Field Theoretical Description of Quantum Hall Edge Reconstruction

Kun Yang
National High Magnetic Field Laboratory and Department of Physics, Florida State University, Tallahassee, Florida 32306

We propose a generalization of the chiral Luttinger liquid theory to allow for a unified description of quantum Hall edges with or without edge reconstruction. Within this description edge reconstruction is found to be a quantum phase transition in the universality class of one-dimensional dilute Bose gas transition, whose critical behavior can be obtained exactly. At principal filling factors $\nu = 1/m$, we show the additional edge modes due to edge reconstruction modifies the point contact tunneling exponent in the low energy limit, by a small and non-universal amount.

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Recently there has been considerable interest in the physics at the edge of a quantum Hall liquid [1]. Our theoretical understanding of the edge physics is mostly based on the chiral Luttinger liquid (CLL) theory advanced by Wen [2]. The CLL theory is a long-wave length, low-energy effective field theory which is closely tied to the fundamental topological features of the bulk quantum Hall liquid, and describes the most robust physical properties of the edges states, including the quantization of Hall conductance. It also makes a number of remarkable prediction about single-particle properties at the edge; for example it predicts that in point contact tunneling between a Fermi liquid metal and a quantum Hall edge, the current-voltage relation follows a power-law $I \sim V^\alpha$, which is characteristic of a Luttinger liquid, and for a whole sequence of bulk filling factors, the exponent $\alpha$ is universal and independent of the details of the edge confining potential and electron-electron interaction. Such power law behavior has been observed in recent tunneling experiments [3–6], and at principal filling factors like $\nu = 1/3$, the exponent $\alpha$ was found to be close to but noticeably different from the CLL prediction [3]. Away from $\nu = 1/3$ however, more significant discrepancy between theory and experiment has been found [4–6].

In the meantime, microscopic theoretical studies have suggested that the interplay between electron-electron interaction and confining potential at the edge at shorter distance (typically of order magnetic length $\ell$ or slightly above that) can give rise to nontrivial low-energy physics. In particular, it was found that the competition between the two can lead to edge reconstruction, both at integer [7,8] and fractional [9,10] bulk filling. In particular, it was argued recently [9,10] that for realistic sample parameters, edge reconstruction is essentially always present for fractional bulk filling, despite the presence of a sharp potential barrier in the samples grown by the cleaved edge overgrowth technique used in the recent tunneling experiments. Edge reconstruction gives rise to additional low-energy edge modes [8–10] that are not described by the original CLL theory [11]; these additional modes can profoundly affect the low-energy physics at the edge.

In this paper we propose a generalization of the CLL theory to accommodate the short-distance physics. In our generalized theory the edge with and without reconstruction can be described on equal footing; in particular, the edge reconstruction transition between these two different phases can be studied. For simplicity and clarity we will focus on principal filling factors $\nu = 1/m$ with $m$ being an odd integer; generalization to other filling factors is conceptually straightforward. We find that the edge reconstruction is a quantum phase transition in the universality class of one-dimensional dilute Bose gas transition, whose critical behavior can be obtained exactly. We also show that the additional edge modes due to edge reconstruction modifies the power-law exponent of the single electron Green’s function, and thus the point contact tunneling exponent in a non-universal way, although the modification is likely to be quantitatively rather small. We will also make contact with recent experiments and existing theories on edge tunneling.

