UNIVERSAL DEFORMATION RINGS OF STRING MODULES OVER A CLASS OF
SELF-INJECTIVE SPECIAL BISERIAL ALGEBRAS

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Abstract. Let $k$ be an algebraically closed field, let $\Lambda$ be a finite dimensional $k$-algebra and let $V$ be a $\Lambda$-module with stable endomorphism ring isomorphic to $k$. If $\Lambda$ is self-injective, then $V$ has a universal deformation ring $R(\Lambda, V)$, which is a complete local commutative Noetherian $k$-algebra with residue field $k$. Moreover, if $\Lambda$ is further a Frobenius $k$-algebra, then $R(\Lambda, V)$ is stable under syzygies. We use these facts to determine the universal deformation rings of string $\Lambda_N$-modules whose corresponding stable endomorphism ring is isomorphic to $k$, and which lie in a connected component of the stable Auslander-Reiten quiver of $\Lambda_N$ containing a module with endomorphism ring also isomorphic to $k$, where $N \geq 1$ and $\Lambda_N$ is a self-injective special biserial $k$-algebra whose Hochschild cohomology ring is a finitely generated $k$-algebra (as proved by N. Snashall and R. Taillefer).

1. Introduction

Let $k$ be a field of arbitrary characteristic, and denote by $\hat{\mathcal{C}}$ the category of all complete local commutative Noetherian $k$-algebras with residue field $k$. Suppose that $\Lambda$ is a fixed finite dimensional $k$-algebra and let $V$ be a finitely generated $\Lambda$-module. We denote by $\text{End}_\Lambda(V)$ (resp., by $\text{End}_\Lambda(V)$) the endomorphism ring (resp., the stable endomorphism ring) of $V$ (see e.g. [3, §IV.1]). Let $R$ be an arbitrary object in $\hat{\mathcal{C}}$. A lift $(M, \phi)$ of $V$ over $R$ is a finitely generated $R \otimes_k \Lambda$-module $M$ that is free over $R$ together with an isomorphism of $\Lambda$-modules $\phi : k \otimes_R M \to V$. If $\Lambda$ is self-injective and the stable endomorphism ring of $V$ is isomorphic to $k$, then there exists a particular object $R(\Lambda, V)$ in $\hat{\mathcal{C}}$ and a lift $(U(\Lambda, V), \phi_{U(\Lambda, V)})$ of $V$ over $R(\Lambda, V)$, which is universal with respect to all isomorphism classes of lifts of $V$ over such $k$-algebras $R$ (see [14] and §2). The ring $R(\Lambda, V)$ and the isomorphism class of the lift $(U(\Lambda, V), \phi_{U(\Lambda, V)})$ are respectively called the universal deformation ring and the universal deformation of $V$. Traditionally, universal deformation rings are studied when $\Lambda$ is equal to a group algebra $kG$, where $G$ is a finite group and $k$ has positive characteristic $p$ (see e.g., [5, 6, 7, 8, 9, 10, 11]). In particular, it was proved in [7] that if $V$ is a finitely generated $kG$-module whose stable endomorphism ring is isomorphic to $k$, then $V$ has a universal deformation ring $R(G, V)$. Observe that $kG$ is an example of a self-injective $k$-algebra (see e.g., [4, Prop. 3.1.2]). This approach has led to the solution of various open problems, e.g., the construction of representations whose universal deformation rings are not local complete intersections (see [5, 8, 9]). On the other hand, in [12, 14, 30], universal deformation rings for certain self-injective algebras, which are not Morita equivalent to a block of a group algebra, are also discussed. More recently, it follows from [13, Prop. 3.2.5] that the isomorphism class of universal deformation rings of modules whose stable endomorphism ring isomorphic to $k$ is also an invariant under stable equivalences of Morita type (as introduced by M. Broué in [15]) between self-injective $k$-algebras.

Deformation of modules over more general and other types of finite dimensional algebras have been studied by various authors in different contexts (see e.g., [2, 22, 31] and their references).

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In this article, we assume that \( k \) is algebraically closed, that \( m \) is an integer with \( m \geq 3 \), and we consider the basic \( k \)-algebra \( \Lambda_N \) as in Figure 1, where \( N \geq 1 \).

\[
Q = \begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

\[
I_N = (a_{i+1}a_i, \bar{a}_{i-1}a_i, (\bar{a}_ia_i)^N - (a_{i-1}\bar{a}_{i-1})^N : i \in \mathbb{Z}/m).
\]

**Figure 1.** The basic \( k \)-algebra \( \Lambda_N = kQ/I_N \).

These algebras \( \Lambda_N \) are all basic self-injective (so Frobenius by e.g. [28, Cor. 4.3]), and have been studied by various authors in different contexts. For example, when \( m = 3 \) and \( N \geq 2 \), it follows from [21, Thm. 3.4] that \( \Lambda_N \) is a derived equivalence class representative for a certain family of dihedral algebras with three simple modules that are Morita equivalent to a block of a group algebra. If \( m = 3 \) and \( N = 1 \), then \( \Lambda_1 \) is an algebra of dihedral type of polynomial growth as discussed by K. Erdmann and A. Skowroński in [20, §4]. Moreover, it follows from [18, §V.2.4.1] that if the characteristic of \( k \) is equal to 2, then \( \Lambda_1 \) is isomorphic to the group algebra of the alternating group \( A_4 \). This situation was also discussed in the context of universal deformation rings by F. M. Bleher and S. N. Talbott in [12, Prop. 3.1(iii)]. In a more general setting, N. Snashall and R. Taillefer proved in [29] that the Hochschild rings of the algebras \( \Lambda_N \) are finitely generated \( k \)-algebras. The reader is invited to look at the references in e.g. [19, 25, 29] for getting further information concerning the algebras \( \Lambda_N \) in different settings.

Since \( \Lambda_N \) is a self-injective special biserial algebra (as introduced in [16]), all the non-projective indecomposable \( \Lambda_N \)-modules can be described combinatorially by using so-called string and bands for \( \Lambda_N \); the corresponding indecomposable \( \Lambda_N \)-modules are called string and band modules. In this article, we are only interested in these string \( \Lambda_N \)-modules.

The aim of this article is to prove the following result (for more specific results and details see Remark 3.3, Corollary 3.4, and Propositions 4.1, 4.4, 4.7 and 4.8).

**Theorem 1.1.** Let \( \Lambda_N \) be the basic \( k \)-algebra as in Figure 1, and let \( \Gamma_s(\Lambda_N) \) denote the Auslander-Reiten quiver of \( \Lambda_N \).

(i) If \( N = 1 \), then there are at most finitely many components \( C \) of \( \Gamma_s(\Lambda_N) \) containing a string \( \Lambda_N \)-module of minimal string length whose endomorphism ring is isomorphic to \( k \).

(ii) Assume that \( N \geq 2 \).

(ii.a) If \( m \) is odd, then there are at most finitely many components \( C \) of type \( \mathbb{Z}\Lambda_N^\infty \) of \( \Gamma_s(\Lambda_N) \) containing a string \( \Lambda_N \)-module \( V \) with \( \text{End}_{\Lambda_N}(V) \cong k \).

(ii.b) If \( m \) is even, then there are infinitely many components \( C \) of type \( \mathbb{Z}\Lambda_N^\infty \) of \( \Gamma_s(\Lambda_N) \) containing a string \( \Lambda_N \)-module \( V \) with \( \text{End}_{\Lambda_N}(V) \cong k \).

(iii) Let \( C \) be a component of \( \Gamma_s(\Lambda_N) \) containing a \( \Lambda_N \)-module \( V \) whose endomorphism ring is isomorphic to \( k \), and let \( W \) be a \( \Lambda_N \)-module lying in \( C \) with \( \text{End}_{\Lambda_N}(W) \cong k \).

(iii.a) If \( N = 1 \), then the universal deformation ring \( R(\Lambda,W) \) is isomorphic to \( k \).
(iii.b) If \( N \geq 2 \), then the universal deformation ring \( R(\Lambda, W) \) is isomorphic either to \( k \), or to \( k[[t]]/(t^N) \), or to \( k[[t]] \).

(iv) Let \( T \) be a tube of \( \Gamma_\Lambda(\Lambda_N) \) containing only string \( \Lambda_N \)-modules, and let \( W \) be a \( \Lambda_N \)-module lying in \( T \) with \( \text{End}_{\Lambda_N}(W) \cong k \). Then \( R(\Lambda_N, W) \) is isomorphic either to \( k \) or to \( k[[t]] \).

This article is organized as follows. In §2, we recall the definitions of deformations and universal deformation rings and summarize some of their properties. In §3, we describe the radical series of the indecomposable projective modules for \( \Lambda_N \), and classify all \( \Lambda_N \)-modules whose endomorphisms rings are isomorphic to \( k \) (see Proposition 3.2). In §4, we prove Theorem 1.1. For the convenience of the reader, in §A we describe some basic aspects concerning the representation theory of \( \Lambda_N \), including a precise description of string modules for \( \Lambda_N \) and of the corresponding components of \( \Gamma_\Lambda(\Lambda_N) \) using hooks and co-hooks (see [16]). Moreover, we also give a description of the homomorphisms between string modules as determined by H. Krause in [23].

We refer the reader to e.g., [1, 3, 4, 18] for getting further information about basic concepts from the representation theory of finite dimensional algebras such as the definition and properties of the syzygy functor \( \Omega \) as well as the definition of the stable Auslander-Reiten quiver of an arbitrary Artinian algebra \( \Lambda \).

