SHIFT-IN-VARIANT HOMOGENEOUS CLASSES OF RANDOM FIELDS

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Abstract: Given an $\mathbb{R}^d$-valued random field (rf) $Z(t), t \in T$ and an $\alpha$-homogeneous mapping $\kappa$ we define the corresponding equivalent class of rfs (denoted by $\mathcal{C}_\alpha[Z]$) which include representers of the same tail measure $\nu_Z$. When $T$ is an additive group, tractable equivalent classes of interest are the shift-invariant ones, which contain in particular all independent random shifts of $Z$. This contribution is mainly concerned with the investigation of the probabilistic properties of shift-invariant $\mathcal{C}_\alpha[Z]$'s. Important objects introduced in our setting are tail and spectral tail rfs. Further, the class of universal maps $\mathcal{U}$ acting on elements of $\mathcal{C}_\alpha[Z]$ turns out to be crucial for properties of functionals of $Z$. Applications of our findings concern max-stable and symmetric $\alpha$-stable rfs, their maximal indices as well as their random shift-representations.

Key words: tail measures; shift-invariant classes of random fields; max-stable random fields; Rosiński representation; extremal index; lattices;

1. Introduction

Homogeneous functionals (hf’s) play a crucial role in the study of max-stable and symmetric $\alpha$-stable random fields (rf’s) as well as tail measures and tail rf’s. This fact has been exploited in [1], see also the later contributions [2–10]. It is simpler to demonstrate the importance of hf’s in the context of max-stable rf’s as we show next. Fix below $d,l$ two positive integers, $\alpha > 0$ and a norm $\| \cdot \|$ on $\mathbb{R}^d$. Let $X(t), t \in T$ be an $\mathbb{R}^d$-valued max-stable rf with $\alpha$-Fréchet max-stable marginal distributions having almost surely (a.s.) càdlàg sample paths being defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this contribution we consider $T = \mathbb{R}^l$ or $T \subset \mathbb{Z}^l$ assumed to be non-empty. In the light of de Haan characterisation, see e.g., [1, 11–13] we have the following representation (in distribution)

$$ X(t) = \max_{i \geq 1} \Gamma_i^{-1/\alpha} Z^{(i)}(t), \quad t \in T, $$

where $\Gamma_i = \sum_{k=1}^i \varepsilon_k$ with $\varepsilon_k$ independent $i.i.d.$ exponential random variables (rv’s) independent of $Z^{(i)}$’s, which are independent copies of a $d$-dimensional rf $Z(t) = (Z_1(t), \ldots, Z_d(t)), t \in T$ with non-negative components having a.s. càdlàg sample paths such that $\mathbb{E}[\|Z(t)\|^\gamma] < \infty$ for all $t \in T$. The rf $Z$ is referred to as the representer of $X$. It is well-known that for all $t \in T, x_i \in (0, \infty)^d, i = 1, \ldots, n, \gamma > 0$

$$ \mathbb{P}\{X(t_i) \leq x_i, i = 1, \ldots, n\}^\gamma = \mathbb{P}\{X(t_i) \leq x_i/\gamma^{1/\alpha}, i = 1, \ldots, n\} $$

$$ = \exp\left(-\gamma \mathbb{E}\left(\max_{1 \leq i \leq d, 1 \leq j \leq n} Z_i^{(1)}(t_j)/x_{ij}^\alpha\right)\right). $$

(1.2)

We shall consider next rf’s $Z(t), \tilde{Z}(t), t \in T$ that satisfy (the backshift operation is $B^{-t}f = f(t)$)

$$ \mathbb{P}\left\{\sup_{t \in T} \kappa(B^{-t}Z) > 0\right\} = \mathbb{P}\left\{\sup_{t \in T} \kappa(B^{-t}\tilde{Z}) > 0\right\} = 1, \quad \kappa(B^{-t}f) = \|f(t)\|^\alpha, $$

(1.3)

where $D(T, \mathbb{R}^d)$ denotes the space of functions $f : T \mapsto \mathbb{R}^d$ equipped with the product (cylindrical) $\sigma$-field $\mathcal{D}$ and

$$ \sup_{t \in A} \kappa(B^t f) := \max_{t \in A \cap T_0} \kappa(B^t f), \quad A \subset T, f \in D(T, \mathbb{R}^d). $$

Throughout this paper $T_0$ consists of all $t \in \mathbb{R}^l \cap T$ which have rational coordinates.

In view of [14, Lem 7.1] the requirement (1.3) is no restriction for a given representer $Z$ of $X$. Suppose hereafter without loss of generality that

$$ \mathbb{E}\{\kappa(Z)\} \in (0, \infty). $$

As shown in [10, Prop 2.1] given two representers $Z$ and $\tilde{Z}$ of the max-stable rf $X$

$$ \mathbb{E}\{\kappa(B^{-h}Z)F(\tilde{Z}/\kappa(B^{-h}Z))^{1/\alpha}\} = \mathbb{E}\{\kappa(B^{-h}\tilde{Z})F(Z/\kappa(B^{-h}Z))^{1/\alpha}\}, \quad \forall F \in \mathcal{H}, \forall h \in T$$

(1.5)

holds with $\mathcal{H}$ the class of all $\mathcal{D}/\mathcal{B}(\mathbb{R}^{\infty})$-measurable maps $F : D(T, \mathbb{R}^d) \mapsto E$ where $E = \mathbb{R}$ or $E = [0, \infty)$. Here $\mathcal{B}(V)$ stands for the Borel $\sigma$-field of a topological space $V$.

Moreover, also the converse is valid, i.e., any non-negative $\mathbb{R}^d$-valued rf $\tilde{Z}$ with càdlàg sample paths that satisfies

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(1.3) and (1.5) is a representer for $X$.

**Definition 1.1.** If $\mathcal{T} = \mathbb{R}^d$, then $D_c \subset D(\mathcal{T}, \mathbb{R}^d)$ consists of only càdlàg functions, while for $\mathcal{T} \subset \mathbb{Z}^d$ we set $D_c = D(\mathcal{T}, \mathbb{R}^d)$. In both cases $D_c$ is equipped with a metric $d_{D_c}$ that turns it into a Polish space.

The functional identity (1.5) is a consequence of Balkema’s Lemma, see [15, Lem. 4.1] and [10, Rem. 3.3]. Moreover, it can be related to the regular variation of rf’s. Specifically, in view of [6, 9, 16] where càdlàg rf’s are considered, $X$ is regularly varying with respect to the boundedness $B_0$, some positive sequence $a_n > 0, n \in \mathbb{N}$ and the tail measure $\nu_2$ defined on $\mathcal{D}$ by

$$
\nu_2[F] = \int_0^{\infty} \mathbb{E}\{F(z)\} a_z^{-\alpha - 1} dz, \quad \forall F \in \mathcal{H}.
$$

Tail measures are first introduced in [17, p. 159] and subsequently studied in [5–7, 9, 16, 18]. Applying Lemma 9.5 in Appendix we obtain

$$
\lim_{n \to \infty} n\mathbb{P}\{\kappa(B^h X) F(X) > b_n\} = \mathbb{E}\{\kappa(B^h Z) F(Z)\} < \infty, \quad \forall h \in \mathcal{T}
$$

for all $F : D_c \mapsto [0, \infty)$ continuous, $F(0) = 0$, and 0-homogeneous with an appropriate choice of $b_n$’s. Since for another representer $\tilde{Z}$ of $X$ necessarily $\nu_2 = \nu_{\tilde{Z}}$, then (1.5) follows. From (1.2) and the assumption on the sample paths of $X$ we obtain

$$
\mathbb{E}\{\sup_{t \in [-c, c] \cap \mathcal{T}} \kappa(B^{-t} Z)\} < \infty, \quad \forall c \in (0, \infty).
$$

In the light of (1.5) and the regular variation considerations above, it becomes clear that hf’s play a crucial role in the characterisation of the representer of $X$.

Two representers $Z$ and $\tilde{Z}$ of $X$ are called in [1, 10] max-zonoid equivalent, while in [19] and papers that refer to it, those are simply called equivalent. A flow representation of a representer of $X$ is derived in [20, Thm 1], see also [1, p.1219] for $d = 1$. In fact, flow representations are commonly discussed for symmetric $\alpha$-stable processes based on results of Rosiński [21], see e.g., [1, 17, 22]. Note that two representers of $X$ can be also related directly to each other, see [1, Eq. (3.1)] for the case $\alpha = d = 1$.

Indeed, the imposed restriction that both $Z, \tilde{Z}$ have non-negative components is not essential. In connection with symmetric $\alpha$-stable rf’s this fact is known from [23, Thm 1.1] and [1, Thm 2.2]. We drop therefore that specific assumption in the discussion below and consider $\nu_2$ on $\mathcal{D}$ for $Z$ that can have negative components. Also for such $Z$, in view of [16, Prop 3.6, Rem 3.11.(ii)] the functional identity (1.5) for $\tilde{Z}$ satisfying (1.3) is equivalent with

$$
\nu_2 = \nu_{\tilde{Z}}.
$$

Hereafter $\kappa : D(\mathcal{T}, \mathbb{R}^d) \mapsto [0, \infty]$ belongs to $\mathcal{H}$ being further $\alpha$-homogeneous i.e.,

$$
\kappa(cf) = c^\alpha \kappa(f), \quad \forall c > 0, \forall f \in D(\mathcal{T}, \mathbb{R}^d).
$$

**Definition 1.2.** $\mathfrak{M}_\kappa$ stands for the class of $\mathbb{R}^d$-valued rf’s $V(t), t \in \mathcal{T}$ defined on a complete probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\kappa(B^{-t} V), t \in \mathcal{T}$ is stochastically continuous. Moreover, $\kappa(B^t V), t \in \mathcal{T}$ is separable with separant $\mathcal{T}_0$ and jointly measurable.

Constructing a new representer $\tilde{Z}$ for a given max-stable rf $X$ or for a given tail measure $\nu_2$ is an important topic discussed in numerous papers. The former task is for instance crucial for simulations, see e.g., [24] and the references therein. In this article, we shall mainly focus on the functional equation (1.5) discussing the properties of representers of $\nu_2$ from $\mathfrak{M}_\kappa$ without making any specific reference to regular variation. In particular

i) instead of a norm $\|\cdot\|$ on $\mathbb{R}^d$, below we shall consider an $\alpha$-homogeneous and $\mathcal{D}/\mathcal{B}([0, \infty])$-measurable map $\kappa : D(\mathcal{T}, \mathbb{R}^d) \mapsto [0, \infty]$;

ii) we shall only assume that the representers $\tilde{Z}$ belong to $\mathfrak{M}_\kappa$ when $\mathcal{T} = \mathbb{R}^d$.

The representer of $\nu_2$ that belong to $\mathfrak{M}_\kappa$ define a class of rf’s. Note that below $\tilde{Z}$ might be defined in another probability space than $Z$. For notational simplicity we write $\mathbb{P}$ and $\mathbb{E}$ instead of $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{E}}$, respectively.

**Definition 1.3.** $\mathfrak{C}_\kappa[Z]$ with representer $Z$ consists of $Z, \tilde{Z} \in \mathfrak{M}_\kappa$ that satisfy (1.3)–(1.5) and (1.7).

The Brown-Resnick max-stable rf’s were introduced for particular instances initially in [25, 26] and discussed in greater generality later in [27]. We introduce below the corresponding $\alpha$-homogeneous classes.

**Example 1.4** (Brown-Resnick $\mathfrak{C}_\kappa[Z]$). For $V(t) = (V_1(t), \ldots, V_d(t)), t \in \mathcal{T}$ a centered $\mathbb{R}^d$-valued Gaussian rf with a.s. continuous sample paths define

$$
Z(t) = (\xi_1 e^{W_1(t)}, \ldots, \xi_d e^{W_d(t)}), \quad W_i(t) = V_i(t) - \alpha \text{Var}(V_i(t))/2, \quad 1 \leq i \leq d, t \in \mathcal{T},
$$
where \(\xi_1, \ldots, \xi_d\) are rv’s that take values \(\pm 1\) being further independent of any other random element. Taking
\[
\|x\|_\alpha = \left(\sum_{i=1}^d |x_i|^\alpha \right)^{1/\alpha}, \quad x \in \mathbb{R}^d,
\]
then \(\mathbb{E}\{\|Z(t)\|_{\alpha}^\alpha\} = 1\) for all \(t \in T\). In light of [28, Cor. 6.1]
\[
d\mathbb{E} \left\{ \sup_{t \in [-c,c]^d} \|Z(t)\|_{\alpha}^\alpha \right\} = \sum_{i=1}^d \mathbb{E} \left\{ \sup_{t \in [-c,c]^d} e^{\alpha V_i(t) - \text{Var}(\alpha V_i(t))/2} \right\} < \infty, \quad \forall c > 0
\]
and thus \(Z \in \mathcal{W}_\alpha\) and it satisfies (1.3), (1.4), (1.7) with \(\kappa(f) = \|f(0)\|_{\alpha}^\alpha\). Hence we can define the corresponding \(\mathcal{C}_\alpha[Z]\). Write \(\gamma\) for the pseudo-variogram matrix-valued function of \(V\) with components
\[
\gamma_{ij}(s,t) = \text{Var}(V_i(s) - V_j(t))/2, \quad s,t \in T, 1 \leq i \leq d, 1 \leq j \leq d.
\]
In view of Example 4.2 below \(\gamma\) uniquely determines this class.

The Brown-Resnick-Lévy max-stable processes have been introduced in [29], see [2, 30] for further results. The corresponding \(\alpha\)-homogeneous class of rf’s is defined next.

**Example 1.5** (Brown-Resnick-Lévy \(\mathcal{C}_\alpha[Z]\)). Let \(W_{ij}(s), s \geq 0, 1 \leq i \leq l, 1 \leq j \leq d\) be Lévy processes with Laplace exponent \(\psi_{ij}(\alpha) = \ln \mathbb{E} \left\{ e^{\alpha W_{ij}^{(1)}(1)} \right\}\) such that \(\psi_{ij}(\alpha) = 0\) for some \(\alpha > 0, 1 \leq i \leq l, 1 \leq j \leq d\). Write \(W_{ij}^{(\alpha)}(s), s \geq 0\) for the exponentially tilted Lévy process with Laplace exponent \(\psi_{ij}(\alpha + \theta)\). Suppose further that \(W_{ij}, W_{ij}^{(\alpha)}\) are all independent with càdlàg sample paths defined in the same probability space and set
\[
Z_j(t) = \xi_j \prod_{i=1}^l e^{\left(t_i \geq 0\right) W_{ij}(t_i) - \left(t_i < 0\right) W_{ij}^{(\alpha)}(-t_i)}, \quad t = (t_1, \ldots, t_l),
\]
with \(\xi_j\)’s as in Example 1.4. Taking \(\kappa(f) = \|f(0)\|_{\alpha}^\alpha\) it follows easily that \(\mathcal{C}_\alpha[Z]\) is well-defined.

In applications, of particular interest are max-stable, symmetric \(\alpha\)-stable, and tail measures that are shift-invariant. A unified approach in their study is the investigation of shift-invariant \(\alpha\)-homogeneous classes of rf’s, which we define below. Whenever we consider the shift-invariance, we shall assume that \(T\) is an additive group and hence \(B^h f, f \in D(T, \mathbb{R}^d)\) is well-defined.

**Definition 1.6.** \(\mathcal{C}_\alpha[Z]\) is shift-invariant if for some \(\tilde{Z} \in \mathcal{C}_\alpha[Z]\) we have
\[
B^h \tilde{Z} \in \mathcal{C}_\alpha[Z], \quad \forall h \in T. \tag{1.9}
\]
By the above definition and (1.5) if \(\mathcal{C}_\alpha[Z]\) is shift-invariant, then
\[
\mathbb{E} \left\{ \kappa(B^{-h} Z^*) F(Z^*) \right\} = \mathbb{E} \left\{ \kappa(B^{-h} \tilde{Z}) F(\tilde{Z}) \right\} = \mathbb{E} \left\{ \kappa(\tilde{Z}) F(B^h \tilde{Z}) \right\} \tag{1.10}
\]
since when \(F \in \mathcal{H}_0\) also \(G(f) = F(B^h f), f \in D(T, \mathbb{R}^d)\) belongs to \(\mathcal{H}_0\) and thus (1.9) holds for all \(Z^* \in \mathcal{C}_\alpha[Z]\).

