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CANONICAL FORMULATION OF THE SPHERICALLY-SYMMETRIC
EINSTEIN-YANG-MILLS-HIGGS SYSTEM FOR A GENERAL GAUGE GROUP

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ABSTRACT

The dynamics of the spherically-symmetric system of gravitation interacting with scalar and Yang-Mills fields is presented in the context of the canonical formalism. The gauge group considered is a general (compact and semisimple) $N$ parameter group. The scalar (Higgs) field transforms according to an unspecified $N$-dimensional orthogonal representation of the gauge group. The canonical formalism is based on Dirac's techniques for dealing with constrained hamiltonian systems.

First the condition that the scalar and Yang-Mills fields and their conjugate momenta be spherically symmetric up to a gauge is formulated and solved in its maximal generality, finding - in a general gauge - the explicit angular dependence of the fields and conjugate momenta. It is shown that if the gauge group does not admit a subgroup (locally) isomorphic to the rotation group, then the dynamical variables can only be manifestly spherically symmetric. If the opposite is the case, then the number of allowed degrees of freedom is connected to the angular momentum content of the adjoint representation of the gauge group.

Once the suitable variables with explicit angular dependence have been obtained, a reduced action is derived by integrating away the angular coordinates. The canonical formulation of the problem is now based on dynamical variables depending only on an arbitrary radial coordinate $r$ and an arbitrary time coordinate $t$. Besides the gravitational variables, the formalism now contains two pairs of $N$-vector variables, $(\mathbf{B}, \mathbf{P})$, $(\mathbf{Q}, \mathbf{T})$, corresponding to the allowed Yang-Mills degrees of freedom and one pair of $N$-vector variables, $(\mathbf{E}, \mathbf{F})$, associated with the original scalar field. The reduced hamiltonian is invariant under a group of $r$-dependent gauge transformations such that $\mathbf{F}$ plays the role of the gauge field (transforming in the typically inhomogeneous way) and in terms of which the gauge covariant derivatives of $\mathbf{F}$ and $\mathbf{F}$ naturally appear. No derivatives of $\mathbf{F}$ appear in the hamiltonian and the gauge freedom allows us to define a gauge in which $\mathbf{F}$ is zero. Also the $r$ and $t$ coordinates are fixed in a way consistent with the equations of motion. Some non-trivial static solutions are found. One of these solutions is given in closed form, it is singular and corresponds to a generalisation of the singular solution found in the literature with different degrees of generality and the geometry is described by the Reissner-Nordstrom metric. The other solution is defined through its asymptotic behaviour. It generalises to curved space the finite energy solution discussed by Julia and Zee in flat space.

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INTRODUCTION

At the classical level the existence of particle-like solutions found by 't Hooft and Polyakov has particularly enhanced the interest in classical non-abelian gauge theories. Many generalizations of such solutions have subsequently been found both in flat and curved space-time, most of which - when explicitly developed - refer only to the case when the gauge group is isomorphic to the rotation group SO(3). In this paper we generalize results of a previous paper considering the spherically symmetric Einstein-Yang-Mills-Higgs system associated with a general gauge group.

We begin by studying a systematic way of reducing the variables of the problem to variables with no angular dependence and show which is the number of degrees of freedom in the spherically-symmetric problem which are present in the formulation before any gauge or coordinate fixation is performed. In the main part of the paper we present a canonical formulation of the dynamics of the spherically-symmetric problem of interacting scalar, Yang-Mills and gravitational fields. The scalar field is considered to be a Higgs field in the sense that it provides mass to some of the gauge fields through the mechanism of spontaneous symmetry breaking. The analysis of the dynamics is an application of Dirac's techniques for constrained Hamiltonian systems to the gauge-invariantly coupled system of scalar, Yang-Mills and gravitational fields to a Hamiltonian "reduced" by the requirement of spherical symmetry (reduced in the sense that the Hamiltonian generates only the motions consistent with spherical symmetry). In this reduced Hamiltonian formalism the dynamical variables also "reduce", they depend only on an arbitrary radial coordinate r and an arbitrary time coordinate t.

By means of Dirac's canonical method we are able to present the whole dynamical problem in a clear and orderly way from a general point of view. There is a clear distinction between the constraints and the actual equations of motion, and the coordinate and gauge fixation is performed in the characteristic way developed by Dirac. The question of quantization is, of course, at the back of our mind when using the canonical formulation.

The concept of spherically-symmetric (scalar or Yang-Mills) fields used in this paper refers to fields whose angular dependence is fixed by the condition that the effect of a rotation be compensated by a gauge transformation. The fields that we consider, then, are not necessarily manifestly spherically symmetric (SSS) but only spherically symmetric up to a gauge (SSUO). (Compare the present definition with that used in Refs. 14 and 15).

In Sec. II we study the implications on the form of the scalar and Yang-Mills fields (and their conjugate momenta) when the condition of being spherically symmetric up to a gauge is imposed. The consequent equations are integrated finding the explicit and most general angular dependence of the dynamical variables. Our treatment, therefore, still has another difference from most of the papers quoted at the beginning and not Ref. 6) in the fact that we derive the most general angular dependence of the fields instead of feeding in an ansatz a particular form for the solutions (the Wu-Yang ansatz) is used in the case when the gauge group is SO(3) and without discussing its generality and the problem of gauge choice.

Also in Sec. II it is shown that it is necessary that the gauge group admits a subgroup isomorphic to the rotation group in order that SSS solutions can exist; otherwise the dynamical variables can only be SSS. It is not difficult to imagine a (non-rigorous) picture of why this has to be so: the effect of rotations can only be compensated by (gauge) "rotations". Furthermore, the number of degrees of freedom is connected to the angular momentum content of the adjoint representation of the gauge group.

