Noncommutative Instantons in 4k Dimensions

Tatiana A. Ivanova† and Olaf Lechtenfeld*

†Bogoliubov Laboratory of Theoretical Physics, JINR
141980 Dubna, Moscow Region, Russia
Email: ita@thsun1.jinr.ru

*Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
Email: lechtenf@itp.uni-hannover.de

Abstract

We consider Ward’s generalized self-duality equations for U(2r) Yang-Mills theory on \( \mathbb{R}^{4k} \) and their Moyal deformation under self-dual noncommutativity. Employing an extended ADHM construction we find two kinds of explicit solutions, which generalize the ’t Hooft and BPST instantons from \( \mathbb{R}^4 \) to noncommutative \( \mathbb{R}^{4k} \). The BPST-type configurations appear to be new even in the commutative case.
1 Introduction and summary

Moyal-type deformations of gauge field theories arise naturally in string theory when D-branes and an NS-NS two-form background are present [1]. Such noncommutative extensions are also interesting by themselves as specific nonlocal generalizations of ordinary Yang-Mills theories. In particular, the question whether and how the known BPS classical solutions (instantons, monopoles, vortices) [2]–[4] can be deformed in this manner has been answered in the affirmative [5]–[11].

In space-time dimensions higher that four, BPS configurations can still be found as solutions to first-order equations, known as generalized self-duality or generalized self-dual Yang-Mills (SDYM) equations. Already more than 20 years ago such equations were proposed [12, 13], and some of their solutions were found e.g. in [13]–[15]. Also the ADHM solution technique [16] was generalized from four-dimensional to 4k-dimensional Yang-Mills (YM) theory [17]. More recently, various BPS solutions to the noncommutative Yang-Mills equations in higher dimensions and their brane interpretations have been investigated e.g. in [18]–[20]. Some of these works employ the (generalized) ADHM construction extended to the noncommutative realm.

In this letter we use the extended ADHM method to construct new generalizations of the `t Hooft as well as of the BPST instanton for U(2r) gauge groups on self-dual noncommutative \( \mathbb{R}^{4k} \). Only for simplicity we restrict ourselves to the gauge groups U(2) on \( \mathbb{R}^{4k+4} \) and U(4) on \( \mathbb{R}^8 \), respectively. In the first (`t Hooft-type) case, our solutions do not have a finite topological charge, but their four-dimensional “slices” coincide with the noncommutative n-instanton configurations of [21] featuring noncommutative translational moduli. In the second (BPST-type) case, we correct a faulty ansatz of [17] and find solutions to the (self-dually) deformed generalized self-duality equations. Even in the commutative limit these configurations are novel instantons in eight dimensions.

2 Quaternionic structure in \( \mathbb{R}^{4k} \)

We consider the 4k-dimensional space \( \mathbb{R}^{4k} \) with the metric \( \delta_{\mu\nu} \), where \( \mu, \nu, \ldots = 1, \ldots, 4k \). Its complexification \( \mathbb{C}^{4k} \) splits into a tensor product, \( \mathbb{C}^{4k} \cong \mathbb{C}^{2k} \otimes \mathbb{C}^2 \). Complex coordinates \( x^\mu \) on \( \mathbb{C}^{4k} \) are related to complex coordinates \( x^{aP} \) on \( \mathbb{C}^{2k} \otimes \mathbb{C}^2 \) by [13, 17]

\[
    x^\mu = V^\mu_{aP} x^{aP} \quad \text{for} \quad a = 1, \ldots, 2k \quad \text{and} \quad P = 1, 2 ,
\]

where \( (V^\mu_{aP}) \) is a nonsingular invertible matrix which defines a mapping from \( \mathbb{C}^{2k} \otimes \mathbb{C}^2 \) to \( \mathbb{C}^{4k} \). Imposing the reality condition \( x^\mu = \overline{x^\mu} \), we obtain coordinates on \( \mathbb{R}^{4k} \). To preserve (2.1) one should impose the following reality conditions on \( V^\mu_{aP} \) and \( x^{aP} \):

\[
    \overline{V^\mu_{aP}} = \varepsilon^{abP} \varepsilon^{PQ} V^\mu_{bQ} \quad \text{and} \quad \overline{x^{aP}} = \varepsilon_{ab} \varepsilon_{PQ} x^{bQ} ,
\]

where

\[
    \varepsilon = (\varepsilon_{PQ}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad (\varepsilon_{ab}) = \begin{pmatrix} \varepsilon & 0_2 \\ 0_2 & \varepsilon \end{pmatrix}
\]

while the inverse matrices are defined by

\[
    \varepsilon^{PQ} \varepsilon_{QR} = \delta^P_R \quad \text{and} \quad \varepsilon^{ab} \varepsilon_{bc} = \delta^a_c .
\]
The space \( \mathbb{R}^{4k} \) with metric
\[
d s^2 = \delta_{\mu\nu} \, dx^\mu dx^\nu = \varepsilon_{ab} \varepsilon_{PQ} \, dx^a dx^b \quad (2.5)
\]
can be decomposed into a direct sum of \( k \) four-dimensional spaces,
\[
\mathbb{R}^{4k} \cong \mathbb{R}^k \otimes \mathbb{R}^4 = \mathbb{R}^k \otimes \mathbb{H} = \mathbb{H}^k \quad (2.6)
\]
with coordinates \( x^\mu \), where \( \mu_i = (4i+1, \ldots, 4i+4) \) is an index-quadruple and \( i = 0, \ldots, k-1 \). Each such subspace is identified with the algebra \( \mathbb{H} \) of quaternions realized in terms of \( 2 \times 2 \) matrices with a basis
\[
(\epsilon_{\mu}) = (\epsilon_{\mu_0}) = (-i \sigma_1, -i \sigma_2, -i \sigma_3, 1_2) \quad (2.7)
\]
where \( \sigma_\alpha, \alpha=1,2,3, \) are the Pauli matrices.