Clearly, if \(Z\) is stationary, then \(\mathcal{C}_\alpha[Z]\) is shift-invariant. More interesting cases are shift-invariant \(\mathcal{C}_\alpha[Z]\)’s generated by some non-stationary \(Z\). Two prominent instances are the Brown-Resnick \(\mathcal{C}_\alpha[Z]\), see Example 4.2, while the shift-invariance of the Brown-Resnick-Lévy \(\mathcal{C}_\alpha[Z]\) follows easily from the definition and the shift-invariance of the corresponding max-stable rf, see [30].

It was shown in [2] for the case \(d = 1\) and \(Z\) being non-negative and in [5–9, 16, 31] for discrete or càdlàg \(Z\) that the functional equation (1.5) is equivalent with the time-change formula discovered in [32]. Therein it is formulated in terms of the so called spectral tail rf \(\Theta\), see below (5.1). As shown in [5] the time-change formula is equivalent to (5.4). Dropping the regular variation context of [32], we introduce a more general object, again denoted by \(\Theta\), which as in [7] is labeled as the local rf.

**Definition 1.7.** Given a \(\mathcal{C}_\alpha[Z]\) we shall define its local rf \(\Theta\) as the rf \(Z/\kappa(Z)^{1/\alpha}\) under the probability law \(\tilde{\mathbb{P}}\), where (recall that we assume (1.4))
\[
\tilde{\mathbb{P}}(A) = \frac{1}{\mathbb{E} \{\kappa(Z)\}} \mathbb{E} \{\kappa(Z) I(A)\}, \quad \forall A \in \mathcal{F}, \tag{1.11}
\]
with \(I(A)\) the indicator function of the set \(A\).

Brief organisation of the paper: Section 2 introduces lattices, some classes of maps and our notation. Extension of (1.5) is discussed in Section 3 followed by a section on the characterisations of shift-invariance \(\mathcal{C}_\alpha[Z]\). Both spectral tail and tail rf’s are introduced in Section 5. Universal maps and their relations with shift-involutions and anchoring maps are introduced in Section 6. Implications of our findings are briefly considered in Section 7. All proofs are displayed in Section 8 followed by some technical results in the concluding part Section 9.
2. Preliminaries

2.1. Lattices. Unless otherwise stated \( \mathcal{L} \) shall denote a discrete subgroup of the additive group \( \mathcal{T} \) (called also a lattice on \( \mathcal{T} \)) which has finite or countably infinite number of elements \( |\mathcal{L}| \). In several instances we shall assume that \( \mathcal{L} \) has full rank, i.e., for some non-singular \( l \times l \) real matrix \( A \) (called base matrix) \( \mathcal{L} = \{ Ax, x \in \mathcal{T} \} \), where \( x \) denotes a \( l \times 1 \) vector. Two base matrices \( A, B \) generate the same lattice on \( \mathcal{T} \) if and only if (iff) \( A = BU \),

where \( U \) is an \( l \times l \) real matrix with determinant \( \pm 1 \). Therefore, when \( \mathcal{T} = \mathbb{R}^l \) the volume of the fundamental parallelepiped \( \{ Ax, x \in [0,1]^j \} \) of the lattice does not depend on the choice of the base matrix and is given by

\[
\Delta(\mathcal{L}) = |\det(A)| > 0.
\]

2.2. Some classes of functionals. Write \( \mathcal{H}_\beta \subset \mathcal{H}, \beta \geq 0 \) for the class of maps (recall the definition of \( \mathcal{H} \) in the Introduction) which are \( \beta \)-homogeneous, i.e., \( F(cf) = c^\beta F(f) \) for all \( f \in D(\mathcal{T}, \mathbb{R}^d) \) and \( c > 0 \). Throughout this paper \( \lambda \) denotes the Lebesgue measure on \( \mathbb{R}^d \) or the counting measure if the integration is over a countable set. Let \( g : \mathcal{T} \mapsto [0,\infty) \) being locally bounded and \( \lambda \)-measurable. When \( \mathcal{T} = \mathbb{R}^d \) it is assumed that \( g \) is almost everywhere positive. Write \( \mathcal{F} \) for the class of such \( g \)'s and write \( \mathcal{F}_\beta, \beta \geq 0 \) for the class of \( \beta \)-homogeneous maps determined for given \( \Gamma \in \mathcal{H}_{\mathcal{F}_\beta}, g_i \in \mathcal{F}, i \leq 3 \) as

\[
F(f) = \frac{\Gamma(f)\mathcal{J}_i(f,g)(\mathcal{J}_2(f,g) \in A)}{\mathcal{J}_3(f,g)}, \quad \mathcal{J}_i(f,g) = \int_{\mathcal{T}} \kappa(B^{-1}f)^{\xi_i} g_i(t) \lambda(dt), \quad f \in D(\mathcal{T}, \mathbb{R}^d).
\]

Here \( \xi_i \)'s are constants and \( A \subset [0,\infty] \) is such that \( F \) is \( \beta \)-homogeneous. If \( F(f) \) is undefined we set \( F(f) = 0 \).

2.3. Anchoring & involution maps. Anchoring maps introduced in [8] play a crucial role in the investigation of tail r.f.'s. See also [5, 6, 8, 31, 33–36] for various results concerning those maps. Given a lattice \( \mathcal{L} \) on \( \mathcal{T} \) we introduce next (positive) shift-involutions and anchoring maps. Below \( (\mathbb{R}^d)^* = (\mathbb{R}^d \cup \{ \infty \}), k \in \mathbb{N} \) is the one-point compactification of \( \mathbb{R}^k \) and \( 0 \) denotes the origin of \( \mathbb{R}^k \) or the zero of \( D(\mathcal{T}, \mathbb{R}^d) \).

Definition 2.1. Let \( \mathcal{J} : D(\mathcal{T}, \mathbb{R}^d) \mapsto (\mathbb{R}^d)^* \) be \( \mathcal{G}/\mathcal{B}(\mathbb{R}^d)^* \)-measurable.

J1 For all \( j \in \mathcal{L}, f \in D(\mathcal{T}, \mathbb{R}^d) \) we have \( \mathcal{J}(B^1 f) = \mathcal{J}(f) + j \).

J2 For all \( f \in D(\mathcal{T}, \mathbb{R}^d) \) if \( \mathcal{J}(f) = j \in \mathcal{L} \), then \( \kappa(B^{-1}f) \geq \min(\kappa(f), 1) \).

J3 For all \( f \in D(\mathcal{T}, \mathbb{R}^d) \) if \( \mathcal{J}(f) = j \in \mathcal{L} \), then \( \kappa(B^{-1}f) > 0 \).

Suppose that \( \mathcal{J} \) satisfies J1. When J2) holds it is referred to as anchoring. If \( \mathcal{J} \) is \( 0 \)-homogeneous it is called a shift-involution and if further J3) is satisfied it is called a positive shift-involution.

Hereafter \( \lt \) stands for some total order on \( \mathcal{T} \) which is shift-invariant, i.e., \( i \lt j \) implies \( i + k \lt j + k \) for all \( i, j, k \in \mathcal{T} \); a canonical instance is the lexicographical order. Both inf and sup are defined with respect to \( \lt \). As pointed out in [8] an interesting anchoring map is the first exceedance functional \( \mathcal{I}_{\mathcal{L},f,e} \) defined by

\[
\mathcal{I}_{\mathcal{L},f,e}(f) = \inf\left\{ j \in \mathcal{L} : \kappa(B^{-1}f) > 1 \right\}, \quad f \in D(\mathcal{T}, \mathbb{R}^d),
\]

where \( \mathcal{I}_{\mathcal{L},f,e}(f) = \infty \) if there are infinitely many exceedance on \( \{ j \in \mathcal{L}, j \lt k \} \) for some \( k \in \mathcal{L} \) with all components positive. Define further the infargmap

\[
\mathcal{I}_{\mathcal{L},argsup}(f) = \inf\left\{ j \in \mathcal{L} : \kappa(B^{-1}f) = \sup_{i \in \mathcal{L}} \kappa(B^{-1}f) \right\}, \quad f \in D(\mathcal{T}, \mathbb{R}^d),
\]

which is a positive shift-involution but not anchoring. The infimum of an empty set is \( \infty \). Also if it is not attained at some element of \( \mathcal{L} \), then the maps defined above are assigned to \( \infty \).

3. Extensions of (1.5)

As in the Introduction consider an \( \alpha \)-homogeneous class \( \mathcal{E}_\alpha[Z] \), which is defined with respect to a fixed \( \alpha > 0 \) and a given non-negative \( \kappa \in \mathcal{H}_\alpha \). Hereafter \( \|x\|_\infty = \max_{1 \leq i \leq k} |x_i|, x \in \mathbb{R}^k \) and set

\[
s^+ = \max(0, s), \quad s^- = \max(-s, 0), \quad s \in \mathbb{R}, \quad x^+ = (x_1^+, \ldots, x_k^+), \quad x^- = (x_1^-, \ldots, x_k^-), \quad \kappa_\infty(f) = \| (f(0))_\infty \|_\infty, \quad f \in D(\mathcal{T}, \mathbb{R}^d).
\]

We show next that a simpler condition than (1.5) can be formulated under weak assumptions.

Lemma 3.1. Fix \( Z \in \mathcal{M}_\kappa, \kappa \in \mathcal{H}_\alpha \) and suppose that (1.3) holds for both \( \kappa \) and \( \kappa_\infty \). If further

\[
\max(\{ \kappa_\infty(B^1 Z) \}, \{ \kappa(B^1 Z) \}) < \infty, \quad \forall t \in \mathcal{T},
\]

then (1.5) is equivalent with

\[
\mathbb{E}\left\{ \max_{1 \leq i \leq n} \kappa_\infty(B^{i_1} Z/i) \right\} = \mathbb{E}\left\{ \max_{1 \leq i \leq n} \kappa_\infty(B^{i_1}/Z/i) \right\} < \infty
\]


for all \((t_1, \ldots, t_n) \in \mathbb{T}_0^n, n \in \mathbb{N}, x \in [0, \infty)^d, \bar{Z} \in \mathcal{C}_n[Z]\).

**Remark 3.2.** As shown in the proof of Lemma 3.1, under the conditions therein \(Z\) and \(\bar{Z} \in \mathcal{C}_n[Z]\) are closely related since \(Z_\ast\) and \(\bar{Z}_\ast\) are representatives of the same max-stable rf \(X_\ast\) and therefore (when \(d = 1, \alpha = 1\)) \([1, \text{Eq. (3.1)}]\) holds.

If \(F : D(\mathcal{T}, \mathbb{R}^d) \to \mathbb{R}\) is such that \(F(\bar{Z})\) is a rv for all \(\bar{Z} \in \mathcal{C}_n[Z]\) and moreover, there exists a sequence of Borel measurable and 0-homogeneous functions \(F_n : \mathbb{R}^n \to [0, \infty]\) such that we have the convergence in probability

\[
F_n(\bar{Z}(t_1), \ldots, \bar{Z}(t_n)) \xrightarrow{p} F(\bar{Z}), \quad n \to \infty,
\]

with \(t_i\)'s in \(\mathcal{T}\), then by (1.7) it follows that (1.5) holds also for such general \(F\).

We show next that this is the case for \(F\) defined in (2.2) justifying further the label \(\alpha\)-homogeneous.

**Theorem 3.3.** For a given \(\mathcal{C}_n[Z]\) (1.5) is equivalent with

\[
\mathbb{E}\{\kappa(B^h Z) F(Z)\} = \mathbb{E}\{\kappa(B^h \bar{Z}) F(\bar{Z})\}, \quad \forall F \in \mathcal{S}_0, \forall h \in \mathcal{T}, \bar{Z} \in \mathcal{C}_n[Z]
\]

and

\[
\mathbb{E}\{F(Z)\} = \mathbb{E}\{F(\bar{Z})\}, \quad \forall F \in \mathcal{S}_0, \forall \bar{Z} \in \mathcal{C}_n[Z].
\]

**Remark 3.4.**

(i) We interpret \(\infty \cdot 0\) and \(0/0\) as 0. This is the case for instance if the maps \(F\) above are products or ratios with one component being the indicator function;

(ii) In view of the above result also \(\kappa \in \mathcal{S}_0\) can be considered. Particular choices of interest (as suggested from one referee) are

\[
\kappa(f) = \max_{t \in \mathbb{T}_0} |f(t)|^\alpha,
\]

with \(\mathbb{T}_0 \subset \mathcal{T}\) having countable number of elements.

(iii) So far the assumption (1.3) has been used in the derivation of (3.5) only. If \(F \in \mathcal{S}_0\) is such that

\[
\mathbb{E}\left\{F(\bar{Z}) \mathbb{1}_{\left(\sup_{t \in \mathcal{T}} \kappa(B^{-1} \bar{Z}) = 0\right)}\right\} = 0, \quad \forall \bar{Z} \in \mathcal{C}_n[Z],
\]

then (3.5) still holds without (1.3). When \(\mathcal{T}\) has countable number of elements instead of (1.3) we can assume \(F(0) = 0\).

(iv) By (3.4) with pdf \(p_N(t) > 0, t \in \mathcal{T}\) we have (set \(I_T(f) = \int_{\mathcal{T}} \kappa(B^{-1} f) p_N(t) \lambda(dt)\))

\[
\mathbb{E}\{F(Z) I_T(Z)\} = \mathbb{E}\{F(\bar{Z}) I_T(Z)\}, \quad \forall F \in \mathcal{S}_0, \bar{Z} \in \mathcal{C}_n[Z].
\]

Hence from (1.3) and applying Theorem 9.1 taking therein \(d = 1, A = \mathbb{R}^I, U = \kappa(B^{-1} \bar{Z}), \gamma = z = 0\) and \(g_1(s) = s^\alpha, \alpha \geq 0, g_2(t) = p_N(t), t \in \mathcal{T}\) we obtain \(I_T(Z) \geq 0\) a.s. and therefore

\[
\mathbb{E}\{F(Z)\} = 0 \implies \mathbb{E}\{F(\bar{Z})\} = 0, \quad \forall \bar{Z} \in \mathcal{C}_n[Z].
\]

If \(\mathcal{C}_n[Z]\) consist of rf’s with c\(\text{id}l\)\(\text{g}\) sample paths, for any measurable cone \(C \subset D(\mathcal{T}, \mathbb{R}^d)\) (see [19] for the definition), by (3.6)

\[
P\{Z \in C\} = 1 \implies P\{\bar{Z} \in C\} = 1, \quad \forall \bar{Z} \in \mathcal{C}_n[Z]
\]

and thus we retrieve [19, Prop 1].

Borrowing the idea of [37, Thm 4.2], the next application of Theorem 3.3 shows how to split \(\mathcal{C}_n[Z]\) into shift-invariant classes.

**Example 3.5** (Splitting of \(\mathcal{C}_n[Z]\)). Given a \(\mathcal{C}_n[Z]\) suppose that \(Z = Z_1 + Z_2\), where \(Z_i\)'s belong to \(\mathfrak{M}_\kappa\) are such that both \(Z_1, Z_2\) satisfy (1.3), (1.4) and (1.7). Hence we can define \(\mathbb{K}_n[Z_i], i = 1, 2\). It follows easily that under the following condition (which is borrowed from [6, Eq. (13.2.12)])

\[
P\left\{\sup_{t \in \mathcal{T}} \kappa(B^{-1} Z_1) = 0, \sup_{t \in \mathcal{T}} \kappa(B^{-1} Z_2) = 0\right\} = 0
\]

we obtain for all \(\bar{Z}_i \in \mathbb{K}_n[Z_i], i = 1, 2\) defined in the same probability space and satisfying (3.8) that

\[
\bar{Z}_1 + \bar{Z}_2 \in \mathcal{C}_n[Z].
\]

If \(Z\) has non-negative components and \(\kappa(f) = ||f(0)||^\alpha\) (maximum is taken component-wise below)

\[
X(t) = \max(X_1(t), X_2(t)), \quad t \in \mathcal{T}
\]

is a max-stable rf as in (1.1) with representers \(Z_k\), if \(X_k\) has the same de Haan representation (1.1) with representers \(Z_k, k = 1, 2\). In view of (1.2) and (3.8) it follows that \(X_1\) and \(X_2\) are independent. This splitting property is well-known, see [38, Lem 5], [6, Lem 13.2.8] and [37, Thm 4.2].
Example 3.6 (Cone splitting of $\mathcal{C}_n[Z]$). Let $C_i \subset D(T, \mathbb{R}^d), i \in I = \{1, \ldots, L\}$ with $L$ a positive integer or equal to infinity such that $H_i(cZ) = \mathcal{L}(cZ \in C_i)$ is a rv for all $c > 0, \tilde{Z} \in \mathcal{C}_n[Z]$ and
\[
P \{ H_i(c\tilde{Z}) = H_i(\tilde{Z}) \} = 1, \quad \forall c > 0, \forall i \in I.
\]
Suppose that $p_i = P\{Z \in C_i\} \in (0, 1), i \in I$ and define $Z_i$ to be equal in law with $Z$ conditioned on $Z \in C_i$. Write $N$ for a rv taking values in $I$ such that $P\{N = i\} = p_i, i = 1, \ldots, L$ and let
\[
\tilde{Z}_i \in \mathcal{C}_n[Z], \quad i \in I
\]
be defined in the same probability space being independent of $N$. If further $P\{\sum_{n \in I} I(Z \in C_n) = 1\} = 1$, then it follows from the choice of $C_i$’s and (3.5) that
\[
(3.10) \quad \tilde{Z}_N \in \mathcal{C}_n[Z].
\]

4. Characterisation of shift-invariance

In order to deal with the shift-invariance property, we shall assume that $T$ is an additive group, whenever this property is mentioned.

