Counting of Shortest Paths in Cubic Grid

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The enumeration of shortest paths in cubic grid is presented here. The cubic grid considers three neighborhoods—6-neighborhood (face connectivity), 18-neighborhood (edge connectivity), and 26-neighborhood (vertex connectivity). The formulation for distance metrics are given here. $L_1$, $D_{18}$, and $L_\infty$ are the three metrics for 6-neighborhood, 18-neighborhood, and 26-neighborhood. The problem is to find the number of shortest paths based on neighborhoods between two given points in 3D cubic grid represented by coordinate triplets. The formulation for the three neighborhoods are presented here. This problem has theoretical importance and practical aspects.

**Keywords:** Cubic grid, digital distances, shortest paths, combinatorics, path counting

1 Introduction

Shortest path problems have ample applications in digital geometry, which works on discrete space, i.e., points can have only integer coordinates. Based on the application, the shortest path problem can be formulated. Shortest path problems in various grids are defined based on digital distances. In digital geometry, two basic neighborhood relations are defined in the square grid [34]—cityblock and chessboard. The cityblock motion allows horizontal and vertical movements only, while in the chessboard motion the diagonal directions are also permitted. So, based on these motions, two kinds of distances are defined in this grid, which are well explained in [21, 33]. Each coordinate of a point in square grid is independent of the other. Generalizing the concepts to $n$ dimensions, $n$ independent coordinates are used. In the $n$-dimensional hyper cubic grid, the structure of the nodes is isomorphic to the structure of the $n$-dimensional hypercubes. The field called ‘Geometry of Numbers’ works on these grids [2, 12, 13, 16, 32, 31]. The concept of ‘lattice’ and ‘array’ were used which have about the same meaning with ‘grid’, which we are using.

The hexagonal and triangular grids are duals of each other which are also analyzed in digital geometry. There is a connection among the cubic grid, hexagonal grid, and triangular grid [14, 23, 26], and thus, coordinate systems with three coordinates are apt for these grids. The relation between square grid and hexagonal grid is explained in [36]. The three kinds of neighborhood criteria of the triangular grid can be

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found in \cite{5}. The three coordinates used in triangular grid are not independent of each other \cite{24}. The digital distances of two points based on a neighborhood relation gives the length of a shortest path connecting the two points where in each step the path moves to next neighborhood point (given a neighborhood type) \cite{25,28,29}.

The general Euclidean Shortest Path (ESP) problem is known to be NP-hard \cite{1}. A polynomial time algorithm for ESP calculations for cases where all obstacles are convex and number of obstacles is small, is stated in \cite{35}. The Euclidean shortest paths within a given cube-curve with arbitrary accuracy is given in \cite{17}. Euclidean shortest paths between two points are stated in \cite{18,19,20} for 2D and 3D using rubberband algorithms. An algorithm to compute an $L_1$-shortest path between two given points that lies on or above a given polyhedral terrain is presented in \cite{22}.

There may exist more than one shortest path as a shortest path is not unique. The recursive formulation for the number of cityblock, chessboard, and octagonal shortest paths between two points in 2D digital plane is presented in \cite{4}. It is to be noted here that the general formulation for chessboard shortest paths between two points was given by a recursive method based on generating function. In this paper, we also give an alternative non-recursive formulation based on enumerative combinatorics in Sec. 3. In \cite{3}, the number of minimal paths in a digital image between every pair of points with respect to a particular neighborhood relation is presented, where the image is considered as matrix and hence the algorithm contains matrix operations. The determination of shortest isothetic path (cityblock) between two points inside a digital object for a given grid size, is presented in \cite{8,7}. Since a shortest isothetic path is not unique, finding the number of shortest isothetic paths between two points is essential. The corresponding problem is presented in \cite{6}. The number of shortest paths in triangular grid is analyzed in \cite{9}. Here, in this paper, we will discuss the path counting problem between two points whose coordinate triplets are given in cubic grid for 6-, 18-, and 26-neighborhoods, i.e., $L_1$, $D_{18}$, and $L_{\infty}$ metrics respectively. The path counting problems in 3D digital geometry for the three neighborhoods are presented in \cite{10,11} in a different way. The formulation of path counting problem in 26-neighborhood in \cite{10} is based on the generating function stated in \cite{3,4}, whereas in this paper we have proposed it in a comprehensive and straightforward way. In \cite{11}, the formulation for path counting problem in 18-neighborhood is divided into three cases whereas the first two cases are based on the generating function stated in \cite{3,4} and the third one is based on induction on the length of minimal paths. The formulation for 18-neighborhood proposed here is much more simpler and definite devoid of any generating function. The computation of number of paths is more directly proposed here compared to the formulae in \cite{10,11}. All these formulae are proved here using combinatorial techniques.

