Quasi-weak equivalences in complicial exact categories

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Abstract
We introduce a notion of quasi-weak equivalences associated with weak-equivalences in an exact category. It gives us a delooping for (idempotent complete) exact categories and a condition that the negative $K$-group of an exact category becomes trivial.

1 Introduction

The negative $K$-theory $K(E)$ for an exact category $E$ is introduced in [6] and [7] by M. Schlichting. This generalizes the definition of Bass, Karoubi, Pedersen-Weibel, Thomason, Carter and Yao. The first motivation of our work is to investigate some vanishing conjectures of such negative $K$-groups:

(a) For any noetherian scheme $X$ of Krull dimension $d$, $K_{−n}(X)$ is trivial for $n > d$ ([11]).
(b) The negative $K$-groups of a small abelian category is trivial ([12]).
(c) For a finitely presented group $G$, $K_{−n}(\mathbb{Z}G) = 0$ for $n > 1$ ([13]).

In [17], it was given a description of $K_{−1}(E)$ and a condition on vanishing of $K_{−1}(E)$ for an (essentially small) exact category $E$ in terms of its unbounded derived category $D(E)$: We have $K_{−1}(E) = K_{0}(D(E))$ and $K_{−1}(E)$ is trivial if and only if $D(E)$ is idempotent complete (= Karoubian in the sense of [10], A.6.1). To extend this observation, we shall introduce the notion of higher derived categories $D_{n}(E)$ and show the following theorem:

Theorem 1.1 (Cor. [14]). For an idempotent complete exact category $E$, we have $K_{−n}(E) = K_{0}(D_{n}(E))$. Moreover, $K_{−n}(E)$ is trivial if and only if $D_{n}(E)$ is idempotent complete.
Although we limit our consideration to idempotent complete exact categories to avoid some technical difficulties, the exact categories in the conjectures (a)-(c) above satisfies this condition. Recall that the derived category $D(\mathcal{E})$ is the triangulated category obtained by formally inverting quasi-isomorphisms in the category of chain complexes $\text{Ch}(\mathcal{E})$. The pair $(\text{Ch}(\mathcal{E}), \text{qis})$ of the category of chain complexes $\text{Ch}(\mathcal{E})$ and qis the class of quasi-isomorphisms forms a complicial exact category with weak equivalences (cf. Def. 3.1). More generally, for a complicial exact category with weak equivalences $\mathcal{E} = (\mathcal{E}, w)$ (cf. Def. 3.3), we define a class of weak equivalences $qw$ in the category of chain complexes $\text{Ch}(\mathcal{E})$, which is called quasi-weak equivalences associated with $w$. If $w$ is the class of isomorphisms in $\mathcal{E}$, then $qw$ is just the class of quasi-isomorphisms on $\text{Ch}(\mathcal{E})$. The derived category $D(\mathcal{E})$ of $\mathcal{E}$ is obtained by formally inverting the quasi-weak equivalences in $\text{Ch}(\mathcal{E})$. Put $\text{Ch}(\mathcal{E}) = (\text{Ch}(\mathcal{E}), qw)$ and one can define the class weak equivalences in $\text{Ch}_n(\mathcal{E}) := \text{Ch}(\text{Ch}_{n-1}(\mathcal{E}))$ inductively. The $n$-th derived category $D_n(\mathcal{E})$ associated with $\mathcal{E}$, is the derived category of $\text{Ch}_n(\mathcal{E})$. We also obtain the following theorems on the negative $K$-theory $K(\mathcal{E})$ for $\mathcal{E}$ (for definition, see [8]):

**Theorem 1.2** (Thm. 4.2). Assume that $\mathcal{E}$ is idempotent complete. Then we have:

(i) (Gillet-Waldhausen theorem) $K(\mathcal{E}) \xrightarrow{\sim} K(\text{Ch}^0(\mathcal{E}))$,

(ii) (Eilenberg swindle) $K(\text{Ch}^+(\mathcal{E})) \xrightarrow{\sim} K(\text{Ch}^-(\mathcal{E})) \xrightarrow{\sim} 0$,

(iii) (Delooping) $K(\text{Ch}(\mathcal{E})) \xrightarrow{\sim} \Sigma K(\mathcal{E})$, where $\Sigma$ is a suspension functor on the stable category of spectra.

The organization of this note is as follows: In Section 2, we list several axioms about weak equivalences in a category with cofibrations and study their implication. In Section 3, after recalling the definition of complicial exact category with weak equivalences, we consider the notion of null classes and investigate the relation to weak equivalences in a complicial exact category. In Section 4, we introduce the quasi-weak equivalences as noted above which is a class of weak equivalences in the exact category of chain complexes $\text{Ch}(\mathcal{E})$ associated with a given weak equivalences $w$ in an exact category $\mathcal{E}$. By using this, we prove the main theorem. Throughout this note, we follow basically the terminologies on algebraic $K$-theory in [10] and [8].

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2 Weak equivalences in categories with cofibrations

In this section, we list several axioms on weak equivalences in a category with cofibrations and study their implications. Let $\mathcal{C}$ be a category with cofibrations and $w$ be a class of morphisms in $\mathcal{C}$. We denote by $\text{Ar}(\mathcal{C})$ the category of arrows $\mathcal{C} \to \mathcal{C}$. The functors $\text{dom}, \text{ran} : \text{Ar}(\mathcal{C}) \to \mathcal{C}$ are defined by $\text{dom}(f) = x$ and $\text{ran}(f) = y$ respectively, for $f : x \to y \in \text{Ar}(\mathcal{C})$.

First, we consider the following axioms on $w$:

(WE 1) Every isomorphisms in $\mathcal{C}$ is in $w$.

(WE 2) For composable morphisms $f$ and $g$ in $\mathcal{C}$, if two of $f$, $g$ and $gf$ are in $w$, then the other one is also in $w$.

(WE 3) For a commutative diagram in $\mathcal{C}$,

$$
\begin{array}{ccc}
x & \rightarrow & y \\
\downarrow^a & & \downarrow^b \\
x' & \rightarrow & y'
\end{array}
\begin{array}{ccc}
y' & \rightarrow & z' \\
\downarrow^c
\end{array}
$$

where the horizontal lines are cofibration sequences, if $a$ and $c$ are in $w$, then so is $b$.

(WE 3)' For the commutative diagram (1) in $\mathcal{C}$, if $a$ and $b$ are in $w$, then so is $c$.

