Dimension-Free Anticoncentration Bounds for Gaussian Order Statistics with Discussion of Applications to Multiple Testing

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Abstract: The following anticoncentration property is proved. The probability that the $k$-order statistic of an arbitrarily correlated jointly Gaussian random vector $X$ with unit variance components lies within an interval of length $\varepsilon$ is bounded above by $2\varepsilon k(1 + \mathbb{E}[\|X\|_\infty])$. This bound has implications for generalized error rate control in statistical high-dimensional multiple hypothesis testing problems, which are discussed subsequently. JEL Codes: C1.

Introduction

Consider a random vector $X$, taking values in $\mathbb{R}^p$ for some positive integer $p$, with components denoted $X_1, ..., X_p$. Let $k$-$\text{max}(X)$ denote the value of the $k$-th largest component of $X$. The following theorem holds.

**THEOREM 1.** Let $X \sim \mathcal{N}(0, \Sigma) \in \mathbb{R}^p$ be a Gaussian random vector with unit variance components, ie. the only restriction on $\Sigma$ is that $\mathbb{E}[X_1^2] = ... = \mathbb{E}[X_p^2] = 1$. Then

$$\sup_{y \in \mathbb{R}} \mathbb{P}(k$-$\text{max}(X) \in [y, y + \varepsilon]) \leq 2\varepsilon k(1 + \mathbb{E}[\|X\|_\infty]).$$

Theorem 1 is proved below after a literature review and general description of key new arguments needed for its proof. Theorem 1 has implications for generalized error rate control in statistical high-dimensional multiple hypothesis testing problems, which are then discussed.

Bounds of the form of Theorem 1—upper bounds on the probability that a random variable falls in intervals of small width—are referred to commonly as anticoncentration bounds. The study of anticoncentration bounds for sums of independent random variables dates back to at least Littlewood and Offord (1943) and Erdős (1945) with the later reference proving that $\sup_{y \in \mathbb{R}} \mathbb{P}(\sum_{i=1}^p z_i R_i \in [y, y + 2]) \leq 2^{-p} p C_{\lfloor p/2 \rfloor}$ for any fixed real $z_j$ with $|z_j| \geq 1$ and $R_j \in \{\pm 1\}$ independent Rademacher random variables.

In the case of $k = 1$, the conclusion of Theorem 1 was proved in Chernozhukov et al. (2015). Note that the bound in Theorem 1 is dimension-free (ie. independent of $p$). Thus it is potentially much tighter than dimension-dependent bounds when components $X_j$ are sufficiently correlated. Prior to Chernozhukov et al. (2015), Nazarov (2003) proved dimension-
dependent anticoncentration bounds in the case \( k = 1 \). Nourdin and Viens (2009) give a dimension-free bound in the case \( k = 1 \) but required that all components of \( \Sigma \) be strictly positive.

The bound for \( k = 1 \) has found various applications to statistics. It is a major component in proving asymptotic family-wise error rate control in a variety of high-dimensional statistical hypothesis testing problems using the multiplier bootstrap as in Chernozhukov et al. (2013) and using the empirical bootstrap as in Chernozhukov et al. (2017). In the case of \( k = 1 \), Chernozhukov et al. (2016) and Chernozhukov et al. (2014) used the dimension-free property in application to studying infinite dimensional Gaussian elements and empirical process theory. Anticoncentration bounds are also used extensively in the course of proving various high-dimensional central limit theorems in Chernozhukov et al. (2019), Lopes (2020), Koike (2021), and Chernozhukov et al. (2021).

