Black Hole Gravito-Thermo-Electrodynamics: How can Kerr black holes manage to self-extract energy?

Isao Okamoto\textsuperscript{1,2} and Yoogeun Song\textsuperscript{3,4}

\textsuperscript{1}National Astronomical Observatory, 2-21-1 Osawa, Mitaka-shi, Tokyo 181-8588, Japan
\textsuperscript{2}Institute of Black Hole Mining, 114-6 Ochikawa, Hino-shi, Tokyo 191-0034, Japan
\textsuperscript{3}Korea Astronomy and Space Science Institute, 776 Daedeok-daero, Yuseong-gu, Daejeon 34055, Korea
\textsuperscript{4}University of Science and Technology, 217 Gajeong-ro, Yuseong-gu, Daejeon 34113, Korea

\textsuperscript{*}E-mail: iokamoto@jcom.zaq.ne.jp, ygsong1004@gmail.com

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Abstract

This paper attempts to elucidate how Kerr black holes manage to radiate energy through their force-free magnetospheres specified by the two conserved quantities; the field-line angular velocity $\Omega_F(\Psi)$ and the current-angular momentum flux function $J(\Psi)$, where $\Psi$ is the stream function. To comply with the first law of thermodynamics, with the help of the frame-dragging angular velocity $\omega_0$, the energy-angular momentum flux relation is modified like $\mathcal{E} = \mathcal{EM} + \mathcal{SD} = \Omega_F J$, where $\mathcal{EM} = \Omega_F \omega_0 S_1$ is the Poynting flux, $\Omega_F = \Omega_F - \omega$ is the ZAMO-measured angular-velocity (ZAMOs are the zero-angular-momentum-observers), and $S_{SD} = \omega S_1$ is the spin-down flux. For $\Omega_F(\Psi)$ and its overall value $\bar{\Omega}_F$, the second law imposes $0 \leq \Omega_F \approx \bar{\Omega}_F < \Omega_H$, where $\Omega_H$ is the hole’s angular-velocity. The null surface $S_N$, where $\Omega_{F,0} = S_{EM} = 0$, always exists to break down the freezing-in and force-free conditions, resulting in severance of both current- and stream-lines there, thereby dividing the magnetosphere into the two domains by $\Omega_{F,0} > 0$. Oppositely to the outer domain $\Omega_{F,0} > 0$, the counter-rotation of the inner domain $\Omega_{F,0} < 0$ gives rise to an influx of negative angular momentum, which couples with the frame-dragging effect, to induce an influx of negative energy, i.e., the outflow of positive spin-down energy. The voltage drop between the two batteries at $S_N$ will kind of spark-discharge to steadily produce pair-particles, making $S_N$ widened to the ‘zero-angular-momentum-Gap’ filled with ZAM-particles. The Poynting flux will be launched both outward and inward from $S_N$. The ‘boundary condition’ of keeping the ZAM-state of the Gap yields the eigenvalue of $\bar{\Omega}_F = 0.5\Omega_H$ for this self-extraction. The magnetosphere will be frame-dragged into rotation with $\omega_N \equiv \Omega_F$ by the hole’s spin.

Key words: stars: black holes\textsubscript{1} — acceleration of particles\textsubscript{2} — magnetic fields\textsubscript{3} — methods: analytical\textsubscript{4}

\section{1 Introduction}

1.1 A brief review on the BZ process

As is well known by the Penrose process, the rotational energy of the Kerr black hole (BH) is in principle extractable, by making use of such property as the presence of negative-energy orbits in the ergosphere (Penrose 1969; Misner et al. 1973; Lasota et al. 2014), although this mechanical process so far is not regarded as viable enough to fuel astrophysical phenomena such as high-energy gamma-ray jets. It is then the course of nature that a notice of an electrodynamic mechanism analogous to that in pulsar electrodynamics (Goldreich & Julian 1969) was taken,
because the presence of the magnetic field is known to be very effective to transfer angular momentum and energy from rapidly rotating objects like a neutron star (NS). A magnetized NS is thought to genetically anchor an active magnetosphere as its belongings, with the field line angular velocity (FLAV) $\Omega_\text{FLAV} = \Omega_{\text{NGS}}$ fixed by the ‘boundary condition’. This means in principle that a unipolar induction battery (Landau et al. 1984) be at work on the stellar surface, driving electric currents through the pulsar magnetosphere, together with the outgoing Poynting flux.

Subsequent to the Penrose process is the Blandford-Znajek (BZ) process, i.e., an electromagnetic process of extracting energy from Kerr BHs. This has been taken to be promising and efficient so far, and its efficiency is given by $\epsilon = \Omega_{\text{FL}} / \Omega_{\text{BH}}$ (Blandford & Znajek 1977; Znajek 1977; Blandford 1979), where $\Omega_{\text{BH}}$ is the hole’s angular velocity. BH electrodynamics was then formulated in the form of the 3+1 formalism by Macdonald & Thorne (Macdonald & Thorne 1982), and Thorne, Price & Macdonald (Thorne et al. 1986) proposed ‘The Membrane Paradigm’, in which the ‘horizon battery’ was explicitly regarded as existent in the event horizon. Phinney 1983a and Phinney 1983b were the first that tried to develop a comprehensive model for ‘BH-driven hydromagnetic flows’ or jets for AGNs, referring to the pulsar wind theory (Okamoto 1978; Kennel et al. 1983). It was thought since then that a ‘magnetized’ Kerr BH would possess not only a battery but an internal resistance $Z_H$ on the horizon, as seen in ‘a little table on BH circuit theory for engineers’ (Fig. 3 in Phinney 1983a). The image in the 1980s looks like the magnetosphere consisting of double wind structures with a negligible violation of the force-free condition for particle production and a single series circuit with a battery on the horizon.

The BZ process and the Membrane Paradigm have, however, invited serious critiques of causality violation (Punsly & Coroniti 1989; Punsly & Coroniti 1990; Blandford 2002). It was, in reality, pointed out that the BZ process is “incomplete and additional physics is needed” (Punsly & Coroniti 1990). It appears that the presence of the causality question and related confusion since the 1990s had been bottlenecking sound development of BH electrodynamics (see Punsly (2008); Beskin (2009); Meier (2012) for general reviews).

We attempt to trace the question exhaustively to its origins. There appear to be two causes inextricably linked: one is on the pulsar-electrodynamics side, and the other is on the general-relativistic side. It can be pointed out in the former that the role of $\Omega_{\text{FLAV}}$ as the FLAV appears to have been well-explored, whereas that of $\Omega_{\text{FL}}$ as the potential gradient has unfortunately not so well, so that in the latter, the frame-dragging angular velocity (FDAV) $\omega$ could not accomplish a smooth coupling with $\Omega_{\text{FL}}$ in the BH’s unipolar induction. The coupling tempts us to define the ZMO-measured FLAV $\Omega_{\text{FLAV}} = \Omega_{\text{FL}} - \omega$, and naturally gives rise to the null surface $S_N$ (Okamoto 1992; see the ‘this surface’ in Blandford & Znajek 1977 and the $C_p$(iv)-statement, section 4.1), which divides the force-free magnetosphere inexorably into the outer and inner domains with the force-free condition broken down in-between. Also, a Kerr BH itself is not an electrodynamic object, distinctly different from a magnetized NS, but basically a thermodynamic object, being fated to obey the four laws of thermodynamics (Thorne et al. 1986; Okamoto & Kaburaki 1990; Okamoto & Kaburaki 1991; Kaburaki & Okamoto 1991). The electrodynamic process of extraction of energy is therefore under severe control of the first law, $c^2 dM = T_{ij} ds + \Omega_{\text{FL}} d\Omega_{\text{FL}}$ and the second law, $dS \geq 0$, because the former defines the efficiency of extraction, and the latter poses an important restriction on the efficiency (see section 3.2).

It became soon apparent that what lacked for in the BZ process is the frame-dragging (FD) effect (Blandford 2002). ‘The dragging of inertial frames’ was indeed mentioned in Blandford & Znajek 1977 and Phinney 1983a, and was in reality perfectly incorporated into the 3+1 formulation for BH electrodynamics in Macdonald & Thorne 1982. The ‘needed additional physics’ really is thermodynamics. It is the FD effect that bridges the event horizon between (BH) thermodynamics and (pulsar) electrodynamics (see sections 2 and 3), and the resulting physics is ‘gravito-thermo-electrodynamics’ unified by coupling the FDAV $\omega$ with the FLAV $\Omega_{\text{FL}}$. The key is to elucidate ‘why, where, and how’ the coupling gives rise to a violation of Ferraro’s law of isotropy and then the breakdown of the freezing-in and force-free conditions.

The path from pulsar electrodynamics to gravito-thermo-electrodynamics (GTED) has already been cut open almost four decades ago (Blandford & Znajek 1977; Macdonald & Thorne 1982; Thorne et al. 1986), but it seems that none of the travelers has so far succeeded in traveling through all the way to the final goal, without going into a maze. The lesson learned so far is always to keep the “physical observers’ (the ZAMOs’) viewpoint” (see section 4.1). This paper describes our track of having been attempting to follow this difficult path toward GTED.

1.2 A brief overview of this paper

The followings are a brief outline of each section. Sections 2 and 3 briefly describe the fundamental properties of pulsar electrodynamics and BH thermodynamics, in order to smoothly unify them with the help of the FD effect toward GTED. It is clarified how the first law indicates the unique way of energy extraction under strict restrictions of the second law. For example, the first law allows us to define the overall efficiency of extraction.

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1 The referee of Okamoto 1992 asked the author to comment on the question of causality then arising about the BZ process. Of course he could not devise a reasonable reply at that time. It unfortunately took more than three decades to find out a good chemistry among relativity, thermodynamics, and electrodynamics toward GTED, but it seems anyway that this might fortunately be done before everything has been forgotten owing to dementia.
by $\xi_{GTD} = \Omega_0/\Omega_4$ and the second law gives a restriction of $0 \leq \xi_{GTD} \leq 1$ (Blandford & Znajek 1977). The FD effect is visualizable by ‘coordinatizing’ $\omega$ and $\Omega_{FL}$ along each field line (see Figure 1).

Section 4 clarifies the nature of the BZ process. Eleven critical statements, both positive from a physical observer’s viewpoint and rather negative from a static observer’s viewpoint, are picked up from Blandford & Znajek (1977) and Macdonald & Thorne (1982). The BZ process may be referred to as the single-pulsar model in circuit theory, in the sense that the single battery in the horizon with an internal resistance in addition to the external one feeds the two wind zones with the pair-production discharge taking place between the two light surfaces, under the condition of a negligible violation of the force-free condition (see, i.e., Beskin et al. 1992; Hirotani & Okamoto 1998). Vital difficulties and contradictions and some crucial suggestions are presented on the negative $C_4$-statements, which are rather contradictory to the positive $C_7$-statements.

In section 5, we introduce some central critical premises, to modify the BZ process. One of them is that the large-scale poloidal magnetic field $B_p$ extending from near the horizon $S_H$ to the infinity surface $S_{\infty}$, with the FLAV $\Omega_0 = \text{constant}$ (Ferraro’s law of iso-rotation). We presume that magnetic field lines need not thread the horizon nor be anchored there, partly from the viewpoint of the no-hair theorem.

In section 6, we make full use of the $3+1$ formulation with the freezing-in and force-free conditions as well as the FD effect taken into account. Physical quantities are measured by the zero-angular-momentum-observers (ZAMOs) or ‘physical observers’ (Blandford & Znajek 1977) circulating in the hole with $\omega$. We reproduce and examine necessary quantities and relations, such as the two outer and inner light surfaces $S_{aD}$ and $S_{aI}$ and the densities of the electromagnetic energy and angular momentum $E_{\ell,i}$ and $J_{\ell,i}$.

It is shown in section 7 that when the FLAV $\Omega_0$ obeys the iso-rotation law throughout the magnetosphere, the ZAMO-FLAV $\Omega_{FL}$ does not by the FD effect $\omega$. Then the violation by $\Omega_{FL}$ leads to breakdown of the freezing-in and force-free conditions at the ‘null surface’ $S_N$, giving rise to $I = \Omega_{FL} = 0$, as well as to the particle velocity $v = 0$ and the current $j = 0$. The null surface $S_N$ must be in-between the two light surfaces, $S_{aD}$ and $S_{aI}$, for the outflow and inflow, and hence some pair-production mechanism must be at work there. The breakdown of the two conditions imposes strong constraints in building a reasonable gap model in section 9.

Section 8 extends the Membrane Paradigm (Thorne et al. 1986) from one membrane on the stretched horizon to three membranes, namely, two resistive membranes on the horizon and infinity surfaces ($S_H$ and $S_{\infty}$), $S_{gD}$ and $S_{gI}$, and one inductive membrane $S_N$ (or Gap $G_N$) on the null surface $S_N$. The force-free magnetosphere is then edged with the resistive membranes of the surface resistivity $\mathcal{R} = 4\pi/c$ (Znajek 1978), where the Ohmic dissipation of the surface currents implies entropy production in $S_{\infty}$ and particle acceleration in $S_{gI}$, whereas the inductive membrane $S_N$ divides the magnetosphere into the two force-free domains, $D_{(out)}$ and $D_{(in)}$, and is installed with a pair of unipolar induction batteries with electromotive forces (EMFs), $E_{(out)}$ and $E_{(in)}$, driving currents to flow through the circuits $C_{(out)}$ and $C_{(in)}$ in $D_{(out)}$ and $D_{(in)}$, respectively. There is a huge voltage drop $\Delta V$ between the two EMFs for particle production at $S_N$.

It is argued in section 9 that one of the most important constraints in constructing the Gap model is $I(\ell, \Psi) = \Omega_{FL}(\ell, \Psi) = 0$ at $S_N$. When we presume $I = 0$ in $|\Omega_{FL}| \leq \Delta \omega$, where $\Delta \omega$ is the half-width of the Gap, the Gap will be filled with zero-angular-momentum-particles (ZAM-particles) pair-produced due to the voltage drop $\Delta V$, and the ‘zero-angular-momentum’ state of the Gap (ZAM-Gap) will be maintained in the steady-state, i.e., $I(\ell, \Psi) = \Omega_{FL}(\ell, \Psi) = 0$ in $|\Omega_{FL}| \leq \Delta \omega$.

It is then shown in section 10 that the ingoing flow of negative angular momentum from the inductive membrane $S_N$ is equivalent to the outgoing flow of positive angular momentum in the inner domain $\mathcal{D}_{(in)}$, and this fact is helpful for understanding a smooth flow of positive angular momentum beyond the Gap from the horizon $S_H$ to infinity surface $S_{\infty}$. The inevitable existence of the ZAM-Gap $G_N$ between the two force-free domains will thus allow imposing the boundary condition to determine the eigenfunction $\Omega_0(\Psi)$ in the eigen-magnetosphere. The counter-rotation $\Omega_{FL} < 0$ of the inner domain $\mathcal{D}_{(in)}$ will, in short, exert braking torque on the hole and, in turn, accelerating torque on the outer domain $\mathcal{D}_{(out)}$ with $\Omega_{FL} > 0$ under existence of poloidal magnetic field lines threading the Gap with $\Omega_0$ = constant.

Section 11 attempts to explain that the null surface $S_N$ (or the ZAM-surface $S_{ZAM}$) is a new kind of rotational-tangential discontinuity (Okamoto 2015a). As opposed to the single-pulsar model for the conventional BZ process, we propose the twin-pulsar model in a modified BZ process. It is conjectured that this rotational-tangential discontinuity involving the voltage drop $\Delta V$ between the two EMFs will lead to a new type of pair-particle creation viable in the general-relativistic setting.

The last section 12 is devoted to discussions and conclusions with some remaining questions listed. One of the conclusions is that any electrodynamic process will not work unless it complies with the first three laws of thermodynamics by the agency of the dragging of inertial frames.

2 The basics of the force-free pulsar magnetosphere

For a force-free pulsar magnetosphere filled with perfectly conductive plasma, with $B_p = -(\mathbf{\nabla} \Psi / 2\pi c)$ for the poloidal component of the magnetic field $B$, we have two integral functions
of $\Psi$, i.e., $\Omega_F$ and $I$, from the induction equation and conservation of the field angular momentum (see section 6): $\Omega_F(\Psi)$ denotes the FLAV in wind theory or the potential gradient in circuit theory, and $I(\Psi)$ denotes the angular momentum flux (multiplied by $-2/\ell c$) or the current function in wind or circuit theory. The toroidal component is given by $B_t = -(2I/\sigma_\infty c)$, and the electric field is given by $E_p = -(\Omega_F/2\pi c)\nabla \Psi$ from the induction equation with the freezing-in condition (with $E_t \equiv 0$ by axisymmetry). Then, the electromagnetic Poynting and angular momentum fluxes and the particle velocity become

$$S_{EM} = \Omega_F S_t, \quad S_t = (I/2\pi c)B_p, \quad (2.1a)$$

$$v = j/\sigma_\ell, \quad (2.1b)$$

where the toroidal component of $S_{EM}$ (and other fluxes) is omitted here, and $j$ and $\sigma_\ell$ denotes the electric current and the charge density for the charge-separated plasma.

Contrary to a force-free BH magnetosphere, there is no reason nor necessity for the force-free condition to break down within a force-free pulsar magnetosphere, because one may determine $\Omega_F$ rather automatically by imposing the 'boundary condition' $\Omega_F = \Omega_{NS}$ for field lines emanating from the magnetized NS, where $\Omega_{NS}$ is the angular velocity of the star. We may usually suppose that the force-free condition has already been broken down in the 'matter-dominated' interior with $I = 0$. On the other hand, in order to determine the eigenfunction $I(\Psi)$, one needs a kind of process to terminate the force-free, i.e., 'field-dominated' domain by restoring particle inertia so far neglected in the force-free domain, which can be expressible by a few equivalent ways (Okamoto & Sigalo 2006). One of them is the 'criticality condition' at the fast magnetosonic surface $S_F$ ($=S_{\ell_0}$) in wind theory, or the infinity resistive membrane $S_{\ell_\infty}$, with the surface resistivity $\mathcal{R} = 4\pi c = 377$ Ohm in circuit theory, containing a layer from $S_F$ at $\ell = \ell_\ell$ to $S_{\ell_\infty}$ at $\ell = \ell_\infty$ where the transfer of field energy to kinetic energy takes place in the form of the MHD particle acceleration (Okamoto 1974);

$$I_{NS}(\Psi) = \frac{1}{2}\Omega_F(\mathcal{B}_\ell \sigma_{\ell_\infty}^2), \quad (2.2)$$

equivalent to the 'radiative' condition and Ohm's law for the surface current on the resistive membrane $S_{\ell_\infty}$. Then, the toroidal field $B_t$ is regarded as the swept-back component of $B_p$ due to inertial loadings on the terminating surface $S_{\ell_\infty}$ of the force-free domain. Then the behavior of $I(\ell, \Psi)$ will be described as follows;

$$I(\ell, \Psi) = \begin{cases} 0 & : \ell \leq \ell_{NS}, \\ I_{NS}(\Psi) & : \ell_{NS} \leq \ell \leq \ell_F, \\ 0 & : \ell_F \leq \ell \leq \ell_\infty \end{cases} \quad (2.3)$$

(see equation (9.3) and Figure 2 for a Kerr hole's force-free magnetosphere). We do not intend to consider complicated interactions of the force-free pulsar wind with the interstellar media permeated by the general magnetic field in this model. We assume simply that $I(\ell, \Psi)$ tends to null for $\ell \to \infty$ and also $\sigma \to \infty$. This presumes that all the Poynting energy eventually is transmitted to the particle kinetic energy.

We consider that the force-free model is not applicable to the interior of the NS and hence $I = 0$. Instead, there will be a kind of 'inductive membrane' on the NS crust, on which a unipolar induction battery is at work to drive currents throughout the pulsar magnetosphere, whose field lines are anchored in the magnetized NS. That is to say, the NS accommodates the sources of the angular momentum flux $S_t$ as well as electromagnetic Poynting energy flux $S_{EM}$, together with the source of charged particles of both signs (at least in principle) at its surface. The force-free pulsar wind consists of charge-separated plasma, e.g., electrons or positrons, and the particle velocity is given by $v = j/\sigma_\ell$, which means that current-field-streamlines in the force-free domain are equipotentials everywhere, and $j_p = \sigma_\ell v_p = -(1/2\pi c)(dS_{NS}/d\Psi)B_p$. The current-closure condition holds along each current line in circuit theory. Thus, there is no reason or necessity of looking for breaking down the force-free condition, and this is crucially different from the hole's force-free magnetosphere, as argued in the following.

The wind theory and circuit theory must be complementary with each other, where $\Omega_F$ and $I$ take two sides of the same coin respectively (see Okamoto 2015a); $\Omega_F$ gives rise to the magneto-centrifugal particle acceleration in the former and to an EMF due to the unipolar induction battery on the NS surface in the latter, as related to the source of the Poynting flux at $S_{NS}$, i.e.,

$$\mathcal{E}_{NS} = -\frac{1}{2\pi c} \int_{\Psi_1}^{\Psi_2} \Omega_F(\Psi)d\Psi, \quad (2.4)$$

which drives currents along the current-field-streamline $\Psi_2$ with $j_p > 0$ and return currents along $\Psi_1$ with $j_p < 0$, where $\Psi_1 < \Psi_c < \Psi_2 < \Psi_f$ and $j_p > 0$ for $\Psi \gg \Psi_c$ (see equation (6.23) and Figure 3), where $\Psi$ is the last limiting field line satisfying $I(\Psi_f) = I(\Psi) = 0$ (see figure 2 in Okamoto & Sigalo 2006 for one example of $I(\Psi)$). The surface return currents flows from $I(\Psi_2)$ to $I(\Psi_1)$, crossing field lines between $\Psi_1$ and $\Psi_2$ on the resistive membrane $S_{\ell_\infty}$, and the Ohmic dissipation thereof formally represents the MHD acceleration taking place in $S_{\ell_\infty}$.

It is stressed that there is no reason, nor necessity in the 'force-free' pulsar magnetosphere for the 'force-free' condition to break down, at least because a magnetized NS is usually regarded as behaving like a unipolar induction battery with e.g. particle acceleration as an external resistance far from the star, but with no internal resistance. The FLAV $\Omega_F$ is given by $\Omega_{NS}$, i.e., the efficiency of the 'extraction of energy' is $\varepsilon_{NS} = \Omega_F/\Omega_{NS} = 1$ in the sense that no dissipation of rotational energy into the irreducible mass energy inside the NS takes place. If we may interpret thermodynamically, this case may be called an 'adiabatic extraction' (see sections 3).

In terms of pulsar thermo-electrodynamics for the 'boundary
condition Ω\textsubscript{r0} = Ω\textsubscript{NS}, no difference between the two angular velocities gives rise to no energy dissipation in the surface layer, and the whole spin-down energy of the star will be delivered without any loss of energy to the magnetosphere. This means that the pulsar magnetosphere is adiabatically rooted in the stellar crust.

3 Black hole thermodynamics

It is an ill-understood phenomenon called the ‘dragging of inertial frames’ that will connect pulsar electrodynamics with black hole thermodynamics beyond the event horizon, thereby leading to ‘gravito-thermo-electrodynamics’ (GTED). There are special people living around the Kerr black hole who are called ‘physical observers’ (Blandford & Znajek 1977), the ‘zero-angular-momentum-observers’ (ZAMOs; Macdonald & Thorne 1982) or ‘fiducial observers’ (FIDOs; Thorne et al. 1986) circulating round the hole with the frame-dragging-angular-velocity (FDAV) \( \dot{\phi} = \omega \). They play an indispensable role in describing the GTED extraction process of energy from the hole. Then the field-line-angular-velocity (FLAV), which the ZAMOS measure, is denoted by \( \Omega_{r\phi} \equiv \Omega_{\text{F}} - \omega \).

3.1 Basic thermodynamic properties of the Kerr hole

The no-hair theorem tells us that Kerr holes possess only two hairs, e.g., two ‘extensive’ variables such as the entropy \( S \) and the angular momentum \( J \), and then all other thermodynamic quantities are expressed as functions of these two. For example, the BH’s mass-energy \( M \) is expressed in terms of \( S \) and \( J \);

\[
M = \sqrt{(hcS/4\pi kG) + (\pi kc J^2/hG S)} \quad \text{(3.1)}
\]

As one can, in principle, utilize a Kerr BH as a Carnot engine (Kaburaki & Okamoto 1991), it is regarded basically as a ‘thermodynamic object,’ but not as an electrodynamic one, because the Kerr hole stores no extractable electromagnetic energy. The four laws of thermodynamics govern the rotational-evolutional process of Kerr holes (see, e.g., Chapter III C3 in Thorne et al. 1986 for a succinct summary; also Okamoto 1992).

