THE MILNOR-WITT MOTIVIC RING SPECTRUM AND ITS ASSOCIATED THEORIES

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Abstract. We build a ring spectrum representing Milnor-Witt motivic cohomology, as well as its étale local version and show how to deduce out of it three other theories: Borel-Moore homology, cohomology with compact support and homology. These theories, as well as the usual cohomology, are defined for singular schemes and satisfy the properties of their motivic analog (and more), up to considering more general twists. In fact, the whole formalism of these four theories can be functorially attached to any ring spectrum, giving finally maps between the Milnor-Witt motivic ones to the classical motivic ones.

Introduction

One of the nice features, and maybe part of the success of Voevodsky’s theory of sheaves with transfers and motivic complexes is to provide a rich cohomological theory. Indeed, apart from defining motivic cohomology, Voevodsky also obtains other theories: Borel-Moore motivic homology, (plain) homology and motivic cohomology with compact support. These four

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theories are even defined for singular $k$-schemes and, when $k$ has characteristic 0 (or with good resolution of singularities assumptions), they satisfy nice properties, such as duality. Finally, again in characteristic 0, they can be identified with known theories: Borel-Moore motivic cohomology agrees with Bloch’s higher Chow groups and homology (without twists) with Suslin homology. This is described in [VSF00, chap. 5, 2.2].

It is natural to expect that the Milnor-Witt version of motivic complexes developed in this book allows one to get a similar formalism. That is what we prove in this chapter. However, we have chosen a different path from Voevodsky’s, based on the tools at our disposal nowadays. In particular, we have not developed a theory of relative MW-cycles, nevertheless an interesting problem which we leave for future work. Instead, we rely on two classical theories, that of ring spectra from algebraic topology and that of Grothendieck’s six functors formalism established by Ayoub in $\mathbb{A}^1$-homotopy along the lines of Voevodsky.

In fact, it is well known since the axiomatization of Bloch and Ogus (see [BO74]) that a corollary of the six functors formalism is the existence of twisted homology and cohomology related by duality. In the present work, we go further that Bloch and Ogus in several directions. First, we establish a richer structure, made of Gysin morphisms (wrong-way functoriality), and establish more properties, such as cohomological descent (see below for a precise description). Second, following Voevodsky’s motivic picture, we show that the pair of theories, cohomology and Borel-Moore homology, also comes with a compactly supported pair of theories, cohomology with compact support and (plain) homology. And finally we consider more general twists for our theories, corresponding to the tensor product with the Thom space of a vector bundle.

Indeed, our main case of interest, MW-motivic cohomology, is an example of non orientable cohomology theory. In practice, this means there are no Chern classes, and therefore no Thom classes so that one cannot identify the twists by a Thom space with a twist by an integer. In this context, it is especially relevant to take care of a twist of the cohomology or homology of a scheme $X$ by an arbitrary vector bundle over $X$, and in fact by a virtual vector bundle over $X$. Our first result is to build a ring spectrum $H_{\text{MW}}R$ (Definition 3.1.2) over a perfect field $k$, with coefficients in an arbitrary ring $R$, which represents MW-motivic cohomology (see Paragraph 3.2.1 for precise statements). In this text, the word ring spectrum (for instance the object $H_{\text{MW}}R$) means a commutative monoid object of the $\mathbb{A}^1$-derived category $D_{\mathbb{A}^1}(k)$ over a field $k$ (see Section 1.1 for reminders). One can easily get from this one a commutative monoid of the stable homotopy category $\text{SH}(k)$ (see Remark 3.1.4). Out of the ring spectrum $H_{\text{MW}}R$, we get as expected the following theories, associated with an arbitrary $k$-variety $X$ (i.e. a separated $k$-scheme of finite type) and a pair $(n, v)$ where $n \in \mathbb{Z}$, $v$ is a virtual vector bundle over $X$:

- MW-motivic cohomology, $H^n_{\text{MW}}(X, v, R)$,
- MW-motivic Borel-Moore homology, $H^n_{\text{MW,BM}}(X, v, R)$,
- MW-motivic cohomology with compact support, $H^n_{\text{MW,c}}(X, v, R)$.

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1. The results of Voevodsky have been generalized in positive characteristic $p$, after taking $\mathbb{Z}[1/p]$-coefficients, using the results of Kelly [Kel72]. See [CD15, Sec. 8].

2. Recall that given an oriented cohomology theory $E^*$ and a vector bundle $V/X$ of rank $n$, the Thom class of $V/X$ provides an isomorphism $E^*(\text{Th}(E)) \simeq E^{*-2n,*-n}(X)$. See [Pan03] for the cohomological point of view or [Deg14] for the ring spectrum point of view.

3. See Paragraph 1.1.5 for this notion. If one is interested in cohomology/homology up to isomorphism, then one can take instead of a virtual vector bundle a class in the K-theory ring $K_0(X)$ of $X$ (see Remark 2.1.2).
• MW-motivic homology, $H_{MW}^n(X, v, R)$.

As expected, one gets the following computations of MW-motivic cohomology.

**Proposition.** Assume $k$ is a perfect field, $X$ a smooth $k$-scheme and $(n, m)$ a pair of integers.

Then there exists a canonical identification:

$$H_{MW}^n(X, m, R) \simeq \text{Hom}_{DM(k, R)}(\mathcal{M}(X), 1(m)[n + 2m]) = H_{MW}^{n+2m, m}(X, R)$$

using the notations of [DF17].

If in addition, we assume $k$ is infinite, then one has the following computations:

$$H_{MW}^n(X, m, \mathbb{Z}) = \begin{cases} CH_m(X) & \text{if } n = 0, \\ H^0_{\text{Zar}}(X, K_0^{MW}) & \text{if } m = 0, \\ H^{2+2m}_{\text{Zar}}(X, \mathbb{Z}(m)) & \text{if } m > 0, \\ H^{n+2m}_{\text{Zar}}(X, \mathbb{W}) & \text{if } m < 0. \end{cases}$$

In fact, the first identification follows from the definition and basic adjunctions (see Paragraph 3.1.3) while the second one was proved in [DF17] (as explained in 3.2.1). In fact, though the ring spectrum $H_{MW}R$ is our main case of interest, because of the previous computation, the four theories defined above, as well as their properties that we will give below, are defined for an arbitrary ring spectrum $E$ — and indeed, this generality is useful as will be explained afterwards. The four theories associated with $E$ enjoy the following functoriality properties (Section 2.2) under the following assumptions:

- $f : Y \to X$ is an arbitrary morphism of $k$-varieties (in fact $k$-schemes for cohomology);
- $p : Y \to X$ is a morphism of $k$-varieties which is either smooth or such that $X$ and $Y$ are smooth over $k$. In any case, $p$ is a local complete intersection morphism and one can define its virtual tangent bundle denoted by $\tau_p$.

| Theory          | natural variance | Gysin morphisms |
|-----------------|------------------|-----------------|
|                | additional assumption on $f$ | additional assumption on $p$ |
| cohomology      | none             | $\mathbb{E}^n(X, v) \overset{f^*}{\longrightarrow} \mathbb{E}^n(Y, f^{-1}v)$ proper | $\mathbb{E}^n(Y, p^{-1}v + \tau_p) \overset{p_*}{\longrightarrow} \mathbb{E}^n(X, v)$ |
| BM-homology     | proper           | $\mathbb{E}^n_{BM}(Y, f^{-1}v) \overset{f^*}{\longrightarrow} \mathbb{E}^n_{BM}(X, v)$ none | $\mathbb{E}^n_{BM}(X, v) \overset{p^*}{\longrightarrow} \mathbb{E}^n_{BM}(Y, p^{-1}v - \tau_p)$ |
| $c$-cohomology  | proper           | $\mathbb{E}^n_c(X, v) \overset{f^*}{\longrightarrow} \mathbb{E}^n_c(Y, f^{-1}v)$ none | $\mathbb{E}^n(Y, p^{-1}v + \tau_p) \overset{p_*}{\longrightarrow} \mathbb{E}^n(X, v)$ |
| homology        | none             | $\mathbb{E}_n(Y, f^{-1}v) \overset{f^*}{\longrightarrow} \mathbb{E}_n(X, v)$ proper | $\mathbb{E}_n(X, v) \overset{p^*}{\longrightarrow} \mathbb{E}_n(Y, p^{-1}v - \tau_p)$ |

Given a closed immersion $i : Z \to X$ between arbitrary $k$-varieties with complement open immersion $j : U \to X$, and a virtual vector bundle $v$ over $X$, there exists the so-called localization long exact sequences (Paragraph 2.2.3):

$$\mathbb{E}^n_{BM}(Z, i^{-1}v) \overset{i^*}{\longrightarrow} \mathbb{E}^n_{BM}(X, v) \overset{j^*}{\longrightarrow} \mathbb{E}^n_{BM}(U, j^{-1}v) \to \mathbb{E}^n_{BM-1}(Z, i^{-1}v),$$

$$\mathbb{E}_n^c(U, j^{-1}v) \overset{i^*}{\longrightarrow} \mathbb{E}_n^c(X, v) \overset{j^*}{\longrightarrow} \mathbb{E}_n^c(Z, i^{-1}v) \to \mathbb{E}_{n+1}^c(U, j^{-1}v).$$

There exists the following products for a $k$-variety $X$ (or simply a $k$-scheme in the case of cup-products), and couples $(n, v), (m, w)$ of integers and virtual vector bundles on $X$:
| Name | pairing | symbol |
|------|---------|--------|
| cup-product | $\mathbb{E}^n(X,v) \otimes \mathbb{E}^m(X,w) \to \mathbb{E}^{n+m}(X,v+w)$ | $x \cup y$ |
| cap-product | $\mathbb{E}^n_{BM}(X,v) \otimes \mathbb{E}^m(X,w) \to \mathbb{E}^{BM}_{n-m}(X,v-w)$ | $x \cap y$ |
| cap-product with support | $\mathbb{E}^n_{BM}(X,v) \otimes \mathbb{E}^n_{BM}(X,w) \to \mathbb{E}^{n-m}(X,v-w)$ | $x \cap y$ |

Given a smooth $k$-scheme $X$ with tangent bundle $T_X$ there exists (Definition 2.3.3) a fundamental class $\eta_X \in \mathbb{E}_{BM}^0(X,T_X)$ such that the following morphisms, called the duality isomorphisms,

$$
\mathbb{E}^n(X,v) \to \mathbb{E}_{BM}^n(X,T_X - v), \ x \mapsto \eta_X \cap y \\
\mathbb{E}_{BM}^n(X,v) \to \mathbb{E}_{-n}(X,T_X - v), \ x \mapsto \eta_X \cap y
$$

are isomorphisms (Theorem 2.3.6).

Finally, the four theories satisfy the following descent properties. Consider a cartesian square

$$
W \xrightarrow{k} V \\
g \downarrow \Delta \downarrow f \\
Y \xrightarrow{i} X
$$

of $k$-varieties (or simply $k$-schemes in the case of cohomology (we refer the reader to Paragraph 1.1.11 for the definition of Nisnevich and cdh distinguished applied to $\Delta$). Put $h = i \circ g = f \circ k$.

