Selecting the best system and multi-armed bandits

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Abstract

Consider the problem of finding a population or a probability distribution amongst many with the largest mean when these means are unknown but population samples can be simulated or otherwise generated. Typically, by selecting largest sample mean population, it can be shown that false selection probability decays at an exponential rate. Lately, researchers have sought algorithms that guarantee that this probability is restricted to a small $\delta$ in order $\log(1/\delta)$ computational time by estimating the associated large deviations rate function via simulation. We show that such guarantees are misleading when populations have unbounded support even when these may be light-tailed. Specifically, we show that any policy that identifies the correct population with probability at least $1 - \delta$ for each problem instance requires infinite number of samples in expectation in making such a determination in any problem instance. This suggests that some restrictions are essential on populations to devise $O(\log(1/\delta))$ algorithms with $1 - \delta$ correctness guarantees. We note that under restriction on population moments, such methods are easily designed, and that sequential methods from stochastic multi-armed bandit literature can be adapted to devise such algorithms.

Keywords: Ordinal optimization, ranking and selection, best arm selection, pure exploration, multi armed bandits

1 Introduction

Suppose that we can sample independently from $d$ different probability distributions or populations $(X(i) : i \leq d)$. Further, from each population $i$, we can generate independent identically distributed (iid) samples $(X(i,j) : j \geq 1)$. The distribution of $(X(i) : i \leq d)$ is not known and our aim is to find the ‘best’ population

$$i^* = \arg \max_{1 \leq j \leq d} EX(j).$$

This and related problems of ascertaining the orders of expectations of populations are referred to as ordinal optimization in simulation literature. The key observation is that ordinals are learnt correctly faster than the values of expectations. More specifically, when the random variables $(X(i) : i \leq d)$ are light-tailed (a distribution is said to be light-tailed if its moment generating
function is finite in a neighbourhood of zero) when $n$ samples are generated from each population and the population with the largest sample mean is selected as the best one, the probability of false selection, $P(FS)$,

$$P(\bar{X}_n(i^*) < \max_{j \neq i^*} \bar{X}_n(j))$$

decays exponentially in $n$, where $\bar{X}_n(i) = \frac{1}{n} \sum_{k=1}^{n} X(i, k)$, while it is well known through central limit theorem that the rate of convergence of $\bar{X}_n(i)$ to $EX(i)$ is a much slower $n^{-1/2}$.

This exponential decay rate of false selection probability suggests that for small but positive $\delta$, one may be able to construct algorithms that, when acting on light-tailed distributions, by generating $O(\log(1/\delta))$ samples of $X(i)'s$, can make a correct selection with the probability of false selection restricted to $\delta$. In this paper we critically examine this proposition. We answer this in negative for algorithms acting on a collection of random variables with unbounded support on the right, under a mild technical condition. This includes algorithms that act on the collection of all light-tailed distributions with unbounded support on the right. In fact, we conclude that expected number of samples generated by such an algorithm would be infinite. We provide a positive answer when further information on moments of the underlying random variables is available.

Applications of ordinal optimisation in simulation arise in many set-ups including in selecting a best design from a set of competing designs via simulation where all of the designs may be modelled as discrete event dynamic systems. Such systems include queueing systems, computer and communications networks, manufacturing systems and transportation networks (see, e.g., Ho, Srinivas and Vakili 1992 for some applications).

1.1 Brief literature review

1.1.1 Simulation

Ho et. al. (1992) observed that determining ordinals amongst population means is faster than estimating the means. Dai (1996) used large deviations theory to show in a fairly general framework that for light-tailed random variables the probability of false selection decays exponentially. Chen, Lin, Yuceesan and Chick (2000) considered the problem of ordinal optimization under the assumption that a fixed but large computation budget $n$ is available and the underlying random variables have a Gaussian distribution. They optimized the budget allocated to each population asymptotically as $n \to \infty$ so that the probability of false selection is minimized. Glynn and Juneja (2004) observed that if fraction $p_i n$ of computational budget is allocated to population $i$ for each $i$ ($p_i > 0$, and $p_i n$ an integer), then the probability of false selection,

$$\lim_{n \to \infty} \frac{1}{n} \log P(X_i^*(p_i, n) < \max_{i \neq i^*} X_i(p_i n)) = -H(p_1, \ldots, p_d),$$

where $H(\cdot)$ is concave. They, then maximized $H(p_1, \ldots, p_d)$ under the constraint $\sum_{i=1}^{d} p_i = 1$ to determine the optimal allocations as $n \to \infty$ even for non-Gaussian distributions. Significant liter-
ature since then has appeared that relies on large deviations analysis (e.g., Hunter and Pasupathy 2013, Szechtman and Yucesan 2008, Broadie, Han, Zeevi 2007, Blanchet, Liu, Zwart 2008, Shin, Broadie and Zeevi 2016).

Substantial literature exists on selecting the best system amongst many alternatives using ranking/selection procedures, see Chernoff (1959), Paulson (1964), Bechhofer, Kiefer and Sobel (1968), Jennison, Johnstone and Turnbull (1982) for some of the earlier references from the statistics literature.

See, e.g., Kim and Nelson (2001, 2003), Branke, Chick and Schmidt (2007) for an overview, Frazier (2014) for a Bayesian inspired approach. Gaussian assumption is critical to most of the analysis here. Typically, an ‘indifference-zone formulation’ is considered where it is assumed that there exists a known $\epsilon > 0$ such that

$$EX(i^*) \geq EX(j) + \epsilon$$

for $j \neq i^*$ (see, e.g., Nelson and Matejcik 1995). Kim and Nelson (2001) propose a number of algorithms for selecting the best system. They conduct asymptotic analysis where the probability of false selection is kept fixed at a specified $\delta > 0$ while the indifference zone is set to $\epsilon$. They show that the proposed algorithms restrict $P(FS)$ to within $\delta$, asymptotically as $\epsilon \to 0$, by generating $O(\epsilon^{-2})$ samples. In contrast, in this paper, we primarily consider settings where the indifference zone is kept fixed and $\delta$ is either fixed, or it decreases to zero.

1.1.2 Stochastic multi-armed bandits

The problem of selecting the best system amongst many via sampling is related to the vast, elegant and evolving literature in learning and statistics community referred to as stochastic multi-armed bandit methods. Typically, this strand of literature refers to sampling from a population as ‘pulling an arm’ and assumes that each such pull leads to a random reward whose distribution depends upon the population. The aim then is to develop optimal or near optimal sequential sampling strategy that maximises the long term total expected reward or equivalently minimises the total expected regret (regret over $n$ trials is referred to as the reward that would have been realised in these trials if the ‘best’ arm was pulled each time versus the actual reward realisation from a given sequential strategy - here best refers to the arm with the highest expected reward. See, e.g., Bubeck and Cesa-Bianchi 2012, Cappe et. al. 2013; see Lai and Robbins 1985 for a seminal paper in this area). Optimal sequential strategies have to carefully manage the exploration-exploitation trade-offs in arm selections. Recently, these methods have also been used in the pure exploration setting where the goal is to identify the arm with the highest expected reward in minimum expected number of trials. See, e.g., Even-Dar, Mannor and Mansour (2002, 2006), Audibert and Bubeck (2010), Jamieson, Malloy, Nowak, & Bubeck (2014), Garivier and Kaufmann (2016) and Kaufmann, Cappe
and Garivier (2016). Thus, this problem is identical to the ordinal optimisation problem that we consider. A standard assumption both in minimising regret and the pure exploration settings is that the rewards from each arm are either Bernoulli, are bounded with known bounds, or have sub-Gaussian tails. Garivier and Kaufmann (2016) and Kaufmann, Cappe and Garivier (2016) consider a somewhat general setting where the rewards come from a single-parameter exponential family. In this setting, Garivier and Kaufmann (2016) develop lower bounds on expected number of samples from each arm, as well as algorithms that match these lower bounds asymptotically as $\delta \to 0$. Bubeck, Cesa-Bianchi and Lugosi (2013), again consider the problem of minimising expected regret under the assumption that the rewards are unbounded but explicit bounds on their moments are known.

