Tight Approximation Ratio for Minimum Maximal Matching

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Abstract

We study a combinatorial problem called Minimum Maximal Matching, where we are asked to find in a general graph the smallest matching that can not be extended. We show that this problem is hard to approximate with a constant smaller than 2, assuming the Unique Games Conjecture.

As a corollary we show, that Minimum Maximal Matching in bipartite graphs is hard to approximate with constant smaller than $\frac{4}{3}$, with the same assumption. With a stronger variant of the Unique Games Conjecture — that is Small Set Expansion Hypothesis — we are able to improve the hardness result up to the factor of $\frac{3}{2}$.

1 Introduction

Matchings are one of the most central combinatorial structures in theory of algorithms. A routine computing them is a basic puzzle used in numerous results in Computer Science (like Christofides algorithm). Various variants of matchings are studied extensively. Their computation complexity status is usually well-known and some techniques discovered when studying matchings are afterwards employed in other problems.

As we know since 1961, all natural variants of perfect matchings and maximum matchings can be found in polynomial time, even in general graphs. Here we study a different problem — Minimum Maximal Matching (MMM). The task is — given graph $G$, to find an inclusion-wise maximal matching $M$ with the smallest cardinality (or weight in the weighted version).

1.1 Related Work

The MMM problem was studied as early as 1980, when Yannakakis and Gavril showed, that it is NP-hard even in some restricted cases [19]. Their paper also presents an equivalence of MMM and Minimum Edge Dominating Set (EDS) problem, where the goal is to find minimum cardinality subset of edges $F$, such that every edge in the graph is adjacent to some edge in $F$. Every maximal matching is already an edge dominating set, and any edge dominating set can be easily transformed to a maximal matching of no larger size. This equivalence does not hold for the weighted variants of the problem.

It is a well known, simple combinatorial fact, that one maximal matching in any graph can not be more than twice as large as another maximal matching. This immediately gives a trivial 2-approximation algorithm for MMM. Coming up with 2-approximation in the weighted variant of
either of the problems is more challenging. In 2003, Carr, Fujito, Konjevod and Parekh presented a $2\frac{1}{10}$-approximation algorithm for a weighted EDS problem [3]. Later the approximation was improved to 2 by Fujito and Nagamochi [7].

Some algorithms aiming at approximation ratio better than 2 were also developed for the unweighted problem. Gotthilf, Lewenstein and Rainschmidt came up with a $2 - \frac{c \log n}{n}$-approximation for the general case [8]. Schmied and Viehmann have a better-than-two constant ratio for dense graphs [18].

Finally, hardness results need to be mentioned. In 2006 Chlebík and Chlebíková proved, that it is NP-hard to approximate the problem within factor better than $\frac{7}{6}$ [4]. The result was later improved to 1.18 by Escoffier, Monnot, Paschos, and Xiao [6]. $\frac{1}{2}$-hardness results depending on UGC were also obtained [6, 18].

1.2 Unique Games Conjecture

Unique Games Conjecture, since being formulated by Khot in 2002 [10], has been used to prove hardness of approximation of many problems. For the survey on UGC results see [11].

Many hardness results obtained from Unique Games Conjecture match previously known algorithms, as is the case, for example, of Vertex Cover, Max Cut or Maximum Acyclic Subgraph. Therefore, it is appealing to use it to obtain new results. While UGC is still open, recently a related 2–2-Games Conjecture has been proved [13], in consequence proving Unique Games Conjecture with partial completeness. This result provides some evidence towards validity of Unique Games Conjecture.

Basing on Unique Games Conjecture we are able to prove the main result of our paper.

**Theorem 1.** Assuming Unique Games Conjecture, it is NP-hard to approximate Minimum Maximal Matching with constant better than 2.

The proof of this theorem relies on the UGC-hardness proof for Vertex Cover of Khot and Regev [14]. In essence, we endeavour to build a matching over the vertices of Vertex Cover.

As a side-effect of our proof, hardness of approximating Total Vertex Cover follows. In this problem the goal is to find a subset $W$ of vertices, which is a Vertex Cover and every vertex in $W$ is incident to at least one other vertex in $W$.

**Corollary 2.** Assuming Unique Games Conjecture, it is NP-hard to approximate Total Vertex Cover with constant better than 2.

The Minimum Maximal Matching problem does not seem to be easier on bipartite graphs. All the algorithms mentioned above are defined for general graphs and we are not aware of any ways to leverage the bipartition of the input graph. At the same time, our hardness proof only works for general graphs. With some observations we are able to achieve a hardness result for bipartite graphs, which, however, is not tight.

