Estimations of the discrete Green’s function of the SDFEM on Shishkin triangular meshes for singularly perturbed problems with characteristic layers

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Abstract
In this technical report, we present estimations of the discrete Green’s function of the streamline diffusion finite element method (SDFEM) on Shishkin triangular meshes for singularly perturbed problems with characteristic layers.

1. Continuous problem, Shishkin mesh, SDFEM
We consider the singularly perturbed boundary value problem
\begin{align*}
-\varepsilon \Delta u + bu_x + cu &= f & \text{in } \Omega = (0,1)^2, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
where $b, c > 0$ are constants, $b \geq \beta$ on $\Omega$ with a positive constant $\beta$ and $\varepsilon \ll b$ is a small positive parameter. It is assumed that $f$ is sufficiently smooth. The solution of (1) typically has an exponential layer of width $O(\varepsilon \ln(1/\varepsilon))$ near the outflow boundary at $x = 1$ and two characteristic (or parabolic) layers of width $O(\sqrt{\varepsilon} \ln(1/\varepsilon))$ near the characteristic boundaries at $y = 0$ and $y = 1$.

Throughout the article, the standard notation for the Sobolev spaces and norms will be used; and generic constants $C, C_i$ are independent of $\varepsilon$ and $N$. The constants $C$ are generic while subscripted constants $C_i$ are fixed.

The Shishkin mesh used for discretizing (1) is a piecewise uniform mesh. The reader is referred to [5, 6, 3] for a detailed discussion of their properties and applications. Mesh changes from coarse to fine are denoted by two mesh transition parameters $\lambda_x$ and $\lambda_y$. They are defined by
\begin{align*}
\lambda_x := \min \left\{ \frac{1}{2}, \rho \frac{\beta}{\epsilon} \ln N \right\} \quad \text{and} \quad \lambda_y := \min \left\{ \frac{1}{3}, \rho \sqrt{\epsilon} \ln N \right\},
\end{align*}
where $N = 6k$ with $k \in \mathbb{Z}^+$ is the number of mesh intervals in each direction and $\rho = 2.5$ in our analysis for technical reasons as in [9] and [7]. Then, the domain $\Omega$ is dissected into four
subdomains as $\hat{\Omega} = \Omega_s \cup \Omega_x \cup \Omega_y \cup \Omega_{xy}$ (see Fig. 1), where

$$
\Omega_z := [0, 1 - \lambda_x] \times [\lambda_y, 1 - \lambda_x], \quad \Omega_s := [0, 1 - \lambda_x] \times ((0, \lambda_x) \cup [1 - \lambda_y, 1]),
$$

$$
\Omega_x := [1 - \lambda_x, 1] \times [\lambda_y, 1 - \lambda_x], \quad \Omega_{xy} := [1 - \lambda_x, 1] \times ((0, \lambda_y) \cup [1 - \lambda_x, 1]).
$$

**Assumption 1.** Assume that $\varepsilon \leq N^{-1}$, as is generally the case in practice. Furthermore we assume that $\lambda_x = \rho \varepsilon \beta^{-1} \ln N$ and $\lambda_y = \rho \sqrt{\varepsilon \ln N}$ as otherwise $N^{-1}$ is exponentially small compared with $\varepsilon$.

We introduce the set of mesh points $\{(x_i, y_j) \in \Omega : i, j = 0, \ldots, N\}$ which divide the subdomains into uniform rectangles and triangles by drawing the diagonal in each rectangle. This triangulation is denoted by $T_N$ (see Fig. 1).

