2-VERMA MODULES AND THE KHOVANOV-ROZANSKY LINK HOMOLOGIES

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ABSTRACT. For a particular choice of parabolic subalgebra of $\mathfrak{gl}_{2n}$ we construct a parabolic 2-Verma module and use it to give a construction of Khovanov-Rozansky’s HOMFLY-PT and $\mathfrak{gl}_N$ link homologies. We use our version of these homologies to prove that Rasmussen’s spectral sequence from the HOMFLY-PT-link homology to the $\mathfrak{gl}_N$-link homology converges at the second page, proving a conjecture of Dunfield, Gukov and Rasmussen.

CONTENTS

1. Introduction 1
2. Parabolic Verma modules and link invariants 4
  2.1. Link invariants 4
  2.2. Parabolic Verma modules 8
  2.3. Parabolic Verma modules and link invariants 9
3. Parabolic 2-Verma modules 10
  3.1. Universal 2-Verma module of $U_q(\mathfrak{gl}_{2n})$ 11
  3.2. A parabolic 2-Verma module I: the case of $\mathfrak{p} \subseteq U_q(\mathfrak{gl}_{2n})$ 24
  3.3. Recovering the finite dimensional irreducible $V((N)^n,(0)^n)$ 30
  3.4. A parabolic 2-Verma module II: the case of $\bar{\mathfrak{p}} \subseteq U_q(\mathfrak{gl}_{2n-1})$ 31
4. Link homology 32
  4.1. Braiding 32
  4.2. Invariance under the Markov moves 33
  4.3. $H(b)$ is isomorphic to the Khovanov-Rozansky HOMFLY-PT link homology 36
  4.4. Khovanov-Rozansky’s $\mathfrak{gl}_N$-link homologies 37
  4.5. Reduced homologies 37
  4.6. HOMFLY-PT to $\mathfrak{gl}_N$ spectral sequences converge at the second page 37
References 41

1. INTRODUCTION

One of the most celebrated homology theories of knots and links in 3-space is Khovanov and Rozansky’s $\mathfrak{gl}_N$-link homology [12] categorifying the $\mathfrak{gl}_N$-link invariant. Soon after the appearance of [12] the existence of a triply-graded link homology categorifying the HOMFLY-PT polynomial was predicted in [5] by Dunfield, Gukov and Rasmussen, who made various conjectures about the structure of such homology theory. One of these conjectures was the existence of
a family of differentials \( d_N \), for \( N \) an integer, acting on the triply-graded homology and whose homology for each link \( L \) coincides with Khovanov and Rozansky’s \( \mathfrak{gl}_N \)-link homology when \( N > 0 \).

A rigorous construction of a triply-graded link homology categorifying the HOMFLY-PT polynomial was given by Khovanov and Rozansky in [13] (see also [9] for a construction using Hochschild homology of Soergel bimodules). The structure of this link homology was studied by Rasmussen in [24]. Rasmussen defined a family of differentials on the KR HOMFLY-PT link homology and showed that, for each \( N > 0 \), these differentials give rise to a spectral sequence starting at the KR HOMFLY-PT homology and converging to the \( \mathfrak{gl}_N \) homology.

In this paper we first give a new interpretation of the HOMFLY-PT polynomial in terms of parabolic Verma modules. Then, using the categorification of these Verma modules from [21], we lift this procedure, yielding a new construction of KR HOMFLY-PT link homology. Using our version of HOMFLY-PT homology, we prove that Rasmussen’s spectral sequence converges at the second page, which implies the Dunfield–Gukov–Rasmussen conjecture for \( N > 0 \).

**Summary of the paper and description of the main results.** In the search for a construction of the HOMFLY-PT polynomial based on representation theory, Queffelec and Sartori [23] provided an algebraic gadget, called the double Schur algebra, which accommodates the Hecke algebra and the Ocneanu trace under the same roof. The double Schur algebra \( \mathcal{S}_{q, \beta}(\ell, k) \) contains two copies of the Schur algebra (the cases \( \ell = 0 \) and \( k = 0 \)), and is defined as a quotient of idempotented quantum \( \mathfrak{gl}_{\ell+k} \), whose weight lattice has been shifted by a formal parameter. For a link \( L \) presented as the closure of a braid \( b \) with \( n \) strands, Queffelec and Sartori constructed an element in the double Schur algebra. This element is a multiple of a certain idempotent and its coefficient coincides with the HOMFLY-PT polynomial of \( L \).

The construction in [23] finds a natural place in the terms of representations of the double Schur algebra. In this paper we extend the notion of Weyl modules to double Schur algebras and translate Queffelec and Sartori’s results to this context. Concretely we show that with the choice of highest weight \( \beta = (\beta, \ldots, \beta, 0, \ldots, 0) \) (there are \( n \) \( \beta \)’s and \( n \) 0’s in \( \beta \)), the HOMFLY-PT polynomial of \( L \) can be obtained from the Weyl module \( W(\beta) \) as a map which is a multiple of the identity, the coefficient being the HOMFLY-PT polynomial of \( L \).

As representations of \( \mathcal{U}_q(\mathfrak{gl}_{\ell+k}) \), Weyl modules \( W(\mu) \) for any highest weight \( \mu \) are isomorphic to parabolic Verma modules \( M^p(\mu) \) for a certain parabolic subalgebra \( p \). The previous paragraph can then be reformulated entirely in terms of parabolic Verma modules.

**Theorem A** (Theorem 2.8). For a link presented as the closure of a braid \( b \) with \( n \) strands, the construction above defines an element \( P_p(b) \in \text{End}_{\mathcal{U}_q(\mathfrak{gl}_{2n})}(M_q(\beta)) \) which is a link invariant. It is a multiple of the identity whose coefficient equals the HOMFLY-PT polynomial of the closure of \( b \).

We construct a version of KLR algebras associated with the pair \( (p, \mathfrak{gl}_{2n}) \) that combines the original KLR algebra for \( \mathfrak{gl}_{2n} \) with the superalgebra \( A_n \) introduced in [19] (see also [20] for a further study of \( A_{nm} \)). This new algebra, denoted \( R_p \), is a particular case of a general family of

\[^1\text{We allow ourselves to harmlessly abuse notation here, which will payoff further ahead.}\]
We pass to a suitable cyclotomic quotient $R_p^\beta$ and make use of the usual procedure to construct functors $E_i^\beta, F_i^\beta$ (for $\alpha_i$ a simple root of $\mathfrak{gl}_{2n}$) from functors of restriction and extension for maps adding strands to diagrams. We prove that $U_q(\mathfrak{gl}_{2n})$ acts on the category of (cone bounded, locally finite dimensional) supermodules for $R_p^\beta$, whose (topological) Grothendieck group is therefore isomorphic to the parabolic Verma module $M^p(\mu)$. Since we work with a bigraded supercategory the Grothendieck group is a module over $\mathbb{Z}[[q, \lambda]]/(\pi^2 - 1)$.

**Theorem B** (Proposition 3.27 and Theorem 3.28). Functors $F_i^\beta$ and $E_i^\beta$ are exact and induce a $U_q(\mathfrak{gl}_{2n})$-action on the Grothendieck group of $R_p^\beta$ which becomes isomorphic to $M^p(\beta)$ after specializing $\pi = -1$.

We upgrade this data into a 2-category $\mathcal{M}^p(\beta)$, which we call a 2-Verma module.

The Rickard complex associated to a braid acts on the homotopy category of complexes of the $\text{Hom}$-categories of $\mathcal{M}^p(\beta)$ (this is a lift of the usual braiding induced by the embedding of the Hecke algebra into a Schur algebra that in turn embeds canonically in a double Schur algebra). For a closure of a braid $b$ this procedure gives a chain complex $C(b)$.

**Theorem C** (Proposition 4.2, Corollary 4.4, Theorem 4.5). The homotopy type of $C(\text{cl}(b))$ is invariant under the Markov moves. Its homology groups are triply-graded link invariants and its bigraded Euler characteristic is the HOMFLY-PT polynomial of the closure of $b$. Moreover, $H(b)$ is isomorphic to Khovanov and Rozansky HOMFLY-PT homology $\text{HKR}(b)$.

Introducing a differential $d_N$ on $R_p^\beta$ turns it into a dg-algebra. We can then recover the usual cyclotomic quotient $R_{\mathfrak{gl}_{2n}}^\Lambda$ of the KLR algebra associated with $\mathfrak{gl}_{2n}$, in the sense that the former is formal and quasi-isomorphic to the latter. The work of Mackaay and Yonezawa in [17] implies that replacing $R_p^\beta$ by $R_{\mathfrak{gl}_{2n}}^\Lambda$ in the construction above produces Khovanov and Rozansky’s $\mathfrak{gl}_N$-link homology $\text{HKR}_{\mathfrak{gl}_N}(\text{cl}(b))$ for the closure of $b$.

The differential $d_N$ descends to $\text{HKR}(b)$ and engenders a spectral sequence starting at $\text{HKR}(b)$ and converging to $\text{HKR}_N(b)$, which coincides with Rasmussen’s spectral sequence in [24]. As a matter of fact, both $R_p^\beta$ and $R_{\mathfrak{gl}_{2n}}^\Lambda$ admit similar dg-enhancements, denoted $(R_b, d_\beta)$ and $(R_b, d_\Lambda)$ for a KLR-like algebra $R_b$ that can be thought as associated to the Borel subalgebra of $\mathfrak{gl}_{2n}$. Moreover, the Rickard complex lifts to this setting. Using this extra structure, one can prove the Dunfield–Gukov–Rasmussen conjecture for $N > 0$:

**Theorem D** (Theorem 4.9). For each link $L$ and for each $N > 0$,

$$\text{HKR}_N(L) \cong H(\text{HKR}(L), d_N).$$

In particular this implies the following:

**Corollary A** (Corollary 4.18). Rasmussen’s spectral sequence converges at the second page.

All the results above have an analogue for the case of reduced homologies.
Plan of the paper. In Section 2 we describe the double Schur algebra following Queffelec–Sartori’s and how to use it to define the HOMFLY-PT polynomial. We interpret their construction in the context of parabolic Verma modules. Section 3 consists of higher representation theory: we introduce parabolic 2-Vermas. We define the superalgebras $R_b$ and $R_{p}^{\beta}$ and develop a categorification of the parabolic Verma modules. These are the ones that will be used to construct link homology theories categorifying the results of Section 2. We introduce a differential $d_{N}$ on $R_{p}^{\beta}$ yielding a dg-algebra quasi-isomorphic to the cyclotomic KLR algebra $R_{\text{gl}_{2n}}^{\Lambda}$. Finally, Section 4 contains the topology. We introduce the braiding on the homotopy category of complexes of the Hom-categories of the 2-Vermas through Rickard complexes, and prove invariance under the Markov moves. We also prove our link homology groups are isomorphic to Khovanov and Rozansky’s. We also prove our link homology groups are isomorphic to Khovanov and Rozansky’s. We also prove our link homology groups are isomorphic to Khovanov and Rozansky’s. We also prove our link homology groups are isomorphic to Khovanov and Rozansky’s.

2. PARABOLIC VERMA MODULES AND LINK INVARIANTS

2.1. Link invariants. In [23] Queffelec and Sartori proposed an algebraic method to construct the HOMFLY-PT and the Alexander polynomials of links in 3-space. They defined a generalization of the idempotented $q$-Schur algebra called the \textit{doubled Schur algebra}. In this section we will briefly recall the basics of the construction in [23] and explain how it fits with parabolic Verma modules.

2.1.1. The doubled Schur algebra. In the following we let $\beta$ be a formal parameter, we write $\lambda$ for $q^{\beta}$, and work over the ring of Laurent polynomials in $\lambda$ with coefficients in $\mathbb{Q}(q)$. We denote by $\Lambda^{\beta}_{\ell,k}$ the set of sequences $(\mu_{-\ell+1}, \ldots, \mu_{0}, \ldots, \mu_{k}) \in (\beta - \mathbb{N}_{0})^{\ell} \times \mathbb{N}_{0}^{k}$ for $\ell \geq 0$ and $k \geq 0$.

Remark 2.1. We follow this convention, slightly different from [23], because we want to relate it later with highest weight, rather than lowest weight, parabolic Verma modules, and it will allow to keep some notations simple.

Let $I = \{-\ell + 1, \ldots, 0, \ldots, k - 1\}$. Let $\alpha_{i} = (0, \ldots, 0, 1, -1, 0, \ldots, 0) \in \mathbb{Z}^{\ell+k}$, the entry 1 being at position $i \in I$. For $\mu_{i} \in \mathbb{Z} \subset \mathbb{Z} + \beta$ let

$$[\mu_{i}]_{q} = \begin{cases} \frac{q^{\mu_{i}} - q^{-\mu_{i}}}{q - q^{-1}}, & \text{if } \mu_{i} \in \mathbb{Z}, \\ \frac{\lambda q^{\mu_{i}-\beta} - \lambda^{-1}q^{\beta-\mu_{i}}}{q - q^{-1}}, & \text{if } \mu_{i} \in \mathbb{Z} + \beta, \end{cases}$$

Acknowledgments. The authors would like to thank Paul Wedrich for helpful discussions, for suggesting us to use our machinery to study the Rasmussen spectral sequences and for valuable comments on a preliminary version of this paper. We also thank Jonathan Grant for helpful discussions and comments on a preliminary version of this paper. G.N. is a Research Fellow of the Fonds de la Recherche Scientifique - FNRS, under Grant no. 1.A310.16. P.V. was supported by the Fonds de la Recherche Scientifique - FNRS under Grant no. J.0135.16.
be the (shifted) quantum number.

**Definition 2.2.** The doubled Schur algebra $\hat{S}_{q,\beta}(\ell, k)$ is the $\mathbb{Q}[q^{\pm 1}]$-linear category defined by the following data:

- Objects: $1_{\mu}$ for $\mu \in \Lambda_{\ell,k}^{\beta}$, together with a zero object.
- Morphisms: generated by morphisms

$$E_i 1_{\mu} = 1_{\mu + \alpha_i}, \quad E_i \in \text{Hom}(1_{\mu}, 1_{\mu + \alpha_i}),$$  
$$F_i 1_{\mu} = 1_{\mu - \alpha_i}, \quad F_i \in \text{Hom}(1_{\mu}, 1_{\mu - \alpha_i}),$$

for $i \in I$, together with the identity morphism of $1_{\mu}$ (denoted by the same symbol). The morphisms are subject to the relations below:

1. $1_{\mu} 1_{\nu} = \delta_{\mu,\nu} 1_{\nu}$,
2. $(E_i F_j - F_j E_i) 1_{\mu} = \delta_{i,j} [\mu_i - \mu_{i+1}]_q 1_{\mu}$,
3. $E_i E_j 1_{\mu} = E_j E_i 1_{\mu}$ and $F_i F_j 1_{\mu} = F_j F_i 1_{\mu}$ if $|i - j| > 1$, and

$$E_i^2 E_{i \pm 1} 1_{\mu} - [2]_q E_i E_{i \pm 1} E_i 1_{\mu} + E_{i \pm 1} E_i 1_{\mu} = 0,$$
$$F_i^2 F_{i \pm 1} 1_{\mu} - [2]_q F_i F_{i \pm 1} F_i 1_{\mu} + F_{i \pm 1} F_i 1_{\mu} = 0.$$

4. $1_{\nu} \in \text{End}(1_{\nu}, 1_{\nu})$ is zero if $\nu \notin \Lambda_{\ell,k}^{\beta}$.

