Higher Heegner points on elliptic curves over function fields

Florian Breuer

Mathematics Division, National Center for Theoretical Sciences
National Tsing-Hua University, Hsinchu, Taiwan

Abstract

Let $E$ be a modular elliptic curve defined over a rational function field $k$ of odd characteristic. We construct a sequence of Heegner points on $E$, defined over a $\mathbb{Z}^\infty_p$-tower of finite extensions of $k$, and show that these Heegner points generate a group of infinite rank. This is a function field analogue of a result of C. Cornut and V. Vatsal.

Key words: elliptic curves, Heegner points, Drinfeld modular curves

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1 Introduction

Heegner points are a way of constructing explicit points of infinite order on modular elliptic curves, and have been used to great effect over number fields. More recently, people have produced analogous constructions on elliptic curves over global function fields. The first such construction is due to M.L. Brown [2], but there seem to be a number of errors in his paper. H.-G. Rück and U. Tipp [15] have proved an analogue of the celebrated Gross-Zagier formula in a special case, and I. Longhi [11] and A. Pál [14] have (independently) constructed Heegner points of infinite order on elliptic curves, using the $p$-adic approach due to M. Bertolini and H. Darmon. D. Ulmer has announced a more general version of the Gross-Zagier formula, which, combined with his non-vanishing result for $L$-functions, yields the full Birch and Swinnerton-Dyer conjecture for elliptic curves over function fields when their analytic rank is $\leq 1$ and the characteristic is $p > 3$. For a nice survey of all this, see [16].

Email address: breuer@math.cts.nthu.edu.tw (Florian Breuer).
In this paper we prove a function field version of a result of C. Cornut [3] and V. Vatsal [17] on higher Heegner points on modular elliptic curves, which had been conjectured by B. Mazur [12]. Their result has some important consequences, including J. Nekovář’s celebrated results concerning the parity of ranks of Selmer groups. See [3] for a discussion.

Cornut actually found two proofs, the second of which [4] uses a known case of the André-Oort conjecture due to B. Moonen, and is much simpler. We use the function field analogue of a special case of the André-Oort Conjecture, proved in [1], and then follow closely Cornut’s second proof.

Let \( k \) be a global function field with field of constants \( \mathbb{F}_q \), where \( q \) is a power of the odd prime \( p \). Let \( E \) be an elliptic curve defined over \( k \) with non-constant \( j \)-invariant (we say \( E \) is non-isotrivial). Then, replacing \( k \) by a finite extension if necessary, we can choose a place \( \infty \) of \( k \) such that \( E \) has multiplicative reduction at \( \infty \). Let \( k_\infty \) denote the completion of \( k \) at \( \infty \), and set \( \mathbb{C}_\infty = \hat{k}_\infty \) the completion of an algebraic closure of \( k_\infty \). Furthermore, we let \( A \) be the ring of functions in \( k \) regular outside \( \infty \). It is a Dedekind domain with finite class number \( h = |\text{Pic}(A)| = \deg(\infty)h_k \), where \( h_k = |\text{Pic}^0(k)| \) denotes the class number of \( k \). By a Drinfeld module we will always mean a Drinfeld \( A \)-module of rank 2, defined over a subfield of \( \mathbb{C}_\infty \) (in particular, we deal only with the case of “generic” characteristic).

The conductor of \( E \) may be written as \( n \cdot \infty \), where \( n \) is an ideal in \( A \). Then, by the work of V.G. Drinfeld (and A. Weil, A. Grothendieck, H. Jacquet-R.P. Langlands, P. Deligne and Y.G. Zarhin) we have a modular parametrization

\[
\pi : X_0(n) \longrightarrow E,
\]

defined over \( k \), where \( X_0(n) \) is the Drinfeld modular curve parametrizing isomorphism classes of pairs \((\Phi, \Phi')\) of Drinfeld modules linked by a cyclic isogeny of degree \( n \).

Now let \( K \) be an imaginary quadratic extension of \( k \) (i.e. such that \( \infty \) does not split in \( K/k \)) with the property that all primes dividing \( n \) split in \( K \) (this is known as the Heegner hypothesis). There exist infinitely many such fields. Denote by \( \mathcal{O}_K \) the integral closure of \( A \) in \( K \), it contains an ideal \( \mathcal{N} \) such that \( \mathcal{O}_K/\mathcal{N} \cong A/n \). Let \( p \subset A \) be a prime not dividing \( n \), and let \( \mathcal{O}_n = A + p^n \mathcal{O}_K \) be the order of conductor \( p^n \) in \( \mathcal{O}_K \). We set \( \mathcal{N}_n = \mathcal{N} \cap \mathcal{O}_n \), so \( \mathcal{O}_n/\mathcal{N}_n \cong A/n \). Then \( \mathcal{O}_n \) and \( \mathcal{N}^{-1}_n \), viewed as rank 2 lattices in \( \mathbb{C}_\infty \), correspond to a pair of Drinfeld modules \((\Phi^{\mathcal{O}_n}, \Phi^{\mathcal{N}^{-1}_n})\), linked by a cyclic isogeny of degree \( n \). Hence they define a Heegner point \( x_n \in X_0(n) \), which is in fact defined over the ring class field \( K[p^n] \) of \( \mathcal{O}_n \), as \( \text{End}(\Phi^{\mathcal{O}_n}) \cong \text{End}(\Phi^{\mathcal{N}^{-1}_n}) \cong \mathcal{O}_n \).
We now set $K^\infty = \cup_{n \geq 1} K[p^n]$. Then $G = \text{Gal}(K^\infty/K) \cong \mathbb{Z}_p^\infty \times G_0$, where $\mathbb{Z}_p^\infty$ denotes the product of countably many copies of $\mathbb{Z}_p$ ($p$ being the characteristic of $k$), and $G_0 = G_{\text{tors}}$ is a finite group (Proposition 2.1). This is in marked contrast to the number field case, where the analogue of $G$ contains only one copy of $\mathbb{Z}_p$, and where $p$ plays the role of $p$, and may be chosen. $G_0$ corresponds to a subfield $H[p^\infty]$ satisfying $\text{Gal}(H[p^\infty]/K) \cong \mathbb{Z}_p^\infty$.