The paper is organized as follows. The preliminaries are discussed in Sec. 2. The formulation of number of shortest paths in cubic grid for 6-neighborhoods, 18-neighborhoods, and 26-neighborhoods are given in Sec. 3, Sec. 4, and Sec. 5 respectively. Section 6 presents concluding remarks.

2 Preliminaries

According to \cite{15}, the cubic grid on $\mathbb{R}^3$ is denoted by $\phi\mathbb{Z}^3$ ($\phi > 0$) and defined as $\phi\mathbb{Z}^3 = \{(\phi c_1, \phi c_2, \phi c_3) \mid c_1, c_2, c_3 \in \mathbb{Z}\}$. Let $G$ be any set of points in $\mathbb{R}^3$. The Voronoi neighborhood of $g \in G$ is defined as $N_G(g) = \{v \in \mathbb{R}^3 \mid \forall h \in G, \|v - g\| \leq \|v - h\|\}$.

The Voronoi neighborhood of $(\phi c_1, \phi c_2, \phi c_3)$ in $\phi\mathbb{Z}^3$ is a cube of volume $\phi^3$ centered in $(\phi c_1, \phi c_2, \phi c_3)$. When perceived as a set of points in $\mathbb{R}^3$, $\phi\mathbb{Z}^3$ is referred to as a cubic grid. The Voronoi neighborhoods in a grid in $\mathbb{R}^3$ are referred to voxels. Figure 1 represents the directions of the three axes in the cubic grid.
Path Counting in Cubic Grid

Fig. 1: The origin and the directions of the three axes.

Fig. 2: The three neighborhoods.

and the origin is also shown. Further we assume that \( \phi = 1 \).

There are three widely used neighborhoods in \( \mathbb{Z}^3 \). Those are 6-neighborhood (face neighbors), 18-neighborhood (face-edge neighbors), and 26-neighborhood (face-edge-vertex neighbors). Let \( r = (x_r, y_r, z_r) \in \mathbb{Z}^3 \) and \( s_i = (x_{s_i}, y_{s_i}, z_{s_i}) \in \mathbb{Z}^3 \) be all points satisfying \( \max \{|x_r - x_{s_i}|, |y_r - y_{s_i}|, |z_r - z_{s_i}|\} \leq 1 \).

- \( N_6(r) = \{s_i : |x_r - x_{s_i}| + |y_r - y_{s_i}| + |z_r - z_{s_i}| \leq 1\} \)
- \( N_{18}(r) = \{s_i : |x_r - x_{s_i}| + |y_r - y_{s_i}| + |z_r - z_{s_i}| \leq 2\} \)
- \( N_{26}(r) = \{s_i : |x_r - x_{s_i}| + |y_r - y_{s_i}| + |z_r - z_{s_i}| \leq 3\} \)

These are shown in Fig. 2. The neighbor voxels \( N_6(r) \), \( N_{18}(r) \), and \( N_{26}(r) \) are shown in red, magenta, and yellow colors respectively.

Let us consider two points \( p \) and \( q \) in cubic grid. The problem is to find the number of shortest paths between \( p \) and \( q \) in a given neighborhood. To formulate the problem, the points have to be translated such that either \( p \) or \( q \) be in origin \((0, 0, 0)\). Let the coordinates of the points be \( p = (x_p, y_p, z_p) \) and \( q = (x_q, y_q, z_q) \). The coordinates of the points after translation will be \( p = (x_p - x_q, y_p - y_q, z_p - z_q) \) and \( q = (0, 0, 0) \).

