Abstract

This work presents a parametrized family of divergences, namely Alpha-Beta Log-Determinant (Log-Det) divergences, between positive definite unitized trace class operators on a Hilbert space. This is a generalization of the Alpha-Beta Log-Determinant divergences between symmetric, positive definite matrices to the infinite-dimensional setting. The family of Alpha-Beta Log-Det divergences is highly general and contains many divergences as special cases, including the recently formulated infinite-dimensional affine-invariant Riemannian distance and the infinite-dimensional Alpha Log-Det divergences between positive definite unitized trace class operators. In particular, it includes a parametrized family of metrics between positive definite trace class operators, with the affine-invariant Riemannian distance and the square root of the symmetric Stein divergence being special cases. For the Alpha-Beta Log-Det divergences between covariance operators on a Reproducing Kernel Hilbert Space (RKHS), we obtain closed form formulas via the corresponding Gram matrices.

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infinite-dimensional Log-Determinant divergences, Alpha divergences, Alpha-Beta divergences, affine-invariant Riemannian distance, Stein divergence, positive definite operators, trace class operators, extended trace, extended Fredholm determinant, Reproducing kernel Hilbert spaces, covariance operators

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1. Introduction

Symmetric Positive Definite (SPD) matrices play an important role in many areas of mathematics, statistics, machine learning, optimization, computer vision, and related fields, see e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The set $\text{Sym}^{++}(n)$ of $n \times n$ SPD matrices is an open convex cone and can also be equipped with a Riemannian manifold structure. Among the most studied Riemannian metrics on $\text{Sym}^{++}(n)$ are the classical affine-invariant metric [1, 2, 3, 5, 12] and the more recent Log-Euclidean metric [4, 9, 13]. The convex cone structure of $\text{Sym}^{++}(n)$, on the other hand, gives rise to distance-like functions such as the Alpha Log-Determinant divergences [14], which have been shown to be special cases of the Alpha-Beta Log-Determinant divergences [15]. These divergences are fast to compute and have been shown to work well in various applications [7, 16, 8]. The present work aims to generalize the Alpha-Beta Log-Determinant divergences to the infinite-dimensional setting.

Finite-dimensional Alpha-Beta Log-Determinant divergences. We recall that for $A, B \in \text{Sym}^{++}(n)$, the Alpha-Beta Log-Determinant (Log-Det) divergence between $A$ and $B$ is a parametrized family of divergences defined by (see [15])

$$D^{(\alpha, \beta)}(A, B) = \frac{1}{\alpha \beta} \log \det \left[ \frac{\alpha(AB^{-1})^\beta + \beta(AB^{-1})^{-\alpha}}{\alpha + \beta} \right], \quad (1)$$

where $\alpha \neq 0, \beta \neq 0, \alpha + \beta \neq 0$.

Remark 1. To keep our presentation compact, in the following we consider the case $\alpha > 0, \beta > 0$, as well as the limiting cases $\alpha = 0, \beta = 0$. Since $D^{(\alpha, \beta)}(A, B) = D^{(-\alpha, -\beta)}(B, A)$, the case $\alpha < 0, \beta < 0$ is essentially identical to the previous case. We do not consider the cases $\alpha, \beta$ have opposite signs, since in those cases the well-definedness and finiteness of $D^{(\alpha, \beta)}(A, B)$ depends on the spectrum of $AB^{-1}$ (see Theorem 2 in [15]), that is it is not a valid divergence on all of $\text{Sym}^{++}(n)$.

The parametrized family of divergences defined by Eq. (1) is highly general and admits as special cases many metrics and distance-like functions on $\text{Sym}^{++}(n)$, including in particular the following:
1. The affine-invariant Riemannian distance [3], corresponding to the limiting case $D^{(0,0)}(A, B)$, with

$$D^{(0,0)}(A, B) = \frac{1}{2} d_{\text{alb}}^2(A, B) = \frac{1}{2} \left\| \log(B^{-1/2} A B^{-1/2}) \right\|^2_F,$$  \tag{2}

where \(\log(A)\) denotes the principal logarithm of the matrix \(A\) and \(\| \cdot \|_F\) denotes the Frobenius norm.

2. The Alpha Log-Determinant divergences [14], corresponding to $D^{(\alpha,1-\alpha)}(A, B)$, \(0 < \alpha < 1\), with

$$D^{(\alpha,1-\alpha)}(A, B) = \frac{1}{\alpha(1-\alpha)} \log \left( \frac{\det[\alpha A + (1-\alpha)B]}{\det(A)^\alpha \det(B)^{1-\alpha}} \right).$$  \tag{3}

A special case of this divergence is the symmetric Stein divergence (also called the Jensen-Bregman LogDet divergence), corresponding to $D^{(1/2,1/2)}(A, B)$, whose square root is a metric on $\text{Sym}^{++}(n)$ [16], with

$$D^{(1/2,1/2)}(A, B) = 4 d_{\text{stein}}^2(A, B) = 4 \log \left( \frac{\det(A+B)}{\sqrt{\det(A) \det(B)}} \right).$$  \tag{4}

3. The limiting cases $\beta = 0$ and $\alpha = 0$ correspond to, respectively,

$$D^{(\alpha,0)}(A, B) = \frac{1}{\alpha^2} \left\{ \text{tr}((A^{-1} B)^\alpha - I) - \alpha \log \det(A^{-1} B) \right\},$$  \tag{5}

$$D^{(0,\beta)}(A, B) = \frac{1}{\beta^2} \left\{ \text{tr}((B^{-1} A)^\beta - I) - \beta \log \det(B^{-1} A) \right\},$$  \tag{6}

with $D^{(1,0)}(A, B) = \text{tr}(A^{-1} B - I) - \log \det(A^{-1} B)$ and $D^{(0,1)}(A, B) = \text{tr}(B^{-1} A - I) - \log \det(B^{-1} A)$.

\textbf{Contributions of this work.} The current work is a continuation and generalization of the author’s recent work [17]. In [17], we generalized the Alpha Log-Det divergences between SPD matrices [14] to the infinite-dimensional Alpha Log-Determinant divergences between positive definite unitized trace class operators in a Hilbert space.

In the current work, we present a formulation for the Alpha-Beta Log-Det divergences between positive definite unitized trace class operators, generalizing the Alpha-Beta divergences between SPD matrices as defined by Eq.(1). As in the finite-dimensional setting, the formulation we present here is general and admits as special cases many
metrics and distance-like functions between positive definite unitized trace class operators, including in particular the following: the infinite-dimensional affine-invariant Riemannian distance \cite{18}; the infinite-dimensional Alpha Log-Det divergences \cite{17}, a special case of which is the infinite-dimensional symmetric Stein divergence. For the divergences between reproducing kernel Hilbert spaces (RKHS) covariance operators, we obtain closed form formulas for the Alpha-Beta Log-Det divergences via the corresponding Gram matrices.

**Organization.** We provide a summary of the main results of the paper in Section 2, including our definition of the infinite-dimensional Alpha-Beta Log-Det divergences. The key concepts involved are described in Section 3. The motivations and derivations leading to our definition of the Alpha-Beta Log-Det divergences are presented in Section 4. We then show in Section 5 that both the affine-invariant Riemannian distance and the Alpha Log-Det divergences are special cases of the Alpha-Beta Log-Det divergences. All mathematical proofs are presented in Appendix A.

2. **Summary of main results**

We present a summary of our main results in this section, with the detailed technical descriptions provided in subsequent sections. Throughout the paper, let \( \mathcal{H} \) denote a separable Hilbert space, with \( \dim(\mathcal{H}) = \infty \), unless explicitly stated otherwise. Let \( \mathcal{L}(\mathcal{H}) \) be the Banach space of bounded linear operators on \( \mathcal{H} \) and \( \text{Sym}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H}) \) be the subspace of self-adjoint, bounded operators on \( \mathcal{H} \). For \( A \in \mathcal{L}(\mathcal{H}) \), we write \( A > 0 \) to denote that \( A \) is a self-adjoint positive definite operator. Let \( \text{Tr}(\mathcal{H}) \) denote the Banach algebra of trace class operators on \( \mathcal{H} \). The set of positive definite unitized trace class operators on \( \mathcal{H} \) is then defined to be

\[
P\text{Tr}(\mathcal{H}) = \{ A + \gamma I > 0 : A = A^*, A \in \text{Tr}(\mathcal{H}), \gamma \in \mathbb{R} \}.
\]

(7)

The main purpose of the current work is the generalization of the Alpha-Beta Log-Det divergence between SPD matrices, as defined in Eq. (1), to that between positive definite unitized trace class operators in \( P\text{Tr}(\mathcal{H}) \). The following is our definition of the Alpha-Beta (Log-Det) divergences in the infinite-dimensional setting.
Definition 1 (Alpha-Beta Log-Determinant Divergences). Assume that \( \dim(\mathcal{H}) = \infty \). Let \( \alpha > 0, \beta > 0 \) be fixed. Let \( r \in \mathbb{R}, r \neq 0 \) be fixed. For \( (A + \gamma I), (B + \mu I) \in \text{PTr}(\mathcal{H}) \), the \( (\alpha, \beta) \)-Log-Det divergence \( D^{(\alpha, \beta)}_r[(A + \gamma I), (B + \mu I)] \) is defined to be

\[
D^{(\alpha, \beta)}_r[(A + \gamma I), (B + \mu I)] = \frac{1}{\alpha \beta} \log \left[ \left( \frac{\gamma}{\mu} \right)^{r \delta - \frac{r \mu}{\alpha \beta}} \right] \det_X \left( \frac{\alpha (A + \gamma I)^{r(1-\delta)} + \beta (A + \gamma I)^{-r \delta}}{\alpha + \beta} \right), \tag{8}
\]

where \( \Lambda + \frac{2}{\mu} I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}, \delta = \frac{\alpha \gamma^r}{\alpha \gamma + \beta \mu^r} \). Equivalently,

\[
D^{(\alpha, \beta)}_r[(A + \gamma I), (B + \mu I)] = \frac{1}{\alpha \beta} \log \left[ \left( \frac{\gamma}{\mu} \right)^{r \delta - \frac{r \mu}{\alpha \beta}} \right] \det_X \left( \frac{\alpha (Z + \gamma I)^{r(1-\delta)} + \beta (Z + \gamma I)^{-r \delta}}{\alpha + \beta} \right), \tag{9}
\]

where \( Z + \frac{2}{\mu} I = (A + \gamma I)(B + \mu I)^{-1} \).

Remark 2. In Definition 1, \( \det_X \) denotes the extended Fredholm determinant defined in [17] (see Section 3 below). For \( \gamma = 1 \), we have \( \det_X(A + \gamma I) = \det(A + I) \), with \( \det \) on the right hand side being the Fredholm determinant. For \( \dim(\mathcal{H}) < \infty \), \( \det_X(A + \gamma I) = \det(A + \gamma I) \), with \( \det \) on the right hand side being the standard matrix determinant.

The quantity \( D^{(\alpha, \beta)}_r[(A + \gamma I), (B + \mu I)] \) where \( \alpha > 0, \beta > 0 \), as stated in Definition 1, can be extended to the cases \( \alpha > 0, \beta = 0 \) and \( \alpha = 0, \beta > 0, \forall r \in \mathbb{R}, r \neq 0 \), via limiting arguments. The following is our definition in these cases.

Definition 2 (Limiting cases - I). Assume that \( \dim(\mathcal{H}) = \infty \). Let \( \alpha > 0, \beta > 0 \), \( r \neq 0 \) be fixed. For \( (A + \gamma I), (B + \mu I) \in \text{PTr}(\mathcal{H}) \), the Log-Det divergence \( D^{(\alpha, 0)}_r[(A + \gamma I), (B + \mu I)] \) is defined to be

\[
D^{(\alpha, 0)}_r[(A + \gamma I), (B + \mu I)] = \frac{r}{\alpha^2} \left[ \left( \frac{\mu}{\gamma} \right)^r - 1 \right] \log \frac{\mu}{\gamma} \tag{10}
\]

\[
+ \frac{1}{\alpha^2} \text{tr}_X([(A + \gamma I)^{-1}(B + \mu I)]^r - I)
\]

\[
- \frac{1}{\alpha^2} \left( \frac{\mu}{\gamma} \right)^r \log \det_X([(A + \gamma I)^{-1}(B + \mu I)]^r).
\]
Similarly, $D^{(0, \beta)} [(A + \gamma I), (B + \mu I)]$ is defined to be

$$
D^{(0, \beta)} [(A + \gamma I), (B + \mu I)] = \frac{r}{\beta^2} \left[ \left( \frac{\gamma}{\mu} \right)^{r} - 1 \right] \log \frac{\gamma}{\mu} - \frac{1}{\beta^2} \text{tr}_X \left( [(B + \mu I)^{-1}(A + \gamma I)]^r - I \right) + \frac{1}{\beta^2} \left( \frac{\gamma}{\mu} \right)^{r} \log \det_X \left( [(B + \mu I)^{-1}(A + \gamma I)]^r \right).$$

The following result confirms that the quantity $D^{(\alpha, \beta)}$, as defined in Definitions 1 and 2, is in fact a divergence on $\text{PTr}(\mathcal{H})$.

**Theorem 1 (Positivity).** Assume the hypothesis stated in Definitions 1 and 2. Then

$$
D^{(\alpha, \beta)} [(A + \gamma I), (B + \mu I)] \geq 0 \quad (12)
$$

$$
D^{(\alpha, \beta)} [(A + \gamma I), (B + \mu I)] = 0 \iff A = B, \gamma = \mu. \quad (13)
$$

**Theorem 2 (Special cases - I).** The following are some of the most important special cases of Definitions 1 and 2.

1. The infinite-dimensional affine-invariant Riemannian distance $d_{\text{aRHS}} [(A + \gamma I), (B + \mu I)]$ [18], which corresponds to the limiting case $\lim_{\alpha \to 0} D^{(\alpha, \alpha)} [(A + \gamma I), (B + \mu I)]$, where $r = r(\alpha)$ is smooth, with $r(0) = 0$, $r'(0) \neq 0$, and $r(\alpha) \neq 0$ for $\alpha \neq 0$. The limit is given by

$$
\lim_{\alpha \to 0} D^{(\alpha, \alpha)} [(A + \gamma I), (B + \mu I)] = \frac{[r'(0)]^2}{8} d_{\text{aRHS}}^2 [(A + \gamma I), (B + \mu I)]. \quad (14)
$$

In particular, for $r = 2\alpha$,

$$
\lim_{\alpha \to 0} D^{(2\alpha, \alpha)} [(A + \gamma I), (B + \mu I)] = \frac{1}{2} d_{\text{aRHS}}^2 [(A + \gamma I), (B + \mu I)]. \quad (15)
$$

This is the content of Theorem 9.

2. The infinite-dimensional Alpha Log-Determinant divergences $d_{\text{logdet}}^{(\alpha)} [(A + \gamma I), (B + \mu I)]$ [17], with

$$
D^{(\alpha, 1-\alpha)} [(A + \gamma I), (B + \mu I)] = d_{\text{logdet}}^{(\alpha)(1-\alpha)} [(A + \gamma I), (B + \mu I)], \quad 0 \leq \alpha \leq 1. \quad (16)
$$

This is the content of Theorem 10.
Since the limit \( \lim_{\alpha \to 0} D_{\alpha,\beta}^{(\alpha,\alpha)}[(A + \gamma I), (B + \mu I)] \) in the first part of Theorem 2 is unique, up to the multiplicative factor \([r'(0)]^2/8,\) we define the quantity \( D_{0,0}^{(0,0)}[(A + \gamma I), (B + \mu I)] \) as follows.

**Definition 3 (Limiting cases - II).** For \((A + \gamma I), (B + \mu I) \in \text{PTr}(\mathcal{H}),\) the Log-Det divergence \( D_{0,0}^{(0,0)}[(A + \gamma I), (B + \mu I)] \) is defined to be

\[
D_{0,0}^{(0,0)}[(A + \gamma I), (B + \mu I)] = \lim_{\alpha \to 0} D_{2\alpha,0}^{(\alpha,\alpha)}[(A + \gamma I), (B + \mu I)] = \frac{1}{2} d_{\text{aiHS}}[(A + \gamma I), (B + \mu I)].
\]  (17)

Since \( d_{\text{aiHS}}[(A + \gamma I), (B + \mu I)] \) is a metric on \( \text{PTr}(\mathcal{H}),\) \( D_{0,0}^{(0,0)}[(A + \gamma I), (B + \mu I)] \) is automatically a symmetric divergence on \( \text{PTr}(\mathcal{H}).\) In fact, it is a member of the parametrized family \( D_{2\alpha,0}^{(\alpha,\alpha)}[(A + \gamma I), (B + \mu I)], \alpha \geq 0,\) of symmetric divergences on \( \text{PTr}(\mathcal{H}),\) as stated in the following result.

**Theorem 3 (Special cases - II).** The parametrized family \( D_{2\alpha,0}^{(\alpha,\alpha)}[(A + \gamma I), (B + \mu I)], \alpha \geq 0,\) is a family of symmetric divergences on \( \text{PTr}(\mathcal{H}),\) with \( \alpha = 0\) corresponding to the infinite-dimensional affine-invariant Riemannian distance above and \( \alpha = 1/2\) corresponding to the infinite-dimensional symmetric Stein divergence, which is given by \( \frac{1}{4} d_{\text{logdet}}^{(0)}[(A + \gamma I), (B + \mu I)].\)

**Finite-dimensional case.** For \( \gamma = \mu,\) we have \( \delta = \frac{\alpha}{\alpha + \beta},\) so that Eq. (9) becomes

\[
D_{\delta}^{(\alpha,\beta)}[(A + \gamma I), (B + \gamma I)] = \frac{1}{\alpha \beta} \log \det X \left( \frac{\alpha [(A + \gamma I)(B + \gamma I)^{-1}]^{\frac{2\alpha}{\alpha + \beta}} + \beta [(A + \gamma I)(B + \gamma I)^{-1}]^{\frac{\alpha}{\alpha + \beta}}}{\alpha + \beta} \right).
\]  (18)

In the finite-dimensional case, where \( A \) and \( B \) are two \( n \times n \) SPD matrices, setting \( \gamma = 0 \) and recalling that \( \det_X = \det \) for finite matrices, we obtain

\[
D_{\delta}^{(\alpha,\beta)}(A, B) = \frac{1}{\alpha \beta} \log \det \left( \frac{\alpha (AB^{-1})^{\frac{2\alpha}{\alpha + \beta}} + \beta (AB^{-1})^{\frac{\alpha}{\alpha + \beta}}}{\alpha + \beta} \right).
\]  (19)

In particular, by setting \( r = \alpha + \beta,\) we recover Eq. (1). For \( \gamma = \mu,\) Eq. (10) becomes

\[
D_{\sqrt{r}}^{(\alpha,0)}[(A + \gamma I), (B + \gamma I)] = \frac{1}{\alpha r} \left\{ \text{tr}_X([(A + \gamma I)^{-1}(B + \gamma I)]^r) - I) - \log \det_X([(A + \gamma I)^{-1}(B + \gamma I)]^r) \right\},
\]  (20)
which reduces to Eq. (5) when $A, B \in \text{Sym}^{++}(n)$, $\gamma = 0$, and $r = \alpha$. Similarly, Eq. (11) becomes

$$D^{(0,\beta)}(\alpha,\beta)$$

$$= \frac{1}{\beta^2} \left\{ \operatorname{tr}_X\left( [(B + \gamma I)^{-1}(A + \gamma I)]^r - I \right) - \log \det X [(B + \gamma I)^{-1}(A + \gamma I)]^r \right\},$$

which reduces to Eq. (6) when $A, B \in \text{Sym}^{++}(n)$, $\gamma = 0$, and $r = \beta$.

**Remark 3.** As in the cases of the Log-Hilbert-Schmidt distance [19], the infinite-dimensional affine-invariant Riemannian distance [18, 20], and the infinite-dimensional Alpha Log-Det divergences [17], we show below that in general, the infinite-dimensional formulation is not obtainable as the limit of the finite-dimensional version as the dimension approaches infinity.

**Remark 4.** Except for the case $r = \alpha + \beta$, the quantity $r$ in $D(\alpha,\beta)$ that we introduce here, to the best of our knowledge, has no equivalence in the existing literature in the finite-dimensional setting.

**Remark 5.** Throughout the paper, we employ the following notations. Using the identity $(B + \mu I)^{-1} = \frac{1}{\mu} I - \frac{B}{\mu}(B + \mu I)^{-1}$, we write the operator $(B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}$ as

$$(B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2} = \Lambda + \frac{\gamma}{\mu} I \in \text{PTr}(H),$$

where $\Lambda = (B + \mu I)^{-1/2}A(B + \mu I)^{-1/2} - \frac{\gamma}{\mu} B(B + \mu I)^{-1} \in \text{Tr}(H)$. This notation is employed in Eq. (8). Similarly, in Eq. (9), we write

$$(A + \gamma I)(B + \mu I)^{-1} = \frac{\mu}{\mu} I + A(B + \mu I)^{-1} - \frac{\gamma}{\mu} B(B + \mu I)^{-1} = Z + \frac{\gamma}{\mu} I,$$

where $Z = A(B + \mu I)^{-1} - \frac{\gamma}{\mu} B(B + \mu I)^{-1} \in \text{Tr}(H)$.

**Metric properties.** Consider now a special case, where $\alpha = \beta$ and $r = \alpha + \beta$. For simplicity, we consider operators $(A + \gamma I)$ and $(B + \mu I)$ with $\gamma = \mu$. For $\gamma > 0, \gamma \in \mathbb{R}$ fixed, we define the following subset of $\text{PTr}(H)$

$$\text{PTr}(H)(\gamma) = \{ A + \gamma I > 0 : A^* = A, A \in \text{Tr}(H) \}.$$
Remark 6. Throughout the paper, we assume, unless stated otherwise, that $\dim(\mathcal{H}) = \infty$, and the condition $A + \gamma I > 0$ automatically implies that $\gamma > 0$. When $\dim(\mathcal{H}) < \infty$, we can set $\gamma = 0$.

Theorem 4 (Metric property). Let $\gamma > 0, \gamma \in \mathbb{R}$ be fixed. The square root function 
\[ \sqrt{D_{2\alpha}^{(\alpha,\alpha)}[(A + \gamma I), (B + \gamma I)]} \]

is a metric on $\text{PTr}(\mathcal{H})(\gamma)$ for all $\alpha \geq 0$.

We thus have a family of metrics between positive definite operators of the form $(A + \gamma I) \in \text{PTr}(\mathcal{H})(\gamma)$, parametrized by the parameter $\alpha \geq 0$. In particular, with $\alpha = 0$ in Theorem 4, we obtain the affine-invariant Riemannian distance, and with $\alpha = \frac{1}{2}$ we obtain the following metric, which is the square root of the infinite-dimensional Stein divergence

\[ \sqrt{D_{1}^{(1/2,1/2)}[(A + \gamma I), (B + \gamma I)]} = 2 \sqrt{\log \left[ \frac{\det_X \left[ (A + \gamma I) + (B + \gamma I) \right]}{\det_X (A + \gamma I)^{1/2} \det_X (B + \gamma I)^{1/2}} \right]} \cdot \] (25)

The corresponding finite-dimensional result [15], where $A, B \in \text{Sym}^{++}(n)$, is recovered by setting $\gamma = 0$ in Theorem 4. In particular, with $\alpha = 1/2$ and $A, B \in \text{Sym}^{++}(n)$, we obtain the corresponding result of [16].

Remark 7. The analysis of $\sqrt{D_{2\alpha}^{(\alpha,\alpha)}[(A + \gamma I), (B + \mu I)]}$, where $\gamma \neq \mu$, is technically more involved and will be presented in a separate work.

3. Positive definite unitized trace class operators

To generalize the Alpha-Beta Log-Determinant divergences from the finite to infinite-dimensional setting, we need to employ the following concepts

- Positive definite operators $\mathbb{P}(\mathcal{H})$.
- Extended (or unitized) trace class operators $\text{Tr}_X(\mathcal{H})$.
- Positive definite unitized trace class operators $\text{PTr}(\mathcal{H})$.
- Extended Fredholm determinant $\det_X$ on $\text{Tr}_X(\mathcal{H})$. 

9
Exponential, logarithm, and power functions for operators in \( \text{PTr}(\mathcal{H}) \) and their products.

We discuss in detail below the logarithm and power functions of products of operators in \( \text{PTr}(\mathcal{H}) \). Other concepts are briefly reviewed and we refer to [17] for the detailed motivations leading to the definitions of these concepts. Throughout the following, we assume that \( \dim(\mathcal{H}) = \infty \), unless stated explicitly otherwise.

**Positive definite operators.** We recall that an operator \( A \in \mathcal{L}(\mathcal{H}) \) is said to be positive definite if there exists a constant \( M_A > 0 \) such that

\[
\langle x, Ax \rangle \geq M_A ||x||^2 \quad \forall x \in \mathcal{H}.
\]

This is equivalent to saying that \( A \) is both strictly positive and invertible. We denote by \( \mathcal{P}(\mathcal{H}) \) the set of all positive definite operators on \( \mathcal{H} \).

