CFTs on Riemann Surfaces of genus $g \geq 1$

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Abstract

$N$-point functions of holomorphic fields in rational conformal field theories can be calculated by methods from complex analysis. We establish explicit formulas for the $2$-point function of the Virasoro field on hyper-elliptic Riemann surfaces of genus $g \geq 1$. $N$-point functions for higher $N$ are obtained inductively, and we show that they have a nice graph representation. We discuss the $3$-point function with application to the $(2,5)$ minimal model.

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1 Introduction

Quantum field theories are a major challenge for mathematicians. Apart from cases without interaction, the theories best understood at present are conformally invariant and do not contain massive particles.

Conformal field theories (CFTs) can be defined over arbitrary Riemann surfaces. A theory is considered to be solved once all of its $N$-point functions are known. The case of the Riemann sphere is rather well understood.

The present paper establishes explicit formulas for the 2-point functions of the Virasoro field over some specific class of genus-$g$ Riemann surfaces $X_g$, where $g \geq 1$. $N$-point functions for $N \geq 3$ are obtained inductively from these. We show that they can be written in terms of a list of oriented graphs with $N$ partially linked vertices which is complete under some natural condition.

Much has been achieved previously for conformal field theories over the torus $X_1$ (e.g., [8]), the case $g > 1$ is technically more demanding though. Some first steps have been made ([3], [4], [5] and more recently, [9]) using operator vertex algebras. Quantum field theory on a compact Riemann surface of any genus can be approached differently using methods from algebraic geometry [1] and complex analysis. $N$-point functions of holomorphic fields are meromorphic functions. That is, they are determined by their poles and their respective behaviour at infinity. By compactness of $X_g$, these functions admit a Laurent series expansion whose principal part has finite length.

2 Rational coordinates

Let $X_1$ be a compact Riemann surface of genus $g = 1$. Such manifold is biholomorphic to the torus $\mathbb{C}/\Lambda$ (with the induced complex structure), for the lattice $\Lambda$ spanned over $\mathbb{Z}$ by 1 and some $\tau \in \mathbb{H}^+$, unique up to an $SL(2, \mathbb{Z})$ transformation. Here $\mathbb{H}^+$ denotes the upper complex half plane. We denote by $z$ the local coordinate on $X_1$. $N$-point functions on $X_1$ are elements of the field $K(X_1) = \mathbb{C}(\wp, \wp')$ of meromorphic functions over $X_1$, which is generated over $\mathbb{C}$ by the Weierstrass function $\wp$ (with values $\wp(z|\tau)$) associated to $\Lambda$ and its derivative $\wp' = \partial \wp/\partial z$. Instead of with $z$ we shall work with the pair of complex coordinates

$$x = \wp(z), \quad y = \wp'(z), \quad (\tau \text{ fixed}),$$

satisfying

$$y^2 = 4(x^3 - 15G_4x - 35G_6). \quad (1)$$

Here $G_{2k}$ for $k \geq 2$ are the holomorphic Eisenstein series. We compactify $X_1$ by including the point $x = \infty$ (corresponding to $z = 0 \mod \Lambda$), and view $x$ as a holomorphic function on $\mathbb{C}/\Lambda$ with values in $\mathbb{P}^1_{\mathbb{C}}$. $y = \wp'(z)$ defines a double cover of $\mathbb{P}^1_{\mathbb{C}}$.

If $g > 1$, one can write $X_g$ as the quotient of $\mathbb{H}$ by a Fuchsian group, but working with a corresponding local coordinate $z$ becomes difficult. We shall consider hyperelliptic Riemann surfaces $X_g$ only, where $g \geq 1$. Such are defined by

$$X_g : \quad y^2 = p(x),$$
where $p$ is a polynomial of degree $n = 2g + 1$ (the case $n = 2g + 2$ is equivalent and differs from the former by a rational transformation of $\mathbb{C}$ only). We assume $p$ has no multiple zeros and so $X_g$ is regular. A generic point on $X_g$ is determined by a tuple $(x, y) \in \mathbb{C}^2$. Locally we will work with one complex coordinate, either $x$ or $y$. A coordinate out of the set of $x$ and $y$ is locally a good coordinate if the other can be recovered from it. The Inverse Function Theorem cannot be applied to non-contractible neighbourhoods or close to any ramification point. $x$ is a good coordinate away from the ramification points, whereas $y$ is a good coordinate away from the locus where $p' = 0$.

3 The Virasoro OPE

Let $X_g$ be a connected Riemann surface of genus $g \geq 1$. We don’t give a complete definition of a meromorphic conformal field theory \[2\] here, but the most important properties are as follows:

1. We consider a vector bundle $F$ over $X_g$ of infinite rank. That is, for any sufficiently small open set $U \subseteq X_g$, $F|_U \cong U \times F$, where $F$ is an infinite dimensional complex vector space. Thus the fiber over $z \in X_g$ is the vector space $F_z$ which locally is $F$. We postulate that every choice of a chart $U \to \mathbb{C}$, together with a complex coordinate on $U$, yields a canonical trivialisation of $F$. Such open set $U$ will be referred to as a coordinate patch. Local sections in $F$ are called holomorphic fields. To every field $\phi$ there is associated a natural number $h(\phi)$, called the dimension of $\phi$. This induces a grading $F = \bigoplus_{h \in \mathbb{N}} F(h)$, where $F(0) = \mathbb{C}$, and we assume that for any $h_0 \in \mathbb{N}$,

$$\dim_{\mathbb{C}} \left( \bigoplus_{h < h_0} F(h) \right) < \infty.$$ 

We postulate that for any $z \in X_g$, the ascending filtration of $F_z$ associated to the grading does not depend on the choice of the coordinate patch containing $z$. Since in a conformal field theory fields of finite dimension only are considered, it is sufficient to deal with finite sums.

2. For $i = 1, 2$, let $X_i$ be a Riemann surface and let $F_i$ be a rank $r_i$ vector bundle over $X_i$. Let $p_i^*F_i$ be the pullback bundle of $F_i$ by the morphism $p_i : X_1 \times X_2 \to X_i$. Let $F_1 \boxtimes F_2 := p_1^*F_1 \otimes p_2^*F_2$ be the rank $r_1r_2$ vector bundle whose fiber at $(z_1, z_2) \in X_1 \times X_2$ is $F_{1,z_1} \otimes F_{2,z_2}$. We are now in position to define $N$-point functions for bosonic fields. Let $F$ be the vector bundle introduced in point \[1\] A state is a multilinear map

$$\langle \quad \rangle : S_c(F) \to \mathbb{C},$$

where $S_c(F)$ denotes the restriction of the symmetric algebra $S(F)$ to fibers away from the partial diagonals

$$\Delta_N := \{(z_1, \ldots, z_N) \in X^N | z_i = z_j, \text{ for some } i \neq j\},$$

3
for any \( N \in \mathbb{N} \). Locally, over any \( U^N \subseteq X^N \setminus \Delta_N \) such that \( U \) admits local coordinates, a state is the data for any \( N \in \mathbb{N} \) of an \( N \)-linear map

\[
\langle \quad \rangle : \quad U^N \times F^{\otimes N} \to \mathbb{C} \\
(z_1, \varphi_1) \otimes \ldots \otimes (z_N, \varphi_N) \mapsto \langle \varphi_1(z_1) \otimes \ldots \otimes \varphi_N(z_N) \rangle
\]

which is compatible with the OPE. This condition will be explained in point 5.

**Remark 1.** The standard physics’ notation for this object is

\[
\langle \varphi_1(z_1) \ldots \varphi_N(z_N) \rangle
\]

(i.e., tensor product omitted). We shall adopt this notation but keep in mind that each \( z_i \) lies in an individual copy of \( U \) whence the \( \varphi_i(z_i) \) are elements in different copies of \( F \) and multiplication is meaningless.

Since each \( \varphi_i \) is defined over \( U \), we may view \( \langle \varphi_1(z_1) \ldots \varphi_N(z_N) \rangle \) as a function of \( (z_1, \ldots, z_N) \in U^N \). We call it the \( N \)-point function of the fields \( \varphi_1, \ldots, \varphi_N \) over \( U \).

3. Fields are understood by means of their \( N \)-point functions. A field \( \varphi \) is zero if all \( N \)-point functions involving \( \varphi \) vanish. That is, for any \( N \in \mathbb{N} \) and any holomorphic fields \( \varphi_2, \ldots, \varphi_N \), \( \langle \varphi(z_1) \ldots \varphi_N(z_N) \rangle = 0 \) for any state. \( \varphi \) is holomorphic if \( \partial_\overline{\varphi} = 0 \).

4. For any \( N \in \mathbb{N} \) and whenever \( \varphi_1, \ldots, \varphi_N \) are holomorphic fields over a coordinate patch \( U \), we postulate that the \( N \)-point function \( \langle \varphi_1(z_1) \ldots \varphi_N(z_N) \rangle \) is meromorphic in \( z_1 \) and has a Laurent series expansion about \( z_1 = z_2 \) given by

\[
\langle \varphi_1(z_1) \ldots \varphi_N(z_N) \rangle = \sum_{m \geq m_0} (z_1 - z_2)^m N_m(\varphi_1, \varphi_2)(z_2) \varphi_3(z_3) \ldots \varphi_N(z_N),
\]

for some \( m_0 \in \mathbb{Z} \). Here \( N_m(\varphi_1, \varphi_2) \) is a holomorphic field in \( z_2 \), of dimension \( h(\varphi_1) + h(\varphi_2) \). Note that \( N_m(\varphi_1, \varphi_2) \) does not depend on the fields \( \varphi_3, \ldots, \varphi_N \) and the positions \( (z_1, \ldots, z_N) \in U^{N-2} \). Symbolically we write

\[
\varphi_1(z_1) \varphi_2(z_2) \mapsto \sum_{m \geq m_0} (z_1 - z_2)^m N_m(\varphi_1, \varphi_2)(z_2),
\]

and call the arrow the operator product expansion (OPE) of \( \varphi_1 \) and \( \varphi_2 \).

**Remark 2.** Physicists write an equality here. Recall however that \( \otimes \) is understood on the l.h.s.

The OPE can be defined wherever local coordinates are available.

