ON TIGHT PROJECTIVE DESIGNS

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Abstract. It is shown that among all tight designs in \( FP^n \neq RP^1 \), where \( F \) is \( \mathbb{R} \) or \( \mathbb{C} \), or \( H \) (quaternions), only 5-designs in \( CP^1 \) \cite{14} have irrational angle set. This is the only case of equal ranks of the first and the last irreducible idempotent in the corresponding Bose-Mesner algebra.

Keywords: projective design, angle set, Bose-Mesner algebra.

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1. Introduction

A well known theorem of Bannai and Hoggar \cite{3} states that there are no tight \( t \)-designs in \( FP^n \neq RP^1 \) if \( t \geq 6 \). Moreover, a theorem of Hoggar \cite{10} states the same for \( t \geq 4 \) if \( F \neq \mathbb{R} \). Surprisingly, a tight 5-design in \( CP^1 \) has been constructed in \cite{14}, so Hoggar’s theorem has to be corrected. The results of \cite{3} and \cite{10} are essentially based on Theorem 2.6(c) \cite{9} that states that the angle set of every tight \( t \)-design in \( FP^n \neq RP^1 \) is rational. But it is not rational for the 5-design constructed in \cite{14}.

In the present paper we investigate this contradiction and prove that the only cases where the angle set is not rational are

1. \( F = \mathbb{C}, n = 1, t = 5 \) and
2. \( F = \mathbb{R}, n = 1, t \neq 1, 2, 3, 5 \).

A fortiori, there are no complications in \cite{3} where \( t \geq 6 \) by assumption.

Our principal observation is that if \( t = 2s - 1, s \geq 2 \) then the last irreducible idempotent \( L_s \) in the corresponding Bose-Mesner algebra is not \( E_s \) from the proof of Theorem 2.6(c) \cite{9} (actually, from \cite{18}). Nevertheless, \( rk L_s \neq rk E_1 \), except for our case (1). This “critical inequality” implies the rationality of the angle set, similarly to the argument in \cite{9}. This material is concentrated in Section 4 of the present paper, while Sections 2 and 3 contain all the necessary background and preliminary analysis.

2. Projective \( t \)-designs

For the reader’s convenience we basically use the same notation as in \cite{8} and other related papers. Let us recall this notation. In particular, let

\[
F \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \}; \quad m = \frac{1}{2}(F : \mathbb{R}) = \begin{cases} 
1/2 & F = \mathbb{R} \\
1 & F = \mathbb{C} \\
2 & F = \mathbb{H}
\end{cases} \quad N = m(n+1).
\]

The number \( 2N \) is nothing but the real (topological) dimension of the \( F \)-linear space \( \mathbb{P}^{n+1} \). The latter consists of all \((n+1) \times 1\) matrices (columns) over \( F \) with the standard addition and multiplication by scalars \( \tau \in F \) from the right (for
definiteness). As usual, the inner product of \( a, b \in \mathbb{F}^{n+1} \) is \( a^* b \) where \( a^* \) is the row conjugate transpose to \( a \). Accordingly, the set

\[
S^{2N-1} = \{ a : a^* a = 1 \}
\]

is the unit sphere in \( \mathbb{F}^{n+1} \). A quotient set of the sphere with respect to the equivalence relation \( a_1 \sim a_2 \iff a_1 = a_2 \lambda, \; \lambda \in \mathbb{F}, \; |\lambda| = 1 \), is the projective space \( \mathbb{F}P^n \). The “inner product” \( (\hat{a}, \hat{b}) = |a^* b|^2 \) in \( \mathbb{F}P^n \) is well-defined through the natural mapping \( a \mapsto \hat{a} \) from \( S^{2N-1} \) onto \( \mathbb{F}P^n \). Obviously, \( (\hat{b}, \hat{a}) = (\hat{a}, \hat{b}) \) and \( 0 \leq (\hat{a}, \hat{b}) \leq 1 \) with the equality \( (\hat{a}, \hat{b}) = 1 \) if and only if \( \hat{a} = \hat{b} \). For every nonempty \( X \subset \mathbb{F}P^n \) its angle set is

\[
A(X) = \{ (x, y) : x, y \in X, x \neq y \}
\]