**Theorem 3.** Assuming Unique Games Conjecture, it is NP-hard to approximate bipartite Minimum Maximal Matching with constant better than $\frac{4}{3}$.

1.3 Obtaining a Stronger Result

The studies on Unique Games Conjecture and hardness of approximation of different problems have led to formulating different hypotheses strengthening upon UGC — among them the Small
Set Expansion Hypothesis proposed by Raghavendra and Steurer [17], and another conjecture — whose name is not yet established and so far the name Strong UGC is used — formulated by Bansal and Khot [1]. A competent discussion on differences between the two conjectures can be found in [15, Appendix C].

To improve our result on bipartite graphs, we construct a reduction from a problem called Maximum Balanced Biclique (MBB), where — given a bipartite graph — the goal is to find a maximum clique with the same number of vertices on each side of the graph. Hardness of approximation results suitable for our reduction have been found starting from both the Small Set Expansion Hypothesis [15] and Strong UGC [2].

Theorem 4. Assuming Small Set Expansion Hypothesis (or Strong Unique Games Conjecture), it is NP-hard to approximate Bipartite Minimum Maximal Matching with a constant better than \( \frac{3}{2} \).

2 Revisiting the Khot-Regev Reduction

In their paper [14] Khot and Regev prove the UGC-hardness of approximating Minimum Vertex Cover within a factor smaller than 2. In this section we look at parts of their proof more closely.

Their reduction starts off with an alternative formulation of UGC\(^1\), which, they show, is a consequence of the standard variant.

2.1 Khot-Regev Formulation of Unique Games Conjecture

This formulation talks about a variant of Unique Label Cover problem described variables, \( E \) are the edges and \( \Psi_{x_1,x_2} \) defines a constraint by a tuple \( \Phi = (X,R,\Psi,E) \). \( X \) is a set of for every pair of variables connected by an edge. A constraint is a permutation \( \Psi_{x_1,x_2} \in R \leftrightarrow R \) meaning that if \( x_1 \) is labelled with a colour \( r \in R \), \( x_2 \) must be labelled with \( \Psi_{x_1,x_2}(r) \).

A \( t \)-labelling is an assignment of subsets \( L(x) \) of size \( |L(x)| = t \) to the variables. A constraint \( \Psi_{x_1,x_2} \) is satisfied by the \( t \)-labelling \( L \) if there exists a colour \( r \in L(x_1) \) such that \( \Psi_{x_1,x_2}(r) \in L(x_2) \).

Conjecture 5 (Unique Games Conjecture). For any \( \xi, \gamma > 0 \) and \( t \in \mathbb{N} \) there exists some \( |R| \) such that it is NP-hard to distinguish, given an instance \( \Phi = (X,R,\Psi,E) \) which category it falls into:

- (YES instance): There exists a labelling (1-labelling) \( L \) and a set \( X_0 \subseteq X \), \( |X_0| \geq (1 - \xi)|X| \), such that \( L \) satisfies all constraints between vertices of \( X_0 \).

- (NO instance): For any \( t \)-labelling \( L \) and any set \( X_0 \subseteq X \), \( |X_0| \geq \gamma |X| \), not all constraints between variables of \( X_0 \) are satisfied by \( L \).

2.2 Weighted Vertex Cover

The next step is a reduction from the UGC to the Minimum Vertex Cover problem. Given an instance \( \Phi = (X,R,\Psi,E) \) of Unique Label Cover problem, as described above, we build a graph \( G_\Phi \).

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\(^1\)In their paper, Khot and Regev call this formulation “Strong Unique Games Conjecture”. Since then, however, the same name has been used to refer another formulation, as in [1], we decided to minimise confusion by not recalling this name.
For every variable in \( x \in X \) we create a cloud \( C_x \) of \( 2^{|R|} \) vertices. Each vertex corresponds to a subset of labels and is denoted by \((x, S) \in |X| \times \mathcal{P}(R)\). The weight of a new vertex \((x, S)\) is equal to

\[
\mu(|S|) = \frac{1}{|X|} \cdot p^{|S|} (1 - p)^{|R| - |S|}
\]

where \( p = \frac{1}{2} - \varepsilon \) (there is a bias towards smaller sets). The total weight of \( G_\Phi \) is thus equal to 1.

Next, we connect the vertices \((x_1, S_1)\) and \((x_2, S_2)\) if the labellings \( S_1 \) and \( S_2 \) do not satisfy the constraint \( \Psi_{x_1, x_2} \).

