LOCAL BEHAVIOR OF SOLUTIONS OF QUASILINEAR PARABOLIC EQUATIONS ON METRIC SPACES

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Abstract. We introduce a notion of quasilinear parabolic equations over metric measure spaces. Under sharp structural conditions, we prove that local weak solutions are locally bounded and satisfy the parabolic Harnack inequality. Applications include the parabolic maximum principle and pointwise estimates for weak solutions.

1. Introduction

In their 1967 paper [2], Aronson and Serrin proved the parabolic Harnack inequality for weak solutions \( u = u(t,x) \) of the quasilinear equation

\[
\int_Q \int_R (-u \phi_t + \phi_x \cdot A(t,x,u,u_x)) \, dx \, dt = \int_Q \int_R \phi \cdot B(t,x,u,u_x) \, dx \, dt,
\]

provided that \( A \) and \( B \) satisfy certain structural conditions. Here, \( x \) is in Euclidean space \( \mathbb{R}^n \), \( u_x \) is the spatial gradient of \( u \), and \( \phi = \phi(t,x) \) is any continuously differentiable test function having compact support in \( Q \).

At about the same time, Trudinger [26] and Ladyzhenskaja, Solonnikov and Uralceva [15] proved very similar results.

The present paper introduces a notion of quasilinear equations over metric measure spaces and proves the parabolic Harnack inequality under certain hypotheses on the structure of the equation and natural conditions on the geometry of the underlying space. In particular, the parabolic Harnack inequality holds for quasilinear equations on metric measure spaces that satisfy volume doubling, Poincaré inequality and the cutoff Sobolev inequality. In the case of the linear heat equation, these are known to be equivalent to the parabolic Harnack inequality, as well as to sharp two-sided heat kernel estimates [12, 22, 25, 4].

Concerning the structural hypotheses, we follow [2, 15, 26] by assuming that a quasilinear equation should be represented in terms of some "divergence form part" \( A \) and a "lower order part" \( B \). This is somewhat contrary to the approach in [24, 16, 17] that was based on the bilinearity and a structural decomposition of bilinear forms, in addition to quantitative inequalities.

Though the main interest of this work is likely to be in the context of a reference Dirichlet space, some of our results - the mean value estimates and the local boundedness of weak solutions - apply also to subelliptic operators such as the Kolmogorov-Fokker-Planck operator.

The hypotheses that we impose on the structure of the equation are, in a certain sense, sharp. This is because we use Lorentz spaces rather than \( L^p \)-spaces. For the special case of quasilinear operators on Euclidean space this means that we recover the parabolic Harnack inequality of Aronson - Serrin but under slightly weaker - and sharp - integrability conditions on the coefficients.
There already exists a quite broad literature that applies the parabolic Moser iteration \[18\] in a non-Euclidean, linear setting, beginning with \[12, 22, 23\] on Riemannian manifolds, \[9, 16\] on Dirichlet spaces that admit a carré du champ, \[17\] on fractal-type Dirichlet spaces. The proof in the present paper is based on some of these earlier works (as well as \[2\]) but self-contained and aims to give full attention to technical issues pertaining to the existence and local boundedness of weak solutions, the appropriate function spaces, and the structural hypotheses.

A different direction concerns the generalization of the equation rather than the underlying space. A class of degenerate elliptic operators (so-called generalized Kimura diffusion operators) is studied in \[11\] and covered by the setting of the present paper. For quasilinear subelliptic operators, the parabolic Harnack inequality is proved in \[5\].

Degenerate subelliptic operators such as those of Kolmogorov-Fokker-Planck type are of interest as they indicate the margin of the wide scope of Moser’s iteration. While mean value estimates follow by Moser iteration (see, e.g., \[5\]), the second part of Moser’s iteration does not seem to apply due to the lack of a Poincaré inequality as well as a lack of the proper structure of the equation (cf. our hypothesis H.2 and Section 5.5).

We mention that there are alternative ways to obtain the parabolic Harnack inequality from volume doubling and Poincaré inequality, for instance by using elliptic Moser iteration, see e.g. \[4\]. This is of interest especially in time-independent settings. A variational approach to the parabolic Harnack inequality on metric measure spaces is taken in \[14\] under the hypothesis that weak solutions (i.e. parabolic minimizers) already satisfy the Cacciopoli-type estimates.

Our main results are in part motivated by an application to the study of heat kernels on inner uniform domains similar to \[13\]. For certain non-symmetric heat kernels, Doob’s transform yields a heat equation whose structure is not covered by \[16\] due to unbounded coefficients, but does satisfy our structural hypotheses H.1 and H.2.

**Structure of the paper.** In the first part of the paper we introduce the notion of quasilinear equations on abstract spaces (Section 2), prove Cacciopoli-type estimates and mean value estimates (Section 3) and the parabolic Harnack inequality (Section 4).

In the second part of the paper (Section 5) we discuss examples. First, we apply our main results to Dirichlet spaces satisfying volume doubling and Poincaré inequality. We show that if a quasilinear form is “adapted” to a reference Dirichlet form (see Definition 5.1), then our structural hypotheses H.1, H.2 are satisfied. Being adapted to a Dirichlet form is a property that should be easy to check in applications. In the Dirichlet space context, we provide several applications of the Harnack inequality: the Hölder continuity of weak solutions, the parabolic maximum principle, and pointwise estimates for weak solutions.

Second, we apply our results to quasilinear operators on Euclidean space and discuss the sharp conditions on the coefficients in comparison to the structural hypotheses in \[2\].

Third, we consider a Kolmogorov-Fokker-Planck operator. This example illustrates that there is an actual difference between our hypothesis H.1 and hypothesis H.2.
Finally, we emphasize the relevance of the metric measure space setting by combining our main result with a metric measure transform.

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2. Quasilinear forms, structural and geometric hypotheses

2.1. Quasilinear forms and weak solutions. Let \( X \) be a locally compact separable Hausdorff space and \( \mu \) a locally finite Borel measure with full support on \( X \).

Let \( \mathcal{F} \) be a linear subspace of \( L^2(X, \mu) \) such that

(i) \( \mathcal{F} \) is dense in \( L^2(X, \mu) \).

(ii) There is a norm \( \| \cdot \|_X \) so that \( (\mathcal{F}, \| \cdot \|_X) \) is a Banach space and \( \| f \|_X \geq \| f \|_{L^2} \) for every \( f \in \mathcal{F} \).

(iii) \( \mathcal{F} \cap \mathcal{C}_c(X) \) is dense in \( (\mathcal{C}_c(X), \| \cdot \|_\infty) \) and dense in \( (\mathcal{F}, \| \cdot \|_X) \).

(iv) \( \mathcal{F}_b := \mathcal{F} \cap L^\infty(X) \) is an algebra.

(v) If \( f \in \mathcal{F} \) then \( (f \circ 0) \in \mathcal{F} \) and \( (f \circ 1) \in \mathcal{F} \).

(vi) If \( f \in \mathcal{F} \) then \( \Phi(f) \in \mathcal{F} \) for any function \( \Phi \in C^1(\mathbb{R}^m) \) with \( \Phi(0) = 0 \), where \( m \) is a positive integer.

(vii) \( \mathcal{F} \) contains cutoff functions: for every open \( U \subseteq X \) and every compact \( K \subseteq U \) there exists a continuous function \( \psi : X \to [0,1] \in \mathcal{F} \) that takes value \( 1 \) on \( K \) and value \( 0 \) on \( X \setminus U \).

Here, \( \mathcal{C} \) denotes the space of continuous functions, \( \mathcal{C}_c \) are continuous functions with compact support, and \( \mathcal{F}_c \) will be the functions in \( \mathcal{F} \) with compact support. Further, \( a \wedge b = \min\{a, b\} \) and \( a \vee b = \max\{a, b\} \).

**Definition 2.1.** We call a collection of maps \( \mathcal{E}_t : \mathcal{F} \times \mathcal{F} \to \mathbb{R} \), \( t \in \mathbb{R} \), a quasi-linear form if

(i) there exist signed measure valued forms \( \mathcal{A}_t \) and \( \mathcal{B}_t \) such that

\[
\mathcal{E}_t(u, g) = \int d\mathcal{A}_t(u, g) + \int d\mathcal{B}_t(u, g), \quad \text{for all } u, g \in \mathcal{F}.
\]

(ii) the maps \( t \mapsto \int d\mathcal{A}_t(u, g) \) and \( t \mapsto \int d\mathcal{B}_t(u, g) \) are measurable for all \( u, g \in \mathcal{F} \).

(iii) (right-linearity) \( \mathcal{A}_t(u, g_1 + g_2) = \mathcal{A}_t(u, g_1) + \mathcal{A}_t(u, g_2) \) and \( \mathcal{A}_t(u, cg_1) = c\mathcal{A}_t(u, g_1) \) for all \( u, g_1, g_2 \in \mathcal{F}, c \in \mathbb{R} \). Similarly for \( \mathcal{B}_t \).

(iv) (\( \mathcal{B}_t \) is right-local) \( \mathcal{B}_t(u, g) = 0 \) whenever \( u, g \in \mathcal{F} \) with \( g = 0 \) on the support of \( u \).

(v) (\( \mathcal{A}_t \) is right-strongly local) \( \mathcal{A}_t(u, g) = 0 \) whenever \( u, g \in \mathcal{F} \) with \( g \) constant on the support of \( u \).

(vi) (right-product rule) \( d\mathcal{A}_t(u, fg) = g d\mathcal{A}_t(u, f) + f d\mathcal{A}_t(u, g) \) and \( d\mathcal{B}_t(u, fg) = g d\mathcal{B}_t(u, f) \) whenever \( u, f, g \in \mathcal{F} \).

(vii) (right-chain rule) for any \( u, g_1, g_2, \ldots, g_m \in \mathcal{F}_b \), \( g = (g_1, \ldots, g_m) \), and \( \Phi \in C^1(\mathbb{R}^m) \) with \( \Phi(0) = 0 \), we have \( \Phi(g) \in \mathcal{F}_b \) and

\[
d\mathcal{A}_t(u, \Phi(g)) = \sum_{i=1}^m \frac{\partial \Phi}{\partial x_i}(g) d\mathcal{A}_t(u, g_i).
\]

(viii) (right-continuity) For every \( t \in \mathbb{R} \) there exists an open interval \( I \ni t \) such that for any \( u \in L^2(I \to \mathcal{F}) \cap \mathcal{C}(I \to L^2) \) and there is a constant \( C(u, t) \)
such that
\[ |\mathcal{E}_s(u(s, \cdot), g)| \leq C(u, t)\|g\|_F \quad \text{for all } s \in I, g \in F_c. \]

Let \( I \) be a bounded open interval and \( U \subset X \) open. Let \( L^2_{\text{loc}}(I \to F; U) \) be the space of all functions \( u : I \times U \to \mathbb{R} \) such that for any open interval \( J \) relatively compact in \( I \), and any open subset \( A \) relatively compact in \( U \), there exists a function \( u^J \in L^2(I \to F) \) such that \( u^J = u \) a.e. in \( J \times A \). If all these \( u^J \) are in \( C(I \to L^2(U)) \), then we write \( u \in L^2_{\text{loc}}(I \to F; U) \cap C_{\text{loc}}(I \to L^2(U)) \).

**Definition 2.2.** Set \( Q = I \times U \). A map \( u : Q \to \mathbb{R} \) is a local weak subsolution of the heat equation for \( \mathcal{E}_t \) in \( Q \) if

(i) \( u \in L^2_{\text{loc}}(I \to F; U) \cap C_{\text{loc}}(I \to L^2(U)) \).

(ii) For almost every \( a, b \in I \) with \( a < b \), and any non-negative \( \phi \in F_c(U) \),

\[
\int_a^b u(b)\phi \, d\mu - \int_a^b u(a)\phi \, d\mu + \int_a^b \mathcal{E}_t(u(t), \phi) \, dt \leq 0.
\]

A map \( u \) is a local weak supersolution if \(-u\) is a local weak subsolution. If both \( u \) and \(-u\) are local weak subsolutions then \( u \) is called a local weak solution.

It is worth to remark that local weak solutions can equivalently be defined using weak time-derivatives, see [16, Proposition 7.8].

### 2.2. Structural hypotheses.

For the rest of the paper, we fix \( \delta^* \in (0, 1) \). Let \( (B_\delta) \) be a collection of relatively compact open subsets of \( X \) such that \( B_{\delta'} \subset B_{\delta} \) whenever \( \delta^* < \delta' < \delta \leq 1 \).

Fix \( a < a' < b' < b \). For \( \delta \in [\delta^*, 1] \), let \( I_\delta^- = (a - a_\delta, b') \) be a strictly increasing sequence of bounded open intervals. Let \( I^- = (a - a_1, b) \), \( Q^- = I^- \times B_1 \) and \( Q^-_\delta = I^-_\delta \times B_{\delta} \). We also define \( I^+ = (a', b + a_\delta) \), \( Q^+ = I^+ \times B_1 \), and \( Q^+_\delta = I^+_\delta \times B_{\delta} \).

We assume there are constants \( C_3, k \in (0, \infty) \) such that
\[
|a_\delta - a_{\delta'}|^{-1} \leq C_3|\delta - \delta'|^{-k}
\]
for all \( \delta^* \leq \delta' < \delta \leq 1 \).

Let \( \kappa \geq 0 \) and define \( \bar{v} = \max(v, 0) + \kappa \) and \( \bar{v}_\varepsilon = \bar{v} + \varepsilon \) for \( \varepsilon \in (0, 1) \). For \( p < 2 \), let
\[
\mathcal{H}(v) := \begin{cases} 
\frac{1}{p} \bar{v}^p, & p \neq 0 \\
\log \bar{v}, & p = 0.
\end{cases}
\]

Then \( \mathcal{H} \) is twice continuously differentiable on \((0, \infty)\).

For \( p \geq 2 \) and positive integers \( n \), define also
\[
\mathcal{H}_n(v) = \frac{1}{2} \bar{v}^2 (\bar{v} \wedge n)^{p-2} + \left( \frac{1}{p} - \frac{1}{2} \right) (\bar{v} \wedge n)^p - \bar{v} \kappa^{p-1} + \frac{p-1}{p} \kappa^p.
\]

Then \( \mathcal{H}_n \) has one continuous derivative
\[
\mathcal{H}_n'(v) = \bar{v} (\bar{v} \wedge n)^{p-2} - \kappa^{p-1}
\]
on \((0, \infty)\). For non-negative functions \( u \) we will write \( u_n = u \wedge n \).

Fix \( \eta \in (0, 1) \). We say that \( \text{H.1a}, \text{H.1b}, \text{or H.2, respectively, hold for } u \in L^2_{\text{loc}}(I \to F; B_1) \), if there is a positive Radon measure \( d\Gamma(u) \) and constants \( \beta \in [1, \infty), a \in [0, \infty) \)
quasi-norms are defined as

\[ \frac{a}{2} \int \bar{u}_n^{p-2} \psi^2 d\Gamma(u) + \frac{p-2}{4} a \int_{\{u \leq n\}} \bar{u}_n^{p-2} \psi^2 d\Gamma(u) \]

for all positive integers \( n \) and all \( p \in [2, \infty) \), and any smooth function \( \chi : I \to [0, 1] \).

