Optical Response and Drift Matrix of Quadratic Optomechanical System

A. Kundu

1Lovely Professional University, Phagwara 144411, Punjab, India.

Nonlinear interactions in optomechanical systems play a crucial role in many emerging number of interesting studies and phenomena such as existence of optomechanical chaos introduced by F. Monifi et al. [Nature Photonics 10, 399405 (2016)] and optomechanical symmetry breaking proposed by Zhong-Peng Liu et al. [Phys. Rev. Lett. 117, 110802 (2016)]. In this article we have theoretically examined quadratically coupled optomechanical system containing two atomic levels. We have first studied the solution of various modes of the system at steady state and later we have observed the variation of Transmission Intensity ($T$) with several parameters of the system. Further we have extended our analyzation to find Drift matrix of the quadratic optomechanical system and stability conditions by adiabatically eliminating atomic degree of freedom.

keywords: Nonlinear interaction . Optomechanical system . Langevin equation . Transmission amplitude . Drift matrix . Adiabetic elimination

1. INTRODUCTION

Optomechanical coupling via radiation pressure is a very successful approach to prepare and manipulate quantum states of mechanical oscillators. A widely used cavity optomechanical system, is represented by a single mode Febry-Pérot cavity with one movable end (i.e. less heavy mirror). The average position of the movable mirror is controlled by radiation pressure of light intensity inside cavity. This circulating intensity introduces an interaction between the cavity and mechanical degree of freedom. Coupling of optical and mechanical degrees of freedom influenced by externally applied radiation pressure has many applications from gravitational-wave detectors[1, 2] to laser cooling[3–5]. A large amount of studies of optomechanical systems deals with linear optomechanical coupling where, the cavity mode couples to the position of the movable mirror. Linear coupling has wide variety of applications such as, it is used for quantum ground state cooling[6–10] of the mechanical mirror, entanglement between light and the mirror[11–13], electromagnetically induced transparency[14, 15], optomechanical induced transparency[16] and studies concerning normal-mode splitting[17, 18]. In several works quadratic optomechanical coupling have also been considered where, the cavity mode is coupled to the square of position of mechanical operator. Quadratic optomechanical coupling has many applications such as it has been used to observe quantization in mechanical energy[19], traditional two phonon laser cooling[20], tunable slow light[21], photon blockade[22], optomechanics at a single photon level[23] etc. In this article we have considered the variation of transmission intensity of an optically driven quadratic optomechanical system containing two atomic levels with various parameters of the following system. Previously, the variation of optical response (optical transmission) within two sideband limits[24] of cavity detuning for a linearly coupled optomechanical system has been discussed analytically as well as numerically in Ref.[25] which we have reproduced on the way for quadratic optomechanical system.
This work is organized as follows, in section (2) we have briefly discussed the constriction of optomechanical system followed by the Hamiltonian in mode as well as quadrature representation. In section (3) we have observed the dynamics of the system using quantum Langevin equation which followed by a steady state treatment. Under the steady state conditions we have analytically found out the dependence of cavity mode on mechanical and atomic modes. A mathematical expression of transmission amplitude and transmission intensity has been obtained by using the results of section (3) in (4) followed by schematic plotting of transmission amplitude with various parameters of the system. In the section (5) we have constructed a drift matrix of the quadratically coupled optomechanical system by observing dynamics of the system and introducing infinitesimal quantum fluctuations to the steady state values. Finally the article has been concluded in section (6).

2. SYSTEM HAMILTONIAN

In our system, we have considered an optomechanical system containing two level electronic system. Our system consists of mode of single frequency of single Febry-Pérot optical cavity and a single two level system inside the cavity. The arrangement of cavity is such that, it has one mirror which is heavier than the other and the heavier mirror is fixed in space hence, the lighter mirror can oscillate while exposed to radiation pressure with variable driven frequency ($\omega_D$). To find the dynamical as well as steady state solutions we have used quantum Langevin equations. The optomechanical system has been driven by an external coherent field, $\alpha_c = e^{-i\omega_D t}$ of single mode driven frequency $\omega_D$. The smaller and less heavier mirror is oscillating at a single mode frequency $\omega_m$ under the radiation pressure and its damping rate denoted by $\Gamma$. Hence our system Hamiltonian will be divided into two parts, $\hat{H}_{om}$ representing the Hamiltonian of the optomechanical system and $\hat{H}_e$ represents the Hamiltonian of two level electronic system. The optomechanical hamiltonian of the system is given by,

$$\hat{H}_{om} = \hbar \omega_a \hat{a}^\dagger \hat{a} + \hbar \omega_m \hat{b}^\dagger \hat{b} + \hbar g_{op} \hat{a}^\dagger \hat{a} \left( \hat{b} + \hat{b}^\dagger \right)^2 - i\hbar g (\hat{a}^\dagger e^{-i\omega_D t} + H.c.) (1)$$

