Bright-dark wave envelopes of nonlinear regularized-long-wave and Riemann wave models in plasma physics

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ABSTRACT

The nonlinear regularized-long-wave (RLW) and the Riemann wave (RW) models are physically significant in plasma physics and in further study of nonlinear dispersive waves, namely shallow water, ion-acoustic and magneto-sound waves in plasmas, anharmonic lattice and pressure waves in liquid-gas bubble mixtures, tidal and tsunami waves in rivers and oceans, longitudinal electromagnetic wave propagation in plasma, etc. The advanced generalized \((G'/G)\)-expansion scheme is facilitated in this article for retrieving the bright-dark exact solitary wave solutions generated in the form of hyperbolic, trigonometric, and rational structures to the above-stated models. To figure out the internal contrivance of the solutions, 3D and contour plots concerning bright parabolic wave shape, compacton wave shape, bell-shaped soliton, kink wave type soliton, dark anti-parabolic wave shape, anti-compacton wave shape, bright propagation of solitary wave shape solutions are depicted for definite free parametric values. The characteristic feature of the wave envelopes fluctuates taking into consideration the changes of free parameters and they are essentially dominated by the linear and nonlinear impact.

Introduction

Exact solitary wave solutions to the nonlinear evolution equations (NLEEs) contribute extensively in the branches of applied sciences and engineering, such as, optical fibers, condensed matter physics, theory of solitons, chaos theory, fluid dynamics, plasma physics etc. [1]. In mathematical physics, namely meteorology, biology, nuclear physics, and other fields, nonlinear waves involving acoustic waves, hydro-magnetic waves, and acoustic gravity waves make significant contributions [2,3]. Many mathematical models have been developed for explaining those wave behaviors.

The water wave propagation in any channel governs the gravitational body forces and surface tension at the interface between the atmosphere and the water [4]. This kind of propagation relies on the water depth, the wavelength and amplitude of the waves, the slopes of the water surface in two orthogonal directions, the capillary length scale etc. Surface tension can be ignored when the ratio of gravitational force to surface tension is very high. Soliton solutions of the evolution equations have a great impact in the direction of pulse propagation via optical fibers for trans-continental and trans-oceanic distances [5].

Dimensionless condition of the dependent and independent variables, boundary conditions and governing equation presents the dynamics of water waves is acknowledged by the anisotropy, amplitude, steepness and topography parameters [6]. The first ratio is between water depth and wave amplitude, and the second ratio is between wavelength and water depth. The parameters of the topography concerned to the slopes of the channel’s topography in two orthogonal directions, when the anisotropy represents the dependence of the velocity of propagation on direction.

Searching exact wave solutions is very important for understanding the internal mechanism of dynamical behaviour of the nonlinear wave phenomena. In accordance with the above discussion, a variety of schemes, namely the finite difference scheme [7], meshless method [8], tanh-coth technique [9], partial Noether’s approach [10], generalized \((G'/G)\)-expansion scheme [11–13], distributed approximating functional method [14], simplified Hirota’s technique [15], Chebyshev-Chebyshev spectral collocation scheme [16], auto-and non-auto-Bäcklund transformations approach [17–19], Lie group analysis [20–22], generalized Kudryashov scheme [23,24], binary Darboux transformation approach [25], Hetero-Bäcklund transformation

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technique [26], Lie symmetry technique [27,28], sine-Gordon expansion [29] and others [30,31] were suggested and established.

For extracting analytic solutions to the NLEEs in nonlinear physical processes, the advanced generalized \( (G'/G) \)-expansion technique is particularly influential and compatible. Such scheme has been adopted to investigate solitary wave solutions to the Vakhnenko-Parkes equation [32], the stochastic new coupled Konno-Oono equation [33], the sine-Gordon and Burgers equation [34], the Korteweg-de Vries equation [35], the coupled Jauntel-Miodek equation [36], the Sharma-Tasso-Olver and Klein–Gordon equation [37], the Boussinesq equation [38] etc.

A group of researchers used several approaches to investigate exact wave solutions to the RLC [39–45] and RW [46,47] equations. Raslan et al. [1] examined exact solutions through the coupled generalized RLC equations by taking advantage of the sine-cosine function algorithm and the Kudryashov scheme. Roshid et al. [3] determined solitary wave in plasma, ion-acoustic plasma and shallow water to the RLC equation via the modified simple equation technique. By the aid of exponential and compact-operator approaches, Garcia-Lopez, and Ramos [4] extracted solitary waves generated by bell-shaped initial conditions in the inviscid and viscous generalized RLC equations. Further, the optimal perturbation iteration approach has been applied by Bildik and Deniz [41] to find the analytic approximate solutions to the generalized RLC model. In Ref. [9] the author explained the dark and bright soliton solutions and computational modeling of nonlinear 1D and 2D RLC models. Barman et al. [2] investigated the competent closed form solition solutions to the RW equation through the generalized Kudryashov technique and accomplished bell-shaped, consolidated bell-shaped, compacton, singular kink shape, flat kink and other soliton solutions. Duran [46] computed the breaking theory of solitary waves for the RW equation in fluid dynamics by utilizing the generalized exponential rational function scheme. Burman [47] established the physically important wave solutions to the RW model using the expanded tanh-function approach and achieved dark soliton, bright soliton, peakon, compacton, periodic and other type soliton solutions in ion-acoustic and magneto-sound waves in plasma. However, no one has used the sophisticated generalized \( (G'/G) \)-expansion technique yet to analyze the above reported models. Therefore, motivated by the recent focus of research, the purpose of this study is to explore the exact soliton solutions by adopting the advanced and generalized \( (G'/G) \)-expansion method having substantial applications in different branches of nonlinear science and engineering. Moreover, we investigate the dynamic wave envelopes which are very realistic and timely, and also they are widely used in plasma physics.

