On the well-posedness of the Ideal MHD equations in the Triebel-Lizorkin spaces

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Abstract

In this paper, we prove the local well-posedness for the Ideal MHD equations in the Triebel-Lizorkin spaces and obtain blow-up criterion of smooth solutions. Specially, we fill a gap in a step of the proof of the local well-posedness part for the incompressible Euler equation in [7].

Key words. Ideal MHD equations, well-posedness, bow-up criterion, particle trajectory mapping, para-differential calculus, Triebel-Lizorkin space

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1 Introduction

In this paper, we are concerned with the Ideal MHD equations in \( \mathbb{R}^d \):

\[
\begin{aligned}
\begin{cases}
    u_t + u \cdot \nabla u &= -\nabla p - \frac{1}{2} \nabla b^2 + b \cdot \nabla b, \\
    b_t + u \cdot \nabla b &= b \cdot \nabla u, \\
    \nabla \cdot u &= \nabla \cdot b = 0, \\
    u(0, x) &= u_0(x), \quad b(0, x) = b_0(x),
\end{cases}
\end{aligned}
\]

(1.1)

where \( x \in \mathbb{R}^d, t \geq 0, u, b \) describes the flow velocity vector and the magnetic field vector respectively, \( p \) is a scalar pressure, while \( u_0 \) and \( b_0 \) are the given initial velocity and initial magnetic field with \( \nabla \cdot u_0 = \nabla \cdot b_0 = 0 \).

Using the standard energy method [14], it can be proved that for \( (u_0, b_0) \in H^s(\mathbb{R}^d) \), \( s > \frac{d}{2} + 1 \), there exists \( T > 0 \) such that the Cauchy problem (1.1) has a unique smooth solution \( (u(t, x), b(t, x)) \) on \([0, T)\) satisfying

\[
(u, b) \in C([0, T); H^s) \cap C^1([0, T); H^{s-1}).
\]

But whether this local solution will exist globally or lead to a singularity in finite time is still an outstanding open problem. Caflisch, Klapper and Steele[4] extended Beale-Kato-Majda criterion [2] for the incompressible Euler equations to the Ideal MHD equations. More
precisely, they showed if the smooth solution \((u, b)\) satisfies the following condition:

\[
\int_0^T (\|\nabla \times u\|_{L^\infty} + \|\nabla \times b\|_{L^\infty}) dt < \infty,
\]

then the solution \((u, b)\) can be extended beyond \(t = T\), namely, for some \(T < \tilde{T}\), \((u, b) \in C([0, \tilde{T}); H^s) \cap C^1([0, \tilde{T}); H^{s-1})\). One can refer to \([5, 23]\) for the other refined criterions, and for the viscous MHD equations, some criterions can be found in \([6, 19, 20, 21, 22]\).

Recently, Chae studied the local well-posedness and blow-up criterion for the incompressible Euler equations in the Triebel-Lizorkin spaces \([7, 8]\). As we know, Triebel-Lizorkin spaces are the unification of several classical function spaces such as Lebesgue spaces \(L^p(\mathbb{R}^d)\), Sobolev spaces \(H^s_p(\mathbb{R}^d)\), Lipschitz spaces \(C^s(\mathbb{R}^d)\), and so on. In \([7]\), the author first used the Littlewood-Paley operator to localize the Euler equation to the frequency annulus \(\{\|\xi\| \sim 2^j\}\), then obtained an integral representation of the frequency-localized solution on the Lagrangian coordinates by introducing a family of particle trajectory mapping \(\{X_j(\alpha, t)\}\) defined by

\[
\left\{
\begin{array}{ll}
\frac{\partial}{\partial t}X_j(\alpha, t) = (S_{j-2}v)(X_j(\alpha, t), t) \\
X_j(\alpha, 0) = \alpha,
\end{array}
\right.
\]

where \(v\) is a divergence-free velocity field and \(S_{j-2}\) is a frequency projection to the ball \(\{\|\xi\| \lesssim 2^j\}\) (see Section 2).

With the integral representation, one can obtain the well-posedness of the Euler equation in the framework of the Besov spaces by standard argument, due to the following important relation

\[
\left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|\Delta_j v(X_j(\alpha, t))\|_{L^p(\cdot d\alpha)}^q\right)^\frac{1}{q} \approx \|v\|_{\dot{B}^s_{p,q}}
\]

by the volume-preserving property of the mapping \(\{X_j(\alpha, t)\}\) which is defined by (1.3).

However, if we work in the framework of the Triebel-Lizorkin spaces, and the trajectory mapping \(\{X_j(\alpha, t)\}\) is taken, we don’t know whether the relation

\[
\left\|\left(\sum_{j \in \mathbb{Z}} 2^{jsq} |\Delta_j v(X_j(\alpha, t))|^q\right)^\frac{1}{q}\right\|_{L^p(\cdot dx)} \approx \left\|\left(\sum_{j \in \mathbb{Z}} 2^{jsq} |\Delta_j v(x)|^q\right)^\frac{1}{q}\right\|_{L^p(\cdot dx)} = \|v\|_{\dot{F}^s_{p,q}}
\]

(1.4)

holds. The reason is that the mapping \(\{X_j(\alpha, t)\}\) depends on the index \(j\), and we can’t find a uniform change of the coordinates independent of \(j\) such that (1.4) holds. On the other hand, the proof of the commutator estimate (the key point of the proof of the local well-posedness part)

\[
\left\|\left(\sum_{j \in \mathbb{Z}} 2^{jsq} |(S_{j-2}v \cdot \nabla)\Delta_j v - \Delta_j ((v \cdot \nabla)v)(X_j(\alpha, t))|^q\right)^\frac{1}{q}\right\|_{L^p} \leq C \|\nabla v\|_{\infty} \|v\|_{\dot{F}^s_{p,q}}
\]

(1.5)

also leads to some trouble due to similar reasons.

The purpose of this paper is to deal with the well-posedness of the Ideal MHD equations (1.1) in the Triebel-Lizorkin spaces. Firstly, we can reduce (1.1) to the transport equations by introducing the symmetrizers. If we still use the trajectory mapping depending on \(j\), the
above-mentioned trouble will occur. In order to overcome this difficulty, we will introduce a
different family of particle trajectory mapping \( \{ X(\alpha, t) \} \) independent of \( j \) defined by
\[
\begin{aligned}
\frac{\partial}{\partial t} X(\alpha, t) &= v(X(\alpha, t), t) \\
X(\alpha, 0) &= \alpha.
\end{aligned}
\]

The price to pay here is that we have to establish the following commutator estimate
\[
\left\| \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \left| \left[ (v \cdot \nabla) \Delta_j u - \Delta_j((v \cdot \nabla)u) \right] \right|^q \right) \right\|_{L^p}^{\frac{1}{q}} \leq C \left( \| \nabla v \|_{L^\infty} \| u \|_{\dot{F}^{s}_{p,q}} + \| u \|_{L^\infty} \| \nabla v \|_{\dot{F}^{s}_{p,q}} \right)
\]
by the paradifferential calculus, whose proof is more complicated since \( v \) is rougher than \( S_j - 2v \)
which is the smooth low frequency cut-off of \( v \). It is necessary to point out that the Maximal
inequality (see Lemma 2.5 in Section 2) plays a key role in the proof of the above inequality,
which helps us to avoid other difficulties arising from the change of the coordinates.