Defining \( V_\mu = \delta_{\mu\nu} V^\nu = (\delta_{\mu\nu} V_{aP}^\nu) \), one can write \( V_\mu \) with \( (\mu) = (\mu_0, \ldots, \mu_{k-1}) \) as a collection of \( k \) quaternionic \( k \times 1 \) column vectors, which we choose as follows,
\[
V_{\mu_0} = \begin{pmatrix} e_{\mu_0}^\dagger \ \ 0_2 \ \ \cdots \ \ 0_2 \end{pmatrix}, \quad \ldots, \quad V_{\mu_i} = \begin{pmatrix} 0_2 \ \ \cdots \ \ e_{\mu_i} \ \ \cdots \ \ 0_2 \end{pmatrix}, \quad \ldots, \quad V_{\mu_{k-1}} = \begin{pmatrix} 0_2 \ \ \cdots \ \ e_{\mu_{k-1}}^\dagger \end{pmatrix} \quad (2.8)
\]
with \( e_{\mu_i}^\dagger \) in the \( \hat{i} \)-th position. With the help of these matrices we can introduce the complex \( 2k \times 2 \) matrix
\[
\mathbf{x} = x^\mu V_\mu = x^{\mu_0} V_{\mu_0} + \cdots + x^{\mu_{k-1}} V_{\mu_{k-1}} = \begin{pmatrix} x^{\mu_0} e_{\mu_0}^\dagger \\ \vdots \\ x^{\mu_{k-1}} e_{\mu_{k-1}}^\dagger \end{pmatrix} = \begin{pmatrix} x_0 \\ \vdots \\ x_{k-1} \end{pmatrix} \quad (2.9)
\]

Note that (2.1), (2.2), (2.5) and (2.9) are invariant only under the subgroup \( \text{Sp}(1) \times \text{Sp}(k)/\mathbb{Z}_2 \) of the \( \mathbb{R}^{4k} \) rotational isometry group \( \text{SO}(4k) \).

3 Generalized self-duality

The matrices (2.7) enjoy the properties
\[
e_{\mu_i}^\dagger e_{\nu_j} = e_{\mu_i}^\dagger e_{\nu_0} = \delta_{\mu_0 \nu_0} \mathbf{1}_2 + \eta_{i0}^\alpha i \sigma_\alpha =: \delta_{\mu_0 \nu_0} \mathbf{1}_2 + \eta_{\mu_0 \nu_0} \quad (3.1)
\]
\[
 e_{\mu_i}^\dagger e_{\nu_j} = e_{\mu_0}^\dagger e_{\nu_j} = \delta_{\mu_0 \nu_0} \mathbf{1}_2 + \bar{\eta}_{i0}^\alpha i \sigma_\alpha =: \delta_{\mu_0 \nu_0} \mathbf{1}_2 + \bar{\eta}_{\mu_0 \nu_0} \]

where \( \eta_{\mu_0 \nu_0} \) and \( \bar{\eta}_{\mu_0 \nu_0} \) denote the self-dual and anti-self-dual \( \text{'t} \) Hooft tensors, respectively, with \( \mu_0 \) and \( \nu_0 \) running from 1 to 4. Due to the identities (3.1) we have
\[
V_\mu V_\nu + V_\nu V_\mu = 2 \delta_{\mu\nu} \mathbf{1}_2 \quad (3.2)
\]
Let us define antihermitian $2k \times 2k$ matrices \[ N_{\mu\nu} := \frac{1}{2}(V_{\mu}V_{\nu}^{\dagger} - V_{\nu}V_{\mu}^{\dagger}) \] (3.3) and antihermitian $2 \times 2$ matrices \[ \bar{N}_{\mu\nu} := \frac{1}{2}(V_{\mu}^{\dagger}V_{\nu} - V_{\nu}^{\dagger}V_{\mu}) , \] (3.4) which in the four-dimensional case (when $k=1$) coincide with the matrices $\eta_{\mu\nu}$ and $\bar{\eta}_{\mu\nu}$, respectively. Note that for any $\mu, \nu = 1, \ldots, 4k$ we have $N_{\mu\nu} \in sp(k) \subset u(2k)$ and $\bar{N}_{\mu\nu} \in sp(1) \subset u(2)$. These matrix-valued antisymmetric tensors are invariant under $Sp(1) \times Sp(k)/\mathbb{Z}_2$ (acting on spacetime indices) since this subgroup preserves the quaternionic structure on $\mathbb{R}^{4k} \cong \mathbb{H}^k$. There exists a third $Sp(1) \times Sp(k)/\mathbb{Z}_2$ invariant tensor $M_{\mu\nu}$ taking values in the complement of $sp(1) \oplus sp(k)$ in $so(4k)$. Its explicit form can be found e.g. in \[17, 15\].

