Integrable boundary conditions for multi-species ASEP

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Received 10 June 2016, revised 26 July 2016
Accepted for publication 1 August 2016
Published 24 August 2016

Abstract
The first result of the present paper is to provide classes of explicit solutions for integrable boundary matrices for the multi-species ASEP with an arbitrary number of species. All the solutions we have obtained can be seen as representations of a new algebra that contains the boundary Hecke algebra. The boundary Hecke algebra is not sufficient to build these solutions. This is the second result of our paper.

Keywords: multi-species ASEP, open boundaries, integrable systems

1. Introduction

The asymmetric simple exclusion process (ASEP) [36, 45] describes particles that hop on a one-dimensional lattice with anisotropic rates and hard core exclusion. Though it is one of the simplest examples of a driven diffusive system, it has become over the last decades a paradigm in out-of-equilibrium statistical physics [19]. It displays indeed a rich phenomenology, including effects such as boundary induced phase transitions [30, 44] and shock waves [18, 24, 38]. This model was introduced in biology [37], but it has found many other applications, for example in traffic flow [11, 12] and in polymer growth [27]. The stationary state of this model can be explicitly constructed using a matrix product ansatz [20]. A particularly remarkable feature of this stochastic process is that it is integrable. It has thus attracted much interest in combinatorics, mathematical physics and probability theory.

The bulk dynamics of the ASEP can be generalized to several species of particles, preserving the integrability property. It has led to many studies, among them can be mentioned the

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computation of the stationary state, for the model with periodic boundary conditions, using a matrix product ansatz [9, 32, 42]. Then, the stationary state has been computed for reflective boundaries [3] and semi-permeable boundaries [47]. The case of the multi-species system with generic boundaries, which is of particular interest in out-of-equilibrium statistical physics for the comprehension of boundary induced phenomenon, appears more complicated.

Fortunately, the integrability property allows one to choose the particle injection and extraction rates at the boundaries which permits the computation of the stationary state [16, 43]. Indeed, the inverse scattering method provides a general framework to determine the boundary conditions that preserve the integrability of the model [46]. However, the price to pay consists in solving a compatibility equation between the dynamics of the bulk and the reflection rates, called the reflection equation. The resolution of this equation is a complicated problem and is at the heart of a lot of research (see e.g. [1, 4, 6, 21, 25]).

Recent progress has been made to classify the integrable boundaries for the two-species ASEP and to compute the associated stationary state in a matrix product form [15]. The goal of this paper is to provide integrable boundaries for the multi-species ASEP with an arbitrary number of species. The integrable boundaries that we find divide the set of all species into five classes, which we call very-slow, slow, intermediate, fast and very-fast species. This division is labeled by four integers. There are also two free real parameters, associated to transition rates on the boundaries. We show also that all these integrable boundaries satisfy a generalization of the boundary Hecke algebra. This generalization is necessary to take into account the whole set of the solutions we found and, to our knowledge, is new in the literature.

The outline of this paper is as follows. In section 2 we recall briefly the stochastic dynamics of the multi-species ASEP and the quantum inverse scattering framework used to determine the integrable boundary conditions. Section 3 is devoted to the presentation of a class of integrable boundaries on the left and on the right. We also point out examples of combination of boundaries for which the Markov chain is irreducible. In section 4 we introduce a novel algebra, which includes all the boundary matrices presented. This algebra then allows construction of $K$-matrices solving the reflection equation through an easy Baxterization procedure. We argue that some of the boundary conditions presented in section 3 do not fit in the standard framework of boundary Hecke algebras and that the algebraic structure defined in this paper is needed to encompass them.

2. Multi-species ASEP

2.1. Presentation of the model

We start by recalling the dynamical rules of the multi-species ASEP. We will call the model multi-species ASEP, with the convention that we consider a model with $(N-1)$ species of particles on a one-dimensional lattice with $L$ sites. Each site on the lattice is occupied by a single particle, or is empty, and we identify this vacancy (or hole) as an additional species, that we call 1. The species of particles shall be labeled $2, 3, \ldots, N$. A configuration on the lattice is thus a $L$-tuplet $(\tau_1, \tau_2, \ldots, \tau_L)$ that belongs to $\{1, \ldots, N\}^L$. To each of the $L$ sites we will associate a $\mathbb{C}^N$ vector space, so that the set of all configurations is embedded into the tensor space $\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N$. The natural basis of this space is given by $|\eta\rangle \otimes \cdots \otimes |\tau_L\rangle$ with $\tau_i = 1, 2, \ldots, N$ and $|\tau\rangle = (0, \ldots, 0, 1, 0, \ldots, 0)^T$.

In the bulk. The dynamics is defined as follows: a bond $(i, i+1)$, with $1 \leq i \leq L-1$, between two neighboring lattice sites, is updated between time $t$ and $t+dt$ by swapping the
particles at \( i \) and \( i + 1 \) with rate 1 or \( q \) depending on the local configuration \( \tau_i \tau_{i+1} \) involved

\[
\begin{align*}
\tau_i \tau_{i+1} & \rightarrow \tau_{i+1} \tau_i \quad \text{if} \quad \tau_i > \tau_{i+1}, \\
\tau_i \tau_{i+1} & \rightarrow q \tau_{i+1} \tau_i \quad \text{if} \quad \tau_i < \tau_{i+1},
\end{align*}
\]

(2.1)

where \( q \) is a free positive parameter. These rules show that particles are ordered by their species: the species \( N \) has the highest priority, followed by species \((N - 1)\), down to particles of species 2, and lastly by holes (i.e. species 1). Species with higher priority will be said to be faster, so that species \( N \) is the fastest species (it is the flash) and species 1 the slowest.\(^4\)

The bulk rules can be encoded in a local Markov matrix acting on two sites, i.e. on \( \mathbb{C}^N \otimes \mathbb{C}^N \). Explicitly, it has the form

\[
m = \sum_{1 \leq i < j \leq N} \left\{ (E_{ij} \otimes E_{ji} - E_{ji} \otimes E_{ij}) + q(E_{ji} \otimes E_{ij} - E_{ii} \otimes E_{jj}) \right\},
\]

(2.2)

where \( E_{ij} \) is the \( N \times N \) elementary matrix with 1 at position \((i,j)\) and 0 elsewhere. This matrix can be obtained from an \( R \)-matrix satisfying the Yang–Baxter equation which allows us to prove integrability of the model: this construction will be briefly recalled in the next section. The complete Markov matrix in the bulk is given by

\[
M_{\text{bulk}} = m_{12} + m_{23} + \cdots + m_{L-1,L},
\]

(2.3)

where the indices on \( m \) indicate on which copies of \( \mathbb{C}^N \) it acts non-trivially.

