A new credit derivatives pricing model under uncertainty process

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ABSTRACT
Due to many uncertainties in the financial market, the pricing process of credit derivatives has not only the characteristic of randomness but also nonrandom uncertainties. Thus, the absence of uncertain factors will make the pricing model and the actual market insufficiently fit for risk analysis and derivative pricing. Following this, we introduce uncertainty theory into pricing derivatives, develop a new Uncertainty form pricing formula for CDS and put forward a One-factor Canonical Copula function, which shows that all kinds of uncertain factors in the market have a significant impact on credit spread. Moreover, it offers some relevant calculating samples of numerical values.

1. Introduction
The International Swaps and Derivatives Association (ISDA) in 1992 first put forward officially an innovative product called credit derivatives which can be used in dispersing, removing and hedging credit risks. Credit derivatives pricing mainly price single assets or default-related multi-asset situations. At the present, a great deal of literatures available can be classified into structural model, reduced model and mixed model. The latest developments about credit derivatives pricing can be seen also from Hao (2011). However, over-the-counter financial derivatives are non standardized face-to-face financial contracts, its trading environment is characterized by less information, information disclosure may be distorted, objectively lagging, no exchange protection, and the floating of stock price will also be greater. It leads to the market information disclosure is faced with the challenge of uncertain. Which result in the uncertainty from market information becomes a decisive factor in risk management and derivatives pricing in the OTC market. Meanwhile, the existence of uncertainty seriously destroys the quality of information disclosure and distorts the market environment. Therefore, in the management of credit risk, the uncertainty of the market information must be taken into consideration.

Thus, with various uncertainties of the market, the factors influencing financial products pricing have not only the characteristic of randomness but also nonrandom uncertainties. Thus, it is necessary to bring uncertainties into the process of credit derivatives pricing. Uncertainty Theory is a powerful tool to deal with all kinds of uncertainties. The research on it offers a new theoretical foundation for financial product pricing. It is a beneficial and necessary supplement to traditional financial product pricing methods.

Based on a great deal of uncertainties in the real world – they may be random and fuzzy even nonrandom and nonfuzzy. Liu (2008) first put forward a mathematical branch based on normality, monotonicity, self-duality, countable subadditivity and product measure axiom – Uncertainty Theory. Then Liu (2009) brought this theory into financial issues, and offered a new European Option pricing model based on his stock price model; following that, Chen (2011) worked out an American Option pricing formula; Chen, Liu, and Ralescu (2013) established some option pricing formulas with the stock model of the periodical dividend. The latest development about Uncertainty Theory can be seen also from Peng and Yao (2011), Zhang and Liu (2014), Li and Peng (2014), Jiao and Yao (2015), Ji and Zhou (2015), Liu, Chen, and Ralescu (2015), etc. Considering the fuzzy uncertainty, Wu and Zhuang (2015) proposed a reduced-form intensity-based model under fuzzy environments, and presented some applications of the methodology for pricing defaultable bonds and credit default swap, the model results change into a closed interval. Wu, Zhuang, and Lin (2015) based on fuzzy process theory, first discussed some pricing formulas of credit derivatives, and put forward a One-Factor Fuzzy Copula function. However, the pricing issues about credit derivatives pricing based on this theory have not
been studied. This paper first brings Uncertainty theory into the model of credit derivatives pricing in expectation to match credit derivatives pricing model and real financial market better.

2. Uncertainty theory

An uncertain process is a sequence of uncertain variables indexed by time or space, which was defined by Liu (2008). As the preparatory part of the following discussion, we will briefly introduce the knowledge of uncertainty theory in this section.

**Definition 2.1:** (Liu, 2008) Let $T$ be an index set and $(\Gamma, L, M)$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, L, M)$ to the set of real numbers, i.e. for $\forall t \in T$ and any Borel set $B$ of real numbers, $\{X_t \in B\} = \{\gamma \in \Gamma | X_t(\gamma) \in B\}$ is an event.