1.2 Observations

Suppose that, as in (1), order $n$ samples are allocated to each population, and as is typically the case when the underlying random variables are light-tailed, the probability of false selection, $P(FS) \leq e^{-nI}$, for some $I > 0$. If such an $I$ is known beforehand, setting

$$n = -\frac{1}{I} \log(\delta)$$

ensures $P(FS) \leq \delta$.

Typically, such an $I$ is not known but needs to be estimated via simulation. One then hopes for algorithms that for $n = O(\log(1/\delta))$ ensure that $P(FS) \leq \delta$, for each $\delta$, or this holds asymptotically, that is,

$$\limsup_{\delta \to 0} P(FS) \delta^{-1} \leq 1,$$

even when an indifference zone formulation is considered so that there exists a fixed and known $\epsilon > 0$ and (2) holds.

Note that $O(\log(1/\delta))$ effort is necessary to achieve (3) in the sense that if $\log(1/\delta)^{1-\tilde{\epsilon}}$ samples are generated, for any small $\tilde{\epsilon} > 0$, then for any $i$ and $A$ such that $P(X(i) \in A) > 0$

$$P(X(i) \in A)^{\log(1/\delta)^{1-\tilde{\epsilon}}} = \delta^{\frac{\text{positive no.}}{\log(1/\delta)^{c}}} > \delta$$

as $\delta \to 0$, so one cannot hope for (3) to hold.

Also observe that order $\log(1/\delta)^{1+\tilde{\epsilon}}$ is sufficient as for $c > 0$

$$\delta^{-1} P(FS) \leq \delta^{-1} e^{-nI} = \delta^{-1} e^{-c \log(1/\delta)^{1+\tilde{\epsilon}}} I = \delta^{c \log(1/\delta)^{c} I - 1}$$

which goes to zero as $\delta \to 0$.

In this paper we shed light on conditions under which algorithms that require $O(\log(1/\delta))$ samples and satisfy $P(FS) \leq \delta$ for each $\delta$, or more generally satisfy the relation (3), exist, as well as when such algorithms do not exist.
1.3 Our contributions

Our key result is negative and is as follows - Consider a collection of probability distributions \( \mathcal{L} \) and an algorithm that given any finite set of distributions belonging to \( \mathcal{L} \), and a small \( \delta > 0 \), adaptively generates samples from these distributions and identifies the distribution with the largest mean with the probability of false selection \( P(FS) \leq \delta \). These finite set of distributions are assumed to be well separated in the sense that the mean of the distribution with the highest mean exceeds the mean of other distributions by at least \( \epsilon > 0 \). We also consider algorithms that only provide asymptotic guarantees in the sense that

\[
\limsup_{\delta \to 0} P(FS)\delta^{-1} \leq 1. \tag{4}
\]

We prove that if \( \mathcal{L} \) comprises populations with unbounded support that are KL right dense (defined in Section 4), the expected number of samples so that \( P(FS) \leq \delta \) for any \( \delta \in (0, 1/2) \), must be infinite. If the algorithm only provides asymptotic guarantees as in (4), then the expected number of samples cannot be \( O(\log(1/\delta)) \). Further, similar results hold when the criteria for selecting the best design may not be the population mean but another function of the population distribution such as its specific quantile.

Our positive contributions - Under explicitly available upper bounds on convex, increasing functions of underlying random variables, we develop random variable truncation as well as capping based \( O(\log(1/\delta)) \) computation time algorithms that guarantee that \( P(FS) \leq \delta \). We also observe that the sequential algorithms proposed in multi-armed bandit regret setting (see Bubeck, Cesa-Bianchi and Lugosi 2013), when heavy tails are involved and bounds on the \( \alpha \in (1, 2] \) moments of the underlying distributions are assumed known, are easily adapted to this pure exploration setting to provide \( O(\log(1/\delta)) \) computation time sequential algorithms. We also develop upper bounds on computational effort under these algorithms and suggest tweaks that lead to minor performance improvement guarantees.

Our attempt in presenting positive results is to illustrate reasonable conditions under which \( O(\log(1/\delta)) \) computation time algorithms are feasible in principle. We do not attempt to excessively optimize the proposed implementations. A theory involving general distributions that characterizes the lower bounds on expected number of samples from each arm, and develops algorithms that match these lower bounds, is an interesting area for future research. Recall that Garivier and Kaufmann (2016) achieve this for single parameter exponential family of distributions.

1.4 Roadmap

Section 2 contains our key negative result illustrating that, under mild regularity conditions, the algorithms that control the probability of false selection to within a small \( \delta \) must require infinite number of samples in expectation. In Sections 3 and 4 we provide some positive results. In Section 3,
we develop $O(\log(1/\delta))$ algorithms when upper bounds on suitable moments of underlying random variables are available. In Section 4, we adapt recent results from multi-armed bandit related research to our ordinal optimisation setting. We end with a brief conclusion in Section 5.

All but the simpler proofs are given in the appendix.

2 The negative result

We consider an algorithm or a policy acting on any finite set of probability distributions taken from a collection $\mathcal{L}$. The policy does not know the underlying set of distributions but it can adaptively sample from them. The objective is to identify the distribution with largest mean while adaptively generating fewest number of samples, and providing guarantees on the probability of error.

Specifically, consider a policy that acts on a set of $d$ distributions ($F_i : i \leq d$) in $\mathcal{L}$. Let

- $D_t$ denote the index of the distribution sampled from at stage $t$.
- $N_i(t) = \sum_{k=1}^{t} I(D_k = i)$ denote the number of samples generated from distribution $F_i$ by stage $t$.
- $X_{i,j}$ denote the sample $j$ generated from distribution $F_i$. This is assumed to be independent of other previously generated samples.
- $\mathcal{F}_t$ denote the sigma-algebra associated with $(D_k, k \leq t)$ and $(X_{i,j}, i \leq d, j \leq N_i(t))$.

Then, at stage $t+1$, the policy selects a distribution $D_{t+1}$ to sample from where $D_{t+1}$ is measurable w.r.t. $\mathcal{F}_t$. Thus, if $D_{t+1} = s$, then the policy generates a sample $X_{s,N_s(t)+1}$ from distribution $F_s$.

Each policy considered is further assumed to have an almost sure finite stopping time $\tau$ with respect to the filtration $\{\mathcal{F}_t\}$. At time $\tau$ the policy announces its decision $J = i$ if it deems distribution $F_i$ to have the largest mean.

**Definition 1** An algorithm or a policy is referred to as $\mathcal{P}(\epsilon, \delta)$, if for any $\epsilon > 0$ and $0 < \delta < 1$, given any finite collection of distributions from $\mathcal{L}$ such that the difference between the highest and the second highest mean amongst these distributions exceeds $\epsilon$, the policy, based on generated samples from these distributions, identifies the correct distribution (the one with the largest mean) with the probability of false selection $P(FS)$ guaranteed to be $\leq \delta$.

**Definition 2** $\mathcal{P}_a(\epsilon)$ denotes a larger class of policies that differ from $\mathcal{P}(\epsilon, \delta)$ in that for any $\epsilon > 0$, they only guarantee that $P(FS)$ is asymptotically bounded from above by $\delta$. That is,

$$\limsup_{\delta \to 0} P(FS)\delta^{-1} \leq 1.$$ (5)
Thus, roughly speaking, a policy $\mathcal{P}_a(\epsilon)$ is $\mathcal{P}(\epsilon, \delta)$ for all positive and sufficiently small $\delta$. In many applications, it may be difficult to design a $\mathcal{P}(\epsilon, \delta)$ policy, however, designing a $\mathcal{P}_a(\epsilon)$ policy may be easier, and often a practitioner may be satisfied with asymptotic guarantees. (This is analogous to accepting central limit theorem based confidence intervals for population means where the coverage guarantees are valid only asymptotically, see, e.g., Glynn and Whitt 1992). Note that if a policy is $\mathcal{P}(\epsilon, \delta)$ on $\mathcal{L}$, then it is also $\mathcal{P}_a(\epsilon)$ on $\mathcal{L}$.