Let us further define a tensor \[ Q_{\mu\nu\rho\sigma} := \text{tr}(V_{\mu}^{\dagger}V_{\nu}V_{\rho}^{\dagger}V_{\sigma}) \] (3.5) and its total antisymmetrization \[ T_{\mu\nu\rho\sigma} := \frac{1}{12}Q_{\mu[\nu\rho\sigma]} = \frac{1}{12}(Q_{\mu\rho\sigma\nu} + Q_{\mu\sigma\nu\rho} - Q_{\mu\rho\nu\sigma} - Q_{\mu\sigma\rho\nu} - Q_{\mu\nu\sigma\rho} - Q_{\mu\sigma\rho\nu}) , \] (3.6) which generalizes $\varepsilon_{\mu\nu\rho\sigma}$ from four to $4k$ dimensions. By direct calculation one finds \[17\] that the matrices $N_{\mu\nu}$ are self-dual in this generalized sense,

\[ \frac{1}{2}T_{\mu\nu\rho\sigma}N_{\rho\sigma} = N_{\mu\nu} , \] (3.7) while

\[ \frac{1}{2}T_{\mu\nu\rho\sigma}\bar{N}_{\rho\sigma} = -\frac{2k+1}{3}\bar{N}_{\mu\nu} . \] (3.8)

4 Linear system and generalized self-dual Yang-Mills equations

We consider a gauge potential $A = A_{\mu}dx^{\mu}$ and the Yang-Mills field $F = dA + A \wedge A$ with components $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + [A_{\mu}, A_{\nu}]$, where $(x^{\mu}) \in \mathbb{R}^{4k}$ and $\partial_{\mu} := \partial/\partial x^{\mu}$. The fields $A_{\mu}$ and $F_{\mu\nu}$ take values in the Lie algebra $u(2r)$.

Let us consider the linear system of equations

\[ \lambda^{P}V_{aP}^{\mu}(\partial_{\mu}\psi + A_{\mu}\psi) = 0 \quad \text{for} \quad a = 1, \ldots, 2k , \] (4.1) where $\lambda^{P}$ with $P = 1, 2$ are homogeneous coordinates on the Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, the matrix elements $V_{aP}^{\mu}$ are given in (2.8), and the auxiliary $2r \times 2r$ matrix $\psi$ depends on $x^{\mu}$ and $\lambda^{P}$. The compatibility conditions of this linear system read

\[ (V_{aP}^{\mu}V_{bQ}^{\nu} + V_{aQ}^{\mu}V_{bP}^{\nu})F_{\mu\nu} = 0 . \] (4.2)

Using the definitions of $V_{\mu}, Q_{\mu\nu\rho\sigma}$ and $T_{\mu\nu\rho\sigma}$, we find that these conditions are equivalent \[13, 17\] to the equations

\[ \frac{1}{2}T_{\mu\nu\rho\sigma}F_{\rho\sigma} = F_{\mu\nu} . \] (4.3)
In four dimensions $T_{\mu\nu\rho\sigma} = \varepsilon_{\mu\nu\rho\sigma}$ and hence (4.3) coincides with the standard self-dual Yang-Mills equations. In higher dimensions these equations can be considered as generalized SDYM equations. Obviously, any gauge field fulfilling (4.3) also satisfies the second-order Yang-Mills equations due to the Bianchi identities.

Solutions to (4.3) arising from the linear system (4.1) can be constructed via a (generalized) twistor approach [13]. Namely, one can introduce a twistor space $\mathbb{R}^{4k} \times \mathbb{C}P^1$ of $\mathbb{R}^{4k}$. As a complex manifold, this space is the holomorphic vector bundle $P^2k+1 \rightarrow \mathbb{C}P^1$ with $P^2k+1 = \mathcal{O}(1) \otimes \mathbb{C}^{2k}$. (4.4)

More precisely, $P^2k+1$ is an open subset of $\mathbb{C}P^2k+1$ which is the twistor space of $\mathbb{H}P^k \subset \mathbb{H}^k = \mathbb{R}^{4k}$. With the help of the linear system (4.1), one can show [13] that there is a one-to-one correspondence between gauge equivalence classes of solutions to the self-duality equations (4.3) and equivalence classes of holomorphic vector bundles $E$ over $P^2k+1$ such that their restriction to any curve $\mathbb{C}P^1_x \hookrightarrow P^2k+1$ is trivial. Employing this correspondence, one can apply Ward’s splitting method [22, 13] for obtaining solutions to the self-duality equations (4.3).

Instead of (4.3) we might ponder about (cf. (3.8))

$$\frac{1}{2} T_{\mu\nu\rho\sigma} F_{\rho\sigma} = -\frac{2k+1}{3} F_{\mu\nu}$$

as generalized anti-self-duality equations. When $k\geq 2$ however, these equations cannot arise as compatibility conditions for a linear system [17]. Note that if one changes the definitions (2.8) by substituting $e^\dagger_{\mu i}$ with $e_{\mu i}$ and redefines $T_{\mu\nu\rho\sigma} \rightarrow - T_{\mu\nu\rho\sigma}$, then (4.3) and (4.5) (each with a sign change) will generalize the four-dimensional anti-self-duality and self-duality equations, respectively [15]. Yet, this does not change the fact that only (4.3) arises from the integrability conditions (4.2) of the linear system (4.1).

