Inhomogeneous superconducting state in quasi-one-dimensional systems

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We report on results of theoretical study of non-uniform superconducting states in quasi-one-dimensional systems, with attractive interactions and Zeeman splitting between electron spins. Using bosonization to treat intrachain electron-electron interactions, and a combination of renormalization group and mean-field approximation to tackle interchain couplings, we obtain the phase diagram of the system, and show that the transition between the uniform and non-uniform superconducting phases is a continuous transition of the commensurate-incommensurate type.

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The possibility of a superconducting state with inhomogeneous order parameter, stabilized by a sufficiently large Zeeman splitting between electrons with opposite spin orientations due to either an external magnetic or internal exchange field, was suggested more than thirty years ago by Fulde and Ferrell and Larkin and Ovchinnikov. Since then this Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state has been the subject of a number of theoretical studies, but no direct evidence of its existence has ever been found in conventional superconductors. More recently it has attracted renewed interest in the context of organic, heavy-fermion, and high-$T_c$ cuprate superconductors as these new classes of superconductors are believed to provide conditions that are favorable to the formation of FFLO state due to their pairing symmetry. Indeed, some experimental evidence of its existence has been reported.

The following picture emerged from the early theoretical studies (mostly of mean-field type) of conventional s-wave superconductors subject to a Zeeman field $B$. For sufficiently high field the system is in the normal state. As the field strength decreases, at low temperatures the system undergoes a second-order phase transition at $B = B_{c2}(T)$ into the FFLO superconducting state. As the field strength further decreases, another phase boundary is encountered at $B_{c1}(T)$, and the system goes through another phase transition into the usual BCS superconducting state with uniform superconducting order parameter. While it is much more difficult to locate the position of $B_{c1}(T)$ than $B_{c2}(T)$ (even in mean-field theory), as well as to address the nature of the transition there, it has been widely assumed that this is a first-order phase boundary, across which the momentum of the order parameter and the magnetization change discontinuously. This viewpoint was disputed in Ref. [3] in which the authors argue that the transition at $B_{c1}(T)$ is of second order. Thus the nature of this transition is an unsettled issue.

In this paper we study quasi-one-dimensional (Q1D) superconductors subject to a Zeeman field, and the possibility of formation of FFLO states in these systems. Our motivation comes from two considerations. First of all, some of the experimental candidates for FFLO state are made of weakly coupled chains and therefore Q1D. Secondly, it is known in that fluctuations are much stronger in low-dimensional systems than in 3D systems, and mean-field theories are much less reliable there. On the other hand the non-perturbative machinery developed for studying one-dimensional interacting electron systems (especially bosonization) allows us to go beyond mean-field theory and treat the intrachain electron-electron correlation exactly in Q1D systems. In this paper we will take an approach that is similar to the one used in Ref. [2] namely to treat the intrachain electron-electron interaction exactly using bosonization, and tackle the interchain couplings using a combination of renormalization group (RG) analysis and mean-field approximation. Using this approach we are able to make a number of quantitative and reliable predictions about the FFLO state in these systems. In particular, we will show that the phase transition at $B_{c1}$ is continuous in these systems, and work out its critical properties. For the sake of simplicity and concreteness, we restrict our discussion to zero temperature throughout the paper.

We start by considering a one-dimensional electron gas with attractive interactions. In the bosonized form, the Hamiltonian reads

$$H = H_c + H_s + H_z,$$  \hspace{1cm} (1)

where $H_c$ and $H_s$ are the Hamiltonian for the charge and spin sectors (which are decoupled, signaling the spin-charge separation).

$$H_\alpha = \int dx \left\{ \frac{v_\alpha}{2} \left[ K_\alpha (\partial_x \theta_\alpha)^2 + \frac{\left( \partial_x \phi_\alpha \right)^2}{K_\alpha} \right] + V_\alpha \cos (\sqrt{2} \pi \phi_\alpha) \right\},$$  \hspace{1cm} (2)

where $\alpha = c$ or $s$, and $H_z$ is the Zeeman coupling:

$$H_z = g \mu_B B S_z^\text{tot} = \sqrt{\frac{1}{2\pi}} g \mu_B B \int dx \partial_x \phi_s(x).$$  \hspace{1cm} (3)