**Lemma 6 ([14, Sec. 4.2]).** If \( \Phi \) was a *YES* instance, the graph \( G_\Phi \) has an independent set of weight at least \( \frac{1}{2} - 2\varepsilon \).

**Proof.** The instance \( \Phi \), being a *YES* instance, has a labelling \( L \) assigning one colour \( r_x \) to each variable \( x \). We know, that there is a large set \( X_0 \) of variables (\( |X_0| \geq (1 - \xi)|X| \)), such that all constraints between variables of \( X_0 \) are satisfied by \( L \).

We now define

\[
\mathcal{IS} = \{(x, S) \mid x \in X_0, r_x \in S\}
\]

and claim, that \( \mathcal{IS} \) is an independent set in \( G_\Phi \). For any two variables \( x_1 \) and \( x_2 \) of \( X_0 \) we know, that

\[
\Psi_{x_1, x_2}(r_{x_1}) = r_{x_2}.
\]

Indeed, if we then take the sets of labels \( S_1 \ni r_1 \) and \( S_2 \ni r_2 \), they do satisfy the constraint for the variables \( x_1, x_2 \). Hence, there is no edge between \((x_1, S_1)\) and \((x_2, S_2)\).

Finally, the weight of \( \mathcal{IS} \) is equal to

\[
\begin{align*}
w(\mathcal{IS}) &= \sum_{x \in X_0} \left( \sum_{S \subseteq R, S \ni r_x} w(x, S) \right) = \sum_{x \in X_0} \left( \frac{1}{|X|} \sum_{k=1}^{|R|} \binom{|R| - 1}{k - 1} \cdot p^k \cdot (1 - p)^{|R| - k} \right) \\
&= \sum_{x \in X_0} \left( p \cdot \frac{1}{|X|} \sum_{k=0}^{|R| - 1} \binom{|R| - 1}{k} \cdot p^k \cdot (1 - p)^{|R| - 1 - k} \right) \\
&= \sum_{x \in X_0} \left( p \cdot \frac{1}{|X|} \cdot (p + (1 - p))^{|R| - 1} \right) \\
&= \frac{|X_0|}{|X|} \cdot p \geq (1 - \xi)(\frac{1}{2} - \varepsilon) > \frac{1}{2} - 2\varepsilon.
\end{align*}
\]

The most of their paper is dedicated to proving the following key lemma.

**Lemma 7 ([14, Sec. 4.3]).** If \( \Phi \) is a *NO* instance, it does not have an independent set of weight larger than \( 2\gamma \).

Since the Minimum Vertex Cover is a complement of the Maximum Independent Set, we see that it is hard to distinguish between graphs with Minimum Vertex Cover of the weight \( \frac{1}{2} + 2\varepsilon \) and those, where Minimum Vertex Cover weights \( 1 - 2\gamma \).
2.3 Notation

Throughout this paper we are going to use $\Phi$ as an instance of Unique Label Cover problem that we are translating to $G_\Phi$. The weight function $w$ on vertices and bias function $\mu$ is going to be recalled, as well as the constants $\varepsilon$ and $\gamma$. When $\Phi$ is a $\text{YES}$ instance, we are going to refer to the set $X_0$ as in Conjecture 5, and use the independent set $IS$ from Lemma 6.

3 Weighted Minimum Maximal Matching

Let us now modify their reduction. The graph $G_\Phi'$ gets additional edges between vertices $(x,S_1), (x,S_2)$ if $S_1 \cap S_2 = \emptyset$ — they do not assign the same colour to the variable $x$. Clearly, the Lemmas 6 and 7 still hold for $G_\Phi'$.

Moreover, we introduce the weight function on the edges.

$$w_+ ((x_1,S_1),(x_2,S_2)) \overset{\text{def}}{=} w(x_1,S_1) + w(x_2,S_2)$$

We will now show the similar statements are true for the Minimum Maximal Matching as for the independent set.

**Lemma 8.** If $\Phi$ was a $\text{YES}$ instance, the Minimum Maximal Matching in $(G_\Phi', w_+)$ weights at most $\frac{1}{2} + 2\varepsilon$.

**Lemma 9.** If $\Phi$ was a $\text{NO}$ instance, the Minimum Maximal Matching in $(G_\Phi', w_+)$ weights at least $1 - 2\gamma$.

These lemmas altogether will give us the theorem.

