ON FINITENESS OF CURVES WITH HIGH CANONICAL DEGREE ON A SURFACE

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Abstract. The canonical degree of a curve $C$ on a surface $X$ is $K_X \cdot C$. Our main result, Theorem 1.1, is that on a surface of general type there are only finitely many curves with negative self–intersection and sufficiently large canonical degree. Our proof strongly relies on results by Miyaoka. We extend our result both to surfaces not of general type and to non–negative curves, and give applications, e.g. to finiteness of negative curves on a general blow–up of $\mathbb{P}^2$ at $n \geq 10$ general points (a result related to Nagata’s Conjecture). We finally discuss a conjecture by Vojta concerning the asymptotic behaviour of the ratio between the canonical degree and the geometric genus of a curve varying on a surface. The results in this paper go in the direction of understanding the bounded negativity problem.

1. Introduction.

Let $C$ be a projective curve on a smooth projective complex surface $X$. By curve we mean an irreducible, reduced 1–dimensional scheme. We denote by $g = g(C)$ its geometric genus and by $p = p_a(C)$ its arithmetic genus, i.e. $C^2 + K \cdot C = 2p_a - 2$, where $K = K_X$ is a canonical divisor of $X$. We set $\delta = \delta(C) = p - g$. We call a curve negative if $C^2 < 0$. The canonical degree of $C$ is $k_C = K \cdot C$, often simply denoted by $k$. If $g(C) \neq 1$, we set

$$\beta_C = \frac{k_C}{g(C) - 1},$$

often simply denoted by $\beta$. For a surface $X$ we set

$$a_X = 3c_2(X) - K_X^2,$$

often simply denoted by $a$. If the Kodaira dimension $\kappa = \kappa(X)$ is non–negative, one has $a \geq 0$.

The main result of this paper concerns negative curves with high canonical degree:

**Theorem 1.1.** (A) Let $C$ be a negative curve not isomorphic to $\mathbb{P}^1$ on a surface $X$ with $\kappa \geq 0$. If $a = 0$, there are no rational curves on $X$, i.e. either $a \neq 0$ or $g > 0$. Moreover

$$k_C \leq 3(g - 1) + \frac{3}{4}a + \frac{1}{4}\sqrt{9a^2 + 24a(g - 1)}. \quad (1.1)$$

Furthermore, if $g > 1$ then

$$\beta_C \leq 3 + \frac{3}{4}a + \frac{1}{4}\sqrt{9a^2 + 24a} \leq 4 + \frac{3}{2}a. \quad (1.2)$$

If, in addition, $\beta = 3 + \epsilon > 3$, then

$$g \leq 1 + \frac{3a(\epsilon + 1)}{2\epsilon^2}. \quad (1.3)$$

(B) Suppose $\kappa(X) = 2$. Then for any $\epsilon > 0$ there are at most finitely many negative curves $C$ on $X$ such that $k_C \geq (3 + \epsilon)(g - 1) > 0$.

From (B) it follows that:

**Corollary 1.2.** Let $X$ be a surface of general type. There is a function $B(\epsilon)$, defined for $\epsilon \in [0, \infty[$ such that for all negative curves $C$ we have

$$k_C \leq (3 + \epsilon)(g - 1) + B(\epsilon) \text{ and } - C^2 \leq (1 + \epsilon)(g - 1) + B(\epsilon).$$

On some smooth quaternionic Shimura surfaces $X$ there are infinitely many totally geodesic curves (see [1]). Such a curve $C$ is also called a Shimura curve and satisfies $k_C = 4(g - 1)$. We thus obtain the following corollary, which was one of the main result of [2]:

**Corollary 1.3.** On a Shimura surface, there exist finitely many (may be none) negative Shimura curves.
The proof of Theorem 1.1 contained in [2] strongly relies on a result by Miyaoka’s (see [11] Cor 1.4] stated as Theorem 2.1 below). In particular, the inequality (1.1) is very similar to [11] formula (3)], which has a slightly lower growth in \( g \), but applies only to minimal surfaces.