By definition, $\mathcal{C}_n[Z]$ is shift-invariant if $B^hZ$ belongs to $\mathcal{C}_n[Z]$ for all $h \in T$. Consequently, by (1.10) and (3.4) $\mathcal{C}_n[Z]$ is shift-invariant iff
\[
E \{ \kappa(B^{-h}Z)F(Z) \} = E \{ \kappa(\tilde{Z})F(B^h\tilde{Z}) \}, \quad \forall F \in \mathcal{F}_0, \forall h \in T, \forall \tilde{Z} \in \mathcal{C}_n[Z],
\]
which in view of (3.5) is also equivalent with
\[
E \{ F(Z) \} = E \{ F(B^h\tilde{Z}) \}, \quad \forall F \in \mathcal{F}_0, \forall h \in T, \forall \tilde{Z} \in \mathcal{C}_n[Z].
\]

In the sequel a shift-invariant $\mathcal{C}_n[Z]$ will be simply denoted by $\mathcal{C}_n[Z]$. Assume next without loss of generality that
\[
E \{ \kappa(Z) \} = 1.
\]

Hence, for a given $\mathcal{C}_n[Z]$
\[
(4.1) \quad 1 = E \{ \kappa(B^h\tilde{Z}) \}, \quad \forall h \in T, \forall \tilde{Z} \in \mathcal{C}_n[Z].
\]

Remark 4.1. Under the assumptions of Lemma 3.1, if further $\mathcal{C}_n[Z]$ is shift-invariant, then in view of the proof of Lemma 3.1 $B^hZ, h \in T$ and $Z_\ast$ are two representatives of the shift-invariant max-stable rf $X_\ast$. These representatives are directly related, see [1, p.1219, case $d = 1$] and [20, Thm 1, case $d \geq 1$]. Consequently, also $B^hZ$ and $Z$ are related, recall the definition of $x_\ast$ in (3.1).

Clearly, by (4.2) all $\mathcal{C}_n[Z]$’s are closed under random shifting, i.e.,
\[
(4.5) Z_N = B^NZ \in \mathcal{C}_n[Z]
\]
for all $T$-valued rv’s $N$ defined in $(\Omega, \mathcal{F}, P)$ being independent of $Z$. For special $Z$ included in Example 1.4 the claim in (4.5) is proved in [24, Thm 2].

Example 4.2 (Shift-invariance of Brown-Resnick $\mathcal{C}_n[Z]$). Consider the settings of Example 1.4 and suppose for simplicity that $\alpha = 1$. In view of [39, Thm 22] the pseudo-variogram matrix-valued function $\gamma$ is conditionally negative definite. Moreover, examples of such vector-valued Gaussian rf’s $V$ do exist. Suppose next that $\gamma(s,t), s, t \in T$ depends only on $t - s$ for all $s, t \in T$. For given positive integers $i, k \leq d$ by Lemma 9.14 the tilted law of $W_j(t) - W_k(h)$ with respect to $\gamma_{B^h}$ is the same as that of (recall $W_j(t) = V_j(t) - \text{Var}(V_j(t))/2$)
\[
W_j(t) - W_k(h) + \text{Cov}(V_j(t), V_k(h)) - \text{Cov}(V_k(h), V_k(h)) = V_j(t) - V_k(h) - \gamma_{jk}(t,h).
\]

By the assumption on $\gamma$ the latter has the same law as $V_j(t-h) - V_k(0) - \gamma_{jk}(t-h,0)$ and moreover (4.6) holds jointly for all $1 \leq j \leq d$. Consequently, since $\xi_i$’s are independent of $V$, for all $F \in \mathcal{F}_0$ using the 0-homogeneity of $F$ in the derivation of the first and the third equality below
\[
dE \{ ||Z(h)||_n^\alpha F(Z) \} = \sum_{k=1}^d E \{ e^{W_k(h)} F(Z/Z_k(h)) \}
\]
\[
\quad = \sum_{k=1}^d E \{ e^{W_k(0)} F(B^hZ/Z_k(0)) \}
\]
\[
\quad = E \left\{ \sum_{k=1}^d |Z_k(0)| F(B^hZ) \right\}
\]
\[
\quad = dE \{ ||Z(0)||_n^\alpha F(B^hZ) \}, \quad \forall h \in T.
\]

Hence $\mathcal{C}_n[Z]$ is shift-invariant, which for $d = 1$ is a direct consequence of [27]. In view of (4.6) and the derivation above the law of the local rf $\Theta$ depends only on $\gamma$ (even without imposing further restrictions on $\gamma(s,t)$), which is
shown for \( d = 1 \) already in \[27\].

Note that when \( \xi_i \)'s are deterministic, the shift-invariance of \( \mathcal{C}_\alpha[Z] \) is equivalent with the stationarity of the max-stable rf \( X \) defined in the Introduction. Under the settings of this example, \[10, \text{Lem 4.2} \] shows the stationarity of \( X \).

**Example 4.3** (\( C_\alpha[Z] \)'s defined from pdf's). Consider \( \xi_i \)'s as in Example 1.4 and let \( \varphi_i, 1 \leq i \leq d \) be pdf's of \( k \)-dimensional centered Gaussian random vectors and set \( \varphi = (\xi_1 \varphi_1, \ldots, \xi_d \varphi_d) \). Assuming that \( N \) is independent of \( \xi_i \)'s, setting

\[
Z_N(t) = \left[p_N(t)\right]^{1/\alpha} \varphi(t - N), \quad t \in \mathcal{T}
\]

and taking \( \kappa \) as in Example 1.4, it follows that \( Z_N \) satisfies (1.3), (1.5) and (1.7) and the corresponding \( \mathcal{C}_\alpha[Z] \) is shift-invariant.

**Definition 4.4.** Write \( \mathcal{P}_{\mathcal{T}_0} \) for the class of functions

\[
p_N: \mathcal{T}_0 \mapsto (0, \infty), \quad \int_{\mathcal{T}_0} p_N(t)\lambda(dt) < \infty,
\]

where \( \lambda \) is the counting measure on countable dense set \( \mathcal{T}_0 \) (defined in the Introduction). We define similarly \( \mathcal{P}_\mathcal{T} \) when \( \mathcal{T} = \mathbb{R}^d \) substituting \( \mathcal{T}_0 \) by \( \mathcal{T} \) and taking \( \lambda \) to be the Lebesques measure on \( \mathcal{T} \) assuming further that \( p_N \) is almost everywhere positive and locally bounded.

We state next equivalent characterisations of shift-invariant \( \mathcal{C}_\alpha[Z] \)'s. Write \( Z_{\mathcal{T}_0}(t) = Z(t), t \in \mathcal{T}_0 \) for the restriction of \( Z \) on \( \mathcal{T}_0 \) and denote the corresponding \( \alpha \)-homogeneous class of rf's by \( \mathcal{C}_\alpha[Z_{\mathcal{T}_0}] \). In the following \( N \) is independent of any random element, being a \( \mathcal{T}_0 \)-valued or \( \mathcal{T} \)-valued rv with density \( p_N \in \mathcal{P}_{\mathcal{T}_0} \) or \( p_N \in \mathcal{P}_\mathcal{T} \), respectively.

In order to define the integral

\[
I_\mathcal{T}(Z) = \int_{\mathcal{T}} \kappa(B^{-\tau}Z)p_N(r)\lambda(dr)
\]

when \( \mathcal{T} = \mathbb{R}^d \) we need the joint measurability of \( \kappa(B^{-\tau}Z), t \in \mathcal{T} \), which is a direct consequence of the stochastic continuity assumption. The latter is also important for the positivity of \( I_\mathcal{T}(Z) \) when applying Theorem 9.1 below.

**Lemma 4.5.** \( \mathcal{C}_\alpha[Z] \) is shift-invariant iff one of the following holds:

(i) \( \mathcal{C}_\alpha[Z_{\mathcal{T}_0}] \) is shift-invariant;
(ii) (4.1) is satisfied for all \( h \in \mathcal{T}_0 \);
(iii) for all rv's \( N \) with \( p_N \in \mathcal{P}_{\mathcal{T}_0} \), the following rf's

\[
Z_N(t) = \left(\frac{\kappa(Z)}{F_{\mathcal{T}_0}(B^{\mathcal{T}'}Z)}\right)^{1/\alpha} B^{\mathcal{T}'}Z(t)
\]

satisfy (3.4);
(iv) Item (iii) holds with \( \mathcal{T}_0 \) substituted by \( \mathcal{T} \) for all \( p_N \in \mathcal{P}_\mathcal{T} \).

**Definition 4.6.** \( F \in \mathcal{S}_0 \) (and similarly for other maps) is shift-invariant with respect to \( \mathcal{L} \subseteq \mathcal{T} \) if

\[
F(B^{\mathcal{H}}f) = F(f), \quad \forall f \in D(\mathcal{T}, \mathbb{R}^d), \forall \mathcal{H} \in \mathcal{L}.
\]

Below shift-invariant maps are considered with respect to \( \mathcal{L} = \mathcal{T}_0 \). Therefore we shall simply write shift-invariant maps without specifying \( \mathcal{L} \).

**Example 4.7** (Shift-invariant maps & new shift-invariant classes of rf's). Consider \( C_\alpha[Z] \) and a shift-invariant non-negative map \( \Gamma \in \mathcal{S}_0 \). If \( E(\kappa(Z)\Gamma(Z)) \in (0, \infty) \) by (4.2) \( \mathcal{K}_\alpha[Z] \) is also shift-invariant with \( \tilde{Z} \) equal \( Z\Gamma(Z) \) conditioned on \( \Gamma(Z) > 0 \).

**Example 4.8** (Cone splitting of \( C_\alpha[Z] \)). Reconsider Example 3.6 assuming additionally that \( H_i \)'s therein are shift-invariant. In view of (4.1) and Lemma 4.5, Item (i) we have that \( \mathcal{C}_\alpha[Z_{\mathcal{T}_0}] \)'s are shift-invariant. Moreover, taking independent \( \tilde{Z}_i \)'s from \( \mathcal{C}_\alpha[Z_{\mathcal{T}_0}] \)'s defined in the same probability space as \( N \) being also independent of \( N \), then we have

\[
(4.7) \quad \tilde{Z}_N \in \mathcal{C}_\alpha[Z].
\]

5. Tail and spectral tail RF's

Recall that given a \( C_\alpha[Z] \) we define a local rf \( \Theta \) as the rf \( \bar{Z}/\kappa(\bar{Z})^{1/\alpha} \) under \( \tilde{\mathbb{P}} \) given by (1.11), where \( \bar{Z} \in \mathcal{C}_\alpha[Z] \). Note that our assumption \( (\Omega, \mathcal{F}, \mathbb{P}) \) is complete yields that also \( (\Omega, \mathcal{F}, \tilde{\mathbb{P}}) \) is complete. For notational simplicity we write \( \tilde{\mathbb{P}} \) instead of \( \mathbb{P} \).

In view of (3.5) and (4.4) for local rf's \( \Theta \) and \( \bar{\Theta} \)

\[
(5.1) \quad E\{\kappa(B^{-\mathcal{H}}\Theta)\Gamma(\bar{\Theta})\} = E\{\kappa(B^{-\mathcal{H}}\Theta)\Gamma(\Theta)\} = E\{\tilde{\kappa}(B^{-\mathcal{H}}\Theta) \neq 0 \} \Gamma(B^{\mathcal{H}}\Theta) \}, \quad \forall \Gamma \in \mathcal{S}_0, \forall \mathcal{H} \in \mathcal{T}.
\]

Consequently, the fidr's of \( \Theta \) and \( \bar{\Theta} \) agree. Hence in view of Lemma 9.3 if \( F \in \mathcal{H}_\beta, \beta \geq 0 \), the law of \( F(\Theta) \) is the same as that of \( F(\bar{\Theta}) \).
For the special case $\kappa(f) = ||f(0)||^a$, $f \in D(\mathbb{T}, \mathbb{R}^d)$ the identity (5.1) is equivalent with the time-change formula discovered in [32]. Therein $\Theta$ is the so-called spectral tail rf of a regularly varying time series. The time-change formula for a radial function $\kappa$ appears in [40, Thm 6.1].

**Remark 5.1.** If $P\{\kappa(Z) > 0\} = 1$, then since we assume (4.3), from (5.1) we conclude that $\Theta \in \mathcal{C}_m[Z]$. For this case a similar formula to (5.1) referred to as the non-singular group action property is shown in [41, Thm S.10].

Motivated by the spectral tail rf’s appearing in the context of time series, we define a corresponding rf without any reference to a particular $\mathcal{C}_m[Z]$.

**Definition 5.2.** $\Theta \in \mathcal{M}_\kappa$ is called a spectral tail rf, if

(i) $P\{\kappa(\Theta) = 1\} = 1$;

(ii) for all $\Gamma \in \mathcal{S}_0, h \in \mathbb{T}$ the identity (5.1) holds;

(iii) for all $p_N \in \mathcal{P}_{\mathbb{T}_0}$ (recall $I_{\mathbb{T}_0}(\Theta) = \int_{\mathbb{T}_0} \kappa(h^{-1}\Theta)p_N(r)\lambda(dr)$)

\[
E\left\{\sup_{s \in [-c,0]} |\kappa(h^{-1}\Theta)\|B^{N-s}\Theta}\| / I_{\mathbb{T}_0}(B^N \Theta)\right\} < \infty, \quad \forall c > 0.
\]

**Remark 5.3.**

(i) If $\mathbb{T} = \mathbb{Z}^d$, then $\mathbb{T}_0 = \mathbb{T}$ and hence (5.2) is satisfied since $[-c, e]^d \cap \mathbb{T}_0$ has only finite number of elements and for all $s \in \mathbb{T}$ by (5.1) and $P\{\kappa(\Theta) = 1\} = 1$

\[
E\left\{\kappa(h^{-1}\Theta)\|B^{N-s}\Theta\| / I_{\mathbb{T}_0}(B^N \Theta)\right\} = \int_{\mathbb{T}_0} E\left\{\kappa(h^{-1}\Theta)\|B^{N-s}\Theta\| / I_{\mathbb{T}_0}(B^N \Theta)\right\} p_N(t)\lambda(dt) = \frac{\int_{\mathbb{T}_0} \kappa(h^{-1}\Theta)p_N(t)\lambda(dt)}{I_{\mathbb{T}_0}(B^N \Theta)} = 1.
\]

Note that $I_{\mathbb{T}_0}(\Theta) \in [p_N(0), \infty)$ a.s. follows by the definition of $p_N$ and

\[E\{\kappa(B^h \Theta)\} = P\{\kappa(B^h \Theta) \neq 0\} \leq 1, \quad h \in \mathbb{T}.
\]

Moreover, $I_{\mathbb{T}_0}(\Theta)$ and in general $F(\Theta), \forall F \in \mathcal{S}_0, \beta > 0$ is well-defined as shown in the proof of Theorem 3.3. An equivalent condition to (5.2) is given in (5.7) below, see for a more restrictive setup [9, 16].

