FLAGGED GROTHENDIECK POLYNOMIALS

TOMOO MATSUMURA

Abstract. We show that the flagged Grothendieck polynomials defined as generating functions of flagged set-valued tableaux of Knutson–Miller–Yong [11] can be expressed by a Jacobi–Trudi type determinant formula generalizing the work of Hudson–Matsumura [9]. We also introduce the flagged skew Grothendieck polynomials in these two expressions and show that they coincide.

1. Introduction

Lascoux–Schützenberger ([12], [14]) introduced the Grothendieck polynomials to represent the $K$-theory classes of the structure sheaves of Schubert varieties and Fomin–Kirillov ([5], [4]) gave their combinatorial description in terms of pipe dreams or rc graphs. Knutson–Miller–Yong [11] expressed the Grothendieck polynomial associated to a vexillary permutation as the generating function of flagged set-valued tableaux, unifying the work of Wachs [16] on flagged tableaux, and Buch [2] on set-valued tableaux (cf. [10]). On the other hand, in the joint work [9] with Hudson, the author proved the Jacobi–Trudi type formula for the vexillary Grothendieck polynomials in the context of degeneracy loci formula, generalizing his joint work [7] and [8] with Hudson, Ikeda, and Naruse for the Grassmannian case (see also [1] and [15]).

Motivated by these results, we study the generating functions of flagged set-valued tableaux in general beyond the ones given by vexillary permutations. For a given partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0)$ of length $r$, a flagging $f$ of $\lambda$ is a nondecreasing sequence of natural numbers $(f_1, \ldots, f_r)$. A flagged set-valued tableau of shape $\lambda$ with a flagging $f$ is nothing but a set-valued tableau of shape $\lambda$ of Buch [2] satisfying extra conditions that the numbers used in the $i$-th row are at most $f_i$ for all $i$. Let $FSVT(\lambda, f)$ be the set of all flagged set-valued tableaux of shape $\lambda$ with the flagging $f$. Let $x = (x_1, x_2, \ldots)$ be a set of infinitely many indeterminants. Following Knutson–Miller–Yong’s work, we define the flagged Grothendieck polynomials $G_{\lambda,f}(x)$ by

$$G_{\lambda,f}(x) := \sum_{T \in FSVT(\lambda,f)} \beta_{|T|-|\lambda|} \prod_{k \in T} x_k.$$ 

The main goal of this paper is to show that $G_{\lambda,f}(x)$ is given by the following Jacobi-Trudi type determinant formula (Theorem 2.8):

$$G_{\lambda,f}(x) = \det \left( \sum_{s=0}^{\infty} \frac{(i-j)^s}{s!} \beta^s G_{\lambda_i+j-i+s}(x) \right)_{1 \leq i,j \leq r}. \quad (1.1)$$
Here $G_m^p(x)$ is defined by the generating function
\[
\sum_{m \in \mathbb{Z}} G_m^p(x)u^m = \frac{1}{1 + \beta u - \prod_{1 \leq i \leq p} \frac{1 + \beta x_i}{1 - x_i u}},
\]
which geometrically corresponds to the Segre classes of vector bundles (see [7]). Since not all flagged Grothendieck polynomials are Grothendieck polynomials of Lascoux and Schützenberger, our result generalizes the work of Knutson–Miller–Yong and Hudson–Matsumura mentioned above.

The flagged Schur polynomials are the specialization of the flagged Grothendieck polynomials $G_{\lambda,f}(x)$ at $\beta = 0$. These polynomials were introduced by Lascoux and Schützenberger in [13] to identify Schubert polynomials associated to vexillary permutations. They are also generalizations of Schur polynomials and satisfy the generalized Jacobi–Trudi formula, due to Gessel and Wachs. In this paper, we closely follow Wachs’ inductive proof in [16], which makes the use of the divided difference operators. In particular, our proof shows that the above determinant formula in terms of the one row Grothendieck polynomials $G_m^p(x)$ behaves nicely under the action of those symmetrizing operators.

In Section 3, we give an inductive proof that the Grothendieck polynomial associated to a vexillary permutation is a flagged Grothendieck polynomial. This gives a combinatorial proof of Knutson–Miller–Yong’s result and of the determinant formula in [9]. We also show that any flagged Grothendieck polynomial can be obtained from a monomial by applying the divided difference operators.

In Section 4, we introduce the flagged skew Grothendieck polynomials. Their tableaux and determinant expressions are natural extension of the (row) flagged skew Schur functions studied by Wachs [16]. We show those two expressions coincide.

It is worth pointing out that the formulas of Knutson–Miller–Yong and Hudson–Matsumura are also for double Grothendieck polynomials defined for two sets of variables. Thus it would be natural to extend our result to its double version (see the work [3] of Chen–Li–Louck for the flagged double Schur functions). This extension will be studied elsewhere.

2. flagged Grothendieck polynomials

A partition is a weakly decreasing finite sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers and $r$ is called its length. We often identify a partition $\lambda$ of length $r$ with its Young diagram $\{(i, j) \mid 1 \leq i \leq r, 1 \leq j \leq \lambda_i\}$ in English notation. A flagging $f$ of a partition $\lambda$ of length $r$ is a weakly increasing sequence $f = (f_1, \ldots, f_r)$ of positive integers. A flagged set-valued tableau $T$ of shape $\lambda$ with a flagging $f$ is a set-valued tableau of shape $\lambda$ such that each filling in the $i$-th row consists of the numbers not greater than $f_i$. Namely, $T$ is a filling of the boxes of $\lambda$ such that the box at $(i, j)$ in $\lambda$ is filled by a non-empty subset of $\{1, \ldots, f_i\}$ and the maximum number at $(i, j)$ is at most the minimum number at $(i, j + 1)$ for $j + 1 \leq \lambda_i$ and less than the minimum number
at \((i + 1, j)\) for \(j \leq \lambda_{i+1}\). In particular, if \(f_1 = \cdots = f_r\), then it coincides with the definition of set-valued tableaux defined by Buch in [2]. We denote by \(FSVT(\lambda, f)\) the set of all flagged set-valued tableaux of shape \(\lambda\) with the flagging \(f\). If \(f_1 = f_2 = 1\) and \(r > 1\), then \(FSVT(\lambda, f) = \emptyset\). Thus we assume that if \(f_1 = 1\) and \(r > 1\), then \(f_2 > 1\).

**Example 2.1.** Let \(\lambda = (2, 1)\) and \(f = (2, 4)\). Then \(FSVT(\lambda, f)\) contains tableaux such as

\[
\begin{array}{cccc}
1 & 1 & 12 & 2 \\
2 & 3 & 4 & 12
\end{array}
\]

If we chance \(f\) to \(f' = (2, 3)\), then \(FSVT(\lambda, f')\) doesn’t contain the second and forth tableaux.

Let \(x = (x_1, x_2, \ldots)\) be the set of infinitely many variables. Let \(\mathbb{Z}[\beta]\) be the polynomial ring of the variable \(\beta\) where we set \(\deg \beta = -1\). Let \(\mathbb{Z}[\beta][[x]]\) be the ring of polynomials and of formal power series in \(x\) respectively. We define the flagged Grothendieck polynomial associated to a partition \(\lambda\) and a flagging \(f\) by

\[
(2.1) \quad G_{\lambda, f}(x) := \sum_{T \in FSVT(\lambda, f)} \beta^{|T| - |\lambda|} \prod_{k \in T} x_k,
\]

where \(|T|\) is the total number of entries in \(T\), \(|\lambda|\) is the number of boxes in \(\lambda\), and \(k \in T\) denotes an entry in \(T\). We also define an element \(\tilde{G}_{\lambda, f}(x)\) of \(\mathbb{Z}[\beta][[x]]\) by

\[
(2.2) \quad \tilde{G}_{\lambda, f}(x) := \det \left( \sum_{s=0}^{\infty} \binom{i - j}{s} \beta^s G_{\lambda_{i+j-i+s}}^{[f]}(x) \right)_{1 \leq i, j \leq r},
\]

where \(G_{m}^{[p]}(x) \in \mathbb{Z}[\beta][[x]]\) is defined by the generating function

\[
(2.3) \quad G^{[p]}(x; u) = \sum_{m \in \mathbb{Z}} G_{m}^{[p]}(x) u^m = \frac{1}{1 + \beta u^{-1}} \prod_{1 \leq i \leq p} \frac{1 + \beta x_i}{1 - x_i u}.
\]

For the empty partition, we set both of the functions \(G_{\lambda, f}(x)\) and \(\tilde{G}_{\lambda, f}(x)\) to be 1. We also remark that \(G_{-m}^{[p]} = (-\beta)^m\) and \(G_{m}^{[1]} = x_1^m\) for all integer \(m \geq 0\), which follow from the direct computation.