In [3] we make an extension of Theorem 1.1 which works also in the case \( \kappa = -\infty \), and we prove a finiteness result for negative curves on a general blow-up of \( \mathbb{P}^2 \) at \( n \geq 10 \) general points. This is a bounded negativity result which is reminiscent of the famous Nagata’s Conjecture, predicting that there is no negative curve on such a surface except for \((-1)\)-rational curves.

In [4] using again Miyaoka’s result, we prove a boundedness theorem for non–negative curves of high canonical degree. In [5] we discuss a conjecture by Vojta concerning the asymptotic behaviour of \( k_C/(g-1) \) when \( C \) varies among all curves on a surface. We introduce an invariant related to Vojta’s conjecture and we prove a bound for it.

The results in this paper go in the direction of understanding bounded negativity (see [2]). The Bounded Negativity Conjecture (BNC) predicts that on a surface of general type over \( \mathbb{C} \) the self–intersection of negative curves is bounded below. Nagata’s conjecture, which we mentioned above, is also a sort of bounded negativity assertion. As a general reference on both bounded negativity and Nagata’s conjecture, see [8]. Also Vojta’s conjecture is related to bounded negativity, as we discuss in [5]. As a consequence of Theorem 1.1 we have the following information on negative curves:

**Corollary 1.4.** Suppose BNC fails, so that there exists a sequence \( (C_n)_{n \in \mathbb{N}} \) of negative curves of genus \( g_n \) with \( \lim g_n = \infty \) and \( \lim C_n^2 = -\infty \). Then

\[
\limsup_n \frac{K \cdot C_n}{g_n - 1} \leq 3.
\]

In conclusion, the authors would like to thank B. Harbourne and J. Roé for useful exchanges of ideas about the application to Nagata’s Conjecture in [3].

2. The proof of the main theorem

Our proof relies on the following result by Miyaoka’s (see [11] Cor 1.4]):

**Theorem 2.1.** Let \( C \) be curve on a surface \( X \) with \( \kappa \geq 0 \). Then for all \( \alpha \in [0, 1] \), we have:

\[
\alpha^2(C^2 + 3k_C - 6g + 6) - 4\alpha(k_C - 3g + 3) + 2\alpha \geq 0.
\]

Suppose \( C \) is not isomorphic to \( \mathbb{P}^1 \) and \( k_C > 3(g-1) \). Then

\[
2(k_C - 3g + 3)^2 - a(C^2 + 3k_C - 6g + 6) \leq 0.
\]

Suppose in addition \( K^2 > 0 \). Then

\[
\left( \frac{C_2}{K^2} - 1 \right) k_C^2 + (4(g-1)+a)k_C - 2(g-1)(3g-1+a) \geq \left( \frac{C_2}{K^2} - 1 \right) [k_C^2 - C^2K^2] \geq 0.
\]

**Proof.** Inequality (2.1) is [11] Thm 1.3, (i)] and (2.2) is [11] Thm 1.3, (ii)]. As for (2.3) this is [11] Thm 1.3, (iii)], which is stated there under the assumption that \( X \) is minimal of general type and \( C \neq \mathbb{P}^1 \). However Miyaoka’s argument works more generally under the weaker assumption \( K^2 > 0 \). \( \square \)

We are now ready for the:

**Proof of Theorem 1.1.** Let us prove (A). Let \( C \) be a curve on a surface \( X \) not isomorphic to \( \mathbb{P}^1 \). Then \(-aC^2 \geq 0\), with equality if and only if \( a = 0 \). If \( k_C \leq 3(g-1) \), there is nothing to prove. Let us suppose \( k = k_C > 3(g-1) \) and set \( g = g - 1 \). By (2.2), one has

\[
P(k) := 2(k - 3g)^2 - a(3k - 6g) \leq 0,
\]

with strict inequality if \( a > 0 \).

If \( a = 0 \) and \( g = 0 \), the polynomial \( P \) is positive, thus this cannot occur. In the remaining cases, \( k_C \) is less than or equal to the largest root of \( P \), whence we get (1.1).

