SPECTRAL INVARIANCE OF DENSE
SUBALGEBRAS OF OPERATOR ALGEBRAS

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Abstract

We define the notion of strong spectral invariance for a dense Fréchet subalgebra \( A \) of a Banach algebra \( B \). We show that if \( A \) is strongly spectral invariant in a C*-algebra \( B \), and \( G \) is a compactly generated polynomial growth Type R Lie group, not necessarily connected, then the smooth crossed product \( G \rtimes A \) is spectral invariant in the C*-crossed product \( G \rtimes B \). Examples of such groups are given by finitely generated polynomial growth discrete groups, compact or connected nilpotent Lie groups, the group of Euclidean motions on the plane, the Mautner group, or any closed subgroup of one of these. Our theorem gives the spectral invariance of \( G \rtimes A \) if \( A \) is the set of \( C^\infty \)-vectors for the action of \( G \) on \( B \), or if \( B = C_0(M) \), and \( A \) is a set of \( G \)-differentiable Schwartz functions \( S(M) \) on \( M \). This gives many examples of spectral invariant dense subalgebras for the C*-algebras associated with dynamical systems. We also obtain relevant results about exact sequences, subalgebras, tensoring by smooth compact operators, and strong spectral invariance in \( L_1(G, B) \).
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Introduction

The theory of differential geometry on a noncommutative space Connes [6] requires the use of “differentiable structures” for these noncommutative spaces, or some sort of algebra of “differentiable functions” on the noncommutative space. Such algebras of functions have usually been provided by a dense subalgebra of smooth functions $A$ for which the $K$-theory $K_*(A)$ is the same as the $K$-theory of the C*-algebra $K_*(B)$ (see for example Baum-Connes [1], Blackadar-Cuntz [3], Bost [4], Ji [9], the recent works of G. Elliott, T. Natsume, R. Nest, P. Jolissaint, V. Nistor and many others). In this paper, we use the algebras constructed in Schweitzer [22] to provide such dense subalgebras for a large class of examples. Let $G$ be any compactly generated polynomial growth Type R Lie group, not necessarily connected. Here Type R means that all the eigenvalues of $Ad$ lie on the unit circle, and polynomial growth means that the Haar measure of $U^n$ is bounded by a polynomial in $n$, where $U$ is a generating neighborhood. For example $G$ can be a a finitely generated polynomial growth discrete group, a compact or a connected nilpotent Lie group, or the group of Euclidean motions on the plane, the Mautner group, or any closed subgroup of one of these. We provide smooth subalgebras $G \rtimes S(M)$ of the C*-crossed product $G \rtimes C_0(M)$, where $M$ is any $G$-space (see Examples 2.6-7, 6.26-7, 7.20). We show that our subalgebras $G \rtimes S(M)$ are all spectral invariant in the C*-crossed product $G \rtimes C_0(M)$ (see Corollary 7.16), which implies that they have the same $K$-theory as the C*-crossed product by [5], VI.3 and [21], Lemma 1.2, Corollary 2.3. For
an example, if $M = H/K$ is a quotient of a compactly generated polynomial growth Type R Lie group $H$ by a closed cocompact subgroup $K$, and $G$ is a closed subgroup of $H$, then we have the spectral invariance of $G \rtimes C^\infty(M)$. If $G = \mathbb{Z}$, then $G \rtimes C^\infty(M)$ is just the standard Fréchet algebra of Schwartz functions from $\mathbb{Z}$ to $C^\infty(M)$, which are special cases of the algebras studied in Nest [14].

We also show that if $B$ is any C*-algebra on which $G$ acts, then the smooth crossed product $G \rtimes B^\infty$ is spectral invariant in the C*-crossed product $G \rtimes B$, where $B^\infty$ is the $C^\infty$-vectors for the action of $G$ on $B$. This generalizes the result Bost [4], Theorem 2.3.3 for elementary Abelian groups.

To prove the spectral invariance of these dense subalgebras, we introduce the notion of strong spectral invariance, which implies spectral invariance (see Definition 1.2 below - this notion is similar to the condition (1.4) of Blackadar and Cuntz [3], 3.1(b)). We show that $S(M)$ is always strongly spectral invariant in $C_0(M)$, and similarly for $B^\infty$ when $B$ is any Banach algebra with an action of a Lie group. If $A$ is strongly spectral invariant in $B$, we show the smooth crossed product $G \rtimes A$, which consists of functions that vanish rapidly with respect to a generalized “word length function”, is strongly spectral invariant in $L_1(G,B)$. (For $G \rtimes A$ to actually be a Fréchet algebra, we require that $G$ be compactly generated and Type R. We also make the third assumption that $G/\text{Ker}(Ad)$ has a cocompact solvable subgroup (this happens, for example, if $G$ is solvable or discrete). We conjecture that this third assumption is unnecessary - see [22], §1.4, Question 1.4.7.) We then imitate a result of Pytlik [19] for the group algebra case to show that if $G$ has polynomial growth, then a certain algebra of weighted $L_1$ functions $L_1^\tau(G,B)$ is in fact spectral invariant in the C*-crossed product $G \rtimes B$. This implies that $G \rtimes A$ is spectral invariant in $G \rtimes B$ - see Corollaries 7.14 and 7.16. (For such polynomial growth groups we do not need the third assumption above about cocompact solvable subgroups, since such subgroups are always present. Hence our final result holds for an arbitrary compactly generated polynomial growth Type R Lie group.)

As a partial converse, for an arbitrary Lie group $G$, if $G$ is not Type R, then none of the smooth Fréchet *-algebras $S(G)$ we defined in [22] are spectral invariant in $L_1(G)$, or in either of the C*-algebras $C^*_r(G)$ or $C^*(G)$ (see Theorem 6.29).

We analyze how the properties of spectral invariance and strong spectral invariance behave with exact sequences, tensoring with $n \times n$ matrices over $\mathbb{C}$ (see also [21]), and tensoring by a smooth version $K^\infty$ of the compact operators $K$. For example, we show that the completed projective tensor product $K^\infty \hat{\otimes}_\pi A$ is spectral invariant in the C*-algebra tensor product $K \hat{\otimes} B$.
if $A$ is strongly spectral invariant in $B$. Related results are obtained in Phillips [18], §4 for the case $A = B$.

The property of spectral invariance is important for the study of how the representation theory of the dense subalgebra relates to the representation theory of the $C^*$-algebra. For example, $A$ is spectral invariant in $B$ iff every simple $A$-module is contained in a *-representation of $B$ on a Hilbert space (Schweitzer [21], Corollary 1.5, Lemma 1.2). Our results on the spectral invariance of crossed products by polynomial growth Lie groups thus generalize the result of J. Ludwig [12] on the algebraically irreducible representations of the Schwartz algebra of a nilpotent Lie group.

Throughout this paper, the notations $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{T}$ shall be used for the natural numbers (with zero), integers, reals, and the circle group respectively. All of our algebras will be over $\mathbb{C}$. The term norm may be used interchangeably with the term seminorm. If the positive definiteness of a norm is important, we shall state it explicitly. The term differentiable will always mean infinitely differentiable. All groups will be assumed locally compact and Hausdorff.

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§1 Strong Spectral Invariance

In the section, we define what it means for a dense Fréchet subalgebra $A$ of a Banach algebra $B$ to be strongly spectral invariant in $B$. We show that strong spectral invariance implies spectral invariance, and also exhibit an example of a spectral invariant dense subalgebra which is not strongly spectral invariant.

Definition 1.1. A topological algebra is a topological vector space over $\mathbb{C}$ with an algebra structure for which the multiplication is separately continuous. Let $A$ be a dense subalgebra of a topological algebra $B$. If $A$ has no unit, let $\tilde{A}$ be $A$ with unit adjoined, and let $\tilde{B}$ be $B$ with the same unit adjoined (even if $B$ is unital already, adjoin a new one). If $A$ has a unit, then $B$ has the same unit, and we let $\tilde{A} = A, \tilde{B} = B$. We say that $B$ is a $Q$-algebra if $\tilde{B}$ has an open group of invertible elements. We shall usually be assuming that $B$ is a $Q$-algebra.

By a Fréchet algebra, we mean a (locally convex) Fréchet space with an algebra structure for which multiplication is jointly continuous Waelbroeck [24], Chap VII. A Fréchet algebra $A$ is $m$-convex if there exists a family of submultiplicative seminorms on $A$ which give the topology of $A$. We say that a Fréchet algebra $A$ is a dense Fréchet subalgebra of $B$ if $A$ is a dense subalgebra of $B$ and the inclusion map $\iota: A \hookrightarrow B$ is a continuous injective algebra
homomorphism. Note that if $A$ is a dense (Fréchet) subalgebra of $B$, then $\tilde{A}$ is a dense (Fréchet) subalgebra of $\tilde{B}$.

If $A$ is any dense subalgebra of $B$, we say that $A$ is spectral invariant in $B$ if the invertible elements of $\tilde{A}$ are precisely those elements of $\tilde{A}$ which are invertible in $\tilde{B}$ [21].

**Definition 1.2.** Let $A$ be a dense Fréchet subalgebra of a Banach algebra $B$. Let $\{\| \cdot \|_m\}$ be a family of seminorms giving the topology of $A$, and arrange that $\| \cdot \|_0$ is a norm giving the topology of $B$. (From now on we shall always assume that $\| \cdot \|_0$ is a norm giving the topology on $B$ (though not always the same one), whatever the choice of seminorms $\{\| \cdot \|_m\}$ topologizing $A$.) We say that $A$ is strongly spectral invariant in $B$ if

$$(1.3) \quad (\exists C > 0)(\forall m)(\exists D_m > 0)(\exists p_m \geq m)(\forall a_1, \ldots, a_n \in A) \left\{ \| a_1 \ldots a_n \|_m \leq D_m C^n \sum_{k_1+\ldots+k_n \leq p_m} \| a_1 \|_{k_1} \ldots \| a_n \|_{k_n} \right\}.$$ 

Notice that in the summand of (1.3), at most $p_m$ of the natural numbers $k_j$ are nonzero, regardless of $n$. This condition appears similar to the condition [22], (3.1.5) for $m$-convexity, and also to [22], (3.1.19). We require that $A$ be dense in $B$ in order to show that strong spectral invariance implies spectral invariance. We choose to have “$\sum_{k_1+\ldots+k_n \leq p}$” on the right hand side of (1.3) instead of “$\max_{k_1+\ldots+k_n \leq p}$” since sums commute with integration (see Theorems 5.4 and 6.7 below). However both ways are equivalent up to a constant.

We say that $A$ satisfies the *Blackadar-Cuntz condition* in $B$ if there exists a family of seminorms $\{\| \cdot \|_m\}$ for $A$ such that

$$(1.4) \quad (\exists C > 0)(\forall a, b \in A) \left\{ \| ab \|_m \leq C \sum_{i+j=m} \| a \|_i \| b \|_j \right\}$$

(see Blackadar-Cuntz [3]). Note that $m$ appears on the right hand side of the inequality (1.4), whereas in (1.3) we replaced $m$ by the possibly larger natural number $p_m$.

**Example 1.5.** Let $B$ be the commutative C*-algebra $c_0(\mathbb{Z})$ of complex valued sequences on $\mathbb{Z}$ which vanish at infinity, with pointwise multiplication. Let $A$ be the dense Fréchet subalgebra $S(\mathbb{Z})$ of sequences which satisfy

$$(1.6) \quad \| f \|_m = \sup_{n \in \mathbb{Z}} (1 + |n|)^m | f(n) | < \infty, \quad m \in \mathbb{N}.$$ 

Then we have $\| f_1 \ldots f_n \|_m \leq \| f_1 \|_m \| f_2 \|_0 \ldots \| f_n \|_0$ for $f_1, \ldots, f_n \in A$. So $A$ is strongly spectral invariant in $B$ (with $C = 1, p_m = m, D_m = 1$), and also satisfies the Blackadar-Cuntz condition in $B$, taking $n = 2$. 

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Proposition 1.7. Let $A$ be a dense Fréchet subalgebra of a Banach algebra $B$. Then $A$ is strongly spectral invariant in $B$ iff (1.3) holds for every family of seminorms \( \| \|_m \) on $A$, or iff (1.3) holds for any one family. The constant $C$ depends only on the choice of the zeroth norm \( \| \|_0 \) on $B$. If $A$ is strongly spectral invariant in some Banach algebra $B$, then $A$ is $m$-convex.

If $A$ satisfies the Blackadar-Cuntz condition in $B$, then $A$ is strongly spectral invariant in $B$. Moreover the constants $D_m$ in (1.3) need not depend on $m$, and we may take $p_m = m$.