In Eq.(1) the first and second term represents free energy of optical cavity and mechanical resonator with $\hat{a}^\dagger (\hat{a})$ and $\hat{b}^\dagger (\hat{b})$ are the creation(annihilation) operators of cavity and mechanical resonator. $g_{op}$ is the optomechanical coupling strength between cavity and mechanical mode(less heavier mirror) of the system.

After inserting the two level system inside the cavity, the contribution of two-level system must be included via the Hamiltonian,

$$\hat{H}_e = \hbar \omega_e \hat{e}^\dagger \hat{e} + \hbar g (\hat{e}^\dagger \hat{a} + \hat{a}^\dagger \hat{e}) (2)$$

where, $g$ is the coupling constant between the cavity field and two level system and, $\hat{\sigma}_+, \hat{\sigma}_-$ and, $\hat{\sigma}_z$ are the Pauli operators. For low excitation probability, the quantity, $\frac{1}{2}(|\langle \hat{\sigma}_z \rangle| + 1) \ll 1$, which allows us to apply the Holstein-Primakoff approximation in which atom is modeled by harmonic oscillator, so $\hat{\sigma}_- \rightarrow \hat{e}$, $\hat{\sigma}_+ \rightarrow \hat{e}^\dagger$ and $\hat{\sigma}_z \approx 2 \hat{e}^\dagger \hat{e}$. Hence the atom Hamiltonian of our system,

$$\hat{H}_e = \hbar \omega_e \hat{e}^\dagger \hat{e} + \hbar g (\hat{e}^\dagger \hat{a} + \hat{a}^\dagger \hat{e}) (3)$$

The total Hamiltonian of the system,

$$\hat{H} = \hat{H}_{om} + \hat{H}_e$$
using Eq.(1) and Eq.(3) we get,

\[ \hat{H} = \hbar \omega_a \hat{a}^\dagger \hat{a} + \hbar \omega_m \hat{b}^\dagger \hat{b} + h g_{op} \hat{a}^\dagger \hat{a} \left( \hat{b} + \hat{b}^\dagger \right)^2 + h \omega_c \hat{c}^\dagger \hat{c} + h g (\hat{c}^\dagger \hat{a} + \hat{c} \hat{a}^\dagger) - i \hbar \delta_{\hat{a}^\dagger \hat{c} - i \omega_D t} + H.c. \] (4)

2.1. Quadrature Representation

We may define our optomechanical Hamiltonian (1) in terms of quadrature of the mechanical mode i.e. \( \hat{b} = \frac{\hat{Q} + i \hat{P}}{\sqrt{2}} \) and \( \hat{b}^\dagger = \frac{\hat{Q} - i \hat{P}}{\sqrt{2}} \) as,

\[ \hat{H}_{om} = \hbar \omega_a \hat{a}^\dagger \hat{a} + \hbar \omega_m \left( \frac{\hat{Q}^2 + \hat{P}^2}{2} - 1 \right) + 2 h g_{op} \hat{a}^\dagger \hat{a} \hat{Q}^2 - i \hbar \delta_{\hat{a}^\dagger \hat{c} - i \omega_D t} + H.c. \] (5)

3. DYNAMICS OF THE SYSTEM

To get valuable information of the optomechanical system defined by Eq. (4) we need to observe the dynamics of the system. For this purpose we used Quantum Langevin Equations given by,

\[
\begin{align*}
\frac{d\hat{a}}{dt} &= i \hbar [\hat{H}, \hat{a}] - \frac{\epsilon}{2} \hat{a} + \sqrt{2} \epsilon \hat{a}_{in} \quad (6a) \\
\frac{d\hat{b}}{dt} &= i \hbar [\hat{H}, \hat{b}] - \frac{\Gamma}{2} \hat{b} + \sqrt{2} \Gamma \hat{b}_{in} \quad (6b) \\
\frac{d\hat{c}}{dt} &= i \hbar [\hat{H}, \hat{c}] - \frac{\gamma}{2} \hat{c} + \sqrt{2} \gamma \hat{c}_{in} \quad (6c)
\end{align*}
\]

