THE DUAL COMPLEX OF LOG CALABI–YAU PAIRS ON MORI FIBRE SPACES

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ABSTRACT. In this paper we show that the dual complex of a dlt log Calabi–
Yau pair \((Y, \Delta)\) on a Mori fibre space \(\pi: Y \to Z\) is a finite quotient of a
sphere, provided that either the Picard number of \(Y\) or the dimension of \(Z\) is
\(\leq 2\). This is a partial answer to Question 4 in [KX16].

CONTENTS

1. Introduction 1
2. Notation: birational dictionary 3
3. Notation: simplicial complexes 4
4. Dual complex of logCY pairs with Picard number 1 6
5. Generalities on the dual complex of logCY pairs on Mori fibre spaces 9
6. Dual complex of logCY pairs on Mori fibre spaces with Picard number 2 10
7. Dual complex of logCY pairs on Mori fibre spaces over a surface 17
References 20

1. INTRODUCTION

A dual complex is a cell complex, encoding the combinatorial data of how the
irreducible components of a simple normal crossing or dlt boundary intersect. These
objects have raised the interest of many scholars in different fields. For instance,
the homeomorphism type of the dual complex of a minimal dlt modification is
an interesting invariant of a singularity, see [dFKX17]. In mirror symmetry, the
dual complex of the special fibre of a good minimal dlt degeneration of Calabi–
Yau varieties has recently been proved to be the basis of a non-archimedean SYZ
fibration, see [NX16] and [NXY18].

In both these examples, a neighbourhood of any cell of the dual complex is a
cone over the dual complex of a new dlt pair \((X, \Delta)\), which satisfies the additional
property that \(K_X + \Delta \sim_{\mathbb{Q}} 0\), provided that the singularity is log canonical and the
degeneration semistable. These pairs are called log Calabi–Yau, in brief logCY.
Their dual complexes have been deeply studied in [KX16]. In that paper, the
authors have posed the question whether the dual complex of a logCY pair of
dimension \(n + 1\) is the quotient of a sphere \(S^k\) of dimension \(k \leq n\) for some finite
group \(G \subset O_{k+1}(\mathbb{R})\). With the techniques developed, they were able to provide
a positive answer in dimension \(\leq 4\), and in dimension \(= 5\) under the additional
hypothesis that \((X, \Delta)\) is simple normal crossing. It is worthy to remark that an
affirmative answer to this question would imply, for instance, that the basis of a
SYZ fibration has the structure of a topological orbifold.

In this paper, we answer positively the question for a special class of dlt logCY
pairs \((Y, \Delta)\), endowed with a morphism \(\pi: Y \to Z\) of relative Picard number one.
This hypothesis is inspired by the following observation. If \((X, \Delta_X)\) is a logCY
pair with maximal intersection, i.e. the pair admits a 0-dimensional lc centre,
then \(X\) is rationally connected, see [KX16, §18]. By [BCHM10], a \(K_X\)-MMP with
scaling \(f: X \dasharrow Y\) terminates with a Mori fibre space \(\pi: Y \to Z\) and the pair
\((Y, \Delta := f_\ast \Delta_X)\) is still logCY. It sounds sensible to the author to check first whether
the dual complexes of these special pairs are finite quotient of spheres, under the
dlt assumption.

In this paper, we describe the dual complex of dlt logCY pairs \((Y, \Delta)\) on Mori
fibre spaces \(\pi: Y \to Z\), subject to the condition that either the Picard number of
\(Y\) or the dimension of \(Z\) is smaller or equal to two, i.e. \(\rho(Y) \leq 2\) or \(\dim Z \leq 2\).
The main results are collected in the following statement.

**Theorem 1.1.** Let \((Y, \Delta)\) be a dlt pair such that:

1. \(Y\) is a \(\mathbb{Q}\)-factorial projective variety of dimension \(n + 1\);
2. (Mori fibre space) \(\pi: Y \to Z\) is a Mori fibre space of relative dimension \(r\);
3. (logCY) \(K_Y + \Delta \sim_{\mathbb{Q}} 0\).

If \(\rho(Y) = 1\), then the dual complex \(\mathcal{D}(\Delta)\) is PL-homeomorphic either to a ball
\(B^m\) of dimension \(m \leq n\) or to the sphere \(S^n\).

If \(\rho(Y) = 2\), then \(\mathcal{D}(\Delta)\) is PL-homeomorphic either to a ball \(B^m\) of dimension
\(m \leq n\) or to a sphere \(S^m\) of dimension \(m = r - 1, n - r\) or \(n\).

If \(\dim Z = 2\), then \(\mathcal{D}(\Delta)\) is PL-homeomorphic to a ball \(B^m\) of dimension \(m \leq n,
to a sphere \(S^m\) of dimension \(m = 1, n - 2, n - 1\) or \(n\), or to the quotient \(\mathbb{P}^2(\mathbb{R}) \ast S^{n-3}\).

All these cases occur.

**Proof of Theorem 1.1.** It is a combination of the following theorems:

1. Theorem 4.1 for \(\rho(Y) = 1\) (cf. Remark 4.2 for a slightly general statement);
2. Theorem 6.1 for \(\rho(Y) = 2\);
3. Theorem 7.1 for \(\dim Z = 2\).

All the cases can be realized as suitable hyperplane arrangements in \(\mathbb{P}^{n+1}, \mathbb{P}^{n+r-1} \times \mathbb{P}^r,\) or \(E \times \mathbb{P}^1 \times \mathbb{P}^{n-1}\), where \(\mathbb{P}^k\) is the projective space of dimension \(k\) and \(E\) is an
elliptic curve, with the only exception of the PL-homeomorphism type \(\mathbb{P}^2(\mathbb{R}) \ast S^{n-3}\),
which is discussed in Example 7.4. □

Theorem 1.1 can be summarised in the following table:

| \(\rho(Y)\) | \(\dim(Z)\) | PL-homeomorphism type of \(\mathcal{D}(\Delta)\) |
|-----|-----|-----------------|
| 1   | \(\mathbb{B}^n, S^n\) | |
| 2   | \(\mathbb{B}^m, S^{r-1}, S^{n-r}, S^n\) | |
| 1   | \(\mathbb{B}^m, S^0, S^{n-1}, S^n\) | |
| 2   | \(\mathbb{B}^m, S^1, S^{n-2}, S^{n-1}, \mathbb{P}^2(\mathbb{R}) \ast S^{n-3}\) | |

Table 1. Dual complex of logCY pairs on Mori fibre spaces.

In particular observe that all these dual complexes are quotients of spheres, compatibly with the prediction [KX16, Question 4].
The main ingredients of the proof of Theorem 1.1 are various connectivity theorems. The first of them is the Hodge index theorem: ample divisors always intersect, provided that they have dimension at least one. This fact allows in §4 to list all the triangulations of $D(\Delta)$ under the assumption $\rho(Y) = 1$. The naive idea for the next step, namely the case of $\rho(Y) = 2$, would be to build $D(\Delta)$ out of the contribution of vertical divisors together with the information provided by horizontal divisors. Indeed, the pushforward of the former determines a logCY pair $(Z, B)$ of Picard number one, while the latter cut out a logCY pair $(F_{\text{gen}}, \Delta_{\text{gen}})$ on a general fibre $F_{\text{gen}}$, which in turn behaves like a logCY pair of Picard number one. The special pairs for which this program works are here called of combinatorial product type. The proof of Theorem 1.1 consists precisely in understanding how far the general pair $(Y, \Delta)$ is from this ideal arrangement.

As a measure of what can go wrong, observe that there could be strata of $(Y, \Delta)$ which do not dominate $Z$, but which are not contained in any vertical divisor of $\Delta$. This instance is analysed in §6.1. Another issue is represented by horizontal strata which map two-to-one to $Z$: section §6.2 accounts for them. These latter strata are also responsible for the occurrence of the homeomorphism type $\mathbb{P}^2(\mathbb{R}) \ast S^{n-3}$, as explained in §7. We point out that the proof of Theorem 1.1 highly relies on the connectivity theorems [Kol13, Proposition 4.37] and [Kol13, Theorem 4.40] and the canonical bundle formula [Kol07, Theorem 8.5.1]. Observe finally that for a statement which does not involve non-trivial quotients of spheres, our hypothesis on the Picard number is the sharpest possible. Indeed, in Example 7.4 we construct logCY pairs on Mori fibre spaces of Picard rank three such that $D(\Delta) \cong \mathbb{P}^2(\mathbb{R}) \ast S^{n-3}$.

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2. Notation: birational dictionary

2.1. We work over an algebraically closed field in characteristic zero. A log pair $(Y, \Delta)$ is the datum of a normal variety $Y$ and a $\mathbb{Q}$-divisor $\Delta$ such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. If all coefficients of $\Delta$ are in $(0, 1]$ (resp. $(-\infty, 1]$), we say that $\Delta$ is a boundary (resp. a sub-boundary). Its support is the union of the prime divisors with non-zero coefficient in $\Delta$. If $\Delta$ is a sub-boundary, then $\Delta = \Delta^{=1} + \Delta^{<1}$, where $\Delta^{=1}$ and $\Delta^{<1}$ are the sums of the irreducible divisors which appear in $\Delta$ with coefficient equal to one or smaller than one respectively.

A $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier if one of its multiples is Cartier. A normal variety $Y$ is $\mathbb{Q}$-factorial if any Weil divisor on $Y$ is $\mathbb{Q}$-Cartier.

