On the representativeness of approximate solutions of discrete optimization problems with interval cost function

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Abstract. We consider discrete optimization problems with interval uncertainty of cost function coefficients. The interval uncertainty models the measurements errors. A possible optimal solution is a solution that is optimal for some possible values of the coefficients. The probability of a possible solution is a probability of obtaining such coefficients that the solution is optimal. Similarly we define the notion of a possible approximate solution and its probability. We consider a possible solution unrepresentative if its probability less than some boundary value. The mean (optimal or approximate) solution is a solution that we obtain for mean values of interval coefficients. We show that the share of instances of a discrete optimization problem with unrepresentative mean approximate solution may be large enough for rather small values of errors.

Keywords: discrete optimization problems, interval uncertainty, approximate solutions.

Introduction

The ultimate goal of scientific investigations is prediction of experimental results. This concerns such mathematical discipline as discrete optimization too. For discrete optimization problems, input data are results of some measurements for the models that exploited by discrete optimization methods. The models include a cost function that it is needed to minimize or, in other formulation, to maximize by choosing some solution from a given set of feasible solutions. For real world applications, the cost of a feasible solution estimates losses that we shall have choosing this solution.

In the most general form, we may formulate a discrete optimization problem (we abbreviate it to DO-problem) as followed.

The DO-problem (I). Suppose we have a discrete set of feasible solutions $\mathcal{D}$ and a cost function $f : \mathcal{D} \rightarrow \mathbb{R}_+$, where $\mathbb{R}_+ = \{a \in \mathbb{R} \mid a > 0\}$. It is needed to find such $\hat{x} \in \mathcal{D}$ that

$$f(\hat{x}) = \min_{x \in \mathcal{D}} f(x)$$

(1)
or such $\hat{x} \in D$ that

$$f(\hat{x}) = \max_{x \in D} f(x).$$

(2)

We shall consider DO-problems with a linear cost function of the form

$$f(x) = \sum_{i=1}^{n} c_i x_i,$$

(3)

where $x = (x_1, \ldots, x_n) \in D \subset \{0,1\}^n$, $c = (c_1, \ldots, c_n) \in \mathbb{R}_+^n$, i.e. $c_i > 0$. A wide share of applied problems may be formulated as DO-problems. Inventory management tasks, placements tasks, scheduling problems and others are examples of such problems.

The questions we shall consider relate to the concept of stability of a solution of DO-problem. Since many of DO-problems may be formulated as integer linear programming problems, we may investigate stability of optimal and approximate solutions using the methods that were elaborated for DO-problems in this setting. However doing this we do not take into account the specificity of combinatorial structure of problems on graphs and hypergraphs that we shall consider. Further, we consider DO-problems of the form (II).

The DO-problem (II). Let $E = \{e_1, \ldots, e_n\}$. $c(e) > 0$ is a cost of an element $e \in E$, $c_i = c(e_i)$. A binary vector $x = (x_1, \ldots, x_n)$ defines the set $E_x \subset E$: $x_i = 1$, if $e_i \in E_x$, and $x_i = 0$, if $e_i \in E \setminus E_x$. The set $D$ of feasible solutions is given. We need to find $\hat{x} \in D$ such that

$$f(x, c) = \sum_{e \in E_x} c(e) = \sum_{i=1}^{n} c_i x_i.$$  

(4)

A lot of DO-problems on graphs may be formulated this way. Namely, for a graph $G = (V, E)$, the set $E$ may be considered as the set of its edges, while the set of feasible solutions $D$ may be considered as a set of some subgraphs of $G$, i.e. we associate binary vectors $x \in D$ with subgraphs $E_x$ of a given type. E.g., it may be the set of paths which connect some two graph vertices or the set of spanning trees, it may be the set of Hamiltonian cycles, matchings in graphs, cuts and so on. Not only problems on graphs (hypergraphs) may be formulated as problems of the form (II). For example, the knapsack problem may be formulated this way too.

Since almost always exact measurements of parameters of DO-problems are impossible, intervals of possible parameters values often represent the only reliable information on them. Considering approximate solutions for problems (II) with interval uncertainties of cost function coefficients, we assume the uniform distribution on the intervals as the most uninformative probability distribution.

We shall call a DO-problem with interval cost function as an IDO-problem. For an IDO-problem, a scenario is a vector $c = (c_1, \ldots, c_n)$ where every $c_i$ belongs
to the interval of its possible values. A possible optimal solution is a feasible solution which is optimal for some scenario.

Generally, considering NP-hard DO-problems, we cannot find optimal solution for reasonable time even for problems with hundreds variables. So we use approximate algorithms with guaranteed accuracy which operates in polynomial time. Greedy algorithms are examples of such algorithms.

An $\alpha$-approximate solution of a DO-problem (II) is a such $\hat{x} \in \mathcal{D}$ that

$$f(\hat{x}, c) \leq \alpha f(x^*, c),$$

where $x^* \in \mathcal{D}$ is an optimal solution for the scenario $c$. Suppose we have some algorithm to obtain $\alpha$-approximate solution for some fixed $\alpha > 0$ and for a fixed scenario. We call $\hat{x} \in \mathcal{D}$ a possible approximate solution of an IDO-problem if we obtain it as a result of using the algorithm for some scenario $c$.

Let us consider the scenario $c_\mu \in \mathbb{R}^n$ which components are equal to midpoints of given intervals. The mean approximate solution is a possible approximate solution that we obtain for the scenario $c_\mu$. The midpoint may be interpreted as the mean value of measurements of a parameter. It is often the case that, using mean values of measurements, for some practical problem, researchers obtain just one of possible solutions (the mean solution) and treat it as a valid solution for the case of uncertain parameters and for interval parameters particularly. Further we show that this approach is not justified quite frequently.

Probability of a possible approximate solution $\hat{x}$ is equal to probability of obtaining such a scenario that we obtain $\hat{x}$ as a result of the approximate algorithm operating. For the case when probability of a possible (optimal or approximate) solution less than some given boundary value, we call the solution as unrepresentative. The solution is representative otherwise. We call this boundary value as boundary of representativeness. We use probability of a possible solution as a measure of its representativeness.

Having an instance of IDO-problem, we must answer to the following questions about the situation that we have.

a). How many possible (optimal or approximate) solutions are there?

b). How much values of costs differ for these solutions?

There were used different ways to answer these questions ever since DO-problems with exact parameters has been studied. The notion of stability of a solution have been used usually to address it. An optimal solution is stable if it remains to be optimal while the values of costs are varied within some predefined intervals.

We present the concept of representativeness of a solution that is close to the concept of stability by its meaning. Admitting uncertainty of DO-problem instance prameters, we have a set of possible solutions for the instance. So we answer the questions of how many possible solutions we have for the instance and, having a possible solution of the instance, e.g. the mean solution, how likely is it that we shall obtain this solution for arbitrary values of uncertain parameters?

