Representation Theory of a W-algebra from generalised DS reduction

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ABSTRACT

We investigate the W-algebra resulting from Drinfel’d- Sokolov reduction of a $B_2$ WZW model with respect to the grading induced by a short root. The quantum algebra, which is generated by three fields of spin-2 and a field of spin-1, is explicitly constructed. A ‘free field’ realisation of the algebra in terms of the zero-grade currents is given, and it is shown that these commute with a screening charge. We investigate the representation theory of the algebra using a combination of the explicit fusion method of Bauer et al. and free field methods. We discuss the fusion rules of degenerate primary fields, and give various character formulae and a Kac determinant formula for the algebra.
1 Introduction

Perhaps the outstanding goal in conformal field theory [1] is a complete classification. At present such a classification seems to remain beyond our scope. We may hope that if we study enough examples of conformal field theory, the patterns that we discover may provide us with clues as to how we should go about a classification.

Until recently, the most general construction of rational conformal field theories available was the coset construction of Goddard, Kent and Olive [2]. This construction has many appealing features; in particular, one can write down the partition function and fusion rules of the theory straightforwardly. On the other hand, the symmetry, or chiral algebra, of the theory seems somewhat obscured in this formulation [3, 4]. For instance, it is not a straightforward matter to see that the coset models $\hat{su}(2)_2 \times \hat{su}(2)_k / \hat{su}_{k+2}$ are in fact superconformal minimal models [2], or that the models $\hat{g}_1 \times \hat{g}_k / \hat{g}_{k+1}$ are minimal models for the algebras $W_g$ [5, 6].

In the last few years another construction of conformal field theory has arisen. This is based on a generalised version of the Drinfel’d-Sokolov reduction [7–12]. Many of the features of the resulting W-algebras can be easily derived from simple finite Lie-algebraic concepts [13, 14], and it has been argued using cohomological techniques that the algebras can always be quantised [16, 17]. However, as yet surprisingly little is known about the representation theory of this new class of algebras, except for the algebras $W_g$ which correspond to the standard reduction of a WZW model based on a simply-laced Lie algebra $g$ [18–20]. As a result, the primary field content, fusion rules and partition function of the conformal field theories are poorly understood.

In this paper, we shall try to further our understanding of the representation theory of W-algebras constructed from generalised Drinfel’d-Sokolov reductions by examining one particular example of a such a W-algebra in rather thorough detail. Our example, which we shall hereafter refer to as $\bar{W}$, is based on the reduction of $B_2$ by the nilpotent algebra associated with the grading induced by the short root of $B_2$. This algebra is a relatively simple object generated by three spin-2 fields and a single spin-1 field, yet it contains all of the important features that we expect the general case to have. The main tool that we shall use to analyze the representation theory of this algebra is the explicit fusion technique pioneered in [23, 24].

The structure of the paper is as follows. In section 2 we give a quick review of generalised Drinfel’d-Sokolov reduction. In the third section we construct the quantum version of our example. In addition we quantise the Miura transformation and give a ‘free field’ construction of the algebra in terms of the non-abelian zero grade currents of $B_2$. These expressions are shown to commute with the appropriate screening charge.

In section 4 we begin our investigation of the representation theory by examining the
zero-grade algebra, which acting on highest-weight states is a finite W-algebra in the terminology of [34]. Coincidentally this algebra has arisen before in the literature. We then use a number of low-lying null vectors in section 5 to explicitly derive the fusion of the basic primary fields of the theory with other fields using the methods of Bauer et al. and Bajnok et al.. This also gives us a recurrence relation which can be used to derive explicit formulae for a subset of all the possible null vectors.

Combining this information with what we might expect from the free field form of the algebra and our experience with other W-algebras, in section 6 we conjecture the general form of degenerate representations, the fusions of the corresponding fields, formulae for the characters of the representations and the Kac determinant formula for this algebra. We conclude with some comments on what we have found and directions for future research.

2 Review of the generalised DS reduction

Both the coset construction and the method of Hamiltonian reduction can be thought of as a gauging of the WZW model [26] whose action is given by

\[
S(u) = \frac{\hbar k}{4} \left[ \frac{1}{2} \int d^2 x \text{Tr}(\partial^\mu u \partial_\mu u^{-1}) + \frac{1}{3} \int d^3 x \text{Tr}(\epsilon^{ijk} u^{-1} \partial_i uu^{-1} \partial_j uu^{-1} \partial_k u) \right] \tag{2.1}
\]

where the field \( u \) takes its value in the group \( G \) with Lie algebra \( g \). When quantised, the modes of the left and right currents

\[
J_L(x_+) = \frac{\hbar k}{4} u^{-1} \partial_+ u, \quad J_R(x_-) = \frac{\hbar k}{4} \partial_- uu^{-1}
\]

of the WZW model form two commuting copies of a Kac-Moody algebra \( \hat{g} \)

\[
[T^a_m, T^b_n] = f^{abc}_{c} T^c_{m+n} + km\delta_{m+n,0}, \tag{2.3}
\]

where \( f^{ab}_{c} \) are the structure constants of the algebra \( g \). By Noether’s theorem, \( J_L, J_R \) generate the transformations \( u \to uv(x_+), \ u \to w(x_-)u \) respectively where \( v, w \) are elements of \( G \). In the coset construction one gauges a vector subgroup of the symmetry of the WZW model by adding a number of terms to the action [27]

\[
S(u, A_R, A_L) = S(u) + \int d^2 x \text{Tr}(A_L J_R - A_R J_L)
+ \frac{\hbar k}{4} \int d^2 x \text{Tr}(A_L u A_R u^{-1} - A_L A_R) \tag{2.4}
\]

where \( A_L, A_R \in \hat{h} \subset \hat{g} \).
Now let us turn our attention to DS reductions. These also are a gauging of the WZW model, but in this case one gauges the currents associated with nilpotent directions of the algebra. This can be achieved by altering the action (2.4) to

\[ S(u, A_R, A_L) = S(u) + \int d^2x \text{Tr}[A_L(J_R - I_-) - A_R(J_L - I_+) + \frac{\hbar}{4} A_L u A_R u^{-1}], \quad (2.5) \]

restricting \( A_\pm \) to nilpotent directions of the algebra [12] and taking \( I_\pm \) to be certain constant elements of \( g \) which we will discuss below. Although this action does not appear naively gauge-invariant, one can use the nilpotency of the gauge group to show that its variation under gauge transformations is a total derivative. In the traditional reduction associated with standard (abelian) Toda theory one gauges the maximal nilpotent algebra generated by all the positive roots of \( g \). It was then realised that one could generalise this construction by gauging some smaller set of currents, and moreover, that this set could be succinctly labelled by some \( su(1, 1) \) embedding [14].

In order to establish notation we shall see how this works in a little more detail. Let us consider some modified Cartan-Weyl basis for \( g \),

\[ g = g^- \oplus h \oplus g^+. \quad (2.6) \]

Here

\[ g^\pm = \bigoplus_{\alpha \in \Delta_+} CE^{\pm \alpha}, \quad h = \bigoplus CH^i, \quad (2.7) \]

where \( \Delta_+ \) is the set of positive roots, and commutation relations

\[ [E^\alpha, E^{-\alpha}] = (2/\alpha^2) \alpha^i H^i, \quad [H^i, E^\beta] = \beta^i E^\beta. \quad (2.8) \]

One can always conjugate any \( su(1, 1) \) subalgebra of \( g \) so that \( I_+ \in g^+, I_- \in g^- \) and \( I_0 \in h \) where \( I_+, I_-, I_0 \) are the usual raising, lowering and diagonal basis of \( su(1, 1) \). We may write \( I_0 = \rho^\vee \cdot H \). If we use the standard normalisation for the \( su(1, 1) \) algebra,

\[ [I_0, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = \sqrt{2} I_0, \quad (2.9) \]

then we may define the characteristic of the \( su(1, 1) \) embedding to be \( (\rho^\vee \cdot e_1, ..., \rho^\vee \cdot e_i) \), where \( e_j \) are the simple roots of \( g \). It is a fact that the entries of the characteristic are 0, 1/2, 1 [28]. We shall restrict our attention to integral embeddings for which they must either be 0 or 1. The standard reduction is associated with the principal embedding whose characteristic contains all ones.

We may grade \( g \) with respect to the \( \rho^\vee \cdot H \) eigenvalue as

\[ g = \bigoplus_{m \in \mathbb{Z}} g_m. \quad (2.10) \]
We denote the subalgebra $\oplus_{n \geq 0} g_n$ by $p^+$ and the subalgebra $\oplus_{n > 0} g_n$ by $n^+ \subset g^+$. We define $p^−, n^−$ in a similar way. The above grading extends to the affine algebra in an obvious way, and so each of the above subalgebras have ‘hatted’ counterparts. We denote the restricted set of roots $\{\alpha \in \Delta_+: E^\alpha \in n^+\}$ by $\delta_+$. For the standard reduction associated with the principal reduction, $n^± = g^±$ and $\delta_+ = \Delta_+$.

Let us now return to the action (2.5). We restrict $A_L \in n^+$, $A_R \in n^−$ and take $I_\pm$ to be the $su(1,1)$ generators as above. Choosing the gauge $A_L = A_R = 0$, the action (2.5) reduces to (2.1), together with the constraints

$$J_L = I_+ + X_L(z) \quad , \quad J_R = I_+ + X_R(z) \quad (2.11)$$

where $X_L, X_R \in \hat{p}^−, \hat{p}^+$ respectively. From now on, we shall focus on the left chirality for ease of exposition. Since the grading of $g$ implies that $[g_m, g_n] \in g_{m+n}$ and $I_+ \in g_1$, it follows that these constraints are first class. Indeed they generate the gauge transformations $u \to uN(x_+)$ under which (2.5) is invariant, where $N \in N^−$, the group generated by the algebra $n^−$. Under such a gauge transformation

$$J_L \to J'_L = \frac{\hbar k}{4} (uN)^{-1} \partial_+(uN) = N^{-1}(J_L + \frac{\hbar k}{4} \partial)N. \quad (2.12)$$

We can remove this gauge freedom by gauge fixing; this is done by further restricting the form of $J_L$. For our purposes, the most natural gauge is given by the highest weight gauge where we choose

$$J_L^{hw} = I_+ + \sum_i W^i E^i \quad (2.13)$$

where $E^i$ are elements of $n^−$ such that $[I_−, E^i] = 0$ and each $E^i$ belongs to a distinct irreducible representation of the grading $su(1,1)$.