On the boundaries. Particles are allowed to enter or to exit from both boundaries and the corresponding entrance/exit rates may depend on the type of the particle that was previously located at the boundary. More precisely, both on the left and on the right boundary, we can have a transition of the type

\[
\tau_1 \xrightarrow{r(\tau_1, \tau_2)} \tau_2,
\]

(2.4)

for \( \tau_1, \tau_2 = 1, 2, \ldots , N \). This leads to \( 2N(N - 1) \) independent rates (that is \( N(N - 1) \) rates on each side). The rates corresponding to the left boundary are gathered in an \( N \times N \) boundary matrix \( B \):

\[
B = \sum_{1 \leq i < j \leq N} r(\tau_i, \tau_j) E_{ji} - \sum_{i=1}^{N} \sum_{j=i}^{N} r(\tau_i, \tau_j) E_{ii}.
\]

(2.5)

Similarly, the rates for the right boundary are gathered in a matrix \( \overline{B} \). An open multi-species ASEP will be defined by the bulk matrix (2.3) and the two boundary matrices \( B \) and \( \overline{B} \). The Markov matrix associated to the model will be

\[
M = M_{\text{bulk}} + B_1 + \overline{B}_L,
\]

(2.6)

and the master equation, governing the time evolution of the probability \( P(\tau_1, \ldots , \tau_L) \) to be in the configuration \((\tau_1, \ldots , \tau_L)\), is written

\[
\frac{d|P\rangle}{dr} = M|P\rangle \quad \text{where} \quad |P\rangle = \sum_{1 \leq \tau_1, \ldots , \tau_L \leq N} P(\tau_1, \ldots , \tau_L)|\tau_1 \otimes \cdots \otimes |\tau_L\rangle.
\]

(2.7)

Although the bulk part \( M_{\text{bulk}} \) corresponds to an integrable model, for arbitrary choices of the boundary rates, the model will not be integrable. The first result of the present paper is to provide classes of explicit solutions for integrable boundary matrices for the multi-species ASEP. They are presented in section 3.1.

\(^4\) Note that this interpretation makes sense when \( q < 1 \).

3
2.2. Integrable approach to open models

We briefly recall the context of the quantum inverse scattering method that allows one to define open integrable models. We refer to the historical paper [46] and to the review [16] for more details.

We introduce an $R$-matrix acting on two copies of $\mathbb{C}^N$. It obeys the Yang–Baxter equation and the unitarity relation:

$$
R_{12}\left(\frac{x_1}{x_2}\right)R_{13}\left(\frac{x_1}{x_3}\right)R_{23}\left(\frac{x_2}{x_3}\right) = R_{23}\left(\frac{x_2}{x_3}\right)R_{13}\left(\frac{x_1}{x_3}\right)R_{12}\left(\frac{x_1}{x_2}\right),
$$

(2.8)

$$
R_{12}(x)R_{21}\left(\frac{1}{x}\right) = 1.
$$

(2.9)

Again, the indices indicate in which copies of $\mathbb{C}^N$ the $R$-matrices act non-trivially. For the multi-species ASEP, the $R$-matrix can be written as follows

$$
R_{i,i+1}(x) = P_{i,i+1}\left(1 + \frac{x - 1}{qx - 1}m_{i,i+1}\right),
$$

(2.10)

where $P$ is the permutation operator that exchanges the two copies of $\mathbb{C}^N$ in $\mathbb{C}^N \otimes \mathbb{C}^N$.

To define an integrable open model, one introduces the transfer matrix [46]:

$$
t_{\text{open}}(x) = t_{01}(R_{01}(x)\ldots R_{01}(x)K_0(x)R_{01}(x)\ldots R_{01}(x)\tilde{K}_0(x)),
$$

(2.11)

where $K(x)$ is a $N \times N$ matrix which obeys the reflection equation and is unitary:

$$
R_{12}\left(\frac{x_1}{x_2}\right)K_1(x_1)R_{21}(x_1,x_2)K_2(x_2) = K_2(x_2)R_{12}(x_1,x_2)K_1(x_1)R_{21}\left(\frac{x_1}{x_2}\right),
$$

(2.12)

$$
K(x)K\left(\frac{1}{x}\right) = 1.
$$

(2.13)

The boundary matrix $\tilde{K}(x)$ in (2.11) satisfies a dual reflection equation. The solutions to this dual reflection equation can be obtained from the solutions $K(x)$ of the reflection equation (2.12) by

$$
\tilde{K}_1(x) = t_{01}(K_0(1/x)((R_{01}(x^2)^-)^{-1})^y)P_{01}),
$$

(2.14)

where

$$
\tilde{K}(x) = UK(1/x)U \quad \text{and} \quad U = \begin{pmatrix}
1 & & \\
& \ddots & \\
& & 1
\end{pmatrix}.
$$

(2.15)

From these properties, usual calculations [46] prove that the transfer matrix $t_{\text{open}}(x)$ defines an integrable model: $[t_{\text{open}}(x), t_{\text{open}}(y)] = 0$. The global Markov matrix is then defined as

$$
M = \frac{q - 1}{2} \frac{d}{dx}t_{\text{open}}(x) \bigg|_{x=1}.
$$

(2.16)

Then the integrable boundaries are obtained from the $K$-matrices by

$$
B = \frac{q - 1}{2} \frac{d}{dx}K(x) \bigg|_{x=1} \quad \text{and} \quad \tilde{B} = -\frac{q - 1}{2} \frac{d}{dx}\tilde{K}(x) \bigg|_{x=1}.
$$

(2.17)
3. Integrable boundary conditions for the multi-species ASEP

Integrability for open systems is grounded in the reflection equation, obeyed by reflection matrices $K(x)$, and the construction of a two-row transfer matrix, as reviewed in section 2.2. However, for the construction of the Markov matrix, one needs only boundary matrices, deduced from the matrices $K(x)$ via equation (2.17). In this section we focus on the physical model, described by the Markov matrix, and present the boundary matrices associated to integrable reflection matrices $K(x)$, postponing the genuine discussion on integrability and the reflection equation to section 4. Yet, one has to keep in mind that the solutions we present are constrained by integrability: it imposes for instance the constraints (3.1) and (3.6).

3.1. Presentation of the left boundary conditions/matrices

We wish to give explicit solutions for integrable Markovian boundary matrices for the multi-species ASEP. These solutions are obtained with relations (2.17) from $K$-matrices obeying the reflection equation (2.12) with the $R$-matrix (2.10). We present here the integrable Markovian boundary conditions. We postpone the presentation of the associated $K$-matrices to section 3.5 and the proof of the integrability to section 4.3.

