Extreme value statistics from the real space renormalization group: Brownian motion, Bessel processes and continuous time random walks

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Abstract. We use the real space renormalization group (RSRG) method to study extreme value statistics for a variety of Brownian motions, free or constrained, such as the Brownian bridge, excursion, meander and reflected bridge, recovering some standard results, and extending others. We apply the same method to compute the distribution of extrema of Bessel processes. We briefly show how the continuous time random walk (CTRW) corresponds to a non-standard fixed point of the RSRG transformation.

Keywords: exact results, disordered systems (theory), fluctuations (theory), stochastic processes (theory)

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1. Introduction

There has recently been a resurgence of interest in extreme value statistics (EVS) and its applications to physics [1]–[5] biology [6, 7] and finance [8, 9]. Characterizing the statistical properties of the maximum $X_{\text{max}}$ (or the minimum $X_{\text{min}}$) of a set of $N$ random variables $X_1, X_2, \ldots, X_N$ has recently found many applications in these various areas of research, particularly in statistical physics. Indeed, being at the heart of optimization problems, EVS plays a major role in the study of disordered and glassy systems.

The EVS of independent, or weakly correlated, and identically distributed random variables is now well understood, thanks to the identification, in the large $N$ (thermodynamical) limit, of three distinct universality classes depending on the parent distribution of the $X_i$s [10]. The difficult cases arise when the random variables are strongly correlated, which happens to be the case of interest in many problems encountered in statistical physics. This is the case for instance for the directed polymer in random media and its connections to the Kardar–Parisi–Zhang equation [11] or the statistical
physics of a single particle in a logarithmically correlated random potential [3], closely related to Derrida’s random energy model [12].

The most important example of a set of strongly correlated random variables is of course that of Brownian motion (BM). And despite the fact that it has been widely studied [13]–[15], there has been renewed interest in the study of functionals involving Brownian motions [16,17], and in particular in the context of extreme statistics [18]–[23]. Consider a Brownian motion \( u(x) \) on the interval \( x \in [0,L] \), subject to \( u(0) = 0 \); it reaches its maximum \( u_m \) at time \( x = x_m \). The basic question that we ask is: what is the joint distribution \( P_L(u_m,x_m) \) of these random variables? For instance, for unconstrained Brownian motion, the (marginal) distribution \( P_L(x_m) \) is given by

\[
P_L(x_m) = \frac{1}{\pi} x_m^{-1/2} (L - x_m)^{-1/2}
\]

or equivalently the cumulative distribution

\[
Prob(x_m \leq x) = \left( \frac{2}{\pi} \right) \sin^{-1} \left( \sqrt{\frac{x}{L}} \right),
\]

which is the classical Lévy arcsine law [24]. These classical results for BM for \( P_L(u_m,x_m) \) have been recently obtained using the Feynman–Kac formula and further extended to a wider class of constrained Brownian motions, including Brownian excursions, meanders and the reflected Brownian bridge [22]. Interestingly, it was shown that this joint distribution \( P_L(u_m,x_m) \) for a Brownian motion and Brownian bridge arises naturally in the study of the convex hull of planar Brownian motions [25]. Recently, such results for the distribution of the maximum \( P_L(u_m) \) have been obtained, using also a path integral formalism, for the case of multidimensional processes, where one considers \( p \) non-intersecting Brownian walkers (vicious walkers) [26,27].

On the other hand, it is interesting to extend these studies to other stochastic processes. A natural one is the Bessel process, which is the radius of \( d \)-dimensional Brownian motion. The distribution of the maximum \( P_L(u_m) \) for a Bessel bridge has actually been widely studied in probability theory and statistics, in particular in the context of the generalization of the Kolmogorov–Smirnov test (corresponding to \( d = 1 \)) [28]. The marginal distribution \( P_L(u_m) \) for such Bessel bridges was first computed by Gikhman and Kiefer [29,30] and then generalized to any real \( d \) by Pitman and Yor [31], while much less is known about \( P_L(x_m) \). In [29]–[31], these results are obtained using rather complicated tools from probability theory, including the so called ‘agreement formulae’ which, although rigorous, are not very physically intuitive (see however [22]). To our knowledge, the extreme statistics of the Bessel process has not been studied using the tools of statistical physics.

In this paper we will use the real space renormalization group (RSRG) to study these problems of extreme value statistics. While the idea of renormalization in real space, or decimation, for pure systems dates back to the 1970s [32,33], the RSRG that we use here was first devised for the study of quantum spin chains with disorder [34]–[37]. It was then applied [38] to the study of the Sinai model [39], i.e. the Arrhenius motion (diffusion) of a particle in a random potential \( u(x) \) which is itself a random walk, i.e. it becomes a one-dimensional Brownian motion in the scaling limit. In both cases the idea is to iteratively decimate the smallest energy levels (for quantum systems) or barriers (for Arrhenius dynamics), defined as variables

\[
F = |u(x_i) - u(x_{i+1})|
\]

where \( (x_i, x_{i+1}) \) are pairs of successive local ‘min–max’. At a given stage, indexed by the ‘decimation scale’ called \( \Gamma \), the landscape does not contain any barrier smaller than \( \Gamma \). While it is only asymptotically exact for the above mentioned problems, it was later realized [40] that it is also a fully exact and quite powerful tool for studying extreme value statistics, and that it leads to solvable equations when applied to a broad class of processes, which includes, but

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is not restricted to, the Brownian motion. In fact it gives much more information than
the frequently asked questions in extreme value statistics as it also allows one to compute
the statistics of $h$-extrema (as introduced in the mathematical literature by Neveu and
Pitman [41]), i.e. the extrema separated by a barrier larger than $h$, and here $h \equiv \Gamma$. In
Sinai’s walk, these are ‘important extrema’ of the energy landscape, which are separated
by large enough barriers such that the process is not allowed to backtrack on more than
$\Gamma$ (see [42] for $h$-extrema in the context of Sinai’s walk). In [40] the RSRG was used to
obtain the extremal statistics of a toy model, i.e. Brownian motion plus a quadratic well.

The aim of this paper is to explain and illustrate how the RSRG allows one to recover
known results about extrema of the most standard constrained Brownian motions, i.e. the
bridge, the meander, the reflected BM and the excursion. It is in a sense overkill, as
we will see that the method is too powerful and we only use it in the limit $\Gamma \to \infty$.
However, in view of the versatility and importance of this method, we think that it
is useful to establish some links with more classical approaches and results. Since we
want to be mostly pedagogical, we restrict to symmetric BM, the case of a bias being a
straightforward extension. Note that even for the Brownian motion we compute a rather
complicated object, depicted in figure 2, at the same price as the standard, well known
result for the distribution of the maximum. Next we study the Bessel process (i.e. the
radius of a $d$-dimensional Brownian motion) and obtain its extremal statistics. Finally we
close with a short section on the extensions to continuous time random walks (CTRW).
Some of our results are standard well known results, derived here in a rather different
fashion; others are extensions and likely to be not so well known.

2. Extreme value statistics of some classical examples of constrained Brownian
motion

2.1. A short review of the RSRG

Let us briefly recall the main results of the RSRG method. We first focus on landscapes
$u(x)$ which perform a random walk which belong to the universality class of Brownian
motion. The general case, introduced in [40], is recalled later. The starting landscape is
chosen discrete, i.e. $u(x)$ is defined on an increasing sequence of points $x_j$ on the real axis
(successive intervals between them are called bonds) where it takes values $u_j = u(x_j)$.
It is by construction alternating, i.e. all $x_{2n}$ are local minima and all $x_{2n+1}$ are local
maxima. The rightmost and leftmost bonds play a special role and are called edge
bonds. The set $x_j, u(x_j)$ is chosen using a Markovian rule such that the bond lengths
$\ell_i = x_i - x_{i-1}$ and (positive) barriers $F_{2n} = u(x_{2n-1}) - u(x_{2n})$ or $F_{2n+1} = u(x_{2n+1}) - u(x_{2n})$ are statistically independent from bond to bond and chosen from a common probability
density $P_{\Gamma=0}(F, \ell)$ (bulk bonds) and $E_{\Gamma=0}(F, \ell)$ (edge bonds). This defines the starting
landscape at $\Gamma = 0$. The renormalized landscape at $\Gamma$ is obtained as follows. The
transformation which increases $\Gamma$ is called the decimation. It iteratively reduces the list
of extrema $x_0 = 0 < x_1 < \cdots < x_{2k+2} < x_L = L$ by removing an adjacent min–max
(or max–min) pair from the list corresponding to the smallest $F$ in the system when $\Gamma$
crosses that value, in effect gluing three bonds into one. The edge bonds play a special
role as they cannot be decimated; they can only grow by decimation of their nearest
neighbor. Because of the statistical independence of successive bonds, this results in a
simple RG (i.e. evolution) equation for $P_{\Gamma}(F, \ell)$ (bulk bonds) and $E_{\Gamma}(F, \ell)$ (edge bonds),

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functions which eventually reach fixed point forms. Under the same conditions as the initial landscape performs a random walk \((x_j, u_j)\) for, belonging at large scale to the universality class of the Brownian motion, this fixed point form describes the extremal statistics of the ‘important extrema’ of the Brownian motion (with the constraint that no return of more than \(\Gamma\) is permitted between two such extrema). It is convenient here to work in an ensemble of fixed total length \(L\); hence the complete measure \(\mathcal{P}_{L,\Gamma}[u]\) which we consider concerns the symmetric Brownian motion \(u(x)\) with \(x \in [0, L]\) and \(u(0) = u_0, u(L) = u_L\), and can be written as the following sum:

\[
\mathcal{P}_{L,\Gamma}[u] = \bar{\Gamma} E_\Gamma(F_1, \ell_1) E_\Gamma(F_2, \ell_2) \delta(L - (\ell_1 + \ell_2)) \delta(u_L - (u_0 + F_1 - F_2)) \\
+ \sum_{k=1}^{\infty} \bar{\Gamma} E_\Gamma(F_1, \ell_1) \prod_{j=2}^{2k+1} P_\Gamma(F_j, \ell_j) E_\Gamma(F_{2k+2}, \ell_{2k+2}) \delta \left( L - \sum_{i=1}^{2k+2} \ell_i \right) \\
\times \delta \left( u_L - \left( u_0 + \sum_{j=1}^{2k+2} (-1)^{j+1} F_j \right) \right)
\]

where \(\bar{\Gamma} = \Gamma^2\). In the RSRG language \([38]\) this is called the finite size measure for the renormalized landscape at scale \(\Gamma\) (as illustrated in figure 1). It evolves under decimation by the same RG flow of the functions \(P_\Gamma\) and \(E_\Gamma\), unaffected by the fixed length constraint. The total measure (1) is a sum of measures for the events where there remain \(2k+2\) bonds.

