Asymptotic affirmative actions: The top trading cycles and the Boston mechanism case

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Abstract

This paper analyzes the asymptotic performance of two popular affirmative action policies, majority quota and minority reserve, under the top trading cycles mechanism (TTCM) and the Boston mechanism (BM). These two affirmative actions induce different matching outcomes with non-negligible probability under the TTCM even if the number of reserved seats for minorities grows relatively slowly in a sequence of random markets. Given the possible preference manipulations under the BM, we further characterize the asymptotically equivalent sets of Nash equilibria of the BM with the majority quota and its corresponding minority reserve when the market becomes large.

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Keywords: school choice; top trading cycles mechanism; Boston mechanism; affirmative actions; large market.

1 Introduction

Together with the public school choice reform, affirmative action policies were introduced in many school districts which offer students from socioeconomically disadvantaged groups (i.e., minority students) preferential treatments to mitigate the ethnic and socio-economic desegregation in schools. The quota-based affirmative action (majority quota, henceforth) and the reserve-based affirmative action (minority reserve, henceforth) are two common affirmative action policy designs. Abdulkadiroğlu and Sönmez (2003) formalize the majority quota policy which limits the number of admissible students from socioeconomically advantaged groups (i.e., majority students) and leaves the difference to minority students. The minority reserve policy proposed by Hafalir et al. (2013), on the other hand, gives higher priority to minority students up to the point that all reserved seats have been assigned to minorities. Hafalir et al. (2013) imply that unlike the student optimal stable mechanism (SOSM, henceforth), the majority quota and its corresponding minority reserve do not present a clear Pareto dominance relationship for minorities under the Shapley and Scarf (1974)’s top trading cycles mechanism (TTCM, henceforth). Under the Boston mechanism, Afacan and Salman (2016) show that different from its majority quota counterpart, the minority reserve will never generate a Pareto inferior outcome for minority students in a finite market when all students report truthful preferences.

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This paper further compares the asymptotic performance of these two affirmative action policies under the TTCM and the BM in a sequence of random markets of different sizes. Our Proposition 1 reveals that the two affirmative actions produce different matching outcomes under the TTCM with non-negligible probability, even if the number of reserved seats grows at a slower rate of $O(n^a)$ in a sequence of random markets, where $0 \leq a < 1/2$ and $n$ is the number of schools in a random market (see Condition (4) of Definition 1). This result differs from the asymptotic outcome equivalence of the majority quota and its corresponding minority reserve under the SOSM illustrated by Liu (2022). Such distinct asymptotic performance of the TTCM and the SOSM essentially comes from the priority trade nature of the TTCM, as blocking possible priority trades under the TTCM with affirmative actions requires that it is unlikely for any students to list a school with affirmative actions in random markets of arbitrary sizes, which cannot be satisfied even if the share of the schools with affirmative actions vanishes with market size. By contrast, the convergence process under the SOSM, as illustrated by Liu (2022), only demands that no two distinct students (either majority or minority) will list the same school with affirmative actions with a high probability.

We then study the asymptotic performance of the majority quota and its minority reserve counterpart under the Boston mechanism (BM, henceforth). Our Proposition 2 implies that although the BM is open to preference manipulations, these two affirmative actions are most likely to induce the same set of matching outcomes with non-profitable deviations from the truthful preferences under the BM when the market becomes sufficiently large. Nevertheless, from both empirical and experimental evidence (Pathak and Sönmez, 2008; Featherstone and Niederle, 2016; Chen et al., 2018), we know that students are not necessarily playing equilibrium strategies under the BM; such off-equilibrium behavior could be further exaggerated by the potentially large equilibria set when the market contains sufficiently many students and schools. Thus, an important policy implication of our results is that the SOSM is more cost-effective compared to the TTCM and the BM in school choice markets with affirmative actions, as it is unnecessary to identify the different welfare effects of these two affirmative actions under the SOSM if the policymaker can assure a sufficient supply of popular schools to the matching markets.

The literature on large matching markets has been growing rapidly in recent years. Most studies nevertheless indicate that many existing impossibility results, ranging from incentives to existence and efficiency in finite matching markets, disappear if we admit an approximate variant of these properties in large market environments.\footnote{Two exceptions include Hatfield et al. (2016) and Che and Tercieux (2019). Hatfield et al. (2016) show that all stable mechanisms approximately respect improvements of school quality, while neither the Boston nor the TTC mechanism satisfies this approximate property; Che and Tercieux (2019) suggest that the inefficiency of the deferred acceptance algorithm (Gale and Shapley, 1962) and instability of the TTCM remain significant when agents’ preferences are correlated in large markets, and propose a variant of deferred acceptance that is asymptotically efficient, asymptotically stable and asymptotically incentive compatible.} Given the fading of many impossibility results in large markets, some researchers have criticized using approximation properties in market design problems in the sense that the large market analytic framework may be too permissive to make market design irrelevant.\footnote{See the discussions in Section 3.4 of Kojima (2015).} The current paper thus also supports the validity of the large market analytic approach, as it still enables us to capture the subtle difference between the matching mechanisms that can asymptotically satisfy some desirable properties from those that cannot.
2 Model