The integrable boundary conditions depend on two free real positive parameters (rates) $\alpha$ and $\gamma$, and four positive integers $s_1$, $s_2$, $f_1$ and $f_2$, that label two special slow ($s$) and two special fast ($f$) species, with the conditions

$$1 \leq s_1 \leq s_2 < f_2 \leq f_1 \leq N \quad \text{and} \quad f_1 - f_2 = s_2 - s_1.$$  \hspace{1cm} (3.1)

The four special species will be essentially created on the boundary, while the remaining species will essentially (but not only) decay onto these four types. Any species in between $s_1$ and $s_2$ will be paired with one species in between $f_2$ and $f_1$, allowing a transmutation (on the boundary) between the pairs. Finally, in between $s_2$ and $f_2$, either nothing happens, or the species decay to $s_2$ and $f_2$.

More specifically, integrability is preserved when we have the following rules and rates on the boundary:

* **Class of very slow species:** for species $\tau$ with $1 \leq \tau < s_1$, we have:

$$\tau \xrightarrow{\gamma} s_1 \quad \text{and} \quad \tau \xrightarrow{\alpha} f_1.$$  \hspace{1cm} (3.2)

* **Class of slow species:** for species $\tau$ with $s_1 \leq \tau \leq s_2$, we have:

$$\tau \xrightarrow{\alpha} \tau = s_1 + f_1 - \tau = s_2 + f_2 - \tau.$$  \hspace{1cm} (3.3)

* **Class of intermediate species:** for species $\tau$ with $s_2 < \tau < f_2$, we have the two possibilities:

1. $\tau \xrightarrow{0} \tau', \forall \tau' \quad \text{(no decay, creation or transmutation)}$.
2. $\tau \xrightarrow{\gamma} s_2$ and $\tau \xrightarrow{\alpha} f_2$.

* **Class of fast species:** for species $\tau$ with $f_2 \leq \tau \leq f_1$, we have:

$$\tau \xrightarrow{\gamma} \tau = s_1 + f_1 - \tau.$$  \hspace{1cm} (3.4)
• Class of very fast species: for species \( \tau \) with \( f_1 < \tau \leq N \), we have:

\[
\tau \xrightarrow{s_1} \text{and} \quad \tau \xrightarrow{f_1}.
\]

We have introduced the following combination of the rates:

\[
\tilde{\alpha} = \frac{(\alpha + \gamma + q - 1)\alpha}{\alpha + \gamma}, \quad \tilde{\gamma} = \frac{(\alpha + \gamma + q - 1)\gamma}{\alpha + \gamma}.
\]

This implies that \( \alpha, \gamma, q \) are constrained such that \( \tilde{\alpha}, \tilde{\gamma} \) are positive.

Note that, depending on the choice of \( s_1, s_2, f_2 \) and \( f_1 \), some classes of species may not occur: for instance if \( s_1 = 1 \), there is no very slow species. In the same way, if \( f_2 = s_2 + 1 \), there are no intermediate species.

Due to the second constraint in (3.1), the number of slow species coincides with the number of fast species, in accordance with the pairing mentioned above. By counting the number of possibilities for \( s_1, s_2, f_1 \) and \( f_2 \) with the constraints (3.1), we can deduce that, for multi-species ASEP there exist \( \left( N + 1 \right) / 3 \) different integrable boundaries, each of them depending on two real parameters.

Note that in any transition, the number of particles for the species in the very slow and very fast classes can only decrease. It may stay constant for the slow, fast and intermediate classes. For the four special types it may increase.

To summarize, these rates are gathered in the two following types of boundary matrices, depending on the two possibilities for intermediate species:

\[
B^0(\alpha, \gamma | s_1, s_2, f_1, f_2) =
\begin{pmatrix}
\begin{array}{cccc}
-\sigma & -\sigma & \gamma & \alpha \\
\gamma & \gamma & -\alpha & -\alpha \\
-\alpha & -\alpha & \gamma & \gamma \\
0 & 0 & \ldots & \ldots \\
0 & 0 & \ldots & \ldots \\
0 & 0 & \ldots & \ldots \\
\end{array}
\end{pmatrix},
\]

\[
(3.7)
\]

We have included in the counting the two possible choices for the intermediate species when \( f_2 > s_2 + 1 \).
We have introduced $\sigma = \alpha + \gamma$, $\tilde{\sigma} = \tilde{\alpha} + \tilde{\gamma}$ and $\sigma' = \alpha + \tilde{\gamma}$. The empty spaces in the matrices above are filled with zeros, and the lines indicate the positions of the four special types of species.

We remark that the rates can be written in a more symmetric way by introducing the combination of parameters\(^6\)
\[
a = -\frac{\alpha}{\gamma}, \quad c = \frac{\alpha + \gamma}{\alpha + \gamma + q - 1},
\]
which have previously appeared as the parameters of the Askey–Wilson polynomials in the context of the ASEP stationary state [48]. Then
\[
\alpha = \frac{ac(1-q)}{(a-1)(c-1)} \quad \text{and} \quad \gamma = -\frac{c(1-q)}{(a-1)(c-1)},
\]
and the parameters $\tilde{\alpha}$ and $\tilde{\gamma}$ become
\[
\tilde{\alpha} = \frac{a(1-q)}{(a-1)(c-1)} \quad \text{and} \quad \tilde{\gamma} = -\frac{(1-q)}{(a-1)(c-1)}.
\]

**More solutions:** One can produce more integrable solutions using conjugation by any diagonal invertible matrix $V$. Indeed, due to the invariance of the $R$-matrix (2.10) by the conjugation by $V_1 V_2 V K(x) V^{-1}$ is solution of the reflection equation if $K(x)$ is also a solution. However, the resulting conjugated matrix may not be Markovian. Nonetheless, we remark that conjugation by the diagonal matrix $\text{diag}(e^{s_1}, e^{s_2}, \ldots, e^{s_N})$ provides a deformed integrable boundary matrix that allows one to compute the cumulants of the currents at the boundary for the different species.

\(^{6}\) Note that in our parameters, $\tilde{\gamma}$ corresponds to $\gamma$ in the one-species ASEP. This choice avoids the presence of a square root in $a$ and $c$ and in the rates appearing in the boundary matrices.
3.2. Construction of the right boundary matrices

A right boundary matrix $B$ is obtained through the relation (2.17), where $K(x)$ is deduced from a solution $K(\lambda)$ thanks to (2.15). Let us stress that the parameters entering $B$ are independent from the ones used in the left boundary $B$. Altogether we will have four real parameters: $\alpha$, $\gamma$ for the left boundary, and $\beta$, $\delta$ for the right one. In the same way, the labels $s_1', s_2', f_1', f_2'$ of the four special species in the right boundary are independent from the four special species labels $s_1, s_2, f_1, f_2$ in the left boundary. Explicitly, the right boundary matrices are obtained from relation (2.15) and take the form

$$B(\beta, \delta | s_1', s_2', f_1', f_2') = U B(\beta, \delta | s_1'', s_2'', f_1'', f_2'') U^{-1},$$

where $U$ is defined in (2.15). The conjugation by $U$ implies $f_j'' = N + 1 - s_j'$ and $s_j'' = N + 1 - f_j'$, $j = 1, 2$.

