On the motion of a rigid body with small diameter in a viscous incompressible fluid

Eduard Feireisl * Arnab Roy1 Arghir Zarnescu1,2,3

September 21, 2022

Abstract

We consider the motion of a rigid particle of small diameter immersed in a viscous incompressible fluid contained in a domain in the Euclidean space $\mathbb{R}^d$, $d = 2, 3$. Assume the particle, represented by a compact set $S_\varepsilon \subset \mathbb{R}^d$, has a constant mass density $0 < \rho_\varepsilon < \rho$, and

$$0 < \lambda \text{diam}[S_\varepsilon]^d \leq |S_\varepsilon| \to 0 \text{ as } \varepsilon \to 0, \ d \leq \beta < \begin{cases} 15 \text{ if } d = 3, \\ \text{arbitrary finite if } d = 2 \end{cases}$$

*The work of E.F. was partially supported by the Czech Sciences Foundation (GAČR), Grant Agreement 21–02411S. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840. A.R and A.Z have been partially supported by the Basque Government through the BERC 2022-2025 program and by the Spanish State Research Agency through BCAM Severo Ochoa excellence accreditation SEV-2017-0718 and through project PID2020-114189RB-I00 funded by Agencia Estatal de Investigación (PID2020-114189RB-I00 / AEI / 10.13039/501100011033). A.Z. was also partially supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS - UEFISCDI, project number PN-III-P4-PCE-2021-0921, within PNCDI III.
We show the fluid flow is not influenced by the presence of the particle in the asymptotic limit as $\varepsilon \to 0$. The result is stated in the class of finite energy weak solutions of the corresponding fluid–structure interaction problem and holds without any restriction of the size on the initial data and smallness of the particle density $\rho_S^\varepsilon$.

Keywords: Navier-Stokes system, body–fluid interaction problem, small light rigid body

1 Introduction

There is a number of studies concerning the impact of a small rigid body immersed in a viscous fluid on the fluid motion. A general approach used so far is based in the idea that if the body is small but “heavy”, meaning its mass density $\rho_S^\varepsilon$ is large, its velocity can be controlled and the resulting situation is therefore close to the rigid obstacle problem. He and Iftimie [10], [11] exploited this idea to handle the case when $\rho_S^\varepsilon \to \infty$, while the rigid body diameter is proportional to a small number $\varepsilon$. More recently, Bravin and Nečasová [2] showed that if the density is “very large”, the rigid object keeps moving with its initial velocity not being influenced by the fluid. Note that these results are slightly at odds with a physically relevant hypothesis that the body density should be at least bounded and also with a commonly accepted scenario that a light particle should not have any major impact on the fluid motion.

To the best of our knowledge, the only available result concerning a body with a constant density was obtained by Lacave and Takahashi [17] in the case of the planar motion. Their technique, similarly to a more recent paper by Tucsnak et al. [3], is based on the $L^p - L^q$ theory for the associated solution semigroup and requires smallness of the initial fluid velocity. Our goal is to extend the main result of [17] in the following sense.

- The fluid is confined to an arbitrary domain $\Omega \subset \mathbb{R}^d$, $d=2,3$.

- The specific shape of the body is irrelevant. The only restriction reads

$$0 < \lambda \text{diam}[S_\varepsilon]^\beta \leq |S_\varepsilon| \to 0 \text{ as } \varepsilon \to 0, \quad d \leq \beta < \begin{cases} 15 \text{ if } d = 3, \\ \text{arbitrary finite if } d = 2, \end{cases}$$

for some $\lambda > 0$ independent of $\varepsilon$, whereas its mass density $\rho_S^\varepsilon$ is a positive constant $0 < \rho_S^\varepsilon \leq \overline{\rho}$ bounded above uniformly for $\varepsilon > 0$. In particular, we allow the density to vanish in the asymptotic limit,

$$\rho_S^\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$
• The result holds in the class of weak solutions and for any finite–energy initial data.

Very roughly indeed, we may conclude that the effect of a single rigid body is negligible as soon as its diameter is small and the body is not very “thin” with respect to its diameter. The mass density of the body, however, is irrelevant. Hypothesis (1.1) allows the body to shrink to zero with different scaling in different directions. Note that $\beta = d$ if $S_\epsilon$ is a ball with a vanishing diameter.

Let us mention that there are some results in the context of rigid obstacles in viscous Newtonian fluids. The flow around a small rigid obstacle was studied by Iftimie et al. [12]. Lacave [16] studies the limit of a viscous fluid flow in the exterior of a thin obstacle shrinking to a curve.

As already pointed out, we use the framework of weak solutions in the spirit of Ginzburger, Lee and Seregin [15] or Galdi [13]. The relevant existence theory for the fluid structure interaction problem was developed by San Martin, Starovoitov, and Tucsnak [18] for $d = 2$ and in [6] for $d = 3$. In both cases, the solutions are global–in–time and allow for possible collisions of the body with the domain boundary.

Similarly to the companion paper [8] concerning compressible fluids, our approach is based on a new restriction operator that assigns a given function its “projection” on the space of rigid motions attached to the body. This improves considerably the error estimates necessary to perform the asymptotic limit. Another new ingredient is that we use the dissipation energy rather than the energy itself to obtain suitable bounds on the rigid body translation velocity. This is why the result is independent of the mass density of the body.

The paper is organized as follows. In Section 2, we formulate the problem and state our main result. Next, in Section 3 we derive uniform bounds on the sequence of solutions to the fluid–structure interaction problem independent of the scaling parameter. In Section 4 we introduce a restriction operator suitable for modifying the test function in the weak formulation of the problem. The convergence analysis and the proof of the main result are done in Section 5.

### 2 Problem formulation, main result

We consider a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, containing a viscous, incompressible Newtonian fluid. Accordingly the fluid velocity $u$ satisfies the Navier–Stokes system of equations

$$\operatorname{div} xu = 0, \quad (2.1) \tag{m1}$$
$$\partial_t u + \operatorname{div} x (u \otimes u) + \nabla x \Pi = \operatorname{div} x S(D_x u) + g, \quad (2.2) \tag{m2}$$
$$S(D_x u) = \mu D_x u, \quad D_x u = \frac{\nabla x u + \nabla x^t u}{2}, \quad \mu > 0, \quad (2.3) \tag{m3}$$
where $\Pi$ is the pressure and the function $g$ denotes an external volume force.

