Abstract. We outline various developments of affine and general Landau Ginzburg models in physics. We then describe the A-twisting and coupling to gravity in terms of Algebraic Geometry. We describe constructions of various path integral measures (virtual fundamental class) using the algebro-geometric technique of cosection localization, culminating in the theory of “Mixed Spin P (MSP) fields” developed by the authors.

CONTENTS

1. Introduction                                           2
Acknowledgments                                         2
2. Mirror Symmetry and Gromov-Witten Invariants of Quintics 2
   2.1. Physics                                            2
   2.2. Mathematics                                        4
3. Witten’s Gauged Linear Sigma Model (GLSM)              5
4. Hyperplane Property, Ghost, and P-field               6
   4.1. Physics: Guffin and Sharpe                         6
   4.2. Mathematics: Hyperplane Property                   7
5. Fields Valued in Two GIT Quotients                    10
   5.1. Physics: GLSM                                      10
   5.2. Mathematics                                        11
6. Affine LG Phase and Spin Structure                    12
   6.1. Physics: SUSY A-twisted LG Theory Coupled To Gravity 12
   6.2. Mathematics: FJRW Invariants                      13
7. The Puzzle to Link Invariants in Opposite Phases       14
   7.1. Mathematics                                        14
8. Master space                                           15
   8.1. Mathematics                                        15
9. Mixed Spin Fields: Quantization of the Master Space    16
   9.1. Mixed Spin P-fields                                16
   9.2. Propersness: Capture Ghost at Infinity             18
10. Vanishing and Polynomial Relations                     21
11. Comparison with Physical Theories                     23
   11.1. Comparison with Witten’s GLSM                    23
   11.2. Compare with B-model                             24
References                                               24
1. Introduction

In this survey we will describe several mathematical and physical theories.

(1) The physical theory of generalized LG model (Guffin-Sharpe), and the mathematical theory of stable maps with P-fields and the hyperplane property in all genera (H.-L. Chang and J. Li).

(2) The Fan-Jarvis-Ruan-Witten (FJRW) theory of affine LG model, and the algebro-geometric construction of Witten’s top Chern class in the narrow case (H.-L. Chang, J. Li and W.P. Li).

(3) Witten’s Gauged Linear Sigma Model (GLSM) which specializes to (1) in the Calabi-Yau (CY) phase and (2) in the Landau-Ginzburg (LG) phase as the Kähler parameter $r \to +\infty$ and $r \to -\infty$ respectively. The two phases are then linked via promoting the Kähler parameter to a $\nu$-field, which leads to moduli spaces of Mixed Spin P (MSP) fields.

All the above three models (stable maps with P-fields, Witten’s top Chern class, MSP fields) require Kiem-Li’s cosection localization. A ghost P-field in the CY phase can be transformed continuously to a field in the LG phase determining the spin structure on the underlying curve. This phenomenon is named Landau-Ginzburg transition, and is responsible for the interaction between fields in the CY and LG phases. In the end we discuss how effective the MSP moduli could be used to attack various problems, including the enumeration of positive-genus curves in the quintic Calabi-Yau threefold.

The article is written by mathematicians, aiming to include the physics involved and the mathematics therefore stimulated, especially the algebro-geometric constructions by authors. In most sections we survey results in mathematics and in physics separately. In physics side, $C$ is a compact Riemann surface (worldsheet), and $K_C$ denotes its canonical line bundle. In mathematical side, $C$ is an orbifold curve with markings with at worst nodal singularities, and $\omega_C$ denotes its dualizing sheaf. In physics part, $L$ is a $C^\infty$ complex line bundle over the compact Riemann surface $C$, while in mathematical part $L$ is an algebraic line bundle over the algebraic curve $C$. Sections of a bundle or maps between spaces, if not being mentioned “$C^\infty$”, are assumed to be holomorphic, i.e. algebraic sections/maps in algebraic geometry. Finally, we let $W_5 = x_1^5 + \cdots + x_5^5$ be the Fermat quintic polynomial.

Acknowledgments. H.-L. Chang is partially supported by Hong Kong GRF grant 600711 and 6301515. J. Li is partially supported by NSF grant DMS-1104553, DMS-1159156, and DMS-1564500. W.-P. Li is partially supported by by Hong Kong GRF grant 602512 and 6301515. C.-C. Liu is partially supported by NSF grant DMS-1206667, DMS-1159416, and DMS-1564497. This article is an expansion of J. Li’s plenary talk at String Math 2015 in Sanya.

2. Mirror Symmetry and Gromov-Witten Invariants of Quintics

2.1. Physics. A 2d supersymmetric sigma model governs maps from a fixed Riemann surface $\Sigma$ to a target manifold $X$. When the target is a Calabi-Yau manifold, Witten [Wi2] introduced two different ways to twist the standard supersymmetric sigma model, known as the A twist and the B twist, and obtained two different topological field theories, the A-model and B-model on $X$, denoted $A(X)$ and $B(X)$.
In the A-model, the path integral over the infinite dimensional space of maps to $X$ can be reduced to an integral over the space of holomorphic maps to $X$. The A-model correlation functions depend on the Kähler structure but not the complex structure on $X$.

In the B-model, the path integral over the infinite dimensional space of maps of $X$ can be reduced to an integral over the space of constant maps to $X$, i.e., an integral over $\mathbb{X}$. The B-model correlation functions depend on the complex structure but not the Kähler structure on $X$.

Given a Calabi-Yau manifold $X$, the mirror $\mathcal{X}$ of $X$ is another Calabi-Yau manifold of the same dimension such that

$$A(X) \cong B(\mathcal{X}), \quad B(X) \cong A(\mathcal{X}).$$

The expected/virtual (complex) dimension of space of holomorphic maps from a closed Riemann surface to $X$ is $\dim \mathcal{X} \cdot (1 - g)$, where $g$ is the genus of the Riemann surface. Therefore, we expect there to be no holomorphic maps from a fixed generic Riemann surface of genus $g > 1$ to $X$. By allowing the complex structure on the domain Riemann surface to vary, we obtain the 2d sigma model coupled with gravity. The A-model (resp. B-model) topological string theory on $X$ is obtained by applying A twist (resp. B twist) to the sigma model on $X$ coupled with gravity. In the rest of this paper we will always consider theories coupled with gravity, still denoted by $A(X)$ and $B(X)$. The mirror symmetry (2.1) is still expected. The equivalence $A(X) \cong B(\mathcal{X})$ implies an equality of genus $g$ topological string amplitudes:

$$F_{A(X)}^g(q(t)) = F_{B(X)}^g(t)$$

where $t \mapsto q(t)$ is the mirror map from the moduli of complex structures on $\mathcal{X}$ to the moduli of (complexified) Kähler classes on $X$. The genus zero B-model is determined by the classical variation of Hodge structures. In 1993, Bershadsky-Cecotti-Ooguri-Vafa (BCOV) developed the Kodaira-Spencer theory of gravity, which is a string field theory of higher genus B-model [BCOV].

A typical example is the quintic Calabi-Yau threefold $Q$, which is a degree 5 hypersurface in $\mathbb{P}^4$. Its mirror $\mathcal{Q}$ is a degree 5 hypersurface in $\mathbb{P}^4/(\mathbb{Z}_5)^3$. In 1991, Candelas-de la Ossa-Green-Parkes [COGP] derived a formula for the genus zero B-model topological string amplitude $F_0^B(\mathcal{Q})$ of $\mathcal{Q}$ and the mirror map in terms of explicit hypergeometric series, and obtained a mirror formula of the genus zero A-model topological string amplitude $F_0^{A(Q)}$ of $Q$, which is a generating function of (virtual) numbers of rational curves in $Q$. Mirror symmetry predictions on higher genus A-model topological string amplitudes $F_g^A(\mathcal{Q})$ have been obtained by Bershadsky-Cecotti-Ooguri-Vafa at genus $g = 1, 2$ ([BCOV], 1993), by Katz-Klemm-Vafa at genus $g = 3, 4$ ([KKV], 1999), and at genus $g \leq 51$ by Huang-Klemm-Quackenbush ([HKQ], 2007).

Using results of BCOV [BCOV] and Yamaguchi-Yau [YY] and assuming mirror symmetry, Huang-Klemm-Quackenbush [HKQ] provide an algorithm to determine $F_g^A(\mathcal{Q})(q(t)) = F_g^{B(\mathcal{Q})}(t)$ for genus $g \leq 51$. When $g \geq 2$, the holomorphic anomaly equation determines $F_g^A(\mathcal{Q})(q)$ up to $3g - 2$ unknowns. The degree zero Gromov-Witten invariant $N_{g,d=0}$ is known, so we are left
with \(3g - 3\) unknowns; the boundary conditions at the orbifold point (which corresponds to Landau-Ginzburg theory of the Fermat quintic polynomial in five variables) impose \(\left\lfloor \frac{2}{5}(g - 1) \right\rfloor\) constraints on the \(3g - 3\) unknowns, whereas the “gap condition” at the conifold point imposes \(2g - 2\) constraints on the \(3g - 3\) unknowns. In summary, the holomorphic anomaly equation and the boundary conditions determine \(\mathbf{F}_A^g(Q)\) up to \(\left\lfloor \frac{2}{5}(g - 1) \right\rfloor\) unknowns. When genus \(g \leq 51\), the Gopakuma-Vafa conjecture (which relates Gromov-Witten invariants and Gopakumav-Vafa invariants) and the Castelnuovo bound (which implies vanishing of low degree Gopakumar-Vafa invariants) are sufficient to fix the remaining \(\left\lfloor \frac{2}{5}(g - 1) \right\rfloor\) unknowns.

2.2. Mathematics. Gromov-Witten theory can be viewed as a mathematical theory of the A-model topological string theory. There are two approaches to Gromov-Witten theory. Here we describe the algebro-geometric definition. For non-negative integers \(d, g\), \(\overline{M}_{g}(Q,d)\) denotes the moduli space of stable maps from genus \(g\) nodal curves to \(Q\) of degree \(d\). Li-Tian [LT] and Behrend-Fantachi [BF] construct a degree zero cycle \([\overline{M}_{g}(Q,d)]^{\text{vir}}\in A_0(\overline{M}_{g}(Q,d);Q)\), which is called the virtual cycle. Note that \(\overline{M}_{g}(Q,d)\) is empty when \((g,d)\in\{(0,0),(1,0)\}\). For \((g,d)\neq(0,0),(1,0)\), define genus \(g\), degree \(d\) Gromov-Witten invariant of \(Q\) by

\[
N_{g,d} := \int_{[\overline{M}_{g}(Q,d)]^{\text{vir}}} 1 \in \mathbb{Q}.
\]

The genus-\(g\) Gromov-Witten potential of \(Q\) is given by

\[
F^A_g(q) := \begin{cases} 
\frac{5}{6}(\log q)^3 + \sum_{d=1}^{\infty} N_{0,d}q^d, & g = 0; \\
-\frac{25}{12}\log q + \sum_{d=1}^{\infty} N_{1,d}q^d, & g = 1; \\
\sum_{d=0}^{\infty} N_{g,d}q^d, & g \geq 2.
\end{cases}
\]

One of the main unsolved problems in Gromov-Witten theory is to determine \(F^A_g(q)\), which is a generating function of genus \(g\) Gromov-Witten invariants of \(Q\).

Using the hyperplane property in genus zero, Kontsevich [Ko] proposed to use torus localization to calculate the genus zero Gromov-Witten invariants \(N_{0,d}\). Givental [Gi] and Lian-Liu-Yau [LLY] proved the mirror formula of \(F^A_0(q)\) predicted in [COGP]. The BCOV mirror formula of \(F^A_1(q)\) was solved in 2000’s. J.Li and A. Zinger [LZ] obtained a formula

\[
(2.3) \quad N_{1,d} = N^\text{red}_{1,d} + \frac{1}{12}N_{0,d}
\]

where \(N^\text{red}_{1,d}\) is the genus one, degree \(d\) reduced GW-invariant of \(Q\). Using (2.3) and \(\mathbb{C}^*\)-localization, Zinger proved the BCOV mirror formula of \(F^A_1(q)\) in [Zi2]. Gathmann [Gath] provided an algorithm for \(N_{1,d}\) using the relative GW-invariant formula.

Using degeneration, Maulik and Pandharipande [MP] found an algorithm which determines \(N_{g,d}\) for all genus \(g\) and degree \(d\): one degenerates the quintic of \(\mathbb{P}^4\) to a quartic and a \(\mathbb{P}^3\),
and than degenerates the quartic to a cubic and a $\mathbb{P}^3$, etc. In [MP] Section 0.6, Maulik-Pandharipande described a second algorithm based on Gathmann’s proposal. The second algorithm only requires one degeneration: one degenerates $\mathbb{P}^4$ to $\mathbb{P}^4$ and a $\mathbb{P}^1$ bundle over $Q$. Therefore, the second algorithm should be significantly more efficient than the first algorithm. Gathmann did the genus 0 and 1 cases. J. Li’s degeneration formula and [MP Theorem 1] (the quantum Leray-Hirsch) allow Maulik and Pandharipande to pursue Gathmann’s proposal in all genera. Maulik-Pandharipande proved that the second algorithm determines all the genus 2 invariants $N_{g,d}$ and conjectured that it determines $N_{g,d}$ for all $g, d$; recently, L. Wu proved this conjecture in the genus 3 case.

