On the Relationship between the Posterior and Optimal Similarity

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Abstract

For a classification problem described by the joint density $P(\omega, x)$, models of $P(\omega = \omega'|x, x')$ (the “Bayesian similarity measure”) have been shown to be an optimal similarity measure for nearest neighbor classification. This paper analyzes demonstrates several additional properties of that conditional distribution. The paper first shows that we can reconstruct, up to class labels, the class posterior distribution $P(\omega|x)$ given $P(\omega = \omega'|x, x')$, gives a procedure for recovering the class labels, and gives an asymptotically Bayes-optimal classification procedure. It also shows, given such an optimal similarity measure, how to construct a classifier that outperforms the nearest neighbor classifier and achieves Bayes-optimal classification rates. The paper then analyzes Bayesian similarity in a framework where a classifier faces a number of related classification tasks (multitask learning) and illustrates that reconstruction of the class posterior distribution is not possible in general. Finally, the paper identifies a distinct class of classification problems using $P(\omega = \omega'|x, x')$ and shows that using $P(\omega = \omega'|x, x')$ to solve those problems is the Bayes optimal solution.

1 Introduction

Statistical models of similarity have become increasingly important in recent work on information retrieval [7], case-based reasoning [5], pattern recognition[1], and computer vision [8, 9]. Of particular interest is Bayesian similarity, a discriminatively trained model of $P(x)$ and $x'$ are in the same class|x, x' which we will abbreviate as $P(\text{same}|x, x')$. These models have been demonstrated to work well in a number of pattern recognition and visual object recognition problems [4, 8, 9].

It is easy to see that nearest neighbor classification using $1 - P(\text{same}|x, x')$ minimizes the risk that the class labels for $x$ and $x'$ differ and therefore is optimal for 1-nearest neighbor classification [8, 9]. However, beyond that observation, there have been several kinds of analysis of Bayesian similarity. The first, presented by Mahamud

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[8] [9] is an analysis considering a single instance of a classification problem, determined by a joint distribution \( P(\omega, x) \) of class labels \( \omega \) and feature vectors \( x \). The authors also argue for the existence of useful invariance properties of Bayesian similarity functions when those functions have a specific form [9]. The second is an analysis based on a hierarchical Bayesian framework presented by Breuel [3], which effectively considers Bayesian similarity in the context of a distribution of related classification tasks.

This paper analyzes the relationship between Bayesian similarity \( P(\text{same}|x, x') \) and the class posterior distribution \( P(\omega|x) \) in both the non-hierarchical and hierarchical cases and uses those results to construct an asymptotically Bayes-optimal classification procedure using Bayesian similarity. It also presents a new statistical model for the kinds of discrimination tasks described in [9] and demonstrates that Bayesian similarity is the Bayes-optimal solution for those tasks. The implications of these results for applications of Bayesian similarity will be discussed at the end.

2 Bayesian Similarity

Consider a classification problem in which feature vectors \( x \in \mathbb{X} = \mathbb{R}^n \) and class variables \( \omega \in \{1, \ldots, c\} \) are jointly distributed according to some distribution \( P(x, \omega) \).

Definition 1 Let \( P(x, \omega) \) be the distribution for a classification problem. Given two samples from this distribution, \((x, \omega)\) and \((x', \omega')\), we define Bayesian similarity as the probability \( P(\omega = \omega'|x, x') \). When \( \omega \) and \( \omega' \) are clear from context, we will usually denote this as \( P(\text{same}|x, x') \).

Let \( x_\omega \) be a sample that has somehow been selected as a “prototype” for class \( \omega \). It is natural to classify some unknown feature vector \( x \) using the rule:

\[
D(x) = \arg \max_{\omega'} P(\omega = \omega'|x, x_\omega)
\]

That is, we classify the unknown feature vector \( x \) using the class associated with the training example \( x_\omega \) that is most similar to it in the sense of Bayesian similarity.

Observe that Equation 1 is analogous to nearest neighbor classification if we use \( d(x, x') = 1 - P(\omega = \omega'|x, x') \) as the similarity function. Because \( P(\omega = \omega'|x, x') \) is, by definition, the probability that \( x \) and \( x' \) have the same class label, it is also the Bayes-optimal misclassification rate using a nearest neighbor rule; therefore, nearest-neighbor classification using \( d(x, x') = -P(\omega = \omega'|x, x') \) is an optimal nearest neighbor classifier.