\textbf{Figure 1.} Schematic representation of the finite size RSRG measure (1) for the renormalized landscape (on the \(x\) axis, the \(\ell_j\)s are the bond lengths, and on the vertical \(y\) axis, the \(F_j\)s are the barriers). It is a sum of terms with \(k = 1, 2, \ldots\) valleys (i.e. sets of two consecutive bonds) not yet decimated at scale \(\Gamma\), i.e. of barriers larger than \(\Gamma\) (in the vertical \(y\) direction). The edge bonds (first and last) cannot be decimated; only the bulk bonds (all the others) are.
in the system, i.e. \( k \) valleys of bulk bonds and two edge bonds, with \( k = 0, 1, \ldots \). At large \( \Gamma \), for \( L \) finite, there remains only the term \( k = 0 \) which plays a special role. The product form reflects the Markovian nature of the landscape, which is preserved by decimation, and hence is either an exact consequence of the choice of a Markovian initial landscape, or a consequence of the convergence to the fixed point landscape. The only constraint is the fixed total length, implemented by the delta functions. The factor \( l\Gamma = \Gamma^2 \) is nothing but the average bond length which ensures the normalization of the total probability to unity. The fixed point form for the bond length probability, i.e. the one corresponding to the BM, takes, in Laplace variables \( P_{\Gamma}(F,p) = \int_0^L e^{-p\ell} P_{\Gamma}(F,\ell) \) the simple form

\[
P_{\Gamma}(F,p) = \frac{\sqrt{p}}{\sinh(\Gamma \sqrt{p})} e^{-(F-\Gamma)\sqrt{p} \coth(\Gamma \sqrt{p})} \theta(F - \Gamma),
\]

(2)

\[
E_{\Gamma}(F,p) = \Gamma^{-1} e^{-F \sqrt{p} \coth(\Gamma \sqrt{p})}.
\]

(3)

As mentioned above, \( P_{\Gamma}(F,\ell) \) can be related to a sum over all BM which are constrained to climb from 0 to \( F \) in a time \( \ell \) while remaining in the interval \( [0, F] \) and with no descent of more than \( \Gamma \). The RSRG equations which yield the fixed point forms (2) and (3) are recalled in appendix A.

2.2. The free Brownian motion and the Brownian bridge

In all cases one considers a Brownian motion \( u_x \) on the interval \([0, L] \), with \( \langle du_x^2 \rangle = 2 \, dx \), with initial condition \( u_0 = 0 \). Note that this corresponds to a diffusion coefficient \( D = 1 \), in contrast to the value \( D = 1/2 \) used in [22] to which we will refer in the following. While the free Brownian motion is unconstrained at \( x = L \) (see figure 2), the Brownian bridge is defined by the constraint \( u_L = 0 \) (see figure 4(a)). It is natural to consider the joint

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probability density $P_L(u_m, x_m, u_L)$ for the free Brownian motion ending at $u_L$ and having maximum value $u_m$ at position $x_m$. From the above considerations it is obtained from the RSRG by considering the last block (see figure 3), and is given by

$$P_L(u_m, x_m, u_L) = \lim_{\Gamma \to \infty} \Gamma^2 E_\Gamma(F_1, x_m) E_\Gamma(F_2, L - x_m), \quad \text{(4)}$$

with $F_1 = u_m$ and $F_2 = u_m - u_L$. Using the result (3) for the edge bond, one finds, in the large $\Gamma$ limit,

$$\lim_{\Gamma \to \infty} \Gamma E_\Gamma(F, p) = e^{-F\sqrt{p}}, \quad \text{(5)}$$

which gives after inverse Laplace transformation:

$$\lim_{\Gamma \to \infty} \Gamma E_\Gamma(F, l) = \frac{F}{2\sqrt{\pi} l^{3/2}} e^{-F^2/(4l)}. \quad \text{(6)}$$

Thus using equation (4) together with the expression for the edge bond in equation (6) one has

$$P_L(u_m, x_m, u_L) = \frac{1}{4\pi x_m^{3/2}} \frac{u_m (u_m - u_L)}{(L - x_m)^{3/2}} e^{-(u_m^2/4x_m) - ((u_m - u_L)^2)/(4(L - x_m))} \theta(u_m - u_L) \theta(u_m). \quad \text{(7)}$$

After integration over $x_m$ and $u_m$ this expression (7) yields the propagator $W(u_L, L)$:

$$W(u_L, L) = \int_{\max(0,u_L)}^{\infty} du_m \int_0^L dx_m P_L(u_m, x_m, u_L) = \frac{1}{\sqrt{4\pi L}} e^{-(u_L^2/4L)}. \quad \text{(8)}$$

We now obtain the result for the free Brownian motion by integrating the formula (7) over the final position $u_L$:

$$P_{\text{free}}(u_m, x_m) = \int_{-\infty}^{u_m} du_L P_L(u_m, x_m, u_L) = \theta(u_m) \frac{u_m}{2\pi \sqrt{L - x_m x_m^{3/2}}} e^{-(u_m^2/4x_m)}. \quad \text{(9)}$$

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Figures 4. Various constrained Brownian motions considered in this paper: (a) the Brownian bridge which is a Brownian motion constrained to start and end at 0, i.e. with \( u_0 = u_L = 0 \); (b) the reflected Brownian bridge, i.e. \( u_x = |u_x^0| \) where \( u_x^0 \) is the standard Brownian motion, and constrained to start and end at 0; (c) the Brownian excursion, i.e. Brownian motion constrained to start and end at 0, remaining positive in between; (d) the Brownian meander i.e. Brownian motion starting at 0, ending anywhere in \( u_L > 0 \) and remaining positive in between.

Figures 5. Bessel process in dimension \( d = 2 \), i.e. \( \nu = 0 \). (a) Brownian motion in dimension \( d = 2 \). (b) The Bessel process with index \( \nu = 0 \) corresponds to the radius of the two-dimensional Brownian motion.
This yields the well known results for the marginal distribution of the position of the maximum \( x_m \) for the free Brownian motion:

\[
\mathcal{P}_L^{\text{free}}(x_m) = \frac{1}{L} \mathcal{Q}_L^{\text{free}} \left( x_m / L \right), \quad \mathcal{Q}_L^{\text{free}}(x) = \frac{1}{\pi \sqrt{x(1-x)}}, \tag{10}
\]

which is the well known Lévy’s ‘arcsine law’ \[24\], as well as the distribution of the maximum \( u_m \):

\[
\mathcal{P}_L^{\text{free}}(u_m) = \frac{1}{L^{1/2}} \mathcal{Q}_L^{\text{free}} \left( u_m / L^{1/2} \right), \quad \mathcal{Q}_L^{\text{free}}(x) = \theta(x) e^{-(x^2/4)}, \tag{11}
\]

where \( \theta(x) \) is the Heaviside step function: \( \theta(x) = 0 \) if \( x < 0 \) and \( \theta(x) = 1 \) if \( x \geq 0 \).

From equation (7) we also immediately get the result for the Brownian bridge, where \( u_L = 0 \) (see figure 4(a)):

\[
P_L^{\text{bridge}}(u_m, x_m) = \mathcal{P}_L(u_m, x_m, u_L = 0) = \theta(u_m) \sqrt{\frac{L}{4\pi x_m^3 (L - x_m)^{3/2}}} \frac{u_m^2}{L - x_m},
\]

This yields the marginal distributions of \( x_m \) on the one hand and of \( u_m \) on the other hand:

\[
P_L^{\text{bridge}}(x_m) = \frac{1}{L}, \tag{13}
\]

\[
P_L^{\text{bridge}}(u_m) = \frac{1}{L^{1/2}} \mathcal{Q}_L^{\text{bridge}} \left( u_m / L^{1/2} \right), \quad \mathcal{Q}_L^{\text{bridge}}(x) = 2\theta(x) xe^{-x^2}. \tag{14}
\]

We thus see that the RSRG method allows one to recover standard results for the relatively easy cases of free Brownian motion and the Brownian bridge. Let us now investigate more complicated instances of other classical constrained Brownian motions.

**Notational remark:** in the following we will use invariably the notation \( P_L(\cdot, \cdot, \cdot) \), independently of the number of its arguments, to denote some joint distribution. Similarly, we will also use the same notation to denote its Laplace transform with respect to one or several of its arguments. The choice of notation for the different variables should be clear enough to avoid any confusion.

### 2.3. Reflected Brownian motion and the Brownian excursion and meander

In this section, we study the cases of the reflected Brownian motion, excursion and meander; see figures 4(b), (c) and (d) respectively. The joint distribution \( P_L(u_m, x_m) \) of the maximum \( u_m \) and its position \( x_m \) for these three cases was recently computed using both path integral techniques as well as the so called rigorous ‘agreement formulae’ in [22]. Using RSRG, all these cases can be obtained from the probability \( P_L(u_1, x_1, u_m, x_m, u_2, x_2, u_L) \) that an unconstrained Brownian motion \( u_x \) starting at \( u_0 = 0 \) ends at \( u_L \), reaches its maximum \( u_m \) at \( x_m \) and that its minimum \( u_1 \) on the segment \([0, x_m]\) is reached at \( x_1 \), while its minimum \( u_2 \) on the segment \([x_m, L]\) is reached at \( x_2 \) (see figure 2).

It can be obtained from the edge bond probabilities (including information on the minima)
From this object (19) we now obtain the probability density

\[ P_L(u_1, x_1, u_m, x_m, u_2, x_2, u_L) = E(u_m, x_m, -u_1, x_1) \]
\[ \times E(u_m - u_L, L - x_m, -u_2 + u_L, L - x_2), \]
\[ E(u_m, x_m, u, x) = \lim_{\Gamma \to \infty} \Gamma E(\Gamma(u_m, x_m, u, x)). \]  

(15)

We show in appendix C that, in Laplace variables, one has

\[ E(u_m, p, u, q) = \int_0^{\infty} dx_m \int_0^{\infty} dx e^{-px_m-qx} E(u_m, x_m, u, x), \]
\[ E(u_m, p, u, q) = \frac{\sinh(\sqrt{p + q u_m})}{\sinh(\sqrt{p + q} u_m)} \frac{\sqrt{p}}{\sinh(\sqrt{p} u_m)}. \]  

(16)

Performing Laplace inversions (see equations (C.11) and (C.13)) one finds

\[ E(u_m, x_m, u, x) = \sum_{n,m=0}^{\infty} 4\pi^3 n m^2 (-1)^{n+m} \frac{\sin((\pi u_m)/(u + u_m) n)}{(u + u_m)^3} \]
\[ \times e^{-(\pi^2)/(2(u + u_m)^2)(n^2 + m^2(x_m - x))}. \]  

(17)

Notice the scaling forms, directly read from equations (16) and (17):

\[ E(u_m, x_m, u, x) = \frac{1}{u_m^3} \tilde{E}(x_m, u_m, x, u_m), \]
\[ E(u_m, p, u, q) = \frac{1}{u_m} \tilde{E}(pu_m, u_m, qu_m), \]
with \( \tilde{E}(\tilde{p}, \tilde{u}, \tilde{q}) = \int_0^{\infty} d\tilde{x}_m \int_0^{\infty} d\tilde{x} e^{-\tilde{p}\tilde{x}_m - \tilde{q}\tilde{x}} \tilde{E}(\tilde{x}_m, \tilde{u}, \tilde{x}). \)  

(18)