2. Preliminaries Results on Deformations and Universal Deformation Rings

Let \( k \) be a field of arbitrary characteristic and denote by \( \hat{C} \) the category of all complete local commutative Noetherian \( k \)-algebras with residue field \( k \). Note that the morphisms in \( \hat{C} \) are continuous \( k \)-algebra homomorphisms that induce the identity map on \( k \). Suppose that \( \Lambda \) is a finite dimensional \( k \)-algebra and \( V \) is a fixed finitely generated \( \Lambda \)-module. As in §1, we denote by \( \text{End}_\Lambda(V) \) (resp., by \( \hat{\text{End}}_\Lambda(V) \)) the endomorphism ring (resp., the stable endomorphism ring) of \( V \). Let \( R \) be an arbitrary object in \( \hat{C} \). A lift \((M, \phi) \) of \( V \) over \( R \) is a finitely generated \( R \otimes_k \Lambda \)-module \( M \) that is free over \( R \) together with an isomorphism of \( \Lambda \)-modules \( \phi : k \otimes_R M \to V \). Two lifts \((M, \phi) \) and \((M', \phi') \) over \( R \) are isomorphic if there exists an \( R \otimes_k \Lambda \)-module isomorphism \( f : M \to M' \) such that \( \phi' \circ (\text{id}_k \otimes f) = \phi \), where \( \text{id}_k \) denotes the identity map on \( k \). If \((M, \phi) \) is a lift of \( V \) over \( R \), we denote by \([M, \phi] \) its isomorphism class and say that \([M, \phi] \) is a deformation of \( V \) over \( R \). We denote by \( \text{Def}_\Lambda(V, R) \) the set of all deformations of \( V \) over \( R \). The deformation functor over \( V \) is the covariant functor \( \hat{F}_V : \hat{C} \to \text{Sets} \) defined as follows: for all objects \( R \) in \( \hat{C} \), define \( \hat{F}_V(R) = \text{Def}_\Lambda(V, R) \); and for all morphisms \( \alpha : R \to R' \) in \( \hat{C} \), let \( \hat{F}_V(\alpha) : \text{Def}_\Lambda(V, R) \to \text{Def}_\Lambda(V, R') \) be defined as \( \hat{F}_V(\alpha)([M, \phi]) = [R' \otimes_{R, \alpha} M, \phi_\alpha] \), where \( \phi_\alpha : k \otimes_{R'} (R' \otimes_{R, \alpha} M) \to V \) is the composition of \( \Lambda \)-module isomorphisms

\[
k \otimes_{R'} (R' \otimes_{R, \alpha} M) \cong k \otimes_R M \overset{\phi}{\to} V.
\]

Suppose there exists an object \( R(\Lambda, V) \) in \( \hat{C} \) and a deformation \([U(\Lambda, V), \phi_{U(\Lambda, V)}] \) of \( V \) over \( R(\Lambda, V) \) with the following property. For each objects \( R \) in \( \hat{C} \) and for all lifts \( M \) of \( V \) over \( R \), there exists a morphism \( v : R(\Lambda, V) \to R \) in \( \hat{C} \) such that

\[
\hat{F}_V(v)([U(\Lambda, V), \phi_{U(\Lambda, V)}]) = [M, \phi],
\]

and moreover \( v \) is unique if \( R \) is the ring of dual numbers \( k[[t]]/(t^2) \). Then \( R(\Lambda, V) \) and \([U(\Lambda, V), \phi_{U(\Lambda, V)}]\) are respectively called the versal deformation ring and versal deformation of \( V \). If for each object \( R \) in \( \hat{C} \) and for all lifts \((M, \phi) \) of \( V \) over \( R \), the morphism \( v \) is unique up to unique canonical isomorphism in \( \hat{C} \), then \( R(\Lambda, V) \) and \([U(\Lambda, V), \phi_{U(\Lambda, V)}]\) are respectively called the universal deformation ring and the universal deformation of \( V \). In other words, the universal deformation ring \( R(\Lambda, V) \) represents the deformation functor \( \hat{F}_V \) in the sense that \( \hat{F}_V \) is naturally isomorphic to the \( \text{Hom} \) functor \( \text{Hom}_k(R(\Lambda, V), -) \). Using Schlessinger’s criteria [26, Thm. 2.11] and using methods similar to those in [24], it is straightforward to prove that the deformation functor \( \hat{F}_V \) is continuous, that every finitely generated \( \Lambda \)-module \( V \) has a versal deformation ring, and that this versal deformation is universal provided that the endomorphism ring \( \text{End}_\Lambda(V) \) is isomorphic to \( k \) (see [14, Prop. 2.1])).

Recall that \( \Lambda \) is said to be self-injective if the regular left \( \Lambda \)-module \( _\Lambda \Lambda \) is injective, and that \( \Lambda \) is called a Frobenius algebra provided that the right \( \Lambda \)-modules \( \Lambda^\vee \) and \((\Lambda \Lambda)^* = \text{Hom}_k(\Lambda \Lambda, k) \) are isomorphic. By [17, Prop. 9.9], every Frobenius algebra is self-injective. It is also well-known that a basic algebra is self-injective.
if and only if it is Frobenius (see e.g. [28, Cor. 4.3]). It was proved in [13, Prop. 3.2.5] that the isomorphism class of versal deformation rings is preserved under stable equivalences of Morita type (as introduced by M. Broué in [15]) between self-injective \( k \)-algebras. Consequently, universal deformation rings of modules whose stable endomorphism ring is isomorphic to \( k \) are also preserved by stable equivalences of Morita type. This was used in [13, §4] to classify universal deformation rings up to derived equivalence for certain family of symmetric algebras of dihedral type (as introduced by K. Erdmann in [18]).

\textbf{Remark 2.1.} If \( \Lambda \) is self-injective and \( (M, \phi) \) is a lift of \( V \) over an object \( R \) in \( \mathcal{C} \) with \( \text{End}_\Lambda(V) \cong k \), then the deformation \( [M, \phi] \) does not depend on the particular choice of the \( \Lambda \)-module isomorphism. More precisely, if \( f : M \to M' \) is an \( R \otimes_k \Lambda \)-module isomorphism with \( (M', \phi') \) a lift of \( V \) over \( R \), then there exists an \( R \otimes_k \Lambda \)-module isomorphism \( \tilde{f} : M \to M' \) such that \( \phi' \circ (\text{id}_R \otimes f) = \phi \). In other words, \( [M, \phi] = [M', \phi'] \) in \( \hat{F}_V(R) = \text{Def}_\Lambda(V, R) \) (see Claim 3 within the proof of [14, Thm. 2.6]).

If \( V \) is an arbitrary \( \Lambda_N \)-module, then we denote the first syzygy of \( V \) by \( \Omega V \), i.e., \( \Omega V \) is the kernel of a projective cover \( P_V \to V \) (see e.g., [3, pp. 124-126]).

\textbf{Example 2.2.} Let \( G \) be a finite group and consider the group algebra \( kG \), which is a self-injective \( k \)-algebra (see e.g., [4, Prop. 3.1.2] and [17, Prop. 9.6]). It was proved in [7] that if \( V \) is a finitely generated \( kG \)-module whose stable endomorphism ring is isomorphic to \( k \), then \( V \) has a universal deformation ring \( R(kG, V) \). Moreover, the stable endomorphism ring of \( \Omega V \) is also isomorphic to \( k \) and the universal deformation rings \( R(kG, V) \) and \( R(\Omega V) \) of \( V \) and \( \Omega V \), respectively, are isomorphic.

The following result generalizes the properties of universal deformation rings mentioned in Example 2.2 to arbitrary Frobenius \( k \)-algebras (see [14, Thm. 2.6]).

\textbf{Theorem 2.3.} Let \( \Lambda \) be a finite dimensional self-injective \( k \)-algebra, and suppose that \( V \) is a finitely generated \( \Lambda \)-module whose stable endomorphism ring \( \text{End}_\Lambda(V) \) is isomorphic to \( k \).

\begin{enumerate}
\item The module \( V \) has a universal deformation ring \( R(\Lambda, V) \).
\item If \( P \) is a finitely generated projective \( \Lambda \)-module, then \( \text{End}_\Lambda(V \oplus P) \cong k \) and \( R(\Lambda, V \oplus P) \cong R(\Lambda, V) \).
\item If \( \Lambda \) is also a Frobenius algebra, then \( \text{End}_\Lambda(\Omega V) \cong k \) and \( R(\Lambda, \Omega V) \cong R(\Lambda, V) \).
\end{enumerate}

3. Classification of the string \( \Lambda_N \)-modules with endomorphism ring isomorphic to \( k \)

For the remainder of this article, let \( k \) be a fixed algebraically closed field of arbitrary characteristic and for all \( N \geq 1 \), we let \( \Lambda_N = kQ/I_N \) be as in Figure 1, where \( m \geq 3 \). We identify the vertices of \( Q \) with elements of \( \mathbb{Z}/m \) (the cyclic group of \( m \) elements). These algebras are all basic self-injective (so Frobenius), and have been studied by various authors in different contexts (see e.g. [29, 25] and their references). A brief description of the representation theory of \( \Lambda_N \) is provided in §A. Recall that \( \Gamma_s(\Lambda_N) \) denotes the stable Auslander-Reiten quiver of \( \Lambda_N \).

\textbf{Remark 3.1.} For all \( m \geq 3 \) and \( N \geq 1 \), the \( k \)-algebra \( \Lambda_N \) is the Brauer graph algebra of a polygon with \( m \) edges and multiplicity \( N \) at every vertex. This implies in particular that the \( k \)-algebras \( \Lambda_N \) are all symmetric (see [27, Thm. 1.1]). Therefore, it follows from [3, Prop. IV.3.8] that for all \( \Lambda_N \)-modules \( V \), \( \tau_{\Lambda_N}V = \Omega^2 V \), where \( \tau_{\Lambda_N} \) denotes the Auslander translation of \( \Gamma_s(\Lambda_N) \) (see [3, Prop. IV.3.8 & §VII.1]). In particular, for all connected components \( \mathcal{C} \) of \( \Gamma_s(\Lambda_N) \), we have \( \Omega^2 \mathcal{C} = \mathcal{C} \).