**Lemma 5.4.** Let $\Theta$ be the local rf of $\mathcal{C}_m[Z]$. If $\Theta$ is a spectral tail rf and for all rv’s $N$ independent of $\Theta$ such that

\[
Z_N(t) = \frac{1}{(I_{\mathbb{T}_0}(B^N \Theta))^{1/\alpha}} B^N \Theta(t) \in \mathcal{C}_m[Z],
\]

$p_N \in \mathcal{P}_{\mathbb{T}_0}, \forall N \in \mathcal{P}_\mathbb{T}$ (with $\mathbb{T}_0$ substituted by $\mathbb{T}$), then $\mathcal{C}_m[Z]$ is shift-invariant. Conversely, if $\mathcal{C}_m[Z]$ is shift-invariant, then its local rf $\Theta$ is a spectral tail rf and moreover (5.2) holds also for $\mathbb{T}_0$ substituted by $\mathbb{T}$ and all $p_N \in \mathcal{P}_\mathbb{T}$.

**Remark 5.5.** Given a spectral tail rf $\Theta$, the class of rf’s $\mathcal{C}_m(\Theta)$ of elements from $\mathcal{M}_\kappa$ that includes all $Z_N$ determined by (5.3) for both $p_N \in \mathcal{P}_{\mathbb{T}_0}$ or $p_N \in \mathcal{P}_\mathbb{T}$ and all $Z \in \mathcal{M}_\kappa$ such that (1.5) holds with $Z = Z_N$ is $\alpha$-homogeneous and shift-invariant.

Hereafter the $\alpha$-Pareto rv $R$ with survival function $t^{-\alpha}, t \geq 1$ is assumed to be independent of any other random element. For $Y = R\Theta$ using [16, Lem 3.16] or as in [5, 8], we obtain

\[E\{\Gamma(xB^hY)\|\kappa(xB^hY) > 1\} = x^\alpha E\{\Gamma(Y)\|\kappa(B^{-h}Y/x) > 1\}, \quad \forall \Gamma \in \mathcal{H}, \forall h \in \mathbb{T}, \forall x > 0.
\]

**Definition 5.6.** $Y \in \mathcal{M}_\kappa$ is called a tail rf, if

(i) $P\{\kappa(Y) > 1\} = 1$;

(ii) (5.4) holds for all $x > 0$;

(iii) Eq. (5.2) holds for all $p_N \in \mathcal{P}_{\mathbb{T}_0}$ with $\Theta$ substituted by $Y$.

Tail and spectral tail rf’s are crucial for the study of stationary regularly varying time series, see [5–7, 32, 42, 43].

**Definition 5.7.** $\mathcal{S}_\kappa$ consists of all maps $F$ defined by

\[
F = F_1F_2, \quad F_1 \in \mathcal{S}_\beta, \quad F_2 \in \mathcal{H}, \quad \forall \beta > 0.
\]

**Remark 5.8.**

(i) When $\kappa(f) = ||f(0)||^a$, then (5.4) is shown in [5, 8] assuming $\mathbb{T} = \mathbb{Z}^d$ and in [9, 16] considering the câdàg case $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{R}^d$, respectively.

(ii) If $\Theta$ is the local rf of some shift-invariant $\mathcal{C}_m[Z]$, then $Y = R\Theta$ is a tail rf. Moreover, as in the proof of [5, Appendix A] (5.1) implies that (5.4) holds for all $\Gamma \in \mathcal{S}_\kappa$.

(iii) $\Gamma$ in (5.4) can be of the form $\infty \cdot 0$ or $0/0$ and both interpreted as 0.

The next result is motivated by [43, Thm 3.2] and [9] which shows (5.7) for $l = 1, \tau = 0$ in the câdàg case. The values of $\tau$ considered below are stated in the next condition:

**Condition 5.1.** $\tau \in \mathbb{R}$ is such that

\[
\sup_{t \in [-c,0] \cap \mathbb{T}} E\{\kappa(B^{-t}\Theta)^l\} < \infty, \quad \forall c > 0.
\]
Theorem 5.9. If \( Y \) is a tail rf, then \( R = \kappa(Y)^{1/\alpha} \) is an \( \alpha \)-Pareto rv being independent of \( \Theta = Y/R \). Further \( \Theta \) is a spectral tail rf satisfying

\[
\int_0^\infty \mathbb{E} \left[ \Gamma(z\Theta) \mathbb{I}(\kappa(zB^{-h}\Theta) > 1) \right] \alpha_z^{-\alpha - 1} dz = \mathbb{E} \{ \Gamma(B^h Y) \mathbb{I}(\kappa(B^h Y) \neq 0) \}, \quad \forall \Gamma \in \mathcal{S}, \forall h \in \mathcal{T}.
\]

Moreover, for \( \tau \) satisfying Condition 5.1 there exists a \( C_\kappa[Z] \) with local rf \( \Theta \) such that

\[
\int_{t \in [-c,0]\cap \mathcal{T}} \mathbb{E} \left[ \frac{1}{\int_{s \in [-c,0]\cap \mathcal{T}} \kappa(B^{s-t} \Theta)^+ \mathbb{I}(\kappa(B^{s-t} Y) > 1) \lambda(ds)} \right] \lambda(dt) < \infty, \quad \forall c > 0.
\]

Conversely, if \( \Theta \in \mathcal{W}_\kappa \) satisfies Definition 5.2, Item (i) and both (5.6), (5.7) hold with \( Y = R\Theta \), then \( Y \) is a tail rf.

6. New \( \alpha \)-Homogeneous Classes & Universal Maps

In this section \( \mathcal{L} \subset \mathcal{T} \) is a discrete subgroup of the additive group \( \mathcal{T} \) (or simply a lattice). Given a \( C_\kappa[Z] \), in the light of our results above, the following approaches lead to new classes of shift-invariant \( \alpha \)-homogeneous rf’s:

(i) \( \mathcal{C}_\kappa[Z] \) is shift-invariant, where \( Z(t) = Z(t), t \in \mathcal{L} \);

(ii) Splitting with respect to \( C_i \)'s as in Example 4.8 leads to new shift-invariant \( \mathcal{C}_\kappa[Z_i] \)'s such that further (4.7) holds;

(iii) Given shift-invariant \( C_\kappa[Z_i], i = 1, \ldots, L \) with \( L \) a positive integer or equal to infinity, then taking \( \tilde{Z}_i \in C_\kappa[Z_i], i = 1, \ldots, L \) in the same probability space independent of a rv \( N \) defined in the same probability space, then \( C_\kappa[Z_N] \) is also shift-invariant;

(iv) Utilising \( N \) and \( Z_N \) as in Lemma 4.5 and then defining the corresponding \( \mathcal{C}_\kappa[Z_N] \) by keeping only \( \tilde{Z}_i \)'s that satisfy (1.3).

The last approach to construct shift-invariant \( C_\kappa[Z] \)'s relies on the fact that by choosing a given \( p_N \), it is possible to show that the corresponding integral functional of \( Z \) is a.s. positive and finite.

It is possible to utilise functionals that do not depend on the choice of some given \( p_N \). A particular class of such functionals, referred below as universal maps, play a crucial role in the construction of random shift-representations which are studies in the subsequent contribution [44].

Hereafter all event inclusions/ equalities are modulo null sets with respect to the corresponding probability measure. Let \( V \subset \mathcal{T} \) be such that \( \lambda(V) > 0 \). For given \( f : \mathcal{T} \to \mathbb{R} \) set

\[
S_V(f) = \int_V \kappa(B^{-t} f) \lambda(dt), \quad B_{V,\tau}(f) = \int_V \kappa(B^{-t} f)^\tau \mathbb{I}(\kappa(B^{-t} f) > 1) \lambda(dt).
\]

When the above are not properly defined we set them equal to 0. If \( \mathcal{V} = \mathcal{T} \) we write \( S \) instead of \( S_V \).

In the rest of this section \( \mathcal{L} \) has countably infinite number of elements. A shift-involution will be denoted by \( \mathcal{J}_1 \), whereas \( \mathcal{J}_2 \) and \( \mathcal{J}_3 \) denote a positive shift-involution and an anchoring map, respectively (with respect to \( \mathcal{L} \)).

Given a tail rf \( Y \) a weak restriction introduced in [14] that we shall impose next is

\[
\mathbb{P} \{ S_{\mathcal{L}}(Z) < \infty \} = \mathbb{P} \{ S_{\mathcal{L}}(Z) < \infty, \mathcal{J}_1(Z) \in \mathcal{L} \}
\]

and

\[
\mathbb{P} \{ S_{\mathcal{L}}(Y) < \infty \} = \mathbb{P} \{ S_{\mathcal{L}}(Y) < \infty, \mathcal{J}_k(Y) \in \mathcal{L} \}, \quad k = 2
\]

and further \( \mathcal{J}_3 \) satisfies (6.2) with \( S_{\mathcal{L}}(Y) \) substituted by \( B_{\mathcal{L},0}(Y) \). These two conditions are satisfied in particular if \( \mathcal{J}_1 = \mathcal{I}_{\mathcal{L}, \text{argsup}} i, i = 1, 2, 3 \) and \( \mathcal{J}_3 = \mathcal{I}_{\mathcal{L}, \text{e}} \). Below we set \( ||t|| = \sum_{i=1}^{d} |t_i|, \quad t = (t_1, \ldots, t_d) \in \mathbb{R}^d \).

Definition 6.1. Denote by \( \mathcal{U} \) the the class of maps \( F : D(\mathcal{T}, \mathbb{R}^d) \to (\mathbb{R}^k)^*, k \in \mathbb{N} \) such that for all \( F_1, F_2 \in \mathcal{U} \) there is a measurable subset \( \mathcal{A} \) of \( (\mathbb{R}^k)^* \) so that for all tail rf’s \( Y \) we have \( \{ F_1(Y) \in \mathcal{A} \} = \{ F_2(Y) \in \mathcal{A} \} \). We call \( \mathcal{U} \) the class of universal maps.

The next lemma shows in particular that \( \mathcal{U} \) contains both \( S_{\mathcal{L}}(\cdot) \) and \( B_{\mathcal{L},\tau}(\cdot) \).

Lemma 6.2. Let \( \mathcal{L} \) be a discrete subgroup of the additive group \( \mathcal{T} \) and suppose that \( |\mathcal{L}| = \infty \). If \( \mathcal{J}_1 \) satisfies (6.1), \( \mathcal{J}_2 \) satisfies (6.2) and \( \mathcal{J}_3 \) satisfies (6.2) with \( S_{\mathcal{L}}(Y) \) being substituted by \( B_{\mathcal{L},0}(Y) \), then

\[
\{ S_{\mathcal{L}}(Z) < \infty \} = \{ \mathcal{J}_1(Z) \in \mathcal{L} \} = \left\{ \lim_{\|t\| \to \infty, t \in \mathcal{L}} \kappa(B^{-t} Z) = 0 \right\},
\]

\[
\{ S_{\mathcal{L}}(Y) < \infty \} = \{ B_{\mathcal{L},\tau}(Y) < \infty \} = \left\{ \lim_{\|t\| \to \infty, t \in \mathcal{L}} \kappa(B^{-t} Y) = 0 \right\},
\]

with \( I \) equal to \( \mathcal{J}_2 \) or \( \mathcal{J}_3 \) and all \( \tau \in \mathbb{R} \).

Since \( \mathbb{P}(\kappa(\Theta) = 1) = 1 \) and \( 0 \in \mathcal{L} \), then clearly \( \mathbb{P}(S_{\mathcal{L}}(\Theta) > 0) = \mathbb{P}(B_{\mathcal{L},\tau}(Y) > 0) = 1 \). In view of Lemma 9.8 it follows further that \( \mathbb{P}(S_{\mathcal{L}}(Z) > 0) = 1 \). When \( \mathcal{L} \) is substituted by \( \mathcal{T} \), these conclusions are not trivial.

We present below the main result of this section.
Theorem 6.3. Under the assumptions of Lemma 6.2, if $\tau$ satisfies Condition 5.1, then we have
\begin{equation}
\mathbb{P}(S(Z) > 0) = \mathbb{P}(S(\Theta) > 0) = \mathbb{P}(B_{\tau,r}(Y) > 0) = 1.
\end{equation}
Moreover, if $\mathcal{L}$ is a full rank lattice on $\mathcal{T}$, then we have
\begin{equation}
\{S(\Theta) < \infty\} = \{\lim_{\|t\| \to \infty} \kappa(B^{-t}(\Theta)) = 0\} = \left\{ \int_{\mathcal{T}} \sup_{t \in K \cap T} \kappa(B^{s-t}(\Theta)) \lambda(ds) < \infty \right\} = \{B_{\tau,r}(Y) < \infty\}
\end{equation}
and for all $\bar{Z} \in \mathcal{C}_n[Z]$ and all compact sets $K \subset \mathbb{R}^l$
\begin{equation}
\{S(\bar{Z}) < \infty\} = \left\{ \lim_{\|t\| \to \infty} \kappa(B^{-t}\bar{Z}) = 0\right\} = \left\{ \int_{\mathcal{T}} \sup_{t \in K \cap T} \kappa(B^{s-t}\bar{Z}) \lambda(ds) < \infty\right\}
\end{equation}
hold with $I_1 = I_{\mathcal{L}, \text{argsup}}$ and $I_2 = I_{\mathcal{L}, \text{f.c.}}$.

Remark 6.4. (i) Theorem 6.3 extends several claims derived for particular choices of $\mathcal{T}, \mathcal{L}, \tau = 0$ and $\kappa(f) = ||f(0)||^\alpha$ in [5-9, 14, 37, 39, 42, 45]. In case of max-stable or symmetric $\alpha$-stable rf's, the results in the literature that we extended above concern also several criteria for pure dissipativity/conservativity. Similar characterisations also appear in the analysis of regularly varying rf's.

(ii) The importance of lattices in the study of dissipativity of symmetric $\alpha$-stable rf's is known from [46].

(iii) The class $\mathcal{U}$ is non-empty, since all functionals in Lemma 6.2 and Theorem 6.3 belong to it (under the conditions therein).

7. DISCUSSIONS

We consider in this section some implications for max-stable rf's and calculation of maximal indices.

7.1. Max-stable rf's. A given $\mathcal{C}_n[Z]$ with $Z$ having non-negative components and local rf $\Theta$ is closely related to max-stable rf's as shown in [10]. For such $Z$, define a max-stable rf $X(t)$, $t \in \mathcal{T}$ as in (1.1). The tractability of $X$ is related to expression (1.2), which can be rewritten as
\begin{equation}
-\ln \mathbb{P}(X(t_i) \leq x_i, 1 \leq i \leq n) = \sum_{1 \leq i \leq n} \mathbb{E} \left\{ ||Z(t_i)/x_i||^\alpha_\infty \mathbb{I}(\inf_{1 \leq j \leq n} ||Z(t_j)/x_j||^\alpha_\infty = l) \right\}
\end{equation}
for all $t_i \in \mathcal{T}, x_i \in (0, \infty)^d$, $i = 1, \ldots, n$. Moreover, if $\kappa$ is such that a.s.
\begin{equation}
\{\kappa(Z) = 0\} = \{||Z(0)|| = 0\}
\end{equation}
and $\mathcal{C}_n[Z]$ is shift-invariant, then another expression for (1.2) is shown in [2]. Namely, as therein for general $\Theta$ defined with respect to this $\kappa$, we obtain by (4.2)
\begin{equation}
-\ln \mathbb{P}(X(t_i) \leq x_i, 1 \leq i \leq n) = \sum_{1 \leq i \leq n} \mathbb{E} \left\{ ||Z(0)/x_i||^\alpha_\infty \mathbb{I}(\inf_{1 \leq j \leq n} ||Z(t_j - t_i)/x_j||^\alpha_\infty = l) \right\}
\end{equation}
and
\begin{equation}
\mathbb{E} \left\{ ||\Theta(0)/x_i||^\alpha_\infty \mathbb{I}(\inf_{1 \leq j \leq n} ||\Theta(t_j - t_i)/x_j||^\alpha_\infty = l) \right\}.
\end{equation}

Note that $X$ is regularly varying, when $X$ has càdlàg sample paths and hence $\Theta$ can be derived also as a weak limit (under conditions on $\kappa$), see [9, 16]. Since in view of [2, 10] $X$ is stationary iff (4.1) holds for all $F \in \mathcal{H}_0$, see also [5, 7] the above derivation implies the following result:

Lemma 7.1. If $\mathcal{C}_n[Z]$ is such that $Z$ has non-negative components, then $\mathcal{C}_n[Z]$ is shift-invariant iff $X$ is stationary. In further (7.1) holds, then $\mathcal{C}_n[Z]$ is shift-invariant iff the finite dimensional distributions (fidi's) of $X$ are given by (7.2).