(H.1b)

\[
\int I \left[ \frac{1-p}{1-p} \mathcal{E}_\ell(u, \mathcal{H}'(u)\psi^2) + \frac{p-1}{4} a \int \bar{u}_n^{p-2} \psi^2 d\Gamma(u) \right] \chi dt \\
\leq (1 \lor |p|)C_1 \|\bar{u}_n^p \psi\|_{2,B_1}^2 + (1 \lor |p|)C_2 |\delta - \delta|^{-k} \int I \int \bar{u}_n^{p-2} \psi \, d\mu \, dt,
\]

for all \( \varepsilon \in (0, 1) \), all \( p \in (-\infty, 0) \cup (0, 1 - \eta) \cup (1 + \eta, 2) \), any smooth function \( \chi : I \to [0, 1] \), provided that \( u \) is non-negative and locally bounded.

(H.2)

\[
\sum_{i=1}^{m} D_i(t) \|\psi\|_{2,r_i',2}^2 + C_2 |1 - \delta|^k \int \psi \, d\mu, \quad \text{for a.e. } t \in I,
\]

for all \( \varepsilon \in (0, 1) \), \( p = 0 \), provided that \( u \) is non-negative and locally bounded. Here, \( m \) is a positive integer and for each \( i \), \( D_i \) is in \( L^q(I) \) and the pair \((r_i', q_i')\) has Hölder conjugates \( (r_i, q_i) \) satisfying (3).

The function \( \psi = \psi_{\gamma, \delta} \) in H.1 and H.2 is a cutoff function for \( B_{\delta'} \) in \( B_{\delta} \) and we will assume throughout the paper that it is the same cutoff function as the one appearing in the weighted Sobolev inequality \((\text{wSI})\), see Definition 2.5. The norm \( \|\psi\| \) is defined below in (7).

A sufficient condition for a quasilinear form to satisfy H.1 and H.2 is that the quasilinear form is adapted to a reference Dirichlet form in the sense of Definition 5.1.1 below. In many applications, the adaptedness is easy to verify. The sufficiency is proved in Section 5.1.2.

2.3. Lorentz spaces. For Borel measurable functions \( f : X \to \mathbb{R} \), the Lorentz quasi-norms are defined as

\[
\|f\|_{r_1} = \left( r_1 \int_0^{\infty} s^{r_1} \mu([f] \geq s))^{r_1/r} \frac{ds}{s} \right)^{r_1}\]

for \( r, r_1 \in (0, \infty) \), and

\[
\|f\|_{r, \infty} = \sup_{s > 0} \left\{ s \mu([f] \geq s)^{1/r} \right\}, \quad \|f\|_{\infty, \infty} = \|f\|_{\infty}.
\]

Observe that \( \|f\|_{r, r} = \|f\|_r \). The Lorentz space \( L^{r, r_1} \) is defined as the collection of Borel measurable functions \( f : X \to \mathbb{R} \) with \( \|f\|_{r, r_1} < \infty \).

For any \( \sigma > 0 \), it holds

\[
\|f^\sigma\|_{r_1}^{\frac{1}{\sigma}} = \|f\|_{\sigma r, \sigma r_1}.
\]
Furthermore,

\[ \|f\|_{r,r_2} \leq 2^{2/r_1} \|f\|_{r,r_1}, \quad \text{for all } 0 < r_1 \leq r_2 \leq \infty, \]

see, e.g., [3, (4.3)].

The Lorentz-Hölder inequality ([19, Theorem 3.5])

\[ \|fg\|_{r,r_1} \leq \|f\|_{a,a_1} \|g\|_{b,b_1} \]

holds whenever

\[ \frac{1}{r} = \frac{1}{a} + \frac{1}{b}, \quad \frac{1}{r_1} = \frac{1}{a_1} + \frac{1}{b_1}. \]

This and [3] imply that

\[ \|f\|_{r,r_1} \leq \|f\|_{\sigma,a_1} \|f\|_{1^{-\sigma}} \]

holds whenever

\[ \frac{1}{r} = \sigma + 1 - \sigma, \quad \frac{1}{r_1} = \sigma + 1 - \sigma, \quad 0 \leq \sigma \leq 1. \]

For any \( r \geq 1 \), we let \( r' \) be its Hölder conjugate, and \( \frac{r''}{2} \) be the Hölder conjugate of \( r \). That is,

\[ \frac{1}{r} + \frac{1}{r'} = 1, \quad \frac{1}{r} + \frac{1}{r''} = \frac{1}{2}. \]

The next lemma is similar to [2, Lemma 1].

**Lemma 2.3.** Let \( w \in L^2(I \to F(U)) \cap L^\infty(I \to L^2(U)) \) and \( \nu > 2 \). Then \( w \) is in \( L^{2q'}(I \to L^{2r'',2}(U)) \) for all exponent pairs \((r', q')\) whose Hölder conjugates \((r, q)\) satisfy \( 1 - \frac{1}{q} - \frac{\nu}{2r'} \geq \gamma \geq 0 \). Moreover,

\[ \left( \int_I \|w\|_{2r'',2}^{2q'} \, dt \right)^{\frac{1}{q'}} \leq |I|^\gamma \left( \int_I \|w\|_{2r'',2}^{2} \, dt \right)^{\frac{2-q'}{2}} \left( \sup_{t \in I} \|w\|_2 \right)^{-\frac{2-q'}{2}}, \]

where the Lorentz quasi-norms are taken over \( U \).

**Proof.** By the Hölder inequality and [6],

\[ \left( \int_I \|w\|_{2r'',2}^{2q'} \, dt \right)^{\frac{1}{q'}} \leq |I|^\gamma \left( \int_I \|w\|_{2r'',2}^{2} \, dt \right)^{\sigma} \]

\[ \leq |I|^\gamma \left( \int_I \left( \|w\|_{2r'',2} \right)^{2-q/2} \|w\|_2^{1-q/2} \, dt \right)^{\sigma} \]

\[ \leq |I|^\gamma \left( \int_I \|w\|_{2r'',2}^{2} \, dt \right)^{\sigma} \sup_{t \in I} \|w\|_2^{2(1-q/2)}, \]

provided that \( \gamma \geq 0, 0 \leq \sigma \leq 1 \), and

\[ \frac{1}{q'} = \sigma + \gamma, \quad \frac{1}{r''} = \frac{\sigma}{\nu - 2} + \frac{1 - \sigma}{2}. \]

These relations imply that \( \gamma = \frac{1}{q'} - \sigma = 1 - \frac{1}{q} - \frac{\nu}{2r'} \) and \( \sigma = \frac{\nu}{2r'} \).

We fix \( \gamma \in [0, 1) \) and define

\[ \|w\|_{I \times U} := \sup_{r', q'} \left( \int_I \|w\|_{L^{2r'',2}(U)}^{2q'} \, dt \right)^{\frac{1}{2q'}}, \]
where the supremum is taken over all pairs \((r', q')\) whose Hölder conjugates \((r, q)\) satisfy
\[
1 - \frac{1}{q} - \frac{\nu}{2r} \geq \gamma, \quad r \geq \frac{1}{1 - \gamma}.
\]

Note that since \(\gamma < 1\),
\[
\|u\|_{\mathcal{I} \times \mathcal{U}} = \sup_{r'' \in \mathcal{R}, q'' \in \mathcal{Q}} \left( \int_{\mathcal{I}} \|u\|_{L^{r''} (\mathcal{U})}^{q''} dt \right)^{\frac{1}{q''}}
\]
where the supremum is taken over all \((r'', q'')\) whose corresponding pair \((r, q)\) satisfies
\[
1 - \frac{1}{q} - \frac{\nu}{2r} \geq \frac{\gamma}{2}, \quad r \geq \frac{2}{1 - \gamma}.
\]

Since \(\nu > 2\), \(8\) implies that \(r > 1\). Also, \(q > 1, r' \geq 1, r'' \geq 2\).

### 2.4. Geometric hypotheses: weighted Sobolev inequality and weighted Poincaré inequality.

**Definition 2.4.** Let \(\delta^* \leq \delta' < \delta \leq 1\). A function \(\psi : X \rightarrow [0, 1]\) is a cutoff function for \(B_{\delta'}\) in \(B_\delta\) if \(\psi \in \mathcal{F} \cap C_c (B_\delta)\) and \(\psi = 1\) in \(B_{\delta'}\).

**Definition 2.5.** We say that the weighted Sobolev inequality holds for a non-negative function \(f\) if there exist constants \(k \geq 0, \nu > 2\) and \(C_{SI}, C_{SI0} \geq 1\) such that, for any \(\delta^* \leq \delta' < \delta \leq 1\), there is a cutoff function \(\psi = \psi_{\delta', \delta}\) for \(B_{\delta'}\) in \(B_\delta\) such that
\[
\|f^{p/2} \psi\|_{2, \frac{p}{2r} \leq \frac{2}{1 - \gamma}}^2 \leq |\delta - \delta'|^{-k} \left( C_{SI} \frac{p^2}{4} \int_{B_\delta} f^{p-2} \psi^2 d \Gamma (f) + C_{SI0} \int_{B_\delta} f^p d \mu \right),
\]
for all \(p \in \mathbb{R}\) with \(f^{p/2} \in \mathcal{F}\). The constants \(C_{SI}, C_{SI0}\) may depend on \(\delta^*\) but not on \(\delta', \delta\).

**Remark 2.6.**

(i) The choice of the Lorentz parameter \(r_1 = 2\) in \(wSI\) is sharp, see [3, Remark 4.5].

(ii) There is no loss of generality in assuming that \(k\) is the same exponent as in \(\delta^*\).

**Definition 2.7.** We say that the weighted Poincaré inequality holds for \(\log f\), where \(f \in \mathcal{F}\) is a uniformly positive function, if there is a positive constant \(C_{wPI}\) such that
\[
\int |\log f - (\log f)_B|^2 \psi^2 d \mu \leq \int C_{wPI}(\delta', \delta) \psi^2 f^{-2} d \Gamma (f),
\]
where \((\log f)_B = \int \log f \psi^2 d \mu / \int \psi^2 d \mu\) is the weighted mean of \(f\) over \(B\), and \(\psi = \psi_{\delta', \delta}\).

The Cacciopoli-type estimates and the mean value estimates rely only on the weighted Sobolev inequality. For the parabolic Harnack we need in addition the weighted Poincaré inequality which is used in the Lemma [4, Lemma 3].

The next lemma is similar to [2, Lemma 3].
Lemma 2.8. Let $f \in L^2(I \to \mathcal{F}(B_1)) \cap L^\infty(I \to L^2(B_1))$. Suppose the weighted Sobolev inequality holds for $f(t)$ uniformly for all $t \in I$. Then for any $\delta^* \leq \delta' < \delta \leq 1$ and any $\sigma \geq 1$,
\[
\|f^\sigma\|_{L^2(B_1)}^{\frac{2}{\delta'}} \leq 2|I|^\gamma \left( \frac{C_{SI}}{\delta - \delta'} \int_I \int \psi^2 \mathrm{d}I(f) \, dt + \frac{C_{SI0}}{\delta - \delta'} \int_I \int_{B_1} f^2 \, d\mu \, dt + \sup_{t \in I} \|f\|^2_{L^2(B_1)} \right),
\]
where $\psi = \psi_{\delta', \delta}$.

Proof. For any pair $(\nu', q')$ whose Hölder conjugates $(\nu, q)$ satisfy \(\square\), the Hölder conjugates of $(\sigma \nu', \sigma q')$ also satisfy \(\square\), and
\[
\|f^\sigma\|_{L^{2q'}(I \to L^{2\nu'}(2\nu', 2\sigma))}^\frac{1}{2q'} = \|f\|_{L^{2q'}(I \to L^{2\nu'}(2\nu', 2\sigma))} \leq 2\|f\|_{L^{2q'}(I \to L^{2\nu'}(2\nu', 2\sigma))},
\]
where the estimate is due to \(\square\). Thus, by Lemma 2.3 and \(\square\),
\[
\|f^\sigma\|_{L^{2q'}(I \to L^{2\nu'}(2\nu', 2\sigma))}^\frac{1}{2q'} \leq 2|I|^\gamma |\delta - \delta'|^{-\frac{1}{2q'}} \left( C_{SI} \int_I \int \psi^2 \mathrm{d}I(f) \, dt + C_{SI0} \int_I \int_{B_1} f^2 \, d\mu \, dt \right)^\frac{\nu'}{2q'} \left( \sup_{t \in I} \|f\|^2_{L^2(B_1)} \right)^{\frac{1}{2q'} - \frac{k}{2q'}},
\]
Now apply Young’s inequality and take the supremum over all pairs $(\nu', q')$ whose Hölder conjugates satisfy \(\square\). \(\square\)

3. Mean value estimates

3.1. Chain rule in the time variable.

Lemma 3.1. Let $u$ be a local weak subsolution of the heat equation for $\mathcal{E}_t$ in $Q^-$. Let $p \geq 2$. Let $\chi : I^- \to [0, 1]$ be any smooth function. Let $\psi \in \mathcal{F}_c(B_1)$. Then, for almost every $a - a_1 < s_0 < t_0 < b'$,
\[
\int_X \mathcal{H}_n(u(t_0)) \psi^2 \, d\mu - \int_X \mathcal{H}_n(u(s_0)) \psi^2 \, d\mu \\
\leq -\int J_t(u(t), \mathcal{H}_n(u(t)) \psi^2) \chi(t) \, dt + \int_X \mathcal{H}_n(u(t)) \psi^2 \chi' \, d\mu \, dt.
\]
where $J = (s_0, t_0)$.

Proof. For a real number $0 < h < b - b'$, let
\[
uh(t) := \frac{1}{h} \int_t^{t+h} u(s) \, ds,
\]
be the Steklov average of $u$ at $t < b'$. Here, the integral is a Bochner integral over functions that take values in the Banach space $(\mathcal{F}, \|\cdot\|)$. By definition, $\nuh(t) \in \mathcal{F}$.

Since $u$ is a local weak subsolution,
\[
\int_X \mathcal{H}_n(u(t_0)) \psi^2 \, d\mu - \int_X \mathcal{H}_n(u(s_0)) \psi^2 \, d\mu \\
= \int_J \frac{d}{dt} \left( \int_X \mathcal{H}_n(uh(t)) \psi^2 \chi \, d\mu \right) \, dt
\]

Proof. First consider the case when \( \psi \) is a non-negative locally bounded local weak supersolution of the heat equation for \( E \) as \( h \to 0 \), at almost every \( t \). Since the \( L^2 \)-norm dominates the \( L^2 \)-norm, it follows that

\[
\int_X \mathcal{H}_n(u_h(t)) \psi^2 \, d\mu \longrightarrow \int_X \mathcal{H}_n(u(t)) \psi^2 \, d\mu \quad \text{at a.e. } t
\]

and

\[
\int_j \int_X \mathcal{H}_n(u_h(t)) \psi' \, d\mu \, dt \longrightarrow \int_j \int_X \mathcal{H}_n(u(t)) \psi' \, d\mu \, dt
\]

as \( h \to 0 \) (passing to a subsequence if necessary).