For principal bulk filling $\nu = 1/m$, there is one chiral edge mode described by the following Hamiltonian within the CLL theory [2]:

$$H = 2\pi mv \sum_{k>0} \rho_k \rho_{-k} = \pi mv \int dx \rho^2(x), \quad (1)$$

where $v$ is the velocity for edge excitations, $\rho_k$ is the momentum space edge electron density operator which satisfies the Kac-Moody algebra: $[\rho_k, \rho_{k'}] = -\frac{k}{\pi m} \delta_{k+k'}$, and $\rho(x)$ is the corresponding edge electron density operator in real space. Eq. (1) describes a single branch of chiral bosons with linear dispersion: $\epsilon_k = vk$, i.e., the bosons propagate with a fixed velocity $v$. In real space Eq. (1) describes a density-density coupling that is completely local. While it is appropriate to neglect the non-locality of the electron-electron interaction in the long-wave length limit, we must also keep in mind that what drives edge reconstruction is the interplay between electron-electron interaction and confining potential at shorter distances; the typical length scale associated with edge reconstruction is the magnetic length $\ell$. We thus generalize Eq. (1) to incorporate the non-local nature of the electron-electron interaction:
In particular, there exist a critical point $a_c$ below which the dispersion is no longer perfectly linear and has a downward curvature for small $k$, indicating edge reconstruction instability. See text for details.

$$H = \int dx \int dx' \rho(x)V(x-x')\rho(x')$$

$$= \pi mv \int dx \{\rho^2(x) - a[\partial_x \rho(x)]^2 + b[\partial^2_x \rho(x)]^2 + \cdots\}. \tag{2}$$

In (2) we have performed a gradient expansion, and $a$ and $b$ are phenomenological constants that depend on details of the electron-electron interaction; for a generic short-range repulsive interaction we have $a > 0$ and $b > 0$, thus we can truncate the gradient expansion to the corresponding order without losing stability of the model.

Once the gradient terms are included, the chiral boson dispersion is no longer perfectly linear and has a downward curvature for small $k$:

$$\epsilon_k = v(k - ak^3 + bk^5 + \cdots), \tag{3}$$

as illustrated in Fig. (1) for different values of $a$; such non-linear dispersion has been seen in our numerical studies [12]. In particular, there exist a critical point $a_c = 2\sqrt{\frac{2}{b}}$ at this point there is a momentum $k_0 = b^{-1/4}$ at which $\epsilon_k = 0$. For $a > a_c$ we have $\epsilon_k < 0$ for $k \approx k_0$; i.e., the ground state of the system is no longer the vacuum of the chiral bosons, and the chiral bosons will condense into states with $k \approx k_0$! This instability is precisely the instability toward edge reconstruction, as it leads to an increase of the momentum of the ground state, and edge density oscillation [13]. In this case we must include a repulsive interaction among the chiral bosons to maintain the stability of the model [14]:

$$H_{int} = \int dx [u_3 \rho^3(x) + u_4 \rho^4(x) + \cdots]. \tag{4}$$

By stopping at the quartic order we are assuming that $u_4 > 0$. Combining Eqs. (2) and (4), we propose the following field theory to describe the edge of a $\nu = 1/m$ quantum Hall liquid:

$$S = \frac{m}{4\pi} \int dt dx \{\partial_t \phi \partial_\xi \phi - v(\partial_\xi \phi)^2 - a(\partial_\xi^2 \phi)^2 + b(\partial^3_\xi \phi)^2]\}
- \int dt dx [u_3 (\partial_\xi \phi)^3 + u_4 (\partial_\xi \phi)^4]. \tag{5}$$

Here $S$ is the action, and the real bosonic field $\phi(x)$ is related to the edge electron density through $\rho(x) = \partial_\xi \phi(x)/(2\pi)$; in the special case $a = b = u_3 = u_4 = 0$ it reduces to the action of the original CLL theory [2]. This model supports two phases: for $a < a_c$ the ground state is the vacuum state of the bosons, which properly describes the edge without reconstruction; the low-energy excitations are the chiral bosonic modes at small $k$. For $a > a_c$ there is a finite density of bosons in the ground state, mostly occupying modes with $k \approx k_0$. This is the phase with edge reconstruction. These 1D bosons with repulsive interaction form an ordinary (or non-chiral) Luttinger liquid which in turn can be mapped onto non-chiral free bosons; thus in addition to the chiral branch of bosons, there is also a non-chiral branch of low-energy bosonic excitations in this case, which we have seen in our numerical studies [9,10].

