Extended Newman-Janis algorithm and rotating and Kerr
Newman de Sitter (anti de Sitter) metrics

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Abstract. The Newman-Janis algorithm is well known to provide rotating black holes solutions to Einstein’s equations from static seeds, through a complexification of a radial and a time coordinates. However, an ambiguity remains for the replacement of the $r^{-1}$ and $r^{-2}$ powers of the radial coordinate. We show here that the two cases are unified by a simple expression which allows its extension to the $r^2$ power, characteristic of the de Sitter (dS) and anti de Sitter (AdS) spacetimes. The formula leads almost automatically to the Kerr and Kerr-Newman-dS and -AdS metrics.

Keywords: Newman-Janis algorithm, rotating (anti) de Sitter metrics, Kerr (anti) de Sitter metrics

1. Introduction

A usual criticism to the Newman-Janis algorithm (NJA) [1] for generating rotating metrics from seed static ones, is its apparent arbitrariness [2] in the replacement of powers or products of the complexified radial coordinate $r$ and its complex conjugate $\bar{r}$. So, $\frac{1}{r}$ is replaced by $\frac{1}{2}(\frac{1}{r} + \frac{1}{\bar{r}})$ and $\frac{1}{r^2}$ by $\frac{1}{|r|^2}$. The de Sitter (dS) [3] and anti de Sitter (AdS) [4] metrics have an $r^2$ in the numerator (see eqs.(7) and (18)) and it is not clear, in this case, which would be the correct replacement.

It is easy to see that after the complexification

$$\mathbb{R} \ni r \rightarrow \mathbb{C} \ni z = r' - ib\cos\theta'$$

with $r' \in (-\infty, +\infty)$, $\theta' = \theta \in [0, \pi]$, and $b = \text{const.} > 0$ to be interpreted as the rotation parameter (angular momentum/unit gravitational mass in the Kerr-de Sitter (KdS), Kerr-anti de Sitter (KAdS), Kerr-Newman de Sitter (KNdS) and Kerr-Newman anti de Sitter (KNAdS) cases),

$$r^p \rightarrow \frac{(Re(r))^{p+2}}{|r|^2} = \frac{r'^{p+2}}{r'^2 + b^2 \cos^2\theta'}$$

reproduces, for $p = -1, -2$, the above replacements:

$$r^{-1} \rightarrow \frac{Re(r)}{|r|^2} = \frac{r'}{|r|^2} = \frac{1}{2}(\frac{1}{r} + \frac{1}{\bar{r}});$$

$$r^{-2} \rightarrow \frac{1}{|r|^2} = \frac{1}{r\bar{r}}.$$  

This suggests the extension to the case $p = 2$:

$$r^2 \rightarrow \frac{(Re(r))^4}{|r|^2} = \frac{r'^4}{|r|^2};$$

which, as we will see, leads to the metrics for $r dS$ (rotating de Sitter) [5] and $r AdS$ (rotating anti de Sitter), and also for $KdS$, $KAdS$, $KNdS$, and $KNAdS$ [6].

In what follows all spacetimes are 4-dimensional, and all quantities are expressed in geometrical units.
2. Schwarzschild anti de Sitter metric $SAdS$

The $SAdS$ metric is given by

$$ds_{SAdS}^2 = f_{SAdS} dt^2 - f_{SAdS}^{-1} dr^2 - r^2 d\Omega_2^2$$

with

$$f_{SAdS} = 1 - \frac{2M}{r} + \frac{r^2}{a^2},$$

where $r > 0$, $t \in (-\infty, +\infty)$, $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2$, $\varphi \in [0, 2\pi)$, $M$ is the mass of the Schwarzschild black hole and $a$ is the curvature radius of the $AdS$ space, corresponding to an attractive cosmological constant

$$\Lambda_{AdS} = -\frac{3}{a^2}. (8)$$

$f_{SAdS}(r)$ has no extrema for positive $r$, while

$$f_{SAdS} \to \{+\infty, \quad r \to +\infty\}, \quad -\infty, \quad r \to 0_+. \quad (7)$$

Then, it has a unique zero which is the position of the horizon:

$$r_h(M, a) = (Ma^2)^{1/3}((1 + \sqrt{1 + \frac{a^2}{27M^2}})^{1/3} + (1 - \sqrt{1 + \frac{a^2}{27M^2}})^{1/3}), \quad f_{SAdS}(r_h) = 0. \quad (9)$$

