Deriving conservation laws for ABS lattice equations from Lax pairs

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Abstract

In this paper we derive infinitely many conservation laws for ABS lattice equations from their Lax pairs. These conservation laws can be algebraically expressed by means of some known polynomials. For each equation, the infinitely many conservation laws are not equivalent and are nontrivial. We also show that the (H1), (H2), (H3), (Q1), (Q2), (Q3) and (A1) equations in the ABS list share a generic discrete Riccati equation.

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1. Introduction

Multidimensional consistency can be treated as an integrability criterion and was presented for the first time in [1]. The celebrated ABS list [2] consists in total of nine quadrilateral lattice equations which are consistent around a cube (CAC) satisfying the extra restrictions of multi-linearity, $D_4$ symmetry and the tetrahedron property. A 3D-consistent equation itself provides a Lax pair as well as an auto Bäcklund transformation (BT) [2–4]. However, it seems hard for such a Lax pair to play as useful a role as in the continuous case, where from Lax pairs one can usually derive solutions either through inverse scattering transform [5] or Darboux transformation [6] and construct BTs [7], evolution equation hierarchies, commutative flows and infinitely many symmetries [8] and conservation laws [7].

With regard to infinitely many conservation laws, which serve as an integrability characteristic, there are many ways of deriving them for continuous and semi-discrete integrable systems [7, 9–13]. Using a Lax pair is a simple way of doing this [7, 12, 13]. For ABS lattice equations, their conservation laws have been derived through a direct approach [14] based on the idea of [15], a symmetry approach [16–18], Gardner’s approach (using BTs and initial conservation laws) [18, 19], and by using quasi-difference operators and recursion operators [20] etc. In this paper, we will start from Lax pairs to derive infinitely
many conservation laws for ABS lattice equations. In [12] (see also [13]) we introduce two kinds of techniques to construct conservation laws for the Toda lattice and Ablowitz–Ladik system, respectively. In fact, the Gardner method used in [19] is closely related to the technique used for the Ablowitz–Ladik system [12]. However, one will see that we can easily write out the so-called initial conservation laws from Lax pairs, and infinitely many conservation laws can be algebraically expressed by means of known polynomials. We also find that many ABS lattice equations share a generic discrete Riccati equation.

This paper is organized as follows. In section 2 we introduce the ABS list and the main idea of our approach. In section 3 the first (H1) equation serves as a detailed example for deriving conservation laws. We then list the main results for the (H2), (H3), (Q1), (Q2), (Q3) and (A1) equations. Finally, also in this section, we derive conservation laws for the (A2) and (Q4) equations in a slightly different way, but still starting from Lax pairs. Section 4 contains conclusions and discussions.

2. Preliminary and general description

Let us start from the following quadrilateral equation

$$Q(u, \tilde{u}, \tilde{u}, p, q) = 0,$$  \hspace{1cm} (2.1)

where

$$u = u(n, m), \quad \tilde{u} = E_{n}u = u(n + 1, m), \quad \hat{u} = E_{m}u = u(n, m + 1), \quad \hat{\tilde{u}} = u(n + 1, m + 1),$$

$E_{n}$ and $E_{m}$ serve as shift operators in directions $n$ and $m$, respectively, and $p$ and $q$ are spacing parameters of direction $n$ and $m$, respectively.

The ABS list reads [2]

\begin{itemize}
  \item [(H1)] \((u - \tilde{u})(\tilde{u} - \hat{\tilde{u}}) + q - p = 0,\)
  \item [(H2)] \((u - \hat{\tilde{u}})(\tilde{u} - \hat{\tilde{u}}) + (q - p)(u + \tilde{u} + \hat{\tilde{u}}) + q^{2} - p^{2} = 0,\)
  \item [(H3)] \(p(u\tilde{u} + \tilde{u}\hat{\tilde{u}}) - q(u\tilde{u} + \tilde{u}\hat{\tilde{u}}) + \delta(p^{2} - q^{2}) = 0,\)
  \item [(A1)] \(p(u + \tilde{u})(\tilde{u} + \hat{\tilde{u}}) - q(u + \tilde{u})(\tilde{u} + \hat{\tilde{u}}) - \delta^{2}pq(p - q) = 0,\)
  \item [(A2)] \((q^{2} - p^{2})(u\tilde{u} + \tilde{u}\hat{\tilde{u}} + 1) + q(p^{2} - 1)(u\tilde{u} + \tilde{u}\hat{\tilde{u}}) - p(q^{2} - 1)(u\tilde{u} + \tilde{u}\hat{\tilde{u}}) = 0,\)
  \item [(Q1)] \(p(u - \tilde{u})(\tilde{u} - \hat{\tilde{u}}) - q(u - \tilde{u})(\tilde{u} - \hat{\tilde{u}}) + \delta^{2}pq(p - q) = 0,\)
  \item [(Q2)] \((q^{2} - p^{2})(u\tilde{u} + \tilde{u}\hat{\tilde{u}} + 1) + q(p^{2} - 1)(u\tilde{u} + \tilde{u}\hat{\tilde{u}}) - p(q^{2} - 1)(u\tilde{u} + \tilde{u}\hat{\tilde{u}})
  - \delta^{2}(p^{2} - q^{2})(p^{2} - 1)(q^{2} - 1)/(4pq) = 0,\)
  \item [(Q3)] \(p(u\tilde{u} + \tilde{u}\hat{\tilde{u}}) - q(u\tilde{u} + \tilde{u}\hat{\tilde{u}}) - \frac{pQ - qP}{1 - p^{2}q^{2}}\left[(\hat{\tilde{u}}\tilde{u} + u\tilde{u}) - pq(1 + u\tilde{u}\tilde{u})\right] = 0,\)
\end{itemize}

where equation (Q4) is of the form given by Hietarinta [21] (see also [22]), and in the equation

\begin{itemize}
  \item \(P^{2} = p^{4} - kp^{2} + 1, \quad Q^{2} = q^{4} - kq^{2} + 1.\)
\end{itemize}

If (2.1) is a CAC equation, then it is easy to write out its BT (which is referred to as the natural auto BT in [23])

$$Q(u, \tilde{u}, \overline{u}, \overline{u}, p, r) = 0,$$  \hspace{1cm} (2.2a)
where \( r \) is referred to as a Bäcklund parameter, and if \( u \) solves (2.1), so does \( \bar{u} \). Replacing \( \bar{u} \) by \( \phi_1/\phi_2 \), the above BT can be rewritten in terms of \( \phi = (\phi_1, \phi_2)^T \) as the following [3],

\[
\begin{align*}
\tilde{\phi} = L_1\phi &= \frac{\beta}{\sqrt{|M|}} M(u, \bar{u}, p, r)\phi, \\
\tilde{\phi} = L_2\phi &= \frac{\gamma}{\sqrt{|N|}} N(u, \bar{u}, q, r)\phi,
\end{align*}
\]

(2.3a)

where \( M \) and \( N \) are \( 2 \times 2 \) matrices, \( \beta \) and \( \gamma \) are constants and factors \( \frac{1}{\sqrt{|M|}} \) and \( \frac{1}{\sqrt{|N|}} \) are chosen to guarantee \( |L_1| \) as constant (cf [3]), which consequently guarantees the compatibility \( |L_1L_2| = |L_2L_1| \). Constants \( \beta \) and \( \gamma \) can be arbitrary but play key roles in obtaining a discrete Riccati equation that can be easily solved. Equations (2.3) can be a Lax pair of equation (2.1) and \( r \) serves as a spectral parameter. A conservation law of (2.1) is defined by (cf [14–16], and particularly, [20])

\[
\Delta_m F(u) = \Delta_m J(u),
\]

(2.4)

where \( \Delta_n = E_n - 1 \), \( \Delta_m = E_m - 1 \) and \( u \) solves equation (2.1). The above conservation law is trivial [20] if it is the gradient of some function \( H(n, m) \), i.e. \( F(u) = \Delta_n H(n, m) \) and \( J(u) = \Delta_m H(n, m) \).

