Abstract: Let $X$ be a smooth projective variety over an algebraically closed field $k$. We repeat Bloch’s construction of a $\mathbb{G}_m$-biextension (torseur) $E$ over $\text{CH}^p_{\text{hom}}(X) \times \text{CH}^q_{\text{hom}}(X)$ for $p+q = \dim(X)+1$. First we show that in characteristic zero $E$ comes via pullback from the Poincaré biextension $P$ over the corresponding product of intermediate Jacobians - as conjectured by Bloch and Murre. Then the relations between $E$ and various equivalence relations for algebraic cycles are studied. In particular we reprove Murre’s theorem stating that Griffiths’ conjecture holds for codimension 2 cycles, i.e. every 2-codimensional cycle which is algebraically and incidence equivalent to zero has torsion Abel-Jacobi invariant.

Key Words: Biextensions, higher Chow groups, algebraic cycles, Abel-Jacobi maps, Deligne-Beilinson cohomology, incidence equivalence, Griffiths’ conjecture.
§ 0. Introduction

Let \( X \) be a compact Kähler manifold of dimension \( d \). If we fix two integers \( 0 \leq p, q \leq d \) with \( p + q = d + 1 \), then the two Griffiths intermediate Jacobians \( \mathcal{J}^p(X) \) and \( \mathcal{J}^q(X) \) are mutually dual. The Poincaré line bundle \( P \) on \( \mathcal{J}^p(X) \times \mathcal{J}^q(X) \) minus its zero section, denoted by \( \mathbb{P} \), is a \( \mathbb{C}^* \)-torseur and in fact a biextension in the sense of Grothendieck (SGA 7) and [Mumford 69].

Any two disjoint homologically trivial cycles \( Z \in CH^p_{\text{hom}}(X) \), \( W \in CH^q_{\text{hom}}(X) \) define an element \( < Z, W > \in \mathbb{P} \), projecting onto the element \( (\psi_p(Z), \psi_q(W)) \in \mathcal{J}^p(X) \times \mathcal{J}^q(X) \), where \( \psi \) is the Abel-Jacobi homomorphism.

Bloch has given another construction for a \( \mathbb{G}_m \)-torseur \( E \) over \( CH^p_{\text{hom}}(X) \times CH^q_{\text{hom}}(X) \) for a projective manifold \( X \) over an arbitrary field [Bloch 89]. He also conjectures, that \( E \) may be obtained as the pullback of \( \mathbb{P} \) via the Abel-Jacobi map in characteristic zero. We prove this:

**Theorem 1.** Let \( X \) be a smooth projective variety over \( \mathbb{C} \). Then \( E \) is the pullback of \( \mathbb{P} \) as torseurs under the Abel-Jacobi homomorphism.

The idea of proof is that the essential point in Bloch’s construction factors through Deligne cohomology.

Theorem 1 can be applied to shed some light on the relation between Abel-Jacobi equivalence and incidence equivalence of algebraic cycles. A cycle \( W \in CH^q_{\text{hom}}(X) \) is called *incidence equivalent* to zero (\( W \sim_{\text{inc}} 0 \)), if for all pairs \( (T, B) \), with a smooth projective variety \( T \) and a cycle \( B \in CH^{d+1-q}(T \times X) \), the divisor \( B(W) = (pr_T)_*(B \cap pr_X^*(W)) \) is linearly equivalent to zero on \( T \) (see [Griffiths 69]).

It is easy to show that \( \psi(W) = 0 \) implies \( W \sim_{\text{inc}} 0 \). Griffiths conjectured, that the opposite is true modulo torsion if \( W \) is assumed to be algebraically equivalent to zero. It was known that this follows from the generalized Hodge conjecture of Grothendieck. Murre ([Murre 85]) has proved the conjecture for codimension two cycles. We show here, that theorem 1 has something to do with this problem. Denote by \( E^\text{alg}_W \) the restriction of \( E \) to the fiber over \( W \in CH^q_{\text{alg}}(X) \):

\[
1 \to \mathbb{C}^* \to E^\text{alg}_W \to CH^p_{\text{alg}}(X) \to 0
\]

We show:
Theorem 2. Let $X$ be smooth projective over $\mathbb{C}$ and $W$ be algebraically equivalent to zero. Then $E^\text{alg}_W$ is a split extension if and only if $W \sim_{\text{inc}} 0$.