We remark that the theories used by mathematicians to approach $N_{g,d}$ as above are essentially (1) “hyperplane property” for $g = 0, 1$, (2) torus localization formula, and (3) degeneration formula. They have intrinsic origin from theory of virtual cycles in mathematics.

It remains a central problem in Gromov-Witten theory to find new effective algorithms to calculate all genus Gromov-Witten invariants of $Q$, with structural properties compatible with physics treatment by mirror symmetry, such as (quasi-)modularity of $F^g_d$ and finitely many holomorphic ambiguities with linear growth in $g$.

3. WITTEN’S GAUGED LINEAR SIGMA MODEL (GLSM)

The same quintic polynomial $W_5 = x_1^5 + \ldots + x_5^5$ defines a map $\mathbb{C}^5 \to \mathbb{C}$. The corresponding physical theory is the Landau-Ginzburg theory for the pair $(\mathbb{C}^5, W_5)$. Since $W_5$ is invariant under the diagonal multiplicative action of $\mathbb{Z}_5$ on $\mathbb{C}^5$, it descends to give an orbifold LG model $([\mathbb{C}^5/\mathbb{Z}_5], W_5)$. In [GLSM], Witten embedded $Q$ into a larger background with superpotential as follows. Let $\mathbb{C}^*$ act on $\mathbb{C}^6 = \mathbb{C}^5 \times \mathbb{C} = \{(x_1, \ldots, x_5, p)\}$ with weights $(1, \ldots, 1, -5)$. The quotient $[\mathbb{C}^6/\mathbb{C}^*]$ has two GIT quotients:

\[
\frac{((\mathbb{C}^5 - \{0\}) \times \mathbb{C}^*)}{\mathbb{C}^*} = K_{\mathbb{P}^4}, \\
\frac{(\mathbb{C}^5 \times (\mathbb{C} - 0))}{\mathbb{C}^*} = \mathbb{C}^5/\mathbb{Z}_5.
\]

Here $K_{\mathbb{P}^4}$ is the total space of the canonical line bundle $O(-5)$ on $\mathbb{P}^4$. The polynomial $p(x_1^5 + \cdots + x_5^5)$ on $\mathbb{C}^6$ is invariant under the above $\mathbb{C}^*$ action, so it descends to a function $\tilde{W} : [\mathbb{C}^6/\mathbb{C}^*] \to \mathbb{C}$. Thus one has a picture relating generalized Landau-Ginzburg models

\[
\begin{array}{ccc}
Q & \subset & (K_{\mathbb{P}^4}, w) \\
\left((\mathbb{C}^6/\mathbb{C}^*), \tilde{W}\right) & (\mathbb{C}^5/\mathbb{Z}_5), W_5)
\end{array}
\]

where the restriction $w$ of $\tilde{W}$ to $K_{\mathbb{P}^4}$ is the function induced by tensoring $x_1^5 + \cdots + x_5^5 \in H^0(\mathbb{P}^4, O_{\mathbb{P}^4}(5))$ under the pairing $O_{\mathbb{P}^4}(-5) \otimes O_{\mathbb{P}^4}(5) \to O_{\mathbb{P}^4}$. The critical locus of the superpotential $w$ on $K_{\mathbb{P}^4}$ is the quintic $Q$ embedded in $K_{\mathbb{P}^4}$ as the subvariety defined by $p = 0$ and $x_1^5 + \cdots + x_5^5 = 0$. The two skew arrows in Diagram (3.1) are open smooth subsets defined by $(x_1, \ldots, x_5) \neq 0$ and $p \neq 0$ respectively.

In 1993 Witten [GLSM] provides a theory called Gauged Linear Sigma Model (GLSM) which can be considered as a sort of “quantization” of $([\mathbb{C}^6/\mathbb{C}^*], \tilde{W})$. Here the word quantization means
promoting variables \((x_1, \cdots, x_5)\) to fields \((\varphi_1, \cdots, \varphi_5)\) on the worldsheet (which is a connected closed Riemann surface), and promoting \(\mathbb{C}^*\) to a principal \(\mathbb{C}^*\)-bundle over the worldsheet, with a gauge field. Witten’s GLSM theory is parameterized by a real number \(r\) called Fayet-Iliopoulos parameter, which is essentially the Kähler parameter of the symplectic quotient of \(\mathbb{C}^6\) by the Hamiltonian \(U(1)\)-action with weights \((1, \ldots, 1, -5)\). When \(r \to -\infty\) the GLSM is contributed by the Landau-Ginzburg model \(([\mathbb{C}^5/\mathbb{Z}_5], W_5)\); when \(r \to +\infty\) the GLSM is contributed by massless instantons in \(Q\), the critical locus of \((K_{\mathbb{P}^4}, w)\). Witten also showed that for \(r \to +\infty\) the GLSM is contributed by instantons of a sigma model on the quintic threefold \(Q\), i.e. holomorphic curves in \(Q\).

Witten’s model suggests a few things. Firstly, a physics theory for the Landau-Ginzburg model \((K_{\mathbb{P}^4}, w)\) should be found to undertake the specialization from GLSM to sigma model on the quintic threefold \(Q\). Secondly, when \(r > 0\) is finite, contribution to GLSM is made by massive instantons in \(Q\), where mass corresponds to common zeros of \(\varphi_1, \cdots, \varphi_5\). In the language of mathematics, it foresees intermediate theories (obtained by varying the stability condition) other than Gromov-Witten (GW) theory of \((K_{\mathbb{P}^4}, w)\) (at \(r \to +\infty\)) or Landau-Ginzburg (LG) theory of \(([\mathbb{C}^5/\mathbb{Z}_5], W_5)\) (at \(r \to -\infty\)).

When \([\text{GLSM}]\) first appeared, it was far from a mathematical theory: as Witten’s \([\text{GLSM}]\) is a gauged field theory defined by using path integral, mathematicians need to find conditions to ensure the convergence of the path integral and then substitute the infinite dimensional path integral measure with certain finite dimensional construction to obtain rigorous mathematical definitions. Moreover, theories for every spaces in \((3.1)\) were neither twisted nor coupled to gravity.

Witten’s GLSM tells us, once mathematicians can possibly achieve finite dimensional constructions which lead to rigorous mathematical definitions, the (massive or massless) theories for \(Q, (K_{\mathbb{P}^4}, w), ([\mathbb{C}^5/\mathbb{Z}_5], W_5)\), and possibly even the universal \((([\mathbb{C}^6/\mathbb{C}^*], \tilde{W})\), should determine each other; namely, they should be “equivalent” theories. However, what are the explicit relations among amplitudes from any two different theories? Could these conjectural equivalences help determine all of them, or just reduce three sorts of mysteries to one that is still mysterious? In later sections of this paper, we will discuss solutions to the above questions on constructions and relations.

In the following we shall call \(([\mathbb{C}^5/\mathbb{Z}_5], W)\) an affine Landau-Ginzburg model as \(\mathbb{C}^5\) is an affine space, to be distinguished from \((K_{\mathbb{P}^4}, w)\), which is a general Landau-Ginzburg model.

4. Hyperplane Property, Ghost, and P-field

4.1. Physics: Guffin and Sharpe. Guffin and Sharpe consider the A-twisting of the LG model \((K_{\mathbb{P}^4}, w)\) suggested by Witten, and show its amplitudes are equal to genus zero GW invariants of \(Q\). This can be viewed as the hyperplane property in physics, at least in genus zero. To describe the matter field \(\varphi : C^* \to \mathbb{P}^4\) (where \(\varphi\) and \(C\) are smooth), in additional to classical \(\varphi_1, \cdots, \varphi_5\) as \(C^\infty\) sections of a line bundle \(L\) on \(C\), the noncompact direction needs to be twisted by the canonical line bundle \(K_C\) of \(C\) and considered as

\[
p \in C^\infty(C, K_C \otimes L^{\otimes -5}),
\]

sec:hyperplane
so a term in the Lagrangian becomes a top form on $C$ and can be integrated to make sense of the action. Guffin and Sharpe showed that, in the genus zero case, their integral (as an invariant associated to enumerating curves mapped to $(K_{\mathbb{P}^4}, w)$) is equal to

$$e(E_d) \cap [X_d] \in H_0(X; \mathbb{Q}) = \mathbb{Q}$$

where $e(E_d)$ is the Euler class of the finite rank complex vector bundle

$$E_d = \bigcup_{\varphi \in X_d} H^0(C, \varphi^*\mathcal{O}_{\mathbb{P}^4}(5))$$

over the smooth compact complex orbifold $X_d = \{\varphi : C \to \mathbb{P}^4 \ | \ deg \varphi = d, g(C) = 0\}$. By Kontsevich’s hyperplane property, $e(E_d) \cap [X_d]$ is the genus zero, degree $d$ GW invariant $N_{g=0,d}$ of $Q$.

4.2. Mathematics: Hyperplane Property. We explain the hyperplane problem in mathematics. Fixing the degree $d$, for each genus $g$, over the finite dimensional compact space

$$Y_{g,d} := \{\varphi : C \to \mathbb{P}^4 \ | \ deg \varphi = d, g(C) = g\}$$

there are two unions of vector spaces

$$V_{g,d} := \bigcup_{\varphi \in Y_g} H^0(C, \varphi^*\mathcal{O}_{\mathbb{P}^4}(5)) \quad \text{and} \quad V'_{g,d} := \bigcup_{\varphi \in Y_g} H^1(C, \varphi^*\mathcal{O}_{\mathbb{P}^4}(5)),$$

defining two sheaves over $Y_{g,d}$. By Riemann-Roch formula, the difference of the dimensions $\dim V_{g,d}|_{\varphi} - \dim V'_{g,d}|_{\varphi}$ is $5d + 1 - g$, which is independent of $\varphi \in Y_{g,d}$.

If $g = 0$, one can show $V'_{0,d} = 0$. Thus $V_{0,d}$ has constant dimensional fiber over $Y_{0,d}$, namely $V_{0,d}$ is a complex vector bundle of rank $5d + 1$ over $Y_{0,d}$. The hyperplane property of Kontsevich says

$$N_{0,d} = \deg (e(V_{0,d}) \cap [Y_{0,d}]) \in \mathbb{Q}, \quad \text{(4.2)}$$

where $\deg : A_0(X_d; \mathbb{Q}) \to \mathbb{Q}$. This reconstructs the enumeration of rational curves in $Q$ from the information of $\mathbb{P}^4$. The identity (4.2) is an easy consequence of virtual cycle theory [KKP]. By (4.2), one may compute $N_{0,d}$ by torus localization as $V_{0,d}$ and $Y_{0,d}$ admit a $(\mathbb{C}^*)^4$-action inherited from that on $\mathbb{P}^4$.

When $g > 0$, everything above fails unfortunately. For example when $g = 1$, $Y_{g=1,d}$ contains essentially two kinds of components. The two components collect maps of different forms (see Figure 1 below).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{graphs for an honest map $\alpha$ and a ghost map $\beta$.}
\end{figure}
The main component of \( Y_{1,d} \) consists of maps looking like \( \alpha \), which has positive degree on the genus one component of the curve. The other component of \( Y_{1,d} \) consists of maps looking like \( \beta \), which contracts the (black) genus zero one component to one point, and has positive degree on the (blank) genus zero \( \mathbb{P}^1 \). For the curve and maps being indicated as \( \alpha \), every element in \( H^1(C, L^{\otimes 5}) = H^0(C, K_C \otimes L^{\otimes -5}) \) vanishes since \( L = \alpha^* \mathcal{O}(1) \) has positive degree and \( K_C \otimes L^{\otimes -5} \) has negative degree. Thus the P-field must vanish for \( \alpha \) (recall that \( P \in H^0(C, K_C \otimes L^{\otimes -5}) \)), or equivalently, \( V'_{1,d} |_{\alpha} = 0 \). However, for \( \beta \) one can have nonzero one form on the elliptic component, which extends by zero to give a section of \( K_C \otimes L^{\otimes -5} \). This corresponds to the fact that \( \beta \) contracts a genus one component of the curve \( C \), for which reason we say \( \beta \) is a ghost map. The black genus one component is a “ghost” component, and \textit{P-field can survive on a ghost.} One easily calculates \( V'_{1,d} |_{\beta} \cong \mathbb{C} \).