From the Lax pair (2.3) we construct a formal conservation law in the following way. First we define

\[
\theta = \frac{\tilde{\phi}_2}{\phi_2}, \quad \eta = \frac{\tilde{\phi}_1}{\phi_2},
\]

(2.5)

Then, noting that

\[
\ln \theta = \Delta_n \ln \phi_2, \quad \ln \eta = \Delta_m \ln \phi_2,
\]

(2.6)

we immediately reach

\[
\Delta_m \ln \theta = \Delta_n \ln \eta,
\]

(2.7)

which is a formal conservation law of the lattice equation related to the Lax pair (2.3). Here by the word ‘formal’ we mean that \( \theta \) and \( \eta \) are still obscure and once their explicit forms are settled, (2.7) can be a starting point to generate infinitely many conservation laws. In a continuous case with a \( 2 \times 2 \) Lax pair, it is also quite natural to write out a formal conservation law that can be easily solved. Equations (2.3) can be a Lax pair of equation (2.1) and \( r \) serves as a spectral parameter. A conservation law of (2.1) is defined by (cf [14–16], and particularly, [20])

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To determine \( \theta \), we find that for lattice equations (H1), (H2), (H3), (A1), (Q1), (Q2) and (Q3) in the ABS list, \( \theta \) satisfies a discrete Riccati equation of the following form,

\[
\tilde{\mu} \tilde{\theta} \theta = (u - \tilde{u})\theta - \varepsilon^2 \mu,
\]

(2.8)

where \( \mu \) is a function of \( u, \tilde{u} \) related to the considered equations and \( \varepsilon \) is a constant related to \( p, r \). In fact, one can suppose in (2.3a) that

\[
M = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

From (2.3a) one can always get

\[
\sqrt{|M|} \tilde{\phi}_2 = \beta \tilde{C} \left( \frac{A}{C} + \frac{\tilde{D}}{C} \right) \tilde{\phi}_2 - \beta^2 \sqrt{|M|} \tilde{C} \frac{\tilde{C}}{C} \tilde{\phi}_2.
\]

(2.9)
For (H1), (H2), (H3), (A1), (Q1), (Q2) and (Q3), the above equation can be written out into a
clear form (also see [26])

\[ \tilde{\mu} \tilde{\phi}_2 = (u - \tilde{u})\tilde{\phi}_2 - \varepsilon^2 \mu \phi_2 \]

with \( \mu \) being functions of \( u \) and \( \tilde{u} \). This equation, divided by \( \phi_2 \), yields the discrete Riccati
equation of the form (2.8).

For the discrete Riccati equation (2.8), it is not difficult to verify the following.

**Proposition 1.** The discrete Riccati equation (2.8) is solved by

\[ \theta = \varepsilon^2 \rho \left( 1 + \sum_{j=1}^{\infty} \theta_j \varepsilon^{2j} \right) \]

with \( \rho = \frac{\mu}{u - \tilde{u}} \).

\[ \theta_{j+1} = \frac{\tilde{\mu} \tilde{\rho}}{u - \tilde{u}} \sum_{i=0}^{j} \tilde{\theta}_i \theta_{j-i}, \quad j = 0, 1, 2, \ldots, (\theta_0 = 1). \]

This gives an explicit form of \( \theta \), but it is not enough to obtain infinitely many conservation
laws from (2.7). We still need an explicit \( \eta \). However, we cannot insert \( \eta \) into a Riccati equation
similar to (2.8) with the same \( \varepsilon \) because \( \varepsilon \) is independent of \( q \). Let us find a relationship between
\( \eta \) and \( \theta \). To do this, we write \( N \) in the Lax pair (2.3) as

\[ N = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}, \]

where \( A', B', C', D' \) are nothing but \( A, B, C, D \) with the replacement \( (p, \tilde{u}) \rightarrow (q, \hat{u}) \). From
the Lax pair (2.3) we can have

\[ \tilde{\phi}_2 = \frac{\beta}{\sqrt{|M|}} (C\phi_1 + D\phi_2), \quad \hat{\phi}_2 = \frac{\gamma}{\sqrt{|N|}} (C'\phi_1 + D'\phi_2), \]

and further

\[ \theta = \frac{\beta}{\sqrt{|M|}} \left( C \frac{\phi_1}{\phi_2} + D \right), \quad \eta = \frac{\gamma}{\sqrt{|N|}} \left( C' \frac{\phi_1}{\phi_2} + D' \right). \]

After eliminating \( \hat{\phi}_2 \) we reach the relation

\[ \eta = \omega (\sigma \theta + 1), \]

where

\[ \omega = \frac{\gamma C (D' - \frac{\mu}{\rho})}{\sqrt{|N|}}, \quad \sigma = \frac{\sqrt{|M|}}{\beta C (D' - \frac{\mu}{\rho})}. \]

We note that both \( \omega \) and \( \sigma \) are functions of \( (u, \tilde{u}, \tilde{u}, p, q) \) and they satisfy

\[ \frac{1}{\omega(u, \tilde{u}, \tilde{u}, p, q)} = -\sigma(u, \tilde{u}, \tilde{u}, q, p), \]

if \( \beta \leftrightarrow \gamma \) under the interchange \( p \leftrightarrow q \), i.e. \( \beta(p, r) \leftrightarrow \gamma(q, r) \).

With both explicit \( \theta \) and \( \eta \) in hand, by substituting them into the formal conservation
law (2.7) it is possible to obtain explicit infinitely many conservation laws. To do that, we
make use of the following expansion formula.
Proposition 2. The following expansion holds,

\[
\ln \left( 1 + \sum_{j=1}^{\infty} t_j k^j \right) = \sum_{j=1}^{\infty} h_j(t) k^j, \tag{2.16a}
\]

where

\[
h_j(t) = \sum_{|\alpha|=j} (-1)^{|\alpha|-1} (|\alpha| - 1)! \frac{t^\alpha}{\alpha!}, \tag{2.16b}
\]

and

\[
t = (t_1, t_2, \ldots), \quad \alpha = (\alpha_1, \alpha_2, \ldots), \quad \alpha_i \in \{0, 1, 2, \ldots\}, \tag{2.16c}
\]

\[
t^\alpha = \prod_{i=1}^{\infty} t_i^{\alpha_i}, \quad \alpha! = \prod_{i=1}^{\infty} (\alpha_i!) \quad \text{and} \quad \sum_{|\alpha|=m} |\alpha| = \sum_{i=1}^{\infty} i \alpha_i. \tag{2.16d}
\]

The first few of \(h_j(t)\) are

\[
h_1(t) = t_1, \tag{2.17a}
\]

\[
h_2(t) = -\frac{1}{2} t_1^2 + t_2, \tag{2.17b}
\]

\[
h_3(t) = \frac{1}{4} t_1^3 - t_1 t_2 + t_3, \tag{2.17c}
\]

\[
h_4(t) = -\frac{1}{8} t_1^4 + \frac{1}{2} t_1^2 t_2 - t_1 t_3 - \frac{1}{2} t_2^2 + t_4. \tag{2.17d}
\]

We note that \(h_j(t)\) also satisfy

\[
\partial_i h_{i+j}(t) = \partial_j h_{i+j}(t), \quad \text{for} \quad i, j, s \in \mathbb{Z}^+, \tag{2.18}
\]

and \(\{h_j(t)\}\) are different from the Schur function (see [27]).

