UNIPOTENT REPRESENTATIONS
AS A CATEGORICAL CENTRE

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INTRODUCTION

0.1. Let $k$ be an algebraic closure of the finite field $\mathbf{F}_p$ with $p$ elements. For any power $q$ of $p$ let $\mathbf{F}_q$ be the subfield of $k$ with $q$ elements. Let $G$ be a reductive connected group over $k$, assumed to be adjoint. Let $\mathcal{B}$ be the variety of Borel subgroups of $G$.

Let $W$ be the Weyl group of $G$ and let $c$ be a two-sided cell of $W$. Let $s \in \mathbb{Z}_{>0}$ and let $F: G \to G$ be the Frobenius map for an $\mathbf{F}_p$-rational structure on $G$. Let $G^F = \{ g \in G; F(g) = g \}$, a finite group. Let $\text{Rep}^\bullet(G^F)$ (resp. $\text{Rep}^c(G^F)$) be the category of representations of $G^F$ over $\overline{\mathbf{Q}}_l$ which are finite direct sums of unipotent representations in the sense of [DL] (resp. of unipotent representations whose associated two-sided cell (see 1.3) is $c$); here $l$ is a fixed prime number invertible in $k$.

In the rest of this subsection we assume for simplicity that the $\mathbf{F}_p$-rational structure on $G$ is split. The simple objects of $\text{Rep}^c(G^F)$ were classified in [L1]. The classification turns out to be the same as that [L4] of unipotent character sheaves on $G$ whose associated two-sided cell is $c$. The fact that

(a) these two classification problems have the same solution

has not until now been adequately explained.

In [L12] we have shown that the category of perverse sheaves on $G$ which are direct sums of unipotent character sheaves with associated two-sided cell $c$ is naturally equivalent to the centre of a certain monoidal category $\mathcal{C}^c\mathcal{B}^2$ of sheaves on $\mathcal{B}^2$ introduced in [L9] for which the induced ring structure on the Grothendieck group is the $J$-ring attached to $c$, see [L10, 18.3]. (The analogous statement for $D$-modules on a reductive group over $\mathbf{C}$ was proved earlier in a quite different way in [BFO].) In this paper we show that $\text{Rep}^c(G^F)$ is also naturally equivalent to the centre of $\mathcal{C}^c\mathcal{B}^2$ (see 6.3). This implies in particular that the simple objects of $\text{Rep}^c(G^F)$ are naturally in bijection with the unipotent character sheaves with associated two-sided cell $c$, which explains (a). It also implies that the set of simple objects $\text{Rep}^c(G^F)$ is “independent” of the choice of $s$; in fact, as we show in 7.1,
it is also independent of the characteristic of \( k \). It follows that to classify the unipotent representations of \( G^F \) it is enough to classify the unipotent character sheaves on \( G \) in sufficiently large characteristic; for the latter classification one can use the scheme of [L11] which uses the unipotent support of a character sheaf.

The methods of this paper are extensions of those of [L12]. We replace \( \text{Rep}^c G^F \) by an equivalent category consisting of certain \( G \)-equivariant perverse sheaves on \( G_s \), the set of all Frobenius maps \( G \to G \) corresponding to split \( F_p \)-rational structures on \( G \); we view \( G_s \) as an algebraic variety in a natural way. We construct functors \( \chi_s, \zeta_s \) between this category and the category \( C^c B^2 \) which are \( q \)-analogues of the truncated induction and truncated restriction \( \chi, \zeta \) of [L12] and we show that most properties of \( \chi, \zeta \) are preserved. We also define a truncated convolution product from our sheaves on \( G_s \) and on \( G_{s'} \) to our sheaves on \( G_{s+s'} \) which is analogous to the truncated convolution of character sheaves in [L12]; we also give a meaning for this even when \( s, s' \) are arbitrary integers. The main application of this truncated convolution product is in the case where \( s' = -s \), the result of the product being a direct sum of character sheaves on \( G \); this is used in the proof of a weak form of an adjunction formula between \( \chi_s, \zeta_s \) which is then used to prove the main result (Theorem 6.3).

0.2. In this paper we also prove extensions of the results in 0.1 to the case where \( F : G \to G \) is the Frobenius map of a nonsplit \( F_p \)-rational structure. In this case the role of unipotent character sheaves on \( G \) is taken by the unipotent character sheaves on a connected component of the group of automorphism group of \( G \). Moreover, in this case the centre of \( C^c B^2 \) is replaced by a slight generalization of the centre (the \( \epsilon \)-centre) which depends on the connected component above.

Many arguments in this paper are very similar to arguments in [L12] and are often replaced by references to the corresponding arguments in [L12].

Our results can be extended to non-unipotent representations and non-unipotent character sheaves; this will be discussed elsewhere.

0.3. Notation. We assume that we are given a split \( F_p \)-rational structure on \( G \) with Frobenius map \( F_0 : G \to G \). Let \( \nu = \dim B, \Delta = \dim(G), \rho = \text{rk}(G) \). We shall view \( W \) as an indexing set for the orbits of \( G \) acting on \( B^2 := B \times B \) by simultaneous conjugation; let \( O_w \) be the orbit corresponding to \( w \in W \) and let \( \overline{O_w} \) be the closure of \( O_w \) in \( B^2 \). For \( w \in W \) we set \( |w| = \dim \overline{O_w} - \nu \) (the length of \( w \)). Let \( w_{\text{max}} \) be the unique element of \( W \) such that \( |w_{\text{max}}| = \nu \).

As in [L1, 3.1], we say that an automorphism \( \epsilon : W \to W \) is ordinary if it leaves stable the set \( \{ s \in W ; |s| = 1 \} \) and for any two elements \( s \neq s' \) in that set which are in the same orbit of \( \epsilon \), the product \( ss' \) has order \( \leq 3 \). Let \( \mathfrak{A} \) be the group of ordinary automorphisms of \( W \).

For \( B \in B \), let \( U_B \) be the unipotent radical of \( B \). Then \( B/U_B \) is independent of \( B \); it is “the” maximal torus \( T \) of \( G \). Let \( \mathcal{X} \) be the group of characters of \( T \).

Let \( \text{Rep} W \) be the category of finite dimensional representations of \( W \) over \( \mathbb{Q} \); let \( \text{Irr} W \) be a set of representatives for the isomorphism classes of irreducible objects.
of $\text{Rep} W$.

The notation $\mathcal{D}(X), \mathcal{M}(X), \mathcal{D}_m(X), \mathcal{M}_m(X)$ is as in [L12, 0.2]. (When $X$ is $G, B, \mathcal{O}_w$ or $\mathcal{O}_v$, the subscript $m$ refers to the $\mathbb{F}_q$-structure defined by $F_0^{s_0}$ for a sufficiently large $s_0 > 0$.) For $K \in \mathcal{D}(X)$, $H^i K$, $H^i_x K$, $K[[m]] = K[n](n/2)$, $\mathcal{D}(K)$ are as in [L12, 0.2]. For $K \in \mathcal{M}_m(X)$, $gr_j K$ is as in [L12, 0.2]. For $K \in \mathcal{D}_m(X)$, $K^{(i)} = gr_i(K^i)/(i/2)$, is as in [L12, 0.2].

If $K \in \mathcal{M}(X)$ and $A$ is a simple object of $\mathcal{M}(X)$ we denote by $(A : K)$ the multiplicity of $A$ in a Jordan-Hölder series of $K$. The notation $C \simeq \{C_i; i \in I\}$ is as in [L12, 0.2].

If $X, X'$ are algebraic varieties over $k$, we say that a map of sets $f : X \to X'$ is a quasi-morphism if for some $\mathbb{F}_q$-rational structure on $X$ and $X'$ with Frobenius maps $F$ and $F'$ and some integer $t \geq 0$, $F^t : X \to X'$ is a morphism equal to $F'^t f$. If, in addition, $fF = F'f$ then we have well defined functors $f_1 : \mathcal{D}_m(X) \to \mathcal{D}_m(X')$, $f^* : \mathcal{D}_m(X') \to \mathcal{D}_m(X)$ such that $f_1$ is the composition of usual functors $(F^t)_!((F^t)'_!) = (F'^t)_!(F'^t f)_!$ and $f^*$ is the composition of usual functors $(F^t)_!(F^t)' = (F^t)'(F'^t)_!$. The usual properties of $f_1, f^*$ for morphisms continue to hold for quasi-morphisms.

We will denote by $p$ the variety consisting of one point. For any variety $X$ let $\mathcal{L}_X = \alpha_! \mathcal{Q}_t \in \mathcal{D}_m X$ where $\alpha : X \times T \to X$ is the obvious projection. We sometimes write $\mathcal{L}$ instead of $\mathcal{L}_X$.

Let $v$ be an indeterminate. For any $\phi \in \mathcal{Q}[v, v^{-1}]$ and any $k \in \mathbb{Z}$ we write $(k; \phi)$ for the coefficient of $v^k$ in $\phi$. Let $A = \mathbb{Z}[v, v^{-1}]$.

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**1. Truncated induction**

1.1. For $y \in W$ let $L_y \in \mathcal{D}_m(B^2)$ be the constructible sheaf which is $\mathcal{Q}_t$ (with the standard mixed structure of pure weight 0) on $\mathcal{O}_y$ and is 0 on $B^2 - \mathcal{O}_y$; let $L^*_y \in \mathcal{D}_m(B^2)$ be its extension to an intersection cohomology complex of $\mathcal{O}_y$ (equal to 0 on $B^2 - \mathcal{O}_y$). Let $L_y = L^*_y[[|y| + v]] \in \mathcal{D}_m(B^2)$.

Let $r \geq 1$. For $w = (w_1, w_2, \ldots, w_r) \in W^r$ we set $|w| = |w_1| + \cdots + |w_r|$. Let $L^*[1, r]_w \in \mathcal{D}_m(B^r+1)$ be as in [L12, 1.1]. For any $J \subset [1, r]$ let $L^J_w \in \mathcal{D}_m(B^r+1)$, $L^J_w \in \mathcal{D}_m(B^r+1)$ be as in [L12, 1.1]. As in [L12, 1.1(a)], we have a distinguished triangle

$$(a) \quad (L^J_w, L^*[1, r]_w, L^J_w)$$
in \( D_m(B^{r+1}) \). For any \( i < i' \) in \([1, r]\) let \( p_{i,i'} : B^{r+1} \to B^2 \) be the projection to the 
\( i, i' \) factors. For \( 1^L, 2^L, \ldots, r^L \) in \( D_m(B^2) \) we set

\[
1^L \bullet 2^L \bullet \cdots \bullet r^L = p_{0!*}(p_{01}^* 1^L \otimes p_{12}^* 2^L \otimes \cdots \otimes p_{r-1,r}^* r^L) \in D_m(B^2).
\]

1.2. Let \( H \) be the free \( A \)-module with basis \( \{T_w; w \in W\} \). It is well known that 
\( H \) has a unique structure of associative \( A \)-algebra with \( 1 = T_1 \) (Hecke algebra)

\( \) such that \( T_wT_w = T_{ww'} \) if \( w, w' \in W \), \( |ww'| = |w| + |w'| \) and \( T_s^2 = 1 + (v-v^{-1})T_s \)

\( \) if \( s \in W; \) \( |s| = 1 \). Let \( \{c_w; w \in W\} \) be the “new” basis of \( H \) defined as in [L10,

\( \) 5.2] with \( L(w) = |w| \).

\( \) For \( x, y \in W \), the relations \( x \leq y, x \sim y \) \( \) and \( x \sim_L y \) on \( W \) are defined as in [L12,

\( \) 1.3]. If \( c \) is a two-sided cell of \( W \) and \( w \in W \), the relations \( w \leq c, c \leq w, w \prec c, \)

\( \) \( c \prec w \) are defined as in [L12, 1.3]. If \( c, c' \) are two-sided cells of \( W \), the relations 
\( \) \( c \leq c', c \prec c' \) are defined as in [L12, 1.3]. Let \( a : W \to N \) be the \( a \)-function in

\( \) [L10, 13.6]. If \( c \) is a two-sided cell of \( W \), then for all \( w \in c \) we have \( a(w) = a(c) \)

\( \) where \( a(c) \) is a constant.

\( \) Let \( J \) be the free \( Z \)-module with basis \( \{t_z; z \in W\} \) with the structure of associative 
\( \) ring (with 1) as in [L12, 1.3]. For a two-sided cell \( c \) of \( W \) let \( J^c \) be the subgroup of \( J \) 
\( \) generated by \( \{t_z; z \in c\} \); it is a subring of \( J \) with unit element \( \sum_{d \in D_c} t_d \) where \( \)
\( \) \( D_c \) is the set of distinguished involutions of \( c \). We have \( J = \bigoplus_c J^c \)
\( \) as rings.

\( \) For \( E \in \text{Irr} W \) we define a simple \( Q \otimes J \)-module \( E_\infty \) and a simple \( Q(v) \otimes A \)
\( \) \( H \)-module \( E(v) \) as in [L12, 1.3]; there is a unique two-sided cell \( c_E \) of \( W \) such that 
\( \) \( J^c \cap E_\infty \neq 0 \).

\( \) Let \( \epsilon \in \mathfrak{a} \). Let \( E \in \text{Irr} W \). We say that \( E \in \text{Irr}_\epsilon W \) if \( \text{tr}(\epsilon(w), E) = \text{tr}(w, E) \)
\( \) for any \( w \in W \). In this case there exists a linear transformation of finite order 
\( \) \( \epsilon_E : E \to E \) such that \( \epsilon_E w \epsilon_E^{-1} = \epsilon(w) : E \to E \) for any \( w \in W \); moreover \( \)
\( \) \( \epsilon_E \) is unique up to multiplication by \(-1\). See [L1, 3.2]]. For each \( E \in \text{Irr}_\epsilon W \) we 
\( \) choose \( \epsilon_E \) as above. As a \( Q \)-vector space we have \( E_\infty = E, E(v) = Q(v) \otimes Q \)
\( \) \( E \) hence, if \( E \in \text{Irr}_\epsilon W, \) \( \tilde{\epsilon} : E \to E \) can be viewed as a \( Q \)-linear map (of finite order) 
\( \) \( \tilde{\epsilon} : E_\infty \to E_\infty \) and as a \( Q(v) \)-linear map (of finite order) \( \tilde{\epsilon} : E(v) \to E(v) \). From the 
\( \) definitions we see that \( \tilde{\epsilon} t_w \epsilon^{-1} = t_{\epsilon(w)} : E_\infty \to E_\infty \) and \( \epsilon \tilde{\epsilon} t_w \epsilon^{-1} = T_{\epsilon(w)} : E(v) \to E(v) \) for any \( w \in W \).

