Energy Spectrum of Secondary Leptons in $e^+e^- \rightarrow t\bar{t}$
— Non-Standard Interactions and CP violation —

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ABSTRACT

The process of top-quark pair production at future high-energy $e^+e^-$ linear colliders has been investigated as a possible test of physics beyond the Standard Model. Non-standard interactions have been assumed both for the production and for the subsequent decay of the top quarks. The energy spectrum of the single lepton $\ell^\pm$ and the energy correlation of $\ell^+$ and $\ell^-$ emerging from the process $e^+e^- \rightarrow t\bar{t} \rightarrow \ell^\pm X/\ell^+\ell^-X$ are calculated. The energy-spectrum asymmetry of $\ell^+$ and $\ell^-$ is considered as a measure of CP violation. An optimal method to determine whether CP violation occurs in the production or in the decay processes is proposed.

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1. Introduction

$CP$ violation is a challenging significant problem in electroweak physics. Decays of $D$ and $B$ mesons have been extensively investigated for this purpose. In the near future we are expecting much more fruitful experimental data from $B$-factories under construction. On the other hand, the top-quark production may be another efficient source of information on $CP$ violation once Large Hadron Collider (LHC) and/or Next Linear Collider (NLC) are constructed, as discussed in [1−6].

It is relevant to notice that the amount of $CP$ violation provided by the Cabibbo-Kobayashi-Maskawa mechanism for the top-quark sector is tiny, therefore $CP$ violation in this sector offers a wide window to look for physics beyond the Standard Model (SM). Furthermore, the top quark is expected to give us a unique opportunity to study quark interactions much more directly thanks to its extremely large mass, $m_t^{exp} = 180 \pm 12 \text{ GeV}$ [7]. Since the top quark is so heavy it decays as a single quark before forming bound states, therefore it is possible to avoid complicated non-perturbative effects brought through fragmentation processes in a case of lighter quarks.

Since $t\bar{t}$ pairs are produced through the vector-boson exchange, the handedness of $t$ and $\bar{t}$ must be the same. Consequently, the helicities of $t\bar{t}$ would be $(+-)$ or $(-+)$ if the top mass were much smaller than $\sqrt{s}$. However, since the observed $m_t$ is far from being negligible at any accelerators in the planning stage, we will also face copious production of $(++)$ and $(--)$. For example, $\sigma_{tot}(e^+e^- \to t\bar{t})$ is estimated to be 0.60 pb for $\sqrt{s} = 500$ GeV (and $m_t = 180$ GeV) within the SM, in which $N(-+) : N(+-) : N(--) : N(++)$ is $4.8 : 3.4 : 0.9 : 0.9$, where $N(\cdots)$ denotes the number of $t\bar{t}$ pairs with the indicated helicities (cf. this ratio would be about $6.1 : 3.9 : O(10^{-5}) : O(10^{-5})$ if $m_t$ were the same as $m_b$).

We can use this fact to explore $CP$ properties of the $t\bar{t}$ state: $|-+\rangle$ and $|+\rangle$ are $CP$ self-conjugate while $|--\rangle$ and $|++\rangle$ transform into each other under $CP$
operation as

$$\hat{CP}|\mp\mp\rangle = \hat{C}|\pm\pm\rangle = |\pm\pm\rangle.$$ 

This indicates that the difference between $N(- -)$ and $N(+ +)$ could be a useful measure of $CP$ violation [4 – 6], although what we can observe in experiments are not the top quarks but products of their subsequent decays. Fortunately we know that the semileptonic decays can serve as an efficient top-quark-spin analyzers [8]. Indeed, the energy spectrum of $\ell^+$ and $\ell^-$ in $e^+e^- \rightarrow t\bar{t} \rightarrow b\bar{b}W^+W^- \rightarrow b\bar{b}\ell^+\nu\ell^-\bar{\nu}$ can be a good measure of $N(- -) - N(+ +)$, as we will see. One can understand it qualitatively since:

1. The large top mass requires a predominantly longitudinal $W$ in $t \rightarrow bW$ since $\bar{b}r_\mu(\gamma_5)t \cdot \varepsilon^\mu \sim m_b \bar{b}(\gamma_5)t$ when $\varepsilon^\mu = \varepsilon_L^\mu \sim k^\mu$ ($\varepsilon$ and $k$ are the polarization and the four-momentum of $W$, respectively).

2. The produced $b$ ($\bar{b}$) is left-handed (right-handed) in the SM since $m_b/\sqrt{s} \ll 1$.

3. Because of (1) and (2), $W^+$’s three-momentum prefers to be parallel (antiparallel) to that of $t(+) (t(-))$, where $t(\cdots)$ expresses a top with the indicated helicity. Consequently $\ell^+$ in the $t(+) \rightarrow \ell$ decay becomes more energetic than in the $t(-)$ decay, while it is just opposite for the $\bar{\ell}$ decay, i.e., $\bar{\ell}(-)$ produces more energetic $\ell^-$ than $\bar{\ell}(+)$ does.