At \(\beta = 0\), \(G_{\lambda, f}(x)\) specializes to the flagged Schur polynomial of Wachs in [16], and \(\tilde{G}_{\lambda, f}(x)\) is nothing but the corresponding Jacobi–Trudi formula since \(G_{m}^{[p]}(x)\) becomes the complete symmetric function of degree \(m\).

The main goal of this section is to show that \(G_{\lambda, f}(x)\) and \(\tilde{G}_{\lambda, f}(x)\) coincide (Theorem 2.8). We closely follow Wachs’ proof of the analogous statement for the flagged Schur polynomials in [16]. The key for the proof is the action of the divided difference operators, which makes the induction proof possible.
2.1. Divided difference operators and basic formulas. We recall the definition of divided difference operators and show a few formulas that will be used in the proof of the propositions to follow.

Let $S_n$ be the permutation group of the set $\{1, \ldots, n\}$. We have the action of $S_n$ on $\mathbb{Z}[\beta][[x]]$ permuting the variables. For example, let $s_i$ denote the $i$-th transposition $i.e.$ $s_i(i) = i + 1, s_i(i + 1) = i$ and $s_i(j) = j$ for $j \neq i, i + 1$, then $s_i(f(x))$ is defined by exchanging $x_i$ and $x_{i+1}$ in $f(x) \in \mathbb{Z}[\beta][[x]]$ and leave other variables unchanged.

**Definition 2.2.** For an element $f(x)$ of $\mathbb{Z}[\beta][[x]]$, we define

$$
\pi_i(f(x)) := \frac{(1 + \beta x_{i+1})f(x) - (1 + \beta x_i)s_i(f(x))}{x_i - x_{i+1}}.
$$

The following Leibniz rule can be checked by a direct computation: for $f(x), g(x) \in \mathbb{Z}[\beta][[x]]$, we have

$$
\pi_i(fg) = \pi_i(f)g + s_i(f)\pi_i(g) + \beta s_i(f)g.
$$

It is also easy to check directly that, if $f(x)$ is symmetric in $x_i$ and $x_{i+1}$, then we have

$$
\pi_i(x_i^k f(x)) = \begin{cases} 
-\beta f(x) & (k = 0) \\
(\sum_{s=0}^{k-1} x_i^s x_{i+1}^{k-1-s} + \beta \sum_{s=1}^{k-1} x_i^s x_{i+1}^{k-s}) f(x) & (k > 0).
\end{cases}
$$

**Lemma 2.3.** For each $m \in \mathbb{Z}$ and $p \in \mathbb{Z}_{\geq 1}$, we have

$$
\pi_i(G_m^{[p]}(x)) = \begin{cases} 
G_{m-1}^{[p+1]}(x) & (i = p), \\
-\beta G_m^{[p]}(x) & (i \neq p).
\end{cases}
$$

*Proof.* If $i = p$, it follows from the identity $\pi_p G_m^{[p]}(x; u) = u G_m^{[p+1]}(x; u)$ which can be proved by a direct computation. If $i \neq p$, then $G_m^{[p]}(x)$ is symmetric in $x_i$ and $x_{i+1}$, and hence (2.5) implies the claim.

**Lemma 2.4.** If $f(x)$ is symmetric in $x_p$ and $x_{p+1}$, then we have $\pi_p(G_m^{[p]} f) = G_m^{[p+1]} f$.

*Proof.* It follows from the Leibniz rule (2.4) and Lemma 2.3.

**Lemma 2.5.** For each $m \in \mathbb{Z}$ and $p \in \mathbb{Z}_{\geq 1}$, we have

$$
G_m^{[p]} - \frac{x_1}{1 + \beta x_1} G_m^{[p]} - \frac{x_1}{1 + \beta x_1} \beta G_m^{[p]} = G_m^{[p]} \bigg|_{x_1=0}.
$$

*Proof.* It follows from the identity

$$
\sum_{m \in \mathbb{Z}} \left( G_m^{[p]} - \frac{x_1}{1 + \beta x_1} G_m^{[p]} - \frac{x_1}{1 + \beta x_1} \beta G_m^{[p]} \right) u^m = G_m^{[p]}(x; u) \bigg|_{x_1=0},
$$

which can be checked by a direct computation.
2.2. The main theorem. We prove the main theorem (Theorem 2.8) below by induction based on the following two propositions.

Proposition 2.6. Let $\lambda$ be a partition of length $r$ with a flagging $f$. If $\lambda_1 > \lambda_2$ and $f_1 < f_2$, then we have

(i) $\pi_{f_1}(\tilde{G}_{\lambda,f}) = \tilde{G}_{\lambda',f'}$,

(ii) $\pi_{f_1}(G_{\lambda,f}) = G_{\lambda',f'}$,

where $\lambda' = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_r)$ and $f' = (f_1 + 1, f_2, \ldots, f_r)$.

Proof. For (i), we recall from [7, §3.6] that we can write

$$
\tilde{G}_{\lambda,f} = \sum_{s \in \mathbb{Z}^r} a_s G_{\lambda_1+s_1}^{[f_1]} \cdots G_{\lambda_r+s_r}^{[f_r]},
$$

where $a_s \in \mathbb{Z}[\beta]$ is the coefficient of $t^s$ in the Laurent series expansion

$$
\prod_{1 \leq i < j \leq r} (1 - \bar{t}_i/\bar{t}_j) = \sum_{s \in (s_1, \ldots, s_r) \in \mathbb{Z}^r} a_s t_1^{s_1} \cdots t_r^{s_r}.
$$

Here we denoted $\bar{t} = \frac{t}{1+\beta t} = -t \sum_{s \geq 0} (-\beta)^s t^s$. Since $f_1 < f_2$, one can apply Lemma 2.4 to the expression (2.6) and obtains (i). Indeed, we have

$$
\pi_{f_1}(\tilde{G}_{\lambda,f}) = \sum_{s \in \mathbb{Z}^r} a_s \pi_{f_1} \left( G_{\lambda_1+s_1}^{[f_1]} \cdots G_{\lambda_r+s_r}^{[f_r]} \right) = \sum_{s \in \mathbb{Z}^r} a_s G_{\lambda_1-1+s_1}^{[f_1+1]} G_{\lambda_2+s_2}^{[f_2]} \cdots G_{\lambda_r+s_r}^{[f_r]} = \tilde{G}_{\lambda',f'}.
$$