We remark that in the approach determining \( N_{1,d} \) in [LZ, Z], a key issue is to locate the contribution of the ghost in the counting. In their formula \( N_{1,d} = N^\text{red}_{1,d} + \frac{1}{12} N_{0,d} \), the term \( N^\text{red}_{1,d} \) is the contribution from maps of type \( \alpha \), and \( \frac{1}{12} N_{0,d} \) is the contribution from maps of type \( \beta \), where the \( \frac{1}{12} \) comes from integrating out all the P-fields living on the ghost (black) elliptic component of \( \beta \). For our ultimate purpose to approach \( N_{g,d} \) for larger \( g \), locating the contribution of P-field (including ghosts) becomes very difficult and out of control.

Since \( V'_{1,d} |_{\alpha} = 0 \) and \( V'_{1,d} |_{\beta} \cong \mathbb{C} \), \( V'_{1,d} \) has fiber rank jumping over \( X_{1,d} \), and by Riemann-Roch \( V_{1,d} \) also does and is not a vector bundle over \( Y_{1,d} \). As the Euler class is only defined for vector bundles, \( e(V_{1,d}) \) no longer makes sense. It is natural to ask how the hyperplane property [1.2] should be modified, so that the information of \( \mathbb{P}^4 \) can be used to reconstruct enumeration of higher genus curves in \( Q \) in mathematics (namely only finite dimensional construction allowed).

After A-twisting, the topological string theory (with supersymmetry) admits a mathematical counterpart called “virtual cycle”\(^1\). As virtual cycle ([LZ]) is governed by tangent-obstruction (deformation) theory (in physics words, after A-twisting, the zero mode of fermions over SUSY fixed loci, even if the loci is singular, recovers the path integral algebraically), we may view the above problem of higher genus hyperplane property in the following way. Let \( f : C \to Q \) be a point in 

\[
\begin{align*}
  f &= [f_1, \cdots, f_5] \in \overline{M}_g(Q, d) := X_{g,d} \subset \overline{M}_g(\mathbb{P}^4, d) = Y_{g,d}.
\end{align*}
\]

The exact sequence \( 0 \to T_Q \to T_{\mathbb{P}^4}|_Q \to \mathcal{O}(5)|_Q \to 0 \) induces the following long exact sequence

\[
\begin{array}{ccccccc}
0 & \to & H^0(C, f^* T_Q) & \to & H^0(C, f^* T_{\mathbb{P}^4}) & \to & H^0(C, f^* \mathcal{O}(5)) \\
& \to & H^1(C, f^* T_Q) & \to & H^1(C, f^* T_{\mathbb{P}^4}) & \to & H^1(C, f^* \mathcal{O}(5)) & \to & 0.
\end{array}
\]

Every vector space above has geometric meanings, namely the sequence \( 4.3 \) is identical to 

\[
\begin{array}{ccccccc}
0 & \to & T_{f,X_{g,d}} & \to & T_{f,Y_{g,d}} & \to & \mathcal{O}_{f,X_{g,d}/Y_{g,d}} \\
& \to & \mathcal{O}_{f,X_{g,d}} & \to & \mathcal{O}_{f,Y_{g,d}} & \to & \mathcal{O}^\text{higher}_{f,X_{g,d}/Y_{g,d}} & \to & 0.
\end{array}
\]

where

- \( T_{f,X_{g,d}} \) and \( T_{f,Y_{g,d}} \) are the first order deformations of \( f \) in \( X_{g,d} \) and \( Y_{g,d} \) (relative to moduli space of genus \( g \) nodal curves) respectively;

\^[1]\text{also called virtual fundamental class}
• $\mathcal{O}b_{f,X_{g,d}}$ and $\mathcal{O}b_{f,Y_{g,d}}$ are obstructions to deforming $f$ in $X_{g,d}$ and $Y_{g,d}$ respectively.

• $H^0(C, f^*\mathcal{O}(5))$, which contains the element $f_1^2 + \cdots + f_5^2$, is the obstruction\textsuperscript{2} of an element in $Y_{g,d}$ to be in $X_{g,d}$, namely the relative obstruction $\mathcal{O}b_{f,X_{g,d}/Y_{g,d}}$.

• $H^1(C, f^*\mathcal{O}(5))$ is the higher obstruction of a point in $Y_{g,d}$ to lie in $X_{g,d}$.

Now recall that the tangent and obstruction theory would determine the virtual cycle (path integral measure), and the two terms in the left column in (4.4) are tangent and obstruction theories of $X_{g,d}$, therefore are responsible for the Gromov-Witten invariant $N_{g,d}$ of the quintic Calabi-Yau threefold $Q$. The two terms in the middle column are tangent and obstruction theories of $Y_{g,d}$ which parametrizes maps to $\mathbb{P}^4$. To solve the hyperplane property problem, one should combine the right column in (4.4) with the datum of $Y_{g,d}$. If this can be done then one may expect to recover $N_{g,d}$.

We observe that the last term (higher obstruction) $H^1(C, f^*\mathcal{O}(5))$ is dual to the space $H^0(C, K_C \otimes f^*\mathcal{O}(-5))$ of algebraic P-fields (c.f. (4.1)), which we may add it to the moduli space $\mathcal{M}_g(\mathbb{P}^4, d)$ of stable maps to $\mathbb{P}^4$ to form the moduli of stable maps to $\mathbb{P}^4$ with P-fields.\textsuperscript{3}

\[ Y_{g,d}^p := \mathcal{M}_g(\mathbb{P}^4, d)^p = \{ [\varphi : C \to \mathbb{P}^4] \in \mathcal{M}_g(\mathbb{P}^4, d), \rho \in \Gamma(C, K_C \otimes \varphi^*\mathcal{O}(-5)) \}, \]

where $\rho$ is called an algebraic “P-field” as its analogue in (4.1). As the obstructions to deforming $\varphi$ and $\rho$ lie in $H^1(C, \varphi^*T_{\mathbb{P}^4})$ and $H^1(C, K_C \otimes \varphi^*\mathcal{O}(5)) = H^0(C, \varphi^*\mathcal{O}(5))^\vee$ respectively, the deformation theory of $Y_{g,d}^p$ is given by the middle and right columns in (4.3). If one is able to define a virtual cycle for $\mathcal{M}_g(\mathbb{P}^4, d)^p$, then it is expected to be “equivalent” to the virtual cycle of $X_{g,d}$, and the hyperplane problem is solved. Now the difficulty appears because $\mathcal{M}_g(\mathbb{P}^4, d)^p$ is non-compact due to the presence of P-fields: for example, over the ghost map $\beta$, the P-field can be any element in $C$ that is unbounded. This difficulty is then overcome by the invention of “cosection localization” by Y.H. Kiem and Jun Li, along with H.L. Chang’s observation that “the supersymmetry variation of the superpotential on worldsheet defines a cosection, which solves Witten’s equation in the general Landau-Ginzburg theory.”

We now describe the algebro-geometric results of H.-L. Chang and J. Li\textsuperscript{[CL1]} discovered based on the above reasoning. In the definition of (4.5), the data $([\varphi, C], \rho)$ is equivalent to the data $(C, L, \varphi_1, \cdots, \varphi_5, \rho)$, since the map $\varphi$ is equivalent to the line bundle $L = \varphi^*\mathcal{O}_{\mathbb{P}^4}(1)$ with five sections $(\varphi_1, \cdots, \varphi_5)$ of $L$. We regard $\mathcal{M}_g(\mathbb{P}^4, d)^p$ as a space of “maps from curve to $K_{\mathbb{P}^4}$.” The moduli stack $\mathcal{M}_g(\mathbb{P}^4, d)^p$ has a perfect obstruction theory relative to the smooth Artin stack $\mathcal{D} = \{(C, L)\}$. At $\xi = [(C, L, \varphi, \rho)] \in \mathcal{M}_g(\mathbb{P}^4, d)^p$, the (relative) obstruction space of deforming $\xi$ is

$\mathcal{O}b_{\mathcal{M}/\mathcal{D}}|_{\xi} = H^1(L)^{\oplus 5} \oplus H^1(L^{\mathbb{P}^4}) \otimes \omega_{C}.$

There exists a cosection

$\sigma : \mathcal{O}b_{\mathcal{M}/\mathcal{D}} \to \mathcal{O}b_{\mathcal{M}_g(\mathbb{P}^4, d)^p}$

constructed as follows. Let

$(\hat{\varphi}_1, \ldots, \hat{\varphi}_5, \hat{\rho}) \in H^1(L)^{\oplus 5} \oplus H^1(L^{\mathbb{P}^4}) \otimes \omega_{C}) = \mathcal{O}b_{\mathcal{M}/\mathcal{D}}|_{\xi}.$

\[ \text{because } f_1^2 + \cdots + f_5^2 = 0 \text{ characterizes the } f \in Y_{g,d} \text{ lies in } X_{g,d} \]

\[ \text{now allow } f : C \to Q \text{ to be more general } \varphi : C \to \mathbb{P}^4 \]
Define

\[ \sigma|_\xi(\dot{\phi}_1, \ldots, \dot{\phi}_5, \dot{\rho}): = \dot{\rho} \sum_{i=1}^{5} \varphi_i^5 + \rho \sum_{i=1}^{5} \varphi_i^4 \dot{\varphi}_i. \]

The degeneracy locus \( D(\sigma) \) of the cosection \( \sigma \) consists of \( \xi \) such that \( \sigma|_\xi \) is zero, i.e.,

\[ \sigma|_\xi(\dot{\phi}_1, \ldots, \dot{\phi}_5, \dot{\rho}) = 0 \text{ for all } \dot{\varphi}_i \text{ and } \dot{\rho}. \]

Thus

\[ D(\sigma) = \{ \xi \in \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p | \rho = 0 \text{ and } \sum_{i=1}^{5} \varphi_i^5 = 0 \} = \overline{\mathcal{M}}_g(Q, d) \subset \overline{\mathcal{M}}_g(\mathbb{P}^4, d). \]

This corresponds to the fixed loci of supersymmetry (SUSY) in path integral. The expression of the cosection \( \sigma \) comes from supersymmetry variation \( \delta \) (in physics) applied to \( p \cdot W_5 = p(x_1^5 + \ldots + x_5^5) \) where \( p \) and \( x_i \) live on the worldsheet, via H.L. Chang’s observation.

Since \( \rho \) is a section, the moduli space \( \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p \) is not proper (when \( g \geq 1 \)) and hence cannot be used to define invariants. However, the degeneracy locus \( D(\sigma) \) is the moduli space of stable maps to the quintic threefold \( Q \) and thus proper. Using cosection localization developed by Y.H. Kiem and J. Li [KL], H.L. Chang and J. Li constructed [CL1] the cosection localized virtual cycle for Landau-Ginzburg theory

\[ \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p_{\text{vir loc}} \in A_*(\overline{\mathcal{M}}_g(Q, d)) = A_* \overline{\mathcal{M}}_g(Q, d). \]

As always one defines the \( P \)-fields GW invariants

\[ N_{g,d}^p = \int_{[\overline{\mathcal{M}}_g(\mathbb{P}^4, d)^p_{\text{vir loc}}]} 1 \in \mathbb{Q}. \]

H.L Chang and J. Li proved the following.

**Theorem 4.1** (H.L. Chang - J. Li [CL1]). The GW invariant of the quintic threefold \( Q \) equals the \( P \)-fields GW-invariant up to a sign:

\[ N_{g,d} = (-1)^{d+g+1} N_{g,d}^p. \]

The advantage of this result is that \( F_g^A(q) = \sum_d N_{g,d} q^d \) now becomes the amplitude of a theory valued in \( K_{\mathbb{P}^4} = (\mathbb{C}^5 - \tilde{0}) \times \mathbb{C} / \mathbb{C}^* \).

In conclusion, the invariant enumerating maps from curves to \( (K_{\mathbb{P}^4}, w) \) is equal to the invariant enumerating maps from curves to \( Q \), up to a sign. This generalizes the genus zero case to the positive genus case, and solves the hyperplane property problem.

5. **Fields Valued in Two GIT Quotients**

5.1. **Physics: GLSM.** From [4], we see the curve-counting theories for \( Q \) and \( (K_{\mathbb{P}^4}, w) \) in Diagram (3.1) are both established in mathematics for all genera. Witten’s [GLSM] suggests the theory for \( ((\mathbb{C}^5 / \mathbb{Z}_5, W_5)) \) at the lower right corner of (3.1) should also exist, and match the physical theory of A-twisted LG model coupled with gravity.
5.2. Mathematics. We now consider the space of maps from curves to each target in (3.1), viewed as a sort of “quantization” of (3.1).

