Designing arrays of Josephson junctions for specific static responses

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Received 12 March 2007, in final form 11 February 2008
Published 10 March 2008
Online at stacks.iop.org/IP/24/025022

Abstract

We consider the inverse problem of designing an array of superconducting Josephson junctions that has a given maximum static current pattern as a function of the applied magnetic field. Such devices are used for magnetometry and as terahertz oscillators. The model is a 2D semilinear elliptic operator with Neuman boundary conditions so the direct problem is difficult to solve because of the nonlinearity and the multiplicity of solutions. For an array of small junctions in a passive region, the model can be reduced to a 1D linear partial differential equation with Dirac distribution sine nonlinearities. For small junctions and a symmetric device, the maximum current is the absolute value of a cosine Fourier series whose coefficients (respectively wave numbers) are proportional to the areas (respectively the positions) of the junctions. The inverse problem is solved by inverse cosine Fourier transform after choosing the area of the central junction. We present several examples and show that the reconstruction is robust and that its accuracy can be controlled. These new devices could then be tailored to meet specific applications.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The coupling of two Type I superconductors across a thin oxide layer is described by the two Josephson equations [1]:

\[ V = \Phi_0 \frac{d\phi}{dt}, \quad I = s J_c \sin(\phi), \]  

where \( \phi \) is the phase difference between the two superconductors in units of \( \Phi_0 = \hbar/2e \), the reduced flux quantum, \( V \) and \( I \) are, respectively, the voltage and current across the layer, \( s \) is the contact surface and \( J_c \) is the critical current density. The Josephson equations and Maxwell’s equations imply the modulation of dc current by an external magnetic field in the static regime (SQUIDs) and the conversion of ac current in microwave radiation [2, 3]. In all these systems there is a characteristic length which reduces to the Josephson length \( \lambda_J \), the
ratio of the electromagnetic flux to the quantum flux $\Phi_0$ for standard junctions. The behavior of a Josephson junction depends on its size compared to $\lambda_J$. In small junctions, the phase will not vary much except for large magnetic fields. Long junctions in contrast enable large variations of the phase accommodating the so-called fluxons or sine-Gordon kinks where the phase varies by $2\pi$ [2].

For many applications and in order to protect the junction, Josephson junctions are embedded in a so-called microstrip line which is the capacitor made by the overlap of the two superconducting layers. This is the ‘window geometry’ where the phase difference satisfies an inhomogeneous 2D damped driven sine-Gordon equation [4] resulting from Maxwell’s equations and the Josephson constitutive relations (1). For resonator applications, this design allows us to couple the junctions in an array to increase the output power and adapt impedance for coupling the device to a transmission line. In addition, one can select some desirable dynamic features like resonances [8] and optimize the frequency response over a given band for wave mixing applications [5].

Parallel arrays of Josephson junctions can be used in the static regime as very fine magnetic field detectors. The maximal current $I_{\text{max}}$ which can cross the device (see figure 1) for a given magnetic field $H$, without any voltage ($V = 0$ the static regime), defines the $I_{\text{max}}(H)$ curve which is important for experimentalists. Despite extensive studies of arrays of identical and equidistant small Josephson junctions [2, 3], the problem of finding $I_{\text{max}}$ remains difficult to solve because of the multiplicity of solutions due to the sine nonlinearity and the Neuman boundary conditions.

For fundamental reasons and applications, it is interesting to work with non-uniform arrays where the junction sizes and their spacings can vary. In [8, 9] we developed a continuous/discrete or long wave model where the phase variation is neglected in the junctions and where the couplings between the junction and surrounding microstrip are correctly taken into account. In particular, we consider the waves between the junctions that are completely neglected by the classical resistive shunted junction (RSJ) lumped models [3]. Our approach allows us to choose the distance between junctions and their area. In the same device we can model junctions with different areas and different current responses, in particular $\pi$-junctions. This simple model allows us to analyze in depth the statics of the device, and this is not possible from the 2D original equations [9]. This long wave approximation can be generalized to 2D to explain the behavior of squids [13]. In addition, we obtain an excellent agreement with the complex experimental $I_{\text{max}}(H)$ curves [10] using the very simple magnetic limit introduced in [9].

Experimentalists can extract some parameters of the array from the $I_{\text{max}}(H)$ curve. For example, it gives information on the quality of the junctions. Recent studies by Itzler and Tinkham examine how defects in the coupling affect this maximum current [14, 15]. This is important because high $T_c$ superconductors can be described as Josephson junctions where the critical current density is a rapidly varying function of the position, due to grain boundaries. Fehrenbacher et al [16] calculated $I_{\text{max}}(H)$ for such disordered long Josephson junctions and for a periodic array of defects. The expressions obtained are complicated, so the inverse problem of determining the junction parameters from the $I_{\text{max}}(H)$ curve is very difficult to solve for arrays or general current densities. However, when the jump of the phase gradient at the junction can be neglected (the magnetic limit) the maximum current $I_{\text{max}}(H)$ has a simple expression. This allows us to extract the sizes and positions of the junctions in an array assuming $I_{\text{max}}(H)$ is a periodic and even function. This solves the longstanding inverse problem and is the purpose of this paper. In particular, we show how one can obtain a cosine profile and multi-cosine profile from a combination of simple three junction arrays. We indicate what parameters can be obtained from a general $I_{\text{max}}(H)$ profile. After presenting the
I − Δφ = 0

Figure 1. The left panel shows the top view of a superconducting microstrip line containing three Josephson junctions. \( H, I \) and \( φ \) are respectively the applied magnetic field, current and the phase difference between the two superconducting layers. The phase difference \( φ \) between the two superconducting layers satisfies 

\[-Δφ = 0 \] in the linear part and 

\[-Δφ + \sin(φ) = 0 \] in the Josephson junctions. The right panel shows the associated 2D domain of size \( l \times w \) containing \( n = 3 \) junctions placed at the positions \( y = w/2 \) and \( x = a_i, i = 1, n \).

general model in sections 2 and 3, we introduce the magnetic limit and give its properties in section 4. Section 5 discusses the direct problem for the magnetic limit. We introduce a simple solution of the inverse problem for a symmetric junction array in section 6 and conclude in section 7.

