Hamiltonian Thermodynamics of Charged Black Holes

by

A.J.M. Medved and G. Kunstatter

♭ Dept. of Physics and Winnipeg Institute of Theoretical Physics
University of Manitoba, Winnipeg, Manitoba
Canada R3T 2N2
[e-mail: joey@theory.uwinnipeg.ca]

♯ Dept. of Physics and Winnipeg Institute of Theoretical Physics
University of Winnipeg, Winnipeg, Manitoba
Canada R3B 2E9
[e-mail: gabor@theory.uwinnipeg.ca]

ABSTRACT

We consider the most general diffeomorphism invariant action in 1+1 space-
time dimensions that contains a metric, dilaton and Abelian gauge field, and
has at most second derivatives of the fields. Our action contains a topological
term (linear in the Abelian field strength) that has not been considered in
previous work. We impose boundary conditions appropriate for a charged
black hole confined to a region bounded by a surface of fixed dilaton field and
temperature. By making some simplifying assumptions about the quantum
theory, the Hamiltonian partition function is obtained. This partition func-
tion is analyzed in some detail for the Reissner-Nordstrom black hole and for
the rotating BTZ black hole.

PACS 04.70.Dy
1 Introduction

The microscopic origin of black hole entropy is currently a subject of intense investigation. The Bekenstein-Hawking entropy\cite{1} of certain extremal and near extremal black holes has been successfully derived by counting states in the large coupling limit of string theory\cite{2}. It is important to keep in mind, however, that several other, very different, approaches have also achieved a measure of success\cite{3}\cite{4}\cite{5}. For example, Carlip\cite{3} has counted edge states in the gauge theory formulation of 2+1 gravity and obtained the correct entropy for the BTZ black hole\cite{6}. This calculation has taken on new importance with the realization that many of the string inspired black holes can be related to the BTZ geometry either by looking at their near horizon geometry\cite{7}, or by using M-theory inspired duality arguments\cite{8}. This suggests that the correct explanation for black hole entropy might not necessarily be tied to a specific microscopic theory, nor to any specific low energy gravity theory: it might in some sense be universal\cite{4}. It is therefore of interest to examine the statistical mechanics of black holes in a large variety of theories, in order to look for model independent features. A particularly useful arena for such investigations is generic dilaton gravity in two spacetime dimensions. This class of theories provides a large number of diffeomorphism invariant, solvable theories of gravity that admit black hole solutions. Moreover, there are several specific models in this class that are of direct physical significance, such as spherically symmetric gravity\cite{9} and Jackiw-Teitelboim gravity\cite{10}. The latter is important because its black hole solutions correspond to the dimensionally reduced BTZ black hole\cite{11}.

The study of the Hamiltonian thermodynamics of black holes in generic vacuum dilaton gravity was started in \cite{12}, generalizing a formalism first applied by Louko and Whiting\cite{13} to spherically symmetric gravity. The purpose of the present work is to extend the results of \cite{12} to include coupling to an Abelian gauge field. In particular we calculate the Hamiltonian partition function for a charged black hole confined to a “box” of fixed dilaton size. Our generic results contain as special cases all the black holes previously analyzed\cite{14} using Louko and Whiting’s formalism, and provides a unified treatment of a large variety of charged black holes. In order to compare our results to previous work and check the validity of our formalism, we will examine in some detail our expression for the partition function in the
case of spherically symmetric gravity. We will also use our results to study the Hamiltonian thermodynamics of the rotating BTZ black hole, which, to the best of our knowledge, has not to date been analyzed.

The paper is organized as follows. In Section 2 we review generic dilaton gravity coupled to an Abelian gauge field. We present the most general solution as well as a description of the thermodynamic properties of black holes in the generic theory. For completeness, we include in the action a topological term involving the Abelian field strength. This term can only be constructed in two spacetime dimensions and has not been considered in previous work. In Section 3, the Hamiltonian analysis of the theory is summarized, while Section 4 derives the boundary terms that must be added to the Hamiltonian when considering a charged black hole in a box. Section 5 presents the Hamiltonian partition function using the results of Section 4 and examines the resulting thermodynamics in the semi-classical, or saddle-point approximation. In Section 6 we analyze in detail two specific examples: spherically symmetric charged black holes in 3+1 Einstein gravity, and the rotating BTZ black hole. Finally, Section 7 summarizes our results and discusses prospects for future work.

2 Generic Dilaton Gravity with Abelian Gauge Field

In two spacetime dimensions, the Einstein tensor vanishes identically. In order to construct a dynamical theory of gravity with no more than two derivatives of the metric in the action, it is necessary to introduce a scalar field, traditionally called the dilaton. In the past, the dilaton was treated as essentially a lagrange multiplier, with no physical or geometrical significance. In recent years, however, it has become clear that the dilaton plays an important role. For example, when the dilaton theory is derived via dimensional reduction by imposing spherical symmetry in n+2 dimensional Einstein gravity, the dilaton has a geometrical interpretation as the invariant radius of the n-sphere. More generally, the dilaton is instrumental in determining both the symmetries and the topology of the solutions.[15]

In the following, we consider the most general action functional depending on the metric tensor $g_{\mu \nu}$, scalar field $\phi$ and Abelian gauge field in two
where $G$ is the dimensionless 2-d Newton constant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $l$ is a fundamental constant with dimensions of length. In addition, $\nabla(\phi)$, $D(\phi)$, $W(\phi)$ and $Z(\phi)$ are arbitrary functions of the dilaton $\phi$. The last term in the action is a topological term that is only possible in two spacetime dimensions.\[1\]

It is convenient to eliminate the kinetic term for the scalar field. This can be done with an invertible field redefinition providing that $D(\phi)$ is a differentiable function of $\phi$ such that $D(\phi) \neq 0$ and $\frac{dD(\phi)}{d\phi} \neq 0$ for any admissable value of $\phi$:\[16, 18\]:

$$g_{\mu\nu} = \Omega^2(\phi)\bar{g}_{\mu\nu}$$

$$\phi = D(\phi)$$

where

$$\Omega^2(\phi) = \exp\left(\frac{1}{2} \int \frac{d\phi}{dD/d\phi}\right)$$

The electromagnetic potential is left unchanged. In terms of the new fields, the action Eq.(1) takes the form:

$$S[g, A] = \frac{1}{2G} \int d^2x \sqrt{-g} \left( R(g) + \frac{1}{\ell^2} V(\phi) \right)$$

$$+ \int d^2x \left( -\frac{1}{4} \sqrt{-g} W(\phi) F_{\mu\nu} F_{\mu\nu} + Z(\phi) \epsilon^{\mu\nu} F_{\mu\nu} \right).$$

where $V$, $W(\phi)$ and $Z(\phi)$ are defined as:

$$V(\phi) = \frac{\nabla(\phi)}{\Omega^2(\phi)}$$

$$W(\phi) = \Omega^2(\phi)\nabla(\phi)$$

$$Z(\phi) = \Omega^2(\phi)\nabla(\phi)$$

\[1\]GK is grateful to R. Jackiw for pointing out this possibility.
We henceforth consider the action only in the form Eq.(8), keeping in mind that the physical metric in general may be different from $g_{\mu\nu}$.

