CLASSES WITH NEGATIVE COEFFICIENTS AND STARLIKE WITH RESPECT TO OTHER POINTS II

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Abstract. A class $S \ast_s T(\alpha, \beta, \sigma, k)$ of functions $f$ which are analytic and univalent in the open unit disk $D = \{z : |z| < 1\}$ given by $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ and satisfying the condition

$$\left| \frac{zf'(z)}{f(z) - f(-z)} - k \right| < \beta \left| \frac{\alpha zf'(z)}{f(z) - f(-z)} - (2\sigma - k) \right|$$

for $0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \sigma \leq \frac{1}{2}, \frac{1}{2} < k \leq 1, z \in D,$ is introduced and studied. An analogous class $S \ast_c T(\alpha, \beta, \sigma, k)$ and $S \ast_sc T(\alpha, \beta, \sigma, k)$ are also examined.

1. Introduction

Let $S$ be the class of functions $f$ which are analytic and univalent in the open unit disk $D = \{z : |z| < 1\}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

and $a_n$ a complex number. Let $S \ast$ be the subclass of $S$ consisting of functions given by (1) satisfying

$$\text{Re} \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in D.$$ 

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [4]. El-Ashwa and Thomas [2] obtained various results concerning functions in $S \ast$ and two other classes namely the class $S \ast_c$ of functions starlike with respect to conjugate points and the class $S \ast_sc$ of functions starlike with respect to symmetric conjugate points.

Now, we denote $T$ the class consisting of functions $f$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (2)$$

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where $a_n$ is a non-negative real number.

For $f \in T$, Halim et al. [3] studied the class $S^*_T(\alpha, \beta)$, $0 \leq \alpha \leq 1, \frac{1}{2} < \beta \leq 1$, consisting of functions $f \in T$ and starlike with respect to symmetric points. An analogous results are also obtained for the class $S^*_cT(\alpha, \beta)$, $0 \leq \alpha \leq 1, \frac{1}{2} < \beta \leq 1$, consisting of functions $f \in T$ and starlike with respect to conjugate points and the class $S^*_scT(\alpha, \beta)$, $0 \leq \alpha \leq 1, \frac{1}{2} < \beta \leq 1$, consisting of functions $f \in T$ and starlike with respect to symmetric conjugate points.

For this paper, we consider a class $S^*_sT(\alpha, \beta, \sigma, k)$, $0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \sigma \leq \frac{1}{2} < k \leq 1$, consisting of functions $f \in T$ and starlike with respect to symmetric points as follows:

**Definition 1.** A function $f \in S^*_sT(\alpha, \beta, \sigma, k)$ if it satisfies

$$\left| \frac{zf'(z)}{f(z) - f(-z)} - k \right| < \beta \left| \frac{zf'(z)}{f(z) - f(-z)} - (2\sigma - k) \right|$$

for some $0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \sigma \leq \frac{1}{2} < k \leq 1$ and $z \in D$.

An analogous results are also obtained for the class of functions $f \in T$ and starlike with respect to conjugate points and functions starlike with respect to symmetric conjugate points. These classes are defined as below:

**Definition 2.** A function $f \in S^*_cT(\alpha, \beta, \sigma, k)$ if it satisfies

$$\left| \frac{zf'(z)}{f(z) + f(\bar{z})} - k \right| < \beta \left| \frac{zf'(z)}{f(z) + f(\bar{z})} - (2\sigma - k) \right|$$

for some $0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \sigma \leq \frac{1}{2} < k \leq 1$ and $z \in D$.

**Definition 3.** A function $f \in S^*_scT(\alpha, \beta, \sigma, k)$ if it satisfies

$$\left| \frac{zf'(z)}{f(z) - f(\bar{z})} - k \right| < \beta \left| \frac{zf'(z)}{f(z) - f(\bar{z})} - (2\sigma - k) \right|$$

for some $0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 \leq \sigma \leq \frac{1}{2} < k \leq 1$ and $z \in D$.

