Spectral triples of holonomy loops

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Abstract

The machinery of noncommutative geometry is applied to a space of connections. A noncommutative function algebra of loops closely related to holonomy loops is investigated. The space of connections is identified as a projective limit of Lie-groups composed of copies of the gauge group. A spectral triple over the space of connections is obtained by factoring out the diffeomorphism group. The triple consist of equivalence classes of loops acting on a hilbert space of sections in an infinite dimensional Clifford bundle. We find that the Dirac operator acting on this hilbert space does not fully comply with the axioms of a spectral triple.
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1 Introduction

The story of noncommutative geometry starts with the idea that instead of studying spaces one studies algebras of functions on the spaces. A concrete result supporting this idea is the Gel’fand-Naimark theorem \([1]\) that states that the world of locally compact Hausdorff spaces is, by taking the corresponding algebra of continuous complex valued functions vanishing at infinity, the same as the world of commutative \(C^*-\)algebras. Hence noncommutative \(C^*-\)algebras can be considered as noncommutative locally compact Hausdorff spaces.

The crucial leap from noncommutative topology to geometry was done by Alain Connes \([2]\). The key observation is that the Dirac operator on a Riemannian manifold gives full information about the metric. This idea provides the definition of a noncommutative geometry, i.e. a spectral triple, by abstractizing a Dirac operator as an operator acting on the same hilbert space as the (non)commutative algebra; satisfying a list of axioms generalizing interaction rules of smooth functions with the Dirac operator.

Prime examples of noncommutative geometries are given by quotient spaces. A conceptually simple case is the set of two points identified. The classical way of identification would be to consider just one point. The noncommutative quotient is to consider two by two matrices. So we regard the two sub-algebras

\[
\begin{pmatrix}
\mathbb{C} & 0 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
0 & \mathbb{C}
\end{pmatrix}
\]

as the function algebras over the two points. These algebras are then identified through the partial isometries

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix},
\]

which not only identify the points but also belong to the algebra. Represented on \(\mathcal{H} = \mathbb{C} \oplus \mathbb{C}\) the algebra of two by two matrices interacts with a Dirac operator given by

\[
\mathcal{P} = \begin{pmatrix}
0 & a \\
-a & 0
\end{pmatrix}, \quad a \in \mathbb{R}.
\]

This noncommutative geometry, when combined with the commutative algebra of smooth functions on a manifold, is related to the Higgs effect in the
Connes-Lott model \cite{3} and to the Higgs effect in Connes’ full formulation of the Standard Model \cite{4}. The crucial point is that exactly the noncommutativity of the algebra generates the entire bosonic sector, including the Higgs scalar, through fluctuations around the Dirac operator. The action of the standard model coupled to gravity comes out \cite{5,6}

$$\langle \xi | \tilde{D} | \xi \rangle + \text{Trace} \varphi \left( \frac{\tilde{D}}{\Lambda} \right),$$

where $\tilde{D}$ is the fluctuated Dirac operator, $\xi$ a hilbert state and $\varphi$ a suitable cutoff function selecting eigenvalues below the cutoff $\Lambda$.

Unfortunately, this beautiful unification of the standard model with general relativity is completely classical. No clear notion of quantization exists within the framework of noncommutative geometry.

The aim of this paper is to explore new ideas on the unification of noncommutative geometry with the principles of quantum field theory. Quantum field theory deals with spaces of field configurations. The central object is the path integral

$$\int \mathcal{D}\Phi \exp \left( \frac{i}{\hbar} S[\Phi] \right),$$

where $\Phi$ denotes the field content of the theory described by the (symmetries of the) classical action $S[\Phi]$. $\mathcal{D}\Phi$ is a formal measure on the space of field configurations. Therefore, rather than dealing with manifolds or algebras of functions hereon, quantum field theory lives on the much larger spaces of field configurations. We now suggest the following: If Connes’ formulation of the standard model and quantum field theory are to be linked, and if the principles of noncommutative geometry are fundamental (which we believe they are), then one should apply the machinery of noncommutative geometry to some space of field configurations. Further, since Connes’ formulation of the standard model is in principle a gravitational theory (pure geometry) we suggest that the correct implementation of quantum theory must involve quantum gravity. Thus, we suggest to study a functional space related to general relativity.

The aim is to find a suitable configuration space on which a generalized Dirac operator exist. A function algebra hereon may very well be naturally
noncommutative (classically). The hope is that the Dirac operator will generate a kind of quantization of the underlying space.

For the space of field configurations we use ideas from loop quantum gravity [7]. Here the space is the space of certain connections modulo gauge equivalence. The function algebra is generated by traced holonomies of connections along loops, i.e. all physical observables can be expressed by these. This gives a commutative algebra. However, the lesson taught by noncommutative geometry is that the noncommutativity of the algebra provides essential structure. The idea is therefore to keep the noncommutativity by taking holonomies without tracing them; a loop $L$ maps connections into group elements of $G$

$$L : \nabla \to \text{Hol}(L, \nabla) \in G,$$  \hspace{1cm} (1)

where $\text{Hol}(L, \nabla)$ is the holonomy along $L$, and $G$ is the gauge group which we, for now, assume to be compact. Loops functions like (1) correspond to an underlying space of gauge connections which includes also gauge equivalent connections. This will also resemble Connes’ construction of standard model, since we get an algebra of matrix valued functions over a configuration space just as Connes’ matrix valued functions over a manifold.

Furthermore, in loop quantum gravity a fibration of the space-time manifold into global space and time directions is considered. This is done in order to apply a canonical quantization scheme. In the present case the aim is to construct a spectral triple over a functional space of connections. For this purpose such a fibration is not needed and we therefore consider the whole manifold. Thus, the connections considered are space-time connections.

The central achievement of loop quantum gravity is its ability to obtain a separable hilbert space of loop functions via diffeomorphism invariance\(^1\) (see [8] and references therein). It is possible to extend these results to the case of a noncommutative algebra; we represent certain equivalent classes of noncommutative loop operators on a diffeomorphism invariant, separable hilbert space. Further, the Dirac operator we construct on the holonomy algebra is diffeomorphism invariant and hence also descends to the diffeomorphism invariant hilbert space. This is important since the Dirac operator stores the full physical information.

\(^1\)In fact, diffeomorphism invariance alone does not give a separable hilbert space. Instead one has to use a generalized notion of diffeomorphisms, see [8].
Let us finally add a note on noncommutativity and quantum theory. Clearly, the noncommutativity suggested is classical: It is simply related to the non-Abelian structure of the group $G$ and therefore carries no quantum aspect. On the other hand, the Dirac operator which we construct will resemble a global functional derivation. As a Dirac operator it carries spectral information of the underlying space – the space of connections – and will enable integration theory. In this sense, the quantum aspect enters through the constructed Dirac operator.

Outline of the paper

The algebra of (untraced) holonomy transformations, which is a central object in this paper, is introduced as the hoop group $\mathcal{HG}$ in section 2. Since a smooth connection in a $G$-bundle maps a loop $L \in \mathcal{HG}$ into $G$ homomorphically via the holonomy transform we define in section 3 the space $\mathcal{A}$ of generalized connections as the set of homomorphisms 

$$\mathcal{A} = \text{Hom}(\mathcal{HG}, G).$$

This is the functional space on which we wish to do geometry. Conversely, since the hoop group acts on $\mathcal{A}$ simply by 

$$H_L(\nabla) = \nabla(L),$$

we interpret $\mathcal{HG}$ as a noncommutative function algebra on $\mathcal{A}$. The key technical tool for dealing with the space $\mathcal{A}$ is described in section 4. Referring to [9] we identify $\mathcal{A}$ as a projective limit over the representations of finite subgroups of the hoop group. This enables us to work with only finitely many loops at a time. The space $\mathcal{A}$ seen from finitely many loops looks like

$$G^n = G \times \ldots \times G,$$

where $n$ is related to the number of loops. Thus, since $G$ is a Lie-group, we are at this level dealing with just an ordinary manifold and we can therefore write down Dirac operators from classical geometry.

A concrete realization of this technique/idea is worked out in section 5. Since we are sitting in a projective system we are not entirely free to choose our Dirac operator; it has to fit with different choices of finitely many
loops. In fact, problems arise from loops with common line segments. We remedy this defect by technically excluding such combinations of loops. Also, for technical reasons, we choose the classical Euler-Dirac instead of the real Dirac operator.

In doing this the link to connections becomes unclear. This is clarified in section 6 where we show that the connection are still contained in the spectrum of the modified algebra.