The bijection between right and left boundaries can be seen in the following identity

$$B(\beta, \delta | s_1, s_2, f_1, f_2) \equiv B(\delta, \beta | s_1, s_2, f_1, f_2)_{z \rightarrow \bar{z}}.$$

where $z \leftrightarrow \bar{z}$ means that we interchange $\beta$ with $\beta$ and $\delta$ with $\delta$. As in the case of left boundaries, we use the notation

$$\bar{\beta} = \frac{(\beta + \delta + q - 1)\beta}{\beta + \delta} \quad \text{and} \quad \bar{\delta} = \frac{(\beta + \delta + q - 1)\delta}{\beta + \delta}.$$

3.3. Examples

For the case $N = 2$, we recover the one-species ASEP. We get only one possible choice for $s_1, s_2, f_1$ and $f_2$ given by $s_1 = s_2 = 1$ and $f_1 = f_2 = 2$. Then, in the language used in this paper, the particle 1 (vacancy) is slow and the particle 2 is fast and the rates at the boundary are given by

$$1 \overset{\gamma}{\rightarrow} 2 \quad \text{and} \quad 2 \overset{\gamma}{\rightarrow} 1.$$

One recovers that for the one-species ASEP, the generic boundary is integrable. The boundary matrix has the form

$$B = \left( \begin{array}{cc} -\alpha & \bar{\gamma} \\ \alpha & -\bar{\gamma} \end{array} \right).$$

One can use Bethe ansatz method to compute the eigenvalues and compute for example the spectral gap [17].

Conjugation by a diagonal matrix provides the non-Markovian boundary matrix used to compute the cumulant of the current [26]:

$$B(s) = \left( \begin{array}{cc} -\alpha & e^{s\bar{\gamma}} \\ e^{-s\alpha} & -e^{s\bar{\gamma}} \end{array} \right).$$

It still corresponds to an integrable boundary.

For the case $N = 3$, we obtain the two-species ASEP. There are four possibilities summarized in Table 1. We recover the boundaries found in [15].

Generic examples. Some of the boundary matrices can be related to former studies of boundary Hecke algebras [22, 33] (see also section 4.1). In our notation, they correspond to the matrices $B(\alpha, \gamma | 1, s_1, N + 1 - s_2, N)$ or $B^0(\alpha, \gamma | 1, s_2, N + 1 - s_2, N)$. Among them, some have been considered: $B^0(\alpha, \gamma | 1, 2, N - 1, N)$ was analyzed in [39], and for the two-species ASEP ($N = 3$) $B^0(\alpha, \gamma | 1, 1, 3, 3)$ was studied in [8, 13, 47].
3.4. Irreducible open multi-species ASEP

Since the boundary matrices we exhibited depend only on two different rates, one can wonder if, when using these boundaries, the open multi-species ASEP trivialises for $N$ big enough. More precisely, one may ask whether there is some limit on the number of species above which a multi-species ASEP can always be mapped (through identification) to an ASEP with a smaller number of species. In fact, it is not the case, thanks to the four types of special species, that can be chosen freely on each of the two boundaries. Indeed it can be shown that there are pairs of boundary matrices for which the number of particles of any given species is not conserved. Moreover, for any given subset of species, the total number of particles whose species is in this subset is not conserved either.

We give below examples of such pairings of boundary matrices. We write them as $B = B(\sigma, \gamma|s_1, s_2, f_1)$ and $\overline{B} = \overline{B}(\delta, \beta|s'_1, s'_2, f'_1)$, where the first matrix represents the left boundary, and the second matrix the right one. The explicit values of the four special species (for each boundary) depends on the parity of $N$ in the multi-species ASEP:

**Table 1.** The four integrable boundaries in the case $N = 3$. The last row corresponds to the names of these boundaries in [15].

| Type of Part. | Part. 1 slow | Part. 1 very slow | Part. 1 slow |
|--------------|-------------|------------------|-------------|
| Part. 2 fast | Part. 2 slow | Part. 2 intermediate |
| Part. 3 very fast | Part. 3 fast |

| Rates | $\alpha$ | $\gamma$ | $\alpha$ | $\gamma$ | $\alpha$ | $\gamma$ |
|-------|---------|---------|---------|---------|---------|---------|
| $s_1 = s_2 = 1$ | $1 \rightarrow 2$ | $1 \rightarrow 2$ | $1 \rightarrow 3$ |
| $s_1 = s_2 = 2$ | $2 \rightarrow 1$ | $1 \rightarrow 3$ | $2 \rightarrow 1$ |
| $s_1 = s_2 = 3$ | $3 \rightarrow 1$ | $2 \rightarrow 3$ | $3 \rightarrow 1$ |

| Name in [15] | $L_1$ | $L_2$ | $L_3$ |
|---------------|-------|-------|-------|

For the multi-species ASEP with $N = 2n + 1$, we can consider the matrices $B = B(\alpha, \gamma|2, n + 1, n + 2, 2n + 1)$ and $\overline{B} = \overline{B}(\beta, \delta|1, n, n + 1, 2n)$. Explicitly, they are given by

$$B = \begin{pmatrix} -\sigma & -\alpha \\ -\gamma & -\alpha & -\gamma \\ \alpha & \alpha & -\gamma \end{pmatrix}$$

with $\sigma = \alpha + \gamma$.

$$\overline{B} = \begin{pmatrix} -\delta & \beta \\ -\delta & -\beta & \beta \\ \overline{\delta} & -\beta & -\overline{\delta} \end{pmatrix}$$

with $\sigma = \beta + \delta$. 


In both cases, the intermediate particles drop out because we choose \( f_2 = s_2 + 1 \) and the very fast (respectively very slow) particles do not exist in \( B \) (respectively in \( \overline{B} \)). Then, we have drawn only the line corresponding to \( s_1 \) in \( B \) and to \( f_1 \) in \( \overline{B} \).

