Scale space Radon transform

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Abstract
An extension of Radon transform by using a measure function capturing the user need is proposed. The new transform, called scale space Radon transform, is devoted to the case where the embedded shape in the image is not filiform. A case study is brought on a straight line and an ellipse where the SSRT behaviour in the scale space and in the presence of noise is deeply analyzed. In order to show the effectiveness of the proposed transform, the experiments have been carried out, first, on linear and elliptical structures generated synthetically subjected to strong altering conditions such blur and noise and then on structures images issued from real-world applications such as road traffic, satellite imagery and weld X-ray imaging. Comparisons in terms of detection accuracy and computational time with well-known transforms and recent work dedicated to this purpose are conducted, where the proposed transform shows an outstanding performance in detecting the above-mentioned structures and targeting accurately their spatial locations even in low-quality images.

1 INTRODUCTION

Radon transform has been widely used for the detection of embedded thin structures in images and other signals [1, 2]. It has been also used to define other transforms such as ridgelet and curvelet transforms [3, 4]. Even if it has been introduced more than one century ago, its properties are still studied [5–8].

Radon transform can be interpreted as a matching between an embedded thin structure in an image and the Dirac distribution (δ) of an implicit parametric shape. If it exists, the parametric model maximizing the matching is the desired one. Although the use of Dirac is motivated by strong arguments, there are some practical limitations. Indeed, the embedded shape is often represented in an image by elongated areas of more than one pixel width. The road in satellite image or a lane marking in a road traffic image are examples. In such cases, the estimated model using the Radon transform can be different from the user expectation, and hence is inexact. Consequently, a trick allowing the control of the parametric shape position is needed. This issue, which seems not be tackled before in the case of parametric shapes, is the scope of this paper. The idea is to express the desired parametric model through a kernel defined on a parametric shape instead of being limited to the Dirac distribution. The proposed transform, which we call scale space Radon transform (SSRT), is a matching of the kernel and an embedded shape in an image. More formally, for an implicit shape (x, μ) and a kernel (g), we define the SSRT by

\[ F(\mu, \alpha) = \int_{D} I(x) g(S(x, \mu), \alpha) dX. \]  

The kernel g must be chosen such that \( F(\mu, \alpha) \) satisfies the following requirements: (1) has at least one maximum with regards to the parameter vector \( \mu \); (2) has a nice behaviour in scale space; (3) is robust to noise; and (4) can be computed efficiently. The second requirement means that if the integral exists, then the displacement of the maxima in scale space is smooth. It follows that, the Radon transform is a particular case of the SSRT; \lim_{\alpha \to 0} g(S(x, \mu), \alpha) = \delta(S(x, \mu)). \) This restriction has three desirable consequences that are: the shape detection is reduced to a maxima detection in the Radon space, Radon transform can be used as a basis when studying the properties of the SSRT, and the user need can be expressed through the scale parameter. The third other requirements do not need additional explanations. In this paper, we will use the Gaussian function as a kernel and study the behaviour of SSRT maximum in the scale space for the straight line and the ellipse. Through the work presented in this paper, we will show that the Gaussian kernel fulfils the four requirements mentioned before.
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method [16], where they show in experiments the robustness of images and the recovering lines parameters by the Prony problem. Then, they have applied it successfully to the deblurring primal-dual splitting algorithm to solve the convex optimization problem for the purpose of recovering lines in degraded images using it is not suitable for irregular or occluded linear shape structures on real-world applications remain insufficient. However, when dealing with perfect straight road segments even if experiments where SSRT is applied and compared with other methods for the detection of linear and elliptical structures the latter are detected by the diagonals (instead by the medians) which does not agree with the reality. Indeed, the best manner to detect a thick line is to determine its median and thickness values, in other words, its centerline and width. Miao et al. [12] presents an automatic approach for 2D road centerline extraction from classified satellite images that integrates tensor voting, principal curves, and the geodesic method. This method consists of three main steps: Tensor voting is first used to extract feature points from the classified image using the subspace constrained mean shift (SCMS) method [13]. Finally, the feature points are then projected onto the principal curves using the voting cells around a peak in the Hough space. With the aim at detecting a final peak accurately, Hough transform is used as mentioned above on the raw data. The experimental results indicate that, in contrast to the other methods, the proposed method can achieve unbiased estimates of the ellipse parameters, in an elegant way, without any further work. Other improvements brought by these methods compared to the state-of-the-art dealing with linear structures detection. The authors in [9] propose a statistical method based on the Hough transform (HT) for line-segment detection. Their method focuses on the voting values distribution in each column around a peak in the Hough space. With the aim at detecting a final peak accurately, they use the normal angle and distance of the peak by using fitting and interpolation techniques. In another work of the same authors [10], a minimum entropy-based Hough transform around a peak in Hough space is proposed a step-by-step method which makes it dependent on the output quality of each processed step. The estimation of the line parameters is done by evaluating the values distribution in each column around a peak in the Hough space. With the aim at detecting a final peak accurately, they use the normal angle and distance of the peak by using fitting and interpolation techniques. In another work of the same authors [11], a minimum entropy-based Hough transform around a peak in Hough space is proposed a step-by-step method which makes it dependent on the output quality of each processed step. The estimation of the line parameters is done by evaluating the values distribution in each column around a peak in the Hough space. With the aim at detecting a final peak accurately, they use the normal angle and distance of the peak by using fitting and interpolation techniques. In another work of the same authors [12], an automatic approach for 2D road centerline extraction from classified satellite images that integrates tensor voting, principal curves, and the geodesic method. This method consists of three main steps: Tensor voting is first used to extract feature points from the classified image using the subspace constrained mean shift (SCMS) method [13]. Finally, the feature points are then projected onto the principal curves using the voting cells around a peak in the Hough space. With the aim at detecting a final peak accurately, Hough transform is used as mentioned above on the raw data. The experimental results indicate that, in contrast to the other methods, the proposed method can achieve unbiased estimates of the ellipse parameters, in an elegant way, without any further work.
The function $RT()$ is the length of the segment within the box. Its maximum is reached when $\rho=0$ and $\theta=\pi/2 \pm \beta$, corresponding to the two diagonals of the box ($(-a,-b), (a,b)$ and $(-a,b), (a,-b)$). Recall that instead of the Dirac, we propose the use of Gaussian along the line. The SSRT for the configuration in Figure 1 is given by

$$F(\rho, \theta, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\rho}^{\rho} \int_{-\sigma}^{\sigma} I(x, y) e^{-\left[\frac{\cos^2 \theta + \sin^2 \theta}{2\sigma} \right]} \, dx \, dy,$$  \hspace{1cm} (3)

Step-by-step manipulations of Equation (3), as shown in Appendix A, leads to:

$$F(\rho, \theta, \sigma) = \frac{\sigma \sqrt{2}}{\sqrt{\pi}} \left[ e^{-\frac{\rho^2}{2\sigma^2}} - e^{-\frac{\rho^2}{2\sigma^2}} - e^{-\frac{\theta^2}{2\sigma^2}} + e^{-\frac{\theta^2}{2\sigma^2}} + d_1 \text{erf} \left( \frac{d_1}{\sqrt{2\sigma}} \right) - d_2 \text{erf} \left( \frac{d_2}{\sqrt{2\sigma}} \right) 
- d_3 \text{erf} \left( \frac{d_3}{\sqrt{2\sigma}} \right),
+ d_4 \text{erf} \left( \frac{d_4}{\sqrt{2\sigma}} \right) \right],$$  \hspace{1cm} (4)

where $d_1 = a \cos \theta + b \sin \theta - \rho$, $d_2 = a \cos \theta - b \sin \theta - \rho$, $d_3 = a \cos \theta - b \sin \theta + \rho$, $d_4 = a \cos \theta + b \sin \theta + \rho$.

