COVERING SEMIGROUPS

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A topological semigroup is a Hausdorff space $S$ together with a continuous associative multiplication $m: S \times S \to S$. The lifting of the group structure of a topological group to its simply connected covering space is a technique used in the theory of Lie groups. In this paper we investigate the lifting of the multiplication of a topological semigroup $S$ to its simply connected covering space $(\tilde{S}, \varphi)$. A general theory is developed and applications to examples are discussed.

1. Covering spaces. Let $\tilde{S}$ and $S$ be locally connected topological spaces and $\varphi: \tilde{S} \to S$ a continuous map. If $C$ is a subset of $S$, then $C$ is evenly covered if $\varphi | C: \tilde{C} \to C$ is a homeomorphism for each component $\tilde{C}$ of $\varphi^{-1}(C)$. If each point in $S$ has an evenly covered open neighborhood, then $\varphi$ is called a covering map. If $\varphi$ is a covering map and $\tilde{S}$ is connected, then $(\tilde{S}, \varphi)$ is called a covering space of $S$. A covering space is called trivial if the covering map is a homeomorphism, and if $S$ admits only trivial covering spaces, then $S$ is called simply connected. If $(\tilde{S}_1, \varphi_1)$ and $(\tilde{S}_2, \varphi_2)$ are simply connected covering spaces of $S$ and $\psi: \tilde{S}_1 \to \tilde{S}_2$ is a homeomorphism such that $\varphi_2 \circ \psi = \varphi_1$, then $\psi$ is called a covering space isomorphism. An automorphism of $(\tilde{S}, \varphi)$ is an isomorphism of $(\tilde{S}, \varphi)$ with itself.

**Lemma 1.** Let $(\tilde{S}, \varphi)$ be a covering space of $S$ and $T$ a connected space. If $\alpha, \beta: T \to \tilde{S}$ are continuous maps with $\varphi \circ \alpha = \varphi \circ \beta$, then $\alpha$ and $\beta$ agree everywhere or nowhere.

**Lemma 2.** Let $P$ be a topological space. Then $P$ is simply connected if and only if (a) $P$ is connected and locally connected and (b) if $\varphi: \tilde{S} \to S$ is a covering map, $\psi: P \to S$ is continuous, $p$ is in $P$, $s$ is in $\tilde{S}$ with $\psi(p) = \varphi(s)$, then there exists unique continuous $\tilde{\psi}: P \to \tilde{S}$ such that $\tilde{\psi} = \varphi \circ \tilde{\psi}$ and $\tilde{\psi}(p) = s$.

**Lemma 3.** Let $(P, \psi)$ and $(\tilde{S}, \varphi)$ be covering spaces of $S$ with $p$ in $P$ and $s$ in $\tilde{S}$ with $\psi(p) = \varphi(s)$. If $P$ is simply connected and $\tilde{\psi}: P \to \tilde{S}$ is the unique lifting of $\psi$ with $\tilde{\psi}(p) = s$, then $\tilde{\psi}$ is a covering map.

**Lemma 4.** If $(\tilde{S}_1, \varphi_1)$ and $(\tilde{S}_2, \varphi_2)$ are simply connected covering spaces of $S$ and $s_i$ is in $\tilde{S}_i$, $i = 1, 2$ with $\varphi_1(s_i) = \varphi_2(s_i)$, then there exists a unique covering space isomorphism $\psi: \tilde{S}_1 \to \tilde{S}_2$ such that $\psi(s_i) = s_i$. 427
LEMMA 5. Let $(\tilde{S}, \varphi)$ be a simply connected covering space of $S$. We define the set of all automorphisms of $(\tilde{S}, \varphi)$ to be the Poincare group or fundamental group of $S$ and denote it by $P(S)$. The orbits of $P(S)$ are the discrete subspaces $\varphi^{-1}(x)$, $x$ in $S$, and $P(S)$ is simply transitive on these orbits, i.e., a given point can be mapped into a given point in the same orbit by precisely one automorphism in $P(S)$.

LEMMA 6. $(\tilde{S}, \varphi)$ be a covering space of $S$. If $A$ is a connected, locally connected subspace of $S$ and $\tilde{A}$ is a component of $\varphi^{-1}(A)$, then $(\tilde{A}, \varphi|\tilde{A})$ is a covering space of $A$.

LEMMA 7. If $S$ and $T$ are topological spaces admitting simply connected covering spaces $(\tilde{S}, \varphi_1)$ and $(\tilde{T}, \varphi_2)$, then $S \times T$ admits the simply connected covering space $(\tilde{S} \times \tilde{T}, \varphi_1 \times \varphi_2)$ and $P(S \times T) \cong P(S) \times P(T)$. It follows that the product of two topological spaces is simply connected if and only if both are.

The proofs of the above lemmas can be found in either Chevalley [2], Hochschild [4], Hofmann [5], or Pontrjagin [10]. Theorem 8 seems to be of a van Kampen type.

THEOREM 8. Let $U$, $V$ be simply connected subsets of a space $A$. If $U \setminus V$ and $V \setminus U$ are separated and if $U \cap V$ is nonvoid and connected, then $U \cup V$ is simply connected.

Proof. We may assume $A = U \cup V$. Then $A$ is trivially connected and is locally connected by a proof identical to the first paragraph of Lemma 1.3 on page 45 of Hochschild [4]. Now let $\varphi: \tilde{S} \to S$ be a covering map, $\alpha$ a continuous map of $A$ into $S$, $a_0$ a point of $A$, $s_0$ a point of $\tilde{S}$ with $\alpha(a_0) = \varphi(s_0)$. We may assume $a_0$ is in $U$. Define $\alpha_1 = \alpha|U: U \to S$. Since $U$ is simply connected and

$$
\begin{array}{ccc}
A & \xrightarrow{\tilde{\alpha}} & \tilde{S} \\
\downarrow{\alpha} & \nearrow{\varphi} & \\
S & & \\
\alpha(a_0) & = & \varphi(s_0), \\
\end{array}
\begin{array}{ccc}
U & \xrightarrow{\tilde{\alpha}_1} & \tilde{S} \\
\downarrow{\alpha_1} & \nearrow{\varphi} & \\
S & & \\
\alpha_1(a_0) & = & \varphi(s_0), \\
\end{array}
\begin{array}{ccc}
V & \xrightarrow{\tilde{\alpha}_2} & \tilde{S} \\
\downarrow{\alpha_2} & \nearrow{\varphi} & \\
S & & \\
\alpha_2(b_0) & = & \varphi(s_0)
\end{array}
$$