In Sec. III the angular coordinates are integrated away in the expression of the action integral to set forth a reduced action principle. Each of the terms appearing in the action integrand (and the Hamiltonian) are written down explicitly in terms of the reduced variables depending only on r and t. These variables are described as vectors to avoid using too many indices. The Yang-Mills degrees of freedom are described by the vectors \( \mathbf{H} \) and \( \mathbf{\hat{H}} \) belonging to the \( \mathcal{H} \)-dimensional space of the adjoint representation of the gauge group and by their conjugate momenta \( \mathbf{\pi}_H \) and \( \mathbf{\pi}_\mathbf{\hat{H}} \). \( \mathbf{H} \) is associated with the r-component of the Yang-Mills field, \( \mathbf{\hat{H}} \), and \( \mathbf{\hat{H}} \) is associated with \( \mathbf{\pi}_H \). The reduced variables associated with the degrees of freedom allowed to the scalar field are the couple of conjugate variables \( (\mathbf{\tilde{H}}, \mathbf{\tilde{\mathbf{\hat{H}}}}) \) belonging to a generic \( \mathcal{H} \)-dimensional representation space corresponding to the orthogonal representation under which the scalar field transforms. In Sec. IV the residual gauge freedom allowed by the reduced action is presented and it is observed that \( \mathbf{\hat{H}} \) behaves like the compensating field (gauge field) - transforming in the typically inhomogeneous way - in terms of which gauge covariant derivatives of \( \mathbf{\hat{H}} \) and \( \mathbf{\hat{\hat{H}}} \) naturally arise. No derivatives of the \( \mathbf{\hat{H}} \) field appear in the Hamiltonian, and the gauge group is shown to have as many r-dependent parameters as \( \mathbf{\hat{H}} \) has components, thus allowing to choose a gauge in which \( \mathbf{\hat{H}} \) is zero. The remaining coordinates (r and t) are also fixed.
From the equations of motion, the constraints and the coordinate and gauge fixations, we derive, in Sec. V, equations for the remaining variables valid for the static case. Two non-trivial solutions of these equations are presented. The first is a generalization of the well known singular solution found by many authors in flat or curved space-time. The geometry corresponds to the Reissner-Nordström metric, the Yang-Mills field has a form which is an N-dimensional generalization of the Wu-Yang ansatz and the Higgs field takes an essentially constant value (corresponding to a minimum of its self-potential) apart from a trivial angular dependence that gives it topological properties as the asymptotic form of the scalar field used by ’t Hooft. The other solution is a generalization of the finite energy solution found by Julia and Lee in flat space-time.

Finally, there are three appendices. Appendix A defines the basic notation, the indices and dynamical variables and it also gives a sketchy presentation of the hamiltonian formulation of the problem of interacting Yang-Mills and scalar fields with gravitation. In Appendix B we give some useful properties of the vectors of the adjoint representation space, the cross product defined between them and some matrices acting on them. The equations of motion and constraints for the spherically-symmetric solution are given in Appendix C.

II. SPHERICAL SYMMETRY

Since we are going to study the spherically-symmetric motions of the system, we must first of all study the general form that the gravitational, Yang-Mills and scalar fields must have in agreement with the space symmetry required. In the following we give separately the angular dependence allowed for the space metric $g_{ij}$, the space components of the Yang-Mills fields $W^A (A = 1, 2, ..., N; i = r, \theta, \phi)$ and the scalar field $\phi$ as a consequence of the requirement of spherical symmetry. As will be seen in Appendix A, there is no need to study the time components $g_00$ and $W^0$ since in the hamiltonian method, they play the role of arbitrary functions and not of dynamical variables.

(a) The most general spherically-symmetric $g_{ij}$ is well known to have the form

$$g_{ij} = \text{diag} [e^{2\lambda(r,t)}, e^{2\lambda(r,t)}, e^{2\lambda(r,t)} \sin^2 \theta].$$

(b) We shall now require that the gauge vector fields $W^A (A = 1, 2, ..., N; i = r, \theta, \phi)$ and their conjugate momenta $\pi^i_A$ be spherically symmetric. We must satisfy the requisite that the effect of any space rotation may be compensated by a gauge transformation, that is, $W^A$ is required to be spherically symmetric up to a gauge. For practical reasons we shall restrict the application of this concept to global gauge transformations.

The gauge group $T$ must be compact if we want to guarantee the positive definiteness of the Yang-Mills field kinetic energy term in the hamiltonian. Further, for simplicity, we take $T$ to be semisimple. Under these conditions we can always choose the metric of the group, in terms of the structure constants $c^A_{BC}$ of $T$, to be

$$g_{AB} = \frac{1}{2} c^A_{BC} c^C_{BD},$$

and therefore we shall not have to worry about the height of the group indices. In the following we also disregard the possibility of having many different gauge coupling constants as it would happen if the group $T$ were expressible as a direct product of smaller groups. (All the simplifying assumptions made in this paragraph can be lifted without great difficulty.)

The condition of spherical symmetry of the gauge fields and their conjugate momenta is obtained equating the effect of a small rotation (generated by $\xi_\alpha)$ with the effect of a global gauge transformation:

$$\delta \alpha^a = C^a_{bc} \delta \alpha^b W^c,$$

or

$$\delta \alpha^a \left[ L_{\xi}, W^a \right] = -C^a_{bc} N^b_{c} W^c.$$
The infinitesimal generators of rotation which we use are:
\[
L_1 = \sin \varphi \frac{2}{3} z + \cos \varphi \cot \theta \frac{2}{3} y, \quad (2.9a)
\]
\[
L_2 = \cos \varphi \frac{2}{3} z - \sin \varphi \cot \theta \frac{2}{3} y, \quad (2.9b)
\]
\[
L_3 = \frac{2}{3} x, \quad (2.9c)
\]
and they satisfy
\[
[L_1, L_2] = e_{abc} L_c. \quad (2.10)
\]

It is straightforward to prove from the Jacobi identity satisfied by \( L_a \) and \( L^A \) that
\[
[Z_a, Z_b] = e_{abc} Z_c, \quad (2.11)
\]
which shows that the three matrices \( Z_a \) close an \( SO(3) \) algebra. But since this algebra is a subalgebra of \( T \), then either the gauge group admits a subgroup isomorphic to \( SO(3) \) or the \( Z_a \) are identically zero. The latter would correspond to MSB gauge fields and we shall not consider it any further. If the \( Z_a \) do not vanish, on the other hand, then it is possible to parametrize \( T \) in accordance with \( SO(3) \) so that
\[
N^\prime_a = \delta^a_b \quad \text{and} \quad N^\prime_b = 0. \quad (2.12)
\]

