THE BI-CANONICAL DEGREE OF A COHEN-MACaulay RING

L. GHEZZI, S. GOTO, J. HONG, H. L. HUTSON, AND W. V. VASCONCELOS

Dedicated to Professor Craig Huneke for his groundbreaking contributions to Algebra, particularly to Commutative Algebra and generosity on the occasion of his birthday.

Abstract. This paper is a sequel to [8] where we introduced an invariant—canonical degree—of Cohen-Macaulay local rings that admit a canonical ideal. Here to each such ring $R$ with a canonical ideal $C$, we attach a different invariant—bicanonical degree—which in dimension 1 appears also in [13] as the trace ideal of $R$. The minimal values of these functions characterize specific classes of Cohen-Macaulay rings. Our expectation is that other values may have similar outcomes. We give a uniform presentation of the three degrees and discuss some computational opportunities offered by the bicanonical degree.

Key Words and Phrases: Anti-canonical degree, bi-canonical degree, canonical degree, Cohen-Macaulay type, analytic spread, roots, reduction number.

1. Introduction

Let $(R, m)$ be a Cohen-Macaulay local ring of dimension $d$ that has a canonical ideal $C$. Our central viewpoint is to look at the properties of $C$ as a way to refine our understanding of $R$. In [8] several metrics are treated aimed at measuring the deviation from $R$ being Gorenstein, that is when $C \simeq R$. Here we explore another pathway but still with the same overall goal. Unlike [8] the approach here is arguably more suited for computation in classes of algebras such as Rees algebras and monomial subrings. First however we outline the general underpins of these developments. The organizing principle to set up a canonical degree is to recast numerically criteria for a Cohen-Macaulay ring to be Gorenstein.

We shall now describe how this paper is organized. For a Cohen-Macaulay local ring $(R, m)$ of dimension $d$ with a canonical ideal $C$, we are going to attach a non-negative integer $c(R)$ whose value reflects divisorial properties of $C$ and provide for a stratification of the class of Cohen-Macaulay rings. We have noted two such functions in the current literature ([13], [8]) and here we will build a third degree.

In Section 2 we recall from the literature the needed blocks to put together the degrees. The next section quickly assembles three degrees and begins the comparison of its properties. These assemblages turn out to provide effective symbolic calculation [we used Macaulay2 ([11]) in our experiments] but turn out useful for theoretical calculations in special classes of rings. The new degree is labelled the bi-canonical degree of $R$ and is given by

$$\text{bideg}(R) = \deg(C^{**}/C) = \sum_{\text{height } p = 1} \text{bideg}(R_p) \deg(R/p) = \sum_{\text{height } p = 1} [\lambda(R_p/C_p^{**}) - \lambda(R_p/C_p^{*})] \deg(R/p).$$

This is a well-defined finite sum independent of the chosen canonical ideal $C$. It leads immediately to comparisons to two other degrees, the canonical degree of [8], $cdeg(R) = \deg(C/(s))$ for a minimal reduction $(s)$ of $C$ [in dimension one and suitably assembled as above to all
Dimensions], and the residue of \( R \) of $[13]$ $tdeg(R) = \deg(R/\text{trace}(C))$, where $\text{trace}(C)$ is the trace ideal of $C$. Arising naturally is a comparison conjecture, that $cdeg(R) \geq bideg(R)$.

We engage in a brief discussion on how to recognize that a codimension one ideal $I$ is actually a canonical ideal. We finally recall the notion of the rootset of $R$ \([8]\), perhaps one of least understood sets attached to $C$ and raise questions on how it affects the values of the degrees.

We begin in Section 4 a study of algebras according to the values of one of the $c(R)$. If $c(R) = 0$, for all canonical degrees, $R$ is Gorenstein in codimension one. It is natural to ask which rings correspond to the minimal values of $c(R)$. In dimension one, $cdeg(R) \geq r(R) - 1$ and $bideg(R) \geq 1$, where equality corresponds to the almost Gorenstein rings of \([2, 9, 10]\) and nearly Gorenstein rings of $[13]$.

We begin in Sections 5, 6, 7 and 8 calculations of $cdeg(R)$ and $bideg(R)$ for various classes of algebras. Unlike the case of $cdeg(R)$, already for hyperplane sections the behavior of $bideg(R)$ is more challenging. Interestingly, for monomial rings $k[t^a, t^b, t^c]$ the technical difficulties are reversed. In two cases, augmented rings and $[\text{tensor}]$ products, very explicit formulas are derived. More challenging is the case of Rees algebras when we are often limited to deciding the vanishing of degrees. The most comprehensive results resolve around $m : m$.

2. Setting up and calculating canonical degrees

In this section we describe the canonical degrees known to the authors.

**Theorem 2.1.** Let $(R, m)$ be a local ring of dimension one and let $Q$ be its total ring of fractions. Assume that $R$ has a canonical module $C$.

1. $R$ has a canonical ideal if and only if the total ring of fractions of $R$ is Gorenstein. $[1, 3, 14]$.
2. The $m$-primary ideal $I$ is a canonical ideal if and only if $I : R m = I : Q m = (I, s)$. $[14, \text{Theorem } 3.3]$.
3. $R$ is Gorenstein if and only if $C$ is a reflexive module. $[14, \text{Corollary } 7.29]$.
4. If $R$ is an integral domain with finite integral closure then $I^{**}$ is integral over $I$. $[5, \text{Proposition } 2.14]$.

We often assume harmlessly that $R$ has an infinite residue field and has arbitrary Krull dimension. For a finitely generated $R$-module $M$, the notation $\text{deg}(M) = e_0(m, M)$ refers to the multiplicity defined by the $m$-adic topology. The Cohen-Macaulay type of $R$ is denoted by $r(R)$.

Among the ways we can set up the comparison of $C$ to a principal ideal we have the following.

1. In dimension one, select a privileged element $c$ of $C$ and define $cdeg(R) = \deg(C/(c))$. The choice should yield the same value for all $C$. In $[8]$ $(c)$ is picked as a minimal reduction of $C$ when then $cdeg(R) = e_0(C, R) - \deg(R/C)$.
2. In this paper the choice is more straightforward. Set $bideg(R) = \deg(C^{**}/C)$, where $C^{**}$ is the bidual of $C$. 

**Divisorial basics of $C$.** We are going to make use of the basic facts expressed in the codimension one localizations of $R$.

- **Theorem 2.1.** Let $(R, m)$ be a local ring of dimension one and let $Q$ be its total ring of fractions. Assume that $R$ has a canonical module $C$.
  - (1) $R$ has a canonical ideal if and only if the total ring of fractions of $R$ is Gorenstein. $[1, 3, 14]$.
  - (2) The $m$-primary ideal $I$ is a canonical ideal if and only if $I : R m = I : Q m = (I, s)$. $[14, \text{Theorem } 3.3]$.
  - (3) $R$ is Gorenstein if and only if $C$ is a reflexive module. $[14, \text{Corollary } 7.29]$.
  - (4) If $R$ is an integral domain with finite integral closure then $I^{**}$ is integral over $I$. $[5, \text{Proposition } 2.14]$.

We often assume harmlessly that $R$ has an infinite residue field and has arbitrary Krull dimension. For a finitely generated $R$-module $M$, the notation $\text{deg}(M) = e_0(m, M)$ refers to the multiplicity defined by the $m$-adic topology. The Cohen-Macaulay type of $R$ is denoted by $r(R)$.

Among the ways we can set up the comparison of $C$ to a principal ideal we have the following.

1. In dimension one, select a privileged element $c$ of $C$ and define $cdeg(R) = \deg(C/(c))$. The choice should yield the same value for all $C$. In $[8]$ $(c)$ is picked as a minimal reduction of $C$ when then $cdeg(R) = e_0(C, R) - \deg(R/C)$.
2. In this paper the choice is more straightforward. Set $bideg(R) = \deg(C^{**}/C)$, where $C^{**}$ is the bidual of $C$. 

**2. Setting up and calculating canonical degrees**

In this section we describe the canonical degrees known to the authors.