Now, we will study the position of the global maximum (if any) of the $F()$ in the scale space for the rectangular structure in Figure 1, starting by the case where $\sigma \to 0$, and increasing its value to follow the movement of this maxima as a function of $\sigma$. When $\sigma \to 0$, the function $F()$ is nothing but the Radon transform; that is, $F() = RT()$. A formal proof is given in Appendix B. Let us now deal with the case where $\sigma \geq 0$. We set the first derivative of $F()$ wrt $\rho$ to zero and we plot it as an implicit function in Figure 2a. We can see that $\rho = 0$, whatever are the values of $\sigma$ and $\theta$. This critical configuration ($\rho = 0$) is a maximum as it is displayed in Figure 3a, and also shown analytically in the Appendix C. This is expected because for fixed values of $\sigma > 0$ and $\theta$, the maximal directional Gaussian volume is within the structure when the line parameterized by $\rho$ and $\theta$ crosses the center of the structure (the axes origin in case of Figure 1). In what follows, we consider only the configuration where $\rho = 0$. The maximum in $\theta$ depends on $\sigma$. In Figure 2b, we set the first derivative of $F()$ wrt $\theta$ to zero and we plot the solution on $F()$. For each small value of $\sigma$, there are three solutions. Two are maxima (in red) and the one in the middle is a minimum (in blue). When $\sigma$ increases, the two maxima move and attract each other in the scale space and merge at some specific value of $\sigma$ related to $b$ as we will see later. This induces a directional Gaussians (the kernel) movement as a function of $\sigma$ in the spacial domain, as illustrated in Figure 4. The movement of a directional Gaussians is a continuous deformation of critical points. When a maximum disappears at a given scale, it will not appear at a higher scale. This desirable property, known as the nice property of a Gaussian in scale space, makes easier the choice of the scale to ensure the existence of only one global maximum and to detect it. In the scale space, Figure 2b, as $\sigma$ increases, the fusion of critical points occurs at some specific value of $\sigma$ (optimal $\sigma$) and the optimal $\theta$ converges to the value $\theta = \pi/2$.

Let us now compute the optimal $\sigma$ maximizing $F()$, that is, $\sigma_{\text{opt}} = \arg \max_{\sigma} F(0, \theta, \sigma)$ and it is unique. This corresponds to the point $(\sigma, \pi/2)$ where $\max_{\sigma} F_{\rho}(0, \pi/2, \sigma)$ starts to decrease. Here, we take a decreasing rate of 0.001% which is expressed as $(\lim_{\sigma \to 0} F(0, \pi/2, \sigma) - F(0, \pi/2, \sigma_{\text{opt}})) / \lim_{\sigma \to 0} F(0, \pi/2, \sigma) = 0.00001$, that is, $(2a - 2d \text{erf}(b(\sigma_{\text{opt}} \sqrt{2})))/(2b) = 0.00001$. Finally, the optimal value of $\sigma$ to detect a $2b$-thickness line is obtained as $\sigma_{\text{opt}} = b/\sqrt{27 \text{erf}^{-1}(0.99999)} = 0.226 \times b$. The length of

*FIGURE 2* (a) $\rho$ evolution as function of $\sigma$ and $\theta$ for the SSRT of the structure of Figure 1. (b) The SSRT $F()$ (the grid) evolution as a function of $\sigma$ and $\theta$ with highlighted scale space.

*FIGURE 3* (a) Evolution of $F()$ as a function of $\theta$ and $\rho$ for $a = 20, b = 5$ and $\sigma = 3$. (b) SNR of the SSRT for a straight line in case of the structure in Figure 1.
In this case, the expression of SNR is given as
\[ \text{SNR}_F = \frac{\sigma^2}{2\sigma^2} \int_{-a}^{+a} \int_{-b}^{+b} I(x,y) e^{-(x \cos \theta + y \sin \theta - \rho)^2 / 2\sigma^2} dxdy. \]
The SNR is given by
\[ \text{SNR}_F (\phi, \theta) = \frac{\sigma^2}{\sqrt{2\pi} \sigma} \left( \int_{-a}^{+a} \int_{-b}^{+b} I(x,y) e^{-(x \cos \theta + y \sin \theta - \rho)^2 / 2\sigma^2} dxdy \right)^2. \]

Figure 3b shows that, for SSRT, the SNR increases with \( \sigma \). It is greater than the one of Radon transform (\( \sigma \) close to 0). Identically, it increases with \( \theta \) reaching a maximum at \( \theta = \pi / 2 \).

In this case, the expression of SNR is given as \( \text{SNR}_F (0, \pi / 2) = [2\pi \sqrt{\sigma} \int_{-b}^{+b} e^{b \sqrt{[a \sqrt{\text{erf}(b / \sqrt{\sigma})}]}}. \) Its plotting in function of \( \sigma \) is illustrated in Figure 3b for \( \theta = \pi / 2 \), \( a = 20 \) and \( b = 5 \). Note that the SSRT’s SNR is proportional to \( 1 / \sigma^2 \). So, the detection of thick lines in noisy images is improved by using the SSRT. It is worth to recall that all these computations are done for the linear structure of Figure 1 to simplify the explanation of the SSRT behaviour.

3 | SCALE SPACE RADON TRANSFORM FOR ELLIPSE

Let us consider an image \( I(x,y) \) that may contain an area bordered by two concentric ellipses [17]. The image support is considered infinite so that we do not have to be concerned about its size. The SSRT of an ellipse is then given by
\[ E(a, b, \sigma) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} I(x,y) e^{-C_{(\xi, \eta), \phi, \psi} / 2\sigma^2} dxdy, \]
where
\[ C_{(\xi, \eta), \phi, \psi} = \frac{((\xi - u) \cos \psi + (\eta - v) \sin \psi)^2}{\sigma^2} + \frac{((\xi - u) \sin \psi - (\eta - v) \cos \psi)^2}{\rho^2} - 1 \]
is the ellipse equation, with \((u, v)\) the ellipse center coordinates, \((a, b)\) the (semi-major, semi-minor) axes and \(\psi\), the ellipse orientation. For the sake of simplification and without a loss of generality, the study of the ellipse SSRT behaviour would be done for an ellipse with \((0,0)\) as center, and a null orientation, that is, \(\psi = 0\). In this case, \(E((a, b), 0) = x^2 / a^2 + y^2 / b^2 - 1\). Here, because the use of SSRT requires to find \((a, b)\) maximizing \(E()\), our goal is to study precisely the behaviour of the maximum (if any) in the scale space. For that purpose, we consider that the image \( I \) as the area between \(C_{(\xi, \eta), \phi, \psi}\) and \(C_{(\xi, \eta), \phi, \psi}\), where \(a_0 < a_1\) and \(b_0 < b_1\); that is, \(I(x,y) = H_{C_{(\xi, \eta), \phi, \psi}} - H_{C_{(\xi, \eta), \phi, \psi}}\) where \(H\) is the Heaviside function. An example of the image \( I \) and the ellipse anchored Gaussian are shown in Figure 5b,c.