The mass \( M \) of the hole is divided into the ‘irreducible’ and ‘rotational’ masses (Thorne et al. 1986), i.e.,

\[
M = M_{\text{irr}} + M_{\text{rot}} \quad \text{(3.2a)}
\]

\[
M_{\text{rot}} = M [1 - 1/\sqrt{1 + h^2}] \quad \text{(3.2b)}
\]

\[
M_{\text{irr}} = \frac{M}{\sqrt{1 + h^2}} = \sqrt{c^4 A_H/16\pi G^2} = \sqrt{hcS/4\pi kG} \quad \text{(3.2c)}
\]

where \( A_H \) is the surface area of the event horizon, and \( h \) is defined as the ratio of \( a \equiv J/Mc \) to the horizon radius \( r_H \), i.e.,

\[
h = \frac{a}{r_H} = \frac{2\pi kJ}{hcS} = \frac{2GM S \Omega_{H}}{e^3} \quad \text{(3.3)}
\]

to specify the evolutional state of the BH, and then \( 0 \leq h \leq 1 \) for the ‘outer-horizon,’ with \( h = 0 \) for a Schwarzschild BH with the same mass and \( h = 1 \) for an extreme-Kerr BH (Okamoto & Kaburaki 1990; Okamoto & Kaburaki 1991) (see also equations (10.4a,b,...,f) in Okamoto 1992).

The hole’s thermo-rotational state is uniquely specified by its entropy \( S \) and angular momentum \( J \), or its mass-energy \( M \) and the spin-parameter \( h \), by equations (3.1) and (3.3).

The zeroth law indicates that two ‘intensive’ variables, \( T_H \) (the surface temperature) and \( \Omega_H \), conjugate to \( S \) and \( J \), respectively, are constant on \( S_H \), e.g., \( \omega \rightarrow \Omega_H \) for \( \alpha \rightarrow 0 \). In passing, the third law indicates that “by a finite number of operations one cannot reduce the surface temperature to the absolute zero with \( h = 1 \).” In turn, “the finite processes of mass accretion with angular momentum cannot accomplish the extreme Kerr state with \( h = 1 \), \( T_H = 0 \) and \( \Omega_H = c^3/2GM^2 \)” (Okamoto & Kaburaki 1991). Also, the ‘inner-horizon’ thermodynamics can formally be constructed analogously to the ‘outer-horizon’ thermodynamics (Okamoto & Kaburaki 1993; Cvetic et al. 2018).

It is the first and second laws that govern the GTED process of extracting energy from a Kerr hole;

\[
c^2 dM = T_H dS + \Omega_H dJ \quad \text{(3.4a)}
\]

\[
T_H dS \geq 0 \quad \text{(3.4b)}
\]

where \( T_H \) and \( \Omega_H \) are uniquely expressed in terms of \( J \) and \( S \) from equation (3.1) or \( M \) and \( h \) (Okamoto & Kaburaki 1990);

\[
T_H = c^2 \left( \frac{\partial M}{\partial S} \right)_J = \frac{hc^3 (1 - h^2)}{8\pi k G M}, \quad S = \frac{4\pi k G M^2}{hc (1 + h^2)} \quad \text{(3.5a)}
\]

\[
\Omega_H = c^2 \left( \frac{\partial M}{\partial J} \right)_S = \frac{c^3 h}{2GM}, \quad J = \frac{2GM^2 h}{c(1 + h^2)^2} \quad \text{(3.5b)}
\]

Then, the ‘overall’ efficiency of extraction, \( \xi_{\text{GTED}} \), is defined as the ratio of “Actual energy extracted/ Maximum extractable energy, when unit angular momentum is removed” (see Blandford & Znajek 1977; the \( C_P(i) \)-statement, section 4.1 later), i.e., from the first law

\[
\xi_{\text{GTED}} = \frac{(dM/dJ)_S}{(dM/dJ)_S} = \frac{c^2}{\Omega_H} \quad \text{(3.6)}
\]

and by the second law \( dS \geq 0 \), we see that always \( 0 \leq \xi_{\text{GTED}} \leq 1 \) (see equation (3.12)).

3.2 Connection of a force-free magnetosphere to the first and second laws

Because the hole’s gravity produces a gravitational redshift of ZAMO clocks, their lapse of proper time \( dt \) is related to the lapse of global time \( dt \) by the lapse function \( \alpha \), i.e., \( dt/dt = \alpha \) (see Macdonald & Thorne 1982). Then, the change in universal time of the hole’s total mass-energy become from the first law (3.4a)

\[
c^2 \frac{dM}{dt} = T_H \frac{dS}{dt} + \Omega_H \frac{dJ}{dt} \quad \text{(3.7)}
\]

When the hole loses angular momentum and energy, i.e., \( dJ < 0 \) and \( c^2 dM < 0 \), through the force-free magnetosphere with
conserved quantities $\Omega_\ell(\Psi)$ and $I(\Psi)$, it is shown in Blandford & Znajek 1977 that the angular momentum and energy fluxes are given by
\[ S_E = \Omega_\ell(\Psi) S_1, \quad S_1 = (I(\Psi)/2\pi r c) B_\rho, \tag{3.8} \]
which are apparently the same as equation (2.1a) for the loss through the pulsar magnetosphere except the redshift factor/ lapse function $\alpha$. When $\Omega_\ell$ and $I$ are determined as the eigenvalue problem due to the criticality-boundary condition (section 10), the loss rate of angular momentum $P_J$ and the resultant output power $P_E$ are given by
\[ P_J = \frac{dJ}{dt} = \int \alpha S_1 \cdot dA = \frac{1}{c} \int_{\Psi_0}^\Psi \rho^\Psi I d\Psi, \tag{3.9a} \]
\[ P_E = -c^2 \frac{dM}{dt} = \int \alpha S_E \cdot dA = \frac{1}{c} \int_{\Psi_0}^\Psi \Omega_\ell I d\Psi, \tag{3.9b} \]
where $B_\rho \cdot dA = \pi d\Psi$ and integration is made over all open field lines in $\Psi_0 \leq \Psi \leq \Psi_0$. When we define the ‘average’ potential gradient, calculated from $\Omega_\ell(\Psi)$ weighted by $I(\Psi)$, i.e.,
\[ \overline{\Omega}_E = \int_{\Psi_0}^\Psi \Omega_\ell(\Psi) I(\Psi) d\Psi \int_{\Psi_0}^\Psi I(\Psi) d\Psi = \frac{P_E}{P_J}, \tag{3.10} \]
the first law (3.4a) reduces to
\[ c^2 dM = \overline{\Omega}_E dJ = \overline{P}_E dt, \tag{3.11a} \]
\[ T_H dS = - (\Omega_H - \overline{\Omega}_E) dJ = (\overline{\Omega}_E P_J - P_E) dt \tag{3.11b} \]
(see equation (3.22)). Then, only if $T_H dS > 0$, i.e., $\overline{\Omega}_E < \Omega_H$, angular momentum and energy are extractable from the hole, i.e., $c^2 dM = \overline{\Omega}_E dJ < 0$. The ‘overall’ efficiency $\overline{\epsilon}_{GTED}$ reduces from equations (3.6), (3.11a), (3.7), (3.9) and (3.10) to
\[ \overline{\epsilon}_{GTED} = \frac{c^2 dM/dt}{\Omega_{H}(dJ/dt)} = \frac{P_E}{\overline{\Omega}_E P_J} = \frac{\overline{\Omega}_E}{\Omega_H} \tag{3.12} \]
The relevant ‘boundary condition’ (Znajek 1977; Blandford & Znajek 1977) used to determine $\overline{\Omega}_E$ at $S_H$ is that $\Psi$ is finite and for $I$ in equation (3.8) or (3.9a)
\[ I_{in}(\Psi) = \frac{1}{2} (\Omega_H - \Omega_\ell_B) (B_\rho \rho^2)^{1/2} \tag{3.13} \]
(see their Eq. (3.15) in Blandford & Znajek 1977), which was regarded as indicating that this, together with appropriate boundary conditions at infinity, determines the angular velocity of field lines crossing the horizon. For example, equation (3.13) for $I_{in}(\Psi)$ combines with that in equation (2.2) or (8.1) for the outgoing wind, to yield $\Omega_\ell \approx 0.5 \Omega_H$, i.e., the efficiency $\epsilon = \Omega_\ell / \Omega_H \approx 0.5$. It appears that this result for $\Omega_\ell$ was interpreted as being at $S_H$ for field lines threading the horizon by the ‘boundary condition’. This procedure is often thought of as an ‘impedance matching’ (e.g., Macdonald & Thorne 1982).

### 3.3 Two non-conserved energy fluxes and their roles

It is the energy flux $S_E$ that corresponds to the term $c^2 dM/dt$ in the first law in equation (3.7); this means that there must be the other two fluxes, which correspond to $T_H dS/dt$ and $T_H dJ/dt$. These two fluxes (say $S_{EM}$ and $S_{SD}$) will be derived formally by the ‘kick-off’ equation of coupling the FD effect with unipolar induction and the two essential conditions, i.e., the force-free and freezing-in conditions (see sections 5 and 6). Here we reproduce these given in Eqs. (4.13) and (5.7) in Macdonald & Thorne 1982 as they are;
\[ S_E = \frac{ac}{4\pi} (E_p \times B_1) + \omega S_1 = \frac{1}{2\pi c} (E_p \times m + \omega B_p), \tag{3.14a} \]
\[ = \Omega_\ell S_J = \Omega_\ell \left( \frac{Gq}{2\pi \alpha c} \right) B_\rho = \Omega_\ell \left( \frac{1}{2\pi \alpha c} \right) B_\rho \tag{3.14b} \]
(see equation (3.8)), where
\[ E_p = -\frac{\Omega_\ell \omega}{2\pi \alpha c} \nabla \Psi, \quad B_1 = -\frac{2I}{\alpha c} t, \tag{3.15a} \]
\[ \Omega_\ell = \omega \ell, \tag{3.15b} \]
and $\Omega_\ell$ denotes the FLAV measured by the ‘physical observers’ (ZAMO) circulating round the hole. We define the electromagnetic Poynting flux $S_{EM}$ and the frame-dragging spin-down energy flux $S_{SD}$ from equation (3.14a) as follows;
\[ S_{EM} = \frac{ac}{4\pi} (E_p \times B_1) = \Omega_\ell \frac{I}{2\pi \alpha c} B_\rho = \Omega_\ell S_J, \tag{3.16a} \]
\[ S_{SD} = \frac{\omega I}{2\pi \alpha c} B_\rho = \omega S_1, \tag{3.16b} \]
and hence a general expression of Macdonald & Thorne 1982 in equation (3.14a,b) is simplified like
\[ S_E = S_{EM} + S_{SD} = \Omega_\ell S_J, \tag{3.17} \]
behind which is a simple and yet significant identity, i.e.,
\[ \Omega_\ell + \omega = \Omega_\ell. \tag{3.18} \]

The ZAMO-FLAV $\Omega_\ell$ changes inwardly from $\Omega_\ell$ at infinity with $\omega = 0$, becoming null at the null surface $S_N$, i.e., $\Omega_\ell(\ell) = 0$, to $-\Omega_H$ at the horizon with $\omega = \Omega_H$ (see equation (3.19)). When $\ell$ denotes distances from the horizon at $\ell = \ell_H$ outwardly along each field line, it is convenient firstly to express $\omega = \omega(\ell, \Psi)$ and $\Omega_\ell \approx \Omega_\ell(\ell, \Psi)$, and next to ‘coordinateize’ $\omega$ and $\Omega_\ell$, instead of $\ell$ along each field line (see Figure 1; Okamoto 2015a).

Note that ‘distant static observers’ may see that Ferraro’s law of iso-rotation holds unconditionally, and $S_\Psi$ is conserved along each field line, i.e., $\nabla \cdot \alpha S_E = \Omega_\ell \nabla \cdot \alpha S_J = 0$, and yet they may not notice the existence of two non-conserved fluxes $S_{EM}$ and $S_{SD}$, whereas ‘physical observers’ or ZAMO circulating the hole with $\omega$ will see that ‘iso-rotation is violated,’ and detect that $S_E$ consists of $S_{EM} = \Omega_\ell S_J$ and $S_{SD} = \omega S_J$, which are not conserved ($S_{EM}$ corresponds to a ‘Poynting flux inside this surface’).
The efficiency of extraction along each field line was defined by $\varepsilon = \Omega_F/\Omega_H$ (Blandford & Znajek 1977; see the $C_p$(i)-statement), and we have a condition $\varepsilon_{\text{GTD}} = \Omega_F(\Psi)/\Omega_H \leq 1$ (see equation (3.23b)).

### 3.4 The second law and its restriction on the efficiency

It will be helpful at first to reproduce general expressions (3.99) and (3.100) in Thorne et al. 1986 on thermodynamic processes taking place in the ‘stretched’ horizon $H^S$, as follows:

$$T_H \frac{dS}{dt} = \oint_{H^S} R_H \vec{j}_H^2 \, dA = \oint_{H^S} \vec{E}_H \cdot \vec{J}_H \, dA$$

$$= \frac{1}{4\pi} \oint_{H^S} \left( -\vec{E}_H \times \vec{B}_H \right) \cdot \vec{n} \, dA,$$  \hspace{1cm} (3.21a)

$$\frac{dJ}{dt} = \oint_{H^S} \left( \sigma_H \vec{E}_H + \vec{J}_H \times \vec{B}_H \right) \cdot \vec{n} \, dA.$$  \hspace{1cm} (3.21b)

The Ohm’s law holds on the stretched horizon $H^S$ (identical to the ‘force-free horizon surface’ $S_{\text{BH}}$; see section 8.1), i.e., $\vec{E}_H = R_H \vec{J}_H$ with the surface resistivity $R_H = 4\pi/c = 377\Omega m$ (equal to $R_{\text{BH}}$) and $\vec{J}_H$ is the surface current (see equation (8.7)). Ohm’s and Ampere’s laws are equivalent to the radiative condition, i.e., $\vec{B}_H = \vec{E}_H \times \vec{n}$. Thus, the inflow of the Poynting flux is equivalent to Joule heating, leading to the BH’s entropy increase, i.e., $dS_{\text{BH}}/dt > 0$ in equation (3.21a). The loss of the hole’s angular momentum due to the surface Lorentz torque in (3.21b) reduces to expression (3.9a) for the outflow of angular momentum.

Inserting $S_{\text{EM}}$ in the last of expressions (3.16a) into equation...
It seems thus that the total available energy resource \( \Omega_F \) and the inner domain \( \text{this surface} \) \( \Omega_{NS} \), for the hole’s ‘force-free’ magnetosphere to be viable, the ‘force-free’ condition must break down somewhere above the horizon, because the ‘adiabatic’ extraction with the ‘perfect’ efficiency \( \epsilon = 1 \) is unattainable (Blandford & Znajek 1977). One of the critical points of this paper is to clarify ‘why, how, and where’ the freezing-in, as well as force-free conditions, must and indeed can be broken down within the ‘force-free’ magnetosphere (cf. Blandford & Znajek 1977; see sections 3, 4 and 7).

The ‘maximum extractable energy’ becomes \( c^2 |dM| = \Omega_{dS} |dJ| \) in the adiabatic case with \( dS = 0 \), and the ‘actual energy extracted’ is given by \( c^2 |dM| = \Omega_F |dJ| \), and hence the ratio reduces to \( \epsilon_{\text{GTED}} \) in equation (3.12) (Blandford & Znajek 1977; see the \( C_p(i,ii) \)-statements in section 4.1).

It is thus the first law of thermodynamics that allows the definition of \( \epsilon_{\text{GTED}} \), while it is the second law that imposes the following restrictions on \( \Omega_F, \Omega_{NS}, \epsilon_{\text{GTED}}, \epsilon_{\text{GTED}}, S_E \) and the power \( \mathcal{P}_E \) (see Eq. (4.6) in Blandford & Znajek 1977):

\[
0 \leq \Omega_F, \Omega_{NS} \leq \Omega_{dS}, \quad 0 \leq \epsilon_{\text{GTED}} = C_{\text{GTED}} \leq 1, \quad S_E = \Omega_F S_I \leq \Omega_{NS} S_I, \quad \mathcal{P}_E = \Omega_{NS} \mathcal{P}_J \leq \Omega_F \mathcal{P}_J \tag{3.23a-d}
\]

for any electromagnetic process of energy extraction. Thus energy extraction is realizable only through an influx of negative angular momentum (or outflow of a positive one, i.e., \( S_I > 0 \) and \( dJ < 0 \)) and ingoing Poynting flux (i.e., \( S_{\text{EM}} < 0 \) and \( T_{\text{HD}} dS > 0 \)).

When the second law \( dS \geq 0 \) is satisfied, such a surface always exists at \( \Omega_{\text{F,0}} = 0 \) above the horizon where \( E_p = S_{\text{EM}} = 0 \), thereby dividing the force-free magnetosphere into the two domains, the outer domain \( \mathcal{D}_{\text{out}} \) of prograde rotation with \( \Omega_{\text{F,0}} > 0, S_{\text{EM}} > 0 \) and the inner domain \( \mathcal{D}_{\text{in}} \) of retrograde rotation with \( \Omega_{\text{F,0}} < 0, S_{\text{EM}} < 0 \) (see the \( C_p(iii,iv) \)-statements in section 4.1). This surface has so far been referred to as the ‘null surface’ \( S_N \) (e.g., Okamoto 1992; Okamoto 2015a) (see also ‘this surface’ in Blandford & Znajek 1977; the \( C_p(iv) \)-statement, section 4.1).

It seems thus that the total available energy resource \( \Omega_{dS} |dJ| \) is split at the null surface \( S_N \) to the outer domain \( \mathcal{D}_{\text{out}} \) with \( S_{\text{EM}} > 0 \) by \( c^2 |dM| = \Omega_E |dJ| \) and to the inner domain \( \mathcal{D}_{\text{in}} \) with \( S_{\text{EM}} < 0 \) by the remaining amount \( T_{\text{H}} dS = (\Omega_{dS} - \Omega_E) |dJ| \), i.e.,

\[
\Omega_{dS} |dJ| = T_{\text{H}} dS + c^2 |dM|, \tag{3.24a}
\]
\[
\Omega_E |dJ| = (\Omega_{dS} - \Omega_E) |dJ| + \mathcal{P}_E \tag{3.24b}
\]

(also see equation (12.1)). There is a simple identity

\[
\Omega_F = (\Omega_{dS} - \Omega_E) + \mathcal{P}_E \tag{3.25}
\]

in equation (3.24). The so-called impedance matching, i.e., \( c^2 |dM| \approx T_{\text{H}} dS \) yields \( \Omega_E = \Omega_{dS} / 2 \) (cf. equation (10.7) for the eigenfunction \( \Omega_E \) due to the criticality-boundary condition).

In passing, the identities with \( \Omega_{NS} \) replaced with \( \Omega_E \) in equations (3.25):

\[
(\Omega_{E,0})_\infty - (\Omega_{E,0})_{\text{H}} = \Omega_{dS} \tag{3.26a}
\]
\[
= \Omega_{dS} + \Omega_E \tag{3.26b}
\]
\[
= -[-(\Omega_{dS} - \Omega_E)] + \Omega_E. \tag{3.26c}
\]

When the ZAMOs measure the FLAV \( \Omega_E \) along each field-line, they find that the FD effect produces the along-field gradient of ‘gravito-electric potential gradient’ \( \Omega_{E,0} \) between different places. The maximum is naturally equal to \( \Omega_{dS} \) between the horizon and infinity (equation (3.24a)). Expression (3.26b) was used to share the total EMF \( AV \) due to a horizon battery to the two drops, i.e., a horizon drop \( \Delta V_H \) and a drop \( \Delta V_L \) for the astrophysical load in a series-circuit model (see section 4.3). On the other hand, in the present dual-circuit model, the ‘along-field potential drop’ \( \Omega_{dS} \) can be seen as the difference between the two EMFs of a pair of batteries existent at the null surface \( S_N \) (see equation (8.12)). This picture is consistent with the difference in the spin rate of two hypothetical magnetic rotators existent back to back at the two surfaces of the Gap (sections 9.2, 10.1 and 11.2; Figure 4).

The ingenious trick of utilizing the hole’s resources under the control of the first law in equation (3.24) or (3.11) is as follows: the entropy term \( T_{\text{H}} dS \) requires an inflow of a Poynting flux from the null surface \( S_N \). This must be followed by an in-flow of negative angular momentum, which is equivalent to an out-flow of positive angular momentum, leading to the decrease of the hole’s, i.e., \( dJ < 0 \). This couples with the FD effect to induce an in-flow of negative energy, which is equivalent to an outward spin-down energy \( \Omega_{dS} |dJ| \), with a part covering the cost of extraction, i.e., the entropy increase \( T_{\text{H}} dS = (\Omega_{dS} - \Omega_E) |dJ| \), and with the rest becoming the outgoing Poynting flux \( c^2 |dM| = \Omega_E |dJ| \) (see section 10.1).

4 The nature of the Blandford-Znajek process

To unlock the nature of the BZ process, we begin at first to classify important statements characterizing its nature into two classes: the first class \( C_p \) from the physical observers’ viewpoint contains positive statements referring to the role of the two energy fluxes \( S_{\text{EM}} \) and \( S_{\text{SD}} \) (see equations (3.14) and (3.17)), and hence implicitly complying with the first and second laws,
4.1 Statements from the ‘physical-observers’ viewpoint

- \( C_p(i) \): In Eqs. (4.8) and (4.10) in Blandford & Znajek 1977, “We can define the efficiency of the energy extraction process to be \( \epsilon = \text{Actual energy extracted}/\text{Maximum extractable energy, when unit angular momentum is removed} \) and then \( \epsilon = \Omega_{\text{E}}/\Omega_{\text{H}} \).

- \( C_p(ii) \): “Inequality (4.7) (i.e., \( S_E \leq \Omega_{\text{H}}S_1 \)) could have been derived using the classical limit of the Second Law of Black Hole Thermodynamics.”

- \( C_p(iii) \): “A physical observer rotating at constant radius close to the horizon will in general see a Poynting flux of energy entering the hole (\( S_{\text{EM}} < 0 \)), but he will also see a sufficiently strong flux of angular momentum leaving the hole to ensure that \( \dot{\mathcal{E}} \geq 0 \) (i.e., \( S_1 \geq 0 \), \( S_E \geq 0 \)).

- \( C_p(iv) \): “Physical observers traveling round the hole at constant \( r \) and \( \theta \) and angular velocity \( d\phi/dt \) will see the electric field reverse direction on the surface \( d\phi/dt = \Omega_{\text{H}} \) (i.e., \( E_{\infty} = \Omega_{\text{E}\infty} = 0 \)). Inside this surface they see a Poynting flux of energy toward the hole. (For a system of observers with time-like world lines \( d\phi/dt = \Omega_{\text{H}} \) on the event horizon and \( d\phi/dt \to 0 \) at infinity. Hence when \( 0 < \Omega_{\text{E}} < \Omega_{\text{H}} \) i.e., when the hole is losing energy electromagnetically, this surface always exists.)” in the caption of Fig. 2 in Blandford & Znajek 1977. (The converse is also true; “the hole cannot lose energy electromagnetically, unless this surface always exists.”)

- \( C_p(v) \): In the footnote at p.443 in Blandford & Znajek 1977; “The outer light surface corresponds to the conventional pulsar light surface and physical particles must travel radially outwards beyond it. Within the inner light surface, whose existence can be attributed to the dragging of inertial frames and gravitational redshift, particles must travel radially inwards.” It was thereby concluded that “the spark gaps discussed in Section 2 must therefore lie between these two surfaces;” but nevertheless “there is no reason to believe that its position is stationary” was written in Sec. 2 of that paper.

