LIN-WANG TYPE FORMULA FOR HAEFLIGER INVARIANT

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Abstract. In this paper we study Haefliger invariant for long embeddings $\mathbb{R}^{4k-1} \hookrightarrow \mathbb{R}^6$ in terms of the self-intersections of their generic projections to $\mathbb{R}^{4k-1}$. We define the notion of “crossing changes” of the embeddings at the self-intersections and describe the change of the isotopy classes under crossing changes using the linking numbers of the double point sets in $\mathbb{R}^{4k-1}$. This formula is a higher dimensional analogue to that of V. Arnold’s plane curve invariant in Lin-Wang theory, but in general our invariant does not coincide with order one invariant of T. Ekholm.

1. Introduction

A long j-embedding in $\mathbb{R}^n$ is an embedding $\mathbb{R}^j \hookrightarrow \mathbb{R}^n$ which is the standard inclusion outside a compact set. We denote by $\mathcal{K}_{n,j}$ the space of long j-embeddings in $\mathbb{R}^n$. Similarly we denote the space of long immersions $\mathbb{R}^j \hookrightarrow \mathbb{R}^n$ by $I_{n,j}$.

In the previous work [20] the author constructed, for some pairs $(n, j)$, a cochain map $I : D^* \to \Omega_{\partial R}(\mathcal{K}_{n,j})$ from a complex $D^*$ of graphs to de Rham complex of $\mathcal{K}_{n,j}$ via configuration space integrals associated with graphs. For other interesting pairs, in particular for $(n, j) = (6k, 4k - 1)$, the map $I$ has not yet been proved to be a cochain map, and it is not clear whether graph cocycles in $D^*$ yield closed forms of $\mathcal{K}_{6k,4k-1}$. But in [20] we found a cocycle $H \in D^*$ and a differential form $c \in \Omega^6_{\partial R}(\mathcal{K}_{6k,4k-1})$ such that $\mathcal{H} := I(H) + c \in \Omega^6_{\partial R}(\mathcal{K}_{6k,4k-1})$ is closed and is equal (up to sign) to Haefliger invariant which gives an isomorphism $\pi_0(\mathcal{K}_{6k,4k-1}) \cong \mathbb{Z}$. This integral expression $\mathcal{H}$ looks very similar to that for the finite type invariant $v_2$ of order two for classical knots [2] [10].

In this paper, based on the integral expression $\mathcal{H}$, we show that Haefliger invariant indeed behaves similarly to $v_2$. To do this, we study $\mathcal{H}(f), f \in \mathcal{K}_{6k,4k-1}$, in terms of generic projections $p \circ f \in \mathcal{I}_{6k-1,4k-1}$, where $p : \mathbb{R}^{6k} \to \mathbb{R}^{4k-1}$ denotes the projection forgetting the last 6k-th coordinate. A generic immersion $g \in \mathcal{I}_{6k-1,4k-1}$ has only (possibly empty) transverse two-fold self-intersection $A = A_1 \sqcup \cdots \sqcup A_m \subset \mathbb{R}^{4k-1}$, where each $A_i$ is a connected, closed oriented $(2k-1)$-dimensional manifold. If $g = p \circ f$ for some $f \in \mathcal{K}_{6k,4k-1}$, then $g : g^{-1}(A_i) \to A_i$ is a trivial double covering and we denote the inverse image by $g^{-1}(A_i) = L^i_{1} \sqcup L^i_{2} \subset \mathbb{R}^{4k-1}$. We define the notion of the crossing changes at the crossings $A_i$, and denote by $f_3 \in \mathcal{K}_{6k,4k-1}$ the embedding obtained from $f$ by crossing changes at $A_i$, $i \in S \subset \{1, \ldots, m\}$. In Theorem 2.5 we show that the difference $\mathcal{H}(f) - \mathcal{H}(f_3)$ can be described using the linking numbers $lk(L^i_j, L^j_{i'})$, $e, e' = 0, 1, i, j = 1, \ldots, m$. This formula is a higher dimensional analogue to those for the order two invariant $v_2$ [11] (4.3), [14] (3.2), [15] (2.6), [18]. As a corollary we see that Haefliger invariant is of order two (Theorem 2.3); this seems reasonable from the view of results in [13] [25] (see Remark 2.3). In this sense Haefliger invariant can be seen as a higher dimensional analogue to $v_2$. It seems that, in some aspects, geometric meaning of Haefliger invariant are understood better (see for example [2] [3] [4] [22] [23]) than those of finite type invariants for classical knots, and
more detailed studies on Haefliger invariant (and other invariants in higher dimensions which can be described by some integrals) might shed light on the geometric meaning of finite type invariants, perhaps in the context of characteristic classes. As another consequence we reprove a result of Murali-Olha [17] concerning the “unknotting numbers” of embeddings $\mathbb{R}^3 \hookrightarrow \mathbb{R}^6$.

Similarly to $v_2$, the invariant $\mathcal{H}$ is essentially the sum of two integrals $I(X), I(Y)$ over some configuration spaces, which correspond respectively to the graphs $X$ and $Y$ (see Figure 3.3). The linking numbers of $L_i^*$'s in Theorem 2.5 arise from $I(X)$ and are thought of as a higher dimensional analogue to the Gauss diagram term in the formulas in [11, 14, 15, 18].

In [11] $I(Y)$ for $v_2$ was proved, by studying $v_2 - I(X)$, to be a linear combination of Arnold invariants for generic plane curves [11] and the number of crossings of knot diagrams. One might expect that Ekholm’s order one invariants [6, 7], which look analogous to Arnold invariants, would appear in our higher dimensional case. We see in Theorem 2.10 that $E_k(\mathcal{H})$ might be expected that Ekholm’s order one invariants [6, 7], which look analogous to Arnold invariants, would appear in our higher dimensional case. We see in Theorem 2.10 that $I(Y)$ is essentially an invariant of generic and liftable immersions $\mathbb{R}^{4k-1} \hookrightarrow \mathbb{R}^{6k-1}$, but in general it is not of order one. The reason is that in a sense any link in $\mathbb{R}^3$ can be realized as the double point set of an immersion $\mathbb{R}^3 \hookrightarrow \mathbb{R}^5$ [16].

This paper is organized as follows. In §2 we fix the notations and state the results. The main results are Theorems 2.5, 2.8 (proved in §3) and 2.10 (proved in §4). In §2 we show an explicit computation using Theorem 2.5. We review our construction of $\mathcal{H}$ in §3.

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2. Notations and results

The self-intersection $A$ of an immersion $g : M \hookrightarrow N$ is $A := \{q \in N \mid g^{-1}(q) \geq 2\}$, where $|S|$ is the cardinality of the set $S$. If $g \in I_{6k-1,4k-1}$ is generic, then $|g^{-1}(q)| = 2$ for any $q \in N$. Moreover $A$ is a $(2k-1)$-dimensional closed submanifold and $g : g^{-1}(A) \to A$ is a double covering. We call $g^{-1}(A) \subset \mathbb{R}^{4k-1}$ the double point set. Suppose that $g$ is liftable to $f \in \mathcal{K}_{6k,4k-1}$, namely $g = p \circ f$ where in general $p : \mathbb{R}^n \to \mathbb{R}^m$ is given by $p(x_1, \ldots, x_n) = (x_1, \ldots, x_{m-1})$, then $g : g^{-1}(A) \to A$ is a trivial double covering. Let $A_i \subset \mathbb{R}^{4k-1}$ be path components of $A$, and we set $g^{-1}(A_i) = L_i'^1 \sqcup L_i'^2$. Each $L_i'^1 \subset \mathbb{R}^{4k-1}$ is a $(2k-1)$-dimensional connected closed submanifold. By convention $f(L_i'^1) \subset \mathbb{R}^{6k}$ sits in the “upper” position than $f(L_i'^2)$, namely if $x' = (x_1, \ldots, x_{6k-1}, x_{6k}^1) \in f(L_i'^1)$, $\epsilon = 0, 1$ (then $p(x^\epsilon) = p(x^1)$), then $x_{6k}^\epsilon < x_{6k}^1$.

Remark 2.1. Any $f \in \mathcal{K}_{6k,4k-1}$ can be moved by an isotopy so that $p \circ f$ is a generic immersion; indeed such an isotopy for a generator $S$ of $\pi_0(\mathcal{K}_{6k,4k-1})$ exists (see [4]).

Remark 2.2. Not all $g \in I_{6k-1,4k-1}$ are regularly homotopic to any liftable immersion, in contrast to the case of plane curves. As shown in [24] §3, the Smale invariants of $g \in I_{5,3}$ regularly homotopic to liftable ones form the subgroup $2\mathbb{Z} \subset \mathbb{Z} \cong \pi_0(I_{5,3})$.

Lemma 2.3 ([6, Lemma 5.1.3], [17, Proposition 3.3]). The submanifolds $A \subset \mathbb{R}^{6k}$ and $L_i'^1 \subset \mathbb{R}^{4k-1}$ admit natural orientations.

Proof. Given a basis $\vec{u} = (u_1, \ldots, u_{2k-1})$ of $T_x A_i$ ($x \in A_i$), we can choose tangent frames $\vec{v} = (v_2, \ldots, v_{4k-1})$ and $\vec{w} = (w_2, \ldots, w_{4k-1})$ of the two sheets of $p \circ f$ meeting at $x \in A_i$ so that $(\vec{u}, \vec{v})$ and $(\vec{u}, \vec{w})$ are the positive bases of these two sheets. We say $\vec{u}$ represents the positive orientation of $A_i$ if $(\vec{u}, \vec{v}, \vec{w})$ is a positive basis of $\mathbb{R}^{6k-1}$. Since the codimension of $p \circ f$ is even, this definition is independent of the order of two sheets. We orient $L_i'^1$, $\epsilon = 0, 1$, so that $p \circ f : L_i'^1 \to A_i$ preserves the orientation. \qed
any nontrivial embedding following result of M. Murai-K. Ohba \[17\], which states that the "unknotting number" of $S$ for $\iota$ where $\iota$ stands for the dissertation of $L$ in $\mathbb{R}^{4k-1}$ such that $N(L) \cap N(L') = \emptyset$ if $(i, \epsilon) \neq (j, \epsilon')$.