**Theorem 10.** Assuming the Unique Games Conjecture, for any $\varepsilon > 0$ it is NP-hard to distinguish between graphs with Maximal Matching of weight $\frac{1}{2} + \varepsilon$ and those, where every Maximal Matching weights at least $1 - \varepsilon$.

This in turn means, that — assuming UGC — a polynomial-time approximation algorithm with a factor better than $2$ can not be constructed.

**Proof of Lemma 8.** Let us construct a matching $M$ in $G_\Phi'$. The matching will only consist of the edges between vertices corresponding to the same variable in $\Phi$. First we define the part of $M$ restricted to $X_0$.

$$M_0 = \{(x,S_1) \sim (x,S_2) \mid x \in X_0 \land S_1 \cup S_2 = R \setminus \{x\}\}$$

For vertices in clouds corresponding to variables outside of $X_0$ we define

$$M_1 = \{(x,S_1) \sim (x,S_2) \mid x \in X_0 \land S_1 \cup S_2 = R\}$$

The matching $M$ will be the union of $M_0$ and $M_1$.

We can observe, that the vertices matched by $M$ are exactly those, that do not belong to $IS$. Hence,

$$w_+(M) \leq w(G_\Phi') - w(IS) \leq 1 - \left(\frac{1}{2} - 2\varepsilon\right) = \frac{1}{2} + 2\varepsilon$$

Moreover, since the vertices of $M$ compose a vertex cover, $M$ is a maximal matching. □
Proof of Lemma 9. Let $M$ be any maximal matching. The vertices matched by $M$, $V(M)$ form a vertex cover. Hence, the weight of $M$ is going to be at least as large as the weight of the Minimum Vertex Cover. From Lemma 7 we know, that if $\Phi$ was a NO instance, $G_\Phi'$’s Minimum Vertex Cover weights at least $1-2\gamma$. \hfill \qed

4 Towards the Unweighted MMM: Fractional Matchings

A natural way to reduce a weighted variant of a problem to the unweighted would often be to assume that the weights are integral (that can be achieved by rounding them first at a negligible cost) and copying every vertex as many times, as its weight would suggest. This simple strategy will not however work with instances from previous section, where we were matching pairs of vertices of different weights. Such a matching does not easily translate to the graph with vertex copies. In order to extend our approximation hardness proof to Minimum Maximal Matching problem in unweighted graphs, we thus need first to modify our weighted reduction a bit. The structure remains the same, but the weight of each edge is now defined to be the minimum of the weights of its endpoints.

\[ w_{\min}((x_1,S_1),(x_2,S_2)) \overset{\text{def}}{=} \min\{w(x_1,S_1),w(x_2,S_2)\} \]

Similarly to the reasoning presented in the previous section, when $G_\Phi'$ is a YES instance, we will want to construct a matching and argue that it is maximal using a known vertex cover.

Definition 11. A fractional matching is an assignment of values to variables $x_e$ corresponding to edges, such that for every edge $e$ $x_e \leq w_{\min}(e)$ and for every vertex $v$, the sum $\sum_{(v,w)\in E} x(v,w) \leq w(v)$.

Definition 12. A fractional matching saturates the vertex $v$ if $\sum_{(v,w)\in E} x(v,w) = w(v)$.

As we know already, when $\Phi$ is a YES instance, there is a vertex cover in $G_\Phi'$ composed of all vertices except those in $IS$.

Lemma 13. If $\Phi$ was a YES instance, a fractional matching exists that leaves all vertices in $IS$ unmatched and saturates all the other vertices.

4.1 Proving Lemma 13

Our matching will again only match vertices in the same clouds. Let us first concentrate on vertices in the cloud $C_x$ corresponding to a variable $x \notin X_0$. The matching needs to saturate every vertex in $C_x$.

The fractional matching $F$ can be viewed as a real-valued vector and will be a sum of three matchings. The first one is defined similarly to $M_1$ in Lemma 8.

\[ F^0((x,S_1),(x,S_2)) = \begin{cases} 
 w_{\min}((x,S_1),(x,S_2)), & \text{if } S_1 \cup S_2 = R \\
 0, & \text{otherwise}
\end{cases} \]

Recalling, that the weight function $w$, defined on vertices, has a bias towards smaller sets, we can state the following.
Observation 14. $F^0$ saturates all vertices $(x, S) \in \mathcal{C}_x$ such that $|S| \geq \frac{|R|}{2}$.