- \( C_p(vi) \): “The fundamental differential equation for the potential \( \Lambda \equiv \Psi/2\pi \) is given by Eq. (3.14) in Blandford & Znajek (1977), which has been reproduced for \( \Psi \) in the 3+1 formulation from the ‘stream equation’ Eq. (6.4) in Macdonald & Thorne (1982), i.e.,

\[
\nabla \cdot \left( \frac{\sigma}{\alpha} \left[ 1 - \frac{\Omega_{\text{E}}^2 \sigma^2}{\alpha^2 c^2} \right] \Psi \right) + \frac{\Omega_{\text{E}\infty}}{\alpha c^2} \frac{d\Omega_{\text{E}}}{d\Psi} \left( \frac{\Psi}{\alpha^2 c^2} \right) \left( \frac{d\Psi}{d\Psi} \right) = 0.
\]

Our comments to the above \( C_p \)-statements are as follows:

\( P(i) \): By equation (3.11a), the ‘Actual energy extracted’ is \( c^2 \delta M = \Omega_{\text{E}} dJ \) and the ‘Maximum extractable energy’ is obtainable in the adiabatic process \( T_0 S_0 = 0 \), i.e., from equation (3.4a) \( c^2 \delta M = \Omega_{\text{E}} dJ \), and hence the ratio gives the overall efficiency \( \Omega_{\text{GTED}} = \Omega_{\text{E}}/\Omega_{\text{H}} \). It is the radiation condition for inflow of a Poynting flux into the hole in equation (3.19), or \( (\Omega_{\text{E}\infty})_H < 0 \) that yield the efficiency \( \Omega_{\text{GTED}} = \Omega_{\text{E}}(\Psi)/\Omega_{\text{H}} \) along each field line.

\( P(ii) \): Not only \( S_E \leq \Omega_{\text{H}}S_1 \) (see Eq. (4.7) in Blandford & Znajek 1977 and equation (3.23c)), but also other inequalities in (3.23a,b) are obtainable from the second law of black hole thermodynamics, as conjectured in the \( C_p(i) \)-statement.

\( P(iii) \): Accompanied by an ingoing Poynting flux, there must be an inflow of negative angular momentum. A physical observer will interpret this as equivalent to an outflow of ‘a sufficiently strong flux of positive angular momentum leaving the hole,’ i.e., \( S_1 > 0 \), and he will also see an outflow of the frame-dragging spin-down energy flux \( S_{\text{SD}} = \omega S_1 \), which ensures the ‘total’ flux \( S_E = \Omega_{\text{E}}S_1 \) keeping positive as seen in equation (3.17).

\( P(iv) \): When the second law ensures inequalities (3.23), ‘this surface’ \( S_N \) where \( \Omega_{\text{E}\infty} \) and \( E_{\infty} \) reverse always exist. Therefore, ‘this surface’ unequivocally divides the force-free magnetosphere into the two domains; the outer Semi-Classical (SC) domain \( D_{\text{out}} \) of prograde-rotation where \( \Omega_{\text{E}\infty} > 0 \), \( S_{\text{EM}} > 0 \), and the inner General-Relativistic (GR) domain of retrograde-rotation \( D_{\text{in}} \) where \( \Omega_{\text{E}\infty} < 0 \), \( S_{\text{EM}} < 0 \). Also, “outside this surface he will see a Poynting flux of energy going outward to infinity,” reversely to inside (see equations (3.16a) and (3.19)). It is at ‘this surface’ where the breakdown of not only the force-free condition but also the freezing-in condition takes place, thereby leading to the severance of both of stream- and current-lines, when field lines are continuous across this surface (see section 7; cf. the \( C_S(i,ii) \)-statement later).

\( P(v) \): When a physical observer (ZAMO) measures the field-line rotational-velocity (FLRV), it is given by \( v_F = \Omega_{\text{E}\infty} \sigma/\alpha \) and the two light surfaces \( S_{\infty} \) and \( S_\infty \) are defined by \( v_F = \pm c \) (see section 6.7). Obviously, ‘this surface’ \( S_N \) where \( E_{\infty} = v_F = \Omega_{\text{E}\infty} \) always exists between \( S_{\infty} \) and \( S_\infty \) (see the \( C_p(iv) \)-statement above). The ZAMOs will see \( S_{\text{EM}} > 0 \) in the outer SC domain with \( v_F > 0 \), and \( S_{\text{EM}} < 0 \) in the inner GR domain with \( v_F < 0 \), and hence “the spark gaps must therefore lie between these two light surfaces” (Znajek 1977), and yet “there is a good reason to believe that its position is stationary at this surface \( v_F = \Omega_{\text{E}\infty} = 0 \).” The existence of ‘this surface’ as well as \( S_{\infty} \) can surely be attributed to the FD effect.

\( P(vi) \): The so-called ‘stream equation’ (4.1) already contains

\( ^2 \) This section is written based on the natural supposition that expert readers are familiar with the original text of Blandford & Znajek 1977.
the three fundamental surfaces, important in GTEQ; the two, outer and inner, light surfaces given by given by \( \Omega_{f,0} \sigma / \alpha c = \pm 1 \) and this surface given by \( \Omega_{f,0} = 0 \) (see also section 6.7). Also, the stream equation (4.1) involves the two unknown functions of \( \Psi, I(\Psi) \) and \( \Omega_{f}(\Psi) \), in a nonlinear way, and yet these two cannot be determined within the force-free domains, but only by terminating the force-free domains in the ‘resistive’ membranes for \( I(\Psi) \) and breaking down the force-free condition in the ‘inductive’ membrane at \( S_N \) for \( \Omega_{f}(\Psi) \), and then solving the criticality-boundary value problem in the steady-state. There is no source nor sink of energy and angular momentum within the force-free domains. Though the force-free field structure is seemingly seamless, the null surface \( S_N \) will actually be widened to a gap accommodating the particle-current sources (see sections 8–10).

Komissarov (2009) also found the two energy fluxes \( \mathbf{S} \) and \( \mathbf{S}^\prime \), corresponding to \( S_E \) and \( S_{EM} \), respectively, and stated that one might understand the direction of \( \mathbf{S} \) as the direction of the electromagnetic wind and that of \( \mathbf{S}^\prime \) as the direction of energy flow, and pointed out that these two vectors are parallel if \( \Omega_{f} > \omega \) and anti-parallel if \( \Omega_{f} < \omega \). The latter counter-flow of energy \( \mathbf{S}^\prime \) will correspond to a Poynting flux (see the \( C_9(\text{iv}) \)-statement). It is in the inner domain \( D_{(\text{in})} \) of counter or retrograde rotation that physical sources will see the counter flow.

4.2 Statements from the ‘static-observers’ viewpoint

- \( C_{9}(\text{i}) \): By Eqs. (4.3) and (4.4) in Blandford & Znajek (1977) (see equation (3.8)), “the direction of energy flow cannot reverse on any given field line unless the force-free condition breaks down. Therefore, the natural radiation condition at infinity requires energy to flow outwards on all the field lines”.
- \( C_{9}(\text{ii}) \): “energy and angular momentum from a rotating hole can indeed be extracted by a mechanism directly analogous to that of Goldreich & Julian (1969)”.
- \( C_{9}(\text{iii}) \): “the massive black hole behaves like a battery with an EMF of up to \( 10^{21} \) V and an internal resistance of about \( 30 \Omega \). When a current flows, the power dissipated within the horizon, manifest as an increase in the reducible mass, is comparable with that dissipated in particle acceleration etc. in the far field” (Blandford 1979; Znajek 1978).
- \( C_{9}(\text{iv}) \): “there must be some source of particles within the near magnetosphere. The currents that pervade the magnetosphere as the sources of the magnetic field are presumably carried by charged particles that are flowing outward a large distances. (... positive outflow at large radii seems unavoidable.) We also know that the particle flux must be directed inwards through the event horizon, and so it cannot be conserved.” Then a vacuum gap will presumably appear between the flows of particles of opposite charges in between the two light surfaces. And thus, “Provided that the potential difference necessary to produce breakdown is much less than the total across the open field lines, an electromagnetic force-free solution should provide a reasonable approximation to time-averaged structure of such a magnetosphere.”
- \( C_{9}(\text{v}) \): The \( C_9(\text{iv}) \)-statement was paraphrased in Macdonald & Thorne 1982 as follows: “... magnetic field lines that thread the hole must get their charges and currents in some other manner. Blandford & Znajek (1977) argue that they come from the Ruderman-Sutherland (1975) ‘spark-gap’ process, a cascade production of electron-positron pairs in the force-free region — a production induced indirectly by a component of \( E \) along \( B \), which again is so weak as to constitute a negligible violation of force-freeness and degeneracy.”

Our comments to the above \( C_9 \)-statements are as follows:

\( S(\text{ii}) \): (a) The \( C_9(\text{ii}) \)-statements are incompatible with the \( C_9(\text{iv}) \)-statement, because this surface which should always exist when the hole is losing energy electromagnetically, will not appear in a classical mechanism directly analogous to Goldreich & Julian (1969). The FD effect \( \omega \), as well as this surface, has lost a crucial role to play in the BZ process and the Membrane Paradigm (see section 4.3). In fact, the force-free condition breaks down at this surface, and the direction of the Poynting flux \( S_{EM} \) and the ‘magneto-centrifugal force’ as well will reverse on any field line at \( S_N \), where \( \Omega_{f,0} \equiv 0 \), \( S_{EM} \equiv 0 \) and \( \mathbf{v} \equiv 0 \) (see section 7).

(b) It is certain that the direction of energy flow \( S_E = \Omega_{f} S_J \) does not reverse on any given field line (except but \( S_E \equiv S_J = 0 \) at this surface), but there must exist the two fluxes \( S_{EM} + S_{SD} \) in-between \( S_E \) and \( \Omega_{f} S_J \), which correspond in the first law to \( T_H dS + \Omega_{f} dJ \) in-between \( c^2 dM \) and \( \Omega_{f} dJ \). When \( E_p \) reverses at this surface, \( S_{EM} \) as well does. It is indeed at this surface between the two light surfaces \( S_{stat} \) and \( S_{magn} \) that the freezing-in condition as well as the force-free condition break down on every open field line. It is in the outer SC domain \( D_{(\text{out})} \) rotating prograde \( (\Omega_{f,0} > 0) \) that a pulsar-type wind flows passing through \( S_{stat} \) outward, and it is in the inner GR domain rotating retrograde \( (\Omega_{f,0} < 0) \) that an anti-pulsar-type wind flows passing through \( S_{stat} \) inward (Okamoto 1992). The point is that influxes of negative angular momentum and energy originating in ‘this surface’ are apparently and essentially equivalent to the outflow of positive ones from a rotating hole.

\( S(\text{iii}) \): If ‘this surface’ were not needed as argued in the \( C_9(\text{ii}) \)-statements, there would be no distinct surface suitable for the place of a battery in the force-free magnetosphere, probably except the horizon. Then a single DC circuit consisting of a battery with an internal resistance in the horizon and an astrophysical load as external resistance at infinity would be a natural outcome. This single-pulsar model of a ‘series circuit’ was tried to justify by the Membrane Paradigm (see section 4.3).
... coupled with unipolar induction and the first and second laws this surface Δ this voltage drop in the horizon was shared for the horizon electric potential gradient and external resistances for the horizon's Joule dissipation and astrophysical loads (see Chapter IV D, Thorne et al. 1986).

In order to verify the C\textsubscript{S}(iv)-statement, with the C\textsubscript{S} process, with the C\textsubscript{S} load resistance is equal to "battery resistance," corresponding to the horizon battery into the load seems to be maximized if the total power carried poloidally outward as Poynting flux from the horizon outer-gap model, and this seems to have brought about some fatal ambiguities: The prime-motive forces leading to the vacuum in-between the two light surfaces as an arena of particle production must drive oppositely-directed flows. The idea suggested appears to be due to the centrifugal force acting outwards and gravitational pull acting inwards, and yet these forces of different origin will have to exist back-to-back in between the two light surfaces, for the spark gaps to take place under a negligible violation (e.g., |E\textsubscript{l}| ≪ |E\textsubscript{⊥}|). However, it is not confident that how much immense 'a negligible violation' of the force-free condition will have to be, to allow an amount of particle production needed to maintain the whole magnetospheric currents.

(b) The C\textsubscript{S}(i,ii)-statements will in fact be equivalent to neglecting the existence of 'this surface' ω = Ω\textsubscript{F} in the C\textsubscript{P}(iii)-statement, a natural consequence resulting from the FD effect coupled with unipolar induction and the first and second laws of thermodynamics (see equations (3.11) and (3.23)). In reality, the vitally important roles of 'this surface' where a complete violation takes place were not discussed in Blandford & Znajek (1977), nor Macdonald & Thorne (1982); Thorne et al. (1986).

Although the C\textsubscript{P}-statements take care of the critical roles of the FD effect ω and the two energy fluxes S\textsubscript{EM} and S\textsubscript{SD}, it seems to be the C\textsubscript{S}-statements that forms the framework of the BZ process, with the C\textsubscript{P}-statements taken supplementally into account except the C\textsubscript{P}(iv)-statement. Actually, 'this surface' does not seem to be found in the BZ process nor in any other later references on that.

4.3 Structure and energetics: the Blandford-Znajec process

In order to verify the C\textsubscript{S}(iii,iii)-statements, let us consider a 'series circuit' C with a battery in the horizon and two internal and external resistances for the horizon’s Joule dissipation and astrophysical loads (see Chapter IV D, Thorne et al. 1986). There is a difference of the ZAMO-measured FLAV due to the gravito-electric potential gradient ω between the horizon and infinity, i.e., (Ω\textsubscript{ZAMO})\textsubscript{0} − (Ω\textsubscript{FLAV})\textsubscript{0} = Ω\textsubscript{H} (see equation (3.26)). This may give rise to the total EMF as given by Eq (4.39) in Thorne et al. 1986

\[ ΔV = \oint_C α\vec{E} \cdot d\vec{l} = \frac{1}{2\pi}Ω_H ΔΨ. \]  

(4.2)

If “this voltage drop could really be thought of as produced by a ‘battery’ with Ω\textsubscript{F} = Ω\textsubscript{H} in the hole’s horizon” (Thorne et al. 1986), the adiabatic extraction might not be unattainable, just as in the pulsar force-free magnetosphere with Ω\textsubscript{F} = Ω\textsubscript{SS}. And yet, this voltage drop in the horizon was shared for the horizon drop ΔV\textsubscript{H} and a drop ΔV\textsubscript{L} in the astrophysical load for a strip with ΔΨ between the two magnetic surfaces (see Fig. 38\textsuperscript{3} in Thorne et al. 1986) as follows;

\[ ΔV = ΔV_H + ΔV_L, \]  

(4.3a)

\[ ΔV_H = \frac{1}{4π}Ω_H (Ω_H - Ω_F)ΔΨ, \quad ΔV_L = \frac{1}{4π}Ω_F ΔΨ \]  

(4.3b)

(see Eqs (4.42), (4.41b) and (4.44)). If these were possible, the massive black hole might behave like a battery (see the C\textsubscript{S}(iv)-statement).

Respective powers from ΔV\textsubscript{H} and ΔV\textsubscript{L} are dissipated in ohmic heating in the horizon and deposited in the astrophysical loads;

\[ T_H \frac{dΔS_H}{dt} = \frac{(Ω_H - Ω_F)^2}{4π}σ^2 B_n ΔΨ, \]  

(4.4a)

\[ ΔP_L = -c^2 \frac{dΔM}{dt} = \frac{Ω_F (Ω_H - Ω_F)}{4π}σ^2 B_n ΔΨ \]  

(4.4b)

(see Eqs (4.47) and (4.48) in Thorne et al. 1986 and also equations (12.3) and (12.4) later). The summation of the two dissipation terms is equal to

\[ T_H \frac{dΔS_H}{dt} - c^2 \frac{dΔM}{dt} = -Ω_H \frac{dΔJ}{dt}, \]  

(4.5)

where

\[ \frac{dΔJ}{dt} = -\frac{Ω_H - Ω_F}{4π}σ^2 B_n ΔΨ. \]  

(4.6)

Relation (4.5) reduces to c\textsuperscript{2}dΔM = T\textsubscript{H} dΔS + Ω\textsubscript{H} dΔJ, which is nothing but the first law of thermodynamics for the strip. Then as relation Ω\textsubscript{H}dΔJ = T\textsubscript{H} dΔS + c\textsuperscript{2}dΔM means that “The powers in (4.47), (4.48) are supplied, of course, by the hole’s rotation; as they are dissipated in the horizon and the load, the hole spins down” (see equations (3.24), (12.1), (12.3) and (12.4)). Then, the total power carried poloidally outward as Poynting flux from the horizon battery into the load seems to be maximized if the load resistance is equal to “battery resistance,” corresponding to Ω\textsubscript{F} \sim 1/2Ω\textsubscript{H}, i.e.,

\[ ΔP_L = \frac{Ω_H^2}{16π}σ^2 B_n ΔΨ. \]  

(4.7)

(see Eq (4.49)). These results would be applicable for the hole’s force-free magnetosphere, only if a battery with the internal resistance were really existent in the hole’s horizon, and a negligible violation of the force-free condition might allow a sufficient amount of particle production, and yet only if current lines would not be severed at this surface due to the ‘incomplete’ violation of the force-free condition (because the ‘complete’ violation leads to \( J = 0 \) at S\textsubscript{N}) (see section 7).

There are several severe misgivings in this ‘series circuit’ model (the single-pulsar model as opposed to the twin-pulsar model; see section 11.3);

(a) The notion of the hole’s battery in the horizon was criticized as against the principle of causality already three decades

\footnote{We can confirm the vector \( \vec{E} \)’s change of direction on the strip showing existence of ‘this surface’ S\textsubscript{N} = S\textsubscript{ZAMO} in the C\textsubscript{P}(iv)-statement. Also see Figure 2 and its caption in Blandford & Znajek (1977) and Fig. 3 in Phinney (1983b).}
ago (Punsly & Coroniti 1989; Punsly & Coroniti 1990; Punsly 2008). Besides the issue of causality violation, there are some fundamental difficulties and inconsistencies in this single-circuit theory.

(b) A battery and its internal resistance in the horizon in the \( C_S(i,ii) \)-statement seem to be the natural outcome from the \( C_S(i,iii) \)-statements, e.g., no breakdown of the force-free condition anywhere in the force-free magnetosphere and hence no place of matter-dominance accommodating a unipolar induction battery. On the other hand, the \( C_P(iv) \)-statement will indicate existence of \textit{this surface} \( S_N \) where the electric field \( E_p \) and the Poynting flux \( S_{EM}^{(in)} \) reverse, giving rise to the breakdown of the force-free and freezing-in conditions and hence the severance of current- and stream-lines at \( S_N \) (see equations (7.1b,c)). This means that there cannot be such a continuous DC ‘series circuit’ as \( C \) used in equation (4.2) allowing connection of the horizon battery with the astrophysical load across \( S_N \).

(c) Both of the outgoing and ingoing Poynting fluxes \( S_{EM,(out)} \) and \( S_{EM}^{(in)} \) will be launched from the null surface \( S_N \) (or the inductive membrane \( S_N \) (see section 8.2)), to be dissipated in the two resistive membranes as the entropy increase \( T_H dS/dt > 0 \) in \( S_{BH} \) and particle acceleration \( \nabla^2 dM/dt = \Omega_S dJ/dt < 0 \) in \( S_{Ego} \). Thus, ohmic heating \( T_H d\Delta S_H/dt \) in equation (4.4a) will not be due to the internal resistance of a battery absent in the horizon, but due to the influx of the Poynting flux (see the \( C_P(iv) \)-statement, equation (3.21a) and Eq (3.99) in Thorne et al. 1986).

It seems that the above \( C_S(i,ii,iii,iv) \)-statements in Blandford & Znajek 1977 have led to such explicit expressions as a ‘battery’ and ‘battery resistance’ in the hole’s horizon in Thorne et al. 1986 (p.142–144). Komissarov (2009) stated that the Blandford-Znajek mechanism did not clash with causality, and if so, the “membrane paradigm” would neither clash with causality. The ‘membrane’ itself is a concept of physical substance, and never \textit{artificial} in GETD as well as pulsar electrodynamics (see section 8).

5 Toward unifying electrodynamics into thermodynamics

The foundation for unifying electrodynamics with gravito-thermodynamics into black-hole ‘gravito-thermo-electrodynamics’ was laid by Macdonald & Thorne (1982) and Thorne et al. (1986), yet nearly four decades ago. The dragging of inertial frames and gravitational redshift, the most fundamental general-relativistic properties of the Kerr BH, are wholly incorporated into an ‘absolute-space/universal-time formulation’ from the ZAMOs’ point of view, including how to couple the FD effect with unipolar induction. If the formulation were correctly applied to the BZ process together with the \( C_P(iv) \)-statement (instead of the \( C_S(i,ii) \)-statements), this should have enabled people to modify the BZ process relevantly with ‘\textit{this surface}’ and hence the first and second laws taken into account. Unfortunately, however, the \( 3+1 \) formulation by Macdonald & Thorne (1982) was not used to correct the \( C_S \)-statements in the light of the \( C_P \)-statements, consequently losing a chance of modifying the BZ process toward the right track of progress. The \( C_S(i,iii,iv,v) \)-statements seem to have stemmed, in fact, from skipping the two non-conserved energy fluxes \( S_{EM} \) and \( S_{SD} \) (see equation (3.14a,b)). It was somewhat conversely used to reinforce the \( C_S(iii) \)-statement (see, e.g., section 7.3 in Macdonald & Thorne 1982 and chapter IV D in Thorne et al. 1986).

The significant premises and propositions of this paper in modifying the BZ process are as follows:

(a) Kerr holes are strictly governed by the no-hair theorem and the first three laws of thermodynamics. We presume that every Kerr hole is incapable of being ‘magnetized’ in the steady-state, and hence “the massive black hole will not behave like a battery with an emf...” (cf. \( C_S(iii,iv) \)-statements). Also, a Kerr hole may be an acceptor of a Poynting flux of external origin at \textit{this surface} \( S_N \), but can never be an emitter of that of internal origin (see Punsly & Coroniti 1990; statements \( C_P(iii,iv) \)). The first and second laws indicate that the hole can lose energy electromagnetically only through an influx of \textit{negative} angular momentum (see the \( C_P(iv) \)-statement).

(b) The Kerr hole is not magnetized, unlike NSs, nor allows field lines to be pinned down at the horizon. The hole’s magnetosphere is connected with the hole’s body itself only through coupling the FD-AV \( \omega \) and the FL-AV \( \Omega_F \). Moreover, we do not presume that interstellar general magnetic fields are involved in the extraction process (cf. III D1, Thorne et al. 1986). By the no-hair theorem, no conservation of magnetic fluxes at the birth of a hole will allow the existence of magnetic fluxes of internal origin, emanating to the outside beyond the horizon, nor threading of field lines of external origin with \( \Omega_E = \text{constant} \), because the non-locality of Ferraro’s law of iso-rotation will not extend inside the hole beyond the surface of causal disconnection. We presume that no ‘spooky action’ will be at work at a distance across \( S_H \), except for quantum entanglement.

(c) We firstly assume the existence of the poloidal magnetic field \( B_p \) as constituting the backbone of a hole’s magnetosphere, which extends from the vicinity of the horizon surface \( S_H \) to the infinity surface \( S_{\infty} \). Secondly, we assume the existence of perfectly conductive plasma around a Kerr hole, which is permeated by the poloidal magnetic field \( B_p \), with the FLAV \( \Omega_F (\Psi) \). Ferraro’s law of iso-rotation holds throughout the stationary, axisymmetric magnetosphere, i.e., \( (B_p \cdot \nabla)\Omega_F = 0 \) (see section 6.5), where \( 0 < \Omega_E < \Omega_H \) is assumed by the second law of thermodynamics (see section 3.4). There will then be such a surface moderately above the horizon, where the FD-AV equals the FLAV, i.e., \( \omega = \Omega_E (\equiv \omega_N) \). This surface is referred to as the null surface \( S_N \) (Okamoto 1992), and equal to ‘\textit{this surface}’ in
6 The 3+1 formulation for a modified BZ process

6.1 Fundamental equations and conditions

The absolute space around a Kerr BH with mass $M$ and angular momentum per unit mass $a = J/Mc$ is described in Boyer-Lindquist coordinates:

$$ds^2 = \left(\frac{\rho^2}{\Delta}\right)dr^2 + \rho^2d\theta^2 + \rho^2\sin^2\theta \, d\phi^2; \quad (6.1a)$$

$$\rho^2 \equiv r^2 + a^2\cos^2\theta, \quad \Delta \equiv r^2 - 2GMr/c^2 + a^2; \quad (6.1b)$$

$$\Sigma^2 \equiv (r^2 + a^2)^2 - a^2\Delta \sin^2\theta, \quad \sigma = (\Sigma/\rho)\sin\theta \quad (6.1c)$$

$$\alpha = \rho\Delta^{1/2}/\Sigma, \quad \omega = 2aGMc/\sqrt{\Sigma^2} \quad (6.1d)$$

(see Macdonald & Thorne 1982; Okamoto 1992), where $\alpha$ is the lapse function/redshift factor and $\omega$ is the FADV. The two parameters $\alpha$ and $\omega$ are reminiscent of the no-hair theorem in specifying the Kerr spacetime. They are given as unique functions of $\sigma$ and $z$ in the Boyer-Lindquist coordinates, and $0 \leq \alpha \leq 1$ and $\Omega_\text{H} \geq \omega \geq 0$. Note that for $\alpha \to 0$, $\omega \to \Omega_\text{H} =$ constant on $S_\text{H}$ by the zeroth law of thermodynamics.