If $f \in \mathcal{K}_{6k-1}$ is such that $p \circ f \in \mathcal{I}_{6k-1}$ is generic, then we can transform $f$ to be almost planar without changing $p \circ f$, by an isotopy in the $x_{6k}$-direction.

Suppose that $f$ is almost planar and that the self-intersection of $p \circ f$ has $m$ components. For $S \subset \{1, \ldots, m\}$, let $f_S \in \mathcal{K}_{6k-1}$ be defined by

$$f_S(x) := \begin{cases} \iota(f(x)) & x \in N(L_i), \ i \in S, \\ f(x) & \text{otherwise}, \end{cases}$$

where $\iota : \mathbb{R}^{6k} \to \mathbb{R}^{6k}$ is given by $\iota(x_1, \ldots, x_{6k}) = (x_1, \ldots, x_{6k-1}, -x_{6k})$. We say $f_S$ is an embedding obtained from $f$ by crossing changes at the crossings $\{A_i\}_{i \in S}$. Notice that $p \circ f = p \circ f_S$ and if we write $(p \circ f_S)^{-1}(A_i) = L^i(S) \sqcup L^i(S) \subset \mathbb{R}^{4k-1}$ then

$$L^i(S) = \begin{cases} L^i_{\epsilon'}^1 & i \in S, \\ L^i_{\epsilon} & i \notin S, \end{cases}$$

here $\epsilon$ is understood to be in $\mathbb{Z}/2 = \{0, 1\}$ and $1 + 1 = 0$.

Let $\mathcal{H} : \mathcal{K}_{6k-1} \to \mathbb{R}$ be Haefliger invariant (see \[3\] for our construction). Our main theorem describes the difference $\mathcal{H}(f) - \mathcal{H}(f_S)$ using the linking numbers of $L^i$'s.

**Theorem 2.5.** Let $f \in \mathcal{K}_{6k-1}$ be such that $p \circ f$ is a generic immersion and has the nonempty self-intersection $A = A_1 \sqcup \cdots \sqcup A_m$. Then for any $S \subset \{1, \ldots, m\}$,

$$(2.1) \quad \mathcal{H}(f) - \mathcal{H}(f_S) = \frac{1}{4} \left( \sum_{(i, \epsilon) \neq (j, \epsilon')} (-1)^{\epsilon \epsilon'} \text{lk}(L^i_{\epsilon}, L^j_{\epsilon'}) - \sum_{(i, \epsilon) \neq (j, \epsilon')} (-1)^{\epsilon \epsilon'} \text{lk}(L^i_{\epsilon}(S), L^j_{\epsilon'}(S)) \right)$$

$$(2.2) \quad = \frac{1}{2} \sum_{\substack{(i, \epsilon) \neq (j, \epsilon') \text{ such that exactly one of } i, j \in S \text{ and } \epsilon, \epsilon' \neq 0, 1 \text{ we write } (i, \epsilon) < (j, \epsilon') \text{ if } i < j \text{ or if } i = j, \epsilon < \epsilon' = 1.}} (-1)^{\epsilon \epsilon'} \text{lk}(L^i_{\epsilon}, L^j_{\epsilon'}).$$

Where \text{lk} stands for the linking number, and for $i, j = 1, \ldots, m$ and $\epsilon, \epsilon'$, $0, 1$ we write $(i, \epsilon) < (j, \epsilon')$ if $i < j$ or if $i = j, \epsilon < \epsilon' = 1$.

(2.2) follows from (2.1) if $i, j \notin S$, then $\text{lk}(L^i_{\epsilon}(S), L^j_{\epsilon'}(S))$ is contained in both sums in (2.1) with the same sign $(-1)^{\epsilon \epsilon'}$ and cancels out. If $i, j \in S$, then $\text{lk}(L^i_{\epsilon}(S), L^j_{\epsilon'}(S))$ is contained in both sums with the same sign $(-1)^{\epsilon \epsilon'}$ and cancels out. If $i \notin S$ and $j \notin S$, then $\text{lk}(L^i_{\epsilon}(S), L^j_{\epsilon'}(S))$ is contained in the first sum with sign $(-1)^{\epsilon \epsilon'}$ while $\text{lk}(L^i_{\epsilon}(S), L^j_{\epsilon'}(S))$ is contained in the second sum with sign $(-1)^{\epsilon \epsilon'}$.

Theorem 2.5 together with a result of Ogasa \[16\] gives an alternative proof for the following result of M. Murai-K. Ohba \[17\], which states that the “unknotting number” of any nontrivial embedding $f \in \mathcal{K}_{6,3}$ is one (see \[6,4\] for the proof).
Corollary 2.6 ([17]). Any nontrivial $f \in \mathcal{K}_{6,3}$ can be unknotted by a crossing change at a single crossing. Namely $f$ is isotopic to some $f'$ with $f'[1]$ isotopic to the trivial inclusion.

Below we introduce the notion of finite type invariants for $\mathcal{K}_{6,4k-1}$. As a consequence of Theorem 2.5 we prove in §5 that $\mathcal{H}$ is an invariant of order two.

Definition 2.7. Let $u : \mathcal{K}_{6,4k-1} \to \mathbb{R}$ be a function and $A = \{A_i\}_{i \leq n}$ a (sub)set of crossings of $p \circ f$, where $f \in \mathcal{K}_{6,4k-1}$ is almost planar. Define

$$V_{s+1}(u)(f) := \sum_{s \in [1 \ldots r+1]} (-1)^s u(f_s).$$

An isotopy invariant $u$ is said to be of order $s$ if $V_{s+1}(u) = 0$.

Theorem 2.8. Haefliger invariant is of order two.

Remark 2.9. Volić proved in [25] that any isotopy invariant $v : \mathcal{K}_{6,1} \to \mathbb{R}$ of order $s$ factors through the $2s$-th stage of the Taylor tower [2] for $\mathcal{K}_{6,1}$, namely $v$ can be written as a composite $\mathcal{K}_{6,1} \to \mathcal{T}_s \mathcal{K}_{6,1} \to \mathbb{R}$. This suggests that one might be allowed to say that an invariant $\mathcal{K}_{6s} \to \mathbb{R}$ is of finite order if it factors through a stage of the Taylor tower. On the other hand, Eliashberg [13] proved that Haefliger invariant factors through $\mathcal{T}_1 \mathcal{K}_{6,4k-1}$. Theorem 2.8 seems reasonable from this viewpoint.

Because the linking numbers are constant unless the isotopy class of the self-intersection of $p \circ f$ changes, the difference $E := \mathcal{H} - \sum_{i} (\text{linking numbers})$ gives rise to an invariant for generic liftable immersions $\mathbb{R}^{4k-1} \hookrightarrow \mathbb{R}^{6k-1}$. This would be a higher dimensional analogue to the Lin-Wang invariant $\mathcal{A}$ [11] Definition 5.4] for generic plane curves, which is in fact equal to a linear combination of Arnold's plane curve invariants $J^s$ and $S^t$ [11]. On the other hand, Eliashberg [6] §6.1, [7] §4] defined invariants for generic immersions $\mathcal{M}^{m-1} \hookrightarrow \mathcal{M}^{m+1-1}$ which behave similarly to Arnold invariants (for us $m = 2, n = 2k$). One may then expect that the invariant $E$ might be a linear combination of Ekholm invariants, but we see that it is not the case.

Theorem 2.10. Let $g \in I_{6,4k-1}$ be a liftable generic immersion, namely $g = p \circ f$ for some $f \in \mathcal{K}_{6,4k-1}$ and define

$$E(g) := \mathcal{H}(f) - \frac{1}{4} \sum_{(x,y,z) \in \mathcal{L}^+ \mathcal{L}^-} (-1)^{x+y} \text{lk}(L^+_x, L^-_y).$$

Then $E$ is independent of the choice of $f$ and is invariant under generic regular homotopies. $E$ varies at the strata of non-generic immersions as described in Lemmas 6.4, 6.5, and 6.7.

In the case $k = 1$, the invariant $E$ is not of order one in the sense of Eliashberg [6, 7].

3. Example

Using Theorem 2.5 we show $\mathcal{H}(S) = \pm 1$ for Haefliger’s generator $S$ of $\pi_0(\mathcal{K}_{6,4k-1}) \cong \mathbb{Z}$ [9]. Fix $\alpha, \beta > 0$ so that $2\beta < \alpha$. Consider the Borromean ring $X \cup Y \cup Z \subset \mathbb{R}^{6k}$, where

$$X := \partial((x, y, z) \in (\mathbb{R}^{2k})^3 \mid |y| \leq \alpha, |z| \leq \beta) \approx S^{4k-1},$$

$$Y := \partial((x, 0, z) \in (\mathbb{R}^{2k})^3 \mid |z| \leq \alpha, |x| \leq \beta) \approx S^{4k-1},$$

$$Z := \partial((x, y, 0) \in (\mathbb{R}^{2k})^3 \mid |x| \leq \alpha, |y| \leq \beta) \approx S^{4k-1}$$

(see Figure 3.1) and smooth their corners to get smooth $(4k-1)$-spheres (denoted by $X, Y, Z$ again). $S$ is defined as the connected-sum $S := X^\# Y^\# Z^\# f_0$, where $f_0 : \mathbb{R}^{4k-1} \subset \mathbb{R}^{6k}$ is (isotopic to) the standard inclusion.

