A minimal classical sequent calculus free of structural rules

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Gentzen’s classical sequent calculus LK has explicit structural rules for contraction and weakening. They can be absorbed (in a right-sided formulation) by replacing the axiom $P, \neg P$ by $\Gamma, P, \neg P$ for any context $\Gamma$, and replacing the original disjunction rule with $\Gamma, A, B$ implies $\Gamma, A \lor B$.

This paper presents a classical sequent calculus which is also free of contraction and weakening, but more symmetrically: both contraction and weakening are absorbed into conjunction, leaving the axiom rule intact. It uses a blended conjunction rule, combining the standard context-sharing and context-splitting rules: $\Gamma, \Delta, A$ and $\Gamma, \Sigma, B$ implies $\Gamma, \Delta, \Sigma, A \land B$. We refer to this system $\mathbf{M}$ as minimal sequent calculus.

We prove a minimality theorem for the propositional fragment $\mathbf{M}_p$: any propositional sequent calculus $S$ (within a standard class of right-sided calculi) is complete if and only if $S$ contains $\mathbf{M}_p$ (that is, each rule of $\mathbf{M}_p$ is derivable in $S$). Thus one can view $\mathbf{M}$ as a minimal complete core of Gentzen’s LK.

1 Introduction

The following Gentzen-Schütte-Tait [Gen39, Sch50, Tai68] system, denoted $\mathbf{GS1p}$ in [TS96], is a standard right-sided formulation of the propositional fragment of Gentzen’s classical sequent calculus $\mathbf{LK}$:

$$\begin{array}{c}
\frac{\Gamma, A}{\Gamma, A \land B} & \frac{\Gamma, B}{\Gamma, A \land B} & \frac{\Gamma, A_i}{\Gamma, A_1 \lor A_2} \\
\frac{P, \neg P}{\Gamma, A \land B} & \frac{\Gamma, A \lor A_2}{\Gamma, A \land B} & \frac{\Gamma, A_i}{\Gamma, A \land B} \\
\frac{\Gamma}{\Gamma, A} & \frac{\Gamma, A, A}{\Gamma, A} & \frac{\Gamma, A, A}{\Gamma, A}
\end{array}$$

Here $P$ ranges over propositional variables, $A, A_i, B$ range over formulas, $\Gamma$ ranges over disjoint unions of formulas, and comma denotes disjoint union $\sqcup$. By defining a sequent as a disjoint union of formulas, rather than an ordered list, we avoid an exchange/permutation

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1We label the conjunction and disjunction rules with $\&$ and $\oplus$ for reasons which will become apparent later.
rule (cf. [TS96, §1.1]). Negation is primitive on propositional variables $P$, and extends to compound formulas by de Morgan duality.

The structural rules, weakening $W$ and contraction $C$, are absorbed in the following variant, a right-sided formulation of the propositional part of the calculus of [Ket44], called $\text{GS3p}$ in [TS96].

### System GS3p

\[
\begin{align*}
\Gamma, P, \neg P & \quad \rightarrow \quad \Gamma, A \land \Gamma, B & \land & \Gamma, A, B \rightarrow \ \gamma
\end{align*}
\]

The new axiom $\Gamma, P, \neg P$ amounts to the original axiom $P, \neg P$ followed immediately by weakenings. This paper presents a propositional classical sequent calculus $\text{Mp}$ which is also free of structural rules:

### System Mp

\[
\begin{align*}
P, \neg P & \quad \rightarrow \quad \Gamma, \Delta, A \land \Gamma, \Sigma, B & \land & \Gamma, A, B \rightarrow \ \gamma
\end{align*}
\]

A distinguishing feature of $\text{Mp}$ is the \textit{blended conjunction rule}:

\[
\begin{align*}
\Gamma, \Delta, A & \land \Gamma, \Sigma, B & \land & \Gamma, A, B \rightarrow \ \gamma
\end{align*}
\]

which combines the standard context-sharing and context-splitting conjunction rules:

\[
\begin{align*}
\Gamma, A & \land \Gamma, B & \land & \Delta, A \land \Delta, \Sigma, A \land B
\end{align*}
\]

We refer to $\text{Mp}$ as (cut-free propositional) \textit{minimal sequent calculus}. In contrast to $\text{GS3p}$, contraction and weakening are absorbed symmetrically: both are absorbed into the conjunction rule, leaving the axiom rule intact.

$\text{Mp}$ is evidently sound, since each of its rules can be derived (encoded) in $\text{GS1p}$. Theorem 1 (page 4) is completeness for formulas: a formula is valid iff it is derivable in $\text{Mp}$.

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2. $\neg(A \lor B) = (\neg A) \land (\neg B)$ and $\neg(A \land B) = (\neg A) \lor (\neg B)$.

3. We label the disjunction rule as $\gamma$ to distinguish it from the disjunction rule $\oplus$ of $\text{GS1p}$. The notation is derived from linear logic [Gir87].

