Conformal maps between pseudo-Finsler spaces

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Abstract

The paper aims to initiate a systematic study of conformal mappings between Finsler spacetimes and, more generally, between pseudo-Finsler spaces. This is done by extending several results in pseudo-Riemannian geometry which are necessary for field-theoretical applications and by proposing a technique which reduces a series of problems involving pseudo-Finslerian conformal vector fields to their pseudo-Riemannian counterparts. Also, we point out, by constructing classes of examples, that conformal groups of flat (locally Minkowskian) pseudo-Finsler spaces can be much richer than both flat Finslerian and pseudo-Euclidean conformal groups.

Keywords: pseudo-Finsler space, Finsler spacetime, conformal mapping, conformal vector field, Killing vector field

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1 Introduction

In field theory, conformal maps are fundamental for our understanding of spacetime. Moreover, the existence of a conformal vector field on a manifold can provide valuable information, which can go up to full classification results, [2], [12], [13], [23], on the metric structure.

Among the applications of (pseudo-)Finsler geometry, field-theoretical ones are the most numerous, e.g., [5], [7], [10], [16], [20], [22], [21], [28], [31]. But these applications typically require metrics to be of Lorentzian signature. And, while on conformal maps between positive definite Finsler spaces there exists quite a rich literature, [1], [4], [6], [9], [18], [24], [32], [33], in pseudo-Finsler spaces, the situation is completely different. Apart from a very few papers dedicated to the particular case of isometries, [19], [30] or to a particular metric, [26], to the best of our knowledge, even basic questions related to conformal transformations have not been tackled yet.

Conformal groups of pseudo-Finsler metrics have a much more complicated - and more interesting - structure than both pseudo-Riemannian and Finslerian conformal groups. To prove this statement, we present in Section 3.2 some
classes of examples of flat (locally Minkowski) pseudo-Finsler spaces whose con-
formal symmetries depend on arbitrary functions. Comparatively, in dimension
\( n \geq 3 \), conformal symmetries of a pseudo-Euclidean can only be similarities,
inversions and compositions thereof, \([17]\), while the only conformal symmetries
of a non-Euclidean flat Finsler space are similarities, \([25]\). This hints at the
fact that extending results from either pseudo-Riemannian or Finsler geometry
to pseudo-Finsler spaces can be far from straightforward - and some of these
results might very well fail when passing to pseudo-Finsler spaces.

In the following sections, we focus on two topics:

1. **The behavior of geodesics under conformal mappings.** Here, we prove
that several results in pseudo-Riemannian geometry (which are fundamental for
general relativity) can still be extended to pseudo-Finsler spaces:
   - In dimension greater than 1, any mapping between two pseudo-Finsler
     structures which is both conformal and projective is a similarity. In other words,
     Weyl’s statement (e.g., \([6]\)) that projective and conformal properties of a metric
     space univocally determine its metric up to a dilation factor remains true in
     pseudo-Finsler spaces.
   - Lightlike geodesics are preserved, up to re-parametrization, under arbitrary
     conformal mappings.
   - A conservation law for conformal vector fields along lightlike geodesics.

2. **Conformal vector fields.** In positive definite Finsler spaces, the technique
of averaged Riemannian metrics allows one to prove profound results regarding
conformal transformations, by reducing the corresponding problems to their
Riemannian counterparts, \([25]\). But, unfortunately, this technique is not avail-
able in pseudo-Finsler spaces, as noticed in \([30]\).

Still, dealing with conformal vector fields, we can find a partial substitute
for this method. Given a pseudo-Finslerian metric tensor \( g \) on some manifold
\( M \), an associated Riemannian metric is a pseudo-Riemannian metric
\( g^\xi := g \circ \xi \),

where \( \xi \) is a vector field on \( M \). Associated Riemannian metrics have a series of
appealing properties (e.g., smoothness, same signature as \( g \)) and behave well
under conformal transformations of \( g \); more precisely, we show (Lemma \([4]\)) that,
if \( \xi \) is a conformal vector field for a pseudo-Finsler metric \( g \), then \( \xi \) is also
a conformal vector field for \( g^\xi \). This way, some results in pseudo-Riemannian
geometry become available in the more general context of Finsler metrics. As
an example, we extend to pseudo-Finsler spaces two results on Killing vector
fields in \([27]\).

Also, we prove that any essential conformal vector field of a pseudo-Finsler
metric has to be lightlike at least at a point.

The paper is organized as follows. Section 2 presents some preliminary
notions and results. Section 3 deals with the basic conformality notions and
examples of pseudo-Finslerian conformal maps. Section 4 is devoted to the
behavior of geodesics under conformal transformations. In Sections 5 and 6, we
discuss pseudo-Finslerian conformal vector fields.
2 Pseudo-Finsler spaces. Finsler spacetimes

Let $M$ be a $C^\infty$-smooth, connected manifold of dimension $n$ and $(TM, \pi, M)$, its tangent bundle. We denote by $(x^i)_{i=0}^{n-1}$ the coordinates of a point $x \in M$ in a local chart $(U, \varphi)$ and consider local charts $(\pi^{-1}(U), \Phi)$, $\Phi = (x^i, y^i)_{i=0}^{n-1}$ on $TM$ induced by the choice of the natural basis $\{\partial_i\}$ in each tangent space. Commas ,, will denote differentiation with respect to $x^i$ and dots ,, differentiation with respect to $y^i$. The set of sections of any fibered manifold $E$ over $M$ will be denoted by $\Gamma(E)$.