2.1 School Choice with Affirmative Actions

Let $S$ and $C$ be two finite sets of students and schools, $|S| \geq 2$. There are two types of students, majority and minority. $S$ is partitioned into two subsets of students based on their types. Denote $S^M$ the set of majority students, and $S^m$ the set of minority students, $S = S^M \cup S^m$ and $S^M \cap S^m = \emptyset$. Each student $s \in S$ has a strict preference order $P_s$ over the set of schools and being unmatched (denoted by $s$). All students prefer to be matched with some school instead of herself, $c P_s s$, for all $s \in S$. Each school $c \in C$ has a total capacity of $q_c$ seats, $q_c \geq 1$, and a strict priority order $\succ$ over the set of students which is complete, transitive, and antisymmetric. Student $s$ is unacceptable by a school if $e \succ c s$, where $e$ represents an empty seat in school $c$.

A school choice market with affirmative actions is a tuple $\Gamma = (S, C, P, \succ, (q^M, r^m))$, where $P = (P_s)_{s \in S}, \succ = (\succ)_{c \in C}$; when comparing the effects of a majority quota policy with its minority reserve counterpart in a market $\Gamma$, we assume $\Gamma$ is either with only majority quota or with only minority reserve, such that $r^M_c + q^M_c = q_c, \forall c \in C$, with $q^M = (q^M_c)_{c \in C}$, $r^m = (r^m_c)_{c \in C}$, and $q = (q_c)_{c \in C}$. For each school $c$ with majority quota affirmative action policy, it cannot admit more majority students than its type-specific majority quota $q^M_c \leq q_c$, for all $c \in C$. Accordingly, the minority reserve policy gives priority to the minority applicants of school $c$ up to its minority reserve $r^m_c \leq q_c, \forall c \in C$, and allows $c$ to accept majority students up to its capacity $q_c$ if there are not enough minority applicants to fill the reserves.

A matching $\mu$ is a mapping from $S \cup C$ to the subsets of $S \cup C$ in market $\Gamma$ such that, for all $s \in S$ and $c \in C$: (i) $\mu(s) \in S \cup \{s\}$; (ii) $\mu(s) = c$ if and only if $s \in \mu(c)$; (iii) $\mu(c) \subseteq S$ and $|\mu(c)| \leq q_c$; and (iv) $|\mu(c) \cap S^M| \leq q^M_c$. That is, a matching specifies the school where each student is assigned or matched with herself, and the set of students assigned to each school; no school admits more students than its capacity, and no school admits more majority students than its majority quota.

A matching $\mu$ is blocked by a pair of student $s$ and school $c$ with majority quota, if $c P_s \mu(s)$ and either $|\mu(c)| < q_c$ and $s$ is acceptable to $c$, or: (i) $s \in S^m, s \succ_c s'$, for some $s' \in \mu(c)$; (ii) $s \in S^M$ and $|\mu(c) \cap S^M| < q^M_c, s \succ_c s'$, for some $s' \in \mu(c)$; (iii) $s \in S^M$ and $|\mu(c) \cap S^M| = q^M_c, s \succ_c s'$, for some $s' \in \mu(c) \cap S^M$. A matching $\mu$ is $Q$-stable, if $\mu(s) P_s s$ for all $s \in S$, and has no blocking pair in $\Gamma$ with majority quota.

Accordingly, a matching $\mu$ is blocked by a pair of student $s$ and school $c$ with minority reserve, if $s$ strictly prefers $c$ to $\mu(s)$ and either $|\mu(c)| < q_c$ and $s$ is acceptable to $c$, or: (i) $s \in S^m, s \succ_c s'$, for some $s' \in \mu(c)$; (ii) $s \in S^M$ and $|\mu(c) \cap S^M| > r^m_c, s \succ_c s'$, for some $s' \in \mu(c)$; (iii) $s \in S^M$ and $|\mu(c) \cap S^M| \leq r^m_c, s \succ_c s'$, for some $s' \in \mu(c) \cap S^M$. A matching $\mu$ is $R$-stable, if $\mu(s) P_S s$ for all $s \in S$, and has no blocking pair in $\Gamma$ with minority reserve.

A mechanism $f$ is a function that produces a matching $f(\Gamma)$ for each market $\Gamma$.