A possible approximate solution may be stable for a scenario $c$, i.e. there exists such $\delta > 0$ that there are no another possible approximate solutions in $\delta$-neighbourhood of $c = (c_1, \ldots, c_n)$ in $\mathbb{R}^n$, but it may be unrepresentative for some
boundary of representativeness and given intervals of coefficients. Further we consider such an example of a DO-problem with a stable but unrepresentative mean approximate solution.

We consider any possible (optimal or approximate) solution as an representative of the whole set of all possible (optimal or approximate) solutions. If we model repeating situations using some probability distribution on intervals of possible costs, then the solution is more representative if we obtain it more often using some exact or approximate algorithm. If we need to take a decision only at once, then one of the two solutions is more representative than another if probability of obtaining this solution is greater than probability of obtaining another solution.

Besides, using the notion of representativeness we attain some shortness of formulations. So, for example, if some value of boundary of representativeness is given, then we may briefly say "the solution is unrepresentative" instead of "probability of the solution less than given boundary value". Also, instead of using the words "the share of instances for which the probability of the mean solution less than some boundary value" we say "the share of problems with unrepresentative mean solutions."

We show that mean approximate solutions of DO-problems may be unrepresentative even for relatively small values of error rate of costs measurements and for rather small boundaries of representativeness. The set of all possible solutions (optimal or approximate), probabilities of the solutions, intervals of the solutions costs, probability distribution on the whole set of possible solutions costs — these are the factors that we may use to predict and minimize costs if we need to take a solution in the situation of interval uncertainty.

1 Discrete optimization problems with interval costs

Interval uncertainty representation. We use bold fonts to represent interval values:

\[ a = [\underline{a}, \overline{a}] = \{ a \in \mathbb{R} \mid \underline{a} \leq a \leq \overline{a} \}, \]

where \( \underline{a} \) is the lower bound of the interval and \( \overline{a} \) is its upper bound, \( \underline{a} \leq \overline{a} \). \( \mathbb{R} \) denotes the set of all such intervals on \( \mathbb{R} \). \( \mathbb{R}_+ = \{ a \in \mathbb{R} \mid a > 0 \} \).

For \( a, b \in \mathbb{R} \),

\[ a + b \overset{df}{=} [\underline{a} + \underline{b}, \overline{a} + \overline{b}] . \]

For \( a \in \mathbb{R}, \alpha \in \mathbb{R}_+ \),

\[ \alpha a \overset{df}{=} [\alpha \underline{a}, \alpha \overline{a}] . \]

An interval vector \( a \) has intervals \( a_i \) as its components:

\[ a = (a_1, \ldots, a_n) = ([\underline{a}_1, \overline{a}_1], \ldots, [\underline{a}_n, \overline{a}_n]) . \]

\( \mathbb{R}^n \) denotes the set of \( n \)-dimensional interval vectors with components from \( \mathbb{R} \). We consider scenarios from \( \mathbb{R}_+^n \).
For DO-problems with interval costs (we call it IDO-problems, i.e. interval DO-problems), we replace the cost function of the form (4) with the interval cost function of the form

\[ f(x, c) = \sum_{e \in E_x} c(e) = \sum_{i=1}^{n} c_i x_i, \]  

(5)

where \( c_i = c(e_i) \in \mathbb{IR}_+ \) are intervals of possible cost values, \( e_i \in E \). We define the cost function \( f \) using the operations of addition and multiplication that we have defined earlier.

\( f(x, c) \) is the interval of possible costs for solution \( x \in \mathcal{D} \). Since (4) contains just once only first powers of every variable , using the main theorem of interval arithmetic \([1]\), we have

\[ f(x, c) = \{ f(x, w) \mid w \in c \} = [f(x, c), f(x, \bar{c})]. \]

2 The interval greedy algorithm for the set cover problem with interval costs

Greedy algorithms utilize a rather common approach to obtain approximate solutions of DO-problems of the form (II). Performing iterations of a greedy algorithm, we form the solution \( x \) by choosing elements \( e \in E \) one by one and putting it into \( E_x \). We choose them considering their costs \( c(e) \) and other parameters of the instance trying to minimize the cost of the solution that we obtain performing the algorithm. The algorithm stops operating when we obtain some \( x \in \mathcal{D} \).

If the set \( \mathcal{D} \) is a matroid, then the greedy algorithm gives an optimal solution. It has been proven that greedy algorithms are asymptotically best polynomial algorithms for some DO-problems ([3,4] et al.).

We shall consider the set cover problem (SCP) as an example of a DO-problem (II). There are given a set \( \mathcal{U} = \{1, \ldots, m\} \) and such a collection \( S \) of its subsets \( S = \{S_1, \ldots, S_n\} \), \( S_i \subseteq \mathcal{U} \), that

\[ \bigcup_{i=1}^{n} S_i = \mathcal{U}. \]

We call a collection of sets \( S' = \{S_{i_1}, \ldots, S_{i_k}\}, S_{i_j} \in S \), a cover of \( \mathcal{U} \) if

\[ \bigcup_{j=1}^{k} S_{i_j} = \mathcal{U}. \]

There are given costs \( c_i = c(S_i), c_i > 0 \) for \( S_i \in S \), i.e. the vector \( c \in \mathbb{R}^+_n \) of costs is given. Cost \( c(S') \) of a collection of sets \( S' = \{S_{i_1}, \ldots, S_{i_k}\} \) is equal to the sum
VI

of costs of its elements:

\[ c(S^i) = \sum_{j=1}^{k_i} c(S_{ij}) \]

We need to find an optimal cover that is the cover with minimum cost.

Formulating SCP as a DO-problem of the form (II), we associate the set \( E \) with the collection \( S \) of subsets from \( U \). For the problem, a binary vector \( x \in D \) of dimensionality \( n \) defines a cover of \( U \): if \( x_i = 1 \), then \( S_i \) belongs to the cover, \( x_i = 0 \) otherwise.

SCP with real costs is NP-hard \[5\]. Let us denote an optimal cover by \( Opt \) and let \( Cvr \) denote the cover produced by some algorithm Alg. Let

\[ \rho(Alg) = \frac{c(Cvr)}{c(Opt)} \]

In \[6\], it have been shown that

\[ \rho(Alg) > (1 - o(1)) \ln m \]

for any polynomial time approximate algorithm Alg for SCP whenever \( P \neq NP \). The results concerned with computational complexity of approximation for unweighted case of the problem \([3,7]\) et al.), i.e. when \( c_i = 1 \), \( i = 1, n \), are hold true for weighted case of SCP too.

Computational complexity of the greedy algorithm for SCP is of order \( O(m^2 n) \). Let us denote the cover produced by the greedy algorithm GreedyAlg as \( Gr \). The following logarithmical estimation of \( \rho(Gr) \) holds for the algorithm \[8\]:

\[ \rho(GreedyAlg) = \frac{c(Gr)}{c(Opt)} \leq H(m) \leq \ln m + 1, \]

where \( H(m) = \sum_{k=1}^{m} \frac{1}{k} \).