We let $v$ be a virtual vector bundle over $X$. Then:

| Assumption on $\Delta$ | descent long exact sequence |
|-------------------------|-----------------------------|
| Nisnevich or cdh | $\mathbb{E}^n(X,v) \xrightarrow{i^* + f^*} \mathbb{E}^n(Y,i^{-1}v) \oplus \mathbb{E}^n(V,f^{-1}v) \xrightarrow{k^* - g^*} \mathbb{E}^n(W,h^{-1}v) \to \mathbb{E}^{n+1}(X,v)$ |
| | $\mathbb{E}_n(W,h^{-1}v) \xrightarrow{k^* - g^*} \mathbb{E}_n(Y,i^{-1}v) \oplus \mathbb{E}_n(V,f^{-1}v) \xrightarrow{i^* + f^*} \mathbb{E}_n(X,v) \to \mathbb{E}_{n-1}(X,v)$ |
| Nisnevich | $\mathbb{E}_{BM}^n(X,v) \xrightarrow{i^* + f^*} \mathbb{E}_{BM}^n(Y,i^{-1}v) \oplus \mathbb{E}_{BM}^n(V,f^{-1}v) \xrightarrow{k^* - g^*} \mathbb{E}_{BM}^n(W,h^{-1}v) \to \mathbb{E}_{BM}^{n+1}(X,v)$ |
| | $\mathbb{E}_c(Y,i^{-1}v) \oplus \mathbb{E}_c(V,f^{-1}v) \xrightarrow{i^* + f^*} \mathbb{E}_c(X,v) \to \mathbb{E}_{c+1}(X,v)$ |
| cdh | $\mathbb{E}_{BM}^n(W,h^{-1}v) \xrightarrow{k^* - g^*} \mathbb{E}_{BM}^n(Y,i^{-1}v) \oplus \mathbb{E}_{BM}^n(V,f^{-1}v) \xrightarrow{i^* + f^*} \mathbb{E}_{BM}^n(X,v) \to \mathbb{E}_{BM}^{n+1}(X,v)$ |
| | $\mathbb{E}_c(Y,i^{-1}v) \oplus \mathbb{E}_c(V,f^{-1}v) \xrightarrow{k^* - g^*} \mathbb{E}_c(W,h^{-1}v) \to \mathbb{E}_{c+1}(X,v)$ |

As the reader can guess, the four theories are functorial in the ring spectrum $\mathbb{E}$. And in fact, out of the construction of $[DF17]$, one gets a canonical morphism of ring spectra (Paragraph 3.4.1):

$$
\varphi : H_{MW}R \to H_M R
$$
from the MW-motivic ring spectrum to Voevodsky's motivic ring spectrum (also simply called the motivic Eilenberg-MacLane spectrum), the two ring spectra considered over $k$ and with coefficients in an arbitrary ring of coefficients $R$. Therefore, one deduces natural maps, compatible with the functorialities described above, from the four MW-motivic theories to their motivic analog, which can be identified with the versions defined by Voevodsky in [VSF00, chap. 5] when the characteristic exponent of $k$ is invertible in $R$ (as we recall in paragraphs 3.2.3 and 3.3.2).

Note finally that we also construct the étale analog of the MW-motivic and motivic ring spectra, which are linked with their classical (Nisnevich) counterparts by canonical morphisms (see again 3.4.1).

**Plan of the paper**

As said previously, this paper is an application of our previous work and of general motivic $A^1$-homotopy. So we have tried to give complete reminders for a non specialist reader.

In Section 1, we first recall the formalism of the $A^1$-derived category, as introduced by Morel, and the associated six functors formalism as constructed by Ayoub following Voevodsky. Then we give a brief account of the theory of ring spectra, specialized in the framework of the $A^1$-derived category.

In Section 2, we construct the four theories associated with an arbitrary ring spectrum and establish the properties listed above.

Finally in Section 3, we apply these results to the particular case of MW-motivic cohomology, as well as its étale version, and seemingly the classical case of motivic cohomology. We consider in more details the case of cohomology and Borel-Moore homology, and conclude this paper with the canonical maps relating these ring spectra.

**Special thanks**

**Conventions**

If $S$ is a base scheme, we will say $S$-varieties for separated $S$-schemes of finite type.

We will simply call a symmetric monoidal category $\mathcal{C}$ monoidal. We generically denote by $1$ the unit object of a monoidal category. When this category depends on a scheme $S$, we also use the generic notation $1_S$.

In the last section, we will fix a perfect base field $k$ and a coefficient ring $R$.

1. Motivic homotopy theory and ring spectra

1.1. Reminder on Grothendieck's six functors formalism.

1.1.1. Let us fix a base scheme $S$. We briefly recall the construction of Morel’s $\mathbb{P}^1$-stable and $A^1$-derived category over $S$ using [CD09] as a reference text. The construction has also been recalled in [DF17] in the particular case where $S$ is the spectrum of a (perfect) field.

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4Interestingly, this map can be compared with Beilinson’s classical regulator.
Let $\text{Sh}(S)$ be the category of Nisnevich sheaves of abelian groups over the category of smooth $S$-schemes $\text{Sm}_S$. This is a Grothendieck abelian category. Given a smooth $S$-scheme $X$, one denotes by $\mathbb{Z}_S(X)$ the Nisnevich sheaf associated with the presheaf $Y \mapsto \mathbb{Z}[\text{Hom}_S(Y, X)]$. The essentially small family $\mathbb{Z}_S(X)$ generates the abelian category $\text{Sh}(S)$. The category $\text{Sh}(X)$ admits a closed monoidal structure, whose tensor product is defined by the formula:

\[
F \otimes G = \lim_{\to X/F,Y/G} \mathbb{Z}_S(X \times_S Y)
\]

where the limit runs over the category whose objects are couples of morphisms $\mathbb{Z}_S(X) \to F$ and $\mathbb{Z}_S(Y) \to G$ and morphisms are given by couples $(x, y)$ fitting into commutative diagrams of the form:

\[
\begin{array}{ccc}
\mathbb{Z}_S(X) & \xrightarrow{x} & \mathbb{Z}_S(X') \\
F & \searrow & \mathbb{Z}_S(Y) \\
& \swarrow & \downarrow \mathbb{Z}_S(Y') \\
& \mathbb{Z}_S(Y) & \searrow \mathbb{Z}_S(X)
\end{array}
\]

Note in particular that $\mathbb{Z}_S(X) \otimes \mathbb{Z}_S(Y) = \mathbb{Z}_S(X \times_S Y)$. We let the reader check that this definition coincides with the one given in [DF17, 1.2.14] when $S = k$.

According to [CD09], the category $C(\text{Sh}(S))$ of complexes with coefficients in $\text{Sh}(S)$ admits a monoidal model structure whose weak equivalences are quasi-isomorphisms and:

- the class of cofibrations is given by the smallest class of morphisms of complexes closed under suspensions, pushouts, transfinite compositions and retracts generated by the inclusions

\[
\mathbb{Z}_S(X) \to C(\mathbb{Z}_S(X) \xrightarrow{\text{Id}} \mathbb{Z}_S(X))[-1]
\]

for a smooth $S$-scheme $X$.

- fibrations are given by the epimorphisms of complexes whose kernel $C$ satisfies the classical Brown-Gersten property: for any cartesian square of smooth $S$-schemes

\[
\begin{array}{ccc}
W & \xrightarrow{k} & V \\
q \downarrow & & \downarrow p \\
U & \xrightarrow{j} & X
\end{array}
\]

such that $j$ is an open immersion, $p$ is étale and induces an isomorphism of schemes $p^{-1}(Z) \to Z$ where $Z$ is the complement of $U$ in $X$ endowed with its reduced subscheme structure, the resulting square of complexes of abelian groups

\[
\begin{array}{ccc}
C(X) & \to & C(V) \\
\downarrow & & \downarrow \\
C(U) & \to & C(W)
\end{array}
\]

is homotopy cartesian.

\[5\text{Recall that an abelian category is Grothendieck abelian if it admits small coproducts, a family of generators and filtered colimits are exact. As usual, one gets that the category } \text{Sh}(S) \text{ is Grothendieck abelian from the case of presheaves and the adjunction } (a, \mathcal{O}) \text{ where } a \text{ is the associated sheaf functor, } \mathcal{O} \text{ the obvious forgetful functor.}\]
Indeed, from Examples 2.3 and 6.3 of op. cit., one gets a descent structure \((\mathcal{G}, \mathcal{H})\) (Def. 2.2 of op. cit.) on \(\text{Sh}(S)\) where \(\mathcal{G}\) is the essential family of generators constituted by the sheaves \(\mathbb{Z}_S(X)\) for \(X/S\) smooth, and \(\mathcal{H}\) is constituted by the complexes of the form
\[
0 \to \mathbb{Z}_S(W) \xrightarrow{q_*-k_*} \mathbb{Z}_S(U) \oplus \mathbb{Z}_S(V) \xrightarrow{j_*+p_*} \mathbb{Z}_S(X) \to 0.
\]
This descent structure is flat (§3.1 of loc. cit.) so that 2.5, 3.5, 5.5 gives the assertion about the monoidal model structure described above. Note moreover that this model structure is proper, combinatorial and satisfies the monoid axiom.

Remark 1.1.2. The descent structure defined in the preceding paragraph is also bounded (§6.1 of loc. cit.). This implies in particular that the objects \(\mathbb{Z}_S(X)\), as complexes concentrated in degree 0, are compact in the derived category \(D(\text{Sh}(S))\) (see Th. 6.2 of op. cit.). Moreover, one can describe explicitly the subcategory of \(D(\text{Sh}(S))\) generated by these objects (see loc. cit.).

1.1.3. Recall now that we get the \(\mathbb{A}^1\)-derived category by first \(\mathbb{A}^1\)-localizing the model category \(\text{C}(\text{Sh}(S))\), which amounts to invert in its homotopy category \(D(\text{Sh}(S))\) morphisms of the form
\[
\mathbb{Z}_S(\mathbb{A}^1_X) \to \mathbb{Z}_S(X)
\]
for any smooth \(S\)-scheme \(X\). One gets the so called \(\mathbb{A}^1\)-local Nisnevich descent model structure (cf. [CD09, 5.2.17]), which is again proper monoidal. One denotes by \(D^\text{eff}_{\mathbb{A}^1}(S)\) its homotopy category. Then one stabilizes the latter model category with respect to Tate twists, or equivalently with respect to the object:
\[
1_S\{1\} = \text{coker}(\mathbb{Z}_S(\{1\}) \to \mathbb{Z}_S(\mathbb{G}_m)).
\]
This is based on the use of symmetric spectra (cf. [CD09, 5.3.31]), called in our particular case Tate spectra. The resulting homotopy category, denoted by \(D^\text{eff}_{\mathbb{A}^1}(S)\) is triangulated monoidal and is characterized by the existence of an adjunction of triangulated categories
\[
\Sigma^\infty : D^\text{eff}_{\mathbb{A}^1}(S) \rightleftarrows D^\text{eff}_{\mathbb{A}^1}(S) : \Omega^\infty
\]
such that \(\Sigma^\infty\) is monoidal and the object \(\Sigma^\infty(\mathbb{Z}_S\{1\})\) is \(\otimes\)-invertible in \(D^\text{eff}_{\mathbb{A}^1}(S)\). As usual, one denotes by \(K\{i\}\) the tensor product of any Tate spectrum \(K\) with the \(i\)-th tensor power of \(\Sigma^\infty(\mathbb{Z}_S\{1\})\). Besides, we also use the more traditional Tate twist:
\[
1_S\{1\} = 1_S\{1\}[-1].
\]

Remark 1.1.4. Extending Remark 1.1.2 let us recall that the Tate spectra of the form \(\Sigma^\infty\mathbb{Z}_S(X)\{i\}\), \(X/S\) is smooth and \(i \in \mathbb{Z}\), are compact and form a family of generators of the triangulated category \(D^\text{eff}_{\mathbb{A}^1}(S)\) in the sense that every object of \(D^\text{eff}_{\mathbb{A}^1}(S)\) is a homotopy colimit of spectra of the preceding form (see [CD09, 5.3.40]).