We may also recall the general definition of \( L_i \) distances in 3D between two points, which is given below.

\[
L_i(p, q) = (|x_p - x_q|^i + |y_p - y_q|^i + |z_p - z_q|^i) ^{\frac{1}{i}}
\]  

(2.1)
The digital distances are discussed in \cite{3,4,27,30}. We recall that the length of shortest path between \(p(x_p, y_p, z_p)\) and \(q(0,0,0)\) in cubic grid in 6-neighborhood, 18-neighborhood, and 26-neighborhood are denoted by metrics— \(L_1\), \(D_{18}\), and \(L_\infty\) respectively.

\[
L_1(p, q) = D_6(p, q) = (|x_p| + |y_p| + |z_p|) \tag{2.2}
\]

\[
D_{18} = \max \left\{ \max\{|x_p|, |y_p|, |z_p|\}, \left\lceil \frac{|x_p| + |y_p| + |z_p|}{2} \right\rceil \right\} \tag{2.3}
\]

\[
L_\infty(p, q) = D_{26}(p, q) = \max \{|x_p|, |y_p|, |z_p|\} \tag{2.4}
\]

3 Number of Shortest Paths in 6-Neighborhood

**Theorem 3.1.** The number of shortest paths from \(q = (0, 0, 0)\) to any point \(p = (i, j, k)\) in 6-neighborhood is

\[
f_{6\mathcal{N}}(i, j, k) = \frac{(i+j+k)!}{i!j!k!} \tag{3.1}
\]

**Proof:** Without loss of generality, we assume that the coordinates of the point \(p\) are nonnegative, that is \(i, j, k \geq 0\). In 3D, two points, \(p'(i_1, j_1, k_1)\) and \(q'(i_2, j_2, k_2)\), are in 6-neighborhood if they share a face, i.e., when only one of the following conditions holds i) \(|i_1 - i_2| = 1\), ii) \(|j_1 - j_2| = 1\), or iii) \(|k_1 - k_2| = 1\). Thus, in 6-neighborhood, in each step of a shortest path only one coordinate changes. The other coordinates of the points coincide respectively. So, the length of a shortest path between \(p(i, j, k)\) and \(q(0,0,0)\) in 6-neighborhood is \(i + j + k\) (Eqn. 2.2), the sum of the movements along three axes. Out of total \(i + j + k\) steps \(i, j,\) and \(k\) steps are taken along the \(x, y,\) and \(z\)-axes respectively. Their order is...
Fig. 4: The number of shortest paths (given inside the white circle) from origin to other points (coordinates are shown in parentheses) in 18-neighborhoods for path length three.

arbitrary, thus the total number of arrangements for a path length of $|i| + |j| + |k|$ is given by $\frac{(|i|+|j|+|k|)!}{|i|!|j|!|k|!}$, which is the total number of paths in 6-neighborhood.

In Fig. 4 the number of shortest paths of length three and four are shown for some of the points along with the coordinate triplets. It is to be noted here that the number of shortest paths in 6-neighborhood in cubic grid is similar for some of the coordinates where $x = 0$ or $y = 0$ or $z = 0$, with the values in 4-neighborhood in 2D, i.e., the cityblock (or $L_1$) distance in 2D, coinciding with the binomial coefficients. Actually, these numbers are the trinomial coefficients: $\frac{(|i|+|j|+|k|)!}{|i|!|j|!|k|!}$ which could play a role in expansions like $(x + y + z)^n$, see, e.g., [4].

4 Number of Shortest Paths in 18-Neighborhood

W.l.o.g. we assume that $i, j, k \geq 0$. The length of a shortest path in 18-neighborhood is either maximum of $i, j, k$ or $\lceil \frac{i+j+k}{2} \rceil$ (Eqn. 2.3). The number of shortest paths is discussed in the following two theorems according to two cases.

Theorem 4.1. The number of shortest paths from $q = (0, 0, 0)$ to any point $p = (i, j, k)$ in 18-neighborhood when $D_{18} = \max \{i, j, k\} = i$ and $a$ and $b$ are right-away (positive $x$ and negative $z$) and right-bottom (positive $x$ and negative $y$) directions (based on the axes in Fig. 4) respectively is
The number of paths from $q(0,0,0)$ to $p(i,j,k)$ in 18-neighborhood is given by

$$f_{18N}(i,j,k) = \sum_{a=0}^{2(a+b) \leq i-j-k} \frac{\binom{i}{a+b} \binom{j}{b} \binom{k}{2(a+b)}!}{\binom{k}{2(a+b)}}.$$  \hspace{1cm} (4.1)