To write down Eqs. (2.14) explicitly it is convenient to use a vector notation for the \( W^A \). \( \vec{W} \) means an \( N \)-vector of components \( \vec{W}^A \). In this notation Eqs. (2.14) become:
\[
\vec{W}^1, \vec{W} = -(Z_2 \csc \varphi + Z_3 \cot \theta) \vec{W}^2, \quad (2.13)
\]
\[
\vec{W}^2, \vec{W} = -(Z_1 \csc \varphi - Z_3 \sin \theta) \vec{W}^1, \quad (2.14)
\]
\[
\vec{W}^3, \vec{W} = -(Z_1 \csc \varphi - Z_3 \sin \theta) \vec{W}^1 - \vec{W}^2, \quad (2.15)
\]

It is not difficult to integrate the system (2.13)-(2.15) after noticing that many relations like
\[
e^{-Z_3 \vec{W}^1} Z_2 e^{Z_3 \vec{W}^1} = Z_2 \csc \varphi + Z_3 \sin \theta \quad (2.16)
\]
hold (they all come from (2.11)).

The most general solution of the above system of equations is (written for the covariant \( \vec{W}_1 \))
\[
\vec{W}_1 = \frac{1}{2} e^{Z_3 \vec{W}^2} e^{Z_3 \vec{W}^1} \left[ \vec{R}(\vec{r}_1), \vec{\Theta}(\vec{r}_1, \vec{N}_2) + (Z_3 \vec{\bar{\Theta}} - \vec{N}_2) \sin \theta \right], \quad (2.11a)
\]

where the \( N \)-vector valued functions \( \vec{R}, \vec{\Theta} \) are only restricted by
\[
Z_3 \vec{R} = 0 \quad \text{and} \quad (Z_3)^* \vec{\Theta} = -\vec{\Theta}, \quad (2.17a)
\]
The presence of the vectors \( \vec{R} \) and \( \vec{\Theta} \) (defined in (2.6) in (2.17a) will prove to be very convenient later on.

A similar solution is found for \( \vec{R} \).

\[
\vec{R} = \sum_{k=1}^{p} \alpha_k(r,t) \vec{S}_k ,
\]

where \( \vec{S}_k = \vec{S}(r,t) \) are arbitrary functions of \( r \) and \( t \) that will play the role of conjugate momenta to \( \vec{R} \) and \( \vec{\Theta} \).

We want to know how many scalar (dynamical) variables enter the SUQ field (2.17). In the standard angular momentum terminology \( \vec{R} \) is an \( n = 0 \) vector while \( \vec{\Theta} \) is an arbitrary (real) linear combination of the (complex) eigenvectors of \( Z_3 \) corresponding to the eigenvalues \( \pm i \). If the representation of \( SO(3) \) defined by the \( Z_3 \) decomposes into \( p \) representations with \( \lambda = 0 \), \( p \) representations with integer \( \lambda \neq 0 \) and \( p' \) representations with half integer \( \lambda \) then:

1) \( Z_3 \) has \( p + p \) linearly independent eigenvectors \( \vec{X}_k \) \((k = 1, \ldots, p, 0 \pm p) \) with eigenvalue zero, and

2) \( \vec{X}_k \) has \( 2p \) linearly independent eigenvectors with eigenvalues \( (\pm i)^2 = -1 \). These we choose to be real vectors \( \vec{X}_k \) such that:

\[
Z_3 \vec{X}_k = \vec{X}_k ,
\]

\[
Z_3 \vec{X}_k = -\vec{X}_k , \quad k = 1, 2, \ldots, p .
\]

Therefore the vectors \( \vec{R} \) and \( \vec{\Theta} \) have as their most general expression

\[
\vec{R} = \sum_{k=1}^{p} \alpha_k(r,t) \vec{S}_k ,
\]

\[
\vec{\Theta} = \sum_{k=1}^{p} \left[ \chi_k(r,t) \vec{X}_k + \nu_k(r,t) \vec{X}_k \right] .
\]
where
\[ J_3 \tilde{H}(r,t) = 0 \]  
(2.23b)

and
\[ \tilde{\pi}_\phi = e^{-j} e^{-j} \tilde{\pi}_h(r,t) \sin \theta \]  
(2.24a)

where
\[ J_3 \tilde{\pi}_h(r,t) = 0 \]  
(2.24b)

\[ \tilde{H}(r,t) \) and \( \tilde{v}_h(r,t) \) being \( M \)-vector valued functions.

The general form of \( \tilde{H} \) and \( \tilde{v}_h \) is again (like \( \tilde{H} \)) a linear combination of the \( n = 0 \) linearly independent vectors existing in the representation of \( \text{SO}(3) \) induced by the \( \Phi \) representation of the gauge group.

Eqs. (2.1), (2.17), (2.18), (2.23) and (2.24) are the main results of this section. To some extent they can be generalized to the case when the \( \text{SO}(3) \) symmetry requirement is replaced by a requirement of \( D \)-symmetry, \( D \) being one of the many non-trivial isometry groups that (curved) 3-space can admit. The dependence on the coordinates on each minimal invariant variety (\( \theta \) and \( \phi \) in the case of \( \text{SO}(3) \)) can be made explicit and eliminated from the dynamical problem.

III. THE HAMILTONIAN OF THE SPHERICALLY-SYMMETRIC MOTIONS

The analysis of the dynamics of the Einstein-Yang-Mills-Higgs system is greatly simplified by restricting the general hamiltonian formalism - summarized in Appendix A - to the fields with spherical symmetry found in Sec. II.

The generators \( \tilde{H}_k \), \( \tilde{H}_m \) and \( \tilde{H}_\phi \) defined in Appendix A take the form,

\[ \tilde{H}_k = \tilde{\tilde{H}}_k(r,t) \sin \theta = \left[ \tilde{\tilde{H}}_k^m + \tilde{\tilde{H}}_k^\phi \right] \sin \theta \approx 0, \]  
(3.1)

\[ \tilde{H}_m = \tilde{\tilde{H}}_m(r,t) \sin \theta = \left[ \tilde{\tilde{H}}_m^k + \tilde{\tilde{H}}_m^\phi \right] \sin \theta \approx 0, \]  
(3.2a)

\[ \tilde{H}_\phi = \tilde{\tilde{H}}_\phi(r,t) \sin \theta = 0, \]  
(3.2b)

\[ \tilde{G} = e^{-j} e^{-j} \tilde{G}(r,t) \sin \theta \equiv e^{-j} e^{-j} \left[ \tilde{G}^m + \tilde{G}^\phi \right] \sin \theta \approx 0. \]  
(3.3a)

They are given in Appendix C.