An expansion in $\frac{M}{a}$ (typically $\ll 1$) gives the deviation of $r_h$ from the Schwarzschild value $2M$:

$$r_h = 2M(1 - \sqrt[3]{\frac{M}{a}}) \to 2M \text{ as } a \to +\infty. \quad (10)$$

The surface gravity $\kappa_{SAdS}$ at $r_h$ can be obtained from the calculation of the 4-accelerations of static observers, or through the use of the Rindler approximation in the neighborhood of the horizon, $r = r_h + \frac{\alpha}{r_h} \rho^2$ [7], with $\alpha \in \mathbb{R}$, and neglecting terms of $O(\rho^4)$; the result is

$$\kappa_{SAdS} = \frac{M}{r_h} + \frac{r_h}{a^2} \to \frac{1}{4M} = \kappa_S \text{ as } a \to +\infty. \quad (11)$$

With the choice $\alpha = \frac{1}{2} \kappa_{SAdS} r_h$, the time-radial part of the metric is the Rindler metric:

$$ds_{SAdS}(\rho)|_{time-radial} = (\kappa_{SAdS} \rho)^2 dt^2 - d\rho^2. \quad (12)$$

The Hawking temperature at $r_h$ is given by

$$T_h = \frac{\kappa_{SAdS}}{2\pi}. \quad (13)$$

(For the global embedding Minkowskian spacetime (GEMS) approach to this calculation, see ref. [8].)

In Eddington-Finkelstein retarded coordinates $(u, r, \theta, \varphi)$ [9], with

$$dt = du + \frac{dr}{f_{SAdS}}, \quad (14)$$

$u \in (-\infty, +\infty)$ and $r, \theta, \varphi$ as before, the $SAdS$ metric is

$$ds_{SAdS}^2 = f_{SAdS} du^2 + 2du dr - r^2 d\Omega_2^2. \quad (15)$$
The anti de Sitter metric is obtained setting $M = 0$ i.e. with
\[ f_{\text{AdS}} = 1 + \frac{r^2}{a^2}. \tag{16} \]

3. Schwarzschild de Sitter metric $SdS$

The $SdS$ metric is given by
\[ ds^2_{SdS} = f_{SdS} dt^2 - f_{SdS}^{-1} dr^2 - r^2 d\Omega_2^2 \tag{17} \]
with
\[ f_{SdS}(r) = 1 - \frac{2M}{r} - \frac{r^2}{a^2}, \tag{18} \]
where now $a$ is the curvature radius of the $dS$ space corresponding to a repulsive cosmological constant
\[ \Lambda_{dS} = + \frac{3}{a^2}. \tag{19} \]

Depending on the relation between $M$ and $a$ the $SdS$ metric has no horizon, one horizon, or two horizons. We shall discuss the latest case: $r_-$: black hole horizon, and $r_+$: cosmological horizon, which occur for
\[ \frac{M}{a} < \frac{1}{3\sqrt{3}} \iff M \sqrt{\Lambda_{dS}} < \frac{1}{3}. \tag{20} \]

$r_{\pm}$ are given by the two positive real zeros of (18):
\[ r_- = \frac{2a}{\sqrt{3}} \cos(\varphi_0 + \frac{4\pi}{3}), \tag{21} \]
\[ r_+ = \frac{2a}{\sqrt{3}} \cos(\varphi_0), \tag{22} \]
with
\[ \varphi_0 = \frac{1}{3} \arccos\left(\frac{-3\sqrt{3}M}{a}\right) \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right). \tag{23} \]

Clearly, $r_- < r_+$. At $r_0 = (Ma^2)^{1/3}$, with $r_- < r_0 < r_+$, $f_{SdS}$ has a relative maximum
\[ f_{SdS}(r_0) = 1 - 3\left(\frac{M}{a}\right)^{2/3} > 0. \tag{24} \]
(An absolute maximum is $+\infty$, but occurs for $r \to 0_-$.)

