Research Article

Statistical Inference of Odd Fréchet Inverse Lomax Distribution with Applications

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In this article, we propose and study a new three-parameter distribution, called the odd Fréchet inverse Lomax (OFIL) distribution, derived by combining the odd Fréchet-G family and the inverse Lomax distribution. Since Fréchet is a continuous distribution with wide applicability in extreme value theory, the new model contains these properties as well as the characteristics of the inverse Lomax distribution which make it more flexible and provide a good alternative for some well-known lifetime distributions. We initially present a linear representation of its functions and discussion on density and hazard rate function. Then, we study its various mathematical properties. Different estimation methods are used to estimate parameters of OFIL. The Monte Carlo simulation study is carried out to compare the efficiencies of different methods of estimation. The maximum likelihood estimation (MLE) method is used to estimate the OFIL parameters by considering three practical data applications. We show that the related model is the best in comparisons based on Akaike information criterion (AIC), Bayesian information criterion (BIC), and other goodness-of-fit measures.

1. Introduction

Fréchet distribution is considered as an extreme value distribution and extensively applied in life study affected by earthquakes, floods, horse riding, vehicle racing, queues in supermarkets, wind speeds, and sea waves. A detailed study of Fréchet distribution and its applications was given by Nadarajah and Kotz [1]. Haq et al. [2] proposed the generalization of the Fréchet distribution named the transmuted Weibull Fréchet (TWFr) distribution and derived its main characteristics including probability-weighted moments, moments, incomplete moments, generating function, order statistics, and stress strength model. Hassan and Nassr [3] proposed the inverse Weibull-G family of distributions, and Hashmi and Gull [4] proposed and studied the Weibull–Lomax distribution. A comprehensive lifetime model is necessary to fit complex real data sets. The inverse Lomax (IL) distribution, being the member of the family of generalized beta distribution, is very useful in the fields of economics, geography, medical sciences, and actuarial sciences (see [5]). Kleiber [6] used the IL distribution in the model fitting of geophysical data especially on different sizes of land fires in forests of California (USA). He also presented Lorenz ordering among order statistics for some distributions including IL distribution. Singh et al. [7] considered the IL distribution and derived reliability estimates under
Type-II censoring using the Markov chain Monte Carlo method. Bayesian estimation of parameters of the IL distribution based on the Type-I censoring scheme was discussed by Reyad and Othman [8]. The model said belongs to an inverted family of distributions with more flexibility to analyze nonmonotone behavior of the real hazard rate function. The IL distribution is obtained by considering a random variable X as X = 1/Y, where Y has a Lomax distribution $Y \sim L(x; \alpha, \beta)$. Rahman et al. [9] presented Bayesian inference of model parameters of the IL distribution. A comparison study was conducted between Bayesian estimators and ML estimators. Yadav et al. [10] used different approximation techniques to obtain Bayesian estimates of model parameters of the IL distribution. In this article, the odd Fréchet-G (OF-G) family of distributions is used to boost the flexibility of the baseline distribution. Here, $\alpha$ is the shape parameter, and $g(x; \xi)$ considers a pdf of the baseline distribution. The random variable $X$ with density (4) is denoted by $X \sim OF - G (x; \alpha, \xi)$.

The hazard rate function (hrf) of the OF-G family is

$$h(x; \theta, \xi) = \frac{\theta g(x; \xi) [1 - G(x; \xi)]^{\theta-1} e^{-[(1-G(x; \xi))/(G(x; \xi))]^\theta}}{G(x; \xi)^{\theta+1}}$$

(5)

This article is planned as follows: in Section 2, we define a new model called the odd Fréchet inverse Lomax (OFIL) distribution. In Section 3, some important mathematical properties are derived. Section 4 deals with characterization of the new model. Some different methods of estimation of model parameters are discussed in Section 5. Numerical study based on Monte Carlo simulations is conducted in Section 6. The analysis of real data sets is carried in Section 7, and conclusions are made in Section 8.

2. The Odd Fréchet Inverse Lomax Distribution

The cdf and pdf of the OFIL distribution are given by inserting (1) and (2) in (3) and (4) as follows:

$$F(x; \alpha, \beta, \theta) = e^{-[(1+(\beta/x))^{\alpha-1}]}$$

(6)

$$f(x; \alpha, \beta, \theta) = \frac{\theta \alpha \beta}{x^2} \left[1 + \frac{\beta}{x}\right]^{\alpha-1} \left[1 - \left(1 + \frac{\beta}{x}\right)^{-\alpha}\right]^{\theta-1} e^{-\left[(1+(\beta/x))^{\alpha-1}\right]}$$

(7)

The survival function, hrf, and cumulative hazard rate function are defined as follows:

$$S(x; \alpha, \beta, \theta) = 1 - e^{-[(1+(\beta/x))^{\alpha-1}]}$$

$$h(x; \alpha, \beta, \theta) = \frac{\theta \alpha \beta (1 + (\beta/x))^{\alpha-1} ((1 + (\beta/x))^{\alpha-1} - 1)^{\theta-1}}{\left(e^{(1+(\beta/x))^{\alpha-1}} - 1\right)x^2}$$

$$H(x; \alpha, \beta, \theta) = -\log \left(1 - e^{-[(1+(\beta/x))^{\alpha-1}]^\theta}\right)$$

(8)

Now, a random variable $X$ that follows density (7) is denoted by $X \sim OFIL(x; \alpha, \beta, \theta)$. Some graphic features which include pdf and hrf plots of $X \sim OFIL(x; \theta, \alpha, \beta)$ are illustrated in Figures 1(a)–1(c).

The impact of values of parameters on density and hazard curve is significant. Figure 1(a) shows density plots for specified values of parameters keeping one parameter fixed in each plot. The corresponding hazard rate functions are plotted in Figure 1(b). Figure 1(c) shows both of these plots for varying values of parameters. It is observed that the pdf and hazard curves start from zero or goes down at the origin.