Proof. We first show that if (1.3) holds for one family of seminorms \( \| \|_m \) on $A$, then it holds for any equivalent increasing family \( \| \|'_m \) of seminorms on $A$. First we have

\[
\| a_1 \ldots a_n \|'_m \leq K \sum_{i \leq r} \left\{ \left( \| a_1 \ldots a_n \|_i \right) \right\}_{\text{equiv. of seminorms}} \\
\leq K \sum_{i \leq r} \left\{ D_i C^n \sum_{k_1 + \ldots + k_n \leq p_i} \| a_1 \|_{k_1} \ldots \| a_n \|_{k_n} \right\},
\]

where $r \in \mathbb{N}$ and $K > 0$ depend only on $m$. Let $t = \max_{i \leq r} p_i$. Let $K_1 > 0$ and $s \geq t$ be such that for any $0 \leq j \leq t$, we have $\| a \|_j \leq K_1 \| a \|'_s$. (Here we have used that the family \( \{ \| \|'_m \} \) is increasing.) Define

\[
\tilde{k}_j = \begin{cases} 0 & k_j = 0 \\ s & k_j \neq 0 \\ \end{cases}
\]

Then the right hand side of (1.8) is bounded by

\[
K \sum_{i \leq r} \left\{ D_i C^n \sum_{k_1 + \ldots + k_n \leq t} K_1^n \| a_1 \|'_{\tilde{k}_1} \ldots \| a_n \|'_{\tilde{k}_n} \right\} \\
\leq K_2 (CK_1)^n \sum_{k_1 + \ldots + k_n \leq st} \| a_1 \|'_{k_1} \ldots \| a_n \|'_{k_n},
\]

where $K_2 = K \sum_{i \leq r} D_i$. This shows that the increasing family \( \{ \| \|'_m \} \) satisfies (1.3). A slightly longer argument shows that (1.3) holds for any family of seminorms for $A$, but we omit it for brevity.

To see that $C$ depends only on the choice of the zeroth seminorm, assume that in our above calculations that $\| \|_0 = \| \|'_0$. Then in (1.9) we could replace $K_1^n$ with $K_1^t$ on the left hand side. We would then have $C^n$ on the right hand side instead of $(CK_1)^n$, and set $K_2 = KK_1^t \sum_{i \leq r} D_i$. 

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For the $m$-convexity, it suffices to show that condition (1.3) implies [22], (3.1.5) for every increasing family $\{\| \cdot \|_m\}$ of seminorms for $A$. But by (1.3),

\[
\| a_1 \ldots a_n \|_m \leq D_m C^n \sum_{k_1+\ldots k_n \leq p} \| a_1 \|_p \ldots \| a_n \|_p
\]

\[
\leq C^n \| a_1 \|_p \ldots \| a_n \|_p,
\]

where $C_1 > 0$ is a sufficiently large constant. This is precisely [22], (3.1.5).

Assume that $A$ satisfies the Blackadar-Cuntz condition in $B$. Let $\{\| \cdot \|_m\}$ be a family of seminorms which satisfy (1.4). Then we have

\[
\| a_1 \ldots a_n \|_m \leq C_{k_1+j=m} \| a_1 \|_{k_1} \| a_2 \ldots a_n \|_j
\]

\[
\leq C^2 \sum_{k_1+k_2+j=m} \| a_1 \|_{k_1} \| a_2 \|_{k_2} \| a_3 \ldots a_n \|_j
\]

\[
\ldots \quad \ldots \quad \ldots
\]

\[
\leq C^{n-1} \sum_{k_1+\ldots k_n=m} \| a_1 \|_{k_1} \ldots \| a_n \|_{k_n}.
\]

Taking $D_m = 1/C$, we have the last statement of the theorem. □

Motivated by [22], Theorem 3.1.4 and [22], Theorem 3.1.18, we ask the following question.

**Question 1.12.** If $A$ is strongly spectral invariant in $B$, then does $A$ satisfy the Blackadar-Cuntz condition (or some appropriate modification of it) in $B$?

**Example 1.13.** We give an example of a spectral invariant dense subalgebra which is not strongly spectral invariant. Let $B$ be the $C^*$-algebra $C^*(\mathbb{Z})$ of the integers $\mathbb{Z}$, with convolution multiplication, and let $A$ be the dense Banach subalgebra $l_1(\mathbb{Z})$ of absolutely summable sequences. Let $\| \cdot \|_0$ be the $C^*$ norm, and let $\| \cdot \|_1$ be the $l_1$ norm. We show that (1.3) cannot be satisfied for the family of norms $\{\| \cdot \|_0, \| \cdot \|_1\}$ for $A$. By the estimate [10], Chap VI, §6, p.82(in the proof of Katzenelson’s theorem), we have

\[
\sup_{\psi = \psi^*, \| \psi \|_1 \leq r} \| \exp(i\psi) \|_1 = e^r.
\]

If $\psi = \psi^*$, note that $\| \exp(i\psi) \|_0$ is just the sup norm of the Fourier transform $e^{i\hat{\psi}}$ in $C(\mathbb{T})$, which is 1 since $\hat{\psi}$ is real valued. Also $\| \exp(i\psi) \|_1 \leq e^{\| \psi \|_1}$. We have

\[
\| \exp(i\psi) \|_1 = \| \exp(i\psi/n)^n \|_1 \leq DC^n \| \exp(i\psi/n) \|_1^p
\]

by (1.3)

\[
\leq DC^n (e^{r/n})^p \quad \text{if} \quad \| \psi \|_1 \leq r.
\]
for all \( n \in \mathbb{N} \). Hence by (1.14),

\[
e^r \leq DC^m e^{rn/n}
\]

for arbitrarily large \( r \in \mathbb{N} \). This is a contradiction if we fix \( n \) larger than \( p \). Hence \( A \) is not strongly spectral invariant in \( B \). However, \( A \) is spectral invariant in \( B \) by Wiener's theorem.

**Theorem 1.17.** If \( A \) is strongly spectral invariant in \( B \), then \( A \) is spectral invariant in \( B \).

**Proof.**

**Lemma 1.18.** If \( A \) is strongly spectral invariant in \( B \), then \( \tilde{A} \) is strongly spectral invariant in \( \tilde{B} \).

**Proof.** It suffices to consider the case when \( \tilde{A} \) and \( \tilde{B} \) are the respective unitizations of \( A \) and \( B \). If \( \{ \| \cdot \|_m \} \) is a family of seminorms for \( A \), we define seminorms \( \| \cdot \|'_m \) for \( \tilde{A} \) by

\[
\| a + \lambda 1 \|'_0 =\| a \|_0 + |\lambda| \quad \text{and} \quad \| a + \lambda 1 \|'_m =\| a \|_m \quad \text{for} \quad m > 0.
\]

Let \( \tilde{a}_1, \ldots, \tilde{a}_n \in \tilde{A} \), where \( \tilde{a}_i = a_i + \lambda_i 1 \), with \( a_i \in A \), \( \lambda_i \in \mathbb{C} \). We estimate \( \| \tilde{a}_1, \ldots, \tilde{a}_n \|'_m \). For \( m = 0 \), the norm is submultiplicative up to a constant, so we assume \( m > 0 \). Then

\[
\| \tilde{a}_1, \ldots, \tilde{a}_n \|'_m =\| (a_1 + \lambda_1 1) \ldots (a_n + \lambda_n 1) \|_m
\]

(1.19)

\[
\leq \sum_{1 \leq i_1 < \ldots < i_r \leq n} \| a_{i_1} \ldots a_{i_r} \lambda_{j_{i_1}} \ldots \lambda_{j_{i_r}} \|_m,
\]

where the sum is over all \( r \)-tuples \( 1 \leq i_1 < \ldots < i_r \leq n \) for all \( 1 < r \leq n \), and \( \{ j_1, \ldots, j_{n-r} \} = \{1, \ldots, n \} - \{ i_1, \ldots, i_r \} \). We estimate one of the summands in (1.19). For simplicity, we set \( D_m = 1 \) in (1.3). We have

\[
\| a_{i_1} \ldots a_{i_r} \lambda_{j_{i_1}} \ldots \lambda_{j_{n-r}} \|_m = |\lambda_{j_{i_1}} \ldots \lambda_{j_{n-r}}| \| a_{i_1}, \ldots, a_{i_r} \|_m
\]

\[
\leq |\lambda_{j_{i_1}}| \ldots |\lambda_{j_{n-r}}| C^r \left\{ \sum_{k_1 + \ldots k_r \leq p} \| a_{i_1} \|_{k_1} \ldots \| a_{i_r} \|_{k_r} \right\} \quad \text{strong spec. inv. (1.3)}
\]

\[
\leq \| \tilde{a}_{j_{i_1}} \|'_0 \ldots \| \tilde{a}_{j_{n-r}} \|'_0 C^r \left\{ \sum_{k_1 + \ldots k_r \leq p} \| \tilde{a}_{i_1} \|'_{k_1} \ldots \| \tilde{a}_{i_r} \|'_{k_r} \right\} \quad \text{def. of} \quad \| \cdot \|'_m.
\]

(1.20)

\[
\leq C^r \left\{ \sum_{k_1 + \ldots k_n \leq p} \| \tilde{a}_{i_1} \|'_{k_1} \ldots \| \tilde{a}_n \|'_{k_n} \right\}
\]

\[
\leq \max(1, C)^n \left\{ \sum_{k_1 + \ldots k_n \leq p} \| \tilde{a}_{i_1} \|'_{k_1} \ldots \| \tilde{a}_n \|'_{k_n} \right\}
\]

Combining (1.19) and (1.20), we have

\[
\| \tilde{a}_1, \ldots, \tilde{a}_n \|_m \leq 2^n \max(1, C)^n \left\{ \sum_{k_1 + \ldots k_n \leq p} \| \tilde{a}_{i_1} \|_{k_1} \ldots \| \tilde{a}_n \|_{k_n} \right\}.
\]
This proves Lemma 1.18. □

Now we are ready to prove Theorem 1.17. By Lemma 1.18, it suffices to consider the case when \( A \) and \( B \) are both unital with the same unit. By Proposition 1.7, we may assume that the seminorms on \( A \) are increasing and submultiplicative. By strong spectral invariance, we have

\[
\| a^n \|_m \leq D_m C^n \sum_{k_1 + \cdots + k_n \leq p} \| a \|_{k_1} \cdots \| a \|_{k_n}
\]

(1.21)

\[
\leq D_m C^n \sum_{k_1 + \cdots + k_n \leq p} \| a \|^{n-p}_0 \| a \|_p^p \quad \text{since norms increase}
\]

\[
\leq D'_m (pC)^n \| a \|^{n-p}_0 \| a \|_p^p
\]

where \( p \) depends only on \( m \), and \( D'_m \) is sufficiently large that \( D_m n^p \leq D'_m p^n \) (note that \( p \) is fixed as \( n \) runs). If \( \| a \|_0 < 1/2pC \), then by (1.21),

\[
\| a^n \|_m \leq D'_m (pC)^n (1/2pC)^{n-p} \| a \|_p^p
\]

(1.22)

\[
= 1/2^n \left( (pC)^p D'_m \| a \|_p^p 2^p \right).
\]

So the series \( 1 + a + a^2 + \ldots \) converges absolutely in the norm \( \| \cdot \|_m \) if \( \| a \|_0 \) is sufficiently small.

Let \( A_m \) be the completion of \( A \) in \( \| \cdot \|_m \). By what we’ve just seen, for each \( m \) there is a neighborhood \( U_m \) of the identity in \( B \) such that if \( a \in A \cap U_m \), then the series \( 1 + (1-a) + (1-a)^2 + \ldots \) converges in \( A_m \). This is just the series for \( a^{-1} \), so \( a^{-1} \in A_m \).

Let \( a \in A \) and assume \( a^{-1} \in B \). We show \( a^{-1} \in A \). Since we have choosen submultiplicative seminorms \( \| \cdot \|_m \) for \( A \), it suffices to show that \( a \) is invertible in \( A_m \) for each \( m \) by Micheal [13], Theorem 5.2 (c).

The set \( a^{-1} U_m \cap U_m a^{-1} \) is open and nonempty (contains \( a^{-1} \)) in \( B \), and so contains an element \( a' \) of \( A \) since \( A \) is dense in \( B \). Then \( aa' \) and \( a'a \) are both in \( U_m \). Since they also lie in \( A \), the construction of the \( U_m \)’s tells us that \((aa')^{-1} \) and \((a'a)^{-1}\) both lie in \( A_m \). It follows that \( a^{-1} \in A_m \). Thus \( a^{-1} \in A_m \) for all \( m \) and we are done. The last part of this argument is similar to Bost [4], Lemme A.2.3. □

§ 2 Strong Spectral Invariance of \( S^\sigma_H(M) \) and \( B^\infty \)

We verify the strong spectral invariance of a space of Schwartz functions \( S^\sigma_H(M) \) on a locally compact \( H \)-space \( M \) as a subalgebra of the commutative C*-algebra \( C_0(M) \) of continuous
functions vanishing at infinity on $M$. Also, we show that the set of $C^\infty$-vectors $B^\infty$ is always strongly spectral invariant in $B$, for an arbitrary Banach algebra $B$.