\[\alpha_i(a_0) = \alpha(a_0) = \varphi(s_0),\] there is continuous $\tilde{\alpha}_i: U \to \tilde{S}$ with $\varphi \circ \tilde{\alpha}_i = \alpha_i$ and $\tilde{\alpha}_i(a_0) = s_0$. Fix $b_0$ in $U \cap V$ and define $y_0 = \tilde{\alpha}_1(b_0)$ in $\tilde{S}$. Then $\varphi(y_0) = \varphi \circ \tilde{\alpha}_1(b_0) = \alpha_1(b_0) = \alpha_2(b_0)$, where $\alpha_2 = \alpha|V: V \to S$. Since $V$ is simply connected, there is continuous $\tilde{\alpha}_2: V \to \tilde{S}$ with $\varphi \circ \tilde{\alpha}_2 = \alpha_2$ and $\tilde{\alpha}_2(b_0) = y_0$. We now define the maps $\beta_i = \tilde{\alpha}_i|U \cap V: U \cap V \to \tilde{S}$, $i = 1, 2$. We note that $\varphi \circ \beta_i = \varphi \circ (\tilde{\alpha}_i|U \cap V) = (\varphi \circ \tilde{\alpha}_i)|U \cap V = \varphi \circ \tilde{\alpha}_i|U \cap V$. The maps $\beta_i$ are continuous and $\beta_i(a_0) = y_0$. Thus $\varphi \circ (\beta_1 \cup \beta_2) = \varphi \circ \tilde{\alpha}_1|U \cup V = \varphi \circ \tilde{\alpha}_2|U \cap V = \varphi \circ \tilde{\alpha}_2|U \cup V = \varphi \circ (\tilde{\alpha}_2|U \cap V)$. Therefore, $U \cup V$ is simply connected.
\[ \alpha_i \cap U \cap V = \alpha_z | U \cap V = (\phi \circ \alpha_i) | U \cap V = \phi \circ \beta_z \text{ and that } \beta_i(b_0) = \alpha_i(b_0) = y_0 = \alpha_z(b_0) = \beta_z(b_0). \] Since \( U \cap V \) is connected, we have \( \alpha_i | U \cap V = \beta_i = \beta_z = \alpha_z | U \cap V \). We can now define \( \tilde{\alpha}: A \to \tilde{S} \) with \( \tilde{\alpha}(a) = \alpha_i(a), \) when \( a \) is in \( U \), and \( = \alpha_z(a), \) when \( a \) is in \( V \). The continuity of \( \tilde{\alpha} \) follows by Exercise 3B of Kelley [7], and it is clear that \( \phi \circ \tilde{\alpha} = \alpha \) and that \( \tilde{\alpha}(a_0) = s_0 \). Finally, the uniqueness of \( \tilde{\alpha} \) follows again by the connectedness of \( U \cap V \).

**Lemma 9.** If \( P \) is a simply connected topological space and \( A \) is a retract of \( P \), then \( A \) is simply connected.

*Proof.* It is clear that \( A \) is connected and locally connected. Let \( \phi: \tilde{S} \to S \) be a covering map, \( \psi: A \to S \) be continuous, \( a \) in \( A \) and \( s \) in \( \tilde{S} \) with \( \psi(a) = \phi(s) \). Moreover, let \( \rho: P \to A \) be the retraction map. Then \( \psi \circ \rho: P \to S \) is continuous and \( \psi \circ \rho(a) = \psi(a) = \phi(s) \).

![Diagram](image)

Since \( P \) is simply connected, there is continuous \( \tilde{\psi}: P \to \tilde{S} \) with \( \psi \circ \rho = \phi \circ \tilde{\psi} \) and \( \tilde{\psi}(a) = s \). It is now straightforward to show that if \( \tilde{\psi} = \tilde{\psi} \mid A \), then \( \phi \circ \tilde{\psi} = \psi \) and \( \tilde{\psi}(a) = s \). Uniqueness of \( \tilde{\psi} \) follows from the connectedness of \( A \).

**Lemma 10.** Let \((\tilde{S}, \phi)\) be a simply connected covering space of \( S \) and \( A \) a retract of \( S \). If \( \tilde{A} \) is a component of \( \phi^{-1}(A) \), then \( \tilde{A} \) is a retract of \( \tilde{S} \) and \((\tilde{A}, \phi \mid \tilde{A})\) is a simply connected covering space of \( A \).

*Proof.* Let \( \rho: S \to S \) be the retract and \( \tilde{a} \) be in \( \tilde{A} \). Since \( \phi(\tilde{a}) \) is in \( A \), we have \( \rho \circ \phi(\tilde{a}) = \phi(\tilde{a}) \) and \( \rho \) lifts to continuous \( \rho: \tilde{S} \to \tilde{S} \) with \( \rho(\tilde{a}) = \tilde{a} \) and \( \phi \circ \rho = \rho \circ \phi \). Now let \( j: \tilde{A} \subseteq \tilde{S} \) and \( \rho \mid \tilde{A}: \tilde{A} \to \tilde{S} \). Then it is straightforward to show that \( \phi \circ (\rho \mid \tilde{A}) = \phi \circ j \) and that \( (\rho \mid \tilde{A})(\tilde{a}) = j(\tilde{a}) \), which implies that \( \rho \mid \tilde{A} = j \). Since \( \phi(\rho(\tilde{S})) = \rho(\phi(\tilde{S})) = \rho(S) = A \), we have \( \rho(\tilde{S}) \) a connected subset of \( \phi^{-1}(A) \). Observing that \( \tilde{a} \) is in \( \tilde{A} \cap \rho(\tilde{S}) \), we have \( \rho(\tilde{S}) \subseteq \tilde{A} \). Therefore, \( \rho \) is a retraction of \( \tilde{S} \) onto \( \tilde{A} \). Moreover, \((\tilde{A}, \phi \mid \tilde{A})\) is a simply connected covering space of \( A \) by Lemmas 6 and 9 of this section.

**Lemma 11.** If the topological product of two spaces admits a simply connected covering space, then so do both of them.
Proof. Let $(P, \varphi)$ be a simply connected covering space of $S \times T$. If $t$ is in $T$ and $\bar{S}$ is a component of $\varphi^{-1}(S \times t)$, then $(\bar{S}, \theta \circ (P \mid \bar{S}))$ is a simply connected covering space of $S$, where $\theta : S \times t \to S$ is the natural homeomorphism. Indeed, $S \times t$ is obviously a retract of $S \times T$, and we apply Lemma 10.