The coordination of the rest parts are as follows: In part 2, we mention briefly the methodology of the advanced and generalized \( (G'/G) \)-expansion scheme. The applications of the RLC and RW models are described smoothly in part 3. Part 4 contains the graphical structures involving 3D and contour plots and discusses the physical interpretations elaborately. Finally, the concluding remark is drawn in details in part 5.

**Methodology**

Let us suppose a general nonlinear partial differential equation (PDE) in the form

\[
\mathcal{F}\left(u, u_x, u_y, u_{xx}, u_{yy}, u_t, u_{xt}, u_{yy}, u_{yy}, u_{xx}, u_{xy}, u_{xt}, \ldots \right) = 0
\]

where \( \mathcal{F} \) is a polynomial in \( u, u(x, y, t) = u(\zeta) \) is an undetermined function, and the derivatives involving nonlinear terms and the subscripts refer to the partial derivatives.

**Step 1:** We compile the real variables \( x, y \) and \( t \) by the traveling coordinate \( \zeta \)

\[
u(x, y, t) = u(\zeta), \quad \zeta = gx + hy - \mu t
\]

where \( g, h \) denote the wave numbers and \( \mu \) be the velocity of the solitary wave. The assumption (2) transmutes Eq. (1) into a nonlinear differential equation (NLDE) due for \( u = u(\zeta) \):

\[
\mathcal{F}(u, \dot{u}, \ddot{u}, \ldots) = 0
\]

where \( \mathcal{F} \) be the polynomial of \( u \) with its derivatives and the superscripts reveal the ordinary derivatives in terms of \( \zeta \).

When \( \zeta = 0 \) and \( h = 1 \), the wave transformation (2) turns into

\[
u(y, t) = u(\zeta), \quad \zeta = y - \mu t
\]

**Step 2:** As per possibility, we integrate Eq. (3) one or more times, arises integration constant(s) that should be set zero due to avoid complexity.

**Step 3:** The exact wave solution of Eq. (3) considered as:

\[
u(\zeta) = \sum_{n=0}^{\infty} a_n (d + \mathcal{H}') + \sum_{n=1}^{\infty} b_n (d + \mathcal{H}')^{-n}
\]

where at either \( a_n \) or \( b_m \) may be zero, but both \( a_m \) and \( b_n \) cannot be zero at the same time, \( a_l(l = 0, 1, 2, \ldots, m) \) and \( b_l(l = 1, 2, \ldots, m) \) and \( d \) are arbitrary constants to be estimated subsequently and \( \mathcal{H}'(\zeta) \) is given by

\[
\mathcal{H}(\zeta) = (G'/G)
\]

where \( G = G(\zeta) \) satisfies the nonlinear DE given underneath:

\[
EG'' - FGG' - QG^2 - P(G')^2 = 0
\]

The prime stands for the ordinary derivatives in terms of \( \zeta; E, F, P \) and \( Q \) are taken as real parameters.

**Step 4:** To estimate the balance number \( m \), we apply the balancing technique between the greatest derivatives and the highest order nonlinear terms arise in Eq. (3).

**Step 5:** Transmuting Eqs. (5 and 7) including Eq. (6) into Eq. (3) having \( m \) achieved in Step 4, it gives a polynomial in \( d + \mathcal{H}' \) \( m = 0, 1, 2, \ldots \) and \( (d + \mathcal{H}')^{-m} \) \( m = 0, 1, 2, \ldots \). It might be congregated each coefficient of the ensued polynomial to zero delivers a cluster of algebraic equations for \( a_l(l = 0, 1, 2, \ldots, m), b_l(l = 1, 2, \ldots, m) \) and \( d \) may be fabricated by evaluating the algebraic equations acquired in Step 5. As the general solutions of Eq. (7) are recognized, inserting the values of \( a_l(l = 0, 1, 2, \ldots, m), b_l(l = 1, 2, \ldots, m), d \) and \( \mu \) into solution (5), we obtain more general type and new exact traveling wave solutions to the nonlinear evolution equation (1).

Introducing the general solution of Eq. (7), we have the next solutions of Eq. (6):

**Family 1:** For \( F \neq 0, \phi = E - P \) and \( \gamma = F^2 + 4Q(E - P) > 0 \),

\[
\mathcal{H}(\zeta) = \frac{G'}{G} = \frac{F}{2\phi} + \frac{\sqrt{\phi}}{2\phi} k_{11} \sinh\left(\frac{\sqrt{\phi}}{2\phi} \zeta\right) + k_{12} \cosh\left(\frac{\sqrt{\phi}}{2\phi} \zeta\right)
\]

**Family 2:** For \( F \neq 0, \phi = E - P \) and \( \gamma = F^2 + 4Q(E - P) < 0 \),

\[
\mathcal{H}(\zeta) = \frac{G'}{G} = \frac{F}{2\phi} - \frac{\sqrt{-\phi}}{2\phi} k_{11} \sinh\left(\frac{\sqrt{-\phi}}{2\phi} \zeta\right) + k_{12} \cosh\left(\frac{\sqrt{-\phi}}{2\phi} \zeta\right)
\]