The case of \( k \geq 2 \) is different from the case \( k = 1 \) in important ways. First, only for \( k = 1 \) can the density \( f_1(y) = \max(X) \) be factored into a product of the form \( \phi(y)G_1(y) \) where \( \phi(y) \) is the standard Guassian density and \( G_1(y) \) is nondecreasing. This factorization was a key innovation in Chernozhukov et al. (2015) and it follows that \( f_1(y) = \phi(y)G_1(y) = \phi(y)G_1(y) \left( \int_{y}^{\infty} \phi(t)dt \right)^{-1} \int_{y}^{\infty} \phi(t)dt \leq \phi(y) \left( \int_{y}^{\infty} \phi(t)dt \right)^{-1} \int_{y}^{\infty} \phi(t)G_1(t)dt = M(y)P(\max(X) \geq y) \) where \( M(y) \) is the univariate Mills ratio. Thus, the univariate Mills ratio and bounds for probabilities of large deviations of \( \max(X) \) can be used to construct anticoncentration bounds. Second, only for \( k = 1 \) is \( \max(X) \) is convex in \( X \), ruling out any method for constructing desired anticoncentration bounds for \( k \geq 2 \) which would have needed to rely on convexity of \( k-\max(X) \).

One potential outline for obtaining anticoncentration bounds for \( k \geq 2 \) which would indeed depend on \( p \) is to appeal to Nazarov’s anticoncentration bounds from Nazarov (2003). To do so, note that there exists a Gaussian random vector \( W \in \mathbb{R}^{C \times k} \) with components denoted by \( W_A \) and indexed by subsets \( A \subseteq \{1, \ldots, p\} \) with \( |A| = k \) such that \( E[W_A] = E[\min_{j \in A} (X_j)] \) and \( \text{cov}(W_A, W_{A'}) = \text{cov}(\min_{j \in A} (X_j), \min_{j \in A'} (X_j)) \), with the intention of using \( \max(W) \) to approximate \( k-\max(X) \). The value of Nazarov’s bound is that through recentering, it can be made applicable to obtain an anticoncentration bound for \( \max(W) \) despite the fact that \( W_A \) typically do not have mean 0. Nazarov’s bounds imply \( P(\max(W) \in [y, y + \varepsilon]) \leq (\varepsilon/\sqrt{\min(\text{var}(W))})(\sqrt{2\log_k C_k} + 2) \), where the minimum in \( \min(\text{var}(W)) \) spans over diagonal elements of \( \text{var}(W) \). This bound is dimension-dependent in that it depends on \( p \). Interestingly, this bound is seen to be slightly tighter in \( k \) than the one in Theorem 1 after noting that \( \log_k C_k \approx k \log p \); though \( \min(\text{var}(W)) \) may be made smaller with increasing \( k \). The analysis would continue by comparing \( P(\max(W) \in [y, y + \varepsilon]) \) to \( P(k-\max(X) \in [y, y + \varepsilon]) \) using Gaussian comparison techniques developed in eg. Chernozhukov et al. (2015). However, this line of analysis results in dimension-dependent bounds is outside the scope of the search for dimension-free bounds.
As seen in the discussion on using Nazarov’s bounds in the previous paragraph, relaxing $E[X_j] = 0$ comes with potential loss of generality. On the other hand, the case of more general diagonal entries in $\Sigma$, ie $E[X_j^2] = \sigma_j^2$, where $\sigma_j^2$ are positive real numbers, can be reduced to the case $E[X_j^2] = 1$ by the same methods as were used Chernozhukov et al. (2015) in the case $k = 1$ with a new bound depending on the minimum of the $\sigma_j^2$ but still dimension-free.

The key idea here for handling the case $k \geq 2$ relative to previous literature is to compare $k$-max($X$) to a newly defined auxiliary random variable, $\tilde{k}$-max($X$), constructed as follows. For nonempty subsets $A \subseteq \{1, \ldots, p\}$ let $\bar{X}_A = \frac{1}{|A|} \sum_{j \in A} X_j$. For each $A \subseteq \{1, \ldots, p\}$, let $\iota(A)$ be a randomly chosen, uniformly distributed, element of $A$, independent of $X$ and of $\iota(A')$ for all other $A' \subseteq \{1, \ldots, p\}$. Let $A^* \in \arg\max_{A \subseteq \{1, \ldots, p\}, |A| = k} \bar{X}_A$, with $A^*$ chosen uniformly at random from the arg max set if it is not a singleton. Let $\iota^* = \iota(A^*)$. Define

$$k\text{-}\max(X) = X_{\iota^*}.$$ 

Heuristically, $k\text{-}\max(X)$ is a randomized relative of $k$-max($X$). Note that there is the coupling inequality $P(k\text{-}\max(X) = k$-max($X$)) $\geq \frac{1}{k}$.