Suppose \( g > 1 \) and let \( \epsilon > 0 \) be such that \( k_C = (3+\epsilon)(g-1) \). By (1.1) we obtain

\[
\epsilon(g-1) \leq \frac{3}{4}a + \frac{1}{4} \sqrt{9a^2 + 24a(g-1)},
\]

hence

\[
\epsilon \leq \frac{3a}{4(g-1)} + \frac{1}{4(g-1)} \sqrt{9a^2 + 24a(g-1)} \leq \frac{3}{4}a + \frac{1}{4} \sqrt{9a^2 + 24a} \leq \frac{3}{2}a + 1.
\]
which yields (1.2). On the other hand, (2.5) reads
\[ 4\epsilon(g - 1) - 3\alpha \leq \sqrt{9\alpha^2 + 24\alpha(g - 1)}, \]
and by squaring one gets (1.3), finishing the proof of (A).

Next we prove (B). Let \( \beta_0 > 3 \). By (1.2) and (1.3), negative curves with \( \beta > \beta_0 \) have bounded genus \( g \), therefore by (1.1) also \( k_C \) is bounded, hence the arithmetic genus \( p \) is bounded.

Suppose \( K \) is big. By [10] Cor. 2.2.7 there exist \( m \in \mathbb{N}^* \), an ample divisor \( A \) and an effective divisor \( Z \) such that
\[ mK \equiv A + Z. \]
Since \( Z \) is effective, the set of integers \( Z \cdot C \), when \( C \) varies among negative curves, is bounded from below, therefore the degree \( A \cdot C = (mK - Z) \cdot C \) of these curves with respect to the ample divisor \( A \) is bounded. Hence, by results of Chow–Grothendieck [7], [12, Lecture 15], one has only finitely many components of the Hilbert scheme containing points corresponding to such curves. Since they are negative, these components contain only one curve, proving the assertion. \( \square \)

3. Surfaces not of general type

We want to deduce from Theorem 1.1 a result valid for any smooth surface.

**Theorem 3.1.** Let \( Y \) be any smooth projective surface. Let \( \eta \in \text{Pic}(Y) \) be such that \( |KY + \eta| \) is big and \( |2\eta| \) contains a base point free pencil. Let \( \beta_0 > 3 \). Then there are finitely many negative curves \( D \) on \( Y \) such that
\[ k_D \geq \beta_0(g - 1) + \frac{\beta_0 - 2}{2} D \cdot \eta. \]

**Proof.** Under the hypotheses there is a smooth curve \( B \equiv 2\eta \) intersecting all negative curves of \( X \) only at smooth points with intersection multiplicity 1. Let us make a double cover \( f : X \to Y \) branched along \( B \). Then for all negative curve \( D \) of \( X \), \( C = f^*(D) \) is irreducible, negative and \( g(C) = 2g(D) - 1 + \eta \cdot D \) by Hurwitz formula. Since \( f_*(K_X) = KY \oplus (KY + \eta) \), then \( \kappa(X) = 2 \) and we finish by applying (B) of Theorem 1.1 to \( X \) and to \( C = f^*(D) \). \( \square \)

As an application, we take \( Y_n \) to be the plane blow-up at \( n \) general points. Then \( \text{Pic}(Y_n) \cong \mathbb{Z}^{n+1} \) generated by the classes of the pull-back \( L \) of a line and of minus the exceptional divisors \( E_1, \ldots, E_n \) over the blown-up points. We write \( D = (d, m_1, \ldots, m_n) \) to denote the class of a curve with components \( d, m_1, \ldots, m_n \) with respect to this basis. We may use exponential notation to denote repeated \( m_i \)'s. Thus \( -K = (3, 1^n) \).

**Proposition 3.2.** Fix \( \beta_0 > 3 \). There are finitely many irreducible curves of class \( D = (d, m_1, \ldots, m_n) \) on \( Y_n \) such that
\[ \frac{D^2}{d} \leq \frac{2 - \beta_0}{\beta_0} (1 + \frac{M}{d}) \text{ where } M = \sum_{i=1}^{n} m_i. \]

**Proof.** We apply Theorem 3.1 by taking \( \eta = 4L \). Indeed \( -K + \eta = (1, -1^n) \) is big. \( \square \)

Recall that
\[ \epsilon_n = \inf \{ \frac{d}{M} \text{ for all effective } D = (d, m_1, \ldots, m_n), \text{ such that } M > 0 \} \]
is the Seshadri constant of \( Y_n \). Nagata’s Conjecture (see [13]) is equivalent to say that \( \epsilon_n = 1/\sqrt{n} \) if \( n \geq 10 \) (see [9]).