2.2 Top Trading Cycles Mechanism

For each market $(S, C, P, \succ, (q^M, r^m))$, the top trading cycles mechanism with affirmative actions algorithm runs as follows:

Step 1: Start with a matching in which no student is matched. For each school $c$, set its capacity counter at $q_c$. If $c$ has a majority quota, set its quota counter at its majority quota $q^M_c$; if $c$ has a corresponding minority reserve, set its reserve counter at its minority reserve $r^m_c$. If the reserve counter of school $c$ is positive, then it points to its most preferred minority student; otherwise it points to its most preferred student. Each student $s$ points to her
most preferred acceptable school that still has a seat for her, and otherwise points to herself; that is, an acceptable school $c$ whose capacity counter is strictly positive and, if $s \in SM$, its quota counter is strictly positive. There exists at least one cycle (if a student points to herself, it is regarded as a cycle). Every student in a cycle is assigned a seat at the school she points to (if she points to herself, then she gets her outside option) and is removed. The capacity counter of each school in a cycle is reduced by one and, if $s \in SM$, its quota counter is strictly positive. There exists at least one cycle (if a student points to herself, it is regarded as a cycle). Every student in a cycle is assigned a seat at the school she points to (if she points to herself, then she gets her outside option) and is removed. The capacity counter of each school in a cycle is reduced by one and, if: (i) the assigned student $s$ is a majority student and the school matched to $s$ has a majority quota, then reduces the quota counter of the matched school by one; (ii) the assigned student $s$ is a minority student and the school matched to $s$ has a minority reserve, then reduces the reserve counter of the matched school by one. If no student remains, terminate. Otherwise, proceed to the next step.

**Step k:** Start with the matching and counter profile reached at the end of Step $k - 1$. For each remaining school $c$, if its reserve counter is positive, then $c$ points to its most preferred minority student among all remaining minority students; otherwise it points to its most preferred student among all remaining students. Each remaining student $s$ points to her most preferred acceptable school that still has a seat for her, and otherwise points to herself; that is, an acceptable school $c$ whose capacity counter is strictly positive and, if $s \in SM$, its quota counter is strictly positive. There exists at least one cycle (if a student points to herself, it is regarded as a cycle). Every student in a cycle is assigned a seat at the school she points to (if she points to herself, then she gets her outside option) and is removed. The capacity counter of each school in a cycle is reduced by one and, if: (i) the assigned student $s$ is a majority student and the school matched to $s$ has a majority quota, then reduces the quota counter of the matched school by one; (ii) the assigned student $s$ is a minority student and the school matched to $s$ has a minority reserve, then reduces the reserve counter of the matched school by one. If no student remains, terminate. Otherwise, proceed to the next step.

The algorithm terminates in a finite number of steps since there is at least one student matched and removed in any step of the algorithm. For a market $(S, C, P, \succ, (q^M, r^m))$, if $r^m_c = 0, \forall c \in C$, i.e., a market with only majority quota, then the above algorithm reduces to the top trading cycles mechanism with majority quota (TTCM-Q henceforth) proposed by Abdulkadiroğlu and Sönmez (2003) and Kojima (2012); accordingly, if $q^M_c = q_c, \forall c \in C$, i.e., a market with only minority reserve, then the above algorithm reduces to the top trading cycles mechanism with minority reserve (TTCM-R henceforth) proposed by Hafalir et al. (2013).

### 2.3 Boston Mechanism

Afacan and Salman (2016) adapt the Boston mechanism to the school choice with affirmative actions. For each market $(S, C, P, \succ, (q^M, r^m))$, their Boston mechanism with affirmative actions algorithm runs as follows:

**Step 1:** Each student applies to her most preferred acceptable school (call it school $c$). The school $c$ first considers minority applicants and permanently accepts them up to its minority reserve $r^m_c$ one at a time following its priority order, if $r^m_c > 0$. School $c$ then considers all the applicants who are yet to be accepted, and one at a time following its priority order, it permanently accepts as many students as up to the remaining total capacity while not admitting more majority students than $q^M_c$. The rest (if any) are rejected.

**Step k:** Each student $s$ who was rejected in Step $(k - 1)$ applies to her next preferred acceptable choice (call it school $c$, if any). If school $c$ still has an available seat, it first considers
minority applicants and permanently accepts them up to its minority reserve $r^m_c$ one at a time following its priority order, if $r^m_c > 0$. School $c$ then considers all the applicants who are yet to be accepted, and one at a time following its priority order, it permanently accepts as many students as up to the remaining total capacity while not admitting more majority students than $q^M_c$. The rest (if any) are rejected.

The algorithm terminates either when every student is matched to a school or every unmatched student has all acceptable offers rejected, which always terminates in a finite number of steps. Similar to the TTCM in the previous section, for a market $(S, C, P, >, (q^M, r^m))$ with only majority quota, the above algorithm reduces to the Boston mechanism with majority quota (BM-Q henceforth). Also, if $q^M_c = q_c$, $\forall c \in C$, i.e., a market with only minority reserve, then the above algorithm reduces to the Boston mechanism with minority reserve (BM-R henceforth).