This randomization is sufficiently regularizing so that the corresponding density $\tilde{f}_k(y)$ can in fact be expressed in the form $\phi(y)\tilde{G}_k(y)$ where $\tilde{G}_k(y)$ is nondecreasing. This recovers the applicability of the techniques in Chernozhukov et al. (2015) described above to obtain an anticoncentration bound for $k\text{-}\max(X)$, which subsequently translates to an anticoncentration bound for $k$-max($X$). This factorization is stated formally in the next lemma.

**LEMMA 1.** Let $X$ be as in the statement of Theorem 1. Then $k\text{-}\max(X)$ is absolutely continuous with respect to Lebesgue measure and has density $\tilde{f}_k(y) = \phi(y)\tilde{G}_k(y)$ where $\phi(y)$ is the standard Gaussian density and where

$$\tilde{G}_k(y) = \frac{1}{k} \sum_{j=1}^p P(j \in A^*|X_j = y).$$

Furthermore, $\tilde{G}_k(y)$ is nondecreasing in $y$.

**Proof of Lemma 1**

Absolute continuity follows from $P(k\text{-}\max(X) \in B) \leq kP(X_j \in B)$ for any $j$ and Borel set $B$. By standard reductions, following reasoning in the proof of Lemma 6 in Chernozhukov et al. (2015), the next expression is well defined and provides a version of the desired density $\tilde{f}_k(y)$:

$$\tilde{f}_k(y) = \phi(y)\sum_{j=1}^p P(j = \iota^*|X_j = y).$$

Next, note that $\sum_{j=1}^p P(j = \iota^*|X_j = y) = \sum_{j=1}^p P(j \in A^*, j = \iota^*|X_j = y) = \sum_{j=1}^p P(j = \iota^*|X_j = y)P(j \in A^*|X_j = y) = \frac{1}{k} \sum_{j=1}^p P(j \in A^*|X_j = y) = \tilde{G}_k(y)$. Thus $\tilde{f}_k(y) = \phi(y)\tilde{G}_k(y)$ as stated in Lemma 1.
To show that $\tilde{G}_k(y)$ is nondecreasing, note $P(j \in A^* | X_j = y) = P(\min_{i \in A} (X_i) \leq X_j$ for all $A \subseteq \{1, \ldots, p\}, |A| = k |X_j = y)$. Let $V_{jl} = X_i - E[X_iX_j]X_j$, i.e., $V_{jl}$ are the residuals from the least squares projection of $X_i$ on $X_j$ and are jointly independent of $X_j$ as $X$ is jointly Gaussian. Furthermore, for any given set $A$,

$$\min_{i \in A} (V_{jl} + E[X_iX_j]|y) \leq y \iff \min_{i \in A} (V_{jl} + (E[X_iX_j] - 1)y) \leq 0.$$ 

Independence of $\{V_{jl}\}_{l=1,...,p}$ from $X_j$ and the fact that $(E[X_iX_j] - 1) \leq 0$ then implies $P(j \in A^* | X_j = y)$ is nondecreasing in $y$. ■

**Proof of Theorem 1**

Given the conclusion of Lemma 1, then

$$\tilde{f}_k(y) = \phi(y)\tilde{G}_k(y) = \phi(y)\tilde{G}_k(y) \left(\int_y^\infty \phi(t)dt\right)^{-1} \left(\int_y^\infty \phi(t)dt\right) \leq \phi(y) \left(\int_y^\infty \phi(t)dt\right)^{-1} \left(\int_y^\infty \phi(t)\tilde{G}_k(t)dt\right) = M(y)P(k\max(X) \geq y).$$

