Dynamical symmetry breaking and the Nambu-Goldstone theorem in the Gaussian wave functional approximation

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Abstract

We analyze the group-theoretical ramifications of the Nambu-Goldstone [NG] theorem in the self-consistent relativistic variational Gaussian wave functional approximation to spinless field theories. In an illustrative example we show how the Nambu-Goldstone theorem would work in the O(N) symmetric $\phi^4$ scalar field theory, if the residual symmetry of the vacuum were lesser than O(N-1), e.g. if the vacuum were O(N-2), or O(N-3),... symmetric. [This does not imply that any of the “lesser” vacua is actually the absolute energy minimum: stability analysis has not been done.] The requisite number of NG bosons would be (2N - 3), or (3N - 6), ... respectively, which may exceed N, the number of elementary fields in the Lagrangian. We show how the requisite new NG bosons would appear even in channels that do not carry the same quantum numbers as one of N “elementary particles” (scalar field quanta, or Castillejo-Dalitz-Dyson [CDD] poles) in the Lagrangian, i.e. in those “flavour” channels that have no CDD poles. The corresponding Nambu-Goldstone bosons are composites (bound states) of pairs of massive elementary (CDD) scalar fields excitations. As a nontrivial example of this method we apply it to the physically more interesting ’t Hooft $\sigma$ model (an extended $N_f = 2$ bosonic linear $\sigma$ model with four scalar and four pseudoscalar fields), with spontaneously and explicitly broken chiral $O(4) \times O(2) \simeq SU_R(2) \times SU_L(2) \times U_A(1)$ symmetry.

1 Introduction

The proof of the Nambu-Goldstone [NG] theorem [1, 2, 3, 4, 5] in the Gaussian wave functional approximation [6, 7, 8, 9, 10, 11, 12, 13] used to be an open problem for over 30 years, see Refs. [14, 15, 16, 17]. The first proof was given in the O(2) symmetric $\phi^4$ theory [17], and then straightforwardly extended to O(4) in [18]; the
crucial assumption was that the ground state be $O(N-1)$ symmetric, i.e. that only one (scalar) field develops a vacuum expectation value [v.e.v.]$^1$. By the standard NG boson counting methods [3, 4], for every spontaneously broken symmetry Lie group generator there is one NG boson. As the Lie algebra $O(N)$ has $N(N-1)/2$ generators, for a ground state (vacuum) with an $O(N-1)$ residual symmetry the number of NG bosons ought to be $N(N-1)/2 - (N-1)(N-2)/2 = N-1$. That is exactly the number of available fields in the Lagrangian. What happens when the residual symmetry the ground state is “smaller” than $O(N-1)$ and there ought to be more than $N-1$ NG bosons?

In this paper we shall extend our proof of the NG theorem in the Gaussian approximation [17, 18] to the $O(N)$ symmetric $\phi^4$ theory when the symmetry of the ground state is dynamically broken to some (proper) subgroup of $O(N-1)$, in this specific case to one of the following symmetries: $O(N-2) \times O(2)$, $O(N-3) \times O(3)$, ..., $O(N/2) \times O(N/2)$ for $N$ even, or $O((N-1)/2) \times O((N+1)/2)$ for $N$ odd. As we shall show, the residual symmetry pattern is dictated by the absolute minimum of the $O(N)$ $\phi^4$ model’s ground state energy, which in turn depends on the dynamics (gap equation) in the Gaussian approximation, as well as the free parameters. Absolute minimization of the vacuum energy will not been done in this paper, only a search for the extremal and/or saddle points. In a different model or approximation, or for different values of the free parameters the residual symmetry might be different. In any case, with such an asymmetric ground state the number of NG bosons must exceed $N-1$, the largest number of “elementary” scalar fields available in the Lagrangian (at least one field must develop the vacuum expectation value and thus cannot create or destroy single NG bosons). Nevertheless the canonical number of massless spinless excitations appears in the spectrum [4]. Our proof ought to leave no doubt as to the composite nature of the NG bosons in the Gaussian approximation.

This paper falls into five Sections and two Appendices. Firstly in Sect. 2 we define the $O(N)$ $\phi^4$ model and the Gaussian approximation. In Sect. 3 we show exactly how the requisite number of massless NG (bosonic bound) states appear and that all the “broken symmetry” Nöther currents remain conserved, as the symmetry of the vacuum is reduced. NG bosons invariably appear in the Gaussian approximation to the $O(N)$ $\phi^4$ model in those channels that also contain CDD poles, so in Sect. 4 we extend our proof to the ’t Hooft $\phi^4$ model which does not have this property. There we show how various bound NG states appear, or disappear as the symmetries of the ground state and/or the Lagrangian change. Finally, in Sect. 5 we draw our conclusions and set them in a wider context. In the Appendices we present some technical details omitted in the main body of text.

## 2 Preliminaries

### 2.1 The $O(N)$ symmetric scalar $\phi^4$ model

At first we confine ourselves to the $O(N)$ symmetric scalar $\phi^4$ theory for the sake of simplicity. All scalar field theories with other spontaneously broken internal

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$^1$That assumption is justified when the remaining $N-1$ fields are pseudoscalars, as in the Gell-Mann–Levy model, otherwise CP symmetry is (spontaneously) broken.
symmetries can be reduced to some subgroup of $O(N)$. Of course, in such cases there will be interaction terms other than the simple $\phi^4$ one shown below. The Lagrangian density of this theory is

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 - V(\phi^2),$$

where

$$\phi = (\phi_0, \phi_1, \phi_2, ..., \phi_{N-1}) = (\sigma, \pi),$$

is a column vector and

$$V(\phi^2) = -\frac{1}{2} \mu_0^2 \phi^2 + \frac{\lambda_0}{4} \left(\phi^2\right)^2.$$

We assume here that $\lambda_0$ and $\mu_0^2$ are not only positive, but such that spontaneous symmetry breakdown (SSB) occurs in the Gaussian approximation [GA] introduced below.

### 2.2 The Gaussian variational method

#### 2.2.1 The Gaussian ground state (“vacuum”)

The Rayleigh-Ritz variational approximation to quantum field theories is based on the (“elliptical”) Gaussian ground state (vacuum) functional Ansatz, Refs. [9, 10, 11, 13],

$$\Psi_0[\phi] = \mathcal{N} \exp \left( -\frac{1}{4\hbar} \int dx \int dy \left[ \phi_i(x) - \langle \phi_i(x) \rangle \right] G_{ij}^{-1}(x,y) \left[ \phi_j(y) - \langle \phi_j(y) \rangle \right] \right),$$

where $\mathcal{N}$ is the normalization constant, and one sums all repeated (roman lettered) indices from 0 to $N-1$. $\langle \phi_i(x) \rangle$ is the vacuum expectation value (v.e.v.) of the $i$-th spinless field which henceforth we will assume to be translationally invariant $\langle \phi_i(x) \rangle = \langle \phi_i(0) \rangle \equiv \langle \phi_i \rangle$ and

$$G_{ij}(x,y) = \frac{1}{2} \delta_{ij} \int \frac{dk}{(2\pi)^3} \frac{1}{\sqrt{k^2 + m_i^2}} e^{ik(x-y)}.$$