The integral in Equation (6) is not tractable and therefore we will produce a rough idea about \(E()\) by using numerical calculus. When \(\sigma \to 0\), the ellipse anchored Gaussian is equal to \(\delta(C_{(\xi, \eta), \phi, \psi})\), which yields the elliptical Radon transform (ERT) [18] as the optimal ellipse maximizing \(E()\) by using numerical calculus. When \(\sigma \to 0\), the ellipse anchored Gaussian is equal to \(\delta(C_{(\xi, \eta), \phi, \psi})\), which yields the elliptical Radon transform (ERT) [18] as the optimal ellipse maximizing \(E()\) by using numerical calculus.
$C(c, a, 0, b)$ towards $C(c, a, 0, b)$ without reaching it. Figure 6a shows the evolution of $b$ in the scale space as a function of $\sigma$. The curve evolution is rapid for small values of $\sigma$, but it becomes slow as $\sigma$ increases. The maximum location converges to a stable location $(a, b)$ when $\sigma$ is greater than some specific value. The last situation occurs when the volume of the ellipse anchored Gaussian within the ellipse fulfills

$$
\int_{a_0}^{a_1} \int_{b_0}^{b_1} (H_{C(c, a, 0, b)} - H_{C(c, a, 0, b)}) e^{-c_2^2 / 2\sigma^2} d\alpha d\beta = 0.
$$

(7)

In this case, the ellipse $C(c, a, 0, b)$ is the one maximizing $E()$ under the constraints: $a_0 \leq a \leq a_1$ and $b_0 \leq b \leq b_1$. The smallest value of $\sigma$ leading to this configuration is the one allowing that the maximal volume of the ellipse anchored Gaussian is contained within the ellipse structure. To establish a relationship between $\sigma$ and $b$ analytically, let us represent the ellipse by polar coordinates $(r, \phi)$ with $r$ and $\phi$ are, respectively, the radial distance and the polar angle, where, a geometric configuration of the two concentric ellipses $Ell_i, i = 0, 1$, is plotted in Figure 5a. The SSRT is then given by

$$
E(a, b, \sigma) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \int_0^\infty r_i(\phi) e^{-{(r_i(\phi) - d(\phi))^2 / 2\sigma^2}} r d\phi d\theta
$$

where $d(\phi) = b / \sqrt{1 - \epsilon^2 \cos^2(\phi)}$ and $r_i(\phi) = b_i / \sqrt{1 - \epsilon^2 \cos^2(\phi)}$, $i = 0, 1$, with $\epsilon = \sqrt{1 - b_i^2 / a_i^2}$ the ellipse eccentricity.

Straightforward manipulations of Equation (8) leads to Equation (9) (see the proof in Appendix D). It is noted that the solutions of the latter are given by a numerical integration method.

$$
E(a, b, \sigma) = \int_0^{\pi / 2} \left[ \frac{4\sigma}{\sqrt{2\pi}} e^{-(r_i(\phi) - d(\phi))^2 / 2\sigma^2} - e^{-(r_i(\phi) - d(\phi))^2 / 2\sigma^2} \right] d\phi
$$

$$
+ 2d(\phi) \left[ \text{erf} \left( \frac{r_i(\phi) - d(\phi)}{\sqrt{2\sigma}} \right) - \text{erf} \left( \frac{r_i(\phi) - d(\phi)}{\sqrt{2\sigma}} \right) \right] d\phi.
$$

(9)

It is obvious that $a$ and $b$ values of the optimal ellipse that maximizes $E$ are equal to the ones maximizing the volumes under the Gaussians within the structure for all $\phi$, including $\phi = 0$ and $\phi = \pi/2$. Taking $\phi = \pi/2$ in Equation (9) and discarding the integral, the mentioned area $A(b, \sigma)$ will be expressed as

$$
A(b, \sigma) = \frac{\sigma}{\sqrt{2\pi}} \left[ e^{-(b_1 - b)^2 / 2\sigma^2} - e^{-(b_1 - b)^2 / 2\sigma^2} \right] + \frac{b}{2} \left[ \text{erf} \left( \frac{b_1 - b}{\sqrt{2\sigma}} \right) - \text{erf} \left( \frac{b_1 - b}{\sqrt{2\sigma}} \right) \right].
$$

The value maximizing $A$ is given by

$$
\frac{\partial A(b, \sigma)}{\partial b} = \frac{1}{\sigma \sqrt{2\pi}} \left[ b \left( \frac{-h_0 - e^2}{2\sigma^2} \right) + b \left( \frac{-h_0 - e^2}{2\sigma^2} \right) \right]
$$

$$
+ \frac{1}{2} \left[ \text{erf} \left( \frac{b_1 - b}{\sigma \sqrt{2\sigma}} \right) - \text{erf} \left( \frac{b_1 - b}{\sigma \sqrt{2\sigma}} \right) \right] = 0.
$$

(10)

The plot of the solution of Equation (10) is illustrated in Figure 6b, where it can be noted that the optimal value of $b$ (where the $b$ values are almost constant) is provided for $\sigma$ exceeding a certain value $\sigma_{opt}$. In our case, where $b_1 = 5$, $b_1 = 7.5$, so, we obtain $\sigma_{opt} = 1.6$ and $b_{opt} = 6.34$.

## 4 | EXPERIMENTS

In order to show the effectiveness of the proposed transform, two experiments have been carried out. The first experiment consists in the detection of linear and elliptical structures generated synthetically and corrupted by blur and noise. Comparisons in terms of detection accuracy and computational time with well-known transforms and a recent work, dedicated to detection of linear structures corrupted by strong blur and noise, are conducted. The second experiment uses linear and elliptical structures issued from real-world application such as road traffic, road detection in satellite images and weld radiographic testing where the robustness to image noise is also analyzed. The implementations are performed in Matlab environment on a workstation HP Z8 with Intel Xeon Silver 4210 as processor, 2.2 GHz as processor clock speed and 64 GB as random access memory (RAM).

### 4.1 | Synthetic linear structure images

To compare the SSRT behaviour to some other known transforms behaviours like the Radon transform (RT) and the Hough transform (HT), a set of synthetic images, consisting of a linear structure with known location and orientation, are firstly used.