Here the idea is that incidence equivalence reduces somehow to the situation of points and divisors, which is better understood than the general case. This result brings together the a priori different definitions of incidence equivalence occurring in the literature. Theorem 2 implies a condition for Griffiths’ conjecture:

Theorem 3. Let $X$ be as above and $W \in CH^q_{\text{alg}}(X)$ be such that $\psi(N \cdot W)$ is contained in the dual of $\mathcal{J}^p_{\text{alg}}(X) = \psi(CH^p_{\text{alg}}(X))$ (an abelian subvariety) for some $N \in \mathbb{N}$. Then $W \sim_{\text{inc}} 0$ implies that $\psi(W)$ is torsion in $\mathcal{J}^q(X)$.

The condition follows again from Grothendieck’s generalized Hodge conjecture. But it holds always for codimension two cycles if and therefore we get another proof of Murre’s theorem ([Murre85]):

Corollary (Murre’s theorem).
Griffiths’ conjecture holds for codimension two cycles on a smooth projective manifold $X$ over $\mathbb{C}$, i.e. for every cycle $W$ on $X$, algebraically and incidence equivalent to zero, $\psi(W)$ is torsion in $\mathcal{J}^2(X)$.

It is easy to see that the proof of Griffiths’ conjecture would go through in any codimension $p$, if the adjoint $\Lambda$ of the Lefschetz operator were again algebraic. This can be verified for several types of varieties e.g. rational-like varieties and abelian varieties and complete intersections in these. For a special case see [Griffiths-Schmid75].

The paper is organized as follows:

§ 1. The Poincaré biextension.
§ 2. Facts about higher Chow groups.
§ 3. Bloch’s construction.
§ 4. The pullback theorem.
§ 5. Abel-Jacobi versus incidence equivalence.

It is a pleasure to thank Jacob Murre for mentioning this problem to me and showing me his unpublished notes ([Murre]) during a visit at Leiden. Furthermore I am grateful to A. Collino, H. Esnault and D. Huybrechts for many helpful remarks and improvements.
Let us first discuss briefly the notion of biextension. We refer to [Mumford 69] and SGA 7 for further details: Let $A, B, C$ be abelian groups. A biextension of $B \times C$ by $A$ is a set $E$ on which $A$ acts freely, together with a quotient map $\pi : E \to B \times C$ and two laws of composition $+_1 : E \times B \to E$, $+_2 : E \times C \to E$ subject to the following conditions:

1. For all $b \in B$, $\pi^{-1}(b \times C)$ is an abelian group under $+_1$ and the following sequence is exact:

   $0 \to A \to \pi^{-1}(b \times C) \to C \to 0$

2. For all $c \in C$, $\pi^{-1}(B \times c)$ is an abelian group under $+_2$ and the following sequence is exact:

   $0 \to A \to \pi^{-1}(B \times c) \to B \to 0$

3. $+_1, +_2$ are compatible, i.e. for suitable $w, x, y, z \in E$ we have

   $$(w +_1 x) +_2 (y +_1 z) = (w +_2 y) +_1 (x +_2 z)$$

There is one prominent example which caused this definition to exist, namely the Poincaré biextension:

Let $T$ be a compact complex torus of dimension $n$. There is a natural line bundle $P$ (the Poincaré line bundle) on $T \times T^\vee$, where $T^\vee = \text{Pic}^0(T)$. As an element of $\text{Pic}(T \times T^\vee)$ it is usually normalized by the two conditions

1. $P|_{\{0\} \times T^\vee} \cong \mathcal{O}_{T^\vee}$
2. $P|_{T \times \{\lambda\}}$ represents $\lambda \in \text{Pic}^0(T) = T^\vee$

$P$ is unique under these assumptions. Let $\mathbb{P} = P \setminus \text{zero section}$. Clearly $\mathbb{C}^*$ acts freely on $\mathbb{P}$. $\mathbb{P}$ is called Poincaré-biextension associated to $T$. The projection map $p : \mathbb{P} \to T \times T^\vee$ makes $\mathbb{P}$ a $\mathbb{C}^*$-torseur over $T \times T^\vee$. For every $\lambda \in T^\vee$, the inverse image $p^{-1}(T \times \{\lambda\})$ is denoted by $\mathbb{P}_\lambda$ and sits in the exact sequence

$$0 \to \mathbb{C}^* \to \mathbb{P}_\lambda \to T \to 0.$$ 

Then $\mathbb{P}_\lambda$ is an extension of abelian groups. It is well known that