Proof. Let us first prove the following expansion:

\[
\left( \sum_{i=1}^{\infty} y_i \right)^s = s! \sum_{|\alpha|=s} \frac{y^\alpha}{\alpha!}, \quad (s \in \mathbb{Z}^+), \tag{2.19a}
\]

\[
y = (y_1, y_2, \ldots), \quad y^\alpha = \prod_{i=1}^{\infty} y_i^{\alpha_i}. \tag{2.19b}
\]

Obviously, (2.19a) is valid for \(s = 1\). Taking the derivative for (2.19a) with respect to \(y_k\) and supposing that (2.19a) is right for \(s - 1\), one finds

\[
\frac{\partial}{\partial y_k} \text{lhs of (2.19a)} = s \left( \sum_{j=1}^{\infty} y_j \right)^{s-1} = s \cdot (s - 1)! \sum_{|\alpha|=s-1} \frac{y^\alpha}{\alpha!} = s! \sum_{|\alpha|=s-1} \frac{y^\alpha}{\alpha!},
\]

and meanwhile

\[
\frac{\partial}{\partial y_k} \text{rhs of (2.19a)} = s! \sum_{|\alpha|=s} \frac{y_k^{\alpha_k-1}}{(\alpha_k - 1)!} \prod_{j \neq k} \frac{y_j^{\alpha_j}}{\alpha_j!} = s! \sum_{|\alpha|=s-1} \frac{y^\alpha}{\alpha!}.
\]

This means that (2.19a) is valid for any \(s \in \mathbb{Z}^+\). Then, noting that

\[
\ln \left( 1 + \sum_{j=1}^{\infty} t_j k^j \right) = s \left( \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1}{j} \left( \sum_{i=1}^{\infty} t_i k^i \right)^{s-1} \right),
\]

using (2.19a) with \(y_i = t_i k^i\), and rearranging the expansion in terms of \(k\), we get (2.16a). \(\Box\)

With this proposition, from the formal conservation law (2.7) we have the following.
Proposition 3. When $\theta$ and $\eta$ are defined by (2.11) and (2.13), respectively, the formal conservation law (2.7) yields infinitely many conservation laws,

$$\Delta_m \ln \rho = \Delta_n \ln \omega,$$

$$\Delta_m h_s(\theta) = \Delta_n h_3(\rho \sigma \theta), \text{ (} s = 1, 2, 3, \ldots \text{)} ,$$

where $h_s(t)$ is defined in (2.16b),

$$\theta = (\theta_1, \theta_2, \ldots), \quad \eta = (1, \bar{\theta}_1, \bar{\theta}_2, \ldots),$$

and $\rho \sigma \theta = (\sigma \rho, \sigma \rho \theta_1, \sigma \rho \theta_2, \ldots)$. 

3. Conservation laws of ABS lattice equations

3.1. Conservation laws for the (H1), (H2), (H3), (Q1), (Q2), (Q3) and (A1) equations

Let us first, taking (H1) equation as an example, give a detailed procedure of deriving infinitely many conservation laws.

The Lax pair of the (H1) equation reads

$$\tilde{\phi} = \frac{\beta}{\sqrt{r-p}} \left( \begin{array}{cc} u & -u\tilde{u} + p - r \\ 1 & -\tilde{u} \end{array} \right) \phi,$$

$$\hat{\phi} = \frac{\gamma}{\sqrt{r-q}} \left( \begin{array}{cc} u & -u\hat{u} + q - r \\ 1 & -\hat{u} \end{array} \right) \phi.$$ 

Taking $\beta = \sqrt{r-p} = _{\varepsilon}$, from (3.1a) we can find

$$\tilde{\phi}_2 = (u - \tilde{u})\tilde{\phi}_2 - \varepsilon^2 \phi_2,$$

which leads to

$$\tilde{\theta} = (u - \tilde{u})\tilde{\theta} - \varepsilon^2,$$

where $\theta$ is defined in (2.5), i.e., $\theta = \tilde{\phi}_2/\phi_2$. This is the discrete Riccati equation (2.8) with $\mu = 1$ and is solved by (2.11). Further, taking $\gamma = \sqrt{r-q}$, from the Lax pair (3.1) we have

$$\theta = \frac{\tilde{\phi}_2}{\phi_2} = \frac{\phi_1}{\phi_2} - \tilde{u}, \quad \eta = \frac{\phi_2}{\phi_2} = \frac{\phi_1}{\phi_2} - \hat{u},$$

which by eliminating $\phi_1/\phi_2$ yields the relation

$$\eta = \theta + \bar{u} - \tilde{u}.$$ 

This is (2.13) with $\omega = \tilde{u} - \bar{u}$ and $\sigma = 1/(\tilde{u} - \bar{u})$. Thus, for the (H1) equation, based on proposition 3 we can write out infinitely many conservation laws (2.20) with $\mu, \omega, \sigma$ obtained above. We note that these conservation laws are the same as those derived via the Gardner method [18].

For the lattice equations (H2), (H3), (A1), (Q1), (Q2) and (Q3), starting from their Lax pairs, we can also derive infinitely many conservation laws through a similar procedure. Let us skip the details and list the main results of these equations together with the (H1) equation.

Proposition 4. For the lattice equations (H1), (H2), (H3), (A1), (Q1), (Q2) and (Q3) in the ABS list, starting from their Lax pairs, one can construct a formal conservation law (2.7) with $\theta$ and $\eta$ defined in (2.5), where $\theta$ satisfies the discrete Riccati equation (2.8) solved by (2.11) and $\eta$ is expressed through (2.13). By means of the polynomials $\{h_3(t)\}$ defined in (2.16b), one can explicitly express the infinitely many conservation laws as (2.20). In the following, for the (H1), (H2), (H3), (A1), (Q1), (Q2), and (Q3) equations, we list out the parametrization of...
3.2. Conservation laws for the (A2) equation

3.2.1. Transformation. The transformation [2]

\[ u = v^{(-1)^{n+m}} \tag{3.7} \]

connects Q3|_{\beta=0} equation

\[ (q^2 - p^2)(u\ddot{u} + a\dot{u}) + q(p^2 - 1)(u\ddot{u} + a\dot{u}) - p(q^2 - 1)(u\ddot{u} + a\dot{u}) = 0 \tag{3.10} \]

and (A2) equation

\[ (q^2 - p^2)(v\ddot{v} + 1) + q(p^2 - 1)(v\ddot{v} + \dot{v}^2) - p(q^2 - 1)(v\ddot{v} + \dot{v}^2) = 0. \tag{3.11} \]

Noting that the conservation law (2.4) of equation (2.1) is a relation that holds for all of \( u \) satisfying (2.1), for the (A2) equation (3.11) what we need is to list out the conservation laws of Q3|_{\beta=0} equation (3.10) and then replace \( u \) by \( v^{(-1)^{n+m}} \).
Proposition 5. The infinitely many conservation laws of the $Q^3|_{\lambda=0}$ equation (3.10) is given by (2.20) with $\theta$, $\eta$ and $[h_j(t)]$ given in (2.11), (2.13) and (2.16b), respectively, and

$$\mu = \sqrt{(u-p\tilde{u})(pu-\tilde{u})},$$  \hspace{1cm} (3.12a)$$
$$\omega = \frac{p(q^2-1)\tilde{u} + (p^2-q^2)u - q(p^2-1)\tilde{u}}{(p^2-1)\sqrt{(u-q\tilde{u})(qu-u)}}, \quad \sigma = \frac{(q^2-1)\sqrt{(u-p\tilde{u})(pu-u)}}{p(q^2-1)\tilde{u} + (p^2-q^2)u - q(p^2-1)\tilde{u}}.$$  \hspace{1cm} (3.12b)

The infinitely many conservation laws of the (A2) equation (3.11) can be given through the infinitely many conservation laws of the $Q^3|_{\lambda=0}$ equation (3.10) by replacing $u$ with $v^{(-1)^{n+1}}$.

