The Gribov problem in Noncommutative gauge theory

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Abstract

After reviewing Gribov ambiguity of non-Abelian gauge theories, a phenomenon related to the topology of the bundle of gauge connections, we show that there is a similar feature for noncommutative QED over Moyal space, despite the structure group being Abelian, and we exhibit an infinite number of solutions for the equation of Gribov copies. This is a genuine effect of noncommutative geometry which disappears when the noncommutative parameter vanishes.
1 Introduction

This article is based on a lecture given at the XXV International Fall Workshop on Geometry and Physics in Madrid and it is aimed at illustrating the appearance of Gribov ambiguity [1], which is a phenomenon related to the topology of the bundle of gauge connections, in the framework of noncommutative gauge theory.

The Gribov ambiguity is better understood in the context of functional quantization of gauge theories. These are theories with first class constraints, the generators of gauge transformations. In such a context physical degrees of freedom have to be identified with a gauge fixing procedure: the physical carrier space of dynamics is defined by picking one representative on each gauge orbit, that is, by considering the quotient of the kinematical carrier space with respect to the gauge group. This is usually realized in the functional formalism through the Faddeev-Popov prescription. The Gribov ambiguity amounts to the fact that there could be different field configurations which obey the same gauge-fixing condition, but which are related by a gauge transformation, that is, they are on the same gauge orbit. As first shown by Singer [2] and independently by Narasimhan and Ramadas [3], it can be given a precise mathematical characterization in the language of fiber bundles. Gribov ambiguity is a manifestation of topological obstructions to the existence of a global section for the relevant principal bundle.

In the first part of the paper we review the problem in the framework of standard gauge theory, stressing the geometric and topological issues. In the second part we approach the problem in the framework of noncommutative gauge theory. We first review the formulation of gauge theories in the noncommutative setting, making use of the derivation based differential calculus. We thus analyze the equation for Gribov copies for noncommutative $U(1)$ gauge theory. The latter is based on the results obtained in [4].

2 Gribov ambiguity in gauge theory

Let $M$ be the space-time manifold and consider a principal fiber bundle over $M$, $P \rightarrow M$ with structure group a unitary group. $M$ is further assumed to be a pseudo-Riemannian manifold.\(^1\) A pure theory of fundamental interactions, without matter fields, is a theory where the dynamical fields are the gauge connections, $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ with $\mathfrak{g}$ the Lie algebra of the structure group. Let $A \in \Omega^1(U) \otimes \mathfrak{g}$, $U \subset M$, a local representative of the gauge connection and $F = dA + A \wedge A$ the local curvature two-form. When $M$ is the Euclidean space-time the classical action describing the dynamics is

$$S = \frac{1}{4} \text{Tr} F \wedge \ast_H F = \frac{1}{4} \int F_{\mu \nu}^a F^{\mu \nu \alpha} d^n x$$

(2.1)

where $F = F_{\mu \nu} \tau_a dx^\mu \wedge dx^\nu$, $\ast_H$ is the Hodge product and $\tau_a$ are the generators of the Lie algebra. The trace is to be intended as a scalar product over both Lie algebra and forms. On integrating by parts we arrive at

$$S = \frac{1}{2} \int d^n x \int d^n y A^a_\mu(x) M^\mu_{ab}(x, y) A^b_\nu(y)$$

(2.2)

\(^1\)Eventually, we shall switch to positive definite metric, since functional quantization is defined within Euclidean quantum field theory.

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with
\[ M_{ab}^{\mu \nu}(x, y) = (-\Box \delta^{\mu \nu} + \partial^\mu \partial^\nu) \delta^{(n)}(x - y) \delta_{ab}. \] (2.3)

Within the functional quantization approach one defines the generating functional of Green’s functions
\[ Z[J] = \int DAE^{-\frac{1}{2} \left( S[A] + S_l[A, J] \right)} \] (2.4)
with \( S[A] \) the Euclidean action and \( S_l = \text{tr} (JA) \). From \( \ln Z[J] \) one obtains the quantum action \( \Gamma[A] \) through Legendre transform. The Gaussian integral in (2.4) can be formally performed:
\[ Z[J] = (\det M)^{-\frac{1}{2}} \exp \left( \frac{1}{2} \int J M^{-1} J \right) \] (2.5)
with \( M^{-1} \) the Euclidean propagator, when the operator \( M_{ab}^{\mu \nu} \) defined by (2.3) is invertible. Unfortunately this is not the case for gauge theories.

When \( A \) is a \( U(N) \) the gauge connection the free action is invariant under gauge transformations
\[ A \to A^g = gAg^{-1} + dgg^{-1} \] (2.6)
with \( g : M \to U(N) \). Thus, on considering field configurations of the form \( dgg^{-1} \) (so called pure gauge terms), we have
\[ M_{\mu \nu} \partial_\nu gg^{-1} = 0 \] (2.7)
showing that, because of gauge invariance, the operator (2.3) has eigenvectors with zero eigenvalue (so called zero modes), hence it is not invertible unless we perform the integral in Eq. (2.4) over equivalence classes of gauge connections.