1.1.5. Thom spaces of virtual bundles. It is important in our context to introduce more general twists (in the sense of [CD09 Définition 1.1.39]). Given a base scheme \(X\), and a vector bundle \(V/X\) one defines the Thom space associated with \(V\), as a Nisnevich sheaf over \(\text{Sm}_X\), by the following formula:
\[
\text{Th}(V) = \text{coker}(\mathbb{Z}_X(V^\times) \to \mathbb{Z}_X(V)),
\]
where \(V^\times\) denotes the complement in \(V\) of the zero section. Seen as an object of \(D^\text{eff}_{\mathbb{A}^1}(X)\), which we still denote by \(\text{Th}(V)\), it becomes \(\otimes\)-invertible — as it is locally of the form
Th(\mathbb{A}^n_X) \simeq \mathbb{1}_X(n)[2n]. Therefore, we get a functor

$$\text{Th} : \text{Vec}(X) \to \text{Pic}(D_{A^1}(X))$$

from the category of vector bundles over \(X\) to that of \(\otimes\)-invertible Tate spectra. According to [Rio10, 4.1.1], given any exact sequence of vector bundles:

$$0 \to V' \to V \to V'' \to 0,$$

one gets a canonical isomorphism

$$\text{Th}(V) \xrightarrow{\sim} \text{Th}(V') \otimes \text{Th}(V'')$$

allowing to canonically extend the preceding functor to a functor from the category of virtual vector bundles over \(X\) introduced in [Del87, §4] to \(\otimes\)-invertible objects of \(D_{A^1}(X)\)

$$\text{Th} : \mathcal{K}(X) \to \text{Pic}(D_{A^1}(X))$$

sending direct sums to tensors products.

**Remark 1.1.6.** The isomorphism classes of objects of \(\mathcal{K}(X)\) give the K-theory ring \(K_0(X)\) of \(X\). In other words, neglecting morphisms, the construction recalled above associates to any element \(v\) of \(K_0(X)\) a canonical isomorphism class of Tate spectra \(\text{Th}(v)\) which satisfies the relation: \(\text{Th}(v + w) = \text{Th}(v) \otimes \text{Th}(w)\).

**1.1.7.** Let us finally recall the basic functoriality satisfied by sheaves and derived categories introduced previously.

Let \(f : T \to S\) be a morphism of schemes. We get a morphism of sites \(f^{-1} : \text{Sm}_S \to \text{Sm}_T\) defined by \(X/S \mapsto (X \times_S T/T)\) and therefore an adjunction of categories of abelian Nisnevich sheaves:

$$(1.1.7.a) \quad f^* : \text{Sh}(S) \to \text{Sh}(T) : f_*$$

such that \(f_*(G) = G \circ f^{-1}\) and

$$f^*(F) = \lim_{\rightarrow \ X/F} \mathbb{Z}_T(X \times_S T)$$

where the colimit runs over the category of morphisms \(\mathbb{Z}_S(X) \to F\). Recall that \(f^*\) is not exact in general.

If in addition \(p = f\) is smooth, one gets another morphism of sites:

$$p_* : \text{Sm}_T \to \text{Sm}_S, (Y \to T) \mapsto (Y \to T \xrightarrow{p} S).$$

One can check that \(p^*(F) = F \circ p_*\) and we get an adjunction of additive categories:

$$(1.1.7.b) \quad p_* : \text{Sh}(T) \to \text{Sh}(S) : p^*$$

such that:

$$p_*(G) = \lim_{\rightarrow \ Y/G} \mathbb{Z}_S(Y \to T \xrightarrow{f} S),$$

this time the colimit runs over the category of morphisms \(\mathbb{Z}_T(Y) \to G\).

Using formulas (1.1.1.a), (1.1.7.a) and (1.1.7.b), one can check the following basic properties:
(1) **Smooth base change formula**.— For any cartesian square of schemes

\[
\begin{array}{ccc}
Y & \xrightarrow{q} & X \\
\downarrow g & \downarrow f & \downarrow p \\
T & \xrightarrow{f} & S
\end{array}
\]

such that \( p \) is smooth, the canonical map

\[ q_! g^* \to f^* p_! \]

is an isomorphism.

(2) **Smooth projection formula**.— For any smooth morphism \( p : T \to S \) and any Nisnevich sheaves \( G \) over \( T \) and \( F \) over \( S \), the canonical morphism:

\[ p_! (G \otimes p^*(F)) \to p_! (G) \otimes F \]

is an isomorphism.

We refer the reader to [CD09, 1.1.6, 1.1.24] for the definition of the above canonical maps. Note that the properties stated above shows that \( \text{Sh} \) is a Sm-premotivic abelian category in the sense of [CD09, 1.4.2] (see also [CD09, Ex. 5.1.4]).

According to the theory developed in [CD09, §5], the adjunctions \((f^*, f_*)\) and \((p_!, p^*)\) for \( p \) smooth can be derived and induce triangulated functors

\[
\begin{align*}
L f^* : D_{\mathbb{A}^1}(S) & \xrightarrow{} D_{\mathbb{A}^1}(T) : R f_*, \\
L p_! : D_{\mathbb{A}^1}(S) & \xrightarrow{} D_{\mathbb{A}^1}(T) : p^*.
\end{align*}
\]

By abuse of notations, we will simply denote the derived functors by \( f^*, f_*, p_!, p^* \). Then the analogues of smooth base change and smooth projection formulas stated above hold (see [CD09, Ex. 5.3.31]). In other words, we get a premotivic triangulated category (cf. [CD09, 1.4.2]) which by construction satisfies the homotopy and stability relation ([CD09, 2.1.3, 2.4.4]).

**Definition 1.1.8.** Consider the notations of 1.1.5 and 1.1.7.

Let \( S \) be a base scheme, \( X/S \) a smooth scheme, and \( v \) be a virtual vector bundle over \( X \). One defines the Thom space of \( v \) above \( S \) as the object:

\[ \text{Th}_S(v) = p_! (\text{Th}(v)). \]

Of course, unless \( X = S \), \( \text{Th}_S(v) \) is in general not \( \otimes \)-invertible and we do not have the relation: \( \text{Th}_S(v \oplus w) = \text{Th}_S(v) \otimes \text{Th}_S(w) \).

**1.1.9.** Consider again the notations of paragraph 1.1.7. One can check the so-called localization property for the fibered category \( D_{\mathbb{A}^1}(-) \) (cf. [CD09, 2.4.26]): for any closed immersion \( i : Z \to S \) with complement open immersion \( j : U \to S \), and any Tate spectrum \( K \) over \( S \), there exists a unique distinguished triangle in \( D_{\mathbb{A}^1}(S) \):

\[ j_! j^*(K) \xrightarrow{i_!} i^* i_*(K) \to j_! j^*(K)[1] \]

where \( j_* \) (resp. \( i^* \)) is the counit (resp. unit) of the adjunction \((j_!, j^*)\) (resp. \((i^*, i_*)\)).

As we have also seen in Remark 1.1.4 that \( D_{\mathbb{A}^1}(S) \) is compactly generated, we can apply to it the cross-functor theorem of Ayoub and Voevodsky (cf. [CD09, 2.4.50]) which we state here for future reference.
Theorem 1.1.10. Consider the above notations. Then, for any separated morphism of finite type \( f : Y \to X \) of schemes, there exists a pair of adjoint functors, the exceptional functors,

\[ f_! : D_{A^1}(Y) \rightleftarrows D_{A^1}(X) : f^! \]

such that:

1. There exists a structure of a covariant (resp. contravariant) 2-functor on \( f \mapsto f_! \) (resp. \( f \mapsto f^! \)).
2. There exists a natural transformation \( \alpha_f : f_! \to f^* \) which is an isomorphism when \( f \) is proper. Moreover, \( \alpha \) is a morphism of 2-functors.
3. For any smooth separated morphism of finite type \( f : X \to S \) of schemes with tangent bundle \( T_f \), there are canonical natural isomorphisms

\[ p_f : f_! \to f^!(Th_X(T_f) \otimes_X) \]
\[ p'_f : f^* \to Th_X(-T_f) \otimes_X f^! \]

which are dual to each other – the Thom premotive \( Th_X(T_f) \) is \( \otimes \)-invertible with inverse \( Th_X(-T_f) \).
4. For any cartesian square:

\[
\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
\downarrow{g'} & & \downarrow{g} \\
Y & \xrightarrow{f} & X,
\end{array}
\]

such that \( f \) is separated of finite type, there exist natural isomorphisms

\[ g^*f_! \sim f'_!* \]
\[ g'_!f^! \sim f^!g_* \]

5. For any separated morphism of finite type \( f : Y \to X \) and any Tate spectra \( K \) and \( L \), there exist natural isomorphisms

\[ \text{Ex}(f^*_!, \otimes) : (f_!K) \otimes_X L \sim f^!(K \otimes_Y f^*L), \]
\[ \text{Hom}_X(f_!(L), K) \sim f_* \text{Hom}_Y(L, f^!(K)), \]
\[ f^! \text{Hom}_X(L, M) \sim \text{Hom}_Y(f^*(L), f^!(M)). \]

1.1.11. This theorem has many important applications. Let us state a few consequences for future use.

- **Localization triangles.** Consider again the assumptions of Paragraph 1.1.9. Then one gets canonical distinguished triangles:

\[
\begin{align*}
(1.1.11.a) & \quad j_*j^!(K) \xrightarrow{j_*} K \xrightarrow{i_*i^*} (K) \to j_*j^!(K)[1] \\
(1.1.11.b) & \quad i_!i^!(K) \xrightarrow{i_*} K \xrightarrow{j_*j^*} (K) \to i_!i^!(K)[1]
\end{align*}
\]

where \( j_* \), \( j^* \), \( i_* \), \( i^* \) are the unit/counit morphism of the obvious adjunctions.

- **Descent properties.** Consider a cartesian square of schemes:

\[
\begin{array}{ccc}
W & \xrightarrow{k} & V \\
\downarrow{g} & & \downarrow{f} \\
Y & \xrightarrow{i} & X
\end{array}
\]
One says $\Delta$ is Nisnevich (resp. cdh) distinguished if $i$ is an open (resp. closed) immersion, $f$ is an étale (resp. proper) morphism and the induced map $(V - W) \to (X - Y)$ of underlying reduced subschemes is an isomorphism.

If $\Delta$ is Nisnevich or cdh distinguished, then for any object $K \in \mathcal{D}_{A^1}(X)$, there exists canonical distinguished triangles:

\begin{align*}
(1.1.11.c) & \quad K \xrightarrow{i^* + f^*} i_* i^*(K) \oplus f_* f^*(K) \xrightarrow{h_* h^*(K)} K[1] \\
(1.1.11.d) & \quad h^! h^!(K) \xrightarrow{k_* - g_*} i^! i^!(K) \oplus f^! f^!(K) \xrightarrow{i_* + f_*} K \to h^! h^!(K)[1]
\end{align*}

where $f^*$ (resp. $f_*$) is the unit (resp. counit) of the adjunction $(f^*, f_*)$ (resp. $(f^!, f^!)$) and $h = i g = f k$.

- **Pairing.**– Let us apply point (5) replacing $K$ by $f^!(K)$; one gets an isomorphism which appears in the following composite map:

$$f_!(f^!(K) \otimes f^*(L)) \xrightarrow{\sim} f_! f^!(K) \otimes L \xrightarrow{f_* \otimes Id_L} K \otimes L,$$

where $f_*$ is the counit map for the adjunction $(f^!, f^!)$). Thus by adjunction, one gets a canonical morphism:

$$f^!(K) \otimes f^*(L) \to f^!(K \otimes L).$$

We will see in Paragraph 2.3.1 that this pairing induces the classical cap-product.