The Rindler approximations outside but close to the black hole horizon: $r = r_- + \frac{\kappa_{\text{SdS}-}}{2} \rho^2 + O(\rho^4)$, and inside but close to the cosmological horizon $r = r_+ + \frac{\kappa_{\text{SdS}+}}{2} \rho^2 + O(\rho^4)$, allow us to compute the surface gravities:
\[ \kappa_{\text{SdS}-} = -\frac{M}{r_-^2} + \frac{r_-}{a^2} < 0, \tag{25} \]
\[ \kappa_{\text{SdS}+} = -\frac{M}{r_+^2} + \frac{r_+}{a^2} > 0, \tag{26} \]
with
\[ ds^2_{SdS}(\rho)|_{\text{time-radial±}} = (\kappa_{\text{SdS}±(\rho)})^2 dt^2 - d\rho^2. \tag{27} \]
The Hawking temperatures at $r_{\pm}$ are given by

$$ T_{\pm} = \frac{|\kappa_{SDS}|}{2\pi}, \quad (28) $$

In Eddington-Finkelstein retarded coordinates,

$$ ds^2_{SdS} = f_{SdS}du^2 + 2dudr - r^2d\Omega_2^2. \quad (29) $$

The de Sitter metric is obtained from (29) setting $M = 0$, i.e. replacing $f_{SdS}$ by

$$ f_{dS} = 1 - \frac{r^2}{a^2}. \quad (30) $$

4. Rotating - de Sitter ($rdS$) and - anti de Sitter ($rAdS$) metrics

We can unify the treatments of both metrics if we denote $ds^2_{dS}$ and $ds^2_{AdS}$ by

$$ ds^2_{\Lambda} = f_{\Lambda}du^2 + 2dudr - r^2d\Omega_2^2 \quad (31) $$

where

$$ f_{\Lambda} = 1 - \frac{\Lambda r^2}{3} \quad (32) $$

with $\Lambda = \Lambda_{dS}$ given by (19) and $\Lambda = \Lambda_{AdS}$ given by (8). The metric corresponding to (31) is given by

$$ g_{\mu\nu} {\Lambda} = \begin{pmatrix} f_{\Lambda} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \quad (33) $$

with inverse

$$ g^{\mu\nu} {\Lambda} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ f_{\Lambda} & 0 & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & -r^{-2}\sin^{-2}\theta \end{pmatrix} \quad (34) $$

It is easily verified that the inverse metric defined by

$$ \tilde{g}^{\mu\nu} = \left( l^{\mu}n^{\nu} + l^{\nu}n^{\mu} \right) - \left( m^{\mu}\bar{m}^{\nu} + m^{\nu}\bar{m}^{\mu} \right), \quad (35) $$

where $(l, n, m, \bar{m})$ is the null tetrad given by

$$ l^{\mu} = (0, 1, 0, 0), \quad n^{\mu} = (1, -\frac{f_{\Lambda}}{2}, 0, 0), \quad m^{\mu} = \frac{1}{\sqrt{2r}}(0, 0, 1, -\frac{i}{\sin\theta}), \quad \bar{m}^{\mu} = \frac{1}{\sqrt{2r}}(0, 0, 1, -\frac{i}{\sin\theta}), \quad (36) $$

with scalar products (with respect to $g_{\mu\nu} {\Lambda} )$

$$ \begin{pmatrix} l \ n \ m \ \bar{m} \\ l & 0 & 1 & 0 & 0 \\ n & 1 & 0 & 0 & 0 \\ m & 0 & 0 & 0 & -1 \\ \bar{m} & 0 & 0 & -1 & 0 \end{pmatrix} \quad (37) $$
reproduces \( g_{\mu\nu}^{dS} \) i.e.

\[
\tilde{g}_{\mu\nu} = g_{\mu\nu}^{dS}.
\]

The complexification given by (1), together with

\[
\mathbb{R} \ni u \rightarrow \mathbb{C} \ni u = u' + ib\cos\theta', u' \in (-\infty, +\infty),
\]