**Extended trace class operators.** Let \( \text{Tr}(\mathcal{H}) \) denote the set of trace class operators on \( \mathcal{H} \), the set of extended (or unitized) trace class operators on \( \mathcal{H} \) is defined to be

\[
\text{Tr}_X(\mathcal{H}) = \{ A + \gamma I : A \in \text{Tr}(\mathcal{H}), \gamma \in \mathbb{R} \}.
\]

Equipped with the extended trace class norm

\[
||A + \gamma I||_{\text{tr}_X} = ||A||_{\text{tr}} + |\gamma| = \text{tr}|A| + |\gamma|,
\]

\( \text{Tr}_X(\mathcal{H}) \) becomes a Banach algebra. For \( (A + \gamma I) \in \text{Tr}_X(\mathcal{H}) \), its extended trace is defined to be

\[
\text{tr}_X(A + \gamma I) = \text{tr}(A) + \gamma.
\]

Thus by this definition \( \text{tr}_X(I) = 1 \), in contrast to usual trace definition, according to which \( \text{tr}(I) = \infty \).

**Extended Fredholm determinant.** For \( (A + \gamma I) \in \text{Tr}_X(\mathcal{H}), \gamma \neq 0 \), its extended Fredholm determinant is defined to be

\[
\det_X(A + \gamma I) = \frac{1}{\gamma} \det \left( \frac{A}{\gamma} + I \right),
\]

where the determinant on the right hand side is the Fredholm determinant. For \( \gamma = 1 \), we recover the Fredholm determinant. In the case \( \dim(\mathcal{H}) < \infty \), we define \( \det_X(A + \gamma I) = \det(A + \gamma I) \), the standard matrix determinant.
Positive definite unitized trace class operators. Having defined both positive definite operators and extended trace class operators, the set of positive definite unitized trace class operators $PTr(\mathcal{H}) \subset Tr_X(\mathcal{H})$ is then defined to be the intersection

$$PTr(\mathcal{H}) = \text{Sym}(\mathcal{H}) \cap \mathcal{P}(\mathcal{H}) = \{ A + \gamma I > 0 : A^* = A, A \in Tr(\mathcal{H}) \, \gamma \in \mathbb{R} \}.$$

Exponential, logarithm, and power functions. Consider the exponential function $\exp : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ defined by

$$\exp(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!}.$$

The following result shows that $\exp$ maps $Tr_X(\mathcal{H})$ to $Tr_X(\mathcal{H})$.

**Lemma 1.** Let $(A + \gamma I) \in Tr_X(\mathcal{H})$. Then $\exp(A + \gamma I) \in Tr_X(\mathcal{H})$.

Consider next the inverse function $\log = \exp^{-1} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$. For any $(A + \gamma I) \in PTr(\mathcal{H})$, $\log(A + \gamma I)$ is always well-defined as follows. Let $\{\lambda_k\}_{k=1}^{\infty}$ be the eigenvalues of $A$ with corresponding orthonormal eigenvectors $\{\phi_k\}_{k=1}^{\infty}$. Then

$$A = \sum_{k=1}^{\infty} \lambda_k \phi_k \otimes \phi_k, \quad \log(A + \gamma I) = \sum_{k=1}^{\infty} \log(\lambda_k + \gamma) \phi_k \otimes \phi_k,$$

where $\phi_k : \mathcal{H} \to \mathcal{H}$ is a rank-one operator defined by $(\phi_k \otimes \phi_k)w = \langle \phi_k, w \rangle \phi_k \forall w \in \mathcal{H}$. Moreover, $\log(A + \gamma I) \in \text{Sym}(\mathcal{H}) \cap Tr_X(\mathcal{H})$ and assumes the form

$$\log(A + \gamma I) = A_1 + \gamma_1 I, \quad A_1 \in \text{Sym}(\mathcal{H}) \cap Tr(\mathcal{H}), \gamma_1 \in \mathbb{R}.$$

By Proposition 6 in [17], for any $\alpha \in \mathbb{R}$, the power function $(A + \gamma I)^\alpha$ is then well-defined via the expression

$$(A + \gamma I)^\alpha = \exp[\alpha \log(A + \gamma I)] \in PTr(\mathcal{H}).$$

For the purposes of the current work, we need to go beyond the set $PTr(\mathcal{H})$. Specifically, for two operators $(A + \gamma I), (B + \mu I) \in PTr(\mathcal{H})$, we show that

$$\log[(A + \gamma I)(B + \mu I)^{-1}], \quad [(A + \gamma I)(B + \mu I)^{-1}]^\alpha, \alpha \in \mathbb{R} \quad (27)$$

are all well-defined and are elements of $Tr_X(\mathcal{H})$, even though they are no longer necessarily self-adjoint.
First, let $B \in \mathcal{L}(\mathcal{H})$ be any invertible operator, then for any $A \in \mathcal{L}(\mathcal{H})$, we have
\[
\exp(BAB^{-1}) = \sum_{j=0}^{\infty} \frac{(BAB^{-1})^j}{j!} = B \left( \sum_{j=0}^{\infty} \frac{A^j}{j!} \right) B^{-1} = B \exp(A) B^{-1}.
\]
Thus for $(A + \gamma I) \in P\text{Tr}(\mathcal{H})$, the logarithm of $B(A + \gamma I)B^{-1} = BAB^{-1} + \gamma I \in \text{Tr}_X(\mathcal{H})$ is also well-defined and is given by
\[
\log[B(A + \gamma I)B^{-1}] = B \log(A + \gamma I)B^{-1} = B(A_{1} + \gamma I)B^{-1} = BA_{1}B^{-1} + \gamma I \in \text{Tr}_X(\mathcal{H}).
\] (28)

Using Eq. (28), we obtain the following results.

**Proposition 1.** Let $(A + \gamma I), (B + \mu I) \in P\text{Tr}(\mathcal{H})$. Let $\Lambda + \gamma \mu I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}$. Then

1. The logarithm $\log[(A + \gamma I)(B + \mu I)^{-1}] \in \text{Tr}_X(\mathcal{H})$ is well-defined and is given by
   \[
   \log[(A + \gamma I)(B + \mu I)^{-1}] = (B + \mu I)^{1/2} \log \left( \Lambda + \frac{\gamma I}{\mu} \right) (B + \mu I)^{-1/2}.
   \] (29)

2. For any $\alpha \in \mathbb{R}$, the power function $[(A + \gamma I)(B + \mu I)^{-1}]^\alpha \in \text{Tr}_X(\mathcal{H})$ is well-defined and is given by
   \[
   [(A + \gamma I)(B + \mu I)^{-1}]^\alpha = (B + \mu I)^{1/2} \left( \Lambda + \frac{\gamma I}{\mu} \right)^\alpha (B + \mu I)^{-1/2}. \] (30)

3. For any $p, q \in \mathbb{R}$, any $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta \neq 0$,
   \[
   \det_X \left[ \frac{\alpha[(A + \gamma I)(B + \mu I)^{-1}]^p + \beta[(A + \gamma I)(B + \mu I)^{-1}]^q}{\alpha + \beta} \right] = \det_X \left[ \frac{\alpha(\Lambda + \frac{\gamma I}{\mu})^p + \beta(\Lambda + \frac{\gamma I}{\mu})^q}{\alpha + \beta} \right]. \] (31)

4. Infinite-Dimensional Alpha-Beta Log-Determinant divergences

We now show the motivations and derivations leading to Definition 1. We recall that in the case $\dim(\mathcal{H}) < \infty$, the Log-Det divergences were motivated by Ky Fan’s
inequality [21] on the log-concavity of the determinant, which states that for $A, B \in \text{Sym}^{++}(n)$, $\det(\alpha A + (1 - \alpha)B) \geq \det(A)^\alpha \det(B)^{1-\alpha}$, $0 \leq \alpha \leq 1$, with equality if and only if $A = B$ ($0 < \alpha < 1$). This inequality has recently been generalized to the infinite-dimensional setting for the extended Fredholm determinant (Theorem 1 in [17]). The following is a further generalization of Theorem 1 in [17].

**Theorem 5.** Let $0 \leq \alpha \leq 1$. For $(A + \gamma I), (B + \mu I) \in \text{PTr(H)}$, for any $p, q \in \mathbb{R}$,

$$
\det_X\left[\alpha(A + \gamma I)^p + (1 - \alpha)(B + \mu I)^q\right] \\
\geq \left(\frac{\gamma p}{\mu q}\right)^{\alpha - \delta} \det_X(A + \gamma I)^{\beta \delta} \det_X(B + \mu I)^{q(1-\delta)},
$$

(32)

where \(\delta = \frac{\alpha \gamma p}{\alpha \gamma p + (1 - \alpha)\mu q}\), \(1 - \delta = \frac{(1 - \alpha)\mu q}{\alpha \gamma p + (1 - \alpha)\mu q}\). For $0 < \alpha < 1$, equality happens if and only if

\[
\left(\frac{A}{\gamma} + I\right)^p = \left(\frac{B}{\mu} + I\right)^q \quad \text{and} \quad \gamma^p = \mu^q \iff (A + \gamma I)^p = (B + \mu I)^q.
\]

(33)

In particular, for $\gamma = \mu \neq 1$, equality happens if and only if simultaneously

\[
p = q \quad \text{and} \quad A = B.
\]

(34)

In particular, for $p = q = 1$, we recover Theorem 1 in [17]. From Theorem 5, we immediately have the following result.

**Corollary 1.** Let $\alpha > 0$, $\beta > 0$. For $(A + \gamma I), (B + \mu I) \in \text{PTr(H)}$, for any $p, q \in \mathbb{R}$,

$$
\det_X\left[\frac{\alpha(A + \gamma I)^p + \beta(B + \mu I)^q}{\alpha + \beta}\right] \\
\geq \left(\frac{\gamma p}{\mu q}\right)^{\alpha - \delta} \det_X(U) \frac{\alpha}{\alpha + \beta} \det_X(A + \gamma I)^{\beta \delta} \det_X(B + \mu I)^{q(1-\delta)},
$$

(35)

where \(\delta = \frac{\alpha \gamma p}{\alpha \gamma p + 2\mu q}\), \(1 - \delta = \frac{2\mu q}{\alpha \gamma p + 2\mu q}\). Equality happens if and only if $(A + \gamma I)^p = (B + \mu I)^q$. For $\gamma = \mu \neq 1$, equality happens if and only if simultaneously $p = q$ and $A = B$.

Motivated by Theorem 5 and Corollary 1, we first define the following quantity.
Definition 4. Let $\alpha > 0$, $\beta > 0$ be fixed. For $(A + \gamma I), (B + \mu I) \in \text{PTr}(H)$, for $p, q \in \mathbb{R}$, define

$$D^{(\alpha, \beta)}_{(p,q)}[(A + \gamma I), (B + \mu I)] = \frac{1}{\alpha \beta} \log \left[ \frac{\alpha (\Lambda + \frac{2}{n} I)^p + \beta (\Lambda + \frac{2}{n} I)^{-q}}{\alpha + \beta} \right],$$

(36)

where $\Lambda + \frac{2}{n} I = (B + \mu I)^{-1/2} (A + \gamma I) (B + \mu I)^{-1/2}$, $\delta = \frac{\alpha (\frac{2}{n})^p + \beta}{\alpha (\frac{2}{n})^p + \beta}$. The following theorem gives sufficient conditions for $p, q \in \mathbb{R}$, with $\alpha > 0$, $\beta > 0$ being fixed, so that for a given pair of operators $(A + \gamma I), (B + \mu I) \in \text{PTr}(H)$, the quantity $D^{(\alpha, \beta)}_{(p,q)}[(A + \gamma I), (B + \mu I)]$ in Definition 4 is nonnegative, with equality if and only if $A = B$ and $\gamma = \mu$.

Theorem 6. Let $\alpha > 0$, $\beta > 0$ be fixed. For $(A + \gamma I), (B + \mu I) \in \text{PTr}(H)$, assume that $p, q \in \mathbb{R}$ satisfy the following conditions

$$p + q \neq 0,$$  

(37)

$$\alpha p \left( \frac{\gamma}{\mu} \right)^{p+q} = \beta q.$$  

(38)

Then the quantity $D^{(\alpha, \beta)}_{(p,q)}[(A + \gamma I), (B + \mu I)]$ satisfies

$$D^{(\alpha, \beta)}_{(p,q)}[(A + \gamma I), (B + \mu I)] \geq 0,$$  

(39)

$$D^{(\alpha, \beta)}_{(p,q)}[(A + \gamma I), (B + \mu I)] = 0 \iff A = B, \gamma = \mu.$$  

(40)

Subsequently, we assume that conditions (37) and (38) are satisfied. We see that $p$ and $q$ are not uniquely determined by (38). One way to enforce the uniqueness of $p$ and $q$ is by fixing the sum $p + q$. This is the approach we adopt in this work, which leads to Definition 1.

Theorem 7. Under the hypothesis of Theorem 6, assume further that $p + q = r$, $r \in \mathbb{R}$, $r \neq 0$, $r$ fixed. Under this condition, in Definition 4, we have

$$\delta = \frac{\alpha (\frac{2}{n})^r}{\alpha (\frac{2}{n})^r + \beta}, \quad p = r(1 - \delta) = \frac{\beta r}{\alpha (\frac{2}{n})^r + \beta}, \quad q = r \delta = \frac{\alpha r (\frac{2}{n})^r}{\alpha (\frac{2}{n})^r + \beta}.$$  

(41)
Plugging the expressions for \( p \) and \( q \) in Eq. (41) into Definition 4, we obtain Definition 1. Furthermore, the two formulas given in Eqs. (8) and (9) in Definition 1 are equivalent.

We now show how \( D_{(p,q)}^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)] \) can be expressed concretely in terms of the Fredholm determinant.

**Theorem 8.** Let \( \alpha > 0, \beta > 0 \) be fixed. For \( (A + \gamma I), (B + \mu I) \in \text{PTr}(\mathcal{H}) \), assume that \( p, q \in \mathbb{R} \) satisfy conditions (37) and (38) in Theorem 6. Then

\[
D_{(p,q)}^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)] = \frac{(p+q)(\delta - \frac{\alpha}{\alpha+\beta})}{\alpha\beta} \left( \log \frac{\gamma}{\mu} \right) + \frac{1}{\alpha\beta} \log \det \left[ \frac{\alpha(\Lambda + \frac{\gamma}{\mu} I)^p + \beta(\Lambda + \frac{\gamma}{\mu} I)^{-q}}{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}} \right].
\]

5. Special cases of the Alpha-Beta Log-Determinant divergences

We now describe several important special cases of Definition 1, including the infinite-dimensional affine-invariant Riemannian distance, the infinite-dimensional Alpha Log-Det divergences [17], and the infinite-dimensional Beta Log-Det divergences.

5.1. Affine-invariant Riemannian distance

Let \( \text{HS}(\mathcal{H}) \) denote the space of Hilbert-Schmidt operators on \( \mathcal{H} \), which is defined by

\[
\text{HS}(\mathcal{H}) = \{ A \in \mathcal{L}(\mathcal{H}) : \|A\|_{\text{HS}}^2 = \text{tr}(A^*A) < \infty \},
\]

where \( \| \cdot \|_{\text{HS}} \) is the Hilbert-Schmidt norm. If \( A \) is Hilbert-Schmidt, then \( A \) is compact and possesses a countable set of eigenvalues \( \{\lambda_k\}_{k=1}^{\infty} \). If \( A \) is furthermore self-adjoint, then the Hilbert-Schmidt norm of \( A \) is given by

\[
\|A\|_{\text{HS}}^2 = \sum_{k=1}^{\infty} \lambda_k^2.
\]

We recall the infinite-dimensional Hilbert manifold of positive definite unitized Hilbert-Schmidt operators on \( \mathcal{H} \), considered in [18]

\[
\Sigma(\mathcal{H}) = \{ A + \gamma I > 0 : A = A^*, A \in \text{HS}(\mathcal{H}), \gamma \in \mathbb{R} \}.
\]
In the case $\dim(\mathcal{H}) = \infty$, the set $\text{PTr}(\mathcal{H})$ of positive definite unitized trace class operators on $\mathcal{H}$ is a strict subset of $\Sigma(\mathcal{H})$. The manifold $\Sigma(\mathcal{H})$ can be equipped with the following Riemannian metric, as formulated by [18]. For each $P \in \Sigma(\mathcal{H})$, on the tangent space $T_P(\Sigma(\mathcal{H})) \cong \mathcal{H}_\mathbb{R} = \{A + \gamma I : A = A^*, A \in \text{HS}(\mathcal{H}), \gamma \in \mathbb{R}\}$, we define the following inner product

$$\langle A + \gamma I, B + \mu I \rangle_P = \langle P^{-1/2}(A + \gamma I)P^{-1/2}, P^{-1/2}(B + \mu I)P^{-1/2} \rangle_{\text{eHS}},$$

where $\langle \cdot, \cdot \rangle_{\text{eHS}}$ is the extended Hilbert-Schmidt inner product, defined by

$$\langle A + \gamma I, B + \mu I \rangle_{\text{eHS}} = \langle A, B \rangle_{\text{HS}} + \gamma \mu.$$  

The Riemannian metric given by $\langle \cdot, \cdot \rangle_P$ then makes $\Sigma(\mathcal{H})$ an infinite-dimensional Riemannian manifold. Under this metric, the geodesic distance between $(A + \gamma I), (B + \mu I)$ is given by

$$d_{\text{aHS}}[(A + \gamma I), (B + \mu I)] = ||\log[(B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}]||_{\text{eHS}}.$$  

(43)

We now show that the affine-invariant distance $d_{\text{aHS}}[(A + \gamma I), (B + \mu I)]$ is a limiting case of $D_{\text{r}}^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)]$, as $\alpha \to 0, \beta \to 0$. In this section, we consider $\beta = \alpha$, in which case Definition 1 reduces to the following.

**Definition 5.** In Definition 1, with $\alpha = \beta$, we have

$$D_{\text{r}}^{(\alpha, \alpha)}[(A + \gamma I), (B + \mu I)] = \frac{1}{\alpha^2} \log \left[ \left( \frac{\gamma}{\mu} \right)^{r(\delta - \frac{1}{2})} \det X \left( \frac{(A + \frac{\gamma}{\mu}I)^{r(1-\delta)} + (A + \frac{\gamma}{\mu}I)^{-r\delta}}{2} \right) \right],$$  

(44)

where $\delta = \frac{\frac{\gamma}{\mu}}{1 + \frac{\gamma}{\mu}}, 1 - \delta = \frac{1}{1 + \frac{\gamma}{\mu}}$.

By Theorem 8, we have the following formula, which expresses $D_{\text{r}}^{(\alpha, \alpha)}[(A + \gamma I), (B + \mu I)]$ concretely in terms of the Fredholm determinant.

$$D_{\text{r}}^{(\alpha, \alpha)}[(A + \gamma I), (B + \mu I)] = \frac{r(\delta - \frac{1}{2})}{\alpha^2} \log \left( \frac{\gamma}{\mu} \right) + \frac{1}{\alpha^2} \log \left( \frac{(\frac{\gamma}{\mu})^p + (\frac{\gamma}{\mu})^{-q}}{2} \right)$$

$$+ \frac{1}{\alpha^2} \log \det \left[ \frac{(A + \frac{\gamma}{\mu}I)^p + (A + \frac{\gamma}{\mu}I)^{-q}}{(\frac{\gamma}{\mu})^p + (\frac{\gamma}{\mu})^{-q}} \right],$$  

(45)

16
where \( \delta = \frac{(2r)^\gamma}{(\pi)^{\frac{\gamma}{2}}} \), \( 1 - \delta = \frac{1}{(\pi)^{\frac{1}{2}}} \), \( p = r(1 - \delta) \), \( q = r\delta \).

The following is the main result in this section.

**Theorem 9 (Affine-Invariant Riemannian Distance).** Let \((A + \gamma I), (B + \mu I) \in P\text{Tr}(\mathcal{H})\). Assume that \( r = r(\alpha) \) is smooth, with \( r(0) = 0 \), \( r'(0) \neq 0 \), and \( r(\alpha) \neq 0 \) for \( \alpha \neq 0 \). Then

\[
\lim_{\alpha \to 0} D^{(\alpha, \alpha)}_{\logdet}[(A + \gamma I), (B + \mu I)] = \frac{|r'(0)|^2}{8} d_{\text{aHS}}^2[(A + \gamma I), (B + \mu I)].
\]

In particular, for \( r = 2\alpha \), we have

\[
\lim_{\alpha \to 0} D^{(\alpha, \alpha)}_{\logdet}[(A + \gamma I), (B + \mu I)] = \frac{1}{2} d_{\text{aHS}}^2[(A + \gamma I), (B + \mu I)].
\]

**Remark 8.** We stress that, as they are currently stated, the limits in Theorem 9 are valid for \((A + \gamma I), (B + \mu I) \in P\text{Tr}(\mathcal{H})\), that is \( A \) and \( B \) must be trace class operators. The generalization of Theorem 9 to the entire Hilbert manifold \( \Sigma(\mathcal{H}) \), where \( A \) and \( B \) are Hilbert-Schmidt operators, will be presented in an upcoming work.

### 5.2. Infinite-dimensional Alpha Log-Determinant divergences

We now show that the formulation for the infinite-dimensional Alpha Log-Determinant divergences in [17] is a special case of the present formulation, with \( \beta = 1 - \alpha \) and \( r = \pm 1 \). Let \( \dim(\mathcal{H}) = \infty \). We recall that for \(-1 < \alpha < 1\), the Log-Det \( \alpha \)-divergence \( d_{\logdet}^{(\alpha)} [(A + \gamma I), (B + \mu I)] \) for \((A + \gamma I), (B + \mu I) \in P\text{Tr}(\mathcal{H})\) is defined in [17] to be

\[
d_{\logdet}^{(\alpha)} [(A + \gamma I), (B + \mu I)] = \frac{4}{1 - \alpha^2} \log \left[ \frac{\det_x \left( \frac{1+\alpha}{2} (A + \gamma I) + \frac{1-\alpha}{2} (B + \mu I) \right) \left( \frac{\gamma}{\mu} \right)^{\frac{1-\alpha}{1+\alpha}}} {\det_x (A + \gamma I)^{\mu} \det_x (B + \mu I)^{1-q}} \right],
\]

where \( q = \frac{(1-\alpha)\gamma}{(1-\alpha)\gamma + (1+\alpha)\mu} \) and \( 1-q = \frac{(1+\alpha)\mu}{(1-\alpha)\gamma + (1+\alpha)\mu} \), with the limiting cases \( \alpha = \pm 1 \) given by

\[
d_{\logdet}^{(1)} [(A + \gamma I), (B + \mu I)] = \left( \frac{\gamma}{\mu} - 1 \right) \log \frac{\gamma}{\mu} + \text{tr}_x [(B + \mu I)^{-1} (A + \gamma I) - I]
- \frac{\gamma}{\mu} \log \det_x [(B + \mu I)^{-1} (A + \gamma I)].
\]

\[
d_{\logdet}^{(-1)} [(A + \gamma I), (B + \mu I)] = \left( \frac{\mu}{\gamma} - 1 \right) \log \frac{\mu}{\gamma} + \text{tr}_x [(A + \gamma I)^{-1} (B + \mu I) - I]
- \frac{\mu}{\gamma} \log \det_x [(A + \gamma I)^{-1} (B + \mu I)].
\]
Definition 6. In Definition 1, with $0 < \alpha < 1$ and $\beta = 1 - \alpha$, we have

\[
D_{r}^{(\alpha,1-\alpha)}[(A + \gamma I), (B + \mu I)] \quad (51)
\]

\[
= \frac{1}{\alpha(1-\alpha)} \log \left[ r^{(\delta-\alpha)} \det_{X} \left( \alpha \left( A + \frac{\gamma}{\mu} I \right)^{r(1-\delta)} + (1-\alpha) \left( A + \frac{\gamma}{\mu} I \right)^{-r\delta} \right) \right].
\]

where $\delta = \frac{\alpha(1-\alpha)}{\mu}$ and $1-\delta = \frac{1-\alpha}{\alpha}$.

The following result shows that $D_{r}^{(\alpha,1-\alpha)}[(A + \gamma I), (B + \mu I)]$ for the cases $r = \pm 1$ are precisely $d_{\text{logdet}}^{1-2\alpha}[(A + \gamma I), (B + \mu I)]$ and $d_{\text{logdet}}^{2\alpha-1}[(A + \gamma I), (B + \mu I)]$, respectively.

Theorem 10 (Alpha Log-Determinant Divergences). Let $0 < \alpha < 1$ be fixed. For $(A + \gamma I), (B + \mu I) \in \text{PTr}(H)$,

\[
D_{r}^{(\alpha,1-\alpha)}[(A + \gamma I), (B + \mu I)] = \delta - \alpha \frac{\log \gamma}{\mu} + \frac{1}{\alpha(1-\alpha)} \log \left[ \frac{\det_{X}[\alpha(A + \gamma I) + (1-\alpha)(B + \mu I)]}{\det_{X}(A + \gamma I)\det_{X}(B + \mu I)^{1-\alpha}} \right] \quad (52)
\]

where $\delta = \frac{\alpha\gamma}{\alpha\gamma + (1-\alpha)\mu}$. Similarly,

\[
D_{1}^{(\alpha,1-\alpha)}[(A + \gamma I), (B + \mu I)] = d_{\text{logdet}}^{2\alpha-1}[(A + \gamma I), (B + \mu I)]. \quad (53)
\]

At the endpoints $\alpha = 0$ and $\alpha = 1$,

\[
\lim_{\alpha \to 1} D_{r}^{(\alpha,1-\alpha)}[(A + \gamma I), (B + \mu I)] = d_{\text{logdet}}^{-1}[(A + \gamma I), (B + \mu I)] \quad (54)
\]

\[
\lim_{\alpha \to 0} D_{r}^{(\alpha,1-\alpha)}[(A + \gamma I), (B + \mu I)] = d_{\text{logdet}}^{1}[(A + \gamma I), (B + \mu I)]. \quad (55)
\]

In particular, in Theorem 10, for $\gamma = \mu$, we have $\delta = \alpha$, and

\[
D_{r}^{(\alpha,1-\alpha)}[(A + \gamma I), (B + \gamma I)] = \frac{1}{\alpha(1-\alpha)} \log \left[ \frac{\det_{X}[\alpha(A + \gamma I) + (1-\alpha)(B + \gamma I)]}{\det_{X}(A + \gamma I)^{\alpha}\det_{X}(B + \gamma I)^{1-\alpha}} \right]. \quad (56)
\]

This is the direct generalization of the finite-dimensional formula given by Eq. (6) in [14].