It can be shown that there exists a unique solution to the equation $J'_L = J_L^{hw}$ for $W^i$ and the gauge transformation $N$ in terms of $J_L$. Since (2.13) specifies a well-defined point on the orbit of $J_L$ under the action of the gauge group, it follows that the components $W^i$ are gauge invariant polynomials in $J_L$. As such their Poisson brackets, calculated using the classical version of the Kac-Moody algebra (2.3), are unaffected by the constraints and form a closed classical W-algebra. The number of generators of the algebra is given by the number of irreducible representations $i$ in the decomposition of the adjoint representation of $g$ with respect to the grading $su(1,1)$. This always contains a copy of the $su(1,1)$ itself, $\{I_+, I_0, I_−\}$, and we label this representation by $i = 1$. The coefficient $W^1$ of $E^1 = I_−$ is the left-moving component of the energy-momentum tensor and the other fields $W^i$ transform as primary fields with respect to $W^1$ [12]; that is, in modes we have

$$\{W^m_1, W^i_n\}_{PB.} = [(h_i - 1)m - n]W^i_{m+n}. \quad (2.14)$$
The conformal weight \( h_i \) of the field \( W^i \) is equal to \( s_i + 1 \) where \( s_i \) is the spin of the representation \( i \). This is the beauty of the DS reduction construction of W-algebras; the number and conformal weight of the generating fields are given by simple Lie-algebraic considerations.

We conclude this review of the general DS reduction by making some brief remarks about the free field representation of W-algebras associated with generalised reduction. It turns out that one does not need to use all the components of \( J_L \) to produce a representation of the W-algebra, as we may have guessed, since the number of such components is greater than the number of irreducible representations \( i \). Instead, if we start with \( J_L \) in the free field gauge,

\[
J_L^{ff} = I_+ + X_0
\]

where \( X_0 \in \hat{g}_0 \) and solve \( (J_L^{ff})' = J_L^{w} \), we find that the polynomials \( W^i(J_L^{ff}) \) obey the same algebra as \( W^i(J_L) \). Thus we can construct \( W^i \) out of currents in \( \hat{g}_0 \). In the standard reduction \( g_0 = h \equiv u(1)^{\text{rank} g} \), the Cartan subalgebra of \( g \), whence the nomenclature ‘free field representation’ derives. In the more general case, \( g_0 \) is non-abelian, and so \( W^i \) is represented using a non-abelian Kac-Moody algebra. Note that \( g_0 \) is never semi-simple, as \( I_0 \) is always a commuting \( u(1) \). Also, the dimension of \( g_0 \) is equal to the number of generating fields \( W^i \), as each irreducible representation of \( su(1,1) \) has one ‘highest weight’ component \( E^i \) and one component of zero charge with respect to \( I_0 \).

Not only can one represent the \( W^i \) in terms of currents associated with \( g_0 \), it is also possible using the constraints and the Polyakov-Weigmann identity to rewrite the action (2.5) in terms of a field \( u_0 \) taking values in the associated group \( G_0 \) [29]. The action is given by

\[
S_{GT} = S(u_0) + \frac{\hbar k}{4} \int d^2x \text{Tr}(I_+ u_0 I_-(u_0)^{-1})
\]

For the standard reduction, the first term corresponds to the kinetic term for rank \( g \) free bosons, while the second term produces the familiar sum of exponential terms associated with the \( g \) Toda theory. One can easily generalise the arguments given in [30] to show that polynomials in the currents constructed from \( u_0 \) will be chiral if and only if they Poisson commute with the the second ‘potential’ term. Thus this term has the interpretation of being a generalised screening charge for the W-algebra. In fact, we have one such screening charge for every irreducible component of \( I_- \) under the action of \( g_0 \). (In the standard case there are rank \( g \) such components, each belonging to one-dimensional representations of \( u(1)^{\text{rank} g} \).)
3 An example of a W-algebra from generalised DS reduction

In this section we shall introduce an example of a W-algebra which arises from the sort of generalised reduction procedure described in the last section. Our example has the virtue of being the simplest such algebra which still retains most of the features of the general case. It is therefore useful to study this algebra, both to verify existing conjectures concerning the quantisation of generalised reductions, and to uncover new properties which can then be generalised.

The example we shall consider arises as the reduction of a $B_2$ WZW model. As explained above, a different model can be constructed for each non-isomorphic embedding of $su(1,1)$ in $B_2$. There are precisely three such, with Dynkin indices of embedding one, two and four respectively. The $su(1,1)$ of index four is the principal three-dimensional subalgebra of $B_2$, associated with the standard reduction of $B_2$. The W-algebra for this case is generated by one field of spin two and one field of spin 4, and has already been studied in some detail in the literature [31, 32]. The $su(1,1)$ of index one corresponds to the three dimensional subalgebra whose root is simply a long root of $B_2$. The embedding is non-integral, and the corresponding algebra has generators which do not obey the usual spin-statistics relation.

We shall concentrate on the third embedding of index two, whose root is a short root of $B_2$. In terms of two-dimensional Cartesian coordinates $(x_1, x_2)$, the positive roots of $B_2$ can be taken to be $\alpha_1 = (-1, 1)$, $\alpha_2 = (1, 0)$, $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = \alpha_1 + 2\alpha_2$. Denote the generator corresponding to the root $\alpha_i$ by $E^{\alpha_i}$, and the Cartan subalgebra element associated with the direction $x_i$ by $H_i$. We identify the $su(1,1)$ generators with $I_+ = E^{\alpha_3}$, $I_- = E^{-\alpha_3}$ and $I_0 = H_2$. The grading induced by $I_0$ is just the projection onto the $x_2$ component. The group that we are gauging is generated by the step operators associated with the set of roots $\delta_+ = \{\alpha_1, \alpha_3, \alpha_4\}$.

The $B_2$ algebra decomposes into three spin one representations and a spin zero representation under the action of this $su(1,1)$. By the results of the previous section it follows that there are four generators of the W-algebra associated with this reduction: three of conformal weight two and a single generator of weight one. This last field forms a $u(1)$ Kac-Moody algebra, and simply corresponds to the current associated with the $H_1$ generator which survives the reduction. The three spin two generators can be taken to have charges $1, 0, -1$ with respect to this global part of this $u(1)$ field, and we can identify the chiral component of the energy-momentum tensor $L(z)$ with the field of zero charge.

These general features of the W-algebra, together with the requirement that the algebra be associative are sufficient to determine its commutation relations, which we give below. We shall call this W-algebra $\bar{W}$. 
3.1 The commutation relations of $\bar{W}$

In what follows we derive the commutation relations for the new W-algebra based on $B_2$ by writing down the general form of the commutation relations and checking Jacobi’s identity. The algebra contains the semi-direct product of a $U(1)$ Kac-Moody algebra with the Virasoro algebra:

\[
[U_m, U_n] = km\delta_{m+n,0} \quad (3.1)
\]
\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} \quad (3.2)
\]
\[
[L_m, U_n] = -nU_{m+n} \quad (3.3)
\]

The algebra has two further spin-2 generators $L^+$ and $L^-$ with $U(1)$ charge $\pm 1$; that is

\[
[L_m, L^\pm_n] = (m - n)L^\pm_{m+n} \quad (3.4)
\]
\[
[U_m, L^\pm_n] = \pm L^\pm_{m+n} \quad (3.5)
\]

By charge conservation it is easy to see that $[L^+_m, L^-_n]$ and $[L^-_m, L^+_n]$ both vanish (since the commutator must close on fields of spin three or less and with charge $\pm 2$, and clearly there are none), so that the one non-trivial commutator that we need to determine is $[L^+_m, L^-_n]$.

We can use Virasoro and Kac-Moody Ward identities to ensure that the operator product expansion of the two generators

\[
L^+(z)L^-(\zeta) = \sum_{n \geq 0} \psi_n(\zeta)(z - \zeta)^{-4+n}, \quad (3.6)
\]

transform covariantly. These identities are respectively

\[
L_m|\psi_n\rangle = (m + n - 2)|\psi_{n-m}\rangle \quad (3.7)
\]
\[
U_m|\psi_n\rangle = |\psi_{n-m}\rangle, \quad (3.8)
\]

for $m > 0$ and the state $|\psi\rangle$ is given by the usual correspondence $|\psi\rangle = \lim_{z \to 0} \psi(z)|\text{vac}\rangle$.

Using these relations and Jacobi’s identity for the double commutator of $L^+, L^+, L^-$ we find the following commutation relation

\[
[L^+_m, L^-_n] = k^2(k - 1)m(m^2 - 1)\delta_{m+n,0} + k(k - 1)(m^2 - mn + n^2 - 1)U_{m+n}
\]
\[
+ (m - n)\left[-k(k + 1)L_{m+n} + (2k - 1)(U^2)_{m+n}\right]
\]
\[
- 2(k + 1)(LU - \frac{1}{2}\partial^2U)_{m+n} + 2U^3_{m+n} \quad (3.9)
\]

together with the relation

\[
c = \frac{-12k^2 + 16k - 2}{k + 1}. \quad (3.10)
\]
3.2 The free-field representation

As mentioned in the preceding section, the classical Miura transformation for a generalised DS reduction is of the form

$$N(I_+ + X_0(z) + \partial_z)N^{-1} = I_+ + \sum_i W^i(z)E^i$$

(3.11)

where $X_0(z) \in \hat{g}_0$, $[I_+, E^i] = 0$, and $N \in N^-$ is some gauge transformation. In the present case, we have

$$\sum_i W^i(z)E^i = U(z)H_1 + L^-(z)E^{-\alpha_1} + L(z)E^{-\alpha_3} + L^+(z)E^{-\alpha_4}$$

(3.12)

and we write $X_0 = j_1 H_2 + j E^{\alpha_2} + j E^{-\alpha_2}$. One can solve (3.11) for $L, L^\pm, U$ in terms of $j_\pm, j_i$. We shall refer to the $su(2)$ subalgebra $E^{\pm \alpha_2}$, $H_1$ associated with the currents $\{j_\pm, j_1\}$ as the horizontal $su(2)$, to distinguish it from the grading $su(1, 1)$ whose generators are $I_\pm = E^{\pm \alpha_3}, I_0 = H_2$. In the case in hand, we start by picking some matrix representation of the algebra $B_2$, and then solve the above matrix equation explicitly. The result is as follows, where we have ignored coefficients, since from experience one expects these to be renormalised on quantisation:

$$U = j_1$$

(3.13)

$$L = (j_2)^2 + \partial j_2 + j_+ j_-$$

(3.14)

$$L^+ = (j_1 + j_2)j_+ + \partial j_+$$

(3.15)

$$L^- = (-j_1 + j_2)j_- + \partial j_-$$

(3.16)

Note that we might have expected a $(j_1)^2$ term in $L$, and indeed we shall find one below. However this ambiguity is already well understood [33]. In fact we could have guessed this answer by considering the most general fields of correct charge and conformal weight.