### 2.4 Large Markets

A random market is a tuple $\Gamma = (\hat{\Gamma} = ((S^n, S^m), C, >, (q^M, r^m), k, (\mathcal{A}, \mathcal{B})))$, where $k$ is a positive integer, $\mathcal{A} = (\alpha_c)_{c \in C}$ and $\mathcal{B} = (\beta_c)_{c \in C}$ are the respective probability distributions on $C$, with $\alpha_c, \beta_c > 0$ for each $c \in C$. We assume that $\mathcal{A}$ for majorities to be different from $\mathcal{B}$ for minorities to reflect their distinct favors for schools. A sequence of random markets is denoted by $(\Gamma^1, \Gamma^2, \ldots)$, where $\Gamma^n = ((S^n, S^m, n), C^n, >_n, (q^M, r^m, n), k^n, (\mathcal{A}^n, \mathcal{B}^n))$ is a random market of size $n$, with $|C^n| = n$ as the number of schools and $|r^m| = n$ the number of seats reserved for minorities.

Each random market induces a market by randomly generated preference orders of each student $s$ according to the following procedure introduced by Immorlica and Mahdian (2005):

**Step 1:** Select a school independently from the distribution $\mathcal{A}$ (resp. $\mathcal{B}$). List this school as the top ranked school of a majority student $s \in S^M$ (resp. minority student $s \in S^m$).

**Step $1 \leq k$:** Select a school independently from $\mathcal{A}$ (resp. $\mathcal{B}$) which has not been drawn from steps 1 to step $l-1$. List this school as the $l^{th}$ most preferred school of a majority student $s \in S^M$ (resp. minority student $s \in S^m$).

Each majority (resp. minority) student finds these $k$ schools acceptable, and only lists these $k$ schools in her preference order. Let $\tilde{P}^n_s$ be the (truthful) preference order of student $s$ generated according to the preceding procedure, and $\tilde{P}^n = (\tilde{P}^n_s)_{s \in S^n}$ be the profile of truthful preferences. We introduce the following regularity conditions to guarantee the convergence of the random markets sequence.

**Definition 1.** Consider majority quotas $q^M$ and minority reserves $r^m$ such that $r^m + q^M = q$. A sequence of random markets $(\hat{\Gamma}^1, \hat{\Gamma}^2, \ldots)$ is regular, if there exist $\alpha \in [0, \frac{1}{2})$, $\lambda, \kappa, \theta > 0$, $r \geq 1$, and positive integers $k$ and $\tilde{q}$, such that for all $n$:

1. $k^n \leq k$;
2. $q_c \leq \tilde{q}$ for all $c \in C^n$;
3. $|S^n| \leq \lambda n, \sum_{c \in C} q_c - |S^n| \geq \kappa n$;
4. $|r^m| \leq \theta n^a$;
5. $\frac{\alpha_c}{a_c} \in [\frac{1}{r}, r], \frac{\beta_c}{b_c} \in [\frac{1}{r}, r]$, for all $c, c' \in C^n$;
6. $a_c = 0$, for all $c \in C^n$ with $q_c^M = 0$.

Condition (1) and (2) assume that the length of students’ preferences and the capacity of each school are bounded across schools and markets. Condition (3) requires that the number of students does not grow much faster than the number of schools, while there is an excess supply of school capacities to accommodate all students.\(^3\) Condition (4) requires that the number of seats reserved for minority students grows at a slower rate of $O(n^a)$, where $a \in [0, \frac{1}{2})$. Condition (5) requires that the popularity of different schools, as measured by the probability of being selected by students from $\mathcal{A}$ for majorities and $\mathcal{B}$ for minorities, does not vary too much. Condition (6) requires that a majority student will not select a school that can only accept minority students (i.e., with majority quota $q_c^M = 0$), as these two affirmative actions trivially induce disparate matching outcomes in any arbitrarily large markets when a majority student applies to a school with zero majority quota.\(^4\)

**Definition 2.** For any random market $\hat{\Gamma}$, let $\eta_c(\hat{\Gamma}; f, f')$ be the probability that $f(\hat{\Gamma}) \neq f'(\hat{\Gamma})$. We say two mechanisms are outcome equivalent in large markets, if for any sequence of random markets $(\hat{\Gamma}_1, \hat{\Gamma}_2, \ldots)$ that is regular, \(\max_{c \in C^n} \eta_c(\hat{\Gamma}_n; f, f') \to 0\), as $n \to \infty$.

### 3 Results

We first present our arguments on the asymptotic performance of the TTCM with affirmative actions, which implies that the majority quota and its corresponding minority reserve will induce different matching outcomes with non-negligible probability under the TTCM, even in arbitrarily large markets with sufficiently many schools and a relatively slow growth of reserved seats.