Now we state our result as follows.

**Theorem 1.1** (i) Local-in-time Existence. Let \((u_0, b_0) \in F^{s}_{p,q}, s > \frac{d}{p} + 1, 1 < p, q < \infty \) satisfying \( \text{div } u_0 = \text{div } b_0 = 0 \). Then there exists \( T = T(\| (u_0, b_0) \|_{F^{s}_{p,q}}) \) such that the IMHD
has a unique solution \((u, b) \in C([0, T); F^{s}_{p,q}) \).

(ii) Blow-up Criterion. The local-in-time solution \((u, b) \in C([0, T); F^{s}_{p,q}) \) constructed in (i) blows up at \( T^* > T \) in \( F^{s}_{p,q} \), i.e.
\[
\limsup_{t \to T^*} \| (u, b) \|_{F^{s}_{p,q}} = +\infty, \quad T^* < \infty,
\]
if and only if
\[
\int_0^{T^*} \| (\nabla \times u, \nabla \times b)(t) \|_{\dot{F}^0_{\infty, \infty}} dt = +\infty. \tag{1.6}
\]

**Remark 1.2** In the case of \( b = 0 \), (IMHD) can be read as the incompressible Euler equations,
and what proved in [7] is a straightforward consequence of Theorem 1.1.

**Remark 1.3** Using the argument in [5], we can also refine blow-up criterion (1.6) to the
following form: there exists a positive constant \( M_0 \) such that if
\[
\limsup_{\varepsilon \to 0} \int_{T^* - \varepsilon}^{T^*} \| (\Delta_j(\nabla \times u), \Delta_j(\nabla \times b))(t) \|_{L^\infty} dt \geq M_0,
\]
then the smooth solution \((u, b)\) blows up at \( t = T^* \).

**Notation:** Throughout this paper, \( C \) stands for a “harmless” constant, and we will use the
notation \( A \lesssim B \) as an equivalent to \( A \leq CB \), \( A \approx B \) as \( A \lesssim B \) and \( B \lesssim A \), and denote \( \| \cdot \|_p \)
by \( L^p(\mathbb{R}^d) \) norm of a function.
2 Preliminaries

Let $\mathcal{B} = \{\xi \in \mathbb{R}^d, |\xi| \leq \frac{3}{4}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{5}{4}\}$. Choose two nonnegative smooth radial functions $\chi, \varphi$ supported respectively in $\mathcal{B}$ and $\mathcal{C}$ such that

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d,$$

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

We denote $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$. Then the dyadic blocks $\Delta_j$ and $S_j$ can be defined as follows

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{jd} \int_{\mathbb{R}^d} h(2^j y) f(x - y) dy,$$

$$S_j f = \sum_{k \leq j-1} \Delta_k f = \varphi(2^{-j}D)f = 2^{jd} \int_{\mathbb{R}^d} \tilde{h}(2^j y) f(x - y) dy.$$

Formally, $\Delta_j = S_j - S_{j-1}$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$, and $S_j$ is a frequency projection to the ball $\{|\xi| \lesssim 2^j\}$. One easily verifies that with our choice of $\varphi$

$$\Delta_j \Delta_k f \equiv 0 \quad \text{if} \quad |j - k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} f \Delta_k f) \equiv 0 \quad \text{if} \quad |j - k| \geq 5. \quad (2.1)$$

With the introduction of $\Delta_j$ and $S_j$, let us recall the definition of the Triebel-Lizorkin space. Let $s \in \mathbb{R}$, $(p, q) \in [1, \infty) \times [1, \infty]$, the homogenous Triebel-Lizorkin space $\dot{F}^s_{p, q}$ is defined by

$$\dot{F}^s_{p, q} = \{f \in \mathcal{Z}'(\mathbb{R}^d) ; \|f\|_{\dot{F}^s_{p, q}} < \infty\},$$

where

$$\|f\|_{\dot{F}^s_{p, q}} = \left\{ \begin{array}{ll} \left\| \left( \sum_{j \in \mathbb{Z}} 2^{jqs} |\Delta_j f|^q \right)^{\frac{1}{q}} \right\|_p, & \text{for} \quad 1 \leq q < \infty, \\
\| \sup_{j \in \mathbb{Z}} 2^{js}|\Delta_j f| \|_p, & \text{for} \quad q = \infty, \end{array} \right.$$ and $\mathcal{Z}'(\mathbb{R}^d)$ denotes the dual space of $\mathcal{Z}(\mathbb{R}^d) = \{f \in \mathcal{S}(\mathbb{R}^d) ; \partial^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^d \text{ multi-index}\}$ and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$.

For $s > 0$, and $(p, q) \in [1, \infty) \times [1, \infty]$, we define the inhomogeneous Triebel-Lizorkin space $F^s_{p, q}$ as follows

$$F^s_{p, q} = \{f \in \mathcal{S}'(\mathbb{R}^d) ; \|f\|_{F^s_{p, q}} < \infty\},$$

where

$$\|f\|_{F^s_{p, q}} = \|f\|_p + \|f\|_{\dot{F}^s_{p, q}}.$$

We refer to [1] [18] for more details.

**Lemma 2.1** (Bernstein’s inequality) Let $k \in \mathbb{N}$. There exist a constant $C$ independent of $f$ and $j$ such that for all $1 \leq p \leq q \leq \infty$, the following inequalities hold:

$$\sup \{ |\xi| \lesssim 2^j \} \Rightarrow \sup_{|\alpha| = k} \|\partial^\alpha f\|_q \leq C 2^{jk + jd(\frac{1}{p} - \frac{1}{q})} \|f\|_p,$$

$$\sup \{ |\xi| \sim 2^j \} \Rightarrow \|f\|_p \leq C \sup_{|\alpha| = k} 2^{-jk} \|\partial^\alpha f\|_p.$$
Lemma 2.2 For any \( k \in \mathbb{N} \), there exists a constant \( C_k \) such that the following inequality holds:

\[
C_k^{-1} \| \nabla^k f \|_{\dot{F}_{p,q}^{s+k}} \leq \| f \|_{\dot{F}_{p,q}^{s+k}} \leq C_k \| \nabla^k f \|_{\dot{F}_{p,q}^{s}}.
\]

The proof can be found in \([18]\).