The field equation obtained from varying $\phi$ is:

$$R + \frac{1}{l^2} \frac{dV}{d\phi} - \frac{G}{2} \frac{dW(\phi)}{\phi} F^{\alpha\beta} F_{\alpha\beta} + \frac{2G}{\sqrt{-g}} \frac{dZ}{d\phi} \epsilon^{\alpha\beta} F_{\alpha\beta} = 0$$  \hspace{1cm} (10)

while minimizing the action with respect to variations of the metric yields:

$$\nabla_{\mu} \nabla_{\nu} \phi - \frac{1}{2l^2} g_{\mu\nu} V(\phi) - \frac{3}{4} G g_{\mu\nu} W(\phi) F^{\alpha\beta} F_{\alpha\beta} +GW(\phi) F_{\mu\nu} F_{\nu\gamma} = 0$$ \hspace{1cm} (11)

Finally, the equation for the Abelian gauge field is:

$$\nabla_{\mu} \left( W(\phi) F^{\mu\nu} - 2 \epsilon^{\mu\nu} Z(\phi) \right)$$ \hspace{1cm} (12)

It follows directly from the above field equations that on shell all the fields are left invariant by Lie derivation along the following Killing vector\[15\]

$$k^{\mu} = l \epsilon^{\mu\nu} \partial_{\nu} \phi / \sqrt{-g}$$ \hspace{1cm} (13)

where $\epsilon^{\mu\nu}$ is the contravariant Levi-Civita symbol: ($\epsilon^{01} = -\epsilon^{10} = 1$, etc.) and the constant $l$ has been included to ensure that the vector components are dimensionless.

The most general solution to the field equations without the topological term has been found in [17]. The procedure required with the extra term is virtually identical, so we will omit most of the details. By going to light cone coordinates $(z_+, z_-)$ in conformal gauge:

$$d s^2 = e^{2\rho} d z_+ d z_-$$ \hspace{1cm} (14)

where $\rho(z_+, z_-)$ is an arbitrary function, one finds that Eq.(14) reduces to:

$$\frac{\partial}{\partial z_{\pm}} (W(\phi) F + 2Z(\phi)) = 0$$ \hspace{1cm} (15)

\[2\]It is crucial in this regard that the black hole thermodynamics are invariant under conformal reparametrizations of the form Eq.(4)
In the above, $F$ is a scalar defined implicitly by $F_{\mu\nu} = F E_{\mu\nu}$, where $E_{\mu\nu} = \epsilon_{\mu\nu} / \sqrt{-g}$ is the fundamental alternating tensor. Thus, we find that
\[
F = \frac{1}{W(\phi)}(q - 2Z(\phi))
\tag{16}
\]
where $q$ is a constant that corresponds to the Abelian charge. This leads to field equations for the metric that are completely analogous to those in [17], with $q \rightarrow q - 2Z(\phi)$. Since these field equations are algebraic in $F$, the solutions for the metric in the present case are precisely the same as in [17], up to this replacement. In particular
\[
e^{2\nu} = \frac{1}{4}(j(\phi) - 2GlM - l^2GK(\phi; q))
\tag{17}
\]
where $M$ is a constant of integration, which will be shown below to be the ADM mass of the solution and we have defined:
\[
\begin{align*}
 j(\phi) &= \int_0^\phi d\tilde{\phi}V(\tilde{\phi}) \\
 K(\phi; q) &= \int_0^\phi d\tilde{\phi}(q - 2Z(\tilde{\phi}))^2/W(\tilde{\phi})
\end{align*}
\tag{18, 19}
\]

It is most convenient to write the final solutions in manifestly static coordinates by exploiting the form of the Killing vector given above. That is, we can choose the spatial coordinate to be proportional to the dilaton field:
\[
\phi = x/l
\tag{20}
\]

In these coordinates, the metric depends only on $x$:
\[
ds^2 = -(j(\phi) - 2GlM - l^2GK(\phi; q))dt^2 + (j(\phi) - 2GlM - l^2GK(\phi; q))^{-1}dx^2
\tag{21}
\]

From the above solution it is easy to see that the norm of the Killing vector is
\[
|k|^2 = -l^2|\nabla\phi|^2 = (j(\phi) - 2GlM - l^2GK(\phi; q)).
\tag{22}
\]

Given the above equation, it is clear that the general solution therefore has an apparent horizon at the surface $\phi = \phi_0 = constant$ for $\phi_0$ given by
\[
f(\phi_0) = 0
\tag{23}\]
where we have defined
\[ f(\phi; M, q) := (j(\phi) - 2GM - l^2GK(\phi; q)) \tag{24} \]

The global form of the solution, and in particular the number of horizons, depends on the specific forms of the function \( j(\phi) \) and \( K(\phi; Q) \).

Before describing the Hamiltonian analysis of the theory, we review briefly the thermodynamic properties of the solutions. Specifically, we assume that \( \phi_0 \) is the value of the dilaton field at an exterior, bifurcative horizon. With the Killing vector normalized as in Eq. (22), a straightforward calculation reveals that the surface gravity at the horizon, defined by:
\[ \kappa^2 := -\frac{1}{2} \nabla^\mu k^\nu \nabla_\mu k_\nu |_{\phi_0} \tag{25} \]
is
\[
\kappa = \frac{f'(\phi_0)}{2l} = \frac{V(\phi_0)}{2l} - \frac{l(q - 2Z(\phi_0))^2 G}{2W(\phi_0)} \tag{26}
\]
where the prime denotes differentiation with respect to \( \phi \).

The Hawking temperature of the horizon can be calculated by analytically continuing the solution exterior to the horizon to Euclidean time, imposing periodicity in the imaginary time direction and requiring the resulting solution to be regular at the horizon. Although this is a standard calculation, we summarize it briefly, since it plays an important role in determining the boundary conditions required for the subsequent Hamiltonian analysis.

The Euclidean form of Eq. (21) is:
\[ ds^2_E = f(\phi; M, q) dt_E^2 + \frac{1}{f(\phi; M, q)} dx^2 \tag{27} \]

We wish to find a coordinate transformation that puts the metric in the form:
\[ ds^2_E = R^2 d\theta^2 + H(R) dR^2 \tag{28} \]
where \(|k^2| = 0\) at \( R = 0 \) and \( \theta \) is an angular coordinate with period \( 2\pi \). This can be accomplished by defining:
\[
\theta = \frac{t_E}{a} \tag{29}
\]
\[
R^2 = a^2 f(\phi) \tag{30}
\]
so that
\[ H(R) = \frac{4l^2}{a^2[f'(\phi)]^2} \] (31)

The metric Eq.(28) will be regular at \( R = 0 \) providing that \( H(0) = 1 \), which requires
\[ a = \frac{2l}{f'(\phi_0; M, q)} = \frac{1}{\kappa} \] (32)
and fixes the periodicity of the Euclidean time coordinate to be
\[ 2\pi a = \frac{2\pi l}{f'(\phi_0; M, q)} = \frac{2\pi}{\kappa} \] (33)
The Hawking temperature is then
\[ T_H = \frac{1}{2\pi a} = \frac{f'(\phi_0; M, q)}{4\pi l} \] (34)

As discussed in [15], the expression for the black hole entropy can most easily be derived by demanding that the first law of thermodynamics be satisfied with respect to infinitesimal variations of the mass and charge of the black hole. In particular, if we vary the parameters \( M \) and \( q \) of the solution while staying on the event horizon, \( f = 0 \), we get the condition on the corresponding variation of \( \phi_0 \) at the horizon:
\[
0 = \frac{\partial f}{\partial \phi_0} \delta \phi_0 + \frac{\partial f}{\partial M} \delta M + \frac{\partial f}{\partial q} \delta q + \left( V(\phi_0) - \frac{l^2 G(q - 2Z(\phi_0))^2}{W(\phi_0)} \right) \delta \phi_0 - 2Gl^2 \delta M - P(\phi_0, q) \delta q \] (35)
where
\[ P(\phi_0, q) = \int_{\phi_0}^{\phi} d\phi \frac{(q - 2Z(\phi))}{W(\phi)} \] (36)
This yields the first law of black hole thermodynamics:
\[ \delta M = T_H \delta S_{BH} - P \delta q \] (37)
where we have defined the Bekenstein-Hawking entropy:
\[ S_{BH}(M, q) = \frac{2\pi}{G} \phi_0(M, q) \] (38)
where $\phi_0(M, q)$ is obtained by solving Eq. (23). Eq. (37) also shows that $\mathcal{P}$ is the generalized force associated with the charge $q$.

