\Omega-RESULTS FOR THE HYPERBOLIC LATTICE POINT PROBLEM

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Abstract. For $\Gamma$ a cocompact or cofinite Fuchsian group, we study the lattice point problem on the Riemann surface $\Gamma \backslash \mathbb{H}$. The main asymptotic for the counting of the orbit $\Gamma z$ inside a circle of radius $r$ centered at $z$ grows like $ce^r$. Phillips and Rudnick studied $\Omega$-results for the error term and mean results in $r$ for the normalized error term. We investigate the normalized error term in the natural parameter $X = 2 \cosh r$ and prove $\Omega_\pm$-results for the orbit $\Gamma w$ and circle centered at $z$, even for $z \neq w$.

1. Introduction

Let $\mathbb{H}$ be the hyperbolic plane and $z, w$ two fixed points in $\mathbb{H}$. We denote by $\rho(z, w)$ their hyperbolic distance. For $\Gamma$ a cocompact or cofinite Fuchsian group, we are interested in the problem of estimating, as $r \to \infty$, the quantity

$$N_r(z, w) = \# \{ \gamma \in \Gamma : \rho(z, \gamma w) \leq r \}.$$

The study of the asymptotic behaviour of $N_r(z, w)$ is traditionally called the hyperbolic lattice point problem. This problem has been studied by many authors, see [4, 6, 7, 12, 15, 19]. For notational reasons let $u(z, w)$ denote the point pair-invariant function

$$u(z, w) = \frac{|z - w|^2}{4 \Im(z) \Im(w)}.$$

Then $\cosh \rho(z, w) = 2u(z, w) + 1$ and, after the change of variable $X = 2 \cosh r$, the problem is equivalent to studying the quantity

$$N(X; z, w) = \# \{ \gamma \in \Gamma : 4u(z, \gamma w) + 2 \leq X \},$$

as $X \to \infty$.

Let $\Delta$ be the Laplacian of the hyperbolic surface $\Gamma \backslash \mathbb{H}$ and let $\{ \lambda_j \}_{j=0}^{\infty}$ be the discrete spectrum of $-\Delta$. Write $\lambda_j = s_j(1 - s_j) = 1/4 + t_j^2$, and let $u_j$ be the $L^2$-normalized eigenfunction (Maass form) with eigenvalue $\lambda_j$. We have the following theorem.

**Theorem 1.1** (Selberg [19], Günther [7], Good [6]). Let $\Gamma$ be a cocompact or cofinite Fuchsian group. Then:

$$N(X; z, w) = \sum_{1/2 < s_j \leq 1} \sqrt{\pi} \frac{\Gamma(s_j - 1/2)}{\Gamma(s_j + 1)} u_j(z) u_j(w) X^{s_j} + E(X; z, w),$$

with

$$E(X; z, w) = O(X^{2/3}).$$

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We are interested on the growth of the error term $E(X; z, w)$. Conjecturally
\begin{equation}
E(X; z, w) = O_\epsilon(X^{1/2+\epsilon})
\end{equation}
for every $\epsilon > 0$.

Let $h(t) = h_X(t)$ be the Selberg/Harish-Chandra transform of the characteristic function $\chi_{[0,(X-2)/4]}$. Then, the error term has a 'spectral expansion', which for compact $\Gamma \backslash \mathbb{H}$ takes the form
\begin{equation}
E(X; z, w) = \sum_{t_j \in \mathbb{R}} h(t_j)u_j(z)\overline{u_j(w)} + O \left( X^{-1} + \sum_{1/2 < s_j \leq 1} \frac{X^{1-s_j}}{2\pi - 1} \right),
\end{equation}
(see section 3). The behavior of the terms coming from the eigenvalue $\lambda_j = 1/4$ (corresponding to $t_j = 0$) is well understood (see [18, p. 86, Lemma 2.2]). It contributes
\[h(0) \sum_{t_j=0} u_j(z)\overline{u_j(w)} = O(X^{1/2} \log X).\]

We subtract this quantity from $E(X; z, w)$ and we define the error term $e(X; z, w)$ to be the difference
\begin{equation}
e(X; z, w) = E(X; z, w) - h(0) \sum_{t_j=0} u_j(z)\overline{u_j(w)}.
\end{equation}

Thus, conjecture (1.1) can be restated as
\begin{equation}
e(X; z, w) = O_\epsilon(X^{1/2+\epsilon})
\end{equation}
for every $\epsilon > 0$. The following result of Patterson supports this conjecture.

**Theorem 1.2** (Patterson, [15], [16]). Let $\Gamma$ be a cocompact or cofinite Fuchsian group. Then the error term $e(X; z, w)$ satisfies the average bound
\[
\frac{1}{X} \int_{2}^{X} e(x; z, w) \, dx = O(X^{1/2}).
\]

Further, Phillips and Rudnick proved mean value results for the error term.

**Theorem 1.3** (Phillips-Rudnick, [18]). a) If $\Gamma$ is cocompact, then
\begin{equation}
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \frac{e(2 \cosh r; z, z)}{e^{r/2}} \, dr = 0.
\end{equation}
b) If $\Gamma$ is cofinite, then
\begin{equation}
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \frac{e(2 \cosh r; z, z)}{e^{r/2}} \, dr = \sum_{a} |E_a(z, 1/2)|^2,
\end{equation}
where $E_a(z, s)$ is the Eisenstein series corresponding to the cusp $a$.

In case b), the limit of the Phillips-Rudnick normalized average error term is positive only when $E_a(z, 1/2) \neq 0$ for at least one cusp $a$. Such an Eisenstein series is called a null-vector for $\Gamma$.

In this paper, by $\Omega$-results we mean lower bounds for the $\limsup |e(X; z, w)|$. That means, if $g(X)$ is a positive function, we write $e(X; z, w) = \Omega(g(X))$ if and only if $e(X; z, w) \neq o(g(X))$, i.e.
\[
\limsup \frac{|e(X; z, w)|}{g(X)} > 0.
\]

Phillips and Rudnick proved also the following $\Omega$-results for the error term $e(X; z, z)$.
Theorem 1.4 (Phillips-Rudnick, [18]). a) If \( \Gamma \) is cocompact or a subgroup of finite index in \( \text{PSL}_2(\mathbb{Z}) \), then for all \( \delta > 0 \),
\[
e(X; z, z) = \Omega \left( X^{1/2} (\log \log X)^{1/4 - \delta} \right).
\]
b) If \( \Gamma \) is cofinite but not cocompact, and either at least one eigenvalue \( \lambda_j > 1/4 \) or a null-vector, then,
\[
e(X; z, z) = \Omega \left( X^{1/2} \right).
\]
c) In any other cofinite case, for all \( \delta > 0 \),
\[
e(X; z, z) = \Omega \left( X^{1/2 - \delta} \right).
\]

We distinguish the two cases of \( \Omega \)-results: we write \( e(X; z, w) = \Omega^+ (g(X)) \) if
\[
\limsup \frac{e(X; z, w)}{g(X)} > 0,
\]
and \( e(X; z, w) = \Omega^- (g(X)) \) if
\[
\liminf \frac{e(X; z, w)}{g(X)} < 0.
\]

Instead of the normalization of Theorem 1.3, we are interested in studying the more natural normalization
\[
M(X; z, w) = \frac{1}{X} \int_{X}^{\infty} e(x; z, w) \frac{dx}{x^{1/2}}
\]
as \( X \to \infty \). Theorem 1.2 and integration by parts imply that \( M(X; z, w) = O(1) \).