For all vertex \( i \in \mathbb{Z}/m \) of \( Q \), the radical series of the projective indecomposable \( \Lambda_N \)-module \( P_i \) can be described as in Figure 2.

\textbf{3.1.} \( \Lambda_N \)-modules with endomorphism ring isomorphic to \( k \). Let \( i \in \mathbb{Z}/m \) be fixed but arbitrary. Let \( Z_i \) be the string representative for \( \Lambda_N \) defined as

\begin{equation}
Z_i = \begin{cases} 
\frac{w_1 w_2 \cdots w_{m-1}}{w_i}, & \text{if } m \text{ is odd,} \\
\frac{w_1 w_2 \cdots w_m}{w_i}, & \text{otherwise,}
\end{cases}
\end{equation}
where $s(Z_i) = i,$

$$t(Z_i) = \begin{cases} 
  i - 1, & \text{if } m \text{ is odd,} \\
  i, & \text{otherwise,} 
\end{cases}$$

and for all $1 \leq j \leq m,$ $w_j$ or $w_j^{-1}$ is in $\{a_k, \bar{a}_k\}$ for a suitable $k \in \mathbb{Z}/m.$ Note that the string length of $Z_i$ is $m - 1$ when $m$ is odd, and $m$ otherwise.

If $m$ is even, then for all $n \geq 0$ we define strings representatives $Z_i^{(n)}$ for $\Lambda_N$ inductively as follows:

$$(3.2) \quad Z_i^{(0)} = Z_i,$$ and
$$Z_i^{(n+1)} = \theta_t(Z_i^{(n)}) Z_i^{(n)} \quad \text{for all } n \geq 0,$$

where for all $k \in \mathbb{Z}/m, \theta_k \in \{\bar{a}_{k+1}^{-1} a_k, a_k a_{k+1}^{-1} \}.$

For all integers $n \geq 0$ and for all $i \in \mathbb{Z}/m,$ we define the string $W_i^{(n)}$ for $\Lambda_N$ as

$$(3.3) \quad W_i^{(n)} = \begin{cases} 
  Z_i, & \text{if } m \text{ is odd,} \\
  Z_i^{(n)}, & \text{otherwise.} 
\end{cases}$$

**Proposition 3.2.** Let $M[S]$ be a string $\Lambda_N$-module. Then $M[S]$ has endomorphism ring isomorphic to $k$ if and only if there exists $i \in \mathbb{Z}/m$ such that the string representative $S$ is equivalent either to $\mathbb{I}_i$ or to $a_i,$ or to $\bar{a}_i,$ or there exists an integer $n \geq 0$ such that $S$ is equivalent to a substring of $W_i^{(n)}$ as in (3.3).

**Proof.** If $S$ is a string representative for $\Lambda_N$ as in the hypothesis of the “if” part of Proposition 3.2, then it follows from §A.3 that $\operatorname{End}_{\Lambda_N}(M[S]) = k.$ Assume then that $S$ is a string representative for $\Lambda_N$ such that $\operatorname{End}_{\Lambda_N}(M[S]) = k,$ and let $n$ be the string length of $S.$ If $n = 0,$ then for some $i \in \mathbb{Z}/m,$ $S = \mathbb{I}_i.$ If $n = 1,$ then $S$ is string equivalent to an arrow or to the formal inverse of an arrow, which implies that there exists $i \in \mathbb{Z}/m$ such that $S \sim a_i$ or $S \sim \bar{a}_i.$ Next assume that $n \geq 2$ and let $l$ be a maximal nonnegative integer such that for some $i \in \mathbb{Z}/m,$ $\bar{a}_i a_i^l$ is string equivalent to a substring of $S.$ If $l > 0,$ then there exist suitable strings $T$ and $T'$ for $\Lambda_N$ such that $S \sim T a_i a_i^l T'$ or $S \sim T a_i a_i^l T$ or $S \sim T a_i a_i^l T' \sim T a_i a_i^l T'$ or $S \sim T a_i a_i^l T' \sim T a_i a_i^l T'.$ If $S \sim T a_i a_i^l T$ (resp. $S \sim T a_i a_i^l T'$), then the maximality of $l$ implies that $(a_i a_i^l T)$ (resp. $a_i a_i^l T'$) starts in a peak and ends in a deep as a substring of $S$ as in §A.2, which implies that there exists a non-trivial canonical endomorphism of $M[S]$ as in (A.1) that factors through the string $\Lambda_N$-module $M[a_i]$ (resp. $M[a_i], M[\mathbb{I}_i]$), and thus $\dim_k \operatorname{End}_{\Lambda_N}(M[S]) \geq 2,$ contradicting the hypothesis. It follows then that $l = 0,$ which implies that neither $a_i a_i$ nor its formal inverse is a substring of $S$ for all $i \in \mathbb{Z}/m.$ Similarly we obtain that neither $a_i a_i$ nor its formal inverse is a substring of $S$ for all $i \in \mathbb{Z}/m.$ Therefore, $S = w_i w_{i-1} \cdots w_1,$ where for each $1 \leq j \leq n,$ $w_j$ or $w_j^{-1}$ is in $\{a_k, \bar{a}_k\}$ for a suitable $k \in \mathbb{Z}/m.$ Assume first that $m$ is odd. If $n \geq m,$ then there exists at least a canonical endomorphism of $M[S]$ as in (A.1) that factors through a simple $\Lambda_N$-module, which again contradicts the hypothesis, and
thus $n < m$. This implies that $S$ is equivalent to a substring of $Z_i$ as in (3.1). Next assume that $m$ is even. If $n \leq m$ then $S$ is equivalent to a substring of $Z_i$ as in (3.1). Assume next that $n > m$ and let $l \geq 0$ be maximal such that $S \sim TZ_i^{(l)}$ for a suitable string $T$, where $Z_i^{(l)}$ is as in (3.2). Assume further that $T$ has positive string length. Since for all $k \in \mathbb{Z}/m$, $S$ does contain a substring equivalent neither to $a_k\bar{a}_k$ nor to $\bar{a}_k a_k$, it follows from the maximality of $l$ that $T$ is equivalent to $a_k$ or to $\bar{a}_k$ for some $k \in \mathbb{Z}/m$. This implies that there exists a non-trivial canonical endomorphism of the string $\Lambda_N$-module $M[S]$ as in (A.1) that factors through a non-simple string $\Lambda_N$-module $M[S']$ with $S'$ a string representative for $\Lambda_N$, and which is equivalent to a proper substring of $S$. This contradicts that $\text{End}_{\Lambda_N}(M[S]) \cong k$, and therefore $T$ has string length equal to zero. Hence in this situation, it follows that $S$ is equivalent to $Z_i^{(l)}$ for some $l \geq 0$. This finishes the proof of Proposition 3.2.

\[ \square \]

Remark 3.3. Let $i \in \mathbb{Z}/m$ be fixed but arbitrary and let $\mathcal{C}$ be a connected component of $\Gamma_s(\Lambda_N)$ containing a string $\Lambda_N$-module $M[S]$ with $\text{End}_{\Lambda_N}(M[S]) \cong k$.

(i) Note that if $N = 1$, then for all $n \geq 0$, the string $\Lambda_1$-module $M[W_i^{(n)}]$ as in (3.3) can be obtained either from $M[a_i]$ or from $M[\bar{a}_i]$ by taking hooks or co-hooks as in §A.2, and thus they belong to the component of the Auslander-Reiten quiver of $\Lambda_1$ containing either $M[a_i]$ or $M[\bar{a}_i]$. In this situation, we obtain that $\mathcal{C}$ contains either a simple $\Lambda_1$-module or contains one of the string $\Lambda_1$-modules $M[a_i]$ or $M[\bar{a}_i]$ with $i \in \mathbb{Z}/m$, and thus there are at most finitely many possibilities for $\mathcal{C}$.

(ii) Assume now that $N \geq 2$. Then for all $n \geq 0$, the string $\Lambda_N$-module $M[W_i^{(n)}]$ cannot be obtained from a string $\Lambda_N$-module of minimal string length by adding hooks or co-hooks, which implies that $M[W_i^{(n)}]$ is of minimal string length.

(ii.a) If $m$ is odd, then $\mathcal{C}$ contains either a simple $\Lambda_N$-module or one of the string $\Lambda_N$-modules $M[a_i]$ or $M[\bar{a}_i]$ with $i \in \mathbb{Z}/m$, or the string $\Lambda_N$-module $M[T]$, where $T$ is a substring of $Z_i$ for $i \in \mathbb{Z}/m$.

In this situation we also have that there are at most finitely many possibilities for $\mathcal{C}$.

(ii.b) Finally, by Proposition 3.2, if $m$ is even then for all $n \geq 0$, $M[Z_i^{(n)}]$ has endomorphism ring isomorphic to $k$, which implies that in this situation, there are infinitely many possibilities for $\mathcal{C}$.

Following Remark 3.3, we obtain the following result.

Corollary 3.4.  

(i) If $N = 1$, then there are at most finitely many components of $\Gamma_s(\Lambda_N)$ containing a string $\Lambda_N$-module of minimal string length whose endomorphism ring is isomorphic to $k$.

(ii) Assume that $N \geq 2$.

(ii.a) If $m$ is odd, then there are at most finitely many components of type $\mathbb{Z}\Lambda_\infty^{\infty}$ of $\Gamma_s(\Lambda_N)$ containing a string $\Lambda_N$-module $V$ with $\text{End}_{\Lambda_N}(V) \cong k$.