**Definition 2.2:** (Liu, 2008) If an uncertain process $X_t$ is satisfied the condition, $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k} - X_{t_{k-1}}$ are independent uncertain variables, then they will be said to have independent increments, where $t_0$ is the initial time and $t_1, t_2, \ldots, t_k$ are any times with $t_1 < t_2 < \ldots < t_k$. If an uncertain process $X_t$ is satisfied the condition, for any given time $t > 0$, the increments $X_{t+s} - X_s$ are identically distributed uncertain variables for all $s > 0$, then they will be said to have stationary increments.

**Definition 2.3:** (Liu, 2009) If an uncertain process $C_t$ is satisfied the conditions,

(i) $C_0 = 0$ and $C_t$ is sample-continuous,

(ii) $C_t$ has the property of stationary and independent increments,

(iii) The increment satisfied the condition $(C_{t+1} - C_t) \sim N(0, \sigma^2)$, whose uncertainty distribution is $\Phi(x) = (1 + \exp \left(-\frac{\pi x}{\sqrt{4t}}\right))^{-1}, x \in \mathbb{R}$.

Then it will be said to be a canonical process.

Let $C_t$ be a canonical process, then the uncertain process $X_t = \exp(\sigma t + \sigma C_t)$ is called a geometric canonical process, where $e$ is called the log-drift and $\sigma$ is called the log-diffusion.

**Definition 2.4:** (Liu, 2007) Let $\xi$ be an uncertain variable, the expected value of $\xi$ is defined by

$$E(\xi) = \int_{-\infty}^{\infty} M(\xi \geq r)dr - \int_{-\infty}^{0} M(\xi \leq r)dr,$$

which provided that at least one of the two integrals is finite.

**Theorem 2.1:** (Liu, 2007) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. If the expected value $E(\xi)$ exists, then we have $E(\xi) = \int_{-\infty}^{\infty} xd\Phi(x)$.

**Definition 2.5:** (Liu, 2009) Let $X_t$ be an uncertain process and let $C_t$ be a canonical process, then the uncertain integral of $X_t$ with respect to $C_t$ is

$$\int_{0}^{b} X_t dC_t = \lim_{\Delta \rightarrow 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i})$$

where provided that the limit exists almost surely and is an uncertain variable.

**Definition 2.6:** (Liu, 2008) Suppose $C_t$ is a canonical process and $X_t$ be an uncertain process, and $f$ and $g$ are two given functions. Then $dX_t = f(t, X_t)dt + g(t, X_t)dC_t$ is called an uncertain differential equation.

Suppose that $X_t$ be the bond price, and $Y_t$ the stock price. Let the process $Y_t$ follows a geometric canonical process (Liu, 2009). Then the stock model can be rewritten as below,

$$\begin{cases}
    dX_t &= rX_tdt \\
    dY_t &= eY_tdt + \sigma Y_tdC_t
\end{cases}$$

Where $e$ is stock drift, $\sigma$ is stock diffusion, $r$ is the riskless interest rate, and $C_t$ is a canonical process.

3. The application of uncertainty theory in derivatives pricing

3.1. Defaultable bond

In finance, a bond is an instrument of indebtedness of the bond issuer to the holders. The zero-coupon defaultable bond pays no regular interest. They are issued at a substantial discount to par value, so that the interest is effectively rolled up to maturity (and usually taxed as such). The bondholder receives the full principal amount on the redemption date. An example of zero-coupon bonds is Series E savings bonds issued by the U.S. government. Zero-coupon bonds may be created from fixed rate bonds by a financial institution separating (‘stripping off’) the coupons from the principal. In other words, the separated coupons and the final principal payment of the bond may be traded separately.

In order to get this defaultable bond price of Liu’s stock model, let the face value of it bond is 1 unit and the maturity date is $T$, the interest rate and recovery rate are $r$ and $\delta$. If $Y_t$ is the price process of the bond, then we can obviously get the result $Y_t = Y_0 \exp(\sigma t + \sigma C_t)$. For the simplification of the following analysis, we assume that the upper bound of breach of contract satisfied the condition $K \in \mathbb{R}^+$, then following Merton’s (1974) structural model, we can derive the result as following,
(i) If $0 < K < Y_0$, then

$$
\Phi_t(K) = P(\tau \leq t) = M[Y_t \leq K]
= M[\exp(\lambda t) \leq K]
= M[C_t \leq \frac{\ln(K/Y_0) - \lambda t}{\lambda}]
= \left[1 + \exp\left(\frac{\pi \lambda t - \pi \ln(K/Y_0)}{\sqrt{3} \lambda t}\right)\right]^{-1}
$$

(ii) If $\Phi_t(K) = 1$, then $K \geq Y_0$.