2. **Coefficient Inequalities**

We first state some preliminary lemmas, required for proving our result.

**Lemma 1.** ([5]) If $f \in T$ then $\sum_{n=2}^{\infty} n |a_n| < 1$.

**Lemma 2.** If $f \in T$ then $\sum_{n=2}^{\infty} (n\alpha + (k - 2\sigma)(1 - (-1)^n)) |a_n| < \alpha + 2(k - 2\sigma)$.
Proof. We note that
\[
\sum_{n=2}^{\infty} \left( n \alpha (k - 2 \sigma)(1 - (-1)^n) \right) |a_n|
\]
\[
= \sum_{n=2}^{\infty} n \alpha |a_n| + \sum_{n=2}^{\infty} (k - 2 \sigma)(1 - (-1)^n) |a_n|
\]
\[
= \alpha \sum_{n=2}^{\infty} n |a_n| + (k - 2 \sigma) \sum_{n=2}^{\infty} (1 - (-1)^n)|a_n|
\]
\[
< \alpha (1) + (k - 2 \sigma)2(1) = \alpha + 2(k - 2 \sigma), \quad \text{by (Lemma 1)},
\]
as required.

Lemma 3. If \( f \in T \) then \( \sum_{n=2}^{\infty} (n \alpha + 2(k - 2 \sigma)) |a_n| < \alpha + 2(k - 2 \sigma) \).

Proof. We note that
\[
\sum_{n=2}^{\infty} (n \alpha + 2(k - 2 \sigma)) |a_n|
\]
\[
= \sum_{n=2}^{\infty} n \alpha |a_n| + \sum_{n=2}^{\infty} 2(k - 2 \sigma) |a_n|
\]
\[
= \alpha \sum_{n=2}^{\infty} n |a_n| + 2(k - 2 \sigma) \sum_{n=2}^{\infty} |a_n|
\]
\[
< \alpha (1) + 2(k - 2 \sigma)(1) = \alpha + 2(k - 2 \sigma), \quad \text{by (Lemma 1)},
\]
as required.

For \( S^*_T(\alpha, \beta, \sigma, k) \), we have the following:

Theorem 1. Let \( f \in T \). A function \( f \in S^*_T(\alpha, \beta, \sigma, k) \) if and only if
\[
\sum_{n=2}^{\infty} \left( \frac{(1 + \beta \alpha)n}{\beta(2k - 2 \sigma) + \alpha - (2k - 1)} + \frac{\beta(k - 2 \sigma)(1 - (-1)^n) - k(1 - (-1)^n)}{\beta(2k - 2 \sigma) + \alpha - (2k - 1)} \right) a_n \leq 1
\]
for \( 0 \leq \alpha \leq 1, \frac{1}{2} \leq \beta \leq 1 \) and \( 0 \leq \sigma \leq \frac{1}{2} < k \leq 1 \).

Proof. First we prove the ‘if’ part. We adopt the method used by Clunie and Keogh.
We write
\[ |zf'(z) - k(f(z) - f(-z))| - \beta|\alpha zf'(z) - (2\sigma - k)(f(z) - f(-z))| \]
\[ = |(1 - 2k)z - \sum_{n=2}^{\infty} (n - k(1 - (-1)^n))a_n z^n| - \beta(2(k - 2\sigma) + \alpha)z \]
\[ - \sum_{n=2}^{\infty} (n\alpha + (k - 2\sigma)(1 - (-1)^n))a_n z^n| \]
\[ \leq \sum_{n=2}^{\infty} (n - k(1 - (-1)^n))|a_n| r^n + (2k - 1)r - \beta(2(k - 2\sigma) + \alpha)r \]
\[ + \sum_{n=2}^{\infty} \beta(n\alpha + (k - 2\sigma)(1 - (-1)^n))|a_n| r^n \]
\[ < \left[ \sum_{n=2}^{\infty} (n - k(1 - (-1)^n))|a_n| + (2k - 1) - \beta(2(k - 2\sigma) + \alpha) \right] r \]
\[ + \sum_{n=2}^{\infty} \beta(n\alpha + (k - 2\sigma)(1 - (-1)^n))|a_n| \]
\[ = \left[ \sum_{n=2}^{\infty} ((1 + \beta\alpha)n + \beta(k - 2\sigma)(1 - (-1)^n) - k(1 - (-1)^n))a_n \right] r \]
\[ - \beta(2(k - 2\sigma) + \alpha - (2k - 1)) \]
\[ \leq 0 \text{ by (3)}. \]