The previous sections tells us the space $\overline{M}_g(\mathbb{P}^4, d)^p$ of all “maps to $(K_{\mathbb{P}^4}, w)$” is the set of all $(C, L, \varphi, \rho)$ where $\varphi = (\varphi_1, \ldots, \varphi_5)$ is section of $L^{\oplus 5}$, $\rho$ is a section of $\omega_C \otimes L^{\otimes -5}$, and

\[ \varphi = (\varphi_1, \cdots, \varphi_5) \text{ has no zeros on } C \]

so that $\varphi_i$s define an honest map to $\mathbb{P}^4$. Without the condition (+), one obtains a huge Artin stack $\text{Art}$ of all $(C, L, \varphi, \rho)$ for arbitrary $\varphi \in \Gamma(C, L^{\oplus 5})$ and $\rho \in \Gamma(C, \omega_C \otimes L^{\otimes -5})$. The stack $\text{Art} = \{(C, L, \varphi, \rho)\}$ should be viewed as the moduli space of maps to $[\mathbb{C}^6/\mathbb{C}^*]_{(1,1,1,1,1,-5)}$. $\overline{M}_g(\mathbb{P}^4, d)^p$ is the open substack of objects in $\text{Art}$ subject to condition (+), which corresponds to the open substack $K_{\mathbb{P}^4} \subset [\mathbb{C}^6/\mathbb{C}^*]$ in (3.1) defined by $(x_1, \ldots, x_5) \neq (0, \ldots, 0)$. After quantizing it translates to the requirement (+), as $\varphi_1, \cdots, \varphi_5$ are the five fields promoted from the five coordinates $x_1, \ldots, x_5$.

Parallelly, since the open substack $[\mathbb{C}^5/\mathbb{Z}_5] \subset [\mathbb{C}^6/\mathbb{C}^*]$ is defined by $p \neq 0$ in (3.1), to define a theory whose target is $([\mathbb{C}^5/\mathbb{Z}_5], W_5)$, one analogously expects to pick up the open substack of $\text{Art}$ subject to the condition

\[ \rho \text{ has no zeros on } C \]

as $\rho$ is the field promoted from coordinate $p$ in (3.1). Namely $\rho$ trivializes $\omega_C \otimes L^{-5}$, or equivalently, gives an isomorphism $L^{\otimes 5} \xrightarrow{\sim} \omega_C$. One then expects the theory of $([\mathbb{C}^5/\mathbb{Z}_5], W_5)$ to start with the moduli space of all $(C, L, \varphi)$ where

1. $L$ is a fifth root\(^4\) of $\omega_C$, and
2. $\varphi = (\varphi_1, \cdots, \varphi_5)$ is an arbitrary section of $L^{\otimes 5}$.

We denote this moduli as $\overline{M}_g^{1/5,5p}$ where $1/5$ denotes the 5-spin structure and $5p$ indicates that an object consists of five sections $\varphi_1, \cdots, \varphi_5$ of $L$ (by abuse of notation).

When one quantizes every space in (3.1), one then obtains two open substacks (subspaces) of the common huge Artin stack as follows:

\[ \overline{M}_g(Q, d) \subset \overline{M}_g(\mathbb{P}^4, d)^p \]

\[ \overline{M}_g(\mathbb{P}^4, d)^p \xrightarrow{\varphi \text{ nowhere } 0} \overline{M}_g^{1/5,5p} \]

\[ \overline{M}_g^{1/5,5p} \xrightarrow{\rho \text{ nowhere } 0} \overline{M}_g(Q, d) \]

Naturally one wonders whether the substack $\overline{M}_g^{1/5,5p}$ at the bottom right corner has a virtual cycle, with which intersections represent invariants of the Landau-Ginzburg model $([\mathbb{C}^5/\mathbb{Z}_5], W_5)$ from physics, as Witten predicted. Coincidentally, around 2010 H. Fan, T. Jarvis, and Y. Ruan carried out a construction of an A-side theory of which the target may be any affine LG space $([\mathbb{C}^n/G], w)$, where $G$ is a finite group and the “superpotential” $w$ is a $G$-invariant polynomial on $\mathbb{C}^n$. Their approach to the affine Landau-Ginzburg model $([\mathbb{C}^n/G], w)$ originates from a different line in history, namely the gauged WZW model, Witten equation, and Hamiltonian Floer theory, which we brief in §6 below.

\(^4\)sometimes called 5-spin structure on $C$;
6. AFFINE LG PHASE AND SPIN STRUCTURE

6.1. Physics: SUSY A-twisted LG Theory Coupled To Gravity. The classical Landau-Ginzburg theory on the A-side follows a different line of development in history. In [Wi] E. Witten conjectured that descendant integrals on moduli spaces of stable curves $M_{g,n}$ satisfy the KdV equations, and the string equation (proved by Witten) and the KdV equations uniquely determine all descendant integrals from the initial value $\int_{M_{0,3}} 1 = 1$. Witten’s conjecture was first proved by Kontsevich [Kon1] by stratification of $M_{g,n}$ and matrix model. For the purpose to generalize above to $N$ matrix model, Witten in [Wi1] considered A-twisted gauged WZW model targeting $SU(2)/U(1)$ coupled with gravity, and obtained a topological theory which he conjectured [Wi3] (refining.twisting the minimal model of [KL] et.al.) to solve the generalized KdV hierarchies ($N$-matrix model).

Witten’s A-twisted theory is localized to the SUSY fixed locus consisting of objects almost definable in algebraic geometry. Let $\mathcal{M}_{g}^{1/r}$ denote the moduli space of Riemann surfaces $C$ (with at worst nodal singularities) together with a line bundle $L$ such that $L^{\otimes r} \cong K_{C}$. Mathematically $\xi = (C, L)$ is referred as a $r$-spin curve. One may also add orbifold marked points on $C$ but we omit them here for simplicity. Witten roughly argued that $\mathcal{M}_{g}^{1/r}$ is smooth and compact. He set

$$M^{\infty} = \bigcup_{(C,L) \in \mathcal{M}_{g}^{1/r}} C^{\infty}(C,L)$$

where $C^{\infty}(C,L)$ is the space of $C^{\infty}$ sections $u$ of $L$.

The topological correlation function of Witten’s theory amounts to counting the intersection number of the zero section of the (infinite rank) bundle

$$E^{\infty} = \bigcup_{(\xi,u) \in M^{\infty}} \Omega_{C}^{(0,1)}(L) \longrightarrow M^{\infty}$$

with the graph of the section

$$(\xi = (C, L), u) \mapsto s_{W}(\xi, u) := \overline{\partial} u + r(\overline{u})^{r-1}$$

and possibly with insertions ([Wi3]) such as gravitational descendents (if one adds markings on each $C$). Note that we may choose a Kähler metric on the Riemann surface $C$ and a Hermitian metric on the line bundle $L$, so that $(\overline{u})^{r-1}$ becomes a section of $(L)^{\otimes (r-1)} \cong L^{\otimes (1-r)} \cong K_{C} \otimes L$, where $\overline{\partial} u$ lives. In short the theory counts solutions of

$$\overline{\partial} u + r(\overline{u})^{r-1} = 0.$$  

The Euler class of $E^{\infty}$ localized by Witten’s section $s_{W}$ is then called “Witten’s top Chern class”, a core object in the definition of the theory.

For the purpose to interprete Witten’s correlation function more directly, one may regard it as the A-twisted (and coupled to gravity) version of the “Landau-Ginzburg theories” defined in [Vafa, Ito] (also c.f. [Ce]). Vafa, et.al.’s model build the Landau-Ginzburg structure directly in the Lagrangian. Namely, it is a path integral whose configuration space of fields is the set of
maps from the worldsheet to \(\left( [\mathbb{C}^n/G], w \right)\), with fermions coupled with terms as
\[
s_w = (\partial u_i + \partial u_i w(u_1, \cdots, u_n))_{i=1}^n,
\]
and the contribution to the theory comes from solutions of
\[
(6.4) \quad \partial u_i + \partial u_i w(u_1, \cdots, u_n) = 0 \quad \text{for all } i = 1, \cdots, n
\]
generalizing (6.3) where \(n = 1, w = x^r\).

However, the theories in \cite{Ito, Vafa} are not coupled with gravity, and the group \(G\) is trivial \(G = \{e\}\). It was then later understood (by Fan-Jarvis-Ruan etc.) that Witten’s model is using \(\left( [\mathbb{C}/\mathbb{Z}_n], w = x^r \right)\), whose state spaces are indexed by the monodromy weights of the \(r\)-spin bundle at markings.

### 6.2. Mathematics: FJRW Invariants

Based on Witten’s infinite dimensional Euler class model (with section to be \(s_w\)), Fan-Jarvis-Ruan \cite{FJR1, FJR2} used analytic methods to construct the Witten’s top Chern class, and defined correlators of a Cohomological Field Theory (CohFT) by capping the Witten’s top Chern class with states of the Landau-Ginzburg model \(\left( [\mathbb{C}^n/G], w \right)\). Fan-Jarvis-Ruan’s pioneer work is now known as FJRW invariants associated to the singularity \(\left( [\mathbb{C}^n/G], w \right)\). FJRW invariants of special ADE type singularities can be enumerated and are governed by the Kac-Wakimoto/Drinfeld-Sokolov hierarchies \cite{LRZ}, generalizing \cite{FSZ}’s proof of Witten’s \(r\)-spin conjecture.

In FJRW theory, Witten’s top Chern class is constructed in differential geometry via perturbing (6.4). It can also be constructed in algebraic geometry without perturbing (6.4). The algebro-geometric constructions (in the narrow case) were carried out by Polishchuk-Vaintrob \cite{PV}, by Chiodo \cite{Chi}, and by H.L. Chang, J. Li and W.P. Li \cite{CLL}. For our purpose to provide a field theory valued in \(\left( [\mathbb{C}/\mathbb{Z}_n], w \right)\), we briefly the construction in \cite{CLL} here, using the version with markings. Recall that the moduli \(\mathcal{M}^{1/5,p}_g\) for \(\left( [\mathbb{C}^5/\mathbb{Z}_5], W_5 \right)\) requires a fifth root \(L\) of \(\omega_C\), which does not exist if \(\text{deg } \omega_C = 2g - 2\) is not divisible by five. One thus extends the setup by allowing \(C\) to be a twisted curve with markings (which can be scheme points or stacky points). Thus our field valued in \(\left( \mathbb{C}^5 \times (\mathbb{C} - 0) \right)/\mathbb{C}^* = [\mathbb{C}^5/\mathbb{Z}_5]\) consists of
\[
\xi = (\Sigma^C, C, L, \varphi_1, \ldots, \varphi_5, \rho)
\]
where \((\Sigma^C, C)\) is a pointed twisted curve with markings \(\Sigma^C\) possibly stacky, \(L\) is an invertible sheaf on \(C\), \(\varphi_i \in H^0(L)\), and \(\rho \in H^0(L^{v^5} \otimes \omega^\log_C)\) with \(\omega^\log_C = \omega_C(\Sigma^C)\), and the corresponding property of \((\mathbb{C}^5/\mathbb{Z}_5, W_5)\) is required. This implies \(L^{v^5} \otimes \omega^\log_C \cong \mathcal{O}_C\), or equivalently \(L^{v^5} \cong \omega^\log_C\). Therefore \((\Sigma^C, C, L)\) is a 5-spin curve and \((\varphi_1, \ldots, \varphi_5)\) gives five fields. We get a moduli space of 5-spin curves with five fields:
\[
\mathcal{M}^{1/5,p}_g = \{ (\Sigma^C, C, L, \varphi_1, \cdots, \varphi_5, \rho) \mid \rho \text{ is nowhere zero} \}.
\]
Here \(\gamma\) is the monodromy data: if \(\Sigma_j\) is a stacky marking on \(C\), then \(\mu_5\) acts on \(L|_{\Sigma_j}\) with weight \(\gamma_j = \exp(2\pi ir_j/5)\) where \(1 \leq r_j \leq 4\) and we call \(\gamma_j\) narrow. If \(\Sigma_j\) is a scheme marking, we call it broad, and it corresponds to \(\gamma_j = 1\).
Similar to the case $\overline{M}_g(\mathbb{P}^1, d)^p$, the moduli stack $\overline{M}_g^{1/5,p}$ has a perfect obstruction theory relative to the smooth Artin stack $\mathcal{D} = \{(\Sigma^C, C, L)\}$. There exists a cosection $\sigma: O_b \to O_{\overline{M}_g^{1/5,p}}$ whose degeneracy locus is

$$D(\sigma) = \{\xi \in \overline{M}_g^{1/5,p} | \phi_i = 0 \text{ for all } i\} = \overline{M}_g^{1/5} = \{(\Sigma^C, C, L) | L^{\otimes 5} \cong \omega_C^{\log}\},$$

which is the moduli space of 5-spin curves.