4. Therefore, we expect larger number of energetic $\ell^+$ ($\ell^-$) for $N(- -) < N(+ +)$ (for $N(- -) > N(+ +)$).

The leptonic energy spectrum has been studied in the existing literature [4, 9]. However, in those articles, $CP$-violating interactions were assumed only in the $t\bar{t}\gamma/Z$ vertices, and the standard-model vertex was used for the $t \rightarrow bW$ decay. In this paper, in order to perform a consistent analysis we compute the spectrum assuming that both the $t\bar{t}\gamma/Z$ vertices and the $tbW$ vertex include non-standard $CP$-violating form factors. Concerning the $W$ decays, we shall treat them as in
the SM since it is known through various charged-current processes that the $W$ couplings with light fermions are successfully described within the SM. That is, we shall assume here that only the top-quark interactions may be modified by physics beyond the SM.

The paper is organized as follows. In sec. 2 we will describe a formalism for the energy spectrum calculation together with some related SM results. In sec. 3 we will consider the top-quark decay with non-standard interactions. Section 4 will contain a derivation of the lepton-energy spectrum with $CP$ violation present both in the production and in the decay. Then, in sec. 5, we will discuss how to measure $CP$ violation effectively and propose an optimal method to disentangle effects originating from the production and from the decay. In the Appendix, explicit forms of some functions used in the text will be presented.

2. The lepton-energy spectrum

Before proceeding to actual study of $CP$ violation, let us briefly describe the formalism which we use in this paper, and show the related standard-model calculations.

We will treat all the fermions except the top-quark as massless and adopt a technique developed by Kawasaki, Shirafuji and Tsai [10]. This is a useful method to calculate distributions of final particles appearing in a process of production and subsequent decay. This technique is applicable when the narrow-width approximation

$$\left| \frac{1}{p^2 - m^2 + im\Gamma} \right|^2 \simeq \frac{\pi}{m\Gamma}\delta(p^2 - m^2)$$

can be adopted for the decaying intermediate particles. In fact, this is very well satisfied for the production and subsequent decays of $t$ and $W$ since $\Gamma_t \simeq 175 \text{ MeV}(m_t/m_W)^3 \ll m_t$ and $\Gamma_W = 2.08 \pm 0.07 \text{ GeV} [11] \ll M_W$.

Adopting this method, one can derive the following formula for the inclusive distribution of the single-lepton $\ell^+$ in the reaction $e^+e^- \rightarrow t\bar{t}$ [9]:

$$\frac{d^3\sigma}{d^3p_t/(2p_t^\ell)}(e^+e^- \rightarrow \ell^+ + \cdots)$$
\[ 4 \int d\Omega_t \frac{d\sigma}{d\Omega_t}(n, 0) \frac{1}{\Gamma^3_t d^3 p_t/(2p_t)} (t \rightarrow b\ell^+ \nu), \]  

where \( \Gamma_t \) is the leptonic width of unpolarized top and \( d\sigma(n, 0)/d\Omega_t \) is obtained from the angular distribution of \( t\bar{t} \) with spins \( s_+ \) and \( s_- \) in \( e^+e^- \rightarrow t\bar{t} \), \( d\sigma(s_+, s_-)/d\Omega_t \),  

by the following replacement:

\[ s_+^\mu \rightarrow n^\mu = \left( g^\mu\nu - \frac{p_t^\mu p_t^\nu}{m_t^2} \right) \frac{m_t}{p_t} p_{\ell\nu} \quad \text{and} \quad s_- \rightarrow 0. \]  

(Exchanging the roles of \( s_+ \) and \( s_- \) and reversing the sign of \( n^\mu \), we get the distribution of \( \ell^- \).)

Following ref. [9], let us introduce the rescaled lepton-energy, \( x \), by

\[ x = \frac{2E_\ell}{m_t} \left( \frac{1 - \beta}{1 + \beta} \right)^{1/2}, \]  

where \( E_\ell \) is the energy of \( \ell \) in \( e^+e^- \) c.m. frame and \( \beta = \sqrt{1 - 4m_t^2/s} \) (\( s \equiv (p_{e^+} + p_{e^-})^2 \)). We also define three parameters \( D_V \), \( D_A \) and \( D_{VA} \) as

\[ D_V = |v_e v_t d - \frac{2}{3}|^2 + |a_e v_t d|^2; \]
\[ D_A = |v_e a_t d|^2 + |a_e a_t d|^2; \]
\[ D_{VA} = v_e a_t d(v_e v_t d - \frac{2}{3})^* + a_e a_t d(a_e v_t d)^*, \]

by using the standard-model neutral-current parameters of \( e \) and \( t \): \( v_e = -1 + 4\sin^2\theta_W, a_e = -1, v_t = 1 - (8/3)\sin^2\theta_W, \) and \( a_t = 1, \) and a \( Z \)-propagator factor

\[ d = \frac{s}{s - M_Z^2 + iM_Z \Gamma_Z 16\sin^2\theta_W \cos^2\theta_W}. \]

Then, the \( x \) spectrum is given in terms of these quantities by

\[ \frac{1}{B_\ell \sigma_{ee-t\bar{t}}} \frac{d\sigma^\pm}{dx} = \frac{1}{B_\ell \sigma_{ee-t\bar{t}}} \frac{d\sigma}{dx}(e^+e^- \rightarrow \ell^\pm + \cdots) = f(x) + \eta g(x). \]  