The evolution of the system given by the Markov chain with these boundaries does not preserve the number of particles of any subset of species. To prove that, we can see that there exists a cycle that connects all the species of the particles and the holes:

\[
\begin{align*}
1 & \xrightarrow{\alpha} 2n + 1 \xrightarrow{\gamma} 2 \xrightarrow{\delta} 2n - 1 \xrightarrow{\gamma} 4 \xrightarrow{\delta} \ldots \\
2n & \xleftarrow{\alpha} 3 \xleftarrow{\beta} 2n - 2 \xleftarrow{\alpha} 5 \xleftarrow{\beta} 2n - 4 \xleftarrow{\alpha} \ldots
\end{align*}
\] (3.18)

For the multi-species ASEP with \( N = 2n + 2 \), we can consider the matrices \( B = B(\alpha, \gamma|2, n + 1, n + 3, 2n + 2) \) and \( \overline{B} = \overline{B}(\beta, \delta|1, n, n + 2, 2n + 1) \), namely

\[
\begin{pmatrix}
\gamma & -\alpha & & & \\
-\alpha & -\gamma & -\gamma & & \\
& -\alpha & -\gamma & -\gamma & \\
\alpha & \alpha & \alpha & -\gamma & -\gamma
\end{pmatrix}
\]

with \( \begin{cases} \sigma = \alpha + \gamma \\ \sigma' = \alpha + \gamma' \\ \delta = \alpha + \gamma, \end{cases} \)

\[
\begin{pmatrix}
-\delta & & & \beta & \\
& -\delta & \beta & \beta & \\
& & -\delta & \beta & \\
\delta & \delta & -\beta & \beta & -\sigma
\end{pmatrix}
\]

with \( \begin{cases} \sigma = \beta + \delta \\ \sigma' = \beta + \delta. \end{cases} \)

One sees that now the very fast species have been dropped from \( B \) and the very slow species from \( \overline{B} \).

Again, there exists a cycle (similar to the one above) that connects all the species and the holes, however its form for the intermediate species depends on the parity of \( n \).

3.5. \( K \)-matrix

To make contact with \( K \)-matrices and integrability, we decompose both matrices (3.7) and (3.8) into three pieces, with

\[
b_0^+ = \begin{pmatrix}
-\gamma & & & \\
& -\gamma & & \\
& & -\gamma & \\
& & & -\gamma
\end{pmatrix}
\]

and \( b_0^- = \begin{pmatrix}
\alpha & \cdots & \alpha \\
& -\alpha & \cdots \\
& & -\alpha
\end{pmatrix} \). (3.19)

where we draw symbolically the lines corresponding to the four special types of particles, to indicate which part of the matrix we picked up in the boundary matrix to construct \( b_0^+ \). Again,
the empty spaces are all filled with zeros. The remaining part is \( b_0 = B - (b_0^+ + b_0^-) \), where \( B \) is either (3.7) or (3.8). Note that the decomposition is done in such a way that each matrix \( b_0, b_0^\pm \) is Markovian.

This decomposition of the boundary \( B = b_0 + b_0^+ + b_0^- \) allows the associated \( K \)-matrix to be written as

\[
K(x) = 1 + k(x) \left( b_0 + x b_0^+ + \frac{1}{x} b_0^- \right),
\]

with

\[
k(x) = \frac{(\alpha + \gamma)(x^2 - 1)}{(\gamma x + \alpha)((\alpha + \gamma)(x - 1) + (q - 1)x)}.
\]

The function \( k(x) \) takes a simpler form in the notations (3.10) and (3.11):

\[
k(x) = \frac{(1 - c)(1 - a)}{(q - 1)} \frac{(x^2 - 1)}{(x - a)(x - c)}.
\]

From (3.20), it is easy to check that

\[
B = \frac{q - 1}{2} \frac{d}{dx} K(x) \bigg|_{x=1}.
\]

In the next section, we prove the integrability of the \( K \)-matrix (3.20) through an algebraic approach.

4. Algebraic construction of the integrable boundaries

The integrability of the one-species ASEP can be understood in terms of an underlying Hecke algebra structure. From representations of the Hecke and boundary Hecke (or cyclotomic) algebras, solutions of the Yang–Baxter and reflection equations are constructed through a Baxterization procedure. This connection has also been noted for the two-species ASEP with a certain choice of open boundary conditions [8]. Indeed, some of the multi-species ASEP boundary matrices given above fall into the boundary Hecke family. But in order to encompass all boundary matrices in the classes (3.7) or (3.8), we introduce a new algebra and then show how it is Baxterized to give solutions of the reflection equation.

4.1. Hecke algebra

Before presenting the algebra to construct the boundary, let us recall the construction for the \( R \)-matrix based on the Baxterization of the Hecke algebra [28].

For \( i = 1, 2, \ldots, L - 1 \), we define the following operators

\[
\hat{R}_i(x) = x e_i - e_i^{-1}.
\]

It is well-known that if the generators \( e_i, 1 \leq i \leq L - 1 \) obey the so-called Hecke relations, for \( i = 1, 2, \ldots, L - 1 \) and \( j = 1, 2, \ldots, L - 2 \)

\[
e_i^2 = \omega e_i + 1, \quad e_j e_{j+1} e_j = e_{j+1} e_j e_{j+1},
\]

then the \( \hat{R}_i(x) \) matrix (4.1) obeys the braided Yang–Baxter equation

\[
\hat{R}_i(x_1) \hat{R}_{i+1}(x_1, x_2) \hat{R}_i(x_2) = \hat{R}_{i+1}(x_2) \hat{R}_i(x_0 x_2) \hat{R}_{i+1}(x_1),
\]

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and is unitary, up to normalization. Using relation (4.2), the braided R-matrix can be written as follows
\[ \hat{R}_i(x) = (x - 1)e_i + \omega. \] (4.4)

One can show that the local Markov matrix \( m \) provides a representation of the Hecke algebra:
\[ m = x e_0 + 1 + 1, \] (4.5)
with \( \omega = q^{1/2} - 1/q^{1/2} \). Then the R-matrix (2.10) is written in terms of (4.1) as
\[ R_i(x) = \frac{1}{x^{1/2} - 1/q^{1/2}} p_{i,i+1} \hat{R}_i(x). \] (4.6)

The extra factor is necessary for unitarity. Then relation (4.3) implies relation (2.8).

To summarize, the idea of the Baxterization (4.1) is to get a solution of the Yang–Baxter equation (i.e. an R-matrix depending on a spectral parameter) from a representation of the Hecke algebra. This idea has been intensively used and generalized to try to classify the solutions of the Yang–Baxter equation \([5, 7, 10, 14, 35, 49]\). Then, it has been extended to the reflection equation \([34]\) through the boundary Hecke algebra \([40]\). However, these algebras are not sufficient to include all the boundary matrices we have constructed. Below, we present a slightly more general algebraic structure that encompasses all the boundary matrices we found in the previous section, ensuring integrability of the corresponding models.

### 4.2. Baxterization of the K-matrix

We give in the following proposition the Baxterization of the K-matrix associated to a Baxterized R-matrix with Hecke algebra.