The input image represented on the right side of Figure 7 is of size $201 \times 201$ and contains a linear structure with a known centerline location and orientation $\rho = -155.968$ and $\theta = 150^\circ$, respectively. This structure is of 21 pixels width ($21 = 2k$) and 125 pixels length ($125 = 2a$) where $a$ and $b$, representing the half of width and length, respectively, are illustrated in Figure 1. Before beginning, it is necessary to present some SSRT implementation aspects we have adopted. The space $(\rho, \theta)$ is sampled with discrete steps of 1 pixel for $\rho$ and 1° for $\theta$. Since
we handle digital images, the integral in the SSRT expressions are replaced, during implementation, by summation. To apply the SSRT on an image, we use an explicit method where $F()$ in Equation (4) is estimated for each cell $(\rho, \theta)$ using a large value of $\sigma$. By large, it means that $\sigma \geq 2b/9$. The SSRT $F()$ of the input image computed with $\sigma = 10$ is illustrated at left of Figure 7a, where $F()$ presents only one maximum. The area borders of the integrals in Equation (3) are induced by the image support limits. It follows that the maximum of $F()$ gives $\rho_{opt}$ and $\theta_{opt}$. Furthermore, to estimate $\sigma_{opt}$, $F(\rho_{opt}, \theta_{opt}, \sigma)$ is computed by decreasing the values of $\sigma$, $F()$ increases as $\sigma$ decreases until reaching some value of $\sigma$, noted $\sigma_{opt}$ where $F()$ becomes almost constant as it can be viewed in Figure 8. It follows that, as shown in Section 2, the approximated structure width and length are given by $2b \approx 8.85\sigma_{opt}$ and $2a \approx F(\rho_{opt}, \theta_{opt}, \sigma_{opt})$, respectively. Thus, the SSRT maximum permits the computation of the structure orientation and location which are equal to $\rho = -155.96^\circ$ and $\theta = 150^\circ$, respectively. The corresponding directional line is depicted on this structure at right of Figure 7a. The estimated structure width $2b$ and length $2a$ for the computed $\sigma_{opt} = 2.43$ are equal to 21.6 ($b = 10.8$) and 125.16 ($a = 62.58$), respectively. Applying SSRT on the same image, for $\sigma = 0.01$, provides two directional lines, as shown in Figure 7b, namely the two structure diagonals which correspond to the two SSRT maxima. Moreover, the two lines issued form the Radon transform RT maxima depicted in Figure 7c, are exactly the same as the ones obtained for the SSRT with small $\sigma$, as the two computed directional lines parameters $\rho_{1,2} = \{-157.771, -151.481\}$ and $\theta_{1,2} = \{142^\circ, 158^\circ\}$ are the same, for both cases. These results were expected since the SSRT, when $\sigma$ tends to 0, is nothing but the Radon transform, as demonstrated in Appendix B. Figure 7d concerns the outcome of Hough transform (HT) applied on the image, where the 3D representation of HT is depicted and the highlighted maximum of HT is used to recover the line segment on the image, of which location and orientation are equal to $\hat{\rho} = -154.4031$ and $\hat{\theta} = 148^\circ$, respectively.

Figures 9–11 show the transform spaces and detected lines when adding blur, noise, or both to the image in Figure 7. The image is blurred by using the convolution of the sharp image in Figure 7 and a Gaussian kernel $\Theta(x,y;\sigma) = (2\pi\sigma^2)^{-1/2} \exp(- (x^2 + y^2)/(2\sigma^2))$ of size $15 \times 15$ with spread $\sigma$ (spatial standard deviation). An additive white Gaussian noise (AWGN) with standard deviation $\xi$ is applied to the blurred image, that is (AWGN) $\varepsilon \rightarrow \mathcal{N}(0, \xi)$. Here, we take $\kappa = 3$ and $\zeta = 100$. Figures 9a, 10a and 11a show the 3D representation of the SSRT for each case, as well as the corresponding estimated lines, of which the parameters computed by the SSRT maximum are equal to $\rho = -155.969$ and $\theta = 150^\circ$ for all cases. Figures 9b, 10b and 11b show the results of applying the RT on these images. There are in Figure 9b two directional lines of equal lengths with location and orientation parameters equal
to $\tilde{\rho}_{1,2} = \{-158.211, -153.054\}$ and $\tilde{\theta}_{1,2} = \{146^\circ, 154^\circ\}$. However, for Figures 10b and 11b, there is only one line for both cases, with locations equal to $\tilde{\rho} = -158.445$ and $\tilde{\theta} = -157.077$, and orientations equal to $\tilde{\theta} = 143^\circ$ and $\tilde{\theta} = 144^\circ$. For noisy and blurred images, the HT and the detected lines are depicted in Figures 9c, 10c and 11c. The line parameters, for the three cases, are $(\rho = 145.042, \theta = 145^\circ)$, $(\rho = -152.02, \theta = 135^\circ)$ and $(\rho = -149.553, \theta = 135^\circ)$. We summarize in Table 1 the lines detection accuracy in terms of absolute error values, where

$$\Delta \rho = |\rho - \tilde{\rho}| \text{ and } \Delta \theta = |\theta - \tilde{\theta}|,$$

and this; for each of the transforms HT, RT and SSRT applied on the input image without blur and noise and when the image is corrupted by blur, noise and the combination of both. When more than one line is detected as with RT for sharp and unnoisy and blurry images, no quantitative performances are reported.

We can remark that, conversely to RT and HT, the SSRT, for all cases, gives the same accuracy even when the input image quality is altered. Furthermore, the lowest errors are provided by the SSRT which reflects its high accuracy in the localisation of a linear structure in an image despite its quality. These outcomes are expectable since the strength of the SSRT arises from its inherent filtering. We can see, from the 3D SSRT representations, the variations are smooth and that is why the maximum does not change in number and location as the image is altered. It is worth to note that, for SSRT, the execution time is around 7 and 4 s when $\sigma$ equals 10 and 0.01, respectively. However, for the RT and HT we have implemented, the execution times are 0.5 and 0.1 s, respectively.

The next experiments are made on an image of size $201 \times 201$, consisting of three crossing linear structures of different widths and lengths. The width of the structures are $2b \in \{19, 25, 13\}$, their length are $2a \in \{176, 160, 171\}$ with $(\rho_1 = -165.966, \theta_1 = 150^\circ)$, $(\rho_2 = 51.382, \theta_2 = 20^\circ)$ and $(\rho_3 = -52.936, \theta_3 = 80^\circ)$. SSRT, RT and HT are applied on the images (binarized version for the HT) for each of the four cases: sharp and unnoisy, with blur ($\xi = 3$), with noise ($\zeta = 100$) and finally with blur and noise. The 3D representation of the transforms and the corresponding lines are shown in Figures 13–16. It should be noted that only the SSRT has provided accurately the three lines, for the three structures, for all cases. It is important to notice that in case of thick structures, RT and HT produce several maxima for each structure and then additional RT and HT processing should intervene to select the maximum of these maxima. In our case, image thresholding has been performed on the sinograms obtained from the RT and HT spaces. An example on RT is illustrated in Figure 12a, where a maximum is researched among the maxima of each connected component (see Figure 12b).

### Table 1

|                | HT  | RT  | SSRT |
|----------------|-----|-----|------|
| Sharp and unnoisy image | $\Delta \rho = 1.565$ | $\Delta \theta = 2$ | $\Delta \rho = 1.565$, $\Delta \theta = 2$ |
| Blurry image     | $\Delta \rho = 0.128$ | $\Delta \theta = 5$ | $\Delta \rho = 0.128$, $\Delta \theta = 5$ |
| Noisy image      | $\Delta \rho = 3.949$ | $\Delta \theta = 15$ | $\Delta \rho = 3.949$, $\Delta \theta = 15$ |
| Blurry & noisy image | $\Delta \rho = 6.415$ | $\Delta \theta = 15$ | $\Delta \rho = 6.415$, $\Delta \theta = 15$ |
FIGURE 12  (a) Radon sinogram and its binarization. (b) All the RT maxima (top image) versus the maxima resulted after binarization (bottom image)

FIGURE 13  3D transform and detected lines by using a) SSRT, b) RT and c) HT

Although the additional processing, the localisation of output lines of RT and HT is not accurate. Worse still, HT provides more than three lines in the presence of blur, noise or both.