In order to quantise the generalised Miura transformation given above, we start by considering the Wakimoto representation of $B_2$. In this approach, one constructs a representation for the algebra $\hat{g}$ in terms of the currents associated with $\hat{g}_0$ and ghosts $\beta^i, \gamma_j$ of weight one and zero respectively, which satisfy the usual relation

$$\beta^i(z)\gamma_j(\zeta) \sim \frac{\delta^i_j}{(z - \zeta)}.$$ 

(3.17)

It is a relatively straightforward matter to write down the expressions for $p^+ = \sum_{m \geq 0} g_m$. They are given as follows:

$$\text{Tr}(JE^{-\alpha_1}) = \beta^1$$

(3.18)
\[
\begin{align*}
\text{Tr}(JE^{-\alpha_3}) &= \beta^3 \\
\text{Tr}(JE^{-\alpha_4}) &= \beta^4 \\
\text{Tr}(JE^{\alpha_2}) &= j_+ + \beta^3 \gamma_1 + \beta^4 \gamma_3 \\
\text{Tr}(JE^{-\alpha_2}) &= j_- + \beta^3 \gamma_4 + \beta^1 \gamma_3 \\
\text{Tr}(JH_1) &= j_1 - \beta^1 \gamma_1 + \beta^4 \gamma_4 \\
\text{Tr}(JH_2) &= \sqrt{k' + 3} j_2 + \beta^1 \gamma_1 + \beta^3 \gamma_3 + \beta^4 \gamma_4
\end{align*}
\] (3.25)

In order for the components of the currents defined above satisfy a Kac-Moody \(\hat{B}_2\) at level \(k'\), we must have that \(j_\pm, j_1\) must be an \(\hat{su}(2)\) Kac-Moody algebra with central term \(k' + 2\) with the length of the root squared equal to one. This is equivalent to a \(\hat{su}(2)\) Kac-Moody algebra at level \(2(k' + 2)\) in the standard normalisation. The current \(j_2\) is taken to be a commuting \(\hat{u}(1)\) with central term normalised to \(k' + 2\). Ensuring that each of the above currents transforms as a spin one field fixes the energy-momentum tensor of \(B_2\) to be

\[
T(z) = -\sum_{i=1,3,4} \beta^i \gamma_i + \mathcal{L}_{2(k' + 2)} + \frac{(j_2)^2}{2} - \frac{3}{2\sqrt{k' + 3}}\partial j_2.
\] (3.25)

where \(\mathcal{L}\) is the Sugawara construction for the \(su(2)\) given by \(j_\pm, j_1\). The central charge of \(T(z)\) is given by

\[
c = 6 + \frac{3(k' + 2)}{k' + 3} + 1 - \frac{27}{k' + 3} = \frac{10k'}{k' + 3}
\] (3.26)

in agreement with what we expect for a \(B_2\) Kac-Moody algebra at level \(k'\).

The effect of the reduction is to remove the ghosts (quartet confinement), and to improve the energy momentum so that the constrained currents \(\{\text{Tr}(JE^{-\alpha_i}) : i = 1, 3, 4\}\) have weight zero. The reduced energy-momentum tensor is

\[
L(z) = \mathcal{L}_{2(k' + 2)} + \frac{(j_2)^2}{2} + \left(\sqrt{k' + 3} - \frac{3}{2\sqrt{k' + 3}}\right)\partial j_2
\] (3.27)

The level of \(U\) and \(j_1\) must coincide so we equate \(k = k' + 2\). It is then easy to show that the central charge of \(L(z)\) coincides with the expression (3.10).

From the classical expressions for \(L^\pm\) we expect the ‘free field representation of these fields to be of the form \(L^\pm = (j_1 + r^\pm j_2)j_\pm + s^\pm \partial j_\pm\) where \(r^\pm, s^\pm\) are coefficients which need to be determined. Demanding that \(L^\pm\) transform appropriately with respect to \(L(z), U(z)\) is in fact sufficient to determine these coefficients, and we find that

\[
L^+ = (j_1 - \sqrt{k + 1} j_2)j_+ - k \partial j_+ \\
L^- = (-j_1 - \sqrt{k + 1} j_2)j_- - k \partial j_-
\] (3.28)
These expressions together with (3.27) and $U = j_1$ satisfy the commutations of $\bar{W}$ given above, at least up to normalisation.

Often the free field realisation can also be constructed as the commutant of a set of screening charges [18]. In the previous section we saw that the classical expression for the screening charge is given by $\int d^2x \text{Tr}(I_- u_0 I_+(u_0)^{-1})$. Since $I_+ = E^\alpha$ transforms as a spin one field under the action of the horizontal $su(2)$, we expect the screening charge(s) to be the integral of a primary field(s) transforming as a $(3, -)$ under $g_0 = su(2) \times u(1)$, with zero horizontal charge and with conformal weight one. If $\psi^{\text{adj}}(z)$ is the zero charge component of a primary field for the horizontal $su(2)$ Kac-Moody algebra transforming in the adjoint representation, then the quantum screening charge $S$ is given by

$$S = \int dz \psi^{\text{adj}}(z) \exp[\alpha i X_2(z)]$$

where $j_2(z) = i \partial X_2(z)$. One determines $\alpha$ from the condition that $S$ must have conformal weight one by using equation (3.27). We find the two solutions

$$\alpha = -\frac{1}{\sqrt{k+1}}, \frac{2k}{\sqrt{k+1}}.$$  

(3.31)

It is then a relatively straightforward if tedious calculation to show that the screening charges $S$ commutes with the free field expressions for the generators of $\bar{W}$ given above if we choose the first of the two solutions for $\alpha$ and we relegate it to an appendix.

4 Basics of $\bar{W}$ representation theory

In order to set up the representation theory for $\bar{W}$, we first have to understand what a ‘highest weight’ representation for this algebra is. Loosely speaking, a highest weight representation is one such that the representation space $V$ can be graded with respect to $L_0$ eigenvalue

$$V = \bigoplus_{m \geq 0} V_m$$

(4.1)

where the space $V_m$ has $L_0$ eigenvalue $h + m$. Here we concentrate on the space of highest weight states $\psi \in V_0$. Since $V_m = \emptyset$ for $m < 0$, it follows that

$$X_m |\psi\rangle = 0 \ , \ m > 0$$

(4.2)

for any generator $X_m$ of $\bar{W}$.

Now let us consider the action of the zero modes $X_0$ on $V_0$. Algebraically, the zero modes of $\bar{W}$ do not close; for instance

$$[L_0^+, L_0^-] = ... - 2(k+1)(LU)_0 = ... - 2(k+1)[\sum_{x>0} U_{2-x}L_{x-2} + \sum_{x>0} L_{-1-x}U_{x+1}].$$

(4.3)
However, on $V_0$, the non-zero modes of $\bar{W}$ vanish because of (4.2), and indeed the algebra restricted to the zero modes is a consistent, associative algebra. The mode $L_0$ commutes with $X_0$, and so simply acts as a central term with value $h$ in $V_0$. The remaining three zero modes of generating fields of $\bar{W}$ have the following commutation relations:

$$[U_0, L_0^\pm] = \pm L_0^\pm$$

(4.4)

$$[L_0^+, L_0^-] = [-k(k-1) - 2(k+1)h]U_0 + 2(U_0)^3.$$  

(4.5)

Thus we see that these three modes form a sort of deformation of the usual $su(2)$. We call this algebra $\bar{W}_0$. This is very like the quantum group $su_q(2)$, but in this case the commutators close on polynomial terms, rather than on hyperbolic functions. Polynomial algebras of this sort have been considered before [34], where they were called finite W-algebras. In fact the algebra (4.5) has a longer history [35, 36] appearing as the algebra of conserved quantities for a Coulombic central force problem on a space of constant curvature.

In analogy with the representation theory of $su(2)$, we shall consider representations built up from a highest weight state $\psi_\Lambda$, with the properties that

$$L_0^+|\psi_\Lambda\rangle = 0$$

(4.6)

$$U_0|\psi_\Lambda\rangle = \Lambda|\psi_\Lambda\rangle$$

(4.7)

Thus we specify representations of $\bar{W}$ by three parameters: $c$ (or equivalently $k$), the weight $h$ of the highest weight state $\psi_\Lambda$, and its charge $\Lambda$.

The space $V_0$ is built up by applying the ‘raising operator’ $L_0^-$ to $\psi_\Lambda$, and the states $(L_0^-)^p|\psi_\Lambda\rangle$ form a basis for $V_0$. For arbitrary $\Lambda$, the space $V_0$ is therefore an infinite-dimensional representation of $\bar{W}_0$. For special $\Lambda$ though, we can arrange that the norm of $(L_0^-)^p|\psi_\Lambda\rangle$ vanishes. To see when this occurs, note that

$$|(L_0^-)^{p+1}|\psi_\Lambda\rangle|^2 = \sum_{i=0}^p P(\Lambda - i, h)|(L_0^-)^p|\psi_\Lambda\rangle|^2$$

(4.8)

where $P(\lambda, h)$ is simply the right hand side of (4.5) evaluated on a state of charge $\lambda$ and weight $h$; i.e.

$$P(\lambda, h) = [-k(k-1) - 2(k+1)h]\lambda + 2\lambda^3.$$  

(4.9)

Since $P$ is odd under $\lambda \rightarrow -\lambda$, it follows that the norm vanishes for $\Lambda \in \frac{Z}{2}$ and $p = 2\Lambda + 1$, just as for standard $su(2)$. There are other solutions though. To express these in a compact form, it turns out it is useful to use a parametrisation of $h$ inspired by the free field representation we found in the last section. Any highest weight state $\psi_{ff}$ for $\hat{g}_0$ is obviously also a highest weight state for $\bar{W}$. If

$$\langle j_1 \rangle_0|\psi_{ff}\rangle = \Lambda|\psi_{ff}\rangle$$

(4.10)

$$\langle j_2 \rangle_0|\psi_{ff}\rangle = \frac{a}{\sqrt{k+1}}|\psi_{ff}\rangle$$

(4.11)
then $\psi_{ff}$ is a highest weight state for $\bar{W}$ with charge $\Lambda$ and

$$h = \frac{\Lambda(\Lambda + 1) + a(a + 1 - 2k)}{2(k + 1)}$$ (4.12)

where we have used (3.27) to find this expression. From now on, we shall use $a$ rather than $h$ to parametrise the conformal weight of the highest weight state. We then find that the norm of descendant of the state $|\psi_{ff}\rangle = |\Lambda, a\rangle$ is

$$|\langle L_0^{-p}|\Lambda, a\rangle|^2 = N^{p-1}|\langle L_0^{-p-1}|\Lambda, a\rangle|^2$$

$$N^{p-1} = -\frac{p}{2}(2\Lambda + 1 - p)(p - \Lambda - k + a)(\Lambda + 1 - p - k + a)$$ (4.13)

so that we can expect $V_0$ to be finite-dimensional if and only if at least one of $2\Lambda + 1$, $\Lambda + k - a$ or $\Lambda + 1 - k + a$ is a positive integer.