Let $f: X \to Y$ be a birational morphism. Given a log pair $(Y, \Delta)$, its log pull-back via $f$ is the log pair $(X, \Delta_X)$ determined by the relations $K_X + \Delta_X \sim_{\mathbb{Q}} f^*(K_Y + \Delta)$ and $f_*\Delta_X = \Delta$. 
The negative of the coefficient of a prime divisor $E$ in $\Delta_X$, labelled $a(E, Y, \Delta)$, is its discrepancy.

2.2. A log pair $(Y, \Delta)$ is log canonical, abbreviated lc, if $a(E, Y, \Delta) \geq -1$ for any exceptional divisor $E$ and for any $f: X \to Y$ birational morphism. An irreducible subvariety $W \subset Y$ is a lc centre if there exists a birational morphism $f: X \to Y$ and an exceptional divisor $E$, called lc place, whose discrepancy $a(E, Y, \Delta)$ equals $-1$ and whose image coincides with $W$.

A log pair $(Y, \Delta)$ is log smooth or simple normal crossing, abbreviated snc, if $Y$ is a smooth variety and the support of $\Delta$ has simple normal crossings. Given any log pair, there is a largest open subset $Y^{\text{snc}} \subset Y$, called simple normal crossing locus, such that $(Y^{\text{snc}}, \Delta|_{Y^{\text{snc}}})$ is snc.

A log canonical pair $(Y, \Delta)$ is dlt, alias divisorial log terminal, if none of the lc centres is contained in $Y \setminus Y^{\text{snc}}$. The lc centres of $(Y, \Delta)$ are also called strata of $\Delta$. A log canonical pair $(Y, \Delta)$ with no lc centre is klt, alias Kawamata log terminal.

A log canonical pair $(Y, \Delta)$ is qdlt, alias quotient divisorial log terminal, if for any lc centre $W$ of codimension $d$ there are $\Q$-Cartier divisors $\Delta_1, \ldots, \Delta_d \in \Delta_{=1}$ containing $W$, see also [dFKX17, Proposition 34]. The lc centres of $(q)dlt$ pair $(Y, \Delta)$ are the connected components of the intersection of the irreducible divisors in the support of $\Delta^{-1}$; see [Kol13, Theorem 4.16]. Equivalently, we call the lc centres of the $(q)dlt$ pair strata of $\Delta$.

If $(Y, \Delta := \sum \Delta_i)$ is a $(q)dlt$ pair, then for every lc centre $W$ there exists a unique $\Q$-divisor $\Diff_W^*\Delta$ on $W$, called different, such that $(K_Y + \Delta)|_W \sim_{\Q} K_W + \Diff^*_W(\Delta)$, see [Kol13, §4.18]. By adjunction, we have that
\[
\Diff^*_W(\Delta) = \sum_{W \not\subset D_i} D_i|_W + \Diff^*_W(\Delta)^{\subset 1}.
\]

2.3. A lc pair $(Y, \Delta)$ is logCY, alias log Calabi–Yau, if $K_Y + \Delta \sim_{\Q} 0$. In particular, let $f: X \to Y$ be a quasi-étale map, i.e. a finite map which is étale away from a codimension $\geq 2$ subset. Then, the pair $(X, \Delta_X := f^*\Delta)$ is logCY if and only if the pair $(Y, \Delta)$ is logCY, see also [KM98, Proposition 5.20].

2.4. A Mori fibre space for the log pair $(Y, \Delta)$ is an algebraic fibre space $\pi: Y \to Z$ satisfying the following properties:
\begin{enumerate}
\item $\dim Z < \dim Y$;
\item the relative Picard number $\rho(Y/Z)$ is one;
\item the $\Q$-divisor $-(K_Y + \Delta)$ is $\pi$-ample.
\end{enumerate}

If no log pair is mentioned, we tacitly assume $\Delta = 0$.

**Proposition 2.1.** [HX09, Proposition 5.5], [dFKX17, Proposition 40]. Let $(Y, \Delta)$ be a $\Q$-factorial $(q)dlt$ pair. If $\pi: Y \to Z$ is a Mori fibre space for the log pair $(Y, \Delta)$ and the support of $\Delta$ does not dominate $Z$, then the pair $(Z, \pi_*\Delta)$ is qdlt. In particular, $Z$ is klt.

3. Notation: simplicial complexes

3.1. We establish the notation:
\begin{itemize}
\item $S^n := \{ (x_0, \ldots, x_n) \in \R^{n+1} \mid \sum x_i^2 = 1 \} \text{ is the sphere of dimension } n$;
\item $B^n := \{ (x_0, \ldots, x_n) \in \R^{n+1} \mid \sum x_i^2 \leq 1 \} \text{ is the ball of dimension } n$;
\item $\P^n(\R)$ is the real projective space of dimension $n$;
\end{itemize}
• $\sigma^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_i x_i = 1, x_i \geq 0\}$ is the (standard) simplex of dimension $n$;
• $\partial \sigma^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_i x_i = 1, \prod_i x_i = 0, x_i \geq 0\}$ is the boundary of $\sigma^n$;
• the join $X \ast Y$ of two topological spaces $X$ and $Y$ is the quotient space of $X \times Y \times [0, 1]$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$, for all $x \in X$ and $y_1, y_2 \in Y$, and $(x_1, y, 1) \sim (x_2, y, 1)$, for all $x_1, x_2 \in X$ and $y \in Y$;
• the suspension $\Sigma X$ of a topological space $X$ is the join $X \ast S^0$.

Definition 3.1. [Hat02, §2.1]

• A $\Delta$-complex is the datum of a topological space $D$ and a triangulation, i.e. a collection of characteristic maps $\alpha_j : \sigma^d \to D$, with $d := d(j)$ depending on $j$, such that:
  (i) the restriction of $\alpha_j$ to the interior $\sigma^d$ of $\sigma^d$ is injective and any point of $D$ is contained in $\alpha_j(\sigma^d)$, called $d$-dimensional cell or face, for a suitable choice of $j$;
  (ii) for each $(d - 1)$-dimensional face of $\partial \sigma^d$, the restriction of the characteristic maps $\alpha_j$ to that face is a characteristic map $\alpha_j^*: \sigma^{d-1} \to D$;
  (iii) the topology of $D$ is the coarsest which makes the maps $\alpha_j$ continuous.
• A regular $\Delta$-complex is a $\Delta$-complex such that in addition the characteristic maps $\alpha_j$ are embeddings.
• A simplicial complex is a regular $\Delta$-complex such that any two $k$-cells have at most a $(k - 1)$-cell in common.

The $i$-th skeleton of $D$ is the subcomplex of $D$ given by the union of all the cells with dimension smaller or equal to $i$. The attaching map of a cell is the restriction $a_j|_{\partial \sigma^d} : \partial \sigma^d \to D$ of its characteristic map.

Note that a $\Delta$-complex is prescribed both by the set of its cells and their attaching maps. However, in order to define a regular $\Delta$-complex the datum of the attaching maps is redundant: it is enough to provide the poset of its cells.

Let $D$ be a regular $\Delta$-complex. If $v \subset D$ is a cell of $D$ we define

• the (open) star of $v$, denoted $St(v)$, as the union of the interiors of the cells whose closure intersects $v$;
• the closed star of $v$, denoted $\overline{St(v)}$, is the closure $St(v)$;
• the link of $v$, denoted $Link(v)$, is the difference $\overline{St(v)} \setminus St(v)$.
A map between \( \Delta \)-complexes is **piecewise-linear**, abbreviated PL, if the restriction of the map to any face is a linear application onto a face of the target, up to refinements of the triangulations.

### 3.2. The dual complex

The dual complex of a reduced snc pair \((Y, \Delta)\), denoted \(D(\Delta)\), is the regular \(\Delta\)-complex whose vertices are in correspondence with the irreducible components of \(\Delta\) and whose \(d\)-faces \(v_W\) correspond to a lc centre \(W\) of codimension \(d + 1\).

The dual complex of the dlt pair \((Y, \Delta)\) is the dual complex of the snc pair \((Y_{\text{snc}}, \Delta)|_{Y_{\text{snc}}}\), see [dFKX17, §2]. Hence, \(D(\Delta) = D(\Delta_{=1})\) by definition. We define the dual complex of a log canonical pair \((Y, \Delta)\) as the PL-homeomorphism type of the dual complex of a dlt modification of \((Y, \Delta)\), see [Kol13, Theorem 1.34].

If \((Y, \Delta)\) is qdlt, the same construction performed in the snc pair case gives a regular \(\Delta\)-complex which coincides with the one provided by a dlt modification, see [dFKX17, Corollary 38].

Let \(v_W\) be a \(d\)-face of \(D(\Delta)\) associated to the lc centre \(W\). Up to baricentrical subdivisions, \(\text{Link}(v_W)\) can be identified with the dual complex of the trace of \(\Delta\) on \(W\), i.e.

\[
\sum_{W \notin D_i} D_i|_W.
\]

In symbols,

\[\text{Link}(v_W) \simeq D(D_{\text{Diff}}^*(\Delta)) \simeq D(\sum_{W \notin D_i} D_i|_Y).\]

**Definition 3.2.** The dlt pair \((Y, \Delta)\) of dimension \(n+1\) has **maximal intersection** if it admits a 0-dimensional lc centre \(W\). Equivalently, the corresponding face \(v_W\) has dimension \(n\) and we say that \(D(\Delta)\) has **maximal dimension**.

