THE COVERING DIMENSION
OF THE SORGENFREY PLANE

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ABSTRACT. It is proved that the square of the Sorgenfrey line has infinite covering dimension.

Covering dimension dim was originally introduced by Lebesgue for domains in Euclidean spaces [1]. In 1933 Čech extended Lebesgue’s definition to completely regular spaces [2], and in 1950 Katětov proposed a different definition, which coincided with Čech’s one for normal spaces [3]. Since then, these two versions of covering dimension have become equally popular. Thus, the books [4], [5], and [6] use Čech’s definition, while, e.g., [7] prefers Katětov’s one. There are also books on dimension theory which consider only normal or even only metrizable spaces [8], [9].

We will denote Čech’s covering dimension by dim and Katětov’s by dim₀ and refer to them as “covering dimension” and “cozero covering dimension,” respectively. The definitions of both of them involve the well-known notion of the order of a cover.

Definition 1. Let X be a set, and let ℱ be a family of its subsets. If there exists an integer n ≥ −1 such that each point x ∈ X belongs to at most n + 1 elements of ℱ, then the least such n is called the order of ℱ and denoted by ord ℱ. If there is no such an integer n, then ord ℱ = ∞.

In other words, ord ℱ is the largest integer n for which there are n + 1 members of ℱ with nonempty intersection if such an integer exists and ord ℱ = ∞ otherwise.

Definition 2 (Čech (Katětov)). The covering dimension dim X (cozero covering dimension dim₀ X) of a topological space X is the least integer n ≥ −1 such that any finite open (cozero) cover of X has a finite open (cozero) refinement of order n, provided that such an integer exists. If it does not exist, then dim X = ∞ (dim₀ X = ∞).

By a strongly zero-dimensional space most authors mean a space X with dim₀ X = 0 (although, e.g., Charalambous [6] uses this term for a space X with dim X = 0, in which case a nonnormal space cannot be strongly zero-dimensional).

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The definitions of the two other most classical dimension functions, ind and Ind, can be found in any book on dimension theory and are defined in more or less the same way. The only minor distinction in their definitions is that some authors consider the ind (Ind) dimension to be undefined for nonregular (nonnormal) spaces, while others define these dimensions to be infinite for such spaces. We follow the latter.

It follows directly from the definitions that
\begin{itemize}
  \item if \( \dim X = 0 \), then \( X \) is normal;
  \item \( \dim_0 X = \dim_0 \beta X = \dim \beta X \).
\end{itemize}

This paper is devoted to the Čech covering dimension of the Sorgenfrey plane \( S \times S \). (We use the standard notation \( S \) for the Sorgenfrey line and assume that the underlying set of \( S \) is \( \mathbb{R} \).) Obviously, \( \text{ind} S = \text{ind} S^{\omega} = 0 \), \( \text{Ind} S \times S = \infty \) (because \( S \times S \) is not normal), and \( \dim S \times S > 0 \) (for the same reason). It is also known that \( \dim S = 0 \), because \( S \) is Lindelöf and \( \dim \leq \text{ind} \) for Lindelöf spaces (see, e.g., [6, Proposition 5.3]), and that \( \dim_0 S^{\kappa} = 0 \) and hence \( \dim_0 S^n = 0 \) for all \( n \in \omega \) [10]. Moreover, \( \dim_0 S^{\kappa} = 0 \) for any cardinal \( \kappa \); Terasawa credited this result to K. Morita in [10], and its proof can be found in [11]. We prove that \( \dim S \times S = \infty \).

**Theorem 1.** \( \dim S \times S = \infty \).

The proof of this theorem uses shrinkings of covers and several notions related to Baire categories.

Recall that a cover \( \{ V_\alpha : \alpha \in A \} \) of a set \( X \) is a **shrinking** of a cover \( \{ U_\alpha : \alpha \in A \} \) if \( V_\alpha \subset U_\alpha \) for each \( \alpha \in A \).

**Fact 1** (see, e.g., [6, Proposition 2.10]). Given any space \( X \) and any integer \( n \geq -1 \), \( \dim X \leq n \) if and only if any finite open cover \( \{ U_1, \ldots, U_k \} \) of \( X \) has an open shrinking \( \{ V_1, \ldots, V_k \} \) of order \( \leq n \).

Sets of first Baire category are said to be **meager** and sets of the second Baire category, **nonmeager**. Complements to meager sets are **comeager**, or **residual**. It is easy to show that any comeager set is nonmeager, but not vice versa.

An example of a nonmeager set in \( \mathbb{R} \) which is not comeager is any Bernstein set, i.e., a set \( B \subset \mathbb{R} \) such that both \( B \) and \( \mathbb{R} \setminus B \) intersect all uncountable closed (or, equivalently, compact) subsets of \( \mathbb{R} \).

**Fact 2** ([12]). There exist \( 2^{\aleph_0} \) disjoint Bernstein sets.

**Fact 3** (see, e.g., [13, Chap. 3, Sec. 40, I, Theorem 3]). The intersection of any Bernstein set with any nontrivial interval is of second category in \( \mathbb{R} \).