To study the critical behavior of this transition, we formally integrate out the high energy modes in Eq. (5), and focus on the low-energy modes near $k \approx 0$ and $k \approx k_0$. These low-energy modes are conveniently described in terms of the following slowly-varying (in space) bosonic fields (time dependence is implicit here):

$$\phi_1(x) = \frac{1}{\sqrt{L}} \sum_{|k|<\Lambda} \phi_k e^{ikx}; \tag{6}$$

$$\phi_2(x) = \frac{1}{\sqrt{L}} \sum_{|k|<\Lambda} \phi_{k_0+k} e^{ikx}. \tag{7}$$

Here $L$ is the length of the edge, and $\Lambda$ is a cutoff in momentum space. We note that while $\phi_1(x)$ is a real field, $\phi_2(x)$ is actually a complex field. In terms of $\phi_1(x)$ and $\phi_2(x)$, the original action in Eq. (5) takes the form:

$$S = S_1 + S_2 + S_{12}, \tag{8}$$

where

$$S_1 = \frac{m}{4\pi} \int dt dx [\partial_t \phi_1 \partial_\xi \phi_1 - v(\partial_\xi \phi_1)^2]; \tag{9}$$

$$S_2 = \int dt dx (i\bar{\psi}_2 \partial_t \psi_2 - \frac{|\partial_\xi \psi_2|^2}{2m^*} + \mu |\psi_2|^2 - \bar{a}|\psi_2|^4); \tag{10}$$

$$S_{12} = -\int dt dx [i\bar{\psi}_2 \partial_t \psi_1 + \bar{u}_4 (\partial_\xi \phi_1)^2 |\psi_2|^2]. \tag{11}$$

Here $\psi_2 = \sqrt{m \kappa_0/2\pi^2} \phi_2$, $\mu \propto -a - a_c$, $1/m^* \approx 8\sqrt{2}b^{1/4}$, $\bar{u}$, $\bar{u}_3$, and $\bar{u}_4$ are proportional to $u$, $u_3$, and $u_4$ at tree level but receive loop renormalization from integrating out higher energy modes, and we have neglected terms that scale to zero in the long-wave length limit (like $(\partial_t \psi_2)(\partial_\xi \bar{\psi}_2)$, $(\partial_\xi \phi_1)^3$, and $(\partial_\xi \phi_1)^4$ etc). We see $S_1$ takes exactly the same form as the original CLL action (but with a much reduced momentum cutoff), while $S_2$ is identical to the action of 1D non-relativistic bosons with repulsive interaction [15]. If we neglect $S_{12}$ that describes interaction between $\phi_1(x)$ and $\psi_2(x)$ for the moment, it is known [15]
that the system undergoes a second order phase transition from the vacuum of bosons (corresponding to edge without reconstruction), to a new ground state with a finite bosons density in it (corresponding to reconstructed edge), at the critical point \( \mu = 0 \) (or \( a = a_c \)). The effect of \( S_{12} \) may be taken into account by integrating out \( \phi_1 \) in \( S \), which results in finite renormalization of \( \tilde{u} \) in \( S_2 \). Assuming that the renormalized value of \( \tilde{u} \) to be positive (i.e., the effective interaction of the bosons described by \( \phi_2 \) remains repulsive), neither the position of the critical point \( \mu = 0 \) (or \( a = a_c \)), nor the critical property of the transition is changed by the coupling between \( \phi_1 \) and \( \phi_2 \) as described by \( S_{12} \). The critical exponents of this transition are known exactly [15]: \( \nu = 1/2 \) and \( z = 2 \), from which all other exponents can be deduced. In particular, the boson density or the change of ground state momentum per unit length scales as: \( n \sim \delta k \propto (d - d^*)^{1/2} \), where \( d \) is a controlling parameter (say, the distance between the dopant layer and the 2D electron gas layer \([9,10]\)) that tunes the system through the transition, and \( d^* \) is the critical point.