**Proof:** Let, without loss of generality, $D_{18} = \max \{i,j,k\} = i$, i.e., there are $i$-steps from $q$ to $p$. $D_{18} = i$, implies that $i \geq j$ and $i \geq k$, moreover $i \geq j + k$ by Eqn. 2.3. In 18N, a path can proceed through either a face-shared neighbor (change in only one coordinate value) or an edge-shared neighbor (change in any two coordinate values). In Eqn. 4.1, $a$ and $b$ refer to the numbers of right-away and right-bottom movements w.r.t. the positive $x$-axis (Fig. 1) respectively. Let $c = k + a$ and $d = j + b$ be the respective numbers of right and right-top movements. In a right-away movement, the path moves to the edge-shared neighbor where the $x$-coordinate increases by 1 and the $z$-coordinate decreases by 1 and in a right-bottom movement, both $x$- and $z$- coordinates increase by 1. Similarly, in a right-bottom movement $x$-coordinate increases and $y$-coordinates decreases while for a right-top movement both $x$ and $y$ coordinates increases. The sum of movements cannot be more than $i - a + b + c + d \leq i$, i.e., $2(a + b) \leq i - j - k$. The right-away and the right movements as well as the right-top and the right-bottom movements have some limits. When a number of right-away movements are there, the right movements will be $c = k + a$ such that the decrease of $z$-coordinate in $a$ right-away moves is compensated by the increase of the $z$-coordinate in $k + a$ right moves in order that the destination point has $z$-coordinate as $k$. Note that in each move the $x$-coordinate always increases by 1. Similarly, $b$ number of right-bottom movements implies $d = j + b$ number of right-top movements. Otherwise, it will not be possible to reach the destination in $i$ steps. Apart from right-away, right, right-top, and right-bottom moves, there can be movements in $x$-direction only. For a given $a$, $b$, $c$, and $d$ (i.e., right-away, right-bottom, right, and right-top respectively) steps, there are $i - (a + b + c + d) = i - j - k - 2(a + b)$ number of steps in the positive $x$-direction to face neighbor. Thus, for a given $a$, $b$, $c$, and $d$ combination, the total number of arrangements for a shortest path of length $i$ is given by $\frac{\binom{i}{a+b}\binom{j}{b}\binom{k}{2(a+b)}!}{\binom{k}{2(a+b)}}$. For different values of $a$ and $b$, values of $c$ and $d$ are computed satisfying the condition that $(a + b + c + d) \leq i$. Thus, total number of paths is the summation over the different possible combinations of $a$, $b$, $c$, and $d$ values, and is given by $\sum_{a=0}^{2(a+b) \leq i-j-k} \frac{\binom{i}{a+b}\binom{j}{b}\binom{k}{2(a+b)}!}{\binom{k}{2(a+b)}}!$. \hfill $\square$

The number of paths from $q(0,0,0)$ to $p$ where the length of path is three, are given in Fig. 4. It is to be noted that when $D_{18} = \max \{i,j,k\} = j$ or $k$, the above formula (Eqn. 4.1) will change accordingly. The number of paths from $(0,0,0)$ to $(0,3,0)$ is 13 and that to $(0,3,1)$ is 12.

**Theorem 4.2.** The number of shortest paths from $q = (0,0,0)$ to any point $p = (i,j,k)$ in 18-neighborhood when $D_{18} = \lceil \frac{i+j+k}{2} \rceil = L$, is

$$f_{18N}(i,j,k) = \begin{cases} L! \left( \frac{L!(L-i)!(L-j)!(L-k)!}{(L-i)!(L-j)!(L-k)!} \right), & \text{when } (i+j+k) \mod 2 = 0 \\ L! \left( \frac{L!(L-i)!(L-j)!(L-k)!}{(L-i)!(L-j)!(L-k)!} \right), & \text{when } (i+j+k) \mod 2 = 1 \end{cases}$$ \hspace{1cm} (4.2)

**Proof:** At each step in 18-neighborhood, at most two coordinates can increase by one and the number of steps is $L$. When $i + j + k$ is even, as $L$ divides $i + j + k$ with the quotient 2 which implies that at
Corollary 4.3. The number of shortest paths from \( q = (0, 0, 0) \) to any point \( p = (i,j,k) \) in 18-
neighborhood when \( D_{18} = \lceil \frac{i+j+k}{2} \rceil = \max \{i, j, k\} = L \), \( f_{18N}(i, j, k) \) is as follows.