(ii.b) If $m$ is even, then there are infinitely many components of type $\mathbb{Z}\Lambda_\infty^{\infty}$ of $\Gamma_s(\Lambda_N)$ containing a string $\Lambda_N$-module $V$ with $\text{End}_{\Lambda_N}(V) \cong k$.

4. Universal deformation rings for $\Lambda_N$

In this section with consider the components $\mathcal{C}$ of the stable Auslander-Reiten quiver of $\Lambda_N$ that contain a module whose endomorphism ring is isomorphic to $k$ as in Proposition 3.2.

Let $M$ and $N$ be arbitrary finitely generated $\Lambda_N$-modules. Since $\Lambda_N$ is self-injective, it follows that for all $j \geq 1$, $\text{Ext}_A^{j}(M,N) = \text{Hom}_{A}(\Omega^j M, N)$, where $\text{Hom}_{A}(\Omega^j M, N)$ is the quotient of $k$-vector spaces of $\text{Hom}_{\Lambda_N}(\Omega^j M, N)$ by the subspace consisting in those morphisms from $\Omega^j M$ to $N$ that factor through a finitely generated projective $\Lambda_N$-module.

For all $i \in \mathbb{Z}/m$, we define

$$c_i = a_{i-1}^{-1}(a_i-1)\ldots a_i 1^{-N},$$

and for all $n \geq 1$, define the string representative

$$C_{i,n} = w_1 w_2 \cdots w_n,$$

(4.1)
where \( t(C_i) = i \) and for all \( 1 \leq j \leq n, w_j = c_k \) for a suitable \( k \in \mathbb{Z}/m \). We also define the string representative

\[
D_{i,n} = v_1 v_2 \cdots v_n,
\]
where \( s(D_i) = i \) and for all \( 1 \leq j \leq n, v_j = d_k \) for a suitable \( k \in \mathbb{Z}/m \). For all \( i \in \mathbb{Z}/m \), we let \( C_{i,0} = 1_i = D_{i,0} \).

4.1. Components of \( \Gamma_s(\Lambda_N) \) of type \( z\Lambda_N^{\infty} \) containing a module whose endomorphism ring is isomorphic to \( k \).

4.1.1. Components containing a simple \( \Lambda_N \)-module.

**Proposition 4.1.** Let \( i \in \mathbb{Z}/m \) be fixed and let \( A_i \) be the component of \( \Gamma_s(\Lambda_N) \) containing the simple \( \Lambda_N \)-module \( M[1_i] \). Then the component \( A_i \) is not \( \Omega \)-stable, and all the \( \Lambda_N \)-modules \( V \) in \( \mathcal{A}_i \cup \Omega \mathcal{A}_i \) have stable endomorphism ring \( \text{End}_{\Lambda_N}(V) \) isomorphic to \( k \). Their universal deformation rings \( R(\Lambda_N, V) \) are also isomorphic to \( k \).

**Proof.** Let \( V_0 = M[1_i] \). Then \( \Omega V_0 = M[\varphi(a_{i-1}a_{i-1})^{N-1}] \), where \( \varphi(a_{i-1}a_{i-1})^{N-1} \) is the left co-hook of \((a_{i-1}a_{i-1})^{N-1}\) as in \( \$A_2 \). This implies that \( \Omega A_i \neq A_i \). On the other hand, for all \( n \geq 1 \), let \( V_n = M[\Omega_{[\Omega \mathcal{A}_i]}] \) and \( V_n = M[\Omega_{[\mathcal{A}_i]}] \). It follows from Remark 3.1 and \( \$A_2 \) that every string \( \Lambda_N \)-module belonging to \( A_i \cup \Omega A_i \) lies in the \( \Omega \)-orbit of \( V_n \) for some \( n \in \mathbb{Z} \). Using the description of the canonical morphisms in \( \$A_3 \) and the shape of the indecomposable \( \Lambda_N \)-modules as in Figure 2 it follows that for all \( n \in \mathbb{Z} \), \( \text{End}_{\Lambda_N}(V_n) \cong k \), which implies that \( V_n \) has a universal deformation ring \( R(\Lambda_N, V_n) \). Since it is straightforward to show by using \( \$A_3 \) that \( \text{Ext}^1_{\Lambda_N}(V_n, V_n) = \text{Hom}_{\Lambda_N}(\Omega V_n, V_n) = 0 \), we obtain that \( R(\Lambda_N, V_n) = k \) for all \( n \in \mathbb{Z} \). This finishes the proof of Proposition 4.1. \( \square \)

**Remark 4.2.** If \( m = 3 \) and \( N = 1 \), then Proposition 4.1 recovers some of the results in [12, Prop. 3.1(iii)] for the algebra \( \Lambda_1 \).

4.1.2. Components containing a string \( \Lambda_N \)-module of the form \( M[a_i] \) or \( M[\bar{a}_i] \). Let \( \kappa_m \) be the non-negative integer defined as follows:

\[
\kappa_m = \begin{cases} 
m, & \text{if } m \text{ is even}, \\
\frac{m-1}{2}, & \text{otherwise}. 
\end{cases}
\]

**Remark 4.3.** If \( N = 1 \), then for all \( i \in \mathbb{Z}/m \), the strings \( a_i \) and \( \bar{a}_i \) for \( \Lambda_1 \) are maximal. Thus the respective components of \( \Gamma_s(\Lambda_1) \) containing the string \( \Lambda_1 \)-modules \( M[a_i] \) and \( M[\bar{a}_i] \) with \( i \in \mathbb{Z}/m \) are tubes, which will be discussed in Proposition 4.8. Thus, for the remainder of \( \S 4.1 \), we assume that \( N \geq 2 \).

**Proposition 4.4.** Let \( i \in \mathbb{Z}/m \) be fixed. Let \( \mathcal{B}_i \) be the component of \( \Gamma_s(\Lambda_N) \) containing the \( \Lambda_N \)-module \( M[\zeta_i] \) with \( \zeta_i \in \{a_i, \bar{a}_i\} \). Then the component \( \mathcal{B}_i \) is not \( \Omega \)-stable unless \( N = 2 \), and there are exactly \( \kappa_m \) \( \Omega \)-orbits of \( \Lambda_N \)-modules in \( \mathcal{B}_i \cup \Omega \mathcal{B}_i \) whose stable endomorphism ring is isomorphic to \( k \). More precisely, the modules in \( \mathcal{B}_i \cup \Omega \mathcal{B}_i \) that have stable endomorphism ring isomorphic to \( k \) are the modules in the \( \Omega \)-orbits of \( V_n = M[D_{i+1,n}^{\zeta_i}] \) with \( 0 \leq n \leq \kappa_m - 1 \). The universal deformation rings \( R(\Lambda_N, V_n) \) are isomorphic to \( k[[t]]/(t^N) \) when \( n = 0 \), and to \( k \) when \( 0 < n < \kappa_m - 1 \). If \( m \) is odd, then \( R(\Lambda_N, V_{\kappa_m-1}) \) is isomorphic to \( k[t] \).

**Proof.** Let \( V_0 = M[a_i] \). By using the shape of the indecomposable \( \Lambda_N \)-modules as in Figure 2, we obtain that \( \Omega V_0 = \Omega M[a_i] = M[\varphi(a_i\bar{a_i})^{N-2}] \), where \( \varphi(a_i\bar{a_i})^{N-2} \) is the left co-hook of the string \( a_i\bar{a_i} \) as in \( \$A_2 \). This implies that \( \mathcal{B}_i \) is not \( \Omega \)-stable unless \( N = 2 \). For all \( n \geq 1 \), let \( V_n = M[D_{i+1,n}a_i] \) and \( V_n = M[a_iC_{i,n}] \). As before, it follows from Remark 2.1 and \( \$A_2 \) that every string \( \Lambda_N \)-module belonging to \( \mathcal{B}_i \cup \Omega \mathcal{B}_i \) lies in the \( \Omega \)-orbit of \( V_n \) for some \( n \in \mathbb{Z} \). If \( N = 2 \), then for all \( n \geq 1 \), the \( \Lambda_2 \)-module \( V_n \) lies in the \( \Omega \)-orbit of \( V_{k,n} \) for some integer \( k \geq 0 \). If \( N \geq 3 \) and \( n \geq 1 \), it follows from the description of the canonical morphisms in \( \$A_3 \) and the shape of the indecomposable \( \Lambda_N \)-modules as in Figure 2 that the \( \Lambda_N \)-module \( V_{n-1} \) has a canonical endomorphism as in (A.1) factoring through the simple \( \Lambda_N \)-module
that there exists a unique surjective morphism through $M$ non-trivial canonical endomorphism as in (A.1) that factors through either a simple $\Lambda_R$ or a string $\Lambda_N$-module whose string representative has string length 1, and which does not factor through a projective $\Lambda_N$-module. Assume that $n = 0$. Then there is a unique canonical morphism in $\text{Hom}_{\Lambda_N}(\Omega V_0, V_0)$ that factors through $M[a_i]$ which does not factor through a projective $\Lambda_N$-module. Since $\dim_k \text{Hom}_{\Lambda_N}(\Omega V_0, V_0) = 1$, it follows that $\text{Ext}^1_{\Lambda_N}(V_0, V_0) \cong \text{Hom}_{\Lambda_N}(\Omega V_0, V_0) \cong k$. Therefore, $R(\Lambda_N, V_0)$ is a quotient of $k[[t]]$.

Claim 4.5. $R(\Lambda_N, V_0) = R(\Lambda_N, P[a_i])$ is isomorphic to $k[[t]]/(t^N)$.