To this, let us define more accurately the configuration space of gauge theories and the group of gauge transformations. As above, let \( P \) a principal \( G \)-bundle over \( M \), smooth manifold representing space-time (which, rigorously, should be compact). The structure group \( G \) is a finite dimensional Lie group, which we choose to be \( U(N) \).

**Definition 2.1.** An automorphism of \( P \) is a diffeomorphism \( \varphi : P \to P \) which is \( G \)-equivariant, that is \( \varphi(p \cdot g) = \varphi(p) \cdot g \) for all \( p \in P \) and \( g \in G \).

Every \( \varphi \in \text{Aut}(P) \) induces a diffeomorphism \( \tilde{\varphi} \) on the basis manifold. The map, \( H \), which associates \( \tilde{\varphi} \in \text{Diff}(M) \) to \( \varphi \in \text{Aut}(P) \) is a group homomorphism. Thus, the kernel of \( H \), given by those automorphisms of \( P \) which are mapped to the identity in \( \text{Diff}(M) \), is a group. This allows for a mathematical definition of gauge transformations:

**Definition 2.2.** The gauge group of \( P \) is \( \mathcal{G}(P) := \ker(H) \). Its elements are called gauge transformations or also vertical automorphisms, because they are such that \( \pi(\varphi(p)) = \pi(p) \).

Gauge transformations of vector and spinor fields are implemented by the action of \( \mathcal{G}(P) \) on the vector and spinor bundles associated to \( P \).

An equivalent definition, more physically oriented, is the following:

The gauge group is homeomorphic to the group of smooth maps from space-time to the structure group \( G \).
For Euclidean space-time $\mathbb{R}^N$, physical considerations\(^2\) impose $g(x) \to 1$ as $|x| \to \infty$ which amounts to compactify the base manifold

$$\mathcal{G} \simeq \text{Map}(S^n \to G).$$

(2.8)

The kinematical configuration space of gauge theory is $\mathcal{A}$, the space of gauge connections of $(P, M, G)$, which are locally represented by Lie algebra valued one-forms on the base manifold $A : M \to \Omega^1(M) \otimes \mathfrak{g}$, transforming under the action of the gauge group according to Eq. (2.6). Physical configurations are therefore equivalence classes with respect to the gauge transformation (2.6), which belong to the quotient space $\mathcal{B} = \mathcal{A}/\mathcal{G}$. In order to perform the functional integral in (2.4), one has to integrate over $\mathcal{B}$ instead than $\mathcal{A}$, that is, choose a representative for each equivalence class, by fixing the gauge.

Mathematically, this amounts to choose a surface $\Sigma_f \subset \mathcal{A}$ which possibly intersects the gauge orbits only once: a section for the principal bundle

$$\mathcal{A}(P) \leftarrow \mathcal{G}$$

$$\mathcal{B}(P)$$

(2.9)

The choice of $\Sigma_f$ is physically rephrased as a gauge fixing, for example $\partial_\mu A^\mu = 0$ or, in general $f(A) = h$, for some chosen functions $f, h$.

Unless the bundle is globally trivial the kinematical configuration space is not a product: $\mathcal{A} \neq \mathcal{B} \times \mathcal{G}$, but let us assume for a moment that the equality holds. In such a case we have for the integration measures

$$[d\mu(\mathcal{A})] = [d\mu(\mathcal{B})] [d\mu(\mathcal{G})]$$

(2.10)

and, for gauge transformations close to the identity, $U(x) \simeq 1 + a^a(x) \tau_a$, the integration measure over the kinematical configuration space $[d\mu(\mathcal{G})]$ can be replaced by $[d\alpha]$. In order to perform a change of variables $[d\alpha] \to [df(A)]$, we need the Jacobian of the transformation which is

$$\text{Det} \Delta_{FP}(x, y) = \left. \frac{\delta f^a(x)}{\delta \alpha^b(y)} \right|_{\alpha=0}$$

(2.11)

yielding

$$[d\mu(\mathcal{A})] \text{Det} \Delta = [d\mu(\mathcal{B})][d\alpha] \text{Det} \Delta = [d\mu(\mathcal{B})] [df]$$

(2.12)

and, finally, integrating over $[df]$ with the insertion of a delta function $\delta(f(A) - h(x))$ which implements the gauge choice, we obtain the measure on the quotient space:

$$[d\mu(\mathcal{A})] \text{Det} \Delta \frac{\delta f(A) - h(x)}{[d\mu(\mathcal{B})]}.$$

(2.13)

The Jacobian in (2.11) is the so called Faddeev-Popov determinant.

