ON INTEGRABILITY OF 2-DIMENSIONAL $\sigma$-MODELS OF POISSON-LIE TYPE

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Abstract. We describe a simple procedure for constructing a Lax pair for suitable 2-dimensional $\sigma$-models appearing in Poisson-Lie T-duality.

1. Introduction

There is a class of 2-dimensional $\sigma$-models, introduced in the context of Poisson-Lie T-duality [5], whose solutions are naturally described in terms of certain flat connections. The target space of such a $\sigma$-model is $D/H$, where $D$ is a Lie group and $H \subset D$ a subgroup. The $\sigma$-model is defined by the following data: an invariant symmetric non-degenerate pairing $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{d}$ such that the Lie subalgebra $\mathfrak{h} \subset \mathfrak{d}$ is Lagrangian, i.e. $\mathfrak{h}^\perp = \mathfrak{h}$, and a subspace $V_+ \subset \mathfrak{d}$ such that $\dim V_+ = (\dim \mathfrak{d})/2$ and such that $\langle \cdot, \cdot \rangle|_{V_-}$ is positive definite. The construction and properties of these $\sigma$-models are recalled in Section 2 (including the Poisson-Lie T-duality, which says that the $\sigma$-model, seen as a Hamiltonian system, is essentially independent of $H$). Let us call them $\sigma$-models of Poisson-Lie type.

The solutions $\Sigma \to D/H$ of equations of motion of such a $\sigma$-model can be encoded in terms of $\mathfrak{d}$-valued 1-forms $A \in \Omega^1(\Sigma, \mathfrak{d})$ satisfying

\begin{align}
(1a) & \quad dA + [A, A]/2 = 0 \\
(1b) & \quad A \in \Omega^{1,0}(\Sigma, V_+) \oplus \Omega^{0,1}(\Sigma, V_-),
\end{align}

where $V_- := (V_+)^\perp \subset \mathfrak{d}$. Namely, the flatness (1a) of $A$ implies that there is a map $\ell : \tilde{\Sigma} \to D$ (where $\tilde{\Sigma}$ is the universal cover of $\Sigma$) such that $A = -d\ell \ell^{-1}$. If the holonomy of $A$ is in $H$ then $\ell$ gives us a well-defined map $\Sigma \to D/H$. The maps $\Sigma \to D/H$ obtained in this way are exactly the solutions of equations of motion.

As first observed by Klimčík [3], and later by Sfetsos [12], and Delduc, Magro, and Vicedo [2], some $\sigma$-models of Poisson-Lie type are integrable. Their integrability is proven by finding a Lax pair, i.e. a 1-parameter family of flat connections (with parameter $\lambda$)

$$ A_\lambda \in \Omega^1(\Sigma, g), \quad dA_\lambda + [A_\lambda, A_\lambda]/2 = 0 $$

where $g$ is a suitable semisimple Lie algebra. Such a family is constructed for every element of the phase space, i.e. for every $A \in \Omega^1(\Sigma, \mathfrak{d})$ satisfying (1).

The aim of this note is to make the construction of $A_\lambda$ transparent. We simply observe that if $A \in \Omega^1(\Sigma, \mathfrak{d})$ satisfies (1) and if $p : \mathfrak{d} \to g$ is a linear map such that $[p(X), p(Y)] = p([X, Y]) \quad \forall X \in V_+, \ Y \in V_-$

then

$$ dp(A) + [p(A), p(A)]/2 = 0. $$

A suitable family $p_\lambda : \mathfrak{d} \to g$ will then give us a family of flat connections

$$ A_\lambda = p_\lambda(A). $$

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As an example, we provide a very simple construction of such families \( p_\lambda \) in the case when \( \mathfrak{d} = \mathfrak{g} \otimes W \), where \( W \) is a 2-dimensional commutative algebra. These families recover the deformations of the principal chiral model from \([2, 3, 12]\). Our purpose is thus modest - it is simply to clarify previously constructed integrable \( \sigma \)-models. There is possibly a less naive construction of families \( p_\lambda \) that might produce new integrable models, but we leave this question open.