\[
f_{18N}(i, j, k) = \sum_{a=0, b=0}^{2(a+b) \leq \min(i-j-k, 0)} \frac{i!}{a!b!(k+a)!(j+b)!(i-j-k-2(a+b))!} = \begin{cases} 
\frac{L!}{(L-i)!(L-j)!(L-k)!} & \text{when } (i+j+k) \mod 2 = 0 \\
\frac{L!}{(L-i)!(L-j)!(L-k)!} \left( \frac{i!}{(0+i+1)(k+i+1)!} + \frac{i!}{(0+j+1)(k+j+1)!} \right) & \text{when } (i+j+k) \mod 2 = 1 
\end{cases}
\tag{4.3}
\]

**Proof:** The proof can be done mathematically in two parts when \( i+j+k \) is even and when it is odd. Let \( L = \frac{i+j+k}{2} \), i.e., \( i + j + k \) is even. Thus, \( i - j - k = 0 \). Putting \( i - j - k = 0 \), in Eqn. 4.1 (see Theorem 4.1) we get, \( f_{18N}(i, j, k) = \sum_{a=0, b=0}^{2(a+b) \leq \min(i-j-k, 0)} \frac{i!}{a!b!(k+a)!(j+b)!(i-j-k-2(a+b))!} \), as there is only one possibility for the values of \( a \) and \( b \), i.e., \( a = b = 0 \) since \( 2(a+b) \leq i - j - k = 0 \). By putting these values, we get \( \frac{i!}{j!k!} \). Hence proved.

For the second part, when \( i + j + k \) is odd, \( L = \frac{i+j+k+1}{2} = i \). Thus, \( i - j - k = 1 \). Putting \( i - j - k = 1 \), in Eqn. 4.1 (see Theorem 4.1) we get, \( f_{18N}(i, j, k) = \sum_{a=0, b=0}^{2(a+b) \leq \min(i-j-k, 0)} \frac{i!}{a!b!(k+a)!(j+b)!(i-j-k-2(a+b))!} \), as there is one possibility for the values of \( a \) and \( b \), i.e., \( a = b = 0 \) since \( 2(a+b) \leq i - j - k = 1 \). By putting these values, we get \( \frac{i!}{j!k!} \). Now, \( L - i = 0 \), \( L - j = k + 1 \), \( L - k = j + 1 \). Thus, \( \frac{L!}{(L-i)!(L-j)!(L-k)!} \left( \frac{i!}{(0+i+1)(k+i+1)!} + \frac{i!}{(0+j+1)(k+j+1)!} \right) \). Hence proved.

Similarly, the above mentioned equation (Eqn. 4.3) can be proved when \( L = j \) or \( k \).

## 5 Number of Shortest Paths in 26-Neighborhood

The formulation for the number of shortest paths in 26-neighborhood in cubic grid is dependent on the number of shortest paths in 8-neighborhood in 2D. The number of shortest paths in 8-neighborhood in 2D had been proposed by Das [3, 4] with recurrence relations, in this paper we show a shorter direct proof with combinatorial tools (Eqn. 5.1).

**Theorem 5.1.** The number of shortest paths from origin \( q = (0, 0) \) to the point \( p = (i, j) \) in 2D in 8-neighborhood, is given by

\[
f_8(i, j) = \sum_{b=0}^{2b \leq |i|-|j|} \frac{|i|!}{b!(|j|+b)!(|i|-|j|-2b)!} \quad \text{where } |i| \geq |j| \quad \tag{5.1}
\]

**Proof:** The length of a shortest path between \( p(i, j) \) and \( q(0, 0) \) in 8-neighborhood is \( \max \{|i|, |j|\} \). By the symmetry of the grid we show the proof for the case \( 0 \leq j \leq i \), in this case the distance is \( i \). With respect to the positive \( x \)-direction, let \( b \) be the number of moves along right-bottom diagonal in the shortest path where \( x \)-coordinate increases by 1 and \( y \)-coordinate decreases by 1, and \( d = |j|+b \) be the number of moves along right-top diagonal in the shortest path where the \( x \)- and \( y \)-coordinates increases by 1. A shortest path involves \( i \) steps, out of which if there are \( b \) right-bottom moves then \( d = j + b \).
moves only in right-top direction, hence, the number of paths is given by $\frac{i!}{b!(j+b)!(i-j-2b)!}$. It may be noted here that $b + d \leq i$, i.e., $i - j - 2b$ as the total number of moves cannot be more than $i$. By summing over the different possible combinations of $b$ and $d$, the total number of shortest paths is given by $f_b(i, j) = \sum_{b=0}^{2i} \frac{i!}{b!(j+b)!(i-j-2b)!}$. 