\( \) If \( E \in \text{Irr}_\epsilon W \) then \( \epsilon(c_E) = c_E \). Let \( \text{Irr}_{\epsilon,c} W = \{E \in \text{Irr}_\epsilon W; c_E = c\} \).

1.3. For any \( \epsilon \in \mathfrak{a}, s \in \mathbb{Z} \) let \( G_{\epsilon,s} \) be the set of bijections \( F : G \to G \) such that

\( \) (i) if \( s > 0 \) then \( F \) is the Frobenius map for an \( F_{p^s} \)-rational structure on \( G \);

\( \) (ii) if \( s < 0 \) then \( F^{-1} \) is the Frobenius map for an \( F_{p^{-s}} \)-rational structure on \( G \);

\( \) (iii) if \( s = 0 \) then \( F \) is an automorphism of \( G \);

\( \) moreover in each case (i)–(iii) we require that the following holds: for any \( w \in W \) and any \( (B, B') \in O_w \) we have \( (F(B), F(B')) \in O_{\epsilon(w)} \).

\( \) (If \( \epsilon = 1, s = 0 \) we can identify \( G \) and \( G_{\epsilon,s} \) by \( g \mapsto \text{Ad}(g) \).) Now \( G \) acts on \( G_{\epsilon,s} \) by 
\( \) \( g : F \to \text{Ad}(g) F \text{Ad}(g^{-1}) \). If \( s \neq 0 \), this action is transitive and the stabilizer of a
point \( F \in G_{\epsilon,s} \) is the finite group \( G^F = \{ g \in G; F(g) = g \} \). For any \( s \in \mathbb{Z} \) and any \( \tilde{F} \in G_{\epsilon,s} \), the maps \( \lambda : G \to G_{\epsilon,s}, g \mapsto \text{Ad}(g)\tilde{F} \) and \( \lambda' : G \to G_{\epsilon,s}, g \mapsto \tilde{F}\text{Ad}(g) \) are bijections (by Lang’s theorem); we use \( \lambda \) (resp. \( \lambda' \)) to view \( G_{\epsilon,s} \) with \( s \geq 0 \) (resp. \( s \leq 0 \)) as an affine algebraic variety isomorphic to \( G \); this algebraic variety structure on \( G_{\epsilon,s} \) is independent of the choice of \( \tilde{F} \). We have \( \dim G_{\epsilon,s} = \Delta \). The \( G \)-action above on \( G_{\epsilon,s} \) is an algebraic group action. When \( X = G_{\epsilon,s} \), then the subscript \( m \) in \( D_m(X), M_m(X) \) refers to the \( F_{p^{2m}} \)-structure with Frobenius map \( F \mapsto F_0^m FF_0^{-s_0} \) (with \( F_0, s_0 \) as in 0.3).

Note that \( \bigcup_{\epsilon, s} \mathbb{Z} G_{\epsilon,s} \) is a group under composition of maps: if \( F \in G_{\epsilon,s}, F'' \in G_{\epsilon',s' + s'} \) (it is enough to show that for some \( F' \in G_{\epsilon,s}, F' \in G_{\epsilon',s} \)) we have \( FF' \in G_{\epsilon',s + s'} \). We take \( F = \text{Ad}(\gamma)F_0^m, F' = \text{Ad}(\gamma')F_0^{s'} \) where \( \gamma \in G_{\epsilon,1} \) and \( \gamma' \in G_{\epsilon',1} \) commute with \( F_0 \); then \( FF' = \text{Ad}(\gamma\gamma')F_0^{s + s'} \) (note \( \gamma\gamma' \in G_{\epsilon',1} \) commutes with \( F_0 \) hence \( FF' \in G_{\epsilon',s + s'} \)). Note that the composition \( G_{\epsilon,s} \times G_{\epsilon',s'} \to G_{\epsilon',s + s'} \) is not in general a morphism of algebraic varieties but only a quasi-morphism (see 0.3), which is good enough for our purposes.

Until the end of Section 2 we fix \( \epsilon \in \mathfrak{A} \).

Let \( s \in \mathbb{Z} \). We consider the maps \( \mathfrak{B}^2 \xrightarrow{\iota} X_{\epsilon,s} \xrightarrow{\pi} G_{\epsilon,s} \) where

\[
X_{\epsilon,s} = \{(B, B', F) \in \mathfrak{B} \times \mathfrak{B} \times G_{\epsilon,s}; F(B) = B'\},
\]

\[
f(B, B', F) = (B, B'), \pi(B, B', F) = F.
\]

Now \( L \mapsto \chi_{\epsilon,s}(L) = \pi!f^*L \) defines a functor \( D_m(\mathfrak{B}^2) \to D_m(G_{\epsilon,s}) \). (When \( \epsilon = 1, s = 0 \), \( c_{\epsilon,s} \) coincides with the functor \( \chi \) defined in [L12, 1.5]). For \( i \in \mathbb{Z}, L \in D_m(\mathfrak{B}^2) \) we write \( \chi^i_{\epsilon,s}(L) \) instead of \( (\chi_{\epsilon,s}(L))^i \). For any \( z \in W \) we set \( R_{\epsilon,s,z} = \chi_{\epsilon,s}(L^z) \in D_m(G_{\epsilon,s}) \). (When \( \epsilon = 1, s = 0 \) this is the same as \( R_z \) in [L12, 1.5].)

Let \( b : G_{\epsilon-1,-s} \xrightarrow{\sim} G_{\epsilon,s} \) be the isomorphism \( F \mapsto F^{-1} \) and let \( b' : \mathfrak{B}^2 \xrightarrow{\sim} \mathfrak{B}^2 \) be the isomorphism \( (B, B') \mapsto (B', B) \). From the definitions we see that for \( L \in D_m(\mathfrak{B}^2) \) we have \( \chi_{\epsilon,s}(b'(L)) = b_1\chi_{\epsilon-1,-s}(L) \).

Let \( CS(G_{\epsilon,s}) \) be a set of representatives for the isomorphism classes of simple perverse sheaves \( A \in \mathcal{M}(G_{\epsilon,s}) \) such that \( (A : R_{\epsilon,s,z}^j) \neq 0 \) for some \( z \in W, j \in \mathbb{Z} \). (When \( \epsilon = 1, s = 0 \) this agrees with the definition of \( CS(G) \) in [L12, 1.5].) Now let \( A \in CS(G_{\epsilon,s}) \). We associate to \( A \) a two-sided cell \( c_A \) as follows.

Assume first that \( s \neq 0 \). Since \( A \) is \( G \)-equivariant and the conjugation action of \( G \) on \( G_{\epsilon,s} \) is transitive, for any \( F \in G_{\epsilon,s} \) we have \( A|_{\{F\}} = r_{A,F}[\Delta] \) where \( r_{A,F} \) is an irreducible \( G^F \)-module. From the definitions, for any \( z \in W \) and any \( F \in G_{\epsilon,s} \) we have

\[
(A : R_{s,z}^j) = (r_{A,F} : IH^{j-\Delta}\{(B; (B, FB) \in \mathcal{O}_z)\})_{G^F}
\]

where the right hand side is the multiplicity of \( r_{A,F} \) in the \( G^F \)-module

\[
IH^{j-\Delta}\{(B; (B, FB) \in \mathcal{O}_z)\};
\]
here \( IH \) denotes intersection cohomology with coefficients in \( \mathbb{Q}_l \). In particular, \( r_{A,F} \) is a unipotent representation of \( G^F \). By [L1, 3.8], for any \( A \in CS(G_{\epsilon,s}) \), any \( F \in G_{\epsilon,s} \), any \( z \in W \) and any \( j \in \mathbb{Z} \) we have

\[
(r_{A,F} : IH^{j,\Delta} \{ (B; (B, FB) \in \mathcal{O}_z) \})_{G^F} = (j - \Delta - |z|; (-1)^{j,\Delta} \sum_{E \in \text{Irr}_W} c_{A,E,\epsilon} \text{tr}(\tilde{e}c_z, E(v)))
\]

or equivalently

\[ (a) \quad (A : R^j_{\epsilon,s,z}) = (j - \Delta - |z|; (-1)^{j,\Delta} \sum_{E \in \text{Irr}_W} c_{A,E,\epsilon} \text{tr}(\tilde{e}c_z, E(v))) \]

where \( c_{A,E,\epsilon} \) are uniquely defined rational numbers; now (a) also holds when \( s = 0 \), see [L12, 1.5(a)] when \( \epsilon = 1 \) and [L6, 34.19, 35.22], [L8, 44.7(e)] for general \( \epsilon \). Moreover, if \( s \neq 0 \) then, by [L1, 6.17], given \( A \) as above, there is a unique two-sided cell \( c_A \) of \( W \) such that \( \epsilon(c_A) = c_A \) and \( c_{A,E,\epsilon} = 0 \) whenever \( E \in \text{Irr}_W \) satisfies \( c_E \neq c_A \). The same holds when \( s = 0 \), see [L12, 1.5] when \( \epsilon = 1 \) and [L7, §41] for general \( \epsilon \).

When \( s \neq 0 \), \( c_A \) differs from the two-sided cell associated to \( r_{A,F} \) in [L1, 4.23] by multiplication on the left or right by \( w_{\text{max}} \). Similarly, when \( s = 0 \), \( c_A \) differs from the two-sided cell associated to \( A \) in [L7, §41] by multiplication on the left or right by \( w_{\text{max}} \).

As in [L12, 1.5(b)], for \( s \in \mathbb{Z} \) we have

\[ (b) \quad (A : R^j_{\epsilon,s,z}) \neq 0 \text{ for some } z \in c_A, j \in \mathbb{Z} \text{ and conversely, if } (A : R^j_{\epsilon,s,z}) \neq 0 \text{ for } z \in W, j \in \mathbb{Z}, \text{ then } c_A \preceq z. \]

For \( s \in \mathbb{Z} \), \( A \in CS(G_{\epsilon,s}) \) let \( a_E \) be the value of the \( a \)-function on \( c_A \). If \( z \in W, E \in \text{Irr}_W \) satisfy \( \text{tr}(\tilde{e}c_z, E(v)) \neq 0 \) then \( c_E \preceq z \); if in addition we have \( z \in c_E \), then

\[
\text{tr}(\tilde{e}c_z, E(v)) = \gamma_{z,E,\epsilon} v^{a_E} + \text{lower powers of } v
\]

where \( \gamma_{z,E,\epsilon} \in \mathbb{Z} \) and \( a_E \) is the value of the \( a \)-function on \( c_E \). Hence from (a) we see that

\[ (c) \quad (A : R^j_{\epsilon,s,z}) = 0 \text{ unless } c_A \preceq z \text{ and, if } z \in c_A, \text{ then}
\]

\[
(A : R^j_{\epsilon,s,z}) = (-1)^{j,\Delta} (j - \Delta - |z|; \sum_{E \in \text{Irr}_W: c_E = c_A} c_{A,E,\epsilon} \gamma_{z,E,\epsilon} v^{a_A} + \text{lower powers of } v)
\]

which is 0 unless \( j - \Delta - |z| \leq a_A \).

In the remainder of this section we fix a two-sided cell \( c \) of \( W \) such that \( \epsilon(c) = c \); we set \( a = a(c) \).

For \( s \in \mathbb{Z} \) and \( Y = G_{\epsilon,s} \) or \( Y = B^2 \) let \( M \mathcal{Y} \) be the category of perverse sheaves
on $Y$ whose composition factors are all of the form $A \in CS(G_{\epsilon,s})$, when $Y = G_{\epsilon,s}$, or of the form $L_z$ with $z \in W$, when $Y = B^2$. Let $\mathcal{M}^\leq Y$ (resp. $\mathcal{M}^< Y$) be the category of perverse sheaves on $Y$ whose composition factors are all of the form $A \in CS(G_{\epsilon,s})$ with $c_A \leq c$ (resp. $c_A < c$), when $Y = G_{\epsilon,s}$, or of the form $L_z$ with $z \leq c$ (resp. $z < c$) when $Y = B^2$. Let $\mathcal{D}^\bullet Y$ (resp. $\mathcal{D}^\leq Y$ or $\mathcal{D}^< Y$) be the category of all $K \in \mathcal{D}(Y)$ such that $K_i \in \mathcal{M}^\bullet Y$ (resp. $K_i \in \mathcal{M}^\leq Y$ or $K_i \in \mathcal{M}^< Y$) for all $i \in \mathbb{Z}$. Let $\mathcal{M}_m^\bullet Y$ (or $\mathcal{M}_m^\leq Y$, or $\mathcal{M}_m^< Y$) be the category of all $K \in \mathcal{M}_m Y$ which are also in $\mathcal{M}^\bullet Y$ (or $\mathcal{M}^\leq Y$ or $\mathcal{M}^< Y$). Let $\mathcal{D}_m^\bullet Y$ (or $\mathcal{D}_m^\leq Y$, or $\mathcal{D}_m^< Y$) be the category of all $K \in \mathcal{D}_m Y$ which are also in $\mathcal{D}^\bullet Y$ (or $\mathcal{D}^\leq Y$ or $\mathcal{D}^< Y$). From (c) we deduce:

(d) If $z \leq c$, then $R^{\bullet}_{\epsilon,s,z} \in \mathcal{M}^\leq G_{\epsilon,s}$ for all $j \in \mathbb{Z}$. If $z \in c$ and $j > a + \Delta + |z|$, then $R^{\bullet}_{\epsilon,s,z} \in \mathcal{M}^< G_{\epsilon,s}$. If $z < c$ then $R^{\bullet}_{\epsilon,s,z} \in \mathcal{M}^< G_{\epsilon,s}$ for all $j \in \mathbb{Z}$.

