THE LOOP SHORTENING PROPERTY AND ALMOST CONVEXITY

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Abstract. We introduce the loop shortening property and the basepoint loop shortening property for finitely generated groups, and examine their relation to quadratic isoperimetric functions and almost convexity.

1. Introduction

In this article we introduce two new properties of groups: the loop shortening property and the basepoint loop shortening property. The properties are natural generalizations of the falsification by fellow traveler property introduced by Neumann and Shapiro [17], which in turn is closely related to the property of being almost convex, introduced by Cannon in [8]. The first part of the article is devoted to proving three facts. In Theorem 3.2 we see that asynchronous and synchronous versions of both properties are equivalent. Theorem 4.1 states that having the loop shortening property implies finite presentability and a quadratic Dehn function, and Theorem 4.2 shows that if a group presentation has the basepoint loop shortening property then it is almost convex. In the second part of the article we examine four group presentations, which exhibit a diverse spectrum of properties. These examples answer several natural questions about the loop shortening properties and the interdependence between them and almost convexity, the falsification by fellow traveler property and quadratic isoperimetric functions.

The author is indebted to Noel Brady, Jon McCammond and Walter Neumann for their ideas and suggestions with this paper. In addition the author wishes to thank an anonymous reviewer for her/his careful reading and suggestions.

2. Preliminaries

Throughout this article let \((G, X)\) denote the pair of a group and a finite generating set. The set \(X^*\) denotes the set of all words in the letters of \(X\), including the empty word. The Cayley graph for the pair is denoted \(\Gamma(G, X)\).

Definition 2.1 (Path,loop). A word \(w \in X^*\) corresponds to a path based at some vertex of the Cayley graph. A loop is a path which starts and ends at the same vertex. A path [loop] can be parameterized by arc length, and we denote the point at distance \(t\) along the path [loop] from the start point by \(w(t)\). For \(t > |w|\), \(w(t)\) is defined to be the endpoint of \(w\).
Definition 2.2 ((Asynchronous) fellow traveling). Two paths \([w, u]\) are said to \([k\)-fellow travel\] if \(d(w(t), u(t)) \leq k\) for all \(t \geq 0\), where \(w\) and \(u\) have different start and end points. Two paths \([w, u]\) are said to \([\text{asynchronously} \; k\)-fellow travel\] if there is a proper monotone increasing function \(\phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}\) such that \(d(w(t), u(\phi(t))) \leq k\) for all \(t \geq 0\), where \(w\) and \(u\) have different start and end points.

Definition 2.3 (Falsification by fellow traveler property). \((G, X)\) enjoys the \([\text{asynchronous}] \; \text{falsification by fellow traveler property}\) if there is a constant \(k > 0\) such that for each non-geodesic path \(w\) in \((G, X)\), there is a path \(u\) in \((G, X)\) with the same endpoints so that \(|u| < |w|\) and \(w, u\) \([\text{asynchronously} \; k\)-fellow travel\].

Neumann and Shapiro introduced this property in \([17]\), where they prove that if a pair \((G, X)\) enjoys the property, then the full language of geodesics in the generators \(X\) is regular. They also show that the property is dependent on choice of generating set. A reasonably simple proof in \([11]\) shows that the asynchronous and synchronous versions of this property are in fact equivalent.

Definition 2.4 (Loop shortening property). \((G, X)\) enjoys the \([\text{asynchronous}] \; \text{loop shortening property}\) if there is a constant \(k > 0\) such that for each loop \(w\) in \((G, X)\), there is a loop \(u\) in \((G, X)\) so that \(|u| < |w|\) and \(w, u\) \([\text{asynchronously} \; k\)-fellow travel\].

Note that \(u\) and \(w\) can be disjoint. A (seemingly) stronger version of the property is the following.

Definition 2.5 (Basepoint loop shortening property). \((G, X)\) enjoys the \([\text{asynchronous}] \; \text{basepoint loop shortening property}\) if there is a constant \(k > 0\) such that for each loop \(w\) in \((G, X)\) based at \(w(0)\), there is a loop \(u\) in \((G, X)\) based at \(w(0)\) so that \(|u| < |w|\) and \(w, u\) \([\text{asynchronously} \; k\)-fellow travel\].

Definition 2.6 (Almost convex). \((G, X)\) is \([\text{almost convex}]\) if there is a constant \(C\) such that every pair of points lying distance at most 2 apart and within distance \(N\) of the identity in \(\Gamma(G, X)\) are connected by a path of length at most \(C\) which lies within distance \(N\) of the identity.

See \([8]\) for properties of almost convex groups. This property also depends on the choice of generating set \([21]\).

Let \(R \subseteq X^*\) denote some set of relators such that \((G, X)\) admits the presentation \(\langle X, R \rangle\). Let \(F(X)\) be the free group generated by \(X\), and let \(N(R)\) be the normal closure of \(R\) in \(F(X)\). A word in \(X^*\) represents the identity in \(G\) if and only if it is freely equal to an expression of the form

\[
\prod_{i=1}^{k} g_i r_i g_i^{-1}
\]

where the \(g_i \in F(X)\) and \(r_i \in R \cup R^{-1}\). Define the \([\text{area}]\) \(A(w)\) of a word \(w \in X^*\) which represents the identity to be the minimum \(k\) in any such expression for \(w\).

