Meander Determinants

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We prove a determinantal formula for quantities related to the problem of enumeration of (semi-) meanders, namely the topologically inequivalent planar configurations of non-self-intersecting loops crossing a given (half-) line through a given number of points. This is done by the explicit Gram-Schmidt orthogonalization of certain bases of subspaces of the Temperley-Lieb algebra.
1. Introduction

The meander problem consists in counting the number $M_n$ of meanders of order $n$, i.e. of inequivalent configurations of a closed non-self-intersecting loop crossing an infinite line through $2n$ points. The infinite line may be viewed as a river flowing from east to west, and the loop as a closed circuit crossing this river through $2n$ bridges, hence the name "meander", although here the river and the road play symmetric roles. Two configurations are considered as equivalent if they are smooth deformations of one another. The meander problem probably first arose in the work of Poincaré about differential geometry. Since then, it has emerged in many different contexts, such as mathematics, physics, computer science and even fine arts. The problem was recently reactualized by Arnold, in relation with Hilbert’s 16th problem. Meanders also emerged in the classification of 3-manifolds. More recently, random matrix model techniques, borrowed from quantum field theory, were applied to this problem.

In the present paper, we rather adopt the purely algebraic approach advocated in [10], based on a pictorial representation of the elements of the Temperley-Lieb algebra (see also P. Martin’s book for an elementary introduction) using strings (each element is a sort of domino with string-ends on its boundary, and elements are multiplied by connecting the string-ends of the corresponding dominos), by means of which the meanders are constructed. A particular basis (set of basic dominos) of the Temperley-Lieb algebra will provide us with the building blocks for the construction of meanders, or some of their generalizations, the semi-meanders, introduced in [8]. Roughly speaking, a (multi-component, i.e. made of possibly several non-intersecting roads) meander is obtained as the concatenation of two dominos, and the identification of their free string-ends: this is exactly the manipulation involved when evaluating the standard bilinear form of the Temperley-Lieb algebra on these two dominos. Moreover, the value of the bilinear form is simply $q^L$, $q$ a given complex number, and $L$ the total number of loops formed by the connection of the strings, namely the number of loops in the corresponding meander. The Theorem 1 below is a formula expressing the determinant of the Gram matrix of this basis of the Temperley-Lieb algebra, and was first derived in [10] (an algorithm for its computation was also given in [3]).

By choosing a particular subset of dominos (hence a particular basis of some subspace of the Temperley-Lieb algebra), we may obtain meanders with more specific details: the semi-meanders with fixed winding numbers are some of these. To obtain the latter, consider
the following semi-meander problem: enumerate the inequivalent planar configurations of a loop crossing a half-line through a given number of points $n$. The loop of such a semi-meander may freely wind around the origin of the half-line (interpreted as the source of the river), and we can define a winding number associated to this. The (multi-component, i.e. with possibly several non-intersecting roads) semi-meanders with fixed winding number $w$ may be obtained as the concatenation of particular elements (dominos) of the Temperley-Lieb algebra, namely those with exactly $w$ of the $n$ strings going across the domino. The Theorem 2 below is the generalization to semi-meanders of the abovementioned meander determinant formula. The latter was only conjectured in [10], in a slightly different form.

The paper is organized as follows. In Sect.2, we recall a few definitions and facts on (semi-) meanders, in particular their various formulations as (i) superpositions of two (open) arch configurations (ii) superposition of two (open) walk diagrams, and we state the main results of the paper (Theorems 1 and 2), in the form of determinant formulas.

Sect.3 is devoted to the proof of the formula for the meander determinant. The proof relies on the interpretation of any meander as the product of two elements of the Temperley-Lieb algebra. After displaying the various mappings between arch configurations, walk diagrams and reduced elements of the Temperley-Lieb algebra, we reformulate the meander determinant as the Gram determinant of a particular basis of the Temperley-Lieb algebra, or rather of one of its ideals. The proof is then carried out, by performing the explicit Gram-Schmidt orthogonalization of this basis (Proposition 1). This appears in fact as the consequence of a stronger statement regarding the orthogonalization of all the products of any two basis elements (Lemma 1). The computation of the meander determinant is then a combinatorial exercise (Proposition 2) in rearranging all the normalization factors introduced in the orthonormalization process, which we carry out by performing some mapping of decorated walk diagrams.

In Sect.4, we turn to the semi-meander generalization. We follow the same strategy as in Sect.3, with a number of complications, due to the fact we now deal with a subspace of the Temperley-Lieb algebra, which is not an ideal, i.e. has no good multiplication properties between its elements. Nevertheless, we are still able to perform the explicit Gram-Schmidt orthogonalization of our initial basis (Proposition 3 and Lemmas 2, 3, 4). The determinant formula then follows from a rearrangement (Propositions 4, 5 and 6) of the normalization factors introduced in the orthonormalization process.

We gather in Sect.5 a few concluding remarks.
2. Meander determinants: the results

2.1. Arch configurations and meanders

A meander of order \(2n\) is a planar configuration of a closed non-self-intersecting loop (road) crossing a line (river) through \(2n\) distinct points (bridges), considered up to smooth deformations preserving the topology of the configuration (i.e., preserving the succession of bridges).

![Diagram of meander configurations](image)

**Fig. 1:** Any meander is obtained as the superimposition of a top (a) and bottom (b) arch configurations of same order \((2n = 10\) here). An arch configuration is a planar pairing of the \((2n)\) bridges through \(n\) non-intersecting arches lying above the river (by convention, we represent the lower configuration \(b\) reflected with respect to the river: this will actually be denoted by \(b^t\) in the following).

The river separates the meander into an upper and a lower planar configuration of \(n\) non-intersecting pieces of road (arches) joining the \(2n\) bridges by pairs (see Fig.1 for an example), respectively contained in the upper and lower half-planes defined by the river. Such a configuration, considered up to the abovementioned equivalence, is called an arch configuration of order \(2n\). Let \(A_{2n}\) denote the set of all arch configurations of order \(2n\).

Let us label the bridges from left to right \(1, 2, ..., 2n\). The total number \(c_n\) of arch configurations of order \(n\) is obtained by considering the leftmost arch, joining say the bridge \(1\) to the bridge \(2j\), \(1 \leq j \leq n\). This arch separates the arch configuration into two pieces: the portion below the leftmost arch, and that to the right of this arch. These are

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1 In this paper, the order will always refer to the total number of bridges in the configuration. A different convention was adopted in refs. [8] [10], where the order of a meander is rather half its number of bridges.
two arbitrary arch configurations of respective orders \(2(j-1)\) and \(2(n-j)\). Hence we have \(c_n = \sum_{1 \leq j \leq n} c_{j-1}c_{n-j}\). With \(c_0 = 1\), we get

\[
|A_{2n}| = c_n = \frac{(2n)!}{(n+1)!n!}
\]

which is the Catalan number of order \(n\).

Superposing two arbitrary arch configurations \(a, b \in A_{2n}\) (after a reflection of \(b\) w.r.t. the river) will in general lead to a multi-component meander, made of several non-intersecting roads. We denote by \(\kappa(a|b)\) the corresponding number of connected components.

2.2. Open arch configurations and semi-meanders

A semi-meander of order \(n\) is a planar configuration of a non-self-intersecting loop (road) crossing a half-line (river with a source) through \(n\) distinct points (bridges), up to smooth deformations of the road preserving the topology of the configuration. The main difference with a meander is that the road may now freely wind around the source of the river, therefore the number of bridges needs not be an even integer. We define the winding number of a semi-meander to be the number of pairs of bridges linked by an arch encircling the source of the river (each such arch contributes 1 to the total winding number). Note that a semi-meander of order \(n\) may only have a winding number \(h = n \mod 2\).

![Diagram](image)

**Fig. 2:** Any semi-meander may be viewed as the superimposition of an upper and a lower open arch configurations. Here the initial semi-meander has order \(n = 5\) and winding \(h = 3\). The two open arch configurations on the right have \(h = 3\) open arches. To recover the initial semi-meander, these open arches must be connected two by two, from the right to the left (the arches number 5,4,3 of the upper configuration are respectively connected to the arches number 3,2,1 of the lower configuration).
In any given semi-meander with winding number \( h \), the river still separates the configuration into an upper and a lower one (see Fig. 2 for an example), corresponding to the portion lying respectively above and below the river, but these are linked by \( h \) arches encircling the source of the river, and connecting \( h \) upper parts of bridges to \( h \) lower parts of bridges. Let us cut these \( h \) arches and extend them so as to form \( h \) vertical half-lines on the upper configuration and \( h \) vertical half-lines on the lower configuration. The resulting objects are called \textit{open arch configurations} of order \( n \) with \( h \) open arches. Such an open arch configuration is formed by a line with \( n \) distinct points (upper half-bridges) either connected by pairs through arches in the upper half-plane (there are \((n-h)/2\) such arches), or connected to “infinity” through a vertical half-line (there are \( h \) such open arches), with a total of \((n+h)/2\) arches. We denote by \( A_n^{(h)} \) the set of open arch configurations of order \( n \) with \( h \) open arches. Note again that \( h = n \mod 2 \). In particular, \( A_{2n}^{(0)} = A_{2n} \).

To compute the cardinal \( c_{n,h} \) of \( A_n^{(h)} \), we concentrate again on the leftmost arch of a given configuration. Two cases may occur:

(i) This arch is open. The configuration lying on the right of this arch is an arbitrary open arch configuration of order \( n-1 \) with \( h-1 \) open arches.

(ii) This arch connects the bridges 1 and say \( 2j \), thus separating the configuration into two parts: the one below the leftmost arch is an arbitrary arch configuration of order \( 2(j-1) \), whereas the one to the right of bridge \( 2j \) is an arbitrary open arch configuration of order \( n-2j \) with \( h \) open arches.

These are summarized in the following recursion relation

\[
c_{n,h} = c_{n-1,h-1} + \sum_{j=1}^{[n/2]} c_{j-1} c_{n-2j,h}
\]

where \( c_n \) denotes the Catalan number \( (2.1) \), and \([x]\) is the largest integer smaller or equal to \( x \). With the initial condition \( c_{2n,0} = c_n \) for all \( n \geq 0 \), this determines the numbers \( c_{n,h} \) completely, and we have

\[
|A_n^{(h)}| = c_{n,h} = \binom{n}{n-h/2} - \binom{n}{n-h/2 -1}
\]

where \( c_{n,h} \) are some generalized Catalan numbers. (Note again that \( A_n^{(h)} \) is only defined if \( n = h \mod 2 \)).

\( ^2 \) As before, the order refers to the total number of bridges in the configuration.
Like in the meander case, given two arbitrary open arch configurations \( a, b \in A_n^{(h)} \), we may consider their superposition (after reflecting \( b \) w.r.t. the river) obtained by gluing their half-bridges, and connecting the upper and lower open arches starting from the rightmost one so as to form \( h \) arches encircling the source of the river. This leads in general to a multi-component semi-meander formed of possibly many non-intersecting roads crossing the river, and possibly winding around its source, with a \textit{total} winding number \( h \). By analogy with the meander case, we still denote by \( \kappa(a|b) \) the resulting number of connected components.

2.3. Meander and semi-meander determinants

With the above definitions, let us introduce, for any given complex number \( q \), the meander and semi-meander matrices \( G_{2n}(q) \) and \( G_{n}^{(h)}(q) \) of respective sizes \( c_n \times c_n \) and \( c_{n,h} \times c_{n,h} \), with entries

\[
[G_{2n}(q)]_{a,b} = q^{\kappa(a|b)} \quad a, b \in A_{2n} \\
[G_{n}^{(h)}(q)]_{a,b} = q^{\kappa(a|b)} \quad a, b \in A_{n}^{(h)}
\]  

(2.4)

As \( A_{2n}^{(0)} = A_{2n} \), we have \( G_{2n}^{(0)}(q) = G_{2n}(q) \), hence the meander matrix is just a particular case of semi-meander matrix with \( h = 0 \) winding number. Nevertheless, for a clearer exposition, we will distinguish between the two cases.

Let us denote by \( U_m(q) \) the Chebishev polynomials of the first kind, namely such that

\[
U_m(2 \cos \theta) = \frac{\sin(m + 1) \theta}{\sin \theta}
\]

(2.5)

for all \( m \geq 0 \). With this definition, and the integer numbers \( c_{n,m} \) defined in (2.3), we have the following compact formulas for the determinants of the matrices \( G_{2n}(q) \) and \( G_{n}^{(h)}(q) \)

\textbf{THEOREM 1:}

\[
\det \left[ G_{2n}(q) \right] = \prod_{m=1}^{n} \left[ U_m(q) \right]^{a_{2n,2m}}
\]

(2.6)

\[ a_{2n,2m} = c_{2n,2m} - c_{2n,2m+2} \]
THEOREM 2:

\[
\det \left[ G_n^{(h)}(q) \right] = \prod_{m=1}^{n-h+1} \left[ U_m(q) \right] a_{n,m}^{(h)}
\]

\[
a_{n,m}^{(h)} = c_{n,2m+h} - c_{n,2m+2+h} + h(c_{n,2m+h-2} - c_{n,2m+h})
\]

The theorem 1 was proved in [10], whereas the formula (2.7) was only conjectured there (in a slightly different, but equivalent form). In the following, for pedagogical reasons, we will first give a simplified proof of the theorem 1, in the same spirit as [10]. We will then show how to generalize this proof to that of the theorem 2. Clearly, the theorem 2 contains the theorem 1 as the particular case \( h = 0 \). Before turning to the proofs of theorems 1 and 2 above, we wish to provide the reader with an alternative picture for (open or closed) arch configurations, which will prove useful in the following. The idea is to view an (open or closed) arch configuration of order \( n \) as a walk of \( n \) steps on a half-line.

2.4. Arch configurations and closed walk diagrams

There is a bijection between the arch configurations of order \( 2n \) and the closed paths of \( 2n \) steps on a half-line, or rather their two-dimensional extent, which we call a walk diagram of \( 2n \) steps. The mapping goes as follows. Let us index by \( i \) the portion of river inbetween two consecutive bridges \( i \) and \( i + 1 \), \( 1 \leq i \leq 2n - 1 \), by 0 the portion to the left of the first bridge, and by \( 2n \) the portion to the right of the last bridge.

![Walk diagram of 18 steps](image)

**Fig. 3:** A walk diagram of 18 steps, and the corresponding arch configuration. Each dot corresponds to a segment of river. The height on the walk diagram is given by the number of arches intersected by the vertical dotted line.
To each of these we associate the height $h(i)$ equal to the number of arches passing at the vertical of the corresponding portion of river. With this definition, we have $h(0) = h(2n) = 0$, $h(i) - h(i - 1) = \pm 1$, according to whether an arch originates or terminates at the bridge $i$, and $h(i) \geq 0$ for all $i$. The function $h(i)$ can be thought of as the coordinate of a walker on the half-line after $i$ steps. The two-dimensional extent of the trajectory is simply obtained by joining the consecutive points $(i, h(i))$ i.e., by plotting the graph of the function $h$, as illustrated in Fig.3.

We denote by $W_{2n}$ the set of such walk diagrams of $2n$ steps, with $h(0) = h(2n) = 0$. In particular, the bijection implies that $|W_{2n}| = |A_{2n}| = c_n$ for all $n \geq 0$. In the following, we will denote indifferently by the same letter $a \in A_{2n}$ or $W_{2n}$ an arch configuration or the corresponding (closed) walk diagram.

2.5. Open arch configurations and open walk diagrams

For all $h \leq n$, $h = n \mod 2$, there is a bijection between the set $A_n^{(h)}$ of open arch configurations of order $n$ with $h$ open arches and the set of open walk diagrams on a half-line, starting at the origin and ending at height $h$ after $n$ steps.

