An Additional Properties of the Graded Prime Spectrum of \((R, G)\)

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Abstract. Let \(G\) be a group and \(R\) be a \(G\)–graded commutative ring, i.e., \(R = \bigoplus_{g \in G} R_g\) and \(R_g R_h \subseteq R_{gh}\) for all \(g, h \in G\). For \(L \subseteq R\), let \(V(L)\) denote the set of all \(G\)-prime ideals of \((R, G)\) containing \(L\). Also, let \(GX\) denote the set of all graded prime ideals of \((R, G)\) and \(GM\) denote the set of all graded maximal ideals of \((R, G)\). Then \(F \{GX - V(L): L \subseteq R\}\) is a topological space \(G - \text{spec}(R)\) on \(GX\). In this paper, we give an additional topological properties of \(G - \text{spec}(R)\).

1. Introduction

Let \(R\) be a ring with unity 1 and \(G\) be a group with identity e. Then \(R\) is a \(G\)-graded ring if there exist additive subgroups \(R_g\) of \(R\) indexed by the elements \(g \in G\) such that \(R = \bigoplus_{g \in G} R_g\) and \(R_g R_h \subseteq R_{gh}\) for all \(g, h \in G\).

Let \(R\) be a \(G\)-graded ring and \(I\) be an ideal of \(R\). We say that \(I\) is a \(G\)-graded ideal of \((R, G)\) if \(I = \bigoplus_{g \in G} (I \cap R_g)\). Clearly \(I = \bigoplus_{g \in G} (I \cap R_g)\) is an ideal of \(R\). Hence \(I\) is graded if \(I \subseteq \bigoplus_{g \in G} (I \cap R_g)\).

In Section two we give basic concepts of graded ring and ideals, some definitions, propositions concerning the nondegenerate, examples and the faithful graded ring. In Section three, we give the definition of the graded prime spectrum of \((R, G)\) and some topological properties of the topological space \(G - \text{spec}(R)\) according to \([1]\), \([6]\) and \([7]\) which are necessary in this paper. In Section four, we will give an additional topological properties of topological space \(G - \text{spec}(R)\), which are the main purpose of this paper.

2- Preliminaries

In this Section we will give the definitions of the \(G\)-graded ring and \(G\)-graded ideal with some properties and examples, further details may be found in \([1]\), \([9]\), \([5]\), \([6]\) and \([7]\).

2.1. Definition. Let \(G\) be a group with identity e and \(R\) be a ring with unity 1 different from 0. Then \(R\) is a \(G\)-graded ring if there exist additive subgroups \(R_g\) of \(R\) indexed by the elements \(g \in G\) such that \(R = \bigoplus_{g \in G} R_g\) and \(R_g R_h \subseteq R_{gh}\) for all \(g, h \in G\).

The \(G\)-graded ring \(R\) is denoted by \((R, G)\).

Note that, not every ideal of a graded ring is graded ideal as we will see in the following example:
2.2. Example. Let \( R = \mathbb{Z}[i] \) and \( G = \mathbb{Z}_2 \). Then \( R \) is \( G \)-graded by \( R_0 = \mathbb{Z} \) and \( R_1 = i\mathbb{Z} \). Let \( I \) be the ideal of \( R \) generated by \( x = 1 + i \). Clearly \( x_0 = 1 \) and \( x_1 = i \) and \( x \in I \) while \( x_0 \notin I \). Now, if \( x_0 \in I \) then there exists \( a + ib \in \mathbb{Z}[i] \) with \( x_0 = 1 = (a + ib)(1 + i) \) which implies \( a - b = 1 \) and \( a + b = 0 \) or \( 2a = 1 \) which is a contradiction. Thus \( I \) is not graded ideal of \((R, G)\).

2.3. Definition. Let \( R \) be a \( G \)-graded ring and let \( I \) be a graded ideal of \((R, G)\). Then

(1) \( I \) is a \textit{graded maximal} ideal (in abbreviation “\( G \)-maximal ideal”) if \( I \neq R \) and there is no graded ideal \( J \) of \((R, G)\) such that \( I \subseteq J \subseteq R \).