In particular, if the dlt pair \((Y, \Delta)\) has maximal intersection, then \(Y\) is rationally connected and \(H^i(Y, \mathcal{O}_Y) = 0\) for any \(i > 0\), see [KX16, Proposition 19].

### 4. Dual complex of logCY pairs with Picard number 1

In this section we describe the explicit structure of \(\Delta\)-complex of the dual complex of a qdlt logCY pair with Picard number one.

**Theorem 4.1.** Let \((Y, \Delta)\) be a qdlt pair such that:

(i) \(Y\) is a projective variety of dimension \(n+1\) with \(\rho(Y) = 1\);

(ii) (logCY) \(K_Y + \Delta \sim_{\mathbb{Q}} 0\).

Then \(D(\Delta)\) is PL-homeomorphic either to a ball \(B^m\) of dimension \(m \leq n\) or to the sphere \(S^n\).

More precisely, \(D(\Delta)\) is isomorphic to one of the following regular \(\Delta\)-complexes:

1. (standard simplex) standard simplex \(\sigma^m\) of dimension \(m \leq n\);
2. (simplicial \(n\)-sphere) boundary \(\partial \sigma^{n+1}\) of the standard simplex \(\sigma^{n+1}\);
3. (non-simplicial \(n\)-sphere) union of two standard simplexes \(\sigma^n\), glued along the boundary.

**Proof.** Let \(\Delta_{=1} = \sum_{i=1}^{m+1} \Delta_i\). Any stratum \(W\) of \(\Delta_{=1}\) is a connected (irreducible) component of \(\Delta_{i_1} \cap \ldots \cap \Delta_{i_r}\) for certain \(I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, m + 1\}\). The restriction to \(W\) of \(\Delta_i\) for \(i \notin I\) is an ample divisor in \(W\). By the Hodge index theorem, \(\Delta_i|_W\) is non-empty and connected, as long as \(\dim W \geq 2\). Indeed, suppose on the contrary that an ample divisor is not connected, by slicing with general hyperplane sections, we can reduce to the case of a (possibly singular) surface and derive a contradiction applying the Hodge Index theorem to the resolution of the normalization of the surface.
Case \((m < n)\). Suppose that \(\Delta = 1\) does not contain any 0-dimensional strata, or equivalently \(m < n\). By the connectedness of \(\Delta\), any collection of \(r + 1\) 0-cells of \(\mathcal{D}(\Delta)\) is the set of vertices of a unique \(r\)-simplex. Hence, \(\mathcal{D}(\Delta)\) is a \(m\)-simplex as in \((1)\).

Case \((m \geq n)\). Denote by \(D_i\) and \(\sigma^m_i\) the \(i\)-th skeleton of \(\mathcal{D}(\Delta)\) and \(\sigma^m\) respectively. Arguing as in the previous case, we note that:

\[
D_{n-1} = \sigma^{m-1}_{n-1}, \quad D_n \supseteq \sigma^n.
\]

The long exact sequences in homology of the pairs \((D_i, \sigma^m_i)\) with integral coefficient give:

1. \(H_i(\mathcal{D}(\Delta), \mathbb{Z}) = H_i(D_{n-1}) = H_i(\sigma^m_{n-1}) = H_i(\sigma^m) = 0\) for \(0 < i < n - 1\);
2. \(H_{n-1}(\mathcal{D}(\Delta), \mathbb{Z}) = H_{n-1}(D_n) = 0\), since the latter group fits into the following exact sequence

\[
0 \to H_{n-1}(\sigma^m) \to H_{n-1}(D_n) \to H_{n-1}(D_n, \sigma^m) = 0,
\]

see also Remark 4.7

3. the long exact sequence of the pair \((\mathcal{D}(\Delta), \sigma^m)\) gives

\[
0 \to H_n(\sigma^m) \to H_n(\mathcal{D}(\Delta)) \to H_n(\mathcal{D}(\Delta), \sigma^m) \to 0.
\]

If \(H_n(\mathcal{D}(\Delta)) = 0\), then

\[
\text{rank } \tilde{H}_n(\sigma^m) = \left(\frac{m}{n+1}\right) = 0 \quad \Rightarrow \quad m \leq n;
\]

\(H_*(\mathcal{D}(\Delta), \sigma^m) = 0 \quad \Rightarrow \quad \mathcal{D}(\Delta) = \sigma^n\).

If \(H_n(\mathcal{D}(\Delta)) \neq 0\), we can show by induction on dimensions that \(\mathcal{D}(\Delta)\) is a closed topological manifold of dimension \(n\), provided that each irreducible component of \(\Delta = 1\) is an ample divisor, \(\text{e.g. if } \rho(Y) = 1\) (cf. also Remark 4.2). The base case, namely \(n \leq 3\), is assured by dimensional argument, see for instance \([KX16, \text{§33}]\). Suppose now that the statement holds for varieties of dimension \(\leq n\). Note that the irreducible components of the different \(\text{Diff}_\Delta(\Delta) = 1\) are ample divisors in \(\Delta\). The link of any cell in \(\mathcal{D}(\Delta)\) is a dual complex \(\mathcal{D}(\text{Diff}^*_\Delta(\Delta))\) for some \(d\)-dimensional stratum \(W\), see \([KX16, \text{§1}],\) and it is a \((d - 1)\)-dimensional sphere by induction hypothesis. Therefore, for \(n > 0\), \(\mathcal{D}(\Delta)\) is a connected topological manifold: if \(H_n(\mathcal{D}(\Delta)) \neq 0\), then \(\text{rank } H_n(\mathcal{D}(\Delta)) = 1\). As a result, the long exact sequence of the pair \((\mathcal{D}(\Delta), \sigma^n)\) gives

\[
0 \to H_n(\sigma^n) \to H_n(\mathcal{D}(\Delta)) \simeq \mathbb{Z} \to H_n(\mathcal{D}(\Delta), \sigma^n) \to 0.
\]

We are left with two options: either \(H_n(\mathcal{D}(\Delta), \sigma^n) = 0\) or \(\neq 0\). In the former case, \(m = n + 1\) and \(\mathcal{D}(\Delta) = \sigma^{n+1} = \partial \sigma^{n+1} \simeq \mathbb{S}^n\). Otherwise, \(H_n(\mathcal{D}(\Delta), \sigma^n) = \mathbb{Z}\) and \(H_n(\sigma^n) = 0\), which implies that \(n = m\). In other words, \(\mathcal{D}(\Delta)\) is obtained by attaching an additional \(n\)-cell to the standard simplex \(\sigma^n\), so it is a non-simplicial sphere of dimension \(n\).

Remark 4.2. In the proof of Theorem 4.1, we could have replaced the hypothesis \(\rho(Y) = 1\) with the weaker assumption that each irreducible component of \(\Delta = 1\) is an ample divisor.

Remark 4.3. In order to prove the first statement of Theorem 4.1, after showing the vanishing of the torsion of the homology of \(\mathcal{D}(\Delta)\), we could have concluded by invoking the (generalized) Poincaré conjecture.
Remark 4.4. The proof of Theorem 4.1 shows that a qlt anti-canonical divisor on a variety \( Y \) with \( \rho(Y) = 1 \) has at most \( \dim Y + 1 \) irreducible components. This is also a consequence of the non-negativity of the complexity of the log pair \((Y, \Delta)\); see [BMSZ18, Corollary 1.3].

Remark 4.5. We bring to the attention of the reader a precedent result by Danilov, [Dan75, Proposition 3], which states that if \((Y, \Delta)\) is a snc pair of dimension \( n + 1 \) such that at least one of the irreducible component of \( \Delta \) is ample, then \( D(\Delta) \) has the homotopy type of a bouquet of \( n \)-dimensional spheres. The virtue of Theorem 4.1 is that it provides a complete description of the \( \Delta \)-complex structure of \( D(\Delta) \), which we will exploit in an essential way in the following.

Remark 4.6. In the last case \( H_n(D(\Delta)) \neq 0 \), one could avoid the induction argument by computing directly \( H_n(D(\Delta)) \simeq \mathbb{Z} \), see [KK16, claim 32.3]. The claim 32.3 in [KK16] also implies that if the dlt logCY pair \((Y, \Delta)\) has maximal intersection, then the following statements are equivalent:

1. \( h_n(D(\Delta)) = 1 \);
2. \( \Delta^{n} = 0 \) and \( K_X + \Delta \sim 0 \).

Indeed, if \( \Delta^{n} \neq 0 \), then \( K_Y + \Delta^{n} \sim_{\mathbb{Q}} -\Delta^{n+1} \) and we have

\[
h_n(D(\Delta), \mathbb{C}) = h^n(Y, \mathcal{O}_{\Delta^{n}}) = h^{n+1}(Y, \mathcal{O}(-\Delta^{n+1})) = h^0(Y, K_Y + \Delta^{n+1}) = 0,
\]

which is a contradiction. Conversely, if \( K_X + \Delta^{n} \sim 0 \), the previous sequence of equalities yields \( h_n(D(\Delta)) = 1 \).