For Theorem 1.4, after choosing \( z = w \), Phillips and Rudnick work with the average of the function
\[
\sum_{0 \neq j \in \mathbb{R}} \frac{|u_j(z)|^2}{t_j^{3/2}} e^{it_j \log X},
\]
which is an almost periodic function in the variable \( s = \log X \). In contrast with Theorem 1.3, we deal with \( M(X; z, w) \) for \( z \neq w \). We prove that, under specific conditions, \( M(X; z, w) \) does not have a limit.

Theorem 1.5. Let \( \Gamma \) be a cocompact or cofinite Fuchsian group and \( z \) a fixed point. Then there exists a fixed \( \delta = \delta_{\Gamma, z} > 0 \) such that for every point \( w \in B(z, \delta) \) we have:

a) if \( \Gamma \) is cocompact, then
\[
M(X; z, w) = \Omega^- (1).
\]
b) if \( \Gamma \) is cofinite and has at least one eigenvalue \( \lambda_j > 1/4 \) with \( u_j(z) \neq 0 \), then
\[
M(X; z, w) - \sum_a |E_a(z, 1/2)|^2 = \Omega^- (1).
\]

Further, for both cases, there exists a constant \( a > 1/4 \) such that if \( \lambda_1 > a \) the limit of \( M(X; z, w) \) as \( X \to \infty \) does not exist.

For the exact value of the constant \( a \) see section 2. In many cases, Theorem 1.5 implies as an immediate corollary \( \Omega \)-results pointwise for \( e(X; z, w) \) with \( w \in B(z, \delta) \).
**Corollary 1.6.** With the notation of Theorem 1.5 we have:

a) if $\Gamma$ is either i) cocompact, or ii) as in case b) of Theorem 1.5 and does not have null-vectors, then
\[ e(X; z, w) = \Omega_-(X^{1/2}) \]
for every $w \in B(z, \delta)$.

b) if $\lambda_1 > a$, then
\[ e(X; z, w) = \Omega(X^{1/2}) \]
for every $w \in B(z, \delta)$.

Corollary 1.6 does not cover all cases of cofinite Fuchsian groups. However, using a more careful analysis of $e(X; z, w)$, there are some more cases of cofinite groups for which we can deduce $\Omega$-results for $e(X; z, w)$. For this purpose, we have the following definition, which is related to Weyl’s law (see Theorem 2.4).

**Definition 1.7.** Let $\Gamma$ be a cofinite Fuchsian group. We say that $\Gamma$ has sufficiently many cusp forms at the point $z$ if
\[ \sum_{|t_j| < T} |t_j(z)|^2 \gg T^2. \]

We prove the following result.

**Theorem 1.8.** Let $\Gamma$ be a cofinite but not cocompact Fuchsian group and $z \in \mathbb{H}$ fixed. Then, there exists a fixed $\delta = \delta_{\Gamma, z} > 0$ such that for every point $w \in B(z, \delta)$ we have:

a) if $\Gamma$ has sufficiently many cusp forms at $z$, then
\[ e(X; z, w) = \Omega_-(X^{1/2}) \]

b) if $\Gamma$ has null vectors, then
\[ e(X; z, w) = \Omega_+(X^{1/2}) \]

Hence, we conclude that:

**Corollary 1.9.** If $\Gamma$ is cofinite but not cocompact, has null vectors and sufficiently many cusp forms at $z$, then
\[ e(X; z, w) = \Omega_\pm(X^{1/2}) \]

The proofs of Theorems 1.5 and 1.8 depend crucially on specific ‘fixed sign’ properties of the $\Gamma$-function. We summarize these properties in section 2 (Lemma 2.1). We prove Theorem 1.5 in section 3 and Theorem 1.8 in section 4. Our detailed analysis of both the discrete and the continuous spectrum in section 4 can be used to give a second proof of case a) of Corollary 1.6.

**Remark 1.10.** In the case of the Euclidean circle problem, Hardy was the first who proved $\Omega$-results for the error term. In fact, in [8] he proved that the error term is $\Omega_\pm(X^{1/4})$ and in [9] that it is $\Omega_-(X^{1/4}(\log X)^{1/4})$. These results have been improved several times, see [1] for a detailed discussion.

In Theorem 1.4, in case a) Phillips and Rudnick prove a $\Omega_-$-result, in case b) the sign of the $\Omega$ depends on the group, whereas in case c), the sign cannot be determined by their method. Although they do not mention anything for the sign of their $\Omega$-results, in case that the group $\Gamma$ is as in Corollary 1.9, their method implies a $\Omega_\pm$-result for $e(X; z, z)$. Our analysis allows us to prove a $\Omega_\pm$-result for $w \in B(z, \delta)$, which is one of the main reasons we choose to emphasize the sign of our $\Omega$-results.
Remark 1.11. There are specific arithmetic groups for which we know they satisfy the conditions of Corollary 1.9. If $\Gamma$ is a subgroup of $\text{SL}_2(\mathbb{Z})$ of finite index, then it follows from [11],[20] that in many cases $\Gamma$ has sufficiently many cusp forms at $z$ for $z$ remaining in a compact subset of the surface. Further, every $\Gamma(N)$ with $N = 5$ or $\geq 7$ has null vectors. For $\Gamma_0(N)$, there are also groups having null vectors, for instance $\Gamma_0(25)$. For further discussion on the null vectors of these arithmetic groups see [17, p. 80-81], [13, p. 151-153].

For $\Gamma = \text{SL}_2(\mathbb{Z})$ and certain other groups it is conjectured that the real Satake parameters $t_j$ are linearly independent over $\mathbb{Q}$. Such a conjecture would allow to apply Kronecker’s theorem [10, p. 510, theorem 444] and find a sequence of $R_m \to \infty$ such that the exponentials $\{e^{it_j R_m}n\}_{j=1}^n$ approach the point $-1$ simultaneously (see Lemma 3.1 in section 3). Using the fixed sign properties of the $\Gamma$-function from section 2, this would allow to substitute $\Omega_-$ and $\Omega_+$ in Theorems 1.5 and 1.8 with $\Omega_+$.  