$$T^\vee = \text{Pic}^0(T) \cong \text{Ext}^1(T, \mathbb{C}^*)$$

in the category of complex analytic groups. $\mathbb{P}_\lambda$ is exactly the extension of $T$ by $\mathbb{C}^*$ given by $\lambda$ in this isomorphism. If $D$ is any divisor on $T$ with $c_1(\mathcal{O}(D)) = \lambda$ then we will write also $\mathbb{P}_D$ instead of $\mathbb{P}_\lambda$ and note that $\mathbb{P}_D$ depends only on the class
on $D$ in $\text{Pic}^0(T)$. Now let us consider a special case: Let $X$ be a Kähler manifold of dimension $d$ and $p$ some integer. If we let $T = J^p(X)$ the $p$-th intermediate Jacobian of $X$

$$J^p(X) = \frac{H^{2p-1}(X, \mathbb{C})}{F^p \oplus H^{2p-1}(X, \mathbb{Z})}$$

then it follows by using Poincaré duality, that $T^\vee$ is given by $J^q(X)$ where $q = d + 1 - p$. Let

$$\mathbb{P} \rightarrow T \times T^\vee = J^p(X) \times J^q(X)$$

be the Poincaré-biextension as defined above. If $W \subset CH^q_{\text{hom}}(X)$ with $\lambda = \psi(W) \in J^q(X)$ and $\mathbb{P}_W := \mathbb{P}_\lambda$ we remark that $\mathbb{P}_W$ depends only on the Abel-Jacobi equivalence class of $W$, i.e. $W$ is Abel-Jacobi equivalent to zero if and only if $\mathbb{P}_W$ splits as an extension in $\text{Ext}^1(T, \mathbb{C}^*)$.

§ 2. Facts about higher Chow groups

Literature: [Bloch 86], [Murre] for this chapter. Before we repeat Bloch’s construction, we recall some properties of higher Chow groups as defined by Spencer Bloch.

Let $X$ be a quasi-projective variety over $\mathbb{C}$. Denote by $\Delta^n = \text{Spec}(k[T_0, \ldots , T_n]/< \sum T_i = 1 >)$ which is isomorphic to affine space $\mathbb{A}^n_{\mathbb{C}}$. There are $n + 1$ natural faces isomorphic to $\Delta^n$ contained in $\Delta^n$, defined by the vanishing of one of the coordinate functions $t_i$. Let $Z^r(X, n)$ be the free abelian group of cycles $Z \subset X \times \Delta^n$ of codimension $r$ meeting all faces $X \times \Delta^m$ ($m < n$) properly. Then $CH^r(X, n)$ is defined as the $n$-th homology group of the complex

$$Z^r(X; \cdot) = (\ldots Z^r(X, n + 1) \xrightarrow{\partial} Z^r(X, n) \xrightarrow{\partial} Z^r(X, n - 1) \xrightarrow{\partial} \ldots \rightarrow Z^r(X, 0)).$$

The maps $\partial$ are given by alternating sums of restriction maps to faces. We will need the following properties:

1. $CH^*(X, *)$ are covariant (contravariant) functorial for proper (flat) morphisms.

2. If $W \subset X$ is closed, we have a long exact sequence

$$\ldots \rightarrow CH^*(X, W, n) \rightarrow CH^*(X, n) \rightarrow CH^*(W, n) \rightarrow CH^*(X, W, n - 1) \rightarrow \ldots$$

3. $CH^*(X, 0) = CH^*(X)$ (ordinary Chowgroups).
(4) There is a product for $X$ smooth

$$CH^p(X, q) \otimes CH^r(X, s) \rightarrow CH^{p+r}(X, q + s).$$

(5) There are natural maps to Deligne-Beilinson cohomology, defined in Bloch’s article in Contemporary Math. 58 (1986):

$$c_D : CH^p(X, q) \rightarrow H_D^{2p-q}(X, \mathbb{Z}(p)).$$

(6) If $X$ is proper (not necessarily smooth) and $\dim X = d$, then there is a natural surjective homomorphism

$$\epsilon : CH^{d+1}(X, 1) \rightarrow \mathbb{C}^*$$

factoring through Deligne-Beilinson cohomology

$$\epsilon : CH^{d+1}(X, 1) \rightarrow H_D^{2d+1}(X, \mathbb{Z}(d + 1)) \rightarrow \mathbb{C}^*$$