**Family 3:** For \( F \neq 0, \phi = E - P \) and \( \gamma = F^2 + 4Q(E - P) = 0 \),

\[
\mathcal{H}(\zeta) = \frac{G'}{G} = \frac{F}{2\phi} + \frac{k_{11}}{k_{11}^2 + k_{12}^2}
\]

**Family 4:** For \( F \neq 0, \phi = E - P \) and \( \epsilon = \phi Q > 0 \),
\( \mathcal{R}(\zeta) = (G / G) = \sqrt{\frac{\phi}{k_{12}}} \sinh \left( \frac{c\tau}{T} \zeta \right) + k_{12} \cosh \left( \frac{c\tau}{T} \zeta \right) \)

\( \mathcal{R}(\zeta) = (G / G) = \sqrt{-\frac{\phi}{k_{11}}} \sinh \left( \frac{c\tau}{T} \zeta \right) + k_{11} \cosh \left( \frac{c\tau}{T} \zeta \right) \)

\( a_1 = -\frac{6b\phi^2}{a(E^2 + 4F^2b + 16Qb\phi)}, \quad b_1 = 0, b_2 = -\frac{3b(4Q\phi + F^2)^2}{8a\phi^2(E^2 + 4F^2b + 16Qb\phi)} \)

\( \text{Set-2: } \mu = \frac{\mu}{E^2 + 4F^2b + 16Qb\phi}, \quad a_0 = \frac{6b(4b^2 - 4Q\phi + F^2)}{a(E^2 + 4F^2b - 4Qb\phi)} a_1 = \frac{6b(4b^2 + F^2)}{a(E^2 + 4F^2b - 4Qb\phi)}, \quad a_1 = -\frac{6b\phi^2}{a(E^2 + 4F^2b - 4Qb\phi)} b_1 = 0, b_2 = 0 \)

\( \text{Set-3: } \mu = \frac{\mu}{E^2 + 4F^2b - 4Qb\phi}, \quad a_0 = \frac{6b(4b^2 - 4Q\phi + 3F \phi + F^2)}{a(E^2 + 4F^2b - 4Qb\phi)}, \quad a_1 = \frac{6b(4b^2 - F^2)}{a(E^2 + 4F^2b - 4Qb\phi)} \)

\( \text{Set-4: } \mu = \frac{\mu}{E^2 + 4F^2b - 4Qb\phi}, \quad a_0 = \frac{6b(4b^2 - 4Q\phi + 3F \phi + F^2)}{a(E^2 + 4F^2b + 4Qb\phi)} a_1 = 0, a_2 = 0, \quad b_1 = \frac{6b(2\phi^2 + EF + 3F \phi + F^2)}{a(E^2 + 4F^2b - 4Qb\phi)} b_2 = \frac{6b(2\phi^2 - 2Q \phi \phi + Q^2 + F^2 \phi^2 - 2Q \phi \phi + 2F \phi^2)}{a(E^2 + 4F^2b - 4Qb\phi)} \)

\( \text{Set-5: } \mu = \frac{\mu}{E^2 + 4F^2b + 4Qb\phi}, \quad a_0 = \frac{6b(4b^2 - 4Q\phi + 3F \phi + F^2)}{a(E^2 + 4F^2b + 4Qb\phi)} a_1 = 0, a_2 = 0, \quad b_1 = \frac{6b(2\phi^2 + EF + 3F \phi + F^2)}{a(E^2 + 4F^2b + 4Qb\phi)} b_2 = \frac{6b(2\phi^2 - 2Q \phi \phi + Q^2 + F^2 \phi^2 - 2Q \phi \phi + 2F \phi^2)}{a(E^2 + 4F^2b + 4Qb\phi)} \)

\( \text{Set-6: } d = -\frac{D}{E^2 + 4F^2b - 16Qb\phi}, \quad a_0 = \frac{3b(4b^2 - F^2)}{a(E^2 + 4F^2b - 16Qb\phi)} a_1 = 0, \quad a_2 = -\frac{6b\phi^2}{a(E^2 + 4F^2b - 16Qb\phi)} b_1 = 0, b_2 = -\frac{3b(4Q\phi + F^2)^2}{8a\phi^2(E^2 + 4F^2b - 16Qb\phi)} \)

Analysis of solutions

In this subdivision, we discuss the application of the approaches through two important examples, the nonlinear RLW and RW models via the advanced and generalized \((G / G)\)-expansion method.

The generalized Regularized-Long-Wave model

Let us consider the generalized RLW model \([8]\) as:

\[ u_t + u_x + 2p(\kappa^x)_x - bu_{yy} = 0 \]  

where \( p \) and \( b \) are positive constants and \( q \) be a positive integer. Choosing \( q = 2 \) in Eq. (13) it takes a special form named regularized long wave (RLW) equation:

\[ u_t + u_x + auu - bu_{yy} = 0 \]  

where \( a = 2p \) is considered. Peregrine was the first who introduced the RLW model for small-amplitude and long-wave on the surface water passing through a channel \([48]\). The model works as an alternative model to the Korteweg-de Vries equation in practical fields like long-crested waves in near-shore zones, unidirectional propagating waves in a water channel and many other similar issues. Applying the transformation \((4) \) into Eq. (14), we obtain a nonlinear equation in the following form:

\[(1 - \mu)u + au^2 + \mu bu'' = 0 \]