When we introduce curvilinear orthogonal coordinates $(\ell, \Psi)$ in the poloidal plane, where $\ell$ stands for the distances measured along each field line $\Psi =$constant, then we express, e.g., $\omega = \omega(\ell, \Psi)$. Just as $\alpha$ was ‘coordinated’ in the stretched horizon (Macdonald & Thorne 1982; Thorne et al. 1986), we ‘coordinatize’ $\omega$ along field lines in the whole magnetosphere (Okamoto 2015a). The ZAMO-measured FLAV $\Omega_{\text{fr}}$ as well is ‘coordinated’ (see, e.g., Figure 1).

From a somewhat pedagogical point of view, we revisit basic expressions for the poloidal and toroidal components of $B$, $E$, the charge density $\varrho_\text{e}$, the particle velocity $v$ and the field line rotational velocity (FLRV) $v_F$ in the steady axisymmetric state (Macdonald & Thorne 1982; Thorne & Macdonald 1982; Thorne et al. 1986; Okamoto 1992; Okamoto 2015a).

For the electric field $E$ in curved spacetime we use Eq. (2.24a) or (4.7) in Macdonald & Thorne 1982

$$E = \frac{1}{\alpha} \left( \nabla A_0 + \frac{\omega}{c} \nabla A_\phi \right), \quad (6.2)$$

where $A_0$ is a scalar potential and $A = (0, 0, A_\phi)$ is a vector potential, and $A_\phi = \Psi/2\pi$. This is the kick-off equation to make the FD effect couple with unipolar induction, by utilizing both the freezing-in and force-free conditions in the ‘force-free’ magnetosphere;

$$E + v/c \times B = 0, \quad (6.3a)$$

$$\varrho_\text{e}E + j/c \times B = 0, \quad (6.3b)$$

$$E \cdot B = j \cdot E = v \cdot E = 0. \quad (6.3c)$$

When the second condition (6.3b) regards inertial forces as negligible compared with the Lorenz force, the first condition (6.3a) implies that ‘force-free’ magnetic field lines are frozen in particles and yet dragged around by the motion $v$ of ‘massless’ particles. The combination of two opposite conditions, i.e. force-freeness and freezing-in, then creates a kind of extreme physical state (Okamoto 2006), with the fields degenerate (Macdonald & Thorne 1982), where current-field-streamlines are equipotentials (see equation (6.22) later). Condition (6.3c) means that no particle acceleration takes place in the force-free domains.

The ‘force-free magnetosphere’ under the above conditions (6.3) possesses the fundamental conserved quantities $\Omega_\text{e}(\Psi)$ and $I(\Psi)$, but cannot be viable unless the two conserved quantities are determined by the criticality-boundary condition formulated by the breakdown of the above conditions (see section 7). The flows of angular-momentum and energy, particles, and currents are described by flux, wind, and circuit theories, respectively, not mentioning that these theories must be consistent with each other.

6.2 The electric current

We decompose the magnetic field $B = \nabla \times A$ as

$$B_\gamma = -(t \times \nabla \Psi)/2\pi \varpi, \quad (6.4a)$$

$$B_\ell = -(2I/\varpi c)t, \quad (6.4b)$$

where the ‘current function’ is denoted with $I = I(\ell, \Psi) = I(\Omega_{\text{fr}}, \Psi)$ in general. From Eq. (2.17c) in Macdonald & Thorne 1982 for $j$ we have

$$j = \frac{c}{4\pi \alpha} \left\{ \nabla \times A B + \frac{1}{c} (E \cdot \nabla \omega) m \right\}, \quad (6.5)$$

where $m = \varpi t$ is a Killing vector, and then for $j_p$

$$j_p = \frac{t \times \nabla I}{2\pi \varpi \alpha}, \quad (6.6)$$

Introducing the two orthogonal unit vectors $p$ and $n$ in the poloidal plane, i.e., $p = B_p/|B_p|$ and $n = -\nabla \Psi/|\nabla \Psi|$, and $n \times p = t$, we have for the current function in general, i.e., $I = I(\ell, \Psi)$,

$$\nabla I = \frac{\partial I}{\partial \ell} p - 2\pi \varpi B_p \frac{\partial I}{\partial \psi} n \quad (6.7)$$

and hence, we express the electric current $j_p$ as follows;

$$j_p = j_\parallel p + j_\perp n, \quad (6.8)$$

(see Okamoto 1999). Also, for $j_\parallel$ we have from equation (6.5)
The velocity $v$ of 'massless' particles

Combining the two conditions (6.3a,b), we have

$$v = j / \varrho_e$$

(6.10)

(Blandford & Znajek 1977). When we denote the number densities of electrons and positrons by $n^{(e)}$ and $n^{(p)}$, the charge density is given by

$$\varrho_e = e(n^{(e)} - n^{(p)})$$

Equation (6.10) implies that the 'force-free' plasma must be charge-separated, i.e., $\varrho_e = -en^{(e)} + en^{(p)}$, and that the role of 'massless' or 'inertia-free' particles is just to carry charges, exerting no dynamical effect.

The 'force-free' domains of no particle acceleration must be terminated by restoration of particle inertia for particles to accelerate, thereby determining the eigenfunction $I(\Psi)$. This requires a change of the volume currents, parallel to the poloidal field $B_p (j_\perp = 0)$ in the force-free domain, into the surface currents, perpendicular to $B_p (j_\parallel \equiv 0)$ on the terminating surfaces of the outer and inner force-free domains $S_{ff0}$ and $S_{ff}$ (see sections 8.1.1, 8.1.2). Moreover, the breakdown of the freezing-in and force-free conditions at 'this surface' $S_N$ imposes $v = j = 0$, because $v > 0$ far outside and $v < 0$ near the horizon and $j$ does not change direction but must vanish, and also $\varrho_e$ must certainly change its sign at the place of the breakdown (see section 7). This implies that the breakdown on 'this surface' $S_N$ must locate the sources of particles and currents there (cf. the C-statement in section 4.2).

The field angular momentum flux $I(\Psi)$

An inner product of equation (6.3b) with $m = \omega t$ yields, with the use of equation (6.6),

$$0 = m \cdot \varrho_e E_p + \frac{\varrho_e}{c} \times B = \frac{\varrho_e}{c} (j_p \times B_p) = \frac{\varrho_e}{c} j_\perp B_p$$

$$= \frac{(B_p \cdot \nabla) I}{2\pi c \alpha} = -\frac{1}{\alpha} \nabla \cdot \frac{(B_p \cdot \nabla) I}{2\pi c} = -\frac{1}{\alpha} \nabla \cdot a S_j$$

(6.11)

where $S_j$ is given by the second of equations (3.8). It turns out that the field angular momentum $-\alpha \varrho_e B = (2/c) I(\Psi)$ is conserved along each field line. From equations (6.4a) and (6.8), we have $j_\perp = 0$ and then

$$j_\parallel = \frac{\sigma c}{8\pi^2 \alpha} \left[ -\nabla \cdot \left( \frac{\alpha \nabla \Psi}{c^2} \right) + \frac{2\pi}{c} E \cdot \nabla \omega \right]$$

(6.9)

When $(B_p \cdot \nabla) I = B_p(\partial X / \partial \ell) = 0$ for an arbitrary function $X$, we have $X = X(\Psi)$, and then it is said that $X$ is conserved along each field line. For example, $I = I(\Psi)$ in the 'force-free' domains (see equation (6.11)). We presume that each current line given by $I(\ell, \Psi) = \text{constant$\,$must$\,$close}$ in circuit theory, starting from one terminal of a unipolar induction battery, to return to the other terminal in the steady-state, after supplying power to the acceleration zone with $j_\perp > 0$ (the current-closure condition).

6.5 The potential gradients $\Omega_{\ell}(\Psi)$ and $\Omega_{\ell, \omega}(\ell, \Psi)$

The coupling of frame-dragging with unipolar induction in BH electrodynamics begins with equation (6.2). Inserting relations $B = B_p + B_{\ell} t$ and $v = v_p + v_\ell t$ into equation (6.3a) yields

$$E = -v/c \times B = -v_p/c \times B_p + \hat{l} \times (v_p B_\ell - v_\ell B_p)$$

and by axial symmetry, $E_\parallel = v_p/c \times B_p = 0$ and hence

$$v_p = \kappa B_p$$

(6.13)

where $\kappa$ is a scalar function (see equation (6.18d)). Then we have

$$E_p = -\alpha (v_\ell - \kappa B_\ell) \nabla \Psi$$

(6.14)

Equating two equations (6.2) and (6.14) for $E_p$ yields

$$\nabla A_0 = -K \nabla \Psi, \quad K \equiv -\frac{\alpha (v_\ell - \kappa B_\ell)}{2\pi c \alpha}$$

(6.15)

and taking the curl of $\nabla A_0$, we get

$$0 = \nabla \times \nabla A_0 = -\nabla \times (K \nabla \Psi) = -\nabla K \times \nabla \Psi = 2\pi c \omega t (B_p \cdot \nabla K)$$

which indicates that $K$ is a function of $\Psi$ only, and hence

$$K = -\frac{dA_0}{d\Psi} \equiv \frac{\Omega_{\ell, \omega}(\ell, \Psi)}{2\pi c}$$

(6.16)

(see Toma & Takahara (2014) for another derivation of $\Omega_{\ell, \omega}$). Equating this $K$ to the one in equations (6.15) yields the FL-AV $v_F$ in equation (6.18c) later. From equations (6.14)-(6.16), we get

$$E_p = -\frac{\Omega_{\ell, \omega}}{2\pi c \alpha} \nabla \Psi = \frac{v_F}{c} B_p n$$

(6.17a)

$$\varrho_e = \frac{1}{8\pi^2 c} \nabla \cdot \left( \frac{\Omega_{\ell, \omega}}{\alpha} \nabla \Psi \right)$$

(6.17b)

where $E_p$ is already given in equation (3.15a). Note that it is the freezing-in condition that ensures Ferraro’s law of isorotation for field lines in the steady axisymmetric state, i.e., $\Omega_{\ell, \omega}(\Psi) = \text{constant}$, but the ZAMOs see that the isorotation law for $\Omega_{\ell, \omega}$ is violated by the FD effect, as shown by the $\ell$-dependence of $\Omega_{\ell, \omega} = \Omega_{\ell, \omega}(\ell, \Psi)$. The importance of ‘this surface’ $S_N$ resulting from violation, where $\Omega_{\ell, \omega} = E_p = 0$, was already pointed out by Blandford & Znajek (1977) (see the
Because \( v_F \) stands for the physical velocity of field lines relative to the ZAMOs, \( E_F \) seen by the ZAMO is entirely induced by the motion of the magnetic field lines, i.e., \( E_F = -(v_F/c) \times B_p \) (Macdonald & Thorne 1982). It is the ‘\( \alpha \omega \) mechanism’ (Okamoto 1992) that one can define the inner light surface \( S_{IL} \) by \( v_F = -c \) and ‘this surface’ \( S_N \) by \( v_F = 0 \), in addition to the outer light surface \( S_A \) by \( v_F = c \) (see the CP(v)-statement and section 6.7). We decompose the Lorentz force (\( \varphi_v E + j/c \times B \)) as

\[
\frac{\partial \varphi_v E}{\partial t} + \frac{1}{c} j \times B = \frac{1}{c} \left[ -j L B_p + j L B_p t \right] + \left( j L B_1 - j L B_p + \frac{\Omega_{F\omega} \varphi}{\alpha} B_p \varphi_c \right) B \cdot \nabla \left( \frac{\partial \varphi_v}{\partial t} - \frac{\varphi}{\alpha} \right),
\]

(6.19)

(see Okamoto 1999). The force-free and torque-free conditions are given simply by \( j_L = 0 \), i.e., \( I = I(\Psi) \) constant along each field line (see equations (6.11) and (6.24)). The \( n \)-component yields

\[
j_t = \varphi_v v_t = \left( \Omega_{F\omega} \varphi / \alpha \right) \varphi_v + \left( 1 / \alpha^2 \varphi c \right) \left( d l / d \Psi \right),
\]

(6.20a)

\[
= \Omega_{F\omega} \varphi / \alpha \nabla \cdot \nabla \left( \frac{\varphi}{\alpha} \right) + \frac{1}{\sigma \alpha^2 c} \frac{d l^2}{d \Psi}, \quad (6.20b)
\]

which accords with the result from \( j_t = \varphi_v v_t \) in equation (6.10), utilizing \( \varphi_v \) in (6.17b), \( v_t \) in (6.18b) and \( v_F \) in (6.18c). By equations (6.9) and (6.17a), we have also

\[
j_t = \frac{\varphi}{\alpha} \left( \nabla \cdot \left( \frac{\varphi}{\alpha^2} \right) \nabla \Psi \right) + \frac{\Omega_{F\omega} \varphi}{\alpha c^2} \left( \nabla \Psi \cdot \nabla \right) \omega \right)
\]

(6.21)

(see Eqs. (2.17c), (5.6b) in Macdonald & Thorne 1982). Equating two expressions (6.20b) and (6.21) for \( j_t \) leads to the stream equation (4.1).

Putting relations among \( v, j, B \) together from equations (6.10), (6.12), and (6.20a), we have

\[
v = \frac{\dot{J}}{\ddot{\varphi}} = -\frac{1}{\varphi} \frac{d l}{d \Psi} B + \frac{\Omega_{F\omega} \varphi \omega t}{\alpha},
\]

(6.22)

which indicates that current-field-streamlines are equipotentials in the force-free domains (see equation (6.3c)).

In passing, we clarify an important constraint imposed by the ‘current closure condition’ in the steady axisymmetric state: no net gain nor loss of charges over any closed surface threaded by current lines in the force-free domains. For a closed surface from the first open field line \( \Psi = \Psi_0 \) to the last open field line \( \Psi = \Psi \) in the poloidal plane, we have

\[
\int_0^\alpha j \cdot dA \propto I(\Psi) - I(\Psi_0) = 0, \quad I(\Psi) = I(\Psi_0) = 0,
\]

(6.23)

when there is no line current at \( \Psi = \Psi_0 \), or \( \Psi = \Psi \). This requires that function \( I(\Psi) \) has at least one extremum at \( \Psi = \Psi_0 \) where \( (dI/d\Psi)_0 = 0 \) (see figure 2 in (Okamoto & Sigal 2006) for one example of \( I(\Psi) \)), and hence \( f_p = \varphi_v v_F / 0 \) for \( \Psi < \Psi_c < \Psi_2 < \Psi \).

6.6 The ‘conserved’ and ‘non-conserved’ energy fluxes

Multiplying equation (6.11) with \( \Omega_F \), we have

\[
0 = -\frac{\Omega_F}{\alpha} \nabla \cdot \varphi S_{F1} = -\frac{1}{\alpha} \nabla \cdot \varphi \omega \psi, \quad (6.24)
\]

which indeed reproduces equation (3.8) for the \( S_E = \Omega_F \psi \) relation. This procedure of derivation, however, does not yield the non-conserved fluxes \( S_{EM} + S_{SD} \) between \( S_E \) and \( \Omega_F \psi \), although one can quickly obtain ‘a Poynting flux’ (see Blandford & Znajek 1977; the CP(iv)-statement), which accords with \( S_{EM} \), from equations (6.17a) and (6.4b). Then, to replace \( \Omega_F \psi \) with \( \Omega_F \psi^2 + \psi \) with the use of the identity in (3.18), one obtains equation (3.17) or (3.14a,b), which shows that the FD effect splits the ‘total’ conserved flux \( S_E \) into the two non-conserved fluxes \( S_{EM} \) and \( S_{SD} \) (see section 3.2).

The conserved energy flux \( S_E \) and the angular momentum flux \( S_\phi \) correspond to terms \( c^2 (dF/dt) \) and \( dJ/dt \), respectively, in the first law (see equations (3.7) and (3.9a,b)). The ‘non-conserved’ energy flux \( S_{EM} \) fits at the horizon to the second term \( \Omega_{HF} dF/dt \) of the first law, because \( \omega \) tends to \( \Omega_{HF} \) on the horizon surface \( S_H \) by the zeroth law of thermodynamics. Another non-conserved flux \( S_{EM} \) corresponds to \( T_F (dS/dt) \) on the horizon. Therefore, we have three energy fluxes indispensable in describing the force-free magnetosphere in total (see Figure 1; cf. figure 3 in Okamoto 2009).

6.7 Two light surfaces \( S_{OL} \) and \( S_{IL} \)

When the hole is losing energy electromagnetically, not only two light surfaces, \( S_{OL} \) and \( S_{IL} \), but also ‘this surface’ \( S_N \) in between always exist (see the CP(iv)-statements). The ‘physical velocity’ of field-lines \( v_F \) by equation (6.18c) approaches \( \pm \infty \) for \( \sigma \rightarrow \infty \) towards \( S_{OL} \) and for \( \sigma \rightarrow 0 \) towards \( S_{IL} \). The outer and inner light surfaces are given by \( v_F = \pm c ; \) (see Figure 1)

\[
(\Omega_{F\omega})_{OL} = +c(\sigma / \sigma^s), \quad (\Omega_{F\omega})_{IL} = -c(\sigma / \sigma^s), \quad (6.25)
\]

respectively, and their positions are obtained by solving

\[
\omega_{OL} = \omega_N - c(\sigma / \omega)_{OL}, \quad \omega_{IL} = \omega_N + c(\sigma / \omega)_{IL}, \quad (6.26)
\]

provided that \( \Omega_F = \omega_N \) is given, and hence

\[
\omega_{OL} > \omega_N = \Omega_F > \omega_{IL}, \quad (6.27)
\]

(see equations (A.7a,b) for the behavior of \( S_{OL} \) and \( S_{IL} \) for the limit of \( h \rightarrow 0 \)). The two non-conserved energy fluxes become from equations (3.16a,b), (6.25) and (6.26).
where \( I_{\text{out}} + I_{\text{in}} = \Omega_{\text{E}} \cdot S_{\Omega} \) and

\[
\begin{align*}
S_{\text{EM}} &= \Omega_{\text{E}} \cdot S_{\Omega} = \begin{cases} I_{\text{out}} (\Psi)(B_p/2\pi\sigma)_{\text{OL}} > 0 & ; S_{\text{OL}}, \\
I_{\text{in}} (\Psi)(B_p/2\pi\sigma)_{\text{IL}} > 0 & ; S_{\text{IL}}.
\end{cases} \\
S_{\text{SD}} &= \omega \cdot S_{\Omega} = \begin{cases} I_{\text{out}} (\Psi)(\omega B_p/2\pi\sigma c)_{\text{OL}} > 0 & ; S_{\text{OL}}, \\
I_{\text{in}} (\Psi)(\omega B_p/2\pi\sigma c)_{\text{IL}} > 0 & ; S_{\text{IL}}.
\end{cases}
\end{align*}
\]

Eliminating the factor \( \nabla \cdot (\alpha \nabla \Psi / \sigma^2) \) between equations (630) and (621), we have

\[
\begin{align*}
\rho_E &= \frac{\rho_{\text{F}}}{c^2} \, \hat{t} = \frac{1}{\alpha^2 \omega c} \left( \nabla \Psi \cdot \nabla \omega - \frac{\alpha^2}{c^2} \nabla \Psi \cdot \nabla \left( \nabla \frac{\rho_{\text{F}}}{\alpha} \right) \right),
\end{align*}
\]

and

\[
\begin{align*}
\hat{t} &= \frac{1}{\alpha^2 \omega c} \left( \nabla \Psi \cdot \nabla \omega - \frac{\alpha^2}{c^2} \nabla \Psi \cdot \nabla \left( \nabla \frac{\rho_{\text{F}}}{\alpha} \right) \right),
\end{align*}
\]

(see Eqs. (29) and (30) in Okamoto 2009), which reduce to Eqs. (45) and (46) in Okamoto 1974 for a pulsar force-free magnetosphere with \( \alpha = 1 \) and \( \omega = 0 \). For both \( \rho_{\text{F}} \) and \( \hat{t} \) not to diverge at \( S_{\text{OL}}/S_{\text{IL}} \) with \( \rho_{\text{F}} = \pm c \), the numerators should vanish, i.e.,

\[
\begin{align*}
\frac{\rho_{\text{F}}}{\alpha^2 \omega c} \left( \nabla \Psi \cdot \nabla \omega - \frac{\alpha^2}{c^2} \nabla \Psi \cdot \nabla \left( \nabla \frac{\rho_{\text{F}}}{\alpha} \right) \right) &= 0, \\
\frac{\rho_{\text{F}}}{\alpha^2 \omega c} \left( \nabla \Psi \cdot \nabla \omega - \frac{\alpha^2}{c^2} \nabla \Psi \cdot \nabla \left( \nabla \frac{\rho_{\text{F}}}{\alpha} \right) \right) &= 0.
\end{align*}
\]

which will automatically be satisfied, when the eigenvalues \( \lambda_{\text{out}} \) and \( \lambda_{\text{in}} \) and \( \Omega_{\text{E}} \) are determined by the criticality-boundary condition (see section 10). This is because, in order to determine the eigenfunctions, the breakdown of the force-free condition and termination of the force-free domains are necessary, while the 'criticality condition' (6.34) has nothing to do with the determination of the eigenfunctions. When \( \Omega_{\text{E}} (\Psi) \) is obtained, we will be aware of not only \( \omega_N = \Omega_{\text{E}} \), but also \( \omega_{\text{OL}} \) and \( \omega_{\text{IL}} \) in equations (6.26).

It will be evident that 'this surface' \( S_N \) with \( S_{\text{EM}} = \rho_F = \Omega_{\text{F}} = 0 \) is located between the two light surfaces, i.e. \( S_{\text{OL}} < S_N < S_{\text{IL}} \), indicating that the particle source of two, oppositely-directed, magneto-centrifugal winds, outwardly and inwardly passing through \( S_{\text{OL}} \) and \( S_{\text{IL}} \), respectively, must be coexistent under the null surface \( S_N \) (or the Gap \( G_\text{T} \); see sections 8.2 and 9). Contrary to the statement "there is no reason to believe that its position is stationary" (Blandford & Znajek 1977), its position must unequivocally be at 'this surface' \( S_N \) (see the Cp(iii,iv)-statements; section 7).  

6.8 The densities of the electromagnetic energy and angular momentum in the force-free domains

Substituting \( E_p, B_p \) and \( B_i \) from equations (3.15a) into Eqs (3.20a) and (2.31a) in Macdonald & Thorne 1982 for the density of the electromagnetic field, i.e.,

\[
\begin{align*}
\rho_E &= (\alpha B_p^2/8\pi) \left( E^2 + B^2 \right) + (\omega/4\pi c) (E \times B) \cdot m, \\
\rho_j &= (1/4\pi c) (E \times B) \cdot m,
\end{align*}
\]

we have

\[
\begin{align*}
\rho_E &= \frac{\alpha B_p^2}{8\pi} \left( 1 + \frac{B_i^2}{B_p^2} + \frac{\omega^2}{\alpha c^2} (\Omega_{\text{F}}^2 - \omega^2) \right), \\
\rho_j &= \frac{\Omega_{\text{F}}(\omega B_p)^2 c}{\alpha c} = \frac{\rho_{\text{F}} B_p^2}{c}.
\end{align*}
\]

(see equation (2.17a) in Okamoto 1992 and Eq (55) in Komissarov (2009)), where \( E_j \) and \( \rho_j \) are a function of \( \omega \) co-ordinated along each field line labeled with \( \Psi \). Note that the density of 'angular momentum about the hole’s symmetry axis' is positive, i.e. \( \rho_j > 0 \) in the outer domain \( D_{\text{out}} \) with \( \Omega_{\text{F}} > 0 \), but turns to negative, i.e., \( \rho_j < 0 \) in the inner domain \( D_{\text{in}} \) counter-rotating with \( \Omega_{\text{F}} < 0 \), where a Poynting flux is directed toward the hole, \( S_{\text{EM}} < 0 \) (see the Cp(iii)-statement).