(2) \( I \) is a \textit{graded prime} ideal (in abbreviation “\( G \)-prime ideal”) if \( I \neq R \) and whenever \( rs \in I \). We have \( r \in I \) or \( s \in I \), where \( r, s \in h(R) \).

3. The Graded Prime Spectra of a Graded Ring

In this Section, we define the graded spectra of \((R, G)\) and give some topological properties of the topological space \( G - \text{spec}(R) \) according to [2], [7] and [8].

Notation. Let \( R \) be a \( G \)-graded ring. For \( L \subseteq R \) let \( V(L) \) denote the set of all \( G \)-prime ideals of \((R, G)\) containing \( L \). Clearly, \( V(L) = V(h(L)) \) where \( h(L) = \{ x \in h(R) : r = x_g \text{ for some } x \in L, g \in G \} \).

Also, let \( GX \) denote the set of all \( G \)-prime ideals of \((R, G)\) and \( GMC \) denote the set of all \( G \)-maximal ideals of \((R, G)\).

3.1. Lemma

(1) If \( X, Y \subseteq R \) with \( X \subseteq Y \), then \( V(Y) \subseteq V(X) \).

(2) If \( L \subseteq R \) and \( P \) is the graded ideal of \((R, G)\) generated by \( h(L) \), then \( V(P) = V(G(P)) \).

(3) \( V(0) = GX \) and \( V(1) = \emptyset \).

(4) If \( \{ L_\alpha : \alpha \in \Delta \} \) is any family of subsets of \( R \), then \( V \left( \bigcup_{\alpha \in \Delta} L_\alpha \right) = \bigcap_{\alpha \in \Delta} V(L_\alpha) \).

(5) \( V(P \cap Q) = V(PQ) = V(P) \cup V(Q) \) for any two graded ideals \( P, Q \) of \((R, G)\).

3.2. Proposition. Let \( F = \{ GX - V(L) : L \subseteq R \} \). Then \( F \) is a topology on \( GX \).

We will call the resulting topology space of any graduation \((R, G)\) of \( R \), the graded prime spectrum of \((R, G)\) and we write \( G - \text{spec}(R) \).

Notation. If \( R \) is a \( G \)-graded ring and \( t \in R \), then we denote by \( GX \), an open set \( GX - V(t) \) of \( G - \text{spec}(R) \).

3.3. Proposition. Let \( R \) be a \( G \)-graded ring. Then the family \( \beta = \{ GX_t : t \in h(R) \} \) is a base for \( G - \text{spec}(R) \).

3.4. Proposition. Let \( R \) be a \( G \)-graded ring. Then

(1) \( GX_r \cap GX_s = GX_{rs} \) for all \( r, s \in h(R) \).

(2) \( GX_r = \emptyset \) iff \( r \) is a nilpotent, where \( r \in h(R) \).

(3) \( GX_r = GX \) iff \( r \) is a unit, where \( r \in h(R) \).

(4) \( GX_r = GX_s \) iff \( Gr^r(I_r) = Gr^s(I_s) \), where \( r, s \in h(R) \) and \( I_r, I_s \) are the ideals generated by \( r \) and \( s \), respectively.

3.5. Proposition. Let \( R \) be a \( G \)-graded ring. Then

(1) \( \{ p \} \) is closed in \( G - \text{spec}(R) \) iff \( p \) is a \( G \)-maximal ideal of \((R, G)\).

(2) \( Cl(\{ p \}) = V(p) \), where \( Cl \) stands for the closure.

(3) \( q \in Cl(\{ p \}) \) iff \( p \subseteq q \).

(4) \( G - \text{spec}(R) \) is a \( T_0 \) - space.
3.6. Proposition. [7]: Set \( R \) be a \( G \)-graded ring. If \( l \in h(R) \) and \( a \) is an idempotent in \( R \), then \( GX_l \) is closed in \( G - \text{spec}(R) \).