\(^2\)See for example [5], cap. 10 where, on coupling gauge fields to matter fields, the request of invariance of physical states under the action of constraints, imposes, at fixed time, that $g(x)$ tend to the identity at spatial infinity.
2.1 Gribov ambiguity

The gauge fixing described above is not enough to remove unphysical degrees of freedom if the theory is non-Abelian. Indeed, let us consider the gauge orbit

\[ A^g = gAg^{-1} + dgg^{-1} \simeq A + D \alpha \]  

(2.14)
with \( D \alpha = d\alpha + \alpha \wedge A = d\alpha + \alpha^a \wedge A^b[\tau_a, \tau_b] \). The gauge fixing condition \( \partial \mu A^g_\mu = 0 \) yields

\[ \partial \mu D^\mu \alpha = 0 \]  

(2.15)

which may have nontrivial solutions, whenever the gauge group is non-Abelian.\(^3\) This is the so called equation of copies and the phenomenon is known as Gribov ambiguity. Notice that \(- (\partial \mu D^\mu) \delta^{(4)}(x - y) \delta^{ab} \) is exactly the FP determinant for this choice of gauge fixing.

Let us return to the global approach and let us show how the existence of Gribov copies (solutions of Eq. (2.15)) is the manifestation of the fact that the bundle \( \mathcal{A} \to \mathcal{B} \) is nontrivial \([2,3]\).

The kinematical configuration space \( \mathcal{A} \) is an affine space. Indeed any convex combination

\[ A_\tau = (1 - \tau)A_1 + \tau A_2 \quad 0 \leq \tau \leq 1 \]  

(2.16)
is a gauge connection, because it satisfies

\[ A^g_\tau = gA_\tau g^{-1} + dgg^{-1} \]  

(2.17)
therefore \( \mathcal{A} \) is topologically trivial. Let us consider the gauge group \( \mathcal{G} = \{ g : S^4 \to G \} \). The fundamental group \( \pi_1(\mathcal{G}) \) may be identified with \( \pi_5(G) = \{ g : S^5 \to G \} \).

\[ \pi_1(\mathcal{G}) \simeq \pi_5(G). \]  

(2.18)

Thus we can use standard results in topology which state that, for \( G = U(N) \)

\[ \pi_5(U(N)) = \mathbb{Z}, N \geq 3; \]
\[ \pi_5(U(N)) = \mathbb{Z}_2, N = 2; \]
\[ \pi_5(U(N)) = 0, N = 1 \]  

(2.19)
showing that, by virtue of (2.18), the group manifold \( \mathcal{G} \) is nontrivial except for the Abelian case. Let us come to the physical configuration space \( \mathcal{B} = \mathcal{A}/\mathcal{G} \). Since \( \mathcal{A} \) is homotopically trivial whereas \( \mathcal{B} \) and \( \mathcal{G} \) in general aren’t,\(^4\) \( \mathcal{A} \) cannot be globally trivialized as the product of \( \mathcal{B} \) and \( \mathcal{G} \) unless \( \mathcal{G} \) is topologically trivial. On the basis of (2.19), both \( \mathcal{G} \) and \( \mathcal{B} \) are only trivial for \( G = U(1) \), which is the case for electrodynamics.

This global analysis translates into the fact that Eq. (2.15), namely \( \Box \alpha = 0 \), only has trivial solutions in the Abelian case. Vice-versa, we can conclude that non-Abelian gauge theories do not admit global sections, which amounts to the FP operator \( \Delta \) having non trivial zero modes.

\(^3\) In the Abelian case we only have trivial solutions, if we further assume that \( \lim_{x \to \infty} \alpha(x) = 0 \).

\(^4\) On considering the long exact sequence

\[ \ldots \to \pi_n(\mathcal{G}) \to \pi_n(\mathcal{A}) \to \pi_n(\mathcal{B}) \to \pi_{n-1}(\mathcal{G}) \to \ldots \to \pi_0(\mathcal{A}) \]

we have that \( \pi_k(\mathcal{B}) = \pi_{k-1}(\mathcal{G}) \).
3 Noncommutative Electrodynamics on $\mathbb{R}^{2n}_\theta$

In this section we shall briefly review the formulation of Electrodynamics in the noncommutative setting of Moyal space-time, $\mathbb{R}^{2n}_\theta$.

This is the simplest noncommutative space, modeled on the phase-space of quantum mechanics, the quantum phase-space. In order to define the latter, one considers the dual description of classical phase-space in terms of its algebra of functions (classical observables) and quantizes it. The algebra of quantum observables represents quantum phase space. This is noncommutative, because the operator product is noncommutative, moreover, it has no underlying, dual notion of smooth manifold anymore.

Equivalently, one can describe quantum observables in terms of smooth functions on classical phase-space with a noncommutative or star product. This is the Moyal-Weyl-Wigner description of quantum mechanics.