Balancing technique between \( u_t \) and \( u'' \) in \((14) \) yields the balance number \( m = 2 \). Thus, the solution structure of the Eq. (15) takes the form:

\[ u(\zeta) = a_0 + a_1 (d + \mathcal{R}) + b_1 (d + \mathcal{R})^{-1} + a_2 (d + \mathcal{R})^2 + b_2 (d + \mathcal{R})^{-2} \]

where \( a_0, a_1, a_2, b_1, b_2 \) are constants to be computed. Replacing \((16) \) along with \((6) \) and \((7) \) into Eq. (15), the left side changed into polynomial in \((d + \mathcal{R})^l (l = 0, 1, 2, \ldots) \) and \((d + \mathcal{R})^{-l} (l = 0, 1, 2, \ldots) \). Taking each coefficient of the attained polynomial and putting them to zero becomes an over-estimated set of algebraic equations for \( a_0, a_1, a_2, b_1, b_2, d, \) and \( \mu \). The algebraic equations are ignored herewith for simplicity. Computing the algebraic equations by means of the Maple computation software, we accomplish the six sets of solutions as appeared underneath:

Set-1: \( d = \frac{E}{2F}, \mu = \frac{E^2 + 4F^2b + 16Qb\phi}{a(E^2 + 4F^2b + 16Qb\phi)} a_0 = \frac{b(4Q\phi + F^2)}{a(E^2 + 4F^2b + 16Qb\phi)} a_1 = 0 \)

\[ u_1(y,t) = -\frac{byr}{a(E^2 + 4F^2b + 16Qb\phi)} \left( 1 + \frac{K_{12}}{K_{11} + K_{12}(y - \mu)} \right)^2 + \frac{3}{8a\phi^2} \left( \frac{K_{12}}{K_{11} + K_{12}(y - \mu)} \right)^{-2} \]
We insert (17) into (16) along with (10) by employing Family-3 and after simplification the solution takes the form

Substituting (17) into (16) accompany with (11) by means of the Family-4 and after simplification, we achieve the wave solutions as \((K_{11} \neq 0, K_{12} = 0 \text{ and } K_{12} \neq 0, K_{11} = 0)\):

\[
u(x, t) = \frac{-b}{a(E^2 + 4F^2b + 16be)} \left( 4e + F^2 \right) + 6\phi^2 \left( -\frac{F}{2\phi} + \frac{\sqrt{e}}{\phi} \tanh \left( \frac{\sqrt{e}}{E} (y - \mu t) \right) \right)^2 + \frac{3(4e + F^2)}{8\phi^2} \left( -\frac{F}{2\phi} + \frac{\sqrt{e}}{\phi} \tanh \left( \frac{\sqrt{e}}{E} (y - \mu t) \right) \right)^{-2}
\]

Moreover, placing (17) into (16) along with (12) by the aid of Family-5 and simplifying, we extract the exact solutions as \((K_{11} \neq 0, K_{12} = 0 \text{ and } K_{12} \neq 0, K_{11} = 0)\):

\[
u(x, t) = \frac{-b}{a(E^2 + 4F^2b + 16be)} \left( 4e + F^2 \right) + 6\phi^2 \left( -\frac{F}{2\phi} + \frac{\sqrt{e}}{\phi} \coth \left( \frac{\sqrt{e}}{E} (y - \mu t) \right) \right)^2 + \frac{3(4e + F^2)}{8\phi^2} \left( -\frac{F}{2\phi} + \frac{\sqrt{e}}{\phi} \coth \left( \frac{\sqrt{e}}{E} (y - \mu t) \right) \right)^{-2}
\]

It should be highlighted that the above estimations have not been accomplished in former studies which are crucial for analyzing nonlinear wave phenomena including shallow water waves, tidal and tsunami waves, ion-acoustic, electromagnetic waves and magnetosound waves, etc.

*The Riemann wave equation*

The \((2+1)\)-dimensional generalized breaking soliton equation \([53]\) is given by

\[u_t + au_{xx} + buw_t + cw_t + duw_x + ew = 0\]

(23)

where \(a, b, c, d, e, f\) be the real arbitrary parameters. The \((2 + 1)\)-dimensional interaction of the Riemann wave is interpreted by the Eq. (23) in which overlapping solutions have been created for the particular case \(a = 0\). The spectral parameter is entitled by the reputed breaking behaviour which is the principal nature of such types of equations. Otherwise, the spectral value is treated as multivalued function. Thus, the solution types of these equations would be multivalued. Therefore, the Eq. (23) is associated with the \((2+1)\) dimensional generalized breaking soliton equations \([49]\) as:

\[u_t + au_{xx} + buw_t + cw_t + dw_x + ew = 0\]

(24a)

\[u_x = w_s\]

(24b)

Xu [49] showed the Eqs. (24a) and (24b) have many special cases investigated by Calogero and Degasperis [50], Bogoyavlenskii [51], Radha and Laksmanan [52], and Zhang et al. [53] for different particular values. Following this technique, we consider the RW model yield from the Eqs. (24a and 24b) is of the structure [47]:

\[u_t + fu_{xx} + aw_x + buw = 0\]

(25a)

\[u_x = w_s\]

(25b)

where into \(a, b, f\) stand for the real numbers. To extract the traveling wave solutions, placing the assumption \((2)\) into (25a and 25b), and decomposed into the ensuing system

\[h \frac{g^2}{l} u'''' + agw' + bwa' = 0\]