The surface at infinity \( S_{\infty} \), where \( \alpha \approx 1, \omega \approx 0, B_i^2 \gg B_p^2 \) and \( \omega^2 \gg \omega_{\text{OL}} \), we have

\[
\begin{align*}
\rho_E &\approx \frac{1}{8\pi} \left( \frac{\Omega_{\text{F}}^2 \omega^2 B_i^2}{c^2} \right), \\
\rho_j &\approx \frac{\Omega_{\text{F}} (\omega B_p)^2}{c}.
\end{align*}
\]

which will be dissipated for particle acceleration in the resistive membrane \( S_{\infty} \) (see section 8.1.1). Near the null surface \( S_N \) where \( \Omega_{\text{F}} \approx 0 \) and \( \sigma B_i \propto I \approx 0, \)

\footnote{In the case of treating the whole magnetohydrodynamic theory of BH winds, not only the outflow but also the inflow must pass smoothly through three critical points; slow, intermediate and fast magnetosonic surfaces (Weber & Davis 1967; Michel 1969; Okamoto 1978; Kennel et al. 1983 ; Pinsky & Coroniti 1990; Okamoto 1999; Okamoto 2002; Okamoto 2003). In the force-free theory, the last two surfaces reduce to \( S_{\text{OL}}, S_{\text{IL}} \) and \( S_{\text{F}}, S_{\text{F}} \), respectively, although the slow surface is usually neglected.}
which will be matter-dominated by charged particles pair-created by the voltage drop, thereby breaking down the force-free condition (see sections 7, 8.2 and 9).

It is the FD term \( \omega^2 \) in equation (6.35a) that builds a region of negative-energy region with \( \varepsilon_E < 0 \) in the inner domain \( \mathcal{D}_{(\text{in})} \).

At the inner light surface \( S_{\text{IL}} \), where \( \nu_F = (\Omega_F \sigma c)_\text{IL} = -c \) and when \( (B_p \sigma^2)_{\text{IL}}/(B_\Omega \sigma^2)_{\text{IL}} \approx 1 \) and \( \Omega_F \approx 0.5 \Omega_{H1} \), we have analytically from equation (6.35b)

\[
\varepsilon_E \approx \frac{\alpha B_p^2}{8\pi \alpha} \left[ 1 - \frac{\Omega_F}{\Omega_{H1}} - \left( \frac{\alpha^2 \sigma^2}{2 \Omega_F \Omega_{H1} \sigma^2} \right)_{\text{IL}} \right].
\]

and then we see (6.39a) \( \varepsilon_E < 0 \), if \( (\Omega_F \sigma c)_\text{IL} < 2 \).

There will be the surface \( S_{\text{IL}0} \) dividing the inner domain \( \mathcal{D}_{(\text{in})} \) into the two regions with \( \varepsilon_E(\omega, \Psi) \geq 0 \) for \( \omega \approx \omega_{\text{IL}} \) between \( S_{\text{IL}} \) and \( S_{\text{IL}0} \). The topology of \( S_{\text{IL}0} \) is depicted by the solution of

\[
\omega^2 = \Omega^2_F + \frac{\alpha^2 \sigma^2}{c^2} \left( 1 + \frac{B_p^2}{2} \right),
\]

where \( \alpha/ \sigma \) and \( B_p/B_\Omega \) are thought of as functions of \( \omega \) and \( \Psi \). This obviously indicates \( \omega_{\text{IL}0} > \omega_N = \Omega_F \) (see equations (6.37) and (6.38)). It is thus the \( \alpha/ \sigma \) mechanism (Okamoto 2009) that produces not only the inner domain \( \mathcal{D}_{(\text{in})} \) of \( \Omega_F \sigma c \leq 0 \) with the inner light surface \( S_{\text{IL}} \) (Blandford & Znajek 1977; see the C_F(\text{v})-statement in section 4.1), but also a region of the negative-energy density of \( \varepsilon_E \leq 0 \) in \( \Omega_{H1} \geq \omega > \omega_{\text{IL}0} \).

Near the force-free horizon surface \( S_{\text{H1}} \) (see section 8.1.2) where \( (B_p \sigma^2)_{\text{H1}}/(B_\Omega \sigma^2) \approx 1 \) and hence \( B_p^2/B_\Omega^2 = (2I_{(\text{H1})}/(\alpha c B_p) \approx (((\Omega_{H1} - \Omega_F) \sigma)/(\alpha c))^2 \) by equation (3.13), we have

\[
\varepsilon_E \approx \frac{\alpha B_p^2}{8\pi \alpha} \left[ 1 - \frac{\Omega_F}{\Omega_{H1}} - \left( \frac{\alpha^2 \sigma^2}{2 \Omega_F \Omega_{H1} \sigma^2} \right)_{\text{H1}} \right].
\]

and for \( \alpha \to 0 \) toward the resistive horizon membrane \( S_{\text{H1}} \)

\[
\approx - \left( 1 - \frac{\Omega_F}{\Omega_{H1}} \right) \frac{\alpha B_p^2}{8\pi \alpha} \left[ \frac{2 \Omega_F \Omega_{H1} \sigma^2}{c^2} \right]_{\text{H1}} < 0.
\]

For the density of angular momentum near the horizon, we have

\[
\varepsilon_3 = - (\Omega_{H1} - \Omega_F) \left( \frac{B_p^2}{\alpha c^2} \right)_{\text{H1}} < 0.
\]

It will be evident in equation (6.40) that \( \Omega_{H1} > \omega_{\text{IL}0} > \omega_N = \Omega_F \), which does not lead to any more robust condition upon \( \Omega_F \) than those from the second law and the radiation condition toward the horizon (see equation (3.23a); cf. Komissarov (2009)).

The above result suggests that the negative-energy region will extend from near the inner light surface to near the horizon, although the situation may be somewhat different for the MHD case (see Takahashi et al. (1990) and Koide (2003)). Irrespective of how much negative energy is falling into the horizon, the amount of extracted energy is uniquely given by

\[ c^2 \Delta M = \Omega_F |\Delta J|, \]

which is smaller than the adiabatic value \( \Omega_{H1} |\Delta J| \). As the first law of thermodynamics shows, any process of extracting the reducible rotational energy must occur only through an influx of negative angular momentum, i.e.

\[ c^2 \Delta M = \Omega_F |\Delta J| < 0, \]

although the existence of the negative-energy region may be suggestive of the possibility of extraction. This means that some surface braking torque must be at work on the resistive membrane \( S_{\text{H1}} \) with some entropy increase following (see equations (3.21) and (3.22)).

7 The breakdown of the iso-rotation law as well as the freezing-in and force-free conditions

When the hole is losing angular momentum, the null surface \( S_N = S_{\text{FAM}} \) always exists. Presuming \( (B_p \nu) \neq 0 \) and \( (\Omega_F \nu) \neq 0 \), from equations (6.3a,b), (6.17a,b), (3.16a,b), (3.17), (6.6), (6.22), (6.20a,b) and (6.36b), we see that following quantities must necessarily vanish at \( S_N \), i.e.,

\[
(\Omega_F \nu) = (\nu) = (\nu) = (\nu) = (\nu) = (\nu). \]

(7.1)

where we denote the value of function \( X(\Omega_F \nu, \Psi) \) on \( S_N \): \( (X) = (X). \)

(7.2)

(see equations (9.1a,b) for the widened Constraints in the widened Gap \( g_N \)). It is the kick-off equation (6.2) that combines with the freezing-in condition in (6.3a), to produce the coupling of the FD effect with unipolar induction, i.e., \( \omega \) and \( \Omega_F \), which in turn gives rise to ‘violation of Ferraro’s law of iso-rotation’ by the \( \epsilon \)-dependence of \( \omega \), thereby yielding ‘this surface’ \( S_N \), and then other Constraints in equation (7.1a). Consequently, Constraint \( (E_p) = 0 \) reacts back to the force-free and freezing-in conditions in (6.3a,b) on ‘this surface’ \( S_N \), thereby breaking them down, to yield the Constraints in (7.1b) and (7.1c).

We classify Constraints at \( S_N \) into three distinctive groups: The first group in equation (7.1a) originates from the violation of the iso-rotation law, i.e., \( (\Omega_F \nu) = 0 \), and contains the quantities that reverse direction or change signs across ‘this surface’ \( S_N \). It is the FD effect \( \omega \) that brings about the reversal of these quantities. The second one in equation (7.1b) contains quantities that vanish, but do not reverse direction nor change sign at first sight (see, e.g., Figure 2 for \( I \)). Finally the third one contains \( (\nu) = 0 \) resulting from the breakdown of the freezing-in condition, and
\((f/\varphi_e)_N = 0\), which, similarly to those in the first group, reverses and changes sign due to the existence of \(\varphi_e\).

The above Constraints uniquely specify the fundamental physical nature of the whole force-free magnetosphere as follows:

1. When the hole is losing angular momentum \((dJ < 0)\), the null surface \(S_N\) is nothing but the ZAM-surface \(S_{ZAM}\) \((\Omega_{\Psi_{\infty}} = \varphi = \varepsilon_J = 0)\), the existence of which is required and ensured by the second law of thermodynamics. This surface naturally divides the force-free BH magnetosphere into a domain \(D_{(\text{out})}\) rotating progradely with \(\varepsilon_J > 0\) and another domain \(D_{(\text{in})}\) rotating retrogradely with \(\varepsilon_J < 0\).

2. The behavior of the Poynting flux like \(S_{\text{EM}} \propto 0\) for \(\Omega_{\Psi_{\infty}} \propto 0\) necessitates the existence of a pair of batteries with each EMF on both the upper and lower sides of \(S_N\), in such a way that each current flows along each closed-circuit, to satisfy the current-closure condition in each domain, \(D_{(\text{out})}\) or \(D_{(\text{in})}\) (see section 8.2; Figures 1, 3).

3. Constraints \((f)_N = (\varphi)_N = 0\) mean that both of current- and stream-lines are not allowed to cross \(S_N\). There will thus be no circuit that allows such a current crossing \(S_N\) due to a ‘single’ battery at any (possible) position (cf. Thorne et al. 1986; section 4.3). Each electric circuit must close in its respective force-free domain, \(D_{(\text{out})}\) or \(D_{(\text{in})}\), with each EMF in the inductive membrane \(S_N\) and with the eigenvalue \(I(\Psi)\), i.e., \(I_{(\text{out})}\) or \(I_{(\text{in})}\) determined as the eigenvalue in the resistive membrane \(S_{\text{ff}_{\infty}}\) or \(S_{\text{ff}_{H}}\) (sections 8.1.1 and 8.1.2).

Relation \(v = f/\varphi_e\) in equations (6.22) no longer holds at \(S_N\).

4. The surface \(S_N = S_{ZAM}\) with \(v \propto 0\) for \(\varepsilon_J \propto \Omega_{\Psi_{\infty}} \propto 0\) will behave like a watershed in a mountain pass for outflows and inflows of ‘massless’ particles pair-created by the voltage drop (see equations (8.11) and (8.12)), and yet both flows are due to the magneto-centrifugal forces at work toward the two opposite directions, inward and outward, respectively. ‘This surface’ \(S_N\) may thus be regarded as equipped with a kind of hypothetical ‘magnetic rotators’ oppositely spinning and having a pair of EMFs with the particle-current sources in-between. As the outer pulsar-type magneto-centrifugal wind flows through \(D_{(\text{out})}\) with \(v_p > 0\), the inner anti-pulsar-type wind will be existent in \(D_{(\text{in})}\) with \(v_p < 0\) (see Figure 3; section 9.4).

5. Constraint \((f)_N = 0\) ensures that no angular momentum is conveyed by the force-free magnetic field directly from \(D_{(\text{out})}\) to \(D_{(\text{in})}\). Then, \((I)_N = (B_i)_N = (S_j)_N = (S_{E_i})_N = 0\), resulting from \((f)_N = 0\), means that there will be no inertial loading upon \(B_p\) in the inductive membrane with no resistance (see section 8.2; Figure 2). It is essential to remind that the toroidal field \(B_t\) is a swept-back component of the poloidal component \(B_p\) due to inertial loadings in the resistive membranes (see Figure 2). At \(S_N\) (or \(G_N\)), \((I)_N = 0\) (or \(I_G = 0\)) means that there must be a jump of \(I(\Psi) = I_{(\text{in})}\) to \(I_{(\text{out})}\), just like in the NS surface (see equations (2.2) and (9.3) and Figure 2).

6. Constraint \((\varphi)_N = 0\) means that the particles pair-created at \(S_N\) (or \(G_N\)) make no macroscopic motions, and hence they are ‘zero-angular-momentum’ particles (ZAM-Ps) circulating with \(\omega_N = \Omega_F\). Moreover, Constraints \((I)_N = (S_j)_N = (S_{E_i})_N = 0\) mean that no angular momentum nor energy is transported directly along the field lines of \(B_p\) with \(B_i = 0\) across \(S_N\).

7. The ZAM-surface \(S_{ZAM}\) \(v_p = \varepsilon_J = \Omega_{\Psi_{\infty}} = 0\) means that ‘this surface’ is a new kind of divider of out- and in-flows due to magneto-centrifugal forces (see section 11.2).

8. Two vectorial quantities \(J_p\) and \(S_i\) are both related to the current/angular-momentum function \(I(\Psi)\) in the force-free domains, and do not reverse direction, despite that \((J_p)_N = (S_i)_N = 0\). This is because an outflow of negative charges means the ingoing current, and an inflow of negative angular momentum means an outflow of positive one (see Figures 2 and 3).

8 An extended Membrane Paradigm

The two conserved functions of \(\Psi\), i.e., \(I(\Psi)\) and \(\Omega_F(\Psi)\), are not freely specifiable parameters in the force-free domains, but must be determined as the eigenfunctions of \(\Psi\): it is the ‘criticality condition’ at the resistive membranes \(S_{\text{ff}_{\infty}}\) and \(S_{\text{ff}_{H}}\) terminating the two force-free domains that determine \(I_{(\text{out})}\) and \(I_{(\text{in})}\) in the outer and inner force-free domains (see equations (8.1) and (8.5)), respectively, and then it is the ‘boundary condition’ in the inductive membrane \(S_N\) in-between that determines the final eigenvalue \(\Omega_F\), to ensure continuity of the angular momentum flux across ‘this surface’ \(S_N = S_{ZAM}\) (see equations (10.1). At the same time, \(S_N\) (or \(G_N\)) thus decided must be the optimum place to build up the power station consisting of a pair of unipolar induction batteries with EMFs for the DC circuits in both domains, as well as to produce pair-particles sufficient enough to get strongly magnetized by anchoring the threading magnetic field lines. This procedure would be impossible to conduct unless the freezing-in and force-free conditions are not negligibly but wholly violated (cf. Blandford & Znajek 1977; Macdonald & Thorne 1982; the \(C_S(v_{/}v_i)\)-statement).

8.1 The resistive membranes

8.1.1 The force-free infinity surface \(S_{\text{ff}_{\infty}}\)

It is astrophysical loads such as MHD particle acceleration that produce the toroidal component \(B_t = -2I_{(\text{out})}/\sigma_{\text{arc}}\) as a swept-back component from the poloidal field \(B_p\). In MHD wind theory, the outgoing plasma flow is generally required to pass smoothly through the Alfvénic surface (or the light surface \(S_{\text{AL}}\),
the outer fast-magnetosonic surface $S_{\text{df}}$ (see, e.g., Kennel et al. 1983; Punsly & Coroniti 1990). In the present force-free case, the ‘criticality condition’ at $S_{\text{df}}$ in $S_{\text{f}}$ yields

$$I_{\text{(out)}}(\Psi) = (1/2)\Omega_\Psi (B_p c^2 \sqrt{\sigma}/\Omega_\Psi),$$  \hspace{1cm} (8.1)

which naturally accords with the force-free limit of that for MHD pulsar wind theory (see Eq. (10.1) in Okamoto 1978). The criticality condition in wind theory also accords with ‘Ohm’s law’ in circuit theory for the surface currents flowing in $S_{\text{df}}$ and the ‘radiation condition’ in flux theory. The outgoing magnetocentrifugal wind is followed by the Poynting flux and the angular momentum flux in $D_{\text{(out)}}$ with $\Omega_\Psi > 0$ and $\epsilon_f > 0$, which are given in terms of $I_{\text{(out)}}$ from equation (3.16a);

$$S_{\text{EM,(out)}} = \Omega_\Psi S_1(\text{(out)}), \quad S_1(\text{(out)}) = I_{\text{(out)}}(\Psi)/(2\pi\alpha c)B_p$$  \hspace{1cm} (8.2)

(see equation (2.1a) for a pulsar wind, and equation (8.6b) for $S_{\text{EM,(in)}}$). The outer force-free domain $D_{\text{out}}$ is then terminated by a membrane $S_{\text{f}}$, which may be regarded in wind theory as containing a sufficiently thin layer onto which the MHD acceleration layer from the outer fast-magnetosonic surface $S_{\text{df}}$ to $S_{\text{f}}$ is compressed, and on which membrane currents transformed from the volume currents in $D_{\text{out}}$ flow across poloidal field lines thereby to dissipate. The membrane current flowing from $\Psi_2$ to $\Psi_1$ is

$$I_{\text{f}} = (\Omega_\Psi^2/2\pi\sigma)\Psi,$$  \hspace{1cm} (8.3)

where $I_{\text{f}} = I_{\text{(out)}}(\Psi_1) - I_{\text{(out)}}(\Psi_2)$ and $\Psi > \Psi_1 < \Psi_2 < \Psi$ (see Figure 3). The resistive membrane $S_{\text{f}}$ may also be interpreted as possessing the same surface resistivity $R = 4\pi/c = 377$ Ohm as on another membrane $S_{\text{m}}$ above $S_{\text{f}}$, and Ohm’s law holds on $S_{\text{f}}$, i.e., $RI_{\text{f}} = (E_p)_{\text{f}}$. This Ohmic dissipation (in circuit theory) implies that the MHD acceleration (in wind theory) takes place. From Eq. (4.14) in Macdonald & Thorne (1982), the rate per unit $t$ time at which electromagnetic fields transfer redshifted energy to particles is

$$-\frac{1}{\alpha} \nabla \cdot \sigma E = \alpha J \cdot E = (\alpha/c^2) \cdot m + \frac{\Omega_\Psi \sigma}{c} j_L B_p$$

$$= -\frac{\Omega_\Psi \sigma}{c^2} \frac{B_p}{2\pi\sigma c} \frac{\partial I}{\partial t} > 0,$$  \hspace{1cm} (8.4)

where equations (6.8) and (6.19) are used. It thus turns out that when the current function $I(t, \Psi)$ is continuously decreasing with $t$ beyond near $S_{\text{df}}$ toward $S_{\text{f}}$, the MHD acceleration occurs (see Figure 2), but the force-free magnetosphere regards the ‘force-free’ domain with $j_L = 0$ formally as extending to compress the acceleration layer $j_L > 0$ to the force-free infinity surface $S_{\text{f}}$ with $|j_L| > |j_L|$. By doing so, the circuit $C_{\text{(out)}}$ closes so as not to violate the current-closure condition.

8.1.2 The force-free horizon surface $S_{\text{ffH}}$

The inner domain $D_{\text{(in)}}$ counter-rotating with $\Omega_\Psi > 0$ and $\epsilon_f < 0$ is terminated by another membrane $S_{\text{ffH}}$ with the surface resistivity $R$; in other words, a sufficiently thin resistive layer, in which the Ohmic dissipation of the surface current is taking place, thereby generates Joule heating, to be easily absorbed as entropy by the hole. The inner resistive membrane $S_{\text{ffH}}$ accords with the ‘stretched horizon’ with a pretty small $\alpha$ above the actual horizon surface $S_{\text{H}}$ (Thorne et al. 1986).

The criticality condition in wind theory requires the ingoing magnetocentrifugal wind to pass smoothly through $S_{\text{f}}$ as well as the inner light surface $S_{\text{L}}$, to yield for the eigenfunction $I_{\text{(in)}}$,

$$I_{\text{(in)}}(\Psi) = -I_{\text{(in)}}(\Psi) = -(1/2)(\Omega_\Psi - \Omega_\Psi)(B_p c^2)^2_{\text{ffH}}$$  \hspace{1cm} (8.5)

at $S_{\text{f}}$ as the outermost surface of $S_{\text{ffH}}$ (see equation (3.13). The reason for taking $I_{\text{(in)}} < 0$ is because the inflow with $v_p < 0$ is due to the magneto-centrifugal force inward-directed, passing through the negative angular momentum density domain ($\epsilon_f < 0$), as opposed to the outflow with $\Omega_\Psi > 0$, $v_p > 0$ and $\epsilon_f > 0$ (see Figures 2, 4, and 5). Then we have the ingoing fluxes of negative angular momentum, electromagnetic Poynting energy and frame-dragging spin-down energy in terms of $I_{\text{(in)}} = -I_{\text{(in)}}$ and $S_{\text{ffH}} = -S_{\text{ffH}}$, i.e.,

$$S_{1,\text{(in)}} = \frac{I_{\text{(in)}}}{2\pi\alpha c}B_p = -\frac{I_{\text{(in)}}}{2\pi\alpha c}B_p = -S_{\text{ffH}} > 0,$$  \hspace{1cm} (8.6a)

$$S_{\text{EM,(in)}} = \Omega_\Psi S_1(\text{(in)}), \quad (\omega - \Omega_\Psi)S_{\text{ffH}} < 0,$$  \hspace{1cm} (8.6b)

$$S_{\text{SD,(in)}} = \omega S_{\text{ffH}} = -S_{\text{ffH}} > 0.$$  \hspace{1cm} (8.6c)

Similarly to the membrane current on $S_{\text{ffH}}$ in equation (8.3), the membrane current on $S_{\text{ffH}}$ from $\Psi_1$ to $\Psi_2$ is

$$I_{\text{ffH}} = (I_{\text{(in)}}/2\pi\sigma)_{\text{ffH}}.$$  \hspace{1cm} (8.7)

Then Ohm’s law holds, i.e., $RI_{\text{ffH}} = (E_p)_{\text{ffH}}$, in the stretched horizon (Macdonald & Thorne 1982; Okamoto 2012a; Okamoto 2015a).

The ‘stretched horizon’ with a depth of $\alpha$ covering the true horizon $S_{\text{H}}$ (Thorne et al. 1986) is almost identical to the ‘force-free horizon surface’ $S_{\text{ffH}}$ here to cover a thin layer of the Ohmic dissipation taking place from $S_{\text{ffH}}$ to the inner fast-magnetosonic surface $S_{\text{ff}}$ in circuit theory. The surface current $\alpha I_{\text{ffH}}(\equiv \dot{S}_{\text{H}}$ in equation (3.21a,b)) flows on the stretched horizon $S_{\text{ffH}}$, crossing field lines threading there, the surface torque will be at work through $S_{\text{ffH}}$ to the hole, consequently extracting positive angular momentum of the hole, i.e. $dJ/dt < 0$. Then, by equations (3.21b), (8.5) and (8.6a), we have

$$\frac{dJ}{dt} = -\int_{S_{\text{ffH}}} (\alpha I_{\text{ffH}}/c \times B_p) \cdot dA$$

$$= \int_{S_{\text{ffH}}} \alpha I_{\text{ffH}}(\Psi) \cdot dA = -\frac{1}{c} \int_{\Psi_1}^{\Psi_0} I_{\text{(in)}}(\Psi) d\Psi$$

$$= -\int_{S_{\text{ffH}}} \alpha I_{\text{(in)}}(\Psi) \cdot dA = -\frac{1}{c} \int_{\Psi_1}^{\Psi_0} I_{\text{(in)}}(\Psi) d\Psi < 0,$$  \hspace{1cm} (8.8)

where $B_p \cdot dA = 2\pi d\Psi$. The first line of equations (8.8) shows that the outflow of positive angular momentum takes place by the surface braking torque on $S_{\text{ffH}}$, which is equivalent to the inflow.
of negative angular momentum due to the ingoing magneto-
centrifugal wind and also to a sufficiently strong flux of angular
momentum leaving the hole... (see the $C_P$-iii)-statement in sec-
section 4.1).

Associated with the outgoing flux $S_{1,(in)}$ is the spin-down
energy flux $S_{SD}$ due to the FD effect in equation (3.16b), and
the surface integral of $S_{SD}$ over $S_{BH}$ yields $-\Omega_H(dJ/dt)$, i.e.,
\[
\int_{S_{BH}} \alpha S_{SD,(in)} \cdot dA = -\Omega_H \int_{S_{BH}} \alpha S_{1,(in)} \cdot dA = -\Omega_H \frac{dJ}{dt},
\]
(8.9)
because $\omega$ approaches $\Omega_H = \text{constant}$, by the zeroth law (see section 3). It seems that it is only when negative
angular momentum is poured from a gap under the null surface $S_{ZAMD}$
that the outflow of spin-down energy takes place from the hole,
although positive angular momentum from the surface $S_{ZAMD}$
must be extracted outwardly to keep the Gap in zero-angular-
momentum-state (see section 9).