On the other hand, if the renormalized quartic coupling turns out to be negative, then the effective interaction between the bosons is attractive. In this case higher order couplings need to be kept, and the transition between the two phases may become first-order. Whether this is the case or not depends on the details of edge confining potential and electron-electron interaction. Hartree-Fock study of edge reconstruction at bulk filling \( \nu = 1 \) appears to suggest the transition is indeed first order in that case, for the type of confining potential that was used \([8]\).

No matter the transition is first-order or second-order, the condensed non-chiral bosons described by \( \psi_2 \) in the reconstructed phase form an ordinary or non-chiral Luttinger liquid. Perhaps the easiest way to obtain the Gaussian field theory (or Luttinger liquid) description of these non-chiral bosons from Eqs. (9-11) is to write \( \psi_2 \) as \( \psi_2(x, t) = \sqrt{\mu(x, t)} \exp[i\varphi(x, t)] \), and then integrate out the fluctuation of the boson density \( n \) about its mean value \( \bar{n} \propto \mu/2\tilde{u} \) in \( S \) \([16]\), after which one obtains

\[
S = S_1 + S_\varphi + S_{\text{int}},
\]

where \( S_1 \) takes the same form as in Eq. (9) with renormalized velocity \( v \), and \( S_\varphi \)

\[
S_\varphi = \int dt dx \frac{\bar{n}}{2m^*} \left( \frac{1}{v_\varphi^2} \left( \partial_t \varphi \right)^2 - \left( \partial_x \varphi \right)^2 \right),
\]

and \( v_\varphi \approx \sqrt{2\mu/\bar{n}m^*} \approx \sqrt{\mu/\bar{m}}. \) Physically \( \partial_t \varphi \) and \( \partial_x \varphi \) are proportional to the density and current of the (non-chiral) bosons described by \( \psi_2 \), through the Josephson relation. \( S_{\text{int}} \) describes the interaction between the chiral and non-chiral bosons through density-density coupling:

\[
S_{\text{int}} = -g \int dt dx (\partial_t \varphi) (\partial_x \varphi),
\]

where \( g \approx \tilde{u}_3/2\tilde{u} \). Thus edge reconstruction adds two more propagating edge modes in the edge spectrum, one propagating in the forward direction and another in the \( \text{backward} \) direction; these new modes are coupled to the original long wave-length chiral boson modes. Mathematically, the action of Eq. (12) is equivalent to that of a single CLL mode coupled to one-dimensional acoustic phonons, a model that has been considered before in very different contexts \([17,18]\).

Obviously, multiple edge reconstructions transitions can occur, if there are multiple local minima in the chiral boson spectrum Eq. (3) that go through zero. The critical behavior of these additional transitions will be the same, and each transition will introduce two more edge modes, propagating in opposite directions.

We now turn our discussion to the effect of edge reconstruction on single electron Green’s function. Within the CLL theory, the charge and statistics of the electron operator uniquely determines its form in terms of the edge density field \( \phi(x) \) to be \( 2 \langle \Psi(x) \rangle \propto e^{-im\varphi(x)} \). We are interested in the long-time or low-energy/frequency behavior of the electron Green’s function, which is dominated by the low-energy modes of \( \phi(x) \). For edges without reconstruction they are the long-wave length modes of \( \phi(x) \) with \( k \approx 0 \), while in the presence of edge reconstruction they are modes with \( k \approx 0 \) and \( k \approx \pm \delta \). Thus in the latter case we write the electron operator as

\[
\Psi(x) \sim \exp\{ -im[\phi_1(x) + \phi_2(x)e^{ik_0x} + \phi_2^*(x)e^{-ik_0x}] \} \\
\quad \approx \exp\{ -im[\phi_1(x) + ce^{i(k_0x - \varphi(x))} + ce^{i(\varphi(x) - k_0x)}] \} \\
= e^{-im\phi_1(x)} \sum_{l=0}^{\infty} (-2imc)^l \cos[\varphi(x) - k_0x].
\]