2.1 Quantile Function and Median. The OFIL distribution can be easily simulated by inverting (6) as follows: if $u \sim U(0, 1)$, then the random variable $X$ can be obtained from
Figure 1: (a) The pdf plots of the OFIL distribution. (b) The hrf plots of the OFIL distribution. (c) The pdf and hrf plots of the OFIL distribution.
\[ Q(u) = \frac{\beta}{\sqrt{1 + \sqrt{\ln(1/u)}} - 1} \]  

(9)

where \( Q(u) \) is the quantile function of the OFIL distribution. The quantile density function is obtained by taking the derivative of the quantile function, which is yet another way of prescribing a probability distribution. It is the reciprocal of the pdf composed with \( Q(u) \). Therefore, the density function of quantile \( Q(u) \) is

\[ q(u) = \frac{\beta}{u} \left( 1 + (\ln(1/u))^{1/\alpha} \right)^{1-1/\alpha} \left( \ln(1/u) \right)^{(1/\alpha) - 1} \left( 1 + (\ln(1/u))^{1/\alpha} \right)^{-1} \left( u \right) \left( 1 + (\ln(1/u))^{1/\alpha} \right)^{-1} - 1 \right]^2. \]

(10)

Particularly, the median of the OFIL distribution is obtained from (9) considering \( u = 0.5 \). The median is defined as follows:

\[ \text{median} = \frac{\beta}{\sqrt{1 + \sqrt{\ln(2)}} - 1}. \]

(11)

### 2.2. Skewness and Kurtosis

The skewness and kurtosis measures based on quantile measures are given, respectively, as follows:

Bowley's skewness measure is

\[ S = \frac{Q(0.75) + Q(0.25) - 2Q(0.5)}{Q(0.75) - Q(0.25)}. \]

(12)

and Moors' kurtosis measure is

\[ K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}. \]

(13)

\( S \) is the measure of the asymmetry of the OFIL distribution, and \( K \) is the measure of the heaviness of tails (in comparison to the normal distribution) of the OFIL distribution. These measures are less sensitive to outliers and defined by mathematical formulas. Moreover, they may exist for any distribution which does not have moments.

Plots of \( S \) and \( K \) for certain values of \( \alpha \) as a function of \( \theta \) and for certain choices of \( \theta \) as a function of \( \alpha \) at \( \beta = 0.5 \) are provided in Figures 2(a) and 2(b).

### 2.3. Linear Representation

In this section, the representation of cdf (5) and pdf (7) of the OFIL distribution is obtained.

Firstly, according to Haq and Elgarhy [16], equation (7) can be written as

\[ f(x; \theta, \xi) = \sum_{k=0}^{\infty} \eta_k g(x; \xi) [G(x; \xi)]^k, \]

(14)

where

\[ \eta_k = \sum_{i,j=0}^{\infty} ((\theta(-1)^{i+j})/i!) \left( \theta(i+1) + j \right) \left( \theta(i+1) + j - 1 \right). \]

By inserting equations (1) and (2) in equation (14), the pdf of the OFIL distribution can be written as a linear combination of the IL distribution as

\[ f(x; \theta, \alpha, \beta) = \sum_{k=0}^{\infty} \eta_k g(x; \alpha(k+1), \beta), \]

(15)

where \( g(x; \alpha(k+1), \beta) \) denotes the pdf of the IL distribution given in equation (2) with parameters \( \alpha(k+1) \) and \( \beta \).

\[ f(x; \theta, \alpha, \beta) = \sum_{k=0}^{\infty} \eta_k g(x; \alpha(k+1), \beta). \]

(16)

Secondly, according to Haq and Elgarhy [16], the expansion of cdf \( [F(x)]^q \) is given as follows:

\[ [F(x)]^q = \sum_{z=0}^{\infty} S_z G(x; \xi)^z, \]

(17)

where

\[ S_z = \sum_{\theta, \xi} ((-1)^{z+1}) \left( q^{\theta + u - 1} \right) \left( q^{\theta + u} \right). \]

\[ [F(x)]^q = \sum_{z=0}^{\infty} S_z \left( \left( 1 + \frac{\beta}{x} \right)^{-\alpha} \right)^z. \]

(18)

### 3. Some Statistical Properties of the OFIL Distribution

#### 3.1. Probability-Weighted Moments (PWMs)

Probability-weighted moments are often considered to be superior to ordinary moments. They are less sensitive to outliers and uniquely defined. They are sometimes used when maximum likelihood estimates are unavailable or difficult to compute. They may also be considered as initial values for maximum likelihood estimates. PWMs are denoted by \( \tau_{r,s} \), which can be defined as

\[ \tau_{r,s} = E[X^rF(x)^s] = \int_{-\infty}^{\infty} x^r (F(x))^s f(x) dx. \]

(19)

The PWMs of OFIL are obtained by substituting (14) and (17) into (19) as follows:

\[ \tau_{r,s} = \sum_{z=0}^{\infty} \sum_{k=0}^{\infty} \eta_k S_z \int_{0}^{\infty} x^r g(x; \xi) G(x; \xi)^z dx, \]

(20)

where \( g(x; \xi) \) and \( G(x; \xi) \) have pdf (1) and cdf (2), so we obtain \( \tau_{r,s} \) as follows:

\[ \tau_{r,s} = \sum_{z=0}^{\infty} \sum_{k=0}^{\infty} \eta_k S_z \int_{0}^{\infty} x^{r-2} \left( 1 + \frac{\beta}{x} \right)^{-\alpha - 1} \left( 1 + \frac{\beta}{x} \right)^{-\alpha(z+k)} dx, \]

(21)
where \( \tau_{r,z+k} = \int_0^\infty x^{r+k} g(x; \xi) d\xi \).

According to Hassan and Mohamed [12], the moments about the origin of the IL distribution are given as

\[
\mu_{r+1} = \alpha \beta \Gamma \left( \frac{r+1}{\gamma} \right) \Gamma \left( \frac{1}{\gamma} \right) \Gamma \left( \Gamma \left( \gamma + \frac{1}{\gamma} \right) \right) \Gamma \left( \gamma + 1 \right) \left( \frac{1}{\gamma} + \frac{1}{\gamma+1} \right) + \frac{1}{\gamma} + \frac{1}{\gamma+1} \right),
\]

where \( \Gamma (0) = -\gamma y^r (1-\gamma)^r \phi (r) \) for \( r = 1, 2, \ldots \), \( y \) denotes Euler’s constant and \( \phi (r) = \sum_{k=1}^\infty \frac{1}{k} \) (see [17])

3.2. Moments. If \( X \) has pdf (16), then the \( r \)th moment can be obtained as

\[
\mu_{r+1} = \alpha \beta \Gamma \left( \frac{r+1}{\gamma} \right) \Gamma \left( \frac{1}{\gamma} \right) \Gamma \left( \gamma + 1 \right) \left( \frac{1}{\gamma} + \frac{1}{\gamma+1} \right) + \frac{1}{\gamma} + \frac{1}{\gamma+1} \right).
\]