2. \( \sigma \)-models of Poisson-Lie type and Poisson-Lie T-duality

In this section we review the properties of the “2-dimensional \( \sigma \)-models of Poisson-Lie type” introduced in \([5]\) (together with their Hamiltonian picture from \([6]\) and using the target spaces of the form \( D/H \), as introduced in \([7]\)).

Let \( \mathfrak{d} \) be a Lie algebra with an invariant non-degenerate symmetric bilinear form \( \langle , \rangle \) of symmetric signature and let \( V_+ \subset \mathfrak{d} \) be a linear subspace with \( \dim V_+ = (\dim \mathfrak{d})/2 \), such that \( \langle , \rangle |_{V_+} \) is positive-definite.

Let \( M = D/H \) where \( D \) is a connected Lie group integrating \( \mathfrak{d} \) and \( H \subset D \) is a closed connected subgroup such that its Lie algebra \( \mathfrak{h} \subset \mathfrak{d} \) is Lagrangian in \( \mathfrak{d} \).

This data defines a Riemannian metric \( g \) and a closed 3-form \( \eta \) on \( M \). They are given by

\[
g(\rho(X), \rho(Y)) = \frac{1}{2} \langle X, Y \rangle \quad \forall X, Y \in V_+\]

\[
p^*\eta = -\frac{1}{2} \eta_D + \frac{1}{2} \delta(L, \theta_L)
\]

Here \( \rho \) is the action of \( \mathfrak{d} \) on \( M = D/H \), \( p : D \to D/H \) is the projection, \( \eta_D \in \Omega^3(D) \) is the Cartan 3-form (given by \( \eta_D(X^L, Y^L, Z^L) = \langle [X, Y], Z \rangle \) \( \forall X, Y, Z \in \mathfrak{d} \)), \( \theta_L \in \Omega^1(D, \mathfrak{d}) \) is the left-invariant Maurer-Cartan form on \( D \) (i.e. \( \theta_L(X^L) = X \)), and \( A \in \Omega^1(D, \mathfrak{h}) \) is the connection on the principal \( H \)-bundle \( p : D \to D/H \) whose horizontal spaces are the right-translates of \( V_+ \).

The metric \( g \) and the 3-form \( \eta \) then define a \( \sigma \)-model with the standard action functional

\[
S(f) = \int_{\Sigma} g(\partial_+ f, \partial_- f) + \int_{Y} f^*\eta
\]

where \( \Sigma \) is (say) the cylinder with the usual metric \( ds^2 = dt^2 + f^2 \) and \( f : \Sigma \to M \) is a map extended to the solid cylinder \( Y \) with boundary \( \Sigma \).

For our purposes, the main properties of these \( \sigma \)-models are the following:

- The solutions of the equations of motion are in (almost) 1-1 correspondence with 1-forms \( A \in \Omega^1(\Sigma, \mathfrak{d}) \) satisfying \([1]\). More precisely, a map \( f : \Sigma \to M \) is a solution iff it admits a lift \( \ell : \hat{\Sigma} \to D \) such that \( A := -d\ell \ell^{-1} \) satisfies \([1]\). Notice that \( A \) is uniquely specified by \( f \) (the lift \( \ell \) is not unique - it can be multiplied by an element of \( H \) on the right).

- When we restrict \( A \) to \( S^1 \subset \Sigma = S^1 \times \mathbb{R} \), we get a 1-form \( j(\sigma) d\sigma \in \Omega^1(S^1, \mathfrak{d}) \). The \( \mathfrak{d} \)-valued functions \( j(\sigma) \) on the phase space of the sigma model satisfy the current algebra Poisson bracket

\[
j_a(\sigma), j_b(\sigma') = f_{ab} j_c(\sigma) \delta(\sigma - \sigma') + t_{ab} \delta'(\sigma - \sigma')
\]

(written using a basis \( e^a \) of \( \mathfrak{d} \), with \( f_{ab}^c \) being the structure constants of \( \mathfrak{d} \) and \( t_{ab} \) the inverse of the matrix of \( \langle e^a, e^b \rangle \)). The Hamiltonian of the
\[ \mathcal{H} = \frac{1}{2} \int_{S^1} \langle j(\sigma), Rj(\sigma) \rangle \, d\sigma \]

where \( R : \mathfrak{d} \to \mathfrak{d} \) is the reflection w.r.t. \( V_+ \).