3.2.2. The Lax pair approach. Conservation laws of the (A2) equation can also be derived directly from its Lax pair which reads

$$\tilde{\phi} = \frac{\beta}{\sqrt{A}} \begin{pmatrix} -r(p^2-1)u & p(r^2-1)\tilde{u} - (r^2-p^2) \end{pmatrix} \phi,$$  \hspace{1cm} (3.13a)$$
$$\tilde{\phi} = \frac{\gamma}{\sqrt{B}} \begin{pmatrix} -r(q^2-1)u & q(r^2-1)\tilde{u} - (r^2-q^2) \end{pmatrix} \phi,$$  \hspace{1cm} (3.13b)

with

$$A = (r^2-1)(r^2-p^2)(p-u\tilde{u})(pu-\tilde{u}), \quad B = (r^2-1)(r^2-q^2)(q-u\tilde{u})(qu-\tilde{u}).$$

In this case, equation (2.9) becomes

$$\sqrt{A} \phi_2 \equiv \beta r p(1-r^2) \frac{(r^2-1)(u-\tilde{u})}{p(1-r^2) + (r^2-p^2)u\tilde{u}} \tilde{\phi}_2 - \beta^2 \sqrt{A} \frac{p(1-r^2) + (r^2-p^2)u\tilde{u}}{p(1-r^2) + (r^2-p^2)u\tilde{u}} \phi_2.$$  \hspace{1cm} (3.14)

However, it is hard to find parametrization for $\beta$ to derive a Riccati equation of the form (2.8), unless we implement the transformation $u = w^{(-1)^n}$ originally in the (A2) equation. In that case the Riccati equation will be the same as that for $Q^3|_{\lambda=0}$ (with $w$ in place of $u$), and the conservation laws will be the same as described in proposition 5.

To derive an explicit $\theta = \tilde{\phi}_2/\phi_2$ without using the transformation $u = w^{(-1)^n}$, we turn to the following equation set

$$\theta = \frac{\tilde{\phi}_2}{\phi_2} = \frac{1}{\mu} [((r^2-p^2)u\tilde{u} - p(r^2-1))\xi + r(p^2-1)\tilde{u}],$$  \hspace{1cm} (3.15a)$$
$$\eta = \frac{\tilde{\phi}_2}{\phi_2} = \frac{1}{\nu} [((r^2-q^2)u\tilde{u} - q(r^2-1))\xi + r(q^2-1)\tilde{u}].$$  \hspace{1cm} (3.15b)

This is derived from the Lax pair (3.13), where

$$\xi = \frac{\phi_1}{\phi_2},$$  \hspace{1cm} (3.15c)$$
$$\mu = \sqrt{(p-u\tilde{u})(pu-\tilde{u})}, \quad \nu = \sqrt{(q-u\tilde{u})(qu-\tilde{u})},$$  \hspace{1cm} (3.15d)

and we have taken

$$\beta = \sqrt{(r^2-1)(r^2-p^2)}, \quad \gamma = \sqrt{(r^2-1)(r^2-q^2)}.$$
Proposition 6. For the (A2) equation $\theta = \frac{\delta \xi}{\delta z}$ and $\eta = \frac{\delta w}{\delta z}$ can be expressed by

$$\theta = \left( f_0 \xi_1 + g_1 \right) \xi \left( 1 + \sum_{j=1}^{\infty} \theta_j \xi^j \right),$$

(3.16a)

with

$$\theta_1 = \frac{f_0 \xi_2 + f_1 \xi_1 + g_2}{f_0 \xi_1 + g_1},$$

(3.16b)

$$\theta_s = \frac{f_0 \xi_{s+1} + f_1 \xi_s + f_2 \xi_{s-1}}{f_0 \xi_1 + g_1}, \quad (s = 2, 3, \ldots),$$

(3.16c)

and

$$\eta = \frac{z_0}{v} \left( 1 + \sum_{j=1}^{\infty} \eta_j \xi^j \right),$$

(3.17a)

with

$$\eta_1 = \frac{1}{z_0} (w_0 \xi_1 + z_1),$$

(3.17b)

$$\eta_2 = \frac{1}{z_0} (w_0 \xi_2 + w_1 \xi_1 + z_2),$$

(3.17c)

$$\eta_s = \frac{1}{z_0} \sum_{j=0}^{2} w_j \xi_{s-j}, \quad (s = 3, 4, \ldots),$$

(3.17d)

where

$$f_0 = p(1 - p), \quad f_1 = 2p(u\tilde{u} - p), \quad f_2 = u\tilde{u} - p,$$

$$g_1 = (2pu\tilde{u} - p^2 - 1)\tilde{u}, \quad g_2 = (u\tilde{u} - p)\tilde{u},$$

$$w_0 = \frac{(p^2 - q^2)}{2} u\tilde{u} - q(p^2 - 1), \quad w_1 = 2p(u\tilde{u} - q), \quad w_2 = u\tilde{u} - q,$$

$$z_0 = \frac{(p^2 - q^2)}{2} u\tilde{u} - pq(u\tilde{u} - q) + q\tilde{u} - p\tilde{u},$$

$$z_1 = 2p\tilde{u}(u\tilde{u} - q) + (q^2 - 1)u\tilde{u}, \quad z_2 = (u\tilde{u} - q)\tilde{u},$$

(3.18)

and $\xi$ is given by

$$\xi = \sum_{j=1}^{\infty} \xi_j \xi^j,$$

(3.19a)

with

$$\xi_1 = -\frac{d_1}{c_0},$$

(3.19b)

$$\xi_2 = -\frac{1}{c_0} (a_0 \xi_1 \tilde{\xi} + b_1 \tilde{\xi}_1 + c_1 \xi_1 + d_2),$$

(3.19c)

$$\xi_3 = -\frac{1}{c_0} [a_0 (\xi_1 \tilde{\xi}_2 + \xi_2 \tilde{\xi}_1) + a_1 \xi_1 \tilde{\xi}_1 + b_1 \tilde{\xi}_1 + b_2 \tilde{\xi}_1 + c_1 \xi_2 + c_2 \xi_1],$$

(3.19d)

$$\xi_s = -\frac{1}{c_0} \left( \sum_{k=0}^{2} a_k \sum_{i=1}^{s-k-1} \xi^i \tilde{\xi}_{s-k-i} + \sum_{k=1}^{2} b_k \tilde{\xi}_{s-k} + \sum_{k=1}^{2} c_k \xi_{s-k} \right), \quad (s = 4, 5, \ldots),$$

(3.19e)
and
\begin{align}
a_0 &= -p(p^2 - 1), \quad a_1 = 2p(u\tilde{u} - p), \quad a_2 = u\tilde{u} - p, \\
b_1 &= (2pu\tilde{u} - p^2 - 1)\tilde{u}, \quad b_2 = (u\tilde{u} - p)\tilde{u}, \\
c_0 &= p(p^2 - 1)(u - \tilde{u}), \quad c_1 = 2p\tilde{u}(u\tilde{u} - p) + (p^2 - 1)u, \quad c_2 = (u\tilde{u} - p)\tilde{u}, \\
d_1 &= 2p(u\tilde{u} + 1)\tilde{u} - (p^2 + 1)(u\tilde{u} + \tilde{u}) + 1 - p(u\tilde{u} + \tilde{u}).
\end{align}

**Proof.** Let us go back to (3.15) in which \( \xi \) is determined by the equation
\[ \tilde{\xi} = \frac{-r(p^2 - 1)u \xi + p(r^2 - 1)u\tilde{u} - (r^2 - p^2)}{(r^2 - p^2)u\tilde{u} - p(r^2 - 1)u}, \]
which is derived from (3.13a). To solve it, we take (cf [19])
\[ \xi = \tilde{u} + \xi, \quad \epsilon = r - p, \]
and we reach
\[ (d_0 + a_1 \epsilon + a_2 \epsilon^2)\tilde{\xi} + (b_1 \epsilon + b_2 \epsilon^2)\tilde{\xi} + (c_0 + c_1 \epsilon + c_2 \epsilon^2)\xi + (d_1 \epsilon + d_2 \epsilon^2) = 0, \]
where \( \{a_i, b_i, c_i, d_i\} \) are given in (3.20). We can verify that \( \xi \) defined in (3.19) solves the above equation. Then, noting that (3.15) can be re-expressed in terms of \( \xi \) as
\begin{align}
\theta &= \frac{1}{\mu} \left[ (f_0 + f_1 \epsilon + f_2 \epsilon^2)\tilde{\xi} + g_1 \epsilon + g_2 \epsilon^2 \right], \\
\eta &= \frac{1}{v} \left[ (w_0 + w_1 \epsilon + w_2 \epsilon^2)\tilde{\xi} + z_0 + z_1 \epsilon + z_2 \epsilon^2 \right],
\end{align}
with (3.18). Rearranging the right-hand sides with respect to the powers of \( \epsilon \) we immediately arrive at the expressions (3.16) and (3.17) for \( \theta \) and \( \eta \).