(26)

\[hu' = gw'.\]

(27)

Integrating the Eq. (27) gives

\[w = \frac{h}{g}\]

(28)

wherein the integral constant is presumed zero. Removing \(w\) and \(w'\) from the equation (26) provide the following nonlinear equation

\[h \frac{g^2}{l} u'''' + h(a + b)uw' - 2mu' = 0\]

(29)

The new form of equation (29) is obtained by integrating first time
2\h f_2 g^2 u^{'}^{'} + h(a + b)u^{'}^{2} - 2\mu u = 0 \tag{30}

The equation (30) consists of linear term \( u^{'}^{'} \) and nonlinear term \( u^{'}^{2} \). Actually, the solitary wave is generated by the combination of two factors. The terms \( u^{'}^{'} \) and \( u^{'}^{2} \) lead to an index number \( m = 2 \). Therefore, the solution structure (3) is written as

\[ u(x, t) = a_0 + a_1(x + \Delta x) + b_1(y + \Delta y)^{-1} + a_2(x + \Delta x)^2 + b_2(y + \Delta y)^{-2} \tag{31} \]

where about \( a_0, a_1, b_1, a_2, b_2 \) are constants to be computed. In accordance with, we achieve the new and rich sets of solutions as follows:

Set-1: \( d = \frac{b}{2P} = \mu = \frac{h(Eb_{a_{10}} + Ea_{a_{10}} - 2(F_{a_{30}} + 2g_{a_{30}}))}{2E}, \quad a_1 = 0, \quad a_2 = \frac{-12g_{a_{30}}}{E(a + b)} \tag{32} \)

where at \( \phi = E - P, d, E, F, P, Q \) are free parameters.

Set-2: \( a_1 = 0, \quad a_2 = 0, \quad \mu = \frac{h(Eb_{a_{10}} + Ea_{a_{10}} + 12g_{a_{30}} + 2d_{a_{30}} + 2g_{a_{30}})}{2E}, \quad b_1 = \frac{-12g_{a_{30}}(-2d_{a_{30}} - 2d_{a_{30}} + 12g_{a_{30}} + 2d_{a_{30}} + 2g_{a_{30}})}{E(a + b)} \tag{33} \)

Set-3: \( \mu = \frac{h(Eb_{a_{10}} + Ea_{a_{10}} + 12g_{a_{30}} + 2d_{a_{30}} + 2g_{a_{30}})}{2E}, \quad a_1 = \frac{12g_{a_{30}}}{E(a + b)} \tag{34} \)

The more appropriate output named set-1 has been taken from the above stated solutions set-1 to set-3 so that we could acquire more new and suitable results.

Now, transferring Eq. (32) into Eq. (31) having (8) and Family-1, and also simplifying, we accomplish the subsequent traveling wave results

\[ (K_{11} \neq 0, K_{12} = 0 \text{ and } K_{12} \neq 0, K_{11} = 0): \]

\[ u_{01}(x, y, t) = a_0 - \frac{3\h f_2 g^2}{E^2(a + b)} \left( \tanh \left( \frac{\sqrt{2E}}{2} (gx + hy - \mu) \right) \right)^{-1} - 8Q \phi \left( \tanh \left( \frac{\sqrt{2E}}{2} (gx + hy - \mu) \right) \right)^{-1} \]

\[ u_{02}(x, y, t) = a_0 - \frac{3\h f_2 g^2}{E^2(a + b)} \left( \coth \left( \frac{\sqrt{2E}}{2} (gx + hy - \mu) \right) \right)^{-1} - 8Q \phi \left( \coth \left( \frac{\sqrt{2E}}{2} (gx + hy - \mu) \right) \right)^{-1} \]

where \( \zeta = gx + hy - \mu \) and \( \mu = \frac{h(Eb_{a_{10}} + Ea_{a_{10}} - 2(F_{a_{30}} + 2g_{a_{30}}))}{2E} \).

Similarly, inserting the values of (32) into (31) accompany with (8) and elucidating, we attain the wave solutions with the aid of Family-2 as follows

\[ (K_{11} \neq 0, K_{12} = 0 \text{ and } K_{12} \neq 0, K_{11} = 0): \]

\[ u_{03}(x, y, t) = a_0 - \frac{3\h f_2 g^2}{E^2(a + b)} \left( \tan \left( \frac{\sqrt{2E}}{2} (gx + hy - \mu) \right) \right)^{-1} - 8Q \phi \left( \tan \left( \frac{\sqrt{2E}}{2} (gx + hy - \mu) \right) \right)^{-1} \]

\[ u_{04}(x, y, t) = a_0 - \frac{3\h f_2 g^2}{E^2(a + b)} \left( \cot \left( \frac{\sqrt{2E}}{2} (gx + hy - \mu) \right) \right)^{-1} - 8Q \phi \left( \cot \left( \frac{\sqrt{2E}}{2} (gx + hy - \mu) \right) \right)^{-1} \]

Similarly, by taking help of the above solutions, we achieve another set of solutions from equation (25),

\[ w_{05}(x, y, t) = \]

\[ w_{06}(x, y, t) = \]

\[ w_{07}(x, y, t) = \]

Putting the values of Eq. (32) in (31) along with (10) and after simplification the result takes the form utilizing Family-3.
Fig. 1. Parabolic wave profile of solution $u_0(y,t)$ for the wave speed $\mu = 0.02$ and unnamed values of $a = 0.12, b = 0.72, F = -0.13, E = 2.72, Q = -0.22, \gamma = 0.02, \phi = -4.7, K_{11} = -0.77, K_{12} = -0.24$.