Denote by

$$\text{Tr}_{G_0} : E(K[p^\infty]) \longrightarrow E(H[p^\infty]); \quad x \mapsto \sum_{\sigma \in G_0} x^\sigma$$

the $G_0$-trace on $E$. We define the Heegner point $y_n = \text{Tr}_{G_0}(\pi(x_n)) \in E(H[p^\infty])$.

Our aim is to prove

**Theorem 1** Suppose $k = \mathbb{F}_q(T)$ and $\deg(\infty) = 1$. Let $I \subset \mathbb{N}$ be an infinite subset. Then the group generated by $\{y_n \mid n \in I\}$ in $E(H[p^\infty])$ has infinite rank.

**Remark.** We have tried to avoid the hypothesis on $k$ and $\infty$ as far as possible in this paper. It is used twice, firstly in the proof of Proposition 4.2 (but which should still hold for general $k$), and at the very end of the proof of Theorem 1, where we invoke an analogue of the André-Oort conjecture which is currently only known in this case. Once a more general case of this conjecture has been proved - which is the object of current efforts - Theorem 1 should become true for general $k$ and $\infty$.

The layout of this paper is as follows. In §2 we describe the group $\text{Gal}(K[p^\infty]/K)$, and show that $E(K[p^\infty])$ has finite torsion. In §3 we describe the map (1.1) in more detail, and construct a family of new modular parametrizations in §4 by means of degeneracy maps between Drinfeld modular curves. Then in §5 we describe a canonical factorization of cyclic isogenies between CM Drinfeld modules, which we will use in §6 to characterize the geometric action of Galois on Heegner points. Finally, we deduce Theorem 1 from the André-Oort conjecture in §7.

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## 2 The class field tower

**Proposition 2.1** $G = \text{Gal}(K[p^\infty]/K) \cong \mathbb{Z}_p^\infty \times G_0$, where $G_0 = G_{\text{tors}}$ is finite.
Proof. First, notice that
\[
G = \text{Gal}(K[p^\infty]/K) = \lim_{n \to \infty} \text{Gal}(K[p^n]/K) \cong \lim_{n \to \infty} \text{Pic}(\mathcal{O}_n). \tag{2.2}
\]

Secondly, we have an exact sequence (see e.g. [13], §I.12)
\[
1 \to \mathcal{O}_K^*/\mathcal{O}_n^* \to (\mathcal{O}_K/p^n\mathcal{O}_K)^*/(\mathcal{O}_n/p^n\mathcal{O}_K)^* \to \text{Pic}(\mathcal{O}_n) \to \text{Pic}(\mathcal{O}_K) \to 1.
\]

As \(\mathcal{O}_K^*/\mathcal{O}_n^*\) and \(\text{Pic}(\mathcal{O}_K)\) are bounded, it remains to examine the behavior of \((\mathcal{O}_K/p^n\mathcal{O}_K)^*/(\mathcal{O}_n/p^n\mathcal{O}_K)^*\) as \(n \to \infty\). We insert it into the following diagram, with exact rows and columns:
\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 + p\mathcal{O}_n & (\mathcal{O}_n/p^n\mathcal{O}_n)^* & (\mathcal{O}_n/p\mathcal{O}_n)^* & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 + p\mathcal{O}_K & (\mathcal{O}_K/p^n\mathcal{O}_K)^* & (\mathcal{O}_K/p\mathcal{O}_K)^* & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 + p\mathcal{O}_K & (\mathcal{O}_K/p^n\mathcal{O}_K)^* & (\mathcal{O}_K/p\mathcal{O}_K)^* & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 + p\mathcal{O}_n & (\mathcal{O}_n/p^n\mathcal{O}_n)^* & (\mathcal{O}_n/p\mathcal{O}_n)^* & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 + p\mathcal{O}_K & (\mathcal{O}_K/p^n\mathcal{O}_K)^* & (\mathcal{O}_K/p\mathcal{O}_K)^* & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 + p\mathcal{O}_n & (\mathcal{O}_n/p^n\mathcal{O}_n)^* & (\mathcal{O}_n/p\mathcal{O}_n)^* & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1
\end{array}
\]

Notice that \(p^n\mathcal{O}_K = p^n\mathcal{O}_n\). As \((\mathcal{O}_K/p\mathcal{O}_K)^*/(\mathcal{O}_n/p\mathcal{O}_n)^*\) is bounded, we are lead to studying the (multiplicative) group \(H_n := (1 + p\mathcal{O}_K)/(1 + p\mathcal{O}_n)\).

By computing the cardinality of various groups in the diagram, one finds that \(H_n\) is a \(p\)-group, of order \(|p|^{n-1}\). Let \(x \in 1 + p\mathcal{O}_K\), we will examine its order in \(H_n\). Let \(s = \lceil \log_p(n + 1) \rceil\), then we find that \(x^{p^s} \in 1 + p^{n+1}\mathcal{O}_K \subset 1 + p\mathcal{O}_n\). It follows that \(H_n\) is annihilated by \(p^s\), hence the number of generators of \(H_n\) is at least \(\log_p(|p|)(n - 1)/\lceil \log_p(n + 1) \rceil \to \infty\) as \(n \to \infty\). On the other hand, suppose the order of \(x\) in \(H_n\) is bounded independently of \(n\), say \(x^{p^r} \in 1 + p\mathcal{O}_n\) for all \(n\). But then \(x^{p^r} \in \cap_{n=1}^{\infty} (1 + p\mathcal{O}_n) \subset A\), and so \(x \in A\) to begin with (recall that \(K = k(\sqrt{D})\) for some square-free \(D \in A\), and that \(p\) is odd), so \(x \in A \cap (1 + p\mathcal{O}_K) \subset 1 + p\mathcal{O}_n\). We have shown that \(\lim_{n \to \infty} H_n = \mathbb{Z}_p^\infty\) and the proposition now follows.

\[\square\]

The following result will be crucial.
Lemma 2.3 \( E_{\text{tors}}(K[p^\infty]) \) is finite.

