New Generalizations of sup-Hesitant Fuzzy Ideals of Semigroups

Uraiwan Jittburus\textsuperscript{1}, Pongpun Julatha\textsuperscript{1,*}, Attaphol Pumila\textsuperscript{1}, Napaporn Chunsee\textsuperscript{2}, Aiyared Iampan\textsuperscript{3}, Rukchart Prasertpong\textsuperscript{4}

\textsuperscript{1}Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand
\textsuperscript{2}Faculty of Science and Technology, Uttaradit Rajabhat University, Uttaradit 53000, Thailand
\textsuperscript{3}Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand
\textsuperscript{4}Division of Mathematics and Statistics, Faculty of Science and Technology, Nakhon Sawan Rajabhat University, Nakhon Sawan 60000, Thailand

*Corresponding author: pongpun.j@psru.ac.th

Abstract. As general concepts of sup-hesitant fuzzy right (resp., left, interior, two-sided) ideals of semigroups, the concepts of sup\textsuperscript{+}\textsubscript{α}-hesitant fuzzy right (resp., left, interior, two-sided) ideals and sup\textsuperscript{−}\textsubscript{β}-hesitant fuzzy right (resp., left, interior, two-sided) ideals are introduced and their properties are investigated. Then, the concepts are established by fuzzy sets, Łukasiewicz fuzzy sets, Łukasiewicz anti-fuzzy sets, Pythagorean fuzzy sets, hesitant fuzzy sets, hybrid sets, interval-valued fuzzy sets and cubic sets. Finally, we characterize which is intra-regular, completely regular, simple semigroups or another type of semigroups in terms of sup\textsuperscript{+}\textsubscript{α}-type and sup\textsuperscript{−}\textsubscript{β}-type of hesitant fuzzy sets.

1. Introduction

The fuzzy set theory presented by Zadeh [46] has been successfully and widely applied in many areas such as robotics, expert, computer science, finite state machine, control engineering, logic theory, automata theory, group theory, graph theory and semigroup theory. Furthermore, in the literature, a number of concepts of fuzzy sets and their generalizations and extensions have been

Received: Aug. 24, 2022.
2010 Mathematics Subject Classification. 03E72, 08A72, 20M12.
Key words and phrases. semigroup; sup-hesitant fuzzy ideal; generalized sup-hesitant fuzzy ideal; Łukasiewicz fuzzy set; fuzzy ideal; hesitant fuzzy ideal; interval-valued fuzzy ideal.

https://doi.org/10.28924/2291-8639-2022-58
ISSN: 2291-8639 © 2022 the author(s).
introduced and studied, for instance, Łukasiewicz fuzzy sets [23], Łukasiewicz anti-fuzzy sets [22],
anti-type of fuzzy sets [25, 37], negative fuzzy sets [18], bipolar fuzzy sets [29, 48], interval-valued
fuzzy sets [47], intuitionistic fuzzy sets [4], Pythagorean fuzzy sets [44, 45], rough sets [2, 34], hesitant
fuzzy sets [41, 42], cubic sets [19] and hybrid sets [1, 24].