The related combinatorial parameters are

\[
s = |A(X)|, \quad e = |A(X) \setminus \{0\}|, \quad \epsilon = s - e = |A(X) \cap \{0\}|.
\]

Let \( P_i^{(\alpha, \beta)}(\tau) \) be the Jacobi polynomials \[20\] such that

\[
\text{deg} P_i^{(\alpha, \beta)} = i, \quad P_i^{(\alpha, \beta)}(1) = \frac{(\alpha + 1)_i}{i!}
\]

where

\[
(\alpha + 1)_i = \prod_{l=1}^{i}(\alpha + l), \quad (\alpha + 1)_0 = 1.
\]

In particular, \( P_0^{(\alpha, \beta)}(\tau) \equiv 1 \). In what follows we fix

\[
\alpha = N - m - 1, \quad \beta = m - 1
\]

and set

\[
P_i(\xi) = P_i^{(\alpha, \beta)}(2\xi - 1),
\]

for short. A finite nonempty subset \( X \subset \mathbb{F}P^n \) is called a \( t \)-design if

\[
\sum_{x \in X} P_i((x, y)) = 0, \quad y \in X, \quad 1 \leq i \leq t.
\]

Let \( X \) be a \( t \)-design and let

\[
R_\epsilon^c(\xi) = \binom{N}{s}\frac{P^{(\alpha+1, \beta+\epsilon)}(\xi)}{(m)_\epsilon!}(2\xi - 1).
\]

In particular,

\[
R_\epsilon^c(1) = \binom{N}{s}(N - m + 1)(2m - 1)_\epsilon
\]

The following theorems are fundamental, see [1], [2], [11]. (Cf. [6] for the spherical designs.)

**Theorem A.** The inequalities

\[
t \leq s + e, \quad |X| \geq R_\epsilon^c(1)
\]

hold, and the equalities

\[
t = s + e, \quad |X| = R_\epsilon^c(1)
\]

are equivalent.

In the latter case the \( t \)-design \( X \) is called tight. Note that \( t = s + e \) is equivalent to \( e = \lfloor t/2 \rfloor \), \( \epsilon = \text{res}_2(t) \).
Theorem B. If \( X \) is a tight \( t \)-design then \( A(X) \) coincides with the set of roots of the polynomial \( \xi^t R_n(\xi) \).

Recall that these roots are simple and lie on \((0,1)\).

Theorem C. Let \( X \) be a subset of \( \mathbb{F}^n \) such that \( |X| = R_n(1) \) and \( A(X) \) coincides with the set of roots of \( \xi^t R_n(\xi) \), then \( X \) is a tight \((2c+\epsilon)\)-design.

The projective \( t \)-designs can be characterized as the averaging sets in the sense of \[19\] for suitable spaces of functions on \( \mathbb{F}^n \). Usually, these spaces are described in terms of harmonic analysis but we prefer a more elementary approach \[15, 16\].

We say that a mapping \( \phi : \mathbb{F}^n \to \mathbb{C} \) is a polynomial function if it is of the form

\[
\phi(a) = \psi(a), \quad a \in S^{2N-1},
\]

where \( \psi \) is a polynomial on \( \mathbb{F}^{n+1} \) in real coordinates. This \( \psi \) must be invariant with respect to the rotations of \( \mathbb{F} \), i.e. \( \psi(a\lambda) = \psi(a) \) for all \( \lambda \in F, |\lambda| = 1 \). It is not unique but becomes unique if it is required to be homogeneous (which is always possible) of minimal degree. The latter is said to be the degree of \( \phi \). The number \( \deg \phi \) is an even integer since \( \psi(-a) = \psi(a) \).

Example 2.1. For every \( t \in \mathbb{N} \) and every \( y \in \mathbb{F}^n \) the function \( \phi_{2t,y}(x) = (x,y)^t \), \( x \in \mathbb{F}^n \), is a polynomial function of degree \( 2t \).