Let $n = (1, \ldots, 1) \in \mathbb{R}^{6k}$ and consider the projection $p : \mathbb{R}^{6k} \to (\mathbb{R}^n)^4$, instead of $\mathbb{R}^{6k} \to \mathbb{R}^{6k-1} \times \{0\}$. Then $p \circ S$ is generic, as seen in Figure 3.1. To detect $p(X) \cap p(Y)$, find $(0, y, z) \in X$ and $t \in \mathbb{R}$ satisfying $(0, 0, z) + t n \in Y$. In fact $p(X) \cap p(Y) = A_1 \cup A_3$ has
If we take the connected-sum in the construction of $D$ points on two components, and the double point sets satisfying $[4, 5, 26]$, and a linear map $L$ is the standard inclusion of $\mathbb{R}^{d-1}$. For example any $(D_i, 1, 2)$ are given as follows; put $\beta' := (1, \ldots, 1) \in \mathbb{R}^k$ and $\beta := \langle \beta, \sqrt{2}k \rangle$, then

\[
\begin{align*}
L_i^0 &= \{(\beta', 0, 0, z) \in X \mid |z + \beta' n_i| = \beta), \\
L_i^1 &= \{(0, \beta', n, z) \in X \mid |z| = \beta), \\
L_i^2 &= \{(0, -\beta', z) \in Y \mid |z - \beta' n_i| = \beta).
\end{align*}
\]

Then for any $\beta > 0$, by (2.2) we have that $|y_i| \leq 2\beta < \alpha$ and $L_0$ does not touch the corner. All $L_i'$ are $S^{d-1}$ and they form six disjoint Hopf links

\[
L_1' \sqcup L_2' \sqcup L_3' \subset X, \quad L_1' \sqcup L_2' \sqcup L_3' \subset Y, \quad L_0' \sqcup L_1' \sqcup L_2' \subset Z,
\]

all of whose linking numbers are by symmetry equal to each other. Since we can unknot $S$ by the crossing change at $A_i$, by (2.2)

\[
\mathcal{H}(S) = \frac{1}{2}(((-1)^{i+1}l(k(L_i^1, L_i^0)) + (-1)^{i+1}l(k(L_i^0, L_i^1))) = \pm 1.
\]

4. A review of an integral expression of Haefliger invariant

In [20] the author defined a cochain complex $D^*$ of graphs, which generalize those in [4, 5, 26], and a linear map $I : D^* \rightarrow \Omega_{dB}(\mathcal{K}_m)$. From the graph cocycle $H = X/2 + Y/6 \in D^*$ (Figure 4.1) we obtain Haefliger invariant as follows.

Let $\text{Conf}^m(M) := M^m \setminus \bigcup_{1 \leq i < j \leq m} \{x_i = x_j\}$ denote the configuration space of ordered $m$ points on $M$. Consider two fiber bundles over $\mathcal{K}_{6,4k-1}$ with as fibers configuration spaces
corresponding to (the vertices of) $X$ and $Y$ respectively; to $X$ corresponds

$$C_{4,0} := \mathcal{K}_{6k,4k-1} \times \text{Conf}_4^0(\mathbb{R}^{3k-1})$$

and to $Y$ corresponds

$$C_{3,1} := \{(f; (x_1, x_2, x_3), x_4) \in \mathcal{K}_{6k,4k-1} \times \text{Conf}_3^0(\mathbb{R}^{3k-1}) \times \mathbb{R}^{6k} \mid f(x_i) \neq x_4, \ i = 1, 2, 3\}.$$ 

Define for $1 \leq i \neq j \leq 4$ the generalized Gauss maps

$$\varphi_{ij}^0 : C_{4,0}^0 \to S^{6k-1}, \ (f; (x_1, \ldots, x_4)) \mapsto \frac{f(x_j) - f(x_i)}{|f(x_j) - f(x_i)|},$$

$$\varphi_{ij}^1 : C_{4,0}^1 \to S^{4k-2}, \ (f; (x_1, \ldots, x_4)) \mapsto \frac{x_j - x_i}{|x_j - x_i|},$$

$$\psi_{ij}^0 : C_{3,1}^0 \to S^{6k-1}, \ (f; (x_1, x_2, x_3), x_4) \mapsto \frac{x_4 - f(x_i)}{|x_4 - f(x_i)|}$$

and consider the maps which correspond to (the edges of) $X$ and $Y$:

$$\varphi_X := \varphi_{12}^0 \times \varphi_{34}^0 \times \varphi_{23}^1 : C_{4,0}^0 \to S^{6k-1} \times S^{6k-1} \times S^{4k-2},$$

$$\varphi_Y := \psi_{31}^0 \times \psi_{23}^0 \times \psi_{13}^1 : C_{3,1}^0 \to (S^{6k-1})^3.$$ 

Let $\text{vol}_{k-1}^\Box \in \Omega^{-1}_{\text{DR}}(S^{n-1})$ $(n = 6k, 4k - 1)$ be a unit volume form of $S^{n-1}$ which is (anti-)invariant under the action of $O(n)$ fixing the set of poles $\{ \pm e_n \} \subset S^{n-1}$, where $e_n := (0, \ldots, 0, 1) \in \mathbb{R}^n$. In particular $\text{vol}_{k-1}^\Box$ is (anti-)invariant under the antipodal map. Define $\omega_X \in \Omega^{-1}_{\text{DR}}(C_{4,0})$ and $\omega_Y \in \Omega^{-1}_{\text{DR}}(C_{3,1})$ by

$$\omega_X := \varphi_X^0 (\text{vol}_{3k-1}^\Box \times \text{vol}_{4k-1}^\Box), \quad \omega_Y := \varphi_Y^0 (\text{vol}_{3k-1}^\Box),$$

where $\text{vol}_{k-1}^\Box \times \text{vol}_{k-1}^\Box$ is the product of volume forms pulled back on $S^{n-1} \times S^{n-1}$ by the projections. Integrating $\omega_X$ and $\omega_Y$ along the fibers of the natural projections

$$\pi_X : C_{4,0}^0 \to \mathcal{K}_{6k,4k-1}, \quad \pi_Y : C_{3,1}^0 \to \mathcal{K}_{6k,4k-1}$$

(which fibers are subspaces of configuration spaces), we obtain 0-forms

$$I(X) := \pi_X^* \omega_X, \quad I(Y) := \pi_Y^* \omega_Y \in \Omega^0_{\text{DR}}(\mathcal{K}_{6k,4k-1}).$$

**Remark 4.1.** The above integrals converge since we may replace $C_{4,0}^0$ and $C_{3,1}^0$ with their Fulton-MacPherson compactifications denoted by $C_{4,0}$ and $C_{3,1}$ (see [41]) over which the generalized Gauss maps are smoothly extended. The boundary faces of the compactifications are stratified according to the “complexity of collisions of points”. The strata in which exactly two points collide are called principal.

In a similar way as above, we obtain a differential forms $I(\Gamma)$ from any graph $\Gamma$; consider a differential form $\omega_{\Gamma}$ associated with $\Gamma$ and integrate it over a configuration space. This gives a linear map $I : D^r \to \Omega^0_{\text{DR}}(\mathcal{K}_n)$.

A condition under which $I$ is a cochain map is given in [20] Theorem 1.2]. Unfortunately $(n, j) = (6k, 4k - 1)$ does not satisfy the condition, but $H := X/2 + Y/6 \in D^r$ is in fact a cocycle. By the generalized Stokes’ theorem, $dI(H)$ is a linear combination of integrals along the codimension one boundary faces of the fibers of $\pi_X$ and $\pi_Y$, which are subspaces of compactified configuration spaces (see Remark 4.1). The author proved in [20] that, if $\text{vol}_{k-1}^\Box$ is (anti-)invariant, almost all such integrals along the boundary faces cancel out or vanish; the integrals along the principal faces cancel because $H$ is a cocycle, and the large part of the the proof of the vanishing of the integrals along other faces follows the arguments in [43] [44] [26]. But at present it is not known whether the integral along the “anomalous boundary face” $\Sigma_{3,1} \subset \partial C_{3,1}$ (where all the four points $f(x_1), f(x_2), f(x_3)$ and $x_4$ collapse to a single point) vanishes or not. In [20] we added a correction term $c$ (defined below) to $I(H)$ to kill the anomalous contribution and to get a closed form $\mathcal{H} := I(H) + c$. 
The correction term \( c \) is defined as follows. The interior \( \text{Int} \Sigma_{3,1} \) of \( \Sigma_{3,1} \) can be described by the following pullback square:

\[
\begin{array}{ccc}
\text{Int} \Sigma_{3,1} & \rightarrow & B \\
\downarrow & & \downarrow \rho \\
\mathbb{R}^{4k-1} \times \mathcal{K}_{6k,4k-1} & \rightarrow & \text{Inj}_{6k,4k-1}
\end{array}
\]

Let us explain the spaces and maps in the above diagram. \( \text{Inj}_{6k,4k-1} \) is the space of linear, injective maps \( \mathbb{R}^{4k-1} \hookrightarrow \mathbb{R}^{6k} \). The space \( B \) is defined as

\[
B := \{(t; (x_1, x_2, x_3); x_4) \in \text{Inj}_{6k,4k-1} \times \text{Conf}\left(\mathbb{R}^{4k-1}\right) \times \mathbb{R}^{6k} \mid t(x_i) \neq x_4, \; 1 \leq i \leq 3\}/\mathbb{R}^1 \rtimes \mathbb{R}^{4k-1}
\]

where \( \mathbb{R}^1 \rtimes \mathbb{R}^{4k-1} \) acts as the positive scalar multiplications and translations along \( t(\mathbb{R}^{4k-1}) \).

The map \( \rho \) is the natural projection, and \( D \) is the differential

\[D(x; f) := (df_x : T_x \mathbb{R}^{4k-1} = \mathbb{R}^{4k-1} \hookrightarrow \mathbb{R}^{6k} = T_{f(x)} \mathbb{R}^{6k}) \]

For \( i = 1, 2, 3 \), the map

\[\psi_i : B \rightarrow S^{6k-1}, \quad \psi_i = (t; (x_1, x_2, x_3); x_4) \rightarrow x_4 - t(x_i) \]

is well-defined. Put \( \psi := \psi_1 \times \psi_2 \times \psi_3 : B \rightarrow (S^{6k-1})^3 \) and consider

\[\omega := \psi^* \text{vol}_{S^{6k-1}} \in \Omega^3_{DR}(B)\]

We can see that \( \rho_* \omega \in \Omega^3_{DR}(\text{Inj}_{6k,4k-1}) \) is closed \([20] \text{ Lemma 5.22}\). Since \( \text{Inj}_{6k,4k-1} \) is homotopy equivalent to the Stiefel manifold \( V_{6k,4k-1} \) of \((4k - 1)\)-frames in \( \mathbb{R}^{6k} \) and hence \( H^3_{DR}(\text{Inj}_{6k,4k-1}) = 0 \), there exists \( \mu \in \Omega^3_{DR}(\text{Inj}_{6k,4k-1}) \) such that

\[\rho_* \omega = d\mu\]

The correction term \( c : \mathcal{K}_{6k,4k-1} \rightarrow \mathbb{R} \) is defined by

\[c(f) := \frac{1}{6} \int_{\mathbb{R}^{4k-1}} (df)^3 \mu \in \mathbb{R}\]

where \( df : \mathbb{R}^{4k-1} \rightarrow \text{Inj}_{6k,4k-1} \) is defined by \( x \mapsto df_x \).