4. By analogy with $\text{GS3}$ and $\text{GS3p}$ in [TS96], we reserve the symbol $M$ for a full system with quantifiers, and use $\text{Mp}$ to denote the propositional system. Following [TS96], we treat cut separately. To maximise emphasis on the blended conjunction rule, we omit quantifiers and cut in this paper.

5. Completeness here refers specifically to formulas, not to sequents. Section 6 discusses completeness for sequents.
1.1 Minimality

The blended conjunction rule $\land$ is critical for the liberation from structural rules: Proposition\(^2\) (page 6) shows that relaxing it to the union of the the two standard conjunction rules $\&$ and $\otimes$ breaks completeness.\(^6\) The main theorem of the paper (page 7) formalises the sense in which Mp is a minimal complete core of classical sequent calculus:

**Theorem 2: Minimality**

A standard sequent calculus $S$ is complete iff $S \sqsubseteq Mp$.

Here $S \sqsubseteq T$ ("$S$ contains $T$") iff every rule of $T$ is derivable in $S$, and a standard sequent calculus is any propositional sequent calculus with the axiom $P, \neg P$ and any subset of the following standard rules:

$\Gamma, A \quad \Gamma, B$ \quad $\Gamma, A \land B$ & $\Gamma, A, B$ \quad $\Gamma, A \lor B$ \quad $\Gamma, A$

$\Delta, A \quad \Sigma, B$ \quad $\Delta, \Sigma, A \land B$ \quad $\Gamma, A_1 \quad \Gamma, A_i$ \quad $\Gamma, A_1 \lor A_2$ \quad $\Gamma, A$

$\otimes$ \quad $\oplus_i$ \quad $\oplus$

2 Notation and terminology

Formulas are built from literals (propositional variables $P, Q, R$ . . . and their formal complements $\overline{P}, \overline{Q}, \overline{R}$ . . .) by the binary connectives and $\land$ and or $\lor$. Define negation or not $\neg$ as an operation on formulas (rather than as a connective): $\neg P = \overline{P}$ and $\neg \overline{P} = P$ for all propositional variables $P$, with $\neg (A \land B) = (\neg A) \lor (\neg B)$ and $\neg (A \lor B) = (\neg A) \land (\neg B)$.

We identify a formula with its parse tree, a tree labelled with literals at the leaves and connectives at the internal vertices. A sequent is a non-empty disjoint union of formulas.\(^7\) Commas denote disjoint union. Throughout the document, $P, Q, . . .$ range over propositional variables, $A, B, . . .$ over formulas, and $\Gamma, \Delta, . . .$ over (possibly empty) disjoint unions of formulas.

A formula $A$ is valid if it evaluates to 1 under all possible 0/1-assignments of its propositional variables (with the usual interpretation of $\land$ and $\lor$ on $\{0, 1\}$). A sequent $A_1, . . . , A_n$ is valid iff the formula $A_1 \lor (A_2 \lor (\ldots \lor (A_{n-1} \lor A_n) \ldots))$ is valid. A sub sequent of a sequent $\Gamma$ is any result of deleting zero or more formulas from $\Gamma$; if at least one formula is deleted, the result is a proper sub sequent.

\(^6\)In other words, if we remove the $\land$ rule and add both the $\&$ and the $\otimes$ rules, the resulting system fails to be complete. The formula $((P \land Q) \lor (\overline{Q} \land P)) \lor \overline{P}$ becomes underivable (see the proof of Proposition\(^2\) page 6).

\(^7\)Thus a sequent is a particular kind of labelled forest. This foundational treatment of formulas and sequents as labelled trees and forests sidesteps the common problem of “formulas” versus “formula occurrences”: disjoint unions of graphs are well understood in graph theory \cite{Bol02}.
3 Completeness

THEOREM 1 (COMPLETENESS) Every valid formula is derivable in Mp.

The proof is via the following auxiliary definitions and lemmas.

A sequent is minimally valid, or simply minimal, if it is valid while no proper subsequent is valid. For example, the sequents $P, \neg P$ and $P \land Q, \neg Q \land P, \neg P$ are minimal, while $P, \neg P, Q$ is not.

LEMMA 1 Every valid sequent contains a minimal subsequent.

Proof. Immediate from the definition of minimality. □

LEMMA 2 Suppose a sequent $\Gamma$ is a disjoint union of literals (i.e., $\Gamma$ contains no $\land$ or $\lor$). Then $\Gamma$ is minimal iff $\Gamma = P, \neg P$ for some propositional variable $P$.

Proof. By definition of validity in terms of valuations, $\Gamma$ is valid iff it contains a complementary pair of literals, i.e., iff $\Gamma = P, \neg P, \Delta$ with $\Delta$ a disjoint union of zero or more literals. Since $P, \neg P$ is valid, $\Gamma$ is minimal iff $\Delta$ is empty. □

Suppose $\Gamma$ and $\Delta$ are each disjoint unions of formulas (so each is either a sequent or empty). Write $\Gamma \subseteq \Delta$ if $\Gamma$ results from deleting zero or more formulas from $\Delta$.