Here the constant \( c \approx \sqrt{2\pi m/\mu k_0} \). We have thus expressed the electron operator in terms of the Gaussian variables \( \phi_1 \) and \( \varphi \), whose correlation functions are controlled by the quadratic low-energy effective action \( S \) in Eq. (12). Thus the long-time behavior of the electron Green’s function can be determined straightforwardly:

\[
G(x = 0, t) = \langle \Psi(0, t)\Psi^\dagger(0, t = 0) \rangle = \sum_{l=\infty} A_l t^{-\gamma_l},
\]

where \( A_l \)’s are some constants, and \( \gamma_l \) is twice the scaling dimension of the operator \( O_l = e^{-im\varphi_1 + il\varphi} \): \( \langle O_l(0, t)O_l(0, 0) \rangle \sim t^{-\gamma_l}. \) The minimum value of \( \gamma_l \) controls Green’s function in the long time limit, and thus the \( I - V \) characteristics of point contact tunneling between the edge and a Fermi liquid metal is \( I \sim V^{\alpha} \) with \( \alpha = \gamma_{\text{min}} \). Using a generalized Bogliubov transformation \([17]\) to obtain the eigen modes of the action Eq. (12) and express \( O_l \) as combination of the eigen modes, one can easily show that all \( \gamma_l \)’s are non-universal and satisfy \( \gamma_l > m \). Thus the tunneling exponent \( \gamma_{\text{min}} \) is non-universal. This is a consequence of the lack of maximum
chirality due to edge reconstruction, and in sharp contrast with the case without edge reconstruction, where the tunneling exponent $\alpha = m$ and is thus universal [2].

In real samples, one expects $v$, the chiral charge mode velocity, to be much larger than all other velocity scales, especially the non-chiral mode velocity $v_\varphi$. This is because $v$ is controlled by the long-range Coulomb interaction and thus diverges as $\log k$ in the long-wave length limit; this divergence may be cut off by metallic gates placed near the sample, but only at very long length scales. On the other hand $v_\varphi \approx \sqrt{\mu/m^*}$ is a neutral mode velocity, and expected to be low since there is no reason for the chiral boson mode near the instabilities to develop very deep minima and have large curvature. In the limit of large $v$, we find

$$\alpha = \gamma_0 = m[1 + v_{\text{coup}} v_\varphi/v^2 + O(1/v^4)],$$

where $v_{\text{coup}} \approx m^2/\pi \tilde{u}$ is a positive velocity scale that parametrizes the strength of coupling between charge and neutral modes. We thus find that the tunneling exponent is increased by a small and non-universal amount due to edge reconstruction. While consistent with experimental findings that $\alpha \approx m$ for $m = 3$, we note that in all experiments $\alpha$ is slight below 3 (typically by about 10%). This, however, may be due to the fact that electrostatic forces provided by nearby gates tend increase the electron density by 20 – 30% over several hundred nanometers in the edge region [19]; thus the the actual value of $\alpha$ that correspond to $\nu = 1/3$ in the edge region may very well be slightly above 3, consistent with our result.

Recent edge tunneling experiments motivated a considerable amount of theoretical work [19–23]. One of the main focuses of these studies is the apparent inverse relation between the tunneling exponent $\alpha$ and bulk filling factor $\nu$ observed in at least one of the experiments [4]: $\alpha \approx 1/\nu$. In one of the proposals put forward to explain this approximate dependence, Lee and Wen [21] made a key assumption that there exist neutral mode(s) in the system whose velocity is extremely low. As discussed above, edge reconstruction can naturally lead to neutral modes with low velocities. Thus edge reconstruction can not only explain the lack of universality in $\alpha$ near principal filling factors, but may also be a key ingredient in the understanding of its general dependence on filling factor.

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