Let \( C_{4,0}(f) \) and \( C_{3,1}(f) \) be the fibers of \( \pi_X : C_{4,0} \rightarrow \mathcal{K}_{6k,4k-1} \) and \( \pi_Y : C_{3,1} \rightarrow \mathcal{K}_{6k,4k-1} \) over \( f \). These fibers are finite dimensional configuration spaces with dim \( C_{4,0}(f) = \deg \omega_X \) and dim \( C_{3,1}(f) = \deg \omega_Y \). \( \mathcal{H}(f) \) is calculated as

\[\mathcal{H}(f) = \frac{1}{2} \int_{C_{4,0}(f)} \omega_X + \frac{1}{6} \int_{C_{3,1}(f)} \omega_Y + c(f)\]

**Theorem 4.2** \([20] \). If \( \text{vol}_{\Sigma_{i-1}} (n = 6k, 4k - 1) \) are (anti-)invariant, then \( \mathcal{H} := I(H) + c : \mathcal{K}_{6k,4k-1} \rightarrow \mathbb{R} \) is a closed 0-form and induces an isomorphism \( \pi_0(\mathcal{K}_{6k,4k-1}) \cong \mathbb{Z} \). In particular \( \mathcal{H} \) is a \( \mathbb{Z} \)-valued invariant.

**Theorem 4.2** is proved by evaluating \( \mathcal{H} \) over a generator of \( \pi_0(\mathcal{K}_{6k,4k-1}) \cong \mathbb{Z} \) given by the spinning construction \([3, 19] \). The computation in \([3] \) gives an alternative proof.

To simplify the computations below, we will take the (anti-)invariant volume forms \( \text{vol}_{\Sigma_{i-1}}, n = 6k, 4k - 1 \), so that their supports are contained in small neighborhoods of the poles \( \pm \varepsilon_n := (0, \ldots, 0, \pm 1) \in \mathbb{R}^n \). We call such a volume form a Dirac-type volume form. The following allows us to use such volume forms.

**Proposition 4.3** \([20] \text{ Propositions 3.5, 3.6}] \). The invariant \( \mathcal{H} \) is independent of the choice of the (anti-)invariant volume forms \( \text{vol}_{\Sigma_{i-1}}, n = 6k, 4k - 1 \).
5. Proofs of Theorems 2.5 and 2.8

It is well known that the linking number of closed oriented submanifolds $M^{2k-1} \sqcup N^{2k-1} \subset \mathbb{R}^{4k-1}$ is an isotopy invariant and can be defined as

$$\text{lk}(M, N) := \int_{M \times N} \varphi^* \text{vol}_{S^{4k-2}},$$

where $\varphi : M \times N \to S^{4k-2}$ is the generalized Gauss map given by

$$\varphi(x, y) := \frac{y - x}{|y - x|}.$$

If $M, N$ are in generic positions, then $p(M \sqcup N)$ is generically immersed in $\mathbb{R}^{4k-2}$ and the pairs $(x, y) \in M \times N$ such that $p(\varphi(x, y)) = \mathbf{0} \in \mathbb{R}^{4k-2}$ form a 0-dimensional submanifold of $M \times N$. If moreover $\text{vol}_{S^{4k-2}}$ is Dirac-type (see the paragraph before Proposition 4.3), then

$$(5.1) \quad \text{lk}(M, N) = \sum_{(x, y) \in (p, p')^{-1}(\mathbf{0})} \int_{\text{a neighborhood of } (x, y)} \varphi^* \text{vol}_{S^{4k-2}} = \frac{1}{2} \sum_{(x, y) \in (p, p')^{-1}(\mathbf{0})} \deg \varphi|_{\text{a neighborhood of } (x, y)}$$

and each $\deg \varphi = \pm 1$. This is because the integrand $\varphi^* \text{vol}_{S^{4k-2}}$ is zero outside neighborhoods of such pairs, and the integral of $\text{vol}_{S^{4k-2}}$ over one component of $\text{supp}(\text{vol}_{S^{4k-2}})$ is $1/2$. This interpretation gives us the following, which we use in §6.

**Lemma 5.1.** Let $M_1, M_2$ and $N_1, N_2$ be $(2k-1)$-dimensional disjoint submanifolds of $\mathbb{R}^{4k-1}$. If the connected-sums $M_1 \# M_2$ and $N_1 \# N_2$ are taken so that $[p(M_i) \cap p(N_j)] (i, j = 1, 2)$ do not increase, then

$$\sum_{i, j=1, 2} \text{lk}(M_i, N_j) = \text{lk}(M_1 \# M_2, N_1 \# N_2).$$

For a single submanifold $L^{2k-1} \subset \mathbb{R}^{4k-1}$ the same formula as (5.1) does not give rise to an isotopy invariant, but if $L$ is almost planar, then such a function can be computed by counting $(x, y) \in \text{Conf}_2(L)$ with $p(\varphi(x, y)) = \mathbf{0}$.

**Definition 5.2.** Let $L^{2k-1} \subset \mathbb{R}^{4k-1}$ be a generic closed submanifold such that $p(L) \subset \mathbb{R}^{4k-2}$ is a generically immersed manifold. Define the write $w(L)$ of $L$ by

$$w(L) = \sum_{(x, y) \in (p, p')^{-1}(\mathbf{0})} \int_{\text{a neighborhood of } (x, y)} \varphi^* \text{vol}_{S^{4k-2}} = \frac{1}{2} \sum_{(x, y) \in (p, p')^{-1}(\mathbf{0})} \deg \varphi|_{\text{a neighborhood of } (x, y)}.$$  

For an almost planar $f \in \mathcal{K}_d, f^{\delta}$ the embedding obtained by a scaling in the $x_d$-direction so that $f^{\delta}(\mathbb{R}^{4k-1}) \subset \mathbb{R}^{4k-1} \times [0, \delta]$ (we often abbreviate $f^{\delta}$ as $f$). We compute $I(X)(f^{\delta})$, $I(Y)(f^{\delta})$ and $ct(f^{\delta})$ in the limit $\delta \to 0$.

**Proposition 5.3** (compared with [[11] Proposition 4.3]). Suppose $\text{vol}_{S^{4k-2}}$ are Dirac-type. If $f \in \mathcal{K}_d$ is almost planar and generic so that $\text{lk}(L_i^{t'}, L_j^{t''})$ and $w(L_i^{t'})$ can be calculated by (5.1) and (5.2), then

$$\lim_{\delta \to 0} I(X)(f^{\delta}) = \frac{1}{2} \sum_{i, j \neq i} (-1)^{i+j} \text{lk}(L_i^{t'}, L_j^{t''}) + \frac{1}{4} \sum_{L_i^{t'}} w(L_i^{t'}).$$

**Proof.** A configuration $\vec{z} = (x_1, \ldots, x_4) \in C_{4,0}(f)$ can nontrivially contribute to $I(X)$ only if $\vec{z} \in \varphi_X^{-1}(\text{supp}(\text{vol}_{S^{4k-2}} \times \text{vol}_{S^{4k-2}}))$. Since $\text{vol}_{S^{4k-2}}$ are Dirac-type, such an $\vec{z}$ must be in a neighborhood of $\varphi_X^{-1}(\pm e_6, \pm e_6, \pm e_{4k-1})$, where $e_i := (0, \ldots, 0, 1) \in S^{4k-1}$. If $\delta$ is close to 0 enough, then no vectors tangent to $f(\mathbb{R}^{4k-1})$ point $\text{supp}(\text{vol}_{S^{4k-2}})$. Thus $\vec{z}$ can be in $\varphi_X^{-1}(\text{supp}(\text{vol}_{S^{4k-2}} \times \text{vol}_{S^{4k-2}}))$ in the limit $\delta \to 0$ only if $(x_1, x_2) \in N(L_i^{t'}) \times N(L_j^{t''})$ and $(x_3, x_4) \in N(L_i^{t'}) \times N(L_i^{t''})$ for some $i, j$, possibly $i = j$ (recall that $N(L_i^{t'}) \subset \mathbb{R}^{4k-1}$ are closed disjoint tubular neighborhoods of $L_i^{t'}$). For any $x_i \in N(L_i^{t'})$ we always find $x_i \in N(L_i^{t'})$ such that $p(f(x_i))$ is close to $\mathbf{0}$ $(s, t) = (1, 2), (4, 3))$. Therefore finding $\vec{z} = (\xi_1, \ldots, \xi_4) \in \varphi_X^{-1}(\pm e_6, \pm e_6, \pm e_{4k-1})$ is equivalent to finding $(\xi_2, \xi_3)$ satisfying...
bases of the tangent spaces of spheres following the “outward normal first” convention, at $g$.

Then by Lemma 2.3, as oriented manifolds, the Jacobian $\frac{\partial}{\partial y} \phi_{X}(\xi)$ is given as in (5.1). Their sum would amount to linking numbers by (5.1).

To compute $\deg \phi_{X}$ at each $\xi \in \varphi_{X}(\pm e_{6k}, \pm e_{6k}, \pm e_{4k-1})$, we recall from [1] the local model for two-fold self-intersection. Let $g : L_{k} \to L_{k-1,4k-1}$ be a generic immersion and $q = g(p_{1}) = g(p_{2})$ be a transverse two-fold self-intersection point. In some local coordinates centered at $p_{1}$, $p_{2}$ and $g$, $g$ is given by

$$g(x_{1}, \ldots, x_{4k-1}) = (x_{1}, \ldots, x_{2k-1}, x_{2k}, \ldots, x_{4k-1}, 0, \ldots, 0) \quad \text{near } p_{1},$$

$$g(y_{1}, \ldots, y_{4k-1}) = (y_{1}, \ldots, y_{2k-1}, 0, \ldots, 0, y_{2k}, \ldots, y_{4k-1}) \quad \text{near } p_{2}.$$

Now consider a configuration $\xi \in C_{4,0}(f)$ such that

$$\begin{cases} 
\xi_{1} \in L_{j}^{1}, 
\xi_{2} \in L_{j}^{1}, 
\xi_{3} \in L_{j}^{1} 
\text{and } \xi_{4} \in L_{j}^{1},
\end{cases}$$

$\varphi_{X}(\xi) = (-e_{6k}, e_{6k}, e_{4k-1})$

(Figure [5.1] (the left). Suppose that $(\xi_{2}, \xi_{3})$ is a positive crossing, that is, $\deg \varphi_{X}(\xi_{2}, \xi_{3}) = +1$.

