Two-temperature logistic regression based on the Tsallis divergence

Ehsan Amid
Department of Computer Science
University of California, Santa Cruz
Santa Cruz, CA 95060
eamid@ucsc.edu

Manfred K. Warmuth
Department of Computer Science
University of California, Santa Cruz
Santa Cruz, CA 95060
manfred@ucsc.edu

Abstract

We develop a variant of multiclass logistic regression that achieves three properties: i) We minimize a non-convex surrogate loss which makes the method robust to outliers, ii) our method allows transitioning between non-convex and convex losses by the choice of the parameters, iii) the surrogate loss is Bayes consistent, even in the non-convex case. The algorithm has one weight vector per class and the surrogate loss is a function of the linear activations (one per class). The surrogate loss of an example with linear activation vector $a$ and class $c$ has the form $-\log t_1 \exp t_2 (a_c - G t_2 (a))$ where the two temperatures $t_1$ and $t_2$ “temper” the log and exp, respectively, and $G t_2$ is a generalization of the log-partition function. We motivate this loss using the Tsallis divergence. As the temperature of the logarithm becomes smaller than the temperature of the exponential, the surrogate loss becomes “more quasi convex”. Various tunings of the temperatures recover previous methods and tuning the degree of non-convexity is crucial in the experiments. The choice $t_1 < 1$ and $t_2 > 1$ performs best experimentally. We explain this by showing that $t_1 < 1$ caps the surrogate loss and $t_2 > 1$ makes the predictive distribution have a heavy tail.

1 Introduction

Consider a classification problem where every instance $x \in \mathbb{R}^d$ is labeled by one class $c \in \{1, \ldots, C\}$. The goal of learning algorithm is to develop a classifier, parameterized by $W$, which correctly predicts the class label $c$ of a given instance $x$. In order to learn the optimal parameter $W^*$ of the classifier, we generally minimize the regularized empirical surrogate loss of a set of i.i.d. examples $\{(x_n, c_n)\}_{n=1}^N$ from the data distribution:

$$W^* = \arg\min_W \mathcal{L}(W) + \mathcal{R}(W), \quad \text{where} \quad \mathcal{L}(W) = \frac{1}{N} \sum_n \xi(x_n, c_n|W). \quad (1)$$

Here $\xi(x_n, c_n|W)$ denotes the surrogate loss, which approximates the 0-1 loss associated with the example $(x_n, c_n)$ and $\mathcal{R}(W)$ is the regularizer. In this paper, we consider the linear activation models where $W$ is a matrix with columns $w_c$ (one per class) and in which, both the surrogate loss $\xi(x, c|W)$ and the classifier can be written as functions of the activation vector defined as $a = W^T x$.

Among different properties of the surrogate functions used in practice, convexity plays an important role since it provides the convergence guarantee of the solution to a global minimum [12]. Additionally, there exist many convex optimization packages for solving the minimization problem efficiently [11, 21]. The main drawback of the convexity is that the loss of an individual example, e.g., for a highly misclassified outlier point, can grow indefinitely (at least linearly) and dominate the objective function. Therefore, it has been shown that the convex functions are not robust to noise [13]. Specifically, Ben-David et al. [4] showed that among the convex surrogate loss functions
for linear predictors, the hinge loss has the lowest expected misclassification error rate and any strongly convex loss has a qualitatively worse guarantee when compared to hinge loss. To alleviate this problem, several strategies have been proposed to introduce non-convexity into the loss function \[13][10][9][18][6]. More recently, Ding et al. \[7\] used heavy-tailed properties of $t$-exponential distributions to define a robust loss function for logistic regression. The main idea behind these techniques is to eventually “bend down” the loss and give up those points that are highly misclassified. More recently, Feng et al. \[8\] introduced a different approach for detecting and removing the outliers in logistic regression problem. However, their method makes strict assumption about the type of the generative distribution and the availability of the noise variance and an upper-bound on the number of outliers, which makes it impractical in real-world applications.

Another fundamental property of the surrogate loss is its Bayes consistency \[3][19\], which indicates that we can alternatively solve for the minimizer of the empirical surrogate loss and still hope to converge (in probability) to the minimizer of the expected 0-1 loss. Bayes consistency also implies that the class label for the point $x$ can be predicted via the $\arg\max$ operator over the margins, i.e., $\arg\max_j a_j$ where $a_j = [a]_j$. While many convex surrogate losses enjoy Bayes consistency, achieving Bayes consistency for non-convex losses is a non-trivial property and thus, is an important consideration in designing the loss functions for classification \[14\].

In this paper, we generalize the ideas in \[7\] for constructing a non-convex as the negative log likelihood of a $t$ exponential distribution. Our approach is based on the Tsallis divergence which is the natural choice of divergence for the family of $t$-exponential distributions \[11\]. By varying the temperature parameters for the generalized logarithm and exponential functions, we transition between the convex and more robust quasi-convex loss functions. Our experiments clearly show that the non-convexity is a crucial property for obtaining robustness to both outlier and (random or adversarial) label noise. The best results are achieved when temperature of the logarithm is less than 1 (which caps the loss) and the temperature of the exponential is larger than 1 (which gives the predictive distribution have a heavy tail).