Finally, let us observe that the phase space of the \( \sigma \)-model depends on the choice of \( H \subset D \) only mildly; when we impose the constraint that \( A \) has unit holonomy, the reduced Hamiltonian system is independent of \( H \). This statement is the Poisson-Lie T-duality (in the case of no spectators). (In more detail, the phase space of the \( \sigma \)-model is the space of maps \( \ell : \mathbb{R} \to D \) which are quasi-periodic in the sense that for some \( h \in H \) we have \( \ell(\sigma + 2\pi) = \ell(\sigma)h \), modulo the action of \( H \) by right multiplication. The reduced phase space is \((LD)/D\) (i.e. periodic maps modulo the action of \( D \)); it is the subspace of \( \Omega^1(S^1, \mathfrak{d}) \) given by the unit holonomy constraint.)

3. Constructing new flat connections

As we have seen, the solutions of our \( \sigma \)-model give rise to flat connections \( A \in \Omega^1(\Sigma, \mathfrak{d}) \) satisfying (1). We can obtain new flat connections out of \( A \) using the following simple observation, which is also the main idea of this paper.

**Proposition 1.** Let \( \mathfrak{g} \) be a Lie algebra and let \( p : \mathfrak{d} \to \mathfrak{g} \) be a linear map such that

\[ p([X, Y]) = p([X, Y]) \quad \forall X \in V_+, \ Y \in V_- \]  

If \( A \in \Omega^1(\Sigma, \mathfrak{d}) \) satisfies (1) then \( p(A) \in \Omega^1(\Sigma, \mathfrak{g}) \) is flat, i.e.

\[ dp(A) + [p(A), p(A)]/2 = 0. \]

**Proof.** Let us use the following notation: for \( \alpha \in \Omega^1(\Sigma) \) let \( \alpha^+ \in \Omega^1(\Sigma) \) and \( \alpha^- \in \Omega^1(\Sigma) \) denote the components of \( \alpha \), i.e. \( \alpha = \alpha^+ + \alpha^- \). In particular, \( A^+ \in \Omega^1(\Sigma, V_+) \) and \( A^- \in \Omega^1(\Sigma, V_-) \). We then have

\[ dp(A) + [p(A), p(A)]/2 = dp(A) + [p(A^+), p(A^-)] = p(dA + [A^+, A^-]) = p(dA + [A, A]/2) = 0. \]

□

Given a 1-parameter family of maps \( p_\lambda : \mathfrak{d} \to \mathfrak{g} \) satisfying (1) we would thus get a 1-parameter family of flat connections \( A_\lambda = p_\lambda(A) \) on \( \Sigma \), which may then be used to show integrability of the model. Let us observe that the Poisson brackets of the “Lax operators” \( L(\sigma, \lambda) := p_\lambda(j(\sigma)) \) are automatically of the form considered in [14] (i.e. containing a \( \delta(\sigma - \sigma') \) and a \( \delta(\sigma - \sigma') \) term), and so one can in principle extract an infinite family of Poisson-commuting integrals of motion out of the holonomy of \( A_\lambda \).

**Remark 1.** The procedure of finding integrable deformations of integrable \( \sigma \)-models, due to Delduc, Magro, and Vicedo [1], can be rephrased in our formalism as follows. Suppose that for some particular pair \( V_+ \subset \mathfrak{d} \) we find a family \( p_\lambda : \mathfrak{d} \to \mathfrak{g} \) showing integrability of the model. Let us deform the Lie bracket on \( \mathfrak{d} \), and possibly the pairing \( \langle , \rangle \), in such a way that the restriction of the Lie bracket to \( V_+ \times V_- \to \mathfrak{d} \) is undeformed. Then the same family \( p_\lambda \) will satisfy (1) also for the deformed structure on \( \mathfrak{d} \) and show integrability of the deformed model. These deformations of \( \mathfrak{d} \) do not change the system (1) (and if \( \langle , \rangle \) is not deformed then they don’t change the Hamiltonian (3) either), but they do change the Poisson structure (2) on the phase space.