Remark 4.7. The reduced singular homology of the \( n \)-th skeleton of a standard simplex \( \sigma \) of dimension \( m \) is concentrated in degree \( n \) and

\[
\tilde{H}_n(\sigma^m_n, \mathbb{Z}) = \mathbb{Z}^{\binom{m}{n+1}}.
\]

Indeed, the long exact sequence of the pair \((\sigma^m_n, \sigma^m_{n-1})\) gives

\[
H_i(\sigma^m_{n-1}) \rightarrow H_i(\sigma^m_n) \rightarrow H_i(\sigma^m_n, \sigma^m_{n-1}) = 0 \quad \text{for} \quad i \neq n.
\]

Recursively, we note that \( H_i(\sigma^m_n) \rightarrow H_i(\sigma^m_n) \) is a surjective map, but since \( H_i(\sigma^m_n) = 0 \) for \( i \neq 0 \), we conclude that \( H_i(\sigma^m_n) = 0 \) for \( 0 < i < n \) and that the reduced homology of \( \sigma^m_n \) is concentrated in degree \( n \).

The long exact sequence of the pair \((\sigma^m_n, \sigma^m_{n-1})\) gives also

\[
0 = \tilde{H}_n(\sigma^m_{n-1}) \rightarrow \tilde{H}_n(\sigma^m_n) \rightarrow \tilde{H}_n(\sigma^m_n, \sigma^m_{n-1}) \rightarrow \tilde{H}_{n-1}(\sigma^m_{n-1}) \rightarrow \tilde{H}_{n-1}(\sigma^m_n) = 0
\]

Since rank \( H_i(\sigma^m_n, \sigma^m_{n-1}) = \binom{m+1}{i+1} \), we obtain the following formula by recursion

\[
\text{rank } \tilde{H}_n(\sigma^m_n) = (-1)^{n+1} \sum_{i=0}^{n+1} (-1)^i \binom{m+1}{i} = \binom{m}{n+1},
\]

which can be easily checked using the following identity

\[
\binom{m}{n+1} = \binom{m-1}{n+1} + \binom{m-1}{n}.
\]
5. Generalities on the dual complex of logCY pairs on Mori fibre spaces

Here and in the following, \((Y, \Delta)\) is a \(\mathbb{Q}\)-factorial dlt logCY pair of dimension \(n + 1\) with \(\Delta = \sum_{i=1}^{m+1} \Delta_i\). Suppose that there exists a morphism \(\pi: Y \to Z\) of relative dimension \(r\) and relative Picard number one. The goal is to identify the PL-homeomorphism type of the dual complex \(D(\Delta)\). In this section we collect some general facts about logCY pairs on Mori fibre spaces.

**Definition 5.1.** A horizontal lc centre (or a horizontal stratum) of the log pair \((Y, \Delta)\) is a lc centre of \((Y, \Delta)\) which dominates \(Z\). A lc centre is said vertical if it is not horizontal.

**Definition 5.2.** A vertical lc centre \(W\) is of maximal dimension if it is not contained in any other vertical lc centre.

**Definition 5.3.** The dual complexes of the horizontal or vertical divisors in the support of \(\Delta = 1\) are denoted \(D^{\text{hor}}\) or \(D^{\text{vert}}\) respectively and they are regular subcomplexes of \(D(\Delta)\).

Let \((F_{\text{gen}}, \Delta|_{F_{\text{gen}}})\) be the restriction of the logCY pair \((Y, \Delta)\) to a general fibre of \(\pi\), denoted \(F_{\text{gen}}\). Any stratum of \(\Delta|_{F_{\text{gen}}}\) is cut out by horizontal strata. Equivalently, the restriction of horizontal strata to \(F_{\text{gen}}\) induces a PL-map

\[(2) \quad r^*: D(\Delta|_{F_{\text{gen}}}) \to D^{\text{hor}}.\]

Properties of the map \(r^*\) are discussed in the following, see in particular next item (iii).

**Definition 5.4.** A log pair \((Y, \Delta)\), equivalently \(D(\Delta)\), has combinatorial product type if

1. any horizontal strata intersect vertical strata in a unique connected component, i.e.

\[D(\Delta) = D^{\text{vert}} \ast D^{\text{hor}}.\]

2. any intersection of horizontal divisors is horizontal, i.e.

\[D^{\text{hor}} = D(\Delta|_{F_{\text{gen}}}).\]

Since \(\rho(Y/Z) = 1\), the following properties hold.

(i) The preimage of an irreducible divisor on \(Z\) via \(\pi\) is an irreducible divisor (cf. for instance [HX09, Lemma 5.2]).

(ii) Horizontal lc centres restrict to ample divisors on any fibre.

(iii) The intersection of irreducible horizontal divisors is always non-empty and horizontal in codimension \(\leq r\), and connected in codimension \(\leq r - 1\), since the restriction of \(\pi\) to this intersection has connected fibres. In particular, the map \(r^*\) defined in (2) is an isomorphism on the \((r - 1)\)-th skeleton and it is injective if any horizontal stratum of codimension \(r\) intersects vertical strata in a unique connected component.

(iv) The \(\Delta\)-complex structure of \(D(\Delta|_{F_{\text{gen}}})\) is one of those listed in Theorem 4.1, see also Remark 4.2. In particular, there are at most \(r + 1\) horizontal divisors, see alternatively Remark 4.4.
(v) A vertical lc centre of maximal dimension is either a stratum of codimension $r + 1$ or a vertical divisor. Suppose that $W$ is a vertical lc centre of maximal dimension but not a vertical divisor. Then, $W$ is an irreducible component of the intersection of horizontal divisors $\Delta_{i_0} \cap \ldots \cap \Delta_{i_r}$ for certain $I = \{i_0, \ldots, i_r\} \subseteq \{1, \ldots, m + 1\}$, with $|I| = r + 1$ by (iii) and (iv).

**Proposition 5.5.** [Kol07, Theorem 8.5.1] (Canonical bundle formula) In the previous hypothesis, we can write $0 \sim_\mathbb{Q} K_Y + \Delta \sim_\mathbb{Q} \pi^*(K_Z + B + J)$, where

1. (moduli b-divisor) $J$ is a pseudo-effective $\mathbb{Q}$-linear equivalence class;
2. (boundary b-divisor) $B$ is a $\mathbb{Q}$-divisor with coefficient along the prime divisor $D$ given by
   $$\text{coeff}_B(D) = \sup \left\{ 1 - \frac{1 + a(Y, \Delta, E)}{\text{mult}_E(\pi^*D)} \right\},$$
   where the supremum is taken over all the divisors $E$ over $Y$ which dominate $D$. In particular, $D$ is dominated by a vertical lc centre if and only if $\text{coeff}_B(D) = 1$.

**Remark 5.6.** If $\rho(Z) = 1$ and $B \neq 0$, then there exists a logCY pair $(Z, B + J')$ such that $B + J'$ is a boundary $\mathbb{Q}$-linearly equivalent to $B + J$ and the $\mathbb{Q}$-divisor $J'$ has coefficient smaller than one. Indeed, under these hypotheses, $Z$ is rationally chain connected with rational singularities, thus $H^1(Z, \mathcal{O}_Z) = 0$, and there exists $r \in \mathbb{Q}$ such that $J \sim_\mathbb{Q} rH$, where $H$ is the ample generator of Pic($Z$).

If $B = 0$ and we are interested in the PL-homeomorphism type of $D(\Delta)$, we can avoid to induce any logCY structure on $Z$. Indeed, in this case $D(\Delta) = D_{\text{hor}}$, as the condition $B = 0$ implies that there are no vertical divisors, and the restriction map $r^* : D(\Delta_{F_{\text{gen}}}) \rightarrow D_{\text{hor}}$ is surjective, since any face in the complement of the image corresponds to vertical intersection of horizontal divisors, which would contribute to $B \neq 0$. Hence, either $r^*$ is injective and $D(\Delta) = D(\Delta_{F_{\text{gen}}})$ can be computed by means of Theorem 4.1, or $r^*$ is not and we conclude as in §6.2.1.

### 6. Dual complex of logCY pairs on Mori fibre spaces with Picard number 2

In this section we achieve the goal of §5 under the additional hypothesis that $\text{rank Pic}(Y) = 2$.

**Theorem 6.1** (rank Pic($Y$) = 2). Let $(Y, \Delta)$ be a dlt pair such that:

1. (Picard number two) $Y$ is a $\mathbb{Q}$-factorial projective variety of dimension $n + 1$ with $\rho(Y) = 2$;
2. (Mori fibre space) $\pi : Y \rightarrow Z$ is a Mori fibre space of relative dimension $r$;
3. (logCY) $K_Y + \Delta \sim_\mathbb{Q} 0$.

Then, $D(\Delta)$ is PL-homeomorphic either to a ball $B^m$ of dimension $m \leq n$ or to a sphere $S^m$ of dimension $m = r - 1, n - r$ or $n$.

**Proof.** We have the following dichotomy: either any vertical strata of maximal dimension is a vertical divisor or not. We study these cases separately in §6.2 and §6.1 (cf. Theorem 6.3 and Theorem 6.5) respectively. □
6.1. Not all vertical strata of maximal dimension are vertical divisors. Denote by $W$ a vertical lc centre of maximal dimension which is not a vertical divisor. The following facts hold.

(i) $W$ is an irreducible component of the intersection of all the horizontal divisors in $\Delta$ due to the bound on the number of horizontal divisors in $\text{Hyp}(\Delta)$ and the description of $W$ in $\text{Hyp}(\Delta)$. In particular, $D(\Delta|_{F_{\text{gen}}}) \simeq \partial \sigma^r$ by Theorem 4.1.1 since the restriction $\Delta|_{F_{\text{gen}}}$ of $\Delta$ to a general fibre $F_{\text{gen}}$ of the morphism $\pi$ consists of $r+1$ irreducible components.