Here \( \sigma_{ee-t\bar{t}} \equiv \sigma_{tot}(e^+e^- \rightarrow t\bar{t}) \), \( B_\ell \) is the branching ratio for \( t \rightarrow \ell + \cdots \) (\( \sim 0.22 \) for \( \ell = e, \mu \)), \( f(x) \) and \( g(x) \) are functions derived in [9], which we give in the Appendix, and \( \eta \) is defined as

\[ \eta = \frac{4 \text{Re}(D_{VA})}{(3 - \beta^2)D_V + 2\beta^2 D_A}. \]
Figure 1: The normalized \((x, \bar{x})\) distribution \(\sigma^{-1}d^2\sigma/(dx d\bar{x})\) without CP violation.

\(f(x)\) and \(g(x)\) satisfy the following normalization conditions:

\[
\int f(x)dx = 1 \quad \text{and} \quad \int g(x)dx = 0.
\]

(5)

Applying the same technique, we get the following energy correlation of \(\ell^+\) and \(\ell^-\):

\[
\frac{1}{B_{\ell^+}\sigma_{ee\rightarrow\ell^+}} \frac{d^2\sigma}{dx d\bar{x}} = S_0(x, \bar{x}),
\]

(6)

where \(x\) and \(\bar{x}\) are the rescaled energies of \(\ell^+\) and \(\ell^-\) respectively, and

\[
S_0(x, \bar{x}) = f(x)f(\bar{x}) + \eta' g(x)g(\bar{x}) + \eta\left[ f(x)g(\bar{x}) + g(x)f(\bar{x}) \right]
\]

with \(\eta'\) being defined as

\[
\eta' \equiv \frac{1}{\beta^2} \frac{(1 + \beta^2)D_V + 2\beta^2 D_A}{(3 - \beta^2)D_V + 2\beta^2 D_A}.
\]
Clearly, the \((x, \bar{x})\) distribution is symmetric in \(x\) and \(\bar{x}\), which is a sign of CP symmetry. The distribution is presented in fig.1 for \(\sqrt{s} = 500\) GeV and the SM parameters \(\sin^2 \theta_W = 0.2325\), \(M_W = 80.26\) GeV, \(M_Z = 91.1884\) GeV, \(\Gamma_Z = 2.4963\) GeV and \(m_t = 180\) GeV.

3. Non-standard interactions and the top-quark decay

We will assume that all non-standard effects in the production process can be represented by the photon and \(Z\)-boson exchange in the \(s\)-channel in the following way:

\[\Gamma^\mu_v = \frac{g^2}{2} \bar{u}(p_t) \left[ \gamma^\mu (A_v - B_v \gamma_5) + \frac{(p_t - p_{\ell})^\mu}{2m_t} (C_v - D_v \gamma_5) \right] v(p_t), \quad (7)\]

where \(v = \gamma\) or \(Z\) and \(g\) is the SU(2) gauge-coupling constant. A non-zero value of \(D_v\) is a signal of CP violation.

For the on-shell \(W\), we will adopt the following parameterization of the \(tbW\) vertex suitable for the \(t \to W^+b\) and \(\bar{t} \to W^-\bar{b}\) decays:

\[\Gamma^\mu = -\frac{g}{\sqrt{2}} V_{tb} \bar{u}(p_b) \left[ \gamma^\mu (f_1^L P_L + f_1^R P_R) - \frac{i\sigma^{\mu\nu} k_\nu}{M_W} \right] u(p_t), \quad (8)\]

\[\bar{\Gamma}^\mu = -\frac{g}{\sqrt{2}} V_{tb}^* \bar{v}(p_t) \left[ \gamma^\mu (\bar{f}_1^L P_L + \bar{f}_1^R P_R) - \frac{i\sigma^{\mu\nu} k_\nu}{M_W} \right] v(p_b), \quad (9)\]

where \(P_{L/R} = (1 \mp \gamma_5)/2\), \(V_{tb}\) is the \((tb)\) element of the Kobayashi-Maskawa matrix and \(k\) is the momentum of \(W\). Again, because \(W\) is on shell, the two additional form factors do not contribute. One can show that

\[f_1^{L,R} = \pm \bar{f}_1^{L,R}, \quad f_2^{L,R} = \pm \bar{f}_2^{R,L}, \quad (10)\]

where upper (lower) signs are those for CP-conserving (-violating) contributions. Therefore any CP-violating observable defined for the top-quark decay must be proportional to \(f_1^{L,R} - \bar{f}_1^{L,R}\) or \(f_2^{L,R} - \bar{f}_2^{R,L}\).

