WALL-CROSSINGS FOR TWISTED QUIVER BUNDLES

BUMSIG KIM AND HWAYOUNG LEE

Abstract. Given a double quiver, we study homological algebra of twisted quiver sheaves with the moment map relation by modifying the short exact sequence of Gothen and King. Then in a certain one-parameter space of the stability conditions, we obtain a wall-crossing formula for the generalized Donaldson-Thomas invariants of the abelian category of framed twisted quiver sheaves on a smooth projective curve. To do so, we closely follow the approach of Chuang, Diaconescu, and Pan in the ADHM quiver case, which makes use of the theory of Joyce and Song. The invariants virtually count framed twisted quiver sheaves with the moment map relation.

1. Introduction

A Nakajima’s quiver variety is a holomorphic symplectic quotient attached to a double quiver $\mathcal{Q}$, i.e., a quiver whose arrows are paired $(a, \bar{a})$ such that $\bar{a}$ is a reverse arrow of $a$. This holomorphic symplectic quotient is a GIT quotient of a locus defined by a moment map relation. In [11], the moduli of stable twisted quasimaps to the symplectic quotient from a fixed smooth projective curve $X$ is obtained as an application of the quasimap construction of [3, 4] and shown to come with a natural symmetric obstruction theory. This result generalizes Diaconescu’s work [6].

The stability in the notion of stable twisted quasimaps turns out to be an asymptotic one in the one dimensional stability parameter space $\mathbb{R}_{>0}$ of the abelian category $\mathcal{A}'$ of framed twisted quiver sheaves on $X$. It is therefore natural to investigate the wall-crossing phenomena of the moduli stack $\mathcal{M}_{ss}^\tau(\gamma)$ of $\tau$-semistable objects with numerical class $\gamma$ in $\mathcal{A}'$ as $\tau$ varies in $\mathbb{R}_{>0}$.

For that study of wall-crossings, there are two theories available: the theory of Joyce and Song ([10]); and the theory of Kontsevich and
Soibelman ([12]). In this paper, we perform our research according to the framework of Joyce and Song.

First, we study the homological algebra of the category of twisted quiver sheaves with the moment map relation (2.5). Without the relation, the homological study is carried out by Gothen and King ([7]). We deduce that a truncated part of the category $\mathcal{A}'$ behaves like a Calabi-Yau 3-category. For example, a suitably defined antisymmetric Euler form is numerical (see Proposition 2.7). This property is the first main result of this paper and originates from the moment map relation.

Next, using the above Euler form on the numerical $K$-group of $\mathcal{A}'$, we define a Lie algebra $L(\mathcal{A}')$ and, using the straightforward generalization of the Chern-Simons functional in [6], we construct a Lie algebra homomorphism to $L(\mathcal{A}')$ from a Ringel-Hall type algebra of stack functions with algebra stabilizers supported on virtual indecomposables, as in [10, Theorems 3.16 and 7.13]. This Lie algebra homomorphism yields the definition of the generalized Donaldson-Thomas invariant for $(\mathcal{Q}, X, \gamma, \tau)$ via the log stack function for $\mathcal{M}_{\tau}^\ast(\gamma)$.

Finally, using the approach of [6, 11, 12], we establish a wall-crossing formula of the invariants (see Theorem 3.8), which is the second main result of this paper. In section 3.6, we show that the invariants vanish when the framing is zero and, at the same time, the curve $X$ is not rational. The wall-crossing correction could be therefore nontrivial only when $X$ is rational.

2. Homological Algebra

The aim of this section is to prove the suitably defined antisymmetric Euler form of framed twisted quiver sheaves is numerically determined in certain cases (see Proposition 2.7). For this, we begin with finding a partial injective resolution (2.2) of double quiver representations with the moment map relation (2.1).

2.1. Double quivers. We set up notations for quivers. Let $Q$ be a finite quiver, i.e., a directed graph whose arrow set $Q_0$ and vertex set $Q_1$ are finite. The tail map and the head map from $Q_1$ to $Q_0$ are denoted by $t$ and $h$, respectively. Define the double quiver $\overline{Q}$ of $Q$ by adjoining a reverse arrow to each arrow of $Q$. A path $p$ is an ordered set $a_1...a_m$ of arrows $a_i$ such that $ta_i = ha_{i+1}$ for $i = 1,..., m-1$. Define the head and the tail of $p$ by $hp = ha_1$ and $tp = ta_m$, respectively. For each vertex $i \in Q_0$, define a trivial path $e_i$. Also set $h(e_i) = t(e_i) = i$. The lengths of $p$ and $e_i$ are by definition $m$ and 0, respectively. Let $R$ be a commutative ring with unity 1. The path algebra $R\overline{Q}$ is the $R$-algebra generated by all paths subject to relations by the following
rules (see for instance [7]): $p \cdot q = pq$ if $tp =hq$, 0 otherwise; $p \cdot e_{tp} = p$; and $e_{hp}p = p$.

2.2. **Quiver representations.** In this section, we let $Q'_0$ be any nonempty subset of $Q_0$. We fix $\lambda_i \in \mathbb{R}$ for each $i \in Q'_0$.

We consider the two-sided ideal $(\mu - \lambda)$ generated by the relation

$$\sum_{i \in Q'_0} \left( \left( \sum_{a \in Q_1 : ha = i} (-1)^{|a|} a\alpha \right) - \lambda_i e_i \right) = 0,$$

where $|a|$ is 0 if $a \in Q_1$ or 1 otherwise. In fact, (2.1) is the functorial expression of the moment map equation (see for example [11]) and hence we call (2.1) the moment map relation for the quiver (or more precisely for $(Q, Q'_0, \lambda)$).

We will study the homological algebra of the quotient algebra $A := \mathbb{R} Q / (\mu - \lambda)$, which was called the deformed preprojective algebra in [5]. Note that every $A$-module can be considered as an $\mathbb{R}$-module by the natural $\mathbb{R}$-module homomorphism $\mathbb{R} \to A, r \mapsto \sum_{i \in Q_0} e_i$.

Let $\mathcal{R}$ be the abelian category of (left) $A$-modules. For $V \in \mathcal{R}$, we construct a sequence in $\mathcal{R}$, which becomes a partial injective resolution of $V$ when $R$ is a field. The sequence is defined to be

$$0 \to V \xrightarrow{\varepsilon} \bigoplus_{i \in Q_0} \text{Hom}_R(e_i A, V_i) \xrightarrow{g} \bigoplus_{a \in Q_1} \text{Hom}_R(e_{ta} A, V_{ha}) \xrightarrow{m} \bigoplus_{i \in Q'_0} \text{Hom}_R(e_i A, V_i),$$

where:

- we view $e_i A$ as an $R$-$A$ bimodule so that $\text{Hom}_R(e_i A, V_j)$ is a left $A$-module;
- $V_i := e_i V$ which is an $A$-submodule of $V$;
- the $A$-homomorphisms $\varepsilon$, $g$, $m$ are defined by: for $p_i \in e_i A \subset A$
  1. $\varepsilon(v)(p_i) = p_i v$;
  2. $g(\alpha)_{ap} = \alpha_{ha}(ap_{ta}) - \alpha_{ta}(p_{ta})$;
  3. $m(\gamma)(p_i) = \sum_{a \in Q_1 : ta = i} (-1)^{|a|} (\pi a(p_i) + \gamma a(ap_i))$.