**Remark 1.1.12.** Let $R$ be a ring of coefficients. One can obviously extend the above considerations by replacing sheaves of abelian groups by sheaves of $R$-modules (as in [CD09]). We get a triangulated $R$-linear category $\mathcal{D}_{A^1}(S, R)$ depending on an arbitrary scheme $S$, and also obtain the six functors formalism described above. In brief, there is no difference between working with $\mathbb{Z}$-linear coefficients or $R$-linear coefficients.

Besides, one gets an adjunction of additive categories:

$$\rho^R_* : \text{Sh}(S) \rightleftarrows \text{Sh}(S, R) : \rho^R_*$$

where $\rho^R_*$ is the functor that forgets the $R$-linear structure. The functor $\rho^R_*$ is obtained by taking the associated sheaf of the presheaf obtained after applying the extension of scalars functor for $R/\mathbb{Z}$. Note that the functor $\rho^R_*$ is monoidal. According to [CD09, 5.3.36], these adjoint functors can be derived and further induce adjunctions of triangulated categories:

\begin{align*}
(1.1.12.a) & \quad L\rho^R_* : \mathcal{D}_{A^1}(S) \rightleftarrows \mathcal{D}_{A^1}(S, R) : R\rho^R_*
\end{align*}

such that $L\rho^R_*$ is monoidal.

1.2. **Ring spectra.** Let us start with a very classical definition.

**Definition 1.2.1.** Let $S$ be a base scheme. A ring spectrum $E$ over $S$ is a commutative monoid of the monoidal category $\mathcal{D}_{A^1}(S)$.

A morphism of ring spectra is a morphism of commutative monoids.

In other words, $E$ is a Tate spectrum equipped with a multiplication (resp. unit) map

$$\mu : E \otimes E \to E, \quad \text{resp. } \eta : 1_S \to E.$$
such that the following diagrams are commutative

\[
\begin{aligned}
\begin{array}{ccc}
E \otimes \eta & \xrightarrow{\mu} & E \\
\mu \otimes 1 & \downarrow & \mu \\
E & \xrightarrow{\mu} & E
\end{array}
\end{aligned}
\]

\[
\begin{aligned}
\begin{array}{ccc}
E \otimes E \otimes E & \xrightarrow{1 \otimes \mu} & E \otimes E \\
\mu & \downarrow & \mu \\
E \otimes E & \xrightarrow{\mu} & E
\end{array}
\end{aligned}
\]

where \(\gamma\) is the isomorphism exchanging factors (coming from the underlying symmetric structure of the monoidal category \(D_{\mathbb{A}^1}(S)\)).

**Example 1.2.2.**

1. The constant Tate spectrum \(\mathbb{1}_S\) over \(S\) is an obvious example of ring spectrum over \(S\). Besides, for any ring spectrum \((E, \mu, \eta)\) over \(S\), the unit map \(\eta: \mathbb{1}_S \to E\) is a morphism of ring spectra.

2. Let \(k\) be a fixed base (a perfect field in our main example). Let \(E\) be a ring spectrum over \(k\). Then for any \(k\)-scheme \(S\), with structural morphism \(f\), we get a canonical ring spectrum structure on \(f^*(E)\) as the functor \(f^*\) is monoidal.

In this situation, we will usually denote by \(E_S\) this ring spectrum. The family of ring spectra \((E_S)\) thus defined is a cartesian section of the fibered category \(D_{\mathbb{A}^1}(S)\) and \(\mathcal{T}\).

It forms what we call an absolute ring spectrum over the category of \(k\)-schemes in \(\text{[Deg17]}\).

1.2.3. We finally present a classical recipe in motivic homotopy theory to produce ring spectra. Let us fix a base scheme \(S\).

Suppose given a triangulated monoidal category \(\mathcal{T}\) and an adjunction of triangulated categories \(\phi^*: D_{\mathbb{A}^1}(S) \rightleftarrows \mathcal{T}: \phi_*\) such that \(\phi^*\) is monoidal.

Then, for a couple of objects \(K\) and \(L\) of \(\mathcal{T}\), we get a canonical map:

\[
\nu_{K,L}: \phi_*(K) \otimes \phi_*(L) \xrightarrow{\phi^*} \phi_*(\phi^*(K) \otimes \phi^*(L)) \xrightarrow{\phi^* \otimes \phi^*} \phi_*(K \otimes L)
\]

where \(\phi_*\) and \(\phi^*\) are respectively the unit and counit of the adjunction \((\phi^*, \phi_*)\). Besides, one easily checks that this isomorphism is compatible with the associativity and symmetry isomorphisms of \(D_{\mathbb{A}^1}(S)\) and \(\mathcal{T}\).

We also get a canonical natural transformation

\[
\nu: \mathbb{1}_S \xrightarrow{\delta^*} \phi_* \phi^*(\mathbb{1}_S) \simeq \phi_*(\mathbb{1}_{\mathcal{T}})
\]

which one can check to be compatible with the unit isomorphism of the monoidal structures underlying \(D_{\mathbb{A}^1}(S)\) and \(\mathcal{T}\). In other words, the functor \(\phi_*\) is weakly monoidal\(^6\).

Then, given any commutative monoid \((M, \mu^M, \eta^M)\) in \(\mathcal{T}\), one gets after applying the functor \(\phi_*\) a ring spectrum with multiplication and unit maps

\[
\begin{aligned}
\mu: \phi_*(M) \otimes \phi_*(M) & \xrightarrow{\mu^M} \phi_*(M \otimes M) \xrightarrow{\phi_*(\mu^M)} \phi_*(M), \\
\eta: \mathbb{1}_S & \xrightarrow{\nu} \phi_*(\mathbb{1}_{\mathcal{T}}) \xrightarrow{\phi_*(\eta^M)} \phi_*(M).
\end{aligned}
\]

\(^6\)Other possible terminologies are weak monoidal functor, lax monoidal functor.
The verification of the axioms of a ring spectrum comes from the fact that $\phi_*$ is weakly monoidal.

**Example 1.2.4.** The main example we have in mind is the case where $M$ is the unit object $1_{\mathcal{T}}$ of the monoidal category $\mathcal{T}$:

$$E = \phi_*(1_{\mathcal{T}}).$$

**Remark 1.2.5.** One can also define a strict ring spectrum over a scheme $S$ as a commutative monoid object of the underlying model category of $D_{\mathbb{A}^1}(S)$ — in other words a Tate spectrum equipped with a ring structure such that the diagrams (1.2.1.a) commute in the category of Tate spectra, rather than in its localization with respect to weak equivalences.

In the subsequent cases where ring spectra will appear in this paper, through an adjunction $\phi_*, \phi^*$ as above, this adjunction will be derived from a Quillen adjunction of monoidal model categories. We can repeat the arguments above by replacing the categories with their underlying model categories. Therefore, the ring spectra of the form $\phi_*(1_{\mathcal{T}})$ will in fact be strict ring spectra. While we will not use this fact here, it is an information that could be useful to the reader.

**Example 1.2.6.** Let $R$ be a ring of coefficients. Consider the notations of Remark 1.1.12. Then we can apply the preceding considerations to the adjunction (1.1.12.a) so that we get a ring spectrum

$$H_{\mathbb{A}^1}R_X := R\rho_*^R(1_X).$$

(1.2.7.a)

$$1_S \to \phi_*(1_{\mathcal{T}}).$$

Moreover, suppose there exists another triangulated monoidal categories $\mathcal{T}'$ with an adjunction of triangulated categories:

$$\psi^* : \mathcal{T} \rightleftarrows \mathcal{T}' : \psi_*$$

such that $\psi^*$ is monoidal.

Then one gets a canonical morphism of ring spectra:

$$\phi_*(1_{\mathcal{T}}) \to \phi_*\psi_*\psi^*(1_{\mathcal{T}}) \simeq \phi_*\psi_*(1_{\mathcal{T}'})$$

which is compatible with the canonical morphism of the form (1.2.7.a).

2. THE FOUR THEORIES ASSOCIATED WITH A RING SPECTRUM

2.0.1. In this section, we will fix a base scheme $k$ together with a ring spectrum $(E, \mu, \eta)$ over $k$. Given any $k$-scheme $X$ with structural morphism $f$, we denote by $E_X = f^*(E)$ the pullback ring spectrum over $X$ (Example 1.2.2). We will still denote by $\mu$ (resp. $\eta$) the multiplication (resp. unit) map of the ring spectrum $E_X$.

When $X/k$ is separated of finite type, it will also be useful to introduce the following notation:

$$E'_X = f^!(E).$$

Note that this is not a ring spectrum in general, but that there is a pairing

(2.0.1.a)

$$\mu' : E'_X \otimes E_X = f^!(E) \otimes f^*(E) \to f^!(E \otimes E) \xrightarrow{\mu} f^!(E) = E'_X$$

using the last point of Paragraph 1.1.11.
In the following subsections, we will show how to associate cohomological/homological theories with \( E \) and deduce the rich formalism derived from the six functors formalism (Theorem 1.1.10).

Note finally that the constructions will be functorial in the ring spectra \( E \). To illustrate this fact, we will also fix a generic morphism of ring spectra 

\[ \phi : E \to F. \]

### 2.1. Definitions and basic properties.

**Definition 2.1.1.** Let \( p : X \to \text{Spec}(k) \) be a \( k \)-scheme, \( n \in \mathbb{Z} \) an integer and \( v \in \mathcal{K}(X) \) a virtual vector bundle over \( X \).

We define the cohomology of \( X \) in degree \( (n,v) \) and coefficients in \( E \) as the following abelian group:

\[
E^n(X, v) := \text{Hom}_{D_A^\infty}(k) \left( 1_k, p_!(p^!(E) \otimes \text{Th}(v))[n] \right).
\]

If \( X \) is a \( k \)-variety, we also define respectively the cohomology with compact support, Borel-Moore homology and homology of \( X \) in degree \( (n,v) \) and coefficients in \( E \) as:

\[
E^n_c(X, v) := \text{Hom}_{D_A^\infty}(k) \left( 1_k, p_!(p^!(E) \otimes \text{Th}(v))[n] \right),
\]

\[
E^{BM}_n(X, v) := \text{Hom}_{D_A^\infty}(k) \left( 1_k, p_!(p_!(E) \otimes \text{Th}(v))[-n] \right),
\]

\[
E_n(X, v) := \text{Hom}_{D_A^\infty}(k) \left( 1_k, p_!(p_!(E) \otimes \text{Th}(v))[-n] \right).
\]

We will sometime use the abbreviations *c-cohomology* and *BM-homology* for cohomology with compact support and Borel-Moore homology respectively.

Finally, when one replaces \( v \) by an integer \( m \in \mathbb{Z} \), we will mean that \( v \) is the (opposite of the) trivial vector bundle of rank \( |m| \).

We will describe below the properties satisfied by these four theories, which can be seen as a generalization of the classical Bloch-Ogus formalism (see [BO74]).

**Remark 2.1.2.** It is clear from the construction of Paragraph 1.1.5 and from the above definition that cohomology and cohomology with compact support (resp. Borel-Moore homology and homology) depends covariantly (resp. contravariantly) upon the virtual vector bundle \( v \) — i.e. with respect to morphism of the category \( \mathcal{K}(X) \). In particular, if one consider these theories up to isomorphism, one can take for \( v \) a class in the \( K \)-theory ring \( K_0(X) \) of vector bundles over \( X \).