Definition 2.7 (Dehn function, Isoperimetric function). A \([\text{Dehn function}]\) for \(\langle X|R \rangle\) is defined to be \(\delta(n) = \max\{A(w) : |w| \leq n\}\). An \([\text{isoperimetric function}]\) for \(\langle X|R \rangle\) is any function which satisfies \(f(n) \geq \delta(n)\).

Two functions \(f, g\) are said to be equivalent if there are constants \(A, A', B, B', C, C', D, D', E, E'\) so that \(f(n) \leq Ag(Bn + C) + Dn + E\), and \(g(n) \leq A'f(B'n + C') + D'n + E'\). With respect to this definition, a Dehn function of a group is generating
In particular, if $G$ has a sub-quadratic isoperimetric function then its Dehn function is linear \[2, 19\]. The class of groups which have a quadratic Dehn function is diverse and not particularly well understood. Examples include CAT(0) groups \[7\], automatic groups and $(2n+1)$-dimensional integral Heisenberg groups for $n \geq 2 \ [14]$. 

3. ASYNCHRONOUS VERSUS SYNCHRONOUS

In this section we prove that the asynchronous and synchronous versions of the two properties are equivalent.

**Lemma 3.1** (Discrete to continuous). Let $w, u$ be paths in $(G, X)$, parameterized by $t \in \mathbb{R}_{\geq 0}$. If for some constant $k$ $d(w(t), u(t)) \leq k$ for all $t \in \mathbb{N}$ then $d(w(t), u(t)) \leq k + 1$ for all $t \in \mathbb{R}_{\geq 0}$.

**Proof.** If $t - |t| \leq \frac{1}{2}$ then there is a path of length at most $k + 1$ from $w(t)$ to $w(|t|)$ to $u(|t|)$ to $u(t)$. If $t - |t| > \frac{1}{2}$ then there is a path of length at most $k + 1$ from $w(t)$ to $w(|t|)$ to $u(|t|)$ to $u(t)$. \(\square\)

It follows that in order to prove that two paths synchronously $k$-fellow travel it is sufficient to show that integer points are within $(k - 1)$ of each other.

**Theorem 3.2.** $(G, X)$ has the asynchronous [basepoint] loop shortening property if and only if $(G, X)$ has the synchronous [basepoint] loop shortening property.

**Proof.** If $(G, X)$ has the synchronous [basepoint] loop shortening property then it clearly has the asynchronous [basepoint] loop shortening property. Let $w$ be a loop of length $n$ in $(G, X)$. If $(G, X)$ has the asynchronous loop shortening property with constant $k > 0$ then there is a shorter loop $u$ and a proper monotone increasing function $\phi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $d(w(t), u(\phi(t))) \leq k$ for all $t \in \mathbb{R}_{>0}$. Without loss of generality we may assume that $\phi(0) = 0$. Fix some constant $\epsilon$ so that $0 < \epsilon < 1$.

Let $E = \{t \in \mathbb{R}_{>0} : \phi(t) = |u|\}$. Note that if $t \in E$ then $s \in E$ for all $s \geq t$. If $n \notin E$ then define $\phi' : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ by

$$\phi'(t) = \begin{cases} \phi(t) & 0 \leq t \leq n - \epsilon \\ \phi(n) + (|u| - \phi(n)) \frac{t - n + \epsilon}{\epsilon} & n - \epsilon \leq t \leq n \\ \phi(n) + (|u| - \phi(n)) & t \geq n \end{cases}$$

Note that all points on $u$ from $u(\phi(n))$ to $u(|u|)$ are at most $k$ from the vertex $w(n)$, as seen in Figure \[1\].

The function $\phi'$ is proper and monotone increasing. In addition, $w$ and $u$ asynchronously $(k + \epsilon)$-fellow travel with respect to $\phi'$ since for $n - \epsilon \leq t \leq n$ we have $d(w(t), u(\phi'(t))) \leq d(w(t), w(n)) + d(w(n), u(\phi'(t))) \leq \epsilon + k$.

So without loss of generality (by possibly choosing a different $\phi$) we may assume $n \in E$. We divide the argument into three cases.

**Case 1:** If $|t - \phi(t)| \leq 2k$ for all $t \in \mathbb{N}$, $t \leq n$ then $d(w(t), u(t)) \leq d(w(t), u(\phi(t))) + d(u(\phi(t)), u(t)) \leq k + 2k = 3k$ so by Lemma \[3.1\] $w, u$ synchronously $(3k + 1)$-fellow travel.

**Case 2:** If $t - \phi(t) > 2k$ for some $t \in \mathbb{N}, t \leq n$ then let $j = \min\{t \in \mathbb{N} : t - \phi(t) > 2k\}$. In particular $j - 0 > \phi(j) - \phi(0) + 2k$. Let $l = \max\{l \in \mathbb{N} : l < j, j - l > \phi(j) - \phi(l) + 2k\}$. Then $j - (l + 1) \neq \phi(j) - \phi(l + 1) + 2k$ since $l$ was chosen to be
maximal, so \( j - (l + 1) \leq \phi(j) - \phi(l + 1) + 2k \leq \phi(j) - \phi(l) + 2k \) since \( \phi \) is monotone increasing. Thus we have \( j - l = \phi(j) - \phi(l) + 2k + \delta \) for some \( \delta \in (0, 1] \).

For all \( t \in \mathbb{N} \) with \( t \in (l, j) \), we have \( t - \phi(t) \leq 2k \) since \( t < j \). If \( \phi(t) - t \geq 0 \) then \( j - t > \phi(j) - \phi(t) + 2k \) and \( l \) is not maximal. Thus \( d(w(t), u(t)) \leq d(w(t), u(\phi(t))) + d(u(\phi(t)), u(t)) \leq k + 2k = 3k \).