- **Theorem 2.1.** Let $(R, m)$ be a local ring of dimension one and let $Q$ be its total ring of fractions. Assume that $R$ has a canonical module $C$.
  - (1) $R$ has a canonical ideal if and only if the total ring of fractions of $R$ is Gorenstein. $[1, 3, 14]$.
  - (2) The $m$-primary ideal $I$ is a canonical ideal if and only if $I : R m = I : Q m = (I, s)$. $[14, \text{Theorem } 3.3]$.
  - (3) $R$ is Gorenstein if and only if $C$ is a reflexive module. $[14, \text{Corollary } 7.29]$.
  - (4) If $R$ is an integral domain with finite integral closure then $I^{**}$ is integral over $I$. $[5, \text{Proposition } 2.14]$.
3. The bi-canonical degree

The approach in [8] is dependent on finding minimal reductions, a particularly hard task. We pick here one that seems more amenable to computation.

Let $C^* = \text{Hom}(C, R)$ be the dual of $C$ and $C^{**}$ its bidual. [In general, in writing $\text{Hom}_R$ we omit the symbol for the underlying ring.] In the natural embedding

$$0 \to C \to C^{**} \to B \to 0$$

$B$ remains unchanged when $C$ is replaced by another canonical module, say $D = sC$ for a regular element $s \in Q$. $B$ vanishes iff $R$ is Gorenstein as indicated above. It is easy to see
that a similar observation can be made if \(d > 1\): \(\mathcal{B} = 0\) iff \(R\) is Gorenstein in codimension 1: \(\mathcal{B}\) embeds into the Cohen-Macaulay module \(R/C\) that has dimension \(d - 1\), and thus \(\mathcal{B}\) either is zero or its associated primes are associated primes of \(C\), all of which have codimension one. Like in [5], we would like to explore the length of \(\mathcal{B}\), \(\text{bideg}(R) = \lambda(B)\) which we view as a degree, in dimension 1 and \(\text{deg}(\mathcal{B})\) in general. We stick to \(d = 1\) for the time being. We would like some interesting examples and examine relationships to the other metrics of \(R\). We do not have a best name for this degree, but we could also denote it by \(\text{ddeg}(R)\) (at least the double ‘d’ as a reminder of ‘double dual’).

Let us formalize these observations as:

**Theorem 3.1.** Let \((R, m)\) be a Cohen-Macaulay local ring of dimension \(d \geq 1\) that has a canonical ideal \(C\). Then

\[
\text{bideg}(R) = \text{deg}(C^{**}/C) = \sum_{\text{height } p = 1} \text{bideg}(R_p) \text{deg}(R/p) = \sum_{\text{height } p = 1} [\lambda(R_p/C_p) - \lambda(R_p/C_p^{**})] \text{deg}(R/p)
\]

is a well-defined finite sum independent of the chosen canonical ideal \(C\). Furthermore, \(\text{bideg}(R) \geq 0\) and vanishes if and only if \(R\) is Gorenstein in codimension 1.

**Comparison of canonical degrees.** If \((c)\) is a minimal reduction of \(C\) how to compare

\[
\text{cdeg}(R) = \lambda(R/(c)) - \lambda(R/C) \Leftrightarrow \lambda(R/C) - \lambda(R/C^{**}) = \text{bideg}(R).
\]

The point to be raised is: which is more approachable, \(e_0(C)\) or \(\lambda(R/C^{**})\)? We will argue, according to the method of computation below, that the latter is more efficient which would be demonstrated if the following conjecture were settled.

**Conjecture 3.2.** [Comparison Conjecture] If \(\dim R = 1\) the following inequality holds: \(\text{cdeg}(R) \geq \text{bideg}(R)\), that is from the diagram

\[
(c) \longrightarrow C \longrightarrow C^{**}
\]

where \(C^{**} = (c) : ((c) : C)\), we have \(\lambda(C/(c)) \geq \lambda(C^{**}/C)\). Alternatively

\[
e_0(C) + \lambda(R/C^{**}) \geq 2 \cdot \lambda(R/C).
\]

This would imply, by the associativity formula, that the inequality holds in all dimensions.

**Computation of duals and biduals.** Let \(I\) be a regular ideal of the Noetherian ring \(R\). If \(Q\) is the total ring of fractions of \(R\) then

\[
\text{Hom}(I, R) = R :_Q I.
\]

A difficulty is that computer systems such as Macaulay2 ([11]) are set to calculate quotients of the form \(A :_R B\) for two ideals \(A, B \subset R\), which is done with calculations of syzygies. This applies especially in the case of the ring \(R = k[x_1, \ldots, x_d]/P\) where \(k\) is an appropriate field. To benefit of the efficient *quotient* command of this system we formulate the problem as follows.

**Proposition 3.3.** Let \(I \subset R\) and suppose \(a\) is a regular element of \(I\). Then

1. \(I^* = \text{Hom}(I, R) = a^{-1}((a) :_R I)\).
2. \(I^{**} = \text{Hom}((\text{Hom}(I, R), R) = (a) :_R (I) = \text{annihilator of Ext}_R^1(R/I, R)\).
3. \(\tau(I) = a^{-1}I \cdot (a) :_R I\).
Proof. (1): If \( q \in \text{Hom}(I, R) = R :_Q I \), then
\[
qI \subset R \simeq qaI \subset (a) \simeq qa \in (a) :_R I \simeq q \in a^{-1}((a) :_R I)
\]
(3): Follows from the calculation
\[
q \in I^{**} = [a^{-1}((a) :_R I)]^* = a[\text{R :}_Q ((a) :_R I)] = a[a^{-1}((a) :_R I)] = (a) :_R ((a) :_R I).
\]
See also [20, Remark 3.3], [13, Proposition 3.1].

Recognition of canonical ideals. These methods permit answering the following question. Given an ideal \( C \), is it a canonical ideal? These observations are influenced by the discussion in [7, Section 2]. For other methods, see [9].

Proposition 3.4. Let \((R, m)\) be a one-dimensional Cohen-Macaulay local ring and let \( Q \) be its total ring of fractions. Then an \( m \)-primary ideal \( I \) is a canonical ideal if the following hold:

1. \( I \) is an irreducible ideal and
2. \( \text{Hom}(I, I) = R \).

Proof. Note first that if \( q \in I :_Q m \) then \( qI \subset I \), that is \( q \in \text{Hom}(I, I) \). Now we invoke Theorem [2.112].

Corollary 3.5. Let \((R, m)\) be a Cohen-Macaulay local ring of dimension one.

1. If \( I \) is an irreducible ideal, then \( I \) is a canonical ideal. ([7, Proposition 2.2])

2. The \( m \)-primary ideal \( I \) is a canonical ideal if and only if both \( I \) and \( xI \) are irreducible for any regular element \( x \in m \).

3. \( m \) is a canonical ideal if and only if \( R \) is a discrete valuation ring.

Proof. (1) If \( q \in I :_Q m, qx \subset I \subset xR \), so \( q \in R \). Thus \( q \in I :_R m \). Since \( I \) is irreducible \( I :_R m = (I, s) \) and we can invoke again Theorem [2.112].

(2) Apply the previous assertion to \( xI \). The converse is well-known.

(3) Let \( x \) be a regular element in \( m \). Since \( \text{dim} R = 1 \) there is a positive integer \( n \) such that \( m^n \subset xR \) but \( m^{n-1} \not\subset xR \). If \( n = 1 \), \( m = xR \) there is nothing else to prove. If \( n > 1 \), we have that \( L = m^{n-1}x^{-1} \) satisfies \( L \cdot m \) is an ideal of \( R \) so that either \( L \cdot m = R \) and hence \( m \) is invertible, or \( L \cdot m \subset m \) that means \( L \subset \text{Hom}(m, m) = R \). Thus \( m^{n-1}x^{-1} \subset R \) so \( m^{n-1} \subset xR \), which is a contradiction.

The roots of \( R \). There are several properties of \( C \)--such as its reduction number and type--that may impact the canonical degree of the ring (see [8]). Another property, even less well understood, is the rootset of \( R \) consisting of the ideals \( L \) such that \( L^n \simeq C \) for some integer \( n \). They are distributed into finitely many isomorphism classes. It was studied in [8] for its role in examining properties of canonical ideals. Here is one instance:

Proposition 3.6. Let \( R \) be a Cohen-Macaulay local ring of dimension one. If \( L \) is an irreducible ideal then \( R \) is a Gorenstein ring.

Proof. Note that if \( q \in Q \) satisfies \( qL \subset L \) then \( qL^n \subset L^n \). This implies that \( qC \subset C \) and so \( q \in R \). By Proposition [3.4] \( L \simeq C \).

