On the center problem for ordinary differential equations

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ON THE CENTER PROBLEM FOR
ORDINARY DIFFERENTIAL EQUATIONS

By Alexander Brudnyi

Abstract. We study the Center Problem for equations \( v' = \sum_{i=1}^{\infty} a_i(x) v^{i+1} \). This problem is closely connected to the classical Poincaré Center-Focus problem about the characterization of planar polynomial vector fields whose integral trajectories are closed curves with interiors containing a fixed point, a center.

1. Introduction and main results.

1.1. Let \( L^\infty(S^1) \) be the Banach space of bounded complex-valued functions on the unit circle \( S^1 \). We will also identify elements of \( L^\infty(S^1) \) with bounded \( 2\pi \)-periodic functions on \( \mathbb{R} \). Let us consider the ordinary differential equation

\[
\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x) v^{i+1},
\]

where all \( a_i \in L^\infty(S^1) \). If the coefficients of (1.1) grow not very fast for a sufficiently small initial value one can solve this equation by the Picard iteration method so that the (generalized) solution is a Lipschitz function on \([0, 2\pi]\). (I.e., in this way we obtain a function \( v \) for which (1.1) holds almost everywhere on \([0, 2\pi]\).) Moreover, there is a unique solution with the prescribed initial value \( v(0) \). We say that equation (1.1) determines a center if for any sufficiently small initial values \( v(0) \) the solution of (1.1) satisfies \( v(0) = v(2\pi) \). The Center Problem for equation (1.1) is to find conditions on the coefficients \( a_i \) under which this equation determines a center. This problem arises naturally in the framework of the qualitative theory of ordinary differential equations created by H. Poincaré. The main purpose of the theory can be described as follows: without explicitly solving a given differential equation, using only certain properties of its right-hand side, to try and give an as complete as possible description of the geometry of the solution curves of this equation (where they are defined). In this paper we study the Center Problem for equation (1.1). It is worth noting that there is a con-
Consider the system of ODEs in the plane

\[
\begin{align*}
\frac{dx}{dt} &= -y + F(x,y), \\
\frac{dy}{dt} &= x + G(x,y)
\end{align*}
\]

where \(F, G\) are real analytic functions in an open neighbourhood of \(P\) whose Taylor expansions at \(P\) do not contain constant and linear terms. Then for \(F, G\) polynomials of a given degree, the classical Poincaré Center-Focus Problem asks about conditions on the coefficients of \(F\) and \(G\) under which all trajectories of (1.2) situated in a small open neighbourhood of the origin are closed. (A similar problem can be posed for the general case.) Poincaré proved that \(P\) is a center if and only if the coefficients of \(F\) and \(G\) satisfy a certain infinite system of algebraic equations \(E_1, E_2, \ldots\) such that the coefficients of \(E_n\) are functions in the solutions of \(E_1, \ldots, E_{n-1}\). Thus in order to solve \(E_n\) one first should solve all the previous equations. In contrast, our approach gives another characterization of centers in terms of an infinite system of algebraic equations \(\tilde{E}_1, \tilde{E}_2, \ldots\) such that each \(\tilde{E}_n\) can be solved independently of the others (see Theorem 1.2). This allows to establish nonexistence of a center using only one successfully chosen equation \(\tilde{E}_n\). Unfortunately, trying to apply either this or Poincaré criterion in the converse direction, gives rise, in the general case, to almost insurmountable difficulties. Therefore it is important to describe some typical cases for which all equations of the system determining centers are satisfied. In this paper we consider one of such cases.

Observe that passing to polar coordinates in (1.2) we rewrite this in the form

\[
\frac{dr}{d\theta} = \frac{P}{1 + Qr^2}
\]

where \(P(r, \phi) := \frac{x F(x,y) + y G(x,y)}{r^2}, \ Q(r, \phi) := \frac{x G(x,y) - y F(x,y)}{r^2}, \ (x = r \cos \phi, \ y = r \sin \phi)\). If the moduli of the coefficients of \(F\) and \(G\) are small enough we can expand the right-hand side of (1.3) as a series in \(r\) to obtain an equation (1.1) whose coefficients are trigonometric polynomials. This reduces the Center Problem for (1.2) to the Center Problem for a class of equations (1.1) whose coefficients depend polynomially on the coefficients of (1.2) explaining the role of equation (1.1) in this area.

1.2. Operator-valued center problems. Let \(X_i := L^\infty(S^1)\) be the space of all coefficients \(a_i\) from (1.1), and \(X\) be the complex Fréchet space of sequences
\( a = (a_1, a_2, \ldots) \in \prod_{i \geq 1} X_i \) satisfying (for some positive \( l \) depending on \( a \))

\[
\sup_{x \in S^1} |a_i(x)| \leq l, \quad i = 1, 2, \ldots.
\]  

(1.4)

From Picard iteration it follows that for any \( a \in X \) the corresponding equation (1.1) is locally solvable for sufficiently small initial values. Let \( C \subset X \) be the center set of equation (1.1), that is, the set of those \( a \in X \) for which the corresponding equations (1.1) determine centers. In our approach we are trying to better understand the influence of the algebraic dependence of the coefficients of equation (1.1) on the center conditions. For this purpose we allow the coefficients and the solutions of this equation to belong to certain associative algebras. Such a formulation can be obtained after performing a natural linearization of equation (1.1) as described in Section 2.2 below (see also [Br2]).

Our first result reveals a connection of the Center Problem for equation (1.1) with some similar operator-valued problems. For its formulation one introduces the associative algebra \( \mathcal{A}(X_1, X_2) \) with unit \( I \) of noncommutative polynomials with complex coefficients in variables \( X_1, X_2 \) and \( I \) satisfying relations

\[
p_1(X_1, X_2, I) = 0, \ldots, p_m(X_1, X_2, I) = 0
\]  

(1.5)

where \( p_i \in \mathbb{C}[z_1, z_2, z_3] \) are holomorphic polynomials. Then \( \mathcal{A}(X_1, X_2)[[t]] \) stands for the associative algebra of formal power series in \( t \) whose coefficients are elements from \( \mathcal{A}(X_1, X_2) \). For an element \( a = (a_1, a_2, \ldots) \in X \) let us consider the equation on \( S^1 \)

\[
\frac{dF}{dx} = \left( \sum_{i=1}^{\infty} a_i(x)t^iX_1X_2^{i-1} \right) F.
\]  

(1.6)

This can be solved locally by Picard iteration to obtain a local solution \( F(x) \) as a function in \( x \) with values in the group \( G(X_1, X_2)[[t]] \) of invertible elements of \( \mathcal{A}(X_1, X_2)[[t]] \) whose coefficients are Lipschitz. As usual, the monodromy of (1.6) is a homomorphism \( \rho: \mathbb{Z} \to G(X_1, X_2)[[t]] \) where \( \mathbb{Z} \) is the fundamental group of the unit circle \( S^1 \). It is the only obstruction to extending local solutions of (1.6) to global ones.

By \( C_A \subset X \) we denote the center set of equation (1.6) that is the set of those \( a \in X \) for which the monodromy of the corresponding equations (1.6) is trivial. The center problem for (1.6) is to describe \( C_A \). E.g., one can show that \( C_A \) is the set of zeros of a polynomial ideal of the algebra of iterated integrals on \( X \). Moreover, for a specific \( \mathcal{A}(X_1, X_2) \) we have the following connection with the Center Problem for equation (1.1).

**Theorem 1.1.** Suppose \( \mathcal{A}(X_1, X_2) \) is determined by the (unique) relation

\[
[X_1, X_2] := X_1X_2 - X_2X_1 = -X_2^2.
\]  

(1.7)

Then \( C_A \) coincides with \( C \).
1.3. An explicit expression for the first return map. Let us consider the iterated integrals

\[ I_{i_1, \ldots, i_k}(a) := \int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_k \leq 2\pi} a_{i_k}(s_k) \cdots a_{i_1}(s_1) \, ds_k \cdots ds_1 \]

defined on \( X \) (for \( k = 0 \) we assume that this equals 1). They can be thought of as \( k \)-linear holomorphic functions on \( X \). By the Ree formula \([R]\) the linear space generated by all such functions is an algebra. A linear combination of iterated integrals of order \( \leq k \) is called an iterated polynomial of degree \( k \).

As an important corollary of Theorem 1.1 we obtain an explicit description of the center set \( C \subset X \) in terms of the iterated integrals. (It is worth noting that iterated integrals appeared recently in a similar context in the study of a version of 16th Hilbert problem, see \([BY]\), \([Fr]\), \([G]\).)

Let \( v(x; r; a) \), \( x \in [0, 2\pi] \), be the Lipschitz solution of equation (1.1) corresponding to \( a \in X \) with (sufficiently small) initial value \( v(0; r; a) = r \). Then \( P(a)(r) := v(2\pi; r; a) \) is the first return map of this equation.

**Theorem 1.2.** For sufficiently small initial values \( r \) the first return map \( P(a) \) is an absolutely convergent power series \( P(a)(r) = r + \sum_{n=1}^{\infty} c_n(a) r^{n+1} \), where

\[ c_n(a) = \sum_{i_1 + \cdots + i_k = n} c_{i_1, \ldots, i_k} I_{i_1, \ldots, i_k}(a), \quad \text{and} \]

\[ c_{i_1, \ldots, i_k} = (n - i_1 + 1) \cdot (n - i_1 - i_2 + 1) \cdot (n - i_1 - i_2 - i_3 + 1) \cdots 1. \]

The center set \( C \subset X \) of equation (1.1) is determined by the system of polynomial equations \( c_n(a) = 0 \), \( n = 1, 2, \ldots \).

**Corollary 1.3.**
(a) \( c_n(a) = I_n(a) + f_n(a) \) where \( I_n(a) := \int_0^{2\pi} a_n(s) \, ds \) and \( f_n \) is an iterated polynomial of degree \( n \) in \( a_1, \ldots, a_{n-1} \);
(b) The set \( \mathcal{C}_n = \{ a \in X : c_1(a) = c_2(a) = \ldots = c_n(a) = 0 \} \)

is a closed complex submanifold of \( X \) of codimension \( n \) containing \( 0 \in X \);
(c) The tangent space to \( \mathcal{C}_n \) at \( 0 \) is determined by equations \( I_1(a) = \cdots = I_n(a) = 0 \).

Another characterization of centers of (1.1) is given by the following theorem.

**Theorem 1.4.** An element \( a = (a_1, a_2, \ldots) \in X \) belongs to the center set \( \mathcal{C} \) if and only if there is a sequence \( u_1, u_2, \ldots \) of 2\( \pi \)-periodic Lipschitz functions such...
that \( u_i(0) = 0 \) for any \( i \) and

\[
\sum_{i=1}^{\infty} a_i(x) t^{i+1} = \frac{-\sum_{k=1}^{\infty} u'_k(x) t^{k+1}}{1 + \sum_{k=1}^{\infty} (k+1) u_k(x) t^k}
\]
as formal power series in \( t \).

This result gives a parametrization of the center set \( C \subset X \) but leaves open the question on existence of such a sequence \( u_1, u_2 \ldots \) for a specific \( a \in X \).

1.4. Universal centers. The above results show that it is natural to study center problems for equations (1.6) somehow related to the Center Problem for (1.1). In this paper we solve one of such problems for the algebra \( A(F_1, F_2) \) with unit \( I \) of complex noncommutative polynomials in \( I \) and free noncommutative variables \( F_1, F_2 \) (i.e., there are no relations between \( F_1 \) and \( F_2 \)). As before \( A(F_1, F_2)[[t]] \) is the associative algebra of formal power series in \( t \) with coefficients from \( A(F_1, F_2) \). Clearly, there is a surjective homomorphism \( \phi: A(F_1, F_2)[[t]] \rightarrow A(X_1, X_2)[[t]] \) onto the algebra determined by (1.7) uniquely defined by conditions \( \phi(F_1) = X_1 \) and \( \phi(F_2) = X_2 \). Next, for \( a = (a_1, a_2, \ldots) \in X \) consider the equation on the circle \( S^1 \)

\[
\frac{dF}{dx} = (\sum_{i=1}^{\infty} a_i(x) t^i F_1 F_2^{i-1}) F.
\]

Let \( \tilde{\rho}: \mathbb{Z} \rightarrow G(F_1, F_2)[[t]] \) be the monodromy of (1.8) and \( \rho \) be the monodromy of (1.6) with \( A(X_1, X_2) \) as in Theorem 1.1. Then it is clear that

\[
\rho = \phi \circ \tilde{\rho}.
\]

Now, from Theorem 1.1 follows:

\textbf{Corollary 1.5.} The center set of equation (1.8) is a subset of \( C \).

We say that equation (1.1) determines a \textit{universal center}, if the monodromy of the corresponding equation (1.8) is trivial. The set \( \mathcal{U} \) of universal centers is, in a sense, a stable part of the center set \( C \). As we will see, \( \mathcal{U} \neq C \), in general. In our paper we describe some classes of equations (1.1) which determine universal centers. Some of our results reveal a connection with the so-called \textit{composition condition} whose role and importance for the Center Problem was studied in [AL], [BFY1], [BFY2], [Y] for the special case of Abel differential equations.