**Lemma 12.** Let $(\bar{S}, \varphi)$ be a simply connected covering space of $S$, $A$ a connected, locally connected subset of $S$, and $\bar{A}$ a component of $\varphi^{-1}(A)$. If $\bar{A}$ is simply connected, and we let $P(S)$ and $P(A)$ be the automorphism groups of $(\bar{S}, \varphi)$ and $(\bar{A}, \varphi \mid \bar{A})$, respectively, then there exists a monomorphism $\theta : P(A) \to P(S)$ such that if $\psi$ is in $P(A)$, then $\theta(\psi) = \psi$ is the unique extension of $\psi$ to $\psi$ in $P(S)$. Moreover, $\theta$ is an isomorphism if and only if $\varphi^{-1}(A)$ is connected, i.e., if and only if $\bar{A} = \varphi^{-1}(A)$.

Proof. Suppose $\psi$ is in $P(A)$. Fix $a_1$ in $\bar{A}$. Let $\psi(a_1) = a_2$ in $\bar{A}$. Now, $\varphi(a_2) = (\varphi \mid \bar{A})(a_1) = (\varphi \mid \bar{A})(a_2) = \varphi(a_2)$. Thus, there exists unique $\psi$ in $P(S)$ such that $\psi(a_1) = a_2$.

We show that $\psi$ is an extension of $\psi$. We first show that $\bar{\psi}(\bar{A}) = \bar{A}$. Clearly, $\bar{\psi}(\varphi^{-1}(A)) = \varphi^{-1}(A)$. We see that $\bar{\psi}(\bar{A})$ is a connected subset of $\varphi^{-1}(A)$ with $a_2$ in $\bar{A} \cap \bar{\psi}(\bar{A})$. Therefore, $\bar{\psi}(\bar{A}) \subseteq \bar{A}$. Let $\eta$ be the inverse of $\psi$ in $P(S)$. As before, we find $\eta$ in $P(S)$ such that $\eta(a_2) = a_1$ and $\eta(\bar{A}) \subseteq \bar{A}$. Now, $\overline{\psi \circ \eta}$ is in $P(S)$ and fixes $a_2$. Thus, $\overline{\psi \circ \eta}$ is the identity of $P(S)$, and $\bar{A} = \overline{\psi \circ \eta}(\bar{A}) \subseteq \bar{\psi}(\bar{A}) \subseteq \bar{A}$. Therefore, $\bar{\psi}(\bar{A}) = \bar{A}$. Since $\bar{\psi} : \bar{S} \to \bar{S}$ is a homeomorphism, so is $\bar{\psi} \mid \bar{A} : \bar{A} \to \bar{A}$. Moreover, $(\varphi \mid \bar{A}) \circ (\varphi \mid \bar{A})(a) = \varphi \circ \overline{\psi \circ \eta}(a) = \varphi(a) = (\varphi \mid \bar{A})(a)$, for all $a$ in $\bar{A}$. So, $\bar{\psi} \mid \bar{A}$ is in $P(A)$. But $\psi$ is in $P(A)$, and $\psi(a_1) = a_2 = (\varphi \mid \bar{A})(a_1)$.

Now that we have $\theta$ a well-defined function, we observe that it is trivially injective. A simple computational argument shows that $\theta$ is a homomorphism.

We next show that $\bar{A} = \varphi^{-1}(A)$ if and only if $\theta$ is surjective. Suppose $\bar{A} = \varphi^{-1}(A)$. Let $\psi$ be in $P(S)$. Then $\psi(\bar{A}) = \psi(\varphi^{-1}(A)) = \varphi^{-1}(A) = \bar{A}$. As above, we see that $\psi \mid \bar{A}$ is in $P(A)$. Moreover, $\theta(\psi \mid \bar{A}) = \psi$. Therefore, $\theta$ is surjective. Conversely, suppose $\theta$ is surjective. Let $\bar{a}_1$ be in $\varphi^{-1}(A)$. Let $\varphi(\bar{a}_1) = a$ in $A$. There exists $\bar{a}_2$ in $\bar{A}$ such that $\varphi(\bar{a}_2) = a = \varphi(\bar{a}_1)$. Thus, there is $\psi$ in $P(S)$ with $\bar{\psi}(\bar{a}_2) = \bar{a}_1$. Since $\theta$ is onto, there is $\psi$ in $P(A)$ with $\theta(\psi) = \bar{\psi}$, i.e., $\psi = \bar{\psi} \mid \bar{A}$. Then $\bar{a}_1 = \bar{\psi}(\bar{a}_2) = \psi(\bar{a}_2)$ in $\bar{A}$. Since $\bar{a}_1$ was arbitrary in $\varphi^{-1}(A)$, we have $\varphi^{-1}(A) \subseteq \bar{A}$, and they are equal.

2. General theory of covering semigroups. Let $\bar{S}$ and $S$ be topological semigroups and $\varphi : \bar{S} \to S$ a homomorphism. If, moreover, $(\bar{S}, \varphi)$ is a covering space of $S$, then we say that $(\bar{S}, \varphi)$ is a covering
semigroup of $S$. The proofs of the first two of the following theorems are omitted, as they are similar to the development of covering groups. See [2], [4], [5].

**Theorem 1.** Let $S$ be a topological semigroup with topological space structure admitting a simply connected covering space $(\tilde{S}, \varphi)$. Let $e$ be an idempotent in $S$ and fix some point $\tilde{e}$ in $\tilde{S}$ such that $\varphi(\tilde{e}) = e$. There exists a unique topological semigroup multiplication on $\tilde{S}$ such that $\tilde{e}$ is an idempotent and $\varphi$ is a homomorphism. If $e$ is an identity for $S$, then $\tilde{e}$ is an identity for $\tilde{S}$. If $S$ is a topological group, then so is $\tilde{S}$.