## 2 Tsallis Entropy and Tsallis Divergence

The $\log_t$ function with temperature parameter $0 < t < 2$ is defined as a generalization of the standard log function \[16][17\].

$$\log_t x = \frac{1}{1-t} (x^{1-t} - 1).$$

(2)

The $\log_t$ function is monotonically increasing and recovers the standard log function in the limit $t \to 1$. However, some properties of the log function do not generalize to $\log_t$. For instance, $\log_t a b \neq \log_t a + \log_t b$ in general. Additionally, unlike the standard log function, the $\log_t$ function is bounded from below for $0 < t < 1$ and bounded from above for $1 < t < 2$ by the value $-1/(1-t)$. This property has been used to design robust loss transformations for metric learning \[2\].

Using the $\log_t$ function, we can generalize the notion of the (Shannon) entropy of a probability distribution. For a probability distribution $p(x)$, the Tsallis entropy \[20\] is defined as

$$H_t(p) = \frac{\int p(x) dx}{1-t} = \int p(x) \frac{1}{p(x)} \log_t \frac{p(x)}{x} dx.$$  \hspace{1cm} (3)

Note that the standard entropy is recovered when $t \to 1$. Similarly, the Tsallis divergence between the distributions $p(x)$ and $q(x)$ can be defined as a generalization of the Kullback-Leibler (KL) divergence, that is,

$$D_t(p||q) = -\int p(x) \log_t \frac{q(x)}{p(x)} dx.$$  \hspace{1cm} (4)

Note that the KL divergence is also recovered in the limit $t \to 1$. Similarly, the $\exp_t$ function is defined as the inverse of $\log_t$,

$$\exp_t(x) = [1 + (1-t)x]^t_+^{1/(1-t)}.$$  \hspace{1cm} (5)

where $[\cdot]_+ = \max(\cdot, 0)$. Note that again the exp function is recovered in the limit $t \to 1$. An important property of the $\exp_t$ function is its heavier tail compared to $\exp$ for values of $1 < t < 2$. 


This property leads to definition of a class of generalized distributions under the exp\(_t\) function, called the \(t\)-exponential family of distributions with vector of sufficient statistics \(x\),
\[
p_t(x|\theta) = \exp_t(\theta^T x - G(\theta)), \quad 0 < t < 2, \tag{6}
\]
where \(\theta\) is called the canonical parameter and the log-partition function \(G(\theta)\) ensures that the distribution is normalized, that is,
\[
\int \exp_t(\theta^T x - G(\theta)) \, dx = 1. \tag{7}
\]

An important distribution related to the \(t\)-exponential distribution \(6\) is called the escort distribution and is defined as
\[
q_t(x|\theta) = \frac{1}{Z(\theta)} \exp'(\theta^T x - G(\theta)) \quad \text{where} \quad Z(\theta) = \int \exp'(\theta^T x - G(\theta)) \, dx. \tag{8}
\]
Here \(Z(\theta)\) is the normalization factor. It can be shown \(^1\) that
\[
\nabla G(\theta) = \mathbb{E}_{q_t}[x] = \frac{1}{Z(\theta)} \int x \exp'(\theta^T x - G(\theta)) \, dx. \tag{9}
\]

When dealing with \(t\)-exponential distributions, the Tsallis entropy and divergence take the role of Shannon entropy and KL divergence respectively, for the exponential distributions. For further details, please see \(^1\).

### 3 Two-temperature Logistic Regression

Let \(a = W^T x\). Following our discussion on the heavy-tail properties of the \(t\)-exponential family of distributions, we model the conditional probability of the class \(c\) given input \(x\) with a \(t\)-exponential distribution with temperature \(t_2\),
\[
\hat{p}_{t_2}(c|x, W) = \exp_{t_2}(w_c^T x - G_{t_2}(W^T x)) = \exp_{t_2}(a_c - G_{t_2}(a)) \tag{10}
\]
where the log-partition function \(G_{t_2}(a(x))\) ensures that the probabilities sum up to 1, that is,
\[
\sum_c \exp_{t_2}(a_c - G_{t_2}(a)) = 1. \tag{11}
\]

This definition for the conditional probabilities is similar to the ones given in \(^7\). The definition \(^{10}\) also contains the softmax probabilities as special case when \(t_2 = 1\),
\[
\hat{p}(c|x, W) = \exp(a_c - \log \sum_j \exp(a_j)) = \frac{\exp(a_c)}{\sum_j \exp(a_j)}. \tag{12}
\]
where we use the fact that for \(t_2 = 1\), \(G(a) = \log \sum_j \exp(a_j)\). In order to adopt the heavy-tail properties of \(t\)-exponential distribution, we are mainly interested in the values of \(1 < t_2 < 2\). However, for values of \(t_2 \neq 1\), the log-partition function \(G_{t_2}(a)\) does not have a closed form solution in general and must be calculated numerically\(^1\).

Given the prediction probabilities \(^{10}\) in the form of a \(t\)-exponential distribution, we can now define the mismatch loss between the empirical label distribution \(p_e(c|x_n) = \mathbb{I}_{c = c_n}\), and the prediction \(\hat{p}_{t_2}(c|x_n)\) using sum of Tsallis divergences with temperature \(t_1\),
\[
\mathcal{L}(W) = -\frac{1}{N} \sum_n \sum_c p_e(c|x_n) \log_{t_1} \frac{\hat{p}_{t_2}(c|x_n, W)}{p_e(c|x_n)} = -\frac{1}{N} \sum_n \sum_c \mathbb{I}_{c = c_n} \log_{t_1} \frac{\hat{p}_{t_2}(c|x_n, W)}{\mathbb{I}_{c = c_n}}. \tag{13}
\]