Next we prove (ii). Let $t := f_1$ and $t' := f_1 + 1$. Define an equivalence relation $\sim$ on $FSVT(\lambda, f)$ as follows: for $T_1, T_2 \in FSVT(\lambda, f)$, let $T_1 \sim T_2$ if the collection of boxes that contain either $t$ or $t'$ is the same for $T_1$ and $T_2$. We have

$$
G_{\lambda,f} = \sum_{\mathcal{A} \in FSVT(\lambda, f)/\sim} \left( \sum_{T \in \mathcal{A}} M(T) \right),
$$

where $M(T) := \beta^{|T| - |\lambda|} \prod_{k \in T} x_k$ for each $T \in FSVT(\lambda, f)$. Let $\mathcal{A}$ be the equivalence class whose tableaux have the configuration of $t$ and $t'$ as shown in Figure 1.
Each rectangle with $\ast$ has $m_i$ boxes and each box contains $t$ or $t'$ so that the total number of entries $t$ and $t'$ in the rectangle is $m_i$ or $m_i + 1$ for $i \geq 2$. Note that $m_i$ and $r_i$ may be 0 and the rectangles in Figure 1 may not be connected. Then we see that

$$
\sum_{T \in \mathcal{A}} M(T) = x_t^{m_1} (x_t x_{t'})^{r_1 + \cdots + r_k} \left( \prod_{i=2}^{k} \left( \sum_{v=0}^{m_i} x_t^v x_{t'}^{m_i-v} + \beta \sum_{v=1}^{m_i} x_t^v x_{t'}^{m_i+1-v} \right) \right) R(\mathcal{A}),
$$

where $R(\mathcal{A})$ is the polynomial contributed from the entries other than $t$ and $t'$. Let

$$
R'(\mathcal{A}) := \left( \prod_{i=2}^{k} \left( \sum_{v=0}^{m_i} x_t^v x_{t'}^{m_i-v} + \beta \sum_{v=1}^{m_i} x_t^v x_{t'}^{m_i+1-v} \right) \right) R(\mathcal{A}).
$$

Observe that the factor $(x_t x_{t'})^{r_1 + \cdots + r_k} R'(\mathcal{A})$ is symmetric in $x_t$ and $x_{t'}$. Thus, by Lemma [2.5] if $m_1 = 0$, then we have

$$
(2.7) \quad \pi_t \left( \sum_{T \in \mathcal{A}} M(T) \right) = -\beta (x_t x_{t'})^{r_2 + \cdots + r_k} R'(\mathcal{A}),
$$

if $m_1 = 1$, we have

$$
(2.8) \quad \pi_t \left( \sum_{T \in \mathcal{A}} M(T) \right) = (x_t x_{t'})^{r_1 + \cdots + r_k} R'(\mathcal{A}),
$$

and if $m_1 \geq 2$, we have

$$
(2.9) \quad \pi_t \left( \sum_{T \in \mathcal{A}} M(T) \right) = \left( \sum_{s=0}^{m_1-1} x_t^s x_{t'}^{m_1-1-s} + \beta \sum_{s=1}^{m_1-1} x_t^s x_{t'}^{m_1-1-s} \right) (x_t x_{t+1})^{r_1 + \cdots + r_k} R'(\mathcal{A}).
$$

We consider the decomposition

$$
FSVT(\lambda, f)/\sim = \mathcal{F}_1 \sqcup \mathcal{F}_2 \sqcup \mathcal{F}_3 \sqcup \mathcal{F}_4
$$

where $\mathcal{F}_1, \ldots, \mathcal{F}_4$ are the sets of equivalence classes whose configurations of the boxes containing $t$ or $t'$ respectively satisfy

1. $m_1 = 0$ (so that $r_1 = 0$),
2. $m_1 = 1$ and the box at $(1, \lambda_1)$ in $\lambda$ contains more than one entry (so that $r_1 = 0$),
3. $m_1 = 1$ and the box at $(1, \lambda_1)$ in $\lambda$ contains contains only $t$,
4. $m_1 \geq 2$.

By the expressions (2.7) and (2.8), we have

$$
\sum_{\mathcal{A} \in \mathcal{F}_1 \sqcup \mathcal{F}_2} \pi_t \left( \sum_{T \in \mathcal{A}} M(T) \right) = 0.
$$

Now we consider the equivalence class $\mathcal{A}'$ in $FSVT(\lambda', f')/\sim$ whose associated skew diagram of boxes containing $t$ or $t'$ is as shown in Figure 2 below.
One can see that, if $m_1 = 1$, then $\sum_{T \in A'} M(T)$ is exactly the right hand side of (2.8) under the condition (3), and if $m_1 \geq 2$, then $\sum_{T \in A'} M(T)$ is exactly the right hand side of (2.9). It is also clear that $F_3$ and $F_4$ are in bijection to the equivalence classes of $FSVT(\lambda', f')$ such that $m_1 = 1$ and $m_1 \geq 2$ respectively. Thus the desired identity holds. □

**Proposition 2.7.** Let $\lambda$ be a partition of length $r$ with a flagging $f$. If $f_1 = 1$, then we have

(i) $\widetilde{G}_{\lambda, f} = x_1^{\lambda_1}(\widetilde{G}_{\lambda', f'}|x_1 = 0)$,

(ii) $G_{\lambda, f} = x_1^{\lambda_1}(G_{\lambda', f'}|x_1 = 0)$,

where $\lambda' = (\lambda_2, \ldots, \lambda_r)$ and $f' = (f_2, \ldots, f_r)$.

**Proof.** First we observe that (ii) holds clearly since $f_1 = 1$. We prove (i). Since $f_1 = 1$, we see that the first row of the determinant for $\widetilde{G}_{\lambda, f}$ is

$$
\begin{pmatrix}
    x_1^{\lambda_1}, x_1^{\lambda_1}, \frac{x_1}{1 + \beta x_1}, \ldots, x_1^{\lambda_1} \left( \frac{x_1}{1 + \beta x_1} \right)^{r-1}
\end{pmatrix}.
$$

Indeed, since $G_{m}^{[1]} = x_1^m$ for $m \geq 0$, we have

$$\sum_{s=0}^{\infty} \binom{1-j}{s} \beta^s G_{1+j-1+s}(x) = x_1^{\lambda_1+j-1} \sum_{s=0}^{\infty} \binom{1-j}{s} \beta^s x_1^s = \frac{x_1^{\lambda_1+j-1}}{(1 + \beta x_1)^{j-1}}.$$ 

We do the column operation to $\widetilde{G}_{\lambda, f}$ by subtracting $\frac{x_1}{1 + \beta x_1}$ times the $(j-1)$-st column from the $j$-th column for each $j = 2, \ldots, r$ so that the first row becomes $(x_1^{\lambda_1}, 0, \ldots, 0)$. Then we observe that the $(i, j)$-entry for $i, j \geq 2$ equals to

$$\sum_{s=0}^{\infty} \binom{i-j}{s} \beta^s G_{\lambda_1+j-i+s}(x) - \frac{x_1}{1 + \beta x_1} \sum_{s=0}^{\infty} \binom{i-j+1}{s} \beta^s G_{\lambda_1+j-1-i+s}(x),$$

$$\sum_{s=0}^{\infty} \binom{i-j}{s} \beta^s G_{\lambda_1+j-i+s}(x) - \frac{x_1}{1 + \beta x_1} \sum_{s=0}^{\infty} \binom{i-j}{s} \beta^s G_{\lambda_1+j-1-i+s}(x),$$

$$\sum_{s=0}^{\infty} \binom{i-j}{s} \beta^s G_{\lambda_1+j-i+s}(x) - \frac{x_1}{1 + \beta x_1} G_{\lambda_1+j-1-i+s}(x) - \frac{x_1}{1 + \beta x_1} \beta G_{\lambda_1+j-i+s}(x),$$

$$\left( \sum_{s=0}^{\infty} \binom{i-j}{s} \beta^s G_{\lambda_1+j-i+s}(x) \right)_{x_1 = 0},$$
where the first equality is by an identity of the binomial coefficients and the last equality follows from Lemma 2.5. Finally the cofactor expansion with respect to the first row gives the desired identity for (i).

\[ \square \]

**Theorem 2.8.** For each partition \( \lambda \) and a flagging \( f \), we have \( G_{\lambda,f}(x) = \bar{G}_{\lambda,f}(x) \). In particular, \( \bar{G}_{\lambda,f}(x) \) is a polynomial in \( x \).