Following the same approach for classical space-time, say $\mathbb{R}^{2n}$, one replaces $\left(\mathcal{F}(\mathbb{R}^{2n}), \cdot \right)$ with a noncommutative algebra, $\mathbb{R}^{2n}_\theta \equiv \left(\mathcal{F}(\mathbb{R}^{2n}), \star \right)$. The Moyal star-product is so defined:

$$f \star \theta g(x) := (2\pi)^{-2n} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} f(x + \frac{1}{2} \theta J u) g(x + v) e^{-iu \cdot v} d^{2n}u d^{2n}v \quad (3.20)$$

with $J$ antisymmetric $2n \times 2n$ matrix such that $J^2 = -1$. Its popular asymptotic expansion reads

$$(f \star g)(x) = f(x) \exp \left\{ \frac{i}{2} \theta^{\rho\sigma} \partial_\rho \partial_\sigma \right\} g(x) \quad (3.21)$$

yielding, for coordinate functions,

$$x^i \star x^j - x^j \star x^i = i \theta^{ij} \quad (3.22)$$

and also

$$x^i \star f = x^i \cdot f + \frac{i}{2} \theta^{ij} \partial_j f \quad (3.23)$$

which defines the Lie algebra of derivations $\partial_j \in \text{Der}(\mathbb{R}^{2n}_\theta)$ as inner, with respect to the product:

$$\partial_j f = i \theta_{jk}^{-1} (x^k \star f - f \star x^k) \quad \text{with} \quad \partial_j(f \star g) = \partial_j f \star g + f \star \partial_j g. \quad (3.24)$$

The Moyal star product possesses an important property: it is cyclic and closed namely

$$\int d^{2n}x f \star g = \int d^{2n}x g \star f = \int d^{2n}x f \cdot g \quad (3.25)$$

which can be shown by integration by parts.\(^5\) The algebraic properties of classical gauge invariant actions on Moyal space are described by a simple version of the derivation-based differential calculus. The latter, introduced long ago [8–10], is a generalization of the de Rham differential calculus. For mathematical details and applications to NCFT, we refer the reader to [11–13]. In what follows we give a short review based on [14].

\(^5\)An instance of a cyclic product which is cyclic but not closed is the Wick-Voros product. Its relation to Moyal product and its application to quantum field theory is discussed in [6]. We follow here the convention of [7], so that a closed star product of two elements in the noncommutative algebra is a product whose integral is equal to the integral of the pointwise commutative product.
3.1 Differential calculus for (noncommutative) associative algebras

Given the commutative associative algebra \( A \) of smooth functions over a manifold \( M \), the usual differential calculus can be equivalently defined algebraically, once a Lie algebra of derivations, \( \text{Der}(A) \), is given (see for example \([8,10]\)). Having defined one-forms as linear maps from \( \text{Der}(A) \) to \( A \), the exterior derivative \( d \) is defined for one forms as

\[
d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])
\]

(3.26)

It is easily verified that \( d^2 = d \circ d \) is zero. Higher forms are defined as skew-symmetric multilinear maps from \( \text{Der}(A) \) to the associative algebra \( A \). Then, the exterior derivative is easily generalized

\[
d\omega(X_1, \ldots, X_{p+1}) := \sum_{i=1}^{p+1} (-1)^{i+1} X_i (\omega(X_1, \ldots \hat{X}_i \ldots, X_{p+1}))
\]

(3.27)

\[
+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([X_i, X_j], \ldots \hat{X}_i \ldots \hat{X}_j \ldots, X_{p+1}),
\]

(3.28)

with \( \hat{X}_i \) meaning that the argument \( i \) is omitted.

This construction can be extended to noncommutative algebras, once we have chosen a set of derivations of \( A \), such that

\[
X(f \star g) = (Xf) \star g + f \star (Xg), \quad X \in \text{Der}(A), \quad f, g \in A
\]

(3.29)

where \( \star \) is the noncommutative product in \( A \). For Moyal algebra \( \mathbb{R}_\theta^{2n} \) the Lie algebra of derivations is the Abelian algebra generated by the derivatives \( \partial_\mu \), \( \mu = 1, \ldots, n \). Zero-forms are identified with the algebra itself, \( \Omega^0 = A \). Then the exterior derivative is implicitly defined by

\[
df(X) = X(f)
\]

(3.30)

It automatically verifies the Leibnitz rule because of Eq. (3.29). Moreover \( d^2 = 0 \) because \( \star \)-derivations close a Lie algebra. The second step consists in defining \( \Omega^1 \) as a left (or right) \( A \)-module that is

\[
gdf(X) = g \star X(f).
\]

(3.31)

To construct \( \Omega^2 \) we observe that

\[
df \wedge_* dg(X, Y) = df(X) \star df(Y) - df(Y) \star df(X)
\]

(3.32)

where \( \wedge_* \) is the deformed wedge product. Because of noncommutativity \( df \wedge_* dg \neq -dg \wedge_* df \). In a similar way to \( \Omega^1 \), \( \Omega^2 \) is defined as a left \( A \)-module with respect to the \( \star \)-multiplication

\[
fdg \wedge_* dh(X, Y) = f \star dg(X) \star dh(Y) - f \star dg(Y) \star dh(X).
\]

(3.33)

Higher \( \Omega^p \) are built along the same lines.