\( \varphi' = \varphi \), and the prescriptions (4) and (5), lead to the transformed tetrad

\[
l'' = \delta_{\mu}^\nu, \quad n'' = \delta_{\nu}^\mu - \frac{1}{2} f_r dS_{AdS} \delta_{\nu}^\mu = (1, -\frac{1}{2} f_r dS_{AdS}, 0, 0),
\]

\[
m'' = m'^\mu, \quad \tilde{m}'' = \tilde{m}'^\mu,
\]

with

\[
f_r dS = 1 - \frac{\Lambda r'^4}{3\Sigma}, \quad \Sigma = r'^2 + b^2 \cos^2 \theta',
\]

and inverse metric

\[
\tilde{g}''_{\mu\nu} = g''_{\mu\nu}^{dS} \equiv g''_{\mu\nu}^{r AdS} = (l''_{\mu} n''_{\nu} + l''_{\nu} n''_{\mu}) - (m''_{\mu} \tilde{m}''_{\nu} + m''_{\nu} \tilde{m}''_{\mu})
\]

\[
= \begin{pmatrix}
\frac{b^2 \sin^2 \theta'}{\Sigma} & 0 & -\frac{b}{\Sigma} \\
0 & \frac{r'^2 + b^2}{\Sigma} - \Lambda r'^4 / 3 & 0 \\
-\frac{b}{\Sigma} & 0 & \frac{b}{\Sigma}
\end{pmatrix}.
\]

Its inverse gives the rdS and the rAdS metrics:

\[
g_{\mu\nu}^{rdS} (u', r', \theta', \varphi') = \begin{pmatrix}
1 - \frac{\Lambda r'^4}{3\Sigma} & 1 & 0 & \frac{b\Lambda \sin^2 \theta' - r'^4}{\Sigma} \\
0 & 0 & -\Lambda \Sigma & 0 \\
\cdot & -\Sigma & 0 & -\frac{b\sin^2 \theta'}{\Sigma} A
\end{pmatrix}
\]

with

\[
A = (r'^2 + b^2) - b^2 \sin^2 \theta' (r'^2 + b^2 - \Lambda r'^4 / 3).
\]

It is interesting to observe that under the interchange \( \frac{\Lambda r'^4}{3} \leftrightarrow 2M \), the Kerr and the rAdS metrics go into each other i.e.

\[
g_{\mu\nu}^{rdS} (u', r', \theta', \varphi') \xrightarrow{\Lambda r'^4 \leftrightarrow 2M} g_{\mu\nu}^{rAdS} (u', r', \theta', \varphi').
\]

5. Kerr-de Sitter (KdS) and Kerr-anti de Sitter (KAdS) metrics
The same complexification and change of coordinates and tetrads used in section 4., produce the change

\[
\frac{fS}{AdS} \dS = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} \quad \rightarrow \quad \frac{fK}{AdS} \dS = 1 - \frac{2Mr'}{\Sigma} - \frac{\Lambda r'^4}{3\Sigma}
\]

and the inverse Kerr-de Sitter (anti de Sitter) metrics

\[
g_{\mu\nu}^{KN} \dS (u', r', \theta', \varphi') = \begin{pmatrix}
\frac{-b^2\sin^2\theta}{\Sigma} & \frac{-r'^2+b^2}{\Sigma} & 0 & -\frac{b}{\Sigma} \\
0 & 0 & \frac{b}{\Sigma} & 0 \\
0 & \frac{b}{\Sigma} & \frac{-1}{\Sigma} & 0 \\
0 & 0 & 0 & -\frac{1}{\Sigma\sin^2\theta'}
\end{pmatrix}
\]

with inverse

\[
g_{\mu\nu}^{KN} \dS (u', r', \theta', \varphi') = \begin{pmatrix}
1 - \frac{2Mr' + \Delta r'^4}{\Sigma} & 1 & 0 & \frac{b\sin^2\theta'}{\Sigma}(2Mr' + \Delta r'^4) \\
0 & 0 & \frac{-b\sin^2\theta}{\Sigma} & 0 \\
0 & \frac{-b\sin^2\theta}{\Sigma} & \frac{-1}{\Sigma} & 0 \\
0 & 0 & 0 & \frac{-1}{\Sigma\sin^2\theta'} A_K
\end{pmatrix}
\]

with

\[A_K = (r'^2 + b^2)^2 - b^2\sin^2\theta'(r'^2 + b^2 - 2Mr' - \Lambda r'^4).\]

The inverse metric (48) is nothing but the inverse metric (43) with the addition of the term \(-2Mr'\) in the numerator of \(-g'^{\mu}\).