**Example 2.1.3.** Let us assume that \( X \) is a smooth \( k \)-scheme, with structural morphism \( p \). Consider a couple of integers \( (n,m) \in \mathbb{Z}^2 \).
Then, one gets the following computations:

\[ E^n(X, m) = \text{Hom}_{D_{X}^{1}(k)} \left( I_{k}, p_{*} \left( p^{*}(E) \otimes \text{Th}(\mathbb{A}^{m}_{X}) \right) \right) [n] \]

\[ = \text{Hom}_{D_{X}^{1}(k)} \left( I_{k}, p_{*}p^{*}(E \otimes \text{Th}(\mathbb{A}^{m}_{k})) \right) [n] \]

\[ \overset{(1)}{=} \text{Hom}_{D_{X}^{1}(k)} \left( p_{1}p^{*}(1_{k}), E \otimes \text{Th}(\mathbb{A}^{m}_{k}) \right) [n] \]

\[ \overset{(2)}{=} \text{Hom}_{D_{X}^{1}(k)} \left( \mathbb{Z}_{k}(X), E(m)[n + 2m] \right) \]

\[ = \mathbb{E}^{n+2m,m}(X). \]

The identification (1) comes from the (derived) adjunctions described in Paragraph 1.1.7 and (2) comes from the definition of \( p^{*} \) (resp. \( p_{*} \)) — see again Paragraph 1.1.7.

So for smooth \( k \)-schemes and constant virtual vector bundles, the cohomology theory just defined agree (up to change of twists) with the classical cohomology represented by \( E \).

**Remark 2.1.4.** Using the conventions stated in the beginning of this section, one can rewrite the previous definitions as follows:

\[ E^{n}(X, v) = \text{Hom}_{D_{X}^{1}(X)} \left( I_{X}, E_{X} \otimes \text{Th}(v)[n] \right) \],

\[ E_{c}^{n}(X, v) = \text{Hom}_{D_{X}^{1}(k)} \left( I_{k}, p_{*}(E_{X} \otimes \text{Th}(v))[n] \right) \],

\[ E_{BM}^{n}(X, v) = \text{Hom}_{D_{X}^{1}(k)} \left( I_{k}, p_{*}(E_{X}' \otimes \text{Th}(-v))[n] \right) \],

\[ E_{n}(X, v) = \text{Hom}_{D_{X}^{1}(k)} \left( I_{k}, p_{*}(E_{X} \otimes \text{Th}(-v))[n] \right) \].

In particular, if one interpret \( E_{X}' \) as a dual of \( E_{X} \), our definition of Borel-Moore homology is analogue to that of Borel and Moore relative to singular homology (see [BM60]).

**2.1.5.** Assume \( p : X \to \text{Spec}(k) \) is separated of finite type. From the natural transformation \( \alpha_{p} : p_{!} \to p_{*} \) of Theorem 1.1.10(2), one gets canonical natural transformations:

\[ E_{c}^{n}(X, v) \to E^{n}(X, v), \]

\[ E_{n}(X, v) \to E_{BM}^{n}(X, v) \]

which are isomorphisms whenever \( X/k \) is proper.

**Remark 2.1.6.** Consider an arbitrary \( k \)-scheme \( X \).

One must be careful about the homotopy invariance property. Indeed, if \( v \) is a virtual bundle over \( \mathbb{A}^{1}_{X} \) which comes from \( X \), that is \( v = \pi^{-1}(v_{0}) \) where \( \pi : \mathbb{A}^{1}_{X} \to X \) is the canonical projection, then one gets:

\[ E^{n}(\mathbb{A}^{1}_{X}, v) \simeq E^{n}(X, v_{0}) \]

from the homotopy property of \( D_{X}^{1}(X) \) — more precisely, the isomorphism \( \text{Id} \simeq \pi_{*}\pi^{*} \).

This will always happen if \( X \) is regular. But in general, \( v \) could not be of the form \( \pi^{-1}(v_{0}) \) and there is no formula as above.

Similarly, if \( v = p^{-1}(v_{0}) \), one gets:

\[ E_{n}(\mathbb{A}^{1}_{X}, v) \simeq E_{n}(X, v_{0}). \]

Note finally there is no such formula for c-cohomology or BM-homology.
2.1.7. It is clear that the morphism of ring spectra $\phi : E \to F$ induces morphisms of abelian groups, all denoted by $\phi^*$:

\[
\begin{align*}
E_n(X, v) &\to F_n(X, v) \\
E^n_c(X, v) &\to F^n_c(X, v) \\
E_{BM}^n(X, v) &\to F_{BM}^n(X, v) \\
E_n(X, v) &\to F_n(X, v).
\end{align*}
\]

2.2. Functoriality properties.

2.2.1. Basic functoriality. Let $f : Y \to X$ be a morphism of $k$-schemes and consider $(n, v) \in \mathbb{Z} \times K(X)$. Letting $p$ (resp. $q$) be the projection of $X/k$ (resp. $Y/k$), we deduce the following maps in $\text{DA}^1(X)$, where in the second one we have assume that $p$ and $q$ are separated of finite type

\[
\begin{align*}
\text{Th}(v) \otimes p^*(E) &\xrightarrow{f^*} f_*(\text{Th}(v) \otimes p^*E) \\
\text{Th}(f^{-1}v) \otimes q^*E &\xrightarrow{f_!} \text{Th}(v) \otimes p^!E
\end{align*}
\]

where $f^*$ (resp. $f_*$) is the unit (resp. counit) map of the adjunction $(f^*, f_*)$ (resp. $(f_!, f^!)$) and the isomorphism (1) (resp. (2)) follows from the fact $f^*$ is monoidal (resp. $\text{Th}(v)$ is $\otimes$-invertible).

Composing respectively with $p_*$ and $p_!$, we get canonical morphisms:

\[
\begin{align*}
p_*\left(\text{Th}(v) \otimes p^*E\right) &\xrightarrow{\pi(f)} q_*\left(\text{Th}(f^{-1}v) \otimes q^*E\right) \\
q_!\left(\text{Th}(f^{-1}v) \otimes q^*E\right) &\xrightarrow{\pi'(f)} p_!\left(\text{Th}(v) \otimes p^!E\right),
\end{align*}
\]

which induces the following pullback and pushforward maps:

\[
\begin{align*}
E^n(X, v) &\xrightarrow{f^*} E^n(Y, f^{-1}v) \\
E_n(Y, f^{-1}v) &\xrightarrow{f_*} E_n(X, v).
\end{align*}
\]

It is straightforward to check that these maps are compatible with composition, turning cohomology (resp. homology) into a contravariant (resp. covariant) functor with respect to all $k$-schemes (resp. all $k$-varieties).

Assume now that $f$ is proper. Then from Theorem 1.1.10(2), we gets a canonical isomorphism $\alpha_f : f_! \simeq f_*$ and the map $\pi(f)$, $\pi'(f)$ respectively induces canonical morphisms:

\[
\begin{align*}
E^n_c(X, v) &\xrightarrow{f^*} E^n_c(Y, f^{-1}v) \\
E_{BM}^n(Y, f^{-1}v) &\xrightarrow{f_*} E_{BM}^n(X, v).
\end{align*}
\]

Again, notably because $\alpha_f$ is compatible with composition, these maps are compatible with composition so that cohomology with compact support (resp. Borel-Moore homology) is a contravariant (resp. covariant) functor with respect to proper morphisms of $k$-varieties.

Remark 2.2.2. With that functoriality at our disposal, we can understand the homotopy property described in Remark 2.1.6 as follows. Given any scheme $X$ and any virtual bundle
\(v_0\) over \(X\), the canonical projection \(\pi: \mathbb{A}^1_X \to X\) induces isomorphisms:
\[
\pi^*: E^n(X, v_0) \to E^n(\mathbb{A}^1_X, \pi^{-1}(v_0)),
\pi_*: E_n(\mathbb{A}^1_X, \pi^{-1}(v_0)) \to E_n(X, v_0).
\]

2.2.3. Localization long exact sequences. One of the main properties of Borel-Moore homology, as well as cohomology with compact support is the existence of the so-called localization long exact sequences. In our case, it follows directly from the localization triangle stated in Paragraph 1.1.11.

Indeed for a closed immersion \(i: Z \to X\) of \(k\)-varieties with complement open immersion \(j: U \to X\), and a virtual vector bundle \(v\) over \(X\), one gets localization sequences:
\[
E^n_{BM}(Z, i^{-1}v) \xrightarrow{j^*} E^n_{BM}(X, v) \xrightarrow{i^*} E^n_{BM}(U, j^{-1}v) \to E^{n-1}_{BM}(Z, i^{-1}v),
E^n_c(U, j^{-1}v) \xrightarrow{j^*} E^n_c(X, v) \xrightarrow{i^*} E^n_c(Z, i^{-1}v) \to E^{n-1}_c(U, j^{-1}v).
\]

More explicitly, the first (resp. second) exact sequence is obtained by using the distinguished triangle (1.1.11.b) with \(K = E^n_X\) (resp. (1.1.11.a) with \(K = E^n_X\)) and applying the cohomological functor \(\text{Hom}_{D^{\ast}}(X)(1, -)\). Note we also use the identifications \(i_! = i_*\) (resp. \(j^! = j^*\)) which follows from Theorem 1.1.10 point (2) (resp. (3)).

2.2.4. Gysin morphisms. Let us fix a morphism \(f: Y \to X\) of \(k\)-schemes which is separated of finite type and consider the notations of Remark 2.1.4.

Assume \(f\) is smooth with tangent bundle \(\tau_f\). Then, according to Theorem 1.1.10(3) and the \(\otimes\)-invertibility of Thom spectra, we get a canonical isomorphism:
\[
p_f^*: f^!(E_X) \simeq f^*(E_X) \otimes \text{Th}(\tau_f) = E_Y \otimes \text{Th}(\tau_f).
\]

Suppose now that \(X\) and \(Y\) are smooth \(k\)-varieties, with respective structural morphisms \(p\) and \(q\). Then \(f\) is a local complete intersection morphism, and has for relative virtual tangent bundle the virtual bundle in \(\mathcal{K}(Y)\):
\[
\tau_f = [T_q] - [f^{-1}(T_p)].
\]

Then one can compute \(q^!\) in two ways:
\[
q^!(E) \overset{(1)}{=} q^*(E) \otimes \text{Th}(T_q) = E_Y \otimes \text{Th}(T_q)
\overset{(2)}{=} f^!p^!(E) \overset{(3)}{=} f^!(p^*(E) \otimes \text{Th}(T_p)) \overset{(3)}{=} f^!(E_X) \otimes \text{Th}(f^{-1}T_p)
\]
where (1) and (2) are given by the relative purity isomorphisms of Theorem 1.1.10(3), respectively for \(p\) and \(q\), and (3) follows from the fact \(\text{Th}(T_p)\) is \(\otimes\)-invertible. Putting the two formulas together, one gets as in the previous case an isomorphism:
\[
(2.2.4.a) \quad \tilde{\eta}_f: f^!(E_X) \simeq E_Y \otimes \text{Th}(\tau_f).
\]

Similarly, using the same procedure but exchanging the role of \(f^*\) and \(f^!\), one gets a canonical isomorphism:
\[
(2.2.4.b) \quad \tilde{\eta}_f^*: f^*(E'_X) \simeq E'_Y \otimes \text{Th}(-\tau_f),
\]
assuming either \(f\) is smooth or \(f\) is a morphism of smooth \(k\)-varieties.
Therefore one gets using adjunctions the following trace maps:

\[ tr_f : f_!(E_Y \otimes \text{Th}(\tau f)) \rightarrow E_X, \]
\[ tr'_f : E'_X \rightarrow f_*(E'_Y \otimes \text{Th}(-\tau f)). \]

We can tensor these maps with the Thom space of an arbitrary virtual vector bundle \( v \) over \( X \), and compose the map with \( p! \) for the first one and \( p* \) for the second one to get the following maps:

\[ q_!(q^*E \otimes \text{Th}(f^{-1}v + \tau f)) \rightarrow p_!(p^*E \otimes \text{Th}(v)), \]
\[ p_*(p_!E_X \otimes \text{Th}(v)) \rightarrow q_*(q_!E \otimes \text{Th}(f^{-1}v - \tau f)). \]

If we assume moreover that \( f \) is proper, then we get using the same procedure and using the identification \( f_* = f_! \) the following maps:

\[ q_*(q^*E \otimes \text{Th}(f^{-1}v + \tau f)) \rightarrow p_*(p^*E \otimes \text{Th}(v)), \]
\[ p_!(p_!E_X \otimes \text{Th}(v)) \rightarrow q_!(q_!E \otimes \text{Th}(f^{-1}v - \tau f)). \]

Let us state the result in term of the four theories in the following proposition.