In view of [5, 38] $\mathbb{P}(S(Z) < \infty) = 1$ is related to the existence of so-called Rosiński (or mixed moving maxima) representation of max-stable rf's, see also [9, 47]. Such representations are also referred to as mixed moving maxima (M3) representation [5, 27] being closely related to dissipative/conservative decompositions, [22, 37, 38, 46, 48, 49]. The findings of Theorem 6.3 give several other equivalent conditions for such representations, which are considered in detail in [44].
Let \( L \) be a full rank lattice on \( T \) with \(|L| = \infty \) or \( L = \mathbb{R}^l \). Consider further a given \( C_\alpha[Z] \) and define the maximal index

\[
\vartheta[Z] = \lim_{n \to \infty} n^{-1} E \max_{t \in [0,n] \cap L} \kappa(B^{-t}Z).
\]

We drop the superscript \( L \) when it is equal to \( T \). It is clear that by (4.2)

\[
\vartheta[Z] = \vartheta[Z], \quad \forall Z \in C_\alpha[Z].
\]

If the stationary max-stable rf \( X \) has representative \( \kappa(B^{-t}Z)^{1/\alpha}, t \in T \) and \( \alpha \)-Fréchet marginals, then \( \vartheta[Z] \) is closely related to the distribution of supremum of \( X \), namely utilising (1.2) and (1.7)

\[
-\ln P \left( \sup_{t \in [0,n] \cap L} X(t) \leq rn^{1/\alpha} \right) = \frac{1}{r\alpha n^d} E \max_{t \in [0,n] \cap L} \kappa(B^{-t}Z) \to \frac{1}{r\alpha} \vartheta[Z]
\]

as \( n \to \infty \) for all \( r > 0 \). Hence in this case \( \vartheta[Z] \) is the extremal index of \( X \) with respect to \( L \).

Calculation of the extremal index of a stationary regularly varying rf continues to be a topic of great research interest. Its existence in general case cannot be guaranteed and therefore \( \vartheta[Z] \) is commonly referred to as the candidate extremal index, see [5, 6]. Given its relation to Pickands constants and related applications in statistics, the derivation of different representations of the maximal indices is of particular interest, see [5, 7, 8, 14, 31, 32, 50–53].

**Proposition 7.2.** Let \( \tau \in \mathbb{R} \) be such that Condition 5.1 holds. If \( L \) is a full rank lattice on \( T \) and \( C_\alpha[Z] \) has local \( rf \) \( \Theta \), then

\[
P(\mathcal{B}_{L,\tau}(Y) = \infty) = 1 \iff \vartheta[Z] = 0 \iff \vartheta[Z] = 0 \iff P(\mathcal{B}_{T,\tau}(Y) = \infty) = 1.
\]

**Remark 7.3.**

(i) Note that \( \vartheta[Z] = 0 \iff P(\mathcal{S}(Y) = \infty) = 1 \) follows from [20, 49, 54, 55] or directly from (6.7).

(ii) In view of Theorem 6.3, several other equivalent conditions can be added to (7.5).

(iii) If \( Z \) has a.s. sample paths in \( D(T, \mathbb{R}^d) \) and \( \kappa(f) = \|f(0)\|^{\alpha} \), then the last two equivalences in (7.5) for \( \tau = 0, \alpha = 1 \) follow from [9, Lem 2.5, Thm 2.9] and for \( \alpha = \infty \) by [14, Thm 3.8], see also [6] for the case \( L = Z = T \).

(iv) Some novel representations for \( \vartheta[Z] \) can be derived by combining (8.11), (8.14) and (8.18).

8. Proofs

**Proof of Lemma 3.1** Let \( p_N(t) > 0, \forall t \in T_0 \) be the pdf of some \( T_0 \)-valued rv and set

\[
I_{T_0}(f) = \int_{T_0} \kappa(B^{-t}f)p_N(t) \lambda(dt).
\]

By (1.3) and the fact that \( p_N \) is a pmf we have a.s.

\[
I_{T_0}(\tilde{Z}) \in (0, \infty), \quad \forall \tilde{Z} \in C_\alpha[Z].
\]

Applying (1.5) for all \( G \in H_\alpha \) yields

\[
E\{G(\tilde{Z})\} = E\left\{ G(\tilde{Z}) \frac{I_{T_0}(\tilde{Z})}{I_{T_0}(\tilde{Z})} \right\} = \sum_{t \in T_0} p_N(t) E\left\{ \kappa(B^{-t}Z) \frac{G(Z)}{I_{T_0}(Z)} \right\} = \sum_{t \in T_0} p_N(t) E\{ \kappa(B^{-t}Z) F(Z) \} = E\left\{ G(\tilde{Z}) \right\}
\]

where we used that \( F \) is \( 0 \)-homogeneous for the derivation of last line above. By assumption (3.2) we obtain thus (3.3). Suppose next that (3.3) is satisfied and set

\[
I_{T_0}^{\infty}(f) = \int_{T_0} \kappa_{\infty}(B^{-t}f)p_N(t) \lambda(dt).
\]

Again by (1.3) a.s. (8.2) holds for \( I_{T_0}^{\infty}(\tilde{Z}) \). As above for all \( F \in H_\alpha \) we obtain (set \( \Gamma(Z)(t) := Z^+(t) - Z^-(t) = Z(t) \) and recall \( x_+ = (x^+, x^-), x \in \mathbb{R}^d \))

\[
E\{\kappa(B^{-h}Z) F(Z)\} = E\left\{ \kappa(B^{-h}Z) \frac{I_{T_0}^{\infty}(Z)}{I_{T_0}^{\infty}(Z)} F(Z) \right\} = \sum_{t \in T_0} p_N(t) E\left\{ \kappa_{\infty}(B^{-t}Z) \frac{1}{I_{T_0}^{\infty}(Z)} F(\Gamma(Z)) \right\} = \sum_{t \in T_0} p_N(t) E\{ \kappa_{\infty}(B^{-t}Z) G(Z) \}
\]
where we used that $G \in \mathcal{H}_0$. Now we justify the second last equality, i.e., we need to show that

$$(8.3) \quad \mathbb{E}\{\kappa_{\infty}(B^{-t}\bar{Z})G(\bar{Z})\} = \mathbb{E}\{\kappa_{\infty}(B^{-t}Z)G(Z)\}, \quad \forall t \in \mathcal{T}, \forall G \in \mathcal{H}_0.$$  

It follows from (1.2) that the max-stable rf $X_\star$ with reprenter $Z_\star$ has also reprenter $\bar{Z}_\star$. Since the law of $X_\star$ is determined by the tail measure $\nu_{\bar{Z}_\star}$ (recall definition (1.6)), which is equal with $\nu_{\bar{Z}}$, then [16, Prop 3.6, Rem 3.11.(ii)] implies that (1.5) is satisfied with $\kappa(B^{-h}Z), \kappa(B^{-h}\bar{Z})$ substituted by $\kappa_{\infty}(B^{-h}Z_\star), \kappa_{\infty}(B^{-h}\bar{Z}_\star)$, respectively and thus (8.3) follows establishing the proof. \hfill \Box

**Proof of Theorem 3.3** (1.5) $\Rightarrow$ (3.4): The claim is clearly valid when $\mathcal{T} = \mathbb{R}^l$. Consider therefore the case $\mathcal{T} = \mathbb{R}^l$. Let $g \in \mathcal{G}$, i.e., $g : \mathcal{T} \mapsto [0, \infty)$ is locally bounded and $\lambda$-measurable and write

$$(8.4) \quad R_{c,n}(f, g) = \left(\frac{2c}{n}\right)^l \sum_{t \in (2/\sqrt{n})^l \cap [-c, c]^l} \kappa(B^{-t}f)^{\xi}g(t), \quad f \in D(\mathcal{T}, \mathbb{R}^d),$$

with $\xi \geq 0$. Set next $\mathcal{J}(f, g) = \int_{\mathcal{T}} \kappa(B^{-t}f)^{\xi}g(t)\lambda(dt)$ and define

$$F(f) = \frac{\mathcal{J}(f, g)}{\mathcal{J}(f, g)} \quad \Gamma_{\xi} \in \mathcal{H}_{\xi}, f \in D(\mathcal{T}, \mathbb{R}^d).$$

We consider only this case for $F$, the other cases follow with the same arguments. Note in passing that $F(\bar{Z}), \bar{Z} \in \mathcal{C}_\xi[Z]$ is a well-defined rv (recall $\kappa(B^tZ), t \in \mathcal{T}$ is jointly measurable). By (1.3), applying Theorem 9.1 taking $d = 1, A = \mathbb{R}^l, U(t) = \kappa(B^{-t}\bar{Z}), \gamma = z = 0$ and $g_1(s) = s^\alpha$, $s \geq 0, g_2(t) = 1, t \in \mathcal{T}$ therein, we obtain

$$(8.5) \quad \mathbb{P}\{\mathcal{J}(Z, g) > 0\} = \mathbb{P}\{\mathcal{J}(\bar{Z}, g) > 0\} = 1, \quad \forall \bar{Z} \in \mathcal{C}_\xi[Z].$$

In fact, even when $\mathbb{P}\{\mathcal{J}(Z, g) = 0\} > 0$, then by Lemma 9.4 $\mathbb{P}\{\mathcal{J}(\bar{Z}, g) = 0\} > 0$ and thus (3.4) obviously holds since both sides are equal to infinity.

We can assume without loss of generality that $F$ is bounded since $F(f)\|\mathcal{J}(f) \leq n), n \in \mathbb{N}$ is also 0-homogeneous and then we can apply the dominated convergence theorem for unbounded case. Hence by (8.5) the rv’s $F(\bar{Z}), F(\bar{Z})$ are strictly positive and finite. Clearly, the following map

$$\mathcal{U}_{k,n}(f) = \Gamma_{\xi}(f)/R_{c,n}(f, g), \quad f \in D(\mathcal{T}, \mathbb{R}^d), k, n > 0$$

is a $\mathcal{G}/\mathcal{B}(\mathbb{R})$-measurable belonging to $\mathcal{H}_0$ for all large integers $k, n$ (we set $\mathcal{U}_{k,n}(f) = 0$ if $f$ is not integrable and interpret $0/0$ as 0). For all $h \in \mathcal{T}, r > 0$ and all positive integers $k, n$ large enough by (4.2) for all $\bar{Z} \in \mathcal{C}_\xi[Z]

$$(8.6) \quad \mathbb{E}\{\kappa(B^{-h}Z)\mathcal{U}_{k,n}(Z)\|\mathcal{U}_{k,n}(Z) \leq r\} = \mathbb{E}\{\kappa(B^{-h}\bar{Z})\mathcal{U}_{k,n}(\bar{Z})\|\mathcal{U}_{k,n}(\bar{Z}) \leq r\} < \infty.$$  

In view of (1.7), Lemma 9.3 yields for some sequence $c_k, n_k, k \in \mathbb{N}$

$$R_{c_k,n_k}(Z, g) \leq \mathcal{J}(Z, g), \quad \forall \bar{Z} \in \mathcal{C}_\xi[Z]$$

as $k \to \infty$. Hence for almost all $r > 0, \bar{Z} \in \mathcal{C}_\xi[Z]$ we have the convergence in distribution

$$\mathcal{U}_{k,n_k}(\bar{Z})\|\mathcal{U}_{k,n_k}(\bar{Z}) \leq r \leq \mathcal{U}(\bar{Z})\|\mathcal{U}(\bar{Z}) \leq r, \quad k \to \infty,$$

where $\mathcal{U}(\bar{Z}) = \Gamma_{\xi}(\bar{Z})/\mathcal{J}(\bar{Z}, g)$. Consequently, by (8.6)

$$(8.7) \quad \mathbb{E}\{\kappa(B^{-h}Z)\mathcal{U}(Z)\|\mathcal{U}(Z) \leq r\} = \mathbb{E}\{\kappa(B^{-h}\bar{Z})\mathcal{U}(\bar{Z})\|\mathcal{U}(\bar{Z}) \leq r\} < \infty.$$  

In view of (8.5) a.s. $\mathcal{U}(Z), \mathcal{U}(\bar{Z}) \in [0, \infty)$. This is important since we can now apply monotone convergence theorem letting $r \to \infty$ to obtain

$$\mathbb{E}\{\kappa(B^{-h}Z)\mathcal{U}(Z)\}$

establishing (3.4).

(3.4) $\Rightarrow$ (3.5): With the notation and the arguments of the proof in Lemma 3.1 for $\bar{Z} \in \mathcal{C}_\xi[Z]$ applying the Fubini-Tonelli theorem and (3.4) for all $F \in \mathcal{H}_0$ we obtain (recall (8.2))

$$\mathbb{E}\{F(Z)\} = \mathbb{E}\{F_{\gamma_0}(Z)/F_{\gamma_0}(Z)F(Z)\} = \sum_{h \in \mathcal{H}_0} p\mathcal{N}(h)\mathbb{E}\{\kappa(B^{-h}Z)F(Z)\}/F_{\gamma_0}(Z) = \mathbb{E}\{F(Z)\}$$  

establishing (3.5).

If (3.5) holds, then clearly (1.5) is satisfied, hence the proof is complete. \hfill \Box

**Proof of Lemma 4.5** $\mathcal{C}_\xi[Z]$ is shift-invariant $\iff$ (i), (ii): If $\mathcal{C}_\xi[Z]$ is shift-invariant, then Item (i)-Item (ii) follows straightforwardly. Suppose next that $\mathcal{C}_\xi[Z]$ is shift-invariant. Note that (1.7) holds with $\mathcal{T}$ substituted by
Since $T_0$ is a separant for $\kappa(B^tZ)$, $t \in \mathcal{T}$, then (1.7) follows. The latter together with the stochastic continuity and the non-negativity of $\kappa$ imply that for all $t \in \mathcal{T}$ and some $c$ sufficiently large
\[ \infty > \mathbb{E}\left\{ \sup_{s \in [-c,c]} \kappa(B^{-t}Z) \right\} \geq \lim_{n \to \infty} \mathbb{E}\left\{ \kappa(B^{-tn}Z) \right\} = \mathbb{E}\{ \kappa(B^{-t}Z) \}, \]
with $t_n \in \mathbb{T}_0$, $n \geq 1$ such that $\lim_{n \to \infty} t_n = t$ (recall $T_0$ is a dense countable set of $T$). In order to finish the proof, we need to show that for all $F \in \mathcal{H}$, $h \in \mathcal{T}$
\[ \mathbb{E}\{F(\Theta_h)\} = \mathbb{E}\{\kappa(B^{-h}Z)F(Z/\kappa(B^{-h}Z)^{1/\alpha})\} = \mathbb{E}\{\kappa(Z)F(B^hZ/\kappa(Z)^{1/\alpha})\} = \mathbb{E}\{F(B^h\Theta)\}. \]

It suffices to show therefore that $\Theta_h$ and $B^h\Theta$ have the same fidis. For $h \in \mathbb{T}_0$ this is an immediate consequence of $\mathcal{E}_\kappa[Z_{\mathbb{T}_0}]$ being shift-invariant. For a general $h \in \mathcal{T}$ it suffices to show that for all $t_1, \ldots, t_n$ in $\mathcal{T}$ and for all bounded continuous $F : (\mathbb{R}^d)^n \to [0, \infty)$
\[ \mathbb{E}\{\kappa(B^{-h}Z)F(Z/\kappa(B^{-h}Z)^{1/\alpha})\} = \mathbb{E}\{\kappa(Z)F(B^hZ/\kappa(Z)^{1/\alpha})\} = \mathbb{E}\{F(B^h\Theta)\}. \]

Since $T_0$ is dense in $\mathcal{T}$ there exists $t_k, k \in \mathbb{T}_0, k \in \mathbb{N}$ such that
\[ \lim_{k \to \infty} t_k = t_1 \in \mathcal{T}, \ 1 \leq i \leq n, \ \lim_{k \to \infty} h_k = h. \]

By the stochastic continuity of $\kappa(B^hZ)$, $t \in \mathcal{T}$ and $t_k - h \in \mathbb{T}_0$ (since $T_0$ is an additive group) using the dominated convergence theorem (recall also (1.7)) the claim follows.