Consider a non-empty open submanifold $A \subset TM$, with $\pi(A) = M$ and $0 \notin A$. We assume that each $A_x := T_x M \cap A$, $x \in M$ is a positive conic set, i.e., $\forall \alpha > 0$, $\forall y \in A_x : \alpha y \in A_x$. Then the triple $(A, \pi|_A, M)$, where $\pi|_A$ is the restriction of $\pi$ to $A$, is a fibered manifold over $M$. For $x \in M$, elements $y \in A_x$ are called admissible vectors at $x$.

Definition 1 ([8]): Fix a natural number $0 \leq q < n$. A smooth function $L : A \to \mathbb{R}$ defines a pseudo-Finsler structure $(M, A, L)$ on $M$ if, at any point $(x, y) \in A$ and in any local chart $(\pi^{-1}(U), \Phi)$ around $(x, y)$:

1) $L(x, \alpha y) = \alpha^2 L(x, y)$, $\forall \alpha > 0$;

2) the matrix $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}(x, y)$ has $q$ negative and $n - q$ positive eigenvalues.

The Finsler Lagrangian (Finslerian energy) $L$ can always be prolonged by continuity to the closure $\bar{A}$. In particular, we can set $L(x, 0) = 0$.

Particular cases.

1) If $q = 0$, then the Finsler structure $(M, A, L)$ is called positive definite. If $A = TM \setminus \{0\}$, then it is called smooth. A smooth and positive definite pseudo-Finsler structure is a Finsler structure.

2) A pseudo-Finsler space $(M, A, L)$ with $q = n-1$, is called a Lorentz-Finsler space or a Finsler spacetime.

In a Finsler spacetime, $ds^2 = L(x, dx)$ is interpreted as spacetime interval and it allows the introduction of the basic causality notions. For any point $x \in M$, an admissible vector $y \in A_x$ will be called: timelike, if $L(x, y) > 0$, spacelike, if $L(x, y) < 0$ and null or lightlike, if $L(x, y) = 0$. Accordingly, a curve $c : [a, b] \to M$, $t \mapsto c(t)$ is called timelike (respectively, null, spacelike) if its tangent vector $\dot{c}$ is everywhere timelike (respectively, null, spacelike).

3) A pseudo-Finsler space $(M, A, L)$ is (pseudo)-Riemannian, if, in any local chart, $g_{ij} = g_{ij}(x)$ and flat (locally Minkowski) if around any point of $A$, there exists a local chart in which $g_{ij} = g_{ij}(y)$ only.

A curve on $M$ is called admissible if its tangent vector is everywhere admissible. In the following, we will assume that all the curves under discussion are admissible.

1This terminology will be actually used not only in Finsler spacetimes, but in pseudo-Finsler spaces of arbitrary signature.
admissible. The arc length of a curve \( c : t \in [a, b] \mapsto (x^i(t)) \) on \( M \) is calculated as
\[
l(c) = \int_a^b F(x(t), \dot{x}(t)) \, dt,
\]
where the Finslerian norm \( F : A \to \mathbb{R} \) is defined as:
\[
F = \sqrt{|L^i|}.
\]
The correspondence \( (x, y) \mapsto g_{(x,y)} \), where
\[
g_{(x,y)} = g_{ij}(x, y) dx^i \otimes dx^j
\]
defines a mapping \( g : A \to T^0_2 M \), called the pseudo-Finslerian metric tensor attached to \( L \). A pseudo-Finsler metric \( g \) can thus be regarded as a section of the pullback bundle \( \pi_A^* (T^0_2 M) \).

In any local chart \( (\pi^{-1}(U), \Phi) \), there hold the equalities:
\[
L^i = 2y_i, \quad y^i_j = g_{ij},
\]
where \( y_i = g_{ij}y^j \).

On \( A^o := \{(x, y) \in A \mid L(x, y) \neq 0\} \), it makes sense the angular metric
\[
h = g - \frac{1}{4L} p \otimes p : A^o \to T^0_2 M
\]
where \( p := \frac{\partial L}{\partial y^i} dx^i \). Using (2), this is written locally as:
\[
h = h_{ij} dx^i \otimes dx^j, \quad h_{ij} = g_{ij} - \frac{y_i y_j}{L}. \tag{4}
\]
The functions \( h_{ij} \) and their contravariant versions \( h^{ij} = g^{ik} g^{jl} h_{kl} \) obey:
\[
h_{ij} y^j = 0, \quad h^{ij} y_i = 0. \tag{5}
\]