![Fig. 4: An open walk diagram of $n = 14$ steps with final height $h = 4$, and the corresponding open arch configuration. The height on the walk diagram is given by the number of arches intersected by the dotted lines, plus that of open arches lying on the left of the point considered.](image)

Starting from some open arch configuration $a \in A_n^{(h)}$, let us label as before by 0, 1, ..., $n$ the portions of river inbetween consecutive bridges of $a$ (including that to the left of the first bridge, 0 and to the right of the last bridge, $n$). To each of these, we associate
the *height* $h(i)$ equal to the number of arches passing at the vertical of the corresponding portion of river, *plus* the total number of open arches originating from the bridges number 1, 2, ..., $i$, namely the total number of open arches lying to the left of the portion $i$ of river. With this definition, the function $i \rightarrow h(i)$ satisfies

$$ h(0) = 0, \quad h(n) = h, \quad h(i) \geq 0, \quad \text{and} \quad h(i+1) - h(i) = \pm 1 \quad (2.8) $$

according to whether an (open or closed) arch originates from the bridge $i$ or a (closed) arch terminates at the bridge $i$. The function $i \rightarrow h(i)$, satisfying the properties (2.8), defines a unique walk on the half-line, starting at the origin (height $h(0) = 0$), and ending after $n$ steps of $\pm 1$ in height at height $h(n) = h$. The graph of the function, $(i, h(i))$, with consecutive points linked by segments of line, is the two-dimensional extent of such a walk, which we call an *open walk diagram* of $n$ steps with final height $h$.

We denote by $W_n^{(h)}$ the set of open walk diagrams of $n$ steps with final height $h$ (note that this is only defined for $h = n \mod 2$). In particular, the above bijection implies

$$ |W_n^{(h)}| = |A_n^{(h)}| = c_{n,h} \quad (2.9) $$

In the following, we will also use indifferently the same letter $a$ to denote an element of $W_n^{(h)}$ or $A_n^{(h)}$ whichever picture is most convenient.

### 3. The meander determinant: proof of theorem 1

In this section, we give a detailed proof of theorem 1. We first recall the equivalence between arch configurations and reduced elements of the Temperley-Lieb algebra $TL_n(q)$, or rather a certain left ideal $I_n(q)$ of $TL_{2n}(q)$, isomorphic to $TL_n(q)$. In the latter language, the meander matrix $G_{2n}(q)$ (2.4) is interpreted as the Gram matrix of a basis (called basis 1) of $I_n(q)$ with respect to the standard bilinear form. The determinant of $G_{2n}(q)$ will be a by-product of the orthogonalization of this matrix.

#### 3.1. Temperley-Lieb algebra and arch configurations

The arch configurations of order $2n$ have a direct interpretation in terms of reduced elements of the Temperley-Lieb algebra $TL_n(q)$, for a given complex number $q$. The latter
is best expressed in its pictorial form, as acting on a “comb” of $n$ strings, with the $n$ generators $1, e_1, e_2, \ldots, e_{n-1}$ defined as

$$1 = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
1 \quad 1 \quad 1 \\
\vdots \\
\vdots \\
\vdots
\end{array}$$

(3.1)

The most general element $e$ of $TL_n(q)$ is obtained by composing the generators (3.1) like dominos. The algebra is defined through the following relations between the generators

$$(i) \quad e_i^2 = q e_i \quad i = 1, 2, \ldots, n - 1$$

$$(ii) \quad [e_i, e_j] = 0 \quad \text{if } |i - j| > 1$$

(3.2)

$$(iii) \quad e_i e_{i+1} e_i = e_i \quad i = 1, 2, \ldots, n - 1$$

The relation (ii) expresses the locality of the $e$’s, namely that the $e$’s commute whenever they involve distant strings. The relations (i) and (iii) read respectively

$$\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
e_i^2 = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
1 \quad 1 \quad 1 \\
\vdots \\
\vdots \\
\vdots
\end{array} \\
i
\end{array}$$

(3.3)

$$\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
e_i e_{i+1} e_i = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
1 \quad 1 \quad 1 \\
\vdots \\
\vdots \\
\vdots
\end{array} \\
i+1
\end{array}$$

In (i), we have replaced a closed loop by a factor $q$. Therefore we can think of $q$ as being a weight per connected component of string. In (iii), we have simply “pulled the string” number $i + 2$.

An element $e \in TL_n(q)$ is said to be reduced if all its strings have been pulled and all its loops removed, and if it is further normalized so as to read $\prod_{i \in I} e_i$ for some minimal finite set of indices $I$. A reduced element is formed of exactly $n$ strings.

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**Fig. 5:** The transformation of a reduced element of $TL_9(q)$ into an arch configuration of order 18. The reduced element reads $e_3 e_4 e_2 e_5 e_3 e_1 e_6 e_4 e_2$. 



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There is a bijection between the reduced elements of the Temperley-Lieb algebra $TL_n(q)$ and the arch configurations of order $2n$. Starting from a reduced element of $TL_n(q)$, we index the left ends of the $n$ strings by $1, 2, \ldots, n$, and the right ends of the strings $2n, 2n - 1, \ldots, n + 1$ from top to bottom (see Fig. 5 for an illustration). Interpreting these ends as bridges, and aligning them on a line, we obtain a planar pairing of bridges by means of non-intersecting strings (arches), hence an arch configuration of order $2n$. Conversely, we can deform the arches of any arch configuration of order $2n$ to form a reduced element of $TL_n(q)$. As a consequence, we have $\dim(TL_n(q)) = c_n$, as vector space with a basis formed by all the reduced elements.

In the following, we will rather use the identification between $TL_n(q)$ and the left ideal $I_n(q)$ of $TL_{2n}(q)$ generated by the element $u_n = e_1e_3\ldots e_{2n-1}$, which goes over to reduced elements.

Fig. 6: The arch configuration of order 18 of Fig. 5 is immediately interpreted as an element of the ideal $I_9(q)$ of $TL_{18}(q)$, by adding a succession of $n = 9$ strings linking the consecutive upper ends of strings by pairs (for simplicity, the element of $TL_{18}(q)$ is now read from bottom to top). The corresponding reduced element of $I_9(q)$ reads, from bottom to top, $(e_3e_9)(e_2e_4e_8e_{10}e_{12}e_{14}e_{16})(e_1e_3e_5e_7e_9e_{11}e_{13}e_{17})$.

Indeed any reduced element of $I_n(q)$ has a pictorial representation as a set of $2n$ strings linking by pairs the $2n$ left and $2n$ right ends of strings, as illustrated in Fig. 6. The pairing of the right ends of strings is very simple, and represents the right factor $e_1e_3\ldots e_{2n-1}$. It consists of $n$ arches connecting the $n$ pairs of successive right ends of strings. Therefore the $2n$ left ends of strings are connected among themselves through the $n$ remaining strings. This gives exactly an arch configuration of order $2n$.

Fig. 7: Example of a walk diagram in $W_8$, expressed as the result of four box additions on the fundamental walk $a_4$. 
The converse construction is best expressed in the walk diagram representation of arch configurations. Let us construct a map $\rho$ from $W_{2n}$ to the set of reduced elements of the ideal $\mathcal{I}_n(q)$ of $TL_{2n}(q)$. Let $a_n$ denote the fundamental walk diagram of $W_{2n}$, such that

$$a_n : h(0) = h(2) = ... = h(2n) = 0 \quad \text{and} \quad h(1) = h(3) = ... = h(2n - 1) = 1 \quad (3.4)$$

To this diagram, we associate the reduced element

$$u_n = \rho(a_n) = e_1e_3...e_{2n-1} \quad (3.5)$$

Now any walk diagram in $W_{2n}$ is obtained from $a_n$ by successive box additions, illustrated in Fig.7. A box addition at a minimum $i$ of $a \in W_{2n}$, i.e. where $h(i+1) = h(i-1) = h(i)+1$, simply consists in shifting $h(i) \to h(i) + 2$, which amounts to formally add a square box which fills the minimum at $i$ and transforms it into a maximum. We will denote by $a \to a + \diamond_i$ this operation on $a \in W_{2n}$ (this notation keeps track of the point $i$ at the vertical of which the box is added). This enables us to define the length of a walk diagram $a \in W_{2n}$ as the number of box additions which have to be performed on the fundamental $a_n$ to build $a$. We set

$$|a| = \# \text{boxes in } a \quad \text{for all } a \in W_{2n} \quad (3.6)$$

In particular, we have $|a + \diamond_i| = |a| + 1$. The mapping $\rho$ is then defined as

$$\rho(a + \diamond_i) = e_i \rho(a) \quad (3.7)$$

where the box addition is made at the point $i$ (necessarily a minimum of $a$).

Fig. 8: Example of the mapping $\mu$ between an element $a \in W_8$ and $e = \rho(a) \in \mathcal{I}_4(q)$. We read the element $e = \rho(a)$ from the various layers of box additions, using the formula (3.7). Here we get $e = (e_3)(e_2e_4e_6)(e_1e_3e_5e_7)$ (the parentheses correspond to the successive layers of boxes added.)
The most general reduced element \( e = \rho(a) \) in \( \mathcal{I}_n(q) \), \( a \in W_{2n} \), is therefore written as the product over all box additions leading from the fundamental walk \( a_n \) to \( a \), of the corresponding \( e_i \)'s. This is illustrated in Fig.8. The reduced elements of \( \mathcal{I}_n(q) \) form a basis (which we call basis 1 from now on) of the corresponding vector space over the complex numbers. We have established that \( |W_{2n}| = \dim(\mathcal{I}_n(q)) = c_n \). For simplicity, we will adopt the following notation for the basis 1 elements: we write

\[
(a)_1 = \rho(a) \quad \text{for any} \ a \in W_{2n}
\]

As an example, the basis 1 for \( \mathcal{I}_3(q) \) is formed by the \( c_3 = 5 \) following elements

\[
\begin{align*}
(1) & = e_1e_3 \\
(2) & = e_2e_1 \\
(3) & = e_4e_1 \\
(4) & = e_2e_4 \\
(5) & = e_3e_2
\end{align*}
\]

indexed by the 5 walk diagrams of \( W_6 \).

We now show how to reconstruct the string-domino pictorial representation attached to an element of \( \mathcal{I}_n(q) \), from the box decomposition of the corresponding walk \( a \in W_{2n} \). The idea is to represent the \( e_i \)'s forming the fundamental element \( u_n \) by boxes as well. Actually each box will have the meaning of a left multiplication by \( e_i \), this time acting on \( 1 \). Starting from some walk diagram \( a \in W_{2n} \), we write it as the result of box additions on the empty diagram. To go to the string-domino picture, we have to draw “arches” using the box configurations. This is done by marking each box with a pair of strings as follows

\[
\begin{array}{c}
\Diamond \\
\rightarrow
\end{array}
\]

and by continuing each string with a vertical line ending at some string-end on the border of the corresponding domino.

This is illustrated in Fig.9, where the strings are represented in thick black lines.
3.2. Gram matrix for the basis $1$ of $\mathcal{I}_n(q)$

The Temperley-Lieb algebra is endowed with a standard trace, defined on the reduced elements as the number

$$\text{Tr}(e) = q^{\kappa(e)} \quad (3.11)$$

where $\kappa(e)$ is the number of connected components of strings after the identification of the left ends of strings with the right ones, as depicted in Fig.10. This definition extends by linearity to any element of $TL_n(q)$. Given a reduced element $e \in TL_n(q)$, we may consider the adjoint $e^t$, obtained by reflecting the corresponding arch configuration w.r.t. the river. The corresponding operation on $TL_n(q)$ satisfies $e_i^t = e_i$ (the generators are self-adjoint), and $(ef)^t = f^te^t$ for any reduced elements $e, f$. Taking the adjoint simply reflects the
string-domino picture of the corresponding reduced element, and exchanges the left and right ends of strings. Again, this extends to any element of $TL_n(q)$ by linearity. We can now introduce the bilinear form

$$ (e, f) = \text{Tr}(ef^t) \quad \text{for any } e, f \in TL_n(q) \quad (3.12) $$

The above definitions extend by restriction to any ideal of the Temperley-Lieb algebra. Let us now concentrate on the ideal $\mathcal{I}_n(q)$ of $TL_{2n}(q)$. Let us consider the Gram matrix of the basis $1$, with respect to the bilinear form $(3.12)$, namely the matrix $\Gamma_{2n}(q)$, with entries

$$ [\Gamma_{2n}(q)]_{a,b} = ((a)_{1}, (b)_{1}) = \text{Tr}((a)_{1}(b)_{1}^t) \quad (3.13) $$

where $a, b$ run over the walk diagrams of $W_{2n}$ which are used to index the corresponding basis $1$ elements $(a)_{1}, (b)_{1}$.

![Fig. 11:](image)

**Fig. 11:** The bilinear form $(e, f)$ is obtained by first multiplying $e$ with $f^t$, and then identifying the upper and lower ends of the strings (The bridges numbered 1, 2, ..., 10 are identified), and counting the number of connected components of strings. Here we have created $n = 5$ simple loops at the connection between the two dominos, and $\kappa(a|b) = 3$ other loops, from the superposition of the arch configurations $a$ and $b$ of order 10, corresponding respectively to $e$ and $f$. Note that $b \rightarrow b^t$ is reflected w.r.t. the river, corresponding to the adjoint in $f^t$. Finally we have $\text{Tr}(ef^t) = q^{n+\kappa(a|b)} = q^8$ here.

But evaluating the matrix element $(3.13)$ just amounts, as illustrated in Fig.11, to connecting the domino corresponding to $(a)_{1}$ and the reflection of the domino corresponding to $(b)_{1}$, and identifying the left ends of strings (on the bottom of the figure) with the right ones (on top of the figure), and counting the number of connected components of strings. Because of the particular form of the elements of $\mathcal{I}_n(q)$ (see Fig.6), this procedure will create $n$ loops along the connection line between the two dominos (these loops are formed by the arches connecting consecutive ends of strings) plus an extra $\kappa(a|b)$ loops, namely
those appearing in the superposition of the arch configurations $a$ and (the reflection of) $b$. Therefore we have

$$[\Gamma_{2n}(q)]_{a,b} = q^{n+\kappa(a|b)} = q^n [\mathcal{G}_{2n}(q)]_{a,b}$$  \hspace{1cm} (3.14)

by comparison with the previous definition (2.4) of the meander matrix $\mathcal{G}_{2n}(q)$.

Hence the meander determinant is simply related to the Gram determinant of the basis 1, through

$$\det(\Gamma_{2n}(q)) = q^{n} \det \mathcal{G}_{2n}(q)$$ \hspace{1cm} (3.15)

The remaining subsections of this section will be devoted to the computation of the Gram determinant of the basis 1 of $\mathcal{I}_n(q)$.