**Theorem 2.** Let $(\tilde{S}_1, \varphi_1)$ and $(\tilde{S}_2, \varphi_2)$ be covering semigroups of $S$ with idempotents $\tilde{e}_1$ in $\tilde{S}_1$ and $\tilde{e}_2$ in $\tilde{S}_2$ such that $\varphi_1(\tilde{e}_1) = \varphi_2(\tilde{e}_2)$. If $\tilde{S}_1$ is simply connected, then there exists a unique homomorphism and covering map $\psi: \tilde{S}_1 \to \tilde{S}_2$ with $\varphi_2 \circ \psi = \varphi_1$ and $\psi(\tilde{e}_1) = \tilde{e}_2$. Moreover, if $\tilde{S}_2$ is also simply connected, then $\psi$ is a covering space and semigroup isomorphism.

**Theorem 3.** Let $[X, G, Y]$ be a topological paragroup (Hofmann and Mostert [6]) where $X(Y)$ is a left (right) zero semigroup and $G$ is a group. If $X$, $G$, and $Y$ admit simply connected covering spaces $(\tilde{X}, \varphi_1)$, $(\tilde{G}, \varphi_2)$ and $(\tilde{Y}, \varphi_3)$, then the left (right) zero multiplication of $X(Y)$ lifts to a left (right) zero multiplication on $\tilde{X}(\tilde{Y})$ and the group multiplication of $G$ lifts to a group multiplication on $\tilde{G}$. Moreover, the sandwich function $\sigma: X \times X \to G$ lifts to a sandwich function $\tilde{\sigma}: \tilde{X} \times \tilde{X} \to \tilde{G}$ such that $([\tilde{X}, \tilde{G}, \tilde{Y}], \varphi_1 \times \varphi_2 \times \varphi_3)$ is a simply connected covering paragroup of $[X, G, Y]$.

**Proof.** Note that $\varphi_1(\varphi_3)$ is automatically a homomorphism if we give $\tilde{X}(\tilde{Y})$ the left (right) zero multiplication. Any lifting of $\sigma$ to $\tilde{\sigma}$ allows us to form the paragraph $[\tilde{X}, \tilde{G}, \tilde{Y}]$. A straightforward computation, making use of the equation $\sigma \circ (\varphi_3 \times \varphi_1) = \varphi_2 \circ \tilde{\sigma}$, shows that $\varphi_1 \times \varphi_2 \times \varphi_3: [\tilde{X}, \tilde{G}, \tilde{Y}] \to [X, G, Y]$ is a homomorphism. We omit further details.

**Theorem 4.** If $(\tilde{S}, \varphi)$ is a covering semigroup of $S$, then $\varphi^{-1}(\text{center } S) = \text{center } \tilde{S}$.

**Proof.** Clearly, $\text{center } \tilde{S} \subseteq \varphi^{-1}(\text{center } S)$. Let $s$ be any element of $\varphi^{-1}(\text{center } S)$. Define $\alpha, \beta: \tilde{S} \to \tilde{S}$ with $\alpha(x) = sx$ and $\beta(x) = xs$. Straightforward computations show that $\varphi \circ \alpha = \varphi \circ \beta$ and that $\alpha(s) = \beta(s)$. Thus, $\alpha = \beta$, i.e., $s$ is in center $\tilde{S}$.

For the rest of this section we assume that $(\tilde{S}, \varphi)$ is a simply
connected covering semigroup of $S$. Moreover, $\bar{S}$ and $S$ have identities $\bar{1}$ and 1, respectively. We define $\text{Ker} \, \varphi$ to be $\varphi^{-1}(1)$. Although this is not standard semigroup terminology, we feel that Theorem 6 of this section is ample motivation.

**Corollary 5.** $\text{Ker} \, \varphi$ is central.

**Proof.** Note that 1 is central.

**Theorem 6.** If $s$ is in $\text{Ker} \, \varphi$ and we define $\psi: \bar{S} \to \bar{S}$ by $\psi(x) = sx$, then $\psi$ is in $P(S)$. This defines an isomorphism between $\text{Ker} \, \varphi$ and $P(S)$. Therefore, $P(S)$ is commutative.

**Proof.** Let $s$ be in $\text{Ker} \, \varphi$ and define $\psi$ as above. There exists $\eta$ in $P(S)$ with $\eta(\bar{1}) = s$. Straightforward computation shows that $\varphi \circ \psi = \varphi \circ \eta$ and $\psi(\bar{1}) = \eta(\bar{1})$. So, $\psi = \eta$, and $\psi$ is in $P(S)$. Since $\bar{S}$ has an identity, we conclude that mapping $s$ into $\psi$ gives a monomorphism of $\text{Ker} \, \varphi$ into $P(S)$. We show that the mapping is onto. Let $\psi$ be in $P(S)$. Define $s = \psi(\bar{1})$. Then $s$ is in $\text{Ker} \, \varphi$, and we define $\eta = \theta(s)$ in $P(S)$. But then $\psi$ and $\eta$ agree at $\bar{1}$ and, therefore, are equal.

**Corollary 7.** If $a$ and $b$ are in $\bar{S}$ with $\varphi(a) = \varphi(b)$, then there exists unique $s$ in $\text{Ker} \, \varphi$ with $sa = b$.

Material from here through Corollary 18 is independent and completely algebraic in nature, providing we define $(\bar{S}, \varphi)$ to be an algebraic covering of $S$ with group $P(S)$ if:

(a) $\bar{S}$ and $S$ are purely algebraic semigroups with identities $\bar{1}$ and 1, respectively.

(b) The map $\varphi: \bar{S} \to S$ is a surmorphism with $\text{Ker} \, \varphi = \varphi^{-1}(1)$ being a central subgroup of $\bar{S}$.

(c) $\text{Ker} \, \varphi$ acts on $\bar{S}$ with orbits $\varphi^{-1}(x)$, $x$ in $S$, and is simply transitive on these orbits.

(d) $P(S)$ is a faithful functional representation of $\text{Ker} \, \varphi$ on $\bar{S}$.

**Lemma 8.** If $x$ is in $S$, $\bar{x}$ is in $\varphi^{-1}(x)$, and $A, B$ are subsets of $S$, then $\varphi^{-1}(AxB) = \varphi^{-1}(A)\bar{x}\varphi^{-1}(B)$. Also $\varphi^{-1}(Ax) = \varphi^{-1}(A)\bar{x}$, $\varphi^{-1}(xB) = \bar{x}\varphi^{-1}(B)$, and $\varphi^{-1}(AB) = \varphi^{-1}(A)\varphi^{-1}(B)$.