Remark 1.12. In [3], Cramér studied the normalized error term of the Chebyshev’s prime counting function $\psi(x)$. He proved that

$$\frac{\psi(e^x) - e^x}{e^{x/2}}$$

has mean square average [3, p. 148, eq. (1)], whereas

$$\frac{\psi(u) - u}{u^a}$$

does not have mean square average for $a < 1$ [3, p. 148, eq. (2)]. For the hyperbolic lattice point problem Theorems 1.3 and 1.5 show that a similar phenomenon appears for the error term $e(X; z, w)$.  

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2. Lemmas

One of the key ingredients in the proofs of our results is the following lemma.

Lemma 2.1. For every $t \in \mathbb{R}$, we have:

a)

$$\Re\left(\frac{\Gamma(it)}{\Gamma(3/2 + it)}\right) < 0,$$

b)

$$\Re\left(\frac{\Gamma(it)}{\Gamma(3/2 + it)(1 + it)}\right) < 0.$$

The proof of Lemma 2.1 uses an elementary result about real functions.

Lemma 2.2. Let $f : (\infty, 0) \to \mathbb{R}$ be a continuous and strictly increasing real valued function such that $f(x)\sin(x)$ is integrable in $(\infty, 0)$. Then

$$\int_{-\infty}^{0} f(x)\sin(x)dx < 0.$$
Proof. (of Lemma 2.2) Since \( f(x) \sin(x) \) is integrable in \((-\infty, 0)\), we split the integral as
\[
\int_{-\infty}^{0} f(x) \sin(x) dx = \sum_{n=0}^{\infty} \int_{-2(n+1)\pi}^{-2n\pi} f(x) \sin(x) dx
\]
\[
= \sum_{n=0}^{\infty} \int_{-2n\pi-\pi}^{-2n\pi} (f(x) - f(x - \pi)) \sin(x) dx.
\]
Since \( f \) is strictly increasing and \( \sin(x) \) is negative in the interval \((-2n\pi - \pi, -2n\pi)\), the statement follows.

Proof. (of Lemma 2.1) a) Since \( \Gamma(z) = \Gamma(z) \), it suffices to prove the Lemma for \( t > 0 \). Using [5, p. 909, eq. (8.384.1)] we get
\[
\frac{\Gamma(it)}{\Gamma(3/2 + it)} = \frac{2}{\sqrt{\pi}} B(it, 3/2),
\]
where \( B(x, y) \) is the Beta function. By the definition of the Beta function [5, p. 908, eq. (8.380.1)] and the formula
\[
B(x + 1, y) = B(x, y) \frac{x}{x+y}
\]
we see that inequality (2.1) is equivalent with
\[
\frac{2t}{3} \int_{0}^{1} \cos(t \log s)(1 - s)^{1/2} ds + \int_{0}^{1} \sin(t \log s)(1 - s)^{1/2} ds < 0.
\]
Using integration by parts, setting \( s = e^{x/t} \) and applying Lemma 2.2 for \( f(x) = \frac{1}{t}(1 - e^{x/t})^{1/2} e^{x/t} - \frac{2}{3}((1 - e^{x/t})^{1/2} e^{x/t})' \), part a) follows.
b) It follows in the same way. Using the formula
\[
\frac{\Gamma(it)}{\Gamma(3/2 + it)(1 + it)} = -\frac{i + t}{\sqrt{\pi t} (1 + i t^2)} B(1 + it, 1/2),
\]
it follows that (2.2) is equivalent with
\[
-t \int_{0}^{1} \cos(t \log s)(1 - s)^{-1/2} ds + \int_{0}^{1} \sin(t \log s)(1 - s)^{-1/2} ds < 0.
\]

Remark 2.3. There exists a positive constant \( c > 0 \) such that for \(|t| > c\) we have
(2.3)
\[
\Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)(1 + it)} \right) < 0.
\]
This can be deduced easily from Stirling’s formula. Using Lemma 2.1 and working as in a) we can get the desired result for \( c \approx 2.30277... \), whereas in fact \( c \) can be numerically found to be around \( \approx 1.59135... \) using Mathematica. For this \( c \), we can choose \( a \) of Theorem 1.5 by \( a = 1/4 + c^2 \approx 2.7823... \) (see section 3). In particular, \( \text{SL}_2(\mathbb{Z}) \) satisfies the condition \( \lambda_1 > a \).

We will also use the following local Weyl’s law for \( L^2(\Gamma \backslash \mathbb{H}) \) (see [18, p. 86, Lemma 2.3]).

Theorem 2.4 (Local Weyl’s law). For every \( z \), as \( T \to \infty \),
\[
\sum_{|j| < T} |u_j(z)|^2 + \sum_a \frac{1}{4\pi} \int_{-T}^{T} |E_a(z, 1/2 + it)|^2 dt \sim cT^2,
\]
where \( c = c(z) \) depends only on the number of elements of \( \Gamma \) fixing \( z \).
When $z$ remains in a bounded region of $\mathbb{H}$ (more specifically in a compact set), the constant $c(z)$ is uniformly bounded, depending only on $\Gamma$.

**Remark 2.5.** Phillips and Rudnick in [18] generalize Theorem 1.3 and case $a)$ of Theorem 1.4 in the case of the $n$-dimensional hyperbolic space $\mathbb{H}^n$ [18, p. 106]. Considering the $n$-th dimensional analogues of Theorems 1.5, 1.8 we notice that in order to make our method work for the $\Gamma$-function. For instance, for Theorem 1.5 we would need the property that

$$\Re \left( \frac{\Gamma(it)}{\Gamma(\frac{n+1}{2} + it)} (1 + it) \right)$$

fixes sign for all $t \in \mathbb{R}$. However, this property fails for $n \geq 4$.

3. **$\Omega$-results for the averaged error term $M(X; z, w)$**

3.1. **Proof of Theorem 1.5 for $\Gamma$ cocompact.** The quantity $N(X; z, w)$ can be interpreted as

$$N(X; z, w) = K(z, w) = \sum_{\gamma \in \Gamma} k(u(z, \gamma w)),$$

for $k(u) = \chi_{[0, (X-2)/4]}(u)$. Because $k$ is not smooth, we cannot apply the pre-trace formula to the kernel $K(z, w)$. Instead of that, we work with $M(X; z, w)$. Using [2, p. 321, eq. (2.7)], the Selberg Harish-Chandra transform $h(t) = h_X(t)$ of $k(u)$ can be expressed as

$$h(t) = 2\pi \sinh r P_{-1/2 + it} (\cosh r),$$

where $r = \cosh^{-1}(X/2)$ and $P_\nu^{\mu}(z)$ is the associated Legendre function. Using the formula [5, p. 971, eq. (8.776.1)], for $t \in \mathbb{R}$ we get

$$h(t) = 2\sqrt{\pi} \Re \left( \frac{\Gamma(it)}{\Gamma(\frac{n}{2} + it)} X^it \right) \left( X^{1/2} + O(X^{-3/2}) \right).$$

We first deal with $M(X; z, w)$ for the cocompact case.