5. While fields and coordinates are local objects, states contain global information. A state is said to be compatible with the OPE (cf. point 4), if for every coordinate patch \( U \) and for every \( N \in \mathbb{N} \), identity (2) holds true, for any choice of holomorphic fields \( \varphi_1, \ldots, \varphi_N \) over \( U \) and any \( (z_1, \ldots, z_N) \in U^N \). In particular, the \( N \)-point function of a compatible state has the poles at \( z_1 = z_2 \) prescribed by the OPE. We postulate that every OPE admits compatible states.
6. When the transition between different coordinate patches (open subsets of \( \mathbb{R}^2 \)) is given by conformal maps, the theory should be conformally invariant. In conformal field theories, one demands the existence of a Virasoro field. It is a holomorphic field \( T \) defined by the condition

\[
N_{-1}(T, \varphi) = \partial \varphi,
\]

for all holomorphic sections \( \varphi \in \Gamma(U \times F) \). Here \( \partial \) denotes the ordinary derivative of fields. \( T \) has dimension \( h = 2 \) (its one-point function is a holomorphic two-form).

The OPE is a local statement that holds in any coordinate patch. When written in local coordinates \( z \) and \( w \) so that its singular part is symmetric, the Virasoro OPE reads

\[
T(z)T(w) \mapsto \frac{c/2}{(z-w)^3} + \frac{1}{(z-w)^2}(T(z)+T(w)) + \Phi(w) + O(z-w). \tag{3}
\]

Thus \( \Phi = N_0(T,T) - \frac{\partial^2 T}{2} \). \( 1 \) is the identity field which is holomorphic of dimension \( h = 0 \) (its one-point function is a complex number). The coefficient \( c \) is referred to as the central charge of the theory,

\[
c = 1 - \frac{6(p-q)^2}{pq}, \quad \text{where } p, q \in \mathbb{Z}, \quad \gcd(p, q) = 1.
\]

**Example 1.** A model is minimal if it has only finitely many non-isomorphic irreducible lowest (or highest) weight representations; for the \((p, q)\) minimal model the number is

\[
\frac{(p-1)(q-1)}{2}.
\]

We will be particularly interested in the \((2,5)\) minimal model. This is the simplest minimal model with just two irreducible representations, namely for the lowest weight \( h = 0 \) (vacuum representation \( \langle 1 \rangle \)) and for \( h = -\frac{1}{5} \). The space of states is spanned by \( \langle 1 \rangle \) and \( \langle T \rangle \).

Let us recapitulate the behaviour of \( T \) under coordinate transformations.

**Definition 3.** Given a holomorphic function \( f \) with derivative \( f' \), we denote by

\[
S(f) := \frac{f'''}{f'} - \frac{3|f'|^2}{2|f|^2}
\]

the Schwarzian derivative of \( f \).

The Schwarzian derivative \( S \) has the following well-known properties:

1. \( S(\lambda f) = S(f), \forall \lambda \in \mathbb{C}, f \in \mathcal{D}(S). \)

2. Suppose \( f : \mathbb{P}^1_C \to \mathbb{P}^1_C \) is a linear fractional transformation,

\[
f : z \mapsto f(z) = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})
\]

Then \( f \in \mathcal{D}(S) \), and \( S(f) = 0 \).
3. Let $f, g \in D(S)$ be such that $f \circ g$ is defined and lies in $D(S)$. Then

$$S(f \circ g) = [g']^2 S(f) \circ g + S(g).$$

**Remark 4.** Let $p, y \in D(S)$ with $y^2 = p(x)$. Then by the properties 7 and 8 of the Schwarzian derivative,

$$S(y) = S(p) + \frac{3}{8} \left[ \frac{p''}{p'} \right]^2,$$

where $S(p) = \frac{p''}{p'} - \frac{3}{2} \left[ \frac{p''}{p'} \right]^2.$

**Lemma 5.** Let $T$ be the Virasoro field in the coordinate $x$. We consider a coordinate change $x \mapsto \hat{x}(x)$ such that $\hat{x} \in D(S)$, and set

$$\hat{T}(\hat{x})[\hat{x}']^2 = T - \frac{c}{12} S(\hat{x}),$$

Then $\hat{T}$ satisfies the OPE in $\hat{x}$.

It can be shown that $\hat{T}(\hat{x})$ is the only such field.

**Proof.** Direct computation. □

**Corollary 6.** Let $X_g$ be a Riemann surface of genus $g \geq 2$. Given a state $\langle ... \rangle$ on $X$, there is a coordinate patch $U \subset X$ with local coordinate $z$ such that $\langle T(z) \rangle$ defines a section in the vector bundle $(T^*U)^{\otimes 2}$.

The theory assumes that this section is holomorphic.

**Proof.** For $g \geq 2$, $X$ can be realised as $\mathbb{H}^+ / \Gamma$, where $\Gamma$ is a Fuchsian group. The Schwarzian derivative of a linear fractional map is zero, (property 2). Eq. (5) shows that $\langle T(z) \rangle$ has the correct transformation behaviour. □

**Example 2.** Let $g = 1$. Then $T^*X$ is trivial and $\langle T(z) \rangle$ is a constant.

Let $X_g$ be a Riemann surface of genus $g \geq 2$. By the Riemann-Roch Theorem,

$$\dim_{\mathbb{C}} H_0((T^*X)^{\otimes 2}) = 3(g - 1).$$

### 4 Calculation of the 1-point function

Associate to the hyperelliptic surface $X$ its field of meromorphic functions $K = \mathbb{C}[x, y]/(y^2 - p(x))$. Then $K$ is a field extension of $\mathbb{C}$ of transcendence degree one, and the two sheets are exchanged by a Galois transformation.

In what follows, we set

$$p(x) = \sum_{k=0}^{n} a_k x^{n-k},$$

where $n = 2g + 1$, or $n = 2g + 2$.

**Theorem 1.** *(On the Virasoro one-point function)*

Let

$$X_g : y^2 = p(x)$$

be a regular Riemann surface of $g \geq 1$.  

1. Suppose \( n \) is even. Then as \( x \to \infty \),
\[
\langle T(x) \rangle \sim x^{-4}.
\]
If \( n \) is odd, then as \( x \to \infty \),
\[
\langle T(x) \rangle = \frac{c}{32}x^{-2}(1) + O(x^{-3}).
\]

2. We have
\[
p(T(x)) = \frac{c}{32} \left[ \frac{p'}{p} \right]^2(1) + \frac{1}{4} p(x, y),
\]  
where \( P(x, y) \) is a polynomial in \( x \) and \( y \). More specifically, we have the Galois splitting
\[
P(x, y) = P^{(G-even)}(x) + yP^{(G-odd)}(x).
\]
Here \( P^{(G-even)}(x) \) is a polynomial of degree \( n - 2 \) with the following property:
(a) If \( n \) is even, \[
\left[ P^{(G-even)} + \frac{c}{32} \left[ \frac{p'}{p} \right]^2 \right] > n - 4 = 0.
\]
(b) If \( n \) is odd, \[
\left[ P^{(G-even)} + \frac{c}{32} \left[ \frac{p'}{p} \right]^2 \right] > n - 3 = 0.
\]
\( P^{(G-odd)} \) is a polynomial of degree \( \frac{n}{2} - 4 \) if \( n \) is even, and \( \frac{n-1}{2} - 3 \) if \( n \) is odd, provided \( g \geq 3 \).

3. Let \( g \geq 2 \). Then the space of \( \langle T(x) \rangle \) has dimension \( 3(g - 1) \).

**Remark 7.** The number of degrees of freedom in Theorem 1.3 for \( g \geq 2 \) equals the dimension of the automorphism group of the Riemann surface \( X_g \), which in genus \( g = 0 \) and \( g = 1 \) is \( \text{dim}_{\mathbb{C}} \text{SL}(2, \mathbb{C}) = 3 \) and \( \text{dim}_{\mathbb{C}}(\mathbb{C}, +) = 1 \), respectively.

**Proof.**  
1. For \( x \to \infty \), we change coordinate \( x \mapsto \tilde{x}(x) := \frac{1}{x} \). By property 2 of the Schwarzian derivative, \( S(\tilde{x}) = 0 \) identically, and
\[
T(x) = \tilde{T}(\tilde{x}) \left[ \frac{d\tilde{x}}{dx} \right]^2,
\]
where \( \left[ \frac{d\tilde{x}}{dx} \right]^2 = x^{-4} \). If \( n \) is even, then \( \tilde{x} \) is a good coordinate, so \( \langle \tilde{T}(\tilde{x}) \rangle \) is holomorphic in \( \tilde{x} \). If \( n \) is odd, then we may take \( \tilde{y} := \sqrt{x} \) as coordinate.
\[
\frac{d\tilde{x}}{dx} = \frac{1}{2} x^{-1.5},
\]
and according to eq. (5) and eq. (4),
\[
T(x) = \frac{c}{32} x^{-2} + \tilde{T}(\tilde{y}) \frac{1}{4} x^{-3},
\]
where \( \langle \tilde{T}(\tilde{y}) \rangle \) is holomorphic in \( \tilde{y} \).

2. \( \langle T(x) \rangle \) is a meromorphic function of \( x \) and \( y \) over \( \mathbb{C} \), whence rational in either coordinate. The ring \( \mathbb{C}[x, y] \) of polynomials in \( x \) and \( y \) is a vector space over the field of rational functions in \( x \), spanned by 1 and \( y \). Thus we have a splitting
\[
\langle T(x) \rangle = \langle T(x) \rangle^{(G-even)} + y\langle T(x) \rangle^{(G-odd)}.
\]
\(T(x)\) is \(O(1)\) in \(x\) iff this holds for \(T(x)^{(G\text{-even})}\) and for \(y(T(x))^{(G\text{-odd})}\) individually, as there can’t be cancellations between Galois-even and Galois-odd terms. We obtain a Galois splitting for \(\hat{T}(y)\) by applying a rational transformation to \(T(x)\). From (5) and (11) follows

\[
p(T(x))^{(G\text{-even})} \left\{  \begin{array}{ll}
        & c \frac{(1)[p'(x)]^2}{p(x)} + \frac{1}{4} P^{(G\text{-even})}(x), \\
\end{array} \right.
\]

where \(P^{(G\text{-even})}\) and \(P^{(G\text{-odd})}\) are rational functions of \(x\). We have

\[
\frac{1}{4} P^{(G\text{-even})} = p(T(x))^{(G\text{-even})} - c \frac{1}{32} \frac{(1)[p'(x)]^2}{p} = \frac{1}{4} [p'(x)]^2 (\hat{T}(y))^{(G\text{-even})} + \frac{c}{12}(1)pS(p).
\]