Proof. Let \( \mathfrak{l} \mid \mathfrak{p} \mathfrak{n} \) be a prime of \( k \) which is inert and principal in \( K \). Then \( E \) has good reduction at \( \mathfrak{l} \), and \( \mathfrak{l} \) splits completely in \( K[p^n] \), hence the residue field of \( K[p^n] \) at \( \mathfrak{l} \) is just \( \mathbb{F}_\mathfrak{l} = \mathcal{O}_K/\mathfrak{l} \) for every \( n \geq 0 \). It follows that reduction mod \( \mathfrak{l} \) induces an injection of the prime-to-\( p \) part of \( E_{\text{tors}}(K[p^\infty]) \) into \( \tilde{E}(\mathbb{F}_\mathfrak{l}) \), which is finite.

Let \( K^{\text{sep}} \) denote the separable closure of \( K \). We complete the proof by showing that \( E(K^{\text{sep}}) \cap E[p^\infty] \) is finite, for any non-isotrivial elliptic curve \( E/K \). Indeed, if \( E[p^n] \subset E(K) \), then \( j = j(E) \) is a \( p^n \)-th power in \( K \), as can be seen by factoring the multiplication by \( p^n \)-map into \( [p^n] = f \circ g \), with \( \ker(g) = E[p^n] \) and \( f \) the \( p^n \)-th power Frobenius. Now, let \( n \) be such that \( j \) is not a \( p^n \)-th power in \( K \). Then \( j \) is a \( p^n \)-th power in \( K' = K[E[p^n]] \), and it follows that \( K'/K \) is not separable. Alternatively, one may apply a general criterion of J.F. Voloch for abelian varieties ([18], §4).

\[ \square \]

3 Modular parametrizations

As the literature already contains excellent expositions of the theory of Drinfeld modular curves and the parametrization (1.1), such as [7], [9], [10], and [11] (§1.4 and §1.5), we will not attempt a detailed account. Instead, we only recall here some of the results and notations that we will need, and refer the reader to [10] for the details.

Every projective \( A \)-module of rank 2 is isomorphic to \( Y_a = A \times a \subset k^2 \) for an ideal \( a \subset A \), so in particular their isomorphism classes correspond to \( \text{Pic}(A) \). The group \( \text{GL}_2(k) \) acts on \( k^2 \) from the right. Let \( x = [a] \in \text{Pic}(A) \) and let \( \Gamma_x = \text{Stab}_{\text{GL}_2(k)}(Y_a) \) be the stabilizer of \( Y_a \). Denote by \( \Omega = \mathbb{C}_\infty \setminus k_\infty \) the Drinfeld upper half-plane, and choose \( z \in \Omega \). We map \( k^2 \) into \( \Omega \) by sending \((a, b)\) to \( az + b \). Then the image of any projective rank 2 \( A \)-module under this map is a lattice in \( \mathbb{C}_\infty \), i.e. a discrete projective \( A \)-submodule of \( \mathbb{C}_\infty \) of rank 2. \( \text{GL}_2(k) \) acts on \( \Omega \) from the left by fractional linear transformations, and on the lattices this action corresponds to the right action on \( k^2 \).

Let \( M_0(n) \) be the coarse moduli scheme for the moduli problem “pairs of Drinfeld modules linked by a cyclic isogeny of degree \( n \)”. This is equivalent to the problem “pairs of lattices \( \Lambda_1 \subset_n \Lambda_2 \)”, where the \( \subset_n \) notation means that \( \Lambda_2/\Lambda_1 \cong A/n \) as \( A \)-modules. We denote by \( Y_0(n) = M_0(n) \times k \) the base extension to \( k \), and by \( X_0(n) \) the smooth projective model of \( Y_0(n) \), which may be obtained from \( Y_0(n) \) by including finitely many cusps. The curve \( X_0(n) \) has \( h = |\text{Pic}(A)| \) irreducible components, each defined over the Hilbert class field.
$H$ of $(k, A)$, and which we denote by $X_x$ for $x = [a] \in \text{Pic}(A)$. The group $\text{Gal}(H/k)$ permutes the components by $(a, H/k)X_{[b]} = X_{[a^{-1}b]}$.

Each component $X_x$ has an analytic parametrization

$$X_x(C_\infty) \cong \Gamma_x(n) \backslash \Omega^*, \quad \text{where } \Omega^* = \Omega \cup \mathbb{P}^1(k), \text{ and}$$

$$\Gamma_x(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_x \mid c \equiv 0 \mod n \right\}. $$

Let $E/k$ be a non-isotrivial elliptic curve with conductor $n \cdot \infty$, where $n \subset A$. Then there is a surjective morphism defined over $k$

$$\pi : X_0(n) \longrightarrow E. \quad (3.1)$$

We may suppose that $E$ is maximal in its $k$-isogeny class with respect to the map (3.1), we call it the strong Weil curve. Then this map can be given explicitly on the $C_\infty$-valued points, which we will describe next.