On semigroups, Kuroki [27, 28] applied fuzzy sets to semigroups. Mordeson et al. [30] explained
semigroup theory to fuzzy semigroup theory and showed their applications in coding theory, languages
and fuzzy finite state machines. Shabir and Nawaz [37], Khan and Asif [25], Julatha and Siripitukdet
[17] studied anti-type of fuzzy sets based on ideal theory in semigroups. Chinnadurai and Arulselvam
[7] introduced Pythagorean fuzzy sets based on ideal theory in semigroups and investigated their
properties. Narayanan and Manikantan [33], and Thillaigovindan and Chinnadurai [40] studied interval-
valued fuzzy sets in semigroups. Jun and Khan [20], Umar et al. [43], and Muhiuddin [31] studied
cubic sets in semigroups. Anis et al. [1], Elavarasan et al. [9] studied hybrid sets in semigroups. Jun et
al. [21] and Talee et al. [39] studied hesitant fuzzy sets in semigroup. Studying hesitant fuzzy sets, in
the meaning of the supremum of their images, on semigroups, Jittburus and Julatha [12] introduced
sup-hesitant fuzzy ideals of semigroups and investigated properties via sets, fuzzy sets, interval-valued
fuzzy sets and hesitant fuzzy sets. Phummee et al. [35] introduced sup-hesitant fuzzy interior ideals of
semigroups and studied its properties by sets, fuzzy sets, interval-valued fuzzy sets and hesitant fuzzy
sets. Julatha et al. [13] introduced sup-hesitant fuzzy right (left) ideals of semigroups and studied
their characterizations in terms of sets, fuzzy sets, Pythagorean fuzzy sets, interval-valued fuzzy sets,
hesitant fuzzy sets, cubic sets and hybrid sets. Many researchers have taken intense and eager interest
in the novel area of hesitant fuzzy sets on algebraic structures in the meaning of the supremum of
their images (see [1, 10, 12, 14–16, 32, 35, 36, 38]).

As previously stated, it motivated us to study hesitant fuzzy sets on semigroups in the meaning of
the supremum of their images. We will introduce concepts of $\sup^+_\alpha$-hesitant fuzzy right (resp., left,
interior, two-sided) ideals and $\sup^-_\beta$-hesitant fuzzy right (resp., left, interior, two-sided) ideals and
investigate their properties. Also, we will show that every sup-hesitant fuzzy right (resp., left, interior,
two-sided) ideal of a semigroup is both a $\sup^+_\alpha$-hesitant fuzzy right (resp., left, interior, two-sided)
ideal and a $\sup^-_\beta$-hesitant fuzzy right (resp., left, interior, two-sided) ideal, but the converse is not true.
Later, the concepts will be established by fuzzy sets, Łukasiewicz fuzzy sets, Łukasiewicz anti-fuzzy
sets, Pythagorean fuzzy sets, hesitant fuzzy sets, hybrid sets, interval-valued fuzzy sets and cubic sets.
Finally, we will characterize which is intra-regular, left (right) regular, completely regular, left (right)
simple and simple semigroups in terms of $\sup^+_\alpha$-type and $\sup^-_\beta$-type of hesitant fuzzy sets.

2. Preliminaries

In this section we first give some basic definitions and results which will be used in this paper.
In what follows, unless otherwise specified, let \( A \) be a semigroup, \( B \) be a nonempty set, \( \wp(B) \) be the power set of \( B \) and \( \nabla, \blacktriangle \in \wp([0, 1]) \). A nonempty subset \( B \) of \( A \) is called a right ideal (resp., a left ideal, an interior ideal) of \( A \) if
\[
BA \subseteq B \quad \text{(resp., } AB \subseteq B, \ ABA \subseteq B) \]
and an ideal of \( A \) if \( B \) is both a right ideal and left ideal of \( A \).

A fuzzy subset (FS) \([46]\) of \( B \) is defined to be a function \( \xi : B \to [0, 1] \) where \([0, 1] \) is the unit interval. For FSs \( \xi \) and \( \eta \) of \( B \), define \( \xi \leq \eta \) if \( \xi(p) \leq \eta(p) \) for all \( p \in B \). A FS \( \xi \) of \( A \) is called
\begin{enumerate}
\item a fuzzy right ideal (FRI) \([30]\) of \( A \) if \( \xi(p) \leq \xi(pq) \) for all \( p, q \in A \),
\item a fuzzy left ideal (FLI) \([30]\) of \( A \) if \( \xi(q) \leq \xi(pq) \) for all \( p, q \in A \),
\item a fuzzy ideal (FI) \([30]\) of \( A \) if it is both a FRL and a FLI of \( A \), that is, \( \max\{\xi(p), \xi(q)\} \leq \xi(pq) \) for all \( p, q \in A \),
\item a fuzzy interior ideal (FII) \([30]\) of \( A \) if \( \xi(w) \leq \xi(pwq) \) for all \( p, q, w \in A \),
\item an anti-fuzzy right ideal (AFRI) \([37]\) of \( A \) if \( \xi(pq) \leq \xi(p) \) for all \( p, q \in A \),
\item an anti-fuzzy left ideal (AFLI) \([37]\) of \( A \) if \( \xi(pq) \leq \xi(q) \) for all \( p, q \in A \),
\item an anti-fuzzy ideal (AFI) \([37]\) of \( A \) if it is both an AFRI and an AFLI of \( A \), that is, \( \xi(pq) \leq \min\{\xi(p), \xi(q)\} \) for all \( p, q \in A \), and
\item an anti-fuzzy interior ideal (AFII) \([25]\) of \( A \) if \( \xi(pwq) \leq \xi(w) \) for all \( p, q, w \in A \).
\end{enumerate}