The l.h.s. is \(O(1)\) in \(x\) for finite \(x\) and away from \(p = 0\) (so wherever \(x\) is a good coordinate) while the r.h.s. is holomorphic in \(y(x)\) for finite \(x\) and away from \(p' = 0\) (so wherever \(y\) is a good coordinate). The r.h.s. does not actually depend on \(y\) but is a function of \(x\) alone. Since the loci \(p = 0\) and \(p' = 0\) do nowhere coincide, we conclude that \(P^{(G\text{-even})}\) is an entire function on \(\mathbb{C}\). It remains to check that \(P^{(G\text{-even})}\) has a pole of the correct order at \(x = \infty\). We have

\[
\frac{[p'(x)]^2}{p} = n^2 a_0 x^{n-2} + n(n-2)a_1 x^{n-3} + O(x^{n-4}). \tag{11}
\]

(a) If \(n\) is even, then \(p(T(x))^{(G\text{-even})} = O(x^{n-4})\) as \(x \to \infty\), by part (4). By eqs (7) and (11), \(P^{(G\text{-even})}(x)\) has degree \(n - 2\) in \(x\). Moreover,

\[
P^{(G\text{-even})}(x) = -\frac{c}{8} \left(n^2 a_0 x^{n-2} + n(n-2)a_1 x^{n-3}\right) (1) + O(x^{n-4}). \tag{12}
\]

(b) If \(n\) is odd, then \(p(T(x))^{(G\text{-even})} = \frac{c}{32} a_0 x^{n-2}(1) + O(x^{n-3})\) as \(x \to \infty\), by eq. (9). Thus \(P^{(G\text{-even})}(x)\) has degree \(n - 2\) in \(x\). Moreover, by eq. (7) and eq. (11),

\[
P^{(G\text{-even})}(x) = -\frac{c}{8} (n^2 - 1) a_0 x^{n-2}(1) + O(x^{n-3}).
\]

Likewise, we have

\[
\frac{1}{4} yP^{(G\text{-odd})}(x) = yp(T(x))^{(G\text{-odd})} = \frac{1}{4} [p'(x)]^2 y(\hat{T}(y))^{(G\text{-odd})};
\]

the l.h.s. is \(O(1)\) in \(x\) wherever \(x\) is a good coordinate while the r.h.s. is holomorphic in \(y\) wherever \(y\) is a good coordinate. Since \(y\) is a holomorphic function in \(x\) and in \(y\) away from \(p = 0\) and away from \(p' = 0\), respectively, this is also true for

\[
\frac{1}{4} pP^{(G\text{-odd})}(x) = p^2(T(x))^{(G\text{-odd})} = \frac{1}{4} [p'(x)]^2 (\hat{T}(y))^{(G\text{-odd})}.
\]
Now the r.h.s. does no more depend on \(y\) but is a function of \(x\) alone, so the above argument applies to show that \(pP^{(G\text{-odd})}(x) =: \hat{P}\) is an entire function and thus a polynomial in \(x\). We have \(p|\hat{P}|\)

\[
\frac{\hat{P}}{y} = yP^{(G\text{-odd})}(x) = y[p]² \langle \hat{T}(y) \rangle^{(G\text{-odd})}
\]

is holomorphic in \(y\) about \(p = 0\). Since \(\hat{P}\) is a polynomial in \(x\), and \(p\) has no multiple zeros, we must actually have \(y^2 = p\) divides \(\hat{P}\). This proves that \(P^{(G\text{-odd})}\) is a polynomial in \(x\). The statement about the degree follows from part 1.

3. This is a consequence of the Riemann Roch Theorem. Let us choose a different approach here: W.l.o.g. \(n\) is even. We show the following:

(a) Let \(g \geq 1\). Then the space of \(\langle T(x) \rangle^{(G\text{-even})}\) has dimension \(2g - 1\).

(b) Let \(g \geq 2\). Then the space of \(\langle T(x) \rangle^{(G\text{-odd})}\) has dimension \(g - 2\).

In the expression for \(\langle T(x) \rangle^{(G\text{-even})}\), only \(P^{(G\text{-even})}\) is unknown. Since \(X\) is non-degenerate, we have

\[
P^{(G\text{-even})} = \langle 1 \rangle \sum_{i=1}^{n} b_i^{(G\text{-even})} \frac{1}{(x - x_i)} = \langle 1 \rangle \sum_{i=1}^{2g+2} b_i^{(G\text{-even})} \left( \frac{1}{x} + \frac{x_i}{x^2} + \frac{x^2}{x^3} + \ldots \right),
\]

where \(b_1^{(G\text{-even})}, \ldots, b_n^{(G\text{-even})} \in \mathbb{C}\). On the other hand, \(\langle T(x) \rangle \sim x^{-4}\). (This is true for \(x \to \infty\). However, \(X\) is a closed surface, so \(\langle T(x) \rangle = O(x^{-4})\) close to any ramification point.) The terms of order \(> -4\) in the equation for \(\langle T(x) \rangle^{(G\text{-even})}\) must drop out. This yields three equations for the set of \(n\) coefficients \(b_i^{(G\text{-even})}\), and claim (iii) follows. A similar argument works for \(\langle T(x) \rangle^{(G\text{-odd})}\), claim (iv). Since \(y^2 = p \sim x^{2g+2}\), we have

\[
\frac{1}{x^4} \sim \frac{y}{x^{g+5}}.
\]

This yields \(g + 4\) conditions on the \(n\) coefficients \(b_i^{(G\text{-odd})}\).

\[\square\]

5 Calculation of the 2-point function

For the polynomial \(P = P^{(G\text{-even})} + yP^{(G\text{-odd})}\) defined by eqs (7) and (8), we set

\[
P^{(G\text{-even})} = \sum_{k=0}^{n-2} A_k x^{n-2-k}, \quad A_k \in \mathbb{C}.
\]

It will be convenient to replace \(P^{(G\text{-even})}(x) =: -\hat{\Pi}(x)\) for which we introduce even polynomials \(\Pi^{(\text{even})}\) and \(\Pi^{(\text{odd})}\) such that

\[
\Pi(x) =: \Pi^{(\text{even})}(x) + x\Pi^{(\text{odd})}(x).
\]

Likewise, there are even polynomials \(p^{(\text{even})}\) and \(p^{(\text{odd})}\) such that

\[
p(x) = p^{(\text{even})}(x) + xp^{(\text{odd})}(x).
\]
Lemma 8. For the polynomials introduced by (14), we have
\[ p^{\text{even}}(x_1) + p^{\text{even}}(x_2) \]
\[ = 2p^{\text{even}}(\sqrt{x_1x_2}) + (x_1 - x_2)^2 \frac{1}{4} \left( \frac{p^{\text{even}}(\sqrt{x_1x_2})}{\sqrt{x_1x_2}} + p^{\text{even}}(\sqrt{x_1x_2}) \right) \]
\[ + O((x_1 - x_2)^4), \]
\[ x_1p^{\text{odd}}(x_1) + x_2p^{\text{odd}}(x_2) \]
\[ = (x_1 + x_2) \left\{ p^{\text{odd}}(\sqrt{x_1x_2}) + (x_1 - x_2)^2 \frac{1}{8} \left( \frac{3p^{\text{odd}}(\sqrt{x_1x_2})}{\sqrt{x_1x_2}} + p^{\text{odd}}(\sqrt{x_1x_2}) \right) \right\} \]
\[ + O((x_1 - x_2)^4). \]

Note that the polynomials \( p^{\text{even}}, p^{\text{odd}}, p^{\text{even}}' \) and \( p^{\text{odd}}'' \) in \( \sqrt{x_1x_2} \) are actually polynomials in \( x_1x_2 \).

Proof. Direct computation. The calculation can be shortened by using
\[ x_1 = (1 + \varepsilon)x, \]
\[ x_2 = (1 - \varepsilon)x, \]
where \( \varepsilon > 0. \]

Abusing notations, for \( j = 1, 2 \), we shall write \( p_j = p(x_j) \) and \( P_j = P(x_j, y_j) \). For any \( k \geq 0 \) and any rational function \( R(x_1, x_2) \) of \( x_1 \) and \( x_2 \), we denote by \( [R(x_1, x_2)]_k \) and \( [R(x_1, x_2)]_k^c \) the projection of \( R(x_1, x_2) \) onto the part of degree strictly larger than \( k \) in \( x_1 \) and \( x_2 \), respectively, at infinity.

Theorem 2. (The Virasoro two-point function)

Let \( X_g \) be the hyperelliptic surface
\[ X : \quad y^2 = p(x), \]
where \( p \) is a polynomial of degree \( \deg p = n = 2g + 1 \), or \( n = 2g + 2 \), and \( g \geq 1 \).

1. For \( n \) odd,
\[ \langle T(x_1)T(x_2) \rangle_{c} p_1p_2 = O(x_1^{n-3}). \quad (15) \]

2. The connected two-point function of the Virasoro-field is given by
\[ \langle T(x_1)T(x_2) \rangle_{c} p_1p_2 - \langle 1 \rangle^{-1} \langle T(x_1) \rangle \langle T(x_2) \rangle p_1p_2 \]
\[ = R(x_1, x_2) + O(1)|_{x_1=x_2}, \]
where \( R(x_1, x_2) \) is a rational function of \( x_1, x_2 \) and \( y_1, y_2 \). \( O(1)|_{x_1=x_2} \) denotes terms that are regular at \( x_1 = x_2 \) in the finite region.
More specifically, we have

\[ R(x_1, x_2) = \frac{c(1)}{4} \frac{p_1 p_2}{(x_1 - x_2)^4} + \frac{c}{4} y_1 y_2 (1) \left( \frac{p_{(\text{even})}^\prime \left( \sqrt{x_1 x_2} \right)}{(x_1 - x_2)^2} + \frac{1}{2} (x_1 + x_2) \frac{p_{(\text{odd})}^\prime \left( \sqrt{x_1 x_2} \right)}{(x_1 - x_2)^2} \right) \]

\[ + \frac{c}{32} y_1 y_2 (1) \left( \frac{p_{(\text{even})}'' \left( \sqrt{x_1 x_2} \right)}{(x_1 - x_2)^2} + \frac{3}{2} (x_1 + x_2) \frac{p_{(\text{odd})}'' \left( \sqrt{x_1 x_2} \right)}{(x_1 - x_2)^2} \right) \]

\[ + \frac{1}{8} (p_{(\text{odd})})_{2} \left( \frac{y_1 p_{(\text{odd})}}{x_1 - x_2} \right) \left( \frac{p_{(\text{even})}}{(x_1 - x_2)^2} + \frac{1}{2} (x_1 + x_2) \frac{p_{(\text{odd})} \left( \sqrt{x_1 x_2} \right)}{(x_1 - x_2)^2} \right) \]

\[ + \frac{c}{32} y_1 y_2 (1) \left( \frac{\Pi_{(\text{even})} \left( \sqrt{x_1 x_2} \right)}{(x_1 - x_2)^2} + \frac{1}{2} (x_1 + x_2) \frac{\Pi_{(\text{odd})} \left( \sqrt{x_1 x_2} \right)}{(x_1 - x_2)^2} \right). \]