3.7. Proposition. Let \( R \) be a \( G \)-graded ring. If \( (R, G) \) is gr-Artinian then \( G - \text{spec}(R) \) is a second countable space.

3.8. Proposition. Assume \( R \) is a trivial \( G \)-graded ring. Then \( G - \text{spec}(R) \) are homeomorphic space.

3.9. Proposition. Set \( W \) be a \( G \)-graded ring. Then

1) Every point in \( \text{spec}(W) \) is closed iff every prime ideal of \( W \) is maximal
2) If \( W = W_1 \times W_2 \) then \( \text{spec}(W_1) \) and \( \text{spec}(W_2) \) are clopen subsets of \( W \).

3.10. Proposition. Let \( W \) be a unitary ring. If \( W \) is trivai \( G \)-graded ring, then \( G - \text{Spec}(W) \) satisfies the finit intersection property.

3.11. Proposition. Let \( A \) be a ring with unit 1 different from 0. If \( A \) is trivial \( G \)-graded ring, then \( \text{spec}(A) \) is connected iff \( A / \text{prim rad}(A) \) has no central idempotent.

3.12. Proposition. \( G - \text{spec}(R) \) is compact space.

3.13. Proposition. \( G - \text{spec}(R) \) is locally compact space.

In general, \( G - \text{spec}(R) \) need not be a \( T_1 \)-space as in the following example:

3.14. Example. Let \( R = \mathbb{C}[x] \), where \( \mathbb{C} \) is the field of complex numbers and let \( G = \mathbb{Z} \). Then \( R \) is a \( G \)-graded ring with \( R_0 = \mathbb{C}, R_i = \mathbb{C} x^i \) for \( i > 0 \). Clearly, \( R_x \in V(0) \) and hence \( \{0\} \neq V(0) = \text{Cl}(\{0\}) \). Therefore, \( G - \text{spec}(R) \) is not a \( T_1 \)-space because \( \{0\} \) is not closed.

3.15. Proposition. \( G - \text{spec}(R) \) is a \( T_1 \)-space iff every \( G \)-prime ideal of \( (R, G) \) is a \( G \)-maximal.

3.16. Definition. \( (R, G) \) is called a graded regular if each \( x \in R \), then there exist a homogeneous element as, \( y \in R \), such that: \( x = xyx \). We say, \( (R, G) \) is graded abelian regular if each idempotent element in \( R \) is central idempotent in \( R \).

It is sufficient to satisfy the following condition: For each homogeneous element in \( R \), there exists a homogeneous element in \( R \) such that: \( x = x^2 y \). According to this definition we will prove the following proposition.

3.17. Proposition. If, \( (R, G) \) is a graded abelian regular ring then, \( GX \subseteq GM \).

Proof: Let \( P \in GX \), to show that \( P \in GM \). Let \( x \notin P \). (because \( P \) is \( G \)-prime and so \( P \neq R \). Now, since \( R \) is a graded regular then we have \( x = xyx \) and \( x \notin P \). Now we have \( x(yx - 1) = 0 \in P \) and \( x \notin P \), this implies that, \( yx - 1 \notin P \) (since, \( P \) is a \( G \)-prime ideal). \( yx - 1 = P \); \( P \neq x \). So, \( 1 = yx - P \in Rx + P \). Thus \( Rx + P = R \), implies \( P \) is \( G \)-maximal ideal. Hence, \( GX \subseteq GM \). (i.e., every \( G \)-prime ideal is a \( G \)-maximal.)

4. An Additional properties for \( G - \text{spec}(R) \)

In this Section we will first list some topological facts according to [3], [4] and [7], after that we will give some additional topological properties of the topological space \( G - \text{spec}(R) \).

4.1 Remark. Let \( \tilde{R} = R/\text{Prime rad}(R) \), with \( - \) denoting the canonical homomorphic image. Then \( V(A) \to V(\tilde{A}) \) defines a homeomorphism from \( \text{spac}(R) \) to \( \text{spec}(\tilde{R}) \).