**Theorem 3.1:** Based on the aforementioned formula (1), the value of a defaultable bond $Y_t$ is following,

$$
Y_t = \exp(-r(T - t))(1 + \delta - 1)(1 + \exp\left(\frac{\pi e}{\sqrt{3} \lambda t}\right))^{-1}
$$

where $Q$ is an equivalent martingale measure.

**Proof:** By Definition 2.4, we have,

$$
Y_t = E^Q[\exp(-r(T - t))1_{(\tau > T)} + \exp(-r(T - t))\delta 1_{[\tau \leq T]}]
= E^Q[\exp(-r(T - t))(1 - 1_{[\tau \leq T]})]
+ \exp(-r(T - t))\delta 1_{[\tau \leq T]}]
= \exp(-r(T - t))
+ \int_{-\infty}^{+\infty} \exp(-r(T - t))d((\delta - 1)\Phi_t(K))
\times \int_0^{Y_0} \exp(-r(T - t))d\Phi_t(K)
= \exp(-r(T - t))
\times \left[1 + (\delta - 1)(1 + \exp\left(\frac{\pi e}{\sqrt{3} \lambda t}\right))^{-1}\right]
$$

**Example 1:** In our numerical example, we set the parameters in the model as follows, $r = 5\%$, $\sigma = 0.25$, $\delta = 0.4$. Then, we can derive the following calculation results.

From Table 1, we can come to the conclusion, with the uncertainties of the financial markets becomes intense, the value of defaultable bond becomes lower with time increasing.

### 3.2. The basic derivatives pricing

A CDS is a particular type of swap designed to transfer the credit exposure of fixed income products between two

| $r$ | 0.5 | 1 | 3 | 5 | 10 |
|-----|-----|---|---|---|----|
| $Y_0$ | 0.8933 | 0.8712 | 0.7883 | 0.7133 | 0.5555 |

or more parties. The transaction involves two parties, the protection buyer and the protection seller, and also a reference entity, usually a bond. The protection buyer pays a stream of premiums (the CDS ‘fee’ or ‘spread’) to the protection seller, who in exchange offers to compensate the buyer for the loss in the bond’s value if a credit event occurs. The stream of premiums is called the premium leg, and the compensation when a credit event occurs is called the protection leg.

Hypothesize that the notional amount of swap be $N$, the recovery rate be $\delta$, and the credit spreads be $s$. At the same time, the discount factor between $t$ and $T$ be $D(t, T)$, the sequence payment dates be $(T_n)_{n=1}^N$, which along with $T_n = T$. The CDS running spread or fee is computed such that the fair price of the CDS equals zero at initiation. In other words, the present value of the protection leg equals the present value of the protection leg. As a result, we obtain the following pricing formula of CDS with noise interference.

**Theorem 3.2:** The spread of CDS can be calculated as,

$$
\sigma = \frac{(1 - \delta)\exp(-r(T - t))}{\Delta T_j \exp\left(\frac{\pi e}{\sqrt{3} \lambda t}\right) \sum_{i=1}^{n} \exp(-r(T_j - t))}
$$

**Proof:** By Definition 2.4, we can have the following,

**PV (protection leg)**

$$
= E^Q[(1 - \delta)N\exp(-r(T - t))1_{[\tau \leq T]}]
\times (1 - \delta)N \int_0^{Y_0} \exp(-r(T - t))d\Phi_t(K)
\times \left(1 + \exp\left(\frac{\pi e}{\sqrt{3} \lambda t}\right)\right)^{-1},
$$

**PV (premium leg)**

$$
= E^Q[sN \sum_{i=1}^{n} 1_{(T_j > T)} \Delta T_i \exp(-r(T_j - t))]
\times \left(1 - \delta\right)N \sum_{i=1}^{n} \exp(-r(T_j - t))
\times \int_0^{Y_0} \exp(-r(T_j - t))d\Phi_t(K)]
\times \int_0^{Y_0} \exp(-r(T_j - t))d\Phi_t(K)]
$$
Then based on the equation, i.e. $PV(\text{premium leg}) = PV(\text{ protection leg})$, we can prove this theorem.

