On Some Variants Of Schinzel’s Theorem For Global Function Fields

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Dedicated to the memory of Andrzej Schinzel

Abstract

In this paper, we obtain global function field versions of the results of Schinzel - Postnikova for multiplicative groups, and of Hahn - Cheon for elliptic curves, which is an analog of the former result.

1 Introduction

In 1974, Schinzel in [Sch74] proved that for any number field $K$, there exists a constant $n(K)$ depending only on $K$ such that for every $x \in K^*$ which is not a root of unity and every integer $n > n(K)$, the number $x^n - 1$ has a primitive divisor. This is more general than the main result of [PS68]. Passing to the elliptic curves, S. Hahn and J. Cheon proved a similar result which says that for any elliptic curve $E$ over a number field $K$, and $P$ is a non-torsion $K$–point on $E$, then for every sufficiently large integer $n$, there exists a prime $p$ of good reduction so that the order of $P$ modulo $p$ is equal to $n$. In this paper, we prove the global function field versions for those above results, see Theorem 4, Theorem 4.3, Proposition 5.1, and Theorem 5.2. We follow a similar strategy in those results, but we need to modify it so that the proof works well in positive characteristics. We note that the result clearly does not hold for unipotent groups.

2 Some Preliminaries

First, we will give proof for the elliptic curve case. In this case, we always assume that the characteristic of the base field is different from 2 and 3. Let $K$ be a global function field

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over \( \mathbb{F}_q \) whose constant field is \( \mathbb{F}_q \), where \( q = p^s \) for some prime number \( p \neq 2, 3 \). We denote by \( M_K \) the set of all pairwise non-equivalent discrete valuations \( v \) on \( K \) defined as in [Lan83b] Chapter 2, §3. So for each \( v \in M_K \), we have \( \mathfrak{p}_v \) the prime divisor associated to \( v \in M_K \), \( \mathcal{O}_v \) the associated discrete valuation ring with the valuation \( v(\cdot) \), and the normalized norm \( |\cdot|_v \). By "log" we always mean the logarithmic function with base \( q \). We remark that the field \( K \) with multiplicities 1 satisfies the product formula as in [Zim76]. So we can use results in [Zim76].

For a point \( P \in \mathbb{P}^n(K) \) with homogeneous coordinates \( P = [x_0 : \ldots : x_n] \), the multiplicative height of \( P \) relative to \( K \) is
\[
H_K(P) := \prod_{v \in M_K} \max\{|x_0|_v, \ldots, |x_n|_v\}.
\]

and the logarithmic height relative to \( K \) is defined as
\[
h_K(P) := \log(H_K(P)) = -\sum_{v \in M_K} \min\{v(x_0), \ldots, v(x_n)\}.
\]

Since \( p \neq 2, 3 \), we may assume that an elliptic curve \( E \) over \( K \) is given by
\[
y^2 = x^3 + ax + b \ (a, b \in K)
\]
with the identity element \( \mathcal{O} \). For \( P = (x, y) \in E(K) \), the local height function \( h_v \) at \( P \) is
\[
h_v(P) := \begin{cases} 
-\min\{0, v(x), v(y)\} & \text{if } P \neq \mathcal{O} \\
0 & \text{if } P = \mathcal{O}
\end{cases}
\]

The Weil height \( h \) at \( P \) is defined by
\[
h(P) := \sum_{v \in M_K} h_v(P).
\]

Further, for \( f \in K(E) \), the height on \( E \) (relative to \( f \)) is the function
\[
h_f : E(K) \to \mathbb{R}, \ h_f(P) = h_K(f(P)).
\]

As in [Zim76], we have the Néron-Tate height \( \hat{h} \) on \( E(K) \). From [Zim76] §2 we have that the function \( \hat{h} \) is a non-negative quadratic form on \( E(K) \) and the difference \( \hat{h} - h \) is bounded on \( E(K) \). Further, \( \hat{h}(P) = 0 \) if and only if \( P \) is a torsion point.

**Proposition 2.1** (Northcott-type finiteness theorem). With above notation, for any positive number \( B \), the set
\[
\{ P \in E(K) : h(P) < B \}
\]
is finite.
Proof. It follows directly from Northcott’s theorem. It also can be proven as follows. Since $E$ is an abelian variety over $K$, $E$ admits a $K/k$-trace $(A, \tau)$ (see Theorem 8 of Chapter VIII, §3 in [Lan83a]). It means that $A$ is an abelian variety defined over $k$ and $\tau$ is a $K$–homomorphism $A \times_k K \to E$ satisfying the usual universal property in the set of pairs of this form. Applying Theorem 5.3 of Chapter 6, §5 in [Lan83b], the set $\{P \in E(K) : h(P) < B\}$ lies in a finite number of cosets of $A(k)$. Since $A$ is of finite type over $k$ and $k$ is finite, the set $A(k)$ is finite. Hence $\{P \in E(K) : h(P) < B\}$ is also finite. 