1.5. Characterization of universal centers in terms of iterated integrals. Let \( \omega(x) := \sum_{i=1}^{\infty} a_i(x) t^i F_1 F_2^{i-1} \). As before we identify functions on the circle with \( 2\pi \)-periodic functions on \( \mathbb{R} \). Recall that the fundamental solution of (1.8) is a
map $F: \mathbb{R} \to G(F_1, F_2)[[t]]$ such that $F(0) = I$ and $F'(x) = \omega(x) \cdot F(x)$. It can be presented by Picard iteration (cf. [Na]) in the form

$$F(x) := I + \sum_{i=1}^{\infty} \int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_k \leq x} \omega(s_k) \cdots \omega(s_1) \, ds_k \cdots ds_1$$

(1.10) Also, it is easy to see that

$$F(x) = \sum_{i=0}^{\infty} f_i(x; F_1, F_2) t^i$$

(1.11)

where $f_0 = I$ and the other $f_i$ are homogeneous polynomials of degree $i$ in $F_1$ and $F_2$ whose coefficients are locally Lipschitz functions in $x \in \mathbb{R}$. Then the monodromy of (1.8) is defined as $\tilde{\rho}(n) := F(2\pi n) = F(2\pi)^n$, $n \in \mathbb{Z}$.

One of the main results related to universal centers is

**Proposition 1.6.** Equation (1.1) determines a universal center (i.e. $F(2\pi) = I$) if and only if for all positive integers $i_1, \ldots, i_k$ and $k \geq 1$

$$\int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_k \leq 2\pi} a_{i_k}(s_k) \cdots a_{i_1}(s_1) \, ds_k \cdots ds_1 = 0.$$  

(1.12) In particular then,

$$\tilde{\alpha}_i(x) := \int_0^x a_i(s) \, ds, \quad i = 1, 2, \ldots,$$

are $2\pi$-periodic Lipschitz functions.

Next, let us consider equations

$$F'(x) = \omega_n(x) \cdot F(x) \quad \text{with} \quad \omega_n := \sum_{i=1}^{n} a_i(x) t^i F_1^{i-1} F_2^{i-1}.$$  

(1.13)

By $\tilde{\rho}_n: \mathbb{Z} \to G(F_1, F_2)[[t]]$ we denote the monodromy of (1.13).

**Proposition 1.7.** Equation (1.1) determines a universal center if and only if all $\tilde{\rho}_n$ are trivial. Moreover, the triviality of $\tilde{\rho}_n$ is equivalent to the fulfilment of equations (1.12) for any integers $1 \leq i_1, \ldots, i_k \leq n$, and any $k$.

Note also that the triviality of $\tilde{\rho}_n$ is equivalent to the fact that the equation

$$\frac{dv}{dx} = \sum_{i=1}^{n} a_i(x) v^{i+1}$$

(1.14)

determines a universal center.
Remark 1.8. There are many ways to determine universal centers. For instance, the set $\mathcal{U}$ of universal centers coincides with the set of centers of the equation

$$\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x)F_i v^{i+1}$$

with free noncommutative variables $F_i$.

1.6. Topological properties of universal centers. Let us introduce the Lipschitz map $A_n: S^1 \to \mathbb{C}^n$, $A_n(x) = (\tilde{a}_1(x), \ldots, \tilde{a}_n(x))$, and set $\Gamma_n := A_n(S^1)$. We require:

Definition 1.9. The polynomially convex hull $\hat{K}$ of a compact set $K \subset \mathbb{C}^n$ is the set of points $z \in \mathbb{C}^n$ such that if $p$ is any holomorphic polynomial in $n$ variables

$$|p(z)| \leq \max_{x \in K} |p(x)|.$$  

It is well known (see e.g. [AW]) that $\hat{K}$ is compact, and if $K$ is connected then $\hat{K}$ is connected.

The central result of this paper is:

Theorem 1.10. If equation (1.14) determines a universal center, then for any domain $U$ containing $\hat{\Gamma}_n$ the path $A_n: S^1 \to U$ is contractible in $U$.

As we will see below in certain cases the converse statement in Theorem 1.10 is also true. These cases include, e.g.,

(A) equations (1.14) with all $a_i$ trigonometric polynomials or
(B) equations (1.14) with all $a_i$ polynomials defined on a finite interval $[a, b) \subset \mathbb{R}$.

(In the latter case identifying $\mathbb{R}/((b - a)\mathbb{Z})$ with the unit circle we think of every $a_i$ as a piecewise continuous function on $S^1$ with a possible discontinuity at the image of $a$ and $b$ in $S^1$.)

Moreover, in these cases the topological condition of Theorem 1.10 is equivalent to a certain composition condition (see Corollaries 1.19, 1.20 below): in the case (A) equation (1.14) determines a universal center if and only if there are a trigonometric polynomial $q$ and polynomials $p_1, \ldots, p_n \in \mathbb{C}[z]$ such that

$$\tilde{a}_i = p_i \circ q, \quad 1 \leq i \leq n;$$

in the case (B) equation (1.14) determines a universal center if and only if there are polynomials $q, p_1, \ldots, p_n \in \mathbb{C}[z]$ such that

$$q(a) = q(b) \quad \text{and} \quad \tilde{a}_i = p_i \circ q, \quad 1 \leq i \leq n.$$  

(Recall that $\tilde{a}_i$ stands for the antiderivative of $a_i$.)
Remark 1.11. A statement similar to Theorem 1.10 is valid if we consider vanishing not all but only some special families of iterated integrals. For instance, let us consider the family of integrals
\begin{equation}
\int_0^{2\pi} (\tilde{a}_{i_1}(s))^{n_1} \cdots (\tilde{a}_{i_k}(s))^{n_k} a_{i_{k+1}}(s) ds
\end{equation}
for all possible nonnegative integers \(n_1, \ldots, n_k\) and \(1 \leq i_1, \ldots, i_{k+1} \leq n\). These integrals are called moments. They play an important role in the study of the Center Problem for Abel differential equations (see, e.g., [AL], [BFY1], [BFY2], [Y]).

One can prove the following result (we retain the notation of Theorem 1.10):

**Theorem.** Suppose that all moments defined in (1.15) equal 0. Then for any domain \(U\) containing \(\hat{\Gamma}_n\) the path \(A_n: S^1 \rightarrow U\) represents 0 in the first homology group \(H_1(U, \mathbb{Q})\).

Similar results are valid if we assume vanishing of families of higher-order moments for coefficients of equation (1.14). In a forthcoming paper we present these results and their converses, where possible.

1.7. Composition conditions. Since \(A_n\) is Lipschitz, \(\Gamma_n\) is of a finite linear measure. Then according to the result of Alexander [A], \(\Gamma_n \setminus \Gamma_n\) is a (possibly empty) pure one-dimensional analytic subset of \(\mathbb{C}^n \setminus \Gamma_n\). In particular, since the covering dimension of \(\Gamma_n\) is 1, the covering dimension of \(\hat{\Gamma}_n\) is 2. However, in general we do not know how to use this to get from Theorem 1.10 more information about \(\Gamma_n\). Thus we restrict our presentation to several special cases.

**Corollary 1.12.** Suppose \(\Gamma_n\) is triangulable and \(\hat{\Gamma}_n = \Gamma_n\). If equation (1.14) determines a universal center then the path \(A_n: S^1 \rightarrow \Gamma_n\) is contractible in \(\Gamma_n\). Moreover, its contractibility is equivalent to the factorization \(A_n = A_{1n} \circ A_{2n}\) where \(A_{2n}: S^1 \rightarrow G_n\) is a continuous map into a finite tree \(G_n \subset \mathbb{R}^N\), and \(A_{1n}: G_n \rightarrow \Gamma_n\) is a finite continuous map.

The converse result requires a much stronger than just triangulability condition on \(\Gamma_n\) that is described by:

**Definition 1.13.** A compact curve \(C \subset \mathbb{R}^N\) is called Lipschitz triangulable if
(a) \(C = \bigcup_{j=1}^i C_i\) and for \(i \neq j\) the intersection \(C_i \cap C_j\) consists of at most one point;
(b) There are Lipschitz embeddings \(f_i: [0, 1] \rightarrow \mathbb{R}^N\) such that \(f_i([0, 1]) = C_i\);
(c) The inverse maps \(f_i^{-1}: C_i \rightarrow \mathbb{R}\) are locally Lipschitz on \(C_i \setminus (f_i(0) \cup f_i(1))\).

**Theorem 1.14.** Suppose \(\Gamma_n\) is Lipschitz triangulable, \(\hat{\Gamma}_n = \Gamma_n\), and \(A_n: S^1 \rightarrow \Gamma_n\) is contractible in \(\Gamma_n\). Suppose also that \(A_n^{-1}(x)\) is countable for any \(x \in \Gamma_n\). Then the corresponding equation (1.14) determines a universal center.
Remark 1.15. We will see from the proof that under the hypotheses of Theorem 1.14 the map $A_{2n}: S^1 \to \mathbb{R}^{N_n}$ is Lipschitz, $G_n$ is Lipschitz triangulable, and $A_{1n}^{-1}: G_n \to \Gamma_n$ is locally Lipschitz outside a finite set.

Example 1.16. (1) If $A_n: S^1 \to \mathbb{C}^n$ is nonconstant analytic, then $\Gamma_n$ is Lipschitz triangulable, and $A_n^{-1}(x)$ is finite for any $x \in \Gamma_n$.
(2) To satisfy the second hypothesis of Corollary 1.12, one can suppose, e.g., that $\Gamma_n$ belongs to a compact $K_n$ in a $C^1$-smooth manifold $M_n$ with no complex tangents such that $\hat{K}_n = K_n$. For instance, one can take any compact $K_n$ in $M_n = \mathbb{R}^n$ (for the proof see e.g. [AW, Th. 17.1]).

Next, we consider maps into one-dimensional complex spaces. Suppose that one of the following two conditions is satisfied:

(1) $\Gamma_n \subset X$ where $X$ is a closed one-dimensional complex analytic subset of a domain $U \subset \mathbb{C}^n$ such that $U = \cup_j K_j$ with $K_j \subset K_{j+1}$, and $\hat{K}_j = K_j$ for any $j$.
(2) $\Gamma_n \subset X$ where $X \subset \mathbb{C}^n$ is a connected one-dimensional complex space with $\dim_{\mathbb{C}} H^1(X, \mathbb{C}) < \infty$, and for each $\delta \in H^1(X, \mathbb{C})$ there is a holomorphic 1-form with polynomial coefficients $\omega_{\delta}$ such that $\delta(c) = \int_c \omega_{\delta}$ for any $c \in H^1(X, \mathbb{C})$.

Corollary 1.17. Let $\Gamma_n \subset X$ where $X$ satisfies either condition (1) or (2). Suppose that there is a continuous map $\tilde{A}_n: R \to X$ of an open neighbourhood $R \subset \mathbb{C}$ of $S^1$ such that $\tilde{A}_n|S^1 = A_n$, and $\tilde{A}_n^{-1}(x)$ is finite for any $x \in \tilde{A}_n(R)$. Then the corresponding equation (1.14) determines a universal center if and only if $A_n = A_{1n} \circ A_{2n}$ where $A_{2n}: S^1 \to \mathbb{D}$ is a continuous map into the unit disk $\mathbb{D} \subset \mathbb{C}$, locally Lipschitz outside a finite set, and $A_{1n}: \mathbb{D} \to \mathbb{C}^n$ is a finite holomorphic map.

Remark 1.18. (1) In the proof of Corollary 1.17 we give separate arguments for assumptions (1) and (2). However, it is possible to show that condition (1) can be reduced to the case of condition (2).
(2) We will also show that if $\tilde{A}_n$ is holomorphic then $A_{2n}$ can be extended to a holomorphic map of an open neighbourhood of $S^1$ into $\mathbb{D}$.
(3) Note that the hypotheses of the corollary are valid for $X$ a complex algebraic curve.
(4) In the case when $A_n: S^1 \to X$ is a Lipschitz embedding and $\Gamma_n = A_n(S^1)$ lies outside singularities of $X$, one can obtain under the hypotheses of Corollary 1.17 that $A_n = A_{1n} \circ A_{2n}$ where $A_{2n}: S^1 \to \mathbb{D}$ is a Lipschitz embedding and $A_{1n}: \mathbb{D} \to \mathbb{C}^n$ is a holomorphic map, one-to-one outside a finite set.

Corollary 1.19. Suppose that the coefficients $a_1, \ldots, a_n$ in (1.14) are trigonometric polynomials. Then (1.14) determines a universal center if and only if there are a trigonometric polynomial $q$ and polynomials $p_1, \ldots, p_n \in \mathbb{C}[z]$ such that

$$\bar{a}_i(x) = p_i(q(x)), \quad x \in S^1, \quad 1 \leq i \leq n.$$
**Corollary 1.20.** Suppose that the coefficients $a_1, \ldots, a_n$ in (1.14) are polynomials defined on an interval $[a, b) \subset \mathbb{R}$ (see Section 1.6 above). Then (1.14) determines a universal center if and only if there are polynomials $q, p_1, \ldots, p_n \in \mathbb{C}[z]$ such that

$$q(a) = q(b) \quad \text{and} \quad \tilde{a}_i(x) = p_i(q(x)), \quad x \in [a, b], \quad 1 \leq i \leq n.$$ 

### 1.8. Finiteness results.

The first result of this section shows that in certain cases triviality of $\tilde{\rho}_N$ for some integer $N$ implies that of $\tilde{\rho}$ (i.e., equation (1.1) determines a universal center if and only if the truncated equation (1.14) with $n := N$ determines such a center).

**Theorem 1.21.** Suppose that the coefficients $a_1, a_2, \ldots$ in equation (1.1) are continuous functions such that each $\tilde{a}_i := \int_0^1 a_i(s)ds$ is $2\pi$-periodic. Let $A_k: S^1 \to \mathbb{C}^k$, $A_k(x) := (\tilde{a}_1(x), \ldots, \tilde{a}_k(x))$, and $\Gamma_k = A_k(S^1)$. Suppose that there is an integer $n$ such that $\Gamma_n$ is a piecewise smooth curve, the number of critical values of $A_n$ is finite, and $A_n^{-1}(x)$ is finite for any $x \in \Gamma_n$. If also $\hat{\Gamma}_k = \Gamma_k$ for any $k$, then there is an integer $N \geq 1$ such that the triviality of $\tilde{\rho}_N$ implies the triviality of $\tilde{\rho}$.