(ii) Viceversa, any stratum of codimension $r+1$, intersection of horizontal divisors, is a vertical stratum of maximal dimension not contained in any vertical divisor of $\Delta$. By dimensional reasons, $W$ is not dominant. Moreover, if $W$ were contained in a vertical divisor of $\Delta$, there would be at least $r+2$ divisors passing through its generic points, contradicting the dlt hypothesis.

(iii) Let $W^+$ be a horizontal stratum of codimension $r$. Then, the restriction map $\pi|_{W^+} : W^+ \to Z$ is a birational morphism. We first claim that $\pi|_{W^+}$ is generically injective. In fact, it is generically finite, since a general fibre $\pi$ of $\pi|_{W^+}$ is a collection of 0-dimensional strata of the log pair $(F_{\text{gen}}, \Delta|_{F_{\text{gen}}})$. Hence, we have that

$$r + 1 = \# \{0\text{-dimensional strata in } (F_{\text{gen}}, \Delta|_{F_{\text{gen}}}) \} = \sum_{W^+} \deg \pi|_{W^+},$$

where the first equality follows from (i) and the last summation runs over all the horizontal strata $W^+$ of codimension $r$. Since any $r$-uple of horizontal divisors intersect along a general fibre, we conclude that any $r$-uple of horizontal divisors intersects in a single connected (irreducible) component (cf. $\text{Hyp}(\Delta)$) and $\deg \pi|_{W^+} = 1$. Finally, the normality of $Z$ and Zariski’s Main Theorem [Har77, cor 11.4] implies the restriction map $\pi|_{W^+}$ is a birational morphism. In particular, $D^{\text{hor}} = D(\Delta|_{F_{\text{gen}}})$.

(iv) The image $B_0 := \pi(W)$ is an irreducible divisor of $B$. The restriction map $\pi|_{W^+} : W^+ \to Z$ is birational by (iii) and $W$ is cut out by horizontal divisors, hence it is effective and $\pi|_{W^+}$-ample. By the negativity lemma [KM98, Lemma 3.39], it cannot be contracted by the morphism $\pi$.

(v) There are at most two disjoint vertical lc centre $W$ and $W'$ of maximal dimension not contained in any vertical divisor of $\Delta$ by the connectedness theorem [Kol93, Proposition 4.37], but there exists at most one of them if $\rho(Z) = 1$ and $\dim Z > 1$. In this case, the subvarieties $\pi(W)$ and $\pi(W')$ are ample divisors by (iv), hence they intersect, although the lc centres $W$ and $W'$ are disjoint in $\Delta$, defined as in (iii). This contradicts Kollár-Shokurov connectedness theorem [KM98, Theorem 5.48].

(vi) The pair $(Z, B^{\text{sing}})$ is qdlt. The pair $(W^+, \text{Diff}_{W^+}^{\text{sing}} \Delta)$ and $(Z, \pi_* \text{Diff}_{W^+}^{\text{sing}} \Delta)$ are crepant birational logCY pairs. Since $\pi_* \text{Diff}_{W^+}^{\text{sing}} \Delta^{\text{gg}} = B^{\text{gg}}$, the pair $(Z, B^{\text{gg}})$ is lc and its lc centres are dominated by lc centres of $\Delta$. We need to check that $(Z, B^{\text{sing}})$ is qdlt. By [HX09, Proposition 5.5] and (iv), the pair $(Z, B - \pi(W))$ is qdlt (or $(Z, B - \pi(W) - \pi(W'))$ is so, if $W^+$ contains two disjoint vertical lc centre): in any case, we can restrict our analysis to a neighbourhood of $B_0$. Note that there are $d - 1$ components of $B - B_0$ passing through any log centre of codimension $d$ of $(B_0, \text{Diff}_{B_0}(B))$, because
the lc centres of the dlt pair \((W, \text{Diff}^+_W(\Delta))\) are cut by vertical divisors. Adding \(B_0\) itself, we conclude that \((Z, B)\) is qdlt.

**Remark 6.2.** Note that section 6.1 does not rely in an essential way on the hypothesis on the Picard number of \(Y\). In the following, this information will be used only to constrain the topology of the dual complex \(D(B)\).

6.1.1. In order to prove Theorem 6.1, we can suppose without loss of generality that \(\text{Diff}^+_W(\Delta) \sim_{\mathbb{Q}} -K_W\) is not trivial, as the following lemma shows.

**Theorem 6.3.** Let \((Y, \Delta)\) be a dlt pair such that:

(i) \(Y\) is a \(\mathbb{Q}\)-factorial projective variety of dimension \(n + 1\);

(ii) (Mori fibre space) \(\pi: Y \to Z\) is a Mori fibre space of relative dimension \(r\);

(iii) (logCY) \(K_Y + \Delta \sim_{\mathbb{Q}} 0\);

(iv) (vertical CY-stratum) there exists a vertical stratum of maximal dimension \(W\) which is not a vertical divisor and such that \(K_W \sim_{\mathbb{Q}} 0\).

Then, \(D(\Delta)\) is PL-homeomorphic either to the ball \(B^r\) or to the sphere \(S^r\).

More precisely, \(D(\Delta)\) is isomorphic to one of the following regular \(\Delta\)-complexes:

1. (standard simplex) \(\sigma^r \simeq B^r\);
2. (simplicial \(r\)-sphere) \(\partial \sigma^{r+1} \simeq S^r\);
3. (non-simplicial \(r\)-sphere) \(\sigma^r \cup \partial \sigma^r \simeq S^r\).

**Proof.** Under this assumption, \(\text{Diff}^+_W(\Delta) \sim_{\mathbb{Q}} 0\). Denote by \(W^+\) a horizontal stratum of codimension \(r\) containing \(W\). Then, \(\text{Diff}^+_W(\Delta)^{=1}\) is either \(W\) or the disjoint union of two vertical strata \(W\) and \(W'\) by the connectedness theorem [Kol13, Proposition 4.37]. Therefore, either no component of \(\Delta^{=1}\) is vertical and one of the following equalities hold:

\[
D(\Delta) = D(\Delta|_{F_{\text{gen}}}) \cup v_W = \sigma^r \simeq B^r
\]
\[
D(\Delta) = D(\Delta|_{F_{\text{gen}}}) \cup v_W \cup v_{W'} = \sigma^r \cup \partial \sigma^r \simeq S^r.
\]

Or, there exists a unique vertical component \(\Delta^{\text{vert}}\) of \(\Delta^{=1}\) and

\[
D(\Delta) = (v_{\Delta^{\text{vert}}} \ast D(\Delta|_{F_{\text{gen}}})) \cup v_W = \partial \sigma^{r+1} \simeq S^r.
\]

\(\square\)

**Remark 6.4.** In the hypothesis of Theorem 6.3, if \(\rho(Z) = 1\) and \(\dim Z > 1\), then \(\text{Diff}^+_W(\Delta)^{=1}\) is connected due to 6.1(v). Hence, \(D(\Delta) = \sigma^r\).

6.1.2. We can finally prove Theorem 6.1 under the additional hypothesis of the existence of \(W\).

**Theorem 6.5.** Let \((Y, \Delta)\) be a dlt pair such that:

(i) \(Y\) is a \(\mathbb{Q}\)-factorial projective variety of dimension \(n + 1\) with \(\rho(Y) = 2\);

(ii) (Mori fibre space) \(\pi: Y \to Z\) is a Mori fibre space of relative dimension \(r\);

(iii) (logCY) \(K_Y + \Delta \sim_{\mathbb{Q}} 0\);

(iv) (vertical CY-stratum) there exists a vertical stratum of maximal dimension \(W\) which is not a vertical divisor, such that \(K_W\) is not \(\mathbb{Q}\)-linearly trivial.

Then, \(D(\Delta)\) is PL-homeomorphic either to a ball \(B^m\) of dimension \(r \leq m \leq n\) or to the sphere \(S^n\).

More precisely, \(D(\Delta)\) is isomorphic to one of the following regular \(\Delta\)-complexes:

1. (standard simplex) standard simplex \(\sigma^m\) of dimension \(r \leq m \leq n\);
If $D$ is simplicial or non-simplicial, respectively of types (1), (2) or (3).

As a result, we obtain $\alpha$ where the attaching map is an embedding. Hence, the dual complex $D(\Delta)$ has the following $\Delta$-complex structure:

$$D(\Delta) = (D^{\text{vert}} \ast D^{\text{hor}}) \cup_{D(\text{Diff}_W^*(\Delta)) \ast \partial v_W} (D(\text{Diff}_W^*(\Delta)) \ast v_W)$$

$$= (D(B - B_0) \ast D(\Delta_{|W_{\text{gen}}})) \cup_{D(\text{Diff}_W^*(\Delta)) \ast \partial v_W} (D(\text{Diff}_W^*(\Delta)) \ast v_W),$$

where the attaching map $\alpha$ is given by the natural map

$$D(\text{Diff}_W^*(\Delta)) \ast \partial v_W = D((B - B_0)|_{B_0}) \ast \partial v_W \rightarrow D(B - B_0) \ast \partial v_W$$

$$= D(B - B_0) \ast D(\Delta_{|W_{\text{gen}}})).$$

Property 6.1(vi) and Theorem 4.1 imply that $D(\Delta)$ is a standard simplex, a simplicial one, or a non-simplicial sphere, respectively of types (1), (2) or (3). If $D(\Delta)$ is a standard simplex $\sigma^k$ with $k \leq n - r$, then

1. $D(\text{Diff}_W^*(\Delta)) \simeq \sigma^{k-1}$;
2. the attaching map $\alpha$ is an embedding.