Let us now pick $0 < k < \frac{|R|}{2}$ and look at the layer $\mathcal{C}_x^k = \{ (x, S) \mid |S| = k \}$. The graph is symmetric, and $F^0$ saturates every vertex by the same amount — $\mu(|R| - k) = \frac{1}{\lambda \mu} p^{\lambda R} (1 - p)^{k}$. In order to build a matching $F^1$, that saturates all vertices in the layer we build a bipartite graph $\mathcal{B}^k$ out of $\mathcal{C}_x^k$.

Definition 15. For every set $S$ of size $k$, $\mathcal{B}^k$ has two vertices, $S^L$ and $S^R$. $S^L$ is connected with $S^R$ if $S_1 \cup S_2 = \emptyset$.

The graph $\mathcal{B}^k$ is in fact a Bipartite Kneser Graph. As proved in [16], it has a Hamiltonian cycle $\mathcal{H}_k$. We are using this cycle to define $F^1$ — for every edge connecting the sets $S_1$ and $S_2$ in $\mathcal{H}_k$ we lay the weight of

$$F^1((x, S_1), (x, S_2)) = \frac{1}{4}(\mu(k) - \mu(|R| - k))$$

on the edge connecting them in $\mathcal{C}_x^k$.

To saturate the vertices $(x, \emptyset)$ (for $x \notin X_0$), we must realize that all these vertices form a clique in which we can find a Hamiltonian Cycle $\mathcal{H}_\emptyset$. Let us define $F^2$

$$F^2((x_1, \emptyset), (x_2, \emptyset)) = \begin{cases} \frac{\mu(0) - \mu(|R|)}{2}, & \text{for } \{x_1, x_2\} \in \mathcal{H}_\emptyset \\ 0, & \text{otherwise} \end{cases}$$

Lemma 16. $F^0 + F^1 + F^2$ saturates all vertices in $\mathcal{C}_x^k$.

Proof. We look at the vertex $(x, S)$. For $0 < |S| < \frac{|R|}{2}$, the Hamiltonian Cycle $\mathcal{H}_k$ visits every set exactly twice (once $S^L$ and once $S^R$), using four edges incident to it. Hence, the total contribution of $F^0$ and $F^1$ is equal to

$$\mu(|R| - k) + 4 \cdot \frac{1}{4}(\mu(k) - \mu(|R| - k)) = \mu(k) = w(x, S).$$

$F^0$ contributes $\mu(|R|)$ to the vertex $(x, \emptyset)$, while $F^2$ contributes $2 \cdot \frac{\mu(0) - \mu(|R|)}{2}$, hence that vertex is also saturated.

Finally, vertices with $S = \emptyset$ are saturated by $F^0 + F^2$. \hfill \Box

4.1.1 When $x \in X_0$

We proceed similarly as for vertices not in $X_0$. For the cloud $\mathcal{C}_x$ when $x \in X_0$, our first matching $F^0$ is taking the labeling of the variable $x$ into account. Similarly to Lemma 8, we match $(x, S_1)$ and $(x, S_2)$ if $S_1 \cup S_2 = R \setminus \{ r_x \}$, thus saturating the larger of the sets.

Again, the layer $\mathcal{C}_x^k$ for $k < \frac{|R| - 1}{2}$, composed of sets not containing $r_x$, is a Bipartite Kneser Graph, and we use its Hamiltonian cycle to define $F^1$.

Also the vertices $(x, \emptyset)$ for $x \in X_0$ form a clique. Once again, we can use the Hamiltonian Cycle in that clique to define $F^2$.

\footnote{A significantly more crude approach is possible, that just uses every edge equally.}
5 Unweighted MMM

Starting with a graph $G_\Phi$ with the weight function $w$ on the vertices, and any precision parameter $\rho > 0$, we are going to construct an unweighted graph $G^\rho_\Phi = (V^\rho, E^\rho)$. The resulting graph size is polynomial in $|\Phi|$ and $\frac{1}{\rho}$.

**Definition 17.** Let $n = |V(G_\Phi)| \cdot \frac{1}{\rho}$. For every $v \in V(G_\Phi)$ we set $n_v = \lfloor n \cdot w(v) \rfloor$. The new set of vertices is going to consist of multiple copies of original vertices; for each vertex $v$, we add $4 \cdot n_v$ copies.

$$V^\rho = \{ (v, i) \mid v \in V(G_\Phi), i \in \{1, \ldots, 4 \cdot n_v\}\}.$$  

The edges are going to connect each pair of copies of vertices connected in $G_\Phi$.

$$E^\rho = \{ \{(v_1, i_1), (v_2, i_2)\} \mid \{v_1, v_2\} \in E(G_\Phi), \quad i_1 \in \{4 \cdot n_{v_1}\}, i_2 \in \{4 \cdot n_{v_2}\}\}.$$  

This construction has been presented in [5]. It is shown, that any vertex cover $C \subseteq G_\Phi$ yields a *product vertex cover* $C^p = \bigcup_{v \in C} \{v\} \times \{4 \cdot n_v\}$. Moreover, every minimal vertex cover in $G^\rho_\Phi$ is a product vertex cover [5, Proposition 8.1].