**Lemma 1.4.** Let $s \in \mathbb{Z}$. Let $r \geq 1$, $J \subset [1, r]$, $J \neq \emptyset$ and $w = (w_1, w_2, \ldots, w_r) \in W^r$. Let $\mathfrak{c} = \Delta + ra$.

(a) Assume that $w_i \in c$ for some $i \in [1, r]$. If $j \in \mathbb{Z}$ (resp. $j > \mathfrak{c}$) then $\chi^j_{\epsilon,s}(p_{0 \mathfrak{c}} L^J_w[[w]])$ is in $\mathcal{M}^\leq G_{\epsilon,s}$ (resp. $\mathcal{M}^< G_{\epsilon,s}$).

(b) Assume that $w_i \in c$ for some $i \in J$. If $j \in \mathbb{Z}$ (resp. $j > \mathfrak{c}$) then $\chi^j_{\epsilon,s}(p_{0 \mathfrak{c}} L^J_w[[w]])$ is in $\mathcal{M}^\leq G_{\epsilon,s}$ (resp. $\mathcal{M}^< G_{\epsilon,s}$).

(c) Assume that $w_i \in c$ for some $i \in J$. If $j \geq \mathfrak{c}$ then the cokernel of the map $\chi^j_{\epsilon,s}(p_{0 \mathfrak{c}} L^J_w[[w]]) \to \chi^j_{\epsilon,s}(p_{0 \mathfrak{c}} L^J_w[[w]])$ associated to 1.1(a) is in $\mathcal{M}^< G_{\epsilon,s}$.

(d) Assume that $w_i \in c$ for some $i \in J$. If $j \in \mathbb{Z}$ (resp. $j > \mathfrak{c}$) then $\chi^j_{\epsilon,s}(p_{0 \mathfrak{c}} L^J_w[[w]])$ is in $\mathcal{M}^\leq G_{\epsilon,s}$ (resp. $\mathcal{M}^< G_{\epsilon,s}$).

(e) Assume that $w_i < c$ for some $i \in J$. If $j \in \mathbb{Z}$ then $\chi^j_{\epsilon,s}(p_{0 \mathfrak{c}} L^J_w[[w]])$ is in $\mathcal{M}^\leq G_{\epsilon,s}$ and $\chi^j_{\epsilon,s}(p_{0 \mathfrak{c}} L^J_w[[w]])$ is in $\mathcal{M}^< G_{\epsilon,s}$.

When $\epsilon = 1, s = 0$ this is just [L12, 1.6]; the proof in the general case is entirely similar (it uses 1.3(b), 1.3(c)).

**1.5.** Let $s \in \mathbb{Z}$. Let $CS_{\epsilon,s,c} = \{ A \in CS(G_{\epsilon,s}); c_A = c \}$. For any $z \in c$ we set $n_z = a + \Delta + |z|$. Let $A \in CS_{\epsilon,s,c}$ and let $z \in c$. We have:

(a) $$(A : R_{s,z}^{n_z}) = (-1)^{a + |z|} \sum_{E \in \text{Irr}_{\epsilon,c} W} c_{A,E,\bar{\epsilon}} \text{tr}(\bar{e}t_z, E_{\infty}).$$

When $\epsilon = 1, s = 0$ this is just [L12, 1.7(a)]. In the general case, from 1.3(a) we have

$$(A : R_{s,z}^{n_z}) = (-1)^{a + |z|} \sum_{E \in \text{Irr}_{\epsilon,c} W} c_{A,E,\bar{\epsilon}} (a; \text{tr}(\bar{e}c_z, E(v)))$$

and it remains to use that $(a; \text{tr}(\bar{e}c_z, E(v))$ is equal to $\text{tr}(\bar{e}t_z, E_{\infty})$ if $E \in \text{Irr}_{\epsilon,c} W$ and to 0, otherwise. We have:

(b) For any $A \in CS_{\epsilon,s,c}$ there exists $z \in c$ such that $(A : R_{s,z}^{n_z}) \neq 0$.

The proof, based on (a), is the same as that in the case $\epsilon = 1, s = 0$ given in [L12, 1.7(b)].
Let $c^0 = \{z \in c; z \sim_L \epsilon(z^{-1})\}$. If $z \in c - c^0$ and $E \in \text{Irr}_{\epsilon,c}W$, then $\text{tr}(\hat{e}t_z, E_\infty) = 0$. (We can write $E_\infty = \bigoplus_{d \in D_c} t_d E_\infty$ and $\hat{e}t_z : E_\infty \to E_\infty$ maps the summand $t_d E_\infty$ (where $z \sim_L d$) into $t_{d'} E_\infty$ (where $d' \in D_c$, $d' \sim_L z^{-1}$) and all other summands to 0. If $\text{tr}(\hat{e}t_z, E_\infty) \neq 0$, we must have $t_d E_\infty = t_{\epsilon(d')} E_\infty \neq 0$ and $d = \epsilon(d')$ and $z \sim_L \epsilon(z^{-1})$.) From this and (a) we deduce

(c) If $z \in c - c^0$, then $R_{\epsilon,s,z}^c = 0$.

1.6. Let $s \in \mathbb{Z}$. For $Y = G_{\epsilon,s}$ or $B^2$ let $\mathcal{C} Y$ be the subcategory of $\mathcal{M} \otimes Y$ consisting of semisimple objects; let $\mathcal{C}_Y^0$ be the subcategory of $\mathcal{M}_m Y$ consisting of those $K \in \mathcal{M}_m Y$ such that $K$ is pure of weight 0 and such that as an object of $\mathcal{M}(Y)$, $K$ belongs to $\mathcal{C} Y$. Let $\mathcal{C}^c Y$ be the subcategory of $\mathcal{M} \otimes Y$ consisting of objects which are direct sums of objects in $CS_{\epsilon,s,c}$ (if $Y = G_{\epsilon,s}$) or of the form $L_z$ with $z \in c$ (if $Y = B^2$). Let $\mathcal{C}_Y^0$ be the subcategory of $\mathcal{C}_0 Y$ consisting of those $K \in \mathcal{C}_0 Y$ such that as an object of $\mathcal{C} Y$, $K$ belongs to $\mathcal{C}^c Y$. For $K \in \mathcal{C} Y$, let $K$ be the largest subobject of $K$ such that, as an object of $\mathcal{C} Y$, we have $K \in \mathcal{C}^c Y$.

For $L \in \mathcal{C}_0 \otimes B^2$ we define $\epsilon \in \mathcal{C}_0 \otimes B^2$ as follows. We have canonically $L = \bigoplus_{y \in W} V_y \otimes L_y$ where $V_y$ are finite dimensional $\bar{Q}_l$-vector spaces; we set $\epsilon L = \bigoplus_{y \in W} V_y \otimes L_{\epsilon^{-1}(y)}$. We show:

(a) Let $s \in \mathbb{N}$. Define $u : G_{\epsilon,s} \times B^2 \to G_{\epsilon,s} \times B^2$ by

$$(F, (B_1, B_2)) \mapsto (F, F(B_1), F(B_2))$$

and let $L \in \mathcal{C}_0 \otimes B^2$. We have canonically $u^*(\bar{Q}_l \otimes L) = \bar{Q}_l \otimes \epsilon L$.

We can assume that $L = L_y$ where $y \in W$; we must show that $u^*(\bar{Q}_l \otimes L_y) = \bar{Q}_l \otimes L_{\epsilon^{-1}(y)}$ or that $u^*(\bar{Q}_l \otimes L_y^s) = \bar{Q}_l \otimes L_{\epsilon^{-1}(y)}^s$. Now $\bar{Q}_l \otimes L_y^s$ is the intersection cohomology complex of $G_{\epsilon,s} \otimes \bar{O}_y$ with coefficients in $\bar{Q}_l$ (extended by 0 on $G_{\epsilon,s} \times (B^2 - \bar{O}_y)$). Hence $u^*(\bar{Q}_l \otimes L_y^s)$ is the intersection cohomology complex of $u^{-1}(G_{\epsilon,s} \otimes \bar{O}_y)$ with coefficients in $\bar{Q}_l$ (extended by 0 on $G_{\epsilon,s} \times u^{-1}(B^2 - \bar{O}_y)$). That is, the intersection cohomology complex of $G_{\epsilon,s} \times \bar{O}_{\epsilon^{-1}(y)}$ (extended by 0 on $G_{\epsilon,s} \times (B^2 - \bar{O}_{\epsilon^{-1}(y)}))$. This is $\bar{Q}_l \otimes L_{\epsilon^{-1}(y)}^s$, as required.

Assume that $s \in \mathbb{Z}_{>0}$ and let $F \in G_{\epsilon,s}$. For any $A \in \mathcal{C} \otimes G_{\epsilon,s}$ we have $A|_{\{F\}} = r_{A,F}[\Delta]$ where $r_{A,F} \in \text{Rep}^{\epsilon}(G^F)$ (see 0.1). Moreover, from the definitions we see that

(b) $A \mapsto r_{A,F}$ is an equivalence of categories $\mathcal{C} G_{\epsilon,s} \sim \text{Rep}^\epsilon(G^F)$ (see 0.1).

Proposition 1.7. Let $s \in \mathbb{Z}$.

(a) If $L \in \mathcal{D} \otimes B^2$ then $\chi_{\epsilon,s}(L) \in \mathcal{D} \otimes G_{\epsilon,s}$. If $L \in \mathcal{D} \otimes B^2$, then $\chi_{\epsilon,s}(L) \in \mathcal{D} \otimes G_{\epsilon,s}$.

(b) If $L \in \mathcal{M} \otimes B^2$ and $j > a + \nu + \rho$ then $\chi_{\epsilon,s}^j(L) \in \mathcal{M} \otimes G_{\epsilon,s}$.

When $\epsilon = 1, s = 0$ this is just [L12, 1.9]; the proof in the general case is entirely similar (it uses 1.4(a), (e)).
1.8. Let \( s \in \mathbb{Z} \). For \( L \in C_0^cB^2 \) we set
\[
\chi_{\epsilon,s}(L) = (\chi_{\epsilon,s}^a(L)(a+\nu+\rho)/2) = (\chi_{\epsilon,s}(L))^{(a+\nu+\rho)} \in C^cG_{\epsilon,s}.
\]
The functor \( \chi_{\epsilon,s} : C_0^cB^2 \to C^cG_{\epsilon,s} \) is called truncated induction. For \( z \in c \) we have
\[
(a) \quad \chi_{\epsilon,s}(L_z) = R_{\epsilon,s,z}^n(n_z/2).
\]
When \( \epsilon = 1, s = 0 \) this is just [L12, 1.10(a)]; the proof in the general case is entirely similar.

We shall denote by \( \tau : J^c \to \mathbb{Z} \) the group homomorphism such that \( \tau(t_z) = 1 \) if \( z \in D_c \) and \( \tau(t_z) = 0 \), otherwise. For \( z, u \in c \) we have:
\[
(b) \quad \dim \text{Hom}_{cG_{\epsilon,s}}(\chi_{\epsilon,s}(L_z), \chi_{\epsilon,s}(L_u)) = \sum_{y \in c} \tau(t_{y-1}t_zt_{y}(y)t_{u-1}).
\]

When \( \epsilon = 1, s = 0 \) this is just [L12, 1.10(b)]. We now consider the general case.

Using (a) and the definitions we see that the left hand side of (b) equals
\[
\sum_{A \in \text{CS}_{\epsilon,s,c}} (A : R_{\epsilon,s,z}^n)(A : R_{\epsilon,s,u}^n),
\]
hence, using 1.5(a) it equals
\[
\sum_{E, E' \in \text{Irr}_{\epsilon,c}W} (-1)^{|z|+|u|} \sum_{A \in \text{CS}_{\epsilon,s,c}} c_{A,E',E} c_{A,E,E'} \text{tr}(\hat{e}t_z, E_\infty) \text{tr}(\hat{e}t_u, E'_\infty).
\]
Replacing in the last sum \( \sum_{A \in \text{CS}_{\epsilon,s,c}} c_{A,E',E} c_{A,E,E'} \) by 1 if \( E = E' \) and by 0 if \( E \neq E' \) (see [L1, 3.9] in the case \( s \neq 0 \) and [L3, 13.12], [L6, 35.18(g)] in the case \( s = 0 \)) we obtain
\[
\sum_{E \in \text{Irr}_{\epsilon,c}W} (-1)^{|z|+|u|} \text{tr}(\hat{e}t_z, E_\infty) \text{tr}(\hat{e}t_u, E_\infty).
\]
This is equal to \( (-1)^{|z|+|u|} \) times the trace of the operator \( \xi \mapsto t_z\epsilon(\xi)t_{u-1} \) on \( Q \otimes J^c \) (see [L6, 34.14(a), 34.17]). The last trace is equal to the sum over \( y \in c \) of the coefficient of \( t_y \) in \( t_zt_{(y)}t_{u-1} \); this coefficient is equal to \( \tau(t_{y-1}t_zt_{(y)}t_{u-1}) \) since for \( y, y' \in c \), \( \tau(t_{y}t_{y'}) \) is 1 if \( y' = y^{-1} \) and is 0 if \( y' \neq y^{-1} \) (see [L10, 20.1(b)]). Thus we have
\[
\dim \text{Hom}_{cG_{\epsilon,s}}(\chi_{\epsilon,s}(L_z), \chi_{\epsilon,s}(L_u)) = (-1)^{|u|+|z|} \sum_{y \in c} \tau(t_{y-1}t_zt_{(y)}t_{u-1}).
\]
Since \( \dim \text{Hom}_{cG_{\epsilon,s}}(\chi_{\epsilon,s}(L_z), \chi_{\epsilon,s}(L_u)) \in \mathbb{N} \) and \( \sum_{y \in c} \tau(t_{y-1}t_zt_{(y)}t_{u-1}) \in \mathbb{N} \), it follows that (b) holds.
Lemma 1.9. Let $s \in \mathbb{N}$. Let $Y_1, Y_2$ be among $G_{\epsilon,s}, B^2$ and let $X \in \mathcal{D}_{m}^{\leq} Y_1$. Let $c, c'$ be integers and let $\Phi : \mathcal{D}_{m}^{\leq} Y_1 \to \mathcal{D}_{m}^{\leq} Y_2$ be a functor which takes distinguished triangles to distinguished triangles, commutes with shifts, maps $\mathcal{D}_{m}^{\leq} Y_1$ into $\mathcal{D}_{m}^{\leq} Y_2$ and maps complexes of weight $\leq i$ to complexes of weight $\leq i$ (for any $i$). Assume that (a),(b) below hold:

(a) $(\Phi(X_0))^h \in \mathcal{M}_{m}^{\leq} Y_2$ for any $X_0 \in \mathcal{M}_{m}^{\leq} Y_1$ and any $h > c$;

(b) $X$ has weight $\leq 0$ and $X^i \in \mathcal{M}^{\leq} Y_1$ for any $i > c'$.