As a result, we obtain

$$D(\Delta) = (\sigma^{k-1} \ast \partial \sigma^r) \cup_{\sigma^{k-1} \ast \partial \sigma^r} (\sigma^{k-1} \ast \sigma^r) =$$

$$= \sigma^{k-1} \ast \sigma^r \simeq \sigma^m \quad \text{with } r \leq m \leq n.$$

If $D(\Delta)$ is a simplicial sphere, then

1. $D(\text{Diff}_W^*(\Delta)) \simeq \partial \sigma^{n-r}$;
2. the attaching map $\alpha$ is an embedding.

As a result, we obtain

$$D(\Delta) = (\sigma^{n-r} \ast \partial \sigma^r) \cup_{\partial \sigma^{n-r} \ast \partial \sigma^r} (\partial \sigma^{n-r} \ast \sigma^r) =$$

$$= \partial (\sigma^{n-r} \ast \sigma^r) \simeq S^n.$$

If $D(\Delta)$ is a non-simplicial sphere, then

1. $D(\text{Diff}_W^*(\Delta)) \simeq \sigma^{n-r-1}(2) \cup_{\partial \sigma^{n-r-1}(2)} \sigma^{n-r-1}(1)$, where we distinguish the two top dimensional cells by a subscript;
2. $D(B - B_0) \simeq \sigma^{n-r-1}$;
3. the attaching map $\alpha: (\sigma^{n-r-1}(1) \cup_{\partial \sigma^{n-r-1}(2)} \sigma^{n-r-1}(2)) \ast \partial \sigma^r \rightarrow \sigma^{n-r-1} \ast \partial \sigma^r$

is two-to-one and glues $\sigma^{n-r-1}(1) \ast \partial \sigma^r$ with $\sigma^{n-r-1}(2) \ast \partial \sigma^r$;
4. the $(n - 1)$-skeleton of $D(\Delta)$ contains

$$(\partial \sigma^{n-r-1} \ast \partial \sigma^r) \cup ((\sigma^{n-r-1}(1) \cup_{\partial \sigma^{n-r-1}(2)} \sigma^{n-r-1}(2)) \ast \partial \sigma^r),$$
which can be reduced modulo the identification induced by \( \alpha \) to

\[
(\partial\sigma^{n-r-1} * \partial\sigma^r) \cup_{\partial\sigma^{n-r-1} * \partial\sigma^r} (\sigma^{n-r-1} * \partial\sigma^r) = \partial(\sigma^{n-r-1} * \sigma^r);
\]

(5) the \( n \)-cells of \( D(\Delta) \) are \( \sigma_r^{n-r-1} * \sigma^r \) and \( \sigma_r^{n-r-1} * \sigma^r \).

As a result, we obtain

\[
D(\Delta) = \sigma_r^{n-r-1} * \partial\sigma^r \cup_{\alpha} ((\sigma_r^{n-r-1} \cup_{\partial\sigma^{n-r-1}} \sigma_r^{n-r-1}) * \sigma^r) = \\
= \sigma_r^{n-r-1} * \sigma^r \cup_{\alpha(\sigma_r^{n-r-1} * \sigma^r)} \sigma_r^{n-r-1} * \sigma^r \simeq S^n.
\]

\[\square\]

6.2. **Vertical strata of maximal dimension are vertical divisors.** Suppose now that any vertical strata of maximal dimension is a vertical divisor. If \((Y, \Delta)\) has combinatorial product type, then

\[
D(\Delta) = D_{\text{vert}} * D_{\text{hor}} = D(B) * D(\Delta_{\text{gen}}).
\]

By Theorem 4.1, \( D(\Delta) \) is then PL-homeomorphic to one of the following \( \Delta \)-complexes:

\[
\sigma^k \star \sigma^l \simeq \sigma^m, \quad \text{if } m := k + l + 1 \leq n;
\]

\[
S^{n-r} \star \sigma^k \simeq \begin{cases} 
\sigma^m, & \text{if } n - r + 1 \leq m := n - r + k + 1 \leq n; \\
S^{n-r}, & \text{if } D_{\text{hor}} = \emptyset;
\end{cases}
\]

\[
\sigma^k \star S^{r-1} \simeq \begin{cases} 
\sigma^m, & \text{if } r \leq m := k + r \leq n; \\
S^{r-1}, & \text{if } D_{\text{vert}} = \emptyset;
\end{cases}
\]

\[
S^{n-r} \star S^{r-1} \simeq S^n.
\]

However, if this is not the case, by the connectivity of ample divisors the only horizontal strata which can fail to intersect a general fibre in a unique connected component have codimension \( r \), namely those which restrict to points onto a general fibre. In this case, \( D(\Delta_{\text{gen}}) \) has maximal dimension but it cannot be simplicial if \( r > 1 \). Indeed, in the simplicial case the restriction of \( \pi \) to any horizontal strata of codimension \( r \) is generically injective due to property 6.1(iii). Hence, by Theorem 4.1, \( D(\Delta_{\text{gen}}) \) is a non-simplicial sphere of type \( \Delta \text{[4]} \). In particular, there is at most one horizontal stratum of codimension \( r \), denoted \( W \), which maps generically two-to-one to \( Z \) via \( \pi \).

6.2.1. If \( B = 1 = 0 \), then \( D(\Delta) = D_{\text{hor}} \). Since \( W \) maps generically two-to-one, the map \( r^* : \sigma^{r-1} \cup_{\partial\sigma^{r-1}} \sigma^{r-1} = D(\Delta_{\text{gen}}) \to D_{\text{hor}}, \) defined in \( \text{[5][2]} \), identifies the two cells of maximal dimensions of the domain. Hence, \( D(\Delta) = \sigma^{r-1} \). We can suppose that \( B = 1 \neq 0 \).

6.2.2. Now, if the branch locus of \( \pi|_W \) is a divisor \( B_0 \) (e.g., when \( Z \) is smooth), then \((Y, \Delta + \pi^*B_0)\) is not lc at the generic point of \( \pi^*B_0 \), which forces \( \text{coeff}_B(B_0) < 1 \) by Proposition 5.5. As \( B < 1 \neq 0 \), \( D(B) \) is a standard simplex \( \sigma^k \) by Remark 4.6. The dual complex \( D(\Delta) \) is no more the join of the dual complex of the vertical and horizontal strata, but we still have a PL-map induced by \( r^* \)

\[
\sigma^k \star D(\Delta_{\text{gen}}) = \sigma^k \star (\sigma_r^{n-r-1} \cup_{\partial\sigma^{n-r-1}} \sigma_r^{n-r-1}) \to D(\Delta),
\]
which identifies the vertical strata contained in $W$ passing through the two 0-dimensional strata of $(W, \text{Diff}_W^*(\Delta))$, namely $\partial \sigma^k \ast \sigma_{(1)}^{-1}$ and $\partial \sigma^k \ast \sigma_{(2)}^{-1}$. As a result, we obtain

$$D(\Delta) = \sigma^k \ast (\sigma_{(1)}^{-1} \cup_{\partial \sigma^{-1}} \sigma_{(2)}^{-1}) \mod \partial \sigma^k \ast \sigma_{(1)}^{-1} \equiv \partial \sigma^k \ast \sigma_{(2)}^{-1}$$

$$= (\sigma^k \ast \sigma_{(1)}^{-1} \cup_{\sigma^k \ast \partial \sigma^{-1}} \sigma_{(2)}^{-1} (\sigma^k \ast \sigma_{(2)}^{-1}))$$

$$= (\sigma^k \ast \sigma_{(1)}^{-1} \cup_{\partial \sigma^k \ast \partial \sigma^{-1}} (\sigma^k \ast \sigma_{(2)}^{-1}) \simeq S^m, \text{ with } r - 1 \leq m \leq n.$$

6.2.3. Suppose that the ramification locus of $\pi_{|W}$ has dimension at least 2. Without loss of generality, we can also assume that $D(\Delta)$ has maximal dimension and $\Delta = \Delta$, since eventual other options could be described as in 6.2.1 and 6.2.2. We claim that this case cannot occur, by showing that $\pi_{|W}$ induces an identification between $D(\text{Diff}_W^*(\Delta))$ and $D(B)$. Indeed, this is a contradiction, since the fibre via $\pi_{|W}$ of any 0-dimensional stratum of $(Z, B)$ is not connected, see next item (i) and the PL-morphism induced by $\pi_{|W}$ cannot be a homeomorphism.

In order to prove the claim, consider first the Stein factorization of the restriction morphism $\pi_{|W}$

$$\xymatrix{ (W, \text{Diff}_W^*(\Delta)) \ar[dr]_{\pi_{|W}} \ar[rr]^\theta & & (Z', B' := \nu^*(B)) \ar[dl]^\nu \ar[rr]_{2:1} & & (Z, B).}$$

In particular, the following facts hold.

(i) The pair $(Z', B')$ is log CY and qdlt. The morphism $\nu$ is quasi-étale and étale at the generic point of the lc centres of $B$ by [KM98, Proposition 5.7].