We recall the definition of $S^\sigma_H(M)$ from [22], §5. Let $H$ be a Lie group, possibly disconnected, and let $M$ be a locally compact space on which $H$ acts. We say that a Borel measurable function $\sigma:M \to [0,\infty)$ is a scale if it is bounded on compact subsets of $M$. We say that a scale $\sigma$ dominates another scale $\gamma$ if there exists $C > 0$ and $d \in \mathbb{N}$ such that $\gamma(m) \leq C(1 + \sigma(m))^d$ for $m \in M$. We say that $\sigma$ and $\gamma$ are equivalent (denoted by $\sigma \sim \gamma$) if they dominate each other. If $h \in H$, define $\sigma_h(m) = \sigma(h^{-1}m)$. We say that $\sigma$ is uniformly $H$-translationally equivalent if for every compact subset $K$ of $H$ there exists $C_K > 0$ and $d \in \mathbb{N}$ such that

\begin{equation}
\sigma_h(m) \leq C_K(1 + \sigma(m))^d, \quad m \in M, \ h \in K.
\end{equation}

If $\sigma$ is a uniformly $H$-translationally equivalent scale on $M$, we may define the $H$-differentiable $\sigma$-rapidly vanishing functions $S^\sigma_H(M)$ by

\[ S^\sigma_H(M) = \{ f \in C_0(M), \ f \ H\text{-differentiable} \mid X^\gamma f \in C_0(M) \text{ and } \| \sigma^d X^\gamma f \|_\infty < \infty \}, \]

where $X^\gamma$ ranges over all differential operators from the Lie algebra of $H$, and $d$ ranges over all natural numbers. We topologize $S^\sigma_H(M)$ by the seminorms

\[ \| f \|_d = \sum_{|\gamma|=d} \| (1 + \sigma)^d X^\gamma f \|_\infty, \]

where we make the convention $|\gamma| = \sum |\gamma_i|$. Then $S^\sigma_H(M)$ is an $m$-convex Fréchet *-algebra under pointwise multiplication, with differentiable action of $H$, and is a dense Fréchet subalgebra of $C_0(M)$ [22], §5.

In the following theorem, $B$ will be any Banach algebra, with a strongly continuous action of a Lie group $G$ by isometric automorphisms. (Throughout this paper, we shall assume that group actions on $B$ are by isometric automorphisms, which means that $\| \alpha_g(b) \| = \| b \|$ for all $b \in B$ and $g \in G$.) We then may form a dense Fréchet subalgebra $B^\infty$ of $C^\infty$-vectors for the action of $G$ on $B$ [22], Theorem A.2.

**Theorem 2.2.** The Fréchet algebra $S^\sigma_H(M)$ is a strongly spectral invariant subalgebra of $C_0(M)$, and moreover satisfies the Blackadar-Cuntz condition in $C_0(M)$. The same is true for the Fréchet algebra $B^\infty$ of $C^\infty$-vectors in $B$. 
Proof. By the product rule,
\[
\sum_{|\gamma|=d} \| (1 + \sigma)^d X^\gamma (f_1 f_2) \|_\infty \leq C \sum_{|\gamma|=d, \beta_1 + \beta_2 = \gamma} \| (1 + \sigma)^d (X^{\beta_1} f_1) (X^{\beta_2} f_2) \|_\infty \leq C \sum_{|\beta_1| + |\beta_2|=d} \| (1 + \sigma)^{d_1} (X^{\beta_1} f_1) \|_\infty \| (1 + \sigma)^{d_2} (X^{\beta_2} f_2) \|_\infty,
\]
where \(d_1 = |\beta_1|\) and \(d_2 = |\beta_2|\). Since the right hand side of (2.3) is bounded by
\[
(2.4) \quad C \sum_{d_1 + d_2 = d} \| f_1 \| d_1 \| f_2 \| d_2,
\]
we see that \(S_H^\sigma(M)\) satisfies the Blackadar-Cuntz condition in \(C_0(M)\).

We show that \(B^\infty\) satisfies the Blackadar-Cuntz condition in \(B\). Define seminorms by
\[
\| b \|_m = \sum_{|\gamma|=m} \| X^\gamma b \|. \quad \text{Then}
\]
\[
\sum_{|\gamma|=m} \| X^\gamma (b_1 b_2) \| \leq C \sum_{|\gamma|=m, \beta_1 + \beta_2 = \gamma} \| X^{\beta_1} b_1 \| \| X^{\beta_2} b_2 \| = C \sum_{i_1 + i_2 = m} \| b_1 \| i_1 \| b_2 \| i_2.
\]
□

Definition 2.5. We say that a locally compact group \(H\) is compactly generated if there exists a relatively compact neighborhood \(U\) of the identity of \(H\) such that \(U^{-1} = U\) and \(H = \bigcup_{n=0}^\infty U^n\). We say that \(H\) has polynomial growth if the Haar measure of \(U^n\) is bounded by a polynomial in \(n\). We say that a scale \(\tau\) on \(H\) is a gauge if \(\tau(e) = 0\), \(\tau(g^{-1}) = \tau(g)\), and \(\tau(gh) \leq \tau(g) + \tau(h)\). We define the word gauge \(\tau_U\) on \(H\) by
\[
\tau_U(g) = \min\{ n \mid g \in U^n \}.
\]
Then \(\tau_U\) is independent up to equivalence of the choice of \(U\) [22], Theorem 1.1.21.

Example 2.6. Assume that \(H\) is a compactly generated polynomial growth Lie group, and let \(K\) be a closed subgroup. Let \(\sigma(h) = \inf_{k \in K} \tau_U(hk)\), and \(M = H/K\). Then \(S_H^\sigma(M)\) is strongly spectral invariant in \(C_0(M)\). Moreover, \(S_H^\sigma(M)\) is also a nuclear Fréchet algebra [22], Proposition 1.5.1, Theorem 6.8.

Example 2.7. Let \(G\) be any other closed subgroup of \(H\). Then \(S_G^\sigma(M)\) is also strongly spectral invariant in \(C_0(M)\). However \(S_G^\sigma(M)\) may not be nuclear. For example, let \(G = K = \{e\}, H = \mathbb{T}\). Then \(S_G^\sigma(M)\) is the infinite dimensional Banach algebra \(C(\mathbb{T})\) of continuous functions on the circle, and hence not nuclear.
To give a familiar example, let $M = H = \mathbb{R}$, $G = K = \{e\}$. Then $\sigma$ is equivalent to the absolute value function on $\mathbb{R}$, and $S^\sigma_H(M)$ is just the standard set of Schwartz functions $S(\mathbb{R})$ on $\mathbb{R}$. The space $S^\sigma_G(M)$ is the set of continuous functions on $\mathbb{R}$, which vanish rapidly with respect to $|r|$.

§3 Subalgebras and Exact Sequences

We look at how the properties of spectral invariance and strong spectral invariance behave in the context of exact sequences. We let $I$ be a two-sided ideal of $A$, and let $J$ be the closure of $I$ in $B$. Note that if we assume that $J \cap A = I$, then $A/I$ is a subalgebra $B/J$.

Example 3.1. For example, for $A = C^\infty(\mathbb{T})$, $B = C(\mathbb{T})$, we could take $I$ and $J$ to be the set of functions in $A$ and $B$ respectively which vanish at some fixed point $p \in \mathbb{T}$. The property $J \cap A = I$ is then satisfied. However, if in place of $I$ we took the ideal of functions which vanish along with all of their derivatives at $p$, we would still have $I = J$ but not $J \cap A = I$.

Results related to part (2) of the following theorem appear in Palmer [15], Corollary 5.6,7 (see also the introduction of that paper). (In that paper, a “spectral invariant subalgebra” is called a “spectral subalgebra”.)

Theorem 3.2. Let $I$ and $J$ be as above, and assume $J \cap A = I$. Let $A_1$ be any subalgebra of $A$, and let $B_1$ be the closure of $A_1$ in $B$. Assume that $B$ is a Banach algebra, so that both $B_1$ and $B/J$ are Banach algebras and hence $Q$-algebras.

(1) Let $A$ be a dense Fréchet subalgebra of $B$, and assume that $I$ and $A_1$ are both closed in the topology of $A$, with Fréchet topology inherited from $A$. Let $A$ be strongly spectral invariant in $B$. Then $A_1$ is strongly spectral invariant in $B_1$, and the ideal $I$ is strongly spectral invariant in $J$. Similar statements hold for the Blackadar-Cuntz condition.

(2) Let $A$ be any dense subalgebra of $B$. Assume that $A$ is spectral invariant in $B$. If $A_1 = A \cap B_1$, then $A_1$ is spectral invariant in $B_1$. The ideal $I$ is spectral invariant in $J$ and $A/I$ is spectral invariant in $B/J$. Conversely, if $I$ is spectral invariant in $J$ and $A/I$ is spectral invariant in $B/J$, then $A$ is spectral invariant in $B$.

Proof. Since seminorms on $A_1$ are given by the restriction of any family of seminorms on $A$ to $A_1$, and the norm on $B_1$ is the restriction of the norm on $B$ to $B_1$, the strong spectral invariance of $A_1$ in $B_1$ is obtained simply by restricting the inequality (1.3) in the definition of strong spectral invariance to elements of $A_1$. As a special case, we also see that $I$ is strongly spectral invariant in $J$. This proves (1).
Assume that $A$ is spectral invariant in $B$. Recall from [21], Theorem 1.4 that $A$ is spectral invariant in $B$ if every simple $A$-module is contained as a dense $A$-submodule of a continuous $B$-module. Let $V$ be simple $A/I$-module. Then $V$ is a simple $A$-module and so extends to a continuous $B$-module $W$ in which $V$ is dense (using [21], Theorem 1.4 $(i) \Rightarrow (iii)$). To see that $W$ factors to a $B/J$-module it suffices to show that $JW = \{0\}$. We know $IV = \{0\}$. So since $J = T$ and $V$ is dense in $W$, $JW = \{0\}$. Thus $W$ is a $B/J$-module extending $V$, and we have shown that $A/I$ is spectral invariant in $B/J$ by [21], Theorem 1.4 $(iii) \Rightarrow (i)$. This did not require $B/J$ to be a $Q$-algebra, so it would suffice for $B$ to be any $Q$-algebra.

Next we show that $A_1$ is spectral invariant in $B_1$ under the assumption that $A_1 = A \cap B_1$ (and hence, as a special case, that $I$ is spectral invariant in $J$). By replacing $A$, $B$ with $\tilde{A}$, $\tilde{B}$, we may assume that $A$ and $B$ are unital with the same unit. If $A_1$ and $B_1$ are unital, let $a \in A_1$, $a^{-1} \in B_1$. Then $aa^{-1} = a^{-1}a = 1_{B_1}$. Let $q$ be the projection $1_B - 1_{B_1}$. Then $(q + a)(q + a^{-1}) = (q + a^{-1})(q + a) = 1_B$, so $q + a$ is invertible in $B$ and hence in $A$. Hence $q + a^{-1} \in A$, and $a^{-1} \in A$. By assumption, $a^{-1} \in A_1$.

Next assume $A_1$ is nonunital. We may assume that $\tilde{A}_1$, $\tilde{B}_1$, $A$, and $B$ all have the same unit. If $a \in \tilde{A}_1$ is invertible in $\tilde{B}_1$, then clearly $a$ is invertible in $B$. Hence $a^{-1}$ lies in both $A$ and $\tilde{B}_1$, so by assumption $a^{-1} \in \tilde{A}_1$ and $A_1$ is spectral invariant in $B_1$.

Assume that $I$ is spectral invariant in $J$ and $A/I$ is spectral invariant in $B/J$. We show that $A$ is spectral invariant in $B$ using [21], Theorem 1.4. Let $V$ be an irreducible $A$-module. One easily checks that either $IV = \{0\}$ or $V$ is a simple $I$-module. So we have two cases.

Case 1. Say $V$ is a simple $I$-module. Then since $I$ is spectral invariant in $J$, we can extend $V$ to a $J$-module $W$. Since $J$ is a two-sided ideal in $B$, we can extend the action of $J$ on $W$ in a unique way to one of $B$ (see, for example, the argument in Fell [7], Proposition 1).