Geodesics of \((M, A, L)\) are described (e.g., [3], [11]), by the equations:
\[
\frac{d^2 x^i}{dt^2} + 2G^i(x, \dot{x}) = 0, \tag{6}
\]
where the geodesic coefficients
\[
2G^i(x, y) = \frac{1}{2} g^{ih} (L_{h,j} y^j - L_{h,j}) \tag{7}
\]
are defined for \((x, y) \in A\). The canonical nonlinear connection \( N \) will be understood as a connection on the fibered manifold \( A \), in the sense of [15], pp. 30-32, i.e., as a splitting
\[
TA = HA \oplus VA,
\]
where \( VA = \ker d\pi|_A \) is called the vertical subbundle and \( HA \), the horizontal subbundle of the tangent bundle \((TA, \pi_A, A)\). The local adapted basis will be
denoted by \((\delta_i, \dot{\partial}_i)\), where \(\delta_i := \frac{\partial}{\partial x^i} - G^j_i \frac{\partial}{\partial y^j}\) and its dual basis, by 
\[(dx^i, \delta y = dy^i + G^j_i dx^j),\]
where
\[G^i_j = G^i \cdot j.\]

Every vector field \(X \in X(M)\) can thus be uniquely decomposed as \(X = hX + vX\), where \(hX = X_i \delta_i \in \Gamma(HA)\) and \(vX = Y_i \dot{\partial}_i \in \Gamma(VA)\).

By \(h : \Gamma(A) \rightarrow \Gamma(HA)\), \(v : \Gamma(A) \rightarrow \Gamma(VA)\), \(v = v^i \partial_i \mapsto v^i \delta_i\) and \(v = v^i \partial_i \mapsto v^i \dot{\partial}_i\), we will mean the corresponding horizontal and vertical lifts of vector fields.

The dynamical covariant derivative, [11], p. 34, determined by the canonical nonlinear connection \(N\) becomes, in a pseudo-Finsler space \((M, A, L)\), a mapping \(\nabla : \Gamma(VA) \rightarrow \Gamma(VA)\), \(X \mapsto \nabla X\) on the vertical subbundle \(VA\); it is given in any local chart by:
\[\nabla X(x, y) := (S(X^j) + G^i_j X^j)(x, y) \dot{\partial}_i, \quad \forall (x, y) \in A,\]
where \(X = X^i \dot{\partial}_i\) and \(S := y^k \delta_k\). The operator \(\nabla\) acts on functions \(f : TM \rightarrow \mathbb{R}\) as: \(\nabla f = S(f)\), it is additive and obeys the Leibniz rule with respect to multiplication with functions.

The complete lift \(\xi^e = \xi^i \partial_i + \xi^j y^j \dot{\partial}_i\) of an admissible vector field \(\xi \in \Gamma(A)\) can be expressed in terms of \(\nabla\) as:
\[\xi^e = \xi^h + \nabla(\xi^v).\]

From the 2-homogeneity in \(y\) of the geodesic coefficients \(2G^i\), it follows that, along geodesics \(c : [a, b] \rightarrow M, t \mapsto (x^i(t))\) of \((M, A, L)\), we have, [11], p. 108: \(\nabla x^i = 0\); equivalently,
\[\left(\nabla \dot{c}^e\right)_{(c(t), \dot{c}(t))} = 0.\]

The canonical nonlinear connection \(N\) is metrical, that is, for the vertical lift \(g^v = g_{ij} \delta^i \otimes \delta^j : \Gamma(VA) \times \Gamma(VA) \rightarrow \mathbb{R}\) of the metric \(g\), there holds [11], p. 98, at any \((x, y) \in A:\)
\[\nabla g^v = 0,\]
where \((\nabla g^v)(X, Y) = \nabla(g^v(X, Y)) - g^v(\nabla X, Y) - g^v(X, \nabla Y), \forall X, Y \in \Gamma(VA)\).

Another known property which will be used in the following is that \(L\) is constant along horizontal curves, [14], that is,
\[X(L) = 0, \quad \forall X \in \Gamma(HA).\]

3 Basic notions and examples

3.1 Conformal maps and conformal vector fields

The notion of conformal map between Finsler spaces is extended in a straightforward way to pseudo-Finsler spaces; we have to just take care to the domains of definition of the involved metric tensors.
Definition 2  A diffeomorphism $f : M \to M'$ is called a conformal map between two pseudo-Finsler spaces $(M, A, L)$ and $(M', A', L')$ if there exists a function $\sigma : M \to \mathbb{R}$ such that:

$$L' \circ df|_A = e^{\sigma} L.$$  \hfill (14)

In Finsler spacetimes, conformal maps preserve the light cones $L = 0$.

For positive definite Finsler spaces, transformations (14) coincide with angle-preserving transformations.

A conformal map is a similarity if $\sigma = \text{const.}$ and an isometry if $\sigma = 1$.

Denoting by $\tilde{A} := A \cap (df^{-1})(A')$ the set where (14) makes sense, (14) reads:

$$L'(f(x), df_x(y)) = e^{\sigma(x)} L(x, y), \quad \forall (x, y) \in \tilde{A}.$$  \hfill (15)

Convention. In the following, we will assume that $\pi(\tilde{A}) = M$ (in particular, this implies that $\tilde{A}$ is a fibered manifold over $A$). Under this assumption, there will be no loss of generality if we consider that $A' = (df)(A)$; in the contrary case, we will restrict our discussion to the sets $\tilde{A}$ and $(df)(\tilde{A}) = A' \cap df(A)$ respectively and re-denote them by $A$ and $A'$. We will denote the restriction $df|_A : A \to A'$ simply by $df$.