Because $k\max(X) \leq \max(X)$ it holds that $P(k\max(X) \geq y) \leq P(\max(X) \geq y)$ and therefore

$$\tilde{f}_k(y) \leq M(y)P(\max(X) \geq y).$$

Next, under the condition that $E[X_j^2] = 1$ for all $j$, Chernozhukov et al. (2015) prove that $P(\max(X) \in [y, y + \epsilon]) \leq 2\epsilon (1 + E[\|X\|_\infty])$ using only the property that the density, $f_1(y)$, of $\max(X)$, satisfies $f_1(y) \leq M(y)P(\max(X) \geq y)$. This same property holds for $\tilde{f}_k(y)$ and thus also $P(k\max(X) \in [y, y + \epsilon]) \leq 2\epsilon (1 + E[\|X\|_\infty])$. Finally, by construction, $P(k\max(X) \in [y, y + \epsilon]) \leq kP(k\max(X) \in [y, y + \epsilon])$, and Theorem 1 follows. ■

**Discussion of applications of Theorem 1 to multiple testing problems**

As an application of Theorem 1, consider the problem of testing a collection of $p$ statistical hypotheses $H_{01}, \ldots, H_{0p}$. When $p \geq 1$, conducting separate testing procedures for each $H_{0j}$ at level $\alpha \in (0, 1)$ independently may lead to the classical multiple testing problem that the family-wise error rate exceeds $\alpha$, i.e. FWER = $P(H_{0j}$ rejected for some true null $j) > \alpha$, even if $P(H_{0j}$ rejected $) \leq \alpha$ individually for all $j$ corresponding to a true $H_{0j}$. To address this, researchers have designed many joint testing procedures aimed at controlling family-wise error rate.

In many applications, FWER is too stringent a notion of error rate control. There are several alternatives to FWER which allow for control in the tradeoff between power and tolerance for false positives in multiple testing problems with many hypotheses. These include $k$-family-wise error rate, $k$-FWER = $P(H_{0j}$ rejected for at most $k$ true nulls $j$). $k$-FWER is a natural starting point for discussion due to its simplicity, and procedures controlling other error rate
notations can sometimes be built \( k \)-FWER-controlling procedures; eg. the FDP-controlling procedure of Romano and Wolf (2007).

One common method for controlling \( k \)-FWER is a step-down-based method in Algorithm 2.1 in Romano and Wolf (2007). Their procedure depends only on test statistics \( T_1, \ldots, T_p \) corresponding to \( H_{01}, \ldots, H_{0p} \) and estimated critical values \( \hat{c}_K(1-\alpha,k) \) for each \( K \subseteq \{1,\ldots,p\}, |K| \geq k, \) which preferably estimate upper bounds over \( 1-\alpha \) quantiles of \( k \)-max\((T_j : j \in K)\) under distributions in the intersection null, \( \cap_{j \in K} H_{0j}. \) Note, \( k \)-max\((T_j : j \in K)\) refers to \( k \)-max applied to the vector with components \( T_j \) with \( j \in K. \) Let \( I \subseteq \{1,\ldots,p\} \) be the set of true nulls. Theorem 2.1 of Romano and Wolf (2007) proves that under an additional monotonicity condition that \( \hat{c}_K(1-\alpha,k) \geq \hat{c}_I(1-\alpha,k) \) for \( K \supseteq I, \) their Algorithm 2.1 results in \( k \)-FWER \( \leq P( k \text{-max}(T_j : j \in I) > \hat{c}_I(1-\alpha,k)) \).