The formal conservation law is still written as
\[ \Delta_m \ln \theta = \Delta_n \ln \eta. \]

By the above proposition and the polynomials \( \{h_j(t)\} \) defined in (2.16b), infinitely many conservation laws can be derived.

**Proposition 7.** The infinitely many conservation laws of the (A2) equation are given by
\begin{align}
\Delta_m \ln \frac{f_0 \tilde{\xi} + f_1}{\mu} &= \Delta_n \ln \frac{\xi}{v}, \\
\Delta_m h_s(\theta) &= \Delta_n h_s(\eta), \quad s = 1, 2, \ldots,
\end{align}
where
\[ \theta = \{\theta_1, \theta_2, \ldots\}, \quad \eta = \{\eta_1, \eta_2, \ldots\}, \]
with \( \mu \) and \( v \) given by (3.15d), and \( \{\theta_j\} \) and \( \{\eta_j\} \) given by proposition 6.

It is not easy to examine the relationship of the conservation laws given by propositions 5 and 7. For the simplest case we can show that they are the same. The simplest conservation laws from these two propositions are respectively
\begin{align}
\Delta_m \ln \frac{\sqrt{(u^{-1})^{y+1} - p\tilde{u}^{-1})^{y+1}})}{(u^{-1})^{y+1} - \tilde{u}^{-1})^{y+1}} + (p^2 - q^2)u^{-1})^{y+1} - q(p^2 - 1)\tilde{u}^{-1})^{y+1} + (p^2 - 1)\tilde{u}^{-1})^{y+1}} \\
&= \Delta_n \ln \frac{p(q^2 - 1)\tilde{u}^{-1})^{y+1} + (p^2 - q^2)u^{-1})^{y+1} - q(p^2 - 1)\tilde{u}^{-1})^{y+1} + (p^2 - 1)\tilde{u}^{-1})^{y+1}} \\
&= \Delta_n \ln \frac{\sqrt{(u^{-1})^{y+1} - q\tilde{u}^{-1})^{y+1}})}{(u^{-1})^{y+1} - \tilde{u}^{-1})^{y+1}} + (p^2 - q^2)u^{-1})^{y+1} - q(p^2 - 1)\tilde{u}^{-1})^{y+1} + (p^2 - 1)\tilde{u}^{-1})^{y+1}}
\end{align}
and
\[ \Delta_m \ln \frac{\sqrt{(p - uu)(p - uu)\beta}}{u - u} + \Delta_n \ln \frac{p(q^2 - 1)\hat{u} + (p^2 - q^2)uu\hat{u} - q(p^2 - 1)\hat{u}}{\hat{u}(q - uu)(q - uu) - q(p^2 - 1)\hat{u}} \] (3.27)
which is given in appendix A. Let us suppose \( n + m \) to be even without loss of generality. Then the lhs of (3.26) reads
\[ \Delta_m \ln \frac{\sqrt{(p - uu)(p - uu)\beta}}{u - u} + \ln(-\hat{u}u), \] (3.28)
while the rhs is
\[ E_n \ln \frac{p(q^2 - 1)\hat{u} + (p^2 - q^2)uu\hat{u} - q(p^2 - 1)\hat{u}}{\hat{u}(q - uu)(q - uu) - q(p^2 - 1)\hat{u}}. \]

Next, we eliminate \( p^2 - q^2 \) from the first term by using the (A2) equation
\[ p^2 - q^2 = (q^2 - p^2)uu\hat{u} + q(p^2 - 1)(uu\hat{u} + \hat{u}u) - p(q^2 - 1)(uu\hat{u} + \hat{u}u), \]
and the rhs is now
\[ \Delta_n \ln \frac{p(q^2 - 1)\hat{u} + (p^2 - q^2)uu\hat{u} - q(p^2 - 1)\hat{u}}{\hat{u}(q - uu)(q - uu) - q(p^2 - 1)\hat{u}} + \ln(-\hat{u}u). \] (3.29)

This, together with (3.28), composes a conservation law which is the same as (3.27).

### 3.3. Conservation laws for the (Q4) equation

For the (Q4) equation, one can use the same method as in section 3.2.2. The Lax pair of (Q4) equation is
\[ \widetilde{\phi} = \frac{\beta}{\sqrt{A}} \left( r(1 - p^2 r^2)u + (pR - rP)\tilde{u} - p(1 - p^2 r^2)uu\tilde{u} - pr(pR - rP)u \right) \phi, \] (3.30a)
\[ \tilde{\phi} = \frac{\gamma}{\sqrt{B}} \left( r(1 - q^2 r^2)u + (qR - rQ)\tilde{u} - q(1 - q^2 r^2)uu\tilde{u} - qr(qR - rQ)u \right) \phi, \] (3.30b)
with
\[ A = r(1 - p^2 r^2)(pR - rP)[2Pu\tilde{u} + p^2(u^2\tilde{u}^2 + 1) - \tilde{u}^2 - u^2], \]
\[ B = r(1 - q^2 r^2)(qR - rQ)[2Qu\tilde{u} + q^2(u^2\tilde{u}^2 + 1) - \tilde{u}^2 - u^2], \]
and \((r, R)\) are formulated by the elliptic curve
\[ R^2 = r^4 - kr^2 + 1. \] (3.30c)

Taking \( \beta = \sqrt{r(1 - p^2 r^2)(pR - rP)} \) and \( \gamma = \sqrt{r(1 - q^2 r^2)(qR - rQ)} \) in (3.30), we have
\[ \theta = \frac{1}{\mu}[p(1 - p^2 r^2) + pr(pR - rP)uu\tilde{u}] - r(1 - p^2 r^2)\tilde{u} - (pR - rP)u], \] (3.31a)
\[ \eta = \frac{1}{\nu}[q(1 - q^2 r^2) + qr(qR - rQ)uu\tilde{u}] - r(1 - q^2 r^2)\tilde{u} - (qR - rQ)u], \] (3.31b)
where
\[ \zeta = \frac{\phi_1}{\phi_2}. \] (3.31c)

and
\[ \mu = \sqrt{2Pu\tilde{u} + p^2(u^2\tilde{u}^2 + 1) - \tilde{u}^2 - u^2}, \quad \nu = \sqrt{2Qu\tilde{u} + q^2(u^2\tilde{u}^2 + 1) - \tilde{u}^2 - u^2}. \] (3.31d)

To get explicit forms for \( \theta \) and \( \eta \), we need to expand (3.30c) in terms of \( \varepsilon = r - p \). For this we have the following.
Proposition 8. R defined in (3.30c) can be expanded as
\[ R = \sum_{i=0}^{\infty} r_i e^i, \quad (r_0 = P), \]  
(3.32a)
in which \( r_i \) is given by
\[ r_i = P \sum_{|\alpha|=i} g^\alpha \prod_{j=0}^{\alpha_j-1} \left( \frac{1}{2} - j \right), \quad (i = 1, 2, \ldots), \]  
(3.32b)
where
\[ g = (g_1, g_2, g_3, g_4), \quad \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad \alpha_j \in \{0, 1, 2, \ldots\} \]
\[ g_1 = \frac{2}{p^2} (2p^3 - kp), \quad g_2 = \frac{1}{p^2} (6p^2 - k), \quad g_3 = \frac{4p}{p^2}, \quad g_4 = \frac{1}{p^2}, \]
\[ |\alpha| = \sum_{j=1}^{4} s\alpha_j, \quad |\alpha| = \sum_{j=1}^{4} \alpha_j, \quad \alpha! = \prod_{j=1}^{4} (\alpha_j!), \quad g^\alpha = \prod_{j=1}^{4} g_j^{\alpha_j}. \]

Then, similar to the case for (A2), we have the following.