For $\ell = 0$ we recover the idempotented $q$-Schur algebra $\hat{S}_q(k)_d$, with $d = \mu_1 + \cdots + \mu_k$, and for $k = 0$ we recover $\hat{S}_q(\ell)_d$, with $d = \ell \beta - (\mu_{-\ell+1} + \cdots + \mu_0)$. There are canonical inclusions

$$\hat{S}_q(k)_d \hookrightarrow \hat{S}_{q,\beta}(\ell, k)$$

sending

$$1_{\mu_1, \ldots, \mu_k} \mapsto 1_{0, \ldots, 0, \mu_1, \ldots, \mu_k}, \quad F_i \mapsto F_{i + \ell + 1}, \quad E_i \mapsto E_{i + \ell + 1},$$

and

$$\hat{S}_q(\ell)_d \hookrightarrow \hat{S}_{q,\beta}(\ell, k)$$

sending

$$1_{\mu_1, \ldots, \mu_0, 0, \ldots, 0} \mapsto 1_{\mu_{-\ell+1}, \ldots, \mu_0, 0, \ldots, 0}, \quad F_i \mapsto F_i, \quad E_i \mapsto E_i.$$

### 2.1.2. Link invariants from the doubled Schur algebra. In [23] it was given a diagrammatic presentation for the doubled Schur algebra in terms of webs. These are generated by the ladder operators below.

$$1_{\mu} \mapsto \begin{array}{cccccc}
\downarrow & \downarrow & \downarrow & \downarrow & \uparrow & \uparrow \\
\mu_{-\ell+1} & \mu_{-2} & \mu_{-1} & \mu_0 & \mu_1 & \mu_k
\end{array}$$
1_{\mu-\alpha_i} F_i 1_{\mu} \mapsto \begin{cases} 
\downarrow \cdots \downarrow \uparrow \cdots \uparrow & \text{if } i > 0, \\
\downarrow \cdots \downarrow \uparrow \cdots \uparrow & \text{if } i = 0, \\
\downarrow \cdots \downarrow \uparrow \cdots \uparrow & \text{if } i < 0, 
\end{cases}

1_{\mu+\alpha_i} E_i 1_{\mu} \mapsto \begin{cases} 
\downarrow \cdots \downarrow \uparrow \cdots \uparrow & \text{if } i > 0, \\
\downarrow \cdots \downarrow \uparrow \cdots \uparrow & \text{if } i = 0, \\
\downarrow \cdots \downarrow \uparrow \cdots \uparrow & \text{if } i < 0. 
\end{cases}

Note that edges labeled with $\mathbb{N}_0$ are oriented upwards while edges labeled with $\beta-\mathbb{N}_0$ are oriented downwards. Multiplication corresponds to concatenation of diagrams, and in our conventions $ab$ consists of placing the diagram of $a$ on the top of the one for $b$ if the labels match, being zero otherwise. On the $\mathbb{Q}(q)[\lambda^{\pm 1}]$-vector space spanned by all such webs we impose the relations of the doubled Schur algebra from Definition 2.2.

We consider links presented in the form of closures of braids. To start with, let $b$ be a braid diagram in $n$ strands. For such a diagram we assign an element of $\hat{S}_{q,\beta}(n)_n = \hat{S}_{q,\beta}(0, n)$ using a well known rule originally due to Lusztig [14, Definition 5.2.1]. This extends immediately to an element of $\hat{S}_{q,\beta}(n, n)$ from the embedding $\hat{S}_{q,\beta}(n)_n \hookrightarrow \hat{S}_{q,\beta}(n, n)$. Denote by $((\beta - 1)^n, (1)^n)$ the label consisting of $n$ entries equal to $\beta - 1$ and $n$ entries equal to 1. For $\sigma_i$ (resp. $\sigma_i^{-1}$) a positive (resp. negative) crossing between the $i$th and the $(i + 1)$th strands counting from the left we have

$\sigma_i \mapsto \begin{array}{c}
\uparrow \cdots \uparrow \uparrow \cdots \uparrow \\
\downarrow \cdots \downarrow \end{array} \iff q^{-1} 1_{((\beta - 1)^n, (1)^n)} - F_i E_i 1_{((\beta - 1)^n, (1)^n)},$

$\sigma_i^{-1} \mapsto \begin{array}{c}
\uparrow \cdots \uparrow \uparrow \cdots \uparrow \\
\downarrow \cdots \downarrow \end{array} \iff -F_i E_i 1_{((\beta - 1)^n, (1)^n)} + q 1_{((\beta - 1)^n, (1)^n)}.$
We consider closing braids on the left as in the diagram below.

\[ \text{cl}(b) = \] 

To \( \text{cl}(b) \) we assign an element of \( \hat{S}_{q,\beta}(n, n) \) obtained by adding cups and the bottom and caps at the top using the following pattern (dotted edges correspond to labels 0 and \( \beta - 0 \), and solid lines to labels 1 and \( \beta - 1 \)):

\[
\text{(1)}
\]

and similar for the top of \( \text{cl}(b) \).

As explained in [23] this procedure gives an element \( P(b) \in S_{q,\beta}(n, n) \) which is an endomorphism of \( 1_{(\beta)^n,(0)^n} \), which in turn implies that \( P(b) \in \mathbb{Q}(q)[\lambda^{\pm 1}] \). One of the main results in [23] is the following.

**Theorem 2.3** (Theorems 3.1 and 3.8 in [23]). For a braid \( b \) the element \( P(b) \) is a framed link invariant which equals the HOMFLY-PT polynomial of the closure of \( b \).

The proof of Theorem 2.3 goes by first verifying that \( P(b) \) is a braid invariant, and then checking invariance under the Markov moves. In the process of showing that it equals the HOMFLY-PT polynomials, it was shown that it gives the value \( \frac{\lambda^{-1}}{q^{-1}} \) for the unknot, and it satisfies

\[
P\left( \begin{array}{c}
\includegraphics[scale=0.5]{unknot1}\end{array} \right) = \lambda P\left( \begin{array}{c}
\includegraphics[scale=0.5]{unknot2}\end{array} \right),
\]

\[
P\left( \begin{array}{c}
\includegraphics[scale=0.5]{unknot3}\end{array} \right) = \lambda^{-1} P\left( \begin{array}{c}
\includegraphics[scale=0.5]{unknot4}\end{array} \right),
\]

and the skein relation

\[
P\left( \begin{array}{c}
\includegraphics[scale=0.5]{unknot5}\end{array} \right) - P\left( \begin{array}{c}
\includegraphics[scale=0.5]{unknot6}\end{array} \right) = (q^{-1} - q)P\left( \begin{array}{c}
\includegraphics[scale=0.5]{unknot7}\end{array} \right).
\]

Multiplying \( P(b) \) by \( \lambda^{w(\text{cl}b)} \), where \( w(\text{cl}b) \) is the writhe of \( \text{cl}(b) \), results in the usual, framing independent, HOMFLY-PT polynomial.

By the usual specializations of \( \lambda \) we recover the \( \mathfrak{gl}_N \) (\( \lambda = q^N \)) and the Alexander polynomials (\( \lambda = 1 \)) of the closure of \( b \). Note that for the latter one needs to cut open one of the strands to avoid getting the value zero associated to the unknot, and therefore to any link, as explained in [23, §4] (see the discussion on normalized invariants in §2.3.1 below for further details).
2.1.3. Weyl modules. Let $\Lambda_+ \subset \Lambda_{q,k}^\ell$ be the subset consisting of sequences satisfying
\[
\begin{aligned}
\mu_i - \mu_{i+1} &\geq 0 & \text{for } i \neq 0, \\
\beta - (\mu_0 - \mu_1) &\geq 0 & \text{for } i = 0,
\end{aligned}
\]
Introduce a total order on $\Lambda_{q,k}^\ell$ by declaring $\mu > \nu$ if $\mu - \nu \in \Lambda_+$.

**Definition 2.4.** For any $\mu \in \Lambda_+$ we define
\[
W(\mu) = \hat{S}_{q,\beta}(\ell,k)1_\mu/\{\nu > \mu\}.
\]
Here $\{\nu > \mu\}$ is the ideal generated by all elements of the form $1_\nu x 1_\mu$, for some $x \in \hat{S}_{q,\beta}(\ell,k)$ and $\nu > \mu$.

For $\ell = 0$, we recover the well-known Weyl modules for the $q$-Schur algebra $\hat{S}_q(k)_{\mu_1 + \cdots + \mu_k}$. As in the case of $\ell = 0$, it is also true that $U_q(\mathfrak{gl}_{k+\ell})$ acts on $W(\mu)$: for $F_i \in U_q(\mathfrak{gl}_{k+\ell})$ and $1_\nu x 1_\mu \in W(\mu)$ we put $F_i 1_\nu x 1_\mu = 1_{\nu - \alpha_i} F_i x 1_\mu$, and similarly for $E_i \in U_q(\mathfrak{gl}_{k+\ell})$. Note that the Chevalley generators of $U_q(\mathfrak{gl}_{k+\ell})$ are indexed from $\{-\ell + 1, \ldots, k - 1\}$, which is the set $I$ introduced in the definition of $\hat{S}_{q,\beta}(\ell,k)$.

From §2.1.2 it is clear that all weights occurring in $P(b) \in \hat{S}_{q,\beta}(n,n)$ are smaller w.r.t. $>$ than $((\beta)^n, (0)^n)$, and thus $P(b)$ is sent to the same word in $E$'s and $F$'s under the quotient map $\hat{S}_{q,\beta}(n,n) \to W((\beta)^n, (0)^n)$.

**Proposition 2.5.** The element $\lambda^{u(\text{cl}(b))} P(b)$ acts on $W((\beta)^n, (0)^n)$ as an endomorphism of the highest weight vector which is multiplication by the HOMFLY-PT polynomial of the closure of $b$.

2.2. Parabolic Verma modules. Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra. Let also $\mathfrak{l}$ and $\mathfrak{n}$ be respectively the Levi factor and the nilpotent radical of $\mathfrak{p}$. Let $V$ be an irreducible representation of $U_q(\mathfrak{l})$. It can be extended to $U_q(\mathfrak{p})$ by setting $U_q(\mathfrak{n})V = 0$.

**Definition 2.6.** The generalized Verma module associated to $\mathfrak{p}$ and $V$ is the induced module
\[
M^\mathfrak{p}(V) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p})} V.
\]
Taking $V$ finite dimensional gives a parabolic Verma module.
irreducible $L(\Lambda_f)$ for $U_q(\mathfrak{g}_f)$ can be extended uniquely to an irreducible for $U_q(\mathfrak{l})$ by declaring that $U_q(\mathfrak{h}_m)$ acts on the highest weight vector as $K_i v_0 = \lambda_i v_0$. We denote this parabolic Verma module by $M^p(\Lambda)$, where $\Lambda_i = n_i$ if $i \in I_f$ or $\Lambda_i = \lambda_i$ if $i \in I_m$. We also allow to take $\Lambda_i = \lambda_i q^{n_i}$, for $i \in I_m$ and $n_i \in \mathbb{Z}$, and in this case, $K_i v_0 = \lambda_i q^{n_i} v_0$. We call such modules shifted Verma modules.

2.3. **Parabolic Verma modules and link invariants.** Let $\mathfrak{p} \subset \mathfrak{g}_{k+\ell}$ be the parabolic subalgebra obtained by adjoining the Chevalley generators $F_i$ for $i \in I \setminus \{0\}$ to the Borel of $\mathfrak{g}_{k+\ell}$, where $I = \{1-k, \ldots, \ell-1\}$. Its Levi factor consists in $(\mathfrak{g}_k \times \mathfrak{gl}_\ell) \oplus \mathfrak{h}_0$, and its nilpotent radical is generated by $E_0$.

**Proposition 2.7.** For $\mu \in \Lambda_+$, the module $W(\mu)$ is isomorphic with the parabolic Verma module $M^p(\mu)$ as modules over $U_q(\mathfrak{g}_{k+\ell})$.

**Proof.** This follows at once from the fact that both $U_q(\mathfrak{g}_{k+\ell})$-modules are generated by a single vector of weight $\mu$ and the PBW theorem, which is also true for $\mathcal{S}_{q,\beta}(\ell, k)$, and shows that the weights occurring in both modules are the same. \end{proof}

**Notation.** From now on, for the sake of keeping the notation simple we denote the highest weight modules $W((\beta)^n, (0)^n)$ and $M^p((\beta)^n, (0)^n)$ by $W(\beta)$ and $M^p(\beta)$ respectively.

In the particular case of $M^p(\beta)$, the irreducible $L(\Lambda_f)$ is the 1-dimensional irreducible module of highest weight $(\beta)^n \cup (0)^n$. Under the isomorphism in **Proposition 2.7**, the element $P(b)$ defines an endomorphism $P^b$ of the highest weight object of the Verma module $M^p(\beta)$ (seen as a linear category with objects indexed by the weights, in the obvious way). Since $P^b$ consists of the same word in $E$’s and $F$’s as $P(b)$, it yields the same element in $\mathbb{Q}(q)[\lambda^{\pm 1}]$.

**Theorem 2.8.** For a braid $b$ the element $\lambda^w(\text{cl}(b)) P^b \in \text{End}_{U_q(\mathfrak{gl}_{2n})}(M^p(\beta))$ is a link invariant which equals the HOMFLY-PT polynomial of the closure of $b$.

2.3.1. **Normalized link invariants.** In order to be able to compute the normalized HOMFLY-PT and $\mathfrak{gl}_N$-link invariants we follow the procedure described in [23, §4]. It consists in cut opening the braid closure diagram $\text{cl}(b)$ into a special type of $(1, 1)$-tangle diagram.

We open the diagram of $\text{cl}(b)$ by cutting the outermost strand, following a pattern as shown in the example in (2) below for a braid with three strands,

(2)

and similarly for the top part. We denote $\text{cl}_o(b)$ the diagram obtained with this procedure.

The procedure described in §2.1.2 gives an element $\overline{P}(b) \in S_{q,\beta}(n-1, n)$ which is an endomorphism of $1_{((\beta)^{n-1}, (0)^{n-1})}$, which in turn implies that $\overline{P}(b) \in \mathbb{Q}(q)[\lambda^{\pm 1}]$ (see [23, §4] for details).
**Theorem 2.9** (Proposition 4.6 in [23]). For a braid $b$ the element $\overline{P}(b)$ is a framed link invariant which equals the reduced HOMFLY-PT polynomial of the closure of $b$.

Notice we could have opened the diagram in a different way, by choosing a different strand to cut it open. We could have equally opened the diagram by cutting it using one of the inner strands at the expense of adding crossings to the original diagram. In [23] the authors proved that the link invariants obtained do not depend on this choice.

In order to parallel the construction of §2.3 using a parabolic Verma module, we consider $\mathfrak{gl}_{2n-1}$ with simple roots $\{\alpha_{2-n}, \ldots, \alpha_{n-1}\}$ (we no longer need the root $\alpha_{1-n}$ since the braid is not completely closed on the left anymore). We form the parabolic subalgebra $\overline{p}$ given by adjoining $\{F_{\pm 1}, \ldots, F_{\pm(n-2)}, F_{n-1}\}$ to the Borel of $\mathfrak{gl}_{2n-1}$ (these are all the Chevalley generators of $\mathfrak{gl}_{2n-1}$ except $F_0$). Then we consider the shifted parabolic Verma module

$$M^\overline{p}(\overline{\beta}) = M^p((\beta)^{n-1}, 1, (0)^{n-1}),$$

where the highest weight is chosen to agree with the bottom of (2)

$$\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta & \beta & 1 & 0 & 0
\end{array}$$

Note that the irreducible module $L(\Lambda_f)$ of $U_q(\mathfrak{g} \times \mathfrak{g})$, with $\Lambda_f = ((\beta)^{n-1}) \sqcup (1, (0)^{n-1})$, used to construct this parabolic Verma module is two-dimensional with $F_1$ acting non-trivially on the highest weight vector $v_0$. Moreover, $K_0$ acts on $v_0$ as $q^{-1}\lambda$ (instead of $\lambda$ in $p$), when extending the action to $U_q(\mathfrak{l})$.

The method described in §2.3 defines an endomorphism $P^\overline{p}(b)$ of the highest weight object of $M^\overline{p}(\overline{\beta})$. The following is an easy consequence of the paragraphs above.

**Theorem 2.10.** For a braid $b$ the element $\lambda w^{(\mathfrak{cl},(b))}P^\overline{p}(b) \in \text{End}_{U_q(\mathfrak{gl}_{2n-1})}(M^\overline{p}(\overline{\beta}))$ is a link invariant which equals the reduced HOMFLY-PT polynomial of the closure of $b$.

**Remark 2.11.** From now on, for the means of higher representation theory, we will consider the parabolic Verma modules $M^p$ and $M^\overline{p}(\overline{\beta})$ as living in $\mathbb{Q}(q, \lambda) \supset \mathbb{Q}(q)[\lambda^{\pm 1}]$ with polynomial fractions viewed as formal power series. See [19, §5.3] for more about rings of formal Laurent series in the context of categorification, see also [2] for a general discussion about these rings.

### 3. Parabolic 2-Verma modules

In this section we describe superalgebras which can be thought of as a mild mix of the Khovanov–Lauda–Rouquier algebras from [10, 25] and the (super)algebra $A_m$ introduced by the authors in [19]. We follow the diagrammatic presentation for $A_m$ given in [20]. These superalgebras contain KLR algebras as subalgebras concentrated in even parity. In [21] we give superalgebras associated to any pair $(p, g)$ where $g$ is a Kac–Moody algebra and $p \subset g$ a parabolic subalgebra.

In this paper we focus on three cases:
• the $b$-KLR algebra (see §3.1.1), which is used to construct the universal 2-Verma module, categorifying the universal Verma module $M^b(\beta_{1-n}, \ldots, \beta_{n-1})$ of highest weight
$(\beta_{1-n}, \ldots, \beta_{n-1})$,
• the $p$-KLR algebra, Definition 3.21, used for the parabolic 2-Verma modules associated to the parabolic subalgebra $p \subseteq U_q(\mathfrak{gl}_{2n})$ from §2.3,
• and the $\bar{p}$-KLR algebra, Definition 3.34, for $\bar{p} \subseteq U_q(\mathfrak{gl}_{2n-1})$ from §2.3.1.