**Proposition 1.** The TTCM-Q and its corresponding TTCM-R are not outcome equivalent in large markets.

**Proof.** See Appendix A.1.

The intuition for the proof is as follows. Different from its majority quota counterpart, the minority reserve policy requires a school to first consider the minority student with highest priority. For a school $c$ with minority reserve policy, we cannot rule out the possibility that a minority student (call it $m$) with lower priority compared to a majority student (call it $M'$) in a school $c$ (with affirmative actions) matched to her most preferred school $c'$ (without affirmative actions), if $m$ can trade priorities with another majority student (call it $M$) who has a higher priority over $m$ and $M'$ in $c'$. Such different orders of priority trades under the majority quota and its minority reserve counterpart remain with non-negligible probability regardless of the market size.

Proposition 1 differs from the asymptotic outcome equivalence of the majority quota and its corresponding minority reserve under the SOSM illustrated by Liu (2022). This is because the SOSM does not permit students to trade their priorities throughout the matching process, the convergence process under the SOSM thus only demands that no two different

\(^3\)Note that we do not distinguish the growth rate between majority and minority students, as minority students are generically treated as the intended beneficial student groups from affirmative action policies rather than race or other single social-economic status; therefore, the number of minority students is not necessarily less than majorities.

\(^4\)See Liu (2022) for more detailed illustrations of each of the regularity conditions.
students (either majority or minority) will list the same school with affirmative actions with a high probability. By contrast, in order to block the possible priority trades under the TTCM, the asymptotic convergence of the TTCM-Q and its corresponding TTCM-R requires that it is very unlikely for any majority students to list a school with affirmative actions, which cannot be satisfied in large markets even under our relatively strong regularity conditions of Definition 1.

**Remark 1.** We can further state that the asymptotic convergence of the TTCM-Q and its corresponding TTCM-R essentially requires that it is very unlikely for any students to list a school with affirmative actions. To see this, we first preserve assumptions in the preceding proof and restate Event 1 as there are exactly one student, who is a minority student denoted by \( m' \), ranks \( c \) first. The probability of this alternative of Event 1 is

\[
\left( \frac{n(1-t_n)}{1} \right) \times \frac{1}{n} \times \left( 1 - \frac{1}{n} \right)^{n-1},
\]

which converges to \( \frac{1-t_n}{e} \) as \( n \) approaches \( \infty \). Keep Event 2 and 3 as in the proof with their corresponding conditional probabilities, we restate Event 4 as the event that apart from other students, \( c \) ranks \( m' \), \( M' \), and \( m \), as

\[ M' >_c m, \quad \text{and} \quad m >_c m'. \]

which has the same conditional probability \( \frac{1}{6} \) as the original Event 4. Thus, when a minority student \( m' \)—instead of a majority student \( M \) as in the preceding proof—lists a school with affirmative actions, the unconditional probability that the TTCM-Q and its corresponding TTCM-R generate disparate matching outcomes becomes \( p_n \geq \frac{t_n(1-t_n)^2\delta_n}{48e^3} \), which remains strictly positive for any arbitrarily large \( n \). Also, given our restrictive regularity conditions, this impossibility result is robust to other alternative approximations of large school choice markets; e.g., correlations in preferences, preference distributions with geographic heterogeneity, unbounded (but sufficiently slow growth of) preference length, as discussed in Kojima et al. (2013) and Hatfield et al. (2016).

Next, we analyze the asymptotic performance of the majority quota and its minority reserve counterpart under the Boston mechanism (BM). Note that given the possible preference manipulations under the BM, students may have different incentives to manipulate their reported preference orders under the BM with either of these two affirmative actions (Afacan and Salman, 2016).5

We first define the BM-Q and its corresponding BM-R as a preference revelation game. Formally, given a regular random market \( \tilde{\Gamma}^n \) of size \( n, n = 1, 2, \ldots \), students’ corresponding truthful preference profile \( \tilde{P}^n = (\tilde{P}_s^c)_{s \in S^n} \), and a mechanism \( f \), the preference revelation game induced by \( f \) is a strategic game \( G_f(\tilde{\Gamma}^n) = (\mathcal{P}^n, \tilde{P}^n, f) \), where \( \mathcal{P}^n \) is the strategy space of each student (i.e., all the possible stated preferences over schools), and \( f \) is the outcome function in which each student evaluates outcomes according to \( \tilde{P}^n \).

**Definition 3.** A strategy profile \( P^* \in \Pi_{s \in S^n} \mathcal{P}^n \) is a Nash equilibrium of \( G_f(\tilde{\Gamma}^n) \), if for each \( s \in S^n \), there is no strategy \( P'_s \in \mathcal{P}^n \) such that \( f_s(P'_s, P^*_s) \tilde{P}_s \neq f_s(P^*_s) \), where \( P^*_s = (P^*_i)_{i \in S^n \setminus s} \). Also, given a Nash equilibrium \( P^* \), its corresponding Nash equilibrium outcome is \( f(P^*) \).