obtained as follows: Let $\pi : X \rightarrow \text{Spec} (\mathbb{C})$ be the natural morphism. Then $\epsilon$ is given by the direct image map $\pi_* : CH^{d+1}(X, 1) \rightarrow CH^1(\text{Spec} (\mathbb{C}), 1)$, since it is a straightforward exercise to show that

$$CH^1(\text{Spec} (\mathbb{C}), 1) \cong H_D^1(\text{Spec} (\mathbb{C}), \mathbb{Z}(1)) \cong \mathbb{C}/\mathbb{Z}(1)$$

via the classes in (5) and the latter group can be identified with $\mathbb{C}^*$. We claim:

(1) $\epsilon$ is surjective.

(2) It factors through Deligne-Beilinson cohomology.

Proof for (1) and (2)

(1): By definition $CH^1(\text{Spec} (\mathbb{C}), 1) = Z^1(\text{Spec} (\mathbb{C}), 1)/\text{Im}(\partial)$, since $Z^1(\text{Spec} (\mathbb{C}), 0) = 0$. In the same way $CH^{d+1}(X, 1) = Z^{d+1}(X, 1)/\text{Im}(\partial)$. Hence in both groups the elements are represented by finite sums of points. It is clear that the map $Z^{d+1}(X, 1) \rightarrow Z^1(\text{Spec} (\mathbb{C}), 1)$ induced by $\pi_*$ is surjective. Hence we deduce (1).

(2): By the functorial properties of the Deligne-Beilinson classes we get a commutative diagram:

$$
\begin{array}{ccc}
CH^{d+1}(X, 1) & \xrightarrow{\pi_*} & CH^1(\text{Spec} (\mathbb{C}), 1) \\
\downarrow c_D & & \downarrow c_D \\
H_D^{2d+1}(X, \mathbb{Z}(d + 1)) & \xrightarrow{\pi_*} & H^1_D(\text{Spec} (\mathbb{C}), \mathbb{Z}(1))
\end{array}
$$

The right vertical arrow being the identity we get (2). \qed
§ 3. Bloch’s Construction

For this chapter we refer to [Bloch 89].

Let us describe Bloch’s construction in detail: Let \( X \) be smooth and projective over \( \mathbb{C} \) with \( \dim X = d \), and \( W \in \mathbb{Z}_{m-1}(X) = \mathbb{Z}^{d+1-m}(X) \) a cycle of dimension \( m - 1 \), which is homologous to zero, denoted by \( W \sim_{\text{hom}} 0 \) as usual. We will construct an extension

\[
1 \to \mathbb{C}^* \to E_W \to CH^m_{\text{hom}}(X) \to 0
\]

such that they can be glued together to give

\[
E = \bigcup E_W \to CH^m_{\text{hom}}(X) \times CH^{d+1-m}_{\text{hom}}(X)
\]

a biextension in the sense of § 1. To do this, Bloch first considers the map

\[
\Theta_W = \epsilon \circ (\cap W) : CH^m(X, 1) \to CH^{d+1}(X, 1) \to \mathbb{C}^*
\]

using properties (4) and (6) of higher Chow groups.

**Lemma 1.** If \( W \sim_{\text{hom}} 0 \), then \( \Theta_W \equiv 1 \).

**Proof.** Recall the following fact from Deligne-Beilinson cohomology ([Esnault-Viehweg 88]):

\( J^* \subset H^*_D(X, \mathbb{Z}(*)) \) is a square zero ideal, where

\[
J^* = \text{Ker}(H^*_D(X, \mathbb{Z}(*)) \to H^*_{\text{Betti}}(X, \mathbb{Z}(*)))
\]

Now look at the diagram

\[
\begin{array}{ccc}
CH^m(X, 1) & \xrightarrow{\cap W} & CH^{d+1}(X, 1) \\
\downarrow c_D & & \downarrow c_D \\
H^{2m-1}_D(X, \mathbb{Z}(m)) & \xrightarrow{\cap \rho(W)} & H^{2d+1}_D(X, \mathbb{Z}(d+1)) \\
\end{array}
\]

But \( W \sim_{\text{hom}} 0 \) implies \( c_D(W) \in J^*(X) \), and also \( H^{2m-1}_D(X, \mathbb{Z}(m)) \subset J^*(X) \). Hence the lemma follows. \( \square \)