Item (ii) $\Rightarrow$ Item (iii): From (8.2) a.s. $I_{T_0}(B^hZ) \in (0, \infty)$. Consequently, the following rf
\[ Z_N(t) = \left( \frac{\kappa(Z)}{I_{T_0}(B^N Z)^{1/\alpha}} \right)^{1/\alpha} B^N Z(t), \quad t \in \mathcal{T} \]
is well-defined and stochastically continuous. By the assumption and the fact that $T_0$ is dense in $\mathcal{T}$ it follows that (4.2) holds for all $h \in \mathbb{T}_0, F \in \mathcal{S}_\alpha$. Next, given $F \in \mathcal{S}_\alpha$, by the independence of $N$ and $Z$, using further (4.2) for the derivation of the third last equality below we obtain
\[ \mathbb{E}\{F(Z_N)\} = \mathbb{E}\left\{ \int_{T_0} \frac{\kappa(Z)}{I_{T_0}(B^hZ)} F(B^hZ)p_N(h)\lambda(dh) \right\} = \int_{T_0} \mathbb{E}\{F(B^hZ)p_N(h)\}\lambda(dh) = \mathbb{E}\{F(Z)\} \int_{T_0} \frac{\kappa(B^{-h}Z)p_N(h)}{I_{T_0}(Z)}\lambda(dh) \]
\[ = \mathbb{E}\{F(Z)\}, \]
hence $Z_N \in \mathcal{E}_\kappa[Z]$.

Item (i) $\Rightarrow$ Item (iv): The proof is the same as above substituting $T_0$ by $\mathcal{T}$ and the counting measure on $T_0$ by the Lebesgue measure on $\mathbb{R}^t$ if $\mathcal{T} = \mathbb{R}^t$.

Item (ii) or (iv) $\Rightarrow$ (i): Clearly, if $Z_N \in \mathcal{E}_\kappa[Z]$ for all $T_0$-valued rv $N$ with $p_N \in \mathcal{P}_{T_0}$, then it follows easily that $B^hZ_N \in \mathcal{E}_\kappa[Z]$ for all $h \in T_0$ implying that $\mathcal{E}_\kappa[Z_{T_0}]$ is shift-invariant. The proof is the same when $p_N \in \mathcal{P}_{\mathcal{T}}$. □

Proof of Lemma 5.4 Since by the assumption $Z_N \in \mathcal{E}_\kappa[Z]$, by Lemma 4.5, Item (i) the claim follows if we show that
\[ \mathbb{E}\{\kappa(B^{-h}Z_N)F(Z_N)\} = \mathbb{E}\{F(B^h\Theta)\}, \quad \forall F \in \mathcal{S}_\alpha, h \in \mathbb{T}_0. \]

If $\Theta$ is a spectral tail rf, then in view of (5.1) and the assumption that $\mathbb{P}\{\kappa(\Theta) = 1\} = 1$ by the Fubini-Tonelli theorem for all $F \in \mathcal{S}_\alpha, h \in \mathbb{T}_0$ we obtain
\[ \mathbb{E}\{\kappa(B^{-h}Z_N)F(Z_N)\} = \int_{T_0} \mathbb{E}\{\kappa(B^{-h}\Theta)\kappa(\Theta)\frac{F(B^h\Theta)}{I_{T_0}(B^h\Theta)}\}p_N(y)\lambda(dy) \]
\[ = \mathbb{E}\left\{ \frac{F(B^h\Theta)}{I_{T_0}(B^h\Theta)} \int_{T_0} \kappa(B^{-h}\Theta)p_N(y)\lambda(dy) \right\} \]
\[ = \mathbb{E}\{F(B^h\Theta)\}, \]
hence (8.9) holds. When $p_N \in \mathcal{P}_{\mathcal{T}}$ the proof follows with the same arguments as above. The converse is consequence of (4.2) and Lemma 4.5. □
Proof of Theorem 5.9 From Remark 5.8, Item (ii) it follows that \( \kappa(\Theta)^{1/\alpha} = R \) is an \( \alpha \)-Pareto rv and \( \Theta = Y/R \) is independent of \( R \). Since \( Y \) satisfies (5.2), then \( \Theta \) also satisfies (5.2). Next, by Remark 5.8, Item (ii) and the \( \alpha \)-homogeneity of \( \kappa \) (recall \( \lambda_\alpha(\{dz\}) = az^{-\alpha-1}dz \))

\[
\int_0^\infty E\{I(\Gamma^\alpha(\Omega(\kappa(zB^{-h}\Theta) > 1))\lambda_\alpha(\{dz\})
\]

\[ = x^{-\alpha} \int_0^\infty E\{I(xz\Omega(\kappa(zB^{-h}\Theta) > 1))\lambda_\alpha(\{dz\})
\]

\[ = x^{-\alpha}E\{I(z\Omega(\kappa(zB^{-h}Y) > 1))
\]

\[ = E\{I(\Gamma^\alpha(\Omega(\kappa(B^{h}Y) > x^{\alpha}))\}
\]

Consequently, letting \( x \downarrow 0 \) the monotone convergence theorem yields (5.6). For all \( F \in \mathcal{F}_h, h \in \mathcal{T} \) by Remark 5.8, Item (ii) and the Fubini-Tonelli theorem using further \( Y = R\Theta, P\{\kappa(\Theta) = 1\} = 1 \) and the \( \alpha \)-homogeneity of \( \kappa \) we obtain

\[
E\{\kappa(B^{-h}\Theta)F(\Theta)\} = E\\left\{\frac{\kappa(B^{-h}Y)}{\kappa(Y)}F(Y)\right\}
\]

\[ = \int_0^\infty \kappa(B^{-\alpha-1}Y)\lambda_\alpha(\{dz\})
\]

\[ = \int_0^\infty \kappa(B^{-\alpha-1}Y)\lambda_\alpha(\{dz\})
\]

\[ = \int_0^\infty \kappa(B^{-\alpha-1}Y)\lambda_\alpha(\{dz\})
\]

Hence \( \Theta \) is a spectral tail rf. Next, let \( \mathcal{C}_\alpha[\Theta] = \mathcal{C}_\alpha(\Theta) \) be as in Remark 5.5 and write \( Y = R\Theta \), which is a tail spectral rf. In view of Condition 5.1 and Theorem 9.1 we have

\[
J(Y) = \int_K \kappa(B^{-\alpha-1}Y)\lambda_\alpha(\{dz\}) < \infty, \quad K = [-c,c] \cap \mathcal{T}.
\]

Since \( \kappa(B^Y), t \in \mathcal{T} \) is jointly measurable, \( J(Y) \) is well-defined and further \( P\{\kappa(Y) > 1\} = 1 \) together with Theorem 9.1 (taking \( d = 1, A = K, U = \kappa(Y), g_1(s) = s^\alpha, s \geq 0, \gamma = z = 1 \) and \( g_2 = 1 \) therein) imply \( P\{J(Y) > 0\} = 1 \). Moreover, applying again Theorem 9.1 for all \( s > 0 \) on the event \( \{\sup_{t \in K} \kappa(sB^Y) > 1\} \) we have that \( J(sB^{N-t}Y) > 0 \) and thus on that event \( J(sB^{N-t}Y)/J(sB^{N-t}Y) = 1 \) for all \( t \in K \). Consequently, borrowing the arguments of [9], using additionally Lemma 4.5 and the \( \alpha \)-homogeneity of \( \kappa \) (with \( N \) and \( p_N \) as therein) we obtain

\[
E\left\{\sup_{t \in K} \kappa(B^{-\alpha-1}Z)\right\} = E\left\{\sup_{t \in K} \kappa(B^{N-t}Y)\right\}
\]

\[ = E\left\{\int_0^\infty \kappa(B^{N-t}Y)\lambda_\alpha(\{dz\})\right\}
\]

\[ = \int_0^\infty \kappa(B^{N-t}Y)\lambda_\alpha(\{dz\})
\]

where we used Remark 5.8, Item (ii) (this is crucial for the proof) to derive the last third line above. Consequently, we have

\[
(8.10) \quad E\left\{\sup_{t \in K} \kappa(B^{-\alpha-1}Z)\right\} = \int_K \frac{1}{\kappa(B^{N-t}Y)} \lambda(\{dx\}) \quad \in (0, \infty)
\]

implying that \( Z \in \mathcal{C}_\alpha(\Theta) \) satisfies (1.7) is equivalent with (5.7).

It is clear that (5.6) implies (5.1) for all \( \Gamma \in \mathcal{F}_h, h \in \mathcal{T} \) and we can write again \( Y = R\Theta \) with \( R \) independent of \( \Theta \) being \( \alpha \)-Pareto. Indeed, if \( Y \) is a spectral tail rf, then (8.10) follows as mentioned above. Consequently, we can
define an \( \alpha \)-homogeneous rf generated by \( Z_N \) as in Lemma 4.5, which is shift-invariant. It follows that its local rf is \( \Theta \) establishing the proof.

**Proof of Lemma 6.2** By the assumption that \( Z \) satisfies (6.1), then a.s.

\[
\{S_L(Z) < \infty\} \subset \{J_1(Z) \in L\}.
\]

Taking \( F(Z) = I(S_L(Z) = \infty) \) and applying (4.2) (recall \( \mathbb{E}(A; B) \) stands for \( \mathbb{E}(A|B) \) and Item J1))

\[
\mathbb{E}\{S_L(Z)F(Z); J_1(Z) = 0\} = \sum_{s \in L} \mathbb{E}\{\kappa(B^s Z)F(Z); J_1(Z) = 0\}
\]

\[
= \sum_{s \in L} \mathbb{E}\{\kappa(Z)F(Z); (J_1(B^{-s} Z) = s)\}
\]

\[
= \mathbb{E}\{\kappa(Z)F(Z); J_1(Z) = s\}
\]

\[(8.11)\]

Since \( \mathbb{E}\{S_L(Z)F(Z); J_1(Z) = L\} \in \{0, \infty\} \) we have

\[
\mathbb{E}\{F(\theta); J_1(\theta) \in L\} = \mathbb{P}\{S_L(\theta) < \infty, J_1(\theta) \in L\} = 0.
\]

Hence Lemma 9.8 implies

\[
\{J_1(Z) \in L\} \subset \{S_L(Z) < \infty\}.
\]

Taking \( J_1 \) equal to the infargs map \( I_{\text{argsup}} \) establishes thus (6.3). As above, substituting \( J_1 \) by \( J_2 \) we have

\[
0 = \mathbb{P}\{S_L(\theta) < \infty, J_2(\theta) \in L\} = \mathbb{P}\{S_L(Y) < \infty, J_2(Y) \in L\} = 0,
\]

hence by (6.2) modulo null sets \( \{S_L(Y) < \infty\} = \{J_2(Y) \in L\} \) and thus the first two equalities in (6.4) follow for \( \mathcal{I} = \mathcal{J}_2 \). Next since \( K \cap \mathcal{L} \) has only finite number of elements for every compact \( K \subset \mathbb{R}^1 \), then

\[
\{S_L(Y) < \infty\} = \{S_L(\theta) < \infty\} \subset \left\{ \sum_{i=1}^{\infty} \kappa(\kappa^{-1}(\theta)) = 0 \right\} \subset \{B_{\mathcal{L}, r}(Y) < \infty\}
\]

holds modulo null sets. Further, we have a.s.

\[
\mathcal{M}_L(Y) = \sup_{i \in \mathcal{L}} \kappa(B^i Y)^{1/\alpha} \geq \kappa(Y)^{1/\alpha} > 1.
\]

Since by Lemma 9.12 \( B_{\mathcal{L}, r}(Y) < \infty \) implies \( \mathcal{M}_L(Y) < \infty \), then in view of (9.9) and Lemma 9.10

\[
\{B_{\mathcal{L}, r}(Y) < \infty\} \subset \{S_L(Y) < \infty\}
\]

modulo null sets implying

\[(8.13)\]

\[
\{S_L(Y) < \infty\} = \{B_{\mathcal{L}, r}(Y) < \infty\}.
\]

Let \( \Gamma \in \mathfrak{H}_0 \) be shift-invariant with respect to \( L \) and recall \( B_{\mathcal{L}, r}(Y) = \sum_{i \in L} \kappa(B^i Y)^{r/2} \mathbb{I}(\kappa(B^i Y) > 1) \). Using the Fubini-Tonelli theorem for any \( \Gamma \in \mathfrak{H}_0 \) and Item J1,Item J2 (recall Remark 5.8, Item (iii))

\[
\mathbb{E}\{B_{\mathcal{L}, r}(Y)\Gamma(Y); J_3(Y) = 0\} = \sum_{i \in \mathcal{L}} \mathbb{E}\{\kappa(B^i Y)^{r/2} \mathbb{I}(\kappa(B^i Y) > 1)\Gamma(Y); J_3(Y) = 0\}
\]

\[
= \sum_{i \in \mathcal{L}} \mathbb{E}\{\kappa(Y)^{r/2} \mathbb{I}(\kappa(B^{-i} Y) > 1)\Gamma(Y); J_3(B^{-i} Y) = 0\}
\]

\[
= \sum_{i \in \mathcal{L}} \mathbb{E}\{\kappa(Y)^{r/2} \Gamma(Y); J_3(Y) = i\}
\]

\[(8.14)\]

\[
= \mathbb{E}\{\Gamma(Y)\kappa(Y)^r; J_3(Y) \in L\}.
\]

Taking \( \Gamma(Y) = I(B_{\mathcal{L}, r}(Y) = \infty)/R^{1/r} \) we obtain

\[
\mathbb{E}\{B_{\mathcal{L}, r}(Y)\Gamma(Y); J_3(Y) = 0\} = \mathbb{E}\{\kappa(\theta)^r I(B_{\mathcal{L}, r}(Y) = \infty); J_3(Y) \in L\} \leq 1.
\]

Since \( \mathbb{P}\{\kappa(\theta) = 1\} = 1 \), then

\[
\mathbb{P}\{B_{\mathcal{L}, r}(Y) = \infty, J_3(Y) \in L\} = 0
\]

and thus by (6.2)

\[
\{J_3(Y) \in L\} = \{B_{\mathcal{L}, 0}(Y) < \infty\};
\]

hence the proof follows utilising further (8.12).

**Proof of Theorem 6.3** Applying Theorem 9.1 establishes (6.5). Assume next that \( \mathbb{P}\{S(Z) < \infty\} = 1 \), which in view of Lemma 9.8 is equivalent with \( \mathbb{P}\{S(Z) < \infty\} = 1 \). Again by Lemma 9.8

\[
q_1 = \mathbb{P}\{0 < S(\theta) < \infty\} = 1.
\]

Taking \( N \) independent of \( Z \) with pdf \( p_N \in \mathcal{P}_r \) we have

\[
(8.15) Z_N = \left( \frac{\kappa(Z)}{p_N(N)S(Z)} \right)^{1/\alpha} B^N Z
\]
By the assumptions on $P$ establishes (and Lemma implies $\check{\varphi}$ and Remark $7.5$)

\[ q_2 = \mathbb{P} \left\{ \int_{T} \sup_{t \in [-c,c]} \kappa(Bs^{-t}\Theta) \lambda(ds) < \infty \right\} = 1, \quad \forall c \in (0, \infty), \]

and thus by Lemma 9.10

\[ \{ S(\Theta) < \infty \} \subset \left\{ \int_{T} \sup_{t \in [-c,c]} \kappa(Bs^{-t}\Theta) \lambda(ds) < \infty \right\}, \quad \forall c \in (0, \infty). \]

In fact the above is an equality since the reverse inclusion clearly holds. When $q_2 = 1$, then also

\[ q_3 = \mathbb{P} \left\{ \lim_{\sum_{i=1}^{|t_i|} \to \infty} \kappa(Bs^{-t}\Theta) = 0 \right\} = 1. \]

Moreover if $q_3 = 1$, then

\[ \mathbb{P}\{ B_{\check{L},r}(Y) < \infty \} = 1. \]

Hence Lemma 9.10 implies

\[ \{ B_{\check{L},r}(Y) < \infty \} = \{ B_{\check{L},r}(Y) < \infty \}, \quad \{ B_{\check{L},r}(Y) < \infty \} \subset \{ S(\Theta) < \infty \}, \]

which together with (8.16), (8.17) and Lemma 6.2 establishes (6.6)-(6.7).