Suppose that the data was generated from distributions $(F_i : i \leq d)$ under probability measure $P$, and from $(\tilde{F}_i : i \leq d)$ under probability measure $\tilde{P}$, where $F_i$ and $\tilde{F}_i$ are mutually absolutely continuous for each $i$. Let $H(G|\tilde{G}) = \int_{x \in \mathbb{R}} \left( \log \frac{dG}{d\tilde{G}}(x) \right) dG(x)$ denote the Kullback-Leibler distance between distributions $G$ and $\tilde{G}$.

Kaufmann, Cappe and Garivier (2016) Lemma 1, arrives at the following elegant inequality (under the assumption that $\tau$ is a.s. finite under $P$),

$$\sum_{i=1}^{d} E_{P} N_i(\tau) H(F_i|\tilde{F}_i) \geq \sup_{\mathcal{E} \in \mathcal{F}_\tau} d(P(\mathcal{E})|\tilde{P}(\mathcal{E})) \tag{6}$$

where for $p, q \in (0, 1)$

$$d(p, q) = p \log \frac{p}{q} + (1 - p) \log \frac{1-p}{1-q},$$

and $\mathcal{F}_\tau$ denotes the sigma algebra associated with stopping time $\tau$. This allows development of lower bounds $E_{P} N_i(\tau)$ under $\mathcal{P}(\epsilon, \delta)$ and $\mathcal{P}_a(\epsilon)$ policies. Essential ideas for deriving such lower bounds go back to Lai and Robbins (1985) and Mannor and Tsitsiklis (2002). See Glynn and Juneja (2015), an earlier version of this paper, for a derivation based on these ideas when $\delta \to 0$. Inequality (6) considerably simplifies the development of lower bounds, and provides non-asymptotic bounds for any $\delta \in (0, 1)$.

2.1 Lower bounds

Our aim in this section is to show that $\mathcal{P}(\epsilon, \delta)$ policies require infinite number of samples in expectation to provide the desired probabilistic guarantees, while $\mathcal{P}_a(\epsilon)$ policies cannot provide the probabilistic guarantees by generating only $O(\log(1/\delta))$ samples when $\mathcal{L}$ comprises distributions with finite mean, unbounded support and additional mild restrictions. To illustrate this simply, we restrict our analysis to $d = 2$. Analysis extends easily to $d > 2$.

Consider distributions $F$, $G$ and $\tilde{G}$ in $\mathcal{L}$ with respective means $\mu_F$, $\mu_G$ and $\mu_{\tilde{G}}$. For $\epsilon > 0$, $\mu_G < \mu_F - \epsilon$ and $\mu_{\tilde{G}} > \mu_F + \epsilon$. Further, the Kullback-Leibler distance $H(G|\tilde{G}) < \infty$, and $G$ and $\tilde{G}$ are mutually absolutely continuous.

2.1.1 $\mathcal{P}(\epsilon, \delta)$ and $\mathcal{P}_a(\epsilon)$ policies

Consider a $\mathcal{P}(\epsilon, \delta)$ policy acting on samples $(X_{i,j}, i = 1, 2, j \geq 1)$ from two distributions in $\mathcal{L}$.
Suppose that under probability measure $P$, distribution of $(X_{1,j}, j \geq 1)$ is $F$, and that of $(X_{2,j}, j \geq 1)$ is $G$. Under $\tilde{P}$, the respective distributions are $F$ and $\tilde{G}$. Then,

$$P(J = 2) \leq \delta$$

while

$$\tilde{P}(J = 2) \geq 1 - \delta.$$  

Applying (6) to this set-up with $E = \{J = 2\}$, it follows that for $\delta \in (0, 1/2)$,

$$E_{P} N_2(\tau) \geq \frac{d(\delta, 1 - \delta)}{H(G|G)}. \quad (7)$$

As noted by Kauffmann, Cappe and Garivier (2016)

$$d(\delta, 1 - \delta) \geq \log \frac{1}{2.4\delta}$$

for all $\delta \in (0, 1)$.

Theorem 1 is the main result of this section. It requires Lemma 1 below.

**Lemma 1** Given $G$ with finite mean $\mu_G$ and unbounded support on the positive real line, for any $\alpha > 0$, and $\beta > \mu_G$ there exists a distribution $G_\beta$, such that the Kullback-Leibler distance between $G$ and $G_\beta$,

$$H(G|G_\beta) \leq \alpha, \quad (8)$$

and

$$\mu_{G_\beta} = \int_{x \in \mathbb{R}} xdG_\beta(x) \geq \beta. \quad (9)$$

**Definition 3** A collection of probability distributions $\mathcal{L}$ is referred to as KL right dense, if each distribution has finite mean and if for every $G \in \mathcal{L}$, and every $\alpha > 0$, $\beta > \mu_G$, there exists a distribution $G_\beta \in \mathcal{L}$ such that (8) and (9) hold.

A necessary condition for $\mathcal{L}$ to be KL right dense is that each member have an unbounded support on the positive real line. Examples of KL right dense class includes collection of distributions, which in addition to unbounded support on the positive real line, are also light tailed (recall that a distribution is said to be light tailed if its moment generating function is finite in a neighbourhood of zero). Another example is collection of distributions that along with having an unbounded support on the positive real line, are in $\mathcal{L}^p$, for some $p \geq 1$. That is, for some $p \geq 1$, their absolute $p$ th moment is finite.

**Theorem 1** Under $\mathcal{P}(\epsilon, \delta)$ policy operating on KL right dense $\mathcal{L}$, for $P$ as above with distributions $F$ and $G$ in $\mathcal{L}$, for $\delta \in (0, 1/2)$,

$$E_{a} N_2(\tau) = \infty. \quad (10)$$
Under $\mathcal{P}_a(\epsilon)$ policy operating on $KL$ right dense $\mathcal{L}$, for $P$ as above with distributions $F$ and $G$ in $\mathcal{L}$,

$$\liminf_{\delta \to 0} \frac{E_P N_2(\tau)}{\log(1/\delta)} = \infty. \quad (11)$$

**Proof of Theorem 1**: Equation (10) follows from (6) and Lemma 1.

To see (11), observe that under $\mathcal{P}_a(\epsilon)$ policy,

$$\limsup_{\delta \to 0} P(\hat{J} = 2) \delta^{-1} \leq 1, \quad (12)$$

and

$$\limsup_{\delta \to 0} (1 - \hat{P}(J = 2)) \delta^{-1} \leq 1. \quad (13)$$

Again applying (6) to this set-up with $E = \{J = 2\}$, it follows that

$$E_P N_2(\tau) \geq \frac{d(P(J = 2) | \hat{P}(J = 2))}{H(G|\tilde{G})}. \quad (14)$$

It is easy to see from (12) and (13) that

$$d(P(J = 2) | \hat{P}(J = 2)) \sim \log \delta^{-1}$$

Hence,

$$\liminf_{\delta \to 0} \frac{E_P N_2(\tau)}{\log(1/\delta)} \geq \frac{1}{H(G|\tilde{G})}.$$

(11) follows from Lemma 1. □

**Remark 1** Observe that under $\mathcal{P}(\epsilon, \delta)$ policy, a bound on $E_P N_1(\tau)$ can be obtained by essentially repeating the analysis:

$$\frac{E_P N_1(\tau)}{\log(1/2.4\delta)} \geq \sup_{\tilde{F} \in \mathcal{L}, \mu_{\tilde{F}} < \mu_G - \epsilon, H(F|\tilde{F}) < \infty} \frac{1}{H(F|\tilde{F})}. \quad (15)$$

If the objective of the proposed policies was to find a population with the smallest mean, one simply alters the sign of the random variables in the above analysis. Then, $\mathcal{L}$ needs to be $KL$ left dense (defined analogously to $KL$ right dense collection) to ensure that under the associated $\mathcal{P}(\epsilon, \delta)$ expected number of samples is infinite, while a $\mathcal{P}_a(\epsilon)$ policy requires more than $O(\log(1/\delta))$ samples.