**Proposition 2.2.5.** Let \( f : Y \rightarrow X \) be a morphism of \( k \)-varieties satisfying one of the following assumptions:

(a) \( f \) is smooth;

(b) \( X \) and \( Y \) are smooth \( k \)-varieties.

Then the maps defined above induce the following Gysin morphisms:

\[ f_* : \mathbb{E}^n(Y, f^{-1}v + \tau f) \rightarrow \mathbb{E}^n(X, v), \]
\[ f^* : \mathbb{E}^BM_n(X, v) \rightarrow \mathbb{E}^BM_n(Y, f^{-1}v - \tau f). \]

Assume moreover that \( f \) is proper. Then using again the previous constructions, one gets the following maps:

\[ f_* : \mathbb{E}^n(Y, f^{-1}v + \tau f) \rightarrow \mathbb{E}^n(X, v), \]
\[ f^* : \mathbb{E}^BM_n(X, v) \rightarrow \mathbb{E}^BM_n(Y, f^{-1}v - \tau f). \]

These Gysin morphisms are compatible with composition.

Under assumption (a), for any cartesian square,

\[ \begin{array}{ccc}
Y' & \xrightarrow{g} & X' \\
\downarrow q & & \downarrow p \\
Y & \xrightarrow{f} & X
\end{array} \]

one has the classical base change formulas:

- \( p^*f_* = g_*q^* \) in case of cohomologies,
- \( f^*p_* = q_*g^* \) in case of homologies.

Indeed, the construction of maps are directly obtained from the maps defined in the paragraph preceding the proposition. The compatibility with composition is a straightforward check once we use the compatibility of the relative purity isomorphism with composition (due to Ayoub, see [CD09, 2.4.52] for the precise statement). The base change formulas in the smooth case is similar and are ultimately reduced to the compatibility of the relative purity isomorphism with base change (see for example the proof of [Deg17, Lemma 2.3.13]).
Remark 2.2.6. Gysin morphisms can be defined under the weaker assumption that $f$ is a global complete intersection. Similarly, the base change formula can be extended to cover also the case of assumption (b), as well as the general case. We refer the reader to [DJK17] for this generality, as well as for more details on the proofs.

Remark 2.2.7. According to these constructions, it is clear that the map associated in Paragraph 2.1.7 with the morphism of ring spectra $\phi$ are natural in $X$ with respect to the basic functoriality and Gysin morphisms of each of the four theories.

2.3. Products and duality.

2.3.1. As usual, one can define a cup-product on cohomology,

$$E^n(X, v) \otimes E^m(X, w) \to E^{n+m}(X, v+w), (x, y) \mapsto x \cup y$$

where, using the presentation of Remark 2.1.4 one defines $x \cup y$ as the map:

$$1_X \xrightarrow{x \otimes y} E_X \otimes \text{Th}(v) \otimes E_X \otimes \text{Th}(w)[n+m] \simeq E_X \otimes E_X \otimes \text{Th}(v+w)[n+m]$$

$$\xrightarrow{\mu \otimes 1d} E_X \otimes \text{Th}(v+w)[n+m].$$

Here we use the fact that $p^*$ is weakly monoidal — as the right adjoint of a monoidal functor; see Paragraph 1.2.3. One can easily check that the pullback morphism on cohomology is compatible with cup product (see for example [Deg14, 1.2.10, (E5)]).

Besides one gets a cap-product:

$$E_{BM}^n(X, v) \otimes E^m(X, w) \to E_{BM}^{n-m}(X, v-w), (x, y) \mapsto x \cap y$$

defined, using again the presentation of Remark 2.1.4 as follows:

$$x \cap y : 1_X[n-m] \xrightarrow{x \otimes y} E'_X \otimes \text{Th}(-v) \otimes E_X \otimes \text{Th}(w) \simeq E'_X \otimes E_X \otimes \text{Th}(-v+w)$$

$$\xrightarrow{\mu'} E'_X \otimes \text{Th}(-v+w).$$

where $\mu'$ is defined in (2.0.1.a).

There is finally a cap-product with support:

$$E_{BM}^n(X, v) \otimes E^m_c(X, w) \to E_{n-m}(X, v-w), (x, y) \mapsto x \cap y$$

defined, using Remark 2.1.4 as follows:

$$x \cap y : 1_k[n-m] \xrightarrow{x \otimes y} p_* (E'_X \otimes \text{Th}(-v)) \otimes p_!(E_X \otimes \text{Th}(w))$$

$$\xrightarrow{(1)} p_! (p^* p_* (E'_X \otimes \text{Th}(-v)) \otimes E_X \otimes \text{Th}(w))$$

$$\xrightarrow{ad} p_! (E'_X \otimes \text{Th}(-v) \otimes E_X \otimes \text{Th}(w)) \simeq p_! (E'_X \otimes E_X \otimes \text{Th}(-v+w))$$

$$\xrightarrow{\mu'} p_! ((E'_X \otimes \text{Th}(-v+w))).$$

where (1) is obtained using the base change isomorphism of Theorem 1.1.10(5) and $\mu'$ is defined in (2.0.1.a).

Remark 2.3.2. These products satisfies projection formulas with respect to Gysin morphisms. We let the formulation to the reader (see also [DJK17]).
Remark 2.3.3. Clearly, the map
\[ \phi_* : E^n(X,v) \to F^n(X,v) \]
defined in \(2.1.4\) is compatible with cup-products. Similarly, the other natural transformations associated with the morphism of ring spectra \(\phi\) are compatible with cap-products.

2.3.4. Fundamental class. Let us fix a smooth \(k\)-variety \(f : X \to \text{Spec}(k)\), with tangent bundle \(\tau_X\).

Applying theorem \(1.1.10(3)\) to \(f\), we get an isomorphism:
\[ p'_f : E_X = f^*(E) \to \text{Th}_X(-T_X) \otimes E'_X \]
Then, in view of Remark 2.1.4, the composite map:
\[ \eta_X : 1_X \xrightarrow{\eta} E_X \xrightarrow{p'_f} \text{Th}_X(-T_X) \otimes E'_X \approx \text{Th}_X(-T_X) \]
corresponds to a class in the Borel-Moore homology group \(E_{BM0}(X,T_X)\).

Definition 2.3.5. Under the assumptions above, we call the class \(\eta_X \in E_{BM0}(X,T_X)\) the fundamental class of the smooth \(k\)-scheme \(X\) with coefficients in \(E\).

The following Poincaré duality theorem is now a mere consequence of the definitions and of part (3) of Theorem \(1.1.10\).

Theorem 2.3.6. Let \(X/k\) be a smooth \(k\)-variety and \(\eta_X\) its fundamental class with coefficients in \(E\) as defined above.

Then the following morphisms
\[ E^n(X,v) \to E_{BM}^n(X,T_X - v), x \mapsto \eta_X \cap y \]
\[ E'_n(X,v) \to E_{BM}^{-n}(X,T_X - v), x \mapsto \eta_X \cap y \]
are isomorphisms.

Proof. Let us consider the first map. Using Remark \(2.1.4\) we can rewrite it as follows:
\[ \text{Hom}_{\text{BM}}(X) (1_X, E_X \otimes \text{Th}(v))[n]) \to \text{Hom}_{\text{BM}}(X) (1_X, E'_X \otimes \text{Th}(v-T_X))[n]. \]
Then it follows from the definition of the fundamental class (Paragraph 2.3.4) and that of cap-products (Paragraph 2.3.1), that this map is induced by the morphism
\[ p'_f : E_X = f^*(E) \to \text{Th}_X(-T_X) \otimes f^!(E) = \text{Th}_X(-T_X) \otimes E'_X \]
which is an isomorphism according to Theorem \(1.1.10(3)\).

The proof in the case of the second isomorphism is the same. \(\square\)

Remark 2.3.7. According to this theorem, and the basic functoriality of the four theories (Paragraph 2.2.1), one gets the following exceptional functoriality for morphisms of smooth \(k\)-varieties:

- cohomology becomes covariant with respect to proper morphisms;
- BM-homology becomes contravariant with respect to all morphisms;
- c-cohomology becomes covariant with respect to all morphisms;
- homology becomes contravariant with respect to proper morphisms.

An application of the projection formulas alluded to in Remark \(2.3.2\) give that this extra functorialities coincide with Gysin morphisms constructed in Proposition \(2.2.6\).
Example 2.3.8. One deduces from the above duality isomorphisms and the localization long exact sequence of Paragraph 2.2.3 that, for a closed immersion \( i : Z \to X \) of smooth \( k \)-varieties with complement open immersion \( j \), one has long exact sequences:

\[
\begin{align*}
\mathbb{E}^n(Z, T_Z - i^{-1}v) & \xrightarrow{i^*} \mathbb{E}^n(X, T_X - v) \xrightarrow{j^*} \mathbb{E}^n(U, j^{-1}(T_X - v)) \to \mathbb{E}^{n+1}(Z, T_Z - i^{-1}v), \\
\mathbb{E}_n(U, j^{-1}(T_X - v)) & \xrightarrow{j^*} \mathbb{E}_n(X, T_X - v) \xrightarrow{i^*} \mathbb{E}_n(Z, T_Z - i^{-1}v) \to \mathbb{E}_{n+1}(U, j^{-1}(T_X - v)).
\end{align*}
\]

Besides, substituting \((T_X - v)\) to \(v\) and using the exact sequence over vector bundles over \(Z\):

\[
0 \to T_Z \to i^{-1}T_X \to N_Z X \to 0
\]

where \(N_Z X\) is the normal bundle of \(Z\) in \(X\), the above exact sequences can be written more simply as:

\[
\begin{align*}
\mathbb{E}^n(Z, i^{-1}v - N_Z X) & \xrightarrow{i^*} \mathbb{E}^n(X, i^{-1}v) \xrightarrow{j^*} \mathbb{E}^n(U, j^{-1}(i^{-1}v - N_Z X)) \\
\mathbb{E}_n(U, j^{-1}(i^{-1}v - N_Z X)) & \xrightarrow{j^*} \mathbb{E}_n(X, i^{-1}v) \xrightarrow{i^*} \mathbb{E}_n(Z, i^{-1}v - N_Z X) \to \mathbb{E}_{n+1}(U, j^{-1}(i^{-1}v - N_Z X)).
\end{align*}
\]

2.4. Descent properties. The following result is a direct application of Paragraph 1.1.11.

Proposition 2.4.1. Consider a cartesian square of \(k\)-schemes:

\[
\begin{array}{c}
W \\
\downarrow g \\
\Delta \\
\downarrow f \\
Y \\
\xrightarrow{i} X
\end{array}
\]

which is Nisnevich or cdh distinguished (see Paragraph 1.1.11). Put \(h = i \circ g\) and let \(v\) be a virtual bundle over \(X\).