A key issue in the construction presented is the implementation of diffeomorphism invariance; the concern of section 7. Using once more ideas from loop quantum gravity we construct diffeomorphism invariant states and a diffeomorphism invariant algebra of loop operators. Finally we are concerned with the question whether the obtained, diffeomorphism invariant triple is spectral in the sense of Connes. It turns out not to be the case since the eigenvalues of the Dirac operator has infinite multiplicity. In particular, this is linked to the kernel of the Euler-Dirac operator on $G$ which has dimension larger than one. Although we are at present unable to solve the problem we suggest some possible solutions.

We provide a final discussion and outlook in section 8 and leave some extra material for the appendices.

2 The hoop group

The starting point is a manifold $M$. Let us for simplicity assume that $M$ is topological trivial. On this manifold we consider first the set $\mathcal{P}$ of piecewise analytic paths

$$\mathcal{P} := \{ P(t) | P : [0, 1] \rightarrow M \},$$

where paths which differ only by a reparameterization are identified. If two paths $P_1, P_2 \in \mathcal{P}$ have coinciding end and start points, $P_1(1) = P_2(0)$, we define their product

$$P_1 \circ P_2(t) = \begin{cases} 
P_1(2t) & t \in [0, \frac{1}{2}] \\
P_2(2t - 1) & t \in [\frac{1}{2}, 1]
\end{cases}.$$

In case $P_1(1) \neq P_2(0)$ we set their product is zero. There is a natural involution on $\mathcal{P}$

$$P^*(t) = P(1 - t) \quad \forall t,$$
since \((P^*)^* = P\), \((P_1 \circ P_2)^* = P_2^* \circ P_1^*\).

Choose an arbitrary basepoint \(x^o \in M\). We call a path which starts and ends at \(x^o\) a based loop. Further, by a simple loop we understand a based loop for which

\[ L(t) = x^o \iff t \in \{0, 1\}. \]

The set of based loops is called loop space and is denoted \(\mathcal{L}_{x^o}\). An equivalence relation on loop space is generated by identifying loops which differ by a simple retracing along a path

\[ L_1 \sim L_2 \iff \begin{cases} L_1 = P_1 \circ P_2 \circ P_2^* \circ P_3 \\ L_2 = P_1 \circ P_3 \end{cases}, \]

where \(L_i \in \mathcal{L}_{x^o}\), and \(P_i \in \mathcal{P}\). An equivalence class \([L]\) is called a hoop \[10\].

The set of hoops is called the hoop group, denoted

\[ \mathcal{H}\mathcal{G} = \mathcal{L}_{x^o}/\sim, \]

since the involution on \(\mathcal{H}\mathcal{G}\) gives an inverse element

\[ [L] \cdot [L]^* = [L_{id}], \]

where \(L_{id}\) is the trivial loop

\[ L_{id}(t) = x^o \quad \forall t \in [0, 1]. \]

To ease the notation we will denote a hoop \([L]\) simply by an representative \(L\) of the equivalence class. Furthermore, for literary reasons we often call \([L]\) a loop.

We emphasize that since \(M\) has no metric any notion of distance between and length of loops and hoops is meaningless.

### 3 Hoop group representations

Consider the space of homomorphisms

\[ \bar{A} = \text{Hom}(\mathcal{H}\mathcal{G}, G), \]

from the hoop group into a matrix representation of a compact Lie group \(G\) (we denote both the group and its representation by \(G\). The group \(G\) is
assumed to have a both left and right invariant metric). That is, for $\nabla \in \tilde{A}$ we have
\[
\nabla(L_1) \cdot \nabla(L_2) = \nabla(L_1 \circ L_2) \quad \forall L_1, L_2 \in \mathcal{H}G .
\]
If we denote by $\mathcal{A}$ the space of smooth connections in a bundle with structure group $G$, then a connection $\nabla \in \mathcal{A}$ clearly gives such a homomorphisms via
\[
\nabla : L \to Hol(L, \nabla) ,
\]where $Hol(L, \nabla)$ is the holonomy of the connection around the loop $L$. Let us recall that the holonomy is the parallel transport of the connection along a path $P$
\[
Hol(P, \nabla) = \mathcal{P} \exp \left(i \int_P \nabla \right) ,
\]where $\mathcal{P}$ is the path ordering symbol. The parallel transport along a closed loop is a non-local, gauge covariant object and the trace hereof, the Wilson loop, is gauge invariant. From (2) we conclude that
\[
\mathcal{A} \subset \tilde{A} .
\]
It is however important to realize that $\tilde{A}$ is much larger\footnote{In fact, $\mathcal{A}$ has, with respect to the Ashtekar-Lewandowski measure, zero measure in $\tilde{A}$ (modulo gauge transformations).}.  

The space $\tilde{A}$ is the general space of field configurations on which we wish to obtain a geometrical structure. We therefore consider an algebra of functions over $\tilde{A}$. To this end we first notice that a hoop $L \in \mathcal{H}G$ gives rise to a function $H_L$ on $\tilde{A}$ into $G$ via
\[
H_L(\nabla) = \nabla(L) ,
\]where $\nabla \in \tilde{A}$. Notice that
\[
H_{L_1} \cdot H_{L_2} = H_{L_1 \circ L_2} , \quad (H_L)^* = H_{L^{-1}} , \quad H_{L_{id}} = 1 ,
\]where $L_i \in \mathcal{H}G$. The set of complex linear combinations of all functions $H_L$ is a $*$-algebra. The norm of a general linear combination
\[
a_1 H_{L_1} + \ldots + a_n H_{L_n}
\]
is defined by
\[ \|a_1 H_{L_1} + \ldots + a_n H_{L_n}\| = \sup_{\nabla \in \mathcal{A}} (\|a_1 H_{L_1}(\nabla) + \ldots + a_n H_{L_n}(\nabla)\|), \]
where \(\|\cdot\|\) on the rhs is the matrix norm. Notice that
\[ \|H_L\| = 1 \quad \forall L \in \mathcal{H} \mathcal{G} \]
if the group \(G\) is orthonormal or unitary. The closure in this norm of the algebra generated by functions \(H_L\) is a \(C^*\)-algebra. Let us denote it \(C^*(\mathcal{L})\).

For now, this is the noncommutative function algebra over \(\tilde{\mathcal{A}}\) which we wish to imbed in a spectral triple. However, as we will explain in the next section, we need to change the algebra slightly to be able to construct a Dirac operator.

4 The space \(\tilde{\mathcal{A}}\) as a projective limit

The space \(\tilde{\mathcal{A}}\) was analyzed in [9] in a somewhat different context. Here, the authors identify \(\tilde{\mathcal{A}}\) with a projective limit (see appendix B for details on projective and inductive limits)

\[ \lim_{\leftarrow} \text{Hom}(F, G), \quad F \in \mathcal{F}, \]
where \(\mathcal{F}\) is the set of all strongly independent, finitely generated subgroups of \(\mathcal{H} \mathcal{G}\) (strongly independent in the sense of [10]). Let \(L_1 \ldots L_{n(F)}\) be the strongly independent generators of \(F \in \mathcal{F}\). We then identify

\[ \text{Hom}(F, G) \simeq G^{n(F)} \]

since we just map \(\phi \in \text{Hom}(F, G)\) into [10]

\[ (\phi(L_1), \ldots, \phi(L_{n(F)})) \in G^{n(F)}. \]  \(\text{(4)}\)

This identification is of great advantage: Since \(G^{n(F)}\) is a Lie group it is now straightforward to construct a spectral triple by choosing a metric on \(G\) and

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\(^3\)The authors of [9] considered the smaller space \(\tilde{\mathcal{A}}/\text{Ad}\) of smooth connections modulo local gauge transformations. This otherwise important difference is not essential for the issues regarding the projective limit.
then using the Euler-Dirac\textsuperscript{4} or the Dirac operator (since $G^n(F)$ is a Lie group it is parallelizable and hence possesses a spin structure). Once a geometrical construction on $G^n(F)$ is obtained we extend this to all of $\mathcal{A}$ by taking the projective limit of the algebra and the inductive limit of the relevant Hilbert space.

Thus, it is tempting to consider the Hilbert space
\[ L^2(G^n(F), M_k(\mathbb{C}) \otimes S) , \]
where $S$ is the Clifford algebra or the spin bundle corresponding to either the Euler-Dirac or the Dirac operator. $L^2$ is with respect to the Haar measure on $G^n(F)$. The problem with this construction is that the Euler-Dirac and the Dirac operators contain all the metric information on the underlying space \textsuperscript{[2]} and the structure maps defining the projective limit are not metric. The problems can be traced back to the definition of the generating hoops. Here, following \textsuperscript{[10]}, we encounter overlapping hoops which leads to structure maps
\[ P_{F_1, F_2} : \text{Hom}(F_1, G) \to \text{Hom}(F_2, G) \]
of the form\textsuperscript{5}
\[ (g_1, g_2, g_3) \to g_1 g_2 , \]
where $F_1 \subset F_2$ lie in $\mathcal{F}$. The problem is that such maps do not have a canonical isometric cross-section.