It remains to show that \(- \int_j \frac{1}{h} \int_t^{t+h} \mathcal{E}_s(u(s), \mathcal{H}_n(u_h(t))) \psi^2 \chi \, ds \, d\mu \) converges to \(- \int_j \mathcal{E}_s(u(t), \mathcal{H}_n(u(t))) \psi^2 \chi \, dt \) as \( h \to 0 \). We have

\[
\mathcal{H}_n(u_h(t)) \psi^2 \longrightarrow \mathcal{H}_n(u(t)) \psi^2
\]

in \( F \) as \( h \to 0 \), at almost every \( t \). Hence, by the right-continuity of \( \mathcal{E}_t \),

\[
\left| \int_j \frac{1}{h} \int_t^{t+h} \mathcal{E}_s(u(s), \mathcal{H}_n(u_h(t))) \psi^2 \chi \, ds \, d\mu \right| \longrightarrow 0,
\]

Applying (10) Theorem 9 with \( f(s) = \mathcal{E}_s(u(s), \mathcal{H}_n(u(t))) \psi^2 \), we see that

\[
\int_j \frac{1}{h} \int_t^{t+h} \left| \mathcal{E}_s(u(s), \mathcal{H}_n(u(t))) \psi^2 \right| \, ds \, d\mu \longrightarrow 0.
\]

Indeed, \( f \) is integrable due to the right-continuity of \( \mathcal{E}_t \). Combining the above and using the right-linearity of \( \mathcal{E}_t \) completes the proof. \( \square \)

Lemma 3.2. Let \( u \) be a non-negative locally bounded local weak supersolution of the heat equation for \( E \) in \( Q^2 \). Let \( p \leq 1 - \eta \). Let \( \chi : I^2 \to [0, 1] \) be any smooth function. Let \( \psi \in F_c(B_1) \). Then, for any interval \( J = (s_0, t_0) \subset I^2 \),

\[
\int_X \mathcal{H}(u_\varepsilon(t_0)) \psi^2 \, d\mu - \int_X \mathcal{H}(u_\varepsilon(s_0)) \psi^2 \, d\mu \\
\geq - \int_J \mathcal{E}_s(u(t), \mathcal{H}'(u_\varepsilon(t)) \psi^2 \chi) \chi \, dt + \int_J \int_X \mathcal{H}(u_\varepsilon(t)) \psi^2 \chi' \, d\mu \, dt.
\]

Proof. First consider the case when \( \varepsilon \) is \( - \). For a real number \( 0 < h < b - b' \), let

\[
(u_\varepsilon)_h(t) := \frac{1}{h} \int_t^{t+h} u_\varepsilon(s) \, ds,
\]

be the (upper) Steklov average of \( u_\varepsilon \) at \( t < b' \).

Since \( u \) is a local weak supersolution,

\[
\int_X \mathcal{H}((u_\varepsilon)_h(t_0)) \psi^2 \, d\mu - \int_X \mathcal{H}((u_\varepsilon)_h(s_0)) \psi^2 \, d\mu \\
= \int_J \frac{d}{dt} \left( \int_X \mathcal{H}((u_\varepsilon)_h(t)) \psi^2 \chi \, d\mu \right) \, dt \\
= \int_j \frac{1}{h} \int_X (u(t + h) - u(t)) \mathcal{H}'((u_\varepsilon)_h(t)) \psi^2 \chi \, d\mu \, dt + \int_J \int_X \mathcal{H}((u_\varepsilon)_h(t)) \psi^2 \chi' \, d\mu \, dt.
\]
\[
\geq - \int_{\frac{1}{h}}^{1} \int_{t}^{t+h} \mathcal{E}_s(u(s), \mathcal{H}'((u_\varepsilon)_h(t))) \psi^2 dt + \int_{X} \mathcal{H}((u_\varepsilon)_h(t)) \psi^2 \chi' d\mu dt.
\]

By [10, Theorem 9], \( u_h(t) \) converges to \( u(t) \) in \( \mathcal{F} \) as \( h \to 0 \), at almost every \( t \). Since the \( \mathcal{F} \)-norm dominates the \( L^2 \)-norm, and since \( (\bar{u}_\varepsilon)_h = u_h + \kappa + \varepsilon \) it follows that

\[
\int_{X} \mathcal{H}((u_\varepsilon)_h(t)) \psi^2 d\mu \rightarrow \int_{X} \mathcal{H}(u_\varepsilon(t)) \psi^2 d\mu \quad \text{at a.e. } t
\]

and

\[
\int_{X} \mathcal{H}((u_\varepsilon)_h(t)) \psi^2 \chi' d\mu dt \rightarrow \int_{X} \mathcal{H}(u_\varepsilon(t)) \psi^2 \chi' d\mu dt
\]
as \( h \to 0 \) (passing to a subsequence if necessary).

It remains to show that \( - \int_{\frac{1}{h}}^{1} \int_{t}^{t+h} \mathcal{E}_s(u(s), \mathcal{H}'((u_\varepsilon)_h(t))) \psi^2 \chi(t) dt ds \) converges to \( - \int \mathcal{E}_s(u(t), \mathcal{H}'(u_\varepsilon(t))) \psi^2 \chi(t) dt ds \) as \( h \to 0 \). We have

\[
\mathcal{H}'((u_\varepsilon)_h(t)) \psi^2 \rightarrow \mathcal{H}'(u_\varepsilon(t)) \psi^2
\]
in \( \mathcal{F} \) as \( h \to 0 \), at almost every \( t \). Hence, by the right-continuity of \( \mathcal{E}_t \),

\[
\left| \int_{\frac{1}{h}}^{1} \int_{t}^{t+h} \mathcal{E}_s(u(s), [\mathcal{H}'((u_\varepsilon)_h(t)) - \mathcal{H}'((u_\varepsilon(t)))]) \psi^2 dt ds \right| \rightarrow 0,
\]

Applying [10, Theorem 9] with \( f(s) = \mathcal{E}_s(u(s), \mathcal{H}'(u_\varepsilon(t))) \psi^2 \), we see that

\[
\int_{\frac{1}{h}}^{1} \int_{t}^{t+h} \mathcal{E}_s(u(s), \mathcal{H}'(u_\varepsilon(t))) \psi^2 | \mathcal{H}'((u_\varepsilon)_h(t)) - \mathcal{H}'((u_\varepsilon(t)))| \psi^2 dt ds \chi(t) dt ds \rightarrow 0.
\]

Indeed, \( f \) is integrable due to the right-continuity of \( \mathcal{E}_t \). Combining the above and using the right-linearity of \( \mathcal{E}_t \) completes the proof in the case \( p \in (-\infty, 0) \). In the case when \( \pm \) is \( + \), we use the (lower) Steklov average of \( \bar{u}_\varepsilon \) at \( t > a' \), defined as

\[
(\bar{u}_\varepsilon)_h(t) := \frac{1}{h} \int_{t-h}^{t} \bar{u}_\varepsilon(s) ds,
\]

where \( 0 < h < a' - a \). Then the proof is as in the previous case.

\[ \square \]

3.2. Estimates for local weak subsolutions.

**Theorem 3.3** (Cacciopoli-type inequality for subsolutions). *Let \( u \) be a local weak subsolution of the heat equation for \( \mathcal{E}_t \) in \( Q^- \). Suppose \( H.1a \) holds for \( u \). Then, for any \( p \geq 2 \),

\[
(10) \quad \frac{1}{2} \sup_{t \in I_{\frac{1}{h}}} \int \bar{u}_\varepsilon \psi^2 d\mu + a \frac{p^2}{4} \int_{I_{\frac{1}{h}}} \int \bar{u}_\varepsilon \psi^2 d\mu dt \\
\leq p^2 C_1 \| \bar{u}_\varepsilon \psi \|_{L^2}^2 + (p^{\beta+1} C_2 + 2 C_3) |\delta' - \delta|^k \int_{I_{\frac{1}{h}}} \int \bar{u}_\varepsilon \psi d\mu dt + (p - 1) \int \kappa \psi^2 d\mu,
\]

provided that the right hand side is finite.

*If, in addition, \( H.1b \) holds for \( u \) and all \( p \in (1 + \eta, 2) \), then [10] also holds for these values of \( p \).*
Proof. Let $\chi = \chi_{\delta', \delta}$ be a smooth function of the time variable $t$ such that $0 \leq \chi \leq 1$, $\chi = 0$ in $(-\infty, a - a_\delta)$, $\chi = 1$ in $(a - a_\delta, \infty)$ and $|\chi'| \leq 2|a_\delta - a_{\delta'}|^{-1}$.

From H.1a and (3), we get
\[
\int_X \mathcal{H}_n(u(t_0)) \psi^2 \, d\mu + \int_{I_0} \int_{I_0} \bar{u} \psi (t_0) \, d\Gamma(u) + \int_{I_0} \int_{I_0} \bar{u} \psi \, d\Gamma(u) \leq pC_1 \|\bar{u} \psi\|_{L^2(I_0 \times B_\delta)} + p^2 C_2 |\delta' - \delta|^{-k} \int_{I_0} \int \bar{u}^2 \psi \, d\mu + \int \int_X \mathcal{H}_n(u) \psi \, d\mu dt.
\]

We multiply each side by $p$, let $n \to \infty$, and take the supremum over $t_0 \in I_0$. By Young’s inequality,
\[
\lim_{n \to \infty} p\mathcal{H}(u_n) = \bar{u}^p - p\bar{u}^{p-1} + (p - 1)\kappa^p \geq \frac{1}{2} \bar{u}^p - (p - 1)\kappa^p.
\]

Hence,
\[
\int_X \left( \frac{1}{2} \bar{u}^p - (p - 1)\kappa^p \right) \psi^2 \, d\mu + \int_{I_0} \int_{I_0} \bar{u} \psi \, d\Gamma(u) \leq p^2 C_1 \|\bar{u} \psi\|_{L^2(I_0 \times B_\delta)} + \int_{I_0} \int (p^2 + C_2 |\delta' - \delta|^{-k} + \bar{u}^p \psi \, d\mu dt,
\]

where we have used that $\kappa \leq \bar{u}$. Finally, apply (2) to estimate $|\chi'|$. \hfill \Box

3.3. Local boundedness and $L^{p, \infty}$ mean value estimates for $p \geq 2$. Define $B_\delta$, $I_0^+$, $I_0^-$, $I_0^+ \times B_\delta$, $Q_\delta$, $Q_\delta^+$ as in Section 2.2.

Lemma 3.4. Let $u$ be a local weak subsolution to the heat equation for $E_t$ in $Q^-$. Suppose (WS1) holds for $\bar{u}(t)$ uniformly for all $t \in I^-$. Then
\[
\sup_{t \in I_0^-} \|\bar{u} \psi\|_{L^2}^2 + \int_{I_0^-} \int_{I_0^-} \psi \, d\Gamma(u) dt \leq C \int_{I_0^-} \int_{B_\delta} \bar{u}^2 \, d\mu dt,
\]

where $\psi = \psi_{\delta, \delta'}$. The constant $C \in (0, \infty)$ depends only on $\beta$, $\gamma$, $a$, $C_1$, and upper bounds for $(C_2 + C_3 + C_\delta) |\delta - \delta'|^{-k}$, $C \bar{u}^p$, and $|I_0^-|$.

Proof. Our proof follows (2) Section 3). Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth function with $\chi = 0$ on $(-\infty, a - a_\delta)$ and $\chi = 1$ on $(a - a_\delta, \infty)$. Due to (2), we may assume that $|\chi'| \leq 2C_3 |\delta - \delta'|^{-k}$. We choose $s_0 \in (a - a_1, a - a_\delta)$ and let $s_n = s_0 + nL$ for some $L > 0$ given below. Let $J = (s_n, s_{n+1})$, assuming that $s_{n+1} \in \Gamma_0^-$. Let $u^+ = \max(u, 0)$ and
\[
X(t) = \frac{1}{2} \int u^+(t)^2 \psi^2 \chi(t) \, d\mu.
\]

As in the proof of Theorem 3.3 with $p = 2$, we have for almost every $t \in J$,
\[
X(t) - X(s_n) + a \int J \psi^2 \, d\Gamma(u) \chi dt \leq 4C_1 \|\bar{u} \psi \chi^{1/2}\|_{L^2(B_\delta)} + 2(2^{\sigma + 1} C_2 + 2 C_3) |\delta' - \delta|^{-k} \int J \bar{u}^2 \psi \chi \, d\mu dt + \int \kappa^2 \psi \, d\mu.
\]

Repeating the proof of Lemma 2.8 with $\sigma = 1$ and $f = \bar{u} \psi \chi^{1/2}$,
\[
\|\bar{u} \psi \chi^{1/2}\|^2_{L^2(B_\delta)} \leq 2L(\frac{C_3}{|\delta - \delta'|^{-k}} \int J \psi^2 \, d\Gamma(u) \chi dt + \frac{C \bar{u}^p}{|\delta - \delta'|^{-k}} \int J \bar{u}^2 \psi \chi \, d\mu dt + \sup_{t \in J} \|\bar{u} \psi\|^2 \chi).
\]
Now we choose
\[ L := \left( \frac{a|\delta - \delta'|^k}{4C_1 \cdot C_{\text{SI}}} \right)^\frac{1}{k}. \]

Then, for almost every \( t \in J \),
\begin{equation}
X(t) + \frac{a}{2} \int J \psi^2 d\Gamma(u) \chi dt \\
\leq X(s_n) + \frac{a}{4} \sup_{t \in J} \|u^+\psi\|_2^2 \chi + \left( \frac{aC_{\text{SI}}}{4C_{\text{SI}}} + \frac{2^{\beta+1}C_2 + 2C_3}{|\delta' - \delta|^{k}} + 2 \frac{1(\kappa > 0)}{L} \right) \int J \int B_s \bar{u}^2 d\mu \chi dt \\
+ 2 \int \kappa^2 \psi^2 d\mu.
\end{equation}

Disregarding the non-negative integral on the left hand side of (11), rearranging, and taking supremum over all \( t \in J \),
\[ \frac{1}{4} \sup_{t \in J} \|u^+\psi\|_2^2 \chi \leq X(s_n) + \left( \frac{aC_{\text{SI}}}{4C_{\text{SI}}} + \frac{2^{\beta+1}C_2 + 2C_3}{|\delta' - \delta|^{k}} + 2 \frac{1(\kappa > 0)}{L} \right) \int J \int B_s \bar{u}^2 d\mu \chi dt. \]

Iterating over the time-intervals \((s_n, s_{n+1})\), we obtain
\begin{equation}
\sup_{t \in I^+_{\delta}} \|u^+\psi\|_2^2 \chi \leq \frac{2^{1+|I^+_{\delta}|}K}{L} + X(s_0),
\end{equation}

where
\[ K := \left( \frac{aC_{\text{SI}}}{4C_{\text{SI}}} + \frac{2^{\beta+1}C_2 + 2C_3}{|\delta' - \delta|^{k}} + 2 \frac{1(\kappa > 0)}{L} \right) \int J \int B_s \bar{u}^2 d\mu \chi dt. \]

By the choice of \( \chi \) and \( s_0 \), we have \( X(s_0) = 0 \).