Let \( w_1 = [w(0), w(l)], w_2 = [w(l), w(j)], w_3 = [w(j), w(n)], u_1 = [u(0), u(\phi(l))], u_2 = [u(\phi(l)), u(\phi(j))], u_3 = [u(\phi(j)), u(\phi(j))], \) \( p_1 \) a path of length \( k_1 \) from \( w(l) \) to \( u(\phi(l)) \), and \( p_2 \) a path of length \( k_2 \) from \( u(\phi(j)) \) to \( w(j) \), as in Figure 2.

\[ \text{Figure 2. Case 2: } j - l > \phi(j) - \phi(l) + 2k \]

Define \( v = w_1p_1u_2p_2w_3 \) which is seen as the bold path in Figure 3. This loop has length \( l + k_1 + (\phi(j) - \phi(l)) + k_2 + (n - j) = \phi(j) - j + l - \phi(l) + k_1 + k_2 + n \leq \phi(j) - j + l - \phi(l) + 2k + n < n \). We will now show that \( w \) and \( v \) synchronously fellow travel. For \( 0 \leq t \leq l \) the paths \( w, v \) 0-fellow travel. For \( l < t \leq l + k_1 \) we can find a path of length at most \( 2k_1 \) from \( w(t) \) back along \( w \) to \( w(l) \) then down \( p_1 \) to \( v(t) \). Thus \( d(w(t), v(t)) \leq 2k_1 \leq 2k \).

The vertex \( u(\phi(l)) = v(l + k_1) \), so \( d(u(l + k_1), u(\phi(l))) = |(l + k_1) - \phi(l)| \leq 2k + k_1 \leq 3k \). For \( t \in \mathbb{N} \) and \( l + k_1 < t \leq l + k_1 + \phi(j) - \phi(l) \) we have \( t < j \) so \( d(u(\phi(t)), u(t)) \leq 2k \) and so \( d(w(t), v(t)) \leq d(w(t), u(\phi(t))) + d(u(\phi(t)), u(t)) + d(u(t), v(t)) \leq k + 2k + 3k = 6k \).

Now \( j - \phi(j) = l - \phi(l) + 2k + \delta \) so \( l - \phi(l) + \phi(j) = j - 2k - \delta \) and so \( l + k_1 - \phi(l) + \phi(j) = j + k_1 - 2k - \delta \). Let \( r = 2k + \delta - k_1 \leq 2k + 1 \).
The vertex \( u(\phi(j)) = v(j-r) \) so for \( t \in \mathbb{N} \) and \( j-r < t \leq j-r+k_2 \) there is a path from \( w(t) \) along \( w \) to \( w(j) \) then down \( p_2^{-1} \) to \( v(t) \) of length at most \( r+k_2 \leq 3k+1 \), so \( d(w(t), v(t)) \leq 3k+1 \).

Now \( d(w(j-k_2), w(j)) = r-k_2 \leq 2k+1 \) and for \( t \in \mathbb{N} \) and \( j-r < t \leq n \) with \( v \) travel at constant speed along \( w \), at distance \( r-k_2 \) apart, so \( d(w(t), v(t)) \leq 2k+1 \). Thus in total, and by Lemma 3.1, Case 3: if \( t - \phi(t) \leq 2k \) for all \( t \in \mathbb{N} \) and \( t \leq n \) but \( \phi(t) - t > 2k \) for some \( t \in \mathbb{N} \), then let \( j = \max\{ t \in \mathbb{N} : t \leq n, \phi(t) - t > 2k \} \).

Then \( \phi(j+1)-(j+1) \leq 2k \) so \( \phi(j) - j \leq \phi(j+1) - j \leq 2k+1 \) so \( \phi(j)-j = 2k+\delta \) for some \( 0 < \delta \leq 1 \).

Let \( w_1 = [w(0), w(j)], w_2 = [w(j), w(n)], u_1 = [u(0), u(\phi(j))], u_2 = [u(\phi(j)), u(|u|)], p_1 \) a path of length \( k_1 \) from \( w(j) \) to \( u(\phi(j)) \), and \( p_2 \) a path of length \( k_2 \) from \( u(\phi(|u|)) \) to \( w(n) \), as in Figure 4.

Define \( v = w_1 p_1 u_2 p_2 \), shown in bold in Figure 5. This loop has length \( j+k_1+(|u| - \phi(j)) + k_2 \leq |u| - (\phi(j) - j) + 2k < |u| \leq n \). We will show that \( w \) and \( v \) synchronously fellow travel.

For \( 0 \leq t \leq j \) the paths \( w \) and \( v \) synchronously 0-fellow travel. For \( j < t \leq j+k_1 \) there is a path from \( w(t) \) back along \( w_2 \) to \( w(j) \) then down \( p_2 \) to \( v(t) \) of length at most \( 2k_1 \), so \( d(w(x), v(x)) \leq 2k_1 \leq 2k \).
The vertex $u(\phi(j)) = v(j + k_1)$, so for $t \in \mathbb{N}$ and $j + k_1 < t \leq j + k_1 + |u| - \phi(j)$ we have $d(u(t), v(t)) = \phi - j - k_1 = 2k + \delta - k_1 \leq 2k + 1$ so $d(w(t), v(t)) \leq d(w(t), u(\phi(t))) + d(u(\phi(t)), u(t)) + d(u(t), v(t)) \leq k + 2k - 1 = 5k + 1$.