A Pythagorean fuzzy set (PFS) \( P \) \([44, 45]\) in \( B \) is an object having the form \( P = \{(p, \xi(p), \eta(p)) | p \in B\} \) where the functions \( \xi : B \to [0, 1] \) and \( \eta : B \to [0, 1] \) denote the degree of membership and the degree of nonmembership, respectively, and \( 0 \leq (\xi(p))^2 + (\eta(p))^2 \leq 1 \) for all \( p \in B \). For the sake of simplicity, we shall use the symbol \( (\xi, \eta) \) of the PFS \( \{(p, \xi(p), \eta(p)) | p \in B\} \). A PFS \( (\xi, \eta) \) in \( A \) is called
\begin{enumerate}
\item a Pythagorean fuzzy right ideal (PFRI) \([7]\) of \( A \) if \( \xi \) is a FRI and \( \eta \) is an AFRI of \( A \),
\item a Pythagorean fuzzy left ideal (PFLI) \([7]\) of \( A \) if \( \xi \) is a FLI and \( \eta \) is an AFLI of \( A \),
\item a Pythagorean fuzzy ideal (PFI) \([7]\) of \( A \) if it is both a PFRI and a PFLI of \( A \), and
\item a Pythagorean fuzzy interior ideal (PFII) \([7]\) of \( A \) if \( \xi \) is a FI and \( \eta \) is an AFII of \( A \).
\end{enumerate}

By an interval number \( \tilde{a} \) we mean an interval \([a^-, a^+]\), where \( a^-, a^+ \in [0, 1] \) and \( a^- \leq a^+ \). We denote \( \tilde{D}([0, 1]) \) for the set of all interval numbers. Then, we obtain \( \tilde{D}([0, 1]) \subseteq \wp([0, 1]) \). For \( \tilde{a} = [a^-, a^+] \), \( \tilde{b} = [b^-, b^+] \subseteq \tilde{D}([0, 1]) \), the operations \( \preceq, = \) and \( \preceq \) in case of two elements in \( \tilde{D}([0, 1]) \) are defined by:
\begin{enumerate}
\item \( \tilde{a} \preceq \tilde{b} \leftrightarrow a^- \leq b^- \) and \( a^+ \leq b^+ \),
\item \( \tilde{a} = \tilde{b} \leftrightarrow a^- = b^- \) and \( a^+ = b^+ \), and
\item \( \tilde{a} \prec \tilde{b} \leftrightarrow \tilde{a} \preceq \tilde{b} \) and \( \tilde{a} \neq \tilde{b} \).
\end{enumerate}

An interval-valued fuzzy set (IVFS) \([47]\) on \( B \) is defined to be a function \( \tilde{\lambda} : B \to \tilde{D}([0, 1]) \), \( \tilde{\lambda}(p) \mapsto [\tilde{\lambda}^L(p), \tilde{\lambda}^U(p)] \) where \( \tilde{\lambda}^L \) and \( \tilde{\lambda}^U \) are FSs of \( B \) such that \( \tilde{\lambda}^L \leq \tilde{\lambda}^U \). For FSs \( \xi \) and \( \eta \) of \( B \) with \( \xi \leq \eta \),
we define the IvFS $[\xi, \eta]$ on $B$ by $[\xi, \eta](p) = [\xi(p), \eta(p)]$ for all $p \in B$. An IvFS $\tilde{\lambda} = [\tilde{\lambda}_L, \tilde{\lambda}_U]$ on $A$ is called