### 2.2 More general setting

It is easy to extend this analysis in a variety of ways. For instance, suppose that given two populations our aim is to identify the one with the largest $p$ th quantile. Recall that for any real
valued random variable \( Z \) with distribution function \( F_Z(\cdot) \), the quantile function is the inverse of \( F_Z(\cdot) \), and in particular, \( p \) th quantile is given by

\[
F_Z^{-1}(p) = \inf_{x \in \mathbb{R}} \{ F_Z(x) \geq p \}.
\]

The analysis easily generalises to handle such cases. To see this, consider random elements taking values in a general state space \( \mathcal{X} \). Let \( h \) denote a mapping from a probability distribution (more generally, a probability measure) on \( \mathcal{X} \) to the real line. The \( p \) th quantile is one example of such a mapping from a probability distribution of a real-valued random variable to the real line. Let \( \mathcal{H} \) denote a collection of probability distributions on \( \mathcal{X} \) such that \( |h(F) - h(G)| > \epsilon > 0 \) the policy \( P(\epsilon, \delta) \), generates samples from the two distributions, adaptively deciding which distribution to sample from next and at some stage selects one of the two distributions as the one with the higher \( h \) value, where \( h \) may be evaluated at the respective empirical distribution. The policy guarantees that \( P(FS) \leq \delta \). As before, \( P_a(\epsilon) \) guarantees this asymptotically.

In the discussion at the beginning of Section 2.1 where the two populations \( (X_{i,j} : i = 1, 2, j \geq 1) \) are compared under the two probability measures \( P_a \) and \( P_b \), replace \( \mu_F, \mu_G \) and \( \mu_{\tilde{G}} \) with \( h(F), h(G) \) and \( h(G) \), where \( F, G, \tilde{G} \in \mathcal{H} \). Then, it follows that under \( P(\epsilon, \delta) \),

\[
\frac{E_a N_2(\tau)}{\log(1/2.4\delta)} \geq \sup_{\tilde{G} \in \mathcal{H}, h(\tilde{G}) > h(F) + \epsilon, H(G|\tilde{G}) < \infty} \frac{1}{H(G|\tilde{G})}. \tag{16}
\]

Under \( P_a(\epsilon) \) policy

\[
\lim_{\delta \to 0} \inf \frac{E_a N_2(\tau)}{\log(1/\delta)} \geq \sup_{\tilde{G} \in \mathcal{H}, h(\tilde{G}) > h(F) + \epsilon, H(G|\tilde{G}) < \infty} \frac{1}{H(G|\tilde{G})}. \tag{17}
\]

**Example 1** In the case where \( \mathcal{X} = \mathbb{R} \), \( h(F) \) denotes the \( p \) th quantile of \( F \), it easy to see that RHS on (16) can be infinite. We show this when \( \mathcal{H} \) includes mixture of Gaussian distributions.

For \( \mu > 0 \), and \( 0 < \epsilon < p < 1/2 \), consider the pdfs

\[
g(x) = \sqrt{\frac{1}{2\pi}} \left( p \exp(-x^2/2) + (1 - p) \exp(-(x - \mu)^2/2) \right),
\]

and

\[
g_\epsilon(x) = \sqrt{\frac{1}{2\pi}} \left( (p - \epsilon) \exp(-x^2/2) + (1 - p + \epsilon) \exp(-(x - \mu)^2/2) \right).
\]

Since,

\[
\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \exp(-x^2/2)dx = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\mu/2} \exp(-(x - \mu/2)^2/2)dx.
\]
and \( p < 1/2 \), the \( p \) th quantile of \( g \), \( m_g < \mu/2 \). The \( p \) th quantile of \( g_\epsilon \), \( m_{g_\epsilon} \geq m(\epsilon) \) where \( m(\epsilon) \) denotes the point where the second Gaussian component has a total mass equal to \( \epsilon \), that is,

\[
\frac{\epsilon}{1-p+\epsilon} = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{m(\epsilon)} \exp\left(-\frac{(x-\mu)^2}{2}\right) dx = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{(m(\epsilon)-\mu)} \exp\left(-\frac{x^2}{2}\right) dx.
\]

Using the well known fact that

\[
\int_{-\infty}^{z} \exp\left(-\frac{x^2}{2}\right) dx \sim \frac{1}{|z|} \exp\left(-\frac{z^2}{2}\right)
\]

as \( z \to -\infty \), we have

\[
\mu - m(\epsilon) \sim \sqrt{2 \log\left(\frac{(1-p+\epsilon)/\epsilon}{1}\right)}
\]

as \( \epsilon \to 0 \). In particular, it follows that for small and fixed \( \epsilon \), \( m_{g_\epsilon} - m_g \) can be made arbitrarily large by increasing \( \mu \).

Now consider, the Kullback-Leibler distance

\[
\int_{-\infty}^{\infty} \log\left(\frac{g(x)}{g_\epsilon(x)}\right) g(x) dx.
\]

Observe that

\[
\left(\frac{p}{p-\epsilon}\right) g_\epsilon(x) \geq g(x),
\]

for all \( x \), for all \( \mu \). Thus, \( \frac{g(x)}{g_\epsilon(x)} \leq \frac{p}{p-\epsilon} \) for all \( x \) and \( \mu \). Hence, for any \( \mu \), (18) can be made arbitrarily close to zero by choosing \( \epsilon \) sufficiently close to zero.

3 Positive results - the non-adaptive algorithms

In this section, we show that under conditions on moments of strictly increasing non-negative convex functions of the underlying populations, for any given \( \epsilon, \delta > 0 \), one can develop non-adaptive \( \mathcal{P}(\epsilon, \delta) \) algorithms that require a deterministic and known \( O(\log(1/\delta)) \) computational effort for any instance of underlying populations. These algorithms rely on truncating or capping the random samples generated and carefully bounding the resulting worst-case bias using explicitly available moment bounds.

Specifically, we suppose (as in the Introduction) that we can sample independently from \( d \) different random variables \( (X(i) : i \leq d) \). Further, from each population \( i \), we can generate independent identically distributed samples \( (X(i, j) : j \geq 1) \). Recall that the distribution of \( (X(i) : i \leq d) \) is unknown and our aim is to find

\[
i^* = \arg\max_{1 \leq j \leq d} \mathbb{E}X(j).
\]

First in Section 3.0.1 we assume that \( X(i) \in [0, b] \) a.s for each \( i \leq d \), for some \( b > 0 \), and review the \( \mathcal{P}(\epsilon, \delta) \) algorithms that require \( O(\log(1/\delta)) \) computational effort in that simple setting.
Analysis is straightforward and is discussed (independently, it appears), e.g., in Even-Dar et. al. (2002, 2006), Glynn and Juneja (2004, 2011). In Section 3.1 we arrive at the maximum expected error that may result through either appropriately truncating or capping a random variable when an upper bound on the moment of strictly increasing, non-negative convex function of a random variable is explicitly known. In Section 3.2 we develop $P(\epsilon, \delta)$ algorithms that require $O(\log(1/\delta))$ computational effort, when appropriate moment upper bounds are available for each population. Such bounds can often be found in simulation models by the use of Lyapunov function based techniques (see, e.g., Glynn and Zeevi 2008).

### 3.0.1 $P(\epsilon, \delta)$ policy for bounded random variables

For $\epsilon, b > 0$, consider $\mathcal{X}_\epsilon(b) = \{(X(i) : i \leq d) : EX(i^*) > EX(j) + \epsilon \ \forall j \neq i^*, X(i) \in [0, b] \ \forall i\}.$

A reasonable algorithm on $\mathcal{X}_\epsilon(b)$ for a well chosen $n$ (discussed later) is:

- Generate independent samples $(X(i, j) : i = 1, \ldots, d$ and $j = 1, \ldots, n)$.
- Let $\bar{X}(i) = \frac{1}{n} \sum_{j=1}^{n} X(i, j)$. Declare
  \[ \hat{i} = \arg \max_{1 \leq i \leq d} \bar{X}(i) \]
  as the best design.

Recall that false selection occurs if $\hat{i} \neq i^*$, with probability

\[ P(\bar{X}(i^*) < \max_{j \neq i^*} \bar{X}(j)) \]

This is bounded from above by

\[ \sum_{j \neq i^*} P(\bar{X}(i^*) < \bar{X}(j)) \]

Using Hoeffding’s inequality, we have

\[ P(\bar{X}(i^*) < \bar{X}(j)) \leq P(\bar{X}(i^*) - \bar{X}(j) - (EX(i^*) - EX(j)) < -\epsilon) \leq \exp \left(-n \frac{\epsilon^2}{2b^2}\right). \]

Thus, $n = \frac{2b^2}{\epsilon^2} \log((d - 1)/\delta)$ provides the desired $P(\epsilon, \delta)$ policy for any set of distributions corresponding to $\mathcal{X}_\epsilon(b)$.