**Proof.** It is trivial that $\varphi^{-1}(A)\bar{x}\varphi^{-1}(B) \subseteq \varphi^{-1}(AxB)$. Conversely, let $y$ be in $\varphi^{-1}(AxB)$. There exists $a$ in $A$, $b$ in $B$ with $\varphi(y) = axb$. If we pick $\bar{a}$, $\bar{b}$, in $\bar{S}$ with $\varphi(\bar{a}) = a$ and $\varphi(\bar{b}) = b$, then $\varphi(\bar{a}\bar{b}) = axb = \varphi(y)$. Thus, there exists $s$ in $\text{Ker} \, \varphi$ with $s(\bar{a}\bar{b}) = y$. Observing
that $s\alpha$ is in $\varphi^{-1}(A)$, we have $y = (s\alpha)\bar{\xi}\bar{\beta}$ in $\varphi^{-1}(A)\bar{\xi}\varphi^{-1}(B)$, as desired.

The remaining equations follow easily from the equation $\varphi^{-1}(AxB) = \varphi^{-1}(A)\bar{\xi}\varphi^{-1}(B)$. Indeed, if $\bar{x} = \bar{1}$ and $x = 1$, we have $\varphi^{-1}(AB) = \varphi^{-1}(A)\varphi^{-1}(B)$, and if $B$ or $A$ is $\{1\}$, then the remaining equations result.

**Theorem 9.** If $H$ is a subgroup of $S$, then $\varphi^{-1}(H)$ is a subgroup of $\bar{S}$. In particular, if $e$ is an idempotent in $S$, then $\varphi^{-1}(e)$ is subgroup of $\bar{S}$. Moreover, if $\theta : \text{Ker } \varphi \to \varphi^{-1}(e)$ by $\theta(s) = se$, where $\bar{e}$ is the identity of $\varphi^{-1}(e)$, then $\theta$ is an isomorphism. Thus, $\varphi^{-1}(e) \cong P(S)$. Note that it follows that $\varphi^{-1}(H)$ is an extension of $P(S)$ by $H$, in the sense of Kurosh [8], p. 76.

**Proof.** Let $\bar{x}$ be in $\varphi^{-1}(H)$, $\varphi(\bar{x}) = x$ in $H$. Then $\bar{x}\varphi^{-1}(H) = \varphi^{-1}(xH) = \varphi^{-1}(H)$ and $\varphi^{-1}(H)\bar{x} = \varphi^{-1}(Hx) = \varphi^{-1}(H)$. Therefore, $\varphi^{-1}(H)$ is a group.

We show $\theta$ is an isomorphism. Since $\bar{e}$ is idempotent and Ker $\varphi$ is central, $\theta(st) = (st)\bar{e} = (se)(te) = \theta(s)\theta(t)$, for all $s, t$ in Ker $\varphi$. Moreover, if $x$ is in $\varphi^{-1}(e)$ then there exists unique $s$ in Ker $\varphi$ with $se = x$, i.e., $\theta(s) = x$. Therefore, $\theta$ is an isomorphism.

**Theorem 10.** If $\bar{E}$ and $E$ are the sets of idempotents of $\bar{S}$ and $S$, respectively, then $\varphi \mid \bar{E} : \bar{E} \to E$ is bijective. In particular, if $S$ has no idempotents other than $1$, then $\bar{S}$ has no idempotents other than $\bar{1}$.

**Proof.** If $e$ is in $E$, then $\varphi^{-1}(e)$ is a group and thus contains exactly one idempotent.

In the next few pages we deal with $L$-, $R$-, $H$-, $D$-, and $J$-classes of a semigroup. Notation and terminology are as in Clifford and Preston [3].

**Lemma 11.** Let $a, b$ be in $S$ and $\bar{a}, \bar{b}$ in $\varphi^{-1}(a), \varphi^{-1}(b)$, respectively. Then $aLb$ if and only if $\bar{a}L\bar{b}$, and similarly for $R, H, D,$ and $J$.

**Proof.** The fact that $\bar{a}L\bar{b}$ implies $aLb$ is automatic algebraically, and likewise for $R, H, D,$ and $J$. All that is needed is that $\bar{S}$ and $S$ be algebraic semigroups and that $\varphi$ be an epimorphism. Conversely, let $aLb$. Then $\bar{S}a = \varphi^{-1}(S)a = \varphi^{-1}(Sa) = \varphi^{-1}(Sb) = \varphi^{-1}(S)\bar{b} = \bar{S}\bar{b}$ gives $\bar{a}L\bar{b}$. Symmetrically, $aRb$ implies $\bar{a}R\bar{b}$. As for $H$-classes, we have $aHb$ if and only if $aLb$ and $aRb$ if and only if $\bar{a}L\bar{b}$ and $\bar{a}R\bar{b}$ if and only if $\bar{a}H\bar{b}$. As for $D$-classes, we use the fact that for any semigroup $S, D = L \circ R$, [3], page 47.
Thus, suppose $a \mathcal{D} b$. Then there is $c$ in $S$ with $a \mathcal{L} c$ and $c \mathcal{R} b$. If $c$ is in $\varphi^{-1}(c)$, then $\bar{a} \mathcal{L} \bar{c}$ and $\bar{c} \mathcal{R} \bar{b}$, i.e., $\bar{a} \mathcal{D} \bar{b}$. Finally, for $\mathcal{J}$-classes we have $a \mathcal{J} b$ implies $\bar{a} \bar{S} \bar{S} = \varphi^{-1}(S \bar{a} \bar{S}) = \varphi^{-1}(S \bar{b} \bar{S}) = \bar{S} \bar{b} \bar{S}$, i.e., $\bar{a} \mathcal{D} \bar{b}$.

**Theorem 12.** $\varphi$ induces a bijective correspondence between the $\mathcal{L}$ classes of $\bar{S}$ and the $\mathcal{L}$-classes of $S$. More precisely, if $\bar{a}$ is in $\bar{S}$ and $a = \varphi(\bar{a})$, then $\varphi^{-1}(L_{\bar{a}}) = L_a$. This holds similarly for $R_a$, $H_a$, $D_a$, and $J_a$.

**Proof.** $x$ is in $\varphi^{-1}(L_{\bar{a}})$ if and only if $\varphi(x)$ is in $L_a$ if and only if $\varphi(x) \mathcal{R} \bar{a}$ if and only if $x \mathcal{L} \bar{a}$ if and only if $x$ is in $L_{\bar{a}}$. Similar proofs hold for $R_a$, $H_a$, $D_a$, and $J_a$.