3.1. Universal 2-Verma module of $U_q(\mathfrak{gl}_{2n})$. Despite the fact the $p$-KLR and $\bar{p}$-KLR algebras admit an independent definition, we will obtain them through $b$-KLR algebras. This extra structure will be particularly precious for the proof of Theorem 4.9 in §4.6.

Recall that we identify the set of simple roots with $I = \{0, \pm 1, \ldots, \pm (n-1)\}$.

3.1.1. The $b$-KLR algebra. For each $\nu = \sum_{i \in I} \nu_i i \in \mathbb{N}_0[I]$, we consider the collection of KLR diagrams $R(\nu)$, for the quiver $A_{2n-1}$, with a bit of additional structure. Besides the KLR generators

\[
\cdots \quad i \quad \cdots \quad \text{and} \quad \cdots \quad i \quad j \quad \cdots
\]

for all $i, j \in I$, regions can be decorated with floating dots, as in the example below,

\[
\cdots \quad i \quad j \quad k \quad \cdots
\]

Floating dots are labeled by a simple root as subscript (e.g. $\ell \in I$ in the diagram above), and by a non-negative integer as superscript ($a \in \mathbb{N}_0$ in the example). By convention, we don’t write a superscript when it is 0 and refer to the respective floating dot as unlabeled.

Floating dots are odd while all other KLR generators are even. This means we endow diagrams with a parity and declare floating dots to have parity one and (KLR) dots and crossings to have parity zero. The diagrams are equipped with a Morse height function that keeps track of the relative height of floating dots and forbids two floating dots to be at the same height. We allow all isotopies with respect to this Morse function without creating any critical point.

Let $\mathbb{k}$ denote the field of rationnals. Denote by $R_b(\nu)$ the $\mathbb{k}$-algebra obtained by linear combinations of these diagrams, together with multiplication given by gluing diagrams on top of each other whenever the labels agree, and zero otherwise. In our conventions $ab$ means stacking the diagram for $a$ atop the one for $b$, whenever they are composable (this is a consequence from the fact that we read diagrams from bottom to top).

Both dots and floating dots form a supercommutative subsuperalgebra. This means that floating dots anticommute among themselves as in (3), while they commute with dots. As usual in
KLR algebras, dots commute with each other.

\[
\cdots \bullet^b_j = - \bullet^a_i \cdots
\]

Note that a region cannot carry more than one floating dot with the same sub- and superscript (otherwise it is zero).

Besides the dot relations above, diagrams are subject to the local relations (4) to (11) below.

- **The KLR relations**: For all \(i, j, k \in I\) we have

\[
\begin{cases} 
0 & \text{if } i = j, \\
\text{ if } |i - j| > 1, \\
\text{ if } |i - j| = 1, \\
\end{cases}
\]

(4)

(5) \quad \begin{align*}
\begin{array}{c}
\begin{array}{c}
 i \\
 j
\end{array}
\end{array}
\end{align*}

(6) \quad \begin{align*}
\begin{array}{c}
\begin{array}{c}
 i \\
 i
\end{array}
\end{array}
\end{align*}

(7) \quad \begin{align*}
\begin{array}{c}
\begin{array}{c}
 i \\
 j \\
 k
\end{array}
\end{array}
\end{align*}

unless \(i = k\) and \(|i - j| = 1,\)
The relations involving floating dots: For all \( i, j \in I \) and \( \alpha \in \mathbb{N}_0 \), we put

\[
\begin{align*}
\begin{tikzpicture} 
    \node (i) at (0,0) [shape=circle,draw] {i};
    \node (j) at (1,0) [shape=circle,draw] {j};
    \node (i) at (2,0) [shape=circle,draw] {i};
    \draw (i) .. controls (0.5,1) and (1.5,1) .. (j);
    \node (i) at (3,0) [shape=circle,draw] {i};
    \draw (i) .. controls (2.5,1) and (3.5,1) .. (j);
\end{tikzpicture}
\end{align*}
\]

\[ = \begin{array}{c}
\begin{tikzpicture} 
    \node (i) at (0,0) [shape=circle,draw] {i};
    \node (j) at (1,0) [shape=circle,draw] {j};
    \node (i) at (2,0) [shape=circle,draw] {i};
    \node (i) at (3,0) [shape=circle,draw] {i};
    \draw (i) .. controls (0.5,1) and (1.5,1) .. (j);
\end{tikzpicture}
\end{array}
\]  

if \( |i-j| = 1 \),

We also impose a floating dot on the leftmost region of a diagram to be zero:

\[
\begin{align*}
\begin{tikzpicture} 
    \node (i) at (0,0) [shape=circle,draw] {i};
    \node (j) at (1,0) [shape=circle,draw] {j};
    \node (i) at (2,0) [shape=circle,draw] {i};
    \node (i) at (3,0) [shape=circle,draw] {i};
    \draw (i) .. controls (0.5,1) and (1.5,1) .. (j);
\end{tikzpicture}
\end{align*}
\]

\[ = \begin{array}{c}
\begin{tikzpicture} 
    \node (i) at (0,0) [shape=circle,draw] {i};
    \node (j) at (1,0) [shape=circle,draw] {j};
    \node (i) at (2,0) [shape=circle,draw] {i};
    \node (i) at (3,0) [shape=circle,draw] {i};
    \draw (i) .. controls (0.5,1) and (1.5,1) .. (j);
\end{tikzpicture}
\end{array}
\]  

We turn \( R_b(\nu) \) into a \( \mathbb{Z} \times \mathbb{Z}^{2n-1} \)-graded superalgebra. We refer to the first grading as the \( q \)-grading and the second as \( \Lambda \)-grading, where \( \Lambda = \{\lambda_i\}_{i \in I} \). For \( u \in R_b(\nu) \) we write \( \deg(u) = (q(u), \Lambda(u)) \), with \( q(u) \) being the \( q \)-degree of \( u \) and \( \Lambda(u) = \sum_{i \in I} \lambda_i(u) \cdot i \). We will also write \( \lambda(u) \) for \( \lambda_0(u) \) and call it the \( \lambda \)-grading. Finally, \( p(u) \) is the parity of \( u \). For the KLR generators
we extend the KLR grading trivially to a multigrading by setting

\[
\deg \left( \begin{array}{c}
\bullet \\
\end{array} \right) = (2, 0), \quad \deg \left( \begin{array}{c}
\bigstar \\
\end{array} \right) = \begin{cases} 
(0, 0) & \text{if } i = j, \\
(1, 0) & \text{if } |i - j| = 1, \\
(2, 0) & \text{else},
\end{cases}
\]

and they have parity zero. In order to define the degree of a floating dot, let \( \ell_i \) be the number of strands labeled \( i \) at its left for any \( i \in I \). We declare that

\[
q \left( \bigcirc^a_i \right) = 2(1 + a - \ell_i + \ell_{i+1}), \quad \lambda_j \left( \bigcirc^a_i \right) = 2\delta_{ij}, \quad p \left( \bigcirc^a_i \right) = 1.
\]

It is immediate that relations (4) to (11) above preserve the multigrading and the parity.

**Definition 3.1.** We define the \( b \)-KLR superalgebra as

\[
R_b = \bigoplus_{\nu \in \mathbb{N}_0[I]} R_b(\nu).
\]

3.1.2. A basis for \( R_b \). We will now construct a basis for \( R_b \), viewed as a \( k \)-module. Let us first introduce some notations borrowed from \cite{10}. We denote by \( \text{Seq}(\nu) \) the set of all ordered sequences \( i = i_1 i_2 \cdots i_m \) with each \( i_k \in I \) and \( i \) appearing \( \nu_i \) times in the sequence. We let \( e_i \in R_b(\nu) \) be the idempotent given by \( m \) vertical strands with labels in the order induced by \( i \),

\[
e_i = \begin{array}{c}
i_1 \\
i_2 \\
\vdots \\
i_m 
\end{array}
\]

**Definition 3.2.** We say that a floating dot is in tight position, or that it is tight, if it is unlabeled and located in the region immediately at the right of the leftmost strand.

By (12) and (10), we can also assume that the left strand next to a tight floating dot is labeled by the same label as the subscript of the floating dot since it would give zero otherwise.

**Example 3.3.** The floating dot with subscript \( i \) in the diagram below is tight, while the one with \( j \) is not.

\[
\text{Proposition 3.4. The following equality holds in } R_b
\]

\[
\bigcirc^a_i = \bigcirc^a_i - \bigcirc^a_i
\]
Proof. It is an immediate consequence of (4), (6) and (9).

Proposition 3.4 together with relations (10) and (11) imply that any diagram in $R_b$ can be written as a linear combination of diagrams involving KLR generators and only tight floating dots.

**Lemma 3.5.** As a $k$-module, $R_b$ is generated by diagrams involving only KLR generators and tight floating dots.

Let $\omega$ denote a tight floating dot in a diagram given by $m$ vertical strands with labels supposed to be given by the context. Let $\tau_a$ be a crossing between the $a$-th and the $(a+1)$-th strands. We also write $\tau_\omega = \tau_{\ell_r} \cdots \tau_{\ell_1}$ for a reduced expression $w = s_{\ell_r} \cdots s_{\ell_1} \in S_m$, and we let $x_i$ denote a dot on the $i$-th strand. Define the tightened floating dot $\theta_a = \tau_{a-1} \cdots \tau_1 \omega \tau_1 \cdots \tau_a$, where the strand pulled to the left is the $a$th one and its label matches the subscript of the floating dot.

Fix $i, j \in \text{Seq}(\nu)$. Since they are both sequences of the same elements, there is a subset $jS_4 \subset S_m$ of permutations $\phi \in S_m$ such that $i_k = j_{\phi(k)}$ for all $k \in \{1, \ldots, m\}$. Suppose we have chosen a reduced expression for a fixed permutation $\phi = s_{\ell_r} \cdots s_{\ell_1} \in jS_4$. Up to choosing another equivalent reduced expression, we can also suppose it is left-adjusted. By this we mean that $\ell_r + \cdots + \ell_1$ is minimal. If we write $O_{s_{\ell_r} \cdots s_{\ell_1}}(k) = \{s_{\ell_t} \cdots s_{\ell_1}(k) | 0 \leq t \leq r\}$, then $s_{\ell_r} \cdots s_{\ell_1}$ is left-adjusted if and only if

$$\min O_{s_{\ell_r} \cdots s_{\ell_1}}(k) \leq \min O_{s_{\ell_r} \cdots s_{\ell_1}'}(k),$$

for all $k$ and reduced expressions $s_{\ell_r'} \cdots s_{\ell_1}' = \phi$.

If we view $s_{\ell_r} \cdots s_{\ell_1}$ as a string diagram where each $s_k$ corresponds to a crossing $\tau_k$, its left-adjusted form is given by the diagram where each strand is “pulled as far as possible to the left without creating double intersections”. For example, the permutation $(13)$ has two reduced expressions and the one on the left is left-adjusted:

In general one can transform a reduced expression into a left-adjusted one by applying a sequence of braid moves, $s_{k+1}s_k s_{k+1} \rightarrow s_k s_{k+1} s_k$, and distant crossing permutations, transforming each triplet of intersections as in the example above from right to left. The left-adjusted reduced expression is unique up to permutation of distant crossings (reduced expressions differ only up to braid moves and distant crossings permutations, but braid moves increase $\ell_r + \cdots + \ell_1$).
Now fix a left-adjusted reduced expression $\phi = s_{\ell_r} \cdots s_{\ell_1} \in jS_i$, and choose $m$-uples $u = (u_1, \ldots, u_m) \in \mathbb{N}_0^m$ and $\alpha = (\alpha_1, \ldots, \alpha_m) \in \{0, 1\}^m$. To such a triplet $(u, \alpha, s_{\ell_r} \cdots s_{\ell_1})$ we associate an element of $e_j R_p(\nu) e_i$ as follows:

- we take the braid-like diagram in which crossings correspond with the simple transpositions in the chosen left-adjusted reduced expression $s_{\ell_r} \cdots s_{\ell_1}$,
- for each $\alpha_k = 1$, we look where the $k$th strand (counting from bottom left) attains its leftmost position (in the sense that there is a minimal number of other strands at its left) and we insert a tightened floating dot at this position: that is we pull the strand to the far left at this height, and add a floating dot with matching label immediately at its right,
- we add $u_k$ dots at the top of the $k$th strand, counting from top left, for all $k$.

**Example 3.6.** For example, with $i = ijk$, $j = kji$, $u = (2, 0, 1)$ and $\alpha = (0, 1, 0)$ we could get

![Diagram](image)

**Example 3.7.** We take $i = ijjii$ and $j = jiii j$. We choose the following permutation

![Diagram](image)

It is not left-adjusted as we can pull the fourth strand to the left, so by applying two braid moves we get the following left-adjusted reduced expression

![Diagram](image)

Now if we choose $u = (0, 0, 1, 0, 2)$ and $\alpha = (1, 0, 0, 1, 0)$, then applying the construction as above yields the following element of $e_j R_p(\nu) e_i$

![Diagram](image)

where we can see that the fourth strand has been pulled to the left to add a tight floating dot.
Let $jB_i$ denote the set of diagrams from $i$ to $j$ given by all choices of $x, \alpha$ and permutations $\phi$, with a fixed choice of presentation $\phi = s_{i_r} \cdots s_{i_1}$ for each of them.

**Lemma 3.8.** Elements in $jB_i$ generate the $\mathbb{k}$-module $e_j R_0(\nu) e_i$.

**Sketch of proof.** By Lemma 3.5, we can assume all floating dots are tight. Then the proof follows by an inductive argument on the number of crossings, using the fact we can apply all braid relations and slide (KLR) dots at the cost of adding diagrams with fewer crossings, thanks to (4) and (7-8). Indeed, this means that we can work up to braid isotopy and assume that all dots are concentrated in the top of the diagram, allowing us to isolate each (tight) floating dot with only left-adjusted reduced type diagrams in-between. Then we observe that a strand can have up to only one tight floating dot at its immediate right, otherwise it would be zero because of the relation

\[
\begin{pmatrix}
\begin{array}{c}
\circ_i
\\
\circ_j
\end{array}
\end{pmatrix}
-
\begin{pmatrix}
\begin{array}{c}
\circ_i
\\
\circ_j
\end{array}
\end{pmatrix}
= -
\begin{pmatrix}
\begin{array}{c}
\circ_i
\\
\circ_j
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
\circ_i
\\
\circ_j
\end{array}
\end{pmatrix}
\]

that can be deduced from (4-11). Finally, for the same reasons, one can show that two tightened floating dots $\theta_a$ and $\theta_b$ anti-commute (up to adding diagrams with fewer crossings), so that we can bring any diagram to a linear combination of diagrams in $jB_i$. See [21, Lemma 3.13] for more details. □

To prove $jB_i$ gives a basis for $e_j R_0 e_i$, we first need to construct an action of $R_0$ on some polynomial-type ring.

### 3.1.3. Polynomial action

We fix $\nu \in \mathbb{N}_0[I]$ with $|\nu| = m$. For each $i \in I$ we define

\[
P_i = \mathbb{k}[x_{1,i}, \ldots, x_{\nu_i,i}] \otimes \mathbb{k} \wedge^{\bigwedge} \langle \omega_{1,i}, \ldots, \omega_{\nu_i,i} \rangle,
\]

where $\wedge^{\bigwedge} \langle \omega_{1,i}, \ldots, \omega_{\nu_i,i} \rangle$ is the exterior algebra of the $\mathbb{k}$-vector space generated by $\omega_{1,i}, \ldots, \omega_{\nu_i,i}$. We write $P_I = \bigotimes_{i \in I} P_i$ where $\bigotimes$ means the supertensor product, so that $P_I$ is a multigraded superring with $\deg(x_{r,i}) = (2, 0)$ and $q(\omega_{r,i}) = 2(1 - r), \lambda_j(\omega_{r,i}) = 2\delta_{ij}$. Then we define

\[
P_\nu = \bigoplus_{i \in \text{Seq}(\nu)} P_I 1_{i},
\]

where the $1_i$’s are central idempotents.