Using the fact that \(0 \times \log_t 0 = 0 \times \log_t \infty = 0\), the loss \(^{13}\) simplifies to
\[
\mathcal{L}(W) = -\frac{1}{N} \sum_n \log_{t_1} \hat{p}_{t_2}(c_n|x_n, W) \]
\[
= -\frac{1}{N} \sum_n \log_{t_1} \exp_{t_2}(w_{c_n}^T x_n - G_{t_2}(W^T x_n)) \]
\[
= -\frac{1}{N} \sum_n \log_{t_1} \exp_{t_2}([a_n|x_n] - G_{t_2}(a_n)) = \frac{1}{N} \sum_n \xi_{t_1}^{t_2}(x_n, c_n|W). \tag{14}
\]

\(^1\)Note that this involves finding the zero of a one-dimensional function, which can be solved efficiently using general purpose solvers, e.g., \texttt{fzero} function in MATLAB.
We refer to the classification algorithm with the loss defined in (14) as Two-Temperature Logistic Regression (TTLR). The gradient of the loss with respect to the \( c \)-th parameter \( w_c \) can be written as

\[
\nabla_{w_c} \mathcal{L}(W) = - \sum_n \hat{p}_{t_2}(c_n | x_n, W)^{t_2-t_1} \left[ \mathbb{I}_{c=c_n} \cdot x_n - \nabla_{w_c} G_{t_2}(a_n) \right],
\]

where

\[
\nabla_{w_c} G_{t_2}(a_n) = \hat{q}_{t_2}(c | x_n, W) \cdot x_n,
\]

and

\[
\hat{q}_{t_2}(c | x, W) = \frac{\exp^{t_2}(a_c - G_{t_2}(a))}{\sum_j \exp^{t_2}(a_j - G_{t_2}(a))} \sim \hat{p}_{t_2}(c | x, W)^{t_2}
\]

is the escort distribution of \( \hat{p}_{t_2}(c | x, W) \).

We are mainly interested in \( 0 < t_1 < 1 \) because the loss of each individual observation becomes capped by a constant \(-1/(1-t_1)\). The boundedness of loss provides significant improvement in handling noisy observations, as we show in the experiments. Note that the gradient of the loss of the \( n \)-th observation contains a importance factor of the form \( \hat{p}_{t_2}(c_n | x_n, W)^{t_2-t_1} \) that depends on the conditional probability of the \( n \)-th observation and the temperature difference \( t_2 - t_1 \), which we call the temperature gap. Note that for \( t_2 > t_1 \), the temperature gap is non-negative and the importance factors damp the gradient of those observations that have small probability towards zero. However, the loss of each observation is bounded only for values of \( 0 < t_1 < 1 \). On the other hand, the importance factors vanish when \( t_1 = t_2 \) and each observation affects the final gradient to the same extent. This corresponds to, e.g., standard logistic regression where \( t_1 = t_2 = 1 \).

Next, we consider the binary classification as a special case and consider several properties of its surrogate loss function.

4 Binary Classification

In the binary case \( C = 2 \), without loss of generality, we can consider \( c \in \{ \pm 1 \} \) and parameterize the classifier by \( W = [w_+, w_-] \) and the vector of margins \( a = [w_+, w_-]^\top = [a_+, a_-]^\top \). Similar to (10), we can define the probabilities

\[
\hat{p}_{t_2}(c = \pm 1 | x) = \exp^{t_2}(w_{c+}^\top x - G_{t_2}(W^\top x)) = \exp^{t_2}(a_c - G_{t_2}(a)).
\]

The log-partition function \( G_{t_2}(a) \) again ensures that the probabilities sum up to 1. Note that from (11), for any constant \( b \), we have \( G_{t_2}(a + b \mathbf{1}) = G_{t_2}(a) + b \mathbf{1} \). Therefore, we can simplify the margin vector \( a \) by subtracting the mean of the inner-products \( \frac{w_+^\top x + w_-^\top x}{2} \), that is, \( a = (w_+ - w_-)^\top x - \frac{w_+^\top x - w_-^\top x}{2} = [\frac{w_+ + w_-}{2}, -\frac{w_+ - w_-}{2}]^\top \), where we define \( w = w_+ - w_- \).

Thus, we can write the probabilities in the following compact form

\[
\hat{p}_{t_2}(c | x, w) = \exp^{t_2}(\frac{c}{2} w^\top x - G_{t_2}(w^\top x)) = \exp^{t_2}(\frac{c}{2} a - G_{t_2}(a)).
\]

The definition (19) also contains the logistic probabilities as special case \( t_2 = 1 \), that is,

\[
\hat{p}(c | x) = \frac{\exp^{\frac{c}{2} a}}{\exp^{\frac{c}{2} a} + \exp^{-\frac{c}{2} a}} = \frac{1}{1 + \exp(-c a)} = \frac{1}{1 + \exp(-c w^\top x)},
\]

where we use the fact that for \( t_2 = 1 \), \( G(a) = \log (\exp^{\frac{a}{2}} + \exp^{-\frac{a}{2}}) \). However, for \( t_2 \neq 1 \), \( G_{t_2}(a) \) also does not have a closed form solution in general.

