DERIVATIONS, LOCAL AND 2-LOCAL DERIVATIONS OF STANDARD OPERATOR ALGEBRAS

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Abstract. Let $X$ be a Banach space over field $F$ ($R$ or $C$). Denote by $B(X)$ the set of all bounded linear operators on $X$ and by $F(X)$ the set of all finite rank operators on $X$. A subalgebra $A$ of $B(X)$ is called a standard operator algebra if $A$ contain $F(X)$. We give a brief proof of a well-known result that every derivation from $A$ into $B(X)$ is inner. There is another classical result that every local derivation on $B(X)$ is a derivation. We extend the result by proving that every local derivation from $A$ into $B(X)$ is a derivation. Based on these two results, we prove that every 2-local derivation from $A$ into $B(X)$ is a derivation.

1. Introduction

Let $A$ be an algebra and $M$ be an $A$-bimodule. Suppose that $\delta$ is a linear mapping from $A$ into $M$. $\delta$ is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for each $x, y$ in $A$, and $\delta$ is called an inner derivation if there exist an element $m$ in $M$ such that $\delta(x) = mx - xm$. Clearly, every inner derivation is a derivation. But the converse is not so trivial. It is a classical problem to identify those algebras on which every derivation is an inner derivation. In [13, 17], R. Kadison and S. Sakai independently prove that every derivation on a von Neumann algebra is an inner derivation. In [5], P. Chernoff proves that every derivation from a standard operator algebra into $B(X)$ is an inner derivation. In this paper, we shall give a brief proof of this result.

In 1990, R. Kadison [14], D. Larson and A. Sourour [15] independently introduce the concept of local derivation. A linear mapping $\delta$ from $A$ into $M$ is called a local derivation if for every $x$ in $A$, there exists a derivation $\delta_x$ (depends on $x$) from $A$ into $M$, such that $\delta(x) = \delta_x(x)$. In [14], R. Kadison proves that every continuous local derivation from a von Neumann algebra into its dual Banach module is a derivation. In [15], D. Larson and A. Sourour prove that every local derivation from $B(X)$ into itself is a derivation. For more information about local derivations, we refer to [12, 7, 10, 11, 16, 20, 21]. In this paper, we shall extend the result of D. Larson and A. Sourour [15] to that every local derivation from a standard operator algebra into $B(X)$ is a derivation.

The concept of 2-local derivation is firstly introduced by P. Šemrl [18] in 1997. A mapping (not necessarily linear) $\delta$ from $A$ into $M$ is called a 2-local derivation
if for each \(x, y\) in \(A\), there exist a derivation \(\delta_{x,y}\) (depends on \(x, y\)) from \(A\) into \(M\), such that \(\delta(x) = \delta_{x,y}(x)\) and \(\delta(y) = \delta_{x,y}(y)\). P. Šemrl [18] proves that every 2-local derivation from \(B(H)\) into itself is a derivation for a separable Hilbert space \(H\). Many other authors study 2 local derivations and there are several important results, we refer to [1, 2, 3, 19, 9, 8]. In this paper, we prove that every 2-local derivation from a standard operator algebra into \(B(X)\) is a derivation.

2. Main results

We shall firstly review some simple properties of \(B(X)\), especially about finite rank operators. Throughout this section, \(X\) is a Banach space over field \(F\) (\(\mathbb{R}\) or \(\mathbb{C}\)), and \(X^*\) is the set of all bounded linear functionals on \(X\). Denote by \(B(X)\) the set of all bounded linear operators on \(X\). An operator \(A \in B(X)\) is said to be of finite rank if the range of \(A\) is a finite dimensional subspace of \(X\). Denote by \(F(X)\) the set of all finite rank operators on \(X\). \(A\) is a standard operator algebra, which means that \(A\) is a subalgebra of \(B(X)\) and \(A \supseteq F(X)\). For each \(x\) in \(X\) and \(f, g\) in \(X^*\), one can define an operator \(x \otimes f\) by \((x \otimes f)y = f(y)x\) for all \(y\) in \(X\). Obviously, \(x \otimes f \in B(X)\). If both \(x\) and \(f\) are nonzero, then \(x \otimes f\) is of rank one.

