Some topics concerning integrals of derivatives and difference quotients on metric spaces

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Some classical inequalities

Suppose that $f(x)$ is a function on $\mathbb{R}^n$ which is, say, continuously differentiable, and which is equal to 0 on the complement of a bounded set, or at least has some kind of reasonable decay. A well-known formula states that

\[ f(x) = \frac{1}{\sigma_{n-1}} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \frac{\partial f}{\partial y_j}(y) \frac{x_j - y_j}{|x - y|^n} \, dy, \]

where $\sigma_{n-1}$ denotes the surface measure of the unit sphere $S^{n-1} = \{ z \in \mathbb{R}^n : |z| = 1 \}$, and $|z|$ denotes the standard Euclidean norm of $z \in \mathbb{R}^n$. This formula is given in equation (18) on p125 of [Ste1], and it is proved by using the Fundamental Theorem of Calculus to first write $f(x)$ as the integral along any ray emanating from $x$ of the directional derivative of $f$ in the direction of the ray, and then averaging over all such rays.

In particular,

\[ |f(x)| \leq \frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} |\nabla f(y)| \frac{1}{|x - y|^{n-1}} \, dy, \]

where $\nabla f$ denotes the gradient of $f$. There are numerous relatives of this inequality, such as Sobolev and Poincaré inequalities, concerning the behavior of a function in terms of integrals of its gradient, and isoperimetric-type estimates, in which the volume of a region (or its complement) is bounded in terms of the surface measure of its boundary.

For instance, if $f$ is a continuously-differentiable function on $S^n$, then one can show that

\[ \int_{S^n} |f(x) - \text{Av}(f)| \, dx \leq c(n) \int_{S^n} |\nabla f(y)| \, dy, \]
where $c(n)$ is a positive constant that depends only on the dimension, the $dx$ and $dy$ in the integrals denote standard surface measure, and $\text{Av}(f)$ is the average of $f$ over $\mathbb{S}^n$. It is understood that the gradient $\nabla f$ of $f$ refers to the gradient in the tangential directions to $\mathbb{S}^n$. Like the previous inequality, this one can be approached in terms of the Fundamental Theorem of Calculus and suitable averages of the estimates that it yields.

One can consider similar matters on spaces which are in some way “approximately Euclidean” but may not be quite standard. Now we shall turn our attention to situations which are fractal.

**A family of (fractal) Carnot–Carathéodory geometries**

Suppose that $n$ is an odd integer, $n = 2m + 1$, and that $m \geq 1$. Let us think of $\mathbb{S}^n$ as being the unit sphere in $\mathbb{C}^{m+1}$, i.e., $\mathbb{S}^n = \{w \in \mathbb{C}^{m+1} : |w| = 1\}$. At each point $z$ in $\mathbb{S}^n$, one has the usual real tangent space to $\mathbb{S}^n$, consisting of vectors pointing in directions tangent to $\mathbb{S}^n$ at $z$. This can be described explicitly as

$$\left\{ v \in \mathbb{C}^{m+1} : \Re \sum_{j=1}^{m+1} v_j z_j = 0 \right\}, \quad (4)$$

where $\Re a$ and $\overline{a}$ denote the real part and complex conjugate of a complex number $a$, and $v_j, z_j$ denote the $j$th components of $v, z \in \mathbb{C}^{m+1}$. This is a real linear subspace of $\mathbb{C}^{m+1}$ of dimension $2m + 1 = n$, which is the real dimension of $\mathbb{C}^{m+1}$ minus 1. Inside this subspace is a complex linear subspace of complex dimension $m$, which is given by

$$\left\{ v \in \mathbb{C}^{m+1} : \sum_{j=1}^{m+1} v_j z_j = 0 \right\}. \quad (5)$$

In other words, the original tangent space consists of the elements of $\mathbb{C}^{m+1}$ which are orthogonal to $z$ with respect to the standard real inner product, and this complex subspace consists of the elements of $\mathbb{C}^{m+1}$ which are orthogonal to $z$ with respect to the standard complex Hermitian inner product. One can account for the difference in terms of $iz$, which lies in the first and not in the second, and which is in fact orthogonal to the second subspace.