\(^{21}\)Two other possible form factors do not contribute in the limit of zero electron mass.
We shall consider here the top-quark decay with the above non-standard-interaction terms. Assuming that \( f^L_1 - 1, f^R_1, f^L_2 \) and \( f^R_2 \) are small and keeping only linear terms, we obtain for the double differential spectrum in \( x \) and \( \omega \equiv (p_t - p_\ell)/m^2_t \) the following result:

\[
\frac{1}{T_t} \frac{d^2 \Gamma_t}{dx d\omega} (t \to b\ell^+\nu) = \frac{1 + \beta}{\beta} 3 B_t \frac{W}{W} \omega \left[ 1 + 2 \text{Re}(f^R_2) \sqrt{r} \left( \frac{1}{1 - \omega} - \frac{3}{1 + 2r} \right) \right],
\]

(11)

where

\[
W \equiv (1 - r)^2 (1 + 2r), \quad r \equiv M^2_W/m^2_t.
\]

An analogous formula for \( \bar{t} \to \bar{b} \ell^- \bar{\nu} \) holds with \( f^R_2 \) replaced by \( \bar{f}^L_2 \).

4. \( CP \) violation in the production and in the decay processes

Combining the results of the previous sections, we obtain the lepton-energy spectrum for \( e^+e^- \to t^\pm + \cdots \) with the non-standard \( CP \)-violating terms as

\[
\frac{1}{B_t \sigma_{e\ell\to st\ell}} \frac{d\sigma}{dx} = F_\pm(x) + (\eta \mp \xi) G_\pm(x),
\]

(12)

where

\[
\xi \equiv \frac{1}{(3 - \beta^2)D_V + 2\beta^2 D_A} - \frac{1}{\sin \theta_W} \text{Re} \left[ \frac{2}{3} D_\gamma + \frac{s^2}{(s - M_Z^2)^2 + M_Z^2 D_Z} \frac{(v_e^2 + \alpha_e^2)v_t}{64 \sin^2 \theta_W \cos^3 \theta_W} \right] D_Z
\]

\[
- \frac{-1}{s - M_Z^2} \left( \frac{v_e v_t}{16 \sin^2 \theta_W \cos^2 \theta_W} \right) \frac{1}{6 \sin \theta_W \cos \theta_W} D_Z
\]

which characterizes the \( CP \) violation in the \( t\bar{t} \) production process\(^{22}\) and \( F_\pm(x) \) and \( G_\pm(x) \) are defined as

\[
F_+(x) = f(x) + \text{Re}(f^R_2) \delta f(x), \quad G_+(x) = g(x) + \text{Re}(f^R_2) \delta g(x),
\]

(13)

\[
F_-(x) = f(x) + \text{Re}(f^L_2) \delta f(x), \quad G_-(x) = g(x) + \text{Re}(f^L_2) \delta g(x),
\]

(14)

\(^{22}\)Since our main interest is in \( CP \) violation, we dropped all the \( CP \)-conserving non-standard terms in eq.(7).

\(^{23}\)This point will become much clearer in later discussions (see eq.(21)).
with $\delta f(x)$ and $\delta g(x)$ being given in the Appendix. Note that $F_\pm(x)$ and $G_\pm(x)$ satisfy the same normalization conditions as $f(x)$ and $g(x)$:

$$
\int F_\pm(x) dx = 1 \quad \text{and} \quad \int G_\pm(x) dx = 0. \tag{15}
$$

![Cross-section with CP-violation](image)

Figure 2: The normalized $(x, \bar{x})$ distribution $\sigma^{-1} d^2 \sigma/(dx d\bar{x})$ with CP violation for $\text{Re}(D_\gamma) = \text{Re}(D_Z) = \text{Re}(f_2^R) = -\text{Re}(f_2^L) = 0.2$.

The $(x, \bar{x})$ distribution receives extra pieces, which are anti-symmetric in $x$ and $\bar{x}$:

$$
\frac{1}{B_\ell^2 \sigma_{ee\rightarrow tt}} \frac{d^2 \sigma}{dx \, d\bar{x}} = S(x, \bar{x}) + \xi A_\xi(x, \bar{x}), \tag{16}
$$

where $S(x, \bar{x})$ is obtained through replacement of $f(x)$ and $g(x)$ ($f(\bar{x})$ and $g(\bar{x})$) by $F_+(x)$ and $G_+(x)$ ($F_-(\bar{x})$ and $G_-(\bar{x})$) in $S_0(x, \bar{x})$ as

$$
S(x, \bar{x}) = F_+(x)F_-(\bar{x}) + \eta' G_+(x)G_-(\bar{x})
$$
\[ + \eta [ F_+(x) G_-(\bar{x}) + G_+(x) F_-(\bar{x}) ] \]

and

\[ A_\xi(x, \bar{x}) = F_+(x) G_-(\bar{x}) - G_+(x) F_-(\bar{x}). \]

The distribution is shown for Re\( D_\gamma \) = Re\( D_Z \) = Re\( f^R_2 \) = -Re\( f^L_2 \) = 0.2 in fig.2, where all the SM parameters are the same as in fig.1. Since we assumed that the non-standard interactions are not strong, the two distributions in figs.1 and 2 look similar to each other at first sight. However, looking carefully at the contour lines, we find that the distribution in fig.2 is not symmetric in \( x \) and \( \bar{x} \), which is a sign of \( CP \) violation. In order to show more explicitly the both \( CP \)-violating

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{The \( CP \)-violating function \( A_\xi(x, \bar{x}) \) given in eq.(18).}
\end{figure}