5 Extended ADHM construction in 4k dimensions

Recall that we consider gauge fields with values in the Lie algebra $u(2r) \supset sp(r)$. A simple extension of the ADHM construction to $4k$ dimension [17] for generating solutions to (4.3) is based on

$$a \quad (2\ell+2r) \times 2r \quad matrix \quad \Psi \quad and\quad (5.1)$$

$$a \quad (2\ell+2r) \times 2\ell \quad matrix \quad \Delta = a + b \quad (x \otimes 1_{\ell}) = a + \sum_{i=0}^{k-1} b_i \quad (x_i \otimes 1_{\ell}) , \quad (5.2)$$

where $a$ and $b_i$ are constant $(2\ell+2r) \times 2\ell$ matrices.\(^1\) The matrices (5.1) and (5.2) are subject to the following conditions:

$$\Delta^\dagger \Delta \quad is \quad invertible , \quad (5.3)$$

$$[\Delta^\dagger \Delta , V_\mu \otimes 1_{\ell}] = 0 , \quad (5.4)$$

$$\Delta^\dagger \Psi = 0 , \quad (5.5)$$

$$\Psi^\dagger \Psi = 1_{2r} , \quad (5.6)$$

$$\Psi \Psi^\dagger + \Delta (\Delta^\dagger \Delta)^{-1} \Delta^\dagger = 1_{2\ell+2r} . \quad (5.7)$$

\(^1\)Note that $b$ is a constant $(2\ell+2r) \times 2k\ell$ matrix, $x$ is the $2k \times 2$ matrix given in (2.9) and $x_i = x^{\mu i} e^\dagger_{\mu i}$ is $2 \times 2$. Correspondingly, $x \otimes 1_{\ell}$ and $x_i \otimes 1_{\ell}$ are $2k\ell \times 2\ell$ and $2\ell \times 2\ell$ matrices, respectively.
The completeness relation (5.7) means that \( \Psi \Psi^\dagger \) and \( \Delta (\Delta^\dagger \Delta)^{-1} \Delta^\dagger \) are projectors on orthogonal complementing subspaces of \( \mathbb{C}^{2^f+2r} \).

For \((\Delta, \Psi)\) satisfying (5.3)–(5.7) the gauge potential is chosen in the form

\[
A = \Psi^\dagger d\Psi \Rightarrow A_\mu = \Psi^\dagger \partial_\mu \Psi.
\] (5.8)

The components of the gauge field \( F \) then take the form

\[
F_{\mu\nu} = \partial_\mu (\Psi^\dagger \partial_\nu \Psi) - \partial_\nu (\Psi^\dagger \partial_\mu \Psi) + \{ \Psi^\dagger \partial_\mu \Psi, \Psi^\dagger \partial_\nu \Psi \}
\]
\[
= \Psi^\dagger \{ (\partial_\mu \Delta)(\Delta^\dagger \Delta)^{-1} \partial_\nu \Delta^\dagger - (\partial_\nu \Delta)(\Delta^\dagger \Delta)^{-1} \partial_\mu \Delta^\dagger \} \Psi
\]
\[
= 2 \Psi^\dagger b N_{\mu\nu}(\Delta^\dagger \Delta)^{-1} b^\dagger \Psi.
\] (5.9)

From (5.4) and (5.9) one easily sees that \( F^\dagger_{\mu\nu} = -F_{\mu\nu} \). Due to (3.7) the gauge field (5.9) indeed solves the generalized SDYM equations (4.3).

### 6 Self-dual noncommutative deformation

A Moyal deformation of Euclidean \( \mathbb{R}^{4k} \) is achieved by replacing the ordinary pointwise product of functions on it by the nonlocal but associative Moyal star product. The latter is characterized by a constant antisymmetric matrix \((\theta^{\mu\nu})\) which prominently appears in the star commutation relation between the coordinates,

\[
[x^\mu, x'^\nu] = i\theta^{\mu\nu}.
\] (6.1)

A different realization of this algebraic structure keeps the standard product but promotes the coordinates (and thus all their functions) to noncommuting operators acting in an auxiliary Fock space \( \mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_{k-1} \), where each \( \mathcal{H}_j \) is a two-oscillator Fock space [1]. The two formulations are tightly connected through the Moyal-Weyl map. When dealing with noncommutative U(2r) Yang-Mills theory from now on, we shall not denote the noncommutativity by either inserting stars in all products or by putting hats on all operator-valued objects, but simply by agreeing that our coordinates are subject to (6.1). The existence of \((\theta^{\mu\nu})\) breaks the Euclidean SO(4k) symmetry but we may employ SO(4k) rotations to go to a basis in which \((\theta^{\mu\nu})\) takes Darboux form, i.e. the only nonzero entries are

\[
\theta^{4i+1,4i+2} = -\theta^{4i+2,4i+3} \quad \text{and} \quad \theta^{4i+3,4i+4} = -\theta^{4i+4,4i+1} \quad \text{for} \quad i = 0, \ldots, n,
\] (6.2)

where we defined \( n := k-1 \) for convenience. Such a matrix is block-diagonal, with each \(4 \times 4\) block labelled by \( i \) being some linear combination of the self-dual 't Hooft tensor \((\eta^{3\mu_0\nu_0})\) and the anti-self-dual one \((\bar{\eta}^{3\mu_0\nu_0})\). In this letter we shall restrict ourselves to the purely self-dual situation,

\[
\theta^{\mu_0\nu_0} = \theta_i \eta^{3\mu_0\nu_0} \quad \text{and zero otherwise},
\] (6.3)

which is characterized by real numbers \((\theta_0, \theta_1, \ldots, \theta_n)\). In this case the extended ADHM construction of the previous section survives the deformation unharmed.
7 Noncommutative 't Hooft-type solution in 4k dimensions