(WE 4) For a commutative diagram in $\mathcal{C}$,

$$
\begin{array}{ccc}
y & \leftarrow & x \\
\downarrow^a & & \downarrow^b \\
y' & \leftarrow & x'
\end{array}
\begin{array}{ccc}
x' & \rightarrow & z' \\
\downarrow^c
\end{array}
$$

$$
\begin{array}{ccc}
\rightarrow & f & \rightarrow \\
\downarrow^i & \downarrow & \downarrow^c \\
\rightarrow & \rightarrow & \rightarrow
\end{array}
\begin{array}{ccc}
y' & \leftarrow & z' \\
\downarrow^f
\end{array}
$$
where $i$ and $i'$ are cofibrations, if $a$, $b$ and $c$ are in $w$, then the induced map $a \sqcup_b c : y \sqcup_x z \to y' \sqcup_{x'} z'$ on pushouts is also cofibration.

(WE 5) For any cofibration $x \to y$ in $C$ and $x \to z$ in $w$, the induced morphism $y \to y \sqcup_x z$ is in $w$.

(WE 6) (Factorization axiom) There is a functor $\text{Cyl} : \text{Ar}C \to C$ and, natural transformations $\alpha : \text{dom} \to \text{Cyl}$ and $\beta : \text{Cyl} \to \text{ran}$ such that for any morphism $f : x \to y$ in $C$, $\alpha(f) : x \to \text{Cyl}(f)$ is a cofibration, $\beta(f) : \text{Cyl}(f) \to y$ is in $w$ and $\beta(f)\alpha(f) = f$.

(WE 7) For a commutative diagram

\[
\begin{array}{c}
x \xrightarrow{i} y \\
\downarrow a \quad \quad \downarrow b \\
x' \xrightarrow{i'} y'
\end{array}
\]

with retractions $p : y \to x$ and $p' : y \to x'$ such that $pi = \text{id}_x$ and $p'i' = \text{id}_{x'}$, if $b$ is in $w$, so is $a$.

Note that the axiom (WE 2) implies that the class $w$ is closed under composition. Next we study logical relations among these axioms as follows:

**Proposition 2.1** (cf. [3]; [2], 8.8). (i) (WE 1) and (WE 3) imply (WE 5).

(ii) (WE 2), (WE 5) and (WE 6) imply (WE 4).

(iii) (WE 1) and (WE 4) imply (WE 3)'.$$

**Proof.** (i) Let $i : x \to y$ be a cofibration in $C$ and $a : x \to x'$ in $w$. It is easy to see that the sequence $x' \to y' := x' \sqcup_x y \to x/y$ is a cofibration sequence. Now applying (WE 3) to the following commutative diagram

\[
\begin{array}{c}
x \xrightarrow{i} y \quad x/y \\
\downarrow a \quad \downarrow b \quad \downarrow \text{id}_{x/y} \\
x' \xrightarrow{i'} y' \quad x/y
\end{array}
\]

we learn that $b : y \to y'$ is also in $w$. 

(iii) Now we consider the following commutative diagram in $C$:

$$
\begin{array}{c}
x \to y \\
a \downarrow \quad b \downarrow \\
x' \to y'
\end{array} \quad \begin{array}{c}
z \\
c
\end{array} \quad \begin{array}{c}
x' \to y' \\
z'
\end{array}
$$

where the horizontal lines are cofibration sequences, and $a$ and $b$ are in $w$. Note that the map $0 \to 0$ is in $w$ by (WE 1). For the commutative diagram (11) in $C$, we assume that $a$ and $b$ are in $w$. The following diagrams are coCartesian:

$$
\begin{array}{c}
x \to y \\
\downarrow \\
0 \to z
\end{array} \quad \begin{array}{c}
x' \to y' \\
\downarrow \\
0 \to z'
\end{array}
$$

By (WE 4), the map $c = 0 \sqcup_a b$ is in $w$.

(ii) Consider the commutative diagram (1) in (WE 4). First we assume the lemma below we intend to conclude the result.

**Lemma 2.2.** Let us assume the axioms (WE 2) and (WE 5). In the diagram (2), suppose that both $f$ and $f'$ are cofibrations or in $w$. Then, $a \sqcup_b c : y \sqcup_x z \to y' \sqcup_{x'} z'$ is in $w$.

Applying (WE 6) to the diagram (2), we get the following diagrams

$$
\begin{array}{c}
y \xleftarrow{i} x \xrightarrow{\alpha(f)} \text{Cyl}(f) \xrightarrow{\beta(f)} z \\
\downarrow \quad \downarrow \quad \downarrow \\
y' \xleftarrow{i'} x' \xrightarrow{\alpha(f')} \text{Cyl}(f') \xrightarrow{\beta(f')} z'
\end{array}
$$

$$
\begin{array}{c}
y \sqcup_x \text{Cyl}(f) \xleftarrow{c} \text{Cyl}(f) \xrightarrow{\beta(f)} z \\
\downarrow \quad \downarrow \quad \downarrow \\
y' \sqcup_{x'} \text{Cyl}(f') \xleftarrow{c} \text{Cyl}(f') \xrightarrow{\beta(f')} z'
\end{array}
$$
By Lemma 2.2 we learn that $e$ is in $w$. Again using Lemma 2.2, we learn that $e \sqcup_d c$ is in $w$. Thus by Lemma 2.3 (i) below, we notice that $e \sqcup_d c$ is $a \sqcup_b c$. □

Lemma 2.3. Let us consider a commutative diagram below in a category.

(i) If the diagrams I and II are coCartesian squares, then the diagram I + II is also.

(ii) If the diagram I + II and I are coCartesian squares, then the diagram II is also.