Then one has canonical long exact sequences of the form:

\[
\begin{align*}
\mathbb{E}_n(X, v) & \xrightarrow{i^*+f_*} \mathbb{E}_n(Y, i^{-1}v) \oplus \mathbb{E}_n(V, f^{-1}v) \xrightarrow{k^*+g_*} \mathbb{E}_n(W, h^{-1}v) \to \mathbb{E}^{n+1}(X, v) \\
\mathbb{E}_n(W, h^{-1}v) & \xrightarrow{k^*+g_*} \mathbb{E}_n(Y, i^{-1}v) \oplus \mathbb{E}_n(V, f^{-1}v) \xrightarrow{i^*+f_*} \mathbb{E}_n(X, v) \to \mathbb{E}_{n-1}(X, v).
\end{align*}
\]

If the square \(\Delta\) is Nisnevich distinguished (in which case all of its morphisms are étale), one has canonical long exact sequences of the form:

\[
\begin{align*}
\mathbb{E}^n_{BM}(X, v) & \xrightarrow{i^*+f_*} \mathbb{E}^n_{BM}(Y, i^{-1}v) \oplus \mathbb{E}^n_{BM}(V, f^{-1}v) \xrightarrow{k^*+g_*} \mathbb{E}^n_{BM}(W, h^{-1}v) \to \mathbb{E}^{BM}_{n-1}(X, v) \\
\mathbb{E}^n_c(W, h^{-1}v) & \xrightarrow{k^*+g_*} \mathbb{E}^n_c(Y, i^{-1}v) \oplus \mathbb{E}^n_c(V, f^{-1}v) \xrightarrow{i^*+f_*} \mathbb{E}^n_c(X, v) \to \mathbb{E}^{n+1}_c(X, v)
\end{align*}
\]

where we have used the Gysin morphisms with respect to étale maps for BM-homology and c-cohomology.

If the square \(\Delta\) is cdh distinguished (in which case all of its morphisms are proper), one has canonical long exact sequences of the form:

\[
\begin{align*}
\mathbb{E}^n_{BM}(W, h^{-1}v) & \xrightarrow{k^*+g_*} \mathbb{E}^n_{BM}(Y, i^{-1}v) \oplus \mathbb{E}^n_{BM}(V, f^{-1}v) \xrightarrow{i^*+f_*} \mathbb{E}^n_{BM}(X, v) \to \mathbb{E}^{BM}_{n-1}(X, v), \\
\mathbb{E}^n_c(X, v) & \xrightarrow{i^*+f_*} \mathbb{E}^n_c(Y, i^{-1}v) \oplus \mathbb{E}^n_c(V, f^{-1}v) \xrightarrow{k^*+g_*} \mathbb{E}^n_c(W, h^{-1}v) \to \mathbb{E}^{n+1}_c(X, v)
\end{align*}
\]

where we have used the covariance (resp. contravariance) of BM-homology (resp. c-cohomology) constructed in Paragraph 2.2.1.
Proof. The proof is a simple application of the descent properties obtained in Paragraph 1.1.11. For example, one gets the case of cohomology by using the distinguished triangles (1.1.11.c) with $K = E_X$:

$$E_X \xrightarrow{i^* + f^*} i_*(E_Y) \oplus f_*(E_V) \xrightarrow{k^* - g^*} h_*(E_W) \to E_X[1]$$

and applying the cohomological functor $\text{Hom}_{\text{D}^+(X)}(\mathbb{1}_X, -)$. The description of the maps in the long exact sequence obtained follows directly from the description of the contravariant functoriality of cohomology (see 2.2.1).

The other exact sequences are obtained similarly. □

We can state the existence of the long exact sequences in the preceding proposition by saying that the four theories satisfies Nisnevich and cdh cohomological descent. This also shows that, under the existence of resolution of singularities, they are essentially determined by their restriction to smooth $k$-varieties.

Let us make a precise statement.

Proposition 2.4.2. Let us assume $k$ is of characteristic 0 or more generally that any $k$-variety admits a non singular blow-up and that $k$ is perfect.

Suppose one has a contravariant functor $H^*$ from $k$-varieties equipped with a virtual vector bundle $(X, v)$ to graded abelian groups, $H^n(X, v)$, and a natural transformation:

$$\phi_X : E^n(X, v) \to H^n(X, v)$$

such that for any cdh distinguished square as in the preceding proposition, one has a long exact sequence:

$$H^n(X, v) \xrightarrow{i^* + f^*} H^n(Y, i^{-1}v) \oplus H^n(V, f^{-1}v) \xrightarrow{k^* - g^*} H^n(W, h^{-1}v) \to H^{n+1}(X, v)$$

which is compatible via $\phi$ with the one for $E^*$. Then, if $\phi_X$ is an isomorphism when $X/k$ is smooth, it is an isomorphism for any $k$-variety $X$.

Proof. The proof is an easy induction on the dimension of $X$. When $X$ has dimension 0, it is necessarily smooth over $k$ as the latter field is assumed to be perfect. The noetherian induction argument follows from the existence of a blow-up $f : V \to X$ such that $V$ is smooth. Let $Y$ be the locus where $f$ is not an isomorphism, $W = V \times_X Y$. Then the dimension of $Y$ and $W$ is strictly less than the dimension of $X$ and $V$. By assumptions, $\phi_Y$ is an isomorphism. By the inductive assumption, $\phi_Y$ and $\phi_W$ are isomorphisms. So the existence of the cdh descent long exact sequences, and the fact $\phi$ is compatible with these, allow to conclude. □

Remark 2.4.3. Similar uniqueness statements, with the same proof, hold for the other three theories. We let the formulation to the reader.

---

One can express cohomological descent for the cdh or Nisnevich topology in the style of [AGV73, Vbis], or [CD09, §3], using the fact the four theories admits an extension to simplicial schemes and stating that cdh or Nisnevich hypercovers induces isomorphisms. The simplification of our formulation comes as these topologies are defined by cd-structure in the sense of [Voe10].
3. The MW-motivic ring spectrum

3.1. The ring spectra.

3.1.1. We will now apply the machinery described in the preceding section to MW-motives. Recall from our notations that $k$ is now a perfect field and $R$ is a ring of coefficients.

Recall also from [DF17] (3.3.6.a) that we have adjunctions of triangulated categories:

\[
\begin{array}{cccc}
D_{k}(k, R) & \overset{L_{\gamma+}}{\longrightarrow} & \sim \text{DM}(k, R) & \overset{L_{\pi+}}{\longrightarrow} & \text{DM}(k, R) \\
\text{RO} & \text{RO} & \text{RO} & \text{RO} & \text{RO}
\end{array}
\]

(3.1.1.a)

\[
\begin{array}{cccc}
D_{k, \text{ét}}(k, R) & \overset{L_{\gamma+\text{ét}}}{\longrightarrow} & \sim \text{DM}_{\text{ét}}(k, R) & \overset{L_{\pi+\text{ét}}}{\longrightarrow} & \text{DM}_{\text{ét}}(k, R) \\
\text{RO} & \text{RO} & \text{RO} & \text{RO} & \text{RO}
\end{array}
\]

such that each left adjoint is monoidal. Therefore, one can apply the general procedure of Paragraph 1.2.3 to deduce ring spectra from these adjunctions.

Definition 3.1.2. Consider the above notations.

We define respectively the $R$-linear MW-spectrum, étale MW-spectrum, motivic Eilenberg-MacLane spectrum and étale Eilenberg-MacLane spectrum as follows:

\[
\begin{align*}
\text{H}_{\text{MW}} & : = \rho_{*}^{R}\gamma_{*}(1), \\
\text{H}_{\text{MW, ét}} & : = \rho_{*}^{R}\text{RO}\gamma_{*\text{ét}}(1), \\
\text{H}_{M} & : = \rho_{*}^{R}\gamma_{*}\pi_{*}(1), \\
\text{H}_{M, \text{ét}} & : = \rho_{*}^{R}\text{RO}\gamma_{*\text{ét}}\pi_{*}(1).
\end{align*}
\]

3.1.3. Each of these ring spectrum represents the corresponding cohomology on smooth $k$-schemes. This follows by adjunction using Example 2.1.3. Explicitly, for a smooth $k$-scheme $X$ and integer $(n, m) \in \mathbb{Z}^{2}$, one gets:

\[
\begin{align*}
\text{H}_{\text{MW}}^{n}(X, m, R) & = \text{Hom}_{D_{k}}(k) \left( \Sigma^{\infty} \mathbb{Z}_{k}(X), \rho_{*}^{R}\gamma_{*}(1)(m)[n + 2m] \right) \\
& = \text{Hom}_{\text{DM}(k, R)}(\tilde{M}(X), 1(m)[n + 2m]), \\
& = \text{H}_{\text{MW}}^{n+2m}(X, R)
\end{align*}
\]

where the first identification follows from Example 2.1.3; the second by adjunction — here, $\tilde{M}(X)$ denotes the MW-motive associated with the smooth $k$-scheme $X$, following the notation of [DF17]. The last group was introduced in [DF17] 4.1.1.

Similarly we get:

\[
\begin{align*}
\text{H}_{M}^{n}(X, m, R) & = \text{Hom}_{\text{DM}(k, R)}(M(X), 1(m)[n + 2m]), \\
\text{H}_{\text{MW, ét}}^{n}(X, m, R) & = \text{Hom}_{\text{DM}_{\text{ét}}(k, R)}(M_{\text{ét}}(X), 1(m)[n + 2m]), \\
\text{H}_{M, \text{ét}}^{n}(X, m, R) & = \text{Hom}_{\text{DM}_{\text{ét}}(k, R)}(\tilde{M}_{\text{ét}}(X), 1(m)[n + 2m])
\end{align*}
\]

where $M(X)$ (resp. $M_{\text{ét}}(X)$, $\tilde{M}_{\text{ét}}(X)$) denotes the motive (resp. étale motive, étale MW-motive) associated with the smooth $k$-scheme $X$.

Remark 3.1.4. (1) The Eilenberg-MacLane motivic ring spectrum defined here, apart the fact that we work in $D_{k}(X)$, agrees with the one defined by Voevodsky ([Voe98] 6.1); see also [CD09] 11.2.17).
(2) The use of the functor $\rho^R_*$ in the above definition can appear rather artificial. We could have equally worked within the $R$-linear $A^1$-derived category $D^{A^1}(X, R)$.

(3) As in algebraic topology, one derive from the Dold-Kan equivalence an adjunction of triangulated categories:

$$N : \text{SH}(k) \rightleftarrows D^{A^1}(k) : K$$

see [CD09, 5.3.35]. The functor $N$ is monoidal (loc. cit.) so that using the arguments of Paragraph 1.2.3, the functor $N$ is weakly monoidal. Therefore, if one apply $K$ to any ring spectrum of the above definition, one obtain a commutative monoid in the stable homotopy category $\text{SH}(k)$. From the point of view of the cohomology theory (and actually, the four theories), this does not change anything as the corresponding object do represent the same cohomology. This is why we have restricted our attention here to the category $D^{A^1}(k)$. Though it does not change anything for our purpose, it is important to know that we have also constructed a ring spectrum in $\text{SH}(k)$ representing MW-motivic cohomology.

3.2. Associated cohomology.

3.2.1. MW-cohomology. Applying Definition 2.1.1, we can associate to the preceding ring spectra four cohomological/homological theories.

Our main interest is in the MW-motivic cohomology spectrum. First considering the associated cohomology theory, our definition allows to extend the definition of the MW-cohomology theory of [DF17, 4.1.1], with its product, to the case of possibly singular $k$-schemes. In characteristic 0, this extension is the unique one satisfying cdh descent (see Proposition 2.4.2). And finally, we have defined Gysin morphisms on cohomology, with respect to proper morphisms of smooth $k$-varieties (or proper smooth morphisms of arbitrary $k$-schemes); see Proposition 2.2.5.