Given \( d \in 2\mathbb{N} \), we denote by \( \text{Pol}_d \) the space of all polynomial functions of degrees \( \leq d \). It has been proven in \[16\] that the family \( \{ \phi_{d,y} : y \in \mathbb{F}^n \} \) spans the whole space \( \text{Pol}_d \). We apply this result to prove the following

**Proposition 2.2.** A finite nonempty set \( X \subset \mathbb{F}^n \) is a tight \( t \)-design if and only if

\[
\frac{1}{|X|} \sum_{x \in X} \phi(x) = \int_{S^{2N-1}} \hat{\phi}(a) \, d\sigma(a), \quad \phi \in \text{Pol}_2(t),
\]

where \( \hat{\phi} \) is induced by the natural mapping \( S^{2N-1} \to \mathbb{F}^n \) and \( \sigma \) is the normalized Lebesgue measure.

**Proof.** The identity (2.6) is equivalent to

\[
\frac{1}{|X|} \sum_{x \in X} F((x,y)) = \int_{S^{2N-1}} F(|a^* b|^2) \, d\sigma(a),
\]

where \( y = \hat{b}, b \in S^{2N-1} \), \( F \) runs over the space \( \Pi_t \) of all univariate polynomials of degrees \( \leq t \). By a known integration formula (see \[8\], Theorem 2.11) one can rewrite (2.7) in the form

\[
\frac{1}{|X|} \sum_{x \in X} F((x,y)) = \int_{-1}^1 F \left( \frac{1 + \tau}{2} \right) \Omega_{\alpha,\beta}(\tau) \, d\tau, \quad \psi \in \Pi_t,
\]

where \( \Omega_{\alpha,\beta}(\tau) \) is the normalized Jacobi weight, i.e.

\[
\Omega_{\alpha,\beta}(\tau) = c_{\alpha,\beta}(1-\tau)^{\alpha}(1+\tau)^{\beta}, \quad -1 < \tau < 1,
\]

with

\[
c_{\alpha,\beta} = \left( \int_{-1}^1 (1-\tau)^{\alpha}(1+\tau)^{\beta} \, d\tau \right)^{-1} = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}.
\]
In turn, (2.8) is equivalent to its restriction to \( F = P_i(\xi), 1 \leq i \leq t \), since these polynomials constitute a basis in \( \Pi_t \). It remains to note that

\[
\int_{-1}^{1} P_i \left( \frac{1 + \tau}{2} \right) \Omega_{\alpha,\beta}(\tau) \, d\tau = \int_{-1}^{1} P_i(\alpha,\beta)(\tau) \Omega_{\alpha,\beta}(\tau) \, d\tau = 0
\]

by (2.3) and the \( \Omega_{\alpha,\beta} \)-orthogonality of the system \( \{P_i(\alpha,\beta) : i \geq 0\} \).

\[ \square \]

Corollary 2.3. Let \( X \subset \mathbb{F}^n \) be a \( t \)-design. Then

\[
\frac{1}{|X|} \sum_{x \in X} P((u, x))Q((x, v)) = \int_{S^2N-1} P(|a^*c|^2)Q(|c^*b|^2) \, d\sigma(c) \quad (2.11)
\]

for \( u = \hat{a}, v = \hat{b} \) and all univariate polynomials \( P, Q \) such that \( \deg P + \deg Q \leq t \).

Proof. The mapping \( x \mapsto P((u, x))Q((x, v)), x \in \mathbb{F}^n \), is a polynomial function of degree \( \leq 2t \). \[ \square \]

Corollary 2.4. Let \( X, P, Q \) be fixed under the conditions of Corollary 2.3. Then the value

\[
\sum_{x \in X} P((u, x))Q((x, v))
\]

depends only on the inner product \( (u, v) \) of \( u, v \in \mathbb{F}^n \).

Proof. Let \( (u_1, v_1) = (u, v) \), i.e. \( |a^*_1b_1|^2 = |a^*b|^2 \) where \( \hat{a}_1 = u, \hat{b}_1 = v \). Without loss of generality one can assume that \( a^*_1b_1 = a^*b \). Then there exists a \( (n+1) \times (n+1) \) matrix \( T \) over \( \mathbb{F} \) such that \( T^*T = \text{id} \) and \( a_1 = Ta^*_1 b_1 = Tb^* \). This substitution in (2.11) is equivalent to the change of variable \( c \mapsto T^*c \). The latter does not affect the integral since the measure \( \sigma \) is orthogonally invariant. \[ \square \]

