The mysterious incoherent metallic states with T-linear resistivity ($\rho_{DC} \propto T$) seen in many strongly correlated materials [1,3] have long puzzled researchers, as such a temperature dependence is inaccessible from Fermi liquid theory [4]. Particularly remarkable is the fact that the slope $d\rho_{DC}/dT$ remains constant as $T$ varies over 2 or 3 orders of magnitude, while the temperature shoots through multiple crossover energy scales: we dub this phenomenon “slope invariance”. Recent successes in building solvable strongly interacting models that yield T-linear resistivity through explicit microscopic calculations [5–10] and through computational and experimental quantum simulation [11] have injected renewed enthusiasm and hope into the community. Given that “solvability” requires unrealistic limits however, theoretical insight into the common or unique mechanisms of the T-linear resistivity obtained in these models is much needed in making contact with experimental observations.

The unusual feature of local self-energy is at play in two of the most studied microscopic models with T-linear resistivity: lattice models of coupled SYK quantum dots [5,8]; and the Hubbard model in single-site dynamical mean-field theory (DMFT). We find distinct mechanisms at intermediate temperatures converging to a unifying picture at high temperature. Further we note how the unifying picture ties to “slope invariance”.

For both models of interest, vertex corrections to the conductivity vanish [6,7,14]. Hence the DC conductivity can be compactly written in the spectral representation of the Kubo formula

$$\sigma_{DC} = 2\pi \int d\epsilon \phi(\epsilon) \int \frac{\beta \omega (A(\epsilon, \omega))^2}{4 \cosh^2(\beta \omega/2)},$$

where $\phi(\epsilon) = \sum_k \left(\frac{\partial \epsilon(k)}{\partial x}\right)^2 \delta(\epsilon - \epsilon(k))$ is the transport function and $A(\epsilon, \omega) = -\frac{1}{2} \text{Im}G(\epsilon, \omega)$ is the spectral function, normalized to $\int d\omega A(\epsilon, \omega) = 1$. Hence different mechanisms of T-linear resistivity arise from different functional forms of the spectral function.

Intermediate Temperatures – We first review recent results finding T-linear resistivity in coupled SYK models [5,8]. These models are the best understood among the solvable models with T-linear resistivity, as their large-N limit (where each SYK dot consists of $N$ flavors of fermions) affords a closed-form self-consistency equation between the single-fermion Green’s function and self-energy. In particular, the imaginary-time Dyson equa-
tions take the general local form,
\[ G(i\omega)^{-1} = i\omega + \mu - \Sigma(i\omega), \]
\[ \Sigma(\tau) = -U^2 G(\tau)^{9/2} G(-\tau)^{9/2-1}, \]
as the single-fermion hopping between the “quantum dots” \[ t \sum_{(i,j)} c_i^\dagger c_j \] is weak at intermediate and high temperatures, regardless of the details of the model. Here \( q \) is the fermionic degree of the interaction term in the Hamiltonian and \( \mu \) is the chemical potential. In the intermediate-temperature regime, where \( T \ll U \), the \( i\omega \) term in the first line can be ignored. At half-filling, these equations then have the solution \( G(\omega) \propto -i3^2\beta^{-2/3}g_q(\beta\omega) \), where we have defined for convenience \( g_q(x) = \Gamma(1/q - ix/2\pi)/\Gamma(1-1/q - ix/2\pi) \).

Due to the local spectral function, the Kubo expression \( \langle \Sigma(i\omega) \rangle = \frac{\pi}{\cosh(\beta\omega/2)} \) of the Dyson equation makes clear that \( T \sim \frac{\pi}{\cosh(\beta\omega/2)} \) vanishes at half-filling \[ T \gg \frac{\pi}{\cosh(\beta\omega/2)} \], in which case the complete expression for the conductivity at intermediate temperatures \( t^2U \ll T \ll U \) is
\[ \sigma_{\text{DC}} \propto t^2 \beta^2 \frac{\sqrt{\pi}}{2U \cosh 2\pi\beta}, \]
where \( E \) is a function solely of the fermion filling that vanishes at half-filling \[ T \gg \frac{\pi}{\cosh(\beta\omega/2)} \]. Two features of this calculation offer clarifying insight: First, the SYK spectral function is a scaling function of the dimensionless parameter \( \beta\omega \), and extends broadly over the entire frequency range below the UV cutoff \(|\omega| < U \). (Fig. 6)

Second, although local self-energy is a generic feature of SYK models, \( T \)-linear resistivity in the intermediate temperature range is a consequence of the particular scaling exponent of the local Green’s function within the “quantum dots” for \( q = 4 \).