where, \( \epsilon, \Gamma \) and \( \gamma \) are cavity, mirror and atomic mode damping. Now by expanding Eq.(6a)-6c, we get,

\[
\begin{align*}
\frac{d\hat{a}}{dt} &= -i \omega_a \hat{a} - i g \hat{c} - i g_{op} \hat{a} \left( \hat{b} + \hat{b}^\dagger \right)^2 - \eta e^{-i \omega_D t} - \frac{\epsilon}{2} \hat{a} + \sqrt{2} \epsilon \hat{a}_{in} \quad (7a) \\
\frac{d\hat{b}}{dt} &= -i \omega_m \hat{b} - 2 i g_{op} \hat{a}^\dagger \hat{a} \left( \hat{b} + \hat{b}^\dagger \right) - \frac{\Gamma}{2} \hat{b} + \sqrt{2} \Gamma \hat{b}_{in} \quad (7b) \\
\frac{d\hat{c}}{dt} &= -i \omega_c \hat{c} - ig \hat{a} - \frac{\gamma}{2} \hat{c} + \sqrt{2} \gamma \hat{c}_{in} \quad (7c)
\end{align*}
\]

The nonlinear terms in Eqs.(7a) and (7b) are arising due to the coupling between cavity and the quadratically moving mirror. Simplification of these type of equations are difficult to handle due to the complexity of calculations but with the help of quadratic representation given in Eq.(5) it can be easily analysed. To simplify the Eqs.(7a)-(7c) we use Rotating Wave Approximation (RWA).
where we replace \( \hat{a} \rightarrow \hat{a}[\exp(-i\omega_D t)] \), \( \hat{b} \rightarrow \hat{b}[\exp(-i\omega_D t)] \) and, \( \hat{e} \rightarrow \hat{e}[\exp(-i\omega_D t)] \) and arrive at,

\[
\begin{align*}
\frac{d\hat{a}}{dt} &= -i\Delta_a \hat{a} - ig\hat{e} - ig_{op}\hat{a}\left(\hat{b} + \hat{b}^\dagger\right)^2 - \eta - \frac{\epsilon}{2}\hat{a} + \sqrt{2}\epsilon\hat{a}_in \quad (8a) \\
\frac{d\hat{b}}{dt} &= -i\Delta_m \hat{b} - 2ig_{op}\hat{a}^\dagger\hat{a}\left(\hat{b} + \hat{b}^\dagger\right) - \frac{\Gamma}{2}\hat{b} + \sqrt{2}\Gamma\hat{b}_in \quad (8b) \\
\frac{d\hat{e}}{dt} &= -i\Delta_e \hat{e} - ig\hat{a} - \frac{\gamma}{2}\hat{e} + \sqrt{2}\gamma\hat{e}_in \quad (8c)
\end{align*}
\]

where, \( \Delta_i = \omega_i - \omega_D \), \( i = a, m \) and, \( e \) representing cavity, mechanical and atomic level detuning. The expectation values of the system’s operators can be used as observables to get proper information of the dynamical behaviour of the system hence, from Eqs. (8a)-(8b) we get,

\[
\begin{align*}
\frac{d\langle\hat{a}\rangle}{dt} &= -i\Delta_a \langle\hat{a}\rangle - ig\langle\hat{e}\rangle - ig_{op}\langle\hat{a}\rangle\left(\hat{b} + \hat{b}^\dagger\right)^2 - \eta - \frac{\epsilon}{2}\langle\hat{a}\rangle \quad (9a) \\
\frac{d\langle\hat{b}\rangle}{dt} &= -i\Delta_m \langle\hat{b}\rangle - 2ig_{op}\langle\hat{a}^\dagger\hat{a}\rangle\left(\hat{b} + \hat{b}^\dagger\right) - \frac{\Gamma}{2}\langle\hat{b}\rangle \quad (9b) \\
\frac{d\langle\hat{e}\rangle}{dt} &= -i\Delta_e \langle\hat{e}\rangle - ig\langle\hat{a}\rangle - \frac{\gamma}{2}\langle\hat{e}\rangle \quad (9c)
\end{align*}
\]

where we have consider the expectation values of the noises are null i.e. \( \langle\hat{a}_in\rangle = \langle\hat{b}_in\rangle = \langle\hat{e}_in\rangle = 0 \)