3. Bose-Mesner algebra

Let \( X \) be a finite nonempty subset of \( \mathbb{F}^n \) and let

\[
A'(X) = A(X) \cup \{1\} = \{(x, y) : x, y \in X\},
\]

so that \( |A'(X)| = s + 1 \). The \( X \times X \) matrices of the form

\[
M_F = [F((x, y))]_{x,y \in X} \quad (3.1)
\]

where \( F \) runs over all functions \( A'(X) \to \mathbb{C} \), constitute a complex linear space \( \mathcal{D}(X) \). Its natural basis consists of the matrices

\[
\Delta_\zeta = [\delta_{\zeta, (x,y)}]_{x,y \in X}, \quad \zeta \in A'(X),
\]

thus, \( \dim \mathcal{D}(X) = s + 1 \). The Lagrange interpolation formula allows us to let \( F \) in (3.1) run over the polynomial space \( \Pi_s \), so that we have the isomorphism \( F \mapsto M_F \) between \( \Pi_s \) and \( \mathcal{D}(X) \). In particular, if \( F|A(X) = 0 \) and \( F(1) = 1 \) then \( M_F = I \), the unit matrix.

According to Corollary 2.3, for \( P, Q \in \Pi_s \), the matrix product \( M_P M_Q \) belongs to \( \mathcal{D}(X) \) if \( \deg P + \deg Q \leq t \). However, this condition is not fulfilled if \( t = 2s - 1 \) and \( \deg P = \deg Q = s \). Moreover, Corollary 2.4 cannot be extended to this situation if \( X \) is tight. Indeed, suppose to the contrary that

\[
\sum_{x \in X} (u, x)^s(x, v)^s = \Phi((u, v)) \quad (u, v \in \mathbb{F}^n)
\]
with a function $\Phi : [0, 1] \to \mathbb{R}_+$. Setting $v = u$ we obtain
$$\sum_{x \in X} (u, x)^{2s} = \Phi(1), \quad u \in \mathbb{F}_q^n.$$  
In other words,
$$\sum_{c \in \hat{X}} |a^*_c|^{4s} = \Phi(1), \quad a \in S^{2N-1},$$
where $\hat{X} \subset S^{2N-1}$ is a complete system of representatives of points $x \in X$, $|\hat{X}| = |X|$. By integration over $a$ we obtain
$$\Phi(1) = \left( \int_{S^{2N-1}} |a^*_c|^{4s} d\sigma(a) \right) \cdot |X|$$
since the integral does not depend on $c$. As a result,
$$\frac{1}{|X|} \sum_{x \in X} \phi_{4s;\mu}(x) = \int_{S^{2N-1}} \hat{\phi}_{4s;\mu}(a) d\sigma(a), \quad (3.3)$$
and by linearity, $\Phi$ extends to the whole space $\text{Pol}_q(4s)$. Thus, $X$ is a 2s-design which is a contradiction since $2s + 1$.

Nevertheless, under the constraint $u, v \in X$, one can extend Corollary 2.3 to $t = 2s - 1$ and $P, Q$ such that $\max(\deg P, \deg Q) = s$. This follows from the construction of a basis in $\mathcal{D}(X)$ using the Jacobi polynomials (cf. [6], Remark 7.6).

**Lemma 3.1.** Let $X$ be a $t$-design in $\mathbb{F}_q^n$ and let $s = \left[ \frac{t+1}{2} \right]$. Then $s + 1$ matrices $M_i = M_{P_i}$, $0 \leq i \leq s$, constitute a basis $\mathcal{M}$ of $\mathcal{D}(X)$ such that
$$M_i M_k = |X| M_{\delta_{ik} \rho_{\mu(i,k)}} \quad (3.4)$$
where $\mu(i, k) = \min(i, k)$ and all $\rho_j > 0$.