In fact, the high SSRT accuracy can be explained by the Gaussian smoothing embedded in the transform.

FIGURE 14  3D transform and detected lines by using a) SSRT, b) RT and c) HT in the presence of blur

FIGURE 15  3D transform and detected lines by using a) SSRT, b) RT and c) HT in the presence of noise

Here, even if the visual comparison is enough to show the outstanding accuracy of the SSRT over Radon and Hough transforms, we provide in Table 2 the quantitative performances of HT, RT and SSRT. We do not report the performances when more than three lines are detected as with RT for blurry image and HT in the presence of blur, noise or both. In this table, we consider $\Delta \rho_i = |\rho_i - \bar{\rho}_i|$ and $\Delta \theta_i = |\theta_i - \bar{\theta}_i|$ with $i = 1, 2, 3$ and root mean square error (RMSE) computed as
FIGURE 16 3D transform and detected lines by using a) SSRT, b) RT and c) HT in the presence of blur and noise.

FIGURE 17 Line detection estimates for Ref. [15] and SSRT for blur and noise parameters $(\kappa, \zeta) = (8,0), (8,100)$ and $(8,200)$.

$\Sigma_{\Delta \rho} = \sqrt{\sum_{i} \Delta \rho_i^2 / n}$ and $\Sigma_{\Delta \theta} = \sqrt{\sum_{i} \Delta \theta_i^2 / n}$ with $n = 3$. We also provide the errors of the estimated structure in case of SSRT without blur and noise.

To compare the SSRT performances with ones of the most recent works [15] aiming at recovering linear structures in images through, among others, detecting their centerlines, we consider an image of size $201 \times 201$, consisting of three structures of centerline parameters $(\rho_1, \theta_1) = (-141.077, 144)$, $(\rho_2, \theta_2) = (65.148, 11)$ and $(\rho_3, \theta_3) = (45.629, 30)$. These structures are created by a Gaussian blurring process with $\kappa = 8$, as presented by the authors in [15]. We synthetise, from the obtained image, two different noisy images with $\zeta = 100$ and $\zeta = 200$. Afterward, each of the SSRT and the method proposed in [15] are applied. The detected lines are illustrated in Figure 17. The errors are reported in Table 3. The computational time is expressed in seconds and noted $R_t$. The following remarks can be addressed: (1) the method of [15] is accurate in the case of unnoisy images but it is time consuming, while important estimation errors occur when an important noise effect is added: for $\zeta = 100$, $200$, $\Sigma_{\Delta \rho} = 8.5 - 10$ and $\Sigma_{\Delta \theta} = 6.5 - 7.5^\circ$ versus $\Sigma_{\Delta \rho} = 0.46 - 0.57$ and $\Sigma_{\Delta \theta} = 0^\circ$, for SSRT, and (2) the SSRT outperforms the method in [15], in all cases, by providing less computational time and less estimation errors where the centerline estimates match almost perfectly the reference image centerlines.

Similarly to the line case, the ellipse parameters space is sampled and the SSRT is estimated at each sample by using Equation (6). For test, we generate synthetically an elliptical annulus with the following parameters $u = 56, v = 56, a_0 = 36, a_1 = 48, b_0 = 27, b_1 = 36$ and $\psi = 10^\circ$. The parameters of the median ellipse, located at the middle of the annulus, are then given by $u_{med} = 56, v_{med} = 56, a_{med} = 42, b_{med} = 31.5$ and $\psi_{med} = 10^\circ$. The maximum of $E(\cdot)$ provides the ellipse semimajor and semi-minor axes $a_{opt}$ and $b_{opt}$, the ellipse center $(u_{opt}, v_{opt})$, and the orientation $\psi_{opt}$. We have used $\sigma = 0.5$ which reveals to be convenient to maximize the volume under the Gaussian fitting the annulus while it discards the image background minimizing by the way the effect of noise as confirmed.

### Table 2 - Error values of line detection and estimated structure dimensions for HT, RT and SSRT for an image containing three thick linear structures and its versions affected by blur ($\kappa = 3$), noise ($\zeta = 100$) and the combination of both.

|          | HT  | RT  | SSRT |
|----------|-----|-----|------|
| Sharp and unnoisy image | $\Delta \rho$ | $[4.71, 16.97, 7.55]$ | $[2.25, 14.31, 8.95]$ | $[0.002, 0.001]$ |
|          | $\Sigma_{\Delta \rho}$ | 11.06 | 9.83 | 0.01 |
|          | $\Delta \theta$ | $[262]$ | $[474]$ | $[000]$ |
|          | $\Sigma_{\Delta \theta}$ | 3.83 | 5.2 | 0.01 |
| Blurry image | $\Delta \rho$ | — | — | $[0.66, 0.02, 0.01]$ |
|          | $\Sigma_{\Delta \rho}$ | — | — | 0.22 |
|          | $\Delta \theta$ | — | — | $[1.00]$ |
|          | $\Sigma_{\Delta \theta}$ | — | — | 0.33 |
| Noisy image | $\Delta \rho$ | — | $[6.62, 6.71, 6.02]$ | $[0.002, 0.001]$ |
|          | $\Sigma_{\Delta \rho}$ | — | 6.46 | 0.01 |
|          | $\Delta \theta$ | — | $[112]$ | $[000]$ |
|          | $\Sigma_{\Delta \theta}$ | — | 1.41 | 0.01 |
| Blurry and noisy image | $\Delta \rho$ | — | $[3.2, 9.53, 0.02]$ | $[0.002, 0.001]$ |
|          | $\Sigma_{\Delta \rho}$ | — | 5.81 | 0.01 |
|          | $\Delta \theta$ | — | $[2.50]$ | $[000]$ |
|          | $\Sigma_{\Delta \theta}$ | — | 3.11 | 0.01 |
TABLE 3 Line parameters estimation errors and runtime for Ref. [15] and SSRT with blur and noise ($\kappa, \zeta$) equal to (8,0), (8,100) and (8,200)

| $\kappa = 8$ | $\zeta = 0$ | $\zeta = 100$ | $\zeta = 200$ |
|---------------|-------------|-------------|-------------|
| $\Delta \rho$ | 0.30 1.23   | 0.27 0.34   | 0.29 0.56   |
| $\Sigma \Delta \rho$ | 0.73        | 0.25        | 0.51        |
| $\Delta \theta$ | 0.01 0.51   | 0.03 0.51   | 0.00 0.10   |
| $\Sigma \Delta \theta$ | 0.29        | 0.03        | 0.10        |
| $R_t$ (sec) | 5630.38     | 4.06        | 5624.93     |

FIGURE 18 Ellipse detection without noise ((a) for ERT and (b) for SSRT) and with AWG noise ($\zeta = 38$) for ERT and (c) for SSRT without noise and with AWG noise ($\zeta = 38$).

by Figure 18 and Table 4. The performance of SSRT to detect an elliptical annulus are better than ERT. However, for large values of $\sigma$, which induce an annulus overfitting by the Gaussian, the optimal ellipse is recovered by SSRT for an annulus image without noise but, in case of noise, the optimal ellipse tends to be very close to the external border of the annulus. This can be explained by the fact that, for a noisy image, the effect of noise on the annulus does not affect the location of $E(\cdot)$ maximum (see Figure 18b,d) since the noise variance is constant over the image and the Gaussian volume of the inner and the outer parts of the annulus are almost equal. That said, even if a graphical method could be useful as illustrated in Figure 6, searching automatically the optimal $\sigma$, providing the best fitting between the Gaussian and the annulus dimensions, constitutes the key direction for further works.