The following properties are evident and will be used frequently in the proof. For each \(x, y\) in \(X\), \(f, g\) in \(X^*\), and \(A, B\) in \(B(X)\),

1. \((x \otimes f)A = x \otimes (fA)\), and \(A(x \otimes f) = (Ax) \otimes f\);
2. \((x \otimes f)(y \otimes g) = f(y)(x \otimes g)\);
3. every operator in \(F(X)\) of rank \(n\) can be written into the form \(\sum_{i=1}^{n} x_i \otimes f_i\), where \(x_i \in X\) and \(f_i \in X^*\);
4. \(F(X)\) is a two-side ideal of \(B(X)\);
5. \(F(X)\) is a separating set of \(B(X)\), which means that \(AF(X) = 0\) implies \(A = 0\) and \(F(X)A = 0\) implies \(A = 0\).

Now we are in position to give our main results. The following Theorem 2.2 can be found in [5]. But we shall give another brief proof here.

**Lemma 2.1.** Every derivation \(\delta\) from standard operator algebra \(A\) into \(B(X)\) is continuous.

**Proof.** Assume that \(\{T_n\} \subseteq A\) is a sequence converging to 0, and \(\{\delta(T_n)\}\) converges to \(T\). According to the closed graph Theorem, to prove \(\delta\) is continuous, it is sufficient to show that \(T = 0\).

For each \(x, y\) in \(X\) and \(f, g\) in \(X^*\), we have that,

\[
\delta(x \otimes f T_n y \otimes g) = \delta(f(T_n y)x \otimes g) = f(T_n y)\delta(x \otimes g).
\]

By \(T_n \to 0\), it follows that \(\delta(x \otimes f T_n y \otimes g) \to 0\). On the other hand, we have that

\[
\delta(x \otimes T_n y \otimes g) = \delta(x \otimes f)T_n y \otimes g + x \otimes f \delta(T_n)y \otimes g + x \otimes f T_n \delta(y \otimes g).
\]

By \(T_n \to 0\) and \(\delta(T_n) \to T\), it follows that

\[
\delta(x \otimes f T_n y \otimes g) \to x \otimes f Ty \otimes g = f(T y)x \otimes g.
\]
It implies that \( f(Ty)x \otimes g = 0 \) for each \( x, y \in X \) and \( f, g \in X^\ast \). Hence we can obtain that \( T = 0 \). The proof is complete. \( \square \)

**Theorem 2.2.** Every derivation \( \delta \) from standard operator algebra \( \mathcal{A} \) into \( B(X) \) is an inner derivation.

**Proof.** Let \( x_0 \) be in \( X \) and \( f_0 \) be in \( X^\ast \) such that \( f_0(x_0) = 1 \). Define a mapping \( T \) from \( X \) into itself by

\[
T x = \delta(x \otimes f_0)x_0
\]

for all \( x \) in \( X \). By Lemma 2.1, we know that \( \delta \) is continuous. So it is not difficult to check that \( T \) is also continuous. Obviously \( T \) is linear. Hence we have \( T \in B(X) \).

For each \( x \) in \( X \) and \( A \) in \( \mathcal{A} \), we have that

\[
T Ax = \delta(Ax \otimes f_0)x_0 = \delta(A)x \otimes f_0x_0 + A\delta(x \otimes f_0)x_0 = \delta(A)x + AT x.
\]

It implies that \( \delta(A) = TA - AT \) for all \( A \) in \( \mathcal{A} \). That is to say, \( \delta \) is an inner derivation. The proof is complete. \( \square \)

**Theorem 2.3.** Every local derivation \( \delta \) from standard operator algebra \( \mathcal{A} \) into \( B(X) \) is a derivation.

**Proof.** We shall firstly consider the case that \( \mathcal{A} \) contains unit \( I \).

By the definition of local derivation, there exists a derivation \( \delta_I \) such that \( \delta(I) = \delta_I(I) = 0 \). Let \( A, B \) and \( C \) be in \( \mathcal{A} \) with \( AB = BC = 0 \). There exists a derivation \( \delta_B \) such that \( \delta_B(B) = \delta(B) \). Thus we have that

\[
A\delta(B)C = A\delta_B(B)C = \delta_B(ABC) - \delta_B(A)BC - AB\delta_B(C) = 0. \tag{2.1}
\]