Case 2. Say $IV = \{0\}$. Then $V$ is an irreducible $A/I$-module. By hypothesis, and since $B/J$ is a $Q$-algebra, $V$ extends to a $B/J$-module $W$. Using the canonical algebra homomorphism from $B$ to $B/J$, we make $W$ a $B$-module. This action of $B$ on $W$ clearly extends the action of $A$ on $V$. This proves (2). $\Box$

**Remark 3.3.** Theorem 3.2(2) and [21], Lemma 1.2 answer Question 3.1.8 of Blackadar [2] in the affirmative, in the case that the local Banach algebra $A$ has a Fréchet topology stronger than the norm topology, and the ideal $I$ is closed in $A$. A general answer to this question is given in Schmitt [20].
§4 Tensoring by $M_l(\mathbb{C})$ and Crossed Products by finite groups

Theorem 4.1.

(1) Let $A$ be a dense Fréchet subalgebra of a Banach algebra $B$. Then the matrix algebra $M_l(A)$ is strongly spectral invariant in $M_l(B)$ iff $A$ is strongly spectral invariant in $B$.

(2) (Schweitzer [21], Theorem 2.1) Let $A$ be a dense subalgebra of a $Q$-algebra $B$. Then the matrix algebra $M_l(A)$ is spectral invariant in $M_l(B)$ iff $A$ is spectral invariant in $B$.

Proof. For part (1), by Theorem 3.2 (1) it suffices to show that if $A$ is strongly spectral invariant in $B$, then $M_l(A)$ is strongly spectral invariant in $M_l(B)$. We define seminorms $\| \cdot \|_m$ on $M_l(A)$ by

$$\| [a] \|_m' = \max_{1 \leq i,j \leq l} \| [a]_{ij} \|_m.$$ 

An inductive argument shows that the $ij$th entry of a product $[a_1] \ldots [a_n]$ is the sum of $l^{n-1}$ products of elements of the form $[a_1]_{i_1 j_1} \ldots [a_n]_{i_n j_n}$. By the strong spectral invariance of $A$ in $B$, we have

$$\| [a_1]_{i_1 j_1} \ldots [a_n]_{i_n j_n} \|_m \leq D_mC^m \sum_{k_1 + \ldots + k_n \leq p} \left\{ \| [a_1]_{i_1 j_1} \|_{k_1} \ldots \| [a_n]_{i_n j_n} \|_{k_n} \right\}$$

$$\leq D_mC^m \sum_{k_1 + \ldots + k_n \leq p} \left\{ \| [a_1] \|'_{k_1} \ldots \| [a_n] \|'_{k_n} \right\}.$$ 

It follows that

$$\| [a_1] \ldots [a_n] \|'_m \leq D_ml^{n-1}C^m \sum_{k_1 + \ldots + k_n \leq p} \left\{ \| [a_1] \|'_{k_1} \ldots \| [a_n] \|'_{k_n} \right\}.$$ 

□

Corollary 4.2. Let $G$ be a finite group acting on $A$ and $B$ by algebra automorphisms, which are continuous on $B$.

(1) Let $A$ be a dense Fréchet subalgebra of a Banach algebra $B$, and assume that $G$ acts continuously on $A$. Then the crossed product $G \rtimes A$ is strongly spectral invariant in $G \rtimes B$ iff $A$ is strongly spectral invariant in $B$.

(2) Let $A$ be a dense subalgebra of a $Q$-algebra $B$. Then the crossed product $G \rtimes A$ is spectral invariant in $G \rtimes B$ iff $A$ is spectral invariant in $B$. 

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Proof. First assume that $G \rtimes A$ is (strongly) spectral invariant in $G \rtimes B$. Then by (Theorem 3.2 (1)) Theorem 3.2 (2), and since $B \cap G \rtimes A = A$, we have that $A$ is (strongly) spectral invariant in $B$.

Next assume that $A$ is (strongly) spectral invariant in $B$. The following argument is similar to one in the appendix of Baum-Connes [1]. Let $l$ be the order of $G$. We may identify $M_l(B)$ with the set of functions from $G \times G$ to $B$, if we define the multiplication

$$S \ast T(g, h) = \sum_{k \in G} S(g, k)T(k, h)$$

on elements $S, T$ of $C(G \times G, B)$. We make similar definitions for $M_l(A)$. Then by Theorem 4.1, we know that $C(G \times G, A)$ is (strongly) spectral invariant in $C(G \times G, B)$.

We embed $G \rtimes B$ in $C(G \times G, B)$ via

$$i(F)(g, h) = \alpha_g(F(g^{-1}h)), \quad F \in G \rtimes B.$$ 

This embedding is easily seen to be an algebra homomorphism, and is a topological embedding since $G$ acts continuously. Similarly $i$ embeds $G \rtimes A$ as a subalgebra of $C(G \times G, A)$, topologically if $A$ is Fréchet.

Let $G$ act on $C(G \times G, B)$ (resp. $C(G \times G, A)$) via

$$\theta_g(S)(k, h) = \alpha_{g^{-1}}(S(gk, gh)).$$

Then $i(G \rtimes A)$ is the set of fixed points for the action of $\theta$ on $C(G \times G, A)$, and similarly for $i(G \rtimes B)$ and $C(G \times G, B)$. Clearly $i(G \rtimes A) = i(G \rtimes B) \cap C(G \times G, A)$. So (Theorem 3.2 (1)) Theorem 3.2 (2) tells us that $G \rtimes A$ is (strongly) spectral invariant in $G \rtimes B$. □

Remark 4.3. There is a nice alternate proof of the spectral invariance part of Corollary 4.2 using extensions of simple modules and Fell [7], Proposition 5.

§5 Tensoring by Smooth Compacts

Throughout this section, we let $B$ be a C*-algebra, and $A \subseteq B$ be a dense Fréchet subalgebra. Let $\mathcal{H} = l_2(\mathbb{Z})$ and let $\mathcal{K}$ be the compact operators on $\mathcal{H}$. We define two dense subalgebras of the C*-tensor product $\mathcal{K} \otimes B$.

If $\mathcal{A}$ is any Fréchet algebra, let $S(\mathbb{Z}^2, \mathcal{A})$ be the set of functions $\varphi$ from $\mathbb{Z}^2$ to $\mathcal{A}$ which satisfy

$$\| \varphi \|_q = \sum_{r, s \in \mathbb{Z}} (1 + |r| + |s|)^q \| \varphi(r, s) \|_q < \infty$$

(5.1)
for every $q \in \mathbb{N}$. We define multiplication by

\begin{equation}
\varphi \ast \psi(r,t) = \sum_{s \in \mathbb{Z}^2} \varphi(r,s)\psi(s,t).
\end{equation}

If $\mathcal{A} = \mathbb{C}$, we denote the resulting nuclear $m$-convex Fréchet algebra by $\mathcal{K}^\infty$, the smooth compact operators. In general, $\mathcal{S}(\mathbb{Z}^2,\mathcal{A})$ is isomorphic to the projective completion $\mathcal{K}^\infty \hat{\otimes}_\pi \mathcal{A}$. Let $\mathcal{L}$ be a Hilbert space on which $B$ is faithfully $*$-represented. Then $\mathcal{K} \hat{\otimes} B$ is faithfully represented on the Hilbert space tensor product $\mathcal{H} \otimes \mathcal{L} = l_2(\mathbb{Z},\mathcal{L})$, and $\mathcal{K}^\infty \hat{\otimes}_\pi \mathcal{A}$ is the dense subalgebra of $\mathcal{K} \hat{\otimes} B$ which acts on $l_2(\mathbb{Z},\mathcal{L})$ via

\begin{equation}
\varphi \xi(r) = \sum_{t \in \mathbb{Z}} \varphi(r,t)\xi(t), \quad \varphi \in \mathcal{K}^\infty \hat{\otimes}_\pi \mathcal{A}, \quad \xi \in l_2(\mathbb{Z},\mathcal{L}).
\end{equation}

We also note that the space $l_1(\mathbb{Z}^2,B) \cong l_1(\mathbb{Z}^2) \hat{\otimes}_\pi B$, topologized by the norm (5.1) with $q = 0$, is a Banach algebra under the multiplication (5.2). It also acts on $l_2(\mathbb{Z},\mathcal{L})$ via (5.3) and is a dense subalgebra of $\mathcal{K} \hat{\otimes} B$, which contains $\mathcal{K}^\infty \hat{\otimes}_\pi \mathcal{A}$ as a dense subalgebra.

**Theorem 5.4.** If $A$ is strongly spectral invariant in $B$, then $\mathcal{K}^\infty \hat{\otimes}_\pi \mathcal{A}$ is strongly spectral invariant in $l_1(\mathbb{Z}^2) \hat{\otimes}_\pi B$.

**Proof.** Let $\varphi_1, \ldots, \varphi_n \in \mathcal{K}^\infty \hat{\otimes}_\pi \mathcal{A}$. We estimate

\begin{equation}
\| \varphi_1 \ast \cdots \ast \varphi_n \|_q \leq \sum_{r_1, \ldots, r_{n+1} \in \mathbb{Z}} (1 + |r_1| + |r_{n+1}|)^q \| \varphi_1(r_1, r_2) \cdots \varphi_n(r_n, r_{n+1}) \|_q
\end{equation}

\begin{equation}
\leq D_q C^n \sum_{k_1 + \cdots + k_n \leq p_q} \left\{ \sum_{r_1, \ldots, r_{n+1} \in \mathbb{Z}} (1 + |r_1| + |r_{n+1}|)^q \| \varphi_1(r_1, r_2) \|_{k_1} \cdots \| \varphi_n(r_n, r_{n+1}) \|_{k_n} \right\}.
\end{equation}

Because of the inequality (see argument after [22], (3.2.5))

\[(1 + |r_1| + |r_{n+1}|)^q \leq 2^q \{(1 + |r_1|)^q + (1 + |r_{n+1}|)^q\},\]

the right hand side of (5.5) is bounded by

\[D_q 2^q C^n \sum_{k_1 + \cdots + k_n \leq p_q} \left\{ \| \varphi_1 \|_{k_1+q} \cdots \| \varphi_n \|_{k_n} + \| \varphi_1 \|_{k_1} \cdots \| \varphi_n \|_{k_n+q} \right\}\]

\[\leq D_q 2^q C^n \sum_{k_1 + \cdots + k_n \leq p_q+q} \left\{ \| \varphi_1 \|_{k_1} \cdots \| \varphi_n \|_{k_n} \right\}.
\]

This gives the strong spectral invariance of $\mathcal{K}^\infty \hat{\otimes}_\pi \mathcal{A}$ in $l_1(\mathbb{Z}^2) \hat{\otimes}_\pi B$. □
Corollary 5.6. If $A$ is strongly spectral invariant in $B$, then $\mathcal{K}^\infty \hat{\otimes}_\pi A$ is spectral invariant in $\mathcal{K}\hat{\otimes} B$.

Proof. By Theorem 5.4, it suffices to show that $C = l_1(\mathbb{Z}^2) \hat{\otimes}_\pi B$ is spectral invariant in $D = \mathcal{K}\hat{\otimes} B$. First we note that $CDC \subseteq C$. For let $\psi_1, \psi_2 \in C$ and $\varphi \in D$. Then, in the above representation (5.3) on $l_2(\mathbb{Z}, \mathcal{L})$, $\varphi$ may be thought of as a $\mathbb{Z} \times \mathbb{Z}$ matrix with entries in $B$, where each entry has norm bounded by $c = \| \varphi \|_{\mathcal{K}\hat{\otimes} B}$. We have

$$\| \psi_1 \ast \varphi \ast \psi_2 \|_0 \leq \sum_{r,s,t,w} \| \psi_1(r,s) \|_B \| \varphi(s,t) \|_B \| \psi_2(t,w) \|_B \leq c \| \psi_1 \|_0 \| \psi_2 \|_0,$$

where $\| \|_0$ is the norm on $C$ (see (5.1) with $q = 0$). So $\psi_1 \ast \varphi \ast \psi_2 \in C$ and $CDC \subseteq C$. We see that $C$ is spectral invariant in $D$ by the following lemma.

Lemma 5.7 (Compare Bost [4], Proposition A.2.8). Let $A$ be any subalgebra of an algebra $B$. If $ABA \subseteq A$, then $A$ is spectral invariant in $B$.