The equations of motion can then be obtained from the extremization of the restricted action
\[ \tilde{S} = \int dt \int dr \left\{ \pi_\mu \dot{\pi}_\mu + \pi_\lambda \dot{\pi}_\lambda + \pi_\phi \dot{\pi}_\phi + \pi_\theta \hat{\pi}_\theta + \pi_\phi \hat{\pi}_\phi - N \tilde{R}_1 - N^\nu \tilde{R}_\nu - W \tilde{G} \right\}. \]  
(3.4)

In writing the reduced action we have assumed that the angular dependence in the integrals on the full action was a common factor \( e^{-j} e^{-j} \) integrated away \( \int \sin \theta d\theta d\phi \) and for this we have assumed that \( \tilde{H} \) and \( \tilde{v}_h \) are functions of \( \tau \) and \( t \) only, while \( \tilde{v}_0 \) has been assumed to have the form
\[ \tilde{W}_0 = e^{-j} e^{-j} \tilde{W}(r,t) \]  
(3.5)

to cancel exactly the angular dependence in the last term of the integrand defining \( S \). These assumptions represent no loss of generality as was discussed in Ref. 8. Furthermore, the explicit form of \( \tilde{G} \) given below is such that
\[ \tilde{Z}_j \tilde{G} = 0 \]  
(3.6)

hence, there is no loss of generality in taking also the arbitrary vector function \( \tilde{W} \) to satisfy the same property
\[ \tilde{Z}_a \tilde{W} = 0. \]  
(3.7)

Next we give the derived explicit expression corresponding to each one of the reduced generators split into the different contributions coming from the gravitational, Yang-Mills and scalar fields,

(a) The functions \( \tilde{H}_k^m \) and \( \tilde{H}_k^\phi \) are \( \tilde{G} \)
\[ \tilde{G}_k^m = e^{-j} e^{-j} \left[ \frac{1}{2} \pi^3 - \frac{1}{2} \pi^\mu \pi^\lambda + 2 e^{j} (2 \lambda' - 2 \lambda \mu' + 3 \lambda^2 - e^{2(\mu^2\lambda)} \right], \]  
(3.8)

(b) To obtain \( \tilde{G}^m \) and \( \tilde{G}^\phi \), it is necessary to evaluate \( \tilde{R} \). The result being
\[ \tilde{R} = \frac{1}{2} e^{-j} e^{-j} \tilde{R} \]  
(3.9a)

where \( \tilde{R} \) is defined in Appendix A and the cross product between \( \tilde{R} \) vectors is defined in terms of the structure constants of the gauge group,
\[ (\tilde{R} \times \tilde{R}) = C_{\tilde{R} \tilde{R}}^{\tilde{R}} \tilde{R} \]  
(3.10)

and
\[ (\tilde{R} \cdot \tilde{R}) = C_{\tilde{R} \tilde{R}}^{\tilde{R}} \tilde{R} \]  
(3.11)
We notice that the expressions in round brackets in (3.10b,c) look like a gauge covariant derivative of $\bar{\mathcal{A}}$, where the role of the compensating field is played by $\bar{\mathcal{R}}$.

Inserting (2.27), (2.28) and (3.10) into (A.3) and (A.4), one finds

$$\bar{\mathcal{H}}^{\mu \nu} = \frac{1}{2} \left[ \epsilon^{\mu \nu - 2 \lambda} \pi_{R}^2 + \frac{1}{2} \epsilon^{\nu \mu - 2 \lambda} \pi_{\Theta}^2 \right] + \frac{1}{2} \left[ \epsilon^{\mu \nu - 2 \lambda} (\bar{\Theta}^2 - \bar{\mathcal{A}}_2) + 2 \epsilon^{\nu \mu - 2 \lambda} (\Theta^2 + \Theta \bar{\mathcal{R}}) \right].$$

(3.12)

$$\bar{\mathcal{H}}^{\nu \mu} = \bar{\pi}_{\Theta} \cdot (\bar{\Theta}^2 + \Theta \bar{\mathcal{R}}).$$

(3.13)

$$\bar{\mathcal{C}}^{\mu \nu} = -\bar{\pi}_{R} \bar{\mathcal{R}} - \frac{1}{2} (\bar{\pi}_{R} \bar{\Theta} + Z_{2} \bar{\pi}_{R} - Z_{2} \bar{\Theta}) - \bar{\pi}_{R}^{-1}.$$ (3.14)

In the derivations necessary to get the above results frequent use is made of the following facts:

i) the $Z_{a}$ matrices are antisymmetric and traceless,

ii) their exponentiation yields orthogonal matrices,

iii) vectors belonging to different eigenspaces of $Z_{3}$ are orthogonal, e.g., $Z_{R} \Theta = 0$,

iv) the results of Appendix B.

Finally, to obtain the functions $\bar{H}_{\mu \nu}$, $\bar{H}_{\mu \nu}$ and $\bar{H}$ we need to know that the gauge covariant derivative $\bar{\nabla} \bar{\Phi}$ (defined in (A.6)) of the scalar field is

$$\bar{\nabla} \bar{\Phi} = \epsilon^{-J \phi} e^{-J \Theta} \left[ \bar{\mathcal{H}} - \bar{R}^{A} \bar{T}_{A} \bar{\mathcal{H}} \right].$$

(3.15a)

$$\bar{\nabla}_{\Theta} \bar{\Phi} = -\epsilon^{-J \phi} e^{-J \Theta} \Theta_{B} e^{\bar{R}^{A} \bar{T}_{A}} \bar{\mathcal{H}}.$$ (3.15b)

$$\bar{\nabla}_{\Theta} \bar{\Phi} = \epsilon^{-J \phi} e^{-J \Theta} \left( Z_{2} \bar{\Theta}^{B} \bar{T}_{B} \bar{\mathcal{H}} \right) \Theta_{C}.$$ (3.15c)

Again in (3.15a) we can see what will turn out to be the gauge covariant derivative of $\bar{\mathcal{H}}$ with $\bar{\mathcal{R}}$ again appearing as the gauge field.