The solution to this problem is to redefine our notion of generating hoops. This, in turn, will affect the projective limit. Let us go into details in the next section.

Before we do that we end this section by mentioning that the identification \textsuperscript{[1]} indirectly chooses an orientation of the hoop. Basically, there are two possible identifications corresponding to either $\varphi(L)$ or $\varphi(L^{-1})$. Therefore, we can identify $\text{Hom}(F, G)$, where $F$ is a subgroup generated by a single hoop, with both $G$ and $G^{-1}$.

\textsuperscript{4}see appendix \textsuperscript{A}

\textsuperscript{5}here $\text{Hom}(F_1, G)$ and $\text{Hom}(F_2, G)$ are, as an example, identified with $G^1$ and $G^3$, respectively. A similar structure map with a $G^2$-subgroup is not possible due to the special construction of independent hoops.
5 Spectral triples over $G^n$ and the projective limit

Let $\mathcal{F}_I$ be the set of finitely generated subgroups of $\mathcal{H}G$ with the property that they are generated by simple, non-selfintersecting loops that do not have overlapping segments or points$^6$. The inclusion of groups $F_1 \subset F_2$ gives an inductive system on $\mathcal{F}_I$ and therefore a projective structure on $\{\text{Hom}(F, G)\}_{F \in \mathcal{F}_I}$. Again, we can identify $\text{Hom}(F, G)$ with $G^{n(F)}$, where $n(F)$, as before, is the number of simple loops in a generating set of $F$.

Since we are looking at subgroups with the property that no two loops have overlapping segments the maps $P_{F_1, F_2} : G^{n(F_2)} \to G^{n(F_1)}$ induced by the inclusion $F_1 \subset F_2$ are just given by deleting some coordinates or inverting some coordinates. This eliminates structure maps of the form $\text{(5)}$ and thus enables the following construction of a spectral triple.

5.1 The hilbert space

We first construct the hilbert space. We choose a left and right invariant metric on $G$. We therefore also have a metric on $G^{n(F)}$ and hence we can construct the Clifford bundle $\text{Cl}(TG^{n(F)})$. Due to the invariance of the metric we get the result

**Proposition 5.1.1** There is an embedding of hilbert spaces

$$P_{F_1, F_2}^* : L^2(G^{n(F_1)}, \text{Cl}(TG^{n(F_1)})) \to L^2(G^{n(F_2)}, \text{Cl}(TG^{n(F_2)})),$$

where the measure on $G^{n(F_i)}$ is the Haar measure.

**Proof.** We will need some notation. Let $e_1, \ldots, e_n$ be an orthonormal basis in $T_{id}G$, the tangent space over the identity in $G$. Due to the invariance property of the metric we get that

$$D_g(id)(e_1), \ldots, D_g(id)(e_n)$$

is an orthonormal basis in $T_gG$. Here $D_g(id)$ denotes the differential of the map

$$m_g : G \to G, \quad m_g(g_1) = gg_1$$

$^6$In contrast to [10] we no-longer require loops to be piece-wise analytic. Nor does the manifold need a real analytic structure.
in the identity. We will also use the notation $e_1, \ldots, e_n$ to denote the corresponding global vector fields in $TG$, i.e. $e_k(g) = D_g(id)(e_k)$.

We will abbreviate $n(F_i)$ by $n_i$. We first consider the case where the projection $P_{F_1, F_2}$ is of the form

$$P_{F_1, F_2}(g_1, \ldots, g_{n_2}) = (g_1, \ldots, g_{n_1}),$$

and denote by

$$e_1^1, \ldots, e_{n_1}^1, e_1^2, \ldots, e_{n_1}^2, \ldots, e^n_1, \ldots, e^n_{n_1},$$

the global vector fields on $G^{n_i}$, where $e^k_1, \ldots, e^k_n$ denote the global vector fields $e_1, \ldots, e_n$ in the $k$'th component of $TG^{n_i}$.

Put $\bar{g}_{n_i} = (g_1, \ldots, g_{n_1})$. We clearly have that

$$\langle e^k_1(\bar{g}_{n_1}), e'^k_1(\bar{g}_{n_1}) \rangle_{T_{\bar{g}_{n_1}}G^{n_1}} = \langle e^k_1(\bar{g}_{n_2}), e'^k_1(\bar{g}_{n_2}) \rangle_{T_{\bar{g}_{n_2}}G^{n_2}},$$

where $k, k' \leq n_1$.

An element in $L^2(G^{n_1}, Cl(TG^{n_1}))$ is a linear combination of elements of the form $fe$, where $e$ is a product of elements in

$$e_1^1, \ldots, e_{n_1}^1, e_1^2, \ldots, e_{n_1}^2, \ldots, e^n_1, \ldots, e^n_{n_1},$$

and $f \in L^2(G^{n_1})$. We define

$$P^*_{F_1, F_2}(fe) = \tilde{f}e,$$

where

$$\tilde{f}(\bar{g}_{n_2}) \equiv f(P_{F_1, F_2}(g_{n_2})) = f(\bar{g}_{n_1}).$$

This map preserves the inner product since

$$\langle fe, f'e' \rangle_{L^2(G^{n_1}, Cl(TG^{n_1}))} = \int f(\bar{g}_{n_1})f'(\bar{g}_{n_1})\langle e, e' \rangle_{T_{\bar{g}_{n_1}}G^{n_1}} d\mu_H(g_1) \cdots d\mu_H(g_{n_1})$$

$$= \int f(\bar{g}_{n_1})f'(\bar{g}_{n_1})\langle e, e' \rangle_{T_{\bar{g}_{n_2}}G^{n_2}} d\mu_H(g_2) \cdots d\mu_H(g_{n_2})$$

$$= \langle f\tilde{e}, f'\tilde{e}' \rangle_{L^2(G^{n_2}, Cl(TG^{n_2}))},$$

where we have used that $\int 1 d\mu_H = 1$.

To finish the construction we only need to consider a map of the form

$$P_{F_1, F_2}(g) = g^{-1},$$

(7)
since any structure map is the composition of maps of the type (6) and (7). However, the map

\[ P^*_{F_1,F_2} : L^2(G, Cl(TG)) \rightarrow L^2(G, Cl(TG)), \]

defined by (with the notation from before)

\[ P^*_{F_1,F_2}(f)(g) = f(g^{-1})D_{P^{-1}_{F_1,F_2}}(e) \]
is, due to the left and right invariance of the metric, a map of hilbert spaces. This completes the proof.

We can now construct the direct limit of these hilbert spaces (see appendix B for a more detailed discussion on inductive limits). This is done in the following way: First define

\[ H_{\text{alg}} = \bigoplus_{F \in \mathcal{F}_I} L^2(G^n(F), Cl(TG^n(F)))/N, \]

where \( N \) is the subspace generated by elements of the form

\[ (\ldots, v, \ldots, -P^*_{F_1,F_2}(v), \ldots). \]

In other words, we identify the vectors \( v \) and \( P^*_{F_1,F_2}(v) \).

The problem is now to define an inner product on \( H_{\text{alg}} \). Decompose \( L^2(G, Cl(TG)) \) into the subspace generated by the function 1 and the orthogonal complement. We will write this as

\[ L^2(G, Cl(TG)) = H_1 \oplus H_2, \]

where \( H_1 = \mathbb{C} \). Given a vector \( v \in L^2(G^n(F), Cl(TG^n(F))) \) this can be uniquely decomposed into vectors of the form

\[ v_1 \otimes \cdots \otimes v_{n(F)} , \]

where each \( v_i \) belong either to \( H_1 \) or \( H_2 \). It is therefore enough to define the inner product of vectors of this type. Further, let \( v_1 \in L^2(G^n(F_1), Cl(TG^n(F_1))) \) and \( v_2 \in L^2(G^n(F_2), Cl(TG^n(F_2))) \) be vectors of this form. We will assume that in the tensor decomposition of \( v_1 \) and \( v_2 \) only elements from \( H_2 \) appear. We can assume this since else \( v_1 \) and/or \( v_2 \) will be the image under one of the


\[ \langle v_1, v_2 \rangle = \langle P_{F_1,F_3}^* (v_1), P_{F_2,F_3}^* (v_2) \rangle_{L^2(G^n(F_3), Cl(TG^n(F_3)))} \]

if there exist a \( F_3 \) with \( F_1, F_2 \subseteq F_3 \) and zero else. The completion of the \( H_{\text{alg}} \) with respect to this inner product is the inductive limit and will be denoted by \( H'_{\text{si}} \) (\( \text{si} \sim \text{segment independent} \)).