Remark 9 (Beta Log-Determinant Divergences). In the finite-dimensional setting in [15], the authors call $D^{1,\beta}(A, B)$ the Beta Log-Determinant divergence between
$A, B \in \text{Sym}^{++}(n)$. Similarly, in the case $\dim(\mathcal{H}) = \infty$, let $\beta > 0$ be fixed and let $r \in \mathbb{R}, r \neq 0$ be fixed. For $(A + \gamma I), (B + \mu I) \in \text{PTr}(\mathcal{H})$, we then have the corresponding infinite-dimensional Beta Log-Determinant divergence

$$D_r^{(1,\beta)}[(A + \gamma I), (B + \mu I)]$$

$$= \frac{1}{\beta} \log \left[ \left( \frac{\gamma}{\mu} \right)^{r(\beta - \frac{1}{1+\beta})} \text{det}_X \left( \frac{(A + \frac{\gamma}{\mu} I)^{r(1-\delta)} + \beta(A + \frac{\gamma}{\mu} I)^{-r\delta}}{1 + \beta} \right) \right],$$

(57)

where $A + \frac{\gamma}{\mu} I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}$, $\delta = \frac{2r}{(1+\beta)^2}, 1 - \delta = \frac{\beta}{(2 + \beta)^2}$. However, we do not explore this divergence in detail in this work.

5.3. Other limiting cases

We consider next two other limiting cases, namely $\beta \to 0$ when $\alpha > 0$ is fixed, and $\alpha \to 0$ when $\beta > 0$ is fixed. In particular, our definitions of $D_r^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)], \alpha > 0$, and $D_r^{(0,\beta)}[(A + \gamma I), (B + \mu I)], \beta > 0$, as given in Definition 2, are based on the respective limits in Theorems 11 and 12 below.

**Theorem 11 (Limiting case $\alpha > 0, \beta \to 0$).** Let $\alpha > 0$ be fixed. Assume that $r = r(\beta)$ is smooth, with $r(0) = r(\beta = 0)$. Then

$$\lim_{\beta \to 0} D_r^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)] = \frac{r(0)}{\alpha^2} \left[ \left( \frac{\mu}{\gamma} \right)^{r(0)} - 1 \right] \log \frac{\mu}{\gamma}$$

$$+ \frac{1}{\alpha^2} \text{tr}_X \left( [(A + \gamma I)^{-1}(B + \mu I)]^{r(0)} - I \right)$$

$$- \frac{1}{\alpha^2} \left( \frac{\mu}{\gamma} \right)^{r(0)} \text{log det}_X [(A + \gamma I)^{-1}(B + \mu I)]^{r(0)}.$$  

(58)

**Theorem 12 (Limit case $\alpha \to 0, \beta > 0$).** Let $\beta > 0$ be fixed. Assume that $r = r(\alpha)$ is smooth, with $r(0) = r(\alpha = 0)$. Then

$$\lim_{\alpha \to 0} D_r^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)] = \frac{r(0)}{\beta^2} \left[ \left( \frac{\gamma}{\mu} \right)^{r(0)} - 1 \right] \log \frac{\gamma}{\mu}$$

$$+ \frac{1}{\beta^2} \text{tr}_X \left( [(B + \mu I)^{-1}(A + \gamma I)]^{r(0)} - I \right)$$

$$- \frac{1}{\beta^2} \left( \frac{\gamma}{\mu} \right)^{r(0)} \text{log det}_X [(B + \mu I)^{-1}(A + \gamma I)]^{r(0)}.$$ 

(59)
Special cases. Let us now describe several special cases of Theorems 11 and 12, including their specialization to the finite-dimensional setting.

(i) For $\gamma = \mu$, we have

$$
\lim_{\beta \to 0} D^{(\alpha,\beta)}_q[(A + \gamma I), (B + \gamma I)] = \frac{1}{\alpha^2} \text{tr}_X([(A + \gamma I)^{-1}(B + \gamma I)]^{(0)} - I)
$$

$$
- \frac{1}{\alpha^2} \log \det X [(A + \gamma I)^{-1}(B + \gamma I)]^{(0)},
$$

$$
\lim_{\alpha \to 0} D^{(\alpha,\beta)}_q[(A + \gamma I), (B + \gamma I)] = \frac{1}{\beta^2} \text{tr}_X([(B + \gamma I)^{-1}(A + \gamma I)]^{(0)} - I)
$$

$$
- \frac{1}{\beta^2} \log \det X [(B + \gamma I)^{-1}(A + \gamma I)]^{(0)}. \tag{66}
$$

In particular, for $r = \alpha + \beta$, we have $r(\beta = 0) = \alpha$, $r(\alpha = 0) = \beta$, so that

$$
\lim_{\beta \to 0} D^{(\alpha,\beta)}_q[(A + \gamma I), (B + \gamma I)]
$$

$$
= \frac{1}{\alpha^2} \{ \text{tr}_X([(A + \gamma I)^{-1}(B + \gamma I)]^\alpha - I) - \alpha \log \det X [(A + \gamma I)^{-1}(B + \gamma I)] \},
$$

$$
\lim_{\alpha \to 0} D^{(\alpha,\beta)}_q[(A + \gamma I), (B + \gamma I)]
$$

$$
= \frac{1}{\beta^2} \{ \text{tr}_X([(B + \gamma I)^{-1}(A + \gamma I)]^\beta - I) - \beta \log \det X [(B + \gamma I)^{-1}(A + \gamma I)] \}. \tag{63}
$$

These are the direct generalizations of the corresponding formulas in the finite-dimensional setting. In fact, for $A, B \in \text{Sym}^+(n)$, $n \in \mathbb{N}$, by setting $\gamma = 0$, we obtain

$$
\lim_{\beta \to 0} D^{(\alpha,\beta)}_q[A, B] = \frac{1}{\alpha^2} \{ \text{tr}([A^{-1}B]^\alpha - I) - \alpha \log \det (A^{-1}B) \}, \tag{64}
$$

$$
\lim_{\alpha \to 0} D^{(\alpha,\beta)}_q[A, B] = \frac{1}{\beta^2} \{ \text{tr}([B^{-1}A]^\beta - I) - \beta \log \det (B^{-1}A) \}. \tag{65}
$$

These are precisely the finite-dimensional expressions given by Eqs. (5) and 6, which are Eqs. (23) and (22) in [15], respectively.

(ii) If $r(0) = r(\beta = 0) = 1$, we have for $\alpha > 0$ fixed,

$$
\lim_{\beta \to 0} D^{(\alpha,\beta)}_q[(A + \gamma I), (B + \mu I)] = \frac{1}{\alpha^2} \left( \frac{\mu}{\gamma} - 1 \right) \log \frac{\mu}{\gamma}
$$

$$
+ \frac{1}{\alpha^2} \left\{ \text{tr}_X[(A + \gamma I)^{-1}(B + \mu I) - I] - \frac{\mu}{\gamma} \log \det X [(A + \gamma I)^{-1}(B + \mu I)] \right\}
$$

$$
= \frac{1}{\alpha^2} d^{-1}\log_{\det X}[(A + \gamma I), (B + \mu I)]. \tag{66}
$$
Similarly, if \( r(0) = r(\alpha = 0) = 1 \), we have for \( \beta > 0 \) fixed,

\[
\lim_{\alpha \to 0} D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)] = \frac{1}{\beta^2} \left( \frac{\gamma}{\mu} - 1 \right) \log \frac{\gamma}{\mu} + \frac{1}{\beta^2} \left\{ \text{tr}[(B + \mu I)^{-1}(A + \gamma I) - I] - \frac{\gamma}{\mu} \log \det[(B + \mu I)^{-1}(A + \gamma I)] \right\}
\]

\[
= \frac{1}{\beta^2} d_1^{d_1 \log \det[(A + \gamma I), (B + \mu I)]}.
\]

(67)

In particular, if \( r \equiv 1 \) as a constant function, then with \( \beta = 1 - \alpha \), we have

\[
\lim_{\alpha \to 1} D_r^{(\alpha, 1 - \alpha)}[(A + \gamma I), (B + \mu I)] = d_1^{d_1 \log \det[(A + \gamma I), (B + \mu I)]}.
\]

(68)

This is precisely the dual symmetry of the infinite-dimensional Alpha Log-Det divergences (Theorem 4 in [17]).

6. Properties of the Alpha-Beta Log-Determinant divergences

The following results establish several important results of \( D_r^{(\alpha, \beta)} \) as defined above, which generalize those from both the finite-dimensional setting [14, 15] and the infinite-dimensional Alpha Log-Det divergences [17].

**Theorem 13 (Dual symmetry).**

\[
D_r^{(\beta, \alpha)}[(B + \mu I), (A + \gamma I)] = D_r^{(\alpha, \beta)}[(A + \gamma I), (B + \mu I)].
\]

(68)

_in particular, for \( \beta = \alpha \), we have

\[
D_r^{(\alpha, \alpha)}[(B + \mu I), (A + \gamma I)] = D_r^{(\alpha, \alpha)}[(A + \gamma I), (B + \mu I)].
\]

(69)

**Special case: Dual symmetry of the infinite-dimensional Alpha Log-Det divergences.** By Theorem 10, we have for \( 0 \leq \alpha \leq 1 \),

\[
D_1^{(\alpha, 1 - \alpha)}[(A + \gamma I), (B + \mu I)] = D_1^{(1 - \alpha, \alpha)}[(B + \mu I), (A + \gamma I)]
\]

\[
\iff d_1^{d_1 \log \det[(A + \gamma I), (B + \mu I)]} = d_1^{d_1 \log \det[(B + \mu I), (A + \gamma I)]}.
\]

(70)

This is precisely the dual symmetry of the infinite-dimensional Alpha Log-Det divergences (Theorem 4 in [17]).
Theorem 14 (Dual invariance under inversion).
\[
D_r^{(\alpha,\beta)}[(A + \gamma I)^{-1}, (B + \mu I)^{-1}] = D_r^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)]
\]  \hspace{1cm} (71)

Special case: Dual invariance under inversion of the infinite-dimensional Alpha Log-Det divergences. By Theorem 10, we have
\[
D_1^{(\alpha,1-\alpha)}[(A + \gamma I)^{-1}, (B + \mu I)^{-1}] = D_{-1}^{(\alpha,1-\alpha)}[(A + \gamma I), (B + \mu I)]
\]
\[
\iff d_{\logdet}^{(1-2\alpha)}[(A + \gamma I)^{-1}, (B + \mu I)^{-1}] = d_{\logdet}^{(1-2\alpha)}[(A + \gamma I), (B + \mu I)].
\]  \hspace{1cm} (72)

This is precisely the dual invariance under inversion of the infinite-dimensional Alpha Log-Det divergences (Theorem 5 in [17]).

Theorem 15 (Affine invariance). For any \((A + \gamma I), (B + \mu I) \in \text{PTr}(\mathcal{H})\) and any invertible \((C + \nu I) \in \text{Tr}_X(\mathcal{H}), \nu \neq 0\),
\[
D_r^{(\alpha,\beta)}[(C + \nu I)(A + \gamma I)(C + \nu I)^*, (C + \nu I)(B + \mu I)(C + \nu I)^*] = D_r^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)].
\]  \hspace{1cm} (73)

Theorem 16 (Invariance under unitary transformations). For any \((A + \gamma I), (B + \mu I) \in \text{PTr}(\mathcal{H})\) and any \(C \in \mathcal{L}(\mathcal{H})\), with \(CC^* = C^*C = I\),
\[
D_r^{(\alpha,\beta)}[C(A + \gamma I)C^*, C(B + \mu I)C^*] = D_r^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)].
\]  \hspace{1cm} (74)

Theorem 17.
\[
D_r^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)] = D_r^{(\alpha,\beta)}\left[\left(A + \frac{\gamma}{\mu} I\right), I\right].
\]  \hspace{1cm} (75)

Theorem 18. Let \(\omega \in \mathbb{R}, \omega \neq 0\) be arbitrary. Then
\[
D_{(\alpha,\omega\beta)}[(A + \gamma I), (B + \mu I)] = \frac{1}{\omega^\alpha} D_r^{(\alpha,\beta)}\left[\left(A + \frac{\gamma}{\mu} I\right)^\omega, I\right].
\]  \hspace{1cm} (76)

The following two properties are important for proving that the square root function
\[
\sqrt{D_{2\alpha}^{(\alpha,\omega)}}[(A + \gamma I), (B + \gamma I)]\text{, is a metric on PTr}(\mathcal{H})(\gamma).\] We focus on the case \(\alpha > 0\), since for \(\alpha = 0\),
\[
\sqrt{D_{2\alpha}^{(\alpha,\omega)}}[(A + \gamma I), (B + \mu I)] = \frac{1}{\sqrt{2}} d_{\text{adH}}[(A + \gamma I), (B + \mu I)\text{ is automatically a metric on PTr}(\mathcal{H}).
Theorem 19 (Convergence in trace norm). Let $\alpha > 0$ be fixed. Let $\mathcal{H}$ be a separable Hilbert space. Let $A, B : \mathcal{H} \to \mathcal{H}$ be self-adjoint, trace class operators such that $(I + A) > 0, (I + B) > 0$. Let $\{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}}$ be sequences of self-adjoint, trace class operators such that $\lim_{n \to \infty} \|A_n - A\|_{\text{tr}} = 0, \lim_{n \to \infty} \|B_n - B\|_{\text{tr}} = 0$. Then

$$\lim_{n \to \infty} D^{(\alpha,\alpha)}_{2n}(A_n, B_n) = D^{(\alpha,\alpha)}_{2n}(A, B).$$

(77)

Theorem 20 (Triangle inequality). Let $\alpha > 0$ be fixed. Let $\mathcal{H}$ be a separable Hilbert space. Let $\gamma > 0, \gamma \in \mathbb{R}$ be fixed. Let $A, B, C : \mathcal{H} \to \mathcal{H}$ be self-adjoint, trace class operators such that $(A + \gamma I) > 0, (B + \gamma I) > 0, (C + \gamma I) > 0$. Then

$$\sqrt{D^{(\alpha,\alpha)}_{2n}[(A + \gamma I), (B + \gamma I)]} \leq \sqrt{D^{(\alpha,\alpha)}_{2n}[(A + \gamma I), (C + \gamma I)]} + \sqrt{D^{(\alpha,\alpha)}_{2n}[(C + \gamma I), (B + \gamma I)]}.$$  

(78)

In particular, for $\alpha = 1/2$ and $\gamma = 1$, we obtain the following triangle inequality.

Theorem 21 (Triangle inequality - square root of symmetric Stein divergence). Let $\mathcal{H}$ be a separable Hilbert space. Let $A, B, C : \mathcal{H} \to \mathcal{H}$ be self-adjoint trace class operators with $A + I > 0, B + I > 0, C + I > 0$. Then

$$\sqrt{\log \frac{\det(\frac{A + B}{2} + I)}{\det(A + I) \det(B + I)}} \leq \sqrt{\log \frac{\det(\frac{A + C}{2} + I)}{\det(A + I) \det(C + I)}} + \sqrt{\log \frac{\det(\frac{C + B}{2} + I)}{\det(C + I) \det(B + I)}}.$$  

(79)

Theorem 22 (Diagonalization). Let $\alpha \geq 0$ be fixed. Let $\mathcal{H}$ be a separable Hilbert space. Let $\gamma > 0, \gamma \in \mathbb{R}$, be fixed. Let $A, B : \mathcal{H} \to \mathcal{H}$ be self-adjoint trace class operators, such that $A + \gamma I > 0, B + \gamma I > 0$. Let $\text{Eig}(A), \text{Eig}(B) : \ell^2 \to \ell^2$ be diagonal operators with the diagonals consisting of the eigenvalues of $A$ and $B$, respectively, in decreasing order. Then

$$D^{(\alpha,\alpha)}_{2n}[(\text{Eig}(A) + \gamma I), (\text{Eig}(B) + \gamma I)] \leq D^{(\alpha,\alpha)}_{2n}[(A + \gamma I), (B + \gamma I)].$$  

(80)

7. Alpha-Beta Log-Det divergences between RKHS covariance operators

Let $\mathcal{X}$ be an arbitrary non-empty set. We now compute the Alpha-Beta Log-Det divergences between covariance operators on an RKHS induced by a positive definite
kernel $K$ on $X \times X$. In this case, we have explicit formulas for $D^{(\alpha,\beta)}$ via the corresponding Gram matrices. We recall that similar formulas exist in the cases of the Log-Hilbert-Schmidt distance [19], the infinite-dimensional affine-invariant Riemannian distance [18, 20], and the infinite-dimensional Alpha Log-Det divergences [17].

We first prove the following result.

**Theorem 23.** Let $\mathcal{H}_1, \mathcal{H}_2$ be separable Hilbert spaces. Let $A, B : \mathcal{H}_1 \to \mathcal{H}_2$ be compact linear operators such that both $AA^* : \mathcal{H}_2 \to \mathcal{H}_2$ and $BB^* : \mathcal{H}_2 \to \mathcal{H}_2$ are trace class operators. Assume that $\dim(\mathcal{H}_2) = \infty$. Let $\alpha, \beta > 0$ be fixed. For any $r \in \mathbb{R}$, $r \neq 0$, for any $\gamma > 0$, $\mu > 0$,

$$D^{(\alpha,\beta)}_r[(AA^* + \gamma I_{\mathcal{H}_2}), (BB^* + \mu I_{\mathcal{H}_2})] = r(\delta - \frac{\alpha}{\alpha + \gamma}) \left( \log \frac{\gamma}{\mu} \right) + \frac{1}{\alpha + \beta} \log \left( \frac{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}}{\alpha + \beta} \right) + \frac{1}{\alpha + \beta} \log \det \left[ \frac{\alpha(\frac{\gamma}{\mu})^p(C + I_{\mathcal{H}_1} \otimes I_3)^p + \beta(\frac{\gamma}{\mu})^{-q}(C + I_{\mathcal{H}_1} \otimes I_3)^{-q}}{\alpha(\frac{\gamma}{\mu})^p + \beta(\frac{\gamma}{\mu})^{-q}} \right],$$

(81)

where $\delta = \frac{\alpha \gamma^r}{\alpha \gamma^r + \beta \mu^r}$, $p = r(1 - \delta)$, $q = r \delta$, and

$$C = \begin{pmatrix}
\frac{A^* A}{\gamma} - \frac{B^* B}{\gamma \mu} & \frac{A^* A B}{\gamma \mu^p} (I_{\mathcal{H}_1} + \frac{B^* B}{\mu})^{-1} - \frac{A^* A B^*}{\gamma \mu^p} (I_{\mathcal{H}_1} + \frac{B^* B}{\mu})^{-1} \\
\frac{B^* A}{\gamma \mu^p} - \frac{B^* A B}{\gamma \mu} & \frac{B^* A B}{\gamma \mu^p} (I_{\mathcal{H}_1} + \frac{B^* B}{\mu})^{-1} - \frac{B^* A B^*}{\gamma \mu^p} (I_{\mathcal{H}_1} + \frac{B^* B}{\mu})^{-1}
\end{pmatrix}.$$  

(82)

For comparison, the following is the corresponding version of $D^{(\alpha,\beta)}_r[(AA^* + \gamma I_{\mathcal{H}_2}), (BB^* + \mu I_{\mathcal{H}_2})]$, using the finite-dimensional formula given in Eq. (19), when $\dim(\mathcal{H}_2) < \infty$.

**Theorem 24.** Let $\mathcal{H}_1, \mathcal{H}_2$ be separable Hilbert spaces. Let $A, B : \mathcal{H}_1 \to \mathcal{H}_2$ be compact linear operators such that both $AA^* : \mathcal{H}_2 \to \mathcal{H}_2$ and $BB^* : \mathcal{H}_2 \to \mathcal{H}_2$ are trace class operators. Assume that $\dim(\mathcal{H}_2) < \infty$. Let $\alpha, \beta > 0$ be fixed. For any
\( r \in \mathbb{R}, r \neq 0, \) for any \( \gamma > 0, \mu > 0, \)
\[
D^{(\alpha, \beta)}_{r}[(AA^* + \gamma I_{H_2}), (BB^* + \mu I_{H_2})] = \alpha \beta \log \det \left[ \frac{\alpha(\gamma \mu)^p + \beta (\gamma \mu)^{-q}}{\alpha + \beta} \right],
\]
where \( p = r_{\alpha + \beta}, q = r_{\alpha + \beta}, \) and \( C \) is as given in Theorem 23.

Let us briefly recall the RKHS covariance operators discussed in [17]. Let \( x = [x_1, \ldots, x_m] \) be a data matrix randomly sampled from \( \mathcal{X} \) according to a Borel probability distribution \( \rho \), where \( m \in \mathbb{N} \) is the number of observations. Let \( K \) be a positive definite kernel on \( \mathcal{X} \times \mathcal{X} \) and \( K_K \) its induced reproducing kernel Hilbert space (RKHS). Let \( \Phi : \mathcal{X} \rightarrow \mathcal{H}_K \) be the corresponding feature map, so that \( K(x, y) = \langle \Phi(x), \Phi(y) \rangle_{\mathcal{H}_K} \) for all pairs \( (x, y) \in \mathcal{X} \times \mathcal{X} \). The feature map \( \Phi \) gives rise to the bounded linear operator
\[
\Phi(x) : \mathbb{R}^m \rightarrow \mathcal{H}_K, \quad \Phi(x)b = \sum_{j=1}^{m} b_j \Phi(x_j), \quad b \in \mathbb{R}^m.
\]  
(84)

The operator \( \Phi(x) \) can also be viewed as the (potentially infinite) mapped data matrix \( \Phi(x) = [\Phi(x_1), \ldots, \Phi(x_m)] \) of size \( \dim(\mathcal{H}_K) \times m \) in the feature space \( \mathcal{H}_K \), with the \( j \)th column being \( \Phi(x_j) \). The corresponding empirical covariance operator for \( \Phi(x) \) is defined to be
\[
C_{\Phi(x)} = \frac{1}{m} \Phi(x)J_m \Phi(x)^* : \mathcal{H}_K \rightarrow \mathcal{H}_K,
\]  
(85)

where \( \Phi(x)^* : \mathcal{H}_K \rightarrow \mathbb{R}^m \) is the adjoint operator of \( \Phi(x) \) and \( J_m \) is the centering matrix, defined by \( J_m = I_m - \frac{1}{m} 1_m 1_m^T \) with \( 1_m = (1, \ldots, 1)^T \in \mathbb{R}^m \).

Let \( x = [x_i]_{i=1}^{m}, y = [y_i]_{i=1}^{m}, m \in \mathbb{N}, \) be two random data matrices sampled from \( \mathcal{X} \) according to two Borel probability distributions and \( C_{\Phi(x)}, C_{\Phi(y)} \) be the corresponding covariance operators induced by the kernel \( K \). Let \( K[x], K[y], \) and \( K[x, y] \) be the \( m \times m \) Gram matrices defined by
\[
(K[x])_{ij} = K(x_i, x_j), \quad (K[y])_{ij} = K(y_i, y_j),
\]
\[
(K[x, y])_{ij} = K(x_i, y_j), \quad 1 \leq i, j \leq m.
\]  
(86)
Let $A = \frac{1}{\sqrt{m}} \Phi(x) J_m : \mathbb{R}^m \to \mathcal{H}_K$, $B = \frac{1}{\sqrt{m}} \Phi(y) J_m : \mathbb{R}^m \to \mathcal{H}_K$, so that

$$AA^* = C \Phi(x), \quad BB^* = C \Phi(y), \quad A^* A = \frac{1}{m} J_{m}[x] J_m, \quad B^* B = \frac{1}{m} J_{m}[y] J_m.$$ (87)

Theorems 23 and 24 can then be applied to give closed form formulas for the divergences between $(C_{\Phi(x)} + \gamma I)$ and $(C_{\Phi(y)} + \mu I)$, as follows.