3 A version of Siegel’s theorem for global function fields.

In this section, we prove the following result that is similar to a classical theorem of Siegel for number fields, see [Sil86] Theorem 3.1.

Theorem 3.1. With the above notation, let $P \in E(K)$ be a non-torsion point. Then for any place $v$, we have

$$\lim_{n \to \infty} \frac{h_v(nP)}{h(nP)} = 0.$$ 

To deal with this, we need the notion of $v$–adic distance between points on a curve. We follow the construction in [Sil86] Chapter IX.

Definition 3.2. Let $C$ be a smooth projective curve over $K$, and let $v \in M_K$. For $P$ and $Q$ in $C(K_v)$, we choose a function $t_Q$ which has a zero of order $e$ at $Q$ and no other zeros for some positive integer $e$. Then we define the $v$-adic distance from $P$ to $Q$ by

$$d_v(P, Q) = \min \left\{|t_Q(P)|_v^{1/e}, 1\right\}.$$ 

We note that Proposition IX.2.2 and Proposition IX.2.3 in [Sil86] also hold for global function fields. To tackle Theorem 3.1, we need the following consequence of Roth’s theorem for global function fields.

Proposition 3.3. Let $\epsilon > 0$, $\alpha \in \bar{K}$ which does not lie in any cyclic extension of degree a power of $p$ over $K$, and let $v \in M_K$. Then for any positive constant $c$, there exist only finitely many $x \in K$ satisfying the inequality

$$|x - \alpha|_v < c.H_K(x)^{-2-\epsilon}.$$ (1)

Proof. We first fix a positive number $\epsilon_1 \in (0, \epsilon)$ and let $\epsilon_2 := \epsilon - \epsilon_1$. Let $x$ be an element satisfying (1). If $c.H_K(x)^{-\epsilon_1} > 1$ then $H_K(x) < c^{1/\epsilon_1}$ which is bounded. Otherwise, $x$ then satisfies $|x - \alpha|_v < H_K(x)^{-2-\epsilon_2}$. We consider two cases.
• If \(|x - \alpha|_v \geq 1\), then \(1 < H_K(x)^{-2-\varepsilon}\) and hence \(H_K(x) < 1\).

• If \(|x - \alpha|_v < 1\), then we have \(\inf(1, |\alpha - x|_v) < H_K(x)^{-2-\varepsilon}\). Apply Roth’s theorem in positive characteristic, see the main theorem in [Arm99], for \(\{v\}\) and \(\alpha_v = \alpha\), we deduce that the height of \(x\) must be bounded.

Thus, these elements \(x\)'s satisfying (1) have bounded height.

Combining results above, we have

Corollary 3.4. Let \(C\) be a smooth projective curve over \(K\), and let \(v \in M_K\). Let \(f \in K(C)\) be a nonconstant function on \(C\), and let \(Q \in C(\bar{K})\) such that \(f(Q)\) does not lie in any cyclic extension of degree a power of \(p\) over \(K\). Then

\[
\liminf_{P \to Q} \frac{\log d_v(P, Q)}{H_K(f(P))} \geq -2.
\]

Proof. The proof is similar to [Sil86] Corollary IX.2.4.

Proposition 3.5. Let \(E/K\) be an elliptic curve with \(#E(K) = \infty\). Fix a point \(Q \in E(K)\), and place \(v \in M_K\). Then

\[
\lim_{P \to \infty} \frac{\log d_v(P, Q)}{h_x(P)} = 0.
\]

Proof. First we there exists a sequence of \(K\)-points \(P_1, P_2, \ldots\) of \(E\) such that

\[
\lim_{i \to \infty} \frac{\log d_v(P_i, Q)}{h_x(P_i)} = \lim_{P \to \infty} \frac{\log d_v(P, Q)}{h_x(P)} = L.
\]

Since \(d_v(P, Q) \leq 1\) and \(h_x(P) \geq 0\), we have \(L \leq 0\). So now we will prove that \(L \geq 0\). Indeed, let \(m\) be a sufficiently large prime number coprime to \(p = \text{char}K\) such that \(p \nmid m - 1\) and \(K\) does not contain any \(m\)th primitive roots of unity (there are infinitely many such \(m\)). By the Weak Mordell-Weil theorem, the group \(E(K)/mE(K)\) is finite. Therefore some coset contains infinitely many of the \(P_i\). We then replace \(P_i\)'s by this subset, and there exists \(P_i, R \in E(K)\) such that

\[P_i = [m]P_i' + R.\]