The rigid body is a compact connected set $S \subset \mathbb{R}^d$. We suppose the body is immersed in the fluid and its position at a time $t > 0$ is determined by a family of affine isometries $(\sigma(t))_{t \geq 0}$,

$$S(t) = \sigma[S], \quad \sigma(t)x = \mathbb{O}(t)x + h(t), \quad \mathbb{O} \in SO(d), \quad t \geq 0.$$ 

In addition, we introduce the associated rigid velocity field,

$$u_S(t, x) = Y + Q(x - h), \quad Y(t) = \frac{d}{dt}h(t), \quad Q(t) = \frac{d}{dt}\mathbb{O}(t) \circ \mathbb{O}^{-1}(t).$$ \hfill (2.4)

Finally, we identify the fluid region

$$\Omega_f(t) = \Omega \setminus S(t), \quad Q_f = \{(t, x) \mid t \in (0, T), \ x \in \Omega_f(t)\}.$$ 

### 2.1 Weak formulation

We suppose that the rigid body is immersed in the fluid. As the fluid is viscous, a natural working hypothesis asserts that both the velocity and the momentum coincide on the body boundary, see e.g. [13]. Accordingly, a suitable weak formulation of the fluid–structure interaction problem (see [6]) is based on the quantities $(\rho, u, h, \mathbb{O})$. It is convenient to consider both the mass density $\rho = \rho(t, x)$ and the velocity $u = u(t, x)$ as functions defined for all $x \in \mathbb{R}^d$.

#### 2.1.1 Regularity

- The mass density $\rho$ is non-negative,

$$\rho \in L^\infty((0, T) \times \mathbb{R}^d) \cap C^1([0, T]; L^1(\Omega)).$$ \hfill (2.5)

- The velocity $u$ belongs to the Ladyzhenskaya class

$$u \in L^\infty(0, T; L^2(\mathbb{R}^d; \mathbb{R}^d)) \cap L^2(0, T; W^{1,2}(\mathbb{R}^d; \mathbb{R}^d)), \quad \text{div}_x u = 0.$$ \hfill (2.6)

- The affine isometries are Lipschitz continuous in time,

$$h \in W^{1,\infty}(0, T; \mathbb{R}^d), \quad \mathbb{O} \in W^{1,\infty}(0, T; SO(d)).$$ \hfill (2.7)
2.1.2 Compatibility

- \( \rho(t, x) = 1 \) for any \( t \in [0, T] \) and a.a. \( x \in \mathbb{R}^d \setminus \mathcal{S}(t) \). (2.8)  

- \( u(t, \cdot) \in W^{1,2}_0(\Omega; \mathbb{R}^d) \) \( (u - u_S)(t, \cdot) \in W^{1,2}_0(\mathbb{R}^d \setminus \mathcal{S}(t); \mathbb{R}^d) \)\( \) (2.9)  

\( m_10 \)

\( (u - u_S)(t, \cdot) \in W^{1,2}_0(\mathbb{R}^d \setminus \mathcal{S}(t); \mathbb{R}^d) \) for a.a. \( t \in (0, T) \). (2.10) \( m_11 \)

2.1.3 Mass conservation

The equation of continuity

\[
\int_0^T \int_{\mathbb{R}^d} \left[ \rho \partial_t \varphi + \rho u \cdot \nabla_x \varphi \right] dt = -\int_{\mathbb{R}^d} \rho_0 \varphi(0, \cdot) \quad (2.12) \quad m_{13}
\]

holds for any \( \varphi \in C^1_c([0, T) \times \mathbb{R}^d) \).

2.1.4 Momentum balance

The momentum equation

\[
\int_0^T \int_{\mathbb{R}^d} \left[ \rho u \cdot \partial_t \varphi + \rho u \otimes u : D_x \varphi \right] dt \\
= \int_0^T \int_{\mathbb{R}^d} \left[ S(D_x u) : D_x \varphi - \rho g \cdot \varphi \right] dt - \int_{\mathbb{R}^d} \rho_0 u_0 \cdot \varphi \quad (2.13) \quad \text{weak:mom}
\]

holds for any function \( \varphi \in C^1_c([0, T) \times \Omega, \mathbb{R}^d) \), \( \text{div}_x \varphi = 0 \) satisfying

\[
D_x \varphi(t, \cdot) = 0 \text{ on an open neighbourhood of } \mathcal{S}(t) \text{ for any } t \in [0, T). \quad (2.14) \quad m_{15}
\]

2.1.5 Total energy dissipation

The energy inequality

\[
\int_{\mathbb{R}^d} \rho |u|^2(\tau, \cdot) + \mu \int_0^\tau \int_{\mathbb{R}^d} S(D_x u) : D_x u \leq \int_{\mathbb{R}^d} \rho_0 |u_0|^2 + \int_0^\tau \int_{\mathbb{R}^d} \rho g \cdot u \ dt \quad (2.15) \quad \text{eq2}
\]

holds for a.a. \( \tau \in (0, T) \).
2.2 Main result: Vanishing body problem

We are ready to state our main result.

**Theorem 2.1 (Asymptotic limit).** Let \( \mathcal{S}_\varepsilon \subset \mathbb{R}^d, d = 2, 3 \) be a family of compact connected sets satisfying

\[
0 < \lambda \text{diam}[\mathcal{S}_\varepsilon]^{\beta} \leq |\mathcal{S}_\varepsilon| \to 0, \quad 2 \leq \beta < \infty \text{ if } d = 2, \quad 3 \leq \beta < 15 \text{ if } d = 3 \tag{2.16}
\]

for some \( \lambda > 0 \) independent of \( \varepsilon \). Let the rigid body densities be constant,

\[
0 < \rho^\varepsilon_S \leq \overline{\rho} \text{ uniformly for } \varepsilon \to 0. \tag{2.17}
\]

Let \( (\rho^\varepsilon, u^\varepsilon, h^\varepsilon, \Omega^\varepsilon)_{\varepsilon > 0} \) be the associated sequence of weak solutions to the fluid–structure interaction problem specified in Section 2.1, with the initial data \( (\rho^\varepsilon_0, u^\varepsilon_0)_{\varepsilon > 0} \) satisfying

\[
\rho^\varepsilon_0(x) = \begin{cases} 
\rho^\varepsilon_S & \text{if } x \in \mathcal{S}_\varepsilon(0), \\
1 & \text{otherwise}
\end{cases} \tag{2.18}
\]

\[
u^\varepsilon_0 \in L^2(\mathbb{R}^d; \mathbb{R}^d), \quad \rho^\varepsilon_0 u^\varepsilon_0 \to u_0 \text{ in } L^2(\mathbb{R}^d; \mathbb{R}^d), \tag{2.19}
\]

where

\[
u_0 \in L^2(\Omega; \mathbb{R}^d), \quad \text{div}_x u_0 = 0. \tag{2.20}
\]