Because large-\( p \) settings are precisely those for which control of generalized error rates is highly relevant, a natural question is: to what extent do \( \hat{c}_K(1-\alpha,k) \) result in favorable statistical properties for \( k \)-FWER as a function of both \( n \) and \( p \) when based on bootstrap estimates of order statistics of vectors with components being subsets of \( T_1, \ldots, T_p? \) Note, Romano and Wolf (2007) discussed conditions in which \( k \)-FWER is asymptotically controlled with bootstrap-based \( \hat{c}_K(1-\alpha,k) \) in a frame in which \( n \rightarrow \infty \) while \( p \) was fixed, thus suppressing dependence of \( k \)-FWER on \( p \) from notation. For illustration, suppose \( U_1, \ldots, U_n \in \mathbb{R}^p \) is a collection of \( n \) independent, identically distributed random Gaussian vectors with unknown mean \( \mu \in \mathbb{R}^p. \) Write \( U_{ij} \) and \( \mu_j \) for the \( j \)th component of \( U_i \) and \( \mu. \) Suppose \( H_{0j} \) are given by \( H_{0j} : \mu_j \leq 0 \) versus \( \mu_j > 0. \) Suppose covariance of the \( U_i \) is unknown with the exception, for the sake of simplicity, that the diagonal elements of the covariances are finite and known (and assumed \( =1. \) Let \( T_j = n^{-1/2} \sum_{i=1}^n U_{ij}. \) Recall that the test that rejects for large \( T_j \) is uniformly most powerful for testing \( H_{0j}. \) Let \( U_{1}^*, \ldots, U_n^* \) be an empirical bootstrap sample from \( U_1, \ldots, U_n, \) and let \( T_j^* = n^{-1/2} \sum_{i=1}^n(U_{ij}^* - n^{-1} \sum_{s=1}^n U_{sj}). \) Let \( \hat{c}_K(1-\alpha,k) \) be the \( 1-\alpha \) quantile of \( k \)-max\((T_j^* : j \in K)\) (conditioned on \( U_1, \ldots, U_n, \) calculated over draws of \( \ast. \) ) Apply Algorithm 2.1 of Romano and Wolf (2007) to yield decisions \( D_j \in \{ \text{Reject}, \text{Fail to Reject} \} \) for each \( H_{0j}. \)

Properties of \( D_j \) follow from analyzing \( P( k \text{-max}(T_j : j \in I) > \hat{c}_I(1-\alpha,k)) \). Let \( \beta = (q_{1-\alpha}(k \text{-max}(T_j : j \in I)) - \hat{c}_I(1-\alpha,k)), \) and let \( \gamma, \delta \geq 0 \) be such that \( P(\beta \geq \gamma) \leq \delta. \) By application of Theorem 1 above, \( P( k \text{-max}(T) \geq \hat{c}_I(1-\alpha,k)) \leq \alpha + 2k\gamma(1 + E[\|U\|_\infty]) + \delta. \) Application of Theorem 2.2 of Romano and Wolf (2007), along with the fact that their monotonicity condition is satisfied in the above construction, gives

\[
k \text{-FWER} \leq \alpha + 2k\gamma(1 + E[\|U\|_\infty]) + \delta.
\]

Due to its use in deriving the above expression, a dimension-free bound for the anticoncentration of Gaussian order statistics can improve finite sample upper bounds on deviations of \( k \)-FWER from \( \alpha. \) The dimension-free property becomes important when there is enough correlation in the components of \( U_i. \) Although an exhaustive treatment of conditions which lead to control of \( \gamma \) and \( \delta \) are outside of the current scope, note that following arguments
like those in Chernozhukov et al. (2017) can then give more explicit bounds on $\gamma, \delta$. Note that $P(\beta \geq \gamma) \leq \delta$ is a concentration property rather than an anticoncentration property, and is implied by $P(\sup_{y \in \mathbb{R}} |P(k-\text{max}(T_j : j \in I) < y) - P(k-\text{max}(T_j^* : j \in I) < \gamma/U_1, ..., U_n)| < \gamma| \geq 1 - \delta$. Note also, the above bound for $k$-max is applied in the same way that Chernozhukov et al. (2013) studied bootstrap estimates $\hat{c}_K(1 - \alpha, 1)$ in the case $k = 1$ leading to bootstrap-based procedures controlling FWER.