### 3.1. Steady State Dynamics

In this section we are about to study the steady state dynamics of the system which can be achieved by equating the RHS of Eqs. (8d)-(8f) to zero,

\[
\begin{align*}
0 &= -i\Delta_a \langle\hat{a}\rangle - ig\langle\hat{e}\rangle - ig_{op}\langle\hat{a}\rangle\left(\hat{b} + \hat{b}^\dagger\right)^2 - \eta - \frac{\epsilon}{2}\langle\hat{a}\rangle \quad (9a) \\
0 &= -i\Delta_m \langle\hat{b}\rangle - 2ig_{op}\langle\hat{a}^\dagger\hat{a}\rangle\left(\hat{b} + \hat{b}^\dagger\right) - \frac{\Gamma}{2}\langle\hat{b}\rangle \quad (9b) \\
0 &= -i\Delta_e \langle\hat{e}\rangle - ig\langle\hat{a}\rangle - \frac{\gamma}{2}\langle\hat{e}\rangle \quad (9c)
\end{align*}
\]

Further analysis leads to,

\[
\begin{align*}
0 &= -\left(\frac{\epsilon}{2} + i\Delta_a\right) \langle\hat{a}\rangle - ig\langle\hat{e}\rangle - ig_{op}\langle\hat{a}\rangle\left(\hat{b} + \hat{b}^\dagger\right)^2 - \eta \quad (10a) \\
0 &= -\left(\frac{\Gamma}{2} + i\Delta_m\right) \langle\hat{b}\rangle - 2ig_{op}\langle\hat{a}^\dagger\hat{a}\rangle\left(\hat{b} + \hat{b}^\dagger\right) \quad (10b) \\
0 &= -\left(\frac{\gamma}{2} + i\Delta_e\right) \langle\hat{e}\rangle - ig\langle\hat{a}\rangle \quad (10c)
\end{align*}
\]
by simplifying the above expressions we get,

\[ \langle \hat{e} \rangle = \frac{-ig\langle \hat{a} \rangle}{\left(\frac{\gamma}{2} + i\Delta_e\right)} \]  
(11)

\[ \langle \hat{b} \rangle = \frac{-2ig_{\text{op}}\langle \hat{a}^\dagger \hat{a} \rangle(\hat{b} + \hat{b}^\dagger)}{\left(\frac{\Gamma}{2} + i\Delta_m\right)} \]  
(12)

\[ -\left[ \left( \frac{\epsilon}{2} + i\Delta_a \right) + ig_{\text{op}}(\hat{b} + \hat{b}^\dagger)^2 \right] \langle \hat{a} \rangle - ig\langle \hat{e} \rangle = \eta \]  
(13)

Using Eq.(11) on Eq.(13) we finally get,

\[ \langle \hat{a} \rangle = \frac{-\eta}{\left(\frac{\epsilon}{2} + i\Delta\right) + \frac{g^2}{\left(\frac{\gamma}{2} + i\Delta_e\right)}} \]  
(14)

where, \( \Delta = \Delta_a + g_{\text{op}}(\hat{b} + \hat{b}^\dagger)^2 \)

4. TRANSMISSION AMPLITUDE AND INTENSITY

To observe the optical response of the cavity with the environment we need to focus on the transmission intensity of the system which can be evaluated after calculating complex transmission amplitude. By using input-output formalism we found the transmission amplitude as[25][27],

\[ \mathcal{A}_T = -\eta\langle \hat{a} \rangle \]  
(15)

by using Eq.(14) we get,

\[ \mathcal{A}_T = \frac{\eta^2}{\left(\frac{\epsilon}{2} + i\Delta\right) + \frac{g^2}{\left(\frac{\gamma}{2} + i\Delta_e\right)}} \]  
(16)

The Transmission Intensity can be evaluated as, \( \mathcal{T} = |\mathcal{A}_T|^2 \).

\[ \mathcal{T} = \frac{\eta^4}{\left(\frac{\epsilon^2}{4} + \Delta^2\right) + \frac{g^4}{\left(\frac{\gamma^2}{4} + \Delta_e^2\right)}} \]  
(17)