Finally, suppose

\[
g = g(x), \quad g \in L^2 \cap L^\infty(\mathbb{R}^d; \mathbb{R}^d). \tag{2.21}
\]

Then, up to a suitable subsequence,

\[
u^\varepsilon \to u \text{ weakly in } L^2(0,T; W^{1,2}_0(\Omega; \mathbb{R}^d)),
\]

and in \( L^2((0,T) \times \Omega; \mathbb{R}^d) \),

\[
\tag{2.22}
\]

where \( u \) is a weak solution of the Navier–Stokes system \((2.1) \rightarrow (2.3)\), with the initial data \( u_0 \) satisfying the energy inequality

\[
\int_{\Omega} |u|^2(\tau, \cdot) \, dx + \int_{0}^{\tau} \int_{\Omega} S(D_x u) : D_x u \, dx \leq \int_{\Omega} |u_0|^2 \, dx + \int_{0}^{\tau} \int_{\Omega} g \cdot u \, dx \, dt \tag{2.23}
\]

for a.a. \( \tau \in (0,T) \).

**Remark 2.2.** In fact, hypothesis \((2.17)\) requiring uniform bounds on the body density is not restrictive. Indeed, if \( \rho^\varepsilon_S \) contains an unbounded sequence, then the relevant results have been already obtained by He and Iftimie \cite{10}, \cite{11}. The main novelty of the present result is allowing the body density to be asymptotically small.
Remark 2.3. We may set diam\([S_\varepsilon]\) = \(\varepsilon\). In accordance with hypothesis (2.16), there is a ball \(B_{r\varepsilon}[y_\varepsilon]\) centred at \(y_\varepsilon \in \mathbb{R}^d\) and of radius \(r\varepsilon\) such that

\[
S_\varepsilon \subset B_{r\varepsilon}[y_\varepsilon].
\]

In addition, choosing \(r > 0\) large enough, we may suppose

\[
y_\varepsilon = \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} x \, dx
\]

coincided with the barycenter of \(S_\varepsilon\). Accordingly, we may infer

\[
S_\varepsilon(t) \subset B_{r\varepsilon}[h^\varepsilon(t)], \quad t \in [0, T],
\]

(2.24) \[1\]

where \((h^\varepsilon)_{\varepsilon > 0}\) is the rigid translation of the barycenter of the body. Without any loss of generality, we consider \(r = 1\) and \(S_\varepsilon(t) \subset B_{\varepsilon}[h^\varepsilon(t)], \quad t \in [0, T]\).

It follows from the weak formulation of the momentum equation (2.13)

\[
\int_0^T \int_{\mathbb{R}^d} \left[ \rho u \cdot \partial_t \varphi + u \otimes u : D_x \varphi \right] \, dx \, dt = \int_0^T \int_{\mathbb{R}^d} \left[ S(D_x u) : D_x \varphi - \rho g \cdot \varphi \right] \, dx \, dt - \int_{\mathbb{R}^d} \rho^\varepsilon_0 u^\varepsilon_0 \cdot \varphi \, dx
\]

(2.25) \[2\]

holds for any function \(\varphi \in C^1_c([0, T) \times \Omega; \mathbb{R}^d), \text{div}_x \varphi = 0\) satisfying

\[
\nabla_x \varphi(t, \cdot) = 0 \quad \text{on} \quad B_\varepsilon[h^\varepsilon(t)] \quad \text{for any} \ t \in [0, T).
\]

(2.26) \[3\]

Obviously, the test functions satisfying (2.26) are constant on the shifted balls containing the rigid body.

3 Uniform bounds

In this section, we derive suitable uniform bounds necessary for passing to the limit in the weak formulation of the momentum equation (2.13). All bounds used in the limit passage follow from the energy inequality (2.15).

Let \((\rho^\varepsilon, u^\varepsilon, h^\varepsilon, \Omega^\varepsilon)_{\varepsilon > 0}\) be the associated sequence of weak solutions to the fluid–structure interaction problem satisfying (2.5)–(2.15). As \(g\) satisfies the hypothesis (2.21), we have

\[
\int_{\mathbb{R}^d} \rho^\varepsilon g \cdot u^\varepsilon \, dx = \int_{S_\varepsilon(t)} \rho^\varepsilon g \cdot u^\varepsilon \, dx + \int_{\mathbb{R}^d \setminus S_\varepsilon(t)} g \cdot u^\varepsilon
\]
where, in accordance with the hypothesis (2.17), (2.21):

\[
\int_{S_\varepsilon(t)} \rho_\varepsilon \mathbf{g} \cdot \mathbf{u}^\varepsilon \, dx \lesssim \int_{S_\varepsilon(t)} \rho_\varepsilon \, dx + \int_{\mathbb{R}^d} \rho^\varepsilon |\mathbf{u}^\varepsilon|^2 \, dx \lesssim 1 + \int_{\mathbb{R}^d} \rho^\varepsilon |\mathbf{u}^\varepsilon|^2 \, dx.
\]

Here and hereafter, the symbol \(a \lesssim b\) means \(a \leq cb\), where \(c\) is a generic constant independent of the scaling parameter \(\varepsilon\). Similarly, by virtue of Cauchy–Schwartz inequality,

\[
\int_{\mathbb{R}^d \setminus S_\varepsilon(t)} \mathbf{g} \cdot \mathbf{u}^\varepsilon \leq c(\delta) + \delta \int_{\mathbb{R}^d} |\mathbf{u}^\varepsilon|^2 \, dx
\]

for any \(\delta > 0\). In addition, using Korn–Poincaré inequality, we get

\[
\int_{\mathbb{R}^d} |\mathbf{u}^\varepsilon|^2 \, dx \lesssim \int_{\mathbb{R}^d} \mathcal{S}(\mathbb{D}_x \mathbf{u}^\varepsilon) : \mathbb{D}_x \mathbf{u}^\varepsilon \, dx + \int_{\mathbb{R}^d} \rho^\varepsilon |\mathbf{u}^\varepsilon|^2 \, dx.
\] (3.1) \(u1a\)