The above sequence (2.2) is obtained from a combination of the short exact sequence [7, (2.1)] and the ‘moment map’. Since $A$ has the moment map relation, $g$ is not surjective.

**Proposition 2.1.** The sequence (2.2) is exact.

**Proof.** We adapt the proof of [7, Proposition 2.4] to our situation. If $\varepsilon(v)(e_i) = e_i v = 0$ for all $i \in Q_0$, then $\sum_i e_i v = 0$. Since $\sum e_i = 1$, we
see that \( v = 0 \). Thus \( \epsilon \) is injective. It is clear that \( \text{Im} \  \epsilon \subset \text{Ker} \  g \). Next we will show that \( \text{Ker} \  g \subset \text{Im} \  \epsilon \). Consider \( \alpha \in \bigoplus_{i \in Q_0} \text{Hom}_R(e_i A, V_i) \subset \text{Hom}_R(A, V) \) such that \( g(\alpha) = 0 \). This implies that \( \alpha \) is \( A \)-linear. Therefore \( \text{Ker} \  g \subset \text{Hom}_A(A, V) = \epsilon(V) \). Next we can check that \( \text{Im} \  g \subset \text{Ker} \  m \) since

\[
(m \circ g(\alpha_i))_i(p_i) = \sum_{a \in \mathcal{Q}: ta = i} (-1)^{|\mathcal{I}|} (\overline{\alpha}g(\alpha_i)_a(p_i) + g(\alpha_i)\pi(ap_i)) = \sum_{a \in \mathcal{Q}: ta = i} (-1)^{|\mathcal{I}|} (\overline{\alpha}(\alpha_{ha}(ap_i)) - a\alpha_i(p_i)) + \alpha_i(\overline{\alpha}ap_i) - \overline{\alpha}\alpha_{\overline{\alpha}}(ap_i) = 0.
\]

The third equality above follows from the \( R \)-linearity of \( \alpha \), the moment map relation, and \( \overline{ta} = ha \).

Finally, let us show the hard part \( \text{Ker} \  m \subset \text{Im} \  g \). Let \( (\gamma_a)_{a \in \mathcal{Q}} \in \text{Ker} \  m \), in other words, for each \( i \in Q'_0 \),

\[
\sum_{a \in \mathcal{Q}: ta = i} (-1)^{|\mathcal{I}|} (\overline{\alpha}(\gamma_a(p_{ta}) + \gamma_{ap_{ta}})) = 0.
\]

Let \( A_k \) be an \( R \)-module generated by arrows \( p \) whose lengths are less than or equal to \( k \). For example, \( A_0 \) is generated by \( e_i, i \in Q_0 \) and \( A_1 \) is generated by \( a \in \mathcal{Q}_1 \) and \( e_i, i \in Q_0 \). Then consider the filtration

\[
A_0 \subset A_1 \subset A_2 \subset \cdots.
\]

For each vertex \( i \), there is the corresponding filtration \( e_i A_0 \subset e_i A_1 \subset e_i A_2 \subset \cdots \) as well. An \( R \)-linear map \( \alpha \) will be constructed by induction on the filtration. Define \( \alpha = (\alpha_i) \in \bigoplus_{i \in Q_0} \text{Hom}_R(e_i A, V_i) \), \( \alpha_i : e_i A \rightarrow V_i \) as

\[
\begin{align*}
\bullet &\quad \alpha_i \mid_{e_i A_0} = 0 \text{ and } \\
\bullet &\quad \alpha_i(ap_{ta}) = a\alpha_{ta}(p_{ta}) + \gamma_{ta}(p_{ta}) \text{ for } a \in \mathcal{Q}_1 \text{ with } ha = i.
\end{align*}
\]

We need to show that \( \alpha \) is well-defined. For that, we consider the path algebra \( F := R\mathcal{Q} \) without any relation and its corresponding filtration \( F_i \). Note that \( \alpha \) is well defined as an element on \( \text{Hom}(F, V) \) and \( \overline{g}(\alpha) = \gamma \) if

\[
\overline{g} : \bigoplus_{i \in Q_0} \text{Hom}_R(e_i F, V_i) \rightarrow \bigoplus_{a \in \mathcal{Q}_1} \text{Hom}_R(e_{ta} F, V_{ha})
\]

is the homomorphism corresponding to \( g \). Now we claim that \( \alpha \) is well-defined on \( A_n \). It is clear that the claim is true when \( n = 0, 1 \). Let \( F_n \) be the \( R \)-submodule of \( F \) spanned by length-\( n \) elements. Suppose that
\(\alpha\) is well-defined on \(A_{n-1}\). Then note that for \(p_i \in e_i F_{n-2}\) and \(i \in Q'_0\),

\[
\alpha_i \left( \sum_{a \in Q'_1, ha = i} (-1)^{|a|} a\bar{a} - \lambda_i \right) p_i
\]

\(= \sum (-1)^{|a|}(a\alpha_h(\bar{a}p_i) + \gamma_a(\bar{a}p_i)) - \lambda_i \alpha_i(p_i) = 0\)

since

\[
\sum (-1)^{|a|}\gamma_a(\bar{a}p_i)
\]

\(= - \sum (-1)^{|a|}a\gamma_\bar{a}(p_i) \quad \text{(by } m(\gamma_a) = 0\text{)}
\]

\(= - \sum (-1)^{|a|}a(\bar{g}(\alpha_i)\gamma(p_i)) \quad \text{(by } \gamma = \bar{g}(\alpha)\text{)}
\]

\(= - \sum (-1)^{|a|}(a\alpha_h\bar{a}\gamma(p_i) - a\bar{a}\alpha\bar{a}(p_i)) \quad \text{(by the definition of } \bar{g} \text{)}.
\]

Combined with the inductive definition of \(\alpha\), the equation (2.3) implies that \(\alpha = 0\) on the two-sided ideal \((\mu - \lambda)\) of \(A_n\). By the definition of \(\alpha\), it is clear that \(g(\alpha) = \gamma\).

Replacing \(V\) by \(W\) in the sequence (2.2) and taking \(\text{Hom}_A(V, \cdot)\) on \(\mathcal{C}(V, W)\), we obtain a sequence

\[
0 \to \text{Hom}_A(V, W) \to \bigoplus_{i \in Q_0} \text{Hom}_R(V_i, W_i)
\]

\[
\to \bigoplus_{a \in Q'_1} \text{Hom}_R(V_{ta}, W_{ha}) \to \bigoplus_{i \in Q'_0} \text{Hom}_R(V_i, W_i)
\]

of \(R\)-modules. This simplification follows from the adjunction

\[
\text{Hom}_{R_1}(N_1, \text{Hom}_{R_2}(N_2, N_3)) = \text{Hom}_{R_2}(N_2 \otimes_{R_1} N_1, N_3),
\]

where \(N_1\) is a left \(R_1\)-module, \(N_2\) is an \(R_2\)-\(R_1\) bimodule, and \(N_3\) is a left \(R_2\)-module (see [14, Proposition (2.6.3)] or [7, (2.2)]). Let \(\mathbf{C}(V, W)\) be the complex consisting of the last three terms of the sequence (2.4) with the first term at degree 0. Then we get the following.