Following similar steps as in (13), we can write the loss for the binary case as

\[
\mathcal{L}(w) = -\sum_n \log \hat{p}_{t_2}(c_n | a_n - G_{t_2}(a_n)) \nonumber
\]

\[
-\sum_n \log \hat{q}_{t_2}(c_n w^\top x_n - G_{t_2}(w^\top x_n)) = \sum_n \frac{c_{t_2}^{c_n}}{2}(x_n, y_n | w)
\]

\footnote{Note that this assumption is equivalent to considering second class as \( c = -1 \).}
We use the results of [3] to show the Bayes consistency of the binary case in the following theorem.

Theorem 2. The binary surrogate loss

\[ \hat{\xi}^{t_2}_{t_1}(x, c) \]

where \( \hat{\xi}^{t_2}_{t_1}(c) \) is Bayes consistent.

4.1 Properties

The curvature of the two-temperature loss function \( \xi^{t_2}_{t_1}(x, c, \theta) \) depends on the choice of the temperature parameters \( t_1 \) and \( t_2 \). For certain choice of temperatures, we still have convex losses while for the others, the loss function shows a quasi-convex behavior. The properties of the loss function are summarized in the following theorem. Without loss of generality, we assume \( c = 1 \).

Theorem 1. The loss function \( \xi^{t_2}_{t_1}(x, c, W) \) has the following properties:

1. For values of \( t_1 \geq t_2 \) and \( t_1 \geq 1 \), the function is convex. Specifically, for \( t_1 = t_2 = t \geq 1 \), we have the following convex function

\[ \xi^{t_2}_{t_1}(x, c, \theta) = G_{t_2}(a) - \frac{a}{2}. \]

Moreover, the curvature of the function increases as the gap between the temperatures \( t_2 - t_1 \) increases.

2. The function is quasi-convex for \( t_1 < t_2 \) or \( t_1 < 1 \). Moreover, the function is locally convex (concave) for large (small) values of margin \( a \). The inflection point of the function happens at the point which satisfies the following equation

\[ \partial^2 G_{t_2}(a) = (t_2 - t_1) \exp_{t_2}(a/2 - g(a))^{t_2 - 1} \left( \frac{1}{2} - \partial G_{t_2}(a) \right)^2 \]

where

\[ \partial G_{t_2}(a) = \frac{1}{2} \sum_c c \exp_{t_2} \left( \frac{a}{2} - G_{t_2}(a) \right)^{t_2} \]

\[ \partial^2 G_{t_2}(a) = t_2 \sum_c \exp_{t_2} \left( \frac{a}{2} - G_{t_2}(a) \right)^{t_2 - 1} \left( \frac{c}{2} - \partial G_{t_2}(a) \right)^2 \]

5 Bayes Consistency

5.1 Binary Case

We use the results of [3] to show the Bayes consistency of the binary case in the following theorem. Note that because of the form of the margin vector \( \mathbf{a} = [a, -a] \) in the binary case, the arg max operator is equivalent to \( \arg \max \text{sign}(a) \).

Theorem 2. The binary surrogate loss \( \xi^{t_2}_{t_1}(x, c, W) \) is Bayes consistent.

Proof. Let us define \( \eta = p(c = +1 | x) \) and thus, \( 1 - \eta = p(c = -1 | x) \). Consider the expected loss of a given observation \( x \),

\[
\mathbb{E}[\xi^{t_2}_{t_1}(x, c, W) | x] = p(c = +1 | x) \xi^{t_2}_{t_1}(x, +1 | w) + p(c = -1 | x) \xi^{t_2}_{t_1}(x, -1 | w)
= -\eta \log t_1 \exp_{t_2} (a/2 - G_{t_2}(a)) - (1 - \eta) \log t_1 \exp_{t_2} (-a/2 - G_{t_2}(a))
= -\eta \log t_1 \exp_{t_2} (a/2 - G_{t_2}(a)) - (1 - \eta) \log t_1 (1 - \exp_{t_2} (a/2 - G_{t_2}(a))).
\]
The value $a^* = \arg\min_a \mathbb{E}[\xi^T_t(x, c | w) | x]$ solves $\exp_{t_2}(a^*/2 - G_{t_2}(a)) = \frac{\eta^{1/t_2}}{\eta^{1/t_2} + (1 - \eta)^{1/t_2}}$. Taking $\log_{t_2}$ of both sides and using the fact that $\exp_{t_2}(-a^*/2 - G_{t_2}(a)) = 1 - \exp_{t_2}(a^*/2 - G_{t_2}(a))$, we have

$$a^* = \log_{t_2} \left( \frac{\eta^{1/t_2}}{\eta^{1/t_2} + (1 - \eta)^{1/t_2}} \right) - \log_{t_2} \left( \frac{(1 - \eta)^{1/t_2}}{\eta^{1/t_2} + (1 - \eta)^{1/t_2}} \right)$$

(29)

Note that $\log_{t_2}$ is a monotonically increasing function. Thus for $\eta > \frac{1}{2}$ (respectively, $\eta < \frac{1}{2}$), we have $a^* > 0$ (respectively, $a^* < 0$). Thus, $\text{sign}(a^*) = \text{sign}(p(c | x) - \frac{1}{2})$, and the loss is Bayes consistent.

5.2 Multiclass Case

Tewari and Bartlett [19] showed that the Bayes consistency of the binary loss does not necessarily imply the Bayes consistency of the multiclass case. In general, showing Bayes consistency for a multiclass loss is more complicated than the binary case and involves geometric properties of the loss function [19]. However, for the class of loss functions satisfying the conditions stated in the following lemma, we can prove the Bayes consistency.