3.3. Orthonormalization of the basis 1

In the following, we will compute the Gram determinant (3.15) by performing an explicit Gram-Schmidt orthonormalization of the basis 1 w.r.t. the bilinear form (3.12). The orthonormalization process consists in a change of basis from the basis 1 to another basis, which we call basis 2, satisfying the following properties

(i) The basis 2 elements are still indexed by the walk diagrams of $W_{2n}$, we denote them by $(a)_2$, $a \in W_{2n}$.
(ii) The basis 2 is orthogonal w.r.t. the bilinear form (3.12), namely $((a)_2,(b)_2) = 0$ whenever $a \neq b$.
(iii) The basis 2 elements have all the same norm 1, namely

$$((a)_2,(a)_2) = 1 \quad \text{for any } a \in W_{2n}$$ \hspace{1cm} (3.16)

The basis 2 elements are constructed as follows. We start from the fundamental element $(a_n)_2$, indexed by the fundamental walk of $W_{2n}$ (3.4), and defined as

$$(a_n)_2 = q^{-n}e_1e_3...e_{2n-1}$$ \hspace{1cm} (3.17)

The normalization factor ensures that the property (iii) above holds for the norm of this element, namely that

$$((a_n)_2,(a_n)_2) = 1$$ \hspace{1cm} (3.18)

(Indeed, $(a_n)_2(a_n)_2^t = (a_n)_2$, and $\text{Tr}(a_n)_2 = q^{-n}q^n = 1$.) As any walk diagram $a \in W_{2n}$ is obtained from the fundamental one $a_n$ by successive box additions, we define the other basis 2 elements by the following box addition rule, which amounts to a recursion. Suppose
we have constructed \((a)_2\) for some \(a \in W_{2n}\). The following rule gives the element \((a + \diamond_{i,\ell})_2\), where a box addition has been performed on a minimum \(i\) of \(a\), with \(h(i + 1) = h(i - 1) = h(i) + 1 = \ell\), the height of the box addition.

\[
(a + \diamond_{i,\ell})_2 = \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}} (e_i - \mu_\ell) (a)_2
\]

\[(3.19)\]

where we have used the notation

\[
\mu_\ell = \frac{U_{\ell-1}(q)}{U_\ell(q)} \quad \text{for } \ell = 1, 2, 3...
\]

\[(3.20)\]

in terms of the Chebishev polynomials \((2.3)\). Note that the recursion relation \(U_{m+1}(q) = qU_m(q) - U_{m-1}(q)\) translates into the relation

\[
\frac{1}{\mu_1} - \mu_m = \frac{1}{\mu_{m+1}} \quad \text{for all } m \geq 1
\]

\[(3.21)\]

The rule \((3.19)\) may be viewed as a deformation of the rule \((3.7)\) used to construct the basis 1. However, two new ingredients have appeared: (i) the box addition now depends on the height \(\ell\) at which it is performed (hence the notation \(\diamond_{i,\ell}\), to keep track of this height) and (ii) there is an overall change of normalization \(\sqrt{\mu_{\ell+1}/\mu_\ell}\). Together with the initial point \((3.17)\), the recursive rule \((3.19)\) determines the basis 2 elements completely.

By construction, these elements all have the right factor \(e_1 e_3 \ldots e_{2n-1}\), hence belong to the ideal \(\mathcal{I}_n(q)\). Moreover, when expressed on basis 1 elements, they read

\[
(a)_2 = \sum_{b \subseteq a, b \in W_{2n}} P_{b,a}(b)_1
\]

\[(3.22)\]

where the matrix elements of \(P\) are products of factors involving the \(\mu_m\)'s, and the sum extends only on the walk diagrams \(b\) included in \(a\), i.e., such that \(a\) is obtained from \(b\) by some box additions (this includes the case \(b = a\) with no box addition). The change of basis \(1 \rightarrow 2\) is therefore triangular, as the walk diagrams \(a \in W_{2n}\) may be arranged by growing length \(|a|\) \((3.6)\), thus making the matrix \(P\) of the change of basis upper triangular (this implies that the basis 2 is indeed a basis of \(\mathcal{I}_n(q)\)).

To distinguish between the two different box additions \((3.7)\) for basis 1 and \((3.19)\) for basis 2, which could both be performed on a given walk diagram \(a \in W_{2n}\), representing
either a basis 1 or a basis 2 element, we decide to represent (3.7) by grey boxes, whereas (3.19) is represented by white boxes, namely

\[ e_i = \boxed{i} \]

(3.23)

(We have indicated the position \( i \) and the height \( m \) at which the box acts.) In this pictorial representation, the basis 2 elements of \( I_3(q) \) read

(3.24)

With these definitions, we have the

**PROPOSITION 1:**

The basis 2 is orthonormal w.r.t. the bilinear form (3.12), namely

\[ ((a)_2, (b)_2) = \delta_{a,b} \quad \text{for all } a, b \in W_{2n} \]

(3.25)

We will first prove by recursion, using box additions, the following

**LEMMA 1:**

\[ (a)_2^t (b)_2 = 0 \quad \text{for all } a, b \in W_{2n} \text{ such that } |a| \leq |b| \text{ and } a \neq b \]

(3.26)

Note that this result is stronger than the one for the trace of \( (a)^t_2(b)_2 \), which is implied in proposition 1. We learn from lemma 1 that the product of any two distinct basis 2
elements vanishes. This stronger result is linked to the property that \( \mathcal{I}_n(q) \) is an ideal. This remark will take its full strength when we study the semi-meander determinant. For an element \((a)_2\) of basis 2, the length of \(|a|\), represents its number of white boxes. We will therefore prove the lemma 1 by recursion on the white box addition.

Suppose that the property \( P \)

\[
(P) : (a)_2^t(b)_2 = 0 \quad \text{for all } b \text{ such that } |b| \geq |a| \text{ and } b \neq a
\]

(3.27)

holds for some \( a \in W_{2n} \). Let us prove that \( P \) holds for \( a + \diamond \), for any white box addition on \( a \). Pick any walk diagram \( b \in W_{2n} \) such that

\[
|b| \geq |a + \diamond| = |a| + 1
\]

We wish to evaluate the quantity

\[
(a + \diamond_{i,\ell})_2^t(b)_2
\]

and show that it vanishes. The idea is simply to transfer the box addition from \( a \) to \( b \), namely use the commutation of \( e_i \) with \( e_j \), \(|j - i| > 1\), to let the white box \( \diamond_{i,\ell} \) act on \( (b)_2 \) (the white box is self-adjoint, and multiplies \((a)_2^t\) to the right, hence we can let it act on \((b)_2\) by left multiplication). There are however two problems associated with this transfer: (i) \( b \) may not have a minimum at \( i \) or (ii) if \( b \) has a minimum at \( i \), it may lie at a different height \( m \neq \ell \). We therefore have to distinguish between the following three possible configurations of \( b \) at \( i \) (maximum, slope, minimum)

1. \( b \) has a maximum at \( i \), namely with \( h(i + 1) = h(i - 1) = h(i) - 1 = m \). This means that \( b \) itself is the result of a box addition at \( i \) on the walk \( b' = b - \diamond_{i,m} \) with \( h(i + 1) = h(i - 1) = h(i) + 1 = m \), and all other \( h(j) \) identical to those of \( b \). Hence the white box addition on \((b)_2\) reads

\[
\sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(e_i - \mu_\ell)(b)_2 = \sqrt{\frac{\mu_{\ell+1}\mu_{m+1}}{\mu_\ell\mu_m}}(e_i - \mu_\ell)(e_i - \mu_m)(b - \diamond_{i,m})_2
\]

\[
= \sqrt{\frac{\mu_{\ell+1}\mu_{m+1}}{\mu_\ell\mu_m}}[(\mu_1^{-1} - \mu_\ell - \mu_m)(e_i - \mu_m) + (\mu_1^{-1} - \mu_m)\mu_m](b - \diamond_{i,m})_2
\]

\[
= \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(\mu_{m+1}^{-1} - \mu_\ell)(b)_2 + \sqrt{\frac{\mu_{\ell+1}\mu_m}{\mu_\ell\mu_{m+1}}}(b - \diamond_{i,m})_2
\]

(3.30)
where we have used the relation (3.21), and the property $e_i^2 = qe_i = \mu_1^{-1} e_i$. In the second line of (3.30), we have reconstructed a white box addition on the minimum of $b - \circ_i, m$ at $i$ (first term) up to an additive constant (second term), resulting respectively in the terms $(b)_2$ and $(b - \circ_i, m)_2$ of the result.

(2) $b$ has an ascending slope (resp. a descending slope) at $i$, namely with $h(i + 1) - 1 = h(i) = h(i - 1) + 1 = m$ (resp. $h(i + 1) + 1 = h(i) = h(i - 1) - 1 = m$). In either case, let us write the white box addition on $(b)_2$ as

$$
\sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}} e_i - \sqrt{\mu_\ell \mu_{\ell+1}} e_i
$$

(3.31)

namely as a term proportional to a grey box addition (multiplication by $e_i$) plus a constant. But the grey box addition on a grey slope of $(b)_2$ has a zero result. Indeed, the slope is itself the result of prior white box additions, hence, in the case of an ascending slope

$$
\begin{align*}
&= e_i \sqrt{\frac{\mu_{m+1}}{\mu_m}} \\
&= \sqrt{\frac{\mu_{m+1}}{\mu_m}} \\
&= -\sqrt{\mu_{m+1}}
\end{align*}
$$

(3.32)

where we have used the relations $e_i^2 = \mu_1^{-1} e_i$ and $e_i e_{i+1} e_i = e_i$, and where (3.21) has implied the vanishing of the coefficient of $e_i$ in the second line of (3.32). We are left with an expression involving the action of a grey box at the point $i + 1$ (factor $e_{i+1}$) on a white slope of the rest of $(b)_2$ (symbolized by the ... in (3.32)). We can therefore repeat the calculation (3.32) with $i \rightarrow i + 1$ (and $m \rightarrow m - 1$), and so on, until the “bottom” of the diagram is reached, namely the situation where the slope is formed by the piling up of a grey and a white box:

$$
\begin{align*}
&= e_{i+m-1} \sqrt{\frac{\mu_{m-1}}{\mu_m}} \\
&= 0
\end{align*}
$$

(3.33)
by using \( e_{i+m-1}e_{i+m}e_{i+m-1} = e_{i+m-1} \) and \( e_{i+m-1}^2 = \mu_1^{-1}e_{i+m-1} \). The same reasoning applies for a descending slope: such a slope may indeed be viewed as the adjoint of an ascending slope, whereas the white box to be added is self-adjoint. The addition of a white box on any slope of \( b \) therefore reduces to the second term of (3.31), namely

\[
\sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}} (e_i - \mu_\ell)(b)_2 = -\sqrt{\mu_{\ell+1}\mu_\ell}(b)_2
\]

(3) \( b \) has a minimum at \( i \), namely with \( h(i+1) = h(i) + 1 = h(i-1) = m \). Then, writing

\[
\sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}} (e_i - \mu_\ell)(b)_2 = \sqrt{\frac{\mu_{\ell+1}\mu_m}{\mu_\ell\mu_{m+1}}} (e_i - \mu_m) + \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(\mu_m - \mu_\ell)(b)_2
\]

where we have reconstructed a white box addition at point \( i \) and height \( m \) in the first term, we simply get

\[
\sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}} (e_i - \mu_\ell)(b)_2 = \sqrt{\frac{\mu_{\ell+1}\mu_m}{\mu_\ell\mu_{m+1}}} (b + \diamond_{i,m})_2 + \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(\mu_m - \mu_\ell)(b)_2
\]

These three situations are summarized in the following recursion relation, according to the configuration of \( b \) at \( i \), respectively denoted \( \delta_{b,\text{max}(i,m)} \) (case (1)) and \( \delta_{b,\text{min}(i,m)} \) (case (3))

\[
\sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}} (e_i - \mu_\ell)(b)_2 = -\sqrt{\mu_{\ell+1}\mu_\ell}(b)_2
\]

\[
+ \delta_{b,\text{max}(i,m)} \left[ \sqrt{\frac{\mu_{\ell+1}\mu_m}{\mu_\ell\mu_{m+1}}} (b + \diamond_{i,m})_2 + \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(b - \diamond_{i,m})_2 \right]
\]

\[
+ \delta_{b,\text{min}(i,m)} \left[ \sqrt{\frac{\mu_{\ell+1}\mu_m}{\mu_\ell\mu_{m+1}}} (b + \diamond_{i,m})_2 + \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}}(b)_2 \right]
\]

In all cases (1-3), this enables us to reexpress

\[
(a + \diamond_{i,\ell})_2(b)_2 = (a)_2 \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}} (e_i - \mu_\ell)(b)_2 = \sum_{b' \in W_{2n}} \lambda_{b'} (a)_2(b')_2
\]

as a linear combination of terms of the form \((a)_2(b')_2\), where \( b' = b, b + \diamond \) or \( b - \diamond \), hence with \( |b'| \geq |b| - 1 \). But, by hypothesis (3.28), we have \( |b| \geq |a| + 1 \), hence \( |b'| \geq |a| \). We can therefore apply the recursion hypothesis \( P \) (3.27) to each of the products \((a)_2(b')_2\) in (3.38), which must then vanish, and we finally get a zero answer for \((a + \diamond)_2(b)_2\). This establishes the property \( P \) (3.27) for \( a + \diamond \), under the assumption that it is satisfied for \( a \).
To complete the recursion, we have to establish the property $P$ \((3.27)\) for the initial point $a = a_n$. It will then hold for any $a \in W_{2n}$. Let us prove that

$$
(a_n)_2^t(b)_2 = 0 \quad \text{for all } b \in W_{2n} \text{ such that } b \neq a_n
$$

(3.39)

(The condition that $|b| \geq |a_n| = 0$ does not give any restriction on $b$.) We have to evaluate the product $e_1 e_3 \ldots e_{2n-1}(b)_2$, namely the addition of a row of $n$ grey boxes to $b$. As those cover the whole width of $b$, there is at least one such grey box which acts on a white slope of $b$ (otherwise, $b$ should have no slope, hence would be equal to $a_n$). But in eqs \((3.32)-(3.33)\) above, we have proved that the addition of a grey box on a white slope of $(b)_2$ yields a zero answer. This completes the proof of \((3.39)\), and the lemma 1 follows by recursion.

To prove the proposition 1, we note that by symmetry of the bilinear form \((3.12)\)

$$
((a)_2, (b)_2) = \text{Tr}((a)_2(b)_2) = \text{Tr}((b)_2^t(a)_2) = ((b)_2, (a)_2) = \text{Tr}((a)_2^t(b)_2)
$$

(3.40)

If $a \neq b$ we immediately get $((a)_2, (b)_2) = 0$ by applying the lemma 1 to $(a, b)$ if $|a| < |b|$ or $(b, a)$ if $|a| > |b|$, and either of the two if $|a| = |b|$. This gives the orthogonality of the distinct basis 2 elements w.r.t. the bilinear form \((3.12)\).

The norm of $(a)_2, a \in W_{2n}$, is easily computed by recursion. We have already seen in \((3.18)\) that the fundamental basis 2 element has norm $((a_n)_2, (a_n)_2) = 1$. Suppose that $((a)_2, (a)_2) = 1$ for some $a \in W_{2n}$. Let us compute the norm of the element $(a + \diamond)_2$, for some white box addition on $a$. We have

$$
(a + \diamond_{i,\ell})_2^t (a + \diamond_{i,\ell})_2 = (a)_2^t \frac{\mu_{\ell+1}}{\mu_\ell} (e_i - \mu_\ell)^2 (a)_2
$$

$$
= (a)_2^t \left(1 + \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}} (e_i - \mu_\ell) \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}} (e_i - \mu_\ell) \right) (a)_2
$$

$$
= (a)_2^t (a)_2 + \sqrt{\frac{\mu_{\ell+1}}{\mu_\ell}} (e_i - \mu_\ell) (a)_2^t (a + \diamond_{i,\ell})_2
$$

$$
= (a)_2^t (a)_2
$$

(3.41)

where we have used $e_i^2 = \mu_1^{-1} e_i$ and the relation \((3.24)\) in the first line, and $(a)_2^t (a + \diamond)_2 = 0$ in the second, by direct application of the lemma 2. Eq.\((3.41)\) implies that $((a + \diamond)_2, (a + \diamond)_2) = ((a)_2, (a)_2) = 1$, by the recursion hypothesis. Together with the initial point \((3.18)\), this proves that $((a)_2, (a)_2) = 1$, for all $a \in W_{2n}$. The proposition 1 follows.
3.4. The meander determinant

The meander determinant (3.13) follows from the Gram determinant for the basis 1. Let us now compute the latter. The basis 2 being orthonormal, its Gram matrix is the $c_n \times c_n$ identity matrix $I$. The change of basis from basis 1 to 2 (with the upper triangular matrix $P$ (3.22)) therefore reads

$$P \Gamma_{2n}(q) P^t = I \quad (3.42)$$

Hence we have $\det \Gamma_{2n}(q) = (\det P)^{-2}$. As $P$ is an upper triangular matrix, only the diagonal elements $P_{a,a}$ enter the determinant formula. From the definition of the basis 2 elements by white box additions (3.19) on the fundamental $(a_n)_2$, we immediately get that the matrix elements $P_{a,a}$ satisfy the recursion

$$P_{a+o_i,\ell,a+o_i,\ell} = \sqrt{\frac{\mu_{\ell+1}}{\mu_{\ell}}} P_{a,a} \quad (3.43)$$

With the initial condition $P_{a_n,a_n} = \mu_1^n$ for the fundamental walk diagram, this determines the $P_{a,a}$ completely. We have

$$P_{a,a}^2 = \mu_1^{2n} \prod_{\text{boxes of } a} \frac{\mu_{\ell+1}}{\mu_{\ell}} \quad (3.44)$$

Fig. 12: The decomposition of a walk $a \in W_{12}$ into strips of white boxes. There are $n = 6$ such strips, with respective lengths 2, 4, 3, 2, 1 and 1 (note that an empty strip has by definition length 1).