Putting (12) into (11), using \( X(t) \geq 0 \), and summing over all \((s_n, s_{n+1})\), we get
\[ \frac{a}{2} \int I^+_{\delta} \int J \psi^2 d\Gamma(u) \chi dt \leq K + \left( \frac{a}{4} + 1 \right) \left( \frac{2^{1+|I^+_{\delta}|}K}{L} \right). \]

Simplifying,
\[ \int I^+_{\delta} \int J \psi^2 d\Gamma(u) dt \leq \frac{2}{a} \left( \frac{2^{1+|I^+_{\delta}|}K}{L} \right). \]

\[ \square \]

**Lemma 3.5** (Gain of integrability). Let \( u \) be a local weak subsolution of the heat equation for \( \mathcal{E}_t \) in \( Q^- \). Suppose (WSI) holds for \( \bar{u}(t) \) uniformly for all \( t \in I^- \). Then
\[ \| \bar{u}^\sigma \|_{L^\infty(I^+_{\delta} \times B'_{\delta})} < \infty \]

for all \( \sigma > 1 \) and all \( \delta' \in [\delta^*, 1) \).

**Proof.** Due to Lemma 3.3 \( \bar{u} \) is in \( L^\infty(I^+_{\delta} \rightarrow L^2(B_{\delta})) \). Therefore we can apply Lemma 2.8 to get
\begin{equation}
\| \bar{u}^\sigma \|_{L^\infty(I^+_{\delta} \times B'_{\delta})} \leq 2|I^+_{\delta}| \left( \int I^+_{\delta} \int J \psi^2 d\Gamma(u) dt + \frac{C_{\text{SI}}}{|\delta' - \delta|^{k}} \int I^+_{\delta} \int B_s \bar{u}^2 d\mu dt + \sup_{t \in I^+_{\delta}} \| \bar{u} \|_{L^2(B_{\delta})}^2 \right),
\end{equation}

where \( \psi = \psi_{\delta', \delta} \). The right hand side is finite by Lemma 3.3. \[ \square \]
Theorem 3.6 (Mean value estimate for subsolutions). Let \( u \) be a local weak subsolution of the heat equation for \( \mathcal{E}_t \) in \( Q_- \). Suppose H.1a holds for \( u \) and the weighted Sobolev inequality \( \text{[wSI]} \) holds for \( \bar{u}(t) \) uniformly for all \( t \in I_- \). Let \( p \geq 2 \). Then there exists a positive constant \( C' = C'(\nu, \beta, k) \) such that, for all \( \delta^* \leq \delta' < \delta \leq 1 \),

\[
\sup_{Q_{\delta'}} \{ \bar{u}^p \} \leq \frac{C' A_0^{\frac{p+2}{p}}}{|\delta - \delta'|^{k(p+2)}} \left( \frac{C_1 + 1_{\{ \kappa > 0 \}}}{|I_\delta|} + \frac{aC_{SL0}}{C_{SI}} + C_2 + C_3 \right). 
\]

where

\[
A_0 := \frac{4|I_\delta|^{\gamma+1}C_{S0}p^{\beta+1}}{a|\delta - \delta'|^{2k}} \left( \frac{C_1 + 1_{\{ \kappa > 0 \}}}{|I_\delta|} + \frac{aC_{SL0}}{C_{SI}} + C_2 + C_3 \right). 
\]

If, in addition, H.1b holds for \( p \in (1 + \eta, 2) \), then (14) also holds for these values of \( p \).

Proof. Let \( \chi \) be a smooth function of the time variable \( t \) such that \( 0 \leq \chi \leq 1 \), \( \chi = 0 \) in \( (-\infty, a - a_\delta), \chi = 1 \) in \( (a - a_\delta, \infty) \) and \( |\chi'| \leq 2|a_\delta - a_\delta'|^{-1} \).

Set \( \hat{\delta}_i = (\delta - \delta')2^{-i-1} \) so that \( \sum_{i=0}^{\infty} \hat{\delta}_i = \delta - \delta' \). Set also \( \delta_0 = \delta, \delta_{i+1} = \delta_i - \hat{\delta}_i = \delta - \sum_{j=0}^{i} \hat{\delta}_j \).

Let \( \theta = \frac{\nu}{\beta}. \) Let \( \psi_i = \psi_{\delta_i, \delta_{i+1}} \) be the cutoff function for \( B_{\delta_{i+1}} \) in \( B_{\delta_i} \) that is given by \( \text{[wSI]} \).

As in the proof of Lemma 2.8 but with \( \bar{u}^{p\delta_i/2} \psi_i \) in place of \( f \) and \( \sigma = \theta \), and then applying Theorem 3.3, we get

\[
\| \bar{u}^{p\delta_i/2} \psi_i \|_{L^{2\nu'}(I_{\delta_i} \rightarrow L^{(2\nu'+2)(B_{\delta_i}))}} 
\]

\[
\leq 2|I_{\delta_i}|^\gamma \left( \frac{C_{S1}}{|\delta_i - \delta_{i+1}|^k} \left( \frac{p\theta_i^2}{4} \right)^4 \int_{I_{\delta_i}} \int_{B_{\delta_i}} \bar{u}^{p\theta_i/2} \psi_i^2 d\Gamma(u) dt + \frac{C_{SL0}}{|\delta_i - \delta_{i+1}|^k} \right) ^{1 - \frac{1}{2\nu'}} \sup_{t \in I_{\delta_i}} (\| \bar{u}^{p\theta_i/2} \|_{L^2(B_{\delta_i})}^2)
\]

\[
\leq 2|I_{\delta_i}|^\gamma \left( \frac{C_{S1}}{a|\delta_i - \delta_{i+1}|^k} \right)^2 + 2 \left[ (p\theta_i^2)C_1|\| \bar{u}^{p\theta_i/2} \|_{L^2(I_{\delta_i} \times B_{\delta_i})}^2\right] + \left( \frac{(p\theta_i^2)^3 + 1}{|\delta_i - \delta_{i+1}|^k} \right) \int_{I_{\delta_i}} \int_{B_{\delta_i}} \bar{u}^{p\theta_i} \psi_i^2 d\mu dt + \left( p\theta_i^2 - 1 \right) \int \kappa^{p\theta_i} \psi_i^2 d\mu \right]
\]

By Hölder’s inequality,

\[
\int_{I_{\delta_i}} \int \bar{u}^{p\theta_i} d\mu dt \leq \| 1 \|_{L^1(I_{\delta_i} \rightarrow L^{\infty}(B_{\delta_i}))} \| \bar{u}^{p\theta_i} \|_{L^\infty(I_{\delta_i} \rightarrow L^1(B_{\delta_i}))}
\]

\[
\leq |I_{\delta_i}| \cdot \| \bar{u}^{p\theta_i} \|_{L^2(I_{\delta_i} \times B_{\delta_i})}.
\]

Similarly, by Hölder’s inequality and the fact that \( \kappa \leq \frac{\kappa}{\bar{\kappa}}, \)

\[
\| \bar{u}^{p\theta_i} \|_{L^2(I_{\delta_i} \times B_{\delta_i})} \leq \| \bar{u}^{p\theta_i} \|_{L^2(I_{\delta_i} \times B_{\delta_i})}.
\]

Combining the above estimates and using that \( \psi_i = 1 \) on \( B_{\delta_{i+1}}, \)

\[
\| \| \bar{u}^{p\theta_i/2} \psi_i \|_{I_{\delta_i+1} \times B_{\delta_{i+1}}}
\]

Theorem 3.6 (Mean value estimate for subsolutions). Let \( u \) be a local weak sub-
\[ \leq \frac{4|I_{\delta_0}|^{\gamma+1}C_{SI}p^{\beta+1}}{a|\delta - \delta'|^{2k}} C^i \left( \frac{C_1 + 1_{\{\kappa>0\}}}{|I_{\delta_0}|} + \frac{aC_{SI}^2}{C_{SI}} + C_2 + C_3 \right) \left\lVert \bar{u}^{\rho \theta/2} \right\rVert^2_{L^2_t \times B_\delta}, \]

where \( C \) depends only on \( \theta, \beta \) and \( k \). Iterating the above inequality,

\[ \left\lVert \bar{u}^{\rho \theta + 1} \right\rVert_{L^2_t \times B_{\delta_{i+1}}} \leq C\sum_{j=0}^{\infty} j\theta^{-2} (A_0|\delta - \delta'|^{-2k}) \sum_{j=0}^{\infty} ||u_j^x||^2_{L^2_t \times B_\delta}. \]

This proves (14) in the case \( p \geq 2 \). Now Theorem 3.7 already follows. In the case \( 1 + \eta < p < 2 \), the assertion can be proved in the same way as above, except that we use Theorem 3.7 instead of Lemma 3.5 to verify that the right hand side of (10) is finite.

**Theorem 3.7** (Local boundedness). Under the same hypotheses as in Theorem 3.6 any non-negative local weak subsolution \( u \) of the heat equation for \( \mathcal{E}_t \) is locally bounded. Moreover, if \( u \) is a local weak solution of the heat equation for \( \mathcal{E}_t \) and the hypotheses in Theorem 3.6 hold for both \( u \) and \( -u \), then \( u \) is locally bounded.

**Proof.** In the proof of Theorem 3.6 we have shown that for any local weak subsolution \( u \), \( \max(u, 0) \) is locally bounded. If \( u \) is a weak solution, then the same reasoning applies to \(-u\). \( \square \)

### 3.4. Estimates for local weak supersolutions

Let \( \varepsilon \in (0, 1) \) and recall that \( \bar{u}_\varepsilon := u + \kappa + \varepsilon \).

**Lemma 3.8** (Caccioppoli-type inequality supersolutions). Let \( u \) be a non-negative locally bounded local weak supersolution of the heat equation for \( \mathcal{E}_t \) in \( Q^\pm \). Suppose H.1b holds for \( u \). Then for any \( p \in (\infty, 0) \cup (0, 1 - \eta) \),

\[ \sup_{t \in I_{\delta_0}} \int I_{\delta} \bar{u}_\varepsilon^p \psi^2 d\mu + a\eta \frac{p^2}{4} \int I_{\delta} \int I_{\delta} \bar{u}_\varepsilon^{p - 2} \psi^2 d\Gamma(u) dt \]

\[ \leq (1 + p^2)C_1 \left\lVert \bar{u}_\varepsilon \right\rVert^2_{L^2_t \times B_\delta} + (1 + |p|^{\beta+1})C_2 + 2C_3 \left| \delta' - \delta \right|^{-k} \int I_{\delta} \int \bar{u}_\varepsilon^p \psi^2 d\mu dt, \]

where \( \psi = \psi_{\delta', \delta} \). Here, the superscript \( \pm \) is + when \( p \in (0, 1 - \eta) \) and - when \( p \in (\infty, 0) \).

**Proof.** In the case \( p \in (\infty, 0) \), we let \( \chi : \mathbb{R} \rightarrow [0, 1] \) be a smooth function with \( \chi = 0 \) on \( (-\infty, a - a_\delta) \), \( \chi = 1 \) on \( (a - a_\delta, \infty) \), and \( |\chi'| \leq 2|a_\delta - a_\delta'|^{-1} \). Let \( s_0 \in (a - a_1, a - a_3) \), \( t_0 \in I_{\delta'} \), and set \( J = (s_0, t_0) \). By Lemma 3.2

\[ \int \mathcal{H}(u_\varepsilon(t_0)) \psi^2 d\mu \]

\[ \geq - \int \mathcal{E}_t(u(t), \mathcal{H}'(u_\varepsilon(t)) \psi^2) \chi(t) dt + \int \int \mathcal{H}(u_\varepsilon(t)) \psi^2 \chi' dt d\mu dt. \]
In the case \( p \in (0, 1 - \eta) \), we let \( \chi : \mathbb{R} \to [0, 1] \) be a smooth function with \( \chi = 0 \) in \((b + a_\delta, \infty)\), \( \chi = 1 \) in \((-\infty, b + a_\delta')\) and \( |\chi'| \leq 2|a_\delta - a_\delta'|^{-1} \). Let \( s_0 \in (b + a_\delta, b + a_\delta') \), \( t_0 \in I_\delta^+ \) and set \( J = (t_0, s_0) \). By Lemma 3.2

\[
\int_X \mathcal{H}(u_\varepsilon(t_0))\psi^2 d\mu \\
\leq \int J \mathcal{E}_t(u, \mathcal{H}'(u_\varepsilon(t))\psi^2)\chi(t) dt - \int J \int_X \mathcal{H}(u_\varepsilon(t))\psi^2 \chi' d\mu dt.
\]

In either case, we get from the above inequalities and H.1b that

\[
p \int_X \mathcal{H}(u_\varepsilon(t))\psi^2 d\mu + \frac{|p||p-1|}{4} \int_{I_\delta^+} \int \tilde{u}_\varepsilon^{-2}\psi^2 d\Gamma(u) \chi dt \\
\leq |p| \int J \mathcal{E}_t(u, \mathcal{H}'(u_\varepsilon(t))\psi^2) + \frac{|p-1|}{4} \int \tilde{u}_\varepsilon^{-2}\psi^2 d\Gamma(u) \chi dt + p \int J \int_X \mathcal{H}(u_\varepsilon(t))\psi^2 |\chi'| d\mu dt \\
\leq (1 + p^2)C_1 ||\tilde{u}_\varepsilon^p \psi||^2_{J \times B_\delta} + ((1 + |p|^{2+1})C_2 + 2C_3)|\delta' - \delta|^{-k} \int J \tilde{u}_\varepsilon^p \psi d\mu dt.
\]

Applying Hölder’s inequality as in (15), we get

\[
\int J \int \tilde{u}_\varepsilon^p \psi d\mu dt \leq |J| \cdot ||\tilde{u}_\varepsilon^p||^2_{L^p_\varepsilon \times B_\delta}.
\]

Taking the supremum over \( t_0 \in I_\delta^+ \) proves (16) with \( a \frac{|p-1|}{|p|} \) in place of \( a\eta \). Note, however, that \( \frac{|p-1|}{|p|} \geq \eta \). \( \square \)

The next theorem can be proved analogously to the proof of Theorem 3.6 by applying Lemma 3.8 instead of Theorem 3.3.

