A Fixed Point Conjecture

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Abstract

Inverse limits, unlike direct limits, can in general be void, [1]. The existence of fixed points for arbitrary mappings \( T : X \rightarrow X \) is conjectured to be equivalent with the fact that related direct limits of all finite partitions of \( X \) are not void.

1. The Setup

Let \( X \) be a nonvoid set and \( T : X \rightarrow X \) a mapping. We denote by

\[
\mathcal{FP}(X)
\]

the set of all finite partitions of \( X \).

Given \( x \in X \) and \( \Delta \in \mathcal{FP}(X) \), then obviously

\[
\exists A \in \Delta : \{ n \in \mathbb{N}_+ | T^n(x) \in A \} \text{ is infinite}
\]

Here and in the sequel, we use the notation \( \mathbb{N} = \{0,1,2,3,\ldots\} \) and \( \mathbb{N}_+ = \{1,2,3,\ldots\} \).

Let us therefore denote
(3) \[ \Delta(x) = \{ A \in \Delta \mid \{ n \in \mathbb{N}_+ \mid T^n(x) \in A \} \text{ is infinite} \} \]

In view of (2) we obtain

(4) \[ \Delta(x) \neq \phi \]

In setting up the fixed point conjecture, it is useful to consider the following two simple instances.

**Example 1**

1) Let \( T = id_X \), that is, the *identity* mapping on \( X \). Then for \( x \in X \) and \( A \subseteq X \), we clearly have

(5) \[ \{ n \in \mathbb{N}_+ \mid T^n(x) \in A \} = \begin{cases} \phi & \text{if } x \notin A \\ \mathbb{N}_+ & \text{if } x \in A \end{cases} \]

hence for \( \Delta \in \mathcal{FP}(X) \), we obtain

(6) \[ \Delta(x) = \{ A \} , \text{ where } x \in A \]

2) Let \( T \) be a *constant* mapping on \( X \), that is, \( T(x) = c \), for \( x \in X \), where \( c \in X \) is given. Then for \( x \in X \) and \( A \subseteq X \), we clearly have

(7) \[ \{ n \in \mathbb{N}_+ \mid T^n(x) \in A \} = \begin{cases} \phi & \text{if } c \notin A \\ \mathbb{N}_+ & \text{if } c \in A \end{cases} \]

hence for \( \Delta \in \mathcal{FP}(X) \), we obtain

(8) \[ \Delta(x) = \{ A \} , \text{ where } c \in A \]

Let us recall now the following natural partial order structure on \( \mathcal{FP}(X) \) given by the concept of *refinement* of partitions. Namely, if \( \Delta, \Delta' \in \mathcal{FP}(X) \), then we denote

(9) \[ \Delta \leq \Delta' \]
if and only if

\( \forall A' \in \Delta' : \exists A \in \Delta : A' \subseteq A \)

and in such a case, we define the mapping

\( \psi_{\Delta', \Delta} : \Delta' \rightarrow \Delta \)

by

\( A' \subseteq A = \psi_{\Delta', \Delta}(A') \)

Then obviously

\( \psi_{\Delta', \Delta}(\Delta'(x)) \subseteq \Delta(x) \)

Indeed, if \( A' \in \Delta'(x) \), then (3) gives

\( \{ n \in \mathbb{N}_+ | T^n(x) \in A' \} \) is infinite

but in view of (10), we have

\( A' \subseteq \psi_{\Delta', \Delta}(A') \)

hence

\( \{ n \in \mathbb{N}_+ | T^n(x) \in \psi_{\Delta', \Delta}(A') \} \) is infinite

thus (13).

We note now that the finite partitions

\( \Delta \in \mathcal{FP}(X) \)

together with the mappings

\( \psi_{\Delta', \Delta}, \Delta, \Delta' \in \mathcal{FP}(X), \Delta \leq \Delta' \)
form an *inverse* family, [1, p.191].

Furthermore, in view of (13), we also have the following stronger version of the above. For every \( x \in X \), let us define the mappings

\[
\psi_{\Delta', \Delta, x} : \Delta'(x) \rightarrow \Delta(x), \quad \Delta, \Delta' \in \mathcal{FP}(X), \Delta \leq \Delta'
\]

by

\[
\psi_{\Delta', \Delta, x} = \psi_{\Delta', \Delta|_{\Delta'(x)}}
\]

Then again, for every \( x \in X \), we obtain the *inverse* family

\[
\Delta \in \mathcal{FP}(X)
\]

\[
\psi_{\Delta', \Delta, x} : \Delta'(x) \rightarrow \Delta(x), \quad \Delta, \Delta' \in \mathcal{FP}(X), \Delta \leq \Delta'
\]

Consequently, for each \( x \in X \), we can consider the *inverse limit*

\[
\lim_{\Delta \in \mathcal{FP}(X)} \Delta(x)
\]

Returning now to the two simple instances in Example 1 above, we further have

**Example 2**

1) In the case 1) of Example 1, it follows easily that, for \( x \in X \), we have

\[
\lim_{\Delta \in \mathcal{FP}(X)} \Delta(x) = \{ (\xi_\Delta \mid \Delta \in \mathcal{FP}(X)) \}
\]

where

\[
\xi_\Delta = x, \quad \Delta \in \mathcal{FP}(X)
\]

2) In the case 2) of Example 1, for \( x \in X \), we easily obtain
lim_{\Delta \in \mathcal{F}(X)} \Delta(x) = \begin{cases} \emptyset & \text{if } x \neq c \\ \{ (\xi_\Delta \mid \Delta \in \mathcal{F}(X)) \} & \text{if } x = c \end{cases}

where

\begin{equation}
\xi_\Delta = c, \quad \Delta \in \mathcal{F}(X)
\end{equation}

**Remark 1**

At this stage, an important fact to note is that, in general, an inverse limit such as in (20) may be void, [1, (c) in Exercise 4, p. 252], even if none of the sets \( \Delta(x) \) is void, and each the mappings \( \psi_{\Delta, \Delta, x} \) is surjective. Therefore, the relations (21) and (23), even if easy to establish, are as inverse limits nontrivial, in view of the arbitrariness of the sets \( X \) and mappings \( T \) involved.

A common feature of the mappings \( T : X \to X \) in both cases above is that they have fixed points. Namely, for the identity mapping \( T = id_X \), each point \( x \in X \) is such a fixed point, while for the constant mapping \( T = c \), the point \( x = c \in X \) is the only fixed point.

Further, as suggested by Scott Kominers, Daniel Litt and Brett Harrison, in view of the fact that a fixed point of a mapping \( T : X \to X \) is but a particular case of a periodic point of that mapping, or equivalently, of a fixed point of the mapping \( T^n \), for some \( n \in \mathbb{N}_+ \), we are led to the

**Conjecture**

Given a nonvoid set \( X \) and a mapping \( T : X \to X \), then for \( x \in X \), we have

\begin{equation}
\lim_{\Delta \in \mathcal{F}(X)} \Delta(x) \neq \emptyset \iff ( \exists \ n \in \mathbb{N}_+ : T^n(x) = x )
\end{equation}

where the issue is whether the implication \( \implies \) holds, since the converse implication is easy to establish.
References

[1] Bourbaki N: Elements of Mathematics, Theory of Sets. Springer, 2004