The boxes of any $a \in W_{2n}$ can be arranged into $n$ strips, as illustrated in Fig.12, namely $n$ diagonal lines of boxes of increasing consecutive heights and positions. Each such line has an upper end, the top of the rightmost box in the line. Let us call the height
of this end the length of the corresponding strip. For instance, the fundamental diagram is formed of \( n \) strips of length 1. With this definition, we simply get

\[
P_{a,a}^2 = \mu_1^n \prod_{\text{strips of } a} \mu_\ell
\]  

(3.45)

where, in the product over the \( n \) strips of \( a, \ell \) stands for the corresponding strip length.

The Gram determinant of the basis 1 reads then

\[
det \Gamma_{2n}(q) = \prod_{a \in W_{2n}} P_{a,a}^{-2} = \mu_1^{-nc_{2n}} \prod_{\text{strips of all } a \in W_{2n}} \mu_\ell^{-1}
\]  

(3.46)

We also get a formula for the meander determinant, using (3.15)

\[
det G_{2n}(q) = \prod_{\text{strips of all walks } \in W_{2n}} \mu_\ell^{-1}
\]  

(3.47)

This can be rewritten as

\[
det G_{2n}(q) = \prod_{m=1}^{n} \left[ \mu_m \right]^{-s_{2n,m}}
\]  

(3.48)

where \( s_{2n,m} \) denotes the total number of strips of length \( m \) in all the walk diagrams of order \( 2n, W_{2n} \). The formula of theorem 1 will follow from the explicit computation of the numbers \( s_{2n,m} \). We have the

**PROPOSITION 2:**

\[
s_{2n,m} = c_{2n,2m} = \left( \frac{2n}{n-m} \right) - \left( \frac{2n}{n-m-1} \right)
\]  

(3.49)

This will be proved by establishing a bijection between the walks \( a \in W_{2n} \) with a marked end of strip at height \( m \), and the walks of \( 2n \) steps on a half-line starting at the origin \( h(0) = 0 \) and ending at height \( h(2n) = 2m \), namely the elements \( b \in W_{2n}^{(2m)} \). The cardinal of the latter set being equal to \( |W_{2n}^{(2m)}| = c_{2n,2m} \) (see eq.(2.49)), the proposition 2 will follow.

Let us consider any walk \( a \in W_{2n}^{(2m)} \). As shown in Fig.13, the line \( h = m \) intersects the walk \( a \) at least once along an ascending slope (at some point \( j \) where \( h(j) = m \) and
Fig. 13: The map from any \( a \in W_{2n}^{(2m)} \) to a walk \( b \in W_{2n} \) with a marked end of strip at height \( m \). \( i \) is the rightmost intersection of the line \( h = m \) with \( a \) at an ascending slope. The walk \( a \) is cut into two parts: the left \( L \in W_i^{(m)} \), and the right \( R \), such that its reflection \( \bar{R} \in W_{2n-i}^{(m)} \). We have \( b = L\bar{R} \in W_{2n} \), with the marked point at \( i \). If \( h(i + 1) = m - 1 \) (not the case in the present figure), this is the desired walk of \( W_{2n} \) with a marked end of strip at height \( m \). If \( h(i + 1) = m + 1 \) (the case of the present figure), we migrate \( i \to i' = \min \{ j > i | h(j) = m = h(j + 1) + 1 \} \), and mark \( i' \). The migration is indicated by an arrow. The corresponding strip of length \( m \) has also been represented.

\[
h(j - 1) = m - 1 \text{ on } a.
\]

Let \( i \) denote the position of the rightmost such intersection, namely \( i = \max \{ j | h(j) = m = h(j - 1) + 1 \} \). Cutting the walk \( a \) at the point \((i, h(i) = m)\) separates the walk into a left part \( L \in W_i^{(m)} \) and a right part \( R \), which may be viewed as an element of \( W_{2n-i}^{(m)} \) (see Fig.13). Indeed, from the definition of \( i \), the walk \( R \) stays above the line \( h = m \) until its end: subtracting \( m \) from all its heights, and counting its steps from 0 to \( 2n - i \) (instead of from \( i \) to \( 2n \)) expresses \( R \) as an element of \( W_{2n-i}^{(m)} \). Reflecting \( R \to \bar{R} \), i.e. describing it in the opposite direction (\( \bar{R} \) is a walk on the half-line starting at height \( m \) and ending at height 0 after \( 2n - i \) steps), and composing \( L \) and \( \bar{R} \), i.e. attaching

\[3\] The fact that we take the rightmost intersection here is responsible for the bijectivity of the mapping.
the origin of \( \bar{R} \) to the end of \( L \), we form a walk \( b = L\bar{R} \in W_{2n} \) (see Fig.13). In this walk, we have \( h(i) = m \). If \( h(i + 1) = m - 1 \), \( i \) is an end of strip of height \( m \), which we mark. If \( h(i + 1) = m + 1 \), \( i \) cannot be an end of strip. Nevertheless, we just have to consider the smallest point \( i' > i \) such that \( h(i') = m = h(i' + 1) + 1 \), which always exists, as the walk \( a \) goes back to height 0 at position \( 2n \). This point \( i' \) is an end of strip at height \( m \), which we mark.

Conversely, let us start from some \( a \in W_{2n} \) with a marked end of strip at position \( i \) and height \( m \). By definition, this end of strip satisfies \( h(i) = m \) and \( h(i + 1) = m - 1 \). If \( i \) is a maximum of \( a \), namely \( h(i - 1) = m - 1 \), it separates the walk \( a \) into a left part \( L \) and a right part \( R \). The left part is a walk on the half-line, ending at height \( m \) after \( i \) steps, hence \( L \in W_i^{(m)} \). The right part \( R \) is a walk on the half-line starting at height \( m \) and ending at the origin, after \( 2n - i \) steps. The reflected walk \( \bar{R} \) is obtained by describing \( R \) in the opposite direction, namely starting from the origin, and ending at height \( m \), after \( 2n - i \) steps. Hence we can write that \( \bar{R} \in W_{2n-i}^{(m)} \). Now if we compose the walks \( L \) and \( \bar{R} \) (attach the origin of \( \bar{R} \) to the end of \( L \)), the resulting walk \( b = L\bar{R} \in W_{2n}^{(2m)} \), and due to the fact that \( i \) was a maximum of \( a \), we have \( h(2n - 1) = 2m - 1 \) and \( h(2n) = 2m \) in \( b \). If \( i \) is not a maximum of \( a \), we first migrate the marked point from \( i \) to the largest value \( i' < i \), such that \( h(i') = m = h(i' - 1) + 1 \) (the closest ascending slope at height \( m \) to the left of \( i \)). Then we apply the previous cutting, reflecting and pasting procedure at the point \( i' \). This produces a walk \( b = L\bar{R} \in W_{2n}^{(2m)} \), with the particular property that \( h(2n - 1) = 2m + 1 \) and \( h(2n) = 2m \) on \( b \).

We have in fact established a more refined mapping between (i) the \( a \in W_{2n} \) with a marked maximum, of height \( m \) (namely at a point \( i \) such that \( h(i) = m = h(i + 1) + 1 = h(i - 1) + 1 \)) and the \( b \in W_{2n-1}^{(2m-1)} \) (ii) the \( a \in W_{2n} \) with a marked descending slope at height \( m \) (i such that \( h(i) = m = h(i - 1) - 1 = h(i + 1) + 1 \)) and the \( b \in W_{2n-1}^{(2m+1)} \). This forms a bijection between the walks \( a \in W_{2n} \) with a marked end of strip (either a maximum or a descending slope) and the walks \( b \in W_{2n}^{(2m)} \) (with either \( h(2n - 1) = 2m - 1 \) or \( h(2n - 1) = 2m + 1 \)). Hence we conclude that

\[
s_{2n,m} = |W_{2n}^{(2m)}| = c_{2n,2m} \tag{3.50}
\]

which proves the proposition 2.
To translate the result (3.49) of proposition 2 into the formula of theorem 1, using (3.48), we simply have to reexpress the meander determinant in terms of the Chebisheshev polynomials $U_m(q)$, using $\mu_m = U_{m-1}/U_m$. Eq (3.48) becomes

$$
\det G_{2n}(q) = \prod_{m=1}^{n} \left( \frac{U_m}{U_{m-1}} \right)^{s_{2n,m}} = \prod_{m=1}^{n} \left[ U_m \right]^{s_{2n,m}-s_{2n,m+1}}
$$

by noting that $s_{2n,n+1} = \binom{2n}{-1} - \binom{2n}{-2} = 0$. This takes exactly the form of (2.6), with

$$a_{2n,2m} = s_{2n,m} - s_{2n,m+1} = c_{2n,2m} - c_{2n,2m+2}
$$

which completes the proof of the theorem 1.

4. The semi-meander determinant: proof of theorem 2

The strategy of the proof of theorem 2 is exactly the same as for theorem 1. It is based on the representation of open arch configurations by a particular set of reduced elements of the Temperley-Lieb algebra, forming the basis (still called basis 1, but not to be confused with that of previous section) of a vector subspace thereof. The semi-meander determinant is then expressed in terms of the Gram determinant of this basis 1. The next step is the explicit Gram-Schmidt orthogonalization of this basis, defining another basis, called basis 2. The semi-meander determinant is then computed by using the change of basis $1 \rightarrow 2$.

4.1. Temperley-Lieb algebra and open arch configurations

The open arch configurations of $A_n^{(h)}$, with order $n$ and with $h$ open arches, can be represented by some particular reduced elements of the Temperley-Lieb algebra $TL_n(q)$.

![Fig. 14: The interpretation of an open arch configuration of order $n = 15$ and with $h = 3$ open arches (right diagram) as a reduced element of $TL_{15}(q)$ (left diagram). Note that exactly $h = 3$ strings go across the domino, namely link three lower to (the three rightmost) upper ends. The linking of the upper ends of the domino is made through $(n - h)/2 = 6$ strings connecting consecutive ends by pairs.](image)
In Fig.14, we have represented in the string-domino pictorial representation the domino corresponding to a reduced element of the Temperley-Lieb algebra, immediately interpretable as an open arch configuration. Starting from $a \in A_n^{(h)}$, let us construct an element, still denoted $\mathcal{I}$ by $(a)_1$ of $TL_n(q)$: representing the corresponding domino as acting from bottom to top, the connection of its $n$ lower ends of strings is realized through the closed arches of $a$, whereas the $h$ open arches just go across the domino, and connect $h$ of the lower ends to the $h$ rightmost upper ends of strings. The remaining $n-h$ ends are then connected by consecutive pairs like in the meander case. This construction establishes a bijection between $A_n^{(h)}$ and the reduced elements of $TL_n(q)$ with exactly $h$ strings connecting lower ends to the $h$ rightmost upper ends, and $(n-h)/2$ strings connecting the remaining $n-h$ upper ends by consecutive pairs. Let us denote by $\mathcal{I}_n^{(h)}(q)$ the vector space spanned by these reduced elements. From now on, we will refer to the basis $\{(a)_1|a \in A_n^{(h)}\}$ as the basis 1.

Like in the meander case, the basis 1 is best expressed in the equivalent language of walk diagrams $a \in W_n^{(h)}$. Let $a_n^{(h)}$ be the fundamental element of $W_n^{(h)}$, with $h(0) = h(2) = ... = h(n-h) = 0$, $h(1) = h(3) = ... = h(n-h-1) = 1$ and $h(n-h+j) = j$ for $j = 1, 2, ..., h$. Any $a \in W_n^{(h)}$ may be viewed as the result of box additions on the fundamental $a_n^{(h)}$. The construction of $(a)_1$, $a \in W_n^{(h)}$ is performed recursively. We first set

$$(a_n^{(h)})_1 = e_1 e_3 ... e_{n-h-1}$$

and then for a box addition at position $i$, we set

$$(a + \phi_i)_1 = e_i (a)_1$$

As an example, the basis 1 elements for $\mathcal{I}_4^{(2)}$ read

$$\begin{pmatrix}
\begin{array}{c}
\text{\includegraphics{fig14a}}
\end{array}
\end{pmatrix}_1 = e_1$$

$$\begin{pmatrix}
\begin{array}{c}
\text{\includegraphics{fig14b}}
\end{array}
\end{pmatrix}_1 = e_3 e_2^3$$

Footnote: Here we adopt the same notation for elements of $TL_n(q)$ corresponding to open arch configurations as that used before for closed arch configurations. These will correspond to another basis $\{(a)_1\}$ for $a \in A_n^{(h)}$, which we will refer to again as the basis 1. This should not be confusing, as we are only dealing with the open arch case from now on.
Fig. 15: The string-domino picture corresponding to the box decomposition of an open walk diagram \( a \in W_{11}^{(3)} \). Note that exactly 3 strings join upper and lower ends. The domino is rather read from top to bottom, as opposed to the case of Fig.14, where it is read from bottom to top.

where we have represented the boxes added on the walk diagrams.

To make direct contact with the string-domino pictorial representation, we may attach to the box decomposition of any walk diagram \( a \in W_n^{(h)} \) a domino using the same rule as in Sect.3.1, namely represent all the boxes corresponding to left multiplications by \( e_i \) (including those of the fundamental element \( a_n^{(h)} \)), and decorate them by a horizontal double line (string), as in (3.10). The picture is then completed by drawing vertical strings joining the string ends on the upper and lower borders of the domino. This is illustrated in Fig.15, where the strings are represented in thick black lines.

The main and new difficulty here, in comparison with the former meander case, is that these reduced elements of \( TL_n(q) \) do not form an ideal. For instance, we have listed in (4.3) the basis 1 elements for \( I_n^{(2)}(q) \). If we multiply the first (fundamental) element by the third one, we find \((e_1)e_3e_2e_1 = e_1e_3\) which does not belong to the space \( I_n^{(2)}(q) \) (there is no string connecting lower and upper ends in \( e_1e_3 \), whereas there must be 2 such strings in any element of \( I_n^{(2)}(q) \)), which is therefore not an ideal.