Proposition 2.3 \([7]\) Let \( s > 0 \), \((p, q) \in (1, \infty) \times (1, \infty]\), or \( p = q = \infty \), then there exists a constant \( C \) such that

\[
\| fg \|_{\dot{F}_{p,q}^{s}} \leq C (\| f \|_{\infty} \| g \|_{\dot{F}_{p,q}^{s}} + \| g \|_{\infty} \| f \|_{\dot{F}_{p,q}^{s}}),
\]

\[
\| fg \|_{F_{p,q}^{s}} \leq C (\| f \|_{\infty} \| g \|_{F_{p,q}^{s}} + \| g \|_{\infty} \| f \|_{F_{p,q}^{s}}).
\]

For a locally integrable function \( f \), the maximal function \( Mf(x) \) is defined by

\[
Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,
\]

where \( |B(x,r)| \) is the volume of the ball \( B(x,r) \) with center \( x \) and radius \( r \).

Lemma 2.4 \([11]\) (Vector Maximal inequality) Let \((p, q) \in (1, \infty) \times (1, \infty]\) or \( p = q = \infty \) be given. Suppose \( \{f_j\}_{j \in \mathbb{Z}} \) is a sequence of function in \( L^p \) with the property that \( \| f_j(x) \|_{L^q(\mathbb{Z})} \in L^p(\mathbb{R}^d) \). Then there holds

\[
\left\| \left( \sum_{j \in \mathbb{Z}} |Mf_j(x)|^q \right)^{\frac{1}{q}} \right\|_p \leq C \left\| \left( \sum_{j \in \mathbb{Z}} |f_j(x)|^q \right)^{\frac{1}{q}} \right\|_p.
\]

Lemma 2.5 Let \( \varphi \) be an integrable function on \( \mathbb{R}^d \), and set \( \varphi_\epsilon(x) = \frac{\varphi(x)}{\varphi(\frac{x}{\epsilon})} \) for \( \epsilon > 0 \). Suppose that the least decreasing radial majorant of \( \varphi \) is integrable; i.e. let

\[
\psi(x) = \sup_{|y| \geq |x|} |\varphi(y)|,
\]

and we suppose \( \int_{\mathbb{R}^d} \psi(x)dx = A < \infty \). Then with the same \( A \), for \( f \in L^p(\mathbb{R}^d), 1 \leq p \leq \infty \)

\[
\sup_{\epsilon>0} |(f * \varphi_\epsilon)(x)| \leq AM(f)(x).
\]

The proof can be found in \([17]\), Chap. III.

3 The proof of Theorem 1.1

We divide the proof of Theorem 1.1 into several steps.

**Step 1.** A priori estimates.

Let us symmetrize the equation \((1.1)\). Set

\[
z^+ = u + b, \quad z^- = u - b,
\]

For the proof, see \([10]\) \([15]\).
\[ \frac{\partial t}{\partial t} + (z^- \cdot \nabla) z^+ = -\nabla \pi, \]
\[ \frac{\partial t}{\partial t} + (z^+ \cdot \nabla) z^- = -\nabla \pi, \]
\[ \nabla \cdot z^+ = \nabla \cdot z^- = 0, \]
\[ z^+(0) = z^+_0 = u_0 + b_0, \quad z^-(0) = z^-_0 = u_0 - b_0, \]

where \( \pi = p + \frac{1}{2}b^2 \). Taking the operation \( \Delta_k \) on both sides of (3.1), we get
\[
\begin{align*}
\{ & \frac{\partial t}{\partial t} \Delta_k z^+ + z^- \cdot \nabla \Delta_k z^+ + \nabla \Delta_k \pi = [z^-, \Delta_k] \cdot \nabla z^+, \\
& \frac{\partial t}{\partial t} \Delta_k z^- + z^+ \cdot \nabla \Delta_k z^- + \nabla \Delta_k \pi = [z^+, \Delta_k] \cdot \nabla z^-, \\
\end{align*}
\]
where we denote the commutators
\[
[z^-, \Delta_k] \cdot \nabla z^+ \triangleq z^- \cdot \nabla \Delta_k z^+ - \Delta_k ((z^- \cdot \nabla) z^+), \\
[z^+, \Delta_k] \cdot \nabla z^- \triangleq z^+ \cdot \nabla \Delta_k z^- - \Delta_k ((z^+ \cdot \nabla) z^-).
\]

Let \( X^+_t(\alpha) \) and \( X^-_t(\alpha) \) be the solutions of the following ordinary differential equations:
\[
\begin{align*}
\{ & \frac{\partial t}{\partial t} X^+_t(\alpha) = z^- (X^+_t(\alpha), t), \\
& \frac{\partial t}{\partial t} X^-_t(\alpha) = z^+ (X^-_t(\alpha), t), \\
& X^+_t(\alpha)|_{t=0} = X^-_t(\alpha)|_{t=0} = \alpha.
\end{align*}
\]

Then, it follows from (3.2) that
\[
\begin{align*}
\frac{d}{dt} \Delta_k z^+ (X^+_t(\alpha), t) &= [z^-, \Delta_k] \cdot \nabla z^+ (X^+_t(\alpha), t) - \nabla \Delta_k \pi (X^+_t(\alpha), t), \\
\frac{d}{dt} \Delta_k z^- (X^-_t(\alpha), t) &= [z^+, \Delta_k] \cdot \nabla z^- (X^-_t(\alpha), t) - \nabla \Delta_k \pi (X^-_t(\alpha), t),
\end{align*}
\]
which implies that
\[
|\Delta_k z^+ (X^+_t(\alpha), t)| \leq |\Delta_k z^+_0(\alpha)| + \int_0^t \left| ([z^-, \Delta_k] \cdot \nabla z^+) (X^+_\tau(\alpha), \tau) \right| d\tau \\
+ \int_0^t \left| \Delta_k \nabla \pi (X^+_\tau(\alpha), \tau) \right| d\tau.
\]