Example 2: In our numerical example, we set the parameters in the model as follows, $r = 7\%$, $e = 0.07$, $\sigma = 0.07$, $\delta = 0.4$. Then, we can derive the following calculation results. (unit of measure is basis point).

From Table 2, we can see that, these results represent a decline of credit curve, which indicating that the market uncertainty increases, in today’s financial crisis.

4. Factor canonical copula function

For basket credit derivatives pricing, its earnings and the default correlation in the investment portfolio are of strong sensibility. Thus, to confirm the default correlation among the basket asset, that is the correlation of default risks, is the core process of basket credit derivatives pricing. With a reference to the definition of Li’s (2000) One-factor Gaussian Copula Model, as well as combined with the uncertainty theory, we can derive the following One-factor Canonical Copula function.

Definition 4.1: Let the factors influencing company asset value can be expressed as system factors and non-system factors; and both of the two are uncertain processes and obey the canonical process $C_t$, then the asset value $V_i$ can be rewritten as the following One-factor Canonical Copula model,

$$V_i = \sqrt{\mu_i}Y + \sqrt{1 - \rho_iZ_i}, i = 1, 2, \cdots, n.$$ 

In which, the system factor $Y$ express the macroeconomic financial state, and non-system factor $Z_i$’s corresponding to the company asset $i$, and they are mutual independent. $\sqrt{\mu_i}$ is corresponding to the correlation coefficient between the two factors. And on the condition that system $Y$ and non-system $Z_i$’s are all obey the canonical process $C_t$, the company asset $V_i$ will also obey the canonical process.

Following Merton (1974), when the company asset $V_i$ is under the upper bound of breach of contract $K_i$, a default event will happen. Through Copula function, the condition default probability of one company asset can be rewritten as,

$$\Phi_i(K^i) = p_i^1Y = P(V_i \leq K^1 | Y) = P\left(\sqrt{\mu_i}Y + \sqrt{1 - \rho_iZ_i} \leq K^1 | Y\right)$$

$$= P\left(Z_i \leq \frac{K^1 - \sqrt{\mu_i}Y}{\sqrt{1 - \rho_i}} | Y\right)$$

$$= \left(1 + \exp\left(\frac{-\pi}{\sqrt{3t}}\frac{(K^1 - \sqrt{\mu_i}Y)}{\sqrt{1 - \rho_i}}\right)\right)^{-1}$$

(5)

Theorem 4.1: Based on the assumed condition of one-factor Canonical Copula model, we can have the following,

$$p = P(V_1 \leq K^1, V_2 \leq K^2, \cdots, V_n \leq K^n)$$

$$= \int_{-\infty}^{+\infty} \prod_{i=1}^{n} p_i^1Y d\Phi(y)$$

(6)

Proof: We assume that the system factor $Y$ is known, then the following default can be derived,

$$p = P(V_1 \leq K^1, V_2 \leq K^2, \cdots, V_n \leq K^n)$$

$$= E(1_{\{V_1 \leq K^1, V_2 \leq K^2, \cdots, V_n \leq K^n\}})$$

$$= E(E(1_{\{V_1 \leq K^1, V_2 \leq K^2, \cdots, V_n \leq K^n\}} | Y) = y)$$

$$= E(E(1_{\{V_1 \leq K^1\}} | Y = y) \cdots E(1_{\{V_n \leq K^n\}} | Y = y))$$

$$= E(E(\prod_{i=1}^{n} p_i^1Y | Y) = \int_{-\infty}^{+\infty} \prod_{i=1}^{n} p_i^1Y d\Phi(y)$$

According to the above conclusions we have obtained, basket credit derivatives can be priced. For example, collateralized debt obligations (CDO), index default swaps, etc.

5. Conclusion

In this paper, we introduce uncertainty theory into pricing derivatives, develop a new Uncertainty form pricing formula for CDS and put forward a One-factor Canonical Copula function, which shows that all kinds of uncertain factors in the market have a significant impact on credit spread. However, due to the limited time, the combinatorial derivatives pricing is not discussed in this paper, which will be an important direction for our future research.

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