

From (21) we see that this is equivalent to
\[
\hat{g}(A_V X, Y) = \hat{g}(X, A_V Y) \text{ for all } X, Y \in \Gamma(S), V \in \Gamma(Tr).
\]

For $X, Y \in \Gamma(S)$ and $V \in \Gamma(Tr)$ we have
\[
\bar{R}(X, V)Y = \nabla^*_X \nabla_V Y - \nabla_V \nabla_X Y - \nabla^*_{[X,Y]} Y \tag{16}
\]
and
\[
\bar{R}(X, V)Y = \bar{R}(X, V)Y + (\nabla^*_X \hat{h})(V, Y) - (\nabla^*_V \hat{h})(X, Y) + \hat{h}(A_V, X) + \hat{h}(\nabla^*_V X, Y), \tag{37}
\]
where
\[
(\nabla^*_X h)(V, Y) = \nabla^*_X (\nabla^*_V h)(V, Y) - \nabla^*_X \nabla^*_V Y - \nabla^*_V \nabla^*_X Y. \tag{38}
\]

From (28) and (37) it follows that
\[
\bar{R}_2(X, V)Y = \bar{R}(X, V)Y \tag{39}
\]
and
\[
\bar{R}_3(X, V)Y = (\nabla^*_X h)(V, Y) - (\nabla^*_V h)(X, Y) + \hat{h}(A_V, X) + \hat{h}(\nabla^*_V X, Y), \tag{40}
\]
for all $X, Y \in \Gamma(S)$ and $V \in \Gamma(Tr)$.

From (35) and (39) it follows that the curvature tensor $R^*$ can be found as follows
\[
\hat{R}(W_1, W_2)X = \text{pr}_S(\hat{R}_1(W_1, W_2)X + \bar{R}_2(W_1, W_2)X) \text{ for all } W_1, W_2 \in \Gamma(TM), X \in \Gamma(S), \tag{41}
\]
where pr$_S$ is the projection on $S$ with respect to decomposition (13).

Let $x \in \bar{M}$ and $M_x$ the lightlike hypersurface through $x$. From (11) it follows that for the curvature tensor of connection $\nabla^*$ on the vector bundle $S(TM_x) = S|_{M_x}$ we have
\[
\hat{R}_{M_x} = \hat{R}_1|_{\Gamma(TM_x) \times \Gamma(TM_x) \times \Gamma(S(TM_x))}, \tag{42}
\]

\[
\nabla^*_X \nabla^*_V Y - \nabla^*_V \nabla^*_X Y - \nabla^*_X \nabla^*_Y Y
\]
and
\[
\nabla^*_X \nabla^*_Y - \nabla^*_Y \nabla^*_X - \nabla^*_X \nabla^*_Y
\]
are the projection on $S$ with respect to decomposition (13).
From (45) and (46) we obtain

\[ R_{M^\perp} = (\bar{R}_1 + \bar{R}_2)|_{\Gamma(TM^\perp)\times\Gamma(TM^\perp)\times\Gamma(TM^\perp)}. \]  

(43)

This shows that the curvature tensors \( \bar{R}_1 \) and \( \bar{R}_2 \) at a point \( x \in \bar{M} \) depend only on the lightlike hypersurface through \( x \), while the curvature tensor \( \bar{R}_3 \) depends on the links between different hypersurfaces.

For \( U \in \Gamma(T^\perp) \) and \( V, V_1 \in \Gamma(Tr) \) we have

\[ \bar{R}(U, V)V_1 = \bar{\nabla}_U \bar{\nabla}_V V_1 - \bar{\nabla}_V \bar{\nabla}_U V_1 - \bar{\nabla}_{[U, V]}V_1 \]

\[ \bar{\nabla}_U(-A_{V_1} V + \nabla^t_{V_1} V) - \bar{\nabla}_V(-A_{V_1} U + \nabla^t_{V_1} U) + A_{V_1} ([U, V]) - \nabla^t_{[U, V]} V_1 \]

\[ = -\nabla_U^t A_{V_1} V - A_{\nabla^t_{V_1} U} + \nabla^t_U \nabla^t_V V_1 + \nabla_V A_{V_1} U + A_{\nabla^t_{V_1} V} \]

\[ - \nabla^t_V \nabla^t_U V_1 + A_{V_1} (\bar{\nabla}_U V - \bar{\nabla}_V U) - \nabla^t_{[U, V]} V_1 \]

\[ = \bar{R}^t(U, V)V_1 - \bar{\nabla}^t_U A_{V_1} V - \bar{\nabla}^t_V A_{V_1} U - A_{\nabla^t_{V_1} V} \]