Lemma 3 (Zhang et al. [22]). A surrogate loss $\xi(a, c)$ w.r.t. a margin $a = [a_1(x), \ldots, a_m]$ with the additional constraint $\sum_c a_c = 0$ is said to be Bayes consistent if for all possible label probability distributions $p(c | x)$ the following conditions are satisfied:

1. The minimization problem $a^* = \arg\min_a \sum_c p(c | x) \xi(a, c)$ has a unique solution for all $x \in \mathbb{R}^d$, and
2. $\arg\max_c a^*_c = \arg\max_c p(c | x)$ for all $x \in \mathbb{R}^d$.

We now proof the following.

Theorem 4. The multiclass surrogate loss $\xi^T_t(x, c | W) = -\log_{t_2} \exp_{t_2}(a_c - G_{t_2}(a))$ is Bayes consistent.

Proof. The minimizer of the expectation

$$-\sum_c p(c | x) \log_{t_2} \exp_{t_2}(a_c - G_{t_2}(a))$$

(30)

has the unique solution $a^*$ such that $\exp_{t_2}(a^*_c - G_{t_2}(a^*)) \propto p(c | x)^{1/t_2}$. Note that the minimizer is unique since, because $\exp_{t_2}$ is an injective function, any other minimizer $a^{**}$ must satisfy the following: $a^*_c - G_{t_2}(a^*) = a^{**}_c - G_{t_2}(a^{**})$ for all $c \in \{1, \ldots, C\}$ and the constraint $\sum_c a^*_c = \sum_c a^{**}_c = 0$ implies $a^{**} = a^{**}$. Finally, monotonicity of $\exp_{t_2}$ function implies

$$\arg\max_c a^*_c = \arg\max_c \exp_{t_2}(a^*_c - G_{t_2}(a^*)) = \arg\max_c p(c | x)^{1/t_2} = \arg\max_c p(c | x).$$

(31)

The result of Theorem 4, i.e. $\hat{p}_{t_2}(x, c | W^*) \propto p(c | x)^{1/t_2}$, is the direct consequence of using the sum of Tsallis divergences between the observed class distributions and the predicted class probabilities (see Appendix B). However, the $\arg\max$ operator is invariant with respect to the positive powers and thus, we still achieve Bayes Consistency.

6 Experiments

We compare the performance of our TTLR with the following classification techniques: 1) logistic regression (LR), 2) Linear SVM (SVM), and 3) $t$-logistic regression ($t$-LR). We use the same temperature $t_2 = t_1 = 1.6$ for both $t$-LR and TTLR and we set $t_1 = 0.6$ for our method. This allows a temperature gap of $t_2 - t_1 = 1$ for our method and $t_2 - 1 = 0.6$ for $t$-LR. We perform binary

Note that we can always enforce the constraint $\sum_c a_c = 0$ by adding and subtracting the constant vector of mean value $\left(\frac{1}{C} \sum_c a_c \right) \mathbf{1}$ without changing the probabilities since $G_{t_2}(a + b \mathbf{1}) = G_{t_2}(a) + b \mathbf{1}$ for any constant $b$. 
classification experiments in the presence of two types of noise: 1) outlier noise, in which a certain portion of training examples is contaminated by Gaussian noise with standard-deviation $\sigma = 10$, 2) labeling noise, where with a certain probability, the label of each instance $c_n$ is flipped. This first noise mimics the effect of bad outliers in the data. For the label noise, we consider two different noises: i) we flip the label of each instance with constant probability, ii) we first run LR on the clean data and calculate the margin of each training point and then, flip the points with larger margins with higher probability (the details are given in the Appendix C). The latter noise is harder than the former one because it targets the training points with larger margins with higher probability. We use the following datasets for the experiments: 1) Splice [4], 2) SVMguide [4], 3) Mushrooms [5], 4) MNIST [6] (even vs. odd), 5) USPS [7] (even vs. odd), and 6) Handwritten Letters [8] (‘a’-‘m’ vs. ‘n’-’z’). For Mushrooms, MNIST, USPS, and Letters we randomly select, respectively, 4062, 5000, 5500, and 10,000 points for training. We used an $L_2$-regularizer for LR and TTLR methods. The values of the regularizer parameters for all methods were selected via cross-validation over the range $[10^{-10}, 10^2]$. The same initial parameter was used for LR $t$-LR and TTLR in each trial and was selected from a zero mean Normal distribution with variance $10^{-10}$. We used the original implementation of $t$-LR in MATLAB and for SVM, we used LIBSVM [5]. For the optimization of our method, we use the L-BFGS implementation in MATLAB.

6.1 Outlier Noise

For the outlier noise, we vary the ratio of the noisy examples from 0 to 0.5 and report the average classification accuracy on the test set over 10 repetitions. The results of the experiments in shown in [7]. As can be seen, TTLR is highly robust to the noise while the performance of the other methods
We developed a generalized loss function for logistic regression which provides two temperatures to tune the properties of the loss. The first temperature allows non-convexity and the boundedness of the loss while the second one controls the tail-heaviness of the probabilities. Our experiments indicate that the non-convexity is a crucial property for obtaining robustness to both outlier and label noise.