Recall finally the results of [DF17], assuming that $k$ is infinite (in addition to be perfect). For a smooth $k$-scheme $X$ and a couple of integers $(n, m) \in \mathbb{Z}^2$ we have the following computations:

$$H^n_{MW}(X, m, \mathbb{Z}) = \begin{cases} \widetilde{CH}^m(X) & \text{if } n = 0, \\ H^0_{\text{Zar}}(X, K^0_{MW}) & \text{if } m = 0, \\ H^{n+2m}_{\text{Zar}}(X, \tilde{Z}(m)) & \text{if } m > 0, \\ H^{n+m}_{\text{Zar}}(X, W) & \text{if } m < 0. \end{cases}$$

See [DF17] Cor. 4.2.6, Prop. 4.1.2].

Example 3.2.2. Motivic cohomology (smooth case). Recall the following classical computations. For a smooth $k$-variety $X$ and a virtual bundle $v$ over $X$ of rank $m$, one has:

$$H^n_M(X, v, \mathbb{Z}) = \begin{cases} \widetilde{CH}^m(X) & \text{if } n = 0, \\ H^{n+2m}_{\text{Zar}}(X, Z(m)) & \text{if } m \geq 0, \\ 0 & \text{if } m < 0, \end{cases}$$

where $Z(m)$ is Suslin-Voevodsky motivic complex. Note that the computation uses the fact that motivic cohomology is an oriented cohomology theory (see for example [Dég14, 2.1.4(2)]).
Motivic cohomology (singular case). For any $k$-scheme $f : X \to \text{Spec} k$, following the conventions of Paragraph 2.0.1 we consider the following ring spectrum over $X$:

$$H_{M,X} := f^*(H_M).$$

Apart from the fact that we are working in $D^A_1(X)$ instead of $SH(X)$, this ring spectrum agrees with the ring spectrum defined in [CD15, 3.8].

Assume further that:

(3.2.3.a) The characteristic exponent of $k$ is invertible in $R$.

Then according to [CD15], one can extend the construction of the category $DM(k, R)$ to an arbitrary $k$-scheme $X$ and obtain a triangulated $R$-linear category $DM_{cdh}(X, R)$ which satisfies the six functors formalism for various $X$, as described in Section 1.1 (see [CD15, 5.11]). Indeed, $DM_{cdh}(-, R)$ form what we called a motivic triangulated category in [CD09, 2.4.45] — here, the assumption (3.2.3.a) is essential. Note in particular that we can define, as in Paragraph 1.1.5, Thom motives of virtual bundles (see [CD09, 2.4.15]). Given a virtual bundle $v$ over $X$, we denote by $M_{Th}(v)$ this Thom motive, an object of $DM_{cdh}(X, R)$.

Moreover, according to [CD09, 11.2.16], one has a natural adjunction of triangulated categories:

$$\gamma^* : D^A_1(X, R) \to DM_{cdh}(X, R) : \gamma_*$$

extending the adjunction $(L\pi^*L\tilde{\gamma}^*, \gamma_*\pi_*)$. In fact, the family of adjunctions $(\gamma^*, \gamma_*)$ for various schemes $X$ form what we called a premotivic adjunction in [CD09, 1.4.6].

As a consequence, $\gamma^*$ is monoidal and commutes with functors of the form $f^*$ and $p_!$ while $\gamma_*$ commutes with functors of the form $f_*$ and $p^!$ (see [CD09, 2.4.53]); here $f$ is any morphism of $k$-schemes while $p$ is separated of finite type.

Note that by construction, $\gamma^*(Th(v)) = M Th(v)$. As the objects $Th(v)$ and $M Th(v)$ are $\otimes$-invertible, one deduces that

$$\gamma_*(M Th(v)) = Th(v).$$

Finally, applying [CD15] Prop. 4.3 and Th. 5.1, we also obtain that $\gamma_*$ commutes with $f^*$. As a consequence

$$H_{M,X} = f^*(\gamma_*(\mathbb{1}_X)) \simeq \gamma_*(f^*(\mathbb{1}_X)) = \gamma_*(\mathbb{1}_X).$$

The argument goes as follows. Consider the functors:

$$Th(v) : K \mapsto Th(v) \otimes K, Th^M(v) : K \mapsto M Th(v) \otimes K.$$}

Then we obtain an isomorphism of functors:

$$\gamma^* \circ Th(v) \simeq Th^M(v) \circ \gamma^*.$$}

Now, as the Thom objects are $\otimes$-invertible, the functors $Th(v)$ and $Th^M(v)$ are equivalences of categories. Their quasi-inverses are respectively: $Th(-v)$ and $Th^M(-v)$. Then from the preceding isomorphism of functors, one deduces an isomorphism of the right adjoint functors:

$$Th(-v) \circ \gamma_* \simeq \gamma_* \circ Th^M(-v).$$

Note this identification is by no means obvious. In fact, it answers a conjecture of Voevodsky (cf. [Voe02 Conj. 17]) in the particular case of the base change map $f : X \to \text{Spec}(k)$. See also [CD15] 3.3, 3.6.
Finally, under assumption (3.2.3.a), we can do the following computation for an arbitrary $k$-scheme $X$ and a virtual vector bundle $v$ over $X$ of rank $m$:

\[
H^n_{\mathcal{M}}(X, v, R) = \text{Hom}_{\mathcal{DA}}(\mathcal{M}(X, v, R), \mathcal{M}(X)) = \text{Hom}_{\mathcal{CDH}}(\mathcal{M}(X, v, R), \mathcal{M}(X)) = \text{Hom}_{\mathcal{DA}}(\mathcal{M}(X, v, R), \mathcal{M}(X))
\]

which uses the identifications recalled above, and for the last one, the fact that motivic cohomology is an oriented cohomology theory (equivalently, $\mathcal{DA}$ is an oriented motivic triangulated category, see [CD09, 2.4.38, 2.4.40, 11.3.2].

We can give more concrete formulas as follows. Assume $X$ is a $k$-variety. Recall Voevodsky has defined in [VSF00, chap. 5, §4.1] a motivic complex $C_\ast(X)$ in $\mathcal{DA}_{\text{eff}}(k, \mathbb{Z})$ by considering the Suslin complex of the sheaf with transfers $L(X)$ represented by $X$. With $R$-coefficients, let us put:

\[
C_\ast(X)_R := C_\ast(X) \otimes \mathbb{Z}^L R.
\]

Then according to [CD15, 8.4, 8.6], one gets:

\[
H^n_{\mathcal{M}}(X, v, R) = \begin{cases} 
\text{Hom}_{\mathcal{DA}}(C_\ast(X)_R, R(m)[n + 2m]) & \text{if } m \geq 0, \\
0 & \text{if } m < 0
\end{cases}
\]

where $R(m)$ is the $R$-linear Tate motivic complex: $R(m) = C_\ast(\mathbb{G}_m^\wedge)^{R}[−m]$. Note also that one can compute the right hand side as the following cdh-cohomology group (see [CD09 (8.3.1)]:

\[
\text{Hom}_{\mathcal{DA}}(C_\ast(X)_R, R(m)[n + 2m]) = H_{\mathcal{DA}}(X, R(m))
\]

where $R(m)$ is seen as a complex of cdh-sheaves on the site of $k$-schemes of finite type.

3.3. Associated Borel-Moore homology.

3.3.1. Borel-Moore motivic MW-homology. We also get the Borel-Moore MW-homology of $k$-varieties, covariant (resp. contravariant) with respect to proper (resp. étale) maps, satisfying the localization long exact sequence (see Paragraph 2.2.3) and contravariant for any smooth maps or arbitrary morphisms of smooth $k$-varieties (Gysin morphisms, Proposition 2.2.5).

Besides, using the duality theorem 2.3.6, we get for a smooth $k$-scheme $X$ with tangent bundle $T_X$ and any couple of integers $(n, m) \in \mathbb{Z}^2$ the following computation:

\[
H^n_{\text{MW}}(X, T_X - m, R) = H^{2m-n, m}_{\text{MW}}(X, R)
\]

with the notation of [DF17].

3.3.2. Borel-Moore motivic homology. Let us consider the situation of Paragraph 3.2.3. We assume further that $R$ is a localization of $\mathbb{Z}$ satisfying condition (3.2.3.a).
Then we can compute Borel-Moore motivic homology, for a \( k \)-variety \( f : X \to \text{Spec}(k) \) and a virtual bundle \( v/X \) of rank \( m \) as follows:

\[
H_{n, \text{BM}}^M(X, v, R) = \text{Hom}_{D_{A^1}(X, R)} \left( 1_X, f^! (H_{M}R) \otimes \text{Th}(-)[-n] \right)
\]

\[
= \text{Hom}_{D_{A^1}(X, R)} \left( 1_X, f^! (\gamma_*(1_k)) \otimes \text{Th}(-v)[-n] \right)
\]

\[
(1) \quad = \text{Hom}_{D_{A^1}(X, R)} \left( 1_X, \gamma_*(f^! (1_k) \otimes M \text{Th}(-v)[-n]) \right)
\]

\[
= \text{Hom}_{D_{A^1}(X, R)} \left( 1_X, f^! (1_k) \otimes M \text{Th}(-v)[-n] \right)
\]

\[
(2) \quad = \text{Hom}_{D_{A^1}(X, R)} \left( 1_X, f^! (1_k)(-m)[-2m-n] \right)
\]

\[
(3) = \text{CH}_m(X, n) \otimes \mathbb{Z} R
\]

where (1) follows from the properties mentioned in Paragraph 3.2.3, (2) as \( DM_{\text{cdh}} \) is oriented and (3) using [CD15, Cor. 8.12].

3.4. Generalized regulators.

3.4.1. As explained in Paragraph 1.2.7, the commutativity of Diagram (3.1.1.a) automatically induces morphisms of ring spectra as follows:

\[
\begin{array}{ccc}
H_{A^1 R} & \xrightarrow{\psi} & H_{MW R} \\
\downarrow & & \downarrow \\
H_{MW, \text{ét}} R & \xrightarrow{\varphi} & H_{M, \text{ét}} R \\
\end{array}
\]

As explained in Paragraph 2.1.7 and Remarks 2.2.7, 2.3.3 these morphisms induce natural transformations of the four associated theories, compatible with products.

In particular, given a \( k \)-scheme \( X \) and virtual bundle \( v \) over \( X \) of rank \( m \), we get morphisms

\[
H_{n, A^1}(X, v, R) \xrightarrow{\psi_*} H_{n, MW}(X, v, R) \xrightarrow{\varphi_*} H_{n+2m, M}(X, R)
\]

where the right hand side is Voevodsky’s motivic cohomology (see 3.2.3). In brief, these maps are compatible with all the structures on cohomology described in Section 2.

Assume finally the conditions of Paragraph 3.3.2 are fulfilled. Then we get natural morphisms:

\[
H_{n, BM, A^1}(X, v, R) \xrightarrow{\psi_*} H_{n, MW, BM}(X, v, R) \xrightarrow{\varphi_*} \text{CH}_m(X, n) \otimes \mathbb{Z} R,
\]

compatible with contravariant and covariants functorialities, and localization long exact sequences (Paragraph 2.2.3).

Of course, when \( X \) is a smooth \( k \)-variety, the two maps \( \varphi_* \) (resp. \( \psi_* \)) that appear above can be compared by duality (Theorem 2.3.6). Besides, if \( v = [m] \) and \( n = 0 \), one can check the latter map \( \varphi_* \) is simply the canonical map:

\[
\overline{\text{CH}}_m(X) \otimes \mathbb{Z} R \to \text{CH}_m(X) \otimes \mathbb{Z} R
\]

from Chow-Witt groups to Chow groups.
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