**Remark 1.22.** The hypotheses of the theorem are fulfilled if, e.g., all $a_i$ are real continuous functions and some $\tilde{a}_n$ is a non-zero analytic function.

Let us formulate another finiteness result.

**Theorem 1.23.** Suppose that there are $2\pi$-periodic Lipschitz functions $b_1, \ldots, b_k$ such that each coefficient $a_i$ in equation (1.1) is the uniform limit of functions of the form $\sum_{j=1}^k p_{ij}(b_1, \ldots, b_k) \cdot b_j'$, where $p_{ij} \in \mathbb{C}[z_1, \ldots, z_k]$ are holomorphic polynomials and $b_j'$ is the derivative of $b_j$. If the equation $v'(x) = \sum_{i=1}^k b_i(x)v_i+1$ determines a universal center, then (1.1) determines such a center, as well.

**Corollary 1.24.** Let now $H(x, y) \in \mathbb{C}[x, y]$ be a homogeneous polynomial. For any holomorphic functions $P_1, P_2$ defined in an open neighbourhood of $0 \in \mathbb{C}$ we define $A(x, y) := P_1(H(x, y))$, and $B(x, y) := P_2(H(x, y))$. Then the vector field

$$\begin{cases}
\dot{x} = -y - xy \frac{\partial A(x, y)}{\partial x} + x^2 \frac{\partial A(x, y)}{\partial y} - yB(x, y) \\
\dot{y} = x - y^2 \frac{\partial A(x, y)}{\partial x} + xy \frac{\partial A(x, y)}{\partial y} + xB(x, y)
\end{cases}$$

determines a center.

The next result deals with symmetries of vector fields. Since the center problem is invariant under rotations, it suffices to consider a particular case of the symmetry. This result first was obtained by Poincaré [P] by a different argument.
COROLLARY 1.25. Let \( F(x, y), G(x, y) \) be analytic functions defined in an open neighbourhood of \( 0 \in \mathbb{R}^2 \) such that their Taylor expansions at 0 do not contain constant and linear terms. Suppose \( F(x, -y) = -F(x, y) \) and \( G(x, -y) = G(x, y) \). Then the vector field \( \dot{x} = -y + F(x, y), \dot{y} = x + G(x, y) \) determines a center.

Example 1.26. The polynomial vector field with homogeneous \( F(x, y) \) and \( G(x, y) \) of degree 2 can be written in the Dulac-Kapteyn form with 5 parameters instead of 6 by using an appropriate rotation of the plane:

\[
\begin{align*}
\dot{x} &= -y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2 \\
\dot{y} &= x + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2.
\end{align*}
\] (1.16)

It has been established by H. Dulac that the center set for (1.16) consists of 4 components described as follows (cf. [Si]):

1. **Lotka-Volterra component:** \( \lambda_3 = \lambda_6 \);
2. **Symmetric component:** \( \lambda_2 = \lambda_5 = 0 \);
3. **Hamiltonian component:** \( \lambda_4 = \lambda_5 = 0 \);
4. **Darboux component:** \( \lambda_5 = \lambda_4 + 5\lambda_3 - 5\lambda_6 = \lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2 = 0 \).

Next, passing to polar coordinates in (1.16) we obtain

\[
\begin{align*}
\frac{dr}{d\phi} &= \frac{f(\phi)r^2}{1 + g(\phi)r} \\
\end{align*}
\] (1.17)

where \( f(\phi) := \frac{x F(x, y) + y G(x, y)}{r} \), \( g(\phi) := \frac{x G(x, y) - y F(x, y)}{r} \), \( x = r \cos \phi, \ y = r \sin \phi \). Let us investigate the universal center conditions for equation (1.17). If (1.17) determines a universal center then the first integrals \( I(f) \) and \( I(fg) \) of \( f \) and \( fg \) satisfy the relations of Corollary 1.19. This easily implies that \( I(f) \) and \( g \) satisfy the same relations. Going back to the functions \( F \) and \( G \) we obtain a planar polynomial vector field of the form of Corollary 1.24. Now \( H \) is a linear homogeneous polynomial, and \( P_1, P_2 \) are linear polynomials without constant terms. A simple computation shows that in this case for the equation (1.16) we have:

(A) \( \lambda_3 - \lambda_6 = 4\lambda_2 + \lambda_5 = 4\lambda_3 + \lambda_4 = 0 \) (a linear two-dimensional subspace in the Lotka-Volterra component);

(B) \( \lambda_2 = \lambda_5 \) (Symmetric component).

Remark 1.27. We can further simplify equation (1.17) by the application of the Cherkas transformation [Che], \( r(\phi) = \frac{\rho(\phi)}{1 - g(\phi)r(\phi)} \). Then we get the Abel differential equation

\[
\frac{dp}{d\phi} = p(\phi)p^2 + q(\phi)p^3
\] (1.18)

where \( p(\phi) = f(\phi) + g'(\phi) \) and \( q(\phi) = -f(\phi)g(\phi) \). Moreover, since the Cherkas transformation and its inverse are regular at \( r = \rho = 0 \), the center sets of
(1.17) and (1.18) coincide. Let \( P(\phi) = \int_0^\phi p(s)ds \) and \( Q(\phi) = \int_0^\phi q(s)ds \). In case equation (1.18) determines a center, \( P \) and \( Q \) are trigonometric polynomials. It was shown by Blinov [Bl] that the Hamiltonian and the Symmetric components after performing the Cherkas transformation determine universal centers for (1.18) and the other two components do not. Here for the Hamiltonian component we have \( Q = -\frac{3}{16}P^2 + \frac{\lambda}{2}P \), and for the Symmetric component \( P \) and \( Q \) are polynomials in \( \sin(\phi) \). Then the required result follows from Corollary 1.19.

Thus the Hamiltonian component (which is a universal center for (1.18)) can be obtained as a result of an additional nonlinear change of variables in (1.17).

In the Appendix following [Br1] we describe an algebraic model for the Center Problem.

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2. Proofs of results of sections 1.2 and 1.3.

2.1. We recall some definitions and results used in the solution of linear ODEs with \( L^\infty \)-coefficients by Picard iteration (see e.g. [Ru, Ch. 7]):

(a) A map \( f: I \to \mathbb{R}^n \) of a closed interval \( I \subset \mathbb{R} \) is called Lipschitz if \( \text{dist}(f(x), f(y)) \leq C|x-y| \) for any \( x, y \in I \). Here \( \text{dist}(\cdot, \cdot) \) is the Euclidean distance on \( \mathbb{R}^n \).

(b) A map \( f: J \to \mathbb{R}^n \) of an open interval \( J \subset \mathbb{R} \) is called locally Lipschitz if it is Lipschitz on each closed subinterval of \( J \).

(c) A function \( f: I \to \mathbb{R} \) is said to be absolutely continuous if \( \text{mes}(f(E)) = 0 \) provided that \( \text{mes}(E) = 0 \). Here \( \text{mes}(V) \) is the Lebesgue measure of \( V \subset \mathbb{R} \).

Any Lipschitz \( f: I \to \mathbb{R} \) is absolutely continuous, differentiable almost everywhere and \( f' \in L^\infty(I) \). The first integral of a function from \( L^\infty(I) \) is a Lipschitz function. If \( f \) is absolutely continuous and \( f' \) is zero almost everywhere, then \( f \) is constant. Thus the derivative of an antiderivative of any function from \( L^\infty(I) \) almost everywhere coincides with this function. Moreover, any Lipschitz \( f \) is an antiderivative of \( f' \). Therefore the Newton-Leibnitz formula \( \int_a^b f'(t)dt = f(b) - f(a) \) holds. Similar results are valid for complex-valued Lipschitz functions.

2.2. Proof of Theorem 1.1. Substituting \( r(x) = t \cdot v(x) \) we transfer (1.1) to

\[
(2.1) \quad v' = \sum_{i=1}^{\infty} a_i(x) v^{i+1}.
\]
Note that equation (1.1) determines a center if and only if (2.1) determines a center for any sufficiently small $t$. Multiplying (2.1) by $\epsilon^{k-1}$ we get

\[(\epsilon^{k})' = \sum_{i=1}^{\infty} (k a_i(x) t^i) \epsilon^{i+k}\]

Let $V$ be the linear space spanned by vectors $e_i = (0, \ldots, 0, 1, 0, \ldots)$ with 1 at the $i$th place. We set $y_i = \epsilon^i e_i$ and $Y = (y_1, y_2, \ldots)$. Combining equations (2.2) for all $k$ we obtain a system of linear ODEs

\[Y' = (\sum_{i=1}^{\infty} A_i a_i(x) t^i) Y\]

where $A_i: V \to V$ are linear operators. We identify $V$ with the algebra $\mathbb{C}[[z]]$ of formal power series with complex coefficients so that $e_n$ coincides with $z^{n-1}$. Let $D, L: \mathbb{C}[[z]] \to \mathbb{C}[[z]]$ be the differentiation and the left translation operators defined on $f(z) = \sum_{k=0}^{\infty} c_k z^k$ by

\[
(Df)(z) := \sum_{k=0}^{\infty} (k+1) c_{k+1} z^k, \quad (Lf)(z) := \sum_{k=0}^{\infty} c_{k+1} z^k.
\]

**Lemma 2.1.** It is true that $A_i = DL^{i-1}$. Thus (2.3) acquires the form

\[Y' = (\sum_{i=1}^{\infty} a_i(x) t^i DL^{i-1}) Y\]

**Proof:** Applying $DL^{i-1}$ to $z^k$ we have

\[DL^{i-1}(z^k) = D(z^{k-1}) = \begin{cases} 0, & \text{if } k \leq i - 1 \\ (k-i+1) z^{k-i}, & \text{if } k > i \end{cases} \]

By definition, $A_i = (a_{i,j}^k)$ is an infinite matrix such that $a_{i,j}^s = s$ for any $s \in \mathbb{Z}_+$ and $a_{i,j}^s = 0$ otherwise. Thus we have

\[A_i(e_{k+1}) = \sum_{j} a_{j,k+1}^i e_j = (k-i+1)e_{k-i+1}.\]

Further, by $A(D, L)[[t]]$ we denote the associative algebra of formal power series $R(t) = \sum_{k=0}^{\infty} p_k(D, L, I) t^k$ where $p_k(D, L, I)$ are complex polynomials in variables $D$, $L$ and $I$, the identity operator on $\mathbb{C}[[z]]$. Let $G(D, L)[[t]]$ be the
group of invertible elements of \( \mathcal{A}(D)[][t] \), and \( \rho': \mathbb{Z} \to G(D)[[t]] \) be the monodromy of (2.4). In what follows \( \mathcal{A}(X_1,X_2) \) is the algebra introduced in Theorem 1.1, \( \mathcal{A}_0(X_1,X_2) \) is its subalgebra of complex polynomials in \( X_1 \) and \( X_2 \) only, and \( \rho \) is the monodromy of the corresponding equation (1.6).

**Proposition 2.2.** There is an isomorphism \( \Phi: \mathcal{A}(D)[[t]] \to \mathcal{A}(X_1,X_2)[[t]] \) uniquely defined by \( \Phi(D) = X_1 \) and \( \Phi(L) = X_2 \). In particular, \( \rho = \Phi \circ \rho' \).

**Proof.** Let \( \mathcal{A}(D,L) \) be the associative algebra of complex polynomials in variables \( D,L,I \). By \( \mathcal{A}_0(D,L) \) we denote the subalgebra of \( \mathcal{A}(D,L) \) generated by \( D \) and \( L \) only. Let \( \mathbb{C}(I) \subset \mathcal{A}(D,L) \) be the one-dimensional central subalgebra generated by \( I \). Then \( \mathcal{A}(D,L) = \mathcal{A}_0(D,L) \oplus \mathbb{C}(I) \).

**Lemma 2.3.**

\[
[D,L] := DL - LD = -L^2.
\]

**Proof.** It suffices to check the identity for elements \( z^k \) with \( k \geq 2 \). Here we have

\[
DL(z^k) = D(z^{k-1}) = (k-1)z^{k-2}, \quad LD(z^k) = L(kz^{k-1}) = kz^{k-2}.
\]

Therefore

\[
[D,L](z^k) = (k-1)z^{k-2} - kz^{k-2} = -z^{k-2} = -L^2(z^k).
\]

Let \( \mathcal{A}_0(F_1,F_2) \) be the algebra of complex polynomials in free noncommutative variables \( F_1,F_2 \). Then there is a homomorphism \( \phi: \mathcal{A}_0(F_1,F_2) \to \mathcal{A}_0(D,L) \) uniquely defined by conditions \( \phi(F_1) = D, \phi(F_2) = L \). From Lemma 2.3 it follows that \( [F_1,F_2] + F_2^2 \in \text{Ker}(\phi) \). Let \( J \subset \mathcal{A}(F_1,F_2) \) be the two-sided ideal generated by \( [F_1,F_2] + F_2^2 \), \( \mathcal{A}_0(X_1,X_2) := \mathcal{A}_0(F_1,F_2)/J \) be the quotient algebra and \( \gamma: \mathcal{A}_0(F_1,F_2) \to \mathcal{A}_0(X_1,X_2) \) be the quotient homomorphism uniquely defined by conditions \( \gamma(F_1) = X_1 \) and \( \gamma(F_2) = X_2 \). Clearly, there is a surjective homomorphism \( \tilde{\phi}: \mathcal{A}_0(X_1,X_2) \to \mathcal{A}_0(D,L) \) such that \( \phi = \tilde{\phi} \circ \gamma \).