The proof of (6.8) and (6.9) follows with the same arguments using Lemma 9.8 and Remark 9.11.

**Proof of Lemma 7.1** If $\mathcal{C}_r[Z]$ is shift-invariant, then the stationarity of $X$ follows from (1.10) and [2, Thm 6.9] and the converse follows from the latter result. □

**Proof of Proposition 7.2** First note that $\check{\varphi}_{\check{L}}$ exists and is finite which follows from the shift-invariance of $\mathcal{C}_r[Z]$ and the subadditivity of supremum, see also [47]. If $\mathcal{L}$ is a full rank lattice on $T$ or $\mathcal{L} = T$, then by (8.10)

\[ \check{\varphi}_{\check{L}} = \frac{1}{\Delta(\mathcal{L})} \mathbb{E} \left\{ \int_{T} \kappa(B^{s^{-t}\Theta})^\ast \mathbb{E}(\kappa(B^{s^{-t}Y}) > 1) \lambda(ds) \right\} < \infty \]

for all $r \in \mathbb{R}$ such that Condition 5.1 holds. In our notation $\Delta(\mathbb{R}) = 1$ and if $\mathcal{L}$ is a full rank lattice on $T$, then $\Delta(\mathcal{L}) = |\det(A)| > 0$ as in (2.1), where $A$ is a non-singular base matrix of the full rank lattice $\mathcal{L}$ i.e., $\mathcal{L} = \{ Ax, x \in \mathbb{Z} \}$. Consequently, $\check{\varphi}_{\check{L}} = 0$ iff

\[ \mathbb{P}\{ B_{\check{L},r}(Y) = \infty \} = 1 \]

and thus the first equivalence in (7.5) is clear.

Suppose next that $\mathbb{P}\{ B_{\check{L},r}(Y) = \infty \} = 1$. By Lemma 6.2 $\mathbb{P}\{ B_{\check{L},0}(Y) = \infty \} = 1$. Since for all $n \in \mathbb{N}$ we have that $\mathbb{P}\{ B_{2^{-n}\check{L},0}(Y) = \infty \} = 1$ and further $2^{-n}\mathcal{L}$ is also a full rank lattice on $\mathbb{R}$, then by the first equivalence in (7.5) and (9.11) we have that $\check{\varphi}_{\check{L}} = 0$, which is equivalent with $\mathbb{P}\{ B_{\check{L},0}(Y) = \infty \} = 1$. If the latter holds, using that $\check{\varphi}_{\check{L}} \leq \check{\varphi}_{\check{L}} = 0$, then $\check{\varphi}_{\check{L}} = 0$ establishing (7.5). □

9. **Appendix**

Recall that in our notation convergence in probability and a.s. are denoted by $\overset{p}{\to}$ and $\overset{a.s.}{\to}$, respectively.

**Theorem 9.1.** Let $g_1 : \mathbb{R}^d \to [0, \infty]$ be Borel measurable and let $g_2 : \mathbb{R}^d \to [0, \infty)$ be Lebesgue measurable, almost everywhere positive and locally bounded. Let further $A \subset \mathbb{R}^d$ be open and $U \in \mathbb{D}_\infty$. If $\mathbb{P}\{ \sup_{t \in A} \|U(t)\| > z \} > 0, z \in \mathbb{R}$, then for all $r \in \mathbb{R}$

\[ \mathbb{P}\left\{ \sup_{t \in A} \|U(t)\| > z, \int_{A} g_1(U(t))\mathbb{E}(\|U(t)\| > \gamma)g_2(t)\lambda(dt) = 0 \right\} = 0, \]

provided that $g_1(f(t)) > 0$ for all $t \in A$ such that $\|f(t)\| > \gamma, f \in D(T, \mathbb{R}^d)$.

**Proof of Lemma 9.1** By the assumptions on $g_1$ we have that $g_1(U(t)), t \in \mathbb{R}$ is measurable and non-negative. By the Fubini-Tonelli Theorem (see e.g., [56, Thm 2.7]) $\mathbb{E}(U) = \int_{A} g_1(U(t))g_2(t)\lambda(dt)$ is a non-negative rv. For $d = 1$ and $g_2$ constant the claim follows from [57, Thm 2.1]. The extension $d > 1$ and $g_2$ non-constant follows with the same argument as the proof of the aforementioned result. □
Lemma 9.2. Let $X_{m,n}, Y_n, m, n \in \mathbb{N}$ be rv’s defined on the same probability space. If
\[ X_{m,n} \overset{p}{\to} Y_m, \quad n \to \infty, \quad Y_m \overset{p}{\to} Y, \quad m \to \infty \]
holds, then there exists a non-decreasing sequence of integers $m_k, n_k \in \mathbb{N}$ such that $X_{m_k,n_k} \overset{a.s.}{\to} Y$ as $k \to \infty$.

Proof of Lemma 9.2 Since the convergence in probability is metrizible by the metric
\[ d(X, Y) = E[|X - Y|/(1 + |X - Y|)], \]
then by [6, Lem A.1.3] there exists a subsequence of non-decreasing integers $m_n, n \in \mathbb{N}$ such that $X_{m_k,n_k} \overset{p}{\to} Y$ as $k \to \infty$ holds in probability. Hence there exists another a subsequence of non-decreasing integers $m_k, n_k \in \mathbb{N}$ such that $X_{m_k,n_k} \overset{a.s.}{\to} Y$ as $k \to \infty$ establishing the proof.

Lemma 9.3. Let $g_1, g_2, U$ be as in Theorem 9.1. If further
\[ \int_A \mathbb{E}\{g_1(U(t))\}g_2(t)\lambda(dt) < \infty \]
is valid for all $A = [-c, c]^l, c > 0$, then $\tilde{R}_{c,n}(g_1(U), g_2) \overset{a.s.}{\to} \mathcal{I}_{g_1}(U), i \to \infty$, where $c_i, n_i, i \in \mathbb{N}$ are two sequences of increasing positive integers converging to $\infty$ as $i \to \infty$ and
\[ \tilde{R}_{c,n}(g_1, g_2) = \frac{2c^l}{n} \sum_{t \in (\mathbb{R}/n)^l \cap [-c, c]^l} g_1(f(t))g_2(t), \quad f \in \mathcal{D}(\mathcal{T}, \mathbb{R}^d). \]

Proof of Lemma 9.3 Since $\mathcal{I}_{[-c, c]^l}(U) \overset{a.s.}{\to} \mathcal{I}_{U}(U), c \to \infty$ in view of Lemma 9.2 it suffices to show that $\tilde{R}_{c,n}(g_1(U), g_2) \overset{a.s.}{\to} \mathcal{I}_{[-c, c]^l}(U), k \to \infty$ for a given $c > 0$ and some non-decreasing sequence of integers $n_k, k \in \mathbb{N}$ such that $\lim_{k \to \infty} n_k = \infty$. The claim for $l = 1$ is consequence of the derivations in [58, p. 329-320]. Borrowing those arguments, leads to the proof for the case $l$ is a positive integer.

Lemma 9.4. Let $\mathcal{C}_a[Z]$ be given and let $A \subset \mathcal{R}$ be a Borel set. If $G \in \mathcal{H}$ is such that $I(G(\cdot) \in A) \in \mathcal{B}_0$, then
\[ P\{G(Z) \in A\} > 0 \text{ is equivalent with } P\{G(\tilde{Z}) \in A\} > 0, \quad \forall \tilde{Z} \in \mathcal{C}_a[Z]. \]

Proof of Lemma 9.4 Using (8.2), in view of (3.5)
\[ \mathbb{E}\{I_{\mathcal{I}_0}(Z)\|I(G(Z) \in A)\} = \sum_{t \in \mathcal{T}_0} \mathbb{E}\{\|Z(t)\|^\alpha I(G(Z) \in A)\} = \sum_{t \in \mathcal{T}_0} \mathbb{E}\{\|\tilde{Z}(t)\|^\alpha I(G(\tilde{Z}) \in A)\} = \mathbb{E}\{I_{\mathcal{I}_0}(\tilde{Z})\|I(G(\tilde{Z}) \in A)\}, \]

hence the claim follows.

Recall the Polish metric space $(D_c, d_{D_c})$ in Definition 1.1 and note that the Borel $\sigma$-field of $D_c$ agrees with the cylindrical $\sigma$-field $\mathcal{D}$, see e.g., [16, 59].

For $X(t), t \in \mathcal{T}$ with a.s. sample paths in $D_c$ that is regularly varying with $\alpha$-homogeneous tail measure $\nu$, in view of [6, 7, 16] we have that $\nu = \nu_2$ for some $\tilde{Z} \in \mathcal{C}_a[Z]$, where $\|\cdot\| : \mathbb{R}^d \to [0, \infty]$ is a norm on $\mathbb{R}^d$ and $\kappa(B f) = \|f(\cdot)|^\alpha$. Let $\mathcal{B}_0$ be a boundedness, which consists only of sets $A \in \mathcal{D}$ such that for all $f \in A$ we have $d_{D_c}(f, 0) > \varepsilon_A$ for some $\varepsilon_A > 0$, see [6, 9, 16]. Consider some positive sequence $a_n, n \geq 1$ such that $n\mathbb{P}\{X/a_n \in \cdot\}$ converges weakly to $\nu_2(\cdot)$ as $n \to \infty$ with respect to the boundedness $\mathcal{B}_0$, see [6, 9, 16] for more details.

We formulate next a lemma on 1-homogeneous maps which is a minor extension of [60, Prop 2.5], see also the related result [61, Lem A.7]. In the following $H : D_c \to [0, \infty]$ is called lower semi-continuous if $A_x = \{f \in D_c : H(f) > x\}$ is an open subset of $D_c$ for all $x > 0$.

Lemma 9.5. Let $F \in \mathcal{H}_1$ with $F(0) = 0$ be lower semi-continuous, non-negative and continuous at 0. If $X$ is as above, then for all $x > 0$
\[ \lim_{n \to \infty} n\mathbb{P}\{F(X) > a_n x\} = x^{-\alpha}\mathbb{E}\{F^\alpha(Z)\} < \infty. \]

Proof of Lemma 9.5 The continuity at zero of $H$ yields that $d_{D_c}(0, f) > \varepsilon$ for some $\varepsilon > 0$ and all $f \in A_x, x > 0$, which implies that for some hypercube $K \subset \mathbb{R}^d$ we have
\[ \sup_{t \in K \cap T} \|f(t)\| > \varepsilon \]
for all $f \in A_x$, see [6, 16]. Consequently, we have $A_x \in \mathcal{B}_0$. Since tail measures on $\mathcal{D}$ are such that $\nu_2(E) < \infty$ for all $E \in \mathcal{B}_0$, then by the $-\alpha$-homogeneity of $\nu_2$ and the 1-homogeneity of $F$
\[ \nu_2(A_x) = x^{-\alpha}\nu_2(A_1) = x^{-\alpha}\mathbb{E}\{F^\alpha(Z)\} < \infty, \]

where $A_1$.


Since $A_ε$ is open its frontier is a subset of $A_ε^+ = \{ f ∈ D_ε : F(f) = x \}$, which also belongs to $B_0$. By the $\overline{α}$-homogeneity of $ν_2$ we have that
\[
ν_2(A_ε^+) = 0
\]
see also [16, Rem 3.2]) and hence the claim follows. See [60, Prop 2.5] or the derivation of [3, (A.1)], where the last arguments appear for the case $d = 1$ and $T$ has a finite number of elements, see also [40, Lem 3.1].

**Remark 9.6.**

(i) In view of [59, Lem 2.2, Prop 2.1]
\[
F(c, f) = \inf_{t ∈ [−ε, c]} ∥f(t)∥, \quad c ∈ (0, ∞]
\]
satisfies the assumptions of Lemma 9.5;

(ii) If $F_1, F_2$ and $X, Z$ are as in Lemma 9.5 such that $F_2(Z) > 0$ a.s., then for $x, y$ positive
\[
\lim_{n → ∞} \mathbb{P}[F_1(X) > a_n x | F_2(X) > a_n y] = \mathbb{E} \left[ \frac{\min(y F_1(Z), x F_2(Z))^{α}}{\mathbb{E} F_2^α(Z)} \right].
\]
In particular, this is applicable for $F_1(c, f) = \sup_{t ∈ [−ε, c]} ∥f(t)∥$ or $F_1(f) = \int_{[−ε, c]} ∥f(t)∥ (dt)$, with $c ∈ (0, ∞)$ and $F_2$ as in (i).

**Lemma 9.7.** The local rf $Θ$ of $C_ε[Z]$ is such that $κ(B^2 Θ), t ∈ T$ is stochastically continuous.

**Proof of Lemma 9.7** Let $t_n → t ∈ T$ as $n → ∞$. By the $\overline{α}$-homogeneity of $κ$
\[
C_n = \mathbb{E} \left\{ \frac{|κ(B^n Θ) − κ(B^2 Θ)|}{1 + |κ(B^n Θ) − κ(B^2 Θ)|} \right\} = \mathbb{E} \left\{ \frac{κ(Z)}{1 + κ(Z)} \frac{|κ(B^n Z) − κ(B^2 Z)|}{κ(Z)} \right\} = \mathbb{E} \left\{ \frac{κ(Z)}{κ(Z)} \frac{|κ(B^n Z) − κ(B^2 Z)|}{κ(Z) + |κ(B^n Z) − κ(B^2 Z)|} \right\}.
\]
Since by assumption $κ(B^2 Θ), t ∈ T$ is stochastically continuous, then $|κ(B^n Z) − κ(B^2 Z)| / κ(Z) ↘ 0$ as $n → ∞$. Further, using that $κ(Z)$ is non-negative and $\mathbb{E} [κ(Z)] < ∞$ we obtain by the dominated convergence theorem $\lim_{n → ∞} C_n = 0$. Using that the convergence in probability is metrizable (recall (9.2)), the claim follows. Note in passing that by [58, Thm 1, p. 171 & Thm 5, p. 169] both $Θ$ and $κ(B^2 Θ), t ∈ T$ have a jointly measurable and separable version with separant $T_0$.

**Lemma 9.8.** Let $C_ε[Z]$ with local rf $Θ$ be given. If $F ∈ S_0$ is non-negative, then the following are equivalent:

(i) $\mathbb{E} \{F(\tilde{Z})\} = 0$ for some (and then all) $\tilde{Z} ∈ C_ε[Z]$;

(ii) $\mathbb{E} \{F(B^h \tilde{Z})\} = 0$ for some (and then all) $\tilde{Z} ∈ C_ε[Z]$ and for all $h ∈ T$;

(iii) $\mathbb{E} \{F(B^h Θ)\} = 0$ for all $h ∈ T$.

Moreover, if $F$ is bounded by some constant $c > 0$, then Item (i)-Item (iii) hold with $c − F$ instead of $F$.

**Proof of Lemma 9.8** Recall the definition of $F_0(Z)$ in (8.1) and let $p_n ∈ P_{T_0}$.

(i) $→$ (ii): Since by the assumption $F ∈ S_0$ is non-negative, then when (i) holds a.s. $F(\tilde{Z}) = 0$. Using (4.2) for all $h ∈ T$, $\tilde{Z}^* ∈ C_ε[Z]$,
\[
\mathbb{E} \{F(B^h \tilde{Z}^*) I_h(\tilde{Z}^*)\} = \mathbb{E} \{F(B^h \tilde{Z}) I_h(\tilde{Z})\} = \sum_{t ∈ T_0} \mathbb{E} \{κ(B^{h−t} \tilde{Z}) F(\tilde{Z})\} p_n(t) = 0.
\]
Consequently, $I_h(\tilde{Z}^*) ∈ (0, ∞)$ a.s. implies $F(B^h \tilde{Z}^*) = 0$.