Proposition 9. For the (Q4) equation \( \theta = \frac{\phi}{\phi_0} \) and \( \eta = \frac{\phi}{\phi_0} \) have the form
\[ \theta = \frac{(c_0 \xi_1 + f_1)\xi}{\mu} \left( 1 + \sum_{j=1}^{\infty} \theta_j e^j \right), \]  
(3.33a)
with
\[ \theta_j = \frac{1}{c_0 \xi_1 + f_1} \left( \sum_{k=0}^{j} c_k \xi_{j+1-k} + f_{j+1} \right), \quad (j = 1, 2, \ldots), \]  
(3.33b)
and
\[ \eta = \frac{z_0}{v} \left( 1 + \sum_{j=1}^{\infty} \eta_j e^j \right), \]  
(3.34a)
with
\[ \eta_j = \frac{1}{c_0} \left( \sum_{k=0}^{j-1} w_k \xi_{j-k} + z_j \right), \quad (j = 1, 2, \ldots) \]  
(3.34b)
where
\[ f_0 = 0, f_i = c_1 \tilde{u} + d_i, \quad (i = 1, 2, \ldots), \]
\[ w_0 = q(1 - q^2 p^2) + q(pQ - pQ)w\tilde{u}, \quad w_1 = -2pq^3 + q(pQ + pqr - pQ)u\tilde{u}, \]
\[ w_2 = -q^3 + q(qpr + qr - Q)u\tilde{u}, \quad w_i = q^2 (pr_1 + r_{i-1})u\tilde{u}, \quad (i = 3, 4, \ldots), \]  
(3.35)
\[ z_0 = -p(1 - q^2 p^2)\tilde{u} - q(pQ - pQ)u + w_0 u\tilde{u}, \quad z_1 = (3p^2 q^2 - 1)\tilde{u} - (qr - Q)u + w_1 u\tilde{u}, \]
\[ z_2 = 3pq^2 \tilde{u} - qr_1 u + w_2 u\tilde{u}, \quad z_3 = q^2 \tilde{u} - qr_2 u + w_3 u\tilde{u}, \quad z_i = -qr_1 u + w_0 u\tilde{u}, \quad (i = 4, 5, \ldots). \]
And \( \xi \) is given by
\[ \xi = \sum_{j=1}^{\infty} \xi_j e^j, \]  
(3.36a)
with
\[ \xi_1 = \frac{-1}{c_0 \tilde{u} - a_0} \left[ (c_1 \tilde{u} + d_1)\tilde{u} - a_1 \tilde{u} - b_1 \right], \]  
(3.36b)
Proposition 10. The infinitely many conservation laws of (Q4) equation are given by

\[ \xi_s = -\frac{1}{c_0 \bar{u} - a_0} \left[ \sum_{k=0}^{s-1} \sum_{i=1}^{s-1} c_i \bar{\xi}_{s-k-1} + \sum_{i=1}^{s-1} \left( (c_i \bar{u} + d_i) \bar{\xi}_{s-1} + (c_i \bar{u} - a_i) \xi_{s-1} \right) + (c_i \bar{u} + d_i) \bar{u} - a_i \bar{u} - b_i \right] \quad (s = 2, 3, \ldots), \]  

(3.36c)

and

\[ a_0 = p(1 - p^4) u, a_1 = (1 - 3p^4) u + (p r_1 - p) \bar{u}, a_2 = -3p^3 u + pr_2 \bar{u}, \]

\[ a_3 = -p^2 u + pr_3 \bar{u}, \]

\[ b_0 = -p(1 - p^4) \bar{u}, b_1 = 2p^4 \bar{u} - p^2 (p r_1 - p), b_2 = p^3 \bar{u} - p(pr_1 - p) - p^3 r_2, \]

\[ b_3 = -p^2 (p r_1 + r_{-1}) \bar{u}, \]

\[ c_0 = p(1 - p^4), c_1 = p^2 (p r_1 - p) \bar{u} - 2p^4, c_2 = [p(p r_1 - p) + p^3 r_2] \bar{u} - p^3, \]

\[ c_3 = p^2 (p r_1 + r_{-1}) \bar{u}, \]

\[ d_0 = -p(1 - p^4) \bar{u}, d_1 = -(1 - 3p^4) \bar{u} - (p r_1 - p) u, d_2 = 3p^3 \bar{u} - pr_2 u, \]

\[ d_3 = p^2 \bar{u} - pr_3 u, \]

(3.37)

Proof. From (3.30a) we have a Riccati equation

\[ \tilde{\xi} (c \xi + d) = a \xi + b, \]

(3.38)

for \( \xi \) where

\[ a = r(1 - p^2 r^2) u + (p R - r P) \bar{u}, \quad b = -p(1 - p^2 r^2) \bar{u} - pr (p R - r P), \]

\[ c = p(1 - p^2 r^2) + pr (p R - r P) \bar{u}, \quad d = -r(1 - p^2 r^2) \bar{u} - (p R - r P) u. \]

Inserting

\[ \xi = \bar{u} + \xi, \quad \varepsilon = r - p, \]

and (3.32a) into the above \( a, b, c, d \) we find

\[ a = \sum_{i=0}^{\infty} a_i \varepsilon^i, \quad b = \sum_{i=0}^{\infty} b_i \varepsilon^i, \quad c = \sum_{i=0}^{\infty} c_i \varepsilon^i, \quad d = \sum_{i=0}^{\infty} d_i \varepsilon^i \]

(3.40)

with \( \{a_i, b_i, c_i, d_i\} \) given in (3.37). Meanwhile, (3.38) is written as

\[ \sum_{i=0}^{\infty} c_i \varepsilon^i \xi + \sum_{i=1}^{\infty} (c_i \bar{u} + d_i) \varepsilon^i \xi + \sum_{i=1}^{\infty} (c_i \bar{u} - a_i) \varepsilon^i \xi + \sum_{i=1}^{\infty} [(c_i \bar{u} + d_i) \bar{u} - a_i \bar{u} - b_i] \varepsilon^i = 0. \]

(3.41)

This is solved by (3.36). Consequently, (3.31) is written into

\[ \theta = \frac{1}{\mu} \left( \sum_{i=0}^{\infty} c_i \varepsilon^i \xi + \sum_{i=0}^{\infty} f_i \varepsilon^i \right), \quad \eta = \frac{1}{\nu} \left( \sum_{i=0}^{\infty} w_i \varepsilon^i \xi + \sum_{i=0}^{\infty} z_i \varepsilon^i \right), \]

with \( \{f_i, w_i, z_i\} \) given in (3.35). This then leads to the expressions (3.33) and (3.34). \( \square \)

Finally, we have the following.