The number of paths from $q(0, 0)$ to other points in 8-neighborhood in 2D are shown in Fig. 5. Figure 6 shows two examples of all possible paths from a source to destination. For a particular path (shown in red) among all possible shortest paths, the $l$ and $r$ values are given for ease of understanding. The formulation for $|j| \geq |i|$, is just reverse (exchanging $i$ with $j$) of the above equation (Eqn 5.1). The values appearing in 8-neighborhood in 2D are also present in 26-neighborhood of cubic grid when $x = y$ or $y = z$ or $z = x$ (see Fig. 5 and Fig. 7).

**Theorem 5.2.** The number of shortest paths from $q = (0, 0, 0)$ to any point $p = (i, j, k)$ in 26-neighborhood
is
\[
\begin{align*}
\text{\small{f}}_{26N}(i, j, k) &= \sum_{b=0}^{2|b|\leq|i|-|j|} \frac{|i|!}{b!(|j| + b)!(|i| - |j| - 2b)!} \\
&\times \sum_{a=0}^{2a\leq|i|-|k|} \frac{|i|!}{a!(|k| + a)!(|i| - |k| - 2a)!}
\end{align*}
\]

i.e., \(\text{\small{f}}_{26N}(i, j, k) = \text{\small{f}}_{8}(i, j) \times \text{\small{f}}_{8}(i, k)\), where \(|i| \geq |j|\) and \(|i| \geq |k|\).

**Proof:** The length of a shortest path between \(p(i, j, k)\) and \(q(0, 0, 0)\) in 26-neighborhood is \(\max \{ |i|, |j|, |k| \} = |i|\) (given in Eqn.2.4) (say). In each step of a shortest path in 26-neighborhood, at most three coordinates can change. A shortest path in 26-neighborhood is a combination of a shortest path in xy-plane, from \(q(0, 0, 0)\) to \(p_{xy}(i, j, 0)\) and a shortest path in xz-plane, from \(q(0, 0, 0)\) to \(p_{xz}(i, 0, k)\). With each shortest path in xy-plane, each shortest path in xz-plane is combined to get the total number of shortest paths in 3D. Thus, the number of shortest paths in 26N is given by \(\text{\small{f}}(i, j) \times \text{\small{f}}(i, k)\) where \(\text{\small{f}}(i, j)\) and \(\text{\small{f}}(i, k)\) are the number of shortest paths in xy- and xz- planes respectively (Theorem 4). Thus, \(\text{\small{f}}_{26N}(i, j, k) = \sum_{b=0}^{2b\leq|i|-|j|} \frac{|i|!}{b!(|j| + b)!(|i| - |j| - 2b)!} \times \sum_{a=0}^{2a\leq|i|-|k|} \frac{|i|!}{a!(|k| + a)!(|i| - |k| - 2a)!}\) where \(a\) and \(b\) indicate the number of steps in right-bottom (a simultaneous move in negative y- and positive x- directions) and right-away (a simultaneous move in negative z- and positive x- directions) directions respectively.

From the Equation 5.2, the formulation for the number of shortest paths between two points for
The number of shortest paths in 8-neighborhood in 2D is stated in this paper using combinatorial tool. This paper, extending the results of Das [4] from 2D to 3D, using $L_\infty$ distances is one of the important feature in this regard. Many studies have already been proposed on it. In Fig. 7, with corresponding.