The mesh sizes are denoted by $h_{x,i} := x_{i+1} - x_i$ and $h_{y,j} := y_{i+1} - y_j$ which satisfy

$$
N^{-1} \leq h_{x,i} := H_x, h_{x,j} =: H_y \leq 3N^{-1} \quad 0 \leq i < N/2, \; N/3 \leq j < 2N/3,
$$

$$
C_1 \varepsilon N^{-1} \ln N \leq h_{x,i} =: h_x \leq C_2 \varepsilon N^{-1} \ln N \quad N/2 \leq i < N,
$$

$$
C_1 \sqrt{\varepsilon} N^{-1} \ln N \leq h_{x,j} =: h_y \leq C_2 \sqrt{\varepsilon} N^{-1} \ln N \quad 0 \leq j < N/3; \; 2N/3 \leq j < N.
$$

For notation convenience, we shall use $K_{ij}^1$ to denote the triangle with vertices $(x_i, y_j)$,
where \(l \) and \(m \) are nonnegative integers.

The choice of \(\delta \) makes the following coercivity hold
\[
a_{SD}(u^N, v^N) = a_{Gal}(u^N, v^N) + a_{stab}(u^N, v^N)
\]
and
\[
a_{stab}(u^N, v^N) = \sum_{K \in \Omega} \left( -\varepsilon \Delta u^N + b u^N + c u^N, \delta K b v^N \right)_K.
\]

Note that \(\Delta u^N = 0\) in \(K\) for \(u^N|_K \in P_1(K)\). Following usual practice [6], the stabilization parameter \(\delta_K = \delta(x, y)|_K\) are defined by
\[
\delta(x, y) = \begin{cases} C^* N^{-1} \\ 0 \end{cases} \quad \text{if } (x, y) \in \Omega_x \cup \Omega_y
\]
where \(C^*\) is a positive constant independent of \(\varepsilon\) and the mesh \(\mathcal{T}_N\). The choice of \(\delta\) makes the following coercivity hold
\[
a_{SD}(v^N, v^N) \geq 1/2 \|v^N\|^2 \quad \forall v^N \in V^N
\]
where
\[
\|v^N\|^2 := \varepsilon \|v^N\|^2 + \|v^N\|^2 + \sum_{K \in \Omega} \delta_K \|b v^N\|^2_K.
\]

Note that existence and uniqueness of the solution to (2) is guaranteed by this coercivity. Also Galerkin orthogonality holds, i.e.,
\[
a_{SD}(u - u^N, v^N) = 0 \quad \forall v^N \in V^N.
\]

For analysis on Shishkin meshes, we need the following anisotropic interpolation error bounds given in [4, Lemma 3.1] and [1, Theorem 1].

**Lemma 1.** Let \(K \in \mathcal{T}_N\) and \(p \in (1, \infty]\) and suppose that \(K\) is \(K^N_{i,j}\) or \(K_{i,j}^2\). Assume that \(w \in W^{l,p}(\Omega)\) and denote by \(w^j\) the linear function that interpolates to \(w\) at the vertices of \(K\). Then
\[
\|w - w^j\|_{L^p(K)} \leq C \sum_{l+m=2} h^l_{x_i,y_j} \|\partial_x^l \partial_y^m w\|_{L^p(K)},
\]
\[
\|(w - w^j)_x\|_{L^p(K)} \leq C \sum_{l+m=1} h^l_{x_i,y_j} \|\partial_x^{l+1} \partial_y^m w\|_{L^p(K)},
\]
\[
\|(w - w^j)_y\|_{L^p(K)} \leq C \sum_{l+m=1} h^l_{x_i,y_j} \|\partial_x^l \partial_y^{m+1} w\|_{L^p(K)}
\]
where \(l\) and \(m\) are nonnegative integers.
2. The discrete Green’s function

The estimates of the discrete Green’s function are similar to ones in [4] and [8]. However, here we present more delicate requirements for the parameters in the weight function, as will be helpful to obtain sharper pointwise bounds.