**Proof.** The matrices $M_i$ are linearly independent because of the linear independence of the polynomials $P_i$. Since $|\mathcal{M}| = s + 1$, this is a basis of $\mathcal{D}(X)$. Now note that
$$\int_{S^{2N-1}} P_i(|a^*_c|^2) P_k(|c^* b|^2) \, d\sigma(c) = 0, \quad i \neq k,$$
by the addition formula for polynomial functions [15] (cf. [7], [13], [17]). The same formula with $i = k$ yields
$$\int_{S^{2N-1}} P_i(|a^*_c|^2) P_i(|c^* b|^2) \, d\sigma(c) = \chi_i P_i(|a^*_c|^2) \quad (3.5)$$
where $\chi_i > 0$. Assuming $\mu(i, k) \leq s - 1$ (a fortiori, $i + k \leq 2s - 1 \leq t$) and using Corollary 2.3 we get (3.4) with
$$\rho_j = \chi_j, \quad 0 \leq j \leq s - 1. \quad (3.6)$$
In particular, $M_i M_s = M_s M_i = 0$ for $0 \leq i \leq s - 1$. It remains to consider the case $i = k = s$.

If $t$ is even the $t = 2s$ and Corollary 2.3 is applicable to $i = k = s$, so $M^2 = |X| M_s \rho_s$ with
$$\rho_s = \chi_s. \quad (3.7)$$
Let $t$ be odd, so $t = 2s - 1$. Then we decompose the unity matrix $I$ for the basis $\mathcal{M}$,
$$I = \sum_{i=0}^s \lambda_i M_i, \quad (3.8)$$
and get \( M_s = \lambda_s M_s^2 \) multiplying Eq. (3.8) by \( M_s \). This yields

\[
\lambda_s = \frac{\text{tr} M_s}{\text{tr} M_s^2} = \frac{|X| P_s(1)}{\sum_{x,y} P_s^2((x,y))} > 0,
\]

and then \( M_s^2 = |X|M_s \rho_s \) with

\[
\rho_s = (\lambda_s |X|)^{-1}.
\]

(3.9)

(3.10)

\[\square\]

Remark 3.2. The formulas (3.6) and (3.7) are joined in

\[
\rho_i = \chi_i, \quad 0 \leq i \leq \lceil t/2 \rceil,
\]

(3.11)

while (3.10) appears only for \( t = 2s - 1 \) in addition to (3.11).

Remark 3.3. The multiplication table (3.4) shows that under conditions of Lemma 3.1 \( \mathcal{D}(X) \) is a commutative matrix algebra, the Bose-Mesner algebra of \( X \) [4], [5], [6].

In what follows the conditions of Lemma 3.1 are assumed to be fulfilled. By setting

\[
L_i = \frac{M_i}{\rho_i |X|} = \frac{1}{\rho_i |X|} [P_i((x,y))]_{x,y \in X}
\]

(3.12)

the basis \( \mathcal{M} \) turns into \( \mathcal{L} = \{L_i\}_{i=0}^s \) consisting of idempotents \( (L_i^2 = L_i) \) which are pairwise orthogonal \( (L_i L_k = 0 \text{ for } i \neq k) \). It is important to calculate their ranks.

We have

\[
\text{rk} L_i = \text{tr} L_i = \rho_i^{-1} P_i(1), \quad 0 \leq i \leq s,
\]

hence,

\[
\text{rk} L_i = \chi_i^{-1} P_i(1) = \frac{P_i^2(1)}{\int_{-1}^1 P_i^2(\alpha^2) \, d\sigma(\alpha)}, \quad 0 \leq i \leq \lceil t/2 \rceil
\]

by (3.11) and (3.5) for \( a = b \). Finally,

\[
\text{rk} L_i = \frac{\left( P_i^{(\alpha,\beta)}(1) \right)^2}{\int_{-1}^1 \left( P_i^{(\alpha,\beta)}(\tau) \right)^2 \Omega^{(\alpha,\beta)}(\tau) \, d\tau}, \quad 0 \leq i \leq \lceil t/2 \rceil.
\]

(3.13)

In particular, \( \text{rk} L_0 = 1 \). In addition to (3.13) we have to find \( \text{rk} L_s \) in the case \( t = 2s - 1 \). Formula (3.10) is not effective to this end since \( \lambda_s \) is unknown. Indeed, in (3.9) we cannot proceed to the formally corresponding integral in the denominator. Instead of this, we return to the decomposition of unity and express \( \text{rk} L_s \) through \( \text{rk} L_i, \ 0 \leq i \leq s - 1 \). We have

\[
I = \sum_{i=0}^s L_i,
\]

whence,

\[
\text{rk} L_s = \text{tr} L_s = |X| - \sum_{i=0}^{s-1} \text{tr} L_i = |X| - \sum_{i=0}^{s-1} \text{rk} L_i.
\]