Consequently, we apply Gronwall’s argument to the energy inequality (2.15) and deduce the following bounds

\[
\text{ess sup}_{t \in (0,T)} \int_{\mathbb{R}^d} \rho^\varepsilon |\mathbf{u}^\varepsilon|^2(t, \cdot) \, dx \lesssim 1, \quad \int_0^T \int_{\mathbb{R}^d} \mathcal{S}(\mathbb{D}_x \mathbf{u}^\varepsilon) : \mathbb{D}_x \mathbf{u}^\varepsilon \, dx \lesssim 1,
\] (3.2) \(u1\)

which, together with (3.1), yields

\[
\int_0^T \int_{\mathbb{R}^d} |\mathbf{u}^\varepsilon|^2 + |\nabla_x \mathbf{u}^\varepsilon|^2 \, dx \, dt \lesssim 1.
\] (3.3) \(u2\)

### 3.1 Bounds on the rigid velocity \(d = 2\)

In view of (2.10), \(\mathbf{u}(t, \cdot) = \mathbf{u}_S(t, \cdot)\) a.a. on \(S_\varepsilon(t)\) for a.a. \(t \in (0,T)\) in particular, as a consequence of (3.3). As the rigid body density is constant, the translational and rotational velocities are orthogonal on \(S_\varepsilon(t)\), specifically

\[
\int_{S_\varepsilon(t)} \mathbf{Y}^\varepsilon(t) \cdot \mathbf{Q}^\varepsilon(\cdot - \mathbf{h}^\varepsilon(t)) \, dx = 0.
\]

Consequently, we have

\[
\int_{S_\varepsilon(t)} |\mathbf{u}^\varepsilon|^2 \, dx = \int_{S_\varepsilon(t)} |\mathbf{u}_S^\varepsilon|^2 \, dx = \int_{S_\varepsilon(t)} |\mathbf{Y}^\varepsilon(t) + \mathbf{Q}^\varepsilon(\cdot - \mathbf{h}^\varepsilon(t))|^2 \, dx \geq \int_{S_\varepsilon(t)} |\mathbf{Y}^\varepsilon(t)|^2 \, dx
\] (3.4) \(u3\)

If \(d = 2\), the standard Sobolev embedding relation yields

\[
\|\mathbf{u}^\varepsilon\|_{L^q(S_\varepsilon(t); \mathbb{R}^2)} \leq c(q) \|\mathbf{u}^\varepsilon\|_{W^{1,2}(\mathbb{R}^2; \mathbb{R}^2)} \text{ for any finite } 1 \leq q < \infty,
\]
where the constant \( c(q) \) is independent of \( \varepsilon \).

Next, by (3.4) and interpolation,
\[
|S_\varepsilon|^{\frac{1}{2}}|Y^\varepsilon| \leq \|u^\varepsilon\|_{L^2(S_\varepsilon(t);\mathbb{R}^2)} \leq \|u^\varepsilon\|_{L^q(S_\varepsilon(t);\mathbb{R}^2)}|S_\varepsilon|^{\frac{1}{2}-\frac{1}{q}} \leq c(q)\|u^\varepsilon\|_{W^{1,2}(\mathbb{R}^2;\mathbb{R}^2)}|S_\varepsilon|^{\frac{1}{2}-\frac{1}{q}}.
\]

Consequently, in view of the bound (3.3), we may infer
\[
\|Y^\varepsilon\|_{L^2(0,T)} \lesssim |S_\varepsilon|^{-\frac{1}{q}} \text{ for any finite } 1 \leq q < \infty, \; d = 2.
\]

### 3.2 Bounds on the rigid velocity \( d = 3 \)

Now, we repeat the arguments of the previous section with \( q = 6 \) obtaining
\[
\|Y^\varepsilon\|_{L^2(0,T)} \lesssim |S_\varepsilon|^{-\frac{1}{6}}.
\]

### 4 Restriction operator

A suitable choice of the restriction operator is crucial in our analysis. In contrast to the overwhelming amount of the available literature, where the test functions are modified to vanish on the body, we take advantage of the freedom allowed by the weak formulation (2.25) and replace the function on a ball of radius \( \varepsilon \) by its integral average over that ball. The same idea has already been exploited in [8], where detailed proofs of the statements collected below are available.

Consider a function
\[
H \in C^\infty(\mathbb{R}), \quad 0 \leq H(Z) \leq 1, \quad H'(Z) = H'(1-Z) \text{ for all } Z \in \mathbb{R},
\]

\[
H(Z) = 0 \text{ for } -\infty < Z \leq \frac{1}{4}, \quad H(Z) = 1 \text{ for } \frac{3}{4} \leq Z < \infty.
\]

For \( \varphi \in L^1_{\text{loc}}(\mathbb{R}^d) \), we define \( E_\varepsilon \),
\[
E_\varepsilon[\varphi](x) = \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} \varphi \ dz \ H \left( 2 - \frac{|x|}{\varepsilon} \right) + \varphi(x)H \left( \frac{|x|}{\varepsilon} - 1 \right),
\]

where \( B_\varepsilon \) denotes the ball centred at zero with the radius \( \varepsilon > 0 \).

### 4.1 Restriction operator in the class of solenoidal functions

The operator \( E_\varepsilon(h) \) does not preserve solenoidality if applied to a solenoidal function. To fix this problem, we introduce the operator
\[
R_\varepsilon[\varphi] = E_\varepsilon[\varphi] + B_{2\varepsilon,\varepsilon} \left[ (\text{div}_x \varphi - \text{div}_x E_\varepsilon[\varphi]) \right]_{B_{2\varepsilon}\setminus B_\varepsilon},
\]

where \( B_{2\varepsilon,\varepsilon} \) denotes the annulus between the balls of radii \( 2\varepsilon \) and \( \varepsilon \).
where $B_{2\varepsilon,\varepsilon}$ is a suitable branch of the inverse of the divergence operator defined on the annulus $B_{2\varepsilon} \setminus B_{\varepsilon}$. A possible construction of $B$ was proposed by Bogovskii [11] and later elaborated by Galdi [14] followed by Diening at al. [4], Geißert [9] at al. among others. The basic properties of the operator $B$ needed for our construction can be summarized as follows:

Given a bounded Lipschitz domain $O \subset \mathbb{R}^d$, $d = 2, 3$, there is a linear operator $B_O$ defined a priori on smooth functions $g \in C^\infty_c(O)$, $\int_O g \ dx = 0$, enjoying the following properties:

- $B_O[g] \in C^\infty_c(O; \mathbb{R}^d)$, $\text{div}_x B_O[g] = g$;

- $\|\nabla_x B_O[g]\|_{L^p(O; \mathbb{R}^d)} \leq c(p, O) \|g\|_{L^p(O)}$ for any $1 < p < \infty$. \hfill (4.5) \tag{E6a}

Thanks to this property, $B_O[g]$ can be extended as a bounded linear operator on the space

$$L^p_0(O) = \left\{ g \in L^p(O) \mid \int_O g \ dx = 0 \right\}$$

ranging in $W^{1,p}_0(O; \mathbb{R}^d)$.