4.2. Definition. A Hausdorff space is locally compact if each point has a relatively compact neighborhood.

4.3 Definition. A topological space is locally compact iff every point in \( X \) has a compact neighborhood.

4.4 Proposition. Every compact space is locally compact.

4.5. Theorem. Every locally compact is completely regular.

4.6. Proposition. A completely regular space is also regular.
4.7. Proposition. Let \((R, G)\) be a finite graded ring and every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal, then \(G = \text{spec}(R)\) is discrete topology.

**Proof:** \((R, G)\) if finite implies \(G\) is finite. Thus, \(\exists \ n \in \mathbb{N} \ ; \ GX = \{p_1, p_2, \cdots, p_n\}\)

Since every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal, then for each \(p_i ; i = 1, 2, \cdots, n\) belong to \(GX\), then \(\{p_i\}\) is closed, by Proposition 2.7. Now, since complement of \(\{p_i\} = \bigcup_{j \neq i} \{p_j\}\) which is finite from closed sets. Thus, complement of \(\{p_i\}\) is closed. Hence, \(\{p_i\}\) is open. Therefore, for each \(i = 1, 2, \cdots, n\) we have \(\{p_i\}\) is open. Thus, \(G = \text{spec}(R)\) is discrete topology.

4.8. Proposition. Let \((R, G)\) be a finite graded ring and every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal then \(G = \text{spec}(R)\) is discrete topology.

**Proof:** Let \(p, q \in GX\) such that \(p \neq q\). Let \(GX_t, GX_s \ ; \ t, s \in R\) be any two open sets such that, \(GX_t = \{p\}\) and \(GX_s = \{q\}\) are open. So, \(GX_t, GX_s \in F\) and \(p \in GX_t, q \in GX_s\). Thus, \(GX_t \cap GX_s = \{p\} \cap \{q\} = \emptyset\). Hence, \(G = \text{spec}(R)\) is \(T_2\) – space.

4.9. Proposition. Let \((R, G)\) be a finite graded ring and every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal, then \(G = \text{spec}(R)\) is completely regular.

**Proof:** By Proposition 2.13, \(G = \text{Spec}(R)\) is locally compact and from Proposition 3.8, \(G = \text{spec}(R)\) is \(T_2\) – space. Thus, by Theorem 3.5, we have \(G = \text{spec}(R)\) is completely regular.

4.10. Corollary. Let \((R, G)\) be a finite graded ring and every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal, then \(G = \text{spec}(R)\) is completely regular.\(T_2\) – space.

**Proof:** By Proposition 3.9, \(G = \text{spec}(R)\) is completely regular which is implies \(G = \text{spec}(R)\) from above. So, by Proposition 2.15 \(G = \text{spec}(R)\) is \(T_1\) – space. Thus, \(G = \text{spec}(R)\) is \(T_3\) – space.

4.11. Proposition. Let \((R, G)\) be a finite graded ring and every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal, then \(G = \text{spec}(R)\) is \(T_3\) – space.

**Proof:** By Proposition 3.12 \(G = \text{spec}(R)\) is compact space. Now, by Proposition 3.8 \(G = \text{spec}(R)\) is \(T_2\) – space. Thus, \(G = \text{spec}(R)\) is normal space (Every compact Hausdorff space is normal) and by Proposition 3.15 \(G = \text{spec}(R)\) is \(T_4\) – space. Thus, \(G = \text{spec}(R)\) is \(T_4\) – space.

4.12. Corollary: [4]: Every locally compact Hausdorff space is a Tychonoff space.

4.13. Proposition. Let \((R, G)\) be a finite graded ring and every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal then \(G = \text{spec}(R)\) is a Tychonoff space.

**Proof:** Direct from Proposition 3.13 and Proposition 4.8.

4.14. Definition. A Hausdorff space is \(2^\ast\) countable (or, satisfies the second axiom of countability) if it has a countable basis.

4.15. Proposition. Let \((R, G)\) be a finite graded ring and every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal, then \(G = \text{spec}(R)\) is \(2^\ast\) countable.