6. Kerr-Newman-de Sitter (KNdS) and Kerr-Newman-anti de Sitter (KNAdS) metrics

6.1. Eddington-Finkelstein coordinates

Incorporating to \(fS_{AdS}\) the Reissner-Nordstrom (RN) term \(\frac{Q^2}{r^2}\), \(Q^2 = g^2 + q^2\) with \(q\): electric charge and \(q\): magnetic Dirac charge, \([q] = [g] = [L]\) in geometric units, defines

\[
f_{RN} \dS = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}.
\]

To simplify, take \(g = 0\); then the gauge potential 1-form associated to \(q\) is

\[A = \frac{q}{r} dt = \frac{q}{r} (du + \frac{dr}{f_{RN} \dS}) = A_u(r) du + A_r(r) dr.
\]

The gauge transformation \([11] A'_\mu(r) = A_\mu(r) + \partial_\mu \psi(r)\) allows us to fix \(A'_r(r) = 0\) with \(A'_u(r) = A_u(r) = \frac{q}{r}\) and contravariant components

\[(A^u, A^r, A^\theta, A^\varphi) = (0, \frac{q}{r}, 0, 0) = \frac{q}{r} \ell^u.\]

Then, following the same strategy as in section 5., one obtains the inverse Kerr-Newman-de Sitter (anti de Sitter) metrics

\[
g_{\mu\nu}^{KN} \dS (u', r', \theta', \varphi') = \begin{pmatrix}
\frac{b^2\sin^2\theta}{\Sigma} & \frac{-r'^2+b^2}{\Sigma} & 0 & -\frac{b}{\Sigma} \\
0 & 0 & \frac{b}{\Sigma} & 0 \\
0 & \frac{b}{\Sigma} & \frac{-1}{\Sigma} & 0 \\
0 & 0 & 0 & -\frac{1}{\Sigma\sin^2\theta'}
\end{pmatrix}
\]
with inverse
\[
g^{\mu\nu}_{\text{KN} dS_{AdS}}(u', r', \theta', \varphi') = \begin{pmatrix} 
1 - \frac{2Mr' - Q^2 + \Lambda r'^4}{\Sigma} & 1 & 0 & \frac{\text{bsin}^2\theta'}{\Sigma} \\
0 & 0 & 0 & -\frac{\text{bsin}^2\theta'}{\Sigma} \\
\cdot & -\Sigma & -\Sigma & \cdot \\
\cdot & \cdot & \cdot & -\Sigma \end{pmatrix}
\]
and
\[
A_{KN} = (r'^2 + b^2)^2 - b^2\text{sin}^2\theta' (r'^2 + b^2 - 2Mr' + Q^2 - \frac{\Lambda r'^4}{3}),
\]
and the gauge vector \( A'_{\mu} = \frac{q_{\mu'}}{\Sigma} \delta_{\mu'} \) with covariant components \( A'_{\mu} = g^{\mu\nu}_{\text{KN} dS_{AdS}} A'_{\nu} = \frac{q_{\mu'}}{\Sigma} (1, 0, 0, -\text{bsin}^2\theta') \)
i.e.
\[
A' = \frac{q_{\mu'}}{\Sigma} (du' - \text{bsin}^2\theta' d\varphi').
\]

6.2. Boyer-Lindquist (B-L) coordinates

The B-L coordinates \((t, r, \theta, \phi)\) [10] are defined by
\[
du' = dt + x dr, \quad d\varphi' = d\phi + y dr, \quad r' = r, \quad \theta' = \theta
\]
with the condition that the coefficients of \(dt dr\) and \(dr d\phi\) vanish. The result is
\[
x = -\frac{r^2 + b^2}{\Delta}, \quad y = -\frac{b}{\Delta}
\]
with
\[
\Delta = r^2 + b^2 - 2Mr + Q^2 - \frac{\Lambda r^4}{3},
\]
and the metric
\[
g^{\mu\nu}_{\text{KN} dS_{AdS}}(t, r, \theta, \phi) = \begin{pmatrix} 
\frac{\Delta - b^2\text{sin}^2\theta}{\Sigma} & 0 & 0 & \frac{\text{bsin}^2\theta}{\Sigma} (r^2 + b^2 - \Delta) \\
\cdot & -\frac{\Delta}{\Sigma} & 0 & 0 \\
\cdot & \cdot & -\Sigma & 0 \\
\cdot & \cdot & \cdot & -\frac{\text{bsin}^2\theta}{\Sigma} (r^2 + b^2 - b^2\text{sin}^2\theta \Delta) \\
\end{pmatrix},
\]
with inverse
\[
g^{\mu\nu}_{\text{KN} dS_{AdS}}(t, r, \theta, \phi) = \begin{pmatrix} 
\frac{r^2 + b^2 + b^2\text{sin}^2\theta \cdot (2Mr - Q^2 + \Lambda r^4)}{\Sigma} & 0 & 0 & \frac{b(2Mr - Q^2 + \Lambda r^4)}{\Sigma \Delta} \\
\cdot & -\frac{\Delta}{\Sigma} & 0 & 0 \\
\cdot & \cdot & -\Sigma & 0 \\
\cdot & \cdot & \cdot & -\Delta - b^2\text{sin}^2\theta \\
\end{pmatrix}
\]