Let $T$ be the Bruhat-Tits tree for $\text{PGL}_2(k_\infty)$, and denote by $H(T, \mathbb{Z})$ the group of harmonic cochains on $T$ with values in $\mathbb{Z}$, and define the subgroup $H_1(T, \mathbb{Z})^{\Gamma_x(n)}$ of cochains invariant under $\Gamma_x(n)$ and with compact support on $\Gamma_x(n) \backslash T$. Then these cochains correspond to certain automorphic forms on $\text{GL}_2$. Moreover, to $E$ one associates, for each $x = [a] \in \text{Pic}(A)$, a primitive Hecke newform $\varphi_x \in H_1(T, \mathbb{Z})^{\Gamma_x(n)}$. To $\varphi_x$ we associate furthermore a holomorphic theta function $u_x : \Omega \to \mathbb{C}_{\infty}^*$ with multiplier $c_x$, (i.e. $u_x(\alpha z) = c_x(\alpha)u_x(z)$ for all $\alpha \in \Gamma_x(n)$). Let $\Delta_x = \{ c_x(\alpha) \mid \alpha \in \Gamma_x(n) \}$, which is a multiplicative lattice in $\mathbb{C}_{\infty}^*$. Then $E$, which has multiplicative reduction at $\infty$, is isomorphic over $k_\infty$ to the Tate curve $\mathbb{C}_{\infty}^*/\Delta_x$. We have the explicit parametrization

$$\pi_x : X_x(C_\infty) \longrightarrow E(C_\infty) \cong \Gamma_x(n) \backslash \Omega \longrightarrow \mathbb{C}_{\infty}^*/\Delta_x$$

$$u_x : \Gamma_x(n) \backslash \Omega \longrightarrow \mathbb{C}_{\infty}^*/\Delta_x$$

$$[z] \longrightarrow u_x(z) \mod \Delta_x.$$

4 Degeneracy maps

We next define degeneracy maps between Drinfeld modular curves. Let $m \subset A$ be an ideal coprime to $n$. A generic point of $X_0(mn)$ can be written as $(\Phi, \Phi/C)$, where $\Phi$ is a Drinfeld module, and $C \cong A/mn$ an $A$-submodule of $\Phi_{\text{tors}}$. For
any divisor $\mathfrak{d}|mn$ we denote by $C[\mathfrak{d}] \cong A/\mathfrak{d}$ the $\mathfrak{d}$-torsion submodule of $C$.

Now, for every $\mathfrak{d}|m$ we define the $\mathfrak{d}$th degeneracy map

$$
\delta_\mathfrak{d} : X_0(mn) \longrightarrow X_0(n)
$$

$$(\Phi, \Phi/C) \longmapsto (\Phi/C[\mathfrak{d}], \Phi/C[\mathfrak{d}n]),$$

which maps the $[a]$-component of $X_0(mn)$ to the $[\mathfrak{d}^{-1}a]$-component of $X_0(n)$.

In this way, we may define the Hecke correspondence $T_m \subset X_0(n)^2$ as the image of $X_0(mn)$ under the map $\delta_1 \times \delta_m$.

More generally, let $\tau(m)$ be the number of divisors of $m$, then we define the full degeneracy map $\delta : X_0(mn) \rightarrow X_0(n)^{\tau(m)}$ as the product $\delta = \prod_{\mathfrak{d}|m} \delta_\mathfrak{d}$.

Composing with $\pi$, we obtain a new parametrization of $E$ by $X_0(mn)$:

$$
\pi' : X_0(mn) \overset{\delta}{\longrightarrow} X_0(n)^{\tau(m)} \overset{\pi}{\longrightarrow} E^{\tau(m)} \overset{\Sigma}{\longrightarrow} E
$$

(4.1)

**Proposition 4.2** The morphism $\pi' : X_0(mn) \rightarrow E$ is defined over $k$. Suppose that $k = \mathbb{F}_q(T)$ and $\deg(\infty) = 1$. Then $\pi' : X_0(mn) \rightarrow E$ is surjective.

**Proof.** It is clear that $\text{Gal}(H/k)$ leaves $\pi'$ invariant, and hence $\pi'$ is defined over $k$.

Now we suppose that $k = \mathbb{F}_q(T)$ and $\deg(\infty) = 1$. Then, after replacing $T$ by another generator if necessary, we may take $A = \mathbb{F}_q[T]$. In particular, $\text{Pic}(A)$ is trivial, and the modular curves $X_0(mn)$ and $X_0(n)$ are irreducible and defined over $k$.

Analytically, the map $\pi'$ is given by

$$
\pi'([z]) = \prod_{\mathfrak{d}|m} u(dz) \mod \Delta \in \mathbb{C}_\infty^*/\Delta = E(\mathbb{C}_\infty),
$$

where $[z]$ denotes the class of $z \in \Omega$ in $Y_0(mn)$, $u = u_\varphi$ is the theta function associated to the newform $\varphi$, and $d \in A$ is the monic generator of the ideal $\mathfrak{d}$, for each $\mathfrak{d}|m$. We need to show that the map $u'(z) = \prod_{\mathfrak{d}|m} u(dz)$ is not constant.

Denote by $\mathcal{O}_\Omega(\Omega)$ the ring of rigid holomorphic functions on $\Omega$, then there is an exact sequence

$$
1 \longrightarrow \mathbb{C}_\infty^* \longrightarrow \mathcal{O}_\Omega(\Omega)^* \overset{r}{\longrightarrow} H(\mathcal{T}, \mathbb{Z}) \longrightarrow 0.
$$

Furthermore, for any $f \in \text{GL}_2(k)$ we have $r(u \circ f) = \varphi \circ f \in H(\mathcal{T}, \mathbb{Z})$, so if $u'$
is constant, then we get

\[ 0 = r(u) = \varphi + \sum_{d|m, \ d \neq 1} \varphi \circ d \in H(T, \mathbb{Z}), \quad (4.3) \]

where \( d \) denotes the matrix \( \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \). Now we examine the Fourier coefficients of these terms. From Proposition 2.10 of [8] follows that the “first” Fourier coefficient of \( \varphi \circ d \) is \( c(\varphi \circ d, (1)) = 0 \) if \( d \not\in \mathbb{F}_q^* \). On the other hand, by a result of Atkin-Lehner (see §1.4 of [6]) follows that every newform \( \varphi \neq 0 \) satisfies \( c(\varphi, (1)) \neq 0 \). Thus (4.3) implies that \( \varphi = 0 \), a contradiction.

\[ \square \]

**Remark.** In the case of general \( k \) and \( \infty \), Proposition 4.2 should still hold. Indeed, for each component \( X_x \) of \( X_0(\mathfrak{m}n) \), one has a similar analytical description of \( \pi' \) and an equality of the form (4.3). But these now involve a combination of different theta functions \( u_y \), one for each component \( X_y \) of \( X_0(n) \) such that \( y = x[\mathfrak{d}^{-1}] \) in \( \text{Pic}(A) \) for some \( \mathfrak{d}|\mathfrak{m} \). This may be simplified by considering only those \( \mathfrak{m} \) for which every \( \mathfrak{d}|\mathfrak{m} \) is principal (in our application in §7 this amounts to choosing \( K \) such that only principal primes of \( k \) ramify in \( K \)), but one must still calculate the Fourier coefficients in this case.