**Theorem 25 (Alpha-Beta Log-Det divergences between RKHS covariance operators - Infinite-dimensional version).** Let $\alpha, \beta > 0$ be fixed. Let $r \in \mathbb{R}$, $r \neq 0$ be fixed. Assume that $\dim(\mathcal{H}_K) = \infty$. For any $\gamma > 0$, $\mu > 0$, the divergence $D_r^{(\alpha, \beta)}[(C_{\Phi(x)} + \gamma I), (C_{\Phi(y)} + \mu I)]$ is given by

$$D_r^{(\alpha, \beta)}[(C_{\Phi(x)} + \gamma I), (C_{\Phi(y)} + \mu I)] = r(\delta - \frac{\alpha}{\alpha + \beta}) \left( \log \frac{\gamma}{\mu} \right) + \frac{1}{\alpha \beta} \log \left( \frac{\alpha(\frac{q}{p})^\mu + \beta(\frac{q}{p})^{-\mu}}{\alpha + \beta} \right) + \frac{1}{\alpha \beta} \log \det \left[ \frac{\alpha(\frac{q}{p})^\mu (C + I_{3m})^\rho + \beta(\frac{q}{p})^{-\mu} (C + I_{3m})^{-\rho}}{\alpha(\frac{q}{p})^\mu + \beta(\frac{q}{p})^{-\mu}} \right],$$ (88)

where $\delta = \frac{\alpha^p r^q}{\alpha^{-\mu} + \beta^q}$, $p = r(1 - \delta)$, $q = r\delta$, and

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \in \mathbb{R}^{3m \times 3m}. \quad (89)$$

Here the sub-matrices $C_{ij}$, $i = 1, 2, j = 1, 2, 3$, each of size $m \times m$, are given by

$$C_{11} = \frac{1}{\gamma m} J_m K[x] J_m, \quad (90)$$

$$C_{12} = -\frac{1}{\sqrt{\gamma m}} J_m K[x, y] J_m \left( I_m + \frac{1}{\mu m} J_m K[y] J_m \right)^{-1}, \quad (91)$$

$$C_{13} = -\frac{1}{\sqrt{\gamma m}^2} J_m K[x] J_m K[x, y] J_m \left( I_m + \frac{1}{\mu m} J_m K[y] J_m \right)^{-1}, \quad (92)$$

$$C_{21} = \frac{1}{\sqrt{\gamma m}} J_m K[y, x] J_m, \quad (93)$$

$$C_{22} = -\frac{1}{\mu m} J_m K[y] J_m \left( I_m + \frac{1}{\mu m} J_m K[y] J_m \right)^{-1}, \quad (94)$$

$$C_{23} = -\frac{1}{\gamma m \mu} J_m K[y, x] J_m K[x, y] J_m \left( I_m + \frac{1}{\mu m} J_m K[y] J_m \right)^{-1}. \quad (95)$$
Theorem 26 (Alpha-Beta Log-Det divergences between RKHS covariance operators - Finite-dimensional version). Let $\alpha, \beta > 0$ be fixed. Let $r \in \mathbb{R}, r \neq 0$ be fixed. Assume that $\dim(H_K) < \infty$. For any $\gamma > 0$, $\mu > 0$, the divergence $D_r^{(\alpha, \beta)}[[C_{\Phi(x)} + \gamma I], (C_{\Phi(y)} + \mu I)]$ is given by

$$D_r^{(\alpha, \beta)}[[C_{\Phi(x)} + \gamma I], (C_{\Phi(y)} + \mu I)] = \frac{1}{\alpha \beta} \log \left( \frac{\alpha (\frac{\gamma}{\mu})^p + \beta (\frac{\gamma}{\mu})^{-q}}{\alpha + \beta} \right) \dim(H_K)$$

$$+ \frac{1}{\alpha \beta} \log \det \left[ \frac{\alpha (\frac{\gamma}{\mu})^p (C + I_{3m})^p + \beta (\frac{\gamma}{\mu})^{-q} (C + I_{3m})^{-q}}{\alpha (\frac{\gamma}{\mu})^p + \beta (\frac{\gamma}{\mu})^{-q}} \right],$$

where $p = r \frac{\beta}{\alpha + \beta}, q = r \frac{\alpha}{\alpha + \beta}$, and $C$ is as given in Theorem 25.

**Remark 10.** The closed form formulas for $D_r^{(\alpha, \beta)}[[C_{\Phi(x)} + \gamma I], (C_{\Phi(y)} + \mu I)]$ given in Eqs. (88) and (96) in Theorems 25 and 26, respectively, coincide if and only if $\gamma = \mu$. If $\gamma \neq \mu$, then the right hand side of Eq. (96) approaches infinity when $\dim(H_K) \to \infty$. Thus in general, the infinite-dimensional version is not obtainable as the limit of the finite-dimensional version as the dimension goes to infinity.

**Remark 11.** The closed form formulas given by Eqs. (88) and (96) in Theorems 25 and 26, respectively, are derived under more general conditions than those in [17] and are consequently more general but more complicated than the corresponding closed form formulas for the Alpha Log-Det divergences in [17] (see Theorems 12, 13, 15, 16 in [17]). Thus for practical applications involving the Alpha Log-Det divergences, the corresponding closed form formulas in [17] should be employed.

**Appendix A. Proofs of main results**

**Appendix A.1. Proofs for the general Alpha-Beta Log-Determinant divergences**

In this section, we prove Lemma 1, Proposition 1, and Theorems 5, 6, 7, and 8.

**Proof of Lemma 1.** Since any bounded operator $A$ commutes with the identity operator $I$, we have

$$\exp(A + \gamma I) = e^\gamma \exp(A) = e^\gamma \left( I + \sum_{j=1}^{\infty} \frac{A^j}{j!} \right) = e^\gamma I + e^\gamma \sum_{j=1}^{\infty} \frac{A^j}{j!},$$
where \( \sum_{j=1}^{\infty} \frac{A_j}{j!} \) is trace class, since

\[
\left\| \sum_{j=1}^{\infty} \frac{A_j}{j!} \right\|_{tr} \leq \sum_{j=1}^{\infty} \frac{||A||_j}{j!} = \exp(||A||_{tr}) - 1 < \infty.
\]

Thus \( \exp(A + \gamma I) \in \text{Tr}_X(\mathcal{H}) \). This completes the proof. \( \square \)

**Proof of Proposition 1.** For \((A + \gamma I), (B + \mu I) \in \text{PTr}(\mathcal{H})\), we have \((B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2} \in \text{PTr}(\mathcal{H})\) and the logarithm \(\log[(B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}] \in \text{Tr}_X(\mathcal{H})\) is well-defined. By the discussion preceding Proposition 1, we have

\[
\log[(A + \gamma I)(B + \mu I)^{-1}] = \log[(B + \mu I)^{1/2}(A + \gamma I)(B + \mu I)^{-1/2}](B + \mu I)^{-1/2}
\]

\[(B + \mu I)^{1/2} \log\left(\Lambda + \frac{\gamma}{\mu} I\right) (B + \mu I)^{-1/2} \in \text{Tr}_X(\mathcal{H}).\]

For the power function, we have

\[\begin{align*}
\alpha[(A + \gamma I)(B + \mu I)^{-1}]^\alpha &= \exp(\alpha \log[(A + \gamma I)(B + \mu I)^{-1}]) \\
&= \exp[(B + \mu I)^{1/2} \alpha \log\left(\Lambda + \frac{\gamma}{\mu} I\right) (B + \mu I)^{-1/2}]
\end{align*}\]

\[(B + \mu I)^{1/2} \left(\Lambda + \frac{\gamma}{\mu} I\right)^\alpha (B + \mu I)^{-1/2}.
\]

For the sum of two power functions, we then have

\[\begin{align*}
\frac{\alpha}{\alpha + \beta}[(A + \gamma I)(B + \mu I)^{-1}]^\alpha + \frac{\beta}{\alpha + \beta}[(A + \gamma I)(B + \mu I)^{-1}]^\beta
\end{align*}\]

\[= (B + \mu I)^{1/2} \left[\frac{\alpha(\Lambda + \frac{\gamma I}{\mu})^\alpha + \beta(\Lambda + \frac{\gamma I}{\mu})^\beta}{\alpha + \beta}\right] (B + \mu I)^{-1/2}.
\]

By Lemma 5 in [17], \(\det_X[C(A + \gamma I)C^{-1}] = \det_X(A + \gamma I)\) for any invertible
operator $C \in \mathcal{L}(\mathcal{H})$. It follows that

$$
\det_X \left[ \frac{\alpha[(A + \gamma I)(B + \mu I)^{-1}]^p + \beta[(A + \gamma I)(B + \mu I)^{-1}]^q}{\alpha + \beta} \right]
$$

This completes the proof.

**Proof of Theorem 5.** By definition of the power function, we have

$$\alpha (A + \gamma I)^p + (1 - \alpha)(B + \mu I)^q = \alpha \exp[p \log(A + \gamma I)] + (1 - \alpha) \exp[q \log(B + \mu I)]$$

$$= \alpha \exp \left[p \log \left( \frac{A}{\gamma} + I \right) + p(\log \gamma)I \right] + (1 - \alpha) \exp \left[q \log \left( \frac{B}{\mu} + I \right) + q(\log \mu)I \right]$$

$$= \alpha \gamma^p \left( \frac{A}{\gamma} + I \right)^p + (1 - \alpha) \mu^q \left( \frac{B}{\mu} + I \right)^q.$$ 

It follows that for $\delta = \frac{\alpha \gamma^p}{\alpha \gamma^p + (1 - \alpha)\mu^q}$, $1 - \delta = \frac{(1 - \alpha)\mu^q}{\alpha \gamma^p + (1 - \alpha)\mu^q}$, we have

$$\det_X [\alpha (A + \gamma I)^p + (1 - \alpha)(B + \mu I)^q]$$

$$= [\alpha \gamma^p + (1 - \alpha)\mu^q] \det \left[ \frac{\alpha \gamma^p}{\alpha \gamma^p + (1 - \alpha)\mu^q} \left( \frac{A}{\gamma} + I \right)^p + \frac{(1 - \alpha)\mu^q}{\alpha \gamma^p + (1 - \alpha)\mu^q} \left( \frac{B}{\mu} + I \right)^q \right]$$

$$\geq [\alpha \gamma^p + (1 - \alpha)\mu^q] \det \left( \frac{A}{\gamma} + I \right)^{p\delta} \det \left( \frac{B}{\mu} + I \right)^{q(1 - \delta)}$$

by Proposition 7 in [17]

$$\geq \gamma^p \mu^{(1 - \alpha)q} \det \left( \frac{A}{\gamma} + I \right)^{p\delta} \det \left( \frac{B}{\mu} + I \right)^{q(1 - \delta)}$$

by Ky Fan’s inequality applied to $\alpha \gamma^p + (1 - \alpha)\mu^q$

$$= \gamma^{p(\alpha - \delta)} \mu^{-q(\alpha - \delta)} \det_X (A + \gamma I)^{p\delta} \det_X (B + \mu I)^{q(1 - \delta)}$$

$$= \left( \frac{\gamma^p}{\mu^q} \right)^{\alpha - \delta} \det_X (A + \gamma I)^{p\delta} \det_X (B + \mu I)^{q(1 - \delta)}.$$ 

For $0 < \alpha < 1$, equality happens if and only if simultaneously, we have

$$\left( \frac{A}{\gamma} + I \right)^{p\delta} = \left( \frac{B}{\mu} + I \right)^{q(1 - \delta)}$$

and $\gamma^p = \mu^q \iff (A + \gamma I)^p = (B + \mu I)^q$.

In particular, for $\gamma = \mu$, the condition $\gamma^p = \mu^q$ becomes

$$\gamma^p = \gamma^q \iff \gamma^{p-q} = 1 \iff p = q \quad \text{if} \quad \gamma \neq 1.$$
With the conditions $\gamma = \mu \neq 1$ and $p = q$, we then have
\[
\left( \frac{A}{\gamma} + I \right)^p = \left( \frac{B}{\gamma} + I \right)^p \iff A = B.
\]
This completes the proof of the theorem.

**Proof of Theorem 6.** Recall that we write the operator $\left( B + \mu I \right)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}$ in the form
\[
(B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2} = \Lambda + (\gamma/\mu) I \in \text{PTr}(H).
\]
Its inverse has the form
\[
(B + \mu I)^{1/2}(A + \gamma I)^{-1}(B + \mu I)^{1/2} = [\Lambda + (\gamma/\mu) I]^{-1} \in \text{PTr}(H).
\]
It follows from Corollary 1 that
\[
\det X \left[ \frac{\alpha [(\Lambda + (\gamma/\mu) I)^p + \beta [(\Lambda + (\gamma/\mu) I)^{-1}] q]}{\alpha + \beta} \right] \geq 1
\iff D(\alpha,\beta)(p,q) \geq 0.
\]

Assuming that this condition holds, then along with the definition of $D_{(p,q)}^{(a,\beta)}$, (A.1) gives
\[
\left[ \frac{\gamma}{\mu} \right]^{(p+q)(\delta - \frac{\alpha}{\alpha + \beta})} \det X \left( \frac{\alpha (\Lambda + \frac{\gamma}{\mu} I)^p + \beta (\Lambda + \frac{\gamma}{\mu} I)^{-q}}{\alpha + \beta} \right) \geq 1
\iff D_{(p,q)}^{(a,\beta)} [(A + \gamma I), (B + \mu I)] \geq 0.
\]
In the inequality in (A.1), the equality sign happens if and only if
\[(\Lambda + (\gamma/\mu)I)p = (\Lambda + (\gamma/\mu)I)^{-q} \iff [(\Lambda + (\gamma/\mu)I)^{p+q} = I].\]

If \(p + q = 0\), then this is always true, so that \(D_{(p,q)}^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)] = 0\) for all pairs \((A + \gamma I), (B + \mu I) \in \mathcal{PTr}(\mathcal{H})\), which is not what we want. In fact, with \(p + q = 0\), the condition \(\alpha p (\gamma/\mu)^r = \beta q\) gives \((\alpha + \beta)p = 0 \Rightarrow p = 0 \Rightarrow q = 0\).

If \(p + q \neq 0\), since \(\Lambda + (\gamma/\mu)I > 0\), this happens if and only if \(\Lambda + (\gamma/\mu)I = I \iff (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2} = I \iff A + \gamma I = B + \mu I \iff A = B\) and \(\gamma = \mu\).

This completes the proof. \(\square\)

**Proof of Theorem 7.** Under the condition \(p + q = r\), by Theorem 6, we have

\[
\alpha p \left(\frac{\gamma}{\mu}\right)^r = \beta (r - p) \Rightarrow p = \frac{\beta r}{\alpha \left(\frac{\gamma}{\mu}\right)^r + \beta}
\]

It follows then that \(q = r - p = \frac{r\alpha (\frac{\gamma}{\mu})^r}{\alpha (\frac{\gamma}{\mu})^r + \beta}\). The equivalence of Eqs. (8) and (9) follows from Proposition 1. \(\square\)

**Proof of Theorem 8.** We have

\[
\frac{\alpha (\Lambda + \frac{\gamma}{\mu}I)^p + \beta (\Lambda + \frac{\gamma}{\mu}I)^{-q}}{\alpha + \beta} = \frac{\alpha (\frac{\gamma}{\mu})^p (\frac{\gamma}{\mu} \Lambda + I)^p + \beta (\frac{\gamma}{\mu})^{-q} (\frac{\gamma}{\mu} \Lambda + I)^{-q}}{\alpha + \beta}
\]

\[
= \frac{\alpha (\frac{\gamma}{\mu})^p (I + C_1) + \beta (\frac{\gamma}{\mu})^{-q} (I + C_2)}{\alpha + \beta}
\]

\[
= \left[\frac{\alpha (\frac{\gamma}{\mu})^p + \beta (\frac{\gamma}{\mu})^{-q}}{\alpha + \beta}\right] I + \left[\frac{\alpha (\frac{\gamma}{\mu})^p C_1 + \beta (\frac{\gamma}{\mu})^{-q} C_2}{\alpha + \beta}\right]
\]

\[
= \frac{\alpha (\frac{\gamma}{\mu})^p + \beta (\frac{\gamma}{\mu})^{-q}}{\alpha + \beta} \left[I + \frac{\alpha (\frac{\gamma}{\mu})^p C_1 + \beta (\frac{\gamma}{\mu})^{-q} C_2}{\alpha (\frac{\gamma}{\mu})^p + \beta (\frac{\gamma}{\mu})^{-q}}\right],
\]

where \(C_1 = \sum_{k=1}^{\infty} \frac{\mu^k}{k!} \left[\log \left(\frac{\mu}{\gamma} \Lambda + I\right)\right]^k \in \mathcal{Tr}(\mathcal{H}), C_2 = \sum_{k=1}^{\infty} \frac{(-1)^k q^k}{k!} \left[\log \left(\frac{\mu}{\gamma} \Lambda + I\right)\right]^k \in \mathcal{Tr}(\mathcal{H})\).
Tr(H). By definition of the detX function, we then have
\[
\log \det_X \left[ \frac{\alpha(\Lambda + \frac{\gamma}{n}I)^p + \beta(\Lambda + \frac{\gamma}{n}I)^{-q}}{\alpha + \beta} \right]
= \log \left( \frac{\alpha(\frac{\gamma}{n})^p + \beta(\frac{\gamma}{n})^{-q}}{\alpha + \beta} \right) + \log \det \left[ I + \frac{\alpha(\frac{\gamma}{n})^p C_1 + \beta(\frac{\gamma}{n})^{-q} C_2}{\alpha(\frac{\gamma}{n})^p + \beta(\frac{\gamma}{n})^{-q}} \right]
= \log \left( \frac{\alpha(\frac{\gamma}{n})^p + \beta(\frac{\gamma}{n})^{-q}}{\alpha + \beta} \right) + \log \det \left[ \frac{\alpha(\Lambda + \frac{\gamma}{n}I)^p + \beta(\Lambda + \frac{\gamma}{n}I)^{-q}}{\alpha(\frac{\gamma}{n})^p + \beta(\frac{\gamma}{n})^{-q}} \right].
\]
This, together with the definition of \(D_{(\alpha,\beta),(p,q)}\), gives us the desired expression. □

Appendix A.2. Proofs for the Affine-invariant Riemannian distance

In this section, we prove Theorem 9. We first need the following preliminary results.

Lemma 2. Let \(\gamma > 0\). Assume that \(r = r(\alpha)\) is smooth, with \(r(0) = 0\). Let \(\delta = \frac{\gamma - 1}{\gamma + 1}\).
Then
\[
\lim_{\alpha \to 0} \frac{r(\delta - \frac{1}{2})}{\alpha^2} = \left[\frac{r'(0)}{4}\right]^2 \log \gamma. \quad (A.2)
\]
In particular, for \(r = 2\alpha\), we have
\[
\lim_{\alpha \to 0} \frac{r(\delta - \frac{1}{2})}{\alpha^2} = \log \gamma. \quad (A.3)
\]

Proof of Lemma 2. By L’Hôpital’s rule applied twice, we obtain
\[
\lim_{\alpha \to 0} \frac{r(\delta - \frac{1}{2})}{\alpha^2} = \lim_{\alpha \to 0} \frac{r'(\gamma - 1)}{2\alpha^2(\gamma + 1)} = \lim_{\alpha \to 0} \frac{r'(\gamma - 1)}{4\alpha^2}
= \lim_{\alpha \to 0} \frac{r''(\alpha)(\gamma - 1) + r'(\gamma)(\gamma + 1) \log \gamma}{8\alpha}
= \lim_{\alpha \to 0} \frac{r''(\alpha)(\gamma - 1) + \gamma r''(\alpha)(\gamma + 1) \log \gamma}{8}
+ \lim_{\alpha \to 0} \frac{r''(r'(\alpha) \log \gamma)^2 + r''(\alpha) \log \gamma}{8}
= \left[\frac{r'(0)}{4}\right]^2 \log \gamma.
\]
This completes the proof. □
Lemma 3. Let $\gamma > 0$ be fixed. Let $\lambda > 0$ be fixed. Assume that $r = r(\alpha)$ is smooth, with $r(0) = 0$. Define $\delta = \frac{\gamma}{\gamma + 1}$, $p = r(1 - \delta)$, $q = r\delta$. Then

$$\lim_{\alpha \to 0} \frac{1}{\alpha^2} \log \left( \frac{\lambda^p + \lambda^{-q}}{2} \right) = \frac{[r'(0)]^2}{4} \left[ -(\log \gamma)(\log \lambda) + \frac{1}{2}(\log \lambda)^2 \right].$$

(A.4)

In particular, if $\gamma = \lambda$, then

$$\lim_{\alpha \to 0} \frac{1}{\alpha^2} \log \left( \frac{\gamma^p + \gamma^{-q}}{2} \right) = -\frac{[r'(0)]^2}{8}(\log \gamma)^2.$$

(A.5)

Proof of Lemma 3. For $p, q$ sufficiently small,

$$\lambda^p = e^{p \log \lambda} = 1 + p \log \lambda + \frac{p^2}{2} (\log \lambda)^2 + o(p^3),$$

$$\lambda^{-q} = e^{-q \log \lambda} = 1 - q \log \lambda + \frac{q^2}{2} (\log \lambda)^2 + o(q^3).$$

Thus for $\alpha$ sufficiently small, so that $p = o(\alpha), q = o(\alpha)$, we have

$$\frac{\lambda^p + \lambda^{-q}}{2} = 1 + \frac{p - q}{2} \log \lambda + \frac{p^2 + q^2}{4} (\log \lambda)^2 + o(p^3, q^3)$$

$$= 1 + r \left( \frac{1}{2} - \delta \right) (\log \lambda) + \frac{r^2}{4} \left[ (1 - \delta)^2 + \delta^2 \right] (\log \lambda)^2 + o(\alpha^2).$$

By Lemma 2, we have

$$\lim_{\alpha \to 0} \frac{r \left( \frac{1}{2} - \delta \right)}{\alpha^2} = -\frac{[r'(0)]^2}{4} \log \gamma.$$

We have by L'Hopital’s rule

$$\lim_{\alpha \to 0} \frac{r^2}{\alpha^2} = \lim_{\alpha \to 0} \frac{2rr'(\alpha)}{2\alpha} = \lim_{\alpha \to 0} [r'(\alpha)]^2 + rr''(\alpha) = [r'(0)]^2.$$

Since $\lim_{\alpha \to 0} \delta = \frac{1}{2}$, it follows then that

$$\lim_{\alpha \to 0} \frac{r^2}{4\alpha^2} \left[ (1 - \delta)^2 + \delta^2 \right] = \frac{[r'(0)]^2}{8}. $$

Combining these limits with $\lim_{x \to 0} \frac{\log(1 + ax)}{x} = a$, we obtain

$$\lim_{\alpha \to 0} \frac{1}{\alpha^2} \log \left( \frac{\lambda^p + \lambda^{-q}}{2} \right) = \frac{[r'(0)]^2}{4} \left[ -(\log \gamma)(\log \lambda) + \frac{1}{2}(\log \lambda)^2 \right].$$

This completes the proof of the lemma. \qed
Lemma 4. Let $\gamma > 0$ be fixed. Let $\lambda \in \mathbb{R}$ be fixed such that $\lambda + \gamma > 0$. Assume that $r = r(\alpha)$ is smooth, with $r(0) = 0$. Define $\delta = \frac{\gamma}{\gamma + \lambda}$, $p = r(1 - \delta)$, $q = r\delta$. Then

$$
\lim_{\alpha \to 0} \frac{1}{\alpha^2} \log \left( \frac{(\lambda + \gamma)^p + (\lambda + \gamma)^{-q}}{\gamma^p + \gamma^{-q}} \right) = \frac{[r'(0)]^2}{8} \left[ \log(\gamma) + \log(\lambda + \gamma) \right]^2.
$$

(A.6)

In particular, for $r = r(\alpha) = 2\alpha$, we have

$$
\lim_{\alpha \to 0} \frac{1}{\alpha^2} \log \left( \frac{(\lambda + \gamma)^p + (\lambda + \gamma)^{-q}}{\gamma^p + \gamma^{-q}} \right) = \frac{1}{2} \left[ \log(\gamma) + \log(\lambda + \gamma) \right]^2.
$$

(A.7)

Proof of Lemma 4. We have by Lemma 3

$$
\lim_{\alpha \to 0} \frac{1}{\alpha^2} \log \left( \frac{(\lambda + \gamma)^p + (\lambda + \gamma)^{-q}}{\gamma^p + \gamma^{-q}} \right) = \lim_{\alpha \to 0} \frac{1}{\alpha^2} \log \left( \frac{\gamma^p + \gamma^{-q}}{2} \right) - \frac{[r'(0)]^2}{4} \left[ -\log(\gamma) \log(\lambda + \gamma) + \frac{1}{2} \left[ \log(\lambda + \gamma) \right]^2 - \frac{1}{2} \left[ \log(\gamma) \right]^2 \right]
$$

$$
= \frac{[r'(0)]^2}{8} \left[ \log(\gamma) + \log(\lambda + \gamma) \right]^2
$$

$$
= \frac{[r'(0)]^2}{8} \left[ \log \left( \frac{\lambda}{\gamma} + 1 \right) \right]^2.
$$

This completes the proof. \qed

Lemma 5. Let $\gamma > 0$ be fixed. Let $\lambda \in \mathbb{R}$ be fixed such that $\lambda + \gamma > 0$. Assume that $r = r(\alpha)$ is smooth, with $r(0) = 0$. Define $\delta = \frac{\gamma}{\gamma + \lambda}$, $p = r(1 - \delta)$, $q = r\delta$. Then

$$
\frac{(\lambda + \gamma)^p + (\lambda + \gamma)^{-q}}{\gamma^p + \gamma^{-q}} \geq 1,
$$

(A.8)

$$
\log \left( \frac{(\lambda + \gamma)^p + (\lambda + \gamma)^{-q}}{\gamma^p + \gamma^{-q}} \right) \geq 0.
$$

(A.9)
Proof of Lemma 5. By Theorem 5, we have
\[
\frac{(\lambda + \gamma)p + (\lambda + \gamma)^{-q}}{\gamma p + \gamma^{-q}} = \frac{\gamma^p}{\gamma p + \gamma^{-q}} \left( \frac{\lambda}{\gamma} + 1 \right)^p + \frac{\gamma^{-q}}{\gamma p + \gamma^{-q}} \left( \frac{\lambda}{\gamma} + 1 \right)^{-q}
\]
\[
= \alpha \left( \frac{\lambda}{\gamma} + 1 \right)^p + (1 - \alpha) \left( \frac{\lambda}{\gamma} + 1 \right)^{-q}
\]
where \( \alpha = \frac{\gamma^p}{\gamma^p + \gamma^{-q}} = \frac{\gamma^p + q}{\gamma^p + q + 1} = \delta \).