### 3.1 Bounding the worst-case truncation or capping bias

Suppose that $\mathcal{X}$ is a class of non-negative random variables and $f$ is a strictly increasing non-negative convex function. Examples include $f(x) = x^\alpha$ for $x \geq 0$ and $\alpha > 1$, and $f(x) = \exp(\theta x)$ for $\theta > 0$. We discuss two formulations to bound worst-case biases resulting from 1) truncating a random variable, and 2) capping it.
Consider the optimization problem $O_1$
\[
\max_{X \in \mathcal{X}} E[X] \mathbb{I}(X \geq u)
\]
subject to
\[
Ef(X) \leq c,
\]
for some positive $u$ and $c$.

Also consider $O_2$, where the objective function is instead set to
\[
\max_{X \in \mathcal{X}} E[X - u] \mathbb{I}(X > u),
\]
again under constraint (20).

Furthermore, since, by Jensen’s inequality,
\[
Ef(X) \geq f(EX) \geq f(0),
\]
we assume that $c > f(0)$.

$O_1$ denotes the worst-case expected truncation bias under constraint (20) when $X$ is replaced by the truncated $XI(X < u)$. $O_2$ denotes the smaller bias when $X$ is replaced by the capped $\min(X, u)$.

We now argue that the solutions to $O_1$ and $O_2$ put their mass at most at two points. To see this, observe that given any non-negative random variable $X$ that satisfies (20), a two-valued random variable $Y$ that takes value
\[
\mathbb{E}[X|X < u] \mathbb{I}(X < u) \quad \text{with probability } P(X < u) \quad (P(X \leq u)),
\]
and
\[
\text{value } \mathbb{E}[X|X \geq u] \mathbb{I}(X \geq u) \quad \text{with probability } P(X \geq u) \quad (P(X > u)),
\]
has the same mean $EY = EX$ and same objective function values under $O_1$ ($O_2$), i.e.,
\[
EY \mathbb{I}(Y \geq u) = EX \mathbb{I}(X \geq u), \quad (E(Y - u) \mathbb{I}(Y > u) = E(X - u) \mathbb{I}(X > u)).
\]
Furthermore, $Ef(Y) \leq Ef(X)$, with equality only if $X = Y$ a.s. Thus, only random variables that take at most two values can solve our optimization problems $O_1$ and $O_2$. It is also easy to see that at the optimal solution in both the cases, the constraint (20) has to be tight. Hence, we restrict our search to random variables taking values $0 \leq x_1 < x_2$ with probability $1 - p$ and $p \in [0, 1]$ where
\[
(1 - p)f(x_1) + pf(x_2) = c,
\]
so that $0 \leq x_1 < f^{-1}(c)$ and $x_2 \geq f^{-1}(c)$. Furthermore,
\[
p = \frac{c - f(x_1)}{f(x_2) - f(x_1)}.
\]

Propositions 1 and 2 below observe that unique (a.s.) solutions $\tilde{X}_1$ and $\tilde{X}_2$, respectively, to the optimization problems $O_1$ and $O_2$ are either degenerate and equal $f^{-1}(c)$ with probability 1, or they take value zero with positive probability.
Proposition 1 The unique optimal solution $\tilde{X}_1$ for $O_1$,

1. equals $f^{-1}(c)$ with probability 1, for $u \leq f^{-1}(c)$.

2. For $u > f^{-1}(c)$, $\tilde{X}_1$ has a two-value distribution. It equals $u$ with probability
   \[ \frac{c - f(0)}{f(u) - f(0)}, \]
   and zero otherwise.

The following assumption considerably eases the analysis of $O_2$.

Assumption 1 Given any $u > 0$, there exists $x_u > u$, the unique solution to
   \[ x - u = \frac{f(x) - f(0)}{f'(x)}. \]

The above assumption can be seen to hold, e.g., for $f(x) = x^{\alpha}$, $\alpha > 1$ and for $f(x) = \exp(\theta x)$, $\theta > 0$. To see this, observe that in these cases the function $f(x) - f(0) - (x - u)f'(x)$ monotonically decreases from a positive value at $u$ to $-\infty$ as $x \uparrow \infty$.

Proposition 2 The unique optimal solution $\tilde{X}_2$ for $O_2$ under Assumption 1

1. a.s. equals $f^{-1}(c)$ for $x_u \leq f^{-1}(c)$.

2. For $x_u > f^{-1}(c)$, $\tilde{X}_2$ has a two-valued distribution. It equals $x_u$ with probability
   \[ \frac{c - f(0)}{f(x_u) - f(0)}, \]
   and zero otherwise.

Remark 2 Suppose that $f(x) = x^\alpha$, $\alpha > 1$. Under $O_1$, the solution corresponds to $\tilde{X}_1 = c^{1/\alpha}$ with probability 1 for $u \leq c^{1/\alpha}$ and the associated objective function value equals $c^{1/\alpha}$. Otherwise, $\tilde{X}_1$ takes two values 0 and $u$ where the probability of the latter equals $cu^{-\alpha}$. The objective function value then equals $cu^{-(\alpha-1)}$.

Under $O_2$, for $u > 0$

\[ x_u = u \left( \frac{\alpha}{\alpha - 1} \right). \]

Then, $\tilde{X}_2 = c^{1/\alpha}$ with probability 1 for $u \leq c^{1/\alpha} \left( \frac{\alpha - 1}{\alpha} \right)$, and the associated objective function value equals $c^{1/\alpha} - u$. Otherwise, $\tilde{X}_2$ takes two values 0 and $x_u$ where the probability of the latter equals

\[ cu^{-\alpha} \left( \frac{\alpha - 1}{\alpha} \right)^{\alpha}. \]

The optimal objective function value equals

\[ cu^{-(\alpha-1)} \left( \frac{\alpha - 1}{\alpha} \right)^{\alpha}. \]

(22)
Thus the worst case bias reduces by factor \((\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha-1}})\) if we use \(\min(X,u)\) instead of \(XI(X < u)\) to get random variables bounded from above by \(u\).

Also note that under \(O_1\) if \(f(x) = \exp(\theta x)\) for some \(\theta > 0\), then \(\bar{X}_1 = \log c/\theta\) for \(u < \log c/\theta\) with probability 1, and \(\bar{X}_1\) takes two values 0 and \(u\) where the probability of the latter equals \(\frac{c-1}{\exp(\theta u)-1}\).

The objective function value in this case equals \(u \frac{c-1}{\exp(\theta u)-1}\).

3.1.1 Threshold function

Going forward, we focus on using capping to bound the random variables, because this has smaller bias compared to truncating. Let \(g(u)\) denote the optimal value of \(O_2\) as a function of \(u\). Since,

\[
E(X - u)I(X > u) = \int_u^\infty P(X > y)dy,
\]

\(g(u)\) is a strictly non-increasing function of \(u\) when \(X\) is unbounded. Let the threshold function \(r(\cdot)\) denote the inverse of \(g(\cdot)\). Thus, if wish to restrict the bias to \(\epsilon\), let the threshold function \(r(\epsilon)\) denote the value of \(u\) in \(O_2\) for which the optimal value \(g(u)\) equals \(\epsilon\).

3.2 Analysis for unbounded random variables

We now return to the problem of finding \(P(\epsilon, \delta)\) algorithms that require a deterministic and known \(O(\log(1/\delta))\) computational effort for any instance of underlying populations, when upper bounds on the moments of strictly increasing non-negative convex functions of the underlying random variables are explicitly known. Specifically, let \(Y\) denote a collection probability distributions corresponding to non-negative random variables \((X(i) : i \leq d)\) such that there exists strictly convex, non-decreasing functions \((f_i(\cdot) : i \leq d)\) and positive constants \((c_i : i \leq d)\) so that

\[
Ef_i(X(i)) \leq c_i
\]

for all \(i\). Further, let \(r_i(\cdot)\) denote the threshold functions associated with each \(f_i(\cdot)\). Set

\[
R(x) = \max_{i \leq d} r_i(x).
\]

Consider the following algorithm for selecting the population with the highest mean. Let \(\beta \in (0,1)\) and \(\epsilon > 0\).