**Proof.** By induction on \((r, |f| := f_1 + \cdots + f_r)\), Proposition 2.6 and 2.7 imply the claim: for the base case \((r, |f|) = (0, 0)\), the claim holds trivially. If \( f_1 = 1 \), apply Proposition 2.7 and if \( f_1 > 1 \), apply Proposition 2.6. In both cases, the claim follows from the induction hypothesis. \( \square \)

3. Grothendieck polynomials

In this section, we show that the Grothendieck polynomials associated to a vexillary permutation is a flagged Grothendieck polynomial, recovering the results of Knutson–Miller–Yong [11] and Hudson–Matsumura [9]. We also show that any flagged Grothendieck polynomial can be obtained from a monomial in the same way as any Grothendieck polynomial is defined.

The Grothendieck polynomial \( G_w = G_w(x_1, \ldots, x_n) \) associated to a permutation \( w \in S_n \) is defined as follows. We use the same convention as in [2]. For the longest element \( w_0 \) in \( S_n \), we set
\[
G_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^2 x_{n-1} = \prod_{i=1}^{n-1} x_i^{n-i}.
\]
If \( w \) is not the longest, there is \( i \) such that \( \ell(ws_i) = \ell(w) + 1 \). We then define
\[
G_w := \pi_i(G_{ws_i}).
\]
This definition is independent of the choice of \( s_i \) because the operators \( \pi_i \) satisfy the Coxeter relations. By the same reason we can define \( \pi_w \) by \( \pi_w = \pi_{i_k} \cdots \pi_{i_1} \) where \( w = s_{i_1} \cdots s_{i_k} \) with \( \ell(w) = k \).

Now we recall how to obtain a partition \( \lambda(w) \) and a flagging \( f(w) \) for each vexillary permutation \( w \in S_n \). We follow [9] and [11] (cf. [9]). Let \( r_w \) be the rank function of \( w \in S_n \) defined by
\[
r_w(p,q) := \sharp\{i \leq p \mid w(i) \leq q\}
\]
and we define the diagram \( D(w) \) of \( w \) by
\[
D(w) := \{(p,q) \in \{1, \ldots, n\} \times \{1, \ldots, n\} \mid \pi(p) > q, \text{ and } \pi^{-1}(q) > p\}.
\]
We call an element of the grid \( \{1, \ldots, n\} \times \{1, \ldots, n\} \) a box. The essential set \( E_{\lambda}(w) \) of \( w \) is the subset of \( D(w) \) given by
\[
E_{\lambda}(w) := \{(p,q) \mid (p+1,q), (p,q+1) \notin D(w)\}.
\]
A permutation \( w \in S_n \) is called vexillary if it avoids the pattern (2143), i.e. there is no \( a < b < c < d \) such that \( w(b) < w(a) < w(d) < w(c) \). In [16], a vexillary permutation was called a single-shape permutations. Fulton showed in [6] that \( w \in S_n \) is vexillary if and only if the boxes in \( E_{\lambda}(w) \) are placed along the direction going from northeast to southwest. We can assign a
partition \( \lambda(w) \) to each vexillary permutation \( w \) as follows: let the number of boxes \((i, i+k)\) in the \( k \)-th diagonal of the Young diagram of \( \lambda(w) \) be equal to the number of boxes in the \( k \)-th diagonal of \( D(w) \) for each \( k \) (see [11][10]). This defines a bijection \( \phi \) from \( D(w) \) to \( \lambda \), namely
\[
\phi(p, q) = (p - r_w(p, q), q - r_w(p, q)) \quad \text{for each } (p, q) \in D(w).
\]
In particular, \( \phi \) restricted to \( E_{ss}(w) \) is a bijection onto the set of the southeast corners of \( \lambda(w) \). Let \( r \) be the length of \( \lambda(w) \). The flagging \( f(w) = (f(w)_1, \ldots, f(w)_r) \) associated to \( w \) is defined as follows. We can choose a subset \( \{(p_i, q_i), i = 1, \ldots, r\} \) of \( \{1, \ldots, n\} \times \{1, \ldots, n\} \) containing \( E_{ss}(w) \) and satisfying
\[
\begin{align*}
(3.1) & \quad p_1 \leq p_2 \leq \cdots \leq p_r, \quad q_1 \geq q_2 \geq \cdots \geq q_r, \\
(3.2) & \quad p_i - r_w(p_i, q_i) = i, \quad \forall i = 1, \ldots, r.
\end{align*}
\]
In [9], we called this subset \( \{(p_i, q_i)\} \) a flagging set of \( w \) and used it to express the double Grothendieck polynomials as a determinant. We set \( f(w) \) by letting \( f(w)_i := p_i \). We can always express \( \lambda(w) \) by \( \lambda_i = q_i - p_i + i \) for each \( i = 1, \ldots, r \). Remark that the set \( FSVT(\lambda(w), f(w)) \) doesn’t depend of the choice of flagging sets.

**Example 3.1.** Consider a vexillary permutation \( w = (w(1) \cdots w(5)) = (23541) \) in \( S_5 \). We represent the corresponding permutation matrix \( M_w \) by \((M_w)_{ij} = \delta_{w(i),j}\). In the picture below, we represent 1 in \( M_w \) by a dot and the boxes in \( D(w) \) by squares. We make hooks by drawing lines from each dot going south and east, and then \( D(w) \) is the collection of boxes that are not on the hooks. We see that \( \lambda(w) = (\lambda_1, \ldots, \lambda_4) = (2, 1, 1, 1) \). For a flagging set of \( w \), we must have \((p_1, q_1) = (3, 4) \) and \((p_4, q_4) = (4, 1) \) since they consist \( E_{ss}(w) \). Then the conditions \((3.1)\) and \((3.2)\) require that we must choose \((p_2, q_2)\) from \((3, 2)\) and \((4, 3)\) and \((p_3, q_3)\) from \((3, 1)\) and \((4, 2)\) in such a way that \( p_2 \leq p_3 \) and \( q_2 \geq q_3 \). Thus \( f(w) = (3, 3, 3, 4), (3, 3, 4, 4) \) or \((3, 4, 4, 4)\). By the column strictness, each of the flaggings gives the same collection of flagging set-valued tableaux.

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
0 & 1 & 2 & 2 \\
0 & 1 & 2 & 2 \\
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 & 5
\end{array}
\]

**Theorem 3.2.** If \( w \) is a vexillary permutation, then \( G_w = G_{\lambda(w), f(w)} \).

**Proof.** By Propositions \( 2.6 \) and \( 2.7 \) the proof is almost identical to the one given by Wachs for the flagged Schur functions in [16]. We write the proof for completeness since we have slightly different terminologies and notations.
First we observe that for the longest element \( w_0 \in S_n \), we have \( \lambda(w_0) = (n - 1, n - 2, \ldots, 1) \) and \( f(w_0) = (1, 2, 3, \ldots, n - 1) \). Therefore by definition we have
\[
G_{\lambda(w_0), f(w_0)} = \prod_{i=1}^{n-1} x_i^{n-i} = G_{w_0}.
\]
We show the claim by induction on \( (n, \ell(w_0) - \ell(w)) \) with the lexicographic order where \( \ell(w) \) denotes the length \( w \in S_n \). We say that \( w \) has a descent at \( i \) if \( w_i > w_{i+1} \). Let \( d \) be the leftmost descent of \( w \). We consider the following three cases.