(ii) $→$ (iii): For all $h ∈ T$, $\tilde{Z} ∈ C_ε[Z]$,
\[
\mathbb{E} \{F(B^h Θ)\} = \mathbb{E} \{κ(Z) F(B^h Z / κ(Z))^{1/α}\} = \mathbb{E} \{κ(Z) F(B^h Z)\} = \mathbb{E} \{κ(Z) F(B^h \tilde{Z})\}
\]
and thus (iii) holds.

(iii) $→$ (i): By the shift-invariance of $C_ε[Z]$ (using (4.2) to derive the last equality below)
\[
0 = \sum_{t ∈ T_0} \mathbb{E} \{F(B^h Θ)\} p_n(t) = \sum_{t ∈ T_0} \mathbb{E} \{κ(Z) F(B^h Z)\} p_n(t) = \mathbb{E} \{F(Z) \sum_{t ∈ T_0} κ(B^{−t} Z) p_n(t)\},
\]

hence (i) follows for $Z$.

Next, if $\mathbb{E} \{F(Z)\} = c$ we have by the assumption that $\tilde{F} = c − F ∈ S_0$ is non-negative and $\mathbb{E} \{\tilde{F}(Z)\} = 0$, which by the above is equivalent with $\mathbb{E} \{\tilde{F}(Θ)\} = 0$, hence the proof is complete.

As an application of Lemma 9.8 we have the following characterisation for such $C_ε[Z]$’s.

**Corollary 9.9.** Given $C_ε[Z]$ such that $\mathbb{E} \{κ(Z)\} = 1$, then the following are equivalent:

(i) $κ(Z) > 0$ a.s.;

(ii) For some (and then for all) $\tilde{Z} ∈ C_ε[Z]$ we have $κ(B^h \tilde{Z}) > 0$ a.s. for all $h ∈ T$;

(iii) $κ(B^h Θ) > 0$ a.s. for all $h ∈ T$;
(iv) \( \Theta \in C_n[Z] \).

We state next a fundamental property of shift-invariant maps.

**Lemma 9.10.** Let \( F_1, F_2 \) be two shift-invariant maps with respect to \( T_0 \) and let \( A \in \{ \{0\}, (0, \infty), \{\infty\} \} \).

(i) If \( F_1, F_2 \in \mathcal{S}_A \) are such that for any tail \( r.f. Y^* \) we have that \( \mathbb{P}\{F_1(Y^*) \in A\} = 1 \) implies \( \mathbb{P}\{F_2(Y^*) \in A\} = 1 \), then

\[
\{F_1(Y) \in A\} \subset \{F_2(Y) \in A\}
\]

is valid for all tail \( r.f.'s \) \( Y \);

(ii) If \( F_1, F_2 \in \mathcal{S}_A \) are such that for any spectral tail \( r.f. \Theta^* \) we have that \( \mathbb{P}\{F_1(\Theta^*) \in A\} = 1 \) implies \( \mathbb{P}\{F_2(\Theta^*) \in A\} = 1 \), then

\[
\{F_1(\Theta) \in A\} \subset \{F_2(\Theta) \in A\}
\]

is valid for all spectral tail \( r.f.'s \) \( \Theta \).

**Remark 9.11.** Utilising Lemma 9.8 also the corresponding result of Lemma 9.10 for elements \( \bar{Z} \in C_n[Z] \) can be shown to hold.

**Proof of Lemma 9.10** Let \( Y \) be a tail rf with respect to \( \kappa \) such that \( \mathbb{P}\{F_1(Y) \in A\} \in (0, 1] \) and define \( Y^* = Y'|F_1(Y) \in A \), which by the assumption on \( A \) and the shift-invariance of \( F_1 \) is a tail rf. Clearly, \( \mathbb{P}\{F_1(Y^*) \in A\} = 1 \).

By the assumption this implies that \( \mathbb{P}\{F_2(Y^*) \in A\} = 1 \) and hence

\[
1 = \mathbb{P}\{F_2(Y^*) \in A\} = \mathbb{P}\{F_2(Y) \in A, F_1(Y) \in A\}/\mathbb{P}\{F_1(Y) \in A\},
\]

which in turn yields

\[
\{F_1(Y) \in A\} \subset \{F_2(Y) \in A\}
\]

modulo null sets establishing thus the first claim. The second claim follows with similar arguments. \( \square \)

**Lemma 9.12.** Let \( C_n[Z] \) be given and let \( \tau \in \mathbb{R} \) be such that Condition 5.1 holds. If \( \mathbb{P}\{B_{L, \tau}(Y) < \infty\} > 0 \) and \( \mathcal{L} \) is a lattice on \( \mathcal{T} \) or \( \mathcal{L} = \mathcal{T} \), then (recall \( M_\mathcal{L}(Y) = \sup_{\mathcal{L}} \kappa(B^{-1}Y)^{1/\alpha} \))

\[
\mathbb{P}\{0 < B_{L, \tau}(Y) < \infty, M_\mathcal{L}(Y) = \infty\} = 0
\]

and if further \( \mathbb{P}\{B_{L, \tau}(Y) < \infty\} = 1 \), we have

\[
\mathbb{E}\left\{ \frac{\kappa(Y)^\tau S_\mathcal{L}(Y)}{[M_\mathcal{L}(Y)]^{\alpha} B_{L, \tau}(Y)} \right\} = 1.
\]

**Proof of Lemma 9.12** Since by Lemma 9.7 \( \kappa(B^{-1}Y) = R^\tau \kappa(B^{-1}\Theta), t \in \mathcal{T} \) is stochastically continuous and hence belongs to \( \mathfrak{X}_\alpha \), then in view of Theorem 9.1 (take therein \( d = 1, A = \mathbb{R}^2, z = \gamma = 1, U = \kappa(Y)^\gamma \) and \( g_1, g_2 \) equal 1)

\[
M_\mathcal{L}(Y) = \sup_{t \in \mathcal{L}} \kappa(B^tY)^{1/\alpha} > 1/z \quad \Rightarrow \quad \mathbb{P}\{B_{L, \tau}(zY) > 1, \ z \in (0, 1] \}.
\]

Moreover, \( B_{L, \tau}(Y) < \infty \) implies \( B_{L, \tau}(zY) < \infty, z \in (0, 1] \). Hence for all \( z \in (0, 1] \) by Remark 5.8, Item (ii) and the Fubini-Tonelli theorem (recall Remark 5.8, Item (iii))

\[
\mathbb{E}\left\{ \frac{1}{B_{L, \tau}(Y)} \right\} = \mathbb{E}\left\{ \frac{B_{L, \tau}(zY)}{B_{L, \tau}(zY)} \right\} = \mathbb{E}\left\{ \frac{B_{L, \tau}(zY)}{B_{L, \tau}(zY)} \right\} = \mathbb{E}\left\{ \frac{B_{L, \tau}(zY)}{B_{L, \tau}(zY)} \right\} = \mathbb{E}\left\{ \frac{1}{B_{L, \tau}(Y)} \right\} = \infty,
\]

where the last inequality follows from (5.7). Note that when \( \mathcal{L} \) is not equal to \( \mathcal{T} \) the above conclusion is obvious. Consequently, letting \( z \to 0 \) yields

\[
\{B_{L, \tau}(Y) < \infty\} \subset \{M_\mathcal{L}(Y) < \infty\}.
\]
For all $s \in (0,1)$ by Theorem 9.1 we have
\[ P\{V_s > 0, M_L(Y) < \infty\} = 1, \quad V_s = B_{L,r}(sM_L(Y))^{-1}Y \]
since for all $s \in (0,1)$ on the event $\{M_L(Y) < \infty\}$
\[ \sup_{t \in \mathcal{L}} I(\{sM_L(Y))^{-1}k(B^{-1}Y) > 1\} = 1. \]

By the Fubini-Tonelli theorem and (recall Remark 5.8, Item (iii)) and $\lambda_n(ds) = \alpha^s a^{-1} ds$
\[
\int_0^1 \mathbb{E}\{V_s, I(V_s = \infty, M_L(Y) < \infty\}\} \alpha^s a^{-1} ds
\]
\[
= \int_L \int_0^1 \mathbb{E}\left\{ I(\kappa(B^{-1}Y/s, sM_L(Y))) > 1)I(V_s = \infty, M_L(Y) < \infty) \right\} \lambda_n(ds) \lambda(dh)
\]
\[
= \int_L \int_0^\infty \mathbb{E}\left\{ \frac{1}{[M_L(Y)^{s}]^\alpha I(\kappa(B^{-1}Y) > 1)I(B_{L,r}(t^{-1}Y) = \infty, M_L(Y) < \infty) \right\} \lambda_n(dt) \lambda(dh)
\]
\[
= \int_L \int_0^\infty \mathbb{E}\left\{ \frac{1}{[M_L(Y)^{s}]^\alpha} I(\kappa(B^{-1}Y > 1)I(B_{L,r}(Y) = \infty, M_L(Y) < \infty) \right\} \lambda_n(dt) \lambda(dh)
\]
\[
= 0,
\]
under the assumption $P\{B_{L,r}(Y) = \infty\} = 0$, which we shall suppose next. Consequently, for all $s \in (0,1)$ up to a set with Lebesgue measure equal zero we have
\begin{equation}
(9.10) \quad P\{V_s \in (0, \infty)\} = 1.
\end{equation}

Using the latter implication, by the Fubini-Tonelli theorem and (recall Remark 5.8, Item (iii)), $\alpha \cdot 0$ as 0) $M_L(Y) \in (0, \infty)$ a.s. since the origin belongs to $L$ and we suppose that $P\{B_{L,r}(Y) = \infty\} = 1$ and
\[ B_{L,r}(z^{-1}Y) = 0 \implies I(\kappa(B^{-1}Y/z) > 1) = 0, \quad t \in L \]
as which are needed for justification of the third equality below (recall here Remark 5.8, Item (iii))
\[
\mathbb{E}\left\{ \frac{\kappa(Y)^s S_L(Y)}{[M_L(Y)]^{s}B_{L,r}(Y)} \right\}
\]
\[
= \int_L \mathbb{E}\left\{ \frac{\kappa(Y)^s \kappa(B^{-1}Y)^{s}}{[M_L(Y)]^{s}B_{L,r}(Y)} \right\} \lambda(dt)
\]
\[
= \int_L \int_0^\infty \mathbb{E}\left\{ \frac{\kappa(Y)^s \kappa(sB^{-1}Y) > 1)I(B_{L,r}(Y) \in (0, \infty))}{[M_L(Y)]^{s}B_{L,r}(Y)} \lambda_n(ds) \lambda(dh)
\]
\[
= \int_L \int_0^\infty \mathbb{E}\left\{ \frac{\kappa(B^{-1}Y/s) \kappa(B^{-1}Y/s) > 1)I(B_{L,r}(s^{-1}Y) \in (0, \infty))}{[M_L(Y)]^{s}B_{L,r}(s^{-1}Y)} \lambda_n(ds) \lambda(dh)
\]
\[
= \int_L \int_0^\infty \mathbb{E}\left\{ \frac{\kappa(B^{-1}Y/s, sM_L(Y))) \kappa(B^{-1}Y/s, sM_L(Y))) > 1)I(V_s \in (0, \infty))}{V_s} \lambda_n(ds) \lambda(dh)
\]
\[
= \int_L \left\{ \int_0^1 \mathbb{E}\left\{ \frac{\kappa(B^{-1}Y/s, sM_L(Y))) \kappa(B^{-1}Y/s, sM_L(Y))) > 1)}{V_s} \lambda_n(ds) \lambda(dh) \right\}
\]
\[
= 1,
\]
where setting 1 in the upper bound of the integrand is justified by the fact that $I(\kappa(B^{-1}Y/(sM_L(Y))) > 1) = 0$ a.s. for all $s > 1$ and further the $\alpha$-homogeneity of $\kappa$ was used for the derivation of the third equality.

\textbf{Lemma 9.13.} If $T = \mathbb{R}^l$ and $L$ is a full rank lattice on $\mathbb{R}^l$, then
\[
\lim_{n \to \infty} \lim_{S \to \infty} S^{-l} \mathbb{E}\left\{ \sup_{t \in [0, S]^l} \kappa(B^{-1}Z) - \sup_{t \in [0, S]^l \cap 2^{-n}L} \kappa(B^{-1}Z) \right\} = 0.
\]

Moreover, with $\Delta(L)$ defined in (2.1) we have
\begin{equation}
(9.11) \quad \lim_{n \to \infty} \frac{\delta_{l,n}}{\Delta(L_n)} = \delta_{l} \in [0, \infty), \quad L_n = 2^{-n}L.
\end{equation}

\textbf{Proof of Proposition 9.13} Assume for simplicity that $S$ is a positive integer and $L = \mathbb{Z}^l$. Set next $L_n = 2^{-n}L, n \in \mathbb{N}$
\[ K_n = \times_{k=1}^l [i_k, i_k + 1] = [i, i + 1], i \in \mathbb{Z}^l, \quad K_0 = \times_{k=1}^l [0, 1].
\]

For all $n$ sufficiently large and any $i \in \mathbb{Z}^l$ we have that $L_n \cap K_i$ is non-empty. Note further that $L_n$ is a full rank lattice for any $n \in \mathbb{N}$ and $T_0 = \cup_{n \geq 1} L_n$ is a countable dense subset of $T$ and thus it is a separant for $\kappa(B^jZ), t \in T$.

Set below
\[ a(K) = \sup_{t \in K} \kappa(B^{-1}Z), \quad K \subset T.
\]
From (1.7) and (4.2) for all $n$ large we obtain
\[
\mathbb{E}\{a(K_0 \cap L_n)\} = \mathbb{E}\{a(K_i \cap L_n)\} < \infty, \quad \mathbb{E}\{a(K_i)\} = \mathbb{E}\{a(K_0)\} < \infty, \quad \forall i \in \mathbb{Z}^d
\]
since $i \in L_n$ and $L_n$ is an additive group. Consequently, for some positive integer $S$
\[
\mathbb{E}\{a([0,S]^d)\} - \mathbb{E}\{a([0,S]^d \cap L_n)\} = \mathbb{E}\left\{ \max_{i \in [0,S]^d \cap \mathbb{Z}^d} a(K_i) - \max_{i \in [0,S]^d \cap \mathbb{Z}^d} a(K_i \cap L_n) \right\} \\
\leq \mathbb{E}\left\{ \max_{i \in [0,S]^d \cap \mathbb{Z}^d} \left[ a(K_i) - a(K_i \cap L_n) \right] \right\} \\
\leq \sum_{i \in [0,S]^d \cap \mathbb{Z}^d} \mathbb{E}\left\{ a(K_i) - a(K_i \cap L_n) \right\} \\
= S^d \mathbb{E}\{a(K_0)\} - \mathbb{E}\{a(K_0 \cap L_n)\}.
\]
Since $\kappa(B^{-1}Z), t \in T$ is stochastically continuous we obtain
\[
(9.12) \quad M_n = \sup_{t \in K_0 \cap L_n} \kappa(B^{-1}Z) \overset{\mathcal{D}}{=} \sup_{t \in K_0} \kappa(B^{-1}Z), \quad n \to \infty.
\]
Hence by the dominated convergence theorem which is justified by (1.7)
\[
\lim_{n \to \infty} \mathbb{E}\{a(K_0) - a(K_0 \cap L_n)\} = 0
\]
and thus the proof follows from (8.18).
\[
\square
\]
**Lemma 9.14.** If $X(t) = (X_1(t), \ldots, X_d(t)), t \in T$ is a centered $\mathbb{R}^d$-valued Gaussian rf and let $(Y, X(t)), t \in T$ be jointly Gaussian defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The law of $X$ under the probability measure $\mathbb{P}_v(A) = \mathbb{E}\{e^{-v/2}(A)\}, A \in \mathcal{F}$ is the same as that of $X + C$ under $\mathbb{P}$, where $C(t) = (\text{Cov}(X_1(t), Y), \ldots, \text{Cov}(X_d(t), Y)), t \in T$.
\begin{proof}
Given $t_1, \ldots, t_n \in T$ we calculate the df $W = (X(t_1), \ldots, X(t_n))$ under $\hat{\mathbb{P}}_v$ by exponential tilting of $(X(t_1), \ldots, X(t_n))$ with respect to $Y$. It is well-known that exponential tilting of multivariate Gaussian df is again a multivariate Gaussian, see for instance [2, Lem 6.1]. The only thing that changes under the exponential tilting is the trend which can be calculated as in the aforementioned lemma for each component separately.
\end{proof}

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