**Corollary 13.** $\varphi$ induces a bijective correspondence between the maximal subgroups of $\bar{S}$ and the maximal subgroups of $S$. More precisely, if $H$ is a maximal subgroup of $\bar{S}$, then $\varphi(\bar{H})$ is a maximal subgroup of $S$; if $H$ is a maximal subgroup of $S$, then $\varphi^{-1}(H)$ is a maximal subgroup of $\bar{S}$.

**Proof.** This is immediate if we observe that the maximal subgroups of a semigroup are precisely the $\mathcal{H}$-classes containing idempotents [3], p. 61.

Let $S$ be a semigroup, $H$ an $\mathcal{H}$-class of $S$, and $s$ an element of $S$ such that $sH \subseteq H$. Then we denote by $\gamma_s$ the element of $\Gamma(H)$, the left Schützenberger group [3] of $H$, such that $\gamma_s(x) = sx$, for all $x$ in $H$. The following theorem generalizes Theorem 9.

**Theorem 14.** If $H$ is an $\mathcal{H}$-class in $S$ and $\bar{H} = \varphi^{-1}(H)$ is the corresponding $\mathcal{H}$-class in $\bar{S}$, then the left Schützenberger group $\Gamma(\bar{H})$ is an extension of $\varphi$-classes by the left Schützenberger group $\Gamma(H)$.

**Proof.** Let $T(\bar{H})$ be the subsemigroup of $\bar{S}$ of all $s$ in $\bar{S}$ with $s\bar{H} \subseteq \bar{H}$, and let $T(H)$ be similar in $S$. Let $\bar{\nu}: T(\bar{H}) \rightarrow \Gamma(\bar{H})$ and $\nu: T(H) \rightarrow \Gamma(H)$ be the natural homomorphisms. It is straightforward to show that $\varphi^{-1}(T(H)) = T(\bar{H})$ and that $\varphi$ induces epimorphisms $\varphi_{H}: T(\bar{H}) \rightarrow T(H)$ and $\varphi_{\bar{H}}: \Gamma(\bar{H}) \rightarrow \Gamma(H)$ with $\varphi_{H} \circ \bar{\nu} = \nu \circ \varphi_{\bar{H}}$. Moreover, $\ker \varphi_{H}$ is contained in $T(\bar{H})$, and $\bar{\nu}(\ker \varphi_{H})$ is contained in $\ker \varphi_{\bar{H}}$. Thus $\bar{\nu}$ induces a homomorphism $\bar{\nu}_{\varphi}: \ker \varphi \rightarrow \ker \varphi_{\bar{H}}$. Since the image of $\bar{\nu}_{\varphi}$ is the restriction of all the functions in $\varphi$ to $\bar{H}$, it follows that $\bar{\nu}_{\varphi}$ is injective. We next show that $\bar{\nu}_{\varphi}$ is surjective. Let $\psi$ be in $\ker \varphi_{\bar{H}}$. There is $s$ in $T(\bar{H})$ with $\psi = \bar{\nu}(s)$. Let $\bar{x}$ be in $\bar{H}$. If $\varphi(\bar{x}) = x$ in $H$, then $\varphi(s\bar{x}) = \varphi(s)x = \gamma_{\varphi(s)}(x) = [\nu \circ \varphi_{\bar{H}}(s)](x) = [\varphi_{\bar{H}} \circ \bar{\nu}(s)](x) = [\varphi_{H}(\psi)](x) = \gamma_{\varphi}(x) = x = \varphi(\bar{x})$. Thus, there is $t$ in $\ker \varphi$.
with \( t\bar{x} = s\bar{x} \), and we have \( \gamma_t \) and \( \gamma_s \) in \( \Gamma(\bar{H}) \) agreeing at \( \bar{x} \). But \( \Gamma(\bar{H}) \) is simply transitive on \( \bar{H} \), and thus \( \bar{\nu}_0(t) = \gamma_t = \gamma_s = \phi \), as desired.

We recall that an element \( a \) of a semigroup \( S \) is called regular if \( axa = a \) for some \( x \) in \( S \), and \( S \) is called regular if every element of \( S \) is regular. Moreover, \( a \) and \( b \) are inverses of each other if \( aba = a \) and \( bab = b \), and \( S \) is an inverse semigroup if every element of \( S \) has a unique inverse. The following are equivalent for an element \( a \) of a semigroup \( S \): (1) the element \( a \) is regular, (2) the element \( a \) has an inverse \( \delta \), (3) the principal left ideal generated by \( a \) has an idempotent generator, and (4) the principal right ideal generated by \( a \) has an idempotent generator [3], p. 27.

**Theorem 15.** If \( a \) is a regular element of \( S \) and \( \bar{a} \) is in \( \varphi^{-1}(a) \), then \( \bar{a} \) is regular. Therefore, if \( S \) is regular then so is \( \bar{S} \).

**Proof.** Since \( a \) is regular, there is an idempotent \( e \) in \( S \) with \( Se = Sa \). Let \( \bar{e} \) be the idempotent in \( \varphi^{-1}(e) \). Then \( S\bar{e} = \varphi^{-1}(Se) = \varphi^{-1}(Sa) = \bar{S}a \), and thus \( \bar{a} \) is regular.

**Theorem 16.** If \( S \) is an inverse semigroup, then so is \( \bar{S} \).

**Proof.** We recall that a semigroup is inverse if and only if every principal right ideal and every principal left ideal has a unique idempotent generator. Let \( S \) be an inverse semigroup. By the above theorem, every principal right ideal and every principal left ideal has at least one idempotent generator. Suppose \( \bar{e} \) and \( \bar{f} \) are idempotents in \( \bar{S} \) with \( \bar{S}e = \bar{S}f \). Then \( \varphi(\bar{e}) \) and \( \varphi(\bar{f}) \) are idempotents generating the same principal left ideal in \( S \). Since \( S \) is an inverse semigroup, we have \( \varphi(\bar{e}) = \varphi(\bar{f}) \), which implies \( \bar{e} = \bar{f} \), by Theorem 10. Principal right ideals are treated symmetrically.

**Theorem 17.** If \( I \) is a left ideal (right ideal) (ideal) in \( S \), then \( \varphi^{-1}(I) \) is a left ideal (right ideal) (ideal) in \( \bar{S} \). If \( \bar{I} \) is a left ideal (right ideal) (ideal) in \( \bar{S} \), then \( \varphi^{-1}(\varphi(\bar{I}) = \bar{I} \). Therefore, \( \varphi \) induces a bijective, inclusion preserving correspondence between the left ideals (right ideals) (ideals) of \( \bar{S} \) and those of \( S \).