\( CP \)-violating

\( CP \)-violating
contributions, we re-express the right-hand side of eq.(16) as

\[ S_0(x, \bar{x}) + \xi A_\xi(x, \bar{x}) + \frac{1}{2} \text{Re}(f_2^R \bar{f}_2^L) A_f(x, \bar{x}), \]  

(17)

and show \(A_\xi, f(x, \bar{x})\) in figs.3 and 4 respectively.

\[ A_\xi(x, \bar{x}) = f(x)g(\bar{x}) - g(x)f(\bar{x}), \]  

(18)

\[ A_f(x, \bar{x}) = \delta f(x)f(\bar{x}) - f(x)\delta f(\bar{x}) + \eta' \left[ \delta g(x)g(\bar{x}) - g(x)\delta g(\bar{x}) \right] \]

\[ + \eta \left[ \delta f(x)g(\bar{x}) - f(x)\delta g(\bar{x}) + \delta g(x)f(\bar{x}) - g(x)\delta f(\bar{x}) \right], \]  

(19)
5. Measurements of $CP$ violation

As mentioned in the Introduction,

$$\delta \equiv \frac{N(-) - N(++)}{N(all)} \quad (20)$$

is a measure of $CP$ violation in the production process. One can show, assuming the dominance of $\gamma$ and $Z$ exchange in the $s$-channel, that $\delta$ is related to the parameter $\xi$ introduced in eq.(12):

$$\delta = -\beta \xi. \quad (21)$$

If there was no $CP$ violation in the $tbW$ vertex, the energy-spectrum asymmetry $a(x)$ defined as

$$a(x) \equiv \frac{d\sigma^-}{dx} - \frac{d\sigma^+}{dx} \quad \frac{d\sigma^-}{dx} + \frac{d\sigma^+}{dx} \quad (22)$$

would be given by a simple form

$$a(x) = -\frac{\delta}{\beta} \frac{g(x)}{f(x) + \eta g(x)},$$

and may serve as a useful observable to measure $CP$ violation. However, when the $CP$-violating contributions to the $tbW$ vertex are taken into account, it becomes

$$a(x) = \frac{-2(\delta/\beta) g(x) + \text{Re}(f^R_2 - \bar{f}^L_2)[\delta f(x) + \eta \delta g(x)]}{2 [f(x) + \eta g(x)]}.$$

Therefore, it turns out that $a(x)$ is not a direct measure of the helicity asymmetry $\delta$. Measuring a differential asymmetry is a challenging task since $a(x)$ has not been integrated over the energy and therefore the expected statistics cannot be high.

We shall find more appropriate observables to measure $CP$ violation in the production (expressed by $\xi$) and that in the decay process (expressed by $\text{Re}(f_2^R - \bar{f}^L_2)$) individually. It shall be useful to write down explicit expressions for the single lepton spectrum:

$$\frac{1}{\sigma^+} \frac{d\sigma^+}{dx} = f(x) + \eta g(x) - \xi g(x) + \text{Re}(f^R_2)[\delta f(x) + \eta \delta g(x)], \quad (23)$$

$$\frac{1}{\sigma^-} \frac{d\sigma^-}{dx} = f(x) + \eta g(x) + \xi g(x) + \text{Re}(\bar{f}^L_2)[\delta f(x) + \eta \delta g(x)]. \quad (24)$$
where

\[ \sigma^\pm \equiv \int dx \frac{d\sigma^\pm}{dx} = B_t \sigma_{e\bar{e}\to t\bar{t}}. \]

Now, following the methods developed in ref. [12] one can show that in order to maximize statistical significance of the CP-violating signal the following observables should be applied:

\[ O_{t\bar{t}V}^\pm = \frac{1}{\sigma^\pm} \int dx \frac{d\sigma^\pm}{dx} \Theta_{t\bar{t}V}(x), \quad O_{t\bar{b}W}^\pm = \frac{1}{\sigma^\pm} \int dx \frac{d\sigma^\pm}{dx} \Theta_{t\bar{b}W}(x), \quad (25) \]

where we shall use

\[ \Theta_{t\bar{t}V}(x) = \frac{g(x)}{f(x) + \eta g(x)}, \quad \Theta_{t\bar{b}W}(x) = \frac{\delta f(x) + \eta \delta g(x)}{f(x) + \eta g(x)}, \quad (26) \]

as the weighting functions. \( O_{t\bar{t}V}^\pm \) and \( O_{t\bar{b}W}^\pm \) are the most sensitive observables for a measurement of \( \xi \) and \( \text{Re}(f_2^R - \bar{f}_2^L) \) respectively. Once those \( O_{t\bar{t}V}, O_{t\bar{b}W} \) are experimentally determined, we are able to obtain \( \xi \) and \( \text{Re}(f_2^R - \bar{f}_2^L) \) through

\[ 2\xi = \frac{-c(O_{t\bar{t}V}^+ - O_{t\bar{t}V}^-) - a(O_{t\bar{b}W}^+ - O_{t\bar{b}W}^-)}{a^2 - bc} \]

\[ = \int dx \left[ \frac{1}{\sigma^+ dx} - \frac{1}{\sigma^- dx} \right] \Omega_\xi(x), \quad (27) \]