If $f$ is a continuously differentiable function on $\mathbb{S}^n$, then let us write $\overline{\nabla} f(z)$ for the version of the gradient of $f$ which includes directional derivatives of $f$ at $z$ in $\mathbb{S}^n$ only in directions in the complex subspace (5) of the tangent
space. It turns out that we again have that

\[ \int_{S^n} |f(x) - \text{Av}(f)| \, dx \leq \tilde{c}(n) \int_{S^n} |\tilde{\nabla} f(y)| \, dy, \]

where \( \tilde{c}(n) \) is a positive constant that depends only on \( n \). There is a related notion of \( \tilde{\nabla} f \) on \( \mathbb{R}^n \), which is connected to this version on \( S^n \) through a “Cayley transform”, and which can also be described in terms of the \( n \)-dimensional Heisenberg group. For this there are natural counterparts of (1) and (2), as well as other inequalities along these lines. See [Ste2], for instance, and also Remark 3 below.

Ordinarily we can measure distances between points in \( S^n \) by simply taking their distance in the Euclidean metric in \( \mathbb{R}^{n+1} \), or, more intrinsically, by minimizing the length of the shortest curve in \( S^n \) between the two points. The first measurement of distance is less than or equal to the second one, and the second measurement of distance is bounded by a constant factor times the first (namely, \( \pi \)). In connection with the complex subspaces of the tangent spaces and the associated version of the gradient \( \tilde{\nabla} f \), let us define another distance function on \( S^n \), where the distance between two points \( p, q \) is the infimum of the lengths of the paths between \( p \) and \( q \) which have the additional feature that their tangent vectors lie in the complex subspace of the tangent space to \( S^n \) at the given point (as in (3)). More precisely, in this definition let us consider curves \( \gamma(t) \) in \( S^n \), where \( t \) runs through a closed interval in \( \mathbb{R} \), which are piecewise-smooth, and which satisfy this condition on the tangent vectors except possibly at finitely many values of \( t \).

It is a classical fact that for each pair of points \( p, q \) in \( S^n \) there is such a path \( \gamma(t) \). This reflects the “nonintegrability” of the complex subspaces of the tangent spaces. Thus the distance between \( p \) and \( q \) defined above is always finite, and of course it is greater than or equal to the distance defined by minimizing the lengths of all curves between \( p \) and \( q \), and not just the ones with this condition on the tangent vectors.

This new distance function defines the same topology on \( S^n \) as the usual one, but it is quite different geometrically. Basically, it looks like the square root of the other distances in some directions, to balance the restriction on the curves. Indeed, if one starts at a point \( z \) in \( S^n \) and tries to go a small distance in the direction \( iz \), which lies in the tangent space (3) to \( S^n \) at \( z \) but not in the complex subspace (4), then one cannot use curves which go directly in this direction. Instead one uses more indirect paths which follow the complex subspaces of the tangent spaces.
The new distance function in fact defines a geometry on $S^n$ which is fractal. For balls of radius $r \leq 1$, the measure of the ball is on the order of $r^{n+1}$, rather than $r^n$, as for the original geometry. In the new geometry, $S^n$ has Hausdorff dimension $n + 1$, rather than $n$.

These changes affect the kind of inequalities for functions $f$ and the modified gradient $\tilde{\nabla} f$ that one gets. This does not come up so much in (1), (2), except for the constant, but the distance function and Hausdorff dimension are explicitly involved in other inequalities.