### 6.2 Label Noise

The performance of different methods in the presence of random label noise and the large-margin label noise is shown in Table 1 and Table 2 respectively. For both types of noise, TTLR is the best performing method in most cases. Especially, the superior performance of TTLR is more evident from the results of large-margin noise where it outperforms the other method on all datasets. This result is a direct consequence higher non-convexity of the loss function and capping of the loss of each example by considering \( t_1 < 1 \).

| Dataset    | SVM     | LR      | \( t \)-LR | TTLR    |
|------------|---------|---------|------------|---------|
| Splice     | 75.45 ± 0.19 | 74.97 ± 0.18 | 75.78 ± 0.17 | 75.70 ± 0.11 |
| SVMguide   | 84.65 ± 0.53 | 94.56 ± 0.22 | 93.80 ± 0.39 | 95.42 ± 0.19 |
| Mushrooms  | 98.93 ± 0.24 | 99.90 ± 0.06 | 99.92 ± 0.04 | 99.85 ± 0.04 |
| MNIST      | 87.39 ± 0.16 | 87.05 ± 0.22 | 87.61 ± 0.14 | 87.71 ± 0.30 |
| USPS       | 87.42 ± 0.25 | 87.66 ± 0.19 | 87.73 ± 0.23 | 88.09 ± 0.17 |
| Letters    | 75.45 ± 0.19 | 74.97 ± 0.18 | 75.78 ± 0.17 | 75.70 ± 0.11 |

Table 1: Classification accuracy with 10% label noise.

| Dataset    | SVM     | LR      | \( t \)-LR | TTLR    |
|------------|---------|---------|------------|---------|
| Splice     | 82.56 ± 0.78 | 78.49 ± 2.67 | 82.76 ± 0.38 | 83.66 ± 0.46 |
| SVMguide   | 87.14 ± 0.60 | 80.30 ± 0.74 | 93.80 ± 0.21 | 96.14 ± 0.03 |
| Mushrooms  | 99.68 ± 0.00 | 99.52 ± 0.17 | 98.19 ± 0.24 | 99.79 ± 0.03 |
| MNIST      | 86.74 ± 0.18 | 85.46 ± 0.24 | 86.50 ± 0.31 | 87.83 ± 0.06 |
| USPS       | 86.33 ± 0.18 | 86.30 ± 0.23 | 86.91 ± 0.32 | 88.16 ± 0.13 |
| Letters    | 74.74 ± 0.14 | 74.41 ± 0.14 | 75.03 ± 0.11 | 76.10 ± 0.05 |

Table 2: Classification accuracy with 10% large-margin label noise.

drops significantly even after a small amount of noise is introduced. Among the other methods, \( t \)-LR has the better performance which can be explained by the heavy-tail properties of the probabilities and non-convexity of the loss. However, using \( t_1 < 1 \) for our method caps the loss of each individual observation and thus, results in a major improvement.

### 7 Conclusion

We developed a generalized loss function for logistic regression which provides two temperatures to tune the properties of the loss. The first temperature allows non-convexity and the boundedness of the loss while the second one controls the tail-heaviness of the probabilities. Our experiments indicate that the non-convexity is a crucial property for obtaining robustness to both outlier and label noise.
References

[1] Shun-ichi Amari, Atsumi Ohara, and Hiroshi Matsuzoe. Geometry of deformed exponential families: Invariant, dually-flat and conformal geometries. *Physica A: Statistical Mechanics and its Applications*, 391(18):4308–4319, 2012.

[2] Ehsan Amid, Nikos Vlassis, and Manfred K. Warmuth. Low-dimensional data embedding via robust ranking. *arXiv preprint arXiv:1611.09957*, 2016.

[3] Peter L Bartlett, Michael I Jordan, and Jon D McAuliffe. Convexity, classification, and risk bounds. *Journal of the American Statistical Association*, 101(473):138–156, 2006.

[4] Shai Ben-David, David Loker, Nathan Srebro, and Karthik Sridharan. Minimizing the misclassification error rate using a surrogate convex loss. *arXiv preprint arXiv:1206.6442*, 2012.

[5] Chih-Chung Chang and Chih-Jen Lin. LIBSVM: A library for support vector machines. *ACM Transactions on Intelligent Systems and Technology*, 2:27:1–27:27, 2011. Software available at [http://www.csie.ntu.edu.tw/~cjlin/libsvm](http://www.csie.ntu.edu.tw/~cjlin/libsvm).

[6] Christophe Croux and Gentiane Haesbroeck. Implementing the bianco and yohai estimator for logistic regression. *Computational statistics & data analysis*, 44(1):273–295, 2003.

[7] Nan Ding and S. V. N. Vishwanathan. t-logistic regression. In *Proceedings of the 23th International Conference on Neural Information Processing Systems*, NIPS’10, pages 514–522, Cambridge, MA, USA, 2010.

[8] Jiashi Feng, Huan Xu, Shie Mannor, and Shuicheng Yan. Robust logistic regression and classification. In *Advances in neural information processing systems*, pages 253–261, 2014.

[9] Yoav Freund. An adaptive version of the boost by majority algorithm. *Machine learning*, 43(3):293–318, 2001.

[10] Yoav Freund. A more robust boosting algorithm. *arXiv preprint arXiv:0905.2138*, 2009.

[11] Michael Grant and Stephen Boyd. Cvx: Matlab software for disciplined convex programming, 2008.

[12] Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal. *Fundamentals of convex analysis*. Springer Science & Business Media, 2012.

[13] Philip M Long and Rocco A Servedio. Random classification noise defeats all convex potential boosters. In *Proceedings of the 25th international conference on Machine learning*, pages 608–615. ACM, 2008.

[14] Hamed Masnadi-Shirazi. *The design of Bayes consistent loss functions for classification*. PhD thesis, University of California, San Diego, 2011.