**Remark 2.** There is a version of \( \sigma \)-models of Poisson-Lie type, introduced in [8], with the target space if \( F \setminus D/H \), where \( f \subset \mathfrak{d} \) is an isotropic Lie algebra (and one needs to suppose that \( F \) acts freely on \( D/H \)). In this case \( V_+ \subset \mathfrak{d} \) is required to
be such that $\langle , \rangle|_{V_+}$ is semi-definite positive with kernel $\mathfrak{f}$ (in particular, $\mathfrak{f} \subset V_+$), and such that $[\mathfrak{f}, V_+] \subset V_+$ (we still have $\dim V_+ = (\dim \mathfrak{d})/2$). The phase space $\Theta$ is the Marsden-Weinstein reduction of $\Omega^1(S^1, \mathfrak{d})$ by $LF$, i.e. $\Omega^1(S^1, \mathfrak{f}^\perp)/LF$. The solutions of equations of motion are still given by the solutions of (1), though this time $A$ is defined only up to $F$-gauge transformations. In this case we can still use Proposition 1 without any changes. This setup should cover, in particular, the discussion of symmetric spaces in [1].

4. Getting a Lax pair in a simple case

In this section we give a simple example of pairs $V_+ \subset \mathfrak{d}$ with natural 1-parameter families $p_\lambda$ satisfying (4).

Let $\mathfrak{g}$ be a Lie algebra with an invariant inner product $\langle , \rangle_\mathfrak{g}$ and let $W$ be a 2-dimensional commutative associative algebra with unit. ($W$ is isomorphic to one of $\mathbb{C}$, $\mathbb{R} \oplus \mathbb{R}$, $\mathbb{R}[e]/(e^2)$.)

Let $\mathfrak{d} := \mathfrak{g} \otimes W$ with the Lie bracket $[X_1 \otimes w_1, X_2 \otimes w_2]_\mathfrak{d} = [X_1, X_2]_\mathfrak{g} \otimes w_1 w_2$.

We choose the following additional data in $W$ to produce a pairing $\langle , \rangle$ on $\mathfrak{d}$ and a subspace $V_+ \subset \mathfrak{d}:

To get the pairing, let $\theta : W \rightarrow \mathbb{R}$ be a linear form such that the pairing on $W$ given by $\langle w_1, w_2 \rangle_W := \theta(w_1 w_2)$ is non-degenerate (i.e. such that it makes $W$ to a Frobenius algebra) and indefinite. The pairing on $\mathfrak{d}$ is then defined via

$\langle X_1 \otimes w_1, X_2 \otimes w_2 \rangle := \langle X_1, X_2 \rangle_\mathfrak{g} \theta(w_1 w_2)$.

To get $V_+ \subset \mathfrak{d}$, let $V_+^0 \subset W$ be a 1-dimensional subspace such that $\langle , \rangle_W$ is positive-definite on $V_+^0$. Let

$V_+ = \mathfrak{g} \otimes V_+^0$.

Then $V_- = \mathfrak{g} \otimes V_0^0$ where $V_0 = (V_+^0)\perp$.

We can now describe the construction of a family $p_\lambda : \mathfrak{d} \rightarrow \mathfrak{g}$ satisfying (4). Let us choose non-zero elements $e_+ \in V_+^0$ and $e_- \in V_0^0$ (this choice is inessential).

**Proposition 2.** If a linear form $q : W \rightarrow \mathbb{R}$ satisfies

\begin{equation}
q(e_+)q(e_-) = q(e_+e_-)
\end{equation}

then the map $p = \text{id}_\mathfrak{g} \otimes q : \mathfrak{d} \rightarrow \mathfrak{g}$ satisfies (4).

**Proof.** If $X = x \otimes e_+ \in V_+$ and $Y = y \otimes e_- \in V_-$ then

$p([X, Y]_\mathfrak{d}) = p([x, y]_\mathfrak{g} \otimes e_+ e_-) = q(e_+ e_-)[x, y]_\mathfrak{g} = [q(e_+)x, q(e_-)y]_\mathfrak{g} = [p(X), p(Y)]_\mathfrak{g}$.