Nearest neighbor classification using \( d(x, x') = 1 - P(\omega = \omega'|x, x') \) is not necessarily Bayes-optimal; in fact, the asymptotic bounds on its performance are no better than those known for traditional nearest neighbor methods [8] [9]. However, when \( x_\omega \) is an unambiguous prototype, that is, \( P(\omega|x_\omega) = 1 \), then classification with Bayesian similarity is Bayes optimal:

\[
P(\omega = \omega'|x, x_{\omega'}) = \sum_{\omega'} P(\omega|x)P(\omega'|x) = \sum_{\omega'} P(\omega|x)\delta(\omega, \omega') = P(\omega'|x)
\]

2
That is, in the case of unambiguous training examples, Equation 1 just reduces to Bayes-optimal classification.

3 Relationship between $P(\text{same} | x, x')$ and $P(\omega | x)$

While we have seen some relationships between nearest neighbor classification and Bayesian similarity in the previous section and in the literature [8, 9], the question arises whether there are better ways of taking advantage of $P(\text{same} | x, x')$ and whether we can achieve Bayes-optimal classification using a Bayesian similarity framework.

Consider a two-class classification problem; that is, $\omega \in \{0, 1\}$. Now, examine the probability $P(\text{same} | x, x)$; that is, the probability that two samples with the same feature vector actually have the same class. This probability is not equal to 1 in general because the both of the class conditional densities $P(x | \omega = 0)$ and $P(x | \omega = 1)$ may be nonzero at $x$. We obtain:

$$P(\text{same} | x, x) = P(\omega = 0 | x)P(\omega = 0 | x) + P(\omega = 1 | x)P(\omega = 1 | x)$$

$$= (P(\omega = 0 | x))^2 + (P(\omega = 1 | x))^2$$

$$= (P(\omega = 0 | x))^2 + (1 - P(\omega = 0 | x))^2$$

We can solve this for $P(\omega = 0 | x)$ up to a sign:

$$P(\omega = 0 | x) = \frac{1}{2} \pm \frac{1}{2} \sqrt{2P(\text{same} | x, x) - 1}$$

Note that $P(\text{same} | x, x) \in [\frac{1}{2}, 1]$, so this is well-defined and real. Given $P(\text{same} | x, x')$, in particular, we have $P(\text{same} | x, x)$, and from Equation 6 we see that we can reconstruct $P(\omega = 0 | x)$ up to a single choice of a sign at each point. Of course, while this gives us a lot of information about $P(\omega | x)$, the unknown sign is crucial for classification.

Now consider the decision regions for the minimum error decision rule: $D_0 = \{x | P(\omega = 0 | x) > \frac{1}{2}\}$ and $D_1 = \{x | P(\omega = 0 | x) < \frac{1}{2}\}$. That is, given an unknown feature vector $x$, we decide $\omega = 0$ when $x \in D_0$ and $\omega = 1$ when $x \in D_1$. If $x$ is not contained in either decision region, we can make an arbitrary choice between classes 0 and 1.

Now consider two points $x$ and $x'$. Assume they both come from $D_0$: Then, for some positive $d$ and $d'$, $P(\omega = 0 | x) = \frac{1}{2} + d$ and $P(\omega = 0 | x') = \frac{1}{2} + d'$. Therefore,

$$P(\text{same} | x, x') = (\frac{1}{2} + d)(\frac{1}{2} + d') + (\frac{1}{2} - d)(\frac{1}{2} - d')$$

$$= \frac{1}{2} + 2dd'$$

If both come from $D_1$, the result is the same. If one comes from $D_0$ and the other comes from $D_1$, then, for some positive $d$ and $d'$,

$$P(\text{same} | x, x') = (\frac{1}{2} + d)(\frac{1}{2} - d') + (\frac{1}{2} - d)(\frac{1}{2} + d')$$

$$= \frac{1}{2} - 2dd'$$
Since the $d$ and $d'$ are both positive, this means that if $x$ and $x'$ are in the same decision region, $P(\text{same}|x, x') > \frac{1}{2}$, and otherwise $P(\text{same}|x, x') < \frac{1}{2}$. Therefore, for any two points $x$ and $x'$, we can decide whether they are in the same decision region by seeing whether $P(\text{same}|x, x') > \frac{1}{2}$.