1. an interval-valued fuzzy right ideal (IvFRI) \[33, 40\] of $A$ if $\tilde{\lambda}(p) \subseteq \tilde{\lambda}(pq)$ for all $p, q \in A$, that is, $\lambda_L$ and $\lambda_U$ are FRIs of $A$,
2. an interval-valued fuzzy left ideal (IvFLI) \[33, 40\] of $A$ if $\tilde{\lambda}(q) \subseteq \tilde{\lambda}(pq)$ for all $p, q \in A$, that is, $\lambda_L$ and $\lambda_U$ are FLIs of $A$,
3. an interval-valued fuzzy ideal (IvFI) \[33, 40\] of $A$ if it is both an IvFRI and an IvFLI of $A$, that is, $\lambda_L$ and $\lambda_U$ are FIs of $A$, and
4. an interval-valued fuzzy interior ideal (IvFII) \[40\] of $A$ if $\tilde{\lambda}(w) \subseteq \tilde{\lambda}(pwq)$ for all $p, q, w \in A$, that is, $\lambda_L$ and $\lambda_U$ are FIIs of $A$.

A cubic set \[19\] in $B$ is defined to be a function $\langle \tilde{\lambda}, \eta \rangle : B \to D([0, 1]) \times [0, 1], p \mapsto (\tilde{\lambda}(p), \eta(p))$ where $\tilde{\lambda} : B \to D([0, 1])$ and $\eta : B \to [0, 1]$. A cubic set $\langle \tilde{\lambda}, \eta \rangle$ in $A$ is called

1. a cubic right ideal (CuRI) \[20\] of $A$ if $\tilde{\lambda}$ is an IvFRI and $\eta$ is an AFRI of $A$,
2. a cubic left ideal (CuLI) \[20\] of $A$ if $\tilde{\lambda}$ is an IvFLI and $\eta$ is an AFLI of $A$,
3. a cubic ideal (CuI) \[31\] of $A$ if $\tilde{\lambda}$ is an IvFI and $\eta$ is an AFII of $A$.

A hesitant fuzzy set (HFS) \[41, 42\] on $B$ is defined to be a function $\tilde{\varepsilon} : B \to \wp([0, 1])$. Note that every IvFS on $B$ is a HFS on $B$. A HFS $\tilde{\varepsilon}$ on $A$ is called

1. a hesitant fuzzy right ideal (HFRI) \[21\] of $A$ if $\tilde{\varepsilon}(p) \subseteq \tilde{\varepsilon}(pq)$ for all $p, q \in A$,
2. a hesitant fuzzy left ideal (HFLI) \[21\] of $A$ if $\tilde{\varepsilon}(q) \subseteq \tilde{\varepsilon}(pq)$ for all $p, q \in A$,
3. a hesitant fuzzy ideal (HFI) \[12, 21\] of $A$ if it is both a HFRL and a HFLI of $A$, that is, $\tilde{\varepsilon}(p) \cup \tilde{\varepsilon}(q) \subseteq \tilde{\varepsilon}(pq)$ for all $p, q \in A$,
4. a hesitant fuzzy interior ideal (HFII) \[35, 39\] of $A$ if $\tilde{\varepsilon}(w) \subseteq \tilde{\varepsilon}(pwq)$ for all $p, q, w \in A$.