In the interval set cover problem (ISCP), there are given interval costs \( c_i = c(S_i) \), i.e. an interval vector of possible scenarios \( c \in IR^n \) is given. The IDO-problem’s united solution set is the set \( \Sigma \) that contains its possible optimal solutions for all scenarios in \( c \):

\[ \Sigma = \{ x \in D \mid (\exists c \in c) (f(x, c) = \min_{y \in D} f(y, c)) \} \]

By now there is no algorithms to obtain \( \Sigma \) unless we not consider some type of exhaustive search on scenarios from \( c \).

The IDO-problem’s united approximate solution set \( \hat{\Sigma}_\alpha \) is a set that contains possible \( \alpha \)-approximate solutions for all scenarios in \( c \):

\[ \hat{\Sigma}_\alpha = \{ x \in D \mid (\exists c \in c) \left(f(x, c) \leq \alpha \min_{y \in D} f(y, c) \right) \} \]

Considering the greedy algorithm for SCP, we have \( \alpha = H(m) \). With that in mind, we denote the united set of approximate solutions as \( \hat{\Sigma} \) further.
For the case of non-interval costs, let us denote the cost of the solution (cover) $x$ as $c(x)$. For the case of interval costs, we define the interval of possible costs $c(x)$ of the solution $x \in D$, i.e. $c(x)$ contains costs of $x$ for all of the possible scenarios for which $x$ is an $\alpha$-approximate solution. Note that for non-interval costs, we have

$$c(x) = \sum_{x_i=1} c(S_i).$$

While for the case of interval costs, generally we have

$$c(x) \neq \sum_{x_i=1} c(S_i),$$

since the interval $c(x)$ often may be refined from the value $\sum_{x_i=1} c(S_i)$ and we may have

$$c(x) \subset \sum_{x_i=1} c(S_i).$$

We solve ISCP using the interval greedy algorithm [9,10,11]. The algorithm is a generalization of the greedy algorithm for the case of interval costs. It gives $\tilde{\Sigma}$, and, performing the algorithm’s iterations, we obtain exact values of intervals $c(\tilde{x})$ of possible costs for $\tilde{x} \in \tilde{\Sigma}$. If probability distributions on intervals of costs are given, using the interval greedy algorithm we obtain the probabilities $P(\tilde{x})$ for $\tilde{x} \in \tilde{\Sigma}$.

The approach presented in [9,10,11] may be applied to other IDO-problems that we may obtain for DO-problems of the form (II). Computational complexity of the interval greedy algorithm depends on cardinality of the set $\tilde{\Sigma}$. The complexity is exponential for the worst case. The interval greedy algorithm is polynomial for the case when cardinality of $\tilde{\Sigma}$ for an IDO-problem of dimensionality of $m$ is bounded by polynomial of $m$.

Let us consider the following problem. Suppose we have some set of economical indicators which we associate with the set $U = \{1, \ldots, m\}$. Given a collection of sets $S = \{S_1, \ldots, S_n\}$, $S_i \subseteq U$, that we associate with activities that may be performed to achieve some needed values of the indicators. There are given cost $c_i \in \mathbb{R}_+$ for realization of activity $S_i$, $i = 1, n$. We need to find such a set of activities that the predefined values of all of the indicators will be achieved by the minimum cost.

If some cover includes such $S_i, S_j \in S$ that $S_i \cap S_j \neq \emptyset$, we may interpret it as achieving of the needed values of the same indicators by performing different activities. For example, both roads repairs and change of the vehicle pool will increase the cargo turnover. For measurements with errors, costs of activities are intervals. For the instance, both the cost of renewal of vehicle fleet and the cost of roads repairs may be estimated with errors.

Let us consider the following instance of ISCP. $m = 7$, $n = 11$. $S_1 = \{3, 5\}$, $S_2 = \{4, 6\}$, $S_3 = \{1, 3\}$, $S_4 = \{2, 3, 4\}$, $S_5 = \{1, 5, 6\}$, $S_6 = \{4, 5, 6\}$, $S_7 = \{1, 4, 6, 7\}$, $S_8 = \{1, 3, 4, 6\}$, $S_9 = \{2, 4, 5, 7\}$, $S_{10} = \{1, 3, 6, 7\}$, $S_{11} = \{1, 2, 4, 6\}$. The vector

$$c_\mu = (119, 117, 124, 135, 128, 130, 143, 144, 144, 142, 141)$$
sets mean values of costs for the sets (activities) from $S = \{S_1, \ldots, S_9\}$. Let us suppose that the radii of intervals $c_i$ are equal to $\delta = 5$. So the relative error of measurements of costs is not greater than 5%.

Let $\tilde{x}^{(i)}$ denotes an element of $\tilde{\Sigma}$. Below we present the set $\tilde{\Sigma}$ for the considered ISCP instance. Also, we present intervals of possible costs and probabilities of possible approximate solutions.

1) $\tilde{x}^{(1)} = (1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0), c(\tilde{x}^{(1)}) = [382, 410], P(\tilde{x}^{(1)}) = 0.1542$;
2) $\tilde{x}^{(2)} = (1, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0), c(\tilde{x}^{(2)}) = [391, 410], P(\tilde{x}^{(2)}) = 0.0007$;
3) $\tilde{x}^{(3)} = (1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1), c(\tilde{x}^{(3)}) = [390, 410], P(\tilde{x}^{(3)}) = 0.1342$;
4) $\tilde{x}^{(4)} = (0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0), c(\tilde{x}^{(4)}) = [278, 295], P(\tilde{x}^{(4)}) = 0.1172$;
5) $\tilde{x}^{(5)} = (0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0), c(\tilde{x}^{(5)}) = [278, 293], P(\tilde{x}^{(5)}) = 0.3166$;
6) $\tilde{x}^{(6)} = (1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0), c(\tilde{x}^{(6)}) = [389, 417], P(\tilde{x}^{(6)}) = 0.0826$;
7) $\tilde{x}^{(7)} = (1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1), c(\tilde{x}^{(7)}) = [387, 417], P(\tilde{x}^{(7)}) = 0.1946$.

Using the greedy algorithm for scenario $c_\mu$, we obtain the mean approximate solution $\hat{x}_\mu = \tilde{x}^{(7)} = (1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0), c(\hat{x}_\mu) = f(\hat{x}_\mu, c_\mu) = 397$.

For the instance, the approximate solution $\hat{x}_\mu$ is stable since it is the only possible approximate solution in $\delta$-neighbourhood of $\mu_\mu$ in $\mathbb{R}^n$ for $\delta \leq 0.5$. But the number of possible approximate solutions grows as the radii of coefficients grow. Having $\delta > 5$, i.e. radius of every $c_i$ is equal to 5, $\hat{x}_{\mu}$ is unrepresentative for a boundary of representativeness that is greater than 0.2.