There is an action of $S_m$ on $P_\nu$, given by

\[
s_k : P_I 1_i \to P_I 1_{s_k i},
\]

sending

\[
x_{p+1,i} 1_{s_k i}, \quad \text{if } i_k = i_{k+1} = j \text{ and } p = \# \{ s \leq k | i_s = j \},
\]

\[
x_{p-1,i} 1_{s_k i}, \quad \text{if } i_k = i_{k+1} = j \text{ and } p = 1 + \# \{ s \leq k | i_s = j \},
\]

\[
x_{p,j} 1_{s_k i}, \quad \text{otherwise},
\]

\[
\text{if } i_k = i_{k+1} = j \text{ and } p = \# \{ s \leq k | i_s = j \},
\]

\[
\text{if } i_k = i_{k+1} = j \text{ and } p = 1 + \# \{ s \leq k | i_s = j \},
\]

\[
\text{otherwise},
\]
and

\[
\omega_{p,j} 1_i \mapsto \begin{cases} 
(\omega_{p,j} + (x_{p,j} - x_{p+1,j})\omega_{p+1,j}) 1_{s_k i}, & \text{if } i_k = i_{k+1} = j \text{ and } p = \# \{ s \leq k | i_s = j \}, \\
\omega_{p,j} 1_{sk i}, & \text{otherwise}. 
\end{cases}
\]

Note that the condition \( p = \# \{ s \leq k | i_s = j \} \) means \( i_k \) is the \( p \)th element (counting from left to right) in \( i \) with value \( i_k = j \).

**Proposition 3.9.** The operations above describe an action of the symmetric group \( S_m \) on \( P_{\nu} \).

**Proof.** It is a straightforward generalization of [20, §2.1] which is itself a direct computation, with the only additional difficulty of keeping track of the labelings. \( \square \)

Then we define

\[
\omega^a_{r,i} = \sum_{\ell=1}^r (-1)^{r+a+\ell} h_{a+\ell-r}(x_{\ell,i}, \ldots, x_{r,i}) \omega_{\ell,i},
\]

where \( h_n(x_{\ell,i}, \ldots, x_{r,i}) \) is the \( n \)th complete homogeneous symmetric polynomial in variables \( (x_{\ell,i}, \ldots, x_{r,i}) \). It is defined so that

\[
\omega^0_{r,i} = \omega_{r,i}, \quad \omega^a_{r,i} = \omega^{a-1}_{r-1,i} - x_{r,i} \omega^{a-1}_{r-1,i}.
\]

We now have all the tools we need to define an action of \( R_{\ell_b}(\nu) \) on \( P_{\nu} \). First we let \( a \in R_{\ell_b}(\nu) e_i \) act as zero on \( P_{1,j} \) whenever \( j \neq i \). Otherwise, we declare that

- the dot acts as

  \[
  \begin{array}{ccc}
  \bullet & \mapsto & x_{\ell_i+1,j} 1_i,
  \end{array}
  \]

  that is, multiplication by \( x_{\ell_i+1,j} 1_i \) with \( \ell_i \) being the number of strands labeled \( i \) at the left,

- the floating dot acts as multiplication

  \[
  a \begin{array}{ccc}
  \bullet & \mapsto & \sum_{r=0}^{\ell_j+1} (-1)^r \omega^{a+r}_{\ell_j}(i) e_{\ell_j+1-r}(x_{1,j-1}, \ldots, x_{\ell_j-1,j-1}, x_{1,j+1}, \ldots, x_{\ell_j+1,j+1}),
  \end{array}
  \]

  where \( \ell_{j \pm 1} \) is the number of strands labeled \( j+1 \) or \( j-1 \) at the left, and \( e_n(\ldots) \) is the \( n \)th elementary symmetric polynomial in variables \( x_{k,j \pm 1} \) for \( k \leq \ell_{j \pm 1} \),

- the crossing between the \( k \)-th and \((k+1)\)-th strands,
acts as
\[
\begin{align*}
    f_{1_i} &\mapsto s_k(f_{i}) & \text{if } |i - j| > 1, \\
    f_{1_i} &\mapsto \frac{f_{1_i} - s_k(f_{1_i})}{x_{\ell_{i+1,j}} - x_{\ell_{i+2,j}}} & \text{if } i = j, \\
    f_{1_i} &\mapsto (x_{\ell_{1,i+1,j}} + x_{\ell_{1,j+1,j}})s_k(f) & \text{if } i = j + 1, \\
    f_{1_i} &\mapsto s_k(f) & \text{if } i = j - 1.
\end{align*}
\]

**Proposition 3.10.** The rules above define an action of $R_b(\nu)$ on $P_{\nu}$.

**Proof.** The proof is a straightforward computation, checking that the construction respects all the relations from the definition of $R_b$. This is done in detail in [21, §3.2].

**Theorem 3.11.** The family $jB_i$ forms a basis for $e_jR_b e_i$ seen as a $k$-module.

**Proof.** The elements of $jB_i$ act as linearly independent endomorphisms on $P_{\nu}$. The proof is similar to [26, Proposition 3.8] and the details are given in [21, §3.3]. It mostly rely on the fact that the action of the symmetric group on $P_{\nu}$ is faithful, and the crossings act as a deformed version of $S_m$. \hfill \square

3.1.4. **Decomposition of $R_b$.** Our next goal is to construct a categorical version of the $\mathfrak{sl}_2$-commutator relations for each simple root. This will be achieved by describing how $R_b(m)$ can be decomposed as a module over $R_b(m - 1)$, where $R_b(m)$ is the subalgebra of $R_b$ given by diagrams with $m$ strands. The following subsection is inspired by the work of Kang–Kashiwara in [8], and is a specialization of [21, §4] to the $\mathfrak{gl}_{2n}$ case.

Denote by $R_b(m, i)$ the subalgebra given by diagrams with $m$ strands together with a vertical strand with label $i$ at the right. Similarly, denote by $R_b(\nu, i)$ the collection of diagrams $D \in R_b(m, i)$ such that the labels at the bottom of $D$ are $\nu j$ with $j \in \text{Seq}(\nu)$. The idempotents $e_{(m,i)}$ and $e_{(\nu,i)}$ are given respectively by the sum of all diagrams with $m + 1$ vertical strands in $R_b(m, i)$ and in $R_b(\nu, i)$:

\[
e_{(\nu,i)} = \sum_{j_1 \cdots j_m \in \text{Seq}(\nu)} |j_1 \cdots j_i \cdots j_m| , \quad e_{(m,i)} = \sum_{j_1 \cdots j_m \in I^m} |j_1 \cdots j_i \cdots j_m|.
\]

These definitions extend in general for $R_b(m, i) \subset R_b(m + |i|)$ with $i$ some sequence in $I$, and for $R_b(i, m) \subset R_b(m + |i|)$. We will write $\otimes$ for $\otimes_k$ and $\otimes_m$ for $\otimes_{R_b(m)}$ throughout this section. We denote by a monomial $q^r\lambda_{1-1}^{\lambda_{1-1}} \cdots \lambda_{n-1}^{\lambda_{n-1}}$ a shift by $r$ units up in the $q$-grading and $s_i$ units up in the $\lambda_i$-grading. We write a shift in parity by $\Pi$.

For the sake of simplicity we now fix an abuse of notation that we will use throughout this section and after. For a left, right or bimodule $M$ and a free multigraded abelian (super)group $A \cong \bigoplus_{x \in X} x\mathbb{Z}$ generated by all (homogeneous) $x$ in some countable set $X$, we will write

\[
M \otimes A = \bigoplus_{x \in X} q^{g(x)}\lambda_{1-1}^{\lambda_{1-1}(x)} \cdots \lambda_{n-1}^{\lambda_{n-1}(x)} \Pi^{p(x)} M.
\]
We will also write $M \otimes x$ with homogeneous $x \in R_b$ to mean we take the shift $M \otimes x\mathbb{Z} = q(x)\lambda_{1-n}(x) \cdots \lambda_{n-1}(x) \prod p(x)M$.

**Proposition 3.12.** As a left $R_b(m)$-module, $R_b(m + 1)$ decomposes as

$$\bigoplus_{a=1}^{m+1} R_p(m) \otimes \tau_m \cdots \tau_{a}(\mathbb{Z}[x_a] \oplus \theta_a[x_1])$$

where $\theta_a[x_1] = \tau_{a-1} \cdots \tau_1 \omega \mathbb{Z}[x_1] \tau_1 \cdots \tau_{a-1}$.

The statement can be informally written in terms of diagrams as

![Diagram](attachment:diagram.png)

where the strand pulled to the upper right is the $a$th one, counting from the bottom left, and the subscript of the floating dot is determined by the label of this strand.

**Proof.** The right coset decomposition of $S_{m+1}$,

$$S_{m+1} = \bigsqcup_{a=1}^{m+1} S_m s_m \cdots s_a$$

gives a choice of left-adjusted reduced presentations to construct $jB_i$. By Theorem 3.11, this is a basis and we get a decomposition

![Diagram](attachment:diagram2.png)

Then an induction on the number of strands proves that the decomposition in the statement of the lemma also generates $R_b(m + 1)$. We conclude by observing the graded dimensions of both decompositions coincide and using the fact $R_b(m + 1)$ is free over $R_b(m)$. 

**Lemma 3.13.** As a right $R_b(m)$-module, $R_b(m + 1)$ admits a decomposition

$$\bigoplus_{a=1}^{m+1} (\mathbb{Z}[x_a] \oplus \theta_a[x_1]) \tau_a \cdots \tau_m \otimes R_b(m),$$

and if $y \in \theta_{m+1}[x_1] \otimes R_b(m)$ then $y - w \in R_b(m) \otimes \theta_{m+1}[x_1]$ for some $w \notin R_b(m) \otimes \theta_{m+1}[x_1]$.

**Proof.** The first result is the symmetric of Proposition 3.12. The second follows from the same arguments as in [20, Lemma 3.5], which basically consist in showing one can slide elements from $R_b(m)$ over $\theta_{m+1}[x_1]$ in $R_b(m + 1)$ at the cost of adding terms not contained in $R_b(m) \otimes \theta_{m+1}[x_1]$.
The second statement of Lemma 3.13 means that the projections of an element in $R_b(m + 1)$ onto the summands $R_b(m)\theta_{m+1}[x_1]$ or $\theta_{m+1}[x_1]R_b(m)$, when viewed either as a left or a right $R_b(m)$-module, give the same result. Hence, the projection $R_b(m + 1) \rightarrow R_b(m)\theta_{m+1}[x_1]$ is a map of bimodules.

We are now able to construct a collection of short exact sequences that will be used to construct the categorical $\mathfrak{sl}_2$-commutator relations.

**Proposition 3.14.** As $(R_b(m), R_b(m))$-bimodules, we have the following:

1. For all $i \in I$, there is a short exact sequence
   
   $0 \rightarrow q^{-2}R_b(m)e_{(m-1,i)} \otimes_{m-1} e_{(m-1,i)}R_b(m) \rightarrow e_{(m,i)}R_b(m)R_b(m + 1)e_{(m,j)} \rightarrow R_b(m) \otimes \mathbb{Z}[\xi] \oplus R_b(m) \otimes \theta_{m+1}\mathbb{Z}[\xi] \rightarrow 0,$
   
   where $\theta_{m+1}e_{(\nu)}$ is a shift $q^{2\nu_1 - 4\nu_2}\lambda^2\Pi$ and $\deg(\xi) = (2,0)$.

2. For $i \neq j \in I$, there are isomorphisms
   
   $e_{(m,i)}R_b(m + 1)e_{(m,j)} \cong q^{-i-j}R_b(m)e_{(m-1,j)} \otimes_{m-1} e_{(m-1,i)}R_b(m).$

**Proof.** There is a $(R_b(m), R_b(m))$-bimodule morphism $s : R_b(m) \otimes_{m-1} R_b(m) \rightarrow R_b(m + 1)$ given by

$$x \otimes y \mapsto x\tau_{my}.$$ 

For the short exact sequence, we define the epimorphism as the projection on the summands $R_b(m) \otimes \mathbb{Z}[x_{m+1}]$ and $R_b(m) \otimes \theta_{m+1}[x_1]$ in the decomposition from Proposition 3.12. Thanks to Lemma 3.13, this yields a bimodule morphism. By Proposition 3.12, as left $R_b(m)$-modules, we have

$$R_b(m)e_{(m-1,i)} \otimes_{m-1} e_{(m-1,i)}R_b(m) \cong \bigoplus_{a=1}^{m} R_b(m)e_{(m-1,i)} \otimes \tau_{m-1} \cdots \tau_a(\mathbb{Z}[x_a] \oplus \theta_a[x_1]),$$

and thus

$$s \left( R_b(m)e_{(m-1,i)} \otimes_{m-1} e_{(m-1,i)}R_b(m) \right) \cong \bigoplus_{a=1}^{m} R_b(m)e_{(m-1,i)} \otimes \tau_{m-1} \cdots \tau_a(\mathbb{Z}[x_a] \oplus \theta_a[x_1]).$$

Therefore, each direct summand of $R_b(m)e_{(m-1,i)} \otimes_{m-1} e_{(m-1,i)}R_b(m)$ is identified with a direct summand of $e_{(m,i)}R_b(m + 1)e_{(m,j)}$, except for $R_b(m) \otimes \mathbb{Z}[x_{m+1}]$ and $R_b(m) \otimes \theta_{m+1}[x_1]$. Since they form the cokernel part of the sequence, we conclude it is a short exact sequence. For the second case, this part vanishes and we obtain an isomorphism. \qed
We can informally view the short exact sequences and the isomorphisms of Proposition 3.14 in terms of diagrams as

with the cokernel vanishing whenever \( i \neq j \). \textit{Nota Bene:} we think of the rightmost picture not as the bimodule generated by those diagrams but as the quotient of \( e_{(\nu,i)}R_b(m+1)e_{(\nu,i)} \) corresponding to such diagrams.

The categorical Serre relations are interpreted in \( R_b \) as the following.

**Proposition 3.15.** We have isomorphisms of \( (R_b(m+2), R_b(m)) \)-bimodules

\[
e_{(\nu,ij)}R_b(m+2)e_{(\nu)} \cong e_{(\nu,ji)}R_b(m+2)e_{(\nu)} \quad \text{for} \ |i-j| > 1,
\]

and of \( (R_b(m+3), R_b(m)) \)-bimodules

\[
q^{-1}e_{(\nu,ijj)}R_b(m+2)e_{(\nu)} \oplus qe_{(\nu,iji)}R_b(m+2)e_{(\nu)} \\
\cong e_{(\nu,ijj)}R_b(m+2)e_{(\nu)} \oplus e_{(\nu,iji)}R_b(m+2)e_{(\nu)} \quad \text{for} \ i = j \pm 1.
\]

**Proof.** The first isomorphism follows from relation (4) and the second from the relations (4) and (7) as in [10, Proposition 2.13]. \( \square \)

3.1.5. \textit{Categorical} \( U_q(\mathfrak{gl}_{2n}) \)-\textit{action.} Let \( R_b(\nu) \)-mod be the category of (graded) \( R_b(\nu) \)-supermodules with morphisms given by degree preserving supermodule maps. Let \( F_i : R_b(\nu) \text{-mod} \to R_b(\nu+i) \text{-mod} \) be the functor of induction given by adding a strand of color \( i \) to the right of a diagram and \( E_i : R_b(\nu+i) \text{-mod} \to R_b(\nu) \text{-mod} \) be a shift of its right adjoint:

\[
F_i(M) = R_b(\nu+i)e_{(\nu,i)} \otimes R_b(\nu) M,
E_i(N) = q^{2\nu_1-\nu_{i+1}+1}\lambda_{i-1}^{-1}e_{(\nu,i)}N,
\]

for \( M \in R_b(\nu) \text{-mod} \) and \( N \in R_b(\nu+i) \text{-mod} \). We also define

\[
Q = \bigoplus_{k \geq 0} \Pi q^{2k+1} \text{Id},
\]

which we think as a categorification of the fraction \( \frac{1}{q-q^{-1}} = -q(1 + q^2 + q^4 + \cdots) \). We also introduce the notations

\[
\bigoplus_{[\alpha]} M = \bigoplus_{\ell=0}^{a-1} q^{-k+1+2\ell} M, \quad \text{and} \quad \bigoplus_{[\beta+k]} M = Q q^k \lambda_i M \oplus \Pi Q q^{-k} \lambda_i^{-1} M,
\]

for \( a \in \mathbb{N} \) and \( k \in \mathbb{Z} \).

A direct consequence of Theorem 3.24 and Proposition 3.25 ahead is the following, which describe how \( F_i \) and \( E_i \) give a categorical action of \( U_q(\mathfrak{gl}_{2n}) \) on \( \bigoplus_{\nu} R_b(\nu) \text{-mod} \).
Theorem 3.16. There is a natural short exact sequence of functors
\[
0 \rightarrow F_i E_i 1_\nu \rightarrow E_i F_i 1_\nu \rightarrow \oplus \beta_{i + n + 1 - 2} \nu_i 1_\nu \rightarrow 0,
\]
for all \(i \in I\), and isomorphisms
\[
E_i F_j 1_\nu \cong F_j E_i 1_\nu,
\]
for \(i \neq j \in I\). There are also isomorphisms
\[
\oplus \left[ \beta_{i + n + 1 - 2} \nu_i \right] \cong F_i F_j 1_\nu \quad \text{for} |i - j| > 1,
\]
\[
\oplus \left[ \beta_i \right] \cong F_i F_j 1_\nu \oplus F_j F_i 1_\nu \quad \text{for} i = j \pm 1,
\]
and the same with \(E_i, E_j\).