**Proposition 4.1.** Let \( e_i (i = 1, \ldots, L - 1) \) be the generators of the Hecke algebra satisfying (4.2) and \( \hat{R}_i(x) \) the associated braided R-matrices (4.1). Let us also define
\[ \hat{K}(x) = (1 - (x - 1)e_0) \left( 1 - \left( \frac{1}{x - 1} \right)e_0 \right)^{-1}, \] (4.7)
with \( e_0 \) a supplementary generator. The inverse in (4.7) is understood as the formal series
\[ \left( 1 - \left( \frac{1}{x - 1} \right)e_0 \right)^{-1} = x(1 - (1 - x)(e_0 + 1))^{-1} = (y + 1) \sum_{n=0}^{\infty} (-y)^n (e_0 + 1)^n, \] (4.8)
where \( y = x - 1 \).

Then \( \hat{K}(x) \) is a solution of the braided reflection equation
\[ \hat{R}_i(x_1/x_2)\hat{K}(x_1)\hat{R}_i(x_1x_2)\hat{K}(x_2) = \hat{K}(x_2)\hat{R}_i(x_1x_2)\hat{K}(x_1)\hat{R}_i(x_1/x_2) \] (4.9)
if and only if the supplementary generator \( e_0 \) satisfies
\[ e_1 e_0 e_1 e_0 e_1 - e_0 e_1 e_0 e_1 = \omega (e_0^2 e_1 e_0 e_0 e_1 e_0^2). \] (4.10)

Moreover the operator \( \hat{K}(x) \) is unitary:
\[ \hat{K}(x)\hat{K}(1/x) = 1. \] (4.11)
Proof. We multiply both sides of the braided reflection equation (4.9) on the left and on the right by
\[
\frac{x_2}{x_1} \left(1 - \left(\frac{1}{x_2} - 1\right)e_0\right) = \frac{1}{x_1}(1 + (x_2 - 1)(e_0 + 1))
\] (4.12)
and use (4.4), (4.7) to get the following equivalent relation
\[
(1 + y_2(e_0 + 1))\left((x_1 - x_2)e_1 + \omega x_2\right) \frac{1}{x_1} \hat{K}(x_1) \left((x_1 - x_2)e_1 + \omega\right) (1 - y_2 e_0) =
\]
\[
(1 - y_2 e_0)\left((x_1 - x_2)e_1 + \omega\right) \frac{1}{x_1} \hat{K}(x_1) \left((x_1 - x_2)e_1 + \omega x_2\right) (1 + y_2(e_0 + 1)),
\]
where \(y_1 = x_i - 1\). Then, we use the expansion (4.8) of \(\frac{1}{x_1} \hat{K}(x_1)\) in terms of \(y_1\). The coefficient of \(y_1^3\) in (4.13) provides relation (4.10), which proves that (4.9) implies (4.10).

To prove the reverse implication, we use the following lemma:

**Lemma 4.2.** Relation (4.10) implies, for \(k = 0, 1, 2, \ldots\),
\[
e_1 e_0 e_1 e_1^k - e_0^k e_1^k e_0 e_1 = \omega(e_0^{k+1} e_1 e_0 - e_0 e_1 e_0^{k+1}),
\]
\[
e_1 e_0^k e_1 e_0 e_1 e_0^k e_1 = \omega(e_0^{k+1} e_1 e_0 - e_0 e_1 e_0^{k+1})
+ e_0^k e_1 e_0 e_0 e_1 e_0^k,
\]
\[
e_1 (e_0 + 1)^k e_1 e_1 e_0 - e_0 e_1 (e_0 + 1)^k e_1 = \omega((e_0 + 1)^{k+1} e_1 e_0 - e_0 e_1 (e_0 + 1)^{k+1}),
\]
\[
e_1 e_0^k (e_0 + 1)^k e_1 e_1 e_0 - e_0 e_1 e_0 (e_0 + 1)^k e_1 = \omega((e_0 + 1)^{k+1} e_1 e_0 - e_0 e_1 (e_0 + 1)^{k+1}).
\]

The first relation of the lemma (4.14) is proven by recursion using (4.10). Relation (4.15) is proven also by recursion with (4.14) and (4.2). The third and the fourth are proven by expanding \((e_0 + 1)^k\) and using (4.15).

The lemma allows us to prove that
\[
e_1 \hat{K}(x)e_1 e_0 e_0 - e_0 e_1 \hat{K}(x)e_1 = \omega((e_0 + 1)\hat{K}(x)e_1 e_0 e_0 - e_0 e_1 (e_0 + 1)\hat{K}(x)).
\]
(4.18)

Finally, by expanding (4.13) and by using relation (4.18), we prove that equation (4.10) implies (4.9) which concludes the proof of the proposition.

**Connection with Baxterization of cyclotomic Hecke algebras.** Another Baxterization for the \(K\)-matrix was proposed in [31], starting from a slightly different algebra. There, the relation (4.10) is replaced by
\[
e_1 \hat{e}_0 e_1 \hat{e}_0 e_0 - \hat{e}_0 e_1 \hat{e}_0 e_1 = 0,
\]
\[
\sum_{k=0}^{m} a_k (\hat{e}_0)^k = 0
\]
(4.19)
(4.20)
for some fixed \(m = 2, 3, \ldots\) and \(a_0, \ldots, a_m\) free parameters. The relation (4.20) is called the cyclotomic relation, for an introduction to cyclotomic algebras, see e.g. [2]. Then, a \(K\)-matrix
can be constructed as a polynomial in \( \bar{e}_0 \) \[31\]. When \( m = 2 \), the cyclotomic Hecke algebra is just the boundary Hecke algebra.

In fact, similarly to proposition 4.1, one can show that

\[
\mathcal{K}(x) = (1 - x\bar{e}_0) \left( 1 - \frac{1}{x} \right)^{-1}
\]

(4.21)
satisfies the reflection equation, provided \( \bar{e}_0 \) satisfies solely the relation (4.19). The polynomial Baxterization of [31] is recovered when one assumes in addition the cyclotomic relation (4.20).

One can match this Baxterization with the one presented in (4.7) in the following way. Starting from the algebra (4.10), and assuming that \((e_0 + 1)\) is invertible, it is possible to prove that the generator

\[
e_0 = e_0(1 + e_0)^{-1}
\]

(4.22)
satisfies the relation (4.19). This can be shown by using relation (4.10) for \( e_0 \) and lemma 4.2. Then, substituting (4.22) into the Baxterized \( K \)-matrix (4.21) yields (4.7) up to a normalization factor.

4.3. Integrability of the multi-species ASEP boundary matrices

The aim of this section is to prove that the boundary matrices presented in section 3.1 fit into the Baxterization procedure of proposition 4.1.