**Theorem 6.1** (H.L. Chang - J. Li - W. P. Li [CLL]). The (narrow) FJRW invariants can be constructed using cosection localized virtual cycles of $\overline{M}_g^{1/5,p}$:

$$[\overline{M}_g^{1/5,p}]_{\text{vir}}^{\text{loc}} \in A_*\overline{M}_g^{1/5,p}.$$

The Witten equation mentioned in (6.4), in this case, becomes

$$\bar{\partial}_s + 5s^4 = 0 \quad \text{for } i = 1, \ldots, 5. \quad \text{(6.5)}$$

This is used to construct Witten’s top Chern class to define invariants on the moduli space of 5-spin curves. From Witten’s equation (6.5), the term $\bar{\partial}_s_i$ gives the obstruction to extending a holomorphic section. Thus the left hand side of (6.5) is a $C^\infty$ section of the obstruction sheaf of the moduli of spin curves with fields. Substituting the complex conjugate in the Witten’s equation by the Serre duality, the left hand side of (6.5) becomes a smooth inverse of cosection. The Mathai-Quillen setup in (6.1) and (6.2) generalize naturally here and the Euler class localized near solution of Witten equations would be equal to the Kiem-Li’s virtual cycle localized via cosection $\sigma$.

We remark here that the form (6.5) indicates the virtual cycle of $([\mathbb{C}^5/\mathbb{Z}_5], W_5)$ is the five self-intersection of the virtual cycle of $([\mathbb{C}/\mathbb{Z}_5], x^5)$ (each defined by using $\bar{\partial}s + 5s^5 = 0$ which is (6.3) for the case $r = 5$). This remarkable property is related to self-tensor product of conformal field theories and is discussed in [FJR1] or [CR].

There is an important subclass of FJRW invariants: those with insertions $-\frac{2}{5}$. Let $C$ have $k$ markings with all $\gamma_j = \zeta^2$ for $1 \leq j \leq k$ where $\zeta = \exp(2\pi i/5)$. For simplicity we write $\gamma = (\gamma_j)_{j=1}^k = 2^k$. Define

$$\Theta_{g,k} = \int_{[\overline{M}_g^{1/5,p}]_{\text{vir}}^{\text{loc}}} 1 \in \mathbb{Q}, \quad \text{for } k + 2 - 2g = 0 \text{ mod } 5.$$

It is shown [CLL2] that $\{\Theta_{g,k}\}_{g,k}$ determine all FJRW invariants with descendents for the quintic LG space $([\mathbb{C}^5/\mathbb{Z}_5], W_5)$, where an explicit formula will be given in [twFJRW]. For this reason we call $\{\Theta_{g,k}\}_{g,k}$ the primary FJRW invariants.

7. **The Puzzle to Link Invariants in Opposite Phases**

7.1. **Mathematics.** We have seen that the three moduli spaces at the bottom of Diagram (5.1) admit virtual fundamental classes, while the moduli space $\text{Art} := \{(C, L, \varphi, \rho)\}$ at the top of (5.1) does not, because $\text{Art}$ is not a Deligne Mumford stack. One can introduce all possible stability conditions to define open substacks of $\text{Art}$ that are Deligne Mumford, just as
$\overline{M}_g(\mathbb{P}^4, d)^p$ and $\overline{M}_g^{1/5,5p}$, and then construct virtual classes (path integral measures) for them as we defined $[\overline{M}_g(\mathbb{P}^4, d)^p]_{\text{vir}}$ and $[\overline{M}_g^{1/5,5p}]_{\text{vir}}$ using P-fields and cosection machinery. This is the step that most groups are taking. The theory of $\epsilon$-stability and quasimaps ([PK1] [MOP]) are developed, for example.

On the other hand, introducing new stability conditions means there are invariants other than the original $N_{g,d}$’s. Whether these new invariants (defined by new stability conditions) can simplify enumeration of $N_{g,d}$’s or give structures for $N_{g,d}$ predicted by the B-side, is not easy at all. Following Witten’s GLSM, we wishfully expect knowing FJRW invariants $\Theta_{g,k}$’s up to a sign) are related to the FJRW invariants $\Theta_{g,k}$’s defined by $[\overline{M}_g^{1/5,5p}]_{\text{loc}}$. We understand that the task is to construct theories that quantitatively link all in

\begin{equation}
\text{GW of } Q \xleftrightarrow{\text{phases}} \text{GSW of } (K_{\mathbb{P}^4}, w) \leftarrow ?> \longrightarrow \text{FJRW of } ([\mathbb{C}^5/\mathbb{Z}_5], W).
\end{equation}

To pursue this goal, we immediately face a specific problem: “the change of phases’ sign”. The topological type of fields in $K_{\mathbb{P}^4}$ is labelled by a pair $(g, d)$, where $g$ is the genus of the curve $C$ and $d = \deg L = \deg f^* O(1)$ is always a non-negative integer; the topological type of fields valued in $([\mathbb{C}^5/\mathbb{Z}_5], W_5)$ is labelled by a pair $(g, k)$, the genus $g$ of the curve and the number $k$ of $2/5$ marking. When $g$ is fixed and when $k$ is general (large) enough, one can show the degree of the line bundle (over the coarse curve) can be arbitrarily negative, namely, in the phase $([\mathbb{C}^5/\mathbb{Z}_5], W_5)$, fields are generally of negative degree. In GLSM this corresponds to the fact that $N_{g,d}$ are invariants near large radius limit point $r \gg 0$ and the LG phase $([\mathbb{C}^5/\mathbb{Z}_5], W_5)$ occurs near the orbifold point $r \ll 0$ (c.f. [GLSM] Section 5.1).

How can a field of positive degree be transformed to a field of negative degree? In which space could this unusual transform happen? How does such transformation – if it exists – change the virtual cycles and counting? We will address these questions in the following sections.

8. Master space

8.1. Mathematics. If one builds a large moduli space containing $\overline{M}_g(\mathbb{P}^4, d)^p$ and $\overline{M}_g^{1/5,5p}$ as its disjoint closed subspace, then intersection theory over the large moduli would give us information relating $\overline{M}_g(\mathbb{P}^4, d)^p$ to $\overline{M}_g^{1/5,5p}$. Recall that the two target spaces $K_{\mathbb{P}^4}$ and $[\mathbb{C}^5/\mathbb{Z}_5]$ are both open subsets of the 5-dimensional stack $[\mathbb{C}^6/\mathbb{C}^*]$, where the two overlap on a large open set $K_{\mathbb{P}^4} - \mathbb{P}^4 = ([\mathbb{C}^5 - \{\bar{0}\}]/\mathbb{Z}_5)$. If one embeds these two open substacks as disjoint closed substacks of a higher dimensional stack $W$, we may consider the space of maps from curves to $W$ as just stated. This higher dimensional stack has a natural construction in various places in “Variation of GIT” (VGIT) before, called the master space after M. Thaddeus. Here is a brief introduction.

Consider the following $\mathbb{C}^*$-action on $\mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1$: for $t \in \mathbb{C}^*$,

$$(x_1, \ldots, x_5, p, [u_1, u_2])^t: = (tx_1, \ldots, tx_5, t^{-5} p, [tu_1, u_2]).$$
There is a GIT quotient
\[ \tilde{W} = \left( \mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1 - S \right)/\mathbb{C}^* \]
where \( S = \{ (x_i = 0 = u_1) \cup (p = 0 = u_2) \} \), which is a 6-dimensional simplicial toric variety. So \( \tilde{W} \) has at most orbifold singularities. Indeed, \( \tilde{W} \) is smooth outside the unique orbifold point given by \( x_i = u_2 = 0 \). The stacky quotient
\[ W = \left[ \left( \mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1 - S \right)/\mathbb{C}^* \right] \]
is a 6-dimensional smooth toric Deligne-Mumford (DM) stack with coarse moduli space \( \tilde{W} \).

Consider a \( \mathbb{C}^* \)-action on \( W \), and call this action \( T \)-action to avoid confusion. For \( t \in T = \mathbb{C}^* \),
\[
(x_1, \ldots, x_5, p, u_1, u_2)^t = (x_1, \ldots, x_5, p, tu_1, u_2).
\]
The \( T \)-fixed locus is a disjoint union of three connected components:
\[ W^T = \mathbb{K}_{\mathbb{P}^4} \times \{ 0 \} \sqcup \mathbb{K}_{\mathbb{P}^1} \times \left( (\mathbb{P}^1 - \{ 0, \infty \})/\mathbb{C}^* \right) \sqcup \left[ \mathbb{C}^5/\mathbb{Z}_5 \right] \times \{ \infty \} \]
where \( 0 = [1, 0] \) and \( \infty = [1, 0] \) in \( \mathbb{P}^1 \), and the middle term \( \left( (\mathbb{P}^1 - \{ 0, \infty \})/\mathbb{C}^* \right) \) is nothing but one single point. The shape of \( W \) and its \( T \) fixed loci is shown in Figure 2, where \( \mathbb{K}_{\mathbb{P}^4} \) is a single point. The single point defined by \( x_i = p = 0 \) is responsible for the conifold point of complexified Kähler moduli space of the quintic.

9. Mixed Spin Fields: Quantization of the Master Space

9.1. Mixed Spin \( P \)-fields. Following the recipe from previous sections, now we consider a field theory valued in \( W \), namely the space of maps from curves to the master space \( W \).

Such an object is called an mixed spin \( P \)-field (MSP for short). It consists of
\[
\xi = (\Sigma^C, \mathcal{C}, \mathcal{L}, N, \varphi_1, \ldots, \varphi_5, \rho, \nu = [\nu_1, \nu_2])
\]
where
1. \( (\Sigma^C, \mathcal{C}) \) is a pointed twisted curve,
2. \( \mathcal{L} \) and \( N \) are invertible sheaves on \( \mathcal{C} \) (\( \mathcal{L} \) is as before but \( N \) is new due to the extra factor \( \mathbb{P}^1 \) in the master space technique),
3. \( \varphi_i \in H^0(\mathcal{L}) \) and \( \rho \in H^0(\mathcal{L}^{\text{v5}} \otimes \omega_C^\text{log}) \) (as before),
4. \( \nu = [\nu_1, \nu_2] \) is a new field, where \( \nu_1 \in H^0(\mathcal{L} \otimes N) \) and \( \nu_2 \in H^0(N) \).

They satisfy the following conditions:
of $L$ with five $P$-fields, we examine the moduli space of MSP fields in details. 

We say $\xi$ is stable if $\text{Aut}(\xi)$ is finite. For simplicity, we will use $\varphi$ to represent $(\varphi_1, \ldots, \varphi_5)$.

In order to understand why the moduli space of MSP fields geometrically contains the moduli space $M_g(P^4, d)^p$ of stable MSP fields of genus $g$ (H.L. Chang - J. Li - W.P. Li - C.C. Liu [CLLL]).

Let $\xi$ be a MSP field.

1. (narrow condition) $\varphi_i|_{\Sigma_c^g} = 0$,
2. (combined GIT-like stability conditions)
   - (a) $(\varphi_1, \ldots, \varphi_5, \nu_1)$ is nowhere vanishing (coming from excluding $\{(x_i = 0 = u_1)\}$),
   - (b) $(\rho, \nu_2)$ is nowhere vanishing (coming from excluding $\{(p = 0 = u_2)\}$),
   - (c) $(\nu_1, \nu_2)$ is nowhere vanishing (coming from $[u_1, u_2] \in P^1$).

We say $\xi$ is stable if $\text{Aut}(\xi)$ is finite. For simplicity, we will use $\varphi$ to represent $(\varphi_1, \ldots, \varphi_5)$.

Theorem 9.1 (H.L. Chang - J. Li - W.P. Li - C.C. Liu [CLLL]). The moduli stack $W_{g,\gamma,d}$ of stable MSP fields of genus $g$, monodromy $\gamma = (\gamma_1, \ldots, \gamma_5)$ of $L$ along $\Sigma_c^g$ and degree $d = (d_0, d_\infty)$ of $L \otimes \mathbb{C}$ and $N$ respectively is a separated DM stack of locally finite type.

The moduli stack $W_{g,\gamma,d}$ admits a natural $\mathbb{C}^*$-action also called $T$-action: for $t \in \mathbb{C}^*$,

$$(\Sigma_c^g, \mathbb{C}, L, N, \varphi, \rho, \nu_1, \nu_2)^t := (\Sigma_c^g, \mathbb{C}, L, N, \varphi, \rho, t\nu_1, \nu_2).$$

It is not proper since $\varphi$ and $\rho$ are sections of invertible sheaves. Thus we cannot do integrations on this stack. However, there exists a cosection of its obstruction sheaf. Using the arguments similar to the GW case and LG case, we have the following theorem.