Proposition 3.17. The functors \(E_i\) and \(F_i\) are exact (and send projectives to projectives).

Proof. The \((R_b(\nu + i), R_b(\nu + i))\)-bimodule \(e(\nu, \nu)\) is bimodule thanks to Proposition 3.12. The same applies for \(e(\nu, \nu)\) seen as \((R_b(\nu), R_b(\nu + i))\)-bimodule. \(\square\)

3.1.6. The categorification theorem. Theorem 3.16 tells us that the functors \(E_i, F_i\) define a categorical action of \(\mathfrak{gl}_{2n}\) on \(R_b\)-mod. However, in order to construct a categorification of the universal Verma module \(M(\beta_1, \ldots, \beta_{n - 1})\), we need to restrict the category of modules we consider, so that the Grothendieck group does not collapse. In fact, since we need to consider infinite dimensional graded modules, we need to take a slightly different notion than the usual definition of Grothendieck group, which we call topological. Basically, it is given by modding out infinite relations corresponding with infinite filtrations. This part is somehow technical and we omit many details for the sake of shortness, referring to [19, §5] for the details. Hopefully, it is not mandatory to fully understand those technicalities in the rest of this paper, which will mostly use the categorical action of Theorem 3.16.

We write \(\mathbb{Z}_n = \mathbb{Z}[[q]]/\left(q^2 - 1\right)\) and \(\mathbb{Q}_n = \mathbb{Z}_n \otimes \mathbb{Q}\). Let \(\mathbb{Z}((q, \Lambda))\) be the ring of formal Laurent series in the variables \(q\) and \(\{\lambda_i\}_{i \in I}\), given by the order \(0 < q < \lambda_1 < \cdots < \lambda_{n - 1}\), as explained in [19, §5.1] (see [2] for a general discussion about rings of formal Laurent series in several variables). It is given by formal series in the variables \(q^{\pm 1}\) and \(\lambda_i^{\pm 1}\) for which the sum of the (multi)degrees of each term is contained in a cone compatible with the additive order \(0 < q < \lambda_1 < \cdots < \lambda_{n - 1}\) on \(\mathbb{Z}_q \oplus \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_{n - 1}\). This ensures that we can multiply two formal Laurent series using only finite sums to determine each coefficient.

Let \(\mathcal{M}_b(\nu)\)-mod be the category of cone bounded, locally finite dimensional, graded left supermodules over \(\mathcal{M}_b(\nu)\), with degree zero morphisms. By locally finite dimensional we mean that the objects, seen as graded \(k\)-vector spaces, must be finite dimensional in each degree. Cone bounded means the graded dimension is contained in some cone, so that the corresponding polynomial is an element of \(\mathbb{Z}((q, \Lambda))\) (e.g. we allow the series \((1 + q^2 + q^4 + \cdots)\) but not \((1 + q^{-2} + q^{-4} + \cdots))\).

In [19, §5], it is explained how the (topological) Grothendieck group of such a category can be defined as a free \(\mathbb{Z}_n((q, \Lambda))\)-module generated by the simple modules up to shift. It is a topological module for the \((q, \Lambda)\)-adic topology. By computing projective resolutions of the simple modules, one can show that it is freely generated by the projectives as well. We denote it \(G_0(\mathcal{M}_b(\nu))\).
We define \( R_b\mod_{\text{lf}} = \bigoplus_{\nu} R_b(\nu)\mod_{\text{lf}} \) and we equip it with the endofunctors \( F_i \) and \( E_i \). It is not hard to see that the functor \( Q \) descends to multiplication by \( \frac{1}{q-q^{-1}} = \frac{q^{-\pi}}{1-q^{-\pi}} \) in the Grothendieck group, after specializing \( \pi = -1 \in \mathbb{Z}_\pi \). Thus we get

\[
[\bigoplus_{[\nu]} M] = [\nu]_q[M], \quad [\bigoplus_{[\lambda_i+k]} M] = \frac{\lambda_i q^k - \lambda_i^{-1} q^{-k}}{q-q^{-1}} [M].
\]

**Theorem 3.18.** The functors \( F_i \) and \( E_i \) induce an action of \( U_q(\mathfrak{gl}_{2n}) \) on the Grothendieck group of \( R_b \) when specializing \( \pi = -1 \). With this action we have a \( \mathbb{Q}(q,\Lambda) \)-linear isomorphism

\[
\mathcal{G}_0(R_b) \cong M^b(\beta_1 - \ldots, \beta_n - 1),
\]

of \( U_q(\mathfrak{gl}_{2n}) \) representations. Moreover, the isomorphism sends classes of projective objects to canonical basis elements and classes of simple objects to dual canonical basis elements.

**Proof.** By Theorem 3.16 and Proposition 3.17, the functors \( F_i \) and \( E_i \) descend to an action of \( U_q(\mathfrak{gl}_{2n}) \) on the Grothendieck group, giving it a structure of a weight module.

Seeing that \( R_b(\nu) \) has the same decomposition into idempotents as \( R(\nu) \) from [10] we can transpose the arguments in [10, §3.2] such that \( R_b(\nu) \) also categorify \( U_q^-(\mathfrak{gl}_{2n}) \) as a \( \mathbb{k} \)-vector space (but a priori not as a bialgebra). Since \( E_i \) acts as zero on \( \mathcal{R}_q^0(\mathcal{O})) \), it is a highest weight module with 1-dimensional highest weight space. Thus, there is a surjective morphism

\[
\gamma : M^b(\beta_1 - \ldots, \beta_n - 1) \twoheadrightarrow \mathcal{G}_0(R_b),
\]

sending the highest-weight vector to \([\mathcal{R}_b(\mathcal{O})] \). Since the parameter \( q \) is generic (i.e. not a root of unity), we can apply the Jantzen’s criterion (see [7, §9.13] or [18] for the non-quantum case) thanks to the results in [1]. Hence the module \( M^b(\beta_1 - \ldots, \beta_n - 1) \) is irreducible, and \( \gamma \) must be an isomorphism. \( \square \)

## 3.2. A parabolic 2-Verma module I: the case of \( p \subseteq U_q(\mathfrak{gl}_{2n}) \).

We now treat the \( \mathfrak{gl}_{2n} \) (the “unreduced”) case. For this, we introduce a differential \( d_\beta \) on \( R_b(\nu) \), turning it into a multigraded dg-algebra, as in [21, §4.2.2] (this can be found already in [19] and [20]). We will see how the category of dg-modules over \((R_b,d_\beta)\) yields a categorification of \( M^P(\beta) \), and how we can actually interpret this more classically in terms of category of modules over \( H^*(R_b,d_\beta) \). In particular, \( H^*(R_b,d_\beta) \) will be presented as a cyclotomic quotient of some \( p \)-KLR algebra \( R_p \).

### 3.2.1. Dg-structure on \( R_b(\nu) \).

For a given floating dot \( \omega^i_a \), the intersection with a horizontal line gives rise to an idempotent \( 1_{\omega^i_a} \) in \( R_b \) consisting of vertical segments formed from the intersection points with the strands at the left \( \omega^i_a \). Denote by \( \mathbb{Z}[\mathcal{L}_i^+] \) the ring in the dots on the \( \ell^+ \) strands labeled \( i \) in \( 1_{\omega^i_a} \), and by \( \mathbb{Z}[\mathcal{L}_{i+1}] \) the ring in the dots on the \( \ell^{-1} \) strands labeled \( i \pm 1 \) in \( 1_{\omega^i_a} \). For \( r \in \mathbb{N} \) we write \( h_r(\ell) \) for the \( r \)th complete homogeneous symmetric polynomial in the variables in \( \mathbb{Z}[\mathcal{L}_i^+] \), and \( e_r(\ell_{i+1}) \) the \( r \)th elementary symmetric polynomial in the variables in \( \mathbb{Z}[\mathcal{L}_{i+1}] \).

We define

\[
d_\beta \left( \begin{array}{c}
i \\
\end{array} \right) = 0, \quad d_\beta \left( \begin{array}{c}
j \\
i \\
\end{array} \right) = 0,
\]
for all $i, j \in I$, and

$$d_\beta (\begin{array}{l} a \\ i \\ \hline j \end{array}) = \begin{cases} 0, & \text{if } i = 0, \\ (-1)^{a-e_i} \left( \sum_{r=0}^{\ell_i} h_{\ell_i} \ell_i \right), & \text{otherwise.} \end{cases}$$

Of course, thanks to **Lemma 3.8**, it is enough to define the differential on tight floating dots, on which $d_\beta$ is given by

$$d_\beta (\begin{array}{l} \hline i \\ \cdots \\ j \hline k \end{array}) = \begin{cases} 0, & \text{if } i = 0, \\ - \begin{array}{l} \hline i \\ \cdots \\ j \hline k \end{array} & \text{otherwise.} \end{cases}$$

Then one can deduce the definition given above for any floating dot using a recursion on the number of strands together with **Proposition 3.4**, and relations (10) and (11).

This differential has degrees

$$q(d_\beta) = 0, \quad \lambda_0(d_\beta) = 0, \quad \lambda_i(d_\beta) = -2,$$

for $i \neq 0$, and has parity 1. Since all relations (4)-(12) preserve the number of floating dots, we can upgrade the parity into a $\mathbb{Z}$-grading, which we will call $p$-grading.

**Definition 3.19.** Let $d_\beta : R_b(\nu) \to R_b(\nu)$ be defined as above together with the graded Leibniz rule, so that $(R_b(\nu), d_\beta)$ becomes a $\mathbb{Z} \times \mathbb{Z}$-graded $(q\text{-} and \lambda = \lambda_0\text{-degrees})$ dg-algebra, with homological degree given by counting the number of floating dots with subscript different from 0. We call this grading the $d_\beta$-grading.

**Proposition 3.20.** ([21, Lemma 4.13]) The dg-algebra $(R_b(\nu), d_\beta)$ is formal with homology concentrated in $d_\beta$-degree 0.

**Sketch of proof.** The proof goes by an induction on the number of strands $m$, using **Proposition 3.12** to compute the homology. $\square$

This means there is a quasi-isomorphism $(R_b(\nu), d_\beta) \cong (H^*(R_b(\nu), d_\beta), 0)$. Gladly, the homology $H^*(R_b(\nu), d_\beta)$ admits a very nice description in terms of KLR-like diagrams.

**Definition 3.21.** Let $R_p(\nu)$ be defined as $R_b(\nu)$ but where we only admit floating dots having subscript 0. Then we define $R_\beta(\nu)$ to be the quotient of $R_p(\nu)$ by the two-sided ideal generated by all diagrams of the form

$$\begin{array}{l} \hline i \\ \cdots \\ j \hline k \end{array}, \quad i \neq 0.$$

In other words, we kill all diagrams which at some height have a strand of label different from 0 at the left.

**Proposition 3.22.** There is an isomorphism $H^*(R_b(\nu), d_\beta) \cong R_\beta^*(\nu)$. 
Proof. This is immediate by Lemma 3.8 and Proposition 3.20.

3.2.2. A long exact sequence. Take $k \in I \setminus \{0\}$ and consider the short exact sequence

\begin{align}
0 & \to q^{-2} R_b(\nu) e_{(m-1,k)} \otimes_{m-1} e_{(m-1,k)} R_b(\nu) \to e_{(\nu,k)} R_b(m+1) e_{(\nu,k)} \\
& \to R_b(\nu) (\mathbb{Z}[\xi] \oplus \theta_{m+1} \mathbb{Z}[\xi]) \to 0,
\end{align}

of $(R_b(\nu), R_b(\nu))$-bimodules, from Proposition 3.14. Following [21, §4.2.3], we equip the cokernel of (17) with a differential, also denoted $d_\beta$, defined by $d_\beta(\xi^a \oplus 0) = 0$, and

$$d_\beta(0 \oplus \xi^a) = (-1)^{\nu_k} \sum_{r=0}^{\nu_{k+1}} \sum_{s=\nu_k}^{a+r-\nu_k} \sum_{t=0}^{s-\nu_k} \xi^t h_{s-\nu_k-t}(\nu_k) h_{a+r-s-\nu_k}(\nu_k) e_{\nu_{k+1}-r}(\nu_{k+1}).$$

This formula is given by the image of $d_\beta(\tau_1 \cdots \tau_{\nu} a_{\nu+1} \tau_1 \cdots \tau_m) = -\tau_1 \cdots \tau_{1} a_{\nu} \tau_1 \cdots \tau_{m}$ in the projection

$$e_{(\nu,k)} R_b(m+1) e_{(\nu,k)} \to R_b(\nu)[\xi],$$

from the short exact sequence (17).

This means that (17) can be enhanced into a short exact sequence of $((R_b(\nu), d_\beta), (R_b(\nu), d_\beta))$-dg-bimodules

\begin{align}
0 & \to \left( q^{-2} R_b(\nu) e_{(m-1,k)} \otimes_{m-1} e_{(m-1,k)} R_b(\nu), d_\beta \right) \to \left( e_{(\nu,k)} R_b(m+1) e_{(\nu,k)}, d_\beta \right) \\
& \to (R_b(\nu) \mathbb{Z}[\xi] \oplus \theta_{m+1} R_b(\nu) \mathbb{Z}[\xi], d_\beta) \to 0,
\end{align}

where $\theta_{m+1}$ is a shift $q^{2\nu_{k+1}-4\nu_k} \lambda^2[1]$. By the snake lemma and Proposition 3.20, it descends in the homology to a long exact sequence

$$\cdots \to H^1(R_b(\nu) \mathbb{Z}[\xi] \oplus \theta_{m+1} R_b(\nu) \mathbb{Z}[\xi], d_\beta) \to H^0(q^{-2} R_b(\nu) e_{(m-1,k)} \otimes_{m-1} e_{(m-1,k)} R_b(\nu), d_\beta) \to H^0(e_{(\nu,k)} R_b(m+1) e_{(\nu,k)}, d_\beta) \to H^0(R_b(\nu) \mathbb{Z}[\xi] \oplus \theta_{m+1} R_b(\nu) \mathbb{Z}[\xi], d_\beta) \to \cdots$$

of $(H^*(R_b(\nu), d_\beta), H^*(R_b(\nu), d_\beta)) \simeq \left(R^{\beta}_p(\nu), R^{\beta}_p(\nu)\right)$-bimodules.

**Proposition 3.23.** We have isomorphisms of $R^{\beta}_p(\nu)$-bimodules for $r_k = \nu_{k+1} - 2\nu_k \geq 0$,

$$H^p(R^{\beta}_b(\nu) \mathbb{Z}[\xi] \oplus \theta_{m+1} R^{\beta}_b(\nu) \mathbb{Z}[\xi], d_\beta) \simeq \begin{cases} 
\bigoplus_{t=0}^{r_k-1} q^{2t} R^{\beta}_p(\nu) & \text{if } p = 0, \\
0 & \text{if } p \neq 0,
\end{cases}$$

where $R^{\beta}_p(\nu)$ denotes the $p$th graded component of $R^{\beta}_p(\nu)$. This proposition provides a detailed description of the homology classes in terms of the $R^{\beta}_p(\nu)$-bimodules, which are crucial for understanding the structure of the derived category. The proof involves a careful analysis of the differential $d_\beta$ and its effects on the bimodules, leading to the identification of the homology groups in terms of the graded components of the bimodules. The result is a significant tool in the study of derived categories and their applications in algebra and geometry.
and for $r_k \leq 0$,

$$H^p(R_b(\nu)\mathbb{Z}[\xi] \oplus \theta_{m+1}R_b(\nu)\mathbb{Z}[\xi], d_\beta) \cong \begin{cases} 0 & \text{if } p \neq 1, \\ \lambda_k^2q^{4\nu_k-2\nu_{k+1}} \bigoplus_{t=0}^{-r_k-1} q^{2t}R^\beta_p(\nu) & \text{if } p = 1. \end{cases}$$

**Proof.** Take $0 \oplus \xi^a \in R_b(\nu)\mathbb{k}[\xi] \oplus \theta_{m+1}R_b(\nu)\mathbb{k}[\xi]$. For $r_k \geq 0$, we observe that $d_\beta(0 \oplus \xi^a)$ yields a monic polynomial, up to sign, with leading term $\xi^{r_k+a}$. For $r_k \leq 0$, we have $d_\beta(0 \oplus \xi^a) = 0$ whenever $a < -r_k$, and $d_\beta(0 \oplus \xi^{-r_k}) = 1 \oplus 0$.