**Proof.** For $z$ fixed, consider a sequence of points $\{w_n\}_{n=1}^\infty$ such that $w_n \to z$. Then, for every $j$ we get

$$u_j(w_n) \to u_j(z),$$

as $n \to \infty$ (where we do not know uniformity in the limit). For $X = e^R$ we define

$$\tilde{M}(R; z, w) := M(X; z, w).$$

Using the spectral theorem for kernels (see [14, pg. 104, Theorem 7.4]), Theorem 1.1, equations (1.3), (1.7), (3.1), estimates about $h(t)$ ([2, pg. 320, Lemma 2.4.b]) and the fact that $h(t)$ is an even function we get

$$M(X; z, w_n) = \sum_{t_j > 0} u_j(z) u_j(w_n) \left( \frac{1}{X} \int_2^X \frac{h_x(t_j)}{x^{s_j/2}} dx \right) + O \left( \sum_{1/2 < s_j \leq 1} \frac{1}{X} \int_2^X \frac{x^{1/2-s_j}}{2s_j - 1} dx \right).$$

Since the $s_j$’s are discrete, there exists a constant $\sigma = \sigma_\Gamma \in (0, 1/2]$, depending only on $\Gamma$, such that $s_j - 1/2 \geq \sigma$ for all small eigenvalues. We conclude

$$\sum_{1/2 < s_j \leq 1} \frac{1}{X} \int_2^X \frac{x^{1/2-s_j}}{2s_j - 1} dx = O(X^{-\sigma}).$$
We use equation (3.3) to obtain
\[ \hat{M}(R; z, w_n) = 2\sqrt{\pi} \sum_{t_j > 0} u_j(z)u_j(w_n) \mathcal{R} \left( \frac{\Gamma(it_j)}{\Gamma(\frac{3}{2} + it_j)} F(R, t_j) \right) + O(e^{-\sigma R}), \]
where
\[ F(R, t_j) = e^{-R} \int_{0}^{\infty} x^{it_j} \left( 1 + O(x^{-2}) \right) dx = \frac{e^{it_j R}}{1 + it_j} + O \left( \frac{e^{-R}}{1 + |t_j|} \right). \]

Using Stirling’s formula [5, p. 895, eq. (8.328.1)] and Theorem 2.4 we see that
\[ \hat{M}(R; z, w_n) = 2\sqrt{\pi} \sum_{t_j > 0} u_j(z)u_j(w_n) \mathcal{R} \left( \frac{\Gamma(it_j)}{\Gamma(\frac{3}{2} + it_j)} e^{it_j R} \right) + O(e^{-\sigma R}). \]

For \( A > 1 \), we split the sum in the intervals \([0, A]\) and \([A, +\infty)\). Stirling’s formula, Theorem 2.4 and Cauchy-Schwarz inequality imply the bound
\[ \sum_{t_j \geq A} u_j(z)u_j(w_n) \mathcal{R} \left( \frac{\Gamma(it_j)}{\Gamma(\frac{3}{2} + it_j)} e^{it_j R} \right) = O(A^{-1/2}). \]

Let \( \epsilon_1 > 0 \). Since for every \( j \) we have \( u_j(w_n) \to u_j(z) \), we can find an integer \( n_0 = n_0(\epsilon_1, A) \) such that
\[ u_j(w_n) = u_j(z) + O(\epsilon_1) \]
for every \( n \geq n_0 \) and for every \( j \) such that \( 0 < t_j < A \). Thus, using Theorem 2.4 and Cauchy-Schwarz inequality, for \( n \geq n_0(\epsilon_1, A) \) we get
\[ \hat{M}(R; z, w_n) = 2\sqrt{\pi} \sum_{0 < t_j < A} |u_j(z)|^2 \mathcal{R} \left( \frac{\Gamma(it_j)}{\Gamma(\frac{3}{2} + it_j)} e^{it_j R} \right) + O \left( A^{-1/2} + \epsilon_1 + e^{-\sigma R} \right). \]

The sum for \( t_j < A \) can be handled by applying Dirichlet’s principle (see [18, p. 96, Lemma 3.3]).

**Lemma 3.1** (Dirichlet’s box principle). Let \( r_1, r_2, ..., r_n \) be \( n \) distinct real numbers and \( M > 0 \), \( T > 1 \). Then, there is an \( R, M \leq R \leq MT^n \), such that
\[ |e^{ir_j R} - 1| < \frac{1}{T} \]
for all \( j = 1, ..., n \).

We apply Dirichlet’s principle to the sequence \( e^{it_j R} \). For any \( M > 0 \) and any \( T > 1 \) sufficiently large we find an \( R \) such that
\[ \hat{M}(R; z, w_n) = 2\sqrt{\pi} \sum_{0 < t_j < A} |u_j(z)|^2 \mathcal{R} \left( \frac{\Gamma(it_j)}{\Gamma(\frac{3}{2} + it_j)} \right) + O \left( T^{-1} + A^{-1/2} + \epsilon_1 + e^{-\sigma R} \right). \]

We apply local Weyl’s law and Lemma 2.1 to the last sum. Local Weyl’s law implies that as \( A \to \infty \) the sum remains bounded and, for \( \Gamma \) cocompact, there exist infinitely many \( j \)’s such that \( u_j(z) \neq 0 \).
Lemma 2.1 implies that all the nonzero terms are negative. Hence, there exists an $A_0$ such that for every $A \geq A_0$:

$$
(3.8) \quad \left| \sum_{0 < t_j < A} |u_j(z)|^2 \Re \left( \frac{\Gamma(it_j)}{\Gamma \left( \frac{3}{2} + it_j \right) (1 + it_j)} \right) \right| \geq 1.
$$

Choosing $T$ sufficiently large, $A$ fixed and sufficiently large and $\epsilon_1$ fixed and sufficiently small, we deduce that we can choose $n_0$ fixed such that $M(X; z, w_n) = \Omega_-(1)$ for every $n \geq n_0$. Hence, $M(X; z, w) = \Omega_-(1)$ for $w$ in a fixed $\delta$-neighbourhood of $z$. This proves case $a)$ of Theorem 1.5.