The expression Eq. (38) for the entropy can also obtained using Wald’s formalism [19]. It is important to keep in mind that the thermodynamic properties defined above are not affected by conformal reparametrizations of the form Eq. (3).

# 3 Hamiltonian Analysis

The Hamiltonian analysis for generic dilaton gravity has been presented in many works. Here we summarize the results, using the notation and conventions of [17]. We start by decomposing the metric as follows:

$$ds^2 = e^{2\rho} \left[ -u^2 dt^2 + (dx + v dt)^2 \right].$$

where $x$ is a local coordinate for the spatial section $\Sigma$ and $\rho$, $u$ and $v$ are functions of spacetime coordinates $(x, t)$. For convenience we work with the form of the action in Eq. (8). In terms of the parametrization Eq. (39), the action Eq. (8) takes the form (up to surface terms):

$$I = \int dt \int_{\sigma_+}^{\sigma_-} dx \left[ \frac{1}{G} \left( \frac{\dot{\phi}}{u} (v \rho' + v' - \dot{\rho}) + \frac{\dot{\phi}'}{u} (uu' - vv' + v \dot{v} + u^2 \rho' - v^2 \rho') \right) + \frac{1}{2} u e^{2\rho} \frac{V(\phi)}{l^2} \right] + e^{-2\rho} W(\phi) (\dot{A}_1 - A_0')^2 + 2Z(\phi) (\dot{A}_1 - A_0')]$$

In the above, dots and primes denote differentiation with respect to time and space, respectively, while $\sigma_+$ and $\sigma_-$ are the outer and inner spatial boundaries. The canonical momenta for the fields $\{\phi, \rho\}$ are:

$$\Pi_\phi = \frac{1}{Gu} (v \rho' + v' - \dot{\rho})$$

$$\Pi_\rho = \frac{1}{Gu} (-\dot{\phi} + v \phi')$$

$$\Pi_{A_1} = \frac{e^{-2\rho}}{u} W(\phi) (\dot{A}_1 - A_0') + 2Z(\phi)$$

$$\Pi_\mu = \Pi_v = \Pi_{A_0} = 0$$

9
As expected, the momenta conjugate to $u,v$ and $A_0$ vanish because these fields play the role of Lagrange multipliers that are needed to enforce the first class constraints associated with diffeomorphism and gauge of the classical action. A straightforward calculation leads to the canonical Hamiltonian (up to surface terms which will be discussed below):

$$H_c = \int dx \left( vF + \frac{u}{2G}G + A_0 \mathcal{J} \right)$$  \hspace{1cm} (45)

where

$$\mathcal{F} = \rho' \Pi_\rho + \phi' \Pi_\phi - \Pi' \rho \sim 0$$  \hspace{1cm} (46)

$$\mathcal{G} = 2\phi'' - 2\phi' \rho' - 2G^2 \Pi_\rho \Pi_\phi - e^{2\rho} \frac{V(\phi)}{l^2}$$

$$+ \frac{Ge^{2\rho}}{W(\phi)} [\Pi_{A_1} - 2Z(\phi)]^2 \sim 0$$  \hspace{1cm} (47)

$$\mathcal{J} = -\Pi'_{A_1}$$  \hspace{1cm} (48)

are secondary constraints. Note that $\mathcal{F}$ and $\mathcal{G}$ generate spatial and temporal diffeomorphisms, while $\mathcal{J}$ is the Gauss law constraint that generates Abelian gauge transformations.

The general solution presented in the previous section suggests that there are two independent, diffeomorphism invariant physical observables, namely the mass of the black hole and its Abelian charge. These observables can easily be expressed in terms of the phase space variables. In particular, define:

$$Q = \Pi_{A_1}$$  \hspace{1cm} (49)

Clearly, $Q$ commutes with all three constraints and hence the entire canonical Hamiltonian, and the Gauss law constraint implies that $Q = q$ is constant on the constraint surface. The constant mode $q$ of $Q$ is therefore a physical observable and corresponds precisely to the Abelian charge in the solution Eq.(16). Similarly, we can define the mass observable:

$$\mathcal{M} = \frac{l}{2G} \left( e^{-2\rho} (G^2 \Pi_\rho^2 - (\phi')^2) + \frac{j(\phi)}{l} - GK(\phi, Q) \right)$$  \hspace{1cm} (50)

where

$$K(\phi, Q) := \int_\phi^\phi d\tilde{\phi} \frac{(Q - 2Z(\tilde{\phi}))^2}{W(\tilde{\phi})}$$  \hspace{1cm} (51)
Once again it is possible to show that $\mathcal{M}$ commutes with the constraints and is spatially constant on the constraint surface. In particular, we find that:

$$\frac{\partial \mathcal{M}}{\partial x} = -l e^{-2\rho} \left( G\Pi_\rho \mathcal{F} + \frac{1}{2G} \phi' G - e^{2\rho} \mathcal{P}(\phi, \mathcal{Q}) \mathcal{J} \right)$$  (52)

where

$$\mathcal{P}(\phi, \mathcal{Q}) = \int d\phi \frac{(\Pi A_1 - 2Z(\phi))}{W(\phi)}$$  (53)

The constant mode of $\mathcal{M}$ is the mass parameter appearing in the solution Eq. (21).

It is useful to note that both $\mathcal{M}$ and $\mathcal{Q}$ can be written as coordinate invariant scalars in terms of the dilaton and the Abelian field strength as follows:

$$\mathcal{M} = \frac{1}{2Gl} \left( |k|^2 + j(\phi) - l^2 G K(\phi, \mathcal{Q}) \right)$$  (54)

$$\mathcal{Q} = 2Z(\phi) + \left( -\frac{W(\phi)}{2} F^{\mu\nu} F_{\mu\nu} \right)^{\frac{1}{2}}$$  (55)

For completeness we also write down the explicit expressions for the momenta canonically conjugate to the mass and charge observables. They are, respectively,

$$\Pi_{\mathcal{M}} = -G \int dx \frac{e^{2\rho}\Pi_\rho}{(G\Pi_\rho)^2 - (\phi')^2}$$  (56)

$$\Pi_{\mathcal{Q}} = -\int dx \left( A_1 + \frac{G e^{2\rho}\Pi_\rho \mathcal{P}(\phi, \mathcal{Q})}{(G\Pi_\rho)^2 - (\phi')^2} \right)$$  (57)

Although the observables $\mathcal{M}$ and $\mathcal{Q}$ are invariant under general diffeomorphisms, their conjugates $\Pi_{\mathcal{M}}$ and $\Pi_{\mathcal{Q}}$ are only invariant with respect to diffeomorphisms that vanish on the boundaries of the system. The Hamiltonian analysis is therefore consistent with the results of the previous section which indicate that, up to general diffeomorphisms, there exists only a two parameter family of physically distinct solutions.
4 Boundary Terms in the Hamiltonian

The previous Section neglected the boundary terms that must be added to the canonical Hamiltonian in order that the variational principle be well defined. These depend on the boundary conditions and define the canonical energy, since the remainder of the Hamiltonian vanishes on the constraint surface. We now derive the boundary terms for boundary conditions corresponding to a charged black hole in a box of fixed, constant “radius” (surface of constant dilaton field). For convenience we rewrite the canonical Hamiltonian as follows:

\[ H_c = \int_{\sigma_-}^{\sigma_+} dx \left( \tilde{u}\tilde{G} + \tilde{v}\tilde{F} + \tilde{A}\tilde{J} \right) + H_+ - H_-(58) \]

where have replaced the original Hamiltonian constraint \( G \) by the linear combination of constraints corresponding to the spatial derivative of the mass observable:

\[ \tilde{G} = -\frac{\partial M}{\partial x} = le^{-2\rho} \left( G\Pi_{\rho}\mathcal{F} + \frac{1}{2G}\phi'\tilde{G} - e^2\rho \mathcal{P} \mathcal{J} \right) (59) \]

and replace the original lagrange multipliers by:

\[ \tilde{u} = \frac{ue^{2\rho}}{l\phi'} \]
\[ \tilde{v} = v - \frac{uG\Pi_{\rho}}{\phi'} \]
\[ \tilde{A} = A_0 + \frac{ue^{2\rho}}{\phi'}\mathcal{P} \]

\( H_+ \) and \( H_- \) are previously neglected boundary terms determined by the requirement that the surface terms in the variation of \( H_c \) vanish for a given set of boundary conditions.