Substituting (16) in (23) gives

\[
\mu_{r+1} = \sum_{k=0}^{\infty} \eta_k \beta \Gamma \left( \frac{r+1}{\gamma} \right) \Gamma \left( \frac{1}{\gamma} \right) \Gamma \left( \gamma + 1 \right) \left( \frac{1}{\gamma} + \frac{1}{\gamma+1} \right) \right) \Gamma \left( \gamma + 1 \right) \left( \frac{1}{\gamma} + \frac{1}{\gamma+1} \right) \right) \right).
\]

where \( \Gamma_{r,k} \) is the PWM of the IL distribution. The mean of \( X \) can be obtained using equation (22) by putting \( r = 1 \) and \( z = 0 \):

\[
\mu = \mu_{1+1} = E(X) = \sum_{k=0}^{\infty} \eta_k \beta \alpha \gamma
\]

where \( \gamma \) denotes Euler’s constant. The \( r \)th central moment \( \mu_r \) of \( X \) is derived as

\[
\mu_r = E(X - \mu_{1})^r = \sum_{s=0}^{r} \left( -1 \right)^s \frac{r!}{s!} \mu_{r-s} \ (\mu_{1})^s
\]
3.3. Rényi Entropy. The Rényi entropy is an important measure to find the amount of uncertainty in the data. The larger the uncertainty, the larger the value of Rényi entropy.

By definition, Rényi entropy is

\[ I_R(\delta) = \frac{1}{1 - \delta} \log[I(\delta)], \]

where \( I(\delta) = \int_0^\infty f^\delta(x)dx, \delta > 0 \) and \( \delta \neq 1 \). From Haq and Elgarhy [16], the Rényi entropy of OF-G is

\[ I_R(\delta) = 1/(1 - \delta) \log \sum_{k=0}^\infty w_k \int_0^\infty g(x, \zeta)\delta G(x, \zeta)^k dx, \]

where \( w_k = (\theta^\delta \phi^\delta (-1)^{i+k})/i! \left( \theta(i+\delta) + \delta + j - 1 \right) \left( \theta(i+\delta) + j - \delta \right). \)

Therefore, the Rényi entropy of OFIL is given by

\[ I(\delta) = \int_0^\infty x^{-\delta} \left[ 1 + \left( \frac{\beta}{x} \right)^{-a} \right]^{-\frac{\alpha}{1+\alpha}} dx \]

\[ = \alpha^\delta \beta^{-\delta-1} \sum_{k=0}^\infty w_k B(2\delta - 1, \alpha(k+\delta) - \delta + 1). \]

Therefore, Rényi entropy of the OFIL distribution is given as

\[ I_R(\delta) = \frac{1}{1 - \delta} \log \left[ \alpha^\delta \beta^{-\delta-1} \sum_{k=0}^\infty w_k B(2\delta - 1, \alpha(k+\delta) - \delta + 1) \right]. \]

where \( B(\ldots) \) is the beta function.

4. Characterizations

Characterization of distribution is a key aspect that has fascinated the researchers, and it is also helpful in finding an appropriate model. In distribution theory, characterizations of probability models are very useful. A characterization of probability models plays a vital role in statistical studies in various fields of natural sciences, physical sciences, and applied sciences. Characterizing a probability distribution is a characteristic or feature which helps in identifying it, or it is the only distribution that satisfies a specified characteristic. Many researchers have investigated the characterizations of absolutely continuous probability distributions over the years. For instance, the characterization of distributions by truncated moments was studied by Glänzel [18, 19]. Hamedani and Ahsanullah [20–23] discussed various techniques of characterizations of probability distributions.

4.1. Characterization Based on Two Truncated Moments. Glänzel [18, 19] presented a theorem based on the ratio of two truncated moments. By utilizing this theorem, we obtain characterization of the OFIL distribution.

Proposition 1. Let \( X: \Omega \longrightarrow (0, \infty) \) be distributed as equation (7), and

\[ q_1(x) = \left( 1 + \frac{\beta}{x} \right)^{-a} \left( \frac{1 - (1 + (\beta/x))^{-a}}{(1 + (\beta/x))^{-a}} \right)^\theta, \]

\[ q_2(x) = q_1(x) \left( 1 - \left( 1 + \frac{\beta}{x} \right)^{-a} \right)^\theta, \quad \text{for } x > 0. \]

The random variable (r. v.) \( X \) follows OFIL distribution if and only if the function \( \eta \) is of the form

\[ \eta(x) = \frac{1}{2} \left( 1 - \left( 1 + \frac{\beta}{x} \right)^{-a} \right)^\theta. \]

Proof. Let \( X \) be a random variable with density equation (7); then,

\[ (1-F(x))E[q_1(X)|X \geq x] = \left( 1 - \left( 1 + \frac{\beta}{x} \right)^{-a} \right)^\theta, \quad x > 0, \]

\[ (1-F(x))E[q_2(X)|X \geq x] = \frac{(1-(1+(\beta/x))^{-a})^{2\theta}}{2}, \quad x > 0, \]

and finally,

\[ \eta(x) = \frac{1}{2} \left( 1 - \left( 1 + \frac{\beta}{x} \right)^{-a} \right)^\theta. \]
The random variable $X$ has pdf (7), then clearly (42) holds. Now, if (42) holds, then

$$\frac{d}{dx} \left[ (g(x; \xi))^{-1} h(x) \right] = \frac{d}{dx} \left[ \frac{(1 + (\beta/x))^{\alpha} ((1 + (\beta/x))^a - 1)^{\theta-2}}{e((1+\theta)(x+\beta)^a) - 1\theta - 1) \right],$$

which can be simplified to $\int_0^x f_{\text{OFIL}}(u) du = F_{\text{OFIL}}(x)$. □

**Corollary 1.** Let $X$ be distributed as equation (7), and let $q_1(x)$ be as Proposition 1. The pdf of $X$ be (6) if and only if there exist functions $q_1(x)$ and $\eta(x)$ defined in Proposition 1 (equations (34) and (35)) satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{\theta (\alpha \beta (x+\beta/x))^{a-1}}{(1 - (1 + (\beta/x))^{-a})}, \quad x > 0. \tag{40}$$

4.2. Characterization Based on the Hazard Function. It is known that the hrf, $h_{wp}$, of a twice differentiable distribution function, $F$, satisfies the first-order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_{wp}(x)}{h_{wp}(x)} = h_{wp}(x). \tag{41}$$

This may be the only characterization available in terms of hrf for many univariate continuous distributions.