Thus,
\[ \frac{|zf'(z) - k|}{|\alpha zf'(z) - (2\sigma - k)|} |< \beta \]

and hence \( f \in S^*_T(\alpha, \beta, \sigma, k) \).

To prove the 'only if' part, we write
\[ \frac{|zf'(z) - k|}{|\alpha zf'(z) - (2\sigma - k)|} = \frac{(1 - 2k) - \sum_{n=2}^{\infty} (n - k(1 - (-1)^n))a_n z^{n-1}}{2(k - 2\sigma) + \alpha - \sum_{n=2}^{\infty} (n\alpha + (k - 2\sigma)(1 - (-1)^n))a_n z^{n-1}} < \beta. \]

Since \( f \) is analytic, continuous and non constant in \( D \), the maximum modulus principle
and hence, we obtain
\[ \frac{1 - 2\kappa}{2(2 - \sigma) + \alpha - \sum_{n=2}^{\infty} (n\alpha + (k - 2\sigma)(1 - (-1)^n))a_n z^{n-1}} \leq \frac{2(2 - \sigma) + \alpha - \sum_{n=2}^{\infty} (n\alpha + (k - 2\sigma)(1 - (-1)^n))a_n z^{n-1}}{2(2 - \sigma) + \alpha - \sum_{n=2}^{\infty} (n\alpha + (k - 2\sigma)(1 - (-1)^n))a_n z^{n-1}} \]

next, similar coefficient properties for functions which belong to \( S^*T(\alpha, \beta, \sigma, k) \) and \( S^*T(\alpha, \beta, \sigma, k) \) are obtained. Similar method of proving is used for Theorem 2 and Theorem 3.
Thus, since states that $0$ and hence $f$ is analytic, continuous and non constant in $D$, the maximum modulus principle states that

\[
\left| \frac{zf'(z) - k(f(z) + \overline{f(z)})}{azf'(z) - (2\sigma - k)(f(z) + \overline{f(z)})} \right| < \beta
\]

and hence $f \in S^*_T(\alpha, \beta, \sigma, k)$.

To prove the 'only if' part, as before we write

\[
\left| \frac{zf'(z) - k(f(z) + \overline{f(z)})}{azf'(z) - (2\sigma - k)(f(z) + \overline{f(z)})} \right| = \left| \frac{(1 - 2k)(n - 2k)a_n z^n}{2(k - 2\sigma) + \alpha - \sum_{n=2}^{\infty} (n\alpha + 2(k - 2\sigma))a_n z^n} \right| < \beta.
\]

Since $f$ is analytic, continuous and non constant in $D$, the maximum modulus principle states that

\[
\left| \frac{(1 - 2k)(n - 2k)a_n z^n}{2(k - 2\sigma) + \alpha - \sum_{n=2}^{\infty} (n\alpha + 2(k - 2\sigma))a_n z^n} \right| < \beta.
\]
Since \( f \in S^*_c(T(\alpha, \beta, \sigma, k)) \) and \(|z| < r < 1\), we obtain
\[
\left\{ \frac{(2k-1) + \sum_{n=2}^{\infty} (n-2k)a_n r^{n-1}}{2(k-2\sigma) + \alpha - \sum_{n=2}^{\infty} (n\alpha + 2(k-2\sigma))a_n r^{n-1}} \right\} < \beta
\] (6)
for any \( z \in D \). Now letting \( r \to 1 \) in (6), we obtain
\[
(2k-1) + \sum_{n=2}^{\infty} (n-2k)a_n \leq \beta \left( 2(k-2\sigma) + \alpha - \sum_{n=2}^{\infty} (n\alpha + 2(k-2\sigma))a_n \right)
\]
and hence, we obtain
\[
\sum_{n=2}^{\infty} \left( \frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma) + \alpha) - (2k-1)} + \frac{2(\beta(k-2\sigma) - k)}{\beta(2(k-2\sigma) + \alpha) - (2k-1)} \right) a_n \leq 1
\]
as required. This completes the proof of the theorem.