We wish to consider the 1+1 dimensional analogue of a charged black hole in a box of fixed radius. We will therefore keep the value of the dilaton at the outer boundary \( \phi_+ := \phi(\sigma_+) \) fixed and independent of time, as well as the component of the metric along the world line of the box \( (g_{\mu\nu} := g_{tt}(\sigma_+)) \). Note that \( \dot{\phi}_+ = 0 \) requires that \( \tilde{v}(\sigma_+) = 0 \) (cf Eq.(42)). The relevant boundary conditions on the vector potential are \( A_1(\sigma_+) = 0 \) and \( A_0(\sigma_+) = A_0^+ = \text{constant} \). Give the above conditions, the boundary variation of the canonical
Hamiltonian Eq. (58) at \( \sigma_+ \) will vanish if:

\[
\delta H_+(\mathcal{M}, q) = \tilde{u}\delta \mathcal{M}|_{\sigma_+} + \tilde{a}\delta Q|_{\sigma_+}
\]

where we have used Eq. (59) and the fact that \( J = -Q' \). Moreover, since

\[
\tilde{u}_+ = \left( \frac{g^+_{tt} + lj(\phi_+)}{2G\mathcal{M}l - j(\phi_+) + l^2 G K(\phi_+, Q)} \right)^{\frac{1}{2}}
\]

\[
\tilde{A}_+ = A_0^+ + \frac{l}{2} \tilde{u}(\sigma_+) \left. \frac{\partial K(\phi_+, Q)}{\partial Q} \right|_{\sigma_+}
\]

Eq. (63) can be directly integrated to yield:

\[
H_+(\mathcal{M}, Q) = \sqrt{-g^+_{tt}j(\phi_+)} \left( \left| 1 - \frac{2G\mathcal{M}}{j(\phi_+) + l^2 G K(\phi_+, Q)} \right| + A_0^+ Q \right)
\]

Note that we have chosen the constant of integration so as to guarantee that \( H_+ = 0 \) when \( \mathcal{M} = Q = 0 \). If \( K(\phi_+, Q) \) remains finite as \( \phi_+ \to \infty \), then

\[
H_+(\mathcal{M}, Q) \to \sqrt{-g^+_{tt}j(\phi_+)} \mathcal{M}
\]

Hence, on the constraint surface, \( \mathcal{M} \) is proportional to the ADM mass. The value of the constant of proportionality will depend on the boundary conditions on the metric and \( \phi_+ \). This will be discussed in more detail below.

We next consider the inner boundary \( \sigma_- \). Following the work of Louko and Whiting[13] we require our spatial slices to approach the bifurcation point \( (k^\mu = 0) \) of the black hole along a static slice. These boundary conditions are natural for the consideration of the thermodynamics of the black hole, since the resulting spacetimes can be analytically continued to the Euclidean spacetime described by the non-singular Gibbons-Hawking instanton. Given the general form of the Killing vector in Eq. (13), for a static slice \( (\phi_- = 0) \), the condition that \( \sigma_- \) be a bifurcation point reduces to:

\[
\phi'(\sigma_-) = 0
\]

From the thermodynamic considerations of Section 2, it follows that the metric on the inner boundary must approach the form:

\[
ds^2 \to -R^2(dt/\tilde{a})^2 + H(R)dR^2
\]
where $R = 0$ at the bifurcation point $\sigma_-$, $H(0) = 1$ and $2\pi \tilde{a}$ equals the periodicity of the Euclidean time required to make the Euclidean solution regular at the horizon.\(^\dagger\) The required boundary conditions on the metric components in $(t, R)$ coordinates are therefore

\begin{align}
e^{2\rho(\sigma_-)} &= 1 \\
v(\sigma_-) &= 0 \\
u(\sigma_-) &= 0 \\
u'(\sigma_-) &= \frac{1}{\tilde{a}}
\end{align}

(70) (71) (72) (73)

Since, in terms of phase space coordinates,

$$|k|^2 = l^2 e^{-2\rho} ((G\pi \rho)^2 - \phi'^2)$$

(74)

we must also impose the condition\(^\dagger\): \[\pi_\rho(\sigma_-) = 0\] (75)

to ensure that $|k|_-^2 = 0$.

Finally, following Louko and Winters-Hilt\([14]\), we choose the boundary conditions on the U(1) vector potential at the bifurcation point to be:

\begin{align}A_1(\sigma_-) &= 0 \\
A_0(\sigma_-) &= A_0^- = \text{constant}
\end{align}

(76) (77)

With the above boundary conditions we find:

\begin{align}\tilde{v}(\sigma_-) &= 0 \hspace{1cm} (78) \\
\tilde{u}(\sigma_-) &= \frac{2l}{\tilde{a}V(\phi_- , Q)} \hspace{1cm} (79) \\
\tilde{A}(\sigma_-) &= \frac{l^2}{\tilde{a}V(\phi_- , Q)} \frac{\partial K(\phi_- , Q)}{\partial Q} + A_0^- \hspace{1cm} (80)
\end{align}

\(^3\)Recall that the time coordinate in this Section is normalized so that $g_{tt}$ is fixed. The parameter $\tilde{a}$ therefore differs from $a$ in Section 2.

\(^4\)One might expect to conclude this from Eq.(42), but this is not possible without further assumptions because $u = 0$.
where we have defined
\[ \tilde{V}(\phi, Q) = V(\phi) - Gl^2 \frac{\partial K(\phi, Q)}{\partial \phi} \] (81)

The boundary value for \( \tilde{u} \) was obtained by applying l’Hopital’s rule and then using the constraint Eq.(47) to eliminate \( \phi''(\sigma_-) \). \( A_0^- \) and \( \tilde{a} \) are considered fixed parameters. On the other hand, \( \phi_- := \phi(\sigma_-) \) is not constrained, but is a dynamical variable. In fact, an implicit equation for \( \phi_- \) in terms of the physical observables \( \mathcal{M} \) and \( \mathcal{Q} \) can be obtained by setting \( |k|^2 = 0 \) in the expression for \( \mathcal{M} \) (i.e. Eq.(54)).