**Proposition 2.** The pdf of the OFIL distribution is (7) if and only if its hrf satisfies the following differential equation:

$$h'(x; \xi) - \frac{2x + (1 - \alpha) \beta}{x(x+\beta)} h(x; \xi) = a \beta (x+\beta/x)^a \left[ (x+\beta/x)^a - 1 \right]^{\theta-2} \left( e^{-1+\theta(x+\beta/x)^a} - 1 \right)^2 (x+\beta)^2 \times \theta \left[ -2 + \left( x + \beta \right)^a \right] (1 + \theta) + e^{-1+\theta(x+\beta/x)^a} \left( 2 + \left( x + \beta \right)^a \theta - 1 \right) \right]. \tag{42}$$

Proof. If $X$ has pdf (7), then clearly (42) holds.

**Proof.** If $X$ has density

$$f(x; \alpha, \beta, \mu_1, \mu_2, \mu_3, \mu_4, \Var, \Skewness, \Kurtosis),$$

which can be simplified to $\int_0^x f_{\text{OFIL}}(u) du = F_{\text{OFIL}}(x)$. □

**Corollary 1.** Let $X$ be distributed as equation (7), and let $q_1(x)$ be as Proposition 1. The pdf of $X$ be (6) if and only if there exist functions $q_1(x)$ and $\eta(x)$ defined in Proposition 1 (equations (34) and (35)) satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{\theta (\alpha \beta (x+\beta/x))^{a-1}}{(1 - (1 + (\beta/x))^{-a})}, \quad x > 0. \tag{40}$$

4.2. Characterization Based on the Hazard Function. It is known that the hrf, $h_{wp}$, of a twice differentiable distribution function, $F$, satisfies the first-order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_{wp}(x)}{h_{wp}(x)} = h_{wp}(x). \tag{41}$$

This may be the only characterization available in terms of hrf for many univariate continuous distributions.

**Proposition 2.** The pdf of the OFIL distribution is (7) if and only if its hrf satisfies the following differential equation:
or equivalently
\[
h(x) = \frac{a\beta(1 + (\beta/x))^n - 1}{\left(e^{(1+(\beta/x))^n - 1} - 1\right)x^2}.
\]

(44)

5. Statistical Inference

The parameters of the OFIL distribution can be estimated using maximum likelihood (ML), least square (LS), percentile (PC), and Anderson–Darling (AD) methods of estimation.

5.1. ML Estimation. Using ML estimation technique, let \(X_1, \ldots, X_n\) indicate observed values from the OFIL distribution, and then log-likelihood function, say \(\ell\), can be written as

\[
\ell = \prod_{i=1}^{n} f(x_i) = n\ln(\alpha) + n\ln(\theta) + n\ln(\beta)
\]

\[
- 2\sum_{i=1}^{n} \log(x_i) + (\alpha \theta - 1) \sum_{i=1}^{n} \ln\left(1 + \frac{\beta}{x_i}\right)
\]

\[
+ (\theta - 1) \sum_{i=1}^{n} \ln\left(1 - \left(1 + \frac{\beta}{x_i}\right)^{-\alpha}\right) - \sum_{i=1}^{n} \left(\frac{1}{x_i} + \frac{\beta}{x_i}\right) + \left(\frac{1}{x_i} - 1\right)^{\alpha - 1} - 1).
\]

(45)

The ML equations of the OFIL distribution are obtained by taking the partial derivative with respect to parameters \(\alpha, \theta, \) and \(\beta\). We have

\[
\frac{\partial \ell}{\partial \alpha} = -n\alpha + \theta \sum_{i=1}^{n} \ln\left(1 + \frac{\beta}{x_i}\right) + \sum_{i=1}^{n} \left(\frac{1 + (\beta/x_i)^{-\alpha}}{1 - (1 + (\beta/x_i)^{-\alpha})}\right) - \theta n \sum_{i=1}^{n} \left(\frac{1 + (\beta/x_i)^{-\alpha}}{1 - (1 + (\beta/x_i)^{-\alpha})}\right)^{\alpha - 1} - \theta \alpha n \sum_{i=1}^{n} \left(\frac{1 + (\beta/x_i)^{-\alpha}}{1 - (1 + (\beta/x_i)^{-\alpha})}\right)
\]

\[
\frac{\partial \ell}{\partial \theta} = -n\theta + \alpha \sum_{i=1}^{n} \ln\left(1 + \frac{\beta}{x_i}\right) + \sum_{i=1}^{n} \ln\left(1 + \frac{\beta}{x_i}\right)^{-\alpha} - \sum_{i=1}^{n} \left(\frac{1 + (\beta/x_i)^{-\alpha}}{1 - (1 + (\beta/x_i)^{-\alpha})}\right)^{\alpha - 1} - \theta \alpha n \sum_{i=1}^{n} \left(\frac{1 + (\beta/x_i)^{-\alpha}}{1 - (1 + (\beta/x_i)^{-\alpha})}\right)
\]

\[
\frac{\partial \ell}{\partial \beta} = -n\beta + (\alpha \theta - 1) \sum_{i=1}^{n} \frac{1}{x_i + \beta} + \alpha (\theta - 1) \sum_{i=1}^{n} \frac{1}{1 + (\beta/x_i)^{-\alpha}} - \theta \alpha \sum_{i=1}^{n} \left(\frac{1 + (\beta/x_i)^{-\alpha}}{1 - (1 + (\beta/x_i)^{-\alpha})}\right)^{\alpha - 1} - \theta \alpha n \sum_{i=1}^{n} \left(\frac{1 + (\beta/x_i)^{-\alpha}}{1 - (1 + (\beta/x_i)^{-\alpha})}\right)
\]

Equating \(\partial \ell/\partial \alpha, \partial \ell/\partial \theta,\) and \(\partial \ell/\partial \beta\) with zeros and solving simultaneously, we obtain the ML estimators of \(\alpha, \theta, \) and \(\beta\).