Additional remarks

There are several potential avenues for future research expanding on the bounds of Theorem 1. First, understanding anticoncentration properties of the $k$th largest local maximum of almost-surely smooth Gaussian processes $X$ with components indexed on a disjoint union of $p$ intervals $\bigcup_{j=1}^{p} [0, 1]_j$ can lead to bootstrap uniform confidence bands which cover an unknown function $g : \bigcup_{j=1}^{p} [0, 1]_j \to \mathbb{R}$ on all but $k$ intervals $[0, 1]_j$ with probability at least $1 - \alpha$ using methods like those developed in Chernozhukov et al. (2014). Second, the randomization technique leading to the definition of $k-\text{max}(X)$ is distinct from other randomization or symmetrization techniques used in the study of empirical processes (eg. Lemma 2.3.1 in van der Vaart and Wellner (1996)), and may have additional applications outside of Theorem 1. Finally, the dimension-free anticoncentration bounds can potentially lead to additional improvements in understanding $k$-FWER control with a bootstrap-based procedure for non-Gaussian data if coupled with interpolation and Gaussian comparison techniques like those developed in Chernozhukov et al. (2016), Chernozhukov et al. (2019) and Chernozhukov et al. (2021). For example, if $U_i$ themselves are not jointly Gaussian, then interpolation techniques can be used to compare the distributions of $k-\text{max}(Q)$ to $k-\text{max}(T)$, where $Q$ is defined to be a Gaussian random vector with components indexed by $j = 1, ..., p$ with $E[Q_j] = 0$ and $E[Q_j Q_l] = E[T_j T_l]$.

References

Chernozhukov, V., Chetverikov, D., and Kato, K. (2013). Gaussian approximations and multiplier bootstrap for maxima of sums of high-dimensional random vectors. *The Annals of Statistics*, 41(6):2786 – 2819.

Chernozhukov, V., Chetverikov, D., and Kato, K. (2014). Gaussian approximation of suprema of empirical processes. *The Annals of Statistics*, 42(4).

Chernozhukov, V., Chetverikov, D., and Kato, K. (2015). Comparison and anticoncentration bounds for maxima of gaussian random vectors. *Probability Theory and Related Fields*, 162(1):47–70.

Chernozhukov, V., Chetverikov, D., and Kato, K. (2016). Empirical and multiplier bootstraps for suprema of empirical processes of increasing complexity, and related gaussian couplings. *Stochastic Processes and their Applications*, 126(12):3632–3651. In Memoriam: Evarist Giné.

Chernozhukov, V., Chetverikov, D., and Kato, K. (2017). Central limit theorems and bootstrap in high dimensions. *The Annals of Probability*, 45(4):2309–2352.
Chernozhukov, V., Chetverikov, D., Kato, K., and Koike, Y. (2019). Improved central limit theorem and bootstrap approximations in high dimensions.

Chernozhukov, V., Chetverikov, D., and Koike, Y. (2021). Nearly optimal central limit theorem and bootstrap approximations in high dimensions.

Erdős, P. (1945). On a lemma of Littlewood and Offord. *Bulletin of the American Mathematical Society*, 51(12):898 – 902.

Koike, Y. (2021). Notes on the dimension dependence in high-dimensional central limit theorems for hyperrectangles. *Japanese Journal of Statistics and Data Science*, 4(1):257–297.

Littlewood, J. and Offord, A. (1943). On the number of real roots of a random algebraic equation (III). *Rec. Math. (Mat. Sbornik)*, 54(12):277–286.

Lopes, M. E. (2020). Central limit theorem and bootstrap approximation in high dimensions with near $1/\sqrt{n}$ rates.

Nazarov, F. (2003). *On the maximal perimeter of a convex set in $\mathbb{R}^n$ with respect to a Gaussian measure*, pages 169–187. Springer, New York.

Nourdin, I. and Viens, F. (2009). Density Formula and Concentration Inequalities with Malliavin Calculus. *Electronic Journal of Probability*, 14(none):2287 – 2309.

Romano, J. P. and Wolf, M. (2007). Control of generalized error rates in multiple testing. *The Annals of Statistics*, 35(4):1378 – 1408.

van der Vaart, A. W. and Wellner, J. A. (1996). *Weak Convergence and Empirical Processes*. Springer Series in Statistics.