A hybrid set in $A$ over a set $B$ is defined to be a function $\langle \tilde{\varepsilon}, \eta \rangle : A \to \wp(B) \times [0, 1], p \mapsto (\tilde{\varepsilon}(p), \eta(p))$ where $\tilde{\varepsilon} : A \to \wp(B)$ and $\eta : A \to [0, 1]$. Note that every cubic set in $A$ is a hybrid set in $A$ over $[0, 1]$. A hybrid set $\langle \tilde{\varepsilon}, \eta \rangle$ in $A$ over $[0, 1]$ is called

1. a hybrid right ideal (HyRI) \[1\] of $A$ over $[0, 1]$ if $\tilde{\varepsilon}$ is a HFRI and $\eta$ is an AFRI of $A$,
2. a hybrid left ideal (HyLI) \[1\] of $A$ over $[0, 1]$ if $\tilde{\varepsilon}$ is a HFLI and $\eta$ is an AFLI of $A$,
3. a hybrid ideal (HyI) \[1\] of $A$ over $[0, 1]$ if it is both a HyRI and a HyLI of $A$ over $[0, 1]$, and
4. a hybrid interior ideal (HyII) \[13\] of $A$ over $[0, 1]$ if $\tilde{\varepsilon}$ is a HFII and $\eta$ is an AFII of $A$.

For a HFS $\tilde{\varepsilon}$ on $B$, a nonempty subset $\mathcal{Z}$ of $B$, $k \in [0, 1]$ and $\nabla \in \wp([0, 1])$, we define

1. the element SUP$\nabla$ \[12, 35\] of $[0, 1]$ by

$$\text{SUP} \nabla = \begin{cases} \sup \nabla & \text{if } \nabla \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases}$$

2. the subset $S[\tilde{\varepsilon}; \nabla]$ \[12, 35\] of $B$ by $S[\tilde{\varepsilon}; \nabla] = \{ p \in B \mid \text{SUP}\tilde{\varepsilon}(p) \geq \text{SUP}\nabla \}$. 
(3) the HFS $\mathcal{H}^{(\xi,\bigtriangledown)}_{\mathcal{S}^{\text{U}}}$ [14, 16] on $\mathcal{B}$ by $\mathcal{H}^{(\xi,\bigtriangledown)}_{\mathcal{S}^{\text{U}}} (p) = \{ k \in \bigtriangledown \mid \text{SUP}^{\xi} (p) \geq k \}$ for all $p \in \mathcal{B}$.

(4) the characteristic hesitant fuzzy set (CHFS) $\chi_{\mathcal{Z}}$ of $\mathcal{Z}$ on $\mathcal{B}$ by

$$\chi_{\mathcal{Z}} : \mathcal{B} \to \varphi([0,1]), p \mapsto \begin{cases} [0,1] & \text{if } p \in \mathcal{Z}, \\ \emptyset & \text{otherwise,} \end{cases}$$

(5) the FS $f^{\xi}$ of $\mathcal{B}$ by $f^{\xi} (p) = \text{SUP}^{\xi} (p)$ for all $p \in \mathcal{B}$,

(6) a supremum complement $\hat{\omega}$ [36] of $\hat{\xi}$ on $\mathcal{B}$ if $\hat{\omega}$ is a HFS on $\mathcal{B}$ such that $\text{SUP}^{\hat{\omega}} (p) = 1 - \text{SUP}^{\hat{\xi}} (p)$ for all $p \in \mathcal{B}$,

(7) the HFS $\hat{\xi}^*$ by $\hat{\xi}^* (p) = \{ 1 - \text{SUP}^{\hat{\xi}} (p) \}$ for all $p \in \mathcal{B}$.

Let $\mathcal{S}C(\hat{\xi})$ be the set of all supremum complements of $\hat{\xi}$. Then, we obtain that