Nevertheless, we can still form the Gram matrix \( \Gamma_n^{(h)}(q) \) for the basis 1, by using the restriction to \( \mathcal{I}_n^{(h)}(q) \) of the bilinear form (3.12). This reads

\[
\left[ \Gamma_n^{(h)}(q) \right]_{a,b} = \langle (a)_1, (b)_1 \rangle \quad \text{for } a, b \in A_n^{(h)}
\] (4.4)

As illustrated in Fig.16, to compute \( \langle (a)_1, (b)_1 \rangle \), we glue the dominos \( (a)_1 \) and the reflected \( (b)'_1 \), identify the upper and lower string ends, and count the number of resulting

\footnote{This will be responsible for the absence of a generalization of the lemma 1 of Sect.3.3 for the present case.}
Fig. 16: Computation of $((a)_1, (b)_1)$. We put the reflected domino $(b)_1^t$ on top of the domino $(a)_1$ (here, $a, b \in W^{(3)}_{11}$). The upper ends are then identified one by one to the lower ends of strings. Counting the loops formed yields: $(n-h)/2 = 4$ central loops formed at the connection between the two dominos, plus $\kappa(a|b) = 3$ loops coming from the superposition of the open arch configurations $a$ and $b^t$ (reflected w.r.t. the river). This gives finally $((a)_1, (b)_1) = q^7$.

connected components. The connection of the two dominos creates $(n-h)/2$ loops, from the strings connecting the upper ends by consecutive pairs on $(a)_1$ and $(b)_1$. The remaining part simply creates $\kappa(a|b)$ loops, from the superposition of the open arch configurations $a$ and $b^t$ (reflected w.r.t. the river), and the connection of their $h$ open arches (see Fig.16). Hence the Gram matrix for the basis 1 of $I_n^{(h)}(q)$ is simply related to the semi-meander matrix (2.4), through

$$
\left[\Gamma_n^{(h)}(q)\right]_{a,b} = q^{\frac{n-h}{2}+\kappa(a|b)} = q^{\frac{n-h}{2}} \left[G_n^{(h)}(q)\right]_{a,b} \quad (4.5)
$$

The semi-meander determinant is therefore related to the Gram determinant of the basis 1 through

$$
\det G_n^{(h)}(q) = \mu_1^{\frac{n-h}{2}} c_{n,h} \det \Gamma_n^{(h)}(q) \quad (4.6)
$$

4.2. Orthogonalization of the basis 1

In this section, we introduce a basis 2 of $I_n^{(h)}(q)$, still indexed by $a \in W_n^{(h)}$, which will be orthonormal with respect to the bilinear form (3.12).

Like in the meander case, the basis 2 will be defined recursively through box additions. We start from the basic element

$$
(a_n^{(h)})_2 = \mu_1^{n/2} (a_n^{(h)})_1 \quad (4.7)
$$
where the normalization ensures that \((a_n^{(h)})_2, (a_n^{(h)})_2) = q^{-n} q^{\frac{n-h}{2}} q^{\frac{n+h}{2}} = 1\), where we have counted the contributions of the \((n-h)/2\) loops formed by the strings pairing upper ends by consecutive pairs on \((a)_1\), and that of the \(\kappa(a|a) = (n+h)/2\) loops created by the superposition of \(a\) with its own reflection \(a^t\).

To proceed, we need to define the concept of floor of a walk diagram \(a \in W_n^{(h)}\). Let us denote by \(h(i), i = 0, 1, 2, ..., n\) the heights of \(a\), with \(h(0) = 0\) and \(h(n) = h\). The floor of \(a\) is yet another diagram \(f(a) \in W_n^{(h)}\), such that \(f(a) \subset a\), and with heights \(h'(i), i = 0, 1, 2, ..., n\), defined as follows. Let us denote by \(J\) the set of integers

\[
J = \{j \in \{0, 1, ..., n\} \text{ such that } h(k) \geq h(j), \forall k \geq j\}
\]  

\[
(4.8)
\]

![Diagram](image)

**Fig. 17:** A diagram \(a \in W^{(6)}_{20}\) (thick black line) and the construction of its floor \(f(a) \subset a\). The segments \(J_0, J_1, ..., J_4\) of positions forming \(J\) are indicated by dotted lines. The floor \(f(a)\) is represented filled with grey boxes. The boxes inbetween \(f(a)\) and \(a\) are represented in white. The floor-ends have positions 0, 4, 6, 12, 14, 16, 17, 19, 20.

As illustrated in Fig. 17, this set \(J\) is clearly the union of ordered segments of positions, of the form \(j = J_0 \cup J_1 \cup ... \cup J_k\), with \(J_i = \{j_i, j_i+1, j_i+2, ..., j_i+n_i\}\), for some integers \(n_i\) and \(j_i, i = 0, ..., k\). These segments correspond to the ascending slopes of \(a\) such that no point on their right has a lower height. With these notations, the floor \(f(a)\) of \(a\) is defined to have the heights \(h'(j), j = 0, 1, ..., n\), according to the following rules

\[
h'(j) = h(j) \quad \forall j \in J
\]

\[
h'(j_i + n_i + 2r) = h'(j_i + n_i) \quad \forall r \geq 0 \text{ with } 2r \leq j_i+1 - j_i - n_i
\]

\[
h'(j_i + n_i + 2r - 1) = h'(j_i + n_i) - 1 \quad \forall r \geq 1 \text{ with } 2r - 1 \leq (j_i+1 - j_i - n_i)
\]  

\[
(4.9)
\]
This is valid for all \( i = 1, 2, \ldots, k \). For \( i = 0 \), we have to be more careful, as the leftmost floor piece has a different status. If \( J_0 \neq \{0\} \) (this leftmost floor piece is empty), then (4.13) is valid for \( i = 0 \) as well. If \( J_0 = \{0\} \) (this leftmost floor piece is not empty: this is the case in Fig.17), we have to add the values

\[
\begin{align*}
  h'(0) &= h'(2) = \cdots = h'(j_1) = 0 \\
  h'(1) &= h'(3) = \cdots = h'(j_1 - 1) = 1
\end{align*}
\]

(4.10)

The floor diagram is represented filled with grey boxes in Fig.17. The floor diagram \( f(a) \) is in fact a succession of horizontal broken lines, with heights alternating \( h(j_i + n_i) = \ell + 1 \), \( h(j_i + n_i + 1) = \ell \), \( h(j_i + n_i + 2) = \ell + 1 \), \ldots, \( h(j_{i+1}) = \ell + 1 \), on the intermediate positions inbetween the segments \( J_i \) and \( J_{i+1} \). These are separated by ascending slopes (along the segments \( J_i \)). For each such intermediate floor \( F_i \), we define the floor height to be the number \( \ell = h(j_i + n_i - 1) = h(j_i + n_i + 1) = \cdots = h(j_{i+1}) - 1 \), for \( i \geq 1 \). The leftmost floor \( F_0 \), of height 0 if \( J_0 = \{0\} \), is a little different as we have \( \ell = 0 = h(j_0 = 0) = h(2) = \cdots = h(j_1) \) from (4.10). We will also refer to these intermediate floors as simply the floors of \( a \), for which this decomposition is implied. The endpoints with positions \( j_i + n_i \) and \( j_{i+1} \) (and equal height \( h(j_i + n_i) = h(j_{i+1}) \) except maybe for the rightmost floor-end) of each of these floors will be called floor-ends in the following.

To define the basis 2 of \( \mathcal{T}_n^{(h)}(q) \), we will need a pictorial representation of the walk diagrams \( a \in W_n^{(h)} \) in which the floor \( f(a) \) is also represented. As in Sect.3, we adopt the representation (3.23) by grey and white boxes of the left multiplication of a reduced element of \( \mathcal{T}_n^{(h)}(q) \) by respectively \( e_i \) at position \( i \) or \( \sqrt{\mu_{m+1}/\mu_m} (e_i - \mu_m) \) on a minimum of height \( m \) and position \( i \). The basis 2 elements then correspond to

(i) grey box additions for all the boxes forming the floor \( f(a) \), including the basic boxes forming \( a_n^{(h)} \) (see below)

(ii) white box additions for all the superstructures of \( a \) above its floor \( f(a) \). There is however a final subtlety with the height of these white boxes, which is counted along strips, w.r.t. the grey floor.

In the case (4.7) of the fundamental diagram, the representation is simply

\[
(a_n^{(h)})_2 = \mu_1^{n/2} \quad \text{\begin{array}{cc}\cdots\end{array}} \quad = \mu_1^{n/2} e_1 e_3 \ldots e_{n-h-1}
\]

(4.11)
as the floor of this element is simply $f(a_n^{(h)}) = a_n^{(h)}$, and we have represented the basic grey boxes under the floor. The other elements of the basis $2$ are obtained by white box additions on $(a_n^{(h)})_2$. The novelty, when compared to the case of Sect.3, is that some box additions may create a new floor, namely change previously added white boxes into grey ones.

In general, the best way to construct the basis $2$ elements, is to first list all the walk diagrams $a \in W_n^{(h)}$, represent them together with their floor $f(a) \subset a$, and then write the corresponding products of grey and white boxes. This is illustrated now in the case of $W_6^{(2)}$.

\[
\begin{align*}
( & \text{Diagram 1})_2 = \mu_1^3, \\
( & \text{Diagram 2})_2 = \mu_1^3, \\
( & \text{Diagram 3})_2 = \mu_1^3, \\
( & \text{Diagram 4})_2 = \mu_1^3, \\
( & \text{Diagram 5})_2 = \mu_1^3, \\
( & \text{Diagram 6})_2 = \mu_1^3, \\
( & \text{Diagram 7})_2 = \mu_1^3, \\
( & \text{Diagram 8})_2 = \mu_1^3, \\
( & \text{Diagram 9})_2 = \mu_1^3, \\
( & \text{Diagram 10})_2 = \mu_1^3, \\
( & \text{Diagram 11})_2 = \mu_1^3, \\
( & \text{Diagram 12})_2 = \mu_1^3.
\end{align*}
\]

(4.12)

where we have represented the grey and white boxes corresponding to each walk diagram. Note e.g. for the last element of (4.12) that the rightmost white box is counted to have height 1 (instead of 2) because this height is the relative height w.r.t. the grey floor on the
same strip, which is already at height 1. This construction results in the following change of basis $1 \rightarrow 2$

$$(a)_2 = \sum_{f(a) \subseteq b \subseteq a} P_{b,a} \ (b)_1$$

(4.13)

with possibly non-vanishing matrix elements $P_{b,a}$ only for the walks $b \in W_n^{(h)}$ such that $b$ is above the floor of $a$ ($f(a) \subset b$) and below $a$ ($b \subset a$). Like in the meander case of Sect.3, we can arrange the walk diagrams by growing length (number of boxes, grey and white), and make the matrix $P$ upper triangular.

With this definition, the basis 2 satisfies the following PROPOSITION 3:

The basis 2 elements are orthonormal with respect to the bilinear form (3.12), namely

$$((a)_2, (b)_2) = \delta_{a,b} \quad \text{for all } a, b \in W_n^{(h)}$$

(4.14)

This result will be proved in the remainder of this section. Note first, in comparison with the meander case (proposition 1), that no stronger statement (generalizing the lemma 1) will hold here for the products of elements of $I_n^{(h)}(q)$. This is because, as mentioned earlier, $I_n^{(h)}(q)$ is no longer an ideal, hence we have no good control of what the product of two elements of $I_n^{(h)}(q)$ can be. Thus, instead of resorting to the multiplication of elements, we will directly consider the bilinear form (3.12). The main forthcoming results (lemmas 2, 3 and 4 below) will deal with reexpressions and simplifications of this bilinear form, when evaluated on two elements of $I_n^{(h)}(q)$. In particular, the lemma 3 will give a reexpression in terms of the form (3.12), evaluated respectively on elements of $I_n^{(h)}_{n-2p}(q)$ and $I_p(q)$, which will enable us to use the results of Sect.3, namely the proposition 1, to eventually compute (4.14).

To prove the proposition 3, we need a few more definitions. As we are basically dealing with elements of the basis 2, it will be useful to trade the usual notion of walk diagram $a \in W_n^{(h)}$ for that of bicolored box diagram, namely the corresponding pictorial representation using grey and white box addition, i.e. the arrangement of grey and white
Fig. 18: The bicolored box diagram corresponding to an element \( a \in W_n^{(n-10)} \) for all \( n \geq 14 \). The width of the diagram is \( w = 5 \). It is decomposed into 5 strips \( s_j(a), j = 1, 2, ..., 5 \).

boxes forming \((a)_2\). For convenience, we still denote by \((a)_2\) the bicolored box diagram corresponding to \((a)_2\), with \( a \in W_n^{(h)} \).

Such a bicolored box diagram may be viewed as the succession of strips \( s_1(a), s_2(a), ..., s_w(a) \), made of a succession of grey, then white boxes of consecutive positions and heights. The number \( w \) stands for the number of these strips, namely the width of the base of \((a)_2\), i.e. the number of grey boxes of height 0 in \((a)_2\). Note that for all \( a \in W_n^{(h)} \), the element \((a)_2\) has width \( w = (n - h)/2 \). Moreover, we have the following identity between elements of \( I_n^{(h)}(q) \)

\[
(a)_2 = \mu_1^{n/2} s_1(a)s_2(a)...s_w(a) \tag{4.15}
\]

by considering the strips (i.e. successions of grey and white boxes) as elements of the Temperley-Lieb algebra.

To proceed with the proof of proposition 3, we will compute the quantity \(((a)_2, (b)_2) = \text{Tr}((a)_2(b)_2)\). The strategy is the following. We will start by comparing the rightmost strips \( s_w(a) \) and \( s_w(b) \) of \((a)_2 \) and \((b)_2 \). Both are a succession of grey boxes, topped by one white box, in the form

\[
s = \sqrt{\frac{\mu_2}{\mu_1}} (e_{2w+j-2} - \mu_1)e_{2w+j-3}e_{2w+j-4}...e_{2w+2w-1} \tag{4.16}
\]

with possibly different values of \( j = j_a \) or \( j_b \), the total size (total number of boxes) of the strip. Note that if \( j_a = 1 \), \( s_w(a) \) is reduced to a single grey box, without white box on top (this is the case when \((a)_2\) only has one floor of height 0). We have the first result

**Lemma 2:**
For all \( a, b \in W_n^{(h)} \), and \( w = (n - h)/2 \), if \( s_w(a) \neq s_w(b) \), then \(((a)_2, (b)_2) = 0 \).
If \( s_w(a) \neq s_w(b) \), then these strips have different size. Let us assume that \( j_a < j_b \).
Writing \((a)_2, (b)_2 = \text{Tr}((b)_2(a)_2^t)\), and \((b)_2 = B s_w(b), (a)_2 = A s_w(a)\), we have
\[
((a)_2, (b)_2) = \text{Tr}(B s_w(b) s_w(a)^t A^t) \tag{4.17}
\]