**Proof of the theorem.** Take any \( n \in \mathbb{N} \). We must prove that \( \dim S \times S \geq n \). To this end, it suffices to produce a finite open cover of \( S \times S \) which has no finite open shrinking of order \( \leq n \). We take disjoint Bernstein sets \( B_1, \ldots, B_{n+1} \) and put \( B_{n+2} = \mathbb{R} \setminus (B_1 \cup \cdots \cup B_{n+1}) \); clearly, \( B_{n+2} \) is a Bernstein set as well.
We set $D = \{(x, -x) : x \in \mathbb{R}\} \subset S \times S$ and consider the cover $\mathcal{U} = \{U_0, U_1, \ldots, U_{n+2}\}$ of $S \times S$, where

$$U_0 = \{(x, y) : y < -x\}$$

and

$$U_i = \bigcup \{(x, \infty) \times (-\infty, y) : x \in B_i\} \text{ for } i \leq n + 2.$$ 

Let $\mathcal{V} = \{V_0, V_1, \ldots, V_{n+2}\}$ be an open shrinking of $\mathcal{U}$.

Obviously, $V_0 = U_0$ and $V_i \cap D = U_i \cap D = \{(x, -x) : x \in B_i\}$ for $i = 1, \ldots, n + 2$.

For $i = 1, \ldots, n + 2$ and $k \in \mathbb{N}$, we set $B_i(k) = \{x \in B_i : [x, x + \frac{1}{k}] \times [-x, -x + \frac{1}{k}] \subset V_i\}$. Obviously, $B_i(k) \subset B_i(m)$ for $m \leq k$. Our immediate goal is to define positive integers $k_1 \leq \cdots \leq k_{n+2}$ and nonempty intervals $(a_1, b_1) \supset \cdots \supset (a_{n+2}, b_{n+2}) \neq \emptyset$ so that the set $B_i(k_j) \cap (a_j, b_j)$ is dense in $(a_j, b_j)$ with respect to the usual topology on $\mathbb{R}$ for each $j \leq n + 2$ and each $i \leq j$.

Since $\bigcup_{i \in \mathbb{N}} B_1(i) = B_1$ and $B_1$, being a Bernstein set, is nonmeager in $\mathbb{R}$, it follows that $B_1(k_1)$ is not nowhere dense in $\mathbb{R}$ for some positive integer $k_1$. This means that $B_1(k_1) \cap U$ is dense in $U$ for some nonempty open set $U \subset \mathbb{R}$; hence there exist $a_1, b_1 \in \mathbb{R}$, $a_1 < b_1$, for which $B_1(k_1) \cap (a_1, b_1) = [a_1, b_1]$ (the closure is in $\mathbb{R}$).

Suppose that $1 < j \leq n + 2$ and we have already found integers $k_i$ and constructed intervals $(a_i, b_i)$ for $i < j$.

The set $B_j$ intersects all uncountable closed subsets of the interval $[a_{j-1}, b_{j-1}]$, because all such sets are also closed in $\mathbb{R}$, and $B_j \cap [a_{j-1}, b_{j-1}]$ is of second Baire category in $[a_{j-1}, b_{j-1}]$ (see Fact 3). Since $B_j(k) \subset B_j(m)$ for $k \leq m$, there exists a positive integer $k_j \geq k_{j-1}$ for which $B_j(k)$ is not nowhere dense in $[a_{j-1}, b_{j-1}]$. Let $(a_j, b_j) \subset (a_{j-1}, b_{j-1})$ be a nonempty interval for which $B_j(k_j) \cap (a_j, b_j) = (a_j, b_j)$.

At the $(n+2)$th step we obtain an interval $(a_{n+2}, b_{n+2})$ and a number $k_{n+2}$. We set $(a, b) = (a_{n+2}, b_{n+2})$ and $k = k_{n+2}$. By construction $(a, b) \subset (a_i, b_i)$ and $B_i(k_i) \cap (a, b) = [a, b]$ for all $i \leq n + 2$. Since $k \geq k_i$ and $B_i(k_i) \subset B_i(k)$ for all $i \leq n + 2$, it follows that $B_i(k) \cap (a, b) = [a, b]$ for all $i \leq n + 2$.

For each $j \leq n + 2$, there is a point $x_j \in (a, b) \cap (a, a + \frac{1}{k}) \cap B_j(k)$. It is easy to see that $[x_j, x_j + \frac{1}{k}] \times [-x_j, -x_j + \frac{1}{k}] \supset (a + \frac{1}{k}, -a)$. By the definition of $B_j(k)$ we have $[x_j, x_j + \frac{1}{k}] \times [-x_j, -x_j + \frac{1}{k}] \subset V_j$. Therefore, $(a + \frac{1}{k}, -a) \in V_j$ for all $j = 1, \ldots, n + 2$, and the order of the cover $\mathcal{V}$ equals $n + 1$.

**Remark.** The same argument proves that the covering dimension of the Niemytski plane is infinite.

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