The usual properties of height functions give us (see [Sil86] Chapter VIII.6, where the proofs given in this section also hold for global function fields)

\[m^2 h_x(P_i') = h_x([m]P_i') + O(1) = h_x(P_i - R) + O(1) \leq 2h_x(P_i) + O(1).\]

Here we note that \(x\) is an even function and \(O(1)\) is independent of \(i\). If there does not exist a subsequence \(P_j\) of \(P_i\) such that \(P_j \xrightarrow{v} Q\), then \(P_i\) is \(v\)-adically bounded away from \(Q\). Therefore,
$d_v(P_i, Q)$ is bounded and then $L = 0$. So, by replacing $P_i$ by its suitable subsequence, we may assume that $P_i \xrightarrow{u} Q$, then $[m]P_i \xrightarrow{u} Q - R$. Because the number of $m$th-roots of $Q - R$ is $m^2$, there must be a subsequence of $P_i'$ which converges to one of the roots. Therefore, there exists $Q' \in E(K)$ such that

$$P_i' \xrightarrow{u} Q' \text{ and } Q = [m]Q' + R.$$  

Because the multiplication-by-$m$ map and the translation map are unramified, their composition $E \to E, P \mapsto [m]P + R$ is also unramified. Therefore

$$\lim_{i \to \infty} \frac{\log d_v(P_i, Q)}{h_x(P_i')} = 1.$$  

Thus

$$L = \lim_{i \to \infty} \frac{\log d_v(P_i, Q)}{h_x(P_i')} \geq \lim_{i \to \infty} \frac{2 \log d_v(P_i', Q')}{m^2 h_x(P_i')} + O(1).$$  

Because $P_i \xrightarrow{u} Q'$, to apply Corollary 3.4, we will show that $x(Q')$ does not lie in any cyclic extension of degree power of $p = \text{char} K$ in the case $K$ is of characteristic $p > 0$ (in the number field case, there is no condition for $x(Q')$). Indeed, if $x(Q')$ lies in an extension of degree a power of $p$ of $K$, then so are $x(U)$ for any $U \in 1/mQ$ because they are $\text{Gal}(K/K)$-conjugate. Therefore $x\left(\frac{1}{m}Q\right)$ also lies in an extension of degree a power of $p$ of $K$. Because of the equation defining $E$, $y\left(\frac{1}{m}Q\right)$ must lie in an extension of degree 2 times a power of $p$ of $K$. So $K\left(E\left(\frac{1}{m}Q\right)\right)$ also lies in an extension of degree 2 times a power of $p$ of $K$. On the other hand, $K\left(E\left(\frac{1}{m}Q\right)\right)$ contains $K(E[m])$, and hence contains $K(\mu_m)$, where $\mu_m$ is the set of $m$th roots of unity in $K$ (due to the standard argument using Weil pairing). Therefore, $[K(\mu_m) : K]$ is a common divisor of $\phi(m) = m - 1$ (here $\phi$ is the Euler’s totient function) and $2p^s$ for some $s$, and by the assumption of $m$, it implies that $[K(\mu_m) : K] = 1$ or 2, which is a contradiction since $K$ does not contain any $m$th primitive roots of unity. Therefore, Corollary 3.4 gives us

$$\lim_{i \to \infty} \frac{\log d_v(P_i', Q')}{h_x(P_i')} \geq -2.$$  

Then the last two inequalities (2) and (3) implies that $L \geq -\frac{4}{m^2}$. Because there are infinitely many primes $m$ such that $p \nmid m - 1$, and $K$ does not contain any primitive $m$th roots of unity for $m$ sufficiently large, it follows that $L \geq 0$ and the proposition follows.