3. When \( n \) is odd,

\[ \langle T(x_1) T(x_2) \rangle p_1 p_2 - (1) \langle T(x_1) \rangle \langle T(x_2) \rangle p_1 p_2 \]

\[ = R(x_1, x_2) \]

\[ - \frac{1}{8} a_0 \left( x_1^{n - 2} P_2 + x_2^{n - 2} P_1 \right) - \frac{c}{64} (1) (n^2 - 1) a_0^2 x_1^{n - 2} x_2^{n - 2} \]

\[ - \frac{1}{8} y_1 a_1 x_1^{n - 2} x_2^{n - 2} p_2^{(\text{odd})} \]

\[ - \frac{1}{16} y_1 a_0 x_1^{n - 2} x_2^{n - 2} p_1^{(\text{odd})} \]

\[ - \frac{1}{16} y_1 a_0 x_1^{n - 2} x_2^{n - 2} p_2^{(\text{odd})} \]

\[ - \frac{1}{16} y_1 a_0 x_1^{n - 2} x_2^{n - 2} p_1^{(\text{odd})} \]

\[ + P_{(0)}(x_1, x_2) + y_1 P_{(1)}(x_1, x_2) + y_2 P_{(2)}(x_1, x_2) + y_1 y_2 P_{(1,2)}(x_1, x_2). \]

Here \( P_{(0)} \), \( P_{(1,2)} \) and for \( i = 1, 2 \), \( P_{(i)} \) are polynomials in \( x_1 \) and \( x_2 \) with

\[ \text{deg } P_{(0)} = n - 3 \text{ in each } x_1, x_2, \]

\[ \text{deg } P_{(1)} = \frac{n - 1}{2} - 3 \text{ in } x_1, \text{ deg } P_{(1)} = n - 3 \text{ in } x_2, \]

\[ \text{deg } P_{(2)} = n - 3 \text{ in } x_1, \text{ deg } P_{(2)} = \frac{n - 1}{2} - 3 \text{ in } x_2, \]

\[ \text{deg } P_{(1,2)} = \frac{n - 1}{2} - 3 \text{ in each } x_1, x_2. \]

Moreover, \( P_{(0)} \), \( P_{(1,2)} \) and \( y_1 P_{(1)} + y_2 P_{(2)} \) are symmetric under flipping \( 1 \leftrightarrow 2 \). These polynomials are specific to the state.
Proof. 1. We have

\[ \langle T(x_1)T(x_2) \rangle_{p_1p_2} = \langle T(x_1)T(x_2) \rangle_{p_1p_2}^{n-2} + \langle T(x_1)T(x_2) \rangle_{p_1p_2}^{\leq n-3}, \]

where according to (9),

\[ \langle T(x_1)T(x_2) \rangle_{p_1p_2}^{n-2} = \frac{c}{32} a_0 x^{n-2} \langle T(x_2) \rangle_{p_2} \]

\[ = (1)^{-1} \langle T(x_1) \rangle \langle T(x_2) \rangle_{p_1p_2}^{n-2}, \] (18)

so

\[ \langle T(x_1)T(x_2) \rangle_{p_1p_2} - (1)^{-1} \langle T(x_1) \rangle \langle T(x_2) \rangle_{p_1p_2} \]

\[ = \langle T(x_1)T(x_2) \rangle_{p_1p_2}^{n-3} - (1)^{-1} \langle T(x_1) \rangle \langle T(x_2) \rangle_{p_1p_2}^{\leq n-3}. \]

This shows (15).

2. The proof is constructive. We build up a candidate and correct it subsequently so as to

- match the singularities prescribed by the OPE,
- behave at infinity according to (15),
- be holomorphic in the appropriate coordinates away from the locus where two positions coincide. X is covered by the coordinate patches \( \{ p \neq 0 \}, \{ p' \neq 0 \} \) and \( \{ |x^{-1}| < \epsilon \} \).

The two-point function is meromorphic on X whence rational. So once the singularities are fixed it is clear that we are left with the addition of polynomials as the only degree of freedom. The key ingredient is the use of the rational function

\[ \frac{y_1 + y_2}{x_1 - x_2} \]

which has a simple pole at \( x_1 = x_2 \) as \( y_1 = y_2 \neq 0 \), and is regular for \( (x_1, y_1) \) close to \( (x_2, -y_2) \).

(a) For finite and fixed but generic \( x_2 \), we have

\[ \frac{c}{32} \frac{1}{p_1p_2} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^4 = \frac{c/2}{(x_1 - x_2)^2} + \frac{c}{16} \frac{[p_2']^2}{p_2(x_1 - x_2)^2} + O(1), \]

where \( O(1) \) includes all terms regular at \( x_1 = x_2 \). Now

\[ \frac{1}{4} \frac{1}{p_1p_2} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2 = \frac{1}{p_2(x_1 - x_2)^2} + O((x_1 - x_2)^{-1}), \] (19)

Thus we make an error of order \( \sim (x_1 - x_2)^{-1} \) only if we replace the term \( \sim (x_1 - x_2)^{-2} \) in the previous expansion by

\[ \frac{c}{16} \frac{[p_2']^2}{4p_1p_2} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2, \]
and symmetrization lifts the error to order $O(1)$. Thus

\[
\frac{c/2}{(x_1 - x_2)^2} + O(1)
\]

\[
= \frac{c}{32 p_1 p_2} \left\{ \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^4 - \frac{1}{4} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2 \left( \frac{[p_1']^2}{p_1} + \frac{[p_2']^2}{p_2} \right) \right\} + O(1). \tag{20}
\]

(b) By eq. (19), we may replace the term $\sim (x_1 - x_2)^{-2}$ in the two-point function obtained from the OPE (3) by

\[
\frac{1}{4} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2 \left\{ \frac{T(x_1)}{p_2} + \frac{T(x_2)}{p_1} \right\}. \tag{21}
\]

The distribution of $p_1, p_2$ is at this stage arbitrary but will be justified in part (23).

(c) $(T(x_1)T(x_2))p_1p_2$ diverges where one of $x_1$ and $x_2$ is not a good coordinate. From (23) and (21) follows

\[
pT(x) = p[y']^2 T(y) + \frac{c}{12} pS(p) + \frac{c [p']^2}{32 p} = \frac{c [p']^2}{32 p} + \text{terms regular where } p = 0.
\]

Thus we eliminate any such singularity by considering the connected two-point function, unless it occurs together with a singularity as $x_1$ and $x_2$ coincide. This happens in eq. (20). However, adding the term (21) removes the singularity at $p_1 = 0$, by eq. (7).

(d) We conclude that in the region where $x_1$ and $x_2$ are finite, we have

\[
\langle 1 \rangle (T(x_1)T(x_2))c p_1p_2
\]

\[
= \langle T(x_1)T(x_2) \rangle p_1p_2 - (1)^{-1} \langle T(x_1) \rangle \langle T(x_2) \rangle p_1p_2
\]

\[
= R^0(x_1, x_2) + O(1) \big|_{x_1 = x_2}, \tag{22}
\]

where

\[
R^0(x_1, x_2) := \frac{c}{32} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^4 + \frac{1}{16} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2 (1)^{-1} (P_1 + P_2). \tag{23}
\]

and all $O(1)|_{x_1 = x_2}$ terms are polynomials.

(e) All terms in (22) which do not comply with (15) must be subtracted. The actual procedure is acted out in part 3 below. In order to reduce the number of correction terms, we shall reformulate the singular part of the two-point function in the finite region by reducing the order of the terms involved whilst keeping their singularities. Since formula
We have

\[ R^{(0)}(x_1, x_2) = \frac{c}{32} \left( \frac{(p_1 - p_2)^2}{(x_1 - x_2)^4} + \frac{c}{8} \frac{y_1 y_2 (x_1 - x_2)^2}{(x_1 - x_2)^4} + \frac{c}{4} \frac{p_1 p_2}{(x_1 - x_2)^3} \right) + \frac{1}{16} \frac{p_1 + 2y_1 y_2 + p_2}{(x_1 - x_2)^2} \left( P^{(\text{even})}_1 + P^{(\text{even})}_2 \right) + \frac{1}{16} \frac{p_1 + 2y_1 y_2 + p_2}{(x_1 - x_2)^2} \left( y_1 P^{(\text{odd})}_1 + y_2 P^{(\text{odd})}_2 \right). \]

The first term on the r.h.s. is replaced by a new term of milder divergency when we subtract \( \frac{c}{16} \) times the symmetric polynomial

\[ \frac{(p_1 - p_2)^2}{(x_1 - x_2)^4} = \frac{p_1^2 + p_2^2}{(x_1 - x_2)^4}. \]

The terms \( \alpha (x_1 - x_2)^2 \) are dealt with as follows: Let \( a, b \) be polynomials in one variable. Then we have

\[ \frac{a_1 b_1 + a_2 b_2}{(x_1 - x_2)^2} = \frac{a_1 b_2 + a_2 b_1}{(x_1 - x_2)^2} + \text{polynomial}. \]

In the present situation, \( y \) generalises to terms including \( y_i \) as follows: We have

\[ \frac{y_1 P^{(\text{odd})}_1 + y_2 P^{(\text{odd})}_2}{(x_1 - x_2)^2} = \frac{y_1 P^{(\text{odd})}_1 + y_2 P^{(\text{odd})}_2}{(x_1 - x_2)^2} + \frac{p_1 - p_2}{x_1 - x_2} \left( \frac{P^{(\text{odd})}_1}{x_1 - x_2} - \frac{P^{(\text{odd})}_2}{x_1 - x_2} \right) \frac{1}{y_1 + y_2}, \]

so

\[ \frac{1}{16} \frac{p_1 + p_2}{(x_1 - x_2)^2} \left( y_1 P^{(\text{odd})}_1 + y_2 P^{(\text{odd})}_2 \right) + \frac{1}{8} \frac{y_1 y_2}{(x_1 - x_2)^2} \left( y_1 P^{(\text{odd})}_1 + y_2 P^{(\text{odd})}_2 \right) \]

\[ = \frac{1}{16} \frac{p_1 + p_2}{(x_1 - x_2)^2} \left( y_1 P^{(\text{odd})}_1 + y_2 P^{(\text{odd})}_1 \right) + \frac{1}{8} \frac{y_1 y_2}{(x_1 - x_2)^2} \left( y_1 P^{(\text{odd})}_2 + y_2 P^{(\text{odd})}_1 \right) + \frac{1}{16} \frac{y_1 + y_2}{(x_1 - x_2)^2} \left( \frac{p_1 - p_2}{x_1 - x_2} \frac{P^{(\text{odd})}_1}{x_1 - x_2} - \frac{P^{(\text{odd})}_2}{x_1 - x_2} \right). \]