LEMMA 3 Suppose $\Gamma, A_1 \land A_2$ is minimal. Choose $\Gamma_1 \subseteq \Gamma$ and $\Gamma_2 \subseteq \Gamma$ such that $\Gamma_1, A_1$ and $\Gamma_2, A_2$ are minimal (existing by Lemma 1 since $\Gamma, A_1$ and $\Gamma, A_2$ are valid). Then every formula of $\Gamma$ is in at least one of the $\Gamma_i$.

Proof. Suppose the formula $B$ of $\Gamma$ is in neither $\Gamma_i$. Let $\Gamma'$ be the result of deleting $B$ from $\Gamma$. Then $\Gamma', A_1 \land A_2$ is a valid proper subsequent of $\Gamma, A_1 \land A_2$, contradicting minimality. (The sequent $\Gamma', A_1 \land A_2$ is valid since $\Gamma_1, A_1$ and $\Gamma_2, A_2$ are valid.) □

LEMMA 4 Suppose $\Gamma, A \lor B$ is minimal and $\Gamma, A$ is valid. Then $\Gamma, A$ is minimal.

Proof. If not, some proper subsequence $\Delta$ of $\Gamma, A$ is valid. If $\Delta$ does not contain $A$, then it is also a proper subsequence of $\Gamma, A \lor B$, contradicting minimality. Otherwise let $\Delta'$ be the result of replacing $A$ in $\Delta$ by $A \lor B$. Since $\Delta$ is valid, so also is $\Delta'$. Thus $\Delta'$ is a valid proper subsequence of $\Gamma, A \lor B$, contradicting minimality. □

LEMMA 5 Suppose $\Gamma, A \lor B$ is minimal and neither $\Gamma, A$ nor $\Gamma, B$ is valid. Then $\Gamma, A, B$ is minimal.

Proof. Suppose $\Gamma, A, B$ had a valid proper subsequence $\Delta$. Since neither $\Gamma, A$ nor $\Gamma, B$ is valid, $\Delta$ must contain both $A$ and $B$. Let $\Delta'$ result from replacing $A, B$ by $A \lor B$ in $\Delta$. Then $\Delta'$ is a valid proper subsequence of $\Gamma, A \lor B$, contradicting minimality. □

Since a formula (viewed as a singleton sequent) is a minimal sequent, the Completeness Theorem (Theorem 1) is a special case of:
**Proposition 1** Every minimal sequent is derivable in \( \text{Mp} \).

**Proof.** Suppose \( \Gamma \) is a minimal sequent. We proceed by induction on the number of connectives in \( \Gamma \).

- **Induction base (no connective).** Since \( \Gamma \) is minimal, Lemma 2 implies \( \Gamma = P, \neg P \), the conclusion of the axiom rule \( \frac{P, \neg P}{\top} \).

- **Induction step (at least one connective).**
  1. **Case:** \( \Gamma = \Delta, A_1 \land A_2 \). By Lemma 3, \( \Gamma = \Sigma, \Delta_1, \Delta_2, A_1 \land A_2 \) for \( \Sigma, \Delta_1, A_1 \) and \( \Sigma, \Delta_2, A_2 \) minimal. Write down the conjunction rule
     \[
     \frac{
     \Sigma, \Delta_1, A_1 \quad \Sigma, \Delta_2, A_2
     }{
     \Sigma, \Delta_1, \Delta_2, A_1 \land A_2 \quad \land
     }
     \]
     and appeal to induction with the two hypothesis sequents.
  2. **Case:** \( \Gamma = \Delta, A_1 \lor A_2 \).
     (a) **Case:** \( \Delta, A_i \) is valid for some \( i \in \{1, 2\} \). Write down the disjunction rule
     \[
     \frac{
     \Delta, A_i
     }{
     \Delta, A_1 \lor A_2 \quad \oplus_i
     }
     \]
     then appeal to induction with \( \Delta, A_i \), which is minimal by Lemma 4.
     (b) **Case:** \( \Delta, A_i \) is not valid for each \( i \in \{1, 2\} \). Thus \( \Delta, A_1, A_2 \) is minimal, by Lemma 5. Write down the disjunction rule
     \[
     \frac{
     \Delta, A_1, A_2
     }{
     \Delta, A_1 \lor A_2 \quad \lor
     }
     \]
     then appeal to induction with \( \Delta, A_1, A_2 \).

(\( \Gamma \) may match both 1 and 2 in the inductive step, permitting some choice in the construction of the derivation. There is choice in case 2(a) if both \( \Delta, A_1 \) and \( \Delta, A_2 \) are valid.) \qed

Note that completeness does not hold for arbitrary valid sequents. For example, the sequent \( P, \neg P, Q \) is valid but not derivable in \( \text{Mp} \). A sequent is valid iff some some subsequent is derivable in \( \text{Mp} \). Thus \( \text{Mp} \) is complete for sequents modulo final weakenings. In this sense, \( \text{Mp} \) is akin to system \( \text{GS5p} \) of [TS96, §7.4] (related to resolution). (See also Section 6.)