For making contact with the results of [21], we choose in the entries of the noncommutativity matrix (6.3) as
\[ \theta_0 = -\theta_i =: \theta \quad \text{for} \quad i = 1, \ldots, n. \] (7.1)

Let us pick the gauge group U(2) (i.e. \( r = 1 \)) and take\(^2\) (see (5.2))
\[ \begin{pmatrix} \Lambda_1 \mathbf{1}_2 & \cdots & \Lambda_n \mathbf{1}_2 \\ 0_2 & \cdots & 0_2 \\ \vdots & \ddots & \vdots \\ 0_2 & \cdots & 0_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 & \cdots & 0_2 \\ b_i^1 \mathbf{1}_2 & \cdots & 0_2 \\ \vdots & \ddots & \vdots \\ 0_2 & \cdots & b_n^2 \mathbf{1}_2 \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \Psi_0 \\ \Psi_1 \\ \vdots \\ \Psi_n \end{pmatrix}, \] (7.2)

where \( \Lambda_i \) and \( b_i^j \) are real constants and we have put \( \ell = n \) (see section 5). Moreover, we choose
\[ b_i^0 = 1 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad b_1^1 = \cdots = b_n^n = -1 \quad \text{but} \quad b_i^j = 0 \quad \text{otherwise}. \] (7.3)

With this selection we obtain
\[ \Delta = \begin{pmatrix} \Lambda_1 \mathbf{1}_2 & \cdots & \Lambda_n \mathbf{1}_2 \\ \tilde{x}_1 & \cdots & 0_2 \\ \vdots & \ddots & \vdots \\ 0_2 & \cdots & \tilde{x}_n \end{pmatrix} \quad \text{and} \quad \Delta^\dagger = \begin{pmatrix} \Lambda_1 \mathbf{1}_2 & \tilde{x}_1^\dagger & \cdots & 0_2 \\ \vdots & \ddots & \vdots \\ \Lambda_n \mathbf{1}_2 & 0_2 & \cdots & \tilde{x}_n^\dagger \end{pmatrix}, \] (7.4)

where
\[ \tilde{x}_i := x_0 - x_i \quad \text{for} \quad i = 1, \ldots, n. \] (7.5)

Using (3.1) and (6.3)–(7.1), we find that
\[ \tilde{x}_i^\dagger \tilde{x}_i = \tilde{x}_i \tilde{x}_i^\dagger = \tilde{r}_i^2 \mathbf{1}_2 \quad \text{with} \quad \tilde{r}_i^2 = \tilde{x}_i^\mu \tilde{x}_i^\nu \quad \text{(no sum over} \ i). \] (7.6)

At this point we observe that (6.3)–(7.1), (7.6) and (7.4)–(7.5) coincide with formulae (4.1), (4.3) and (2.18) from [21] (where noncommutative 't Hooft instants in four dimensions were discussed) if we identify our \((x_0, x_i)\) with their \((x, a_i)\). Therefore, all computations in [21] also extend to our case. In particular, the operator \( \tilde{r}_i^2 \) is invertible on the Fock space \( \mathcal{H} \), and the task to solve (5.5) and (5.6) is accomplished with (cf. (4.9) of [21])
\[ \Psi_0 = \phi_n^{-\frac{1}{2}} \mathbf{1}_2 \quad \text{and} \quad \Psi_i = -\tilde{x}_i \frac{\Lambda_i}{\tilde{r}_i^2} \phi_n^{-\frac{1}{2}}, \] (7.7)

where
\[ \phi_n = 1 + \sum_{i=1}^{n} \frac{\Lambda_i^2}{\tilde{r}_i^2}. \] (7.8)

Furthermore, by direct calculation one can show that the completeness relation (5.7) is satisfied. Therefore, we can define a gauge potential via (5.8) and obtain from (5.9) a self-dual gauge field on \( \mathbb{R}^{4k} \).

\(^2\)The same ansatz can be considered for the gauge group U(2r). For a more general 't Hooft-type ansatz see [17].
We remark that the fields \( A_{\mu_0} = \Psi^\dagger \partial_{\mu_0} \Psi \) and

\[
F_{\mu_0\nu_0} = 2\Psi^\dagger b N_{\mu_0\nu_0}(\Delta^\dagger \Delta)^{-1} b^\dagger \Psi = 2\Psi^\dagger b \begin{pmatrix} \eta_{\mu_0\nu_0} & \cdots & 0_2 \\ \vdots & \ddots & \vdots \\ 0_2 & \cdots & 0_2 \end{pmatrix} (\Delta^\dagger \Delta)^{-1} b^\dagger \Psi \tag{7.9}
\]

do not contain any derivatives with respect to \( x_1, \ldots, x_n \). Hence, (7.9) exactly reproduces the noncommutative \( n \)-instanton solution in four dimension as derived in [21], if only their translational moduli \( a_i \) are identified with our coordinates \( x_i \) on \( \mathbb{R}^{4n} \), which complements \( \mathbb{R}^n_0 \) in \( \mathbb{R}^k_0 \) (recall \( k = n+1 \)). Since \( x_0 \) and \( x_i \) with \( i = 1, \ldots, n \) are operators acting on the Fock space \( \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n \otimes \mathbb{C}^2 \) one may interpret the configuration (7.9) as a quantum ‘t Hooft \( n \)-instanton. Note, however, that the scale parameters \( \Lambda_2^i \) entering \( \Psi \) and (7.9) are not quantized.