In this expression, we now transfer the boxes of \( s_w(b) \) onto \((a)_2^t\), starting from the lowest one, up to the top of \( s_w(b) \). These boxes now act on \( s_w(a)^t \) from below. Thanks to the relation \( e_i e_{i-1} e_i = e_i \), the first \( j_a - 2 \) grey boxes of the strip \( s_w(a)^t \) are annihilated by the action of the first \( j_a - 1 \) grey boxes of \( s_w(b) \), namely
\[
\begin{align*}
  s_w(b)s_w(a)^t &= \frac{\mu_2}{\mu_1} (e_{2w+j_b-2} - \mu_1) e_{2w+j_b-3} e_{2w+1} e_{2w+2} e_{2w+1} \\
  &\quad \times e_{2w} e_{2w+j_a-3} (e_{2w+j_a-2} - \mu_1) \\
  &= \frac{\mu_2}{\mu_1} (e_{2w+j_b-2} - \mu_1) e_{2w+j_b-3} e_{2w+1} e_{2w+2} e_{2w+1} \\
  &\quad \times \ldots e_{2w+j_a-3} (e_{2w+j_a-2} - \mu_1) \\
  &= \frac{\mu_2}{\mu_1} (e_{2w+j_b-2} - \mu_1) e_{2w+j_b-3} e_{2w+j_a-3} (e_{2w+j_a-2} - \mu_1)
\end{align*}
\tag{4.18}
\]
(Note that the last factor \((e_{2w+j_a-2} - \mu_1)\) must be replaced by \(e_{2w+j_a-2} = e_{2w-1}\) in the case \( j_a = 1 \), but this does not alter the following discussion.) Let us now transfer in the same way all the boxes of \( s_{w-1}(b) \), \( s_{w-2}(b) \), ..., \( s_1(b) \) onto \((a)_2^t\). But these occupy only positions \( k \leq 2w + j_b - 3 \), and the largest position \( k = 2w + j_b - 3 \) may only be occupied by a white box. Hence, after the transfer of \((b)_2\) onto \((a)_2^t\) is complete, the resulting element is a linear combination of the form
\[
(b)_2(a)_2^t = \alpha \ C' \ (e_{2w+j_b-2} - \mu_1) e_{2w+j_b-3} C' \\
+ \beta \ D' \ e_{2w+j_b-3} (e_{2w+j_b-2} - \mu_1) e_{2w+j_b-3} D
\tag{4.19}
\]
where \( C, D, C', D' \) are elements of the Temperley-Lieb algebra only involving the generators \( e_k, k < 2w + j_b - 3 \), and \( \alpha \) and \( \beta \) two complex coefficients, coming from the various normalization factors. The second term in \((4.19)\) vanishes identically, thanks to the identity \( e_i (e_{i+1} - \mu_1) e_i = 0 \). We are therefore left with
\[
((a)_2, (b)_2) = \alpha \ \text{Tr}(C'(e_{2w+j_b-2} - \mu_1) e_{2w+j_b-3} C') \\
= \alpha \ \text{Tr}((e_{2w+j_b-2} - \mu_1) e_{2w+j_b-3} C' C') \tag{4.20}
\]
To show that this expression vanishes, let us use the string representation of the \( e_i \), and the definition of the trace as computing \( q^L \), where \( L \) is the number of loops of the string.
representation of the element, after identification of the upper and lower ends of its strings. In this picture (setting $i = 2w + j_b - 2$, $CC' = E$, and taking the adjoint of the expression in the trace, which does not change its value), we have

$$\text{Tr}(E(e_1, e_2, ..., e_{i-2})e_{i-1}(e_i - \mu_1))$$

= \sum_r \alpha_r (q^L - \mu_1 q^{L+1})

= 0 \quad (4.21)$$

where we have expanded $E(e_1, e_2, ..., e_{i-2})$ as a linear combination of diagrams involving only grey boxes with positions $k \leq i - 2$. In each of these diagrams, the second term has always one more loop than the first one, hence the cancellation, with the factor $\mu_1 = q^{-1}$. This completes the proof of the lemma 2.

The lemma 2 guarantees that $((a)_2, (b)_2) = 0$ as soon as the last strips $s_w(a)$ and $s_w(b)$ are distinct. In the latter case, proposition 3 is therefore proved. Let us assume now that $(a)_2$ and $(b)_2$ have the same last strip, say with $j$ boxes. Then both $a$ and $b$ have a rightmost floor of height $H = j - 1$. Let $p_a$ and $p_b$ denote their respective widths, namely the respective numbers of grey boxes of height $j - 1$ forming this floor in $a$ and $b$. Two situations may occur for these floors

(i) they have the same width $p_a = p_b$. In this case, we will show that $((a)_2, (b)_2)$ is factored into the bilinear form (3.12) evaluated on smaller diagrams, obtained by cutting $(a)_2$ and $(b)_2$ into two pieces (lemma 3 below).

(ii) the width of the rightmost floor of $a$ is strictly smaller that that of the rightmost floor of $b$ $p_a < p_b$. In this case, we will show that $((a)_2, (b)_2) = 0$ (lemma 4 below).

Let us treat these cases separately.

**CASE (i)**: the two rightmost floors of $a$ and $b$ have the same width $p_a = p_b = p$. We will simply *grind* the $j - 2$ consecutive layers of grey boxes underlying the floor of height $j - 1$, and detach the corresponding portions of $a$ and $b$, so that the quantity $((a)_2, (b)_2)$
will factorize into a product of analogous terms, for smaller diagrams (see lemma 3 below).

More precisely, let us compute the quantity

\[ S(a)S(b)^t = s_{w-p+1}(a)s_{w-p+2}(a)...s_w(a)s_w(b)^t...s_{w-p+1}(b)^t \]  \hspace{1cm} (4.22)

involved in the computation of \((a)_2(b)_2^t\). In \(S\) of (4.22), all the strips involved have a floor of height \(j - 1\), i.e. have the form

\[ s_{w-m+1} = \Diamond \Diamond \ldots \Diamond e_{2w-2m+j-1}e_{2w-2m+j-2}\ldots e_{2w-2m+1} \]

where each strip \(\tilde{s}\) has a floor of only one grey box, topped by white boxes. The idea is to transfer the grey boxes from \(s_w(a)\) to \(s_w(b)\), from below, just like we did in (4.18), and do it again for \(s_{w-1}(a)\) and \(s_{w-1}(b)\), etc... until we are left only with the amputated strips \(\tilde{s}\).

The final result simply reads

\[ S(a)S(b)^t = \tilde{S}(a)\tilde{S}(b)^t = s_{w-p+1}(a)s_{w-p+2}(a)...s_w(a)s_w(b)^t...s_{w-p+1}(b)^t \]  \hspace{1cm} (4.24)

This result implies the following

**Lemma 3**: If \(a\) and \(b\) \(\in W_n^{(h)}\) have identical rightmost floors of width \(p\), then

\[ ((a)_2, (b)_2) = ((a')_2, (b')_2) ((a'')_2, (b'')_2) \]  \hspace{1cm} (4.25)

where

\[ (a')_2 = \mu_1^{n-2p} s_1(a)s_2(a)...s_{w-p}(a) \]

\[ (b')_2 = \mu_1^{n-2p} s_1(b)s_2(b)...s_{w-p}(b) \]  \hspace{1cm} (4.26)

\[ (a'')_2 = \mu_1^p \tilde{s}_{w-p+1}(a)...\tilde{s}_w(a) \]

\[ (b'')_2 = \mu_1^p \tilde{s}_{w-p+1}(b)...\tilde{s}_w(b) \]

The normalizations in (4.26) are chosen to guarantee that all the elements \((a')_2, (a'')_2, \ldots\) have norm 1, as we will see below. The lemma 3 will follow from the application of (4.24) to the computation of \(((a)_2, (b)_2) = \text{Tr}((a)_2(b)_2^t)\), Indeed, we simply write

\[ ((a)_2, (b)_2) = \mu_1^{2p} \text{Tr}((a')_2S(a)S(b)^t(b')_2^t) \]

\[ = \mu_1^{2p} \text{Tr}((a')_2\tilde{S}\tilde{S}^t(b')_2^t) \]

\[ = \text{Tr}((a'')_2(b'')_2^t(b'')_2^t(a')_2) \]

\[ = ((a'')_2, (b'')_2) \times ((a)_2, (b)_2) \]  \hspace{1cm} (4.27)
In the last step, we have noted that \((a''_2)(b''_2)^t\) involves only generators \(e_k\) with positions \(k \geq 2w - 2p + j - 1\) (position of the leftmost grey box in \(\bar{S}(a) = (a''_2)\)), whereas \((a'_2)(b'_2)^t\) involves only generators \(e_k\) with \(k \leq 2w - 2p + j - 3\) (maximum position of the rightmost (white) box in \((a'_2)\)). The last line of (4.27) follows then from the locality of the trace, namely that for any two sets of positions \(I, J\), with \(i < j - 1\) for all \(i \in I, j \in J\)

\[
\text{Tr}(\prod_{i \in I} e_i \prod_{j \in J} e_j) = \text{Tr}(\prod_{i \in I} e_i) \text{Tr}(\prod_{j \in J} e_j) \quad (4.28)
\]

which follows from the definition of the trace (the loops arising from the two terms are independent).

In (4.25), the bilinear forms \(((a)_2, (b)_2), ((a')_2, (b')_2)\) and \(((a'')_2, (b'')_2)\), are respectively evaluated in the spaces \(I_{n}^{(h)}(q), I_{n-2p}^{(h)}(q)\) and \(I_{2p+1}^{(1)}(q)\). Let us concentrate on the last term \(((a'')_2, (b'')_2)\). There is a simple morphism \(\varphi\) of algebras between \(I_{2p+1}^{(1)}(q)\) and \(I_{2p+2}(q) = I_p(q)\)

\[
\varphi(E) = \sqrt{\mu_1} E e_{2p+1} \in I_p(q) \quad \forall \ E \in I_{2p+1}^{(1)}(q) \quad (4.29)
\]

The morphism \(\varphi\) consists simply in adding the missing rightmost grey box \((e_{2p+1})\) to complete the floor of \(E\) into that of the ideal \(I_p(q)\). Moreover, we have added an ad-hoc multiplicative normalization factor \(\mu_1\). With this normalization, we have the following simple correspondence between traces over the two spaces

\[
\text{Tr}(\varphi(E)\varphi(F)^t) = \mu_1 \mu_1^{-1} \text{Tr}(E e_{2p+1} F^t) \quad (4.30)
\]

where we have used \(e_{2p+1}^2 = \mu_1^{-1} e_{2p+1}\), then transferred all the boxes of \(F^t\) onto \(E\), and represented the result in the pictorial string-domino representation of the trace (see Fig.15), to show that the presence of the grey box does not change the value of the trace (it does not affect the structure of the loops). Using this fact, we can now apply the result of the
proposition 1 of Sect.3.3 above to the factor \(((a'')_2, (b'')_2)\) of (4.24), by simply interpreting \(\varphi((a'')_2)\) and \(\varphi((b'')_2)\) as elements of \(\mathcal{I}_p(q)\). We conclude that

\[
((a'')_2, (b'')_2) = \left(\varphi((a'')_2), \varphi((b'')_2)\right) = \delta_{a''b''}
\]

Hence, if \(a'' \neq b''\), the bilinear form vanishes, and the proposition 3 follows. If \(a'' = b''\), we go back to the beginning of our study, with now \((a)_2\) and \((b)_2\) replaced with \((a')_2\) and \((b')_2 \in \mathcal{I}_{n-2p}^{(h)}(q)\). If only the case (i) occurs, we will dispose successively of each portion of \(a\) and \(b\) above their common successive floors (as above), and get an expression

\[
((a)_2, (b)_2) = \prod_{\text{portions } a''b'' \text{ above successive floors}} \delta_{a''b''}
\]

If the case (ii) occurs, the result will vanish, as we will see now.

**CASE (ii)**: The diagrams \((a)_2\) and \((b)_2\) have a rightmost floor, of same height \(j - 1\), but with different widths \(p = p_a < p_b\). As in the case (i), we concentrate on the portions of \(a\) and \(b\) above this rightmost floor, over a width \(p\), namely consider

\[
S(a) = s_{w-p+1}(a)s_{w-p+2}(a)...s_w(a)
\]

\[
S(b) = s_{w-p+1}(b)s_{w-p+2}(b)...s_w(b)
\]

To compute the quantity \(((a)_2, (b)_2)\), we now write \((a)_2 = (a')_2 S(a)\) and \((b)_2 = (b')_2 S(b)\), and get

\[
((a)_2, (b)_2) = \text{Tr}((a')_2 S(a) S(b) t (b')_2)
\]

\[
= \text{Tr}((b')_2 t (a')_2 \tilde{S}(a) \tilde{S}(b) t)
\]

Using the cyclicity of the trace and the symmetry of \([3,12]\), we may also write

\[
((a)_2, (b)_2) = \text{Tr}(\tilde{S}(a) t (a')_2 t (b')_2 \tilde{S}(b))
\]

\[
= \text{Tr}(\tilde{S}(a) t (a')_2 \tilde{S}(b))
\]

We have used the commutation of \(\tilde{S}(a)\), which involves only boxes of positions \(\geq \alpha = 2w - 2p + j - 1\), with \((a')_2\), which only involves boxes of positions \(\leq \alpha - 2\) (as its rightmost floor has now an height \(< j - 1\)). Let us now compute \((4.33)\) by transferring the white boxes of \(\tilde{S}(a) t\) onto \((b')_2 \tilde{S}(b)\). Once this transfer is complete, \((b')_2 \tilde{S}(b)\) is replaced by a linear combination of diagrams \((c')_2 \tilde{S}(c)\) with all possible box additions/subtractions induced by
the process of transfer. Note that the left portion \((b')_2\) of \((b)_2\) is also affected, as these boxes act on both \((b')_2\) and \(\tilde{S}(b)\). For notational simplicity, we have used the denomination \(c'\) for the left part of \(c\), so that we still have \((c)_2 = (c')_2S(c)\). We are then left with the transfer of the first layer of grey boxes of \(\tilde{S}(a)^t\), namely those of height \(j - 1\). To get a non-zero result, those must only hit minima or maxima on \((c')_2\tilde{S}(c)\) (the action of a grey box on a white slope vanishes, according to (3.32) (3.33)). Concentrating on the configuration of \((c')_2\tilde{S}(c)\) above the position \(\alpha = 2w - 2p + j - 1\) (namely the configuration of \((c')_2\) above the leftmost grey box in \(\tilde{S}(c)\)), only two situations may yield a non-zero answer

(a) \((c')_2\) has no white box above the position \(\alpha\).

(b) \((c')_2\) has a white maximum at \(\alpha\). In this case, this maximum is necessarily at height \(j + 2\) (white box of height \(j + 1\), hence of relative height 2 w.r.t. the floor), because no white slope is allowed at any of the positions \(\alpha, \alpha + 2, \ldots, \alpha + 2p - 2 = 2w + j - 3\), and the rightmost white box of \(\tilde{S}(c)\) has an height \(\leq j\), hence a relative height \(\leq 1\). Let us transfer this white box \textit{back} onto what is left of \(\tilde{S}(a)^t\) (call it \(\tilde{S}(d)^t = e_\alpha e_{\alpha+2} \ldots e_{\alpha+2p-2}\)). Actually, this diagram has now a grey maximum at the position \(\alpha\) (this is the position of the leftmost grey box in the floor of \(S(a)\)). The white box acts on this grey maximum as \(\sqrt{\mu_3/\mu_2(e_\alpha - \mu_2)e_\alpha} = e_\alpha/\sqrt{\mu_2\mu_3}\), hence is eliminated up to some multiplicative constant. We therefore end up in a situation where \((c')_2 \rightarrow (c' - \circ_\alpha)_2\) has no white box above the position \(\alpha\) hence in the case (a) above.

In either case, we end up in a situation where \(\alpha\) is a floor-end on both diagrams \((a')_2\tilde{S}(d)\) and \((c'')_2\tilde{S}(c)\), where \(c'' = c'\) in the case (a) and \(c'' = c' - \circ_\alpha\) in the case (b). So we can
reexpress

\[
((a)_2, (b)_2) = \sum_c \lambda_c \text{Tr}((a')_2^\dagger (c'')_2 S(c) S(d)^\dagger)
\]

\[
= \sum_c \lambda_c
\]

\[
= \mu_1 \sum_c \lambda_c
\]

\[
= \mu_1 \sum_c \lambda_c ((a')_2, (c'')_2) \text{Tr}(S(c) S(d)^\dagger)
\]

(4.36)

where, by using the string-domino picture, we have removed the grey box linking the left and right parts of the operator in the trace, at the expense of creating a new loop, hence the extra factor of \(\mu_1 = q^{-1}\). Note that in this argument it was crucial that there should be no white box above the grey box we have removed, further linking the left and right parts: this is why we had to go through the case (b) above and modify \(c' \rightarrow c'' = c' - \cdot\) to get back to the situation (a).