### 4 The Minimality Theorem

Relaxing blended conjunction to the pair of standard conjunction rules (context-sharing \& and context-splitting \( \otimes \)) breaks completeness. Let \( \text{Mp}^- \) be the following subsystem of \( \text{Mp} \)\footnote{This precursor of \( \text{Mp} \) is (cut-free) multiplicative-additive linear logic [Gir87] with tensor \( \otimes \) and with \& collapsed to \( \land \), and plus \( \oplus \) and par \( \forall \) collapsed to \( \lor \).}
**System Mp⁻**

\[
\begin{array}{ccc}
\Gamma, A & \Gamma, B & \Gamma, A \& B \\
\Gamma, A \land B & \Gamma, A \lor B & \Gamma, A \& B \\
\Delta, A & \Sigma, B & \Delta, \Sigma, A \land B \\
\Delta, \Sigma, A \land B & \Gamma, A_i & \Gamma, A_1 \lor A_2 \\
\end{array}
\]

**Proposition 2** System Mp⁻ is incomplete.

**Proof.** We show that the valid formula \( A = ((P \land Q) \lor (\overline{Q} \land P)) \lor \overline{P} \) is not derivable in Mp⁻. The placement of the two outermost \( \lor \) connectives forces the last two rules of a potential derivation to be disjunction rules. Since \( P \land Q, \overline{Q} \land P, \overline{P} \) is minimal (no proper subsequent is valid), the two disjunction rules must be \( \land \) rather than \( \oplus \):

\[
\begin{array}{c}
P, Q \land P, \overline{P} \\
(P \land Q) \lor (\overline{Q} \land P), \overline{P} \\
((P \land Q) \lor (\overline{Q} \land P)) \lor \overline{P}
\end{array}
\]

It remains to show that \( P \land Q, \overline{Q} \land P, \overline{P} \) is not derivable in Mp⁻. There are only two connectives, both \( \land \), so the last rule must be a conjunction.

1. **Case: the last rule is a context-sharing \( \& \)-rule.**
   (a) **Case: The last rule introduces \( P \land Q \).**

\[
P, \overline{Q} \land P, \overline{P} \quad Q, \overline{Q} \land P, \overline{P} \\
P \land Q, \overline{Q} \land P, \overline{P} \quad \land
\]

The left hypothesis \( P, \overline{Q} \land P, \overline{P} \) cannot be derived in Mp⁻, since there is no \( Q \) to match the \( \overline{Q} \) (and no weakening).

(b) **Case: The last rule introduces \( \overline{Q} \land P \).** The same as the previous case, by symmetry, and exchanging \( Q \leftrightarrow \overline{Q} \).

2. **Case: the last rule is a context-splitting \( \otimes \)-rule.**
   (a) **Case: The last rule introduces \( P \land Q \).**

\[
P, \Gamma \quad Q, \Delta \\
P \land Q, \overline{Q} \land P, \overline{P} \quad \otimes
\]

We must allocate each of \( \overline{Q} \land P \) and \( \overline{P} \) either to \( \Gamma \) or to \( \Delta \). If \( \overline{Q} \land P \) is in \( \Gamma \), then \( P, \Gamma \) is not derivable in Mp⁻, since it contains no \( Q \) to match the \( \overline{Q} \). So \( \overline{Q} \land P \) is in \( \Delta \). But then the \( \overline{P} \) is required in both \( \Gamma \) and \( \Delta \).
(b) Case: The last rule introduces $\overline{Q} \land P$. The same as the previous case, by symmetry, and exchanging $Q \leftrightarrow \overline{Q}$.

□

A **standard system** is any propositional sequent calculus containing the axiom $P, \neg P$ and any of the following **standard rules**:

\[
\begin{align*}
\Gamma, A & \quad \Gamma, B \\
\Gamma, A \land B & \\
\Gamma, A \lor B & \quad \nabla
\end{align*}
\]

\[
\begin{align*}
\Delta, A & \quad \Sigma, B \\
\Delta, \Sigma, A \land B & \quad \otimes
\end{align*}
\]

\[
\begin{align*}
\Gamma, A, B & \quad \nabla
\end{align*}
\]

\[
\begin{align*}
\Gamma, A_i & \quad \oplus_i
\end{align*}
\]

\[
\begin{align*}
\Gamma, A, A & \quad C
\end{align*}
\]

Thus there are $2^6 = 64$ such systems (many of which will not be complete).