To prove that if $\lambda_1 > a$ the limit does not exist, we consider the finite sum

$$
S_{z, A}(R) = 2\sqrt{\pi} \sum_{0 < t_j < A} |u_j(z)|^2 \Re \left( \frac{\Gamma(it_j)}{\Gamma \left( \frac{3}{2} + it_j \right) (1 + it_j)} e^{it_j R} \right)
$$

for $A$ chosen finite and sufficiently large. We first prove that it attains at least two different values. We differentiate $S_{z, A}(R)$. Since $A$ is finite, we compute

$$
\frac{\partial S_{z, A}}{\partial R}(R) = 2\sqrt{\pi} \sum_{0 < t_j < A} |u_j(z)|^2 \Re \left( \frac{\Gamma(it_j)}{\Gamma \left( \frac{3}{2} + it_j \right) (1 + it_j)} \frac{it_j}{1 + it_j} e^{it_j R} \right) + O(A^{1/2}T_0^{-1}).
$$

Applying again Dirichlet’s principle we find a sufficiently large $T_0$ and a $R_0$, depending on $T_0$, such that

$$
\frac{\partial S_{z, A}}{\partial R}(R_0) = 2\sqrt{\pi} \sum_{0 < t_j < A} |u_j(z)|^2 \Re \left( \frac{\Gamma(it_j)}{\Gamma \left( \frac{3}{2} + it_j \right) (1 + it_j)} e^{it_j R_0} \right) + O(A^{1/2}T_0^{-1}).
$$

Assume $t_1 > c$, with $c$ as in Remark 2.3 (hence $\lambda_1 > 1/4 + c^2 = a$). We conclude that $\frac{\partial S_{z, A}}{\partial R}(R_0) \neq 0$, hence $S_{z, A}(R)$ is not constant. In particular, it admits at least two different values $B_1, B_2$. Assume we express $B_\nu$ as

$$
B_\nu = 2\sqrt{\pi} \sum_{0 < t_j < A} |u_j(z)|^2 \Re \left( \frac{\Gamma(it_j)}{\Gamma \left( \frac{3}{2} + it_j \right) (1 + it_j)} e^{it_j R_\nu} \right)
$$

for $\nu = 1, 2$. We estimate

$$
\left| \hat{M}(R; z, w_n) - B_\nu \right| \ll \sum_{0 < t_j < A} |u_j(z)|^2 \left| \Re \left( \frac{\Gamma(it_j)}{\Gamma \left( \frac{3}{2} + it_j \right) (1 + it_j)} (e^{it_j(R-R_\nu)} - 1) \right) \right| + O \left( A^{-1/2} + \epsilon_1 + e^{-\sigma R_\nu} \right).
$$

Applying Dirichlet’s principle, for $\nu = 1, 2$ we find sequences $T_{\mu, \nu} \to \infty$ and sequences $R_{\mu, \nu} \to \infty$ as $\mu \to \infty$ such that for all $t_j \in (0, A)$:

$$
e^{it_j(R_{\mu, \nu}-R_\nu)} = 1 + O(T_{\mu, \nu}^{-1}).
$$

Hence, we conclude that

$$
\left| \hat{M}(R_{\mu, \nu}; z, w_n) - B_\nu \right| = O \left( T_{\mu, \nu}^{-1} + A^{-1/2} + \epsilon_1 + e^{-\sigma R_{\mu, \nu}} \right).
$$

Since $R_{\mu, \nu} \to \infty$, we conclude that $M(X, z, w)$ approaches both values $B_1, B_2$ infinitely many times as close as we want as $X \to \infty$. Since $B_1 \neq B_2$, we conclude that the $M(X, z, w)$ does not have a limit as $X \to \infty$. $\square$
Remark 3.2. In order to prove the lower bound (3.8), it is enough to assume that there exists at least one \( j \) such that \( u_j(z) \neq 0 \). Thus, in any such case, the contribution of the discrete spectrum in \( M(X; z, w_n) \) is \( \Omega_-(1) \) for \( n \geq n_0 \). The same argument holds for the last statement of Theorem 1.5.

3.2. Proof of Theorem 1.5 for \( \Gamma \) cofinite. For \( \Gamma \) cofinite but not cocompact, the hyperbolic Laplacian \( -\Delta \) has also continuous spectrum which corresponds to the Eisenstein series \( E_a(z, 1/2 + it) \) (see [14, chapters 3, 6 and 7]). To deal with the cofinite case (part b) of Theorem 1.5, we have to study the contribution of the continuous spectrum in \( M(X; z, w_n) \).

Proof. Let \( \Gamma \) be cofinite but not cocompact. Working as in the proof of the cocompact case we get

\[
M(X; z, w_n) = \sum_{t_j > 0} u_j(z) u_j(w_n) \left( \frac{1}{X} \int_2^X \frac{h_x(t_j)}{x^{1/2}} dx \right) + O(X^{-\sigma})
\]

\[+
\sum_a \frac{1}{4\pi} \int_{-\infty}^\infty E_a(z, 1/2 + it) E_a(w_n, 1/2 + it) \left( \frac{1}{X} \int_2^X \frac{h_x(t)}{x^{1/2}} dx \right) dt,
\]

where the second sum is over the cusps \( a \) of \( \Gamma \). Hence, the contribution of the continuous spectrum for the cusp \( a \) in \( M(X; z, w_n) \) is equal to

\[
\frac{1}{4\pi} \int_{-\infty}^\infty E_a(z, 1/2 + it) E_a(w_n, 1/2 + it) \left( \frac{1}{X} \int_2^X \frac{h_x(t)}{x^{1/2}} dx \right) dt.
\]

Set

\[A(X) = \int_{-\infty}^\infty \frac{1}{X} \int_2^X \frac{h_x(t)}{x^{1/2}} dx dt.
\]

We have the following lemma, which is analogous to Lemma 2.4 in [18].

Lemma 3.3. As \( X \to \infty \) we have

\[\lim_{X \to \infty} A(X) = 4\pi.
\]

Proof. The Selberg/Harish-Chandra transform \( h(t) \) of \( \chi_{[0, (\cosh r - 1)/2]} \) can be written in the form

\[h(t) = 2\sqrt{2} \int_{-\infty}^\infty e^{itu} (\cosh r - \cosh u)^{1/2} \chi_{[-r, r]}(u) du,
\]

see [18, p. 84, 85, eq. (2.9), (2.10)]. Hence

\[A(X) = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{itu} \Phi_X(u) du dt,
\]

where \( \Phi_X(u) \) is given by

\[\Phi_X(u) = \frac{4}{X} \int_{|u|}^{\cosh^{-1}(X/2)} \sinh r \sqrt{1 - \frac{\cosh u}{\cosh r}} dr.
\]

Using the Fourier inversion formula and easy estimates we get

\[A(X) = 2\pi \Phi_X(0) = \frac{8\pi}{X} \left( \frac{X}{2} - 1 \right) + O \left( \frac{\log X}{X} \right) \to 4\pi.
\]

\[\square\]
Let $\phi_{a,n}(t), \phi_a(t)$ be defined as:

\begin{equation}
\phi_{a,n}(t) = E_a(z, 1/2 + it)E_a(w_n, 1/2 + it) - |E_a(z, 1/2)|^2,
\phi_a(t) = |E_a(z, 1/2 + it)|^2 - |E_a(z, 1/2)|^2.
\end{equation}

Thus, the contribution of cusp $a$ to (3.9) can be written in the form

\begin{equation}
\frac{1}{4\pi} |E_a(z, 1/2)|^2 A(X) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{a,n}(t) \left( \frac{1}{X} \int_{2}^{X} h(t) \, dt \right) \, dt.
\end{equation}