Recall that $n \in E$ by the argument at the start of this proof, so $\phi(n) = |u|$. Now since $n > |u|$ then $n - \phi(n) > 0$ so it must be that $j < n$, and so $n - \phi(n) \leq 2k$.

We have $j + k_1 + |u| - \phi(j) = |u| + k_1 + j - \phi(j) = \phi(n) + k_1 - (2k + \delta)$. Now for $t \in \mathbb{N}, j + k_1 + |u| - \phi(j) < t \leq j + k_1 + |u| - \phi(j) + k_2$ there is a path from $w(t)$ along $w$ to $w(n)$ of length at most $n - (j + k_1 + |u| - \phi(j))$ and from $w(n)$ there is a path down $p_2^{-1}$ to $v(t)$ of length at most $k_2$. Thus $d(w(t), v(t)) \leq n - (j + k_1 + |u| - \phi(j)) + k_2 = n - \phi(n) + \phi(j) - j - k_1 + k_2 = n - \phi(n) + 2k + \delta - k_1 + k_2 \leq 2k + 2k + 1 + k = 5k + 1$.

Thus in total and by Lemma 3.1 $w$ and $v$ synchronously $(5k + 2)$-fellow travel.

Finally for the basepoint case, we merely repeat the argument with $d(w(0), u(0)) = 0$.

\section{Quadratic Isoperimetric Function and Almost Convexity}

In this section we establish connections between the two loop properties, quadratic isoperimetric functions and almost convexity.

\textbf{Theorem 4.1.} If $(G, X)$ has the loop shortening property then $G$ is finitely presented, and has a quadratic isoperimetric function.

\textbf{Proof.} Let $w =_G 1$. Define $w = w_0$. While $w_i$ is not the empty word, there is a shorter loop $w_{i+1}$ that $k$-fellow travels $w_i$. After at most $|w|$ iterations we get the trivial word. The space between $w_i$ and $w_{i+1}$ can be filled by $|w_i|$ relations of length at most $2k + 2$, so it follows that $G$ is finitely presented as $\langle X \rangle \{ u \in X^* : |u| \leq 2k + 2 \}$. Moreover, the number of such relations needed to fill $w$ is at most $\sum_{i=1}^{|w|} |w_i|^2$.

\textbf{Theorem 4.2.} If $(G, X)$ has the basepoint loop shortening property then $(G, X)$ almost convex.

\textbf{Proof.} Let $w$ and $u$ be two geodesics of length $N$ such that $d(\overline{w}, \overline{u}) \leq 2$, realized by a path $\gamma$. Let $k$ be the basepoint loop shortening constant, and without loss of
generality assume it is an even integer. The word $w^{-1}$ is a loop based at the identity vertex, of length at most $2N + 2$. Applying basepoint loop shortening we get a loop $y$ based at the identity of length at most $2N + 1$. Applying the property once more we get a loop $v$ based at the identity of length at most $2N$, so $v \subseteq B(N)$.

The path that retraces $w$ back to $w(N - \frac{k}{2})$, then travels across to $y(N - \frac{k}{2})$ then to $v(N - \frac{k}{2})$, then travels along $v$ to $v(N + \frac{k}{2} + 2)$, over to $y(N + \frac{k}{2} + 2)$ then to $u(N + \frac{k}{2} + 2)$, then along $u$ to its end lies in $B(N)$ and has length at most $6k + 2$. See Figure 6.

\[ \square \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{The basepoint loop shortening property implies almost convex}
\end{figure}

The theorem provides an easy route to proving almost convex for some examples, and is potentially an extremely useful tool. In the next section we show this by “reproving” a theorem of the author in [13].

We summarize the results so far in Figure 7. The non-reversible implications (in grey) will be proved by counterexamples below.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\end{tikzpicture}
\caption{Implication diagram}
\end{figure}
5. Multiple HNN extensions

In [13] the author proves that a certain class of multiple HNN extension group presentations are almost convex. We show that the same hypothesis implies the basepoint loop shortening property. This gives a rapid proof of almost convexity for some examples of interest.

**Definition 5.1** (Multiple HNN extension). Let \((A, Z)\) be a group with finite generating set \(Z\) and relations \(R\), let \(U_1, \ldots, U_n, V_1, \ldots, V_n\) be subgroups of \(A\) and let \(\phi_i : U_i \rightarrow V_i\) be an isomorphism for each \(i\). The group \((G, X)\) with presentation

\[
\langle Z, s_1, \ldots, s_n | R, s_i^{-1}u_is_i = \phi_i(u_i) \forall u_i \in U_i, \forall i \rangle
\]

is a *multiple HNN extension* of \((A, Z)\). The generators \(s_i\) are called *stable letters*, and the pairs of \(U_i, V_i\) are called *associated subgroups*.

If each \(U_i\) is finitely generated by \(\{u_{i_j}\}\) and \(\phi_i(u_{i_j}) = v_{i_j}\) then \(V_i\) is finitely generated by \(\{v_{i_j}\}\). Thus \((G, X)\) has the finite presentation

\[
\langle Z, s_1, \ldots, s_n | R, s_i^{-1}u_is_i = v_{i_j} \forall i, \forall j \rangle.
\]

**Theorem 5.2** (Britton’s Lemma). Let \((G, X)\) be a multiple HNN extension with the presentation in **Definition 5.1** above. If \(w \in X^*\) is freely reduced and \(w =_G 1\) then \(w\) contains a sub-word of the form \(s_i^{-1}u_is_i\) or \(s_iv_is_i^{-1}\) for some non-trivial \(u_{i_j} \in U_i\) or \(v_{i_j} \in V_i\).