Proof. For all $0 \leq l \leq N - 1$, let $S_l = (a_i, b_i)^l a_i$. Let $l \in \{1, \ldots, N\}$ be fixed. Then there exists a non-trivial canonical endomorphism $\sigma_l$ of the $\Lambda_N$-module $M[S_l]$ as in (A.1) that factors through $M[S_{l-1}]$, namely

$$\sigma_l : M[S_l] \to M[S_{l-1}] \to M[S_l].$$

Observe that the kernel of $\sigma_l$ as well as the image of $\sigma_l^{-1}$ are isomorphic to $V_0 = M[a_i]$, and that $\sigma_l^2 = 0$. Thus, the $\Lambda_N$-module $M[S_l]$ is naturally a $k[[t]]/(t^1 + 1) \otimes_k \Lambda_N$-module, where the action of $t$ on $x \in M[S_l]$ is given as $t \cdot x = \sigma_l(x)$. In particular, $tM[S_l] \cong M[S_{l-1}]$.

Let $\{r_1, r_2\}$ be a $k$-basis of $V_0$. Using the isomorphism $M[S_l]/tM[S_l] \cong V_0$, we can lift the elements $r_1$ and $r_2$ to corresponding elements $r_1, r_2 \in M[S_l]$. It follows that $\{r_1, r_2\}$ is linearly independent over $k$ and that $\{t^s r_1, t^s r_2 : 0 \leq s \leq l\}$ is a $k$-basis of $tM[S_l] \cong M[S_{l-1}]$. Therefore, $\{r_1, r_2\}$ is a $k[[t]]/(t^1 + 1)$-basis of $M[S_l]$, i.e., $M[S_l]$ is free over $k[[t]]/(t^1 + 1)$. Observe that $M[S_l]$ lies in a short exact sequence of $\Lambda_N$-modules

$$0 \to tM[S_l] \to M[S_l] \to k \otimes k[[t]]/(t^1 + 1) M[S_l] \to 0.$$

Then there exists an isomorphism of $\Lambda_N$-modules $\phi_1 : k \otimes k[[t]]/(t^1 + 1) \otimes k \Lambda_N$-module, which defines the lift of $V_0$ over $k[[t]]/(t^1 + 1)$. Consider the lift $(M[S_{l+1}], \phi_{l+1})$ of $V_0$ over $k[[t]]/(t^N)$. Since $\text{End}_{\Lambda_N}(V_0) \cong k$, it follows from Theorem 2.3(i) that there exists a unique morphism $\alpha : R(\Lambda_N, V_0) \to k[[t]]/(t^2)$ in $\hat{C}$ such that

$$M[S_{l+1}] \cong k[[t]]/(t^N) \otimes R(\Lambda_N, V_0), U(\Lambda_N, V_0),$$

where $R(\Lambda_N, V_0)$ and $[U(\Lambda_N, V_0), \phi_{U(\Lambda_N, V_0)}]$ are respectively the universal deformation ring and the universal deformation of the $\Lambda_N$-module $V_0$. Since $(M[S_1], \phi_1)$ is not the trivial lift of $V_0$ over $k[[t]]/(t^2)$, it follows that there exists a unique surjective morphism $\alpha' : R(\Lambda_N, V_0) \to k[[t]]/(t^2)$ in $\hat{C}$ such that

$$M[S_1] \cong k[[t]]/(t^2) \otimes R(\Lambda_N, V_0), \alpha' U(\Lambda_N, V_0).$$

By considering the natural projection $\pi_{N,2} : k[[t]]/(t^N) \to k[[t]]/(t^2)$ and the lift $(U', \phi_{U'})$ of $V_0$ over $k[[t]]/(t^2)$ corresponding to the morphism $\pi_{N,2} \circ \alpha$, we obtain

$$U' \cong k[[t]]/(t^2) \otimes R(\Lambda_N, V_0), \pi_{N,2} \circ \alpha \ U(\Lambda_N, V_0)$$

$$\cong k[[t]]/(t^2) \otimes k[[t]]/(t^N), \pi_{N,2} \circ \alpha \ U(\Lambda_N, V_0)$$

$$\cong k[[t]]/(t^2) \otimes k[[t]]/(t^N), \pi_{N,2} \ M[S_{N-1}]$$

$$\cong M[S_{N-1}]/t^2 M[S_{N-1}] \cong M[S_1].$$

It follows from Remark 2.1 that $(U', \phi_{U'}) = [M[S_1], \phi_1]$ in $\hat{F}_0(k[[t]]/(t^2))$. The uniqueness of $\alpha'$ implies that $\alpha' = \pi_{N,2} \circ \alpha$.

Since $\alpha'$ is surjective, $\alpha$ is also surjective. We want to prove that $\alpha$ is an isomorphism. Suppose this is false. Then there exists a surjective $k$-algebra homomorphism $\alpha_0 : R(\Lambda_N, V_0) \to k[[t]]/(t^{N + 1})$ in $\hat{C}$ such that $\pi_{N+1,1} \circ \alpha_0 = \alpha$, where $\pi_{N+1,1} : k[[t]]/(t^{N + 1}) \to k[[t]]/(t^N)$ is the natural projection. Let $M_0$ be a $k[[t]]/(t^{N + 1}) \otimes k \Lambda_N$-module, which defines the lift of $V_0$ over $k[[t]]/(t^{N + 1})$ corresponding to $\alpha_0$. Since the kernel of $\pi_{N+1,1}$ is $t^N/(t^{N + 1})$, then $M_0/t^N M_0 \cong M[S_{N-1}]$. Consider the $k[[t]]/(t^{N + 1}) \otimes k \Lambda_N$-module homomorphism $g : M_0 \to t^N M_0$ defined by $g(x) = t^N x$ for all $x \in M_0$. Since $M_0$ is free over $k[[t]]/(t^{N + 1})$,
then the kernel of $g$ is isomorphic to $tM_0$. Thus, $M_0/tM_0 \cong t^N M_0$ for $g$ is a surjection. Therefore $t^N M_0 \cong V_0$, which implies that there exists a non-split short exact sequence of $k[[t]]/(t^{N+1}) \otimes_k \Lambda_N$-modules

$$0 \to V_0 \to M_0 \to M[S_{N-1}] \to 0.$$  

Since $\Omega M[S_{N-1}] = \Omega M[(a_i \bar{a}_i)^N - a_i] = M[(\bar{a}_i - a_i)^N - \bar{a}_i]$, then

$$\text{Ext}_\Lambda^1(M[S_{N-1}], V_0) = \text{Hom}_\Lambda(\Omega M[S_{N-1}], V_0) = 0.$$ 

It follows that the sequence (4.5) splits as a sequence of $\Lambda_N$-modules. Hence, $M_0 = V_0 \oplus M[S_{N-1}]$ as $\Lambda_N$-modules. Identifying the elements of $M_0$ as $(v, x)$ with $v \in V_0$ and $x \in M[S_{N-1}]$, we see that the $t$ acts on $(v, x) \in M_0$ as $t \cdot (v, x) = (\mu(x), \sigma_{N-1}(x))$, where $\mu : M[S_{N-1}] \to V_0$ is a surjective $\Lambda_N$-module homomorphism and $\sigma_{N-1}$ is as in (4.4). Since the canonical homomorphism $c : M[S_{N-1}] \to M[a_i]$ generates $\text{Hom}_{\Lambda_N}(M[S_{N-1}], V_0)$, then there exists $c \in k^*$ such that $\mu = cc$, which implies that the kernel of $\mu$ is $tM[S_{N-1}]$. Therefore, $t^N(v, x) = (\mu(t^{N-1}x), \sigma_{N-1}(x)) = (0, 0)$ for all $v \in V_0$ and $x \in M[S_{N-1}]$, which contradicts the fact that $t^N M_0 \cong V_0$. Thus $\alpha : R(\Lambda_N, V_0) \to k[[t]]/(t^N)$ is an isomorphism and $R(\Lambda_N, M[a_i]) = R(\Lambda_N, V_0) \cong k[[t]]/(t^N)$. This finishes the proof of Claim 4.5.

Assume now that $1 \leq n < \kappa_m - 1$. Then it is straightforward to show that $\text{Ext}_\Lambda^1(V_n, V_n) = 0$, which implies that $R(\Lambda_N, V_n) \cong k$. Finally assume that $n = \kappa_m - 1$. If $m$ is odd, then $\text{Ext}_\Lambda^1(V_{\kappa_m-1}, V_{\kappa_m-1}) = 0$, which implies that $R(\Lambda_N, V_{\kappa_m-1}) \cong k$. Assume next that $m$ is even. Then $\text{Hom}_{\Lambda_N}(\Omega V_{\kappa_m-1}, V_{\kappa_m-1})$ has only a non-trivial canonical morphism as in (A.1) which does not factor through a projective $\Lambda_N$-module. Therefore

$$\text{Ext}_\Lambda^1(V_{\kappa_m-1}, V_{\kappa_m-1}) = \text{Hom}_{\Lambda_N}(\Omega V_{\kappa_m-1}, V_{\kappa_m-1}) = k.$$ 

This implies that $R(\Lambda_N, V_{\kappa_m-1})$ is isomorphic to a quotient of $k[[t]]$.

Claim 4.6. If $m$ is even then $R(\Lambda_N, V_{\kappa_m-1})$ is isomorphic to $k[[t]]$.