By substitution from (3.13) the last sum can be written as \( c_{\alpha,\beta}^{-1} K_s^{(\alpha,\beta)}(1,1) \), where \( K_s^{(\alpha,\beta)}(\cdot,\cdot) \) is the reproducing kernel of the Jacobi polynomials with respect to the
weight \((1 - \tau)\alpha(1 + \tau)^\beta\), see [20], Section 4.5. According to (2.10) and formula (4.5.8) from [20] we obtain
\[
\text{rk} \, L_s = |X| - \frac{\Gamma(s + \alpha + \beta + 1)\Gamma(s + \alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)\Gamma(\alpha + 2)\Gamma(s)}
\]
With our \(\alpha, \beta\) defined by (2.2)
\[
\text{rk} \, L_s = |X| - \frac{\Gamma(N + s - 1)\Gamma(N - m + s)\Gamma(m)}{\Gamma(N)\Gamma(N - m + 1)\Gamma(m + s - 1)\Gamma(s)} = |X| - \frac{(N - 1)(N - m + 1)s - 1}{(m)_s(s - 1)!} \frac{1}{(N - 1)(N - m + 1)s - 1}
\]
(3.14)

**Lemma 3.4.** Let \(X\) be a tight \(t\)-design in \(\mathbb{F}^n\) with \(t = 2s - 1\). Then
\[
\text{rk} \, L_s = \frac{(N)_{s-1}(N - m)_{s}}{(m)_s(s - 1)!} \quad (3.15)
\]
**Proof.** In this case \(e = s - 1\), \(\epsilon = 1\), so (2.5) yields
\[
|X| = R_{s-1}(1) = \frac{(N)_{s}(N - m + 1)_{s - 1}}{(m)_s(s - 1)!}
\]
\[
\square
\]

The ranks of the other \(L_i\) (including \(L_s\)) can be explicitly calculated by (3.13), (2.10) and (2.1) combined with (4.33) of [20]. This results in

**Lemma 3.5.** Let \(X\) be a \(t\)-design in \(\mathbb{F}^n\) with \(s = \left\lfloor \frac{t-1}{2} \right\rfloor\). Then
\[
\text{rk} \, L_i = \frac{(N)_{i-1}(N - m)_{i}(N + 2i - 1)}{(m)_i!}, \quad 0 \leq i \leq [t/2]. \quad (3.16)
\]

**Remark 3.6.** Formula (3.10) yields the true value \(\text{rk} \, L_0 = 1\) by setting \((\gamma - 1)(\gamma - 1) = 1\) for all \(\gamma\).

**Corollary 3.7.** The inequality
\[
\text{rk} \, L_i > \text{rk} \, L_{i-1}, \quad 1 \leq i \leq [t/2], \quad (3.17)
\]
holds, except for \(X \subseteq \mathbb{F}^1\). In the latter case
\[
\text{rk} \, L_i = 2, \quad 1 \leq i \leq [t/2]. \quad (3.18)
\]
Now note that our idempotents \(L_i\) coincide with the matrices \(E_i\) from [9] for \(0 \leq i \leq [t/2]\) but \(L_s \neq E_s\) if \(t = 2s - 1\), \(X \not\subseteq \mathbb{F}^1\). Indeed, according to (2.5) from [9],
\[
E_i((x, y)) = \frac{1}{|X|}[Q_i((x, y))]_{x,y \in X}, \quad 0 \leq i \leq s, \quad (3.19)
\]
where \(Q_i(\xi)\) is proportional to \(P_i(\xi)\) and
\[
Q_i(1) = \frac{(N)_{i-1}(N - m)_{i}(N + 2i - 1)}{(m)_i!}, \quad i \geq 0. \quad (3.20)
\]
Hence, \(E_i\) are proportional to \(L_i\) for all \(i\), \(0 \leq i \leq s\). Moreover, if \(0 \leq i \leq [t/2]\) then \(\text{tr} \, E_i = Q_i(1) = \text{tr} \, L_i\) by (3.20) and (3.10). Hence, \(E_i = L_i\) for \(0 \leq i \leq [t/2]\). However, if \(t = 2s - 1\) (so \(s = [t/2] + 1\)) and \(X \not\subseteq \mathbb{F}^1\) then \(\text{tr} \, E_s = Q_s(1) > \text{tr} \, L_s\), see (3.15). In this case \(\text{tr} \, E_s > \text{rk} \, E_s\), so \(E_s\) is not an idempotent. This is an obstacle to the full proof of Theorem 2.6 [9] of the rationality of \(A(X)\). To overcome this difficulty, it suffices to change \(E_s\) for \(L_s\) (when \(t = 2s - 1\), \(s \geq 2\)) but then the “critical inequality” \(\text{rk} \, L_s \neq \text{rk} \, L_1\) is needed. However, the latter is not always true.