**Corollary 2.2.** Let \(R\) be a field.

1. \(\text{Hom}_R(e_i A, V_j)\) is an injective \(A\)-module.
2. For \(l = 0, 1,\)

\[
\text{Ext}^l_A(V, W) \cong H^l(\mathbf{C}(V, W)).
\]

**Proof.** By the above adjunction, we note the equivalence of functors:

\[
\text{Hom}_A(\bullet, \text{Hom}_R(e_i A, V_j)) \cong \text{Hom}_R(e_i A \otimes_A \bullet, V_j).
\]
The latter is an exact functor since \( e_i A \) is a projective right \( A \)-module and \( V_j \) is an injective \( R \)-module. This proves (1). Now (2) follows since (2.2) is a partial injective resolution of \( V \).

\[ \Box \]

2.3. Quiver sheaves. We carry out a similar procedure for twisted quiver sheaves in place of quiver representations. To introduce twisted quiver sheaves, let \( X \) be a Gorenstein projective variety, let \( \omega_X \) be the dualizing sheaf for \( X \), and let \( \lambda_i \in \Gamma(X, \omega_X) \). Suppose also that we choose an invertible sheaf \( M_a \) on \( X \) for each \( a \in \overline{Q}_1 \) and an isomorphism \( f_{b,\bar{b}} : M_b \otimes M_{\bar{b}} \to \omega_X^\vee \) for each \( b \in Q_1 \). We will set \( f_{\bar{b},b} = f_{b,\bar{b}} \) for \( b \in Q_1 \) using the natural isomorphism \( M_{\bar{b}} \otimes M_b \cong M_b \otimes M_{\bar{b}} \).

Provided with the above data, we define an \( \mathcal{O}_X \)-algebra structure on the sheaf

\[ \bigoplus \text{all paths } p \ M_p \]

by making:

- \( M_p := M_{a_1} \otimes \ldots \otimes M_{a_n} \) if \( p = a_1 \ldots a_n \) with \( a_i \in \overline{Q}_1 \), and \( M_{e_i} := \mathcal{O}_X \);
- for \( x_p \in M_p \), \( x_q \in M_q \), let \( x_p x_q := x_p \otimes x_q \) if \( tp = hq \), and 0 otherwise; and in \( \bigoplus_p M_p \) we have natural identifications \( M_p M_{e_{hp}} = M_p \otimes M_{e_{hp}} = M_p = M_{e_{hp}} M_p = M_{e_{hp}} \otimes M_p \).

We denote by \( M\overline{Q} \) this \( \mathcal{O}_X \)-algebra graded by lengths.

We want to define an ideal sheaf from the moment map relation. First, for every local section \( \xi \in \omega_X^\vee \), let

\[(\mu - \lambda)(\xi) := \sum_{i \in Q_0} \left( \sum_{a \in \overline{Q}_1 : ha = i} (-1)^{|a|} \xi_a \otimes \xi_{\overline{a}} \right) - \langle \xi, \lambda_i \rangle e_i ,\]

where \( e_i \) stands for the constant 1 in \( M_{e_i} = \mathcal{O}_X \) and \( \xi_a \otimes \xi_{\overline{a}} \) is required to satisfy \( f_{a,\pi}(\xi_a \otimes \xi_{\overline{a}}) = \xi \). Then since \( (\mu - \lambda)(f \cdot \xi) = f \cdot (\mu - \lambda)(\xi) \) for \( f \in \mathcal{O}_X \), we can define the ideal sheaf \((\mu - \lambda)\) of \( M\overline{Q} \) generated by \((\mu - \lambda)(\xi)\) for all \( \xi \in \omega_X^\vee \) and hence the quotient sheaf \( B := M\overline{Q}/(\mu - \lambda) \).

We will consider homological algebra of the abelian category \( A \) of \( B \)-modules, which is a subcategory of \( \mathcal{O}_X \)-modules. For the study, we use also an alternative and concrete description of a \( B \)-module by a collection of \( \mathcal{O}_X \)-sheaves \( E_i, i \in Q_0 \) and \( \mathcal{O}_X \)-homomorphisms \( \phi_a : M_a \otimes E_{ia} \to E_{ha}, a \in \overline{Q}_1 \). The collection will be called a \textit{\( M \)-twisted quiver sheaf} on \( X \) in our context if the following moment map relation
holds:

$$\sum_{i \in Q'_0} \left( \sum_{a \in Q_1, ha = i} (-1)^{|a|} \phi_a \circ (\text{Id}_{Ma} \otimes \phi_{\pi}) - \lambda_i \otimes \text{Id}_{E_i} \right) = 0.$$  

**Proposition 2.3.** The category of $M$-twisted quiver sheaves is equivalent to $A$.

**Proof.** If $\{E_i\}$ is an $M$-twisted quiver sheaf, then it is obvious how to give a $B$-module structure on $\bigoplus E_i$. Conversely, for a $B$-module $E$, define $E_i := e_i E = M_{e_i} E$. Then it is simple to check the collection $\{E_i\}$ has an induced $M$-twisted quiver sheaf structure. $\square$

For $E \in A$, there is an exact sequence in $A$

$$0 \to E \xrightarrow{\epsilon} \bigoplus_{i \in Q_0} \text{Hom}_{O_X}(e_i B, E_i)$$

$$\xrightarrow{g} \bigoplus_{a \in Q_1} \text{Hom}_{O_X}(M_a \otimes_{O_X} e_{ta} B, E_{ha})$$

$$\xrightarrow{m} \bigoplus_{i \in Q'_0} \text{Hom}_{O_X}(\omega_X^\vee \otimes_{O_X} e_i B, E_i),$$

where:

- $e_i$ denotes the constant 1 in $M_{e_i} = O_X$;
- we view $e_i B$ as an $O_X$-$B$ bimodule so that $\text{Hom}_R(e_i B, V_j)$ is a left $B$-module;
- $E_i := e_i E$ which is a $B$-submodule of $E$;
- the $B$-homomorphisms $\epsilon, g, m$ are defined by: for $p_i \in e_i B \subset B$,

$$\xi = f_{\pi,a}((\xi_\pi \otimes \xi_a),$$

(1) $\epsilon(e)(p_i) = p_i e$;

(2) $g(\alpha)_{a}(x_a \otimes p_{ta}) = \alpha_{ha}(x_a \otimes p_{ta}) - \phi_a(x_a \otimes \alpha_{ta})(p_{ta});$

(3) $m(\gamma)(\xi \otimes p_i) = \sum_{a \in Q:ta = i} (-1)^{|a|} (\phi_a(\xi_\pi \otimes \gamma_{a}(\xi_a \otimes p_i)) + \gamma_{\pi}(\xi_\pi \otimes \phi_a(\xi_a \otimes p_i))).$

**Proposition 2.4.** The sequence (2.6) is exact.

**Proof.** The proof is parallel to the proof of Proposition 2.1. $\square$

As before, we replace $E$ by $F$ in the sequence (2.2) and take $\text{Hom}_B(E, \cdot)$ to obtain a sequence

$$0 \to \text{Hom}_B(E, F) \to \bigoplus_{i \in Q_0} \text{Hom}_{O_X}(E_i, F_i)$$

$$\to \bigoplus_{a \in Q_1} \text{Hom}_{O_X}(M_a \otimes E_{ta}, F_{ha}) \to \bigoplus_{i \in Q'_0} \text{Hom}_{O_X}(\omega_X^\vee \otimes E_i, F_i)$$
of $\mathcal{O}_X$-modules.