In these equations $\phi_\alpha$ and $\phi_s$ are bosonic charge and spin fields related to the (coarse-grained) charge and spin densities:

$$\rho(x) = \sqrt{\frac{2}{\pi}} \partial_x \phi_c(x), \quad S_z(x) = \sqrt{\frac{1}{\pi}} \partial_x \phi_s(x);$$  \hspace{1cm} (4)
while \( \theta_n \) are their dual fields satisfying
\[
[\phi_n(x), \partial_x \theta_n(x')] = i \delta(x - x').
\] (5)

For attractive interactions, we typically have the Luttinger liquid parameters \( K_c > 1 \) and \( K_s < 1 \) (for non-interacting electrons, we have \( K_c = K_s = 1 \)). If the 1D electron gas is sufficiently far away from lattice commensuration, which we assume to be the case here, \( V_c \) (which measures the strength of 4\( k_f \) Umklapp scattering) may be set to zero. Thus \( H_c \) takes the form of free massless bosons. On the other hand, in the spin sector \( H_s \) has the form of 1+1D quantum sine-Gordon model, and for \( K_s < 1 \), \( V_s \) (which measures the strength of back scattering between electrons with opposite spins) is relevant in the RG sense; at low energies it opens up a gap \( \Delta_s \sim v_s \Lambda |\sigma/V_s \Lambda^2|^{1/2(2K_s)} \) (\( \Lambda \) is the ultraviolet cutoff) for spin excitations.\(^8\) The elementary spin excitations are massive solitons (kinks and anti-kinks) of the \( \phi_s \) field, which carry spin \( \pm 1/2 \).\(^9\) This spin gap \( \Delta_s \) is the analog of quasi-particle gap in the BCS theory of higher dimensional superconductors. The fundamental difference here, however, is that in 1D there is no long-range superconducting order; instead the correlation function of the Cooper pair operator decays with a power law. The power law exponent can be calculated using the explicit representation of electron operators in terms of boson fields:
\[
\psi_{\lambda,\sigma} = N_\sigma \exp[\pm i \Phi_{\lambda,\sigma}(x)],
\] (6)
where \( \lambda = \pm 1 \) represents left/right movers, \( \sigma = \pm 1 \) represents up/down spin particles, \( N_\sigma \) is the Klein factor that also includes a normalization constant, and
\[
\Phi_{\lambda,\sigma} = \sqrt{\pi/2}(\theta_c - \lambda \phi_c) + \sigma(\theta_c - \lambda \phi_s).
\] (7)
Thus the singlet pair correlation function (at \( T = 0 \))
\[
\langle \psi_{+1+1}^\dagger(\hat{x}) \psi_{-1-1}^\dagger(\hat{x}') \psi_{-1-1}(\hat{x}) \psi_{+1+1}(\hat{x}') \rangle \propto \langle \psi_{+1+1}(\hat{x}) \psi_{-1-1}(\hat{x}) \psi_{-1-1}(\hat{x}') \psi_{+1+1}(\hat{x}') \rangle \propto |x - x'|^{-2\xi_{sc}},
\] (8)
where the scaling dimension
\[
\xi_{sc} = \frac{1}{2K_c}.
\] (9)

Here we have used the fact that the spin field \( \phi_s(x) \) is long-range ordered in the spin-gapped phase. Let us now consider the effect of \( H_Z \). In \( H_Z \) the Zeeman field couples to the soliton density and plays the role of chemical potential of spin solitons. In fact, \( H_s + H_Z \) takes exactly the form of the Pokrovsky-Talapov model,\(^4\) which was introduced to study the two-dimensional classical commensurate-incommensurate (CIC) transition. In our context, we thus expect a continuous CIC transition at beyond which spin solitons start to proliferate in the ground state. Eq. (6) is an exact result because the Zeeman field couples to \( S_{tot}^z \) which is a conserved quantity.\(^9\) In the incommensurate phase, the spin solitons form a spinless Luttinger liquid with its own bosonized Hamiltonian, which describes the low-energy spin excitations of the system:
\[
H_{sol} = \int dx \frac{v_{sol}}{2} (\partial_x \theta_{sol})^2 + (\partial_x \phi_{sol})^2 / K_{sol}. \] (11)
In the long-wave length limit, the soliton density field \( \phi_{sol}(x) \) is related to the spin field \( \phi_s \) through
\[
\phi_s(x) = \phi_{sol}(x) / \sqrt{2 + \sqrt{\pi/2n_{sol}(B)x + \text{const.}}, \] (12)
where \( n_{sol} \) is the soliton density of the ground state. In the limit \( B \to B_c + 0^+ \), the solitons become extremely dilute and the repulsive interaction among them becomes irrelevant; they can be treated as spinless free fermions.\(^9\) As a consequence of this we have (i) \( K_{sol} = 1 \) and (ii) \( n_{sol}(B) \propto (B - B_c)^{1/2} \) in this limit. Using these results we find in the incommensurate phase the superconducting correlation function
\[
\langle \psi_{+1+1}(\hat{x}) \psi_{-1-1}(\hat{x}) \psi_{-1-1}(\hat{x}') \psi_{+1+1}(\hat{x}') \rangle \propto \exp[iQ(B)(x - x')] / |x - x'|^{-2\xi_{sc}}, \] (13)
where \( Q(B) = \pi n_{sol}(B) \); approaching the phase boundary: \( B \to B_c + 0^+ \), we have \( Q(B) \propto (B - B_c)^{1/2} \) and
\[
\xi_{sc} = \frac{1}{2K_c} + \frac{1}{4} = \xi_{sc} + \frac{1}{4}. \] (14)