Then

(c) $(\Phi(X))^j \in \mathcal{M}^{\leq} Y_2$ for any $j > c + c'$,

and we have canonically

(d) $(\Phi(X)^{\{c\}})^{\{c\}} = (\Phi(X))^{\{c+c'\}}$.

When $\epsilon = 1, s = 0$ this is just [L12, 1.12]; the proof in the general case is entirely similar.

1.10. Let $s \in \mathbb{Z}$. Let $L \in \mathcal{C}_0^{c} B^2$. We have $\mathcal{D}(L) \in \mathcal{C}_0^{c} B^2$. Moreover we have canonically:

(a) $\chi_{\epsilon,s}(\mathcal{D}(L)) = \mathcal{D}(\chi_{\epsilon,s}(L))$.

When $\epsilon = 1, s = 0$ this is just [L12, 1.13]; the proof in the general case is entirely similar.

2. TRUNCATED RESTRICTION

2.1. Recall that $\epsilon \in \mathfrak{A}$ is fixed. In this section we fix $s \in \mathbb{Z}$. Let $\pi, f$ be as in 1.3. Now $K \mapsto \zeta_{\epsilon,s}(K) = f_! \pi^* K$ defines a functor $\mathcal{D}_{m}(G_{\epsilon,s}) \to \mathcal{D}_{m}(B^2)$. (When $\epsilon = 1, s = 0$, $\zeta_{\epsilon,s}$ is the same as $\zeta$ of [L12, 2.5].) For $i \in \mathbb{Z}, K \in \mathcal{D}_{m}(G_{\epsilon,s})$ we write $\zeta_{\epsilon,s}^i(K)$ instead of $(\zeta_{\epsilon,s}(K))^i$.

Let $b : G_{\epsilon^{-1},-s} \to G_{\epsilon,s}, b' : B^2 \to B^2$ be as in 1.3. From the definitions we see that for $K \in \mathcal{D}_{m}(G_{\epsilon^{-1},-s})$ we have

(a) $\zeta_{\epsilon,s}(b_! K) = b'_! \zeta_{\epsilon^{-1},-s}(K)$. 

Proposition 2.2. For any $L \in \mathcal{D}_m(B^2)$ we have

(a) $\zeta_{\epsilon, s}(\chi_{\epsilon, s}(L)) \simeq \{ \oplus_{y \in W : |y| = k} L_y \cdot L \cdot L_{\epsilon(y) - 1} \otimes \mathcal{L}[[2k - 2\nu]]; k \in \mathbb{N}\};$

(b) $\zeta_{\epsilon, s}(\chi_{\epsilon, s}(L)) \simeq$

$\{ \oplus_{y \in W : |y| = k} L_y \cdot L \cdot L_{\epsilon(y) - 1} \otimes \mathcal{L}[[2k - 2\nu - 2\rho]] \otimes \Lambda^d \mathcal{X}[[d/(d/2)]; k \in \mathbb{N}, d \in [0, \rho]].\}$

where $\mathcal{L}, \mathcal{X}$ are as in 0.3.

When $\epsilon = 1, s = 0$ this is proved in [L12, 2.6]. The proof in the general case will be quite similar to that in the case $\epsilon = 1, s = 0.$ Let

$Y = \{(B_1, B_2, B_3, B_4, F) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B} \times \mathcal{B} \times G_s; F(B_1) = B_4, F(B_2) = B_3\}.$

For $ij = 14$ or $23$ we define $h'_{ij} : Y \to \chi_{\epsilon, s}$ by $(B_1, B_2, B_3, B_4, F) \mapsto (B_i, B_j, F)$ and $h_{ij} : Y \to B^2$ by $(B_1, B_2, B_3, B_4, F) \mapsto (B_i, B_j).$ We have $\pi^* \pi^! = h'_{14} h'_{23}^*$ hence

$\zeta_{\epsilon, s}(\chi_{\epsilon, s}(L)) = f_1^* \pi^* \pi^! f^*(L) = f_1^* h'_{14} h'_{23}^* f^*(L) = h_{14} h_{23}^* L.$

For $k \in \mathbb{N}$ let $Y^k = \bigcup_{y \in W : |y| = k} Y_y$ where

$Y_y = \{(B_1, B_2, B_3, B_4, F) \in Y; (B_1, B_2) \in \mathcal{O}_y, (B_3, B_4) \in \mathcal{O}_{\epsilon(y) - 1}\}$

and let $Y^s_k := \bigcup_{k' \geq k} Y^k.$ an open subset of $Y_s; let h_{ij}^k : Y^k \to B^2, h_{ij}^{\geq k} : Y^k \to B^2$ be the restrictions of $h_{ij}.$ For any $k \in \mathbb{N}$ we have a distinguished triangle

$(h_{14}^{k+1}, h_{23}^{k+1}, L), h_{14}^{k+1} h_{23}^{k+1} L, L_{14} h_{23}^{k+1} L).$

It follows that we have

$\zeta_{\epsilon, s}(\chi_{\epsilon, s}(L)) \simeq \{ h_{14}^k h_{23}^k L; k \in \mathbb{N}\}.$

For $k \in \mathbb{N}$ let $Z^k = \bigcup_{y \in W : |y| = k} Z_y$ where

$Z_y = \{(B_1, B_2, B_3, B_4) \in \mathcal{B}^4; (B_1, B_2) \in \mathcal{O}_y, (B_3, B_4) \in \mathcal{O}_{\epsilon(y) - 1}\};$

for $i, j \in [1, 4]$ we define $\tilde{h}_i^k : Z^k \to B^2$ and $\tilde{h}_j^k : Z^k \to B^2$ by $(B_1, B_2, B_3, B_4) \mapsto (B_i, B_j).$ We have an obvious morphism $u : Y^k \to Z^k.$ The fibre of $u$ at $(B_1, B_2, B_3, B_4) \in Z^k$ can be identified with the set of all $F \in G_{\epsilon, s}$ such that $F(B_1) = B_4, F(B_2) = B_3.$ Since $(B_1, B_2) \in \mathcal{O}_y, (B_3, B_4) \in \mathcal{O}_{\epsilon(y) - 1}$ for some
$y \in W$, we can find $\tilde{F} \in G_{\epsilon,s}$ such that $\tilde{F}(B_1) = B_4, \tilde{F}(B_2) = B_3$; hence the fibre above can be identified with

$$\{g \in G; \text{Ad}(g)\tilde{F}(B_1) = B_4, \text{Ad}(g)\tilde{F}(B_2) = B_3\}$$

which is quasi-isomorphic to $k^{\nu-k}$ times the $\rho$-dimensional torus $T$. We have a commutative diagram

$$\begin{array}{ccc}
\mathcal{B}^2 & \xleftarrow{h_{23}^k} & Y_{s}^k \\
1 & \downarrow & u \\
\mathcal{B}^2 & \xleftarrow{h_{23}^k} & Z^k \\
& \downarrow & 1 \\
& \mathcal{B}^2 & \xleftarrow{h_{14}^k} & \mathcal{B}^2
\end{array}$$

We have

$$h_{14}^k h_{23}^k L = h_{14}^k u_1 u^* h_{23}^k L = h_{14}^k (h_{23}^k L \otimes u_1 Q_1) = (h_{14}^k h_{23}^k L) \otimes \mathcal{L}[-2\nu + 2k].$$

We deduce that

$$\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)) \simeq \{(h_{14}^k h_{23}^k L) \otimes \mathcal{L}[-2\nu + 2k]; k \in \mathbb{N}\}.$$

Since $Z^k$ is the union of open and closed subvarieties $Z_y, |y| = k$, we have

$$h_{14}^k h_{23}^k L = \bigoplus_{y \in W; |y| = k} h_{14}^y h_{23}^y L.$$

From the definitions we have

$$h_{14}^y h_{23}^y L = L_y \cdot L \cdot L_{\epsilon(y)^{-1}}.$$

This completes the proof of (a). Now (b) follows from (a) just as in the case where $\epsilon = 1, s = 0$.

In the remainder of this section we fix a two-sided cell $c$ of $W$ such that $\epsilon(c) = c$; we set $a = a(c)$.

**Proposition 2.3.** Let $w \in W$ and let $j \in \mathbb{Z}$. We set $S = \zeta_{\epsilon,s}(R_{\epsilon,s,w})[2\rho + 2\nu + |w|] \in \mathcal{D}_m(\mathcal{B}^2)$.

(a) If $w \preceq c$ then $S^j \in \mathcal{M}^{\leq} \mathcal{B}^2$.

(b) If $w \preceq c$ and $j > \nu + 2a$ then $S^j \in \mathcal{M}^{<} \mathcal{B}^2$.

(c) If $w \preceq c$ then $S^j \in \mathcal{M}^{<} \mathcal{B}^2$.

(d) $S^j$ is mixed of weight $\leq j$.

(e) If $j \neq \nu + 2a$ and $w \preceq c$ then $gr_{\nu + 2a} S^j \in \mathcal{M}^{<} \mathcal{B}^2$.

(f) If $k > \nu + 2a$ and $w \preceq c$ then $gr_k S^j \in \mathcal{M}^{<} \mathcal{B}^2$.

When $\epsilon = 1, s = 0$ this is just [L12, 2.7]. The proof in the general case is entirely similar; it uses 2.2.
Proposition 2.4. (a) If $K \in \mathcal{D}^{\geq}G_{\epsilon,s}$ then $\zeta_{\epsilon,s}(K) \in \mathcal{D}^{\geq}B^2$. If $K \in \mathcal{D}^{\leq}G_{\epsilon,s}$, then $\zeta_{\epsilon,s}(K) \in \mathcal{D}^{\leq}B^2$.

(b) If $K \in \mathcal{M}^{\geq}G_{\epsilon,s}$ and $j > \rho + \nu + a$ then $\zeta_{\epsilon,s}^j(K) \in \mathcal{M}^{\leq}B^2$.

When $\epsilon = 1$, $s = 0$ this is just [L12, 2.8]. The proof in the general case is entirely similar; it uses 1.5(b) and 2.3.

2.5. For $K \in C_0^eG_{\epsilon,s}$ we set

$$\zeta_{\epsilon,s}(K) = (\zeta_{\epsilon,s}(K))_{\{\rho + \nu + a\}} \in C_0^eB^2.$$  

We say that $\zeta_{\epsilon,s}(K)$ is the truncated restriction of $K$.

Proposition 2.6. Let $K \in D_m(G_{\epsilon,s})$ and let $L \in C_0^eB^2$. Then there is a canonical isomorphism $^eL \bullet \zeta_{\epsilon,s}(K) \sim \zeta_{\epsilon,s}(K) \bullet L$.

When $\epsilon = 1$, $s = 0$ this follows from [L12, 2.10(a)]. We now consider the general case. Let $u : G_{\epsilon,s} \times B^2 \to G_{\epsilon,s} \times B^2$ be as in 1.6(a). We have $\zeta_{\epsilon,s}(K) \bullet L = c_1d^*(K \boxtimes L)$ where $Z = \{(F, (B, B'', B')) \in G_{\epsilon,s} \times B^3; F(B) = B''\}$, $d : Z \to G_{\epsilon,s} \times B^2$ is $(F, (B, B'', B')) \mapsto (F, (B''', B'))$, $c : Z \to B^2$ is $(F, (B, B'', B')) \mapsto (B, B')$. We have $^eL \bullet \zeta_{\epsilon,s}(K) = c'_1d'^*(K \boxtimes L)$ where $Z' = \{(F, (B, B'', B')) \in G_{\epsilon,s} \times B^3; F(B''') = B'\}$, $d' : Z' \to G_{\epsilon,s} \times B^2$ is $(F, (B, B'', B')) \mapsto (F, (B, B'''))$, $c' : Z' \to B^2$ is $(F, (B, B'', B')) \mapsto (B, B')$. Using 1.6(a) we have $K \boxtimes ^eL = u^*(K \boxtimes L)$ hence it is enough to show that $c_1d^*(K \boxtimes L) = c'_1d'^*u^*(K \boxtimes L)$. We have $c_1d^*(K \boxtimes L) = c_1d_1^*(K \boxtimes L) = c'_1d'_1u^*(K \boxtimes L)$ where $d_1 : G_{\epsilon,s} \times B^2 \to G_{\epsilon,s} \times B^2$ is $(F, (B, B')) \mapsto (F, (F(B), B'))$, $c_1 : G_{\epsilon,s} \times B^2 \to B^2$ is $(F, (B, B')) \mapsto (B, B')$. The proposition follows.

Proposition 2.7. (a) If $L \in M^{\geq}B^2$ and $j > 2a + 2\nu + 2\rho$ then $(\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)))^j \in M^{\leq}B^2$.