With these boundary conditions there will be no boundary terms at \( \sigma_- \) from the variation of the Hamiltonian if:
\[ \delta H_- = \frac{2l}{aV(\phi_-, Q)} \delta \mathcal{M}_{\sigma_-} + \frac{l^2}{aV(\phi_-, Q)} \frac{\partial K(\phi_-, Q)}{\partial Q} \bigg|_{\sigma_-} \delta Q + A_0^- \delta Q \] (82)

Next we use the fact that the norm of the Killing vector is constrained to vanish at the inner boundary to obtain (via Eq.(54))
\[ \delta \mathcal{M} = \frac{1}{2Gl} \tilde{V}(\phi_-, Q) \delta \phi_- - \frac{l^2}{2} \frac{\partial K(\phi_-, Q)}{\partial Q} \delta Q \] (83)

Substituting this into Eq.(82) and simplifying gives:
\[ \delta H_- = \frac{1}{aG} \delta \phi_- + A_0^- \delta Q \] (84)

which can be trivially integrated to yield:
\[ H_-(\mathcal{M}, \mathcal{Q}) = \frac{1}{aG} \phi_- (\mathcal{M}, \mathcal{Q}) + A_0^- \mathcal{Q} \] (85)

By using Eq.(38) our final expression for the canonical Hamiltonian on the constraint surface takes the simple form:
\[ H_c = E(M, q; \phi_+) - \frac{1}{2\pi a} S_{B.H.}(M, q) - \gamma q \] (86)

where
\[ E(M, q; \phi_+) = \frac{\sqrt{-g_+^r j(\phi_+)} - \sqrt{1 - \frac{2GMl}{\sqrt{\jmath(\phi_+)} - \sqrt{l^2GK(\phi_+)} \jmath(\phi_+)}}}{Gl} \left( 1 - \sqrt{1 - \frac{2GMl}{\sqrt{\jmath(\phi_+)} - \sqrt{l^2GK(\phi_+)} \jmath(\phi_+)}} \right) \] (87)
is the quasilocal energy and \( \gamma \equiv A^+_0 - A^-_0 \). We have also used the fact that on the constraint surface \( M = M \) and \( Q = q \), where \( M \) and \( q \) correspond to the physical mass and charge appearing in the general solution Eq. (21).

In addition to the dynamical variables \( M \) and \( q \), the canonical Hamiltonian appears to depend on four fixed external parameters, \( g^{+}_{tt}, \phi_+, \tilde{a} \) and \( \gamma \). \( \phi_+ \) plays the role of the effective box size, while \( \gamma \) is analogous to a chemical potential. \( g^{+}_{tt} \) and \( \tilde{a} \) on the other hand must be fixed by imposing further boundary conditions. In particular, the metric \( g^{+}_{tt} \) is related to the choice of time coordinate along the boundary. This is normally chosen to equal the proper time as measured with respect to a given physical metric. In vacuum dilaton gravity, the choice of physical metric is subtle since one can always do conformal reparametrizations involving the dilaton. One must therefore define the “physical metric” to be the one which determines the geodesics of massive test particles. This cannot be determined \textit{a priori}, but must ultimately be settled by experiment. Since \( g_{\mu\nu} \) was arrived at from the original metric \( \bar{g}_{\mu\nu} \) by a conformal reparametrization designed to make the action simpler(cf Eq.(2)), the physical metric might be \( \bar{g} = \Omega^{-2}g \) as given in Eq.(3). In this case, we would set \( \bar{g}^{+}_{tt} = -1 \) so that

\[
g^{+}_{tt} = -\Omega^2(\phi_+) \tag{88}
\]

On the other hand, the metric

\[
\bar{g}_{\mu\nu} = \frac{g_{\mu\nu}}{f(\phi)} \tag{89}
\]

has the desirable property that it approaches the Minkowski metric as \( \phi \to \infty \), so one might be tempted to define this as the physical metric. Thus, if we set \( \bar{g}^{+}_{tt} = -1 \), then

\[
g^{+}_{tt} = -j(\phi_+) \tag{90}
\]

With this choice of normalization, the quasilocal energy \( E \to M \) as \( \phi_+ \to \infty \). For now we will consider the most general case and write

\[
g_{\mu\nu} = h(\phi)g^{\text{phys}}_{\mu\nu} \tag{91}
\]

\footnote{It is interesting to note that for spherically symmetric gravity in n+2 dimensions, \( \tilde{g} \) and \( \bar{g} \) are equal, and coincide with the projection onto two spacetime dimensions of the \( n + 2 \) dimensional physical metric.}
where \(h(\phi)\) is an arbitrary function of \(\phi\) that must ultimately be determined experimentally. If \(g_{tt}^{\text{phys}} = -1\), then

\[
g_{tt}^+ = -h(\phi_+) \tag{92}
\]

The constant \(\tilde{a}\) must be fixed by thermodynamic considerations. We have already shown that \(2\pi \tilde{a}\) must be equal to the period of the corresponding Euclidean time in order for the Euclideanized solution to be regular at the horizon. In the Euclidean formulation of black hole thermodynamics, the inverse temperature \(\beta\) at the boundary of the system is

\[
\beta = \sqrt{-g_{tt}^{\text{phys}}(\sigma_+)2\pi \tilde{a}} \tag{93}
\]

Thus, if as discussed above \(g_{tt}^{\text{phys}}(\sigma_+) = -1\), we find that

\[
\tilde{a} = \frac{\beta}{2\pi} \tag{94}
\]

The final form of the canonical Hamiltonian is therefore:

\[
H_c = E(M, q, \phi_+) - \beta^{-1} S_{B.H.}(M, q) - \gamma q \tag{95}
\]

where

\[
E(M, q; \phi_+) = \frac{\sqrt{h(\phi_+)j(\phi_+)}}{Gl} \left(1 - \sqrt{1 - \frac{2GMl}{j(\phi_+)}} - \frac{l^2G^2K(\phi_+, q)}{j(\phi_+)}\right) \tag{96}
\]

We will now examine the properties of the resulting partition function.

### 5 Hamiltonian Partition Function

The quantum partition function of interest is formally defined as:

\[
Z[\beta, \phi_+, \gamma] = Tr[\exp(-\beta \hat{H})] \tag{97}
\]

where the trace is over all physical states and \(\beta\) corresponds to the (fixed) temperature at the boundary of the system. This trace is most easily expressed in term of the eigenstates \(|M, Q >\) of the mass and charge operators:

\[
Z(\beta, \phi_+, \gamma) = \int dM \int dQ \mu(M, Q) < M, Q |e^{-\beta \hat{H}}| M, Q > \tag{98}
\]
In the above, \( \mu(M, Q) \) is an as yet unknown measure on the space of observables. In principle both the spectrum of observables and the measure should be derivable from a rigorous quantization procedure. Exact eigenstates of the mass and charge operators can be found within a Dirac quantization scheme in which the constraints annihilate physical states. This procedure yields a continuous and unbounded mass spectrum for Lorentzian black holes\(^6\). In this case the inner product \( \langle M, Q | M, Q \rangle \) and the choice of measure is problematic. Following Louko and Whiting\(^{[13]}\), we will make the simplest, physically reasonable assumptions about the measure and the allowed values of \( M \) and \( Q \). A more rigorous derivation of the measure will be addressed in future work. First of all, we restrict the ADM mass \( M \) to be positive. Secondly, we allow only those value of \( M \) and \( Q \) for which at least one bifurcative horizon exists where \( f(\phi) \) has a simple zero (i.e. no extremal black holes or naked singularities). Finally, we require the value of the dilaton at the horizon to be less than its value at the boundary of the system (i.e the box must lie outside the horizon) so that equilibrium is in fact possible. With these assumptions the space of allowed values for the observables is finite. This will be made explicit for specific examples in the next section.

As in\(^{[13]}\) (see also \(^{[20]}\)) we assume that

\[
\mu(M, Q) = \frac{1}{\mathcal{V}}
\]  

where \( \mathcal{V} \) is the volume of the allowed space of observables. The final expression for the partition function is therefore:

\[
Z(\beta, \phi_+, \gamma) = \mathcal{V}^{-1} \int \! dMdqe^{S_{BH}(M,q)}e^{-\beta(E(M,q,\phi_+)-\gamma q)}
\]  

Note that the Bekenstein-Hawking entropy enters the partition function as the logarithm of an apparent degeneracy of the physical mass and charge eigenstates. Moreover, \( q \) is thermodynamically analogous to particle number, while \( \gamma \) plays the role of a chemical potential.