From equations (15) and (17) one gets the probability density of the event depicted in figure 2:

\[ P_L(u_1, x_1, u_m, x_m, u_2, x_2, u_L) = \theta(u_L - u_2) \theta(u_m - u_L) \]
\[ \times \sum_{n_1, n_2, m_1, m_2 = 0}^{\infty} 16\pi^6 n_1 n_2 m_1^2 m_2^2 (-1)^{n_1 + m_1 + n_2 + m_2} \]
\[ \times \frac{\sin((\pi u_m)/(u_m - u_1)n_1) \sin((\pi u_m - u_L)/(u_m - u_2)n_2)}{(u_m - u_1)^3 (u_m - u_2)^3} \]
\[ \times e^{-(\pi^2)/(2(u_m - u_1)^2)(n_1^2 + m_1^2(x_m - x_1)) - (\pi^2)/(2(u_m - u_2)^2)(n_2^2 + m_2^2(x_2 - x_m))}. \]  

(19)

From this object (19) we now obtain the probability density \( P_L(u_m, x_m, u, x, u_L) \) for an unconstrained Brownian \( u_x \), starting at \( u_0 = 0 \) and ending at \( u_L \), reaching its maximum \( u_m \) at \( x_m \) and its minimum \( u \) at \( x \). It is given by

\[ P_L(u_m, x_m, u, x, u_L) = \theta(x_m - x) \int_u^{u_L} du_2 \int_{x_m}^L dx_2 P_L(u, x, u_m, x_m, u_2, x_2, u_L) \]
\[ + \theta(x - x_m) \int_u^{u_2} du_1 \int_0^{x_m} dx_1 P_L(u_1, x_1, u_m, x_m, u, x, u_L). \]  

(20)
After some algebra, left to appendix D, one obtains

\[
P_L(u_m, x_m, u, x, u_L) = \theta(x_m - x) \sum_{n_1}^{\infty} 4\pi^3 n_1 m_1^2 (-1)^{n_1+m_1} \frac{\sin((\pi u_m/(u_m - u)) n_1)}{(u_m - u)^5} \times e^{-((\pi^2/2)(n_1^2 x + m_1^2 (x_m - x)))} \times \sum_{n_2=0}^{\infty} 2\pi (-1)^{n_2+1} n_2 \frac{\sin((\pi u_m/(u_m - u)) n_2)}{(u_m - u)^3} e^{-((\pi^2/2)(n_2 + m_2^2 (x_m - x)))} \times \theta(x - x_m) \sum_{n_2,m_2}^{\infty} 4\pi^3 n_2 m_2^2 (-1)^{n_2+m_2} \frac{\sin((\pi u_m/(u_m - u)) n_2)}{(u_m - u)^5} e^{-((\pi^2/2)(n_1^2 + m_1^2 (x_m - x)))}.
\]

(21)

In particular, one can check explicitly the following symmetry:

\[
P_L(u_m, x_m, u, x, u_L = 0) = P_L(-u, x, -u_m, x_m, u_L = 0).
\]

(22)

From the structure of equation (15) one obtains the useful expression for \( \hat{P}_L(u_m, x_m, u, x, u_L, u) \) which is the joint probability of the maximum \( u_m \) and of its position \( x_m \) for a Brownian motion starting at \( u_0 = 0 \) and ending at \( u_L \) and with a minimum value \( u \) larger than \(-a\). It reads

\[
\hat{P}_L(u_m, x_m, u_L, u) = \int_{-a}^{0} du \int_{0}^{u_L} dx \; P_L(u_m, x_m, u, x, u_L)
= \int_{0}^{u_L-a} du \int_{0}^{L-x_m} dx \; E(u_m - u_L, L - x_m, u, x)
\times \int_{0}^{u} du' \int_{0}^{x_m} dx' \; E(u_m, x_m, u', x').
\]

(23)

2.3.1. The reflected Brownian bridge. The reflected Brownian is defined by \( u_x = |u_x^0| \) where \( u_x^0 \) is the usual Brownian motion, and initial condition \( u_0 = 0 \). We focus on the reflected Brownian bridge, for which \( u_L = 0 \); see figure 4(b). Therefore one has

\[
P_{L, \text{ref. bridge}}(u_m, x_m) = \sqrt{4\pi L} \left( \int_{-u_m}^{0} du \int_{0}^{L} dx \; P_L(u_m, x_m, u, x, u_L = 0)
+ \int_{0}^{u_m} du_m \int_{0}^{L} dx \; P_L(\bar{u}_m, \bar{x}_m, -u_m, x_m, u_L = 0) \right),
\]

(24)

where the prefactor \( \sqrt{4\pi L} \) comes from the fact that we are considering bridges (like for equation (12) above). Using the above symmetry (22) one sees that these two terms are actually identical, yielding

\[
P_{L, \text{ref. bridge}}(u_m, x_m) = 2\sqrt{4\pi L} \int_{-u_m}^{0} du \int_{0}^{L} dx \; P_L(u_m, x_m, u, x, u_L = 0).
\]

(25)

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Therefore one has \( P_L^{\text{ref. bridge}}(u_m, x_m) = 2\sqrt{4\pi L} \hat{P}_L(u_m, x_m, u_L = 0, u_m) \), and the above identity in equation (23) yields

\[
P_L^{\text{bridge}}(u_m, x_m) = 2\sqrt{4\pi L} \int_0^{u_m} du \int_0^{L-x_m} dx \, E(u_m, L-x_m, u, x) \\
\quad \times \int_0^{u_m} du \int_0^{x_m} dx \, E(u_m, x_m, u, x),
\]

(26)

From the scaling forms in equation (18) one obtains that

\[
P_L^{\text{ref. bridge}}(u_m, x_m) = 2\sqrt{4\pi L} P_c \left( \frac{u_m}{x_m^2} \right) P_c \left( \frac{L-u_m}{x_m^2} \right),
\]

(27)

where the Laplace transform of \( P_c(z) \) is given by

\[
\int_0^\infty dz \, e^{-pz} P_c(z) = \int_0^1 du \, \hat{E}(p, u, q = 0) = \int_0^1 du \, \frac{\sqrt{p} \sinh(\sqrt{p})}{(\sinh(\sqrt{p}(u+1)))^2} = \frac{1}{2 \cosh(\sqrt{p})}.
\]

(28)

By inverting this Laplace transform, one obtains

\[
P_c(z) = \pi \sum_{n=0}^\infty (-1)^n (n + \frac{1}{2}) e^{-\pi^2(n+1/2)^2} z.
\]

(29)

And finally, one has

\[
P_L^{\text{ref. bridge}}(u_m, x_m) = 4\pi^{5/2} L^{1/2} \frac{1}{u_m^4} \sum_{m,n=0}^\infty (-1)^{n+m} \left( n + \frac{1}{2} \right) \left( m + \frac{1}{2} \right) \\
\quad \times e^{-\pi^2(m+1/2)^2(x_m/u_m^2)} e^{-\pi^2(n+1/2)^2((L-x_m)/u_m^2)},
\]

(30)

which returns the formula obtained by Majumdar et al in [22] (see their formula (28)). In particular, by integrating over \( u_m \), one obtains the marginal distribution of \( x_m \) as

\[
P_L^{\text{ref. bridge}}(x_m) = \frac{1}{L} \hat{\mathcal{P}}_{\text{ref. bridge}} \left( \frac{x_m}{L} \right),
\]

(31)

\[
\hat{\mathcal{P}}_{\text{ref. bridge}}(x) = 2 \sum_{n,m=0}^\infty (-1)^{m+n} \frac{(2m+1)(2n+1)}{[(2m+1)^2 + (2n+1)^2(1-x)]^{3/2}}.
\]

(32)

2.3.2. The Brownian excursion. Here one considers the case of Brownian excursions, constrained to start and end at 0, remaining positive in between; see figure 4(c). In that case, one would naively impose that the minimum is zero exactly and \( u_L = 0 \). However, one immediately sees from equation (20) that \( P_L(u_m, x_m, u = 0, x = 0, u_L = 0) = 0 \). This is expected since the Brownian motion has an infinite density of zero crossings. Therefore, to compute the joint probability \( P_L^{\text{ex}}(u_m, x_m) \) from \( P_L(u_m, x_m, u, x, u_L = 0) \) in equation (20), one uses a limiting procedure (see [19, 22] for a similar procedure used with the Feynman–Kac formula):

\[
P_L^{\text{ex}}(u_m, x_m) = \lim_{\epsilon \to 0} \frac{\int_0^\infty du \int_0^L dx \, P_L(u_m, x_m, u, x, u_L = 0)}{\int_0^\infty du_m \int_0^L dx_m \int_0^L dx \, P_L(u_m, x_m, u, x, u_L = 0)}.
\]

(33)
Following the same analysis as above, and using again the identity in equation (23), one obtains
\[
\int_0^\infty du \int_0^L dx \, P_L(u_m, x_m, u, x, u_L = 0)
= \epsilon^2 \int_0^L dx \, E(u_m, L - x_m, 0, x) \int_0^x dx \, E(u_m, x_m, 0, x) + O(\epsilon^3). \tag{34}
\]
Using the above scaling forms (18) one obtains that
\[
\int_0^\infty du \int_0^L dx \, P_L(u_m, x_m, u, x, u_L = 0) = \epsilon^2 \frac{1}{u_m^6} P_s \left( \frac{u_m}{x_m^2} \right) P_s \left( \frac{L - u_m}{x_m^2} \right) + O(\epsilon^3), \tag{35}
\]
where the Laplace transform of \( P_s(z) \) is given by
\[
\int_0^\infty dz \, e^{-pz} P_s(z) = \tilde{E}(p, u = 0, q = 0) = \frac{\sqrt{p}}{\sinh (\sqrt{p})}. \tag{36}
\]
By inverting the Laplace transform, one obtains
\[
P_s(z) = 2\pi^2 \sum_{m=0}^{\infty} (-1)^{m+1} m^2 e^{-m^2z}. \tag{37}
\]
Finally, after normalization, one obtains
\[
P_{\text{ex}}^L(u_m, x_m) = \frac{8\pi^{9/2} L^{3/2}}{u_m^6} \sum_{m,n=0}^{\infty} (-1)^{m+n} m^2 n^2 e^{-m^2x^2 + n^2u_m^2} e^{-n^2((L - x_m)/u_m^2)}, \tag{38}
\]
which returns the formula obtained by Majumdar et al in [22] (see their formula (8)). In particular, by integrating over \( u_m \), one obtains the marginal distribution of \( x_m \) as
\[
P_{\text{ex}}^L(x_m) = \frac{1}{L} \tilde{p}_{\text{ex}}^L \left( \frac{x_m}{L} \right), \tag{39}
\]
\[
\tilde{p}_{\text{ex}}^L(x) = 3 \sum_{n,m=0}^{\infty} (-1)^{m+n} \frac{m^2n^2}{\left[ n^2x + m^2(1 - x) \right]^{5/2}}. \tag{40}
\]