If we let $C(E, F)$ be a complex consisting of the last three terms of (2.7) and we assume that $E$ is locally free, i.e., by definition, $E_i$ are locally free for all $i \in Q_0$, then we will observe that $C(E, F)$ is quasi-isomorphic to a complex computing $\text{Ext}^i_B(E, F)$ for $i = 0, 1$.

**Corollary 2.5.** Assume that $E$ is locally free. Then, for $i = 0, 1$,

$$\text{Ext}^i_B(E, F) \cong H^i(X, C(E, F)).$$

**Proof.** Note that a partial injective resolution $J^\bullet$ of $F$ can be obtained from injective resolutions $I^k_i$ of $F_i$ and (2.6) since $\oplus_i I^k_i$ has an induced $B$-module structure (see [7, Section 3] for detail). Note that $H^b_B(E, J^\bullet)$ as an $\mathcal{O}_X$-complex is quasi-isomorphic (at 0, 1) to

$$0 \to \bigoplus_{i \in Q_0} E_i^\vee \otimes F_i \to \bigoplus_{a \in Q_1} M_a^\vee \otimes E^\vee_{ta} \otimes F_{ha} \to \bigoplus_{i \in Q_0'} \omega_X \otimes E_i^\vee \otimes F_i$$

which is $C(E, F)$. □

**Definition 2.6.** Define $\chi(E, F)$ to be

$$\dim \text{Ext}^0_B(E, F) - \dim \text{Ext}^1_B(E, F) + \dim \text{Ext}^1_B(F, E) - \dim \text{Ext}^0_B(F, E).$$

This alternating sum $\chi(E, F)$ is called the Euler form of $E, F$.

If $E_0$ denotes the $\mathcal{O}_X$-coherent sheaf $\bigoplus_{i \in Q_0 \setminus Q'_0} E_i$ (when $Q_0 = Q'_0$, always $E_0 = 0$), we finally come to the first main result of this paper.

**Proposition 2.7.** Suppose that $X$ is a smooth projective curve and $E$ and $F$ are locally free. Assume either $E_0 = 0$ or $F_0 = 0$.

(1) For $i = 0, 1, 2, 3$,

$$H^i(X, C(E, F)) \cong H^{3-i}(X, C(F, E))^\vee.$$

(2) The Euler form is numerically determined as follows:

$$\chi(E, F) = \sum_{a \in Q_1} (d(E_{ta})r(F_{ha}) - d(F_{ha})r(E_{ta}) - d(M_a)r(E_{ta})r(F_{ha})$$

$$+ (1 - g)r(E_{ta})r(F_{ha}) + 2 \sum_{i \in Q_0} (-d(E_i)r(F_i) + d(F_i)r(E_i)),$$

where $d(E_i)$ and $r(E_i)$ stand for the degree and the rank of the locally free sheaf $E_i$, respectively.
Proof. (1): Note that $\mathbf{C}(F, E)^\vee \otimes \omega_X = \mathbf{C}(E, F)[2]$. Combined with the Serre duality, this implies $H^i(X, \mathbf{C}(E, F)) \cong H^i(X, \mathbf{C}(F, E)^\vee \otimes \omega_X[-2]) \cong H^{3-i}(X, \mathbf{C}(F, E))^\vee$.

(2): Note that $\chi(X, \mathbf{C}(E, F)) = \chi(E, F)$ by (1) above. On the other hand, $\chi(X, \mathbf{C}(E, F))$ is equal to $\chi(X, \mathbf{C}^0(E, F)) + \chi(X, \mathbf{C}^1(E, F)) + \chi(X, \mathbf{C}^2(E, F))$ and hence the topological expression for $\chi(E, F)$ follows from the Riemann-Roch formula.

3. Wall-crossings

From now on, let $X$ be a smooth projective curve, let $\lambda_i = 0$ for all $i \in Q_0$, and let $Q_0 \setminus Q_0' = \{0\}$. In the abelian category $\mathcal{A}'$ of twisted quiver sheaves, we will consider stability conditions for $\tau \in \mathbb{R}_{>0}$ and, in the framework of Joyce-Song theory [10], we will define the generalized Donaldson-Thomas invariants using the moduli space of $\tau$-semistable objects in $\mathcal{A}'$. We will derive a wall-crossing formula following the approach of Chuang, Diaconescu, and Pan ([6, 1, 2]).

3.1. Chamber structures. In this subsection, for $\tau \in \mathbb{R}_{>0}$ we introduce the notion of a $\tau$-stability on twisted quiver sheaves and show that for each fixed numerical class with a minimal framing the stability space $\mathbb{R}_{>0}$ has a finite number of critical values. The precise definition of critical values are not important. The relevant required property will be only that there are no strictly $\tau$-semistable quiver sheaves for every noncritical value $\tau$.

Let $K$ be a nonzero complex vector space. Denote by $\mathcal{A}'$ the abelian category of $\mathbb{M}$-twisted quiver sheaves $E$ with $E_0 = K^S \otimes \mathcal{O}_X$ for some finite set $S$ (depending on $E$). In this category $\mathcal{A}'$, a morphism from $(E_i, \phi_a)$ to $(E'_i, \phi'_a)$ is by definition a usual morphism as $\mathbb{M}$-twisted quiver sheaves with the framing condition that the attached $\mathcal{O}_X$-homomorphism $K^S \otimes \mathcal{O}_X \to K^{S'} \otimes \mathcal{O}_X$ is a block matrix $(c^{s,s'})_{(s,s') \in S \times S'}$, $c^{s,s'} \in \mathbb{C}$. In [11], the category $\mathcal{A}'$ is denoted by $\text{Rep}_C(\mathbb{Q}(\mathbb{M}, K))$. The category $\mathcal{A}'$ is an abelian category (see [11] Proposition 5.3]).