With the notation

$$\tilde{L} := L' \circ df,$$  \hfill (16)

and with the above convention, (14) becomes:

$$\tilde{L}(x, y) = e^{\sigma} L(x, y), \quad \forall (x, y) \in \tilde{A};$$  \hfill (17)

this is equivalent to:

$$\tilde{g}(x, y) = e^{\sigma} g(x, y), \quad \forall (x, y) \in \tilde{A}.$$  \hfill (18)

Assume that $f : M \to M'$ is given with respect to two arbitrary local charts on $M$ and $M'$ as: $\tilde{x}^i = \tilde{x}^i(x^j)$; the differential $df : A \to A'$, $(x, y) \mapsto (\tilde{x}, \tilde{y})$ is locally expressed as: $\tilde{x}^i = \tilde{x}^i(x^j)$, $\tilde{y}^j = \frac{\partial \tilde{x}^i}{\partial x^j} y^j$, therefore, differentiating (16) twice with respect to $y^i$, we find:

$$\tilde{g}_{ij}(x, y) = \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^l}{\partial x^j} g'_{kl}(\tilde{x}, \tilde{y}), \quad \forall (x, y) \in \tilde{A}.$$  \hfill (19)

In coordinate-free writing, this is:

$$\tilde{g} = T^0_2 f \circ g' \circ df,$$  \hfill (20)

where $T^0_2 f : T^0_2 M' \to T^0_2 M$ is the mapping naturally induced by $f$ on the respective tensor powers (giving the multiplication by the Jacobian matrix of $f$ in (19)); we will write this also as:

$$\tilde{g} := (df)^* g'.$$  \hfill (21)
On a pseudo-Finsler space \((M, A, L)\), an admissible vector field \(\xi \in \Gamma(A)\) is called conformal if its 1-parameter group \(\{\varphi_\varepsilon\}_{\varepsilon \in I}\) consists of conformal transformations, i.e., for any \(\varepsilon \in I\):

\[
L \circ d\varphi_\varepsilon = e^{\sigma_\varepsilon} L,
\]

where \(\sigma_\varepsilon : M \to M\) are smooth functions.

Assume that \(\xi \in \Gamma(A)\) is a conformal vector field. Since \(d\varphi_\varepsilon\) is generated by the complete lift \(\xi^c\), we get, by differentiating (22) at \(\varepsilon = 0\):

\[
\mathcal{L}_{\xi^c} L = \frac{d}{d\varepsilon}|_{\varepsilon=0}(e^{\sigma_\varepsilon} L) = \mu L,
\]

where \(\mu := \frac{d\sigma_\varepsilon}{d\varepsilon}|_{\varepsilon=0}\).

In particular, if \(\sigma_\varepsilon = 1\) for all \(\varepsilon\), i.e., \(\xi\) is a Killing vector field for \(L\), then:

\[
\mathcal{L}_{\xi^c} L = 0.
\]

**Examples.** If \(L = L(y) : T\mathbb{R}^n \to \mathbb{R}\) is locally Minkowski, then:

1) The radial vector field \(\xi(x) = x^i \partial_i\) is a conformal vector field. This can be checked easily, as \(\xi^c = x^i \partial_i + x^j y^i \partial_j = x^i \partial_i + y^j \partial_j\) and, using the homogeneity of degree 2 of \(L\), we obtain:

\[
\mathcal{L}_{\xi^c} L = x^i L_{,i} + y^j L_{,i} = 0 + 2L = 2L.
\]

The flow of \(\xi\) consists of the dilations (homotheties) \(\varphi_\varepsilon : (x^i) \mapsto (\varepsilon x^i)\).

2) Any constant vector field \(\xi_0\) is a Killing vector field for \(L = L(y)\). This follows from: \(\xi^c_0 = \xi_0^i \partial_i\) and:

\[
\mathcal{L}_{\xi_0^c} L = \xi_0^i L_{,i} = 0.
\]

The vector field \(\xi_0\) generates the translations \((x^i) \mapsto (x^i + \varepsilon \xi_0^i)\).

### 3.2 Conformal maps between locally Minkowski spaces

In Euclidean spaces, Liouville’s Theorem states that any conformal transformation relating two domains of \(\mathbb{R}^n\), \(n > 2\), is a similarity or the composition between a similarity and an inversion; passing to pseudo-Euclidean spaces, one has to only add to the picture, \([17]\), compositions of two inversions.

In Finsler spaces, the situation is even more rigid; it was proven in \([25]\) that any conformal map between two non-Euclidean locally Minkowski Finsler spaces is a similarity. Taking all these into account, one could reasonably expect that conformal groups of pseudo-Finsler spaces could not be too rich.

Still, as we will show in the following, one can create whole families of pseudo-Finsler metrics with conformal symmetries which are not only non-similarities, but they depend on arbitrary functions. This gives an affirmative answer, in indefinite signature, to an old and famous question raised by M. Matsumoto, \([25]\), namely whether there exist two locally Minkowski structures which are conformal to each other.
For \( \text{dim} M = 4 \), a first example is actually known from [26]. This example can be extended to any dimension, as follows.