2.3.3. The meander. Here we consider Brownian meanders, which are similar to excursions except that in that case the endpoint \( u_L \) is free; see figure 4(d). Thus one would impose the minimum to be 0 and integrate over \( u_L \). Following the same limiting procedure as above one has
\[
P_{\text{mea}}^L(u_m, x_m) = \lim_{\epsilon \to 0} \frac{\int_0^{u_m} du_L \int_0^L du \int_0^L dx \, P_L(u_m, x_m, u, x, u_L)}{\int_0^{u_m} du_L \int_0^\infty du_m \int_0^L dx_m \int_0^L dx \, \int_0^L dx \, P_L(u_m, x_m, u, x, u_L)}. \tag{41}
\]
Performing the same analysis as before, and using again the identity in equation (23) one obtains that
\[ \int_x^{u_m} du_L \int_0^0 du \int_0^L dx \, P_L(u_m, x_m, u, x, u_L) \]
\[ = \epsilon \int_x^{u_m} du_L \int_0^{L-x_m} dx \, E(u_m-u_L, L-x_m, 0, x) \]
\[ \times \int_0^{x_m} dx \, E(u_m, x_m, 0, x) + O(\epsilon^2). \tag{42} \]

Therefore, after straightforward algebra, one has here
\[ \int_x^{u_m} du_L \int_0^0 du \int_0^L dx \, P_L(u_m, x_m, u, x, u_L) = \epsilon \frac{1}{u_m^4} P_s \left( \frac{u_m}{x_m^2} \right) P_t \left( \frac{L-u_m}{x_m^2} \right), \tag{43} \]
where the Laplace transform of \( P_t(z) \) is given by
\[ \int_0^\infty dz \, e^{-pz} P_t(z) = \sqrt{p} \int_0^1 dv \sinh (\sqrt{p}(1-v)) \int_0^v \frac{du}{(\sinh (\sqrt{p}(1-v+u)))^2} \]
\[ = \frac{\tanh (\sqrt{p}/2)}{\sqrt{p}}. \tag{44} \]

Inverting the Laplace transform yields finally
\[ P_L^\text{mea}(u_m, x_m) = \frac{4\pi^{5/2} L^{1/2}}{u_m^4} \sum_{m,n=1}^\infty ((-1)^{m+n} - (-1)^n) n^2 e^{-n^2 \pi^2 (x_m/u_m^2)} e^{-m^2 \pi^2 ((L-x_m)/u_m^2)}, \tag{45} \]
which returns the formula obtained by Majumdar et al in [22] (see their formula (19)).

In particular, by integrating over \( u_m \), one obtains the marginal distribution of \( x_m \) as
\[ P_L^\text{mea}(x_m) = \tilde{P}_L^\text{mea} \left( \frac{x_m}{L} \right), \tag{46} \]
\[ \tilde{P}_L^\text{mea}(x) = \sum_{n,m=1}^\infty ((-1)^{m+n} - (-1)^n) \frac{n^2}{n^2 x + m^2 (1-x)} \tag{47} \]
\[ = 2 \sum_{m=0, n=1}^\infty ((-1)^{m+n} - (-1)^n) \frac{n^2}{n^2 x + (2m+1)^2 (1-x)} \tag{48} \]
where in the last equation we have used that only the terms where \( m \) is odd contribute, i.e. \( (-1)^{m+n} - (-1)^n = 0 \) when \( m \) is even.

We have thus shown that the RSRG allows one to recover in a rather simple way standard results for various constrained Brownian motions. We now extend this method to other stochastic processes and consider the case of Bessel processes.

### 3. Bessel processes and more

#### 3.1. The RSRG for more general processes

The RSRG was extended to more general processes in [40], and we refer the reader to that paper for all details. The generalization of the finite size measure \( \mathcal{P}_{L,\Gamma}[u] \) in equation (1)

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now reads

\[ \mathcal{P}_{L,\Gamma}[u] = E^-_1(0, u_0; u_1, x_1)E^+_1(u_1, x_1, L, u_L) + \sum_{k=1}^{\infty} E^-_1(0, u_0; u_1, x_1) \]

\[ \times \prod_{j=1}^k B^+_1(u_{2j-1}, x_{2j-1}; u_{2j}, x_{2j})B^-_1(u_{2j}, x_{2j}; u_{2j+1}, x_{2j+1}) \]

\[ \times E^+_1(u_{2k+1}, x_{2k+1}, L, u_L), \]

which still has a block structure, but with slightly more general blocks. The index ‘-’ refers to an ascending bond, i.e. where \( u_x \) is on average an increasing function, while ‘+’ refers to descending bonds. This distinction was not needed and was suppressed in (1) for the symmetric BM, but it is useful in general. As compared to [40], we have reversed the orientation \( \pm \). This amounts to the reflection \( u(x) \rightarrow -u(x) \) w.r.t. the notation there, or, in other words, to considering here \( u(0^-) = u(L^+) = -\infty \). This implies that the \( k = 0 \) term (the first one) in the large \( \Gamma \) limit yields the distribution of the maximum (rather than the minimum there). RSRG equations for these blocks were derived there, which we will not reproduce here, and only a small class of solutions were identified (presumably many more remain to be discovered).

As an example, the following class of the real valued Langevin process \( u_x \) was found to provide solutions of the RSRG equations, and was studied in [40]:

\[ du_x = -W'(u_x) + dB_x, \]

with \( dB_x = 2dx \) a Brownian motion. It represents processes which undergo diffusion in a given potential \( W(u) \) with \( u \in ] -\infty, +\infty [ \). The explicit form of the blocks, for any \( \Gamma \), was obtained in [40], and we need here only the form of the edge blocks, given in Laplace variable w.r.t. \( x_1 - x_0 \) as

\[ E^-_1(0, u_1, p) = e^{-\frac{1}{2}(W(u_1)-W(u_0))} \tilde{E}^-_1(u_0, u_1, p), \]

\[ \tilde{E}_1^- (u_0, u, p) = \tilde{E}_1^+(u, u_0, p) = \exp \left( \int_u^{u_0} dv \partial_1 \ln K(v, v - \Gamma, p) \right), \]

where

\[ K(u, v, p) = \frac{1}{w(p)}(\Phi_1(u, p)\Phi_2(v, p) - \Phi_2(u, p)\Phi_1(v, p)), \]

where \( \Phi_1(u, p) \) and \( \Phi_2(u, p) \) are the two independent solutions of the associated Schrödinger problem:

\[ \partial_1^2 - (p + V(u))\Phi(u, p) = 0, \quad V(u) = \frac{1}{4}W'(u)^2 - \frac{1}{2}W''(u), \]

and \( w(p) \) their Wronskian, \( w(p) = \Phi_1(u, p)\partial_0\Phi_2(u, p) - \Phi_2(u, p)\partial_0\Phi_1(u, p) \). Bulk bonds have a similar expression in terms of the kernel \( K(u, v, p) \). Note that in the limit \( \Gamma \rightarrow \infty \) the quantity

\[ \partial_1 \ln K(v, v - \Gamma, p) = \frac{\Phi_1'(v, p)\Phi_2(v - \Gamma, p) - \Phi_2'(v, p)\Phi_1(v - \Gamma, p)}{\Phi_1(v, p)\Phi_2(v - \Gamma, p) - \Phi_2(v, p)\Phi_1(v - \Gamma, p)}, \]

automatically becomes equal to \( \Phi'(v, p)/\Phi(v, p) \) where \( \Phi(v, p) \) is the linear combination of \( \Phi_1 \) and \( \Phi_2 \) orthogonal to the one which blows up at \( v \rightarrow -\infty \). This is what is expected for a process \( u_x \in ] -\infty, +\infty [ \).

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3.2. Extreme value statistics of the Bessel process

We now call $u_x$ the Bessel process (see figure 5), i.e. the radius of the $d$-dimensional Brownian motion $u_x = \sqrt{\sum_{i=1}^{d} (B^i_x)^2}$ where the $B^i_x$ are $d$ independent BM with $(dB^i_x)^2 = 2\,dx$, $i = 1, \ldots, d$. As is well known, it satisfies the one-dimensional Langevin equation $du_x = dB_x + (d-1)/u_x$, $u_x > 0$, with $dB_x^2 = 2\,dx$. Hence it is of the type (50) with a potential $W(u) = -r \ln |u|$ with $r = d - 1$. This allows a generalization for real values of $d$.

Consider now a Bessel process (BP) starting at $u = u_0$ at $x = 0$, and ending at $u_L$ at $x = L$. Let us call $P_L(u_m, x_m, u_L | u_0)$ the joint probability distribution for the BP $u_x$ starting at $u_0$ having minimum $u_m$ at position $x_m$ and ending up at $u_L$. It is given by the last RSRG block in (49):

$$
P_L(u_m, x_m, u_L | u_0) = \lim_{\Gamma \rightarrow \infty} E^-_{\Gamma}(u_0, u_m, x_m) E^+_{\Gamma}(u_m, u_L, L - x_m).
$$

It satisfies the normalization condition

$$
\int_0^L dx_m \int_{-\infty}^{u_0} du_m \int_{u_m}^0 du_L P_L(u_m, x_m, u_L | u_0) = 1,
$$

which we will check below explicitly for our solution. In the case of the BP, there is a subtlety with applying the recipe given in [40] and the solution (51). Indeed the process remains on the interval $u_x \in [0, +\infty]$. Hence $u = 0$ has a special role, related to the returns to the origin.

Let us first examine the naive solutions of (53) for the potential $V(u) = 1/2 W'(u)^2 - 1/2 W''(u) = (d-1)(d-3)/(4u^2)$ for the BP. The two independent solutions read

$$
\Phi_1(u, p) = \sqrt{u}K_{-(d/2)}(u\sqrt{p}), \quad \Phi_2(u, p) = \sqrt{u}I_{-(d/2)}(u\sqrt{p}),
$$

where $I_p(x)$ and $K_p(x)$ are modified Bessel functions of the first and second kind respectively. Note that for $d = 1$ one recovers the result for the symmetric BM, in which case $W = 0$ and $\Phi_1(u, p) \sim e^{-u\sqrt{p}}$, $\Phi_2(u, p) \sim \sinh(u\sqrt{p})$, up to unimportant $p$-dependent normalization which is canceled by the Wronskian. For $d < 2$ the above solutions behave as $u \rightarrow 0^-$ and for fixed $p > 0$,

$$
\Phi_1(u, p) = 2^{-d/2}p^{(d-2)/4}u^{(d-1)/2}\Gamma(1 - d/2),
\Phi_2(u, p) = 2^{-1+d/2}p^{(2-d)/4}u^{(3-d)/2}\Gamma(2 - d/2),
$$

which, upon a rescaling by $p^{(d-1)/2}$ gives also the $p = 0$ limit. In the above expression, $\Gamma(x)$ is the Gamma function. In that limit $p = 0$ these two solutions correspond to the general expression [40]

$$
\Phi_1(u, p = 0) = e^{-(1/2)|W(u)|}, \quad \Phi_2(u, p = 0) = e^{-(1/2)|W(u)|}\int_0^u du' e^{W(u')}.
$$