Fig. 2. Anti-parabolic wave profile of solution $u_0(y,t)$ for the wave speed $\mu = 0.02$ and unnamed values of $a = 0.12, b = 0.72, F = -0.13, Q = -0.22, \gamma = -0.02, E = 2.72, \phi = -4.7, K_{11} = -0.77, K_{12} = -0.24$.

Fig. 3. Bell type soliton of solution $u_0(y,t)$ regarding the wave speed $\mu = 1.02$ and unfamiliar values of $a = 1.32, b = 0.73, F = -1.75, \epsilon = 0.47, E = 2.72, \phi = -1.16$.

Fig. 4. Anti-bell type soliton of solution $u_0(y,t)$ regarding the wave speed $\mu = 1.02$ and unfamiliar values of $a = 1.32, b = -0.31, F = -1.75, \epsilon = 0.47, E = 2.72, \phi = -1.16$. 
By means of the above stated results, using equation (25) the another set of solutions become,

$$ w_{05}(x, y, t) = \frac{a_0}{E^2(a + b)} \left( \frac{K_1}{K_1 + K_2(gx + hy - \mu t)} \right)^2 - \frac{6Qg^2\varphi}{E^2(a + b)\varphi} \left( \frac{K_2}{K_1 + K_2(gx + hy - \mu t)} \right)^2 $$

Setting Eq. (32) into (31) together with (11) by means of Family-4 and after calculation, we achieve the following exact soliton solutions ($K_{11} \neq 0, K_{12} = 0$ and $K_{13} \neq 0, K_{11} = 0$):

$$ w_{06}(x, y, t) = a_0 - \frac{12Qg^2\varphi^2}{E^2(a + b)} \left( \frac{K_1}{K_1 + K_2(gx + hy - \mu t)} \right)^2 - \frac{6Qg^2\varphi}{E^2(a + b)\varphi} \left( \frac{K_2}{K_1 + K_2(gx + hy - \mu t)} \right)^2 $$

$$ u_{06}(x, y, t) = a_0 - \frac{12Qg^2\varphi^2}{E^2(a + b)} \left( - \frac{F}{2\varphi} + \frac{\sqrt{\varepsilon}}{\varphi} \tanh \left( \frac{\sqrt{\varepsilon}}{E} (gx + hy - \mu t) \right) \right)^2 - \frac{6Qg^2(F^2 + 2\varepsilon)}{E^2(a + b)\varphi} \left( - \frac{F}{2\varphi} + \frac{\sqrt{\varepsilon}}{\varphi} \tanh \left( \frac{\sqrt{\varepsilon}}{E} (gx + hy - \mu t) \right) \right)^2. $$

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Fig. 5. Kink wave evolution of solution $u_{06}(y, t)$ regarding the wave speed $\mu = 3.46$ and unfamiliar values of $a = 0.6, b = -0.04, F = -1.41, \varepsilon = 0.47, E = 2.72, \varphi = 0.32$.

Fig. 6. Solitary wave evolution of solution $u_{07}(y, t)$ corresponding to the wave speed $\mu = 0.50$ and anonymous values of $a = 0.47, b = 0.07, F = 0.92, \varepsilon = -0.01, E = 2.72, \varphi = 1.4$.

Fig. 7. Flat kink soliton of solution $u_{07}(y, t)$ corresponding to the wave speed $\mu = 0.01$ and anonymous values of $a = 1.19, b = 1.06, F = 1.62, E = 2.72, \varphi = -0.97, \varepsilon = 0.01$. 
Fig. 8. Compacton shape of solution $u_{05}(x,y,t)$ at $a = a_0 = -2.05, g = h = Q = 2, \gamma = 0.2, E = 2.72, \phi = 0.02, K_1 = 0.24, K_2 = 1.35$.

Fig. 9. Anti-compacton shape of solution $w_{05}(x,y,t)$ at $b = a = a_0 = -2.05, h = Q = 2, \gamma = 0.2, E = 2.72, \phi = 0.02, K_1 = 0.24, K_2 = 1.35$.

Fig. 10. The interaction solution $u_{06}(x,y,t)$ regarding the standards $a = 0.4, b = 0.5, f = 1.04, F = -1.8, g = -0.3, h = -1.8, Q = -0.9, \epsilon = 0.71, E = 2.72, \phi = -0.2, a_0 = -0.5$.

Fig. 11. Bell type soliton of solution $w_{06}(x,y,t)$ regarding the standards $t = 1, a = -1.2, b = 0.4, f = h = -2, F = 2, g = -1.8, Q = -0.08, \epsilon = 0.4, \mu = 0.01, E = 2.72, \phi = -1.9, a_0 = 0.4$. 
Fig. 12. Smooth kink wave of solution $u_{08}(x,y,t)$ due to the values of $a = 0.4, b = 0.07, f = 0.6, F = 2, g = -1.8, h = a_0 = -2, Q = -1.2, \epsilon = 0.26, E = 2.72, \phi = 0.1$.

Fig. 13. Kink wave profile of solution $w_{08}(x,y,t)$ due to the values of $t = 1, a = a_0 = b = h = \phi = -2, f = 1.6, F = 1.2, Q = 0.2, \epsilon = 0.3, E = 2.72, g = \mu = 2$.