Multiplying \( 2^{ks} \), taking \( \ell^q(\mathbf{Z}) \) norm on both sides of (3.5), we get by using Minkowski inequality that
\[
\left( \sum_k |2^{ks} \Delta_k z^+ (X^+_t(\alpha), t)|^q \right)^{\frac{1}{q}} \leq \left( \sum_k |2^{ks} \Delta_k z^+_0(\alpha)|^q \right)^{\frac{1}{q}} + \int_0^t \left( \sum_k |2^{ks} \Delta_k \nabla \pi (X^+_\tau(\alpha), \tau)|^q \right)^{\frac{1}{q}} d\tau \\
+ \int_0^t \left( \sum_k |2^{ks} ([z^-, \Delta_k] \cdot \nabla z^+) (X^+_\tau(\alpha), \tau)|^q \right)^{\frac{1}{q}} d\tau.
\]
Next, taking the $L^p$ norm with respect to $\alpha \in \mathbb{R}^d$ on both sides of (3.5), we get by using the Minkowski inequality that
\[
\left( \int_{\mathbb{R}^d} \left| \left( \sum_k a_k \nabla \Delta z^+(X^+_t(\alpha), t) \right)^q \right|^p \, d\alpha \right)^{\frac{1}{p}} 
\leq \|z^+_0\|_{F^p_{s,q}} + \int_0^t \left( \int_{\mathbb{R}^d} \left| \left( \sum_k a_k \nabla \Delta z^+(X^+_t(\alpha), \tau) \right)^q \right|^p \, d\alpha \right)^{\frac{1}{p}} \, d\tau 
+ \int_0^t \left( \int_{\mathbb{R}^d} \left| \sum_k a_k \nabla \Delta z^+(X^+_t(\alpha), \tau) \right| \right|^p \, d\alpha \right)^{\frac{1}{p}} \, d\tau. \tag{3.7}
\]
Using the fact that $X^+_t(\alpha)$ is a volume-preserving diffeomorphism due to $\text{div} z^+ = 0$, we get from (3.7) that
\[
\|z^+(t)\|_{F^p_{s,q}} \leq \|z^+_0\|_{F^p_{s,q}} + \int_0^t \|\nabla \pi\|_{F^p_{s,q}} \, d\tau 
+ \int_0^t \left( \|2^{ks}(\nabla \Delta z^+)\|_{\ell^q(k \in \mathbb{Z})} \right)^p \, d\tau. \tag{3.8}
\]
Thanks to Proposition 4.1, the last term on the right side of (3.8) is dominated by
\[
\int_0^t (\|\nabla z^+\|_{\infty} + \|\nabla z^-\|_{\infty})(\|z^-\|_{F^p_{s,q}} + \|z^+\|_{F^p_{s,q}}) \, d\tau. \tag{3.9}
\]
Next, we estimate the second term on the right side of (3.5). Taking the divergence on both sides of (3.11), we obtain the following representation of the pressure
\[
\pi = (-\Delta)^{-1}(\partial_j z^- \partial_i z^+_j) = (-\Delta)^{-1} \partial_i \partial_j (z^- \partial^*_j z^+_+). \tag{3.10}
\]
For $l, m \in [1, d]$, we have
\[
\partial_l \partial_m \pi = (-\Delta)^{-1} \partial_l \partial_m (\partial_j z^- \partial_i z^+_j) = R_l R_m (\partial_j z^- \partial_i z^+_j),
\]
where $R_l$ denotes the Riesz transform. Thanks to the boundedness of the Riesz transform in the homogeneous Triebel-Lizorkin spaces $[12]$, Lemma 2.2 and Proposition 2.3, we get
\[
\|\nabla \pi\|_{F^p_{s,q}} \leq C \sum_{l,m=1}^d \|\partial_l \partial_m \pi\|_{F^p_{s-1,q}} \leq C \|\partial_j z^- \partial_i z^+_j\|_{F^p_{s-1,q}} 
\leq C (\|\nabla z^-\|_{\infty} \|\nabla z^+\|_{F^p_{s-1,q}} + \|\nabla z^-\|_{\infty} \|\nabla z^+\|_{F^p_{s-1,q}}) 
\leq C (\|\nabla z^-\|_{\infty} \|z^+\|_{F^p_{s,q}} + \|\nabla z^+\|_{\infty} \|z^-\|_{F^p_{s,q}}). \tag{3.11}
\]
Plugging (3.9) and (3.11) into (3.8) yields that
\[
\|z^+(t)\|_{F^p_{s,q}} \leq \|z^+_0\|_{F^p_{s,q}} + C \int_0^t (\|\nabla z^+\|_{\infty} + \|\nabla z^-\|_{\infty})(\|z^-\|_{F^p_{s,q}} + \|z^+\|_{F^p_{s,q}}) \, d\tau. \tag{3.12}
\]
Similar argument also leads to
\[
\|z^-(t)\|_{F^p_{s,q}} \leq \|z^-_0\|_{F^p_{s,q}} + C \int_0^t (\|\nabla z^+\|_{\infty} + \|\nabla z^-\|_{\infty})(\|z^-\|_{F^p_{s,q}} + \|z^+\|_{F^p_{s,q}}) \, d\tau. \tag{3.13}
\]
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In order to get the inhomogeneous version of (3.12) and (3.13), we have to estimate the \( L^p \) norm of \((z^+, z^-)\). Multiplying the first equation of (3.11) by \(|z^+|^{p-2}z^+\) and the second one by \(|z^-|^{p-2}z^-\), integrating the resulting equations over \( \mathbb{R}^d \), we obtain

\[
\|z^+\|_p + \|z^-\|_p \leq \|z_0^+\|_p + \|z_0^-\|_p + C \int_0^t \|\nabla \pi(\tau)\|_p d\tau. \tag{3.14}
\]

Using (3.10) and the \( L^p \)-boundedness of the Riesz transform, we get

\[
\|\nabla \pi\|_p \leq C \|z^- \cdot \nabla z^+\|_p \leq C \|\nabla z^+\|_\infty \|z^-\|_p. \tag{3.15}
\]

Summing up (3.12) - (3.15) yields that

\[
\|z^+(t)\|_{F^s_{p,q}} + \|z^-(t)\|_{F^s_{p,q}} \leq \|z_0^+\|_{F^s_{p,q}} + \|z_0^-\|_{F^s_{p,q}} + C \int_0^t (\|\nabla z^+\|_\infty + \|\nabla z^-\|_\infty) (\|z^-\|_{F^s_{p,q}} + \|z^+\|_{F^s_{p,q}}) d\tau, \tag{3.16}
\]

which together with the Gronwall inequality gives

\[
\|(z^+(t), z^-(t))\|_{F^s_{p,q}} \leq \|(z_0^+, z_0^-)\|_{F^s_{p,q}} \exp \left( C \int_0^t (\|\nabla z^+\|_\infty + \|\nabla z^-\|_\infty) d\tau \right). \tag{3.17}
\]

**Step 2.** Approximate solutions and uniform estimates.

We construct the approximate solutions of (3.11). Define the sequence \( \{u^{(n)}, b^{(n)}\}_{n=0}^{\infty} \) by solving the following systems:

\[
\begin{aligned}
\partial_t u^{(n+1)} + u^{(n)} \cdot \nabla u^{(n+1)} - b^{(n)} \cdot \nabla b^{(n+1)} &= -\nabla z_1^{(n+1)}, \\
\partial_t b^{(n+1)} + u^{(n)} \cdot \nabla b^{(n+1)} - b^{(n)} \cdot \nabla u^{(n+1)} &= -\nabla z_2^{(n+1)}, \\
\nabla \cdot b^{(n+1)} &= \nabla \cdot u^{(n+1)} = 0,
\end{aligned}
\]

\[
\{ (u^{(n+1)}, b^{(n+1)}) \}_{t=0}^{\infty} = \begin{cases} S_{n+2}(u_0, b_0). \end{cases} \tag{3.18}
\]