## 5 Canonical factorizations of isogenies

In this section, we will describe the following canonical factorization of cyclic isogenies between CM Drinfeld modules.

**Proposition 5.1** Let \( f : \Psi_1 \to \Psi_2 \) be a cyclic isogeny of degree \( \mathfrak{d} \) between Drinfeld modules with complex multiplication by orders \( \text{End}(\Phi_i) \) of conductor \( \mathfrak{c}_i \) in \( K \), for \( i = 1, 2 \). Then we have a commutative diagram of cyclic isogenies

\[
\begin{array}{ccc}
\Psi_1 & \xrightarrow{f} & \Psi_2 \\
\downarrow f_1 & & \downarrow f_2 \\
\Psi'_1 & \xrightarrow{f'} & \Psi'_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathfrak{c}_1 & \xrightarrow{\mathfrak{d}} & \mathfrak{c}_2 \\
\downarrow \mathfrak{d}_1 & & \downarrow \mathfrak{d}_2 \\
\mathfrak{c} & \xrightarrow{\mathfrak{d}'} & \mathfrak{c}' \\
\end{array}
\]

Here \( \text{End}(\Psi'_1) = \text{End}(\Psi'_2) = \mathcal{O}_\mathfrak{c} \) is an order of conductor \( \mathfrak{c} \) in \( K \), and \( f, f_1, f_2, f' \) are of degree \( \mathfrak{d}, \mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}' \), respectively. This data is summarized in the right-hand diagram. Furthermore, we have \( \mathfrak{d} = \mathfrak{d}_1\mathfrak{d}_2\mathfrak{d}' \), \( \mathfrak{c} = \mathfrak{c}_1/\mathfrak{d}_1 = \mathfrak{c}_2/\mathfrak{d}_2 \) and \( \mathfrak{c} + \mathfrak{d}' = A \). Lastly, there is an ideal \( \mathfrak{D} \subset \mathcal{O}_K \) with \( \mathcal{O}_K/\mathfrak{D} \cong A/\mathfrak{d}' \) such that \( f' \) corresponds to \( \mathfrak{D} \cap \mathcal{O}_\mathfrak{c} \).
Proof. We follow closely the appendix of [4]. We will use the following notation. Let \( a \subset b \) be lattices (all lattices here have rank 2 and are contained in \( K \subset \mathbb{C}_\infty \), then we write \( a \subset_m b \) if \( b/a \cong A/m \). Also, the conductor \( c(a) \) of the lattice \( a \) is the conductor of the order \( \text{End}(a) \) in \( \mathcal{O}_K \).

Suppose \( a \subset_0 b \). Following §9 of [4] word for word, one arrives at an inclusion of lattices

\[
a \subset_{b_1} \mathcal{O}_a \subset_{b'} d_2 \mathcal{O}_{b} \subset_{b_2} b.
\]  

(5.2)

Here \( \mathcal{O}_a \) is an order (of conductor \( c \)), which is maximal with respect to the property \( \mathcal{O}_a \subset b \). Let \( c_1 = c(a), c_2 = c(b) \), then \( d_1 = c_1/c, d_2 = c_2/c \) and \( d' = d/d_1 d_2 \). To continue the argument, we need the following folklore result.

**Lemma 5.3** Let \( b \) be a lattice of conductor \( c \), and let \( q \subset A \) be a prime dividing \( c \). Then there exactly \( |q| + 1 \) sublattices \( a \subset q \ b \). One of them (given by \( a = q \mathcal{O}_{q/q} b \)) has conductor \( c(a) = c/q \) and the other \( |q| \) lattices have conductor \( c(a) = c q \).

\[ \square \]

We want to show that \( c \) and \( d' \) are coprime. Let \( q \subset A \) be a prime. If \( q|c \), then by Lemma 5.3, \( a' = \mathcal{O}_c q \mathcal{O}_a \) is the unique \( \mathcal{O}_c \)-stable lattice satisfying \( qa' \subset q \mathcal{O}_a \), or, equivalently, \( \mathcal{O}_a q \subset q \mathcal{O}_a \). Now suppose \( q \) divides \( d' \). Then \( d_2 \mathcal{O}_c b \mathcal{O}_a \cong A/d' \) has a unique \( A \)-submodule isomorphic to \( A/q \), namely its \( q \)-torsion submodule:

\[
(d_2 \mathcal{O}_c b \mathcal{O}_a)[q] = (d_2 \mathcal{O}_c b \cap q^{-1} \mathcal{O}_a)/\mathcal{O}_a,
\]

hence \( a'' = d_2 \mathcal{O}_c b \cap q^{-1} \mathcal{O}_a \) is also \( \mathcal{O}_c \)-stable and satisfies \( \mathcal{O}_a q \subset a'' \), and is furthermore contained in \( b \). Thus, if \( q \) divides both \( c \) and \( d' \), then \( \mathcal{O}_{c/q} a = a' = a'' \subset b \), which contradicts the maximality of \( \mathcal{O}_c \). So \( c \) and \( d' \) are coprime.