In the above use has been made of the identity

$$\left( \epsilon^{\nu \phi} e^{\nu \phi} \right)_{\mu} T_{B}^{\mu} \left( \epsilon^{\nu \phi} e^{\nu \phi} \right)_{\nu} = \left( \epsilon^{\nu \phi} e^{\nu \phi} \right)_{\nu} T_{3} \bar{\Theta}.$$ (3.16)

And the results of Sec. II together with (3.15) yield,

$$\bar{\mathcal{H}}^{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu - 2 \lambda} \frac{1}{\pi_{R}^{2}} + \frac{1}{2} \left\{ \epsilon^{\mu \nu - 2 \lambda} \left( \Theta_{2} - 2 \Theta \bar{\mathcal{R}} \right) \right\}^{2} + e^{\mu \nu - 2 \lambda} \left( \Theta_{2} - 2 \Theta \bar{\mathcal{R}} \right) \bar{\mathcal{H}}.$$ (3.17)

$$\bar{\pi}_{\Theta} \cdot \left( \Theta_{2} - 2 \Theta \bar{\mathcal{R}} \right) \bar{\mathcal{H}}.$$ (3.18)

(3.19)

In Eq. (3.19) the arrow has been put on top of the $\pi \times N$ matrix $T_{A}$ because it is the factor that carries the $N$ vector index $A$, i.e.

$$\bar{G} \bar{\Phi} = \frac{1}{2} \pi_{R}^{A} T_{A} \bar{T}_{A} \bar{R}.$$ (3.19)

In Eqs. (3.17)-(3.18), on the other hand, the arrow is used to indicate $M$ vectors.

It was said in (3.2) that $\Theta_{0} = 0$. This is proved by first obtaining that

$$\bar{\mathcal{H}}_{\Theta} = \bar{\pi}_{R} \cdot \Theta_{B} e^{\hat{R}^{A} \bar{T}_{A}} \Theta_{C}.$$ (3.20)

and then showing that the $M$ vector $\Theta_{B} e^{\hat{R}^{A} \bar{T}_{A}} \Theta_{C}$ belongs to the eigenspace of $R_{3}^{2}$ associated with the eigenvalue $-1$, while $\Theta_{0}$ satisfies $R_{3}^{2} \Theta_{0} = 0$.

IV. GAUGE FREEDOM AND GAUGE FIXATION

Now that we have the explicit form of the terms that appear in the definition of the action (3.4) it is possible to check that the reduced action $\bar{S}$ is invariant under the gauge group defined by,

$$R^{A} C_{a} \longrightarrow e^{-f^{A} C_{a}} R^{A} C_{a} e^{-f^{A} C_{a}} + (e^{-f^{A} C_{a}})^{r} e^{-f^{A} C_{a}},$$

(4.1a)

$$\pi_{R}^{A} C_{a} \longrightarrow e^{-f^{A} C_{a}} \pi_{R}^{A} C_{a} e^{-f^{A} C_{a}},$$

(4.1b)

$$\Theta^{A} C_{a} \longrightarrow e^{f^{A} C_{a}} \Theta^{A} C_{a} e^{f^{A} C_{a}},$$

(4.1c)

$$\pi_{0}^{A} C_{a} \longrightarrow e^{f^{A} C_{a}} \pi_{0}^{A} C_{a} e^{f^{A} C_{a}},$$

(4.1d)
\[ T_h \rightarrow e^{i\Gamma^A T_h} T_h \]  
\[ \tilde{T}_h \rightarrow e^{i\Gamma^A \tilde{T}_h} \tilde{T}_h \]  
(4.1a)

(4.1b)

where \( \Gamma^A \) is an arbitrary \( r \) dependent \( N \)-vectorial function satisfying
\[ Z_3^A(r) = 0. \]  
(4.2)

This local gauge group is characterized by \( p_0 + p \) arbitrary functions \( \Gamma^A(r), \ T_0 \) and \( p \) being the numbers defined in the discussion following Eq. (2.18) and it is obviously a subgroup of the original local gauge group. In Ref. 8, for example, where \( p = 0 \) and \( p = 1 \) there appeared just one function (called \( \psi(r) \) ) characterizing the residual gauge freedom.

The gauge freedom we have equals then the number of independent components of \( \mathbf{R} \) and it is natural to choose the gauge in which \( \mathbf{R} \) is absent, i.e. being in a general gauge we can go to the gauge \( \mathbf{R} = 0 \) by first solving the equation for \( T \) (coming from (4.1a)),
\[ (e^{-i\Gamma^A C_A})^* e^{i\Gamma^A C_A} = -e^{-i\Gamma^A C_A} R^A C_A e^{i\Gamma^A C_A}. \]  
(4.3)

The formal solution of this equation - defined up to a constant - can be found by iteration, the first terms of the series being
\[ f^A = \int \Gamma(r) \, dr_1 + \frac{1}{2} \int \int \Gamma(r_1, r_2) \, dr_1 \, dr_2 + \ldots \]  
(4.4)

Given this \( T \) we can now transform to our gauge
\[ \mathbf{R} = 0 \quad (\text{gauge fixation}). \]  
(4.5)

A convenient coordinate fixation is:
\[ e^\lambda = r \quad (\text{space coordinate fixation}), \]  
(4.6)

\[ \eta^\mu = 0 \quad (\text{time coordinate fixation}). \]  
(4.7)

These conditions must be preserved in time and they must fix the value of the Lagrange multipliers \( \lambda, \eta^\mu, \mathbf{R} \). In fact, to guarantee that (4.6) is valid for all time we require that \( \lambda = 0 \) which implies, together with (4.7), that \( \eta^\mu = 0 \) (see Eq. (C.7)). Requiring that \( \eta^\mu = 0 \) and that \( \tilde{\mathbf{R}} = 0 \) implies (see Eqs. (C.9) and (C.11)),
\[ N[2e^\mu (1-2r^\mu) + 2e^\mu - \frac{1}{2} e^\mu r^2 - \frac{1}{4} e^\mu r^2 \tilde{T}_h - 2e^\mu \tilde{T}_h - \frac{1}{2} e^\mu \tilde{T}_h^2 + \frac{1}{4} r^2 e^\mu \tilde{T}_h^2 \tilde{T}_h \tilde{T}_h - + \frac{1}{4} r^2 e^\mu \tilde{T}_h^2 \tilde{T}_h \tilde{T}_h \tilde{T}_h \tilde{T}_h] - 4(N e^\mu)^2 = 0 \]  
(4.8)

and
\[ \mathbf{W}' + Ne^\mu r^2 \tilde{T}_h = 0. \]  
(4.9)

The last two equations can indeed be used together with the rest of the equations of motion to determine \( \mathbf{W} \) and \( \mathbf{R} \).