Proof. If $A$ is unital, then $A = B$ and we are done. So assume that $A$ is nonunital. Let $a + \lambda 1 \in \hat{A}$ for $\lambda \neq 0$, and assume that $(a + \lambda 1)^{-1} \in \hat{B}$. Then there is a $b \in B$ such that $b + 1/\lambda = (a + \lambda 1)^{-1} \in \hat{B}$. Since $(a + \lambda 1)(b + 1/\lambda) = (b + 1/\lambda)(a + \lambda 1) = 1$, we have $b = -a/\lambda^2 - ab/\lambda = -a/\lambda^2 - ba/\lambda$. Substituting the first into the second gives $b \in ABA \subseteq A$. So $a + \lambda 1$ is invertible in $\hat{A}$ and $A$ is spectral invariant in $B$. □

This proves Corollary 5.6. □

Lemma 5.7 can also be used to show directly that $C = \mathcal{K}^\infty \hat{\otimes}_\pi B$ is spectral invariant in $D = \mathcal{K}\hat{\otimes} B$, since $CDC \subseteq C$.

Remark 5.8. The proof of the spectral invariance part of Corollary 4.2 above is easily generalized from crossed products of finite groups to crossed products by compact Lie groups $G$, if we assume that $A$ is strongly spectral invariant in $B$, and use Corollary 5.6, with $G$ in place of $\mathbb{Z}$. We omit the details, since we will be obtaining this same result using other methods in §7 (see Corollary 7.16).

§6 Crossed Products by Type R Lie Groups
and Spectral Invariance in $L^1(G, B)$

We show that strong spectral invariance is preserved by taking crossed products, if we use $L_1(G, B)$ as the Banach algebra crossed product, and use a subadditive scale on $G$ to define
the smooth crossed product $G \rtimes \tau A$. Throughout this section, $A$ will be a dense Fréchet subalgebra of a Banach algebra $B$.

**Definition 6.1.** We recall some definitions from [22], §2. Let $\tau$ be a scale on a locally compact group $G$ (see §2 above). Let $E$ be any Fréchet space. We define the $\tau$-rapidly vanishing $L_1$ functions $L_1^\tau(G, E)$ from $G$ to $E$ to be the set of measurable functions $\varphi$ from $G$ to $E$ such that

$$\| \varphi \|_{d,m} = \int_G \| \tau^d \varphi(g) \|_m \, dg < \infty,$$

where $\| \|_m$ ranges over a family of seminorms for $E$, and $d$ ranges over the natural numbers. We shall always use $dg$ to designate a left Haar measure on $G$. We topologize $L_1^\tau(G, E)$ by the seminorms (6.2).

To make our definitions sufficiently general, we let $H$ be any Lie group containing $G$ as a subgroup with differentiable inclusion map. We say that $\tau$ bounds $Ad$ on $H$ if there exists a polynomial $\text{poly}$ such that

$$\| \text{Ad}_g \| \leq \text{poly}(\tau(g)), \quad g \in G,$$

From the estimates of [22], Theorem 2.2.6, we see that if $A$ is a Fréchet algebra with $\tau$-tempered action of $G$, and $\tau$ is sub-polynomial, then $L_1^\tau(G, A)$ is a Fréchet algebra under convolution.

Next, we assume in addition that $G$ is a Lie group, possibly disconnected. We define the differentiable $\tau$-rapidly vanishing functions $S_1^\tau(G, E)$ from $G$ to $E$ to be the set of differentiable functions $\varphi$ from $G$ to $E$ such that

$$\| \varphi \|_{d,\gamma,m} = \int_G \| \tau^d X^\gamma \varphi(g) \|_m \, dg < \infty,$$

where $X^\gamma$ is any differential operator from the Lie algebra of $G$ acting by left translation, and $d$ and $\| \|_m$ are as in (6.2). We topologize $S_1^\tau(G, E)$ by the seminorms (6.4).

We say that $\tau$ is sub-polynomial if there exists a polynomial $\text{poly}$ such that

$$\tau(gh) \leq \text{poly}(\tau(g), \tau(h)), \quad g, h \in G.$$
where \(\| Ad_g \|\) is the operator norm of \(Ad_g\) as an operator on the Lie algebra of \(H\). The inverse scale \(\tau_-\) is defined by \(\tau_-(g) = \tau(g^{-1})\). And finally, if \(\tau\) is a sub-polynomial scale on \(G\) such that \(\tau_-\) bounds \(Ad\) on \(H\), and \(\sigma\) is an \(H\)-translationally equivalent scale on a locally compact \(H\)-space \(M\) (see §2), and we have

\[
(6.6) \quad \sigma(gm) \leq \text{poly}(\tau(g), \sigma(m)), \quad g \in G, m \in M,
\]

for some polynomial \(\text{poly}\), then we say that \((M, \sigma, H)\) is a scaled \((G, \tau)\)-space. By [22], Theorem 2.2.6, Theorem 5.17 it follows that if \(\tau\) is a sub-polynomial scale on \(G\) such that the action of \(G\) on the Fréchet algebra \(A\) is \(\tau\)-tempered, and either \(\tau_-\) bounds \(Ad\) on \(G\) or \(G\) acts differentiably on \(A\), then \(S^\tau_1(G, A)\) is a Fréchet algebra under convolution, which we denote by \(G \rtimes \tau A\). Moreover, if \((M, \sigma, H)\) is a scaled \((G, \tau)\)-space, then the action of \(G\) on \(S^\sigma_H(M)\) is \(\tau\)-tempered. In particular, \(G \rtimes \tau S^\sigma_H(M)\) is a Fréchet algebra.

We warn the reader that if \(G\) acts differentiably on \(A\), and the action of \(G\) on \(A\) is \(\tau\)-tempered, it often is a prerequisite that \(\tau_-\) must bound \(Ad\) on \(G\). Hence we often gain nothing by having the either/or hypothesis in the previous paragraph. On the other hand, in the group algebra case, when \(A = \mathbb{C}\), there is no requirement on \(\tau\) except that \(\tau\) be sub-polynomial.

In light of these comments, we shall often require the Lie group \(G\) to have a gauge that bounds \(Ad\) (see Definition 2.5 for the definition of a gauge), in order to apply the second and third paragraphs of Theorem 6.7 below. We say that \(G\) is Type R if for every \(g \in G\), \(Ad_g\), as an operator on the Lie algebra of \(G\), has eigenvalues on the unit circle. Assume that \(G\) is compactly generated, and that the group \(G/Ker(Ad)\) has a cocompact solvable subgroup (this holds, for example, when \(G\) is solvable or discrete). Then \(G\) has a gauge that bounds \(Ad\) iff \(G\) is Type R [22], Theorem 1.4.3. If \(G\) is Type R, then in fact the word gauge (see Definition 2.5) bounds \(Ad\).

Examples of Type R Lie groups are given by discrete groups, and closed subgroups of connected polynomial growth Lie groups. See the introduction and [22], §1.4 for more examples.

**Theorem 6.7.** Let \(G\) be a locally compact group and let \(\tau\) be a subadditive scale on \(G\). (\(\tau\) is subadditive if \(\tau(gh) \leq \tau(g) + \tau(h)\).) Assume that the action of \(G\) on \(A\) is \(\tau\)-tempered. Then the Fréchet algebra \(L^\tau_1(G, A)\) is strongly spectral invariant in \(L^\tau_1(G, B)\) if \(A\) is strongly spectral invariant in \(B\).

If, in addition, \(G\) is a Lie group, and either \(\tau_-\) bounds \(Ad\) on \(G\) or \(G\) acts differentiably
on $A$, then $G \rtimes \tau A$ is a Fréchet algebra which is strongly spectral invariant in $L_1(G, B)$ if $A$ is strongly spectral invariant in $B$.

In particular, if $(M, \sigma, H)$ is a scaled $(G, \tau)$-space, then the smooth crossed product $G \rtimes \tau S^r_H(M)$ is strongly spectral invariant in $L_1(G, C_0(M))$. Also, $G \rtimes \tau B^\infty$ is strongly spectral invariant in $L_1(G, B)$, where $B^\infty$ denotes the set of $C^\infty$-vectors for the action of $G$ on $B$. (G must often be a Type R Lie group - see remarks preceding the theorem.)

**Corollary 6.8.** If $G$ is any Lie group and $\tau$ is any subadditive scale on $G$, then the group Schwartz algebra $S^r_1(G)$ is strongly spectral invariant in $L_1(G)$. In particular, this holds if $\tau$ is the word gauge on a compactly generated Lie group $G$.

**Proof.** By Theorem 2.2, it suffices to prove the first two paragraphs of the theorem. We first prove the inequality

\[(a_1 + \ldots + a_n)^r \leq 2^r n (a_1^r + \ldots + a_n^r)\]

where $a_1, \ldots, a_n \geq 0$ and $n \in \mathbb{N}$, and $r$ is some fixed natural number. Recall from the argument after [22], (3.2.5) that $(a + b)^r \leq 2^r (a^r + b^r)$. Hence, assuming (6.9) holds for $n - 1$,

\[(a_1 + \ldots + a_n)^r \leq 2^r ((a_1 + \ldots + a_{n-1})^r + a_n^r) \leq 2^r 2^{r(n-1)} (a_1^r + \ldots + a_n^r),\]

which proves (6.9) by induction on $n$.

For convenience, we replace $\tau$ with the equivalent subadditive scale $1 + \tau$, so that we have $\tau \geq 1$ and $\tau$ subadditive.

Assume that $A$ is strongly spectral invariant in $B$. We verify the strong spectral invariance of $L_1(G, A)$ in $L_1(G, B)$. Let $\{ \| \cdot \|_m \}$ be a family of increasing seminorms for $A$. We topologize $L^r_1(G, A)$ by the family of increasing seminorms

\[(6.10) \quad \| \psi \|_{m-\infty} = \| \psi \|_{m, m}.\]

(See (6.2).) Note that $\| \cdot \|_0$ is the norm on $L_1(G, B)$. We show that these seminorms satisfy (1.3).

Let $\psi_1, \ldots, \psi_n \in G \rtimes \tau A$. To prepare to estimate $\| \psi_1 \ast \ldots \ast \psi_n \|_{m, m}$, we write $\psi_1 \ast \ldots \ast \psi_n(g)$ as

\[(6.11) \quad \int \ldots \int \alpha_{\eta_1}(\psi_1(h_1)) \ldots \alpha_{\eta_{n-1}}(\psi_{n-1}(h_{n-1})) \alpha_{\eta_n}(\psi_n(h_n)) dh_1 \ldots dh_{n-1}\]
where \( h_1, \ldots, h_{n-1} \) are the variables of integration, \( h_n = h_{n-1}^{-1} \cdots h_1^{-1} g, \eta_1 = e, \eta_k = h_1 \cdots h_{k-1} \). We proceed to estimate. Using (6.11) and the left invariance of Haar measure, 

\[
\| \psi_1 \cdots \psi_n \|_{m,m} = \int_G \tau^m(g) \| \psi_1 \cdots \psi_n(g) \|_m \, dg
\]

(6.12)

\[
\leq \int \cdots \int \tau^m(g) \| \alpha_{\eta_1}(\psi_1(h_1)) \cdots \alpha_{\eta_n}(\psi_n(h_n)) \|_m \, dh_1 \cdots dh_{n-1} dh_n
\]

Since \( A \) is strongly spectral invariant in \( B \), we may bound the normed expression in the integrand of (6.12):

(6.13)

\[
\| \alpha_{\eta_1}(\psi_1(h_1)) \cdots \alpha_{\eta_n}(\psi_n(h_n)) \|_m \leq D_1 C_1^n \sum_{k_1 + \cdots + k_n \leq p} \left\{ \| \alpha_{\eta_1}(\psi_1(h_1)) \|_{k_1} \cdots \| \alpha_{\eta_n}(\psi_n(h_n)) \|_{k_n} \right\}
\]

for some constants \( D_1, C_1 > 0 \) and \( p \geq m \) all depending only on \( m \). If \( k_i \neq 0 \), then the temperedness of the action of \( G \) on \( A \) gives

(6.14)

\[
\| \alpha_{\eta_i}(\psi_i(h_i)) \|_{k_i} \leq D_2 \tau^d(\eta_i) \| \psi_i(h_i) \|_s
\]

where \( s, D_2 \) and \( d \) depend only on \( p \), since \( k_i \leq p \). If \( \{i_1, \ldots, i_p\} \) contains all the \( i_j \)'s for which \( k_{i_j} \neq 0 \) (note that there are at most \( p \) because \( k_1 + \cdots + k_n \leq p \)) in the bracketed expression of (6.13), then that expression is bounded by

(6.15)

\[
D_2^p \tau^d(\eta_{i_1}) \cdots \tau^d(\eta_{i_p}) \| \psi_1(h_1) \|_{k_1} \cdots \| \psi_{i_1}(h_{i_1}) \|_s \cdots \| \psi_{i_p}(h_{i_p}) \|_s \cdots \| \psi_n(h_n) \|_k
\]

where we used the temperedness condition (6.14) \( p \) times, and the fact that \( \alpha \) leaves the norm \( \| \cdot \|_0 \) on \( B \) invariant. By the subadditivity of \( \tau \), we have

