On the (Non)Equivalence of the Schrödinger and Heisenberg Pictures of Quantum Mechanics

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Abstract

The aim of this short Note is to show that the Schrödinger and Heisenberg pictures of quantum mechanics cannot be equivalent unless one uses a quantization rule clearly stated by Born and Jordan in their famous 1925 paper. This rule is sufficient and necessary to ensure energy conservation in Heisenberg’s matrix mechanics. It follows, in particular, that Schrödinger and Heisenberg mechanics yield different theories if one quantizes the Hamiltonian using the Weyl prescription.

1 Introduction

In the Schrödinger picture of quantum mechanics, the operators are constant (unless they depend explicitly on time), and the states evolve in time; this evolution is governed by Schrödinger’s equation

\[ i\hbar \frac{\partial}{\partial t} |\psi\rangle = H|\psi\rangle. \] (1)

In the Heisenberg picture, the state vectors are time-independent operators that incorporate a dependency on time, while an observable \( A_S \) in the Schrödinger picture becomes a time-dependent operator \( A_H(t) \) in the Heisenberg picture; this time dependence satisfies the Heisenberg equation

\[ i\hbar \frac{dA_H}{dt} = i\hbar \frac{\partial A_H}{\partial t} + [A_H, H]. \] (2)
Both theories are related as follows \[9, 10\]: Let \((U(t, t_0))\) be the propagator determined by Schrödinger’s equation:

\[
i\hbar \frac{d}{dt} U(t, t_0) = \hat{H} U(t, t_0), \quad U(t_0, t_0) = I_d; \tag{3}
\]

the operators \(U(t, t_0)\) are unitary on \(L^2(\mathbb{R})\); a ket

\[
|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle \tag{4}
\]

in the Schrödinger picture becomes, in the Heisenberg picture, the constant ket

\[
|\psi_H\rangle = U(t, t_0)^*|\psi_S(t)\rangle = |\psi_S(t_0)\rangle \tag{5}
\]

whereas an observable \(A_S\) becomes

\[
A_H(t) = U(t, t_0)^*A_SU(t, t_0); \tag{6}
\]

in particular the Hamiltonian is

\[
H_H(t) = U(t, t_0)^*H_SU(t, t_0). \tag{7}
\]

We are going to see that this relation implies that \(H_H\) and \(H_S\) are identical and must thus be quantized using the same rules. The proof of this essential property is based on Born and Jordan’s foundational paper \[5\]. The idea is that the Heisenberg picture requires that the Hamiltonian operator \(H_H(t)\) must be a constant of the motion, which implies, taking (7) into account, that \(H_H(t) = H_S\) for all times \(t\). A consequence of this property is that if we believe that Heisenberg’s “matrix mechanics” is correct and is equivalent\(^1\) to Schrödinger’s theory, then the Hamiltonian operator appearing in the Schrödinger equation (1) must be quantized using the Born–Jordan rules, which are different from (for instance) the Weyl quantization rule.

We begin by shortly exposing the main arguments in Born and Jordan’s paper \[1\].

2 The Born–Jordan paper in a nutshell

We are working in a one-dimensional configuration space; the discussion is generalized mutatis mutandis in \[2\]. Following Heisenberg’s paper \[5\] Born

\(^1\)See however \[7, 8\] for a discussion of the “myth of equivalence”.
and Jordan considered in \[1\] square infinite matrices
\[
a = (a(n, m)) = \begin{pmatrix}
a(00) & a(01) & a(02) & \cdots \\
a(10) & a(11) & a(12) & \cdots \\
a(20) & a(21) & a(22) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\] (8)

where the \(a(nm)\) are what they call “ordinary quantities”, \textit{i.e.} scalars; we will call these infinite matrices (for which we always use boldface letters) \textit{observables}. In particular Born and Jordan introduce momentum and position observables \(p\) and \(q\) and matrix functions \(H(p, q)\) of these observables, which they call “Hamiltonians”. Following Heisenberg, they assume that the equations of motion for \(p\) and \(q\) are formally the same as in classical theory, namely
\[
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q};
\] (9)

limiting themselves deliberately to Hamiltonians which are polynomials in the observables \(p\), \(q\), that is linear combinations of monomials which are products of terms
\[
H = p^s q^t
\] (10)

they define the derivatives in (9) by the formulas, and show that the observables \(p\) and \(q\) satisfy the commutation relation
\[
pq - qp = \frac{\hbar}{2\pi i} 1
\] (11)

where 1 is the identity matrix; more generally,
\[
p^m q^n - q^n p^m = m \frac{\hbar}{2\pi i} \sum_{\ell=0}^{n-1} q^{n-\ell} p^{m-\ell} q^\ell.
\] (12)