We note that the function $y$ is not an even function for general elliptic curves, so we cannot apply the proof of this theorem for $y$. However, when $E$ is given by

$$y^2 = x^3 + ax + b,$$  

the function $y^2$ is then an even function. Therefore, similarly as above, we also have
Proposition 3.6. Let \( E \) be an elliptic curve over \( K \) given by \( y^2 = x^3 + ax + b \) with \( \#E(K) = \infty \). Fix a point \( Q \in E(K) \), and a valuation \( v \in M_K \). Then

\[
\lim_{\substack{P \in E(K) \\
h_y^2(P) \to \infty}} \log \frac{d_v(P, Q)}{h_y^2(P)} = 0.
\]

Proof of Theorem 3.1. Apply Proposition 3.5 for \( Q = \emptyset \), and \( t_\emptyset = \frac{1}{x} \) (\( \emptyset \) is the only one zero of \( x \) and \( \text{ord}_\emptyset x = 2 \)), we have

\[
\lim_{\substack{R \in E(K) \\
h_x(R) \to \infty}} \log \frac{\min\{|x(R)|^{-1/2}, 1\}}{h_x(R)} = 0
\]

since \( d_v(R, \emptyset) = \min\{|x(R)|^{-1/2}, 1\} \). Thus we obtain

\[
\lim_{\substack{R \in E(K) \\
h_x(R) \to \infty}} \frac{-\min\{v(x(R)), 0\}}{h_x(R)} = 0.
\]

In addition, because \( \min\{w(x(R)), 0\} \geq \min\{w(x(R)), w(y(R)), 0\} \) for all \( w \in M_K \), we have \( 0 \leq h_x(R) \leq h(R) \). Consequently,

\[
\frac{-\min\{v(x(R)), 0\}}{h_x(R)} \geq \frac{-\min\{v(x(R)), 0\}}{h(R)} \geq 0.
\]

Therefore

\[
\lim_{\substack{R \in E(K) \\
h_x(R) \to \infty}} \frac{-\min\{v(x(R)), 0\}}{h(R)} = 0.
\]

Similarly, we have

\[
\lim_{\substack{R \in E(K) \\
h_y(R) \to \infty}} \frac{-\min\{v(y(R)), 0\}}{h(R)} = \frac{1}{2} \lim_{\substack{R \in E(K) \\
h_y^2(R) \to \infty}} \frac{-\min\{v(y^2(R)), 0\}}{h(R)} = 0.
\]

Here we note that \( h_y^2(R) = 2h_y(R) \).

Claim. \( h_x(nP) \) tends to \( \infty \) as \( n \) tends to \( \infty \).

Indeed, assume the contrary that there exists a positive number \( N \) and positive integers \( n_1 < n_2 < \ldots \) such that \( h_x(n_iP) < N \) for all \( n_i \), i.e., \( h(x(n_iP)) < N \), \( \forall i \in \mathbb{N} \). Hence \( \{x(n_iP)|i \in \mathbb{N}\} \subseteq K \) takes only finitely many values by Proposition 2.1, and then so does \( \{y(n_iP)|i \in \mathbb{N}\} \) thanks to the equation defining \( E \). It follows that the set \( \{n_iP|i \in \mathbb{N}\} \) is finite, whilst \( P \) is non-torsion, a contradiction. This concludes the claim.

Similarly, \( h_y(nP) \) also tends to \( \infty \) as \( n \to \infty \). Let \( R \) run over the set \( \{nP|n \in \mathbb{N}\} \), then the above limits give

\[
\lim_{n \to \infty} \frac{-\min\{v(x(nP)), 0\}}{h(nP)} = \lim_{n \to \infty} \frac{-\min\{v(y(nP)), 0\}}{h(nP)} = 0.
\]
Since \(-\min\{v(x(nP)),0\} - \min\{v(y(nP)),0\} \geq -\min\{v(x(nP)),v(y(nP)),0\} \geq 0\), we obtain
\[
\lim_{n \to \infty} -\min\{v(x(nP)),v(y(nP)),0\} = 0, \quad \text{i.e.,} \quad \lim_{n \to \infty} \frac{h_v(nP)}{h(nP)} = 0.
\]

\[
\Box
\]

4 Elliptic curve analogue

Now we are ready to state and prove the main theorem.

**Theorem 4.1.** Let \(E\) be an elliptic curve over some global function field \(K\) of characteristic \(p \neq 2,3\) and let \(P \in E(K)\) be a non-torsion point. Then for every sufficiently large integer \(n\) prime to \(p\), there exists a prime \(p\) of good reduction so that the order of \(P\) in the group of points of \(E\) modulo \(p\) is equal to \(n\). Moreover, for all but finitely many \(n\) there exists such a prime \(p\) for all positive integer \(n\) prime to \(p\).

The strategy of the proof is similar to the one in [CH99]. We may assume that \(E\) is given by the equation \(y^2 = x^3 + ax + b\). Let \(S\) be a finite set containing all the places at which \(E\) has bad reduction, all places dividing \(\infty\), and all places at which either \(a\) or \(b\) has the nonzero valuation, i.e.,
\[
S = \{v \in M_K : S \text{ has bad reduction at } v\} \cup \{v \in M_K : v(a) \neq 0\} \cup \{v \in M_K : v(b) \neq 0\}.
\]

The set \(S\) is finite and we denote by \#\(S\) its cardinality. First, we need a result that helps us to detect whether a non-torsion point is trivial or not after taking reduction.