3.2 Gauge connection

We then consider a natural noncommutative extension of the notion of connection, as introduced in \([9]\) where one replaces complex vector bundles of physical fields over space-time, with fiber
\(\mathbb{C}^n\), with right-modules, \(\mathbb{M}\) over \(\mathcal{A}\). A connection on \(\mathbb{M}\) can be conveniently defined by a linear map \(\nabla : \text{Der}(\mathcal{A}) \times \mathbb{M} \rightarrow \mathbb{M}\) satisfying
\[
\nabla_X(mf) = mX(f) + \nabla_X(m)f, \quad \nabla_X(m) = m\nabla_X(c) = c\nabla_X(m), \quad \nabla_{X+Y}(m) = \nabla_X(m) + \nabla_Y(m) \tag{3.34}
\]
for any \(X, Y \in \text{Der}(\mathcal{A})\), \(f \in \mathcal{A}\), \(m \in \mathbb{M}\), \(c \in \mathcal{Z}(\mathcal{A})\), the center of the algebra. Hermitian connections satisfy for any real derivation \(X \in \text{Der}(\mathcal{A})\)
\[
X(h(m_1, m_2)) = h(\nabla_X(m_1), m_2) + h(m_1, \nabla_X(m_2)), \forall m_1, m_2 \in \mathbb{M}, \tag{3.35}
\]
where \(h : \mathbb{M} \otimes \mathbb{M} \rightarrow \mathcal{A}\) denotes a Hermitian structure on \(\mathcal{A}\). The curvature is the linear map \(R(X, Y) : \mathbb{M} \rightarrow \mathbb{M}\) defined by
\[
R(X, Y)m = [\nabla_X, \nabla_Y]m - \nabla_{[X,Y]}m, \forall X, Y \in \text{Der}(\mathcal{A}). \tag{3.36}
\]

The group of gauge transformations of \(\mathbb{M}\), \(\mathcal{U}(\mathbb{M})\), is defined \cite{11} as the group of automorphisms of \(\mathbb{M}\) compatible both with the structure of right \(\mathcal{A}\)-module and the Hermitian structure, i.e
\[
g(mf) = g(m)f, \quad h(g(m_1), g(m_2)) = h(m_1, m_2) \quad \forall g \in \mathcal{U}(\mathbb{M}), \quad \forall m_1, m_2 \in \mathbb{M} \tag{3.37}
\]
This definition is the natural algebraic counterpart of Def. 2.2.

For any \(g \in \mathcal{U}(\mathbb{M})\) we have
\[
\nabla_X^g : \mathbb{M} \rightarrow \mathbb{M}, \quad \nabla_X^g = g^{-1} \circ \nabla_X \circ g \tag{3.38}
\]
\[
R(X, Y)^g : \mathbb{M} \rightarrow \mathbb{M}, \quad R(X, Y)^g = g^{-1} \circ R(X, Y) \circ g. \tag{3.39}
\]

Since we shall eventually consider a gauge theory with structure group \(U(1)\), namely electrodynamics, the relevant vector bundle in the commutative case is a complex line bundle. This is generalized by means of a one-dimensional \(\mathcal{A}\)-module \(\mathbb{M} = \mathbb{C} \otimes \mathcal{A}\). As Hermitian structure we choose \(h(f_1, f_2) = f_1 \dagger f_2\) and take real derivations. Then a Hermitian connection is entirely determined \cite{11} by its action on the one-dimensional basis \(\nabla_X(1)\). We have \(\nabla_X(f) = \nabla_X(1)f + X(f)\), with \(\nabla_X(1)\dagger = -\nabla_X(1)\). This defines in turn the 1-form connection \(A\) by means of
\[
A : X \rightarrow A(X) := \nabla_X(1), \forall X \in \text{Der}(\mathcal{A}) \tag{3.40}
\]
From the compatibility condition with the hermitian structure, Eq. (3.37), one obtains that gauge transformations are the group of unitary elements of the algebra. Indeed, on using \(g(f) = g(1)f = g(1) \ast f\) and imposing compatibility, we get \(h(g(f_1), g(f_2)) = h(f_1, f_2)\) which implies \(g(1)\dagger \ast g(1) = 1\). We pose \(g(1) = U \in \mathcal{U}(\mathbb{R}^2_{\theta})\) the group of unitary elements of the algebra \(\mathbb{R}^2_{\theta}\), acting multiplicatively on the left of \(\mathbb{R}^2_{\theta}\). From Eqs. (3.38), (3.39) we obtain
\[
A^g_\mu = g \ast A_\mu \ast g^\dagger + i\partial_\mu g \ast g^\dagger, \quad F^g_{\mu\nu} = g \ast F_{\mu\nu} \ast g^\dagger, \quad \forall g \in \mathcal{U}(\mathbb{R}^2_{\theta}) \tag{3.41}
\]
where, to make contact with usual notation, we have set \(iR_{\mu\nu} = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]\ast\).