We now make use of a technique of [8]. From \( C \simeq C^n \) we have
\[
C \simeq C^{n-1} \cdot C \simeq C^{n-1} \cdot C^n = C^{2n-1}.
\]
By iteration we get that \( C \simeq C^m \) for arbitrarily large values of \( m \), and for all of them we have \( C^m : C^m = R \).

We may assume that the residue field of \( R \) is infinite and obtain a minimal reduction \( (c) \) for \( C \), that is an equality \( C^{r+1} = cC^r \) for all \( r \geq s \) for some \( s \). This gives \( (c^r)^2 = c^r C^r \), and therefore \( (C^r c^{-r}) C^r = C^r \). Since \( C^r : C^r = R \) the equality gives that \( C^r \subset (c^r) \subset C^r \) and therefore \( C^r = (c^r) \). Thus \( C \) is invertible, as desired.

\begin{question}
Suppose there is an ideal \( L \) such that \( L^2 = C \). Sometimes they imply that \( \text{cdeg}(R) \) is even; can it be the same with \( \text{bideg}(R) \)?
\end{question}

\begin{question}
If \( L^n \simeq C \) and \( L \) is reflexive must \( R \) be Gorenstein?
\end{question}

\begin{exercise}
Let \((R, m)\) be a Cohen-Macaulay local ring of dimension \( d \geq 2 \). Let \( C \) be a Cohen-Macaulay ideal of codimension 1. Goal: To derive a criterion for \( C \) to be a canonical ideal. Let \( x \) be regular mod \( C \). If \( C/xC \) is a canonical ideal of \( R/xR \) then \( C \) is canonical ideal of \( R \). We make use of two facts:

1. \( C \) is a canonical module iff \( \text{Ext}_R^j(R/m, C) = R/m \) for \( j = d \) and 0 otherwise (13 Theorem 6.1).\)
2. The change of rings equation asserts \( \text{Ext}_R^j(R/m, C) \simeq \text{Ext}_{R/(x)}^{j-1}(R/m, C/xC) \), \( j \leq d \).
3. More generally, how do we tell when a Cohen-Macaulay ideal \( I \) is irreducible?

\end{exercise}

4. Minimal values of canonical degrees

Now we begin to examine the significance of the values of \( \text{bideg}(R) \). We focus on rings of dimension 1.

### Almost Gorenstein rings

First we recall the definition of almost Gorenstein rings \((2, 9, 10)\).

\begin{definition}
\((10 \text{ Definition 3.3})\) A Cohen-Macaulay local ring \( R \) with a canonical module \( \omega \) is said to be an almost Gorenstein ring if there exists an exact sequence of \( R \)-modules \( 0 \rightarrow R \rightarrow \omega \rightarrow X \rightarrow 0 \) such that \( \nu(X) = e_0(X) \). In particular if \( R \) has dimension one \( X = (R/m)^{r-1} \), \( r = r(R) \).

\end{definition}

\begin{theorem}
Let \((R, m)\) be a Cohen-Macaulay local ring of dimension 1 with a canonical ideal \( C \). If \( R \) is almost Gorenstein then \( \text{bideg}(R) = 1 \).
\end{theorem}

\begin{proof}
In dimension 1, from
\begin{equation}
0 \rightarrow (c) \longrightarrow C \longrightarrow X \rightarrow 0,
\end{equation}
\( X \) is a vector space \( k^{r-1} \), \( r = r(R) \). To determine \( C^{**} \) apply \( \text{Hom}(-, (c)) \) to \((1)\) to get \( \text{Hom}(C, (c)) = m \). On the other hand, by Proposition 3.3, \( C^{**} = \text{Hom}(m, (c)) = L_0 \), the socle of \( R/(c) \) [which is generated by \( r \) elements], properly containing \( C \) that is \( C^{**} = L \), the socle of \( C \). Therefore \( \text{bideg}(R) = \lambda(L/C) = 1 \).
\end{proof}

The example below shows that the converse does not holds true.

\begin{example}
Consider the monomial ring \((\text{called to our attention by Shiro Goto}) \) \( R = \mathbb{Q}[t^5, t^7, t^9], m = (x, y, z) \). We have a presentation \( R = \mathbb{Q}[x, y, z]/P \), with \( P = (y^2 - xz, x^5 - yz^2, z^3 - x^4y) \). Let us examine some properties of \( R \). For simplicity we denote the images of \( x, y, z \) in \( R \) by the same symbols. An explanation for these calculations can be found in the proof of Theorem 6.2.
\end{example}
(1) Let $C = (x, y)$. A calculation with Macaulay2 shows that if $D = C : \mathfrak{m}$, then $\lambda(D/C) = 1$. Therefore $C$ is a canonical ideal by Corollary 3.5.

(2) $(c) : C \neq \mathfrak{m}$ so $R$ is not almost Gorenstein. However $C^{**} = (c) : [(c) : C]$ satisfies [by another Macaulay2 calculation] $\lambda(C^{**}/C) = 1$, so $C^{**} = L$. This shows that $\text{bideg}(R) = 1$.

(3) This example shows that $\text{bideg}(R) = 1$ holds [for dimension one] in a larger class rings than almost Gorenstein rings.

**Goto rings.** We now examine the significance of a minimal value for $\text{bideg}(R)$. Suppose $R$ is not a Gorenstein ring.

**Definition 4.4.** A Cohen-Macaulay local ring $R$ of dimension $d$ is a Goto ring if it has a canonical ideal and $\text{bideg}(R) = 1$.

**Questions 4.5.**

1. What other terminology should be used? Nearly Gorenstein ring has already been used in [13]. We will examine its relationship to Goto rings.

2. Almost Gorenstein rings of dimension one have $\text{red}(C) = 2$. What about these rings?

3. What are the properties of the Cohen-Macaulay module $X$ defined by
$$0 \to (c) \to C \to X \to 0,$$

*nearly Ulrich or pre-Ulrich bundles?*

4. $\lambda(C^{**}/C) = 1$ implies that
$$C^{**}/C = L/C = C : \mathfrak{m}/C \simeq R/\mathfrak{m}.$$

5. If $L_0 = (c) : \mathfrak{m}$, the socle of $(c)$, is equal to $L$, then $C/(c)$ is a vector space, so $R$ is almost Gorenstein and conversely.

6. In all cases $L^2 = CL$, $mL = mC$, so $C/mC \hookrightarrow L/mL$ ([4] Theorem 3.7). Therefore if $R$ is not almost Gorenstein, $c$ cannot be a minimal generator of $L$ and thus $L = (\mathcal{C}, \alpha)$, $\nu(L) = r + 1$, with $\alpha \in L_0$, or $L = (L_0, \beta)$, with $\beta \in C$.

7. This says that
$$L = (x_1, \ldots, x_r, \beta), \quad x_i \in (c) : \mathfrak{m},$$
$$C = (c, x_2, \ldots, x_{r-2}, \beta), \quad c \in mL$$
$$C^* = c^{-1}[(c) : C]$$
$$C^{**} = (c) : [(c) : C] = L$$
$$C^{***} = c^{-1}[(c) : L]$$
$$(c) : C = (c) : L = (c) : \beta.$$

Let us attempt to get $\text{bideg}(R)$ for $R$ almost Gorenstein but of dimension $\geq 2$. Note that $\text{cdeg}(R) = r - 1$. Assume $d = r = 2$: can we complete the calculation?

**Proposition 4.6.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with a canonical ideal $C$. The following hold

1. **From the**
$$0 \to (c) \to C \to X \to 0$$

*applying $\text{Hom}(\cdot, (c))$, we get*
$$0 \to \text{Hom}(C, (c)) \to R \to \text{Ext}(X, (c)) \to \text{Ext}(C, (c)) \to 0.$$
The image of $\text{Hom}(C, (c))$ in $R$ is a proper ideal $N$, and so $R/N$ is a submodule of $\text{Ext}(X, (c))$ which is annihilated by $\text{ann}(X)$ (which contains $(c)$).

We note (see [16, p. 155]) that

$$\text{Ext}(X, (c)) \approx \text{Ext}_{R/(c)}^{1}(X, R/(c)) = ((c) : N)/(c), \quad \text{if } r = 2.$$ 

Suppose $C$ is equimultiple and $\text{bideg}(R) = 1$. Then $R$ is Gorenstein at all primes of codimension one with one exception, call it $p$. This means that $C^{**}/C$ is $p$-primary and

$$\text{deg}(C^{**}/C) = \text{bideg}(R_{p}) \cdot \text{deg}(R/p) = 1.$$ 

What sort of modules are $C/(c)$ and $C^{**}/C$? The first we know is Cohen-Macaulay, how about the second?