**Theorem 3.9** (Mean value estimate for supersolutions). Let \( u \) be a non-negative locally bounded local weak supersolution of the heat equation for \( \mathcal{E}_t \) in \( Q_\delta^+ \). Suppose H.1b holds for \( u \) and the weighted Sobolev inequality \( \text{(WS1)} \) holds for \( \tilde{u}_\varepsilon(t) \) uniformly for all \( t \in I_\delta^+ \). Then there is a positive constant \( C' = C'(\nu, \beta, k, \eta) \) such that the following holds for all \( \delta^* \leq \delta' \leq \delta \leq 1 \).

\[
\sup_{Q_{\delta'}^+} \{\tilde{u}_\varepsilon^p\} \leq \frac{C' A_0^{-\frac{\nu+2}{2(\delta - \delta')k(\nu + 2)}} ||\tilde{u}_\varepsilon^p||^2_{L^p_\varepsilon \times B_\delta}}{\delta - \delta'} \|
\]

where

\[
A_0 := \frac{4|I_\delta^+|^{\nu+1}C_0(1 + |p|)^{\beta+1}}{a\eta|\delta - \delta'|^{2k}} \left( \frac{C_1}{|I_\delta^+|} + \frac{a\eta C_{ SL0} C_{ SI}}{C_{ SI}} + C_2 + C_3 \right)
\]

Here, the superscript \( \pm \) is + when \( p \in (0, 1 - \eta) \) and - when \( p \in (-\infty, 0) \).

4. **Proof of the parabolic Harnack inequality**

4.1. **The abstract lemma of Bombieri - Giusti**. The following lemma extends the “abstract John-Nirenberg inequality” that was first proved by Bombieri and Giusti [5, Theorem 4]. Our proof closely follows [23, Lemma 2.2.6].

We will write \( d\bar{\mu} = d\mu \times dt \).
Lemma 4.1. Let $k_1, k_2 \geq 0$, $\eta \in (0, 1)$, $C \in (0, \infty)$. Let $f$ be a non-negative measurable function on $I_{B_1}^+ \times B_3$ which satisfies
\[
\sup_{I_{B_1}^+ \times B_3} f^p \leq \frac{A_1}{(\delta - \delta')^{k_1}} ||f^2||^{2}_{I_{B_1}^+ \times B_3},
\]
for all $\delta' \leq \delta < 1$, $0 < p < 1 - \eta$. Suppose further that
\[
(17) \quad \tilde{\mu} \left( Q_1^+ \cap \{ \log f > \lambda \} \right) \leq \frac{A_2}{(\delta - \delta')^{k_2}} \frac{\tilde{\mu}(Q_1^+)}{\lambda}, \quad \forall \lambda > 0.
\]
Then there is a constant $A_3 \in [1, \infty)$, depending only on $\delta^*, \eta, \gamma, k_1, k_2, A_1, A_2$, such that
\[
\sup_{I_{B_1}^+ \times B_3} f^p \leq A_3.
\]

Proof. If $(r', r'_1)$ have Hölder conjugates $(r, r_1)$ satisfying (N), then $r', r'_1 \leq \frac{1}{\gamma}$.
Therefore, at the expense of multiplying $A_1(\delta - \delta')^{-k_1}$ by $(||f^2|| \mu(B_1))^2$, we may assume that $|I_{B_1}^+| \mu(B_1) = 1$. Because $|I_{B_1}^+| \mu(B_1) = 1$, increasing the exponent $r$ increases the $L^r$ norm and the $L^{r, \infty}$ quasi-norm, so
\[
||f^2||^{2}_{I_{B_1}^+ \times B_3} \leq ||f^2||^{2}_{L^{r, \gamma}(I_{B_1}^+ \to L^{2, 2\gamma}(B_3))}.
\]
For each Lorentz space $L^{r, \gamma}$ there is a constant constant $K(r, r_1) > 0$ such that the quasi-norm satisfies
\[
(18) \quad ||u + v||_{r, r_1} \leq K(r, r_1) (||u||_{r, r_1} + ||v||_{r, r_1})
\]
for all $u, v \in L^{r, \gamma}$. Define
\[
\phi = \phi(\delta) := \sup_{I_{B_1}^+ \times B_3} f.
\]
Decomposing $I_{B_1}^+ \times B_3$ into the sets where $\log f > \frac{1}{2} \log(\phi)$ and where $\log f \leq \frac{1}{2} \log(\phi)$, we get from (N) and (17) that
\[
||f^2||^{2}_{L^{2, 2\gamma}(I_{B_1}^+ \to L^{2, 2\gamma}(B_3))} \leq K \sup_{I_{B_1}^+ \times B_3} f^p ||1_{f > \phi^{1/2}}||^{2}_{L^{2, 2\gamma}(I_{B_1}^+ \to L^{2, 2\gamma}(B_3))} + K\phi^{p/2} ||1_{f \leq \phi^{1/2}}||^{2}_{L^{2, 2\gamma}(I_{B_1}^+ \to L^{2, 2\gamma}(B_3))}
\]
\[
\leq K\phi^p \left( \frac{2A_2}{(\delta - \delta')^{k_2} \log(\phi)} \right)^{\gamma} + K\phi^{p/2},
\]
for some $K$ depending only on $\gamma$. The two terms on the right hand side are equal if
\[
p = \frac{2}{\log(\phi)} \log \left( \frac{(\delta - \delta')^{k_2} \log(\phi)}{2A_2} \right)^{\gamma}.
\]
We have $p < 1 - \eta$ if $\phi$ is sufficiently large, that is, if
\[
(19) \quad \phi \geq C
\]
for some $C$ depending on $\eta, \gamma, A_2$. Hence, for $\phi \geq C$, the first hypothesis of the lemma yields
\[
\log(\phi(\delta')) \leq \frac{1}{p} \log(2KA_1(\delta - \delta')^{-k_1}) + \frac{\log(\phi)}{2}
\]
\[
\leq \frac{\log(\phi)}{2} \left[ \log \left( \frac{2KA_1(\delta - \delta')^{-k_1}}{2A_2} \right) + 1 \right].
\]
If
\[(20) \quad \left(\frac{(\delta - \delta')k_2 \log \phi}{2A_2}\right)^\gamma \geq (2KA_1(\delta - \delta')^{-k_1})^2,\]
then
\[
\log \phi(\delta') \leq \frac{3}{4} \log \phi.
\]
On the other hand, if (20) or (19) is not satisfied, then
\[
\log \phi(\delta') \leq \log \phi \leq \log C + \frac{2A_2}{(\delta - \delta')k_2} (2KA_1(\delta - \delta')^{-k_1})^{2/\gamma} \leq \frac{A}{(\delta - \delta')k_2 + 2k_1/\gamma},
\]
for some \(A\) depending on \(A_1, A_2, \eta, \gamma\). In all cases, we have
\[(21) \quad \log \phi(\delta') \leq \frac{3}{4} \log \phi(\delta) + \frac{A}{(\delta - \delta')k_2 + 2k_1/\gamma}.
\]
Let \(\delta_j = 1 - \frac{4j}{j+1}\). Iterating (21), we get
\[
\log \phi(\delta^*) \leq \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j \frac{A}{(\delta_{j+1} - \delta_j)k_2 + 2k_1/\gamma} =: A_3 < \infty.
\]
\[\square\]

In order to verify (17) in our context, we need the following “log lemma” which is based on the weighted Poincaré inequality \([\text{wPI}]\). Our proof of the log lemma roughly follows \([\text{wPI}]\) Lemma 5.4.1.

**Lemma 4.2.** Let \(u\) be a non-negative locally bounded local weak solution of the heat equation for \(E_t\) in \(Q^+\). Suppose H.2 holds for \(u\). Suppose \([\text{wPI}]\) holds for \(f = \log \bar{u}_\varepsilon(t)\) uniformly for all \(t\) in \(I^+\), respectively. Then there exists a constant \(c \in (0, \infty)\) depending on \(u(a', \cdot)\) or \(u(b', \cdot)\), respectively, such that, for all \(\lambda > 0\), \(\delta \in [\delta^*, 1]\),
\[
\bar{\mu}(\{(t, z) \in Q^+_\delta : \pm \log \bar{u}_\varepsilon < -\lambda - (\pm c)\}) \leq \frac{3}{\lambda} (1 + \mu(B_1)) \|I^+_\delta\|_1 \left(\frac{C_{\text{int}}}{\lambda|I^+_\delta|} \|D_i\|_q \|I^+_\delta\|_\gamma \lor \|I^+_\delta\|_\gamma \lor \frac{C_2|I^+_\delta|}{\|I^+_\delta\|_1} \right).
\]

Proof. Let \(p = 0\) and \(\psi = \psi_{b,1}\). Hence Lemma 3.2 applied with \(\chi \equiv 1\) yields
\[
\int \log \bar{u}_\varepsilon(t)\psi^2 d\mu - \int \log \bar{u}_\varepsilon(t - h)\psi^2 d\mu = \int \mathcal{H}(u_\varepsilon(t))\psi^2 d\mu - \int \mathcal{H}(u_\varepsilon(t - h))\psi^2 d\mu
\]
\[= -\int_{t-h}^t \mathcal{E}_\varepsilon(u(s), \mathcal{H}'(u_\varepsilon(s))\psi^2) ds,
\]
for any \(t \in I^+_1\) and \(h < a' - a\). Multiplying each side by \(\frac{1}{h}\) and letting \(h \to 0\),
\[
\frac{d}{dt} \int \log \bar{u}_\varepsilon(t)\psi^2 d\mu = -\mathcal{E}_1(u(t), \mathcal{H}'(u_\varepsilon(t))\psi^2),
\]
where \(\frac{d}{dt}\) denotes taking the left-derivative in \(t\). Thus, by H.2,
\[
\frac{d}{dt} \int \log \bar{u}_\varepsilon(t)\psi^2 d\mu + a \int \bar{u}_\varepsilon(t)^{-2}\psi^2 d\mathcal{H}(u(t))
\]
\[\leq \sum_{i=1}^m D_i(t)\|\psi\|_{2\nu_{t',2}}^2 + C_2(1 - \delta)^{-k} \int \psi d\mu =: A_{30}(t)
\]
\[\square\]
for a.e. $t \in I_1^\pm$. Let

$$W(t) := \frac{\int \log \bar{u}_\varepsilon(t) \psi^2 d\mu}{\int \psi^2 d\mu}.$$  

By (WP1),

$$\int |\log \bar{u}_\varepsilon(t) - W(t)|^2 \psi^2 d\mu \leq C_{wp1} \int \bar{u}_\varepsilon^{-2}(t) \psi^2 d\Gamma(u(t)),$$

for a.e. $t \in I_1^\pm$. Hence,

$$\frac{d}{dt} W(t) + \frac{a}{C_{wp1} \int \psi^2 d\mu} \int_{B_3} |\log \bar{u}_\varepsilon(t) - W(t)|^2 \psi^2 d\mu \leq \frac{A_30(t)}{\int \psi^2 d\mu}.$$  

Writing

$$\bar{w}(t, z) = \log \bar{u}_\varepsilon(t, z) + \frac{\int t^{b+a_1} A_{30} ds}{\int \psi^2 d\mu},$$

$$\bar{W}(t) = W(t) + \frac{\int t^{b+a_1} A_{30} ds}{\int \psi^2 d\mu},$$

we obtain for a.e. $t \in I_1^\pm$ that

$$(22) \quad \frac{d}{dt} \bar{W}(t) + \frac{a}{C_{wp1} \int \psi^2 d\mu} \int_{B_3} |\bar{w} - \bar{W}|^2 \psi^2 d\mu \leq 0.$$  

Integrating over $(t, b + a_1)$, we find that $\bar{W}(b + a_1) - \bar{W}(t) \leq 0$. For $\lambda > 0$, set

$$\Omega^+_1(\lambda) = \{ z \in B_3 : \bar{w}(t, z) < -\lambda + \bar{W}(b + a_1) \}.$$  

Then, for a.e. $t \in I_1^\pm$, $z \in \Omega^+_1(\lambda)$,

$$(23) \quad \bar{w}(t, z) - \bar{W}(t) < -\lambda + \bar{W}(b + a_1) - \bar{W}(t) \leq -\lambda.$$  

Applying (23) in the inequality (22),

$$\frac{d}{dt} \bar{W}(t) + \frac{a}{C_{wp1} \int \psi^2 d\mu} \| \lambda - \bar{W}(b + a_1) + \bar{W}(t) \| \mu(\Omega^+_1(\lambda)) \leq 0.$$  

Dividing by $|\lambda - \bar{W}(b + a_1) + \bar{W}(t)|^2$, we can rewrite this inequality as

$$-\frac{d}{dt} |\lambda - \bar{W}(b + a_1) + \bar{W}(t)|^{-1} + \frac{a}{C_{wp1} \int \psi^2 d\mu} \mu(\Omega^+_1(\lambda)) \leq 0,$$

or, equivalently,

$$(24) \quad \mu(\Omega^+_1(\lambda)) \leq \frac{C_{wp1} \int \psi^2 d\mu}{a} \left( \frac{d}{dt} |\lambda - \bar{W}(b + a_1) + \bar{W}(t)|^{-1} \right).$$  

Integrating over $I_1^\pm$,

$$\bar{\mu} \left( \left\{ (t, z) \in Q_3^\pm : \log \bar{u}_\varepsilon(t, z) + \frac{\int t^{b+a_1} A_{30} ds}{\int \psi^2 d\mu} < -\lambda + \bar{W}(b + a_1) \right\} \right) \leq \frac{C_{wp1} \int \psi^2 d\mu}{a \lambda}.$$  

On the other hand,

$$\bar{\mu} \left( \left\{ (t, z) \in Q_3^\pm : \frac{\sum_{i=1}^m D_i(s) \| \psi \|^2_2 \psi \, ds}{\int \psi^2 d\mu} > \frac{\lambda}{3} \right\} \right)$$
where reasoning but uses right-derivatives and the upper Steklov average instead of the Parabolic Harnack inequality.

4.2. The constant $C$ and $C_\pm$ and $C$ of Theorem 4.3 (Parabolic Harnack inequality) be an open interval containing $\tau$ and let $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4 \leq 1$. Set

$$Q^- = (a + \tau_1, a + \tau_2) \times B_\delta,$$

$$Q^+ = (a + \tau_3, a + \tau_4) \times B_\delta.$$

Let $I$ be an open interval containing $[a, a + \tau_4]$ and let $Q = I \times B$.