**Case 1.** Assume \( d > 1 \). Let \( w' = ws_{d-1} \), then it is easy to see that \( w' \) is vexillary. Since \( \ell(w') > \ell(w) \), we have \( G_{w'} = G_{\lambda(w'), f(w')} \) by the induction hypothesis. We can also observe that \( \lambda(w')_1 = \lambda(w)_1 + 1 \) and \( \lambda(w')_i = \lambda(w)_i \) for \( i \geq 2 \). Furthermore \( f_1(w') = d - 1 = f_1(w) - 1 \) and \( f(w')_i = f(w)_i \) for \( i \geq 2 \). Thus, by Proposition \ref{2.6} we have
\[
G_w = \pi_{d-1} G_{w'} = \pi_{d-1} G_{\lambda(w'), f(w')} = G_{\lambda(w), f(w)}.
\]
This completes the case \( d > 1 \).

**Case 2.** Assume \( d = 1 \) and \( w(1) < n \). Let \( w' := s_{w(1)}w \) so that \( w'(1) = w(1) + 1 \) and \( w'(l) = w(1) + 1 \) for \( l \) such that \( w(l) = w(1) + 1 \). It is clear that \( w' \) is vexillary. Since the leftmost descent of \( w \) is 1, we have \( \ell(w') = \ell(w) + 1 \). Consider the vexillary permutation
\[
u = (w(1) + 1, w(1), w(1) + 2, w(1) + 3, \ldots, n, w(1) - 1, w(1) - 2, \ldots, 1).
\]
Since \( w \) is vexillary, all numbers greater than \( w(1) \) appear in ascending order in \( w \), and therefore we can find integers \( i_1, \ldots, i_k \) greater than 1, satisfying \( w's_1s_2 \cdots s_k = u \) and \( \ell(w') + k = \ell(w) \). For such a choice, we also have \( ws_is_2 \cdots s_k s_1 = u \). Thus by definition we have
\[
G_w = \pi_{i_1} \cdots \pi_{i_k} G_u \quad \text{and} \quad G_{w'} = \pi_{i_1} \cdots \pi_{i_k} G_u.
\]
By induction we have \( G_u = G_{\lambda(u), f(u)} \) and \( G_{w'} = G_{\lambda(w'), f(w')} \). We observe that
\[
\lambda(u) = (w(1), w(1) - 1, w(1) - 1, \ldots, w(1) - 1, w(1) - 2, w(1) - 3, \ldots, 2, 1)
\]
and \( f(u) = (1, 2, 3, \ldots, n) \), from which we find
\[
G_u = x_1^{w(1)} x_2^{w(1) - 1} \cdots x_{n-w(1)+1}^{w(1) - 2} \cdots x_{n-2}^{2} = (1/x_1) G_u.
\]
Lemma \ref{2.5} implies that \( \pi_{i_1} G_u = (1/x_1) G_u \). Since \( i_1, \ldots, i_k \) are greater than 1, again Lemma \ref{2.5} implies that
\[
G_w = \pi_{i_1} \cdots \pi_{i_k} (1/x_1) G_u = (1/x_1) G'_w = G_w = \pi_{i_1} G_{\lambda(w'), f(w')}.
\]
Furthermore, we observe that
\[
\lambda(w')_1 = \lambda(w)_1 + 1, \quad \lambda(w')_i = \lambda(w)_i, \quad \forall i = 2, \ldots, r,
\]
\[
f(w')_1 = f(w)_1 = 1, \quad f(w')_i = f(w)_i, \quad \forall i = 2, \ldots, r.
\]
This implies that \( G_{\lambda(w), f(w)} = (1/x_1) G_{\lambda(w'), f(w')} \). Therefore we conclude that \( G_w = G_{\lambda(w), f(w)} \).
Case 3. Assume $d = 1$ and $w(1) = n$. Let $w_0w = s_{i_1} \cdots s_{i_k}$ where $\ell(w) + k = \ell(w_0)$. Since $w(1) = n$, it follows that $i_1, \ldots, i_k$ are greater than 1. Consequently,

$$G_w = \pi_{i_k} \pi_{i_{k-1}} \cdots \pi_{i_1} (x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^{2} x_{n-1}^1) = x_1^{n-1} \pi_{i_k} \pi_{i_{k-1}} \cdots \pi_{i_1} (x_2^{n-2} \cdots x_{n-2}^{2} x_{n-1}^1).$$

Consider the vexillary permutation $w' := (w(2), \ldots, w(n))$ in $S_{n-1}$. By induction, we have $G_{w'} = G_{\lambda(w'),f(w')}$. Moreover, by definition we have

$$\pi_{i_k} \pi_{i_{k-1}} \cdots \pi_{i_1} (x_2^{n-2} \cdots x_{n-2}^{2} x_{n-1}^1) = (G_{w'})^\dagger.$$

where $f(x)^\dagger$ denotes the function obtained from $f(x)$ by replacing $x_i$ with $x_{i+1}$. Thus $G_w = x_1^{n-1}(G_{w'})^\dagger$. Since $(f(w')_2, \ldots, f(w')_n) = (f(w)_2 - 1, \ldots, f(w)_n - 1)$, it follows that

$$(G_{\lambda(w'),f(w')})^\dagger = G_{\lambda(w'),\tilde{f}(w)}|_{x_1=0},$$

where $\tilde{f}(w) = (f_2(w), \ldots, f_n(w))$. Thus Proposition 2.7 implies that

$$G_w = x_1^{n-1}(G_{\lambda(w'),f(w')})^\dagger = x_1^{n-1} \cdot \left( G_{\lambda(w'),\tilde{f}(w)}|_{x_1=0} \right) = G_{\lambda(w),f(w)}.$$

This completes the proof. \qed

**Theorem 3.3.** Let $\lambda$ be a partition of length $r$ with a flagging $f$. Then $G_{\lambda,f}$ is equal to $\pi_w(x_1^{a_1} \cdots x_r^{a_r})$ where

$$a_i = \lambda_i + f_i - i, \quad and \quad w = s_{r}s_{r+1} \cdots s_{f_r-1} \cdot s_{f_r-2} \cdots s_{f_r-1} \cdot \cdots \cdot s_{1}s_{2} \cdots s_{f_1-1}.$$

*Proof.* We prove by induction on the sum of the flagging $|f| = f_1 + \cdots + f_r$. If $|f| = 1$, then $f_1 = 1$ and $r = 1$. In this case, we have $G_{\lambda,f} = x_1^{\lambda_1} = \pi_{id} x_1^{a_1}$. Assume that $|f| > 1$. If $f_1 > 1$, consider $\lambda' = (\lambda_1 + 1, \lambda_2, \ldots, \lambda_r)$ and $f' = (f_1 - 1, f_2, \ldots, f_r)$. The induction hypothesis implies that $G_{\lambda',f'} = \pi_w s_{f_1-1}(x_1^{a_1} \cdots x_r^{a_r})$. By Proposition 2.6 and the Coxeter relation of divided differences, we obtain

$$G_{\lambda,f} = \pi_{f_1-1} G_{\lambda',f'} = \pi_{f_1-1} \pi_w s_{f_1-1}(x_1^{a_1} \cdots x_r^{a_r}) = \pi_w(x_1^{a_1} \cdots x_r^{a_r}).$$

If $f_1 = 1$, consider $\lambda' = (\lambda_2, \ldots, \lambda_r)$ and $f' = (f_2 - 1, \ldots, f_r - 1)$. Then we have $G_{\lambda,f} = x_1^{\lambda_1}(G_{\lambda',f'})^\dagger = x_1^{a_1}(G_{\lambda',f'})^\dagger$. On the other hand, by induction, we have $G_{\lambda',f'} = \pi_{w'}(x_1^{a_1} \cdots x_r^{a_r})$ where $a'_1 = a_{i+1}$ and $w' = s_{r-1}s_{r-2} \cdots s_{f_r-2}s_{f_r-1} \cdots s_{f_r-2} \cdots s_{f_2-2}$. We can see that $(\pi_{w'}(x_1^{a_1} \cdots x_r^{a_r}))^\dagger = \pi_{w'}(x_2^{a_2} \cdots x_r^{a_r})$ and therefore we have

$$G_{\lambda,f} = x_1^{a_1} \pi_w(x_2^{a_2} \cdots x_r^{a_r}) = \pi_w(x_1^{a_1} x_2^{a_2} \cdots x_r^{a_r}),$$

since none of the transpositions in the expression of $w$ is equal to $s_1$. \qed
4. Flagged skew Grothendieck polynomials