- Generate independent samples \((X(i,j) : i = 1,\ldots,d\text{ and } j = 1,\ldots,n_{\beta})\). Let \(Y(i,j) = \min(X(i,j), R(\beta \epsilon))\) for all \(i, j\).
• Let \( \bar{Y}(i) = \frac{1}{n_\beta} \sum_{j=1}^{n_\beta} Y(i, j) \). Declare

\[
\hat{i} = \arg \max_{1 \leq i \leq d} \bar{Y}(i)
\]

as the best design.

**Proposition 3** The above algorithm is \( \mathcal{P}(\epsilon, \delta) \) for distributions in \( \mathcal{Y} \) with

\[
n_\beta = \left\lceil \frac{2R(\beta \epsilon)^2}{\epsilon^2(1 - \beta)^2 \log((d - 1)/\delta)} \right\rceil
\]

providing the desired guarantees.

**Proof of Proposition 3:** The probability of false selection corresponds to

\[
P(\bar{Y}(i^*) < \max_{j \neq i^*} \bar{Y}(j)). \tag{23}
\]

Repeating the analysis in Section 3.0.1 keeping in mind that for \( j \neq i^* \),

\[
EY(i^*) - EY(j) = E \min(X(i^*), R(\beta \epsilon)) - E \min(X(j), R(\beta \epsilon)) \\
\geq EX(i^*) - \beta \epsilon - EX(j) \\
\geq (1 - \beta) \epsilon,
\]

we conclude that (23) is bounded from above by \( \delta \). □

**Remark 3** The \( \beta \) that minimises \( n_\beta \) corresponds to that minimising \( \frac{R(\beta \epsilon)^2}{1 - \beta} \).

We solve this when \( f_i(x) = x^\alpha, \alpha > 1 \) for all \( i \). Then, it can be seen from Remark 2 that

\[
r_i(x) = \left( \frac{c_i}{x} \right)^{1/(\alpha-1)} \left( \frac{\alpha - 1}{\alpha^{\alpha/(\alpha-1)}} \right)
\]

and we can set

\[
R(x) = \frac{1}{x^{1/(\alpha-1)}} \left( \frac{\alpha - 1}{\alpha^{\alpha/(\alpha-1)}} \right) \max_{i \leq d} c_i^{1/(\alpha-1)}.
\]

Then minimising \( n_\beta \) corresponds to maximising

\[
\beta^{2/(\alpha-1)}(1 - \beta)^2,
\]

and is achieved at \( \beta = 1/\alpha \). Thus the bound on the number of samples needed equals \( n_{1/\alpha} \) or

\[
\frac{2R(\beta \epsilon)^2}{\epsilon^2(1 - \beta)^2 \log((d - 1)/\delta)} = 2 \left( \frac{\max_{i \leq d} c_i}{\epsilon^\alpha} \right)^{2/(\alpha-1)} \log((d - 1)/\delta).
\]
4 Sequential pure exploration algorithm

We now review some of the related literature that comes under the broad topic of stochastic multi-arm bandit methodology. We discuss the elegant sequential sampling strategy referred to as the successive elimination algorithm proposed by Even-Dar et al. (2002, 2006). Although they also propose a slightly more effective median elimination algorithm in that paper, and there have been significant developments on the pure exploration problem since then (see, e.g., Audibert and Bubeck 2010, Jamieson et al. 2013, Garivier and Kaufmann 2016), the sequential algorithm of Even-Dar et al. (2006) is particularly simple and lends to an easier laconic discussion. They considered the setting where the underlying random variables were Bernoulli, while we allow generally distributed random variables when explicit bounds on the moments are available. As mentioned in the introduction, we use one of the proposed methods from Bubeck et al. (2013) for this purpose that relies on careful truncation of underlying random variables. Their analysis focuses on the regret minimisation objective but is easily adapted to our pure exploration setting. We present a minor tweak - while they considered truncations of the form $XI(X < u)$ to bound $rv X$, we note the minor performance benefits from using instead the capped random variable $\min(X, u)$. These may also extend to the regret minimisation objective. In addition, we compute bounds on the expected number of samples generated under the pure exploration algorithm for general random variables.

4.1 Sequential algorithm

We refer to populations as arms in this section. Thus, we have $d$ arms. Arm $i$ when pulled gives a reward distributed as $X(i)$ and our aim is to find the best arm $i^* = \max_{i \leq d} EX(i)$. Let the maximum be achieved by a unique arm and let $\Delta_i = EX(i^*) - EX(i) > 0$ for all $i \neq i^*$.

Suppose that there exists a non-negative function $\alpha_m$ with the property that for $i \leq d$,

$$P(|\bar{X}_m(i) - EX(i)| > \alpha_m) \leq \frac{c\delta}{m^2d}$$

(24)

where $\bar{X}_m(i)$ denotes the average of $m$ i.i.d. samples of $X(i)$, and

$$c = \left(\sum_{m=1}^{\infty} 1/m^2\right)^{-1} = 6/\pi^2.$$

In the examples that we consider later $\alpha_m$ is seen to be decreasing for all $m \geq e$. In our discussions, we will ignore this minor issue and assume that $\alpha_m$ is a decreasing, non-negative function of $m$.

Equation (24) ensures that

$$P(E_\delta) \geq 1 - \delta,$$

where

$$E_\delta = \{|\bar{X}_m(i) - EX(i)| \leq \alpha_m, \forall m, \forall i \leq d\},$$
and is the rationale for the successive elimination algorithm outlined below.

**Successive elimination algorithm**

1. Set $m = 1$ and $S = \{1, 2, \ldots, d\}$.
2. Set for each arm $i$, $\bar{X}_1(i) = 0$;
3. Repeat
   - Sample every arm $i \in S$ once and let $\bar{X}_m(i)$ be the average reward of arm $i$ by trials or pulls $m$;
   - Let $\bar{X}_m(\text{max}) = \max_{i \in S} \bar{X}_m(i)$;
   - For each arm $i \in S$ such that $\bar{X}_m(\text{max}) - \bar{X}_m(i) \geq 2\alpha_m$ do
     - set $S = S - \{i\}$;
   - end
   - $m = m + 1$;

   Until $|S| > 1$;

It is easy to see that on the set $E_\delta$ the best arm is never eliminated and that all other arms are eventually eliminated (see Even-Dar et. al. 2002, 2006). Also, It can be easily checked using Hoeffding’s inequality that $\alpha_m = b\sqrt{\frac{2m \log \left( \frac{dm^2}{c^3} \right)}{m}}$ works for random variables $X(i) - EX(i) \in [-b, +b]$, $i \leq d$ in (24).

### 4.1.1 Explicit moment bounds on random variables

The key step above was simply to arrive at the set $E_{i,\delta}$

$$E_{i,\delta} = \{|\bar{X}_m(i) - EX(i)| < \alpha_m \text{ for all } m\},$$

that has probability at least $1 - \delta/d$.

Bubeck et. al. (2013) achieve this when each $X(i)$ has an explicit bound on its moment. They focus on the on the more interesting case where the bound is on $\alpha$ moment for $\alpha \in (1, 2]$. We also restrict our discussion to this case. Suppose that $K$ upper bounds an $\alpha$ absolute moment of a random variable for $\alpha > 2$. Then, $K^{\alpha/2}$ upper bounds its second moment. Similarly, an upper bound on an exponential moment of a random variable can be reduced to an upper bound on the second moment.

A key to analysis of Bubeck et. al. (2013) is Lemma 2 below that relies on using truncation and Bernstein inequality (shown with the proof of Lemma 2). In Lemma 2 we also state the result when the random variables are capped instead of truncated. For this case, we assume that the underlying random variables are non-negative. For $\alpha > 1$, let $\tilde{\alpha} = \left(\frac{\alpha - 1}{\alpha}\right)^{\alpha - 1}$. 