Using these two results, we can now state the following theorem:

**Theorem 1** We can reconstruct either $P(\omega|x)$ or $1 - P(\omega|x)$ from $P(\text{same}|x, x')$.

Proof. Compute the two possible values for $P(\omega = 0|x)$ using Equation 6. Pick a point $x$ at which $P(\omega = 0|x) \neq \frac{1}{2}$, i.e., where $P(\text{same}|x, x') \neq \frac{1}{2}$. $P(\omega = 0|x)$ is then either less than $\frac{1}{2}$ or greater than $\frac{1}{2}$. Arbitrarily pick one of these; this is a choice of membership of $x$ in $D_0$ or $D_1$. Use the constraint $P(\omega|x, x') > \frac{1}{2}$ for points in the same decision region to assign all other points to decision regions. Given the decision regions and the values from Equation 6 we have reconstructed either $P(\omega|x)$ or $1 - P(\omega|x)$, depending on whether our arbitrary choice above was correct or not. ∗

This means that if we have an estimate of the Bayesian similarity function $P(\text{same}|x, x')$, we have already identified the class posterior distribution up to a choice of two: $P(\omega|x)$ (the correct class posterior distribution), and $1 - P(\omega|x)$.

Once we have $P(\text{same}|x, x')$, training samples only serve to distinguish the two possibilities for the reconstructed class posterior distributions. Since the prior probability for either choice is $\frac{1}{2}$, we can determine which of the two possibilities applies by considering the ratio of the probability of the samples given the models. That is, if we write $P_A(\omega|x)$ and $P_B(\omega|x)$ for the two possibilities, then we evaluate

$$r = \frac{\prod_i P_A(\omega_i, x_i)}{\prod_i P_B(\omega_i, x_i)} = \frac{\prod_i P_A(\omega_i|x_i)P(x_i)}{\prod_i P_B(\omega_i|x_i)P(x_i)} = \frac{\prod_i P_A(\omega_i|x_i)}{\prod_i P_B(\omega_i|x_i)}$$

(11)

If $r > 1$, then $P_A$ is the more likely possibility, otherwise $P_B$ is the more likely possibility.

So, if we take this together, we have a Bayes-optimal classification procedure given $P(\text{same}|x, x')$ and a set of prototypes or samples $(\omega_i, x_i)$: first, we compute the two possible values of $P(\omega|x)$ at each point using Equation 6 then we use Equation 7 to assign those values to the two possible branches, and then finally use the prototypes to identify which of the two branches is the more likely using Equation 11. Finally, we classify using the reconstructed class conditional distribution $P(\omega|x)$.

The only purpose that training samples obtained in addition to the Bayesian similarity function $P(\text{same}|x, x')$ serve in this procedure is to determine which of the two possible choices of the reconstructed $P(\omega|x)$ is the correct one. Asymptotically, the above procedure for making the choice between the two possibilities, can be seen to be correct with probability one. Therefore, this classification procedure is asymptotically Bayes-optimal.

Compare that with the proposed use of Bayesian similarity in a nearest neighbor classification procedure. First, the approach described above is very different from a nearest neighbor classifier, because it integrates information from all samples. Second, given $P(\text{same}|x, x')$ and labeled training examples, a nearest neighbor classifier using Bayesian similarity, even asymptotically, is not guaranteed to come within more than a factor of two of the Bayes-optimal error rate [8, 9], while the procedure described above will almost always reach the Bayes-optimal error rate.
4 Multi-Class Case

The previous section showed that for one large class of classification problems (namely, two-class classification problems), knowledge of the Bayesian similarity function is essentially equivalent to knowledge of the class posterior distributions. That already demonstrates that, given $P(\text{same}|x, x')$, 1-NN classification is not an admissible classification procedure (i.e., there is a procedure that is uniformly better). However, while it is not central to the main argument, it is an interesting question to ask whether that approach generalizes to the multi-class case. Let us sketch the argument here without making a full, formal proof.

As before,

$$P(\text{same}|x, x') = \sum_i P(\omega = i|x)P(\omega = i|x')$$

(12)

Now, assume that are looking at $c$ classes and $n$ points $x_j$ and write $p_{ij} = P(\omega = i|x_j)$. Also, write $s_{ij}$ for $P(\text{same}|x_i, x_j)$. Then, we have

$$s_{ij} = \sum_k p_{ik}p_{kj}$$

(13)

The $s_{ij}$ are $\binom{n}{2}$ given quantities, and there are $(c - 1)$ $n$ unknown quantities $p_{ij}$. We have enough equations to solve for the unknowns when $n \geq 2c - 1$.