Let $E \in \mathcal{A}'$ and $\tau \in \mathbb{R}_{>0}$ be the stability parameter. For a nonzero $\mathbb{M}$-twisted quiver sheaf $E$ we define the $\tau$-slope of $E$ to be

$$\mu_{\tau}(E) := \frac{\deg(\bigoplus_{i \neq 0} E_i)}{\text{rank}(\bigoplus_{i \neq 0} E_i)} + \frac{\tau \cdot \text{rank} E_0}{\text{rank}(\bigoplus_{i \neq 0} E_i)} \in (-\infty, \infty].$$

Definition 3.1. A nonzero object $E$ of $\mathcal{A}'$ is called $\tau$-(semi)stable if $\mu_{\tau}(F)(\leq) \leq \mu_{\tau}(E)$ for any nonzero proper subobject $F$ of $E$.

Let $\tau = \text{rank}(\bigoplus_{i \neq 0} E_i)$, $v = \text{rank} E_0$, and $d = \deg(\bigoplus_{i \neq 0} E_i)$. If $E$ is strictly $\tau$-semistable, that is, $\tau$-semistable but not $\tau$-stable, then $\tau$
must be of form
\[
\tau = \frac{rd' - rd}{r' v} \quad \text{or} \quad \tau = \frac{r'd - rd'}{(r - r')v}
\]
for some \( r', d' \in \mathbb{Z} \) with \( 1 \leq r' \leq r - 1 \).

Let us consider the set \( C(A') = \{0, \dim K\} \times (\mathbb{N} \times \mathbb{Z})^{Q_0'} \) of numerical classes of twisted quiver bundles. The class of \( E\) has rank \( E_0 \) at the first entry, rank \( E_i \) at the middle one for \( i \in Q_0' \), and deg \( E_i \) at the last one for \( i \in Q_0' \).

Let \( \gamma \in C(A') \) and \( v_0(\gamma) \) be the first entry of \( \gamma \). Suppose that \( v_0(\gamma) = \dim K \). Then by \([11, \text{Proposition } 5.11]\) there is a number \( N(\gamma) \) such that there are no strictly \( \tau \)-semistable objects with numerical class \( \gamma \) if \( \tau \geq N(\gamma) \). Let \( C(\gamma) \) be the set of all possible positive values \( \tau \leq N(\gamma) \) in (3.1) so that for \( \tau \notin C(\gamma) \) there are no strictly \( \tau \)-semistable objects with \( \gamma \)-class. Note that \( C(\gamma) \) has no accumulation points in \( \mathbb{R} \). Hence, \( C(\gamma) \) is a finite set. We call an element of \( C(\gamma) \) a critical value.

3.2. Chern-Simons functionals. Let \( \mathcal{M} \) be the moduli stack parameterizing all objects \( E \) of \( \mathcal{A}' \). In order to apply the Joyce-Song theory, we need a local description of \( \mathcal{M} \) as a critical locus of a holomorphic function on a complex domain (see \([10, \text{Theorems } 5.2 \text{ and } 5.3]\) which makes use of Miyajima’s results in \([13]\)). The theorems below are straightforward generalizations of \([6, \text{Theorems } 7.1 \text{ and } 7.2]\).

Let \( \mathcal{M}^{si} \) be the coarse moduli space of simple objects in \( \mathcal{A}'_{\leq 1} \).

**Theorem 3.2.** For every \([E] \in \mathcal{M}^{si}(\mathbb{C})\), the analytic germ of \( \mathcal{M}^{si}(\mathbb{C}) \) at \([E]\) is isomorphic to \((\text{Crit}(f), u)\) for some holomorphic function \( f : U \to \mathbb{C} \) on a finite dimensional complex manifold \( U \), where \( u \) is a point of \( U \).

Let \( S \) be an \( \text{Aut}(E) \)-invariant subscheme of \( \text{Ext}^1_{A'}(E, E) \) parameterizing a versal family of objects in \( \mathcal{M}(\mathbb{C}) \) near \( E \).

**Theorem 3.3.** For every \( E \in \mathcal{M}(\mathbb{C}) \) and a maximal compact subgroup \( G \) of \( \text{Aut}(E) \), the analytic germ of \((S, 0)\) is \( G^\mathbb{C} \)-equivariantly isomorphic to \((\text{Crit}(f), 0)\) for some \( G^\mathbb{C} \)-invariant holomorphic function \( f : (\text{Ext}^1_{A'}(E, E), 0) \to (\mathbb{C}, 0) \), where \( G^\mathbb{C} \) is the complexification of \( G \) in \( \text{Aut}(E) \).

The proofs of \([6, \text{Theorems } 7.1 \text{ and } 7.2]\) work for the general case after the replacement of the Chern-Simons functional \([9, (7.7)]\) according to the double quiver \( \mathcal{Q}' \). In what follows, we describe the Chern-Simons functional for the general case.

Let \( E = (E_i, \phi_a)_{i \in Q_0, a \in Q_1} \) be a framed twisted quiver bundle on \( X \) and let \( \hat{X} \) denote the complex manifold associated to \( X \). Then there is
the gauge-theoretical interpretation \((\hat{E}_i, \bar{\partial}_{E_i}, \phi_0^a)_{i \in Q'_0, a \in \mathcal{Q}_1}\) of \(E\), i.e., \(\hat{E}_i\) is \(E_i\) regarded as a \(C^\infty\) complex vector bundle on \(\hat{X}\),

\[
\bar{\partial}_{E_i} : C^\infty(\hat{E}_i) \rightarrow C^\infty(\hat{E}_i \otimes T^{0,1}_{\hat{X}})
\]

is the unique semiconnection on \(\hat{E}_i\) such that local holomorphic sections of \(E_i\) are translated into horizontal sections of \(\bar{\partial}_{E_i}\), and \(\phi_0^a \in C^\infty(\hat{M}_i^\vee \otimes \hat{E}_i^\vee \otimes \hat{E}_ia)\) corresponds to \(\phi_a\). Here the \((0,1)\)-part of a usual connection is called a semiconnection (see \([10, \text{Definition 9.1}])\). Note that the flatness of \(\bar{\partial}_{E_i}\) automatically holds since \(X\) is a curve.

Now, the Chern-Simons functional \(CS\) ‘near \(E\)’ is defined by

\[
CS(A_i, \varphi_a) = \int_X \text{Tr}(\sum_{a \in Q_1} \varphi_a \bar{\partial}_{a} \varphi_a + \sum_{a \in Q_1, ha \in Q'_0} (-1)^{|a|} A_{ha} \bar{\varphi}_a \bar{\varphi}_a)
\]

for

\[
(A_i, \varphi_a) \in \prod_{i \in Q'_0, a \in \mathcal{Q}_1} C^\infty(\text{End}(\hat{E}_i) \otimes T^{0,1}_{\hat{X}}) \times C^\infty(\hat{M}_i^\vee \otimes \hat{E}_i^\vee \otimes \hat{E}_ia).
\]

Here \(\bar{\partial}_a\) is the semiconnection on \(\hat{E}_i^\vee \otimes \hat{E}_ha\), \(\bar{\varphi}_a = \phi_0^a + \varphi_a\), and the products in the integrand are naturally given by compositions and cup products so that after all they are considered as elements in \(\text{End}(\hat{E}_i) \otimes T^{1,1}_{\hat{X}}\). The Chern-Simons functional is gauge-invariant and its critical equations are

\[
\begin{align*}
(3.2) \quad & \bar{\partial}_a \varphi_a - \bar{\varphi}_a A_{ta} + A_{ha} \bar{\varphi}_i = 0, \forall a \in \mathcal{Q}_1; \\
(3.3) \quad & \sum_{ha = i} (-1)^{|a|} \bar{\varphi}_a \bar{\varphi}_i = 0, \forall i \in Q'_0.
\end{align*}
\]

We note that \((3.2)\) is the holomorphic condition on \(\bar{\varphi}_a\) with respect to the new semiconnection \((\bar{\partial}_{E_i} + A_{ia})^\vee \otimes (\bar{\partial}_{E_i} + A_{ha})\) and \((3.3)\) is the moment map relation on \(\bar{\varphi}_a\).