**Lemma 4.2.** Let \(v \in M_K \setminus S\), and let \(P\) be a non-torsion point of \(E(K)\). Then

- If \(P\) modulo \(p_v\) does not equal \(O\), we have \(h_v(P) = 0\).
- If \(P\) modulo \(p_v\) equals \(O\), we have \(h_v(nP) = h_v(P) > 0\) for any positive integer \(n\) prime to \(p\).

**Proof.** We may write \(P = (x,y)\) and \(P = [X : Y : Z]\) in the corresponding projective closure of \(E\), where \(X,Y,Z \in \mathcal{O}_K\). The condition \(P \mod p_v = 0\) means that \(v(X) > v(Y), v(Z) > v(Y)\), and hence \(v(y) < 0\). Therefore, the condition \(P \mod p_v \neq 0\) is equivalent to either \(v(X) \leq v(Y)\) or \(v(Z) \leq v(Y)\). If \(v(Z) \leq v(Y)\), then \(v(y) \geq 0\), and from \(y^2 = x^3 + ax + b\) we obtain \(v(x) \geq 0\) (since if \(v(x) < 0\), then \(2v(y) = v(x^3 + ax + b) = 3v(x) < 0\), a contradiction), i.e., \(h_v(P) = 0\). If \(v(X) \leq v(Y)\) and \(v(Y) < v(Z)\), then \(v(X) < v(Z)\). But then from the homogeneous Weierstrass equation \(Y^2Z = X^3 + aXZ^2 + bZ^3\) we obtain
\[
2v(Y) + v(Z) = 3v(X),
\]
a contradiction. Therefore the first statement is proven. For the second one, from the Weierstrass equation \( y^2 = x^3 + ax + b \), we have \( 3v(x) = 2v(y) < 0 \), and then \( h_v(P) = -v(y) > 0 \).

Moreover, if we let
\[
E_1(K_v) := \{ M \in E(K_v) : M \mod p_v = \emptyset \},
\]
we then have an isomorphism of groups (see [Sil86], Proposition VII.2.2),
\[
E_1(K_v) \rightarrow \hat{E}(p_v), \quad M = (x(M), y(M)) \mapsto z(M) = \frac{-x(M)}{y(M)},
\]
where \( \hat{E} \) is the formal group associated to \( E \).

Further, this isomorphism gives us the formula \( v(y(M)) = -3v(z(M)) \). Thus, via this isomorphism, \( nP \) maps to \([n].\left(\frac{-x}{y}\right) = n.\left(\frac{-x}{y}\right) + \) (higher-order terms), here \([n].\left(\frac{-x}{y}\right) = \left(\frac{-x}{y}\right) + \left(\frac{-x}{y}\right) + \cdots + \left(\frac{-x}{y}\right) (n \text{ times}) \) in \( \hat{E}(p_v) \). Since \( v(x) > v(y) \), \( v(n) = 0 \) and \( \hat{E}(p_v) \) is defined over \( p_v \), we obtain
\[
v(z(nP)) = v\left([n].\left(\frac{-x}{y}\right)\right) = v\left(\frac{-x}{y}\right) = v(x) - v(y).
\]

Consequently, we get
\[
v(y(nP)) = -3v(z([n].P)) = 3v(y) - 3v(x) = 3v(y) - 2v(y) = v(y) < 0.
\]

So \( 3v(x(nP)) = 2v(y(nP)) < 0 \), and we obtain
\[
h_v(nP) = -v(y(nP)), \quad \text{and hence} \quad h_v(nP) = h_v(P) > 0.
\]

\[\Box\]

Now, to prove the main theorem, we will give estimates for places in \( S \) (as in Theorem 3.1) and places in \( M_K \setminus S \) (as in the above lemma) and combine them together to deduce a contradiction.