2. The model

The device we model (see figure 1) is a so-called microstrip cavity (gray area in figure 1) between two superconducting layers containing small regions (junctions) where the oxide layer is very thin (\( \sim 10 \) Angstrom) enabling Josephson coupling between the top and bottom superconductors. The microstrip is about 100 \( \mu \)m long and 20 \( \mu \)m wide and the length and width of the junctions is about \( w_j = 1 \mu \)m. In the static regime, the phase difference \( ϕ \) between the top and bottom superconducting layers obeys the following semilinear elliptic partial differential equation [4]:

\[-Δφ + g(x; y) \sin ϕ = 0, \]  \( (2) \)

where \( g(x; y) = 1 \) in the Josephson junctions and 0 outside and where we have neglected the difference in surface inductance between the junction and passive region. This formulation guarantees the continuity of the normal gradient of \( ϕ \), the electrical current on the junction interface. The space unit is the Josephson length \( λ_J \), the ratio of the flux formed with the critical current density and the surface inductance to the flux quantum \( Φ_0 \).

The boundary conditions representing an external current input \( I \) or an applied magnetic field \( H \) (along the y-axis) are

\[
\frac{∂φ}{∂y} \bigg|_{y=0} = -\frac{I}{2I}ν, \quad \frac{∂φ}{∂y} \bigg|_{y=w} = \frac{I}{2I}ν, \\
\frac{∂φ}{∂x} \bigg|_{x=0} = H - \frac{I}{2w}(1 - ν), \quad \frac{∂φ}{∂x} \bigg|_{x=l} = H + \frac{I}{2w}(1 - ν),
\]

where \( 0 \leq ν \leq 1 \) gives the type of current feed. The case \( ν = 1 \) shown in figure 1 where the current is only applied to the long boundaries \( y = 0, w \) is called overlap feed, while \( ν = 0 \) corresponds to the inline feed.
We consider long and narrow strips containing a few small junctions of area $w^2_i$ placed on the line $y = w/2$ and centered on $x = a_i, i = 1, n$, as shown in figure 1. Then we search $\phi$ in the form

$$\phi(x; y) = \frac{vI}{2L} \left( y - \frac{\omega}{2} \right)^2 + \sum_{n=0}^{\infty} \phi_n(x) \cos \left( \frac{n \pi y}{w} \right),$$

(4)

where the first term takes care of the $y$ boundary condition. For narrow strips $w < \pi$, only the first transverse mode needs to be taken into account [6] because the curvature of $\phi$ due to current remains small. Inserting (4) into (2) and projecting on the zero mode, we obtain the following equation for $\phi_0$ where the index 0 has been dropped for simplicity:

$$-\phi'' + g \left( x, \frac{w}{2} \right) \frac{w_i}{w} \sin \phi = v \frac{\gamma}{l},$$

(5)

and where $\gamma = I/l$ and the boundary conditions $\phi'(0) = H - (1 - v)\gamma/2$ and $\phi'(l) = H + (1 - v)\gamma/2$. The factor $w_i/w$ is exactly the ‘rescaling’ of $\lambda J = 1$ into $\lambda_{\text{eff}} = \sqrt{w/w_i} > 1$ due to the presence of the lateral passive region [7].

As the area of the junction is reduced, the total Josephson current is reduced and tends to zero. To describe small junctions where the phase variation can be neglected but which can carry a significant current, we introduce the following function $g_h$:

$$g_h(x) = \frac{w_i}{2h}, \quad \text{for} \quad a_i - h < x < a_i + h, \quad \text{and} \quad g_h(x) = 0 \text{ elsewhere},$$

(6)

where $i = 1, \ldots, n$. In the limit $h \to 0$, we obtain our final delta function model [8]

$$-\phi'' + \sum_{i=1}^{n} d_i \delta(x - a_i) \sin \phi = v \frac{\gamma}{l},$$

(7)

where

$$d_i = \frac{w_i^2}{w},$$

(8)

and the boundary conditions are

$$\phi'(0) = H - (1 - v)\gamma/2, \quad \phi'(l) = H + (1 - v)\gamma/2.$$  

(9)

Despite its crude character the delta function approximation is a good model for arrays with short junctions as long as the magnetic field is small compared to $1/w_i$, where $w_i$ is the size of the junctions [10]. It allows simple calculations and in-depth analysis that are out of reach for the 2D full model. In particular in [9] we analyze how the maximum static current $\gamma_{\text{max}}(H)$ is affected by the geometry, length, type of current feed, position and area of the junctions. In addition when $d_i < 0$, the model can describe the so-called $\pi$-junctions. For these, the tunneling current is $\sin(\phi + \pi) = -\sin(\phi)$ instead of the usual sine term in the second Josephson equation (1). This new type of coupling occurs in some materials [11, 12] and it is hoped to be incorporated in the design of arrays. It is then natural to associate negative $d_i$ coefficients with $\pi$ junctions in the device.

3. General properties of the reduced model

For the model defined by equations (7) and (9), we can establish the following properties.

(1) Integrating twice (7) shows that the solution $\phi$ is continuous at the junctions $x = a_i, i = 1, \ldots n$.  

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(2) Almost everywhere (in the mathematical sense), \( -\phi''(x) = v'\gamma/l \), so that outside the junctions, \( \phi \) is a piece-wise second-degree polynomial,

\[
\phi(x) = -\frac{v'\gamma}{2l}x^2 + B_i x + C_i, \quad \forall \, x \in [a_i, a_{i+1}].
\]  

(10)

(3) At each junction \( (x = a_i) \), \( \phi' \) is not defined, but choosing \( \epsilon_1 > 0 \) and \( \epsilon_2 > 0 \), we note that

\[
\lim_{\epsilon_1 \to 0} \lim_{\epsilon_2 \to 0} \int_{a_i - \epsilon_1}^{a_i + \epsilon_2} \phi''(x) \, dx = \int_{a_i}^{a_{i+1}} \phi''(x) \, dx = [\phi'(x)]_{a_i}^{a_{i+1}}.
\]

Since the phase is continuous at the junction \( x = a_i \), \( \phi_i \equiv \phi(a_i) \), we get

\[
[\phi'(x)]_{a_i}^{a_{i+1}} = d_i \sin(\phi_i).
\]  

(11)

(4) Integrating (7) over the whole domain,

\[
[\phi']_0^n = \int_0^l \phi' \, dx = \sum_{i=1}^n d_i \sin(\phi_i) - v'\gamma,
\]

and taking into account the boundary conditions, we obtain

\[
\gamma = \sum_{i=1}^n d_i \sin(\phi_i),
\]  

(12)

which indicates the conservation of current.