1. $\hat{\xi}^* \in \mathcal{S}C(\hat{\xi})$,
2. $f^{\hat{\omega}} (p) = 1 - \text{SUP}^{\hat{\xi}} (p)$ for all $\hat{\omega} \in \mathcal{S}C(\hat{\xi})$ and $p \in \mathcal{B}$,
3. $\text{SUP}^{\hat{\xi}^*} (p) = \text{SUP}^{\hat{\xi}} (p) = f^{\xi} (p)$ for all $p \in \mathcal{B}$,
4. $\text{SUP}^{\hat{\omega}} (p) = 1 - (1 - \text{SUP}^{\hat{\xi}} (p)) = 1 - \text{SUP}^{\hat{\omega}} (p)$ for all $\hat{\omega} \in \mathcal{S}C(\hat{\xi})$ and $p \in \mathcal{B}$,
5. $\text{SUP}^{\hat{\xi}^*} (p) = \text{SUP}^{\hat{\xi}^*} (p) = \hat{\omega}^{\bigtriangledown} (p)$ for every IvFS $\hat{\omega}$ on $\mathcal{B}$ and for all $p \in \mathcal{B}$,
6. $\mathcal{H}^{(\xi,\{0,1\})}_{\mathcal{S}^{\text{U}}}$ is both a HFS and an IvFS on $\mathcal{B}$.

Jittburus and Julatha [12] introduced a sup-hesitant fuzzy ideal, which is a generalization of the concepts of an IvFI and a HFI, of a semigroup and studied its properties via sets, FSs, HFSs and IvFSs in the following.

**Definition 2.1.** [12] A HFS $\hat{\xi}$ on $\mathcal{A}$ is called a sup-hesitant fuzzy ideal of $\mathcal{A}$ related to $\bigtriangledown$ (briefly, $\bigtriangledown$-sup-hesitant fuzzy ideal) of $\mathcal{A}$ if the set $\mathcal{S}[\hat{\xi}; \bigtriangledown]$ is an ideal of $\mathcal{A}$. We say that $\hat{\xi}$ is a sup-hesitant fuzzy ideal (sup-HFI) of $\mathcal{A}$ if $\hat{\xi}$ is a $\bigtriangledown$-sup-hesitant fuzzy ideal of $\mathcal{A}$ for all $\bigtriangledown \in \varphi([0,1])$ when $\mathcal{S}[\hat{\xi}; \bigtriangledown] \neq \emptyset$.

**Theorem 2.1.** [12] Every HFI of $\mathcal{A}$ is a sup-HFI of $\mathcal{A}$.

**Theorem 2.2.** [12] Every IvFI of $\mathcal{A}$ is a sup-HFI of $\mathcal{A}$.

**Theorem 2.3.** [12] Let $\hat{\xi}$ be a HFS on $\mathcal{A}$. The follows are equivalent:

1. $\hat{\xi}$ is a sup-HFI of $\mathcal{A}$,
2. $f^{\xi}$ is a FI of $\mathcal{A}$,
3. $\text{SUP}^{\hat{\xi}} (pq) \geq \max\{\text{SUP}^{\hat{\xi}} (p), \text{SUP}^{\hat{\xi}} (q)\}$ for all $p, q \in \mathcal{A}$,
4. $\mathcal{H}^{(\xi,\bigtriangledown)}_{\mathcal{S}^{\text{U}}}$ is a HFI of $\mathcal{A}$ for all $\bigtriangledown \in \varphi([0,1])$.

**Theorem 2.4.** [12] Let $\mathcal{B}$ be a nonempty subset of $\mathcal{A}$. Then $\mathcal{B}$ is an ideal of $\mathcal{A}$ if and only if the CHFS $\chi_\mathcal{B}$ is a sup-HFI of $\mathcal{A}$.

Phummee et al. [35] introduced a sup-hesitant fuzzy interior ideal, shown a generalization of the concepts of a sup-HFI, an IvFII and a HFII, of a semigroup and studied its properties via sets, FSs, HFSs and IvFSs.
Definition 2.2. [35] A HFS $\hat{\varepsilon}$ on $A$ is called a sup-hesitant fuzzy interior ideal of $A$ related to $\nabla$ (briefly, $\nabla$-sup-hesitant fuzzy interior ideal) of $A$ if the set $S[\hat{\varepsilon}; \nabla]$ is an interior ideal of $A$. We say that $\hat{\varepsilon}$ is a sup-hesitant fuzzy interior ideal (sup-HFII) of $A$ if $\hat{\varepsilon}$ is a $\nabla$-sup-hesitant fuzzy interior ideal of $A$ for all $\nabla \in \wp([0, 1])$ when $S[\hat{\varepsilon}; \nabla] \neq \emptyset$.