Let \( x^* = (x^*, y^*) \) be a mesh node in \( \Omega \). The discrete Green’s function \( G \in V^N \) associated with \( x^* \) is defined by

\[
a_{SD}(v^N, G) = v^N(x^*) \quad \forall v^N \in V^N.
\]

The bound of the discrete Green’s function in the energy norm relies on weight arguments. To start with, we define a weight function \( \omega_{x^*}^{y^*} \):

\[
\omega(x) := g \left( \frac{x - x^*}{\sigma_x} \right) g \left( \frac{y - y^*}{\sigma_y} \right) g \left( - \frac{y - y^*}{\sigma_y} \right)
\]

with \( g(r) = 2/(1 + e^r) \) for \( r \in (-\infty, \infty) \) and

\[
\sigma_x = k \max \{N^{-1}, e \ln^2 N\},
\]

\[
\sigma_y = \begin{cases} 
  kN^{-1/2} & \text{if } e \leq N^{-2} \\
  k \max \{N^{-3/2}e^{-1/2}, e^{1/2}\} & \text{if } N^{-2} \leq e \leq N^{-1}
\end{cases}
\]

We shall choose \( k > 0 \) later.

**Remark 1.** The definition (5) is different from one in [8]. It is more delicate and is helpful for our pointwise estimations. If \( N^{-2} \leq e \leq N^{-1} \), \( \sigma_y \) satisfies \( kN^{-3/4} \leq \sigma_y \leq kN^{-1/2} \) and is smaller than one in [8].

For the following analysis, we collect some basic properties of the weight function which can be obtained by some elementary calculations.

**Lemma 2.** Let \( \sigma_x \geq N^{-1} \) and \( \sigma_y \geq N^{-1} \). The following estimates hold true for the weight function \( \omega(x) \):

(i) \( 0 < \omega < 8 \) on \( \Omega \);
(ii) \( (\omega^{-1})_x > 0 \) on \( \Omega \);
(iii) for any \( K \in T_N \),

\[
\frac{\max_k \omega^{-1}}{\min_k \omega^{-1}} \leq C \quad \text{and} \quad \frac{\max_k (\omega^{-1})_x}{\min_k (\omega^{-1})_x} \leq C;
\]

(iv) for all \( l \geq 0 \) and \( m \geq 0 \),

\[
\left| \frac{\partial^{l+m} \omega(x)}{\partial x^l \partial y^m} \right| \leq C \sigma_x^{-l} \sigma_y^{-m} \omega(x) \quad \text{on} \quad \Omega;
\]

(v) for all \( l \geq 1 \) and \( m \geq 0 \),

\[
\left| \frac{\partial^{l+m} \omega(x)}{\partial x^l \partial y^m} \right| \leq C \sigma_x^{-l} \sigma_y^{-m} |\omega_x(x)| \quad \text{on} \quad \Omega;
\]

(vi) on any triangle \( K^* \) that contains \( x^* \), \( \omega(x) \geq C > 0 \).
Now we define a weighted energy norm
\[ \| G \|_\omega^2 := \| \omega^{-1/2} G \|_\omega^2 + \| \omega^{-1/2} G_r \|_\omega^2 + \frac{b}{2} \| (\omega^{-1})_s \|_\omega^2 \]
\[ + c \| \omega^{-1/2} G \|_\omega^2 + \sum_k b^2 \delta_k \| \omega^{-1/2} G \|_k^2. \]  
(6)

Note that \((\omega^{-1})_s > 0\). For any subdomain \( D \) of \( \Omega \), let \( \| G \|_{\omega, D} \) mean that the integrations in (6) are restricted to \( D \). The equalities (2), (6) and integration by parts yield
\[ \| G \|_\omega^2 = a_{SD}(\omega^{-1}G, G) = a_{SD}(\omega^{-1}G - (\omega^{-1})_s G, G) = a_{SD}(\omega^{-1}G - (\omega^{-1})_s G, G) \]
\[ + \sum_k (b(\omega^{-1})_s G + c\omega^{-1} G, \delta_k b G)_K. \]
Considering (4) we have
\[ a_{SD}(\omega^{-1}G, G) = a_{SD}(\omega^{-1}G - (\omega^{-1})_s G, G) \]
\[ = a_{SD}(\omega^{-1}G - (\omega^{-1})_s G, G) + (\omega^{-1}G)(x^*). \]
With the above two equalities, the weighted energy estimate of \( G \) will be obtained by means of the next three Lemmas.