We clarify this intricate situation in the next section.
4. THE CRITICAL INEQUALITY AND RATIONALITY THEOREM

We prove the following

**Theorem 4.1.** With $t = 2s - 1$, $s \geq 2$, the inequality

\[
\text{rk} L_s \neq \text{rk} L_1
\]

holds for every tight $t$-design $X \subset F\mathbb{P}^n$, except for a tight 5-design in $\mathbb{CP}^1$.

**Proof.** From (3.15) it follows that

\[
\text{rk} L_s \geq \frac{(N)_1(N - m)_2}{(m)_2 \cdot 1!} = \frac{N(N - m)(N - m + 1)}{m(m + 1)}
\]

since the right side of (3.15) increases with $s$. On the other hand, (3.16) yields

\[
\text{rk} L_1 = \frac{(N)_0(N - m)_1(N + 1)}{(m)_1 \cdot 1!} = \frac{(N - m)(N + 1)}{m}
\]

Hence,

\[
\text{rk} L_s - \text{rk} L_1 \geq \frac{(N - m)((N - m)^2 - (m^2 + m + 1))}{m(m + 1)}
\]

Since $N - m = mn$ we obtain $\text{rk} L_s > \text{rk} L_1$ for $n^2 > 1 + m^{-1} + m^{-2}$, i.e. for $n \geq 3$ if $F = \mathbb{R}$ and for $n \geq 2$ if $F = \mathbb{C}$ or $\mathbb{H}$. It remains to consider two cases.

1. $F = \mathbb{R}$, $n = 2$. Then $m = 1/2$, $N = 3/2$, so $\text{rk} L_1 = 5$ by (4.2), while $\text{rk} L_s = 2$ by (3.15).

2. $F$ is arbitrary, $n = 1$. Then $N = 2m$, hence

\[
\text{rk} L_s - \text{rk} L_1 = \frac{(2m)_{s-1}}{(s - 1)!} - (2m + 1) = \begin{cases} -1 & F = \mathbb{R} \\ s - 3 & F = \mathbb{C} \\ \frac{1}{6} s(s + 1)(s + 2) - 5 & F = \mathbb{H} \end{cases}
\]

We see that $\text{rk} L_s \neq \text{rk} L_1$ with the only exception $F = \mathbb{C}$, $s = 3$, so $t = 5$. □

A tight 5-design in $\mathbb{CP}^1$ should contain 12 points since its parameters are $m = 1$, $N = 2$, $s = 3$, $e = 2$, $\epsilon = 1$. Accordingly, (2.5) becomes

\[
R^1_1(1) = \frac{(2)_{13}(2)_2}{(1)_3 \cdot 2!} = 12.
\]

Such a design has been constructed in [14] as the projective image of an orbit of the binary icosahedral group that is a subgroup of $SU(2)$. Its representatives on the unit sphere $S^3 \subset \mathbb{C}^2$ are

\[
a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad a_k = \begin{pmatrix} \mu \eta^{k-3} \\ \mu \eta^{k-3} \\ -\lambda \end{pmatrix}, \quad 3 \leq k \leq 7,
\]

\[
a_k = \begin{pmatrix} \mu \eta^{k-3} \\ \mu \eta^{k-3} \\ -\lambda \end{pmatrix}, \quad 8 \leq k \leq 12,
\]