(b) If $L \in C_0^eB^2$, we have canonically

$$\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)) = (\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)))_{\{2a + 2\nu + 2\rho\}} \in C_0^eB^2.$$  

We apply 1.9 with $\Phi = \zeta_{\epsilon,s} : D_m(G_{\epsilon,s}) \to D_m(B^2)$ and with $X = \chi_{\epsilon,s}(L)$, $(c, c') = (a + \nu + \rho, a + \nu + \rho)$, see 2.4, 1.7. The result follows.

2.8. For $L, L' \in C_0^eB^2$, we set (as in [L12, 3.2])

(a) $L \bullet L' = (L \bullet L')_{\{a - \nu\}} \in C_0^eB^2$.

This defines an associative tensor product structure on $C_0^eB^2$. 

Proposition 2.9. Let $K \in C_0 G_{\epsilon, s}$, $L \in C_0 B^2$. There is a canonical isomorphism

(a) $\epsilon L \cdot \zeta_{\epsilon, s}(K) \sim \zeta_{\epsilon, s}(K) \cdot L$.

Applying 1.9 with $\Phi : D_{m}^{\infty} B^2 \to D_{m}^{\infty} B^2$, $L' \mapsto L' \cdot L$, $X = \zeta_{\epsilon, s}(K)$, $(c, c') = (a - \nu, a + \rho + \nu)$ (see [L12, 3.1] and 2.4), we deduce that we have canonically

(b) $((\zeta_{\epsilon, s}(K))^{(a+\rho+\nu)} \cdot L)^{(a-\nu)} = (\zeta_{\epsilon, s}(K) \cdot L)^{(2a+\rho)}$.

Using 1.9 with $\Phi : D_{m}^{\infty} B^2 \to D_{m}^{\infty} B^2$, $L' \mapsto \epsilon L \cdot L'$, $X = \zeta_{\epsilon, s}(K)$, $(c, c') = (a - \nu, a + \rho + \nu)$ (see [L12, 3.1] and 2.8), we deduce that we have canonically

(c) $((\epsilon L \cdot (\zeta_{\epsilon, s}(K))^{(a+\rho+\nu)})^{(a-\nu)} = (\epsilon L \cdot \zeta_{\epsilon, s}(K))^{(2a+\rho)}$.

We now combine (b), (c) with 2.6; we obtain the isomorphism (a).

2.10. Define $c : G_{\epsilon, s} \times B^2 \to B^2$ by $(F, B, B') \mapsto (F(B), F(B'))$. We show that for $K \in C_0 G_{\epsilon, s}$ we have canonically

(a) $c^* \zeta_{\epsilon, s} K = Q_l \boxtimes \zeta_{\epsilon, s} K$.

We have a commutative diagram with cartesian left squares

$$
\begin{array}{ccc}
G_{\epsilon, s} \times B^2 & \xrightarrow{f''} & X''_{\epsilon, s} \\
\downarrow d & & \downarrow d' & \downarrow d'' & \downarrow d \\
G_{\epsilon, s} \times B^2 & \xrightarrow{f'} & X'_{\epsilon, s} & \xrightarrow{\pi'} & G_{\epsilon, s} \\
\end{array}
\begin{array}{ccc}
G_{\epsilon, s} \times B^2 & \xrightarrow{f} & X_{\epsilon, s} & \xrightarrow{\pi} & G_{\epsilon, s} \\
\end{array}
$$

where $f, g$ are as in 1.3,

- $X'_{\epsilon, s} = \{ (\tilde{F}, B, B', F) \in G_{\epsilon, s} \times B \times B \times G_{\epsilon, s}; F \tilde{F} (B) = \tilde{F} (B') \}$,
- $X''_{\epsilon, s} = \{ (\tilde{F}, B, B', F) \in G_{\epsilon, s} \times B \times B \times G_{\epsilon, s}; F (B) = B' \}$,
- $f'(\tilde{F}, B, B', F) = (\tilde{F}, B, B')$, $f''(\tilde{F}, B, B', F) = (\tilde{F}, B, B')$,
- $\pi'(\tilde{F}, B, B', F) = F$, $\pi''(\tilde{F}, B, B', F) = (\tilde{F}, F)$,
- $c'(\tilde{F}, B, B', F) = (F, \tilde{F} (B), \tilde{F} (B'))$, $c''(F) = F$, $d(\tilde{F}, B, B') = (\tilde{F}, B, B')$,
- $d'(\tilde{F}, B, B', F) = (\tilde{F}, \tilde{F}^{-1} F \tilde{F}, B', B')$, $d''(\tilde{F}, F) = \tilde{F}^{-1} F \tilde{F}$, $e(\tilde{F}, F) = F$.

It is enough to show that $d^* f' c^* \pi^* K = f'' \pi'' e^* K$. or that $f'' d^* \pi^* K = f'' \pi'' e^* K$. It is enough to show that $d^* \pi^* K = \pi'' e^* K$, or that $\pi'' d^* \pi^* K = \pi'' e^* K$. Hence it is enough to show that $d^* \pi^* K = e^* K$. We identify $G \times G_{\epsilon, s} \leftrightarrow G_{\epsilon, s} \times G_{\epsilon, s}$ by $(g, F) \leftrightarrow (F \text{Ad}(g), F)$. Then $d^*, e : G_{\epsilon, s} \times G_{\epsilon, s} \to G_{\epsilon, s}$ become the maps
Lemma 3.2. Let $d_1, e_1 : G \times G \to G$ given by $(g, F) \mapsto \text{Ad}(g^{-1} F \text{Ad}(g))$, $(g, F) = F$ respectively and we have $d_1^\ast K = e_1^\ast K$ by the $G$-equivariance of $K$. Hence $d_1^\ast K = e_1^\ast K$ as required.

Using (a) and the definitions we see that for any $K \in \mathcal{C}_0^G G_{\epsilon, s}$ we have canonically

$$(b) \quad e_1^\ast \zeta_{\epsilon, s} K = \overline{Q}_l \boxtimes \zeta_{\epsilon, s} K.$$

From the definitions (see 1.6) for any $L \in \mathcal{C}_0^B B^2$ we have $e_1^\ast L = \overline{Q}_l \boxtimes e_1 L$. Comparing with (b) we deduce that we have canonically

$$(c) \quad e_1^\ast (\zeta_{\epsilon, s} K) = \zeta_{\epsilon, s} K$$

for any $K \in \mathcal{C}_0^G G_{\epsilon, s}$.

3. TRUNCATED CONVOLUTION FROM $G_{\epsilon, s} \times G_{\epsilon', s'}$ TO $G_{\epsilon s + s'}$

3.1. Let $\epsilon, \epsilon' \in \mathfrak{A}$, $s, s' \in \mathbb{Z}$. We define $\mu : G_{\epsilon, s} \times G_{\epsilon', s'} \to G_{\epsilon s + s'}$ by $(F, F') = FF'$ (composition of maps $G \to G$); this is a quasi-morphism, see 1.3. For $K \in \mathcal{D}_m(G_{\epsilon, s})$, $K' \in \mathcal{D}_m(G_{\epsilon', s'})$ we define the convolution $K \ast K' \in \mathcal{D}_m(G_{\epsilon s + s'})$ by $K \ast K' = \mu_1(K \boxtimes K')$. If $\epsilon'' \in \mathfrak{A}$, $s'' \in \mathbb{Z}$ then for $K, K'$ as above and $K'' \in \mathcal{D}_m(G_{\epsilon'' s''})$, we have canonically $(K \ast K') \ast K'' = K \ast (K' \ast K'') \in \mathcal{D}_m(G_{\epsilon s + s'} \times \mathcal{D}_m(G_{\epsilon'' s''})$ (and we denote this by $K \ast K' \ast K''$).

Lemma 3.2. Let $\epsilon, \epsilon' \in \mathfrak{A}$, $s, s' \in \mathbb{Z}$. Let $K \in \mathcal{D}_m(G_{\epsilon, s})$, $L \in \mathcal{D}_m(B^2)$. We have canonically $K \ast \chi_{\epsilon', s'}(L) = \chi_{\epsilon s + s'}(L \bullet \zeta_{\epsilon, s}(K))$.

Let

$$Z = \{(F_1, F_2, B, B') \in G_{\epsilon, s} \times G_{\epsilon', s'} \times B \times B; F_2(B) = B'\}.$$ 

Define $c : Z \to G_{\epsilon, s} \times B^2$ by $(F_1, F_2, B, B') \mapsto (F_1, (B, B'))$ and $d : Z \to G_{\epsilon s + s'}$ by $(F_1, F_2, B, B') \mapsto F_1 F_2$. From the definitions we see that both

$$K \ast \chi_{\epsilon', s'}(L) \ast \chi_{\epsilon s + s'}(L \bullet \zeta_{\epsilon, s}(K))$$

can be identified with $d_1 c^\ast (K \boxtimes L)$. The lemma follows. (In the case where $\epsilon = \epsilon' = 1$ and $s = s' = 0$ this reduces to [L12, 4.2].)

Proposition 3.3. Let $\epsilon, \epsilon' \in \mathfrak{A}$, $s, s' \in \mathbb{Z}$. For any $L, L' \in \mathcal{D}_m(B^2)$ we have

$$\chi_{\epsilon, s}(L) \ast \chi_{\epsilon', s'}(L') [2\rho + 2\nu]$$

$$\ni \{\chi_{\epsilon s + s'}(L' \bullet L_y \bullet L \bullet L_{\epsilon(y)^{-1}}) [2\nu \nu] \otimes \Lambda^d \mathcal{A}([d](d/2); d \in [0, \rho], y \in W)\}.$$ 

From 2.2(b) we deduce

$$L' \bullet \zeta_{\epsilon, s}(\chi_{\epsilon, s}(L)) [2\nu + 2\rho]$$

$$\ni \{L' \bullet L_y \bullet L \bullet L_{\epsilon(y)^{-1}} [2\nu \nu] \otimes \Lambda^d \mathcal{A}([d](d/2); y \in W, d \in [0, \rho]\}$$
and
\[ \chi_{é, s', s} (L' \bullet \zeta_{é, s} (\chi_{é, s} (L))) [2\nu + 2\rho] \]
\[ \cong \{ \chi_{é, s', s} (L' \bullet L_y \bullet L \bullet L_{\nu y}) [2\nu y] \} \otimes \Lambda^d [d/2]; y \in W, d \in [0, \rho] \} \].

It remains to show that \( \chi_{é, s', s} (L' \bullet \zeta_{é, s} (\chi_{é, s} (L))) = \chi_{é, s} (L) \ast \chi_{é', s'} (L') \). This follows from 3.2 with \( K, L \) replaced by \( \chi_{é, s} (L), L' \).

In the remainder of this section we fix a two-sided cell \( c \) of \( W \); we set \( a = a(c) \).

**Proposition 3.4.** Let \( é, é' \in A, s, s' \in Z \). Assume that \( é(c) = c, é'(c) = c \). Let \( w, w' \in W \) and let \( j \in Z \). We set \( C = R_{é, s, w} \ast R_{é', s', w'} [2\rho + 2\nu + |w| + |w'|] \in D_m (G_{é, s, s'} ) \).

(a) If \( w \leq c \) or \( w' \leq c \) then \( C \in M^\leq G_{é, s, s'} \).
(b) If \( j > \Delta + 4a \) and either \( w \in c \) or \( w' \in c \) then \( C \in M^\leq G_{é, s, s'} \).
(c) If \( w < c \) or \( w' < c \) then \( C \in M^\leq G_{é, s, s'} \).
(d) \( C \) is mixed of weight \( \leq j \).
(e) If \( j \neq \Delta + 4a \) and either \( w \in c \) or \( w' \in c \) then \( \text{gr}_{\Delta + 4a} C \in M^\leq G_{é, s, s'} \).
(f) If \( k > \Delta + 4a \) and \( w \in c \) or \( w' \in c \) then \( \text{gr}_k C \in M^\leq G_{é, s, s'} \).

When \( é = é' = 1, s = s' = 0 \), this is just [L12, 4.4]. The proof in the general case is entirely similar; it uses 3.3 and 1.4(d),(e).

**Proposition 3.5.** Let \( é, é' \in A, s, s' \in Z \). Assume that \( é(c) = c, é'(c) = c \). Let \( K \in D_m (G_{é, s}, K' \in D_m (G_{é', s'}) \).

(a) If \( K \in D^\leq G_{é, s} \) or \( K' \in D^\leq G_{é', s'} \) then \( K \ast K' \in D^\leq G_{é, s, s'} \); if \( K \in D^\leq G_{é, s} \) or \( K' \in D^\leq G_{é', s'} \) then \( K \ast K' \in D^\leq G_{é, s, s'} \).
(b) If \( K \in M^\leq G_{é, s}, K' \in M^\leq G_{é', s'}, j > \rho + 2a \) then \( (K \ast K') \in M^\leq G_{é, s, s'} \).

When \( é = é' = 1, s = s' = 0 \), this is just [L12, 4.5]. The proof in the general case is entirely similar; it uses 3.4.

3.6. Let \( é, é' \in A, s, s' \in Z \). Assume that \( é(c) = c, é'(c) = c \). For \( K \in C_0^c G_{é, s}, K' \in C_0^c G_{é', s'} \), we set
\[ K \ast K' = (K \ast K')^{2a + \rho} \in C_0^c G_{é, s, s'} \).

We say that \( K \ast K' \) is the truncated convolution of \( K, K' \).

**Proposition 3.7.** Let \( é, é', é'' \in A, s, s', s'' \in Z \). Assume that \( é(c) = c, é'(c) = c, é''(c) = c \). Let \( K, K', K'' \) be in \( C_0^c G_{é, s}, C_0^c G_{é', s'}, C_0^c G_{é'', s''} \) respectively. There is a canonical isomorphism
\[ (K \ast K') \ast K'' \cong K \ast (K' \ast K'') \].