- Suppose that the domain $\tilde{O} = rO$ was obtained by the rescaling $x \mapsto rx$, $r > 0$. Then the operator $B_{\tilde{O}}$ can be constructed satisfying (4.5) with the same constant $c(p, O)$, see Galdi [14, Lemma III.3.3]. In particular, for our choice $B_{2\varepsilon,\varepsilon} \equiv B_{B_{2\varepsilon} \setminus B_\varepsilon}$, we obtain

$$\|B_{2\varepsilon,\varepsilon}[g]\|_{W^{1,p}_0(B_{B_{2\varepsilon} \setminus B_\varepsilon}; \mathbb{R}^d)} \leq c(p) \|g\|_{L^p(B_{B_{2\varepsilon} \setminus B_\varepsilon})}$$

for any $1 < p < \infty$ \hfill (4.6) \tag{E6b}

uniformly for $0 < \varepsilon \leq 1$.

- If, in addition, $g \in W^{l,p}_0$, then

$$\|B_O[g]\|_{W^{l+1,p}_0(O; \mathbb{R}^d)} \leq c(l, p, O) \|g\|_{W^{l,p}_0(O; \mathbb{R}^d)}$$

for $l = 0, 1, \ldots$, $1 < p < \infty$,

see Galdi [14] Theorem III.3.3.

Finally, we set

$$R_\varepsilon(h)[\varphi] = S_{-h}R_\varepsilon \left[ S_h[\varphi] \right], \quad (4.7) \tag{E10}$$

where the shift operator $S_h$ is given by

$$S_h[f](x) = f(h + x), \quad h \in \mathbb{R}^d$$

The basic properties of the operator $R_\varepsilon(h)$ are summarized below, see [8, Proposition 5.1].
Proposition 4.1. The operator \( R_\varepsilon(h) \) is well defined for any function \( \varphi \in W^{1,p}_{loc}(\mathbb{R}^d;\mathbb{R}^d), \) \( p > 1, \) satisfying

\[
\text{div}_x \varphi = 0 \text{ for a.a. } x, \ |x - h| < \varepsilon. \tag{4.8} \]

Moreover,

\[
\begin{align*}
R_\varepsilon(h)[\varphi] &= \begin{cases} 
\frac{1}{|B_\varepsilon(h)|} \int_{B_\varepsilon(h)} \varphi \, dx & \text{if } |x - h| < \varepsilon, \\
\varphi & \text{if } |x - h| > 2\varepsilon;
\end{cases} \tag{4.9} \quad \text{E11}
\end{align*}
\]

\[
\text{div}_x R_\varepsilon(h)[\varphi] = \text{div}_x \varphi; \tag{4.10} \quad \text{E12}
\]

\[
\|R_\varepsilon(h)[\varphi]\|_{W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)} \lesssim \|\varphi\|_{W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)} \tag{4.11} \quad \text{E13}
\]

for any \( 1 < p < \infty \) independently of \( \varepsilon > 0. \)

- If \( \varphi \) is compactly supported in an open set \( \mathcal{U} \subset \mathbb{R}^d, \) then so is \( R_\varepsilon(h)[\varphi] \) provided \( \varepsilon > 0 \) is small enough. Precisely,

\[
R_\varepsilon(h)[\varphi](x) = \varphi(x) = 0 \text{ for a.a. } x \in \mathbb{R}^d, \text{dist}[x, \text{supp}[\varphi]] > 3\varepsilon.
\]

Finally, we evaluate the differential of \( \nabla_h R_\varepsilon(h)[\varphi] \) for a given function \( \varphi. \) Note that if \( \varphi \) is smooth, then so is \( \nabla_h R_\varepsilon(h)[\varphi], \) as a consequence of the properties of the operators \( E_\varepsilon, B_{2\varepsilon,\varepsilon} \) stated above. Consequently, as both \( E_\varepsilon \) and \( B_{2\varepsilon,\varepsilon} \) are linear, we compute the desired derivative directly from formula (4.7):

\[
\nabla h R_\varepsilon(h)[\varphi] = \nabla_x (R_\varepsilon(h)[\varphi]) - R_\varepsilon(h)[\nabla_x \varphi]. \tag{4.12} \quad \text{formula}
\]

Note carefully that if \( \varphi \) is solenoidal, meaning \( \text{div}_x \varphi = 0, \) then so is \( \nabla_x \varphi \) (component–wise) so the right–hand side of (4.12) is well defined.

### 4.2 Approximating the test functions

For \( \varphi \in C_\infty([0,T) \times \Omega; \mathbb{R}^d), \) \( \text{div}_x \varphi = 0 \) consider its approximation

\[
\varphi \approx R_\varepsilon(h^\varepsilon)[\varphi].
\]

In accordance with (4.9), (4.10) and the regularity properties of the operators \( E_\varepsilon, B_{2\varepsilon,\varepsilon}, \) we have for any \( t \in [0,T): \)

\[
R_\varepsilon(h^\varepsilon)[\varphi](t, \cdot) \in C_\infty(\Omega; \mathbb{R}^d), \text{ div}_x R_\varepsilon(h^\varepsilon)[\varphi](t, \cdot) = 0, \nabla_x R_\varepsilon(h^\varepsilon)[\varphi](t, \cdot) = 0 \text{ on } B_\varepsilon[h^\varepsilon(t)].
\]
To compute the time derivative, we use formula (4.12) obtaining
\[ \partial_t R_\varepsilon (h^\varepsilon)[\varphi] = R_\varepsilon (h^\varepsilon) [\partial_t \varphi] + \left( \nabla_x (R_\varepsilon (h)[\varphi]) - R_\varepsilon (h)[\nabla_x \varphi] \right) \cdot Y^\varepsilon (t), \quad Y^\varepsilon (t) = \frac{d}{dt} h^\varepsilon (t), \tag{4.13} \]
for a.a. \( t \in (0, T) \). In accordance with (2.7), the function \( h^\varepsilon \) is merely Lipschitz and so is the time derivative of \( R_\varepsilon (h^\varepsilon)[\varphi] \). Given the integrability properties of \( \rho^\varepsilon, u^\varepsilon \), this is enough for \( R_\varepsilon (h^\varepsilon)[\varphi] \) to be a legitimate test function for the momentum balance (2.25).