Lastly, we set \( \mathcal{D} = d_2^{-1} \mathcal{O}_K b^{-1} \mathcal{O}_a \). Then we see easily that \( \mathcal{D} \subset \mathcal{O}_K \). Also, \( \mathcal{D}_c = \mathcal{D} \cap \mathcal{O}_c \) is invertible in \( \mathcal{O}_c \) and we have \( d_2 \mathcal{O}_c b = \mathcal{D}_c^{-1} \mathcal{O}_a \). Now, using the equivalence between lattices and Drinfeld modules, Proposition 5.1 follows.

\[ \square \]

6 Geometric action

Let \( \Phi_n = \Phi^{Q_n} \). We say an element \( \sigma \in G \) is “geometric” if there exists a cyclic isogeny \( f_\sigma : \Phi_n \rightarrow \Phi_{n'}^\sigma \) of fixed degree \( d \) for infinitely many \( n \in \mathbb{N} \). In particular, if \( d + n = A \), then each \( (x_n, x_n^\sigma) \) lies in the Hecke correspondence \( T_0 \subset X_0(n)^2 \).

Let \( p_1, \ldots, p_g \) be the primes \( \neq \mathfrak{p} \) of \( k \) which ramify in \( K/k \), and let \( \Psi_1, \ldots, \Psi_g \) be the primes of \( K \) lying above them. Denote by \( \sigma_i = (\Psi_i, K[p^\infty]/K) \) the
Frobenius elements, and by \( G_1 = \langle \sigma_1, \ldots, \sigma_g \rangle \) the group they generate. Suppose \( [p_i] \) has order \( e_i \) in \( \Pic(A) \). Then \( \sigma_i \) has order \( 2e_i \) in \( G \), and it follows that \( G_1 \supset G_0 \). Let \( m = p_1^{2e_1 - 1} \cdots p_g^{2e_g - 1} \), which is coprime to \( n \) (recall that every prime in \( n \) splits in \( K/k \)). Then the elements of \( G_1 \) are in a one-to-one correspondence with the divisors \( \mathfrak{d} \) of \( m \), via \( \mathfrak{d} = \prod_{i=1}^g p_i^{e_i} \mapsto \sigma_\mathfrak{d} = \prod_{i=1}^g \sigma_i^{e_i} \).

**Proposition 6.1** \( G_1 \) is the subgroup of geometric elements of \( G_0 \). More precisely,

1. Let \( \sigma = \sigma_\mathfrak{d} \in G_1 \) for some \( \mathfrak{d} \mid m \). Then \( (x_n, x_n^\mathfrak{d}) \in T_\mathfrak{d} \subset X_0(n)^2 \) for all \( n \in \mathbb{N} \). In particular, \( (x_n^\mathfrak{d})_{\sigma \in G_1} \subset X_0(n)^{[G_1]} \) lies in the image of \( X_0(mn) \) under the full degeneracy map \( \delta \) described in §4.
2. Conversely, let \( \sigma \in G_0 \) and suppose that there exist cyclic isogenies \( f_n : \Phi_n \to \Phi_n^\sigma \) of fixed degree \( \mathcal{D}_\sigma \) for infinitely many \( n \in \mathbb{N} \). Then \( \sigma \in G_1 \).

**Proof.** From (2.2) follows that each \( \sigma \in G \) can be written in the form \( \sigma = (\sigma_1, \sigma_2, \ldots) \), where for each \( n \in \mathbb{N} \) \( \sigma_n \in \Gal(K[p_n]/K) \) corresponds to an invertible ideal \( \mathfrak{A}_n = \mathfrak{A} \cap \mathcal{O}_n \) in \( \mathcal{O}_n \), for some \( \mathfrak{A} \subset \mathcal{O}_K \).

For each \( m \geq n \) the theory of complex multiplication gives an isogeny \( f_m : \Phi_n \to \Phi_n^{\mathfrak{d}_m} \) with \( \ker(f_m) \cong \mathcal{O}_m/\mathfrak{A}_m \). Now let \( \mathfrak{a} \subset A \) lie under \( \mathfrak{A} = \mathfrak{A}_n \mathcal{O}_K \). If \( \sigma \in G_1 \), then \( \mathfrak{a} \) is a product of primes which ramify in \( K/k \). Thus each \( \ker(f_m) \cong \mathcal{O}_m/\mathfrak{A}_m \cong A/\mathfrak{a} \) is cyclic. It follows that \( (\Phi_n, \Phi_n^\sigma) \) lies on the curve \( Y_0(\mathfrak{a}) \). Furthermore, \( \mathfrak{a} \) is prime to \( n \), hence \( \sigma \) is compatible with \( n \)-isogenies, and it follows that the pair of Heegner points \( (x_n, x_n^\sigma) \) lies on the Hecke correspondence \( T_n \subset X_0(n)^2 \). This proves part (1) of Proposition 6.1.

To prove part (2), we let \( I \subset \mathbb{N} \) be an infinite subset, \( \sigma \in G \), and we suppose that there exists a cyclic isogeny \( f_n : \Phi_n \to \Phi_n^\sigma \) of degree \( \mathfrak{d} \) for all \( n \in I \). One potential source of trouble is the fact that \( p \) might divide \( \mathfrak{d} \). Write \( \mathfrak{d} = p^r \mathfrak{d}' \) with \( p \nmid \mathfrak{d}' \). We apply Proposition 5.1 to the case where \( \Psi_1 = \Phi_n \) and \( \Psi_2 = \Phi_n^\sigma \), and \( n \in I \) satisfies \( n \geq t/2 \). Then we see that \( t = 2r \) is even, and we have a commutative diagram (with the relevant conductors and degrees shown on the right)

![Diagram](attachment:image.png)