Using equation (3.3), the second summand of (3.11) takes the form

\begin{equation}
\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \phi_{a,n}(t) \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it} X^it \right) \, dt + O \left( \frac{1}{X} \int_{-\infty}^{\infty} \phi_{a,n}(t) \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{2it}{1 + it} \right) \, dt \right)
\end{equation}

(3.12) 

For any $A > 1$ we split the first integral:

\begin{equation}
\int_{-A}^{A} \phi_{a,n}(t) \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it} X^it \right) \, dt + \int_{|t| > A} \phi_{a,n}(t) \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it} X^it \right) \, dt.
\end{equation}

Since $w_n \to z$, using Theorem 2.4 and Cauchy-Schwarz inequality, the integral for $|t| > A$ is bounded independently of $n$:

\begin{equation}
\int_{|t| > A} \phi_{a,n}(t) \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it} X^it \right) \, dt = O(A^{-1/2}).
\end{equation}

In $[-A, A]$, we approximate $\phi_{a,n}(t)$: for every $\epsilon_1 > 0$ there exists a $n_0 = n_0(\epsilon_1, A)$ such that for every $n \geq n_0$:

\begin{equation}
\phi_{a,n}(t) = \phi_a(t) + O(\epsilon_1)
\end{equation}

for every $t \in [-A, A]$. Thus, we get

\begin{align*}
\int_{-A}^{A} \phi_{a,n}(t) \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it} X^it \right) \, dt &= \int_{-\infty}^{\infty} \phi_a(t) \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it} X^it \right) \, dt \\
&\quad - \int_{|t| > A} \phi_a(t) \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it} X^it \right) \, dt \\
&\quad + O \left( \epsilon_1 \int_{-A}^{A} \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it} X^it \right) \, dt \right).
\end{align*}

Using the bound $\phi_a(t) = O(t)$ for small $t$ and Theorem 2.4 for $t \to \infty$ we get that the function

$$
\theta_a(t) = \phi_a(t) \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it}
$$

is in $L^1(\mathbb{R})$. Applying the Riemann-Lebesgue Lemma to the first term we conclude that it converges to 0 as $X \to \infty$. We work as for (3.14) to see that the second term is bounded by $O(A^{-1/2})$. Using that the function

\begin{equation}
g_a(t) = \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it} X^it \right)
\end{equation}

is in $L^1(\mathbb{R})$. Applying the Riemann-Lebesgue Lemma to the first term we conclude that it converges to 0 as $X \to \infty$. We work as for (3.14) to see that the second term is bounded by $O(A^{-1/2})$. Using that the function

$$
g_a(t) = \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it} X^it \right)
$$

is in $L^1(\mathbb{R})$. Applying the Riemann-Lebesgue Lemma to the first term we conclude that it converges to 0 as $X \to \infty$. We work as for (3.14) to see that the second term is bounded by $O(A^{-1/2})$. Using that the function

$$
g_a(t) = \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it} X^it \right)
$$

is in $L^1(\mathbb{R})$. Applying the Riemann-Lebesgue Lemma to the first term we conclude that it converges to 0 as $X \to \infty$. We work as for (3.14) to see that the second term is bounded by $O(A^{-1/2})$. Using that the function

$$
g_a(t) = \Re \left( \frac{\Gamma(it)}{\Gamma(3/2 + it)} \frac{1}{1 + it} X^it \right)
$$
is in $L^1(\mathbb{R})$ uniformly in $X$, we see that the third term is $O(\varepsilon_1)$. Using trivial estimates instead of the Riemann-Lebesgue Lemma in (3.12) for the O-terms, we conclude that for every $\varepsilon_1 > 0, A > 1$ there exists a $n_0 = n_0(\varepsilon_1, A)$ such that for every $n \geq n_0$:

$$
\frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{a,n}(t) \left( \frac{1}{X} \int_{2}^{X} \frac{h(t)}{x^{1/2}} dx \right) dt = O(\varepsilon_1 + A^{-1/2}) + o(1).
$$

Thus, choosing $\varepsilon_1 = A^{-1/2}$ we conclude that for every $\varepsilon_1 > 0$ there exists a $n_0 = n_0(\varepsilon_1)$ such that for every $n \geq n_0$ the contribution of the continuous spectrum to $M(X; z, w_n)$ is equal to

$$
(3.15) \sum_{n} |E_n(z, 1/2)|^2 + O(\varepsilon_1) + o(1).
$$

Case b) of Theorem 1.5 follows for $\varepsilon_1$ sufficiently small and fixed. \(\square\)

When there is no contribution from the eigenvalues $\lambda_j > 1/4$, the following proposition follows immediately from (3.15).

**Proposition 3.4.** If either $\Gamma$ does not have eigenvalues $\lambda_j > 1/4$ or $u_j(z) = 0$ for every $\lambda_j > 1/4$, then we have

$$
\lim_{X \to \infty} M(X; z, z) = \sum_{n} |E_n(z, 1/2)|^2.
$$

4. $\Omega$-results for the error term $e(X; z, w)$

We now prove Theorem 1.8. Assume that $\Gamma$ is cofinite but not cocompact. In order to deal with the error term $e(X; z, w)$ we mollify it as in Phillips and Rudnick [18]. Let $\psi \geq 0$ be a smooth even function compactly supported in $[-1, 1]$, such that

$$
(4.1) \int_{-\infty}^{+\infty} \psi(x)e^{-itx} dx := \hat{\psi}(t) \geq 0
$$

and $\int_{-\infty}^{\infty} \psi(x) dx = 1$. For every $\varepsilon > 0$ we also define the family of functions $\psi_\varepsilon(x) = e^{-1/\varepsilon} \psi(x/\varepsilon)$. We have $0 \leq \hat{\psi}_\varepsilon(x) \leq 1$ and $\hat{\psi}_\varepsilon(0) = 1$. We study the contribution of the discrete spectrum first.

4.1. The contribution of the discrete spectrum. For $z$ fixed, we pick again a sequence $\{w_n\}_{n=1}^{\infty}$ converging to $z$. For every $n \geq 1$ we define

$$
(4.2) \hat{\psi}_n(R, z) = \frac{\hat{\psi}(e^{R}; z, w_n)}{e^{R/2}},
$$

and we consider the convolution

$$
\hat{\psi}_n(R, z) = (\psi_\varepsilon \star \hat{\psi}_n)(R) := \int_{-\infty}^{\infty} \psi_\varepsilon(R - Y) \hat{\psi}_n(Y, z) dY.
$$