Then we can choose some local coordinates $x, y$ and $z$ centered at $\xi_{1}, \xi_{2}$ and $\xi_{3}$ such that $\xi_{1} = (0, \ldots, 0, 1)$ in the $y$-coordinate (the same coordinate as for $\xi_{2}$), and we can also choose local coordinates in $\mathbb{R}^{6k}$ in which $f$ is given by

$$f(x) = (x_{1}, \ldots, x_{2k-1}, x_{2k}, \ldots, x_{4k-1}, 0, \ldots, 0, 1) \quad \text{near } \xi_{1},$$

$$f(y) = (y_{1}, \ldots, y_{2k-1}, 0, \ldots, 0, y_{2k}, \ldots, y_{4k-1}, 0) \quad \text{near } \xi_{2},$$

and

$$f(z) = (z_{1}, \ldots, z_{2k-1}, 0, \ldots, 0, z_{2k}, \ldots, z_{4k-1} \pm 1, 1) \quad \text{near } \xi_{3},$$

Then by Lemma 2.3 as oriented manifolds, $L_{j}$’s are given by

- $L_{j}^{3} \cap (x$-coordinate) $= +\mathbb{R}^{2k-1} \times \{0\}^{2k}$,
- $L_{j}^{3} \cap (y$-coordinate) $= +\mathbb{R}^{2k-1} \times \{0\}^{2k}$,
- $L_{j}^{3} \cap (y$-coordinate) $= 0^{2k-1} \times (-\mathbb{R}^{2k-1} \times \{1\}$, and
- $L_{j}^{3} \cap (z$-coordinate) $= 0^{2k-1} \times (-\mathbb{R}^{2k-1} \times \{0\}$

(see Figure [5.2]). Using this local model, we can compute the Jacobian $J(\varphi_{X})_{\xi}$ of

$$\varphi_{X} : N(L_{j}^{3}) \times N(L_{j}^{3}) \times N(L_{j}^{3}) \to S^{6k-1} \times S^{6k-1} \times S^{4k-2}$$

at $\xi$ explicitly. Let $e_{i} := (0, \ldots, 0, 1, 0, \ldots, 0)$ be the $i$-th unit vector. With respect to the natural positive basis $e_{1}, \ldots, e_{6k-4}$ of $T_{\xi}^{\mathbb{R}^{6k-4}} S^{6k-1}$ and the natural positive bases of the tangent spaces of spheres $e_{1}, \ldots, e_{6k-4} \in T_{\xi}^{\mathbb{R}^{6k-4}} S^{6k-1}$ and $e_{1}, \ldots, e_{4k-2} \in T_{\xi}^{\mathbb{R}^{4k-2}} S^{4k-2}$, following the “outward normal first” convention, $J(\varphi_{X})_{\xi}$ is given as in [4.3] and its determinant is $-1$. 

![Figure 5.1](image-url) Two configurations with the same contributions to $\text{lk}(L_{j}^{3}, L_{j}^{3})$: the left shows a neighborhood of $\xi$ satisfying (5.3), and the right shows a neighborhood of $\xi'$ satisfying (5.4).
A local model for a negative crossing \((\xi_2, \xi_3)\) (namely \(\deg \varphi_{23}^{\text{II}}(\xi_2, \xi_3) = -1\)) is obtained from the above model by reversing the orientation of the \(z\)-sheet, and in this case \(J(\varphi)_{\xi_3} = +1\). Thus the integral of \(\omega_Y\) over a neighborhood of \(\vec{\xi}\) satisfying (5.3) with \(\deg \varphi_{23}^{\text{II}}(\xi_2, \xi_3) = \pm 1\) is \(\mp (1/2)^3\), and by (5.1) their sum is equal to \(-\text{lk}(L_0, L_1)/4\). By symmetry of \(X\), the configurations \(\vec{x}\) near \(\vec{\xi}\) satisfying

\begin{align}
\{\xi_1' \in L_0, \xi_2' \in L_1, \xi_3' \in L_1, \xi_4' \in L_0, \xi_5' = (e_{0k}, e_{bk}, -e_{0k(-1)})\}
\end{align}

(see Figure 5.1, the right) also contribute to \(I(X)(f)\) by \(-\text{lk}(L_0, L_1)/4\). Thus a link \(L_0 \cup L_1\) contributes to \(I(X)(f)\) by \(-\text{lk}(L_0, L_1)^3/4\).

A similar computation shows that \(L_0^0\) and \(L_1^0\) contribute to \(I(X)(f)\) by \(+\text{lk}(L_0^0, L_1^0)/2\); we have the same Jacobian matrix as above, but in this case \(\varphi_{23}^{\text{II}}(\vec{\xi}) = e_{0k}\) and \((e_1, \ldots, e_{0k(-1)})\) represent the negative orientation of \(T_{\text{en}}S^{d-1}\), and the sign of the degree changes. This observation shows that in general the link \(L_0^1 \cup L_1^1\), \((i, \epsilon) \neq (j, \epsilon')\), contributes to \(I(X)(f)\) by

\[(-1)^{(j+1)}\text{lk}(L_0^1, L_1^1)/2.\]

In the case \((i, \epsilon) = (j, \epsilon')\), if \(f\) is generic so that \(w(L_0^1)\) can be calculated by (5.2), then the same computation as above shows that the configurations \(x \in N(L_0^{d+1}) \times N(L_1^d) \times N(L_1^d) \times N(L_1^{d+1})\) contributes to \(I(X)(f)\) by \(+\text{lk}(L_0^1, L_1^1)/4\) (no sign appears in this case since \(\epsilon = \epsilon'\)).

**Lemma 5.4.** Let \(f \in \mathcal{K}_{d_0, d_1, -1}\) be almost planar and suppose that \(\text{vol}_{S^{d_1}}\) is Dirac-type. Then

\[\lim_{\delta \to 0} I(Y)(f) = \lim_{\delta \to 0} I(Y)(f_5) \quad \text{and} \quad \lim_{\delta \to 0} c(f) = \lim_{\delta \to 0} c(f_5).\]

**Proof.** The correction term \(c(f^0)\) is defined and continuous when \(\delta = 0\) (then \(f^0 = f_5\) and \(f_5 \) collapse down to an immersion \(f^0 = f_5\) if \(\delta = 0\) because \(c(f)\) is determined by the differential of \(f\), and hence \(\lim_{\delta \to 0} c(f) = c(f^0) = c(f_5)\).

To compute \(\lim_{\delta \to 0} I(Y)(f)\), we may assume that \(\delta < 1\). Choose open neighborhoods \(f(\mathcal{N}(L_1^d)) \subset U' \subset V' \subset \mathbb{R}^{d_1-1}\) (we are assuming \(f\) is almost planar) so that \(U' \subset V' \subset V\), \(V_0 \cap V' = \emptyset (i \neq f)\) and \(f(\mathcal{N}(L_1^d)) \subset U_i := V_i \times [-1, 1]\) (Figure 5.3). Let \(C_{3,1}(f)\) be the subspace of \(C_{3,1}(f)\) consisting of \(x \in C_{3,1}(f)\) with \(x_4 \in V_i := V_i \times [-1, 1]\) for some \(i \in S\). Now we compare the integrals of \(\omega_Y\) over \(C_{3,1}(f)\) and \(C_{3,1}(f_5)\). Since \(\text{vol}_{S^{d_1}}\) is Dirac-type, we only need to consider the set of \(x\)'s with \(x_1, x_2, x_3 \in f^{-1}(V_i)\) for the same \(i\) as for \(x_4\), since \(\omega_Y\) vanishes on other \(x\)'s. The local diffeomorphism \(\Phi : C_{3,1}(f) \to C_{3,1}(f_5)\),

\[\Phi(x_1, x_2, x_3, x_4) := (x_1, x_2, x_3, \tau(x_4)),\]

reverses the orientation, while \(\Phi \circ \omega_Y = -\omega_Y\) since \(\psi^0 \circ \Phi = \tau \circ \psi^0\) and \(f_5 = \tau \circ f\) on \(f^{-1}(V_i)\) (\(i \in S\)) and \(\tau \circ \text{vol}_{S^{d_1}} = -\text{vol}_{S^{d_1}}\) (because \(\text{vol}_{S^{d_1}}\) is anti-invariant). Thus the integrations of \(\omega_Y\) over \(C_{3,1}(f)\) and \(C_{3,1}(f_5)\) are equal to each other.

Next consider the subspace \(C_{3,1}(f)\) of \(C_{3,1}(f)\) consisting of \(x\) with \(x_4 \in \mathbb{R}^{d_1-1} \times [-1, 1]\) but \(x_4 \not\in V_i\) for any \(i \in S\). Since \(\text{vol}_{S^{d_1}}\) is Dirac-type and since \(x_4 \notin V_i\) for any \(i \in S\),
Proof of Theorem 2.8. The side of (2.1). The second and the third terms converge to zero by Lemma 5.4.

Let

\[ \delta > 0 \]

Lastly consider the subspace \( C_{3,1}(f) \) consisting of \( x \) with \( x \notin \mathbb{R}^{4k-1} \times [-1, 1] \).