Proof of Lemma 2.2. First let us assume that both $f$ and $f'$ in the diagram (2) are in $w$. Then in the diagram below

\[
g : y \rightarrow y \sqcup_x z,
\]

\[
d : y' \rightarrow y' \sqcup_{x'} z'.
\]

$g$ and $g'$ are in $w$ by (WE 5). Therefore by (WE 2), $a \sqcup_b c$ is also. Next let us suppose that $f$ and $f'$ in the diagram (2) are cofibrations in $C$. Consider the following diagram:
Claim. The commutative diagram below is coCartesian.

\[
\begin{array}{c}
x' \sqcup_x z \\
\downarrow \\
y' \sqcup_y y \sqcup_x z \\
\downarrow \\
y' \sqcup_{x',} z'
\end{array}
\]

Proof. Consider the following commutative diagram:

\[
\begin{array}{c}
x' \\
\downarrow I \\
y' \sqcup_y y \sqcup_x z \\
\downarrow \\
y' \sqcup_{x',} z'
\end{array}
\]

The squares I + II and I are coCartesian. Therefore by Lemma \[\text{2.3} (\text{ii})\], the diagram II is also coCartesian. \(\square\)

By (WE 5), \(y \sqcup_x z \to y' \sqcup_y y \sqcup_x z\) and \(z \to x' \sqcup_x z\) are in \(w\). Therefore by (WE 2), \(x' \sqcup_x z \to z'\) is in \(w\). Hence by (WE 5) again, \(y' \sqcup_y y \sqcup_x z \to y' \sqcup_{x',} z'\) is in \(w\). Finally by (WE 2), the composition \(a \sqcup_b c : y \sqcup_x z \to y' \sqcup_y y \sqcup_x z \to y' \sqcup_{x',} z'\) is in \(w\). \(\square\)

In a category with fibrations \(C\) which is the dual concept of categories with cofibrations, we consider the following axioms:

(WE 3)\(^{op}\) For the commutative diagram (1) in \(C\), if \(b\) and \(c\) are in \(w\), then so is \(a\).

(WE 4)\(^{op}\) For a commutative diagram in \(C\)

\[
\begin{array}{c}
\downarrow a \\
y' \downarrow \quad \downarrow b \\
x' \sqcup_{x'} z'
\end{array}
\]

where \(p\) and \(p'\) are fibrations, if \(a\), \(b\) and \(c\) are in \(w\), then \(a \times_b c : y \times_x z \to y' \times_{x'} z'\) is also.

(WE 5)\(^{op}\) \(w\) is stable under base change by fibrations. That is, for any fibration \(y \to x\) in \(C\) and \(z \to x\) in \(w\), the induced morphism \(y \times_x z \to y\) is in \(w\).
There are a functor $M : \text{Ar} \mathcal{C} \to \mathcal{C}$ and natural transformations $\gamma : \text{dom} \Rightarrow M$ and $\delta : M \Rightarrow \text{ran}$ such that for any morphism $f : x \to y$ in $\mathcal{C}$, $\gamma(f) : x \to M(f)$ is in $\mathcal{W}$, $\delta(f) : M(f) \to y$ is a fibration and $\delta(f)\gamma(f) = f$.

Thus, one can establish the dual statement of Proposition 2.1. In particular, since an exact category $\mathcal{E}$ is an additive category with bifibrations, we can apply $\mathcal{E}$ to Proposition 2.1 and its dual statement.

3 Complicial exact category

We recall the theory of complicial exact categories with weak equivalences following [8]. Let $\text{Ch}^b(\mathbb{Z})$ be the exact category of bounded chain complexes of finitely generated free $\mathbb{Z}$-modules. Its exact structure is given by the degree-wise split sequences. There is a symmetric monoidal tensor product $\otimes : \text{Ch}^b(\mathbb{Z}) \times \text{Ch}^b(\mathbb{Z}) \to \text{Ch}^b(\mathbb{Z})$ which extends the usual tensor product of free $\mathbb{Z}$-modules defined by $(a \otimes b)^n := \bigoplus_{i+j=n} a^i \otimes b^j$. Its differential $d^n : (a \otimes b)^n \to (a \otimes b)^{n+1}$ is

$$d^n = i^n \otimes \text{id}_b + (-1)^j \text{id}_a \otimes d^n_b + i^n \otimes d^n_b : a^i \otimes b^j \to (a^{i+1} \otimes b^j) \oplus (a^i \otimes b^{j+1})$$

on $a^i \otimes b^j \subset (a \otimes b)^n$ ($i + j = n$). The unit is the chain complex $\mathbb{1}$ which is $\mathbb{Z}$ in degree 0 and 0 elsewhere. The complex $C$ is $\mathbb{Z}$ in degrees 0 and $-1$ and is 0 otherwise. The only non-trivial differential is $d^{-1} = \text{id}_z$. The complex $T$ is $\mathbb{Z}$ in degree $-1$ and 0 elsewhere. Note that we have a conflation of chain complexes $\mathbb{1} \rightarrow C \rightarrow T$.

**Definition 3.1** ([8], Def. 6.2). An exact category $\mathcal{E}$ is said to be *complicial* if it is equipped with a bi-exact action $\otimes : \text{Ch}^b(\mathbb{Z}) \times \mathcal{E} \to \mathcal{E}$ of the symmetric monoidal category $\text{Ch}^b(\mathbb{Z})$ on $\mathcal{E}$. For an object $x \in \mathcal{E}$, put $Cx := C \otimes x$ and $Tx := T \otimes x$. A sequence $x \rightarrow Cx \rightarrow Tx$ forms a conflation.

For any morphism $f : x \to y$ in a complicial exact category $\mathcal{E}$, the *cone* $\text{Cone}(f)$ is defined by the push-out of $f$ along the inflation $x \rightarrow Cx$. We call $	ext{Cyl}(f) := y \oplus Cx$ the *cylinder* of $f$. These make the following conflations: $y \rightarrow \text{Cone}(f) \rightarrow Tx$, and $x \rightarrow \text{Cyl}(f) \rightarrow \text{Cone}(f)$. We associate a triangulated category $\mathcal{E}$ called the *stable category* for a complicial exact category $\mathcal{E}$ as follows: An inflation $i : x \rightarrow y$ in $\mathcal{E}$ is called a *Frobenius inflation* if for any object $u$ and a morphism $f : x \rightarrow Cu$ in $\mathcal{E}$, there is a morphism
$g : y \to Cu$ such that $f = gi$. A deflation $p : x \to y$ in $\mathcal{E}$ is called a Frobenius deflation if for any object $u$ and a morphism $f : Cu \to y$ in $\mathcal{E}$, there is a morphism $g : Cu \to Cx$ such that $f = pg$. The category $\mathcal{E}$ endowed with the class of Frobenius inflations and Frobenius deflations is a Frobenius exact category. That is, it has enough projective and injective objects and the class of projective objects and the injective objects coincide. An object $x$ is a projective-injective object if and only if it is a direct summand of $Cu$ for some object $u$ in $\mathcal{E}$ ([8], Lem. B.16). For any morphism $f : x \to y$ in $\mathcal{E}$, $x \mapsto Cx$ is a Frobenius inflation, thus the conflations $x \mapsto Cx \to Tx$ and $x \mapsto \text{Cyl}(f) \to \text{Cone}(f)$ are Frobenius conflations. From now on, we always consider the complicial exact category $\mathcal{E}$ as the Frobenius category. Recall that two maps $f, g : x \to y$ in $\mathcal{E}$ are called homotopic, if their difference factors through a projective-injective object.