The result in Theorem 2 is sharp for function \( f_n \) given by
\[
f_n(z) = z - \frac{\beta(2(k-2\sigma) + \alpha) - (2k-1)}{(1+\beta\alpha)n + 2(\beta(k-2\sigma) - k)} z^n.
\]

**Corollary 2.** If \( f \in S^*_c(T(\alpha, \beta, \sigma, k)) \) then
\[
a_n \leq \frac{\beta(2(k-2\sigma) + \alpha) - (2k-1)}{(1+\beta\alpha)n + 2(\beta(k-2\sigma) - k)}, \quad n \geq 2.
\]

For completeness, we state the following result with regards to the class \( S^*_c(T(\alpha, \beta, \sigma, k)) \).

**Theorem 3.** Let \( f \in T \). A function \( f \in S^*_c(T(\alpha, \beta, \sigma, k)) \) if and only if
\[
\sum_{n=2}^{\infty} \left( \frac{(1+\beta\alpha)n}{\beta(2(k-2\sigma) + \alpha) - (2k-1)} + \frac{\beta(k-2\sigma)(1 - (-1)^n) - k(1 - (-1)^n)}{\beta(2(k-2\sigma) + \alpha) - (2k-1)} \right) a_n \leq 1
\] (7)
for \( 0 \leq \alpha \leq 1, 0 < \frac{1}{2} < \beta \leq 1 \) and \( 0 \leq \sigma \leq \frac{1}{2} < k \leq 1 \).

The result in Theorem 3 is sharp for functions given by
\[
f_n(z) = z - \frac{\beta(2(k-2\sigma) + \alpha) - (2k-1)}{(1+\beta\alpha)n + \beta(k-2\sigma)(1 - (-1)^n) - k(1 - (-1)^n)} z^n, \quad n \geq 2.
\]

**Corollary 3.** If \( f \in S^*_c(T(\alpha, \beta, \sigma, k)) \) then
\[
a_n \leq \frac{\beta(2(k-2\sigma) + \alpha) - (2k-1)}{(1+\beta\alpha)n + \beta(k-2\sigma)(1 - (-1)^n) - k(1 - (-1)^n)}, \quad n \geq 2.
\]
3. Growth Theorem

A growth property for functions in the class \( S^*_{\alpha, \beta, \sigma, k} \) is given as follows.

**Theorem 4.** Let the functions \( f \) be defined by (2) and belongs to the class \( S^*_{\alpha, \beta, \sigma, k} \). Then for \( \{ z : 0 < |z| = r < 1 \} \),

\[
r - \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)} r^2 \leq |f(z)| \leq r + \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)} r^2.
\]

The result is sharp.

**Proof.** First, it is obvious that

\[
\sum_{n=2}^{\infty} a_n \leq \frac{2(1+\beta\alpha)}{\beta(1+2\sigma)+\alpha-(2k-1)} \sum_{n=2}^{\infty} \frac{\alpha}{\beta(1+2\sigma)+\alpha-(2k-1)} a_n
\]

and as \( f \in S^*_{\alpha, \beta, \sigma, k} \), using the inequality in Theorem 1 yields

\[
\sum_{n=2}^{\infty} a_n \leq \frac{2(1+\beta\alpha)}{\beta(1+2\sigma)+\alpha-(2k-1)} \sum_{n=2}^{\infty} a_n \frac{(1+\beta\alpha)n}{\beta(1+2\sigma)+\alpha-(2k-1)} + \frac{\beta(2-2\sigma)(1-(-1)^n) - k(1-(-1)^n)}{\beta(1+2\sigma)+\alpha-(2k-1)} a_n.
\]

From (2) with \( |z| = r (r < 1) \), we have

\[
|f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + \sum_{n=2}^{\infty} a_n r^2
\]

and

\[
|f(z)| \geq r - \sum_{n=2}^{\infty} a_n r^n \geq r - \sum_{n=2}^{\infty} a_n r^2.
\]

Finally, using (8) in the above inequalities, give the result in Theorem 4.