**Example 1:** Conformal symmetries of Berwald-Moor metrics. Consider, on \( M = \mathbb{R}^n, n > 1 \):

\[
A = \{ (x^i, y^i)_{i=0}^{n-1} \mid y^0 y^1 \ldots y^{n-1} \neq 0 \} \subset TM \setminus \{0\}
\]

and the \( n \)-dimensional Berwald-Moor pseudo-Finsler function ([24], pp. 155-156) on \( A \):

\[
L(y) = \varepsilon |y^0 y^1 \ldots y^{n-1}|^{\frac{2}{n}}, \quad (24)
\]

where \( \varepsilon := \text{sign}(y^0 y^1 \ldots y^{n-1}) \).

For an arbitrary diffeomorphism of the form

\[
f: \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad x = (x^0, x^1, \ldots, x^{n-1}) \rightarrow (f^0(x^0), f^1(x^1), \ldots, f^{n-1}(x^{n-1})) \quad (25)
\]

the Jacobian determinant \( J(x) := \frac{df^0}{dx^0} \frac{df^1}{dx^1} \ldots \frac{df^{n-1}}{dx^{n-1}} \) is always nonzero, hence, there is no loss of generality if we assume that \( J(x) > 0, \forall x \in \mathbb{R}^n \). We find:

\[
\tilde{L}(y) = L(df(y)) = J(x)^{\frac{2}{n}} L(y), \quad \forall y \in A, \quad (26)
\]

i.e., \( \tilde{L} \) is also defined on \( A \). Moreover, \( f \) is a conformal map, with conformal factor:

\[
\sigma(x) = \frac{2}{n} \ln J(x). \quad (27)
\]

The Finsler function (26) is locally Minkowski; more precisely, the coordinate transformation on \( \pi^{-1}(U) \) induced by: \( (x^i) = f^{-1}(x'^i) \) brings \( \tilde{L} \) to the form \( \tilde{L}(y) = (y^0 y^1 \ldots y^{n-1})^{\frac{2}{n}} \cdot \). Yet, \( \sigma(x) \) is not only non-constant, but it depends on \( n \) arbitrary functions.

Berwald-Moor metrics are not the only such examples. Here is a much more general class of flat pseudo-Finsler metrics on \( \mathbb{R}^n, n \geq 2 \), which admit nontrivial conformal symmetries.

**Example 2:** Weighted product Finsler functions. Consider \( M = \mathbb{R}^k \times \mathbb{R}^{n-k} \) and a pseudo-Finsler metric function \( L: A \rightarrow \mathbb{R} \) (with \( A \subset A_1 \times A_2, A_1 \subset T\mathbb{R}^k, A_2 \subset T\mathbb{R}^{n-k} \)), of the form:

\[
L = L_1^\alpha L_2^{1-\alpha}, \quad (28)
\]

where \( L_1 : A_1 \rightarrow \mathbb{R} \) and \( L_2 : A_2 \rightarrow \mathbb{R} \) are pseudo-Finsler functions and \( \alpha \in (0,1) \).

Assume that \( f_1: \mathbb{R}^k \rightarrow \mathbb{R}^k, \ (x^0, \ldots, x^{k-1}) \rightarrow (\tilde{x}^0, \ldots, \tilde{x}^{k-1}) \) is a conformal transformation with non-constant factor \( \sigma = \sigma(x) \), such that \( \tilde{L}_1 = L_1 \circ df \) is locally Minkowski - and let \( L_2 \) be completely arbitrary. Then, the transformation

\[
f: \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad f := (f_1, id_{\mathbb{R}^{n-k}}) \quad (29)
\]

The sign \( \varepsilon \) is meant to ensure the existence of spacelike vectors. Our \( L \) is, up to a sign, the square of the one in [24] (the latter would be, in our notations, \( F \)).
leads to: \( \tilde{L}(y) := L(df(y)) = e^{\alpha(x)}L(y), \forall y = (y_0, \ldots, y^{k-1}, y^k, \ldots, y^{n-1}) \in \mathbb{R}^n, \) i.e., \( f \) is a conformal symmetry (which is not a similarity) of \( L \). The obtained Finsler function \( \tilde{L} \) is obviously locally Minkowski - and it depends on the choice of the function \( f_1 \).

**Particular cases** for the choice of \( L_1 \) in (28) include:

a) *The case* \( k = 1 \). In this case, \( L_1 = \lambda(y_0)^2 \), for some \( \lambda \in \mathbb{R} \) and therefore, any diffeomorphism \( f_1 : \mathbb{R} \to \mathbb{R}, x^0 \mapsto f_1(x^0) \), serves the purpose, since: \( L_1(df_1(y_0)) = L_1(f_1(x^0)y^0) = f_1^2(x^0)L_1(y^0) \).

b) *The k-dimensional Minkowski metric*:
\[
L_1(y_0, \ldots, y^{k-1}) = (y_0)^2 - (y_1)^2 - \ldots - (y_0)^2;
\]
then, \( f_1 \) can be, e.g., an inversion.

c) *The k-dimensional Berwald-Moor metric* can also be chosen as \( L_1 \). In this case, \( f_1 \) can be any mapping of the form (25).