We have explicitly kept $\hbar$ (while setting the velocity of light $c = 1$) to keep track of quantum corrections and count the number of “loops” in our calculation. Then the “vacuum” (ground state) energy density becomes

$$\mathcal{E}(m_i, \langle \phi_i \rangle) = -\frac{1}{2} \mu_0^2 \langle \phi \rangle^2 + \frac{\lambda_0}{4} \left[ \langle \phi \rangle^2 \right]^2$$

$$+ \hbar \sum_{i=0}^{N-1} \left[ I_1(m_i) - \frac{1}{2} \left( \mu_0^2 + m_i^2 \right) I_0(m_i) \right]$$

$$+ \frac{\lambda_0}{4} \left\{ 6\hbar \sum_{i=0}^{N-1} \langle \phi_i \rangle^2 I_0(m_i) + 2\hbar \sum_{i \neq j=0}^{N-1} \langle \phi_i \rangle^2 I_0(m_j) \right.$$  

$$+ \hbar^2 \sum_{i=0}^{N-1} I_0^2(m_i) + \hbar^2 \sum_{i \neq j=0}^{N-1} I_0(m_i) I_0(m_j) \right\},$$

(3)
where

\[ I_0(m_i) = \frac{1}{2} \int \frac{dk}{(2\pi)^3} \frac{1}{\sqrt{k^2 + m_i^2}} = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_i^2 + i\epsilon} = G_{ii}(x,x) \quad (4) \]

\[ I_1(m_i) = \frac{1}{2} \int \frac{dk}{(2\pi)^3} \sqrt{k^2 + m_i^2} = -i \int \frac{d^4k}{(2\pi)^4} \log \left(k^2 - m_i^2 + i\epsilon\right) + \text{const.} \quad (5) \]

We may identify \( \bar{h}I_1(m_i) \) with the familiar “zero-point” energy density of a free scalar field of mass \( m_i \).

The divergent integrals \( I_{0,1}(m_i) \) are understood to be regularized via an UV momentum cutoff \( \Lambda \). Thus we have introduced a new free parameter into the calculation. This was bound to happen in one form or another, since even in the renormalized perturbation theory one must introduce a new dimensional quantity (the “renormalization scale/point”) at the one loop level. We treat this model as an effective theory and thus keep the cutoff without renormalization 2.

### 2.2.2 The vacuum energy minimization equations

We vary the energy density with respect to the field vacuum expectation values \( \langle \phi_i \rangle \) and the “dressed” masses \( m_i \). The extremization condition with respect to the field vacuum expectation values (v.e.v.) reads:

\[ \left( \frac{\partial \mathcal{E}(m_i, \langle \phi_i \rangle)}{\partial \langle \phi_j \rangle} \right)_{\text{min}} = 0; \quad j \in 0, 1, \ldots, N - 1; \quad (6) \]

or explicitly

\[ \langle \phi_j \rangle \left[ -\mu_0^2 + \lambda_0 \left( \langle \phi \rangle^2 + 3hI_0(m_j) + h \sum_{j \neq i=0}^{N-1} I_0(m_i) \right) \right]_{\text{min}} = 0. \quad (7) \]

The second set of energy extremization equations reads

\[ \left( \frac{\partial \mathcal{E}(m_i, \langle \phi_i \rangle)}{\partial m_j} \right)_{\text{min}} = 0, \quad j \in 0, 1, \ldots, N - 1, \quad (8) \]

or

\[ m_j^2 + \mu_0^2 = \lambda_0 \left( 3\langle \phi_j^2 \rangle + h \sum_{j \neq k=0}^{N-1} \langle \phi_k^2 \rangle + 3hI_0(m_j) + h \sum_{j \neq k=0}^{N-1} I_0(m_k) \right)_{\text{min}}. \quad (9) \]

Eqs. (7),(9) can be identified with the (truncated) Schwinger-Dyson (SD) equation [13, 19] for the one- and two-point Green functions, see Ref. [17, 18]. The solutions to these equations plus the additional minimization requirements (positive-definite second derivatives matrix, i.e., the positive definiteness of its principal minors, which will not be discussed here) determine the symmetry of the ground state (vacuum).

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2There are several renormalization schemes for the Gaussian approximation, but they show signs of instability and ultimately seem to lead to “triviality” [12].
2.2.3 The vacuum symmetry

Now one ordinarily assumes that only one of the scalar fields \( \phi_i \) [by convention the \( i = 0 \) one] develops a nonzero v.e.v., i.e. \( \langle \phi_0 \rangle \neq 0 \) and then one proceeds with the proof of the NG theorem.

In the conventional case the scalar field masses are \( m_0 = M; m_1 = m_2 = m_3 = \ldots = \mu \), i.e., the \((N - 1)\) fields \( \phi_i; i \in 1, 2, \ldots, N - 1 \), of mass \( m_i = \mu \), form an \((N - 1)\)-plet and the residual symmetry of the vacuum is O(N-1). That, however, is not the only logical possibility: one may assume that more than one field develops v.e.v.. This, of course, means that the (residual) symmetry of the vacuum is “lesser” than the one in the (“canonical”) case with only one v.e.v. 4.

The decision which of these possibilities actually takes place can be made on the basis of comparing their respective ground state energies. In the case with an O(N-1) symmetric ground state, the vacuum energy, or effective potential is

\[
E(M, \mu, \langle \phi_0 \rangle)_{\text{min}} = -\frac{1}{2} \mu_0^2 \langle \phi_0 \rangle^2 + \frac{\lambda_0}{4} \langle \phi_0 \rangle^4 
+ \hbar \left[ I_1(M) - \frac{1}{2} (\mu_0^2 + M^2) I_0(M) \right] 
+ (N - 1) \hbar \left[ I_1(\mu) - \frac{1}{2} (\mu_0^2 + \mu^2) I_0(\mu) \right] 
+ \frac{\lambda_0}{4} \left\{ 2 \hbar \langle \phi_0 \rangle^2 \left[ 3 I_0(M) + (N - 1) I_0(\mu) \right] + 3 \hbar^2 I_0^2(M) 
+ (N - 1) \hbar^2 I_0(\mu) \left[ 2 I_0(M) + (N + 1) I_0(\mu) \right] \right\} .
\]

(10)

Similar expressions for ground states with lesser symmetries than O(N-1) can be derived by applying the corresponding gap equations (7),(9) to Eq. (3). The question of the absolute energy minimum will not be pursued in this paper, however. For the purpose of argument we shall assume that alternative ground states exist and are stable, i.e., energetically favourable to the standard one.

If two (or more) scalar fields’ v.e.v. are simultaneously nonzero, e.g., if \( \langle \phi_0 \rangle = v_1 \neq 0 \) and \( \langle \phi_j \rangle = v_2 \neq 0; j \in 1, 2, \ldots, N - 1 \) \( \langle \phi_i \rangle = 0; i \in 1, 2, \ldots, j - 1, j - 2, \ldots, N - 1 \), we may change the scalar field labels such that the fields with nonzero v.e.v.s are labeled successively 0,1,...,k without loss of generality. In such a case the masses \( m_i \) may also take on different values.