*Prove of Theorem 4.1.* Assume that for any sufficiently large \( n > 1 \) not divisible by \( p \), the order of \( P \) modulo \( p_v \) is not equal to \( n \) for any \( v \in M_K \). In other words, if \( nP \) modulo \( p_v \) equals \( \emptyset \), then there exists some prime divisor \( r \) of \( n \) such that \( \frac{n}{r}P \) modulo \( p \) equals \( \emptyset \), and hence, Lemma 4.2 gives us
\[
h_v(nP) = h_v\left(\frac{n}{r}P\right).
\]
It follows that for \( v \in M_K \setminus S \), we have
\[
h_v(nP) \leq \sum_r h_v\left(\frac{n}{r}P\right), \quad (4)
\]
where \( r \) runs over the set of prime divisors of \( n \). For \( v \in S \), Theorem 3.1 yields
\[
\lim_{n \to \infty} \frac{h_v(nP)}{h(nP)} = 0.
\]
Since \( \#S \) is finite, it follows that for any \( \epsilon > 0 \),

\[
h_v(nP) \leq \epsilon h(nP)
\]  

(5)

for all sufficiently large integers \( n \). Combining inequalities (4) and (5), we get

\[
h(nP) = \sum_v h_v(nP) \leq \sum_{v \not\in S} \sum_{r \mid n} h_v \left( \frac{n}{r} P \right) + \sum_{v \in S} \epsilon \cdot h(nP) \leq \sum_{r \mid n} h \left( \frac{n}{r} P \right) + \#S \cdot \epsilon \cdot h(nP).
\]

So

\[
(1 - \#S \cdot \epsilon) h(nP) \leq \sum_{r \mid n} h \left( \frac{n}{r} P \right).
\]

(6)

Now because there exists a constant \( c \) such that

\[
\hat{h}(Q) - c < h(Q) < \hat{h}(Q) + c
\]

for all \( Q \in E(K) \), combining this with (6) implies

\[
(1 - \#S \cdot \epsilon)(\hat{h}(nP) - c) < \sum_{r \mid n} \hat{h} \left( \frac{n}{r} P \right) + c.n
\]

since \( \# \{ \text{prime divisors of } n \} < n \). Because of the quadraticity of \( \hat{h} \), it follows that

\[
(1 - \#S \cdot \epsilon)(n^2 \cdot \hat{h}(P) - c) < \sum_{r \mid n} \frac{n^2}{r^2} \hat{h}(P) + c.n < \frac{n^2}{2} \hat{h}(P) + cn
\]

since \( \sum_{r \mid n} \frac{1}{r^2} < \frac{1}{2} \). Therefore

\[
\left(1 - \#S \cdot \epsilon \right) n^2 \cdot \hat{h}(P) < (n + 1 - \#S \cdot \epsilon) \cdot c
\]

We choose \( \epsilon < \frac{1}{2 \#S} \) and let \( n \) tend to \( \infty \), then we obtain \( \hat{h}(P) = 0 \), a contradiction. The second claim is proved similarly to the main result in [CH99].

**Remark 4.2.1.** We note that the condition \( \gcd(n, p) = 1 \) is necessary. For example, consider a supersingular elliptic curve \( E \) over \( K \). Equivalently, the Hasse invariant \( A(E, \omega) \) is equal to 0, where \( \omega = \frac{dx}{y} \), see [KM85]. We note that when \( p > 3 \), \( A(E, \omega) \) is the coefficient of \( x^{p-1} \) in \( (x^3 + ax + b)^{(p-1)/2} \). Therefore, for \( v \not\in S \) where \( S \) is the set in the previous proof, the reduction modulo \( p_v \) of \( E \) is an elliptic curve \( E_v \) whose Hasse invariant is also 0. Thus \( E_v \) is supersingular. So for such \( v \), \( E_v[n] = E_v[np] \) for all integer \( n \). Therefore, for all but finitely many \( v \), the order of the reduction of the point \( P \) modulo \( p_v \) must be prime to \( p \).
When $E$ is ordinary, we have

**Theorem 4.3.** Let $E$ be an ordinary elliptic curve over some global function field $K$ of characteristic $p \neq 2, 3$ and let $P \in E(K)$ be a non-torsion point. We fix a positive integer $t$. Then for every sufficiently large integer $n$ prime to $p$, there exists a prime $p$ of good reduction so that the order of $P$ in the group of points of $E$ modulo $p$ is equal to $np^t$.

**Proof.** Because $E$ is ordinary, there exists a point $Q \in E(K)$ of order $p^t$. We set $L := K(E[p^t])$ the $p^t$-division field of $E$, and denote $E_L$ the base change of $E$ to $L$. Then $P - Q \in E_L(L)$ is also a point of infinite order. We note the following properties of the reduction of points (for more details on reductions and integral models, we refer to [Per08]).

1. (see [Per08] Lemma 1.2.3) For all but finitely many primes $p$ of $K$ the following holds: for any prime $q$ of $L$ lying over $p$, the order of $P$ (as an $L$-point of $E_L$) modulo $q$ equals the order of $P$ modulo $p$.