Now the main feature of \((c'')_2\) is that it has still a rightmost grey floor of height \(j - 1\), whereas by definition \((a')_2\) has a rightmost grey floor of height \(< j - 1\). Hence the rightmost strips in both diagrams are distinct: \(s_{w-p}(a') \neq s_{w-p}(c'')\). We can therefore apply the lemma 2, to conclude that

\[
((a)_2, (c'')_2) = 0
\]

(4.37)

in (4.36), so that finally \((a)_2, (b)_2) = 0\). Hence we deduce the

**LEMMA 4:**

For any two bicolored box diagrams \((a)_2\) and \((b)_2\), with rightmost floors of same height \(j - 1\), but of different widths \(p_a < p_b\), we have

\[
((a)_2, (b)_2) = 0
\]

(4.38)
The proof of the proposition 3 is now straightforward. We start with the two bicolored box diagrams \((a)_2\) and \((b)_2\). If their rightmost strips are distinct, then \(((a)_2, (b)_2) = 0\) by the lemma 2. Otherwise, we focus our attention to their rightmost floors, which have the same height \(j - 1\). If they have different widths, the lemma 4 above implies that \(((a)_2, (b)_2) = 0\). If they have the same width, the lemma 3 expresses \(((a)_2, (b)_2) = \prod \delta_{a''_b''}\), hence we finally get that \(((a)_2, (b)_2) = 0\) unless \(a\) and \(b\) are identical, in which case \(((a)_2, (a)_2) = 1\). This completes the proof of the proposition 3.

4.3. The semi-meander determinant: a preliminary formula

The semi-meander determinant \((4.6)\) follows from the Gram determinant of the basis 1. The latter is best expressed through the change of basis \(1 \to 2\), in which the Gram matrix is sent to the \(c_{n,h} \times c_{n,h}\) identity matrix \(I\). With the upper triangular matrix \(P\) defined in \((4.13)\), this reads

\[
P \Gamma_{n}^{(h)}(q) P^t = I \tag{4.39}
\]

Hence \(\det \Gamma_{n}^{(h)}(q) = (\det P)^{-2}\). The diagonal elements of \(P\) are linked by the recursion relation

\[
P_{a + \diamond_{i,\ell}, a + \diamond_{i,\ell}} = \sqrt{\frac{\mu_{\ell+1}}{\mu_{\ell}}} P_{a, a} \tag{4.40}
\]

where the box addition \(\diamond_{i,\ell}\) is performed at the point \(i\), and at relative height \(\ell\), with respect to the grey box floor in \(a\). With the initial condition \(P_{a_{n}^{(h)}}, a_{n}^{(h)} = \mu_{n/2}^{1/2}\) \((4.7)\) for the fundamental walk diagram of \(W_{n}^{(h)}\), this gives

\[
P_{a, a}^2 = \mu_{1}^{n} \prod_{\text{white boxes of } a} \frac{\mu_{\ell+1}}{\mu_{\ell}} \tag{4.41}
\]

where \(\ell\) denotes the height of the white box addition, relative to the grey floor in \(a\).

Fig. 19: The strips of white boxes on a walk \(a \in W_{n}^{(h)}\) with \(n = 20\) and \(h = 6\). The walk is represented in a thick black line. We have also represented the floor of grey boxes for this walk. We have \((n - h)/2 = 7\) strips of white boxes, of respective lengths 2, 1, 3, 2, 2, 2.
Like in the meander case, let us arrange the white boxes of any bicolored box diagram corresponding to an \( a \in W_n^{(h)} \) into strips of white boxes, namely sequences of white boxes with consecutive positions and heights, added on top of the grey floor of \( a \) (see Fig.19 for an illustration). There are exactly \((n - h)/2\) such white strips. The strip length is now defined as the relative height of the top of the white box sitting on top of the strip (hence an empty strip has length 1). With this definition, we simply have

\[
P_{a,a}^2 = \mu_1^{-\frac{n+h}{2}} \prod_{\text{white strips of } a} \mu_\ell \tag{4.42}
\]

where, in the product over the \((n - h)/2\) strips of \( a \in W_n^{(h)} \), \( \ell \) stands for the strip length (all denominators have been cancelled along the strips, except for the \( \mu_1 \) ones, which have rebuilt the prefactor). This yields the determinant of the basis 1

\[
det \Gamma_n^{(h)}(q) = \prod_{a \in W_n^{(h)}} P_{a,a}^{-2} = \mu_1^{-\frac{(n+h)c_{n,h}}{2}} \prod_{\text{white strips of all } a \in W_n^{(h)}} \mu_\ell^{-1} \tag{4.43}
\]

and thanks to (4.3), the semi-meander determinant

\[
det G_n^{(h)}(q) = \mu_1^{-hc_{n,h}} \prod_{\text{white strips of all } a \in W_n^{(h)}} \mu_\ell^{-1} \tag{4.44}
\]

The latter can be recast into

\[
det G_n^{(h)}(q) = \mu_1^{-hc_{n,h}} \prod_{m=1}^{\frac{n-h+1}{2}} [\mu_m]^{-s_{n,m}^{(h)}} \tag{4.45}
\]

where \( s_{n,m}^{(h)} \) denotes the total number of white strips of length \( m \) in all the bicolored box diagrams corresponding to the walk diagrams of \( W_n^{(h)} \) (the notation is such that \( s_{2n,m}^{(0)} = s_{2n,m} \) (3.50)). Note also that the strips have all length \( \leq (n - h)/2 + 1 \), hence the upper bound in the product in (4.45). The formula of theorem 2 will follow from the explicit computation of the numbers \( s_{n,m}^{(h)} \).

This will be done in two steps. The first step (Proposition 4, Sect.4.4 below) consists in arranging the \( s_{n,m}^{(h)} \) walks above according to their floor configuration (namely their configuration of grey boxes). The second step (Proposition 5, Sect.4.5 below) consists in enumerating the walks with minimal floor configurations (namely made of only one layer of grey boxes). Finally in the proposition 6, Sect.4.6, the combination of these two results will eventually lead to a formula for \( s_{n,m}^{(h)} \), which will complete the proof of the theorem 2.
4.4. Enumeration of the floor configurations

In this first step, we note that many different diagrams \( a \in W_{n}^{(h)} \) have the same contribution to (4.44), namely those with identical white strips, but different floors of grey boxes. Assembling all these contributions leads to the following formula for \( s_{n,m}^{(h)} \)

**PROPOSITION 4 :**

\[
s_{n,m}^{(h)} = \sum_{k \geq 0} \binom{h+k-1}{k} f_{n-h+2,m,k} \tag{4.46}
\]

where \( f_{2n,m,k} \) denotes the total number of walk diagrams \( a \in W_{2n} \) with \( k + 1 \) floors of height 0, and with a marked top of strip of length \( m \).

---

**Fig. 20:** A typical walk diagram \( a \in W_{n}^{(h)} \) is represented in thick black line on the upper diagram. We have also represented its floor of grey boxes, and the white boxes topping it. The floor of grey boxes in \( a \) is a succession of a number \( k + 1 \) of horizontal floors, \( F_0, F_1, ..., F_k \), with respective heights \( H_0 = 0, H_1, H_2, ..., H_k \geq 0 \). The conjugates of \( a \) are obtained by varying these heights, without changing the white strips of \( a \) (this is done by letting the floors slide along the dashed lines separating them). The minimal conjugate \( \hat{a} \in W_{n-h+2} \) of \( a \) is represented below it: it has \( H_1 = H_2 = ... = H_k = 0 \). The floor-ends are indicated by arrows.

Indeed, as illustrated in Fig.20, in any walk diagram \( a \in W_{n}^{(h)} \), the floor of grey boxes may be viewed as a succession of a number, say \( k + 1 \) of consecutive horizontal floors of grey boxes \( F_0, F_1, ..., F_k \), with respective heights \( H_0 = 0, H_1, ..., H_k \), and \( H_j \geq 0 \) for all \( j \geq 1 \). The leftmost floor \( F_0 \), of height \( H_0 = 0 \), is made of one layer of grey boxes of the
form $e_1e_3e_5...$, and occupies a segment $I_0 = \{i_0 = 0, 1, 2, ..., i_1 - 1\}$ of positions (we include here the case when $F_0 = \emptyset$, i.e. $I_0 = \{0\}$, corresponding to the case $J_0 \neq \{0\}$ of (4.8)). It is topped by white strips of arbitrary lengths. Any horizontal floor $F_j$, $j = 1, 2, ..., k$, of height $H_j \geq 0$, is a parallelogram made of $H_j + 1$ horizontal layers of grey boxes, whose base occupies a segment of positions $I_j = \{i_j, i_j + 1, i_j + 2, ..., i_{j+1}\}$, with $i_{k+1} = n - h + 1$.

What distinguishes these floors from $F_0$ is that they are necessarily topped with at least two layers of white boxes, resulting in white strips of lengths $m \geq 2$ only. The separation between two consecutive floors of height $H_j \geq 0$ is formed by the strips of length 2 (with one white box), as illustrated in the upper diagram of Fig.20, where the floor separations are indicated by dashed lines. The various floor heights are subject to the constraint

$$0 \leq H_1 \leq H_2 \leq ... \leq H_k \leq h - 1$$

(4.47)

arising from the original definition of the floor of a walk $a \in W_n^{(h)}$ (the floor $F_j$ is always of lesser or equal height than $F_{j+1}$).

By varying only the heights $H_1$, $H_2$, ..., $H_k$ subject to (4.47), and by keeping the white strips fixed, we describe the set of all conjugates of a given walk diagram $a \in W_n^{(h)}$. There are therefore

$$|\{(H_1, ..., H_k) \in \mathbb{N} \text{ s.t. } 0 \leq H_1 \leq H_2 \leq ... \leq H_k \leq h - 1\}| = \binom{h + k - 1}{k}$$

(4.48)

such conjugates for each diagram $a \in W_n^{(h)}$ with $k + 1$ floors. We now choose among the conjugates of $a$, the minimal one, namely that with $H_1 = H_2 = ... = H_k = 0$, which we denote by $a$ (the bottom diagram of Fig.20). We may amputate this diagram from the final slope with positions $n - h + 2$, $n - h + 3$, ..., $n$, and view it as a diagram $\hat{a} \in W_{n-h+2}$. Indeed, the diagram $\hat{a}$ has $h(n - h + 1) = 1$, the height of the rightmost floor-end, hence we may complete it by $h(n - h + 2) = 0$ into an element of $W_{n-h+2}$. Denoting by $f_2n,k,m$ the total number of walks of $W_{2n}$ with $k + 1$ floors of height 0, and with a marked top of strip of length $m$, the proposition 4 follows, by enumerating these $f_{n-h+2,k,m}$ walk diagrams with a marked top of strip of length $m$, and weighing each of them by the number of its conjugates (4.48).
4.5. The mapping of walk diagrams

The second step of the calculation of \( s_{n,m}^{(h)} \) is the computation of the numbers \( f_{2n,k,m} \) appearing in (4.46). The result reads

**PROPOSITION 5:**

The total number \( f_{2n+2,k,m} \) of walks in \( W_{2n+2} \), with \( k+1 \) floors of height 0, and with a marked top of strip of length \( m \) reads

\[
\begin{align*}
 f_{2n+2,k,m} &= c_{2n-k,2m+k} + kc_{2n-k,2m+k-4} & \text{for } m \geq 2, n \geq 1 \\
 f_{2n+2,k,1} &= c_{2n-k,k+2} & \text{for } m = 1, n \geq 1
\end{align*}
\]

(4.49)

where the numbers \( c_{n,h} \) are defined in (2.3). Here we have excluded the trivial case \( n = 0 \), for which no strip appears, hence

\[
 f_{2,k,m} = 0 \quad \text{for all } k \text{ and } m
\]

(4.50)

To prove this proposition, we will construct a bijection from the set of walk diagrams of \( W_{2n+2} \) with \( k+1 \) floors of height 0, and with a marked top of strip of length \( m \geq 2 \) to (i) the set \( W_{2n-k}^{(2m+k)} \) (ii) \( k \) copies of the set \( W_{2n-k}^{(2m+k-4)} \), which will prove (4.49), as \( |W_n^{(h)}| = c_{n,h} \). (In the case \( m = 1 \), only the part (i) will apply, namely, we will construct a bijection between the walks of \( W_{2n+2} \) with a marked end of empty strip (length 1) and \( W_{2n-k}^{(k+2)} \).)

We start from \( a \in W_{2n+2} \), with \( k+1 \) floors, all of height 0, and with a marked top of strip of length \( m \). Two cases may occur:

(i) The marked top of strip lies above the leftmost floor \( (F_0) \). In this case, we will construct a walk \( b \in W_{2n-k}^{(2m+k)} \) by a cutting-reflecting-pasting procedure on \( a \), analogous to that used in the meander case. This will produce the first term in (4.49).

(ii) The marked top of strip lies above one of the \( k \) other floors \( (F_1, F_2, ..., F_k) \). This is possible only if \( m \geq 2 \), as there is no empty strip above these floors, by definition. By a circular permutation of the \( k \) floors, we can always bring the block containing the marked point to the right. We therefore have a \( k \)-to-one mapping to the situation where the marked strip is above the rightmost floor. This \( k \)-fold circular permutation symmetry is responsible for the factor \( k \) in the second term of (4.49). The diagrams with the marked top of strip of length \( m \) above the rightmost \( (F_k) \) floor are then
mapped to the walk diagrams with a marked top of strip at height \( m - 2 \) above the leftmost \( (F_0) \) floor considered in the case (i), hence to the set \( W_{2n-k}^{(2(m-2)+k)} = W_{2n-k}^{(2m+k-4)} \) (the case \( m = 2 \) will have to be treated separately). Together with the multiplicity factor \( k \) this will produce the second term of (4.49).

Let us now construct the maps for the cases (i) and (ii) above.

**CASE (i)**: We start from a walk diagram \( a \in W_{2n+2}^{2(m+k)} \), with \( k + 1 \) floors of height 0, and with a marked top of strip of length \( m \) above its leftmost floor \( F_0 \), say at position \( i \).

The point \((i, h(i) = m)\) separates \( a \) into a left \( L \) and right \( R \) parts, respectively such that \( L \in W_i^{(m)} \) and \( \bar{R} \in W_{2n+2-i}^{(m)} \). Reflecting \( L \) and pasting it again at the left end of \( R \), we create a walk diagram \( b' \), whose reflection \( \bar{b}' \in W_{2n+2}^{(2m)} \). To construct the eventual image \( b \in W_{2n-k}^{(2(m+k))} \) of \( a \), we perform the following amputations of the walk \( b' \). We will suppress some pieces of \( b' \) at each separation of floor, according to the following rules

\[
\begin{align*}
(1) & \quad \text{with } (i, h(i) = m) \Rightarrow (i, h(i) = m + 3) \quad (o \rightarrow o - 1) \\
(2) & \quad \text{with } (i, h(i) = m - 1) \Rightarrow (i, h(i) = m + 1) \quad (o \rightarrow o - 1) \\
(3) & \quad \text{with } (i, h(i) = m) \Rightarrow (i, h(i) = m - 2) \quad (o \rightarrow o - 2)
\end{align*}
\]

where we have represented the floor end by an empty circle, and where we indicate the change in final height \( (h) \) and in the order \( (o) \) resulting from the amputation. Considering that the rules in (4.51) apply respectively (1) to the first floor separation only (between \( F_0 \) and \( F_1 \)), (2) to the \( k - 1 \) intermediate floor separations (between \( F_j \) and \( F_{j+1}, j = 1, 2, \ldots, k - 1 \)), and (3) to the rightmost floor end (right end of \( F_k \)), we get an overall change from the initial values \( (h = 2m, o = 2n + 2) \) of the height and order of \( b' \) to the amputed \( b'' \) with

\[
\begin{align*}
h \rightarrow h + 3 + (k - 1) - 2 = 2m + k \\
o \rightarrow o - 1 - (k - 1) - 2 = 2n - k
\end{align*}
\]

Hence taking \( b = \bar{b}'' \), we get an element of \( W_{2n-k}^{(2m+k)} \).