A sub-word \(s_i^{-1}u_is_i\) or \(s_iv_is_i^{-1}\) is called a *pinch*, and if a word admits no pinches it is called *stable letter reduced*.

**Definition 5.3** (Strip equidistant). Let \((G, X)\) be a multiple HNN extension with the presentation in **Definition 5.1** above. If \([u_{i}] = |\phi(u_{i})|\) for all \(i\) then we say \((G, X)\) has a *strip equidistant* presentation.

Note that if \((G, X)\) has a strip equidistant presentation then if a word \(w \in X^*\) admits a pinch then it can be shortened by \(2\), so geodesics are stable letter reduced.

**Definition 5.4** (Totally geodesic). Let \((G, X)\) be any group with generating set \(X\). A subgroup \(A\) of \(G\) with generating set \(Y \subseteq X^*\) is *totally geodesic* in \((G, X)\) if every geodesic word \(w \in X^*\) evaluating to an element of \(A\) is an element of \(Y^*\).

**Theorem 5.5.** Let \((G, X)\) be a multiple HNN extension of \((A, Z)\) as in **Definition 5.1** with a strip equidistant presentation, such that associated subgroups are totally geodesic and \((A, Z)\) enjoys the falsification by fellow traveler property. Then \((G, X)\) enjoys the basepoint loop shortening property.

**Proof.** Let \(k\) be the falsification by fellow traveler property constant for \((A, X)\). Let \(w\) be a loop based at \(w(0)\) in \((G, X)\). If \(w\) has no stable letters, then there is a shorter loop in \((A, X)\) that \(k\)-fellow travels \(w\) by the falsification by fellow traveler property in \((A, X)\).

If \(w\) has stable letters, then by Britton’s Lemma it admits a pinch. Let \(sw_2s^{-1}\) be an inner-most pinch, that is, \(w_2 \in Z^*\), and \(w = w_1sw_2s^{-1}w_3\). If \(w_2\) is not a geodesic then apply the falsification by fellow traveler property in \((A, X)\) to get a shorter sub-word \(u\) so that \(w_1suw_2s^{-1}w_3\) \(k\)-fellow travels \(w\). If \(w_2\) is geodesic then by total geodesity of associated subgroups, \(w_2\) is a word in \(\{u_{i}\}^*\) [respectively \(\{v_{i}\}^*\)]. Then \(sw_2s^{-1}\) is \(2\)-fellow traveled by \(\phi(w_2)\) [respectively \(\phi^{-1}(w_2)\)]. □
One might think that the preceding proof can be strengthened to show that \((G, X)\) in fact has the falsification by fellow traveler property. The first example of the next section shows that this is not possible. Moreover the last example of the next section shows that the totally geodesic hypothesis cannot be relaxed.

6. Examples

In this section we consider four group presentations which display a diverse range of properties. In particular we will fill in Table 1 below of examples and their properties.

**Table 1.** Examples and their properties

|                  | falsification by fellow traveler property | basepoint loop shortening property | loop shortening property | almost convex | automatic | CAT(0) | quadratic Dehn function |
|------------------|------------------------------------------|-----------------------------------|--------------------------|--------------|----------|--------|-------------------------|
| \((G, X)\)       | ?                                        | yes                               | yes                      | yes          | yes      | yes    | yes                     |
| \((G, X)\)       | no                                       | yes                               | yes                      | yes          | ?        | yes    | yes                     |
| \((G, X)\)       | no                                       | no                                | no                       | yes          | no       | no     | yes                     |
| \((G, X)\)       | no                                       | no                                | ?                        | no           | no       | no     | yes                     |

**Example 1.** \((G, X) = \langle a, b, c, d, s, t | c = ab, c = ba, d = c^2, s^{-1}as = d, t^{-1}bt = d \rangle\).

This group was considered by Wise [22], who proved it is \(\text{CAT}(0)\) and non-Hopfian. The group is a double HNN extension of \(\langle \mathbb{Z}, \{a, b, c, d\} \rangle\), with associated subgroups \(\langle a \rangle, \langle b \rangle, \langle d \rangle\) totally geodesic in \(\langle \mathbb{Z}^2, \{a, b, c, d\} \rangle\). Neumann and Shapiro prove that any finite generating set for an abelian group has the falsification by fellow traveler property [17]. It follows from Theorem 5.5 that \((G, X)\) enjoys the basepoint loop shortening property and is consequently almost convex. The author has shown that this example does not enjoy the falsification by fellow traveler property (see [10]), and so the basepoint loop shortening property does not imply the falsification by fellow traveler property. It is not known whether this group is automatic.

**Example 2.** \((G, X) = \langle a, b, \gamma, s, t | c = aba^{-1}b^{-1}, \gamma a \gamma^{-1} = a^{-1}, \gamma b \gamma^{-1} = b^{-1}, sas^{-1} = c, tbt^{-1} = c \rangle\).