If \( \delta > 0 \) is small enough, then under the diffeomorphism \( C_{3,1}(f) \rightarrow C_{3,1}(f_{0}) \) given by \( x' \mapsto x \), the differences between the vectors \( \psi_{i}(x) (i = 1, 2, 3) \) are small. This is because

\[ f \]

differs from \( f_{0} \) only near \( N(L_{1}) \) and the difference is small relative to \( |x| \). Thus the difference between the integrals of \( \omega_{Y} \) over \( C_{3,1}(f) \) and \( C_{3,1}(f_{0}) \) converges to zero in the limit \( \delta \rightarrow 0 \).

\[ \square \]

Proof of Theorem 2.5. Any \( f \) with \( p \circ f \) generic can be transformed by an ambient isotopy of \( \mathbb{R}^{4k-1} \) so that \( f \) satisfies the condition in Proposition 5.3 without changing the isotopy class of \( \bigcup L_{i}' \). Thus we may assume that \( f \) satisfies the condition in Proposition 5.3. Notice

\[ \mathcal{H}(f) - \mathcal{H}(f_{0}) = \frac{I(X)(f) - I(X)(f_{0})}{2} + \frac{I(Y)(f) - I(Y)(f_{0})}{6} + (c(f) - c(f_{0})). \]

The left hand side does not depend on \( \delta \). In the limit \( \delta \rightarrow 0 \) (for generic \( f \) remains unchanged), the first term of the right hand side is computed in Proposition 5.3 and gives the right hand side of (2.1). The second and the third terms converge to zero by Lemma 5.4.

\[ \square \]

Proof of Theorem 2.8. Choose three components \( A_{1}, A_{2}, A_{3} \) of \( A = A_{1} \sqcup \cdots \sqcup A_{m}, m \geq 3 \). Let \( W_{T}(\mathcal{H})(f) := \mathcal{H}(f_{T}) - \mathcal{H}(f_{T,1}) \) for any \( T \subset \{2, 3\} \). Then by (2.2),

\[ 2W_{T}(\mathcal{H})(f) = \sum_{j \neq k, l, m, n, \delta} (-1)^{\epsilon_{j}^{\epsilon_{k}^{\epsilon_{l}^{\epsilon_{m}^{\epsilon_{n}^{\epsilon_{\delta}}}}}} l_{1}^{j}(T), l_{1}^{j}(T)) \]

Because \( L_{j}^{j}(T) = L_{1}^{j+1} \) if \( j \in T \) and \( L_{j}^{j}(T) = L_{1}^{j} \) otherwise,

\[ 2W_{0}(\mathcal{H})(f) = \sum_{\epsilon_{1}, \epsilon_{2}, \epsilon_{3}} (-1)^{\epsilon_{1}^{\epsilon_{2}^{\epsilon_{3}}} l_{1}^{1}(L_{1}^{1}, L_{1}^{1}) + l_{1}^{1}(L_{1}^{1}, L_{1}^{1})) \]

\[ 2W_{2}(\mathcal{H})(f) = \sum_{\epsilon_{1}, \epsilon_{2}, \epsilon_{3}} (-1)^{\epsilon_{1}^{\epsilon_{2}^{\epsilon_{3}}} l_{1}^{1}(L_{1}^{1}, L_{1}^{1}) + l_{1}^{1}(L_{1}^{1}, L_{1}^{1})) \]

\[ 2W_{3}(\mathcal{H})(f) = \sum_{\epsilon_{1}, \epsilon_{2}, \epsilon_{3}} (-1)^{\epsilon_{1}^{\epsilon_{2}^{\epsilon_{3}}} l_{1}^{1}(L_{1}^{1}, L_{1}^{1}) - l_{1}^{1}(L_{1}^{1}, L_{1}^{1})) \]

Substituting them into \( V_{3}(\mathcal{H})(f) = \sum_{T \subset \{2, 3\}} (-1)^{\epsilon_{j}^{\epsilon_{k}^{\epsilon_{l}^{\epsilon_{m}^{\epsilon_{n}^{\epsilon_{\delta}}}}}} W_{T}(\mathcal{H})(f) \), we obtain \( V_{3}(\mathcal{H})(f) = 0 \).

\[ \square \]
6. Proof of Theorem 2.10

6.1. Well-definedness and invariance of $E$. Suppose $g \in I_{6k-1,4k-1}$ is generic and liftable, and let $f, f' \in \mathcal{K}_{6k,4k-1}$ be lifts of $g$. We can transform $f'$ by an isotopy in the $x_{6k}$-direction (without changing $p \circ f$) so that $f' = f_\varepsilon$ for some index set $S$ of the self-intersection of $g$. Then (2.4) implies that $E(g)$ does not depend on the choice of $f$.

Let $g_\varepsilon \in I_{6k-1,4k-1}$ ($\varepsilon \in [0,1]$) be a generic regular homotopy with each $g_\varepsilon$ liftable. We show that $g_\varepsilon$ can be lifted (at least locally) to an isotopy $f_\varepsilon \in \mathcal{K}_{6k,4k-1}$, namely $g_\varepsilon \equiv p \circ f_\varepsilon$. For $t_0 \in [0,1]$, let $f_{0t}$ be a lift of $g_{0t}$. Then $f_{0t}$ can be written as $f_0 = (g_\varepsilon, h)$ by using some $h : \mathbb{R}^{4k-1} \to \mathbb{R}$. Define $G_\varepsilon : \text{Conf}_{X}^2(\mathbb{R}^{4k-1}) \to \mathbb{R}^{6k-1}$ by

$$G_\varepsilon(x, y) := g_\varepsilon(x) - g_\varepsilon(y).$$

The first projection $\text{Conf}_{X}^2(\mathbb{R}^{4k-1}) \to \mathbb{R}^{4k-1}$ restricts to a diffeomorphism $G_\varepsilon^{-1}(0) \cong g_\varepsilon^{-1}(A_\varepsilon)$, where $A_\varepsilon$ is the self-intersection of $g_\varepsilon$. Since $g_\varepsilon$ is a generic regular homotopy, $G_\varepsilon^{-1}(0)$ gives an isometry of a closed submanifold of $\text{Conf}_{X}^2(\mathbb{R}^{4k-1})$. Because $f_\varepsilon = (g_\varepsilon, h) \in \mathcal{K}_{6k,4k-1}$, there exists an open neighborhood $W$ of $G_\varepsilon^{-1}(0)$ such that $h(x) \neq h(y)$ for any $(x, y) \in W$. The compactness of $G^{-1}_\varepsilon(0)$ (for any $t$) implies that there exists $\varepsilon > 0$ such that $G^{-1}_\varepsilon(0) \subset W$ for $|t - t_0| < \varepsilon$. Then $f_\varepsilon := (g_\varepsilon, h) : \mathbb{R}^{4k-1} \to \mathbb{R}^{6k}$ is in $\mathcal{K}_{6k,4k-1}$ for $|t - t_0| < \varepsilon$ and is a lift of $g_\varepsilon$.

Because $\mathcal{H}(f_\varepsilon)$ is constant and the linking part of (2.3) is invariant unless the double point set varies, $E(g_\varepsilon)$ is also constant. Thus for any $t_0$ there exists $\varepsilon > 0$ such that $E(g_\varepsilon)$ is constant on $(t_0 - \varepsilon, t_0 + \varepsilon)$, and hence $E(g_{0t})$ is constant on $[0,1]$.

Remark 6.1. By Proposition 5.3 and (2.3), for a generic liftable $g \in I_{6k-1,4k-1}$,

$$E(g) = \lim_{\delta \to 0} \frac{1}{6} I(Y) + c(f) + \frac{1}{8} \sum_{i,k} w(L_\varepsilon^i).$$

This gives a geometric interpretation of $I(Y)$ (added by $c$ and the writhes), and is a higher dimensional analogue to [11] Definition 5.4].

6.2. Local models of non-generic self-intersections. As explained in [6, 7], the set of generic immersions $g : \mathbb{R}^{4k-1} \to \mathbb{R}^{6k-1}$ is an open dense subspace of $I_{6k-1,4k-1}$ and the complement is a stratified hypersurface. To characterize an invariant of generic immersions, we must study its jumps at non-generic strata. The codimension one strata (in $I_{6k-1,4k-1}$) consist of immersions with a single self-tangency point or a single triple point [7] Lemma 3.4. The local picture of the versal deformation [6] §5.3, [7] §3.2] of an immersion with a self-tangency or a triple point is given in [6, 7].

Proposition 6.2 ([7] Lemma 3.5)]. Let $g_0 \in I_{6k-1,4k-1}$ be an immersion with a single self-tangency point. Then the versal deformation $g_\varepsilon$ of $g_0$ is constant far from the self-tangency point, and in some local coordinates near the self-tangency point, $g_\varepsilon$ is given by

$$g_\varepsilon(x) = (x_1, \ldots, x_{2k-1}, x_{2k}, x_{2k+1}, \ldots, x_{4k-1}, 0, \ldots, 0),$$

$$g_\varepsilon(y) = (y_1, \ldots, y_{2k}, 0, \ldots, 0, Q(y_1, \ldots, y_{2k}) + t, y_{2k+1}, \ldots, y_{4k-1})$$

where $Q$ is a non-degenerate quadratic form on $2k$ variables.

We say a self-tangency point definite (resp. indefinite) if the quadratic form $Q$ is definite (resp. indefinite).

Proposition 6.3 ([7] Lemma 3.6). Let $g_0 \in I_{6k-1,4k-1}$ be an immersion with a single triple point. Then the versal deformation $g_\varepsilon$ of $g_0$ is constant far from the triple point, and in some local coordinates near the triple point, $g_\varepsilon$ is given by

$$g_\varepsilon(x) = (x_1, \ldots, x_{2k-1}, x_{2k}, x_{2k+1}, \ldots, x_{4k-1}, 0, 0, \ldots, 0),$$

$$g_\varepsilon(y) = (y_1, \ldots, y_{2k}, 0, 0, 0, y_{2k}, y_{2k+1}, \ldots, y_{4k-1}),$$

$$g_\varepsilon(z) = (0, \ldots, 0, z_{2k}, z_1, \ldots, z_{2k-1}, z_{2k}, t, z_{2k+1}, \ldots, z_{4k-1}).$$
6.3. The jump of $E$ at a non-generic liftable immersion. Suppose that $g_0 \in \mathcal{I}_{6k−1,4k−1}$ has a single self-tangency point or a single triple point, and let $g_t$ be its versal deformation.