Of course, as in the two-class case, given any solution $p_{ij}$, any permutation of class labels remains a solution, and as before, this is expressed as an uncertainty of signs in the system of equations given by Equation 13. But, as in the two-class case, there is only a finite number of possibilities, and we can distinguish among them by computing the likelihoods of the actual set of training samples for each of the different possible solutions. Therefore, we see that, as in the two-class case, we can reconstruct the class conditional density up to permutation. As before, any additional training examples or prototypes we use merely serve to pick the most likely possibility among this finite set.

5 Batched Hierarchical Bayesian Similarity

In the previous sections, we have seen that knowledge of the Bayesian similarity function $P(\text{same}|x, x')$ is mostly equivalent to knowledge of the class posterior distribution $P(\omega|x)$. In effect, Bayesian similarity is a suboptimal application of $P(\omega|x)$. This raises the question of whether using Bayesian similarity for nearest neighbor classification is of any use at all. Both this and the next section answer that question in the affirmative. While Bayesian similarity is not useful for simple classification problems, it is useful for hierarchical Bayesian problems and actually Bayes-optimal for certain discrimination problems. In fact, all previous applications of Bayesian similarity in the literature, including [9] are probably better analyzed in one of these two frameworks than as simple classification problems.

One way of understanding learning similarity measures for nearest neighbor classifiers is to think of the problem as learning a similarity measure for a collection of related task. For example, in an OCR problem, a similarity function might generally be able
to evaluate the similarity of different character shapes to one another, but when applied to a specific classification problem, the identity of individual characters is given by a set of training examples. See [6, 10] for further information. The idea of a collection of related classification problems can be formalized in its most general form as that of hierarchical Bayesian methods. After describing hierarchical Bayesian classification, we will return to its relationship with Bayesian similarity.

In a hierarchical Bayesian framework, we assume that the distribution governing the classification problem is parameterized by some parameter vector $\theta$, which is itself distributed according to some prior $P(\theta)$. We write $P(\theta|x|\omega)$ or, equivalently, $P(x|\omega, \theta)$ for the parameterized class conditional density. If we are just given individual samples from such a hierarchical Bayesian model, the model is merely a particular representation of a non-hierarchical density using an integral [2]:

$$P(x|\omega) = \int P(x|\omega, \theta)P(\theta)d\theta$$

(14)

In a batched hierarchical Bayesian problem, a classifier faces a collection of batches, where the samples $(\omega_i, x_i)$ within each batch are drawn using the same parameter $\theta$. The Bayes-optimal classification for a batch of samples $B = \{\ldots, (\omega_i, x_i), \ldots\}$ can be derived from the class conditional density for that batch:

$$P(x|\omega) = \int \prod_i P(x_i|\omega_i, \theta)P(\theta)d\theta$$

(15)

Note that this differs from a non-batched hierarchical Bayesian model, for which the class conditional density for the same batch would be $P(x|\omega) = \prod_i P(x_i|\omega_i, \theta)P(\theta)d\theta$.

Let us now return to the question of how a hierarchical Bayesian approach relates to Bayesian similarity. Trivially, we have

$$P(\text{same}|x, x') = \int P(\text{same}|x, x', \theta)P(\theta)d\theta$$

(16)

This function can be approximated by taking pairs of samples $(\omega, x)$ and $(\omega', x')$ from the same batch $\theta$ and training a classifier with it. We refer to this as batched training. That is, it is learned analogously to Bayesian similarity in the non-hierarchical cases, but all pairs of feature vectors $x$ and $x'$ used for training are taken from the same batch.

What is the equivalent to Equations [12] and [13]? Those equations relied on the relationship $P(\omega = \omega'|x, x') = P(\omega|x)P(\omega|x')$. But the equivalent relationship is not true in the hierarchical Bayesian case. While $P(\omega = \omega'|x, x', \theta) = P(\omega|x, \theta)P(\omega|x', \theta)$, the same is not true in general for the corresponding marginal distributions after integration over $\theta$: $P(\omega = \omega'|x, x') \nless P(\omega|x)P(\omega|x')$. Therefore, given $n$ sample points $x_1, \ldots, x_n$, in general, we may have to estimate the values for all $c^n$ combinations of classifications $P(\omega_1, \ldots, \omega_n|x_1, \ldots, x_n)$, and for that, the $\frac{1}{2}n(n-1)$ Bayesian similarity values $P(\text{same}|x_i, x_j)$ do not provide sufficient information in general.