**Lemma 3.** If \( \sigma_s \geq kN^{-1} \) and \( \sigma_s \geq k\varepsilon^{1/2} \) for \( k > 1 \) sufficiently large and independent of \( N \) and \( \varepsilon \), we have
\[ a_{SD}(\omega^{-1}G, G) \geq \frac{1}{4} \| G \|_\omega^2. \]

**Proof.** See [4, Lemma 4.2].

**Lemma 4.** Assume \( \sigma_s \geq kN^{-1} \) with \( k > 0 \) independent of \( N \) and \( \varepsilon \). Then for each mesh point \( x^* \in \Omega_1 \cup \Omega_2 \), we have
\[ |(\omega^{-1}G)(x^*)| \leq \frac{1}{16} \| G \|_\omega^2 + \begin{cases} CN^2 \sigma_s & \text{if } x^* \in \Omega_1 \\ CN \ln N & \text{if } x^* \in \Omega_2 \end{cases}, \]
where \( C \) is independent of \( N, \varepsilon \) and \( x^* \).

**Proof.** See [4, Lemma 4.3].

**Lemma 5.** If \( \sigma_s \) and \( \sigma_s \) satisfy (5), where \( k > 1 \) is sufficiently large and independent of \( N \) and \( \varepsilon \), then
\[ a_{SD}( (\omega^{-1}G)' - \omega^{-1}G, G) \leq \frac{1}{16} \| G \|_\omega^2. \]

**Proof.** For convenience we set \( \tilde{E}(x) := (\omega^{-1}G)' - \omega^{-1}G(x) \). Integration by parts yields \( (bE_s, G) = -(b\tilde{E}, G) \), and Cauchy-Schwarz inequalities yield
\[ |a_{SD}(\tilde{E}, G)| \leq C \left( \left( \| (\varepsilon + b^2 \delta)^{1/2} \omega^{1/2} \tilde{E}_s \| + \varepsilon^{1/2} \| \omega^{1/2} \tilde{E}_s \| \right) + \| (\varepsilon + b^2 \delta)^{-1/2} \omega^{1/2} \tilde{E} \| \right) \| G \|_\omega. \]  
(7)
Step 1. To analyze different kinds of interpolation bounds, we first estimate the derivatives of the weighted discrete Green’s functions. Note that \( G_{xx} = G_{yy} = G_{xy} = 0 \) on \( K \) because of \( G|_K \in P_1(K) \). For convenience, we set \( M_k := \max_K \omega^{-1/2} \).

Using Lemma 2 (iii)–(v), we obtain

\[
\|(\omega^{-1}G)_{xx}\|_K \leq \|(\omega^{-1})_{xx}G\|_K + 2\|(\omega^{-1})_{x}G_x\|_K \\
\leq CM_k \left( \sigma_x^{-3/2}\|(\omega^{-1})_{x}G\|_K + \sigma_x^{-1}\|\omega^{-1/2}G_x\|_K \right) \\
\leq CM_k \left( \sigma_x^{-3/2} + \sigma_x^{-1}(\epsilon + h^2\delta)^{-1/2} \right) \|G\|_{\omega,K}.
\]  

(8)