In order to prove a lower bound for $\hat{\psi}_n(R, z)$ it suffices to prove a lower bound for $\hat{\psi}_n(R, z)$. Using the pre-trace formula for $L^2(\Gamma \backslash \mathbb{H})$ ([14, pg. 104, Theorem 7.4]), the expression (3.3), the bound

$$
(4.3) \hat{\psi}_\varepsilon(t_j) = O_k ((\varepsilon|t_j|)^{-k})
$$

for every $k \in \mathbb{N}$ and working as in [18] we conclude that the contribution of the discrete spectrum in $\hat{\psi}_n(R, z)$ is equal to:

$$
(4.4) 2\sqrt{\pi} \sum_{t_j > 0} u_j(z) u_j(w_n) \Re \left( \frac{\Gamma(it_j)}{\Gamma\left(\frac{3}{2} + it_j\right)} e^{it_jR} \hat{\psi}_\varepsilon(t_j) \right) + O(e^{-\sigma R}).
$$
Let $A > 1$. Clearly for $k \in \mathbb{N}$ we have $k > 1/2$ and thus, using the estimate (4.3), we can bound the tail of the series for $t_j > A$. Applying (4.3), Theorem 2.4 and Stirling’s formula, we conclude that (4.4) takes the form

$$2\sqrt{\pi} \sum_{0 < t_j < A} |u_j(z)|^2 \Re \left( \frac{\Gamma(it_j)}{\Gamma(\frac{3}{2} + it_j)} e^{it_j R} \right) \hat{\psi}_R(t_j) + O \left( A^{1/2-k} e^{-k} + e^{-\sigma R} \right).$$

Let $\epsilon_1 > 0$. We find again an integer $n_0 = n_0(\epsilon_1, A)$ such that

$$u_j(w_n) = u_j(z) + O(\epsilon_1)$$

for every $n \geq n_0$ and for every $j$ such that $0 < t_j < A$. Since $\hat{\psi}_R(x)$ is bounded, Cauchy-Schwarz inequality, weak Weyl’s law (i.e. $\{ j : |t_j| \leq T \} \ll T^2$) and Theorem 2.4 yield that the quantity in (4.5) is

$$2\sqrt{\pi} \sum_{0 < t_j < A} |u_j(z)|^2 \Re \left( \frac{\Gamma(it_j)}{\Gamma(\frac{3}{2} + it_j)} e^{it_j R} \right) \hat{\psi}_R(t_j) + O \left( T^{-1} A^{1/2} + A^{1/2-k} e^{-k} + A^{1/2} \epsilon_1 e^{-\sigma R} \right).$$

Applying Dirichlet’s principle for the exponentials $e^{it_j R}$, for any $T > 1$ sufficiently large we find an $R$ such that $e^{it_j R} = 1 + O(T^{-1})$, thus concluding that the contribution of the discrete spectrum to $\hat{e}_{n, \epsilon}(R, z)$ takes the form

$$2\sqrt{\pi} \sum_{0 < t_j < A} |u_j(z)|^2 \Re \left( \frac{\Gamma(it_j)}{\Gamma(\frac{3}{2} + it_j)} \right) \hat{\psi}_R(t_j) + O \left( T^{-1} A^{1/2} + A^{1/2-k} e^{-k} + \epsilon_1 + e^{-\sigma R} \right).$$

By part a) of Lemma 2.1, the coefficients in the sum are all negative, whereas the balance $\epsilon^{-1} = A^{1-3/(2k+2)}$, $\epsilon_1 = A^{-1/2}$ implies that the error term is $O(\epsilon^{-1} A^{1/2} + \epsilon + e^{-\sigma R})$. For the function $\psi$, there exists one $\tau \in (0, 1)$ such that $\psi(x) \geq 1/2$ whenever $|x| \leq \tau$. Using this, local Weyl’s law and the fact that $\hat{\psi}_R(t_j) = \hat{\psi}(it_j)$ we bound the modulus of the above main term from below by

$$\sum_{0 < t_j < A} |u_j(z)|^2 |\hat{\psi}_R(t_j)| \Re \left( \frac{\Gamma(it_j)}{\Gamma(\frac{3}{2} + it_j)} \right) \gg \sum_{0 < t_j < \tau / \epsilon} |u_j(z)|^2 \Re \left( \frac{\Gamma(it_j)}{\Gamma(\frac{3}{2} + it_j)} \right).$$

If $\Gamma$ has sufficiently many cusp forms, we obtain the bound

$$\sum_{0 < t_j < \tau / \epsilon} |u_j(z)|^2 \Re \left( \frac{\Gamma(it_j)}{\Gamma(\frac{3}{2} + it_j)} \right) \gg \epsilon^{-1/2}.$$

Since the error is $O(T^{-1} A^{1/2} + \epsilon + e^{-\sigma R})$, there exists a fixed, sufficiently small $\epsilon_0 > 0$ such that, for $T$ and $R$ sufficiently large, $\epsilon_0^{-1/2}$ dominates the error. Therefore there exists a fixed integer $n_0 = n_0(\epsilon_0)$ such that for every $n \geq n_0$ the contribution of the discrete spectrum in $\hat{e}_{n, \epsilon}(R, z)$ is $\Omega_-(1)$, i.e. for every $n \geq n_0$ the contribution in $e(X ; z, w_n)$ is $\Omega_-(X^{1/2})$.

### 4.2. The contribution of the continuous spectrum

We have to consider the contribution of the continuous spectrum in $\hat{e}_{n, \epsilon}(R, z)$, which is given by