4.2 Real-world applications

As first real-world application, the SSRT is applied for the detection of longitudinal lane markings (solid or broken lines). In Figure 19, we can see a road traffic image with a white line (solid and broken). The road lines thus detected by both transforms RT and SSRT are illustrated in red color on Figure 19a and Figure 19b, respectively. In case of SSRT, $\sigma$ is set to 7. Indeed, for the given thickness of the longitudinal marking which does not exceed 44 pixels in the image, the chosen value of $\sigma$ is enough to guarantee that the detected maximum location will not move any more. We can see that close to the camera, the lines detected by SSRT are located in the middle of the lane markings, while the ones detected by RT are located at the right side of the latter. Numerical evaluation in Table 5 using the reference lane centerlines (computed with medial axis transform of the structures after binarization) ascertains the best road lane estimation provided by SSRT compared to RT. This is expected because we have no control over the position of the lines detected by RT which, by its nature, tracks the longest path inside these structures. We give other examples in
TABLE 5 Error values of line detection for RT and SSRT for road lane markings in Figure 19

|         | RT       | SSRT     |
|---------|----------|----------|
| Δφ      | [3.47, 5.22] | [0.88, 1.45] |
| ΣΔφ     | 4.43     | 1.2      |
| Δθ      | [1.41, 1.36] | [0.41, 0.64] |
| ΣΔθ     | 1.39     | 0.53     |

Figure 20 SSRT-based lane detection for traffic road images

Figure 20 where SSRT maxima are all detected on the middle of the lane markings: solid or broken.

A second application considered for road and path detection in satellite images consists in two images: the first image represents an urban block separated by straight roads (see Figure 21) whereas the second represents linear paths in an archaeological site (Figure 22). For the purpose previously mentioned, SSRT is applied on the grey-level version of the image and its variant corrupted by an AWG noise of standard deviation ($\xi = 25$) and then compared with RT. The detected roads and paths for both methods are drawn in yellow color for Figure 21 and in red color for Figure 22. First of all, the first image has been inverted, that is, the grey levels are replaced by $255 - $ Input grey level, since

Figure 21 Road detection for Buckeye city (AR, USA) by (a) RT and (b) SSRT; and, with noise ($\xi = 25$) by (c) RT and (d) SSRT

Figure 22 Road detection for Leptis Magna site (Libya) by (a) RT and (b) SSRT; and, with noise ($\xi = 25$) by (c) RT and (d) SSRT

Figure 23 Example of reference structure centerline computation

the roads in the first image are darker than the rest of the image. Then, to perform a comparison between SSRT and RT, the road and the path centerlines of reference are drawn using an image binarization of the regions of interest, followed by computing the medial axis of the binarized structure, as illustrated in Figure 23. On the basis of the outcomes provided by the figures and Table 6, it appears clearly that SSRT outperforms RT in the detection of the underlined structures with or without noise.

Another real-world application which can take advantage from SSRT is the weld defect detection in non-destructive testing by radiography [19]. One of the various types of weld defects, called lack of penetration (LP), occurs because of incomplete penetration of the melted metal at the root of the weld groove where, it appears in a radiographic film image as a thick dark line along the welded butt joint with a thickness equal to the weld root opening. Lack of penetration can occur on all the length of the welded joint or just on a part of this one and, depending on the weld groove shape, the lack of penetration could be located at the middle of welded joint (for grooves in ‘V’, ‘U’, ‘X’, ...) or near the border of the latter (for grooves in ‘J’, ‘K’, ...). Motivated by the SSRT properties and all the above-mentioned features for the lack of penetration, we opt to use SSRT with the aim of detecting this kind of weld defects. Since low grey-level values characterize the lack of penetration,
### TABLE 6

Error values of line detection for RT and SSRT for satellite images of Buckeye and Leptis Magna and their versions affected by AWG noise ($\zeta = 25$)

|        | RT       | SSRT      |
|--------|----------|-----------|
|        | $\Delta \rho$ | $\Sigma \Delta \rho$ | $\Delta \theta$ | $\Sigma \Delta \theta$ | $\Delta \rho$ | $\Sigma \Delta \rho$ | $\Delta \theta$ | $\Sigma \Delta \theta$ |
| **Buckeye** | [1.93 5 4 3.17] | 3.7 | [1 0 2] | 1.12 | [1.5 1 3.57 1] | 2.06 | [0 0 1 0] | 0.5 |
| **Noisy image** | [5.5 4 8.85] | 6.11 | [0 0 0 3] | 1.5 | [1.5 1 3.57 1] | 2.06 | [0 0 1 0] | 0.5 |
| **Leptis Magna** | [1.07 1.09 2.26 1.54 9.05] | 4.28 | [0.44 0.04 0.28 0.23 1.78] | 0.83 | [1.07 1.09 1.26 1.54 3.74] | 2.01 | [0.44 0.04 0.28 0.23 0.78] | 0.43 |
| **Noisy image** | [1.07 1.09 0.74 1.54 9.05] | 4.18 | [0.44 0.04 0.23 0.23 1.78] | 0.83 | [1.07 1.09 1.26 1.54 3.74] | 2.01 | [0.44 0.04 0.28 0.23 0.78] | 0.43 |

As illustrated in Figures 24a, 25a, while the detection is provided by SSRT maximization, we have to invert the radiographic image, as shown in Figure 24b. Then, the SSRT is computed for the inverted image of which the 3D representation is displayed in Figure 24c and finally, the SSRT maxima are drawn, as yellow lines, in Figure 24d. The same above-mentioned operations are performed on Figure 25a, where the outcomes are depicted in Figure 25b and Figure 25d, respectively.

In the light of the lines obtained which we have visually examined, it is worthy to point out the success of SSRT to recover the defect indications for LP which are located around the middle of the welded joint. Moreover, the SSRT permits to demarcate the borders of LP in case of Figures 24a and 25a. We recall that, for the SSRT, the value of $\sigma$ is chosen empirically. The $\sigma$ values of 4 and 2 reveal to be performing to detect roads and paths in Figures 21 and 22, respectively, while $\sigma$ set to 1 reveals to be sufficient to detect all LP indications used, for the weld X-ray imaging experiment. Unsurprisingly, the above-mentioned values of $\sigma$ are suitable for the images in question since it is proportional to the thickness of the structure to be detected.

As with SSRT for linear structures, the SSRT for an ellipse $E(\cdot)$ is also applied in a real-world application dealing with road traffic. Here, the purpose is to detect a roundabout since its appearance in the image looks like a thick ellipse. In Figure 26, the optimal detected ellipse, provided by the maximum of $E(\cdot)$, is drawn as black solid ellipse while the blue ellipse is the result of applying ERT on the roundabout. For SSRT, the value of $\sigma$ is taken equal to 1 which reveals to be sufficient to ensure the solution stability, as explained in Figure 6b. The results given in Table 7 show that SSRT outcomes match better the estimated roundabout median ellipse in comparison to the ERT since the deviation values to the median ellipse are lower.