Now we turn to the DMFT study of the repulsive \( U \) Hubbard model on the two-dimensional square lattice:
\[ H = -t \sum_{(i,j),\sigma} c_i^\dagger c_j + U \sum_i n_i^\dagger n_i - \mu \sum_i n_i^\dagger n_i. \]
Despite intense efforts at computational studies of the model and its extensions and observations of \( T \)-linear resistivity upon certain approximations \[ \text{DMFT} \text{+ED} \], an analytical understanding of the mechanism of \( T \)-linear resistivity is largely lacking. Here we carry out single-site DMFT using an exact diagonalization (ED) impurity solver that frees us from the need of analytic continuation.

We explored the range of \( 7.5 < U < 12 \) and \( 0.2 < T < 9 \) in units of hopping \( t \), with the electron density per site \( n = 0.825 \). In Fig. 1 we plot a representative result. Here the width of the band represents the dependence of \( \rho_{\text{dc}}(T) \) on the ED broadening parameter \( \eta \). The resulting curve is clearly linear in the intermediate temperature range \( t \ll T < U \) as it has been seen earlier \[ \text{DMFT} \text{+ED} \].

In search of analytic insight into \( T \)-linear resistivity at intermediate temperatures \( t \ll T \ll U \), we examine the DMFT spectral function \( A(\epsilon, \omega) \) in detail for a suitable analytic ansatz. The DMFT spectral function is largely \( T \)-independent at the intermediate temperatures being shown, and consists of a lower and an upper band of widths \( \sim t \) that are separated in frequency by \( \approx U \) (see Fig. 2). Hence we approximate the DMFT spectral function using the following ansatz:
\[ A(\epsilon, \omega) = a_1 h(\omega; \omega_l, \eta_l) + a_u h(\omega; \omega_u, \eta_u), \]
where the lower and upper bands \( A_{l,u}(\epsilon, \omega) \) have weights \( a_{l,u} \), satisfying \( a_l + a_u = 1 \), and are localized in frequencies at \( \omega_{l,u} \) with widths \( \eta_{l,u} \ll T \). For simplicity, we model each band \( h(\omega) \) by a normalized Gaussian, centered at \( \omega_{l,u} \) with standard deviation \( \eta_{l,u} \). Then the parameters for the lower band \( a_l, \omega_l, \eta_l \) are fit to the DMFT spectral function by \( \int d\omega A_l(\epsilon, \omega) = a_l \), \( \int d\omega \omega A_l(\epsilon, \omega) = a_l \epsilon_l \), and \( \int d\omega (A_l(\epsilon, \omega))^2 = (1/2\sqrt{\pi}) \epsilon^2_l / \eta_l \), and likewise for the upper band parameters \[ \text{DMFT} \text{+ED} \]. The parameters depend on \( \epsilon \) but are independent of \( T \). With this ansatz, we arrive at a simplified explicit expression for DC conductivity
\[ \sigma_{\text{DC}} = \beta \left( \frac{C_l}{4 \cosh^2(\beta \mu/2)} + \frac{C_u}{4 \cosh^2(\beta(U - \mu)/2)} \right), \]
where \( C_{l,u} = \sqrt{\pi} \int d\epsilon \phi(\epsilon) a_{l,u}^2 / \eta_{l,u} \). In the intermediate temperature range, Eq. 7 yields \( T \)-linear resistivity by the first term in the paranthesis approaching a constant \( (\mu \ll T) \) and second term vanishing \( (T \ll \mu) \).
Fig. 2. \(A(\epsilon = 0, \omega)\), from DMFT with \(U = 12\) at electron filling \(n = 0.825\), using ED impurity solver with \(n_s = 8\) and \(\eta = 0.08\). Frequencies have been shifted by the chemical potential.

Fig. 1 demonstrates that the ansatz Eq. (7) (shown in orange) captures the DMFT results on resistivity excellently over the whole temperature range. This fit reveals that key mechanism for \(T\)-linear resistivity in DMFT is the highly localized, temperature-independent structure of the spectral function \[22\]. This is in contrast to the inner workings of coupled SYK, where the spectral function is a broad, singly-peaked scaling function of \(\beta\omega\) and \(T\)-linear resistivity is specific to \(q = 4\) SYK. Although it can be violated even in a Fermi liquid in the presence of inelastic scattering \[22\], it is nevertheless a useful quantity to evaluate. The thermal and electrical conductivities can be expressed in terms of kinetic coefficients