The process of entropy generation in the stretched horizon
$S_{BH}$ is formally described by ‘Ohmic dissipation’ of the surface
currents due to the surface resistivity given by $R = 377\Omega$.m.
Other than exerting the surface torque to extract angular mo-
mentum, the surface current on the resistive membrane gives
rise to Ohmic dissipation, which generates the Joule heating of
the surface current on $S_{BH}$ into a digestible form of irreducible
mass or entropy, i.e., from equation (3.21a) and (3.13) or (8.5),
\[
T
\frac{dS}{dt} = \int_{Sm} R(\alpha I_{in})^2 dA = -(\Omega_H - \Omega_F) \frac{dJ}{dt},
\]
(8.10)
which is naturally equivalent to the Poynting flux flowing into
$S_{BH}$ in expression (3.22) for entropy production in the stretched
horizon (see equation (12.3) later). This is an inevitable result
due to the second law in the causal extraction of the rotational
energy. This cannot be due to the battery’s internal resistance in
the horizon (cf. the $C_S$-iv)-statement).

8.2 The inductive membrane $S_N$

When the hole is losing energy electromagnetically, the inductive
membrane $S_N$ does not fail to exist at this surface between the
two light surfaces, so as to break down the force-free condition
(cf. the $C_S$-i)-statement), and to construct the dual-power station
for a pair of batteries to launch the Poynting fluxes both outward
and inward. The two resistive membranes in both ends can in
truth never be an emitter of a Poynting energy flux (Punsly &
Coroniti 1989). Then, by dividing the force-free magnetosphere
at this surface $S_N = S_{ZAM}$ into the two domains of prograde
rotation $\Omega_{pro} > 0$ or retrograde rotation $< 0$, the Kerr hole can
accommodate the inductive membrane in between and install a
pair of batteries, one for the outgoing Poynting flux toward
astrophysical loads in $S_{BH}$ and the other for the ingoing Poynting
flux toward Ohmic dissipation in $S_{H}$.

In terms of circuit theory, there will be no DC ‘series circuit’
consisting of a battery and its internal resistance in the horizon
and the astrophysical load at infinity (see Thorne et al. 1986;
section 4.3). This is because an inevitable breakdown of the
force-free and freezing-in conditions at the surface $S_{ZAMD}$
leads to severance of current- and stream-lines at the null surface $S_N$,
\[
(j) N = (\Psi)_N = 0.
\]
The extremum value $\Omega_F$ of the gravito-electric potential gra-
dient $\omega$ produces a voltage drop $\Delta V = (\Omega_{H}/2\pi c)\Delta \Psi$ between $S_{BH}$
and $S_{H}$, which will appear as not only the difference of the two
EMFs $E_{\text{out}}$ and $E_{\text{in}}$ for a pair of circuits (Okamoto 2015a),
but also the difference of a pair of hypothetical magnetic axes
spinning with $\Omega_{H}$ and $-\Omega_{H}$ for the two domains $D_{\text{out}}$
and $D_{\text{in}}$ across the inductive membrane $S_N$ (see equations
(8.11a,b) and (3.26a,b,c)). Then, we think of the two circuits
$C_{\text{out}}$ and $C_{\text{in}}$ closed in $D_{\text{out}}$ and $D_{\text{in}}$, respectively (see
Okamoto 2015a). These circuits are existent back-to-back, but
disconnected, by the null surface $S_N$ (see Figure 3). For elec-
tric currents to flow in the closed-circuit $C_{\text{out}}$ and $C_{\text{in}}$, there
must naturally be an EMF due to the unipolar induction bat-
tery for each circuit, resulting from the gravito-electric potential
gradient $\Omega_{\text{out}} = \Omega_{F} - \omega$.

Let us then pick up such two current-field-streamlines $\Psi_1$
and $\Psi_2$ in each force-free domain as the two roots of an algebraic
equation $I(\Psi) = I_{\text{in}}$, i.e., $I(\Psi_1) = I(\Psi_2) \equiv I_{\text{in}}$ in the range of
$0 < \Psi_1 < \Psi_c < \Psi_2 < \Psi$ (see equation (6.23); Figs. 2, 3 in Okamoto
2015a), where $(dl/d\Psi)_0 = 0$ and $j_p \neq 0$ for $\Psi \neq \Psi_c$. Note, however,
that current- and stream-lines do not connect between the outer
and inner domains along field-lines $\Psi_1$ and $\Psi_2$.

The Faraday path integrals of $E_p$ in equation (6.17a) along
two circuits, $C_{\text{out}}$ and $C_{\text{in}}$, yield
\[
E_{\text{out}} = \oint_{C_{\text{out}}} \alpha E_p \cdot dl = -\frac{1}{2\pi c} \int_{\Psi_1}^{\Psi_2} \Omega_F(\Psi) d\Psi,
\]
(8.11a)
\[
E_{\text{in}} = \oint_{C_{\text{in}}} \alpha E_p \cdot dl = +\frac{1}{2\pi c} \int_{\Psi_1}^{\Psi_2} (\Omega_H - \Omega_F) d\Psi.
\]
(8.11b)
There is no contribution to EMFs from integrating along $\Psi_1$
and $\Psi_2$ and on the null surface, because of $E_p \cdot dl = (E_p)_N = 0$.
The difference between the two EMFs across $S_N$ is
\[
[E]_N = E_{\text{out}} - E_{\text{in}} = -\Omega_H \Delta \Psi / 2\pi c = -\Delta V,
\]
(8.12)
where $\Delta \Psi = \Psi_2 - \Psi_1$, and the difference of a quantity $X$ across
the (infinitely thin) interface $S_N$ is denoted with
\[
[X]_N = (X)_{\text{out}} - (X)_{\text{in}}.
\]
Expression (8.12) is derivable simply by integrating the identity
(3.26c) from $\Psi_1$ to $\Psi_2$, just as the potential difference between
the two equipotential lines.

Note that the voltage drop $\Delta V$ is not a series sum of the two
drops $\Delta V_H$ and $\Delta V_L$ in equation (4.3a), but a discontinuous jump
between the two EMFs, $E_{\text{out}}$ and $E_{\text{in}}$, which are partitioned by
the infinitely thin surface $S_H$ in the force-free limit. The advan-
tage of this circuit theory is to concentrate a voltage drop
$\Delta V = (\Omega_{H}/2\pi c)\Delta \Psi$ between $S_{BH}$ and $S_{H}$ in-between the two
The driving source of $j_p$ or the non-ideal MHD region exists outside the horizon in the time-dependent state, while it is not required in the steady-state. We show here that the coupling of unipolar induction with the frame-dragging effect indeed gives rise to a pair of batteries at work as the driving source of currents separately in the outer SC and inner GR domains. Any electric current flowing in the steady-state will need the driving force.
When the null ZAM-surface $S_{ZAM}$ has relaxed to the ZAM-Gap in the steady-state, a pair of batteries will be at work on both the upper and lower surfaces of the Gap (see section 9.2).

9 The zero-angular-momentum gap $\mathcal{G}_N$

The Gap $\mathcal{G}_N$ under the inductive membrane $S_N$ will be in the zero-angular-momentum state $\varepsilon = \Omega_{\psi} = 0$, so that the particles and the field carry no angular momentum within the Gap across the Gap. Also, the Gap will be magnetized in almost the same way as a magnetized neutron star, because the poloidal magnetic field lines threading the Gap will naturally be pinned down in the ZAM-particles pair-created and circulating round the hole with $\omega_N = \Omega_f$ (see section 9.3). Therefore, the magnetized ZAM-particles will ensure $\Omega_f = \omega_N$ (see section 10). The non-force magnetized ZAM-Gap $\mathcal{G}_N$ will thus be formed in the steady-state, with its surfaces $S_{G(out)}$ and $S_{G(in)}$ respectively, at $\Omega_{\psi} \approx \pm \Delta \omega$ (Figure 2), where $\Delta \omega \approx |(\partial \omega/\partial \ell)| \Delta \ell$ stands for the Gap half-width (see equation (9.3)), and for $\Delta \omega \rightarrow 0$, $\mathcal{G}_N \rightarrow S_N$ (see Figure 4 in Okamoto (2015a) for the interplay of microphysics with macrophysics in the magnetized, matter-dominated Gap).

9.1 A plausible Gap structure with $I(\Omega_{\psi}, \Psi)$

In the two force-free domains, $I$ has a constant value $I_{(out)}(\Psi)$ or $I_{(in)}(\Psi)$ for $\Omega_{\psi} > 0$ or $0 \leq \omega < 0$ along each field line, whereas Constraints $(f)_{G} = (I)_{G} = 0$ in-between require $I(\Omega_{\psi}, \Psi)$ to vanish, due to the breakdown of the freezing-in and force-free conditions within $\mathcal{G}_N$. In reality, the voltage drop, $\Delta V = -[E]_{N}$ at $S_N$ (see equation (8.12)), will produce particle-particles copious enough, and the plasma pressure in the steady-state will expand $S_N$ to $\mathcal{G}_N$ with a half-width $\Delta \omega$ in $|\Omega_{\psi}| \leq \Delta \omega$ (see section 8.2). We have already denoted the widened null Gap with $\mathcal{G}_N$, and then we replace Constraints in (7.1) at $S_N$ with the following ‘widened’ Constraints in $\mathcal{G}_N$ in $|\Omega_{\psi}| \leq \Delta \omega$:

\[
(\Omega_{\psi})_{G} = (E_{G})_{G} = (x_{G})_{G} = (f)_{G} = (I)_{G} = 0, \quad (9.1a)
\]

\[
(\epsilon_{G})_{G} = (S_{G})_{G} = (S_{EM})_{G} = (S_{GD})_{G} = (S_{E})_{G} = (P_{J})_{G} = (P_{E})_{G} = 0. \quad (9.1b)
\]

In passing, $[X]_{G}$ denotes the difference of $X$ across the Gap $|\Omega_{\psi}| \leq \Delta \omega$ (cf. $[X]_{N}$ in equation (8.13));

\[
[X]_{G} = X(\Delta \omega, \Psi) - X(\Delta \omega, \Psi) \equiv (X)_{G}^{(out)} - (X)_{G}^{(in)}. \quad (9.2)
\]

In the following, we suppose $\mathcal{G}_N \rightarrow S_N$ in the force-free limit of $\Delta \omega \rightarrow 0$.

Then we presume such a simple form of $I = I(\Omega_{\psi}, \Psi)$ along a typical field line in $0 \leq \Psi \leq \Psi$ in the force-free magnetosphere;

\[
\begin{align*}
I(\Omega_{\psi}, \Psi) = & \begin{cases}
0 & \text{if } (\Omega_{\psi})_{G} \leq \Delta \omega,
\end{cases} \\
& \begin{cases}
I_{(out)}(\Delta \omega \leq \Omega_{\psi} \leq \Omega_{\psi}(\Delta \omega)),
\end{cases} \\
& \begin{cases}
I_{(in)}(\Omega_{\psi}(\Delta \omega) \leq \Omega_{\psi} < -\Omega_{\psi}(\Delta \omega)),
\end{cases} \\
& \begin{cases}
0 & \text{if } \Omega_{\psi}(\Delta \omega) \leq \Omega_{\psi} \leq \Omega_{\psi}(\Delta \omega),
\end{cases}
\end{align*}
\]

(9.3)

similarly to the pulsar case (see equations (2.3), (2.2); Figure 2). The behavior of $I(\Omega_{\psi}, \Psi)$ in the outer domain $D_{(out)}$ is more or less similar to that of the force-free pulsar magnetosphere. Note that there is a jump from $I_{(out)}$ to $I_{(in)}$ beyond $(\Omega)_{N} = 0$. Because “the particle-production mechanism described in Blandford & Znajek (1977) must operate between the two light surfaces” (see Znajek 1977; the $\mathcal{C}_p(iv,v)$-statements), the Gap $\mathcal{G}_N$ must exist in between $S_{N}$ and $S_{D}$ (see equation (6.27)), and also the particle source will have to be situated well within the Gap $\mathcal{G}_N$, and from equation (6.25) we have

\[
|\Omega_{\psi}| \leq \Delta \omega < c(\alpha/\gamma)\omega \approx c(\alpha/\gamma)\omega. \quad (9.4)
\]

It is not clear how helpful or rather indispensable is the above condition, in constructing a reasonable gap model, but we suppose that the particle production will eventually take place by the voltage drop across the Gap $\mathcal{G}_N$, $\Delta V = -[\Omega_{\psi}(2/3c)\Delta \Psi$, almost independently of the presence of the light surfaces in wind theory.

The outer half of the Gap is thought to play a role like a normal NS, while the inner half is done to play a role like an abnormal NS. Each half of hypothetical magnetized spinning NSs is stuck together, back to back, reverse to each other. In the force-free limit, the dual EMFs, $E_{(out)}$ and $E_{(in)}$, make a rotational-tangential discontinuity with the voltage drop $\Delta V$ (see equation (8.12); section 11.2). This voltage drop will lead to a new type of pair-production mechanism quite different from the one described in Blandford & Znajek (1977), and make the thin $S_{N} \rightarrow$ a gap $\mathcal{G}_N$ with a finite $\Delta \Psi$ or $\Delta \omega$. Then the outer half of the Gap will launch the outgoing magneto-centrifugal wind with $\nu = j/\Omega_{e} > 0$, the angular momentum flux $S_{j, (out)} > 0$ and the Poynting flux $S_{EM, (out)} = \Omega_{\psi}S_{j, (out)} > 0$, while the counter-rotating inner half will launch the ingoing magneto-centrifugal wind with $\nu = j/\Omega_{e} < 0$, the angular momentum flux $S_{j, (in)} < 0$ and the Poynting flux $S_{EM, (in)} = \Omega_{\psi}S_{j, (in)} = (\omega - \Omega_{e})S_{j, (in)} < 0$. The inward ‘negative’ angular momentum flux $S_{j, (in)} < 0$ is equivalent to the outward ‘positive’ angular momentum flux $S_{j, (in)} > 0$, and quite similarly, the inflow of negative energy is equivalent to the outflow of positive energy (see Figure 2). It must be remarked that when the boundary condition of $\Omega_{e} = \Omega_{NS}$ is used at $S_{NS}$ for the pulsar force-free magnetosphere, the behavior of $I = I(\ell, \Psi)$ in the vicinity of $\ell \approx \ell_{NS}$ in equation (2.3) seems to be ill-understood so far, and the same is true here, so that the above treatment of $I = I(\Omega_{\psi}, \Psi)$ in the vicinity of $\Omega_{\psi} \approx \pm \Delta \omega$ will be allowable (see the caption of Figure 2).
9.2 A pair of batteries installed at the Gap surfaces

The two force-free domains $D_{(out)}$ and $D_{(in)}$ will adjoin the magnetized ZAM-Gap filled with ZAM-particles at the surfaces (say) $S_{G(out)}$ and $S_{G(in)}$ with $\Omega_{Fi,\omega} \approx \pm \Delta \omega$, where the unipolar induction batteries are installed with EMFs $E_{(out)}$ and $E_{(in)}$, respectively (see equation (8.11a,b)). The situation is somewhat similar to the force-free pulsar magnetosphere attached to a magnetized NS (see section 2). A magnetized spinning NS will possess its own unipolar induction battery with an EMF $E_{NS}$ (see equation (2.4)), which drives currents flowing through the force-free magnetosphere with astrophysical loads on the resistive membrane $S_{fin}$ (see equation (2.3)). Then we may suppose likewise that the EMF $E_{(out)}$ at $S_{G(out)}$ supplies electricity to the outer circuit $C_{(out)}$, like a 'normal' NS spinning with $\Omega_F = \omega_N$, whereas the EMF $E_{(in)}$ at $S_{G(in)}$ supplies electricity to the inner circuit $C_{(in)}$, like an 'abnormal' NS reversely spinning with $-(\Omega_{Hi} - \Omega_F)$. At the same time, the Poynting flux $S_{EM}$ flows towards both of the directions, related to $E_{(out)}$ and $E_{(in)}$, respectively, i.e., $S_{EM, (out)} = (\Omega_F - \omega) S_{J, (out)} > 0$ to $S_{fin}$ and $S_{EM, (in)} = (\omega - \Omega_F) S_{J, (in)} < 0$ to $S_{fin}$ (see Figure 4). That is to say, $S_{G(out)}$ and $S_{G(in)}$ of the Gap will behave as if they were the surfaces of two hypothetical NSs; the outer one is like a normal NS spinning with $\Omega_{NS} = \Omega_F$, and the inner one is like an 'abnormal' NS counter-spinning with $\Omega_{NS} = -(\Omega_{Hi} - \Omega_F)$. This does not insist that the matter state of the particles pair-produced with the voltage drop $\Delta V = [E]_G$ in the Gap may be the same as that of NSs, and rather will be distinctly different from that, but what we suggest here is just that the role of the two surfaces of the Gap $G_N$, $S_{G(out)}$ and $S_{G(in)}$, will be similar to those of two surfaces $S_{NS}$ of NSs oppositely spinning with each other (see the twin-pulsar model in section 11). Also, the BH circuit theory consists of a superposition of an infinite number of a pair of unipolar induction batteries connected by a pair of field lines and, in addition, a corresponding pair of external resistances connected by current-streamlines to respective batteries (see section 8.2).

9.3 Pinning-down of threading field lines on ZAM-Ps and magnetization of the Gap

When we regard the Gap surfaces $S_{G(out)}$ and $S_{G(in)}$ as being equipped with EMFs $E_{(out)}$ and $E_{(in)}$, respectively, these EMFs not only drive currents in the respective circuits $C_{(out)}$ and $C_{(in)}$, but also produce an extreme voltage drop $\Delta V = [-E]_G$ across the Gap, which will produce plenty of ZAM-particles necessary to pin the field lines down on. The ZAM-Gap filled with ZAM-particles will then circulate the hole with $\omega_N$, and the poloidal field lines threading the Gap $G_N$ will surely be pinned down on ZAM-particles, circulating with $\Omega_F = \omega_N$. The ZAM-Gap will thus be in the perfectly magnetized state, without going with the toroidal component $B_{(t)} = (I) = (J) = 0$. To keep the force-free magnetosphere active, the two EMFs $E_{(out)}$ and $E_{(in)}$ must supply enough currents to the two circuits $C_{(out)}$ and $C_{(in)}$, connected to each other by the field lines $\Psi_1$ and $\Psi_2$. 

Fig. 3. A schematic picture illustrating a pair of circuits $C_{(out)}$ and $C_{(in)}$ closed in the force-free domains $D_{(out)}$ and $D_{(in)}$, which are separated by the non-force-free Gap $G_N$ in $|\Delta \omega| \leq \Delta \omega$, where $\Omega_{Fi,\omega} = (E_F) = (S_{in}) = 0$ and $\psi_{(in)} = (J) = (I) = (S) = 0$ (see equation (9.1a,b)). There will be the dual unipolar inductors with EMFs $E_{(out)}$ and $E_{(in)}$ at work with the spin axes of two hypothetical magnetic rotators oppositely directed. The angular velocities of the axes are $\Omega_F$ and $-(\Omega_{Hi} - \Omega_F)$, respectively, and the difference is $\Omega_{Hi}$ (see equation (3.26c); Figures 2, 4 and 5). Note that $\psi_{(in)} = J_p / \Omega_F > 0$ in $D_{(out)}$, and $< 0$ in $D_{(in)}$. There will be a huge voltage drop of $\Delta V \propto \Omega_{Hi}$ (see equation (8.12)), which corresponds to the difference of the two hypothetical rotators, and will lead to viable particle production of ample plasma particles towards the development of a thick Gap with the half-width $\Delta \omega$. The particles with $\psi_{(in)} = 0$, circulating around the hole’s axis with $\omega_N$, are ZAM-Ps, dense enough to pin down magnetic field lines, to fix $\Omega_F = \omega_N$ and make the Gap magnetized, thereby enabling the dual batteries to drive currents in each circuit (see Figures 4 and figure 4 in Okamoto 2015a).
production mechanism come from the counter-rotation of the models different from any existing models based on pulsar outer-gap a pair of batteries and a strong voltage drop is fundamentally

It is actually across the Gap that the ‘boundary condition’ of no jump of the angular momentum transport rate, i.e., \( [J]_G = 0 \), will determine the eigenfunction \( \Omega_F \) (see section 10).

9.4 Plasma-shed on the ZAM-surface \( S_{ZAM} \)

Just as ZAMOs literally are ‘zero-angular-momentum’ observers, plasma particles pair-created by the voltage drop due to discontinuity \( \Delta V = -[E]_G \) within the Gap \( G_N \) (see equation (8.12)) as well will be ZAM-particles, circulating with ZAMOs at \( \omega = \omega_N = \Omega_F \), but may not behave as force-free particles with negligible inertia in the Gap. These particles will soon become 

charge-separated owing to zero angular momentum (and also \( (v)_G = 0 \), to flow from the Gap out to the force-free domains as ‘force-free’ and charge-separated, with \( v_p > 0 \) in \( D_{(out)} \) and \( v_p < 0 \) in \( D_{(in)} \) (see Figures 3 and 4). These ZAM-particles pair-produced will easily be blown out of the Gap by the magneto-centrifugal forces at work towards both directions as follows: The null surface \( S_N = S_{ZAM} \) within the ZAM-Gap \( G_N \) will then redefine quite a new general-relativistic type of divider due to magneto-centrifugal force modified by the FD effect \( (\Omega_F, \bar{z} \bar{z})_0 \), to outward and inward, for particles pair-created in the ZAM-Gap. That is to say, ‘this surface’ will behave like a magneto-centrifugal plasma-shed, similarly (and dissimilarly) to a gravitational watershed of a mountain ridge on the Earth. Just as for the Poynting flux \( S_{EM} = \Omega_F c_o S_J \bar{z} \bar{z} \), this will be quite a natural way to launch the winds from the ZAM-Gap for both directions to infinity and the horizon (see item 4, section 7).

The models so far used for pair-production discharges are based mainly on an extension from a ‘negligible violation’ of the force-free condition at this surface (see the C$_\delta$-statements and section 4.3; Phinney 1983a; Beskin et al. 1992; Hirotani & Okamoto 1998; Song et al. 2017; Hirotani et al. 2018; etc.) It is argued here that the ‘perfect violation’ of the force-free condition due to the FD effect at ‘this surface’ leads to a unique gap model in the present modified BZ process. It was stated already (Okamoto 2015a) that “The present gap model with a pair of batteries and a strong voltage drop is fundamentally different from any existing models based on pulsar outer-gap models.” The significant differences from the previous particle production mechanism come from the counter-rotation of the inner domain giving rise to the voltage drop \( \Delta V = -[E]_G \) between the two domains.

10 The Eigen-magnetosphere

For a viable force-free magnetosphere, we have referred to the condition by which to finally determine the eigenfunction \( \Omega_F(\Psi) \) for the potential gradient and the location of it as the ‘boundary condition,’ distinguishing from the ‘criticality condition’ for another eigenfunction \( I(\Psi) \). The situation seems to be much more complicated than for the force-free pulsar magnetosphere, which consists of magnetic fluxes emanating from the rotating magnetized NS. For the latter, it is usually taken for granted that \( \Omega_F \) is given at the NS surface \( S_{NS} \) as the surface angular velocity \( \Omega_{NS} \), and hence the ‘\( \Omega_F \) problem’ will not be an eigenfunction problem (see section 2). For the Kerr hole’s force-free magnetosphere, on the other hand, if we had to consider that the force-free condition does not break down anywhere in the force-free magnetosphere, we could reach no unique procedure on where and how \( \Omega_F \) should be determined (see section 4).

10.1 The ‘boundary condition’ for the eigenfunction \( \Omega_F \)

One of vital roles of the ZAM-Gap is to pin the poloidal field \( B_p \) down onto the ZAM-particles pair-created in there, and to accomplish magnetization of the ZAM-Gap, thereby ensuring \( \Omega_F = \omega_N \) for threading field lines. Then, the ZAM-state of the Gap will be maintained, with the magnetosphere circulating with \( \omega_N \) dragged by the hole. But the actual position of the Gap and the specific value \( \Omega_F = \omega_N \) per se remain still undetermined. In order to determine the final eigenfunction \( \Omega_F(\Psi) = \omega_N \) in terms of \( \Omega_F \), we formulate the ‘boundary condition’ rightly, with Constraints (7.1) or (9.1) appropriately taken into account, in particular, \( (I)_G = [I]_G = 0 \) at the place of the ZAM-Gap.