The process being only defined for $u_x \geq 0$, the question arises of the correct solution which is selected in the limit $\Gamma \rightarrow +\infty$. First of all the above formulae in equation (51) must be replaced by

$$
E_1^\pm(u_0, u_1, p) = e^{-(1/2)(W(u_1) - W(u_0))} E_1^\pm(u_0, u_1, p),
$$

$$
\tilde{E}_1^\pm(u_0, u, p) = \tilde{E}_1^\pm(u, u_0, p) = \exp \left( \int_{u_0}^u dv \, \partial_v \ln K(v, \max(0, v - \Gamma), p) \right),
$$

\[\text{do}\text{i}:10.1088/1742-5468/2010/01/P01009\]
and the question of the value for $K(u, 0, p)$ arises. One way to regularize the problem would be to add near $u_0 > 0$ a very steep barrier (e.g. $W(u) = e^{a_0/|u|}$) and consider $K(u, v, p)$ in the limit $|v| \gg u_0$, both going to zero. A short-cut for treating that barrier problem is to notice that this is equivalent to forgetting the barrier and asking for reflecting boundary conditions at $v = 0$ for the process $u_\epsilon$. We must also ask that as $u \to 0$ the current vanishes so that

$$J_S = \left( \partial_u - \frac{1}{2} W'(u) \right) K(u, 0, p) = \left( \partial_u - \frac{d-1}{2u} \right) K(u, 0, p) = 0. \quad (62)$$

It is then easy to find that the proper solution is $K(u, 0, p) \sim \cosh(u \sqrt{p})$ for the symmetric BM (i.e. when $W(u) = 0$; i.e. the $d = 1$ BP becomes the reflected BM—see below), while for the BP we find

$$K(u, 0, p) = A \left( \frac{2}{\pi} \sin \left( \frac{d\pi}{2} \right) \Phi_1(u) + \Phi_2(u) \right) = \sqrt{u} I_{1+(d/2)}(u \sqrt{p}), \quad (63)$$

where $A$ is some constant. We may understand this from a limit case of the regularization by a barrier at scale $u_0$, writing

$$K(u, v, p) \sim \Phi_{1,2}^{u_0}(u) \Phi_{1,2}^{u_0}(v) - \Phi_2^{u_0}(u) \Phi_1^{u_0}(v), \quad (64)$$

where $\Phi_{1,2}^{u_0}(u)$ are respectively decaying and exploding at $u = +\infty$, and showing that, irrespective of the form of the barrier, it reflects as

$$\lim_{|v| \gg u_0 \to 0} \frac{\Phi_2^{u_0}(v)}{\Phi_1^{u_0}(v)} = -\frac{2}{\pi} \sin \left( \frac{d\pi}{2} \right). \quad (65)$$

Using this result we now find that the large $\Gamma$ limit is given by

$$\lim_{\Gamma \to -\infty} E_\Gamma^-(u_0, u_m, p) = e^{-\frac{1}{2}(W(u_m) - W(u_0))} \left( \frac{2}{\pi} \right) \sin((\pi d)/2) \Phi_1(u_0, p) + \Phi_2(u_0, p) \quad (66)$$

$$= \left( \frac{u_m}{u_0} \right)^{(d-2)/2} \left( \frac{2}{\pi} \right) \sin((\pi d)/2) K_{1-(d/2)}(u_0 \sqrt{p}) + I_{1-(d/2)}(u_0 \sqrt{p}) \quad (67)$$

$$= \left( \frac{u_m}{u_0} \right)^{(d-2)/2} \frac{I_{1+(d/2)}(u_0 \sqrt{p})}{I_{1-(d/2)}(u_m \sqrt{p})}, \quad (68)$$

which is found to hold for any $d$. One finds similarly

$$\lim_{\Gamma \to -\infty} E_\Gamma^+(u_m, u_L, p) = \left( \frac{u_L}{u_m} \right)^{d/2} \frac{I_{1-(d/2)}(u_L \sqrt{p})}{I_{1-(d/2)}(u_m \sqrt{p})}. \quad (69)$$

One can check the normalization (56) in Laplace variables, if one writes the product

$$\lim_{\Gamma \to -\infty} E_\Gamma^-(u_0, u_m, p) E_\Gamma^+(u_m, u_L, p) = 1 \left( \frac{u_L}{u_0} \right)^{d/2} \frac{I_{1+(d/2)}(u_0 \sqrt{p})}{I_{1-(d/2)}(u_m \sqrt{p})}. \quad (70)$$

Integration over $u_L$ and $u_m$ as described in (56) yields $1/p$, as required.
The Inverse Laplace transform now gives, for any positive zeros of \(d\) and writes

\[
\lim_{\Gamma \to \infty} E^{-}\Gamma(u_0, u_m, p) = \frac{\cosh(u_0 \sqrt{\Gamma})}{\cosh(u_m \sqrt{\Gamma})}, \quad \lim_{\Gamma \to \infty} E^{+}\Gamma(u_m, u_L, p) = \frac{\cosh(u_L \sqrt{\Gamma})}{\cosh(u_m \sqrt{\Gamma})}. \tag{71}
\]

Similarly for \(d = 3\), one finds

\[
\lim_{\Gamma \to \infty} E^{-}\Gamma(u_0, u_m, p) = \frac{u_m \sinh(u_0 \sqrt{\Gamma})}{u_0 \sinh(u_m \sqrt{\Gamma})}, \quad \lim_{\Gamma \to \infty} E^{+}\Gamma(u_m, u_L, p) = \frac{u_L \sinh(u_L \sqrt{\Gamma})}{u_m \sinh(u_m \sqrt{\Gamma})}. \tag{72}
\]

The Inverse Laplace transform now gives, for any \(d\),

\[
\lim_{\Gamma \to \infty} E^{-}\Gamma(u_0, u_m, x) = \left(\frac{u_m}{u_0}\right)^{(d-2)/2} \sum_n \frac{2}{u_m^2} J_{\nu,n}(u_0/u_m) I_{\nu}(j_{\nu,n}) e^{-((j_{\nu,n}^2)u_m^2)}, \tag{73}
\]

with \(I_{\nu}(z) = e^{-(1/2)\nu \pi} J_{\nu}(iz)\), \(\nu = -1 + d/2\), and \(0 < j_{\nu,1} < j_{\nu,2} < \cdots\) is the sequence of positive zeros of \(J_{\nu}\), i.e. \(J_{\nu}(j_{\nu,n}) = 0\), where \(J_{\nu}(x)\) is a Bessel function of the first kind. It can be further simplified using standard relations between Bessel functions:

\[
\lim_{\Gamma \to \infty} E^{-}\Gamma(u_0, u_m, x) = \left(\frac{u_m}{u_0}\right)^{(d-2)/2} \sum_n \frac{2}{u_m^2} J_{\nu,n}(u_0/u_m) J_{\nu+1}(j_{\nu,n}) e^{-((j_{\nu,n}^2)u_m^2)}, \tag{74}
\]

and, similarly,

\[
\lim_{\Gamma \to \infty} E^{+}\Gamma(u_m, u_L, x) = \left(\frac{u_L}{u_m}\right)^{d/2} \sum_n \frac{2}{u_L^2} J_{\nu,n}(u_L/u_m) J_{\nu+1}(j_{\nu,n}) e^{-((j_{\nu,n}^2)u_m^2)}. \tag{75}
\]

Combining these two expressions (74) and (75) yields our final result for the joint distribution of the maximum, its position and the endpoint of the Bessel process:

\[
P_L(u_m, x_m, u_L|u_0) = \frac{u_L^{d/2}}{u_0^{d-2}} \sum_{n,m} j_{\nu,n} j_{\nu,m} \frac{J_{\nu}(j_{\nu,n}(u_0/u_m)) J_{\nu}(j_{\nu,m}(u_L/u_m))}{J_{\nu+1}(j_{\nu,n}) J_{\nu+1}(j_{\nu,m})} \times e^{-(j_{\nu,n}^2u_m^2)/u_m^2} e^{-(j_{\nu,m}^2(x_m-L)/u_m^2)}, \tag{76}
\]

with \(\nu = -1 + d/2\).

We now focus on Bessel bridges for which \(u_0 = u_L = 0\). Here again one uses a limiting procedure as before and writes

\[
P_L^{\text{Bessel bridge}}(u_m, x_m) = \lim_{\epsilon \to 0} P_L(u_m, x_m, u_L = \epsilon|u_0 = \epsilon). \tag{77}
\]

Using the small \(x\) behavior \(J_{\nu}(x) \sim x^{\nu}/(2^\nu \Gamma(\nu + 1))\) one obtains

\[
P_L^{\text{Bessel bridge}}(u_m, x_m) = \frac{8L^{\nu+1}}{\Gamma(1 + \nu) u_0^{5+2\nu}} \sum_{n,m} j_{\nu,n} j_{\nu,m} \frac{J_{\nu+1}(j_{\nu,n}) J_{\nu+1}(j_{\nu,m})}{J_{\nu+1}(j_{\nu,n}) J_{\nu+1}(j_{\nu,m})} \times e^{-(j_{\nu,n}^2u_m^2)/u_m^2} e^{-(j_{\nu,m}^2(x_m-L)/u_m^2)}, \tag{78}
\]

For \(d = 1\), one has \(\nu = -1/2\) and \(j_{-1/2,n} = (n + 1)\pi\); one checks that this formula returns the result for the reflected Brownian motion in equation (30). For \(d = 3\) one has \(\nu = 1/2\) and \(j_{-1/2,n} = n\pi\), and one checks that this formula returns the result for the excursion in equation (38).

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On the other hand, if one integrates over $x_m$ one recovers the distribution of $u_m$ as obtained by Gikhman [29] and Kiefer [30] for integer values of the dimension $d$ and later generalized by Pitman and Yor in [31] for non-integer $d$ (see appendix E). It reads

$$\tilde{P}_{\text{Bessel bridge}}(x) = \frac{d}{dx} \left( \frac{4}{\Gamma(\nu+1)x^{2\nu+2}} \sum_{n=1}^{\infty} \frac{j_{\nu,n}^2}{J_{\nu+1}(j_{\nu,n})^2} \exp\left( -\frac{j_{\nu,n}^2 x^2}{x^2} \right) \right). \quad (79)$$

Now if one integrates instead over $u_m$ one obtains the marginal distribution of $x_m$ as

$$P_{\text{Bessel bridge}}(x_m) = \frac{1}{L} \tilde{P}_{\text{Bessel bridge}}(\frac{x_m}{L}) \quad (80)$$

$$\tilde{P}_{\text{Bessel bridge}}(x) = 4(1+\nu) \sum_{n,m} \frac{j_{\nu+1,n}^2 j_{\nu+1,m}^2}{J_{\nu+1}(j_{\nu,n}) J_{\nu+1}(j_{\nu,m})} \left( j_{\nu,n}^2 x + j_{\nu,m}^2 (1-x) \right)^{\nu+2}. \quad (81)$$

Many more quantities could be computed and it remains also to extend these considerations to finite $\Gamma$ and bulk bonds. We leave this to future studies.