Theorem 9.2 (CLLL). The moduli stack $W_{g,\gamma,d}$ has a $T$-equivariant perfect obstruction theory, a $T$-equivariant cosection $\sigma$ of its obstruction sheaf, and thus carries a $T$-equivariant cosection localized virtual cycle

$$[W_{g,\gamma,d}]^{\text{vir}}_{\text{loc}} \in A^*_T W_{g,\gamma,d}^{-}$$

where $W_{g,\gamma,d}^{-}$ is the degeneracy locus of $\sigma$, i.e.,

$$W_{g,\gamma,d}^{-} = (\sigma = 0) = \{\xi \in W_{g,\gamma,d} | \mathbb{C} = (\varphi = 0) \cup (\varphi^5 + \ldots + \varphi^5 = 0 = \rho)\}.$$

The cycle enumerates “maps to the master space $W$”. Figure 3 is an example where the domain curve is represented as a union of one dimension lines (which is the standard notation in algebraic geometry).
In Figure 3, the component $C_e$ is mapped to single point $b$, and is what we call a “ghost” (over which $\rho$ can be nonvanishing) in Figure 1. Note that considering $\xi$ as a map is just for easiness of understanding: indeed the map cannot be realized due to the presence of $\omega_c$ in the definition of $(\rho$ in) $\xi$.

9.2. Properness: Capture Ghost at Infinity. In order to integrate, we need properness of $W_{g,\gamma,d}^-$. In fact, we have

**Theorem 9.3 ([CLLL]).** The degeneracy locus $W_{g,\gamma,d}^-$ is a proper T-DM stack of finite type.

The proof of the properness reveals an important phenomenon transforming fields of different phases in the MSP moduli. Under the transformation, the spin structure of line bundles arises naturally in the LG-phase as a limit of a family of P-fields in CY-phase. We call this phenomenon the “Landau-Ginzburg transition”. As mentioned before, the contribution from ghosts is one of the difficulties to approach positive genus Gromov-Witten invariants. The LG transition phenomenon enables the FJRW theory to capture the ghost contribution in the GW theory, inside the MSP moduli space.

9.2.1. LG-Transition: An Example. For any positive integer $d$, we construct a simple example where $g = 1$, $\gamma = \emptyset$, and $d = (d,0)$ to illustrate the phenomenon of LG-transition and explain why FJRW theory comes into the picture naturally when we consider GW theory with a P-field. The argument below is also a part of the procedure to prove Theorem 9.3 (properness of the degeneracy locus).

1. A point in the degeneracy locus $W_{1,\emptyset,(d,0)}^-$.

We give an MSP-field which looks like the picture in the left of Figure 4. Given a positive integer $d$, define an MSP-field

$$\xi = (C, L, N, \varphi, \rho, \nu = [\nu_1, \nu_2])$$

over a point as follows. The curve $C$ is a union of a smooth elliptic curve $C_1$ and a smooth rational curve $C_0$, intersecting at a node $p$. Under the isomorphism $C_0 \cong \mathbb{P}^1$ we have

$$L|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}(d), \quad N|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}, \quad \varphi|_{C_0} = (x^d, -x^d, 0, 0, 0), \quad \rho|_{C_0} = 0, \quad \nu_1|_{C_0} = y^d, \quad \nu_2|_{C_0} = 1$$
where \([x,y]\) are homogeneous coordinates on \(C_0 = \mathbb{P}^1\), and \(p = [0,1]\). In particular \(\varphi(p) = 0\) and \(\nu_1(p) = \nu_2(p) = 1\). On \(C_1\), we have
\[
\mathcal{L}|_{C_1} \cong \mathcal{O}_{C_1}, \quad \mathcal{N}|_{C_1} \cong \mathcal{O}_{C_1}, \quad \varphi|_{C_1} = (0, \ldots, 0), \quad \nu_1|_{C_1} = 1, \quad \nu_2|_{C_1} = 1.
\]
In particular, \(\mathcal{N} \cong \mathcal{O}_C\). Finally, we extend \(\rho|_{C_0} = 0\) to a non-zero section \(\rho \in H^0(\mathcal{L}^{5} \otimes \omega^\log_{C})\) as follows: \(\rho|_{C_1}\) is a non-zero section of \(H^0((\mathcal{L}^{5} \otimes \omega_C)|_{C_1}) = H^0(\omega_{C_1}(p))\) vanishing at \(p\) only. The choice of \(\rho|_{C_1}\) is unique up to multiplication by a nonzero constant. Then \(\xi\) represents a point in the degeneracy locus \(W_{g=1,\gamma=0,d=(d,0)} \subset W_{g=1,\gamma=0,d=(d,0)}\).

2. A morphism from \(\mathbb{C}^*\) to \(W^-_{1,0,(d,0)}\).

We describe a one-parameter deformation of the MSP field \(\xi\), depicted by Figure 4. Let \(S_* = \mathbb{C} \times \mathbb{C}^*\) and let \(\pi_1 : S_* \rightarrow \mathbb{C}\) be the projection to the first factor. We consider a family of MSP-fields over \(\mathbb{C}^*\)
\[
\xi_* = (S_*, \mathcal{L}_* = \pi_*^1 \mathcal{L}, \mathcal{N}_* = \pi_*^1 \mathcal{N} = \mathcal{O}_{S_*}, \varphi_* = \pi_*^1 \varphi, \rho_* = t^{-1} \pi_*^1 \rho, [\nu_* = \pi_*^1 \nu_1, \nu_* = \pi_*^1 \nu_2 = 1])
\]
where \(t\) is the parameter of \(\mathbb{C}^* = \text{Spec}[t, t^{-1}]\). This family over \(\mathbb{C}^*\) defines a morphism
\[
\phi : \mathbb{C}^* \longrightarrow W^-_{1,0,(d,0)}.
\]
By abuse of notation, let \(\pi_1\) also denote the projection from \(S*_{si} := C_i \times \mathbb{C}^*\) to the first factor, where \(i = 0, 1\). The restriction of the family \(\xi_*\) to \(S_{0*}\) is a constant family over \(\mathbb{C}^*\):
\[
\xi_{0*} = (\Sigma_{0*} = p \times \mathbb{C}^*, S_{0*} = C_0 \times \mathbb{C}^*, \pi_*^1 \mathcal{O}_{\mathbb{P}^1}(d), \mathcal{O}_{S_{0*}}, (x^d, -x^d, 0, 0, 0, 0), [y^d, 1])
\]
which defines a constant map \(\phi_0 : \mathbb{C}^* \longrightarrow W^-_{g=0,\gamma=(1),d=(d,0)}\). The restriction of the family \(\xi_*\) to \(S_{1*}\) is
\[
\xi_{1*} = (\Sigma_{1*} = p \times \mathbb{C}^*, S_{1*} = C_1 \times \mathbb{C}^*, \mathcal{O}_{S_{1*}}, \mathcal{O}_{S_{1*}}, (0, \ldots, 0), t^{-1} \pi_*^1 (\rho|_{C_1}), [1, 1])
\]
which defines a morphism \(\phi_1 : \mathbb{C}^* \longrightarrow W^-_{g=1,\gamma=(1),d=(0,0)}\).

3. The limits \(t \rightarrow 0\) and \(t \rightarrow \infty\)

We will see that the morphism \([9.2]\) extends to a morphism
\[
\bar{\phi} : \mathbb{P}[1, 5] \longrightarrow W^-_{1,0,(d,0)}
\]
where the embedding \(\mathbb{C}^* \hookrightarrow \mathbb{P}[1, 5]\) is given by \(t \mapsto [1, t^{-1}]\). The image \(\bar{\phi}([0, 1])\) (resp. \(\bar{\phi}([1, 0])\)) is the limit in \(W^-_{1,0,(d,0)}\) when \(t \rightarrow 0\) (resp. \(t \rightarrow \infty\)). It is easy to see that
\[
\bar{\phi}([1, 0]) = (\mathbb{C}, \mathcal{L}, \mathcal{N}, \varphi, 0, \nu = [\nu_1, \nu_2])
\]
where \(\mathbb{C}, \mathcal{L}, \mathcal{N}, \varphi, \nu_1, \nu_2\) are defined as in Step 1.
Once we work with orbifolds, let \( \rho \) exist. Indeed, they do not exist in the ordinary sense, but their existence again will be resolved by replacing blow up the intersection of these two divisors to separate them.

If we cannot be zero simultaneously at any point, we don’t get an MSP extension. Thus we need to provide detailed construction of the family, which may be technical.

Then we can have the extension

\[
\text{extension of } \nu \text{ of } \tau \text{ of } \phi
\]

where \( \Sigma = \tau S, \rho \).

The extension of the constant map \( \phi_0 : \mathbb{C}^* \to W_{0,\gamma,d=(0,0)}^- \) is the constant map \( \tilde{\phi}_0 : \mathbb{P}[1,5] \to W_{0,\gamma,d=(0,0)}^- \).

To find the limit \( \tilde{\phi}([0,1]) \), it suffices to find \( \tilde{\phi}_1([0,1]) \) where \( \tilde{\phi}_1 \) is the extension of \( \phi_1 \) to \( \mathbb{P}[1,5] \).

Let \( S_1 = C_1 \times \mathbb{C}^* \) and \( \rho_1 = \rho|_{C_1} \). If we naively take \( \mathcal{L} = \mathcal{O}_{S_1}, \mathcal{N} = \mathcal{O}_{S_1}, \varphi_1 = 0, \nu_1 = 1 \) and \( \nu_2 = 1 \), the section \( t^{-1} \pi_1^* \rho_1 \) cannot be extended to a regular section of \( \mathcal{L} \circ \omega_{S_1/\mathbb{C}}^* \). Here by abuse of notations, \( \pi_i \) is the projection from \( S_1 \) to the \( i \)-th factor. One way to solve this problem is to use the equivalence of \( \xi_1 \) with the following \( \xi_1^{1*} \):

\[
(S_1, \Sigma_1, \mathcal{L} = \mathcal{O}_{S_1}, \mathcal{N} = \mathcal{O}_{S_1}, \varphi_1 = 0, \ldots, \varphi_5 = 0, \rho_{S_1} = \pi_1^n \rho_1, \nu_1 = 1, \nu_2 = t^{1/5})
\]

Then we can have the extension

\[
(S_1, \Sigma_1, \mathcal{L} = \mathcal{O}_{S_1}, \mathcal{N} = \mathcal{O}_{S_1}, \varphi_1 = 0, \ldots, \varphi_5 = 0, \rho_{S_1} = \pi_1^n \rho_1, \nu_1 = 1, \nu_2 = t^{1/5})
\]

where \( \Sigma = p \times C \).

The term \( t^{1/5} \) may look troublesome. Let’s just treat this as indicating that the zero locus of \( \nu_2 \) is \( \frac{1}{5} C_1 \times 0 \). The issue of fractional divisor will be resolved once we work in the world of twisted curves.

Let’s assume that we can work with fractional divisors. The zero divisor of \( \rho_{S_1} \) is \( \Sigma_1 \) and that of \( \nu_2 \) is \( \frac{1}{5} C_1 \times 0 \). Since these two divisors intersect, by MSP requirement that \( \rho_{S_1} \) and \( \nu_2 \) cannot be zero simultaneously at any point, we don’t get an MSP extension. Thus we need to blow up the intersection of these two divisors to separate them.

Let \( \tau : S_1' \to S_1 \) be the blow up of \( S_1 \) at \( P = \Sigma_1 \cap (C_1 \times 0) = p \times 0, E \) be the exceptional divisor, \( \Sigma_1' \subset S' \) the strict transform of \( \Sigma_1 \), and \( C_1' \) be the strict transform of \( C_1 \times 0 \). The zero divisor of \( \tau^* \nu_2 \) is \( \frac{1}{5} C_1' + \frac{1}{5} E \). The zero divisor of \( \tau^* \rho_{S_1} \) is \( \Sigma_1' + E \). Now we need to modify \( \mathcal{L} \) and \( \mathcal{N} \) by replacing \( \mathcal{L} \) by \( \mathcal{L}' = \mathcal{O}_{S_1'}(E/5) \) and \( \mathcal{N} \) by \( \mathcal{N}' = \mathcal{O}_{S_1'}(-E/5) \). Here we pretend that \( \mathcal{L}' \) and \( \mathcal{N}' \) exist. Indeed, they do not exist in the ordinary sense, but their existence again will be resolved once we work with orbifolds. Let \( \rho' \) be the section in \( H^0(\mathcal{L}' \circ \omega_{S_1'/\mathbb{C}}^* \log (-E)) \) whose
image is \( \tau^* \rho S_1 \) under the natural map \( H^0(\omega_{S'_1/\mathbb{C}}^{\log}(-E)) \to H^0(\omega_{S'_1/\mathbb{C}}^{\log}) \). The zero divisor of \( \rho' \) is \( \Sigma' \). Let \( \nu_2' \) be the section of \( \mathcal{O}_{S'_1}(-E/5) \) whose image under the natural map \( \mathcal{O}_{S'_1}(-E/5) \to \mathcal{O}_{S'_1} \) is \( \tau^* \nu_2 \). Then the zero divisor of \( \nu_2' \) is \( \frac{1}{5} C_1' \). Since \( C_1' \) and \( \Sigma' \) don’t intersect, we can get an MSP extension by taking \( \mathcal{L}' = \mathcal{O}_{S'_1}(E/5) \), \( N' \cong \mathcal{L}'^\vee \), \( \nu_1' \) a nonzero constant section \( \mathcal{L}' \otimes N' \), \( \varphi'_1 = \ldots = \varphi'_5 = 0 \), \( \rho' \in H^0(\mathcal{L}'^{5/5} \otimes \omega_{S'_1/\mathbb{C}}^{\log}) \) with its zero divisor being \( \Sigma'_1 \) which is the marking of \( S'_1/\mathbb{C} \), and \( \nu'_2 \in H^0(N') \) whose zero divisor is \( \frac{1}{5} C_1' \). As we mentioned earlier, to make this construction rigorous, we have to do base change and introduce stacky structures at nodes of \( C_1' \cup E \) (see [AGV CLMN]).