(ii) The lc centres of $(Z', B')$ are sent to lc centres of $(Z, B)$ via $\nu$, see [KM98, Proposition 5.20]. It means that there exists a PL-map

$$\nu_* : D(B') \to D(B).$$

(iii) The morphism $\nu_* : D(B') \to D(B)$ has finite fibres of cardinality at most two. Let $S_{d}$ be a stratum of $(Z, B)$ of codimension $d$. Then $\nu^{-1}(S_{d})$ is either irreducible of dimension $d$ or the union of two strata of $(Z', B')$ of dimension $d$, because $\nu$ is finite two-to-one. To conclude, we need to show that in the latter case the two strata are disjoint. If this is not the case, their intersection would be a stratum $S^*_{d+1}$ of codimension $d + 1$, which is contained in the pull-back of a lc centre $S_{d+1}$ of $(Z, B)$ of codimension $d + 1$, again by [KM98, Proposition 5.20]. Since $S_{d+1}$ is contained in $d + 1$ strata of codimension $d$ in view of the hypothesis $(Z, B)$ qdlt, so does $S^*_{d+1}$. However, $S^*_{d+1}$ is contained in the two irreducible components pull-back of $S_{d}$ and in the pull-back of the other lc centres of codimension $d$ containing $S_{d}$, thus at least in $d + 2$ strata of dimension $d$, which is a contradiction, since the pair $(W, \text{Diff}_W^*(\Delta)) = \varphi(B')$ is dlt.

(iv) The morphism $\nu_* : D(B') \to D(B)$ is a topological covering map. Let $v_d$ be a cell of maximal dimension, say $d$, in the ramification locus of $\nu_*$. By maximality, $\nu_*$ is a topological covering of degree two onto $\text{Link}(v_d, D(B)) \simeq S^{\dim Z - d - 2}$. We claim that the existence of $v_d$ yields a contradiction, so that $\nu_*$ must be a topological covering map.
(a) If \( \dim Z - d - 2 > 1 \) or \( \dim Z - d - 2 = 0 \), the link is simply-connected or the union of two points, so that
\[
\text{Link}(\nu_*^{-1}(v_d), \mathcal{D}(B^\nu)) \cong \mathcal{S}^{\dim Z - d - 2} \sqcup \mathcal{S}^{\dim Z - d - 2}.
\]
This is a contradiction by the connectedness theorem \cite[Proposition 4.37]{Kol13}.

(b) If \( \dim Z - d - 2 = 1 \), then \( \text{Link}(v_d, \mathcal{D}(B)) \) is isomorphic to a circle with two or three vertices, due to Theorem 4.1. Comparing with the previous connectivity argument, we can suppose that \( \text{Link}(\nu_*^{-1}(v_d), \mathcal{D}(B^\nu)) \) is a connected topological covering space of \( \text{Link}(v_d, \mathcal{D}(B)) \). Therefore, the \( \text{Link}(\nu_*^{-1}(v_d), \mathcal{D}(B^\nu)) \) is a circle with four or six vertices.

This implies that there exist two 2-dimensional lc centres
\[
(W, \text{Diff}_W^*(B)) := \sum_i B_i,
\]
\[
(W^\nu, \text{Diff}_{W^\nu}^*(B^\nu)) := \sum_i B^\nu_{i,0} + \sum_i B^\nu_{i,1}
\]
of the log pairs \((Z, B)\) and \((Z^\nu, B^\nu)\) respectively, such that:

1. \( W^\nu = \nu^{-1}(W) \);
2. \( \text{Link}(v_d, \mathcal{D}(B)) = \mathcal{D}(\text{Diff}_W^*(B)) \);
3. \( \text{Link}(\nu_*^{-1}(v_d), \mathcal{D}(B^\nu)) = \mathcal{D}(\text{Diff}_{W^\nu}^*(B^\nu)) \).

Note that \( B_1 \) is cut out on \( W \) by an ample divisor of \( Z \), as \( \rho(Z) = 1 \). Hence, the curve \( \nu^*(B_1) = B^\nu_{1,0} + B^\nu_{1,1} \) is a non-connected ample divisor, which yields a contradiction by the Hodge index theorem.

(c) If \( d = \dim Z - 1 \), we can suppose that there are two adjacent cells of maximal dimension in \( \mathcal{D}(B) \) on which \( \nu_* \) is respectively one-to-one and two-to-one. The link of a common face \( w \) contains three points, which is a contradiction by \cite[Proposition 4.37]{Kol13}.

All these facts imply that \( \mathcal{D}(B^\nu) = \mathcal{D}(B) \). Indeed, if \( \dim Z > 2 \), then \( \mathcal{D}(B) \) is a simply-connected sphere. Hence, \( \nu_* \) has degree one and \( \mathcal{D}(B^\nu) = \mathcal{D}(B) \). Instead, if \( \dim Z = 2 \) and \( \nu_* : \mathcal{D}(B^\nu) \to \mathcal{D}(B) \) is a non-trivial covering map, then as before the pull-back of an ample boundary divisor is not connected, which yields a contradiction by the Hodge index theorem (cf. (iv)\( \delta \)).

Finally, observe that \( \mathcal{D}(\text{Diff}_W^*(\Delta)) = \mathcal{D}(B^\nu) \), because \( \text{Supp} \text{Diff}_W^*(\Delta) \) is all vertical and the morphism \( W \to Z^\nu \) is a fibre space. We conclude that \( \pi|_W \) induces a PL-homeomorphism between \( \mathcal{D}(\text{Diff}_W^*(\Delta)) \) and \( \mathcal{D}(B) \).
7. Dual complex of log CY pairs on Mori fibre spaces over a surface

In this section we adapt the arguments of [6.1] and [6.2] to describe the dual complex of log CY pairs on Mori fibre spaces whose base has dimension two.

**Theorem 7.1** (dim $Z = 2$). Let $(Y, \Delta)$ be a dlt pair such that:

(i) $Y$ is a $\mathbb{Q}$-factorial projective variety of dimension $n + 1$;

(ii) (Mori fibre space) $\pi : Y \to Z$ is a Mori fibre space of relative dimension $r$

(iii) $(\log CY)$ $K_Y + \Delta \sim_{\mathbb{Q}} 0$.

Then $D(\Delta)$ has one of the following PL-homeomorphism types:

(1) $\mathbb{S}^1, \mathbb{S}^{n-2}, \mathbb{S}^{n-1}$ or $\sigma^n$ with $m < n$ if $(Y, \Delta)$ has not maximal intersection;

(2) $\sigma^n, \mathbb{S}^n$ or $\mathbb{P}^2(\mathbb{R}) \times \mathbb{S}^{n-3}$ if $(Y, \Delta)$ has maximal intersection.

**Proof.** We distinguish two cases: either any vertical stratum of maximal dimension is a vertical divisor or not.

First, suppose that there exists a vertical stratum $W$ of codimension $n$ not contained in any vertical divisor of $\Delta$. The properties [6.1](i)-(vi) of $W$ continue to hold. If there exists a unique such vertical stratum, then $D(B)$ is computed as in Theorem 6.3 and Theorem 6.5.

Otherwise, assume that there are two of them, say $W_1$ and $W_2$, see [6.1](v).

Since $\pi(W_1)$ and $\pi(W_2)$ are disjoint, $D(\Delta|_{F_{\text{gen}}})$ is simplicial and the attaching map $\alpha_{W_1} : D(Diff_{W_1}(\Delta)) \ast \partial v_{W_1} \to D(\Delta)$ and $\alpha_{W_2} : D(Diff_{W_2}(\Delta)) \ast \partial v_{W_2} \to D(\Delta)$ are embeddings. As a result, we have that

1. If $D(B) \simeq \mathbb{S}^1$, then

$$D(\Delta) = (D(\Delta|_{F_{\text{gen}}}) \ast D^{\text{vert}}) \cup D(Diff_{W_1}(\Delta)) \ast \partial v_{W_1} \cup D(Diff_{W_2}(\Delta)) \ast \partial v_{W_2}$$

2. If $D(B) \simeq \sigma^1$, then in a similar fashion

$$D(\Delta) \simeq_{\text{homeom. collapse}} \partial \sigma^{n-1} \ast \sigma^1 \simeq \mathbb{S}^n.$$
Suppose now that all the vertical strata are vertical divisors. The only places in §6.2 where we have exploited the hypothesis of $\rho(Z) = 1$ is to prescribe the homeomorphism type of the dual complex $D_{\text{vert}} = D(B^{1})$ and to exclude the case detailed in §6.2.3. Under the hypothesis $\dim Z = 2$, $D_{\text{vert}}$ is empty, a point, a segment or a circle by dimensional reason. However, we cannot avoid the second issue anymore.

As in §6.2, we can reduce to the case that the ramification locus of $\pi|_W$ has codimension $\geq 2$, $D(\Delta)$ has maximal dimension and $\Delta^{-1} = \Delta$. Consider again the Stein factorization of the restriction morphism $\pi|_W$ and the cartesian diagrams given respectively by the pairs of morphism $(\pi|_W, \nu)$ and $(\pi, \nu)$:

\[
\begin{array}{c}
(W^\nu := W \times Z \nu', \nu'^*\text{Diff}_W^*(\Delta) = \pi'|_W(B^\nu)) \xrightarrow{\nu'|_W} (W, \text{Diff}_W^*(\Delta)) \\
(Z^\nu, B^\nu := \nu^*(B)) \xrightarrow{\nu} (Z, B).
\end{array}
\]

Since $\nu$ is quasi-étale, the log pair $(W^\nu, \nu'^*\text{Diff}_W^*(\Delta))$ is logCY, but in general it fails to be dlt over the ramification locus of $\nu$. One can check it for instance from the computation for $W^\nu$ below. As in §6.2(iv), the PL-map $\nu_* : D(B^\nu) \to D(B)$

is a topological covering of degree at most two. This means that no 0-dimensional stratum of $(Z, B)$ is contained in the branch locus of $\nu$, denoted $\text{Branch}(\nu)$. In particular, up to shrinking $Z$, namely removing $\text{Branch}(\nu)$, we can suppose that $\nu$ is étale. In fact, the dual complexes are unchanged:

\[
\begin{align*}
D(B|_{Z \setminus \text{Branch}(\nu)}) &= D(B) ; \\
D(B^\nu|_{Z^\nu \setminus \nu^{-1}\text{Branch}(\nu)}) &= D(B^\nu) ; \\
D(\text{Diff}_W^*(\Delta)|_{W \setminus \nu^{-1}(\text{Branch}(\nu))}) &= D(\text{Diff}_W^*(\Delta)) .
\end{align*}
\]

The advantage is that now $(Y^\nu, \nu'^*\Delta)$ is dlt, as pullback of a dlt pair via the étale morphism $\nu'$, and the dual complex $D(Y^\nu, \nu'^*\Delta)$ is well-defined.