We conclude our discussion of zero modes by mentioning that it is possible to define a Casimir operator $C$ for $\bar{W}_0$, given by [35]

$$C = L_0^- L_0^+ - \frac{1}{2}[k(k - 1) + 2(k + 1)h]U_0(U_0 + 1) + \frac{1}{2}U_0^2(U_0 + 1)^2$$ (4.15)

In particular we find using the parametrisation (4.12) that

$$C|\Lambda, a\rangle = -\frac{1}{2}\Lambda(\Lambda + 1)(a - k)(a - k + 1)|\Lambda, a\rangle .$$ (4.16)

The pair $(C, h)$ are an invariant way of specifying an irreducible representation of $\bar{W}$. Let us compare this with the Casimirs of the other two reductions of $B_2$ with which we are familiar: the ‘null’ reduction or just the Kac-Moody algebra $\tilde{\mathfrak{so}}(5)$ and the standard reduction with algebra $WB_2$. All three algebras possess two Casimirs which are polynomials of order two and four in terms of the free field parametrisation $\lambda$ of their representations [31]. Furthermore these polynomials exhibit a symmetry under a shifted $B_2$ Weyl group action $\lambda - \rho \rightarrow w(\lambda - \rho)$ for some constant vector $\rho$. For $\bar{W}$, $\rho = (-1/2, -1/2 + k)$ and setting $(\Lambda', a') = (\Lambda + 1/2, a + 1/2 - k)$ we see that $C$ and $h$ as given in (4.12), (4.16) are indeed invariant under the $B_2$ Weyl group action

$$(\Lambda', a') \rightarrow (\pm \Lambda', \pm a') . (\pm a', \pm \Lambda')$$ (4.17)

Note that if we express the formula for the norm of $(L_0^-)^p|\Lambda, a\rangle$ (4.8) in terms of its charge $q = \Lambda - p$ then it is also invariant under the action of the shifted Weyl group.
5 Representation theory of $\bar{W}$ from fusion

In the last section we saw how to construct highest weight representations of $\bar{W}$ as Verma modules built up from a highest weight state $|\Lambda, a\rangle$. For generic values of $\Lambda$ and $a$ these representations will be irreducible. However, past experience teaches us that the physically interesting cases are the special cases where the Verma module is not an irreducible representation of $\bar{W}$ but instead contains one or more singular vectors. Such representations are called degenerate. Indeed, we have already entertained the possibility that $|\Lambda, a\rangle$ has a singular vector at level zero (4.8), since this avoids the rather unpleasant infinite degeneracy in energy that would arise otherwise.

In general we expect that each independent singular vector (i.e. a singular vector which is not a descendent of other singular vectors) in the Verma module requires an equation to be satisfied by the parameters specifying the representation. Therefore the maximum number of such independent singular vectors is equal to the number of parameters specifying the representation, and in analogy with Lie algebras of finite dimension we refer to this as the rank of the $W$-algebra. Although we could (and perhaps should) include the central charge $c$ (or in our case the value of $k$) in the parametrisation, in this paper we choose to exclude it from the set and leave it as an arbitrary irrational number, since the representation theory is simplest to work out in this ‘quasi-rational’ case. We expect that for rational $c$, representations can have many more singular vectors, and that the fusion rules acquire a quantum-group like structure. We call representations containing the maximum number of independent and dependent singular vectors maximally degenerate representations (although not all representations with the maximum number of independent null vectors are maximally degenerate). We reserve the nomenclature completely degenerate for the equivalent concept for rational $c$. From previous experience maximally degenerate representations should have the special property that the fusions of the corresponding primary fields can be completely determined by demanding that their null descendents decouple and so we shall be primarily interested in such representations.

The representations of $\bar{W}$ are specified by the pair $(\Lambda, a)$ or equivalently by the Casimirs $(C, h)$ (where as mentioned above we take $k$ or $c$ to be irrational so that this parameter does not play a role in the singular vector structure). Thus the rank of $\bar{W}$ is two, and so maximally degenerate representations have two independent singular vectors. If we demand that the highest weight space $V_0$ form a finite dimensional representation of $\bar{W}$, then we must have an independent singular vector in $V_0$, leaving in general one more independent singular vector in $V_m$, $m > 0$. In the remainder of this section we shall investigate the fusion of various basic representations which are of this type.
5.1 **Using null vectors to determine fusion**

We follow the approach developed in [23–25]. This involves choosing representations whose independent singular vector is in $V_m$ for $m$ small. This vector is explicitly constructed by brute force and then used to solve for fusion of the corresponding primary field with any other primary field. As a side product, we also find an infinite set of representations which are degenerate, and find a recurrence relation which can be used to explicitly construct the singular vector associated with these representations.

We begin by considering the classical form of null vectors, for the simple case of the $B_2$ WZW model. Let us recall the form of the currents (2.2). This can be re-written

$$ (\partial_z - J_a^a T^a) g = 0 \quad a \in g $$

(5.1)

where we take $T^a$ to be some particular representation $\lambda$ of $g$. Although $g$ is a matrix, the equation (5.1) acts on each column of $g$ independently, so we can think of $g$ as being a column vector (we ignore the other chirality for the moment). From now on we write $g = g_j$ where $j$ runs over the different elements of the representation $\lambda$. The trick is to think of (5.1) as the classical version of the Kniznik-Zamolodchikov equation

$$ L_{-1} - J_{-1}^a T^a |\psi\rangle $$

(5.2)

where $\psi$ is some highest weight state for $\hat{g}$. Here and below we ignore constant coefficients for simplicity. This is a singular vector which is in any highest weight representation of the semi-direct product of a Kac-Moody and Virasoro algebra.

When we reduce these equations, after gauge fixing the equation (5.1) becomes

$$ (\partial_z \delta_{jk} - (I_+)_{jk} - \sum_i W_i (E_i)_{jk}) g_k = 0. $$

(5.3)

This set of coupled differential equations can be thought of as the classical equivalent of some null vector condition for a representation of the W-algebra. The order of the system of equations is simply equal to the dimension of the representation that $g$ is in, and one can think of different fusions of $g$ as being associated with different solutions of this equation. It is therefore very clear at the classical level that there is a close correspondence between representations of finite Lie algebras and the representations of W-algebras obtained by DS reduction.

The quantised version of (5.3) is the set of equations

$$ (L_{-1} \delta_{jk} - (I_+)_{jk} - \sum_i W_i (E_i)_{jk}) |\psi_k\rangle = 0, $$

(5.4)

where we have ignored some constant coefficients which experience shows need to be introduced. This equation simplifies if we choose a basis for the representation $\lambda$ which
diagonalises the action of the Cartan subalgebra $H^i \in h$, so that we have

$$H^i |\psi_k\rangle = \lambda^i_k |\psi_k\rangle$$

(5.5)

We can extend the grading of the Lie algebra $g$ introduced in section two to the states $|\psi_k\rangle$ by defining

$$G(H^i) = 0$$

(5.6)

$$G(E^{\alpha_i}) = \rho^\vee \cdot \alpha_i$$

(5.7)

$$G(|\psi_k\rangle) = \rho^\vee \cdot \lambda_k .$$

(5.8)

Consistency of (5.4) with respect to $L_0$ shows that

$$L_0 |\psi_k\rangle = (\kappa - G(|\psi_k\rangle)) |\psi_k\rangle = (\kappa - \lambda_k \cdot \rho) |\psi_k\rangle .$$

(5.9)

where $\kappa$ is a constant. We take $|\psi_k\rangle \in V_m$ if $(\lambda - \lambda_k) \cdot \rho = m$. In particular if $|\psi_\lambda\rangle$ is the state corresponding to the highest weight $\lambda$ then $|\psi_\lambda\rangle \in V_0$ and is also a highest weight state for $g_0$, so we identify this as the highest weight state for the W-algebra.

The null vectors implied by (5.4) can be used to solve explicitly for the operator product expansion of $|\psi_\lambda\rangle$ with some other primary field $\phi$. Corresponding to each null state $|\chi\rangle$ is a null field $\chi(z)$ and we simply demand that the operator product expansion of this field with $\phi$ is zero or a sum of null fields. Using the methods of [23–25], we can reexpress this condition as an equation in terms of the operator product of $\psi_\lambda$ with $\phi$, and this enables us to solve for the latter. As an example, consider the operator product of the descendant field corresponding to $(L^+)_m |\psi_\lambda\rangle$ with some other primary field $\phi$. This can be written

$$\hat{L}^+_{-p} \psi_\lambda(z) |\phi\rangle = \oint z d\zeta L^+(\zeta) \psi_\lambda(z)(\zeta - z)^{1-p} |\phi\rangle$$

(5.10)

$$= \int_{\zeta > z} - \int_{\zeta < z} d\zeta L^+(\zeta) \psi_\lambda(z)(\zeta - z)^{1-p} |\phi\rangle$$

(5.11)

$$= \left[ \sum_{s=0}^{\infty} \left( \begin{array}{c} p - 2 - s \rule{0cm}{0.5cm} \\ s \end{array} \right) z^s L^+_{-p-s} \psi_\lambda(z) + (-1)^p z^{1-p-s} \psi_\lambda(z) L^+_{-1+s} \right] |\phi\rangle$$

(5.12)

$$= \left[ \sum_{s=0}^{\infty} \left( \begin{array}{c} p - 2 - s \rule{0cm}{0.5cm} \\ s \end{array} \right) z^s L^+_{-p-s} - (-1/z)^{p-1} (L^+_{-1} - \hat{L}^+_{-1}) \right] \psi_\lambda(z) |\phi\rangle$$

(5.13)

By similar manipulations it is possible to rewrite the operator product expansion of any descendant field $\hat{X}^-_{-m} \psi_\lambda |\phi\rangle$ as a sum of terms involving operators acting on terms of the form

$$O(i,p,q,r;z) = (\hat{L}^+_{-1})^p (\hat{L}^-_{-1})^q \psi_\lambda (\hat{L}^-_{-1})^r |\phi\rangle,$$

(5.14)

where $p + q \leq m$ and $p - q - r$ equals the charge of the operator $\hat{X}^-_{-m}$. The $O(p,q,r;z)$ can be thought of as forming a basis of independent operator product expansions. For
maximally degenerate representations, we conjecture that there are precisely enough null vectors in the Verma module to solve for the $O(i, p, q, r; z)$.

We shall now apply this technique to the two fundamental representations of $B_2$, that is the spinor and vector representations, and solve for the fusions of the primary fields corresponding to these representations. In addition, we shall also find the weights of a subset of all degenerate representations, and an explicit expression for their null vectors.