The term \( \sim y_1 y_2 \) fits well with that in line \( (24) \) and provides \( (17) \), while the last summand is absorbed in a redefinition of \( y_1 P^{(1)}(x_1, x_2) + y_2 P^{(2)}(x_1, x_2) \) (with too high order terms cut off).
To lower the number of correction terms, we shall make use of the even polynomials $p^{(\text{even})}$ and $p^{(\text{odd})}$ introduced in \([14]\) and the preceding Lemma. Thus we shall replace

\[
\frac{c}{8} y_1 y_2 (1) \frac{p_1 + p_2}{(x_1 - x_2)^4} = \frac{c}{8} y_1 y_2 (1) \frac{p^{(\text{even})}(\sqrt{x_1 x_2})}{(x_1 - x_2)^4} + \frac{c}{8} y_1 y_2 (1) \frac{p^{(\text{odd})}(\sqrt{x_1 x_2})}{(x_1 - x_2)^4}
\]

Similarly, to lower the degree of the term $\sim y_1 y_2 (x_1 - x_2)^{-2} (P_1^{(\text{even})} + P_2^{(\text{odd})})$ in the two-point function, we make use of the even polynomials $\Pi^{(\text{even})}$ and $\Pi^{(\text{odd})}$ introduced by \([13]\). Then

\[
16 (x_1 - x_2)^2 \left( y_1 P_2^{(\text{odd})} + y_2 P_1^{(\text{odd})} \right) = \frac{1}{8} \left( P_1^{(\text{even})}(\sqrt{x_1 x_2}) + \frac{1}{2} (x_1 + x_2) P_1^{(\text{odd})}(\sqrt{x_1 x_2}) \right) \left( y_1 P_2^{(\text{odd})} + y_2 P_1^{(\text{odd})} \right)
\]

Likewise, to lower the degree of the term $\sim y_1 y_2 (x_1 - x_2)^{-2} (P_1^{(\text{even})} + P_2^{(\text{odd})})$ in the two-point function, we make use of the even polynomials $\Pi^{(\text{even})}$ and $\Pi^{(\text{odd})}$ introduced by \([13]\). Then

\[
\frac{1}{8} y_1 y_2 \frac{p^{(\text{even})}}{(x_1 - x_2)^2} = \frac{c}{32} y_1 y_2 \left( \frac{\Pi^{(\text{even})}(\sqrt{x_1 x_2})}{(x_1 - x_2)^2} + \frac{1}{2} (x_1 + x_2) \Pi^{(\text{odd})}(\sqrt{x_1 x_2}) \right)
\]

Note that the $O(1)$ terms in \([25]\), \([29]\) and \([30]\) are all polynomials in $x_1, x_2$ and $y_1, y_2$.

Formula \([23]\) with the replacements performed using \([25]\), \([28]\), \([26]\), \([27]\) and \([30]\) yields the singular part $R(x_1, x_2)$ of the claimed formula.

3. We first subtract all terms from $R$ which are of non-admissible order in $x_1$. These depend polynomially on $x_2$ because this is true for $[(x_1 - x_2)^{-\ell}]_{\ell \in \mathbb{N}, \ k \in \mathbb{Z}, \ (x_1 \text{ large})}$, and may depend on $y_2$. From the result
we subtract all terms of order \(> n - 3\) in \(x_2\). Thus the corrected rational function reads

\[
R = [R]_{>n-3} - [R - [R]_{>n-3}]_{>n-3}
\]

\[
= R - [R]_{>n-3} - [R]_{>n-3} + [[R]_{>n-3}]_{>n-3},
\]

where we must have

\[
[[R]_{>n-3}]_{>n-3} = [R]_{>n-3}. \tag{31}
\]

In addition we allow for a symmetric contribution of the form

\[
P^{(0)}(x_1, x_2) + y_1 P^{(1)}(x_1, x_2) + y_2 P^{(2)}(x_1, x_2) + y_1 y_2 P^{(1,2)}(x_1, x_2)
\]

which is specific to the state. The degree and symmetry requirements for the \(P^{(i)}\) are immediate (noting that \(\frac{2}{2} = 1\) is an integer).

In the following we list the correction terms:

\[
- \frac{1}{8} P_2 \left[ \frac{p_1}{(x_1 - x_2)^2} \right]_{>n-3} - \frac{1}{8} p_2 \left[ \frac{p_2}{(x_1 - x_2)^2} \right]_{>n-3} \tag{32}
\]

\[
+ \frac{1}{8} P_2^{(even)} \left[ \frac{p_1}{(x_1 - x_2)^2} \right]_{>n-3} - \frac{1}{8} y_2 P_2^{(odd)} \left[ \frac{p_1}{(x_1 - x_2)^2} \right]_{>n-3} \tag{33}
\]

In addition, we have (we only list the \(-y_1 [R]_{>n-3}\) contribution here):

\[
- \frac{3c}{64} y_1 y_2 \left[ x_1 (1) \frac{\sqrt{x_1 x_2} a^{(odd)'}(\sqrt{x_1 x_2})}{(x_1 - x_2)^2} \right]_{>n-3} \tag{34}
\]

\[
+ \frac{1}{8} y_1 P_2^{(even)} \left[ \frac{p_1}{(x_1 - x_2)^2} \right]_{>n-3} \tag{35}
\]

\[
+ \frac{1}{16} y_1 P_2^{(odd)} \left[ x_1 \frac{p_1}{(x_1 - x_2)^2} \right]_{>n-3} \tag{36}
\]

\[
+ \frac{1}{16} y_1 x_2 P_2^{(odd)} \left[ \frac{p_1}{(x_1 - x_2)^2} \right]_{>n-3} \tag{37}
\]

\[
- \frac{c}{64} y_1 y_2 \left[ x_1 (1) \frac{\sqrt{x_1 x_2}}{(x_1 - x_2)^2} \right]_{>n-3} \tag{38}
\]

\[
+ \frac{c}{64} y_1 y_2 \left[ x_1 \frac{\Pi^{(odd)}(\sqrt{x_1 x_2})}{(x_1 - x_2)^2} \right]_{>n-3} \tag{39}
\]

Terms originating from (17):

When \(n\) is odd, then \(A_0 = -\frac{c}{8} (n^2 - 1)a_0(1)\), so (32) and (33) yield

\[
- \frac{1}{8} a_0 \left( x_1^{n-2} P_2 + x_2^{n-2} P_1 \right) - \frac{c}{64} (n^2 - 1)a_0^{2} x_1^{n-2} x_2^{n-2}.
\]
Terms originating from (27):

(35) yields
\begin{align*}
&\frac{1}{8} y_1 a_1 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} - \frac{1}{2} P_2^{(\text{odd})} + \frac{1}{8} y_2 a_1 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} - \frac{1}{2} P_1^{(\text{odd})}.
\end{align*}

(36) yields:
\begin{align*}
&\frac{1}{16} y_1 a_0 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} - \frac{1}{2} P_2^{(\text{odd})} + \frac{1}{16} y_2 a_0 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} - \frac{1}{2} P_1^{(\text{odd})} \\
&\frac{1}{8} y_1 a_0 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} + \frac{1}{2} P_2^{(\text{odd})} + \frac{1}{8} y_2 a_0 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} + \frac{1}{2} P_1^{(\text{odd})} \\
&\frac{1}{16} y_1 a_2 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} - \frac{1}{2} P_2^{(\text{odd})} + \frac{1}{16} y_2 a_2 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} - \frac{1}{2} P_1^{(\text{odd})}.
\end{align*}

(37) yields:
\begin{align*}
&\frac{1}{16} y_1 a_0 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} - \frac{1}{2} P_2^{(\text{odd})} + \frac{1}{16} y_2 a_0 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} - \frac{1}{2} P_1^{(\text{odd})}.
\end{align*}

Terms originating from (28):

(34) and (38) yield
\begin{align*}
\frac{c}{64} y_1 y_2 (n^2 - 1) a_0 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} - \frac{1}{2} x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} - \frac{1}{2} = -\frac{1}{8} y_1 y_2 A_0 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} - \frac{1}{2} x_1^{\frac{n}{2}} x_2^{\frac{n}{2}}. \quad (40)
\end{align*}

Term originating from (30):

(39) yields
\begin{align*}
\frac{1}{8} y_1 y_2 A_0 x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} - \frac{1}{2} x_1^{\frac{n}{2}} x_2^{\frac{n}{2}} + \frac{1}{2}.
\end{align*}

The term cancels against (40).

4. (15) determines the degree of the polynomials $P^{(0)}(x_1, x_2)$, $P^{(1)}(x_1, x_2)$, $P^{(2)}(x_1, x_2)$ and $P^{(1,2)}(x_1, x_2)$; the symmetry requirements are immediate.
5.1 Application to the $(2, 5)$ minimal model, for $n = 5$

In Section 3 we introduced the normal ordered product

$$N_0(\varphi_1, \varphi_2)(x_2) = \lim_{x_1 \to x_2} [\varphi_1(x_1)\varphi_2(x_2)]_{\text{reg.}}$$

of two fields $\varphi_1, \varphi_2$, where $[\varphi_1(x_1), \varphi_2(x_2)]_{\text{reg.}}$ is the regular part of the OPE for $\varphi_1, \varphi_2$. In particular, $\langle N_0(T, T)(x) \rangle$ can be determined from Theorem 2.3.

Lemma 9. $N_0(T, T) \propto \partial^2 T$ implies $c = -\frac{22}{5}$ and

$$N_0(T, T)(x) = \frac{3}{10} \partial^2 T(x). \quad (41)$$

Proof. The statement is local, so we may assume w.l.o.g. $g = 1$. In this case,

$$P^{(\text{even})} = -4c x(1) + A_1, \quad P^{(\text{odd})} = 0,$$

by Theorem 1.(2b). Using Corollary 6 and the transformation rule (5), we find

$$\langle T(x) \rangle = \frac{c}{32} \frac{[p']^2}{p^2}(1) - c(1) \frac{x}{p} + \frac{T}{p},$$

where by (7), $\langle T \rangle = \frac{1}{4}$. Direct computation shows that

$$\langle N_0(T, T)(x) \rangle = \alpha \partial^2(T(x))$$

iff $\alpha = \frac{3}{10}$ and $c = -\frac{22}{5}$. Since by assumption the two fields are proportional to one another, the claim follows.