To conclude, the detected lines and ellipses comply with the SSRT behaviour presented in previous sections. The SSRT
and its behaviour in the scale space make it possible to detect the lines and the ellipses representing thick regions while having the ability of targeting their spatial locations, even in noisy images.

### 5 Conclusion

In real images, thin structures are not filiform. When using Radon transform for their detection, there can be infinite possible locations. The Radon transform does not provide to a user a mechanism allowing to choose the desired location. This weakness can be overcome by using a measure function instead of Dirac distribution. Here, we have highlighted this issue where we propose the use of a Directional Gaussian distribution as a measure function. We focused on the straight line and the ellipse. The behaviour in scale space shows that the location of the desired shape is scale dependent. The longest shape is detected when the scale tends to zero (i.e., Radon transform). The location of the detected shape moves to the middle when the scale increases. Compared to Radon transform, the new transform is more robust to noise and blur and should lead to more accurate detection of shapes, as given by the experiments on synthetic and real-world application images. In order to study deeply the SSRT properties, more investigations should be carried out on this transform. For instance, with the aim of speeding up the SSRT during implementation, future works devoted to SSRT algorithm optimization will soon be undertaken.

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APPENDIX A

From Figure 1, SSRT, with \( I(x,y) = 1 \) and the integration done respect to \( x \) first, is given by

\[
F(\rho, \theta, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-b}^{b} \int_{-a}^{a} e^{-\left(x\cos\theta+y\sin\theta-\rho\right)^2/2\sigma^2} \, dx \, dy.
\]

With the following change of variables \( X = x\cos\theta \) and \( Y = y\sin\theta-\rho \), the differentials \( dx \) and \( dy \) will become \( dX/\cos\theta \) and \( dY/\sin\theta \), respectively; while the integral limits \( -a, a \), \( -b \) and \( b \) will be equal, for the new double integral, to \( -a\cos\theta \), \( a\cos\theta \), \( -b\sin\theta-\rho \) and \( b\sin\theta-\rho \), respectively.

Then the SSRT in the \( X-Y \) axes system is given by

\[
F(\rho, \theta, \sigma) = \frac{\sqrt{2}}{\sqrt{\pi}\sigma \sin 2\theta} \int_{-\sin\theta}^{\sin\theta} \int_{-\cos\theta}^{\cos\theta} e^{-\left(X+Y\right)^2/2\sigma^2} \, dX \, dY.
\]

Let us put

\[
J(\rho, \theta, \sigma) = \int_{-\sin\theta}^{\sin\theta} \int_{-\cos\theta}^{\cos\theta} I(\theta, \sigma) \, dX \, dY,
\]

where

\[
I(\theta, \sigma) = \int_{-\cos\theta}^{\cos\theta} e^{-\left(X+Y\right)^2/2\sigma^2} \, dX.
\]

Let us calculate \( I(\theta, \sigma) \)

\[
I(\theta, \sigma) = \frac{\sigma\sqrt{\pi}}{\sqrt{2}} \left[ \text{erf} \left( \frac{a\cos\theta + Y}{\sqrt{2}\sigma} \right) - \text{erf} \left( -\frac{-a\cos\theta + Y}{\sqrt{2}\sigma} \right) \right].
\]

\( J \) is then given by \( J(\rho, \theta, \sigma) = \frac{\sigma\sqrt{\pi}}{\sqrt{2}} \left[ J_1(\rho, \theta, \sigma) - J_2(\rho, \theta, \sigma) \right] \)

where

\[
J_1(\rho, \theta, \sigma) = \int_{-\sin\theta}^{\sin\theta} \text{erf} \left( \frac{a\cos\theta + Y}{\sqrt{2}\sigma} \right) \, dY,
\]

and

\[
J_2(\rho, \theta, \sigma) = \int_{-\sin\theta}^{\sin\theta} \text{erf} \left( -\frac{-a\cos\theta + Y}{\sqrt{2}\sigma} \right) \, dY.
\]

Using the following property of the erf integral

\[
\int \text{erf} \left( \frac{\mu + t}{\nu} \right) \, dt = \frac{\nu e^{-\frac{\mu^2}{\nu}}}{\sqrt{\nu}} + (\mu + t) \text{erf} \left( \frac{\mu + t}{\nu} \right) + C \nu,
\]

we obtain

\[
j_1 = \frac{\sqrt{2\sigma}}{\sqrt{\pi}} e^{-\frac{a^2}{2\sigma^2}} + d_1 \text{erf} \left( \frac{d_1}{\sqrt{2\sigma}} \right) - \frac{\sqrt{2\sigma}}{\sqrt{\pi}} e^{-\frac{d_1^2}{2\sigma^2}} - d_2 \text{erf} \left( \frac{d_2}{\sqrt{2\sigma}} \right),
\]

and

\[
j_2 = \frac{\sqrt{2\sigma}}{\sqrt{\pi}} e^{-\frac{a^2}{2\sigma^2}} + d_2 \text{erf} \left( \frac{d_2}{\sqrt{2\sigma}} \right) - \frac{\sqrt{2\sigma}}{\sqrt{\pi}} e^{-\frac{d_2^2}{2\sigma^2}} - d_1 \text{erf} \left( \frac{d_1}{\sqrt{2\sigma}} \right),
\]

where \( d_1 = a\cos\theta + b\sin\theta - \rho, d_2 = a\cos\theta - b\sin\theta - \rho, d_3 = a\cos\theta - b\sin\theta + \rho \) and \( d_4 = a\cos\theta + b\sin\theta + \rho \).

Finally, we have

\[
F(\rho, \theta, \sigma) = \frac{\sqrt{2}}{\sqrt{\pi}\sigma \sin 2\theta} J(\rho, \theta, \sigma) = \frac{\sqrt{2}}{\sqrt{\pi}\sigma \sin 2\theta} \left( J_1 - J_2 \right)
\]

\[
= \frac{1}{\sin 2\theta} \left[ \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \left( e^{-d_1^2/2\sigma^2} - e^{-d_2^2/2\sigma^2} - e^{-d_3^2/2\sigma^2} + e^{-d_4^2/2\sigma^2} \right) \right]
\]

\[
+ d_1 \text{erf} \left( \frac{d_1}{\sqrt{2\sigma}} \right) - d_2 \text{erf} \left( \frac{d_2}{\sqrt{2\sigma}} \right) - d_3 \text{erf} \left( \frac{d_3}{\sqrt{2\sigma}} \right)
\]

\[
+ d_4 \text{erf} \left( \frac{d_4}{\sqrt{2\sigma}} \right).
\]

APPENDIX B

Let examine the behaviour of the SSRT for a straight line when \( \rho = 0 \) and \( \sigma \) is the neighbourhood of zero, that is, \( \sigma \to 0 \). In this case, \( F() \) is expressed as \( \lim_{\sigma \to 0} F(0, \theta) = \frac{2}{\sin 2\theta} \left| a\cos\theta + b\sin\theta \right| = \left| a\cos\theta - b\sin\theta \right| \) which is obtained using the equations

\[
\lim_{\sigma \to 0} \sigma e^{-\frac{(a\cos\theta \pm b\sin\theta)^2}{2\sigma^2}} = 0 \quad \text{and} \quad \lim_{\sigma \to 0} \frac{a\cos\theta \pm b\sin\theta}{\sqrt{2\sigma}} = \text{sgn}(a\cos\theta \pm b\sin\theta). \]