Consider two partitions \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0) \) and \( \mu = (\mu_1 \geq \cdots \geq \mu_r \geq 0) \) such that \( \mu_i \leq \lambda_i \) for all \( i = 1, \ldots, r \) and two sequence of positive integers \( f = (f_1, \ldots, f_r) \) and \( g = (g_1, \cdots, g_r) \) such that

(4.1) \[ g_i \leq g_{i+1}, \quad \text{and} \quad f_i \leq f_{i+1}, \quad \text{whenever} \quad \mu_i < \lambda_{i+1}. \]

In this case, we say that \( f/g \) is a flagging of the skew shape \( \lambda/\mu \). A flagged skew set-valued tableau of the skew shape \( \lambda/\mu \) with a flagging \( f/g \) is a set-valued tableau of the skew shape \( \lambda/\mu \) such that each filling in the \( i \)-th row is a subset of \( \{g_i, g_i + 1, \ldots, f_i\} \). Let \( \text{FSVT}(\lambda/\mu, f/g) \) denote the set of all flagged skew tableaux of shape \( \lambda/\mu \) with a flagging \( f/g \). If \( g = (1, \ldots, 1) \) and \( f_1 = \cdots = f_r \), then the associated flagged skew set-valued tableaux are nothing but the set-valued tableaux of skew shape \( \lambda/\mu \) considered by Buch in [2].

**Example 4.1.** Consider the skew shape \( \lambda/\mu \) with a flagging \( f/g \) where \( \lambda = (4, 3, 1, 1), \mu = (2, 2, 1, 0), f = (2, 4, 2, 1), \) and \( g = (1, 2, 3, 1) \). The following tableaux, for example, are in \( \text{FSVT}(\lambda/\mu, f/g) \).

\[
\begin{array}{cccc}
12 & 2 & 34 \\
1 & 234 \\
\end{array} 
\begin{array}{cccc}
12 & 2 & 34 \\
1 & 234 \\
\end{array} 
\begin{array}{ccc}
2 & 2 \\
1 \\
\end{array} 
\begin{array}{cc}
4 \\
1 \\
\end{array}
\]

The goal of this section is to show \( G_{\lambda/\mu,f/g}(x) = \tilde{G}_{\lambda/\mu,f/g}(x) \) where these functions are defined as follows. We define

\[
G_{\lambda/\mu,f/g}(x) := \sum_{T \in \text{FSVT}(\lambda/\mu,f/g)} M(T), \quad M(T) := \beta^{|T|-|\lambda/\mu|} \prod_{k \in T} x_k,
\]

where \( |T| \) is the total number of entries in \( T \), \( |\lambda/\mu| \) is the number of boxes in the skew shape \( \lambda/\mu \), and \( k \in T \) denotes an entry in \( T \). We also define

\[
\tilde{G}_{\lambda/\mu,f/g}(x) := \det \left( \sum_{s \geq 0} \binom{i-j}{s} \beta^s G^{[f_i/g_j]}_{\lambda_i-\mu_j+j-i+s} \right)_{1 \leq i, j \leq r},
\]

where, for \( p, q \in \mathbb{N} \), the function \( G^{[p/q]}_m = G^{[p/q]}_m(x) \) is defined by the generating function

\[
\sum_{m \in \mathbb{Z}} G^{[p/q]}_m u^m = \frac{1}{1 + \beta^{-1} u} \prod_{q \leq i \leq p} \frac{1 + \beta x_i}{1 - x_i u}.
\]

Note that \( G^{[p/q]}_m = (-\beta)^m \) for all integer \( m \geq 0 \) and, if \( q > p \), then \( G^{[p/q]}_m = 0 \) for all \( m > 0 \).

If we specialize at \( \beta = 0 \), \( G_{\lambda/\mu,f/g}(x) \) becomes the (row) flagged skew Schur polynomial in [16]. Furthermore, under this specialization, \( \tilde{G}_{\lambda/\mu,f/g}(x) \) gives nothing but the corresponding Jacobi–Trudi formula also in [16], since \( G^{[p/q]}_m(x) \) becomes the complete symmetric function of degree \( m \) in variables \( x_q, x_{q+1}, \ldots, x_p \).
First of all, we show the following basic formula.

**Lemma 4.2.** If \( q \leq p \), we have

\[
G_m^{[p/q]} = x_q G_{m-1}^{[p/q]} + (1 + \beta x_q) G_m^{[p/q+1]}, 
\]
(4.2)

\[
G_m^{[p/q]} + \beta G_m^{[p/q+1]} = \frac{x_q}{1 + \beta x_q} (G_{m-1}^{[p/q]} + \beta G_m^{[p/q]}), 
\]
(4.3)

for each \( m \in \mathbb{Z} \).

**Proof.** Equation (4.2) follows by comparing the coefficient of \( u^m \) of the identity

\[(1 - x_q u) \sum_{m \in \mathbb{Z}} G_m^{[p/q]} u^m = (1 + \beta x_q) \sum_{m \in \mathbb{Z}} G_m^{[p/q+1]} u^m.\]

For (4.3), we compute the generating function of \( G_m^{[p/q]} + \beta G_m^{[p/q+1]} \):

\[
\sum_{m \in \mathbb{Z}} (G_m^{[p/q]} + \beta G_m^{[p/q+1]}) u^m = \frac{u}{1 + \beta u} \prod_{q \leq i \leq p} \left( \frac{1 + \beta x_i}{1 - x_i u} \right) + \frac{1 + \beta x_i}{1 + \beta u} \prod_{q \leq i \leq p} \left( \frac{1 + \beta x_i}{1 - x_i u} \right) 
\]

\[
= \left( \sum_{m \in \mathbb{Z}} G_m^{[p/q]} u^m \right) \left( \frac{u + \beta}{1 + \beta x_q} \right) 
\]

\[
= \frac{1}{1 + \beta x_q} \left( \sum_{m \in \mathbb{Z}} (G_m^{[p/q]} + \beta G_m^{[p/q]}) u^m \right). 
\]

Thus Equation (4.3) holds. \( \square \)

We will prove that \( G_{\lambda/\mu,f/g} = \tilde{G}_{\lambda/\mu,f/g} \) using the following four propositions.

**Proposition 4.3.** Let \( k \) be such that \( \mu_k \geq \lambda_{k+1} \). Then we have

1. \( \tilde{G}_{\lambda/\mu,f/g} = \tilde{G}_{\lambda/\mu,f/g} \cdot \tilde{G}_{\lambda/\mu,f/g} \).
2. \( G_{\lambda/\mu,f/g} = G_{\lambda/\mu,f/g} \cdot G_{\lambda/\mu,f/g} \).

where \( \cdot \) and \( \cdot \) applied to a sequence \( (t_1, \ldots, t_r) \) denote the sequences \((t_1, \ldots, t_k)\) and \((t_{k+1}, \ldots, t_r)\) respectively.