To construct \( E_W \) consider the long exact sequence for the pair \((X, |W|)\):

\[
\ldots \to CH^m(X, 1) \to CH^m(|W|, 1) \to CH^m(X, |W|, 0) \to CH^m(X, 0) \to 0
\]

where \( |W| = \text{supp}(W) \) as usual. Since \( |W| \) is proper of dimension \( m - 1 \), we have a surjective map \( \epsilon : CH^m(|W|, 1) \to \mathbb{C}^* \) by property (6). But \( \Theta_W \equiv 1 \), hence \( \epsilon \) factors through

\[
A := \frac{CH^m(|W|, 1)}{\text{im} CH^m(X, 1)} \to \mathbb{C}^*.
\]
Consider the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & A & \rightarrow & CH^m(X, |W|, 0) & \rightarrow & CH^m(X) & \rightarrow & 0 \\
& & \downarrow \varepsilon & & \downarrow & & \downarrow & & \\
1 & \rightarrow & C^* & \rightarrow & E_W & \rightarrow & CH^m(X) & \rightarrow & 0 \\
\end{array}
\]

where we have defined \( E_W \) as

\[
CH^m(X, |W|, 0) / i(\text{Ker}(\varepsilon))
\]

as the pushout of the upper line via \( \varepsilon \). By abuse of language, we also denote by \( E_W \) the restriction

\[
1 \rightarrow C^* \rightarrow E_W \rightarrow CH^m_{\text{hom}}(X) \rightarrow 0
\]

to \( CH^m_{\text{hom}}(X) \). Bloch proves several results for \( E_W \):

If \( W \) is rationally equivalent to \( W' \), then \( E_W \) is canonically isomorphic to \( E_W' \). In particular, if \( W \sim_{\text{rat}} 0 \), then \( E_W \) splits as an extension of groups. We prove a little bit more in § 4.

The \( E_W \) may be glued together, to obtain a \( C^* \)-torseur (biextension)

\[
E \rightarrow CH^m_{\text{hom}}(X) \times CH^{d+1-m}_{\text{hom}}(X)
\]

satisfying the axioms of § 1.

We refer the reader to [Bloch 89] for this construction, since we don’t use the full torseur \( E \) for our applications. In fact by our theorem 1 this follows from the statement for \( P \).

§ 4. The Pullback Theorem

Here we prove the main result of this paper. Let \( X \) be a smooth projective variety over \( \mathbb{C} \).

**Lemma 2.** If \( W \sim_{A,J} 0 \) (Abel-Jacobi equivalent to zero), then \( E_W \) splits.

**Proof.** We use the functorial Deligne-Beilinson classes \( c_D \) from (5), §2. Consider the commutative diagram

\[
\begin{array}{ccccccccc}
A & \rightarrow & CH^m_{\text{hom}}(X, |W|, 0) & \rightarrow & CH^m_{\text{hom}}(X) & \rightarrow & 0 \\
\downarrow c_D & & \downarrow & & \downarrow & & \\
H^{2m-1}_D(|W|; \mathbb{Z}(m)) & \rightarrow & H^{2m}_D(X, |W|; \mathbb{Z}(m))_{\text{hom}} & \rightarrow & J^m(X) & \rightarrow & 0 \\
\end{array}
\]

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$\epsilon : A \to \mathbb{C}^*$ is surjective and by §2.(6) factors through

$$H^{2m-1}_D(|W|; \mathbb{Z}(m)) \to \bigoplus \mathbb{C}^* \to \mathbb{C}^*$$

by summation over all irreducible components of $|W|$. Define $K := \text{Ker}(A \to \mathbb{C}^*)$ and recalling that $E_W = CH^{m}_X(|W|, 0)/i(K)$ we obtain the diagram

$$\begin{array}{ccc}
T & \rightarrow & T' \\
\downarrow & & \downarrow \\
0 & \rightarrow & E_W \\
\downarrow \epsilon & & \downarrow \\
1 & \rightarrow & \mathbb{C}^* \\
\end{array}$$

where $\mathbb{P}_W$ is as in § 1, $T = \text{Ker}(E_W \to \mathbb{P}_W)$ and $T' = \text{Ker}(CH^{m}_X \to \mathcal{J}^{m}(X))$. Since $\epsilon$ is an isomorphism, $T \cong T'$ by the snake lemma. It follows that $E_W$ is - as an extension - the pullback of $\mathbb{P}_W$ via the Abel-Jacobi map and therefore is split if $W$ is Abel-Jacobi equivalent to zero. This proves the lemma.