**Proposition 4.3.** For any matrix \( B = B(\alpha, \beta; s_1, s_2, f_1, f_2) \) or \( B = B^0(\alpha, \beta; s_1, s_2, f_1, f_2) \), the generators

\[
e_0 = \frac{B + \alpha + \gamma + q - 1}{1 - q} \quad \text{and} \quad e_1 = (m + q)/\sqrt{q}
\]

(4.23)

\( B \) acts non trivially in space 1. Proof. The matrices \( e_1 \) and \( e_0 \) given in (4.5) and (4.23) act on two site multi-species ASEP configurations. For a given start state, \( \tau_1 \tau_2 \), we can find a subset of the particle species \( S = \{ \tau_1, \tau_2, \tau_3, \ldots \} \) such that for any polynomial in \( e_1 \) and \( e_0 \) acting on this state, these are the only species involved in the resulting configurations.

For all of the boundary matrices we consider, the subset \( S \) turns out to be small, and related to the different classes of particles we introduced above: the non-diagonal part of \( e_1 \) exchanges particles on sites 1 and 2, as allowed by bulk matrix \( m \); the non-diagonal part of \( e_0 \) injects and removes particles at site 1 as allowed by the boundary transitions given in section 3.1. The idea of the proof is then to project the ‘global’ matrices \( e_0, e_1 \) down to the smaller number of species in \( S \). If for every starting state we can show that the resulting projected \( e_0, e_1 \) satisfy (4.10), then this implies that the ‘global’ matrices also satisfy (4.10).

At this point, the proof decomposes into different steps:

- We remark that for any start state \( \tau_1 \tau_2 \), the set \( S \) falls into one of three categories:

\[
S = \{ \tau_1, \tau_2, s_1, s_2, f_1, f_2 \}, \quad (4.24)
\]

\[
S = \{ \tau_1, s_1 + f_1 - \tau_1, \tau_2, s_1 + f_1 - \tau_2 \}, \quad (4.25)
\]
\[
\mathcal{S} = \{ \tau_1, s_1 + f_1 - \tau_1, \tau_2, s, f \}, \quad \text{with} \quad (s, f) = (s_1, f_1) \text{ or } (s_2, f_2).
\] (4.26)

Note that these sets can be reduced depending on the class of the species \( \tau_1 \) and \( \tau_2 \). For instance, if \( \tau_1 \) and \( \tau_2 \) are of very slow class, then \( \mathcal{S} = \{ \tau_1, \tau_2, s, f \} \). Note also that the ordering of the start state does not change \( \mathcal{S} \) so \( \tau_1, \tau_2 \) are interchangeable in (4.26).

- Projecting the boundary matrix, \( B \), corresponding to \( e_0 \) down to the species in \( \mathcal{S} \) results in a boundary matrix of size \( |\mathcal{S}| \) of type (3.7) or (3.8). To see this, we perform the projection by ‘deleting’ species from \( B \) by removing the corresponding row and column: we use the following operations which preserve the forms (3.7) or (3.8):
  - Deleting any species in the very slow, intermediate, or very fast class;
  - Deleting a species, \( \tau \), in the slow or fast class with \( \tau = s_1 + f_1 \) if we also delete the species \( s_1 + f_1 - \tau \);
  - Deleting species \( s_1 \) and \( f_1 \) together, if \( s_1 = 1, f_1 = N \), and \( f_1 - f_2 = s_2 - s_1 > 0 \);
  - Deleting species \( s_2 \) and \( f_2 \) together, if \( f_2 = s_2 + 1 \), and \( f_1 - f_2 = s_2 - s_1 > 0 \).

These operations are always sufficient to project down to any subsets \( \mathcal{S} \) as defined above. The projected \( e_0 \) is then obtained from the projected \( B \) through (4.23).

- For the local bulk matrix \( m \) (giving \( e_1 \)) we can delete any number of species, preserving the form (2.2).

To complete the proof all we need to do is to verify that all boundary matrices in this family give \( e_0 \) matrices which satisfy (4.10) for size 2 up to 6 (the maximum \( |\mathcal{S}| \)). We have done this by a direct computation with a formal mathematical software package.

To illustrate the projection on \( \mathcal{S} \), we consider the following boundary matrix

\[
B = \begin{pmatrix}
-\sigma & \gamma & -\alpha & -\alpha & -\alpha & -\alpha \\
-\alpha & -\alpha & \gamma & \gamma & \gamma & \gamma \\
-\alpha & -\sigma & \gamma & \gamma & \gamma & \gamma \\
\alpha & \alpha & -\alpha & -\alpha & -\alpha & -\alpha \\
\alpha & \alpha & -\alpha & -\alpha & -\alpha & -\alpha \\
\alpha & \alpha & -\alpha & -\alpha & -\alpha & -\alpha \\
\end{pmatrix}
\] (4.27)

and give some examples of start state \((\tau_1, \tau_2)\) and the resulting subset \( \mathcal{S} \) and corresponding reduced matrix. In the case where \((\tau_1, \tau_2) = (1, 4)\), we obtain \( \mathcal{S} = \{ 1, 4, s_1 = 2, s_2 = 3, f_1 = 6, f_2 = 5 \} \) and the reduced matrix reads

\[
\begin{pmatrix}
-\sigma & \gamma & -\alpha & -\alpha & -\alpha & -\alpha \\
-\alpha & -\alpha & \gamma & \gamma & \gamma & \gamma \\
-\alpha & -\sigma & \gamma & \gamma & \gamma & \gamma \\
\alpha & \alpha & -\alpha & -\alpha & -\alpha & -\alpha \\
\alpha & \alpha & -\alpha & -\alpha & -\alpha & -\alpha \\
\alpha & \alpha & -\alpha & -\alpha & -\alpha & -\alpha \\
\end{pmatrix}
\] (4.28)

In the case where \((\tau_1, \tau_2) = (2, 6)\), we obtain \( \mathcal{S} = \{ 2, 6 \} \) and the reduced matrix reads

\[
\begin{pmatrix}
-\alpha & \gamma & -\alpha & -\alpha \\
\alpha & \alpha & -\alpha & -\alpha \\
\end{pmatrix}
\] (4.29)
Finally, in the case where $\gamma_1, \gamma_2 = (3, 4)$, we obtain $S = \{3, 4, 5\}$ and the reduced matrix reads
\[
\begin{pmatrix}
-\alpha & \bar{\gamma} \\
0 & -\sigma' \\
\alpha & \alpha - \bar{\gamma}
\end{pmatrix}.
\tag{4.30}
\]

From proposition 4.1, the generator $e_0$ defined above provides a $K(x)$ matrix that obeys the reflection equation and is unitary. The connection with the expression (3.20) is given by the following proposition.