**Theorem 4.3** (Parabolic Harnack inequality). Let $u$ be a non-negative local weak solution of the heat equation for $E_t$ in $Q$. Suppose $H.1$, $H.2$ hold for $u$. Suppose the weighted Sobolev inequality (wSI) holds for $\bar{u}(t)$ and $\bar{u}_\varepsilon(t)$ uniformly for all $t \in I$ and all small $\varepsilon > 0$. Suppose the weighted Poincaré inequality (wPI) holds for $(\log \bar{u}_\varepsilon(t))$ uniformly for all $t \in I$ and all small $\varepsilon > 0$. Then there is a constant $C_{\text{PHI}} \in (0, \infty)$ such that

$$\sup_{Q^-} \bar{u} \leq C_{\text{PHI}} \inf_{Q^+} \bar{u}.$$

The constant $C_{\text{PHI}}$ depends only on $\tau_1$, $\tau_2$, $\tau_3$, $\tau_4$, $\delta$, $\gamma$, $k$, $\nu$, and upper bounds on $C_{\text{wPI}}(\tau_1 - \tau_2)^{-1}$, $C_{\text{wPI}}(\tau_4 - \tau_2)^{-1}$, $C_\delta$, $C_\gamma$, $C^2_3$, $C_4 T^\gamma$, $\left(C_2 + 2C_3 + \frac{C_\delta}{C_4} \right)^{T^\gamma + 1}$, $\|D_1\|_{q_1}(T^\gamma \vee T)$, where $T = \tau_2 \vee (\tau_4 - \tau_2)$.\]
Proof. By Theorem 3.7, $u$ is locally bounded, so the mean value estimates of Theorem 3.9 hold. Let $A_1 = C'A_0^2\frac{2}{2}$ and $k_1 = k(\nu + 2)$. By Lemma 4.2, there exists a positive constant 

$$c = \int \log \bar{u}(a + \tau_2)\psi^2 d\mu$$

such that the hypotheses of Lemma 4.1 are satisfied with $f = (\bar{u}_\varepsilon e^c)$ on $I_1^+ = (a, a + \tau_2)$ and with $f = (\bar{u}_\varepsilon e^c)^{-1}$ on $I_2^- = (a + \tau_2, a + \tau_4)$. We obtain that there exist positive constants $A_3, A'_3$ such that

$$\sup_{Q^+} (\bar{u}_\varepsilon e^c)^{-1} \leq A_3,$$

and

$$\sup_{Q^-} (\bar{u}_\varepsilon e^c) \leq A_3'$$

for any $\varepsilon \in (0, 1)$.

Hence,

$$\sup_{Q^-} \bar{u}_\varepsilon \leq e^{-c} A_3 \leq A_3 \frac{A'_3}{\sup_{Q^+} \bar{u}_\varepsilon} \leq A_3 A'_3 \inf_{Q^+} \bar{u}_\varepsilon.$$

Letting $\varepsilon \to 0$ on both sides finishes the proof. \hfill $\square$

5. Examples

5.1. Quasilinear forms adapted to a Dirichlet form.

5.1.1. Dirichlet spaces with induced metric. Let $(X, d)$ be a locally compact separable metric space and $\mu$ a locally finite Borel measure on $X$ with full support. Any symmetric strongly local regular Dirichlet form $(E, F)$ on $L^2(X, \mu)$ induces a pseudo-metric

$$d_E(x, y) := \sup \{ f(x) - f(y) : f \in F_{loc} \cap C(X), d\Gamma(f, f) \leq d\mu \},$$

where $d\Gamma(f, f)$ is the energy measure of $(E, F)$, $C(X)$ is the space of continuous functions on $X$, and

$$F_{loc}(U) := \{ f \in L^2_{loc}(U) : \forall \text{ compact } K \subset U, \exists f^K \in F, f^K |_K = f |_K \mu\text{-a.e.} \}.$$ 

For an open subset $Y \subset X$, we consider

(A1) $d_E$ is a (finite, non-degenerate) metric which generates the original topology on $X$,

(A2) for every $B(x, 2R) \subset Y$, the ball $B(x, R)$ is relatively compact.

If (A1) and (A2) are satisfied on $Y$, then there exists a cutoff function for $B(x, R)$ in $B(x, R + r)$ such that

$$d\Gamma(\psi, \psi) \leq 2r^{-2} d\mu,$$

provided that $0 < r \leq R$ and $B(x, 2R) \subset Y$.

For instance, (A1) and (A2) are satisfied by the canonical Dirichlet forms on $\mathbb{R}^n$, Riemannian manifolds $(M^n, g)$ with Ricci curvature bounded below, or Riemannian complexes (see [21]). These spaces are known to satisfy the volume doubling property and the scale-invariant Poincaré inequality up to some scale $R_0 \in (0, \infty)$ which depends on a lower curvature bound. Volume doubling and Poincaré inequality
imply that for any \( x \in X, R \in (0,R_0), B_\delta = B(x,\delta R) \), and any \( f \in \mathcal{F}_*(B(x,R)) \), the weighted Sobolev inequality \((\text{wSI})\) holds with \( k = 0 \) and

\[
C_{\text{SI}} = CR^2 \mu(B(x,R))^{-2/\nu},
\]

\[
C_{\text{Slb}} = C' \mu(B(x,R))^{-2/\nu}
\]

and the weighted Poincaré inequality \((\text{wPI})\) holds with

\[
C_{\text{wPI}}(\delta', \delta) = C'' R^2.
\]

for some constant \( C, C', C'' \in (0, \infty) \) that may depend on \( \delta^* \) but not on \( \delta, \delta' \).

The parabolic Harnack inequality on Dirichlet spaces satisfying (A1) and (A2) is studied in [25] under the hypothesis that the scale-invariant Poincaré inequality and the doubling property hold locally on a subset \( Y \) up to scale \( R_0 > 0 \), that is, for balls \( B(x,R) \) with \( B(x,4R) \subset Y \) and \( R \leq R_0/4 \). Then a scale-invariant parabolic Harnack inequality holds on \( Y \) up to scale \( R_0 \). Though Sturm does not present the proof of this result in reasonably full detail (cf. the discussion in [16]) and particularly an argument like the chain rules for weak time-derivatives in Section 3.1 are not given in [24, 25], we would like to mention that, in the special case of a symmetric strongly local regular (time-dependent) Dirichlet form as considered in [25], it was communicated to the author by K.-T. Sturm that it is possible to give a simpler proof by replacing \( \mathcal{H}_n \) by a twice continuously differentiable function. More precisely, the author has verified that the argument works with

\[
\mathcal{H}_n(v) = \frac{1}{2} v^2 (v \wedge n)^{p-2} - \left( 1 - \frac{1}{p-1} \right) v(v \wedge n)^{p-1} + \left( \frac{1}{2} - \frac{1}{p-1} + \frac{1}{p-1} \right) v_n^p.
\]

Unfortunately, it seems that this simpler argument does not extend beyond the special case of symmetric strongly local Dirichlet forms.

5.1.2. Adapted parabolic forms satisfy \( H.1 \) and \( H.2 \). In this subsection we show that parablic forms that are adapted to a reference Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) satisfy hypotheses \( H.1 \) and \( H.2 \), provided that the underlying space admits appropriate cutoff functions.

**Definition 5.1.** We say that a parabolic form \( \mathcal{E}_t \) is adapted to \( (\mathcal{E}, D(\mathcal{E})) \) if the domain of \( \mathcal{E}_t \) is \( \mathcal{F} = D(\mathcal{E}) \), and there is a positive integer \( m \) such that

(i) (Generalized uniform coerciveness) for all \( u \in \mathcal{F} \),

\[
\text{(26)} \quad d\mathcal{A}_t(u,u) \geq \alpha d\Gamma(u,u) - \sum_{i=1}^m b_i^2 u_i^2 d\mu - \sum_{i=1}^m w_i^2 d\mu
\]

(ii) (Generalized sector condition) for all \( u,v \in \mathcal{F}, f : X \to \mathbb{R} \) bounded Borel measurable, \( g \in L^2(X, d\Gamma(v,v)) \),

\[
\int fg d\mathcal{A}_t(u,v) \leq \left( \bar{a} \left( \int f^2 d\Gamma(u,u) \right)^{1/2} + \sum_{i=1}^m \| e_i \|_{r_i,\infty} \| fu \|_{r_i''} + \sum_{i=1}^m \| w_i \|_{r_i,\infty} \| f \|_{r_i''} \right) \left( \int g^2 d\Gamma(v,v) \right)^{1/2},
\]

where \( \bar{a} \) is the \( \bar{a} \) in \( \alpha d\Gamma \).
and, for all \( u, v \in \mathcal{F} \), and all bounded Borel measurable functions \( f \) and \( g \) on \( X \),

\[
\left| \int fg \, dB_t(u,v) \right| \leq \sum_{i=1}^{m} \|c_i\|_{r_i,\infty} \left( \int f^2 \, d\Gamma(u,u) \right)^{1/2} \|g\|_{r'_i,2} + \sum_{i=1}^{m} \|d_i\|_{r_i,\infty} \|fu\|_{2r'_i,2} \|gv\|_{2r'_i,2} + \sum_{i=1}^{m} \|w_{2,i}\|_{r_i,\infty} \|f\|_{2r'_i,2} \|g\|_{2r'_i,2}.
\]

Here, \( a \) and \( \tilde{a} \) are positive constants and the “coefficients” \( b_i, c_i, d_i, e_i, w_{1,i}, w_{2,i}, w_{3,i} \) are non-negative functions of \((x,t)\) and each coefficient is in \( L^6(I \to L^{r_i,\infty}(B)) \) for some \((r_i,q_i)\). The pair \((r_i,q_i)\) may be different for each coefficient but, for some fixed \( \gamma > 0 \), all pairs \((r_i,q_i)\) must satisfy \([5]\).

**Proposition 5.2.** Suppose the reference Dirichlet form \((\mathcal{E}, \mathcal{F})\) satisfies (A1)-(A2). Let \( I \) be a bounded open time-interval and \( U \subset X \) open. If \( \mathcal{E}_i \) is a quasilinear form adapted to \((\mathcal{E}, \mathcal{F})\) then \( \mathcal{E}_i \) satisfies H.1a, H.1b and H.2 for all \( u \in L^2_{loc}(I \to L^2(U)) \) with \( \kappa = \sum_i \|w_{1,i}\| + \|w_{2,i}\| + \|w_{3,i}\| \).

In the following results we assume the volume doubling property and the Poincaré inequality “locally up to scale \( R_0 > 0 \)”. For the precise definitions of these properties, we refer to [10].

**Theorem 5.3** (Scale-invariant parabolic Harnack inequality). Suppose the reference Dirichlet form \((\mathcal{E}, \mathcal{F})\) satisfies (A1)-(A2), volume doubling and the scale-invariant Poincaré inequality on \( Y \subset X \) up to scale \( R_0 > 0 \). Let \( \mathcal{E}_i \) be a quasilinear form adapted to \((\mathcal{E}, \mathcal{F})\). Then \( \mathcal{E}_i \) satisfies the scale-invariant parabolic Harnack inequality up to scale \( R_0 \): There is a positive constant \( C_{PHI} \) such that for any \( s \in \mathbb{R} \), any ball \( B(x,R) \) with \( B(x,2R) \subset Y \) and \( R \leq R_0 \), and for any non-negative local weak solution \( u \) for \( \mathcal{E}_i \) in \( Q = (s,s+\tau R^2) \times B(x,R) \), it holds

\[
\sup_{Q^-} u + \kappa \leq C_{PHI} (\inf_{Q^+} u + \kappa),
\]

where \( Q^- = (s + \frac{1}{4}\tau R^2, s + \frac{1}{2}\tau R^2) \times B(x,\delta R) \) and \( Q^+ = (s + 3\tau R^2, s + \tau R^2) \times B(x,\delta R) \), and \( \kappa = \sum_i \|w_{1,i}\| + \|w_{2,i}\| + \|w_{3,i}\| \).

The constant \( C_{PHI} \) depends only on \( \tau, \delta, a, \tilde{a} \), the norms of the coefficients in their respective spaces, the volume doubling constant, the Poincaré constant, and - unless \( \gamma, b, c, d, e, w_1, w_2, w_3 \) all vanish - also on an upper bound on \( R_0^2 \).

For the proofs of Proposition 5.2 and Theorem 5.3 we need two lemmas stated below.

**Theorem 5.4.** Suppose the reference Dirichlet form \((\mathcal{E}, \mathcal{F})\) satisfies (A1)-(A2), volume doubling and the scale-invariant Poincaré inequality on \( Y \subset X \) up to scale \( R_0 > 0 \). Let \( \mathcal{E}_i \) be a quasilinear form adapted to \((\mathcal{E}, \mathcal{F})\).

Let \( u \) be a non-negative local weak solution of the heat equation for \( \mathcal{E}_i \) in \( Q = (s,s+\tau R^2) \times B(x,R) \) where \( s \in \mathbb{R} \), \( B(x,2R) \subset Y \) and \( R \leq R_0 \). Then \( u \) has a continuous version which satisfies

\[
\sup_{(t,y),(t',y') \in Q'} \left\{ \frac{|u(t,y) - u(t',y')|}{|t - t'|^{1/2} + d(y,y')^{\alpha}} \right\} \leq C \sup_{Q} |\bar{u}|^{\alpha}
\]
where \( Q' = (s + (1 - \delta)\pi R^2, s + \tau R^2) \times B(x, \delta R) \). The constant \( C > 0 \) and the Hölder exponent \( \alpha > 0 \) depend at most on \( \tau, \delta, \gamma, a, \bar{a} \), the norms of the coefficients \( b_i, c_i, d_i, e_i, w_{1,i}, w_{2,i}, w_{3,i} \) in their respective spaces, the volume doubling constant, the Poincaré constant, and - unless the coefficients all vanish - also on an upper bound on \( R_0^2 \).

Proof. We omit the proof because it is a standard application of the parabolic Harnack inequality which we proved in Theorem 4.3. See [2] Theorem 4] for details.

\[ \square \]

Remark 5.5. Assumptions (A1)-(A2) in Proposition 5.3 Theorem 5.4 and in the maximum principle of Theorem 5.3 can be relaxed: We may instead assume that (A2) holds for metric balls in \((X, d)\), and the cutoff Sobolev inequality on annuli, CSA(\( \Psi \)), holds (see [1] for the definition). In this case, the time-space scaling has to be changed in the obvious way from \( R^2 \) to \( \Psi(R) \) in the Poincaré inequality, the Sobolev inequality and in Theorem 5.3, and from \( |t - t'|^{1/2} \) to \( \Psi^{-1}(|t - t'|) \) in Theorem 5.4. The constants \( C \) will then depend also on the constants and exponents appearing in CSA(\( \Psi \)).