Proposition 10. The infinitely many conservation laws of (Q4) equation are given by

\[ \Delta_m \ln c_0 \xi + f_1 = \Delta_n \ln c_0 \xi, \]

(3.42a)

\[ \Delta_m \theta_s = \Delta_n \theta_s (\eta), \quad s = 1, 2, \ldots, \]

(3.42b)

where

\[ \theta = (\theta_1, \theta_2, \ldots), \quad \eta = (\eta_1, \eta_2, \ldots), \]

with \( \mu \) and \( \nu \) given by (3.31d), and \( \theta_j \) and \( \eta_j \) described by proposition 9.
4. Conclusions and discussions

We have shown that infinitely many conservation laws of ABS lattice equations can be derived from their Lax pairs. We generalized the approach used in [12]. From a discrete $2 \times 2$ Lax pair it is easy to write out a formal conservation law. We found a generic discrete Riccati equation (2.8) which is shared by the (H1), (H2), (H3), (Q1), (Q2), (Q3) and (A1) equations. This generic Riccati equation is derived from the generic scalar spectral problem (2.10) already derived in [26] from Lax pairs in different forms (see table 2 in [26], except for the (Q3) equation, of which the Lax pair is the same as the one we list in proposition 4 of our paper). In fact, these Lax pairs in different forms are gauge equivalent, and more precisely, the transform matrix\(^2\) $T$ is $\lambda(I - \frac{1}{T})$ with either $\lambda = 1$ or $\lambda = i$. A series form was provided for $\theta$ as a solution to the generic Riccati equation (2.8), and with the help of polynomials $\{h_i(t)\}$ defined in (2.16b), infinitely many conservations laws can be expressed both algebraically and explicitly. We also want to emphasize that the value of $\beta$ that we choose is important for reaching the generic discrete Riccati equation (2.8), while in the Gardner method $\beta$ is canceled in the ratio form $\bar{\tau} = \phi_1/\phi_2$. Besides, we also note that if we conduct the same procedure starting from the ($q'_\tau$) part of Lax pairs, we only need to switch ($p'_\tau$ and $q'_\tau$) in the present results and this is guaranteed by the symmetric property (2.15). The (A2) and (Q4) equations seem to be special and so far we do not know whether their Riccati equations fall in the same generic form (2.8). For them we derive their conservation laws by using the approach used for the Ablowitz–Ladik system [12]. This is closely related to the Gardner method because for a CAC equation, its Lax pair is obtained by just taking $\bar{\tau} = \phi_1/\phi_2$ in its natural auto BT. However, starting from Lax pairs gives naturally the formal (initial) conservation law. In appendix A for the lattice equations in the ABS list we list the first one or few conservation laws.

Next, we discuss the relations of the conservation laws. Let us employ some notions used in [20]. $U_s = \{u(n + j, m) | j \in \mathbb{Z}\}$ and $\mathcal{F}_s = \mathbb{C}(U_s)$ denotes the field of rational functions defined on $U_s$. For a function $f = f(u(n + j_1, m), u(n + j_2 + 1, m), \ldots, u(n + j_2, m)) \in \mathcal{F}_s$ and $\frac{\partial f}{\partial u(n + j_1, m)} \neq 0$, $\frac{\partial f}{\partial u(n + j_2, m)} \neq 0$, its order is defined as $\text{ord}_s(f) = (j_1, j_2)$. For the function $f \in \mathcal{F}_s$ with order $(j_1, j_2)$, its variational derivative $\delta_s$ of $f$ is defined as

$$\delta_s(f) = \sum_{k=j_1}^{j_2} E_{-k}^{-1} \left( \frac{\partial f}{\partial u(n + k, m)} \right).$$

(4.1)

For all the conservation laws obtained in our paper, the densities $F(u)$ belong to $\mathcal{F}_s$. A density $F(u) \in \mathcal{F}_s$ is trivial if $\delta_s(F(u)) = 0$. The order of a density $F(u) \in \mathcal{F}_s$ is defined as $\text{ord}_s(F(u)) = j_2 - j_1$, where $(j_1, j_2) = \text{ord}_s(F(u)))$. Two densities $F_1(u), F_2(u) \in \mathcal{F}_s$ are equivalent if and only if $\delta_s(F_1(u)) = \delta_s(F_2(u))$. Equivalent densities have the same order.

Since in the paper infinitely many conservations laws are expressed both algebraically and explicitly, it is possible to check their orders and obtain the following result.

**Proposition 11.** For each equation in the ABS list, the infinitely many conservation laws obtained in the paper are nontrivial and not equivalent.

We give a short proof in appendix B. The same result was also provided in [17].

In [20] infinitely many canonical conservation laws were constructed by using a recursion operator and the first three explicit conservation laws were given (see equations (53) and (54) in [20]; see also [17]). Compared with our results listed in appendix A for each equation in

\(2\times2\) discrete spectral problems $\tilde{\phi} = M\phi$ and $\tilde{\psi} = M_1\psi$ being gauge equivalent means there is an invertible $2 \times 2$ transform matrix $T$ such that $\psi = T\phi$ and $M_1 = TMT^{-1}$.\(^2\)}
the ABS list (except (Q4) because different parametrization is used), we find that they are equivalent by checking the variational derivative $\delta s$ of the densities $F_j(u)$ in appendix A and $\rho_j(u)$ in [20].

We also compared our formal conservation laws with the initial conservation laws given in [19]. For the (H1), (H2), (H3), (Q2) and (Q3) equations they are same, while for the (A1), (A2) and (Q1) equations, they are different. We do not make any comparison for (Q4) due to the different parametrization. The first conservation laws for the (A1), (A2) and (Q1) equations derived from [19] were also listed in appendix A. By calculation one can find for the (A1), (A2) and (Q1) equations, respectively,

(A1) $\delta_s(\Phi_1) = \delta_s(2F_1)$, (A2) $\delta_s(\Phi_1) = \delta_s(2F_1)$,

(Q1) $\delta_s(\Phi_1) = \delta_s \left( F_1 + \frac{1}{2} \ln \frac{u}{u-p} \right)$,

where the $F_i$ are our obtained densities listed in appendix A and $\ln \frac{u}{u-p}$ is a density of the (Q1) equation found in [16].

Besides, we note that it is preferable to define $\theta$ and $\eta$ in terms of $\phi_2$ as in (2.5), and not in terms of $\phi_1$. This is because in those Lax pairs given in proposition 3, $C$ (in place of $M_{21}$) is always constant, which makes the scalar spectral problem (2.9) as simple as (2.10). This simplifies the derivation of solutions to the related Riccati equation and infinitely many conservation laws. Of course we can replace $\phi_2$ with $\phi_1$, define $\theta' = \phi_1^2, \eta' = \phi_1^2$, and implement the procedure as we described in section 2. Taking the (H1) equation as an example, the Riccati equation corresponding to $\theta'$ is (with $\varepsilon = r - p$)

$$(u\varepsilon + r)\theta' = u^2 (u - r) \theta' - \varepsilon u^2 - \varepsilon^2,$$

which is more complicated than (3.3). We can use the above Riccati equation to derive conservation laws for the (H1) equation. We have checked the first two conservation laws and found that they are equivalent to the ones listed in appendix A.

Based on the above discussion on the equivalences of the explicit conservation laws obtained from different approaches, we can say that they are all equivalent (sometimes combinations of known conservation laws are needed), and all of them correspond to the canonical conservation laws given by the integrability conditions derived in [20].

Our approach can also apply to other multidimensionally consistent systems such as the NQC equation [29, 30], discrete Boussinesq-type equations [31–33] and so on; see [34].

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Appendix A. First few conservation laws for the lattice in the ABS list

If we number the conservation laws for each equation in the ABS list as

$$\Delta_m F_i = \Delta_n G_i, \quad i = 1, 2, \ldots,$$

here we list out some $F_i$ and $G_i$. 