Therefore, for hierarchical Bayesian classification, knowledge of the Bayesian similarity is not, in general, equivalent to knowledge of the class posterior distributions. However, even in a hierarchical Bayesian framework, it is still true that Bayesian similarity is the optimal similarity function for nearest neighbor classification: for any
1 − \(P(\omega = \omega' | x, x') = 1 - \int P(\omega = \omega' | x, x', \theta)P(\theta)d\theta\) is the risk that the class labels associated with \(x\) and \(x'\) differ, and minimizing that risk minimizes the overall risk of misclassification in a 1-nearest neighbor framework (this is the analogous argument to that made in [8, 9]). We can therefore state:

**Theorem 2** Batch-trained Bayesian similarity is the optimal distance function for 1-nearest neighbor classification in a batched hierarchical Bayesian classification problem.

### 6 Discrimination Tasks

In the previous section, we looked at a hierarchical Bayesian classification task. Let us now look at a closely related problem.

Mahamud [8, 9] considers the problem of determining whether two image patches in different images come from the same object or different objects. For this, they train a Bayesian similarity model \(P(\text{same} | x, x')\) and use it to make this decision for real images.

They analyze this by formulating it as a non-hierarchical classification problem and postulate an underlying joint distribution \(P(\omega, x)\) between class labels and feature vectors. That presupposes that some class structure exists over the image patches; that is, that image patches can be classified into a fixed set of categories and that the purpose of nearest neighbor classification is to recover those categories. But the authors do not demonstrate that such a class structure actually exist, and its existence does not appear particularly plausible.

Consider, for example, feature vectors consisting of color histograms over image patches. While it is meaningful to ask whether two such color histograms are sufficiently similar between two images to have come from the same object, there is no obvious classification of color histograms that is independent of the specific problem instance.

There are two different condition, the **same** condition and the **different** condition. Let us write \(S = 1\) and \(S = 0\) for the two conditions, respectively. Under the \(S = 1\) condition, two unknown feature vectors \(x\) and \(x'\) are produced by the same, parameterized as \(\theta\). Under the \(S = 0\) condition, two unknown feature vectors are produced by different patches, parameterized as \(\theta\) and \(\theta'\). The task Mahamud [8, 9] set out to solve is whether a given pair of feature vectors \(x\) and \(x'\) was produced under the \(S = 1\) or \(S = 0\) conditions. In order to solve this problem, they postulate the existence of an underlying classification problem \(P(\omega, x)\) and then address it using non-hierarchical Bayesian similarity. Their justification for using Bayesian similarity is that \(\omega\) is unobservable, so training a traditional classifier would be impossible.

If we don’t invoke an underlying, unobservable class structure, how should we analyze this kind of discrimination problem? Let us say that the possible surface patches on a 3D object are parameterized by some parameter vector \(\theta\). Furthermore, let the viewing parameters for that surface patch be given as \(\phi\) and that there is some random noise variable \(\nu\). Then, the feature vector representing the appearance of the surface
patch in the image, for unknown viewing parameters and noise, is given by

\[ P(x|\theta) = \int P(x|\theta, \phi, \nu) P(\phi) P(\nu) d\phi d\nu \quad (17) \]

The problem is now to determine whether two samples \( x \) and \( x' \) come from the same distribution \( P(x|\theta) \).

For concreteness, let us write down the distributions involved in this problem. The class conditional density under the same condition is

\[ P(x, x'|S = 1) = \int P(x|\theta) P(x'|\theta) P(\theta) d\theta \quad (18) \]

For the \( S = 0 \) condition, it is given by

\[ P(x, x'|S = 0) = \int P(x|\theta) P(\theta) d\theta \int P(x'|\theta') P(\theta') d\theta' = P(x) P(x') \quad (19) \]

The joint distribution is just the mixture:

\[ P(x, x', S) = P(x, x'|S = 1) P(S = 1) + P(x, x'|S = 0) P(S = 0) \quad (20) \]