To prove that this mapping is bijective, let us compute its inverse. Starting from \( b \in W_{2n-k}^{(2m+k)} \), let \( i \) be the position of the *rightmost* intersection between \( b \) and the line \( h = m+k \) at an ascending slope \((h(i-1)+1) = h(i) = h(i+1)-1\). This point separates the walk \( b \) into a left part \( L \) and a right part \( R \). Let us reflect \( R \) and paste it again to the right end of \( L \). This produces a walk \( a' \in W_{2n-k}^{(k)} \). As before, if \( h(i+1) - 1 = h(i) = m + k \), we mark the point \( i \), which will be an end of strip (in the eventually reflected walk). Otherwise,
\( h(i + 1) = h(i) - 1 \), and we migrate the mark to the point \( i' = \max\{j < i | h(j + 1) = h(j) - 1 = m + k\} \). Let us now mark (by black dots) the rightmost intersections between \( a' \) and the lines \( h = k, h = k - 1, \ldots, h = 1 \) at ascending slopes of \( a' \), and also the left end \((i = 0, h = 0)\) of \( a' \). We reconstruct the \( k + 1 \) separations of floors using the following rules (inverse of (4.51))

\[
\begin{align*}
(1) & \quad \bullet \rightarrow \begin{array}{c}
\circ \\
\circ
\end{array} \\
(2) & \quad \begin{array}{c}
\circ \\
\circ
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ
\end{array} \\
(3) & \quad \begin{array}{c}
\circ \\
\circ
\end{array} \rightarrow \begin{array}{c}
\circ \\
\circ
\end{array}
\end{align*}
\]

The corresponding separations have been represented by empty circles. They all lie at height \( h = 1 \) in the resulting final walk \( a'' \). The three rules of (4.53) apply respectively (1) to the left end of \( a' \), (2) to any of the \( k - 1 \) intermediate points of intersection with the lines \( h = 1, \ldots, h = k - 1 \), and (3) to the rightmost intersection with the line \( h = k \). The rules (4.53) therefore result in a change of final height and order \((h = k, o = 2n - k) \rightarrow (h + 2 - (k - 1) - 3 = 0, o + 2 + (k - 1) + 1 = 2n + 2)\), hence \( a'' \in W_{2n+2} \). The last step consists simply in reflecting \( a'' \), to produce \( a = \tilde{a}' \), with a marked top of strip of height \( m + k - k = m \) above the leftmost \((F_0)\) floor, and \( a \) has a total of \( k + 1 \) floors, all of height 0. As before, the bijectivity of the map follows from the fact that we considered rightmost points of intersection, which makes the construction unique. This bijection yields the number \( c_{2n-k,2m+k} \) of walks in \( W_{2n+2} \) with \( k + 1 \) floors of height 0, and with a marked top of strip of length \( m \) above \( F_0 \). This is the first term of (4.49).

**CASE (ii) :** We start from a walk \( a \in W_{2n+2} \), with \( k + 1 \) floors \( F_0, F_1, \ldots, F_k \), all of height zero, and with a marked top of strip of length \( m \) above its rightmost floor \( F_k \). By definition, we necessarily have \( m \geq 2 \), and in fact there is one and only one strip of length 2 above the floor \( F_k \) (the one just above the right floor-end), and all other strips have length \( \geq 3 \). We now construct a bijection between these walks and the \( b \in W_{2n+2} \) with \( k + 1 \) floors \( F_0', F_1', \ldots, F_k' \), all of height 0, and with a marked top of strip of length \( m - 2 \) above their leftmost floor \( F_0' \).

If \( m = 2 \), the above remark shows that the number of walks \( a \) with \( k + 1 \) floors of height 0, and with a marked top of strip of length 2 above \( F_k \) is equal to the number of such walks, without marked top of strip (there is exactly one such strip of length 2 per walk). Skipping the cutting-reflecting-pasting procedure of the case (i) (we have no more marked
top of strip), we can still apply the amputation rules (4.51) on the walk $b' = \bar{a} \in W_{2n+2}$: this results in a walk $b \in W_{2n-k}^{(k)}$. Conversely, starting from any $b \in W_{2n-k}^{(k)}$, let us apply to it the inverse of the amputation rules (4.53), after marking the rightmost intersections at ascending slopes with the lines $h = k, h = k - 1, \ldots, h = 1$. This produces a walk $a' \in W_{2n+2}$, and finally $a = \bar{a}' \in W_{2n+2}$ has $k + 1$ floors of height 0. This bijection yields the number $c_{2n-k,k}$ of walks in $W_{2n+2}$ with $k + 1$ floors of height 0, and a marked top of strip of height $m = 2$ above $F_k$. Together with the $k$-fold cyclic degeneracy of the case (ii) this gives the second term of (4.49), for $m = 2$.

\[ \text{Fig. 21: The exchange map on walk diagrams of } W_n^{(h)}, \text{ maps the walks with a marked strip of length } m \text{ above their rightmost floor onto those with a marked strip of length } m - 2 \text{ above the leftmost floor (the corresponding strip of length } m = 3 \text{ is marked with a black dot on the figure). We have indicated by a thick broken line the portions exchanged. The double-layer of white boxes on the rightmost floor is adapted to fit the exchange.} \]

If $m \geq 3$, we simply exchange the floors $F_0$ and $F_k$ in the following way. The floor $F_k$ is by definition topped by at least two layers of white boxes (see Fig.21). Let $i_k, i_k + 1, \ldots, i_{k+1}$ denote the positions occupied by $F_k$, the ends $i_k$ and $i_{k+1}$ being at height 1. Let us cut out the portion $a_k$ of $a$ inbetween the positions $i_k + 2$ and $i_{k+1} - 2$, both at height 3 (the level of the second layer of white boxes). Let us also cut the portion $a_0$ of $a$ above the leftmost floor $F_0$, inbetween the positions $i_0 = 0$ and $i_1 - 1$, both at height 0. We form a walk $b \in W_{2n+2}$ by simply exchanging the portions $a_0$ and $a_k$ in $a$, as depicted in Fig.21. The marked top of strip on $a_k$ has been therefore transferred above the leftmost floor of $b$, but as two layers of white boxes have been suppressed, all the lengths of strips have been decreased by 2. Hence the walk $b \in W_{2n+2}$ has $k + 1$ floors of height 0, and a marked top of
strip of length \( m - 2 \) above its leftmost floor \( F_0' \). This construction is clearly invertible, by just exchanging again \( a_k \) and \( a_0 \). From the case (i) above, we learn that the walk \( b \) can be mapped onto an element of \( W_{2n-k}^{(2(m-2)+k)} = W_{2n-k}^{(2m+k-4)} \), in a bijective way. This bijection yields the number \( c_{2n-k,2m+k-4} \) of walks \( a \in W_{2n+2} \) with \( k+1 \) floors of height 0, and with a marked top of strip of length \( m \) above its rightmost floor \( F_k \). With the overall \( k \)-fold cyclic degeneracy mentioned above, this gives the second term in \((4.49)\) for all \( m \geq 3 \).

The mappings of the cases (i) and (ii) above complete the proof of proposition 5, with the understanding that the case \( m = 1 \) only gives rise to the case (i), hence the different answer.

4.6. The semi-meander determinant: the final formula

Combining the results of propositions 4 and 5, namely eqs.\((4.46)\) and \((4.49)\), we get the following formula for the numbers \( s_{n,m}^{(h)} \) of walk diagrams in \( W_n^{(h)} \) with a marked top of strip of length \( m \) above its floor

\[
\begin{align*}
  s_{n,m}^{(h)} &= \sum_{k \geq 0} \binom{h + k - 1}{k} (c_{n-h-k,2m+k} + kc_{n-h-k,2m+k-4}) & \text{for } m \geq 2 \\
  s_{n,1}^{(h)} &= \sum_{k \geq 0} \binom{h + k - 1}{k} c_{n-h-k,h+2} & \text{for } m = 1
\end{align*}
\]

(4.54)

This is valid for \( h \leq n - 1 \). If \( h = n \), \((4.56)\) yields \( s_{n,m}^{(n)} = 0 \) for all \( m \). By a direct calculation, we find

**PROPOSITION 6:**

The numbers of walks in \( W_{2n+2} \) with \( k + 1 \) floors of height 0 and a marked end of strip of length \( m \) read

\[
\begin{align*}
  s_{n,m}^{(h)} &= c_{n,h+2m} + hc_{n,h+2m-2} & \text{for } m \geq 2, \ h \leq n - 1 \\
  s_{n,1}^{(h)} &= c_{n,h+2} & \text{for } m = 1, \ h \leq n - 1 \\
  s_{n,m}^{(n)} &= 0 & \text{for } h = n \text{ and all } m \geq 1
\end{align*}
\]

(4.55)

The proof relies on the following classical identity for binomial coefficients

\[
\sum_{k=b-a}^{c-d} \binom{k+a}{b} \binom{c-k}{d} = \binom{a+c+1}{b+d+1}
\]

(4.56)
for all integers $a$, $b$, $c$, $d$. This is easily proved by use of generating functions. We now simply have to apply (4.56) to the various sums appearing on the r.h.s. of (4.54)

$$\sum_{k \geq 0} \binom{k + h - 1}{h - 1} \binom{n - h - k}{\frac{n-h}{2} + m} = \binom{n}{\frac{n+h}{2}} = \binom{n}{\frac{n-(h+2m)}{2}}$$

$$\sum_{k \geq 0} \binom{k + h - 1}{h - 1} \binom{n - h - k}{\frac{n-h}{2} + m + 1} = \binom{n}{\frac{n+h}{2}} + m + 1) = \binom{n}{\frac{n-(h+2m)}{2} - 1}$$

(4.57)

hence

$$\sum_{k \geq 0} \binom{k + h - 1}{k} c_{n-h-k,k+2m} = c_{n,h+2m}$$

(4.58)

and, noting that $k\binom{k+h-1}{k} = h\binom{k+h-1}{k-1}$, we also get

$$\sum_{k \geq 0} k \binom{k + h - 1}{k} c_{n-h-k,k+2m-4} = hc_{n,h+2m-2}$$

(4.59)

The propositions 6 follows from (4.58) and (4.59).

Substituting the result (4.58) above into (4.43), we finally get the semi-meander determinant

$$\det G_n^{(h)}(q) = \prod_{m=1}^{n-h+1} [\mu_m]^{-(c_{n,h+2m} + hc_{n,h+2m-2})}$$

(4.60)

where we have absorbed the prefactor $\mu_1^{-hcn,h}$ of (4.45) into the $m = 1$ term of the product. Finally, using the fact that $\mu_m = U_{m-1}(q)/U_m(q)$, for $m \geq 1$, the theorem 2 follows.

5. Conclusion

In this paper, we have proved two determinant formulas for meanders and semi-meanders. This has been done by the Gram-Schmidt orthogonalization of the corresponding bases 1 of the Temperley-Lieb algebra or some of its subspaces. The main philosophy of the construction leading to the Gram-Schmidt orthogonalization of these bases 1 lies in the concept of box addition, the building block of the definition of the bases 2 elements. We believe that this type of construction should be much more general and apply to many other cases of algebra-related Gram-Schmidt orthogonalization.

An important remark about Theorems 1 and 2 above is that they implicitly give the structure (including multiplicity) of the zeros of the Gram determinants, considered as
functions of the variable $q$. Due to the definition of the Chebyshev polynomials, the zeros of the Gram determinant always take the form

$$q = 2 \cos\left(\pi \frac{m}{p+1}\right)$$

with $1 \leq m \leq p \leq \frac{n-h}{2} + 1$ in the semi-meander case. These zeros actually correspond to the cases when the corresponding subspace of the Temperley-Lieb algebra is reducible (there are linear combinations of the basis 1 elements which are orthogonal to all the basis 1 elements: i.e. there may be vanishing linear combinations of the basis 1 elements, the basis 1 being therefore no longer a basis at these values of $q$). The multiplicity of these zeros is linked to the degree of reducibility (namely to how many such independent linear combinations exist).

Unfortunately, the information we obtain from these determinant formulas on the meanders and the semi-meanders themselves is very difficult to exploit. Indeed, quantities such as asymptotics (for large order) of the meander and semi-meander numbers and distributions are only indirectly related to the Gram determinants, as they would rather involve the exact knowledge of the asymptotics of the Gram matrices, or at least of their eigenvalue spectra. However, the exact orthogonalization performed above is useful to derive new asymptotic formulas for the meander numbers, as sums over walk diagrams (see [10] for the meander example). We hope to return to this question in a later publication.

The Theorems 1 and 2 above can probably be generalized in many directions. A first possibility relies on the fact that there exists a canonical Temperley-Lieb algebra attached to any non-oriented, connected graph (see [13] and references therein), which may still be interpreted as the image of a walk diagram on that graph. The only constraint is that the number $q$ must be an eigenvalue of the adjacency matrix of the graph (a matrix $G_{a,b}$ made of 1’s and 0’s according to whether the couple $(a, b)$ of vertices of the graph is joined by a link or not). In the examples treated here, this graph is simply the set of heights, namely the integer points on the (infinite) half-line, linked by segments between consecutive pairs (hence $G_{i,j} = \delta_{j,i+1} + \delta_{j,i-1}$ for $i, j > 0$ and $G_{0,j} = \delta_{j,1}$, with the eigenvalue $q$ for the (infinite) eigenvector $\vec{v} = (U_0(q), U_1(q), U_2(q), ...)$). But nothing prevents us from considering more complicated graphs. We believe that there exists a general determinant formula, associated to each such graph, expressing the result in terms of features of the graph only (with $c_{2n,2m}$ replaced by a corresponding number of paths of given length and given origin and end on the graph, and $U_m(q)$ replaced by the components of the eigenvector of the adjacency matrix for the eigenvalue $q$).
Another direction of generalization has to do with replacing the Temperley-Lieb algebra by a more general quotient of the Hecke algebra. Indeed, recall that the Temperley-Lieb algebra $TL_n(q)$ is nothing but a simple quotient of the Hecke algebra, defined as follows. The Hecke algebra $H_n(q)$ is defined by generators $1, e_1, e_2, \ldots, e_{n-1}$ satisfying the following relations

\begin{align*}
(i) \quad & e_i^2 = q \ e_i \\
(ii) \quad & [e_i, e_j] = 0 \quad \text{if } |i - j| > 1 \\
(iii) \quad & e_i e_{i+1} e_i - e_i = e_{i+1} e_i e_{i+1} - e_i + 1
\end{align*}

hence the Temperley-Lieb algebra is the quotient of the Hecke algebra by the ideal generated by the elements $e_i e_{i\pm 1} e_i - e_i$. This quotient was identified as the commutant of the quantum enveloping algebra $U\hat{q}(sl_2)$ acting on the fundamental representation of $H_n(q)$, with $q = \hat{q} + \hat{q}^{-1}$. More quotients are found by considering the commutants of other quantum enveloping algebras (such as $U\hat{q}(sl_k)$ for instance) \[14\]. These quotients await a good combinatorial interpretation, but should lead to a natural generalization of meanders and semi-meanders. We believe that many Gram determinants can still be computed exactly in this framework.
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