Note \( \|G\|_K \leq C h^{-1}_{y,K} \|G\|_K \) or \( \|G\|_K \leq C \epsilon^{-1/2} \cdot \epsilon^{1/2} \|G\|_K \), then one has

\[
\|G\|_K \leq C \min\{h^{-1}_{y,K}, \epsilon^{-1/2}\} \|G\|_K,
\]

and

\[
\|(\omega^{-1})_{x}G_y\|_K \leq CM_k\sigma_y^{-1}\omega^{-1/2} \cdot \min\{h^{-1}_{y,K}, \epsilon^{-1/2}\} \|G\|_K \\
\leq CM_k\sigma_y^{-1} \|h^{-1}_{y,K}, \epsilon^{-1/2}\| \|G\|_{\omega,K},
\]

where we have used (iii) in Lemma 2, for example

\[
\max_K \omega^{-1/2} \cdot \|G\|_K \leq \frac{\max_K \omega^{-1/2}}{\min_K \omega^{-1/2}} \|G\|_K \leq \min \|G\|_K \leq C \|\omega^{-1/2}G\|_K.
\]

Similarly, we have \( \|(\omega^{-1})_{yy}G\|_K \leq CM_k\sigma_y^{-2}\|\omega^{-1/2}G\|_K \) and

\[
\|(\omega^{-1})_{yy}G\|_K \leq \|(\omega^{-1})_{yy}G\|_K + \|(\omega^{-1})_{y}G\|_K \\
\leq CM_k \left( \sigma_y^{-2} + \sigma_y^{-1}\min\{h^{-1}_{y,K}, \epsilon^{-1/2}\} \right) \|G\|_{\omega,K}.
\]  

(9)

Recalling (iii) in Lemma 2 and inverse estimates \[3\] Theorem 3.2.6], we have

\[
\|(\omega^{-1})_{y}G\|_K \leq C\max_K (\omega^{-1})_{y} \cdot \|G\|_K \leq C\max_K (\omega^{-1})_{x} \cdot h^{-1}_{x,K} \|G\|_K \\
\leq C h^{-1}_{y,K} \left( \max_K (\omega^{-1})_{x} \right)^{1/2} \left( \min_K (\omega^{-1})_{x} \right)^{1/2} \|G\|_K \\
\leq CM_k \cdot h^{-1}_{y,K} \sigma_x^{-1/2} \cdot \|(\omega^{-1})_{x} \|_K \|G\|_K.
\]  

(10)

Also, we have

\[
\|(\omega^{-1})_{y}G\|_K \leq CM_k \cdot \epsilon^{-1/2} \sigma_x^{-1} \cdot \epsilon^{1/2} \|\omega^{-1/2}G\|_K.
\]  

(11)

Then from (10) and (11), one has

\[
\|(\omega^{-1})_{y}G\|_K \leq CM_k \min\{h^{-1}_{x,K}, \epsilon^{-1/2}\} \|G\|_{\omega,K}.
\]
and then
\[
\|(\omega^{-1}G)_x\|_K \leq \|(\omega^{-1})_xG\|_K + \|(\omega^{-1})_yG\|_K + \|(\omega^{-1})_zG\|_K
\]
\[
\leq CM_k \left( \sigma_x^{-1/2} \sigma_y^{-1} \|(\omega^{-1})_xG\|_K + \sigma_y^{-1} \|(\omega^{-1})_yG\|_K \right) + \|(\omega^{-1})_zG\|_K
\]
\[
\leq CM_k \left( \sigma_x^{-1/2} \sigma_y^{-1} + \sigma_y^{-1}(\varepsilon + b^2 \delta)^{-1/2} \right.
\]
\[
+ \min\{h_y,x\}, \epsilon^{-1/2} \sigma_y^{-1}\} \frac{1}{\varepsilon} \|G\|_{\omega,K}.
\]