Lemma 5.6. If \( \mathcal{E}_t \) is a quasilinear form adapted to \( (\mathcal{E}, \mathcal{F}) \) with \( m = 1 \) then, for any \( t \in \mathbb{R} \), any non-negative \( u \in \mathcal{F} \), \( \kappa > 0 \), \( n \geq \kappa \) positive integer, \( p \in [2, \infty) \),

\[
\begin{align*}
- \mathcal{E}_t(u(t), \mathcal{H}'(u(t))\psi^2) &+ \frac{\alpha}{2} \int \bar{u}^{p-2}\psi^2 d\Gamma(u, u) + (p - 2)a \int_{\{\bar{u} \leq n\}} \bar{u}^{p-2}\psi^2 d\Gamma(u, u) \\
&\leq \left( p - 1 \right) \left( \frac{\|\bar{b}\|_{r, \infty}^2}{\kappa} + \frac{\|\bar{w}_1\|_{r, \infty}^2}{\kappa} + \frac{\|\bar{w}_2\|_{r, \infty}^2}{\kappa} + \frac{\|\bar{w}_3\|_{r, \infty}^2}{\kappa} + \frac{\|\bar{d}\|_{r, \infty}^2}{\kappa} \right) \|\bar{u}^{p-2}\psi\|_{r, 2}^2 \\
&+ 2 \left( \|\bar{d}\|_{r, \infty} + \frac{\|\bar{w}_2\|_{r, \infty}}{\kappa} \right) \|\bar{u}^{p-2}\bar{u}\psi\|_{r, 2}^2 + 4 \left( \frac{4\bar{a}^2 + 1}{\alpha} \right) \int \bar{a}^2 \bar{u}^{p-2} d\Gamma(\psi, \psi).
\end{align*}
\]

Proof. It suffices to give the proof in the case \( m = 1 \). We use the decomposition \( \mathcal{E}_t(f, g) = \int dA_t(f, g) + \int dB_t(f, g) \) and estimate each integral separately. We write \( u \) for \( u(t) \) and \( u_n \) for \( u_n(t) \). By the chain rule, right strong locality and right linearity, we have

\[
- \int dA_t(u, \mathcal{H}'(u)\psi^2)
= - \int dA_t(u, (\bar{u} \bar{u}^{p-2} + \kappa^{p-1})\psi^2)
= - \int \bar{u}^{p-2}\psi^2 dA_t(u, u) - \int 2(\bar{u} \bar{u}^{p-2} + \kappa^{p-1})\psi dA_t(u, \psi) - (p - 2) \int \bar{u} \bar{u}^{p-3}\psi^2 dA_t(u, \bar{u}).
\]

By right strong locality and right linearity,

\[
\int \bar{u} \bar{u}^{p-3}\psi^2 dA_t(u, \bar{u}) = \int_{\{\bar{u} \leq n\}} \bar{u} \bar{u}^{p-3}\psi^2 dA_t(u, \bar{u}) = \int_{\{\bar{u} \leq n\}} \bar{u} \bar{u}^{p-2}\psi^2 dA_t(u, u).
\]

Thus, by (26) and (27),

\[
- \int dA_t(u, \mathcal{H}'(u(t))\psi^2)
= - \int \bar{u} \bar{u}^{p-2}\psi^2 dA_t(u, u) - \int 2(\bar{u} \bar{u}^{p-2} + \kappa^{p-1})\psi dA_t(u, \psi) - (p - 2) \int \bar{u} \bar{u}^{p-2}\psi^2 dA_t(u, u)
\]
\[ \leq -a \int \tilde{u}_n^{p-2} \psi^2 d\Gamma(u, u) - (p - 2)a \int_{\{\tilde{u} \leq n\}} \tilde{u}_n^{p-2} \psi^2 d\Gamma(u, u) \\
+ (p - 1) \left[ \int b^2 u^2 \tilde{u}_n^{p-2} \psi^2 d\mu + \int w_1^2 \tilde{u}_n^{p-2} \psi^2 d\mu \right] \\
+ 2 \left( \tilde{a} \left( \int \tilde{u}_n^{p-2} \psi^2 \Gamma(u, u) \right)^{1/2} + \|e\|_{r, \infty} \left\| \tilde{u}_n^{p-2} \psi \right\|_{r', 2}^{p-2} + \|w_3\|_{r, \infty} \left\| \tilde{u}_n^{p-2} \psi \right\|_{r', 2}^{p-2} \right) \\
\left( \int \tilde{u}_n^{2} \tilde{u}_n^{p-2} \Gamma(\psi, \psi) \right)^{1/2} \\
+ 2 \left( \tilde{a} \left( \int \kappa^{p-2} \psi^2 \Gamma(u, u) \right)^{1/2} + \|e\|_{r, \infty} \left\| \kappa \tilde{u}_n^{p-2} \psi \right\|_{r', 2}^{p-2} + \|w_3\|_{r, \infty} \left\| \kappa \tilde{u}_n^{p-2} \psi \right\|_{r', 2}^{p-2} \right) \\
\left( \int \kappa^{2} \Gamma(\psi, \psi) \right)^{1/2} . \]

By right linearity, the chain rule, and (28),

\[- \int d\mathcal{B}_t(u, \mathcal{H}_n'(u)) \psi^2 \]
\[= - \int d\mathcal{B}_t(u, (\tilde{u} \tilde{u}_n^{p-2} + \kappa^{p-1}) \psi^2) \]
\[= - \int \tilde{u}_n^{p-2} \psi d\mathcal{B}_t(u, \tilde{u} \psi) - \int \kappa^{p-2} \psi d\mathcal{B}_t(u, \kappa \psi) \]
\[\leq \|c\|_{r, \infty} \left( \int \tilde{u}_n^{p-2} \psi^2 \Gamma(u, u) \right)^{1/2} \left\| \tilde{u}_n^{p-2} \tilde{u} \psi \right\|_{r', 2}^{p-2} \\
+ \left( \|d\|_{r, \infty} \left\| \tilde{u} \tilde{u}_n^{p-2} \psi \right\|_{2r', 2}^{p-2} + \|w_2\|_{r, \infty} \left\| \tilde{u}_n^{p-2} \psi \right\|_{2r', 2}^{p-2} \right) \left\| \tilde{u}_n^{p-2} \tilde{u} \psi \right\|_{2r', 2}^{p-2} \]
\[+ \|c\|_{r, \infty} \left( \int \kappa^{p-2} \psi^2 \Gamma(u, u) \right)^{1/2} \left\| \kappa \tilde{u}_n \psi \right\|_{r', 2}^{p-2} \\
+ \left( \|d\|_{r, \infty} \left\| \kappa \tilde{u}_n^{p-2} \psi \right\|_{2r', 2}^{p-2} + \|w_2\|_{r, \infty} \left\| \kappa \tilde{u}_n^{p-2} \psi \right\|_{2r', 2}^{p-2} \right) \left\| \kappa \tilde{u}_n \psi \right\|_{2r', 2}^{p-2} . \]

Combining the above estimates and using the fact that \( \kappa \leq \tilde{u}_n \) for \( n \geq \kappa \),

\[- \mathcal{E}_t(u(t), \mathcal{H}_n'(u(t))) \psi^2 + a \int \tilde{u}_n^{p-2} \psi^2 \Gamma(u, u) + (p - 2)a \int_{\{\tilde{u} \leq n\}} \tilde{u}_n^{p-2} \psi^2 \Gamma(u, u) \]
\[\leq (p - 1) \int \left( b^2 + \frac{w_1^2}{\kappa^2} \right) \tilde{u}_n^{2} \tilde{u}_n^{p-2} \psi^2 d\mu \]
\[+ 4 \left( \tilde{a} \left( \int \tilde{u}_n^{p-2} \psi^2 \Gamma(u, u) \right)^{1/2} + \|e\|_{r, \infty} \left\| \tilde{u} \tilde{u}_n^{p-2} \psi \right\|_{r', 2}^{p-2} + \|w_3\|_{r, \infty} \left\| \tilde{u}_n^{p-2} \psi \right\|_{r', 2}^{p-2} \right) \]
\[\left( \int \tilde{u}_n^{2} \tilde{u}_n^{p-2} \Gamma(\psi, \psi) \right)^{1/2} \\
+ 2 \|e\|_{r, \infty} \left( \int \tilde{u}_n^{p-2} \psi^2 d\Gamma(u, u) \right)^{1/2} \left\| \tilde{u}_n^{p-2} \tilde{u} \psi \right\|_{r', 2}^{p-2} \]
\[+ 2 \left( \|d\|_{r, \infty} \left\| \tilde{u} \tilde{u}_n^{p-2} \psi \right\|_{2r', 2} + \|w_2\|_{r, \infty} \left\| \tilde{u}_n^{p-2} \psi \right\|_{2r', 2} \right) \left\| \tilde{u}_n^{2} \tilde{u} \psi \right\|_{2r', 2} . \]
By (26) and (27),
\[ \| b^2 u^2 \hat{u}_u^2 \|_{1,1} \leq \| b \hat{u} u^2 \psi \|_{2,2}^2 \leq \| b \|_{r,\infty} \| \hat{u} u^2 \psi \|_{r',2}^2, \]
and similarly for \( \frac{a}{\kappa} \) in place of \( b \). Now the assertion follows from Young’s inequality.  

Lemma 5.7. If \( \mathcal{E}_t \) is a quasilinear form adapted to \((\mathcal{E}, F)\) with \( m = 1 \) then, for any \( t \in \mathbb{R} \),
\[
\frac{1 - p}{1 - p} \mathcal{E}_t(u(t), \mathcal{H}_t(u(t))) = (p - 1) \int \hat{u}_x^p \psi^2 d\Gamma(u, u)
\]
\[
\leq \left( |p - 1| \left( \| b \|_{r,\infty}^2 + \frac{\| w_1 \|_{r,\infty}^2}{\kappa} \right) + \| e \|_{r,\infty}^2 + \frac{\| w_3 \|_{r,\infty}^2}{\kappa} + \frac{1}{|p - 1|} \| e \|_{r,\infty}^2 \right) \| \hat{u}_x^2 \psi \|_{r',2}^2
\]
\[
+ \left( \| d \|_{r,\infty} + \frac{\| w_2 \|_{r,\infty}}{\kappa} \right) \| \hat{u}_x \psi \|_{2',2}^2 + \left( \frac{4 \tilde{a}^2}{a|p - 1|} + 1 \right) \int \hat{u}_x^p d\Gamma(\psi, \psi).
\]
for all non-negative locally bounded \( u \in \mathcal{F}, t \in \mathbb{R}, p \in (-\infty, 1 - \eta) \cup (1 + \eta, 2) \).

Proof. By the chain rule, right strong locality and right linearity, we have
\[
\int d\mathcal{A}_t(u, \mathcal{H}_t(u)) = (p - 1) \int \hat{u}_x^p \psi^2 d\mathcal{A}_t(u, u) + \int 2\hat{u}_x^p \psi d\mathcal{A}_t(u, \psi).
\]
Thus, by (20) and (27),
\[
\frac{1 - p}{1 - p} \int d\mathcal{A}_t(u, \mathcal{H}_t(u)) \psi^2
\]
\[
\leq |p - 1| \left[ -a \int \hat{u}_x^p \psi^2 d\Gamma(u, u) + \int b^2 u^2 \hat{u}_x^p \psi^2 d\mu + \int w_3^2 \hat{u}_x^p \psi^2 d\mu \right]
\]
\[
+ 2 \left( \tilde{a} \left( \int \hat{u}_x^p \psi^2 d\Gamma(u, u) \right)^{1/2} + \| e \|_{r,\infty} \| \hat{u}_x \psi \|_{r',2} + \| w_3 \|_{r,\infty} \| \hat{u}_x \psi \|_{r',2} \right) \int \hat{u}_x^p d\Gamma(\psi, \psi)^{1/2}.
\]
By the chain rule and (28),
\[
\left| \int d\mathcal{B}_t(u, \mathcal{H}_t(u)) \psi^2 \right| = \left| \int \hat{u}_x^p \psi d\mathcal{B}_t(u, \psi) \right|
\]
\[
\leq \| e \|_{r,\infty} \left( \int \hat{u}_x^p \psi^2 d\Gamma(u, u) \right)^{1/2} \| \hat{u}_x \psi \|_{r',2}
\]
\[
+ \left( \| d \|_{r,\infty} \| \hat{u} u^2 \hat{u}_x \psi \|_{2',2} + \| w_2 \|_{r,\infty} \| \hat{u} u^2 \hat{u}_x \psi \|_{2',2} \right) \| \hat{u}_x \psi \|_{2',2}.
\]
Now the assertion follows from the fact that \( \kappa \leq \tilde{u} \), (3), (4), and Young’s inequality.  

Remark 5.8. It is clear that Lemma 5.6 and Lemma 5.7 generalize in the obvious way to the case \( m > 1 \).

Proof of Proposition 5.3. This is immediate from Lemma 5.6 Lemma 5.7 Remark 5.8 Hölder’s inequality and (28).
Proof of Theorem 5.8. Apply Proposition 5.2, Lemma 5.6, Lemma 5.7, Remark 5.8 and Theorem 4.3 with $\kappa = 0$. □

Theorem 5.9 (Maximum Principle). Let $\mathcal{E}$ be a quasilinear form adapted to $(\mathcal{E}, \mathcal{F})$. Suppose $(\mathcal{E}, \mathcal{F})$ satisfies (A1)-(A2), volume doubling and Poincaré inequality. Let $u$ be a local weak solution of the heat equation for $\mathcal{E}$ in $Q = (s, T) \times U$ where $U \subset X$ is an open subset. Let $M \in \mathbb{R}$ and suppose $(u + M)^+(t) \in \mathcal{F}(U)$ for every $t \in (s, T)$ and $(u + M)^+(t) \to 0$ in $L^2(U)$ as $t \to 0$. Then

$$u(t, x) \leq M + C((\|b\| + \|d\|)|M| + \kappa) \quad \text{a.e. in } Q,$$

where $\kappa = \|w_1\| + \|w_2\|$ and the constant $C$ depends only on $(T - s)$, $\mu(U)$, $\gamma$, $\nu$, $C_{\text{SI}}$, and the norms of the coefficients in their respective spaces.

Proof. We first prove the maximum principle in the case $M = 0$. Let $(t, x) \in Q$. Choose an appropriate increasing sequence of neighborhoods $(x, t) \in Q_{\delta'} \subseteq Q_{\delta} \subset Q$ satisfying (2), for all $\frac{1}{2} \leq \delta' < \delta \leq 1$. Applying the mean value estimate of Theorem 3.2 and Lemma 2.3,

$$\bar{u}^2(t, x) \leq C'(\nu)A_0^{\frac{\nu+2}{2}} \|\bar{u}\|^2_{Q_{\delta'}} \leq 2C'(\nu)A_0^{\frac{\nu+2}{2}} |I_{\delta'}| \left( C_{\text{SI}} \int_{I_{\delta'}} \int \psi^2 d\Gamma(u) dt + \sup_{t \in I_{\delta'}} \|\bar{u}\|^2_{L^2(B_{\delta})} \right)$$

where

$$A_0 := \frac{32|I_{\delta'}|\gamma C_{\text{SI}}}{a}(C_1 + 1_{\{\kappa > 0\}}).$$

To estimate the right hand side, we repeat the reasoning in the proof of Lemma 3.4 except that we can omit $\psi$ and $\chi$ due to the boundary condition and therefore $K = (T - s)\mu(B_1)\kappa^2$.

$$\sup_{Q_{\delta'}} \bar{u}^2 \leq C(T - s)\mu(B_1)\kappa^2,$$

where the constant $C$ depends only on $(T - s)$, $\mu(U)$, $C_{\text{SI}}$, $\nu$, $\gamma$, and the norms of the coefficients in their respective spaces. This completes the proof in the case $M = 0$.

If $M \neq 0$, notice that $u - M$ satisfies the zero boundary conditions, and $u - M$ is a local weak subsolution to the heat equation for the quasilinear form

$$\mathcal{E}^M_t(f, g) := \mathcal{E}_t(f + M, g).$$

Since $(\mathcal{E}, \mathcal{F})$ is also adapted to $(\mathcal{E}, \mathcal{F})$ we can now apply the case $M = 0$ to $u - M$. Just note that $\kappa$ must be replaced by $\kappa^M = (\|b\| + \|d\|)|M| + \|w_1\| + \|w_2\|$, see [2].