In equation (8), since \( v_1 \) and \( v_2 \) are, per definition, decomposed into tensor-powers in \( H_2 \), the inner product will be different from zero only when \( F_1 = F_2 \).

We can give a more concrete description of \( H'_{\text{si}} \) in terms of the hilbert space \( H_2 \), namely

\[ H'_{\text{si}} = \mathbb{C} \oplus (\oplus_l H_2) \oplus (\oplus_l H_2 \otimes H_2) \oplus \ldots , \]

where \( l^k \) is the set of all products of \( k \)-nonintersecting simple loops, and where \( \oplus \) means orthogonal sum. The first \( \mathbb{C} \) corresponds to the trivial loop. For each simple loop we get a copy of \( L^2(G, Cl(TG)) \); however the constant functions are identified in the inductive limit, and we hence only get a copy of \( H_2 \) for each simple loop. This picture continues for products of two simple loops and so on.

### 5.2 The Euler-Dirac operator

On each of the hilbert spaces \( L^2(G^n(F), Cl(TG^n(F))) \) we have a canonical Euler-Dirac operator

\[ \mathcal{P}(\xi) = \sum e_i \cdot \nabla e_i (\xi) , \]

where \( \{ e_i \} \) are global, orthonormal sections in the tangent bundle of \( G^n(F) \) and \( \nabla \) is the Levi-Civita connection. It is clear that this Euler-Dirac operator commutes with the structure maps \( P_{F_1,F_2}^* \) not involving inversions. According to [14], \( \mathcal{P} \) can, under the identification of \( Cl(TM) \) with \( \wedge^*(T^* M) \) (differential forms), be identified with \( d + d^* \). The exterior derivative \( d \) is invariant under all diffeomorphisms, and since \( d^* \) only additionally depends on the metric and the metric on \( G \) is invariant under inversions, the Euler-Dirac operator also commutes with structure maps involving inversions. Therefore get an Euler-Dirac operator \( \mathcal{P} \) on \( H'_{\text{si}} \).
The reason why we choose the Euler-Dirac operator instead of the classical Dirac operator is that the former has better functorial properties. In particular, it is invariant under inversions of loops. If we consider for example the Abelian case, $G = S^1$, and parameterize $S^1$ by $\theta \in [0, 2\pi]$, then the Dirac operator reads

$$\mathcal{D} = i \frac{\partial}{\partial \theta},$$

which, under inversion of the underlying loop

$$G \rightarrow G^{-1}$$

picks up a minus sign. On the other hand, we have just argued that the Euler-Dirac operator is invariant under inversions.

It is of course desirable to work out a construction that works for the classical Dirac operator, but for now we choose to work with the easier Euler-Dirac operator.

The particular choice of “Dirac” operator in (11) is motivated by its resemblance to a (integrated) functional derivation. Heuristically: A (smooth) connection is determined by holonomies along hoops. In the projective system described here we consider first a finite number of hoops and a connection is thus described ’coarse-grained’ by assigning group elements to each of the finitely many elementary hoops. The Euler-Dirac operator (10) takes the derivative on each of these copies of the group $G$ and throws it into the Clifford bundle. In this way the Dirac operator resembles a functional derivation operator.

We interpret this Euler-Dirac operator as intrinsically ’quantum’ since it bears some resemblance to a canonical conjugate of the connection. Heuristically, we write

$$\mathcal{D} \sim \frac{\delta}{\delta \nabla}$$

and

$$H_L \sim 1 + \nabla$$

due to $H_L$’s relation to the holonomy map. Here $\nabla$ is a connection. From (13) and (14) the non-vanishing commutator

$$[\mathcal{D}, H_L] \neq 0$$
obtains, on a very heuristical level, a resemblance to a commutation relation of canonical conjugate variables. Thus, it is not the noncommutativity of the algebra of holonomy loops (to be defined rigorously below) which is ‘quantum’ but rather the Dirac operator and its interaction with the algebra. This is an essential point for the interpretation of the geometrical construction presented.

5.3 The algebra

We will construct our algebra as an algebra of operators on $\mathcal{H}_{si}^\prime$, or rather a variant of hereof denoted $\mathcal{H}_{si}$. This algebra will be similar, but not equal, to the group algebra $C^*(\mathcal{L}_{x^0})$ of hoops.

The hilbert space $\mathcal{H}_{si}$ is constructed the same way as $\mathcal{H}_{si}^\prime$ but where

$$L^2(G^{n(F)}, Cl(TG^{n(F)}) \otimes M_n(\mathbb{C}))$$

is used instead of $L^2(G^{n(F)}, Cl(TG^{n(F)}))$. Here $n$ is the size of the representation of $G$. The reason for the additional matrix factor is that we wish to represent the holonomy loops by left matrix multiplication.

The decomposition analogous to (9) looks like

$$\mathcal{H}_{si} = M_n(\mathbb{C}) \oplus (\oplus_{n^2} \mathcal{H}_2 \otimes M_n(\mathbb{C})) \oplus (\oplus_{n^2} \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes M_n(\mathbb{C})) \ldots.$$  

If we are given a simple hoop $L$, we construct an operator $H_L$ on $\mathcal{H}_{si}$ in the following way: For a subgroup $F \in \mathcal{F}_I$ we make use of the identification (4) of $G^{n(F)}$ with $\text{Hom}(F, G)$ and hence define

$$H_L(s)(\varphi) = (id \otimes \varphi(L))(s(\varphi)),$$

where $s \in L^2(G^{n(F)}, Cl(TG^{n(F)}) \otimes M_n(\mathbb{C}))$ and where $\varphi(L) = id$ when $L \notin F$. Since $H_L$ respects the maps $P_{F_1,F_2}$ we get an operator $H_L$ on $\mathcal{H}_{si}$.

For a general hoop $L$, using the unique decomposition of $L$ into simple hoops $L_1 \circ \ldots \circ L_n$ define

$$H_L = H_{L_1} \circ \ldots \circ H_{L_n}.$$ 

Our algebra, which we denote $A$, is the $C^*$-algebra generated by the operators $H_L, L \in \mathcal{H}_G$.  

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It is important to realize that the algebra $A$ is not identical to the $C^*$-algebra $C^*(\mathcal{L}_{x^o})$ introduced in section 3. That is, we have not obtained a representation of the group algebra of hoops on $M$. To illustrate this consider the following two situations:

1. **Loops with common line segment.** We consider for example two loops $L_1$ and $L_2$ where

   \[ L_1 = P_1 \circ P_2 , \quad L_2 = P_2^* \circ P_3 , \]

   with $P_i \in \mathcal{P}$. Hence

   \[ L_3 \equiv L_1 \circ L_2 = P_1 \circ P_3 . \]

2. **Intersecting loops.** Consider two loops $L_4$ and $L_5$ where

   \[ L_4(t_1) = L_5(t_2) \neq x^o . \]

   In the first case $L_1$, $L_2$ and $L_3$ cannot belong to the same subgroup $F \in \mathcal{F}_I$ since they all have common line segments. Thus, their associated operators $H_{L_i}$ act on different parts of the hilbert space. This means that they commute

   \[ H_{L_1} \cdot H_{L_2} = H_{L_2} \cdot H_{L_1} . \]

   In particular, it means that

   \[ H_{L_1} \cdot H_{L_2} \neq H_{L_3} . \]

   In the second case, the product $L_4 \circ L_5$ does not even belong to any subgroup $F \in \mathcal{F}_I$. Thus, the operator $H_{L_4 \circ L_5}$ only exist as the composition $H_{L_4} \cdot H_{L_5}$.

### 5.4 An extended Euler-Dirac operator

The Euler-Dirac operator defined in equation (10) acts, basically, on the hilbert space $\mathcal{H}'_{si}$. When acting on $\mathcal{H}_{si}$ it does not ‘see’ the matrix part of the hilbert space. This need not be so. We can for example define an extended Euler-Dirac operator by

\[
\mathcal{P}_{\text{ext}}(\xi \otimes m)(g) = \mathcal{P}(\xi(g)) \otimes m + \xi(g) \otimes m_n(g) \cdot m ,
\]

where $m_n(g)$ is a matrix valued function on $G^{n(F)}$ and $\xi \otimes m \in \mathcal{H}_{si}$. The form of the operator in equation (15) is similar to the Dirac operators of the almost commutative geometries (including the standard model). See for example [3].
6 The space of connections

So far, we have considered a geometrical structure over spaces related to certain loop group homomorphisms. We now want to describe in more detail the role of connections in this construction.