\[ \text{Re}(f_2^R - \bar{f}_2^L) = \frac{-a(O_{t\bar{t}V}^+ - O_{t\bar{t}V}^-) - b(O_{t\bar{b}W}^+ - O_{t\bar{b}W}^-)}{a^2 - bc} \]

\[ = \int dx \left[ \frac{1}{\sigma^+ dx} - \frac{1}{\sigma^- dx} \right] \Omega_f(x), \quad (28) \]

for

\[ \Omega_\xi(x) = \frac{c \Theta_{t\bar{t}V}(x) - a \Theta_{t\bar{b}W}(x)}{a^2 - bc}, \quad \Omega_f(x) = \frac{a \Theta_{t\bar{t}V}(x) - b \Theta_{t\bar{b}W}(x)}{a^2 - bc}, \]

where

\[ a \equiv \int dx \frac{g(x)[\delta f(x) + \eta \delta g(x)]}{f(x) + \eta g(x)}, \]

\[ b \equiv \int dx \frac{g^2(x)}{f(x) + \eta g(x)}, \]

\[ c \equiv \int dx \frac{[\delta f(x) + \eta \delta g(x)]^2}{f(x) + \eta g(x)}. \]
Using eqs. (27, 28) one can calculate the statistical errors for $2\xi$ and $\text{Re}(f_2^R - \bar{f}_2^L)$ measurements:

$$\Delta_i = \left[\Delta_{i+}^2 + \Delta_{i-}^2\right]^{1/2} \quad (i = \xi, f),$$

(29)

where $\Delta_{i\pm}$ denotes the statistical error for $(\sigma^\pm)^{-1}\int dx \Omega_i(x)(d\sigma^\pm/dx)$ measurement given by

$$\Delta_{i\pm} = \frac{1}{\sqrt{N_\ell}} \left[\frac{1}{\sigma^\pm} \int dx \Omega_i^2(x) \frac{d\sigma^\pm}{dx} - \left(\frac{1}{\sigma^\pm} \int dx \Omega_i(x) \frac{d\sigma^\pm}{dx}\right)^2\right]^{1/2},$$

(30)

with $N_\ell$ being the total number of events with one lepton $\ell^\pm$ for the integrated luminosity $L$. Therefore the statistical significances $N_{SD}^{thV}$ and $N_{SD}^{thW}$ with which the presence of a nonzero value of $\xi$ and $\text{Re}(f_2^R - \bar{f}_2^L)$ may be ascertained are

$$N_{SD}^{thV} = |2\xi|/\Delta_\xi \quad \text{and} \quad N_{SD}^{thW} = |\text{Re}(f_2^R - \bar{f}_2^L)|/\Delta_f.$$

(31)

Within the approximation adopted in this paper, we may use the standard-model formula to estimate the size of $\Delta$’s, consequently we have $\Delta_{i+} = \Delta_{i-}$. Eventually we obtain for the errors:

$$\Delta_\xi = 17.44/\sqrt{N_\ell} \quad \text{and} \quad \Delta_f = 10.06/\sqrt{N_\ell}.$$

(32)

There are two quantities relevant for the experimental potential of NLC, namely the total integrated luminosity $L$ and the tagging efficiency for an observation of $t\bar{t}$ pairs $\epsilon_{tt}$ in various decay channels. Since they enter the statistical significance in a combination $\sqrt{\epsilon_{tt}L}$, it will be useful to adopt a notation $\epsilon_L \equiv \sqrt{\epsilon_{tt}L}$ and parameterize our results in terms of $\epsilon_L$. Table 1 shows $\epsilon_{tt}$ (in %) necessary to achieve a desired $\epsilon_L$ corresponding to a given luminosity $L$ (note that $\epsilon_{tt} \leq \epsilon_{L} \approx 22\%$ for the single-lepton-inclusive final state).

| L (fb$^{-1}$) | $\epsilon_{tt}$ (in %) |
|--------------|------------------------|
| 40           | 15                     |
| 40           | 20                     |

For instance, $\epsilon_{tt} = 15\%$ may be obtained for 4 jets + one charged lepton. If $L = 40\text{ fb}^{-1}$ is achieved, we will obtain $\epsilon_L = 77.5\text{ pb}^{-1/2}$. Since $\sigma_{e\bar{e}\to tt} = 0.60\text{ pb}$

$L = 10 - 100\text{ fb}^{-1}$ is used in, e.g., [14].
for $m_t = 180$ GeV, we have $\sqrt{N_\ell}=\sqrt{\epsilon_{tt}}/\sqrt{\epsilon_{e\bar{e}\rightarrow t\bar{t}}}=0.77 \epsilon_L/ pb^{-1/2}$. Therefore we can compute the minimal values for $|\xi|$ and $|\text{Re}(f^R_2 - \bar{f}^L_2)|$ observable at a desired statistical significance for a given $\epsilon_L$ as

$$|\xi|^{\text{min}} = 11.3(N_{SD}^{ttV}/\epsilon_L \text{ pb}^{1/2}) \text{ and } |\text{Re}(f^R_2 - \bar{f}^L_2)|^{\text{min}} = 13.1(N_{SD}^{tbV}/\epsilon_L \text{ pb}^{1/2}).$$

(33)

Having $\epsilon_L = 77.5 \text{ pb}^{-1/2}$ we will be able to test $\xi$ down to 0.44 and $\text{Re}(f^R_2 - \bar{f}^L_2)$ down to 0.51 at three standard deviations. Some other typical values are given in tables 2 and 3.