Therefore, if $r_k = \nu_{k+1} - 2\nu_k \geq 0$, then the long exact sequence gives a short exact sequence

$$q^{-2}R^\beta_p(\nu)e_{(m-1,k)} \otimes_{m-1} e_{(m-1,k)}R^\beta_p(\nu) \hookrightarrow e_{(\nu,k)}R^\beta_p(m+1)e_{(\nu,k)} \quad \rightarrow \quad q^{-r_k-1} \bigoplus_{t=0}^{r_k-1} q^{2t}R^N_p(\nu),$$

and if $r_k \leq 0$ it yields, taking the degree of the connecting homomorphism into account,

$$q^{-2-2}R^\beta_p(\nu) \hookrightarrow q^{-2}R^\beta_p(\nu)e_{(m-1,k)} \otimes_{m-1} e_{(m-1,k)}R^\beta_p(\nu) \quad \rightarrow \quad e_{(\nu,k)}R^\beta_p(m+1)e_{(\nu,k)}.$$

Clearly the map on the left in the first sequence splits, since the projection $e_{(\nu,k)}R_b(m+1)e_{(\nu,k)} \rightarrow R_b(\nu)\mathbb{Z}[\xi]$ was already invertible (this is obvious from the diagrammatic point of view: the splitting morphism is given by adding a strand labeled $k$ with dots on it corresponding to the power of $\xi$). For the second sequence, it is not that easy to see that the injection morphism splits, however it can be done, for example using the same arguments as in [8, Proof of Theorem 5.2] (see also [21, §4.2]). In conclusion, both short exact sequences split into direct sum decompositions.

The short exact sequence (17) for the simple root $\alpha_0$ can also be enhanced in terms of dg-bimodules by equipping the cokernel with a trivial differential, so that it induces a similar short exact sequence on the homology.

All this adds up to the following:

**Theorem 3.24.** There is a short exact sequence of $(R^\beta_p(m), R^\beta_p(m))$-bimodules

$$0 \rightarrow q^{-2}R^\beta_p(m)e_{(m-1,0)} \otimes_{m-1} e_{(m-1,0)}R^\beta_p(m) \rightarrow e_{(m,0)}R^\beta_p(m+1)e_{(m,0)} \quad \rightarrow \quad R^\beta_p(m) \otimes \mathbb{k}[\xi] \oplus R^\beta_p(m)\theta_{m+1} \otimes \mathbb{k}[\xi] \rightarrow 0,$$

and the following isomorphisms for $k \in \Gamma \setminus \{0\}$:

$i)$ if $\nu_{k+1} \geq 2\nu_k$,

$$e_{(\nu,k)}R^\beta_p(m+1)e_{(\nu,k)} \cong q^{-2}R^\beta_p(\nu)e_{(m-1,k)} \otimes_{m-1} e_{(m-1,k)}R^\beta_p(\nu) \oplus R^\beta_p(\nu) \otimes \mathbb{k}[\xi]/(\xi^{r_k-2\nu_k}),$$
ii) if $\nu_{k+1} \leq 2\nu_k$,
$$q^{-2}R_p^\beta(\nu)e_{(m-1,k)} \otimes R_p^\beta(\nu) \cong e_{(m,k)}R_p^\beta(m+1)e_{(\nu,k)} \otimes q^{2(\nu_{k+1}-2\nu_k-1)}R_p^\beta(\nu) \otimes k[\xi]/(\xi^{2\nu_k-\nu_{k+1}}),$$

iii) for $i \neq j$,
$$e_{(m,i)}R_p^\beta(m+1)e_{(m,j)} \cong q^{-i\beta}R_p^\beta(m)e_{(m-1,j)} \otimes R_p^\beta(m)\text{ for }|i-j|>1,$$

The direct sums of Proposition 3.25 can be interpreted as direct sums of dg-modules, so that they descend to the homology. Then, the categorical Serre relations are interpreted in $R_p^\beta$ as the following.

**Proposition 3.25.** We have isomorphisms of $(R_p^\beta(m+2), R_p^\beta(m))$-bimodules
$$e_{(\nu,ij)}R_p^\beta(m+2)e_{(\nu)} \cong e_{(\nu,ij)}R_p^\beta(m+2)e_{(\nu)} \text{ for }|i-j|>1,$$
and of $(R_p^\beta(m+3), R_p^\beta(m))$-bimodules
$$q^{-1}e_{(\nu,ij)}R_p^\beta(m+2)e_{(\nu)} \oplus qe_{(\nu,ij)}R_p^\beta(m+2)e_{(\nu)} \cong e_{(\nu,ij)}R_p^\beta(m+2)e_{(\nu)} \oplus e_{(\nu,ji)}R_p^\beta(m+2)e_{(\nu)} \text{ for }i=j \pm 1.$$

### 3.2.3. Categorical action.

From now on, we will consider $R_p^\beta(\nu)$ as a bigraded superalgebra, with a quantum grading $q$ and a single $\lambda$-grading (corresponding to $\lambda_0$).

Again, let $F^\beta_i : R_p^\beta(\nu) \mod \rightarrow R_p^\beta(\nu+i) \mod$ be the functor of induction given by adding a strand of color $i$ to the right of a diagram, and $E^\beta_i : R_p^\beta(\nu+i) \mod \rightarrow R_p^\beta(\nu) \mod$ be a shift of its right adjoint:

$$F^\beta_i(M) = R_p^\beta(\nu+i)e_{(\nu,i)} \otimes R^\beta(\nu) M,$$
$$E^\beta_i(N) = q^{2\nu_i-\nu_i+1}\lambda^{-\delta_{i,0}}e_{(\nu,i)}N,$$

for $M \in R_p^\beta(\nu) \mod$ and $N \in R_p^\beta(\nu+i) \mod$.

A direct consequence of Theorem 3.24 and Proposition 3.25 is the following, which describes how $F^\beta_i$ and $E^\beta_i$ give a categorical action of $U_q(\mathfrak{gl}_{2n})$ on $\bigoplus_{\nu} R_p^\beta(\nu) \mod$.

**Theorem 3.26.** There is a short exact sequence of functors
$$0 \rightarrow E^\beta_0 F^\beta_0 1_\nu \rightarrow E^\beta_0 F^\beta_1 1_\nu \rightarrow \bigoplus_{[\beta_0+\nu_{k+1}-2\nu_k]} 1_\nu \rightarrow 0,$$

and isomorphisms

$$E^\beta_k F^\beta_k 1_\nu \cong F^\beta_k E^\beta_k 1_\nu \oplus [\nu_{k+1}-2\nu_k] 1_\nu \text{ if } \nu_{k+1} > 2\nu_k,$$

$$F^\beta_k E^\beta_k 1_\nu \cong E^\beta_k F^\beta_k 1_\nu \oplus [2\nu_k-\nu_{k+1}] 1_\nu \text{ if } \nu_{k+1} < 2\nu_k,$$

$$E^\beta_j F^\beta_j 1_\nu \cong F^\beta_j E^\beta_j 1_\nu,$$
for $i \neq j$ and $k \in I \setminus \{0\}$. There are also isomorphisms

\begin{align}
F_i^\beta F_j^\beta 1_\nu & \simeq F_j^\beta F_i^\beta 1_\nu & \text{for } |i - j| > 1, \\
\bigoplus_{j \in I} F_i^\beta F_j^\beta 1_\nu & \simeq F_i^\beta F_j^\beta F_i^\beta 1_\nu \oplus F_j^\beta F_i^\beta F_i^\beta 1_\nu & \text{for } i = j \pm 1.
\end{align}

As in Proposition 3.17:

**Proposition 3.27.** The functors $E_i^\beta$ and $F_i^\beta$ are exact (and send projectives to projectives).

3.2.4. **The categorification theorem.** Let $R_p^\beta(\nu) \text{-mod}_{lf}$ be the category of cone bounded, locally finite dimensional, left supermodules over $R_p^\beta(\nu)$, with degree zero morphisms. We define $R_p^\beta \text{-mod}_{lf} = \bigoplus_{\nu} R_p^\beta(\nu) \text{-mod}_{lf}$, and we equip it with the endofunctors $F_i^\beta$ and $E_i^\beta$. Recall that $M^p(\beta) = M^p((\beta)^n, (0)^n)$.

**Theorem 3.28.** The functors $F_i^\beta$ and $E_i^\beta$ induce an action of $U_q(\mathfrak{gl}_{2n})$ on the Grothendieck group of $R_p^\beta$ when specializing $\pi = -1$. With this action we have a $\mathbb{Q}((q, \lambda))$-linear isomorphism

$$G_0(R_p^\beta) \cong M^p(\beta),$$

of $U_q(\mathfrak{gl}_{2n})$ representations. Moreover, the isomorphism sends classes of projective objects to canonical basis elements and classes of simple objects to dual canonical basis elements.

**Proof.** By Theorem 3.26 and Proposition 3.27, the functors $F_i^\beta$ and $E_i^\beta$ descend to an action of $U_q(\mathfrak{gl}_{2n})$ on the Grothendieck group, giving it a structure of a weight module.

Seeing that $R_p(\nu)$ has the same decomposition into idempotents as $R(\nu)$ from [10] we can transpose the arguments in [10, §3.2] such that $R_q(\nu)$ also categorify $U_q^- (\mathfrak{gl}_{2n})$ as $k$-vector spaces (but not as a bialgebra). Combining this with the arguments from [8, §6], we conclude that $G_0(R_p^\beta)$ is generated by a highest weight vector $[R_p^\beta(\emptyset)] = [\mathbb{Q}]$.

Since $E_i$ and $F_{\pm k}$ act as zero on $[R_p^\beta(\emptyset)]$, it is a highest weight module with 1-dimensional highest weight space, isomorphic to $V(0, \ldots, 0)$ as $U_q(1)$-module. Thus, there is a surjective morphism

$$\gamma : M^p(\beta) \to G_0(R_p^\beta),$$

sending the highest-weight vector to $[R_p^\beta(0)]$. The module $M^p(\beta)$ being irreducible by the Jantzen’s criterion (see [7, §9.13] or [18] for the non-quantum case), $\gamma$ must be an isomorphism.

**Remark 3.29.** A dg-analogue of this theorem exists and is presented in [21, §6]. However it is not needed for our purposes here.

3.2.5. **A parabolic 2-Verma module.** Let $\text{Fun}(R_p^\beta \text{-mod}_{lf})$ be the 2-category with objects being the categories $R_p^\beta(\nu) \text{-mod}_{lf}$ for all $\nu$, and 1-hom being the abelian categories of functors, with grading preserving natural transformations. The parabolic 2-Verma module $\mathcal{M}^p(\beta)$ is the completion under extensions of the full sub 2-category of $\text{Fun}(R_p^\beta \text{-mod}_{lf})$ whose

- objects $1_\mu = R_p^\beta(\mu) \text{-mod}_{lf}$ are indexed by weights $\mu \in \Lambda^\beta_{n,n}$,
• 1-morphisms are compositions of locally finite, left bounded direct sums of grading shifts of the various functors \( \text{Id}, F^\beta_i \) and \( E^\beta_i \) (in particular \( Q \) is a 1-morphism as being a cone bounded sum of the identity).

The completion under extensions means that whenever there is a short exact sequence of functors \( A \to B \to C \) in \( \text{Fun}(R^\beta_\mathfrak{p}\text{-mod}_\mathfrak{t}) \) with \( A \) and \( C \) in \( \mathcal{M}^\beta(\mathfrak{p}) \), then \( B \) is also in \( \mathcal{M}^\beta(\mathfrak{p}) \). In particular, this means the 1-hom of \( \mathcal{M}^\beta(\mathfrak{p}) \) are Quillen exact (see [19, §6.3]).

Let \( \hat{\mathcal{U}}(\mathfrak{sl}_n) \) denote Khovanov–Lauda and Rouquier’s 2-Kac–Moody algebra from [11, 25] (which are the same by [3]). The following result is immediate, thanks to Theorem 3.26 and the fact that \( E_i \) and \( F_i \) are adjoint.

**Lemma 3.30.** There is a 2-action of \( \hat{\mathcal{U}}(\mathfrak{sl}_n \times \mathfrak{sl}_n) \) on \( \mathcal{M}^\beta(\mathfrak{p}) \).

The lemma implies that in particular, the categorified \( q \)-Schur algebra \( S(0, n) \) from [16] acts on \( \mathcal{M}^\beta(\mathfrak{p}) \).

### 3.3. Recovering the finite dimensional irreducible \( V((N)^n, (0)^n) \)

For each \( N \geq 0 \) we introduce a differential \( d_N \) on \( R^\beta_\mathfrak{p} \) by declaring that \( d_N \) acts trivially on crossings and on dots, and sends a labeled floating dot \( \omega^a_0 \) to the following polynomial on dots

\[
d_N(\omega^a_0) = (-1)^{N+a-\ell_0} \left( \sum_{r=0}^{\ell_\pm} h_{\alpha-\ell_0+\ell_\pm+N+1-r}(\ell_0) e_r(\ell_\pm) \right),
\]

(24)

to be placed at the same height as the floating dot. Again, it is enough to define it on tight floating dots:

\[
d_N \left( \begin{array}{c|c|c|c}
0 & \cdots & 0 \\
1 & i & k & 1 \end{array} \right) = -N \begin{array}{c|c|c|c}
1 & i & k & 1 \end{array}
\]

This differential has degree \( \text{deg}(d_N) = (2N, -2) \) and parity 1.

**Proposition 3.31.** ([21, §4.2.4]) The differential \( d_N \) gives \( R^\beta_\mathfrak{p} \) the structure of a \( \mathbb{Z} \)-graded dg-algebra, with homological grading, called \( d_N \)-grading, given by counting the number of floating dots with subscript 0.

**Lemma 3.32.** ([21, Proposition 4.19]) The DG-algebra \( (R^\beta_\mathfrak{p}, d_N) \) is formal. Moreover, it is quasi-isomorphic to the cyclotomic KLR algebra\(^2\) \( R^\Lambda_\mathfrak{gl}_{2n} \), with \( \Lambda = ((N)^n, (0)^n) \).

Passing to the bounded derived category \( \mathcal{D}^r(R^\beta_\mathfrak{p}, d_N) \) of the category of bigraded, left, compact \( (R^\beta_\mathfrak{p}, d_N) \)-modules and defining functors \( F^N_i, E^N_i, i = -k + 1, \ldots, k - 1 \), in the same way as \( F^\beta_i \) and \( E^\beta_i \) we have the following.

**Proposition 3.33.** The functors \( F^N_i \) and \( E^N_i \) induce an action of \( U_q(\mathfrak{gl}_{2n}) \) on the Grothendieck group of \( \mathcal{D}^r(R^\beta_\mathfrak{p}, d_N) \). With this action, \( K_0(\mathcal{D}^r(R^\beta_\mathfrak{p}, d_N)) \) is isomorphic to the irreducible representation of highest weight \( ((N)^n, (0)^n) \).

\(^2\)We use \( \mathfrak{gl}_{2n} \)-weights rather than \( \mathfrak{sl}_{2n} \)-weights in our version of KLR algebras.
Proof. The proof is given by the Lemma 3.32 and Kang–Kashiwara’s results [8, Theorem 6.2] (this is also reproved in [21] using similar arguments as in §3.2.2). □

3.4. A parabolic 2-V erma module II: the case of $\bar{p} \subseteq U_q(\mathfrak{gl}_{2n-1})$. We now treat the $\mathfrak{gl}_{2n-1}$ (the “reduced”) case. Recall the $\mathfrak{gl}_{2n}$-weight $((\beta)^{n-1}, 1, (0)^{n-1})$ is translated into the $\mathfrak{sl}_{2n}$-weight $((0)^{n-2}, \beta - 1, 1, (0)^{n-2})$. Shifted Verma modules are given by changing the minimal integer superscript of the floating dots. Indeed, it is clear from Theorem 3.24 that shifting up the $q$-degree of the tight floating dots by $2r$ (which is equivalent to shift the superscript by $r$) will change the cokernel of the short exact sequence of Theorem 3.26 into

$$\bigoplus_{[\beta_0 + r + \nu^*_1 - 2\nu_0]} 1_{\nu}.$$ 

We also know that the choice of cyclotomic quotient for the $\bar{p}$-KLR algebra determines the highest weight. Therefore, we define the following.