Theorem 2.5. [35] Every sup-HFI of $A$ is a sup-HFII of $A$.

Theorem 2.6. [35] Every HFII of $A$ is a sup-HFII of $A$.

Theorem 2.7. [35] Every IvFII of $A$ is a sup-HFII of $A$.

Theorem 2.8. [35] Let $\hat{\varepsilon}$ be a HFS on $A$. The followings are equivalent:

1. $\hat{\varepsilon}$ is a sup-HFII of $A$,
2. $\hat{\varepsilon}$ is a FII of $A$,
3. $\text{SUP} \hat{\varepsilon}(pq) \geq \text{SUP} \hat{\varepsilon}(w)$ for all $p, q, w \in A$,
4. $H_{\text{SUP}}(\hat{\varepsilon}, \nabla)$ is a HFII of $A$ for all $\nabla \in \wp([0, 1])$.

Theorem 2.9. [35] If $B$ is a nonempty subset of $A$, then $B$ is an interior ideal of $A$ if and only if $\hat{\chi}_B$ is a sup-HFII of $A$.

Julatha et al. [13] introduced a sup-hesitant fuzzy right (left) ideal, shown a generalization of the concept of a HFRI (HFLI) and an IvFRI (IvFLI), of a semigroup and studied its properties via sets, FSs, PFSs, HFSs, IvFSs, cubic sets and hybrid sets.

Definition 2.3. [13] Let $\hat{\varepsilon}$ be a HFS on $A$.

1. $\hat{\varepsilon}$ is called a sup-hesitant fuzzy left ideal (sup-HFLI) of $A$ if $\forall p, q \in A \left(\text{SUP} \hat{\varepsilon}(q) \leq \text{SUP} \hat{\varepsilon}(pq)\right)$.
2. $\hat{\varepsilon}$ is called a sup-hesitant fuzzy right ideal (sup-HFRI) of $A$ if $\forall p, q \in A \left(\text{SUP} \hat{\varepsilon}(p) \leq \text{SUP} \hat{\varepsilon}(pq)\right)$.

Theorem 2.10. [13] Let $\hat{\varepsilon}$ be a HFS on $A$. The followings are equivalent:

1. $\hat{\varepsilon}$ is a sup-HFRI (sup-HFLI) of $A$,
2. $\hat{\varepsilon}$ is a FRI (FLI) of $A$,
3. $H_{\text{SUP}}(\hat{\varepsilon}, \nabla)$ is a HFRI (HFLI) of $A$ for all $\nabla \in \wp([0, 1])$.

Theorem 2.11. [13] If $B$ is a nonempty subset of $A$, then $B$ is a right ideal (resp., left ideal) of $A$ if and only if $\hat{\chi}_B$ is a sup-HFRI (resp., sup-HFLI) of $A$.

3. Generalized sup-hesitant fuzzy ideals

In what follows, let $\alpha$ and $\beta$ be elements of $[0, 1]$, unless otherwise specified. We introduce the concepts of $\text{sup}^+_\alpha$-hesitant fuzzy left (resp., right, interior, two-sided) ideals and $\text{sup}^-_{\beta}$-hesitant fuzzy left
We denote \( \sup \) (resp., right, interior, two-sided) ideal of a semigroup as a generalization of the concept of a \( \sup \) of a semigroup, and investigate some of their properties. Also, it is shown that a \( \sup \) hesitant fuzzy left ideal, a \( \sup \) hesitant fuzzy right ideal, a \( \sup \) hesitant fuzzy interior ideal and a \( \sup \) hesitant fuzzy two-sided ideal of a semigroup, and investigate some of their properties. Also, it is shown that a \( \sup \) hesitant fuzzy left (resp., right, interior, two-sided) ideal of a semigroup is a generalization of the concept of a \( \sup \) hesitant fuzzy left (resp., right, interior, two-sided) ideal.