5Also, see Kojima and Pathak (2009) (their online Appendix A.8) and Hatfield et al. (2016) (their Proof of Theorem 2) for the illustration of students’ incentives to manipulate the BM in large markets.
The following result characterizes the asymptotically equivalent sets of Nash equilibrium outcomes of the BM-Q and its corresponding BM-R in a sequence of regular random markets.

**Proposition 2.** The sets of Nash equilibria of the BM-Q and its corresponding BM-R are outcome equivalent in large markets.

**Proof.** See Appendix A.2.

Although the BM is open to preference manipulations, Proposition 2 implies that when the market becomes sufficiently large, the BM-Q and its corresponding BM-R are most likely to have the same set of matching outcomes with non-profitable deviations from the truthful preference order profile \(\tilde{P}^n\). We construct the proof based on the asymptotic outcome equivalence of the sets of Q-stable and R-stable matchings under the BM-Q and its corresponding BM-R (Lemma 2 in Appendix A.2), and the market-wise outcome equivalence between the respective sets of Nash equilibrium outcomes and the set of stable matchings under the BM with these two affirmative actions (Lemma 3).

Nevertheless, since students (and their parents) may have different levels of strategic sophistication, both empirical and experimental evidence suggest that students will not always adopt their best responses and consequently, a low percentage of equilibrium outcomes can be realized under the BM (Pathak and Sönmez, 2008; Featherstone and Niederle, 2016; Chen et al., 2018). Our characterization of the asymptotic equivalence of Nash equilibrium outcomes under the BM with these two affirmative actions is thus less robust compared to the corresponding asymptotic outcome equivalence under the SOSM with affirmative actions of Liu (2022), as truthful reporting is a (weakly) dominant strategy for all students under the SOSM (Dubins and Freedman, 1981; Roth, 1982).

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**A Appendix**

**A.1 Proof of Proposition 1**

Consider a sequence of random markets \((\tilde{\Gamma}^1, \tilde{\Gamma}^2, \ldots)\), where there are \(n\) schools and \(\lambda n\) students, \(\lambda \geq 1\), in each random market \(\tilde{\Gamma}^n\). Assume that the preferences of all students are generated according to the preference generation procedure defined in Section 2.4, with uniform distribution over all schools and preference length \(k = 1\). Also, assume that school priorities are drawn identically and independently from the uniform distribution over students such that all students are acceptable. For each random market \(\tilde{\Gamma}^n\), denote \(t_n \in (0, 1)\) the portion of majority students, while \(1 - t_n\) the corresponding portion of minority students. Also, assume that \(q_c = 1\) or \(2\) for every school \(c\) in \(\tilde{\Gamma}^n\), denote \(\delta_n \in (0, 1)\) the portion of schools with one seat, while \(1 - \delta_n\) the corresponding portion of schools with two seats. The preceding assumptions guarantee that the regularity conditions of Definition 1 are satisfied.
Let $p_n$ be the probability that the two affirmative actions produce different outcomes under the TTCM in market $\tilde{\Gamma}^n$. We construct the proof by showing that the probability that $p_n$ is strictly bounded away from zero in a sequence of random markets of different sizes ($\tilde{\Gamma}^1, \tilde{\Gamma}^2, \ldots$).

$p_1 > 0$ is trivially satisfied when $\tilde{\Gamma}^1$ contains one majority student $M$ and one minority student $m$, while the exact school $c$ has one seat, $\delta_1 \in (0,1)$, and $M >_c m$. For $n \geq 2$. Let $c$ be an arbitrary school, $q_c = 2$, with either a majority quota $q_c^M = 1$ or its corresponding minority reserve policy $r_c^m = 1$. Let Event 1 be the event that there are exactly one student, who is a majority student denoted by $M$, ranks $c$ first. The probability of Event 1 is

$$\left( \frac{n \cdot t_n}{1} \right) \times \frac{1}{n} \times \left( 1 - \frac{1}{n} \right)^{n-1},$$

where $t_n \in (0,1)$ for any arbitrarily large $n$. We can derive its limit when $n$ approaches $\infty$ as

$$\lim_{n \to \infty} t_n \times \left( 1 - \frac{1}{n} \right)^{n-1} = \lim_{n \to \infty} t_n \times \frac{1}{n} \times \left( 1 - \frac{1}{n} \right)^{n-1} = t_n \times \frac{1}{e} \times 1 = \frac{t_n}{e}.$$