For simplicity’s sake we assume that only two fields have nonvanishing vacuum expectation values (v.e.v.) 5, \( \phi_0, \phi_1 \neq 0 \) (we label their masses \( m_0 = M, m_1 = \mu_1 \)).

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3The validity of this assumption, of course, depends on the values of the bare parameters in the Lagrangian and the cutoff \( \Lambda \), or the values of the renormalized parameters, if one insists on renormalization. One may also have an unbroken symmetry: with all the fields having a zero v.e.v., \( \langle \phi_i \rangle = 0, i \in 0, 1, \ldots, N - 1 \), the gap equations (7),(9) lead to all the masses \( m_i \) being equal. In other words the \( N \) fields form an O(N) multiplet, so we may say that the symmetry of the vacuum is O(N), i.e. not broken.

4In the Born approximation such a vacuum might be reducible to the “one v.e.v.” vacuum by means of an O(N) transformation (if it lies in the same “orbit” of the symmetry group [20]), but in the Gaussian approximation dynamical symmetry breaking may lead to an irreducibly different ground state.

5Extension to three, or more v.e.v.s follows by straightforward analogy.
The remaining \((N - 2)\) fields \(\phi_i; i \in 2, \ldots, (N - 1)\), with mass \(m_i = \mu_2\), form an \((N - 2)\)-plet. In that case, it is clear that the residual symmetry of the ground state is (at least) \(O(N - 2)\), i.e., that the \(O(N)\) symmetry has been dynamically (spontaneously) broken to (at least) \(O(N - 2)\). We say “at least” because, when the masses \(m_0 = M = m_1 = \mu_1\) are equal there is additional \(O(2)\) symmetry of the vacuum. Similar comments are valid in the case when more than two fields develop v.e.v.s. Of course, the residual symmetry of the vacuum determines the number of the NG bosons. Next we turn to the case of two fields with v.e.v. as worked out in Appendix A. We have shown in Appendix A that there are only two distinct mass solutions \(M = \mu_1, \neq \mu_2\) to Eqs. (55-58). Thus the residual symmetry of the vacuum is \(O(N - 2) \times O(2)\), and the corresponding number of NG bosons must be \((2N - 4)\). In the following we shall see how all these NG bosons come about.

3 The Nambu-Goldstone theorem

3.1 The two-body equation

In Ref. [17, 18] it was shown that in the Gaussian approximation to the \(O(2)\) and \(O(4)\) \(\phi^4\) model, the Nambu-Goldstone particles appear in the two-particle spectra, i.e. that they are massless bound states of two different massive elementary excitations with an admixture of the (massive elementary) one-body state with the same quantum numbers (the CDD pole). This admixture of the one-body state is crucial for the masslessness of the NG state [it also proved to be a source of confusion]. As there are at most \((N - 1)\) such elementary particles/CDD poles, it appears that there can be at most \((N - 1)\) NG bosons in this theory. There are \(N(N - 1)/2\) (distinct) pairs of elementary particles in this model, however, and an equal number of distinct (potentially bound) two-body states. Thus there can be at most \(N(N - 1)/2\) NG bosons, precisely the maximum number allowed by the \(O(N)\) Lie group generator counting. Next we must show that the number of “broken symmetry generators” corresponds precisely to the number of massless bound states and that the corresponding Noether currents are conserved.

The two-body equation of motion in the Gaussian approximation is equivalent to the four-point Schwinger-Dyson (or Bethe-Salpeter) equation [for proof of this equivalence, see Ref. [9, 21]]. All the NG channels obey a generic four-point SD equation that reads

\[
D_{ij,ij}(s) = V_{ij,ij}(s) + V_{ij,kl}(s)\Pi_{kl,mn}(s)D_{mn,ij}(s)
\]

where \(D(s)\) is the four-point Green function \((N \times N)\) matrix (scattering amplitude), \(\Pi(s)\) is the polarization function matrix and \(V(s)\) is the potential matrix; \(ij\) denote the \(O(N)\) indices of the two constituents, with the generic solution

\[
D(s) = V(s)[1 - V(s)\Pi(s)]^{-1}
\]

where \(s = (p_1 + p_2)^2 \equiv P^2\) is the center-of-mass (CM) energy. These matrices may reduce to a direct sum of submatrices depending on the residual symmetry of the

\[6\text{One field adopts a v.e.v. so it cannot create or destroy single NG bosons.}\]
system. Such effective two-body propagators can also be written in the following form (see Ref. [22])

$$D_\alpha(s) \simeq \frac{g_\alpha^2}{s - m_\alpha^2},$$

(13)

where $m_\alpha$ is the effective mass in channel $\alpha$. The difference between various $O(N)$ “flavour” sectors appears in the polarization functions $\Pi(s)$ and potentials $V(s)$: more specifically in $(N - 2)$ channels $[(i = 0, j \subset 2, \ldots, N - 1) \equiv M\mu_2]$ the said matrices are diagonal and have the form

$$\Pi_{M\mu_2}(s) = I_{M\mu_2}(s) = i\hbar \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - M^2 + i\epsilon][(k - P)^2 - \mu_2^2 + i\epsilon]}$$

(14)

$$V_{M\mu_2}(s) = 2\lambda_0 \left[ 1 + \left( \frac{2\lambda_0(\phi_0)^2}{s - \mu_2^2} \right) \right] = 2\lambda_0 \left[ 1 + \frac{M^2}{s - \mu_2^2} \right]$$

(15)

and another $(N - 2)$ channels $[(i = 1, j \subset 2, \ldots, N - 1) \equiv \mu_1\mu_2]$ have the form

$$\Pi_{\mu_1\mu_2}(s) = I_{\mu_1\mu_2}(s) = i\hbar \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - \mu_2^2 + i\epsilon][(k - P)^2 - \mu_1^2 + i\epsilon]}$$

(16)

$$V_{\mu_1\mu_2}(s) = 2\lambda_0 \left[ 1 + \left( \frac{2\lambda_0(\phi_1)^2}{s - \mu_2^2} \right) \right] = 2\lambda_0 \left[ 1 + \frac{\mu_1^2}{s - \mu_2^2} \right].$$

(17)

The poles $\frac{1}{s - \mu_{1,2}^2}$ in Eqs. (15), (17) are called the Castillejo-Dalitz-Dyson [CDD] poles and correspond to one-particle states in the theory. Now due to Eqs. (55-58) these two channels happen to be equivalent in this case, but that degeneracy is accidental (for a different example see Sect. 4).

### 3.2 Massless (Nambu-Goldstone) two-body states

With an $O(N - 2) \times O(2)$ residual symmetry there ought to be $(2N - 4) = (N - 2) + (N - 2)$ NG bosons. This number of NG bosons quickly exceeds the number of available fields in the Lagrangian: $2(N - 2) > (N - 1)$ for $N > 3$. Thus, there aren’t enough scalar fields to provide for all the NG bosons in case of nonstandard symmetry breaking. Does this mean that the NG theorem breaks down in such a case?