4.1. Numerical Plotting

Fig. [1] shows the variation of \( \mathcal{T} \) with \( \Delta \) where, we can observe for \( \Delta_e = 0 \), the \( \mathcal{T} \) shows maximum value 0.04 at \( \Delta = 0 \) hence, under this specific condition \( \Delta = 0 \) is the resonance position.
For, $\Delta_e = 1$ the $T$ shows maximum value of $\approx 0.112$ at $\Delta \approx 1.95$ hence, $\Delta = 1.95$ is the resonance position. For, $\Delta_e = 5$ the $T$ shows maximum value of $\approx 0.76$ at $\approx \Delta = 0.75$ hence, $\Delta = 0.75$ is the resonance position. For, $\Delta_e = 10$ the $T$ shows maximum value of $\approx 0.93$ at $\approx \Delta = 0.4$ hence, $\Delta = 0.4$ is the resonance position. Finally, for $\Delta_e = 100$ the $T$ shows maximum value of 1.0 at $\Delta = 0$ hence, $\Delta = 0$ is the resonance position again. So, it can be observed from the plot that as, we increase $\Delta_e$ from 0 to 100 the resonance position varies from 0 to $\approx 2$ and gradually comes back to 0 near $\Delta_e \approx 100$. It has been observed that for a variation of $\Delta_e$ from 0.0 to 5; $T$ varies rapidly from 0.04 to 0.76 but this variation rapidity drastically reduces when we vary $\Delta_e$ from 5 to 100.

In Fig. 2 we have plotted the variation of Transmission Intensity with coupling coefficient $g$ for (1) $\Delta = 0$, (2) $\Delta = 0.5$, (3) $\Delta = 1$ and, (4) $\Delta = 2$ by considering $\epsilon = 2$, $\gamma = 2$, $\eta = 1$ and, $\Delta_e = 0$. The figure shows variation of $T$ with coupling between the cavity photons and two level system $g$ and it can be observed that, as the coupling varies from 0 to 0.2, $T$ remains constant.
and, an increment in \( g \) from 0.3 to 1.0 rapidly decreases \( T \) and the it gradually tends to saturate. Further increase in \( g \) above 1.5 puts \( T \) to saturation at a value \( \approx 0.05 \). But at the same time an increase in \( \Delta \) from 0 to 2 reduces \( T \) at position \( g = 0 \) i.e. the maximum achievable value of \( T \) for a particular value of \( \Delta \).

In Fig. 3 we have reproduced the results demonstrated in Ref. [25] (see Fig. 3 of Ref. [25]) by plotting transmission intensity \( T \) with the optically driven field frequency \( \omega_D \) for Upper Sideband of cavity detuning i.e. \( \Delta = \Delta_a + \omega_m \) which describes Upper and Lower Sideband of cavity detuning. Now the distinguishable factor of our outcome with the previous results described as follows, when, \( g_{op} > 0 \) we can only consider the Upper sideband of cavity detuning i.e. \( (\Delta_a + \omega_m) \) which makes \( g_{op}(\hat{b} + \hat{b}^\dagger)^2 = \omega_m \) or, \( 2\hat{Q}^2 = \frac{g_{op}}{\omega_m} \) and the mirror displacement \( \hat{Q} = \pm \sqrt{\frac{g_{op}}{2\omega_m}} \) which completely describes the to and fro motion of the mechanical mirror due to the impact of radiation pressure but, for the Lower Sideband i.e. \( (\Delta_a - \omega_m) \) and, \( -g_{op}(\hat{b} + \hat{b}^\dagger) = \omega_m \) mirror displacement \( \hat{Q} \) represented by a completely imaginary entity which is impossible as \( \hat{Q} \) is real.

5. DRIFT MATRIX

In this section we will construct the Drift Matrix for our system by considering optical pumping frequency \( \omega_D \gg \omega_e \) which allows us to adiabatically eliminate the ionic degree of freedom i.e. \( H_e \) can be neglected. Hence our system Hamiltonian under RWA takes the form,

\[
\mathcal{H} = \hbar \Delta_a \hat{a}^\dagger \hat{a} + \hbar \Delta_m \hat{b}^\dagger \hat{b} + \hbar g_{op} \hat{a}^\dagger \hat{a} \left( \hat{b} + \hat{b}^\dagger \right)^2 - i\hbar (\hat{a}^\dagger + \hat{a}) \tag{18}
\]
for the sake of simplicity, we will proceed with the quadrature representation of the Hamiltonian as,

\[ H = \hbar \Delta a \hat{a}^\dagger \hat{a} + \hbar \frac{\Delta m}{2} \left( \hat{Q}^2 + \hat{P}^2 - 1 \right) + 2i \hbar g_{op} \hat{a} \hat{a}^\dagger \hat{Q}^2 - i \hbar (\hat{a}^\dagger + \hat{a}) \] (19)