As before, we are now going to prove two lemmas witnessing the completeness and soundness of our reduction.

**Lemma 18 (Soundness).** If $\Phi$ was a NO instance, for every maximal matching $M$ in $G^\rho_\Phi$

$$2 \cdot |M| > |V(G^\rho_\Phi)| \left(1 - 2\gamma - \rho\right).$$

*Proof.* Take any maximal matching $M$. The $2 \cdot |M|$ vertices matched by it form a vertex cover $C$. Let $C_-$ be a minimal vertex cover obtained by removing unneeded vertices from $C$. As presented in [5], $C_-$ is a product vertex cover, which means, there is a vertex cover $C_w$ in $G_\Phi$ with weight

$$w(C_w) < \frac{|C_-|}{V(G^\rho_\Phi)} + \rho \leq \frac{|C|}{V(G^\rho_\Phi)} + \rho.$$  

On the other hand, from Lemma 7 we have, that $w(C_w) > 1 - 2\gamma$. \hfill \Box

**Lemma 19 (Completeness).** If $\Phi$ was a YES instance, a maximal matching $M$ exists in $G^\rho_\Phi$ with

$$2 \cdot |M| < |V(G^\rho_\Phi)| \left(\frac{1}{2} + 2\epsilon + \rho\right).$$

*Proof.* Take $F$, a fractional matching on $(G_\Phi, w_{\min})$ constructed in Lemma 13. When $F^0$ matches vertices $u = (x, S_1)$ and $v = (x, S_2)$ with some weight $F^0(u, v)$, we are going to match $4 \cdot |F^0(u, v)|$ copies of $u$ and $v$ using parallel edges.

Let us focus on a vertex $u = (x, S) \notin IS$ belonging to a vertex cover of $G_\Phi$, with $0 < |S| < \frac{|R|}{2}$. It is matched by $F^0$ to $(x, S')$, which leaves $4 \lfloor w(x, S) - |w(x, S')| \rfloor$ vertices in $G^\rho_\Phi$ unmatched. This number is divisible by 4, which allows us to match all the copies of vertices in the Bipartite Kneser Graph according to $F^1$ (see Fig 1).
Finally, the number of unmatched copies of the \( (x,\emptyset) \) vertices is divisible by 2. We can thus replicate \( F^2 \) to match all the remaining copies of these vertices.

Since we are matching every node in a vertex cover of the graph \( G_\Phi \), our matching is maximal and its cardinality is half of the cardinality of the vertex cover.

\[
|M| = \frac{1}{2} \left( V(G^p_\Phi) - |IS|^p \right) < \frac{1}{2} V(G^p_\Phi) \left( 1 - \left( \frac{1}{2} - 2\epsilon - \rho \right) \right)
\]

\[\square\]

### 5.1 Hardness of Total Vertex Cover

In Lemmas 18 and 19 we proved UGC-hardness of the following problem. For any \( \epsilon > 0 \), given a graph \( G \) with \( n \) vertices it is hard to distinguish if:

- (YES instance) \( G \) has a Maximal Matching of size smaller than \( n \left( \frac{1}{2} + \epsilon \right) \).
- (NO instance) \( G \) has no Vertex Cover of size smaller than \( n \left( 1 - \epsilon \right) \).

Vertices matched in MMM form a Total Vertex Cover, so in the YES case there is a Total Vertex Cover of size smaller than \( n \left( \frac{1}{2} + \epsilon \right) \). On the other hand, every Total Vertex Cover is a Vertex Cover, so in the NO case there is no Total Vertex Cover of size smaller than \( n \left( 1 - \epsilon \right) \).

Therefore, Total Vertex Cover is UGC-hard to approximate with constant better than 2.

### 6 Hardness of Bipartite MMM

In this section we will perform a natural reduction to prove the following theorem.
Theorem 20. Assuming the Unique Games Conjecture, for any \( \epsilon > 0 \) it is NP-hard to distinguish between balanced bipartite graphs of \( 2n \) vertices:

- (YES instance) with a Maximal Matching of size smaller than \( n \left( \frac{1}{2} + \epsilon \right) \).
- (NO instance) with no Maximal Matching of size smaller than \( n \left( \frac{3}{2} - \epsilon \right) \).