Remark 1 In [LutYZ] “affine” versions of Sobolev inequalities are discussed, i.e., inequalities which only involve the vector space structure and Lebesgue measure on $\mathbb{R}^n$, and not an inner product or norm. To my knowledge analogous matters have not been investigated for the Heisenberg groups or other (noncommutative simply-connected) nilpotent Lie groups. That is, one would be interested in inequalities that use the underlying Haar measure (which is Lebesgue measure in standard coordinates), group structure, and dilations, but not otherwise a choice of norm or metric. To look at it another way, in place of invariance under invertible linear transformations on $\mathbb{R}^n$ that preserve volume, one could try to get invariance under automorphisms of a nilpotent Lie group which preserve volume. On the one hand, noncommutativity makes the group more tricky, and on the other hand, the group of automorphisms is smaller.

Other geometries

What about other kinds of geometry? As a basic set-up, let $(M, d(x, y))$ be a metric space, and let $\mu$ be a reasonable measure on $M$. If $f$ is a function on $M$, define $D_\epsilon f$ for $\epsilon > 0$ by

$$D_\epsilon f(x) = \sup \left\{ \frac{|f(y) - f(x)|}{\epsilon} : d(x, y) \leq \epsilon \right\}.$$  

(8)

We would like to know about inequalities concerning the behavior of $f$ and integrals involving $D_\epsilon f$.

For example, suppose that $M$ is a fractal set in a Euclidean space, like the Sierpinski gasket or carpet, with the usual Euclidean metric restricted to the set as our metric. For fractals like these there are natural measures $\mu$ so that the whole set has positive finite measure, and so that the measure assigns equal values to different “pieces” of the set that are essentially the same in
the construction, such as intersections of the set with certain triangles or squares from the construction. Thus there are well-behaved ingredients for the basic set-up.

However, the inequalities do not work in the same way as before. One can have functions $f$ on the Sierpinski gasket or carpet, for instance, which are equal to 0 on one fixed (nontrivial) part of the set, are equal to 1 on another fixed part of the set, and for which the integral of $D_\epsilon f$ tends to 0 as $\epsilon$ tends to 0. This happens by concentrating the oscillations of $f$ in sufficiently-small sets, which is possible for these fractal spaces. For these types of spaces adjustments are needed in the inequalities to be considered in order to take the geometry into account. See [Kig].

In [BouP1, Laa1], remarkable families of metric spaces are discussed in which one does have inequalities as before, i.e., with suitable bounds for a function $f$ in terms of integrals of $D_\epsilon f$ which remain uniform as $\epsilon \to 0$.

It may seem surprising that there are genuinely fractal metric spaces with approximately the same kind of inequalities concerning the behavior of a function $f$ and integrals of $D_\epsilon f$ (with uniform bounds in $\epsilon$ in a suitable range). However, there are basic differences in the “internal” structure of the spaces, regarding intermediate dimensions (compared to the dimension of the space). See [Gro5] in this connection, for the fractal geometries on the spheres $S^n$ described earlier, and other analogous geometries.

There are related issues in non-fractal contexts, such as weighted inequalities on $\mathbb{R}^n$. For (nonsmooth) conformal deformations associated to “strong-$A_\infty$ weights”, as in [DavS1], there is good behavior internally, and a form of this is described in [Sem6]. On the other hand, there are weights for which there are plenty of nice inequalities for functions and their derivatives, but which do not behave so well internally. For instance, the weight might be a continuous function on $\mathbb{R}^3$ which vanishes on a circle and is positive otherwise.

**Graphs**

Suppose that we have a graph, which is to say a nonempty set $V$ of vertices and a set $E$ of edges, which can be represented by unordered pairs of distinct vertices in $V$. It is natural to that ask the graph has locally bounded geometry, in the sense that there is a constant $C_0$ so that each element of $V$ is adjacent to at most $C_0$ elements of $V$. If $C_0$ is not too large, then this is also a significant condition when $V$ is finite.
In addition we ask that the graph is connected, which means that for any pair of points \( x, y \) in \( V \) there is a path in \( V \) which goes from \( x \) to \( y \), i.e., a finite sequence of successively adjacent elements of \( V \) connecting \( x \) to \( y \). Thus we can define the distance \( d(x, y) \) between \( x \) and \( y \) to be the length of the shortest such path, where the length of the path is the number of elements of \( V \) in the sequence minus 1.