The above expression, can in principle be integrated to yield the partition function describing the thermodynamics charged black holes in a box for any particular dilaton gravity theory. We will now show that it gives the correct

\(^6\)Interestingly, a discrete spectrum has been obtained via this procedure for Euclidean black holes in generic dilaton gravity

18
classical thermodynamic behaviour in the saddle-point approximation. In this approximation, the choice of measure is irrelevant except in the unlikely event that it is exponential in the observables. Thus, we have

\[ Z(\beta, \phi_+, \gamma) \approx e^{-I(\overline{M}, \overline{\eta}, \beta, \phi_+, \gamma)} \]  

(101)

where we have defined:

\[ I(M, q, \beta, \phi_+, \gamma) = \beta(E(M, q, \phi_+) - \gamma q) - S_{BH}(M, q) \]  

(102)

and \( \overline{M} \) and \( \overline{\eta} \) are the values of the mass and charge at the minimum of \( I \) (if one exists). The equation obtained by extremizing with respect to \( M \) is:

\[ 0 = \frac{\partial I}{\partial M} \bigg|_{\overline{M}, \overline{\eta}} = \left( \beta \frac{\partial E}{\partial M} - \frac{\partial S_{BH}}{\partial M} \right) \bigg|_{\overline{M}, \overline{\eta}} = \left( \beta \sqrt{h(\phi_+)} \frac{M}{\sqrt{f(\phi_+, M, q)}} - \beta_H(M, q) \right) \bigg|_{\overline{M}, \overline{\eta}} \]  

(103)

where \( \beta_H = 1/T_H = 4\pi l/f'(\phi_-, M, q) \). This implies that, semi-classically, the temperature at the boundary is the Hawking temperature \( T_H \) red-shifted with respect to the physical metric \( g_{phys} \):

\[ \beta = \sqrt{\frac{f(\phi_+, \overline{M}, \overline{\eta})}{h(\phi_+)}} \beta_H(\overline{M}, \overline{\eta}) \]  

(104)

Variation with respect to \( q \) gives:

\[ 0 = \frac{\partial I}{\partial Q} = \beta \left( \frac{\partial E}{\partial q} - \gamma \right) - \frac{\partial S_{BH}}{\partial q} = \frac{\beta_H l}{2} \left( \frac{\partial K(\phi_+, q)}{\partial q} - \frac{\partial K(\phi_-, q)}{\partial q} \right) - \beta \gamma \]  

(105)

where as \( \phi_- = \phi_-(M, q) \) as determined by Eq. (23). This then yields an expression for the chemical potential in terms of the \( \overline{M}, \overline{\eta} \) and the inverse temperature, \( \beta \):

\[ \gamma = \frac{l}{2} \frac{\beta_H(\overline{M}, \overline{\eta})}{\beta} \left( \frac{\partial K(\phi_+, q)}{\partial q} - \frac{\partial K(\phi_-, q)}{\partial q} \right) \bigg|_{\overline{M}, \overline{\eta}} \]  

(106)
Using Eq.\((101)\) we can evaluate the mean energy, mean charge and entropy of the system:

\[
\langle E \rangle = -\frac{\beta}{\partial \ln(\mathcal{Z})} + \frac{\gamma}{\partial \beta} \frac{\partial \ln(\mathcal{Z})}{\partial \mathcal{M}, q} \approx E(M, q, \phi_+) \tag{107}
\]

\[
\langle q \rangle = \beta^{-1} \frac{\partial \ln(\mathcal{Z})}{\partial \gamma} \approx \bar{q} \tag{108}
\]

\[
S = \left(1 - \beta \frac{\partial}{\partial \beta}\right) \ln(\mathcal{Z}) = S_{BH}(M, \bar{q}) \tag{109}
\]

A straightforward calculation verifies that the above expressions for the mean energy, charge and entropy automatically obey the generalized first law

\[
\delta \langle E \rangle = \frac{\delta E}{\delta M} \delta M + \frac{\delta E}{\delta q} \delta q + \frac{\delta E}{\delta \phi_+} \delta \phi_+ \nonumber
\]

\[
= \beta^{-1} \delta S_{BH} + \gamma \delta \langle q \rangle - W \delta \phi_+ \tag{110}
\]

where

\[
W := -\frac{\partial E(M, q, \phi_+)}{\partial \phi_+} \tag{111}
\]

is a generalized surface pressure: it is the rate of change of quasilocl energy with “box size”. The final expression in Eq.\((110)\) was obtained by using the mean field equations Eq.\((103)\) and Eq.\((105)\) to express \(\partial E/\partial M\) and \(\partial E/\partial q\) in terms of the derivatives of \(S_{BH}\) with respect to \(M\) and \(E\).

### 6 Examples

#### 6.1 Spherically Symmetric Gravity

The action for four dimensional Einsten-Maxwell theory is:

\[
I^{(4)} = \frac{1}{16\pi G^{(4)}} \int d^4x \sqrt{-g^{(4)}} \left( R(g^{(4)}) - F^{AB}F_{AB} \right) \tag{112}
\]

where the indices \(A, B = 0, 1, 2, 3\) and \(G^{(4)}\) is the four dimensional Newtonian constant. For convenience we have rescaled the vector potential by a multiple
of the 4-D Planck length \( l_p = \sqrt{G^{(4)}} \) in order to make it dimensionless. We impose spherical symmetry via the ansatz

\[
\begin{align*}
    ds^2_{(4)} &= \bar{g}_{\mu\nu}dx^\mu dx^\nu + \frac{l^2\phi^2}{2}(d\theta^2 + \sin^2\theta d\phi^2) \\
    A_A dx^A &= A_\mu(x^{\mu})dx^\mu
\end{align*}
\]

with \( \mu, \nu = 0, 1 \). Note that \( l^2\phi^2/2 = r^2 \) where \( r \) is the usual radial coordinate. Here, \( \phi \) and \( r \) are taken to be functions of the coordinates \( x^\mu \). \( l \) is an arbitrary constant of dimension length, and without loss of generality we take it to equal the four dimensional Planck length \( l_p \). After integrating over the angular variables, the reduced action takes the form of a dilaton gravity theory in two spacetime dimensions:

\[
I = \frac{1}{2} \int d^2x \sqrt{-g} \left( \frac{\phi^2}{4}R(\bar{g}) + \frac{1}{l_p^2} + \frac{1}{2}|\nabla\phi|^2 \right) - \frac{1}{4} \int d^2x \sqrt{-g} \frac{\phi^2}{2} F^{\mu\nu}F_{\mu\nu}
\]

This is the same form as Eq.(11) with \( D(\bar{\phi}) = \bar{\phi}^2/4, W(\bar{\phi}) = \bar{\phi}^2/2 \) and \( G = 1 \). We now make the field redefinitions

\[
\begin{align*}
    g_{\mu\nu} &= \Omega^2(\bar{\phi})\bar{g}_{\mu\nu} \\
    \phi &= \frac{\bar{\phi}}{4}
\end{align*}
\]

with

\[
\Omega^2(\bar{\phi}) = \exp \frac{1}{2} \int \frac{d\bar{\phi}}{(dD(\bar{\phi})/d\bar{\phi})} = \frac{\bar{\phi}}{\sqrt{2}}
\]