### 5.2 The spinor representation of $B_2$

The basic representation of $B_2$ is the spinor representation, with the four weights $(\pm 1/2, \pm 1/2)$. It is easiest to work in the eigenvalue basis of this representation, so again ignoring constants since we expect these to be renormalised on quantisation anyway, for this case (5.3) reads

$$
\begin{pmatrix}
U & 0 & 1 & 0 \\
0 & -U & 0 & 1 \\
L & L^+ & U & 0 \\
L^- & L & 0 & -U
\end{pmatrix}
\begin{pmatrix}
g_1 \\
g_2 \\
g_3 \\
g_4
\end{pmatrix}
= \partial
\begin{pmatrix}
g_1 \\
g_2 \\
g_3 \\
g_4
\end{pmatrix}
$$

(5.15)

The first and second equation can be used to eliminate $g_3$ and $g_4$ in favour of $g_1$ and $g_2$. Substituting into the last two equations we find the following coupled differential equations:

$$
[(\partial + U)^2 - L]g_2 = L^- g_1
$$

(5.16)

$$
[(\partial - U)^2 - L]g_1 = L^+ g_2.
$$

(5.17)

The components $g_1$, $g_2$ have $U_0$ charge 1/2 and $-1/2$ respectively. We label the corresponding quantum states $|\psi_+\rangle$ and $|\psi_-\rangle$. As above, we take $|\psi_+\rangle$ to be a highest weight state for $\bar{W}$ and $|\psi_-\rangle = L_0^- |\psi_+\rangle$. We label the conformal weight of these states $\delta_s$ so that $L_0 |\psi_\pm\rangle = \delta_s |\psi_\pm\rangle$. From the classical constraint equation (5.17), we expect there to be a null state of the form

$$
|\chi_+\rangle = (L^+)_{-2}|\psi_-\rangle + (\beta_1 L_{-1} + \beta_2 L_{-2})|\psi_+\rangle
$$

(5.18)

Applying $L_1$ to this vector we see that we also expect a null vector of the form

$$
(L^+)_{-1}|\psi_-\rangle + (\gamma_1 U_{-1} + \gamma_2 L_{-1})|\psi_+\rangle = 0.
$$

(5.19)

There are two solutions to this being a null vector: $(\gamma_1, \gamma_2, \delta_s) = (k - 1, 1 - k^2, 3 - 2k + \frac{1}{4k+1})$ or $(\frac{k+1}{8}, \frac{(k+1)(2k+1)}{4}, \frac{8k+3}{8(2k+1)})$. The second solution is a highest weight state for $\bar{W}_0$, or in other words is annihilated by $L_0^+$. The first solution belongs to a null spin-3/2 multiplet of $\bar{W}_0$, and is a descendent of the state $L_{-1}^- |\psi_+\rangle$ which is also null in this case. In fact, only the first solution has a null descendent of the form (5.18). Explicitly this is

$$
\beta_1 = \frac{k^2 - 1}{2}, \beta_2 = (k + 1)^2(1 - k), \beta_3 = k^2 - 1, \beta_4 = \frac{1 - k}{2}, \beta_5 = \frac{(k + 2)(k - 1)}{2}.
$$

(5.20)
There is also a null vector $\chi_-$ of spin $-1/2$ which is the charge conjugate of $\chi_+$.

Let us write

$$\psi_+(z)(L_0^-)^p|\phi\rangle = \sum_{n \in \mathbb{Z}_+} \mu_n^p z^{n-y}$$  \hspace{1cm} (5.21)$$
$$\psi_-(z)(L_0^+)^p|\phi\rangle = \sum_{n \in \mathbb{Z}_+} \nu_n^p z^{n-y}.$$  \hspace{1cm} (5.22)

We now use the null vectors (5.18), (5.19) together with the relations at level 0,

$$L_0^+ |\psi_+\rangle = 0,$$  \hspace{1cm} (5.23)
$$L_0^- |\psi_+\rangle = |\psi_-\rangle,$$  \hspace{1cm} (5.24)
$$L_0^- |\psi_-\rangle = 0,$$  \hspace{1cm} (5.25)

to solve for the above operator product expansions. Performing the sort of contour manipulations described in the previous section we arrive at the following recurrence relations:

$$\alpha_n^p \beta_n^p = \frac{2N^{p-1}}{(k-1)} \nu_n^{p-1} + \sum_{x > 0} U_{-x} \mu_{n-x}^p [2(N - Y - k - 1) - k(x - 1) - 2(\Lambda - p)]$$
$$+ \frac{2}{(k-1)} \sum_{x > 0} L_{-x}^+ \nu_{n-x}^p + \sum_{x > 0} L_{-x}^- \mu_{n-x}^p (1 + k) - \sum_{x,y > 0} U_{-x} U_{-y} \nu_{n-x-y}^p$$  \hspace{1cm} (5.26)

$$\beta_n^p \nu_n^p = (1 - k) \sum_{x > 0} L_{-x}^+ \mu_{n-x}^p + \sum_{x > 0} L_{-x}^- \nu_{n-x}^p (1 + k) - \sum_{x,y > 0} U_{-x} U_{-y} \nu_{n-x-y}^p$$
$$+ \sum_{x > 0} U_{-x} \nu_{n-x}^p [2(N - Y - k - 1) - k(x - 1) - 2(p - 1 - \Lambda)]$$  \hspace{1cm} (5.27)

where

$$\alpha_n^p = 2(k + 1)^2(y - n)^2 + (2k - 1 + 2(\Lambda - p))(k + 1)(y - n) + (\Lambda - p)(\Lambda - p + k)$$
$$- (1 + k)h_\phi$$  \hspace{1cm} (5.28)

$$\beta_n^p = 2(k + 1)^2(y - n)^2 + (2k - 1 - 2(\Lambda - p))(k + 1)(y - n) + (\Lambda - p)(\Lambda - p - k)$$
$$- (1 + k)h_\phi$$  \hspace{1cm} (5.29)

and $N = n(k + 1), Y = y(k + 1)$. Alternatively, the equation (5.27) can be expressed as

$$\beta_n^p \nu_n^p = (1 - k) \nu_n^{p+1} + (1 - k) \sum_{x > 0} L_{-x}^+ \mu_{n-x}^p + \sum_{x > 0} L_{-x}^- \nu_{n-x}^p (1 + k) - \sum_{x,y > 0} U_{-x} U_{-y} \nu_{n-x-y}^p$$
$$+ \sum_{x > 0} U_{-x} \nu_{n-x}^p [2(N - Y - k - 1) - k(x - 1) - 2(p - \Lambda)]$$  \hspace{1cm} (5.30)

Consistency of (5.30) with (5.26) for $n = 0$ implies that $2N^{p-1} = -\alpha_0^p \beta_0^{p-1}$. Substituting in for $N^p, \alpha_0^p$ and $\beta_0^p$, we see that this equation is equivalent to

$$0 = \alpha_0^p \beta_0^{p-1} = (2Y^2 + (2k - 1 + 2\Lambda)Y + \Lambda(\Lambda + k) - (1 + k)h_\phi) \times$$
$$= (2Y^2 + (2k - 3 - 2\Lambda)Y + (\Lambda + 1)(\Lambda + 1 - k) - (1 + k)h_\phi)$$  \hspace{1cm} (5.31)
With some more work we can use this equation to solve for the allowed fusions $\psi^\pm \times \phi \rightarrow \phi'$. Taking the free field parametrisation for $\psi^+$, $\phi$, $\phi'$ to be $(1/2,1/2)$, $(\Lambda,a)$, $(\Lambda',a')$ respectively we find that

$$(1/2,1/2) \times (\Lambda,a) \rightarrow (\Lambda',a') = (\Lambda \pm \frac{1}{2}, a \pm \frac{1}{2})$$

(5.32)

The fusion rule for the field $\psi^+$ is exactly the same that selection rules for the finite dimensional spinor representation from which it is derived. We shall see that this property holds for more general representations below.

We can extract more information from (5.26). For any of the above allowed fusions (5.32), we can substitute in (5.26) the corresponding solution for $y$, and solve explicitly for $\mu_n^p$, $\nu_n^p$ in terms of the highest weight state ($\mu^0_0$ or $\nu^0_0$ depending on whether $\Lambda' = \Lambda + 1/2$ or $\Lambda - 1/2$). However, for special values of $(\Lambda,a)$ this iterative process breaks down because one of the coefficients $\alpha_n^p, \beta_n^p$ vanishes. The right hand side of the corresponding equation (5.26) can be interpreted as a null vector. For instance, consider the fusion $(1/2,1/2) \times (\Lambda,a) \rightarrow (\Lambda + 1/2, a + 1/2)$. In this case $y = -(\Lambda+a)/2$ and

$$\alpha_n^p = N(2N + 2a - 2k + 1 + 2p) + p(p - \Lambda - k - a)$$

(5.33)

$$\beta_n^p = N(2N + 2a - 2k + 1 + 4\Lambda - 2p) + (2\Lambda - p)(\Lambda - p - k - a).$$

(5.34)

Note that $\alpha_0^0 = 0$, as required by (5.31). If in addition $a = k - 1/2 - n(k+1)$, then $\alpha_n^0$ also vanishes, indicating the presence of a null vector at level $n$ with $U_0$ charge $\Lambda + 1/2$ in the module labelled by $(\Lambda',a') = (\Lambda + 1/2, k - n(k+1))$. Similarly, if $a' = k - 1/2 - n(k+1) - \Lambda$, then $\beta_{n-1}$ vanishes, indicating that there is a null vector at level $n$ with $U_0$ charge $\Lambda - 1/2$. In both cases, we can solve the recurrence relations (5.26,5.27) up to level $n$, and use the level $n$ equation to find an explicit expression for the null vector, much as was done in [25].

One might be tempted to argue that the vanishing $\alpha_n^p, \beta_n^p$ for $p > 0$ gives null vectors with lower $U_0$ charge, but it turns out that the equations (5.26), (5.27) conspire to give zeroes on the right hand side which can be cancelled with those on the left in this case.

5.3 The vector representation of $B_2$

The vector representation of $B_2$ is five-dimensional with weights $(0,1), (-1,0), (0,0), (1,0)$ and $(0,-1)$, to which we associate an eigenvalue basis $g_1$, $g_2$, $g_3$, $g_4$ and $g_5$ respectively. In this basis, the classical equation of motion (5.3) reads

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
L^- & U & 0 & 0 & 0 \\
L & 0 & 0 & 0 & 1 \\
L^+ & 0 & 0 & U & 0 \\
0 & L^+ & L & L^- & 0
\end{pmatrix} \begin{pmatrix}
g_1 \\
g_2 \\
g_3 \\
g_4 \\
g_5
\end{pmatrix} = \partial \begin{pmatrix}
g_1 \\
g_2 \\
g_3 \\
g_4 \\
g_5
\end{pmatrix}$$

(5.35)
From general considerations we identify the state $|\psi_1\rangle$ associated with $g_1$ as a highest weight state for $\hat{W}$, which has zero $U_0$ charge and therefore is a singlet under the action of $\hat{W}_0$. The first and third of these equations imply that $|\psi_3\rangle \sim L_{-1}|\psi_1\rangle$, $|\psi_5\rangle \sim (L_{-2} + \zeta L_{-1} L_{-1})|\psi_1\rangle$, while the weight and charge of $|\psi_2\rangle$ and $|\psi_4\rangle$ imply that $|\psi_2\rangle = L_{-1}^-|\psi_1\rangle$ and $|\psi_4\rangle = L_{-1}^+|\psi_1\rangle$. The remaining three equations imply the existence of three null vectors of the following form

$$|n^+\rangle = (\gamma_1 L_{-2}^+ + \gamma_2 U_{-1} L_{-1}^+ + \gamma_3 L_{-1} L_{-1}^+)|\psi_1\rangle$$

$$|n^-\rangle = (\gamma_1 L_{-2}^- - \gamma_2 U_{-1} L_{-1}^- + \gamma_3 L_{-1} L_{-1}^-)|\psi_1\rangle$$

$$|n^f\rangle = (L_{-2}^- L_{-1}^+ + L_{-2}^+ L_{-1}^- + \beta_1 L_{-1} U_{-1} U_{-1} + \beta_2 U_{-1} U_{-2} + \beta_3 L_{-3} + \beta_4 L_{-1} L_{-2} + \beta_5 (L_{-1})^3)|\psi_1\rangle$$

Note that we are allowed to add terms of the form $U^2$ to the last null vector, much as the classical and quantum energy momentum tensor by terms of this way.