If \( x \) lies in \( V \), let us write \( N(x) \) for the set of elements of \( V \) which are adjacent to \( x \). For a real or complex-valued function \( f \) on \( V \), define \( D(f) \) on \( V \) by
\[
D(f)(x) = \max\{|f(w) - f(x)| : w \in N(x)\}.
\]
This plays the role of the absolute value of the gradient in the present discussion, and is the same as \( D_1(f) \) in (8). In place of integrals before, one can now use sums.

For example, for \( V \) one can take \( \mathbb{Z}^n \), the set of \( n \)-tuples of integers, and one can say that two elements of \( \mathbb{Z}^n \) are adjacent when their difference has one component equal to \( \pm 1 \) and the others equal to 0. One might instead take \( V \) to be the set of \( n \)-tuples of integers which lie in a box, with the same condition for adjacency.

Another class of examples comes from the discrete version of the Heisenberg groups. Fix a positive integer \( n \), and let \( H_n \) be the group with generators \( a_1, \ldots, a_n, b_1, \ldots, b_n \), subject to the relations that the \( a_i \)'s commute with each other, the \( b_i \)'s commute with each other, the \( a_i \)'s commute with the \( b_j \)'s when \( i \neq j \), and the commutators \( b_i a_i b_i^{-1} a_i^{-1} \) are equal to each other and commute with all generators of the group. Let us write \( c \) for the common value of the commutators \( b_i a_i b_i^{-1} a_i^{-1} \).

There is a standard normal form for the elements of \( H_n \), namely
\[
a_1^{r_1} \cdots a_n^{r_n} b_1^{s_1} \cdots b_n^{s_n} c^t,
\]
where \( r_1, \ldots, r_n, s_1, \ldots, s_n, \) and \( t \) are integers. That is, every element of \( H_n \) can be written in this form in a unique manner.

Let us say that two elements \( x, y \) of \( H_n \) are adjacent if \( y \) can be written as \( xu \), where \( u \) is either a generator or its inverse. This defines the Cayley graph associated to \( H_n \), and we obtain from it a distance function \( d(x, y) \) on \( H_n \). This distance function is invariant under left translations on \( H_n \).

One can show that the number of elements of \( H_n \) whose distance to the identity element is less than or equal to \( \ell \) is of the order \( \ell^{2n+2} \) when \( \ell \) is a positive integer (rather than \( \ell^{2n+1} \), as might be suggested by the normal form...
A very nice approach to various inequalities for $H_n$ and other groups is given in [ConS].

### Isoperimetric inequalities

The classical isoperimetric inequality states that if $\Omega$ is a region in $\mathbb{R}^n$ with finite volume, then

$$\text{Vol}_n(\Omega) \leq c_n \text{Vol}_{n-1}(\partial \Omega)^{n/(n-1)},$$

where $c_n$ is a positive constant which is chosen so that equality holds when $\Omega$ is a ball. Here $\partial \Omega$ denotes the boundary of $\Omega$, and $\text{Vol}_n$, $\text{Vol}_{n-1}$ denote the $n$ and $(n-1)$-dimensional volumes (of a given set). One might as well assume that $\Omega$ is a bounded open set with reasonably-nice boundary, so that the volumes in question can be defined as in vector calculus, but there are more elaborate versions using Hausdorff measures or generalized derivatives and perimeter which apply more generally.