\square

The solutions $q \in W^*$ of (5) form a curve in $W^*$, which is either a hyperbola or a union of two lines. If

$e_+ e_- = ae_+ + be_- \quad a, b \in \mathbb{R},$

we rewrite (5) as

$(q(e_+) - b)(q(e_-) - a) = ab.$

We thus have a hyperbola if $ab \neq 0$ and a union of two straight lines if $ab = 0$.

One can easily check that $ab = 0$ iff one of $V_+^0$ is of the form $\mathbb{R}e$ where $e \in W$ satisfies $e^2 = e$. This means that one of $V_+^0 = \mathfrak{g} \otimes V_+^0 \subset \mathfrak{d}$ is a Lie subalgebra isomorphic to $\mathfrak{g}$ and thus, according to [3], for any Lagrangian $\mathfrak{h} \subset \mathfrak{d}$, the corresponding $\sigma$-model is simply the WZW model given by $G$.

Let us now choose a rational parametrization $\lambda \mapsto q_\lambda$ of the hyperbola (5). The standard parametrization in this context seems to be the one sending $\lambda = \pm 1$ to
the two points at the infinity of the hyperbola, and \( \lambda = \infty \) to 0 (though any other
parametrization would do). This gives
\[
q_\lambda = \frac{q_+}{1 + \lambda} + \frac{q_-}{1 - \lambda}
\]
where \( q_+, q_- \in W^* \) are given by
\[
q_+(e_-) = q_-(e_+) = 0, \quad q_+(e_+) = 2b, \quad q_-(e_-) = 2a.
\]
(If the curve a union of two lines then this parametrizes only one of the lines, or possibly
just a single point.)

**Corollary.** If \( A \in \Omega^1(\Sigma, \mathfrak{d}) \) satisfies (1), i.e. if \( A = A_+ \otimes e_+ + A_- \otimes e_- \) with
\( A_+ \in \Omega^1(\Sigma, \mathfrak{g}) \) and \( A_- \in \Omega^0(\Sigma, \mathfrak{g}) \) and if \( A \) is flat, then the \( \mathfrak{g} \)-connections
\[
A_\lambda = (1 \otimes q_\lambda)(A) = \frac{2b}{1 + \lambda} A_+ + \frac{2a}{1 - \lambda} A_-
\]
are flat.

The \( \mathfrak{g} \)-valued Lax operator obtained in this way is thus
\[
L(\sigma, \lambda) = \frac{2b}{1 + \lambda} j_+(\sigma) + \frac{2a}{1 - \lambda} j_-(\sigma)
\]
where we decomposed \( j(\sigma) \) as \( j(\sigma) = j_+(\sigma) \otimes e_+ + j_-(\sigma) \otimes e_- \). For completeness,
the Hamiltonian (3) is
\[
\mathcal{H} = \frac{1}{2} \int_{S^1} \left( \langle \theta(e_+)(j_+(\sigma), j_+(\sigma))_\mathfrak{g} - \langle \theta(e_-)(j_-(\sigma), j_-(\sigma))_\mathfrak{g} \rangle d\sigma.
\]

5. EXAMPLES OF THE EXAMPLE

In this section \( \mathfrak{g} \) is a compact Lie algebra and \( G \) the corresponding compact
1-connected Lie group.

Let start with the case of \( W = \mathbb{R} \oplus \mathbb{R} \), i.e. \( \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} \). The only admissible \( \theta \in W^* \), up to rescaling (which can be absorbed to \( \langle \cdot, \cdot \rangle_\mathfrak{g} \) and exchange of the two components of \( W \), is \( \theta(x, y) = x - y \). (Here the main limiting factor is existence of a lagrangian Lie subalgebra \( \mathfrak{h} \subset \mathfrak{d} \): if \( \theta(x, y) = cx + dy \) with \( cd \neq 0 \) (the non-degeneracy condition), it forces \( c = -d \).) The pairing \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{d} \) is \( \langle (X_1, X_2), (Y_1, Y_2) \rangle = (X_1, X_2)_\mathfrak{g} - (Y_1, Y_2)_\mathfrak{g} \).