When \( é = é' = é'' = 1, s = s' = s'' = 0 \), this is just [L12, 4.7]. The proof in the general case is entirely similar; it uses 1.9, 3.5.
**Proposition 3.8.** Let $\epsilon, \epsilon' \in \mathfrak{A}$, $s, s' \in \mathbb{Z}$. Assume that $\epsilon(c) = c$, $\epsilon'(c) = c$. Let $K \in C_0G_{\epsilon,s}$, $K' \in C_0G_{\epsilon',s'}$. There is a canonical isomorphism (in $C_0B^2$):

$$\zeta_{\epsilon',s'}(K') \circ \zeta_{\epsilon,s}(K) \overset{\sim}{\to} \zeta_{\epsilon\epsilon',s+s'}(K \ast K').$$

When $\epsilon = \epsilon' = 1$, $s = s' = 0$ this is just [L12, 5.2]. The proof in the general case is entirely similar.

4. Analysis of the composition $\zeta_{\epsilon,s}\chi_{\epsilon,s}$

4.1. In the remainder of this paper we fix a two-sided cell $c$ of $W$; we set $a = a(c)$. We also fix $\epsilon \in \mathfrak{A}$ such that $\epsilon(c) = c$. In this section we fix $s \in \mathbb{Z}$. Let $e, f, e'$ be integers such that $e \leq f \leq e' - 3$ and let $e = e' - e + 1$; we have $e \geq 4$. We set

$$\mathcal{Y} = \{(B_e, B_{e+1}, \ldots, B_{e'}), F) \in \mathcal{B}^e \times G_s; F(B_f) = B_{f+3}, F(B_{f+1}) = B_{f+2}\}.$$

Define $\vartheta : \mathcal{Y} \to \mathcal{B}^e$ by $((B_e, B_{e+1}, \ldots, B_{e'}), F) \mapsto (B_e, B_{e+1}, \ldots, B_{e'})$. For $i, j$ in $\{e, e+1, \ldots, e'\}$ let $\rho_{ij} : \mathcal{B}^e \to \mathbb{B}^2$ be the projection to the $i, j$ coordinate; define $h_{ij} : \mathcal{Y} \to \mathcal{B}^2$ by $h_{ij} = \rho_{ij}\vartheta$. Now $G^{e-2}$ acts on $\mathcal{Y}$ by

$$(g_{e}, \ldots, g_f, g_{f+3}, \ldots, g_{e'}) : (B_e, B_{e+1}, \ldots, B_{e'}), F) \mapsto \left(\operatorname{Ad}(g_e)(B_e), \operatorname{Ad}(g_{e+1})(B_{e+1}), \ldots, \operatorname{Ad}(g_f)(B_f)\right), \operatorname{Ad}(g_f)B_{f+1}, \ldots, \operatorname{Ad}(g_{e'})(B_{e'})),$$

$$\operatorname{Ad}(g_{f+3})B_{f+2}, \operatorname{Ad}(g_{f+3})B_{f+3}, \operatorname{Ad}(g_{f+4})B_{f+4}, \ldots, \operatorname{Ad}(g_{e'})B_{e'}),$$

this induces a $G^{e-2}$-action on $\mathcal{B}^e$ so that $\vartheta$ is $G^{e-2}$-equivariant.

Let $E = \{e, e+1, \ldots, e' - 1\} = \{f, f + 2\}$. Assume that $x_n \in c$ are given for $n \in E$. Let $P = \otimes_{n \in E} h_{n+1}L_{x_n} \in \mathcal{D}_m\mathcal{B}^e$, $\hat{P} = \otimes_{n \in E} h_{n+1} L_{x_n} = \vartheta^* P \in \mathcal{D}_m\mathcal{Y}$. In 4.1-4.7 we will study

$$h_{ee'}\hat{P} \in \mathcal{D}_m\mathbb{B}^2.$$

Setting $\Xi = \vartheta|_{\mathcal{Y}} Q_{l} \in \mathcal{D}_m\mathcal{B}^e$, we have

$$h_{ee'}\hat{P} = p_{ee}(\Xi \otimes P).$$

Clearly, $\Xi^j$ is $G^{e-2}$-equivariant for any $j$. For any $y, y'$ in $W$ we set

$$Z_{y, y'} := \{(B_e, B_{e+1}, \ldots, B_{e'}) \in \mathcal{B}^e; (B_f, B_{f+1}) \in O_y, (B_{f+2}, B_{f+3}) \in O_{y'}\}.$$

These are the orbits of the $G^{e-2}$-action on $\mathcal{B}^e$. Note that the fibre of $\vartheta$ over a point of $Z_{y, y'}$ is isomorphic to $T \times k^{\nu-|y|}$ if $y' = \epsilon(y)^{-1}$ and is empty if $y' \neq \epsilon(y)^{-1}$. Thus

(a) $\Xi|_{Z_{y, y'}}$ is 0 if $y' \neq \epsilon(y)^{-1}$

and for any $y \in W$ we have

(b) $\mathcal{H}^h\Xi|_{Z_{y, \epsilon(y)^{-1}}} = 0$ if $h > 2\nu - 2|y| + 2\rho$, $\mathcal{H}^{2\nu - 2|y| + 2\rho}\Xi|_{Z_{y, \epsilon(y)^{-1}}} = Q_l(-\nu + |y| - \rho)$.

The closure of $Z_{y, y'}$ in $\mathcal{B}^e$ is denoted by $\mathcal{Z}_{y, y'}$. We set $k_{e} = \epsilon\nu + 2\rho$. We have the following result.
Lemma 4.2. (a) We have $\Xi^j = 0$ for any $j > k_e$. Hence, setting $\Xi' = \tau_{\leq k_e-1}\Xi$, we have a canonical distinguished triangle $(\Xi', \Xi, \Xi^{k_e}[−k_e])$.

(b) If $\xi \in \mathbb{Z}_y, y'$ and $i = 2\nu − |y| − |y'| + 2\rho$, the induced homomorphism $\mathcal{H}_\xi^i \Xi \rightarrow \mathcal{H}_\xi^{i−k_e}(\Xi^{k_e})$ is an isomorphism.

When $\epsilon = 1$, $s = 0$ this is just [L12, 6.2]. The proof in the general case is entirely similar; it uses 4.1(a), (b).

4.3. For any $y, y'$ in $W$ let $\mathfrak{T}_{y, y'}$ be the intersection cohomology complex of $\mathcal{Z}_y, y'$ extended by 0 on $B^e − \mathcal{Z}_y, y'$, to which $[(\mathfrak{e} − 2\nu + |y| + |y'|)]$ is applied. Note that

(a) $\mathfrak{T}_{y, y'} = p_{f, f+1}^* L_2 \otimes p_{f+2, f+3}^* L_{y'y'}[[\mathfrak{e} − 4\nu]]$.

We have the following result.

Lemma 4.4. We have canonically $\text{gr}_0(\Xi^{k_e}(k_e/2)) = \bigoplus_{y \in W} \mathfrak{T}_{y, \epsilon(y)^{-1}}$.

When $\epsilon = 1$, $s = 0$ this is just [L12, 6.4]. The proof in the general case is entirely similar; it uses 4.2(b) and 4.1.

4.5. Let $y, \tilde{y} \in W$. Using the definitions and 1.2(a) we have

$$p_{ee'}!(\mathfrak{T}_{y, \tilde{y}} \otimes P[[\mathfrak{e} − 2\nu]])$$

(a) $= L^2_{x_1} \cdots L^2_{x_{f−1}} \cdots L^2_{y} \cdot L^2_{x_{f+1}} \cdots L^2_{y'} \cdot \cdots \cdot L^2_{x_{e}, [\nu + |y| + |\tilde{y}| + \sum_{n \in E}|x_n|]}$. [\]

Lemma 4.6. The map $\Xi \rightarrow \Xi^{k_e}[-k_e]$ (coming from $(\Xi', \Xi, \Xi^{k_e}[-k_e])$ in 4.2(a)) induces a morphism

$$(p_{ee'}!(\Xi \otimes P))^{(\mathfrak{e}−2)a+(\mathfrak{e}−2)\nu+2\rho} \rightarrow (p_{ee'}!(\Xi^{k_e} \otimes P))^{(\mathfrak{e}−2)a+(\mathfrak{e}−2)\nu+2\rho−k_e}$$

whose kernel and cokernel are in $\mathcal{M}_m B^2$.

When $\epsilon = 1$, $s = 0$ this is just [L12, 6.6]. The proof in the general case is entirely similar; it uses 4.5(a), (b) and [L12, 2.2(a)].

Lemma 4.7. We have canonically

$$\frac{(h_{ee'}! P)^{(\mathfrak{e}−2)a+(\mathfrak{e}−2)\nu+2\rho}}{=} = \bigoplus_{y \in E} Q_y$$

where

$$Q_y = p_{ee'}!(\mathfrak{T}_{y, \epsilon(y)^{-1}} \otimes P)^{(\mathfrak{e}−2)a+(\mathfrak{e}−2)\nu}$$

$$= L^2_{x_1} \cdots L^2_{x_{f−1}} \cdots L^2_{y} \cdot L^2_{x_{f+1}} \cdots L^2_{x_{e}, (\mathfrak{e}−1) \cdot \cdots \cdot L^2_{x_{e′}}}$.

When $\epsilon = 1$, $s = 0$ this is just [L12, 6.7]. The proof in the general case is entirely similar; it uses 4.6, 4.5(a), (b) and [L12, 2.2(a), 2.3, 3.2].
Theorem 4.8. Let \( x \in \mathfrak{c} \). We have canonically

\[
\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L_x)) = \oplus_{y \in \mathfrak{c}} L_y \bullet L_x \bullet L_{\epsilon(y)}^{-1}.
\]

When \( \epsilon = 1, s = 0 \) this is just [L12, 6.8]. The proof in the general case is entirely similar; it uses 4.7, the proof of 2.2 and 2.7(b).

4.9. Using [L12, 2.4] we see that 4.8(a) implies

\[
\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L_x)) \cong \oplus_{z \in \mathfrak{c}} (L_x) \oplus \psi_x(z)
\]

in \( \mathcal{C}^c \mathcal{B}^2 \) where \( \psi_x(z) \in \mathbb{N} \) are given by the following equation in \( \mathcal{J}^c \):

\[
\sum_{y \in \mathfrak{c}} t_y t_x t_{\epsilon(y)}^{-1} = \sum_{z \in \mathfrak{c}} \psi_x(z) t_z.
\]

5. Adjunction formula (weak form)

Proposition 5.1. Let \( \epsilon' \in \mathfrak{A}, s, s' \in \mathbb{Z} \). We assume that \( \epsilon'(\mathfrak{c}) = \mathfrak{c} \). Let \( K \in \mathcal{C}^c_c(G_{\epsilon,s}), L \in \mathcal{C}^c_c(B^2) \). We have canonically

\[
K \times_{\epsilon,s'}(L) = \chi_{\epsilon',s+s'}((L \circ \zeta_{\epsilon,s}(K)).
\]

When \( \epsilon = \epsilon' = 1, s = s' = 0 \) this is just [L12, 8.1]. The proof in the general case is entirely similar; it uses 3.2.

5.2. Let \( s \in \mathbb{Z} \). When \( \epsilon = 1, s = 0 \), the arguments in this subsection reduce to arguments in [L12, 8.8]. Let \( u' : G_{\epsilon-1,-s} \to \mathfrak{p} \) be the obvious map. From [L2, 7.4] we see that for \( K, K' \in \mathcal{M}^c_m G_{\epsilon-1,-s} \) we have canonically

\[
(u'_t(K \otimes K'))^0 = \hom_{\mathcal{M}_G_{\epsilon-1,-s}}(\mathfrak{D}(K), K'), \quad (u'_t(K \otimes K'))^j = 0 \text{ if } j > 0.
\]

We deduce that if \( K, K' \) are also pure of weight 0 then \( (u'_t(K \otimes K'))^0 \) is pure of weight zero that is, \( (u'_t(K \otimes K'))^0 = gr_0(u'_t(K \otimes K'))^0 \). Let \( t : \mathfrak{p} \to G = G_0 \) be the map with image 1. From the definitions we see that we have \( u'_t(K \otimes K') = t^*(b_t(K) \ast K') \) where \( b : G_{\epsilon-1,-s} \to G_{\epsilon,s} \) is given by \( F \mapsto F^{-1} \). Hence for \( K, K' \in \mathcal{C}^c_c G_{\epsilon-1,-s} \) we have

\[
\hom_{\mathcal{C}_c G_{\epsilon-1,-s}}(\mathfrak{D}(K), K') = (t^*(b_t(K) \ast K'))^0 = (t^*(b_t(K) \ast K'))^{\{0\}}.
\]

Applying [L12, 8.2] with \( \Phi : \mathcal{D}^c_m G_0 \to \mathcal{D}_m \mathfrak{p}, K_1 \mapsto t^* K_1, c = -2a - \rho \) (see [L12, 8.3(a)]), \( K \) replaced by \( b_t(K) \ast K' \in \mathcal{D}_m(G_{1,0} \text{ and } c' = 2a + \rho \) we see that we have canonically

\[
(t^*(b_t(K) \ast K'))^{\{-2a-\rho\}} \subset (t^*(b_t(K) \ast K'))^{\{0\}}.
\]
In particular, if $L, L' \in \mathcal{C}_0^s \mathcal{B}^2$ then we have canonically

$$(t^*(\mathcal{X}_{\epsilon,s}(L) \cdot \mathcal{X}_{\epsilon-1,-s}(L)))^{[-2a_\rho]} \subset (t^*(\mathcal{X}_{\epsilon,s}(L') \cdot \mathcal{X}_{\epsilon-1,-s}(L)))^{[0]}.$$ 