Let \( A_0 \) and \( B_0 \) be in \( \mathcal{A} \) with \( A_0B_0 = 0 \). Define a bilinear mapping \( \phi_1 \) from \( \mathcal{A} \times \mathcal{A} \) into \( B(X) \) by

\[
\phi_1(X, Y) = X\delta(YA_0)B_0
\]

for each \( X \) and \( Y \) in \( \mathcal{A} \). By (2.1), we see that \( XY = 0 \) implies \( \phi_1(X, Y) = 0 \). Let \( A \) be in \( \mathcal{A} \) and \( R \) be in \( F(X) \). By [4, Theorem 4.1], we have that

\[
\phi_1(R, A) = \phi_1(RA, I),
\]

that is,

\[
R\delta(AA_0)B_0 = RA\delta(A_0)B_0.
\]

Since \( F(X) \) is a separating set of \( B(X) \), we obtain that \( \delta(AA_0)B_0 = A\delta(A_0)B_0 \) whenever \( A_0B_0 = 0 \).

Define a bilinear mapping \( \phi_2 \) from \( \mathcal{A} \times \mathcal{A} \) into \( B(X) \) by

\[
\phi_2(X, Y) = \delta(AX)Y - A\delta(X)Y
\]

for each \( X \) and \( Y \) in \( \mathcal{A} \). By the previous discussion, we see that \( XY = 0 \) implies \( \phi_2(X, Y) = 0 \). Let \( B \) be in \( \mathcal{A} \) and \( R \) be in \( F(X) \). Again by [4, Theorem 4.1], we have that

\[
\phi_2(B, R) = \phi_2(I, BR).
\]

By \( \delta(I) = 0 \), it implies that

\[
\delta(AB)R - A\delta(B)R = \delta(A)BR.
\]
Since $F(X)$ is a separating set of $B(X)$, we obtain that
\[ \delta(AB) = A\delta(B) + \delta(A)B \]
for each $A, B$ in $\mathcal{A}$. That is to say, $\delta$ is a derivation.

Next we shall consider the case that $\mathcal{A}$ do not contain unit $I$. In this case, denote the unital algebra $\mathcal{A} \oplus \mathbb{F}I$ by $\tilde{\mathcal{A}}$.

For every linear mapping $\phi$ from $\mathcal{A}$ into $B(X)$, we can define a linear mapping $\tilde{\phi}$ from $\tilde{\mathcal{A}}$ into $B(X)$ by $\tilde{\phi}(A + \lambda I) = \phi(A)$ for all $A \in \mathcal{A}$ and $\lambda \in \mathbb{F}$. For all $A, B \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{F}$, we have
\[
\tilde{\phi}((A + \lambda I)(B + \mu I)) = \tilde{\phi}(AB + \lambda B + \mu A + \lambda \mu I) = \phi(AB) + \lambda \phi(B) + \mu \phi(A)
\]
and
\[
\tilde{\phi}(A + \lambda I)(B + \mu I) + (A + \lambda I)\tilde{\phi}(B + \mu I) = \phi(A)B + A\phi(B) + \lambda \phi(B) + \mu \phi(A).
\]
It implies that $\phi$ is a derivation if and only if $\tilde{\phi}$ is a derivation.

Since $\delta$ is a local derivation from $\mathcal{A}$ into $B(X)$. For each $A \in \mathcal{A}$ and $\lambda \in \mathbb{F}$, there exists a derivation $\delta_A$ such that $\delta(A) = \delta_A(A)$. Moreover, we have $\tilde{\delta}(A + \lambda I) = \delta(A) = \delta_A(A) = \tilde{\delta}_A(A + \lambda I)$. It means that $\tilde{\delta}$ is a local derivation from $\tilde{\mathcal{A}}$ into $B(X)$. By the result of the case that $\mathcal{A}$ contains unit, $\tilde{\delta}$ is a derivation. Hence $\delta$ is also a derivation. The proof is complete. \[ \Box \]

**Theorem 2.4.** Every 2-local derivation $\delta$ from standard operator algebra $\mathcal{A}$ into $B(X)$ is a derivation.