**Proof.** Let \( I \) be a left ideal in \( S \). Then \( \bar{S}\varphi^{-1}(I) = \varphi^{-1}(SI) \subseteq \varphi^{-1}(I) \), i.e., \( \varphi^{-1}(I) \) is a left ideal in \( \bar{S} \). Now, let \( x \) be in \( \varphi^{-1}(\bar{I}) \) where \( \bar{I} \) is a left ideal in \( \bar{S} \). There is \( y \) in \( \bar{I} \) with \( \varphi(x) = \varphi(y) \). So, there is \( s \) in \( \text{Ker } \varphi \) with \( x = sy \) in \( \bar{I} \). The proof for right ideals or ideals is similar.
COROLLARY 18. If I is a minimal left ideal (right ideal) (ideal) in S, then $\varphi^{-1}(I)$ is a minimal left ideal (right ideal) (ideal) in $\bar{S}$.

THEOREM 19. If S has a minimal ideal K then $P(S) \cong P(K)$.

Proof. By Proposition 1.9 of [1] we have that K is a retract of S, and thus K is connected and locally connected. Let $\bar{K} = \varphi^{-1}(K)$. By Corollary 18, $\bar{K}$ is the minimal ideal of $\bar{S}$ and, hence, is connected. By Lemma 10 of the previous section, $\bar{K}$ is simply connected. Then by Lemma 12 of that section $P(K) \cong P(S)$.

THEOREM 20. Let S have a minimal ideal K. Moreover, let e be a primitive idempotent in K. Let $X = E(Se)$, $Y = E(eS)$ be the sets of idempotents in $Se$ and $eS$, respectively, and let $G = eSe$, a maximal subgroup of K. Let $\sigma: Y \times X \to G$ such that $\sigma(y, x) = yx$. Let $\theta: [X, G, Y] \to K$ be the canonical map, i.e., $\theta(x, g, y) = xgy$. Now, $\theta$ is an algebraic isomorphism and continuous [6]. If $\theta$ is also a homeomorphism, then X and Y are simply connected and thus $P(K) \cong P(G)$.

Proof. From Proposition 1.9 of [1], p. 47, we have that K is a retract of S. Let $\bar{K} = \varphi^{-1}(K)$. By Lemma 10 of the previous section, $(\bar{K}, \varphi|\bar{K})$ is a simply connected covering space of K. The topological space structure of $[X, G, Y]_e$ is $X \times G \times Y$ with the product topology. By Lemma 11 of the previous section and the fact that $\theta$ is a homeomorphism, X, G, and Y have simply connected covering spaces $(\bar{X}, \varphi_1)$, $(\bar{G}, \varphi_2)$, and $(\bar{Y}, \varphi_3)$. By Theorem 3, $([\bar{X}, \bar{G}, \bar{Y}]_e, \varphi')$ is a simply connected covering paragroup of $[X, G, Y]_e$, where $\varphi' = \varphi_1 \times \varphi_2 \times \varphi_3$. In lifting $\sigma$ to $\bar{\sigma}$ we

\[
\begin{array}{ccc}
\bar{Y} \times \bar{X} & \xrightarrow{\bar{\sigma}} & \bar{G} \\
\varphi_3 \times \varphi_1 & \downarrow & \varphi_2 \\
Y \times X & \xrightarrow{\sigma} & G 
\end{array}
\]

can choose $\bar{\sigma}$ such that $\bar{\sigma}(\bar{e}_3, \bar{e}_1) = \bar{e}_2$, where $\bar{e}_2$ is the identity of $\bar{G}$ and $\bar{e}_3$ and $\bar{e}_1$ are fixed in $\bar{Y}$ and $\bar{X}$, respectively, such that $\varphi_3(\bar{e}_3) = e$ and $\varphi_1(\bar{e}_1) = e$.

Now $\theta \circ \varphi'(\bar{e}_1, \bar{e}_2, \bar{e}_3) = \theta(e, e, e) = e^3 = e = (\varphi|\bar{K})(\bar{e})$, where $\bar{e}$ is the idempotent of $\bar{K}$ such that $\varphi(\bar{e}) = e$. By Theorem 2, we can lift $\theta$ to a semigroup and covering space isomorphism $\bar{\theta}$ so that $\bar{\theta}(\bar{e}_1, \bar{e}_2, \bar{e}_3) = \bar{e}$ and $(\varphi|\bar{K}) \circ \bar{\theta} = \theta \circ \varphi'$.
We now show that all the elements of $\tilde{X} \times \tilde{e}_2 \times \tilde{e}_3$ are idempotent. Now, $\phi_2(\tilde{e}_2 \times \tilde{X}) = \sigma(\phi_3 \times \phi_1(\tilde{e} \times \tilde{X})) = \sigma(\tilde{e} \times \tilde{X}) = e\tilde{X} = e$, since $\tilde{X}$ is a left zero semigroup. This means that $\tilde{e}(\tilde{e}_2 \times \tilde{X})$ is a connected subset of the discrete set $\ker \phi_2$. Moreover, $\tilde{e}_2 = \tilde{e}(\tilde{e}_2, \tilde{e}_3)$ is in $\tilde{e}(\tilde{e}_2 \times \tilde{X})$. Thus, if $x$ is in $\tilde{X}$, then $(x, \tilde{e}_2, \tilde{e}_3)^2 = (x, \tilde{e}_2 \tilde{e}(\tilde{e}_2, x)\tilde{e}_2, \tilde{e}_3) = (x, \tilde{e}_2^2, \tilde{e}_3) = (x, \tilde{e}_2, \tilde{e}_3)$, as desired.