In the above we constructed the Hilbert space

\[ \mathcal{H}_{si} = \lim_{\rightarrow} L^2(G^m, Cl(TG^m) \otimes M_n(\mathbb{C})) . \]

Let us for simplicity now consider the same Hilbert space but without the spin structure and the matrix factor:

\[ \mathcal{H} = \lim_{\rightarrow} L^2(G^n) . \]

Hence, \( \mathcal{H}_{si} \) is \( \mathcal{H} \) with coefficients in an infinite dimensional Clifford algebra tensored with \( n \) by \( n \) matrices. If we backtrack our line of reasoning we first make the identification

\[ \mathcal{H} = L^2(\lim_{\leftarrow} \text{Hom}(F, G)) , \]

where \( F \in \mathcal{F}_I \). Let \( \nabla \) be a fixed, smooth connection in \( \mathcal{A} \). As already mentioned, for a given \( F \), \( \nabla \) gives rise to a homomorphism into \( G \) via the holonomy loop

\[ \nabla : L \rightarrow \text{Hol}(L, \nabla) \in G , \]

where \( L \in F \). It is easy to see that this commutes with the structure map and hence that we get a map

\[ \mathcal{A} \rightarrow \lim_{\leftarrow} \text{Hom}(F, G) . \]

Clearly, this map is injective. We therefore conclude that \( \mathcal{H} \) is a Hilbert space over a space which contains all smooth connections.

6.1 Distances on \( \bar{\mathcal{A}} \)

On a Riemannian spin-geometry the Dirac operator \( D \) contains the geometrical information of the manifold \( \mathcal{M} \). In particular, distances can be formulated in a purely algebraic fashion due to Connes [2]. Given two points \( x, y \in \mathcal{M} \) their distance is given by

\[
    d(x, y) = \sup_{f \in C^\infty(M)} \{ |f(x) - f(y)| : \|\|D, f\|\| \leq 1 \} \quad (16)
\]
On a noncommutative geometry the state space replaces the notion of points. It is possible to extend the notion of distance to the state space by generalizing (16) in an obvious manner.

For the present case, however, it is quite unclear in what sense a Dirac operator incorporates a distance. Further, the usefulness of such a notion is in the present situation not obvious. Clearly, if the Dirac operator (16) is interpreted as a metric it will give rise to distances on the space $\mathcal{A}$.

For example, it is not difficult to see that for the $G = U(1)$ case the distance between two smooth connections will be infinite. This can be seen by first noting that the distance induced by the Dirac operator on $U(1)^n$ is just the sum of distances on each copy of $U(1)$. This product distance of two smooth connections will differ on infinitely many non-intersecting loops. Further, summing these differences will give an infinite distance between the points. Perhaps this is not so surprising considering the fact that our geometry is infinite dimensional.

### 7 Diffeomorphism invariance

Clearly, the construction considered so far is very large. In fact, the Hilbert space $\mathcal{H}_{si}$ is not separable and it is unclear how to extract physical quantities in a well-defined manner. What is missing is of course the implementation of diffeomorphism invariance relative to the underlying manifold $M$. Invariance under arbitrary coordinate transformations is the defining symmetry of general relativity and it is therefore an essential ingredients in the formalism. It turns out that the ‘size’ of the construction can indeed be drastically reduced by taking diffeomorphism invariance into account\(^7\). First we write down transformation laws of Hilbert states and operators. Next, we define diffeomorphism invariant states via a formal sum over states connected via diffeomorphisms. We are able to represent loop operators on such ‘smeread states albeit not as a representation of the operator algebra $\mathcal{A}$. In a subsequent subsection we investigate an alternative approach where we introduce an equivalence of spectral triples to cut down the size of both the Hilbert space and the algebra as well as the corresponding Euler-Dirac operator si-

\(^7\)The construction on this section works both for diffeomorphisms and for extended diffeomorphisms, and we will therefore notationally not distinguish between them. But only the latter case give a seperable Hilbert space, and hence the verification of (or lack of) the axioms of a spectral triple only makes sense for the extended diffeomorphisms.
multaneously. We find that the two approaches are in fact equivalent. Finally we look at the spectrum of the relevant Euler-Dirac operators and show that it is not fully a Dirac operator in the sense of Connes.

We assume that the space-time dimension of the manifold $M$ is larger than three. Since there exist no knot theory outside 3 dimensions we hereby avoid considering different “knot states” etc.

7.1 Transformations of states and operators

We first consider states in $L^2(G^n(F), Cl(TG^n(F)) \otimes M_n(\mathbb{C}))$ which are polynomial in $g_1, \ldots, g_n(F)$ tensored with constant elements in $Cl(TG^n(F))$. A diffeomorphism $d \in Diff(M)$ which maps

$$d : L_i \to L'_i$$

has a natural action on such polynomials

$$d : p(g_1, \ldots, g_n(F)) \to p(g'_1, \ldots, g'_n(F)),$$

where $g'_i \in G'_i$ is the group corresponding to the new loop $L'_i$. Because we interpret states in $H_{si}$ as (polynomials in) holonomy loops we can really only state how polynomials and their closure should transform under diffeomorphisms. However, we can simply extend the transformation law (17) to all of $L^2(G^n(F), Cl(TG^n(F)) \otimes M_n(\mathbb{C}))$

$$d : \xi(g_1, \ldots, g_n(F)) \to \xi(g'_1, \ldots, g'_n(F)),$$

and via the inductive limit to all of $H_{si}$.

The action of the diffeomorphism group on the algebra $A$ is straightforward, simply taken from (17).

Above and in the following we only consider diffeomorphisms in $Diff(M)$ which preserve the basepoint $x^0$.

7.2 Diffeomorphism invariant states

From one point of view we need to solve the diffeomorphism constraint

$$d\xi = \xi, \quad \forall d \in Diff(M) ; \quad \xi \in H_{si}.$$

(18)
Let us start by investigating this. The following is inspired by [7, 12]. Equation (18) has, of course, the formal solution
\[ \tilde{\xi} = \sum_{d \in \text{Diff}(M)} d(\xi). \] (19)

This, however, makes no sense in \( \mathcal{H}_{si} \). Instead we need to consider the dual of \( \mathcal{H}_{si} \). So, given a vector \( \eta \in \mathcal{H}_{si} \) we let the formal sum (19) act on \( \eta \) like
\[ \tilde{\xi}(\eta) = \sum_{d \in \text{Diff}(M)} \langle d(\xi) | \eta \rangle. \] (20)

Strictly speaking this does not make sense either, since the sum on the right hand side need not be convergent. If we however define the action of \( \tilde{\xi} \) only on the algebraic part of \( \mathcal{H}_{si} \), i.e. only finite sums of elements in the sum (9), the sum (20) becomes finite if the summation over \( \text{Diff}(M) \) is understood correctly. We will now describe how this works:

First we define the projection onto symmetrized states. Given a state \( \xi \in \mathcal{H}_2^{\otimes n(F)} \otimes M_n \) we denote by \( \text{Diff}(M|F) \) diffeomorphisms which preserve form as well as orientation of all loops in \( F \). Consider next diffeomorphisms \( F \to F \) which do not lie in \( \text{Diff}(M|F) \). We denote these by \( \text{Diff}(F \to F) \). The symmetry group of \( F \), denoted \( \text{SG}_F \), is the quotient
\[ \text{SG}_F = \text{Diff}(F \to F)/\text{Diff}(M|F). \] (21)

They consist of certain permutations and inversions. The projection is defined by
\[ P(\xi) = \frac{1}{N_F} \sum_{d \in \text{SG}_F} d(\xi), \] (22)
where \( N_F \) is the number of elements in \( \text{SG}_F \). Next, consider the remaining diffeomorphisms which moves the loops in \( F \) outside \( F \). We define the sum (20) by
\[ \tilde{\xi}(\eta) = \sum_{d \in \text{Diff}(M)/\text{Diff}(F \to F)} \langle d(P\xi) | \eta \rangle, \] (23)
where the sum is interpreted as an effective sum, i.e. if \( d_1(F) = d_2(F) \) we identify \( d_1 \) and \( d_2 \). If \( \eta \in \mathcal{H}_2^{\otimes n(F)} \otimes M_n \) we find \( N_F \) contributions on the rhs of (23). Else it is zero.
The vector space of linear combinations of sums \(23\) is given the inner product
\[
\langle \tilde{\xi}_1 | \tilde{\xi}_2 \rangle = \tilde{\xi}_1(\xi_2) .
\]
(24)

The crucial point is that this sum has finitely many non-vanishing terms (see above).