8 Noncommutative BPST-type solution in eight dimensions

In the commutative case, as it was explained in [17], the ‘t Hooft-type ansatz of the previous section produces solutions of the YM equations on \( \mathbb{R}^{4k} \) which cannot be extended to the quaternionic projective space \( \mathbb{H}P^k \) if \( k \geq 2 \). Hence, these solutions cannot have finite topological charges beyond \( k=1 \). However, their four-dimensional “slice” describes ‘t Hooft instantons both in the commutative and noncommutative cases. We shall now consider a different kind of ansatz for \( a \) and \( b \) in (5.2) which generates true instanton-type configurations (with finite Pontryagin numbers) since in the commutative limit \( \theta_i \to 0 \) they can be extended to \( \mathbb{H}P^k \).

For simplicity, we restrict ourselves to eight dimensions \((k=2)\) and to the gauge group \( U(4) \). For the commutative case this kind of ansatz was discussed in Appendix B of [17], but the matrix \( \Psi \) proposed there fails to obey (5.5). By properly solving the extended ADHM equations (5.3)--(5.7) for the eight-dimensional \( U(4) \) setup we shall derive a smooth noncommutative instanton configuration which in the commutative limit generalizes the finite action solutions on \( \mathbb{R}^4 \) to \( \mathbb{R}^8 \).

We consider the noncommutative space \( \mathbb{R}^8_0 \) with coordinates \( (x^\mu) = (x^{\mu_0}, y^{\mu_0}) \) such that

\[
[x^{\mu_0}, x^{\nu_0}] = i \theta_0 \eta^{\mu_0\nu_0}, \quad [x^{\mu_0}, y^{\nu_0}] = 0 \quad \text{and} \quad [y^{\mu_0}, y^{\nu_0}] = i \theta_1 \eta^{\mu_0\nu_0}, \tag{8.1}
\]

with \( \theta_0, \theta_1 > 0 \). We introduce

\[
x = x^{\mu_0} e_{\mu_0}^\dagger, \quad y = y^{\mu_0} e_{\mu_0}^\dagger \quad \text{and} \quad x = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{8.2}
\]

which obey

\[
x^\dagger x = x^{\mu_0} x^{\mu_0} \mathbf{1}_2 = r_0^2 \mathbf{1}_2 \quad \text{and} \quad y^\dagger y = y^{\mu_0} y^{\mu_0} \mathbf{1}_2 = r_1^2 \mathbf{1}_2, \tag{8.3}
\]

\[
xx^\dagger = \begin{pmatrix} r_0^2 - 2\theta_0 & 0 \\ 0 & r_0^2 + 2\theta_0 \end{pmatrix} \quad \text{and} \quad yy^\dagger = \begin{pmatrix} r_1^2 - 2\theta_1 & 0 \\ 0 & r_1^2 + 2\theta_1 \end{pmatrix}. \tag{8.4}
\]