Now we see that the central fiber of \( S'_1 \) over \( \mathbb{C} \) at \( 0 \in \mathbb{C} \) is set-theoretically \( C_1 \cup E \), a union of the elliptic curve \( C_1 \) with the smooth rational curve \( E \). The section \( \nu_2' \) vanishes on \( C_\infty = C'_1 \cong C_1 \) and \( \rho' \) is nowhere vanishing on \( C_\infty \). Hence \( C_\infty \) is a 5-spin twisted curve, and \( E \) is a rational smooth twisted curve with a marking where \( \rho' \) vanishes.

Then we can glue the MSP field on \( S'_1 \) with the MSP field on \( C_0 \times \mathbb{C} \) by identifying the marking \( \Sigma'_1 \) with the marking \( p \times \mathbb{C} \in C_0 \times \mathbb{C} \) after possibly a base change. Thus the central fiber of the extension is a union of a smooth rational curve \( C_0 \), an elliptic curve \( C_1 \) which is a 5-spin curve, and a rational twisted curve \( E \) intersecting with \( C_1 \) at the stacky point and with \( C_0 \) at another point where the nonzero \( \rho \)-section vanishes.

We can also deform the MSP field \( (9.1) \) to a MSP field in GW sector as follows.

Consider \( S = C \times \mathbb{C} \). Let \( \pi_1 \) be the projection of \( S \) to its first factor. Take, for \( t \in \mathbb{C} \), we have a family of MSP over \( \mathbb{C} \),

\[
(S, \pi^*_1 L, \pi^*_1 N, \varphi_S, \pi^*_1 \rho, \left[ t \pi^*_1 \nu_1, \pi^*_1 \nu_2 \right]).
\]

Here \( \varphi_S \) is defined as follows.

\[
\varphi_S|_{C_0 \times \mathbb{C}} = (x^d, -x^d, (1-t)y^d, -(1-t)y^d, 0), \quad \varphi_S|_{C_1 \times \mathbb{C}} = (0, 0, 1-t, t-1, 0).
\]

When \( t = 1 \), we get the MSP field \( \xi \) in \( (9.1) \). When \( t = 0 \), we get an MSP lying in GW sector since \( t \pi^*_1 \nu_1|_{t=0} = 0 \).

\[ \square \]

10. Vanishing and Polynomial Relations

How to extract information of GW and/or FJRW invariants from the cycle \( [W_{g, \gamma, d}]_{\text{loc}}^{\text{vir}} \). In this section, we consider a less general case \( \gamma = \emptyset \) (i.e. no markings) to illustrate the key ideas. By virtual dimension counting, we have

\[
[W_{g, d}]_{\text{loc}}^{\text{vir}} \in H^0_{\overline{\mathcal{G}}^g(d_0 + d_{\infty} + 1 - g)}(W_{g, d}, \mathbb{Q}).
\]

When \( d_0 + d_{\infty} + 1 - g > 0 \), letting \( u = c_1(1)_{\text{vir}} \), i.e. \( u \) is the parameter for \( H^0_{\overline{\mathcal{G}}^g}(pt) \), we have

\[
[u^{d_0 + d_{\infty} + 1 - g} \cdot [W_{g, d}]_{\text{loc}}^{\text{vir}}]_{0} = 0.
\]

Here \( [\cdot]_0 \) is the degree zero term in the variable \( u \).

Let \( \Gamma \) be a graph associated to fixed points of the \( T \)-action of \( W_{g, d} \) and \( F_T \) be a connected component of \( W_{g, d}^T \) of the graph type \( \Gamma \). Applying the cosection localized version of the virtual
localization formula of Graber-Pandaripande [GP] proved by Chang-Kiem-J.Li in [CKL], we obtain
\[ \sum_{\Gamma} \left[ u_{d_0+d_\infty+1-g} \frac{[F_{\Gamma}]_{\text{loc}}^{\text{vir}}}{e(N_{\Gamma})} \right]_0 = 0. \]

To deal with \([F_{\Gamma}]_{\text{loc}}^{\text{vir}}\), we need a decomposition result to be explained below.

Let \(\xi = (C, L, N, \varphi, \rho, \nu_1, \nu_2) \in (W_g, d)^T\) be an MSP field fixed by the \(T\)-action. We set

1. \(C_0\) to be the part of \(C\) where \(\nu_1 = 0\);
2. \(C_1\) to be the part of \(C\) where \(\varphi = 0 = \rho\) and hence \(\nu_1 = 1 = \nu_2\), i.e., \(\nu_1\) and \(\nu_2\) are nowhere zero;
3. \(C_\infty\) to be the part of \(C\) where \(\nu_2 = 0\).

Thus

- \(\xi|_{\text{connected component of } C_0}\) is in \(\overline{M}_{g', n'}(\mathbb{P}^d, d)\) which gives GW invariants of \(Q\).
  Here marked points appear coming from some nodes on \(C_0\).
- \(\xi|_{\text{connected component of } C_1}\) is in \(\overline{M}_{g', n'}\) which gives Hodge integrals.
- \(\xi|_{\text{connected component of } C_\infty}\) is in \(\overline{M}_{g', 5p, \gamma'}\) which gives FJRW invariants of \((\mathbb{C}^5/\mathbb{Z}_5, W_5)\) where \(\gamma'\) appears because of some stacky nodes on \(C_\infty\).

We have the following decomposition result:
\[ [F_{\Gamma}]_{\text{loc}}^{\text{vir}} = c \prod [\text{moduli of } \xi|_{C_0}]_{\text{loc}}^{\text{vir}} \cdot [\text{moduli of } \xi|_{C_1}]_{\text{loc}}^{\text{vir}} \cdot [\text{moduli of } \xi|_{C_\infty}]_{\text{loc}}^{\text{vir}} \]

where \(c\) is a constant. The first factor gives GW invariants of stable maps to \(\mathbb{P}^d\) with P-fields, i.e. \(N_{g', d'}\). The second factor gives Hodge integrals on \(\overline{M}_{g', n'}\). The third factor gives FJRW invariants of insertions \(-\frac{2}{5}\) (after using a vanishing). After \(e(N_{\Gamma})\)'s are calculated, using the polynomial relations (10.1), we obtain the following results about GW invariants of the quintic.

**Theorem 10.1** ([CLLZ2]). Letting \(d_\infty = 0\), the relations (10.1) provide an effective algorithm to evaluate the GW invariants \(N_{g,d}\) provided the following are known

1. \(N_{g', d'}\) for \((g', d')\) such that \(g' < g\) and \(d' \leq d\);
2. \(N_{g, d'}\) for \(d' < g\);
3. \(\Theta_{g', k}\) for \(g' \leq g - 1\) and \(k \leq 2g - 4\);
4. \(\Theta_{g, k}\) for \(k \leq 2g - 2\).

Recall that \(\Theta_{g,k}\) is the genus \(g\) FJRW invariants of insertions \(-\frac{2}{5}\) and \(\Theta_{g,k}\) may be non-zero only when \(k + 2 - 2g \equiv 0(5)\). We can see that when \(g = 2\) only \(\Theta_{2,2}\) is needed, and when \(g = 3\) only \(\Theta_{3,4}\) is needed.

**Remark 10.2.** As we know, on using mathematical induction, upon more numerical datum the induction is, the less effective the computation will be. We can see from Theorem ?? that MSP induction for GW invariants is carried out on two numbers, genus and the degree only. Thus this provides a rather effective way to facilitate the induction procedure.

We can also use the vanishings (10.1) for \(d = (0, d_\infty)\), to determine quintic’s FJRW invariants up to finite many initial data.
Theorem 10.3 ([CLLL2]). For a fixed positive genus \( g \), the finite set \( \{ \Theta_{g,k} \}_{k<7g-2} \) determine all genus \( g \) FJRW invariants \( \{ \Theta_{g,k} \}_{k=0}^{\infty} \).

These relations are effective in calculating FJRW invariants. For example, for the case of genus 2, \( \{ \Theta_{2,k} \}_{k} \) can be inductively derived from only two unknowns \( \Theta_{2,2} \) and \( \Theta_{2,7} \).

We end this section by some speculations.

Let us look at Theorem 10.1 from a different aspect. Inductively we may suppose all GW/FJRW invariants for genus less than \( g \) are known. Then for genus \( g \), Theorem 10.1 reduces the problem of determining the infinitely many GW invariants \( \{ N_{g,d} \}_{d=1}^{\infty} \) to two finite sets of initial datum
\[
\{ N_{g,1}, \ldots, N_{g,g-1} \} \quad \text{and} \quad \{ \Theta_{g,k} \}_{k<2g-2}.
\]

We formulate the following speculation:

By suitable choice of positive \( d_0 \) and \( d_{\infty} \), the relations (10.1) provide an effective algorithm to determine the first set of initial data \( \{ N_{g,1}, \ldots, N_{g,g-1} \} \).

If this is true, then one is left to determine the second set of initial data \( \{ \Theta_{g,k} \}_{k<2g-2} \). We propose another conjecture about fully determining all FJRW invariants for the quintic,

**Conjecture 10.4.** The equations (10.1) using \( d_0 = 0 \) and nonempty \( \gamma \)'s (i.e. with markings) give relations that effectively evaluate all \( \Theta_{g,k} \).

11. Comparison with Physical Theories

11.1. Comparison with Witten’s GLSM. In [GLSM], Witten introduced a family of theories using path integrals, called the Gauged Linear Sigma Model (GLSM), linking a non-linear sigma models on a Calabi-Yau hypersurface to a sigma model targeting in a Landau-Ginzburg space. The GLSM is parameterized by a complexified Kähler parameter
\[
t := r - i\theta
\]
where \( r \) is “Fayet-Iliopoulos parameter” and \( \theta \) is called the theta angle\(^5\). Witten [GLSM Sect 3.1] argued that the GLSM specializes to GW path integral when \( r \to +\infty \), and specializes to LG model path integral when \( r \to -\infty \). This is known as the Calabi-Yau/Landau-Ginzburg correspondence.

The Mixed-Spin-P fields (MSP fields) introduced in [CLLL] is a field theory designed to capture “phase space transition” in one cage\(^6\). An MSP field can be viewed as an interpolation between fields valued in \( K_{\mathbb{P}^4} \) and fields valued in \( [\mathbb{C}^5/\mathbb{Z}_5] \), and the interpolation is governed by the “\( \nu \) field”. Over the part of worldsheet (curve) where \( \nu = 0 \), the MSP field is a pure field taking values in \( K_{\mathbb{P}^4} \), and, over \( \nu = \infty \), the MSP field is a pure field taking values in \( [\mathbb{C}^5/\mathbb{Z}_5] \).

In a nutshell, by promoting the phase parameter \( r - i\theta \) into a field \( \nu \) on worldsheet (curve), we transform Witten’s family of theories parametrized by \( r - i\theta \) into a single new field theory. Also, an advantage of MSP moduli is it works for higher loop (in physics terms) or for higher genus (in math terms), while GLSM in physics literature does not\(^7\).

---

\(^5\) we use notations in [Wi2 Sect 15.2.2];
\(^6\) in the MSP cage (proper) integral does not diverge because the cage is proper and separated by [CLLL];
\(^7\) GLSM only treat genus zero worldsheet
We recall the question raised by Witten [GLSM, Page 28]: “Are Calabi-Yau and Landau-Ginzburg separated by a true phase transition, at or near \(r = 0\)? There is no reason that the answer to this question has to be universal, that is, independent of the path one follows in interpolating from Calabi-Yau to Landau-Ginzburg in a multiparameter space of not necessarily conformally invariant theories. Along a suitable path, there may well be a sharply defined phase transition, while along another path there might not be one. This seems quite plausible.” Though our \(\nu\) field does not have a definite value of which we can vary “the theories”, \(\nu\) allows us to introduce a \(T = \mathbb{C}^*\) action, and by localizing to \(T\) fixed locus, we obtain (possibly as referred to by Witten) a precise phase transition between the quintic Calabi-Yau threefold and the LG of the Fermat quintic. Furthermore, such phase transitions are multi-fold: for each \(d = (d_0, d_\infty)\) that provides a vanishing, we get an interpolation. And when we vary \(d\), we obtain a class of such interpolations. Thus in physics terms, we may say that each \((d_0, d_\infty) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}\), being a mode describing how worldsheet is wrapped to Kähler moduli spaces, provides a path linking CY point to LG point in the phase spaces.