### 3.3. Ringel-Hall type algebras.

Recall that \(\mathcal{M}\) denotes the moduli stack parameterizing all objects \(E\) of \(\mathcal{A}'\). We call a pair \((\mathcal{X}, \rho)\) a \(\mathcal{M}\)-valued stack function if \(\mathcal{X}\) is an Artin stack over \(\mathbb{C}\) and \(\rho : \mathcal{X} \rightarrow \mathcal{M}\) is a representable 1-morphism. Let \(SF(\mathcal{M})\) be the ‘Grothendieck group’ of \(\mathcal{M}\)-valued stack functions, i.e., the quotient group of the free abelian group generated by stack functions, whose quotient is given by the subgroup spanned by all elements of form

\[
(\mathcal{X}, \rho) - (\mathcal{Y}, \rho|_\mathcal{Y}) + (\mathcal{X} \setminus \mathcal{Y}, \rho|_{\mathcal{X} \setminus \mathcal{Y}})
\]

for a closed substack \(\mathcal{Y}\) of \(\mathcal{X}\).
There is a multiplication structure on $\text{SF}(\mathcal{M})$ for which the multiplication

$$(\mathcal{X}_1, \rho_1) \ast (\mathcal{X}_2, \rho_2)$$

is defined to be the fiber product $(\mathcal{X}, \rho)$ in diagram

$$
\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \text{Exact}(\mathcal{M}) \quad \longrightarrow \quad \mathcal{M} \\
\downarrow \quad & & \downarrow \pi_2 \times \pi_3 \\
\mathcal{X}_1 \times_{\mathcal{C}} \mathcal{X}_2 & \longrightarrow & (\mathcal{M}, \mathcal{M})
\end{array}
$$

where:

- $\text{Exact}(\mathcal{M})$ is an Artin stack parameterizing short exact sequences in $\mathcal{M}$ and $\pi_i$ is the obvious $i$-th projection;
- the square is the fiber product and $\rho$ is the composition of the upper arrows.

The multiplication is associative by [8, Theorem 5.2]. The induced algebra $\text{SF}(\mathcal{M})$ is called the Ringel-Hall type algebra.

Denote by $\mathcal{M}_{\geq 2}$ the moduli stack parameterizing all objects $E$ of $\mathcal{A}'$ with rank $E_0 \geq 2 \dim K$. Let $\text{SF}(\mathcal{M}_{\leq 1})$ be the quotient algebra of $\text{SF}(\mathcal{M})$ factored by the ideal generated by all $\rho : \mathcal{X} \to \mathcal{M}$ which factor though $\mathcal{M}_{\geq 2}$. Finally we consider the subalgebra $\text{SF}(\mathcal{M}_{\leq 1})$ generated by all $\rho : \mathcal{X} \to \mathcal{M}$ which factor though the moduli stack of locally free objects.

By Proposition 2.7 we may define an antisymmetric bilinear form $\chi : (\mathbb{N} \times (\mathbb{N} \times \mathbb{Z})_{Q_0})^2 \to \mathbb{Z}$. Let $L(\mathcal{A}')$ be the $\mathbb{Q}$-vector space with basis $e_\gamma, \gamma \in \{0, \dim K\} \times (\mathbb{N} \times \mathbb{Z})_{Q_0}$, equipped with a Lie algebra structure given by

$$
[e_\gamma, e_\tilde{\gamma}] := \begin{cases} 
(-1)^{\chi(\gamma, \tilde{\gamma})} \chi(\gamma, \tilde{\gamma}) e_{\gamma + \tilde{\gamma}} & \text{if } \gamma_0 + \tilde{\gamma}_0 \leq \dim K, \\
0 & \text{otherwise}.
\end{cases}
$$

In the below, we let $B\mathbb{C}^*$ denote the classifying stack of the multiplicative group $\mathbb{C}^*$. By the local descriptions, Theorems 3.2 and 3.3 of the moduli spaces we will have this.

**Theorem 3.4.** There is a Lie algebra homomorphism

$$
\Psi : \text{SF}_{\text{alg}}^{\text{ind}}(\mathcal{M}_{\leq 1}) \to L(\mathcal{A}')
$$

satisfying

$$
\Psi([Z \times B\mathbb{C}^*, \rho]) = -\chi(Z, \rho^* \nu_{\mathcal{M}_{\leq 1}}) e_\gamma,
$$

where:
Proof. See [1, section 2.2 and Theorem 3.4]. \hfill \square

3.4. Harder-Narasimhan filtrations. In this section, using Harder-
Narasimhan filtrations we express the stack function representing
in terms of those representing \[ \mathfrak{M}^{\mathfrak{s}}_{\tau_1,} \] and unframed moduli spaces. By
a purely algebraic Lemma in [1] this expression induces a wall-crossing
formula. From now on, we let \( \tau_0 \in C(r, d), \tau_+ > \tau_0 \) and \( \tau_- < \tau_0 \) such
that there are no critical values between intervals \( (\tau_0, \tau_) \) and \( (\tau_-, \tau_0) \).

For \( E \in \mathcal{A}' \) and \( \tau \in \mathbb{R}_{>0} \), it is easy to see that there is a unique
filtration

\[ 0 = E_0 \subset E_1 \subset E_2 \ldots \subset E_n = E \]

such that \( E_k/E_{k-1} \) is \( \tau_- \)-semistable and \( \mu_{\tau}(E_{k-1}/E_{k-2}) > \mu_{\tau}(E_k/E_{k-1}) \)
for \( k = 1, \ldots , n \). This so-called Harder-Narasimhan filtration will lead us to the following.

Lemma 3.5. Let \( E \in \mathcal{A} \) with \( \langle E \rangle_0 = K \otimes \mathcal{O}_X \). TFAE.