Fig. 14. Propagation of solitary wave of solution $u_{09}(x,y,t)$ when $t = 1, b = -1.4, f = -0.08, F = -1.93, g = 0.001, h = 1.84, Q = 0.1, \epsilon = 0.97, \mu = 0.12, E = 2.72, \phi = -0.5, a - a_0 = -2$.

Fig. 15. Propagation of solitary wave of solution of solution $w_{09}(x,y,t)$ when $t = 1, F = -1.93, g = 0.001, h = 1.84, Q = 0.1, \epsilon = 0.97, \mu = 0.12, E = 2.72, \phi = f = 2, a_0 = b = a = -2$. 
have been widely used in the communication industry with optical fibers, nuclear physics with nuclear fission and fusion reactions, and oceanography with natural disasters such as cyclones, tornadios, and tidal bores, etc.

**Graphical representations of the results**

The Eq. (25) becomes by getting help of the above solutions as:

This segment analyzes the graphical representations and discussions

The established results are illustrative, further generic, and they have not been examined in the previous works. The dynamic wave solutions have been widely used in the communication industry with optical fibers, nuclear physics with nuclear fission and fusion reactions, and of a variety of solitary wave representations of the computed results to the generalized RLW equation and RW equation. The attained results concerned with some unnamed parameters and the graphical structures have effect of those parameters (which are involved in linear and
nonlinear term). That is for the definite values of the free parameter, we accomplish several knowledgeable exact solitary wave shapes. First of all, we have portrayed several types of 3D and contour surfaces for each of the solutions by means of the mathematical software like Wolfram Mathematica and then explained the behavior of those results. The 3D plot shows a relationship between designated dependent and independent variables. Otherwise, the contour graphics demonstrates the 3D wave envelopes in the format of 2D shape.

**Graphical outlines of the results to the generalized RLW equation**

The generalized RLW equation delivers new forms of exact solutions with the combination of trigonometric and hyperbolic structures. Some distinct wave profiles, such as, bell-shaped, anti-bell shaped, kink shaped, parabolic shaped and other soliton profiles are developed from these solutions by choosing the appropriate values of the unfamiliar parameters. These types of wave forms cover different nonlinear phenomena. For example, kink wave interprets the fiber-optic-communication process. It is a technique of transmitting information from one state to another state by sending the pulses of infrared light through the optical fiber. In addition, bell shape profile explains the Rogue wave propagations. Rogue waves are such types of waves whose height are more than twice the height of the significant waves. Rogue waves appear to be caused by a combination of physical aspects, such as tremendous storms and robust currents, which bring about these waves to combine into an extremely unique long wave. The behavior of the solutions \(u_{03}(\xi, \eta, t)\), \(u_{09}(\eta, t)\) and \(u_{09}(\eta, t)\) have been focused on Figs. 1-7.

The solution \(u_{03}(\eta, t)\) represents a parabolic soliton structure for the wave speed \(\mu = 0.02\) and other unknown values \(\alpha = 0.12\), \(b = 0.72\), \(F = -0.13\), \(Q = -0.22\), \(\gamma = 0.02\), \(E = 2.72\), \(\phi = -4.7\), \(K_{11} = -0.77\), \(K_{12} = -0.24\). The 3D envelope is outlined in Fig. 3 within the interval \(-2 \leq \gamma \leq 2\) and the contour envelope is outlined for \(t = 0\). But only the change of the other parameter \(\gamma = -0.02\), this solution reforms a general soliton profile visualized in Fig. 2.

The solution \(u_{09}(\eta, t)\) shows various types of wave profiles, such as, bell shape, anti-bell shape and kink shape regarding the effect of standard associated with the linear and nonlinear factors. The wave velocity \(\mu = 1.02\) and unfamiliar values \(a = 1.32\), \(b = 0.73\), \(F = -1.75\), \(\gamma = 0.47\), \(E = 2.72\), \(\phi = -1.16\) construct the bell type soliton. The 3D profile is portrayed in Fig. 3 within the boundary \(-10 \leq \eta \leq 10, 0 < t \leq 8\) and the contour profile is portrayed for \(t = 0\).

Again, only the effects of parameters \(b = -0.31\) connected with linear terms, this solution develops an anti-bell soliton structure. The 3D and contour profiles are portrayed in Fig. 4 within the same boundary.

Another particular value of the wave speed \(\mu = 3.46\) and unfamiliar parameters \(a = 0.6, b = -0.04, F = -1.41, \gamma = 0.47, E = 2.72, \phi = 0.32\), we take the smooth kink soliton shape. The movement of this profile is portrayed in Fig. 4 within the same boundary.

The solution \(u_{07}(\eta, t)\) indicates a general soliton form corresponding to the wave speed \(\mu = 0.50\) and other anonymous values \(a = 0.47, b = 0.07, F = 0.92, \gamma = -0.01, E = 2.72, \phi = 1.4\). The 3D plot is delineated in Fig. 6 in the region \(0 \leq \eta \leq 2.0 < t \leq 2\) and the contour plot is delineated for \(t = 0\). For other values of the wave speed \(\mu = 2\) and anonymous parameters \(a = b = F = \phi = -2, E = 2.72, e = -1.64\), this solution yields a periodic wave structure whose behaviors are outlined in Fig. 7 in the region \(-8 \leq \eta \leq 8.0 < t \leq 5\).