- $C^{**} = C : p$: both are divisorial ideals that agree in codimension one.
- If $\text{dim } R = 2$, $C^{**}/C$ is Cohen-Macaulay of dimension one and multiplicity one so what sort of module is it? It is an $R/p$-module of rank one and $R/p$ is a discrete valuation domain so $C^{**}/C \simeq R/p$.
- In all dimensions, $C^{**}/C$ has only $p$ for associated prime and is a torsion-free $R/p$-module of rank one. Thus it is isomorphic to an ideal of $R/p$. $C^{**}/C$ has also the condition $S_2$ of Serre from the exact sequence

$$0 \rightarrow C^{**}/C \rightarrow R/C \rightarrow R/C^{**} \rightarrow 0,$$

$R/C^{**}$ has the condition $S_1$. If $R$ is complete and contains a field by [18] $R/p$ is a regular local ring and therefore $C^{**}/C \simeq R/p$.

- Some of these properties are stable under many changes of the rings. Check for generic hyperplane section.

**Questions 4.7.** (1) How to pass from $\text{bideg}(A)$ of a graded algebra $A$ to $\text{bideg}(B)$ of one of its Veronese subalgebras?

(2) If $S$ is a finite injective extension of $R$, is $\text{bideg}(S) \leq |S : R| \cdot \text{bideg}(R)$?

**5. Change of rings**

Let $\varphi : R \rightarrow S$ be a homomorphism of Cohen-Macaulay rings. We examine a few cases of the relationship between $\text{cdeg}(R)$ and $\text{cdeg}(S)$ induced by $\varphi$. We skip polynomial, power series and completion since flatness makes it straightforward.

**Finite extensions.** If $R \rightarrow S$ is a finite injective homomorphism of Cohen-Macaulay rings and $C$ is a canonical ideal of $R$, then $D = \text{Hom}(S, C)$ is a canonical module for $S$. Recall how $S$ acts on $D$: If $f \in D$ and $a, b \in S$, then $a \cdot f(b) = f(ab)$.
Augmented rings. A case in point is that of the so-called augmented extensions. Let \((R, m)\) be an 1-dimensional Cohen-Macaulay local ring with a canonical ideal. Assume that \(R\) is not a valuation domain. Suppose \(A\) is the augmented ring \(R \times m\). That is, \(A = R \oplus me, e^2 = 0\). [Just to keep the components apart in computations we use \(e\) as a place holder.]

Let \((c)\) be a minimal reduction of the canonical ideal \(C\). We may assume that \(C \subset m^2\) by replacing \(C\) by \(\alpha C\) if necessary. Then a canonical module \(D\) of \(A\) is \(D = \text{Hom}_R(A, C)\). Let us identify \(D\) to an ideal of \(A\). For this we follow [3].

Let \(R\) be a commutative ring with total quotient ring \(Q\) and let \(F\) denote the set of \(R\)-submodules of \(Q\). Let \(M, K \in F\). Let \(M^\vee = \text{Hom}_R(M, K)\) and let

\[ A = R \times M \]

denote the idealization of \(M\) over \(R\). Then the \(R\)-module \(M^\vee \oplus K\) becomes an \(A\)-module under the action

\[ (a, m) \circ (f, x) = (af, f(m) + ax), \]

where \((a, m) \in A\) and \((f, x) \in M^\vee \times K\). We notice that the canonical homomorphism

\[ \varphi : \text{Hom}_R(A, K) \to M^\vee \times K \]

such that \(\varphi(f) = (f \circ \lambda, f(1))\) is an \(A\)-isomorphism, where \(\lambda : M \to A, \lambda(m) = (0, m)\), and that \(K : M \in F\) and \((K : M) \times K \subset Q \times Q\) is an \(A\)-submodule of \(Q \times Q\), the idealization of \(Q\) over itself. When \(Q \cdot M = Q\), identifying \(\text{Hom}_R(M, K) = K : M\), we therefore have a natural isomorphism

\[ \text{Hom}_R(A, K) \cong (K : M) \times K \]

of \(A\)-modules.

Using this observation, setting \(K = C, M = m\), we get the following

\[ D = \text{Hom}_R(R \oplus me, C) = \text{Hom}_R(m, C) \oplus C e = L + Ce \]

because \(C : Q m = C : R m \subset C : Q C = R\). Denote \(L = \text{Hom}_R(m, C)\). Then \(L \cong C : m \subset m\) so that \(D\) is an ideal of \(A\).

Let us determine \(D^{**}\). The total ring of fractions of \(A\) is \(Q \times Q e\). If \((a, b e) \in Q \times Q e\) is in

\[ \text{Hom}(D, A) = \text{Hom}((C \oplus Le), (R \oplus me)), \]

then \(aC \subset R, aL \subset m\) and \(aC \neq R\) as \(R\) is not Gorenstein. Thus \(a \in C^*\). On the other hand, \(bC \subset m\). Thus \(b \in C^*\) and conversely. Thus

\[ D^* = C^* \oplus C^*. \]

Suppose \((a, b e) \in D^{**}\),

\[ D^{**} = \text{Hom}((C^* \oplus C^* e), (R \oplus me)). \]

Then \(aC^* \subset R\) and \(C^{**} \neq R\). In turn \(bC^* \subset me\) and so \(b \in C^{**}\), and conversely. Thus

\[ D^{**} = C^{**} \oplus C^{**} e. \]

Let us summarize this calculation. This gives

\[ D^{**}/D = C^{**}/C \oplus C^{**} e/L e. \]

**Proposition 5.1.** Let \((R, m)\) be a Cohen-Macaulay local ring with a canonical ideal and let \(A = R \times m\). Then

\[ \text{bideg}(A) = 2 \text{bideg}(R) - 1. \]

In particular if \(R\) is a Goto ring then \(A\) is also a Goto ring.
Products. Let \( k \) be a field, and let \( A_1, A_2 \) be two finitely generated Cohen-Macaulay \( k \)-algebras. Let us look at the canonical degrees of the product \( A = A_1 \otimes_k A_2 \).

- As a rule, if \( B_i, C_i \) are \( A_i \)-modules, we use the natural isomorphism
  \[
  \text{Hom}(B_1 \otimes B_2, C_1 \otimes C_2) = \text{Hom}_{A_1 \otimes_k A_2}(B_1 \otimes_k B_2, C_1 \otimes_k C_2),
  \]
  which works out to be
  \[
  \text{Hom}_{A_1}(B_1, C_1) \otimes_k \text{Hom}_{A_2}(B_2, C_2).
  \]
- If \( A_i, i = 1, 2 \), are localizations [of finitely generated \( k \)-algebras] then \( B \) is an appropriate localization. [If \( m_i \) are the maximal ideals of \( A_i \), pick primes \( M_i \) in \( B_i \) and a prime \( B \) over both \( m_i \).] If \( B_i \) are finite \( A_i \)-modules and \( F_i \) are minimal resolutions over \( A_i \) [or over \( S_i \), a localization in the next item], then \( F_1 \otimes_k F_2 \) is a resolution whose entries lie in appropriate primes.
- If \( S_i \to A_i, i = 1, 2 \), are presentations of \( A_i \), \( S = S_1 \otimes S_2 \to A_1 \otimes A_2 = A \) gives a presentation of \( A \) and from it we gather the invariants [all \( \otimes \) over \( k \)].