In the above, the integration constant was chosen to be \( 1/\sqrt{2} \). As we will see in the subsequent analysis, this choice guarantees that the physical metric \( \bar{g} \) has the correct asymptotic behaviour. The final action is then of the same form as Eq.(11), with \( G = 1, Z(\phi) = 0, V(\phi) = 1/(\sqrt{2}\bar{\phi}) \) and \( W(\phi) = (2\phi)^{3/2} \). The solution for the metric \( g \) is therefore:

\[
\begin{align*}
    ds^2 &= -\left(\sqrt{2\bar{\phi}} - 2Ml_p + \frac{Q^2l_p^2}{\sqrt{2}\bar{\phi}}\right)dt^2 + \left(\sqrt{2\bar{\phi}} - 2Ml_p + \frac{Q^2l_p^2}{\sqrt{2}\bar{\phi}}\right)^{-1}dx^2
\end{align*}
\]

In terms of the radial coordinate \( r = l\sqrt{2\bar{\phi}} \), the solution takes the form:

\[
\begin{align*}
    ds^2 &= \frac{r}{l_p} \left( -\left(1 - 2Ml_p^2 + \frac{Q^2l_p^4}{r^2}\right)dt^2 + \left(1 - 2Ml_p^2 + \frac{Q^2l_p^4}{r^2}\right)^{-1}dr^2 \right)
\end{align*}
\]

21
The physical metric is therefore the usual Reissner-Nordstrom solution:

\[ ds^2 = - \left( 1 - \frac{2Ml_p^2}{r} + \frac{Q^2 l_p^4}{r^2} \right) dt^2 + \left( 1 - \frac{2Ml_p^2}{r} + \frac{Q^2 l_p^4}{r^2} \right)^{-1} dr^2 \]  

(121)

The solution for the scalar \( F \) Eq.(16) is:

\[ F = \frac{Q l_p^3}{r^3} \]  

(122)

so that the electromagnetic field strength is

\[ F_{01} = \frac{Q l_p^2}{r^2} \]  

(123)

as expected. The solution Eq.(120) has event horizons at:

\[ r_{o,i} = l_p^2 (M \pm \sqrt{M^2 - Q^2}) \]  

(124)

where \( r_o \) and \( r_i \) denotes the outer and inner horizon, respectively.

From the formulae Eq.(34) and Eq.(38), we can calculate the Hawking temperature associated with the outer horizon:

\[ T_H = \frac{1}{4\pi r_o} - \frac{l_p^4 Q^2}{4\pi r_o^3} \]  

(125)

\[ = \frac{\sqrt{M^2 - Q^2}}{2\pi l_p^2 (M + \sqrt{M^2 - Q^2})^2} \]  

(126)

and the associated Bekenstein-Hawking entropy:

\[ S_{BH} = \frac{\pi r_o^2}{l_p^2} \]  

(127)

which is one quarter the area of the outer horizon as required.

We now discuss the thermodynamics implied by the partition function Eq.(100). Since the physical metric is \( g \) we must choose \( h(\phi) = \Omega^2(\phi) = r/l_p \). The thermodynamic energy Eq.(10) in the semi-classical approximation is:

\[ E(M, Q, r_+) = \frac{r_+}{l_p^2} \left( 1 - \sqrt{1 - \frac{2Ml_p^2}{r_+} + \frac{Q^2 l_p^4}{r_+^2}} \right) \]  

(128)
which matches the expression for the quasilocal energy obtained in previous work\cite{21}.

Using Eq. (106), the semi-classical chemical potential for charged black holes is:

\[
\gamma = \frac{l_p^2 Q}{r_o} \sqrt{\frac{r_+}{l_p}} \left[ \frac{1 - r_o/r_+}{\sqrt{r_+ l_p - 2M l_p + Q^2 l_p^3/r_+}} \right]
\]  

(129)

This expression can be simplified significantly by expressing $M$ in terms of the radius $r_1$ of the outer horizon:

\[
M = \frac{r_o}{2l_p^2} + \frac{Q^2 l_p^2}{2r_o}
\]  

(130)

which yields

\[
\gamma = \frac{l_p^2 Q}{r_o} \sqrt{1 - \frac{r_o}{r_+}} \left[ \frac{1}{\sqrt{1 - \frac{Q^2 l_p^2}{r_o r_+}}} \right]
\]  

(131)

Note that as the box size, $r_+$ goes to infinity, $\gamma$ approaches the usual expression for the electrostatic potential at a distance $r_o$ from a charge $Q$.

Finally we examine in more detail the exact expression Eq. (100) for the quantum partition function. In particular we will evaluate an explicit expression for the volume of the allowed space of observables $V$. Recall that we wish to restrict the values of $M$ and $Q$ so that there is always at least one positive and non-degenerate root to $f(r, M, Q) = 0$. Given the expression Eq. (124), this requires:

\[
M > 0 \quad (132)
\]

\[
M^2 > Q^2 \quad (133)
\]

Moreover, the outer boundary must lie exterior to the outer horizon, so that

\[
r_+ > l_p^2 (M + \sqrt{M^2 - Q^2})
\]  

(134)

For any given value of charge $Q$, this puts an upper bound on the mass:

\[
M < \frac{Q^2 l_p^2}{2r_+} + \frac{r_+}{2l_p^2}
\]  

(135)
The constraints Eq.(133) and Eq.(135) define the region of observable space illustrated in Figure(1). The volume of this space can be readily obtained:

\[ V = \int_{r_+ \to r_+} dQ \int_{|Q|} \left( \frac{Q^2}{r_+^2} + \frac{r_+}{2r_+^3} \right) dM \]

(136)

\[ = r_+^2 / 3l_p^4 \]

(137)

A numerical analysis of the partition function Eq.(100) for SSG will be treated elsewhere.

6.2 Dimensionally Reduced BTZ

Starting with the Einstein action with cosmological constant in 2+1 dimensions:

\[ I^{(3)} = \frac{1}{16\pi G^{(3)}} \int d^3x \sqrt{-g^{(3)}} (R(g^{(3)}) + \Lambda) \]

(138)

In 2+1 dimensions, the gravitational constant \( G^{(3)} \) has dimensions of length. We now impose axial symmetry by considering metrics of the form

\[ ds^{(3)} = g_{\mu\nu} dx^\mu dx^\nu + \phi(x)^2 (a d\theta + A_\mu dx^\mu)^2 \]

(139)

where \( a \) is an arbitrary constant with dimensions of length which, without loss of generality we take to be proportional to the 2+1 dimensional Planck length \( a = 8G^{(3)} \). The one-form components \( A_\mu \) are dimensionless. Unless the one-form \( A = A_\mu dx^\mu \) is closed, the metric is not static so that the field strength \( F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \) is proportional to the angular momentum of the solution. With the above metric ansatz the reduced action is that of Jackiw-Teitelboim dilaton gravity coupled to an abelian gauge field:

\[ I^{(2)} = \int d^2x \sqrt{-g} \left( \phi R(g) + \phi \Lambda - \frac{1}{4}\phi^3 F_{\mu\nu} F^{\mu\nu} \right) \]

(140)

This action is already of the generic form Eq.(8) without the need for further definitions. In particular, \( G = 1/2, l = \Lambda^{-1/2}, V(\phi) = \phi \) and \( W(\phi) = \phi^3 \).