**Proof.** The identity (ii) is trivial from the definition. We prove (i). If \( j \leq k < i \), we have \( \lambda_i - \mu_j \leq 0 \). This implies that \((i,j)\)-entry of the determinant of \( G_{\lambda/\mu,f/g} \) is 0 for \( j \leq k < i \). Indeed, since \( G_m^{[p/q]} = (-\beta)^m \) for \( m \geq 0 \) and by an identity of binomial coefficients, we have

\[
\sum_{s \geq 0} \binom{i-j}{s} \beta^s G_{\lambda_i - \mu_j + j - i + s} = \sum_{0 \leq s \leq i-j} \binom{i-j}{s} \beta^s (-\beta)^{-(\lambda_i - \mu_j + j - i + s)} 
\]

\[
= (-\beta)^{-(\lambda_i - \mu_j + j - i)} \sum_{0 \leq s \leq i-j} (-1)^s \binom{i-j}{s} 
\]

\[
= 0. 
\]

Thus the determinant of \( \tilde{G}_{\lambda/\mu,f/g} \) is the product of the determinants of \( \tilde{G}_{\lambda/\mu,f/g} \) and \( \tilde{G}_{\lambda/\mu,f/g} \). \( \square \)
 Proposition 4.4. Let \( k \) be such that \( \mu_k < \lambda_k \) and \( g_k \leq f_k \). If \( g_k < g_{k+1} \) (or \( k = r \)) and \( \mu_{k-1} > \mu_k \) (or \( k = 1 \)), then we have

\[
\begin{align*}
(i) \quad & \tilde{G}_{\lambda/\mu,f/g} = x_{g_k} \tilde{G}_{\lambda/\mu,f/g} + (1 + \beta x_{g_k}) \tilde{G}_{\lambda/\mu,f/g'}, \\
(ii) \quad & G_{\lambda/\mu,f/g} = x_{g_k} G_{\lambda/\mu,f/g} + (1 + \beta x_{g_k}) G_{\lambda/\mu,f/g'},
\end{align*}
\]

where \(^{'} \) applied to a sequence denotes adding 1 to the \( k \)-th element of the sequence.

Proof. The assumption guarantees that \( f/g \) and \( f/g' \) are flaggings of the skew shapes \( \lambda/\mu' \) and \( \lambda/\mu \) respectively. First we prove (i). The columns in the determinants of both sides are identical except for the \( k \)-th one. Thus the equality (i) holds if, for all \( i = 1, \ldots, r \), we have

\[
\sum_{s \geq 0} \binom{i-k}{s} \beta^s G_{\lambda-\mu_k+k-i+s}^{[f_i/g_k]}
= x_{g_k} \sum_{s \geq 0} \binom{i-k}{s} \beta^s G_{\lambda-\mu_k+k-i+s}^{[f_i/g_k]} + (1 + \beta x_{g_k}) \sum_{s \geq 0} \binom{i-k}{s} \beta^s G_{\lambda-\mu_k+k-i+s}^{[f_i/g_k+1]}.
\]

If \( g_k \leq f_i \), Equation (2) proves the claim. Suppose that \( g_k > f_i \). This implies that \( f_i < f_k \) and \( i \neq k \). If \( i < k \), then since \( \mu_k < \lambda_k \leq \lambda_i \), we have \( \lambda_i - \mu_k + k - i \geq 2 \). In this case, the both sides of (4.4) are zero since \( G_{\lambda m}^{[f_i/g_k]} = 0 \) for all \( m > 0 \). Thus it remains to show (4.4) for the case when \( i > k \) and \( \lambda_i - \mu_k + k - i \leq 1 \). Since \( f_i < f_k \), the condition (4.1) implies \( \mu_k \geq \lambda_i \). Now we claim that (4.4) follows by evaluating the right hand side using the identity \( G_{\lambda m}^{[f_i/g_k]} = (-\beta)^m \) for all \( m \geq 0 \). Indeed, let \( a := i-k > 0 \) and \( b := \mu_k - \lambda_i \geq 0 \), then we have

\[
x_{g_k} \sum_{s \geq 0} \binom{a}{s} \beta^s G_{b-1-a+s}^{[f_i/g_k]} + (1 + \beta x_{g_k}) \sum_{s \geq 0} \binom{a}{s} \beta^s G_{b-a+s}^{[f_i/g_k+1]}
= x_{g_k} \sum_{0 \leq s \leq a} \binom{a}{s} \beta^s (-\beta)^{b+1+a-s} + (1 + \beta x_{g_k}) \sum_{0 \leq s \leq a} \binom{a}{s} \beta^s (-\beta)^{b+a-s}
= \sum_{0 \leq s \leq a} \binom{a}{s} \beta^s G_{b-a+s}^{[f_i/g_k]}
= \sum_{s \geq 0} \binom{a}{s} \beta^s G_{b-a+s}^{[f_i/g_k]}.
\]

This finishes the proof of (i).

To prove (ii), we partition the set \( \text{FSVT}(\lambda/\mu, f/g) \) into three subsets \( \mathcal{F}_1, \mathcal{F}_2 \) and \( \mathcal{F}_3 \): \( \mathcal{F}_1 \) consists of those tableaux such that the filling in the leftmost box in the \( k \)-th row is exactly \( \{g_k\} \), \( \mathcal{F}_2 \) consists of those tableaux such that the filling in the leftmost box in the \( k \)-th row does not contain \( x_{g_k} \), and \( \mathcal{F}_3 \) consists of the rest. Note that by the condition (4.1) the set \( \mathcal{F}_1 \) is non-empty. Since \( g_{k+1} > g_k \), there is a bijection from \( \mathcal{F}_1 \) to \( \text{FSVT}(\lambda/\mu', f/g) \) sending \( T \) to \( T' \) obtained from removing the leftmost box together with its filling \( \{g_k\} \) from the \( k \)-th row. Thus \( \sum_{T \in \mathcal{F}_1} M(T) = x_{g_k} G_{\lambda/\mu', f/g} \). It is clear that \( \mathcal{F}_2 = \text{FSVT}(\lambda/\mu, f/g') \) so that \( \sum_{T \in \mathcal{F}_2} M(T) = G_{\lambda/\mu, f/g'} \). Furthermore, since \( g_{k+1} > g_k \), we have a bijection from \( \mathcal{F}_3 \) to \( \text{FSVT}(\lambda/\mu, f/g') \) by

\[
\begin{align*}
\sum_{T \in \mathcal{F}_1} M(T) & = x_{g_k} G_{\lambda/\mu', f/g} + \sum_{T \in \mathcal{F}_2} M(T) + \sum_{T \in \mathcal{F}_3} M(T)
& = \sum_{T \in \mathcal{F}_1} M(T) + \sum_{T \in \mathcal{F}_2} M(T) + \sum_{T \in \mathcal{F}_3} M(T).
\end{align*}
\]
Proof. For (ii), it suffices to observe that
\[ \sum_{T \in F_k} M(T) = \beta x_{g_k} G_{\lambda/{\mu}, f/g}. \]
This proves (ii).

Proposition 4.5. Let \( k \) be such that \( g_{k-1} = g_k \) and \( \mu_{k-1} = \mu_k \). Then

(i) \( \widetilde{G}_{\lambda/{\mu}, f/g} = \widetilde{G}_{\lambda/{\mu}, f/g} \),

(ii) \( G_{\lambda/{\mu}, f/g} = G_{\lambda/{\mu}, f/g} \),

where \( \beta \) is as in Proposition 4.4.