We set \((u^{(0)}, b^{(0)}) = (0, 0)\), and

\[
z^{+(n)} = u^{(n)} + b^{(n)}, \quad z^{-(n)} = u^{(n)} - b^{(n)}.
\]

Then (3.18) can be reduced to

\[
\begin{aligned}
\partial_t z^{+(n+1)} + (z^{-(n)} \cdot \nabla) z^{+(n+1)} &= -\nabla z_1^{(n+1)}, \\
\partial_t z^{-(n+1)} + (z^{+(n)} \cdot \nabla) z^{-(n+1)} &= -\nabla z_2^{(n+1)}, \\
\nabla \cdot z^{+(n+1)} &= \nabla \cdot z^{-(n+1)} = 0, \quad \forall n \in \mathbb{N}.
\end{aligned}
\]

\[
\{ z^{+(n+1)}(0) = S_{n+2} z_0^+, \quad z^{-(n+1)}(0) = S_{n+2} z_0^- \}, \tag{3.19}
\]

where \((z^{+(0)}, z^{-(0)}) = (0, 0)\). Similar to the proof of (3.16), we conclude that

\[
\begin{aligned}
\&(\|(z^{+(n+1)}(t), z^{-(n+1)}(t))\|_{F^s_{p,q}} \\
\leq &\|(z_0^+, z_0^-)\|_{F^s_{p,q}} + C \int_0^t \left( \|\nabla z^{+(n)} \|_\infty + \|\nabla z^{-(n)} \|_\infty \right) \\
&\times \left( \|\nabla z^{+(n)} \|_{F^s_{p,q}} + \|\nabla z^{-(n+1)} \|_{F^s_{p,q}} \right) d\tau, \tag{3.20}
\end{aligned}
\]

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where we used the fact that
\[\|(S_{n+2\delta z_0}, S_{n+2\delta z_0})\|_{F_{p,q}} \leq \|(z_0^+, z_0^-)\|_{F_{p,q}}.\]

Note that \(F_{p,q}^{s-1} \hookrightarrow L^\infty\) for \(s - 1 > \frac{2}{p}\). (3.20) ensures that there exists \(T_0 = T_0(\|(z_0^+, z_0^-)\|_{F_{p,q}})\) such that for any \(n, t \in [0, T_0]\)
\[\|(z^{+(n)}(t), z^{-(n)}(t))\|_{F_{p,q}} \leq 2\|(z_0^+, z_0^-)\|_{F_{p,q}}.\]  

(3.21)

**Step 3. Existence.**

We will show that there exists a positive time \(T_1(\leq T_0)\) independent of \(n\) such that \([z^{+(n)}, z^{-(n)}]\) is a Cauchy sequence in \(X^{s-1} \triangleq C([0, T_1]; F_{p,q}^{s-1}).\) For this purpose, we set
\[\delta z^{+(n+1)} = z^{+(n+1)} - z^{+(n)}, \quad \delta z^{-(n+1)} = z^{-(n+1)} - z^{-(n)}, \quad \delta \pi_j^{(n+1)} = \pi_j^{(n+1)} - \pi_j^{(n+1)}, \quad j = 1, 2.\]

Using (3.19), it is easy to verify that the difference \((\delta z^{+(n+1)}, \delta z^{-(n+1)}, \delta \pi^{(n)})\) satisfies
\[
\begin{aligned}
\partial_t \delta z^{+(n+1)} + z^{-(n)} \cdot \nabla \delta z^{+(n+1)} &= -\delta z^{-(n)} \cdot \nabla z^{+(n)} - \nabla \delta \pi_1^{(n+1)}, \\
\partial_t \delta z^{-(n+1)} + z^{+(n)} \cdot \nabla \delta z^{-(n+1)} &= -\delta z^{+(n)} \cdot \nabla z^{-(n)} - \nabla \delta \pi_2^{(n+1)}, \\
(\delta z^{+(n+1)}, \delta z^{-(n+1)})|_{t=0} &= \Delta_{n+1}(z_0^+, z_0^-).
\end{aligned}
\]

(3.22)

Applying \(\Delta_k\) to the first equation of (3.22), we get
\[
\partial_t \Delta_k \delta z^{+(n+1)} + z^{-(n)} \cdot \nabla \Delta_k \delta z^{+(n+1)} = [z^{-(n)}, \Delta_k] \cdot \nabla \delta z^{+(n+1)} - \Delta_k (\delta z^{-(n)} \cdot \nabla z^{+(n)}) - \nabla \Delta_k \delta \pi_1^{(n+1)}.
\]

(3.23)

Exactly as in the proof of (3.19), we get
\[
\|\delta z^{+(n+1)}\|_{F_{p,q}^{s-1}} \leq C \|\Delta_{n+1} z_0^+\|_{F_{p,q}^{s-1}} + \int_0^t \|2^{k(s-1)}(z^{-(n)}, \Delta_k) \cdot \nabla \delta z^{+(n+1)}(\alpha, \tau)\|_{L^p(Z)} \|d\tau\|_p + \int_0^t \|\nabla \delta \pi_1^{(n+1)}(\tau)\|_{F_{p,q}^{s-1}} d\tau + \int_0^t \|\nabla \delta \pi_2^{(n+1)}(\tau)\|_{F_{p,q}^{s-1}} d\tau.
\]

(3.24)

Thanks to the Fourier support of \(\Delta_{n+1} z_0^+\), we have
\[
\|\Delta_{n+1} z_0^+\|_{F_{p,q}^{s-1}} \leq C 2^{-(n+1)} \|z_0^+\|_{F_{p,q}^s}.
\]

(3.25)

Using Proposition 4.1 and the embedding \(F_{p,q}^{s-1} \hookrightarrow L^\infty\), the second term on the right side of (3.24) is dominated by
\[
\|\nabla z^{-(n)}\|_\infty \|\delta z^{+(n+1)}\|_{F_{p,q}^{s-1}} + \|\delta z^{+(n+1)}\|_\infty \|\nabla z^{-(n)}\|_{F_{p,q}^{s-1}} \leq C \|z^{-(n)}\|_{F_{p,q}^{s-1}} \|\delta z^{+(n+1)}\|_{F_{p,q}^{s-1}} + \|\delta z^{+(n+1)}\|_{F_{p,q}^{s-1}}.
\]

(3.26)

Thanks to Proposition 2.1, the third term on the right hand side of (3.24) is dominated by
\[
\|\delta z^{-(n)}\|_\infty \|\nabla z^{+(n)}\|_{F_{p,q}^{s-1}} + \|\delta z^{-(n)}\|_{F_{p,q}^{s-1}} \|\nabla z^{+(n)}\|_\infty \leq C \|\delta z^{-(n)}\|_{F_{p,q}^{s-1}} \|z^{+(n)}\|_{F_{p,q}^s}.
\]