System $S$ **contains** system $T$, denoted $S \supseteq T$, if each rule of $T$ is a derived rule of $S$. For example, system $\text{GS1p}$ (page 1) contains $\text{Mp}$ since the blended conjunction rule $\land$ and the disjunction rule $\nabla$ of $\text{Mp}$ can be derived in $\text{GS1p}$:

\[
\begin{align*}
\Gamma, \Delta, A \quad \Gamma, \Sigma, B & \quad \nabla \\
\Gamma, \Delta, \Sigma, A \land B & \quad \leftarrow
\end{align*}
\]

\[
\begin{align*}
\Gamma, \Delta, A \quad \Gamma, \Sigma, B & \quad W^* \\
\Gamma, \Delta, \Sigma, A \quad \Gamma, \Delta, \Sigma, A \land B & \quad \otimes_i
\end{align*}
\]

\[
\begin{align*}
\Gamma, A, B & \quad \nabla
\end{align*}
\]

\[
\begin{align*}
\Gamma, A_i & \quad \oplus_i
\end{align*}
\]

\[
\begin{align*}
\Gamma, A, A & \quad C
\end{align*}
\]

where $W^*$ denotes a sequence of zero or more weakenings.

**Theorem 2 (Minimality Theorem)** A standard system is **complete** iff it contains $\text{Mp}$.

**4.1 Proof of the Minimality Theorem**

Two systems are **equivalent** if each contains the other. For example, it is well known that $\text{GS1p}$ (page 1) is equivalent to $\text{9}$

\[
\begin{align*}
P, \neg P & \quad \Delta, A \quad \Sigma, B \quad \otimes \\
\Delta, \Sigma, A \land B & \quad \nabla
\end{align*}
\]

\[
\begin{align*}
\Gamma, A_1, A_2 & \quad \nabla
\end{align*}
\]

\[
\begin{align*}
\Gamma, A & \quad W
\end{align*}
\]

\[
\begin{align*}
\Gamma, A, A & \quad C
\end{align*}
\]

\[\text{9} \text{This system is multiplicative linear logic [Gir87] plus contraction and weakening (with the connectives denoted } \land \text{ and } \lor \text{ instead of } \otimes \text{ and } \nabla).}\]
via the following rule derivations:

\[
\frac{\Delta, A}{\Delta, \Sigma, A \land B} \otimes \quad \frac{\Delta, A \land B}{\Delta, \Sigma, A \land B}
\]

\[
\frac{\Gamma, A_i}{\Gamma, A_1 \lor A_2} \oplus_i \quad \frac{\Gamma, A_1 \lor A_2}{\Gamma, A_i}
\]

\[
\frac{\Gamma, A, B}{\Gamma, A \lor B} \otimes \quad \frac{\Gamma, A \lor B}{\Gamma, A, A \lor B}
\]

\[
\frac{\Gamma, A \land B}{\Gamma, A \lor B} \otimes \quad \frac{\Gamma, A \lor B}{\Gamma, \Gamma, A \land B}
\]

We shall abbreviate these four rule derivations as follows, and write analogous abbreviations for other rule derivations.

\[
\otimes \leftarrow \& W \quad \& \leftarrow \otimes C
\]

\[
\oplus \leftarrow \otimes W \quad \otimes \leftarrow \oplus C
\]

### 4.1.1 The three complete standard systems

As a stepping stone towards the Minimality Theorem, we shall prove that, up to equivalence, there are only three complete standard systems.

We abbreviate a system by listing its non-axiom rules. For example, \( \text{GS1p} = (\&, \oplus, W, C) \) and \( \text{Mp} = (\land, \oplus, \otimes) \). Besides \( \text{GS1p} \), we shall pay particular attention to the systems

\[
\begin{align*}
\text{Pp} &= (\otimes, \oplus, C) \quad \text{Positive calculus} \\
\text{Np} &= (\&, \otimes, W) \quad \text{Negative calculus}
\end{align*}
\]

(Our terminology comes from polarity of connectives in linear logic [Gir87]: tensor \( \otimes \) and plus \( \oplus \) are positive, and with \( \& \) and par \( \otimes \) are negative.)
PROPOSITION 3  Up to equivalence:

1. $\text{GS1p} = (\&, \oplus, C, W)$ is the only complete standard system with both contraction C and weakening W;

2. $\text{Pp} = (\otimes, \oplus, C)$ is the only complete standard system without weakening W;

3. $\text{Np} = (\&, \&y, W)$ is the only complete standard system without contraction C.

The proof is via the following lemmas.

**Lemma 6** $\text{Mp} = (\&y, \oplus) \leq \text{Pp} = (\otimes, \oplus, C), \text{Np} = (\&, \&y, W)$ and $\text{GS1p} = (\&, \oplus, C, W)$.