Denote by $\tau$ the involution of the covering map $\nu'$. Observe that over the generic point of $Z^\nu$

\[
W^\nu = \{(w, z) \in W \times Z^\nu | \pi(w) = \nu(\nu(z)) = \nu(z)\} = \{(w, g(w)) | w \in W \} \sqcup \{(w, \tau g(w)) | w \in W\},
\]

i.e. $W^\nu$ is the union of two irreducible components of the boundary $\nu'^*\text{Diff}_W^*(\Delta)$, exchanged by the involution $\tau$. By §6.2, $D(\nu'^*\Delta)$ has combinatorial product type

\[
D(\nu'^*\Delta) = D_{\text{vert}} \ast D_{\text{hor}} \simeq S^1 \ast S^{n-1} .
\]

The involution $\tau$ descends to a PL-involution $\tau_*$ of $D(Y^\nu, \nu'^*\Delta)$: $\tau_*$ fixes the horizontal strata of codimension $\leq n$, which form a subcomplex isomorphic to $S^{n-3}$.
We conclude that
\[ D(\Delta) = D^{\text{vert}} \ast D^{\text{hor}} / \tau_* \simeq S^1 \ast S^{n-1} / \tau_* \]
\[ = S^1 \ast S^0 \ast S^{n-3} / \tau_* \simeq S^2 \ast S^{n-3} / \tau_* \simeq \mathbb{P}^2(\mathbb{R}) \ast S^{n-3}. \]

\[ \square \]

**Corollary 7.2.** In the hypothesis of Theorem 7.1, \( D(\Delta) \simeq \mathbb{S}^n \) if and only if \((Y, \Delta)\) has maximal intersection, \( \Delta = \Delta = 1 \) and \( K_Y + \Delta \) is Cartier.

**Proof.** The conditions are necessary. The argument in Remark 4.6 implies that if \( D(\Delta) \simeq \mathbb{S}^n \), then \( \Delta = \Delta = 1 \) and \( K_Y + \Delta \sim \mathcal{O}_Y \), thus Cartier.

The conditions are also sufficient. Indeed, by Theorem 7.1 the maximality implies that \( D(\Delta) \) is isomorphic to \( \sigma^n \) or \( \mathbb{P}^2(\mathbb{R}) \ast S^{n-3} \) or \( S^n \). However, as in Remark 4.6, we have
\[ h_n(D(\Delta), \mathbb{C}) = h^0(Y, K_Y + \Delta = 1) \neq 0, \]
which excludes the first two cases.

\[ \square \]

**Remark 7.3.** Without assuming that \( Y \) carries a structure of Mori fibre space over a surface (or a curve), the additional hypotheses of Corollary 7.2, namely \( \Delta = \Delta = 1 \), \((Y, \Delta)\) has maximal intersection and \( K_Y + \Delta \) is Cartier, do not imply that \( D(\Delta) \simeq \mathbb{S}^n \). Indeed, we cannot exclude the occurrence of finite quotient of spheres. See also [KX16, §33, n=4].

We describe an example of this phenomenon. Let \( Y' := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \tau \) the involution which swaps the homogeneous coordinates \([x : y] \rightarrow [y : x]\) on each factor. Let \((Y, \Delta)\) be the dlt logCY pair where \( Y := Y' / \tau \) and \( \Delta \) is the pushforward of the toric boundary of \( Y' \). The dual complex \( D(\Delta) \) is isomorphic to \( \mathbb{P}^3(\mathbb{R}) \); see also [KX16] for detailed computations. However, \( \Delta = \Delta = 1 \) and \((Y, \Delta)\) has maximal intersection by construction and also \( K_Y + \Delta \) is Cartier. Indeed, notice that the singularities of \( Y' \) are quotient singularities of type \( 1/2(1, 1, 1, 1) \), thus cones over the projective space \( \mathbb{P}^3 \) polarized with the line bundle \( \mathcal{O}_{\mathbb{P}^3}(2) \). Since the canonical class of \( \mathbb{P}^3 \) is a multiple of the polarization, the singularities are Gorenstein, see [Kol13, Proposition 3.14]. Together with the fact that \( \Delta \) is supported on the smooth locus of \( Y \), we conclude that \( K_Y + \Delta \) is Cartier.

**Example 7.4.** We construct examples of dlt logCY pairs \((Y, \Delta)\) in any dimension with the following properties:

1. \( Y \) is a \( \mathbb{Q} \)-factorial projective variety of dimension \( n + 1 \) with \( \rho(Y) = 3 \);
2. \( \pi : Y \rightarrow Z \) is a morphism with \( \rho(Y / \mathbb{P}^2) = 1 \) and \( \dim Z = 2 \);
3. \( D(\Delta) \simeq \mathbb{P}^2(\mathbb{R}) \ast S^{n-3} \).

The following construction generalizes [KX16, Example 60]. Consider the logCY pair \((Y', \Delta')\) defined by
\[ Y' := \mathbb{P}^1_{[x_0 : x_1]} \times \mathbb{P}^1_{[y_0 : y_1]} \times \mathbb{P}^{n-1}_{[z_0 : \ldots : z_{n-1}]} \]
\[ \Delta_x := \{ x_0 x_1 = 0 \} \subseteq \mathbb{P}^1_{[x_0 : x_1]} \]
\[ \Delta_y := \{ y_0 y_1 = 0 \} \subseteq \mathbb{P}^1_{[y_0 : y_1]} \]
\[ \Delta := \left\{ (z_0z_1 + z_2^2 + \ldots + z_{n-1}^2) \prod_{i=2}^{n-1} z_i = 0 \right\} \subseteq \mathbb{P}^{n-1}_{[z_0 : \ldots : z_{n-1}]} \]

\[ \Delta' := \Delta_x \boxtimes \Delta_y \boxtimes \Delta_z = \left\{ x_0x_1y_0y_1(z_0z_1 + z_2^2 + \ldots + z_{n-1}^2) \prod_{i=2}^{n-1} z_i = 0 \right\} \]

The following facts hold.

1. The projection \( \pi' : Y' \rightarrow Z' := \mathbb{P}^1 \times \mathbb{P}^1 \)
   onto the first two factors of \( Y' \) is a Mori fibre space.

2. The involution \( \tau : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n-1} \)
   \(([x_0 : x_1], [y_0 : y_1], [z_0 : z_1 : \ldots : z_{n-1}]) \rightarrow ([x_1 : x_0], [y_1 : y_0], [z_1 : z_0 : \ldots : z_{n-1}])\)
   preserves the boundary \( \Delta' \) and descends to an involution onto \( Z' \).

3. By construction, the following diagram commutes:

\[
\begin{array}{ccc}
(Y', \Delta') & \xrightarrow{q_Y} & (Y, \Delta) := (Y'/\tau, q_Y, \Delta') \\
\pi' \downarrow & & \downarrow \pi \\
Z' & \xrightarrow{q_Z} & Z := Z'/\tau.
\end{array}
\]

4. The quotient map \( q_Z \) and \( q_Y \) are quasi-étale. In particular, the two morphisms are étale along \( \Delta_x \boxtimes \Delta_y \) and its preimage via \( \pi' \) respectively.

The pair \((Y, \Delta)\) is dlt logCY by (4) and the morphism \( \pi : Y \rightarrow Z \) is a Mori fibre space by the commutativity of the diagram (3).

The dual complex \( \mathcal{D}(\Delta') \) has combinatorial product type

\[ \mathcal{D}(\Delta') = \mathcal{D}(\Delta_x) \ast \mathcal{D}(\Delta_y) \ast \mathcal{D}(\Delta_z) \]

\[ = \mathbb{S}^0 \ast \mathbb{S}^0 \ast (\sigma^{n-2} \cup \partial \sigma^{n-2} \sigma^{n-2}) \]

\[ \simeq_{PL} \mathbb{S}^0 \ast \mathbb{S}^0 \ast \mathbb{S}^0 \ast \partial \sigma^{n-2} \]

\[ = \mathbb{S}^2 \ast \partial \sigma^{n-2}. \]

Since the involution \( \tau \) preserves \( \Delta' \), it defines a PL-involution of the dual complex \( \mathcal{D}(\Delta') \). It acts as the antipodal map on \( \mathbb{S}^2 \) and it fixes \( \partial \sigma^{n-2} \). We conclude that

\[ \mathcal{D}(\Delta) \simeq \mathbb{P}^2(\mathbb{R}) \ast \mathbb{S}^{n-3}. \]

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