$$\sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} E_a(z, 1/2 + it) E_a(w_n, 1/2 + it) \left( \int_{-\infty}^{\infty} \psi_R(R - Y) \frac{h_\epsilon(t)}{e^{\epsilon R/2}} dY \right) dt.$$
Let $φ_{a,n}(t)$ be as in equation (3.10). Hence, for $h(t) = h_{cγ}(t)$ the contribution of cusp $a$ in (4.6) takes the form
\begin{equation}
\frac{1}{4π} |E_a(z, 1/2)|^2 \int_{-∞}^{∞} \left( \int_{-∞}^{∞} \psi(R-Y) \frac{h(t)}{e^{Y/2}} dY \right) dt + \frac{1}{4π} \int_{-∞}^{∞} φ_{a,n}(t) \left( \int_{-∞}^{∞} \psi(R-Y) \frac{h(t)}{e^{Y/2}} dY \right) dt.
\end{equation}
Using [18, p. 98, eq. (3.30)] we get
\begin{equation}
\frac{1}{4π} |E_a(z, 1/2)|^2 \int_{-∞}^{∞} \left( \int_{-∞}^{∞} \psi(R-Y) \frac{h(t)}{e^{Y/2}} dY \right) dt = |E_a(z, 1/2)|^2 + O(e^{-R}).
\end{equation}
Using that $ψ(x)$ has support in $[-1, 1]$ and quoting expansion (3.3) we see that the second summand of (4.7) takes the form
\begin{equation}
\frac{1}{2\sqrt{π}} \Re \left( \int_{-∞}^{∞} φ_{a,n}(t) \frac{Γ(it)}{Γ(3/2 + it)} e^{itR} \hat{ψ}_c(t) dt \right) + O \left( e^{-2R} \int_{-∞}^{∞} φ_{a,n}(t) \Re \left( \frac{Γ(it)}{Γ(3/2 + it)(-2 + it)} \right) dt \right).
\end{equation}
For the first term of (4.8), we work as in subsection 3.2: we split the integral for $t \in [-A, A]$ and $|t| > A$. For $|t| > A$ we apply Theorem 2.4 and the bound (4.3) to get
\begin{equation}
\Re \left( \int_{|t| > A} φ_{a,n}(t) \frac{Γ(it)}{Γ(3/2 + it)} e^{itR} \hat{ψ}_c(t) dt \right) = O(ε^{-1}A^{-1/2}),
\end{equation}
independently of $n$. We approximate $φ_{a,n}(t)$ uniformly $ε_1$-close to $φ_a(t)$: $φ_{a,n}(t) = φ_a(t) + O(ε_1)$ for every $n \geq n_0 = n_0(ε_1)$ and for every $t \in [-A, A]$. The function $φ_a(t)$ satisfies the bounds $φ_a(t) = O(t)$ for small $t$ and $O(t^2)$ for large $t$. Using $ψ_c(0) = 1$, $ψ_c(t) = O((ε|t|)^{-k})$ we deduce that, for any fixed $ε > 0$, the function
\begin{equation}
φ_a(t) = \frac{Γ(it)}{Γ(3/2 + it)} e^{2it} \hat{ψ}_c(t)
\end{equation}
is in $L^1(ℝ)$ independently of $ε$. Applying the Riemann-Lebesgue Lemma and local Weyl’s law we deduce that
\begin{equation}
\frac{1}{2\sqrt{π}} \Re \left( \int_{-A}^{A} φ_a(t) \frac{Γ(it)}{Γ(3/2 + it)} e^{itR} \hat{ψ}_c(t) dt \right) = e^{-2G(R)} + O(ε^{-1}A^{-1/2}),
\end{equation}
with $G(R) = o(1)$, independently of $ε$. Since the function
\begin{equation}
\Re \left( \frac{Γ(it)}{Γ(3/2 + it)} e^{itR} \right) = \Re \left( \frac{Γ(it)}{Γ(3/2 + it)} \right) \cos(tR) - \Im \left( \frac{Γ(it)}{Γ(3/2 + it)} \right) \sin(tR)
\end{equation}
is bounded as $t → 0$ we conclude it is in $L^1(ℝ)$, hence
\begin{equation}
\frac{1}{2\sqrt{π}} \Re \left( ε_1 \int_{-A}^{A} \frac{Γ(it)}{Γ(3/2 + it)} e^{itR} \hat{ψ}_c(t) dt \right) = O(ε_1).
\end{equation}
Working similarly we get
\begin{equation}
e^{-2R} \int_{-∞}^{∞} φ_{a,n}(t) \Re \left( \frac{Γ(it)}{Γ(3/2 + it)(-2 + it)} \right) dt = O(e^{-2R})
\end{equation}
uniformly, i.e. independently of $n$. Using the balance $ε^{-2} = A^{1/2}$ and $ε_1 = ε$, we conclude that the contribution of the continuous spectrum in $c_{n,ε}(R, z)$ can be finally written in the form
\begin{equation}
\sum_a |E_a(z, 1/2)|^2 + ε^{-2}G(R) + O(ε + ε^{-R}).
\end{equation}
4.3. **Proof of part a) of Theorem 1.8.** Since \( \Gamma \) has sufficiently many cusp forms, the contribution of the discrete spectrum in \( \hat{c}_{n,\epsilon}(R,z) \) is of the form \( k(\epsilon) + O(T^{-1}A^{1/2} + \epsilon + e^{-\sigma R}) \) with \( k(\epsilon) = \Omega_-(\epsilon^{-1/2}) \). Pick a fixed \( \epsilon_0 \) such that \( \epsilon_0^{-1/2} \) dominates the \( \sum_a |E_a(z,1/2)|^2 \) and the \( O(\epsilon_0) \)-terms. Let \( T, R \to \infty \). Since \( \epsilon_0 \) is fixed, \( \epsilon_0^{-2}G(R) \to 0 \) as \( R \to \infty \). Thus, for every \( n \geq n_0(\epsilon_0) \):

\[
\hat{c}_{n,\epsilon_0}(R,z) = \Omega_-(1),
\]

which implies \( e(X; z, w_n) = \Omega_-(X^{1/2}) \) for every \( n \geq n_0 \).

4.4. **Proof of part b) of Theorem 1.8.** Assume that \( \Gamma \) has null-vectors. We have to prove that \( e(X; z, w) = \Omega_+(X^{1/2}) \) for \( w \) in a small neighborhood \( B(z, \delta) \) of \( z \). By Theorem 1.3 of Phillips and Rudnick in [18] we have \( e(X; z, z) = \Omega_+(X^{1/2}) \). Hence, in order to prove part b), it suffices to prove the following Proposition.

**Proposition 4.1.** If \( \Gamma \) has null-vectors, then there exists a \( \delta = \delta_{\Gamma, z} > 0 \) such that for every \( w \in B(z, \delta) \)

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{e(\epsilon; z, w)}{e^{\sigma/2}} dr > \frac{1}{2} \sum_a |E_a(z,1/2)|^2.
\]

Since the proof is a routine using the ideas used in the proof of Theorem 1.3 in [18] and in section 3, we only sketch the basic steps.

**Proof.** (sketch) For any \( w_n \to z \), the contribution of Maass forms in

\[
\frac{1}{T} \int_0^T \frac{e(\epsilon; z, w_n)}{e^{\sigma/2}} dr
\]

is estimated using expression (3.3), the local Weyl’s law and the Cauchy-Schwarz inequality:

\[
2\sqrt{\pi} \sum_{\ell_j > 0} u_j(z) u_j(w_n) \Re \left( \frac{\Gamma(it_j)}{\Gamma(1/2 + it_j)} \right) \frac{1}{T} \int_0^T e^{i\ell_j R} dr + O \left( T^{-1} + \sum_{1/2 < s_j \leq 1} \frac{1}{T} \int_0^T e^{\epsilon(1/2 - s_j)} dr \right)
\]

\[
= O \left( T^{-1} + T^{-1} \sum_{\ell_j > 0} \frac{u_j(z) u_j(w_n)}{|t_j|^{5/2}} \right) = O \left( T^{-1} \right).
\]

The contribution of Eisenstein series is equal to

\[
\sum_a \frac{1}{4\pi} \int_{-\infty}^\infty E_a(z,1/2 + it) \overline{E_a(w_n,1/2 + it)} \left( \frac{1}{T} \int_0^T h(t) dt \right) dr dt.
\]

Let \( \phi_{a,n}(t) \) be as in section 3. Using [18, p. 87, Lemma 2.4] and working as in section 3 for \( \phi_{a,n}(t) \), for every \( \epsilon_1 > 0 \) there a \( n_0 \) such that for every \( n \geq n_0 \) the contribution of the continuous spectrum is

\[
\sum_a |E_a(z,1/2)|^2 + O(\epsilon_1).
\]

Thus, for \( \epsilon_1 \) sufficiently small and fixed the proposition follows. \( \square \)
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