**Lemma 2.4.** \( \tilde{\phi}: \mathcal{A}_0(X_1,X_2) \to \mathcal{A}_0(D,L) \) is an isomorphism.

**Proof.** Note that any polynomial in \( X_1 \) and \( X_2 \) can be transformed by the basic relation (1.7) to the canonical form \( \sum_{1 \leq i,j \leq n} a_{ij}X_1^iX_2^j, a_{ij} \in \mathbb{C} \). Suppose, to the contrary, that \( \text{Ker}(\tilde{\phi}) \neq 0 \). Then there is a polynomial \( P(X_1,X_2) \) whose coefficients in the canonical form not all are zeros such that \( P(D,L) = 0 \). We write \( P(X_1,X_2) = \sum_{i=1}^n F_i(X_1,X_2) \), where \( F_i(X_1,X_2) = \sum_{k=0}^k a_{ik}X_1^iX_2^{k-1} \) is the homogeneous component of degree \( k \). Using the identity

\[
D^jL^j(z^s) = D^j(z^{s-j}) = (s-j)(s-j-1) \cdots (s-j+i-1)z^{s-i-j}
\]
we obtain for $s \geq n$

$$0 = P(D, L)(z^s) = \sum_{k=1}^{n} \left[a_{0k} + \sum_{j=0}^{k-1} a_{k-jj}(s-j) \cdots (s-k+1)\right]z^{s-k}.$$ 

In particular, for any $1 \leq k \leq n$ and any sufficiently big positive integers $s$ we have

$$a_{0k} + \sum_{j=0}^{k-1} a_{k-jj}(s-j) \cdots (s-k+1) = 0.$$ 

Then the same identity is valid for any $s \in \mathbb{R}$. Substituting in this identity $s = k - 1$ we get $a_{0k} = 0$. Then dividing it by $(s-k+1)$ and substituting $s = k - 2$ we obtain $a_{1k-1} = 0$. Proceeding similarly we finally have $a_{ij} = 0$ for all $i, j$. This means that $P(X_1, X_2) = 0$. This contradiction proves the lemma.

In order to finish the proof of the proposition it remains to determine

$$\Phi(\sum_{k=0}^{\infty} p_k(D, L, I)^{tk}) = \sum_{k=0}^{\infty} p_k(X_1, X_2, I)^{tk}$$

for $\sum_{k=0}^{\infty} p_k(D, L, I)^{tk} \in A(D, L)[[t]]$.

Then Lemma 2.4 implies that $\Phi$ is an algebraic isomorphism. The factorization $\rho = \Phi \circ \rho'$ easily follows from Picard iteration for solutions of (2.4) and (1.6).

The proof of Proposition 2.2 is complete.

This proposition implies that it suffices to prove Theorem 1.1 for equation (2.4) and its monodromy $\rho'$. So let $\omega(x) := \sum_{i=1}^{\infty} a_i(x)L_i^{-1}$ with $a = (a_1, a_2, \ldots) \in X$. The fundamental solution $Y$ of (2.4) can be obtained by Picard iteration:

$$Y(x; a) = I + \sum_{i=1}^{\infty} \int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_n \leq x} \omega(s_n) \cdots \omega(s_1) \, ds_n \cdots ds_1.$$ (2.5)

Let us introduce the (locally Lipschitz) functions

$$I_{i_1, \ldots, i_k}(x; a) := \int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_k \leq x} a_{i_k}(s_k) \cdots a_{i_1}(s_1) \, ds_k \cdots ds_1, \quad x \in \mathbb{R}. \quad (2.6)$$

Then it is easy to see that

$$Y(x; a) = I + \sum_{i=1}^{\infty} p_i(x; a; D, L)t^i$$

where

$$p_i(x; a; D, L) = \sum_{i_1+\cdots+i_k=i} I_{i_1, \ldots, i_k}(x; a)(DL^{i-1}) \cdots (DL^{i-1}), \quad i \geq 1. \quad (2.7)$$
Now, the solution of (2.4) with an initial value \( f \in \mathbb{C}[[z]] \) equals \( Y(x; a)(f) \). Also, by definition, the solution of (2.4) with the initial value \( \sum_{i=0}^{\infty} r_i z^i \) equals

\[
Y(x; a) \cdot (\sum_{i=0}^{\infty} r_i z^i) = \sum_{i=0}^{\infty} y(x; t; r; a) r_i z^i
\]

where \( y(x; t; r; a) \) is the solution of (2.1) with \( y(0; t; r; a) = r \).

We are ready to prove Theorem 1.1. Suppose first that equation (1.1) corresponding to \( a = (a_1, a_2, \ldots) \in X \) determines a center. Then \( y(2\pi; t; r; a) = 0 \) for any sufficiently small \( r \). In particular, for such \( r \) we have

\[
(Y(2\pi; a) - I) (\sum_{i=0}^{\infty} r_i z^i) = 0.
\]

For a fixed \( \epsilon > 0 \) let \( W_\epsilon \subset \mathbb{C}[[z]] \) be the complex vector space generated by series \( \sum_{i=0}^{\infty} r_i z^i \) for all \( |r| < \epsilon \).

**Lemma 2.5.** For any \( h = \sum_{i=0}^{\infty} h_i z^i \) and any \( N \in \mathbb{N} \) there is an element \( h_N \in W_\epsilon \) such that the first \( N \) terms in the series decomposition of \( h - h_N \) are zeros.

**Proof.** Let \( r_1, \ldots, r_N \) be pairwise distinct numbers satisfying \( |r_k| < \epsilon \) for any \( k \). We set \( v(r_k) = (r_k, r_k^2, \ldots, r_k^N) \). It is well known that the vectors \( v(r_k) \) are linearly independent. In particular, there are \( c_1, \ldots, c_N \in \mathbb{C} \) such that \( \sum_{k=1}^{N} c_k v(r_k) = h_N \) where \( h_N = (h_0, \ldots, h_{N-1}) \). Let us define \( h_N(z) = \sum_{k=1}^{N} c_k (\sum_{i=0}^{\infty} r_k^i z^i) \). Then \( h_N \) satisfies the required property. \( \square \)

Now, to prove that the monodromy \( \rho' \) of equation (2.4) corresponding to \( a \in X \) is trivial we should check that \( (Y(2\pi; a) - I)(h) = 0 \) for any \( h \in \mathbb{C}[[z]] \). Suppose, to the contrary, that there is \( h(z) = \sum_{i=0}^{\infty} h_i z^i \) such that \( (Y(2\pi; a) - I)(h) / h = 0 \). Then by definition \( (Y(2\pi; a) - I)(h) = \sum_{0 \leq p, q \leq \infty} b_{pq} z^p \in \mathbb{C}[[z, t]] \) and there are integers \( p, q \) such that \( b_{pq} \neq 0 \). We set \( N := p + q + 1 \). Applying Lemma 2.5 we can find \( h_N \in W_\epsilon \) (with a sufficiently small \( \epsilon \)) such that the first \( N \) terms in the decomposition of \( h - h_N \) are zeros. Moreover, condition (2.9) implies that \( (Y(2\pi; a) - I)(h_N) = 0 \). Thus using formula (2.7) we get \( p_n(2\pi; a, D, L)(h - h_N) = \sum_{k \geq (N-n)_+} c_k z^k \) for some \( c_k \in \mathbb{C} \); here \( (N-n)_+ := \max (N-n, 0) \). Finally, we have

\[
\sum_{0 \leq k, n \leq \infty} b_{kn} z^n t^p = (Y(2\pi; a) - I)(h) = (Y(2\pi; a) - I)(h - h_N) = \sum_{n=1}^{\infty} \sum_{k=0}^{(N-n)_+} c_k z^k t^p.
\]

In particular, \( b_{kn} = c_k \) if \( k \geq (N-n)_+ \) and is 0 otherwise. This implies that \( b_{pq} = 0 \) which contradicts to our assumption.

Thus we proved that if \( a \in \mathcal{C} \) then it belongs also to \( \mathcal{C}_A \) (cf. Theorem 1.1). The converse implication follows directly from (2.8).

The proof of Theorem 1.1 is complete. \( \square \)
Proof of Theorem 1.2. Using (2.7), (2.8) and the fact that the triviality of the monodromy ρ is equivalent to the identities y(2π; τ; r; a) − r = 0 for any sufficiently small r, we conclude that in order to find equations determining the center set for (1.1) we should apply each pk(2π; a; D, L) to \( \sum_{i=0}^{\infty} r^{i+1} z^i \), then substitute z = 0 and equate the latter to 0. Now, it is clear that we obtain the same equations if we apply \( p_n(2\pi; a; D, L) \) to \( z^n \). From here and (2.7) we obtain for \( i_1 + \cdots + i_k = n \),

\[
c_{i_1, \ldots, i_k} := (DL^{k-1}) \cdots (DL^{i_1-1})(z^n) = (n - i_1 + 1)(n - i_1 - i_2 + 1) \cdots 1.
\]

Also, the expression \( P(a)(r) := y(2\pi; 1; r; a) \) is the first return map for equation (1.1). From above it follows that

\[
P(a)(r) = r + \sum_{n=1}^{\infty} c_n r^{n+1} \quad \text{where} \quad c_n = \sum_{i_1 + \cdots + i_k = n} c_{i_1, \ldots, i_k} I_{i_1, \ldots, i_k}(a). \quad \square
\]

Proof of Corollary 1.3. The proof of part (a) follows directly from the above formula for the first return map. Let us prove part (b).

By \( \pi_k: X \rightarrow \prod_{j=1}^{k} X_j \) we denote the natural projection to the first k coordinates. According to part (a) the set \( \tilde{C}_n \) is given by equations \( I_k(a) = f_k(a), 1 \leq k \leq n \), where \( I_k \) depends on \( a_k \) and \( f_k \) depends on \( a_1, \ldots, a_{k-1} \). Let \( \tilde{C}_n \) be the subset of \( \prod_{j=1}^{n} X_j \) defined by these equations. Then clearly \( \tilde{C}_n = \pi_n^{-1}(\tilde{C}_n) \). Thus in order to prove the corollary it suffices to prove that \( \tilde{C}_n \subset \prod_{j=1}^{n} X_j \) is a closed complex submanifold of codimension \( n \) containing 0. We prove this by induction on \( n \).

If \( n = 1 \), then \( \tilde{C}_1 \) is determined by the equation \( I_1(a) := \int_0^{2\pi} a_1(s) \, ds = 0 \). Since \( I_1 \) is a continuous linear functional, \( \tilde{C}_1 \) coincides with the complex hyperplane \( \text{Ker}(I_1) \subset X_1 \). Thus the required statement holds. Suppose now that the statement is proved for \( k - 1 \), let us prove it for \( k \). It is easy to see that \( \tilde{C}_k \) is a subset of \( \tilde{C}_{k-1} \times X_k \) determined by the equation \( I_k(a) = f_k(a) \). Since \( I_k|_{X_k} \) is a continuous linear functional, we can decompose \( X_k = E_k \oplus I_k \), where \( E_k = \text{Ker}(I_k|_{X_k}) \) and \( I_k \) is the one-dimensional vector space generated by a vector \( e_k \) such that \( I_k(e_k) = 1 \). Further, let us consider the set \( R_k := \tilde{C}_{k-1} \times E_k \). It is clear that \( R_k \subset \text{Ker}(I_k) \) and \( \tilde{C}_{k-1} \times X_k = R_k \oplus I_k \). In particular, for any \( a \in \tilde{C}_{k-1} \times X_k \) we have \( a = w + ve_k \) for some \( w \in R_k, v \in \mathbb{C} \). Thus in the definition of \( \tilde{C}_k \) we have

\[
v = I_k(a) = -f_k(a) = -f_k(w).
\]

This shows that \( \tilde{C}_k \) is the graph of the function \(-f_k: R_k \rightarrow \mathbb{C}\). But according to the induction hypothesis, \( R_k \subset (\prod_{j=1}^{k-1} X_j) \times E_k \) is a closed complex submanifold of codimension \( (k - 1) \) containing 0. From here it follows that \( \tilde{C}_k \subset \tilde{C}_{k-1} \times X_k \) is a closed complex submanifold of codimension 1 containing 0. This implies the required result and proves part (b). Let us prove part (c).
By definition the linear term of the Taylor expansion of \( c_n \) at \( 0 \in X \) equals \( I_n \). Note that the tangent space to \( C_n \) at \( 0 \in X \) is determined as the set of zeros of the linear terms of \( c_1, \ldots, c_n \), that is, by equations \( I_1(a) = \ldots = I_n(a) = 0 \). \( \square \)

**Proof of Theorem 1.4.** Let \( v(x; r; a) \), \( x \in [0, 2\pi] \), be the Lipschitz solution of equation (1.1) corresponding to \( a = (a_1, a_2, \ldots) \in X \) with initial value \( v(0; r; a) = r \). From Picard iteration it follows that

\[
(2.10) \quad v(x; r; a) = r + \sum_{i=1}^{\infty} v_i(x; a) r^{i+1}
\]

where each \( v_i(x; a) \) is a Lipschitz function on \([0, 2\pi]\) and the series converges uniformly in the domain \( 0 \leq x \leq 2\pi, |r| \leq \bar{r} \) for a sufficiently small positive \( \bar{r} \). Assuming that \( a \in C \) we get \( v_i(0; a) = v_i(2\pi; a) = 0 \) for any \( i \). Now the inverse function theorem implies that there is a function \( u(x; \rho; a) := \rho + \sum_{i=1}^{\infty} u_i(x; a) r^{i+1} \) where each \( u_i(x; a) \) is a \( 2\pi \)-periodic Lipschitz function, \( u_i(0; a) = 0 \), and the series converges uniformly in the domain \(-\infty < x < \infty, |ho| \leq \bar{\rho} \) for a sufficiently small positive \( \bar{\rho} \), such that \( u(x; v(x; r; a); a) \equiv r \) for all sufficiently small \( r \). Differentiating this equation in \( x \) we obtain

\[
v'(x; r; a) = -\frac{\sum_{k=1}^{\infty} u_k'(x; a) v(x; r; a)^{k+1}}{1 + \sum_{k=1}^{\infty} (k+1) u_k(x; a) v(x; r; a)^k}.
\]

This and equality \( v'(x; r; a) = \sum_{i=1}^{\infty} a_i(x) v(x; r; a)^{i+1} \) imply the required identity of power series.