As in Ref. [17], we shall prove that at zero CM energy $P = 0$, the matrix $[1 - V_{M\mu_2}(0)\Pi_{M\mu_2}(0)]$ vanishes. We use Eq. (49) to write

$$V_{M\mu_2}(0) = 2\lambda_0 \left[ 1 - \frac{M^2}{\mu_2^2} \right]$$

$$\Pi_{M\mu_2}(0) = \hbar \left( \frac{I_0(M) - I_0(\mu_2)}{M^2 - \mu_2^2} \right)$$

(18)

then use Eq. (33) to obtain the final result

$$V_{M\mu_2}(0)\Pi_{M\mu_2}(0) = 1.$$

(19)
Thus the inverse propagator Eq. (12) evaluated at zero momentum vanishes:

\[ D_{M\mu_2}^{-1}(0) = [1 - V_{M\mu_2}(0)\Pi_{M\mu_2}(0)]V_{M\mu_2}(0)^{-1} = 0, \]  

(20)

which is equivalent to the result \( m_{M\mu_2}^2 = 0 \). Similarly for the remaining \((N - 2)\) \( \mu_1\mu_2 \) channels. \textit{q.e.d.}

Note that the proof in no way depended on the fact that \( \mu_1 = M \). In other words, if the gap Eqs. (55-58) allowed a solution such that \( \mu_1 \neq M \) and \( \langle \phi_1 \rangle = 0 \), the NG theorem would still hold. But, then the CDD poles would decouple in the latter \((N - 2)\) channels, i.e., there would be no CDD poles in these channels. This shows that, at least in principle, \((N - 2)\) NG bosons could be pure bound states with no single particle admixtures. In Sect. 4 we shall give an example of a model with a gap equation that allows such solutions.

\[ 3.3 \text{ Conservation of Nöther currents} \]

There are \( a \subset 2, ..., N - 1 \) dynamically broken O(N) symmetry Nöther current matrix elements corresponding to

\[ J_{\mu_5}^a(p', p) = \langle \phi^a(p')|J_\mu(0)|\phi_0(p)\rangle \]

\[ = (p' + p)_\mu + q_\mu \left( \frac{M^2}{q^2 - \mu_2^2} \right) \]

\[ - \Gamma_{\mu_5}^{M\mu_2}(q)D_{M\mu_2}(q). \]  

(21)

where \( \Gamma_{\mu_5}^{M\mu_2}(q) \) is

\[ \Gamma_{\mu_5}^{M\mu_2}(q) = \frac{q_\mu}{q^2} \left[ \frac{\mu_2^2}{2\lambda_0} \left( V_{M\mu_2}(0)\Pi_{M\mu_2}(0) - V_{M\mu_2}(q^2)\Pi_{M\mu_2}(q^2) \right) \right]. \]  

(22)

Inserting the vertex \( \Gamma_{\mu_5}^{M\mu_2}(q) \), Eq. (22) together with the two-body propagator \( D_{M\mu_2}(q^2) \), Eq. (12) into Eq. (21) one finds

\[ J_{\mu_5}^a(p', p) = (p' + p)_\mu + q_\mu \left( \frac{M^2}{q^2 - \mu_2^2} \right) \]

\[ - \frac{q_\mu}{q^2} \left[ \frac{\mu_2^2}{2\lambda_0} V_{M\mu_2}(q^2) \right] \]

\[ = (p' + p)_\mu + q_\mu \left( \frac{M^2 - \mu_2^2}{q^2} \right), \]  

(23)

where \( q^\nu = (p' - p)^\nu \). This current is manifestly devoid of a pole at \( q^2 = \mu_2^2 \). The composite state plays precisely the role of the Nambu-Goldstone boson in the conservation of the dynamically broken O(N) symmetry Nöther currents [1], i.e., in the basic O(N) symmetry Ward-Takahashi identity, c.f. Refs. [23, 24],

\[ q^\nu J_{\mu_5}^a(p', p) = \left( p'^2 - \mu_2^2 \right) - \left( p^2 - M^2 \right), \]  

(24)
that follows directly from Eq. (23). Similarly for the remaining N-2 \([a \subset 2, ..., N-1]\) dynamically broken O(N) symmetry Nöther current matrix elements corresponding to

\[
J_{\mu_5}^a(p', p) = \langle \phi^a(p') | J_\mu(0) | \phi_1(p) \rangle \\
= (p' + p)_\mu + q_\mu \left( \frac{\mu_1^2}{q^2 - \mu_2^2} \right) \\
- \Gamma_{\mu_1\mu_2}(q) D_{\mu_1\mu_2}(q). 
\tag{25}
\]

The identity of the two “gap”, or CDD masses \(\mu_1 = M\) and the concomitant “excess” O(2) vacuum symmetry are consequences of the simplicity of the vacuum equation (7) that only depends on one O(N) algebraic invariant [4, 20]. That, in turn, is a consequence of the fact that we are dealing with fields in the fundamental irrep. of O(N) and the requirement that the Lagrangian (1) be renormalizable, i.e. at most of the fourth power in the fields. The assumption of a second v.e.v. \(\langle \phi_3 \rangle = \langle \pi^0 \rangle \neq 0\) is particularly unrealistic in the Gell-Mann–Levy model (O(N=4) \(\phi^4\) model) [25], because of the negative parity of the \(\pi\) fields: their nonzero v.e.v. would imply spontaneous breaking of P and CP “parities”. In Sect. 4 we give an example of a \(\phi^4\) model with an internal symmetry that leads to a gap equation with distinct mass solutions and potentially exotic NG bound states.

4 The ’t Hooft model

4.1 Definition of the model

’t Hooft’s [26] extension of the linear sigma model Lagrangian reads

\[
L_{tH} = \frac{1}{2} \left[ (\partial_\mu M \partial^\mu M^\dagger) + \mu^2 M M^\dagger \right] \\
- \frac{1}{2} (\lambda_1 - \lambda_2) \left[ \text{tr} \left( M M^\dagger \right) \right]^2 - \lambda_2 \text{tr} \left[ \left( M M^\dagger \right)^2 \right] \\
+ 2\kappa \left[ e^{i\theta} \text{det} M + c.c. \right], \tag{26}
\]

where

\[
M = \frac{1}{\sqrt{2}} (\Sigma + i\Pi) \\
\Sigma = \frac{1}{\sqrt{2}} (\sigma + \alpha \cdot \tau) \\
\Pi = \frac{1}{\sqrt{2}} (\eta + \pi \cdot \tau). \tag{27}
\]

Eq.(26) is equivalent to the following

\[
L_{tH} = \frac{1}{2} \left[ (\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2 + (\partial_\mu \eta)^2 + (\partial_\mu \alpha)^2 \right] \\
+ \frac{\mu^2}{2} \left[ \sigma^2 + \pi^2 + \eta^2 + \alpha^2 \right] \\
+ 2\kappa \cos \theta \left[ \sigma^2 + \pi^2 - \eta^2 - \alpha^2 \right].
\]
\[
- 4\kappa \sin \theta [\sigma \eta - \pi \cdot \alpha] \\
- \frac{\lambda_1}{8} [\sigma^2 + \pi^2 + \eta^2 + \alpha^2]^2 \\
- \frac{\lambda_2}{2} \left[ (\sigma \alpha + \eta \pi)^2 + (\pi \times \alpha)^2 \right]
\]

(28)

which describes the dynamics of the two chiral meson quartets, \((\sigma, \pi)\) and \((\alpha, \eta)\), in this model, and \(\lambda_1, \lambda_2, \kappa, \theta\) are the bare coupling constants. Nonvanishing angle \(\theta\) leads to the explicit (not spontaneous) CP violation in this model, so we set it equal to zero. Thus we see that the 't Hooft model consists of two coupled Gell-Mann–Lévy (GML) linear sigma models \([25]\), one with a light and the other with a heavy quartet of mesons.