As discussed above, maximally degenerate representations with a finite-dimensional highest weight space $V_0$ have only one independent null vector at higher level; therefore we expect that $|n^-\rangle \sim (L_0^-)^2|n^+\rangle$. We label the neutral null vector $L_0^-|n^+\rangle$ by $|n^0\rangle$. The charge conjugation invariance of $|n^f\rangle$ implies that

$$|n^f\rangle \sim \epsilon_1 (L_{-1}^- |n^-\rangle + L_{-1}^- |n^+\rangle) + \epsilon_2 L_{-1}^- |n^0\rangle$$

We find two possible solutions for the above null vectors which we consider in turn below.

**Case (i):** $L_0^-|\psi_1\rangle = \frac{1+k}{1-k^2}|\psi_1\rangle$

This case closely parallels the results for the spinor field in the previous section. We find that

$$|n^+\rangle = (L_{-2}^+ + U_{-1} L_{-1}^+ - (1 + k)L_{-1} L_{-1}^+)|\psi_1\rangle$$

$$|n^0\rangle = (L_{-1}^- L_{-1}^+ + L_{-1}^- L_{-1}^+ - 4U_{-1} U_{-1} + 4(k + 1)L_{-2} - 2(k + 1)^2(L_{-1})^2)|\psi_1\rangle$$

$$|n^-\rangle = (L_{-2}^- - U_{-1} L_{-1}^- - (1 + k)L_{-1} L_{-1}^-)|\psi_1\rangle$$

$$|n^f\rangle = (L_{-2}^- L_{-1}^+ + L_{-2}^+ L_{-1}^- - (k + 1)L_{-1} U_{-1} U_{-1} + (k - 3)U_{-1} U_{-2} - k(k + 1)L_{-3} + 2(k + 1)^2 L_{-1} L_{-2} - (k + 1)^3 (L_{-1})^3)|\psi_1\rangle$$

Much as in the spinor case, where the null vectors $\chi_-, \chi_+$ were part of a larger multiplet of null vectors with ‘highest weight’ $L_{-1}^+|\psi_+\rangle$, in this case $|n^+\rangle, |n^0\rangle, |n^-\rangle$ form part of a spin-2 multiplet with highest weight $(L_{-1}^+)^2|\psi_1\rangle$. We let

$$\psi_1(z)(L_0^-)^p|\phi\rangle = \psi_n^p z^{n-y}$$

$$\hat{L}_{-1}^- \psi_1(z)(L_0^-)^p|\phi\rangle = (\phi^+)^p_n z^{n-y-1}$$

$$\hat{L}_{-1}^- \psi_1(z)(L_0^-)^p|\phi\rangle = (\phi^-)^p_n z^{n-y-1}$$
Since $L^\pm_0 |\psi_1\rangle = 0$ we have that

\begin{align}
(\phi^+)_n^p &= L^+_0 \psi_n^p - N^{p-1}\psi_{n-1}^p \\
(\phi^-)_n^p &= L^-_0 \psi_n^p - \psi_n^{p+1}.
\end{align}

\tag{5.47}
\tag{5.48}

Then we can use the vectors $|n^+\rangle$, $|n^-\rangle$ to derive the following relations:

\begin{align}
\alpha^p_n (\phi^+_n)^p &= N^{p-1}\psi_{n-1}^p + \sum_{x>0} L^+_x \psi_{n-x}^p + U^-_x (\phi^+_n)_{n-x}^p \\
\beta^p_n (\phi^-)_n^p &= \psi_n^{p+1} + \sum_{x>0} L^-_x \psi_{n-x}^p - U^-_x (\phi^-)_n^p.
\end{align}

\tag{5.49}
\tag{5.50}

where

\begin{align}
\alpha^p_n &= N - Y - k - \Lambda + p \\
\beta^p_n &= N - Y - k + \Lambda - p.
\end{align}

\tag{5.51}
\tag{5.52}

We can also write (5.50) alternatively using that $L_0^- |\psi_1\rangle = 0$ as

\begin{align}
\beta_n^{p-1} (\phi^-)^p_n &= \sum_{x>0} L^-_x \psi_{n-x}^p - \sum_{x>0} U^-_x (\phi^-)^p_{n-x}.
\end{align}

\tag{5.53}

Combining the information in $|n^l\rangle$ and $|n^0\rangle$ gives the following equation:

\begin{align}
\gamma^p_n \psi_n^p &= L_0^- (\phi^+_n)^p + N^{p-1}(\phi^-)^p_n \\
&\quad + \sum_{x>0} (k+1)[2(N-Y) - kx - 2(k-1)]L^-_x \psi_{n-x}^p \\
&\quad + [-3(N-Y) + 2x(k-2) - (k-1)]U^-_x \psi_{n-x}^p \\
&\quad + \sum_{x,w>0} [(N-Y) + (k-3)w + (k-1)]U^-_x U^-_w \psi_{n-x-w}^p
\end{align}

\tag{5.54}

where

\begin{align}
\gamma^p_n &= \{(N-Y) - k + 1\}(N-Y)^2 + (1-2k)(N-Y) + (\Lambda - p)^2 + (\Lambda - p) - 2(k+1)h_{\phi}\]
\end{align}

\tag{5.55}

Using (5.54) for $n = 0$ together with equations derived from null vectors at level zero, and expressing $h_{\phi}$ in terms of $a$ and $\Lambda$ we find that

\begin{align}
(Y+k-1)[(Y+k-1 - \Lambda + p)(Y+k + \Lambda - p)(Y+a)(Y-a - 1 + 2k) - N^{p-1}]\psi_0^p &= 0
\end{align}

\tag{5.56}

The $p$ dependence cancels in these equations and since by definition $\psi_0^p \neq 0$ for some $p$, it follows that $Y$ must satisfy

\begin{align}
(Y+k-1)(Y+k-1 - \Lambda)(Y+k + \Lambda)(Y+a)(Y-a - 1 + 2k) &= 0
\end{align}

\tag{5.57}
One can also evaluate the fourth order Casimir $C$ on the states $\psi_0^p$ and show that only the following fusions are allowed:

$$\psi_1(z) \times \left(\Lambda, a\right) \rightarrow \left(\Lambda', a'\right) = \left(\Lambda \pm 1, a\right), \left(\Lambda, a\right), \left(\Lambda, a \pm 1\right) \quad (5.58)$$

Just as in the spinor case, the allowed fusions for the degenerate primary field derived from the vector representation correspond to the tensor decomposition rules of the vector representation. Note that though $\psi_1(z)$ is a singlet under $\hat{W}_0$ its fusion with other primary fields changes the value of $\Lambda$. This stems from the simple observation that

$$\left[L_0^+, \psi_1(z)\right] = z\hat{L}_0^+ \psi_1(z) + z\hat{L}_{-1}^+ \psi_1(z) = z\hat{L}_{-1}^+ \psi_1(z) \neq 0 \quad (5.59)$$

**Case(ii):** $L_0|\psi_1\rangle = \frac{3k}{2}|\psi_1\rangle$

Although the calculation is similar to case (i), the results require a more subtle interpretation. In this case, the null vectors are given by

$$|n^+\rangle = ((k + 1)L_{+2}^+ - U_{-1}^+ L_{-1}^+ - L_{-1}^+ L_{-2}^+)|\psi_1\rangle \quad (5.60)$$

$$|n^0\rangle = \left(\hat{L}_{-1}^+ L_{-1}^- + L_{-1}^+ L_{-2}^- - (k + 1)[(k - 2)U_{-1}^- U_{-1}^- - 2k(k + 1)L_{-2}^- + k(L_{-1}^-)^2]\right)|\psi_1\rangle \quad (5.61)$$

$$|n^-\rangle = ((k + 1)L_{-2}^- + U_{-1}^- L_{-1}^- - L_{-1}^- L_{-2}^-)|\psi_1\rangle \quad (5.62)$$

$$|n^f\rangle = \left(\hat{L}_{-2}^+ L_{-1}^- + L_{-2}^+ L_{-2}^- - U_{-1}^- U_{-1}^- + 5U_{-1}^- U_{-2}^- - 2(k + 1)L_{-3}^- + 2k(k + 1)L_{-2}^- - (L_{-1}^-)^3\right)|\psi_1\rangle \quad (5.63)$$

The recurrence relations derived from the two charged null vectors $|n^\pm\rangle$ are

$$\alpha_n^p(\phi^+)^p_n = N^{p-1}(k + 1)\psi_n^{p-1} + (k + 1)\sum_{x>0} L_{-x}^+ \psi_{n-x}^p - U_{-x}(\phi^+)^p_{n-x} \quad (5.64)$$

$$\beta_n^p(\phi^-)^p_n = (k + 1)\psi_n^{p+1} + (k + 1)\sum_{x>0} L_{-x}^- \psi_{n-x}^p + U_{-x}(\phi^-)^p_{n-x}, \quad (5.65)$$

where

$$\alpha_n^p = n - y + k + \Lambda - p \quad (5.66)$$

$$\beta_n^p = n - y + k - \Lambda + p. \quad (5.67)$$

As before we can derive a further relation from $|n^0\rangle$ and $|n^f\rangle$. For brevity we only quote the relation for $n = 0$ which is

$$\gamma_0^p\psi_0^p = N^{p-1}(\phi^-)^{p-1}_0 + L_0^+ (\psi^+)^p_0 \quad (5.68)$$

where

$$\gamma_n^p = (k + 1)(n - y + \frac{3k}{2})[(n - y)(n - y + 2k - 1) + (\Lambda - p)(\Lambda - p + 1) - 2(k + 1)h_\phi]. \quad (5.69)$$
Using (5.48), (5.65) and (5.68), we can show that

\[(y - \frac{3k}{2})[(y - k - \Lambda + p)(y + 1 - k + \Lambda - p)(y - a)(y + a + 1 - 2k) - N^p] \psi_0^p = 0 \] (5.70)

Solving this equation for \( y \), and calculating the fourth order Casimir on \( \psi_0^p \), we find that the following fusions are allowed;

\[\psi_1(z) \times (\Lambda, a) \rightarrow (\Lambda', a') = (\Lambda \mp [1 + k], a), (\Lambda, a), (\Lambda, a \mp [1 + k]). \] (5.71)

Once more, the fusion rules for the field \( \psi_1 \) resemble the tensor decomposition rule of the vector representation, with the important difference that the weight lattice has been scaled by \( 1 + k \). Of course, the co-existence of lattices at two different scales occurs in the representation theory of the standard \( \mathcal{W}A_n \) algebras, where the scales are usually labelled \( \alpha_+ \) and \( \alpha_- \). However, in the present case there seems to be some problem in interpreting the first two of the above fusions. The charge of the highest weight state \( (\Lambda', a') \) has been shifted by \( \pm (1 + k) \) which by assumption is irrational and certainly non-integral. The states \( \psi_0^p \) have charge \( \Lambda - p \) and thus are of the form \( (L'_0)^q |\Lambda', a'\rangle \) where \( q = \Lambda' - \Lambda + p = \pm (1 + k) + p \) which is non-integral. It seems therefore that the representations corresponding to \( (\Lambda', a') \) are necessarily unbounded above and below in these cases.