(3.27)
Taking the divergence on both sides of (3.22), we get
\[ \delta \pi_1^{(n+1)} = \partial_j (-\Delta)^{-1} (\delta z_i^{-(n)} \partial_i z_j^{+(n)}) + \partial_i (-\Delta)^{-1} (\partial_j z_i^{-(n)} \delta z_j^{+(n+1)}). \]
Hence, we have
\[ \partial_i \delta \pi_1^{(n+1)} = R_i R_j (\delta z_i^{-(n)} \partial_j z_j^{+(n)}) + R_i R_i (\partial_j z_i^{-(n)} \delta z_j^{+(n+1)}), \]
which together with Proposition 2.3 and the boundedness of the Riesz transform in the homogeneous Triebel-Lizorkin spaces gives
\[
\| \nabla \delta \pi_1^{(n+1)} \|_{F_p^{s,q}} \lesssim \| \delta z_i^{-(n)} \partial_i z_j^{+(n)} \|_{F_p^{s,q}} + \| \partial_j z_i^{-(n)} \delta z_j^{+(n+1)} \|_{F_p^{s,q}}
\lesssim \| \delta z_i^{-(n)} \|_{\infty} \| \nabla z_j^{+(n)} \|_{F_p^{s,q}} + \| \delta z_i^{-(n)} \|_{F_p^{s,q}} \| \nabla z_j^{+(n)} \|_{\infty}
+ \| \nabla z_i^{-(n)} \|_{\infty} \| \delta z_j^{-(n+1)} \|_{F_p^{s,q}} + \| \delta z_i^{-(n)} \|_{F_p^{s,q}} \| \delta z_j^{-(n+1)} \|_{F_p^{s,q}}.
\]
By summing up (3.24)-(3.28), we get
\[
\| \delta z_j^{+(n+1)} \|_{F_p^{s,q}} \leq C 2^{-((n+1)/p)} \| \delta z_0 \|_{F_p^{s,q}} + C \int_0^t \left( \| \delta z_i^{-(n)} \|_{F_p^{s,q}} \| \delta z_j^{+(n+1)} \|_{F_p^{s,q}} \right) dt.
\] (3.29)
Now, we estimate the $L^p$ norm of $\delta z_j^{+(n+1)}$. Multiplying $|\delta z_j^{+(n+1)}|^{p-2} \delta z_j^{+(n+1)}$ on both sides of the first equation of (3.22), and integrating the resulting equations over $\mathbb{R}^d$, we obtain
\[
\| \delta z_j^{+(n+1)}(t) \|_p \leq \| \Delta z_j^{+(n+1)} \|_p + \int_0^t \| \delta z_i^{-(n)} \|_p \| \nabla z_j^{+(n+1)}(\tau) \|_p d\tau + \int_0^t \| \nabla \delta \pi_1^{(n+1)}(\tau) \|_p d\tau
\leq \| \Delta z_j^{+(n+1)} \|_p + C \int_0^t \| \delta z_i^{-(n)} \|_p \| \nabla z_j^{+(n+1)}(\tau) \|_p d\tau
+ C \int_0^t \| \nabla z_i^{-(n)} \|_p \| \delta z_j^{+(n+1)} \|_p d\tau,
\]
which together with (3.29) gives
\[
\| \delta z_j^{+(n+1)} \|_{F_p^{s,q}} \leq C 2^{-((n+1)/p)} \| \delta z_0 \|_{F_p^{s,q}} + C \int_0^t \left( \| \delta z_i^{-(n)} \|_{F_p^{s,q}} \| \delta z_j^{+(n+1)} \|_{F_p^{s,q}} \right) dt.
\] (3.30)
Exactly as in the proof of (3.30), we also have
\[
\| \delta z_i^{-(n+1)} \|_{F_p^{s,q}} \leq C 2^{-((n+1)/p)} \| \delta z_0 \|_{F_p^{s,q}} + C \int_0^t \left( \| \delta z_j^{+(n)} \|_{F_p^{s,q}} \| \delta z_i^{-(n+1)} \|_{F_p^{s,q}} \right) dt.
\] (3.31)
Adding up (3.30) and (3.31), we obtain
\[
\| (\delta z^{+ (n+1)}, \delta z^{- (n+1)}) \|_{F^{s-1}_{p,q}} \leq 2^{-(n+1)} (\| z_0^+ \|_{F^{s}_{p,q}} + \| z_0^- \|_{F^{s}_{p,q}}) + T \sup_{t \in [0,T]} \| (z^{+ (n)}, z^{- (n)}) \|_{F^{s}_{p,q}} \| (\delta z^{+ (n+1)}, \delta z^{- (n+1)}) \|_{F^{s-1}_{p,q}}
\]
plus
\[
+ T \sup_{t \in [0,T]} \| (z^{+ (n)}, z^{- (n)}) \|_{F^{s}_{p,q}} \| (\delta z^{+ (n)}, \delta z^{- (n)}) \|_{F^{s-1}_{p,q}},
\]
which together with (3.21) yields that
\[
\| (\delta z^{+ (n+1)}, \delta z^{- (n+1)}) \|_{X^{s-1}_T} \leq C_1 2^{-(n+1)} + C_1 T \| (\delta z^{+ (n+1)}, \delta z^{- (n+1)}) \|_{X^{s-1}_T}
+ C_1 T \| (\delta z^{+ (n)}, \delta z^{- (n)}) \|_{X^{s-1}_T},
\]
(3.32)
where \( C_1 = C_1(\| (z_0^+, z_0^-) \|_{F^{s}_{p,q}}) \). Thus, if \( C_1 T \leq \frac{1}{2} \), then
\[
\| (\delta z^{+ (n+1)}, \delta z^{- (n+1)}) \|_{X^{s-1}_T} \leq C_1 2^{-n} + 2C_1 T \| (\delta z^{+ (n)}, \delta z^{- (n)}) \|_{X^{s-1}_T}.
\]
This implies that
\[
\| (\delta z^{+ (n+1)}, \delta z^{- (n+1)}) \|_{X^{s-1}_T} \leq 2C_1 2^{-(n+1)}.
\]
Thus, \( \{ z^{+ (n)}, z^{- (n)} \}_{n \in \mathbb{N}_0} \) is a Cauchy sequence in \( X^{s-1}_{T_1} \). By the standard argument, for \( T_1 \leq \min \{ T_0, \frac{1}{4C_1} \} \), the limit \( (z^+, z^-) \in X^{s}_{T_1} \) solves the equation (3.1) with the initial data \( (z_0^+, z_0^-) \). Moreover, \( (z^+, z^-) \) satisfies
\[
\| (z^+, z^-) (t) \|_{L^\infty_{T_1} (F^{s}_{p,q})} \leq C \| (z_0^+, z_0^-) \|_{F^{s}_{p,q}},
\]
which implies \( (u, b) \) is a solution of (1.1) with the initial data \( (u_0, b_0) \in F^{s}_{p,q} \), and
\[
\| (u, b) (t) \|_{L^\infty_T (F^{s}_{p,q})} \leq C \| (u_0, b_0) \|_{F^{s}_{p,q}}.
\]