Let us now consider the maximum current \( \gamma_{\text{max}}(H) \) for a given magnetic field.

3.1. The maximum static current \( \gamma_{\text{max}}(H) \): periodicity

An important property of \( \gamma_{\text{max}}(H) \) is that for a harmonic array of junctions, the curve is periodic. To show this, let us introduce

\[
l_j \equiv a_{j+1} - a_j,
\]

the distance between two consecutive junctions. Let \( l_{\text{min}} \) be the smallest distance \( l_j \). We define the array as harmonic if \( l_j \) is a multiple of \( l_{\text{min}} \) for all \( i \).

**Proposition 3.1** (Periodicity of \( \gamma_{\text{max}}(H) \)). For a harmonic array, the \( \gamma_{\text{max}}(H) \) curve is periodic with a period \( \frac{2\pi}{l_{\text{min}}} \).

**Proof.** Let \( \phi \) be a solution of (7) for a current \( \gamma \) and a magnetic field \( H \). We introduce \( f(x) = (2\pi/l_{\text{min}})(x-a_i) \) and \( \psi(x) = \phi(x) + f(x) \). So \( \psi \) verifies

\[
-\psi''(x) + \sum_{i=1}^n \delta(x-a_i) \sin(\psi - f) = v'j,
\]  

(13)

with \( \psi'(0) = H + 2\pi/l_i - (1 - v)\gamma'2 \) and \( \psi'(l) = H + 2\pi/l_i + (1 - v)\gamma'2 \). Since \( f(a_i) = 2k\pi, \forall \, i \in \{1, \ldots, n\} \), then \( \psi \) is a solution of (7) for \( H + H_p \equiv H + 2\pi/l_{\text{min}} \) and the same \( \gamma \), so \( \gamma_{\text{max}}(H + H_p) \geq \gamma_{\text{max}}(H) \).

Conversely, by subtracting \( f \) from a solution associated with \( H + H_p \) and a current \( \gamma \), we obtain a solution for \( H \) and the same current \( \gamma \) so \( \gamma_{\text{max}}(H + H_p) \leq \gamma_{\text{max}}(H) \). From the two inequalities, we get

\[
\gamma_{\text{max}}(H + H_p) = \gamma_{\text{max}}(H),
\]  

(14)

with \( H_p = 2\pi/l_{\text{min}} \). □
In the non-harmonic case, if the junctions are set such that \( l_j = p_j/q_j \), where \( p_j \) and \( q_j \) are integers, prime with each other, then \( \gamma_{\text{max}} \) is periodic with period \( H_p \) such that

\[
H_p = 2\pi \frac{\text{LCM}(q_1, \ldots, q_{n-1})}{\text{HCF}(p_1, \ldots, p_{n-1})},
\]

see figure 4, where \( \text{LCM} \) is the lowest common multiple and \( \text{HCF} \) is the highest common factor. To prove this, write \( f(x) = p(x - a_i) \) and again use the previous argument.

In the general case, we only have an approximate periodicity of \( \gamma_{\text{max}}(H) \) which can be estimated using (15). Also, real junctions have a finite size which causes \( \gamma_{\text{max}}(H) \to 0 \) when \( H \to +\infty \). Our model is thus valid as long as the dimensionless magnetic field \( H \) is not larger than \( 1/\omega_j \).

### 3.2. The solution as a piece-wise polynomial

We developed two numerical methods to solve the boundary value problem (7) and find the \( \gamma_{\text{max}}(H) \) curve for the device using the properties established above. These are based on a shooting method and a Newton iteration. We now describe the first method because it will be helpful in the analysis. For details on the Newton method, see [9].

In the following, we fix the current feed so that \( \nu \) has a constant value. Let \( \phi \) be a solution of (7) and \( \phi_1 = \phi(a_1) \). From remark (10), \( \phi \) is a polynomial by parts. We define \( P_{k+1}(x) \), the second-degree polynomial, such that

\[
P_{k+1}(x) = -\frac{\nu\gamma}{2l} (x^2 - a_1^2) + \left( H - \frac{1 - \nu}{2} \right) (x - a_1) + \phi_1.
\]

At the junctions, (11) tells us that \( \forall k \in \{1; \ldots; n\} \),

\[
P'_{k+1}(a_k) - P_k'(a_k) = dk \sin(P_k(a_k)).
\]

Considering that \( \phi'' = -\nu\gamma/l \) on each interval, the previous relation and the continuity of the phase at the junction, we can give a first expression for \( P_{k+1} \):

\[
P_{k+1}(x) = -\frac{\nu\gamma}{2l} (x - a_k)^2 + [P'_k(a_k) + dk \sin(P_k(a_k))](x - a_k) + P_k(a_k).
\]

So, \( \phi \) is entirely determined by \( \phi_1, \gamma \) and \( H \).

The solution given by the polynomials (16) and (18) needs to satisfy the boundary conditions. The first one,

\[
\phi'(0) = P'_1(0) = H - (1 - \nu)\gamma/2,
\]

is true by construction. The second one (for an \( n \) junction circuit) implies

\[
P'_{n+1}(l) = H + (1 - \nu)\gamma/2.
\]

If this relation holds, we have the solution for a given \( H \) and \( \gamma \). To obtain the maximal current we use equation (19) which defines a constraint between \( \phi_1 \) and \( \gamma \) for fixed \( H \). Since the solutions \( \phi \) are defined modulo \( 2\pi \) (see equation (7)), and because of equation (12) we assume \( \phi_1 \in [-\pi; \pi] \) and \( \gamma \in [0; \sum_i d_i] \). We then solve this problem with Maple©, by plotting the implicit function (the constraint)

\[
P'_{n+1}|_{x=l}(\phi_1, \gamma, \nu, H) - H - \frac{1 - \nu}{2} \gamma = 0,
\]

of the two variables \( \phi_1 \) and \( \gamma \) with \( H \) and \( \nu \) fixed. The program searches in an exhaustive way the biggest value of \( \gamma \) of this implicit curve. Incrementing \( H \), we obtain the whole \( \gamma_{\text{max}}(H) \) curve.
4. Small current densities: the magnetic limit

Since $[\phi']_i^{a_i} = d_i \sin(\phi_i)$, equation (11), and $\gamma \leq \sum_i d_i$, we note that for small $d_i$, $\phi_i \approx 0$ and $\phi$ tends to the linear function $\phi(x) = f(x) = Hx + c$. This is what we call the magnetic limit. We generalize here what was done in [18] for arrays of equidistant junctions.