Note that the symmetries of various parts of the interaction Lagrangian also vary, see App. B: (i)

\[
\lambda_1 \neq 0; \lambda_2 = \kappa = \theta = 0
\]

implies \(O(8)\) symmetry. (ii)

\[
\lambda_1 \neq 0 \neq \lambda_2; \kappa = \theta = 0
\]

implies \(O(4) \times O(2)\) symmetry. (iii)

\[
\lambda_1 \neq 0 \neq \lambda_2; \kappa \neq 0 = \theta
\]

implies \(O(4)\) symmetry. And the number of NG bosons must change accordingly.

### 4.2 The gap equations

The first set of energy minimization equations (7) reads

\[
0 = -v \left( \mu_0^2 + 4\kappa \right) + \frac{\lambda_1}{2} v \left[ v^2 + 3I_0(m_\sigma) + 3I_0(m_\alpha) + 3I_0(m_\pi) + I_0(m_\eta) \right] \\
+ 3\lambda_2 v I_0(m_\alpha) \\
\]

(32)

\[
m_\sigma^2 = -\mu_0^2 - 4\kappa + \left( \frac{\lambda_1}{2} + \lambda_2 \right) v^2 + \frac{\lambda_1}{2} \left[ I_0(m_\sigma) + 5I_0(m_\alpha) + 3I_0(m_\pi) + I_0(m_\eta) \right] \\
+ \lambda_2 \left[ I_0(m_\alpha) + 2I_0(m_\pi) \right] \\
\]

(33)

\[
m_\alpha^2 = -\mu_0^2 - 4\kappa + \left( \frac{\lambda_1}{2} + \lambda_2 \right) v^2 + \frac{\lambda_1}{2} \left[ I_0(m_\sigma) + 5I_0(m_\alpha) + 3I_0(m_\pi) + I_0(m_\eta) \right] \\
+ \lambda_2 \left[ I_0(m_\alpha) + 2I_0(m_\pi) \right] \\
\]

(34)

\[
m_\pi^2 = -\mu_0^2 - 4\kappa + \left( \frac{\lambda_1}{2} + \lambda_2 \right) v^2 + \frac{\lambda_1}{2} \left[ v^2 + I_0(m_\sigma) + 3I_0(m_\alpha) + 5I_0(m_\pi) + I_0(m_\eta) \right] \\
+ \lambda_2 \left[ I_0(m_\alpha) + 2I_0(m_\pi) \right] \\
\]

(35)

\[
m_\eta^2 = -\mu_0^2 - 4\kappa + \left( \frac{\lambda_1}{2} + \lambda_2 \right) v^2 + \frac{\lambda_1}{2} \left[ v^2 + 3I_0(m_\sigma) + 3I_0(m_\alpha) + 3I_0(m_\pi) + I_0(m_\eta) \right] \\
+ 3\lambda_2 I_0(m_\pi) \\
\]

(36)

where the divergent integral \(I_0(m_i)\) is given by Eq. (4). For simplicity in the following we use the following short-hand notation \(\alpha = m_\alpha^2; \eta = m_\eta^2; \sigma = m_\sigma^2; \pi = \)
\(m_\pi^2\). Eqs. (32-36) can be solved for

\[
\Pi_{\alpha\eta}(0) = \frac{I_0(\alpha) - I_0(\eta)}{\alpha - \eta} = \frac{1}{(\lambda_1^2 - \lambda_2^2)} \left( \lambda_1 + \lambda_2 \left( \frac{\pi - \sigma}{\alpha - \eta} \right) \right)
\]

\[
\Pi_{\pi\sigma}(0) = \frac{I_0(\alpha) - I_0(\eta)}{\alpha - \eta} = \frac{-1}{(\lambda_1^2 - \lambda_2^2)} \left( \lambda_1 \left( \frac{\pi}{\sigma - \pi} + \frac{\alpha - \eta - \frac{3\lambda_2}{\lambda_1} \sigma}{\sigma - \pi} \right) \right)
\]

\[
\Pi_{\pi\alpha}(0) = \frac{I_0(\alpha) - I_0(\pi)}{\alpha - \pi} = \frac{1}{(\lambda_1^2 - 2\lambda_1\lambda_2 - 3\lambda_2^2)} \left( \lambda_1 \left( 1 - \frac{8\kappa}{\alpha - \pi} \right) + \lambda_2 \frac{\eta - 8\kappa}{\alpha - \pi} \right)
\]

\[
\Pi_{\sigma\eta}(0) = \frac{I_0(\sigma) - I_0(\eta)}{\sigma - \eta} = \frac{1}{(\lambda_1^2 - 2\lambda_1\lambda_2 - 3\lambda_2^2)} \left( (\lambda_1 - 2\lambda_2) \left( \frac{8\kappa - \eta}{\sigma - \eta} \right) - 3\lambda_2 \frac{\alpha - \pi - 8\kappa - \frac{3\lambda_2}{\lambda_1} \sigma}{\sigma - \eta} \right)
\]

whence follows

\[
\Pi_{\pi\eta}(0) = \frac{I_0(\pi) - I_0(\eta)}{\pi - \eta} = \frac{1}{\lambda_1 (\pi - \eta)} (8\kappa + \pi - \eta + \lambda_2 (\alpha - \eta) \Pi_{\eta\alpha}(0) - 3\lambda_2 (\alpha - \pi) \Pi_{\pi\alpha}(0))
\]

\[
\Pi_{\alpha\sigma}(0) = \frac{I_0(\sigma) - I_0(\alpha)}{\sigma - \alpha} = \frac{1}{\lambda_1 (\sigma - \alpha)} (8\kappa + \frac{\lambda_2}{\lambda_1} \sigma - \alpha + \lambda_2 (\sigma - \pi) \Pi_{\pi\sigma}(0) - 3\lambda_2 (\alpha - \pi) \Pi_{\pi\alpha}(0))
\]

**4.3 The Nambu-Goldstone theorem in the isotensor pseudoscalar sector**

Note that there are \(8 \times 8 = 64\) possible initial or final two-body states here. Thus there are \(64 \times 64 = 4096\) possible channels, but only \(4 \times 7 = 28\) distinct pairs of particles, or equivalently at most 28 possible NG bosons. The last statement holds under the proviso of exact \(O(8)\) symmetry being broken to a discrete (non Lie) symmetry, however. Otherwise there are fewer than 28 NG bosons, and when \(O(8)\) is explicitly broken down to \(O(4) \times O(2)\), or \(O(4)\) by one of the terms in the Lagrangian Eq. (26), there are even fewer than that. This fact tells us that many channels must be coupled and that the residual symmetry plays a crucial role in this coupling. We shan’t look at every possible channel in this paper, but rather concentrate only on the exotic ones, in this case the isotensor, so as to show the existence of pure bound state NG bosons without CDD admixtures.