As a corollary we get:

**Theorem 1.** $E_W$ is the pullback of $\mathbb{P}_W$ as extensions via the Abel-Jacobi map. In particular $E$ is the pullback of $\mathbb{P}$ as torseurs via the Abel-Jacobi map.

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**§ 5. Abel-Jacobi versus Incidence Equivalence**

The notion of “incidence equivalence” was introduced by Griffiths ([Griffiths 69]). Let $X$ be projective, smooth of dimension $d$ over $\mathbb{C}$.

**Definition:** A cycle $W \in CH^{i}_X$ is called incidence equivalent to zero, i.e. $W \sim_{\text{inc}} 0$, if for all pairs $(T, B)$ with $T$ a smooth, projective variety and $B \in CH^{d+1-i}(T \times X)$,

$$B(W) := (pr_T)_*(B \cap pr_X^*W) = \{t \in T | B_t \cap W \neq \emptyset\}$$

is linearly equivalent to zero in $\text{Pic}^0(T)$.

**Remark:** $B(W)$ is a divisor by a dimension count and in $\text{Pic}^0(T)$, since $W \sim_{\text{hom}} 0$. If $i = 1$ we get back the notion of rational equivalence for divisors (= Abel-Jacobi equivalence).

Bloch defines $W \sim_{\text{inc}} 0$ if $E_W$ is a split extension. However it is not a priori clear that this gives the same definition. See the discussion below.
Lemma 3. If $W \sim_{AJ} 0$, then also $W \sim_{inc} 0$.

Proof. $B$ is a correspondence

$$CH^i_{hom}(X) \to CH^1_{hom}(T) = \text{Pic}^0(T)$$

via $W \mapsto B(W)$. Consider the diagram ($\psi =$ Abel-Jacobi homomorphism)

$$
\begin{array}{ccc}
CH^i_{hom}(X) & \xrightarrow{B} & CH^1_{hom}(T) = \text{Pic}^0(T) \\
\downarrow \psi & & \downarrow S \\
J^i(X) & \xrightarrow{J^B} & J^1(T)
\end{array}
$$

The claim follows. \[\square\]

Remark: Griffiths has conjectured, that the opposite is true modulo torsion for cycles algebraically equivalent to zero: If $W \sim_{inc} 0$, then for some $N \in \mathbb{N}$:

$$N \cdot W \sim A_{AJ} 0.$$ 

This is true for $i = 1, d$ without torsion and for $i = 2$ by [Murre 85]. In any case it follows from the generalized Hodge conjecture of Grothendieck, as we will see below.

In the remaining discussion let $i = d + 1 - m$ and $W$ a cycle of dimension $m - 1$, hence codimension $d + 1 - m$ which is homologous to zero.

We want to investigate the relation between $W \sim_{inc} 0$ and the splitting of $\mathbb{B}_W$ in the sense of [Bloch89]. To do this consider the action of the correspondence $B$ on the higher Chow groups. $B(W) \in \text{Pic}^0(T)$ gives rise to the exact sequence

$$
A_{B(W)} = \frac{CH^i([B(W)], 1)}{\text{Im} \ CH^i(T, 1)} \to CH^i(T, |B(W)|, 0) \to CH^i(T) \to 0
$$

where $t = \dim T$ and the corresponding extension $\mathbb{E}_{B(W)}$ is sitting in

$$
1 \to \mathbb{C}^* \to \mathbb{E}_{B(W)} \to CH^i_{hom}(T) \to 0
$$

and is pulled back via the Albanese map from $\mathbb{P}_W$:

$$
1 \to \mathbb{C}^* \to \mathbb{P}_{B(W)} \to \text{Alb}(T) \to 0.
$$

The correspondence $B$ induces maps $B^\sharp = B^{-1} : CH^i_{hom}(T) \to CH^m_{hom}(X)$ then giving rise to a commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & A_{B(W)} \rightarrow CH^i(T, |B(W)|, 0) \rightarrow CH^i(T) \rightarrow 0 \\
\downarrow B^\sharp & & \downarrow B^\sharp \\
0 & \rightarrow & A_W \rightarrow CH^m(X, |W|, 0) \rightarrow CH^m(X) \rightarrow 0
\end{array}
$$

This can be seen as follows: By Chow’s moving lemma, cycles in $CH^m(X)$ and $CH^m(X, |W|)$ may be assumed to have support disjoint from $W$. The same holds for $T$ and $B(W)$. Now if a zero cycle on $T$ does not meet $B(W)$ then $B^\sharp(Z)$ will
not meet $W$. This explains the maps in the diagram. Everything commutes by the functorial properties of § 2.