We will start with the graph \( G^\rho_\Phi \) defined in Section 5. The bipartite graph \( H_\Phi \) has two copies \( v^l \) and \( v^r \) of every vertex \( v \in G^\rho_\Phi \). The vertices \( u^l \) and \( u^r \) are connected with an edge if there is an edge \( (u, v) \) in \( G^\rho_\Phi \). \( n \) is going to be equal to \( |V(G^\rho_\Phi)| \). We will call this construction bipartisation of an undirected graph.

It is easy to see, that if \( \Phi \) is a YES instance of the Unique Label Cover problem, we can use the matching from Lemma 19 (\( M \) in \( G^\rho_\Phi \)) to produce a maximal matching in \( H_\Phi \). For every edge \( (u, v) \in M \) we will put its two copies, \( (u^l, v^r) \) and \( (v^l, u^r) \) into the matching. The resulting matching size is thus equal to \( 2 \cdot |M| < n(\frac{1}{2} + \epsilon) \).

### 6.1 Covering with Paths

In order to analyse the NO case, we need to look at the bipartite instance and its matchings from another angle. For any matching in \( H_\Phi \), we will view its edges as directed edges in \( G^\rho_\Phi \) — the vertices on the left will be viewed as out vertices, and those on the right as in vertices. The graph \( G^\rho_\Phi \) will thus be covered with directed edges. Every vertex will be incident to at most one outgoing and one incoming edge, which means that the edges will form a structure of directed paths and cycles. The set of these paths and cycles will be called \( P(M) \) for a matching \( M \).

**Observation 21.** If \( M \) is a maximal matching, every path \( P \in P(M) \) has a length \( |P| \geq 2 \).

**Proof.** Assume, that for a maximal matching \( M \) in \( H_\Phi \) there is a length-one path \( P = (u, v) \in P(M) \). This means, that the vertices \( v^l \) and \( u^r \) are unmatched in \( M \) — yet, they are connected with an edge, that can be added to the matching (that would form a length-2 cycle in \( P(M) \)). \qed

We will now use this observation to prove the relation between maximal matchings in \( H_\Phi \) and vertex covers in \( G^\rho_\Phi \).

**Lemma 22.** For any maximal matching \( M \) in \( H_\Phi \), there exists a vertex cover \( C \) in \( G^\rho_\Phi \) of size \( |C| \leq \frac{2}{3}|M| \).

**Proof.** We will construct the vertex cover using paths and cycles of \( P(M) \). For every \( P \in P(M) \) we add all the vertices of \( P \) into \( C \). When \( P \) is a cycle, it contains as many vertices as edges. A path has at most \( \frac{1}{2} \) as many vertices as edges, since its length is at least 2. \qed

As shown in Lemma 18, when \( \Phi \) is a NO instance, the Minimum Vertex Cover in \( G^\rho_\Phi \) has at least \( n(1 - \epsilon) \) vertices. The Minimum Maximal Matching in \( H_\Phi \) must in this case have at least \( \frac{2}{3}n(1 - \epsilon) > n\left( \frac{2}{3} - \epsilon \right) \) edges.

The hardness coming from Theorem 20 is, that assuming UGC, no polynomial-time algorithm will provide approximation for Minimum Maximal Matching with a factor \( \frac{3}{2} - \epsilon \) for any \( \epsilon > 0 \).
7 Stronger Result for Bipartite Graphs

We are able to obtain a stronger hardness of approximation result for bipartite graphs, but it assumes a slightly stronger conjecture. In this Section we will show how to prove hardness assuming Small Set Expansion Hypothesis [17], but the same result can be obtained from Strong Unique Games Conjecture — it requires replacing Lemma 23 with a corresponding hardness of MBB from [2]. In order to describe the reasoning, let us now characterise Minimum Maximal Matching solutions in bipartite graphs and focus on the conjecture later.

Imagine, our balanced bipartite graph $H$ has a perfect matching. Clearly, its Minimum Maximal Matching $M$ has at least $\frac{n}{2}$ edges (where $n$ is a number of vertices on either side). $M$ allows us to divide vertices of $H$ into the sets $M_L, M_R$ (matched in $M$) and $I_L, I_R$ (unmatched). In previous sections we used the fact, that the set $I_L \cup I_R$ is an independent set to bound the size of Minimum Maximal Matching in the NO case. Here, we notice that $I_L, I_R$ is a balanced anti-biclique — its complement is a bipartite clique. Clearly, if a bipartite graph of $2n$ vertices has no $K_{\delta n, \delta n}$ anti-biclique, its Minimum Maximal Matching must be larger than $(1 - \delta)n$.