With this simple approximation, it is very easy to obtain the maximum current. For that, we take the maximum of

$$\gamma = \sum_{i=1}^n d_i \sin(Ha_i + c),$$

with respect to $c$. The values of $c$ such that $\partial \gamma / \partial c = 0$ are

$$c_{\text{max}}(H) = \arctan\left(\frac{\sum_{i=1}^n d_i \cos(Ha_i)}{\sum_{i=1}^n d_i \sin(Ha_i)}\right),$$

and as we want a maximal (not only an extremal) current, we obtain

$$\gamma_{\text{max}}(H) = \left| \sum_{i=1}^n d_i \sin(Ha_i + c_{\text{max}}(H)) \right|,$$

Note that this result holds even when the argument of the arctan is infinite. This procedure can also be applied to a long Josephson junction with a weak critical current density $d_c(x)$. In this case, formulae (23) and (22) become

$$\gamma_{\text{max}}(H) = \left| \int_0^l dx d_c(x) \sin(Hx + c_{\text{max}}) \right|,$$

where $c_{\text{max}}$ is such that

$$\int_0^l dx d_c(x) \cos(Hx + c_{\text{max}}) = 0.$$

We now proceed to estimate the error between this simple linear solution $f(x)$ and the solution $\phi$ of the full problem (7). In the appendix, we compute these errors both for the functions and their derivatives

$$\Delta_1' = \phi'(a_i) - f'(a_i), \quad \Delta_1 = \phi(a_i) - f(a_i).$$

In particular, for $\Delta_1$ we get the recursion (A.13)

$$\Delta_{i+1} = -\frac{l}{2} \left( \sum_{i=1}^n \gamma_i \right) \left( v \frac{a_{i+1} + a_i}{l} + 1 - v \right) + l \sum_{k=1}^i \gamma_k + \Delta_i.$$

Having computed $\Delta_1$ and $\Delta_1'$, we can estimate the difference between the exact maximum critical current $\gamma_{\text{max}}(H)$ and the maximum critical current $\gamma_{\text{max}}^m(H)$ for the magnetic limit. The currents are given by

$$\gamma_{\text{max}}(H) = \sum_{i=1}^n d_i \sin(f(a_i)),$$

$$\gamma_{\text{max}}^m(H) = \sum_{i=1}^n d_i \sin(f(a_i) + \Delta_i),$$

where $\sum_{i=1}^n d_i \cos(f(a_i)) = 0$ because of the condition that the current is maximal.
Table 1. Parameters for device A.

| Parameter | In μm | In units of λ_J |
|-----------|-------|-----------------|
| l         | 200   | 65.72           |
| w         | 20    | 6.572           |
| a_1       | 2     | 0.6572          |
| a_2       | 22    | 7.23024         |
| a_3       | 64    | 21.0334         |
| a_4       | 76    | 24.9772         |
| a_5       | 82    | 26.9491         |
| di        | w_2^2/w | 1/20 | 0.01643 |

The difference can be written as
\[ \gamma_{\text{max}}(H) - \gamma_{\text{max}}^m(H) = \sum_{i=1}^{n} d_i (\sin(f(a_i) + \Delta_i) - \sin(f(a_i))) , \]

\[ = \sum_{i=1}^{n} d_i (\cos(f(a_i)) \sin(\Delta_i) + \sin(f(a_i)) \cos(\Delta_i) - \sin(f(a_i))) , \]

\[ = \sum_{i=1}^{n} d_i \left( \cos(f(a_i)) \sin(\Delta_i) + \sin(f(a_i)) \cos(\Delta_i) - 1 \right) \approx \sum_{i=1}^{n} d_i \left( \frac{a_i}{\Delta_1} \right) \approx \Delta_1^2/2 . \]

4.1. An example: the device A studied at the Observatory of Paris

This particular array was originally built as a signal mixer. It consists in five junctions of area 1 μm x 1 μm embedded in a stripline of dimensions 200 μm x 20 μm with inline current feed. The critical current density \( J_c = 5 \text{ kA cm}^{-2} \) and surface inductance \( L \) give the following:

\[ \lambda_J = \sqrt{\Phi_0 L J_c} \approx 3.024 \times 10^{-6} \text{m}. \]

All the geometric parameters of the device are given in table 1. This shows that \( d_i \) are very small. To show that indeed the magnetic approximation is valid for this real device, we calculated the maximum current for three arbitrary values of \( d_i \) by solving (7) and (9) for the inline and overlap current feeds respectively. This is shown in figure 2 for \( d_i = 50w_2^2/w, 20w_2^2/w \) and \( d_i = 10w_2^2/w \). One can see that even for a \( d_i \) ten times larger than the real one, the magnetic limit is a good approximation.

For the inline geometry \( \nu = 0 \), the recursion (A.13) becomes

\[ \Delta_{i+1} = -\frac{l_i}{2} \left( \sum_{i=1}^{n} \gamma_i \right) + \frac{l_i}{2} \sum_{k=1}^{i} \gamma_k + \Delta_i . \]

This recursion can be solved, yielding

\[ \Delta_{i+1} = -\frac{a_{i+1} - a_1}{2} \left( \sum_{i=1}^{n} \gamma_i \right) + \frac{1}{2} \left( \sum_{j=1}^{i} l_j \sum_{k=1}^{j} \gamma_k \right) + \Delta_1 . \]

Assuming \( \Delta_1 = 0 \), we need to compute four terms:

\[ \Delta_2 = -0.3286 \gamma_{\text{max}} + 0.6572 \gamma_1 , \]

\[ \Delta_3 = -3.286 \gamma_{\text{max}} + 0.6572 \gamma_1 + 6.572(\gamma_1 + \gamma_2) . \]
Figure 2. Plot of the maximum current in the magnetic limit $\gamma_{\text{max}}^m(H)$ and of the maximum current $\gamma_{\text{max}}(H)$ calculated by solving (7) and (9) for the inline and overlap current feeds respectively. The parameters are clockwise $d_i = 50w_i^2/w$, $d_i = 20w_i^2/w$ and $d_i = 10w_i^2/w$.