Therefore, for any sufficiently large $n$, the probability of Event 1 is at least, say, $\frac{t_n}{e_0} > 0$. Given Event 1, consider Event 2 such that except school $c$, there is exactly one school $c'$, $q_{c'} = 1$, lists $M$ over all the rest students in its priority order. The conditional probability of Event 2 is given by

$$\left( \frac{n \cdot \delta_n}{1} \right) \times \frac{1}{n} \times \left( 1 - \frac{1}{n} \right)^{n-2},$$

where $\delta_n \in (0,1)$ for any arbitrarily large $n$. The above expression converges to $\frac{\delta_n}{e}$, as $n$ approaches $\infty$. Thus, for any sufficiently large $n$, the conditional probability of Event 2 given Event 1 is at least, say, $\frac{\delta_n}{2e} > 0$. Given Event 1 and 2, consider Event 3 such that except the majority student $M$, there is exactly one majority student $M'$ and one minority student $m$ rank $c'$ first. The conditional probability of Event 3 is

$$\left( \frac{(n-1) \cdot t_n}{1} \right) \times \left( \frac{n \cdot (1 - t_n)}{1} \right) \times \frac{1}{n^2} \times \left( 1 - \frac{1}{n} \right)^{n-3}.$$

The limit of this expression when $n$ approaches $\infty$ is $\frac{t_n}{e_0} \times \frac{t_n}{e_1}$. For any sufficiently large $n$, the conditional probability of Event 3 given Event 1 and 2 is at least, say, $\frac{t_n}{2\cdot e_0}$. Given Events 1, 2, and 3, let Event 4 be the event that apart from other students in $\tilde{\Gamma}^n$, $c$ ranks $M$, $M'$, and $m$, as

$$M' >_c m, \quad \text{and} \quad m >_c M.$$

Since Events 1-3 do not impose any restrictions on the rankings of $M'$, $m$, and $M$ in $c$’s priorities, the conditional probability of Event 4 is $\frac{1}{e}$. Given Events 1-4 and the assumption that $k = 1$, the event that school $c'$ is matched with $M'$ or $m$ while being contained in a cycle involving another agent than $c'$, $M'$, and $m$ occurs with conditional probability $1$. Thus, the unconditional probability that $c'$ is matched with $M'$ when $c$ has the majority quota $q_c^M = 1$ while matched with $m$ when $c$ has the corresponding minority reserve $r_c^m = 1$, is at least $\frac{t_n}{48e^2} > 0$.

Therefore, for any sufficiently large $n$, there is an $n$ such that $p_n \geq \frac{t_n}{48e^2}$, for any $n \geq n$. Together with $p_1 > 0$, we can see that $p_n$ is at least $\min \{ p_1, p_2, \ldots, p_{n-1}, \frac{t_n}{48e^2} \}$, which is bounded away from below by 0. This completes the proof.
A.2 Proof of Proposition 2

Lemma 1. (Liu, 2022) The probability that no two distinct students (either majority or minority) will list the same school $c \in C^n$ with nonzero reserve seats in their preference orders converges to one, as $n \to \infty$.

Under the same preference generation procedure and the regular conditions defined in Section 2.4, Liu (2022) writes his result to state that it is very unlikely for any two different students to list the same school with nonzero reserved seats under the SOSM with either of these two affirmative actions when the market becomes large. Since the underlying matching mechanisms will not affect the probability of listing any particular school (from either $\mathcal{A}$ or $\mathcal{B}$) in students’ preference orders, we omit the proof of Lemma 1 for brevity.6

For any regular random market $\hat{\Gamma}^n$ in the sequence of markets $(\hat{\Gamma}^1, \hat{\Gamma}^2, \ldots)$, let $\hat{P}^n_s$ be the truthful preference order of student $s$ generated according to the procedure defined in Section 2.4, and $\xi^q(\hat{P}^n)$ (resp. $\xi^r(\hat{P}^n)$) be the set of Q-stable (resp. R-stable) matchings under the true preferences $\hat{P}^n_s$ in $\hat{\Gamma}^n$ with majority quota (resp. minority reserve), where $\hat{P}^n = (\hat{P}^n_s)_{s \in S^n}$.

Lemma 2. The probability that $\xi^q(\hat{P}^n) = \xi^r(\hat{P}^n)$ converges to one, as $n \to \infty$.

Proof. (i) $\xi^r(\hat{P}^n) \subseteq \xi^q(\hat{P}^n)$. Given a regular random market $\hat{\Gamma}^n$, let $\mu \not\in \xi^q(\hat{P}^n)$ be a non R-stable matching in it, i.e., $\mu$ is blocked by a pair of student and school $(s, c) \in (S^n, C^n)$ when $\hat{\Gamma}^n$ has the majority quota $q^{M,n}$. By the definition of R-stability, $(s, c)$ also blocks $\mu$ in $\hat{\Gamma}^n$ with the corresponding minority reserve $r^{m,n} = q^n - q^{M,n}$. Thus, we have $\mu \not\in \xi^r(\hat{P}^n)$ in $\hat{\Gamma}^n$.

(ii) $\xi^q(\hat{P}^n) \subseteq \xi^r(\hat{P}^n)$. Let $\mu \in \xi^q(\hat{P}^n)$ be a Q-stable matching in a given regular random market $\hat{\Gamma}^n$ of size $n$. We demonstrate that $\mu$ is asymptotically R-stable when the size of the market grows up; that is, the probability that $\mu \in \xi^r(\hat{P}^n)$ converges to one in the sequence of markets $(\hat{\Gamma}^1, \hat{\Gamma}^2, \ldots)$, as $n \to \infty$.