We claim that \( \Psi'_2 = \Psi'_1^\sigma \). Indeed, we have maps \( f_1^\sigma : \Psi_2 = \Psi_1^\sigma \to \Psi_1^\sigma \), and \( f_2 : \Psi_2 \to \Psi'_2 \), both of which correspond to lattice inclusions of the form \( \Lambda_1, \Lambda_2 \subseteq_{p^r} \Lambda \), where \( c(\Lambda_1) = c(\Lambda_2) = p^{n-r} = c(\Lambda)/p^r \). It follows from Lemma 5.3 (and induction on \( r \)) that \( \Lambda_1 = \Lambda_2 \), which proves the claim.
Now we restrict our attention to $f': \Psi'_1 \to \Psi'_1$. As $f'$ has degree $d'$, which is prime to $p$, it follows that $f'$ corresponds, from Proposition 5.1, to an invertible ideal $D_n \subset \mathcal{O}_K$ such that $\mathcal{O}_K/D_n \cong A/d'$ for all $n$. This leaves only finitely many possibilities for $D_n$, hence, by restricting $I$, we may assume $D_n = \mathcal{D} \subset \mathcal{O}_K$ for all $n \in I$. It now follows that $(\mathcal{D}, K[p^\infty]/K) = \sigma$. In particular, if $\sigma \in G_0$, then $\sigma$ has finite order, and there exists an integer $s$ such that $\mathcal{D}^s$ is principal in $\mathcal{O}_K$. Moreover, $\mathcal{D}^s \cap \mathcal{O}_n$ is principal for each $n \in I$, and so $\mathcal{D}^s$ is generated by an element $d \in \cap_{n \in I} \mathcal{O}_n = A$. Denote by $z \mapsto \bar{z}$ the non-trivial element of $\text{Gal}(K/k)$. Then for every prime $\mathfrak{p}$ of $\mathcal{O}_K$ dividing $\mathcal{D}$ we see that $\bar{\mathfrak{p}}$ also divides $\mathcal{D}$. But as $\mathcal{O}_K/\mathcal{D} \cong A/d'$ is cyclic, this is only possible if $\mathfrak{p}$ is ramified in $K/k$, and hence $\sigma \in G_1$. This concludes the proof of Proposition 6.1.

□

**Remark.** We have also shown that the geometric elements of $G$ are the elements of the form $(\mathcal{D}, K[p^\infty]/K)$ for some $\mathcal{D} \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathcal{D}$ cyclic. Thus they form a countable subgroup of $G$.

## 7 Proof of Theorem 1

Lemma 2.3 says that $E_{\text{tors}}(K[p^\infty])$ is finite, so in particular, all these torsion points are defined over a finite extension of $k$. Now Theorem 1 will follow if the fields of definition of the $y_n$'s grow with $n$. We prove this as follows.

Let $m = p_1 p_2 \cdots p_g \subset A$ as in §6, and recall the full degeneracy map $\delta : X_0(mn) \to X_0(n)[G_1]$ from §4. From Proposition 6.1 follows that the point $(x_n^\sigma)_{\sigma \in G_1}$ lies in the image of $\delta$, and we denote by $x'_n \in X_0(mn)(K[p^\infty])$ its preimage, which is given by $(\Phi^{\mathcal{O}_n}, \Phi^{\mathcal{N}_n^{-1}\mathcal{M}_n^{-1}})$, where $\mathcal{M}_n = \mathfrak{P}_1 \cdots \mathfrak{P}_g \cap \mathcal{O}_n$. We combine $\delta$ with $\pi$ to obtain the new modular parametrization $\pi' : X_0(mn) \to E$ of §4. Let $R \subset G_0$ be a set of representatives for $G_0/G_1$. Notice that

$$y_n = \text{Tr}_{G_0}(\pi(x_n)) = \text{Tr}_R \left( \sum_{\sigma \in G_1} \pi(x_n)^{\sigma} \right) = \sum_{\sigma \in R} \pi'(x'_n)^{\sigma}.$$

For each $m \in \mathbb{N}$ we choose $\theta_m \in \text{Gal}(K[p^\infty]/K[p^m])$ such that none of the
elements \(\theta_m \sigma\) is geometric, for \(\sigma \in R\). Then we consider the composite map

\[
\begin{align*}
  f_m : \mathbb{N} & \rightarrow X_0(\mathbb{m}n)^{|R|} \times X_0(\mathbb{m}n)^{|R|} \xrightarrow{\pi'} E^{|R|} \times E^{|R|} \\
  & \xrightarrow{\Sigma} E \times E \xrightarrow{\partial} E
\end{align*}
\]

\[
\begin{align*}
  n & \mapsto (x_n^\sigma, x_n^{\theta_m \sigma})_{\sigma \in R} \\
  & \mapsto (\pi'(x_n')^\sigma, \pi'(x_n')^{\theta_m \sigma})_{\sigma \in R} \\
  & \mapsto (y_n, y_n^{\theta_m}) \mapsto y_n - y_n^{\theta_m}.
\end{align*}
\]

We will show that \(\rho\) is dominant, hence \(f_m : \mathbb{N} \rightarrow E; \quad f_m(n) = y_n - y_n^{\theta_m}\) has finite fibres. In particular, \(f_m^{-1}(0)\) is finite, and the proof will be complete.

Here the André-Oort conjecture (see e.g. [5]) enters the picture, for which we state the following characteristic-\(p\) analogue (see [1]).

**Conjecture 7.1 (André-Oort)** Let \(X = X_1 \times \cdots \times X_n\) be a product of Drinfeld modular curves, and let \(Z \subset X\) be an irreducible algebraic subvariety for which every projection \(Z \rightarrow X_i\) is dominant. Suppose \(Z\) contains a Zariski-dense set of CM points. Then \(Z\) is a “modular” subvariety, which means the following. There exist \(g_1, \ldots, g_n \in \text{GL}_2(k)\) and a partition \(\{1, \ldots, n\} = \prod_{j=1}^m S_j\) such that \(Z = \prod_{j=1}^m Z_j\), and each \(Z_j(\mathbb{C}_\infty) \subset \prod_{i \in S_j} X_i(\mathbb{C}_\infty)\) is the image of \(\Omega^*\) under the map \(z \mapsto ([g_i(z)])_{i \in S_j}\).