\[ + A_{\nabla^t_{V_1} V} V + A_{V_1} \bar{\nabla}^t_U V - A_{V_1} \bar{\nabla}^t_V U \]

\[ = \bar{R}^t(U, V)V_1 - (\bar{\nabla}^t_U A_{V_1}) V + (\bar{\nabla}^t_V A_{V_1}) U + \bar{\nabla}^t_U A_{V_1} U + \bar{\nabla}^t_V A_{V_1} U + \bar{\nabla}^t_U A_{V_1} U + \bar{\nabla}^t_V A_{V_1} U, \]

where

\[ (\bar{\nabla}^t_U A_{V_1}) V = \bar{\nabla}^t_U A_{V_1} V - A_{\nabla^t_{V_1} V} V - A_{V_1} \nabla^t_U V \]

and

\[ (\bar{\nabla}^t_V A_{V_1}) U = \bar{\nabla}^t_V A_{V_1} U - A_{\nabla^t_{V_1} U} U - A_{V_1} \nabla^t_V U. \]

From (31) it follows that

\[ \bar{R}_5(U, V)V_1 = (\bar{\nabla}^t_V A_{V_1} U - (\bar{\nabla}^t_U A_{V_1}) V. \]

(44)

Using (30) and (29) we obtain

\[ \bar{R}_4(U, V)U_1 = \bar{R}^t(U, V)U_1 \text{ and } \bar{R}_5(X, V)U = \bar{R}^t(X, V)U. \]

(45)

From (27) it follows that

\[ \bar{R}^t(X, Y) = 0. \]

(46)

From (15) and (16) we obtain

\[ \bar{R}^t(W_1, W_2)U = \bar{R}_4(W_1, W_2)U + \bar{R}_5(W_1, W_2)U \]

(47)

for all \( W_1, W_2 \in \Gamma(\bar{M}) \) and \( U \in \Gamma(T^\perp) \).
6 Types of holonomy algebras

Let $\mathfrak{hol}_x$ be the holonomy algebra of $\tilde{M}$ at a point $x \in \tilde{M}$. We suppose that $\mathfrak{hol}_x$ is weakly irreducible. In this section we give a criterion how to find the type of the holonomy algebra in terms of our global operators.

From section 3, equations (47) and (44) we obtain the following

**Proposition 1** The following conditions are equivalent

1) $\mathfrak{hol}_x$ is of type 2 or 4;
2) $\tilde{R} = \tilde{R}_5 = 0$;
3) $\tilde{R} = 0$, i.e. the connection $\tilde{\nabla}$ is flat;
4) $\tilde{R} = 0$ and $(\nabla_V A)_{V_1} U = (\nabla_U A)_{V_1} V = 0$ for all $U \in \Gamma(T^\perp)$, $V, V_1 \in \Gamma(Tr)$.

In the following proposition we use the vector bundle $\text{Hom}(\mathfrak{so}(S), \mathbb{R})$ over $\tilde{M}$ such that $\text{Hom}(\mathfrak{so}(S), \mathbb{R})_y = \text{Hom}(\mathfrak{so}(S_y), \mathbb{R})$ for all $y \in \tilde{M}$. For a curve $\gamma$ in $\tilde{M}$ we will denote by $\tau(\gamma)$ the parallel transport along $\gamma$.

**Proposition 2** The holonomy algebra $\mathfrak{hol}_x$ is of type 3 if and only if the following conditions hold

1. $\tilde{R}(U, V) = 0$ for all $U \in \Gamma(T^\perp)$ and $V \in \Gamma(Tr)$.

2. There exists a section $\varphi \in \Gamma(\text{Hom}(\mathfrak{so}(S), \mathbb{R}))$ such that

   2.1. $\tilde{R}(X_y, Y_y) \in \ker \varphi_y$ for all $y \in \tilde{M}$ and $X_y, Y_y \in S_y$.

   2.2. $\tilde{R}(V, X) = \varphi(\tilde{R}(V, X)) \cdot \text{id}_{\Gamma(T^\perp)}$ for all $X \in \Gamma(S)$ and $V \in \Gamma(Tr)$.

   2.3. There exist $y \in \tilde{M}$, $X_y \in S_y$ and $V_y \in Tr_y$ such that $\tilde{R}_y(V_y, X_y) \notin \ker \varphi_y$.