### 4. Continuous time random walks: a new RSRG fixed point

We now come to the study of the so called continuous time random walks (CTRW), which was first introduced by Montroll and Weiss in [45]. Within this model of diffusion, the walker performs a usual random walk but has to wait for a certain ‘trapping time’ $\tau$ before each jump. The trapping times between jumps are independent and identically distributed random variables with a common density function $\psi(\tau)$ which has a power law tail, $\psi(\tau) \propto \tau^{-1-\alpha}$ with $0 < \alpha < 1$, while the jumps themselves are distributed according to a narrow distribution. This type of model was suggested by Scher and Montroll [46] for modeling non-Gaussian transport of electrons in disordered systems and since then it has been widely used for describing phenomenologically anomalous dynamics in various complex systems [43,44]. Indeed, for $\alpha < 1$, the mean trapping time between two successive jumps is infinite and hence the CTRW is characterized by a subdiffusive behavior, with a dynamical exponent $z = 2/\alpha > 1$, and non-Gaussian statistics. While the CTRW has been widely studied [43,44], it seems that the extreme statistics of such processes have not been studied in detail. It is the purpose of this section to study them using the RSRG.

We now trade the variable $\tau$ for $x$. The CTRW described above generates a landscape $u(x)$. Performing the first stages of the RSRG method as described in [40], one finds that the bonds in the renormalized landscape at scale $\Gamma$ acquire a broad distribution, with a typical length $\Gamma^{2/\alpha}$, while the jumps in $u$ remain of order $\Gamma$. One can now search for a fixed point of the RSRG which has precisely this scaling with $\Gamma$. This is done in appendix B and we find that these new fixed points can be simply obtained (up to some subtleties explained below) from the Brownian case by changing $p$ to $p^\alpha$ ($\alpha = 1$ corresponding to Brownian motion) in the RSRG blocks of the BM.

Therefore it is natural to consider the joint probability density $Q_L(u_m, x_m, u_L)$ for the free CTRW just arriving at $u_L$ at time $L$ and having maximum value $u_m$ at position $x_m$. From similar considerations as for equation (4), and the discussion in appendix B, it is
given by
\[ Q_L(u_m, x_m, u_L) = \lim_{\Gamma \to \infty} \Gamma^{2/\alpha} E_{\Gamma}(F_1, x_m) E_{\Gamma}(F_2, L - x_m), \]
where \( F_1 = u_m \) and \( F_2 = u_m - u_L \). Using the result (3) for the edge blocks together with replacing \( p \) by \( p^\alpha \) (see also equation (B.6) in appendix B) one obtains in the large \( \Gamma \) limit
\[ \lim_{\Gamma \to \infty} \Gamma^{1/\alpha} E_{\Gamma}(F, p) = e^{-F p^{\alpha/2}}, \]
which gives after inverse Laplace transformation
\[ \lim_{\Gamma \to \infty} \Gamma^{1/\alpha} E_{\Gamma}(F, l) = F^{-2/\alpha} \mathcal{L}_{\alpha/2}(I^{\Gamma-2/\alpha}), \]
where \( \mathcal{L}_\nu(x) \) is a one-sided Lévy stable pdf of index \( \nu = \alpha/2 \), whose Laplace transform is \( \int_0^\infty e^{-px} \mathcal{L}_\nu(x) = e^{-\nu x} \). The function \( \mathcal{L}_\nu(x) \) can be represented in terms of Fox \( \mathcal{H} \) functions [47]–[49] or, alternatively, can be written as a power series [50]:
\[ \mathcal{L}_\nu(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} x^{-\nu n-1} \frac{\Gamma(n\nu + 1)}{n!} \sin(n\pi\nu), \quad x > 0, \]
from which one obtains the asymptotic behavior for large argument \( x \gg 1 \) as \( \mathcal{L}_\nu(x) \sim \nu \Gamma(1 - \nu)^{-1} x^{-(1+\nu)} \), where we have used \( \Gamma(\nu)\Gamma(1 - \nu) = \pi / \sin(\nu\pi) \). For small argument, it behaves as [47]
\[ \mathcal{L}_\nu(x) \sim x^{-\sigma} e^{-\kappa x^{-\tau}}, \]
where \( \tau = \nu/(1 - \nu), \quad \kappa = (1 - \nu)\nu^{\nu/(1-\nu)}, \quad \sigma = (2 - \nu)/(2(1 - \nu)). \)
For \( \nu = 1/2 \), one has
\[ \mathcal{L}_{1/2}(x) = \frac{1}{2\sqrt{\pi} x^{3/2}} e^{-1/(4x)}, \]
and thus the formula (84) returns the result for Brownian motion in equation (6). For other values of \( \nu \), a representation of \( \mathcal{L}_\nu(x) \) in terms of other simpler special functions is usually obtained by examining the series expansion in equation (85) and using properties of gamma functions. For any rational value of \( \nu \), it was shown in [46] that \( \mathcal{L}_\nu(x) \) can be expressed in terms of a finite sum of hypergeometric functions (for instance, \( \mathcal{L}_{1/4}(x) \) is given in [49]). Here we simply quote another interesting simple case corresponding to \( \nu = 1/3 \), i.e. \( \alpha = 2/3 \), for which one has
\[ \mathcal{L}_{1/3}(x) = \frac{1}{(3x^3)^{1/3}} \text{Ai}[(3x)^{-1/3}], \]
where \( \text{Ai}(x) \) is the Airy function.

Now the probability that the free CTRW ends in \( u_L \) at time \( L \) and has maximum value \( u_m \) at position \( x_m \) is obtained from \( Q_L(u_m, x_m, u_L) \) as a convolution:
\[ P_L(u_m, x_m, u_L) = \int_{x_m}^{L} dL' Q_L'(u_m, x_m, u_L) \Psi(L - L'), \]
where \( \Psi(l) = \int_0^l \psi(x') dx' \) is the probability of no jump in the time interval \([0, l]\). Its Laplace transform \( \Psi(p) \) behaves, for small \( p \), as \( \Psi(p) = Ap^{\alpha-1} \) with \( A = 1 \); see appendix B.
And therefore one obtains the double Laplace transform of \( P_L(u_m, x_m, u_L) \) with respect to \( x_m \) and \( L \) as
\[
\int_0^L dx_m e^{-q x_m} \int_0^\infty dL e^{-pL} P_L(u_m, x_m, u_L) = e^{-(p+q)r}u_m e^{-p'(u_m-u_L)} p^{2\nu-1}, \tag{90}
\]
where \( \nu = \alpha/2 \). From equation (90), one checks that one recovers the correct expression for the propagator \( W(u_L, L) \), namely the probability that the free CTRW ends in \( u_L \) at time \( L \). It is given by
\[
W(u_L, L) = \int_{\max(u_L, 0)}^\infty du_m \int_0^L dx_m P_L(u_m, x_m, u_L). \tag{91}
\]
One obtains its Laplace transform with respect to \( L \) as
\[
W(u_L, p) = \int_0^\infty dL e^{-pL} W(u_L, L) = \frac{1}{2} p^{\nu-1} e^{-p' |u_L|}, \tag{92}
\]
which yields the correct expression for the Laplace transform (with respect to time) of the propagator [44]. One can invert the Laplace transform to obtain
\[
W(u_L, L) = \frac{L}{2\nu |u_L|^{1+1/\nu}} L^\nu \left( \frac{L}{|u_L|^{1/\nu}} \right), \quad \nu = \alpha/2. \tag{93}
\]
For \( \alpha = 1 \), it returns the expression in equation (8) and for \( \alpha = 2/3 \) equation (93) takes the simple form
\[
W(u_L, L) = \frac{3^{2/3}}{2L^{1/3}} \text{Ai} \left( \frac{|u_L|}{(3L)^{1/3}} \right), \tag{94}
\]
with \( \text{Ai}(x) \sim (3^{2/3} \Gamma(2/3))^{-1} - x(3^{1/3} \Gamma(1/3))^{-1} \) for \( x \to 0 \) and \( \text{Ai}(x) \sim (2x^{1/2} \pi^{1/2})^{-1} e^{-2x^{1/3}} \) for \( x \to \infty \).

We consider the case of the free CTRW where one integrates over the final position \( u_L \) between \(-\infty\) and \( u_m \). In that case one immediately gets from equation (90) the expression for the joint distribution \( P^\text{free}_L(u_m, x_m) \) as
\[
P^\text{free}_L(u_m, x_m) = \theta(u_m) \theta(L - x_m) \frac{1}{\Gamma[1 - \nu]} \frac{1}{u_m^{1/\nu}} \frac{1}{(L - x_m)^\nu} \nu \left( \frac{x_m}{u_m} \right). \tag{95}
\]
In particular, from equation (95) one obtains the marginal distribution of \( x_m \) as
\[
P^\text{free}_L(x_m) = \frac{1}{L} \nu \nu \left( \frac{x_m}{L} \right), \quad \nu(x) = \frac{\sin \nu \pi}{\pi} \frac{1}{x_1^{1-\nu}(1 - x_m)^\nu}, \tag{96}
\]
where we have used \( \int_0^\infty dy y^{-\nu} L_\nu(y) = 1/\Gamma[1 + \nu] \). Similarly, one has the marginal distribution of the maximum \( u_m \) as
\[
P^\text{free}_L(u_m) = \frac{1}{L^\nu} \nu \nu \left( \frac{u_m}{L^\nu} \right), \quad \nu(x) = \theta(x) \frac{1}{\nu} \nu \left( \frac{x^{1+1/\nu}}{(1-x_m)^\nu} \right). \tag{97}
\]
Of course, following the same lines as previously for Brownian motion, one could also consider various constrained CTRW; this is left for future investigations.
5. Conclusion

In conclusion, we have shown that the RSRG is a quite powerful method for computing the extreme statistics of various physically relevant stochastic processes. By exploiting the solution of the RSRG equations found in [38, 40], we have studied the extreme statistics of the one-dimensional Brownian motion (BM) as well as the Bessel process (BP), i.e. the radius of the $d$-dimensional Brownian motion. For the BM, we have shown that it allows us to recover, in a rather different way, standard results for the Brownian motion and its variants, including the Brownian bridge, excursion, meander as well as for the reflected Brownian motion. For the BP, we have recovered in a simpler way the results of Pitman and Yor for the distribution of the maximum, and obtained also the distribution of its position.

We have then extended this method to study the extreme statistics of the continuous time random walk (CTRW), which we have shown to correspond to a new fixed point of the RSRG transformation. Although we have restricted our analysis to the extreme statistics of the free CTRW, various cases of constrained CTRW can be straightforwardly studied following the analysis presented above. Similarly, the study of the dynamics in a disordered energy landscape generated by a CTRW could be done in principle from the analysis of the new fixed point of the RSRG exhibited here.