Table 1: $\epsilon_{tt}$ (in %) necessary to achieve a desired $\epsilon_L$ corresponding to a given luminosity $L$ for the single-lepton-inclusive final state.

| $\epsilon_L \text{ (pb}^{-1/2})$ | $L \text{ (10}^4\text{ pb}^{-1})$ |
|-----------------------------|-----------------------------|
| 50                         | 12.5                        |
| 100                        | 20.0                        |
| 200                        | 20.0                        |

Table 2: Minimal values of $|\xi|$ observable at a desired statistical significance $N_{SD}^{ttV}$ for a given $\epsilon_L$. 

| $\epsilon_L \text{ (pb}^{-1/2})$ | $N_{SD}^{ttV}$ |
|-----------------------------|----------------|
| 50                         | 0.23 0.45 0.68 0.90 1.13 |
| 100                        | 0.11 0.23 0.34 0.45 0.57 |
| 200                        | 0.06 0.11 0.17 0.45 0.28 |

We have not considered any background yet. However, since majority of the single-lepton-inclusive final states is made of 4 jets + one charged lepton + missing energy, the background seems to be easy under control. In this case the final state
could be fully reconstructed since there is only one neutrino, 3 jets must add up to a priori known top-quark mass, and two of them must have the $M_W$ invariant mass. Because of those constraints we would assume that the background could be neglected.

*Within the SM non-zero $\xi$ and $\text{Re}(f_2^R - \bar{f}_2^L)$ may appear at the two-loop level.* Therefore, *an observation of non-zero $\xi$ or $\text{Re}(f_2^R - \bar{f}_2^L)$ would be a strong indication for non-standard physics.*

6. Summary

Next-generation linear colliders of $e^+e^-$ will provide a cleanest environment for studying top-quark interactions. There, we shall be able to perform detailed tests of the top-quark couplings to the vector bosons and either confirm the SM simple generation-repetition pattern or discover some non-standard interactions.

In this paper, we have studied the non-standard $CP$-violating interactions in the $t\bar{t}$ productions and their subsequent decays. $CP$ violation has been parameterized by $\xi$ (eqs.(12, 20, 21)) and $\text{Re}(f_2^R - \bar{f}_2^L)$ (eq.(10)) for the production and decay process, respectively. If the top-quark decay was described by the SM interactions (as it was done in the previous study [4, 5, 9]), then we would have a compact useful formula for a measurement of $CP$ violation in the $t\bar{t}\gamma/Z$ vertices via the final-lepton energy-asymmetry (22). However, in general, $CP$ violation may also enter through the top-decay process at the same strength as it does for the production. Therefore,
we have assumed the most general \( CP \)-violating interactions both in the production and in the decay vertices in order to perform a consistent analysis.

We have defined four optimal observables \( O_{t\bar{t}V}^\pm \) and \( O_{t\bar{t}W}^\pm \) in eq.(25) which yield minimal statistical errors in the determination of \( CP \)-violation parameters. Therefore the statistical significance of the non-standard signal is maximal. Adopting those observables, we have presented in eq.(33), tables 2, and 3 the minimal values of the \( CP \)-violation parameters which can be observed, as a function of the luminosity \( (L) \) of the NLC and the achieved tagging efficiency \( (\epsilon_{tt}) \). For \( L = 40 \text{ fb}^{-1} \) and \( \epsilon_{tt} = 15\% \), one will be able to measure the \( CP \)-violating parameters in the \( t\bar{t} \) production and \( t \) decay, i.e. \( \xi \) and \( \text{Re}(f_2^R - \bar{f}_2^L) \) respectively, at the 3\( \sigma \) level if they are larger than 0.44 and 0.51, respectively. It should be emphasized that an observation of a non-zero signal would be a strong evidence of physics beyond the SM.

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Appendix

The functions \( f(x) \), \( g(x) \), \( \delta f(x) \) and \( \delta g(x) \) are defined as follows:

\[
 f(x) = C_1 \left\{ r(r-2) + 2x \frac{1+\beta}{1-\beta} - x^2 \left( \frac{1+\beta}{1-\beta} \right)^2 \right\},
\]

(for the interval \( I_1, I_4 \))

\[
 = C_1 (1-r)^2;
\]

(for the interval \( I_2 \))

\[
 = C_1 (1-x)^2;
\]