[15] Robert Cameron Mitchell and Richard T Carson. *Using surveys to value public goods: the contingent valuation method*. Resources for the Future, 1989.

[16] Jan Naudts. Deformed exponentials and logarithms in generalized thermostatistics. *Physica A*, 316:323–334, 2002.

[17] Jan Naudts. Generalized thermostatistics and mean-field theory. *Physica A*, 332:279–300, 2004.

[18] Seo Young Park and Yufeng Liu. Robust penalized logistic regression with truncated loss functions. *Canadian Journal of Statistics*, 39(2):300–323, 2011.

[19] Ambuj Tewari and Peter L Bartlett. On the consistency of multiclass classification methods. *Journal of Machine Learning Research*, 8(May):1007–1025, 2007.

[20] Constantino Tsallis. Possible generalization of boltzmann-gibbs statistics. *Journal of statistical physics*, 52(1):479–487, 1988.

[21] Madeleine Udell, Karanveer Mohan, David Zeng, Jenny Hong, Steven Diamond, and Stephen Boyd. Convex optimization in Julia. *SC14 Workshop on High Performance Technical Computing in Dynamic Languages*, 2014.

[22] Zhihua Zhang, Michael Jordan, Wu-Jun Li, and Dit Yan Yeung. Coherence functions for multicategory margin-based classification methods. In *Twelfth International Conference on Artificial Intelligence and Statistics (AISTATS)*, Clearwater Beach, Florida, USA, 2009.
A Proof of Theorem 1

For the surrogate loss $\ell_t^z(a) = -\log_t \exp_t(a/2 - G_tz(a))$, we have

$$\frac{\partial \ell_t^z(a)}{\partial a} = -\hat{p}_t(a)t_z - \left(\frac{1}{2} - \partial G_tz(a)\right)$$

(A.1)

$$\frac{\partial^2 \ell_t^z(a)}{\partial a^2} = \hat{p}_t(a)t_z \left[ \partial^2 G_tz(a) - (t_z - t_1)\hat{p}_t(a)t_z - 1 \left(\frac{1}{2} - G_tz(a)\right)^2 \right]$$

(A.2)

where we defined $\hat{p}_t(a) = \exp_t(a/2 - G_tz(a))$ and $\partial G_tz(a)$ and $\partial^2 G_tz(a)$ are given in (26) and (27), respectively. For $t_z = t_1 \geq 1$, we have

$$\frac{\partial^2 \ell_t^z(a)}{\partial a^2} = \partial^2 G_tz(a) \geq 0$$

(A.3)

which can be verified from (27). Moreover, for $t_z \geq 1$ and $t_1 \geq t_z$, we have

$$\frac{\partial^2 \ell_t^z(a)}{\partial a^2} = \frac{1}{\hat{p}_t(a)t_z - t_1} \left[ \partial^2 G_tz(a) + (t_z - t_1)\hat{p}_t(a)t_z - 1 \left(\frac{1}{2} - G_tz(a)\right)^2 \right]$$

$$\geq \partial^2 G_tz(a) + (t_z - t_1)\hat{p}_t(a)t_z - 1 \left(\frac{1}{2} - G_tz(a)\right)^2 \geq \partial^2 G_tz(a) \geq 0$$

(A.4)

Thus, the loss is more convex than the latter case.

Now, consider the case $t_z \geq t_1$. Suppose $\hat{p}_t(-a) = (1 - \hat{p}_t(a)) = \lambda \hat{p}_t(a)$ for some $\lambda \geq 0$. Substituting for $\hat{p}_t(-a)$ in (26) and (27), we can write (A.2) as

$$\frac{\partial^2 \ell_t^z(a)}{\partial a^2} = \hat{p}_t(a)t_z - 1 \left(\frac{1}{2} - G_tz(a)\right)^2 \left[ t_z \left(\frac{1 + \lambda}{1 + \lambda t_z}\right) - (t_z - t_1) \right]$$

(A.5)

For sufficiently small (respectively, large) value of $\lambda$, we have $\frac{\partial^2 \ell_t^z(a)}{\partial a^2} > 0$ (respectively, $\frac{\partial^2 \ell_t^z(a)}{\partial a^2} < 0$). The inflection point happens when $t_z(1 + \frac{1}{\lambda}) = (t_z - t_1)(1 + \lambda t_z)$, i.e., (25) is satisfied.

Finally, we show the case $t_1 < 1$. We only need to consider the case $t_z \leq t_1 < 1$. Note that for the binary case,

$$\exp_tz(a/2 - G_tz(a)) + \exp_tz(-a/2 - G_tz(a)) = 1$$

(A.6)

Using the definition of $\exp_tz$, we can write (A.6) as

$$[1 + (1 - t_z)\exp_tz(a)]^{1/(1-t_z)} + [1 + (1 - t_z)\exp_tz(-a)]^{1/(1-t_z)} = 1$$

(A.7)

For $a = 0$, (A.7) yields

$$[1 + (1 - t_z)(-G_tz(0))]^{1/(1-t_z)} = \frac{1}{2}$$

(A.8)

From $t_z < 1$, we have $(1 - t_z) > 0$ and therefore, $G_tz(0) > 0$. From convexity and symmetry ($G_tz(a) = G_tz(-a)$) conditions, we conclude $G_tz(a) \geq G_tz(0) \geq 0$, $\forall a$. Consequently, for values of $a \leq -\frac{1}{1-t_z}$, $G_tz(a) = -\frac{1}{2}$ satisfies (A.6). This implies that for $a \leq -\frac{1}{1-t_z}$, we have $\hat{p}_t(a) = 0$ and thus, $\ell_t^z(a) = -\log_t(0) = -\frac{1}{t_z}$ is a constant. From (A.4), we conclude that the loss is convex for $a > -\frac{1}{1-t_z}$ and is a constant for $a \leq -\frac{1}{1-t_z}$. Thus, it is quasi-convex.