Let us choose

\[
a = \begin{pmatrix} \Lambda_1 \mathbf{1}_2 \\ 0_2 \\ 0_2 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0_2 \\ \mathbf{1}_2 \\ 0_2 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0_2 \\ \mathbf{1}_2 \\ 0_2 \end{pmatrix} \Rightarrow b = \begin{pmatrix} 0_2 & 0_2 \\ \mathbf{1}_2 & 0_2 \\ 0_2 & \mathbf{1}_2 \end{pmatrix}. \tag{8.5}
\]
and parametrize
\[
\Psi = \begin{pmatrix}
\psi_0 & \phi_0 \\
\psi_1 & \phi_1 \\
\psi_2 & \phi_2
\end{pmatrix},
\]
where \( \Lambda \) is a scale parameter and all blocks \( \psi_0, \ldots, \phi_2 \) of the 6\times4 matrix \( \Psi \) are 2\times2 matrices. Then we obtain (cf. [17])
\[
\Delta = a + bx = \begin{pmatrix}
\Lambda_2 x \\
y
\end{pmatrix} \Rightarrow \Delta^\dagger = (\Lambda_2 x^\dagger y^\dagger) \Rightarrow \Delta^\dagger \Delta = (r^2 + \Lambda^2) \cdot 1_2,
\]
where \( r^2 = r_0^2 + r_1^2 \), and the extended ADHM equations (5.5) become
\[
\Lambda \psi_0 + x^\dagger \psi_1 + y^\dagger \psi_2 = 0 \quad \text{and} \quad \Lambda \phi_0 + x^\dagger \phi_1 + y^\dagger \phi_2 = 0.
\]
We choose a solution to these equations in the form
\[
\Psi = \begin{pmatrix}
x^\dagger \\
-\Lambda_2 & 0 \\
0 & -\Lambda_2
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} = \begin{pmatrix}
x^\dagger a \\
x^\dagger b + y^\dagger c
\end{pmatrix} \Rightarrow \Psi^\dagger = \begin{pmatrix}
a^\dagger x \\
b^\dagger x + c^\dagger y \\
-\Lambda a^\dagger & -\Lambda b^\dagger & -\Lambda c^\dagger
\end{pmatrix},
\]
where the 2\times2 matrices \( a, b \) and \( c \) are fixed by the normalization condition (5.6) to be
\[
a = \begin{pmatrix}
(r_0^2 + \Lambda^2 - 2\theta_0)^{-\frac{1}{2}} & 0 \\
0 & (r_0^2 + \Lambda^2 + 2\theta_0)^{-\frac{1}{2}}
\end{pmatrix}, \quad b = -a^\dagger x y^\dagger c,
\]
\[
c = \frac{(r_0^2 + \Lambda^2)^{\frac{1}{2}}}{\Lambda} \begin{pmatrix}
(r^2 + \Lambda^2 - 2\theta_1)^{-\frac{1}{2}} & 0 \\
0 & (r^2 + \Lambda^2 + 2\theta_1)^{-\frac{1}{2}}
\end{pmatrix}.
\]
To be sure that our solution (8.9)–(8.10) contains all zero modes of the operator \( \Delta^\dagger \), we should check the completeness relation (5.7). It is known that in the noncommutative case the latter may be violated [9] unless an additional effort is made. After a lengthy calculation we obtain
\[
\Psi \Psi^\dagger = \begin{pmatrix}
x^\dagger \Lambda_2 & -x \Lambda_2 & -x \Lambda_2 & y^\dagger \\
x \Lambda_2 & 1 - y \Lambda_2 & x \Lambda_2 & x \Lambda_2 \\
y \Lambda_2 & 1 - y \Lambda_2 & y \Lambda_2 & y \Lambda_2 \\
y^\dagger & x^\dagger \Lambda_2 & x^\dagger \Lambda_2 & 1 - y^\dagger \Lambda_2
\end{pmatrix}
\]
and
\[
\Delta (\Delta^\dagger \Delta)^{-1} \Delta^\dagger = \begin{pmatrix}
\Lambda_2 & -x \Lambda_2 & -x \Lambda_2 & y^\dagger \\
x \Lambda_2 & 1 - y \Lambda_2 & x \Lambda_2 & x \Lambda_2 \\
y \Lambda_2 & 1 - y \Lambda_2 & y \Lambda_2 & y \Lambda_2 \\
y^\dagger & x^\dagger \Lambda_2 & x^\dagger \Lambda_2 & 1 - y^\dagger \Lambda_2
\end{pmatrix},
\]
and the latter two matrices add up to \( 1_6 \), as the completeness relation (5.7) demands. So, for our solution \( (\Delta, \Psi) \) of (5.4)–(5.7) the gauge field (5.9) satisfies the self-duality equations (4.3) (and the full YM equations) on \( \mathbb{R}^8 \).

In the commutative limit \( \theta_0, \theta_1 \to 0 \) our \( (\Delta, \Psi) \) produces a novel solution to the generalized SDYM equations (4.3) on \( \mathbb{R}^8 \cong \mathbb{H}^2 \). In [17] it was argued that such kind of solutions can be
extended to the compact space $\mathbb{H}P^2$. Briefly, the arguments are as follows [17]. Homogeneous coordinates on $\mathbb{H}P^2$ are
\[
\begin{pmatrix}
z' \\
x' \\
y'
\end{pmatrix} \in \mathbb{H}^3.
\] (8.13)

The patch $U_1$ on $\mathbb{H}P^2$ is defined by the condition $\det(z') \neq 0$. Coordinates $(x, y) = (x_1, y_1)$ on this patch are obtained from (8.13) by multiplying with $(z')^{-1}$,
\[
U_1 : \begin{pmatrix} z' \\ x' \\ y' \end{pmatrix} \mapsto \begin{pmatrix}
1 \\
x' (z')^{-1} \\
y' (z')^{-1}
\end{pmatrix} =: \begin{pmatrix} x_1 \\ y_1 \\
y_1
\end{pmatrix}, \quad (8.14)
\]
and a scale parameter $\Lambda$ can be introduced by redefining $x_1 \mapsto \Lambda^{-1} x_1$, $y_1 \mapsto \Lambda^{-1} y_1$ and multiplying the last column in (8.14) by $\Lambda$. This is the patch on which we worked. One can also consider two other patches, which are defined by conditions $\det(x') \neq 0$ and $\det(y') \neq 0$:
\[
U_2 : \begin{pmatrix} z' \\ x' \\ y' \end{pmatrix} \mapsto \begin{pmatrix}
1 \\
x' (y')^{-1} \\
y' (y')^{-1}
\end{pmatrix} =: \begin{pmatrix} x_2 \\ y_2 \\
y_2
\end{pmatrix} \quad \text{and} \quad U_3 : \begin{pmatrix} z' \\ x' \\ y' \end{pmatrix} \mapsto \begin{pmatrix}
1 \\
x' (y')^{-1} \\
y' (y')^{-1}
\end{pmatrix} =: \begin{pmatrix} x_3 \\ y_3 \\
y_3
\end{pmatrix}, \quad (8.15)
\]
with coordinates $(x_2, y_2)$ and $(x_3, y_3)$ and construct solutions $(\Delta, \Psi)$ and $(A, F)$ on them by reproducing all steps we performed on the patch $U_1$. On the intersections of these patches (local) solutions will be glued by transition functions. For more details see [17].