Conversely, from the identity of Theorem 1.4 arguing in the reverse order we find a formal solution of (1.1) corresponding to \( a \in X \) of the form (2.10) where \( v_i \) are polynomials in \( u_1, \ldots, u_i, v_1, \ldots, v_{i-1} \) without constant terms. In particular, each \( v_i \) is a \( 2\pi \)-periodic Lipschitz function equals 0 at 0. But since \( a \in X \), there is also the strong solution of the corresponding equation (1.1) given by (2.10). It is easy to see that these two solutions coincide. This implies the identity \( v(0; r; a) = v(2\pi; r; a) = r \) for any sufficiently small \( r \). \( \square \)

**3. Proofs of Results of Sections 1.4 and 1.5.**

**Proof of Corollary 1.5.** It follows directly from the factorization \( \rho = \phi \circ \bar{\rho} \). \( \square \)

**Proof of Proposition 1.6.** We set \( R_i := t F_1 F_2^{i-1} \). Let \( \mathcal{A} \) be the algebra of complex polynomials in \( R_1, R_2, \ldots \). The main point of the proof is the following:

**Lemma 3.1.** Suppose \( P(R_1, \ldots, R_k) = \sum a_{i_1, \ldots, i_k} R_{i_1} \cdots R_{i_k} \in \mathcal{A} \) is zero. Then all \( a_{i_1, \ldots, i_k} \) are zeros.

**Proof.** The maximal integer \( l = \sum_{i=1}^{k} i_s \) for indices in the expansion of \( P \) will be called the degree of \( P \). We prove the lemma by induction on \( l \).
For \( l = 1 \) we have \( P = c_1R_1 \) and the statement is obvious. Suppose that we already proved the lemma for \( l \leq s \), let us prove it for \( l = s + 1 \). We can write \( P = P_1 + \cdots + P_k \) where \( P_i \) contains all terms of \( P \) with \( R_i \) on the right. Since \( F_1, F_2 \) are free noncommutative variables, the identity \( P = 0 \) implies that \( P_1 = \ldots = P_k = 0 \). Now every \( P_i = Q_i \cdot R_i \) where \( Q_i \) is a polynomial of degree \( \leq s \) satisfying the hypothesis of the lemma. Thus by the induction hypothesis all coefficients of every \( P_i \) are zeros.

Further, it follows from formula (1.10) for the fundamental solution \( F(x) \) of equation (1.8) that \( F(2\pi) = \sum_{i=0}^{\infty} q_i(2\pi, F_1, F_2) \) where \( q_0 = I \) and others \( q_i \) are polynomials in \( R_1, R_2, \ldots \) of the degree \( i \). Moreover, every iterated integral

\[
\int \cdots \int_{0 \leq s_1 \leq \ldots \leq s_k \leq 2\pi} a_{i_k}(s_k) \cdots a_{i_1}(s_1) \, ds_k \cdots ds_1, \quad \sum_{i=1}^{k} i_k = i,
\]

is the coefficient of one of monomials of \( q_i \). Now, the triviality of the monodromy of (1.8) is equivalent to the fact that all \( q_i \in A, i \geq 1, \) are zeros. In turn, according to Lemma 3.1 this implies that all coefficients of all \( q_i, i \geq 1, \) are zeros.

**Proof of Proposition 1.7.** The proof repeats word-for-word the proof of Proposition 1.6. We leave the details to the reader. \( \square \)

### 4. Proof of Theorem 1.10.

**4.1.** We refer to the book of Hirzebruch [Hi] for an exposition about fibre bundles.

Let \( U \subset \mathbb{C}^n \) be a domain containing \( \tilde{\Gamma}_n \). Since \( \tilde{\Gamma}_n \) is polynomially convex, for any domain \( V \supset \tilde{\Gamma}_n \) there is a Stein domain \( V' \) such that \( \tilde{\Gamma}_n \subset V' \subset V \). Thus without loss of generality we may assume that \( U \) is Stein. Let \( \pi_1(U) \) be the fundamental group of \( U \). Without loss of generality we may assume that \( \pi_1(U) \) is infinite. (For otherwise we naturally consider \( \tilde{\Gamma}_n \) as a polynomially convex subset of \( \mathbb{C}^{n+1} := \mathbb{C}^n \times \mathbb{C} \). Taking a neighbourhood \( U \times W \subset \mathbb{C}^{n+1} \) of \( \tilde{\Gamma}_n \) with \( \pi_1(W) = \mathbb{Z} \), we prove that the path \( A_n: S^1 \to U \) determining \( \Gamma_n \) is contractible in \( U \times W \). This implies its contractibility in \( U \), as well.)

The universal covering \( p: \tilde{U} \to U \) is a discrete bundle on \( U \) with the fibre \( \pi_1(U) \). It is defined on an open acyclic cover \( U = (U_i) \) of \( U \) by a locally constant cocycle \( \{c_{ij}\} \in Z^1(U, \pi_1(U)) \). By definition, \( c_{ij}: U_i \cap U_j \to \pi_1(U) \) is a constant map such that \( c_{ij}c_{jk} = c_{ik} \) on \( U_i \cap U_j \cap U_k \). Then \( U \) is biholomorphic to the quotient space of \( \sqcup U_j \times \pi_1(U) \) by the equivalence relation:

\[
U_j \times \pi_1(U) \ni (z \times g) \sim (z \times g \cdot c_{ij}^{-1}) \in U_i \times \pi_1(U).
\]

Let \( l_U := l_2(\pi_1(U)) \) be the Hilbert space of complex-valued sequences on \( \pi_1(U) \) with the \( l_2 \)-norm. By \( L(l_U) \) we denote the Banach space of bounded com-
plex linear operators \( l_U \to l_U \), and by \( GL(l_U) \) the group of invertible operators from \( L(l_U) \). Let us define the homomorphism \( \xi: \pi_1(U) \to Iso(l_U) \) by

\[
\xi(g)(v)(x) := v(x \cdot g^{-1}), \quad g, x \in \pi_1(U), \ v \in l_U.
\]

Here \( Iso(l_U) \subset GL(l_U) \) is the subgroup of unitary operators. It is clear that \( \xi \) is a faithful representation, that is, \( Ker(\xi) = \{1\} \).

Next, let us construct the holomorphic Banach vector bundle \( E_\xi \) on \( U \) with fibre \( l_U \) associated with \( \xi \). It is defined as the quotient of \( \sqcup_j U_j \times l_U \) by the equivalence relation \( U_j \times l_U \ni x \times w \sim x \times \xi(c_{ij})(w) \in U_i \times l_U \).

Since \( l_U \) is an infinite dimensional Hilbert space, the group \( GL(l_U) \) is contractible (see [K]). In particular, \( E_\xi \) is a topologically trivial Banach vector bundle. Now according to the result of Bungart [B] applied to the Stein manifold \( U \) we obtain that \( E_\xi \) is a holomorphically trivial Banach vector bundle on \( U \). This means that there is a family of holomorphic functions \( F_i: U_i \to GL(l_U) \) satisfying

\[
F_i^{-1}(z) \cdot F_j(z) = \xi(c_{ij}) \text{ on } U_i \cap U_j.
\]

Note that cocycle \( \{\xi(c_{ij})\} \) is locally constant, i.e. \( d(\xi(c_{ij})) = 0 \) for any \( i, j \). Then we define the global holomorphic 1-form \( \omega \) on \( X \) with values in \( L(l_U) \) by the formula

\[
\omega|_{U_i} := dF_i \cdot F_i^{-1}.
\]

Clearly \( \omega \) satisfies the Frobenius condition \( d\omega - \omega \wedge \omega = 0 \), which is equivalent to the fact that equation

\[
dF = \omega \cdot F
\]

is locally solvable on \( U \). Let \( p^*\omega \) be the pullback of \( \omega \) to \( \bar{U} \). Consider the pullback to \( \bar{U} \) of (4.1)

\[
dF = p^*\omega \cdot F.
\]

Then from simply connectedness of \( \bar{U} \) it follows that (4.2) has a global holomorphic solution \( F: \bar{U} \to GL(l_U) \). On the connected component \( U_i \times \{g\} \) of the open set \( p^{-1}(U_i) := \sqcup_{g \in \pi_1(U)} U_i \times \{g\} \subset \bar{U} \) the solution \( F \) equals \( p^*F_i \cdot \xi(g^{-1}) \).

Another way to obtain a solution of (4.2) is by using Picard iteration. To this end, we fix a point \( z_0 \in \bar{U} \) and for any piecewise smooth path \( \gamma \) joining \( z_0 \) with...
a point \( z \) we set

\[
S(z) = I + \sum_{i=1}^{\infty} \int \cdots \int_\gamma \omega \cdots \omega.
\]

The \( k \)th term of this series is the \( k \)-iterated integral of \( \omega \) over \( \gamma \) defined similarly to (1.10). It is well known (cf. [Na]) that the series converges uniformly on compact subsets in \( \bar{U} \) to a \( GL(l_U) \)-valued function \( S \) satisfying (4.2), and \( S(z_0) = I \).

Therefore \( S(z) = F(z) \cdot F^{-1}(z_0) \) where \( F \) is the solution of (4.2) constructed above.

Moreover, by the definition of \( F \),

\[
S(gz) = S(z) \cdot \xi(g^{-1}) \quad \text{for any} \quad g \in \pi_1(U), \ z \in \bar{U}.
\]

In particular, for \( z = z_0 \) we have \( S(gz_0) = \xi(g^{-1}) \).

Let \( \gamma_g \subset U \) be a loop based at \( p(z_0) \) which represents an element \( g \in \pi_1(U, p(z_0)) \). Then the path on \( \bar{U} \) with the origin at \( z_0 \) and the endpoint at \( gz_0 \) represents the lifting of \( \gamma_g \) to \( \bar{U} \). Now the correspondence \( \gamma_g \mapsto \xi(g^{-1}) \) determines a homomorphism \( \tilde{\xi} : \pi_1(U, p(z_0)) \to Aut(GL(l_U)) \) defined by

\[
\tilde{\xi}(g)(H) = H \cdot \xi(g^{-1}), \quad g \in \pi_1(U, p(z_0)), \ H \in GL(l_U).
\]

By the definition of \( \xi \) we obtain that \( \text{Ker}(\tilde{\xi}) = \{1\} \).

4.2. For basic facts of complex analysis in domains of holomorphy see, e.g., the book of Henkin and Leiterer [HL].

Let us write the form \( \omega \) in (4.1) as \( \omega(z) = \sum_{j=1}^{\infty} R_j(z) \, dz_j \) where \( z_1, \ldots, z_n \) are standard coordinates on \( \mathbb{C}^n \), and \( R_1, \ldots, R_n \) are holomorphic \( L(l_U) \)-valued functions on \( U \). Let \( V \subset U \) be a domain containing \( \Gamma \). Then there is a Weil polynomial polyhedron

\[
D = \{ z \in U : |P_k(z)| < 1, \ P_k \in \mathbb{C}[z_1, \ldots, z_n], \ k = 1, \ldots, N \} \subset V
\]

containing \( \Gamma_n \). Note that for \( L(l_U) \)-valued holomorphic functions defined in an open neighbourhood of \( V \) we have exactly the same integral representations as for scalar holomorphic functions. In particular, any such function \( f \) inside \( D \) can be represented as a finite sum of holomorphic \( L(l_U) \)-valued functions \( f_j \) such that each

\[
f_j(z) = \int_{\sigma_j} f(\xi) K_j(\xi, z) \eta(\xi)
\]

where \( \eta(\xi) = d\xi_1 \wedge \cdots \wedge d\xi_n \), \( K_j(\xi, z) \) is a rational holomorphic function in \( z \) and \( \xi \), and \( \sigma_j \) is an \( n \)-dimensional part of the Silov boundary of \( D \). Moreover, one can uniformly approximate on compacts in \( D \) each \( K_j(\xi, z) \) by holomorphic functions.
Proposition 4.1. Any $L(l_U)$-valued function holomorphic in an open neighbourhood $V \subseteq U$ of $\hat{\Gamma}_n$ can be uniformly approximated on $\hat{\Gamma}_n$ by $L(l_U)$-valued holomorphic polynomials.

This proposition implies that there is a sequence of polynomial 1-forms

$$\omega_i(z) = \sum_{j=1}^n R_{ji}(z) \, dz_j$$

such that

$$R_{ji}(z) = \sum_{0 \leq |\alpha| \leq d_{ji}} R_{ji,\alpha} z^\alpha, \quad R_{ji,\alpha} \in L(l_U), \quad \text{and} \quad \lim_{i \to \infty} \sup_{z \in \hat{\Gamma}} ||R_j(z) - R_{ji}(z)|| = 0.$$

Here $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for any $\alpha := (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_+)^n$, $|\alpha| := \sum_{j=1}^n \alpha_j$, and $\| \cdot \|$ denotes the Banach norm on $L(l_U)$.