We show that $g_t$ is liftable for $|t|$ small. Let $f_0 = (g_0,h)$ be a lift of $g_0$ and $f_t := (g_t,h) \in \mathcal{I}_{6k−1,4k−1}$. Similarly to the argument in 6.1 choose an open neighborhood $W$ of $G^{−1}_0(0)$ such that $h(x) \neq h(y)$ for any $(x,y) \in W$. Then there exists $\epsilon > 0$ such that $G^{−1}_t(0) \subset W$ for $|t| < \epsilon$; this follows from the explicit description of the change of the multiple point set of $g_t$ (see below). Thus $f_t \in \mathcal{K}_{6k−1,4k−1}$ for $|t| < \epsilon$ and is a lift of $g_t$.

By the definition (6.2) of the invariant $E$, its jump $E(g_t) = E(g_{−t}) (t \neq 0)$ is described by the change of linking numbers of $L^*_t$'s because $H(f_t)$ remains unchanged.

6.3.1. Definite self-tangencies. First we study the jump of $E$ at a positive definite self-tangency point (the argument needs no change for negative definite case). It is clear from Proposition 6.2 that in some local coordinate near the tangency point, the double point set is given by

$$K^0 = \{(x_1, \ldots, x_{2k}, 0^{2k−1}) \in \mathbb{R}^{4k−1} \mid x_1^2 + \cdots + x_{2k}^2 = t\} \text{ in the } x\text{-sheet},$$

$$K^1 = \{(y_1, \ldots, y_{2k}, 0^{2k−1}) \in \mathbb{R}^{4k−1} \mid y_1^2 + \cdots + y_{2k}^2 = t\} \text{ in the } y\text{-sheet},$$

which is empty when $t > 0$, and the trivial link when $t < 0$. This link $K^0 \cup K^1$ is separated from the other links since each $K^t$ is contained in a small open set which intersects no other components of double point set. Thus we have the following.

**Lemma 6.4.** If $g_0$ has a definite self-tangency point, then $E(g_0) = E(g_{−t}).$

6.3.2. Indefinite self-tangencies. Next suppose that $g_0$ is liftable and has an indefinite self-tangency point. Let $0 < \lambda < 2k$ be the index of $Q$. By Proposition 6.2, in some local coordinates the double point set of $g_t$ near the self-tangency point is given as follows;

$$\{(x_1, \ldots, x_{2k}, 0^{2k−1}) \in \mathbb{R}^{4k−1} \mid x_1^2 + \cdots + x_{\lambda+1}^2 - \cdots - x_{2k}^2 = t\} \text{ in the } x\text{-sheet},$$

$$\{(y_1, \ldots, y_{2k}, 0^{2k−1}) \in \mathbb{R}^{4k−1} \mid y_1^2 + \cdots + y_{\lambda+1}^2 - \cdots - y_{2k}^2 = t\} \text{ in the } y\text{-sheet}.$$
intersection, then suppose that the components of the self-intersection; namely 1 when two arcs get tangent to each other with opposite velocity vectors; by Lemma 2.3 no positive (Figure 6.1, the right). Similarly in the KEIICHI SAKAI connected one and the followings hold; the connected-sums Lemma 5.1 =

\[
\begin{align*}
\sum_{(p,q)\neq(q',q')} (-1)^{\epsilon + \epsilon'} lk(L^0_p(g_j), L^0_j(g_i)) &\quad - \sum_{(p,q)\neq(q',q')} (-1)^{\epsilon + \epsilon'} lk(L^0_p(g_{-j}), L^0_j(g_{-i})) \\
= (lk(L^0_p, L^0_q) + lk(L^1_p, L^1_j) - \sum_{p,q=i,j} lk(L^0_p, L^0_q) + \sum_{p,q=i,j} (-1)^{\epsilon + \epsilon'} lk(L^0_p, L^0_j) \quad - (-lk(K^0, K^1) + \sum_{m\neq i,j, \epsilon, \epsilon' = 0, 1} (-1)^{\epsilon + \epsilon'} lk(K^e, L^e_m)).
\end{align*}
\]

The connected-sums $K^e = L^e_m \# L^e_j$ are taken near the tangency point and by a small isotopy we may assume that $L^e_j (\epsilon = i, j; \epsilon = 0, 1)$ satisfy the condition in Lemma 5.1. Thus by Lemma 5.1

\[
\sum_{p,q=i,j} lk(L^0_p, L^0_q) = lk(K^0, K^1), \quad lk(L^1_p, L^1_m) + lk(L^1_p, L^1_m) = lk(K^e, L^e_m).
\]

By the above three equations, we have

\[
E(g_j) - E(g_{-j}) = \frac{lk(L^0_p, L^0_q) + lk(L^1_p, L^1_j)}{4}.
\]

**Case 2.** Consider the case when the versal deformation does not change the number of the components of the self-intersection; namely $k > 1$ and $A_i = A_j$. Since $L^0_j = L^1_j$ turns into $K^e$ and other components $L^0_m (m \neq i; \epsilon = 0, 1)$ are unchanged, the jump of the sum of linking numbers is

\[
\sum_{(p,q)\neq(q',q')} (-1)^{\epsilon + \epsilon'} lk(L^0_p(g_i), L^0_j(g_j)) - \sum_{(p,q)\neq(q',q')} (-1)^{\epsilon + \epsilon'} lk(L^0_p(g_{-i}), L^0_j(g_{-j}))
\]

\[
= (-lk(L^0_i, L^1_j) + \sum_{m\neq i, \epsilon, \epsilon' = 0, 1} (-1)^{\epsilon + \epsilon'} lk(L^1_i, L^1_m) - (-lk(K^0, K^1) + \sum_{m\neq i, \epsilon, \epsilon' = 0, 1} (-1)^{\epsilon + \epsilon'} lk(K^e, L^e_m))
\]

and the followings hold;

\[
lk(L^0_i, L^1_j) = lk(K^0, K^1), \quad lk(L^1_i, L^1_m) = lk(K^e, L^e_m).
\]

By the above three equations, we have $E(g_i) = E(g_{-i})$.

**Case 3.** When $k = 1$ and a negative self-tangency occurs at $t = 0$ in $A_i = A_j$ (namely two arcs get tangent to each other with opposite velocity vectors; by Lemma 2.3 no positive self-tangency occurs), the number of the components of the self-intersection increases by 1 when $t$ changes from $t > 0$ to $-t$. This case is similar to Case 1.

The case $A = 2k - 1$ is similar, and we have the following.

**Lemma 6.5.** Suppose that $g_0$ has an indefinite self-tangency point. If the index of $Q$ is 1 or $2k - 1$, and if the versal deformation changes the number of the components of self-intersection, then

\[
E(g_j) - E(g_{-j}) = \frac{lk(L^0_p, L^0_q) + lk(L^1_p, L^1_j)}{4},
\]
where $L_i^0 \cup L_j^1 = g_i^{-1}(A_i)$ and $L_i^0 \cup L_j^0 = g_i^{-1}(A_j)$ are the double point set corresponding to $A_i$ and $A_j$, the distinct components of the self-intersection of $g_i$ which are joined into a single component after the versal deformation. Otherwise $E(g_i) = E(g_{-i}).$

6.3.3. **Triple points.** Suppose $g_0$ is liftable and has a triple point. By Proposition 6.3 in some local coordinates, the double point set near the triple point in the $x$-sheet is given by

$$S_i^1 := \{(x_1, \ldots, x_{2k-1}, 0, 0, \ldots, 0)\} = +\mathbb{R}^{2k-1} \times \{0\}^{2k},$$

$$S_i^0 := \{(0, 0, \ldots, 0, t, x_{2k+1}, \ldots, x_{4k-1})\} = \{0\}^{2k-1} \times \{t\} \times (+\mathbb{R}^{2k-1})$$

as oriented manifolds (see Figure 6.2). The orientations are direct consequences of Lemma 2.3. Similarly the double point set in the $y$-sheet is given by

$$S_i^1 := \{(0, 0, \ldots, 0, -t, y_{2k+1}, \ldots, y_{4k-1})\} = \{0\}^{2k-1} \times \{-t\} \times (-\mathbb{R}^{2k-1}),$$

$$S_i^0 := \{(y_1, \ldots, y_{2k-1}, 0, 0, \ldots, 0)\} = +\mathbb{R}^{2k-1} \times \{0\}^{2k},$$

and in the $z$-sheet

$$S_i^0 := \{(z_1, \ldots, z_{2k-1}, t, 0, 0, \ldots, 0)\} = +\mathbb{R}^{2k-1} \times \{t\} \times \{0\}^{2k-1},$$

$$S_i^0 := \{(0, 0, \ldots, 0, 0, z_{2k+1}, \ldots, x_{4k-1})\} = \{0\}^{2k} \times (-\mathbb{R}^{2k-1}).$$

Here without loss of generality we assume that the lift $f_i$ of $g_i$ maps the $x$-sheet (in Proposition 6.3) to the “highest position”, the $y$-sheet to the “middle” and the $z$-sheet to the “lowest”. The following holds by the above descriptions.

**Lemma 6.6** (see [6] Remark 6.2.3). The versal deformation changes three crossing $S_i^1 \cup S_j^1 \cup S_p^1$ and $S_i^0 \cup S_p^0$, changing their linking numbers or writhes by the common value $\pm 1$ (the signs are same for all the three crossings).

Let $L_i^\epsilon$ ($\epsilon = i, j, p; \epsilon = 0, 1$) be the component of the double point set containing $S_i^\epsilon$.

**Case 1.** If all the six components $L_i^\epsilon$ are different, then by Lemma 6.6 the versal deformation changes $(-1)^{i+1}lk(L_i^0, L_i^1) + (-1)^{j+1}lk(L_j^0, L_j^1) + (-1)^{p+1}lk(L_p^0, L_p^1)$ by $\pm 1$. Other linking numbers do not change. Thus $E(g_i) - E(g_{-i}) = \pm 1/4$.

**Case 2.** If $L_i^\epsilon = L_j^\epsilon \neq L_p^\epsilon$ ($\epsilon = 0, 1$), then the crossing change at $S_i^0 \cup S_j^1$ and $S_i^1 \cup S_p^0$ changes $(-1)^{i+1}lk(L_i^0, L_i^1) + (-1)^{j+1}lk(L_j^0, L_j^1) + (-1)^{j+1}lk(L_j^0, L_j^1)$ by $\pm 1 = 0$. Other linking numbers do not change, and the change of $w(L_i^1)$ (by $\pm 1$) arising from the crossing change at $S_i^1 \cup S_j^1$ does not change $E$. Thus $E(g_i) = E(g_{-i})$.