**Proof.** $\text{Pp}$ contains $\text{Mp}$ since $\&y \leftarrow C \otimes.

\[
\begin{array}{ccc}
\Gamma, \Delta, A & \Gamma, \Sigma, B \\
\hline
\Gamma, \Delta, \Sigma, A \& B
\end{array}
\]

$\otimes$ $\leftarrow$

\[
\begin{array}{ccc}
\Gamma, \Delta, A & \Gamma, \Sigma, B \\
\hline
\Gamma, \Delta, \Sigma, A \& B
\end{array}
\]

(where $C^\ast$ denotes zero or more consecutive contractions) and $\&y \leftarrow \oplus C$:

\[
\begin{array}{ccc}
\Gamma, A, B \\
\hline
\Gamma, A \vee B
\end{array}
\]

$\oplus_1$ $\leftarrow$

\[
\begin{array}{ccc}
\Gamma, A, A \vee B \\
\hline
\Gamma, A \vee B
\end{array}
\]

$\oplus_2$ $\leftarrow$

\[
\begin{array}{ccc}
\Gamma, A, A \vee B \\
\hline
\Gamma, A \vee B
\end{array}
\]

$\text{Np}$ contains $\text{Mp}$ since $\&y \leftarrow W \&y$, and $\oplus \leftarrow W \&y$ (see page 8). $\text{GS1p} = (\&, \oplus, C, W)$ is equivalent to $(\otimes, \&, \oplus, \&y, C, W)$ since $\otimes$ and $\&y$ are derivable. Thus $\text{GS1p}$ contains $\text{Pp}$ (and $\text{Np}$), hence $\text{Mp}$. □

**Lemma 7** $\text{Pp} = (C, \otimes, \oplus)$ and $\text{Np} = (\&, \&y, W)$ are complete.\[11\]

**Proof.** Each contains $\text{Mp}$ by Lemma 6 which is complete (Theorem I). □

**Lemma 8** Up to equivalence, system $\text{GS1p} = (\&, \oplus, C, W)$ is the only complete standard system with both contraction C and weakening W.

**Proof.** $\text{GS1p}$ is complete (see e.g. [TS96], or by the fact that $\text{GS1p}$ contains $\text{Mp}$ which is complete). Any complete system must have a conjunction rule ($\otimes$ or $\&$) and a disjunction rule ($\oplus$ or $\&y$). In the presence of $C$ and $W$, the two conjunctions are derivable from one other, as are the two disjunctions (see page 8). □

**Lemma 9** A complete standard system without weakening W must contain $\text{Pp} = (\otimes, \oplus, C)$.

\[11\text{Recall that completeness refers to formulas, not sequents in general.}\]
Proof. System $\mathsf{Mp}^- = (\otimes, \oplus, \&, \wedge)$, with both conjunction rules and both disjunction rules, is incomplete (Proposition 2 page 6), therefore we must have contraction $C$.

Without the $\oplus$ rule, the valid formula $(P \lor \overline{P}) \lor \overline{Q}$ is not derivable: the last rule must be $\neg$, leaving us to derive $P \lor \overline{P}, Q$, which is impossible without weakening $W$ (i.e., with at most $\neg, \&,$ and $\otimes$ available), since, after a necessary axiom $P, \overline{P}$ at the top of the derivation, there is no way to introduce the formula $Q$.

Without the context-splitting $\otimes$ rule, the valid formula $P \lor (Q \lor (P \land \overline{Q}))$ is not derivable. The last two rules must be $\neg$, for if we use a $\oplus$ we will not be able to match complementary literals in the axioms at the top of the derivation. Thus we are left to derive $P, Q, \overline{P} \land \overline{Q}$, using $\&$ and $C$. The derivation must contain an axiom rule $P, \overline{P}$. The next rule can only be a $\&$ (since $P, \overline{P}$ cannot be the hypothesis sequent of a contraction $C$ rule). Since the only $\land$-formula in the final concluding sequent $P, Q, \overline{P} \land \overline{Q}$ is $P \land Q$, and the $\&$ rule is context sharing, the $\&$-rule must be

$$
\begin{array}{c}
\vdots \\
P, \overline{P} \\
\hline \\
P, \overline{P} \land \overline{Q}
\end{array}
\&$$

but $P, \overline{Q}$ is not derivable. □

Lemma 10 Up to equivalence, $\mathsf{Pp} = (\otimes, \oplus, C)$ is the only complete standard system without weakening $W$.

Proof. By Lemma 7 $\mathsf{Pp}$ is complete. By Lemma 9 every $W$-free complete standard system contains $\mathsf{Pp}$. All other $W$-free standard systems containing $\mathsf{Pp}$ are equivalent to $\mathsf{Pp}$, since the standard rule derivations $\& \leftarrow \otimes C$ and $\neg \leftarrow \oplus C$ yield $\&$ and $\neg$ (see page 8). □

Lemma 11 A complete standard system without contraction $C$ must contain $\mathsf{Np} = (\&, \neg, W)$.

Proof. System $\mathsf{Mp}^- = (\otimes, \oplus, \&, \neg)$, with both conjunction rules and both disjunction rules, is incomplete (Proposition 2 page 6), therefore we must have weakening $W$.