So the mean approximate solution $\hat{x}_{\mu} = \tilde{x}^{(7)}$ is just one of seven possible solutions from $\tilde{\Sigma}$. It has a probability of 0.1946 while the most probable solution $\tilde{x}^{(5)}$ has a probability of 0.3166. This example shows, that the mean approximate solution may be unrepresentative even for rather small boundaries of representativeness.

3 ISCP instances with unrepresentative mean approximate solutions

3.1 The sample of ISCP instances

For the experiment considered further, we generate the samples of ISCP instances using the algorithm that we present below. Implementing the algorithm, we generate sets $S_i \subseteq \mathcal{U}$ with random elements. We generate these sets until we obtain such a collection $S$ that every element of $\mathcal{U}$ is covered at least $k$ times by the sets from $S$. As a result, we obtain an SCP instance. For the instance, we model measurements errors using interval values of costs for the sets from $S$. Thus we obtain an ISCP instance.
For given \( m \) and \( \delta \), we obtain a sample of the pairs \( (\mathcal{P}^{(i)}(m, \delta), d^{(i)}(m, \delta)) \), where

1) \( \mathcal{P}^{(i)}(m, \delta) \) is a set that contains 1000 of ISCP instances which we generate using the algorithm below;
2) the vector \( d^{(i)}(m, \delta) \) characterize distribution of the instances according to probabilities of their mean approximate solutions:

\[
d^{(i)}(m, \delta) = (d_1^{(i)}(m, \delta), \ldots, d_{10}^{(i)}(m, \delta)),
\]

where \( d_k^{(i)}(m, \delta) \) is a share of problems from \( \mathcal{P}^{(i)}(m, \delta) \) (in percent) for which the probability of its mean approximate solution belongs to the interval \( [(k-1)/10, k/10], \ k = 1, 10 \).

Further we show vectors of mean values \( d(m, \delta) \) of components of the vectors \( d^{(i)}(m, \delta) \) for the samples

\[
\left\{ (\mathcal{P}^{(i)}(m, \delta), d^{(i)}(m, \delta)) \right\}_{i=1}^{100}
\]

that we obtain for \( m = 5, 10, 15, 20 \) and for \( \delta = 1, 5 \). The modeled relative error is not greater than 5% for \( \delta = 5 \).

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**Generating ISCP instances**

**Input:** \( m, k, \delta \).

1. \( i := 0 \).
2. Do the following iterations while all elements of \( \mathcal{U} \) are covered less than \( k \) times.
   2.1 \( i := i + 1 \).
   2.2 Generate cardinality \( p_i \) of the set \( S_i \):
      \( p_i \) is a random value uniformly distributed on \( \mathcal{U} = \{1, \ldots, m\} \).
   2.3 Assuming uniform distribution on the set \( \mathcal{U} \), randomly select \( p_i \) elements from \( \mathcal{U} \) to obtain \( S_i \).
   2.4 Generate cost \( c_i \):
      \[
      c_i := 100 + 10 \cdot p_i + \eta
      \]
      where \( \eta \) is an integer random value distributed uniformly on the interval \([-5, 5]\).
   2.5 Form the interval cost \( c_i \): \( c_i := [c_i - \delta, c_i + \delta] \).

**Output:** ISCP instance with obtained \( \mathcal{U}, S = \{S_1, \ldots, S_n\}, c \in \mathbb{R}_+^n \).

Cardinality of the set of all ISCP instances that we may obtain using the presented algorithm is substantially exceeds cardinality of the set of all SCP instances that we may generate in the curse of its implementation which grows
x

exponentially with respect to $m$. \[12\] Even though we cannot say that the sample of 100 elements $P^{(i)}(m, \delta)$ is large enough for some fixed $m$ and $\delta$, small dimensionalities of considered instances give us statistically stable results as it justified by standard deviation of $d^{(i)}(m, \delta)$.

### 3.2 The experiments

**The share of ISCP instances with unrepresentative mean approximate solution.** Components of the vectors $d(5, \delta), d(10, \delta), d(15, \delta), d(20, \delta), \delta = 1, 5$, are placed in the rows of tables 1, 3, 5, 7 respectively. You may see standard deviation of $d^{(i)}(m, \delta)$ in tables of the Appendix. It is not greater than 5.92% for all of the instances of $m$ and $\delta$, $m = 5, 20, \delta = 1, 5$.

In particular, the results we present show that, having the boundary of representativeness $b = 0.9$ and the relative error that is not greater than 5%, the mean approximate solution is unrepresentative for more than half of the generated ISCP instances. It also shows that the share of ISCP instances with unrepresentative mean approximate value grows as dimensionality of the instances grows. It holds for all values of $b = k/10, k = 1, 9$.

**Deviation of possible approximate solutions costs from cost of the mean approximate solution.** Let us turn back to the ISCP instance that we considered earlier. For the instance, there are two possible optimal solutions $\hat{x}^{(1)} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0)$ and $\hat{x}^{(2)} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0)$. Costs of the solutions belong to the interval $[276, 296]$ considering all of the possible scenarios for the instance. $\hat{x}^{(2)}$ is the mean optimal solution ($\hat{x}^{(2)} = \tilde{x}^{(4)}$). $P(\hat{x}^{(2)}) = 0.667$, i.e. $\hat{x}^{(2)}$ has maximum probability. Its cost equal to the mean value of all possible optimal solutions costs for all scenarios: $c(\hat{x}^{(2)}) = 286$. Possible deviation from the mean value is equal to 10.

Hystogram of frequencies (axis $p$) of costs of possible approximate solutions (axis $c$) of the ISCP instance is presented on Fig. 1. We build it based upon $10^6$ random scenarios. The hystogram approximate graph of costs distribution density. The mean value of costs for all scenarios and the cost of the mean approximate solution are marked on the axis $f$ as the square and the circle respectively. We may see that there is a significant deviation of the mean approximate solution cost from the mean value of costs for all scenarios. The maximum possible value of deviation of the value of possible approximate solution cost from the mean approximate solution cost is equal to 119 for its minimum value and to 20 for its maximum value. The set of all possible approximate solution costs is disjoint for the instance.

