Comment on “Non-intersecting Brownian Bridges in the Flat-to-Flat Geometry”

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Abstract
This is a comment on our recent paper (Grela et al. in J Stat Phys 183:49, 2021). In this comment we provide an easier derivation of the effective Langevin equation for vicious Brownian bridges in the flat-to-flat geometry. This derivation shows that it is not necessary to invoke the intermediate step of mapping to a Dyson Brownian bridge. The result can be directly derived using the Karlin–McGregor formula.

Keywords Conditioned Brownian motions · Rare events · Effective Langevin equation

In a recent paper [1], we studied a system of $N$ vicious Brownian bridges (VBB) on an infinite line, defined as follows. Consider a system of $N$ Brownian motions on a line with coordinates $\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t)\}$ at time $t$ that evolve independently via the Langevin equation
\[
\frac{d\lambda_i(t)}{dt} = \frac{1}{\sqrt{N}} \eta_i(t),
\]
where $\eta_i$’s are Gaussian white noises with zero mean and correlator $\langle \eta_i(t) \eta_j(t') \rangle = \delta_{i,j} \delta(t-t')$. The walkers start at the initial positions $\vec{a} = \{a_1, a_2, \ldots, a_N\}$ with $a_1 < a_2 < \ldots < a_N$ and are conditioned to stay ordered, i.e., non-intersecting at all subsequent times $t$. Such a system of interacting walkers is called the vicious Brownian motions (VBM). A VBB is a VBM that, in addition to the non-crossing condition, is conditioned to reach the fixed final positions $\vec{b} = \{b_1, b_2, \ldots, b_N\}$ with $b_1 < b_2 < \ldots < b_N$ at a future fixed time $t_f$. One of the

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main questions addressed in Ref. [1] was how to generate such a VBB configuration (for fixed \( \bar{a}, \bar{b} \) and \( t_f \)) numerically by using a rejection-free algorithm. A principal result of Ref. [1] was to derive, for the special flat final configuration \( b_i = (i - 1)/N \) (equidistant points), an exact effective Langevin equation for the VBB coordinates \( \{ \lambda_i(t) \} \) at any intermediate time \( t \in [0, t_f] \)

\[
\frac{d\lambda_i}{dt} = -\frac{\lambda_i}{t_f - t} + \frac{1}{N} \left( \frac{1}{t_f - t} \sum_{j \neq i} \frac{e^{\frac{\lambda_i - \lambda_j}{t_f - t}}}{e^{\frac{\lambda_j}{t_f - t}} - e^{\frac{\lambda_i}{t_f - t}}} \right) + \frac{1}{\sqrt{N}} \xi_i(t),
\]

where \( \xi_i(t) \)'s are are Gaussian white noises with zero mean and correlator \( \langle \eta_i(t)\eta_j(t') \rangle = \delta_{i,j} \delta(t - t') \). The Langevin equation (2) starts from arbitrary ordered initial positions \( a_1 < a_2 < \ldots < a_N \) at \( t = 0 \) and automatically forces the walkers to reach the final flat configuration \( b_i = (i - 1)/N \) at time \( t_f \), while staying non-intersecting at all intermediate times \( t \in [0, t_f] \). This effective Langevin equation (2) is then very useful as one can discretize this continuous-time equation in units of small \( \Delta t \) and generate the VBB trajectories in a completely rejection-free way.

The derivation of Eq. (2) in Ref. [1] involved two steps: (i) an exact mapping between the coordinates of the VBB and those of the Dyson Brownian bridges (DBB) with parameter \( \beta = 2 \) and (ii) a subsequent derivation of the effective Langevin equation for the \( \beta = 2 \) DBB. Let us briefly recall the definition of a DBB with parameter \( \beta \). Consider a system of \( N \) interacting particles on a line with coordinates \( \{ \lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t) \} \) at time \( t \) that evolve via the Langevin equation

\[
\frac{d\lambda_i(t)}{dt} = \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i(t) - \lambda_j(t)} + \sqrt{\frac{2}{\beta N}} \eta_i(t),
\]

where \( \eta_i \)'s are again Gaussian white noises with zero mean and correlator \( \langle \eta_i(t)\eta_j(t') \rangle = \delta_{i,j} \delta(t - t') \). The walkers start at the initial positions \( \bar{a} = \{ a_1, a_2, \ldots, a_N \} \) at \( t = 0 \) with \( a_1 < a_2 < \ldots < a_N \) and automatically stay ordered at all future times due to the explicit interaction (repulsive) term on the right hand side (rhs) of Eq. (3). Such a system of interacting particles is called the Dyson Brownian motion (DBM) with parameter \( \beta > 0 \) [2]. Note that \( \beta \) appears only in the noise amplitude in Eq. (3). A DBB is just a DBM which is conditioned to reach a fixed final configuration \( \bar{b} = \{ b_1, b_2, \ldots, b_N \} \) at a future time \( t_f \) with \( b_1 < b_2 < \ldots < b_N \). It was shown in Ref. [1] that for arbitrary ordered initial positions \( \bar{a} \) and final positions \( \bar{b} \), the coordinates of the VBB and that of the DBB with parameter \( \beta = 2 \) are statistically equivalent

\[
\{ \lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t) \}_{\text{VBB}} \equiv \{ \lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t) \}_{\text{DBB}}, \beta = 2,
\]

where the symbol \( \equiv \) denotes an equivalence in law. Consequently, if one can generate a DBB configuration (with \( \beta = 2 \)) in a rejection-free way, that gives also a VBB configuration with the correct statistical weight. Finally, in Ref. [1], the effective Langevin equation (2), valid for the final flat configuration \( b_i = (i - 1)/N \), was actually derived for the DBB (with \( \beta = 2 \)) which, using the exact mapping, thus also generates a VBB configuration in a rejection-free way.