### 4 Behavior of geodesics under conformal maps

1. **Projective-and-conformal mappings.** A diffeomorphism \( f : M \to M \) between is called a *projective map* if geodesics of \( L \) coincide, up to re-parametrization, with geodesics of \( \tilde{L} := L \circ df \). In a completely similar manner to the positive definite case ([3], pp. 110-111), it follows that the mapping \( f \) is projective if and only if there exists a 1-homogeneous scalar function \( P : A \to \mathbb{R} \) such that, in any local chart,
\[
2\tilde{G}^i = 2G^i + P(x, y)y^i, \quad \forall (x, y) \in A. \tag{30}
\]
Assume that the projective map \( f \) is also conformal, with conformal factor \( e^\sigma \). Then, a direct calculation using (7) shows that:
\[
2\tilde{G}^i = 2G^i + \frac{1}{2}g^{ih}(\sigma_{,k}y^kL_{,h} - \sigma_{,h}L);
\]
using (2), this is:
\[
2\tilde{G}^i = 2G^i + \sigma_{,k}y^k y^i - \frac{1}{2}g^{ih}\sigma_{,h}L. \tag{31}
\]
Based on the properties of the angular metric tensor (3), we can extend to arbitrary signature a result known in positive definite Finsler spaces from [6], [29]:

**Theorem 3** If a mapping \( f : M \to M \), relating two pseudo-Finsler structures on a manifold \( M \) with \( \dim M \geq 2 \) is both conformal and projective, then \( f \) is a similarity.
**Proof.** Denote by \((M, A, L)\) and \((M, A, \tilde{L})\) the two Finsler structures; that is, \(\tilde{L} = L \circ df\). As \(f\) is both conformal and projective, equalities (31) and (30) are both satisfied. Therefore, at any \((x, y) \in A\) and in any local chart around \((x, y)\),

\[
\sigma_{,k} y^k y^j - \frac{1}{2} g^{ik} \sigma_{,k} L = Py^i. \tag{32}
\]

Now, fix an arbitrary \(x \in M\) and an arbitrary open region of \(A_x\) where \(L \neq 0\); on such a region, it makes sense the angular metric tensor (3). Contracting (32) with \(h_{ij}\) and using (5), it remains: \(h_{ij} g^{ik} \sigma_{,k} L = 0\). Taking into account that \(h_{ij} g^{ik} = \delta^k_j - y^k y_j L\), this becomes:

\[
L \sigma_{,j} - \sigma_{,k} y^k y_j = 0. \tag{33}
\]

Differentiating with respect to \(y^i\), we find, by (2):

\[
2 y_i \sigma_{,j} - \sigma_{,i} y_j - \sigma_{,k} y^k g_{ij} = 0.
\]

Now, contract both hand sides of the above equality with \(h_{ij}\). Using again (5), we get rid of the first and of the second term. Further, noticing that \(h^{ij} g_{ij} = n - 1\), we obtain: \((n - 1) \sigma_{,k} y^k = 0\). But, by hypothesis, \(n = \dim M \geq 2\), therefore:

\[
\sigma_{,h} y^h = 0,
\]

which, by differentiation with respect to \(y^k\), gives that: \(\sigma_{,k}(x) = 0\). As the point \(x\) was arbitrarily chosen, we obtain \(\sigma(x) = \text{const.}\), q.e.d. ■

**Remark.** Substituting \(\sigma = \text{const.}\) into (32), we obtain \(P = 0\). That is, if two pseudo-Finsler metrics \(L\) and \(\tilde{L}\) are both conformally and projectively related, then, \(2 \tilde{G}^i = 2 G^i\) - meaning that their parametrized geodesics coincide.

2. **Conformal changes and null geodesics.** Generally, conformal maps do not preserve geodesics. Still, for null geodesics, we can extend a remarkable result from the semi-Riemannian case:

**Proposition 4** Null geodesics of two conformally related pseudo-Finsler metrics coincide up to parametrization.

**Proof.** Denote by \(L\) and \(\tilde{L}\) the two conformally related pseudo-Finsler metrics. Taking into account that, along null geodesics, \(L = 0\) and substituting into (31), we find that, along these curves, \(2 \tilde{G}^i = 2 G^i + \sigma_{,k} y^k y^i\). Setting \(P := \sigma_{,k} y^k\), we get: \(2 \tilde{G}^i = 2 G^i + Py^i\), which means that null geodesics of the two spaces coincide up to re-parametrization. ■

Another result in pseudo-Riemannian geometry, (23), which can be extended to pseudo-Finsler spaces is:
Proposition 5 Let $\xi \in \Gamma(A)$ be a conformal vector field for a pseudo-Finsler space $(M, A, L)$. Along any lightlike geodesic $c : [a, b] \to M, t \mapsto c(t)$ the quantity $g_{(c(t),\dot{c}(t))}(\dot{c}(t), \xi(t))$ is conserved.

Proof. Take an arbitrary lightlike geodesic $c$ on $M$ and denote by $C = (c, \dot{c})$, the lift of $c$ to $TM$. Under the above made assumption that $c$ is admissible, we can write $C : [a, b] \to A$.