(6.16)

\[
\tau^d(\eta_k) \leq (\tau(h_1) + \cdots \tau(h_{k-1}))^d \\
\leq (\tau(h_1) + \cdots \tau(h_n))^d.
\]

So (6.13) is bounded by

(6.17)

\[
D_1 C_1^n D_2^p (\tau(h_1) + \cdots \tau(h_n))^{dp} \sum_{k_1 + \cdots + k_n \leq p+ps} \left\{ \| \psi_1(h_1) \|_{k_1} \cdots \| \psi_n(h_n) \|_{k_n} \right\}.
\]

Plugging this bound on (6.13) back into the integrand of (6.12), and also using (6.16) with \( \eta_i \) replaced with \( g \) and \( d = m \), we see that the integrand of (6.12) is bounded by

(6.18)

\[
D_1 D_2^p C_1^m (\tau(h_1) + \cdots \tau(h_n))^{dp+m} \sum_{k_1 + \cdots + k_n \leq p+ps} \left\{ \| \psi_1(h_1) \|_{k_1} \cdots \| \psi_n(h_n) \|_{k_n} \right\}
\]
which by (6.9) is bounded by (here \(C_3 = 2^r\) from (6.9) with \(r = dp + m\))

\[
D_1D_2^p(C_1C_3)^n(\tau(h_1)^{dp+m} + \ldots + \tau(h_n)^{dp+m})
\times \sum_{k_1 + \ldots + k_n \leq p + ps} \left\{ \| \psi_1(h_1) \|_{k_1} \ldots \| \psi_n(h_n) \|_{k_n} \right\}
\]

(6.19)

\[
\leq D_4C_4^n \sum_{k_1 + \ldots + k_n \leq t} \left\{ \tau(h_1)^{k_1} \| \psi_1(h_1) \|_{k_1} \ldots \tau(h_n)^{k_n} \| \psi_n(h_n) \|_{k_n} \right\}
\]

where \(t = p + ps + dp + m\). Therefore (6.12) is bounded by

(6.20)

\[
\| \psi_1 \ast \ldots \ast \psi_n \|_{m,m} \leq D_4C_4^n \sum_{k_1 + \ldots + k_n \leq t} \int \ldots \int \tau(h_1)^{k_1} \| \psi_1(h_1) \|_{k_1} \ldots \tau(h_n)^{k_n} \| \psi_n(h_n) \|_{k_n} \, dh_1 \ldots dh_n
\]

\[
= D_4C_4^n \sum_{k_1 + \ldots + k_n \leq t} \| \psi_1 \|_{k_1,k_1} \ldots \| \psi_n \|_{k_n,k_n}
\]

\[
= D_4C_4^n \sum_{k_1 + \ldots + k_n \leq t} \| \psi_1 \|'_{k_1} \ldots \| \psi_n \|'_{k_n},
\]

which, since \(D_4\) and \(C_4\) do not depend on \(n\), gives the strong spectral invariance of \(L^\tau_1(G, A)\) in \(L_1(G, B)\).

The strong spectral invariance of \(G \rtimes^\tau A\) in \(L_1(G, B)\) follows from the estimate (6.20) and the following lemma.

**Lemma 6.21.** Let \(\tau\) be any sub-polynomial scale on \(G\). Topologize \(G \rtimes^\tau A\) by the increasing seminorms

(6.22)

\[
\| \psi \|''_m = \sum_{|\gamma| \leq m} \| \psi \|_{m,\gamma,m}.
\]

If \(G\) acts differentiably on \(A\), then \(G \rtimes^\tau A\) is a dense right ideal in \(L^\tau_1(G, A)\) and moreover for all \(m \in \mathbb{N}\) there exists \(D > 0\) and \(k, l \in \mathbb{N}\) such that

(6.23)

\[
\| \varphi \ast \psi \|''_m \leq D \| \varphi \|'_{k} \| \psi \|'_{l},
\]

for all \(\varphi, \psi \in G \rtimes^\tau A\). Similarly, if \(\tau_-\) bounds \(Ad\) on \(G\), then \(G \rtimes^\tau A\) is a dense left ideal in \(L^\tau_1(G, A)\), and we have the inequalities (6.23) holding, but with " and ' switched on the right hand side.
Proof. The second statement is just [22], (2.2.7). Assume that $G$ acts differentiably on $A$, and replace $\tau$ with an equivalent scale satisfying $\tau \geq 1$. We have

\[(6.24) \quad \| X^\gamma \varphi \ast \psi(g) \|_m = \| X^\gamma \int_G \varphi(gh)\alpha_{gh}(\psi(h^{-1}))dh \|_m \]

\[\leq K \sum_{\beta + \beta = \gamma} \int_G \| X^\beta \varphi(gh) \|_p \| (X^{\tilde{\beta}} \text{ on } g)\alpha_{gh}(\psi(h^{-1})) \|_q dh \quad \text{prod rule, A Fréch alg} \]

\[\leq \tilde{K} \sum_{\beta + \beta = \gamma} \int_G \| \tau^d X^\beta \varphi(gh) \|_p \| (\psi(h^{-1})) \|_r dh. \quad \text{action diff, then tempered} \]

To see the inequality (6.23), place a $\tau^m(g)$ in front of (6.24), use that $\tau$ is sub-polynomial, and integrate over $g \in G$. □

This proves Theorem 6.7. □

Remark 6.25. The strong spectral invariance of $G \rtimes^\tau B^\infty$ in $L_1(G, B)$ generalizes Bost [4], Theorem 2.3.3(a), which proves a similar theorem for the case of elementary Abelian groups. See also the last example in [22], §5.

For a specific example, let $G$ be $SL_2(\mathbb{Z})$. Then $G$ is discrete, finitely generated and Type R, but does not have polynomial growth. Let $\tau$ be the word gauge. The group $G$ has a natural action on the irrational rotation algebra $A_\theta$ [22], end of §5, and Theorem 6.7 tells us that $G \rtimes^\tau A_\theta = L^r_1(G, A_\theta)$ is strongly spectral invariant in $L_1(G, A_\theta)$.

Example 6.26. Next let $H$ be a compactly generated polynomial growth Type R Lie group (for example a closed subgroup of a connected nilpotent Lie group). Let $G$ and $K$ be closed subgroups of $H$, and let $\tau$ be the word gauge on $H$. Then $\tau$ restricts to a gauge on $G$ which bounds $Ad$ on $H$ [22], Corollary 1.5.12. Define a scale $\sigma$ on $H/K$ as in Example 2.6. Let $M = H/K$. Then $(M, \sigma, H)$ is a scaled $(G, \tau)$-space. We may form the nuclear Fréchet algebra $G \rtimes^\tau S^\sigma_H(M)$, which is strongly spectral invariant in $L_1(G, C_0(M))$ by Theorem 6.7.

Example 6.27. Alternatively, in the preceding example $(M, \sigma, G)$ is a scaled $(G, \tau)$-space, and the (possibly nonnuclear) Fréchet algebra $G \rtimes^\tau S^\sigma_G(M)$ is strongly spectral invariant in $L_1(G, C_0(M))$.

To give a familiar example, let $H = G = \mathbb{Z}, K = \{ e \}$. Then $\tau$ is the absolute value function on $\mathbb{Z}$ and $G \rtimes^\tau S^\tau_H(M)$ is $\mathbb{Z} \rtimes S(\mathbb{Z})$ with $\mathbb{Z}$ acting by translation, which is isomorphic to the smooth compact operators defined in §5.

For other examples, see [22], §5, or Example 7.20 below. We remark that Theorem 6.7
gives another proof that strong spectral invariance is preserved by taking crossed products with finite groups (see Corollary 4.2(1)).

**Remark 6.28.** In the definition of $G \rtimes^\tau A$ (6.4), we could have let the differential operator $X^\gamma$ act via right translation $g\varphi(h) = \varphi(hg)$ instead of left translation. Then $G \rtimes^\tau A$ would be a Fréchet algebra as long as $\tau$ is sub-polynomial, and the action of $G$ is $\tau$-tempered, with no requirement about $\tau_-$ bounding $Ad$ or the action of $G$ being differentiable. The proof of (the appropriate modified versions of [22], Theorem 2.1.5 and Theorem 2.2.6 and) Theorem 6.7 would still go through to give a strongly spectral invariant smooth crossed product $G \rtimes^\tau A$ in $L_1(G,B)$ if $A$ is strongly spectral invariant in $B$ (in fact, the only changes in Theorem 6.7 would be in the estimate (6.24) of Lemma 6.21). This would allow us to form strongly spectral invariant smooth crossed products in many cases when $G$ has no gauge which bounds $Ad$ (for example, when $G$ is not Type R).

One shortcoming of this approach is that the left “covariant differentiable representations” of $(G,A)$ would not necessarily be in one to one correspondence with left “differentiable representations” of the crossed product $G \rtimes^\tau A$, since the action of $G$ on $G \rtimes^\tau A$ on the left would not necessarily be differentiable (see [23], §5 and Theorem 5.3). Another shortcoming is that if $A$ is a *-algebra, in order for the crossed product $G \rtimes^\tau A$ to be a *-algebra, we must require $\tau_- \sim \tau$, $\tau_-$ bound $Ad$, and that the action of $G$ on $A$ be differentiable [22], §4 and Corollary 4.9. So, if we want *-algebras, we may as well stick with our original definition of $G \rtimes^\tau A$.

We remark that if $\tau_- (\tau)$ bounds $Ad$ on $G$, then left (right) differentiable operators can be turned into right (left) ones.

We investigate what happens when $G$ is not Type R. We already know by [22], §1.4 that $G$ has no gauge that bounds $Ad$. In fact, we have the following more decisive result.

**Theorem 6.29.** Let $G$ be any Lie group, and let $\tau$ be any sub-polynomial scale on $G$. Assume also that $\tau_- \sim \tau$ and that $\tau_-$ bounds $Ad$. Under these conditions, if the Lie group $G$ is not Type R, then the group algebra $S^\tau(G)$ is never spectral invariant in $L_1(G)$, or in either of the $C^*$-algebras $C^r_\tau(G)$ or $C^*(G)$, whatever the choice of $\tau$ satisfying the above conditions.

The theorem basically says that, via all the ways of showing that $S^\tau(G)$ is a Fréchet *-algebra I know of, in general we never get spectral invariance if $G$ is not Type R.
Proof. By [22], Theorem 5.17, and since \( \tau_\sim \sim \tau \), we know that \( \mathcal{S}(G) \) is a Fréchet *-algebra. We define a simple \( \mathcal{S}(G) \)-module which is not contained in an \( L_1(G) \)-module.

We define a \( G \)-module \( V \) by taking \( V \) to be the Lie algebra of \( G \), and letting \( gv \equiv \text{Ad}_g v \). Let \( X \in V \) have eigenvalue \( \lambda \) not on the unit circle for some \( g \in G \). If \( V \) has a non-trivial invariant subspace \( W \), then either \( X \in W \), or \( X \) has nonzero image in \( V/W \). Continuing, we eventually reach a simple, finite dimensional \( G \)-module \( \tilde{V} \) with an eigenvector \( \tilde{X} \) with eigenvalue \( \lambda \).

Since \( \tau_\sim \sim \tau \), we know \( \tau \) bounds \( \text{Ad} \). Thus we may integrate the original representation of \( G \) on \( V \) to a representation of \( A = \mathcal{S}(G) \). This representation of \( A \) goes through the argument of the previous paragraph to give an irreducible representation of \( A \) on \( \tilde{V} \).

By [21], Theorem 1.4, if \( A \) is spectral invariant in any of the algebras \( L_1(G) \), \( C^*_r(G) \), or \( C^*(G) \) - call them \( B \) - then \( \tilde{V} \) must have a continuous irreducible \( B \)-module structure extending the action of \( A \) on \( \tilde{V} \).