Using the equality

$$(t^*(\mathcal{X}_{\epsilon,s}(L) \cdot \mathcal{X}_{\epsilon-1,-s}(L)))^{[-2a_\rho]} = (t^*(\mathcal{X}_{1,0}(\mathcal{L} \cdot \mathcal{X}_{\epsilon,s}(L'))))^{[-2a_\rho]}$$

which comes from 5.1, we deduce that we have canonically

$$(t^*(\mathcal{X}_{1,0}(\mathcal{L} \cdot \mathcal{X}_{\epsilon,s}(L'))))^{[-2a_\rho]} \subset (t^*(\mathcal{X}_{\epsilon,s}(L') \cdot \mathcal{X}_{\epsilon-1,-s}(L)))^{[0]}$$

or equivalently, using (a) with $K, K'$ replaced by $b^* \mathcal{X}_{\epsilon,s}(L'), \mathcal{X}_{\epsilon-1,-s}(L)$:

$$(t^*(\mathcal{X}_{1,0}(\mathcal{L} \cdot \mathcal{X}_{\epsilon,s}(L'))))^{[-2a_\rho]} \subset \text{Hom}_{\mathcal{C}_e \mathcal{G}_{\epsilon-1,-s}}(\mathcal{D}(b^* \mathcal{X}_{\epsilon,s}(L')), \mathcal{X}_{\epsilon-1,-s}(L))$$

$$= \text{Hom}_{\mathcal{C}_e \mathcal{G}_{\epsilon,s}}(\mathcal{D}(b \mathcal{X}_{\epsilon-1,-s}(L)), \mathcal{X}_{\epsilon,s}(L')).$$

Using now [L12, 8.6(c)], we deduce that we have canonically

$$\text{Hom}_{\mathcal{C}_e \mathcal{B}^2}(1', \mathcal{L} \cdot \mathcal{X}_{\epsilon,s}, \mathcal{X}_{\epsilon,s}, L') \subset \text{Hom}_{\mathcal{C}_e \mathcal{G}_{\epsilon,s}}(\mathcal{D}(b \mathcal{X}_{\epsilon-1,-s}(L)), \mathcal{X}_{\epsilon,s}(L'))$$

where $1'$ is as in [L12, 8.6] or equivalently (see [L12, 8.7]):

$$\text{Hom}_{\mathcal{C}_e \mathcal{B}^2}(\mathcal{D}(b'_L), \mathcal{X}_{\epsilon,s}, L') \subset \text{Hom}_{\mathcal{C}_e \mathcal{G}_{\epsilon,s}}(\mathcal{D}(b \mathcal{X}_{\epsilon-1,-s}(L)), \mathcal{X}_{\epsilon,s}(L'))$$

where $b' : \mathcal{B}^2 \to \mathcal{B}^2$ is $(B, B') \mapsto (B', B)$. We now set $1L = \mathcal{D}(b'_L)$ and note that

$$\mathcal{D}(b \mathcal{X}_{\epsilon-1,-s}(L)) = \mathcal{D}(\mathcal{X}_{\epsilon,s}(b'_L)) = \mathcal{X}_{\epsilon,s}(\mathcal{D}(b'_L)) = \mathcal{X}_{\epsilon,s}(1L),$$

see 1.3, 1.10(a). We obtain

(b) \quad \text{Hom}_{\mathcal{C}_e \mathcal{B}^2}(1L, \mathcal{X}_{\epsilon,s}, L') \subset \text{Hom}_{\mathcal{C}_e \mathcal{G}_{\epsilon,s}}(\mathcal{X}_{\epsilon,s}(1L), \mathcal{X}_{\epsilon,s}(L'))

for any $1L, L' \in \mathcal{C}_0^s \mathcal{B}^2$.

We have the following result which is a weak form of an adjunction formula, of which the full form will be proved in 6.6.

**Proposition 5.3.** Let $s \in \mathbb{Z}$. For any $1L, L' \in \mathcal{C}_0^s \mathcal{B}^2$ we have canonically

(a) \quad \text{Hom}_{\mathcal{C}_e \mathcal{B}^2}(1L, \mathcal{X}_{\epsilon,s}, L') = \text{Hom}_{\mathcal{C}_e \mathcal{G}_{\epsilon,s}}(\mathcal{X}_{\epsilon,s}(1L), \mathcal{X}_{\epsilon,s}(L'))

We can assume that $1L = \mathcal{L}_z, L' = \mathcal{L}_u$ where $z, u \in \mathfrak{c}$. By 4.9(a) and 1.8(b), both sides of the inclusion 5.2(b) have dimension $\sum_{y \in \mathfrak{c}} \tau(t_{y-1}tz\tau(y)t_{u-1})$. Hence that inclusion is an equality. The proposition is proved. (The case where $\epsilon = 1, s = 0$ is treated in [L12, 8.9].)
6. Equivalence of $\mathcal{C}^cG_{e,s}$ with the $\epsilon$-centre of $\mathcal{C}^c\mathcal{B}^2$

6.1. For $\epsilon' \in \mathfrak{A}$ such that $\epsilon'(c) = c$ and $s, s' \in \mathbb{Z}$, the functor $\mathcal{C}^cG_{e,s} \times \mathcal{C}^cG_{e', s'} \rightarrow \mathcal{C}^cG_{e\epsilon', s+s'}$, $K, K' \mapsto K \times K'$ in 3.6 defines a bifunctor $\mathcal{C}^cG_{e,s} \times \mathcal{C}^cG_{e', s'} \rightarrow \mathcal{C}^cG_{e\epsilon', s+s'}$ denoted again by $K, K' \mapsto K \times K'$ as follows. Let $K \in \mathcal{C}^cG_{e,s}$, $K' \in \mathcal{C}^cG_{e', s'}$; we choose mixed structures of pure weight 0 on $K, K'$ (this is possible if $s_0$ in 0.3 is large enough), we define $K \times K'$ in 3.6 in terms of these mixed structures and we then disregard the mixed structure on $K \times K'$. The resulting object of $\mathcal{C}^cG_{e\epsilon', s+s'}$ is denoted again by $K \times K'$; it is independent of the choices made.

In the same way, the bifunctor $\mathcal{C}^c\mathcal{B}^2 \times \mathcal{C}^c\mathcal{B}^2 \rightarrow \mathcal{C}^c\mathcal{B}^2$, $L, L' \mapsto L \cdot L'$ gives rise to a bifunctor $\mathcal{C}^c\mathcal{B}^2 \times \mathcal{C}^c\mathcal{B}^2 \rightarrow \mathcal{C}^c\mathcal{B}^2$ denoted again by $L, L' \mapsto L \cdot L'$; the functor $\chi_{\epsilon,s} : \mathcal{C}^c\mathcal{B}^2 \rightarrow \mathcal{C}^cG_{e,s}$ gives rise to a functor $\mathcal{C}^c\mathcal{B}^2 \rightarrow \mathcal{C}^c\mathcal{G}_{s}$ denoted again by $\chi_{\epsilon,s}$ (it is again called truncated induction); the functor $\zeta_{\epsilon,s} : \mathcal{C}^cG_{e,s} \rightarrow \mathcal{C}^c\mathcal{B}^2$ gives rise to a functor $\mathcal{C}^cG_{e,s} \rightarrow \mathcal{C}^c\mathcal{B}^2$ denoted again by $\zeta_{\epsilon,s}$ (it is again called truncated restriction).

The operation $K \times K'$ is again called truncated convolution. It has a canonical associativity isomorphism (deduced from that in 3.7) which satisfies the pentagon property.

The operation $L \cdot L'$ makes $\mathcal{C}^c\mathcal{B}^2$ into a monoidal abelian category (see also [L9]) which has a unit object (see [L12, 9.2]) and is rigid (see [L12, 9.3]).

Note that $L \mapsto \epsilon L$ (see 1.6) can be regarded as a functor $\mathcal{C}^c\mathcal{B}^2 \rightarrow \mathcal{C}^c\mathcal{B}^2$.

6.2. Extending slightly a definition in [Mu, 3.1] we define an $\epsilon$-half braiding for an object $\mathcal{L} \in \mathcal{C}^c\mathcal{B}^2$ as a collection $e_{\mathcal{L}} = \{e_{\mathcal{L}}(L); L \in \mathcal{C}^c\mathcal{B}^2\}$ where $e_{\mathcal{L}}(L)$ are isomorphisms $\epsilon L \cdot \mathcal{L} \overset{\sim}{\rightarrow} \mathcal{L} \cdot L$ such that (i),(ii) below hold:

(i) If $L \overset{\phi}{\rightarrow} L'$ is any morphism in $\mathcal{C}^c\mathcal{B}^2$ then the diagram

$$
\begin{array}{ccc}
\epsilon L \cdot \mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L)} & \mathcal{L} \cdot L \\
\epsilon \cdot \phi \downarrow & & \downarrow \phi \\
\epsilon L' \cdot \mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L')} & \mathcal{L} \cdot L'
\end{array}
$$

is commutative.

(ii) If $L, L' \in \mathcal{C}^c\mathcal{B}^2$ then $e_{\mathcal{L}}(L \cdot L') : \epsilon(L \cdot L') \cdot \mathcal{L} \rightarrow \mathcal{L} \cdot (L \cdot L')$ is equal to the composition

$$
\epsilon L \cdot e_{L'} \cdot \mathcal{L} \xrightarrow{\epsilon \cdot e_{\mathcal{L}}(L')} \epsilon L \cdot \mathcal{L} \cdot L' > e_{\mathcal{L}}(L) \cdot 1 > \mathcal{L} \cdot \mathcal{L} \cdot L'.
$$

When $\epsilon = 1$, this reduces to the definition of a half-braiding for $\mathcal{L}$ given in [Mu, 3.1]. Let $\mathcal{Z}^c_{\epsilon}$ the category whose objects are the pairs consisting of an object $\mathcal{L}$ of $\mathcal{C}^c\mathcal{B}^2$ and an $\epsilon$-half braiding for $\mathcal{L}$. For $(\mathcal{L}, e_{\mathcal{L}}), (\mathcal{L}', e_{\mathcal{L}'})$ in $\mathcal{Z}^c_{\epsilon}$ we define
\[ \text{Hom}_{\mathcal{C}^e}((\mathcal{L}, e_{\mathcal{L}}), (\mathcal{L}', e_{\mathcal{L}'})) \] to be the vector space consisting of all \( t \in \text{Hom}_{\mathcal{C}^eB^2}(\mathcal{L}, \mathcal{L}') \) such that for any \( L \in \mathcal{C}^eB^2 \) the diagram

\[
\begin{array}{ccc}
\epsilon L \otimes \mathcal{L} & \xrightarrow{\epsilon_{\mathcal{L}}(L)} & \mathcal{L} \otimes L \\
\downarrow \ & \ & \downarrow \\
\epsilon L \otimes \mathcal{L}' & \xrightarrow{\epsilon_{\mathcal{L}'}(L)} & \mathcal{L}' \otimes L
\end{array}
\]

is commutative. We say that \( \mathcal{Z}^e \) is the \( \epsilon \)-centre of \( \mathcal{C}^eB^2 \). (When \( \epsilon = 1 \), it reduces to the centre of \( \mathcal{C}^eB^2 \), see [Mu, 3.2].)

If \( s \in \mathbb{Z} \) and \( K \in \mathcal{C}^eG_s \) then the isomorphisms 2.9(a) provide an \( \epsilon \)-half braiding for \( \zeta_{\epsilon,s}(K) \in \mathcal{C}^eB^2 \) so that \( \zeta_{\epsilon,s}(K) \) can be naturally viewed as an object of \( \mathcal{Z}^e \) denoted by \( \overline{\zeta_{\epsilon,s}(K)} \). (Note that 2.9 is stated in the mixed category but, as above, it implies the corresponding result in the unmixed category.) Then \( K \mapsto \overline{\zeta_{\epsilon,s}(K)} \) is a functor \( \mathcal{C}^eG_{\epsilon,s} \to \mathcal{Z}^e \). We have the following result.

**Theorem 6.3.** Let \( s \in \mathbb{Z} \). The functor \( \mathcal{C}^eG_{\epsilon,s} \to \mathcal{Z}^e \), \( K \mapsto \overline{\zeta_{\epsilon,s}(K)} \) is an equivalence of categories.

When \( \epsilon = 1, s = 0 \) this reduces to [L12, 9.5]. The general case will be proved in 6.5.

Note that, when combined with 1.6(b), the theorem yields for any \( F \in G_{\epsilon,s} \) (with \( s > 0 \)) a category equivalence

(a) \[ \text{Rep}^c(F) \xrightarrow{\sim} \mathcal{Z}^e. \]

**6.4.** By a variation of a general result on semisimple rigid monoidal categories in [ENO, Proposition 5.4], for any \( L \in \mathcal{C}^eB^2 \) one can define directly an \( \epsilon \)-half braiding on the object \( I_{\epsilon}(L) := \oplus_{y \in c} L_y \otimes L \otimes I_{\epsilon(y)-1} \) of \( \mathcal{C}^eB^2 \) such that, denoting by \( \overline{I_{\epsilon}(L)} \) the corresponding object of \( \mathcal{Z}^e \), we have canonically

(a) \[ \text{Hom}_{\mathcal{C}^eB^2}(L, \mathcal{L}) = \text{Hom}_{\mathcal{Z}^e}(\overline{I_{\epsilon}(L)}, (\mathcal{L}, e_{\mathcal{L}})) \]

for any \( (\mathcal{L}, e_{\mathcal{L}}) \in \mathcal{Z}^e \).