The aim of this section is to determine at least some of the constants in the Virasoro two-point function in the $(2, 5)$ minimal model for $g = 2$.

We will restrict our considerations to the case when $n$ is odd. (Though we know more about $P^{(G-\text{even})}$ when $n$ is even this knowledge doesn’t actually provide more information, it just leads to longer equations.)

For $n = 5$, all Galois-odd terms are absent. Specializing to Galois-even terms, condition (41) reads as follows:
Lemma 10. In the $(2,5)$ minimal model for $g \geq 1$, we have

\[
\frac{7c}{10 \cdot 64} \left( \frac{p''}{p^2} \right)^2 - \frac{7c}{15 \cdot 64} \left( \frac{p'''}{p^2} \right) + \frac{c}{24 \cdot 64} \frac{p^{(4)}}{p} \\
+ \frac{1}{5 \cdot 4} \frac{p''}{p^2} p^{(G\text{-even})} + \frac{3}{5 \cdot 16} \frac{p'}{p^2} p^{(even)'} - \frac{3}{5 \cdot 32} \frac{p^{(even)''}}{p} \\
- \frac{1}{16} \left( \frac{p^{(even)^2}}{p^2} + \frac{p^{(odd)^2}}{p} \right) \\
+ \frac{1}{4a_0} \frac{1}{p^2} \left( \Pi^{(even)'} + x \Pi^{(odd)'} \right) \\
- \frac{c}{8 \cdot 32} \frac{1}{xp} \left( \Pi^{(even)'} + x \Pi^{(odd)'} \right) \\
- \frac{c}{64} \frac{1}{xp} \left( - \frac{1}{4} p^{(3)} \right) \\
- \frac{1}{8} \left( \frac{p^{(even)''}}{x} - \frac{p^{(odd)''}}{x} \right) \\
+ \frac{1}{8x} \left( \frac{p^{(even)''}}{x} + 5p^{(odd)'} \right) \\
= \frac{P^{(0)}(x,x)}{p^2} + \frac{P^{(1,2)}(x,x)}{p}.
\]

In particular, the terms on the l.h.s. of order $-k$ for $k = 4,5$ drop out.

Note that the equation makes good sense since the l.h.s. is regular at $x = 0$. For instance, $\Pi^{(even)'}$ is an odd polynomial of $x$, so its quotient by $x$ is regular.

Proof. Direct computation. Since according to Theorem 2

\[ \deg P^{(0)}(x,x) = 2n - 6, \quad \deg P^{(1,2)}(x,x) = n - 7, \]

the terms on the l.h.s. of order $-k$ for $k = 4,5$ must drop out.

Example 3. We consider $n = 5$. Here

\[ \deg P^{(0)}(x,x) = 4, \]

and $P^{(1,2)}(x,x)$ is absent. Thus we have 5 degrees of freedom. One of them is the complex number $(1)$, and according to Theorem 13 at most 3 of them are given by $(T(x))$. Set

\[
P^{(0)}(x_1, x_2) = B_{2,2}x_1^2x_2^2 \\
+ B_{2,1}(x_1^2x_2 + x_1x_2^2) \\
+ B_2(x_1^3 + x_2^3) + B_{1,1}x_1x_2 \\
+ B_1(x_1 + x_2) \\
+ B_0, \quad B_0, B_1, B_{i,j} \in \{1\} \mathbb{C}, \ i, j = 1, 2.
\]

The additional constraint provides knowledge of

\[ P^{(0)}(x, x) = B_2x_2x_1^2 + 2B_{2,1}x_1^3 + (2B_2 + B_{1,1})x_2^2 + 2B_1x + B_0 \]
only, so we are left with one unknown. We will see later that all constants can be fixed using \([41]\) when the three-point function is taken into account.

## 6 The connected Virasoro \(N\)-point function

Let \(X\) be the hyperelliptic surface

\[
X : \ y^2 = p(x).
\]

We now give a recursive definition of the connected \(N\)-point function of fields \(\varphi_1, \ldots, \varphi_N\).

### Definition 11.

For \(N \geq 1\), we denote by \(\langle \varphi_1(x_1) \ldots \varphi_N(x_N) \rangle\) the **connected \(N\)-point function** of the fields \(\varphi_1, \ldots, \varphi_N\). It is defined recursively by

\[
(1)^{-1}\langle \varphi_1(x_1) \ldots \varphi_N(x_N) \rangle = \sum_{\text{partition of }\{1, \ldots, N\}} \prod_{r=0}^{m-1} \langle \varphi_{s_{2r+1}}(x_{s_{2r+1}}) \ldots \varphi_{s_{2r+1}}(x_{s_{2r+1}}) \rangle_c.
\]

(42)

For \(N = 1\) and \(N = 2\), we have

\[
\langle \varphi(x) \rangle_c = (1)^{-1}\langle \varphi(x) \rangle,
\]

\[
\langle \varphi_1(x_1)\varphi_2(x_2) \rangle_c = (1)^{-1}\langle \varphi_1(x_1)\varphi_2(x_2) \rangle - (1)^{-2}\langle \varphi_1(x_1) \rangle \langle \varphi_2(x_2) \rangle,
\]

respectively. Thus \(\langle T(x_1)T(x_2) \rangle\) has a fourth order pole at \(x_1 = x_2\) which comes from the OPE (3) for \(\langle T(x_1)T(x_2) \rangle\), and no further such pole.

### Theorem 3.

Let \(N \geq 3\). Then the connected \(N\)-point function \(\langle T(x_1) \ldots T(x_N) \rangle_c\) has no fourfold pole.

**Proof.** (by induction) We show that for \(N = 3\), \(\langle T(x_1) \ldots T(x_3) \rangle_c\) has no fourfold pole at \(x_1 = x_2\):

\[
\langle T(x_1)T(x_2)T(x_3) \rangle_c = (1)^{-1}\langle T(x_1)T(x_2)T(x_3) \rangle - \langle T(x_1)T(x_2) \rangle_c \langle T(x_3) \rangle_c
\]

\[
- \langle T(x_2)T(x_3) \rangle_c \langle T(x_1) \rangle_c
\]

\[
- \langle T(x_3)T(x_1) \rangle_c \langle T(x_2) \rangle_c - \langle T(x_1) \rangle_c \langle T(x_2) \rangle_c \langle T(x_3) \rangle_c,
\]

so the fourth order pole of \(\langle T(x_1)T(x_2)T(x_3) \rangle\) at \(x_1 = x_2\) cancels against that of \(\langle T(x_1)T(x_2) \rangle_c \langle T(x_3) \rangle_c\).

As the inductive hypothesis, all \(i\)-point functions for \(2 < i < k\) have no fourth order pole at \(x_1 = x_2\). The term \(\langle T(x_1)T(x_2) \rangle_c \langle T(x_3) \ldots T(x_{k+1}) \rangle_c\) makes the fourth order singularity of \(\langle T(x_1) \ldots T(x_{k+1}) \rangle_c\) at \(x_1 = x_2\) drop out. But all other connected \(i\)-point functions containing the pair \(T(x_1)\) and \(T(x_2)\) have \(i \leq k\).

While it has no fourth order pole, \(\langle T(x_1) \ldots T(x_N) \rangle_c\) \((N \geq 3)\) has second order poles. On \(X\), a pole at \(x_1 = x_2\) occurs only when \(y_1 = y_2\).
Lemma 12. For $N \geq 1$, on any chart of $X$ where $x$ is a good coordinate, we have the Galois splitting

$$\langle T(x_1) \ldots T(x_N) \rangle_c = R^{(G\text{-even})}(x_1) + y_1 R^{(G\text{-odd})}(x_1),$$

(43)

where $R^{(G\text{-even})}$ and $R^{(G\text{-odd})}$ are rational functions of $x_1$.

Proof. Eqs (10) and (42). \qed

6.1 The connected Virasoro 3-point function

Within this section, let $\langle T(x_1)T(x_2) \rangle_{\text{reg.}} + \langle T(x_1) \rangle_c \langle T(x_2) \rangle_c$ be the regular part of the OPE on $X$,

$$T(x_1)T(x_2) p_1 p_2 - \langle T(x_1) \rangle_c \langle T(x_2) \rangle_c p_1 p_2$$

$$= \frac{c}{32} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^4 1$$

$$+ \frac{1}{4} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2 \left\{ T(x_1) p_1 \frac{c}{32} \frac{[p_1]^2}{p_1^2} 1 + T(x_2) p_2 \frac{c}{32} \frac{[p_2]^2}{p_2^2} 1 \right\}$$

$$+ \langle T(x_1)T(x_2) \rangle_{\text{reg.}} p_1 p_2.$$ 

(44)

Theorem 4. Let $X_g$ be the hyperelliptic surface of genus $g \geq 1$, defined by

$$X : \quad y^2 = p(x),$$

where $p$ is a polynomial of degree $\deg p = n = 2g + 1$, or $n = 2g + 2$.