**Step 2.** Now, we will analyze $\nabla \tilde{E}$ and $\tilde{E}$ respectively.

(a) From Lemma 1 we obtain
\[
\|\tilde{E}_x\|_K \leq C \left( h_{x,K} \|(\omega^{-1}G)_x\|_K + h_{y,K} \|(\omega^{-1}G)_y\|_K \right),
\]
\[
\|\tilde{E}_y\|_K \leq C \left( h_{x,K} \|(\omega^{-1}G)_y\|_K + h_{y,K} \|(\omega^{-1}G)_y\|_K \right).
\]
Substituting (8), (9) and (12) into (13) and (14), we have
\[
(\varepsilon + b^2 \delta)\|\omega^{1/2} \tilde{E}_e\|^2 \leq CK^{-2} \|G\|^2_{\omega},
\]
\[
\varepsilon \|\omega^{1/2} \tilde{E}_y\|^2 \leq CK^{-2} \|G\|^2_{\omega}.
\]

More precisely, we have
\[
\|\omega^{1/2} \tilde{E}_x\|_{\Omega} \leq CK^{-1} (\varepsilon^{-1/2} \ln^{-1} N + \varepsilon^{-1/2} N^{-1} \sigma_x^{-1}) \|G\|_{\omega}.
\]
where we have used $\sigma_x \geq k \varepsilon \ln^2 N$.

(b) Lemma 1 yields
\[
\|\tilde{E}_x\|_K \leq C \left( h_{x,K} \|(\omega^{-1}G)_x\|_K + h_{x,K} \|(\omega^{-1}G)_y\|_K + h_{x,K} \|(\omega^{-1}G)_y\|_K \right).
\]
Substituting (8), (9) and (12) into the above inequality, for $K \subset \Omega_x \cup \Omega_y$ we have
\[
\|\omega^{1/2} \tilde{E}_x\|_K \leq CK^{-1} N^{-1/2} \|G\|_{\omega,K}.
\]
and for $K \subset \Omega_y$
\[
\|\omega^{1/2} \tilde{E}_x\|_K \leq CK^{-1} e^{-1} \|G\|_{\omega,K}.
\]

For what follows we need a sharper bound of $\|\omega^{1/2} \tilde{E}\|_{\Omega}$. Similar to [4, Lemma 4.4] we consider (3), (18) and (17) and obtain
\[
\|\omega^{1/2} \tilde{E}_x^2\|_{\Omega} \leq CR_x^2 \left\{ \|\omega^{1/2} \tilde{E}_x^2\|_{\Omega}^2 + \|\omega^{1/2} \tilde{E}_x\|_{\Omega}^2 \right\}
\]
\[
\leq C \varepsilon \ln^2 N \cdot \left\{ \sigma_x^{-2} \|\omega^{1/2} \tilde{E}_x^2\|_{\Omega}^2 + \|\omega^{1/2} \tilde{E}_x\|_{\Omega}^2 \right\}
\]
\[
\leq C \varepsilon \ln^2 N \|\omega^{1/2} \tilde{E}_x^2\|_{\Omega}^2 + \|\omega^{1/2} \tilde{E}_x\|_{\Omega}^2 \}
\]
\[
\leq C \varepsilon \ln^2 N \|G\|_{\omega}^2.
\]
where we have used $N^{-1} \ln^2 N \leq C$ for $N \geq 2$.

Substituting (15), (16) and (18)–(20) into (7) and recalling the definition of $\delta$, we obtain
\[
|a_{SD}(\tilde{E}, G)| \leq CK^{-1} \|G\|_{\omega}^2.
\]
Choosing $k$ sufficiently large independently of $\varepsilon$ and $N$, we are done. □
Lemmas 3, 4 and 5 yield the following bound of the discrete Green function in the energy norm.

**Theorem 1.** Assume that $\sigma_x$ and $\sigma_y$ satisfy (5), where $k$ is chosen so that Lemmas 3, 4 and 5 hold. For $x^* \in \Omega_x \cup \Omega_y$ we have

$$\| G \|_2^2 \leq 8 \| G \|_{\infty}^2 \leq \begin{cases} CN^2 \sigma_x & \text{if } x^* \in \Omega_x, \\ CN \ln N & \text{if } x^* \in \Omega_y. \end{cases}$$

**Proof.** See [4, Theorem 4.1].

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