   2.4. For any $y \in \tilde{M}$ and any curve $\gamma : [a, b] \to \tilde{M}$ with $\gamma(a) = x$ and $\gamma(b) = y$ we have

   $$\varphi_y(\tilde{R}_y(V_y, X_y)) = \varphi_x(\text{pr}_{S_x} \circ \tau(\gamma)^{-1} \circ \tilde{R}_y(V_y, X_y) \circ \tau(\gamma)|_{S_x})$$

   for all $X_y \in S_y, V_y \in Tr_y$.

**Proof.** Suppose that $\mathfrak{hol}_x$ is of type 3. Since the holonomy algebras at different points of $\tilde{M}$ are isomorphic, for any $y \in \tilde{M}$ we have $\mathfrak{hol}_y = h^y \oplus \mathfrak{so}(S_y)$, where $h_y \subset \mathfrak{so}(S_y)$ and $\varphi_y : h_y \to \mathbb{R}$ is a linear map. From this, (23) and (33) follows the first statement of the proposition. We have $\mathfrak{so}(S_y) = h_y \oplus \mathfrak{h}_y^\perp$, where $\mathfrak{h}_y^\perp$ is the orthogonal complement to $h_y$ with respect to the Cartan-Killing form on $\mathfrak{so}(S_y)$. Extend $\varphi_y$ to $\varphi_y : \mathfrak{so}(S_y) \to \mathbb{R}$ by setting $\varphi_y|_{\mathfrak{h}_y^\perp} = 0$. This gives a section $\varphi \in \Gamma(\text{Hom}(\mathfrak{so}(S), \mathbb{R}))$. Statements 2.1 and 2.2 follow from (23), (33), (39) and (15). Suppose that 2.3 does not hold, then from statement 1, (15) and proposition 1 it follows that
the holonomy algebra of \( \bar{M} \) is of type 2 or 4, i.e. we obtain a contradiction. Let \( y \in \bar{M} \) and let \( \gamma : [a, b] \to \bar{M} \) be a curve with \( \gamma(a) = x \) and \( \gamma(b) = y \). For any \( W_1, W_2 \in T_x \bar{M} \) consider the operator

\[
R^\gamma(W_1, W_2) = \tau(\gamma)^{-1} \circ \bar{R}_y(\tau(\gamma)W_1, \tau(\gamma)W_2) \circ \tau(\gamma) : T_x \bar{M} \to T_x \bar{M}.
\]

From the theorem of Ambrose and Singer it follows that \( R^\gamma \in R(\mathfrak{hol}_x) \). Let \( X_y \in S_y, V_y \in T_{r_y} \), and let \( W_1 = \tau(\gamma)^{-1}V_y \in T_x \bar{M}, W_2 = \tau(\gamma)^{-1}X_y \in T_x \) (since \( S \subset T \) and \( T \) is parallel). We can decompose \( W_1 \) and \( W_2 \) as follows \( W_1 = \text{pr}_{T_x} W_1 + \text{pr}_{S_x} W_1 + \text{pr}_{T_{r_y}} W_1 \) and \( W_2 = \text{pr}_{T_x} W_2 + \text{pr}_{S_x} W_2 \). From section \( \mathbf{4} \) it follows that

\[
R^\gamma_4(W_1, W_2)|_{T_x} = R^\gamma_4(\text{pr}_{T_{r_y}} W_1, \text{pr}_{S_x} W_2)|_{T_x}
\]

\[
= \varphi_x(R^\gamma_2(\text{pr}_{T_{r_y}} W_1, \text{pr}_{S_x} W_2)) \cdot \text{id}_{T_x} = \varphi_x(\text{pr}_{S_x} \circ R^\gamma_2(W_1, W_2)|_{S_x}) \cdot \text{id}_{T_x} \cdot (48)
\]

From section \( \mathbf{4} \) it follows that the components of any abstract curvature tensor \( R \in R(\mathfrak{hol}_x) \) can be obtained by applying to \( R \) some restrictions and projections. We have

\[
\varphi_x(\text{pr}_{S_x} \circ R^\gamma_2(W_1, W_2)|_{S_x}) = \varphi_x(\text{pr}_{S_x} \circ R^\gamma(W_1, W_2)|_{S_x})
\]

\[
= \varphi_x(\text{pr}_{S_x} \circ \tau(\gamma)^{-1} \circ \bar{R}_y(V_y, X_y)) \circ \tau(\gamma)|_{S_x} = \varphi_x(\text{pr}_{S_x} \circ \tau(\gamma)^{-1} \circ \bar{R}_y2(V_y, X_y)) \circ \tau(\gamma)|_{S_x} \quad (49)
\]

and

\[
R^\gamma_4(W_1, W_2)|_{T_x} = R^\gamma(W_1, W_2)|_{T_x}
\]

\[
= \tau(\gamma)^{-1} \circ \bar{R}_y(V_y, X_y) \circ \tau(\gamma)|_{T_x} = \tau(\gamma)^{-1} \circ \bar{R}_{y4}(V_y, X_y) \circ \tau(\gamma)|_{T_x}.
\]