5 Convergence: Proof of Theorem 2.1

Our ultimate goal is to plug \( R_\varepsilon (h^\varepsilon)[\varphi] \) as a test function in the momentum balance (2.25) and perform the limit \( \varepsilon \to 0 \).

5.1 Error estimates for the test functions

To derive the error estimates on the difference
\[ \varphi - R_\varepsilon (h^\varepsilon)[\varphi], \quad \varphi \in C^\infty_c ([0, T) \times \Omega; \mathbb{R}^d), \quad \text{div}_x \varphi = 0 \]
we use essentially two facts:

- For any fixed \( t \in [0, T) \):
  \[ \varphi - R_\varepsilon (h^\varepsilon)[\varphi] = 0 \text{ outside the ball } B_{2\varepsilon}[h^\varepsilon (t)]; \tag{5.1} \]

- \( \varphi \) is smooth, in particular Lipschitz in \([0, T) \times \Omega\).

In view of the bound (4.11), we get
\[ \| R_\varepsilon (h^\varepsilon)[\varphi] \|_{W^{1,p} (\mathbb{R}^d; \mathbb{R}^d)} \lesssim \| \varphi \|_{C^1_c (\Omega; \mathbb{R}^d)}; \]
whence, by virtue of (5.1)
\[ \| (\varphi - R_\varepsilon (h^\varepsilon)[\varphi])(t, \cdot) \|_{W^{1,p} (\mathbb{R}^d; \mathbb{R}^d)} \to 0 \text{ uniformly for } t \in [0, T] \text{ for any } 1 \leq p < \infty \tag{5.2} \]
for any \( \varphi \in C^1_c ([0, T) \times \Omega; \mathbb{R}^d), \quad \text{div}_x \varphi = 0 \).

As for the time derivative, we use formula (4.13) obtaining
\[ \partial_t \varphi - \partial_t R_\varepsilon (h^\varepsilon)[\varphi] = \partial_t \varphi - R_\varepsilon (h^\varepsilon)[\partial_t \varphi] + \left( \nabla_x (R_\varepsilon (h^\varepsilon)[\varphi]) - R_\varepsilon (h^\varepsilon)[\nabla_x \varphi] \right) \cdot Y^\varepsilon (t), \]
where, similarly to the above,
\[\|(\partial_t \varphi - R_\varepsilon (h^\varepsilon) [\partial_t \varphi]) (t, \cdot)\|_{W^{1,p}(\mathbb{R}^d; \mathbb{R}^d)} \rightarrow 0 \text{ uniformly for } t \in [0, T] \text{ for any } 1 \leq p < \infty \tag{5.3}\]

Finally, the second error can be written as
\[\left| \left( \nabla_x (R_\varepsilon (h^\varepsilon) [\varphi]) - R_\varepsilon (h^\varepsilon) [\nabla_x \varphi] \right) \cdot Y^{\varepsilon} (t) \right| \leq 1_{B_{2\varepsilon} (h^\varepsilon)} | Y^{\varepsilon} (t) | \left| \nabla_x (R_\varepsilon (h^\varepsilon) [\varphi]) - R_\varepsilon (h^\varepsilon) [\nabla_x \varphi] \right|, \tag{5.4}\]

where
\[\left\| \nabla_x (R_\varepsilon (h^\varepsilon) [\varphi]) - R_\varepsilon (h^\varepsilon) [\nabla_x \varphi] \right\|_{L^p(\mathbb{R}^d; \mathbb{R}^d)} \lesssim \| \varphi \|_{W^{2,p}(\mathbb{R}^d; \mathbb{R}^d)} \text{ uniformly for } t \in (0, T) \tag{5.5}\]

and any \(1 \leq p < \infty\).

### 5.2 Convergence

We know from estimates (3.2)–(3.3) that
\[\sqrt{\rho_\varepsilon} u^\varepsilon \text{ is bounded in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^d)), \tag{5.6}\]
\[u^\varepsilon \text{ is bounded in } L^2(0, T; W^{1,2}_0(\Omega; \mathbb{R}^d)).\]

Thus there exists \(u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}_0(\Omega))\) such that, up to a subsequence,
\[\sqrt{\rho_\varepsilon} u^\varepsilon \rightharpoonup u \text{ weak-* in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^d)), \tag{5.7}\]
\[u^\varepsilon \rightharpoonup u \text{ weakly in } L^2(0, T; W^{1,2}_0(\Omega; \mathbb{R}^d)). \tag{5.8}\]

#### 5.2.1 Limit in the momentum equation

Our ultimate goal is to perform the limit in the momentum equation (2.25), with the test function \(R_\varepsilon (h^\varepsilon) [\varphi]\).

**Viscous term.**

In view of (5.8) and the error estimate (5.2), it is easy to see
\[\int_0^T \int_\Omega S(D_x u^\varepsilon) : D_x (R_\varepsilon (h^\varepsilon) [\varphi]) \, dx \, dt \rightarrow \int_0^T \int_\Omega S(D_x u) : D_x \varphi \, dx \, dt \tag{5.9}\]
for any \(\varphi \in C^1_c ([0, T] \times \Omega; \mathbb{R}^d), \text{ div}_x \varphi = 0\).