We may use the residual vacuum symmetry, e.g. the \(O(3)\) isospin invariance to split this \(64 \times 64\) matrix equation into six invariant subspaces: three flavour channels [(a) isoscalar; (b) isovector, and (c) isotensor] of either parity. In the two-body, or Bethe-Salpeter (BS) equation for the four-point Green functions \(D_{ij}(s)\), the indices \(i, j\) denote the isospin of the two-body initial and final states, respectively.
The negative parity (pseudoscalar) isotensor two-body equation is a single-channel one and straightforward to solve, see Eq. (12). We look at the zero CM energy $P = 0$ function $V_{\pi\alpha}(0)\Pi_{\pi\alpha}(0)$; we use

$$V^{(I=2)}_{\pi\alpha} = \lambda_1 + \lambda_2$$

and Eq. (37) to obtain the final result

$$V^{(I=2)}_{\pi\alpha}(0)\Pi_{\pi\alpha}(0) = -\left(\frac{\lambda_1 + \lambda_2}{\lambda_1^2 - 2\lambda_1\lambda_2 - 3\lambda_2^2}\right)\left(\lambda_1 \left(1 - \frac{8\kappa}{\alpha - \pi}\right) + \lambda_2 \frac{\eta - 8\kappa}{\alpha - \pi}\right)$$

Now set $\lambda_2 = \kappa \to 0$ and find

$$\lim_{\lambda_2=\kappa\to 0} V^{(I=2)}_{\pi\alpha}(0)\Pi_{\pi\alpha}(0) = 1.$$ (41)

The propagator Eq. (12) evaluated at zero momentum can be written as

$$D^{(I=2)}_{\pi\alpha}(0) = \frac{g^2_{\pi\alpha}}{-m^2_{\pi\alpha}} = \frac{V_{\pi\alpha}(0)}{1 - V_{\pi\alpha}(0)\Pi_{\pi\alpha}(0)},$$

whence it follows that

$$m^2_{\pi\alpha} = \left[\frac{1}{\lambda_1 + \lambda_2} - \left(\frac{1}{\lambda_1^2 - 2\lambda_1\lambda_2 - 3\lambda_2^2}\right)\left(\lambda_1 \left(1 - \frac{8\kappa}{\alpha - \pi}\right) + \lambda_2 \frac{\eta - 8\kappa}{\alpha - \pi}\right)\right]$$

$$\times \left(\frac{\alpha - \pi}{v}\right)^2$$

$$= O(\lambda_2) + O(\kappa)$$

Thus we see that the effective (pseudo) NG boson mass in this channel is proportional to $\lambda_2$, and/or $\kappa$, the two O(8) symmetry breaking parameters. q.e.d.

5 Summary and Conclusions

In summary, we have: 1) proven the NG theorem in the variational Gaussian wave functional approximation to the O(N) symmetric $\phi^4$ model when the symmetry of the ground state is a proper subgroup of O(N-1); 2) proven conservation of Nöther currents corresponding to the dynamically broken symmetries; 3) proven the same NG theorem in the exotic isotensor channel of the ’t Hooft model in the limit $\lambda_2 = \kappa \to 0$. The NG bosons are massless bound states of two massive constituents. We emphasize that our proofs do not depend on the specific values of the bare parameters, or of the cutoff in the theory, so long as the system is in the spontaneously broken phase with appropriate symmetry.

We should like to put these results into their proper logical and chronological setting. The variational method in quantum field theory (QFT) is based on the Schrödinger representation and goes by the name of Gaussian approximation to the ground state wave functional. This method, in its various guises, was pioneered by Schiff, Rosen and Kuti [6, 7, 8] in the 1960’s and 70’s, and later revived and elaborated in the 1980’s by Barnes and Ghandour [9, 13], and by Symanzik [10] and by Consoli, Stevenson and collaborators [11, 12, 16]. Related formalisms based on effective potentials and other functional methods were discussed in Ref. [8, 16, 12]
and references cited therein. Most of these studies addressed the $\phi^4$ scalar field theory that is also the prime example of the Nambu-Goldstone (NG) theorem [1, 2, 3, 4, 5], an exact result in the O(N) symmetric $\phi^4$ scalar theory with spontaneous internal symmetry breaking.

The NG theorem was first shown not to be satisfied by the solutions to the mass, or “gap” equations in the Gaussian approximation by Kamefuchi and Umezawa in 1964 [14], practically simultaneously with the general proofs of the NG theorem [3, 4, 5]. This fact was subsequently rediscovered several times [15, 16] and this unsatisfactory situation persisted until 1994 [17]. Various conjectures as to the reasons for this failure and as to potential remedies were advanced during this period of time. It was first shown in Ref. [17] that this apparent breakdown of the NG theorem is not an intrinsic shortcoming of the Gaussian wave functional approximation, but rather a consequence of incomplete previous analyses. In other words, the NG theorem is satisfied, but the NG bosons are not excitations of the elementary scalar fields, as initially expected. Rather, they are massless bound states of two massive elementary scalar excitations, in close analogy with Nambu’s [1] proof of the NG theorem in a self-interacting fermion theory. NG bosons are solutions to the two-body (or Bethe-Salpeter [BS]) equation in the Gaussian approximation. That equation was only rarely considered in the literature [7, 9], and never before Ref. [17] in the context of spontaneous symmetry breaking of purely bosonic models.

In this light the result is simple enough to understand, yet it drew strong, albeit unpublished criticism and affirmation [27]. Perhaps the underlying reason for the misunderstanding by some was the implication of the proof that the $\phi^4$ scalar field theory could have bound states, which, as “everybody knew” [27], disagrees with various “rigorous no-go” and “triviality” theorems in the same theory [28]. The said theorems hold only in the limit of an infinite cutoff, however, in which the Gaussian approximation also becomes trivial [11]. For finite cutoffs, on the other hand, this is a nontrivial theory that may contain bound states.

Soon after the first proof in Ref. [17] it was also shown along the lines of Ref. [3] that the NG theorem also follows from the Gaussian effective potential, Ref. [30], but that proof did not shed much light on the mechanisms that made the NG bosons come about. Only later, in Ref. [31, 32], it was explicitly shown how this formal proof relates to the Gaussian two-body equations of motion. Another source of confusion was the apparent doubling of degrees of freedom, at least in some “flavour” (internal symmetry) channels, viz. the existence of massive “elementary” and massless bound states in the same channel. This problem was resolved in Ref. [31], wherein the Källen-Lehmann spectral function was calculated in appropriate channels of the model. This spectral function clearly shows the presence of massless NG states and the absence of the massive single-particle excitations. That also constitutes a proof of the NG theorem along Gilbert’s lines, Ref. [5], within the Gaussian wave functional approximation. Thus we have confirmed all the well known proofs of the NG theorem in the Gaussian approximation.