Clearly for \( C^*(G) \) this is impossible since this would imply that the original representation of \( G \) on \( \tilde{V} \) were unitary, contradicting that \( g \) has eigenvalue \( \lambda \). We show that in fact \( \tilde{V} \) cannot have an \( L_1(G) \)-module structure (which proves the theorem). For assume that it does. Then \( G \) acts as left multipliers on \( L_1(G) \) via \( k \varphi(h) = \varphi(k^{-1}h) \), and we have \( \| k \|_{\text{mult}} = 1 \) for all \( k \in G \). If \( w \) is a nonzero element of \( \tilde{V} \), any \( v \in \tilde{V} \) can be written \( \varphi w \) for some \( \varphi \in L_1(G) \). We then define an action of \( G \) on \( \tilde{V} \) by \( kv = (k\varphi)w \), which must agree with the original representation of \( G \) on \( V \). Since \( \| k \|_{\text{mult}} = 1 \), there is some constant \( C \) such that \( \| k \|_{B(\tilde{V})} \leq C \). But for large \( n \), \( g^n \) has arbitrarily large eigenvalues, and hence arbitrarily large norm in \( B(\tilde{V}) \). This is a contradiction, so there can be no \( L_1(G) \)-module structure on \( \tilde{V} \). □

§7 Crossed Products by Polynomial Growth Groups

We use the results of §6 to show that our dense subalgebras are spectral invariant in the \( C^* \)-crossed product \( G \ltimes B \), at least when \( G \) has polynomial growth.

Throughout this section, unless otherwise stated, \( B \) will be a \( C^* \)-algebra. We shall use \( \| \| \) to denote the norm on the reduced \( C^* \)-crossed product \( G \ltimes_r B \). By Paterson [16], Proposition 0.13, \( G \) is amenable if \( G \) has polynomial growth (see Definition 2.5). So in this case, \( G \ltimes B = G \ltimes_r B \). If \( G \) is compactly generated, let \( \tau \) be the word gauge on \( G \) (see Definition 2.5). We replace \( \tau \) with \( 1 + \tau \), so \( \tau \geq 1 \) and \( \tau \) is subadditive and submultiplicative (namely \( \tau(gh) \leq \tau(g) + \tau(h) \) and \( \tau(gh) \leq \tau(g)\tau(h) \)). Let \( L_1(G, \tau^q) \) and \( L_1(G, B, \tau^q) \) be the Banach *-algebras of \( L_1 \) functions corresponding to the measure \( \tau^q dg \) on \( G \). Note that by
Theorem 6.7 (or the estimates in its proof), we know that $L_1(G, B, \tau^q)$ is strongly spectral invariant in $L_1(G, B)$, so we have equality of the two spectral radii

$$(7.1) \quad \lim_{n \to \infty} \| \varphi^\tau \|^1_{1/n} = \lim_{n \to \infty} \| \varphi^n \|^1_{1/n} = \nu(\varphi), \quad \varphi \in L_1(G, B, \tau^q).$$

Here we let $\nu(\varphi)$ denote the spectral radius of $\varphi$ in $L_1(G, B)$. We shall imitate the argument of Pytlik [19], generalizing the proof from the case $L_1(G, \tau^q)$ to the case $L_1(G, B, \tau^q)$, and then use this to show that $L_1(G, B, \tau^q)$ is spectral invariant in the C*-crossed product $G \rtimes B$.

We shall be using the star operation $f^* (g) = \Delta(g) \alpha(g)(f(g^{-1}))$ for functions $f : G \to B$.

Lemma 7.2 (Compare Pytlik [19], Lemma 4). Let $G$ be any locally compact group. Let $\varphi = \varphi^* \in L_1(G, B)$, and let $D$ be any dense subset of $L_1(G, B)$. Then there exists functions $f_1^*, f_2^* \in D$ such that

$$(7.3) \quad \nu(\varphi) \leq \limsup_{n \to \infty} \| f_1^* \varphi^n f_2^* \|^1_{1/n}.$$

Proof. For $\varphi \equiv 0$, this is clear. Let $\varphi \not \equiv 0$. Then $\nu(\varphi) \not = 0$, since, for example, $L_1(G, B)$ has a faithful *-representation on a Hilbert space.

Let $a_n = \| \varphi^{n+2} \|^1_1 \| \varphi^n \|^{-1}_1$. Then $\limsup_{n \to \infty} a_n = \lim_{n \to \infty} \| \varphi^n \|^2_{1/n} = \nu(\varphi)^2 \not = 0$ by manipulations of limits of positive real numbers.

Let $0 < a < \limsup_{n \to \infty} a_n$. Let $\epsilon < \max(1, a(6 \| \varphi \|^1_1))$, and choose $f_1 \in D^*$, $f_2 \in D$ such that $\| f_1 - \varphi \|_1 < \epsilon$. Then $\| f_i \|_1 \leq 2 \| \varphi \|_1$ and we have

$$\| \varphi^{n+2} \|^1_1 \leq \epsilon \| \varphi^{n+1} \|^1_1 + \| f_1^* \varphi^{n+1} \|^1_1$$

$$\leq \epsilon \| \varphi^{n+1} \|^1_1 + \epsilon \| f_1^* \varphi^n \|^1_1 + \| f_1^* \varphi^n f_2 \|^1_1$$

$$\leq a/2 \| \varphi^n \|^1_1 + \| f_1^* \varphi^n f_2 \|^1_1.$$

Hence

$$\| f_1^* \varphi^n f_2 \|^1_1 \geq \| \varphi^{n+2} \|^1_1 - a/2 \| \varphi^n \|^1_1 - \| \varphi^n \|^1_1 (a_n - a/2).$$

Since $a_n - a/2 \geq a/2$ for infinitely many $n$,

$$\limsup_{n \to \infty} \| f_1^* \varphi^n f_2 \|^1_{1/n} \geq \lim_{n \to \infty} \| \varphi^n \|^1_{1/n} = \nu(\varphi).$$

□
Lemma 7.4 (Compare Bost [4], (7.3.10)). Let $G$ be any unimodular locally compact group. Let $D$ be the vector space of measurable, compactly supported, step functions from $G$ to $B$. Then there exists a norm $\| \cdot \|_D$ on $D$ such that

$$\| f_1 \ast \psi \ast f_2 \|_\infty \leq \| f_1^* \|_D \| \psi \| \| f_2 \|_D, \quad f_1^*, f_2 \in D, \quad \psi \in L_1(G, B). \quad (7.5)$$

Proof. We first consider the case $f_1^* = \xi_i \otimes b^*_i$, $f_2 = \xi_j \otimes b_j$, where $\xi_i$ is a characteristic function of a relatively compact measurable subset of $G$, and $b_i \in B$. For $g \in G$, we bound $\| (f_1 \ast \psi \ast f_2)(g) \|_B$. Let $B$ be faithfully *-represented on a Hilbert space $\mathcal{H}$, and let $\eta_1, \eta_2 \in \mathcal{H}$. By changes of variables, we have

$$< \eta_1, (f_1 \ast \psi \ast f_2)(g) \eta_2 >_\mathcal{H} = < \eta_1, (\xi_i \ast (b_1 \psi b_2) \ast \xi_j)(g) \eta_2 >_\mathcal{H} = < \xi_i \otimes \eta_1, (b_1 \psi b_2)((\xi_j=g) \otimes \eta_2) >_{L_2(G, \mathcal{H})},$$

where $(\xi_j=g) \in L_2(G)$ is the function $(\xi_j=g)(h) = \xi_j(hg)$, and $(b_1 \psi b_2) \in L_1(G, \mathcal{H})$ acts on $((\xi_j=g) \otimes \eta_2) \in L_2(G, \mathcal{H})$ via the regular representation induced from the representation of $B$ on $\mathcal{H}$. Pedersen [17], §7.7. It follows that

$$| < \eta_1, (f_1 \ast \psi \ast f_2)(g) \eta_2 >_\mathcal{H} | \leq \| \xi_i \otimes \eta_1 \|_{L_2(G, \mathcal{H})} \| b_1 \psi b_2 \| \| (\xi_j=g) \otimes \eta_2 \|_{L_2(G, \mathcal{H})} \quad (7.6)$$

$$= \left( \| \xi_i \|_{L_2(G)} \| b_1 \psi b_2 \| \| \xi_j \|_{L_2(G)} \right) \| \eta_1 \|_{\mathcal{H}} \| \eta_2 \|_{\mathcal{H}} \leq \left( \| \xi_i \|_{L_2(G)} \| b_1 \|_B \| \psi \| \| \xi_j \|_{L_2(G)} \| b_2 \|_B \right) \| \eta_1 \|_{\mathcal{H}} \| \eta_2 \|_{\mathcal{H}}.$$

Let $\| \cdot \|_D$ be the norm on $D$ inherited from the projective tensor product $L_2(G) \hat{\otimes}_\pi B$. Then by (7.6) and the definition of the projective topology, we have

$$| < \eta_1, (f_1 \ast \psi \ast f_2)(g) \eta_2 >_\mathcal{H} | \leq \left( \| f_1^* \|_D \| \psi \| \| f_2 \|_D \right) \| \eta_1 \|_{\mathcal{H}} \| \eta_2 \|_{\mathcal{H}}, \quad (7.7)$$

for $f_1^*$ and $f_2$ in $L_2(G) \hat{\otimes}_\pi B$. Taking the sup over $\| \eta_i \|_{\mathcal{H}} \leq 1$ we have

$$\| (f_1 \ast \psi \ast f_2)(g) \|_B \leq \| f_1^* \|_D \| \psi \| \| f_2 \|_D. \quad (7.8)$$

Taking the sup over $g \in G$, we get (7.5). \qed

Lemma 7.9 (Compare Pytlik [19], Lemma 5). Let $G$ be a compactly generated polynomial growth group. Let $D$ be as in Lemma 7.4, and let $f_1^* \in D$, $f_2 \in D$, and $\psi \in L_1(G, B, \tau^q)$. Then there exists constants $M, N > 0$ (not depending on $\psi$) such that for $m \in \mathbb{N}^+$

$$\| f_1 \ast \psi \ast f_2 \|_1 \leq \| \psi \| Mm^r + \| \psi \tau^q \|_1 Nm^{-q}, \quad (7.10)$$
where $r > 0$ is the growth constant of the group.

**Proof.** Let $U$ be a generating set for $G$. We have $\| f_1 \ast \psi \ast f_2 \|_1 = \| (f_1 \ast \psi \ast f_2)\chi_{U^m} \|_1 + \| (f_1 \ast \psi \ast f_2)\chi_{G - U^m} \|_1$, where $\chi_{U^m}$ and $\chi_{G - U^m}$ are characteristic functions of the sets $U^m$ and $G - U^m$ respectively. But by Lemma 7.4 and since $G$ is unimodular [16], Proposition 6.9,6.6,

$$\| (f_1 \ast \psi \ast f_2)\chi_{U^m} \|_1 \leq \| f_1 \ast \psi \ast f_2 \|_\infty \| \chi_{U^m} \|_1 \leq \| f_1 \|_D \| f_2 \|_D \| \psi \| m^r = M \| \psi \| m^r.$$

Also

$$\| (f_1 \ast \psi \ast f_2)\chi_{G - U^m} \|_1 \leq \| (f_1 \ast \psi \ast f_2)\tau^q \|_1 \| \chi_{G - U^m} \|_\infty \leq \| \psi \tau^q \|_1 \| f_1 \|_1 \| f_2 \tau^q \|_1 (1 + m)^{q} \leq \| \psi \tau^q \|_1 Nm^{-q}.$$

□

**Theorem 7.11 (Compare Pytlik [19], Theorem 6).** Let $G$ be a compactly generated polynomial growth group. For every $\varphi = \varphi^* \in L_1(G, B, \tau^q)$ we have $\nu(\varphi) = \| \varphi \|.$

**Proof.** It suffices to prove $\nu(\varphi) \leq \| \varphi \|$. Let $a \geq 1$ be arbitrary, and let $m_n$ be a sequence of integers such that $\lim_{n \to \infty} m_n^{1/n} = a$. Putting $\varphi^n$ for $\psi$ and $m_n$ instead of $m$ in (7.10) we get

$$(7.12) \quad \| f_1 \ast \varphi^n \ast f_2 \|_1^{1/n} \leq \left( \| \varphi \|^n Mm_n^r + \| \varphi^n \tau^q \|_1 Nm_n^{-q} \right)^{1/n}.$$

If $n$ tends to infinity, the right side of (7.12) tends to a limit, which by (7.1) is equal to

$$\max \{ \| \varphi \| a^r, \nu(\varphi)a^{-q} \}.$$

Therefore

$$\limsup_{n \to \infty} \| f_1 \ast \varphi^n \ast f_2 \|_1^{1/n} \leq \max \left\{ \| \varphi \| a^r, \nu(\varphi)a^{-q} \right\},$$

which for $a = (\| \varphi \|^{-1} \nu(\varphi))^{\frac{1}{r+q}} \geq 1$ and for $f_1$ and $f_2$ as in Lemma 7.4 yields

$$\nu(\varphi) \leq \| \varphi \|^{\frac{q}{r+q}} \nu(\varphi)^{\frac{r}{r+q}}$$

and so

$$\nu(\varphi) \leq \| \varphi \|.$$

This proves Theorem 7.11 □

The following theorem is essentially Hulanicki [8], Proposition 2.5.
Theorem 7.13. Let $\mathcal{A}$ be a Banach $^*$-algebra. Assume that $\mathcal{A}$ is faithfully $^*$-represented in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, such that the $C^*$-norm $\| a \|_{\mathcal{B}(\mathcal{H})}$ is equal to the spectral radius of $a$ in $\mathcal{A}$, for all $a = a^*$ in $\mathcal{A}$. Then $\text{spec}_\mathcal{A}(a) = \text{spec}_\mathcal{B}(\mathcal{H})(a)$ for all $a \in \mathcal{A}$.