\[
\geq \left( \frac{\lambda}{\gamma} + 1 \right)^{\rho \delta} \left( \frac{\lambda}{\gamma} + 1 \right)^{-q(1 - \delta)} = \left( \frac{\lambda}{\gamma} + 1 \right)^{(p + q)\delta - q} = \left( \frac{\lambda}{\gamma} + 1 \right)^{r \delta - q} = 1,
\]
since \( q = r \delta \). This completes the proof. \( \square \)

Proof of Theorem 9. For \( \alpha = \beta \), we have
\[
\delta = \frac{(\frac{\lambda}{\gamma})^p}{(\frac{\lambda}{\gamma})^p + 1}, \quad p = r(1 - \delta), \quad q = r \delta.
\]

Let \( \{\lambda_j\}_{j \in \mathbb{N}} \) be the eigenvalues of \( \Lambda \). By Theorem 8, we have
\[
D_r^{(\alpha, \alpha)}[(A + \gamma I), (B + \mu I)] = \frac{r(\delta - \frac{1}{2})}{\alpha^2} \log \left( \frac{\gamma}{\mu} \right) + \frac{1}{\alpha^2} \log \left( \frac{(\frac{\lambda}{\gamma})^p + (\frac{\gamma}{\mu})^{-q}}{2} \right)
\]
\[
+ \frac{1}{\alpha^2} \log \det \left[ \frac{(\Lambda + (\frac{\lambda}{\gamma})^p) + (\Lambda + (\frac{\lambda}{\gamma})^{-q})}{(\frac{\lambda}{\gamma})^p + (\frac{\gamma}{\mu})^{-q}} \right]
\]
\[
= \frac{r(\delta - \frac{1}{2})}{\alpha^2} \log \left( \frac{\gamma}{\mu} \right) + \frac{1}{\alpha^2} \log \left( \frac{(\frac{\lambda}{\gamma})^p + (\frac{\gamma}{\mu})^{-q}}{2} \right)
\]
\[
+ \frac{1}{\alpha^2} \sum_{j=1}^{\infty} \log \left( \frac{(\lambda_j + (\frac{\lambda}{\gamma})^p) + (\lambda_j + (\frac{\lambda}{\gamma})^{-q})}{(\frac{\lambda}{\gamma})^p + (\frac{\gamma}{\mu})^{-q}} \right).
\]

By Lemma 2, we have
\[
\lim_{\alpha \to 0} \frac{r(\delta - \frac{1}{2})}{\alpha^2} \log \left( \frac{\gamma}{\mu} \right) = \frac{[r'(0)]^2}{4} \left[ \log \frac{\gamma}{\mu} \right]^2.
\]

By Lemma 3, we have
\[
\lim_{\alpha \to 0} \frac{1}{\alpha^2} \log \left( \frac{(\frac{\lambda}{\gamma})^p + (\frac{\gamma}{\mu})^{-q}}{2} \right) = -\frac{[r'(0)]^2}{8} \left[ \log \frac{\gamma}{\mu} \right]^2.
\]
By Lemma 4, we have
\[
\lim_{\alpha \to 0} \frac{1}{\alpha^2} \log \det \left[ \frac{(\Lambda + \gamma I)^p + (\Lambda + \gamma I)^{-q}}{(\frac{\gamma}{\mu})^p + (\frac{\gamma}{\mu})^{-q}} \right] = \lim_{\alpha \to 0} \frac{1}{\alpha^2} \sum_{j=1}^{\infty} \log \left[ \frac{(\Lambda_j + \frac{\gamma}{\mu})^p + (\Lambda_j + \frac{\gamma}{\mu})^{-q}}{(\frac{\gamma}{\mu})^p + (\frac{\gamma}{\mu})^{-q}} \right]
\]
by Lebesgue’s Monotone Convergence Theorem, since
\[
\log \left[ \frac{(\Lambda_j + \frac{\gamma}{\mu})^p + (\Lambda_j + \frac{\gamma}{\mu})^{-q}}{(\frac{\gamma}{\mu})^p + (\frac{\gamma}{\mu})^{-q}} \right] \geq 0 \quad \forall j \in \mathbb{N}
\] by Lemma 5
\[
= \frac{[r'(0)]^2}{8} \sum_{j=1}^{\infty} \left[ \log \left( \Lambda_j + \frac{\gamma}{\mu} \right) - \log \left( \frac{\gamma}{\mu} \right) \right]^2 = \frac{[r'(0)]^2}{8} \sum_{j=1}^{\infty} \left[ \log \left( \Lambda_j \frac{\mu}{\gamma} + 1 \right) \right]^2.
\]
Summing up these three expressions, we obtain
\[
\lim_{\alpha \to 0} D^{(\alpha, \alpha)}_{L}(A + \gamma I), (B + \mu I) = \frac{[r'(0)]^2}{8} \left[ \log \left( \frac{\gamma}{\mu} \right)^2 + \sum_{j=1}^{\infty} \log \left( \Lambda_j \frac{\mu}{\gamma} + 1 \right)^2 \right] = \frac{[r'(0)]^2}{8} \left[ \log \left( \Lambda + \frac{\gamma}{\mu} I \right) \right]^{2} = \frac{[r'(0)]^2}{8} ||\log((B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2})||^{2}_{\text{HS}} = \frac{[r'(0)]^2}{8} d^{2}_{\text{aHS}}[(A + \gamma I), (B + \mu I)].
\]
This completes the proof. \[\square\]

**Appendix A.3. Proofs for the Alpha Log-Determinant divergences**

In this section, we prove Theorem 10.

**Proof of Theorem 10.** The proof for the cases \(\alpha = 0\) and \(\alpha = 1\) is a special case of the results discussed at the end of Section 5.3.

Consider now the case \(0 < \alpha < 1\). We first note that
\[
d_{\log \det}^{1-2\alpha}(A + \gamma I), (B + \mu I) = \frac{1}{\alpha(1-\alpha)} \log \left[ \frac{\det_X(a(A + \gamma I) + (1-\alpha)(B + \mu I))}{\det_X(A + \gamma I)\det_X(B + \mu I)^{1-q}} \right] + \frac{q-\alpha}{\alpha(1-\alpha)} \log \frac{\gamma}{\mu},
\]
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where \( q = \frac{\alpha \gamma}{\alpha \gamma + (1-\alpha)\mu} \).

By Definition 6, we have

\[
D_r^{(\alpha, 1-\alpha)}[\{A + \gamma I\}, \{B + \mu I\}]
\]

\[
= \frac{1}{\alpha(1-\alpha)} \log \left[ \left( \frac{\gamma}{\mu} \right)^{r(\delta-\alpha)} \det_X \left( \alpha \left( \Lambda + \frac{\gamma}{\mu} I \right)^{r(\delta-\alpha)} + (1-\alpha) \left( \Lambda + \frac{\gamma}{\mu} I \right)^{-r\delta} \right) \right]
\]

\[
= \frac{r(\delta-\alpha)}{\alpha(1-\alpha)} \log \left( \frac{\gamma}{\mu} \right)
\]

\[
+ \frac{1}{\alpha(1-\alpha)} \log \det_X \left( \alpha \left( \Lambda + \frac{\gamma}{\mu} I \right)^{r(\delta-\alpha)} + (1-\alpha) \left( \Lambda + \frac{\gamma}{\mu} I \right)^{-r\delta} \right).
\]

By Proposition 1, we have

\[
\det_X \left( \alpha \left( \Lambda + \frac{\gamma}{\mu} I \right)^{r(\delta-\alpha)} + (1-\alpha) \left( \Lambda + \frac{\gamma}{\mu} I \right)^{-r\delta} \right)
\]

\[
= \det_X \left[ \alpha [(A + \gamma I)(B + \mu I)^{-1}]^{r(\delta-\alpha)} + (1-\alpha) [(A + \gamma I)(B + \mu I)^{-1}]^{-r\delta} \right]
\]

\[
= \det_X [(A + \gamma I)(B + \mu I)^{-1}]^{-r\delta} \det_X \left[ \alpha [(A + \gamma I)(B + \mu I)^{-1}]^{r(\delta-\alpha)} + (1-\alpha) I \right].
\]

In particular, for \( r = 1 \), we have

\[
\det_X \left[ \alpha [(A + \gamma I)(B + \mu I)^{-1}] + (1-\alpha) \right] = \frac{\det_X \left[ \alpha (A + \gamma I) + (1-\alpha)(B + \mu I) \right]}{\det_X (B + \mu I)}.
\]

Thus it follows that

\[
\det_X \left( \alpha \left( \Lambda + \frac{\gamma}{\mu} I \right)^{(1-\delta)} + (1-\alpha) \left( \Lambda + \frac{\gamma}{\mu} I \right)^{-\delta} \right)
\]

\[
= \frac{\det_X \left[ \alpha (A + \gamma I) + (1-\alpha)(B + \mu I) \right]}{\det_X (A + \gamma I)^{\delta} \det_X (B + \mu I)^{1-\delta}}.
\]

Also for \( r = 1 \), in Definition 6, we have \( \delta = \delta(r = 1) = \frac{\alpha \gamma}{\alpha \gamma + (1-\alpha)\mu} \). Combining all of these expressions and comparing with the expressions for \( d_{1-2\alpha} \) \( \log \det \), we obtain the first desired statement.

For \( r = -1 \), we have

\[
D_{-1}^{(\alpha, 1-\alpha)}[\{A + \gamma I\}, \{B + \mu I\}]
\]

\[
= -\frac{(\delta-1-\alpha)}{\alpha(1-\alpha)} \log \left( \frac{\gamma}{\mu} \right)
\]

\[
+ \frac{1}{\alpha(1-\alpha)} \log \det_X \left( \alpha \left( \Lambda + \frac{\gamma}{\mu} I \right)^{-(1-\delta-1)} + (1-\alpha) \left( \Lambda + \frac{\gamma}{\mu} I \right)^{\delta-1} \right).
\]

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where \( \delta_{-1} = \delta(r = -1) = \frac{\alpha \gamma}{\alpha + \gamma + (1 - \alpha) \mu} = \frac{\alpha \mu}{\alpha \mu + (1 - \alpha) \gamma} \).

Similar to the case \( r = 1 \), we have

\[
\det X \left[ \alpha \left( (A + \gamma I)(B + \mu I)^{-1} - 1 + (1 - \alpha) \right) \right] = \frac{\det X \left[ (1 - \alpha)(A + \gamma I) + \alpha (B + \mu I) \right]}{\det X (A + \gamma I)}.
\]

Thus it follows that

\[
\det X \left( \alpha \left( A + \frac{\gamma I}{\mu} \right)^{-(1 - \delta_{-1})} + (1 - \alpha) \left( A + \frac{\gamma I}{\mu} \right)^{\delta_{-1}} \right) = \frac{\det X \left[ (1 - \alpha)(A + \gamma I) + \alpha (B + \mu I) \right]}{\det X (A + \gamma I)^{1 - \delta_{-1}} \det X (B + \mu I)^{\delta_{-1}}}
\]

On the other hand, we have

\[
d_{\log \det}^{2\alpha - 1} [(A + \gamma I), (B + \mu I)] = \frac{1}{\alpha (1 - \alpha)} \log \left[ \frac{\det X \left[ (1 - \alpha)(A + \gamma I) + \alpha (B + \mu I) \right]}{\det X (A + \gamma I)^p \det X (B + \mu I)^{1 - p}} \right] + \frac{p - (1 - \alpha)}{\alpha (1 - \alpha)} \log \frac{\gamma}{\mu},
\]

where \( p = \frac{(1 - \alpha) \gamma}{(1 - \alpha) \gamma + \alpha \mu} = 1 - \delta_{-1} \). Combining all of these expressions, we obtain the second desired statement, namely

\[
D_{-1}^{(\alpha, 1 - \alpha)} \left[ (A + \gamma I), (B + \mu I) \right] = d_{\log \det}^{2\alpha - 1} [(A + \gamma I), (B + \mu I)]
\]

This completes the proof. \( \square \)

**Appendix A.4. Proofs for the other limiting cases**

In this section, we prove Theorems 11 and 12. We need the following preliminary results.

**Lemma 6.** Let \( \mathcal{H} \) be a separable Hilbert space. Let \( A \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \) be such that \( A + I > 0 \). Then \( \forall \alpha \in \mathbb{R} \), the operator \( (A + I)^\alpha \) is well defined and \( (A + I)^\alpha - I \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \). Equivalently, let \( \{\lambda_k\}_{k \in \mathbb{N}} \) be the eigenvalues of \( A \), then

\[
\text{tr}[(A + I)^\alpha - I] = \sum_{k=1}^{\infty} [(\lambda_k + 1)^\alpha - 1] \quad (A.10)
\]

has a finite value.
Proof of Lemma 6. By Lemma 3 in [17], if $A \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$ and $A + I > 0$, then $\log(A + I) \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$. By definition of the power function, we have

$$(A + I)^{\alpha} = \exp[\alpha \log(A + I)] = I + \sum_{j=1}^{\infty} \frac{\alpha^j}{j!} [\log(A + I)]^j.$$  

Since $\text{Tr}(\mathcal{H})$ is a Banach algebra under the trace norm, we have

$$||(A + I)^{\alpha} - I||_{\text{tr}} = \left|\sum_{j=1}^{\infty} \frac{\alpha^j}{j!} [\log(A + I)]^j \right|_{\text{tr}} \leq \sum_{j=1}^{\infty} |\alpha|^j ||\log(A + I)||_{\text{tr}}^j \leq \exp(||\alpha|| ||\log(A + I)||_{\text{tr}}) - 1 < \infty.$$  

Thus $(A + I)^{\alpha} - I \in \text{Tr}(\mathcal{H})$. The equivalent statement is then obvious. This completes the proof.

Lemma 7. Let $\mathcal{H}$ be a separable Hilbert space. Assume that $(A + \gamma I) \in \text{PTr}(\mathcal{H})$. Then for any $\alpha \in \mathbb{R}$, we have $(A + \gamma I)^{\alpha} - \gamma^{\alpha} I \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$ and

$$\text{tr}[(A + \gamma I)^{\alpha} - \gamma^{\alpha} I] = \gamma^{\alpha} \text{tr}\left[\left(\frac{A}{\gamma} + I\right)^{\alpha} - I\right],$$  \hspace{1cm} (A.11)  

$$\text{tr}_X[(A + \gamma I)^{\alpha}] = \gamma^{\alpha}\left(1 + \text{tr}\left[\left(\frac{A}{\gamma} + I\right)^{\alpha} - I\right]\right).$$  \hspace{1cm} (A.12)

Proof of Lemma 7. By definition of the power function, we have

$$(A + \gamma I)^{\alpha} = \exp[\alpha \log(A + \gamma I)] = \exp\left[\left(\alpha \log(\gamma) I + \alpha \log\left(\frac{A}{\gamma} + I\right)\right)\right]$$  

$$= \gamma^{\alpha}\left(\frac{A}{\gamma} + I\right)^{\alpha} = \gamma^{\alpha}\left[\left(\frac{A}{\gamma} + I\right)^{\alpha} - I\right] + \gamma^{\alpha} I,$$

where $\left(\frac{A}{\gamma} + I\right)^{\alpha} - I \in \text{Tr}(\mathcal{H})$ by Lemma 6. Thus it follows that $(A + \gamma I)^{\alpha} - \gamma^{\alpha} I \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H})$ and

$$\text{tr}[(A + \gamma I)^{\alpha} - \gamma^{\alpha} I] = \gamma^{\alpha}\text{tr}\left[\left(\frac{A}{\gamma} + I\right)^{\alpha} - I\right],$$

which is the first identity. By definition of the extended trace

$$\text{tr}_X[(A + \gamma I)^{\alpha}] = \text{tr}_X\left[(A + \gamma I)^{\alpha} - \gamma^{\alpha} I\right] + \gamma^{\alpha}\text{tr}\left[\left(\frac{A}{\gamma} + I\right)^{\alpha} - I\right] + \gamma^{\alpha},$$

which is the second identity. This completes the proof. \hfill \Box
Lemma 8. Let $(A + \gamma I), (B + \mu I) \in PTr(H)$. Let $\Lambda + \frac{\gamma}{\mu} I = (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}$. Then for any $\alpha \in \mathbb{R}$,

$$
\text{tr}_X[(A + \gamma I)(B + \mu I)^{-1}]^\alpha = \text{tr}_X \left[ \left( \Lambda + \frac{\gamma}{\mu} \right)^\alpha \right] = \text{tr}_X[(B + \mu I)^{-1}(A + \gamma I)]^\alpha. \tag{A.13}
$$

$$
\text{det}_X[(A + \gamma I)(B + \mu I)^{-1}]^\alpha = \text{det}_X \left[ \left( \Lambda + \frac{\gamma}{\mu} \right)^\alpha \right] = \text{det}_X[(B + \mu I)^{-1}(A + \gamma I)]^\alpha. \tag{A.14}
$$

**Proof of Lemma 8.** By Proposition 1, we have

$$
[(A + \gamma I)(B + \mu I)^{-1}]^\alpha = (B + \mu I)^{1/2} \left( \Lambda + \frac{\gamma}{\mu} \right)^\alpha (B + \mu I)^{-1/2}.
$$

Similarly,

$$
[(B + \mu I)^{-1}(A + \gamma I)]^\alpha = (B + \mu I)^{-1/2} \left( \Lambda + \frac{\gamma}{\mu} \right)^\alpha (B + \mu I)^{1/2}.
$$

By the commutativity of the $\text{tr}_X$ operation (Lemma 4 in [17]), we then have

$$
\text{tr}_X[(A + \gamma I)(B + \mu I)^{-1}]^\alpha = \text{tr}_X \left[ \left( \Lambda + \frac{\gamma}{\mu} \right)^\alpha \right] = \text{tr}_X[(B + \mu I)^{-1}(A + \gamma I)]^\alpha.
$$

Similarly, by the product property of the $\text{det}_X$ operation (Proposition 4 in [17]),

$$
\text{det}_X[(A + \gamma I)(B + \mu I)^{-1}]^\alpha = \text{det}_X \left[ \left( \Lambda + \frac{\gamma}{\mu} \right)^\alpha \right] = \text{det}_X[(B + \mu I)^{-1}(A + \gamma I)]^\alpha.
$$

This completes the proof. \[ \Box \]

Lemma 9. Assume that $\lambda > 0, \gamma > 0, \alpha > 0$ are fixed. Assume that $r = r(\beta)$ is smooth. Then for $\delta = \frac{\alpha\gamma}{\alpha\gamma + \beta}, \ p = r(1 - \delta), \ q = r\delta$, we have

$$
\lim_{\beta \to 0} \frac{1}{\alpha\beta} \log \left( \frac{\alpha\lambda^p + \beta\lambda^{-q}}{\alpha + \beta} \right) = \frac{1}{\alpha^2} \left( (\log \lambda) \frac{r(0)}{\gamma r(0)} + \lambda^{-r(0)} - 1 \right). \tag{A.15}
$$

In particular, for $\lambda = \gamma$, we have

$$
\lim_{\beta \to 0} \frac{1}{\alpha\beta} \log \left( \frac{\alpha\gamma^p + \beta\gamma^{-q}}{\alpha + \beta} \right) = \frac{1}{\alpha^2} \left( (\log \gamma) r(0) + 1 |\gamma^{-r(0)} - 1 \right). \tag{A.16}
$$
Proof of Lemma 9. We have for \( \alpha > 0 \), \( \lim_{\beta \to 0} \delta = 1 \), \( \lim_{\beta \to 0} p = 0 \), \( \lim_{\beta \to 0} q = r(0) \), so that \( \lim_{\beta \to 0} (\alpha \lambda p + \beta \lambda^{-q}) = \alpha \). With \( p = r(1 - \delta) = \frac{r \beta}{\alpha \gamma + \beta} \), we have \[
\frac{\partial p}{\partial \beta} = \frac{\left( \frac{r \beta}{\alpha \gamma + \beta} + r(\alpha \gamma r + \beta) - r \beta (\alpha \gamma r \log \gamma \frac{\alpha \gamma}{r \gamma} + 1) \right)}{(\alpha \gamma r + \beta)^2}.
\]
With \( q = r \delta = \frac{r \gamma r}{\alpha \gamma + \beta} \), we have \[
\frac{\partial q}{\partial \beta} = \frac{\left( \frac{r \gamma r}{\alpha \gamma + \beta} + r \gamma r \log \gamma \frac{\alpha \gamma}{r \gamma} + r \gamma r (\alpha \gamma r \log \gamma \frac{\alpha \gamma}{r \gamma} + 1) \right)}{(\alpha \gamma r + \beta)^2}.
\]
The required limit is of the form \( \frac{0}{0} \) and L’Hopital’s rule can be applied to give \[
\lim_{\beta \to 0} \frac{1}{\alpha \beta} \log \left( \frac{\alpha \lambda p + \beta \lambda^{-q}}{\alpha + \beta} \right) = \frac{1}{\alpha \beta} \alpha \lim_{\beta \to 0} \left( \frac{\lambda p (\lambda \lambda \log \lambda \frac{\alpha \gamma}{r \gamma} + \lambda^{-q} - \beta \lambda^{-q} \log \lambda \frac{\alpha \gamma}{r \gamma} \alpha + \beta - (\alpha \lambda p + \beta \lambda^{-q})}{(\alpha + \beta)^2} \right) = \frac{1}{\alpha \beta} \log \left( \frac{\alpha \lambda p (\lambda \lambda \log \lambda \frac{\alpha \gamma}{r \gamma} + \lambda^{-q} - \beta \lambda^{-q} \log \lambda \frac{\alpha \gamma}{r \gamma} \alpha + \beta - (\alpha \lambda p + \beta \lambda^{-q})}{(\alpha + \beta)^2} \right).
\]
This completes the proof. \( \square \)

Lemma 10. Assume that \( \gamma > 0 \), \( \alpha > 0 \) are fixed. Assume that \( \lambda \in \mathbb{R} \) is also fixed, such that \( \lambda + \gamma > 0 \). Assume that \( r = r(\beta) \) is smooth. Then for \( \delta = \frac{\alpha \gamma r}{\alpha \gamma + \beta} \), \( p = r(1 - \delta) \), \( q = r \delta \), we have \[
\lim_{\beta \to 0} \frac{1}{\alpha \beta} \log \left( \frac{\alpha (\lambda + \gamma) p + \beta (\lambda + \gamma)^{-q}}{\alpha \gamma p + \beta \gamma^{-q}} \right) = \frac{1}{\alpha \beta} \log \left( \frac{\alpha (\lambda + \gamma) p + \beta (\lambda + \gamma)^{-q}}{\alpha \gamma p + \beta \gamma^{-q}} \right).
\]
Proof of Lemma 10. By Lemma 9, we have \[
\lim_{\beta \to 0} \frac{1}{\alpha \beta} \log \left( \frac{\alpha (\lambda + \gamma) p + \beta (\lambda + \gamma)^{-q}}{\alpha \gamma p + \beta \gamma^{-q}} \right) = \lim_{\beta \to 0} \frac{1}{\alpha \beta} \log \left( \frac{\alpha (\lambda + \gamma) p + \beta (\lambda + \gamma)^{-q}}{\alpha \gamma p + \beta \gamma^{-q}} \right) - \frac{1}{\alpha \beta} \log \left( \frac{\alpha \gamma p + \beta \gamma^{-q}}{\alpha + \beta} \right) = \frac{1}{\alpha \beta} \left( \log (\lambda + \gamma) \frac{r(0)}{r(0)} + (\lambda + \gamma)^{-r(0) - 1} - \frac{1}{\alpha \beta} \left( \log (\lambda) \frac{r(0)}{r(0)} + \gamma^{-r(0) - 1} \right) \right) = \frac{1}{\alpha \beta} \left[ \log \left( \frac{\lambda}{\gamma} + 1 \right) \right] \frac{r(0)}{r(0)} + (\lambda + \gamma)^{-r(0) - 1} - \frac{1}{\alpha \beta} \left( \log (\lambda) \frac{r(0)}{r(0)} + \gamma^{-r(0) - 1} \right)\right].
\]

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This completes the proof.

**Lemma 11.** Assume that $\gamma > 0$, $\alpha > 0$ are fixed. Assume that $\lambda \in \mathbb{R}$ is also fixed, such that $\lambda + \gamma > 0$. Assume that $r = r(\beta)$ is smooth. Then for $\delta = \frac{\alpha \gamma^r}{\alpha \gamma^r + \beta}$, $p = r(1 - \delta)$, $q = r\delta$, we have

\[
\alpha(\lambda + \gamma)p + \beta(\lambda + \gamma)^{-q} \geq 1, \tag{A.18}
\]

\[
\log \left( \frac{\alpha(\lambda + \gamma)p + \beta(\lambda + \gamma)^{-q}}{\alpha \gamma^p + \beta \gamma^{-q}} \right) \geq 0. \tag{A.19}
\]

**Proof of Lemma 11.** We proceed as in the proof of Lemma 5, by applying Theorem 5 as follows

\[
\alpha(\lambda + \gamma)p + \beta(\lambda + \gamma)^{-q} = \alpha \gamma^p \alpha \gamma^p + \beta \gamma^{-q} \geq \frac{\alpha \gamma^p + \beta \gamma^{-q}}{\alpha \gamma^p + \beta \gamma^{-q}} \tag{A.20}
\]

Since the required limit has the form $\frac{0}{0}$, we apply L’Hopital’s rule to get

\[
\lim_{\beta \to 0} \frac{r(\delta - \frac{\alpha \gamma^r}{\alpha \gamma^r + \beta})}{\alpha \beta} = \frac{1}{\alpha^2} r(0)[-\gamma^{-r(0)} + 1].
\]

This completes the proof.