4.3. Next, we recall the Rees formula [R] for the product of iterated integrals.

A permutation $\sigma$ of $\{1, 2, \ldots, r+s\}$ is a shuffle of type $(r, s)$ if

$$\sigma^{-1}(r+1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(r)$$

and

$$\sigma^{-1}(r+1) < \sigma^{-1}(r+2) < \cdots < \sigma^{-1}(r+s).$$

Let $f_1, f_2, \ldots, f_{r+s} \in L^\infty([0, t])$. As before,

$$I_{i_1, \ldots, i_k}(x) := \int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_k \leq t} f_{i_k}(s_k) \cdots f_{i_1}(s_1) \, ds_k \cdots ds_1.$$

Then

$$I_{1,2,\ldots,r}(x) \cdot I_{r+1, r+2, \ldots, r+s}(x) = \sum_{\sigma} I_{\sigma(1),\sigma(2),\ldots,\sigma(r+s)}(x)$$

where $\sigma$ runs over the shuffles of type $(r, s)$.

Going back to the proof of the theorem we set $\tilde{R}_j(x) := A_n^*(R_j)(x), j = 1, \ldots, n$. Then $\tilde{R}_j$ are $L(l_U)$-valued continuous functions on $S^1$.

Lemma 4.2. For any positive integers $k$ and $i_1, \ldots, i_k$, $1 \leq i_k \leq n$, we have

$$\int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_k \leq 2\pi} \tilde{R}_{i_1}(s_k) a_{i_1}(s_k) \cdots \tilde{R}_{i_k}(s_1) a_{i_k}(s_1) \, ds_k \cdots ds_1 = 0.$$
Proof. By Proposition 4.1 it suffices to prove the result for $\tilde{R}_j$ holomorphic $L(l_U)$-valued polynomials. Moreover, it suffices to consider only scalar monomials. Thus without loss of generality we may assume that $\tilde{R}_g(s_i) = \tilde{a}_1(s_i)^{\alpha_1 i} \cdots \tilde{a}_n(s_i)^{\alpha_n i}$. Here $\tilde{a}_i$ is the antiderivative of $a_i$. Now, by the Ree formula each integral on the left-hand-side of (4.4) can be written as a finite sum of integrals of the form

$$\int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_r \leq 2\pi} a_{jr}(s_r) \cdots a_{ji}(s_i) \, ds_r \cdots ds_i \quad \text{where} \quad r = \sum_{k=1}^n (\alpha_k i_k + 1).$$

Thus the required result follows from equations (1.12).

4.4. Let us finish the proof of the theorem.

Let $A_n: S^1 \to U$ represent an element $h \in \pi_1(U, p(z_0))$. Let $\tilde{A}_n: S^1 \to \tilde{U}$ be the lifting of $A_n$ with the origin at $z_0$. Then its endpoint is $h z_0$. Let $S(z)$ be the solution of (4.2) from Section 4.1. Then $S(h z_0) = \xi(h^{-1})$. On the other hand, this value can be obtained by formula (4.3) where the integrals are taken over the path $\tilde{A}_n: S^1 \to \tilde{U}$. To calculate these integrals we take the pullback of the form $p^* \omega$ by $\tilde{A}_n$, and then calculate usual iterated integrals of $A_n^* \omega := A_n^* (p^* \omega)$ over $[0, 2\pi]$. But each of such iterated integrals is zero by Lemma 4.2. This means that $\xi(h^{-1}) = 1$. Since $\xi$ is a faithful representation, the latter implies that $h = 1 \in \pi_1(U, p(z_0))$.

The proof of the theorem is complete.

5. Proofs of Theorem 1.14 and Corollaries 1.12, 1.17, 1.19 and 1.20.

Proof of Corollary 1.12. Suppose $\Gamma_n$ is triangulable, $\tilde{\Gamma}_n = \Gamma_n$, and $\Gamma_n$ satisfies the hypothesis of Theorem 1.10. From the triangulability of $\Gamma_n$ it follows that $\Gamma_n$ is homeomorphic to a one-dimensional simplicial complex (i.e., a finite graph). Moreover, by Borsuk’s theorem (see, e.g. [Hu]) there is an open connected neighbourhood $U$ of $\Gamma_n$ and a retraction $r: U \to \Gamma_n$. Let $i: \Gamma_n \hookrightarrow U$ be the embedding, and $i_*: \pi_1(\Gamma_n) \to \pi_1(U)$, $r_*: \pi_1(U) \to \pi_1(\Gamma_n)$ be the induced homomorphisms of fundamental groups. Then $r_* \circ i_*: \pi_1(\Gamma_n) \to \pi_1(\Gamma_n)$ is the identity homomorphism. Now, condition $\tilde{\Gamma}_n = \Gamma_n$ and Theorem 1.10 imply that the path $i \circ A_n: S^1 \to U$ represents $1 \in \pi_1(U)$. Thus from above we get that $A_n: S^1 \to \Gamma_n$ represents $1 \in \pi_1(\Gamma_n)$, i.e., $A_n$ is contractible in $\Gamma_n$.

Further, let $p: \Gamma_{nu} \to \Gamma_n$ be the universal covering of $\Gamma_n$. Then $\Gamma_{nu}$ can be thought of as an infinite tree. Since $A_n: S^1 \to \Gamma_n$ is contractible, by the covering homotopy theorem (see, e.g. [Hu, Ch.III]) there is a map $A_{2n}: S^1 \to \Gamma_{nu}$ such that $A_n = p \circ A_{2n}$. The image $G_n := A_{2n}(S^1) \subset \Gamma_{nu}$ is a connected compact subset. Therefore $G_n$ is homeomorphic to a finite tree. Finally, we set $A_{1n} := p|_{G_n}$.

Conversely, if $A_n$ admits the factorization of Corollary 1.12 then $A_{2n}: S^1 \to G_n$ is contractible in $G_n$. Hence $A_n: S^1 \to \Gamma_n$ is contractible in $\Gamma_n$.  \[\square\]
\textbf{Proof of Theorem 1.14.} Suppose $\Gamma_n$ is Lipschitz triangulable, $\tilde{\Gamma}_n = \Gamma_n$, and $A_n: S^1 \to \Gamma_n$ is contractible in $\Gamma_n$. By the covering homotopy theorem, there is a covering $s: \mathcal{U} \to U$ and a Lipschitz embedding $i: \Gamma_{nu} \hookrightarrow \mathcal{U}$ such that $p = s \circ i$. Without loss of generality we consider $U$ as a submanifold of some $\mathbb{R}^{N_n}$ so that $i(\Gamma_{nu}) \subset \mathbb{R}^{N_n}$ has an exhaustion by Lipschitz triangulable compact subsets. It follows from the Whitney embedding theorem. We identify $\Gamma_{nu}$ with $i(\Gamma_{nu})$.

Now, the contractibility of $A_n$ implies that there is a map $A_{2n}: S^1 \to \Gamma_{nu} \subset \mathbb{R}^{N_n}$ such that $A_n = s \circ A_{2n}$. Since $A_n$ is Lipschitz, the map $A_{2n}$ is also Lipschitz by the definition of $\mathcal{U}$. Let us consider the equation on $U$

\begin{equation}
\frac{dF}{d\xi} = \eta \cdot F, \quad \text{where} \quad \eta := \sum_{i=1}^n i^j F_1 F_2^{i-1} \, d\xi.
\end{equation}

It is not locally solvable but its lifting $dF = s^* \eta \cdot F$ restricted to $\Gamma_{nu}$ is. This follows from the fact that $\Gamma_{nu}$ outside a countable discrete set of points is the disjoint union of one-dimensional Lipschitz manifolds $M_i$, $i = 1, 2, \ldots$ (this is a consequence of Lipschitz triangulability). Moreover, each $M_i$ is the image of the interval $(0, 1)$ under some Lipschitz map $g_i$: $(0, 1) \to \mathbb{R}^{N_n}$ so that $g_i^{-1}$: $M_i \to (0, 1)$ is locally Lipschitz. Now, since $\Gamma_{nu}$ is a tree, we can solve the equation $dF = s^* \eta \cdot F$ on $\Gamma_{nu}$ by Picard iteration (as in (4.3)). Indeed, since $g_i$ is Lipschitz, we can solve the pullback of this equation by $g_i$ to $(0, 1)$. Then we transfer this solution to $M_i$ by $g_i^{-1}$ to obtain a solution of $dF = s^* \eta \cdot F$ on $M_i$. Finally, we apply the Picard method successively on the edges of $\Gamma_{nu}$ to sew the local solutions to a global one. Let us denote this solution by $F$. Then $F(x)$, $x \in \Gamma_{nu}$, is an element of the group $G(F_1, F_2)[[t]]$. Moreover, by the construction of $F$ and since every $A_n^{-1}(x)$, $x \in \Gamma_{nu}$, is countable, the coefficients of the series expansion of $A_{2n}^* F$ are locally Lipschitz on an open subset $O = S^1 \setminus T$ of $S^1$ where $T \subset S^1$ is countable. So $A_{2n}^* F$ has the derivative almost everywhere on $S^1$. Then clearly

\[ \frac{d}{dx}(A_{2n}^* F) = (A_{2n}^* s^* \eta)) \cdot A_{2n}^* F = \left( \sum_{i=1}^n a_i(x) i^j F_1 F_2^{i-1} \right) \cdot A_{2n}^* F. \]

(Without loss of generality we assume that $A_{2n}^* F(0) = I$.) On the other hand, there is a global solution $H$ of the above equation (that is equation (1.13)) on $[0, 2\pi)$ such that $H(0) = I$, and the coefficients of $H$ in the series expansion are Lipschitz functions. In particular, the coefficients in the series expansion of $H^{-1} \cdot A_{2n}^* F$ are locally Lipschitz on $O$. Thus we have

\[ \frac{d}{dx}(H^{-1} \cdot A_{2n}^* F) = 0 \quad \text{almost everywhere on} \quad S^1. \]

Since $H^{-1} \cdot A_{2n}^* F$ is continuous and $(H^{-1} \cdot A_{2n}^* F)(0) = I$, the results of Section
2.1 imply that $H(x) = A_{2n}^2F(x)$ on $[0, 2\pi)$. But $A_{2n}^2F(2\pi) = A_{2n}^2F(0)$ showing that the monodromy $\rho_n$ of (1.13) is trivial.

Proof of Corollary 1.17. (A) Let $\nu: \tilde{X} \to X$ be the normalization of $X$. Then $\tilde{X}$ is a noncompact (possibly disconnected) complex Riemann surface. Since every $\tilde{A}_n^{-1}(x), x \in A(R)$, is finite, the definition of the normalization implies that there is a continuous map $\tilde{A}_n: R \to \tilde{X}$ such that $\tilde{A}_n = \nu \circ A_1$. In particular, $\tilde{A}_n(R)$ belongs to a connected component of $\tilde{X}$. Thus $\Gamma_n := \tilde{A}_n(S^1)$ belongs to an irreducible component of $\tilde{X}$. (Here $A_n := \tilde{A}_n|_{S^1}$.) In what follows without loss of generality we assume that $X$ itself is irreducible.

LEMMA 5.1. Suppose $X$ satisfies condition (1) of Corollary 1.17. Then $\tilde{\Gamma}_n \subset X$.

Proof. By the definition of $U$, $\tilde{\Gamma}_n$ belongs to a polynomially convex compact set $K_n$. Since $X$ is a closed subspace of the Stein domain $U$, it is the zero set of a family $\{f_i\}_{i \in I}$ of holomorphic on $U$ functions (see e.g. [GR, Ch. 5, Sect. 4]). Since $K_n$ is polynomially convex, by the Oka-Weil approximation theorem each $f_i$ can be uniformly approximated in an open neighbourhood $O \subset U$ of $K_n$ by holomorphic polynomials. Assume, to the contrary, that there exists $z \in \tilde{\Gamma}_n$ such that $z \notin X \cap K_n$. Then there are $i \in I$ and $\epsilon > 0$ such that $|f_i(z)| > \epsilon$. Moreover, the polynomial approximation of $f_i$ gives a holomorphic polynomial $p_i$ such that

$$\max_{K_n \cap \tilde{X}} |p_i| \leq \frac{\epsilon}{2} \quad \text{but} \quad |p_i(z)| > \frac{\epsilon}{2}.$$ 

This contradicts to the fact that $z \in \tilde{\Gamma}_n \subset K_n \cap \tilde{X}$. 

This lemma implies that there is an open connected neighbourhood $Y \subset X$ of the compact connected set $\tilde{\Gamma}_n \subset X$. Since $Y$ is an analytic space, by the Lojasiewicz theorem [Lo] it is triangulated. In particular, there is an open connected neighbourhood $O_Y \subset U$ of $Y$ and a retraction $r: O_Y \to Y$.

Suppose first that the monodromy $\rho_n$ of equation (1.13) is trivial. We should prove that $A_n$ admits the required factorization.

From the triviality of $\rho_n$ and the above properties of $O_Y$, $Y$ and $r$ we obtain (as in the proof of Corollary 1.12) that under hypothesis (1) of Corollary 1.17 the path $A_n: S^1 \to Y$ is contractible in $Y$ (and also in $X$). Further, for one-dimensional Stein spaces $X$ and $\tilde{X}$ the normalization map $\nu: \tilde{X} \to X$ induces an injective homomorphism of the fundamental groups. In particular, $A_1\nu: S^1 \to \tilde{X}$ is contractible in $\tilde{X}$. Let $p: \tilde{X}_u \to \tilde{X}$ be the universal covering. Then by the covering homotopy theorem there is a covering map $A_2\nu: S^1 \to \tilde{X}_u$ such that $A_1\nu = p \circ A_2\nu$. Since $\tilde{X}$ is hyperbolic, $\tilde{X}_u$ is biholomorphic to the unit disk $\mathbb{D}$. Moreover, $A_2(S^1) \subset \tilde{X}_u$ is compact. Thus we can choose a domain $D \subset \tilde{X}_u$ so that $A_2(S^1) \subset D$ and $D$ is biholomorphic to $\mathbb{D}$. Now we set $A_{1u} := (\nu \circ p)|_D$.