11.2. Compare with B-model. For the quintic Calabi-Yau threefold, the modularity of the generating function \(F^A_g\) is suggested by physicists, but is a mystery in mathematics. In physics literature as [BCOV, HKQ], the modularity of \(F^B_g\) and the mirror symmetry \(F^B_g = F^A_g\) When \(g > 0\), the holomorphic anomaly equation determines \(F^B_g(q)\) up to \(3g - 2\) unknowns. The degree zero Gromov-Witten invariant \(N_{g,d=0}\) is known, so we are left with \(3g - 3\) unknowns. The boundary conditions at the orbifold point (which corresponds to Landau-Ginzburg theory of the Fermat quintic polynomial in five variables) impose \(\left\lceil\frac{3}{2}(g - 1)\right\rceil\) constraints on the \(3g - 3\) unknowns, whereas the “gap condition” at the conifold point imposes \(2g - 2\) constraints on the \(3g - 3\) unknowns. In summary, the holomorphic anomaly equation and the boundary conditions determine \(F^A_g(Q)\) up to \(\left\lfloor\frac{2}{5}(g - 1)\right\rfloor\) unknowns. Coincidently, granting Conjecture A, the number of initial data needed to determine \(F_g\) are the FJRW invariants \(\Theta_{g,k}\) up to \(2g - 2 \equiv k(5)\). Thus \(\left\lfloor\frac{2(g-1)}{5}\right\rfloor + 1\) many FJRW invariants are needed to determine \(N_{g,d}\) via MSP moduli, provided all lower genus invariants \(\{F_{g'} : g' < g\}\) are known. We hope there is more geometric explanation than viewing it just as a coincidence.

References

[ACV] D. Abramovich, A. Corti, and A. Vistoli, “Twisted bundles and admissible covers,” Special issue in honor of Steven L. Kleiman. Comm. Algebra 31, no. 8, 3547–3618 (2003).

[AF] D. Abramovich and B. Fantechi, “Orbifold techniques in degeneration formulas,” preprint, math.AG.

[AGV] D. Abramovich, T. Graber, and A. Vistoli, “Gromov-Witten theory of Deligne-Mumford stacks,” Amer. J. Math. 130, no. 5, 1337–1398 (2008).

[AJ] D. Abramovich, T. J. Jarvis, “Moduli of twisted spin curves,” Proc. Amer. Math. Soc. 131, no. 3, 685–699 (2003).

[BF] K. Behrend and B. Fantechi, “The intrinsic normal cone,” Invent. Math. 128, no. 1, 45–88 (1997).

[BCOV] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Holomorphic Anomalies in Topological Field Theories,” Nucl.Phys. B 405 279–304 (1993); “Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes,” Comm. Math. Phys. Volume 165, no. 2, 311–427 (1994).

\(^8\) another mystery in mathematics
[COGP] P. Candelas, X. dela Ossa, P. Green, and L. Parkes, “A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory,” Nucl. Phys. B 359 21–74 (1991).

[Ce] S. Cecotti, “N=2 Landau-Ginzburg Vs. Calabi-Yau models: Non-Perturbative Aspects,” Int. J. Mod. Phys. A6 (1991) 1749.

[CL1] H.-L. Chang and J. Li, “Gromov-Witten invariants of stable maps with fields,” Int. Math. Res. Not. 2012, 18, 4163–4217 (2012).

[CL2] H.-L. Chang and J. Li, “A vanishing for localizing MSP moduli of quintic,” in preparation

[CLLL2] H.-L. Chang, J. Li, W.-P. Li, and C.-C. Melissa Liu, “Mixed-Spin-P fields of Fermat quintic polynomials,” math.AG. arXiv:1505.07532

[CLL2] H.-L. Chang, J. Li, W.-P. Li, C.-C. Melissa Liu, ‘An effective theory of GW and FJRW invariants of quintics Calabi-Yau manifolds,” math.AG. arXiv:1603.06184

[twFJRW] H.-L. Chang, J. Li, W.-P. Li, C.-C. Melissa Liu, “Dual twisted FJRW invariants of quintic singularity via floating MSP fields,” in preparation.

[ChK] J.-W. Choi and Y.-H. Kiem, “Landau-Ginzburg/Calabi-Yau correspondence via quasi-maps, I,” arXiv:1103.0833

[Ch] A. Chiodo, “Towards an enumerative geometry of the moduli space of twisted curves and r-th roots, Compos. Math. 144, no. 6, 1461–1496 (2008).

[CR] A. Chiodo and Y.B Ruan, “Landau-Ginzburg/Calabi-Yau correspondence for quintic three-folds via symplectic transformations,” Invent. Math. 182, no. 1, 117–165 (2010).

[FK1] Ionut Ciocan-Fontanine, Bumsig Kim, “Moduli stacks of stable toric quasimaps,” Advances in Mathematics 225 (2010), 3022–3051.

[FK2] Ionut Ciocan-Fontanine, Bumsig Kim, “Wall-crossing in genus zero quasimap theory and mirror maps,” Algebraic Geometry 4 (2014) 400–448.

[Cad] C. Cadman, “Using stacks to impose tangency conditions on curves,” Amer. J. Math. 129, no. 2, 405–427 (2007).

[Ch] A. Chiodo, “The Witten top Chern class via K-theory,” J. Algebraic Geom. 15, no. 4, 681–707 (2006).

[CZ] A. Chiodo and D. Zvonkine, “Twisted r-spin potential and Givental’s quantization,” Advances in Theoretical and Mathematical Physics 13, no. 5, 1335–1369 (2009).

[FJR1] H. Fan, T. J. Jarvis, Y. Ruan, “The Witten equation, mirror symmetry, and quantum singularity theory,” Ann. of Math (2) 178, no. 1, 1–106 (2013).

[FJR2] H. Fan, T. J. Jarvis and Y. Ruan, “The Witten equation and its virtual fundamental cycle,” math.AG. arXiv:0712.4025

[FJR3] H. Fan, T. J. Jarvis and Y. Ruan, “A Mathematical Theory of the Gauged Linear Sigma Model,” math.AG. arXiv:1506.02109

[FSZ] C. Faber, S. Shadrin and D. Zvonkine, “Tautological relations and the r-spin Witten conjecture,” Annales Scientifiques de l’École Normale Supérieure. Quatrième Série. 43, no. 4 (2010), 621–658.

[Ga] A. Gathmann, “Absolute and relative Gromov-Witten invariants of very ample hypersurfaces,” Duke, 115, no. 2, 171–203 (2002)

[Gi] A. Givental, “Equivariant Gromov-Witten invariants,” Internat. Math. Res. Notices 1996, no. 13, 613–663 (1996).

[GP] T. Graber, R. Pandharipande, “Localization of virtual classes,” Invent. Math. 13, no. 2, 487-518 (1999).

[GS] J. Guffin and E. Sharpe, “A-twisted Landau-Ginzburg models,” hep-th/arXiv:0801.3836

[HKQ] M.X. Huang, A. Klemm, and S. Quackenbush, “Topological String Theory on Compact Calabi-Yau: Modularity and Boundary Conditions,” Lecture Notes in Phys. 757, 45-102 (2009).

[Huy] D. Huybrechts, Fourier-Mukai transforms in algebraic geometry. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford (2006)
[JK] T. Jarvis and T. Kimura, “Orbifold quantum cohomology of the classifying space of a finite group,” Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math. 310, 123–134 Amer. Math. Soc., Providence, RI, (2002).

[KKP] B. Kim, A. Kresch, and T. Pantev, “Functoriality in intersection theory and a conjecture of Cox, Katz, and Lee,” J. Pure Appl. Algebra 179, no. 1-2, 127–136 (2003).

[KKV] S. Katz, A. Klemm, and C. Vafa, “M-theory, topological strings and spinning black holes,” Adv. Theor. Math. Phys. 3 (1999), no. 5, 1445–1537.

[KL] Y.H. Kim and J. Li, “Localized virtual cycle by cosections,” J. Amer. Math. Soc. 26, no. 4, 1025–1050 (2013).

[Kon1] M. Kontsevich, “Intersection theory on the moduli space of curves and the matrix Airy function,” Comm. Math. Phys. 147 (1992), no. 1, 1–23.

[Ko] M. Kontsevich, “Enumeration of rational curves via torus actions,” The moduli space of curves. (Texel Island, 1994), 335-368, Progr. Math. 129, Birkhäuser Boston, Boston, MA, (1995).

[Kr2] A. Kresch, “Cycle groups for Artin stacks,” Invent. Math. 138, no. 3, 495-536 (1999)

[LM] G. Laumon and L. Moret-Bailly, Champs algébriques. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 39, Berlin: Springer-Verlag, (2000)

[LRZ] Liu, Si-Qi, Ruan, Yongbin, Zhang, Youjin “BCFG Drinfeld-Sokolov hierarchies and FJRW-theory,” Invent. Math. 201 (2015), no. 2, 711772.

[KLi] K. Li, “Topological gravity with minimal matter,” Caltech preprint CALT-68-1662; “Recursion relations in topological gravity with minimal matter,” Caltech preprint CALT-68-1670.

[LT] J. Li and G. Tian, “Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties,” J. Amer. Math. Soc. 11, no. 1, 119-174 (1998)

[LZ] J. Li and A. Zinger, “On the Genus-One Gromov-Witten Invariants of Complete Intersections,” J. of Differential Geom. 82 (2009), no. 3, 641-690

[LLY] B. Lian, K.F. Liu and S.T. Yau, “Mirror principle. I,”Asian J. Math. 1, no. 4, 729-763 (1997)

[MOP] A. Marian, D. Oprea and R. Pandharipande, “The moduli space of stable quotients,” math.AG. arXiv:0904.2992

[MP] D. Maulik, and R. Pandharipande, “A topological view of Gromov-Witten theory,” Topology 45, no. 5, 887-918 (2006)

[Mo] T. Mochizuki, “The virtual class of the moduli stack of stable r-spin curves,” Comm. Math. Phys. 264, no. 1, 1-40 (2006)

[PV] A. Polishchuk and A. Vaintrob, “Algebraic construction of Witten’s top Chern class,” Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), Contemp. Math. 276, 229-249, Amer. Math. Soc., Providence, RI, (2001)

[RR] D. Ross and Y. Ruan, “Wall-crossing in genus zero Landau-Ginzburg theory,” arXiv:1402.6688

[Ito] K. Ito, “Topologival phase of $N=2$ superconformal field theory and topological Landau Ginzburg field theory,” Harvard preprint (May, 1990) Physics Letters B Volume 250, number 1,2, 1 November 1990, Pages 91–95.

[Vafa] C. Vafa, “Topological Landau-Ginzburg models,” Modern Physics Letters A, Vol. 6, No. 4(1991) 337-346.

[Wi] E. Witten. “Two-dimensional gravity and intersection theory on the moduli space,” Surveys in Diff. Geom. 1 (1991), 243-310.

[Wi1] E. Witten. “The N matrix model and gauged WZW models,” Nuclear Physics B Volume 371, Issues 1-2, 2 March 1992, 191–245.

[Wi2] E. Witten, “Mirror manifolds and topological field theory,” in Essays on mirror manifolds (S.-T. Yau, ed.), Internat. Press, Hong Kong, 1992, pp. 121–160.

[GLSM] E. Witten, “Phases of $N=2$ theories in two dimensions,” Nuclear Physics B 403, no. 1-2, 159-222 (1993).

[Wi3] E. Witten, “Algebraic geometry associated with matrix models of two dimensional gravity,” Topological models in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, 1993, 235–269.
[YY] S. Yamaguchi and S.-T. Yau, “Topological string partition functions as polynomials,” JHEP 0407 (2004), 047.

[Zi] A. Zinger, “Standard versus reduced genus-one Gromov-Witten invariants,” Geom. Topol. 12, no. 2, 1203–1241, (2008).

[Zi2] A. Zinger, “The reduced genus 1 Gromov-Witten invariants of Calabi-Yau hypersurfaces,” J. Amer. Math. Soc. 22, no. 3, 691–737 (2009).

HUAI-LIANG CHANG, DEPARTMENT OF MATHEMATICS, HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, HONG KONG
E-mail address: mahlchang@ust.hk

JUN LI, SHANGHAI CENTER FOR MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, CHINA;
DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, USA
E-mail address: jli@math.stanford.edu

WEI-PING LI, DEPARTMENT OF MATHEMATICS, HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, HONG KONG
E-mail address: mawpli@ust.hk

CHIU-CHU MELISSA LIU, DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, USA
E-mail address: ccliu@math.columbia.edu