The proof of the uniqueness. Consider \( (z^{+}, z^{-}) \in C_{T_1} (F^{s}_{p,q}) \) is another solution to (3.1) with the same initial data. Let \( \delta z^+ = z^+ - z^+ \) and \( \delta z^- = z^- - z^- \). Then \( (\delta z^+, \delta z^-) \) satisfies the following equations
\[
\begin{align*}
\partial_t \delta z^+ + (z^- \cdot \nabla) \delta z^+ = & - (\delta z^- \cdot \nabla) z^+ - \nabla (\pi - \pi'), \\
\partial_t \delta z^- + (z^+ \cdot \nabla) \delta z^- = & - (\delta z^+ \cdot \nabla) z^- - \nabla (\pi - \pi'), \\
\nabla \cdot \delta z^+ = & \nabla \cdot \delta z^- = 0.
\end{align*}
\]
In the same way as deriving in (3.32), we obtain
\[
\| (\delta z^+, \delta z^-) \|_{X^{s-1}_T} \leq C_2 T \| (\delta z^+, \delta z^-) \|_{X^{s-1}_T}
\]
for sufficiently small \( T \). This implies that \( (\delta z^+, \delta z^-) \equiv 0 \), i.e., \( (z^+, z^-) \equiv (z^+, z^-) \).
Blow-up Criterion. By means of Proposition 1.1 in [7] and
\[
\|(\nabla z^+, \nabla z^-)\|_{\mathcal{F}_{\infty, \infty}^0} \lesssim \|(\nabla \times z^+, \nabla \times z^-)\|_{\mathcal{F}_{\infty, \infty}^0},
\]
we have
\[
\|(\nabla z^+, \nabla z^-)\|_{\infty} \lesssim \left(1 + \|(\nabla z^+, \nabla z^-)\|_{\mathcal{F}_{\infty, \infty}^0} \left(\log \left(1 + \|(\nabla \times z^+, \nabla \times z^-)\|_{\mathcal{F}_{p,q}^{s-1}}\right) + 1\right)\right)
\]
Plugging the above estimates into (3.16) then by Gronwall’s lemma yields that
\[
\|(z^+, z^-)\|_{\mathcal{F}_{p,q}^s} \leq \|(z_0^+, z_0^-)\|_{\mathcal{F}_{p,q}^s} \exp\left[C \int_0^t \left(1 + \|(\nabla \times z^+, \nabla \times z^-)\|_{\mathcal{F}_{\infty, \infty}^0} \right) d\tau\]
\]
which implies the blow-up criterion. This finishes the proof of the Theorem 1.1.

4 Appendix

Let us recall the para-differential calculus which enables us to define a generalized product between distributions, which is continuous in many functional spaces where the usual product does not make sense (see [3]). The para-product between \(u\) and \(v\) is defined by
\[
T_u v \triangleq \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v.
\]

We then have the following formal decomposition:
\[
uv = T_u v + T_v u + R(u, v), \tag{4.33}
\]
with
\[
R(u, v) = \sum_{j \in \mathbb{Z}} \Delta_j u \tilde{\Delta}_j v \quad \text{and} \quad \tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}.
\]
The decomposition (4.33) is called the Bony’s para-product decomposition.

**Proposition 4.1** Let \((p, q) \in (1, \infty) \times (1, \infty)\), or \(p = q = \infty\), and \(f\) be a solenoidal vector field. Then for \(s > 0\)
\[
\left\|\|2^k [f, \Delta_k] \cdot \nabla g]\|_{L^p(\mathbb{Z})}\right\|_p \lesssim \left(\|\nabla f\|_\infty \|g\|_{\mathcal{F}_{p,q}^s} + \|\nabla g\|_\infty \|f\|_{\mathcal{F}_{p,q}^s}\right). \tag{4.34}
\]

or for \(s > -1\)
\[
\left\|\|2^k (f, \Delta_k) \cdot \nabla g]\|_{L^p(\mathbb{Z})}\right\|_p \lesssim \left(\|\nabla f\|_\infty \|g\|_{\mathcal{F}_{p,q}^s} + \|g\|_\infty \|\nabla f\|_{\mathcal{F}_{p,q}^s}\right). \tag{4.35}
\]

**Proof.** By the Einstein convention on the summation over repeated indices \(i \in [1, d]\), and the Bony’s paraproduct decomposition we decompose
\[
[f, \Delta_k] \cdot \nabla g = [f_1, \Delta_k] \partial_i g + T_{\Delta_k} \partial_i g + T_{\Delta_k \partial_i g} f_i - \Delta_k (T_{\partial_i g} f_i) - \Delta_k (R(f_i, \partial_i g)) \triangleq I + II + III + IV,
\]
where $T'_uv$ stands for $T_u v + R(u, v)$. Thank to the support condition (2.1), we rewrite

$$
|I| = \left| \sum_{k' \sim k} [S_{k' - 1} f_i, \Delta_k] \partial_i \Delta_k' g \right|
$$

$$
= \left| \sum_{k' \sim k} \int_{\mathbb{R}^d} \left( S_{k' - 1} f_i(x) - S_{k' - 1} f_i(y) \right) 2^{kd} h(2^k (x - y)) \partial_i \Delta_k' g(y) dy \right|, \quad (4.36)
$$

where $k' \sim k$ stands for $|k' - k| \leq 4$. Integrate by part and use $\text{div} f = 0$, the integrand in (4.36) is

$$
\left( S_{k' - 1} f_i(x) - S_{k' - 1} f_i(y) \right) 2^{k(d+1)} (\partial_i h)(2^k (x - y)) \Delta_k' g(y)
$$

which was dominated by

$$
\|\nabla S_{k' - 1} f \|_\infty 2^k |x - y| 2^{kd} |\nabla h(2^k (x - y))| \|\Delta_k' g(y)\|.
$$

(4.37)