18
Lemma 2 Let $\delta \in (0, 1)$, $\alpha \in (1, 2]$, $K > 0$, and $(X_i : i \leq n)$ be iid samples of rv $X$. Suppose that $E|X|^\alpha \leq K$ and let

$$B_m = \left( \frac{Km}{\log(\delta^{-1})} \right)^{\alpha^{-1}}.$$

Then, with probability at least $1 - \delta$,

$$\bar{X}_n \geq EX - (2 + \alpha)K^{1/\alpha} \left( \frac{\log(\delta^{-1})}{n} \right)^{\frac{\alpha-1}{\alpha}},$$

where

$$\bar{X}_n = \frac{1}{n} \sum_{m=1}^{n} X_m I(|X_m| \leq B_m)$$

denotes the empirical mean of truncated samples.

When $X$ is also assumed to be non-negative, then again with probability at least $1 - \delta$,

$$\tilde{X}_n \geq EX - (2 + \tilde{\alpha})K^{1/\alpha} \left( \frac{\log(\delta^{-1})}{n} \right)^{\frac{\alpha-1}{\alpha}},$$

where

$$\tilde{X}_n = \frac{1}{n} \sum_{m=1}^{n} \min(X_m, B_m)$$

denotes the capped empirical mean.

The reverse of (25)

$$\bar{X}_n \leq EX + (2 + \alpha)K^{1+(1/\alpha)} \left( \frac{\log(\delta^{-1})}{n} \right)^{\frac{\alpha-1}{\alpha}}.$$

also holds with probability at least $1 - \delta$ following essentially identical proof. Similarly, the reverse of (26).

4.2 Expected number of samples generated

Lemma below is useful to our analysis.

Lemma 3 Suppose that $a \geq e$, $b \geq 1$ and $t = t^* \geq 1$ solves

$$a + b \log t = t.$$

Then,

$$t^* \leq a + b \log a + \frac{2b^2}{a} \log(a + b).$$

To compute $ET(i)$ the expected number of times arm $i \neq i^*$ is pulled, note that

$$ET(i) = \sum_{m=1}^{\infty} P(\bar{X}_m(\max) - \bar{X}_m(i) < 2\alpha_m) \leq \sum_{m=1}^{\infty} P(\bar{X}_m(i^*) - \bar{X}_m(i) < 2\alpha_m).$$
For $i \neq i^*$, let

$$\tau_i^* = \inf \{m : 4\alpha_m \leq \Delta_i\}$$

(recall that $\Delta_i = EX(i^*) - EX(i)$). Then,

$$ET(i) \leq \tau_i^* + \sum_{m=\tau_i^*+1}^{\infty} P(\bar{X}_m(i^*) - EX(i^*) - (\bar{X}_m(i) - EX(i)) < -2\alpha_m),$$

$$\leq \tau_i^* + \sum_{m=\tau_i^*+1}^{\infty} \left( P(\bar{X}_m(i^*) \leq EX(i^*) - \alpha_m) + P(\bar{X}_m(i) \geq EX(i) + \alpha_m) \right)$$

$$\leq \tau_i^* + 2\delta/d.$$

Thus, the total expected number of samples generated for all arms $i \neq i^*$ is bounded from above by

$$\sum_{i \neq i^*} \tau_i + 2\delta$$

and the total number of samples generated is bounded from above by twice this amount.

### 4.2.1 Bounded random variables

In particular, when $X(i) - EX(i) \in [-b, +b]$, $\alpha_m = b \sqrt{\frac{2m \log \left( \frac{dm^2}{c\delta} \right)}{m}}$ works. Let $m^*$ be the solution to

$$4b \sqrt{\frac{2 \log \left( \frac{dm^2}{c\delta} \right)}{m}} = \Delta_i.$$

Then, $\tau_i^* \leq m^* + 1$.

Hence, using Lemma 3,

$$\tau_i^* \leq \frac{32b^2}{\Delta_i^2} \log \left( \frac{d}{c\delta} \right) + \frac{64b^2}{\Delta_i^2} \log \left( \frac{32b^2}{\Delta_i^2} \log \left( \frac{d}{c\delta} \right) \right) + 1$$

plus terms that become small as $\delta$ decreases to zero.

Recall that the total number of samples generated is bounded from above by

$$2 \sum_{i \neq i^*} \tau_i^* + 4\delta.$$

Hence the dominant terms in the upper bound for total number of expected samples for small $\delta$ are

$$64b^2 \log \left( \frac{d}{c\delta} \right) \sum_{i \neq i^*} \frac{1}{\Delta_i^2}.$$
4.2.2 Explicit bound on moments

When $E|X(i)|^\alpha \leq K$ for all $i \leq d$, it is easily seen that

$$\alpha_m = (2 + \tilde{\alpha})K^{1/\alpha} \left(\log \left(\frac{2m^2d}{c\delta}\right) \right)^{(\alpha-1)/\alpha}$$

satisfies (21).

Again, the number of times all the arms $i \neq i^*$ are pulled is bounded from above by

$$\sum_{i \neq i^*} \tau_i^* + 2\delta$$

where now $\tau_i^*$ is bounded from above by $m^* + 1$ and $m^*$ solves the equation

$$4(2 + \tilde{\alpha})K^{1/\alpha} \left(\log \left(\frac{2m^2d}{c\delta}\right) \right)^{(\alpha-1)/\alpha} = \Delta_i.$$

From Lemma 3 it follows that

$$m^* \leq a + b \log a + \frac{2b^2}{a} \log(a + b)$$

where

$$a = \left(\frac{4(2 + \tilde{\alpha})K^{1/\alpha}}{\Delta_i}\right)^{\frac{\alpha}{\alpha-1}} \log \left(\frac{2d}{c\delta}\right)$$

and

$$b = 2 \left(\frac{4(2 + \tilde{\alpha})K^{1/\alpha}}{\Delta_i}\right)^{\frac{\alpha}{\alpha-1}}.$$

Hence, the dominant terms in the upper bound for total number of expected samples for small $\delta$ are

$$2 \left(4(2 + \tilde{\alpha})K^{1/\alpha}\right)^{\frac{\alpha}{\alpha-1}} \log \left(\frac{2d}{c\delta}\right) \sum_{i \neq i^*} \left(\frac{1}{\Delta_i}\right)^{\frac{\alpha}{\alpha-1}}.$$

5 Conclusions

We considered the well studied classical problem of identifying the best population amongst many when the populations are ranked by the expectations of associated random variables. These expectations may not be known, but samples of the underlying random variables can be generated, e.g., via simulation. In the existing literature, when the underlying random variables are light-tailed, the probability of false selection is known to decay at an exponential rate. The question that we addressed in the paper was whether this convergence rate could be exploited to build fast algorithms with desired probabilistic guarantees. Specifically, can algorithms that require $O(\log(1/\delta))$
computational effort be constructed that restrict the probability of false selection to a small \( \delta \). The existing literature had suggested that by empirically estimating large deviations rate function of underlying random variables, one may be able to develop such algorithms. We answered this question in negative when the underlying random variables belong to a class of distributions whose support on the right is unbounded and that are right KL-dense.

The key conclusion from our analysis is that further restrictions are needed to develop algorithms that require \( O(\log(1/\delta)) \) computational effort and provide explicit or asymptotic guarantees of restricting the probability of false selection to a small \( \delta \). We showed that explicit bounds on expectations of convex functions of underlying random variables suffices. We also showed that sequential methods developed in multi-armed bandit literature for regret minimization can be adapted to our pure exploration settings once moment bounds on underlying random variables are available.

6 Appendix: Proofs

Proof of Lemma 1 Consider a large \( b \) whose value will be fixed later. Furthermore, take \( \gamma \in (0, 1) \). Construct a probability distribution \( G_k \) as follows: Set

\[
G_k(x) = (1 - \gamma) G(x)
\]

for all \( x \leq b \), and,

\[
\bar{G}_k(x) = \beta \bar{G}(x)
\]

for \( x > b \), where, \( \bar{G}_k(x) = 1 - G_k(x) \) and \( \bar{G}(x) = 1 - G(x) \).