When the Gap stays in the ZAM-state with Constraints \( (S)_{(in)} = (I)_{(in)} = 0 \) in \( |\Omega_F| \omega \leq \Delta \omega \), we may conjecture that the Poynting fluxes \( S_{EM_{(out)}} > 0 \) and \( S_{EM_{(in)}} < 0 \) will be launched from the plasma-shed, both toward to the resistive membranes \( S_{EM_0} \) and \( S_{EM_1} \), together with \( S_{J_{(out)}} > 0 \) and \( S_{J_{(in)}} < 0 \), under the condition that \( S_{J_{(out)}} + S_{J_{(in)}} = 0 \), or equivalently \( S_{J_{(out)}} = S_{J_{(in)}} \). This ensures the continuity of angular momentum flow across the Gap, thereby keeping the ZAM-state.

Thus, under the Major Premise of \( (B_p)_G \neq 0 \) and \( [B_p]_G = 0 \), we may impose no discontinuity between \( I_{(in)} = -I_{(in)}^0 \) and \( I_{(out)} \) across \( G_N \) in the flux-base along each field line:

\[
[I]_G = I_{(out)}(\Psi) - I_{(in)}(\Psi) = I_{(out)}(\Psi) + I_{(in)}^0(\Psi) = 0
\]

(10.1a)

(10.1b)

(see Figure 2), and then by equations (3.8) we have

\[
[S_{EM}]_G = \Omega_F[S_J]_G = 0
\]

(10.2)

for the energy and angular momentum fluxes as well. Likewise, we have for the loss rate of angular momentum by equation (3.9a)

\[
[P_J]_G = P_{J_{(out)}} - P_{J_{(in)}} = P_{J_{(out)}} + P_{J_{(in)}}^0 = 0
\]

(10.3a)

(10.3b)

where
\[ \mathcal{P}_{E,\text{out}} = \frac{1}{2c} \int_{\Psi_0}^{\Psi} \Omega E (B_p \sigma^2) \text{d}\Psi, \quad (10.4a) \]

\[ \mathcal{P}_{E,\text{in}} = \frac{1}{2c} \int_{\Psi_0}^{\Psi} (\Omega - \Omega E) (B_p \sigma^2) \text{d}\Psi = -\mathcal{P}_{E,\text{in}}. \quad (10.4b) \]

The second expression in (10.1b) implies that the outward rate of positive angular momentum at \( S_{\text{G(out)}} \) is offset by the inward rate of negative angular momentum at \( S_{\text{G(in)}} \). At the same time, the first in (10.1a) shows equivalently that the outward transport rate of positive angular momentum at \( S_{\text{G(out)}} \) must be equal to that leaving outward at \( S_{\text{G(out)}} \), although the energy-angular momentum flow does not take place actually inside the ZAM-Gap.

The boundary condition in (10.1) ensures no discontinuity of the power \( \mathcal{P}_E \) as well across the ZAM-Gap, i.e., by equations (3.9b), (3.13) and (8.1)

\[ [\mathcal{P}_E]_G = \mathcal{P}_{E,\text{out}} - \mathcal{P}_{E,\text{in}} = \mathcal{P}_{E,\text{out}} + \mathcal{P}_E = 0, \quad \text{(10.5a)} \]

\[ [\mathcal{P}_E]_G = \frac{1}{c} \int_{\Psi_0}^{\Psi} \Omega E I_{\text{out}} \text{d}\Psi = \frac{1}{c} \int_{\Psi_0}^{\Psi} \Omega E I_{\text{in}} \text{d}\Psi, \quad \text{(10.5b)} \]

where

\[ \mathcal{P}_{E,\text{out}} = \frac{1}{c} \int_{\Psi_0}^{\Psi} \Omega E I_{\text{out}} \text{d}\Psi \quad \text{(10.6a)} \]

\[ \mathcal{P}_{E,\text{in}} = \frac{1}{c} \int_{\Psi_0}^{\Psi} \Omega E I_{\text{in}} \text{d}\Psi, \quad \mathcal{P}_{E,\text{in}} = \frac{1}{c} \int_{\Psi_0}^{\Psi} \Omega E I_{\text{in}} \text{d}\Psi \quad \text{(10.6b)} \]

similarly to \([\mathcal{P}_F]_G = 0 \) for the loss rate of angular momentum (see section 12.1 for relation \([\mathcal{P}_F]_G = -\mathcal{P}_{E,\text{in}} \)).

It is, in reality, the ZAM-state of the Gap that makes it possible to use expression (10.1) as the ‘boundary condition’ for \( \Omega E \) even in the finite value of \( \Delta \omega \) (see section 9), and, conversely, the ‘boundary condition’ (10.1) is necessary to ensure the ZAM-state of the Gap.

### 10.2 The final eigenfunctions \( I(\Psi) \) and \( \Omega E(\Psi) \) in the force-free magnetosphere

From equations (8.1), (8.5) and (10.1) we have

\[ \Omega E(\Psi) = \omega N = \frac{\Omega H}{1 + \xi}, \quad (10.7a) \]

\[ I = I_{\text{out}} = I_{\text{in}} = -I_{\text{in}} = \frac{\Omega H}{2(1 + \xi)} (B_p \sigma^2) \text{d}H, \quad (10.7b) \]

\[ \xi(\Psi) = (B_p \sigma^2) \text{d}H = (B_p \sigma^2) \text{d}H \quad (10.7c) \]

(Okamoto 2009; Okamoto 2012a; Okamoto 2015a). Note that \((B_p)_G \neq 0, (\Omega)_G \neq 0, (B_p)_G = (\Omega E)_G = 0 \) in our Major Premises (section 5), and also \((I)_G = |I|_G = 0 \) across the supposed Gap \( \mathcal{G} \) in \([\Omega E]_G \leq \Delta \omega \) with \([\Omega E]_G = 2\Delta \omega \) (see Figure 2).

Constraints \((f)_G = (B)_G = (I)_G = (r)_G = 0 \) in equation (9.1) imply that no transport of angular momentum and energy by the field, the current and particles is possible within the ZAM-Gap, i.e., \((S)_G = (S)_G = 0 \). These indicate a disconnection of current- and stream-lines between the two force-free domains, and hence the necessity of the current-particle sources and related EMFs in the Gap. It will be ensured in equation (10.1) that the copious charged ZAM-particles pair-produced in \([\Omega E]_G \leq \Delta \omega \) serve to connect and equate both \( J_{\text{out}} \) and \( J_{\text{in}} \) across the Gap \( \mathcal{G} \), i.e., \([\mathcal{P}_E]_G = \mathcal{P}_{E,\text{in}} = 0 \) and also \([\Omega E]_G = 0 \) from equations (3.9a,b).

The eigen-efficiency of gravito-thermo-electrodynamic extraction is given from equations (3.20) and (10.7a) by

\[ \epsilon_{\text{GTED}} = \frac{\Omega E}{\Omega H} = \frac{1}{1 + \xi}. \quad (10.8) \]

When the plausible field configuration allows us to put \( \xi \approx 1 \) and hence \( \epsilon_{\text{GTED}} \approx 0.5 \), we have from equations (10.7a) and (3.24),

\[ \Omega E = \omega N = \frac{1}{2} \Omega H, \quad (10.9a) \]

\[ c^2 |dM| \approx \mathcal{S}_G \approx \frac{1}{2} \Omega H |dJ|. \quad (10.9b) \]

The average value \( \Omega E \approx \omega N \), together with \( \xi \), in the eigen-state will be given from equations (10.7a,b,c), and also the average null surface \( \mathcal{S}_N \) and the overall efficiency \( \epsilon_{\text{GTED}} \) will be given from equation (10.8).

### 11 A twin-pulsar magnetosphere model

#### 11.1 A dual magnetosphere

When we calculate the Faraday path integral of \( E_p \) along rightly chosen closed-current circuits \( C_{\text{out}} \) and \( C_{\text{in}} \) with a cut at ‘this surface’ between the two circuits, this integration will certainly yield a needed pair of batteries and their \( \mathcal{E}_{\text{out}} \) and \( \mathcal{E}_{\text{in}} \) at the right position. A bit puzzling thing here is that the ‘continuous’ variation of \( \Omega_{E,\text{out}}(\omega, \Psi) \) and \( E_p \) as a function of \( \omega = \omega(\xi, \Psi) \) along with two circuits \( C_{\text{out}} \) and \( C_{\text{in}} \) has produced a ‘discontinuous voltage drop’ \( \Delta V = -[\mathcal{E}]_N \propto \Omega H \) across \( \mathcal{S}_N \) between the two EMFs for the two circuits. Also, this situation seems to correspond to two hypothetical magnetic rotators for each domain (see Figure 3). The outer domain \( \mathcal{D}_{\text{out}} \) will then behave like a normal pulsar magnetosphere around a hypothetical NS spinning with \( \Omega_E = \omega N \), and the inner domain \( \mathcal{D}_{\text{in}} \) will do like an abnormal pulsar magnetosphere around another hypothetical NS counter-spinning with \( -[\mathcal{E}]_N \propto \Omega H \). The difference of spin rates in the twin-pulsar model (section 11.3) is \( \Omega_E - [\mathcal{E}]_N \propto \Omega H \), corresponding to the voltage drop \( \Delta V = -[\mathcal{E}]_N \propto \Omega H \) (see equation (3.26)).

The point is that the Kerr hole does not seem to mind about ‘continuous’ or ‘discontinuous’ accommodation of the spin-down energy to the Poynting fluxes in the self-extraction of energy, as long as the ‘actual’ energy flux \( S_E = \Omega H S_1 \) is conserved across the ZAM-Gap along each current-field-streamlines. This is because the conserved energy flux \( S_E = \Omega H S_1 \) alone flows
outward in the pulsar-type force-free magnetosphere. In contrast, in the hole’s force-free magnetosphere the conserved energy flux \( S_E \) is split into the two nonconserved fluxes \( S_{EM} \) and \( S_{SD} \), to comply with the first law of thermodynamics, but these energy flows run along the same equipotential current-field-streamlines, so that the Kerr hole cannot discriminate between the sum of \( S_{EM} = -(\omega - \Omega_F)S_J \) and \( S_{SD} = \omega S_J \) and that of \( \nabla S_{EM} = -(\Omega_H - \Omega_F)S_J \) and \( \nabla S_{SD} = \Omega_H S_J \) in the inner GR domain and \( \nabla S_{EM} = \nabla S_{SD} = 0 \) in the outer SC domain (see equations (11.1), (11.2) and (11.5)).

This suggests that the null surface \( S_N = S_{ZAM} \) will be regarded as a kind of rotational-tangential discontinuity as discussed in the following, although it is pretty different from any of the ordinary MHD discontinuities (Landau et al. 1984; Okamoto 2015a).

11.2 Rotational-tangential discontinuity at the null ZAM-surface

We discuss a fundamental feature of ‘this surface’ \( S_N \) in the force-free limit, where the force-free and freezing-in conditions must break down inevitably, to distinguish the GR domain from the SC one. Calculation of Faraday path integrals of \( E_p \) in equation (8.11a,b) along the two circuits \( C_{out} \) and \( C_{in} \) reveals the sharp potential drop \( \Delta V \) between the EMFs for the two circuits, as if \( \omega \) were such a step function \( \overline{\omega} \) as

\[
\overline{\omega} = \begin{cases} 0 & ; D_{out}(\Omega_{F,\omega} > 0), \\ \Omega_F & ; S_N (\Omega_{F,\omega} = 0), \\ \Omega_H & ; D_{in}(\Omega_{F,\omega} < 0). \end{cases}
\]

This means that \( \omega = 0 \) in the null surface \( S_N \) and \( = \Omega_F \) in the inner GR domain. Note \( \overline{\omega}_N = \overline{\omega}_F \).

Likewise \( \Omega_{F,\omega} \), \( v_F \) and \( e_j \) are also replaced by the following step-functions, i.e., \( \overline{\Omega}_{F,\omega} \equiv \overline{\omega} - \overline{\Omega}_F \).

\[
\overline{\Omega}_{F,\omega} = \begin{cases} \Omega_F & = \overline{\Omega}_{F,\omega}(out) ; D_{out}(\Omega_{F,\omega} > 0), \\ 0 & = \overline{\Omega}_{F,\omega}(N) ; S_N (\Omega_{F,\omega} = 0), \\ -(\Omega_H - \Omega_F) & = \overline{\Omega}_{F,\omega}(in) ; D_{in}(\Omega_{F,\omega} < 0). \end{cases}
\]

(11.2)

\( \overline{\Gamma} = \overline{\Omega}_{F,\omega} \sigma/\alpha \) and \( \overline{\tau}_j = \overline{\Gamma}_F (\sigma B_j^2/c) \) (see equation (9.3) and Figure 2 for \( I(\Omega_{F,\omega}, \Psi) \)). Then, we have the same discontinuity for \( \overline{\Omega}_{F,\omega} \) at \( S_N \)

\[
[\overline{\Omega}_{F,\omega}]= \overline{\Omega}_{F,\omega}(out) - \overline{\Omega}_{F,\omega}(in) = \Omega_H
\]

(see equation (3.26a)). The related electric field \( \overline{E}_F \) and its discontinuity at \( S_N \) become

\[
\overline{E}_F = \frac{\overline{\Omega}_{F,\omega}}{2\pi \alpha c} \nabla \Psi, \quad \text{and}
\]

\[
[\overline{E}_F] = -\frac{\overline{\Omega}_{F,\omega}}{2\pi c} \frac{\nabla \Psi}{\alpha}(N) = \frac{\Omega_H}{2\pi c} \frac{\nabla \Psi}{\alpha}(N). \]

(11.4b)
Expression for $\mathbf{F}_{\Omega}$ naturally reproduces the same results for $E_{(\text{out})}$ and $E_{(\text{in})}$ in Faraday path integrals of $\mathbf{E}_P$ along the circuits $C_{\text{out}}$ and $C_{\text{in}}$, respectively, as given in expressions (8.11a,b).

When $\Omega$ and $\Delta \Omega_{\text{rot}}$ are replaced by step-functions $\dot{\omega}$ and $\Delta \Omega_{\text{rot}}$, the related energy fluxes $S_{\text{EM}}$ and $S_{\text{SD}}$ are also replaced with step-functions $\tilde{S}_{\text{EM}}$ and $\tilde{S}_{\text{SD}}$, i.e.,

$$\tilde{S}_{\text{EM}} = \bar{\omega}_{\text{rot}} S_1, \quad \tilde{S}_{\text{SD}} = \dot{\omega} S_1 \quad (11.5)$$

(see Figure 5; Okamoto 2015a). There is, of course, no discontinuity in the ‘total’ energy and angular momentum flux across the null surface $S_N$ with $[\mathbf{B}_P]_N = 0$, i.e.,

$$[S_E]_N = \tilde{S}_{\text{EM}} + \tilde{S}_{\text{SD}}]_N = \Omega_F [S_1]_N = 0, \quad (11.6)$$

which is similar to $[I]_N = 0$ from the boundary condition for $\Delta V$, although there are the jumps proportional to $[\bar{\Omega}_{\text{rot}}]_N = \Omega_0$ in the EMFs, i.e., $[E]_N = -\Delta V$ in equation (8.12) and non-conserved energy fluxes $\tilde{S}_{\text{EM}}$ and $\tilde{S}_{\text{SD}}$ in equations (11.7a,b).

We compute the differences of the Poynting flux $\tilde{S}_{\text{EM}}$ and the spin-down flux $\tilde{S}_{\text{SD}}$ across $S_N$ from equations (11.3) and (11.5);

$$\tilde{S}_{\text{EM}} | N = \tilde{S}_{\text{EM}}(N) - \tilde{S}_{\text{EM}}(C) = \bar{\omega}_{\text{rot}} S_1 = \Omega_F S_1, \quad (11.7a)$$

$$\tilde{S}_{\text{SD}} | N = -\tilde{S}_{\text{SD}}(N) - \tilde{S}_{\text{EM}}(C) = -\Omega_0 S_1, \quad (11.7b)$$

which combine to yield $\tilde{S}_{\text{SD}} | N = -[\tilde{S}_{\text{EM}} | N]$, or

$$\tilde{S}_{\text{SD},(n)} = -\tilde{S}_{\text{EM},(n)} + \tilde{S}_{\text{EM},(o)} \quad (11.8)$$

where $[\tilde{S}_{\text{EM}} | N] = [\tilde{S}_{\text{EM}}(C)] = 0$. Integration of this over all open fieldlines from $\Psi_0$ to $\Psi$ yields equation (3.24a), which is equal to the first law in (12.1).

When Constraints in (7.1b), i.e., $[S_E]_N = [S_1]_N = 0$ hold, relations $[S_E]_N = [S_1]_N = 0$ also hold at ‘this surface’, which shows that the energy-angular momentum flux will be conserved across the ZAM-surface with the rotational-tangential discontinuity. Therefore, the same relation $S_E = \Omega_0 S_1$ is useful both in the Kerr hole force-free magnetosphere and in the pulsar force-free magnetosphere. The ZAMOs will see that ‘this surface always exists’ and hence the force-free and freezing-in conditions always break down at ‘this surface’ (see the C-iv-statement; cf. the C-ii-statement). Thus, the basic properties of the energy fluxes in the curved space with $\omega$ and $\Omega_{\text{rot}},(\omega, \Psi)$ will be fully reproduced in the pseudo-flat space with $\mathbf{F}$ and $\Omega_{\text{rot}}(\mathbf{F}, \Psi)$.

11.3 The twin-pulsar magnetosphere

Probably a kind of inevitable relaxation of rotational-tangential discontinuity due to, e.g., the particle production by the voltage drop $\Delta V$ will take place, leading to widening from the ZAM-surface $S_{\text{ZAM}}$ to a ZAM-Gap $G_N$ with a finite thickness. This Gap may be regarded as effectively consisting of a hybrid of two halves of hypothetical magnetized NSS; the outer one spinning prograde with $\Omega_{\text{rot},(\omega)} = \Omega_F$ and the inner retrograde with $\Omega_{\text{rot},(\omega)} = -|\Omega_H - \Omega_F|$, reversely packed together, and threaded by the poloidal field $\mathbf{B}_P \neq 0$ with no toroidal component ($\mathbf{B}_T = (I)_G = 0$), and yet with $\mathbf{B}_P \parallel \mathbf{F}$ pinned down in the ZAM-particles pair-produced in the Gap.

The above conjecture is depicted schematically in Figures 4 and 5; the pulsar-type wind is slung outward from the outer magnetic hypothetical rotator spinning with $\Omega_{\text{rot}}(\text{out}) = \Omega_f$ through the outer domain $D_{\text{out}}$, in which the related Poynting flux $\tilde{S}_{\text{EM}}$ is equal to $\Omega_0 S_1(\text{out}) > 0$ with no frame-dragging spin-down energy flux followed. In contrast, the anti-pulsar-type wind is reversely slung inward from the inner magnetic hypothetical rotator counter-spinning with $\Omega_{\text{rot}}(\text{in}) = -|\Omega_H - \Omega_F|$ through the inner domain $D_{\text{in}}$, in which the positive-valued Poynting flux is directed inward, i.e., $S_{\text{EM}}^{(n)} = (\Omega_H - \Omega_F) S_1^{(n)} < 0$, while the negative-valued frame-dragging spin-down flux is directed inward as well, and hence $\tilde{S}_{\text{SD},(n)} = -\Omega_0 S_1^{(n)} > 0$, which may be understood as equivalent to an in-flow of the negative energy, i.e., $\tilde{S}_{\text{EM}}^{(n)} = \Omega_0 S_1^{(n)} = -\tilde{S}_{\text{SD},(n)} < 0$, related to the in-flow of negative angular momentum. Then the ‘total’ energy flux becomes

$$S_{E,(n)} = -\tilde{S}_{\text{EM}}^{(n)} + \tilde{S}_{\text{SD},(n)} = -\Omega_0 S_1^{(n)} = \Omega_0 S_1(\text{in}), \quad (11.9)$$

which must be equal to $S_{E,(\text{out})} = \Omega_F S_1(\text{out})$. To justify the ‘boundary condition’ $S_E|_G = \Omega_0 |S_1|_G = 0$ across the Gap (see equations (10.2); section 12.1).

This new type of rotational-tangential discontinuity in the general-relativistic setting is distinctly different from ordinary tangential or rotational discontinuities in classical magnetohydrodynamics, in the point that relaxation of it due to particle production will lead to widening (see, e.g., Sec. 70, Landau et al. 1984). Also, the present gap model with a pair of batteries and a substantial voltage drop is fundamentally different from any existing pulsar outer-gap models deduced from the C-iv-(v,i)-statements in section 4.2 for a charge-starved magnetosphere (Okamoto 2015a).

It seems that “a mechanism directly analogous to Goldreich & Julian (1969)” (see the C-iii-statement) is undoubtedly applicable to the outer domain $D_{\text{out}}$, prograde-rotating, but it is “a mechanism anti-analogous to Goldreich & Julian (1969)” that is also applicable to the inner domain $D_{\text{in}}$, retrograde-rotating. It is the ZAM-Gap $G_N$ covered by the inductive membrane $S_N$ where the force-free condition breaks down, i.e., $(I)_G = 0$ in $|\Omega_{\text{rot}}| \leq \Delta \omega$ that allows us to partition the force-free magnetosphere off into the two domains with oppositely directed winds and Poynting fluxes, i.e., $\Psi \neq 0$ and $S_{\text{EM}} \neq 0$ for $\Omega_{\text{rot}} \neq 0$. This will be possible only when the inner domain is counter-rotating to the outer, thanks to the FD effect. Then, one of the crucial questions remaining will be how to determine the Gap width $\Delta$ or $\Delta \omega$ in terms of $\Omega_0$, the magnetic flux threading the Gap $G_N$, the details of pair-production, etc.
28 Discussion and conclusion

12.1 Energetics of the hole’s self-extraction of energy

The first law in equation (3.24) seems to show that the energy extracted through the spin-down energy flux will be shared at the inductive membrane \( S_N \) between the two Poynting fluxes toward the two resistive membranes \( S_{ffH} \) and \( S_{ffL} \).

Integrating equation (11.8) over all open field lines from \( \Psi_0 \) to \( \Psi \) yields

\[
\int_{S_{ffH}} aS_{SD, \text{(in)}} \cdot dA = - \int_{S_{ffH}} aS_{EM, \text{(in)}} \cdot dA + \int_{S_{ffL}} aS_{EM, \text{(out)}} \cdot dA,
\]

which seems to explain that the power \( \Omega_{ffH} p_j \) self-extracted from the horizon is shared between \( (\Omega_{ffH} p_j - \Omega_{ffL}) \equiv \mathcal{P}_{EM, \text{(in)}} \) dissipated in the resistive membrane \( S_{ffH} \) (i.e., the stretched horizon) and \( \mathcal{P}_{ffL} \equiv \mathcal{P}_{EM, \text{(out)}} \) in another resistive membrane \( S_{ffL} \) (see equation (3.24)), where, by equation (3.24b),

\[
\Omega_{ffH} p_j = \mathcal{P}_{EM, \text{(in)}} + \mathcal{P}_{EM, \text{(out)}},
\]

which explains that “the power dissipated in the horizon and that dissipated in particle acceleration in the far field” (see the Footnotes of [30]); Blandford 1979; Macdonald & Thorne 1982, § 7.3; Thorne et al. 1986, Ch. IV D; section 4.3).

Then, the two terms of the right-hand side become, by equations (8.1), (3.13), (3.22), and (10.6a,b),

\[
\frac{dS}{dt} = \mathcal{P}_{EM, \text{(in)}} = \frac{1}{2c} \int_{S_{ffH}} (\Omega_{ffH} - \Omega_{ffL})^2 (B_p \sigma^2)_{ffH} d\Psi,
\]

and

\[
-\frac{c^2}{dM} = \mathcal{P}_{EM, \text{(out)}} = \frac{1}{2c} \int_{S_{ffL}} \Omega_{ffL}^2 (B_p \sigma^2)_{ffL} d\Psi.
\]

Summing up equations (12.3) and (12.4a) with the use of the ‘boundary condition’ \( I_{out} = I_{in} \) yields \( -\Omega_{ffH} (dJ/dt) = \Omega_{ffL} p_j \).