5.2. Ordinary LS Estimators (LSEs). Suppose \(X_1, X_2, \ldots, X_n\) be a random sample (r.s.) of size \(n\) from OFIL distribution with a corresponding ordered sample which is \(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\). The mean and variance of OFIL are independent of the unknown parameters, given by

\[
E(F(X_{(i)})) = \frac{i}{n + 1}
\]

\[
\text{Var}(F(X_{(i)})) = \frac{i(n - i + 1)}{(n + 1)^2(n + 2)}
\]

where \(F(X_{(i)})\) is the cdf of \(i^{th}\) order statistic \(X_{(i)}\). The LS estimators can be obtained by minimizing the following sum of square of errors:

\[
\sum_{i=1}^{n} \left[F(X_{(i)}) - \frac{i}{n + 1}\right]^2,
\]

(48)

with respect to \(\alpha, \theta, \) and \(\beta\). So, the LS estimators of \(\alpha, \theta, \) and \(\beta\) of the LS model are obtained by minimizing the following:

\[
\sum_{i=1}^{n} \left[e^{-\left(1+(\beta/x_{(i)})\right)^{-\alpha}} - \frac{i}{n + 1}\right]^2.
\]

(49)

5.3. PC Estimators (PCEs). Let \(X_1, \ldots, X_n\) be a r.s. from the OFIL distribution with the corresponding order statistics \(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\). Using PC technique of estimation, the estimators of \(\alpha, \theta, \) and \(\beta\) are derived by minimizing the following:

\[
\sum_{i=1}^{n} \left[\ln\left(\frac{i}{n + 1}\right) - \ln\left(e^{-\left(1+(\beta/x_{(i)})\right)^{-\alpha}}\right)\right]^2,
\]

(50)

with respect to \(\alpha, \theta, \) and \(\beta\).

5.4. Anderson–Darling Estimators (ADEs). Anderson and Darling [25] introduced the technique of Anderson–Darling estimation. By utilizing it in the OFIL model, the Anderson–Darling estimators (ADEs) of \(\alpha, \theta, \) and \(\beta\) can be obtained by minimizing the function given by

\[
AD = -n - \frac{1}{n} \sum_{i=1}^{n} (2i - 1) \left\{\log\left[F(X_{(i)}; \alpha, \theta, \beta)\right] + \log\left[1 - F(X_{(n+1-i)}; \alpha, \theta, \beta)\right]\right\},
\]

(51)

with respect to \(\alpha, \theta, \) and \(\beta\).

6. Simulation Study

This section consists of the comparison of the proposed estimators MLE, LSE, PCE, and ADE for OFILD in terms of simulated risks. Comparisons among the estimators have
been made through simulation using 5000 simulated samples. We took samples of size $n = 10, 20, 50, 100, 200,$ and 500 from the OFIL distribution with different variations of parameters: set $1 = (\alpha = 2, \theta = 2, \beta = 0.5)$, set $2 = (\alpha = 2, \theta = 3, \beta = 0.5)$, set $3 = (\alpha = 1, \theta = 3, \beta = 0.5)$, set $4 = (\alpha = 0.5, \theta = 2, \beta = 0.5)$, set $5 = (\alpha = 3, \theta = 2, \beta = 1.5)$, and set $6 = (\alpha = 0.5, \theta = 2, \beta = 1.5)$. The MLEs, LSEs, PCEs, and ADEs of $\alpha$, $\theta$, and $\beta$ are determined. Then, the estimates of all methods and their mean square errors (MSEs) are documented in Tables 2–7.

In Table 8, ranks are assigned to different methods of estimation based on MSEs of the corresponding method for specified combinations of the parameters. From Table 8, we can conclude that the ML estimation method outperforms all other estimation methods (overall score is 73.5). Therefore, based on our simulation study, we can conclude that the ML estimation method is the best for the OFIL distribution.

7. Applications

In this section, we give two examples to illustrate the performance of the proposed model. To show the better performance of the OFIL distribution and numerical calculations, we use R software. For comparison purpose, we consider the following distributions:
Marshall–Olkin inverse Lomax (MOIL) distribution \[ \text{[13]}: \]
\[ f(x) = \frac{\theta \alpha \beta (1 + (\beta / x))^{-\alpha - 1}}{x^2 (1 - (1 - \theta) (1 + (\beta / x))^{-\alpha})^2} \]  
\[ \theta, \alpha, \beta > 0, x > 0. \]

(ii) Alpha-power Weibull (APW) distribution \[ \text{[26]}: \]
\[ f(x) = \frac{\log \alpha \alpha - 1}{\lambda} \beta x^{(\beta - 1)} e^{-\lambda x} \alpha^{a - e^{-\lambda x}} \]  
\[ \lambda, \alpha, \beta > 0, x > 0. \]

(iii) Alpha-power inverted exponential (APIE) distribution \[ \text{[27]}: \]
\[ f(x) = \frac{\log \alpha \lambda}{\alpha - 1} x^{a - e^{-\lambda x}} \]  
\[ \lambda, \alpha > 0, x > 0. \]

(iv) Alpha-power transformed IL (APTIL) distribution \[ \text{[14]}: \]
\[ f(x) = \frac{\log \alpha a b (1 + \frac{b}{x})^{-\alpha - 1}}{\alpha - 1} x \]  
\[ a, a, b > 0, x > 0. \]

Table 4: Estimates and MSEs of the OFIL distribution for MLEs, LSEs, PCEs, and ADEs for set 3 (\( \alpha = 1 \), \( \theta = 3 \), and \( \beta = 0.5 \)).

| n | MLEs Estimates | MLEs MSEs | LSEs Estimates | LSEs MSEs | PCEs Estimates | PCEs MSEs | ADEs Estimates | ADEs MSEs |
|---|---|---|---|---|---|---|---|---|
| 10 | 1.224 | 0.231 | 1.183 | 0.064 | 1.154 | 0.146 | 1.148 | 0.112 |
| 20 | 1.203 | 0.157 | 1.119 | 0.055 | 1.178 | 0.099 | 1.13 | 0.06 |
| 50 | 0.823 | 0.034 | 1.130 | 0.040 | 1.165 | 0.074 | 1.117 | 0.042 |
| 100 | 0.826 | 0.033 | 1.110 | 0.032 | 1.101 | 0.039 | 1.093 | 0.027 |
| 200 | 0.819 | 0.027 | 1.096 | 0.025 | 1.106 | 0.037 | 1.082 | 0.024 |
| 500 | 0.812 | 0.024 | 1.087 | 0.023 | 1.064 | 0.017 | 1.099 | 0.024 |