Choosing \( r = l\phi \) as the spatial coordinate the general solution takes the form:

\[ ds^2 = -f(r, M, J) dt^2 + \frac{1}{f(r, M, J)} dr^2 \]

(141)

\footnote{In 2+1 dimensions, there is a generalized Birkhoff theorem which states that all solutions have axial symmetry, and are stationary.}
where
\[ f(r, M, J) = \left( \frac{r^2}{2l^2} - Ml + \frac{J^2l^4}{4r^2} \right) \]  
(142)

As mentioned above, the abelian charge \( J \) in this case is the angular momentum of the black hole. For non-zero \( J \) there are again two event horizons, at
\[ r_{o,i} = l \left( Ml \pm \sqrt{(Ml)^2 - (Jl)^2/2} \right)^{\frac{3}{2}} \]  
(143)
where \( r_o \) (\( r_i \)) is the outer (inner) horizon. The associated entropy is
\[ S = 4\pi r_o \frac{l}{\ell} = 4\pi \phi(r_o) = \frac{A}{4G^{(3)}} \]  
(144)
where \( A = 2\pi a\phi(r_o) = 16\pi G^{(3)}\phi(r_o) \) is the invariant circumference of the outer horizon, as calculated from Eq.(139). The Bekenstein Hawking entropy can also be calculated directly from Eq.(34) to be
\[ T_{BH} = \frac{1}{4\pi l^2} \left( \frac{r_o^2 - r_i^2}{r_o} \right) \]  
(145)

In the semi-classical approximation, the mean energy of a black hole in a box of fixed temperature and radius is:
\[ < E > = \frac{r_o^2}{l^3} \left( 1 - \sqrt{1 - \frac{2\overline{M}l^3}{r_o^2} + \frac{\overline{J}^2l^6}{2r_o^4}} \right) \]  
(146)
where \( \overline{M} \) and \( \overline{J} \) are the mean mass and angular momentum. Note that we have used the fact that the physical metric is \( g_{\mu\nu} \) in this case, so that \( h(\phi_+) = j(\phi_+) \). The physical metric is not asymptotically flat (it is in fact a metric of constant curvature) which accounts for the strange asymptotic behaviour of the mean energy as the box size goes to infinity. One can invert this relation to express the mass in terms of the mean energy:
\[ \overline{M} = < E > - \frac{< E >^2 l^3}{2r_+^2} + \frac{< J >^2 l^3}{4r_+^2} \]  
(147)

It is also straightforward to calculate the chemical potential. It is:
\[ \gamma = -\frac{Jl^3}{2r_o^2} \sqrt{1 - \frac{r_o^2}{r_+}} \]  
(148)
which approaches
\[ \gamma \to -\frac{J l^3}{2 r_+^2} \]
(149)
as \( r_+ \to \infty. \)

Finally, we calculate the allowed volume \( V \) of the physical configuration space. As in the case of spherically symmetric gravity we restrict \( M > 0 \) and, in order that the horizons be non-degenerate \( M > J/\sqrt{2}. \) For the box size to be greater than the radius of the outer horizon, we also require,
\[ M < \frac{J^2 l^3}{4 r_+^2} + \frac{r_+^2}{2 l^3} \]
(150)

The shape of the allowed configuration space is qualitatively as in Fig.(1), but the slope of the straight lines is \( 1/\sqrt{2} \) and the parabola has a different dependence on \( r_+. \) These conditions again put a bound on the allowed range of \( J^2, \) namely: \( J^2 < 2 r_+^4/l^6. \) The volume of the shaded region in this case is:
\[ V = \int_{-\sqrt{2 r_+^4/l^3}}^{\sqrt{2 r_+^4/l^3}} dJ \int_{\sqrt{J/l^2}}^{\sqrt{J^2 + r_+^2}} dM \]
(151)
\[ = \frac{\sqrt{2} r_+^4}{3} \frac{1}{l^6} \]
(152)

### 7 Conclusions

We have calculated the Hamiltonian partition function for generic dilaton gravity coupled to an Abelian gauge field. The class of theories considered contains many specific charged black holes of physical interest. We verified that our formalism gives the correct partition function in the saddle point approximation for spherically symmetric gravity. We then used our generic results to obtain the partition function for a rotating BTZ black hole confined to a box of fixed radius and temperature.

In principle the partition function that we derived can be integrated exactly. In practice, however, a numerical analysis is required in order to go beyond the semi-classical approximation. In a subsequent paper, we will do such a numerical analysis for specific theories, such as the BTZ black hole, in order to gain further information about phase structure, specific heats, etc. The ansatz that we used is, however, only rigorous in the semi-classical
approximation. In particular, the integration measure, although motivated by plausibility arguments, was not derived from the fundamental quantum theory, so it is likely that there are further quantum corrections that we have not been able to encorporate. A detailed analysis of the possible quantum corrections is currently in progress.

8 Acknowledgements

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada. G.K. would like to thank J. Gegenberg for helpful conversations.

References

[1] J. D. Bekenstein, Nuovo cimento Let. 4, 737 (1972); S.W. Hawking, Comm. Math. Phys. 43, 199 (1975).

[2] A. Strominger and C. Vafa, Phys. Lett. B379 (1996) 99.

[3] For a recent critical review of the program see S. Carlip, hep-th/9806026.

[4] V.P. Frolov and D.V. Fursaev, Phys.Rev. D56 (1997) 2212 (hep-th/9703178).

[5] A. Ashtekar, J. Baez, A. Corichi, K. Krasnov, Phys.Rev.Lett. 80 (1998) 904

[6] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1849 (1992); M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D48, 1506 (1993).

[7] A. Strominger, J.High Energy Phys. 02 (1998) 009 (hep-th/9712252).

[8] K. Sfetsos and K. Skenderis, hep-th/9711138 (1997).
[9] B.K. Berger, D.M. Chitre, V.E. Moncrief and Y. Nutku, Phys. Rev. D5, 2467 (1973). W.G. Unruh, Phys. Rev. D14, 870 (1976); G.A. Vilkovisky and V.F. Frolov, Proc. 2nd Seminar on Quantum Gravity (Moscow 1981), ed. M.A. Markov and P.C. West (London: Plenum 1983), p267; P. Thomi, B. Isaak and P. Hajicek, Phys. Rev. D30, 1168 (1984); P. Hajicek, Phys. Rev. D30, 1178 (1984); T. Thiemann and H.A. Kastrup, Nucl. Phys. B399, 211 (1993).

[10] R. Jackiw in Quantum Theory of Gravity, edited by S. Christensen (Hilger, Bristol 1984), p403; C. Teitelboim ibid, p327.

[11] A. Achucarro and M.E. Ortiz, Phys. Rev. D48, 3600 (1993).

[12] G. Kunstatter, R. Petryk and S. Shelemy, Phys. Rev. D57, 3537-3547 (1998).

[13] J. Louko and B.F. Whiting, Phys. Rev. D51, 5583 (1995).

[14] S. Bose, J. Louko, L. Parker and Y. Peleg, Phys. Rev. D53, 5708 (1996); J. Louko and S. N. Winters-Hilt, Phys. Rev. D54, 2647 (1996); J. Louko, Jonathan Z. Simon, Stephen N. Winters-Hilt, Phys. Rev. D55, 3525-3535 (1997).

[15] J. Gegenberg and G. Kunstatter, D. Louis-Martinez, Phys. Rev. D51, 1781 (1995).

[16] T. Banks and M. O’Loughlin, Nucl. Phys. B362, 649 (1991). See also R.B. Mann, Phys. Rev. D47, 4438 (1993) and S. Odintsov and I. Shapiro, Phys. Letts. B263, 183 (1991);

[17] D. Louis-Martinez and G. Kunstatter, Phys. Rev. D52, 3494 (1995).

[18] D. Louis-Martinez, J. Gegenberg and G. Kunstatter, Phys. Letts. B321, 193 (1994).

[19] R. M. Wald, Phys. Rev. D48, R3427 (1993); V. Iyer and R. M. Wald, ibid, 50, 846 (1994).

[20] S. Bose, L. Parker and Y. Peleg, Phys.Rev. D56, 987 (1997).
[21] H.W. Braden, J.D. Brown, B.F. Whiting and J.W. York, Phys. Rev. D42, 3376 (1990).
Figure 1: Allowed space of observables for spherically symmetric gravity