To finish this part of the proof let us check that $A_{2n}$ is locally Lipschitz outside a finite set in $S^1$. 


Let $X_s$ be the finite set of singular points of $X$ containing in $\Gamma_n$. By our hypotheses, the preimage of each point of $A_n$: $S^1 \to X$ is finite. Thus $Y := A_n^{-1}(X_s) \subset S^1$ is finite. Now by the definition of the normalization, there is an open connected neighbourhood $V \subset X$ of $\Gamma_n$ such that $V \setminus X_s$ is smooth, and there is the holomorphic inverse map $\nu^{-1}: V \setminus X_s \to \tilde{X}$. Also, $p: \tilde{X}_n \to \tilde{X}$ is a locally biholomorphic map. Finally, $A_{2n}$ outside $Y$ is locally the composite $p^{-1} \circ \nu^{-1} \circ A_n$. Since $A_n$ is Lipschitz, $A_{2n}$ is locally Lipschitz outside $Y$.

**Remark 5.2.** If $\tilde{A}_n: R \to X$ is a holomorphic map of an open annulus containing $S^1$, the lifted map $\tilde{A}_{1n}: R \to \tilde{X}$ is holomorphic, as well. Also, since $A_{1n}: S^1 \to \tilde{X}$ is contractible, the covering homotopy theorem produces a holomorphic map $A'_{2n}: R \to \tilde{X}_n, A'_{2n}|_{S^1} = A_{2n}$, which covers $\tilde{A}_{2n}$. Thus there is an open annulus $R_1 \subset R$ containing $S^1$ such that $A'_{2n}(R_1) \subset D$. This proves part (2) of Remark 1.18.

Conversely, suppose $A_n = A_{1n} \circ A_{2n}$ where $A_n, A_{1n}$ and $A_{2n}$ satisfy hypotheses of Corollary 1.17. Let us check that the monodromy $\tilde{\rho}_n$ of the corresponding equation (1.13) is trivial.

As in the proof of Theorem 1.14 we consider equation (5.1) defined on $U$. Its lifting to $\mathbb{D}$ by $A_{1n}$ is locally solvable, because the lifted form $A_{1n}^*\eta$ is a holomorphic 1-form on $\mathbb{D}$. Since $\mathbb{D}$ is simply connected, the lifted equation $dF = A_{1n}^*\eta \cdot F$ has a global holomorphic solution $F$ on $U$ (obtained, e.g., by Picard iteration.) Let us consider $A_{2n}^*F$ on $S^1$. By the hypothesis, it is continuous, locally Lipschitz outside a finite set $Y$. Therefore $A_{2n}^*F$ has the derivative almost everywhere on $S^1 \setminus Y$ and

$$\frac{d}{dx} (A_{2n}^*F) = (A_{2n}^*(A_{1n}^*\eta)) \cdot A_{2n}^*F = \left( \sum_{i=1}^n a_i(x) F_i F_2^{-1} \right) \cdot A_{2n}^*F.$$ 

From here as in the proof of Theorem 1.14 we obtain that $A_{2n}^*F$ is Lipschitz and the monodromy $\tilde{\rho}_n$ of (1.13) is trivial.

The proof of the corollary for $X$ satisfying condition (1) is complete.

(B) Suppose that $X$ satisfies condition (2). As above there is a lifting $\tilde{A}_{1n}: R \to \tilde{X}$ of $A_n: R \to X$. We will show that if $\tilde{\rho}_n$ is trivial, $\tilde{A}_{1n}: S^1 \to \tilde{X}$ is contractible. Then the further proof repeats literally the proof in (A).

First, note that groups $\pi_1(X)$ and $\pi_1(\tilde{X})$ are free (because any one-dimensional Stein space is homotopically equivalent to a one-dimensional CW-complex [H]). Then condition (2) implies that $\pi_1(X)$ has a finite number of generators. Further, $\nu_s: H_1(\tilde{X}, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ is an embedding (because $\nu$ is one-to-one outside a discrete set). Thus $\pi_1(\tilde{X})$ is finitely generated, as well. Let $\Omega_p$ denote the set of holomorphic 1-forms with polynomial coefficients on $\mathbb{C}^n$. By $\nu^*\Omega_p$ we denote its pullback by $\nu$ to $\tilde{X}$, and by $\Omega(\tilde{X})$ the set of all holomorphic 1-forms on $\tilde{X}$. Since $\tilde{X}$ is a Stein manifold, the de Rham 1-cohomology group $H^1(\tilde{X}, \mathbb{C})$ can be
computed by $\Omega(\tilde{X})$. Namely, for each $\delta \in H^1(\tilde{X}, \mathbb{C})$ there is an $\omega \in \Omega(\tilde{X})$ such that
\[
\delta(\gamma) = \int_\gamma \omega \quad \text{for any } \gamma \in H_1(\tilde{X}, \mathbb{C}).
\]

Since $\nu_*: H_1(\tilde{X}, \mathbb{C}) \to H_1(X, \mathbb{C})$ is an injection and $\dim \mathbb{C} H_1(X, \mathbb{C}) < \infty$, for any $\delta \in H^1(\tilde{X}, \mathbb{C})$ there is a $\xi \in H^1(X, \mathbb{C})$ such that $\nu^*\xi = \delta$. In particular, condition (2) implies that each $\delta \in H^1(\tilde{X}, \mathbb{C})$ can be defined by (5.2) with $\omega \in \nu^*\Omega_p$. Thus for any $\omega \in \Omega(\tilde{X})$ there is $\omega' \in \nu^*\Omega_p$ such that $\omega - \omega' = df$ for a holomorphic $f \in \mathcal{O}(\tilde{X})$. The rest of the proof can be deduced from Sullivan’s theory of minimal models (see [Su]). For the sake of completeness we sketch the proof.

Let $N_n \subset GL_n(\mathbb{C})$ be the complex Lie subgroup of upper triangular unipotent matrices. By $n_n \subset gl_n(\mathbb{C})$ we denote the corresponding Lie algebras. For any homomorphism $\rho: \pi_1(\tilde{X}) \to N_n$ by $V_p$ we denote the flat vector bundle on $\tilde{X}$ associated with $\rho$ (see e.g. [KN, Ch. II]). Since $N_n$ is contractible, $V_p$ is topologically trivial. Thus, since $\tilde{X}$ is Stein, by the Grauert theorem [Gr] $V_p$ is also holomorphically trivial. Therefore it is determined by a holomorphic flat connection on the trivial bundle $\tilde{X} \times \mathbb{C}^n$, that is, by an $n_n$-valued holomorphic 1-form $\omega$ on $\tilde{X}$. (Note that $\dim \mathbb{C} \tilde{X} = 1$, so the Frobenius condition $d\omega - \omega \wedge \omega = 0$ is valid.)

**Lemma 5.3.** There is a holomorphic function $\tilde{F}: \tilde{X} \to N_n$ and a holomorphic $n_n$-valued 1-form $\eta$ with entries from $\nu^*\Omega_p$ such that
\[
F^{-1} \cdot \omega \cdot F - F^{-1} \cdot dF = \eta.
\]

**Proof.** We write $\omega = (\omega_{kr})$ as $\omega_1 + (\omega_1 - \omega_1)$ where $\omega_1 = (\omega_{1,kr})$, $\omega_{1,k1+k} = \omega_{k1+k}$, $k = 1, \ldots, n-1$, and $\omega_{1,kr} = 0$ otherwise. By condition (2) and the above arguments there is a holomorphic matrix 1-form $\eta_1 = (\eta_{1,kr})$ such that $\eta_{1,kr} = 0$ for $r-k \neq 1$, all $\eta_{1,kr} \in \nu^*\Omega_p$, and $\omega_1 - \eta_1 = df_1$ where $f_1 = (f_{1,kr})$ is a holomorphic matrix such that $f_{1,kr} = 0$ if $r-k \neq 1$. We set $F_1 = I_n + f_1$ where $I_n \in GL_n(\mathbb{C})$ is the unit matrix. Then for
\[
\omega'_1 = F_1^{-1} \cdot \omega \cdot F_1 - F_1^{-1} \cdot dF_1
\]
we have $\omega'_{1,k1+k} = \eta_{1,k1+k} \in \nu^*\Omega_p$ for any $k$. We continue this process. Next, we find a holomorphic matrix $f_2 = (f_{2,kr})$ such that $f_{2,kr} = 0$ if $r-k \neq 2$, and $\omega'_1 - df_2$ has two diagonals over the main one consisting of elements from $\nu^*\Omega_p$. Thus for $F_2 = I_n + f_2$ we have that
\[
\omega'_2 = F_2^{-1} \cdot \omega'_1 \cdot F_2 - F_2^{-1} \cdot dF_2
\]
has two diagonals over the main one with entries from $\nu^*\Omega_p$ etc. Finally, after $n-1$ steps we obtain the required matrix $\eta := \omega'_{n-1}$ with entries from $\nu^*\Omega_p$. It remains to determine $F := F_1 \cdot F_2 \cdots F_{n-1}$. \[\square\]
Next, let \( \tilde{\rho} : \pi_1(\tilde{X}) \to N_n \) be the representation constructed by the above flat connection \( \eta \). Since by Lemma 5.3 connections \( \eta \) and \( \omega \) are \( d \)-gauge equivalent, representations \( \rho \) and \( \tilde{\rho} \) are conjugate (see e.g. [O, Sec. 4,5]). Namely, there is a matrix \( C \in N_n \) such that \( C^{-1} \cdot \rho \cdot C = \tilde{\rho} \). In particular, \( \text{Ker}(\rho) = \text{Ker}(\tilde{\rho}) \). Now as in the proof of Theorem 1.10 we obtain that if the monodromy of equation (1.13) is trivial the element \( \gamma \) representing the path \( A_{1n} : S^1 \to \tilde{X} \) belongs to \( \text{Ker}(\tilde{\rho}) \) (because \( \tilde{\rho} \) is defined by Picard iteration applied to \( \eta \)). Let \( G \) be the intersection of kernels of all homomorphisms \( \rho : \pi_1(\tilde{X}) \to N_n \) for all \( n \). Then the above argument shows that \( \gamma \in G \). But \( \pi_1(\tilde{X}) \) is a free group with a finite number of generators. In particular, it is residually torsion free nilpotent, that is, \( G = \{1\} \). This shows that \( A_{1n} : S^1 \to \tilde{X} \) is contractible. \( \square \)

**Proof of Corollary 1.19.** For basic facts from the algebraic geometry see e.g. the book of Shafarevich [Sh].

In what follows \( C^* := \mathbb{C} \setminus \{0\} \). If the coefficients \( a_1, \ldots, a_n \) in (1.13) are trigonometric polynomials, the map \( A_n : S^1 \to \mathbb{C}^n \) can be extended to a holomorphic map \( A_n : C^* \to \mathbb{C}^n \) whose components are Laurent polynomials. If all components of \( A_n \) are either polynomials in \( z \) or in \( \frac{1}{z} \) the required factorization of \( A_n \) trivially exists. Thus we may assume without loss of generality that at least one component of \( A_n \) is a polynomial in both \( z \) and \( \frac{1}{z} \).

Let \( \mathbb{CP}^n \) denote the complex projective space. Then \( \mathbb{C}^n = \mathbb{CP}^n \setminus H \), where \( H \) is the hyperplane at infinity. Now, Zariski closure \( \overline{X} \) of the image \( X = \tilde{A}_n(C^*) \) is a (possibly singular) rational curve, and according to our assumption \( X = \overline{X} \setminus H \subset C^n \) is a closed algebraic subvariety. Further, if the monodromy \( \tilde{\rho}_n \) of (1.13) is trivial (i.e., the corresponding equation (1.14) determines a universal center), Corollary 1.17 implies that \( A_n : S^1 \to X \) is contractible. Let \( \nu : \mathbb{CP}^1 \to \overline{X} \) be the normalization of \( X \) and \( \tilde{X} = \nu^{-1}(X) \subset \mathbb{CP}^1 \) be the normalization of \( X \). Then there is an algebraic map \( \tilde{A}_{1n} : C^* \to \tilde{X} \) such that \( \tilde{A}_n = \nu \circ \tilde{A}_{1n} \cdot \). In particular, \( \tilde{A}_{1n} : S^1 \to \tilde{X} \) is contractible. Since according to our assumption \( \tilde{A}_{1n} : C^* \to \tilde{X} \) is a finite proper surjective map, the image of the homomorphism \( (\tilde{A}_{1n})_* : \pi_1(C^*) \to \pi_1(\tilde{X}) \) is a subgroup of a finite index in \( \pi_1(\tilde{X}) \). Now the contractibility of \( \tilde{A}_{1n} : S^1 \to \tilde{X} \) implies that \( \pi_1(\tilde{X}) = \{1\} \). Thus \( \overline{X} \cong C \subset \mathbb{CP}^1 \).

Since both maps \( \nu : \tilde{X} \to \mathbb{C}^n \) and \( \tilde{A}_{1n} : C^* \to \mathbb{C}^n \) are algebraic, the latter implies that there are polynomials \( p_1, \ldots, p_n \in \mathbb{C}[z] \) and a Laurent polynomial \( q \) such that \( \nu(z) = (p_1(z), \ldots, p_n(z)), z \in C, \) and \( \tilde{A}_{1n}(z) = q(z) \), \( z \in C^* \). Thus we have \( \tilde{a}_i(x) = p_i(q(x)), x \in S^1, 1 \leq i \leq n, \) as required.