Bridson shows that this group cannot act on any 2-dimensional \(\text{CAT}(0)\) space, but is the fundamental group of a 3-dimensional non-positively curved cube complex [6]. It follows from Niblo and Reeves [18] that \(G_B\) is biautomatic. The pair is a triple HNN extension of \(F_2\), the free group on two letters. Since \(F_2\) is word-hyperbolic, it enjoys the falsification by fellow traveler property with respect to any generating set. It is easy to see that the presentation is strip equidistant, and the associated subgroups are totally geodesic, and so this pair has the basepoint loop shortening property by Theorem 5.5 and consequently is almost convex. It is not known whether it enjoys the falsification by fellow traveler property.

One might ask whether every group with quadratic isoperimetric function is almost convex. The following example shows this is not the case, and in addition shows that the basepoint loop property is not equivalent to the enjoyment of a quadratic isoperimetric function.
Example 3. Let $\phi : (F_2)^3 \to \mathbb{Z}$ be a homomorphism which sends each word in $(F_2)^3 = \langle a,b,c,d,e,f \rangle$ to its exponent sum. Define the group $G_S = \text{Ker}(\phi)$.

Stallings showed that $G_S$ is finitely presented but is not of type $\text{FP}_3$ [20], and so not of type $F_3$. Recently Bridson has shown this group has a quadratic isoperimetric function [5]. Since $G_S$ is not of type $F_3$, it is not automatic, not CAT(0), and does not enjoy the falsification by fellow traveler property for any generating set (See [16, 7, 11] respectively). Finding a generating set for which $G_S$ is almost convex or has the [basepoint] loop shortening property would prove that the respective property does not imply $F_3$.

There is a standard way of associating a right-angled Artin group to a finite flag complex (See Bestvina and Brady [1] for details and references). Dicks and Leary give the following description of a presentation for $G_S$, based on work of Bestvina and Brady [1]. Consider an octahedron with opposite vertices labeled by generators of each free factor of $(F_2)^3$. See Figure 8.

![Figure 8. Flag complex encoding $(G_S, X)$](image)

Each directed edge of the octahedron defines a generator of $G_S$, where $ac^{-1}$ is the edge from $c$ to $a$. For convenience we denote the inverse of a generator of $(F_2)^3$ in upper case, so the edge $ac^{-1}$ is written $aC$, and so on. Each 2-cell of the octahedron defines two relations, so the 2-cell with vertices $a, c, e$ defines two relations $aCcEeA$ and $aCeAcE$. Dicks and Leary prove in [9] that these 12 generators and 16 relations are a presentation for $G_S$. We will denote this generating set by $X$.

Define a homomorphism $\rho : G_S \to (F_2)^3$ by $\rho(yZ) = yZ$. It is clear that if two words $u, v$ evaluate to the same element of $G_S$ then $\rho(u) = \rho(v)$ in $(F_2)^3$.

**Lemma 6.1.** Let $w \in X^*$ and $z \in \{c,d,e,f\}$. If $\rho(w) = f^n d^n E^n C^{n-1} Z$ in $(F_2)^3$ then $|w| \geq 3n$.

**Proof.** The first $E$ or $C$ in $w$ that does not freely cancel with an earlier letter must occur after $w(n)$, since $f^n$ or $d^n$ must be read before $E$ or $C$ respectively. After $w(n)$ we must read $E^n C^{n-1} Z$ so we need at least $2n$ letters of $X$. So $|w| \geq 3n$. \Box

It follows that $\alpha = (fA)^n (dE)^n (aC)^{n-1}$, $\beta = (fB)^n (dE)^n (bC)^{n-1}$ are geodesic since they are sub-words of $aaC, \beta bC$ which are geodesic by the lemma.

**Lemma 6.2.** Let $w \in X^*, z \in \{c,d,e,f\}$, and $v \in X^*$ such that $\rho(v) = 1$ in $\langle a,b \mid - \rangle$. If $v$ is of length less than $n - 1$ and $\rho(w) = f^n d^n E^n C^{n-1} v Z$ in $(F_2)^3$ then $|w| \geq 3n$. 
Proof. $v$ is shorter than $n - 1$, so cannot freely cancel all the $E^n$ and $C^{n-1}$. So again the first $E$ or $C$ in $w$ must occur after $w(n)$. If $v$ freely cancels some of the $E^n, C^{n-1}$ (and $Z$) $v$ must contain $i+j$ (and $z$) so must contain $i+j+1$ upper case letters, since $v \in X$. In sum total $E^n C^{n-1} v Z$ has at least $2n$ upper case letters (plus possibly more upper and lower pairs). So again there must be at least $2n$ letters in $X$ after $w(n)$, so $|w| \geq 3n$. □

Theorem 6.3. $(G_S, X)$ is not almost convex.

Proof. Let $\alpha = (fA)^n(dE)^n(aC)^{n-1}, \beta = (fB)^n(dE)^n(bC)^{n-1}, \gamma = aCcB$. It is easily checked using Lemma 6.1 that $\alpha, \beta$ are geodesics, each of length $3n-1$ ending distance 2 apart in $\Gamma(G_S, X)$, realized by $\gamma$. Assume by way of contradiction that $(G_S, X)$ is almost convex, so there is a path $p$ from $\alpha$ to $\beta$ inside $B(3n-1)$ of bounded length. See Figure 9. We have $pbCcA =_{G_S} 1$ so $pbCcA =_{(F_2)^3} pbA =_{(F_2)^3}$

![Figure 9. Paths in $\Gamma(G_S, X)$](image)