5.1. Dynamics of the System

The evolution of Hamiltonian given by Eq.(19) is given by the set of evolution equations obtained using quantum Langevin equation,

\[ \frac{d \hat{a}}{dt} = -i \Delta a \hat{a} - 2i g_{op} \hat{a} \hat{Q}^2 - \frac{\epsilon}{2} \hat{a} + \sqrt{2} \epsilon \hat{a}_{in} - \eta \] (20)

\[ \frac{d \hat{P}}{dt} = -\Delta_m \hat{Q} - 4 g_{op} \hat{a}^\dagger \hat{a} \hat{Q} - \frac{\Gamma}{2} \hat{P} + \sqrt{2} \Gamma \hat{P}_{in} \] (21)

\[ \frac{d \hat{Q}}{dt} = \Delta_m \hat{P} + \sqrt{2} \Gamma \hat{Q}_{in} \] (22)

where, \( \hat{P}_{in} \) and \( \hat{Q}_{in} \) represents the quadrature input noise of mechanical oscillator.

5.2. Quantum Fluctuations

As the fluctuations in a quantum system is relatively negligible then the steady state values hence we can neglect the nonlinear terms arises due to the fluctuations i.e. \( \hat{a} \rightarrow \hat{a} + \lambda \delta \hat{a} \), \( \hat{P} \rightarrow \hat{P} + \lambda \delta \hat{P} \) and, \( \hat{Q} = \hat{Q} + \lambda \delta \hat{Q} \) where we neglect nonlinear terms in \( \lambda \),

\[ \frac{d}{dt} \delta \hat{a} = -i \Delta_a \delta \hat{a} - 4i g_{op} \hat{a} \delta \hat{Q} + 2i g_{op} \hat{Q}_s^2 \delta \hat{a} - \frac{\epsilon}{2} \delta \hat{a} + \sqrt{2} \epsilon \delta \hat{a}_{in} \] (23)

\[ \frac{d}{dt} \delta \hat{P} = - (\Delta_m + 4 g_{op} |a_s|^2) \delta \hat{Q} + \frac{\Gamma}{2} \delta \hat{P} - 4 g_{op} a_s Q_s \left( \frac{\hat{a}_s + \hat{a}_s^*}{\sqrt{2}} \right) + \sqrt{2} \Gamma \delta \hat{P}_{in} \] (24)

\[ \frac{d}{dt} \delta \hat{Q} = \Delta_m \delta \hat{P} + \sqrt{2} \Gamma \delta \hat{Q}_{in} \] (25)

Now using the expression, \( \delta \hat{a} = \frac{\delta \hat{X}_a + i \delta \hat{P}_a}{\sqrt{2}} \); \( \delta \hat{a}^\dagger = \frac{\delta \hat{X}_a - i \delta \hat{P}_a}{\sqrt{2}} \) and, \( \frac{\hat{a}_s + \hat{a}_s^*}{\sqrt{2}} = X_s \), \( \frac{\hat{a}_s - \hat{a}_s^*}{i \sqrt{2}} = P_s \) we get,

\[ \frac{d}{dt} \delta \hat{X}_a = (\Delta_a + 2 g_{op} Q_s^2) \delta \hat{P}_a + 4 g_{op} Q_s P_s \delta \hat{Q} - \frac{\epsilon}{2} \delta \hat{X}_a + \sqrt{2} \epsilon \delta \hat{X}_{in} \] (26a)

\[ \frac{d}{dt} \delta \hat{P}_a = - (\Delta_a + 2 g_{op} Q_s^2) \delta \hat{X}_a - 4 g_{op} X_s Q_s \delta \hat{Q} - \frac{\epsilon}{2} \delta \hat{P}_a + \sqrt{2} \epsilon \delta \hat{P}_{in} \] (26b)
\[ \frac{d}{dt} \delta \dot{Q} = \Delta_m \delta \dot{P} + \sqrt{2 \Gamma} \delta \dot{Q}_{\text{in}} \] (26c)

\[ \frac{d}{dt} \delta \dot{P} = - (\Delta_m + 4g_{op} |a_s|^2) \delta \dot{Q} - \frac{\Gamma}{2} \delta \dot{P} - 4g_{op} X_s Q_s \delta X_a + \sqrt{2 \Gamma} \delta \dot{P}_{\text{in}} \] (26d)