As an example, we consider the case $H = 0$ for which $\gamma_{\text{max}} = 5d_i = 0.08215$ so that $\gamma_i \approx d_i = 0.01643$. Then we get

$$
\Delta_4 = -10.1881 \gamma_{\text{max}} + 0.6572 \gamma_1 + 6.572(\gamma_1 + \gamma_2) + 13.80316 \sum_{i=1}^{3} \gamma_i,
$$

$$
\Delta_5 = -13.14595 \gamma_{\text{max}} + 0.6572 \gamma_1 + 6.572(\gamma_1 + \gamma_2) + 13.80316 \sum_{i=1}^{3} \gamma_i + 3.9438 \sum_{i=1}^{4} \gamma_i.
$$

As can be seen, the error is very small. It can be made smaller by choosing $\Delta_1$.

Over the whole range of magnetic field, the error becomes very small as the junction size is reduced $w_i \ll w$ so that $d_i \ll 1$. Figure 3 shows that as $d_i$ is reduced, the maximum current obtained in the magnetic limit is very close to the maximum current obtained by solving the full problem (7), (9) with inline or overlap current feed.
5. The direct problem for \( \gamma_{\text{max}}(H) \) in the magnetic limit

The magnetic limit and the maximum current obtained (23), (22) considerably simplify the estimation of \( \gamma_{\text{max}}(H) \). In particular for a given \( H, \gamma \) is single valued. This allows us to consider the inverse problem of determining the device from a given \( \gamma_{\text{max}}(H) \). To do this, let us examine the main properties of the direct problem (23), (22).

We start with a few definitions.

- **Junction unit.** We call a junction unit the set of distances between \( l_i = a_{i+1} - a_i \) between consecutive junctions. We denote it \( \{l_1; l_2; \ldots; l_{n-1}\} \).

We define the position of the junction unit as \( a_1 \), the position of the first junction.

- **\( l_b \):** For an \( n \)-junction device, \( l_b = a_n - a_1 \) is the length of the junction unit.

- **Symmetric unit.** We call a symmetric unit an \( n \)-junction circuit such that \( \forall i \in \{1; \ldots; n\}, \ l_b/2 - a_i = a_n - i - l_b/2 \), and \( d_i = d_{n-i} \).

We plot \( \gamma_{\text{max}}(H) \) and \( c_{\text{max}}(H) \) in figure 4, for a uniform device and for a non-uniform one. In the second case, we have chosen the position of the junction to have a long period (\( H_p = 12\pi \)). We note that the length \( l \) of the device does not appear in equation (23).

We have the following three propositions.

**Proposition 5.1** (Invariance by translation). The \( \gamma_{\text{max}}(H) \) curve obtained within the magnetic limit does not depend on the position of the junction unit.

**Proof.** Let us assume two devices with the same junction unit \( \{l_1; l_2; \ldots; l_n\} \) with the first junction placed respectively at \( x = a_1 \) and \( x = a_1 + c \). We note \( \gamma_{\text{max}}^1(H) \) (respectively \( \gamma_{\text{max}}^2(H) \)), the \( \gamma_{\text{max}}(H) \) curve of the first (respectively the second) device. In the same way we note \( c_{\text{max}}^1(H) \) (respectively \( c_{\text{max}}^2(H) \)), the \( c_{\text{max}}(H) \) function of the first (respectively the second) device:

\[
\gamma_{\text{max}}^2(H) = \left| \sum_{i=1}^{n} d_i \sin \left( H a_i + H c + c_{\text{max}}^2(H) \right) \right| ;
\]
we note \( c^1(H) = Hc + c^2_{\text{max}}(H) \). As we do not know if \( c^1(H) = c^1_{\text{max}}(H) \), \( c^1_{\text{max}}(H) \) being the best value (if it exists) to obtain the maximal \( \gamma \). Consequently, \( \gamma_{\text{max}}(H) \leq \gamma_{\text{max}}(H) \). On the other side, considering

\[
\gamma^1_{\text{max}}(H) = \left| \sum_{i=1}^{n} d_i \sin (H(a_i + c) + c_{\text{max}}(H)) - Hc \right|,
\]

noting \( c^2(H) = c^1_{\text{max}}(H) - Hc \) and using the previous argument, we obtain \( \gamma^2_{\text{max}}(H) \geq \gamma^1_{\text{max}}(H) \). From the two previous inequalities, we obtain \( \gamma^1_{\text{max}}(H) = \gamma^2_{\text{max}}(H) \).

Thus, the \( \gamma_{\text{max}}(H) \) curve for the magnetic limit depends only on the junction unit.

**Proposition 5.2** (Parity of the \( \gamma_{\text{max}} \) curve). For all devices, \( \gamma_{\text{max}}(H) = \gamma_{\text{max}}(-H) \).

**Proof.** Since \( \sin \) and \( \arctan \) are odd functions and \( \cos \) is an even function, then \( c_{\text{max}}(-H) = -c_{\text{max}}(H) \) (see (22)). Finally,

\[
\gamma_{\text{max}}(-H) = \left| \sum_{i=1}^{n} d_i \sin (-Ha_i - c_{\text{max}}(H)) \right| = \gamma_{\text{max}}(H).
\]

Note that \( c_{\text{max}} \) is an odd function and \( \gamma_{\text{max}} \) is an even function (see figure 4).

**Proposition 5.3** (Particular solution for a symmetric device). For all symmetric units such that \( a_n = -a_1 \), \( c_{\text{max}}(H) = \pm \pi/2 \).

**Proof.** To see this, relabel the junctions so that the central one corresponds to \( i = 0 \), the first on the left to \( i = -1 \), the first on the right to \( i = +1 \) and so on. Using the first proposition, we can shift the junction unit so that \( a_0 = 0 \). Then the total current is

\[
\gamma(H) = \sum_{i=-n/2}^{n/2} d_i \sin (Ha_i + c)
\]

\[
= d_0 \sin (c) + 2 \sum_{i=1}^{n/2} d_i \sin (Ha_i + c)
\]

\[
= \sin(c) \left( d_0 + 2 \sum_{i=1}^{n/2} d_i \cos (Ha_i) \right).
\]
Then \( c_{\text{max}} = \pi / 2 \) when \( d_0 + 2 \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} d_i \cos(Ha_i) \geq 0 \) and \( c_{\text{max}} = -\pi / 2 \) otherwise. Thus,

\[
\gamma_{\text{max}}(H) = \left| d_0 + 2 \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} d_i \cos(Ha_i) \right|. \tag{28}
\]

This last result is not obvious from figure 4. □

In these propositions, we establish that the \( \gamma_{\text{max}} \) curve for a symmetric junction unit can be calculated simply by centering the junction unit so that \( c_{\text{max}}(H) = \pm \pi / 2 \). More precisely, consider an \( n + 1 \) symmetric junction unit where \( n \) is even. We can always choose this by setting the \( d \) of the central junction \( (a_1 + a_n)/2 \) to 0. Then we shift the junction unit by \( a(n+1)/2 \) so that the central junction is placed at \( x = 0 \). We relabel the junctions by setting \( i' = i - (n+1)/2 \). Then the central one is for \( i = 0 \), the first one to the right is \( i = 1 \), the first one to the left is \( i = -1 \) and so on. Then the maximum current is given by (28) In the rest of the paper, we will consider the array to be symmetric.