**Graphical outlines of the results to the RW equation**

This paragraph covers the graphical depictions of certain derived results to the RW equation. As each result contains some unfamiliar parameters, some well-known profiles, such as, compacton, parabolic soliton, bell type soliton, kink soliton and other solitons are established by receiving the arbitrary values of parameters associated with the linear and nonlinear factors. Such types of waves conduct with like many nonlinear wave phenomena, as for instance, ion-acoustic and magneto-sound waves in plasma, cluster of hydrodynamic model, fission, and fusion interaction etc. In the following, we have discussed about the behavior of solutions \(u_{05}(\eta, \xi, t), u_{06}(\xi, \eta, t), u_{06}(\eta, \xi, t), u_{08}(\xi, \eta, t), u_{09}(\eta, \xi, t)\) and \(w_{09}(\xi, \eta, t)\) whose propagations are delineated in Figs. 9-15.

The solution \(u_{05}(\xi, \eta, t)\) demonstrates an evolutionary shape defined as compacton for assigning parametric values \(a = b = a_0 = -2, f = -0.03, g = 2, \gamma = 0.2, E = 2.72, \phi = 0.02, K_1 = 0.24, K_2 = 1.35\). Compactons are such type of solitary waves with finite wave-length, compact support, and free of exponential tails. Moreover, this solution is stable in view of satisfying the stability requirements of compacton solution. The 3D and contour envelopes are outlined in Fig. 8 within the limit \(-10 \leq \xi \leq 10\). But only the effect of parameter in linear term \(g = -2\), solution \(w_{05}(\xi, \eta, t)\) behaves an anti-compacton shape portrayed in Fig. 9.

The solution \(u_{06}(\xi, \eta, t)\) reveals a solitary structure for appointing the parametric values of \(a = 0.4, b = 0.5, f = 1.04, E = -1.8, g = -0.3, h = -1.8, Q = -0.9, \gamma = 0.71, E = 2.72, \phi = -0.2, a_9 = -0.5\). The behavior of this shape is sketched in Fig. 10 within the interval \(-10 \leq \xi \leq 10\). At the time of \(t = 1\), the solution \(w_{06}(\eta, \xi, t)\) represents a bell type soliton for other parametric values of \(a = -1.2, b = 0.4, f = h = -2, E = 2.72, g = -1.8, Q = -0.08, \gamma = 0.4, \mu = 0.01, E = 2.72, \phi = -1.9, a_0 = 0.4\) sketched in Fig. 11.

The solution \(u_{08}(\xi, \eta, t)\) exposes a kink wave profile relating to the values \(a_0 = 0.4, b = 0.07, f = 0.6, F = 2.72, g = -0.3, E = 2.72, \phi = -0.1\). The wave starts from the ground level and after a while it moves with a fixed velocity. The 3D and contour surfaces are visualized in Fig. 12 in the region \(-8 \leq \eta, \xi \leq 8\). Moreover, choosing the temporal value \(t = 1\) and other parametric value \(a = a_0 = b = \phi = -2, f = 1.6, E = 1.2, Q = 0, \gamma = 0.3, E = 2.72, g = \mu = 0.2\), we receive the identical profile, but the direction is diverse visualized in Fig. 13.

The solution \(u_{09}(\xi, \eta, t)\) denotes a general soliton structure for allocating the parametric values \(a = -1.4, f = -0.08, F = -1.93, h = 0.001, g = 1.84, Q = 0.1, \gamma = 0.97, \mu = 0.12, E = 2.72, \phi = -0.5, \alpha = a_0 = -2\). The movement of this wave is displayed in Fig. 14 within the boundary \(-12 \leq \xi, \eta \leq 12\). At the time \(t = 1\) and other values of standards \(F = -1.93, g = 0.001, h = 1.84, Q = 0.1, \gamma = 0.97, \mu = 0.12, E = 2.72, \phi = -0.2, a_0 = b = a = -2, \) solution \(w_{09}(\xi, \eta, t)\) represents another soliton profile shown in Fig. 15.

**Conclusion**

Bright-dark, parabolic, anti-parabolic, compacton, anti-compacton, bell-shaped, anti-bell-shaped, kink, flat kink, smooth kink-shaped solutions have been determined in this article which are originated in plasma, rivers and oceans, transmission lines, optical fibers, and other sources. The fiber-optical communication process, energy dissipation, superconductivity, rogue wave propagations, super deformed nuclei, phenomena of nuclear fission and internal fusion etc. can effectively be explained by means of the 3D and contour portrayals. The advanced and generalized \((G'/G)\)-expansion scheme is capable to yield such types of solutions. It is established that the changes in the nature of the wave are explicitly related to the free parameters involving the linear and nonlinear effects. The computed results comprise the trigonometric,
hyperbolic, and rational structures. The results are extremely effective in several disciplines of plasma physics where solitary wave theories are studied. The method is further reliable and applicable due to the small size of computational domain.

CRediT authorship contribution statement

Ripan Roy: Conceptualization, Methodology, Formal analysis, Resources, Writing - original draft, Visualization. Hemontoma Kumar Barman: Software, Investigation, Data curation, Validation, Writing - review & editing. M. Nurul Islam: Software, Formal analysis, Writing - review & editing, Visualization. M. Ali Akbar: Project administration, Supervision, Writing - review & editing, Funding acquisition.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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