Without the $\neg$ rule, the valid formula $P \lor (\overline{P} \land \overline{P})$ would not be derivable. The last rule must be a $\neg$ (rather than a $\oplus$, otherwise we lack either $P$ or $\overline{P}$), so we are left to derive $P, \overline{P} \land \overline{P}$. The last rule cannot be a $\otimes$ or $\oplus$, as the only connective is $\land$. It cannot be $W$, or else we lack either $P$ or $\overline{P}$. It cannot be $\otimes$, as one of the two hypotheses will be the single formula $\overline{P}$. □

Lemma 12 Up to equivalence, $\mathsf{Np} = (\&, \neg, W)$ is the only complete standard system without contraction $C$.

Proof. By Lemma 7 $\mathsf{Np}$ is complete. By Lemma 11 every $C$-free complete standard system contains $\mathsf{Np}$. All other $C$-free standard systems containing $\mathsf{Np}$ are equivalent to $\mathsf{Np}$, since the standard rule derivations $\otimes \leftarrow \& W$ and $\oplus \leftarrow \neg W$ yield $\otimes$ and $\oplus$ (see page 8). □

Proof of Proposition 8 Parts (1), (2) and (3) are Lemmas 8, 10 and 12 respectively. □
Lemma 13  Every standard complete system has contraction C or weakening W.

Proof. Otherwise it is contained in \( \text{Mp}^- = (\otimes, \oplus, \& \&, \#) \), which is incomplete (Prop. [2]). □

Theorem 3  Up to equivalence, there are only three complete standard systems:

1. The Gentzen-Schütte-Tait system \( \text{GS1p} = (\& \&, \oplus, C, W) \).
2. Positive calculus \( \text{Pp} = (\otimes, \oplus, C) \).
3. Negative calculus \( \text{Np} = (\& \&, \# \#, W) \).

Proof. Proposition [3] and Lemma [13]. □

Proof of Minimality Theorem (Theorem [2]). Each of the three complete standard systems contains \( \text{Mp} \) (Lemma [5]). □

The three inequivalent complete standard systems \( \text{GS1p}, \text{Pp} \) and \( \text{Np} \), together with propositional minimal sequent calculus \( \text{Mp} \), sit in the following Hasse diagram of containments:

| Contains of complete inequivalent systems |
|-------------------------------------------|
| Propositional right-sided LK \( (\& \&, \oplus, W, C) \) |
| Propositional Positive Seq. Calc. \( (\otimes, \oplus, C) \) |
| \( \text{GS1p} \) |
| Propositional Negative Seq. Calc. \( (\& \&, \# \#, W) \) |
| \( \text{Pp} \) |
| Propositional Min. Seq. Calc. \( (\& \&, \oplus, \# \#) \) |
| \( \text{Np} \) |
| \( \text{Mp} \) |

Thus we can view propositional minimal sequent calculus \( \text{Mp} \) as a minimal complete core of \( \text{GS1p} \), hence of (propositional) Gentzen’s LK.

5  Extended Minimality Theorem

Define an extended system as one containing the axiom rule \( \overline{P, \overline{P}} \) and any of the following rules. (We have extended the definition of standard system by making blended conjunction available.)
The Minimality Theorem (Theorem 2, page 7) extends as follows.

**Theorem 4 (Extended Minimality Theorem)** An extended system is complete iff it contains propositional minimal sequent calculus $\text{Mp}$.

To prove this theorem, we require two lemmas.

**Lemma 14** Suppose $S$ is a complete extended system with the blended conjunction rule $\land$, and with at least one of contraction $C$ or weakening $W$. Then $S$ is equivalent to a standard system.

*Proof.* If $S$ has weakening $W$, let $S'$ be the result of replacing the blended conjunction rule $\land$ in $S$ by context-sharing conjunction $\&$; otherwise $S$ has contraction, and let $S'$ result from replacing $\land$ by context-splitting $\otimes$. Then $S'$ is equivalent to $S$, since $\land \leftarrow \otimes C$ (page 9) and $\land \leftarrow \& W$ (page 7). □

**Lemma 15** Suppose $S$ is a complete extended system with neither contraction $C$ nor weakening $W$. Then $S$ is equivalent to propositional minimal sequent calculus $\text{Mp}$.

*Proof.* Since $\text{Mp}^- = (\otimes, \& , \oplus, \land)$ is incomplete (Proposition 2, page 6), $S$ must have the blended conjunction rule $\land$ either directly or as a derived rule. Since $S$ is complete, it must have a disjunction rule, therefore it could only fail to be equivalent to $\text{Mp} = (\land, \oplus, \land)$ if (a) it has $\oplus$ and $\land$ is not derivable, i.e., $S$ is equivalent to $(\land, \oplus)$, or (b) it has $\land$ and $\oplus$ is not derivable, i.e., $S$ is equivalent to $(\land, \land)$. In case (a), the valid formula $P \lor \overline{P}$ would not be derivable, and in case (b) the valid formula $(P \lor \overline{P}) \lor Q$ would not be derivable, either way contradicting the completeness of $S$. □

*Proof of the Extended Minimality Theorem (Theorem 4).* Suppose $S$ is a complete extended system. If $S$ has contraction $C$ or weakening $W$ then it is equivalent to a standard system by
Lemma 14, hence contains $M_p$ by the original Minimality Theorem. Otherwise $S$ is equivalent to $M_p$ by Lemma 15 hence in particular contains $M_p$.