5.1. A device such that \( \gamma_{\text{max}}(H) = \cos(H) \)

In figure 5 we present from left to right \( \gamma_{\text{max}}(H) \) for a SQUID (2 junctions), a uniform three-junction unit, a \( d_1 = 1 = d_3, d_2 = 2 \) (termed 1-2-1) junction unit that is discussed in [3] and a 1-3-1 junction unit. In all cases, the junctions are equidistant. The first two panels represent well-known devices.

Applying equation (28) to the following case \( d_2 = 2 \), we obtain

\[
\gamma_{\text{max}}(H) = |d_0 + d_1 \cos(Ha_1)| = 2 + 2 \cos(Ha_1). \tag{29}
\]

This is an exact cosine function shifted by a constant. With the last case \( d_2 = 3 \), we obtain \( \gamma_{\text{max}}(H) = 3 + 2 \cos(Ha_1) \).

Comparing all the panels, we understand the role of the central junction. We can have an exact representation of \( \gamma_{\text{max}} \) for this type of circuit, if we imagine \( \gamma_{\text{max}} \) as an absolute value of a simple cos function translated by the value \( d_0 \) (which is equal to zero if there is no junction).

Equation (28) shows that we can sum cosine functions, with a chosen amplitude and period. Remark that if \( d_0 = -2 \) (\( \pi \) junction), then \( \gamma_{\text{max}}(H) = 2 - 2 \cos(Ha_1) \).

5.2. A multi-cosine \( \gamma_{\text{max}}(H) \)

For arrays with more than two junctions, experimentalists can play on the set of distances \( l_i \) separating the junctions as well as on the strength \( d_i \) (proportional to the area) of each junction.
We now show the influence of each set of parameters starting from \( d_i \)'s. Figure 6 presents on the left panel \( \gamma_{\text{max}}(H) \) for a symmetric set of five equidistant junctions \( a_1 = 1, a_2 = 2 \). The dashed line corresponds to \( d_i = 1, i = -2 \cdot \cdots \cdot 2 \), giving a maximum current of 5. Here one sees the typical interference pattern between the main bumps. The small oscillations can be eliminated by choosing \( d_0 = 1.82025, d_1 = d_{-1} = 1.25 \) and \( d_2 = d_{-2} = 0.3425 \) as seen from the continuous line on the left panel of figure 6. This ‘pulse’ profile could be very useful for specific applications because of the large region where \( \gamma_{\text{max}}(H) = 0 \). The right panel of figure 6 presents what would be the device for this set of \( a_i \) and \( d_i \). We chose a critical current density of \( 10^4 \text{ A cm}^{-1} \) so that \( \lambda_J \approx 10 \mu \text{m} \). We chose a transverse width \( w = 14 \mu \text{m} \). Assuming the area of the smallest junction to be \( 1 \mu \text{m}^2 \), we get the scheme shown, where the central junction has an area \( 5.32 \mu \text{m}^2 \).

The second parameter that can be changed is the position \( a_i \) of each junction in the array. The spacing between the junctions can be changed. As an example in figure 7, we show in the left panel the so-called ‘triangle’ \( \gamma_{\text{max}}(H) \) obtained by setting \( d_0 = 2.4888, d_1 = 1.1234 \) and \( d_2 = 0.121 \) with \( a_1 = 1 \) and \( a_2 = 3 \). The dashed line presents \( \gamma_{\text{max}}(H) \) for equal strengths \( d_i = 1 \). Changing \( d_i \)'s allows us to eliminate the oscillations in the minima of \( \gamma_{\text{max}}(H) \) and obtain an almost linear behavior. The right panel shows the arrangement of the junctions in the microstrip; note how the spacing is non-uniform. We have chosen the same physical parameters as for figure 6. With this last example, we see the practical way to solve the inverse problem of finding the device from a given \( \gamma_{\text{max}}(H) \). First, note that the period \( H_p \) of \( \gamma_{\text{max}}(H) \) is

\[
H_p = \frac{2\pi}{a_1} = 2\pi.
\]

Then \( \gamma_{\text{max}} \) can be written as

\[
\gamma_{\text{max}}(H) = d_0 + 2d_1 \cos \frac{2\pi H}{H_p} + 2d_2 \cos \frac{4\pi H}{H_p} + 2d_3 \cos \frac{6\pi H}{H_p},
\]

(30)

where \( d_0 = 2.4888, d_1 = 1.1234, d_2 = 0 \) and \( d_3 = 0.121 \). So \( d_i \)'s are the cosine Fourier coefficients of \( \gamma_{\text{max}}(H) \) and the wave numbers are \( a_i = (2\pi i)/H_p \). Note that the absent \( j = 2 \) junction is described by \( d_j = 0 \).
6. The inverse problem for a given $\gamma_{\text{max}}(H) = \gamma_g(H)$

We now show how to design an $n+1$-junctions circuit ($n$ is an even integer) to obtain a given periodic $\gamma_g(H)$ curve. We restrict ourselves to this case because a real device can always be assumed to be almost harmonic; then we can calculate $H_p$ from (14). As discussed above, we can solve the inverse problem using cosine Fourier transforms. To avoid ambiguities, we assume a symmetric array and a positive and periodic $\gamma(H)$. We have the following result.