Proposition 5.2. Let \( A_i, i = 1, 2 \), be as above. Then

1. \( A \) is Cohen-Macaulay
2. \( C = C_1 \otimes C_2, C^{**} = C_1^{**} \otimes C_2^{**} \), \( r(A) = r(A_1) \cdot r(A_2) \)
3. If \((c_i)\) is a minimal reduction of \( C_i, i = 1, 2 \), then \((c) = (c_1) \otimes (c_2)\) is a minimal reduction for \( C \) and
   \[
   C/(c) = C_1/(c_1) \otimes C_2 \oplus C_1 \otimes C_2/(c_2),
   \]
   \[
   C^{**}/C = C_1^{**}/C_1 \otimes C_2 \oplus C_1 \otimes C_2^{**}/C_2,
   \]
   \[
   \text{cdeg}(A) = \text{cdeg}(A_1) \cdot \text{deg}(A_2) + \text{deg}(A_1) \cdot \text{cdeg}(A_2),
   \]
   \[
   \text{bideg}(A) = \text{bideg}(A_1) \cdot \text{deg}(A_2) + \text{deg}(A_1) \cdot \text{bideg}(A_2).
   \]
4. If \( A_i \) is almost Gorenstein, that is \( \text{cdeg}(A_i) = r(A_i) - 1 \), then
   \[
   \text{cdeg}(A) = (r_1 - 1) \cdot \text{deg}(A_2) + \text{deg}(A_1) \cdot (r_2 - 1).
   \]

Suppose further, \( A_1 = A_2 \), that is \( A \) is the square. Then
\[
\text{cdeg}(A) = 2 \text{deg}(A_1)(r_1 - 1).
\]

In this case, \( A \) is almost Gorenstein if \( \text{cdeg}(A) = r_1^2 - 1 \), that is
\[
2 \text{deg}(A_1) = r_1 + 1.
\]

Questions 5.3. (1) How to pass from \( \text{bideg}(A) \) of a graded algebra \( A \) to \( \text{bideg}(B) \) of one of its Veronese subalgebras?

2. If \( S \) is a finite injective extension of \( R \), is \( \text{bideg}(S) \leq [S : R] \cdot \text{bideg}(R) \)?
Hyperplane sections. A desirable comparison is that between \( \text{bideg}(R) \) and \( \text{bideg}(R/(x)) \) for appropriate regular element \( x \). [The so-called ‘Lefschetz type’ assertion.] We know that if \( C \) is a canonical module for \( R \) then \( C/(x)C \) is a canonical module for \( R/(x) \) with the same number of generators, so type is preserved under specialization. However \( C/(x)C \) may not be isomorphic to an ideal of \( R/(x) \). Here is a case of good behavior. Suppose \( x \) is regular modulo \( C \). Then for the sequence
\[
0 \to C \to R \to R/C \to 0,
\]
we get the exact sequence
\[
0 \to C/(x)C \to R/(x) \to R/(C, x) \to 0,
\]
so the canonical module \( C/(x)C \) embeds in \( R/(x) \). We set \( S = R/(x) \) and \( D = (C, x)/(x) \) for the image of \( C \) in \( S \). We need to compare \( \text{bideg}(R) = \deg(R/C) - \deg(R/C^{**}) \) and \( \text{bideg}(S) = \deg(S/D) - \deg(S/D^{**}) \).

We don’t know how to estimate the last term. We can choose \( x \) so that \( \deg(R/C) = \deg(S/D) \), but need, for instance to show that \( C^{**} \) maps into \( D^{**} \). Let \( c \) be as in Proposition \( \boxed{3.3} \) and pick \( x \) so that \( c, x \) is a regular sequence. Set \( C_1 = (c) : C \), \( C_2 = (c) : C_1 \), \( D_1 = (c) : D \), \( D_2 = (c) : D_1 \). We have \( C_1D \subset (c)S \) and thus \( C_1S \subset D_1 \). This shows that \( D^{**} = (c) :S \subset D_1 \subset (c) :S \subset C_1 \), and thus \( D_1C_2 \subset (c) \) but not enough to show \( D_2 \subset C_2S \).

A model for what we want to have is [8 Proposition 6.12] asserting that if \( C \) is equimultiple then \( \text{cdeg}(R) \leq \text{cdeg}(R/(x)) \). For \( \text{bideg}(R) \), in consistency with Conjecture \( \boxed{3.2} \).

Conjecture 5.4. Under the conditions above, \( \text{bideg}(R) \geq \text{bideg}(R/(x)) \).

6. Monomial subrings

Let \( H \) be a finite subset of \( \mathbb{Z}_{\geq 0}, \{0 = a_0 < a_1 < \ldots < a_n\} \). For a field \( k \) we denote by \( R = k[H] \) the subring of \( k[t] \) generated by the monomials \( t^{a_i} \). We assume that \( \text{gcd}(a_1, \ldots, a_n) = 1 \). We also use the bracket notation \( H = \langle a_1, a_2, \ldots, a_n \rangle \) for the exponents. For reference we shall use [6 p. 553] or [21 Section 8.7].

There are several integers playing roles in deriving properties of \( R \), with the emphasis on those that lead to the determination of \( \text{bideg}(R) \).

1. There is an integer \( s \) such that \( t^n \in R \) for all \( n \geq s \). The smallest such \( s \) is called the conductor of \( H \) or of \( R \), which we denote by \( c \) and \( \mathfrak{c}k[t] \) is the largest ideal of \( k[t] \) contained in \( k[H] \). \( c-1 \) is called the Frobenius number of \( R \) and reads its multiplicity, \( c-1 = \deg(R) \).
2. For any integer \( a > 0 \), say \( a = c \) of \( H \), the subring \( A = k[t^a] \) provides for a Noether normalization of \( R \). This permits the passage of many properties from \( A \) to \( R \), and vice-versa. \( R \) is a free \( A \)-module and taking into account the natural graded structure we can write
\[
R = \bigoplus_{j=1}^{m} At^{\alpha_j}.
\]

Note that \( s = \sum_{j=1}^{m} \alpha_j \).
3. How to read other invariants of \( R \) such as its canonical ideal \( C \) and \( \text{red}(C) \) and its canonical degrees \( \text{cdeg}(R) \) and \( \text{bideg}(R) \)?
Monomial curves. Let $R = k[t^a, t^b, t^c]$, $a < b < c$, $\gcd(a, b, c) = 1$. Assume $R$ is not Gorenstein, $\omega = (1, t^s)$.

It would be helpful to have available descriptions of the canonical ideal and attached invariants. Some of the information is available from the Frobenius equations which can be expressed as the $2 \times 2$ minors of the matrix (12)

$$
\varphi = \begin{bmatrix}
  x^{a_1} & z^{c_2} \\
  z^{c_1} & y^{b_2} \\
  y^{b_1} & x^{a_2}
\end{bmatrix}.
$$

Calculating the canonical ideal and its bidual. Let $R = A/P$ be a Cohen-Macaulay integral domain and $A$ a Gorenstein local ring. If $\text{codim} P = g$, $\omega = \text{Ext}^g_A(R, A)$ is a canonical module for $R$. Since $R$ is an integral domain, $\omega$ may be identified to an ideal of $R$. A canonical module of $R$ is obtained by deleting one row and a column according to the following comments and mapping the remaining entries to $R$.

1. Let $L = (x_1, \ldots, x_g)$ be a regular sequence contained in $P$. Then

$$
\omega = \text{Ext}^g_A(A/P, A) = \text{Hom}_{A/L}(A/P, A/L) = (L : P)/L.
$$

The simplest case to apply this formula is when $P = (L, f)$ so this becomes

$$
\omega = (L : f)/L.
$$

2. Suppose $P = I_2(\varphi)$ is a prime ideal of codimension two where

$$
\varphi = \begin{bmatrix}
  a_1 & c_2 \\
  c_1 & b_2 \\
  b_1 & a_2
\end{bmatrix}
$$

We are going to argue that if the $2 \times 2$ minors $x_1, x_2$,

$$
\begin{align*}
x_1 &= a_1a_2 - b_1c_2 \\
x_2 &= a_1b_2 - c_1c_2
\end{align*}
$$

that arise from keeping the first row form a regular sequence then for $f = b_1b_2 - c_1a_2$, we have

$$
(x_1, x_2) : f = (a_1, c_2).
$$

With these choices we have

$$
(L : f)/L \simeq (a_1, c_2, P)/P.
$$

Indeed the nonvanishing mapping

$$
\omega = (L : f)/L \mapsto R/P
$$

of modules of rank one over the domain $R/P$ must be an embedding.