**Lemma 3.2.** For any object $x$ in $\mathcal{E}$, $Cx = C \otimes x$ is contractible.

**Proof.** In $\text{Ch}^b(\mathbb{Z})$, the identity map $C \to C$ factor through $CC$. Hence $Cx$ is contractible. \hfill $\square$

The stable category $\mathcal{E}_\ast$ of the Frobenius category $\mathcal{E}$ is the category whose objects are the objects of $\mathcal{E}$ and whose morphisms are the homotopy classes of maps in $\mathcal{E}$. It is known that the stable category $\mathcal{E}_\ast$ is a triangulated category. Distinguished triangles in $\mathcal{E}$ are those triangles which are isomorphic in $\mathcal{E}$ to sequences of the form

$$x \xrightarrow{f} y \to \text{Cone}(f) \to Tx$$

for $f : x \to y$ in $\mathcal{E}$.

**Definition 3.3** ([8], Def. 6.9). A class of morphisms $w$ in a complicial exact category $\mathcal{E}$ is called a class of weak equivalences if $w$ satisfies the following conditions:

(WE 0) The tensor product preserves weak equivalences in both variables, that is, if $f$ is a homotopy equivalence in $\text{Ch}^b(\mathbb{Z})$ and $g$ is in $w$, then $f \otimes g$ is in $w$,

(WE 1)' Every homotopy equivalence is in $w$,

(WE 2), (WE 3) and (WE 7) in the last section, namely, $w$ satisfies the 2 out of 3 and and is closed under extensions and retracts.
We denote the class of all classes of weak equivalences by \( \text{WE}(\mathcal{E}) \). For a class of weak equivalences \( w \) in \( \mathcal{E} \), we say that the pair \( \mathcal{E} := (\mathcal{E}, w) \) is a complicial exact category with weak equivalences.

Every class of weak equivalences \( w \) in a complicial exact category \( \mathcal{E} \) satisfies (\text{WE 6}). If fact, for a morphism \( f : x \to y \) in \( \mathcal{E} \), we have a conflation \( x \twoheadrightarrow \text{Cyl}(f) \to \text{Cone}(f) \). By Lemma 3.2 \( \text{Cyl}(f) = Cx \oplus y \) and \( y \) are homotopy equivalence. Hence (\text{WE 1}') implies (\text{WE 6}). Proposition 2.1 says that \( w \) satisfies (\text{WE 4}) and (\text{WE 5}). Furthermore, it is easy to verify that \( w \) satisfies (\text{WE 4})^{\text{op}} - (\text{WE 6})^{\text{op}}. An object \( x \) in \( \mathcal{E} \) is said to be \( w \)-trivial if the canonical map \( 0 \to x \) is in \( w \). Note that by the axioms (\text{WE 1}') and (\text{WE 2}), this condition is equivalent to the condition that the canonical map \( x \to 0 \) is in \( w \). We denote the class of \( w \)-trivial objects by \( \mathcal{E}^w \) and sometimes consider \( \mathcal{E}^w \) as the full subcategory of \( \mathcal{E} \) of the \( w \)-trivial objects.

**Lemma 3.4.** A morphism \( f : x \to y \) in \( \mathcal{E} \) is in \( w \) if and only if its mapping cone \( \text{Cone}(f) \) is in \( \mathcal{E}^w \).

**Proof.** Let us consider the following diagram:

\[
\begin{array}{ccc}
x & \xrightarrow{\alpha(f)} & \text{Cyl}(f) \\
| & & | \\
f & & \beta(f) \\
y & \xrightarrow{\text{id}_y} & y \\
& & \downarrow \\
& & 0 
\end{array}
\]

where \( \beta(f) \) is in \( w \) by (\text{WE 6}). Assume that \( f \) is in \( w \). Then applying (\text{WE 4}) to the diagram above, we learn that \( \text{Cone}(f) \) is \( w \)-trivial. Next suppose that \( \text{Cone}(f) \) is \( w \)-trivial. Then applying (\text{WE 4})^{\text{op}} to the diagram above, we learn that \( f \) is in \( w \).

Next we introduce a null class associated with a class of weak equivalences in a complicial exact categories and establish a bijective correspondence between null classes and classes of weak equivalences in the complicial exact category. This is an analogue of a bijective correspondence between thick subcategories and localizing systems in a triangulated category.

**Definition 3.5.** An additive full subcategory \( \mathcal{N} \) of \( \mathcal{E} \) is called a null class if it satisfies the following axioms:
(NC 1) If \( x \) is an object in \( \mathcal{N} \) and \( y \) is an object in \( \mathcal{E} \) which is homotopy equivalent to \( x \), then \( y \) is also in \( \mathcal{N} \).

(NC 2) For \( a \in \text{Ch}^b(\mathbb{Z}) \) and \( x \in \mathcal{N} \), we have \( a \otimes x \in \mathcal{N} \).

(NC 3) For any conflation \( x \rightarrow y \rightarrow z \) in \( \mathcal{E} \), if \( x, z \) are in \( \mathcal{N} \), so is \( y \).

(NC 7) If there are maps \( i : x \rightarrow y \) and \( p : y \rightarrow x \) with \( pi = \text{id}_x \) and \( y \in \mathcal{N} \), then \( x \in \mathcal{N} \).

We denote the class of all null classes on \( \mathcal{E} \) by \( \text{NC}(\mathcal{E}) \).

For any null class \( \mathcal{N} \) on \( \mathcal{E} \), by (NC 3), it becomes an exact category in the natural way and by (NC 2), it is complicial. As in the last section, we can consider the stable category \( \overline{\mathcal{N}} \) of \( \mathcal{N} \) which is a full triangulated subcategory of \( \mathcal{E} \).

**Lemma 3.6.** Let \( \mathcal{E} \) be a complicial exact category.

(i) For any class of weak equivalences \( w \) on \( \mathcal{E} \), the class of \( w \)-trivial objects \( \mathcal{E}^w \) is a null class.

(ii) Conversely, for any null class \( \mathcal{N} \) on \( \mathcal{E} \), let us define the class of morphisms \( w_{\mathcal{N}} \) by

\[
w_{\mathcal{N}} := \{ f \in \text{Mor} \mathcal{E} \mid \text{Cone}(f) \in \mathcal{N} \}.
\]

Then \( w_{\mathcal{N}} \) is a class of weak equivalences.