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References

[1] N. Seiberg and E. Witten, JHEP **9909** (1999) 032 [hep-th/9908142];
M.R. Douglas and N.A. Nekrasov, Rev. Mod. Phys. **73** (2001) 977 [hep-th/0106048];
R.J. Szabo, Phys. Rept. **378** (2003) 207 [hep-th/0109162].
[2] A.A. Belavin, A.M. Polyakov, A.S. Schwarz and Y.S. Tyupkin, Phys. Lett. B **59** (1975) 85.
[3] E.B. Bogomolny, Sov. J. Nucl. Phys. **24** (1976) 449;
M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. **35** (1975) 760.
[4] C.H. Taubes, Commun. Math. Phys. **72** (1980) 277; Commun. Math. Phys. **75** (1980) 207.
[5] N. Nekrasov and A. Schwarz, Commun. Math. Phys. **198** (1998) 689 [hep-th/9802068].
[6] D.J. Gross and N.A. Nekrasov, JHEP **0103** (2001) 044 [hep-th/0010090].
[7] D.P. Jatkar, G. Mandal and S.R. Wadia, JHEP **0009** (2000) 018 [hep-th/0007078];
D. Bak, Phys. Lett. B **495** (2000) 251 [hep-th/0008204];
D. Bak, K.M. Lee and J.H. Park, Phys. Rev. D **63** (2001) 125010 [hep-th/0011099].
[8] A.P. Polychronakos, Phys. Lett. B 495 (2000) 407 [hep-th/0007043];
D.J. Gross and N.A. Nekrasov, JHEP 0010 (2000) 021 [hep-th/0007204].

[9] D.H. Correa, G.S. Lozano, E.F. Moreno and F.A. Schaposnik,
Phys. Lett. B 515 (2001) 206 [hep-th/0105085];
C.S. Chu, V.V. Khoze and G. Travaglini, Nucl. Phys. B 621 (2002) 101 [hep-th/0108007];
O. Lechtenfeld and A.D. Popov, JHEP 0203 (2002) 040 [hep-th/0109209];
Y. Tian and C.J. Zhu, Phys. Rev. D 67 (2003) 045016 [hep-th/0210163].

[10] K. Furuuchi, JHEP 0103 (2001) 033 [hep-th/0010119];
Z. Horváth, O. Lechtenfeld and M. Wolf, JHEP 0212 (2002) 060 [hep-th/0211041].

[11] M. Hamanaka, Noncommutative solitons and D-branes, hep-th/0303256;
O. Lechtenfeld and A.D. Popov, JHEP 0401 (2004) 069 [hep-th/0306263];
D.H. Correa, P. Forgacs, E.F. Moreno, F.A. Schaposnik and G.A. Silva,
JHEP 0407 (2004) 037 [hep-th/0404015].

[12] E. Corrigan, C. Devchand, D.B. Fairlie and J. Nuyts, Nucl. Phys. B 214 (1983) 452.

[13] R.S. Ward, Nucl. Phys. B 236 (1984) 381.

[14] D.B. Fairlie and J. Nuyts, J. Phys. A 17 (1984) 2867;
S. Fubini and H. Nicolai, Phys. Lett. B 155 (1985) 369;
A.D. Popov, Europhys. Lett. 17 (1992) 23; Europhys. Lett. 19 (1992) 465;
T.A. Ivanova and A.D. Popov, Lett. Math. Phys. 24 (1992) 85;
E.G. Floratos and G.K. Leontaris, math-ph/0011027.

[15] A.D. Popov, Mod. Phys. Lett. A 7 (1992) 2077;
T.A. Ivanova and A.D. Popov, Theor. Math. Phys. 94 (1993) 225.

[16] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, Phys. Lett. A 65 (1978) 185.

[17] E. Corrigan, P. Goddard and A. Kent, Commun. Math. Phys. 100 (1985) 1.

[18] M. Mihailescu, I.Y. Park and T.A. Tran, Phys. Rev. D 64 (2001) 046006 [hep-th/0011079];
E. Witten, JHEP 0204 (2002) 012 [hep-th/0012054];
P. Kraus and M. Shigemori, JHEP 0206 (2002) 034 [hep-th/0110035];
N.A. Nekrasov, Lectures on open strings, and noncommutative gauge fields, hep-th/0203109.

[19] A. Fujii, Y. Imaizumi and N. Ohta, Nucl. Phys. B 615 (2001) 61 [hep-th/0105079];
M. Hamanaka, Y. Imaizumi and N. Ohta, Phys. Lett. B 529 (2002) 163 [hep-th/0112050];
D.S. Bak, K.M. Lee and J.H. Park, Phys. Rev. D 66 (2002) 025021 [hep-th/0204221];
Y. Hiraoka, Phys. Rev. D 67 (2003) 105025 [hep-th/0301176].

[20] A.D. Popov, A.G. Sergeev and M. Wolf, J. Math. Phys. 44 (2003) 4527 [hep-th/0304263];
T.A. Ivanova and O. Lechtenfeld, Phys. Lett. B 567 (2003) 107 [hep-th/0305195];
O. Lechtenfeld, A.D. Popov and R.J. Szabo, JHEP 0312 (2003) 022 [hep-th/0310267].

[21] T.A. Ivanova, O. Lechtenfeld and H. Müller-Ebhardt,
Mod. Phys. Lett. A 19 (2004) 2419 [hep-th/0404127].

[22] R.S. Ward, Phys. Lett. A 61 (1977) 81.