**Case 3.** If $L_i^\epsilon = L_j^\epsilon \neq L_p^\epsilon$ ($\epsilon = 0, 1$), then the versal deformation changes $lk(L_i^0, L_j^1) - lk(L_j^0, L_i^1)$ by $\pm 1 = \pm 1$, and $E(g_i) - E(g_{-i}) = \pm 1/4$.

**Case 4.** The case $L_i^\epsilon = L_j^\epsilon \neq L_p^\epsilon$ is similar to the Case 2 by symmetry, and $E(g_i) = E(g_{-i})$.

**Case 5.** If $L_i^\epsilon = L_j^\epsilon = L_p^\epsilon$, then the versal deformation changes $-lk(L_i^0, L_j^1)$ by $\pm 1$, and the changes of $w(L_i^1), w(L_j^1)$ do not affect $E$. Thus in this case $E(g_i) - E(g_{-i}) = \pm 1/4$.

Putting them all together, we obtain the following.

**Lemma 6.7.** Suppose that $g_0$ has a triple point. Then $E(g_i) = E(g_{-i})$ if $L_i^\epsilon = L_j^\epsilon \neq L_p^\epsilon$ or $L_i^\epsilon = L_p^\epsilon \neq L_j^\epsilon$, $\epsilon = 0, 1$ (in Figure 6.2). Otherwise $E(g_i) - E(g_{-i}) = \pm 1/4$. 

![Figure 6.2. Double point sets near the triple point](image-url)
6.4. The case \( k = 1 \). For \( g \in \mathcal{K}_{3,3} \) the invariant \( E \) is essentially the Smale invariant;

**Proposition 6.8** ([6, 24]). If \( g \in \mathcal{K}_{3,3} \) (also regarded as \( g \in \mathcal{K}_{0,3} \) by composing \( \mathbb{R}^3 \hookrightarrow \mathbb{R}^5 \)), then \( \mathcal{H}(g) = E(g) = -\Omega(g)/12 \), where \( \Omega : \pi_0(\mathcal{I}_{3,3}) \to \mathbb{Z} \) is the Smale invariant.

**Proof.** \( \mathcal{H} = E \) follows from [6.2.4], since \( g \) has no self-intersection. As explained in [6.4], there exists a “Seifert surface” for \( g \), that is an embedding \( V^4 \hookrightarrow \mathbb{R}^5 \) which restricts to \( g : \partial V \hookrightarrow \mathbb{R}^3 \), and \( \Omega(g) = 3\sigma(V)/2 \), where \( \sigma \) denotes the signature. [24 Corollary 2.4] states that \( \mathcal{H}(g) = -\sigma(V)/8 \) (\( e_f = 0 \) for \( g \in \mathcal{K}_{3,3} \)).

**Remark 6.9.** If \( g \in \mathcal{K}_{3,3} \), then \( E(g) = I(Y)(g)/6 + c(g) \) by (6.1). Thus \( \Omega(g) = -2I(Y)(g) - 12c(g) \).

The double point set of generic \( g \in I_{3,3} \) is a classical link. A result of Ogasa [15] characterizes which link can be realized as a self-intersection of an immersion \( \mathbb{R}^3 \hookrightarrow \mathbb{R}^3 \).

**Theorem 6.10** (A special case of [16 Theorem 1.1]). For any link \( L \subset S^3 \), there exist embeddings \( g^\epsilon : S^3 \hookrightarrow \mathbb{R}^5 \) (\( \epsilon = 0, 1 \)) such that \( (g^\epsilon)^{-1}(g^\epsilon(S^3) \cap g^\epsilon(S^3)) \) \( (\epsilon = 0, 1) \) are isotopic to the given \( L \). Moreover we can choose \( g^\epsilon \) to be isotopic to the natural inclusion.

Using this we show the second half of Theorem 2.10 and Corollary 2.6.

**Proof of Theorem 2.10** the second half. We show that an arbitrarily large jump of \( E \) at an indefinite self-tangency can be realized. Any linear combination of Ekholm invariants \( J \) and \( S \) cannot satisfy this property, because their jumps are bounded [6].

For any two-component link \( L = K_1 \cup K_2 \) in \( S^3 \), choose \( g^\epsilon : S^3 \hookrightarrow \mathbb{R}^5 \), \( \epsilon = 0, 1 \) as in Theorem 6.10. Taking a suitable connected-sum of the standard inclusion \( f_0 : \mathbb{R}^3 \hookrightarrow \mathbb{R}^5 \) with \( g^0 \) and \( g^1 \), we obtain \( g := f_0 g^0 g^1 \in I_{3,3} \) which satisfies the following conditions:

(i) the self-intersection \( A = A_1 \cup A_2 \) of \( g \) satisfies \( g^{-1}(A_1) = K_1^0 \cup K_1^1 \) with each \( K_1^0 \cup K_1^1 \) included in the \( g^-\text{-part} \( (g^-)^{-1}(A) \) and isotopic to the given link \( L \).

(ii) \( g \) is liftable so that \( g^-\text{-part} \) is lifted into \( \mathbb{R}^5 \times \mathbb{R}_+ \) and \( g^0\text{-part} \) remains inside \( \mathbb{R}^5 \times \{0\} \).

\( K_1^0 \cup K_2^0 \) is separated from \( K_1^1 \cup K_2^1 \) by (i) above. Take \( q_i \in A_i \) and \( p_i \in K_i^1 \) so that \( g^\epsilon(p_i) = q_i \). Choose paths \( \gamma^\epsilon : [0, 1] \to \mathbb{R}^3 \setminus (g^-)^{-1}(A) \) from \( p_i \) to \( p_i \) in \( g^-\text{-part} \). Then \( C := g(\gamma^0([0, 1]) \cup \gamma^1([0, 1])) \) is a circle in \( \mathbb{R}^3 \), which can be seen as the image of a trivial knot. Thus \( C \) bounds an embedded 2-dimensional disk in \( \mathbb{R}^3 \) whose interior transversely intersects \( g \) at finitely many points outside \( K_i^* \) (Figure 6.5). There is a homotopy which transform \( g \) near \( \gamma^0([0, 1]) \) so that \( g(\gamma^0([0, 1])) \) gets close to \( g(\gamma^1([0, 1])) \) along \( D \) and eventually \( D \) does not intersect \( g \) in its interior. During this homotopy a number of non-generic self-intersection may appear, but \( A_1 \cup A_2 \) remains unchanged. Thus we get \( g' \sim g \) for which \( \text{Int } D \cap g' = \emptyset \) and we can choose a local coordinate of \( \mathbb{R}^3 \) around the tubular neighborhood of \( D \) so that

1. a tubular neighborhood of \( g(\gamma^0([0, 1])) \) in \( \mathbb{R}^3 \) corresponds to \( \{(x_1, x_2, x_3, 0, 0) \mid x_1^2 + x_2^2 < r, \ |x_3| \leq 1 \} \) \( (r > 0 \text{ small}) \),
(2) a tubular neighborhood of \( g'(\gamma^1([0, 1])) \) in \( \mathbb{R}^3 \) corresponds to \( \{(x_1, x_2, 0, x_1^2 - x_2^2 + 1, x_5) \mid x_1^2 + x_2^2 < r, |x_5| \leq 1 \}. \)

(3) \( D \) corresponds to \( \{(0, x_2, 0, x_4, 0) \mid 0 \leq x_4 \leq 1 - x_2^2 \} \)

(see Figure 6.4). This coordinate coincides with that in Proposition 6.2 and there exists an indeﬁnite self-tangency between \( A_1 \) and \( A_2 \) occurs. By Lemma 6.5, \( E \) jumps by \( \pm (lk(K_0^1, K_0^2) + lk(K_1^1, K_1^2))/4 = \pm lk(K_1, K_2)/2 \) at this self-tangency. This jump can be arbitrarily large. \( \square \)

**Proof of Corollary 2.6** The immersion \( g \) constructed in the proof of Theorem 2.10 can be lifted to \( f \in \mathcal{K}_{6,3} \) by lifting the \( g' \)-part into \( \mathbb{R}^3 \times \mathbb{R}_+ \). Since \( g' \)'s are isotopic to the standard embedding, we see that \( \mathcal{H}(f) = 0 \). Changing the crossing at \( A_1 \), by (2.28) we have

\[
0 - \mathcal{H}(f_{(1)}) = \frac{1}{2}(lk(K_0^1, K_0^2) - lk(K_0^1, K_1^2) - lk(K_1^1, K_0^2) + lk(K_0^1, K_1^2)) = lk(K_1, K_2).
\]

Here we use the fact that \( K_0^1 \) and \( K_0^2 \) are separated and that \( K_0^1 \sqcup K_0^2 \) is isotopic to the given link \( L = K_1 \sqcup K_2 \). This means that an embedding \( \mathbb{R}^3 \hookrightarrow \mathbb{R}^6 \) with arbitrary Haefliger’s invariant can be obtained by a single crossing change from the trivial embedding. \( \square \)

**Appendix A. The Jacobian in the proof of Proposition 5.3**

The Jacobian matrix \( J(\varphi_x) \) at \( \vec{z} \) in the proof of Proposition 5.3 is given by

\[
\begin{pmatrix}
I_{2k-1} & -I_{2k-1} & \cdots & -I_{2k-1} \\
-I_{2k} & I_{2k} & \cdots & -I_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
-I_{2k-1} & -I_{2k-1} & \cdots & I_{2k-1} \\
I_{2k-1} & 0 & \cdots & -I_{2k-1}
\end{pmatrix}
\]

where \( I_N \) is the \( N \times N \)-identity matrix and \( \vec{0} \in \mathbb{R}^{2k-1} \) is the zero vector. The rows correspond to the bases of \( T_{(x_1, x_2, \ldots, x_{2k-1})}(S^{6k-1} \times S^{6k-1} \times S^{4k-2}) \) and the columns correspond to the natural basis of \( T_{\mathcal{C}} \text{Conf}_4(\mathbb{R}^{6k-4}) \cong T_{\mathcal{C}}\mathbb{R}^{16k-4} \). Its determinant is \(-1\).
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