The completion of this vector space in the norm (24) is a diffeomorphism invariant hilbert space which we denote by \(H_{diff}\).

The problem with this construction is that it is somewhat unclear how the algebra of hoops should be represented on \(H_{diff}\). Since our goal is to find a spectral triple involving not only a separable hilbert space but also a (separable) algebra and a well defined Dirac operator, this is clearly a crucial point. The difficulty stems from the fact that the algebra is not diffeomorphism invariant but rather co-variant. The Dirac operator, on the other hand, is diffeomorphism invariant and therefore causes no problems.

Essentially, we need to make sense of a 'smearing' of algebra elements according to
\[
\tilde{H}_L = \sum_{d \in \text{Diff}(M)} H_d(L) ,
\]
(25)
similar to equation (19). As it stands, equation (25) is meaningless. Instead we do the following: Given a hoop operator \(H_L \in A\) define the symmetrized operator by
\[
P_F(H_L) = \frac{1}{N_F} \sum_{d \in \text{SG}_F} H_d(L) ,
\]
(26)
where \(\text{SG}_F\) is the symmetry group of a subgroup \(F\) including \(L\). \(N_F\) is again the total number of elements in \(\text{SG}_F\). For example, if \(L\) is simple and \(F\) is the algebra generated by \(L\), we have
\[
P_F(H_L) = \frac{1}{2} \left( L + L^{-1} \right) .
\]

For a 'smeared' state \(\tilde{\xi} \in H_{diff}\) we define the action of \(H_L\) on \(\tilde{\xi}\) by
\[
H_L(\tilde{\xi}) = \sum_{d \in \text{Diff}(M)/\text{Diff}(F \rightarrow F)} d(P_F(H_L) \cdot P(\xi)) ,
\]
where we choose the representative $\xi$ so that $L$ and $\xi$ have coinciding domains and where $P_F$ is taken with respect to the subgroup $F$ defined by the domain of $\xi$ and $L$.

Note that we no longer deal with a representation of loops. For example, given a simple loop $L$ acting on a state $\xi$ with domain on a single copy of $G$ we find that (using a somewhat sloppy notation)

$$H_L \cdot H_L = \frac{1}{4}(H_{L^2} + H_{L^{-2}} + 2).$$

This relation, however, changes according to what states in $\mathcal{H}_{diff} H_L$ acts on.

### 7.3 Diffeomorphism invariance via equivalent triples

In the previous subsection we implemented diffeomorphism invariance by constructing diffeomorphism invariant states and defining an action of loop operators hereon. In fact, there is another option which, however, only works for extended diffeomorphisms. As explained above, the diffeomorphism group acts not only on the hilbert space but also on the algebra. We can therefore define an equivalence on the level of sub-triples; algebra, hilbert space and Euler-Dirac operator. This identification happens at the level of subgroups $F \in \mathcal{H}G$.

If we consider a single, simple loop $L$, the spectral triple associated to this is just

$$(< L >, L^2(G, Cl(TG) \otimes M_n), \mathcal{P}),$$

where $< a, b, \ldots >$ is the $C^*$-algebra generated by $\{a, b, \ldots \}$. Since all single, simple loops are diffeomorphic, at this level we just get expression (27) when we identify spectral sub-triples which are diffeomorphic. At the level of two nonintersecting simple loops $L_1$ and $L_2$ the spectral triple associated to this is

$$(< L_1, L_2 >, L^2(G^2, Cl(TG^2) \otimes M_n), \mathcal{P}).$$

Again, by identifying spectral triples of diffeomorphic loops we get at this level just expression (28). This picture simply continues for all finitely generated subgroups and taking the limit hereof gives us an equivalence class of spectral triples represented by the infinite dimensional triple

$$(< L_1, L_2, \ldots >, L^2(G^\infty, Cl(TG^\infty) \otimes M_n), \mathcal{P}).$$
Further, not only are all subgroups of nonintersecting loops with \( n \) generators diffeomorphic. There are also internal diffeomorphisms which shuffle the generators. One can factor out this symmetry by symmetrizing operators and states, just as we did in the previous subsection.

Therefore, the result is, in fact, identical to the result of the previous subsection.

Instead of symmetrizing one could also make the noncommutative quotient of the action of the internal diffeomorphism group \( SG_F \) (and the limit), i.e. consider the crossed product \( A_F \times SG_F \), where \( A_F \) is the part of our algebra acting on the \( F \) part. This would be more in the spirit of noncommutative geometry and Connes. We will investigate this alternative elsewhere.

**7.4 Spectral in the sense of Connes?**

It remains to clarify whether the spectral triple \( \mathcal{T} \) satisfy the conditions put forward by Connes \([2]\), see also \([11]\). A confirmative answer will permit us the full power of noncommutative geometry.

Clearly, on each level in the projective/inductive limit, the relevant Dirac (Euler-Dirac) operator satisfy the conditions for a spectral triple, simply per construction. The question remains whether it also holds in the limit.

There are three conditions. First, the operator

\[
[\mathcal{D}, a],
\]

where \( a \) belongs to the subspace of \( \mathcal{A} \) of finite linear combinations of loop operators, has to be bounded. A simple loop operator \( a = H_L \), acts, according to \( (26) \), on a state via

\[
\frac{1}{N_F} (H_{L_1} + H_{L_1^{-1}} + \ldots + H_{L_{n_F}} + H_{L_{n_F}^{-1}}),
\]

where the number \( n(F) \) refers to the domain of \( L \) and the state on which it acts (see section \( \S 2 \)). We can estimate the commutator of \( (31) \) with the Dirac operator by

\[
\frac{1}{N_F} \| [\mathcal{D}, H_{L_1} + H_{L_1^{-1}} + \ldots + H_{L_{n_F}} + H_{L_{n_F}^{-1}}] \| \leq \frac{1}{N_F} (\| [\mathcal{D}, H_{L_1}] \| + \| [\mathcal{D}, H_{L_1^{-1}}] \| + \ldots + \| [\mathcal{D}, H_{L_{n_F}}] \| + \| [\mathcal{D}, H_{L_{n_F}^{-1}}] \|) = \| [\mathcal{D}, H_L] \|. \tag{32}
\]
Because the operator
\[ H_L : G \to G ; \quad g \to g \]
is bounded we conclude that the operator \((30)\) is bounded for a simple loop operator. For compositions of simple loop operators the arguments is repeated and therefore we conclude that the first condition is satisfied.

Second, we need to investigate whether the operator
\[ \frac{1}{\mathcal{D} - l} ; \quad l \in \mathbb{C}/\mathbb{R} \]
is compact. In fact, this turns out not to be the case. Let us explain. For simplicity we leave out the matrix part of the hilbert space and simply consider the space
\[ L^2(G^n, Cl(TG^n)) = L^2(G, Cl(TG))^\oplus n, \]
where we only consider symmetrized (un-ordered) elements according to \((22)\). Given a set of eigenfunctions \(\{\xi_1, \ldots, \xi_m\}\) in \(L^2(G, Cl(TG))\) of the Dirac operator, the product
\[ \xi_{i_1} \otimes \ldots \otimes \xi_{i_n} \quad (33) \]
is an eigenfunction of the Dirac operator in \(L^2(G^n, Cl(TG^n))\). The problem is that if we find a function \(\xi_0\) in \(L^2(G, Cl(TG))\) with eigenvalue zero and which differs from the function 1, then we will automatically have an infinite dimensional eigenspace associated to any eigenvalue. To see this simply consider the function \((33)\) (remember that we consider only symmetrized products)
\[ \xi_0 \otimes \xi_{i_1} \otimes \ldots \otimes \xi_{i_n} \]
in \(L^2(G^{(n+1)}, Cl(TG^{(n+1)}))\). This is again an eigenfunction with the same eigenvalue as \((33)\).