For a careful discussion on gauge transformations and gauge choice see Ref. 20.

V. STATIC SOLUTIONS

The equations of motion for our variables \( \mu, \eta^\mu, \lambda, \eta^r, \tilde{T}_h, \tilde{T}_h, \tilde{T}_h, \tilde{T}_h \) are obtained without any difficulty by evaluating the Poisson brackets of every one of these variables with the total reduced Hamiltonian
\[ \tilde{H} = \int [N \tilde{\mathbf{U}}_L + N^* \tilde{\mathbf{U}}_R + r\mathbf{W} \cdot \mathbf{E}^\mu] \, dr. \]  
(5.1)
They can be found in Appendix C. To these equations one must add the
constraints $\varphi = 0$, $\varphi_r = 0$ and $\varphi_t = 0$, also given in Appendix C.

We have already mentioned at the end of Sec.II how the equations
and $\hat{r} = 0$ and $\hat{\theta} = 0$ determine the functions $H$, $N$ and $V$. The
equations of motion ensure that the constraints $\varphi = 0$, $\varphi_r = 0$ and $\varphi_t = 0$ are preserved in time and therefore that there are no new constraints in the
formalism. The first two of these constraints themselves can be used to
obtain $u$ and $\tau$, in terms of the matter fields and it can be shown that
the equations for $\tau$ and $\hat{\varphi}$ become identities.

In the present section we look for static solutions. A solution is
static if all gauge and coordinate transformation invariant quantities do not
change in time. In our case it means that all quantities left after requiring that
of Sec.II be satisfied, must be time independent. It is possible
to reduce the problem to the following set of equations for $\varphi$, $\varphi_r$, $\varphi_t$ and
$\varphi$:

\begin{equation}
( N \varphi_t \varphi_r)' - \frac{N \varphi_t}{r} \partial^2 \varphi_t \varphi_r - \frac{r^2}{N} \partial \varphi_t \varphi_r\end{equation}

\begin{equation}
- \frac{\partial^2 \varphi_t}{2} \left[ \partial^\alpha \partial^\beta \varphi_t \partial^\gamma \varphi_t \right] = 0,
\end{equation}

\begin{equation}
( N r^2 \varphi_t \varphi_r)' + N \varphi_t \left[ \partial^\alpha \partial^\beta \varphi_t \right] + N \varphi_r \left[ \partial^\alpha \partial^\beta \varphi_r \right] - 2 \varphi_t \varphi_r \partial \varphi_t \varphi_r = 0.
\end{equation}

\begin{equation}
( N r^2 \varphi_t \varphi_r)' - \frac{N \varphi_t}{r} \partial^2 \varphi_t \varphi_r - \frac{r^2}{N} \partial \varphi_t \varphi_r = 0.
\end{equation}

\begin{equation}
2 \varphi_t + \frac{e^{2\varphi}}{r} \left[ 1 + \frac{1}{2} \frac{1}{N^2} \left( \partial^\alpha \partial^\beta \partial^\gamma \right) + \frac{\left( \partial^\alpha \partial^\beta \partial^\gamma \right)^2}{4} + \frac{(\partial^\alpha \partial^\beta \partial^\gamma)^3}{6}
\right] + \frac{\partial^2 \varphi_t}{2} + \frac{\partial^2 \varphi_t}{2} - \frac{\partial^2 \varphi_t}{2} = 0.
\end{equation}

(5.6)

where $\tau$ in (5.3) is given by

\begin{equation}
\tau = \frac{e^{2\varphi}}{N} \left( W^2 \right).
\end{equation}

(5.7)

(1) The trivial solution to these equations corresponds to no fields
except for the gravitational one, $\theta = \theta_0$, $h = 0$, $W = 0$. The geometry will
be that of the Schwarzschild solution.

(2) The second simplest solution is

\begin{equation}
\theta = 0, \quad h = \text{constant}, \quad \tau = 0.
\end{equation}

(5.8)

In fact, (5.3) shows that

\begin{equation}
\frac{N}{N} \left( \frac{N}{N} \right) = 0
\end{equation}

(5.9)

i.e. $\theta_0$ is a field value which minimizes the symmetry breaking potential,
while from (5.4) and the vanishing of $\tau$ it follows that

\begin{equation}
\frac{N}{N} \left( \frac{N}{N} \right) = \text{const.}
\end{equation}

(5.10)

which implies

\begin{equation}
W = \frac{N}{N} \left( \frac{N}{N} \right) d\rho.
\end{equation}

(5.11)
Inserting the last result in (5.7) it is seen that

\[ q^A T_A^\Delta \Delta h_0^\Delta = (q^A T_A^\Delta h_0^\Delta)^\Delta = 0, \]  

(5.12)

showing that there is a connection between the orientation of the charge \( q \) in \( W \) space and the orientation of the "vacuum expectation value" of the scalar field \( h_0^\Delta \). This orientation can be fixed making use of the surviving gauge freedom with constant gauge functions mentioned after (4.3).--See however Ref.20.