We have
\[
e_+ = (1, t) \quad e_- = (t, 1)
\]
for some \(-1 < t < 1\). The Lax operator (6), written in terms of \( j = (j_1, j_2) \), is
\[
L(\sigma, \lambda) = \frac{2t}{(1 + t)(1 - t)^2} \left( \frac{1}{1 + \lambda} (j_1(\sigma) - tj_2(\sigma)) + \frac{1}{1 - \lambda} (j_2(\sigma) - tj_1(\sigma)) \right)
\]
The Poisson brackets (2) of \( j_{1, 2} \) are
\[
\{j_1(\sigma), j_{1b}(\sigma')\} = f^a_{ab} j_1 c(\sigma) \delta(\sigma - \sigma') + \delta_{ab} \delta'(\sigma - \sigma') \langle
\{j_2(\sigma), j_{2b}(\sigma')\} = f^a_{ab} j_2 c(\sigma) \delta(\sigma - \sigma') - \delta_{ab} \delta'(\sigma - \sigma') \langle
\{j_{1a}(\sigma), j_{2b}(\sigma')\} = 0
\]
and the Hamiltonian (3) is
\[
\mathcal{H} = \frac{1}{2(1 - t^2)} \int_{S^1} \left( (1 + t^2) (\langle j_1(\sigma), j_1(\sigma) \rangle_\mathfrak{g} + \langle j_2(\sigma), j_2(\sigma) \rangle_\mathfrak{g} - 4t \langle j_1(\sigma), j_2(\sigma) \rangle_\mathfrak{g} \right) d\sigma.
\]
The degenerate case \( t = 0 \) (when a = b = 0) corresponds to the WZW-model on \( \mathfrak{g} \).

The natural choice for a Lagrangian Lie subalgebra \( \mathfrak{h} \subset \mathfrak{d} \) is the diagonal \( \mathfrak{g} \subset \mathfrak{d} \).

The target space of the \( \sigma \)-model is \( D/G \cong G \). It is the so-called \( \lambda \)-deformed \( \sigma \)-model" introduced by Sfetsos in [12] (Sfetsos’s \( \lambda \) is our \( t \)).
Let us now consider the case $W = \mathbb{C}$, which is the richest one. In this case any non-zero $\theta \in W^*$ is suitable. Let $\theta(z) = \text{Im}(e^{2i\alpha}z)$ for some $\alpha \in \mathbb{R}$. We thus have $\mathfrak{d} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_C$ (seen as a real Lie algebra) with the pairing $\langle X, Y \rangle = \text{Im}(e^{2i\alpha} \langle X, Y \rangle_{\mathfrak{g}_C})$ where $\langle \cdot, \cdot \rangle_{\mathfrak{g}_C}$ is the $\mathbb{C}$-bilinear extension of $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

In this case

$$e_+ = e^{-i\alpha + \phi} \quad e_- = e^{-i\alpha - i\phi}$$

for some $\phi \in (0, \pi/2)$. If $e^{2i\alpha} = e^{\pm 2i\phi}$ then the resulting $\sigma$-model (regardless of the choice of $\mathfrak{h} \subset \mathfrak{d}$) is the WZW-model on $G$.

The Lax operator \([1]\) is

$$L(\sigma, \lambda) = \frac{1}{2\sin^2 2\phi} \left( \frac{e^{2i\alpha} - e^{-2i\phi}}{1 + \lambda} + \frac{e^{2i\alpha} - e^{2i\phi}}{1 - \lambda} \right) J(\sigma) + \text{c.c.}$$

(where c.c. stands for “complex conjugate”). Here $J = j_{1e} + ij_{1m}$ and $\bar{J} = j_{1e} - ij_{1m}$ where $j_{1e}$ and $j_{1m}$ are given by $j = j_{1e} \otimes 1 + j_{1m} \otimes i$, their Poisson brackets \([2]\) (written in an orthonormal basis of $\mathfrak{g}$) are

$$\{J_a(\sigma), J_b(\sigma')\} = f_{ab}^c J_c(\sigma) \delta(\sigma - \sigma') + 2ie^{-2i\alpha} \delta_{ab} \delta(\sigma - \sigma')$$