Conversely, the above factorization of \( A_n \) implies that the monodromy \( \tilde{\rho}_n \) of the corresponding equation (1.13) is trivial. This can be shown as in the proof of part (A) of Corollary 1.17 based on the fact that \( \tilde{X} = C \) is contractible. \( \square \)

**Proof of Corollary 1.20.** Suppose that the coefficients \( a_1, \ldots, a_n \) in (1.14) are polynomials defined on a finite interval \( [a, b] \subset \mathbb{R} \). In this case \( \tilde{a}_i(x) := \int_a^x a_i(s) \, ds, \) \( 1 \leq i \leq n \). If (1.14) determines a universal center, then as in Proposition 1.6 all iterated integrals (defined over \( [a, b] \)) from \( a_1, \ldots, a_n \) are zeros. In particular,
\( \tilde{a}_i(a) = \tilde{a}_i(b) = 0 \) for any \( i \). Identifying \( \mathbb{R}/((b-a)\mathbb{Z}) \) with the unit circle \( S^1 \) we think of \( A_n(x) := (\tilde{a}_1(x), \ldots, \tilde{a}_n(x)) \), \( x \in [a, b] \), as a Lipschitz map \( S^1 \to \mathbb{C}^n \). Substituting to the last formula \( x \in \mathbb{C} \) we extend \( A_n \) to a holomorphic polynomial map \( \tilde{A}_n: \mathbb{C} \to \mathbb{C}^n \). Then \( X = \tilde{A}_n(\mathbb{C}) \) is a (possibly singular) rational curve. Since (1.14) determines a universal center, by Corollary 1.17 \( A_n: S^1 \to X \) is contractible. Further, it is easy to see that the normalization of \( X \) is \( \mathbb{C} \). Let \( \nu: \mathbb{C} \to X \) be the normalization map. Then there is a polynomial \( q \in \mathbb{C}[z] \) such that \( \tilde{A}_n = \nu \circ q \). Since \( \nu \) is also algebraic, \( \nu(z) := (p_1(z), \ldots, p_n(z)) \), \( z \in \mathbb{C} \), for some \( p_1, \ldots, p_n \in \mathbb{C}[z] \). Suppose that \( q(a) \neq q(b) \). Since \( A_n(a) = A_n(b) \), the normalization map sews together \( q(a) \) and \( q(b) \). In this case the definition of \( \nu \) implies easily that \( A_n: S^1 \to X \) is not contractible. Thus \( q(a) = q(b) \) as required.

Conversely, suppose that \( \tilde{a}_i = p_i \circ q \), \( q, p_i \in \mathbb{C}[z] \), \( 1 \leq i \leq n \), and \( q(a) = q(b) \). Then \( q|_{[a,b]} \) can be considered as a map \( S^1 \to \mathbb{C} \). Since \( A_n = \nu \circ q \) with \( \nu : (p_1, \ldots, p_n), \) the proof follows from Corollary 1.17.

6. Proofs of Results of Section 1.8.

Proof of Theorem 1.21. By the hypothesis any curve \( \Gamma_k \subset \mathbb{C}^k \), \( k \geq n \), is piecewise smooth. Let \( S_k \) be the finite set consisting of singularities of \( \Gamma_k \) and critical values of \( A_k \). Without loss of generality we assume that each \( S_k \) is not empty. By \( p_k: \Gamma_{k+1} \to \Gamma_k \) we denote the map induced by the projection \( \mathbb{C}^{k+1} \to \mathbb{C}^k \) to the first \( k \) coordinates. Then by the hypothesis \( p_k \), \( k \geq n \), is a finite surjective map. Clearly, \( S_{k+1} \subset p_k^{-1}(S_k) \). Now, the finiteness condition implies that there is an integer \( K \geq n \) such that \( A_k^{-1}(S_k) = A_m^{-1}(S_m) \) for any \( k, m \geq K \). We set \( Y := A_k^{-1}(S_k) \subset S^1 \). Then \( S^1 \setminus Y \) is the disjoint union of a finite number of open intervals \( I_1, \ldots, I_s \) such that each \( I_i \) is diffeomorphic by \( A_k \) (with \( k \geq K \)) to one of the smooth connected components of \( \Gamma_k \setminus S_k \). Moreover, we have \( p_k^{-1}(S_k) = S_{k+1} \) and \( p_k: S_{k+1} \to S_k \) is a bijection. Based on this and using finiteness of every \( p_k \) we find a number \( N \geq K \) such that \( p_k: \Gamma_{k+1} \to \Gamma_k \) is a homeomorphism for every \( k \geq N \) (moreover, \( p_k: \Gamma_{k+1} \setminus S_{k+1} \to \Gamma_k \setminus S_k \) is a diffeomorphism).

Suppose now that \( \Gamma_k = \Gamma_k \) for any \( k \), and that the monodromy \( \rho_N \) of equation (1.13) is trivial. According to Corollary 1.12 and Theorem 1.14 this is equivalent to contractibility of \( A_N: S^1 \to \Gamma_N \). But \( p_N \circ p_{N+1} \circ \cdots \circ p_k: \Gamma_k \to \Gamma_N \) is a homeomorphism for any \( k \geq N \), and \( A_N = p_N \circ p_{N+1} \circ \cdots \circ p_k \circ A_k \). Therefore \( A_k: S^1 \to \Gamma_k \), \( k \geq N \), is contractible. Now, from Corollary 1.12 and Theorem 1.14 we obtain that the monodromy \( \rho_k \) is trivial for any \( k \geq N \). In particular, according to Proposition 1.7, the monodromy \( \rho \) of (1.8) is trivial.

Proof of Theorem 1.23. Suppose that equation \( v'(x) = \sum_{i=1}^k b_i'(x) v^{i+1} \) determines a universal center. Then according to Proposition 1.6

\begin{equation}
(6.1) \quad \int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_k \leq 2\pi} b_k'(s_k) \cdots b_1'(s_1) \, ds_k \cdots ds_1
\end{equation}
is zero for any $i_1, \ldots, i_k, k$. Since every $a_i$ is the uniform limit of functions of the form $\sum_{j=1}^k p_{ij}(b_1, \ldots, b_k) \cdot b'_j$ where $p_{ij} \in \mathbb{C}[z_1, \ldots, z_k]$ are holomorphic polynomials, the Ree formula (see Section 4.3) implies that every iterated integral

$$\int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_l \leq 2\pi} a_{i_1}(s_1) \cdots a_{i_l}(s_1) \, ds_1 \cdots ds_l$$

is the limit of integrals of the form (6.1). Thus all integrals in (6.2) are zeros. Then Proposition 1.6 implies that the corresponding equation (1.1) determines a universal center.

Proof of Corollary 1.24. Passing to polar coordinates we obtain the equation

$$\frac{dr}{d\phi} = \frac{P}{1 + Q} r \quad \text{where} \quad P(r, \phi) = \frac{\partial A(r, \phi)}{\partial \phi}, \quad Q(r, \phi) = B(r, \phi).$$

Let us write $H(r, \phi) = h(\phi) r^k$ where $h$ is a trigonometric polynomial of degree $k$. Then we have $P(r, \phi) = \sum_{i=1}^k a_i h^{-1}(\phi) h'(\phi) r^{k-i}$, $Q(r, \phi) = \sum_{i=0}^\infty b_i h(\phi) r^i$, where $a_i, b_i \in \mathbb{C}$. If we now expand the right-hand side of the above differential equation as a series in $r$, then every coefficient of this series has the form $(p \circ h) \cdot h'$ with $p \in \mathbb{C}[z]$. In particular, the first integrals of these coefficients satisfy the hypotheses of Corollary 1.19. From this corollary we obtain that the required vector field determines a center.

Proof of Corollary 1.25. Passing to polar coordinates we get

$$\frac{dr}{d\phi} = \frac{P}{1 + Q} r \quad \text{where} \quad P(r, -\phi) = -P(r, \phi), \quad Q(r, -\phi) = Q(r, \phi).$$

In particular $P(r, \phi) = \sum_{j=1}^\infty p_j(\phi) r^j$ where $p_j$ are odd trigonometric polynomials, and $Q(r, \phi) = \sum_{j=0}^\infty q_j(\phi) r^j$ where $q_j$ are even trigonometric polynomials. Then every $p_j$ is a linear combination of functions $\sin(n\phi)$, $n = 1, 2, \ldots$, and the first integrals $\tilde{p}_j(\phi) = \int_0^\phi p_j(s) ds$ are even trigonometric polynomials. Thus $\tilde{p}_j$ and $q_j$ are polynomials in $\cos \phi$. Now the required result follows from Corollary 1.19.

7. Appendix. In this part following [Br1] we describe an algebraic model for the Center Problem. First, we introduce a multiplication $*$: $X \times X \to X$:

Given $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$ from $X$ we define

$$a * b = (a_1 * b_1, a_2 * b_2, \ldots) \in X \quad \text{and} \quad a^{-1} = (a_1^{-1}, a_2^{-1}, \ldots) \in X$$
where for any $i$

$$(a_i * b_i)(t) = \begin{cases} 2a_i(2t) & \text{if } 0 < t \leq \pi \\ 2b_i(2t - 2\pi) & \text{if } \pi < t \leq 2\pi \end{cases}$$

and

$$a_i^{-1}(t) = -a_i(2\pi - t), \quad 0 < t \leq 2\pi.$$ 

We say that $a, b \in X$ are equivalent (written, $a \sim b$) if $a * b^{-1} \in U$. In [Br1] we show that $\sim$ is an equivalence relation, that is $X$ partitions into mutually disjoint equivalence classes. Let $G(X)$ be the set of these classes. Then $*$ induces a multiplication $\cdot: G(X) \times G(X) \to G(X)$ such that the pair $(G(X), \cdot)$ is a group. Moreover, the iterated integrals are constant on any equivalence class. Thus we can consider them as functions on $G(X)$. We prove that these functions separate points on $G(X)$.

Next, equip $G(X)$ with the weakest topology $\tau$ in which all iterated integrals (considered as functions on $G(X)$) are continuous. One of the central results states that $(G(X), \cdot, \tau)$ is a separable topological group satisfying the following properties:

1. $(G(X), \tau)$ is contractible, arcwise connected, locally arcwise and simply connected;

2. $G(X)$ is residually torsion free nilpotent (that is, the set of all finite-dimensional unipotent representations separates elements of $G(X)$).

Consider now the set $G_c[[r]]$ of complex power series $f(r) = r + \sum_{i=1}^{\infty} d_i r^{i+1}$ each convergent in some open neighbourhood of $0 \in \mathbb{C}$. Let $d_i: G_c[[r]] \to \mathbb{C}$ be such that $d_i(f)$ equals the $i$-th coefficient of the Taylor expansion of $f$ at 0. Let $\tau'$ be the weakest topology on $G_c[[r]]$ in which all $d_i$ are continuous functions. We consider $G_c[[r]]$ with the multiplication $\circ$ defined by the composition of series. Then one can show that $(G_c[[r]], \circ, \tau')$ is a separable topological group satisfying the properties

3. $(G_c[[r]], \tau')$ is contractible, arcwise connected, locally arcwise and simply connected;

4. $G_c[[r]]$ is residually torsion free nilpotent.

Now, for any $a \in X$ let $v(x; r; a)$, $x \in [0, 2\pi]$, be the Lipschitz solution of equation (1.1) corresponding to $a$ with initial value $v(0; r; a) = r$. It is clear that for every $x$ we have $v(x; r; a) \in G_c[[r]]$. Let $P(a)(r) := v(2\pi; r; a)$ be the first return map. Then we have

$$P(a * b) = P(b) \circ P(a).$$

This together with the fact that $P(a)(r) \equiv r$ for any $a \in U$ imply that there exists a map $\tilde{P}: G(X) \to G_c[[r]]$ such that $\tilde{P}([a]) := P(a)$ where $[a]$ denotes the equivalence class containing $a \in X$. Then we prove that:
(5) $\hat{P}$ is a surjective homomorphism of topological groups;
(6) The kernel $\hat{C} \subset G(X)$ of $\hat{P}$ coincides with the image of the center set $C \subset X$ in $G(X);
(7) (\hat{C}, \tau)$ is contractible, arcwise connected, locally arcwise and simply connected.

Finally, consider the quotient group $Q(X) = G(X)/\hat{C}$. By $\pi: G(X) \to Q(X)$ we denote the quotient homomorphism. It follows from above that every function $c_i$ in Theorem 1.2 satisfies $c_i(a) = c_i(b)$ for $a \sim b$. Therefore every $c_i$ can be considered as a continuous function on $G(X)$. In fact, one can show that these functions are constant on any fibre of the map $\pi$ and hence they determine functions $\overline{c}_i: Q(X) \to \mathbb{C}$. Let $\tau''$ be the weakest topology on $Q(X)$ in which all $\overline{c}_i$ are continuous. Then

(8) $(Q(X), \tau'')$ is a topological group;
(9) The homomorphism $\hat{P}$ determines an isomorphism $\overline{P}: Q(X) \to G_c[[r]]$ of topological groups defined by $\hat{P} = \overline{P} \circ \pi$.

Also, we show that there is a continuous map $T: G_c[[r]] \to G(X)$ such that $\hat{P} \circ T = id$. In particular, we obtain that:

(10) The map $\overline{T}: G_c[[r]] \times \hat{C} \to G(X)$, $\overline{T}(f, g) := T(f) \cdot g$, is a homeomorphism.

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