Being unitary elements of \(\mathbb{R}^2_{\theta}\) gauge transformations may be written as star exponentials
\[
g[\alpha] = \exp_\ast(i\alpha), \tag{3.42}
\]

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and the star exponential is by definition

$$\exp_x(i\alpha) \equiv \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \alpha \ast \cdots \ast \alpha.$$  \hfill (3.43)

with $\alpha$ is some function of $x$ considered as a parameter of the transformation. In the next section we shall study the infinitesimal form of Eq. (3.42).

4 Gribov ambiguity in noncommutative QED

The infinitesimal form of the gauge transformation reads

$$A'_\mu[\alpha] = A_\mu + D_\mu \alpha + O(\alpha),$$  \hfill (4.44)

where the appearance of the covariant derivative $D_\mu$, despite the gauge group being associated to an Abelian structure group, is an effect of non commutativity and is given by

$$D_\mu \alpha = \partial_\mu \alpha + i (\alpha \ast A_\mu - A_\mu \ast \alpha).$$  \hfill (4.45)

In the commutative limit $\theta \to 0$, the covariant derivative reduces to the ordinary one and the gauge transformation Eq. (3.41) gives back the standard Abelian gauge transformation

$$A'_\mu[\alpha] = A_\mu + \partial_\mu \alpha + O(\theta)$$  \hfill (4.46)

4.1 The gauge action

The natural generalization of the gauge action Eq.(2.1) with structure group $U(1)$ to noncommutative space-time $R^{2n}_\theta$

$$S[A] = (F,F)_*$$  \hfill (4.47)

with a suitably defined scalar product, is obtained as follows. The wedge product is defined for the noncommutative case in Eq. (3.32). In turn, the noncommutative Hodge product can be easily defined starting from the commutative coordinate free definition

$$\ast_H \eta = \iota_\eta \omega,$$  \hfill (4.48)

where $\omega$ is the volume form and $\eta^p$, for each given p-form $\eta$, is the p-vector field associated to $\eta$ through the metric. In local coordinates, $\eta = \eta_{j_1\ldots j_p} dx^{j_1} \wedge \ldots \wedge dx^{j_p}$, $g = g_{jk} \partial_j \otimes \partial_k$ it reads

$$\eta^p = g[\eta] := \eta_{j_1\ldots j_p} g^{rs} \left( \partial_r \otimes \partial_s (dx^{j_1}) \wedge \ldots \wedge \partial_r \otimes \partial_s (dx^{j_p}) \right).$$  \hfill (4.49)

On replacing wedge products with star-wedge products and vector fields with star-derivations we arrive at

$$S = \int F \wedge \ast_H F = \int d^{2n} x \ F_{\mu\nu} \ast F^{\mu\nu}.$$  \hfill (4.50)

The action is easily checked to be gauge invariant, because of the second of Eqs. (3.41) and the ciclicity of Moyal product under integration, but it yields new pathologies with respect to the commutative case, the most studied being the Ultraviolet/Infrared mixing (UV/IR), which
affects noncommutative QFT \cite{15, 16} and, in particular, noncommutative QED \cite{17, 18}. Such a mixing is one of the most important open problems in noncommutative QFT as it spoils the renormalizability of the theory. A less investigated problem is the problem of Gribov ambiguity. Indeed it has been shown \cite{4} that noncommutative QED similarly to commutative non-Abelian gauge theories, exhibits Gribov copies.

These two fundamental problems of noncommutative QED, although at a first glance have nothing to do with each other, share some similarities. The first hint that these two apparently unrelated issues share the same physical origin can be found in \cite{19}, \cite{20}, \cite{21} where the authors, adapting an interesting result in scalar field theory \cite{22}, argued that in order to cure the UV-IR mixing in noncommutative QED one may add the term

\[ S_{\text{fix}} \equiv \int d^4 x A_\mu \left( \frac{\tilde{\gamma}^2}{-\partial^2} \right) A_\mu \]  

(4.51)

to the classical U(1) action

\[ S_{\text{ph}} = A_\mu (-\partial^2) A_\mu, \]  

(4.52)

that leads to the propagator of the IR-UV *improved* theory

\[ G^{\text{imp}}(p) \sim \frac{p^2}{p^4 + \tilde{\gamma}^4}. \]  

(4.53)

This has precisely the structure of the Gribov-Zwanziger propagator introduced to eliminate Gribov copies in the Landau gauge. A less investigated problem is the problem of Gribov ambiguity. Indeed it has been shown \cite{4} that noncommutative QED similarly to commutative non-Abelian gauge theories, exhibits Gribov copies. For a review of noncommutative gauge theories see \cite{27} and refs. therein. However, in principle, in the noncommutative case the dimensional constant \( \tilde{\gamma} \) knows nothing about Gribov copies. Thus, unless one shows that noncommutative geometry induces Gribov copies also in the \( U(1) \) case, the above prescription to eliminate the UV/IR mixing would appear quite *ad hoc* and unnatural.