We thus obtain a diagram

$$
\begin{array}{cccccc}
1 & \to & \mathbb{C}^\ast & \to & \mathbb{E}_{B(W)} & \to & CH^1_{\text{hom}}(T) & \to & 0 \\
\| & \downarrow & B^\sharp & \downarrow & B^\sharp & \downarrow & & \\
1 & \to & \mathbb{C}^\ast & \to & \mathbb{E}_W & \to & CH^m_{\text{hom}}(X) & \to & 0
\end{array}
$$

inducing the identity on $\mathbb{C}^\ast$, where we assumed that $B$ is reduced without loss of generality.

This is the main input in order to prove:

**Lemma 4.** (a) If $\mathbb{E}_W$ splits then also $\mathbb{E}_{B(W)}$ splits for all pairs $(T, B)$.

(b) Assume additionally that $W \sim_{\text{alg}} 0$. Then:

$\mathbb{E}_{B(W)}$ splits for all pairs $(T, B) \iff \mathbb{E}_{W}^{\text{alg}}$ splits, where $\mathbb{E}_{W}^{\text{alg}}$ is the subextension

$$
1 \to \mathbb{C}^\ast \to \mathbb{E}_{W}^{\text{alg}} \to CH^m_{\text{alg}}(X) \to 0
$$

**Proof.** (a) follows directly from the commutative diagram for every pair $(T, B)$. To prove (b) suppose $W \sim_{\text{alg}} 0$ and that $\mathbb{E}_{B(W)}$ splits for all pairs $(T, B)$. We have to show that $\mathbb{E}_{W}^{\text{alg}}$ splits. Take $(T, B)$ such that

$$
B^\sharp : \text{Alb}(T) \to J^m(X)
$$

parametrizes the whole image $J^m_{\text{alg}}(X)$ of the Abel-Jacobi map for cycles algebraically equivalent to zero. $T$ can be chosen to be an abelian variety itself such that $B^\sharp$ becomes an isogeny. It follows that the subextension

$$
1 \to \mathbb{C}^\ast \to \mathbb{P}_{W}^{\text{alg}} \to J^m_{\text{alg}}(X) \to 0
$$

of $\mathbb{P}$ splits by the Deligne cohomology version of the commutative diagram above. Now by theorem 1 and its proof, the extension class of $\mathbb{E}_{W}^{\text{alg}}$ is pulled back from $\text{Ext}^1(\psi(CH^m_{\text{alg}}(X)), \mathbb{C}^\ast) = \text{Ext}^1(J^m_{\text{alg}}(X), \mathbb{C}^\ast)$. (b) follows.

As a corollary we get:

**Theorem 2.** Assume $W \sim_{\text{alg}} 0$. Then $W \sim_{\text{inc}} 0$ if and only if $\mathbb{E}_{W}^{\text{alg}}$ splits.

**Proof.** $W \sim_{\text{inc}} 0$ iff $B(W)$ is linearly equivalent to zero by definition, hence iff $\mathbb{P}_{B(W)}$ splits. But for divisors the splittings of $\mathbb{E}$ and $\mathbb{P}$ are equivalent (as one can see by restricting to suitable curve) and hence this holds if and only if $\mathbb{E}_{B(W)}$ splits. Hence by lemma 4 (b) we are finished.
It remains to discuss Griffiths’ conjecture: Let $W \sim_{\text{alg}} 0$. If $W \sim_{\text{inc}} 0$ then some multiple of $W$ is Abel-Jacobi equivalent to zero. With the help of theorem 2 it is sufficient to show: If $E_W$ splits, then for some $N \in \mathbb{N}$: $\mathbb{P}_{N \cdot W}$ splits.

**Lemma 5.** Suppose the generalized Hodge conjecture (GHC) holds for $J^m(X)$, i.e. the largest abelian subvariety $J^m_{\text{ab}}(X)$ of $J^m(X)$ is parametrized by algebraic cycles. Then for $W \in CH_{m-1}(X)$, $W \sim_{\text{alg}} 0$, we have: If $E_W$ splits, then for some $N \in \mathbb{N}$, $\mathbb{P}_{N \cdot W}$ splits.