(This interpretation seems sensible for the case \( \Lambda' = \Lambda - 1 - k \). Since in this case \( \alpha_0^0 \) vanishes we can have \( (\phi^+)_0^0 \neq 0 \) and so from (5.48) \( L_0^+ \psi_0^0 \neq 0 \). In general we have that

\[ (L_0^+)^r \psi_1(z)|\phi\rangle = z^r (\hat{L}_1^+)^r \psi_1(z)|\phi\rangle \] (5.72)

In this case the states \( (L_0^+)^r |\psi_1\rangle \) are not generally null, and so we have no reason to believe that the left hand side of the above equation should vanish. Note that this is in contrast to case (i) where \( (L_1^+)^2 |\psi_1\rangle \) is null, and in that case one sees from the allowed fusions that indeed \( (L_0^+)^2 \psi_0^0 \) vanishes in all cases. However the case \( \Lambda' = \Lambda + 1 \) the interpretation seems more problematic yet. In this case \( \alpha_0^0 \) does not vanish and it would seem that \( L_0^+ \psi_0^0 \) must vanish. However treating \( \psi_0^0 \) as a highest weight for \( \bar{\mathcal{W}} \) seems to be inconsistent with the values of the Casimir \( C \) associated with \( (\Lambda', a') \), and indeed the calculation of \( C \) breaks down on \( \psi_0^0 \) because of the vanishing of \( \beta_0^{-1} \). One way out of this is to insist that we only fuse with representations which are neither highest nor lowest weight, so that \( p = d + Z, d \notin Z \) in all the above expressions.)

### 6 General degenerate representation theory of \( \bar{\mathcal{W}} \)

In this section we shall combine the information obtained in the previous section with our expectations based on the free field representation given in section 3 to conjecture some results about general degenerate representations of \( \bar{\mathcal{W}} \).
We begin by reconsidering the free field parametrisation of the charge and weight of a highest weight state $|\Lambda, a\rangle$. If we write the vector $(\Lambda, a)$ as $\beta$ then the weight of the state $|\Lambda, a\rangle$ is given by (4.12) in terms of $\beta$ by

$$h(\Lambda, a) = \frac{\beta \cdot (\beta - 2\rho)}{2(k+1)}$$

(6.1)

where $\rho = (-1/2, k - 1/2)$. We define the vector $\tilde{\beta} = \beta - \rho$, so that

$$h(\Lambda, a) = \frac{\tilde{\beta}^2 - \rho^2}{2(k+1)}.$$  

(6.2)

As we already remarked in section 2, $h$ is invariant under $\tilde{\beta} \to w(\tilde{\beta})$ where $w \in W$, the Weyl group of $B_2$.

Suppose that the representation built up from the primary state $|\beta\rangle$ contains some singular vector $|\chi\rangle$. $|\chi\rangle$ is also highest weight for $\bar{W}$, and we label its charge and weight as above by $\tilde{\beta}'$. We should like to argue that

$$\tilde{\beta}' - \tilde{\beta} = -a_i \alpha_i$$

(6.3)

where $a_i$ is a positive integer and $\alpha_i$ is a root of $B_2$. It comes as no surprise that the charge of the primary state and its singular descendent is an integer, since the charge of all descendents of the primary states have this property. That the value of $a$ should differ in this way is more difficult to see. This can be checked in all the examples considered in the previous section. For example, $|\beta\rangle$ has a singular state of the form $(L_0)^p|\beta\rangle$ if $a = \Lambda + k - p$. It is easy to check that $\tilde{\beta}' - \tilde{\beta} = (-p, p)$ in this case. Similarly, we showed that for $a = k - n(k+1)$, $|\beta\rangle$ has a singular vector at level $n$ and charge $\Lambda$. In this case, $\tilde{\beta}' - \tilde{\beta} = (0, -1)$.

Further evidence that (6.3) holds comes from considering the screening charge defined at the end of section 2. The standard construction of singular vectors using a screening charge $S$ involves the expression $(S)^n|\beta'\rangle$ where $(S)^n$ is a multiple integral with suitably defined contours, such that for certain values of $\beta'$ this expression does not vanish, but is a singular vector. Since $\bar{W}$ commutes with $S$, the charge and weight of the singular vector are given by $\beta'$, but given the form of the screening charge (3.30) we have

$$(S)^n|\beta'\rangle = \{X_{-m,...}\}|\beta' - (n, 0)\rangle$$

(6.4)

where $\{X_{-m,...}\}$ represent creation operators of $\bar{W}$. Thus we expect from the form of the screening charge that $\beta' - \beta = \tilde{\beta}' - \tilde{\beta} = (-n, 0)$ in this case. One can generalise this argument to yield other roots of $B_2$ in (6.3) by introducing unreduced screening charges associated with the horizontal $su(2)$ currents.
From now on we shall merely assume the validity of (6.3). The weight of \(\beta\) and its descendent \(\chi\) differ by an integer, so we have that

\[
2(k + 1)(h' - h) = \tilde{\beta}'^2 - \tilde{\beta}^2 = (\tilde{\beta}' - \tilde{\beta}) \cdot (\tilde{\beta}' + \tilde{\beta}) = 2N(k + 1) \tag{6.5}
\]

If we use (6.3), and put \(N = a_i b_i\) where \(b_i\) is a non negative integer, and as before \(a_i\) is a positive integer, we find that

\[
\tilde{\beta} \cdot \alpha_i = a_i \frac{(\alpha_i)^2}{2} - b_i(k + 1). \tag{6.6}
\]

If \(\tilde{\beta}\) satisfies this equation, then there is a singular vector in the Verma module built up from \(|\tilde{\beta}\rangle\) with

\[
\tilde{\beta}' = \tilde{\beta} - a_i \alpha_i \tag{6.7}
\]

at level \(a_i b_i\) and with charge \(\Lambda - a_i (\alpha_i \cdot \alpha_2)\). A more convenient parametrisation of \(\tilde{\beta}\) is given by

\[
\tilde{\beta} = \sum_{i=2}^{i=2} a_i \lambda_i - (1 + k)b_i \lambda^\vee_i \tag{6.8}
\]

where \(\lambda_i \cdot \alpha_j = \delta_{ij}(\alpha_j)^2/2\), and \(\lambda^\vee_i \cdot \alpha_j = \delta_{ij}\) are the fundamental weights and coweights of \(B_2\). With this parametrisation (6.6) is satisfied for \(\alpha_1, \alpha_2\), the simple roots of \(B_2\). We shall call the corresponding primary state

\[
\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \tag{6.9}
\]

Equation (6.6) is the most important in this section. From it we shall now write down character formulae, general formulae for the fusion of maximally degenerate representation and give a determinant formulae for \(\bar{W}\).

6.1 Character formulae for representations of \(\bar{W}\)

In the following paragraphs we give character formulae for representations of \(\bar{W}\). As in the rest of this paper we shall restrict ourselves to the case that \(k\) is irrational for simplicity, and for brevity’s sake, we shall further restrict ourselves to some interesting examples.

If \(\beta\) is such that (6.6) is not satisfied for any \(\alpha_i\), then it follows that the representation built up from \(|\beta\rangle\) contains no singular vectors and is irreducible. The character for a highest weight representation is given by

\[
\chi_{(\Lambda,a)}(x,q) = \text{Tr}_{(\Lambda,a)}(q^{L_0} x^{U_0}) = q^h x^\Lambda F(x,q) \tag{6.10}
\]

\[
F(x,q) = \frac{1}{(1 - x^{-1})} \prod_{n=1}^{\infty} \frac{1}{(1 - q^n x^{-1})(1 - q^n)^2(1 - q^n x)} \tag{6.11}
\]

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At the opposite extreme, suppose that $\beta$ is such that with the parametrisation (6.8), $a_i, b_i$ are all positive integers. We calculate the embedding pattern for the singular vectors in this case by checking for descendent singular states using the condition (6.6), using (6.3) to determine the weight and charge of the singular states so found, and then iterating the process.

![Null Vector embedding diagram for integral $a_i, b_i$](image)

**Fig1: Null Vector embedding diagram for integral $a_i, b_i$**

Strictly speaking if we find two singular states with the same charge and weight by two different paths we cannot be sure that they are the same vector, but we shall assume this is true, relying on our experience with other W-algebras. For $a_i, b_i$ all positive integers, we get the embedding diagram of singular vectors given in fig. 1. Each singular vector is labelled in this diagram by its charge and weight, which are given in terms of the charge and weight $(\Lambda, h)$ of the highest weight state. The character in this case is given by

$$
[(1 - q^{(2a_1+b_1)(b_1+b_2)}) - x_1^a (q^{a_1b_1} - q^{a_1b_1+a_2b_2+a_2b_1}) - x^{-a_2} (q^{a_2b_2} - q^{2a_1b_1+2a_2b_2+a_2b_1})] F(x, q)
$$

$$
+ x^{-a_1-a_2} (q^{a_1b_1+a_2b_2+2a_1b_2} - q^{(a_1+a_2)(b_1+2b_2)}) F(x, q)
$$

(6.12)

Finally let us turn our attention to representations of $\bar{W}$ with finite dimensional highest weight space. We need a singular state at level zero with charge $p$ less than that of the highest weight state. This implies that we must have $a_i b_i = 0$ and $-a_i(\alpha_i \cdot \alpha_2) = p$ There
are three solutions: \( a_i = p, b_i = 0 \) and \( \alpha_i = -\alpha_1, \alpha_2, \alpha_4 \). These correspond to the three solutions given by \( N^{p-1} = 0 \) [c.f. (4.14)]. Let us concentrate on the solution \( 2\Lambda \in \mathbb{Z} \). These correspond to

\[
\begin{pmatrix}
a_1 \\
a_2
\end{pmatrix}
\begin{pmatrix}
b_1 \\
ob_2
\end{pmatrix}
\]

(6.13)

where \( a_2 = p = 2\Lambda + 1 \). The most complicated embedding arises if we take \( a_1 \) and \( b_1 \) also to be positive integers; this amounts to a special case of the embedding diagram we just considered with \( b_2 \) set to zero. The expression (6.12) in this case reduces to

\[
[(1 - x^{-2\Lambda-1})(1 - q^{(2a_1+2\Lambda+1)(b_1+b_2)}) - (x_1^a - x^{-a_1-2\Lambda-1})(q^{a_1b_1} - q^{(a_1+2\Lambda+1)b_1})] F(x, q) .
\]

(6.14)

Embedding patterns for other possible values of \( a_1, b_1 \) are given in fig.2 to give the reader some idea of the different possibilities. The diagram is labelled similarly to fig.1. From the above we can see that not all degenerate representations with two independent singular vectors have the same singular vector structure. This is also true for standard \( W \)-algebras. In analogy with that case, we shall reserve the terminology ‘maximally degenerate’ for those representations with the singular vector structure given in figure 1. It turns out that all of the representations considered in the previous sections are of this type; that is they are labelled by (6.13) with \( a_1, a_2 \) and \( b_1 \) given by positive integers. The vacuum representation, the spinor representation and case (i) and (ii) of the vector representation considered in the previous section are given by

\[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 \\
2 & 0
\end{pmatrix}, \quad \begin{pmatrix}
2 & 1 \\
1 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 2 \\
1 & 0
\end{pmatrix}
\]

(6.15)

respectively.