Appendix A. RSRG equations and fixed points for bulk and edge bonds

Here we recall the RSRG equation for the probability distributions of barriers and lengths in a symmetric landscape denoted as $P_\Gamma(F,l)$ for bulk bonds and $E_\Gamma(F,\ell)$ for edge bonds. When convenient we use the notation $\zeta = F - \Gamma$. For bulk bonds it reads

$$
(\partial_\ell - \partial_\zeta)P_\Gamma(\zeta,\ell) = \int_{\ell_1 + \ell_2 + \ell_3 = \ell} P_\Gamma(0,\ell_2) \int_0^\zeta d\zeta' P_\Gamma(\zeta',\ell_1) P_\Gamma(\zeta - \zeta',\ell_3),
$$

(A.1)

and for edge bonds one has

$$
\partial_\ell E_\Gamma(F,\ell) = -P_\Gamma(0)E_\Gamma(F,\ell) + \int_{\ell_1 + \ell_2 + \ell_3 = \ell} \int_0^\infty dF_1 \int_0^\infty d\zeta_3 E_\Gamma(F_1,\ell_1)P_\Gamma(0,\ell_2)P_\Gamma(\zeta_3,\ell_3)\delta(F - (F_1 + \zeta_3)).
$$

(A.2)

In terms of Laplace variable $p$ with respect to $l$, one obtains

$$
\partial_\ell E_\Gamma(F,p) = -P_\Gamma(0)E_\Gamma(F,p) + \int_0^F dF_1 E_\Gamma(F_1,p)P_\Gamma(0,p)P_\Gamma(F - F_1,p),
$$

(A.3)

and we recall that

$$
P_\Gamma(\zeta,p) = a_\Gamma(p)e^{-b_\Gamma(p)}
$$

(A.4)

$$
a(p) = \sqrt{\frac{p}{\sinh(\Gamma\sqrt{p})}}, \quad b_\Gamma(p) = \sqrt{p}\coth(\Gamma\sqrt{p}).
$$

(A.5)

It is straightforward to check that

$$
E_\Gamma(F,p) = \Gamma^{-1}e^{-F\sqrt{p}\coth(\Gamma\sqrt{p})}
$$

(A.6)

is a solution of equation (A.3).

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Appendix B. A new fixed point for the RSRG

We now seek a solution of (A.1) with the scaling form

\[ P_\Gamma(\zeta, \ell) = \Gamma^{-1-2/\alpha} Q_\Gamma \left( \eta = \frac{\zeta}{\Gamma}, \lambda = \frac{\ell}{\Gamma^{2/\alpha}} \right), \quad (B.1) \]

which, as discussed in the main text, corresponds to a CTRW energy landscape with index \( \alpha \) (we recall that one recovers the standard BM for \( \alpha = 1 \)). Consider \( Q_\Gamma(\eta, \lambda) \), the Laplace transform of \( Q_\Gamma(\eta, \lambda) \) w.r.t. \( \lambda \) only. It satisfies the flow and fixed point equation

\[ 0 \equiv \Gamma \partial_{\Gamma} Q = Q + (1 + \eta) \partial_{\eta} \hat{Q} - \frac{2}{\alpha} \rho \partial_{\rho} \hat{Q} + Q(0, p) Q(\cdot, p) * \eta Q(\cdot, p). \quad (B.2) \]

One can check that the solution of the fixed point equations take the form

\[ Q(\eta, \tilde{p}) = a_1(\tilde{p}^\alpha) e^{-\eta \tilde{p}^{\alpha-1}(\tilde{p}^\alpha)} \quad (B.3) \]

where the functions \( a_1(p) \) and \( b_1(p) \) are given in (A.5). Re-expressed in the Laplace variable \( p \) associated with \( \ell \) we thus find

\[ P_\Gamma(\zeta, p) = \frac{p^{\alpha/2}}{\sinh(\Gamma p^{\alpha/2})} e^{-\tilde{Q}^{\alpha/2} \coth(\Gamma p^{\alpha/2})}. \quad (B.4) \]

This is a new class of RSRG fixed points which corresponds to broad distributions of bond length:

\[ P_\Gamma(\ell) = LT^{-1}_{\ell} \frac{1}{\cosh(\Gamma p^{\alpha/2})}. \quad (B.5) \]

While there is a typical bond length, \( \ell_{\text{typ}} \approx \Gamma^{2/\alpha} \), the average bond length is infinite as the distribution does not have a first moment. Expanding at small \( p \) one has \( P_\Gamma(p) = 1 - \frac{1}{2} \Gamma^2 p^\alpha + \cdots \) from which we find the tail \( P_\Gamma(\ell) \approx \Gamma^{2/\ell^{1+\alpha}} \) at large \( \ell \).

For the edge bonds one finds similarly that

\[ E_\Gamma(F, p) = \Gamma^{-1/\alpha} e^{-F^{\alpha/2} \coth(\Gamma p^{\alpha/2})} \quad (B.6) \]

is a solution (in Laplace variable) of equation (A.3). Hence we see, that, as claimed in the text, the CTRW fixed points are obtained from the BM ones (i.e. the standard RSRG fixed points) by making the substitution \( p \to p^\alpha \).

There is however a subtlety concerning the finite size measure, i.e. the analogue of (1). As explained in the text, it is still valid provided one interprets \( L \) as the ‘first arrival time’ of the process at \( u_L \). Indeed, if one considers \( Z_L \) as

\[ Z_L = \int_{-\infty}^{\infty} du_L \int_{\ell_1, \ell_2} \int_{F_1, F_2} \tilde{t}_{\Gamma} E_\Gamma(F_1, \ell_1) E_\Gamma(F_2, \ell_2) \delta(L - (\ell_1 + \ell_2)) \delta(u_L - (u_0 + F_1 - F_2)) \]

\[ + \int_{-\infty}^{\infty} du_L \sum_{k=1}^{\infty} \int_{\ell_1, F_1} \tilde{t}_{\Gamma} E_\Gamma(F_1, \ell_1) \prod_{j=2}^{2k+1} \int_{\ell_j, F_j} P(F_j, \ell_j) \]

\[ \times \int_{\ell_{2k+2}, F_{2k+2}} E_\Gamma(F_{2k+2}, \ell_{2k+2}) \delta \left( L - \sum_{i=1}^{2k+2} \ell_i \right) \]

\[ \times \delta \left( u_L - \left( u_0 + \sum_{j=1}^{2k+2} (-1)^{j+1} F_j \right) \right) \quad (B.7) \]
then one has \[38\]
\[\int_0^\infty dL e^{-pL} Z_L = \tilde{I}_1 \frac{E_T(p)^2}{1 - P_T(p)^2} = \frac{1}{p^\alpha}, \tag{B.8}\]
where we have used \(\tilde{I}_1 = \Gamma^{2/\alpha}\) (the typical bond length) and the above forms for the fixed point (B.4) and (B.6). Therefore, \(Z_L \neq 1\) for \(\alpha \neq 1\) and this measure is not normalized (B.7) if \(\alpha \neq 1\). The reason for this is that if one considers a fixed time \(L\), one must then convolute the finite size measure with the waiting time function \(\Psi(\ell)\), which, in the language of the stochastic process corresponding to the CTRW (discussed in the main text), is the probability of no jump in the time interval \([\ell, \infty]\). It is straightforward to see that its Laplace transform behaves as \(\Psi(p) = A p^{\alpha-1}\) and the condition of normalization \(\int_0^L e^{-pL} Z_L(\Psi(p) = 1/p\) yields simply \(A = 1\).

Another way to present our result for the finite size measure is to state that the correct generalization of (1) for models with no first moment in the bond length distribution, and statistical independence of successive bonds, reads

\[
\tilde{Z}_L = \int_{-\infty}^\infty du_L \int_{\ell_1,\ell_2,F_1,F_2} \tilde{I}_1 E_T(F_1, \ell_1) E_T(F_2, \ell_2) \sum_{k=1}^{2k+1} P(F_k, \ell_k) \prod_{j=2}^{2k+1} P(F_j, \ell_j) \Psi \left( \sum_{i=1}^{2k+2} \ell_i - L \right) \tag{B.9}\]

with the ‘waiting time’ function \(\Psi\) discussed above, such that \(\tilde{Z}_L\) is then correctly normalized to unity. This measure is clearly invariant under the RSRG procedure, up to the flow of \(P_T\) and \(E_T\) as described above. This means that the \(\delta\) function constraint on total bond length in the finite size measure is only possible for the BM class. It would be very interesting to study whether a starting landscape, e.g. with a fixed total length and number of bonds, will indeed flow, and in which sense, to this asymptotic form. At this stage our main argument is based on (i) invariance of the form (B.9) under RSRG and (ii) the CTRW interpretation given above. More generally, convergence to finite size measures has not, to our knowledge, been studied and is a fascinating subject. This however is beyond the scope of this paper.

**Appendix C. The RSRG method for the extremum on the edge bond**

Let us now call \(E_T(F, \ell, u, x)\) the joint probability that the edge bond has \(F, \ell\) and a maximum at \(x\) of value \(u\). Then it satisfies

\[
\partial_T E_T(F, \ell, u, x) = -P_T(0) E_T(F, \ell, u, x)
+ \int_{F_1,\ell_1,\ell_2,\ell_3,\zeta_3} E_T(F_1, \ell_1, u_1, x_1) P_T(0, \ell_2) P_T(\zeta_3, \ell_3) \delta(x - (F_1 + \zeta_3))
\times \left[ \theta(u_1 - (\Gamma - F_1)) \delta(u - u_1) \delta(x - x_1) + \theta(\Gamma - F_1 - u_1) \times \delta(u - (\Gamma - F_1)) \delta(x - (l_1 + l_2)) \right]. \tag{C.1}\]
This is equivalent to
\[
\partial_\Gamma E_\Gamma(F, \ell, u, x) = -P_\Gamma(0)E_\Gamma(F, \ell, u, x) + \int_0^F dF_1 \times \int_0^{\ell_1+\ell_2+\ell_3=\ell} E_\Gamma(F_1, \ell_1, u, x)P_\Gamma(0, \ell_2)P_\Gamma(F - F_1, \ell_3) + \theta(\Gamma - u)
\times \int_0^u du_1 \int_0^{\ell_1} d\ell_1 \int_0^{\ell_1} dx_1 E_\Gamma(\Gamma - u, \ell_1, u_1, x_1)
\times P_\Gamma(0, x - \ell_1)P_\Gamma(F + u - \Gamma, \ell - x).
\]

In Laplace variables w.r.t. \(\ell\) and \(x\) we get
\[
\partial_\Gamma E_\Gamma(F, p, u, q) = -P_\Gamma(0)E_\Gamma(F, p, u, q) + \int_0^F dF_1 E_\Gamma(F_1, p, u, q)P_\Gamma(0, p)P_\Gamma(F - F_1, p)
+ \theta(\Gamma - u) \int_0^u du_1 E_\Gamma(\Gamma - u, p + q, u_1, q = 0)P_\Gamma(0, p + q)P_\Gamma(F + u - \Gamma, p).
\]

From the Markov property of the BM the solution of this equation must take the form
\[
E_\Gamma(F, \ell, u, x) = A_\Gamma(F, u, x)B_\Gamma(u + F, \ell - x), \quad (C.3)
E_\Gamma(F, p, u, q) = A_\Gamma(F, u, p + q)B_\Gamma(u + F, p), \quad (C.4)
\]
where \(A_\Gamma\) is the sum over paths starting at 0, 0 ending at \(x, u\) constrained to remain on interval \([-F, u]\) and with no return of more than \(\Gamma\), while the \(B_\Gamma\) are paths starting at \(x, u\) ending at \(\ell, -F\) constrained to remain on the interval \([-F, u]\) with no return of more than \(\Gamma\). For \(\Gamma > u\) the return constraint does not play a role for \(A\); for \(\Gamma > u + F\) it does not play a role for \(B_\Gamma\). Whenever the return constraint does not play a role one has
\[
A_\Gamma(F, u, p) = \frac{\sinh(\sqrt{p}F)}{\sinh(\sqrt{p}(u + F))}, \quad (C.5)
B_\Gamma(u + F, p) = \frac{\sqrt{p}}{\sinh(\sqrt{p}(u + F))}, \quad (C.6)
\]
i.e. independent of \(\Gamma\). For smaller \(\Gamma\) one must solve the full equation above with the initial condition \(E_\Gamma(F = \Gamma, \ell, u, x) = \delta(u)\delta(x)E_\Gamma(F = \Gamma, \ell)\). We will not attempt this here as we are mostly interested in the large \(\Gamma\) limit.