Proof. Let $W_i = \mathcal{D}_{i+1, \kappa_m-1} a_i$. Note that $s(W_i) = i, t(W_i) = i-1$ and $V_{\kappa_m-1} = M[W_i]$. Let $T_0 = W_i$ and for all $l \geq 1$, let $T_l$ be the string

$$T_l = T_{l-1} \bar{a}_{l-1}^{-1} W_i$$ 

By using similar arguments as in the proof of Claim 4.5, for each $l \geq 1$ we get lifts $(M[T_l], \varphi_l)$ of $M[W_i]$ over $k[[t]]/(t^{l+1})$, where $t$ acts on $x \in M[T_l]$ as $t \cdot x = \delta_l(x)$ and where $\delta_l$ is the non-trivial canonical endomorphism of $M[T_l]$ that factors through $M[T_l]$, namely

$$\delta_l : M[T_l] \to M[T_{l-1}] \to M[T_l].$$

Note that for all $l \geq 1$, we have natural projections $\pi_{l,t-1} : M[T_l] \to M[T_{l-1}]$. Let $N_0 = \lim_{l \to \infty} M[T_l]$ and let $t$ act on $N_0$ as $\lim_{l \to \infty} \pi_{l,t-1}$. In particular, $N_0$ is a $k[[t]] \otimes_k \Lambda_N$-module and $k \otimes_k [t] N_0 \cong N_0/tN_0 \cong M[W_i]$, which implies that there exists an isomorphism of $\Lambda_N$-modules $\varphi_0 : k \otimes_k [t] N_0 \to M[W_i]$. Let $k = \dim_k M[W_i]$ and let $\{b_j\}_{1 \leq j \leq k}$ be a $k$-basis of $N_0/tN_0$. For all $1 \leq j \leq k$, we are able to lift these elements $b_j$ in $N_0/tN_0$ to elements $\tilde{b}_j$ in $N_0/tN_0$ such that $\{b_j\}_{1 \leq j \leq k}$ is a generating set of the $k[[t]] \otimes_k \Lambda_N$-module $N_0$. It follows that $\{\tilde{b}_j\}_{1 \leq j \leq n}$ is a $k[[t]]$-basis of $N_0$, i.e., $N_0$ is free over $k[[t]]$. Therefore, $(N_0, \varphi_0)$ is a lift of $M[W_i]$ over $k[[t]]$ and there exists a unique $k$-algebra homomorphism $\beta : R(\Lambda_N, M[W_i]) \to k[[t]]$ in $\tilde{\mathcal{C}}$ corresponding to the deformation defined by $(N_0, \varphi_0)$. Since $N_0/t^2 N_0 \cong M[T_l]$ as $\Lambda_N$-modules, we can see, as in the proof of Claim 4.5, that $N_0/t^2 N_0$ defines a non-trivial lift of $M[W_i]$ over $k[[t]]/(t^2)$ and that $\beta$ is a surjection. Since $R(\Lambda_N, M[W_i])$ is a quotient of $k[[t]]$, it follows that $\beta$ is an isomorphism. Hence $R(\Lambda_N, V_{\kappa_m-1}) = R(\Lambda_N, M[W_i]) \cong k[[t]]$. This finishes the proof of Claim 4.6.

Note that the above results concerning the string $\Lambda_N$-module $M[a_i]$ can be adjusted to obtain those for $M[\bar{a}_i]$. In particular, $R(\Lambda_N, M[\bar{a}_i]) \cong k[[t]]/(t^N)$. This finishes the proof of Proposition 4.4.
4.1.3. Components containing a string \( \Lambda_N \)-module whose endomorphism ring is isomorphic to \( k \), and whose string representative have string length greater than one. For all integers \( l \geq 1 \) and \( i \in \mathbb{Z}/m \), define inductively the string representative \( \Theta_{i,l} \) for \( \Lambda_N \) as follows:

\[
\Theta_{i,1} = \{ a_{i+1}^{-1} a_i, a_{i+1} a_i^{-1} \}, \\
\Theta_{i,l+1} = \Theta_{i}(\Theta_{i,l}), \quad \text{if } l \geq 1.
\]

**Proposition 4.7.** Let \( S \) be a string representative for \( \Lambda_N \) with string length greater than one, and which equivalent to a substring of \( W_i^{(n)} \) as in (3.3) for some \( i \in \mathbb{Z}/m \) and \( n \geq 0 \). Let \( C \) be the component of \( \Gamma_s(\Lambda_N) \) containing the string \( \Lambda_N \)-module \( M[S] \).

(i) If \( S \) is string equivalent to \( \Theta_{j,l} \) (as in (4.7)) for some \( j \in \mathbb{Z}/m \) and \( l \geq 1 \), then every \( \Lambda_N \)-module \( V \) in \( \mathcal{C} \cup \mathcal{O} \mathcal{C} \) has stable endomorphism ring isomorphic to \( k \). In this situation, the universal deformation ring \( R(\Lambda_N, V) \) is isomorphic to \( k \).

(ii) If \( S \) is not string equivalent to \( \Theta_{j,l} \) for all \( j \in \mathbb{Z}/m \) and \( l \geq 1 \), then there are exactly \( 2\kappa_m \Omega \)-orbits of \( \Lambda_N \)-modules \( V \) in \( \mathcal{C} \cup \mathcal{O} \mathcal{C} \) that have stable endomorphism ring isomorphic to \( k \). In this situation, the universal deformation ring \( R(\Lambda_N, V) \) is isomorphic to \( k \) or to \( k[[t]] \).

**Proof.** First assume that \( S \) is equivalent to \( \Theta_{i,l} \) for some \( i \in \mathbb{Z}/m \) and \( l \geq 1 \). Note that every \( \Lambda_N \)-module in \( \mathcal{C} \cup \mathcal{O} \mathcal{C} \) lies in the \( \Omega \)-orbit of either \( M[S] \), or \( M[SC_{1,n}] \), or \( M[\mathcal{D}_4(S),n] \) for some \( n \geq 1 \). By using the shape of the indecomposable projective \( \Lambda_N \)-modules as in Figure 2 together with the description of the canonical endomorphisms in §A.3, it is straightforward to show that \( \text{End}_{\Lambda_N}(M[SC_{1,n}]) \cong k \cong \text{End}_{\Lambda_N}(M[\mathcal{D}_4(S),n]) \).

Since \( \Lambda_N \) is self-injective, it follows that \( \Omega \) induces a self-equivalence of the stable module category of \( \Lambda_N \). Then \( \text{End}_{\Lambda_N}(\Omega M[SC_{1,n}]) \cong k \cong \text{End}_{\Lambda_N}(\Omega M[\mathcal{D}_4(S),n]) \), which shows that every \( \Lambda_N \)-module in \( \mathcal{C} \cup \mathcal{O} \mathcal{C} \) has stable endomorphism ring isomorphic to \( k \). Assume next that

\[
S \sim \cdots \bar{a}_{i+1} a_i.
\]

It follows that \( \Omega M[S] \cong M[\Delta S] \), where

\[
\Delta S \sim \cdots (a_{i+1} a_i - 1)^{N-1} a_{i+1} (a_i a_i^{-1})^{1-N} (a_{i-1} a_i^{-1})^{N-1} a_{i-1}^{-1}.
\]

Again by using the shapes of the indecomposable projective \( \Lambda_N \)-modules as in Figure 2 together with the description of the canonical homomorphisms as in §A.3, it follows that

\[
\text{Ext}^1_{\Lambda_N}(M[S], M[S]) = \text{Hom}_{\Lambda_N}(\Omega M[S], M[S]) = \text{Hom}_{\Lambda_N}(M[\Delta S], M[S]) = 0,
\]

which implies that \( R(\Lambda_N, M[S]) = k \). Similarly, since for all \( n \geq 1 \),

\[
\text{Ext}^1_{\Lambda_N}(M[SC_{1,n}], M[SC_{1,n}]) = 0 = \text{Ext}^1_{\Lambda_N}(M[\mathcal{D}_4(S),n], M[\mathcal{D}_4(S),n]),
\]

we obtain that \( R(\Lambda_N, M[SC_{1,n}]) = k = R(\Lambda_n, M[\mathcal{D}_4(S),n]) \). Note that we can adjust the above arguments for the case

\[
S \sim \cdots a_{i+1} a_i^{-1}.
\]

Since the isomorphism class of universal deformation rings is invariant under \( \Omega \) by Theorem 2.3(iii), we obtain Proposition 4.7(i).

To prove Proposition 4.7(ii), assume that \( s(S) = i \) and that \( S \) is not equivalent to \( \Theta_{i,l} \) for all \( l \geq 1 \). Let \( n_1 \geq 0 \) maximal such that the string \( s_{i,n_1} \) is string equivalent to a substring of \( S \). Then there exists a string \( \zeta \) of length 1 with \( s(\zeta) = s(\Theta_{i,n_1}) \) such that \( S \sim (\zeta) \Theta_{i,n_1} \). Let \( \kappa_m \) be as in (4.3). By using the shape of the indecomposable projective \( \Lambda_N \)-modules as in Figure 2 together with the description of the canonical endomorphisms in §A.3, it is straightforward to show that for all \( 0 \leq j < \kappa_m \), \( \text{End}_{\Lambda_N}(M[\mathcal{D}_4(S),j]) \cong k \cong \text{End}_{\Lambda_N}(M[SC_{1,j}]) \), and that if \( j \geq \kappa_m \), \( M[\mathcal{D}_4(S),j] \) (resp. \( M[SC_{1,j}] \)) has a canonical endomorphism ring, which does not factor through a projective \( \Lambda_N \)-module. Thus there are exactly \( 2\kappa_m \Omega \)-orbits in \( \mathcal{C} \cup \mathcal{O} \mathcal{C} \).
2 together with the description of the canonical endomorphisms in §A.3 that \( \text{Ext}^1_{\Lambda_N}(M[S'], M[S']) = 0 = \text{Ext}^1_{\Lambda_N}(M[S''], M[S'']) \). This implies \( R(\Lambda_N, M[S']) \cong \mathbb{k} \cong R(\Lambda_N, M[S'']) \).