According to Hodge theory (see theorem II.5.15 in [14]) the kernel of a Euler-Dirac operator on a compact manifold \(M\) is related to the cohomology group:
\[ \ker(\mathcal{D}) = \bigoplus \mathbb{H}^p , \quad \mathbb{H}^p = H^p(M; \mathbb{C}) , \]
and the cohomology group is, at least on an orientable manifold as the Lie group \(G\), not empty (the volume form is an example). Therefore we conclude that the Euler-Dirac operator in \((29)\) does not satisfy Connes’ second condition.
In principle, it is possible to correct this “flaw” in the construction of the Dirac operator in (29) by adding a bounded perturbation to $\mathcal{D}$ on each level in the projective/inductive limit. Such a perturbation will, in general, not be bounded in the limit itself. Indeed, if the perturbation is constructed in a way so that the perturbed Dirac operator satisfy condition two, then the full perturbation will be unbounded.

Changing the operator on each level of the projective/inductive limit does not change the K-homology class at each level. In the limit, however, the operator will be changed (the original Euler-Dirac operator in (29) does not have a K-homology class).

The third condition is self-adjointness. That $\mathcal{P}$ is self-adjoint is secured by construction.

Let us end this subsection by noting that the fact that the Dirac operator in the triple (29) does not satisfy Connes second condition may be interpreted as a hint that there exist some extra symmetries that have not been (and should be) factored out.

8 Discussion & outlook

In the present paper we presented new ideas on the unification of noncommutative geometry – in particularly Connes formulation of the standard model – and the principles of quantum field theory. We apply the machinery of noncommutative geometry to a general function space of connections related to gravity. A noncommutative algebra of holonomy loops is represented on a separable, diffeomorphism invariant hilbert space. An Euler-Dirac operator is constructed. The whole setup relies on techniques of projective and inductive limits of algebras, hilbert spaces and operators.

What comes out is a geometrical structure, including integration theory, on a space of field configurations modulo diffeomorphism invariance. A global notion of differentiation (the Dirac operator) is obtained. We find it remarkable that the whole construction boils down to the study of Dirac operators on various copies of some Lie-group.

Whereas the noncommutativity of the algebra is intrinsically classic we interpret the Dirac operator, which resembles a functional derivation, as ‘quantum’.
Certain problems arose during the analysis. First, we were unable to represent the full hoop group in a manner compatible with the Euler-Dirac operator. The solution proposed and analyzed is to consider only finite subgroups of non-intersecting loops (in the projective system). This modification has important consequences; instead of graphs (spin-networks) we deal with polynomials on various copies of the group. It is, however, not clear to us whether this is an important point.

More seriously, the final Dirac operator does not fulfill the conditions formulated by Connes. In particular, it has infinite-dimensional eigenspaces. Thus, we did not succeed to construct a spectral triple which satisfy the conditions put forward by Connes.

A prime concern to further development is to understand why our constructed Dirac operator is not spectral in the sense of Connes. We suspect that what is missing is a symmetry related to the infinite dimensional Clifford algebra, i.e. $Cl(T_{id}(G^\infty))$. Another possible solution is to use the ordinary Dirac operator instead of the Euler-Dirac operator. To account for lack of invariance under inversions of loops one can double the hilbert space: Instead of for each simple loop to assign the hilbert space of square integrable functions over $G$ we can assign two copies of this hilbert space; one for each orientation of the loop. The diffeomorphism associated with inversion of the loop will then act by interchanging the two hilbert spaces. There will however still be some problems, for example embedding properties when we increase the the number of copies of $G$'es.

Another concern is to extend the present construction to work for non-compact groups, since gravity involves $SO(3,1)$. The main problem will be the embeddings in the projective limit. For example, $L^2(G)$ is not naturally embedded in $L^2(G^2)$ whenever $G$ is non-compact. We believe, however, that this is a technical and solvable problem. Also, loops will no longer occur as states in the hilbert space; a priori not necessarily a problem.

Also, we would like to understand in what sense the noncommutativity of the holonomy algebra generates a bosonic sector and, if so, what it is. Clearly, noncommutativity permits inner automorphisms and nontrivial fluctuations of the Dirac operator. If we assume that we succeed to construct a Dirac operator $D$ satisfying all of Connes’ conditions, and if we consider fluctuations
around $D$ of the form

$$D \rightarrow \tilde{D} = D + A + JA J^\dagger,$$

where $J$ is Tomita’s anti-linear isometry \cite{13} and $A$ is a noncommutative one-form $^8$, $A \in \Omega^1_D$, then we can apply Chamseddine and Connes’ spectral action principle \cite{5, 6}. Thus, we can write down automorphism invariant quantities like

$$\langle \tilde{\xi} | \tilde{D} | \tilde{\xi} \rangle, \quad \text{Tr} \varphi(\tilde{D}), \quad \ldots.$$

Such terms can be interpreted as integrated quantities, schematically, of the form

$$\int_{A/\text{Diff}} d\nabla \ldots$$

which resembles a Feynman path integral and contains both fermionic and bosonic degrees of freedom. Here the integration is defined, modulo diffeomorphisms, on a space of connections.

In the introduction we motivated our analysis by stating that Connes formulation of the standard model coupled to gravity is intrinsically classical. With the aim of combining noncommutative geometry and the principles of quantum field theory, we have found a spectral triple which a priori appears to be quite far from field theory. It is clearly of prime concern to investigate whether the construction does contain a field theory limit and, if so, what it is.

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A Clifford algebras and Dirac operators

Here we give a brief review of Clifford algebras and the Euler-Dirac operator. For a detailed account see for example \cite{14}. Since we are interested in Clifford algebras over Lie-groups we only treat the Euclidian case.

$^8$elements of $\Omega^1_D$ are of the form $a_0[D, a_1] \ldots [D, a_n]$ where the $a_i$’s are elements of the algebra \cite{3}.
Let $V$ be a real vector-space. We define the tensor-algebra $T(V)$ as

$$T(V) = \sum_{i \geq 0} V^\otimes i$$

with multiplication

$$v_1 \otimes \ldots \otimes v_n \cdot u_1 \otimes \ldots \otimes u_m = v_1 \otimes \ldots \otimes v_n \otimes u_1 \otimes \ldots \otimes u_m.$$

Given a metric $\langle \cdot, \cdot \rangle$ on $V$ one defines the Clifford algebra as

$$Cl(V) = T(V)/(v \otimes u + u \otimes v = -2\langle v, u \rangle).$$

If $e_1, \ldots, e_n$ is an orthonormal basis of $V$ the Clifford algebra $Cl(V)$ consists of elements on the form

$$e_{i_1} \cdots e_{i_k},$$

where $i_1 < \cdots < i_k$ and with the product rules

$$e_i e_j = -e_j e_i, \quad i \neq j, \quad e_i^2 = -1.$$

There is an inner product on $Cl(V)$ given by

$$\langle e_{i_1} \cdots e_{i_k}, e_{j_1} \cdots e_{j_l} \rangle = 1$$

if $k = l$ and $i_1 = j_1, \ldots, i_k = j_l$ and zero else.

The group $O(n)$ acts on $Cl(V)$ by

$$o(e_{i_1} \cdots e_{i_k}) = o(e_{i_1}) \cdots o(e_{i_k}), \quad o \in O(n)$$

as automorphisms preserving the inner product. In particular one also gets an action of $\mathfrak{so}(n)$ on $Cl(V)$.

For a manifold $M$ with a metric, one defines the Clifford bundle $Cl(TM)$ as the bundle

$$M \ni m \to Cl(T_m M),$$

where the inner product on $T_m M$ is the one given by the metric.

Let $\nabla$ denote the Levi-Civita connection associated to the metric. Via the extension of the action of $O(n)$ from $V$ to $Cl(V)$, the Levi-Civita connection extends to a connection in $Cl(TM)$ via the formula

$$\nabla(e_{i_1} \cdots e_{i_k}) = \sum_l e_{i_1} \cdots \nabla(e_{i_l}) \cdots e_{i_k},$$

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where $e_i$ are local orthonormal sections in $TM$.

One defines the Euler Dirac operator $D$ by

$$L^2(M, Cl(TM)) \ni s \rightarrow D(s) = \sum e_i \cdot \nabla e_i s,$$

where $\{e_i\}$ is a local orthonormal sections in $TM$.

Since one wants to work with hilbert spaces one complexifies the space $L^2(M, Cl(TM))$ leaving the notion unchanged.

## B Projective and inductive limits

Here we review the concepts of projective and inductive limits. For a different treatment we refer to [9].