Proof. For (i), first we see that the determinants in the equation are identical except for the \( k \)-th column. We compute the difference of their \((i, k)\)-entries by using (4.2) and (4.3):

\[
\begin{align*}
\sum_{s \geq 0} \binom{i-k}{s} \beta^s G_{\lambda_i - \mu_k + k - i + s} & - \sum_{s \geq 0} \binom{i-k}{s} \beta^s G_{\lambda_i - \mu_k + k - i + s} \\
= \frac{x_{g_k}}{1 + \beta x_{g_k}} \sum_{s \geq 0} \binom{i-k}{s} \beta^s \left( G_{\lambda_i - \mu_k + k - i + s - 1} + \beta G_{\lambda_i - \mu_k + k - i + s} \right) \\
= \frac{x_{g_k}}{1 + \beta x_{g_k}} \sum_{s \geq 0} \binom{i-k}{s} \beta^s G_{\lambda_i - \mu_k + k - i + s - 1} + \frac{x_{g_k}}{1 + \beta x_{g_k}} \sum_{s \geq 0} \binom{i-k}{s-1} \beta^s G_{\lambda_i - \mu_k + k - i + s - 1} \\
= \frac{x_{g_k}}{1 + \beta x_{g_k}} \sum_{s \geq 0} \binom{i-k+1}{s} \beta^s G_{\lambda_i - \mu_k + k - i + s - 1}
\end{align*}
\]

Here the last equality follows from the identity of the binomial coefficients \( \binom{n}{s} + \binom{n}{s-1} = \binom{n+1}{s} \) for \( n, s \in \mathbb{Z} \). This shows that in the determinant \( \widetilde{G}_{\lambda/{\mu}, f/g} - \widetilde{G}_{\lambda/{\mu}, f/g} \), the \( k \)-th column coincides with \( \frac{x_{g_k}}{1 + \beta x_{g_k}} \) times the \((k-1)\)-st column. Thus \( \widetilde{G}_{\lambda/{\mu}, f/g} - \widetilde{G}_{\lambda/{\mu}, f/g} = 0 \).

For (ii), it suffices to observe that \( FSVT(\lambda/{\mu}, f/g) = FSVT(\lambda/{\mu}, f/g) \) which follows from the column strictness. \( \square \)

Proposition 4.6. If \( f_k < g_k \) and \( \mu_k < \lambda_k \) for some \( k \), then

(i) \( \widetilde{G}_{\lambda/{\mu}, f/g} = 0 \),

(ii) \( G_{\lambda/{\mu}, f/g} = 0 \).

Proof. For (ii), it suffices to observe that \( FSVT(\lambda/{\mu}, f/g) = \emptyset \). We prove (i). If \( r = 1 \), then \( k = 1 \) and \( \widetilde{G}_{\lambda/{\mu}, f/g}(x) = G_{\lambda_i - \mu_i}^{[f_i/g_i]} = 0 \). Suppose \( r > 1 \). If there is \( i \) such that \( \mu_i \geq \lambda_{i+1} \), then the claim follows from Proposition 4.3 and the induction hypothesis. Suppose that \( \mu_i < \lambda_{i+1} \) for all \( i \). In this case, (4.1) and the assumption imply that \( \mu_i < \lambda_j \) and \( f_i < g_j \) for all \( i \leq k \leq j \).

Then we can see that the \((i, j)\)-entry of the determinant of \( \widetilde{G}_{\lambda/{\mu}, f/g} \) is 0 for all \( i \) and \( j \) such that \( i \leq k \leq j \) since \( G_{m}^{[f_i/g_j]} = 0 \) for all \( m > 0 \). Thus the claim follows. \( \square \)

Theorem 4.7. We have \( \widetilde{G}_{\lambda/{\mu}, f/g}(x) = G_{\lambda/{\mu}, f/g}(x) \).

Proof. With the help of Proposition 4.3, 4.4, 4.5, 4.6, the proof is exactly the same as in Theorem 3.5 [16]. We write the proof below for completeness. We prove this by induction on \((r, \lambda - \mu, f - g)\) ordered lexicographically where \( r \) is the length of the partition \( \lambda \). If \( r = 1 \), \( \widetilde{G}_{\lambda/{\mu}, f/g} \) and \( G_{\lambda/{\mu}, f/g} \)
are the same one row Grothendieck polynomial of degree $\lambda_1 - \mu_1$ with the shifted variables, thus the claim holds. Suppose $r > 1$. If $\lambda_i - \mu_i = 0$ for some $i$, then we have $\lambda_{i+1}$$\leq$$\lambda_i = \mu_i$ or $\lambda_1 = \mu_1$$\leq$$\mu_{i-1}$. We apply Proposition 4.3 for $k = i - 1$ or $i$, and then the claim follows from the induction hypothesis. Suppose $\lambda_i - \mu_i > 0$ for all $i = 1, \ldots, r$. If $f_k - g_k < 0$ for some $k$, then the claim follows from Proposition 4.6. Suppose that $f_i - g_i > 0$ for all $i = 1, \ldots, r$. Let $k$ be such that $g_1 \geq g_2 \geq \cdots \geq g_k < g_{k+1}$ (or set $k = r$). If $\mu_{k-1} > \mu_k$ (or $k = 1$), we can apply Proposition 4.4 and the claim follows by induction. If $\mu_{k-1} = \mu_k$, then $\mu_{k-1}$$\leq$$\lambda_k$ and hence $g_{k-1} = g_k$. Now we can apply Proposition 4.5 and the claim follows from the induction hypothesis.

Acknowledgements. The author would like to thank Takeshi Ikeda for useful discussions. The author is supported by Grant-in-Aid for Young Scientists (B) 16K17584.

References

[1] Anderson, D. K-theoretic Chern class formulas for vexillary degeneracy loci. ArXiv e-prints (Jan. 2017).
[2] Buch, A. S. A Littlewood-Richardson rule for the $K$-theory of Grassmannians. Acta Math. 189, 1 (2002), 37–78.
[3] Chen, W. Y. C., Li, B., and Louck, J. D. The flagged double Schur function. J. Algebraic Combin. 15, 1 (2002), 7–26.
[4] Fomin, S., and Kirillov, A. N. The Yang-Baxter equation, symmetric functions, and Schubert polynomials. In Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993) (1996), vol. 153, pp. 123–143.
[5] Fomin, S., and Kirillov, A. N. Grothendieck polynomials and the Yang-Baxter equation. In Formal power series and algebraic combinatorics/Séries formelles et combinatoire algébrique. DIMACS, Piscataway, NJ, sd, pp. 183–189.
[6] Fulton, W. Flags, Schubert polynomials, degeneracy loci, and determinantal formulas. Duke Math. J. 65, 3 (1992), 381–420.
[7] Hudson, T., Ikeda, T., Matsumura, T., and Naruse, H. Degeneracy Loci Classes in $K$-theory - Determinantal and Pfaffian Formula -. ArXiv e-prints (Apr. 2015).
[8] Hudson, T., and Matsumura, T. Segre classes and Kempf-Laksov formula in algebraic cobordism. ArXiv e-prints (Feb. 2016).
[9] Hudson, T., and Matsumura, T. Vexillary degeneracy loci classes in $K$-theory and algebraic cobordism. ArXiv e-prints (Jan. 2017).
[10] Knutson, A., Miller, E., and Yong, A. Tableau complexes. Israel J. Math. 163 (2008), 317–343.
[11] Knutson, A., Miller, E., and Yong, A. Gröbner geometry of vertex decompositions and of flagged tableaux. J. Reine Angew. Math. 630 (2009), 1–31.
[12] Lascoux, A. Anneau de Grothendieck de la variété de drapeaux. In The Grothendieck Festschrift, Vol. III, vol. 88 of Progr. Math. Birkhäuser Boston, Boston, MA, 1990, pp. 1–34.
[13] Lascoux, A., and Schützenberger, M.-P. Polynômes de Schubert. C. R. Acad. Sci. Paris Sér. I Math. 294, 13 (1982), 447–450.
[14] Lascoux, A., and Schützenberger, M.-P. Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux. C. R. Acad. Sci. Paris Sér. I Math. 295, 11 (1982), 629–633.
[15] Matsumura, T. An algebraic proof of determinant formulas of Grothendieck polynomials. ArXiv e-prints (Nov. 2016).

[16] Wachs, M. L. Flagged Schur functions, Schubert polynomials, and symmetrizing operators. J. Combin. Theory Ser. A 40, 2 (1985), 276–289.