Next we substitute the previous results in Eq. (5.5) obtaining

\[ \varepsilon^2 2 \mu = | - m^2 + \frac{1}{4} \varepsilon^2 |, \]  

(5.13)

where \( m \) is an integration constant and

\[ \varepsilon^2 = \frac{1}{4} \varepsilon^2. \]  

(5.14)

Had we not assumed in Appendix A that \( W \) vanishes at its minimum (\( \tilde{e} = \tilde{e}_0 \)), there would be an extra term proportional to \( \varepsilon^2 \) in the right-hand side of (5.13). This is the well-known effect of the presence of a cosmological term in the theory (which makes the geometry not asymptotically flat). With or without such extra term Eq. (5.6) gives

\[ N = e^{\varepsilon^2 \mu}, \]  

(5.15)

and henceforth the solution is (without the extra cosmological term)

\[ \tilde{e} = \frac{1}{4} \varepsilon + 3 \varepsilon + 3 \varepsilon + \frac{1}{4} \varepsilon^3, \]  

(5.16a)

\[ \tilde{W} = e^{\varepsilon^2 \varepsilon + 2 \varepsilon \theta} \left[ \frac{2}{r}, 0, \frac{1}{2} \tilde{N}_2, \frac{1}{4} \tilde{N}_1, \sin \theta \right], \]  

(5.16b)

\[ d\sigma^2 = \left( 1 - \frac{2m}{r} + \frac{\varepsilon^2}{4} \right) dr^2 + \left( 1 - \frac{2m}{r} + \frac{\varepsilon^2}{4} \right) d\theta^2 + r^2 (d\phi^2 + \sin^2 \phi d\psi^2), \]  

(5.16c)

where \( \varepsilon^2 \) and \( \tilde{e}_0 \) are constrained to satisfy (5.12). This spherically symmetric singular solution does not have surviving massive components of the gauge field \( \tilde{W} \). In fact, it exists even without introducing a scalar field at all.

(iii) The finite energy solutions which include massive gauge fields, cannot be written explicitly even in the simplest cases, such as those analyzed for example, by 't Hooft\(^1\) and by Julia and Zee\(^3\) in flat space-time. It is not difficult to check, however, that the following asymptotic forms are consistent with Eqs. (5.2)-(5.4):

a) As \( r \to 0 \)

\[ \tilde{e} \to (c r^2 - 1) \tilde{N}_2, \]  

(5.17a)

\[ \tilde{W} \to (a r) \tilde{N}_2, \]  

(5.17b)

\[ \tilde{h} \to (b r) \tilde{N}_2, \]  

(5.17c)

where \( \tilde{N}_2 \) is an \( \lambda = 1, m = 0 \) vector in the \( M \)-dimensional representation space, i.e. \( (\rho_1^2 + \rho_2^2 + J_2^2) \tilde{N}_2 = -\tilde{e}_0 \) and \( J_2 \tilde{N}_2 = 0 \), while \( a, b \) and \( c \) are constant.

b) As \( r \to \infty \)

\[ \tilde{e} \to \tilde{e}_0 \]  

(5.18a)

minimum of \( \tilde{U}(\tilde{e}) \),

(5.18b)

with \( \varepsilon^2 \tilde{e}_0 = 0 \) and

\[ \tilde{e} \to \left( e^{-\varepsilon^2} \right) \tilde{e}_0 \]  

(5.18c)

where \( \tilde{U}^2 \) is a matrix of the form

\[ \Omega_{\tilde{A}} = P_{\tilde{A}} - P_{\tilde{B}} + Q_{\tilde{A}} Q_{\tilde{B}} - S_{\tilde{A}} S_{\tilde{B}}, \]  

(5.19)

\[ \tilde{P}_{\tilde{A}} = \tilde{e} T_{\tilde{A}} \tilde{h}_0 \]  

(5.20a)
The vectors $\vec{h}$, $\vec{M}$, $\vec{g}$ are constants which must be such that (5.18c) tends to zero to have finite energy.

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APPENDIX A

ROTATION AND GENERAL HAMILTONIAN FORMULATION

We use the following indices:

- $a, b = 0, 1, 2, 3$ - space indices
- $i, j = 0, 1, 2, 3$ - space polar indices
- $A, B, C, \ldots = 1, 2, \ldots , N$ - indices of the adjoint representation of the gauge group
- $\Gamma, \Delta, \Sigma = 1, 2, \ldots , M$ - indices of an arbitrary real and orthogonal representation of the gauge group.

We deal with the Hamiltonian formulation of the gauge invariant interacting system of a real scalar field $\phi$, a Yang-Mills field $A_{\mu}^A$ and the gravitational field described by the metric tensor $g_{\mu\nu}$. The canonical variables are the spatial components of the metric $g_{ij}$, the space components of the gauge potentials $A_{\mu}^A$ and the scalar fields $\phi_i$ together with their respective conjugate momenta $\pi_{ij}$, $\pi_{\mu}^A$ and $\pi_{\phi}^A$. The Hamiltonian of the system is

\[ H = \int d^3x (N [N_{ij}^1 + N_{\mu}^1 + N_{\phi}^1 , g_{\mu\nu}) \]  

where the arbitrary functions $N$ and $N^1$ are the combinations of the $g_{\mu\nu}$ that define the space-time line element in the form

\[ ds^2 = -N^2 dt^2 + g_{ij}(dx^i + dt)^2 (dx^j + dt)^2 \]  

The $N_{\mu}^1$ is the arbitrary time component of the gauge field. The generators of normal and tangential deformations ($H_L$ and $H_T$) and of gauge transformations $g_A$ are

\[ H_L = H_T = H_{\phi}^A \]  

\[ = g^{-1/2} (a_{ij} s_{ij} - \frac{1}{2} f^2) - g^{1/2} \frac{1}{2} \]  

\[ + \frac{1}{2} \left[ g^{1/2} \frac{\partial s_{ij}}{\partial t} + g^{1/2} \frac{\partial + \frac{1}{2} g^{1/2} \frac{\partial s_{ij}}{\partial t} \right] \]  

\[ + \frac{1}{2} \left[ g^{1/2} \frac{\partial + \frac{1}{2} g^{1/2} \frac{\partial s_{ij}}{\partial t} \right] \]  

\[ = 0 \]  

where $g = \sum_{\mu} g_{\mu\nu}$.
\[ \mathcal{H}_1 = \mathcal{V}^A_1 + \mathcal{V}^{\theta M}_1 + \mathcal{H}_1 \]
\[ = -2\mathcal{V}^A_1 + \delta_{A1} \mathcal{V}^{\theta A}_1 \mathcal{V}^{\theta A}_1 \eta_{i\xi} \mathcal{V}_{i\xi} \Psi \mathcal{K} D \mathcal{K} , \]
\[ \mathcal{G}_A = \mathcal{X}^{\theta M}_A + \mathcal{V}^A \Phi \]
\[ = -\frac{1}{2} \delta_{A1} \mathcal{V}^{\theta A}_1 \mathcal{V}^{\theta A}_1 - \varepsilon_{A B C} \mathcal{V}^{\theta B}_1 \mathcal{V}^{\theta C}_1 \mathcal{G}_A = 0 \] (A.4) ,
\[ \text{all of which vanish weakly} \] (A.5).