$$\begin{align*}
&(-3, 3, 3) \quad (-2, 3, 3) \quad (-1, 3, 3) \quad (0, 3, 3) \quad (1, 3, 3) \quad (2, 3, 3) \quad (3, 3, 3) \\
&(-3, 2, 3) \quad (-2, 2, 3) \quad (-1, 2, 3) \quad (0, 2, 3) \quad (1, 2, 3) \quad (2, 2, 3) \quad (3, 2, 3) \\
&(-3, 1, 3) \quad (-2, 1, 3) \quad (-1, 1, 3) \quad (0, 1, 3) \quad (1, 1, 3) \quad (2, 1, 3) \quad (3, 1, 3) \\
&(-3, 0, 3) \quad (-2, 0, 3) \quad (-1, 0, 3) \quad (0, 0, 3) \quad (1, 0, 3) \quad (2, 0, 3) \quad (3, 0, 3) \\
&(-3, 0, 0) \quad (-2, 0, 0) \quad (-1, 0, 0) \quad (0, 0, 0) \quad (1, 0, 0) \quad (2, 0, 0) \quad (3, 0, 0) \\
&(-3, 0, -1) \quad (-2, 0, -1) \quad (-1, 0, -1) \quad (0, 0, -1) \quad (1, 0, -1) \quad (2, 0, -1) \quad (3, 0, -1) \\
&(-3, 0, -2) \quad (-2, 0, -2) \quad (-1, 0, -2) \quad (0, 0, -2) \quad (1, 0, -2) \quad (2, 0, -2) \quad (3, 0, -2) \\
&(-3, -1, 0) \quad (-2, -1, 0) \quad (-1, -1, 0) \quad (0, -1, 0) \quad (1, -1, 0) \quad (2, -1, 0) \quad (3, -1, 0) \\
&(-3, -1, -1) \quad (-2, -1, -1) \quad (-1, -1, -1) \quad (0, -1, -1) \quad (1, -1, -1) \quad (2, -1, -1) \quad (3, -1, -1) \\
&(-3, -2, 0) \quad (-2, -2, 0) \quad (-1, -2, 0) \quad (0, -2, 0) \quad (1, -2, 0) \quad (2, -2, 0) \quad (3, -2, 0) \\
&(-3, -2, -1) \quad (-2, -2, -1) \quad (-1, -2, -1) \quad (0, -2, -1) \quad (1, -2, -1) \quad (2, -2, -1) \quad (3, -2, -1) \\
&(-3, -3, 0) \quad (-2, -3, 0) \quad (-1, -3, 0) \quad (0, -3, 0) \quad (1, -3, 0) \quad (2, -3, 0) \quad (3, -3, 0) \\
&(-3, -3, -1) \quad (-2, -3, -1) \quad (-1, -3, -1) \quad (0, -3, -1) \quad (1, -3, -1) \quad (2, -3, -1) \quad (3, -3, -1)
\end{align*}$$

Fig. 7: The number of shortest paths from origin to other points in 26-neighborhoods for $Length = 3$. (The coordinate triplets are written in parantheses and the corresponding number of shortest paths are also mentioned.) Observe that the results are not only symmetric, but they are according to a multiplication table, by Equ. [5.2] where the elements of the border rows and columns are specified by the formula for the 2D $L_\infty$ distance, i.e., the chessboard distance (Equ. [5.1]). Obviously, the diagonals contain the squares of the numbers shown at the borders.

$$|j| \geq |k|, |i| \text{ and } |k| \geq |i|, |j|$$

can be derived similarly. The number of shortest paths of length three from $q(0, 0, 0)$ to other points are shown in Fig. 7. Figure 8 shows an example path from $q(0, 0, 0)$ to $p(7, 4, 2)$ which has 2 right-away movements with corresponding 4 right movements and 1 right-bottom with corresponding 5 right-top movements.

6 Conclusions

The shortest path problem has various applications in several fields, specially in image processing. Digital distances is one of the important feature in this regard. Many studies have already been proposed on it. In this paper, extending the results of Das [4] from 2D to 3D, using $L_1$, $D_{18}$ and $L_\infty$ distances, the number of shortest paths between any point pair in the cubic grid are presented for 6-, 8-, and 26-neighborhood where the coordinate triplets of the two points are provided. It is also to be noted that the formulation for the number of shortest paths in 8-neighborhood in 2D is stated in this paper using combinatorial tool.
Fig. 8: One of the shortest paths of length 7 from (0, 0, 0) to (7, 4, 2) is shown and correspondingly the values of \( l \), \( r \), \( b \), and \( t \) are 2, 4, 1, and 5 respectively. The projection of the paths in \( xy \)-plane is shown at back in green color and that in \( xz \)-plane in blue color at the bottom.

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