Note that

\[
\beta = 1 + \gamma \frac{G(b)}{G(b)} > 1.
\]

Then,

\[
0 \leq \int_{x \in \mathbb{R}} \log \left( \frac{dG(x)}{dG_k(x)} \right) dG(x) \leq -G(b) \log(1 - \gamma).
\]

By selecting \( \gamma = 1 - \exp(-\alpha) \), we get

\[
-G(b) \log(1 - \gamma) \leq \alpha.
\]

Also, for \( b \) such that \( G(b^+) = G(b^-) \),

\[
\mu_{G_k} = (1 - \gamma) \int_{-\infty}^{b} x dG(x) + \left(1 + \gamma \frac{G(b)}{G(b)} \right) \int_{b}^{\infty} x dG(x) \geq \exp(-\alpha) \mu_G + (1 - \exp(-\alpha)) G(b) b
\]

Since, RHS increases to infinity as \( b \to \infty \), one can select \( b \) sufficiently large so that \( \mu_{G_k} \geq k \). \( \square \)

Proof of Proposition 1 The case \( u \leq f^{-1}(c) \), follows by noting that

\[
f(EX) \leq Ef(X) \leq c,
\]

22
so that $EXI(X \geq u) \leq EX \leq f^{-1}(c)$.

Now consider $u > f^{-1}(c)$. Suppose that $\tilde{X}_1$ takes values $x_1^* \in [0, f^{-1}(c))$ and $x_2^* \geq u$. ($x_2^* < u$ is not possible as then the optimal objective function value is zero.)

The objective function (19) at optimal value then equals

$$x_2^* \left( \frac{c - f(x_1^*)}{f(x_2^*) - f(x_1^*)} \right).$$

Note that

$$\frac{c - f(x)}{f(x_2^*) - f(x)}$$

for $x < f^{-1}(c)$ and $f(x_2^*) > c$, is a strictly decreasing function of $x$. Thus, $x_1^* = 0$.

Now observing that

$$f(y) - f(0) < yf'(y)$$

from simple calculus, it follows that

$$y \left( \frac{c - f(0)}{f(y) - f(0)} \right)$$

is non-increasing in $y$ for $y \geq u$ and the result follows. □

**Proof of Proposition 2** Consider a solution that puts mass at two points $x_2^*$ and $x_1^* \in [0, x_2^*)$ with probabilities $p$ and $1 - p$, respectively. Clearly, then $x_1^* < f^{-1}(c)$ and $x_2^* \geq f^{-1}(c)$.

First suppose that an optimal solution has $x_1^* \geq u$ with positive probability. We show that this leads to a contradiction. Note that $x_1^* \geq u$ only if $u < f^{-1}(c)$. In that case, the objective function equals

$$\left( \frac{x_2^* - x_1^*}{f(x_2^*) - f(x_1^*)} \right) \left( \frac{c - f(x_1^*)}{f(x_2^*) - f(x_1^*)} + (x_1^* - u) \right) \tag{28}$$

Consider the function

$$\left( \frac{x - x_1^*}{f(x) - f(x_1^*)} \right) \left( \frac{c - f(x_1^*)}{f(x_2^*) - f(x_1^*)} \right).$$

This is clearly non-increasing in $x$. Thus $x_1^*$ in (28) equals $f^{-1}(c)$ with probability 1 providing the desired contradiction.

Now suppose that $x_1^* < u$. Then, to achieve a positive value of the objective function, we need $x_2^* > u$. In particular, the optimal objective function equals

$$\left( \frac{x_2^* - u}{f(x_2^*) - f(x_1^*)} \right) \left( \frac{c - f(x_1^*)}{f(x_2^*) - f(x_1^*)} \right) \tag{29}$$

Note that

$$\frac{c - f(x)}{f(x_2^*) - f(x)}$$

for $x < \min(x_2^*, f^{-1}(c))$ and $f(x_2^*) \geq c$, is a strictly decreasing function of $x$. Thus, $x_1^* = 0$ in (29).

Now consider the function

$$\left( \frac{x - u}{f(x) - f(0)} \right) \left( \frac{c - f(0)}{f(x_2^*) - f(0)} \right)$$

23
for $x > u$. Due to Assumption 1, this is maximised at $x_u$. Thus, if $x_u > f^{-1}(c)$, then clearly $x_2^* = x_u$ in (29). Else, $x_2^* = f^{-1}(c)$ with probability 1. The proposition stands proved. □

The Bernstein inequality below is useful for the proof of Lemma 2.

**Lemma 4 (Bernstein Inequality)** Suppose that $(Z_i : i \geq 1)$ are mean zero, independent rv and there exists $M > 0$:

$$|Z_i| \leq M.$$  

Then,

$$P\left(\frac{1}{n} \sum_{i=1}^{n} Z_i > t\right) \leq \exp\left(-\frac{n^2t^2/2}{\sum_{i=1}^{n} EZ_i^2 + Mnt/3}\right).$$

**Proof of Lemma 2**

Proof for (25) below is adapted from Bubeck et. al. 2013.

We would like to show that

$$P\left(EX - \frac{1}{n} \sum_{m=1}^{n} X_m I(|X_m| \leq B_m) \geq (2 + \alpha)K^{\alpha-1}\left(\frac{\log(n)}{n}\right)^{(\alpha-1)\alpha^{-1}}\right) \leq \delta. \quad (30)$$

The LHS under probability in (30) equals

$$\frac{1}{n} \sum_{m=1}^{n} (EX - E(XI(|X| \leq B_m)) + \frac{1}{n} \sum_{m=1}^{n} (E(XI(|X| \leq B_m)) - X_m I(|X_m| \leq B_m))$$

The first term above is bounded from above by

$$\frac{1}{n} \sum_{m=1}^{n} K \frac{\alpha^{-1}\log(n)}{\log(n)} \frac{1}{m^{(\alpha-1)\alpha^{-1}}} \leq \alpha K^{\alpha-1}\left(\frac{\log(n)}{n}\right)^{(\alpha-1)\alpha^{-1}}.$$ 

Hence, the probability in (30) is bounded from above by

$$P\left(\frac{1}{n} \sum_{m=1}^{n} (E(XI(|X| \leq B_m)) - X_m I(|X_m| \leq B_m)) \geq 2K^{\alpha-1}\left(\frac{\log(n)}{n}\right)^{(\alpha-1)\alpha^{-1}}\right)$$

This probability can be bounded from above by $\delta$ using Bernstein inequality above with, $Z_m = E(XI(|X| \leq B_m)) - X_m I(|X_m| \leq B_m)$, $M = B_n$ and $EZ_m^2 \leq KB_n^{2-\alpha}$, and

$$t = 2K^{\alpha-1}\left(\frac{\log(n)}{n}\right)^{(\alpha-1)\alpha^{-1}}.$$ 

Equation (25) follows.

To see (26), in the above analysis, replace $X_m I(|X_m| \leq B_m)$ and $X I(|X| \leq B_m)$ by $\min(X_m, B_m)$ and $\min(X, B_m)$, respectively. From (22), the bias term

$$\frac{1}{n} \sum_{m=1}^{n} (EX - E(\min(X, B_m))) \leq \left(\frac{\alpha - 1}{\alpha}\right)^{\alpha^{-1}} \frac{1}{n} \sum_{m=1}^{n} \frac{K}{B_m^{\alpha-1}} \leq \left(\frac{\alpha - 1}{\alpha}\right)^{\alpha^{-1}} K^{\alpha-1}\left(\frac{\log(n)}{n}\right)^{(\alpha-1)\alpha^{-1}}.$$
The result follows by essentially the same analysis as for the truncated rv case. □

**Proof of Lemma 3:** First note that $t^* \leq (a + b)^2$ for if not, then

$$(a + b) \log t^* > (a + b)^2$$

so that $\log t^* > (a + b)$. Since, $(a + b) \geq \frac{t^*}{\log t^*}$, it follows that $\log^2 t^* > t^*$ providing the desired contradiction (since $\log^2 t < t$ for $t \geq 1$).

Now

$$t^* = a + b \log t^* = a + b \log(a + b \log t^*) \leq a + b \log(a + 2b \log(a + b)).$$

and the result easily follows. □.

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25
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