Applying Bayes rule gives us

\[ P(S = 1|x, x') = \frac{P(x, x'|S = 1) P(S = 1)}{\sum_{S \in \{0,1\}} P(x, x', S)} \quad (21) \]

Nowhere in this derivation of the posterior distribution was it necessary to postulate an underlying class structure. Furthermore, if we obtain a model of \( P(S|x, x') \) from training data and use it for deciding whether \( x \) and \( x' \) were generated under \( S = 0 \) or \( S = 1 \) conditions, our decision procedure will be Bayes-optimal because \( P(S|x, x') \) is the optimal discriminant function for \( S \). Suboptimality of the use of \( P(S|x, x') \) for classification was a result of the fact that in classification, we are trying to make a decision about \( \omega \), not \( S \).

### 7 Discussion

In this paper, we have seen three distinct uses of Bayesian similarity: as a similarity measure for non-hierarchical classification problems, as a similarity measure for batched hierarchical classification problems, and as a similarity measure for discrimination tasks.

The paper has shown that for non-hierarchical classification problems, models of \( P(\text{same}|x, x') \) are equivalent to models of \( P(\omega|x) \), up to permutation of the class labels. That makes the use of Bayesian similarity for individual classification problems merely a variation of learning a classifier. In a sense, \( P(\text{same}|x, x') \) is too problem specific: it “knows so much” about the particular classification problem \( P(\omega, x) \) that we might as well use \( P(\omega|x) \) directly. Although this paper did not show it formally,
that is likely to be a problem with any optimal similarity measure for nearest neighbor classification.

Intuitively, what we would like is a similarity measure that works well across an entire class of related problems. We can formalize this notion of a class of related problems in a hierarchical Bayesian framework [2, 1, 6, 10]. When we consider Bayesian similarity in such a framework, it is not equivalent to knowledge of the class posterior distributions anymore. However, the property that it is an optimal similarity function for nearest neighbor classification remains. This means that in a hierarchical Bayesian setting, Bayesian similarity is a procedure that is distinct from other methods and may have useful applications; unlike more direct or generative implementations of hierarchical Bayesian models [6, 10]. Bayesian similarity models appear to be easier to implement and train. It is important to remember that such hierarchical models are trained differently from the non-hierarchical models: for non-hierarchical models, samples $x$ and $x'$ used for training $P(\text{same}|x, x')$ are taken from the entire distribution, while for hierarchical models, such samples are only taken from within a batch that was sampled using the same distributional parameters $\theta$.

Finally, the paper has presented a novel analysis of $P(\text{same}|x, x')$ for discrimination tasks like those considered in [8, 9] and demonstrated that the use of $P(\text{same}|x, x')$ in such tasks is, in fact, Bayes-optimal. This is an important result because those kinds of discrimination class are quite common in computer vision applications.

References

[1] Jonathan Baxter. The canonical distortion measure for vector quantization and function approximation. In Proc. 14th International Conference on Machine Learning, pages 39–47. Morgan Kaufmann, 1997.

[2] James O. Berger. *Statistical Decision Theory and Bayesian Analysis*. Springer Verlag, 1980.

[3] T. M. Breuel. Character recognition by adaptive statistical similarity. In *International Conference for Document Analysis and Recognition (ICDAR)*, 2003.

[4] T.M. Breuel. Classification by probabilistic clustering. In *ICASSP 2001*, 2001.

[5] Boi Faltings. Probabilistic indexing for case-based prediction. In *ICCBR*, pages 611–622, 1997.

[6] T. Heskes. Solving a huge number of similar tasks: a combination of multi-task learning and a hierarchical bayesian approach. In *Proceedings of the International Conference on Machine Learning*, 1998.

[7] Thomas Hofmann, Jan Puzicha, and Michael I. Jordan. Learning from dyadic data. In *Advances in Neural Information Processing Systems (NIPS'98)*, pages 466–472, 1999.

[8] S. Mahamud. *Discriminative Distance Measures for Object Detection*. PhD thesis, School of Computer Science, Computer Science Department, 2002.
[9] S. Mahamud and M. Hebert. The optimal distance measure for object detection. In *IEEE Conf. on Computer Vision and Pattern Recognition (CVPR)*, 2003.

[10] C. Mathis and T. M. Breuel. Classification using a hierarchical bayesian approach. In *Proceedings of the International Conference on Pattern Recognition (ICPR’02), Quebec City, Quebec, Canada*, 2002.