Conversely, suppose $S$ is an extended system containing $M_p$. Then $S$ is complete since $M_p$ is complete. □

We also have the following extension of Theorem 3 (page 11), which stated that, up to equivalence, there are only three complete standard systems, $GS1p$, $Pp$ and $Np$.

**Theorem 5** Up to equivalence, there are only four complete extended systems:

1. The Gentzen-Schütte-Tait system $GS1p = (\&, \oplus, C, W)$.
2. Positive calculus $Pp = (\otimes, \oplus, C)$.
3. Negative calculus $Np = (\&, \exists, W)$.
4. Propositional minimal sequent calculus $M_p = (\land, \oplus, \exists)$.

**Proof.** Theorem 3 together with Lemmas 14 and 15 □

### 6 Degrees of completeness

We defined a system as *complete* if every valid formula (singleton sequent) is derivable. To avoid ambiguity with forthcoming definitions, let us refer to this default notion of completeness as *formula-completeness*. Define a system as *minimal-complete* if every minimal sequent is derivable, and *sequent-complete* if every valid sequent is derivable. (Thus *sequent-complete* implies *minimal-complete* implies *formula-complete*.)

For a minimal-complete system $S$, a sequent $\Gamma$ is valid iff a subsequent of $\Gamma$ is derivable in $S$. Thus a minimal-complete system $S$ can be viewed as sequent-complete, modulo final weakenings. (Cf. system $GS5p$ of [TS96, §7.4] (related to resolution).)

**Proposition 4** $Pp = (\otimes, \oplus, C)$ and $M_p = (\land, \oplus, \exists)$ are formula-complete and minimal-complete, but not sequent-complete.

**Proof.** We have already proved that $M_p$ (hence also $Pp$, by containment) is minimal-complete (Proposition 1).

We show that the valid (non-minimal) sequent $P, \overline{P}, Q$ is not derivable in $Pp$ (hence also in $M_p$). A derivation must contain an axiom rule $P, \overline{P}$. This cannot be followed by a $\otimes$ or $\oplus$ rule, otherwise we introduce a connective $\land$ or $\lor$ which cannot subsequently be removed by any other rule before the concluding sequent $P, \overline{P}, Q$. Neither can it be followed by contraction $C$, since there is nothing to contract. □

**Proposition 5** $Np = (\&, \exists, W)$ is formula-, minimal- and sequent-complete.

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12 Recall that a valid sequent is minimal if no proper subsequent is valid.
Proof. \( \text{Np} \) is minimal-complete since it contains \( \text{Mp} \). Suppose \( \Gamma \) is a valid but not minimal sequent. Choose a minimal subsequent \( \Delta \) of \( \Gamma \) (see Lemma [1] page 4). By minimal-completeness, \( \Delta \) has a derivation. Follow this with weakenings to obtain \( \Gamma \). □

Below we have annotated our Hasse diagram with completeness strengths.

### 7 Possible future work

1. **Cut.** Chapter 4 of [TS96] gives a detailed analysis of cut for Gentzen systems. One could pursue an analogous analysis of cut for minimal sequent calculus. Aside from context-splitting and context-sharing cut rules

   \[
   \Delta, A \quad \Sigma, \neg A \\
   \frac{}{\Delta, \Sigma} \quad \text{cut}_\oplus \\
   \Gamma, A \quad \Gamma, \neg A \\
   \frac{}{\Gamma} \quad \text{cut}_\&
   \]

   one might also investigate a blended cut rule:

   \[
   \frac{}{\Gamma, \Delta, A} \\
   \frac{\Gamma, \Sigma, \neg A}{\Gamma, \Delta, \Sigma} \quad \text{cut}
   \]

2. **Quantifiers.** Explore the various ways of adding quantifiers to \( \text{Mp} \), for a full first-order system \( \text{M} \).

3. **Mix (nullary multicut).** Gentzen’s multicut rule
\[
\Delta, A_1, \ldots, A_m, \Sigma, \neg A_1, \ldots, \neg A_n \quad \frac{}{\Delta, \Sigma}
\]

in the nullary case \(m = n = 0\) has been of particular interest to linear logicians [Gir87], who call it the mix rule. One could investigate context-splitting, context-sharing and blended incarnations:

\[
\begin{align*}
\Delta & \quad \Sigma & \quad \text{mix}_\otimes \\
\Delta, \Sigma & \quad \Gamma, \Gamma & \quad \text{mix}_\& \\
\Gamma & \quad \Gamma, \Delta, \Sigma & \quad \text{mix}_\boxtimes
\end{align*}
\]

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