This incommensurate phase (in the spin sector) is the 1D analog of the FFLO phase in higher dimensional systems, as the appearance of the spin solitons in the ground state induces an oscillatory phase in the superconducting correlation function, Eq. (13). Also the ground state now has a finite magnetization as in the FFLO phase, and the additional fluctuation due to the soliton liquid makes the superconducting correlation function decays faster in the incommensurate phase, in (loose) analogy to the fact that appearance of unpaired quasiparticles reduces the size of the superconducting order parameter in the FFLO phase. We emphasize again that here there is no long-range superconducting order in either the commensurate or incommensurate phases; also the CIC transition is continuous as \( n_{sol} \) increases continuously from zero as \( B \) crosses \( B_c \); both the magnetization and wave vector of oscillation \( Q \) are proportional to \( n_{sol} \).

We now turn to the discussion of interchain couplings. The three leading potentially relevant perturbations to the decoupled Luttinger liquid fixed (dLL) point are single electron hopping \( H_e \), Cooper pair hopping (or Josephson tunneling) \( H_J \), and interchain 2\( k_f \) back scatterings \( H_{C/SDW} \).\(^{11}\) For attractive interactions (\( K_c > 1 \),
$H_{c/SDW}$ is less relevant than $H_J$, in both the commensurate and incommensurate phases. $H_c$ is irrelevant in the commensurate phase due to the presence of a spin gap. Since in this case the scaling dimension for $H_J$ is

$$\xi_J = 2\xi_{sc} = 1/K_c < 2,$$

we conclude Cooper pair hopping is the leading relevant perturbation at the dLL fixed point, and the system flows toward a superconducting phase with long-range superconducting order once interchain coupling is turned on in the commensurate phase.

Now let us consider the incommensurate phase. Right after the system enters the incommensurate phase ($B \rightarrow B_{c1} + 0^+$), we have

$$\xi_J' = 2\xi_{sc} = 1/K_c + 1/2 < 2,$$  

thus $H_J$ is still relevant, albeit having a higher scaling dimension than that in the commensurate phase. However in this case $H_c$ may also be relevant, as there is no longer a spin gap in this case. We find in this case the scaling dimension of $H_c$ to be

$$\xi_c' = \frac{1}{4}(K_c + 1/K_c) + \frac{5}{8}.$$  

We thus find that for $K_c > 3/2$, $\xi_J' < \xi_c'$, and $H_J$ is the leading relevant perturbation at the dLL fixed point which drives the system to the Q1D superconducting FFLO phase once interchain coupling is turned on. On the other hand, for $1 < K_c < 3/2$, $\xi_c' < \xi_J' < 2$, and $H_c$ is the leading relevant perturbation at the dLL fixed point; in this case the system flows toward the high-dimensional Fermi-liquid fixed point. These results are summarized in a schematic phase diagram, Fig. 1. The phase boundary separating the Fermi liquid and the two superconducting phases are likely to be first-order since they are determined by the crossing of the scaling dimensions of two different relevant operators at the dLL fixed point; on the other hand as we will argue below, the transition from uniform to FFLO superconducting phases is continuous. We emphasize in this phase diagram we assume the Zeeman field $B$ is not too strong; if the Zeeman splitting is so strong as to be comparable to, say the Fermi energy, the continuum Luttinger liquid description of Q1D systems breaks down.