Since \( \tau(\gamma)T_x \subset T_y \), the last equality shows that \( R^\gamma_4(W_1, W_2)|_{T_x} \) and \( \bar{R}_{y4}(V_y, X_y)|_{T_y} \) act on \( T_x \) and \( T_y \), respectively, as the multiplication on the same real number. Since \( \bar{R}_{y4}(V_y, X_y)|_{T_y} = \varphi_y(\bar{R}_{y2}(V_y, X_y)) \cdot \text{id}_{T_y} \), we see that

\[
R^\gamma_4(W_1, W_2)|_{T_x} = \varphi_y(\bar{R}_{y2}(V_y, X_y)) \cdot \text{id}_{T_x} \cdot (50)
\]

Now statement 2.3 follows from \( \mathbf{18}, \mathbf{49}, \mathbf{50} \) and \( \mathbf{39} \).

Let us prove the inverse. From \( \mathbf{39}, \mathbf{15} \) and proposition \( \mathbf{1} \) it follows that \( \mathfrak{hol}_x \) is of type 1 or 3. By the Ambrose and Singer theorem, the vector space \( \mathfrak{hol}_x \) is spanned by the elements \( R^\gamma(W_1, W_2) \), where \( W_1, W_2 \in T_x \bar{M} \) and \( \gamma \) is a curve in \( \bar{M} \) with the beginning at the point \( x \).

To show that \( \mathfrak{hol}_x \) is not of type 1, we must prove the claim that if for some natural number \( k \), \( \alpha_i \in \mathbb{R}, W_{i1}, W_{i2} \in T_x \bar{M} \) and curves \( \gamma_i \), where \( 1 \leq i \leq k \), holds \( \sum_{i=1}^{k} \alpha_i R^\gamma(W_{i1}, W_{i2})|_{T_x} \neq 0 \), then \( \sum_{i=1}^{k} \alpha_i \text{pr}_{S_x} \circ R^\gamma(W_{i1}, W_{i2})|_{S_x} \neq 0 \). From section \( \mathbf{4} \) and statement 1 it follows that if for some \( i \) holds \( R^\gamma(W_{i1}, W_{i2})|_{T_x} \neq 0 \), then \( W_{i1} = V_i \in T_{r_x} \) and \( W_{i2} = X_i \in S_x \) (or vice versa). As above we can show that from statement 2.4 it follows that

\[
R^\gamma(V_i, X_i)|_{T_x} = \varphi_x(\text{pr}_{S_x} \circ R^\gamma(W_{i1}, W_{i2})|_{S_x})
\]
This proves the claim and the proposition. □

For any two sub-distributions \( S_1, S_2 \subset S \) denote by \( \text{Hom}(\mathfrak{so}(S_1), S_2) \) the vector bundle over \( \bar{M} \) such that \( \text{Hom}(\mathfrak{so}(S_1), S_2)_y = \text{Hom}(\mathfrak{so}(S_{1y}), S_{2y}) \) for all \( y \in \bar{M} \).

**Proposition 3** The holonomy algebra \( \text{hol}_x \) is of type 4 if and only if it is of type 2 or of type 4 (see proposition 1) and the following conditions hold

1. There exist two parallel sub-distributions \( S_1, S_2 \subset S \) such that \( S = S_1 \perp S_2 \) and the induced connection in \( S_2 \) is flat.

2. There exists a section \( \psi \in \Gamma(\text{Hom}(\mathfrak{so}(S_1), S_2)) \) such that

   2.1. \( R^*(X_y, Y_y) \in \ker \psi_y \) for all \( y \in \bar{M} \) and \( X_y, Y_y \in S_y \).

   2.2. \( R^3(V, Y)|_{S_2} = 0 \) and \( R^3(V, X)Y = g(\psi(R(V, X)), Y)U \) for all \( X \in \Gamma(S_1), Y \in \Gamma(S_2), U \in \Gamma(T^\perp) \) and \( V \in \Gamma(Tr) \) such that \( g(U, V) = 1 \).