Let us note that the power $n/(n-1)$ on the right side of (12) is determined by considerations of scaling. That is, if we apply a dilation $x \mapsto \lambda x$ on $\mathbb{R}^n$, where $\lambda$ is a positive real number, then the $n$-dimensional volume of a set after dilation is equal to $\lambda^n$ times the $n$-dimensional volume of the set before dilation, and similarly the $(n-1)$-dimensional volume of a set after dilation is equal to $\lambda^{n-1}$ times the $(n-1)$-dimensional volume of the set before dilation. The power $n/(n-1)$ on the right side of (12) ensures that we get the same factor of $\lambda^n$ on both sides of (12) after dilation.

If we do not ask for the sharp constant, then (12) can be obtained from (2), as follows. Let $f(x)$ be the characteristic function of $\Omega$, which is to say that $f(x) = 1$ when $x$ lies in $\Omega$ and $f(x) = 0$ when $x$ lies in the complement of $\Omega$ in $\mathbb{R}^n$. Of course $f$ is not a continuous function, let alone smooth, but its “generalized gradient” is given by the unit normal vector on $\partial \Omega$ pointing into $\Omega$ times surface measure on $\partial \Omega$. The appropriate version of (2) becomes

$$|f(x)| \leq \frac{1}{\sigma_{n-1}} \int_{\partial \Omega} \frac{1}{|x-y|^{n-1}} d\text{Vol}_{n-1}(y).$$

(13)
This inequality can be derived through the same kind of computation as before, or it can be viewed as a consequence of the previous one, using suitable approximations of \( f \) by smooth functions.

If \( \mu \) is any nonnegative measure on \( \mathbb{R}^n \) with finite total mass, then there is a “weak-type” inequality to the effect that for each \( t > 0 \) the ordinary volume of the set

\[
\left\{ x \in \mathbb{R}^n : \frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} d\mu(y) \geq t \right\}
\]

is bounded by the product of \( t^{-n/(n-1)} \), the total mass of \( \mu \) raised to the power \( n/(n-1) \), and a constant that depends only on \( n \). This follows from the argument on p120 of [Ste1]. Alternatively, the normability of the “weak-type” space \( L^{n/(n-1),\infty} \) can be used, as in Theorem 3.21 on p204 of [SteW].

In our situation, where \( f \) is the characteristic function of \( \Omega \) and satisfies (13), we can apply the weak-type inequality for (14) with \( t = 1 \) to get a bound for the volume of \( \Omega \) by a constant times \( \text{Vol}_{n-1}(\partial \Omega) \).

These are standard types of arguments which are applied in many contexts. In particular, there are numerous inequalities for potentials such as the weak-type inequality just mentioned.

Let us return now to the matter of the exponent \( n/(n-1) \) on the right side of (12). If \( \Omega \) is a ball of radius \( r \), then the volume of \( \Omega \) is equal to a constant depending only on \( n \) times \( r^n \), and \( \text{Vol}_{n-1}(\partial \Omega) \) is equal to another constant that depends only on \( n \) times \( r^{n-1} \). This again determines the exponent \( n/(n-1) \).

In general, the exponent in an isoperimetric inequality (or other factors for related inequalities) is connected to volume growth in the space. In some cases, including suitable conditions of negative curvature, there are isoperimetric inequalities with the exponent equal to 1 (and no additional factors), and the volume grows exponentially. Here we are more concerned with spaces in which the volume of balls grows at most polynomially, and for which the exponent in an isoperimetric inequality would be larger than 1 (or there would be other adjustments, such as an extra factor of the diameter of \( \Omega \)).

**Hyperbolic groups**

Suppose that \( G \) is a countably-infinite group which is generated by a finite set \( S \). It will be convenient to assume that \( S \) is *symmetric*, by which we
mean that \( S \) contains the inverses of its elements. As in the case of the discrete versions of the Heisenberg groups, the Cayley graph associated to \( G \) and \( S \) consists of the elements of \( G \) as vertices with an edge from \( x \) to \( y \) in \( G \) whenever there is a generator \( u \) in \( S \) such that \( y = x u \).