1. When $n$ is odd,

$$\langle T(x_1)T(x_2)T(x_3) \rangle_c p_1 p_2 p_3 = O(x_1^n - 3).$$

(46)

2. In the region where $x_1, x_2, x_3$ are finite, the connected Virasoro three-point function is given by

$$\langle T(x_1)T(x_2)T(x_3) \rangle_c p_1 p_2 p_3 = R^{(0)}(x_1, x_2, x_3) + O(1)_{|x_1, x_2, x_3},$$

where the $O(1)$ terms are polynomials in $x_1, x_2, x_3$ and $y_1, y_2, y_3$, and $R^{(0)}$
is the rational function of $x_1, x_2, x_3$ and $y_1, y_2, y_3$ given by

$$R^{(0)}(x_1, x_2, x_3) = \frac{c}{64} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2 \left( \frac{y_1 + y_3}{x_1 - x_3} \right)^2 \left( \frac{y_2 + y_3}{x_2 - x_3} \right)^2$$

$$+ \frac{1}{64} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2 \left( \frac{y_1 + y_3}{x_1 - x_3} \right)^2 \langle 1 \rangle^{-1}(P_2 + P_3)$$

$$+ \frac{1}{64} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2 \left( \frac{y_2 + y_3}{x_2 - x_3} \right)^2 \langle 1 \rangle^{-1}(P_1 + P_3)$$

$$+ \frac{1}{64} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2 \left( \frac{y_2 + y_3}{x_2 - x_3} \right)^2 \langle 1 \rangle^{-1}(P_1 + P_2)$$

$$+ \frac{1}{4} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2 \langle 1 \rangle^{-1}(\langle [T(x_1)T(x_3)]_{\text{reg.}} \rangle p_1 p_3$$

$$+ (1)^{-1}(\langle [T(x_2)T(x_3)]_{\text{reg.}} \rangle p_2 p_3)$$

$$+ \frac{1}{4} \left( \frac{y_1 + y_3}{x_1 - x_3} \right)^2 \langle 1 \rangle^{-1}(\langle [T(x_1)T(x_2)]_{\text{reg.}} \rangle p_1 p_2$$

$$+ (1)^{-1}(\langle [T(x_2)T(x_3)]_{\text{reg.}} \rangle p_2 p_3)$$

$$+ \frac{1}{4} \left( \frac{y_2 + y_3}{x_2 - x_3} \right)^2 \langle 1 \rangle^{-1}(\langle [T(x_1)T(x_2)]_{\text{reg.}} \rangle p_1 p_2$$

$$+ (1)^{-1}(\langle [T(x_1)T(x_3)]_{\text{reg.}} \rangle p_1 p_3)$$

3. For odd $n$,

$$(T(x_1)T(x_3))_{\text{reg.}} \langle p_1 p_2 p_3$$

$$= R^{(0)}(x_1, x_2, x_3)$$

$$- R^{(0)}_1 + R^{(0)}_{1,2}$$

$$- R^{(0)}_2 + R^{(0)}_{2,3}$$

$$- R^{(0)}_3 + R^{(0)}_{1,3} - R^{(0)}_{1,2,3}$$

$$+ P^{(0)}(x_1, x_2, x_3)$$

$$+ y_1 P^{(1)}(x_1, x_2, x_3) + y_2 P^{(2)}(x_1, x_2, x_3) + y_3 P^{(3)}(x_1, x_2, x_3)$$

$$+ y_1 y_2 P^{(1,2)}(x_1, x_2, x_3) + y_1 y_3 P^{(1,3)}(x_1, x_2, x_3) + y_2 y_3 P^{(2,3)}(x_1, x_2, x_3)$$

$$+ y_1 y_2 y_3 P^{(3)}(x_1, x_2)$$

Here for $i = 1, 2, 3$, and for any rational function $Q$ of $x_i$ and $y_i$,

$$Q_i := \pi_{n-3,i}(Q^{(G-\text{even})}) + y_i \pi_{n-3,i}(Q^{(G-\text{odd})})$$

where the Galois splitting refers to $x_i$, $\pi_{k,i}$ is the projection onto the part of order strictly larger than $k$ in $x_i$ as $x_i \to \infty$, and $Q_{i,j} := [Q_{ij}]$. $P^{(i)}$, $P^{(i,j)}$ ($i < j$), and $P^{(0)}$, $P^{(3)}$ are state-specific polynomials in $x_1, x_2, x_3$. 

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Proof. 1. By the argument (18), applied to the three-point function ,

\[ \deg P^{(0)} = n - 3 \quad \text{in each } x_1, x_2, x_3 \]

\[ \deg P^{(i)} = \frac{n-1}{2} - 3 \quad \text{in } x_i, \quad \deg P^{(j)} = n - 3 \quad \text{in } x_k, \quad k \neq i, \]

\[ \deg P^{(i,j)} = \frac{n-1}{2} - 3 \quad \text{in } x_i, x_j \quad \deg P^{(i,j)} = n - 3 \quad \text{in } x_k, \quad k \neq i, j \]

\[ \deg P^{(3)} = \frac{n-1}{2} - 3 \quad \text{in each } x_1, x_2, x_3. \]

Moreover, we have \( R_{13} = R_{31} \) and \( R_{123} = R_{231} = R_{312} \), and \( P^{(0)}, P^{(4)} \) as well as

\[ y_1 P^{(1)}(x_1, x_2, x_3) + y_2 P^{(2)}(x_1, x_2, x_3) + y_3 P^{(3)}(x_1, x_2, x_3), \]

and

\[ y_1 y_2 P^{(1,2)}(x_1, x_2, x_3) + y_1 y_3 P^{(1,3)}(x_1, x_2, x_3) + y_2 y_3 P^{(2,3)}(x_1, x_2, x_3) \]

are invariant under permutations of \( x_1, x_2, x_3 \).

Proof. 1. By the argument (13), applied to the three-point function,

\[ [(T(x_1)T(x_2)T(x_3))_{c} p_1 p_2 p_3]_{n-2} = 0. \]

2. We refer to general discussion in the proof of Theorem 2. In order to establish the singular part, we shall consider the locus \( x_1 = x_2 \) and symmetrize the resulting formula by adding terms that are \( O(1) \) but may be singular in other pairs of coordinates. So

\[ \langle T(x_1)T(x_2)T(x_3) \rangle_{c} p_1 p_2 p_3 = R^{(1)}(x_1, x_2, x_3) + O(1) \mid_{x_1 = x_2}, \]

where by (12) and (15),

\[ R^{(1)}(x_1, x_2, x_3) = c \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^{4} (T(x_3))_{c} p_3 \]

\[ + \frac{1}{4} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^{2} \left\{ (1)^{-1} \langle T(x_1)T(x_3) \rangle p_1 p_3 + (1)^{-1} \langle T(x_2)T(x_3) \rangle p_2 p_3 \right. \]

\[ - \frac{c}{32} \left( x_1 - x_2 \right)^{2} \langle T(x_3) \rangle_{c} p_3 - \frac{c}{32} \left( x_1 - x_2 \right)^{2} \langle T(x_3) \rangle_{c} p_3 \]

\[ - (1)^{-1} \langle T(x_1)T(x_2) \rangle \langle T(x_3) \rangle_{c} p_1 p_2 p_3. \]

The term \( \sim \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^{4} \) drops out. According to eq. (47), we have

\[ \langle T(x_1)T(x_2) \rangle p_1 p_2 - \langle T(x_1) \rangle_{c} \langle T(x_2) \rangle_{c} \langle 1 \rangle p_1 p_2 \]

\[ = c \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^{4} (1) + \frac{1}{16} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^{2} (P_1 + P_2) \]

\[ + \langle [T(x_1)T(x_2)]_{reg} \rangle p_1 p_2. \quad (47) \]
Using eq. (45), applied to the tensor product of the Virasoro field at the position \((x_1, x_3)\) and \((x_2, x_3)\), respectively, and eq. (7), we obtain

\[
R^{(i)}(x_1, x_2, x_3) = \frac{1}{4} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2 \left\{ \frac{c}{32} \left( \frac{y_1 + y_3}{x_1 - x_3} \right)^4 + \frac{c}{32} \left( \frac{y_2 + y_3}{x_2 - x_3} \right)^4 + \frac{1}{16} \left( \frac{y_1 + y_3}{x_1 - x_3} \right)^2 (1)^{-1}(P_1 + P_3) 
+ \frac{1}{16} \left( \frac{y_2 + y_3}{x_2 - x_3} \right)^2 (1)^{-1}(P_2 + P_3) 
+ (1)^{-1}([T(x_1)T(x_3)]_{\text{reg.}}) p_1 p_3 + (1)^{-1}([T(x_2)T(x_3)]_{\text{reg.}}) p_2 p_3 
+ \frac{1}{4} (1)^{-1}(P_1 + P_2)(T(x_1))_{c.p} \right\} 
- \frac{1}{16} \left( \frac{y_1 + y_2}{x_1 - x_2} \right)^2 (1)^{-1}\left( \frac{P_1 + P_2}{T(x_3)} \right)_{c.p},
\]

where the last two lines cancel out. Write \(P_2 = P_1 + O(x_1 - x_2)\) and add

\[
\frac{1}{64} \left( \frac{y_1 + y_3}{x_1 - x_3} \right)^2 \left( \frac{y_2 + y_3}{x_2 - x_3} \right)^2 (1)^{-1}(P_1 + P_2).
\]

Moreover, add

\[
\frac{1}{4} \left( \frac{y_1 + y_3}{x_1 - x_3} \right)^2 (1)^{-1}\left( \frac{[T(x_1)T(x_2)]_{\text{reg.}}}{\text{reg.}} p_1 p_2 + \langle [T(x_2)T(x_3)]_{\text{reg.}} \rangle p_2 p_3 \right), \\
\frac{1}{4} \left( \frac{y_2 + y_3}{x_2 - x_3} \right)^2 (1)^{-1}\left( \frac{[T(x_1)T(x_2)]_{\text{reg.}}}{\text{reg.}} p_1 p_2 + \langle [T(x_1)T(x_3)]_{\text{reg.}} \rangle p_1 p_3 \right).
\]

We replace

\[
\left( \frac{y_1 + y_3}{x_1 - x_3} \right)^4 + \left( \frac{y_2 + y_3}{x_2 - x_3} \right)^4 = 2 \left( \frac{y_1 + y_3}{x_1 - x_3} \right)^2 \left( \frac{y_2 + y_3}{x_2 - x_3} \right)^2 + O((x_1 - x_2)^2),
\]

where we may omit the \(O((x_1 - x_2)^2)\) term since the resulting expression is already symmetric in all coordinates. This yields the function \(R^{(0)}(x_1, x_2, x_3)\) defined in the claim.

3. The corrected rational function reads

\[
R^{(0)} - R^{(0)}_{1} = -[R^{(0)} - R^{(0)}_{1}]_2 = -[R^{(0)} - R^{(0)}_1] - [R^{(0)} - R^{(0)}_{1}]_2.
\]

The expression must be symmetric in all three variables. Since there is no preferred coordinate, requiring \(R^{(0)}_{1,3} = R^{(0)}_{3,1}\) ensures \(R^{(0)}_{i,j} = R^{(0)}_{j,i}\) for any \(i, j \in \{1, 2, 3\}\). Moreover, requiring \(R^{(0)}_{1,2,3} = R^{(0)}_{2,3,1} = R^{(0)}_{3,1,2}\) suffices to establish invariance under any permutation of indices. The symmetry requirements for the \(P^{(1)}_i, P^{(1)}(i,j) \ (i < j)\), and \(P^{(0)}_i, P^{(0)}_{(i)} \) are immediate. The degree requirements for all polynomials listed in the claim follow from (40).
6.2 Application to the $(2,5)$ minimal model, for $n = 5$

**Theorem 5.** We consider the $(2,5)$ minimal model on a genus $g = 2$ hyperelliptic Riemann surface

$$X : \quad y^2 = p(x).$$

There are exactly 4 parameters, given by \((1)\) and \((T(x))\), and all other constants in the two- and three-point function are known.