**Proposition 6.1** (Solution of the inverse problem for $\gamma_{\text{max}}(H)$). Assume a $\gamma_g(H)$ even, periodic of period $H_p$ and strictly positive. The array is harmonic and the positions of the junctions are given by $a_i = i\pi / H_p$, where $i$ is an integer. Their strengths $d_i$ are given by

$$d_{-i} = d_i = \frac{1}{H_p} \int_0^{H_p} \gamma_g(H) \cos(Ha_i) \, dH, \quad \forall i \in \{0, \ldots, n/2\}.$$  \hspace{1cm} (31)

This gives the positions $a_i$ and coefficients $d_i$ of an array that will have a $\gamma_{\text{max}}(H)$ that is the truncation to order $n$ of the cosine Fourier series of $\gamma_g(H)$.

As in approximation theory, we can decide to keep the first $k$ modes corresponding to $2k + 1$ junctions. The magnitude of the $(k+1)$th coefficient will give us an estimate of the error between $\gamma_{\text{max}}(H)$ for the device and the desired $\gamma_g(H)$.

An important issue is that of structural stability, i.e. how sensitive is our solution to a small perturbation of $\gamma_g(H)$. To answer this, we should consider the number of solutions of the direct problem. For the original boundary value problem (7), there are in general many solutions because of the sine nonlinearity and Neuman boundary conditions. In the magnetic limit and assuming $\gamma_g(H)$ is given by (28), the solution is unique if $d_0$ is large enough. In this limit, the inverse problem reduces to inverting a cosine Fourier series. Then a small continuous variation of $\gamma_g(H)$ will result in a small change in the $d_i$ junction distribution. All junctions will be affected, but the change will remain small. In this sense, the solution of the inverse problem that we present is structurally stable.

Note that for a small central junction such that $d_0 \ll 1$, $\gamma_{\text{max}}(H)$ will present jumps of the derivative when $\gamma = 0$. These are due to the absolute value in formula (28). In this situation, as long as there are not too many jumps per period, it is possible to ‘rectify’ $\gamma_g(H)$, get the cosine Fourier series and apply the above-mentioned result. A simple way to do this from a
time series \((\gamma_g(H_i), H_i = (i - 1) \, dH)\) is to estimate the first derivative numerically:

\[
\gamma'_g(H_i) = \frac{\gamma_g(H_{i+1}) - \gamma_g(H_i)}{dH},
\]

and look for the values of \(H_i\) where it has a finite jump from a positive value to a negative value or vice versa. At these points, one should ‘rectify’ \(\gamma_g(H)\). We have tested this on a noisy time series and the ‘jumps’ are easily detectable, even with 20% added noise. Then proposition (6.1) can be applied to find \(d_i\) and the junction areas.

6.1. Examples: the pulse and the square \(\gamma_g(H)\)

To gain insight into the problem, let us first review the ‘pulse’ example studied in the previous section. Assume \(\gamma_g(H)\) to be the \(2\pi\) periodic extension of \(e^{-\alpha H^2}\), where \(\alpha\) is large enough. The coefficients \(d_i\) are given by

\[
d_i = \frac{1}{2\pi} \int_0^{2\pi} e^{-\alpha H^2} \cos(i \pi H) \, dH + \frac{1}{2\pi} \int_0^{2\pi} e^{-\alpha (2\pi - H)^2} \cos(i \pi H) \, dH
\]

\[
= \frac{1}{\pi} \int_0^{\pi} e^{-\alpha H^2} \cos(i \pi H) \, dH.
\]

(32)

These Fourier coefficients decay exponentially as expected [17] because \(\gamma_g(H)\) is \(C^\infty\) over the interval \([0, 2\pi]\) and satisfies the boundary conditions. This means that \(i \leq 2\) is enough to get a good approximation of \(\gamma_g(H)\). In fact, figure 6 corresponds to \(\gamma_g(H) \approx 5e^{-0.6H^2}\) and formula (32) gives the values \(d_0 = 1.82025, d_1 = d_{-1} = 1.25\) and \(d_2 = d_{-2} = 0.3425\) that were obtained in the previous section. The next coefficients \(d_3 = 0.043, d_4 = 0.0023\) are very small. They indicate that with the three junctions, we will get \(\gamma_{\text{max}}(H)\) very close to \(\gamma_g(H)\).

Let us now consider a square \(\gamma_{\text{max}}(H)\) curve which could make a very fine magnetic detector because of its strong response over a given interval and zero response elsewhere. For that, we assume the square profile

\[
\gamma_g(H) = 1 \quad \text{for} \quad \pi \left(1 - \frac{h_1}{2}\right) < H < \pi \left(1 + \frac{h_1}{2}\right) \quad \text{and} \quad 0 \quad \text{elsewhere},
\]

(33)

and extend it periodically every \(2\pi = H_p\). To compute the parameters \(a_i\) and \(d_i\), we apply the previous result (see equation (31)) to obtain

\[
d_i = \frac{1}{2\pi} \int_0^{2\pi} \gamma_g(H) \cos \left(\frac{i \pi H}{2\pi}\right) \, dH = \frac{2}{i\pi} \sin \left(\frac{i \pi h_1 + h_2}{2}\right) \cos \left(\frac{i \pi}{2} h_2 - h_1\right).
\]

(34)

This gives the following values of \(d_i\) for \(h_1 = h_2 = h\):

\[
\begin{array}{cccccccc}
i & 0 & 1 & 2 & 3 & 4 & \ldots \\
a_i & 0 & 1 & 2 & 3 & 4 & \ldots \\
2\pi d_i & 2h & 0 & \frac{\sin(i\pi h_1)}{i\pi} & 0 & \frac{\sin(2i\pi h_1)}{2i\pi} & \ldots \\
\end{array}
\]

Note the decay in \(1/i\) of \(d_i\) because \(\gamma_g(H)\) is only continuous. Another interesting fact is that some \(d_i\) are negative so that some junctions are \(\pi\) junctions. So we obtain an array of 11 junctions whose positions are given above together with their strengths \(d_i\) (positive for a normal junction and negative for a \(\pi\) junction). In figure 8, we plot the magnetic limit and the exact solution of equation (7) for \(\nu = 0\) (inline current feed). The values are \(l = 20\), the junction unit is shifted by 10, the Josephson characteristic length is \(\lambda_J = 5.6 \, \mu\text{m}\) so that all \(d_i\) are multiplied by 0.035 714 285. We see that for this type of junction (about 1 \(\mu\text{m}^2\) of area), inline current feed for (7) and magnetic limit give close results. Differences appear when
Figure 8. We compare $\gamma_{\text{max}}(H)$ of the magnetic limit to the exact solution of equation (7) for $\nu = 0$ (inline current feed). The parameters are given in the text.

the maximal current is larger, but the Gibbs phenomenon is less important in the solution of equation (7) than the magnetic limit.