6.2 Fusion of maximally degenerate representations

The examples in the preceding section suggest the following generalisations of the fusion rules proved there for primary fields with \( b_1 = 1 \). In this case we may write

\[
\beta = (a_1 - 1)\lambda_1 + (a_2 - 1)\lambda_2
\]

(6.16)
We can interpret $\beta$ as a highest weight of a (finite) $B_2$ representation. A natural set of fusion rules for $\phi_\beta$ with a non-degenerate primary field are

$$\phi_\beta \times \phi_{\beta'} \rightarrow \phi_{\beta' + \lambda}$$

(6.17)

where $\lambda$ ranges over the different weights of the representation with highest weight $\beta$, so that the number of possible fusions is simply the dimension of this representation. This is consistent with what we found for the spinor representation and case (i) of the vector representation in the previous section.

Some support for this conjecture can be found by considering the number of constraints on the operator product expansion that the null vectors of $\phi_\beta$ imply [25]. If we let $B(n)$
be the difference between the number of independent operator product expansions of the form (5.14) and the number of independent constraints arising from null vectors up to level $n$, then we find that

$$\sum_{n\geq 0} B(n)q^n = \lim_{x\to 1} x^{-\Lambda} q^{-h} \frac{\chi_{\Lambda_1}(x, q)}{\chi_{\text{vac}}(x, q)}$$

$$= \frac{a_2(1 - q^{2a_1+a_2}) - (2a_1 + a_2)(q^{a_1} - q^{a_1+a_2})}{(1 - q)^3}$$

This turns out to be a finite polynomial in $q$ of order $2a_1 + a_2 - 3$. So for $n > 2a_1 + a_2 - 3$, the number of constraints from null vectors exactly matches the number of basis vectors of OPE’s, so adding weight to the claim that we can always solve for the fusion of such representations. Moreover it can be argued that the number of independent fusions is given by

$$\sum_{n\geq 0} B(n) = \lim_{x, q \to 1} x^{-\Lambda} q^{-h} \frac{\chi_{\Lambda_1}(x, q)}{\chi_{\text{vac}}(x, q)} = \frac{a_1a_2(a_1 + a_2)(2a_1 + a_2)}{6}$$

which is the dimension of a representation of $B_2$ with highest weight $\beta$.

Unfortunately, these arguments do not seem to make much sense in the case where $b_1 \neq 1$. For example, let us consider taking $a_1 = a_2 = 1$. In this case we may write

$$\beta = (1 + k)[(b_1 - 1)\lambda_1 + b_2\lambda_2]$$

and we might guess that we should interpret $\beta$ as a highest weight for the Lie algebra with $\alpha_i^\vee$ as roots. This is $C_2$ which is isomorphic to $B_2$ again, but with simple roots $(-1, 1), (2, 0)$. However, with this interpretation, case (ii) of the vector representation considered in section 5 corresponds to $b_1 = 2, b_2 = 0$ which in turn should correspond to the spinor representation of $C_2$. This would only produce four of the five fusions found in the previous section. The conditions established there however were only necessary, not sufficient, so it is not inconceivable that the extra fusion (which corresponds to no shift in $(\Lambda, a)$) is disallowed by conditions coming from higher level null vectors. Worse still, the calculation used above to calculate the number of fusions when $b_1 = 1$ is of no help here. It gives the answer $b_1^3$ for the number of possible fusions of the field corresponding to $\begin{pmatrix} 1 & b_1 \\ 1 & 0 \end{pmatrix}$, which disagrees with both the above answers for $b_1 = 2$. Clearly this deserves a more careful treatment. Nonetheless, the results we have obtained do not seem incompatible the elegant conjecture that if

$$\beta = (a_1 - 1)\lambda_1 + (a_2 - 1)\lambda_2 + (1 + k)[(b_1 - 1)\lambda_1 + b_2\lambda_2]$$

then the fusion of the corresponding field $\phi_\beta$ is given by

$$\phi_\beta \times \phi_\beta' \to \phi_{\beta' + \lambda_1 + (1+k)\lambda_2}$$
where $\lambda$ is a weight of the $B_2$ representation with highest weight $(a_2 - 1)\lambda_1 + (a_2 - 1)\lambda_2$ and $\lambda'$ is the weight of the $C_2$ representation with highest weight $(b_1 - 1)\lambda'_1 + b_2\lambda'_2$.

### 6.3 Determinant formula for $\bar{W}$

It is straightforward to see what the generalisation of the determinant formulae for standard $W$-algebras [37] is, given (6.3). It is convenient to Taylor expand $F(x, q)$ as follows

$$F(x, q) = \sum_{a,b} U_{cd} q^c x^d \quad (6.24)$$

If we consider the space of descendent states of $|\beta\rangle = |\Lambda, a\rangle$ with charge $\Lambda + M$, and at level $N$, then we expect that the determinant of the matrix of inner products of states in this space is given by

$$\det_{MN}(\beta) = C \prod_{j=0}^{N} \prod_{\{r,s\geq 0: rs=j\}} \prod_{\alpha\in \Delta} (\beta \cdot \alpha_i - r(\alpha_i)^2/2 + s(k + 1))^{U_{cd}} \quad (6.25)$$

where $C$ is a constant, and $c = N - rs$, $d = M + r(\alpha \cdot \alpha_2)$.

### 7 Conclusions

In this paper we have constructed a $W$-algebra of remarkable simplicity which we believe has most of the features of $W$-algebras arising from Drinfel’d-Sokolov reductions. This makes it a useful laboratory for studying many of the unresolved issues concerning such algebras. In particular the representation theory and modular properties of the characters are questions that need to be addressed in order to construct the Hilbert space of the corresponding statistical systems.

Some steps towards understanding the representation theory have been made in this work. The zero mode algebra was found to be an interesting polynomial deformation of the zero-grade algebra $g_0$ ($su(2) \times u(1)$ in this case). We expect this to hold in the general case since we know that in the $c \to \infty$ limit, the zero mode algebra coincides with $g_0$ [15]. Many of the formulae which were derived by free field techniques in the standard reduction case can be taken over simply by adjusting the ‘shift vector’ $\rho$. Maximally degenerate representations could be labelled by four integers

$$\left( \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \end{array} \right) \quad (7.1)$$

exactly as in the case of the standard reduction [31]). However, only some of these representations have a finite highest weight space $V_0$. We considered the case where $b_2 = 0$. 

29
which corresponds to half-integrally charged state. Interestingly, only a subset of these representations, corresponding to those with $b_1 = 1$ close under fusion to give states with finite highest weight spaces.

There are many questions that need to be addressed by future work. We have not had time to consider rational values of the central charge $c$ (or rather the parameter $k$). We expect that as in the standard case, for rational $k$ we will have a periodic identification of completely degenerate representations [1], and this will lead to some truncation in the fusion rules. It should also be fairly straightforward to derive character formulae for these representations by analogy to the standard case, and to find how the characters transform under the modular group. It will then be possible to write down modular invariant partition functions for $\bar{W}$ models. Also the issue of unitarity of representations has not been touched in this paper. A necessary condition for unitarity is that the the central charge (3.10) and the parameter $k$ be simultaneously positive. This only occurs for $4 - \sqrt{10} < 6k < 4 + \sqrt{10}$, a fairly restrictive range. This ties in with the prejudice that the standard reduction of simply-laced algebras are rather special in that they each admit an infinite sequence of unitary models. For more general reductions, it seems rather harder to find unitary examples. Many of the formulae considered in the paper seem to have elegant generalisations for all reductions, and give important clues as to how a unified treatment of the general case would look. In particular, the free field parametrisation of the weights of degenerate fields involves both weights and coweights, as we might have anticipated from the duality arguments of [30]. By comparing the primary field content and fusion rules that we obtain, we can compare the conformal field theories constructed by reductions and by the coset method, and this may give us some prescription for going between the two (which already exists for the theories with symmetry algebra $W_g$ if $g$ is simply-laced [5]). This would provide us with two descriptions of the same conformal field theory, one in which the properties of the chiral algebra were transparent, and the other in which the modular properties of characters were clear. We intend to pursue this further elsewhere. Finally, $\bar{W}$ is such a simple algebra that it would be surprising if the corresponding conformal models, or their integrable perturbations did not have some physical significance.

Acknowledgements

It gives me great pleasure to thank G.M.T.Watts for many stimulating conversations. I should also like to thank Adrian Kent, Koos de Vos and Cosmas Zachos for discussions. Part of this work was completed at the University of Chicago and was supported by the following grants: U.S. DOE grant DEFG02-90-ER-40560 and NSF grant PHY900036
8 Appendix

In this appendix we demonstrate that the screening charge \( S \) given by (3.30) commutes with the generators of \( \bar{W} \). By construction \( S \) is the integral of a current of conformal weight one and zero charge with respect to \( U_0 \). This ensures that it commutes with both \( L(z) \) and \( U(z) \). We now show that it commutes with the expressions for \( L^\pm(z) \) given by (3.29).

The operator product of \( L^+ \) with the screening current \( s \) is of the form

\[
L^+(z)s(\zeta) = \frac{\hat{L}_0^+ s(\zeta)}{(z - \zeta)^2} + \frac{\hat{L}_{-1}^+ s(\zeta)}{(z - \zeta)}
\]  

(8.1)

For \( L^+(z) \) to commute with \( S = \oint dz s(z) \), the right hand side of this equation must be a total derivative in \( \zeta \), that is

\[
L^+_{-1}|s\rangle = L^{-1}L^+_0 |s\rangle
\]  

(8.2)

Substituting in the free field representation (3.29) for \( L^+_0 |s\rangle \), we see that

\[
L^+_{-1}|s\rangle = [j^+_1 j^+_0 + j^+_1 j^+_0 + r^+(j^+_2 j^+_0 + j^+_1 j^+_0)]|s\rangle
\]  

(8.3)

\[
L^+_0 |s\rangle = [j^+_0 j^+_0 + r^+ j^+_0 - s^+ j^+_0]|s\rangle
\]  

(8.5)

where \( |s^+\rangle = j^+_0 |s\rangle \). Also using (3.27) we have that

\[
L^{-1}|s^+\rangle = [\mathcal{L}^{-1} + j^2_{-1} j^2_0]|s^+\rangle
\]  

(8.7)

\[
= \frac{1}{k + 1} (j^1_{-1}|s^+\rangle + j^+_1|s\rangle) + \alpha j^2_{-1}|s^+\rangle
\]  

(8.8)

Substituting in \( r^+ = -\sqrt{(k+1)} \), \( s^+ = -k \), one easily reads off that (8.2) is satisfied for \( \alpha = -1/\sqrt{(k+1)} \), but not for \( \alpha = 2k/\sqrt{(k+1)} \). The same result holds if we consider \( L^{-}(z) \) instead.
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