Then, we have

\[
|a\cos\theta + b\sin\theta| = \begin{cases} 
  a\cos\theta + b\sin\theta & \text{if } \theta \leq -\frac{\pi}{2} + \beta, \\
  -a\cos\theta - b\sin\theta & \text{if } \theta \geq -\frac{\pi}{2} + \beta,
\end{cases}
\]

\[
|a\cos\theta - b\sin\theta| = \begin{cases} 
  a\cos\theta - b\sin\theta & \text{if } \theta \leq -\frac{\pi}{2} - \beta, \\
  -a\cos\theta + b\sin\theta & \text{if } \theta \geq -\frac{\pi}{2} - \beta.
\end{cases}
\]

So, one retrieves the following limits values in function of \( \theta \):

\[
\text{for } 0 \leq \theta \leq -\frac{\pi}{2} - \beta \quad \lim_{\sigma \to 0} F(0, \theta) = \frac{2}{\sin 2\theta} (2b\sin\theta) = \frac{2\beta}{\cos\theta},
\]
for $\frac{\pi}{2} - \beta \leq \theta \leq \frac{\pi}{2}$ + $\beta$ \hspace{1em}$\lim_{\sigma \to 0} F(0, \theta) = \frac{2}{\sin 2\theta} (2\sigma \cos \theta) = \frac{2\beta}{\sin \theta}$.

for $\frac{\pi}{2} + \beta \leq \theta \leq \pi$ \hspace{1em}$\lim_{\sigma \to 0} F(0, \theta) = \frac{2}{\sin 2\theta} (-2\beta \sin \theta) = -\frac{2\beta}{\cos \theta}$.

The variations of $\lim_{\sigma \to 0} F(0, \theta)$ in function of $\theta$ look like the ones given in Figure 2b. Here (i.e. when $\sigma \to 0$), the maxima, given for $\theta = \pi/2 \pm \beta = \pi/2 \pm \arctan(b/a)$, are equal to $2a/\sin(\pi/2 \pm \beta)$ or $2a/\cos(\beta')$, which correspond to the two diagonals of the rectangle illustrated in Figure 1. We have demonstrated in this example that the SSRT of a straight line when $\sigma \to 0$ is nothing but the Radon transform when $\rho = 0$.

APPENDIX C

Let us now examine the case where $\sigma > 0$. The derivative of $F(\rho, \theta)$ wrt $\rho$ is given by

$$\frac{\partial F}{\partial \rho} = -\frac{1}{\sin 2\theta} \left[ \text{erf} \left( \frac{d_1}{\sqrt{2\sigma}} \right) - \text{erf} \left( \frac{d_2}{\sqrt{2\sigma}} \right) + \text{erf} \left( \frac{d_3}{\sqrt{2\sigma}} \right) - \text{erf} \left( \frac{d_4}{\sqrt{2\sigma}} \right) \right].$$

This equation is equal to zero when $\rho = 0$ and $\sigma > 0$. The second derivative wrt $\rho$ equals

$$\frac{\partial^2 F}{\partial \rho^2} = -\frac{\sqrt{2}}{\sqrt{\pi} \sigma \sin 2\theta} \left[ -e^{-\frac{d_1^2}{2\sigma}} + e^{-\frac{d_2^2}{2\sigma}} + e^{-\frac{d_3^2}{2\sigma}} - e^{-\frac{d_4^2}{2\sigma}} \right].$$

At $\rho = 0$, we have

$$\frac{\partial^2 F}{\partial \rho^2} = -2\frac{\sqrt{2/\pi}}{\sigma \sin 2\theta} e^{-\frac{\sqrt{2} d_1^2 + \sqrt{2} d_2^2 + \sqrt{2} d_3^2}{2\sigma}} \left[ e^{-\frac{\sqrt{2} d_1^2 + \sqrt{2} d_2^2 + \sqrt{2} d_3^2}{2\sigma}} - e^{-\frac{\sqrt{2} d_1^2 + \sqrt{2} d_2^2 + \sqrt{2} d_3^2}{2\sigma}} \right].$$

Since $\sin 2\theta = 2\sin \theta \cos \theta$ and $\sin \theta \geq 0 \ \forall \ \theta \in [0, \pi]$, we have

$$\text{sgn} \left( \frac{\partial^2 F}{\partial \rho^2} \right) = -\text{sgn} \left( \frac{1}{\cos \theta} \times \text{sgn}(e^{\alpha \sin \theta} - e^{-\alpha \sin \theta}) \right) \in \{0, -1\}$$

where $\alpha = \frac{\sqrt{2} d_1^2 + \sqrt{2} d_2^2 + \sqrt{2} d_3^2}{2\sigma} \geq 0$. Consequently, when $\rho = 0$, the second derivative is negative for $\theta \in [0, \pi]$ and then, $F(\rho)$ presents a maximum.

APPENDIX D

Equation (8) represents the SSRT of an ellipse of which a quarter is illustrated in Figure 5a. Let us make the integration wrt $r$ where the expressions is here noted $I_\phi$, that is,

$$I_\phi = \int_0^{2\pi} F(\phi) \int_0^{\frac{\pi}{4}} e^{-\left(\frac{d_\phi}{2}\right)^2 / \sigma^2} r dr,$$

where $d_\phi = b/\sqrt{1 - \epsilon^2 \cos^2(\phi)}$ and $r_\phi = b/\sqrt{1 - \epsilon^2 \cos^2(\phi)}$, $\epsilon = 1 - b^2/a^2$ the ellipse eccentricity. Then, the original inner integral $I_\phi$ could be rewritten as

$$I_\phi = \int_0^{\frac{\pi}{4}} (r - d_\phi) e^{-\left(\frac{r-d_\phi}{2\sigma^2}\right)^2} dr + \int_{d_\phi}^{\frac{\pi}{4}} e^{-\left(\frac{r-d_\phi}{2\sigma^2}\right)^2} dr$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left[ \int_{d_\phi}^{\frac{\pi}{4}} e^{-\left(\frac{r-d_\phi}{2\sigma^2}\right)^2} dr + \int_0^{d_\phi} e^{-\left(\frac{r-d_\phi}{2\sigma^2}\right)^2} dr \right]$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{d_\phi} e^{-\left(\frac{r-d_\phi}{2\sigma^2}\right)^2} dr$$

$$= \sigma^2 \left[ \text{erf} \left( \frac{r-d_\phi}{\sigma \sqrt{2}} \right) - \text{erf} \left( \frac{d_\phi}{\sigma \sqrt{2}} \right) \right].$$

Then, the SSRT of an ellipse is nothing but

$$F(\nu, b, \sigma) = \frac{1}{\sqrt{2\pi} \sigma} \int_0^{2\pi} I_\phi d\phi$$

$$= \int_0^{\pi/2} \left[ \frac{4\sigma}{\sqrt{\pi}} \left[ e^{-\left(\frac{r_\phi-d_\phi}{2\sigma^2}\right)^2} - e^{-\left(\frac{r_\phi-d_\phi}{2\sigma^2}\right)^2} \right] + 2d(\phi) \left[ \text{erf} \left( \frac{r_\phi-d_\phi}{\sigma \sqrt{2}} \right) - \text{erf} \left( \frac{d_\phi}{\sigma \sqrt{2}} \right) \right] \right] d\phi.$$