The NG theorem is the simplest example of a Ward-Takahashi identity, which follows from the underlying internal symmetry of the $\phi^4$ model. Ward-Takahashi identities typically relate (n-1)-point Green functions to n-point functions and/or

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7For more recent results comparing constructive QFT to the Gaussian approximation in 1+1 dimensional field theories, see Ref. [29]
matrix elements of Noether currents. These identities were developed by Lee in the linear sigma model at the perturbative one loop level [23], and by Symanzik for arbitrary orders of perturbation theory [24], so we shall call them the Lee-Symanzik [LS] identities.

The exact, i.e. nonperturbative Green functions satisfy an infinite set of coupled integro-differential equations called the Schwinger-Dyson equations [19]. The iterative/perturbative solutions to the SD equations form (infinite/finite) sets of Feynman diagrams. If one decouples the SD equations for higher order Green functions from the lower order ones (in popular jargon, if one truncates the SD equations), one may obtain tractable equations and find their solutions that sum infinite, albeit incomplete sets of Feynman diagrams. It has been known at least since 1980, Ref. [9], that the Gaussian approximation to the unbroken symmetry $\phi^4$ theory corresponds to one such truncation of the SD equations. But, truncated SD equations need not obey the conservation laws of the original SD equations of motion, i.e. LS identities may be violated by the truncation. To our knowledge, no proof of LS identities had been given for truncated SD equations, i.e. for infinite classes of diagrams in the bosonic linear sigma model prior to Ref. [17], although a similar proof had been given by Nambu and Jona-Lasinio in their fermionic model some 30 years before [1]. Thus we have shown that the Gaussian functional approximation constitutes a closed, self-consistent symmetry-preserving approximation to the Schwinger-Dyson equations.

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A The gap equations with two v.e.v.s

The first set of energy minimization equations (7) read

$$\mu_0^2 = \lambda_0 \left[ v^2 + 3hI_0(M) + hI_0(\mu_1) + (N - 2)hI_0(\mu_2) \right]$$  \hspace{1cm} (44)

$$\mu_0^2 = \lambda_0 \left[ v^2 + hI_0(M) + 3hI_0(\mu_1) + (N - 2)hI_0(\mu_2) \right]$$  \hspace{1cm} (45)

$$v^2 = \langle \phi_0 \rangle^2 + \langle \phi_1 \rangle^2 = \langle \phi \rangle^2$$ \hspace{1cm} (46)

$$\langle \phi_i \rangle = 0 \quad i = 2, \ldots, N - 1 \quad ,$$ \hspace{1cm} (47)

where the divergent integral $I_0(m_i)$, Eq. (4) is understood to be regularized via an UV momentum cut-off $\Lambda$, either three-, or four dimensional. The second set of gap equations, (7) read:

$$M^2 = -\mu_0^2 + \lambda_0 \left[ 2\langle \phi_0 \rangle^2 + \langle \phi \rangle^2 + 3hI_0(M) + hI_0(\mu_1) + (N - 2)hI_0(\mu_2) \right]$$ \hspace{1cm} (48)

$$\mu_1^2 = -\mu_0^2 + \lambda_0 \left[ 2\langle \phi_1 \rangle^2 + \langle \phi \rangle^2 + hI_0(M) + 3hI_0(\mu_1) + (N - 2)hI_0(\mu_2) \right]$$ \hspace{1cm} (49)

$$\mu_2^2 = -\mu_0^2 + \lambda_0 \left[ \langle \phi \rangle^2 + hI_0(M) + hI_0(\mu_1) + NhI_0(\mu_2) \right] \quad .$$ \hspace{1cm} (50)
Upon inserting Eqs. (44,b) into Eqs. (48,b), the following coupled “gap” equations emerge:

\[ M^2 = 2\lambda_0 (\phi_0)^2 + 2\lambda_0 \bar{h} [I_0(M) - I_0(\mu_1)] = 2\lambda_0 (\phi_0)^2 \quad \text{(51)} \]

\[ \mu_1^2 = 2\lambda_0 (\phi_1)^2 - 2\lambda_0 \bar{h} [I_0(M) - I_0(\mu_1)] = 2\lambda_0 (\phi_1)^2 \quad \text{(52)} \]

\[ M^2 - \mu_1^2 = 2\lambda_0 \left( (\phi_0)^2 - (\phi_1)^2 \right) = 2\lambda_0 \bar{h} [I_0(M) - I_0(\mu_1)] \quad \text{(53)} \]

\[ \mu_2^2 = 2\lambda_0 \bar{h} [I_0(\mu_2) - I_0(M)] = 2\lambda_0 \bar{h} [I_0(\mu_2) - I_0(\mu_1)] \quad \text{(54)} \]

Note that these equations lead to

\[ I_0(M) - I_0(\mu_1) = 0 \quad \text{(55)} \]

\[ M^2 - \mu_1^2 = 2\lambda_0 \left( \langle \phi_0 \rangle^2 - \langle \phi_1 \rangle^2 \right) = 0 \quad \text{(56)} \]

which, in turn have at least one solution: \( \mu_1 = M \) and \( \langle \phi_0 \rangle^2 = \langle \phi_1 \rangle^2 \), that are solutions to a single nontrivial gap equation

\[ M^2 = 2\lambda_0 \langle \phi_1 \rangle^2 \quad \text{(57)} \]

\[ \mu_2^2 = 2\lambda_0 \bar{h} [I_0(\mu_2) - I_0(M)] \quad \text{(58)} \]

with two unknowns one of which is kept fixed, c.f. Ref. [18]. This gap equation has been solved numerically in Ref. [18]: it admits only massive solutions \( M > \mu > 0 \), however, for real, positive values of \( \lambda_0, \mu_0^2 \) and real ultraviolet cut-off \( \Lambda \) in the momentum integrals \( I_0(m_i) \). In other words the “would be NG boson fields” \( \phi_{1,2,...,N-1} \) excitations are all massive \( \mu > 0 \) in the MFA. This looks like a breakdown of the NG theorem in this approximation, but, as discussed in Ref. [17, 18], there is a solution by way of the two-body (Bethe-Salpeter) equation.