Let \(I \subset \mathbb{N}\) be an infinite subset. We suppose for the moment that Conjecture 7.1 holds. In our case \(X = X_0(\mathbb{m}n)^2|R|\) and we take \(Z\) to be a positive-dimensional irreducible component of the Zariski-closure of \(\rho(I)\) in \(X\). Then Conjecture 7.1 implies that either \(Z = X\), or there exist some \(\sigma_i \neq \sigma_j \in R \cup \theta_m R\) such that the projection of \(Z\) onto the factor \(X_0(\mathbb{m}n)^2\) indexed by \((\sigma_i, \sigma_j)\) is contained in some Hecke correspondence \(T_0\). But the latter case is impossible, as this would mean that \(\sigma_i \sigma_j^{-1}\) is geometric, contrary to the definitions of \(R\) and \(\theta_m\). Thus \(Z = X\), and we see that \(\rho\) is dominant. The result follows. More precisely, we have shown:

**Theorem 2** Suppose Conjecture 7.1 holds in the case where \(X = X_0(\mathbb{m}n)^m\) and \(Z \subset X\) contains a Zariski-dense set of CM points \(w = (w_1, \ldots, w_m)\) for which \(\text{End}(w_1) = \cdots = \text{End}(w_m) = \mathcal{O}_n\) for some \(n \in \mathbb{N}\). Suppose further that the map \(\pi' : X_0(\mathbb{m}n) \rightarrow E\) is surjective. Then for every infinite subset \(I \subset \mathbb{N}\) the group generated by \(\{y_n \mid n \in I\}\) in \(E(H[\mathfrak{p}^\infty])\) has infinite rank.

In the special case where \(k = \mathbb{F}_q(T)\), \(A = \mathbb{F}_q[T]\) and \(q\) is odd, Conjecture 7.1 is known [1], and \(\pi'\) is surjective (Proposition 4.2), and so Theorem 1 follows.  

\[\square\]
Remark. Theorems 1 and 2 have two very easy mild generalizations: Firstly, fix an ideal \( c \subset A \) prime to \( n \). Then we can construct Heegner points corresponding to the orders \( \mathcal{O}_{cn} = A + cp^n \mathcal{O}_K \), for which similar results hold. Secondly, let \( \chi : G_0 \rightarrow \{\pm 1\} \) be a character, and let \( K_{\chi}[p^\infty] \) be the subfield of \( K[p^\infty] \) corresponding to \( \ker(\chi) \). Note that \( K_{\chi}[p^\infty] = H[p^\infty] \) if \( \chi \) is trivial. Then we may replace the \( G_0 \)-trace on \( E \) by the \((G_0, \chi)\)-trace

\[
\text{Tr}_{G_0, \chi} : E(K[p^\infty]) \longrightarrow E(K_{\chi}[p^\infty]): \quad x \longmapsto \sum_{\sigma \in G_0} \chi(\sigma)x^\sigma,
\]

and obtain a similar result. We leave the details to the dedicated reader.

References

[1] F. Breuer, The André-Oort conjecture for products of Drinfeld modular curves, preprint available at http://arxiv.org/abs/math.NT/0303038.

[2] M.L. Brown, On a conjecture of Tate for elliptic surfaces over finite fields, Proc. London Math. Soc. (3) 69 (1994), 489-514.

[3] C. Cornut, Mazurs’s Conjecture on higher Heegner points, Invent. Math. 148 (2002) no. 3, 495-523.

[4] C. Cornut, Non-trivialité des points de Heegner, C. R. Acad. Sci. Paris, Ser. I 334 (2002) no. 12, 1039-1042.

[5] S.J. Edixhoven, Special points on products of modular curves, preprint available at http://arxiv.org/abs/math.NT/0302138.

[6] E.-U. Gekeler, Automorphe Formen über \( \mathbb{F}_q(T) \) mit kleinem Führer, Abh. Math. Sem. Univ. Hamburg 55 (1985), 111-146.

[7] E.-U. Gekeler, Drinfeld modular curves, Lecture Notes in Mathematics 1231, Springer-Verlag, Berlin-Heidelberg, 1986.

[8] E.-U. Gekeler, Improper Eisenstein series on bruhat-Tits trees, Manuscripta Math. 86 (1995), 367-391.

[9] E.-U. Gekeler, Analytical construction of Weil curves over function fields, J. Th. Nomb. Bordeaux 7 (1995), 27-49.

[10] E.-U. Gekeler and M. Reversat, Jacobians of Drinfeld modular curves, J. reine angew. Math. 476 (1996), 27-93.

[11] I. Longhi, Non-Archimedean integration and elliptic curves over function fields, J. Number Theory 94 (2002), 375-404.

[12] B. Mazur, Modular curves and arithmetic, Proceedings of the International Congress of Mathematicians, Warszawa, 1983.
[13] J. Neukirch, Algebraische Zahlentheorie, Springer-Verlag, Berlin-Heidelberg, 1992.

[14] A. Pál, Drinfeld modular curves, Heegner points and interpolation of special values, Thesis, Columbia University (2000).

[15] H.-G. Rück and U. Tipp, Heegner points and L-series of automorphic cusp forms of Drinfeld type, Doc. Math. 5 (2000), 365-444 (electronic).

[16] D. Ulmer, Elliptic curves and analogies between number fields and function fields, to appear in: Heegner points and L-series, MSRI Publications 48. Preprint available at http://arxiv.org/abs/math.NT/0305320.

[17] V. Vatsal, Uniform distribution of Heegner points, Invent. Math. 148 (2002), no. 1, 1-46.

[18] J.F. Voloch, Diophantine approximation on abelian varieties in characteristic p, Amer. J. Math. 117 (1995), no. 4, 1089-1095.