It seems undoubtedly plausible that positive angular momentum and energy extracted by the surface magnetic torque through \( S_{ffH} \) from the hole will be transported beyond the inner domain \( D_{in} \) and then the ZAM-Gap with \( (J) = (v) = (e) = 0 \), to the outer domain \( D_{out} \) with astrophysical loads in \( S_{ffL} \) (see Figure 2). Actually, the angular momentum flux does not pass through the ZAM-state in the Gap with \( e = 0 \). The point is that ZAM-particles are spinning with \( \omega = \omega_{ZAM} \) dragged by the hole’s rotation, literally with no angular momentum, so that they will be feasible to flow out of the Gap, flung effortly both outwards and inwards from the surfaces \( S_{ffH} \) and \( S_{ffL} \) on the plasma-shed, with positive and negative angular momenta, by the respective magneto-centrifugal forces, to keep the ZAM-state of the Gap. This corresponds to the situation that the outgoing Poynting flux \( S_{EM} > 0 \) is related with the outer EMF \( E_{out} \) and the ingoing Poynting flux \( S_{EM} < 0 \) is so with the inner EMF \( E_{in} \). The spin-down energy extracted through the stretched horizon \( S_{ffH} \) seems to be shared between the out- and in-going Poynting fluxes reaching the two resistive membranes \( S_{ffH} \) and \( S_{ffL} \), respectively, to dissipate in particle acceleration and entropy generation (see equation (12.1)).

The above picture of energy-sharing on the ZAM-Gap is one of possible interpretations of the extraction process. This is in a sense a reversal of the same phenomenon that the hole can be an acceptor of negative angular momentum, thereby spinning down, eventually and equivalently, to self-extracting positive angular momentum and energy from the hole.
As seen above already, the ZAM-Gap \( \mathcal{G}_N \) covered by the inductive membrane \( \mathcal{S}_N \) will be installed with a pair of batteries \( E_{(\text{out})} \) and \( E_{(\text{in})} \) on both upper and lower sides, launching the Poynting fluxes outward and inward, i.e., \( P_{E_{(\text{out})}} = \Omega_l P_{J,\text{(out)}} > 0 \) for particle acceleration on \( \mathcal{S}_{\text{BH}} \) and \( T_{00}(dS/dt) = P_{E_{(\text{in})}} = -(\Omega_l - \Omega_i) P_{J,\text{(in)}} > 0 \) for entropy production on \( \mathcal{S}_{\text{BH}} \). Both of these Poynting fluxes emerging from the ZAM-Gap will carry positive \( (P_{J,\text{(out)}} > 0) \) and negative \( (P_{J,\text{(in)}} < 0) \) angular momentum. Followed by influx of negative angular momentum is influx of negative energy of the FD-induced spin-down energy \( \Omega_l P_{J,\text{(in)}} < 0 \), which is equivalent to outflow of positive spin-down energy \( \Omega_i P_{J,\text{(out)}} > 0 \). The ingoing Poynting flux is equal to influx of positive electromagnetic energy \( P_{\text{EM},\text{(in)}} = (\Omega_l - \Omega_i) P_{J,\text{(in)}} > 0 \) to dissipate to increase the irreducible mass on the resistive membrane \( \mathcal{S}_{\text{BH}} \). The total influx of negative energy is then \( \Omega_l P_{J,\text{(in)}} - (\Omega_l - \Omega_i) P_{J,\text{(in)}} = \Omega_i P_{J,\text{(in)}} = -P_{E_{\text{(in)}}} < 0 \), which is equivalent to the power in total passing outward through the inner domain given by \( \Omega_i P_{J,\text{(in)}} = P_{E_{\text{(in)}}} = -P_{E_{\text{(in)}}} < 0 \). Because the ZAM-Gap \( \mathcal{G}_N \) must always be in the zero-angular momentum state \( (\epsilon_i)_{\Omega_i} = 0 \) in the steady-state, i.e., \( (P_{J})_{\text{G}} = [P_{J}]_{\text{G}} = 0 \), the ZAMOs will see \( [P_{E}]_{\text{G}} = [P_{E}]_{\text{G}} = 0 \) and hence \( P_{E_{\text{(out)}}} = P_{E_{\text{(out)}}} \) (see equations (10.5) and (12.4a,b)). Therefore, the Kerr hole skillfully makes use of the properties of the inner domain \( \mathcal{D}_{\text{(in)}} \) counter-rotating to pass negative angular momentum downward from the ZAM-Gap for the sake of defending the first and second laws, thereby self-extracting energy smartly and successfully. Thus, the powers of the two dissipations in equation (12.2) cannot be due to any battery on the horizon.

12.2 An integral role of the dragging of inertial frames

The dragging of inertial frames becomes an integral part of making pulsar electrodynamics adaptive to black hole thermodynamics. The FD effect is actually committed to every details in energetics and structure of the hole’s force-free magnetosphere. By nature, the FD\( \mathcal{E} \) reduces to the constant value \( \Omega_l \) by the zeroth law, which is the intensive variable conjugate to an extensive variable \( \Omega_l \) in the first law. The coupling of \( \omega \) with \( \Omega_l \) also endows \( \omega \) with a role of the gravito-electric potential gradient. Also, the simplest defmition of deformation of \( \Omega_l \) to \( \Omega_l = \Omega_l + \omega \) and multiplication with \( S_{J} \) yield an energy-angular momentum flux relation \( S_{E} = S_{\text{EM}} + S_{\text{SD}} \), easily complying with the first law \( c^2 dM = T_{00} dS + \Omega_i dJ \). The FD spin-down flux \( S_{\text{SD}} = \omega S_{J} \) connects at the horizon with the term \( \Omega_l dJ/dt \) by \( \omega \to \Omega_l \). While \( S_{J} \) does not reverse direction (except \( S_{J} = 0 \)), the Poynting flux \( P_{\text{EM}} = \Omega_l dS_{J} \) changes sign at this surface, to adapt to the second law at the horizon, i.e., \( T_{00} dS > 0 \) (see equations (3.11a,b) and (3.22)), which ensures existence of ‘this surface’ \( \omega = \Omega_l \) for the electromagnetic self-extraction with \( c^2 dM = \Omega_l dJ < 0 \). The inevitable breakdown of the force-free as well as freezing-in condition takes place at the ZAM-surface \( S_{\text{ZAM}} = S_{N} \), i.e., ‘this surface’, which divdes the force-free magnetospheres into the GR and SC domains (the twin-pulsar model).

The outer SC domain \( D_{\text{(out)}} \) rotates with \( \Omega_{p\omega} > 0 \) and \( \epsilon_j > 0 \), whereas the inner GR domain \( D_{\text{(in)}} \) counter-rotates with \( \Omega_{p\omega} < 0 \) and \( \epsilon_j < 0 \). The Poynting flux flows outward and inward, following \( \Omega_{p\omega} \neq 0 \). The current \( j \) keeps sign along each current-field-streamlines in the force-free domains, but vanishes at the null surface \( S_{N} = S_{\text{ZAM}} \), whereas the charge density changes sign \( \rho_{\omega} \neq 0 \) for \( \Omega_{p\omega} \neq 0 \). This is because the current system cannot constitute a single series circuit owing to sevevation of current lines at \( S_{N} \). Then the velocity of the ‘force-free, massless’ particles also changes signs, i.e., \( v = j / \rho_{\omega} \neq 0 \). Consistently, this means that the magneto-centrifugal force works to both directions outward and inward at the plasma-shed on the ZAM-surface \( S_{\text{ZAM}} = S_{N} \), driving ‘force-free’ winds passing \( S_{\text{d}}, \) and \( S_{\text{f}}, \) respectively.

The ingoing wind from the ZAM-Gap \( \mathcal{G}_N \) carries negative angular momentum \( S_{J,\text{(in)}} < 0 \) and the FD effect induces inward negative spin-down energy flux \( S_{\text{SD}} = \omega S_{J,\text{(in)}} < 0 \), and also the inward positive Poynting flux \( S_{\text{EM},\text{(in)}} = (\Omega_l - \omega) S_{J,\text{(in)}} = (\omega - \Omega_l) S_{J,\text{(in)}} < 0 \). The point is that these two ingoing fluxes \( S_{J,\text{(in)}} < 0 \) and \( S_{\text{SD}} < 0 \) originating from the Gap with \( \epsilon_j = 0 \) are equivalent to the outgoing ones \( S_{J,\text{(in)}} > 0 \) and \( S_{\text{SD},\text{(in)}} > 0 \), while the ingoing Poynting flux carries a positive field energy to ohmic-dissipate for the hole’s entropy increase. The outgoing fluxes \( S_{J,\text{(in)}} \) and \( S_{\text{SD}} \) in reality do not come from under the horizon, but by nature from the ZAM-Gap \( \mathcal{G}_N \) with the pair-particle production due to the voltage drop between \( E_{(\text{out})} \) and \( E_{(\text{in})} \). The ZAM-particles pair-created in the widening Gap will anchor threading field lines to circulate with \( \omega_{\text{N}} \), as well as keeping the particles themselves magnetized, to ensure \( \Omega_{p\omega} = \omega_{\text{N}} \). One of vital roles of the FD effect will help the magnetized Gap \( \mathcal{G}_N \) kept in the ZAM-state, thereby the whole magnetosphere frame-dragged into circulation around the hole, and yet making electromagnetic self-extraction of energy from the hole.

The ZAM-Gap \( \mathcal{G}_N \) may thus be one of strong candidates for the power sources of AGN activities. It may then be said that the whole process of self-extraction of energy from the Kerr black hole takes place under supervision of the ZAMOs.

12.3 Concluding remarks

In this paper, we have attempted to unify pulsar electrodynamics to black-hole thermodynamics into black-hole gravito-thermo-electrodynamics, by coupling the dragging of inertial frames with unipolar induction. Fundamental concepts and expressions as well as the basic formulation of the 3 + 1-formulation had been given in the almost complete form by Blandford & Znajek (1977); Phinney (1983b); Macdonald & Thorne (1982); Thorne et al. (1986), more than three decades ago. The theory for GTED
must naturally contain the correct procedure of combining pul-
sar electrodynamics with additional physics needed in the BZ
process (Punsly & Coroniti 1990), i.e., thermodynamics inclusive
of 'physics on this surface'. It is the dragging of inertial
frames, despite the presence of the event horizon, that enables
the Kerr hole to manipulate its magnetosphere to radiate the
Poynting flux, complying with the first three laws.

We will no longer need to rely on the concept "magnetic
field lines threading and anchored in the horizon" (Blandford
& Znajek 1977). We may instead consider "poloidal magnetic
field lines not only threading but also pinned down in the ZAM-
Gap circulating round the hole with \( \omega_N = \Omega_\ell \) dragged by the hole rotation." The resulting twin-pulsar model based on a modified
BZ process with an extended Membrane Paradigm will be a
natural outcome from the unification of pulsar electrodynamics
with BH thermodynamics. The secret of physics of the magne-
tized ZAM-Gap, its structure, particle production therein, etc.,
are waiting to be unveiled as one of the ultimate central engines
in the universe (Okamoto 2015a; Okamoto 2015b).

If observed large-scale high-energy \( \gamma \)-ray jets from AGNs
are really originated from quite near the event horizon of the
central super-massive BH, it appears to be plausible that these
jets are a magnificent manifestation of collaboration of relativity,
thermodynamics and electrodynamics; more precisely speaking,
the frame-dragging effect, the first and second laws, and unipolar
induction. The heart of the black hole’s central engine may lie
in the Gap \( G_N \) between the two outer and inner domains above
the horizon, and the embryo of a jet will be born in the Gap \( G_N \)
under the null surface \( S_N \) in the range of \( 2^{1/3} r_H \leq r_N \leq 1.6433 r_H \)
quite near the horizon (see section A).

The confirmation of this postulate awaits a further illumina-
tion of Gap physics. Some of the critical questions left to solve
may include:

(a) When the gravito-electric potential gradient of the hole
\( \Omega_H \) and the strength of magnetic field \( B_p \) threading and pinned
down in the Gap are given, from observations, how effi-
ciently does the voltage drop due to the EMFs, \( \Delta V = [E]_G \),
contribute to particle production? The pinning-down of the
poloidal field \( B_p \) by the ZAM-Ps pair-produced in the Gap will
yield complete magnetization of the plasma to ensure field lines
possess \( \Omega_\ell = \omega_N \). Then, how much density of ZAM-particles is
needed for pinning-down and magnetization to ensure \( \Omega_\ell = \omega_N \)?
How large is the Gap width \( \Delta \omega = |\partial \omega/\partial t| \Delta t \), not only to provide neutral plasma particles with \( \omega_0 \approx 0 \), being charge-separated to
both the outflow and inflow in the force-free domains?

(b) While the force-free domains \( D_{(out)} \) and \( D_{(in)} \) are filled with the ‘massless’ particles, the Gap in between will be regarded
as the extreme-opposite, i.e., non-force-free. That is to say, pair-
produced ZAM-Ps will acquire the rest mass energy probably
greater than the field energy, to be able to anchor the poloidal
field \( B_p \) and ensure \( \Omega_\ell = \omega_N \). That being said, if the ZAM-
particles are massive, they may not be free from the hole’s
gravitational and tidal forces. In that case, how are the ZAM-
particles in the Gap able to be sustained against gravity from
the hole? Or can we neglect the gravitational force by the
hole on these particles frame-dragged by the hole’s rotation into
circulation around the hole?

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Appendix A The place and shape of the null surface \( S_N \)

It is the final eigenvalue \( \Omega_\ell(\Psi) \) that determines not only
the efficiency \( \epsilon_{GTED}(\Psi) \) of energy extraction, but the place
and shape of the null surface \( S_N \), which hides a magnetized ZAM-
Gap \( G_N \) under it. Some basic properties of the structure of force-
free eigen-magnetospheres had been already clarified in some
details in section 7 of Okamoto 1992 and section 2 in Okamoto
2009 (see figure 3 in Okamoto 2006 for a schematic shape; also
see Okamoto 2009; Okamoto 2012a for the monopolar exact
solution in the slow-rotation limit). Then from equation (6.1d)
for the FDAV \( \omega \) we have

\[
\omega = \frac{(1 + h^2)^2 x}{(x^2 + h^2)^2 - h^2 (x - 1)(x - h^2) \sin^2 \theta} = \frac{1}{1 + \zeta^2}, \tag{A.1}
\]

where \( x = r/r_H \) and \( \Omega_H = c^2 h/2GM \). When we use \( \omega_N = \Omega_H / (1 + \zeta(\Psi)) \) from equation (10.7a), the expression of \( x_N = x_N(\theta) \) for
the shape of \( S_N \) reduces to an algebraic equation, i.e.,

\[
F_N(x, \theta, \zeta; h) = (x^2 + h^2)(x^2 + h^2 \cos^2 \theta) - (1 + h^2)[(1 + h^2 \cos^2 \theta) + (1 + h^2) \zeta] x = 0. \tag{A.2}
\]

When \( \zeta(\Psi) \approx 1 \), it is useful to define a 'mid-surface’ \( S_M \) with
\( \omega_M = 0.5\Omega_H \), and to examine topological features of \( S_M \), with the use of

\[
F_M(x, \theta; h) = (x^2 + h^2)(x^2 + h^2 \cos^2 \theta) - (1 + h^2)[(2 + h^2 (1 + \cos^2 \theta)] x = 0. \tag{A.3}
\]

In addition, we introduce the static-limit surface as the surface
limiting the ergosphere from \( g_{tt} = -(\Lambda - a^2 \sin^2 \theta)/r^3 = 0 \),

\[
F_E(x, \theta; h) = (x - 1)(x - h^2) - h^2 \sin^2 \theta, \tag{A.4}
\]

and its solution is expressed as

\[
x_E(\theta; h) = \frac{1}{2} \left( 1 + h^2 + \sqrt{(1 - h^2)^2 + 4h^2 \sin^2 \theta} \right). \tag{A.5}
\]

From equations (A.3) and (A.5), for \( h \ll 1 \), we have
\[ x_{\text{E}} = 1 + h^2 \sin^2 \theta, \quad (A.6a) \]
\[ x_{\text{M}} = 2^{1/3} \left[ 1 + \frac{h^2}{6} \left( 2(2 - 2^{1/3}) + (2^{1/3} - 1) \sin^2 \theta \right) \right], \quad (A.6b) \]

while by Eqs. (7.7a,b) in Okamoto 1992, the two light surfaces, \( S_{\text{ol}} \) and \( S_{\text{il}} \), become
\[ x_{\text{ol}} = \frac{2}{h} \left( 1 - \frac{\sin \theta}{4} \right), \quad (A.7a) \]
\[ x_{\text{il}} = 1 + \frac{h^2}{4} \sin^2 \theta. \quad (A.7b) \]

Then, \( x_{\text{il}} < x_{\text{E}} < x_{\text{ol}} \), and for \( h \to 0 \) it turns out that both of \( x_{\text{il}} \) and \( x_{\text{E}} \) \( \to 1 \) and \( x_{\text{ol}} \to \infty \), but also \( x_{\text{M}} \to 2^{1/3} = 1.2599 \). Thus, when \( \zeta \approx 1 \) and hence \( S_N \supset S_M \), \( S_N \) interestingly keeps a position of \( x_N \to 2^{1/3} \) above the horizon between \( x_{\text{il}} = x_{\text{E}} = 1 \) and \( x_{\text{ol}} \to \infty \) (i.e., \( S_E \supset S_{\text{il}} < S_N < S_{\text{ol}} \to S_{\infty} \)), even for \( h \to 0 \).

There is a certain surface \( S_{\text{Mc}} \), which contacts with \( S_E \) from the outside at the equator, i.e., \( x_M = x_E \). This takes place when
\[ h_\ast = \sqrt[3]{2} - 1 = 0.6436, \quad \text{and then} \quad x_M = 1.3960 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad x_E = x_M = 1 + h_\ast^2 = \sqrt[3]{2} \quad \text{at} \quad \theta = \pi/2 \quad \text{(see Figs. 1 and 2 in Okamoto 2009).} \]

For the extreme-Kerr state with \( h \to 1 \), equation (A.3) reduces to
\[ F_M(x, \theta; 1) = (x^2 + 1)(x^2 + \cos^2 \theta) - 2(3 + \cos^2 \theta)x = 0, \quad (A.8) \]
which yields \( x_M = 1.6085 \) for \( \theta = 0 \) at the pole and \( x_M = 1.6344 \), while by \( F_E(x, \pi/2; 1) = 0 \), we have \( x_E = 2 \) at the equator (see figure 3 in Okamoto 1992).

When \( \zeta \approx 1 \), from the above analysis one can read such interesting features at \( \theta = \pi/2 \) for \( 0 \leq \theta \leq 1 \) that
\[ 1 \leq x_E(h) \leq 2, \quad \text{for} \quad S_E, \quad 2^{1/3} = 1.2599 \leq x_N(h) \leq 1.6433, \quad \text{for} \quad S_N, \quad (A.9) \]
and that \( x_N \geq x_E \) for \( h \geq h_\ast = (2^{1/3} - 1)^{1/2} = 0.6436 \). This shows that for \( 1 \geq h \geq h_\ast \), the equatorial portion of the null surface \( S_N \) lies within the ergosphere \( S_E \), while, for \( h < h_\ast \), the whole of the ergosphere \( S_E \) lies within the null surface. It turns out that the ergosphere changes from a spherical shape at \( h = 0 \) to a spheroidal one at \( h = 1 \), while when \( \zeta \approx 1 \) the null surface keeps an almost spherical shape from \( h = 0 \) to \( h = 1 \). In any case, it appears that mechanical properties in the ergosphere have no direct connection with electrodynamic properties of the null surface and the inner domain \( D_{(\text{in})} \).

References

Beskin V. S., 2009, MHD Flows in Compact Astrophysical Objects. Springer-Verlag, Berlin
Beskin V. S., Isomin Ya. N., Par’ev V. I., 1992, Sov. Astron., 36(6), 642
Blandford R. D., 1979, Accretion disc and black hole electrodynamics, in Active Galactic Nuclei, eds C. Hazard and S. Mitton. p.241, Cambridge University Press, Cambridge
Blandford R. D., 2002, To the Lighthouse, in Lighthouse of the Universe, eds Gilfanov, M. et al. (Springer: Berlin)

Blandford R. D., Znajek R. L., 1977, MNRAS, 179, 433
Cvetič M., Gibbons G.W., Lü H., Pope C.N., 2018, Phys. Rev. D 98, 106015
Goldreich P., Julian W. H., 1969, ApJ, 157, 869
Hirota K., Okamoto I., 1998, ApJ, 497, 563
Hirota K., Put H.-Y., Ootmani S., Huang H., Kim D., Song Y., Matsushita S., Kong A. K. H., 2018b, ApJ, 867, 120
Kaburaki O., Okamoto I., 1991, Phys. Rev. D, 43, 340
Kennef C. F., Fujimura F. S., Okamoto I., 1983, Geophys. Ap. Fluid Dyn., 26, 147
Koide S., 2003, 1991, Phys. Rev. D, 67, 104010
Koide S., Kudoh T., Shibata K., 2006, Phys. Rev. D, 74, 044005
Komissarov S. S., 2009, J. Korean Phys. Soc., 54, 2503
Landau L. D., Lifshitz E. M., 1996, The Classical Theory of Fields, 6th edition, Butterworth-Heinemann, Oxford
Landau L. D., Lifshitz E. M., Pitaevskii L. P., 1984, Electrodynamics of Continuous Media, second edition, Butterworth-Heinemann, Oxford
Lasota J. -P., Gourgoulhon, E., Abramowicz, M., Tchekhovskoy, A., Narayan, R., 2014, Phys. Rev. D, 89, 024041
Macdonald D. A., Thorne K., 1982, MNRAS, 198, 345
Meier D. L., 2012, Black Hole astrophysics: The Engine Paradigm, Berlin: Springer
Michel F. C.,1969, ApJ, 158, 727
Misner C. W., Thorne K. S., Wheeler J. A., 1973, Gravitation, Freeman, San Francisco
Okamoto I., 1974, MNRAS, 167, 457
Okamoto I., 1978, MNRAS, 167, 457
Okamoto I., 1992, MNRAS, 254, 192
Okamoto I., 1999, MNRAS, 254, 192
Okamoto I., 2002, ApJ, 573, L31
Okamoto I., 2003, ApJ, 589, 671
Okamoto I., 2006, PASJ, 58, 1047
Okamoto I., 2009, PASJ, 61, 971
Okamoto I., 2012, PASJ, 64, 50
Okamoto I., 2015a, PASJ, 67, 89
Okamoto I., 2015b, Frame dragging, unipolar induction and jet source, in Proceedings of the Texas Symposium on Relativistic Astrophysics, 2015.12.13–18, Geneva, Switzerland
Okamoto I., Kaburaki O., 1990, MNRAS, 247, 244
Okamoto I., Kaburaki O., 1991, MNRAS, 250, 300
Okamoto I., Kaburaki O., 1993, MNRAS, 225, 539
Okamoto I., Sigalo B. F., 2006, PASJ, 58, 987
Okamoto I., Song, Y., 2019, arXiv:1904.11978 v1[gr-qc]
Penrose R., 1969, Nuovo. Cim., 1, 252
Phinney S., 1983a, in Proc. Torino Workshop on Astrophysical Jets, ed. A. Ferrari & A. Pacholczyk, Reidel, Dordrecht
Phinney S., 1983b, A theory of radio sources, Ph.D. thesis, Univ. of Cambridge
Punsly B., 2008, Black Hole Gravitohydromagnetics, 2nd Ed., Springer, New York
Punsly B., Coroniti F. V., 1989, Phys. Rev. D, 40, 3843
Punsly B., Coroniti F. V., 1990, ApJ, 350, 518
Ruderman M., Sutherland P.G. 1975, MNRAS, 196, 51
Song Y., Pu H.-Y., Hirota K., Matsushita S., Kong A. K. H., Chang H.-K.,...
2017, MNRAS, 471, L135
Takahashi M., Nitta S., Tatematsu Y., Tomimatsu A., 1990, ApJ, 363, 206

Thorne K., Macdonald D. A., 1982, MNRAS, 198, 339
Thorne K. S., Price R. H., Macdonald D. A. 1986, Black Holes: The
Membrane Paradigm, Yale University Press, New Haven

Toma K., Takahara F., 2014, MNRAS, 442, 2855
Toma K., Takahara F., 2016, in Proc. IAU Symp.324 on “New Frontiers
in Black Hole Astrophysics”, Cambridge University Press

Weber E. J., Davis L. J., 1967, ApJ, 148, 217

Znajek R. L., 1977, MNRAS, 179, 457
Znajek R. L., 1978, MNRAS, 185, 833