Note that, considering this ISCP instance, even though the mean optimal solution is sufficiently stable to measurements errors, the mean approximate solution may be unrepresentative for rather small values of the boundary of representativeness. For the instance, the mean approximate solution become unrepresentative for the boundary that is greater than 0.155. Its cost closer to the value of maximum possible cost than to the mean value of costs for all scenarios. Also,
its far away from the value of minimum possible cost.

| \( \delta \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0.05 | 0.18 | 0.53 | 1.59 | 5.04 | 0.10 | 1.49 | 10.13 | 80.89 |
| 2          | 0.06 | 0.35 | 1.20 | 3.64 | 4.04 | 3.65 | 7.70 | 7.39 | 71.96 |
| 3          | 0.13 | 0.65 | 1.98 | 5.27 | 4.57 | 7.70 | 7.13 | 5.32 | 67.26 |
| 4          | 0.20 | 1.11 | 3.31 | 6.20 | 7.39 | 7.21 | 6.17 | 5.62 | 62.77 |
| 5          | 0.35 | 1.71 | 4.23 | 7.58 | 9.29 | 9.00 | 5.09 | 4.09 | 60.65 |

| \( \delta \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0.05 | 0.23 | 0.48 | 1.72 | 6.08 | 0.13 | 1.82 | 11.87 | 77.61 |
| 2          | 0.07 | 0.42 | 1.25 | 3.55 | 5.09 | 3.73 | 9.48 | 8.84 | 67.57 |
| 3          | 0.15 | 0.81 | 2.31 | 5.47 | 5.67 | 9.90 | 8.22 | 6.92 | 59.94 |
| 4          | 0.23 | 1.32 | 3.84 | 7.09 | 8.57 | 9.09 | 8.46 | 8.87 | 52.53 |
| 5          | 0.40 | 2.10 | 4.95 | 9.17 | 11.36 | 9.21 | 7.84 | 8.12 | 46.81 |

| \( \delta \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0.03 | 0.22 | 0.39 | 1.39 | 5.85 | 0.71 | 1.70 | 11.54 | 78.16 |
| 2          | 0.05 | 0.35 | 1.08 | 3.07 | 5.46 | 4.06 | 9.28 | 9.45 | 67.19 |
| 3          | 0.10 | 0.60 | 1.88 | 4.83 | 6.10 | 10.31 | 9.42 | 8.47 | 58.29 |
| 4          | 0.18 | 1.08 | 3.32 | 6.91 | 8.72 | 9.91 | 9.64 | 10.54 | 49.67 |
| 5          | 0.35 | 1.84 | 4.53 | 8.73 | 12.26 | 10.18 | 9.11 | 10.96 | 41.99 |

| \( \delta \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0.00 | 0.10 | 0.29 | 1.01 | 5.35 | 1.42 | 1.89 | 9.92 | 80.00 |
| 2          | 0.04 | 0.19 | 0.79 | 2.43 | 5.69 | 4.12 | 9.37 | 9.46 | 67.91 |
| 3          | 0.06 | 0.46 | 1.67 | 4.64 | 7.04 | 10.71 | 10.02 | 10.55 | 54.84 |
| 4          | 0.11 | 0.78 | 2.74 | 6.46 | 9.48 | 10.87 | 11.10 | 12.94 | 45.52 |
| 5          | 0.20 | 1.33 | 3.98 | 8.62 | 12.92 | 11.89 | 11.46 | 12.97 | 36.63 |

4 Conclusions

The results we present show that the mean approximate solution may be unrepresentative for the set of all possible approximate solutions of IDO problem and there are may be other possible approximate solutions with higher probabilities. It follows that we must justify the representativeness of possible (optimal
or approximate) solution that we obtain solving DO-problem. If we cannot give such a justification then we must solve the problem considering uncertainties in its parameters (e.g. costs). For example, we may obtain and analyze the problem’s united solution set or the united approximate solution set and the values of cost function for them.

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Appendix

Vectors $d(m, \delta)$

Here we present the mean values for the samples

$$\{(P^{(i)}(m, \delta), d^{(i)}(m, \delta))\}_{i=1}^{100}$$

for all values of $m = 5, 20, \delta = 1, 5$.

For given $m$ and $\delta$, (6) is a sample of the pairs $(P^{(i)}(m, \delta), d^{(i)}(m, \delta))$, where

1) $P^{(i)}(m, \delta)$ is a set that contains 1000 of ISCP instances which we generate using the presented algorithm;
2) the vector $d^{(i)}(m, \delta)$ characterize distribution of the instances according to probabilities of their mean approximate solutions:

$$d^{(i)}(m, \delta) = (d^{(i)}_1(m, \delta), \ldots, d^{(i)}_{10}(m, \delta)),$$

where $d^{(i)}_k(m, \delta)$ is a share of problems from $P^{(i)}(m, \delta)$ (in percent) for which the probability of its mean approximate solution belongs to the interval $[(k - 1)/10, k/10]$, $k = 1, 10$.

Components of the vectors $d(m, \delta)$, $\delta = 1, 5$, are placed in the rows of tables with odd numbers. Its standard deviation values are presented in accompanied tables with even numbers.

| $\delta$/b | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1           | 0.05| 0.18| 0.39| 0.59| 0.84| 1.09| 1.49| 1.94| 2.48| 3.10|
| 2           | 0.06| 0.35| 1.20| 3.64| 4.04| 3.65| 7.70| 7.39| 7.19| 6.73|
| 3           | 0.15| 0.65| 1.98| 5.27| 4.57| 7.70| 7.39| 7.19| 6.73| 6.72|
| 4           | 0.20| 1.11| 3.31| 6.20| 7.39| 7.21| 6.17| 5.32| 6.72| 6.77|
| 5           | 0.35| 1.71| 4.23| 7.58| 9.29| 7.00| 5.09| 4.09| 6.05| 6.65|

Table 5. $d(5, \delta)$.

| $\delta$/b | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1           | 0.07| 0.12| 0.27| 0.72| 0.97| 1.15| 1.53| 1.99| 2.45| 3.12|
| 2           | 0.08| 0.20| 0.42| 0.88| 0.97| 1.07| 1.58| 1.66| 5.50|
| 3           | 0.12| 0.27| 0.72| 0.97| 1.15| 1.53| 1.99| 1.10| 5.55|
| 4           | 0.13| 0.39| 0.94| 1.27| 1.28| 1.44| 1.27| 1.00| 6.03|
| 5           | 0.18| 0.47| 1.13| 1.57| 1.51| 1.10| 0.95| 0.83| 5.92|

Table 6. Standard deviation of $d(5, \delta)$. 

Table 7. $d(6, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0.01| 0.06| 0.28| 0.64| 2.04| 5.83| 0.19| 1.83| 11.48| 77.67|
| 2          | 0   | 0.08| 0.50| 1.57| 4.26| 4.76| 4.30| 8.52| 8.09  | 67.92|
| 3          | 0.15| 0.95| 2.77| 6.23| 5.66| 8.83| 7.62| 5.83| 61.96 |       |
| 4          | 0.27| 1.62| 4.49| 7.53| 8.33| 7.71| 6.76| 5.70| 57.60 |       |
| 5          | 0.01| 0.48| 2.65| 5.96| 9.42| 10.30| 7.60| 5.35| 4.31  | 53.90 |

Table 8. Standard deviation of $d(6, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0.02| 0.08| 0.17| 0.24| 0.50| 1.08| 0.13| 0.48| 1.45  | 2.63 |
| 2          | 0.02| 0.09| 0.22| 0.44| 0.82| 1.76| 0.84| 1.35| 1.40  | 4.14 |
| 3          | 0.02| 0.14| 0.32| 0.62| 0.90| 1.05| 1.40| 1.12| 1.17  | 4.83 |
| 4          | 0.02| 0.18| 0.45| 0.80| 1.15| 1.12| 1.17| 1.05| 0.96  | 4.66 |
| 5          | 0.03| 0.29| 0.63| 0.98| 1.13| 1.31| 1.18| 0.93| 0.98  | 5.31 |