The Drift Matrix written as,

\[
M = \begin{pmatrix}
-\frac{\epsilon}{2} & \hat{\Delta}_a & G P_s & 0 \\
-\hat{\Delta}_a & -\frac{\epsilon}{2} & -G X_s & 0 \\
0 & 0 & 0 & \Delta_m \\
-G X_s & 0 & -\hat{\Delta}_m & -\frac{\Gamma}{2}
\end{pmatrix}
\]

where,

\[ \hat{\Delta}_a = \Delta_a + 2g_{op} Q_s^2 \]
\[ \hat{\Delta}_m = \Delta_m + 4g_{op} |a_s|^2 \]
\[ G = 4g_{op} Q_s \]

and, the evolution equation of the optomechanical system due to quantum fluctuations,

\[ \dot{X} = MX + N \] (27)

where, \( X^T = (\delta X_a \quad \delta P_a \quad \delta Q \quad \delta P) \) representing the infinitesimal change in the system parameters and, \( N^T = (\sqrt{2 \epsilon} \delta X_a^{\text{in}} \quad \sqrt{2 \epsilon} \delta P_a^{\text{in}} \quad \sqrt{2 \Gamma} \delta Q_{\text{in}} \quad \sqrt{2 \Gamma} \delta P_{\text{in}}) \) the noise matrix of the system.

### 5.3. Stability Conditions

The stability conditions can be deduced by applying the Routh-Hurwitz\[28]\ criteria by finding the \( n \)th polynomial equation of eigenvalues. And, the conditions for stability are,

\[ l_1 = \left( \frac{\epsilon^2}{4} + \hat{\Delta}_m^2 \right) + \frac{\epsilon \Gamma}{2} + \Delta_m \hat{\Delta}_m > 0 \] (28)

\[ l_2 = \epsilon \Delta_m \hat{\Delta}_m + G^2 X_s P_s \Delta_m + \left( \frac{\epsilon^2}{4} + \hat{\Delta}_m^2 \right) \frac{\Gamma}{2} > 0 \] (29)

\[ l_3 = \Delta_m \hat{\Delta}_m \left( \frac{\epsilon^2}{4} + \hat{\Delta}_m^2 \right) + \frac{G^2 \epsilon}{2} P_s X_s \Delta_m - G^2 X_s^2 \Delta_m \hat{\Delta}_m > 0 \] (30)

\[ \left( \epsilon + \frac{\Gamma}{2} \right) > 0 \quad ; l_1 \left( \epsilon + \frac{\Gamma}{2} \right) > l_2 \quad ; l_1 l_2 \left( \epsilon + \frac{\Gamma}{2} \right) > \left( \epsilon + \frac{\Gamma}{2} \right)^2 l_3 + l_2^2 \] (31)

which takes the same form as represented in Ref.\[29\] while discussing the effects of linear as well as quadratic coupling in an optomechanical system simultaneously.
6. CONCLUSION

In this article we have studied a quadratically coupled optomechanical system and observed the variation of Transmission Intensity $T$ with coupling strength $g$ and $\Delta$. A brief discussion of the variation has been given in section (4). Later, we have constructed Drift Matrix by adiabatically eliminating the atomic degree of freedoms. Finally we have given the stability conditions of the quadratically coupled optomechanical system. In Fig.1 at $\Delta_e = 0$ we can observe the transmission amplitude is minimum because, the pumping frequency is at resonance with the two level atom frequency; which amplifies the probability of absorption of incident photons and excitement of the atomic system. In Fig.2 we have discussed the variation of $T$ with coupling between cavity mode and two level atomic mode $g$. As the coupling decreases, due to less absorption of photons by two level system we expect a better optical response as an output through $T$. The variation of optical response with different parameter of the system definitely gives a better insight of optical response of a quadratically coupled optomechanical system; along with the construction of drift matrix we gave an enhanced insight about the effect of quantum noise on the system. In Fig.3 we have reproduced the results discussed in Ref.25 for quadratically coupled system.

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