### B.1 Projective limits

To illustrate the concept of a projective limit we will consider the index set $\mathbb{N}$ and for each $n \in \mathbb{N}$ the space $\mathbb{R}^n$. If $n_1 \leq n_2$ there are projection

$$P_{n_2, n_1} : \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_1}$$

given by

$$P_{n_2, n_1}(x_1, \ldots, x_{n_2}) = (x_1, \ldots, x_{n_1}).$$

We define the product

$$\prod_{n \in \mathbb{N}} \mathbb{R}^n = \{(X_n)_{n \in \mathbb{N}} | X_n \in \mathbb{R}^n\},$$

i.e. an element in $\prod_{n \in \mathbb{N}} \mathbb{R}^n$ is just where we pick an element in each $\mathbb{R}^n$ for all $n$. An element can thus be written as

$$(x_1^1, (x_1^2, x_2^2), (x_1^3, x_2^3, x_3^3), \ldots).$$

The projective limit is defined as those elements in $\prod_{n \in \mathbb{N}} \mathbb{R}^n$ where

$$(x_1^{n_1}, \ldots, x_{n_1}) = P_{n_2, n_1}(x_1^{n_2}, \ldots, x_{n_2}^{n_2}).$$

or written out

$$x_1^1 = x_1^2, \quad (x_1^2, x_2^2) = (x_1^3, x_2^3), \quad (x_1^3, x_2^3, x_3^3) = (x_1^4, x_2^4, x_3^4), \ldots$$
In other words, the projective limit, also written
\[ \lim_{\leftarrow} (\mathbb{R}^n, P_{n_2, n_1}) , \]
is just \( \mathbb{R}^\infty \), the set of all sequences in \( \mathbb{R} \).

Another example which is more relevant to our case, comes from group theory. Let \( G \) be a group. We let \( \mathcal{F} \) be the set of finitely generated subgroups of \( G \). If \( F_1, F_2 \in \mathcal{F} \) and \( F_1 \subset F_2 \) we have the inclusion map \( \iota_{F_1, F_2} : F_1 \to F_2 \). If we therefore consider group homomorphism from each of these finitely generated subgroups to a fixed group \( G_1 \) we get, by dualizing, restriction maps
\[ \iota_{F_1, F_2}^* : \text{Hom}(F_2, G_1) \to \text{Hom}(F_1, G_1) . \]
As in the case of \( \mathbb{R}^n \) we can consider the product
\[ \prod_{F \in \mathcal{F}} \text{Hom}(F, G_1) = \{ (\varphi_F)_{F \in \mathcal{F}} | \varphi_F \in \text{Hom}(F, G_1) \} , \]
and the projective limit is defined as the subset of the product of sequences that are consistent with the restriction maps, i.e. a sequence \( (\varphi_F)_{F \in \mathcal{F}} \) is in the projective limit if
\[ \iota_{F_1, F_2}^* (\varphi_{F_2}) = \varphi_{F_1} , \]
for all \( F_1, F_2 \in \mathcal{F} \) with \( F_1 \subset F_2 \).

We note that we have a map
\[ \Phi : \text{Hom}(G, G_1) \to \lim_{\leftarrow} (\text{Hom}(F, G_1), \iota_{F_1, F_2}^*) \]
just by restricting a homomorphism from \( G \) to \( G_1 \) to its finite subgroups. It is easy to see that this map is a bijection, and we can hence identify \( \text{Hom}(G, G_1) \) with the projective limit. This might seem like we have just expressed something easy, namely \( \text{Hom}(G, G_1) \) with something complicated, namely the projective limit. However the description as a projective limit turns out to be very useful.

**B.2 Inductive limits**

Inductive limit is the dual concept of projective limit. For simplicity we take \( \mathbb{T}^\infty \), the infinite torus (easier than \( \mathbb{R}^\infty \) since \( \mathbb{T}^n \) is compact). This means that we have a projective system
\[ P_{n_2, n_1} : \mathbb{T}^{n_2} \to \mathbb{T}^{n_1} , \quad n_1, n_2 \in \mathbb{N}, \quad n_1 \leq n_2 , \]

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where $\mathbb{T}^n$ is the $n$-torus and $P_{n_2,n_1}$ are the natural projections.

The dual of a space is the functions on the space. There are of course several candidates for functions. In this example we will take the space of square integrable functions on $\mathbb{T}^n$ with respect to the Haar measure, i.e. $L^2(\mathbb{T}^n,d\mu_H)$. The dual map of $P_{n_2,n_1}$ gives a map

$$P^*_{n_2,n_1} : L^2(\mathbb{T}^{n_1}) \to L^2(\mathbb{T}^{n_2})$$

defined by

$$P^*_{n_2,n_1}(\xi)(x) = \xi(P_{n_2,n_1}(x)), \quad x \in \mathbb{T}^{n_2}.$$ 

These maps are embeddings and are maps of Hilbert spaces since

$$\int 1 d\mu_H = 1.$$ 

The inductive limit of these Hilbert spaces are constructed in the following way: We take the direct sum

$$\bigoplus_n L^2(\mathbb{T}^n),$$

i.e. sequences $\{\xi_n\}_{n \in \mathbb{N}}$ with $\xi_n \in L^2(\mathbb{T}^n)$ such that $\{\xi_n\}$ is zero from a certain step. In this space we consider the subspace $N$ generated by elements of the form

$$(0,\ldots,0,\xi_{n_1},0,\ldots,0,-P^*_{n_2,n_1}(\xi_{n_1}),0,\ldots),$$

and form the quotient space $\bigoplus_n L^2(\mathbb{T}^n)/N$. This quotient just means that we consider all vectors lying in some $L^2(\mathbb{T}^n)$, and identify two vectors $\xi_{n_1},\xi_{n_2}$ if $P^*_{n_2,n_1}(\xi_{n_1}) = \xi_{n_2}$. The space

$$\bigoplus_n L^2(\mathbb{T}^n)/N$$

is the algebraic inductive limit

$$\lim_{\rightarrow} L^2(\mathbb{T}^n).$$

Naively we are considering $L^2(\mathbb{T}^1)$ as a subspace of $L^2(\mathbb{T}^2)$, $L^2(\mathbb{T}^2)$ as a subspace of $L^2(\mathbb{T}^3)$, $L^2(\mathbb{T}^3)$ as a subspace of $L^2(\mathbb{T}^4)$ and so on, and the limit space as $n$ tends to infinity is the direct limit. Or in a picture

$$L^2(\mathbb{T}^1) \subset L^2(\mathbb{T}^2) \subset L^2(\mathbb{T}^3) \subset \ldots \subset \lim_{\rightarrow} L^2(\mathbb{T}^n).$$
We have used the word algebraic inductive limit, since we want to put some Hilbert space structure on the inductive limit.

If we have two vectors in the inductive limit, let us say \( \xi_{n_1} \in L^2(\mathbb{T}^{n_1}) \) and \( \xi_{n_2} \in L^2(\mathbb{T}^{n_2}) \) we define the inner product by:

\[
< \xi_{n_1}, \xi_{n_2} > = < P_{n_2,n_1}^* (\xi_{n_1}), \xi_{n_2} >_{L^2(\mathbb{T}^{n_2})}.
\]

Since the embeddings \( P_{n_2,n_1}^* \) are Hilbert space maps, this inner product is well defined.

The definition of the Hilbert space inductive limit of \( \{ L^2(\mathbb{T}^n), P_{n_2,n_1}^* \} \) is therefore the completion of \( \oplus_n L^2(\mathbb{T}^n)/N \) in the inner product \( < \cdot, \cdot > \). We will also denote this limit with

\[
\lim_{\rightarrow} L^2(\mathbb{T}^n).
\]

### B.3 Constructing operators on inductive limits of Hilbert spaces

The main advantage of giving a description of spaces as projective or inductive limits is that one can work on each copy, and then extend to the whole space if the construction is compatible with the structure maps, i.e. \( P_{n_2,n_1}^* \) for example.

As an example of this, let us take \( \lim_{\rightarrow} L^2(\mathbb{T}^n) \). On \( L^2(\mathbb{T}^n) \) we have the Laplacian \( \Delta_n : L^2(\mathbb{T}^n) \rightarrow L^2(\mathbb{T}^n) \) defined by

\[
\Delta_n = -(\partial^2_{\theta_1} + \partial^2_{\theta_2} + \ldots + \partial^2_{\theta_n}).
\]

Note that

\[
P_{n_2,n_1}^*(\Delta_{n_1}(\xi_{n_1})) = \Delta_{n_2}(P_{n_2,n_1}^*(\xi_{n_1})),
\]

and therefore \( \Delta = \sum_n \Delta_n \) on \( \oplus L^2(\mathbb{T}^n) \) has the property

\[
\Delta(N) \subset N,
\]

i.e. \( \Delta \) descends to a densely defined operator on the quotient space, i.e. the inductive limit \( \lim_{\rightarrow} L^2(\mathbb{T}^n) \).
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