Example 6.1. Let $R = k[t^a, t^b, t^c]$, $b - a = c - b = d$: Then $b = a + d, c = a + 2d$ and

$$
\varphi = \begin{bmatrix}
  x & y & z^p \\
  y & x^q & z
\end{bmatrix},
$$
Note $p(a + 2d) - qa = d$ from $(p + 1)(a + 2d) = qa + (a + d)$ and $(x, y) \mapsto C = (t^a, t^b) = t^a(1, t^d)$. By Proposition 3.3 we have

$$(x): (x, y) = (x, y, z^p),$$

$$C^{**} = (x): (x, y, z^p) = (x, z, y),$$

$$\text{bideg}(R) = \lambda(R/C) - \lambda(R/C^{**}) = \lambda(R/(x, y)) - \lambda(R/(x, y, z)) = p - 1.$$ 

**Proposition 6.2.** Let $R = k[t^a, t^b, t^c]$ be a monomial ring and denote by $\varphi$ its Herzog matrix

$$\varphi = \begin{bmatrix} x^a & z^c \\ z^c & y^b \\ y^b & x^a \end{bmatrix}.$$ 

Then

$$\text{bideg}(R) = a_1 \cdot b_2 \cdot c_2.$$ 

**Proof.** From $\varphi$ we take $C = (x^{a_1}, z^{c_2})$, where harmlessly we chose $a_1 \leq a_2$. Then

$$(x^{a_1}): (x^{a_1}, z^{c_2}) = (x^{a_1}, z^{c_1}, y^{b_1}),$$

$$C^{**} = (x^{a_1}): (x^{a_1}, y^{b_1}, z^{c_1}) = (x^{a_1}, y^{b_2}, z^{c_2}),$$

$$\text{bideg}(R) = \lambda(R/C) - \lambda(R/C^{**}) = \lambda(R/(x^{a_1}, z^{c_2})) - \lambda(R/(x^{a_1}, y^{b_1}, z^{c_2})).$$

and since

$$(x^{a_1}, z^{c_2}) = (x^{a_1}, y^{b_1}y^{b_2}, z^{c_2})$$

we have

$$\text{bideg}(R) = a_1 \cdot (b_1 + b_2) \cdot c_2 - a_1 \cdot b_1 \cdot c_2$$

$$= a_1 \cdot b_2 \cdot c_2.$$ 

\[\square\]

**Remark 6.3.** For the monomial algebra $R = k[t^a, t^b, t^c]$ the value of $\text{bideg}(R)$ is also calculated in [13 Proposition 7.9]. According to [9 Theorem 4.1], $\text{cdeg}(R) = a_2 \cdot b_2 \cdot c_2$, which supports the Comparison Conjecture 3.2

### 7. Rees algebras

Let $R$ be a Cohen-Macaulay local ring and $I$ an ideal such that the Rees algebra $S = R[It]$ is Cohen-Macaulay. We consider a few classes of such algebras. We denote by $C$ a canonical ideal of $R$ and set $G = \text{gr}_I(R)$.

**Rees algebras with expected canonical modules.** (See [15] for details.) This means $\omega_S = \omega_R(1, t)^m$, for some $m \geq -1$. This will be the case when $G = \text{gr}_I(R)$ is Gorenstein ([15 Theorem 2.4, Corollary 2.5]). We actually assume $I$ of codimension at least 2. We first consider the case $C = R$. Set $D = (1, t)^m$, pick $a$ such that $a$ is a regular element in $I$ and its initial form $\overline{a}$ is regular on $G$, and finally replace $D$ by $a^mD \subset S$.

**Proposition 7.1.** Let $R$ be a Gorenstein local ring, $I$ an ideal of codimension at least two and $S$ its Rees algebra. If the canonical module of $S$ has the expected form then $S$ is Gorenstein in codimension less than $\text{codim}(I)$, in particular $\text{bideg}(S) = 0$. 

Proof. It is a calculation in [15, p. 294] that $S : (1, t)^m = I^m S$. It follows that

$$(I^m, I^m t) \subset I^m S \cdot (1, t)^m.$$ 

Since codim$(I^m, (I^m t)) = \text{codim}(I, It) = \text{codim}(I) + 1$, the assertion follows. Note that this implies that $\omega_S$ is free in codimension one and therefore it is reflexive by a standard argument. Finally by Proposition 3.1, $\text{bideg}(S) = 0$, and therefore that $\text{cdeg}(S) = 0$. \qed

Rees algebras of minimal multiplicity. $R$ Cohen-Macaulay, $I$ $m$-primary, $J$ minimal reduction, $I^2 = JI$. Then $IR[It] = IR[It]$ is Cohen-Macaulay, and if $\dim R > 1$ then $R[It]$ is Cohen-Macaulay.

A source of these ideals arises from irreducible ideals in Gorenstein rings, for instance $J$ is a system of parameters and $I = J : m$.

- Let $S_0 = R[It], S = R + ItS_0$. If $R$ is Gorenstein, $\omega_{S_0} = (1, t)^{d-2} S_0$ and by the change of rings formula

$$\omega_S = \text{Hom}_{S_0}(S, (1, t)^{d-2} S_0) = (1, t)^{d-2} S_0 : S.$$

- $d = 2$. Then $\omega_S$ is the conductor $L$ of $S$ relative to $S_0$. In the case $mS_0$ is a prime ideal of $S_0$ so the conductor could not be larger as it would have grade at least two and then $S = S_0$: $\omega_S = mS_0$. If $d > 2$ it seems that $m(1, t)^{d-2} S_0$ will work.

- $S_0$ is Gorenstein in codimension one. If $P$ is such prime and $P \cap R = q \neq m$, then $(S_0)_q = S_q$, so $S_P = (S_0)_P$ is Gorenstein in codimension one. Thus we may assume $P \cap R = m$, so that $P = mS_0$. What is $S_P$ like? To be Gorenstein (see next Example) would mean $\text{Hom}(S, S_0) \simeq S$ at $P$, that is

$$S_0 : S = P Su.$$ 

- Example Let $R = k[x, y], I = (x^3, x^2y^2, y^3)$. Then for the reduction $Q = (x^3, y^3), I^2 = QI$. The Rees algebra $S = R[It] = k[x, y, u, v, w]/L$, $L = (x^2u - xv, y^2w - yv, v^2 - xyuw)$. $L$ is given by the $2 \times 2$ minors of

$$\varphi = \begin{bmatrix} v & xw \\ yu & v \\ x & y \end{bmatrix},$$

whose content is $(x, y, v)$. It follows that $S$ is not Gorenstein in codimension one [it would require a content of codimension at least four].

- $D = \text{Hom}(S, S_0) = mS_0 = mS$. Thus $\text{deg}(S/D) = 2$ and since $D^{**} \neq D, \text{deg}(S/D^{**}) = 1$. It follows that $\text{bideg}(S) = 1.$
Rees algebras of ideals with the expected defining relations. Let $I$ be an ideal of the Cohen-Macaulay local ring $(R, m)$ or a polynomial ring $R = k[t_1, \ldots, t_d]$ over the field $k$ with a presentation

$$R^m \xrightarrow{\varphi} R^n \longrightarrow I = (b_1, \ldots, b_n) \to 0.$$  
Assume that $\text{codim}(I) \geq 1$ and that the entries of $\varphi$ lie in $m$. Denote by $L$ its ideal of relations

$$0 \to L \longrightarrow S = R[T_1, \ldots, T_n] \xrightarrow{\varphi} R[I] \to 0, \quad T_i \mapsto b_i t.$$
$L$ is a graded ideal of $S$ and its component in degree 1 is generated by the $m$ linear forms

$$f = [f_1, \ldots, f_m] = [T_1, \ldots, T_n] : \varphi.$$

Sylvester forms. Let $f = \{f_1, \ldots, f_s\}$ be a set of polynomials in $L \subset S = R[T_1, \ldots, T_n]$ and let $a = \{a_1, \ldots, a_s\} \subset R$. If $f_i \in (a)S$ for all $i$, we can write

$$f = [f_1, \ldots, f_s] = [a_1, \ldots, a_q] \cdot A = a \cdot A,$$
where $A$ is an $s \times q$ matrix with entries in $S$. We call (a) a $R$-content of $f$. Since $a \not\subset L$, then the $s \times s$ minors of $A$ lie in $L$. By an abuse of terminology, we refer to such determinants as Sylvester forms, or the Jacobian duals, of $f$ relative to $a$. If $a = I_1(\varphi)$, we write $A = B(\varphi)$, and call it the Jacobian dual of $\varphi$. Note that if $\varphi$ is a matrix with linear entries in the variables $x_1, \ldots, x_d$, then $B(\varphi)$ is a matrix with linear entries in the variables $T_1, \ldots, T_n$.