The result in Theorem 4 is sharp for functions given by

\[
f_2(z) = z - \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)} z^2
\]

at \( z = \pm r \).

Next, similar growth results for functions which belong to \( S^*_{\alpha, \beta, \sigma, k} \) and \( S_{\alpha, \sigma, \beta, k}^* \) are obtained. Similar method of proving is used for Theorem 5 and Theorem 6.

**Theorem 5.** Let the functions \( f \) be defined by (2) and belongs to the class \( S^*_{\alpha, \beta, \sigma, k} \). Then for \( \{ z : 0 < |z| = r < 1 \} \),

\[
r - \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)+2(\beta(k-2\sigma)-k)} r^2 \leq |f(z)| \leq r + \frac{\beta(2(k-2\sigma)+\alpha)-(2k-1)}{2(1+\beta\alpha)+2(\beta(k-2\sigma)-k)} r^2.
\]
The result is sharp.

Proof. First, it is obvious that

\[
\frac{2(1 + \beta \alpha) + 2(\beta(k - 2\sigma) - k)}{\beta(2(k - 2\sigma) + \alpha) - (2k - 1)} \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \left( \frac{(1 + \beta \alpha)n}{\beta(2(k - 2\sigma) + \alpha) - (2k - 1)} + \frac{2(\beta(k - 2\sigma) - k)}{\beta(2(k - 2\sigma) + \alpha) - (2k - 1)} \right) a_n,
\]

and as \( f \in S^*_c(T(\alpha, \beta, \sigma, k)) \), using the inequality in Theorem 2 yields

\[
\sum_{n=2}^{\infty} a_n \leq \frac{\beta(2(k - 2\sigma) + \alpha) - (2k - 1)}{2(1 + \beta \alpha) + 2(\beta(k - 2\sigma) - k)}.
\]

From (2) with \(|z| = r \) \( (r < 1) \), we have

\[
|f(z)| \leq r + \sum_{n=2}^{\infty} a_n r^n \leq r + \sum_{n=2}^{\infty} a_n r^2
\]

and

\[
|f(z)| \geq r - \sum_{n=2}^{\infty} a_n r^n \geq r - \sum_{n=2}^{\infty} a_n r^2.
\]

Finally, using (9) in the above inequalities, give the result in Theorem 5.

The result in Theorem 5 is sharp for functions given by

\[
f_2(z) = z - \frac{\beta(2(k - 2\sigma) + \alpha) - (2k - 1)}{2(1 + \beta \alpha) + 2(\beta(k - 2\sigma) - k)} z^2
\]

at \( z = \pm r \).

For completeness, we state the following result with regards to the class \( S^*_c(T(\alpha, \beta, \sigma, k)) \).

**Theorem 6.** Let the functions \( f \) be defined by (2) and belongs to the class \( S^*_c(T(\alpha, \beta, \sigma, k)) \). Then for \( \{ z : 0 < |z| = r < 1 \} \),

\[
r - \frac{\beta(2(k - 2\sigma) + \alpha) - (2k - 1)}{2(1 + \beta \alpha)} r^2 \leq |f(z)| \leq r + \frac{\beta(2(k - 2\sigma) + \alpha) - (2k - 1)}{2(1 + \beta \alpha)} r^2.
\]

The result is sharp.

The result in Theorem 6 is sharp for functions given by

\[
f_2(z) = z - \frac{\beta(2(k - 2\sigma) + \alpha) - (2k - 1)}{2(1 + \beta \alpha)} z^2
\]

at \( z = \pm r \).
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