Every pair of vertices in the graph can be connected by a path, and this gives rise to a metric on \( G \), in which the distance between two points is the length of the shortest path that connects them. This distance function is invariant under left translations on the group, which are transformations of the form \( x \mapsto a x \), where \( a \) is any element of \( G \).

Hyperbolic groups can be defined in terms of certain negative curvature properties. See [CooP, GhyH, Gro3]. An important aspect of these groups are their “spaces at infinity”, and indeed these groups lead to many interesting geometries this way.

A special case concerns cocompact (also known as uniform) lattices associated to rank-1 symmetric spaces. The spaces at infinity corresponding to these have Euclidean geometry for classical hyperbolic spaces (with constant negative curvature), and otherwise have Carnot–Carathéodory geometry. The geometry on the unit sphere in \( \mathbb{C}^{m+1} \) using the complex subspaces of the ordinary tangent spaces discussed earlier corresponds to \((m + 1)\)-dimensional complex hyperbolic space, for instance.

The fractal spaces considered in [Bou, BouP1] are also connected to hyperbolic groups, and to buildings instead of symmetric spaces. Concerning the general subject of buildings, see [Bro].

Norms on \( \mathbb{R}^n \)

Fix a positive integer \( n \), and suppose that \( \| \cdot \| \) is a norm on \( \mathbb{R}^n \). Thus \( \| x \| \) is a nonnegative real number for each \( x \) in \( \mathbb{R}^n \) which is 0 exactly when \( x = 0 \), \( \| t x \| = |t| \| x \| \) for all real numbers \( t \) and \( x \) in \( \mathbb{R}^n \), and \( \| x + y \| \leq \| x \| + \| y \| \) for all \( x, y \) in \( \mathbb{R}^n \). Associated to this norm is the distance function \( d(x, y) = \| x - y \| \) on \( \mathbb{R}^n \).

Erwin Lutwak once explained to me how there are many interesting questions in this setting related to the sort of topics and questions that we have been considering here. The distance function just defined and the ordinary Euclidean distance are each bounded by constant multiples of the other, so that similar inequalities are around as in the Euclidean case, but of course the constants do not have to be the same, nor the geometry of various optimizing objects, such as for isoperimetry. For that matter, there are numerous ques-
tions about how to make basic measurements, such as area of submanifolds, and different choices lead to different geometric possibilities.

A basic reference related to these matters is [Tho].

In addition to $\mathbb{R}^n$ itself, one might consider lattices contained therein. Of course the geometry of a norm in connection with a lattice can be quite remarkable. See [Cas1].

$p$-Adic numbers and absolute values

Let $p$ be a prime number. The $p$-adic absolute value $|x|_p$ of a rational number $x$ is defined as follows. If $x = 0$, then $|x|_p = 0$. If $x = (a/b)p^k$, where $a$, $b$, and $k$ are integers, with $a, b \neq 0$ and neither $a$ nor $b$ divisible by $p$, then $|x|_p = p^{-k}$. Two basic properties of $| \cdot |_p$ are

\[(15) \quad |x + y|_p \leq \max(|x|_p, |y|_p), \quad |xy|_p = |x|_p |y|_p\]

for all $x, y \in \mathbb{Q}$. These are not difficult to verify. Notice that

\[(16) \quad (p - 1) \sum_{j=0}^{k} p^j = -1 + p^{k+1}\]

for $k \geq 0$, so that $-1$ can be approximated in the $p$-adic absolute value by positive integers.

Just as the real numbers can be viewed as a completion of the rational numbers with respect to the ordinary absolute value function $|x|$, the $p$-adic numbers can be viewed as a completion of the rational numbers with respect to the $p$-adic absolute value function $|x|_p$. For simplicity, let us restrict our attention to rational numbers here, with measurements in terms of an absolute value function. See [Cas2, Gou] for more information about $p$-adic numbers and absolute values.

Suppose that we have a graph again, consisting of a nonempty set $V$ of vertices which is at most countable, and a set $E$ of edges which can be represented by unordered pairs of distinct vertices in $V$. We ask that the graph has at most finitely many edges attached to any fixed vertex, and, as before, one might also ask for a bound on the number of such edges.