Table 9. $d(7, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0   | 0.04| 0.25| 0.57| 2.06| 6.02| 0.12| 1.85| 11.92| 77.16|
| 2          | 0   | 0.07| 0.50| 1.48| 2.43| 4.91| 4.15| 8.94| 8.76  | 66.96|
| 3          | 0.13| 0.96| 2.87| 6.30| 5.77| 9.66| 8.14| 6.08| 60.08 |       |
| 4          | 0.01| 0.25| 1.55| 4.47| 7.76| 8.74| 8.31| 6.89| 5.93  | 65.69|
| 5          | 0.01| 0.45| 2.63| 5.74| 9.54| 10.55| 8.09| 5.87| 5.01  | 52.07|

Table 10. Standard deviation of $d(7, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0.01| 0.07| 0.15| 0.26| 0.49| 0.86| 0.11| 0.43| 1.51  | 2.52 |
| 2          | 0.01| 0.09| 0.23| 0.38| 0.62| 0.92| 0.70| 1.22| 1.10  | 3.54 |
| 3          | 0.01| 0.12| 0.33| 0.61| 0.94| 0.89| 1.62| 1.32| 1.14  | 4.94 |
| 4          | 0.04| 0.17| 0.37| 0.73| 1.13| 1.18| 1.43| 1.13| 1.25  | 5.25 |
| 5          | 0.03| 0.26| 0.63| 1.06| 1.25| 1.49| 1.09| 1.22| 1.33  | 6.41 |
| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|
| 1         | 0   | 0.07| 0.27| 0.56| 2.00| 6.10| 1.76| 11.99| 77.08|   |
| 2         | 0   | 0.09| 0.48| 1.47| 3.94| 4.91| 4.21| 8.90 | 8.47 | 67.53|
| 3         | 0.01| 0.16| 0.90| 2.77| 6.03| 5.90| 9.74| 8.35 | 6.08 | 60.06|
| 4         | 0.02| 0.31| 1.56| 4.29| 7.26| 8.56| 8.21| 7.07 | 6.76 | 55.96|
| 5         | 0.04| 0.51| 2.40| 5.29| 8.85| 10.46|7.15| 5.71 | 5.39 | 53.63|

**Table 11.** $d(8, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|
| 1         | 0   | 0.01| 0.08| 0.16| 0.25| 0.45| 0.97| 0.12| 0.43| 1.46|
| 2         | 0   | 0.09| 0.21| 0.37| 0.63| 0.98| 0.72| 1.23| 1.10| 3.10|
| 3         | 0.03| 0.13| 0.30| 0.57| 0.75| 0.89| 1.33| 1.24| 1.17| 4.07|
| 4         | 0.04| 0.18| 0.34| 0.70| 0.85| 0.93| 1.11| 1.03| 1.28| 3.62|
| 5         | 0.07| 0.22| 0.48| 0.74| 1.02| 0.87| 0.82| 0.74| 0.84| 2.01|

**Table 12.** Standard deviation of $d(8, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|
| 1         | 0.05| 0.20| 0.56| 1.88| 5.74| 0.13| 1.81| 11.73| 77.90|   |
| 2         | 0   | 0.09| 0.41| 1.33| 3.91| 4.75| 3.81| 8.95 | 8.67 | 68.09|
| 3         | 0.12| 0.75| 2.57| 5.77| 5.60| 9.84| 8.59| 6.64 | 60.11|   |
| 4         | 0.01| 0.23| 1.34| 0.89| 7.10| 8.62| 8.84| 7.82 | 7.48 | 54.67|
| 5         | 0.03| 0.43| 2.26| 5.19| 9.27|11.24| 9.91| 8.83 | 6.77 | 49.05|

**Table 13.** $d(9, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|
| 1         | 0   | 0.01| 0.08| 0.14| 0.24| 0.41| 0.75| 0.11| 0.43| 0.91|
| 2         | 0.02| 0.10| 0.19| 0.38| 0.73| 0.76| 0.64| 0.98| 0.93| 1.42|
| 3         | 0.02| 0.12| 0.29| 0.60| 0.77| 0.78| 1.00| 1.09| 1.06| 2.46|
| 4         | 0.03| 0.16| 0.39| 0.62| 0.79| 0.88| 0.89| 1.10| 1.49| 2.35|
| 5         | 0.05| 0.21| 0.46| 0.73| 0.96| 1.39| 1.12| 1.28| 1.38| 4.13|

**Table 14.** Standard deviation of $d(9, \delta)$. 
### Table 15. $d(10, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0   | 0.05| 0.23| 0.48| 1.72| 6.08| 0.13| 1.82| 11.87| 77.61|
| 2          | 0   | 0.07| 0.42| 1.25| 3.55| 5.09| 3.73| 9.48| 8.84  | 67.57 |
| 3          | 0.01| 0.15| 0.81| 3.41| 5.67| 9.90| 5.67| 9.90| 8.46  | 8.87  |
| 4          | 0.01| 0.23| 1.32| 3.84| 7.09| 8.57| 9.09| 8.46| 8.87  | 52.53 |
| 5          | 0.02| 0.40| 2.10| 4.95| 9.17| 11.36|9.21| 7.84| 8.12  | 46.81 |

### Table 16. Standard deviation of $d(10, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0.01| 0.07| 0.15| 0.21| 0.45| 0.88| 0.12| 0.42| 1.11 | 1.46 |
| 2          | 0   | 0.08| 0.20| 0.36| 0.69| 0.79| 0.71| 1.00| 1.01 | 1.72 |
| 3          | 0.03| 0.15| 0.31| 0.54| 0.80| 0.77| 0.98| 0.82| 0.94 | 1.65 |
| 4          | 0.03| 0.17| 0.42| 0.69| 0.83| 0.91| 1.08| 1.34| 1.20 | 2.65 |
| 5          | 0.04| 0.20| 0.42| 0.79| 0.88| 1.18| 1.17| 1.07| 1.53 | 3.54 |

### Table 17. $d(11, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0   | 0.04| 0.18| 0.43| 1.53| 5.71| 0.16| 1.98| 11.55| 78.43|
| 2          | 0   | 0.06| 0.41| 1.24| 3.49| 5.09| 3.99| 9.16| 8.46  | 67.72 |
| 3          | 0   | 0.13| 0.81| 2.26| 3.48| 5.87| 9.73| 8.88| 7.46  | 59.38 |
| 4          | 0.01| 0.23| 1.19| 3.75| 6.88| 8.91| 9.29| 8.47| 9.56  | 51.71 |
| 5          | 0.02| 0.40| 2.13| 4.94| 9.09| 11.38|9.38| 8.00| 8.15  | 46.49 |