The purpose of this comment is to point out that while the mapping between the VBB and the \( \beta = 2 \) DBB is beautiful, this mapping is actually not necessary for the derivation of the exact effective Langevin equation (2) for the VBB. One can derive Eq. (2) for the VBB directly without having to use the mapping to the \( \beta = 2 \) DBB. This derivation is rather
simple and follows from a straightforward generalisation of the $N = 1$ (an ordinary Brownian bridge) case studied in [3]. Consider a VBB of $N$ particles and let $P_{\text{VBB}}(\bar{\lambda}, t|\bar{a}, \bar{b}, t_f)$ denote its propagator at time $t$, i.e., the probability density for the VBB to be at $\bar{\lambda}$ at any intermediate time $0 \leq t \leq t_f$, given the initial and the final coordinates and the duration $t_f$. Following [3], we split the interval $[0, t_f]$ into two parts: the left $[0, t]$ and the right $[t, t_f]$. Using the Markov property of the bridge, one can then write

$$P_{\text{VBB}}(\bar{\lambda}, t|\bar{a}, \bar{b}, t_f) = \frac{P_{\text{VBM}}(\bar{\lambda}, t|\bar{a}, 0) P_{\text{VBM}}(\bar{b}, t_f|\bar{\lambda}, t)}{P_{\text{VBM}}(\bar{b}, t_f|\bar{a}, 0)}, \quad (5)$$

where $P_{\text{VBM}}(\bar{\lambda}, t|\bar{a}, 0)$ denotes the propagator of the VBM at time $t$, starting from $\bar{a}$ at time 0. The VBM propagator just satisfies diffusion equation

$$\partial_t P_{\text{VBM}}(\bar{\lambda}, t) = \frac{1}{2N} \sum_{i=1}^{N} \partial_{\bar{\lambda}_i}^2 P_{\text{VBM}}(\bar{\lambda}, t). \quad (6)$$

in the ordered sector $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ with absorbing (Fermionic) boundary conditions $P_{\text{VBM}} = 0$ when $\lambda_i = \lambda_j$ for $i \neq j$ (where we used the shorthand notation $P_{\text{VBM}}$ for $P_{\text{VBM}}(\bar{\lambda}, t)$). Let us also denote $P_{\text{VBM}}(\bar{b}, t_f|\bar{\lambda}, t) = Q_{\text{VBM}}(\bar{\lambda}, t|\bar{b}, t_f)$. Then $Q_{\text{VBM}}$ satisfies the backward Fokker–Planck equation (with a negative sign since $t$ gets replaced by $t_f - t$)

$$\partial_t Q_{\text{VBM}}(\bar{\lambda}, t) = -\frac{1}{2N} \sum_{i=1}^{N} \partial_{\bar{\lambda}_i}^2 Q_{\text{VBM}}(\bar{\lambda}, t). \quad (7)$$

Thus $P_{\text{VBB}}(\bar{\lambda}, t|\bar{a}, \bar{b}, t_f) \propto P_{\text{VBM}}(\bar{\lambda}, t|\bar{a}, 0) Q_{\text{VBM}}(\bar{\lambda}, t|\bar{b}, t_f)$. Using the two diffusion equations satisfied by $P_{\text{VBM}}$ and $Q_{\text{VBM}}$ (respectively in (6) and (7)), it is easy to show that $P_{\text{VBB}}$ satisfies the Fokker–Planck equation

$$\partial_t P_{\text{VBB}}(\bar{\lambda}, t) = \frac{1}{2N} \sum_{i=1}^{N} \partial_{\bar{\lambda}_i}^2 P_{\text{VBB}}(\bar{\lambda}, t) - \frac{1}{N} \sum_{i=1}^{N} \partial_{\bar{\lambda}_i} \left[ \partial_{\bar{\lambda}_i} \ln Q_{\text{VBM}} \right] P_{\text{VBB}}(\bar{\lambda}, t). \quad (8)$$

From this Fokker–Planck equation, one can read off the associated Langevin equation

$$\frac{d\lambda_i(t)}{dt} = \frac{1}{N} \partial_{\lambda_i} \ln Q_{\text{VBM}} + \sqrt{\frac{1}{N}} \xi_i(t), \quad (9)$$

where $\xi_i(t)$’s are Gaussian white noises with zero mean and correlator $\langle \eta_i(t)\eta_j(t') \rangle = \delta_{i,j} \delta(t - t')$. Thus the only undetermined quantity on the rhs in Eq. (9) is the backward VBM propagator $Q_{\text{VBM}}(\bar{\lambda}, t|\bar{b}, t_f)$. This propagator can however be computed using the Karlin–McGregor formula [4] (see also Eq. (27) of Ref. [1]) and can be evaluated explicitly for the special flat final configuration $b_i = (i - 1)/N$ (see Eqs. (27)–(31) of Ref. [1]). Substituting this result in Eq. (9) immediately reproduces the effective Langevin equation (2).

This clearly shows that we do not need the intermediate mapping of the VBB to the $\beta = 2$ DBB in order to derive the Langevin equation (2) for the VBB. The derivation is much more straightforward. However, one merit of the somewhat roundabout derivation of Eq. (2) in Ref. [1] is that it provided an exact equivalence between the VBB and the $\beta = 2$ DBB. Thus,
Eq. (2) also provides a rejection-free way to generate a $\beta = 2$ DBB which also appears in many applications.

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**Declarations**

**Conflict of interest** The authors declare that there is no conflict of interest.

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