To this, let us choose the Landau gauge, \( \partial^\mu A_\mu = 0 \) and replace for \( A'_\mu \). The gauge condition \( \partial^\mu A'_\mu[\alpha] = 0 \) implies the equation of copies

\[ \partial^\mu D_\mu \alpha = 0 \]  

(4.54)

which is similar in form to the one obtained in the commutative case, Eq. (2.15) and may now have non trivial solutions, since the remark in footnote 3 does not apply. Let us show that, indeed, it has an infinite number of solutions.

On replacing the expression of the covariant derivative and the asymptotic form of the Moyal product in Eq. (4.54) we arrive at

\[ - \partial^2 \alpha + i A_\mu \exp \left\{ \frac{i}{2} \theta^{\rho\sigma} \partial_\rho \partial_\sigma \right\} (\partial^\mu \alpha) - i (\partial^\mu \alpha) \exp \left\{ \frac{i}{2} \theta^{\rho\sigma} \partial_\rho \partial_\sigma \right\} A_\mu = 0. \]  

(4.55)

The presence of nonlocal terms implies that, differently form QCD, this is not a differential equation and its resolution is a very hard task. However, in order to say whether we have Gribov copies or not we only need to understand whether it has nontrivial solutions \( \alpha \neq 0 \).
After some simple manipulation and upon Fourier transformation it is possible to recast Eq. (4.55) as a homogeneous Fredholm equation of second kind

$$\hat{\alpha}(k) = \int d^d q \ Q(q,k) \ \hat{\alpha}(q)$$

(4.56)

with the kernel $Q$ given by

$$Q(q,k) = \frac{-2i k^\mu \hat{A}_\mu(k-q)}{k^2} \sin \left( \frac{1}{2} \theta^{\rho\sigma} q_\rho k_\sigma \right)$$

(4.57)

The existence of Gribov copies has been reformulated into an eigenvalue equation for the operator $Q$. It is possible to show that the operator $Q$ is symmetric. In principle self adjoint operators have an infinite set of eigenfunctions and eigenvalues, however since we are in the infinite dimensional situation a lot depends on the properties of the kernel. For an analysis of this equation we refer the reader to [4]. Here we shall only exhibit specific gauge potentials for which this equation has solutions. To this, we notice that if we consider gauge potentials $\hat{A}_\mu$ which are proportional to derivatives of $\delta(k)$, Eq. (4.57) becomes a differential equation for $\hat{\alpha}(k)$.

### 4.2 The gauge invariant connection

First we try the following Ansatz

$$A_\mu = K \theta^{-1}_\mu x^\nu$$

(4.58)

with $K$ some constant to be fixed. This potential is easily verified to satisfy the gauge fixing condition $\partial^\mu A_\mu$. In order to get rid of trivial solutions we then look for solutions $\alpha(x)$ of (4.56) which belong to Schwarz space.

The Fourier transform reads

$$\hat{A}_\mu(k) = i K \theta^{-1}_\mu \partial^\nu \delta(k).$$

(4.59)

Substituting (4.59) in the equation (4.57) we arrive at

$$-2K k^\mu (\theta^{-1}_\mu)^{\nu}_{\rho} \int d^d q \ \sin \left( \frac{1}{2} \theta^{\rho\sigma} q_\rho k_\sigma \right) \hat{\alpha}(q) \partial^\nu \delta(k-q) = Q k^2 \hat{\alpha}(k),$$

(4.60)

namely the following algebraic equation

$$(1 + K) k^2 \hat{\alpha}(k) = 0,$$

(4.61)

which exhibits nontrivial solutions. Indeed if and only if $K = -1$, for arbitrary even space-time dimension, any arbitrary function $\hat{\alpha}(k)$ is a solution. Unfortunately, although we found nontrivial solutions of Eq. (4.57), this particular gauge potential has a peculiar feature. One may show [11] that it is invariant under gauge transformations (3.42) and therefore we do not have Gribov copies.

Nevertheless this potential is of interest. First of all, the existence of such a gauge invariant connection is a purely noncommutative feature [11] (also see [12] where such a connection has been used to study NCQED as a nonlocal matrix model) and does not exist in the commutative limit. Second, its smooth approximations may be used in principle to search solutions of the integral equation Eq. (4.57).
4.3 Next to the simplest situation