\[
L_n = 2\pi \int d\epsilon \, \phi(\epsilon) \int \frac{\beta d\omega(\beta \omega)^n (A(\epsilon, \omega))^2}{4 \cosh^2(\beta \omega/2)}
\]

as \(\sigma = L_0\) and \(\kappa = T(L_2 - (L_1)^2/L_0)\). Once again employing our approximate spectral function Eq. (6), we find the violation of the Wiedemann-Franz law with the Lorenz ratio vanishing as a power-law in temperature for \(t < T\). In contrast, the coupled SYK model of \[5\] violates the Wiedemann-Franz law only through modification of the Lorenz ratio \(L = 0.375 \times L_{WF}\) \[5\] \[6\] in the intermediate temperature regime with \(T\)-linear resistivity.

\textbf{High Temperature Limit} – We now consider the high-temperature regime, where \(T\) is the largest scale in the problem. Resistivity at high-temperature is often overlooked, as all Hamiltonians with a bounded spectrum of states are known to have \(T\)-linear resistivity in this temperature regime \[24\]. This argument is useful from a formal perspective but offers little insight into \(T\)-linear resistivity at intermediate temperatures, or the lack of slope change across the crossover \(T \sim U\). In this “weak coupling” limit, we approximate single-site DMFT using self-consistent second-order perturbation theory (GF2) \[24\] \[27\]. Unlike bare second order perturbation theory, GF2 is \(\Phi\)-derivable \[28\] and therefore thermodynamically consistent and symmetry conserving \[29\] \[30\], implying that thermodynamic relations and conservation laws are intrinsically satisfied. We find, within this approximation, that the DMFT equations converge to the self-consistency equations of \(q = 4\) SYK!

To see the convergence note that the self-energy \(\Sigma(\omega)\) is the sum of a tadpole diagram and a sunset diagram in GF2, i.e., \(\Sigma(\omega) = \Sigma^{(1)}(\omega) + \Sigma^{(2)}(\omega)\), where \(\Sigma^{(1)}(\omega) = \sum(\tau = \beta^-) = U n/2\) and \(\Sigma^{(2)}(\tau) = -U^2 G(\tau)^2 G(-\tau)\). Now the Dyson equations take a closed form:

\[
G(\omega)^{-1} = \omega + \Sigma^{(2)}(\omega),
\]

\[
\Sigma^{(2)}(\tau) = -U^2 G(\tau)^2 G(-\tau),
\]

which are identical to the \(q = 4\) SYK equations \[2\], with a shifted chemical potential \(\mu_{eff} = \mu - nU/2\). This discovery allows us to simultaneously treat single-site DMFT and \(q = 4\) SYK.

Interestingly, with \(T \gg U\), we find, by numerical solution of the real-time version of (9), that the self-energy is temperature-independent and Gaussian to a very good approximation (Fig. 3). By using a high-temperature expansion in imaginary-time \[31\] and the “maximum entropy” ansatz for analytic continuation \[32\], we further determine, for \(\mu_{eff} = 0\),

\[
-\frac{1}{\pi} \text{Im} \Sigma(\omega) \approx \frac{U}{2\pi} \sqrt{\frac{\pi}{6}} \exp \left( -\frac{2\omega^2}{3U^2} \right).
\]

Indeed the GF2 result of a Gaussian form of the self-energy in single-site DMFT was suggested in an earlier small-\(\beta\) expansion approach \[32\]. For \(\mu_{eff} \neq 0\) we find a similar function with its peak shifted to \(\omega = \mu_{eff}\). A Gaussian self-energy of width \(\sim U\) leads to a spectral
function also of width $\sim U$. In the high temperature regime $U \ll T$, this leads to $T$-linear resistivity in the same manner as (7), but with only one band at $-\mu_{\text{eff}}$.

The divergence of the two models at intermediate temperatures is visible in how the asymptotic form of self-energy responds to the lowering of temperature. In DMFT, the self-energy remains Gaussian as the temperature is lowered across the crossover scale $T \sim U$, but becomes increasingly sharply peaked at lower temperatures. This trend is visible in Fig. 4, which displays the self-energy in DMFT across a range of intermediate temperatures for $U = 12$ and $n = 0.825$. We have also plotted a Gaussian fit to the primary peak in dashed lines in Fig. 4. In sharp contrast, the self-energy in $q = 4$ coupled SYK undergoes a dramatic change in form, departing from the Gaussian form at high-temperatures to $-\text{Im} \Sigma(\omega < U) \approx \max(\sqrt{U T}, \sqrt{U} \omega)$ (Fig. 6). Surprisingly, despite this qualitative shift in the single-particle properties, the slope change in resistivity $d\rho_{\text{DC}}/dT$ remains negligible across $T \sim U$ for all numerically accessible values of the filling $\nu$, and actually goes to zero within numerical tolerances at fillings $\approx 0.5 \pm 0.283$ (Fig. 5).