**Remark 3.3.** Proposition 3.2 can be seen as weak form of Nagata’s Conjecture. Indeed, let us look at the homogenous case \( D = (d, m^n) \) with \( n \geq 10 \). Nagata’s Conjecture predicts that, if the \( n \) blown-up points are in very general position, there is no irreducible such curve with \( D^2 < 0 \) (see [6, 13]), i.e., with \( d < \sqrt{nm} \). The conclusion of Proposition 3.2 is not absence of curves, but finiteness of their set, under a stronger assumption than Nagata’s. Let us look at the difference between the two assumptions. In the \( (m, d) \)-plane (3.2) applies to pairs \( (m, d) \) in the first quarter below the hyperbola with equation
\[ \beta_0 d^2 + d(\beta_0 - 2) - n\beta_0 m^2 + (\beta_0 - 2)nm = 0 \]
drawn in black in Figure 1. One of its asymptotes (drawn in blue) is parallel to the Nagata line \( d = \sqrt{7nm} \) (drawn in green).
Since for all effective divisor $D = (d, m_1, \ldots, m_n)$ one has $d/M \geq \epsilon_n$, one has approximations $d/M \geq e_n$ to Nagata’s conjecture for any lower approximation $e_n$ of $\epsilon_n$. The best known in general is the one in [9]

$$\epsilon_n \geq e_n = \sqrt{1/n (1 - 1/f(n))} \quad (3.4)$$

where $f(n)$ is, for most $n$, an explicitly given quadratic function of $n$ (see [9, Corollary 1.2.3]). For $n = 10$ in the homogeneous case the best result is $e_{10} = 228/721$ (see [14]).

The hyperbola (3.3) meets the line $d = e_n m$, therefore Proposition 3.2 gives some information in an unlimited region where the above approximations to Nagata do not work.

**Remark 3.4.** Proposition 3.2 implies that there are finitely many irreducible curves of class $D = (d, m_1, \ldots, m_n)$ on $Y_n$ such that

$$\frac{D^2}{d} \leq \frac{2 - \beta_0}{\beta_0} (1 + \frac{1}{\epsilon_n}), \quad (3.5)$$

where $\epsilon_n$ can be replaced by $e_n$ in Harbourne–Roé’s approximation (3.4). This result is not surprising. Indeed, J. Roé pointed out to us an easy argument which shows that there is no irreducible curve of class $D = (d, m_1, \ldots, m_n)$ on $Y_n$ such that

$$\frac{D^2}{d} < -\frac{1}{n \epsilon_n}$$

which is better than (3.5), and the difference

$$\frac{\beta_0 - 2}{\beta_0} (1 + \frac{1}{\epsilon_n}) - \frac{1}{n \epsilon_n}$$

tends to $\frac{\beta_0 - 2}{\beta_0} \sim \frac{1}{3}$ for $n \to \infty$.

### 4. A boundedness result for non–negative curves

With the usual notation, for a curve $C$ on the surface $X$ with $C^2 \neq 0$, we set $x_C := \frac{\delta_C}{C^2}$, with the usual convention that the index $C$ can be dropped if there is no ambiguity.

**Theorem 4.1.** Consider real numbers $x_0 > \frac{1}{2}$ and $\beta_0 > 3$. Let $C$ be a curve on $X$, with $\kappa(X) \geq 0$, satisfying the following conditions:

1. $C^2 > 0$, $k_C = \beta (g - 1)$ with $\beta > \beta_0$ and $g > 1$;
2. $x_C > x_0$.