Next let \( S' = D_{\text{tr}(S), \kappa_m - 1} S \) (resp. \( S'' = SC_{\gamma, \kappa_m - 1} \)). Then

\[
\text{Ext}^1_{\Lambda_N}(M[S'], M[S']) \cong \mathbb{k} \cong \text{Ext}^1_{\Lambda_N}(M[S''], M[S'']),
\]

which implies that \( R(\Lambda_N, M[S']) \) and \( R(\Lambda_2, M[S'']) \) are quotients of \( \mathbb{k}[t] \). Let \( T_0 = S' \) (resp. \( T_0'' = S'' \)) and for all \( l \geq 1 \), let \( T_l' = T_{l-1}' \gamma' S' \) (resp. \( T_l'' = T_{l-1}'' \gamma'' S'' \)) where \( \gamma' \) (resp. \( \gamma'' \)) is either an arrow or the formal inverse of an arrow joining \( \mathbb{T} \) with \( s(S') - 1 \) (resp. \( t(S'') - 1 \)) mod \( \mathbb{m} \). By using similar arguments as in the proof of Claim 4.5, for each formal inverse of an arrow joining \( \mathbb{T} \) we obtain that \( \mathbb{N} \) by using the description of the indecomposable \( \Lambda_N \)-modules in \( \mathbb{T} \). Their universal deformation rings are isomorphic either to \( \mathbb{N} \)-modules as in Figure 2 and the definition of the string \( \Lambda_N \)-module, and where \( \mathbb{N} \)-module in \( \mathbb{V} \) as in Figure 2 and the definition of the string \( \Lambda_N \)-module.

4.2. Tubes of \( \Gamma_s(\Lambda_N) \) containing string \( \Lambda_N \)-modules. Using the description of the indecomposable projective \( \Lambda_N \)-modules as in Figure 2 and the definition of the string \( \Lambda_N \)-modules in §A.1, it follows that if \( m \) is odd, then there are exactly two \( m \)-tubes in \( \Gamma_s(\Lambda_N) \), namely, one containing \( M[(a_0 a_0)^{-1} a_0] \) and another one containing \( \Omega M[(a_0 a_0)^{-1} a_0] = M[(a_m^{-1} a_m)^{-1} a_m^{-1}] \).

On the other hand if \( m \) is even, then there are exactly four \( \mathbb{T} \) tubes in \( \Gamma_s(\Lambda_N) \), namely, one containing the string \( \Lambda_N \)-module \( M[(a_0 a_0)^{-1} a_0] \), another containing \( \Omega M[(a_0 a_0)^{-1} a_0] = M[(a_1^2 a_1)^{-1} a] \), and another one containing \( \Omega M[(a_1 a_1^{-1} a_1)^{-1} a] = M[(a_m^{-1} a_m)^{-1} a_m^{-1}] \).

As noted already in Remark 4.3, if \( N = 1 \), then for all \( i \in \mathbb{Z}/m \), the component of \( \Gamma_s(\Lambda_4) \) containing the string \( \Lambda_4 \)-module \( M[a_i] \) (resp. \( \mathbb{M}[a_i] \)) is a tube.

**Proposition 4.8.** Let \( \mathcal{T} \) be a tube of \( \Gamma_s(\Lambda_N) \) containing only \( \mathbb{S} \)-modules, and let \( \kappa_m \) be as in (4.3). Then there are exactly \( \kappa_m \) \( \mathcal{O}_2 \)-orbits of \( \mathbb{S} \)-modules in \( \mathcal{T} \) whose stable endomorphism ring is isomorphic to \( \mathbb{k} \). Their universal deformation rings are isomorphic either to \( \mathbb{k} \) or to \( \mathbb{k}[t] \).

**Proof.** By using Theorem 2.3 together with Remark 3.1, we can prove Proposition 4.8 by just considering the tube \( \mathcal{T} \) containing the string \( \Lambda_N \)-module \( V_0 = M[(a_0 a_0)^{-1} a_0] \) which lies on the boundary of \( \mathcal{T} \), for \( (a_0 a_0)^{-1} a_0 \) is a maximal string for \( \Lambda_N \). For all \( n \geq 1 \), let \( V_n = M[D_{\text{tr}(V), \kappa_n - 1} (a_0 a_0)^{-1} a_0] \), where \( D_{\text{tr}(V), \kappa_n} \) is as in (4.2). Note that every \( \Lambda_N \)-module in \( \mathcal{T} \cup \mathcal{O}_2 \) lies in the \( \Omega \)-orbit of \( V_n \) for some \( n \geq 0 \). If \( 0 \leq j \leq \kappa_m - 1 \), then by using the description of the indecomposable \( \Lambda_N \)-modules together with that of the canonical morphisms in §A.3, it is straightforward to show that \( \text{End}_{\Lambda_N}(V_j) \cong \mathbb{k} \). If \( j \geq \kappa_m \), then the string \( \Lambda_N \)-module \( V_j \) has an endomorphism ring as in (4.1) that factors through either a simple \( \Lambda_N \)-module or a string \( \Lambda_N \)-module corresponding to a maximal string for \( \Lambda_N \), and which does not factor through a projective \( \Lambda_N \)-module. In this situation we have that \( \text{End}_{\Lambda_N}(V_j) \cong \mathbb{k} \) and therefore there are exactly \( \kappa_m \) \( \mathcal{O}_2 \)-orbits of \( \Lambda_N \)-modules in \( \mathcal{T} \cup \mathcal{O}_2 \) whose stable endomorphism ring is isomorphic to \( \mathbb{k} \). If \( 0 \leq j < \kappa_m - 1 \) we have \( \text{Ext}^1_{\Lambda_N}(V_j, V_j) = \text{Hom}_{\Lambda_N}(\Omega V_j, V_j) = 0 \), which implies \( R(\Lambda_N, V_j) \cong \mathbb{k} \). If \( m \) is odd, then we obtain that

\[
\text{Ext}^1_{\Lambda_N}(V_{\kappa_m - 1}, V_{\kappa_m - 1}) = \text{Hom}_{\Lambda_N}(\Omega V_{\kappa_m - 1}, V_{\kappa_m - 1}) = 0,
\]

which implies \( R(\Lambda_N, V_{\kappa_m - 1}) \cong \mathbb{k} \). Assume that \( m \) is even. Then there is a canonical morphism in \( \text{Hom}_{\Lambda_N}(\Omega V_{\kappa_m - 1}, V_{\kappa_m - 1}) \) as in (A.1) that factors through a simple \( \Lambda_N \)-module, and which does not factor through a projective \( \Lambda_N \)-module. This shows that \( \text{Ext}^1_{\Lambda_N}(V_{\kappa_m - 1}, V_{\kappa_m - 1}) \cong \mathbb{k} \), which implies that \( R(\Lambda_N, V_{\kappa_m - 1}) \) is a quotient of \( \mathbb{k}[t] \). By using analogous arguments to those in the proof of Claim 4.6, we obtain that \( R(\Lambda_N, V_{\kappa_m - 1}) \cong \mathbb{k}[t] \). This finishes the proof of Proposition 4.8.

**Remark 4.9.** If \( m = 3 \) and \( N = 1 \), then Proposition 4.8 recovers the results in [12, Prop. 3.2(iii)] for the algebra \( A_1 \).
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APPENDIX A. SOME REMARKS ABOUT THE REPRESENTATION THEORY OF \( \Lambda_N \)

Let \( \Lambda_N \) be as in Figure 1. Since \( \Lambda_N \) is a special biserial algebra in the sense of [16], all the non-projective indecomposable \( \Lambda_N \)-modules can be described combinatorially by using so-called string and band for \( \Lambda_N \). In the following, we describe these strings for \( \Lambda_N \) and the corresponding indecomposable \( \Lambda_N \)-modules, which are called string modules.

A.1. String modules for \( \Lambda_N \). For all \( i \in \mathbb{Z}/m \) and for each arrow \( a_i, \bar{a}_i \) of \( Q \), we define a formal inverse by \( a_i^{-1}, \bar{a}_i^{-1} \), respectively, and we let \( s(a_i) = t(\bar{a}_i) = (a_i^{-1}) = s(\bar{a}_i) \). By a word of length \( n \geq 1 \) we mean a sequence \( w_1 \cdots w_n \), where the \( w_j \) is either an arrow or a formal inverse of an arrow and where \( s(w_j) = t(w_{j+1}) \) for \( 1 \leq j \leq n-1 \). We define \( (w_1 \cdots w_n)^{-1} = w_n^{-1} \cdots w_1^{-1} \), \( s(w_1 \cdots w_n) = s(w_n) \) and \( t(w_1 \cdots w_n) = t(w_1) \). If \( i \in \mathbb{Z}/m \) is a vertex of \( Q \), we define an empty word \( 1_i \) of length zero with \( t(1_i) = i = s(1_i) \) and \( 1_i^{-1} = 1_i \). Denote by \( W \) the set of all words and let \( J = \{a_{i+1}a_i, \bar{a}_{i-1}\bar{a}_i, (a_ia_i)^N, (\bar{a}_i\bar{a}_i)^N : i \in \mathbb{Z}/m \} \).