(for the interval \( I_3, I_6 \))
\[ g(x) = C_2 \left[ -rx + x^2 \frac{1 + \beta}{1 - \beta} - x \ln \frac{x(1 + \beta)}{r(1 - \beta)} + \frac{1}{2(1 + \beta)} \left\{ r(r - 2) + 2x \frac{1 + \beta}{1 - \beta} - x^2 \left( \frac{1 + \beta}{1 - \beta} \right)^2 \right\} \right], \]
(for the interval \( I_1, I_4 \))

\[ = C_2 \left\{ (1 - r + \ln r)x + \frac{1}{2(1 + \beta)}(1 - r)^2 \right\}, \]
(for the interval \( I_2 \))

\[ = C_2 \left\{ (1 - x + \ln x)x + \frac{1}{2(1 + \beta)}(1 - x)^2 \right\}, \]
(for the interval \( I_3, I_6 \))

\[ = C_2 x \left[ \frac{2\beta x}{1 - \beta} - \ln \frac{1 + \beta}{1 - \beta} + \frac{1}{2(1 + \beta)} \left\{ x + \frac{4\beta}{1 - \beta} - x \left( \frac{1 + \beta}{1 - \beta} \right)^2 \right\} \right], \]
(for the interval \( I_5 \))

where

\[ C_1 \equiv \frac{3}{2W} \frac{1 + \beta}{\beta}, \quad C_2 \equiv \frac{3}{W} \frac{(1 + \beta)^2}{\beta}, \]

and \( \beta, r \) and \( W \) are defined in the text (\( \beta \) is given after eq.(3), and \( r \) and \( W \) are after eq.(11)). The intervals \( I_i \) \((i = 1 \sim 6)\) of \( x \) are given by

\[ I_1 : \quad r(1 - \beta)/(1 + \beta) \leq x \leq (1 - \beta)/(1 + \beta), \]
\[ I_2 : \quad (1 - \beta)/(1 + \beta) \leq x \leq r, \]
\[ I_3 : \quad r \leq x \leq 1, \]
\[ (I_{1,2,3} \text{ are for } r \geq (1 - \beta)/(1 + \beta)) \]
\[ I_4 : \quad r(1 - \beta)/(1 + \beta) \leq x \leq r, \]
\[ I_5 : \quad r \leq x \leq (1 - \beta)/(1 + \beta), \]

\[ -18 - \]
\[ I_6 : \quad (1 - \beta)/(1 + \beta) \leq x \leq 1. \]

\[(I_{4,5,6} \text{ are for } r \leq (1 - \beta)/(1 + \beta))\]

\[ \delta f(x) = C_3 \left\{ \frac{1}{2} r(r + 8) - 2x(r + 2) \frac{1 + \beta}{1 - \beta} + \frac{3}{2} x^2 \left( \frac{1 + \beta}{1 - \beta} \right)^2 \right. \]
\[ + (1 + 2r) \ln \frac{x(1 + \beta)}{r(1 - \beta)} \bigg\}, \]

(for the interval \( I_1, I_4 \))

\[ = C_3 \left\{ \frac{1}{2} (r - 1)(r + 5) - (1 + 2r) \ln r \right\}, \]

(for the interval \( I_2 \))

\[ = C_3 \left\{ \frac{1}{2} (x - 1)(5 + 4r - 3x) - (1 + 2r) \ln x \bigg\}, \]

(for the interval \( I_3, I_6 \))

\[ = C_3 \left\{ (1 + 2r) \ln \frac{1 + \beta}{1 - \beta} - \frac{4\beta x}{1 - \beta} (r + 2) + \frac{6\beta}{(1 - \beta)^2} x^2 \right\}, \]

(for the interval \( I_5 \))

\[ \delta g(x) = C_3 \left[ 1 - \beta + 2(3 - \beta)r + \frac{1}{2} r^2 - \frac{3}{2} (1 - 2\beta) \left( \frac{1 + \beta}{1 - \beta} \right)^2 x^2 \right. \]
\[ + (1 + \beta) x \left\{ \frac{1}{r}(r - 1)(3r + 1) - \frac{2(r + 2)}{1 - \beta} \right\} \]
\[ + \left\{ 1 + 2r + 2(1 + \beta)(r + 2) x \ln \frac{x(1 + \beta)}{r(1 - \beta)} \right\}, \]

(for the interval \( I_1, I_4 \))

\[ = C_3 \left[ \frac{1}{2} (r - 1)(r + 5) - (1 + 2r) \ln r \right. \]
\[ + (1 + \beta) x \left\{ \frac{1}{r}(r - 1)(5r + 1) - 2(r + 2) \ln r \right\} \bigg\],

(for the interval \( I_2 \))

\[ = C_3 \left[ -\frac{7}{2} - 4r - \beta(2r + 1) + 2x \{ 1 - \beta + r(2 + \beta) \} \right. \]
\[ + \frac{3}{2} (1 + 2\beta) x^2 - \left\{ 2r + 1 + 2(1 + \beta)(r + 2) \ln x \right\}, \]

(for the interval \( I_3, I_6 \))

\[ = C_3 \left[ -(1 + 2r) \left( 2\beta - \ln \frac{1 + \beta}{1 - \beta} \right) + \frac{6\beta^3}{(1 - \beta)^2} x^2 \right]. \]
\[ -2(r+2)x\left\{ \frac{2\beta}{1-\beta} - (1+\beta) \ln \frac{1+\beta}{1-\beta}\right\}, \]
(for the interval $I_5$)

where

\[ C_3 \equiv \frac{6}{W} \frac{1+\beta}{\beta} \frac{\sqrt{r}}{1+2r}. \]

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