**Proof.** Remark that $J^m_{\text{ab}}(X)$ and $J^{d+1-m}_{\text{ab}}(X)$ are always dual to each other modulo isogeny, since the dual of a Hodge substructure is again one. Therefore GHC implies that $J^m_{\text{alg}}(X) = J^m_{\text{ab}}(X)$ and that $\psi(N \cdot W)$ lies in the dual abelian variety of $J^m_{\text{alg}}(X) = \psi(CH^m_{\text{alg}}(X))$ for some $N \in \mathbb{N}$. Hence it is sufficient to show that the extension

$$1 \to \mathbb{C}^* \to \mathbb{P}_{N \cdot W, \text{alg}} \to J^m_{\text{alg}}(X) \to 0$$

splits. This in turn is implied by the splitting of $E_W$ and hence of $E^m_{N \cdot W}$ by the proof of lemma 4.

Actually we have proved more:

**Theorem 3.** Let $W \in CH_{m-1}(X)$ be algebraically equivalent to zero. Then if $\psi(N \cdot W)$ is contained in the dual of $J^m_{\text{alg}}(X)$, the splitting of $E_W$ implies the splitting of $\mathbb{P}_{N \cdot W}$.

A little generalization of the argument in Lemma 5.2. of [Murre85] gives another proof of Murre’s theorem:

**Corollary (Murre’s theorem).**

Griffiths’ conjecture holds for codimension two cycles on a smooth projective manifold $X$ over $\mathbb{C}$, i.e. for every cycle $W$ on $X$, algebraicly and incidence equivalent to zero, $\psi(W)$ is torsion in $J^2(X)$.

**Proof.** We have to verify the assumptions of theorem 3. Let $T$ be the universal cover (tangent space) of $J^2_{\text{alg}}(X)$ and $H^3_a(X) = T \oplus \tilde{T}$. This is a sub Hodge structure of $H^3(X, \mathbb{C})$ contained in $H^{1,2}(X) \oplus H^{2,1}(X)$. Now $\psi(W)$ is contained in the dual of $J^{d-1}_{\text{alg}}(X)$ ($d = \dim(X)$) if we can show that the cup product

$$H^3_a(X) \otimes H^{2d-3}_a(X) \to \mathbb{C}$$

has no left kernel. Let $L$ be the Lefschetz operator, i.e. cup product with the hyperplane class. It is algebraic and hence maps $H^k_a(X)$ to $H^{k+2}_a(X)$. 

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Let us repeat the proof of Murre’s Lemma 5.2. for convenience. Assume there is a nonzero element $v \in H^3_a(X)$ in the left kernel and decompose it as $v = v_0 + L v_1$ according to Lefschetz decomposition. The $v_k$ are primitive. In particular $v_1 \in H^1_a(X)$ which is the whole of $H^1(X, \mathbb{C})$. Hence $L v_1$ and $v_0$ are in $H^3_a(X)$. Now decompose $v_0 = w_1 + w_2$ into types $(1, 2)$ and $(2, 1)$. Since $H^{2p-1}_a(X)$ is a sub Hodge structure, $w_1$ and $w_2$ and their complex conjugates are again in $H^{2p-1}_a(X)$. By symmetry we may assume $w_1 \neq 0$ unless $v_0 = 0$. In the first case set $z := L^{d-3}(\bar{w}_1) \in H^{2d-3}_a(X)$, otherwise decompose $v_1 = y_1 + y_2$ into types with $y_1 \neq 0$ (by symmetry) and set $z := L^{d-2}(\bar{y}_1)$. Then for type reasons in the first case $v \cup z = v_0 \cup z = L^{d-3}(w_1 \cup \bar{w}_1) \neq 0$ by primitivity of $w_1$, a contradiction. In the second case $v \cup z = L v_1 \cup z = L^{d-1}(y_1 \cup \bar{y}_1) \neq 0$ by primitivity of $y_1$, a contradiction. □

**Remark:** It is easy to see that the proof of Griffiths’ conjecture would go through in any codimension $p$, if in the Lefschetz decomposition of $v \in H^{2p-1}_a(X)$ every term were again in $H^{2p-2k-1}_a(X)$. This can be verified for several types of varieties e.g. rational-like varieties and abelian varieties and complete intersections in these. For a special case see [Griffiths-Schmid75].
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