**Lemma 12.** Assume that $\gamma > 0$, $\alpha > 0$ are fixed. Assume that $r = r(\beta)$ is smooth. Then for $\delta = \frac{\alpha \gamma^r}{\alpha \gamma^r + \beta}$,

\[
\lim_{\beta \to 0} \frac{r(\delta - \frac{\alpha \gamma^r}{\alpha \gamma^r + \beta})}{\alpha \beta} = \frac{1}{\alpha^2} r(0)[-\gamma^{-r(0)} + 1].
\]

This completes the proof.
Proof of Theorem 11. Let \( \{\lambda_j\}_{j=1}^{\infty} \) be the eigenvalues of \( \Lambda \). By Theorem 8, we have

\[
D_{r}^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)] = \sum_{j=1}^{\infty} \frac{r(\delta - \frac{\alpha}{\alpha + \beta})}{\alpha \beta} \log \left( \frac{\lambda_j + \gamma}{\mu} \right) + \frac{1}{\alpha \beta} \log \left( \frac{\alpha(\frac{\lambda_j}{\mu})^p + \beta(\frac{\lambda_j}{\mu})^{-q}}{\alpha + \beta} \right)
\]

By Lemma 12

\[
\log(\gamma) \lim_{\beta \to 0} \frac{r(\delta - \frac{\alpha}{\alpha + \beta})}{\alpha \beta} = \frac{1}{\alpha^2} \left[ \log \left( \frac{\lambda_j + \gamma}{\gamma} \right) \right] \log(\gamma).
\]

where \( p = p(\beta) = r(1 - \delta) = \frac{r^2}{\alpha(\frac{\lambda_j}{\mu})^p + \beta(\frac{\lambda_j}{\mu})^{-q}} \), \( q = q(\beta) = r\delta = \frac{r\alpha^2(\frac{\lambda_j}{\mu})^p}{\alpha(\frac{\lambda_j}{\mu})^p + \beta(\frac{\lambda_j}{\mu})^{-q}} \).

For \( \alpha > 0 \) fixed, as functions of \( \beta \), we have

\[
\lim_{\beta \to 0} p(\beta) = 0, \quad \lim_{\beta \to 0} q(\beta) = r(0).
\]

For simplicity, in the following, we replace \( \frac{\gamma}{\mu} \) by \( \gamma \). By Lemma 9,

\[
\lim_{\beta \to 0} \frac{1}{\alpha \beta} \log \left( \frac{\alpha \gamma^p + \beta \gamma^{-q}}{\alpha + \beta} \right) = \frac{1}{\alpha^2} \left( [\log(\gamma)r(0) + 1]\gamma^{-r(0)} - 1 \right).
\]

By Lemma 10,

\[
\lim_{\beta \to 0} \frac{1}{\alpha \beta} \log \left( \frac{\alpha(\lambda_j + \gamma)^p + \beta(\lambda_j + \gamma)^{-q}}{\alpha \gamma^p + \beta \gamma^{-q}} \right) = \frac{1}{\alpha^2} \left[ \log \left( \frac{\lambda_j + 1}{\gamma} \right) \right] \frac{r(0)}{\gamma^{r(0)}} + (\lambda_j + \gamma)^{-r(0)} - \gamma^{-r(0)} \right]
\]

By Lemma 11, we have \( \log \left( \frac{\alpha(\lambda_j + \gamma)^p + \beta(\lambda_j + \gamma)^{-q}}{\alpha \gamma^p + \beta \gamma^{-q}} \right) \geq 0 \forall j \in \mathbb{N} \), so that by Lebesgue’s Monotone Convergence Theorem, we obtain

\[
\lim_{\beta \to 0} \frac{1}{\alpha \beta} \sum_{j=1}^{\infty} \log \left( \frac{\alpha(\lambda_j + \gamma)^p + \beta(\lambda_j + \gamma)^{-q}}{\alpha \gamma^p + \beta \gamma^{-q}} \right)
= \sum_{j=1}^{\infty} \frac{1}{\alpha \beta} \log \left( \frac{\alpha(\lambda_j + \gamma)^p + \beta(\lambda_j + \gamma)^{-q}}{\alpha \gamma^p + \beta \gamma^{-q}} \right)
\]

\[
= \frac{1}{\alpha^2} \sum_{j=1}^{\infty} \left[ \log \left( \frac{\lambda_j + 1}{\gamma} \right) \right] \frac{r(0)}{\gamma^{r(0)}} + (\lambda_j + \gamma)^{-r(0)} - \gamma^{-r(0)} \right]
\]

By Lemma 12

\[
\log(\gamma) \lim_{\beta \to 0} \frac{r(\delta - \frac{\alpha}{\alpha + \beta})}{\alpha \beta} = \frac{1}{\alpha^2} r(0)[\gamma^{-r(0)} + 1] \log(\gamma).
\]
Combining all three expressions, we obtain the desired limit as the sum
\[
\frac{1}{\alpha^2} \left[ \gamma^{-r(0)} + r(0) \log(\gamma) - 1 \right] + \frac{1}{\alpha^2} \left\{ r(0) \sum_{j=1}^{\infty} \log \left( \frac{\lambda_j}{\gamma} + 1 \right) + \sum_{j=1}^{\infty} \left[ \frac{1}{(\lambda_j + \gamma)^{r(0)}} - \frac{1}{\gamma^{r(0)}} \right] \right\}. \tag{A.21}
\]
By Lemmas 6 and 7, we have
\[
\sum_{j=1}^{\infty} \left[ \frac{1}{(\lambda_j + \gamma)^{r(0)}} - \frac{1}{\gamma^{r(0)}} \right] = \gamma^{-r(0)} \sum_{j=1}^{\infty} \left[ \left( \frac{\lambda_j}{\gamma} + 1 \right)^{-r(0)} - 1 \right] = \gamma^{-r(0)} \text{tr} \left[ \left( \frac{A}{\gamma} + I \right)^{-r(0)} - I \right] = \text{tr}[(A + \gamma I)^{-r(0)} - \gamma^{-r(0)} I].
\]
Thus it follows that
\[
\gamma^{-r(0)} - 1 + \sum_{j=1}^{\infty} \left[ \frac{1}{(\lambda_j + \gamma)^{r(0)}} - \frac{1}{\gamma^{r(0)}} \right] = \gamma^{-r(0)} - 1 + \text{tr}[(A + \gamma I)^{-r(0)} - \gamma^{-r(0)} I] = \text{tr}[(A + \gamma I)^{-r(0)} - I].
\]
Furthermore,
\[
\frac{r(0)}{\gamma^{r(0)}} \sum_{j=1}^{\infty} \log \left( \frac{\lambda_j}{\gamma} + 1 \right) = r(0) \gamma^{-r(0)} \log \det \left( \frac{A}{\gamma} + I \right) = r(0) \gamma^{-r(0)} \log \det_X(A + \gamma I) - r(0) \gamma^{-r(0)} \log \gamma = -\gamma^{-r(0)} \log \det_X(A + \gamma I)^{-r(0)} - r(0) \gamma^{-r(0)} \log \gamma.
\]
Plugging the last two expressions into (A.21), we obtain the desired limit as
\[
\frac{1}{\alpha^2} \left\{ r(0)(1 - \gamma^{-r(0)}) \log \gamma \right\} + \frac{1}{\alpha^2} \left\{ \text{tr}_X[(A + \gamma I)^{-r(0)} - I] - \gamma^{-r(0)} \log \det_X(A + \gamma I)^{-r(0)} \right\}. \tag{A.22}
\]
We now replace \( \gamma \) by \( \gamma \mu \). We have by Lemma 8,
\[
\text{tr}_X \left[ \left( A + \frac{\gamma}{\mu} I \right)^{-r(0)} \right] = \text{tr}_X[(B + \mu I)^{-1}(A + \gamma I)^{-r(0)} = \text{tr}_X[(A + \gamma I)^{-1}(B + \mu I)^{r(0)}],
\]
\[
\det_X \left( A + \frac{\gamma}{\mu} I \right)^{-r(0)} = \det_X[(B + \mu I)^{-1}(A + \gamma I)^{-r(0)} = \det_X[(A + \gamma I)^{-1}(B + \mu I)^{r(0)}].
\]
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Then (A.22) becomes
\[
\frac{\mu}{\gamma} - \left( \frac{\mu}{\gamma} \right)^r(0) \log \frac{\mu}{\gamma} + \frac{1}{\alpha^2} \text{tr}_X \left( [(A + \gamma I)^{-1}(B + \mu I)]^r(0) - I \right)
- \frac{1}{\alpha^2} \left( \frac{\mu}{\gamma} \right)^r(0) \log \text{det}_X \left( [(A + \gamma I)^{-1}(B + \mu I)]^r(0) \right).
\]

This completes the proof of the theorem. □

**Proof of Theorem 12.** The dual symmetry in Theorem 13 gives
\[
\lim_{\alpha \to 0} D_{\alpha}(\alpha, \beta)[(A + \gamma I), (B + \mu I)] = \lim_{\alpha \to 0} D_{\beta}(\beta, \alpha)[(B + \mu I), (A + \gamma I)].
\]
The limit on the right hand side then follows from Theorem 11. □

**Appendix A.5. Proofs of the properties of the Alpha-Beta Log-Determinant divergences**

In this section, we prove Theorems 13, 14, 15, 16, 17, and 18. For the case \(\alpha = \beta = 0\), we have
\[
D_{0,0}(\alpha, \beta)[(A + \gamma I), (B + \mu I)] = \frac{1}{2}d_{\text{alHS}}[A + \gamma I), (B + \mu I)],
\]
with \(d_{\text{alHS}}\) being the affine-invariant Riemannian distance on \(\text{PTr}(\mathcal{H})\). Thus these properties are either automatic or straightforward to verify. We thus focus on the three cases \((\alpha > 0, \beta > 0), (\alpha > 0, \beta = 0),\) and \((\alpha = 0, \beta > 0)\).

**Proof of Theorem 13 (Dual symmetry).** For the case \((\alpha > 0, \beta = 0)\) and \((\alpha = 0, \beta > 0)\), from Eqs. (10) and (11), we immediately have
\[
D_{\alpha,0}(\alpha, \beta)[(A + \gamma I), (B + \mu I)] = \frac{\mu}{\gamma} - \left( \frac{\mu}{\gamma} \right)^r(0) \log \frac{\mu}{\gamma} + \frac{1}{\alpha^2} \text{tr}_X \left( [(A + \gamma I)^{-1}(B + \mu I)]^r(0) - I \right)
- \frac{1}{\alpha^2} \left( \frac{\mu}{\gamma} \right)^r(0) \log \text{det}_X \left( [(A + \gamma I)^{-1}(B + \mu I)]^r(0) \right).
\]
Consider now the case \((\alpha > 0, \beta > 0)\). Write \(\delta = \delta(\alpha, \beta)\) to emphasize its dependence on \(\alpha\) and \(\beta\), we have \(\delta(\alpha, \beta) = \frac{\alpha \gamma^r}{\alpha \gamma^r + \beta \mu^r} \) in \(D_{\alpha,0}(\alpha, \beta)[(A + \gamma I), (B + \mu I)]\). Then for \(D_{\beta,0}(\beta, \alpha)[(B + \mu I), (A + \gamma I)]\), we have
\[
\delta(\beta, \alpha) = \frac{\beta \mu^r}{\alpha \gamma^r + \beta \mu^r} = 1 - \delta(\alpha, \beta), \quad 1 - \delta(\beta, \alpha) = \delta(\alpha, \beta),
\]
\[
\delta(\beta, \alpha) - \frac{\beta}{\alpha + \beta} = 1 - \delta(\alpha, \beta) - \frac{\beta}{\alpha + \beta} = - \left( \delta(\alpha, \beta) - \frac{\alpha}{\alpha + \beta} \right).
\]
By Definition 1, we have

\[ D_r^{(\beta,\alpha)}[(B + \mu I), (A + \gamma I)] \]

\[ = \frac{1}{\alpha \beta} \log \left( \frac{\mu}{\gamma} \right)^{r(\delta(\beta,\alpha) - \frac{\beta}{2})} 
+ \frac{1}{\alpha \beta} \log \det_X \left( \frac{\beta[(B + \mu I)(A + \gamma I)^{-1}]^{r(1 - \delta(\beta,\alpha))} + \alpha[(B + \mu I)(A + \gamma I)^{-1}]^{-r\delta(\beta,\alpha)}}{\alpha + \beta} \right) \]

\[ = \frac{1}{\alpha \beta} \log \left( \frac{\gamma}{\mu} \right)^{r(\delta(\alpha,\beta) - \frac{\alpha}{2})} 
+ \frac{1}{\alpha \beta} \log \det_X \left( \frac{\alpha[(A + \gamma I)(B + \mu I)^{-1}]^{r(1 - \delta(\alpha,\beta))} + \beta[(A + \gamma I)(B + \mu I)^{-1}]^{-r(1 - \delta(\alpha,\beta))}}{\alpha + \beta} \right) \]

\[ = D_r^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)]. \]

This completes the proof of the theorem. \(\square\)

**Proof of Theorem 14 (Dual invariance under inversion).** We have

\[(A + \gamma I)^{-1} = \frac{1}{\gamma} I - \frac{A}{\gamma}(A + \gamma I)^{-1}, \quad (B + \mu I)^{-1} = \frac{1}{\mu} I - \frac{B}{\mu}(B + \mu I)^{-1},\]

\[(B + \mu I)^{1/2}(A + \gamma I)^{-1}(B + \mu I)^{1/2} = [(B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2}]^{-1}.\]

Consider the case \(\alpha > 0, \beta > 0\). By Definition 1, we have

\[ D_r^{(\alpha,\beta)}[(A + \gamma I)^{-1}, (B + \mu I)^{-1}] \]

\[ = \frac{1}{\alpha \beta} \log \left( \frac{1/\gamma}{1/\mu} \right)^{r(\delta_2 - \frac{\alpha}{2})} 
+ \frac{1}{\alpha \beta} \log \det_X \left( \frac{\alpha(A + \gamma I)^{-r(1 - \delta_2)} + \beta(A + \gamma I)^{r\delta_2}}{\alpha + \beta} \right) \]

where \(\delta_2 = \frac{\alpha(1/\gamma)^r}{\alpha(1/\gamma)^r + \beta(1/\mu)^r} = \frac{\alpha \mu^r}{\alpha \mu^r + \beta \nu^r} = \delta(-r).\) Thus

\[ D_r^{(\alpha,\beta)}[(A + \gamma I)^{-1}, (B + \mu I)^{-1}] = D_r^{(\alpha,\beta)}[(A + \gamma I), (B + \mu I)]. \]

Consider the case \(\alpha = 0, \beta > 0\) (the case \(\alpha > 0, \beta = 0\) then follows by dual symme-
try). We have

$$D_r^{(0,\beta)}[(A + \gamma I)^{-1}, (B + \mu I)^{-1}] = \frac{r}{\beta^2} \left[ \left( \frac{1}{\gamma} \right)^r - 1 \right] \log \left( \frac{1}{\gamma} \right) + 1 \beta^2 tr_X[(B + \mu I)(A + \gamma I)^{-1}]$$

$$+ \frac{1}{\beta^2} tr_X([B + \mu I](A + \gamma I)^{-1}] - I) - \frac{1}{\beta^2} \left[ \left( \frac{1}{\gamma} \right)^r - 1 \right] \log \det_X[(B + \mu I)(A + \gamma I)^{-1}]^-r.$$

By Lemma 8, we have

$$tr_X[(A + \gamma I)(B + \mu I)^{-1}]^-r = tr_X\left[ \left( \frac{\Lambda + \gamma I}{\mu} \right)^{-r} \right] = tr_X[(B + \mu I)^{-1}(A + \gamma I)]^-r.$$

$$det_X[(A + \gamma I)(B + \mu I)^{-1}]^-r = det_X\left[ \left( \frac{\Lambda + \gamma I}{\mu} \right)^{-r} \right] = det_X[(B + \mu I)^{-1}(A + \gamma I)]^-r.$$

Thus it follows that

$$D_r^{(0,\beta)}[(A + \gamma I)^{-1}, (B + \mu I)^{-1}]$$

$$= -\frac{r}{\beta^2} \left[ \left( \frac{\gamma}{\mu} \right)^{-r} - 1 \right] \log \left( \frac{\gamma}{\mu} \right) + 1 \beta^2 tr_X([B + \mu I](A + \gamma I)^{-1}] - I)$$

$$- \frac{1}{\beta^2} \left( \frac{\gamma}{\mu} \right)^-r \log \det_X[(B + \mu I)^{-1}(A + \gamma I)]^-r.$$

This completes the proof. \(\blacksquare\)

**Proof of Theorem 15 (Affine-invariance).** We have for \((A + \gamma I) \in \text{PTr}(\mathcal{H})\) and \((C + \nu I) \in \text{Tr}_X(\mathcal{H}), \nu \neq 0,$$

$$(C + \nu I)(A + \gamma I)(C + \nu I)^* = CAC^* + \nu(CA + AC^*) + \nu^2 A + \gamma CC^* + \gamma \nu(C + C^*) + \gamma \nu^2 I \in \text{Tr}_X(\mathcal{H}).$$

Since \((C + \nu I)\) is assumed to be invertible, the operator \((C + \nu I)(A + \gamma I)(C + \nu I)^*\) is also invertible, with inverse \([C + \nu I]^*(A + \gamma I)^{-1}(C + \nu I)^{-1}\). Furthermore,
∀x ∈ H,
\langle x, (C + νI)(A + γI)(C + νI)^* x \rangle = \langle (C + νI)^* x, (A + γI)(C + νI)^* x \rangle 
\geq MA\| (C + νI)^* x \| \geq 0,
with equality if and only if \((C + νI)(A + γI)(C + νI)^* = D(α, β)\)
\[(A + γI), (B + µI)\].
This completes the proof.

For two operators \((A + γI), (B + µI) \in \text{PTr}(H)\), we then have
\[ ((C + νI)(A + γI)(C + νI)^*)^{-1} = (C + νI)[(A + γI)(B + µI)^{-1}][(C + νI)^{-1}]. \]
Then for any \(p \in \mathbb{R}\), we have
\[ ((C + νI)(A + γI)(C + νI)^*)^p = (C + νI)[(A + γI)(B + µI)^{-1}]^p(C + νI)^{-1}. \]
Thus for any \(a, b > 0\) and any \(p, q \in \mathbb{R}\).
\[ a(\langle (C + νI)(A + γI)(C + νI)^* \rangle \| (C + νI)(B + µI)(C + νI)^* \|)^p 
+ b(\langle (C + νI)(A + γI)(C + νI)^* \rangle \| (C + νI)(B + µI)(C + νI)^* \|)^q = (C + νI)[a[[A + γI](B + µI)^{-1}]^p + b[[A + γI](B + µI)^{-1}]^q](C + νI)^{-1}. \]
By the definition of \(D^{(α, β)}_v\) and the following invariances of the extended Fredholm determinant \(\det_X\) as well as of the extended trace operation \(\text{tr}_X\), namely,
\[ \det_X[C(A + γI)C^{-1}] = \det_X[(A + γI)], \]
\[ \text{tr}_X[C(A + γI)C^{-1}] = \text{tr}_X[(A + γI)], \]
for \(A + γI \in \text{Tr}_X(H), γ \neq 0\), and \(C \in \mathcal{L}(H)\) invertible (Lemma 5 in [17]), we then obtain the desired affine invariance for \(D^{(α, β)}_v\), namely
\[ D^{(α, β)}_v[(C + νI)(A + γI)(C + νI)^*, (C + νI)(B + µI)(C + νI)^*] = D^{(α, β)}_v[(A + γI), (B + µI)]. \]
This completes the proof.
\[ \square \]
Proof of Theorem 16 (Invariance under unitary transformations). The proof of this theorem is similar to that of the proof for Theorem 15, using the fact that $C^* = C^{-1}$ and the properties
\[
\det_X [C(A + \gamma I)C^{-1}] = \det_X [(A + \gamma I)],
\]
\[
\operatorname{tr}_X [C(A + \gamma I)C^{-1}] = \operatorname{tr}_X [(A + \gamma I)],
\]
of the operations $\det_X$ and $\operatorname{tr}_X$.

Proof of Theorem 17. For the case $\alpha > 0, \beta > 0$, this follows immediately from Definition 1. For the case $\alpha > 0, \beta = 0$, by Definition 2 and Lemma 8, we have
\[
D^{(\alpha,0)}[(A + \gamma I), (B + \mu I)] = \frac{r}{\alpha^2} \left[ \left( \frac{\mu}{\gamma} \right)^r - 1 \right] \log \left( \frac{\mu}{\gamma} \right)
+ \frac{1}{\alpha^2} \operatorname{tr}_X (\Lambda + \frac{\gamma}{\mu})^{-r} - I - \frac{1}{\alpha^2} \left( \frac{\mu}{\gamma} \right)^r \log \det_X (\Lambda + \frac{\gamma}{\mu})^{-r}
= D^{(\alpha,0)}[(\Lambda + \frac{\gamma}{\mu}), I].
\]
The case $\alpha = 0, \beta > 0$ is entirely similar.

Proof of Theorem 18. We first note that $(\Lambda + \frac{\gamma}{\mu} I)^\omega = (\frac{\gamma}{\mu} \Lambda + I)^\omega$. Then for $\alpha > 0, \beta > 0$, the statement of the theorem follows immediately from Definition 1. For the case $\alpha > 0, \beta = 0$, by Definition 2 and Lemma 8, we have
\[
D^{(\omega,\alpha,0)}[(A + \gamma I), (B + \mu I)] = \frac{r}{\omega^2 \alpha^2} \left[ \left( \frac{\mu}{\gamma} \right)^{\omega r} - 1 \right] \log \left( \frac{\mu}{\gamma} \right)^\omega
+ \frac{1}{\omega^2 \alpha^2} \operatorname{tr}_X (\Lambda + \frac{\gamma}{\mu})^{-\omega r} - I - \frac{1}{\omega^2 \alpha^2} \left( \frac{\mu}{\gamma} \right)^{\omega r} \log \det_X (\Lambda + \frac{\gamma}{\mu})^{-\omega r}
= \frac{1}{\omega^2} D^{(\alpha,0)}[(\Lambda + \frac{\gamma}{\mu})^\omega, I].
\]
The case $\alpha = 0, \beta > 0$ is entirely similar.

Appendix A.6. Proofs of Theorems 1, 2, and 3

We are now ready to provide the proofs for Theorems 1, 2, and 3.

For the proof of positivity, we first need the following technical result.
Lemma 13. (i) Let \( r \neq 0 \) be fixed. The function \( f(x) = x^r - 1 - r \log(x) \) for \( x > 0 \) has a unique global minimum \( f_{\text{min}} = f(1) = 0 \). In other words, \( f(x) \geq 0 \) \( \forall x > 0 \), with equality if and only if \( x = 1 \).

(ii) Let \( \nu > 0, r \neq 0 \) be fixed. For \( r \neq 0 \), the function \( g(x) = \left( \frac{x}{\nu} + 1 \right)^r - 1 - r \log\left( \frac{x}{\nu} + 1 \right) \) for \( x > -\nu \) has a unique global minimum \( g_{\text{min}} = g(0) = 0 \). In other words, \( g(x) \geq 0 \) \( \forall x > -\nu \), with equality if and only if \( x = 0 \).

Proof of Lemma 13. (i) We have \( f'(x) = \frac{r(x^r - 1)}{x} \). When \( r > 0 \), we have \( x^r < 1 \) for \( 0 < x < 1 \) and \( x^r > 1 \) for \( x > 1 \). When \( r < 0 \), we have \( x^r > 1 \) for \( 0 < x < 1 \) and \( x^r < 1 \) for \( x > 1 \). Thus, for all \( r \neq 0 \), we have \( f'(x) < 0 \) when \( 0 < x < 1 \) and \( f'(x) > 0 \) when \( x > 1 \). Hence \( f \) has a unique global minimum \( f_{\text{min}} = f(1) = 0 \).

(ii) The proof for \( g \) follows that for \( f \) by the change of variable \( y = \frac{x}{\nu} + 1 \).