We show that $\phi_1 : \tilde{X} \to X$ is one-to-one. Let $x_1, x_2$ be in $\tilde{X}$ with $\phi_1(x_1) = \phi_1(x_2)$. Then $\phi(\tilde{e}(x_1, \tilde{e}_2, \tilde{e}_3)) = (\tilde{e} \times X) = \tilde{e}(x_1, \tilde{e}_2, \tilde{e}_3) = \sigma(\tilde{e}_2 \times \tilde{X}) = (\phi_1(x_1), e, e) = \phi_1(x_1)e = \phi_1(x_2)$, since $\phi_1(x_i)$ and $e$ are in $X$, a left zero semigroup. Hence, $\phi(\tilde{e}(x_1, \tilde{e}_2, \tilde{e}_3)) = \phi_1(x_1) = \phi_1(x_2) = \phi(\tilde{e}(x_2, \tilde{e}_2, \tilde{e}_3))$. Since $(x_1, \tilde{e}_2, \tilde{e}_3)$ and $(x_2, \tilde{e}_2, \tilde{e}_3)$ are idempotents, so are $\tilde{e}(x_1, \tilde{e}_2, \tilde{e}_3)$ and $\tilde{e}(x_2, \tilde{e}_2, \tilde{e}_3)$. By Theorem 10, $\tilde{e}(x_1, \tilde{e}_2, \tilde{e}_3) = \tilde{e}(x_2, \tilde{e}_2, \tilde{e}_3)$. Hence, $(x_1, \tilde{e}_2, \tilde{e}_3) = (x_2, \tilde{e}_2, \tilde{e}_3)$ and $x_1 = x_2$.

Therefore, $X$ is simply connected, and symmetrically, $Y$ is simply connected. Moreover, $P(K) \cong P(X \times G \times Y) \cong P(X) \times P(G) \times P(Y) \cong P(G)$.

Let $(\bar{G}, \beta)$ be a simply connected covering group of a compact Lie group $G$. It is known [4] that the following are equivalent: (a) $G$ is semisimple, (b) $P(G)$ is finite, (c) $\bar{G}$ is compact. The following corollary follows easily.

**Corollary 21.** Using the hypotheses and notation of Theorem 20 and assuming that $S$ is compact and that $G$ is a Lie group, we have that the following are equivalent: (a) $G$ is semisimple, (b) $P(S)$ is finite, (c) $\tilde{S}$ is compact.

3. Applications and examples.

(A) **Semigroups on the cylinder.** Mostert and Shields [9] proved that a topological semigroup on the plane with an identity and no other idempotents must be a group. The cylinder can be handled as follows.

**Theorem.** Let $S$ be a topological semigroup with identity $1$ and with the cylinder $S^1 \times R$ as topological space structure. Here $R$ is the line and $S^1 = \{(x, y) : (x, y) \in R^2$ and $x^2 + y^2 = 1\}$. If $S$ has no idempotents other than $1$, then $S$ is a group.

**Proof.** $S$ has a simply connected covering semigroup $(\tilde{S}, \phi)$ with identity $\tilde{1}$ and space the plane. Moreover, $\tilde{S}$ has no other idempotents.
By Mostert and Shields, $\bar{S}$ is a group. Being the homomorphic image of a group, $S$ is a group.

(B) A non-locally connected example. In this section we discuss one type of cylindrical semigroup [6], p. 67. Following [6], we define $H = [0, \infty)$ and $H^* = [0, \infty)$, both under addition.

**Theorem 1.** Let $(\bar{A}, \varphi)$ be a covering group of the group $A$, and let $f: H \to A$ be a continuous homomorphism. Define $f^+: H \to H^* \times A$ by $f^+(p) = (p, f(p))$. Since $H$ is simply connected, there exists a unique homomorphism $\bar{f}: H \to \bar{A}$ such that $\varphi \circ \bar{f} = f$. Now define $\bar{f}^+: H \to H^* \times \bar{A}$ by $\bar{f}^+(p) = (p, \bar{f}(p))$. Let $S = f^+(H) \cup \infty \times A$ and $\bar{S} = \bar{f}^+(H) \cup \infty \times \bar{A}$.

Then $S$ and $\bar{S}$ are closed subsemigroups of $H^* \times A$ and $H^* \times \bar{A}$, respectively, and $\bar{f}^+(H)$ is the component of $(1 \times \varphi)^{-1}(f^+(H))$ that contains $(0, 1)$, where $1 \times \varphi: H^* \times \bar{A} \to H^* \times A$. Moreover, $(\bar{S}, (1 \times \varphi) | \bar{S})$ is a sort of "not necessarily connected (at most two components) covering semigroup" of $S$ in the sense that $(\bar{f}^+(H), (1 \times \varphi) | \bar{f}^+(H))$ is a trivial covering semigroup of $f^+(H)$ and $(\infty \times \bar{A}, (1 \times \varphi) | \infty \times \bar{A})$ is a covering semigroup of $\infty \times \bar{A}$.

**Proof.** The fact that $S$ and $\bar{S}$ are closed subsemigroups of $H^* \times A$ and $H^* \times \bar{A}$ follows as in [6], as does the fact that $f^+(H)$ and $\bar{f}^+(H)$ are copies of $H$ as subsemigroups of $S$ and $\bar{S}$. Observing that $(1 \times \varphi)^{-1} \circ \bar{f}^+ = f^+$, we have that $\bar{f}^+(H)$ is a connected subsemigroup of $(1 \times \varphi)^{-1}(f^+(H))$. Let $C$ be the component of $(1 \times \varphi)^{-1}(f^+(H))$ containing $\bar{f}^+(H)$. Then $(C, (1 \times \varphi) | C)$ is a covering semigroup of the simply connected $f^+(H)$. Thus $C$ is a copy of $H$, and we must have $\bar{f}^+(H) = C$. The rest of the theorem is now obvious.

**Theorem 2.** Let $A$ be a connected topological group and $f: H \to A$ a continuous homomorphism. Define $f^+: H \to H^* \times A$ and $S$ as in Theorem 1. Then $S$ is not connected if and only if $f$ is an imbedding onto a closed subset of $A$.

**Proof.** $S$ is not connected if and only if $f^+(H)$ is closed in $S$ and, therefore, if and only if $f^+(H)$ is closed in $H^* \times A$. This means that for each point $a$ in $A$, there is a $p_a$ in $H$ and a neighborhood $N_a$ of $a$ such that $(p_a, \infty] \times N_a$ is disjoint from $f^+(H)$, i.e., $(p, f(p))$ is not in $(p_a, \infty] \times N_a$ for all $p$ in $H$. Thus, $S$ is not connected is equivalent to the existence of a neighborhood $N_a$ of each point $a$ of $A$ such that $f(p)$ is not in $N_a$ for sufficiently large $p$. This last is equivalent to $f(H)$ being closed in $A$ and the local finiteness of the collection of all sets of the form $f([k, k + 1])$, $k$ a non-negative integer. The remainder of the proof is straightforward.
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