2. (see [Per08] Corollary 2.3.4) Since $Q$ is torsion, the order of $Q$ modulo $q$ equals the order of $Q$, which is $p^t$, for all but finitely many primes $q$ of $L$.

We note that the proofs of those properties given in [Per08] also work well over global function fields. We call $V$ the set of exceptional primes of $K$ in (1) and call $U$ the set of exceptional primes of $L$ in (2) and primes of $L$ lying above primes in $V$. Then both $V$ and $U$ are finite. Now we apply Theorem 4.1 for $P - Q \in E_L(L)$, we have for every sufficiently large integer $n$ prime to $p$, there exists a prime $q$ of good reduction so that the order of $P - Q$ modulo $q$ equals $n$. Since $U$ is finite, the prime $q$ does not lie in $U$ for $n$ sufficiently large. Since $np^t P = np^t (P - Q) + np^t Q = np^t (P - Q)$, the point $np^t P$ modulo $q$ equals $\emptyset$. So the order of $P$ is of the form $mp^s$ where $m|n$ and $s \leq t$. Then

$$\emptyset = mp^s P \mod q = mp^s Q + mp^s (P - Q) \mod q = mp^s (P - Q) \mod q,$$

and hence, $n|mp^s$ which implies that $m = n$. Similarly, we have

$$\emptyset = np^s P \mod q = np^s Q + np^s (P - Q) \mod q = np^s Q \mod q.$$

Thus $p^s|np^s$ which means that $s = t$. Therefore the order of $P$ modulo $q$ is $np^t$. Since $p$, the prime of $K$ lying under $q$, does not lie in $V$, the order of $P$ modulo $p$ is also equal to $np^t$. The theorem is then proven.

**5 One-dimensional torus version**

In this section, we prove an analogous result for one-dimensional tori. First, we establish an analogous result for the multiplicative group $G_m$ over $K$ in arbitrary characteristic (not necessarily different from 2 and 3).
Proposition 5.1. Let $K$ be a global function field, and let $x \in K \setminus \{0\}$ be not a root of unity. Then for every positive integer $n > 1$ prime to $p$, there exist a place $v \in M_K$ such that the order of the reduction of $x$ modulo $p_v$ in the group $\mathbb{G}_m(O_v/p_v)$ is equal to $n$.

Remark 5.1.1. The condition $n$ being prime to $p$ is necessary. Indeed, for any $x \in K$, we have $x^{np} - 1 = (x^n - 1)^p$. Therefore, if $p_v|x^{np} - 1$ for some place $v$, then $p_v|x^n - 1$.

Now we prove Proposition 5.1

Proof of Proposition 5.1. We denote by $\mu$ the Möbius function. Let $W$ be the set $\{v \in M_K : v(x^n - 1) > 0\}$. We consider four following cases.

1. $v \in W$ such that $n$ is not the order of $x$ modulo $p_v$, we call this order by $n_0$. Then $n = n_0k$ for some positive integer $k$ and

$$x^n - 1 = (x^{n_0} - 1)(x^{n_0(k-1)} + x^{n_0(k-2)} + ... + x^{n_0} + 1).$$

Since $x^{n_0(k-1)} + x^{n_0(k-2)} + ... + x^{n_0} + 1 \equiv k \neq 0 \mod p_v ((k, p) = 1)$, we have $v(x^n - 1) = v(x^{n_0} - 1)$. Thus

$$v(\Phi_n(x)) = \sum_{m|n} \mu\left(\frac{n}{m}\right)v(x^m - 1) = \sum_{n_0|m} \mu\left(\frac{n}{m}\right)v(x^{n_0} - 1) = \sum_{n_0|m} \mu\left(\frac{n}{m}\right)v(x^{n_0} - 1) = 0 \text{ since } n_0 < n. \quad (7)$$

2. $v \in M_K$ satisfying $v(x) > 0$. Then $v(x^m - 1) = 0$ for all positive integer $m$. It implies that $v(\Phi_n(x)) = 0$.

3. $v \in M_K$ satisfying $v(x) < 0$. Then $v(x^m - 1) = v(1 - x^{-m}) + v(x^m) = mv(x)$. Hence

$$v(\Phi_n(x)) = \sum_{m|n} \mu\left(\frac{n}{m}\right)v(x^m - 1) = \sum_{m|n} \mu\left(\frac{n}{m}\right)mv(x) = \phi(n).v(x).$$

4. $v \in M_K$ satisfying $v(x) = 0$ and $v \not\in P$. Then $v(x^m - 1) = 0$ for all $m|n$, and hence $v(\Phi_n(x)) = 0$.