**Convective term.**
As \( \rho^\varepsilon \) is bounded and the uniform bounds (3.2), (3.3) hold, it is easy to check that
\[
\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon \to \overline{u} \otimes \overline{u} \text{ weakly in } L^p((0, T) \times \Omega; \mathbb{R}^{d \times d})
\]
for some \( p > 1 \). Consequently, in view of (5.2),
\[
\int_0^T \int_\Omega (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) : \nabla_x (R_{\varepsilon}(h^\varepsilon)[\varphi]) \, dx \, dt \to \int_0^T \int_\Omega (u \otimes u) : \nabla_x \varphi \, dx \, dt \text{ as } \varepsilon \to 0 \quad (5.10)
\]
for any \( \varphi \in C^1_c([0, T) \times \Omega; \mathbb{R}^d) \), \( \text{div}_x \varphi = 0 \).

**Time derivative.**
Our next goal is to establish the limit
\[
\int_0^T \int_\Omega \rho^\varepsilon u^\varepsilon \cdot \partial_t R_{\varepsilon}(h^\varepsilon)[\varphi] \, dx \, dt \to \int_0^T \int_\Omega u \cdot \partial_t \varphi \, dx \, dt. \quad (5.11)
\]
In view of the estimates (5.3), (5.4), this amounts to showing
\[
\int_0^T \int_\Omega \rho^\varepsilon u^\varepsilon \cdot \left( \nabla_x (R_{\varepsilon}(h^\varepsilon)[\varphi]) - R_{\varepsilon}(h^\varepsilon)[\nabla_x \varphi] \right) \cdot Y^\varepsilon(t) \, dx \, dt \to 0 \quad (5.12)
\]
\[
\text{Let us start with the case } d = 3. \text{ In view of (5.4) and uniform boundedness of the density, we have}
\]
\[
\left| \int_\Omega \rho^\varepsilon u^\varepsilon \cdot \left( \nabla_x (R_{\varepsilon}(h^\varepsilon)[\varphi]) - R_{\varepsilon}(h^\varepsilon)[\nabla_x \varphi] \right) \cdot Y^\varepsilon(t) \, dx \right| \lesssim c(\delta) |Y^\varepsilon(t)||u^\varepsilon||_{L^6(\Omega; \mathbb{R}^d)}|B_2[|h^\varepsilon|]^{\frac{5}{2}}| \text{ for any } \delta > 0. \quad (5.13)
\]
Consequently, (5.12) follows from (3.6) and the hypothesis (2.16). The same result can be obtained in the case \( d = 2 \).

### 5.2.2 Limit in the convective term
The only thing remaining is to establish the identity:
\[
\overline{u} \otimes \overline{u} = u \otimes u. \quad (5.14)
\]
We consider the quantity
\[
\varphi = \psi(t)R_{\varepsilon}[h^\varepsilon(t)](\phi), \ \psi(t) \in C^1_c(0, T)
\]
as a test function in the momentum equation (2.25). It follows that the time distributional derivative of
\[ t \in [0, T] \mapsto \int_{\mathbb{R}^d} \rho^\varepsilon u^\varepsilon R_x[h^\varepsilon](\phi) \, dx \]
belongs to \( L^q(0, T) \) for any \( 1 \leq q < 2 \) and Arzelà–Ascoli theorem yields
\[ t \in [0, T] \mapsto \int_{\mathbb{R}^d} \rho^\varepsilon u^\varepsilon R_x[h^\varepsilon](\phi) \, dx \]
is precompact in \( C[0, T] \) for any \( \phi \in C^1_c(\Omega; \mathbb{R}^d), \) \( \text{div}_x \phi = 0. \) \( (5.15) \)

Next, as \( \rho^\varepsilon u^\varepsilon \) is bounded in \( L^\infty(0, T; L^2(\mathbb{R}^d; \mathbb{R}^d)) \), we use the error estimates for \( R_x \) to show
\[ \int_{\mathbb{R}^d} (\rho^\varepsilon u^\varepsilon)(t, \cdot) \left( R_x[h^\varepsilon(t)](\phi) - \phi \right) \, dx \to 0 \quad \text{in} \quad L^\infty(0, T), \]
which, combined with (5.15), implies
\[ \int_{\mathbb{R}^d} (\rho^\varepsilon u^\varepsilon)(t, \cdot) \phi \, dx \to \int_{\mathbb{R}^d} u(t, \cdot) \phi \, dx \quad \text{in} \quad L^\infty(0, T), \]
for any \( \phi \in C^1_c(\Omega; \mathbb{R}^d), \) \( \text{div}_x \phi = 0. \) \( (5.16) \)

Using a density argument, we deduce from (5.16) that
\[ \int_\Omega (\rho^\varepsilon u^\varepsilon)(t, \cdot) \phi \, dx \to \int_\Omega u(t, \cdot) \phi \, dx \quad \text{in} \quad L^\infty(0, T), \]
for any \( \phi \in L^2(\Omega; \mathbb{R}^d), \) \( \text{div}_x \phi = 0 \) in \( \Omega, \) \( \phi \cdot n = 0 \) on \( \partial \Omega. \)

For \( \phi \in C^1_c(\Omega; \mathbb{R}^d), \) consider its Helmholtz projection in \( \Omega, \)
\[ \phi = H[\phi] + \nabla_x \Psi \quad \text{in} \quad \Omega \quad \text{with} \quad \nabla_x \Psi \cdot n = 0 \quad \text{on} \quad \partial \Omega. \]

Accordingly, we get
\[ \int_\Omega (\rho^\varepsilon u^\varepsilon)(t, \cdot) \phi \, dx = \int_\Omega (\rho^\varepsilon u^\varepsilon)(t, \cdot)H[\phi] \, dx + \int_\Omega (\rho^\varepsilon u^\varepsilon)(t, \cdot) \nabla_x \Psi \, dx, \]
where, in accordance with (5.16),
\[ \int_\Omega (\rho^\varepsilon u^\varepsilon)(t, \cdot)H[\phi] \, dx \to \int_\Omega u(t, \cdot)H[\phi] \, dx = \int_\Omega u(t, \cdot) \phi \, dx \quad \text{in} \quad L^\infty(0, T). \]