where

\[
\eta = \exp \frac{2\pi i}{5}, \quad \lambda = 2 \cos \frac{2\pi}{5} = \frac{\sqrt{5} - 1}{2}, \quad \mu = \frac{1}{2 \sin \frac{3\pi}{5}} = \sqrt{\frac{5 + \sqrt{5}}{10}}.
\]
We omit an elementary calculation of the inner products \( (x_j, x_k) = |a_j^* a_k|^2 \), only noting that
\[
\lambda^2 + \lambda - 1 = 0, \quad (2 - \lambda)\mu^2 = 1, \quad |\eta^r - 1|^2 = 0, 2 - \lambda, 3 + \lambda,
\]
the latter for \( r \equiv 0, \pm 1, \pm 2 \mod 5 \), accordingly. A calculation yields,
\[
A(X) = \left\{ 0, \frac{5 - \sqrt{5}}{10}, \frac{5 + \sqrt{5}}{10} \right\}, \tag{4.3}
\]
so \( s = 3, e = 2, \epsilon = 1 \). The polynomial
\[
\xi R_2^1(\xi) = 4\xi P^{(1,1)}(2\xi - 1) = 6\xi(5\xi^2 - 5\xi + 1)
\]
annihilates \( A(X) \). By Theorem C of Section 2 our \( X \) is indeed a tight 5-design.

The angle set \( \{3, 3\} \) is not rational. This occurs because of the equality \( \text{rk} L_3 = \text{rk} L_1 \). In fact, \( \text{rk} L_0 = 1, \text{rk} L_1 = 3, \text{rk} L_2 = 5, \text{rk} L_3 = 3 \), by (3.15) and (3.16).

In the following corrected form of Theorem 2.6 \[9\] the case \( X \subseteq L \) is also included for completeness. In this case, (3.15) makes it possible for \( A(X) \) to be not rational, though \( \text{rk} L_s \neq \text{rk} L_1 \) under the conditions of Theorem 4.1.

**Theorem 4.2.** Let \( X \) be a tight \( t \)-design in \( \mathbb{F}P^n \). Then the angle set \( A(X) \) is rational, except for two cases: 1) \( X \subseteq \mathbb{C}P^1 \), \( t = 5 \); 2) \( X \subseteq \mathbb{R}P^1 \), \( t \neq 1, 2, 3, 5 \).

**Proof.** For \( X \not\subseteq \mathbb{R}P^1 \) the proof is the same as in [9] but with \( L_s \) instead of \( E_s \) when \( t = 2s - 1, s \geq 2 \), and using our Theorem 4.1 in this case.

Now let \( X \subseteq \mathbb{R}P^1 \). Then it is the projective image of a regular \((2t + 2)\)-gon as easily follows from [12]. Therefore,
\[
A(X) \setminus \{0\} = \left\{ \cos^2 \frac{\theta}{t + 1} \right\}^e_1 = \left\{ \frac{1}{2} \left( 1 + \cos \frac{2\pi e}{t + 1} \right) \right\}^e_1
\]
where \( e = [t/2] \). Since \( \cos n\theta \) is a polynomial of \( \cos \theta \) with integer coefficients, the set \( A(X) \) is rational if and only if the number \( \rho = \cos \frac{2\pi e}{t + 1} \) is rational. Obviously, the latter is true if \( t = 1, 2, 3, 5 \). Conversely, let \( \rho \in \mathbb{Q} \). Then the complex number \( w = \exp(2\pi i/(t + 1)) \) satisfies the equation \( w^2 - 2\rho w + 1 = 0 \). On the other hand, this is a primitive root of 1 of degree \( t + 1 \). It is known that irreducible (over \( \mathbb{Q} \)) equation for \( w \) is of degree \( \varphi(t + 1) \) where \( \varphi \) is the Euler function. Hence, \( \varphi(t + 1) \leq 2 \). If \( \varphi(t + 1) = 1 \) then \( t = 1 \). If \( \varphi(t + 1) = 2 \) then \( t \in \{2, 3, 5\} \) as easily follows from the classical formula
\[
\varphi \left( \prod_{i=1}^r q_i^{\nu_i} \right) = \prod_{i=1}^r q_i^{\nu_i - 1}(q_i - 1)
\]
where \( q_1, \ldots, q_r \) are prime divisors of \( t + 1 \). \( \square \)

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