**Definition 7.2.** Let $I = (b_1, \ldots, b_n)$ be an ideal with a presentation as above and let $\phi = a(a_1, \ldots, a_q) = I_1(\varphi)$. The Rees algebra $R[It]$ has the expected relations if

$$L = (T : \varphi, I_s(B(\varphi))).$$

There will be numerous restrictions to ensure that $R[It]$ is a Cohen-Macaulay ring and that it is amenable to the determination of its canonical degrees. We consider some special cases grounded on [17, Theorem 1.3] and [19, Theorem 2.7]

**Theorem 7.3.** Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over an infinite field, let $I$ be a perfect $R$-ideal of grade 2 with a linear presentation matrix $\varphi$, assume that $\nu(I) > d$ and that $I$ satisfies $G_d$ (meaning $\nu(I_p) \leq \dim R_p$ on the punctured spectrum). Then $\ell(I) = d$; $r(I) = \ell(I) - 1$; $R[It]$ is Cohen-Macaulay, and $L = (T : \varphi, I_d(B(\varphi)))$.

The canonical ideal of these rings is described in [19, Theorem 2.7] (for $g = 2$).

**Theorem 7.4.** Let $J$ be a minimal reduction of $I$. If $K = J : I$, $C = KtR[It]$.

Our task becomes to calculate canonical degrees for these rings for $g = 2$, starting with the dual and the bi-dual of

$$C \simeq KR[It] = K + KIt + KI^2t^2 + \cdots.$$

Let $a$ be a regular element of $K$ and set $e = a$. Then

$$(e) : C = \sum L_i t^i \subset R[It], \quad bt^i \in L_i \quad \text{iff} \quad bKI^j \subset aI^{i+j}.$$  
Thus for $b \in L_0$, $b \cdot K \subset (a)$ so $b = r \cdot a$ since $\text{codim}(K) \geq 2$. Hence $L_0 = (a)$. For $bt \in L_1$, $bK \subset aI$, $b = ra$ with $r \in I : K$. As $rKI \subset I^2$, we have $L_1 = a(I : K)t$. For $i \geq 2$, $b \in L_i$ we have $b = rat^i$ with $rK \subset I^i$. Hence $L_i = a(I^i : K)t^i$. In general we have $L_i = a(I^i : K)t^i$.

$$(e) : C = a(R + (I : K)t + (I^2 : K)t^2 + \cdots).$$
This should enable us to write \( \text{trace}(C) \):

\[
C \cdot C^* = C \cdot e^{-1}((e : C) = KR[I] \cdot (R + (I : K)t + (I^2 : K)t^2 + \cdots
\]

\[
= K + (K(I : K) + KI)t + (K(I^2 : K) + KI(I : K))t^2 + \cdots
\]

\[
= K(I : K)t + K(I^2 : K)t^2 + \cdots
\]

Corollary 7.5. For these ideals \( \text{bideg}(R[I]) \neq 0 \) (and similarly \( \text{cdeg}(R[I]) \neq 0 \)).

Proof. We need to calculate

\[
C^{**} = (a) : ((a) : C) = \sum_j L_j t^j = R[I] : (R + (I : K)t + (I^2 : K)t^2 + \cdots)
\]

For \( b \in L_0 \) and \( i \geq 1, b(I^i : K) \subset I^i \) thus \( b \in \bigcap_i I^i : (I^i : K) = L_0 \). In general it is clear that \( KI^j \subset L_I \). Note that \( L_0 = R : (R : K) = R \). It follows that \( C \neq C^{**} \) and hence \( \text{bideg}(R[I]) \neq 0 \).

8. Canonical degrees of \( A = m : m \)

Let \( (R, m) \) be a Cohen Macaulay local ring of dimension 1, and set \( A = \text{Hom}(m, m) = m :_Q m \). Assume that \( R \) is not a DVR. Let \( C \) be its canonical ideal. (We can also discuss some of the same questions by replacing \( m \) by a prime ideal \( p \) such that \( R_p \) is not a DVR.)

General properties. We begin by collecting elementary data on \( A \).

- \( A = m : m \subset R : m \) and since \( m \cdot (R : m) \neq R \) [as otherwise \( m \) is principal] \( R : m = A \).

- If \( R \) is a Cohen-Macaulay local ring of dimension one that is not a DVR, then \( A = R : m \) as \( m \cdot \text{Hom}(m, R) \subset m \) and therefore

\[
A = m :_Q m = R :_Q m = 1/\cdot (\cdot : R \cdot m).
\]

Indeed, if \( q \in m :_Q m \) and \( x \) is a regular element of \( m \), let \( a = qx \in m \). Then \( q = (1/x)a \) and

\[
am = qx = xqm \subset xm.
\]

Thus \( A = (1/x)((x) : m) \).

This makes \( A \) amenable by calculation using software such as Macaulay2 (II).

- A relevant point is to know when \( A \) is a local ring. Let us briefly consider some cases. Let \( L \) be an ideal of the local ring \( R \) and suppose \( L = I \oplus J, L = I + J, I \cap J = 0 \), is a non-trivial decomposition. Then \( I : (J : I) = 0 \) and \( J : (J : I) = 0 \) and thus if \( \text{grade}(I, J) = 1 \) [maximum possible by the Abhyankar–Hartshorne Lemma], then \( I : J = 0 \). It follows that

\[
\text{Hom}(L, L) = \text{Hom}(I, I) \times \text{Hom}(J, J),
\]

and therefore \( \text{Hom}(L, L) \) is not a local ring.
• Suppose $R$ is complete, or at least Henselian. If $A$ is not a local ring, by the Krull-Schmidt Theorem $A$ admits a non-trivial decomposition of $R$-algebras

$$A = B \times C.$$ 

Since $m = mA$, we have a decomposition $m = mB \oplus mC$. If we preclude such decompositions then $A$ is a local ring. Among these cases are: analytically irreducible rings, in particular they include the localization of any monomial ring.

**Number of generators of $A$.**

• Since $m$ is an ideal of both $R$ and $A$, $R : A = m$. Thus $A/R \simeq (R/m)^n = k^n$. The exact sequence

$$0 \to R \to A \to k^n \to 0,$$

yields

$$0 \to R/m \to A/m = A/mA \to k^n \to 0,$$

which gives $\nu(A) = n + 1$ since $R \not\subset mA = m$.

• $D = C : A$ is the canonical ideal of $A$. Applying $\text{Hom}(\cdot, C)$ to the exact sequence above we get

$$0 \to D \to C \to \text{Ext}^1(k^n, C) \simeq k^n \to 0.$$

Thus $C/D \simeq k^n$ and $mC \subset D$. As

$$0 \to mC \to C \to k^r \to 0,$$

$$D/mC \simeq k^{r-n}.$$

**Theorem 8.1.** Let $(R, m)$ be a Cohen-Macaulay local ring of dimension one and type $r(R) = r$. Then

$$A = \text{Hom}(m, m) = m : q \quad m = x^{-1} \cdot ((x) : R)$$

for any regular element $x \in m$. Furthermore

$$\nu_R(A) = r + 1.$$ 

**Proof.** Since $A = x^{-1}((x) : m)$ for any regular element of $m$, we have

• $A = 1/x \cdot ((x) : m)$ gives $\nu(A) = \lambda((x) : m)$ and since

$$((x) : m)/xR \simeq k^r$$

then

$$r \leq \nu((x) : m) \leq r + 1.$$

• Writing $(x) : m = (x, y_1, \ldots, y_r)$, $n = r - 1$ would mean that $x$ is not a minimal generator of $(x) : m$, so

$$x \in m((x) : R \setminus m).$$

In particular $x \in m^2$. If you preclude this with an initial choice of $x \in m \setminus m^2$, we would have $r = n$ always.

\[ \square \]

**Corollary 8.2.** If $(R, m)$ is a local ring and $C$ is the canonical ideal of $R$ then $D = mC$ is the canonical ideal of $A$. 
Canonical invariants of $A$. The driving questions are what are $r(A)$, $cdeg A$ and $bideg A$ in relation to the invariants of $R$. Our main calculation is the following result:

**Theorem 8.3.** Suppose $(A, M)$ is a local ring. Then

\[ cdeg(A) = e^{-1}[cdeg(R) + e_0(m) - 2r]. \]