Table 5: Estimates and MSEs of the OFIL distribution for MLEs, LSEs, PCEs, and ADEs for set 4 (\( \alpha = 2 \), \( \theta = 2 \), and \( \beta = 1.5 \)).

| n | MLEs Estimates | MLEs MSEs | LSEs Estimates | LSEs MSEs | PCEs Estimates | PCEs MSEs | ADEs Estimates | ADEs MSEs |
|---|---|---|---|---|---|---|---|---|
| 10 | 1.918 | 0.243 | 1.826 | 0.368 | 1.911 | 0.341 | 1.835 | 0.139 |
| 20 | 1.918 | 0.229 | 1.825 | 0.062 | 1.931 | 0.259 | 1.891 | 0.063 |
| 50 | 1.821 | 0.043 | 1.788 | 0.06 | 2.06 | 0.075 | 1.802 | 0.025 |
| 100 | 1.824 | 0.039 | 1.788 | 0.055 | 2.11 | 0.07 | 1.816 | 0.052 |
| 200 | 1.835 | 0.029 | 1.79 | 0.054 | 2.106 | 0.063 | 1.814 | 0.046 |
| 500 | 1.831 | 0.032 | 1.795 | 0.05 | 2.115 | 0.062 | 1.832 | 0.039 |

*Indicate that the value multiply \( 6.33 \times 10^{-3} \).

(iii) Alpha-power inverted exponential (APIE) distribution \[ \text{[27]}: \]
\[ f(x) = \frac{\log \alpha \lambda}{\alpha - 1} x^{a - e^{-\lambda x}} \]  
\[ \lambda, \alpha > 0, x > 0. \]

(iv) Alpha-power transformed IL (APTIL) distribution \[ \text{[14]}: \]
\[ f(x) = \frac{\log \alpha a b (1 + \frac{b}{x})^{-\alpha - 1}}{\alpha - 1} x^{a - e^{-\lambda x}} \]  
\[ a, a, b > 0, x > 0. \]
(v) Weibull IL (WIL) distribution [12]:

\[ f(x) = ab\lambda x^{\lambda - 1}(1 + \frac{x}{\beta})^{-1}\left(1 - \left(1 + \frac{x}{\beta}\right)^{-1}\right)^{(b-1)} \cdot e^{-a\left((1+\frac{x}{\beta})\right)^{-1}} \quad a, b, \lambda, \beta > 0, x > 0. \]  

The data sets are reported in Table 9. The first data set consists of 48 observations of maximum annual flood discharges of North Saskatchewan in units of 1000 cubic feet per second of the North Saskatchewan River at Edmonton over a period of 47 years [28]. The second data set consists of the monthly actual tax revenue in Egypt from January 2006 to November 2010 [29]. The third data set consists of a sample of 30 failure times of the air-conditioning system of an airplane [30].

Some descriptive measures for all three data sets are given in Table 10. TTT plots and box plots for three data sets are given in Figures 3 and 4, respectively. From both TTT plots, we see a concave curve, indicating that the hrf behind the data is possibly increasing. This specificity also belongs to the hrf for the OFIL distribution for some values, justifying

| Table 6: Estimates and MSEs of the OFIL distribution for MLEs, LSEs, PCEs, and ADEs for set 5 (\(\alpha = 3, \theta = 2, \beta = 1.5\)). |
|---|---|---|---|---|---|---|---|
| n | Estimates | MSEs | Estimates | MSEs | Estimates | MSEs | Estimates | MSEs |
|---|---|---|---|---|---|---|---|---|
| 10 | 2.349 | 0.955 | 2.293 | 1.532 | 1.312 | 2.309 | 1.153 |
| 20 | 2.404 | 0.557 | 2.336 | 0.627 | 2.328 | 0.85 | 2.399 | 0.735 |
| 50 | 2.456 | 0.434 | 2.41 | 0.556 | 2.54 | 0.362 | 2.443 | 0.492 |
| 100 | 2.538 | 0.312 | 2.522 | 0.436 | 2.57 | 0.337 | 2.45 | 0.492 |
| 200 | 2.626 | 0.214 | 2.626 | 0.413 | 2.619 | 0.326 | 2.482 | 0.481 |
| 500 | 2.793 | 0.054 | 2.684 | 0.379 | 2.539 | 0.313 | 2.526 | 0.413 |

| Table 7: Estimates and MSEs of the OFIL distribution for MLEs, LSEs, PCEs, and ADEs for set 6 (\(\alpha = 0.5, \theta = 2, \beta = 1.5\)). |
|---|---|---|---|---|---|---|---|
| n | Estimates | MSEs | Estimates | MSEs | Estimates | MSEs | Estimates | MSEs |
|---|---|---|---|---|---|---|---|---|
| 10 | 0.674 | 0.105 | 0.713 | 0.112 | 0.686 | 0.105 | 0.833 | 0.328 |
| 20 | 0.605 | 0.049 | 0.66 | 0.037 | 0.611 | 0.075 | 0.647 | 0.058 |
| 50 | 1.986 | 0.038 | 1.811 | 0.069 | 1.619 | 0.041 | 1.897 | 0.139 |
| 100 | 1.938 | 0.031 | 1.775 | 0.065 | 1.618 | 0.034 | 1.856 | 0.055 |
| 200 | 0.546 | 4.68* | 0.646 | 0.023 | 0.683 | 0.034 | 0.631 | 0.022 |
| 500 | 0.551 | 4.337* | 0.646 | 0.022 | 0.683 | 0.037 | 0.635 | 0.029 |

*Indicate that the value multiply \(8.235 \times 10^{-3}\).
its consideration for these data sets. Further details on the TTT plots can be found in [31].

TTT plots and box plots for all three data sets are given in Figures 3 and 4.

From Figure 3, it is revealed that the TTT plots show an increasing trend of the hazard rate function for data sets I and II, whereas data set III shows a decreasing trend.

From Figure 4, box plots show that data sets are positively skewed, and some outliers also exist.