Then

$$g \leq a \frac{(\beta - 2)}{(\beta - 3)^2} \left( \frac{3x_0 - 1}{2x_0 - 1} \right) + 1, \quad (4.1)$$

$$k_C \leq a \frac{(\beta - 2)}{\beta (\beta - 3)^2} \left( \frac{3x_0 - 1}{2x_0 - 1} \right), \quad (4.2)$$
\[ k_C \leq 2(g - 1) + a \frac{(\beta - 2)^2}{(\beta - 3)^2} \cdot \frac{3x_0 - 1}{2x_0 - 1}. \] (4.3)

If \( \kappa(X) = 2 \), then the Hilbert scheme of curves on \( X \) satisfying (1) and (2) has finitely many irreducible components.

**Proof.** One has

\[ k_C - 2(g - 1) = 2\delta - C^2 = (\beta - 2)(g - 1) > 0. \]

Hence by (2.1), we have

\[ P(\alpha) := \alpha^2(3\delta - C^2) + \alpha \frac{2(\beta - 3)}{\beta - 2}(C^2 - 2\delta) + a \geq 0 \] (4.4)

for \( \alpha \in [0, 1] \). Since the coefficient of the leading term of \( P \) is positive, the minimum of \( P(\alpha) \) is attained for

\[ \alpha_0 = \frac{(\beta - 3)(2\delta - C^2)}{(\beta - 2)(3\delta - C^2)}. \]

Since \( \beta > 3 \), we have \( \alpha_0 \in ]0, 1[ \), and, by (4.4) we have

\[ P(\alpha_0) = -\frac{(\beta - 3)^2(2\delta - C^2)^2}{(\beta - 2)^2(3\delta - C^2)} + a \geq 0. \]

Thus

\[ \frac{a}{\mu} \geq \frac{(2\delta - C^2)^2}{(3\delta - C^2)} \text{ where } \mu = \frac{(\beta - 3)^2}{(\beta - 2)^2}, \]

hence

\[ \frac{a}{\mu} \frac{3\delta - C^2}{2\delta - C^2} \geq 2\delta - C^2. \]

We have

\[ \frac{3\delta - C^2}{2\delta - C^2} = \frac{3x_0 - 1}{2x_0 - 1} < \frac{3x_0 - 1}{2x_0 - 1} \]

because \( \frac{3x - 1}{2x - 1} \) is decreasing for \( x > x_0 > \frac{1}{2} \), hence

\[ (\beta - 2)(g - 1) = k_C - 2(g - 1) = 2\delta - C^2 \leq \frac{a}{\mu} \frac{3x_0 - 1}{2x_0 - 1}, \]

which implies (4.1), (4.2) and (4.3). Moreover both \( g \) and \( k_C \) are bounded from above and, if \( \kappa(X) = 2 \), we conclude with the same argument at the end of the proof of Theorem 1.1 \( \square \)

**Corollary 4.2.** Let be \( \beta_0 > 3 \) and let \( (C_n)_{n \in \mathbb{N}} \) be a sequence of curves on \( X \) with \( \kappa = 2 \) such that \( k_{C_n} > \beta_0(g(C_n) - 1), C_n > 0 \) and \( \lim g(C_n) = \infty \). Then

\[ \lim_n \frac{\delta C_n}{C_n^2} = \frac{1}{2}, \] (4.5)

moreover \( \lim_n \frac{g(C_n)}{C_n^2} = \lim_n \frac{k_{C_n}}{C_n^2} = 0. \)

**Proof.** Let \( C \) be a curve with \( k_C = (3 + \epsilon)(g - 1), \epsilon > 0 \). Since \( (1 + \epsilon)(g - 1) = 2\delta - C^2 \), we get \( \frac{\delta}{C^2} = \frac{1}{2} \geq 0 \). Therefore \( \lim_{n} \frac{\delta}{C_n^2} \geq \frac{1}{2} \). On the other hand, by Theorem 1.1 we obtain \( \lim_{n} \sup \frac{\delta_{C_n}}{C_n^2} \leq \frac{1}{2} \). The remaining limits are readily computed. \( \square \)

**Example 4.3.** For Shimura curves on Shimura surfaces, we have \( K \cdot C = 4(g - 1) \) and, if there is one, there are infinitely many of them, with the genus going to infinity.