Together with Condition (6) of Definition 1, a Q-stable matching $\mu$ in $\hat{\Gamma}^n$ with majority quota $q^{M,n}$ is blocked by a pair of $(s, c)$ in $\hat{\Gamma}^n$ with the corresponding minority reserve $r^{m,n}$, can only occur when $c \in \hat{P}^n_s, |\mu(c) \cap S^n| = q^{M,n}, |\mu(c)| < q^c$, and $s' > c$ for all $s' \in \mu(c) \cap S^{M,n}$ and $s \in S^{M,n} \setminus \mu(c)$; that is, school $c$ has excessive majority applicants and insufficient number of minority applicants in $\hat{\Gamma}^n$. By Lemma 1, we know that it is very unlikely for any two distinct students (i.e., $s$ and $s'$ here) to list the same school $c$ with affirmative actions in their preference orders when $n$ becomes sufficiently large. This implies that the probability for any pair $(s, c) \in (S^n, C^n)$ forming a blocking pair in $\hat{\Gamma}^n$ with $r^{m,n}$ but not in $\hat{\Gamma}^n$ with the corresponding $q^{M,n}$ also converges to zero, as $n \to \infty$. Thus, we have the probability that $\mu \in \xi^r(\hat{P}^n)$ converges to one, as $n \to \infty$. \[ \Box \]

Lemma 3. (1) The set of Nash equilibrium outcomes of the BM-Q is equal to the set of Q-stable matchings under the true preferences $\hat{P}^n_s$ in each $\hat{\Gamma}^n$, $n = 1, 2, \ldots$.

(2) The set of Nash equilibrium outcomes of the BM-R is equal to the set of R-stable matchings under the true preferences $\hat{P}^n_s$ in each $\hat{\Gamma}^n$, $n = 1, 2, \ldots$.

Proof. (1.) The market-wise equivalence between the set of Nash equilibrium outcomes of the BM-Q and the set of Q-stable matchings generated under the true preferences in a given market of size $n$, has been given by Theorem 3 of Ergin and Sönmez (2006). We thus only need to prove the second part.

(2.1.) Given a regular random market $\hat{\Gamma}^n$ of size $n$, $n = 1, 2, \ldots$, and a corresponding preference revelation game of BM-R, let $P^r$ be an arbitrary strategy profile and matching $\mu$ be

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6See Expression (A.3) and the succeeding arguments in the Proof of Proposition 2 of Liu (2022) for details.
implies that the sets of Nash equilibria of the profile ˜P^n, we can thus find a pair of student and school (s, c) \in (S^n, c^n) such that c\tilde{P}^n_s \mu(s), and either s \succ c s' for some s' \in \mu(c), or |\mu(c)| < q_c and s is acceptable to c. This implies that c is not at the top in P^n_s, because otherwise student s would be assigned to school c. Let P''^n be a preference order of student s in which c is positioned as her first choice. Clearly, s will be assigned to c under the strategy profile \( (P''^n, P'') \), where P' = (P'_i)_{i \in S^n \setminus s}. Thus, P'' is not a Nash equilibrium, as P''^n is a profitable deviation for student s at P' given c\tilde{P}^n_s \mu(s). Also, since P' is arbitrarily given, the non R-stable matching \mu cannot be obtained in Nash equilibrium.

(2.ii.) Let \mu be a R-stable matching under ˜P^n in the regular random market ˜\Gamma^n. We show that there exists a Nash equilibrium P', such that its associated outcome is \mu. For each student s \in S^n, let P^*_s be the preference order of student s such that school \mu(s) is positioned at the top, i.e., \mu(s) \mu^*_s c', \forall c' \in C^n \setminus \mu(s). Thus, at P' the BM-R will terminate at Step 1 and assign each student s to \mu(s). To show that P' is a Nash equilibrium, consider a pair of student and school (s, c) such that c\mu^*_s \mu(s). As \mu is R-stable, we know that |\mu(c)| = q_c and each student who is matched with school c under \mu is more preferred to s; also, for each s' \in \mu(c), \mu(c) is her top ranked school at P'. Thus, s cannot be matched to c by misreporting her preferences. Since s is arbitrarily chosen, the preceding argument suffices the non-existence of profitable deviations at P'. We conclude that P' is a Nash equilibrium with \mu as its associated Nash equilibrium outcome.

Together with the asymptotically equivalent sets of stable matchings of the BM-Q and its corresponding BM-R of Lemma 2, Lemma 3 implies that the sets of Nash equilibria of the BM-Q and its corresponding BM-R are outcome equivalent in large markets, as n \rightarrow \infty. This completes the proof.

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