**Proof:** Since \((R, G)\) is finite implies that, \(GX\) is finite. Thus, \(F\) is finite. But \(F\) is a base for \(F\). Hence, \(F\) has a finite base, so it has a countable base. Therefore, \(G = \text{spec}(R)\) is \(2^\ast\) countable.

4.16. Corollary (P. Urysohn). In \(2^\ast\) countable space, regularity is equivalent to metrizability.

Proposition 3.17: Let \((R, G)\) be a finite graded ring and every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal, then \(G = \text{spec}(R)\) is Metrizable space.

**Proof:** By Proposition 3.15 \(G = \text{spec}(R)\) is \(2^\ast\) countable space and by Proposition 3.9 \(G = \text{spec}(R)\) is completely regular. Therefore, by Proposition 3.16 \(G = \text{spec}(R)\) is metrizable space.
4.18. Proposition. Let \((R, G)\) be a finite graded ring and every \(G\)-prime ideal of \((R, G)\) is \(G\)-Maximal ideal then \(G - \text{spec}(R)\) is disconnected space.

Proof: \((R, G)\) is finite implies that, \(GX\) is finite. Since every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal then for each \(p \in GX\), \(\{p\}\) is closed in \(G - \text{spec}(R)\) \((\{p\}\) is closed iff \(p\) is \(G\)-maximal). Therefore, \(GX\) is closed (the union of any finite number of closed sets is closed). Let \(A = \{p_1\}\) is closed set, \(B = \{p_2\} \cup \{p_3\} \cup \cdots \cup \{p_n\}\) is closed set (the union of finite closed sets is closed.) Thus, \(GX = A \cup B\)

\[A \cap B = A \cap B = \emptyset \quad (A \cap B \text{ closed})\]

Hence, \(A\) and \(B\) are separated sets. Therefore, \(GX\) is disconnected space.

4.19. Proposition. If \((R, G)\) be a finite graded ring and every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal then \(G - \text{spec}(R/G - \text{nil}(R))\) is disconnected space.

Proof: By Proposition \(G - \text{spec}(R)\) and \(G - \text{spec}(R/G - \text{nil}(R))\) are homeomorphic space. Hence, \(G - \text{spec}(R/G - \text{nil}(R))\) is disconnected space.

4.20. Proposition. Let \((R, G)\) be a finite graded ring and every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal then \(G - \text{spec}(R)\) is \(T_2\) space.

Proof: By Proposition 3.17 \(G - \text{spec}(R)\) is metrizable space. This implies that \(G - \text{spec}(R)\) is \(T_2\) space (every metrizable space is \(T_2\) space).

4.21. Definition [5]. A ring \(R\) is called a graded local ring if it has exactly one maximal graded ideal.

Proposition 3.22: Let \((R, G)\) be a \(G\)-local ring then \(G - \text{spec}(R)\) is \(T_2\) space iff every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal.

Proof: \(\Rightarrow\) Suppose that \(G - \text{spec}(R)\) is \(T_2\) space. To show that, every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal. Since, \(G - \text{spec}(R)\) is \(T_2\) space implies that \(G - \text{spec}(R)\) is \(T_1\) space. But, \(G - \text{spec}(R)\) is \(T_1\) space iff every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal. Thus, every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal.

\(\Leftarrow\) Suppose that every \(G\)-prime ideal of \((R, G)\) is a \(G\)-maximal ideal. To show that, \(G - \text{spec}(R)\) is \(T_2\) space. Let \(p, q\) are two \(G\)-prime ideals of \(GX\), which implies that, \(p, q\) are \(G\)-maximal ideal (from hypothesis). But, \(R\) is \(G\)-local ring (i.e., \(R\) has exactly one \(G\)-maximal ideal) which implies that \(GX\) consists of at most one point \((G\)-prime ideal). Thus, either \(GX = \emptyset\) or \(GX = \{p\}\). Since, \(R\) is \(G\)-local ring and every \(G\)-prime ideal is \(G\)-maximal we have \(GX \neq \emptyset\) and thus, \(GX = \{p\}\). So trivially \(G - \text{spec}(R)\) is \(T_2\) space.

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