To simplify the presentation let us consider the two dimensional case. Here we have only one noncommutative parameter, $\theta_{12} = -\theta_{21} = \theta$. The next to the simplest gauge potential leading to a viable differential equation is the following one:

$$A_\mu(x) \propto \theta_\mu^1 x^\nu x^2,$$

which, being in two dimensions, can be further simplified to the form

$$A_\mu(x) = K \varepsilon_{\mu\nu} x^\nu x^2,$$

with $K$ some constant to be determined and $\varepsilon_{\mu\nu}$ the Levi-Civita tensor in two dimensions. It is easily seen to satisfy the Landau gauge fixing condition. The corresponding Fourier transform reads

$$\hat{A}_\mu(k) = i K \varepsilon_{\mu\nu} \Box \partial^\nu \delta(k).$$

On substituting in the integral equation Eq. (4.57) we obtain

$$K k^\mu \varepsilon_{\mu\nu} \int d^d q (\Box q \partial^\nu \delta(q - k)) \sin \left( \frac{1}{2} \theta^\rho q^\kappa k^\sigma \right) \hat{\alpha}(q) =$$

$$-K k^\mu \varepsilon_{\mu\nu} \Box q \partial^\nu \left[ \sin \left( \frac{1}{2} \theta^\rho q^\kappa k^\sigma \right) \hat{\alpha}(q) \right] = \frac{K \theta}{8} \left( \theta^2 k^4 \hat{\alpha} - 4k^2 \Box \hat{\alpha} - 8 \varepsilon^{\mu\nu\eta\lambda} k_\mu k_\eta \partial_\nu \partial_\lambda \hat{\alpha} \right)$$

hence the zero modes $\hat{\alpha}(k)$ have to satisfy the partial differential equation given below:

$$\left( -4k^2 \Box - 8 \varepsilon^{\mu\nu} \varepsilon_{\eta\lambda} k_\mu k_\eta \partial_\nu \partial_\lambda - \frac{4k^2}{Q^\theta} + \theta^2 k^4 \right) \hat{\alpha}(k) = 0.$$

Passing to polar coordinates $(r, \phi)$ with $k_1 = r \cos \phi, k_2 = r \sin \phi$, Eq. (4.66) reads

$$r^2 \hat{\alpha}_{rr} + 3r \hat{\alpha}_r + \frac{1}{r^2} \hat{\alpha} - \frac{\theta^2}{4} r^4 \hat{\alpha} + 3 \hat{\alpha}_{\phi\phi} = 0.$$

which can be solved by separation of variables. It is shown in [4] that Eq. (4.67) admits solutions when the amplitude $K$ takes one of the discrete values

$$K_{nm} = \frac{1}{\theta^2(\sqrt{3n^2 + 1} + 2m + 1)}, \quad n = 0, \pm 1, \pm 2, ..., \quad m = 0, 1, 2, ...$$

in such a case the general form of the zero modes is found to be [4]

$$\hat{\alpha}_{nm}(r, \phi) = (C_1 \cos(n\phi) + C_2 \sin(n\phi)) r^{\sqrt{3n^2 + 1} - 1} \exp \left( -\frac{r^2 \theta}{4} \right) L^\sqrt{3n^2 + 1}_m \left( \frac{\theta r^2}{2} \right)$$

where in order for $\alpha(x)$ to be real, $C_1, C_2$, are real if $n$ is even and $C_1, C_2$ are purely imaginary if $n$ is odd. $L^\alpha_m(z)$ are the generalized Laguerre polynomials. The four-dimensional case may be analysed by a similar procedure.
5 Discussion

Having generalized the QED action to the noncommutative case of Moyal type, we have studied the equation of Gribov copies and found for simple forms of the gauge potential, an infinite number of solutions. This is a genuine noncommutative effect, which disappears when $\theta \to 0$. The role played by the matrix $\theta^{\mu\nu}$ is similar to the introduction of a background curvature of space-time, whose effect for Abelian gauge theory has been already studied in relation to Gribov problem [24, 25]. To this respect, let us notice that $\theta^{\mu\nu}$ is precisely the curvature of the gauge invariant connection discussed in section 4.2, namely it behaves as a background field affecting space-time geometry. It has been suggested [4] that the problem could be dealt with by a modification of the propagator, as it is done for non-Abelian gauge theories in the Gribov-Zwanziger-dell’Antonio approach [26].

The problem shares some similarities with the UV/IR problem of noncommutative gauge theory [27]. Indeed, a propagator of the form of the Gribov-Zwanziger-dell’Antonio propagator has already been proposed [hand] in the NC field theory framework, emerging from the necessity of curing the IR/UV phenomenon in scalar translation invariant models on the Moyal plane [22] and it has later been argued (see [27] for an up to date review) that the same modification could be applied to NC gauge models, which are known to present the same kind of problem. Thus, the Gribov-Zwanziger restriction would solve, at the same time, the problem of the zero-modes of the noncommutative Faddeev-Popov operator and the UV/IR mixing, clarifying the common origin of both problems.

As a final remark, it is worth emphasizing that in the scalar case where, as we already recalled, the mixing is already present and cured through a modification of the action of the Gribov-Zwanziger type, there is actually a large local symmetry of the Moyal star product at work (see [28] for details) which might be responsible for the existence of copies and the demonstration could be done along the same lines as in previous sections. Thus, if the analysis of [23] can be extended to the noncommutative case, this local symmetry of the star product could be the explanation for the UV/IR mixing for the scalar case as well.

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