   2.3. For any \( y \in \bar{M} \) and any curve \( \gamma : [a, b] \to \bar{M} \) with \( \gamma(a) = x \) and \( \gamma(b) = y \) we have

\[
\bar{g}_x(\psi_y(R_y(V_y, X_y)), \tau(\gamma)_Y) = g_y(\psi_x(\text{pr}_{S_1x} \circ \tau(\gamma)^{-1} \circ R_y(V_y, X_y) \circ \tau(\gamma)|_{S_1y}), Y_x)
\]

for all \( X_y \in S_{1y}, Y_y \in S_{2y}, \) and \( V_y \in Tr_y \).

The proof of proposition 3 is similar to the proof of proposition 2.

Note that the statement 1 is equivalent to

\[
\check{R}(W_1, W_2)\Gamma(S_1) \subset \Gamma(S_1) \text{ and } \check{R}(W_1, W_2)|_{\Gamma(S_2)} = 0 \text{ for all } W_1, W_2 \in \Gamma(T_x\bar{M}).
\]

Using (10), we can rewrite statement 2.2 in terms of \( A, h \) and \( \nabla \).

**Example 1.** Consider the case when \( (\bar{M}, \bar{g}) \) has an Abelian holonomy. It is known (see [11]) that locally there exist coordinates \( x^0, x^1, \ldots, x^n, x^{n+1} \) such that the metric \( \bar{g} \) has the form

\[
\bar{g} = 2dx^0dx^{n+1} + \sum_{i=1}^n(dx^i)^2 + f \cdot (dx^{n+1})^2,
\]

where \( f \) is a function of \( x^1, \ldots, x^n, x^{n+1} \). The nonzero Christoffel symbols are the following

\[
\begin{align*}
\Gamma^0_{n+1n+1} &= \frac{1}{2} \frac{\partial f}{\partial x^{n+1}}, \\
\Gamma^i_{n+1n+1} &= \frac{1}{2} \frac{\partial f}{\partial x^i}, \quad i = 1, \ldots, n, \\
\Gamma^0_{n+1} &= \frac{1}{2} \frac{\partial f}{\partial x^1},
\end{align*}
\]
where $1 \leq i \leq n$.

We see that for all $x$ from this coordinate neighborhood we have $T_x^\perp = \mathbb{R}(\frac{\partial}{\partial x^0})_x$ and we choose $S_x = \text{span}(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n})$. Then $T_r x = \mathbb{R} N_x$, where $N = -\frac{1}{2} f \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^{n+1}}$.

Since for $1 \leq i, j \leq n$ holds $\nabla \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \nabla \frac{\partial}{\partial x^0} \frac{\partial}{\partial x^j} = 0$, we have

$$\nabla \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \nabla \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} = 0$$

and

$$\nabla \frac{\partial}{\partial x^0} \frac{\partial}{\partial x^j} = \nabla \frac{\partial}{\partial x^0} \frac{\partial}{\partial x^0} = 0.$$

For $1 \leq j \leq n$ we have $\nabla \frac{\partial}{\partial x^j} = \frac{1}{2} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^j}$, hence

$$\nabla \frac{\partial}{\partial x^j} = 0 \text{ and } \nabla \left( N, \frac{\partial}{\partial x^j} \right) = \frac{1}{2} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^0}.$$

Furthermore,

$$\nabla N = \nabla \left( -\frac{1}{2} f \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^{n+1}} \right) = -\frac{1}{2} f \frac{\partial}{\partial x^0} \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^{n+1}} + \frac{\partial f}{\partial x^0} \frac{\partial}{\partial x^0} - \sum_{i=1}^n \frac{1}{2} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}$$

and

$$\nabla \frac{\partial}{\partial x^i} N = 0.$$

Hence, $\nabla^W N = 0$ for all $W \in \Gamma(T\bar{M})$,

$$A_N \frac{\partial}{\partial x^0} = A_N \frac{\partial}{\partial x^i} = 0 \text{ and } A_N N = \sum_{i=1}^n \frac{1}{2} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

Let $x \in \bar{M}$ be a point of this coordinate neighborhood and $M_x$ be the lightlike hypersurface through $x$. Then for the shape operator and the second fundamental form of $M_x$ we have

$$A = 0 \text{ and } \nabla^* = 0$$

respectively.

For the curvature tensors we have

$$\bar{R}_1 = \bar{R}_2 = \bar{R} = 0 \text{ and } R_3 \left( \frac{\partial}{\partial x^i}, N \right) \frac{\partial}{\partial x^j} = -\frac{1}{2} \frac{\partial^2 f}{\partial x^i \partial x^j} \frac{\partial}{\partial x^0}.$$

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