Proof of Theorem 1 (Positivity). For the case \( \alpha > 0, \beta > 0 \), this is a special case of Theorem 6, with \( p + q = r \). Consider now the case \( \alpha = 0, \beta > 0 \) (the case \( \alpha > 0, \beta = 0 \) then follows by dual symmetry). For the proof of positivity, we can ignore the positive factor \( \beta^2 \) and thus it suffices to consider \( D_\nu^{(0,1)} \). We recall that we define \( \Lambda + \nu I = (B + \mu I)^{-1/2} (A + \gamma I) (B + \mu I)^{-1/2} \), where \( \nu = \frac{\alpha}{\beta} \). Then, since \( \det_X[(B + \mu I)^{-1/2} (A + \gamma I) (B + \mu I)^{-1/2}] = \det_X[(B + \mu I)^{-1} (A + \gamma I)] \) and \( \text{tr}_X[(B + \mu I)^{-1/2} (A + \gamma I) (B + \mu I)^{-1/2}] = \text{tr}_X[(B + \mu I)^{-1} (A + \gamma I)] \), we have

\[
D_\nu^{(0,1)}[(A + \gamma I), (B + \mu I)] = r(\nu^r - 1) \log \nu + \text{tr}_X[(A + \nu I)^r - I] - \nu^r \log \det_X(A + \nu I)^r
\]

By Lemma 7,

\[
\text{tr}_X[(A + \nu I)^r - I] = \nu^r - 1 + \nu^r \text{tr} \left[ \left( \frac{A}{\nu} + I \right)^r - I \right].
\]

Also

\[
\log \det_X(A + \nu I)^r = \log \left[ \nu^r \det \left( \frac{A}{\nu} + I \right)^r \right] = r \log \det \left( \frac{A}{\nu} + I \right) + r \log \nu.
\]

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Thus we have

\[
D_r^{(0,1)}[(A + \gamma I), (B + \mu I)]
\]

\[
= \nu^r - 1 - r \log \nu + \nu^r \left[ \text{tr} \left( \left( \frac{\Lambda}{\nu} + I \right)^r - I \right) - r \log \det \left( \frac{\Lambda}{\nu} + I \right) \right]
\]

\[
= \nu^r - 1 - r \log \nu + \nu^r \left[ \sum_{k=1}^{\infty} \left( \frac{\lambda_k}{\nu} + 1 \right)^r - 1 - r \log \left( \frac{\lambda_k}{\nu} + 1 \right) \right].
\]

By the first part of Lemma 13, we have for all \( \nu > 0 \)

\[
\nu^r - 1 - r \log \nu \geq 0,
\]

with equality if and only if \( \nu = 1 \). By the second part of the Lemma 13, we have for all \( k \in \mathbb{N} \)

\[
\left( \frac{\lambda_k}{\nu} + 1 \right)^r - 1 - r \log \left( \frac{\lambda_k}{\nu} + 1 \right) \geq 0,
\]

with equality if and only \( \lambda_k = 0 \). Combining these two inequalities, we obtain

\[
D_r^{(0,1)}[(A + \gamma I), (B + \mu I)] \geq 0,
\]

with equality if and only if \( \nu = \frac{\gamma}{\mu} = 1 \) and \( \lambda_k = 0 \forall k \in \mathbb{N} \) \( \iff \Lambda = I \), that is if and only \( (B + \mu I)^{-1/2}(A + \gamma I)(B + \mu I)^{-1/2} = I \iff A + \gamma I = B + \mu I \iff A = B \) and \( \gamma = \mu \). This completes the proof. \( \square \)

**Proof of Theorem 2 (Special cases - I).** The first statement of the theorem is the content of Theorem 9. The second statement is the content of Theorem 10. \( \square \)

**Proof of Theorem 3 (Special cases - II).** This theorem follows from Theorems 9 and 10 as well as the symmetry of \( D_r^{(\alpha, \alpha)}[(A + \gamma I), (B + \mu I)] \) as proved in Theorem 13. \( \square \)

**Appendix A.7. Proofs for the divergences between RKHS covariance operators**

In this section, we prove Theorems 23, 24, 25, and 26. We first need the following preliminary results.
Lemma 14. Let $\mathcal{H}_1, \mathcal{H}_2$ be separable Hilbert spaces. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $B : \mathcal{H}_2 \to \mathcal{H}_1$ be compact linear operators such that both $AB : \mathcal{H}_2 \to \mathcal{H}_2$ and $BA : \mathcal{H}_1 \to \mathcal{H}_1$ are trace class operators. Let $\alpha, \beta > 0$ be fixed. For any $p, q \in \mathbb{R}$,

$$
\det \left[ \frac{\alpha(AB + I_{\mathcal{H}_2})^p + \beta(AB + I_{\mathcal{H}_2})^q}{\alpha + \beta} \right] = \det \left[ \frac{\alpha(BA + I_{\mathcal{H}_1})^p + \beta(BA + I_{\mathcal{H}_1})^q}{\alpha + \beta} \right]. \quad (A.23)
$$

Proof of Lemma 14. Since the nonzero eigenvalues of $AB : \mathcal{H}_2 \to \mathcal{H}_2$ and $BA : \mathcal{H}_1 \to \mathcal{H}_1$ are the same, we have for any $p \in \mathbb{R}$

$$
\det [(AB + I_{\mathcal{H}_2})^p] = \det [(BA + I_{\mathcal{H}_1})^p].
$$

For any $p, q \in \mathbb{R}$,

$$
\det \left[ \frac{\alpha(AB + I_{\mathcal{H}_2})^p + \beta(AB + I_{\mathcal{H}_2})^q}{\alpha + \beta} \right] = \det \left[ \frac{\alpha(BA + I_{\mathcal{H}_1})^p + \beta(BA + I_{\mathcal{H}_1})^q}{\alpha + \beta} \right].
$$

In the above equality, we have used the fact that a zero eigenvalue of $AB$ and $BA$ corresponds to an eigenvalue equal to 1 for $\frac{\alpha(AB + I_{\mathcal{H}_2})^p + \beta(AB + I_{\mathcal{H}_2})^q}{\alpha + \beta} : \mathcal{H}_2 \to \mathcal{H}_2$ and $\frac{\alpha(BA + I_{\mathcal{H}_1})^p + \beta(BA + I_{\mathcal{H}_1})^q}{\alpha + \beta} : \mathcal{H}_1 \to \mathcal{H}_1$, respectively, which does not change the determinant. This completes the proof. \qed

Lemma 15. Let $\mathcal{H}_1, \mathcal{H}_2$ be separable Hilbert spaces. Let $A, B : \mathcal{H}_1 \to \mathcal{H}_2$ be compact linear operators such that both $AA^* : \mathcal{H}_2 \to \mathcal{H}_2$ and $BB^* : \mathcal{H}_2 \to \mathcal{H}_2$ are trace class operators. Let $\alpha, \beta > 0$ be fixed. For any $p, q \in \mathbb{R}$,

$$
\det \left[ \frac{\alpha((AA^* + I_{\mathcal{H}_2})(BB^* + I_{\mathcal{H}_2})^{-1})^p + \beta((AA^* + I_{\mathcal{H}_2})(BB^* + I_{\mathcal{H}_2})^{-1})^q}{\alpha + \beta} \right] = \det \left[ \frac{\alpha(C + I_{\mathcal{H}_1} \otimes I_3)^p + \beta((C + I_{\mathcal{H}_1} \otimes I_3)\otimes I_3)^q}{\alpha + \beta} \right], \quad (A.24)
$$

where

$$
C = \begin{pmatrix}
A^*A - A^*B(I_{\mathcal{H}_1} + B^*B)^{-1} - A^*AA^*B(I_{\mathcal{H}_1} + B^*B)^{-1} \\
B^*A - B^*B(I_{\mathcal{H}_1} + B^*B)^{-1} - B^*AA^*B(I_{\mathcal{H}_1} + B^*B)^{-1} \\
B^*A - B^*B(I_{\mathcal{H}_1} + B^*B)^{-1} - B^*AA^*B(I_{\mathcal{H}_1} + B^*B)^{-1}
\end{pmatrix}. \quad (A.25)
$$

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Proof of Lemma 15. We make use of the following notation. Let $A, B, C : \mathcal{H}_1 \to \mathcal{H}_2$ be three bounded linear operators. Consider the operator $(A \ B \ C) : \mathcal{H}_1^3 \to \mathcal{H}_2$, with

$$(A \ B \ C)^* = \begin{pmatrix} A^* \\ B^* \\ C^* \end{pmatrix} : \mathcal{H}_2 \to \mathcal{H}_1^3.$$ Here $\mathcal{H}_1^3 = \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_1$ denotes the direct sum of $\mathcal{H}_1$ with itself, that is

$$\mathcal{H}_1^3 = \{ (v_1, v_2, v_3) : v_1, v_2, v_3 \in \mathcal{H}_1 \},$$
equipped with the inner product

$$\langle (v_1, v_2, v_3), (w_1, w_2, w_3) \rangle_{\mathcal{H}_1^3} = \langle v_1, w_1 \rangle_{\mathcal{H}_1} + \langle v_2, w_2 \rangle_{\mathcal{H}_1} + \langle v_3, w_3 \rangle_{\mathcal{H}_1}.$$ If $\{ e_i \}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}_1$, then $\{ (e_i, 0, 0) \}_{i=1}^{\infty} \cup \{ (0, e_i, 0) \}_{i=1}^{\infty} \cup \{ (0, 0, e_i) \}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}_1^3$.

We now utilize this notation in our setting. By the Sherman-Morrison-Woodbury formula, we have

$$(BB^* + I_{\mathcal{H}_2})^{-1} = I_{\mathcal{H}_2} - B(I_{\mathcal{H}_1} + B^*B)^{-1}B^*.$$ Thus it follows that

$$(AA^* + I_{\mathcal{H}_2})(BB^* + I_{\mathcal{H}_2})^{-1} = I_{\mathcal{H}_2} + AA^* - B(I_{\mathcal{H}_1} + B^*B)^{-1}B^* \quad - AA^*B(I_{\mathcal{H}_1} + B^*B)^{-1}B^* \quad = I_{\mathcal{H}_2} + C_1C_2.$$

Here the operators $C_1, C_2$ are defined as follows.

$$C_1 = \begin{pmatrix} A^* \\ - B(I_{\mathcal{H}_1} + B^*B)^{-1} \\ - AA^*B(I_{\mathcal{H}_1} + B^*B)^{-1} \end{pmatrix} : \mathcal{H}_1^3 \to \mathcal{H}_2,$$

$$C_2 = \begin{pmatrix} A^* \\ B^* \\ B^* \end{pmatrix} : \mathcal{H}_2 \to \mathcal{H}_1^3.$$ The operator $C_2C_1 : \mathcal{H}_1^3 \to \mathcal{H}_1^3$ is given by

$$C_2C_1 = \begin{pmatrix} A^*A - A^*B(I_{\mathcal{H}_1} + B^*B)^{-1} - A^*AA^*B(I_{\mathcal{H}_1} + B^*B)^{-1} \\ B^*A - B^*B(I_{\mathcal{H}_1} + B^*B)^{-1} - B^*AA^*B(I_{\mathcal{H}_1} + B^*B)^{-1} \\ B^*A - B^*B(I_{\mathcal{H}_1} + B^*B)^{-1} - B^*AA^*B(I_{\mathcal{H}_1} + B^*B)^{-1} \end{pmatrix}.$$
It follows from Lemma 14 that
\[
\det \left[ \frac{\alpha((AA^* + I_{H_2})(BB^* + I_{H_2})^{-1})^p + \beta((AA^* + I_{H_2})(BB^* + I_{H_2})^{-1})^q}{\alpha + \beta} \right] = \ldots
\]
by replacing \( AA^* \) and \( BB^* \) in Lemma 15 with \( AA^* \gamma \) and \( BB^* \mu \), respectively. This completes the proof of the theorem.

Proof of Theorem 23. Let \( \Lambda + \frac{2}{\mu} I = (BB^* + \mu I_{H_2})^{-1/2}(AA^* + \gamma I)(BB^* + \mu I)^{-1/2} \) and \( Z + \frac{2}{\mu} I = (AA^* + \gamma I)(BB^* + \mu I)^{-1} \), with \( \frac{2}{\mu} Z + I = (\frac{AA^*}{\mu} + I)(\frac{BB^*}{\mu} + I)^{-1} \).

By Theorem 8, we have
\[
D[\alpha, \beta][\Lambda, (BB^* + \mu I_{H_2})] = \frac{\log r(\delta - \frac{\gamma}{\alpha})}{\alpha\beta} \left( \log \frac{\gamma}{\mu} \right) + \frac{1}{\alpha\beta} \log \left( \frac{\alpha(\frac{2}{\mu})^p + \beta(\frac{2}{\mu})^q}{\alpha + \beta} \right)
\]
\[
+ \frac{1}{\alpha\beta} \log \det \left[ \frac{\alpha(\Lambda + \frac{2}{\mu} I)^p + \beta(\Lambda + \frac{2}{\mu} I)^{-q}}{\alpha + \beta} \right],
\]
with \( p = r(1 - \delta) \) and \( q = r\delta \). The determinant in the last term is
\[
\det \left[ \frac{\alpha(\Lambda + \frac{2}{\mu} I)^p + \beta(\Lambda + \frac{2}{\mu} I)^{-q}}{\alpha + \beta} \right] = \det \left[ \frac{\alpha(\frac{2}{\mu})^p(\frac{2}{\mu} \Lambda + I)^p + \beta(\frac{2}{\mu})^{-q}(\frac{2}{\mu} \Lambda + I)^{-q}}{\alpha + \beta} \right]
\]
\[
= \det \left[ \frac{\alpha(\frac{2}{\mu})^p(C + I_{H_1} \otimes I_3)^p + \beta(\frac{2}{\mu})^{-q}(C + I_{H_1} \otimes I_3)^{-q}}{\alpha + \beta} \right]
\]
by Lemma 15, where
\[
C = \begin{pmatrix}
\frac{\Lambda^*}{\gamma} & -\frac{\Lambda^*}{\gamma} B(I_{H_2} + \frac{B^* B}{\mu})^{-1} & -\frac{\Lambda^*}{\gamma} A A^* B(I_{H_2} + \frac{B^* B}{\mu})^{-1} \\
-\frac{B^*}{\mu} A & -\frac{B^*}{\mu} B(I_{H_2} + \frac{B^* B}{\mu})^{-1} & -\frac{B^* A A^* B}{\gamma\mu}(I_{H_2} + \frac{B^* B}{\mu})^{-1} \\
-\frac{B^*}{\gamma\mu} & -\frac{B^*}{\mu} B(I_{H_2} + \frac{B^* B}{\mu})^{-1} & -\frac{B^* A A^* B}{\gamma\mu}(I_{H_2} + \frac{B^* B}{\mu})^{-1}
\end{pmatrix},
\]
which is obtained by replacing \( AA^* \) and \( BB^* \) in Lemma 15 with \( \frac{AA^*}{\gamma} \) and \( \frac{BB^*}{\mu} \), respectively. This completes the proof of the theorem. \( \square \)
Proof of Theorem 24. Let $Z + \frac{\gamma}{p} I = (AA^* + \gamma I)(BB^* + \mu I)^{-1}$. By the finite-dimensional formula given in Eq. (19), we have

$$D_r^{(\alpha, \beta)}[(AA^* + \gamma I_{H_2}), (BB^* + \mu I_{H_2})] = \frac{1}{\alpha \beta} \log \det \left[ \frac{\alpha(Z + \frac{\gamma}{p} I)^p + \beta(Z + \frac{\gamma}{p} I)^{-q}}{\alpha + \beta} \right] \dim(H_2)$$

$$= \frac{1}{\alpha \beta} \log \left( \frac{\alpha(Z + \frac{\gamma}{p} I)^p + \beta(Z + \frac{\gamma}{p} I)^{-q}}{\alpha + \beta} \right) \dim(H_2)$$

$$+ \frac{1}{\alpha \beta} \log \det \left[ \frac{\alpha(Z + \frac{2}{p} I)^p + \beta(Z + \frac{2}{p} I)^{-q}}{\alpha(Z + \frac{2}{p} I)^p + \beta(Z + \frac{2}{p} I)^{-q}} \right].$$

As in the proof of Theorem 23, the determinant in last term in the above expression is

$$\det \left[ \frac{\alpha(Z + \frac{2}{p} I)^p + \beta(Z + \frac{2}{p} I)^{-q}}{\alpha(Z + \frac{2}{p} I)^p + \beta(Z + \frac{2}{p} I)^{-q}} \right] = \det \left[ \frac{\alpha(\Lambda + \frac{2}{p} I)^p + \beta(\Lambda + \frac{2}{p} I)^{-q}}{\alpha(\Lambda + \frac{2}{p} I)^p + \beta(\Lambda + \frac{2}{p} I)^{-q}} \right].$$

This gives us the final expression.

Proof of Theorem 25. We consider the linear operators

$$A = \frac{1}{\sqrt{m}} \Phi(x) J_m : \mathbb{R}^m \rightarrow H_K, \quad B = \frac{1}{\sqrt{m}} \Phi(y) J_m : \mathbb{R}^m \rightarrow H_K.$$

The desired expression then follows from Theorem 23.

Proof of Theorem 26. This is proved in the same way as Theorem 25, except that we invoke Theorem 24.

Appendix A.8. Proofs for the metric properties

In this section, we prove Theorems 19, 20, 21, which lead to the proofs of Theorems 4 and 22. We present two sets of separate proofs for Theorems 4 and 22, one simpler proof for the particular case $\alpha = 1/2$, which corresponds to the infinite-dimensional symmetric Stein divergence, and one general proof for any $\alpha > 0$. The former case utilizes Theorem 28 and the latter case utilizes Theorem 30, both of which should be of interest in their own right.
Appendix A.8.1. The case of the infinite-dimensional symmetric Stein divergence

Consider the first case $\alpha = 1/2$, which corresponds to the infinite-dimensional symmetric Stein divergence.

**Lemma 16.** Let $\mathcal{H}$ be a separable Hilbert space. Let $A, B, C : \mathcal{H} \to \mathcal{H}$ be self-adjoint finite-rank operators, such that $A + I > 0$, $B + I > 0$, $C + I > 0$. Then

$$\sqrt{\frac{\log \det(A + B + I)}{\det(A + I) \det(B + I)}} \leq \sqrt{\frac{\log \det(A + C + I)}{\det(A + I) \det(C + I)}} + \sqrt{\frac{\log \det(C + B + I)}{\det(C + I) \det(B + I)}}.$$  (A.26)

**Proof of Lemma 16.** Since $A, B, C$ are all finite-rank operators, there exists a finite-dimensional subspace $\mathcal{H}_n \subset \mathcal{H}$, with $\dim(\mathcal{H}_n) = n$ for some $n \in \mathbb{N}$, such that $\text{range}(A) \subset \mathcal{H}_n$, $\text{range}(B) \subset \mathcal{H}_n$, and $\text{range}(C) \subset \mathcal{H}_n$. Let

$$A_n = A|_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_n, \quad B_n = B|_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_n, \quad C_n = C|_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_n.$$  

Then $A_n, B_n, C_n$ are linear operators on the finite-dimensional space $\mathcal{H}_n$ and thus are represented by $n \times n$ matrices, which we denote by the same symbols. We also have

$$(A + B)_n = (A + B)|_{\mathcal{H}_n} = A_n + B_n, \quad (A + C)_n = A_n + C_n, \quad (C + B)_n = B_n + C_n.$$  

Applying the finite-dimensional result in [16], we then obtain

$$\sqrt{\log \frac{\det(A_n + B_n + I_n)}{\det(A_n + I_n) \det(B_n + I_n)}} \leq \sqrt{\log \frac{\det(A_n + C_n + I_n)}{\det(A_n + I_n) \det(C_n + I_n)}} + \sqrt{\log \frac{\det(C_n + B_n + I_n)}{\det(C_n + I_n) \det(B_n + I_n)}}.$$  

It is clear that the non-zero eigenvalues of $A$ and $A_n$ are the same, so that $\det(A + I) = \det(A_n + I_n)$ and the same holds true for the other operators. This gives us the final result.

**Proof of Theorem 21 (Triangle inequality- square root of symmetric Stein divergence).**

Let $\{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}}, \{C_n\}_{n \in \mathbb{N}}$ be sequences of finite-rank operators with

$$\|A_n - A\|_{tr} \to 0, \quad \|B_n - B\|_{tr} \to 0, \quad \|C_n - C\|_{tr} \to 0,$$  

as $n \to \infty$.  

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By Lemma 16, we have
\[
\sqrt{\log \frac{\det(A_n + B_n + I)}{\det(A_n + I) \det(B_n + I)}} \leq \sqrt{\log \frac{\det(A_n + C_n + I)}{\det(A_n + I) \det(C_n + I)}}
\]
\[
+ \sqrt{\log \frac{\det(C_n + B_n + I)}{\det(C_n + I) \det(B_n + I)}}.
\]
By Theorem 3.5 in [22], as \(n \to \infty\), we have
\[
det(A_n + I) \to det(A + I), \quad det(B_n + I) \to det(B + I),
\]
\[
det(A_n + B_n + I) \to det\left(\frac{A + B}{2} + I\right),
\]
and the same holds true for the other operators. Thus by taking the limit as \(n \to \infty\) in the above triangle inequality for \((A_n + I), (B_n + I)\) and \((C_n + I)\), we obtain the final triangle inequality for \((A + I), (B + I), (C + I)\).

The following is the specialization of Theorem 4 when \(\alpha = 1/2\).

**Theorem 27 (Metric property - square root of symmetric Stein divergence).** Let \(\gamma > 0, \gamma \in \mathbb{R}\) be fixed. The square root of the infinite-dimensional symmetric Stein divergence
\[
\sqrt{D_{1}^{(1/2,1/2)}}[(A + \gamma I), (B + \gamma I)]
\]

is a metric on \(\text{PTr}(\mathcal{H})(\gamma)\).

**Proof of Theorem 27.** We have already shown the positivity and symmetry of \(D_{1}^{(1/2,1/2)}[(A + \gamma I), (B + \gamma I)]\). It remains for us to show the triangle inequality, namely
\[
\sqrt{D_{1}^{(1/2,1/2)}}[(A + \gamma I), (B + \gamma I)] \leq \sqrt{D_{1}^{(1/2,1/2)}}[(A + \gamma I), (C + \gamma I)]
\]
\[
+ \sqrt{D_{1}^{(1/2,1/2)}}[(C + \gamma I), (B + \gamma I)],
\]
for any three operators \((A + \gamma I), (B + \gamma I), (C + \gamma I) \in \text{PTr}(\mathcal{H})\). We have
\[
D_{1}^{(1/2,1/2)}[(A + \gamma I), (B + \gamma I)] = 4 \log \frac{\det(A + B + \gamma I)}{\det(\frac{A + B}{2} + \gamma I)^{1/2} \det(\frac{B + \gamma I}{2})^{1/2}},
\]
Thus the triangle inequality for \(D_{1}^{(1/2,1/2)}[(A + \gamma I), (B + \gamma I)]\) follows that stated in Theorem 21. \(\square\)
Lemma 17. Let $\mathcal{H}$ be a separable Hilbert space. Let $A, B : \mathcal{H} \to \mathcal{H}$ be self-adjoint finite-rank operators, with maximum rank $n, n \in \mathbb{N}$, such that $A + I > 0, B + I > 0$. Then
\[
\prod_{j=1}^{n} \left[ \frac{\lambda_j(A) + \lambda_j(B)}{2} + 1 \right] \leq \det \left( \frac{A + B + I}{2} \right). \tag{A.27}
\]

Proof of Lemma 17. Since $A, B$ are both finite-rank operators, there exists a finite-dimensional subspace $\mathcal{H}_n \subset \mathcal{H}$, with $\dim(\mathcal{H}_n) = n$, such that $\operatorname{range}(A) \subset \mathcal{H}_n, \operatorname{range}(B) \subset \mathcal{H}_n$. Let
\[
A_n = A|_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_n, \quad B_n = B|_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_n.
\]
Then $A_n, B_n$ are linear operators on the finite-dimensional space $\mathcal{H}_n$ and thus are represented by $n \times n$ matrices, which we denote by the same symbols. We also have
\[
(A + B)_n = (A + B)|_{\mathcal{H}_n} = A|_{\mathcal{H}_n} + B|_{\mathcal{H}_n} = A_n + B_n.
\]
Thus we can apply the following inequality for finite-dimensional SPD matrices ([23])
\[
\prod_{j=1}^{n} \left[ \frac{\lambda_j(A_n) + \lambda_j(B_n)}{2} + 1 \right] = \prod_{j=1}^{n} \left[ \frac{\lambda_j(A_n + I_n) + \lambda_j(B_n + I_n)}{2} \right] \leq \det \left( \frac{A_n + B_n + I_n}{2} \right).
\]
We note that the non-zero eigenvalues of $A_n, B_n$ are the same as those of $A, B$, respectively, with the maximum number being $n$, and $\det(\frac{A + B}{2} + I) = \det(\frac{A_n + B_n}{2} + I_n)$. Together with the previous inequality, this gives us the final result.

Theorem 28. Let $\mathcal{H}$ be a separable Hilbert space. Let $A, B : \mathcal{H} \to \mathcal{H}$ be self-adjoint trace class operators, such that $A + I > 0, B + I > 0$. Then
\[
\prod_{j=1}^{\infty} \left[ \frac{\lambda_j(A) + \lambda_j(B)}{2} + 1 \right] \leq \det \left( \frac{A + B + I}{2} \right). \tag{A.28}
\]

Proof of Theorem 28. Let $A = \sum_{j=1}^{\infty} \lambda_j(A) \phi_j \otimes \phi_j$ denote the spectral decomposition for $A$. For each $n \in \mathbb{N}$, define
\[
A_n = \sum_{j=1}^{n} \lambda_j(A) \phi_j \otimes \phi_j.
\]
Then $A_n$ is a finite-rank operator with the eigenvalues being the first $n$ eigenvalues of $A$ and $\lim_{n \to \infty} ||A_n - A||_{\text{tr}} = 0$. In the same way, we construct a sequence of finite-rank operators $B_n$ with $\lim_{n \to \infty} ||B_n - B||_{\text{tr}} = 0$, so that
\[
\lim_{n \to \infty} ||(A_n + B_n) - (A + B)||_{\text{tr}} = 0.
\]
By Theorem 3.5 in [22], as $n \to \infty$, we then have
\[
\lim_{n \to \infty} \det\left(\frac{A_n + B_n}{2} + I\right) = \det\left(\frac{A + B}{2} + I\right).
\]
Applying Lemma 17 to $A_n$ and $B_n$, we have
\[
\prod_{j=1}^{n} \left[\lambda_j(A_n) + \lambda_j(B_n) + 1\right] \leq \det\left(\frac{A_n + B_n}{2} + I\right). \quad (A.29)
\]
The final result is then obtained by taking the limit as $n \to \infty$, noting that the eigenvalues of $A_n$, $B_n$, are precisely the first $n$ eigenvalues of $A$, $B$, respectively.