5. **ON A CONJECTURE BY VOJTA**

The results in [1] are reminiscent of the following conjecture (see [1]), which predicts that curves of bounded geometric genus on a surface of general type form a bounded family:

**Conjecture 5.1.** Let \( X \) be a smooth projective surface. There exist constants \( A, B \) such that for any curve \( C \) we have

\[ k_C \leq A(g - 1) + B. \]

If this conjecture is satisfied for \( X \) with \( \kappa(X) = 2 \), then \( X \) contains finitely many curves of genus 0 or 1. This is known to hold for minimal surfaces with big cotangent bundle (see [3, 4]).

A stronger version of Conjecture 5.1 is the following conjecture by Vojta (see again [1]):
Conjecture 5.2. For any real number $\epsilon > 0$, we can take $A = 4 + \epsilon$ in Conjecture 5.1 (and $B = B(\epsilon)$ a function of $\epsilon$).

An even stronger, more recent version, predicts that $A = 2 + \epsilon$ (see [1]).

Remark 5.3. If $C$ is a smooth curve on $X$, then $k_C = 2(g - 1) - C^2$, therefore if BNC holds, then Vojta’s conjecture holds for smooth curves with $A = 2$. This suggests a close relationship between Vojta’s conjecture and BNC.

Miyaoka proves in [3] that Conjecture 5.1 also holds if $K^2 > c_2$ and he gives explicit values for $A$ and $B$, but they are far away from the ones predicted by Conjecture 5.2. Moreover Miyaoka proves that $k_C \leq 3(g - 1)$ for (smooth) compact ball quotient surfaces on which the equality is attained by an infinite number of curves, i.e., Shimura curves, if they exists.

In [1] one proves that for surfaces whose universal cover is the bi–disk, one has

$k_C \leq 4(g - 1)$.

This is sharp since for Shimura curves on Shimura surfaces, one has $k_C = 4(g - 1)$.

For $X$ a surface, we define

$$\Lambda_X = \sup_{(C_n)_{n \in \mathbb{N}}} \left\{ \limsup_n \frac{K \cdot C_n}{g_n - 1} \right\}$$

where $(C_n)_{n \in \mathbb{N}}$ varies among all sequences of curves $C_n$ in $X$ of genus $g_n = g(C_n) > 1$ with $\lim_n g_n = \infty$. Conjecture 5.1 says that $\Lambda_X < \infty$.

If $X$ has trivial canonical bundle, then $\Lambda_X = 0$. Apart form this case, and the aforementioned cases studied in [1][3][4], nothing is known about $\Lambda_X$. The following result gives us a piece of information:

Theorem 5.4. Let $X$ be a surface of general type. Let $L$ be a very ample divisor on $X$, and let $\gamma$ be the arithmetic genus of curves in $|L|$. Then

$$\Lambda_X \geq \frac{K \cdot L}{L^2 + \gamma - 1} > 0.$$  

Proof. Look at the surface $X$ embedded in $\mathbb{P}^r$, with $r \geq 3$, via $L$. Then take a general projection $\pi : X \to \mathbb{P}^2$. Consider a general rational curve of degree $n$ in $\mathbb{P}^2$ and let $C_n$ be its pull–back via $\pi$. Then $C_n \in |nL|$.

By Hurwitz formula, the ramification divisor $R$ of $\pi$ is such that $R \equiv K + 3L$. So Hurwitz formula again, implies that the geometric genus $g_n$ of $C_n$ satisfies

$$2g_n - 2 = nL \cdot K + (3n - 2)L^2.$$  

Therefore

$$\frac{K \cdot C_n}{g_n - 1} = \frac{L \cdot K}{L^2 + \gamma - 1 - \frac{L^2}{\pi}}$$

and this proves the assertion. \hfill \Box

Example 5.5. Suppose that, in the setting of Theorem 5.4 one has $K = mL$, with $m > 0$. Then

$$\Lambda_X \geq \frac{2m}{m + 3}.$$  

So there are sequences $(X_n)_{n \in \mathbb{N}}$ of surfaces, e.g., complete intersections of increasing degree in projective space, with $m \to \infty$, and therefore $\Lambda_{X_n} \to 2$ (from below).

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