Denote, for simplicity: $g := g_{(c(t),\dot{c}(t))}$, $\nabla X := (\nabla X)_{(c(t),\dot{c}(t))}$ for $X \in \Gamma(VA)$ and $\nabla f := \nabla f_{(c(t),\dot{c}(t))}$ for smooth functions on $M$. As $c$ is a geodesic, we have, by (11), $\dot{C} = \dot{x}^i \delta_i$; hence, 

$$
\frac{df}{dt} = \dot{C} f = \dot{x}^i \delta_i f = \nabla f, \quad \forall f : TM \to \mathbb{R}.
$$

Applying the above equality to: 

$$
f := g(\dot{c}, \xi) = g^v(\dot{c}^v, \xi^v),
$$

we get: 

$$
\frac{d}{dt} (g(\dot{c}, \xi)) = \nabla (g^v(\dot{c}^v, \xi^v))
$$

and therefore, 

$$
\frac{d}{dt} (g(\dot{c}, \xi)) = (\nabla g^v) (\dot{c}^v, \xi^v) + g^v (\nabla \dot{c}^v, \xi^v) + g^v (\dot{c}^v, \nabla \xi^v).
$$

The first term in the right hand side is zero by (12). The second one is also zero since $c$ is a geodesic. It remains to evaluate $g(\dot{c}^v, \nabla \xi^v)$.

Since $\xi$ is a conformal vector field, it obeys: $L_{\xi^v} L = \mu L$ for some function $\mu$. But, by hypothesis, $c$ is lightlike, i.e., $L$ vanishes along $C$. It follows:

$$
L_{\xi^v} L = 0 \quad \text{on } C.
$$

Further, using (10) for the Lie derivative $L_{\xi^v} = \xi^v(L)$, relation (36) becomes:

$$
\xi^h(L) + (\nabla \xi^v)(L) = 0.
$$

The term $\xi^h(L)$ vanishes by (13), which leads to $(\nabla \xi^v)(L) = 0$. In coordinates, this is: $(\nabla \xi^v)_i = 0$. Taking into account (2), we can write it as: $2g_{ij} y^j \nabla \xi^i = 0$.

Along $C$, this is equivalent to:

$$
g^v (\dot{c}^v, \nabla \xi^v) = 0.
$$

Substituting the latter relation into (35), we get: 

$$
\frac{d}{dt} g(\dot{c}, \xi) = 0, \text{ q.e.d.}
$$

5 Associated Riemannian metrics a useful lemma

Consider a pseudo-Finsler space $(M, A, L)$, with metric tensor $g : A \to T^0_0 M$. For any admissible vector field $\xi \in \Gamma(A)$, the mapping 

$$
g^\xi := g \circ \xi : M \to T^0_0 M
$$

(37)
defines a pseudo-Riemannian metric on $M$, called an associated (pseudo-)Riemannian metric or, [28], an osculating (pseudo)-Riemannian metric.

Here are some immediate properties of metrics $g^\xi$, $\xi \in \Gamma(A)$:

1. $g^\xi$ is defined and smooth on the whole base manifold $M$ (even if $g$ cannot be defined on the entire $TM \setminus \{0\}$).
2. $g^\xi$ has the same signature as $g$.
3. If, in particular, $g = g(x)$ is pseudo-Riemannian, then, all the metrics $g^\xi$, $\xi \in \Gamma(A)$ coincide (up to projection onto $M$) with $g$, i.e., $g^\xi \circ \pi = g$.
4. If $g, \tilde{g} : A \to T^0_0 M$ are conformally related, with conformal factor $\sigma = \sigma(x)$, then
   \[ \tilde{g}^\xi = e^{\sigma} g^\xi, \quad \forall \xi \in \Gamma(A). \] (38)

Let us analyze conformal point transformations associated with (38). With this aim, consider an arbitrary pseudo-Finsler metric tensor $g$ and - for the moment - an arbitrary diffeomorphism $f : M \to M$. Assume, as above, that $df(A) = A$, denote $df := df|_A$ and consider
\[ \begin{align*}
\tilde{\xi} & : = df \circ \xi \circ f^{-1} : M \to A, \\
\tilde{g} & : = (df)^* g = T^0_0 f \circ g \circ df : A \to T^0_0 M,
\end{align*} \] (39)
the corresponding deformations of $\xi$ and $g$. Noticing that the pullback $f^*(g^\tilde{\xi})$ of the pseudo-Riemannian metric $g^\tilde{\xi}$ can be written as: $f^*(g^\tilde{\xi}) = T^0_0 f \circ g^\tilde{\xi} \circ f$, we get:
\[ f^*(g^\tilde{\xi}) = T^0_0 f \circ (g \circ \tilde{\xi}) \circ f = T^0_0 f \circ g \circ df \circ \xi = \tilde{g} \circ \xi, \]
i.e., the left hand side of (38) is:
\[ \tilde{g}^\xi = f^*(g^\tilde{\xi}). \] (41)
In particular, if $f$ is a conformal map, then:
\[ f^*(g^\tilde{\xi}) = e^{\sigma} g^\xi. \] (42)

Using (42), we obtain:

**Lemma 6** If $\xi : M \to A$ is a conformal vector field for a pseudo-Finsler metric structure $(M, A, L)$, with 1-parameter group $\{\varphi_\varepsilon\}$, then:

(i) $\xi$ is a conformal vector field for the pseudo-Riemannian metric $g^\xi$.