Combining these equalities, we obtain that if for every $v \in W$, $n$ is not the order of $x$ modulo $p_v$, then we have

$$0 = \sum_{v \in M_K} v(\Phi_n(x)) = \sum_{v \in M_K : v(x) < 0} v(\Phi_n(x)) = \sum_{v \in M_K : v(x) < 0} \phi(n).v(x).$$

This equality holds if and only if there is no place $v$ such that $v(x) < 0$, which means that $x$ must lie in the constant field $\mathbb{F}_q$, which is a contradiction. Thus, there exists some $v$ in $P$ such that $n$ is the order of $x$ modulo $p_v$, this is what we want.
To prove the same result for an arbitrary one-dimensional torus, we just extend it to a multiplicative group and use some arguments on reductions.

**Theorem 5.2.** Let $G$ be a one-dimensional torus over a global function field $K$, and let $x \in G(K)$ be a point of infinite order. Then for every sufficiently large integer $n$ prime to $p$, there exists a place $v \in M_K$ such that the reduction of $x$ modulo $p_v$ exists, and the order of $x$ modulo $p_v$ is $n$.

**Proof.** This theorem can be proven using the method of Lemma 1.2.3 in [Per08]. Firstly, after discarding a finite set of places $S$, we may assume that the coefficients of $x$ are in $O_S$, the ring of $S$–integers in $K$. After discarding finitely more places (we still denote this set of places by $S$), we can assume that $G$ admits an integral model $\mathcal{G}$ over $O_S$ and $x \in G(O_S)$. Further, there exists a finite Galois extension $K'$ of $K$ such that $G \times_K K' = \mathbb{G}_{m,K'}$. After discarding finitely more places (we still denote this set of places by $S$), one can assume that $\mathbb{G}_{m,K'}$ admits an integral model $\mathcal{G}_{m,O_S'}$ which is an extension $\mathcal{G}$, i.e., $\mathcal{G}_{m,O_S'} = \mathcal{G} \times_{O_S} O_{S'}$ (thanks to the uniqueness of integral models). Now for any $v \notin S$ and $w \notin S'$ above $v$, let $P$ be an $O_v$–point of $\mathcal{G}$. Then $P$ is also an $O_v$–point of $G$, and it can be lifted to a point $O_w$–point $P'$ of $\mathbb{G}_{m,K'}$, since $O_v \otimes_K K' \cong \prod_{w|v} O_w$. Since $O_{S'} \subset O_w$, the point $P$ is also an $O_w$–point of $\mathbb{G}_{m,S'}$, i.e., we have the commutative diagram

\[
\begin{array}{ccc}
\text{Spec } O_w & \longrightarrow & \mathbb{G}_{m,O_{S'}} \\
\downarrow & & \downarrow \\
\text{Spec } O_v & \longrightarrow & \mathbb{G}_{m,K'}
\end{array}
\]

In other words, we have $\mathcal{G}(O_v) \subset \mathbb{G}_{m,O_{S'}}(O_w)$. Taking reduction, we see that the group of reduction points modulo $p_v$ in $\mathcal{G}$ is injected in the group of reduction points modulo $p_w$ in $\mathbb{G}_{m,O_{S'}}$, and hence the order of $P$ modulo $p_v$ is equal to the order of $P'$ modulo $p_w$ for every $w|v$ and every $P \in \mathcal{G}(O_v)$. Now we take $P$ to be $x \in \mathcal{G}(O_S)$, then $x$ lifts to $x' \in \mathbb{G}_{m,O_{S'}}(O_{S'}) = O_{S'}^\times \subset K$, and we have the following commutative diagram

\[
\begin{array}{ccc}
\text{Spec } O_w & \longrightarrow & \text{Spec } O_{S'} \\
\downarrow & & \downarrow \\
\text{Spec } O_v & \longrightarrow & \text{Spec } O_S
\end{array}
\begin{array}{ccc}
& & \longrightarrow \\
& & \downarrow \\
& & \text{Spec } O_v & \longrightarrow & \mathbb{G}_{m,O_{S'}} \\
& & \downarrow & & \downarrow \\
& & \text{Spec } O_S & \longrightarrow & \mathbb{G}
\end{array}
\]

Applying Proposition 5.1, for every sufficiently large integer $n$ prime to $p$, since $S'$ is finite, there exists a place $w_0 \notin S'$ such that the order of the reduction of $x'$ modulo $p_{w_0}$ equals $n$. Thus the order of the reduction of $x$ modulo $p_{v_0}$, with $v_0 \notin S$ is the place that lies under $w_0$, also equals $n$. \qed
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