$$\{J_a(\sigma), J_b(\sigma')\} = f_{ab}^c J_c(\sigma) \delta(\sigma - \sigma') - 2ie^{2i\alpha} \delta_{ab} \delta(\sigma - \sigma')$$

$$\{J_a(\sigma), \bar{J}_b(\sigma')\} = 0.$$

The Hamiltonian \([3]\) is

$$\mathcal{H} = \frac{1}{2} \int_{S^1} \left( \frac{\sin 2\phi}{2} \langle J(\sigma), \bar{J}(\sigma) \rangle_{\mathfrak{g}} - \frac{\sin 4\phi}{4} \left( e^{2i\alpha} \langle J(\sigma), J(\sigma) \rangle_{\mathfrak{g}} + e^{-2i\alpha} \langle \bar{J}(\sigma), \bar{J}(\sigma) \rangle_{\mathfrak{g}} \right) \right) d\sigma$$

A suitable Lagrangian Lie subalgebra $\mathfrak{h} \subset \mathfrak{d}$ can be found as follows. Let $\mathfrak{n} \subset \mathfrak{g}_C = \mathfrak{d}$ be the complex nilpotent Lie subalgebra spanned by the positive root spaces and let $t \subset \mathfrak{g}$ be the Cartan Lie subalgebra. Let $0 \neq z \in \mathbb{C}$ be such that $\theta(z^2) = 0$; up to a real multiple we have $z = e^{-i\alpha}$ or $z = ie^{-i\alpha}$. Then

$$\mathfrak{h} = zt + n \subset \mathfrak{g}_C = \mathfrak{d}$$

is a real Lie subalgebra of $\mathfrak{d}$ which is clearly Lagrangian. If $z \notin \mathbb{R}$ then $\mathfrak{h}$ is transverse to $\mathfrak{g} \subset \mathfrak{d}$ and we have an identification $D/H \cong G$ for the target space of the $\sigma$-model.

The case of $\alpha = 0$ corresponds to Klimčík’s Yang-Baxter $\sigma$-model \([3]\). The general case is the Yang-Baxter $\sigma$-model with WZW term introduced in \([2]\) and reinterpreted as a $\sigma$-model of Poisson-Lie type in \([4]\).

The final case is $W = \mathbb{R}[\epsilon]/(\epsilon^2)$. After rescaling $\epsilon$ and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ we can suppose that $\theta(x + ye) = 2tx + y$ for some $t \in \mathbb{R}$ and that

$$e_+ = 1 + (1 - t)\epsilon \quad e_- = 1 - (1 + t)\epsilon.$$

Using the notation $j = j_1 \otimes 1 + j_2 \otimes \epsilon$, we get

$$L(\sigma, \lambda) = \frac{1 + t}{2(1 + \lambda)} ((1 + t)j_1 + j_\epsilon) + \frac{1 - t}{2(1 - \lambda)} ((1 - t)j_1 - j_\epsilon).$$

The Poisson brackets are

$$\{j_1\alpha(\sigma), j_1\alpha'(\sigma')\} = f_{ab}^c j_1c(\sigma) \delta(\sigma - \sigma')$$

$$\{j_2\alpha(\sigma), j_2\alpha'(\sigma')\} = f_{ab}^c j_2c(\sigma) \delta(\sigma - \sigma') + \delta_{ab} \delta'(\sigma - \sigma')$$

$$\{j_2\alpha(\sigma), j_2\alpha'(\sigma')\} = -2t \delta_{ab} \delta'(\sigma - \sigma').$$
and the Hamiltonian
\[ \mathcal{H} = \frac{1}{2} \int_{S^1} (1 + t^2) \langle j_0, j_0 \rangle_g + t \langle j_0, j_\epsilon \rangle_g + \langle j_\epsilon, j_\epsilon \rangle_g \]

In this case the natural \( \mathfrak{h} \subset \mathfrak{d} = g[\epsilon]/(\epsilon^2) \) is \( \mathfrak{h} = \epsilon g \), which gives \( D/H = G \). When \( t = 0 \) the \( \sigma \)-model is the principal chiral model on \( G \), when \( t = \pm 1 \) we get the WZW model, and for other values of \( t \) we get models given by the invariant metric on \( G \) and by a multiple of the Cartan 3-form.

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