Let $(a_{u,v})$ be a matrix of rational numbers, where $u$ and $v$ run through the set $V$ of vertices. We ask that $a_{u,v}$ be zero unless either $u = v$ or there is an edge in the graph between $u$ and $v$. From this matrix we get an operator
$A$ on functions on $V$ with values in $\mathbb{Q}$, given by
\begin{equation}
A(f)(v) = \sum_{u \in V} a_{u,v} f(u).
\end{equation}

Under our assumptions, the sum on the right is a finite sum for each $v$ in $V$.

A fundamental feature of the operator $A$ is that
\begin{equation}
\sup_{v \in V} |A(f)(v)| \leq \left( \sup_{v \in V} \sum_{u \in V} |a_{u,v}| \right) \left( \sup_{w \in V} |f(w)| \right).
\end{equation}

Here we are using the classical absolute values on $\mathbb{Q}$. It is not hardy to verify (18), using the triangle inequality. This estimate is sharp, in the sense that for each $v$ in $V$ there is a function $f$ on $V$ such that
\begin{equation}
A(f)(v) = \sum_{u \in V} |a_{u,v}|
\end{equation}
and $f(w) = \pm 1$ for all $w$ in $V$.

Now let $p$ be a prime number again, and consider the $p$-adic absolute value function $| \cdot |_p$. For this we have that
\begin{equation}
\sup_{v \in V} |A(f)(v)|_p \leq \left( \sup_{v \in V} \sup_{u \in V} |a_{u,v}|_p \right) \left( \sup_{w \in V} |f(w)|_p \right).
\end{equation}

This estimate can be verified using the ultrametric version of the triangle inequality for the $p$-adic absolute value function. We also have that (20) is sharp, in that for each $v$ in $V$ there is a function $f$ on $V$ such that
\begin{equation}
|A(f)(v)|_p = \sup_{u \in V} |a_{u,v}|_p
\end{equation}
and $f(w) = 1$ for one $w$ in $V$, $f(z) = 0$ at all other points in $V$.

Another feature of the operator $A$ is that
\begin{equation}
\sum_{v \in V} |A(f)(v)| \leq \left( \sup_{v \in V} \sum_{u \in V} |a_{u,v}| \right) \left( \sum_{w \in V} |f(w)| \right)
\end{equation}
and
\begin{equation}
\sum_{v \in V} |A(f)(v)|_p \leq \left( \sup_{v \in V} \sum_{u \in V} |a_{u,v}|_p \right) \left( \sum_{w \in V} |f(w)|_p \right)
\end{equation}
for rational-valued functions on $V$. These inequalities are not difficult to verify. Notice that if $y$ is an element of $V$ and $f_y$ is the function on $V$ defined by $f_y(y) = 1$, $f_y(z) = 0$ when $z \neq y$, then $A(f_y)(v) = a_{y,v}$,
\begin{equation}
\sum_{v \in V} |A(f_y)(v)| = \sum_{v \in V} |a_{y,v}|, \quad \sum_{v \in V} |A(f_y)(v)|_p = \sum_{v \in V} |a_{y,v}|_p,
\end{equation}
and $\sum_{w \in V} |f_y(w)| = 1, \sum_{w \in V} |f_y(w)|_p = 1$. This shows the sharpness of (22), (23).
Other types of computations on graphs

There are contexts in which one looks at different kinds of operations associated to vertices or edges in a graph. An instance of this is given by Boolean circuits, with the operations $\land$, $\lor$, and $\neg$ (and, or, and negation). See [LewP, Pap], for example.

Connectedness is a basic issue, i.e., is there a path from a vertex $p$ to a vertex $q$, perhaps satisfying some auxiliary condition. There can also be questions of having a number of paths from one point to another, and not just a single path. See [CarS].

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