B Implications of Using Tsallis Divergence

Consider modeling the (unknown) posterior distribution $p(y|x)$ for the set of random variables $(x, y) \in \mathcal{X} \times \mathcal{Y}$ using a discriminative model $\hat{p}_o(y|x)$. For this purpose, we can minimize the expected Tsallis divergence between the class posterior distribution of the data and the predicted
we recover the maximum-likelihood estimation.

Algorithm 1 Algorithm for Generating the Large-margin Label Noise

| Input: Dataset \( \{x_n, c_n\} \), noise ratio \( r \) |
| Output: Noisy dataset \( \{x_n, \hat{c}_n\} \) |

Train \( \mathbf{w} \) by running LR on \( \{x_n, c_n\} \)
for \( n \in \{1, \ldots, N\} \) do
  Compute \( u_n = c_n (\mathbf{w} \cdot x_n) \)
end for
Compute \( u_{\text{max}} = \max_n u_n \)
for \( n \in \{1, \ldots, N\} \) do
  Update \( u_n \leftarrow u_n - u_{\text{max}} \)
end for
Compute \( u_{\text{min}} = \min_n u_n \)
for \( n \in \{1, \ldots, N\} \) do
  Update \( s_n \leftarrow \exp(- \frac{10 u_n}{u_{\text{min}}} ) \)
end for
\( Z \leftarrow \sum_n s_n \)
\( I_s \leftarrow \text{Draw } |rN| \text{ samples without replacement from the distribution } \sim \frac{1}{Z} s_n \)
for \( n \in \{1, \ldots, N\} \) do
  if \( n \in I_s \) then
    \( \hat{c}_n = - c_n \)
  else
    \( \hat{c}_n = c_n \)
  end if
end for

\( p \)
\( p_{\text{max}} \) are drawn from the tempered conditional distribution \( \sim \)

However, indeed, minimizing the sum in (B.1d) involves the implicit assumption that the second sum in (B.1c) and only keep the terms corresponding to the observed labels, as in (B.1d). Then, approximate the class posteriors \( \hat{p}_{\text{max}} \) in which \( H_t(y) = - \int \sum_{c \in \mathcal{Y}} p(c|x)^t \log_t p(c|x) p(x) dx = \mathbb{E}_x \left[ \sum_{c \in \mathcal{Y}} p(c|x) \log_t \left( \frac{1}{p(c|x)} \right) \right] \) is the expected Tsallis entropy of the posterior distribution \( p(y|x) \) and is a constant. Note that from (B.1b) to (B.1c) we perform a Monte Carlo approximation of the integral using a set of samples \( \{x_n, y_n\} \) and then, approximate the class posteriors \( p(c|x) \) by the 0/1 probability \( \mathbb{I}_{c=y}. \) Therefore, we can eliminate the second sum in (B.1c) and only keep the terms corresponding to the observed labels, as in (B.1d). However, indeed, minimizing the sum in (B.1d) involves the implicit assumption that the \( y \) samples are drawn from the tempered conditional distribution \( \sim p(y|x)^t \) and thus, the minimizer solves \( \hat{p}_{\text{max}}(y|x) \sim p(y|x)^{1/t}. \) In the case of \( t = 1, \) the Tsallis divergence reduces to the KL-divergence and we recover the maximum-likelihood estimation \( - \sum_n \log \hat{p}_{\text{max}}(y_n|x_n) = - \log \prod_n \hat{p}_{\text{max}}(y_n|x_n) \) and \( \hat{p}_{\text{max}}(y|x) \sim p(y|x). \)

C Algorithm for Generating the Large-margin Label Noise

The Algorithm illustrates the process for generating large-margin label noise, used in the experiments.
Table 3: Running time on the noise-free datasets.

| Dataset | SVM       | LR        | t-LR      | TTLR     |
|---------|-----------|-----------|-----------|----------|
| Splice  | 8.63 ± 0.87 | 0.18 ± 0.05 | 0.80 ± 0.12 | 4.00 ± 0.20 |
| SVMguide| 0.04 ± 0.00 | 0.06 ± 0.00 | 2.29 ± 0.00 | 8.58 ± 0.00 |
| Mushrooms| 0.13 ± 0.01 | 0.75 ± 0.02 | 9.47 ± 0.11 | 25.46 ± 0.63 |
| MNIST   | 6.72 ± 0.49 | 8.14 ± 0.71 | 44.92 ± 3.40 | 575.45 ± 96.05 |
| USPS    | 7.38 ± 0.38 | 1.48 ± 0.25 | 79.63 ± 10.05 | 39.77 ± 6.89 |
| Letters | 24.75 ± 1.32 | 0.97 ± 0.06 | 8.17 ± 0.43 | 52.32 ± 3.54 |

D Running Time

Table 3 illustrates the running time of the algorithms on different datasets. In most cases, our method TTLR is the slowest one among different methods because of the non-convexity of the optimization problem and higher complexity of calculating the log-partition functions.