### Table 18. Standard deviation of $d(11, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0   | 0.07| 0.13| 0.23| 0.46| 0.83| 0.14| 0.47| 1.01 | 1.34 |
| 2          | 0.02| 0.08| 0.19| 0.36| 0.71| 0.65| 0.68| 0.87| 1.02 | 1.40 |
| 3          | 0.02| 0.13| 0.30| 0.51| 0.85| 0.80| 0.99| 0.94| 0.97 | 1.76 |
| 4          | 0.04| 0.14| 0.37| 0.71| 0.94| 0.88| 1.05| 1.19| 1.54 | 2.28 |
| 5          | 0.05| 0.19| 0.49| 0.74| 0.78| 1.27| 1.08| 1.01| 1.63 | 2.97 |
| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|
| 1        | 0   | 0.05 | 0.20 | 0.50 | 1.71 | 5.82 | 0.54 | 1.92 | 11.27 | 77.98 |
| 2        | 0   | 0.08 | 0.39 | 1.27 | 3.48 | 5.19 | 4.14 | 9.32 | 9.16 | 66.97 |
| 3        | 0   | 0.14 | 0.76 | 2.32 | 5.45 | 6.20 | 10.03 | 9.01 | 7.78 | 58.32 |
| 4        | 0.01 | 0.27 | 1.35 | 3.86 | 7.05 | 8.73 | 9.23 | 8.72 | 9.15 | 51.63 |
| 5        | 0.01 | 0.42 | 2.18 | 4.83 | 9.03 | 11.58 | 9.62 | 8.56 | 8.71 | 45.01 |

Table 19. $d(12, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|
| 1        | 0   | 0.08 | 0.14 | 0.25 | 0.48 | 0.86 | 0.26 | 0.49 | 0.95 | 1.10 |
| 2        | 0   | 0.10 | 0.30 | 0.39 | 0.71 | 0.79 | 0.93 | 1.55 |       |       |
| 3        | 0   | 0.12 | 0.32 | 0.59 | 0.95 | 0.77 | 0.97 | 0.92 | 0.93 | 1.50 |
| 4        | 0   | 0.17 | 0.39 | 0.66 | 0.84 | 0.94 | 1.02 | 1.01 | 1.37 | 2.30 |
| 5        | 0.04 | 0.25 | 0.56 | 0.91 | 0.94 | 1.19 | 1.05 | 1.10 | 1.60 | 2.76 |

Table 20. Standard deviation of $d(12, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|
| 1        | 0   | 0.04 | 0.22 | 0.41 | 1.63 | 6.02 | 0.42 | 1.63 | 11.56 | 78.06 |
| 2        | 0   | 0.05 | 0.38 | 1.09 | 3.38 | 5.35 | 3.87 | 9.84 | 9.63 | 66.41 |
| 3        | 0   | 0.12 | 0.68 | 2.20 | 5.29 | 6.20 | 10.33 | 9.43 | 7.96 | 57.82 |
| 4        | 0   | 0.19 | 1.16 | 3.35 | 6.79 | 8.32 | 9.68 | 9.24 | 9.82 | 51.45 |
| 5        | 0.01 | 0.36 | 1.87 | 4.45 | 8.6 | 11.86 | 9.79 | 8.27 | 9.58 | 43.18 |

Table 21. $d(13, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|
| 1        | 0.02 | 0.06 | 0.13 | 0.21 | 0.44 | 0.79 | 0.2 | 0.44 | 1.05 | 1.65 |
| 2        | 0.01 | 0.08 | 0.21 | 0.37 | 0.74 | 0.72 | 0.68 | 0.92 | 0.92 | 2.08 |
| 3        | 0.01 | 0.11 | 0.31 | 0.59 | 0.92 | 0.79 | 0.89 | 0.98 | 0.95 | 2.17 |
| 4        | 0.02 | 0.17 | 0.46 | 0.91 | 1.23 | 1.06 | 0.99 | 1.09 | 1.66 | 3.22 |
| 5        | 0.03 | 0.25 | 0.59 | 1.21 | 1.61 | 1.08 | 1.14 | 1.14 | 1.74 | 4.59 |

Table 22. Standard deviation of $d(13, \delta)$.
### Table 23. $d(14, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0.03| 0.19| 0.38| 1.42| 5.11| 0.87| 2.02| 10.41| 79.57|
| 2          | 0.04| 0.32| 1.13| 2.98| 5.08| 4.04| 9.07| 8.96| 68.38|
| 3          | 0.10| 0.69| 2.22| 4.99| 6.49| 9.72| 8.72| 8.31| 58.75|
| 4          | 0.01| 0.23| 1.25| 3.60| 6.95| 8.81| 9.71| 9.04| 10.21| 50.17|
| 5          | 0.02| 0.39| 2.14| 4.72| 8.80| 11.73| 10 | 8.96| 9.74| 43.29|

### Table 24. Standard deviation of $d(14, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0.05| 0.14| 0.23| 0.52| 0.87| 0.31| 0.53| 1.38| 2.52|
| 2          | 0.01| 0.07| 0.20| 0.47| 0.94| 0.69| 1.01| 1.13| 0.95| 3.53|
| 3          | 0.02| 0.10| 0.36| 0.68| 1.11| 0.89| 1.29| 1.05| 0.97| 3.97|
| 4          | 0.03| 0.15| 0.50| 0.94| 1.35| 1.23| 1.05| 1.11| 1.12| 4.36|
| 5          | 0.05| 0.23| 0.69| 1.21| 1.73| 1.17| 1.13| 1.11| 1.04| 4.86|

### Table 25. $d(15, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0.03| 0.22| 0.39| 1.39| 5.85| 0.71| 1.70| 11.54| 78.16|
| 2          | 0.05| 0.35| 1.08| 3.07| 5.46| 4.06| 9.28| 9.45| 67.19|
| 3          | 0.10| 0.60| 1.88| 4.83| 6.10| 10.31| 9.42| 8.47| 58.29|
| 4          | 0.01| 0.18| 1.08| 3.32| 6.91| 8.72| 9.91| 9.64| 10.54| 49.67|
| 5          | 0.01| 0.35| 1.84| 4.53| 8.73| 12.26| 10.18| 9.11| 10.96| 41.99|

### Table 26. Standard deviation of $d(15, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1          | 0.05| 0.14| 0.22| 0.47| 0.92| 0.22| 0.46| 1.39| 2.55|
| 2          | 0.01| 0.09| 0.25| 0.41| 0.85| 0.87| 0.99| 1.11| 1.15| 3.98|
| 3          | 0.01| 0.10| 0.33| 0.79| 1.14| 0.86| 1.18| 1.36| 0.87| 4.16|
| 4          | 0.03| 0.15| 0.51| 1.07| 1.32| 1.26| 1.11| 1.06| 1.73| 3.59|
| 5          | 0.03| 0.26| 0.65| 1.20| 1.68| 1.00| 1.20| 1.19| 1.70| 4.31|
Table 27. \(d(16, \delta)\).