**Proof.** The equality $n = r$ means that $D = mC$. In particular if $(c)$ is a minimal reduction of $C$ and $(x)$ is a minimal reduction of $m$ then $(cx)$ is a minimal reduction of $D$. This gives that if $e = [A/M : R/m]$ is the relative degree then

\[ e_0(D, A) = e^{-1} \cdot \lambda_R(A/xcA) = e^{-1}[\lambda_R(A/xA) + \lambda_R(xA/xcA)] \]
\[ = e^{-1} \cdot [\lambda_R(R/xcR) + \lambda_R(R/cR)] \]
\[ = e^{-1}[e_0(C) + e_0(m)]. \]

On the other hand

\[ \lambda_A(A/D) = e^{-1} \cdot \lambda_R(A/mC) = e^{-1}[\lambda_R(A/R) + \lambda_R(R/C) + \lambda_R(C/mC)] \]
\[ = e^{-1} \cdot [2r + \lambda_R(R/C)]. \]

These equalities give

\[ cdeg(A) = e^{-1} \cdot [e_0(C) - \lambda_R(R/C) + e_0(m) - 2r] = e^{-1}[cdeg(R) + e_0(m) - 2r], \]

as desired. \qed

**Corollary 8.4.** $A$ is a Gorenstein ring if and only if

\[ cdeg(R) + e_0(m) - 2r = 0. \]

In particular $m$ and $CA$ are principal ideals of $A$.

**Remark 8.5.** If $m$ is a maximal ideal but $A$ is semilocal with maximal ideals $\{M_1, \ldots, M_s\}$, we can still obtain a formula for $cdeg(A)$ as a summation of the $cdeg(A_{M_i})$, as

\[ cdeg(A) = \sum_i cdeg(A_{M_i}) = [cdeg(R) + e_0(m) - 2r] \cdot (\sum_i e_i^{-1}). \]

**Example 8.6.** Let $(R, m)$ be a Stanley-Reisner ring of dimension one. If $R = k[x_1, \ldots, x_n]/L$, $L$ is generated by all the binomials $x_ix_j$, $i \neq j$. Note that $m = (x_1) \oplus \cdots \oplus (x_n)$ and since for $i \neq j$ the annihilator of $\text{Hom}((x_i), (x_j))$ has grade positive, as the observation above

\[ A = \text{Hom}(m, m) = \text{Hom}((x_1), (x_1)) \times \cdots \times \text{Hom}((x_n), (x_n)) = k[x_1] \times \cdots \times k[x_n]. \]

To determine $cdeg(R)$, we already have that $e_0(m) = n$ so let us calculate $r = r(R)$. The Hilbert series of $R$ is easily seen to be

\[ \frac{1 + (n-1)t}{1-t}, \]

so $r = n - 1$, which yields

\[ cdeg(R) = 2r - e_0(m) = 2(n-1) - n = n - 2 = r - 1. \]

Thus $R$ is almost Gorenstein ([8 Theorem 5.2]). This also follows from [9 Theorem 5.1].

Let us determine other invariants of $A$:

**Corollary 8.7.** For $A$ as above
The type of $A$ is

$$r(A) = \nu_A(D) = e^{-1}\nu_R(D) = e^{-1}\nu_R(mC) = e^{-1}\lambda(mC/m^2C).$$

For $A$ to be an almost Gorenstein local ring requires

$$c\deg(A) = r(A) - 1 = e^{-1}\lambda(mC/m^2C) - 1 = e^{-1}[c\deg(R) + e_0(m) - 2r].$$

Additional invariants. Next we would like to calculate under the same conditions

$$\text{bideg}(A) = \lambda_A(A/D) - \lambda_A(A/D^{**}).$$

The first term is as above

$$\lambda_A(A/D) = 1/e \cdot [2r + \lambda(R/C)].$$

Example 8.8. We return to the ring $R = \mathbb{Q}[t^5, t^7, t^9], m = (x, y, z)$. We have a presentation $R = \mathbb{Q}[x, y, z]/P$, with $P = (y^2 - xz, x^5 - yz^2, z^3 - x^4y)$. Let us examine some properties of $R$ and $A = m : m$.

1. The canonical module is $C = (x, y)$, and a minimal reduction $(c) = (x)$. It gives $\text{red}(C) = 4$.
2. $(c) : C \neq m$ so $R$ is not almost Gorenstein. However $C^{**} = (c) : [c] : C$ satisfies [by another Macaulay2 calculation] $\lambda(C^{**}/C) = 1$, so $C^{**} = L$. This shows that

$$\text{bideg}(R) = \lambda(C^{**}/C) = 1 \text{ and } c\deg(R) = \lambda(C/c) = 2.$$  

3. Since $e_0(m) = 5$, $r = 2$ and $e = 1$, we have

$$c\deg(A) = 2 - 4 + 5 = 3.$$  

4. $\text{red}(D) = \text{red}(mC) = 2$. Note that $(x)$ is a minimal reduction of both $C$ and $m$.
5. To calculate $D^{**}$ we change $C$ to $xC$. We then get $\lambda(A/D) = 11$ and $\lambda(A/D^{**}) = 9$, and so $\text{bideg}(A) = \lambda(A/D) - \lambda(A/D^{**}) = 2$.

References

[1] Y. Aoyama, Some basic results on canonical modules, J. Math. Kyoto Univ. 23 (1983), 85–94.
[2] V. Barucci and R. Fröberg, One-dimensional almost Gorenstein rings, J. Algebra 188 (1997), 418–442.
[3] M. P. Brodmann and S. Y. Sharp, Local Cohomology, Cambridge Studies in Advanced Mathematics 136, Cambridge University Press, 1998, 2011
[4] A. Corso, C. Huneke and W. V. Vasconcelos, On the integral closure of ideals, Manuscripta Math. 95 (1998), 331–347.
[5] A. Corso, C. Huneke, D. Katz and W. V. Vasconcelos, Integral closure of ideals and annihilators of homology, Commutative Algebra, Lecture Notes in Pure and Applied Mathematics 244, 33–48, Chapman & Hall/CRC, Boca Raton, FL, 2006.
[6] D. Eisenbud, Commutative Algebra with a view toward Algebraic Geometry, Springer, Berlin Heidelberg New York, 1995.
[7] J. Elias, On the canonical ideals of one-dimensional Cohen-Macaulay local rings, Proc. Edinburgh Math. Soc. 59 (2016), 77-90.
[8] L. Ghezzi, S. Goto, J. Hong and W. V. Vasconcelos, Invariants of Cohen-Macaulay rings associated to their canonical ideals, J. Algebra 589 (2017), 506–528.
[9] S. Goto, M. Matsuoka and T. T. Phuong, Almost Gorenstein rings, J. Algebra 379 (2013), 355–381.
[10] S. Goto, R. Takahashi and N. Taniguchi, Almost Gorenstein rings – towards a theory of higher dimension, J. Pure & Applied Algebra 219 (2015), 2666-2712.
[11] D. Grayson and M. Stillman, Macaulay 2, a software system for research in algebraic geometry, 2006. Available at http://www.math.uiuc.edu/Macaulay2/.
[12] J. Herzog, Generators and relations of abelian semigroups and semigroup rings, Manuscripta Math. 3 (1970), 175–193.
[13] J. Herzog, T. Hibi and D. I. Stamade, The trace of the canonical module, Preprint, 2016.
[14] J. Herzog and E. Kunz, *Der kanonische Modul eines Cohen-Macaulay Rings*, Lect. Notes in Math. **238**, Springer, Berlin–New York, 1971.

[15] J. Herzog, A. Simis and W. V. Vasconcelos, On the canonical module of the Rees algebras and the associated graded ring of an ideal, *J. Algebra* **105** (1987), 285–302.

[16] I. Kaplansky, *Commutative Rings*, The University of Chicago Press, Chicago, 1974.

[17] S. Morey and B. Ulrich, Rees algebras of ideals with low codimension, *Proc. Amer. Math. Soc.* **124** (1996), 3653–3661.

[18] M. Nagata, *Local Rings*, Interscience, New York, 1962.

[19] B. Ulrich, Ideals having the expected reduction number, *Amer. J. Math.* **118** (1996), 117–138.

[20] W. V. Vasconcelos, Computing the integral closure of an affine domain, *Proc. Amer. Math. Soc.* **113** (1991), 633–638.

[21] R. H. Villarreal, *Monomial Algebras*, 2nd Edition, C.R.C. Press, New York, 2015.