We recall a recent result by Manurangsi, who has proved the following lemma.

**Lemma 23 ([15, Lemma 2]).** Assuming the Small Set Expansion Hypothesis, for every $\epsilon > 0$ it is NP-hard to distinguish, given a bipartite graph $G = (A \cup B, E)$ with $|A| = |B|$, which category it falls into:

- (YES case) $\exists K_A \subseteq A, K_B \subseteq B$ such that $E_{K_A \cup K_B} = K_{|V(G)|(\frac{1}{2} - \epsilon), |V(G)|(|\frac{1}{2} - \epsilon|)}$. Namely, there is a balanced biclique in $G$ using almost half of vertices.
• (NO case) \( \forall K_A \subset A, K_B \subset B \mid |K_A| = |K_B| > e|A| \implies \exists a \in K_A, b \in K_B (a, b) \notin E(G) \). Namely, there is no balanced biclique using more than \( e \)-fraction of vertices.

We will now use this lemma to prove \( \frac{3}{2} \) approximation hardness for Minimum Maximal Matching problem in bipartite graphs. Take an instance \( G \) from Lemma 23. Let \( n \) be the number of vertices on one side. Our modified graph \( G' = ((A \cup A') \cup (B \cup B'), \hat{E} \cup E') \) is created by adding \( n(\frac{1}{2} + e) \) vertices each to both sides, so \( |A'| = |B'| = (\frac{1}{2} + e)n \). To produce the set of edges we first take complement of \( E \) on \( A \) and \( B \). This way bicliques become anti-bicliques. Next we add \( E' = A' \times B \cup A \times B' \cup A' \times B' \) (we connect the new vertices with every vertex from \( G \) and with each other).

**Lemma 24.** If \( G \) has a \( K_{n(\frac{1}{2} - e),n(\frac{1}{2} - e)} \) biclique \( K_A \times K_B \), there is a maximal matching with \( n(1 + 2e) \) edges in \( G' \).

**Proof.** We can match vertices of \( A \setminus K_A \) with \( B' \) and \( A' \) with \( B \setminus K_B \). All the remaining vertices, \( K_A \cup K_B \), form an anti-biclique in \( G' \) so the matching is maximal.

**Lemma 25.** If \( G \) has no biclique \( K_{en, en} \), every maximal matching in \( G' \) contains at least \( n(\frac{3}{2} - e) \) edges.

**Proof.** It suffices to argue that \( G' \) does not have a large anti-biclique. Since all vertices in \( A' \) are connected with everyone, only one of them can belong to the anti-biclique. The same applies to \( B' \). The remainder of the anti-biclique would form a biclique in \( G \). The largest anti-biclique in \( G' \) can therefore have \( en \) vertices on each side.

## 8 Conclusion

We would like to finish by discussing potentially interesting open problems. Natural question following our result on MMM is whether other hardness results for Vertex Cover also hold for MMM. In particular, it is known that Vertex Cover on \( k \)-hypergraphs is hard to approximate with a constant better than \( k \) [14]. Also, the best known NP-hardness of Vertex Cover is \( \sqrt{2} \), following the reduction from 2–2 Games Conjecture [12], which has been recently proven [13].

Both of these reductions are very similar to Khot and Regev’s UGC-hardness of Vertex Cover. As such they can be used to prove corresponding hardesses of weighted MMM, by following similar approach as in Section 3. They differ, however, in the choice of the weight function of vertices, which turns out to be crucial in terms of unweighted MMM. These weight functions have bias towards bigger sets, so construction described in Section 4 can not be used for these problems.

As such, the best known NP-hardness of MMM remains 1.18 by Escoffier, Monnot, Paschos, and Xiao [6] and it is an open problem, whether it can be improved using 2–2 Games Conjecture.

In case of bipartite MMM, there remains a gap between our \( \frac{3}{2} \)-hardness and best known constant approximation algorithm, which has ratio 2. Showing that bipartite MMM is hard to approximate with a constant better than 2 would immediately imply tight hardness of Maximum Stable Matching with Ties [9]. On the other hand, there are no results for MMM leveraging restriction to bipartite graphs. Thus, a potential better than 2 approximation algorithm for bipartite graphs would be interesting for showing structural difference between MMM in bipartite and general graphs.
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