| \(\delta/b\) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0           | 0.03| 0.15| 0.32| 1.23| 5.20| 1.26| 1.97| 9.90| 79.93|
| 1           | 0.05| 0.28| 0.98| 2.72| 5.55| 4.25| 9.15| 9.17| 67.85|
| 2           | 0.08| 0.48| 1.71| 4.41| 6.83| 9.62| 9.38| 9.17| 58.31|
| 3           | 0.13| 0.90| 2.89| 6.24| 8.65|10.00| 9.77| 9.17| 49.86|
| 4           | 0.28|1.60| 4.14| 8.12|12.06|10.37|10.21|11.94|41.26|

Table 28. Standard deviation of \(d(16, \delta)\).

| \(\delta/b\) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0           | 0.05| 0.14| 0.51| 0.79|0.45|0.54|1.49|2.81|     |
| 1           | 0.07| 0.19| 0.40| 0.79|0.88|1.14|0.98|3.06|     |
| 2           | 0.02| 0.10| 0.68| 1.26|0.95|0.98|1.22|2.87|     |
| 3           | 0.01| 0.12| 0.52| 1.21|1.65|1.38|1.15|1.73|4.71|
| 4           | 0.03| 0.24| 0.67| 1.26|1.67|1.21|1.22|1.36|4.94|

Table 29. \(d(17, \delta)\).

| \(\delta/b\) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0           | 0.01| 0.17| 0.31| 1.17|5.71|0.90|1.65|10.49|79.58|
| 1           | 0.03| 0.26| 0.97| 2.77|5.59|3.79|9.69|9.58|67.31|
| 2           | 0.08| 0.56| 2.01| 5.06|6.59|10.51|9.69|8.78|56.71|
| 3           | 0.14| 1.02| 3.20| 6.83|8.85|10.26|10.08|11.64|47.97|
| 4           | 0.01| 0.30| 1.72| 4.43|8.78|12.52|10.82|10.00|39.59|

Table 30. Standard deviation of \(d(17, \delta)\).

| \(\delta/b\) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|-------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0           | 0.01| 0.15| 0.20| 0.54|1.02|0.32|0.49|1.58|3.00|     |
| 1           | 0.05| 0.18| 0.42| 0.86|0.77|0.84|1.26|1.05|3.76|     |
| 2           | 0.01| 0.09| 0.29| 0.67|1.20|1.07|1.44|1.01|1.17|4.38|
| 3           | 0.02| 0.13| 0.48| 1.01|1.47|1.38|1.11|1.25|2.04|4.82|
| 4           | 0.02| 0.22| 0.69| 1.37|1.85|1.09|1.08|1.09|1.33|5.06|
Table 31. $d(18, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1         | 0.02| 0.17| 0.35| 1.20| 5.07| 1.43| 2.09| 10.30| 79.35|     |
| 2         | 0.03| 0.31| 0.93| 2.69| 5.48| 4.42| 8.78| 9.59 | 67.78|     |
| 3         | 0.09| 0.53| 1.84| 4.80| 6.79|10.04| 9.75 | 9.46 | 56.71|     |
| 4         | 0.14| 0.98| 3.11| 6.68| 9.62|10.41|10.53 |11.33 |47.20 |     |
| 5         | 0.27| 1.73| 4.39| 8.73|12.60|11.04|10.71 |12.10 |38.40 |     |

Table 32. Standard deviation of $d(18, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1         | 0.05| 0.13| 0.20| 0.46| 0.80| 0.44| 0.5  | 1.24 |2.59 |     |
| 2         | 0.05| 0.22| 0.42| 0.80| 0.82| 0.88| 1.11 |1.34  |3.69 |     |
| 3         | 0.11| 0.29| 0.57| 1.30| 0.98| 1.27 |1.02 |1.25  |4.28 |     |
| 4         | 0.11| 0.52| 0.87| 1.36| 1.25| 1.17 |1.06 |1.15  |4.79 |     |
| 5         | 0.19| 0.72| 1.29| 1.72|1.16 |1.19  |1.16 |1.36  |5.46 |     |

Table 33. $d(19, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1         | 0.02| 0.13| 0.32| 1.21| 4.95| 1.78 |2.58 | 9.52 |79.46|     |
| 2         | 0.05| 0.26| 0.84| 2.57| 5.58| 4.46 |9.18 | 9.14 |67.93|     |
| 3         | 0.11| 0.44| 1.65| 4.46| 7.09|10.06| 9.35 |10.18 |56.72|     |
| 4         | 0.11| 0.88| 2.79| 6.53| 9.55|10.72|10.53 |12.54 |46.33|     |
| 5         | 0.22| 1.45| 3.89| 8.42|12.75|11.60|11.83 |12.68 |37.15|     |

Table 34. Standard deviation of $d(19, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1   |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1         | 0.05| 0.12| 0.20| 0.41| 0.77| 0.49| 0.66| 1.17 |2.59 |     |
| 2         | 0.01| 0.06| 0.17| 0.38| 0.86| 0.81| 1.05 |1.20 |0.98 |3.50 |
| 3         | 0.01| 0.07| 0.24| 0.63| 1.29 |1.16 |1.23 |1.09 |1.00 |4.15 |
| 4         | 0.02| 0.12| 0.41| 0.87| 1.49 |1.35 |1.00 |1.17 |1.25 |4.67 |
| 5         | 0.01| 0.16| 0.57|1.14 |1.80 |1.37 |1.32 |0.94 |1.36 |5.07 |
Table 35. $d(20, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1     |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-------|
| 1          | 0   | 0.01| 0.10| 0.29| 1.01| 5.35| 1.42| 1.89| 9.92| 80.00 |
| 2          | 0   | 0.04| 0.19| 0.79| 2.43| 5.69| 4.12| 9.37| 9.46| 67.91 |
| 3          | 0   | 0.06| 0.46| 1.67| 4.64| 7.04| 10.71| 10.02| 10.55| 54.84 |
| 4          | 0   | 0.11| 0.78| 2.74| 4.64| 9.48| 10.87| 11.10| 12.94| 45.52 |
| 5          | 0.01| 0.20| 1.32| 3.98| 8.62| 12.92| 11.89| 11.46| 12.97| 36.63 |

Table 36. Standard deviation of $d(20, \delta)$.

| $\delta/b$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1     |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-------|
| 1          | 0.01| 0.03| 0.11| 0.22| 0.40| 0.91| 0.37| 1.72| 2.38|       |
| 2          | 0   | 0.06| 0.15| 0.33| 0.76| 0.71| 1.31| 1.08| 3.50|       |
| 3          | 0   | 0.07| 0.25| 0.57| 1.32| 0.90| 1.31| 0.98| 0.88| 3.64  |
| 4          | 0.01| 0.12| 0.42| 0.95| 1.65| 1.52| 1.29| 1.15| 1.13| 5.66  |
| 5          | 0.02| 0.15| 0.58| 1.25| 2.09| 1.44| 1.40| 1.24| 1.31| 5.85  |