Definition 3.34. Let $R_{\bar{p}}$ (resp. $R_{\tilde{p}}$) be defined as $R_p$ (resp. $R_0$) but where floating dots with subscript 0 are allowed to have label down to $-1$, and $I = \{-n + 2, -n + 3, \ldots, n - 1\}$. In this context, a tight floating dot is a floating dot with label $-1$ at the immediate right of the leftmost strand. We define $R_{\bar{p}}^{\beta}$ to be the quotient of $R_{\bar{p}}$ by the two-sided ideal generated by all diagrams of the form

\[
\begin{array}{ccc}
& i & j & \cdots & k \\ 
\end{array}
\quad i \neq 0, i \neq 1,
\begin{array}{ccc}
& 1 & j & \cdots & k \\ 
\end{array}
\]

As before, we can obtain $R_{\bar{p}}^{\beta}$ as the homology of $R_{\bar{p}}$ equipped with a differential $d_{\beta}$ sending everything to zero, except tight floating dots:

$$d_{\beta}(\omega_i) = \begin{cases} 
0 & \text{if } i = 0, \\
-x_1 & \text{if } i = 1, \\
-1 & \text{otherwise.}
\end{cases}$$

Define $\mathcal{F}_{\beta}^{\beta} : R_{\bar{p}}^{\beta}(\nu) \text{-mod} \rightarrow R_{\bar{p}}^{\beta}(\nu + i) \text{-mod}$ and $\mathcal{E}_{\beta}^{\beta} : R_{\bar{p}}^{\beta}(\nu + i) \text{-mod} \rightarrow R_{\bar{p}}^{\beta}(\nu) \text{-mod}$ as before

$$\mathcal{F}_{i}^{\beta} (M) = R_{\bar{p}}^{\beta}(\nu + i)e(\nu, i) \otimes R_{\bar{p}}^{\beta}(\nu) M,$$

$$\mathcal{E}_{i}^{\beta} (N) = q^{2\nu_i - \nu_i + 1 - \delta_{i,0} + \delta_{i,1}} \lambda^{\delta_{i,0}} e(\nu, i) N.$$

All results from §3.2 can be applied almost unchanged to this context. Therefore we obtain the following two theorems.

Theorem 3.35. There is a short exact sequence of functors

\[0 \rightarrow \mathcal{F}_{0}^{\beta} \mathcal{E}_{0}^{1_\nu} \rightarrow \mathcal{E}_{0}^{\beta} \mathcal{F}_{0}^{1_\nu} \rightarrow \bigoplus_{[\beta_0 + \nu^*_1 - 2\nu_0 - 1]} 1_{\nu} \rightarrow 0,\]
and isomorphisms

\begin{align}
&\mathcal{E}_k^\beta F_k^\beta 1_{\nu} \simeq F_k^\beta \mathcal{E}_k^\beta 1_{\nu} \oplus [\nu_{k+1} - 2\nu_k + \delta_{k,1}]_{q^\nu} 1_{\nu} \quad \text{if } \nu_{k+1} + \delta_{k,1} \geq 2\nu_k, \\
&\mathcal{F}_k^\beta E_k^\beta 1_{\nu} \simeq E_k^\beta \mathcal{F}_k^\beta 1_{\nu} \oplus [2\nu_k - \nu_{k+1} - \delta_{k,1}]_{q^\nu} 1_{\nu} \quad \text{if } \nu_{k+1} + \delta_{k,1} \leq 2\nu_k, \\
&\mathcal{E}_i^\beta F_j^\beta 1_{\nu} \simeq F_j^\beta \mathcal{E}_i^\beta 1_{\nu},
\end{align}

for \( i \neq j \) and \( k \in \mathbb{N} \setminus \{0\} \). There are also isomorphisms

\begin{align}
&\mathcal{F}_i^\beta F_j^\beta 1_{\nu} \simeq F_j^\beta \mathcal{F}_i^\beta 1_{\nu} \quad \text{for } |i - j| > 1, \\
&\mathcal{F}_i^\beta F_j^\beta 1_{\nu} \simeq F_j^\beta \mathcal{F}_i^\beta 1_{\nu} \oplus F_j^\beta \mathcal{F}_i^\beta 1_{\nu} \quad \text{for } i = j \pm 1.
\end{align}

Recall the parabolic Verma module \( M^\beta(\beta) = M^\beta((\beta)^{n-1}, 1, (0)^{n-1}) \) from §2.3.1.

**Theorem 3.36.** The functors \( \mathcal{F}_i^\beta \) and \( \mathcal{E}_i^\beta \) induce an action of \( U_q(\mathfrak{gl}_{2n-1}) \) on the Grothendieck group of \( R_\beta^\beta \) when specializing \( \pi = -1 \). With this action we have an isomorphism

\[ G_0(R_\beta^\beta) \cong M^\beta(\beta), \]

of \( U_q(\mathfrak{gl}_{2n-1}) \) representations.

Now equip \( R_\beta^\beta \) with the differential \( d_N(\omega_0^{-1}) = -x_1^N \). This gives a formal dg-algebra quasi-isomorphic to the cyclotomic KLR algebra \( R_\Lambda \) with \( \Lambda = ((0)^{n-2}, N, 1, (0)^{n-2}) \). Applying the same arguments as in §3.3 and defining \( \mathcal{E}_i^N \) and \( \mathcal{F}_i^N \) in the obvious way, we get the following proposition.

**Proposition 3.37.** The functors \( \mathcal{F}_i^N \) and \( \mathcal{E}_i^N \) induce an action of \( U_q(\mathfrak{gl}_{2n-1}) \) on the Grothendieck group of \( \mathcal{D}^\beta(R_\beta^\beta, d_N) \). With this action, \( K_0(\mathcal{D}^\beta(R_\beta^\beta, d_N)) \) is isomorphic to the irreducible representation of highest weight \( ((N+1)^{n-1}, 1, (0)^{n-1}) \).

4. **Link homology**

4.1. **Braiding.** By a well-known construction due to Cautis [4], we know how to associate a chain complex in the 2-category \( Kom(\mathcal{U}(\mathfrak{sl}_n)) \) of complexes of the Hom-categories of \( \mathcal{U}(\mathfrak{sl}_n) \), called a RICKARD COMPLEX, which satisfies the braid relations up to homotopy.

For a positive crossing \( \mathcal{T}_i \) between the \( i \)th and \( (i+1) \)th strands the Rickard complex is

\[
\mathcal{T}_i \mathbb{1}_{\mu} = F_i^{(\mu_i)} \mathbb{1}_{\mu} \xrightarrow{d_1} q F_i^{(\mu_i+1)} E_i \mathbb{1}_{\mu} \xrightarrow{d_2} \cdots \xrightarrow{d_r} q^r F_i^{(\mu_i+r)} E_i \mathbb{1}_{\mu} \xrightarrow{d_{r+1}} \cdots,
\]

with \( F_i^{(\mu_i)} \mathbb{1}_{\mu} \) in homological degree zero, if \( \mu_i > 0 \). The Rickard complex of a negative crossing is the left adjoint to \( \mathcal{T}_i \mathbb{1}_{\mu} \) in \( Kom(\mathcal{U}(\mathfrak{sl}_n)) \). There are similar formulas for \( \mu_i \leq 0 \) but we will not use them. An explicit description of the differentials can be obtained easily from Lemma 3.30 by noticing that as usual, the differential \( d_r \) in the Rickard complex (31) is given by the counit of the adjunction \( F_i \dashv E_i \).

In our context the Rickard complex is always truncated at the second term, so that it looks like

\[
\mathbb{1}_{\mu} \xrightarrow{q^r} q^{-1} F_i^\beta E_i^\beta \mathbb{1}_{\mu},
\]
for a positive crossing, with \( \eta' \) being the unit of the adjunction \( E_i^\beta \to F_i^\beta \). For a negative crossing, it is

\[
qF_i E_i^\beta 1_\mu \overset{\varepsilon}{\to} 1_\mu,
\]

where \( \varepsilon \) is the counit of the adjunction \( F_i^\beta \to E_i^\beta \), and \( 1_\mu \) is in homological degree 0. In our case the \( \mathfrak{gl}_{2n} \) weights around a crossing are always \( (\ldots, 1, 1, \ldots) \), hence we have that \( F_i^\beta E_i^\beta 1_\mu \cong E_i^\beta F_i^\beta 1_\mu \).

This means we can use instead the complex

\[
1_\mu \overset{\eta}{\to} q^{-1} E_i^\beta F_i^\beta 1_\mu,
\]

for a positive crossing, where \( \eta \) is now the unit of the adjunction \( F_i^\beta \to E_i^\beta \), thanks to [3]. The same applies for negative crossings. This point of view will be particularly useful to lift the Rickard complex to a dg-enhancement in \( \S 4.6 \).

Following Cautis’s construction in [4], we associate a Rickard complex \( C'(b) \) in the 2-category \( Kom(\mathcal{M}^p(\beta)) \) of complexes in the Hom-categories of \( \mathcal{M}^p(\beta) \), to each braid diagram \( b \) on \( k \) strands (see \( \S 3.2.5 \) for the definition of \( \mathcal{M}^p(\beta) \)). This gives a braiding on the homotopy category of \( Kom(\mathcal{M}^p(\beta)) \).

In the following we denote by \([n]\) a shift in the homological grading of the Rickard complex, which consists of shifting \( n \) units to the right the complex in consideration.

4.2. Invariance under the Markov moves. Closing the diagram for \( b \) consists of precomposing \( C'(b) \) with the appropriate word on functors \( F_{n+1}^\beta, \ldots, F_{n-1}^\beta \) and composing it with the appropriate word from \( E_{n+1}^\beta, \ldots, E_{n-1}^\beta \), following the patterns in (1). This results in a chain complex \( C'(\text{cl}(b)) \) in \( Kom(\mathcal{M}^p(\beta)) \), which is a complex of endofunctors of the block corresponding to the highest weight in \( M^p((\beta)^n, (0)^n) \), that is, a complex of \( k \)-vector spaces. For the sake of simplifying notations, we will forget the \( \beta \) superscript in \( F_i^\beta \) and \( E_i^\beta \) from now on.

**Lemma 4.1.** The homotopy type of the chain complex \( C'(\text{cl}(b)) \) is invariant under isotopy of ladder diagrams:

\[
\begin{align*}
\cdots & \quad \cdots = \cdots \quad \cdots, \\
1 \quad 0 & \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1
\end{align*}
\]

*Proof.* These are straightforward consequences of respectively (19) and (20).

**Proposition 4.2.** The homotopy type of the chain complex \( C'(\text{cl}(b)) \) is invariant under the Markov moves, up to parity and an overall grading shift.
Proof. We start with the first Markov Move. Suppose we have a web in the doubled Schur algebra $\tilde{S}_{q,\beta}(n-1, n)$ coming from a closure of a braid diagram. We need to prove that

\begin{equation}
\cdots \quad B_i \quad \cdots = \cdots \quad B_i \quad \cdots,
\end{equation}

where $B_i = F_i E_i \cong E_i F_i$ involve the $i$th and the $(i + 1)$th strands in the original braid. We have similar relations for either downward oriented strands and for the bottom part of the braid closure. Then, the first Markov move can be decomposed in a sequence of moves like

\begin{equation}
\cdots \quad B_1 \quad \cdots = \cdots \quad B_1 \quad \cdots,
\end{equation}

(to avoid cluttering we have dropped the $\beta$'s from the pictures, since it is clear where to place them), and similar for the bottom part of the closure.

Relation (32) requires an isomorphism of 1-morphisms in $\mathcal{M}_p(\beta)$

\[ E_i E_{i-1} F_i \mathbb{1}_{(...,1,1,...)} \cong E_{i-1} F_{i-1} E_i E_{i-1} \mathbb{1}_{(...,1,...)} \]

which is proved in [22, Lemma 3.19], using Lemma 4.1.

To prove relation (33) we write $L_{(\beta-1,\beta-1,1,1)}$ and $R_{(\beta-1,\beta-1,1,1)}$ for the left-hand-side and right-hand side of (33), respectively. We write also $E_{\pm}$ instead of $E_{\pm 1}$, and the same for $F_{\pm}$. Using Theorem 3.26 we have the following isomorphisms

\[ L_{(\beta-1,\beta-1,1,1)} = E_0 E_+ E_- E_0 F_+ E_+ \mathbb{1}_{(\beta-1,\beta-1,1,1)} \]

\[ \cong \bigoplus_{[2]} E_0 E_+ E_0 E_- \mathbb{1}_{(\beta-1,\beta-1,1,1)} \]

\[ \cong E_0 E_0 E_+ \mathbb{1}_{(\beta-1,\beta-1,1,1)} \],

and

\[ R_{(\beta-1,\beta-1,1,1)} = E_0 E_+ E_- E_0 F_- E_- \mathbb{1}_{(\beta-1,\beta-1,1,1)} \]

\[ \cong \bigoplus_{[2]} E_0 E_+ E_0 E_- \mathbb{1}_{(\beta-1,\beta-1,1,1)} \]

\[ \cong E_0 E_0 E_+ \mathbb{1}_{(\beta-1,\beta-1,1,1)} \].

We now prove the second Markov move. Consider diagrams $D_0$ and $D_1^-$ that differ as below.

\[ D_1^- = \quad \quad \quad \quad , \quad \quad D_0 = \quad \quad \quad \quad \uparrow \quad \quad . \]
The complex for $D^-_1$ induced from the Rickard complex (31) is $E_0 T F_0 \mathbb{I}_{(\beta-0,0,1)}$, which is

$$C'(D^-_1) \cong q E_0 F_1 E_1 F_0 \mathbb{I}_{(\beta-0,0,1)} \xrightarrow{d} E_0 F_0 \mathbb{I}_{(\beta-0,0,1)},$$

with the second term in homological degree zero.

Switching $E_0$ with $F_1$ and $E_1$ with $F_0$ on the term in homological degree $-1$ and applying the exact sequence (18) to $E_0 F_0 \mathbb{I}_{(\beta-0,1,0)}$ gives an isomorphism (note that in this case $F_0 E_0 \mathbb{I}_{(\beta-0,1,0)}$ is zero)

$$q E_0 F_1 F_0 \mathbb{I}_{(\beta-0,0,1)} \cong \bigoplus_{[\beta_0-1]} q F_1 E_1 \mathbb{I}_{(\beta-0,0,1)}$$

with $\deg_q(\xi) = 2$, and using the fact that $F_1 E_1 \mathbb{I}_{(\beta-0,0,1)} \cong \mathbb{I}_{(\beta,0,0,1)}$.

Applying the short exact sequence (18) to the term in homological degree 0 gives

$$E_0 F_0 \mathbb{I}_{(\beta-0,0,1)} \cong q \lambda \Pi k[\xi] \mathbb{I}_{(\beta-0,0,1)} \oplus q \lambda^{-1} k[\xi] \mathbb{I}_{(\beta-0,0,1)}.$$

Computing with the maps used above we get that $C'(D^-_1)$ is isomorphic to the complex

$$C'(D^-_1) \cong \begin{pmatrix} q \lambda \Pi k[\xi] \mathbb{I}_{(\beta-0,0,1)} \\ q^3 \lambda^{-1} k[\xi] \mathbb{I}_{(\beta-0,0,1)} \end{pmatrix} \xrightarrow{\psi} \begin{pmatrix} q \lambda \Pi k[\xi] \mathbb{I}_{(\beta-0,0,1)} \\ q \lambda^{-1} k[\xi] \mathbb{I}_{(\beta-0,0,1)} \end{pmatrix},$$

with differential

$$\psi = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix},$$

where $f : q^3 \lambda^{-1} k[\xi] \mathbb{I}_{(\beta-0,0,1)} \cong q \lambda^{-1} k[\xi] \mathbb{I}_{(\beta-0,0,1)} \to q \lambda^{-1} k[\xi] \mathbb{I}_{(\beta-0,0,1)}$ is the inclusion map. After the removal of all acyclic subcomplexes we get that $C'(D^-_1)$ is homotopy equivalent to the complex

$$C'(D^-_1) \cong 0 \to q \lambda^{-1} k \mathbb{I}_{(\beta-0,0,1)},$$

with $q \lambda^{-1} k \mathbb{I}_{(\beta-0,0,1)}$ in homological degree 0. We have therefore that the complexes $C'(D^-_1)$ and $q \lambda^{-1} C'(D_0)$ are homotopy equivalent.

Similarly for a diagram $D^+_1$ containing a positive crossing, we show that the complex

$$E_0 F_0 \mathbb{I}_{(\beta-0,0,1)} \to q^{-1} E_0 F_1 E_1 F_0 \mathbb{I}_{(\beta-0,0,1)} \cong$$

$$q \lambda \Pi k[\xi] \oplus q \lambda^{-1} k[\xi] \mathbb{I}_{(\beta-0,0,1)} \to q^{-1} \lambda \Pi k[\xi] \oplus q \lambda^{-1} k[\xi] \mathbb{I}_{(\beta-0,0,1)},$$

is homotopy equivalent to the complex

$$0 \to q^{-1} \lambda \Pi k \mathbb{I}_{(\beta-0,0,1)},$$

with 0 in homological degree 0. Thus, the complexes $C'(D^+_1)$ and $q^{-1} \lambda \Pi C'(D_0)[1]$ are homotopy equivalent.