Born and Jordan next proceed to derive the fundamental laws of quantum mechanics. In particular, pursuing their analogy with classical mechanics, they want to prove that energy is conserved; identifying the values of the Hamiltonian \(H\) with the energy of the system, they impose the condition \(\dot{H} = 0\) and show that this condition requires that
\[
\dot{q} = \frac{2\pi i}{\hbar} (Hq - qH), \quad \dot{p} = \frac{2\pi i}{\hbar} (Hp - pH).
\] (13)

\[^2\text{English translation in [11].}\]
Comparing with the Hamilton-like equations (9) this condition is in turn equivalent to

\[ Hq - qH = \frac{\hbar}{2\pi i} \frac{\partial H}{\partial p}, \quad Hp - pH = -\frac{\hbar}{2\pi i} \frac{\partial H}{\partial q}. \]  

(14)

Now comes the crucial step. Given a classical Hamiltonian \( H(p, q) = p^s q^r \) they ask how one should choose the observable \( H(p, q) \). Using the commutation formula (12) Born and Jordan show that the only possible choice satisfying the conditions (14) is

\[ H(p, q) = \frac{1}{s+1} \sum_{\ell=0}^{s} p^{s-\ell} q^\ell p^\ell \]  

(15)

3 Quantum mechanics and Born–Jordan quantization

Born and Jordan thus proved—rigorously—that the only way to quantize polynomials in a way consistent with Heisenberg’s ideas was to use the rule

\[ p^s q^r \xrightarrow{\text{BJ}} \frac{1}{s+1} \sum_{\ell=0}^{s} p^{s-\ell} q^\ell p^\ell = \frac{1}{r+1} \sum_{j=0}^{r} q^{r-j} p^s q^j \]  

(16)

(the equality following from the commutation relation (12). In their subsequent publication [2] with Heisenberg they show that their constructions extend mutatis mutandis to systems with an arbitrary number of degrees of freedom. We will call this rule (and its extension to higher dimensions) the Born–Jordan quantization rule. Weyl [12] proposed, independently, some time later (1926) another rule leading, for monomials, to the replacement of (16) with

\[ p^s q^r \xrightarrow{\text{Weyl}} \frac{1}{2^s} \sum_{\ell=0}^{s} \binom{s}{\ell} p^{s-\ell} q^\ell p^\ell. \]  

(17)

It turns out that both rules coincide when \( m + n \leq 2 \), but they do not for higher values of \( m + n \). For instance

\[ p^2 q \xrightarrow{\text{BJ}} \frac{1}{3} (p^2 q + pqp + q^2p), \quad p^2 q \xrightarrow{\text{Weyl}} \frac{1}{4} (p^2 q + 2pqp + q^2p). \]

Both quantizations are thus not equivalent; as Kauffmann [6] observes, Weyl’s rule is the single most symmetrical operator ordering, whereas the

\[ ^3\text{English translation in [11].} \]
Born–Jordan quantization is the equally weighted average of all the operator orderings.

These facts have the following consequence: if we insist that the Heisenberg and Schrödinger pictures be equivalent, then we must quantize the Hamiltonian in Schrödinger’s equation using Born–Jordan quantization. In fact, recall from formula (7) that the Heisenberg and Schrödinger Hamiltonians $H_H$ and $H_S$ are related by

$$H_H(t) = U(t, t_0)^* H_S U(t, t_0).$$

Since $H_H(t)$ is a constant of the motion we have $H_H(t) = H_H(t_0)$ and hence $H_H(t) = H_S$ so the Heisenberg and Schrödinger Hamiltonians $H_H(t)$ and $H_H$ must be identical. But the condition $H_H(t) = H_H(t_0) = H_H$ means that $H_H$ and hence $H_S$ must be quantized using the Born–Jordan prescription.

An obvious consequence of these considerations is that if one uses in the Schrödinger picture the Weyl quantization rule (or any other quantization rule), we obtain two different renderings of quantum mechanics. This observation seems to be confirmed by Kauffmann’s interesting discussion of the non-physicality of Weyl quantization.

We have been considering the quantization of polynomials for simplicity; in de Gosson and Luef and de Gosson we have shown in detail how to Born–Jordan quantize arbitrary functions of the position and momentum variables.

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