The vector density \( \mathcal{G}^A \) is
\[ \mathcal{G}^A = \frac{1}{2} \delta_{A1} \mathcal{V}^{\theta A}_1 \mathcal{V}^{\theta A}_1 \]
\[ = \frac{1}{2} \delta_{A1} \left( \mathcal{V}^{\theta A}_1 - \mathcal{V}^{\theta A}_1 \right) \mathcal{V}^{\theta A}_1 \mathcal{V}^{\theta A}_1 \] (A.6).

\( \delta_{A1} \) is the inverse of \( \mathcal{G}^A \). The scalar self-coupling \( U(\phi) \) is not specified but, to be in accordance with our denomination of Higgs field, it should be thought of as being
\[ U(\phi) = b(\phi^2 - a^2)^2 \] (A.7a),
which is zero at its minimum to avoid introducing an undesired cosmological term. Another possible choice for the potential could be the popular
\[ U(\phi) = b(1 - \cos \phi)^{1/2} \] (A.7b).

The gauge derivative of \( \phi \) is given by
\[ \mathcal{D}^\Gamma = \phi^\Gamma_{j1} - \mathcal{A}^j \mathcal{D}^\Gamma_{AB} \phi^A \] (A.8),
where the matrices \( \mathcal{T}_B \) were already introduced in (2.20).
Vectors and Matrices of the Adjoint Representation

In Sec. II we have introduced the \( N \)-dimensional \( \tilde{\mathbf{h}}_a \) of components \( h_a^A \) and the \( N \times N \) antisymmetric matrices

\[
(e_a^b)^{BC} = \mathbf{h}_a^{BC} \tag{B.1}
\]

(We remind the reader that the group indices are raised and lowered with the metric (2.31)).

Defining the antisymmetric cross product of any two \( N \) vectors \( \mathbf{v} \) by

\[
(\mathbf{v} \times \mathbf{w})^A = c_{ABC} \mathbf{v}^B \mathbf{w}^C \tag{B.2}
\]

it is immediate that

\[
2 e_a = \tilde{\mathbf{h}}_a \times \tilde{\mathbf{e}} \tag{B.3}
\]

The last result together with (2.11) yields

\[
\tilde{\mathbf{h}}_a \times \tilde{\mathbf{h}}_b = \varepsilon_{abc} \tilde{\mathbf{h}}_c \tag{B.4}
\]

which, in its turn, implies that the three vectors \( \tilde{\mathbf{h}}_a \) have the same norm,

\[
(\tilde{\mathbf{h}}_1) = (\tilde{\mathbf{h}}_2) = (\tilde{\mathbf{h}}_3) \tag{B.5}
\]

The square of the \( \tilde{\mathbf{h}}_a \) vectors must be understood as \( \sum_a h_a^A h_a^A \).

Two other properties which are useful are

\[
(\mathbf{v} \times \mathbf{w}) \times \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = 0 \tag{B.6}
\]

and

\[
\tilde{\mathbf{h}}_a (\mathbf{v} \times \mathbf{w}) = (\tilde{\mathbf{h}}_a \mathbf{v}) \times \mathbf{w} + \mathbf{v} \times (\tilde{\mathbf{h}}_a \mathbf{w}) \tag{B.7}
\]
\[ \dot{\pi}_\lambda = N \left[ 2 e^\mu (1 - 2 \mu') + 2 e^\mu - \frac{1}{2} f^2 e^\mu r^2 e^\nu n_\nu^2 + \frac{1}{2} e^\mu r^2 e^\nu n_\nu - \frac{1}{2} f^2 e^\mu (\Theta \cdot z_2 \Theta - \bar{N}_2)^2 + \frac{1}{2} e^\mu \Theta^2 + \frac{1}{2} r^2 e^\mu n_\lambda^2 + \frac{1}{2} r^2 e^\mu h^2 - r e^\mu U - \frac{1}{2} e^\mu \left[ (\Theta \bar{\Theta})^2 + (\bar{N}_2 \bar{\Theta})^2 \right] \right] - 4 \left( N r e^{-\lambda} \right)' . \] (C.9)

\[ \pi_\lambda = N \left[ -4 e^\mu (1 - 2 \mu') + f^2 e^\mu n_\lambda^2 + f^2 e^\mu (\Theta \cdot z_2 \Theta - \bar{N}_2)^2 + r^2 e^\mu h^2 + 2 r^2 e^\mu U \right] + 4 \left[ (N r e^{-\lambda})' - (N' r e^{-\lambda})' \right] . \] (C.10)

The constraints (3.1), (3.2) and (3.3), respectively, become:

\[ 2 \left[ e^\mu (1 - 2 \mu') - e^\mu \right] + \frac{1}{2} f^2 e^\mu r^2 \left[ e^\mu n_\lambda^2 + \frac{1}{2} r^2 e^\mu n_\lambda^2 \right] + f^2 \left[ \frac{1}{4} r^2 e^\mu (\Theta \cdot z_2 \Theta - \bar{N}_2)^2 + e^\mu \Theta^2 \right] + \frac{1}{2} r^2 e^\mu n_\lambda^2 + \frac{1}{2} r^2 e^\mu h^2 + \frac{1}{2} e^\mu \left[ (\Theta \bar{\Theta})^2 + (\bar{N}_2 \bar{\Theta})^2 \right] + e^\mu r^2 U = 0 . \] (C.11)

\[ \bar{\pi}_\Theta \dot{\Theta} + \bar{\pi}_h \dot{h} + r^{-1} \pi_\lambda = 0 , \] (C.12)

\[ \bar{\pi}_R + \bar{\pi}_h \dot{\Theta} + \bar{\pi}_h \dot{h} = 0 . \] (C.13)