7. Conclusion

We established that the maximum static current $\gamma_{\text{max}}(H)$ of an array of small Josephson junctions is given by the simple expressions (22) and (23) with a good accuracy. Using this simple approximation, we solved the inverse problem of designing the symmetric array of Josephson junctions which has a given periodic $\gamma_{\text{max}}(H)$. The positions of the junctions are $a_i = i2\pi/H_p$, where $H_p$ is the period of $\gamma_{\text{max}}(H)$. Their sizes are given by the coefficients $d_i$ of the cosine Fourier transform of $\gamma_{\text{max}}(H)$. When $d_j = 0$, the junction is absent so that we can obtain nonuniform arrays both in junction sizes and in junction spacings. A curve with $n$ Fourier coefficients will give rise to $2n + 1$ junctions.

Using this formalism, we presented several examples and showed that the reconstruction is robust and that its accuracy can be controlled.

This analysis could lead to the development of new multi-junction devices for specific applications such as special magnetometers. The magnetic limit—where the jump of the gradient of the phase at the junctions is neglected—can be transposed to the dynamical regime. As in the static case, this simplifies the problem considerably. It could be very useful for such applications as new terahertz oscillators.

Acknowledgments

The authors thank Faouzi Boussaha and Morvan Salez for useful discussions. LL thanks Delphine and Damien Belmessieri for their comments.

Appendix. Error estimate for the magnetic limit

In the magnetic limit, the phase $\phi$ is given by $f(x) = Hx + C$. Let us estimate the errors between $\phi$ and $f$ and their $x$ derivatives:

$$\Delta_i = \phi'(a_i) - f'(a_i), \quad \Delta_i = \phi(a_i) - f(a_i).$$

(A.1)
We introduce the supercurrent at the junction $i$:
\[ \gamma_i = d_i \sin \phi_i. \] (A.2)
With this notation, the piece-wise polynomials (16) and (18) become
\[ P_i(x) = -\frac{\nu}{2l} \sum_{i=1}^{n} \gamma_i \left( x^2 - a_i^2 \right) + \left( H - \frac{1 - \nu}{2} \sum_{i=1}^{n} \gamma_i \right) (x - a_i) + \phi_1, \] (A.3)
\[ P_{i+1}(x) = -\frac{\nu}{2l} \sum_{i=1}^{n} \gamma_i (x - a_i) + (\gamma_i + P'_i(a_i)) (x - a_i) + P_i(a_i). \] (A.4)

From this, we obtain the derivatives at the junctions
\[ P'_i(a_i) = -\frac{\nu}{2l} \sum_{i=1}^{n} \gamma_i a_i + H - \frac{1 - \nu}{2} \sum_{i=1}^{n} \gamma_i, \] (A.5)
\[ P'_{i+1}(a_i) = -\frac{\nu}{2l} \sum_{i=1}^{n} \gamma_i (a_i - a_i) + \gamma_i + P'_i(a_i). \] (A.6)

From this, we compute
\[ P'_2(a_2) = -\left( \sum_{i=1}^{n} \gamma_i \right) \left( \frac{a_2}{l} + \frac{1 - \nu}{2} \right) + \gamma_1 + H. \] (A.7)

We can calculate $P'_{i+1}(a_{i+1})$ using a recursion on $i$. We then assume that for $i \geq 1$,
\[ P'_{i+1}(a_{i+1}) = -\left( \sum_{i=1}^{n} \gamma_i \right) \left( \frac{a_{i+1}}{l} + \frac{1 - \nu}{2} \right) + \sum_{k=1}^{i} \gamma_k + H. \] (A.8)

This property holds for $i = 1$. Let us show that it holds for $i + 1$. For this, we form $P'_{i+2}(a_{i+2})$ and use equation (A.6),
\[ P'_{i+2}(a_{i+2}) = -\frac{\nu}{2l} \sum_{i=1}^{n} \gamma_i (a_{i+2} - a_{i+1}) + P_{i+1} - \left( \sum_{i=1}^{n} \gamma_i \right) \left( \frac{a_{i+1}}{l} + \frac{1 - \nu}{2} \right) + \sum_{k=1}^{i} \gamma_k + H.
\]
\[ = -\left( \sum_{i=1}^{n} \gamma_i \right) \left( \frac{a_{i+2}}{l} + \frac{1 - \nu}{2} \right) + \sum_{k=1}^{i+1} \gamma_k + H. \] (A.9)

Equation (A.9) shows that the implication $i \Rightarrow i + 1$ is true, (A.8) is true for $i = 1$; therefore (A.8) holds $\forall i \geq 1$.

It is then easy to compute $\Delta_{i+1}$ for $i \geq 1$:
\[ \Delta'_{i+1} = P'_{i+1}(a_{i+1}) - H = -\left( \sum_{i=1}^{n} \gamma_i \right) \left( \frac{a_{i+1}}{l} + \frac{1 - \nu}{2} \right) + \sum_{k=1}^{i} \gamma_k. \] (A.10)

Let us now compute $\Delta_{i+1}$. For that, consider equation (A.4):
\[ P_{i+1}(a_{i+1}) = -\frac{\nu}{2l} \sum_{i=1}^{n} \gamma_i l_i^2 + (\gamma_i + P'_i(a_i)) l_i + P_i(a_i). \] (A.11)

Using the result (A.10) and rearranging terms, we get
\[ P_{i+1}(a_{i+1}) = -\left( \sum_{i=1}^{n} \gamma_i \right) \left( \frac{a_i}{l} + \frac{1 - \nu}{2} \right) + l_i \sum_{k=1}^{i} \gamma_k + H l_i + P_i(a_i), \]
\[ = -l_i \left( \sum_{i=1}^{n} \gamma_i \right) \left( \frac{a_i}{l} + \frac{1 - \nu}{2} \right) + l_i \sum_{k=1}^{i} \gamma_k + H l_i + f(a_i) + P_i(a_i) - f(a_i). \] (A.12)
From this, we get the estimate of $\Delta_{i+1}$:

$$\Delta_{i+1} = -\frac{l_i}{2} \left( \sum_{i=1}^{n} \gamma_i \right) \left( \nu \frac{a_{i+1}}{l} + 1 - \nu \right) + l_i \sum_{k=1}^{i} \gamma_k + \Delta_i. \quad (A.13)$$

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