### B Symmetries of the ’t Hooft model

The two field quartets, \((\sigma, \pi)\) and \((\alpha, \eta)\), have different “chiral” \( O(4) = O(3) \times O(3) \simeq SU_L(2) \times SU_R(2) \)

\[ \delta_5 \sigma = \beta \cdot \pi \quad \text{(59)} \]

\[ \delta_5 \eta = -\beta \cdot \alpha \quad \text{(60)} \]

\[ \delta_5 \alpha = \beta \eta \quad \text{(61)} \]

\[ \delta_5 \pi = -\beta \sigma \quad \text{(62)} \]

isospin

\[ \delta \sigma = 0 \quad \text{(63)} \]

\[ \delta \eta = 0 \quad \text{(64)} \]

\[ \delta \alpha = -\varepsilon \times \alpha \quad \text{(65)} \]

\[ \delta \pi = -\varepsilon \times \pi \quad \text{(66)} \]

and \( U_A(1) \simeq O(2) \) transformation properties

\[ \delta_5^0 \sigma = \beta \eta \quad \text{(67)} \]

\[ \delta_5^0 \eta = -\beta \sigma \quad \text{(68)} \]

\[ \delta_5^0 \alpha = \beta \pi \quad \text{(69)} \]

\[ \delta_5^0 \pi = -\beta \alpha \quad \text{(70)} \]
The Lie algebra $O(4)$ has two Casimir operators, but there is only one invariant in the $(\frac{1}{2}, \frac{1}{2})$ representation with one meson quartet, viz. (i) $[\sigma^2 + \pi^2]$, whereas with two quartets one has three invariants: (i) above; (ii) $[\eta^2 + \alpha^2]$, and (iii) $[\eta\sigma - \alpha \cdot \pi]$. Any odd power of the third invariant (iii) violates CP. Even without the third invariant the algebraic structure of the Lagrangian is rich enough to allow for multiple vacua [solutions to the energy minimization equations (6)] even in the (first) Born approximation. For example, equations (6) applied directly to the 'tHooft interaction (26) in the Born approximation allow two nonzero v.e.v.s $[\langle \sigma \rangle_{0B} = v_{0B} \neq 0$ and $\langle \alpha_3 \rangle_{0B} = v_{1B} \neq 0]$:

$$-\mu_0^2 - 4\kappa + \frac{\lambda_1}{2} [v_{0B}^2 + v_{1B}^2] + \lambda_2 v_{1B}^2 = 0$$ (71)

$$-\mu_0^2 + 4\kappa + \frac{\lambda_1}{2} [v_{0B}^2 + v_{1B}^2] + \lambda_2 v_{1B}^2 = 0.$$ (72)

Their solutions are

$$\langle \sigma \rangle_{0B} = v_{0B} = \frac{\mu_0^2}{\lambda_1 + \lambda_2} - 4\frac{\kappa}{\lambda_2}$$ (73)

$$\langle \alpha_3 \rangle_{0B} = v_{1B} = \frac{\mu_0^2}{\lambda_1 + \lambda_2} + 4\frac{\kappa}{\lambda_2}$$ (74)

leading to a nontrivially broken ground state. Note, however, that (i) $\lambda_2 \to 0$ is a singular limit point, i.e. these vacua are not continuously connected with the more conventional vacua at $\lambda_2 = 0$, and (ii) the two kinds of vacua/energy minima coincide whenever $\kappa = 0$ and $\lambda_2 \neq 0$. Hence it is possible for the system to be in an unconventional vacuum even with infinitesimally small $\lambda_2$ and $\kappa$ when they vanish with a fixed nonzero ratio. This makes it plausible that unusual vacua may also appear in this model when the symmetry is broken dynamically, i.e. by “loop” effects. This doubling of vacua is a consequence of multiple (two) independent algebraic invariants in the Lagrangian (26) which in turn led to multiple (two) v.e.v.s, in agreement with the general theory [4, 20].

References

[1] Y. Nambu, Phys. Rev. Lett. 4, 380 (1960), and Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961); Phys. Rev. 124, 246 (1961).

[2] J. Goldstone, Nuov. Cim. 19, 154 (1961).

[3] J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[4] S.A. Bludman and A. Klein, Phys. Rev. 131, 2364 (1963).

[5] W. Gilbert, Phys. Rev. Lett. 12, 713 (1964).

[6] L.I. Schiff, Phys. Rev. 130, 458 (1963).

[7] G. Rosen, J. Math. Phys. 9, 804 (1968), and Phys. Rev. 173, 1680 (1968).

[8] J.M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D 10, 2428 (1974).

[9] T. Barnes and G.I. Ghandour, Phys. Rev. D 22, 924 (1980).

[10] K. Symanzik, Nucl. Phys. B 190, 1 (1981).
[11] P. M. Stevenson, Phys. Rev. D 32 (1985) 1389; P. M. Stevenson and R. Tarrach, Phys. Lett. B 176, 436 (1986); P. M. Stevenson, Z. Phys. C 35, 467 (1987).

[12] P. M. Stevenson, B. Allès and R. Tarrach, Phys. Rev. D 35, 2407 (1987).

[13] R.J. Rivers Path Integral Methods in Quantum Field Theory, Cambridge University Press, Cambridge (1987), and B. Hatfield, Quantum Field Theory of Point Particles and Strings, Addison-Wesley, Reading (1992); see also eds. L. Polley and D.E.L. Pottinger, Variational Calculations in Quantum Field Theory, World Scientific, Singapore (1987).

[14] S. Kamefuchi, and H. Umezawa, Nuov. Cim. 31, 429 (1964).

[15] G. Domokos and P. Surányi, Sov., Journ. Nucl. Phys. 2, 361 (1966).

[16] Y. Brihaye and M. Consoli, Phys. Lett. B157, 48 (1985); M. Consoli and A Ciancitto, Nucl. Phys. B 254, 653 (1985).

[17] V. Dmitrašinović, J. A. McNeil and J. Shepard, Z. Phys. C 69, 359 (1996) [hep-th/9406151].

[18] Issei Nakamura and V. Dmitrašinović, Prog. Theor. Phys. 106, 1195 (2001).

[19] M. Chaichian, M. Hayashi, N.F. Nelipa and A.E. Pukhov, J. Math. Phys. 20, 1783 (1979).

[20] G. Cicogna and P. de Mottoni, J. Math. Phys. 15, 1538 (1974).

[21] A. K. Kerman and Chi-Yong Lin, Ann. Phys. (N.Y.) 269, 55 (1998).

[22] F. Gross, Relativistic Quantum Mechanics and Field Theory, J. Wiley, N.Y., (1993).

[23] B. W. Lee, Nucl. Phys. B 9, 649 (1969).

[24] K. Symanzik, Comm. Math. Phys. 16, 48 (1970).

[25] M. Gell-Mann and M. Levi, Nuov. Cim. 16, 705 (1960).

[26] G. ‘t Hooft, Phys. Rep. 142, 357 (1986).

[27] Private communications from H. Georgi’s Phys. Lett. B referees and from M. Consoli and P.M. Stevenson (1994-5).

[28] J. Glimm and A. Jaffe, Quantum Physics, (Springer, New York 1981), see also section 3 in J. Glimm, A. Jaffe and T. Spencer, p. 132 in Constructive Quantum Field Theory, Lecture Notes in Physics 25, (Springer, Berlin, Heidelberg, New York 1973);

[29] E. Prodan, quant-ph/9905088, J. Math. Phys. 41, 787 (2000) and hep-th/0204126.

[30] A. Okopińska, Phys. Lett. B 375, 213 (1996).

[31] V. Dmitrašinović, Phys. Lett. B 433, 362 (1998).

[32] Issei Nakamura and V. Dmitrašinović, Nucl. Phys. A 713, 133 (2003).