(1) \( E \) be \( \tau_0 \)-semistable

(2) \( E \) is \( \tau_+ \)-semistable or there is a unique subobject \( E' \) of \( E \) satisfying:
\( E', E/E' \) are \( \tau_+ \)-semistable, \( \langle E' \rangle_0 = K \otimes \mathcal{O}_X, \mu_{\tau_+}(E') > \mu_{\tau_+}(E/E'), \) and \( \mu_{\tau_0}(E') = \mu_{\tau_0}(E/E'). \)

(3) \( E \) is \( \tau_- \)-semistable or there is a unique subobject \( E' \) of \( E \) satisfying:
\( E', E/E' \) are \( \tau_- \)-semistable, \( \langle E/E' \rangle_0 = K \otimes \mathcal{O}_X, \mu_{\tau_-}(E') > \mu_{\tau_-}(E/E'), \) and \( \mu_{\tau_0}(E') = \mu_{\tau_0}(E/E'). \)

Proof. Note that \( E \) is \( \tau_0 \)-semistable.

(1) \( \Rightarrow \) (2). Let \( 0 = E_0 \subset E_1 \subset \ldots \subset E_n = E \) be the \( \tau_+ \) Harder-
Narasimhan filtration of \( E \). We take \( E' := E_1 \). If \( n = 2, \langle E_1 \rangle_0 \neq 0 \)
for otherwise, \( \mu_{\tau_0}(E_1) = \mu_{\tau_+}(E_1) > \mu_{\tau_+}(E_2/E_1) \geq \mu_{\tau_0}(E_2/E_1) \) which is
a contradiction to the \( \tau_0 \)-semistability of \( E \). We prove that \( n \) cannot
be larger than 2. Suppose that \( n \geq 3 \), then there are \( i, j \) such that
\( 1 \leq i < j \leq n \) and

\[ \mu_{\tau_+}(E_i/E_{i-1}) = \mu_{\tau_0}(E_i/E_{i-1}) > \mu_{\tau_+}(E_j/E_{j-1}) = \mu_{\tau_0}(E_j/E_{j-1}). \]

This induces a contradiction as follows.
a) When $(E_1)_0 = 0$, then $\mu_{\tau_0}(E_1) = \mu_{\tau_+}(E_1) > \mu_{\tau_+}(E) \geq \mu_{\tau_0}(E)$. This is a contradiction to the $\tau_0$-semistability of $E$.

b) When $(E_1)_0 = K \otimes O_X$, then $(E_i)_0 = K \otimes O_X$ for all $i \geq 1$ so that the inequality $\mu_{\tau_+}(E_i) > \mu_{\tau_+}(E)$ implies that $\mu_{\tau_0}(E_i) = \mu_{\tau_0}(E)$ for all $i$. This contradicts to (3.4).

Now, by the uniqueness of Harder-Narasimhan filtrations, the proof of (1) $\Rightarrow$ (2) follows.

(2) $\Leftarrow$ (1). Let $F$ be a nontrivial subobject of $E$ and let $F' := \ker(F \to E/E')$. Then $\mu_{\tau_+}(F') \leq \mu_{\tau_+}(E')$ (because $E'$ is $\tau_+$-semistable) and $\mu_{\tau_+}(F/F') \leq \mu_{\tau_+}(E/E')$ (because $E/E'$ is $\tau_+$-semistable). Now take the limit $\tau_+ \to \tau_0$ to the both inequalities in order to conclude that $\mu_{\tau_0}(F) \leq \mu_{\tau_0}(E)$ since $\mu_{\tau_0}(E') = \mu_{\tau_0}(E/E')$.

(1) $\iff$ (3). This follows by an argument similar to the proof of (1) $\iff$ (2). \qed

Let $\delta_\tau(\gamma)$ denote the stack function $[\mathcal{M}_\tau^{ss}(\gamma), \rho] \in \text{SF}(\mathcal{M}_{\leq 1})$ for the natural open embedding $\rho$ of the moduli stack $\mathcal{M}_\tau^{ss}(\gamma) \subset \mathcal{M}$ of $\tau$-semistable objects of $\mathcal{A}'$ with the numerical class $\gamma \in C(\mathcal{A}')$. We use notation $\delta(\gamma)$ for $\delta_\tau(\gamma)$ if $v_0(\gamma) = 0$. Then the previous lemma will induce relationships between $\delta_{\tau_+}(\gamma)$, $\delta_{\tau_0}(\gamma)$, and $\delta(\gamma)$ in the Ringel-Hall type algebra as in Lemma 3.6 below. Before describing the lemma, we will need the following index sets. For $l \geq 1$, let

$$HN_+(\gamma, \tau_0, l) = \{ (\gamma_1, \ldots, \gamma_l) \mid \gamma_i \in C(\mathcal{A}) \},$$

$$\sum_i \gamma_i = \gamma, v_0(\gamma_1) = v_0(\gamma), \mu_{\tau_0}(\gamma_i) = \mu_{\tau_0}(\gamma) \ \forall i\}$$

and

$$HN_-(\gamma, \tau_0, l) = \{ (\gamma_1, \ldots, \gamma_l) \mid \gamma_i \in C(\mathcal{A}) \},$$

$$\sum_i \gamma_i = \gamma, v_0(\gamma_1) = v_0(\gamma), \mu_{\tau_0}(\gamma_i) = \mu_{\tau_0}(\gamma) \ \forall i\}.$$

**Lemma 3.6.** In $\text{SF}(\mathcal{M}_{\leq 1})$, the followings hold.

(1)

$$\delta_{\tau_0}(\gamma) = \delta_{\tau_+}(\gamma) + \sum_{(\gamma_1, \gamma_2) \in HN_+(\gamma, \tau_0, 2)} \delta_{\tau_+}(\gamma_1) * \delta(\gamma_2).$$

$$\delta_{\tau_0}(\gamma) = \delta_{\tau_-}(\gamma) + \sum_{(\gamma_1, \gamma_2) \in HN_-(\gamma, \tau_0, 2)} \delta(\gamma_1) * \delta_{\tau_-}(\gamma_2).$$
\[ \delta_{\tau_+}(\gamma) = \sum_{l \geq 1} (-1)^{l-1} \sum_{H N_+(\gamma, \tau_0, l)} \delta_{\tau_0}(\gamma_1) \ast \delta(\gamma_2) \ast \ldots \ast \delta(\gamma_l). \]

\[ \delta_{\tau_-}(\gamma) = \sum_{l \geq 1} (-1)^{l-1} \sum_{H N_-(\gamma, \tau_0, l)} \delta(\gamma_1) \ast \delta(\gamma_2) \ast \ldots \ast \delta_{\tau_0}(\gamma_l). \]

\[ \delta_{\tau_-}(\gamma) = \delta_{\tau_+}(\gamma) + \sum_{l \geq 2} (-1)^{l-1} \sum_{H N_-(\gamma, \tau_0, l)} \delta(\gamma_1) \ast \ldots \ast \delta(\gamma_{l-2}) \ast [\delta(\gamma_{l-1}), \delta_{\tau_+}(\gamma_l)]. \]

**Proof.** There are only finite nontrivial terms in each summation of (1), (2), and (3). For example, when \( l = 2 \), let us consider an exact sequence

\[ 0 \rightarrow E^1 \rightarrow E^2 \rightarrow E^2/E^1 \rightarrow 0 \]

whose factors \( E^1, E^2/E^1 \) are \( \tau_0 \)-semistable with \( \mu_{\tau_0}(E^1) = \mu_{\tau_0}(E^2/E^1) \). This implies that \( E^2 \) is \( \tau_0 \)-semistable. Therefore, \( \deg(E^1)_j \) is bounded above by a number \( N \) (depending only on class \( \gamma \)) according to Lemma 5.8 in [11].