To address the nature of the transition between uniform and FFLO superconducting phase, we focus on the pair hopping process and neglect other perturbations that are less relevant:

$$H_J = -\tilde{t}_J \sum_{\langle ij \rangle} \int dx \psi_{i+1}^{\dagger} \psi_i \psi_{j-1}^{\dagger} \psi_j + h.c.$$

$$= -\tilde{t}_J \sum_{\langle ij \rangle} \int dx \cos[\sqrt{2}\pi(\theta_c - \theta_c')] \cos[\sqrt{2}\pi(\phi_s - \phi_s')],$$

where $i$ and $j$ are chain indices, $\tilde{t}_J$ is the pair hopping matrix element (or Josephson coupling strength), $\langle ij \rangle$ stands for neighboring chains, $t_{ij} \propto \tilde{t}_J$, and $h.c.$ stands for Hermitian conjugate. In the case of decoupled Luttinger liquids, there is spin-charge separation and the CIC transition occurs in the spin sector. As we see in Eq. (5), interchain pair hopping couples the spin and charge fields. On the other hand since the system is in the superconducting phase (uniform or non-uniform) in which the charge field $\theta_c$ is long-range ordered, in studying the transition driven by $B$ we may use a mean-field approximation and replace $\cos[\sqrt{2}\pi(\theta_c' - \theta_c)]$ in Eq. (13) by its expectation value: $\langle \cos[\sqrt{2}\pi(\theta_c' - \theta_c)] \rangle = C$. Clearly this expectation value depends on $B$ and it will also develop a dependence on $x$ in the incommensurate phase; however as long as the dependence is smooth across the transition (which would be the case if the transition is continuous as we will show to be the case), we can treat it as a constant. Thus in the mean-field approximation $H_J$ becomes

$$H_J^{MF} = -C\tilde{t}_J \sum_{\langle ij \rangle} \int dx \cos(\sqrt{2}\pi(\phi_s' - \phi_s)).$$

Eq. (19) can also be obtained more formally by integrating out the fluctuations of the $\theta$ field on top of its expectation value in the Lagrangian formalism, which will yield a slightly renormalized coupling $C$. The quantum Hamiltonian of $H_s + H_Z + H_J^{MF}$ can be mapped onto the problem of classical CIC transition driven by $B$ at finite temperatures, in $d + 1$ dimensions ($d$ is the physical dimension of the quantum problem we study here). It is known that the CIC transition in higher dimensions is still continuous, but the critical behavior is very different from the $d = 1$ case considered earlier; in this case the density of domain walls (that consist of solitons of individual chains aligned with true long-range order) depends logarithmically on the distance from criticality. We note that while we obtained these results by making a mean-field approximation to the (long-range ordered) charge fields, the main conclusion that the transition is continuous should be robust; this follows simply from the fact that the domain walls (whose appearance drives the transition) repel each other, which is clearly the case here. The logarithmic dependence of $n_{wall}$ on $B$ near criticality, among our results include a phase diagram in terms of

$$n_{wall} \propto 1/\log(|B - B_c|^{-1})$$

as $B \rightarrow B_c + 0^+$. The wave vector of the inhomogeneous superconducting order parameter $Q$ and the magnetization are both proportional to $n_{wall}$ and thus have the same dependence on $B$ near criticality. We note that while we obtained these results by making a mean-field approximation to the (long-range ordered) charge fields, the main conclusion that the transition is continuous should be robust; this follows simply from the fact that the domain walls (whose appearance drives the transition) repel each other, which is clearly the case here. The logarithmic dependence of $n_{wall}$ on $B - B_c$ then follows from the exponentially weak repulsion between the domain walls. These in turn justify the validity of the mean-field approximation employed.

To summarize, we studied formation of non-uniform superconducting state in quasi-one-dimensional systems. Among our results include a phase diagram in terms of
the Zeeman field and Luttinger liquid parameter. We also showed that the transition between the uniform and non-uniform superconducting states is continuous.

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FIG. 1. Schematic phase diagram of coupled Luttinger liquids subject to a Zeeman field. The solid lines are first order phase boundaries while the dashed line is a second-order phase boundary.