**Proof.** W.l.o.g. $n = 5$. In this case the two-point function in the $(2,5)$ minimal model has been determined previously, up to one constant, cf. Example 3. In the three-point function, only $P^{(0)}(x_1, x_2, x_3)$ is present. Set

$$P^{(0)}(x_1, x_2, x_3) = B_{2,2,2} x_1^2 x_2^2 x_3^2$$

$$+ B_{2,2,1}(x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3^2)$$

$$+ B_{2,1,1}(x_1^2 x_2 x_3 + x_1 x_2 x_3^2 + x_1 x_2^2 x_3)$$

$$+ B_{2,2,0}(x_1^2 x_2^2 + x_1^2 x_3^2 + x_3^2 x_3^2)$$

$$+ B_{2,1,0}(x_1^2 x_2 + x_1 x_2^2 + x_3^2 x_3)$$

$$+ B_{1,1,1}(x_1 x_2 x_3 + x_1 x_2 x_3)$$

$$+ B_{1,0,0}(x_1 + x_2 + x_3)$$

$$+ B_{i,j,k} \quad B_{i,j,k} \in \{1\} \mathcal{C}, \quad i, j, k \in \{1, 2\}, \quad k \leq j \leq i.$$

The constraint eq. (11) provides the knowledge of

$$P^{(0)}(x_2, x_2, x_3) = B_{2,2,2} x_2^4 x_3^2$$

$$+ B_{2,2,1}(x_1^2 x_3 + 2 x_1^2 x_3^2)$$

$$+ 2B_{2,1,1}(x_1^2 x_3 + (B_{2,1,1} + 2B_{2,2,0}) x_3^2 x_3 + B_{2,2,0} x_1^2)$$

$$+ 2B_{2,1,0}(x_1^2 + x_2 x_3^2) + (B_{1,1,1} + 2B_{2,1,0}) x_3^2 x_3$$

$$+(B_{1,1,0} + 2B_{2,0,0}) x_2^2 + 2B_{2,0,0} x_2 x_3$$

$$+ B_{1,1,0}(2x_2 + x_3)$$

$$+ B_{0,0,0},$$

(obtained in the limit as $x_1 \to x_2$), and thus of all 9 coefficients. Thus the three-point function is uniquely determined. Since \(\langle T(x_1)T(x_2) \rangle_{\text{reg.}} p_1 p_2\) is just the $O(1)|_{x_1 = x_2}$ part in the connected two-point function, eqs (22) and (17), the remaining unknown constant in the two-point function is determined using Theorem 4.

7 Graph representation of the Virasoro $N$-point function

**Theorem 6.** For $N \geq 1$, let $S(x_1, ..., x_N)$ be the set of graphs with $N$ partially linked vertices $x_1, ..., x_N$, where $(i,j)$ is an oriented edge between $(x_i, x_j)$, subject to the condition that every vertex has at most one ingoing and at most one outgoing line. There is a $\mathcal{C}(U^N)$ linear map

$$\langle (\ldots) \rangle_r : \quad \cup_N \left( U^N \times F^{\otimes N} \right) \to \mathcal{C}(U^N).$$

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satisfying $(1...)_r = (\ldots)_r$ such that the $N$-point function can be written as
\[
\langle T(x_1) \ldots T(x_N) \rangle_{p_1 \ldots p_N} = \sum_{\Gamma \in S(x_1, \ldots, x_N)} \Phi(\Gamma),
\]
where for $\Gamma \in S(x_1, \ldots, x_N)$,
\[
\Phi(\Gamma) = \left(\frac{c}{2}\right)^{\text{loops}} \prod_{(i,j) \in \Gamma} \left(\frac{1}{4} f(x_i, x_j)\right) \times
\]
\[
\times \left\langle \prod_{k \in A_N \cap E_N} \mathcal{P}(x_k) \prod_{\ell \in (A_N \cup E_N)^c} T(x_\ell)p_\ell \right\rangle_r.
\]
Here $A_N$ and $E_N$ are the sets
\[
A_N := \{i \mid \exists j \text{ such that } (i, j) \in \Gamma\},
\]
\[
E_N := \{j \mid \exists i \text{ such that } (i, j) \in \Gamma\}.
\]
Moreover,
\[
f(x_i, x_j) := \left(\frac{x_i + x_j}{x_i - x_j}\right)^2, \quad \mathcal{P}(x) := T(x)p - \frac{c}{32} \frac{[p']^2}{p}.
\]

Outside the zero set of $p$, $(\ldots)_r$ is regular and takes values in the polynomials.

**Proof.** The function $f(x_i, x_j)$ reproduces the correct type of singularity on $X$. Since every vertex may have only one ingoing and one outgoing line, for every $(i, j) \in \Gamma$ and every $\Gamma \in S(x_1, \ldots, x_N)$, $f(x_i, x_j)$ occurs to zero’th, first or second power, as required by the OPE of the Virasoro field. We will show that $\sum_{\Gamma \not= \Gamma_0} \Phi(\Gamma)$ reproduces the correct singular part of $\langle T(x_1) \ldots T(x_N) \rangle_{p_1 \ldots p_N}$ provided that for the graph $\Gamma_0 \in S(x_1, \ldots, x_N)$ whose vertices are all isolated,
\[
\Phi(\Gamma_0) = \langle T(x_1) \ldots T(x_N) \rangle_{r, p_1 \ldots p_N}
\]
is regular and equals the regular part of $\langle T(x_1) \ldots T(x_N) \rangle_{p_1 \ldots p_N}$. Here by regularity we refer only to coinciding positions, while we allow for singularities where $x$ is not a good coordinate. In fact for $N = 1$, $\Gamma_0$ is the only graph, and
\[
\langle T(x) \rangle_{p} = \langle T(x) \rangle_{r, p}.
\]  
(48)

For $N = 2$, the allowed graphs form a closed loop, a single line segment with two possible orientations, and two isolated points. According to the above formula, writing $f_{ij} = f(x_i, x_j)$ and using $f_{ij} = f_{ji}$,
\[
\langle T(x_1) T(x_2) \rangle_{p_1 p_2} = \frac{c}{2} \frac{1}{16} f_{12}^2(1)_r
\]
\[
+ \frac{1}{4} f_{12} \left(\langle \mathcal{P}_1 \rangle_r + \langle \mathcal{P}_2 \rangle_r\right)
\]
\[
+ \langle T(x_1) T(x_2) \rangle_{r, p_1 p_2}
\]
So $\langle T(x_1) T(x_2) \rangle_{r, p_1 p_2}$ is regular iff the remainder displays the singular part correctly. By linearity of $\langle \ldots \rangle_r$,
\[
\langle \mathcal{P}(x) \rangle_r = \langle T(x) \rangle_{r, p} - \frac{c}{32} \frac{[p']^2}{p} \langle 1 \rangle_r,
\]

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Then the obvious generalisation of (49) shows that the singular part is correct provided (48) holds and \( \langle P \rangle \) is known and regular for \( k < N \). By the OPE, we have for any field \( \phi \),

\[
\langle T_1 T_2 \phi \rangle = \frac{c}{32} f_{12}^2 \langle \phi \rangle + \frac{1}{4} f_{12} \left( \langle P_1 \phi \rangle + \langle P_2 \phi \rangle \right) + \text{reg.}
\]

The set of graphs of \( N \) vertices decomposes as

\[ S(x_1, ..., x_N) = S_{(12)} \cup S_{(1,2)} \cup S_{(2,1)} \cup S_{(1), (2)}. \]

Here \( S_{(12)} \approx S(x_3, ..., x_N) \) is the set of all graphs with a loop between \( x_1 \) and \( x_2 \), \( S_{(1,2)} \) (resp. \( S_{(2,1)} \)) is the set of all \( \Gamma \in S(x_1, ..., x_N) \) containing the link from \( x_1 \) to \( x_2 \) (resp. from \( x_2 \) to \( x_1 \)) but not the one in the opposite direction. Eventually \( S_{(1), (2)} \) is the set of all graphs which have no link between \( x_1 \) and \( x_2 \) at all. The equality

\[
\sum_{\Gamma \in S_{(12)}} F(\Gamma) = \frac{c}{16} f_{12} f_{21} \langle T(x_3) ... T(x_N) \rangle
\]

is immediate. It remains to verify

\[
\sum_{\Gamma \in S(x_1, ..., x_N) \setminus S_{(12)}} F(\Gamma) = \frac{1}{2} f_{12} \langle P_2 T(x_3) ... T(x_N) \rangle p_3 ... p_N + O((x_1 - x_2)^{-1}).
\]
Given the invertible contraction maps

\[ \varphi: S(1,2) \to S(x_2,\ldots,x_N), \]
\[ \bar{\varphi}: S(2,1) \to S(x_2,\ldots,x_N), \]

we have to show that

\[
\sum_{\Gamma \in S(x_2,\ldots,x_N)} F(\varphi^{-1}(\Gamma)) + F(\bar{\varphi}^{-1}(\Gamma)) = \frac{f_{12}}{2} \sum_{\Gamma \in S(x_2,\ldots,x_N)} F(\Gamma) + O((x_1 - x_2)^{-1}).
\]

Let

\[ S(x_2,\ldots,x_N) = S_{(2)}(x_2,\ldots,x_N) \cup S_{(2)}(x_2,\ldots,x_N)^c \]

be the decomposition into the set \( S_{(2)}(x_2,\ldots,x_N) \) of graphs containing \( x_2 \) as an isolated point, and its complement. If \( \Gamma \in S_{(2)}(x_2,\ldots,x_N)^c \), then

\[ F(\varphi^{-1}(\Gamma)) + F(\bar{\varphi}^{-1}(\Gamma)) = \frac{f_{12}}{2} F(\Gamma). \]

Otherwise

\[ F(\varphi^{-1}(\Gamma)) + F(\bar{\varphi}^{-1}(\Gamma)) = \frac{f_{12}}{2} F(\Gamma) - \frac{f_{12}}{2} \frac{c}{32} \frac{[p'_2]^2}{p_2} F(\chi(\Gamma)), \]

by the definition of \( P \). Here \( \chi \) is the isomorphism

\[ \chi: S_{(2)}(x_2,\ldots,x_N) \to S(x_3,\ldots,x_N) \]

given by omitting the vertex \( x_2 \) from the graph. \( \square \)
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