(iii) For the associations \( w \mapsto \mathcal{E}^w \) and \( \mathcal{N} \mapsto w_{\mathcal{N}} \) gives a bijective correspondence between \( \text{WE}(\mathcal{E}) \) and \( \text{NC}(\mathcal{E}) \).

**Proof.** (i) Since 0 is \( w \)-trivial, it is in \( \mathcal{E}^w \). In particular \( \mathcal{E}^w \) is non-empty.

(NC 1) Let \( x \) be a \( w \)-trivial object in \( \mathcal{E} \) and \( f : x \rightarrow y \) is a homotopy equivalent. Then by (WE 1)', \( f \) is in \( w \). Therefore by (WE 2), \( y \rightarrow 0 \) is in \( w \).

(NC 2) Let \( x \) be a \( w \)-trivial object in \( \mathcal{E} \) and \( a \) an object in \( \text{Ch}^b(\mathbb{Z}) \). Since the end-functor \( a \otimes ? \) on \( \text{Ch}^b(\mathbb{Z}) \) is additive, there is a canonical isomorphism \( 0 \cong a \otimes 0 \). Since \( \otimes \) preserves \( w \) (WE 0), the canonical morphism \( a \otimes x \rightarrow a \otimes 0 \) is in \( w \). Therefore by (WE 2), we learn that \( a \otimes x \rightarrow 0 \) is in \( w \).

(NC 3) In the diagram below

\[
\begin{array}{ccc}
x & \rightarrow & y \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\rightarrow & z \\
\downarrow & \\
0 & \rightarrow & 0 \\
\end{array}
\]
assume that $x$ and $z$ are in $\mathcal{E}^w$. Then by (WE 3), $y$ is also in $\mathcal{E}^w$.

(WE 3) Assume that there are maps $i : x \to y$ and $p : y \to x$ with $pi = id_x$ and $y \in \mathcal{E}^w$. By (WE 7), we have $x \in \mathcal{E}^w$.

(ii) (WE 0) Let $f : a \to b$ be a homotopy equivalence in $\text{Ch}^b(\mathbb{Z})$ and $g : x \to y$ in $w_N$. Note that $f \otimes id_y$ is a homotopy equivalence in $\mathcal{E}$. By (NC 1), we have $\text{Cone}(f \otimes id_y)$ in $\mathcal{N}$. From the isomorphism $\text{Cone}(id_a \otimes g) \cong a \otimes \text{Cone}(g)$ and (NC 2), we have $\text{Cone}(id_a \otimes g) \in \mathcal{N}$. From the equality $f \otimes g = (id_a \otimes g)(f \otimes id_y)$, there is a distinguished triangle in $\mathcal{E}$

$$\text{Cone}(id_a \otimes g) \to \text{Cone}(f \otimes g) \to \text{Cone}(f \otimes id_y) \rightarrow$$

by the octahedral axiom. Since $\mathcal{N}$ is triangulated, $\text{Cone}(f \otimes g)$ in $\mathcal{N}$. Then by (NC 1), it is in $\mathcal{N}$.

(WE 1)' Let $f : x \to y$ be a homotopy equivalence in $\mathcal{E}$. Then $\text{Cone}(f)$ is contractible (that is, $\text{Cone}(f) \to 0$ is a homotopy equivalence). Therefore by (NC 1), $\text{Cone}(f)$ is in $\mathcal{N}$.

(WE 2) Let us consider morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathcal{E}$. Then by the octahedral axiom, we have a distinguished triangle in $\mathcal{E}$

$$\text{Cone}(f) \to \text{Cone}(g) \to \text{Cone}(gf) \rightarrow$$

If we assume two of $f$, $g$ and $gf$ are in $w_N$, then their mapping cones are in $\mathcal{N}$. Since $\mathcal{N}$ is triangulated, the mapping cone of the third one is also in $\mathcal{N}$. Then the assertion follows from (NC 1).

(WE 3) For the commutative diagram (1)

$$\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow{a} & & \downarrow{b} \\
  x' & \xrightarrow{f'} & y'
\end{array} \begin{array}{ccc}
  y & \xrightarrow{g} & z \\
  \downarrow{b} & & \downarrow{c} \\
  y' & \xrightarrow{g'} & z'
\end{array}$$

in $\mathcal{E}$. We have a conflation in $\mathcal{E}$ $\text{Cone}(a) \rightarrow \text{Cone}(b) \rightarrow \text{Cone}(c)$. If we assume $\text{Cone}(a)$ and $\text{Cone}(c)$ are in $\mathcal{N}$, then by (NC 3), $\text{Cone}(b)$ is also in $\mathcal{N}$.

(WE 7) For the commutative diagram (3), with $b \in w_N$. We have a retraction $\text{Cone}(a) \rightarrow \text{Cone}(b)$ with $\text{Cone}(b) \in \mathcal{N}$. Therefore $\text{Cone}(a) \in \mathcal{N}$ by (NC 7).
(iii) For any class of weak equivalences \( w \) on \( \mathcal{E} \), by Lemma 3.4 we have
\[
w_{\mathcal{E}^w} = \{ f \in \text{Mor} \mathcal{E} \mid \text{Cone}(f) \in \mathcal{E}^w \} = w.
\]
For any null class \( \mathcal{N} \) on \( \mathcal{E} \), we have \( \mathcal{E}^w_{\mathcal{N}} = \{ x \in \text{Ob} \mathcal{E} \mid Tx = \text{Cone}(x \to 0) \in \mathcal{N} \} = \mathcal{N} \), where the last equality follows from (NC 2).