From these expressions, following the lines of the diagram (62) below, one obtains the \(B - L\) form of the metrics for the indicated spaces:
where \( rQ(A)dS \) is a rotating charged anti de Sitter or de Sitter cosmological universe, \( KN \) is the Kerr-Newman metric, and \( Mink \) denotes Minkowski space.

In \( B - L \) coordinates, the gauge potential is given by

\[
A' = \frac{qr}{\Sigma} (dt - \frac{\Sigma}{\Delta} dr - \sin^2 \theta d\phi).
\]

(63)

Again, the term proportional to \( dr \) can be set equal to zero through a gauge transformation since \( A'_r = \frac{qr}{\Sigma} \Delta(r) = A'_r(r) \), and one ends with the usual form

\[
A' = \frac{qr}{\Sigma} (dt - \sin^2 \theta d\phi).
\]

(64)

For the electromagnetic (electric) field tensor one has:

**Covariant components**:

\[
F_{\mu \nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & F_{tr} & F_{t\theta} & 0 \\ -F_{tr} & 0 & 0 & F_{r\phi} \\ -F_{t\theta} & 0 & 0 & F_{\theta\phi} \\ 0 & -F_{r\phi} & -F_{\theta\phi} & 0 \end{pmatrix},
\]

(65)

with

\[
F_{tr} = \frac{2r^2 - \Sigma}{\Sigma^2}, \quad F_{t\theta} = -\frac{qb^2r \sin(2\theta)}{\Sigma^2}, \quad F_{r\phi} = b \sin^2 \theta F_{tr}, \quad F_{\theta\phi} = -\frac{qb \sin(2\theta)(r^2 + b^2)}{\Sigma^2}.
\]

(66)

\[
[F_{tr}] = [L]^{-1}, \quad [F_{t\theta}] = [L]^0, \quad [F_{r\phi}] = [L]^1, \quad [F_{\theta\phi}] = [L]^0;
\]

**Contravariant components**:

\[
F^{\mu \nu} = \begin{pmatrix} 0 & F_{tr} & F_{t\theta} & 0 \\ -F_{tr} & 0 & 0 & F_{r\phi} \\ -F_{t\theta} & 0 & 0 & F_{\theta\phi} \\ 0 & -F_{r\phi} & -F_{\theta\phi} & 0 \end{pmatrix},
\]

(67)

with

\[
F^{tr} = -\frac{q(2r^2 - \Sigma)(r^2 + b^2)}{\Sigma^3}, \quad F^{t\theta} = \frac{qb^2r \sin(2\theta)}{\Sigma^3}, \quad F^{r\phi} = \frac{qb(2r^2 - \Sigma)}{\Sigma^3}, \quad F^{\theta\phi} = -\frac{qb \sin(2\theta)}{\Sigma^3 \sin^2 \theta}.
\]

(68)

\[
[F^{tr}] = [L]^{-1}, \quad [F^{t\theta}] = [L]^{-2}, \quad [F^{r\phi}] = [L]^{-2}, \quad [F^{\theta\phi}] = [L]^{-3};
\]

**Mixed components**:

\[
F^{\mu \nu} = \begin{pmatrix} 0 & F_{tr} & F_{t\theta} & 0 \\ F_{tr} & 0 & 0 & F_{r\phi} \\ F_{t\theta} & 0 & 0 & F_{\theta\phi} \\ 0 & F_{r\phi} & F_{\theta\phi} & 0 \end{pmatrix},
\]