Proof of Theorem 1. □

Further standard applications of the parabolic Harnack inequality apply to the present setting, for instance, the elliptic Harnack inequality, and various pointwise estimates for weak solutions. Since these applications are well-known and to avoid repetition we keep this section short and only state the following pointwise estimate. For further results see, e.g., [2, Theorem 5'] and [23, Section 5.4.3].
Theorem 5.10 (Pointwise estimate). Let $\mathcal{E}_t$ be a quasilinear form adapted to $(\mathcal{E}, \mathcal{F})$. Suppose $(\mathcal{E}, \mathcal{F})$ satisfies (A1)-(A2), volume doubling and Poincaré inequality up to scale $R > 0$. Then there is a constant $C \in (0, \infty)$ such that the following pointwise inequality holds. Suppose there is a continuous curve of length $d$ joining two points $x, y \in X$. Let $U$ be a $\delta$-neighborhood of this curve where $\delta > 0$. Let $0 < s < t < T$ and let $u$ be a non-negative local weak solution of the heat equation for $\mathcal{E}_t$ in $Q = (0, T) \times U$. Then

$$
\log \frac{u(s, x)}{u(t, y)} + \kappa \leq C \left( 1 + \frac{t-s}{R^2} + \frac{t-s}{s} + \frac{t-s}{d^2} + \frac{d^2}{t-s} \right),
$$

where $\kappa = \sum_i \|w_{1,i}\| + \|w_{2,i}\| + \|w_{3,i}\|$.

Proof. This follows by applying the parabolic Harnack inequality of Theorem 5.3 successively along a Harnack chain connecting $x$ to $y$ within $U$. For details, we refer to [23] Proof of Corollary 5.4.4. $\square$

5.2. The structural hypotheses of Aronson-Serrin. Let $(M^n, g)$ be a smooth complete Riemannian manifold without boundary with Riemannian volume element $d\mu$. Let $\mathcal{E}(u, g) = \int_M \nabla u \nabla g \, d\mu$ for $u, g \in \mathcal{F} = W^{1,2}(M^n)$. Suppose that $M^n$ has a lower Ricci curvature bound. Then the volume doubling property and the Poincaré inequality are known to hold locally. It is also clear that suitable cutoff functions exist in the present setting. In particular, the weighted Sobolev inequality $w$S1 and the weighted Poincaré inequality $w$P1 hold locally.

We define

$$
\mathcal{E}_t(u, g) := \int_{M^n} \mathcal{A}(x, t, u, \nabla u) \nabla g \, d\mu(x) + \int_{M^n} \mathcal{B}(x, t, u, \nabla u) g \, d\mu(x),
$$

where $\mathcal{A}(x, t, u, \bar{p})$ is a vector function, $\mathcal{B}(x, t, u, \bar{p})$ is a scalar function, defined and measurable for all $t \in \mathbb{R}$, $x \in M^n$, and all values of $u$ and $\bar{p}$. We require $\mathcal{A}$ and $\mathcal{B}$ to satisfy the structural inequalities [2] (2), that is,

$$
\bar{p} \cdot \mathcal{A}(x, t, u, \bar{p}) \geq a|\bar{p}|^2 - b^2 u^2 - w_1^2,
$$

$$
|\mathcal{B}(x, t, u, \bar{p})| \leq c|\bar{p}| + d|u| + w_2,
$$

$$
|\mathcal{A}(x, t, u, \bar{p})| \leq a|\bar{p}| + e|u| + w_3,
$$

where $a$ and $\bar{a}$ are positive constants and $b, c, d, e, w_1, w_2, w_3$ are non-negative functions of $(x, t)$ each contained in an $L^q(I \to L^{r,\infty}(M^n))$ space, where the pair $(r, q)$ may be different for each coefficient but must satisfy

$$
r > 2 \quad \text{and} \quad \frac{n}{2r} + \frac{1}{q} < \frac{1}{2} \quad \text{for } b, c, e, w_1, w_3,
$$

$$
r > 1 \quad \text{and} \quad \frac{n}{2r} + \frac{1}{q} < 1 \quad \text{for } d, w_2.
$$

Then $\mathcal{E}_t$ is adapted to the Dirichlet form generated by the Laplace-Beltrami operator on $M^n$. Therefore, the scale-invariant parabolic Harnack inequality of Theorem 5.3 as well as all results of Section 6 hold.

In the special case $M^n = \mathbb{R}^n$, we recover the parabolic Harnack inequality of [2] Theorem 3 but under weaker conditions on the coefficients: Indeed, the original conditions [2] (2) involved $L^r$ in place of the Lorentz space $L^{r,\infty} \subset L^r$. 

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5.3. **Bilinear forms.** In this subsection, we relate the notion of quasilinear forms to the bilinear forms considered in [17].

Let $\mathcal{E}_t$ be a bilinear form satisfying Assumption 0 in [17] with respect to a reference form $(\mathcal{E}, \mathcal{F})$. Suppose the reference form satisfies (A1) and (A2) of Section 5.1.1. Formally, write

$$
\int f d\mathcal{A}_t(u,g) = \int f d\Gamma_t(u,g) + \mathcal{R}_t(fu,g),
\int f d\mathcal{B}_t(u,g) = \mathcal{L}_t(fu,g) + \mathcal{E}_t^{\text{sym}}(fu,g).
$$

If $\mathcal{A}_t$ and $\mathcal{B}_t$ are signed measures and if $|\mathcal{E}_t^{\text{sym}}(fg,1)| \leq C^* \|f\|_2 \|g\|_F$ for all $f, g \in \mathcal{F} \cap C_c$ then $\mathcal{E}_t$ is indeed a quasilinear form in the sense of Definition 2.1. If in addition $\mathcal{E}_t$ satisfies Assumption 1 and Assumption 2 of [16] uniformly in $t$, then our structural hypotheses H.1 and H.2 are satisfied. This is remarkable because it seems that Assumptions 0, 1, 2 do not imply that $\mathcal{E}_t$ would be adapted to $(\mathcal{E}, \mathcal{F})$ in the sense of Definition 5.1.

5.4. **Doob’s transform.** Consider a non-symmetric divergence form operator on $\mathbb{R}^n$,

$$
L = \sum_{i,j=1}^n \partial_i (a_{ij} \partial_j),
$$

with bounded measurable coefficients $a_{ij}$. Assume that its symmetric part is uniformly elliptic, that is, there exists a constant $c > 0$ such that

$$
c|\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j,
\forall \xi, \zeta \in \mathbb{R}^n.
$$

It is clear that the bilinear form associated with $L$ satisfies H.1 and H.2. We also have the Poincaré inequality and the localized Sobolev inequality.

Let $U$ be an unbounded inner uniform domain in $\mathbb{R}^n$ with harmonic profile $h > 0$ for the Dirichlet Laplacian on $U$. By [13], the Doob’s transform $\mathcal{E}_U^{h^2}(f,f) = \sum_{i=1}^n \int_U |\partial_i f|^2 h^2 dx$ with domain $\mathcal{F}^h(U) = \frac{1}{h} \mathcal{F}(U)$ satisfies volume doubling and the Poincaré inequality.

Let

$$
\mathcal{E}_h(f,g) = \sum_{i,j=1}^n a_{ij} \partial_i (hf) \partial_j (hg) dx.
$$

**Proposition 5.11.** The $h$-transformed bilinear form $\mathcal{E}_h$ is adapted to the reference Dirichlet form $(\mathcal{E}_U^{D,h^2}, \mathcal{F}^h(U))$.

Similar results hold for bounded inner uniform domains and for locally inner uniform domains in Euclidean space, and more generally in Harnack-type Dirichlet spaces. The proof will be presented in a forthcoming paper by the author, along with new and sharp two-sided estimates for the Dirichlet heat kernel on $U$ associated with $L$.

5.5. **Kolmogorov-Fokker-Planck operator.** Consider the operator

$$
Lu = \sum_{i,j=1}^m \partial_{x_i}(a_{ij} \partial_{x_j} u) + \langle Bx, \nabla u \rangle,
$$
where \( m \leq n \), the coefficients \( a_{ij} \) are real-valued measurable functions of \((t, x) \in \mathbb{R} \times \mathbb{R}^n \) satisfying \( a_{ij} = a_{ji} \) and

\[
e |\xi|^2 \leq \sum a_{ij} \xi_i \xi_j \leq C|\xi|^2, \quad \forall \xi \in \mathbb{R}^m,
\]

and \( B \) is a constant \( n \times n \) real matrix such that there is a basis of \( \mathbb{R}^n \) in which \( B \) takes the form

\[
B = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
B_1 & 0 & \ldots & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & B_\tilde{k} & 0
\end{pmatrix},
\]

where \( B_j \) is an \( m_j \times m_{j+1} \) matrix of rank \( m_j+1 \), \( j = 1, 2, \ldots, \tilde{k} \) with

\[
m =: m_1 \geq m_2 \geq \ldots \geq m_{\tilde{k}+1} \geq 1 \quad \text{and} \quad m_1 + m_2 + \ldots + m_{\tilde{k}+1} = n.
\]

Then \( L \) is associated with a quasilinear form \( E_t \) which satisfies H.1 with \( \Gamma(u, u) = \sum_{i=1}^m |\partial_x u|^2 \). Indeed, integrating by parts we can treat \( < Bx, \nabla u > \) like a zero order term. However, H.2 is apparently not satisfied, indicating that H.2 has a structural content that is not already captured by H.1.

The Kolmogorov-Fokker-Planck operator \( L \) is an example of a class of subelliptic operators to which the Moser iteration applies, see [8, Example 1.2] and [7]. By [8, Theorem 3.3], a localized Sobolev inequality holds for weak solutions of the heat equation associated with \( L \) in \( Q = (−1, 1) \times B(x, 1) \), for any \( x \in \mathbb{R}^n \), \( B_\delta = B(x, \delta) \). The localized Sobolev inequality implies the weighted Sobolev inequality (wSI) of Definition 2.5 with \( k = 2 \).

A weighted Poincaré inequality for \( L \) is not known. This is possibly related to the failure of H.2.

Nevertheless, H.1 and the Sobolev inequality are sufficient to obtain the mean value estimates of Theorem 3.6 and Theorem 3.9. For the operator \( L \) given above, these mean value estimates are already known from [20, Theorem 1.2 and Corollary 1.4] and [8, Theorem 1.4]. However, Theorems 3.6 and 3.9 also apply to Kolmogorov-type operators on more general spaces, such as Euclidean complexes or Riemannian manifolds. For instance, if \((M^n, g)\) is a smooth Riemannian manifold then we can define a Kolmogorov-type operator on \( M^n \times M^n \) as

\[
Lu = L_\nabla u - v \nabla_{\nabla} u,
\]

where \( L_\nabla \) is a vertical uniformly elliptic diffusion operator and \( \nabla_{\nabla} \) is the horizontal gradient.

References

[1] S. Andres and M. T. Barlow, Energy inequalities for cutoff functions and some applications, J. Reine Angew. Math., 699 (2015), pp. 183–215.
[2] D. G. Aronson and J. Serrin, Local behavior of solutions of quasilinear parabolic equations, Arch. Rational Mech. Anal., 25 (1967), pp. 81–122.
[3] D. Bakry, T. Coulhon, M. Ledoux, and L. Saloff-Coste, Sobolev inequalities in disguise, Indiana Univ. Math. J., 44 (1995), pp. 1033–1074.
[4] M. T. Barlow, R. F. Bass, and T. Kumagai, Stability of parabolic Harnack inequalities on metric measure spaces, J. Math. Soc. Japan, 58 (2006), pp. 485–519.

\(^1\)We remark that the notion of weak solutions in [8] is slightly stronger than considered here since [8] requires the existence of a weak time-derivative that is locally in \( L^2 \).
[5] E. Bombieri and E. Giusti, *Harnack’s inequality for elliptic differential equations on minimal surfaces*, Invent. Math., 15 (1972), pp. 24–46.

[6] L. Capogna, G. Citti, and G. Rea, *A subelliptic analogue of Aronson-Serrin’s Harnack inequality*, Math. Ann., 357 (2013), pp. 1175–1198.

[7] C. Cinti, A. Pascucci, and S. Polidoro, *Pointwise estimates for a class of non-homogeneous Kolmogorov equations*, Math. Ann., 340 (2008), pp. 237–264.

[8] C. Cinti and S. Polidoro, *Pointwise local estimates and Gaussian upper bounds for a class of uniformly subelliptic ultraparabolic operators*, J. Math. Anal. Appl., 338 (2008), pp. 946–969.

[9] G. De Leva, *Parabolic Harnack inequality on metric spaces with a generalized volume property*, Tohoku Math. J. (2), 63 (2011), pp. 303–327.

[10] J. Diestel and J. J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[11] C. L. Epstein and R. Mazzeo, *Harnack inequalities and heat kernel estimates for degenerate diffusion operators arising in population biology*, Appl. Math. Res. Express. AMRX, (2016), pp. 217–280.

[12] A. A. Grigor’yan, *The heat equation on noncompact Riemannian manifolds*, Mat. Sb., 182 (1991), pp. 55–87.

[13] P. Gyrya and L. Saloff-Coste, *Neumann and Dirichlet heat kernels in inner uniform domains*, Astérisque, (2011), pp. viii+144.

[14] J. Kinnunen, N. Marola, M. Miranda, Jr., and F. Paronetto, *Harnack’s inequality for parabolic De Giorgi classes in metric spaces*, Adv. Differential Equations, 17 (2012), pp. 801–832.

[15] O. Ladyzenskaja, V. Solonnikov, and N. Uralceva, *Linear and quasilinear equations of parabolic type*, vol. 23 of Translations of Mathematical Monographs, American Mathematical Society, Providence, R.I., 1968.

[16] J. Lierl, *Parabolic Harnack inequality for time-dependent non-symmetric Dirichlet forms*. submitted.

[17] J. Lierl, *Parabolic Harnack inequality on fractal-type metric measure Dirichlet spaces*. arXiv:1509.04804, accepted for publication in Rev. Mat. Iberoam.

[18] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math., 17 (1964), pp. 101–134.

[19] R. O’Neil, *Convolution operators and L(p, q) spaces*, Duke Math. J., 30 (1963), pp. 129–142.

[20] A. Pascucci and S. Polidoro, *The Moser’s iterative method for a class of ultraparabolic equations*, Commun. Contemp. Math., 6 (2004), pp. 395–417.

[21] M. Pivarski and L. Saloff-Coste, *Small time heat kernel behavior on Riemannian complexes*, New York J. Math., 14 (2008), pp. 459–494.

[22] L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*, Internat. Math. Res. Notices, (1992), pp. 27–38.

[23] L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, vol. 289 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2002.

[24] K.-T. Sturm, *Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations*, Osaka J. Math., 32 (1995), pp. 275–312.

[25] K.-T. Sturm, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. (9), 75 (1996), pp. 273–297.

[26] N. S. Trudinger, *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math., 21 (1968), pp. 205–226.

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