**Proposition 4.4.** For any matrix $B = B(\alpha, \beta; s_1, s_2, f_2, f_1)$ or $B = B^0(\alpha, \beta; s_1, s_2, f_2, f_1)$, let $B = b_0 + b_0^+ + b_0^-$ be the decomposition described in section 3.5. The matrix
\[
K(x) = 1 + k(x)\left(b_0 + x b_0^+ + \frac{1}{x} b_0^-\right),
\tag{4.31}
\]
with
\[
k(x) = \frac{(x^2 - 1)(\alpha + \gamma)}{(\xi + \alpha)((\alpha + \gamma)(x - 1) + (q - 1)x)}
\tag{4.32}
\]
can be expressed as a Baxterized $\tilde{K}(x)$ matrix
\[
K(x) = \frac{(\alpha + \gamma + q - 1)\left(\frac{1}{x} - 1\right) + q - 1}{(\alpha + \gamma + q - 1)(x - 1) + q - 1}\left(1 - (x - 1)e_0\right),
\tag{4.33}
\]
where
\[
e_0 = \frac{B + \alpha + \gamma + q - 1}{1 - q}.
\tag{4.34}
\]

Thus, it satisfies the reflection equation and is unitary.

**Proof.** Note that $K(x)$ in (4.33) has the form
\[
K(x) = \frac{f(x)}{f(1/x)}\tilde{K}(x)
\]
with $\tilde{K}(x)$ as in (4.7) so that it remains unitary and satisfies the reflection equation.

To show that the $K$-matrix (4.33) is equivalent to the form (4.31) we will need the following relations for the matrices $b_0$, $b_0^+$ and $b_0^-:
\begin{align*}
  b_0^2 & = -(\alpha + \bar{\gamma})b_0 + \bar{\alpha}b_0^+ + \gamma b_0^-, & (b_0^\gamma)^2 &= -\gamma b_0^+, & (b_0^-)^2 &= -\bar{\alpha}b_0^-, \\
  b_0 b_0^+ &= b_0^+ b_0, & b_0 b_0^- &= b_0^- b_0, & b_0^+ b_0^- &= b_0^- b_0^+ &= 0.
\end{align*}
\tag{4.35}

They involve the combination of parameters defined in (3.6). These relations are proven using the same method of projecting down to a set $S$ of all species involved from a given starting state. In this case, the matrices act on a single site configuration, and it is sufficient to check that the relations hold for size 2 and 3.
Using (4.35) it is straightforward to check that

\[
\left(1 - \left(\frac{1}{x} - 1\right)e_0\right)\left(q - 1 + \left(\frac{x - 1}{x} + \frac{(x-1)(\alpha + \gamma)}{(x - 1) + q - 1}\right)\left(b_0^+ + b_0^- + b_0^0\right)\right),
\]

\[
= \frac{1}{x}(\alpha + \gamma + q - 1 - (\alpha + \gamma)x),
\]

which allows \(1 - \left(\frac{1}{x} - 1\right)e_0\) to be expressed as a polynomial in \(b_0, b_0^\pm\). Then using (4.35) again we can show that

\[
\frac{(\alpha + \gamma + q - 1)\left(\frac{1}{x} - 1\right) + q - 1}{(\alpha + \gamma + q - 1)(x - 1) + q - 1}\left(1 - \left(\frac{1}{x} - 1\right)e_0\right)\left(1 - \left(\frac{1}{x} - 1\right)e_0\right)^{-1},
\]

\[
= 1 + \frac{(x^2 - 1)(\alpha + \gamma)}{(\gamma x + \alpha)(\alpha + \gamma)(x - 1) + (q - 1)x^2}\left(b_0 + x b_0^+ + \frac{1}{x} b_0^-\right),
\]

which concludes the proof.

**Polynomial relations.** Using relations (4.35) and the expression (4.23) for \(e_0\), we get

\[
e_0(e_0 + 1)\left(e_0 + \frac{\alpha}{\alpha + \gamma}\right)\left(e_0 + \frac{\alpha + \gamma + q - 1}{q - 1}\right) = 0.
\]

We stress that the factor \((e_0 + 1)\) is present in (4.40). Moreover, we find that for particular choices of \(b_0, b_0^+\) and \(b_0^-\), the polynomial (4.40) becomes minimal for \(e_0\). Then, in these cases, \((e_0 + 1)\) is not invertible and we cannot use (4.22) to Baxterize the \(K\)-matrix for the multi-species ASEP from the construction of [31]. However, the model is still integrable thanks to the Baxterization (4.7).

Note that when \(b_0^+ = b_0^- = 0\), relations (4.35) reduce to

\[
b_0(b_0 + \alpha + \gamma) = 0 \quad \text{i.e.} \quad \left(e_0 + \frac{\alpha}{\alpha + \gamma}\right)\left(e_0 + \frac{\alpha + \gamma + q - 1}{q - 1}\right) = 0
\]

and we recover the boundary Hecke algebra.

### 5. Conclusion and perspectives

In this paper, we present integrable boundary matrices for the multi-species asymmetric exclusion process. We believe that the solutions presented here are the only Markovian solutions of the reflection equation with at least two free parameters\(^7\). This conjecture is supported by two facts: (i) we recover the classification done for the one and the two-species models; (ii) we also checked that for the three-species case, and assuming that the \(K\)-matrix entries are polynomials of degree 4 w.r.t. the spectral parameter, the only solutions to the reflection equation are the ones presented in equation (3.20) which are of degree 3 (up to a normalization).

As explained previously, these solutions allow us to define integrable Markovian stochastic processes. Then, we believe that the associated stationary state can be expressed with a matrix ansatz following the generic idea developed in [16, 43] and already exploited for the two-species case in [15]. Let us stress that the existence of a matrix ansatz is not linked with

\(^7\) We have found other distinct solutions, but they have only one free parameter and are physically less interesting.
integrability. In fact it is known [29] that a matrix ansatz is always possible, but that it can be as complicated as the diagonalization of the Markov matrix. Yet, the general viewpoint of [16, 43] is that integrability leads to a tractable matrix ansatz, as shown in the case presented in [15]. The integrability of these models should permit also the computation of other physical quantities such as correlation functions, the spectral gap, and fluctuations of the density and of the current.

The second result of this paper provides an algebraic framework for these solutions that generalizes the boundary Hecke algebra. We hope that this algebra can be exploited in other contexts such as quantum integrable spin chains or the $O(1)$ loop model, allowing one to generalize the results described in [23, 41]. The boundary Hecke algebra has also been used to relate some stationary weights of open semi-permeable two-species ASEP to Koornwinder polynomials [8]. The algebraic structure presented here may be relevant to extend this study to the case of models with permeable integrable boundaries.

Acknowledgments

We are grateful to Jan de Gier for helpful discussions.

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