1. This means $p$ must contain a $B$ to cancel with the $b$ in this word, so $p = uzBv$ with $z \in \{c, d, e, f\}$ and $\rho(v) = 1$ in $\langle a, b \mid - \rangle$, having bounded length since $p$ is of bounded length. We choose $n$ to be greater than this bound. Let $g$ be a geodesic to $\overline{\alpha}a$, as in Figure 10. Now $\rho(g) = (fB)^n(dE)^n(bC)^{n-1}v^{-1}bZ = f^n d^n E^n C^{n-1} (v^{-1})Z$ in $(F_2)^3$. By Lemma 6.2 we have $|g| \geq 3n$, which contradicts the fact that $p \subseteq B(3n-1)$. □

The author has considered alternate finite presentations for $G_S$; see 10. It may be that this example is almost convex for another (possibly weighted) generating set. The boundary loop shown in Figure 11 does not appear to be fellow traveled by a shorter loop for any constant independent of $n$. It is likely (but not proved) that $(G_S, X)$ does not have the loop shortening property.

The final example is another multiple HNN extension, but does not satisfy the totally geodesic associated subgroup hypothesis of Theorem 5.5. It is almost convex however, and has a quadratic isoperimetric function, so one might suspect that it would have the loop shortening property.

Example 4. $(G_G, X) = \langle a, b, c, d, s, t \rangle|c = ab, c = ba, d = ab^{-1}, s^{-1}as = c, t^{-1}at = d \rangle$. 
Figure 10. A geodesic $g$ to $\overline{\alpha u}$

Gersten proves that this group is not CAT(0) \[15\]. Brady and Bridson showed that the group has quadratic isoperimetric function \[3\] and is not biautomatic \[4\]. The author proves the pair is almost convex and fails the falsification by fellow traveler property in \[12\]. Recent work of Bridson and Reeves shows that $G_4$ is not automatic. The group is free-by-cyclic.

In \[12\] the Cayley graph of $(G_G, X)$ is described as being made up of copies of the Cayley graph for $(\mathbb{Z}^2, \{a, b, c, d\})$, which we call “planes”, glued together along bi-infinite lines $a^i, c^i, d^i$ by stable letter “strips”. We now give a more technical definition of the idea of a strip.

**Definition 6.4.** An $s$-strip is the set of open edges of the form $\{(w^i, wa^i s) : i \in \mathbb{Z}\}$ for some arbitrary word $w$. We denote this strip by $(w\langle a \rangle, s)$. The three other possible strips are $(w\langle a \rangle, t)$, $(w\langle c \rangle, s^{-1})$ $(w\langle d \rangle, t^{-1})$.

**Lemma 6.5.** A strip divides the Cayley graph into two connected half spaces.
Proof. Let \((w(x), r)\) be a strip in \(\Gamma(G, X)\). Since \((G, X)\) is strip equidistant, a geodesic crosses each strip at most once. Let \(H_-\) be the set of all points in \(\Gamma(G, X)\) so that a geodesic from it to \(w\) does not cross the strip. Let \(H_+\) be the set of all points in \(\Gamma(G, X)\) so that a geodesic from it to \(wr\) does not cross the strip. It is easily seen that the Cayley graph is the (disjoint) union of \(H_-\) and \(H_+\), the two components are each path connected, and \(H_- \cap H_+ = \emptyset\). □

As a consequence we can say that two points lie on the same side of a strip if they lie in the same half space.

**Theorem 6.6.** \((G, X)\) does not enjoy the loop shortening property.

**Proof.** Assume by way of contradiction that \((G, X)\) has the loop shortening property with constant \(k\). Let \(w = d^n s t^{-1} c^n d^n s t^{-1} c^{-n} d^{-n} s t^{-1} c^{-n} d^{-n} s t^{-1} c^n\) for \(n > k\). See Figure 12. It is easy to check algebraically that \(w\) is a loop.

![Figure 12. The loop \(w\) in \((G, X)\)]

Now by assumption there is a loop \(u\) of length \(|u| < 8n + 8\) that synchronously \(k\)-fellow travels \(w\). The point \(u(0)\) lies on the same side of the strip \(S_1\) as \(w(0)\). To see this, let \(g\) be a geodesic from \(w(0)\) to \(u(0)\). If they lie in different half-spaces, let \((p, ps)\) be the first edge that \(g\) crosses on the strip \(S_1\). Then \(|g| = d(w(0), p) + 1 + d(ps, u(0)) \geq n + 1\). This is a contradiction since \(k < n\).

Repeating the argument, we have that \(u(2n + 2), u(4n + 4), u(6n + 6)\) lie on the same side of the strips \(S_2, S_3, S_4\) as \(w(2n + 2), w(4n + 4), w(6n + 6)\) respectively.
Now the path $u$ must go between these four points by passing through the base plane, shown in Figure 13. Let $p_1$ be the first point on the base plane that $u$ crosses after $u(0)$, $p_2$ the last point on the base plane before $u(2n + 2)$, $p_3$ be the first point on the base plane that $u$ crosses after $u(2n + 2)$, $p_4$ the last point on the base plane before $u(4n + 4)$, and so on up to $p_8$.

Let $m_1$ be the distance between $w(n + 1)$ and $p_1$, $m_2$ be the distance between $w(n + 1)$ and $p_2$, $m_3$ be the distance between $w(3n + 3)$ and $p_3$, $m_4$ be the distance between $w(3n + 3)$ and $p_4$, and so on up to $m_8$.