Moreover, as \( u^\varepsilon \) is solenoidal
\[ \int_\Omega (\rho^\varepsilon u^\varepsilon)(t, \cdot) \nabla_x \Psi \, dx = \int_\Omega (\rho^\varepsilon u^\varepsilon - u^\varepsilon)(t, \cdot) \cdot \nabla_x \Psi \, dx = (\rho^\varepsilon - 1) \int_{B_x[h^\varepsilon(t)]} u^\varepsilon \cdot \nabla_x \Psi \, dx. \]
for a.a. $t \in (0,T)$. Thus it follows from the uniform bounds established in (3.3) that

$$
\int_\Omega (\rho^\varepsilon \mathbf{u}^\varepsilon)(t,\cdot) \nabla_x \Psi \ d\mathbf{x} \to 0 \text{ in } L^2(0,T).
$$

and we may infer that

$$
\int_\Omega (\rho^\varepsilon \mathbf{u}^\varepsilon)(t,\cdot) \phi \ d\mathbf{x} \to \int_\Omega \mathbf{u}(t,\cdot) \phi \ d\mathbf{x} \text{ in } L^2(0,T) \text{ for any } \phi \in C^1_c(\Omega;\mathbb{R}^d).
$$

By density, we extend the conclusion to square integrable function,

$$
\int_\Omega (\rho^\varepsilon \mathbf{u}^\varepsilon)(t,\cdot) \phi \ d\mathbf{x} \to \int_\Omega \mathbf{u}(t,\cdot) \phi \ d\mathbf{x} \text{ in } L^2(0,T) \text{ for any } \phi \in L^2(\Omega;\mathbb{R}^d). 
$$

(5.18) c10

Equivalently, we can extend the function $\mathbf{u}^\varepsilon$ by zero in $\mathbb{R}^d \setminus \Omega$ and obtain:

$$
\rho^\varepsilon \mathbf{u}^\varepsilon \to \mathbf{u} \text{ in } L^2(0,T;L^2_{\text{weak}}(\mathbb{R}^d;\mathbb{R}^d)).
$$

(5.19) c11

Since $L^2_{\text{weak}}(K;\mathbb{R}^d)$ is compactly embedded in the dual $W^{-1,2}(K;\mathbb{R}^d)$ for any compact $K \subset \mathbb{R}^d$, the desired conclusion

$$
\int_0^T \int_{\mathbb{R}^d} (\rho^\varepsilon \mathbf{u}^\varepsilon) \otimes \mathbf{u}^\varepsilon : \nabla_x \varphi \ d\mathbf{x} \ dt \to \int_0^T \int_{\mathbb{R}^d} \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi \ d\mathbf{x} \ dt \text{ for any } \varphi \in C^1_c([0,T) \times \Omega;\mathbb{R}^d)
$$

(5.20) c12

follows.

We infer from the above discussion on the passing to the limit as $\varepsilon \to 0$ that the limit velocity of $\mathbf{u}^\varepsilon$ is given by $\mathbf{u}$ where $\mathbf{u}$ is a weak solution of the Navier–Stokes system (2.1)–(2.3) satisfying the energy inequality (2.23).

6 Concluding Remarks

The idea of using the dissipation rather than energy bounds to obtain estimates on the rigid velocity can be exploited also in the context of compressible viscous fluids. In particular, the improved estimates on the rigid velocity could be used to extend our result [8, Theorem 2.3] to the case of bounded body density $\rho^\varepsilon_S$ and the “full” range of the adiabatic exponent $\gamma > \frac{3}{2}$.

References

[1] M. E. Bogovskii. Solutions of some problems of vector analysis, associated with the operators div and grad. Theory of cubature formulas and the application of functional analysis to problems of mathematical physics, Trudy Sem. S. L. Soboleva, No. 1, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 5–40, 149, 1980.
[2] M. Bravin and Š. Nečasová. On the velocity of a small rigid body in a viscous incompressible fluid in dimension two and three, 2022. arXiv preprint arXiv:2208.12351.

[3] M. Dashti and J. C. Robinson. The motion of a fluid-rigid disc system at the zero limit of the rigid disc radius. Arch. Ration. Mech. Anal., 200(1):285–312, 2011.

[4] L. Diening, M. Růžička, and K. Schumacher. A decomposition technique for John domains. Ann. Acad. Sci. Fenn. Math., 35(1):87–114, 2010.

[5] S. Ervedoza, D. Maity, and M. Tucsnak. Large time behaviour for the motion of a solid in a viscous incompressible fluid, Math. Ann., 1–61, 2022.

[6] E. Feireisl. On the motion of rigid bodies in a viscous incompressible fluid, Journal of Evolution Equations, 3:419–441, 2003.

[7] E. Feireisl, A. Roy and A. Zarnescu. On the motion of a nearly incompressible viscous fluid containing a small rigid body. 2022. Arxiv preprint No. 2206.02931.

[8] E. Feireisl, A. Roy and A. Zarnescu. On the motion of a small rigid body in a viscous compressible fluid. 2022. ArXiv preprint No. 2208.07933.

[9] M. Geißert, H. Heck and M. Hieber. On the equation \( \text{div} u = g \) and Bogovskiĭ’s operator in Sobolev spaces of negative order, Partial differential equations and functional analysis, Oper. Theory Adv. Appl., Birkhäuser, Basel, 168:113–121, 2006.

[10] J. He and D. Iftimie. A small solid body with large density in a planar fluid is negligible. J. Dynam. Differential Equations, 31(3):1671–1688, 2019.

[11] J. He and D. Iftimie. On the small rigid body limit in 3D incompressible flows. J. Lond. Math. Soc. (2), 104(2):668–687, 2021.

[12] D. Iftimie, M. C. Lopes Filho, and H. J. Nussenzveig Lopes. Two-dimensional incompressible viscous flow around a small obstacle. Math. Ann., 336(2):449–489, 2006.

[13] G. P. Galdi. On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications. Handbook of mathematical fluid dynamics, Vol. I, North-Holland, Amsterdam, 653–791, 2002.

[14] G. P. Galdi. An introduction to the mathematical theory of the Navier-Stokes equations, Springer Monographs in Mathematics, Second Edition, 2011.

[15] M. D. Gunzburger, H. C. Lee and A. Seregin. Global existence of weak solutions for viscous incompressible flow around a moving rigid body in three dimensions. J. Math. Fluid Mech., 2, 219–266, 2000.
[16] C. Lacave. Two-dimensional incompressible viscous flow around a thin obstacle tending to a curve. Proc. Roy. Soc. Edinburgh Sect. A, 139(6):1237–1254, 2009.

[17] C. Lacave and T. Takahashi. Small moving rigid body into a viscous incompressible fluid. Arch. Ration. Mech. Anal., 223(3):1307–1335, 2017.

[18] J. A. San Martín, V. Starovoitov and M. Tucsnak, Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid, Arch. Rational Mech. Anal., 161(2): 113–147, 2002.