Let \( \mathcal{E} = (\mathcal{E}, w) \) be a complicial exact category with weak equivalences. Since the \( w \)-trivial objects \( \mathcal{E}^w \) is also a complicial in \( \mathcal{E} \), we have its stable category \( \mathcal{E}^w \) which is a full triangulated subcategory of \( \mathcal{E} \). The two Frobenius categories \( \mathcal{E} \) and \( \mathcal{E}^w \) have the same injective-projective objects. The inclusion \( \mathcal{E}^w \subseteq \mathcal{E} \) induces a fully faithful triangulated functor \( \mathcal{E}^w \to \mathcal{E} \) (6.15). The triangulated category \( \mathcal{T}(\mathcal{E}) \) associated with \( \mathcal{E} \) is the Verdier quotient \( \mathcal{T}(\mathcal{E}) = \mathcal{E} / \mathcal{E}^w \). The distinguished triangles in \( \mathcal{T}(\mathcal{E}) \) are triangles which are isomorphic to triangles of the form \( (\mathcal{E}, \mathcal{N}) \). The canonical projection \( \pi : \mathcal{E} \to \mathcal{T}(\mathcal{E}) \) from the Frobenius category \( \mathcal{E} \) induces an isomorphism \( w^{-1} \mathcal{E} \cong \mathcal{T}(\mathcal{E}) \), where \( w^{-1} \mathcal{E} \) is obtained from \( \mathcal{E} \) by formally inverting the weak equivalences (cf. [8], Def. 6.17). Note that \( \pi \) sends a (Frobenius) conflation to a distinguished triangle in \( \mathcal{T}(\mathcal{E}) \).

**Lemma 3.7** ([8], Exerc. 6.16, 6.18). (i) \( \mathcal{E}^w \) is closed under retract in \( \mathcal{E} \). In particular, objects of \( \mathcal{E} \) which are isomorphic to object of \( \mathcal{E}^w \) in \( \mathcal{E} \) are already in \( \mathcal{E}^w \).
(ii) A morphism \( f : x \to y \) in \( \mathcal{E} \) is a weak equivalence if and only if \( \pi(f) \) is an isomorphism in \( \mathcal{T}(\mathcal{E}) \).

**Proof.** (i) Let \( a \) be an object in \( \mathcal{E}^w \) and \( x \) an object in \( \mathcal{E} \). Assume that there are morphisms \( p : a \to x \) and \( i : x \to a \) such that \( pi = \text{id}_x \) in \( \mathcal{E} \). Hence we have a morphism \( H : Cx \to x \) such that \( H\iota_x = pi - \text{id}_x \) in \( \mathcal{E} \), where \( \iota_x : x \to Cx \) is the inflation. Then there are morphisms
\[
x \xrightarrow{(i \iota_x)} a \oplus Cx \xrightarrow{(p H)} x
\]
such that \( (p H)(i \iota_x) = pi - H\iota_x = pi - (pi - \text{id}_x) = \text{id}_x \). Since \( \mathcal{E}^w \) is closed under homotopy equivalences, we have \( a \oplus Cx \) is in \( \mathcal{E}^w \) and therefore \( x \) is in \( \mathcal{E}^w \).

(ii) By Lemma 3.4 \( f \) is in \( w \) if and only if \( \text{Cone}(f) \) is in \( \mathcal{E}^w \). Since \( \mathcal{E}^w \) is closed under homotopy equivalences, this condition is equivalent to that in \( \mathcal{E} \). \( \text{Cone}(f) \) is in \( \mathcal{E}^w \). By (i), \( \mathcal{E}^w \) is thick in \( \mathcal{E} \). Therefore this condition is
equivalent to that $\pi(\text{Cone}(f))$ is isomorphic to 0 in $\mathcal{T}(E)$. Since there is a distinguished triangle $x \xrightarrow{\pi(f)} y \rightarrow \text{Cone}(f) \xrightarrow{+1}$ in $\mathcal{T}(E)$, the final condition is equivalent to that $\pi(f)$ is an isomorphism in $\mathcal{T}(E)$.

If a full subcategory $\mathcal{N}$ in $\mathcal{E}$ satisfies (NC 2) and (NC 3), $\mathcal{N}$ is also complicial, and the image of $\mathcal{N}$ in the derived category $\mathcal{T}(E)$ by the canonical projection $\pi: \mathcal{E} \rightarrow \mathcal{T}(E)$ is a triangulated subcategory. We define the $w$-closure $\mathcal{N}_w$ of $\mathcal{N}$ by the kernel of the composition $\mathcal{E} \xrightarrow{\pi} \mathcal{T}(E) \rightarrow \mathcal{T}(E)/\pi(\mathcal{N})$; the full subcategory of $\mathcal{E}$ whose objects are isomorphic to 0 in $\mathcal{T}(E)/\pi(\mathcal{N})$.

4 Quasi-weak equivalences

Let $E = (\mathcal{E}, w)$ be an idempotent complete exact category with weak equivalences. We shall define a class of weak equivalences $qw$ in the category of chain complexes $\text{Ch}^\#(E)$ ($\# \in \{b, +, -, \emptyset\}$) as follows: The quasi-isomorphisms closure of $\text{Ch}^b(E_w)$ in $\text{Ch}^\#(E)$ is denoted by $\text{Ac}^\#_w(E)$ which is a null class in $\text{Ch}^\#(E)$. The objects in $\text{Ac}^\#_w(E)$ are called $w$-acyclic complexes. The class of weak equivalences associated with the null class $\text{Ac}^\#_w(E)$ is denoted by $qw$ (Lem. 3.6) which is called the class of quasi-weak equivalences. The category of chain complexes with quasi-weak equivalences $\text{Ch}^\#(E) = (\text{Ch}^\#(E), qw)$ forms a complicial exact category with weak equivalences. The null class $\text{Ac}^\#_w(E)$ is the qis-closure of $\text{Ch}^b(E_w)$. Hence $qw$ contains qis the class of quasi-isomorphisms in $\text{Ch}^\#(E)$.

**Lemma 4.1.** Assume that $\mathcal{E}$ is complicial.

(i) The inclusion functor $i: \mathcal{E} \rightarrow \text{Ch}^\#(E)$ sends a weak equivalence to a quasi-weak equivalence.

(ii) The class $qw$ contains chain maps $f: x \rightarrow y$ in $\text{Ch}^b(E)$ which are degree-wise weak equivalences.

**Proof.** (i) Let $f: x \rightarrow y$ be a weak-equivalence in $w$. Note that the mapping cone $\text{Cone}(f)$ in $\mathcal{E}$ is in $\mathcal{E}^w$ (Lem. 3.4). Now we have a biCartesian square

\[
\begin{array}{ccc}
x & \xrightarrow{a} &Cx \\
f \downarrow & & \downarrow p \\
y & \xrightarrow{b} & \text{Cone}(f)
\end{array}
\]

(5)
The mapping cone of \( i(f) \) in \( \text{Ch}^\#(\mathcal{E}) \) is \( \text{Cone}(i(f)) = \cdots \to 0 \to x \xrightarrow{f} y \to 0 \to \cdots \). On the other hand, we have a complex \( z := \cdots \to 0 \to Cx \xrightarrow{p} \text{Cone} f \to 0 \to \cdots \) in \( \text{Ch}^b(\mathcal{E}^w) \). The above diagram \((5)\) gives a chain map \( \phi : \text{Cone}(i(f)) \to z \). Its mapping cone in \( \text{Ch}^\#(\mathcal{E}) \) is an acyclic complex

\[
\cdots \to 0 \to x \to \text{Cyl}(f) \to \text{Cone}(f) \to 0 \to \cdots
\]

Hence the chain map \( \phi \) is a quasi-isomorphism and thus we have \( \text{Cone}(i(f)) \in \text{Ch}^b(\mathcal{E}^w)_{\text{qis}} = \text{Ch}^\#(\mathcal{E})^{qw} \).