The following is the specialization of Theorem 22 when $\alpha = 1/2$.

**Theorem 29.** Let $\mathcal{H}$ be a separable Hilbert space. Let $A, B : \mathcal{H} \to \mathcal{H}$ be self-adjoint trace class operators, such that $A + I > 0$, $B + I > 0$. Let $\text{Eig}(A), \text{Eig}(B) : \ell^2 \to \ell^2$ be diagonal operators with the diagonals consisting of the eigenvalues of $A$ and $B$, respectively, in decreasing order. Then
\[
D_1^{(1/2,1/2)}[(\text{Eig}(A) + I), (\text{Eig}(B) + I)] \leq D_1^{(1/2,1/2)}[(A + I), (B + I)]. \quad (A.30)
\]

**Proof of Theorem 29.** By definition, we have
\[
D_1^{(1/2,1/2)}[(\text{Eig}(A) + I), (\text{Eig}(B) + I)] = 4 \log \left[ \frac{\det(\text{Eig}(A) + I)}{\sqrt{\det(\text{Eig}(A) + I) \det(\text{Eig}(B) + I)}} \right]
\]
\[
= 4 \log \left[ \prod_{j=1}^{\infty} \left[ \frac{\lambda_j(A) + \lambda_j(B) + 1}{\sqrt{\det(A + I) \det(B + I)}} \right] \right]
\]
\[
\leq 4 \log \left[ \frac{\det(A + B + I)}{\sqrt{\det(A + I) \det(B + I)}} \right]
\]
by Theorem 28
\[
= D_1^{(1/2,1/2)}[(A + I), (B + I)].
\]
This completes the proof.
Appendix A.8.2. The general case

We now consider the general case $\alpha > 0$. We need the following results.

In the following, let $C_p(H)$ denote the class of $p$th Schatten class operators on $H$, under the norm $\| \cdot \|_p$, $1 \leq p \leq \infty$, which is defined by

$$
\|A\|_p = \left[ \sum_{k=1}^{\infty} \lambda_k^p(A^*A)^{1/2} \right]^{1/p},
$$

(A.31)

with $\mathcal{C}_1(H)$ being the space of trace class operators Tr$(H)$, $\mathcal{C}_2(H)$ being the space of Hilbert-Schmidt operators HS$(H)$, and $\mathcal{C}_\infty(H)$ being the set of compact operators under the operator norm $\| \cdot \|$. 

Theorem 30. Let $r \in \mathbb{R}$ be fixed but arbitrary. Assume that $1 \leq p \leq \infty$. Let $\{A_n\}_{n \in \mathbb{N}} \subset \text{Sym}(H) \cap C_p(H)$, $A \in \text{Sym}(H) \cap C_p(H)$ be such that $I + A > 0$, $I + A_n > 0 \ \forall n \in \mathbb{N}$. Assume that $\lim_{n \to \infty} \|A_n - A\|_p = 0$. Then

$$
\lim_{n \to \infty} \|(I + A_n)^r - (I + A)^r\|_p = 0.
$$

(A.32)

Proof of Theorem 30. (i) We first prove that

$$
\lim_{n \to \infty} \|(I + A_n)^r - (I + A)^r\|_p = 0, \quad 0 \leq r \leq 1.
$$

(A.33)

The case $r = 0$ is trivial. Let us prove this for $0 < r \leq 1$. For this limit, we make use of the following result from [24] (Corollary 3.2), which states that for any two positive operators $A, B$ on $H$ such that $A \geq c > 0$, $B \geq c > 0$, and any operator $X$ on $H$,

$$
\|A'X - XB'\|_p \leq r c^{r-1} \|AX - XB\|_p,
$$

(A.34)

where $0 < r \leq 1$ and $\| \cdot \|_p$, $1 \leq p \leq \infty$, denotes the Schatten $p$-norm.

By the assumption $I + A > 0$, there exists $M_A > 0$ such that

$$
\langle x, (I + A)x \rangle \geq M_A \|x\|^2 \quad \forall x \in H.
$$

By the assumption $\lim_{n \to \infty} \|A_n - A\|_p = 0$, for any $\epsilon$ satisfying $0 < \epsilon < M_A$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $\|A_n - A\|_p < \epsilon \ \forall n \geq N$. Then $\forall x \in H$,

$$
\|\langle x, (A_n - A)x \rangle\| \leq \|A_n - A\| \|x\|^2 \leq \|A_n - A\|_p \|x\|^2 \leq \epsilon \|x\|^2.
$$
It thus follows that \( \forall x \in \mathcal{H}, \)

\[
\langle x, (I + A_n)x \rangle = \langle x, (I + A)x \rangle + \langle x, (A_n - A)x \rangle \geq (MA - \epsilon)\|x\|^2.
\]

Thus we have \( I + A \geq MA > 0, \) \( I + A_n \geq MA - \epsilon > 0 \) \( \forall n \geq N = N(\epsilon). \) Then, applying Eq. (A.34), we have for all \( n \geq N = N(\epsilon), \)

\[
\|((I + A_n)^r - (I + A)^r)\|_p \leq r(MA - \epsilon)^{r-1}\| (I + A_n) - (I + A)\|_p
\]

\[
= r \left( \frac{1}{MA - \epsilon} \right)^{1-r} \|A_n - A\|_p,
\]

which implies

\[
\lim_{n \to \infty} \|((I + A_n)^r - (I + A)^r)\|_p = 0.
\]

This completes the proof of the first limit.

(ii) For \( r > 1, \) we proceed by induction as follows. We have

\[
\|((I + A_n)^r - (I + A)^r)\|_p \leq r(MA - \epsilon)^{r-1}\| (I + A_n) - (I + A)\|_p
\]

\[
= r \left( \frac{1}{MA - \epsilon} \right)^{1-r} \|A_n - A\|_p \|A_n - A\|_p
\]

Thus this case follows from the case \( 0 \leq r \leq 1 \) by induction.

(iii) We next prove that

\[
\lim_{n \to \infty} \|((I + A_n)^{-r} - (I + A)^{-r})\|_p = 0.
\]
We have
\[(I + A)^{-1} \geq \frac{1}{\max\{(1 + \lambda_k(A)) : k \in \mathbb{N}\}} = \frac{1}{||I + A||} > 0.\]
From the limit \(\lim_{n \to \infty} ||A_n - A|| = 0\), it follows that for any \(\epsilon\) satisfying \(0 < \epsilon < ||I + A||\), there exists \(M = M(\epsilon) \in \mathbb{N}\) such that \(\forall n \geq M\),
\[||I + A|| - \epsilon \leq ||I + A_n|| \leq ||I + A|| + \epsilon.
\]
It follows that \(\forall n \geq M\),
\[(I + A_n)^{-1} \geq \frac{1}{\max\{(1 + \lambda_k(A_n)) : k \in \mathbb{N}\}} = \frac{1}{||I + A_n||} \geq \frac{1}{||I + A|| + \epsilon}.
\]
Hence invoking Eq. (A.34) again, we obtain \(\forall n \geq M\)
\[||(I + A_n)^{-r} - (I + A)^{-r}||_p \leq r(||I + A|| + \epsilon)^{1-r}||(I + A_n)^{-1} - (I + A)^{-1}||_p,\]
which implies that
\[\lim_{n \to \infty}||(I + A_n)^{-r} - (I + A)^{-r}||_p = 0\]
by the previous limit, when \(r = 1\).
(iv) By an induction argument as in step (ii), we then obtain that
\[\lim_{n \to \infty}||(I + A_n)^{-r} - (I + A)^{-r}||_p = 0, \ \forall r > 1. \quad (A.37)\]
This completes the proof. □

**Lemma 18.** Let \(\mathcal{H}\) be a separable Hilbert space. Assume that \(\{A_n\}_{n \in \mathbb{N}}\), \(A\) are trace class operators on \(\mathcal{H}\) such that \((I + A) > 0\), \((I + A_n) > 0\) \(\forall n \in \mathbb{N}\). Assume that \(||A_n - A||_{tr} = 0\) as \(n \to \infty\). Then \(A_n(I + A_n)^{-1}\) and \(A(I + A)^{-1}\) are trace class operators and
\[\lim_{n \to \infty}||A_n(I + A_n)^{-1} - A(I + A)^{-1}||_{tr} = 0. \quad (A.38)\]

**Proof of Lemma 18.** It is obvious that, given that \(A_n\) and \(A\) are trace class operators, both \(A_n(I + A_n)^{-1}\) and \(A(I + A)^{-1}\) are trace class operators. We have
\[||A_n(I + A_n)^{-1} - A(I + A)^{-1}||_{tr} = ||(I + A_n)^{-1}A_n - A(I + A)^{-1}||_{tr}\]
\[= ||(I + A_n)^{-1}[A_n(I + A) - (I + A_n)A](I + A)^{-1}||_{tr}\]
\[= ||(I + A_n)^{-1}[A_n - A](I + A)^{-1}||_{tr} \leq ||(I + A_n)^{-1}|| \ ||A_n - A||_{tr} \ ||(I + A)^{-1}||.
\]
By the assumption $I + A > 0$, there exists $M_A > 0$ such that
\[
\langle x, (I + A)x \rangle \geq M_A \|x\|^2 \quad \forall x \in \mathcal{H}.
\]

By the assumption \(\lim_{n \to \infty} \|A_n - A\|_{\text{tr}} = 0\), for any \(\epsilon\) satisfying \(0 < \epsilon < M_A\), there
exists \(N = N(\epsilon) \in \mathbb{N}\) such that \(\|A_n - A\|_{\text{tr}} < \epsilon\ \forall n \geq N\). Then \(\forall x \in \mathcal{H},\)
\[
|\langle x, (A_n - A)x \rangle| \leq \|A_n - A\| \|x\|^2 \leq \|A_n - A\|_{\text{tr}} \|x\|^2 \leq \epsilon \|x\|^2.
\]

It thus follows that \(\forall x \in \mathcal{H},\)
\[
\langle x, (I + A_n)x \rangle = \langle x, (I + A)x \rangle + \langle x, (A_n - A)x \rangle \geq (M_A - \epsilon) \|x\|^2.
\]

Thus we have \(I + A \geq M_A > 0\), \(I + A_n \geq M_A - \epsilon > 0 \ \forall n \geq N = N(\epsilon),\) from
which it follows that
\[
\|(I + A_n)^{-1}\| \leq \frac{1}{M_A - \epsilon} \forall N \geq N(\epsilon), \quad \|(I + A)^{-1}\| \leq \frac{1}{M_A}.
\]

Combining this with the first inequality, we have
\[
\|A_n(I + A_n)^{-1} - A(I + A)^{-1}\|_{\text{tr}} \leq \frac{1}{M_A(M_A - \epsilon)} \|A_n - A\|_{\text{tr}} \forall n \geq N,
\]
which implies that
\[
\lim_{n \to \infty} \|A_n(I + A_n)^{-1} - A(I + A)^{-1}\|_{\text{tr}} = 0.
\]

This completes the proof.

\[\square\]

**Lemma 19.** Let \(\mathcal{H}\) be a separable Hilbert space. Let \(\{A_n\}_{n \in \mathbb{N}}, A, \{B_n\}_{n \in \mathbb{N}}, B,\) be
self-adjoint, trace class operators on \(\mathcal{H}\), with \(\lim_{n \to \infty} \|A_n - A\|_{\text{tr}} = 0, \lim_{n \to \infty} \|B_n - B\|_{\text{tr}} = 0\). Assume that \(I + A > 0, I + B > 0, I + A_n > 0, I + B_n > 0 \ \forall n \in \mathbb{N}\). Then
\((I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2} - I\) and \((I + B)^{-1/2}(I + A)(I + B)^{-1/2} - I\) are self-adjoint, trace class operators on \(\mathcal{H}\) and
\[
\lim_{n \to \infty} \|(I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2} - (I + B)^{-1/2}(I + A)(I + B)^{-1/2}\|_{\text{tr}} = 0.
\]

(A.39)
Proof of Lemma 19. We write

\[(I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2} = I - B_n(I + B_n)^{-1} - (I + B_n)^{-1/2}A_n(I + B_n)^{-1/2},\]

\[(I + B)^{-1/2}(I + A)(I + B)^{-1/2} = I - B(I + B)^{-1} - (I + B)^{-1/2}A(I + B)^{-1/2}.\]

It follows immediately that \([I + B_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2} - I] and \([I + B)^{-1/2}(I + A)(I + B)^{-1/2} - I] are self-adjoint, trace class operators on \(\mathcal{H}\).

By Lemma 18, we have

\[\lim_{n \to \infty} ||B_n(I + B_n)^{-1} - B(I + B)^{-1}||_{tr} = 0.\]

Consider next the difference between the third terms of the above two expressions

\[||\frac{1}{2}A_n(I + B_n)^{-1} - \frac{1}{2}A(I + B)^{-1}||_{tr}\]

\[\leq ||(I + B_n)^{-1/2}A_n(I + B_n)^{-1/2} - (I + B)^{-1/2}A(I + B)^{-1/2}||_{tr}\]

\[+ ||(I + B_n)^{-1/2}A(I + B_n)^{-1/2} - (I + B_n)^{-1/2}A(I + B)^{-1/2}||_{tr}\]

\[+ ||(I + B_n)^{-1/2}A(I + B)^{-1/2} - (I + B)^{-1/2}A(I + B)^{-1/2}||_{tr}.\]  \hspace{1cm} (A.40)

By the assumption \(I + A > 0, I + B > 0\), there exist constants \(M_A > 0, M_B > 0\) such that \(I + A \geq M_A, I + B \geq M_B\). As in the proof of Lemma 18, since \(\lim_{n \to \infty} ||A_n - A|| = 0, \lim_{n \to \infty} ||B_n - B|| = 0\), for any \(0 < \epsilon < \min\{M_A, M_B\}\), there exist \(N_A = N_A(\epsilon) \in \mathbb{N}, N_B = N_B(\epsilon) \in \mathbb{N}\), such that

\[I + A_n \geq M_A - \epsilon, \forall n \geq N_A, I + B_n \geq M_B - \epsilon \forall n \geq N_B.\]

The first term on the right hand side of the inequality in Eq. (A.40) is

\[||A_n - A||_{tr} \leq \frac{1}{M_B - \epsilon} ||A_n - A||_{tr} \forall n \geq N_B.\]

The second term is

\[\leq \frac{1}{\sqrt{M_B - \epsilon}} ||A||_{tr} ||(I + B_n)^{-1/2} - (I + B)^{-1/2}||.\]
Similarly, for the third term, we have

\[
\| (I + B_n)^{-1/2} A (I + B)^{-1/2} - (I + B)^{-1/2} A (I + B_n)^{-1/2} \|_{\text{tr}}
\]

\[
\leq \| A (I + B)^{-1/2} \|_{\text{tr}} \| [(I + B_n)^{-1/2} - (I + B)^{-1/2}] \|.
\]

By Theorem 30, we have

\[
\| (I + B_n)^{-1/2} - (I + B)^{-1/2} \| \leq \| (I + B_n)^{-1/2} - (I + B)^{-1/2} \|_{\text{tr}} \to 0
\]
as \( n \to \infty \). The final result is obtained by combining all of the above inequalities.

**Lemma 20.** Let \( \mathcal{H} \) be a separable Hilbert space. Let \( A, B, C : \mathcal{H} \to \mathcal{H} \) be self-adjoint, finite-rank operators such that \( (I + A) > 0 \), \( (I + B) > 0 \), \( (I + C) > 0 \). Then

\[
D_2^{(\alpha, \alpha)}[(I + A), (I + B)] \leq D_2^{(\alpha, \alpha)}[(I + A), (I + C)] + D_2^{(\alpha, \alpha)}[(I + C), (I + B)].
\]  

(A.41)

**Proof of Lemma 20.** Since \( A, B, C \) are all finite-rank operators, there exists a finite-dimensional subspace \( \mathcal{H}_n \subset \mathcal{H} \), with \( \dim(\mathcal{H}_n) = n \) for some \( n \in \mathbb{N} \), such that \( \text{range}(A) \subset \mathcal{H}_n \), \( \text{range}(B) \subset \mathcal{H}_n \), and \( \text{range}(C) \subset \mathcal{H}_n \). Let

\[
A_n = A|_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_n, \quad B_n = B|_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_n, \quad C_n = C|_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_n.
\]

Then \( A_n, B_n, C_n \) are linear operators on the finite-dimensional space \( \mathcal{H}_n \) and thus are represented by \( n \times n \) matrices, which we denote by the same symbols. We have

\[
(I + A_n)(I + B_n)^{-1} = (I + A_n)[I - B_n(I + B_n)^{-1}]
\]

\[
= I + A_n - B_n(I + B_n)^{-1} - A_n B_n(I + B_n)^{-1},
\]

\[
(I + A)(I + B)^{-1} = I + A - B(I + B)^{-1} - AB(I + B)^{-1},
\]

where \( A - B(I + B)^{-1} - AB(I + B)^{-1} \) is of finite rank, since both \( A \) and \( B \) are, with range in \( \mathcal{H}_n \). It is clear that

\[
[A - B(I + B)^{-1} - AB(I + B)^{-1}]|_{\mathcal{H}_n} = A_n - B_n(I + B_n)^{-1} - A_n B_n(I + B_n)^{-1}.
\]

Thus the nonzero eigenvalues of \((I + A)(I + B)^{-1} - I = [A - B(I + B)^{-1} - AB(I + B)^{-1}]\) and \((I + A_n)(I + B_n)^{-1} - I = [A_n - B_n(I + B_n)^{-1} - A_n B_n(I + B_n)^{-1}]\)
are the same. It follows that
\[ D(\alpha, \alpha) = \frac{1}{\alpha^2} \log \det \left[ \frac{[(I + A)(I + B)^{-1}]^\alpha + [(I + A + B)^{-1}]^\alpha}{2} \right] \]

Similarly, we have
\[ D(\alpha, \alpha) = \frac{1}{\alpha^2} \log \det \left[ \frac{[(I + A_n)(I + B_n)^{-1}]^\alpha + [(I + A_n + B_n)^{-1}]^\alpha}{2} \right] = D(\alpha, \alpha) \]

Applying the triangle inequality from the finite-dimensional setting [15], we get
\[ D(\alpha, \alpha) \leq D(\alpha, \alpha) + D(\alpha, \alpha) \]

Together with the above expressions, this gives us the final result.

**Proof of Theorem 19 (Convergence in trace norm).** Let \( I + \Lambda = (I + A)^{-1/2}(I + A)(I + B)^{-1/2} \) and \( I + \Lambda_n = (I + A_n)^{-1/2}(I + A_n)(I + B_n)^{-1/2} \), with \( \Lambda, \Lambda_n \in \text{Sym}(\mathcal{H}) \cap \text{Tr}(\mathcal{H}) \).

By Lemma 19, we have \( \lim_{n \to \infty} ||\Lambda_n - \Lambda||_\text{tr} = 0 \).

Thus by Theorem 30, we have
\[ \lim_{n \to \infty} ||(I + \Lambda_n)^\alpha - (I + \Lambda)^\alpha||_\text{tr} = 0 \ \forall \alpha \in \mathbb{R} \]

By Definition 5, we have
\[ D(\alpha, \alpha) = \frac{1}{\alpha^2} \log \det \left[ \frac{(I + \Lambda_n)^\alpha + (I + \Lambda_n)^{-\alpha}}{2} \right] \]

Taking limit as \( n \to \infty \) and applying the continuity of the Fredholm determinant in the trace norm (e.g. Theorem 3.5 in [22]), we obtain
\[ \lim_{n \to \infty} D(\alpha, \alpha) = \frac{1}{\alpha^2} \log \det \left[ \frac{(I + \Lambda)^\alpha + (I + \Lambda)^{-\alpha}}{2} \right] = D(\alpha, \alpha) \]
This completes the proof. \hfill \square

**Proof of Theorem 20 (Triangle inequality).** For a fixed $\gamma > 0$, we have
\[
D_{2\alpha}^{(\alpha,\alpha)}[(A + \gamma I), (B + \gamma I)] = \frac{1}{\alpha^2} \log \det X \left( \frac{[(A + \gamma I)(B + \gamma I)^{-1}]^\alpha + (A + \gamma I)(B + \gamma I)^{-1\alpha}}{2} \right) = \frac{1}{\alpha^2} \log \det \left( \frac{[(A + I)(B + I)^{-1}]^\alpha + (A + I)(B + I)^{-1\alpha}}{2} \right),
\]
which thus reduces to the case $\gamma = 1$. Thus it suffices for us to prove in triangle inequality for $\gamma = 1$.

Let $\{A_n\}_{n \in \mathbb{N}}$, $\{B_n\}_{n \in \mathbb{N}}$, and $\{C_n\}_{n \in \mathbb{N}}$ be sequences of finite-rank operators such that
\[
\lim_{n \to \infty} \|A_n - A\|_{\text{tr}} = 0, \quad \lim_{n \to \infty} \|B_n - B\|_{\text{tr}} = 0, \quad \lim_{n \to \infty} \|C_n - C\|_{\text{tr}} = 0.
\]
By Lemma 20, we have the triangle inequality
\[
\sqrt{D_{2\alpha}^{(\alpha,\alpha)}[(I + A_n), (I + B_n)]} \leq \sqrt{D_{2\alpha}^{(\alpha,\alpha)}[(I + A_n), (I + C_n)]} + \sqrt{D_{2\alpha}^{(\alpha,\alpha)}[(I + C_n), (I + B_n)]}.
\]
Taking limits on both side as $n \to \infty$ and invoking Theorem 19, we then obtain
\[
\sqrt{D_{2\alpha}^{(\alpha,\alpha)}[(I + A), (I + B)]} \leq \sqrt{D_{2\alpha}^{(\alpha,\alpha)}[(I + A), (I + C)]} + \sqrt{D_{2\alpha}^{(\alpha,\alpha)}[(I + C), (I + B)]}.
\]
This completes the proof of the theorem. \hfill \square

**Proof of Theorem 4 (Metric property).** The case $\alpha = 0$ corresponds to the affine-invariant Riemannian distance on the Hilbert manifold $\Sigma(\mathcal{H})$ [18], which is still a metric when restricted to $\text{PTr}(\mathcal{H})$.

Consider the case $\alpha > 0$. The positivity and symmetry of the divergence $D_{2\alpha}^{(\alpha,\alpha)}[(A + \gamma I), (B + \gamma I)]$ are from Theorems 1 and 13, respectively. The triangle inequality for $\sqrt{D_{2\alpha}^{(\alpha,\alpha)}[(A + \gamma I), (B + \gamma I)]}$ is from Theorem 20. Thus $\sqrt{D_{2\alpha}^{(\alpha,\alpha)}[(A + \gamma I), (B + \gamma I)]}$ is a metric on $\text{PTr}(\mathcal{H})(\gamma)$. \hfill \square
**Proof of Theorem 22 (Diagonalization).** Consider first the case $\alpha > 0$. As in the proof of Theorem 20, it suffices for us to prove this theorem for the case $\gamma = 1$. Let $A = \sum_{j=1}^{\infty} \lambda_j(A) \phi_j \otimes \phi_j$ denote the spectral decomposition for $A$. For each $n \in \mathbb{N}$, define

$$A_n = \sum_{j=1}^{n} \lambda_j(A) \phi_j \otimes \phi_j.$$

Then $A_n$ is a finite-rank operator with the eigenvalues being the first $n$ eigenvalues of $A$ and $\lim_{n \to \infty} \|A_n - A\|_{tr} = 0$. In the same way, we construct a sequence of finite-rank operators $B_n$ with $\lim_{n \to \infty} \|B_n - B\|_{tr} = 0$. By construction, we also have

$$\lim_{n \to \infty} \|\text{Eig}(A_n) - \text{Eig}(A)\|_{tr} = 0, \quad \lim_{n \to \infty} \|\text{Eig}(B_n) - \text{Eig}(B)\|_{tr} = 0.$$

Thus by Theorem 19, we have

$$\lim_{n \to \infty} D_{2\alpha}^{(\alpha, \alpha)}[(\text{Eig}(A_n) + I), (\text{Eig}(B_n) + I)] = D_{2\alpha}^{(\alpha, \alpha)}[(\text{Eig}(A) + I), (\text{Eig}(B) + I)],$$

$$\lim_{n \to \infty} D_{2\alpha}^{(\alpha, \alpha)}[(A_n + I), (B_n + I)] = D_{2\alpha}^{(\alpha, \alpha)}[(A + I), (B + I)].$$

Since $A_n, B_n$ can be identified with finite-dimensional matrices, as in the proof of Lemma 16, we can apply the corresponding finite-dimensional result in [15] to obtain

$$D_{2\alpha}^{(\alpha, \alpha)}[(\text{Eig}(A_n) + I), (\text{Eig}(B_n) + I)] \leq D_{2\alpha}^{(\alpha, \alpha)}[(A_n + I), (B_n + I)].$$

Thus taking limits as $n \to \infty$ gives

$$D_{2\alpha}^{(\alpha, \alpha)}[(\text{Eig}(A) + I), (\text{Eig}(B) + I)] \leq D_{2\alpha}^{(\alpha, \alpha)}[(A + I), (B + I)].$$

Letting $\alpha \to 0$ on both sides of the above expression, we also obtain the result for the case $\alpha = 0$. This completes the proof of the theorem.

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