The parameters are estimated using the MLE approach. Furthermore, for appropriate model selection, we used the following goodness-of-fit measures: log-likelihood ($l$), Akaike information criterion (AIC), Bayesian information criterion (BIC), Anderson–Darling statistic

| Table 8: Partial and overall ranks of all the methods of estimation for some combinations of parameters. |
|-----------------------------------------------|---|---|---|---|---|
| Parameters | $n$ | ML | LS | PC | AD |
| $(\alpha = 2, \theta = 2, \beta = 0.5)$ | | | | | |
| 10 | 1.5 | 3.5 | 1.5 | 3.5 | 3.5 |
| 20 | 2 | 3 | 1 | 4 | 4 |
| 50 | 1 | 2 | 3 | 4 | 4 |
| 100 | 1 | 3 | 2 | 4 | 4 |
| 200 | 2 | 3 | 1 | 4 | 4 |
| 500 | 1.5 | 4 | 1.5 | 3 | 3 |
| $(\alpha = 2, \theta = 3, \beta = 0.5)$ | | | | | |
| 10 | 1 | 3 | 2 | 4 | 4 |
| 20 | 2.5 | 2.5 | 4 | 1 | 1 |
| 50 | 2 | 3.5 | 1 | 3.5 | 3.5 |
| 100 | 3.5 | 3.5 | 1 | 2 | 2 |
| 200 | 4 | 3 | 1 | 2 | 2 |
| 500 | 2.5 | 4 | 1 | 2.5 | 2.5 |
| $(\alpha = 1, \theta = 3, \beta = 0.5)$ | | | | | |
| 10 | 4 | 1.5 | 3 | 1.5 | 1.5 |
| 20 | 3.5 | 1.5 | 4 | 1.5 | 1.5 |
| 50 | 3 | 1.5 | 4 | 1 | 1 |
| 100 | 3 | 2 | 4 | 1 | 1 |
| 200 | 4 | 2 | 3 | 1 | 2 |
| 500 | 4 | 3 | 1 | 2 | 2 |
| $(\alpha = 2, \theta = 2, \beta = 1.5)$ | | | | | |
| 10 | 4 | 3 | 2 | 1 | 1 |
| 20 | 3 | 1 | 4 | 2 | 2 |
| 50 | 1 | 2 | 4 | 3 | 3 |
| 100 | 1 | 3 | 4 | 2 | 2 |
| 200 | 1 | 3.5 | 3.5 | 2 | 2 |
| 500 | 1 | 3.5 | 3.5 | 4 | 4 |
| $(\alpha = 2.5, \theta = 2, \beta = 1.5)$ | | | | | |
| 10 | 3.5 | 3.5 | 1.5 | 3.5 | 1.5 |
| 20 | 2 | 3 | 4 | 1 | 1 |
| 50 | 2 | 4 | 1 | 3 | 3 |
| 100 | 1 | 4 | 2.5 | 1 | 3 |
| 200 | 1 | 3 | 2 | 4 | 4 |
| 500 | 1 | 4 | 2 | 3 | 3 |
| $(\alpha = 0.5, \theta = 2, \beta = 0.5)$ | | | | | |
| 10 | 1 | 2 | 3 | 4 | 4 |
| 20 | 1 | 2 | 4 | 3 | 3 |
| 50 | 1 | 3 | 4 | 2 | 2 |
| 100 | 1 | 3 | 4 | 2 | 2 |
| 200 | 1 | 3 | 4 | 2 | 2 |
| 500 | 1 | 3 | 4 | 2 | 2 |
| $\Sigma$ Ranks | 73.5 | 103 | 95.5 | 90 |
| Overall rank | 1 | 4 | 3 | 2 |

| Table 9: Maximum annual flood discharges, actual taxes, and failure times of the air-conditioning data sets. |
|-----------------------------------------------|---|---|---|---|
| Data set I | 19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.200, 40.000, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560 |
| Data set II | 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17.0, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10.0, 4.1, 36.0, 8.5, 8.0, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7.0, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11.0, 11.6, 11.9, 5.2, 6.8, 8.9, 7.3, 10.8 |
| Data set III | 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95. |
(A*), and its p-value, and Cramer–von Mises statistic (W*) and its p value. The lower the values of these criteria, the better the fit. We also provide the value for the Kolmogorov–Smirnov (KS) statistic and its p value.

MLEs and goodness-of-fit measures are presented in Tables 11–13 for data sets I, II, and III, respectively. The likelihood equations are plotted in Figures 5(c), 6(c), and 7(c), respectively. These figures confirm the MLEs of the OFIL distribution. The goodness-of-fit measures are least for the OFIL distribution when compared with other fitted models for all three cases.

The empirical and fitted pdfs and empirical cdfs are shown in Figures 5(a), 6(a), and 7(a), respectively, for three data sets. The empirical and fitted sfs and empirical
### Table 11: ML estimates along with goodness of fit for data set I.

| Model | MLE | $-\log L$ | AIC | BIC | $A^*$ | $W^*$ | KS |
|-------|-----|-----------|-----|-----|------|------|----|
| OFIL  | $\hat{\theta} = 2.52944$  
$\hat{a} = 0.89524$  
$\hat{\beta} = 41.2755$  
$\hat{\alpha} = 0.00044$  
| 215.112 | 436.224 | 441.838 | 0.14619 (0.9989) | 0.02085 (0.996) | 0.06974 (0.9737) |
| APTIL | $\hat{a} = 79.021$  
$\hat{b} = 0.14638$  
$\hat{\alpha} = 7.83735$  
| 216.492 | 438.984 | 444.597 | 0.27286 (0.9569) | 0.03900 (0.9400) | 0.07224 (0.9636) |
| WIL   | $\hat{b} = 0.11073$  
$\hat{\lambda} = 2359.93$  
$\hat{\beta} = 0.40060$  
$\hat{\theta} = 0.03115$  
| 216.536 | 441.072 | 448.557 | 0.27011 (0.9586) | 0.03803 (0.9430) | 0.07251 (0.9624) |
| MOIL  | $\hat{a} = 1132.43$  
$\hat{\beta} = 0.12629$  
$\hat{\lambda} = 6.31 \times 10^{11}$  
| 216.599 | 439.198 | 444.812 | 0.21208 (0.9868) | 0.02745 (0.9847) | 0.07569 (0.9462) |
| APTW  | $\hat{\beta} = 0.63123$  
$\hat{\lambda} = 0.33636$  
| 217.398 | 440.796 | 446.410 | 0.47026 (0.7765) | 0.06620 (0.7774) | 0.09011 (0.8306) |
| APIE  | $\hat{a} = 5.102 \times 10^{10}$  
$\hat{\lambda} = 1.62081$  
| 236.609 | 477.218 | 480.960 | 6.46850 (0.0006) | 1.22190 (0.0007) | 0.28348 (0.0009) |