Proof. Let $\mathcal{B}$ be the closure of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$. We wish to show that $\mathcal{A}$ is spectral invariant in $\mathcal{B}$. By Hulanicki [8], Proposition 2.5, we have that $\text{spec}_\mathcal{A}(a) = \text{spec}_\mathcal{B}(\mathcal{H})(a)$ for all $a = a^*$ in $\mathcal{A}$.

It follows that for every $a = a^*$ in $\tilde{\mathcal{A}}$, $a$ is invertible in $\tilde{\mathcal{A}}$ iff $a$ is invertible in $\tilde{\mathcal{B}}$. Assume for a contradiction that $\mathcal{A}$ is not spectral invariant in $\mathcal{B}$. Then by [21], Theorem 1.4, there is a maximal left ideal $I$ in $\tilde{\mathcal{A}}$ which is dense in $\tilde{\mathcal{B}}$. Hence $I$ contains an invertible element $a$ of $\tilde{\mathcal{B}}$ which is not invertible in $\tilde{\mathcal{A}}$. But then $a^*a$ is a self-adjoint element of $\tilde{\mathcal{A}}$ which is in $I$ and hence not invertible in $\tilde{\mathcal{A}}$, but is invertible in $\tilde{\mathcal{B}}$. This is a contradiction and completes the proof. □

Corollary 7.14. Let $G$ be a compactly generated polynomial growth group. Then the Banach $^*$-algebra $L_1(G, B, \tau^q)$ is spectral invariant in the $C^*$-crossed product $G \rtimes B$ for any $q \in \mathbb{N}$. Hence $L_1^\tau(G, B)$ is spectral invariant in $G \rtimes B$.

Proof. By Theorem 7.11 and (7.1), the spectral radius of a self-adjoint element of $L_1(G, B, \tau^q)$ is equal to it’s norm in $G \rtimes B$. Hence by Theorem 7.13, $L_1(G, B, \tau^q)$ is spectral invariant in $G \rtimes B$. Since

$$L_1^\tau(G, B) = \cap_{q \in \mathbb{N}} L_1(G, B, \tau^q),$$

we have the spectral invariance of $L_1^\tau(G, B)$ in $G \rtimes B$. □

The following corollary generalizes Bost [4], Corollary 2.3.4, which gives the same result for elementary Abelian groups.

Corollary 7.15. Let $G$ be a compactly generated polynomial growth group. Then the inclusion map $L_1(G, B) \hookrightarrow G \rtimes B$ is an isomorphism of K-theory.

Proof. By Corollary 7.14, $L_1^\tau(G, B)$ is spectral invariant and dense in $G \rtimes B$. As we noticed at the beginning of §7, it is also spectral invariant and dense in $L_1(G, B)$. Hence by [5], VI.3 and [21], Lemma 1.2, Corollary 2.3, or [4], Appendix, all three algebras have the same K-theory. □

Corollary 7.16. Let $G$ be a compactly generated polynomial growth group, and let $\tau$ be the word gauge on $G$ (Definition 2.5). Assume that the action of $G$ on $A$ is $\tau$-tempered. Then the Fréchet algebra $L_1^\tau(G, A)$ is spectral invariant in the $C^*$-crossed product $G \rtimes B$ if $A$ is strongly spectral invariant in $B$. 29
If, in addition, \( G \) is a Lie group, and either \( \tau \) bounds \( \text{Ad} \) on \( G \) or \( G \) acts differentiably on \( A \), then \( G \rtimes \tau A \) is a Fréchet algebra which is spectral invariant in \( G \rtimes B \) if \( A \) is strongly spectral invariant in \( B \).

In particular, if \((M,\sigma,H)\) is a scaled \((G,\tau)\)-space, then the smooth crossed product \( G \rtimes \tau \mathcal{S}_{\mathcal{H}}(M) \) is spectral invariant in the \( C^*\)-crossed product \( G \rtimes C_0(M) \). Also, \( G \rtimes \tau B^\infty \) is spectral invariant in \( G \rtimes B \), where \( B^\infty \) denotes the set of \( C^\infty \)-vectors for the action of \( G \) on \( B \).

(For these results to apply, \( G \) may have to be a Type R Lie group - see remarks preceding Theorem 6.7, and Theorems 6.7 and 6.29. In general, a compactly generated polynomial growth Lie group need not be Type R [11], Example 1. However, the word gauge always bounds \( \text{Ad} \) if such a group is Type R [22], Corollary 1.5.12.)

Let \( G \) be a compactly generated polynomial growth Lie group. As noted in the corollary, there are examples when \( G \) is not Type R. By Theorem 6.29 above, we therefore also have examples of \( G \)-spaces \( M \) for which the smooth crossed product \( G \rtimes \tau \mathcal{S}_{\mathcal{H}}(M) \) is never spectral invariant in \( G \rtimes C_0(M) \), for any choice of \( \sigma \) and \( \tau \) which makes \( G \rtimes \tau \mathcal{S}_{\mathcal{H}}(M) \) a Fréchet \(*\)-algebra. However, if \( G \) is discrete, a closed subgroup of a connected polynomial growth Lie group, or if the connected component of the identity \( G_0 \) of \( G \) is simply connected, then \( G \) is Type R [22], Theorem 1.5.13, so for large classes of groups we do not have this problem and Corollary 7.16 applies (see also the examples mentioned in the introduction and the abstract).

**Proof.** By Theorem 6.7, \( G \rtimes \tau A \) and \( L^1(G,A) \) are strongly spectral invariant and hence spectral invariant in \( L_1(G,B) \). Also, \( L^1(G,B) \) is spectral invariant in \( L_1(G,B) \) by Theorem 6.7, so \( G \rtimes \tau A \) and \( L^1(G,A) \) are both spectral invariant in \( L^1(G,B) \). Since the latter algebra is spectral invariant in \( G \rtimes B \) by Corollary 7.14, this completes the proof. \( \square \)

**Corollary 7.17.** If \( G \) is any compactly generated polynomial growth Lie group and \( \tau \) is the word gauge on \( G \), then the group Schwartz algebra \( \mathcal{S}^*_1(G) \) is spectral invariant in \( C^*(G) \).

**Remark 7.18.** Corollary 7.17 generalizes the corresponding results Ludwig [12], Proposition 2.2 for the Schwartz algebra of a nilpotent Lie group, and Ji [9], Corollary 1.4 for the Schwartz algebra of a finitely generated polynomial growth discrete group.

**Remark 7.19.** The statement about the spectral invariance of \( G \rtimes \tau B^\infty \) in Corollary 7.16 generalizes the corresponding result Bost [4], Theorem 2.3.3(b) for elementary Abelian groups \( G \).

**Example 7.20.** Let \( H, G \) and \( K \) be as in Examples 6.26-7, with \( M = H/K \). Then \( G \rtimes \tau \)}
$\mathcal{S}_H^\tau(M)$ is spectral invariant in the $\mathrm{C}^*$-crossed product $G \rtimes C_0(M)$. Similarly, $G \rtimes^\tau \mathcal{S}_H^\tau(M)$ is spectral invariant in $G \rtimes C_0(M)$.

As a special case, the smooth crossed product $\mathbb{Z} \rtimes \mathcal{S}(\mathbb{Z})$ of Example 6.27 is spectral invariant in the compact operators $\mathbb{Z} \rtimes c_0(\mathbb{Z})$. This also follows from Corollary 5.6 above.

Other examples lie in [22], §5. For example, if $H$ is any closed subgroup of $G = GL(n, \mathbb{R})$ which consists of upper triangular matrices with $\pm 1$'s on the diagonal, then [22], Example 5.23 gives spectral invariant dense subalgebras $H \rtimes_\tau \mathcal{S}_G(M)$ of the $\mathrm{C}^*$-crossed product $H \rtimes C_0(M)$, where $M = \mathbb{R}^n$ and $H$ and $G$ act by matrix multiplication, or $M = M(n, \mathbb{R})$ and $H$ and $G$ act by conjugation.

**Remark 7.21.** If we defined the smooth crossed product $G \rtimes^\tau A$ with differential operators acting on the right instead of the left as in Remark 6.28, Corollary 7.16 would still give the spectral invariance without requiring a gauge that bounds $\mathrm{Ad}$ to form the crossed product. Hence we would have spectral invariant dense subalgebras of smooth functions for $\mathrm{C}^*$-crossed products by arbitrary compactly generated polynomial growth Lie groups, with no assumption about $G$ being Type R. However, see the shortcomings of such algebras mentioned in Remark 6.28.

**Remark 7.22.** We describe an alternate proof of Corollary 7.14 in the case that $G$ is a finitely generated discrete polynomial growth group. This proof, along with Pytlik [19], is what initially suggested to me that Corollary 7.14 might be true. We show that $L_1^\tau(G, B)$ is spectral invariant in $G \rtimes B$. Let $B$ be faithfully *-represented on a Hilbert space $\mathcal{H}$. We have the standard representation

$$\varphi \xi(g) = \sum_{h \in G} \alpha_{g^{-1}}(\varphi(h)) \xi(h^{-1}g), \quad \varphi \in G \rtimes B, \quad \xi \in L_2(G, \mathcal{H}).$$

Define a self-adjoint unbounded operator $D$ on $L_2(G, \mathcal{H})$ by $D \xi(g) = \tau(g) \xi(g)$. Define a derivation $\delta$ on $B(L_2(G, \mathcal{H}))$ by $\delta(T) = i [D, T]$. Then by Ji [9], Theorem 1.2, the set $(G \rtimes B)^\infty_1$ of $C^\infty$-vectors for the action of $\delta$ on $B(L_2(G, \mathcal{H}))$, which lie in $G \rtimes B$, is a spectral invariant subalgebra of $G \rtimes B$, as long as it is dense. It is straightforward to show that $L_1^\tau(G, B) \subseteq (G \rtimes B)^\infty$, so we have the density. We show that $(G \rtimes B)^\infty \subseteq L_1^\tau(G, B)$.

If $\varphi \in (G \rtimes B)^\infty$, then we have

$$\| (\delta^k \varphi) \|_{L_2(G, \mathcal{H})} \leq C_{k, \varphi},$$
for $\| \xi \|_{L^2(G, \mathcal{H})} \leq 1$. Define $\xi = \delta_e \otimes \eta$, where $\delta_e$ is the delta function at $e$, and $\eta \in \mathcal{H}$. Then a simple inductive argument shows that

$$(\delta^k \varphi) \xi(g) = i^k \tau^k(g) \alpha_{g^{-1}}(\varphi(g)) \eta.$$ 

This is similar to the formula for $(\delta^k \varphi) \xi$ in [9], §1. By definition of the $L^2$-norm on $L^2(G, \mathcal{H})$, we have

$$C^2_{k, \varphi} \geq \| (\delta^k \varphi) \xi \|_{L^2(G, \mathcal{H})}^2 = \sum_{g \in G} \tau^{2k}(g) \| \alpha_{g^{-1}}(\varphi(g)) \eta \|_{\mathcal{H}}^2 \geq \tau^{2k}(g) \| \alpha_{g^{-1}}(\varphi(g)) \eta \|_{\mathcal{H}}^2$$

for each $g \in G$. (The last inequality is what uses the discreteness of $G$.) Taking the sup over $\| \eta \|_{\mathcal{H}} \leq 1$, and using the fact that $\alpha$ is an isometry on $B$, we have

$$\tau^k(g) \| \varphi(g) \|_B \leq C_{k, \varphi}.$$ 

It follows that for $p \in \mathbb{N}$,

$$\tau^k(g) \| \varphi(g) \|_B \leq \frac{C_{k, \varphi} + C_{k+p, \varphi}}{1 + \tau^p(g)}.$$ 

Since the right hand side is summable over $g \in G$ for some $p \in \mathbb{N}$ [22], Proposition 1.5.1, we have $\varphi \in L^1_\tau(G, B)$. So $(G \rtimes B)_{\infty} \subseteq L^1_\tau(G, B)$, and the two sets are equal. By our remarks above, it follows that $L^1_\tau(G, B)$ is spectral invariant in $G \rtimes B$.

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