(ii) We show the assertion by induction on the length of the complexes \( x \) and \( y \). The brutal truncation \( \sigma^{\geq k} x \) is \( \sigma^{\geq k} x := \cdots \to 0 \to 0 \to x^k \to x^{k+1} \to \cdots \) and put \( \sigma^{\leq k} x := x/\sigma^{\geq k} x \). Consider the following commutative diagram:

\[
\begin{array}{ccc}
\sigma^{\geq k} x & \xrightarrow{\sigma^{\geq k} f} & \sigma^{\leq k} x \\
\downarrow{\sigma^{\geq k} f} & & \downarrow{\sigma^{\leq k} f} \\
\sigma^{\geq k} y & \xrightarrow{\sigma^{\leq k} f} & \sigma^{\leq k} y
\end{array}
\]

Here, the horizontal sequences are conflations. By the assumption on the induction, we have \( \sigma^{\geq k} f \) and \( \sigma^{\leq k} f \) are in \( qw \). Thus the assertion follows from \((\text{WE 3})\). \( \square \)

We have a triangulated category \( \mathcal{D}^\#(\mathcal{E}) = \mathcal{T}(\text{Ch}^\#(\mathcal{E}), \text{qis}) \) for \( \mathcal{E} \), \( (\# \in \{ b, +, -, \emptyset \}) \) that is the derived category associated with the complicial exact category \( \text{Ch}^\#(\mathcal{E}) \) with quasi-isomorphisms as its weak equivalences. Similarly, when \( \mathcal{E} = (\mathcal{E}, w) \) is complicial, we denote by \( \mathcal{D}^\#(\mathcal{E}) = \mathcal{T}(\text{Ch}^\#(\mathcal{E}), \text{qw}) \) the derived category of \( \mathcal{E} \), which is defined by the derived category of the complicial exact category with weak equivalence \( \text{Ch}^\#(\mathcal{E}) := (\text{Ch}^\#(\mathcal{E}), \text{qw}) \).

The non-connective \( K \)-theory spectrum \( \mathbb{K}(\mathcal{E}) \) is defined in \([\mathcal{E}]\). Recall that the associated \( K \)-groups in positive degree are agree with Waldhausen \( K \)-groups \( K_i(\mathcal{E}) \) and \( \mathbb{K}_0(\mathcal{E}) = K_0(\mathcal{T}(\mathcal{E})^\sim) \), where \( \mathcal{T}(\mathcal{E})^\sim \) is the idempotent completion of \( \mathcal{T}(\mathcal{E}) \). Note also \( \mathcal{T}(\mathcal{E})^\sim \) becomes a triangulated category \((\mathcal{T})\).

**Theorem 4.2.** Assume that \( \mathcal{E} = (\mathcal{E}, w) \) is complicial. Then we have the following isomorphisms for each \( i \in \mathbb{Z} \):

(i) \( \mathbb{K}_i(\mathcal{E}) \cong \mathbb{K}_i(\text{Ch}^b(\mathcal{E})) \).
(ii) \( \mathbb{K}_i(\text{Ch}^-(\mathcal{E})) = \mathbb{K}_i(\text{Ch}^+(\mathcal{E})) = 0 \),
(iii) \( \mathbb{K}_i(\mathcal{E}) \cong \mathbb{K}_{i+1}(\text{Ch}(\mathcal{E})) \).
Proof. (i) The complicial exact categories $E$ and $\text{Ch}^b(E)$ admit (WE 6). Hence they satisfy the factorization axiom, namely, any morphism is a composition of a cofibration followed by a weak equivalence ([7], Appendix). By the very definition of the quasi-weak equivalences, we have $A_c^b(E) = \text{Ch}^b(E)^{qw}$ the $w$-acyclic objects in $\text{Ch}^b(E)$ as null classes. Hence we have the following diagram:

$$
\begin{array}{ccc}
\mathbb{K}(E^w) & \longrightarrow & \mathbb{K}(E) & \longrightarrow & \mathbb{K}(E) \\
\downarrow f & & \downarrow g & & \downarrow h \\
\mathbb{K}(A_c^b(E), \text{qis}) & \longrightarrow & \mathbb{K}(\text{Ch}^b(E), \text{qis}) & \longrightarrow & \mathbb{K}(\text{Ch}^b(E)).
\end{array}
$$

Here, the horizontal sequences are homotopy fibration of spectra by the fibration theorem ([7], Thm. 11) and the vertical map $h$ is induced from the inclusion map $E \to \text{Ch}^b(E)$ (Lem. 1.1). The vertical map $g$ is a homotopy equivalence by the Gillet-Waldhausen theorem again. Recall that $A_c^b(E) = \text{Ch}^b(E)^{qw}$ is the qis-closure of $\text{Ch}^b(E)$. We have $D^b(E)/D^b(E^w) \simeq D^b(E)/T(\text{Ch}^b(E^w), \text{qis})$. By comparing the localization sequences, $\mathbb{K}(\text{Ch}^b(E^w), \text{qis}) \simeq \mathbb{K}(A_c^b(E), \text{qis})$ and the assertion follows from it.

(ii) Let us consider the endofunctor $F = \bigoplus_{n \in \mathbb{N}} [2n]$ on $\text{Ch}^+(E)$. From the identities $F[2] \oplus \text{id} \simeq F$ and $\mathbb{K}(F[2]) = \mathbb{K}(F)$, we notice that $\mathbb{K}(\text{id}) = 0$ by the additivity theorem.