(ii) The conformal factor relating the pseudo-Finsler metrics $g$ and $\tilde{g} = (d\varphi_\varepsilon)^* g$ is the same as the conformal factor relating $g^\xi$ and $\varphi_\varepsilon^*(g^\xi)$.

**Proof.** (i) Set: $\tilde{g} := (d\varphi_\varepsilon)^* g$, $\tilde{\xi} := d\varphi_\varepsilon \circ \xi \circ \varphi_\varepsilon^{-1}$. By (12), we have: $\varphi_\varepsilon^*(g^\xi) = e^{\sigma} g^\xi$. But, the vector field $\xi$ is invariant under its own flow, that is, $\xi = \xi$. We find:
\[ \varphi_\varepsilon^*(g^\xi) = e^{\sigma} g^\xi, \] (43)
that is, $\xi$ is a conformal vector field for $g^\xi$.

(ii) The statement follows from (38).
6 Conformal and Killing vector fields

Here is another property in pseudo-Riemannian geometry, [23], which can be extended to pseudo-Finsler spaces:

**Proposition 7** If a conformal vector field $\xi : M \to A$ for a pseudo-Finsler space $(M, A, L)$ is nowhere lightlike, then, $\xi$ is a Killing vector field for a conformally related pseudo-Finsler structure.

**Proof.** As $\xi$ is a conformal vector field for $L$, we have, at any $(x, y) \in A$:

$$(\mathcal{L}_{\xi^c}L)(x, y) = \mu L(x, y). \quad (44)$$

Using the hypothesis that $L$ is nowhere lightlike, the quantity $\alpha(x) := L(x, \xi(x))$ does not vanish. Set:

$$\tilde{L}(x, y) := \frac{1}{\alpha(x)} L(x, y) : A \to \mathbb{R}.$$  \hspace{1cm} \text{(45)}

Taking the Lie derivative of $\tilde{L}$: $$(\mathcal{L}_{\xi^c}\tilde{L}) = \mathcal{L}_{\xi^c}(\frac{1}{\alpha})L + \frac{1}{\alpha} \mathcal{L}_{\xi^c}(L)$$ and noticing that $\mathcal{L}_{\xi^c}(a) = -\mu \alpha$, $\mathcal{L}_{\xi^c}L = \mu L$, we get:

$$(\mathcal{L}_{\xi^c}\tilde{L}) = -\frac{1}{\alpha^2} \mu \alpha L + \frac{1}{\alpha} \mu L(y) = 0,$$

i.e., $\xi$ is a Killing vector field for $\tilde{L}$. \hspace{1cm} $\blacksquare$

**Remark.** A conformal vector field for a pseudo-Finsler metric $L$ is called essential if it is not a Killing vector field for any conformally related metric to $L$. That is: any essential pseudo-Finslerian conformal vector field must be lightlike at least at a point.

Passing to Killing vector fields, let us mention the following results due to Sanchez, [27], in Riemannian geometry:

**Proposition 8**, [27]: Let $(M, g)$ be a Lorentzian manifold with a non-spacelike (at any point) Killing vector field $\xi$. If $\xi_p = 0$ for some $p \in M$, then $\xi$ vanishes identically.

**Theorem 9**, [27]: If $\xi$ is a Killing vector field on a Lorentzian manifold $(M, g)$, admitting an isolated zero at some point $p \in M$, then, the dimension of $M$ is even and $\xi$ becomes timelike, spacelike and null on each neighborhood of $p$.

Now, using Lemma 6, the extensions to pseudo-Finsler spaces of the above results become simple corollaries:

**Proposition 10** Let $(M, A, L)$ be a Finsler spacetime, with a non-spacelike (at any point) Killing vector field $\xi$. If $\xi = 0$ at some point $p \in M$, then $\xi$ vanishes identically.
**Proof.** Since $\xi$ is a Killing vector field for $L$, it follows from Lemma 6 that $\xi$ is a Killing vector field for the pseudo-Riemannian metric $g^\xi$. But, since the signature of $g^\xi$ coincides with the one of $L$, $g^\xi$ is Lorentzian. The statement now follows from Proposition 8. ■

**Theorem 11** If $\xi$ is a Killing vector field for a Finsler spacetime $(M, A, L)$, admitting an isolated zero at some point $p \in M$, then, the dimension of $M$ is even and $\xi$ becomes timelike, spacelike and null on each neighborhood of $p$.

**Proof.** Assume $\xi$ is a Killing vector field for $(M, A, L)$, with an isolated zero at some $p \in M$. Then, $\xi$ is also a Killing vector for the Lorentzian metric $g^\xi$ on $M$ and

$$L(\xi) = g^\xi(\xi, \xi),$$

which means that $\xi$ is timelike (respectively, null, spacelike) for $L$ iff it is timelike (resp., null, spacelike) for $g^\xi$. The result now follows from Theorem 9. ■

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