Define the normalized chain complex $C(\text{cl}(b)) = C'(\text{cl}(b)) \langle n_+ - n_-, n_- - n_+ \rangle$ where as usual, $n_\pm$ is the number of positive/negative crossings in $\text{cl}(b)$, and where we merge the homological grading of the Rickard complex, denoted $r$, with the $p$-grading to get a total homological degree

$$\deg_{h} = \deg_{r} - \deg_{p}.$$
Remark 4.3. We use (34) instead of $\deg_h = \deg_r + \deg_p$ because $d_N$ decreases the homological degree, while $d_r$ increases it.

Corollary 4.4. The homology groups $H(b)$ of $C(\mathfrak{cl}(b))$ are triply-graded link invariants and their bigraded Euler characteristic is the HOMFLY-PT polynomial of the closure of $b$.

A version of $\mathfrak{M}^p(\beta)$ for divided powers of the $F_i$’s and $E_i$’s could be used to construct a version of HOMFLY-PT homology for links colored by minuscule representations of $\mathfrak{gl}_N$, as the one constructed by Mackaay-Stošić-Vaz in [15] and Webster-Williamson [27]. Moreover, the differential $d_N$ would give rise to a spectral sequence to colored $\mathfrak{gl}_N$-Khovanov-Rozansky link homology, as the one constructed by Wedrich in [28]. However, proving a version of the first exact sequence from Theorem 3.24 for divided powers might be a nontrivial problem.

4.3. $H(b)$ is isomorphic to the Khovanov-Rozansky HOMFLY-PT link homology. We now show that our link homology is equivalent to the HOMFLY-PT link homology by Khovanov and Rozansky [13, 9] by proving that $H(L)$ is isomorphic with Rasmussen’s version of HOMFLY-PT homology in [24].

Theorem 4.5. For every braid $b$ the homology $H(b)$ is isomorphic with Khovanov-Rozansky HOMFLY-PT link homology $\text{HKR}(b)$.

Theorem 4.5 gives us an equivalence in a weak sense. We conjecture the equivalence is in fact stronger, in the following sense. The Soergel category $\mathcal{SC}_n$ from [6] acts on $\mathfrak{M}^p(\beta)$, in particular, on its $((\beta - 1)^n, (1)^n)$-weight space (this action goes through the categorified $q$-Schur algebra $S(n,n)$ (see [16, §6]), which also acts on $\mathfrak{M}^p(\beta)$, as explained at the end of §3.2.5). Composing with the operation of closing the braid on the top with the correct sequence of $E_i$’s, following the pattern in (1), gives a functor $\Phi$ from $\mathcal{SC}_n$ to the category $\text{Ab}_{s,\lambda}$ of bigraded abelian groups (after forgetting the parity). Notice that the same pattern, but with $F_i$’s, has to be used on the bottom of the diagram to create the weight on which $\mathcal{SC}_n$ acts.

Conjecture 4.6. The functor $\Phi$ is isomorphic with the Hochschild homology functor.

We now prove Theorem 4.5. We assume the reader is familiar with [24]. Let $L$ be a link presented as the closure of a braid $b$ in $n$ strands. Recall that the process of closing $b$ amounts to composing a word in $E_i$’s with the Rickard complex for $b$ (after adding $n$ parallel strands at its right) and with a word in $F_i$’s. Of course, the closure procedure extends canonically to webs. Let $S$ be the (polynomial) ring in the dots on the $F_i$’s used to form the closure of a web $\Gamma$.

Lemma 4.7. For every web $\Gamma$, $H(\Gamma)$ is a free module over $S$.

Proof. The proof follows the same reasoning as the proof of Rasmussen of an analogous result using matrix factorizations [24, Proposition 4.8] which is based on an induction scheme introduced by Wu [29, §3]. The only thing we need to check are the MOY relations 0 to III from §4.2 in [24]. MOY relations II and III are already satisfied in $\mathcal{U}(\mathfrak{sl}_n)$, and MOY relations 0 and I are a direct consequence of the short exact sequence (18) in Theorem 3.26, when applied to the weights $(\ldots, \beta - 0, 0, \ldots)$ and $(\ldots, \beta - 0, 1, \ldots)$, since one of the terms in the exact sequence always act as the zero functor on these weights. □
Note that in [24] the total homological grading (the powers of $t$ in the Euler characteristic) is a linear combination of the gradings $gr_h$ and $gr_v$ introduced there. The minus sign occurring in the numerator of the homology of the unknot in [24] comes from a $t$. Since we work with bigraded superrings we can concentrate the homology of the unknot in homological degree 0. To ease the conversion between the several gradings used here and the ones used by Rasmussen in [24] and to make the terminology consistent with his, we will refer and treat the parity in our construction as a $\mathbb{Z}/2\mathbb{Z}$-grading denoted $p$. Recall the total homological grading $deg_h$ in our version of $H(b)$.

**Proof of Theorem 4.5.** Since the braiding in both constructions is the Rickard complex, the proof follows at once from Lemma 4.7, together with the fact that both constructions satisfy the MOY relations. Our $q$, $\lambda$ and $h$ gradings agree with the gradings $q$, $2\, deg_h$ and $deg_v$ in [24]. The only difference is how Rasmussen defines the Euler characteristic, which is translated in the fact that our $h$-grading is the difference of $deg_v$ with $deg_p$. □

4.4. **Khovanov-Rozansky’s $\mathfrak{gl}_N$-link homologies.** Using the 2-representation of $\mathfrak{gl}_{2n}$, constructed from the cyclotomic KLR algebra $R((0)^{n-1},N,(0)^{n-1})$ as input instead of $\mathfrak{gl}_N^p(\beta)$, results in Khovanov and Rozansky’s $\mathfrak{gl}_N$-link homology $H_N(L)$ from [12]. This follows at once from the work of Mackaay and Yonezawa [17].

4.5. **Reduced homologies.** Using the parabolic subalgebra $\mathfrak{p} \subseteq U_q(\mathfrak{gl}_{2n-1})$ and the highest weight $((\beta)^{n-1}, 1, (0)^{n-1})$ in the construction above yields reduced versions $\overline{H}$ (resp. $\overline{H}_N$) of $H$ (resp. $H_N$). All the results in the preceding subsections have analogues for the case of reduced homologies, and are proved essentially in the same way as above.

There is one subtlety to take into account when claiming the equivalence with reduced Khovanov-Rozansky homologies. Recall that in the case of the reduced versions from [12, 13, 24] the abelian groups $\overline{H}(L, i)$ and $\overline{H}_N(L, i)$ are invariants of the link $L$ together with a marked component $i$. With our choice of cutting out and open a diagram of a braid closure in §2.3.1, the outermost strand in our version (the one that is cut) corresponds to the preferred component $i$ of $\mathfrak{cl}(b)$ (as described in [24]) under the isomorphism between our reduced homologies and Khovanov and Rozansky’s. For the sake of completeness we state the main result for the record.

**Theorem 4.8.** The homology groups $\overline{H}(b)$ of $\overline{C}(\mathfrak{cl}(b))$ are triply-graded link invariants which are isomorphic to the reduced Khovanov-Rozansky homology groups when the outermost strand of $\mathfrak{cl}(b)$ is taken as the preferred one. The bigraded Euler characteristic of $\overline{H}(b)$ is the normalized HOMFLY-PT polynomial of the closure of $b$.

Using the cyclotomic KLR algebra $R((N)^{n-1},1,(0)^{n-1})$ for $\mathfrak{gl}_{2n-1}$ results in a reduced version of Khovanov and Rozansky’s $\mathfrak{gl}_N$-link homology.

4.6. **HOMFLY-PT to $\mathfrak{gl}_N$ spectral sequences converge at the second page.** Throughout this section we assume the reader is familiar with [24].

The differential $d_N$ introduced in §3.2.4 can be made to anticommute with the differentials in the Rickard complex (this consists of introducing a factor of $-1$ to the power of the homological degree in the definition of $d_N$) and therefore to induce a differential on $H(b)$, also denoted $d_N$. 
Recall that the differentials $d_N$ and $d_r$ are homogeneous with respect to the $q$, $\lambda$, $r$ and $p$ gradings and
\begin{equation}
\deg(d_N) = (2N, -2, 0, -1), \quad \deg(d_r) = (0, 0, 1, 0),
\end{equation}
where $(i, j, k, \ell)$ are the various gradings in the order as above.

Recall that to any double complex $(M, d', d'')$ we can associate two spectral sequences $\{E^I_q, d^I_q\}$ and $\{E^{II}_q, d^{II}_q\}$, which are induced by the two canonical filtrations. Moreover, we have that $E^I_2 = H(H(M, d''), d')$ and $E^{II}_2 = H(H(M, d'), d'')$, and if the double complex is bounded, then both spectral sequences converge to the total homology $H(M, d' + d'')$. We will also heavily use the fact that if $H(M, d'')$ (resp. $H(M, d')$) is concentrated in a single $d'$-degree (resp. $d''$-degree), then $E^I$ (resp. $E^{II}$) converges at the second page.

For a link $L$ presented in the form of a closure of a braid $b$, we form the bounded double complex $(C(L), d_r, d_N)$, with total grading $\deg_r - \deg_p$. Let $\{E^I_q, d^I_q\}$ and $\{E^{II}_q, d^{II}_q\}$ be respectively the spectral sequences induced by the $p$ and $r$-filtration. Since $(R^0_b, d_N)$ is formal with homology concentrated in a single homological degree, $E^{II}$ converges at the second page. Moreover, we know that $H(H(C(L), d_N), d_r) \cong \text{HKR}_N(L)$ thanks to §4.4. Therefore, $\{E^I_q, d^I_q\}$ is a spectral sequence whose $E^I_1$-page is $H(C(L), d_r) \cong \text{HKR}(L)$, which converges to $H(C(L), d + d_N) \cong \text{HKR}_N(L)$. We will now focus on proving the main theorem:

**Theorem 4.9.** For each link $L$ and for each $N > 0$,
\[ \text{HKR}_N(L) \cong H(\text{HKR}(L), d_N). \]

The proof of Theorem 4.9 is done in §4.6.1 below.

4.6.1. **Lifting the Rickard complex to a dg-enhancement.** To prove the main theorem we need some preparation. For starters, we observe that we can define $d_N$ from §3.3 already on $R_b$. Moreover, from the diagrammatic description of the algebras, it is clear that $d_N$ commutes with $d_\beta$ from §3.2.1.

**Proposition 4.10.** ([21, Proposition 4.13]) The homologies of the dg-algebras $(R_b, d_N + d_\beta)$ and $(R_b, d_N)$ are both concentrated in homological degree 0.

**Proof.** Similar as Proposition 3.20. \hfill \Box

The differentials $d_N$ and $d_\beta$ on $R_b$ define a double complex and we have:

**Proposition 4.11.** ([21, Proposition 4.19]) The two spectral sequences induced by $d_N$ and $d_\beta$ on $R_b$ both converge at the second page, and
\[ H(H(R_b, d_N), d_\beta) \cong H(H(R_b, d_\beta), d_N) \cong H(R_b, d_\beta + d_N) \cong R^\Lambda_{2n}. \]

Now we observe that we can lift the maps
\[ \eta : 1_{\mu} \to q^{-1}E^\beta_{1,d_\beta}1_{\mu} \quad \text{and} \quad \varepsilon' : qE_i^\beta E^d_i \to 1_{\mu}, \]
used to construct the Rickard complex, to maps of $R_b$-bimodules. In particular their lifts are given by (a shift) of the maps $\eta_0$ and $\varepsilon'_0$ below. We have
\[ \eta_0 : R_b(m) \to e_{(m,i)}R_b(m + 1)e_{(m,i)}, \quad x \mapsto x \otimes e_i, \]
which is the map that adds a vertical strand with label \( i \) at the right of a diagram (recall that \( F_i \) is given by induction that adds a vertical strand with label \( i \)). Also,

\[ \varepsilon'_0 : e_{(m,i)} R_b(m + 1) e_{(m,i)} \to R_b(m) , \]

is the map given by the projection onto \( R_b(m) \xi^0 \subset R_b(m) k[\xi] \) in Proposition 3.14. By §3.2.2, this map descends to homology w.r.t. \( d_\beta \) to the one that realizes the isomorphism \( F_i^\beta E_i^\beta \cong E_i^\beta F_i^\beta \), which we know is the counit thanks to the 2-action of \( \mathcal{U}(\mathfrak{sl}_n \times \mathfrak{sl}_n) \).

Diagrammatically, we view \( \eta_0 \) and \( \varepsilon'_0 \) as

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\vcenter{\hbox{\includegraphics[scale=0.5]{diagram2.png}}}
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All of this allows us to lift the Rickard complex to a complex of \((R_b, R_b)\)-bimodules (i.e. to an object in \( Kom(\mathfrak{m}^0(\beta)) \)), which we write \((C_0(L), d_r)\) for a link \( L \). Moreover, it is clear that \( d_r \) commutes with both \( d_\beta \) and \( d_N \), since both \( \eta_0 \) and \( \varepsilon'_0 \) are maps of \( dg \)-bimodules.

**Remark 4.12.** We do not expect the homology of the complex \((C_0(L), d)\) to be an invariant of links.

Moreover, by Proposition 3.14, \( \varepsilon'_0 \) is an epimorphism, and by Proposition 3.12, \( \eta_0 \) is a monomorphism. For the same reasons, we also get the following:

**Lemma 4.13.** The map \( \varepsilon'_0 \) is the left inverse of \( \eta'_0 \). Hence there is a decomposition

\[ e_{(m,i)} R_b(m + 1) e_{(m,i)} \cong R_b(m) \oplus \ker(\varepsilon'_0) \]

of \((R_b(m), R_b(m))\)-bimodules, and thus a decomposition of functors

\[ E_i F_i 1_\mu \cong 1_\mu \oplus \ker(\varepsilon'_0) . \]

**Proposition 4.14.** The complex \((C_0(L), d_r)\) is homotopy equivalent to a complex concentrated in a single \( d_r \)-degree.

**Proof.** The complex \((C_0(L), d_r)\) is obtained as a composition (think tensor product over \( R_b \)) of smaller complexes of the form \( \eta_0 : 1_\mu \leftarrow q E_i F_i 1_\mu \) and \( \varepsilon'_0 : q^{-1} E_i F_i 1_\mu \rightarrow 1_\mu \) (and with the cup and cap that close the braid, but those are complexes concentrated in a single homological degree). By Lemma 4.13, they are both homotopy equivalent to the complexes \( 0 \rightarrow \ker(\varepsilon'_0) \) and \( \ker(\varepsilon'_0) \rightarrow 0 \), respectively. Hence, \((C_0(L), d_r)\) is homotopy equivalent to a composition of those complexes, and is concentrated in a single \( d_r \)-degree. \( \square \)

**Corollary 4.15.** The homology of the complex \((C_0(L), d_r)\) is concentrated in a single homological degree.
**Lemma 4.16.** We have isomorphisms

\[ H(H(C_0(L), d_r), d_\beta) \cong H(H(C_0(L), d_\beta), d_r) \cong \text{HKR}(L), \]

\[ H(H(C_0(L), d_r), d_\beta + d_N) \cong H(H(C_0(L), d_\beta + d_N), d_r) \cong \text{HKR}_N(L). \]

**Proof.** For both cases we have two spectral sequences. By Proposition 3.20, Proposition 4.11 and Corollary 4.15, all homology groups appearing in the first pages are concentrated in single homological degree, i.e. on a single horizontal or vertical line. Therefore, we conclude the four spectral sequences all converge at the second page. \(\square\)

**Remark 4.17.** Note also that Lemma 4.16 could potentially brings a new tool to compute the HOMFLYPT homology since \(H(C_0(L), d_r)\) can be described explicitly, and thus one can construct the complex \(H(H(C_0(L), d_r), d_\beta)\). Diagrammatically, for a braid \(\sigma_{i_r} \ldots \sigma_{i_1}\) with \(\ell\) strands, it is given by a shift of the dg-bimodules (over \((\mathbb{Z}, 0)\)) induced by \(d_\beta\) on the quotient

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by the first isomorphism of Lemma 4.16, followed by Proposition 4.11, and the second isomorphism of Lemma 4.16.

We have proved the $\mathfrak{gl}_N$-part of Conjecture 3.1 from [5], for $N > 0$. As a direct consequence, we also get

**Corollary 4.18.** Both spectral sequences $\{E^1_k, d^1_k\}$ and $\{E^{II}_k, d^{II}_k\}$ converge at the second page.

The same arguments hold for reduced homologies. We state the result for the record.

**Theorem 4.19.** For a link $L$ presented as the closure of a braid, $\overline{H}_N(L)$ is the homology of the differential $d_N : \overline{H}(L) \to \overline{H}(L)$.

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