Recall $h(x) \in S(\mathbb{R}^d)$, it is easy to see that $|x \nabla h(x)|$ satisfies Lemma 2.5, so (4.37) is less than

$$
C \|\nabla S_{k' - 1} f \|_\infty M(|\Delta_k' g(\cdot)|)(x). \quad (4.38)
$$

Multiplying $2^{ks}$ on both sides of (4.36), taking $\ell^q(Z)$ norm then taking $L^p$ norm and putting (4.38) into the resulting inequality, we have

$$
\left\| 2^{ks} |I(x)| \right\|_{\ell^q(Z)} \lesssim \|\nabla S_{k' - 1} f \|_\infty \left\| \sum_{k' \sim k} 2^{(k-k')s} M(2^{k' s} |\Delta_k' g(\cdot)|)(x) \right\|_{\ell^q(Z)}
$$

$$
\lesssim \|\nabla f \|_\infty \left\| M(2^{k' s} |\Delta_k' g(\cdot)|)(x) \right\|_{\ell^q(Z)}
$$

$$
\lesssim \|\nabla f \|_\infty \|2^{ks} |\Delta_k g(x)|\|_{\ell^q(Z)} \lesssim \|\nabla f \|_\infty \|g\|_{F^s_{p,q}}. \quad (4.39)
$$

where we used Lemma 2.4 in the third inequality. Let us turn to the term $II$, thanks to the definition of $II$,

$$
|II| = \left| \sum_{k' \geq k - 2} S_{k' - 2} \partial_i \Delta_k g(x) \right| \leq \sum_{k' \geq k - 2} \|\nabla \Delta_k g\|_\infty |\Delta_k' f(x)|. \quad (4.40)
$$

Then thanks to the convolution inequality for series, we get for $s > 0$,

$$
\left\| 2^{ks} |II(x)| \right\|_{\ell^q(Z)} \lesssim \|\nabla \Delta_k g\|_\infty \left\| \sum_{k' \geq k - 2} 2^{(k-k')s} 2^{ks} |\Delta_k' f(x)| \right\|_{\ell^q(Z)}
$$

$$
\lesssim \|\nabla \Delta_k g\|_\infty \left\| \sum_{k' \geq k - 2} 2^{-ks} \chi_{k' \geq 2} \|\Delta_k' f(x)\|_{\ell^q(Z)} \right\|_{\ell^q(Z)}
$$

$$
\lesssim \|\nabla g\|_\infty \left\|2^{ks} |\Delta_k f(x)|\right\|_{\ell^q(Z)} \lesssim \|\nabla g\|_\infty \|f\|_{F^s_{p,q}}. \quad (4.41)
$$

For the term $III$,

$$
|III| = \left| \sum_{k' \sim k} \Delta_k (S_{k' - 1} \partial_i g \Delta_k' f_i) \right| \lesssim \sum_{k' \sim k} \|M(S_{k' - 1} \partial_i g \Delta_k' f_i)(x)\|
$$

$$
\lesssim \sum_{k' \sim k} \|M(|\Delta_k' f|)(x)\| \|S_{k' - 1} \nabla g\|_\infty. \quad (4.42)
$$
Using (4.42) and in the same way as leading to (4.39) yields
\[
\left\| 2^{ks} |III(x)| \right\|_{\ell^q(Z)} \lesssim \left\| \nabla S_{k'-1} g \right\|_{\infty} \left\| \sum_{k' \sim k} 2^{(k-k')s} M(2^{k'} f)(x) \right\|_{\ell^q(Z)} \lesssim \left\| \nabla g \right\|_{\infty} \left\| M(2^{k'} f)(x) \right\|_{\ell^q(Z)} \lesssim \left\| \nabla g \right\|_{\infty} \left\| 2^{k'} |\Delta_k f(x)| \right\|_{\ell^q(Z)} \lesssim \left\| \nabla g \right\|_{\infty} \| f \|_{\tilde{F}_{p,q}}. \tag{4.43}
\]

In view of $\text{div} f = 0$ and integrating by part, we have
\[
|IV| = \left| \sum_{k' \geq k-3} \Delta_k (\Delta_{k'} f_i \partial_i \Delta_{k'} g) \right| = \left| \sum_{k' \geq k-3} \int_{\mathbb{R}^d} 2^{kd}(2^k(x - y)) \Delta_{k'} f_i(y) \partial_i \Delta_{k'} g(y) dy \right|
\leq \left\| \Delta_k f \right\|_{\infty} \left\| M(2^{k'} \tilde{\Delta}_k g)(x) \right\|_{\ell^q(Z)} \lesssim \sum_{k' \geq k-3} 2^k M(\tilde{\Delta}_k g(x)) \| \Delta_{k'} f \|_{\infty}. \tag{4.44}
\]
The convolution inequality for series and Lemma 2.4 allow us to give that for $s + 1 > 0$,
\[
\left\| 2^{ks} |IV(x)| \right\|_{\ell^q(Z)} \lesssim \left\| \nabla \Delta_k f \right\|_{\infty} \left\| \sum_{k' \geq k-3} 2^{(k-k')(s+1)} M(2^{k'} \tilde{\Delta}_k g)(x) \right\|_{\ell^q(Z)} \lesssim \left\| \nabla f \right\|_{\infty} \left\| M(2^{k'} \tilde{\Delta}_k g)(x) \right\|_{\ell^q(Z)} \lesssim \left\| \nabla f \right\|_{\infty} \| g \|_{\tilde{F}_{p,q}}. \tag{4.45}
\]

Summing up (4.39), (4.41), (4.43) and (4.45), we get the desired inequality (4.34).

In order to prove the inequality (4.35), we only indicate how to get the bound on II and III since I and IV can be treated as above. We estimate the term II as
\[
|II| = \left| \sum_{k' \geq k-2} S_{k'+2} \partial_i \Delta_k g \Delta_{k'} f_i(x) \right| \lesssim \sum_{k' \geq k-2} 2^k \| \Delta_k g \|_{\infty} |\Delta_{k'} f(x)|.
\]
Then thanks to the convolution inequality for series, we get for $s + 1 > 0$,
\[
\left\| 2^{ks} |II(x)| \right\|_{\ell^q(Z)} \lesssim \left\| \Delta_k g \right\|_{\infty} \left\| \sum_{k' \geq k-2} 2^{(k-k')(s+1)} 2^{k'(s+1)} |\Delta_{k'} f(x)| \right\|_{\ell^q(Z)} \lesssim \left\| g \right\|_{\infty} \left\| 2^{-k(s+1)} \chi_{\{k \geq 2\}} \| \ell^1(Z) \right\| 2^{k'(s+1)} |\Delta_{k'} f(x)| \right\|_{\ell^q(Z)} \lesssim \left\| g \right\|_{\infty} \| f \|_{\tilde{F}_{p,q+1}}.
\]

Let’s turn to the term III,
\[
|III| \lesssim \sum_{k' \sim k} |M(|\Delta_{k'} f|)(x)| 2^{k'} |S_{k'-1} g|_{\infty}.
\]
Arguing similarly as in deriving (4.43) yields that
\[
\left\| 2^{ks} |III(x)| \right\|_{\ell^q(Z)} \lesssim \left\| S_{k'-1} g \right\|_{\infty} \left\| \sum_{k' \sim k} 2^{(k-k')s} M(2^{k'} f)(x) \right\|_{\ell^q(Z)} \lesssim \left\| g \right\|_{\infty} \| f \|_{\tilde{F}_{p,q+1}}.
\]
Thus the desired inequality (4.35) is obtained.
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