**Proof.** For all $A \in \mathcal{A}$ and $S \in F(x)$, by the definition of 2-local derivation, there exists a derivation $d$ from $\mathcal{A}$ into $B(X)$ such that $\delta(A) = d(A)$ and $\delta(S) = d(S)$. By Theorem 2.2, $d$ is an inner derivation. Thus we have
\[
\delta(A)S + A\delta(S) = d(A)S + Ad(S) = d(AS) = AST - TAS
\]
for some $T \in B(X)$. It follows that $tr(\delta(A)S + A\delta(S)) = 0$, where $tr$ is the trace mapping on $F(X)$. Moreover, $tr(\delta(A)S) = -tr(A\delta(S))$.

Now, for each $A, B \in \mathcal{A}$ and $S \in F(X)$, we have
\[
tr(\delta(A + B)S) = -tr((A + B)\delta(S)) = -tr(A\delta(S)) - tr(B\delta(S))
= tr(\delta(A)S) + tr(\delta(B)S)
= tr((\delta(A) + \delta(B))S).
\]

Let $C = \delta(A + B) - \delta(A) - \delta(B)$, we obtain $tr(CS) = 0$. By taking $S = x \otimes f$, where $x \in X$ and $f \in X^*$ are chosen arbitrarily, we have $tr(Cx \otimes f) = f(Cx) = 0$. It follows that $C = 0$, i.e. $\delta(A + B) = \delta(A) + \delta(B)$. That is to say $\delta$ is additive. In addition, by the definition of 2-local derivation, it is easy to see that $\delta$ is homogeneous. Hence $\delta$ is linear, moreover, a local derivation. Then by Theorem 2.3, $\delta$ is a derivation. The proof is complete. \[ \Box \]
References

1. S. Ayupov, K. Kudaybergenov, 2-local derivations and automorphisms on $B(H)$, J. Math. Anal. Appl., 395(2012), 15-18.
2. S. Ayupov, K. Kudaybergenov, 2-local derivations on von Neumann algebras, Positivity, 19 (2014), 445-455.
3. S. Ayupov, K. Kudaybergenov, A. Peralta, A survey on local and 2-local derivations on C*- and von Neumann algebras, Topics in functional analysis and algebra, 73-126, Contemp Math., 672, 2016.
4. M. Brešar, Multiplication algebra and maps determined by zero products. Linear Multilinear Algebra, 60 (2012), 763–768.
5. P. Chernoff, Representations, automorphism and derivation of some operator algebras, J. Funct. Anal., 12 (1973), 275-289.
6. E. Christensen, Derivations of nest algebras, Math. Ann., 229(1977), 155-161.
7. R. Crist, Local derivations on operator algebras, J. Funct. Anal., 135 (1996), 72-92.
8. J. He, J. Li, G. An, W. Huang, Characterizations of 2-local derivations and local Lie derivations on some algebras. Sib. Math. J., 59(2018), 721-730.
9. S. Kim, J. Kim, Local automorphisms and derivations on $M_n(\mathbb{C})$, Proc. Amer. Math. Soc., 132(2004), 1389-1392.
10. D. Hadwin, J. Li, Local derivations and local automorphisms, J. Math. Anal. Appl., 290(2004), 702-714.
11. D. Hadwin, J. Li, Local derivations and local automorphisms on some algebras, J. Operator Theory, 60(2008), 29-44.
12. B. Johnson, Local derivations on C*-algebras are derivations, Trans. Amer. Math. Soc., 353(2001), 313-325.
13. R. Kadison, Derivations of operator algebras, Ann. Math., 83(1966), 280-293.
14. R. Kadison, Local derivations, J. Algebra, 130(1990), 494-509.
15. D. Larson, A. Sourour, Local derivations and local automorphisms, Proc. Sympos. Pure Math., 51(1990), 187-194.
16. J. Li, Z. Pan, Annihilator-preserving maps, multipliers and local derivations, Linear Algebra Appl., 432(2010), 5-13.
17. S. Sakai, Derivations of $W^*$-algebras, Ann. Math., 83(1966), 273-279.
18. P. Šemrl, Local automorphisms and derivations on $B(H)$, Proc. Amer. Math. Soc., 125(1997), 2677-2680.
19. J. Zhang, H. Li, 2-local derivations on digraph algebras, Acta Math. Sinica(Chin. Ser.), 49(2006), 1401-1406.
20. J. Zhu, Local derivation of nest algebras, Proc. Amer. Math. Soc., (123)1995, 739-742.
21. J. Zhu, C. Xiong, Bilocal derivations of standard operator algebras, Proc. Amer. Math. Soc., 125(1997), 1367-1370.

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