(69)

with

\[
F^{tr} = g^{tt} F_{tr} + g^{t\phi} F_{\phi r} = \frac{q(2r^2 - \Sigma)(r^2 + b^2)}{\Sigma^2 \Delta}, \quad [F^{tr}] = [L]^{-1},
\]

(70)

\[
F^{t\theta} = g^{tt} F_{t\theta} + g^{t\phi} F_{\phi \theta} = -\frac{qb^2r \sin(2\theta)}{\Sigma^2}, \quad [F^{t\theta}] = [L]^0,
\]

(71)
\[ F^r_t = g^{rt}F_{rt} = \frac{q\Delta(2r^2 - \Sigma)}{\Sigma^3}, \quad [F^r_t] = [L]^{-1}, \] (72)

\[ F^\theta_t = g^{\theta t}F_{\theta t} = -\frac{qb^2r\sin(2\theta)}{\Sigma^3}, \quad [F^\theta_t] = [L]^{-2}, \] (73)

\[ F^r_\phi = g^{rt}F_{r\phi} = -\frac{q\Delta\sin^2\theta(2r^2 - \Sigma)}{\Sigma^3}, \quad [F^r_\phi] = [L]^0, \] (74)

\[ F^\phi_r = g^{\phi r}F_{\phi r} = q\sin(2\theta)(r^2 + b^2), \quad [F^\phi_r] = [L]^{-2}, \] (75)

\[ F^\theta_\phi = g^{\theta \phi}F_{\theta \phi} = -\frac{qb\sin(2\theta)}{\Sigma^2\sin^2\theta}, \quad [F^\theta_\phi] = [L]^{-1}, \] (76)

These expressions allow us to compute the electromagnetic part of the energy-momentum tensor \( T^E_{\mu\nu} \):

\[
T^E_{\mu\nu} = \frac{1}{4\pi} \left( F_{\mu\nu}F^{\mu\nu} + \frac{1}{4} g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \right) = \begin{pmatrix}
T_{tt} & 0 & 0 & T_{t\phi} \\
T_{tr} & 0 & 0 & 0 \\
T_{\theta t} & 0 & 0 & 0 \\
T_{\phi\phi} & 0 & 0 & 0
\end{pmatrix}
\]

\[ = \frac{q^2}{8\pi\Sigma} \begin{pmatrix}
\Delta + b^2 \sin^2\theta & 0 & 0 & A_{t\phi} \\
0 & -\frac{1}{\Sigma^3} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & A_{\phi\phi}
\end{pmatrix}, \] (78)

with

\[ A_{t\phi} = r^2 + b^2 + \Delta \] (79)

and

\[ A_{\phi\phi} = \frac{\sin^2\theta}{\Sigma^2}(r^2 + b^2)^2 + b^2\Delta\sin^2\theta). \] (80)

Clearly, for \( q = 0 \), \( T^E_{\mu\nu} = 0 \) independently of the values for \( M, b \) and \( \Lambda \), while for \( q \neq 0 \), \( b = 0 \) and \( \Lambda = 0 \) one recovers the RN energy-momentum tensor.

### 7. Conclusion

A simple trick (eq.(2)) which unifies the usual treatment of the \( r^{-1} \) and \( r^{-2} \) terms after complexification of the radial coordinate \( r \) in the Newman-Janis approach to the Kerr and Kerr-Newman metrics, allows us to consider under the same scheme terms proportional to \( r^2 \) appearing in the de Sitter (dS) and anti de Sitter (AdS) cases (cf. ref. [11], eq.(2.6c)). In particular, for the massive rotating cosmological cases (K(A)dS and K(N(A)dS) our solution, eq.(55), (or (49) for \( Q^2 = 0 \)), coincides with that of Ibohal [6], eq.(6.41), but, as is the case of this author, is different from those of Carter [12], Gibbons and Hawking [13], Mallett [14], Koberlein [15], and others. Some details of the calculations and a complete study of the geometry associated with the metric (60), like Kruskal coordinates, Penrose diagram, horizons, ergospheres, etc., and the complete energy-momentum tensor, will be published elsewhere.

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