### Table 12: ML estimates along with goodness of fit for data set II.

| Model | MLE | $-\log L$ | AIC | BIC | $A^*$ | $W^*$ | KS |
|-------|-----|-----------|-----|-----|------|------|----|
| OFIL  | $\hat{\theta} = 5.09654$  
$\hat{a} = 0.22799$  
$\hat{\beta} = 185.518$  
$\hat{\alpha} = 0.00060$  
| 188.297 | 382.594 | 388.827 | 0.23206 (0.9790) | 0.03594 (0.9541) | 0.06401 (0.9690) |
| APTIL | $\hat{a} = 656.496$  
$\hat{b} = 0.04073$  
$\hat{\alpha} = 9.34279$  
| 189.036 | 384.072 | 390.305 | 0.34829 (0.8976) | 0.06196 (0.8036) | 0.06549 (0.9620) |
| WIL   | $\hat{b} = 0.25068$  
$\hat{\lambda} = 41.1497$  
$\hat{\beta} = 3.28888$  
$\hat{\theta} = 0.01297$  
| 189.833 | 387.666 | 395.976 | 0.47459 (0.7722) | 0.08094 (0.6882) | 0.08074 (0.8365) |
| MOIL  | $\hat{a} = 9.07896$  
$\hat{\beta} = 6.74118$  
$\hat{\alpha} = 3.94 \times 10^{11}$  
| 189.899 | 392.031 | 392.031 | 0.32147 (0.9209) | 0.05019 (0.8769) | 0.08001 (0.8445) |
| APTW  | $\hat{\beta} = 0.62236$  
$\hat{\lambda} = 0.79610$  
$\hat{\alpha} = 0.00062$  
| 189.422 | 384.844 | 391.077 | 0.53069 (0.7150) | 0.08913 (0.6425) | 0.08714 (0.7615) |
| APIE  | $\hat{\alpha} = 0.66433$  
| 189.030 | 382.060 | 386.215 | 0.34695 (0.8988) | 0.06188 (0.8041) | 0.06530 (0.9629) |

### Table 13: ML estimates along with goodness of fit for data set III.

| Model | MLE | $-\log L$ | AIC | BIC | $A^*$ | $W^*$ | KS |
|-------|-----|-----------|-----|-----|------|------|----|
| OFIL  | $\hat{\theta} = 1.82147$  
$\hat{a} = 0.21103$  
$\hat{\beta} = 430.774$  
$\hat{\theta} = 2.50484$  
| 152.235 | 310.470 | 314.674 | 0.43739 (0.8098) | 0.06749 (0.7708) | 0.12544 (0.7327) |
| APTIL | $\hat{a} = 2.00719$  
$\hat{\beta} = 7.87555$  
$\hat{\alpha} = 3.19731$  
| 152.974 | 311.848 | 315.852 | 0.45508 (0.8019) | 0.07796 (0.7735) | 0.14875 (0.8700) |
| WIL   | $\hat{b} = 0.44466$  
$\hat{\lambda} = 43.8688$  
$\hat{\beta} = 30.6925$  
$\hat{\theta} = 10.0722$  
| 154.471 | 316.942 | 322.547 | 0.71912 (0.5418) | 0.12031 (0.4966) | 0.13307 (0.6628) |
| MOIL  | $\hat{a} = 40.6762$  
$\hat{\beta} = 0.06644$  
$\hat{\alpha} = 84.2552$  
| 152.957 | 311.915 | 316.119 | 0.53144 (0.7136) | 0.07532 (0.723) | 0.13013 (0.6899) |
| APTW  | $\hat{\beta} = 0.91642$  
$\hat{\lambda} = 3.32433$  
| 153.147 | 312.294 | 316.497 | 0.49192 (0.7537) | 0.08907 (0.7410) | 0.14207 (0.7255) |
| APIE  | $\hat{\alpha} = 101.393$  
$\hat{\lambda} = 3.93214$  
| 153.372 | 310.744 | 313.547 | 0.59654 (0.6499) | 0.08915 (0.6441) | 0.12508 (0.7259) |

### Table 14: Complexity
Figure 5: (a) Plots of the estimated pdf and cdf for data set I. (b) Plots of the estimated sf and PP plot for data set I. (c) Log-likelihood plots for data set I.
Figure 6: Continued.
Figure 6: (a) Plots of the estimated pdf and cdf for data set II. (b) Plots of the estimated sf and PP plot for data set II. (c) Log-likelihood plots for data set II.

Figure 7: Continued.
Based on accuracy measures and these plots, it can be seen that the OFIL distribution provides a better fit than other models for all three data sets.

8. Conclusion

In this paper, we introduced a new, three-parameter lifetime distribution called the odd Fréchet inverse Lomax distribution. To complete the practical aspect, we provided the main mathematical properties accompanied with characterizations based on truncated moments and hrf of the proposed distribution, mixture representations for the probability density and cumulative distribution functions, quantile function, Bowley skewness and Moors kurtosis, ordinary moments, and probability-weighted moments. The parameters of the OFIL distribution were estimated using ML, LS, PC, and AD methods of estimation. The simulation study confirmed that the MLE method is the best for this distribution. Three applications on practical data sets showed that the OFIL distribution provides a better fit than several serious competitors, validating its potential in terms of applicability.
Appendix

Theorem A.1. Let $X: Ω \rightarrow (0, \infty)$ be a continuous random variable with the distribution function $F$, and let $q_1(x)$ and $q_2(x)$ be two real functions defined on $H$ (let $H = [d, e]$ be an interval for some $d < e$ ($d = -\infty$ and $e = \infty$ might as well be allowed)) such that

$$E[q_1(X) | X \geq x] = E[q_1(X) | X \geq x] \eta(x), \quad x \in H,$$

(A.1)

is defined with some real function $\eta$. Assume that $q_1, q_2 \in C^1(H), \eta \in C^2(H)$, and $F$ is a twice continuously differentiable and strictly monotone function on set $H$. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of $H$. Then, $F$ is uniquely determined by the functions $q_1, q_2, \eta$, and $\eta$, particularly

$$F(x) = C \int_x^\infty \frac{\eta'(u)}{\eta'(u) q_1(u) - q_2(u)} \exp(-s(u))du,$$

(A.2)

where the function $s$ is a solution of the differential equation $s' = \eta q_1/(\eta q_1 - q_2)$ and $C$ is the normalization constant such that $\int_0^\infty dF = 1$. This is the characterization.

Data Availability

The data used to support the findings of this study are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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