Hence our final result for the solution at large \(\Gamma\) reads
\[
E_\Gamma(F, p, u, q) = \frac{1}{\Gamma} \frac{\sinh(\sqrt{p + q}F)}{\Gamma \sinh(\sqrt{p + q}(u + F)) \sinh(\sqrt{p}(u + F))} \sqrt{p}.
\]

One can check that
\[
\int_0^\infty du E_\Gamma(F, p, u, 0) = \frac{1}{\Gamma} e^{-\sqrt{p}F}, \quad (C.7)
\]
which is the correct result.

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We can now compute \( E_L(F, \ell, u, x) \) by performing a double inverse Laplace transform. One has indeed

\[
LT_{q \to x}^{-1} \left( \frac{\sinh (\sqrt{p + qF})}{\sinh (\sqrt{p + q(u + F)})} \right) = 2\pi \sum_{n=0}^{\infty} (-1)^{n+1} n \frac{\sin((\pi F/(u + F))n)}{(u + F)^2} \times e^{-px - (\pi^2 n^2/(u+F)^2)x}.
\]

Note that this yields the identity (setting \( p = 0 \) and taking the Laplace transform w.r.t. to \( x \) of both sides)

\[
2\pi \sum_{n=0}^{\infty} (-1)^{n+1} n \frac{\sin((\pi F/(u + F))n)}{(u + F)^2} = \frac{\sinh (\sqrt{qF})}{\sinh (\sqrt{q(u + F)})}.
\]

which we also checked with Mathematica.

Next we have

\[
LT_{p \to y}^{-1} \left( \frac{\sqrt{p}/e^{-px}}{\sinh (\sqrt{p}(u + F))} \right) = 2\pi^2 \frac{1}{(u + F)^3} \sum_{m=0}^{\infty} (-1)^{m+1} m^2 e^{-((\pi^2 m^2)/(u+F)^2)(y-x)}.
\]

Note that this yields trivially the identity (setting \( x = 0 \) and taking the Laplace transform w.r.t. to \( y \) of both sides)

\[
2\pi^2 \frac{1}{(u + F)^3} \sum_{m=0}^{\infty} (-1)^{m+1} \frac{m^2}{p + (m^2\pi^2/(u+F)^2)} = \frac{\sqrt{p}}{\sinh (\sqrt{p}(u + F))},
\]

which we also checked with Mathematica. Note that the sum in the left-hand side has to be understood as

\[
2\pi^2 \frac{1}{(u + F)^3} \sum_{m=0}^{\infty} (-1)^{m+1} \frac{m^2}{p + (m^2\pi^2/(u+F)^2)} = \lim_{\alpha \to -1^+} 2\pi^2 \frac{1}{(u + F)^3} \times \sum_{m=0}^{\infty} \alpha^{m+1} \frac{m^2}{p + (m^2\pi^2/(u+F)^2)}.
\]

Finally, combining the both Laplace inversions in equations (C.11) and (C.13) one obtains the formula (17) given in the text.

Appendix D. Details about reflected BM, excursions and meanders

Here we give the details about the computation of the joint distribution \( P_L(u_m, x_m, u, x, u_L) \) of the global minimum \( u \) and maximum \( u_m \) and their positions \( x \) and \( x_m \). We start with the expression given in the text in equation (20):

\[
P_L(u_m, x_m, u, x, u_L) = \theta(x_m - x) \int_u^{u_m} du_2 \int_{x_m}^L dx_2 P_L(u, x, u_m, x_m, u_2, x_2, u_L)
+ \theta(x - x_m) \int_u^0 du_1 \int_0^{x_m} dx_1 P_L(u_1, x_1, u_m, x_m, u, x, u_L).
\]

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Let us first focus on the first term. The integral over the space variable \( x_2 \) can be dealt with by noticing that it can be written, formally, as

\[
\int_{x_m}^{u_x} \int_{x_m}^{L} dx_2 \int_{x_m}^{L} dx_2 P_L(u, x, u_m, x_m, u_2, x_2, u_L) = \int_{u}^{u_L} du_2 \int_{x_m}^{L} dx_2 F(L - x_2) G(x_2 - x_m)
\]

\[
= \int_{u}^{u_L} du_2 \int_{0}^{L-x_m} dy_2 F(L - x_m - y_2) G(y_2),
\]

(D.2)

where the functions \( F, G \) can be read straightforwardly from equation (19). The integral over \( y_2 \) can be performed by taking the Laplace transform with respect to \( \tilde{L} = L - x_m \). This yields

\[
\int_{0}^{L} dy \, e^{-pL} \int_{u}^{u_L} du_2 \int_{0}^{L} dy_2 F(\tilde{L} - y_2) G(y_2)
\]

\[
= \sum_{n_1, n_1} 4 \pi^3 n_1 m_1^2 (1 - n_1 + m_1) \sin((\pi u_m/(u_m - u)) n_1) \frac{1}{(u_m - u)^5} \ e^{-(\pi^2/(u_m-u)^2)(n_1^2 x + m_1^2 (x_m - x))}
\]

\[
\times \int_{u}^{u_L} du_2 \int_{0}^{L} dy_2 \frac{(4 \pi^3)(1 - n_2 + m_2) m_2 \sin((\pi (u_m - u_L)/(u_m - u_2)) n_2)}{(u_m - u_2)^5}
\]

\[
\times \frac{1}{p + (m_2^2 \pi^2/(u_m - u_2)^2)}
\]

(D.3)

Now one can use the identities in equations (C.10) and (C.12) to perform the sums over \( n_1, m_1 \) to obtain

\[
\int_{0}^{L} dy \, e^{-pL} \int_{u}^{u_L} du_2 \int_{0}^{L} dy_2 F(\tilde{L} - y_2) G(y_2)
\]

\[
= \sum_{n_1, n_1} 4 \pi^3 n_1 m_1^2 (1 - n_1 + m_1) \sin((\pi u_m/(u_m - u)) n_1) \frac{1}{(u_m - u)^5} \ e^{-(\pi^2/(u_m-u)^2)(n_1^2 x + m_1^2 (x_m - x))}
\]

\[
\times \frac{\sinh((\sqrt{p} (u_m - u_L)))}{\sinh((\sqrt{p} (u_m - u)))}.
\]

(D.4)

One can then invert the Laplace transform (see equation (C.11)) to obtain

\[
\int_{u}^{u_L} du_2 \int_{x_m}^{L} dx_2 P_L(u, x, u_m, x_m, u_2, x_2, u_L)
\]

\[
= \sum_{n_1, n_1} 4 \pi^3 n_1 m_1^2 (1 - n_1 + m_1) \sin((\pi u_m/(u_m - u)) n_1) \frac{1}{(u_m - u)^5} \ e^{-(\pi^2/(u_m-u)^2)(n_1^2 x + m_1^2 (x_m - x))}
\]

\[
\times \frac{2 \pi (1 - n_2 + m_2) \sin((\pi (u_L - u)/(u_m - u)) n_2)}{(u_m - u)^2} \ e^{-(\pi^2/(u_m-u)^2)(L - x_m)}.
\]

(D.6)
The second term in equation (20) can be computed in a similar way to get finally the expression given in equation (21).

Appendix E. The marginal distribution of \( x_m \) for the Bessel bridge: the link with the result of Pitman and Yor

We start from the joint distribution \( P_{L}^{\text{Bessel bridge}}(u_m, x_m) \) given in the text in equation (78). We recall it here:

\[
P_{L}^{\text{Bessel bridge}}(u_m, x_m) = \frac{8 L^{\nu+1}}{\Gamma(1+\nu)u_m^{\nu+2\nu}} \sum_{n,m} \frac{J_{\nu+1}^{\nu+1} J_{\nu,m}^{\nu+1}}{J_{\nu+1}^{\nu} J_{\nu,m}^{\nu}} e^{-(j_{\nu,n} x_m)/u_m^2} \times e^{-(j_{\nu,n} (L-x_m))/u_m^2}.
\]

The marginal distribution of the maximum \( u_m \) is obtained after integration over \( x_m \). It reads

\[
P_{L}^{\text{Bessel bridge}}(u_m) = \frac{1}{\sqrt{L}} \tilde{P}_{L}^{\text{Bessel bridge}} \left( \frac{u_m}{\sqrt{L}} \right)
\]

while Pitman and Yor found in [31] (note that the value of the diffusion constant used in their paper is smaller than ours by a factor of 2)

\[
\tilde{P}_{L}^{\text{Bessel bridge}}(x) = \frac{d}{dx} \left( \frac{4}{\Gamma(\nu+1)x^{2\nu+2}} \sum_{n=1}^{\infty} \frac{j_{\nu,n}^2}{J_{\nu+1}^{\nu}(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2}{x^2} \right) \right).
\]

We want to show that these two expressions in equations (E.2) and (E.3) are identical and therefore we want to show the identity

\[
\frac{2}{x^{2\nu+5}} \sum_{n=1}^{\infty} \frac{j_{\nu,n}^{2\nu+2}}{J_{\nu+1}^{\nu}(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2}{x^2} \right) = 2 \sum_{n=1}^{\infty} \frac{j_{\nu,n}^{2\nu}}{J_{\nu+1}^{\nu}(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2}{x^2} \right)\]

Now in the double sum over \( n, m \) in the left-hand side of equation (E.4) we separate out the \( m = n \) term since it exactly cancels the first term in the right-hand side of equation (E.4). Finally using a formula obtained by Pitman and Yor in [31] (see their formula (125); notice however that there is a misprint in their formula (124)) one obtains

\[
2 \sum_{n \neq m=1}^{\infty} \frac{j_{\nu,n}^{\nu+1} j_{\nu,m}^{\nu+1}}{J_{\nu+1}^{\nu}(j_{\nu,n}) J_{\nu+1}^{\nu}(j_{\nu,m})} \exp \left( -\frac{j_{\nu,n}^2}{x^2} \right) = -2(\nu+1) \sum_{n=1}^{\infty} \frac{j_{\nu,n}^{2\nu}}{J_{\nu+1}^{\nu}(j_{\nu,n})} \exp \left( -\frac{j_{\nu,n}^2}{x^2} \right),
\]

which shows finally the identity in equation (E.4).
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