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\[\begin{align*}
(\text{H1}): \\
F_1 &= -\ln(u - \tilde{u}), \\
F_2 &= \frac{1}{(u - \tilde{u})(\bar{u} - \tilde{u})}, \\
F_3 &= \frac{2u - \bar{u} - \tilde{u}}{(u - \bar{u})^2(\tilde{u} - \bar{u})^2(\bar{u} - \tilde{u})}, \\
J_1 &= \ln(\bar{u} - \tilde{u}), \\
J_2 &= \frac{1}{(u - \bar{u})(u - \tilde{u})}, \\
J_3 &= \frac{\bar{u} - 2\tilde{u} + \bar{u}}{(u - \bar{u})^2(\tilde{u} - \bar{u})^2(\bar{u} - \tilde{u})};
\end{align*}\]

\[\begin{align*}
(\text{H2}): \\
F_1 &= \ln \frac{\sqrt{\bar{u}u + p\delta}}{u - \bar{u}}, \\
F_2 &= \frac{\bar{u}u + p\delta}{(u - \bar{u})(\tilde{u} - \bar{u})}, \\
F_3 &= \frac{(p + \bar{u} + \tilde{u})[2(u - \bar{u})(p + \bar{u} + \tilde{u}) + (p + \bar{u} + \tilde{u})(\tilde{u} - \bar{u})]}{(u - \bar{u})(\tilde{u} - \bar{u})(\bar{u} - \tilde{u})}, \\
J_1 &= \ln \frac{p - q + \tilde{u} - \bar{u}}{\sqrt{q + u + \tilde{u}}}, \\
J_2 &= \frac{p + u + \tilde{u}}{(u - \bar{u})(\tilde{u} - \bar{u})}, \\
J_3 &= \frac{(p + u + \tilde{u})[2(p - q + \tilde{u} - \bar{u})(p + \tilde{u} + \bar{u}) - (p + u + \tilde{u})(\tilde{u} - \bar{u})]}{(p - q + \tilde{u} - \bar{u})^2(u - \tilde{u})^2(\bar{u} - \tilde{u})};
\end{align*}\]

\[\begin{align*}
(\text{H3}): \\
F_1 &= \ln \frac{\sqrt{\tilde{u}u + p\delta}}{u - \tilde{u}}, \\
F_2 &= \frac{\tilde{u}u + p\delta}{(u - \tilde{u})(\bar{u} - \tilde{u})}, \\
F_3 &= \frac{(\tilde{u}u + p\delta)[2(u - \tilde{u})(\tilde{u}u + p\delta) + (\tilde{u}u + p\delta)(\bar{u} - \tilde{u})]}{(u - \bar{u})(\tilde{u} - \bar{u})(\bar{u} - \tilde{u})}, \\
J_1 &= \ln \frac{q\tilde{u} - p\tilde{u}}{p\sqrt{\tilde{u}u + q\delta}}, \\
J_2 &= \frac{q(\tilde{u}u + p\delta)}{(u - \tilde{u})(q\tilde{u} - p\tilde{u})}, \\
J_3 &= \frac{(u\tilde{u} + p\delta)[2q(q\tilde{u} - p\tilde{u})(\tilde{u}u + p\delta) - q^2(\tilde{u}u + p\delta)(\bar{u} - \tilde{u})]}{(q\tilde{u} - p\tilde{u})^2(u - \tilde{u})^2(\bar{u} - \tilde{u})};
\end{align*}\]

\[\begin{align*}
(\text{A1}): \\
F_1 &= \ln \frac{\sqrt{(u + \tilde{u})^2 - \delta^2p^2}}{u - \tilde{u}}, \\
F_2 &= \frac{(u + \tilde{u})^2 - \delta^2p^2}{(u - \tilde{u})(\bar{u} - \tilde{u})}, \\
J_1 &= \ln \frac{q(u + \tilde{u}) - p(u + \tilde{u})}{p\sqrt{(u + \tilde{u})^2 - \delta^2q^2}}, \\
J_2 &= \frac{q([u + \tilde{u}]^2 - \delta^2p^2)}{(u - \tilde{u})[q(u + \tilde{u}) - p(u + \tilde{u})]};
\end{align*}\]

\[\begin{align*}
(\text{A2}): \\
F_1 &= \ln \frac{\sqrt{(p - u\tilde{u})(1 - pu\tilde{u})}}{u - \tilde{u}}, \\
J_1 &= \ln \frac{p(q^2 - 1)u + (p^2 - q^2)u\tilde{u} - q(p^2 - 1)\tilde{u}}{u\sqrt{(q - u\tilde{u})(q\tilde{u} - 1)}},
\end{align*}\]

\[\begin{align*}
(\text{Q1}): \\
F_1 &= \ln \frac{\sqrt{(u - \bar{u})^2 - \delta^2p^2}}{u - \bar{u}}, \\
F_2 &= \frac{(u - \bar{u})^2 - \delta^2p^2}{(u - \bar{u})(\bar{u} - \tilde{u})}, \\
J_1 &= \ln \frac{q\bar{u} - u\bar{u}}{p\sqrt{(u - \bar{u})^2 - \delta^2q^2}}, \\
J_2 &= \frac{q([u - \bar{u}]^2 - \delta^2p^2)}{(u - \bar{u})[q(u - \bar{u}) - p(u - \bar{u})]};
\end{align*}\]
Recalling the results of propositions 1, 2, 3 and 4, for the (H1), (H2), (H3), (Q1), (Q2), (Q3) Appendices B. Proof for proposition 11

Thus, for the infinitely many conservation laws given in (2.20), we have infinitely many conservations are nontrivial and not equivalent.

Additionally, for the (A1), (A2) and (Q1) equations, we list out their first conservation laws which are derived following the initial conservation laws given in [19]. (Let us use the form \( \Delta_\alpha \Phi_1 = \Delta_\alpha \Psi_1 \))

\[
\begin{align*}
\Phi_1 &= \ln \frac{[\alpha - \widetilde{\alpha}](\alpha + \widetilde{\alpha} - \beta^2 p^2) - \beta p^2 (\alpha - \widetilde{\alpha})^2}{p^2 (\alpha - \widetilde{\alpha})^2 [(\alpha + \widetilde{\alpha})^2 - \beta p^2]}, \\
\Psi_1 &= \ln \frac{(\alpha - \widetilde{\alpha})^2 - \beta (\rho - q)^2}{(\alpha + \widetilde{\alpha})^2 - \beta (\rho - q)^2}, \\
\Phi_2 &= \ln \frac{4(\alpha - \widetilde{\alpha}) (\alpha + \widetilde{\alpha} - \beta p^2) - \beta p^2 (\alpha - \widetilde{\alpha})^2}{(p^2 - 1)^2 (\alpha - \widetilde{\alpha})^2}, \\
\Psi_2 &= \ln \frac{(\alpha - \widetilde{\alpha}) (\beta p - q \alpha - \beta q \alpha)}{(\alpha + \widetilde{\alpha}) (\beta p - q \alpha)}.
\end{align*}
\]

Appendix B. Proof for proposition 11

Recalling the results of propositions 1, 2, 3 and 4, for the (H1), (H2), (H3), (Q1), (Q2), (Q3) and (A1) equations, we can find \( \text{ord}_\rho (\rho) = 4 \) and

\[
\theta_1 = \frac{\beta^2}{(\alpha - \widetilde{\alpha})(\alpha + \widetilde{\alpha})}
\]

is irreducible. For \( \theta_1 \) one finds \( \text{ord}_\rho (\theta_1) = (0, 3) \) and \( \text{ord}_\beta (\theta_1) = 6 \). Then, from the recursive relation (2.11c), we can find that in \( \theta_{j+1} \) there is a term \( \prod_{k=0}^{j-1} E_{\rho}^k \theta_1 \) which indicates

\[ \text{ord}_\rho (\theta_{j+1}) = (0, j + 3), \text{ord}_\beta (\theta_{j+1}) = 2(j + 3). \]

Next, from the definition of \( h_j (\theta) \) (2.16b) (for examples see (2.17)) we know that in \( h_j (\theta) \) there is a term \( \theta_j \) which indicates

\[ \text{ord}_\rho (h_j (\theta)) = \text{ord}_\rho (\theta_j) = (0, j + 2), \text{ord}_\beta (h_j (\theta)) = 2(j + 2). \]

Thus, for the infinitely many conservation laws given in (2.20), we have

\[ \text{ord}_\rho (\ln \rho) = 4, \text{ord}_\beta (h_j (\theta)) = 2(j + 2), (j = 1, 2, \ldots) \]

which means that each density has different (nonzero) order and this guarantees that the infinitely many conservations are nontrivial and not equivalent.
In a similar way, it can be also proved that the conservation laws given in propositions 7 and 10, for the (A2) and (Q4) equations, respectively, are nontrivial and not equivalent.

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