Discussion – At intermediate temperatures $T \ll U$, we observed that single-site DMFT achieves $T$-linear resistivity by grouping the spectrum of states into two narrow bands of widths far below the interaction scale $U$. In the same temperature regime, SYK lattice models have a broad spectrum that extends over the entire range $|\omega| \lesssim U$, and give $T$-linear resistivity for models with $q = 4$, for which $A(\omega) \sim 1/\sqrt{U \omega}$ for $T < \omega \ll U$. Although the intermediate-temperature mechanisms behind $T$-linear resistivity in these models are seemingly unrelated, we have made the surprising observation that the models seem to converge to an identical form at high temperature $T \gg U$. We further observe that the $q = 4$ SYK model shows almost no slope change in its $T$-linear resistivity across the crossover scale $T \sim U$ (Fig. 5).

The mysterious lack of a visible slope change despite major changes in the single-particle spectral properties in coupled SYK is reminiscent of observations made in strongly correlated materials. Interestingly, the model of Ref. [9] is another explicitly solvable model with $T$-linear resistivity without a slope change. Our calculation of compressibility $\chi$ in DMFT (Fig. 7) also shows $\chi^{-1}$ to deviate slightly from the $a + bT/U$ form expected from the high temperature expansion. Such slight deviation was noted in Ref. [9] in the context of Nernst-Einstein relation $\sigma_{\text{DC}} = \chi D$ and the temperature dependence of the diffusivity $D$. We further note that our temperature dependence of $\chi$ can be traced to local physics by comparing the result to a single site problem. One direction for further exploring these connections might be a study of eigenstates and level statistics in the $q = 4$ SYK model and the model of Ref. [9] which are more analytically accessible.

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We have selected a Gaussian form for $h$.

Interestingly, Berg et al. [35] used a phenomenological model with a localized peak in the density of states for “hot fermions” to obtain $T$-linear scattering rate in the context of Sr$_3$Ru$_2$O$_7$.

Consider the spectral representation of conductivity

$$\sigma(\omega) = \sum_{nm} \frac{1}{\omega - E_{nm}} \delta(\omega - E_{nm})$$

where $n, m$ labels the many-body eigenstates and $E_{nm} = E_n - E_m$. When $E_n \ll T$ for all $n$, the Boltzmann factors can be expanded to give a single factor of $\beta$ in the expression to leading order, which shows $T$-linear resistivity.

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Appendix

Comparative Plot

Here we show typical plots of the self-energies and spectral functions in the two temperature regimes from the DMFT calculation and the coupled SYK calculation.

Compressibility

Recent works have proposed looking for insight into $T$-linear resistivity by studying compressibility $\chi$ and diffusivity $D$, related to transport by the Nernst-Einstein relation $\sigma_{DC} = \chi D$, at intermediate and high temperatures. [9][18]. We obtain compressibility in DMFT using continuous-time interaction-expansion (CTI) QMC as the impurity solver. We computed the DMFT solution...
for multiple values of $\mu$ near the value corresponding to $n = 0.825$ and took the numerical derivative to find $\chi^{-1} = n^2 \frac{d \mu}{dn}$. No analytical continuation is necessary as $n = -2G(\tau = \beta^-)$.

Figure 7 presents inverse compressibility vs $T$. Blue dots represent data from DMFT. To gain further insight we consider two limits where we can obtain compressibility analytically. Firstly, we consider the local limit of the Hubbard atom, i.e., Hubbard model with $t \to 0$. In this limit the chemical potential takes the form

$$\mu = T \log \frac{\sqrt{\delta^2 + (1 - \delta^2)e^{-U/T}} - \delta}{(1 + \delta)e^{-U/T}}. \quad (14)$$

From this, one can obtain an analytic expression for the compressibility plotted in Fig. 7 in orange solid line. Secondly, we consider the high temperature expansion which yields

$$\chi^{-1} = \frac{n^2 U}{2} + \frac{2n}{2 - n} T, \quad (15)$$

plotted in green solid line (see Fig. 7). Comparison between these three plots show that the slight deviation in the compressibility calculated within DMFT from the high temperature linear behavior of $\chi^{-1}$ is well-captured by the local physics of the Hubbard atom in the intermediate temperature range.