Let \( \sim \) be the equivalence relation on \( W \) defined by \( w \sim w' \) if and only if \( w = w' \) or \( w^{-1} = w' \). A string is a representative of \( C \) of an equivalence class under the relation \( \sim \), where either \( C = 1_i \) for some vertex \( i \) of \( Q \), or \( C = w_1 \cdots w_n \) with \( n \geq 1 \) and \( w_j \neq w_j^{-1} + 1 \) for \( 1 \leq j \leq n-1 \) and no sub-word of \( C \) or its formal inverse belong to \( J \). If \( C \) is a string such that \( s(C) = t(C) \), then we let \( C^0 = 1_{t(C)} \). If \( C = w_1 \cdots w_n \) and \( D = v_1 \cdots v_m \) are strings of length \( n, m \geq 1 \), respectively, we say that the composition of \( C \) and \( D \) is defined provided that \( w_1 \cdots w_nv_1 \cdots v_m \) is a string and write \( CD = w_1 \cdots w_nv_1 \cdots v_m \). Observe that \( C1_{s(C)} \sim C \) and \( 1_{t(C)}C \sim C \). Moreover, if \( C = w_1 \cdots w_n \) is a string of length \( n \geq 1 \), then \( C \sim w_1 \cdots w_j 1_{t(w_{j+1})} w_{j+1} \cdots w_n \) for all \( 1 \leq j \leq n-1 \). If \( C = w_1 \cdots w_2 \) is a string of length \( n \geq 1 \), then there exists an indecomposable \( \Lambda_N \)-module \( M[C] \), called the string module corresponding to the string representative \( C \), which can be described as follows. There is an ordered \( k \)-basis \( \{z_0, z_1, \ldots, z_n \} \) of \( M[C] \) such that the action of \( \Lambda_N \) on \( M[C] \) is given by the following representation \( \varphi_C : \Lambda_N \to \text{Mat}(n+1, k) \). Let \( \varphi(j) = t(w_{j+1}) \) for \( 0 \leq j \leq n-1 \) and \( \varphi(n) = s(w_n) \). Then for each vertex \( i \in \mathbb{Z}/m \), for each arrow \( \gamma \in \{a_i, \bar{a}_i : i \in \mathbb{Z}/m \} \) in \( Q \), and for all \( 0 \leq j \leq n \) define

\[
\varphi_C(i)(z_j) = \begin{cases} 
    z_j, & \text{if } \varphi(j) = i, \\
    0, & \text{otherwise,}
\end{cases} \quad \varphi_C(\gamma)(z_j) = \begin{cases} 
    z_{j-1}, & \text{if } w_j = \gamma, \\
    z_{j+1}, & \text{if } w_{j+1} = \gamma^{-1}, \\
    0, & \text{otherwise.}
\end{cases}
\]

We call \( \varphi_C \) the canonical representation and \( \{z_0, z_1, \ldots, z_n \} \) a canonical \( k \)-basis for \( M[C] \) relative to the string representative \( C \). Note that \( M[C] \cong M[C^{-1}] \). If \( C = 1_i \) with \( i \in \mathbb{Z}/m \) then \( M[C] = M[1_i] \) is the simple \( \Lambda_N \)-module corresponding to the vertex \( i \).

A.2. The stable Auslander-Reiten quiver of \( \Lambda_N \). Recall that in this article we denote by \( \Gamma_s(\Lambda_N) \) the stable Auslander-Reiten quiver of \( \Lambda_N \) (see [3, VII]). In the following, we describe the irreducible morphisms between string \( \Lambda_N \)-modules.

Assume \( C = w_1w_2 \cdots w_n \) with \( n \geq 1 \) is a string. We say that \( C \) is directed if all \( w_j \) are arrows and we say that \( C \) is a maximal directed string if \( C \) is directed and for every arrow \( \gamma \) in \( Q \), \( \gamma C \in J \). Let \( M \) be the set of all maximal directed strings, i.e.,
\[ \mathcal{M} = \{(a_ia_i^{-1})^{N-1}a_{i+1}(a_{i-1}a_{i-1})^{N-1}a_{i-1} : i \in \mathbb{Z}/m\}. \]

Let \( C \) be a string for \( \Lambda_N \). We say that \( C \) starts on a peak (resp., starts in a deep) provided that there is no arrow \( \varphi \) in \( Q \) such that \( C\varphi \) (resp., \( C\varphi^{-1} \)) is a string; we also say that \( C \) ends on a peak (resp., ends in a deep) provided that there is no arrow \( \gamma \) in \( Q \) such that \( \gamma^{-1}C \) (resp., \( \gamma C \)) is a string. If \( C \) is a string not starting on a peak (resp., not starting in a deep), say \( C\varphi \) (resp., \( C\varphi^{-1} \)) is a string for some arrow \( \varphi \), then there is a unique directed string \( D \in \mathcal{M} \) such that \( C\varphi = C\varphi D^{-1} \) (resp., \( C\varphi = C\varphi D^{-1} \)) is a string. We say \( C_h \) (resp., \( C_c \)) is obtained from \( C \) by adding a hook (resp., a co-hook) on the right side. Dually, if \( C \) is a string not ending on a peak (resp., not ending in a deep), say \( \gamma^{-1}C \) (resp., \( \gamma C \)) is a string for some arrow \( \gamma \) in \( Q \), then there is a unique directed string \( E \in \mathcal{M} \) such that \( E\gamma = E\gamma^{-1}C \) (resp., \( E\gamma = E\gamma^{-1}C \)) is a string. We say \( hC \) (resp., \( cC \)) is obtained from \( C \) by adding a hook (resp., a co-hook) on the left side.

By [16], all irreducible morphisms between string modules are either canonical injections \( M[C] \to M[C_h] \), \( M[C] \to M[hC] \), or canonical projections \( M[C_c] \to M[C] \), \( M[C] \to M[cC] \).

Let \( S \) be a substring of \( C \). We say that \( S \) starts on a peak (resp., starts in a deep) in \( C \) provided that there is no arrow \( \varphi \) in \( Q \) such that \( S\varphi \) (resp., \( S\varphi^{-1} \)) is a substring of \( C \); we also say that \( S \) ends on a peak (resp., ends in a deep) in \( C \) provided that there is no arrow \( \gamma \) in \( Q \) such that \( \gamma^{-1}S \) (resp., \( \gamma S \)) is a substring in \( C \).

Suppose \( M[C] \) is a string module of minimal string length such that \( M[C] \) belongs to a component \( \mathcal{C} \) of \( \Gamma_s(\Lambda_N) \) of type \( \mathbb{Z}\Lambda_{\infty} \). Since none of the projective \( \Lambda_N \)-modules is uniserial then near \( M[C] \) the component \( \mathcal{C} \) looks as in Figure 3.

![Figure 3: The stable Auslander-Reiten component near \( M[C] \).](image)

### A.3. Homomorphisms between string \( \Lambda_N \)-modules.

Let \( S \) and \( T \) be strings for \( \Lambda_N \) and let \( M[S] \) and \( M[T] \) their respective string \( \Lambda_N \)-modules with respective canonical \( k \)-basis \( \{x_i\}_{i=0}^{t_1} \) and \( \{y_i\}_{i=0}^{t_2} \). Suppose that \( C \) is a substring of both \( S \) and \( T \) such that the following conditions (i) and (ii) are satisfied.

(i) \( S \sim S'CS'' \), with \( S' \) of length zero or \( S' = \hat{S}'\xi_1 \) and \( S'' \) of length zero or \( S'' = \xi_2^{-1}\hat{S}'' \), where \( S', \hat{S}', S'', \hat{S}'' \) are strings and \( \xi_1, \xi_2 \) are arrows in \( Q \); and

(ii) \( T \sim T'CT'' \), with \( T' \) of length zero or \( T' = \hat{T}'\xi_1^{-1} \) and \( T'' \) of length zero or \( T'' = \xi_2\hat{T}'' \), where \( T', \hat{T}', T'', \hat{T}'' \) are strings and \( \xi_1, \xi_2 \) are arrows in \( Q \).

Then by [23] there exists a composition of \( \Lambda_N \)-module homomorphisms

\[ \sigma_C : M[S] \to M[C] \to M[T], \]

```plaintext
and which sends each element of \( \{x_u\}_{0 \leq u \leq l_1-1} \) either to zero or to an element of \( \{y_v\}_{0 \leq v \leq l_2-1} \), according to the relative position of \( C \) in \( S \) and \( T \), respectively. Suppose e.g. that \( S = w_1 \cdots w_{l_1}, \ T = \tilde{w}_1 \cdots \tilde{w}_{l_2} \) and 
\[
C = w_{j+1}w_{j+2} \cdots w_{j+l_3} = \tilde{w}_{k+l_3}^{-1} \tilde{w}_{k+l_3-1}^{-1} \cdots \tilde{w}_{k+1}^{-1},
\]
then 
\[
\sigma_C(x_{j+t}) = y_{k+l_3-t} \quad \text{for } 0 \leq t \leq l_3, \quad \text{and} \quad \sigma_C(x_u) = 0 \text{ for all other } u.
\]

We call \( \sigma_C \) a canonical homomorphism from \( M[S] \) to \( M[T] \). Note that there may be several choices for \( S', S'' \) (resp. \( T', T'' \)) in (i) (resp. (ii)). In other words, there may be several \( k \)-linearly independent canonical homomorphisms factoring through \( M[C] \). By [23], every \( \Lambda_N \)-module homomorphism \( \sigma : M[S] \rightarrow M[T] \) can be written as a unique \( k \)-linear combination of canonical homomorphisms which factor through string modules corresponding to strings \( C \) satisfying (i) and (ii). In particular, if \( S = T \), then every \( \Lambda_N \)-module endomorphism of \( M[S] \) can be written as a unique \( k \)-linear combination of the identity homomorphism and of canonical endomorphisms which factor through string \( \Lambda_N \)-modules \( M[C] \) for suitable choices of \( C \) satisfying (i) and (ii). We call \( \sigma_C \) a canonical homomorphism from \( M[S] \) to \( M[T] \) that factors through \( M[C] \). It follows from [23] that each \( \Lambda_N \)-module homomorphism from \( M[S] \) to \( M[T] \) can be written uniquely as a \( k \)-linear combination of canonical \( \Lambda_N \)-module homomorphisms as in (A.1). In particular, if \( M[S] = M[T] \) then the canonical endomorphisms of \( M[S] \) generate \( \text{End}_{\Lambda_N}(M[S]) \).

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