Notice that we impose no restrictions on where the path $u$ enters and exits the plane, just that it does so at least eight times, via the appropriate strips.

Now $d(p_i, p_{i+1}) = m_i + m_{i+1}$ for $i = 1, 3, 5, 7$ since $c^i d^j$ is a geodesic in the base plane.

The path $u$ must go from $p_2$ to $p_3$ via $u(2n + 2)$, so it must cross the strip $S_2$. The distances $m_i$ are the same on either side of the strip. That is, $d(p_{2i-1}, w(n)) = m_2$ and so on. Then $d(p_{2i-1}, p_{3i-1}) \geq |2n - m_2 - m_3|$, $d(p_{4i-1}, p_{5i-1}) \geq |2n - m_4 - m_5|$, $d(p_{6i-1}, p_{7i-1}) \geq |2n - m_6 - m_7|$ and $d(p_{8i-1}, p_{9i-1}) \geq |2n - m_8 - m_1|$.

Thus $|u| \geq (m_1 + m_2) + (m_3 + m_4) + (m_5 + m_6) + (m_7 + m_8) + 8 + |2n - m_2 - m_3| + |2n - m_4 - m_5| + |2n - m_6 - m_7| + |2n - m_8 - m_1|$. 

**Figure 13.** The “base plane”
\[ = 8 + |2n - (m_2 + m_3)| + (m_2 + m_3) + |2n - (m_4 + m_5)| + (m_4 + m_5) + |2n - (m_6 + m_7)| + (m_6 + m_7) + |2n - (m_8 + m_1)| + (m_8 + m_1) \geq 8 + 8n \text{ and this contradicts the fact that } u \text{ must be shorter than } w. \]

Perhaps \( G_G \) has the loop shortening property for another generating set.

7. Open questions

The three question marks in Table [1] are open. We have seen that the loop properties are closely related to the generating set dependent properties of almost convexity and the falsification by fellow traveler property, so it is possible that this unfortunate family trait is inherited. Can we find an example of a group that enjoys the [basepoint] loop shortening property with respect to one generating set and not another? Also, is there an example of a group presentation that has the loop shortening property but not the basepoint loop shortening property?

References

[1] Mladen Bestvina and Noel Brady. Morse theory and finiteness properties of groups. *Invent. Math.*, 129(3):445–470, 1997.
[2] B. H. Bowditch. A short proof that a subquadratic isoperimetric inequality implies a linear one. *Michigan Math. J.*, 42(1):103–107, 1995.
[3] N. Brady and M. R. Bridson. There is only one gap in the isoperimetric spectrum. *Geom. Funct. Anal.*, 10(5):1053–1070, 2000.
[4] Noel Brady and Martin Bridson. On the absence of bi-automaticity for graphs of abelian groups. Unpublished.
[5] Martin R. Bridson. Doubles, finiteness properties of groups, and quadratic isoperimetric inequalities. *J. Algebra*, 214(2):652–667, 1999.
[6] Martin R. Bridson. Length functions, curvature and the dimension of discrete groups. *Math. Res. Lett.*, 8(4):557–567, 2001.
[7] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*. Springer-Verlag, Berlin, 1999.
[8] James W. Cannon. Almost convex groups. *Geom. Dedicata*, 22(2):197–210, 1987.
[9] Warren Dicks and Ian J. Leary. Presentations for subgroups of Artin groups. *Proc. Amer. Math. Soc.*, 127(2):343–348, 1999.
[10] Murray Elder. Automaticity, almost convexity and falsification by fellow traveler properties of some finitely generated groups. PhD Dissertation, University of Melbourne, 2000.
[11] Murray J. Elder. Finiteness and the falsification by fellow traveler property. In *Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part II* (Haifa, 2000), volume 95, pages 103–113, 2002.
[12] Murray J. Elder. Patterns theory and geodesic automatic structure for a class of groups. *Internat. J. Algebra Comput.*, 13(2):203–230, 2003.
[13] Murray J. Elder. A non-Hopfian almost convex group. *J. Algebra*, 271(1):11–21, 2004.
[14] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston. *Word processing in groups*. Jones and Bartlett Publishers, Boston, MA, 1992.
[15] S. M. Gersten. The automorphism group of a free group is not a CAT(0) group. *Proc. Amer. Math. Soc.*, 121(4):999–1002, 1994.
[16] S. M. Gersten. Finiteness properties of asynchronously automatic groups. In *Geometric group theory (Columbus, OH, 1992)*, pages 121–133. de Gruyter, Berlin, 1995.
[17] Walter D. Neumann and Michael Shapiro. Automatic structures, rational growth, and geometrically finite hyperbolic groups. *Invent. Math.*, 120(2):259–287, 1995.
[18] G. A. Niblo and L. D. Reeves. The geometry of cube complexes and the complexity of their fundamental groups. *Topology*, 37(3):621–633, 1998.
[19] A. Yu. Ol’shanskii. Hyperbolicity of groups with subquadratic isoperimetric inequality. *Internat. J. Algebra Comput.*, 1(3):281–289, 1991.
[20] John Stallings. A finitely presented group whose 3-dimensional integral homology is not finitely generated. *Amer. J. Math.*, 85:541–543, 1963.

[21] Carsten Thiel. *Zur fast-Konvexität einiger nilpotenter Gruppen*. Universität Bonn Mathematisches Institut, Bonn, 1992. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1991.

[22] Daniel T. Wise. A non-Hopfian automatic group. *J. Algebra*, 180(3):845–847, 1996.

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