The statement (1) follows from Lemma 3.5.

For (2), we rewrite the first equation of (1) as

\[ (3.5) \quad \delta_{\tau_+}(\gamma) = \delta_{\tau_0}(\gamma) - \sum_{(\gamma_1, \gamma_2) \in H N_+(\gamma, \tau_0, 2)} \delta_{\tau_+}(\gamma_1) \ast \delta(\gamma_2) \]

and apply (3.5) to \( \delta_{\tau_-}(\gamma_1) \). This iterated procedure must stop since there are only finite nontrivial terms in the summation. This proves (2).

To prove (3), we start with the second equation of (2) and replace \( \delta_{\tau_0}(\gamma_l) \) by the first equation of (1). \( \square \)

### 3.5. Log stack functions.

Following [9, Definition 8.1], we define the log stack function for \( \gamma \in C(\mathcal{A}') \) as

\[ \epsilon_{\tau}(\gamma) := \sum_{l \geq 1} \frac{(-1)^{l-1}}{l} \sum_{\gamma = \gamma_1 \ast \mu_{\tau_0}(\gamma_1) = \mu_{\tau}(\gamma)} \delta_{\tau}(\gamma_1) \ast \ldots \ast \delta_{\tau}(\gamma_l). \]

As in the proof of Lemma 3.6, there are only finite nontrivial terms in the sum expression of \( \epsilon_{\tau}(\gamma) \). According to [10, Theorem 3.11], the log stack function \( \epsilon_{\tau}(\gamma) \) is an element in \( \text{SF}_{\text{alg}}(\mathcal{M}_{\leq 1}) \). In the below, if \( v_0(\gamma) = 0 \), we let \( \epsilon(\gamma) \) denote \( \epsilon_{\tau}(\gamma) \).

**Lemma 3.7.**

\[ \epsilon_{\tau_-}(\gamma) - \epsilon_{\tau_+}(\gamma) = \sum_{l \geq 2} \frac{(-1)^{l-1}}{(l-1)!} \sum_{H N_-(\gamma, \tau_0, l)} [\epsilon(\gamma_1), \ldots, [\epsilon(\gamma_{l-1}), \epsilon_{\tau_+}(\gamma_l)]]. \]
Proof. Use the abstract form of [1, Lemma 2.6]. See also [2, Lemma 3.4]. □

Theorem 3.4 implies that for fixed $\gamma$, the $\tau$-invariant $J_\tau(\gamma) \in \mathbb{Q}$ can be defined as

$$\Psi(\epsilon_\tau(\gamma)) = -J_\tau(\gamma)e_\gamma.$$ 

We call $J_\tau(\gamma)$ the generalized DT-invariant for $(Q, X, \gamma, \tau)$. When the rank $v_0(\gamma)$ of the frame is 0, then the invariant will be denoted simply by $J(\gamma)$ since it does not depend on $\tau$. Now by combining Lemma 3.7 and Theorem 3.4, we conclude the following wall-crossing formula.

**Theorem 3.8.**

$$\left( J_{\tau_-}(\gamma) - J_{\tau_+}(\gamma) \right)e_\gamma = \sum_{l \geq 2} \frac{1}{(l-1)!} \sum_{HN_{-}(\gamma, \tau, 0, l)} [ J(\gamma_1)e_{\gamma_1}, \ldots, J(\gamma_l)e_{\gamma_l} ] \cdots ].$$

3.6. The action by the Jacobian variety. Suppose that $v_0(\gamma) = 0$ and the genus of $X$ is $g \geq 1$. In this case, by the similar argument for the proof of [10, Proposition 6.19], the generalized DT invariant $J(\gamma)$ vanishes as follows. Let $J(X)$ be the Jacobian variety of $X$ and let $L$ be the universal line bundle on $X \times J(X)$. Then the torus group $J(X)$ acts on $\mathcal{M}^{ss}_\tau(\gamma)$ by $t \cdot E := E \otimes L_t$ for $t \in J(X)$. Note that this action yields a torus fibration on $\mathcal{M}^{ss}_\tau(\gamma)$ and the Behrend function on $\mathcal{M}^{ss}_\tau(\gamma)$ is constant on each $J(X)$-orbit. Hence we conclude that $J(\gamma) = 0$ using the expression of $J(\gamma)$ by the weighted Euler characteristics (see [10, Section 5.3]).

Acknowledgments. We would like to express our deep gratitude to Kukak Chung for useful discussions and to Emanuel Diaconescu and Jae-Hyouk Lee for their invaluable comments. This work is financially supported by KRF-2007-341-C00006.

References

[1] W-E. Chuang, D.E. Diaconescu, and G. Pan, Chamber structure and wallcrossing in the ADHM theory of curves II, arXiv:0908.1119.
[2] W-E. Chuang, D.E. Diaconescu, and G. Pan, Rank two ADHM invariants and wallcrossing, arXiv:1002.0579.
[3] I. Ciocan-Fontanine and B. Kim, Moduli stacks of stable toric quasimaps, Advances in Mathematics 225 (2010), no. 6, 3022–3051.
[4] I. Ciocan-Fontanine, B. Kim, and D. Maulik, Stable quasimap to GIT quotients, in preparation.
[5] W. Crawley-Boevey and M. Holland, Noncommutative deformations of Kleinian singularities, Duke Math. J. 92 (1998), 605–635.
[6] D.E. Diaconescu, Chamber structure and wallcrossing in the ADHM theory of curves I, arXive:0904.4451.
[7] P.B. Gothen and A.D. King, *Homological algebra of twisted quiver bundles*, J. London Math. Soc. (2) 71 (2005), no. 1, 85–99.

[8] D. Joyce, *Configurations in abelian categories. II. Ringel-Hall algebras*, Advances in Mathematics 210 (2007), no. 2, 635–706.

[9] D. Joyce, *Configurations in abelian categories. III. Stability conditions and identities*, Advances in Mathematics 215 (2007), no. 1, 153–219.

[10] D. Joyce and Y. Song, *A theory of generalized Donaldson-Thomas invariants*, arXiv:0810.5645v4.

[11] B. Kim, *Stable quasimaps to holomorphic symplectic quotients*, arXiv:1005.4125.

[12] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435.

[13] K. Miyajima, *Kuranishi family of vector bundles and algebraic description of the moduli space of Einstein-Hermitian connections*, Publ. Res. Inst. Math. Sci. 25 (1989), no. 2, 301–320.

[14] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994.

School of Mathematics, Korea Institute for Advanced Study, 87 Hoegiro, Dongdaemun-gu, Seoul, 130-722, Korea

E-mail address: bumsig@kias.re.kr

School of Mathematics, Korea Institute for Advanced Study, 87 Hoegiro, Dongdaemun-gu, Seoul, 130-722, Korea

E-mail address: hlee014@kias.re.kr