The \( \epsilon \)-half braiding on \( I_{\epsilon}(L) \) can be described as follows: for any \( L' \in \mathcal{C}^eB^2 \) we have canonically

\[
\begin{align*}
\epsilon L' \otimes I_{\epsilon}(L) & = \oplus_{y \in c} \epsilon L' \otimes L_y \otimes L \otimes I_{\epsilon(y)-1} = \oplus_{y,z \in c} \text{Hom}_{\mathcal{C}^eB^2}(L_z, \epsilon L' \otimes L_y) \otimes L_z \otimes L \otimes I_{\epsilon(y)-1} \\
& = \oplus_{y,z \in c} \text{Hom}_{\mathcal{C}^eB^2}(L_y^{-1}, L_{z^{-1}} \epsilon L') \otimes L_z \otimes L \otimes I_{\epsilon(y)-1} \\
& = \oplus_{y,z \in c} \text{Hom}_{\mathcal{C}^eB^2}(L_{\epsilon(y)}^{-1}, L_{\epsilon(z)-1} \epsilon L') \otimes L_z \otimes L \otimes I_{\epsilon(y)-1} \\
& = \oplus_{z \in c} L_z \otimes L \otimes L_{\epsilon(z)-1} \epsilon L' = I_{\epsilon}(L) \otimes L'.
\end{align*}
\]
By a variation of results in [Mu, 3.3], [ENO, 2.15], we see that $Z_c^\epsilon$ is a semisimple $\bar{Q}_l$-linear category with finitely many simple objects up to isomorphism. Note that

(b) if $\sigma = (L, e_L)$ is a simple object of $Z_c^\epsilon$ then $\sigma$ is a summand of $I_\epsilon(L_z)$ for some $z \in c$.

Indeed, let $z \in c$ be such that $L_z$ is a summand of $L$ in $C^cB^2$; then by (a), $\sigma$ is a summand of $I_\epsilon(L_z)$.

6.5. Let $s \in Z$. For $x \in c$ we have canonically $\zeta_{\epsilon,s}(x, L_x) = I_\epsilon(L_x)$ as objects of $C^cB^2$, see Theorem 4.8. This identification is compatible with the $\epsilon$-half braidings (see 6.2, 6.4). (When $\epsilon = 1, s = 0$ this follows from the last commutative diagram in [L12, 7.9]; in the general case we have an analogous commutative diagram, which is established using the results in Section 4.) It follows that

(a) $\zeta_{\epsilon,s}(x, L_x) = I_\epsilon(L_x)$.

Using this and 6.4(a) with $L = \zeta_{\epsilon,s}(x, L_x)$, $\tilde{L} \in C^cB^2$, we see that

$$\text{Hom}_{C^cB^2}(L_x, \zeta_{\epsilon,s}(x, L_x)) = \text{Hom}_{Z_c^\epsilon}(\zeta_{\epsilon,s}(x, L_x), \zeta_{\epsilon,s}(x, L_x)).$$

Combining this with 5.3 we obtain for $\tilde{L} = L_{x'}$ (with $x' \in c$):

(b) $A_{x,x'} = A'_{x,x'}$

where

$$A_{x,x'} = \text{Hom}_{C^cG_{\epsilon,s}(L_x, L_x)}(\chi_{\epsilon,s}(L_x)), A'_{x,x'} = \text{Hom}_{Z_c^\epsilon}(\zeta_{\epsilon,s}(x, L_x), \zeta_{\epsilon,s}(x, L_{x'})).$$

Note that the identification (b) is induced by the functor $K \mapsto \zeta_{\epsilon,s}(K)$. Let $A = \bigoplus_{x,x' \in c} A_{x,x'}, A' = \bigoplus_{x,x' \in c} A'_{x,x'}$. Then from (b) we have $A = A'$. Note that this identification is compatible with the obvious algebra structures of $A, A'$.

For any $A \in CS_{\epsilon,s,c}$ we denote by $A_A$ the set of all $f \in A$ such that for any $x, x'$, the $(x, x')$-component of $f$ maps the $A$-isotypic component of $\chi_{\epsilon,s}(L_x)$ to the $A$-isotypic component of $\chi_{\epsilon,s}(L_{x'})$ and any other isotypic component of $\chi_{\epsilon,s}(L_x)$ to 0. Then $A = \bigoplus_{A \in CS_{\epsilon,s,c}} A_A$ is the decomposition of $A$ into a sum of simple algebras (each $A_A$ is $\neq 0$ since, by 1.5(b) and 1.8(a), any $A$ is a summand of some $\chi_{\epsilon,s}(L_x)$).

Let $\mathcal{S}$ be a set of representatives for the isomorphism classes of simple objects of $Z_c^\epsilon$. For any $\sigma \in \mathcal{S}$ we denote by $A'_{\sigma}$ the set of all $f' \in A'$ such that for any $x, x'$, the $(x, x')$-component of $f'$ maps the $\sigma$-isotypic component of $\zeta_{\epsilon,s}(x, L_x)$ to the $\sigma$-isotypic component of $\zeta_{\epsilon,s}(x, L_{x'})$ and any other isotypic component of $\chi_{\epsilon,s}(L_x)$.
\[ \zeta_{\epsilon, s}(L_{\mathfrak{z}}) \] to 0. Then \( A' = \bigoplus_{\sigma \in \mathcal{G}} A'_\sigma \) is the decomposition of \( A' \) into a sum of simple algebras (each \( A'_\sigma \) is \( \neq 0 \) since any \( \sigma \) is a summand of some \( \zeta_{\epsilon, s}(L_{\mathfrak{z}}) \) (we use 6.4(b), 6.5(a)).

Since \( A = A' \), from the uniqueness of decomposition of a semisimple algebra as a direct sum of simple algebras, we see that there is a unique bijection \( CS_{\epsilon, s, c} \leftrightarrow \mathcal{G}, A \leftrightarrow \sigma \) such that \( A' = \bigoplus_{\sigma \in \mathcal{G}} A'_\sigma \) is the decomposition of \( A \) into a sum of simple algebras (each \( A'_\sigma \) is \( \neq 0 \) since any \( \sigma \) is a summand of some \( \zeta_{\epsilon, s}(L_{\mathfrak{z}}) \) (we use 6.4(b), 6.5(a)).

Since \( A = A' \), from the uniqueness of decomposition of a semisimple algebra as a direct sum of simple algebras, we see that there is a unique bijection \( CS_{\epsilon, s, c} \leftrightarrow \mathcal{G}, A \leftrightarrow \sigma \) such that \( A' = \bigoplus_{\sigma \in \mathcal{G}} A'_\sigma \) is the decomposition of \( A \) into a sum of simple algebras (each \( A'_\sigma \) is \( \neq 0 \) since any \( \sigma \) is a summand of some \( \zeta_{\epsilon, s}(L_{\mathfrak{z}}) \) (we use 6.4(b), 6.5(a)).

From 6.3, 6.5, we see that for any \( \zeta_{\epsilon, s}(L_{\mathfrak{z}}) \) we have \( \chi_{\epsilon, s}(L_{\mathfrak{z}}) \sim \zeta_{\epsilon, s}(K) \). Therefore Theorem 6.3 holds.

### Theorem 6.6

Let \( s \in \mathbb{Z} \). Let \( L \in \mathcal{C}^c \mathcal{B}^2, K \in \mathcal{C}^c \mathcal{G}_{\epsilon, s} \). We have canonically

(a) \[ \text{Hom}_{\mathcal{C}^c \mathcal{B}^2}(L, \zeta_{\epsilon, s}(K)) = \text{Hom}_{\mathcal{C}^c \mathcal{G}_{\epsilon, s}}(\chi_{\epsilon, s}(L), K). \]

Moreover, in \( \mathcal{C}^c \mathcal{B}^2 \) we have \( \zeta_{\epsilon, s}(K) \sim \bigoplus_{z \in c^0} L_z \oplus m_z \) where \( c^0 \) is as in 1.5 and \( m_z \in \mathbb{N} \).

From 6.3, 6.5, we see that

\[
\text{Hom}_{\mathcal{C}^c \mathcal{G}_{\epsilon, s}}(\chi_{\epsilon, s}(L), K) = \text{Hom}_{\mathcal{Z}^c}(\zeta_{\epsilon, s}(L), \zeta_{\epsilon, s}(K)) = \text{Hom}_{\mathcal{Z}^c}(I_{\epsilon}(L), \zeta_{\epsilon, s}(K)).
\]

Using 6.4(a) we see that

\[
\text{Hom}_{\mathcal{Z}^c}(I_{\epsilon}(L), \zeta_{\epsilon, s}(K)) = \text{Hom}_{\mathcal{C}^c \mathcal{B}^2}(L, \zeta_{\epsilon, s}(K))
\]

and (a) follows. To prove the second assertion of the theorem it is enough to show that for any \( z \in c - c^0 \) we have \( \text{Hom}_{\mathcal{C}^c \mathcal{B}^2}(L_z, \zeta_{\epsilon, s}(K)) = 0 \); by (a), it is enough to show that \( \chi_{\epsilon, s}(L_z) = 0 \) and this follows from 1.5(c). (The case where \( \epsilon = 1, s = 0 \) is just [L12, 9.8].)

### 6.7

Let \( s \in \mathbb{Z} \). For \( K \in \mathcal{C}^c \mathcal{G}_{\epsilon, s} \) we have canonically

(a) \[ \mathcal{D}(\zeta_{\epsilon, s}(\mathcal{D}(K))) = \zeta_{\epsilon, s}(K). \]

When \( \epsilon = 1, s = 0 \) this is proved in [L12, 9.9]. The proof in the general case is entirely similar; it uses 6.6(a), 1.10(a).

### 6.8

In this subsection we assume that \( \epsilon = 1 \). The monoidal structure on \( \mathcal{C}^c \mathcal{B}^2 \) induces a monoidal structure on \( \mathcal{Z}^c_{\mathfrak{f}} \). Moreover, the category

(a) \[ \sqcup_{s \in \mathbb{Z}} \mathcal{C}^c \mathcal{G}_{1, s} = \mathcal{C}^c \mathcal{G}_{1, 0} \sqcup_{s \in \mathbb{Z}, s \neq 0} \text{Rep}^c(G^{F_0^+}) \]

(see 1.6(b)) has a monoidal structure given by truncated convolution, see 6.1. Moreover, 6.3 provides a functor from (a) to \( \mathcal{Z}^c_{\mathfrak{f}} \) which is an equivalence when restricted to any \( \mathcal{C}^c \mathcal{G}_{\epsilon, s} \). This functor is compatible with the monoidal structures (this can be deduced from 3.8 and from the fact that the monoidal structure of \( \mathcal{Z}^c_{\mathfrak{f}} \) is equivalent to its opposite). Note that \( \mathcal{C}^c \mathcal{G}_{1, 0} \) is a monoidal subcategory of (a), whose unit object, described in [L12, 9.10], is also a unit object for the monoidal category (a).
6.9. The functor $L \mapsto {}^\epsilon L$ from $\mathcal{C}^cB^2$ into itself induces a functor $\mathcal{Z}_c^c \to \mathcal{Z}_c^c$ which carries any simple object $(L,e_L)$ of $\mathcal{Z}_c^c$ into an object isomorphic to $(L,e_L)$; this follows from 2.10(c), using Theorem 6.3.

6.10. Let $s \in \mathbb{Z}$. For any $A \in \mathcal{C}^cG^c_{s,s,c}$ and any $x \in c$ we denote by $n_{A,x}$ the multiplicity of $A$ in $\chi_{c,s}(L_x) \in \mathcal{C}^cG^c_{s,s,c}$. From Theorem 6.3 and its proof we see that if $\sigma$ is the simple object of $\mathcal{Z}_c^c$ corresponding to $A$, then $n_{A,x}$ is equal to the multiplicity of $\sigma$ in $I_{\epsilon}(L_x) \in \mathcal{Z}_c^c$. In particular, $n_{A,x}$ is independent of $s$.

7. Relation with Soergel bimodules

7.1. Let $R$ be the algebra of polynomials functions on a fixed reflection representation of $W$ (over $\mathbb{Q}_l$). Then for each $x \in W$, the indecomposable Soergel graded $R$-bimodule $B_x$ is defined as in [So, 6.16]. Let $C_c$ be the category of graded $R$-bimodules which are isomorphic to finite direct sums of graded $R$-bimodules of the form $B_x$ ($x \in c$) without shift. There is a well defined functor $M \mapsto {}^\epsilon M$ from $C_c$ to $C_c$ which is linear and satisfies ${}^\epsilon B_x = B_{\epsilon^{-1}(x)}$ for $x \in c$. Now $C_c$ has a natural monoidal structure (see [L12, 10.1] defined purely in terms of $R,W,c$). (Its definition makes use of the results in [EW].) From the definition we see that $C_c$ is equivalent to $\mathcal{C}^cB^2$ as monoidal categories so that $M \mapsto {}^\epsilon M$ corresponds to $L \mapsto {}^\epsilon L$ from $\mathcal{C}^cB^2$ to itself. Then the $\epsilon$-centre of $C_c$ is defined as in 6.2. It is naturally equivalent to $\mathcal{Z}_c^c$. Thus we can restate Theorem 6.3 as follows.

(a) For any $s \in \mathbb{Z}$, the category $\mathcal{C}^cG^c_{s,s,c} \to \mathcal{Z}_c^c$ is naturally equivalent to the $\epsilon$-centre of the monoidal category $C_c$.

This, combined with 1.6(b), shows that for $F \in G^c_{s,s}$ (with $s > 0$), the category $\text{Rep}^\epsilon(G^F)$ is equivalent to the $\epsilon$-centre of the monoidal category $C_c$; thus, the set of simple objects of $\text{Rep}^\epsilon(G^F)$ is not only independent of $s$ but also independent of the characteristic of $k$, since the $\epsilon$-centre of $C_c$ is so. (Here we identify $\mathbb{Q}_l$ with the complex numbers.)

7.2. As mentioned in [L12, 10.1], the definition of the monoidal category $C_c$ makes sense even when $W$ is replaced by any (say finite, irreducible) Coxeter group and $c$ is a two-sided cell in $W$. Assume now that $\epsilon : W \to W$ is an automorphism of $W$ which leaves stable the set of simple reflections and leaves stable $c$. Then the definition of the $\epsilon$-centre of $C_c$ makes sense even if $W$ is noncrystallographic. We expect that the indecomposable objects of the $\epsilon$-centre of $C_c$ are in bijection with the “unipotent characters” associated to $W,\epsilon,c$ in [L5]. (For $\epsilon = 1$ this expectation has already been stated in [L12, 10.1].)

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