(iii) The truncation functor is defined by $\tau^{\geq k} x := \cdots \to 0 \to \text{Im}(\partial^{k-1}) \to x^k \to x^{k+1} \to \cdots$, for any $x \in \text{Ch}(E)$, where $\partial^{k-1} : x^{k-1} \to x^k$ is the differential map. The kernel of the quotient map $x \to \tau^{\geq k} x$ is denoted by $\tau^{< k} x$. For any $x \in \text{Ch}^b(E) \cap \text{Ch}^+(E)^{qw}$, we have quasi-isomorphisms $x \simeq y \simeq z$ with $y \in \text{Ch}^+(E)$ and $z \in \text{Ch}^b(E^w)$. Since $E$ is idempotent complete, the canonical map $\tau^{< k} y \to y$ becomes a quasi-isomorphism for some $k$ ([11], Lem. 2.6). Hence, we have $\text{Ch}^b(E)^{qw} = \text{Ch}^b(E) \cap \text{Ch}^+(E)^{qw}$. On the other hand, for any $x \in \text{Ch}^b(E) \cap \text{Ch}^-(E)^{qw}$, we have quasi-isomorphisms $x \simeq y \simeq z$ with $y \in \text{Ch}^-(E)$ and $z \in \text{Ch}^b(E^w)$. The canonical map $y \to \tau^{\geq k} y$ becomes a quasi-isomorphism and thus $\text{Ch}^+(E)^{qw} = \text{Ch}^b(E) \cap \text{Ch}^+(E)^{qw}$. Similarly, we have $\text{Ch}^+(E)^{qw} = \text{Ch}^+(E) \cap \text{Ch}^+(E)^{qw}$, and $\text{Ch}^-(E)^{qw} = \text{Ch}^-(E) \cap \text{Ch}^-(E)^{qw}$. We have the square of fully faithful inclusions which induces on category
equivalences on the quotient ([7], Proof of Lem. 7):

\[
\begin{array}{c}
\mathcal{D}^b(\mathcal{E}) \longrightarrow \mathcal{D}^+(\mathcal{E}) \\
\downarrow \downarrow \\
\mathcal{D}^-(\mathcal{E}) \longrightarrow \mathcal{D}(\mathcal{E})
\end{array}
\]

The diagram extends to

\[
\begin{array}{c}
\mathcal{D}^b(\mathcal{E}) \longrightarrow \mathcal{D}^+(\mathcal{E}) \\
\downarrow \downarrow \\
\mathcal{D}^-(\mathcal{E}) \longrightarrow \mathcal{D}(\mathcal{E})
\end{array}
\]

Here, all functors are fully faithful (Lem. 4.3 below) and the induced functors on quotients are equivalences. Thus the assertion follows from the localization theorem and (i) and (ii).

\[\text{Lemma 4.3 ([5], Lem. 10.3). Let } \mathcal{T} \text{ be a triangulated category and } \mathcal{S} \text{ and } \mathcal{N} \text{ full triangulated subcategories of } \mathcal{T}. \text{ If each morphism } x \to y \text{ in } \mathcal{T} \text{ with } x \in \mathcal{N} \text{ and } y \in \mathcal{S} \text{ factors through some object in } \mathcal{S} \cap \mathcal{N}, \text{ then the canonical functor } \mathcal{S}/\mathcal{S} \cap \mathcal{N} \to \mathcal{T}/\mathcal{N} \text{ is fully faithful.}\]

Let us denote \(\text{n-th times iteration of Ch}^\#\) for \(\mathcal{E}\) by \(\text{Ch}^\#_n(\mathcal{E})\) and \(\mathcal{D}^\#_n(\mathcal{E}) := \mathcal{T}(\text{Ch}^\#_n(\mathcal{E}))\) the \(n\)-th higher derived category of \(\mathcal{E}\). As the following corollary, we can consider the negative \(K\)-groups as obstruction group of idempotent completeness of the higher derived categories:

\[\text{Corollary 4.4. We assume that } \mathcal{E} \text{ is complicial or } w \text{ is just the class of isomorphisms in } \mathcal{E}. \text{ For any positive integer } n, \text{ we have}
\]

(i) \(\mathbb{K}_{-n}(\mathcal{E}) \simeq \mathbb{K}_0(\mathcal{D}_n(\mathcal{E}))\).

(ii) \(\mathbb{K}_{-n}(\mathcal{E})\) is trivial if and only if \(\mathcal{D}_n(\mathcal{E})\) is idempotent complete.

\[\text{Proof. Suppose that } \mathcal{E} = (\mathcal{E}, w) \text{ is complicial. By Theorem 4.2 (iii), we have } \mathbb{K}_{-n}(\mathcal{E}) \simeq \mathbb{K}_0(\text{Ch}_n(\mathcal{E})) \simeq \mathbb{K}_0(\mathcal{D}_n(\mathcal{E})). \text{ Then Proposition 4.5 below leads the desired assertion. If } \mathcal{E} = \mathcal{E} \text{ is an exact category; it may not be complicial, but } w = \text{ the class of isomorphisms). From the Gillet-Waldhausen and Lemma 7 and Corollary 6 in [7] as already refered in Introduction, we have } \mathbb{K}_{-n}(\mathcal{E}) \xrightarrow{\sim} \mathbb{K}_{-n}(\text{Ch}^b(\mathcal{E}), \text{qis}) \xrightarrow{\sim} \mathbb{K}_{-n+1}(\text{Ch}(\mathcal{E}), \text{qis}). \text{ Hence, one reduce to the case of complicial.}\]

\[\square\]
Proposition 4.5. (i) For an essentially small triangulated category $T$, if $K_0(T) = K_0(T^\sim)$ is trivial, then $T$ is idempotent complete.

(ii) The derived category $D(E)$ is idempotent complete if and only if the Grothendieck group $K_0(D(E)) = K_0(D(E)^\sim)$ is trivial.

Proof. (i) Since the map $K_0(T) \to K_0(T^\sim)$ is injective by [9] Corollary 2.3, now $K_0(T)$ is also trivial. Applying the Thomason classification theorem of (strictly) dense triangulated subcategories in essentially small triangulated categories [9] Theorem 2.1 for $T^\sim$, the inclusion functor $T \to T^\sim$ must be an equivalence.

(ii) From the diagram (6) in the proof of Theorem 4.2, we have a surjection $0 = K_0(D^+(E)) \oplus K_0(D^-(E)) \to K_0(D(E))$. Therefore $K_0(D(E)) = 0$. If $D(E)$ is idempotent complete, that is, $D(E) \cong \overset{\sim}{D(E)}$, then we have $K_0(D(E)) = K_0(D(E)^\sim) = K_0(D(E)) = 0$. The converse is followed from (i).

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