For a HFS \( \hat{\varepsilon} \) on \( A \), and \( \nabla, \Delta \in \wp([0, 1]) \), we define

1. \( \sup_0^+ \nabla = \min\{\sup_0^+ \nabla + \alpha, 1\} \),
2. \( \nabla \subseteq_0^+ \Delta \) if and only if \( \sup_0^+ \nabla \leq \sup_0^+ \Delta \),
3. \( \nabla \subseteq_0^+ \Delta \) if and only if \( \sup_0^+ \nabla < \sup_0^+ \Delta \),
4. \( \nabla \cong_0^+ \Delta \) if and only if \( \sup_0^+ \nabla = \sup_0^+ \Delta \).

We denote \( \nabla \subseteq \Delta \) (resp., \( \nabla \sqsubseteq_0^+ \Delta \), \( \nabla \cong_0^+ \Delta \)) for \( \nabla \subseteq_0^+ \Delta \) (resp., \( \nabla \subseteq_0^+ \Delta \), \( \nabla \cong_0^+ \Delta \)). Then we have

1. \( \sup_0^+ \nabla = \sup \nabla \),
2. \( \nabla \subseteq_0^+ \Delta \) if and only if \( \sup \nabla \leq \sup \Delta \),
3. \( \nabla \subseteq_0^+ \Delta \) if and only if \( \sup \nabla < \sup \Delta \),
4. \( \nabla \cong_0^+ \Delta \) if and only if \( \sup \nabla = \sup \Delta \),
5. \( \nabla \cong_0^+ \Delta \) if and only if \( \nabla \subseteq_0^+ \Delta \) and \( \Delta \subseteq_0^+ \nabla \).

For elements \( \tilde{a} = [a^-, a^+] \) and \( \tilde{b} = [b^-, b^+] \) in \( \mathcal{D}([0, 1]) \), then the following are true:

1. if \( \tilde{a} \preceq \tilde{b} \), then \( \tilde{a} \subseteq \tilde{b} \), and
2. if \( \tilde{a} = \tilde{b} \), then \( \tilde{a} \cong \tilde{b} \).

**Definition 3.1.** A HFS \( \hat{\varepsilon} \) on \( A \) is called

1. a sup\(^+\) hesitant fuzzy right ideal (sup\(^+\) HFRI) of \( A \) if \( (\forall p, q \in A)(\hat{\varepsilon}(p) \subseteq_0^+ \hat{\varepsilon}(pq)) \),
2. a sup\(^+\) hesitant fuzzy left ideal (sup\(^+\) HFLI) of \( A \) if \( (\forall p, q \in A)(\hat{\varepsilon}(q) \subseteq_0^+ \hat{\varepsilon}(pq)) \),
3. a sup\(^+\) hesitant fuzzy two-sided ideal (or a sup\(^+\) hesitant fuzzy ideal (sup\(^+\) HFI)) of \( A \) if it is both a sup\(^+\) HFRI and a sup\(^+\) HFLI of \( A \),
4. a sup\(^+\) hesitant fuzzy interior ideal (sup\(^+\) HI) of \( A \) if \( (\forall p, q, w \in A)(\hat{\varepsilon}(w) \subseteq_0^+ \hat{\varepsilon}(pwq)) \).

**Example 3.1.** Let \( A = \{(1, 1), (0, 1), (0, 0), (1, 0)\} \). Then \( A \) is a semigroup with respect to multiplication defined as follows: \( (p_1, p_2)(p_3, p_4) = (p_1, p_4) \) for all \( p_1, p_2, p_3, p_4 \in \{0, 1\} \).

1. A HFS \( \hat{\varepsilon}_1 \) of \( A \) is defined by

\[
\hat{\varepsilon}_1((0, 0)) = (0, 0.8), \hat{\varepsilon}_1((0, 1)) = \{0.2, 0.4, 0.9\}, \hat{\varepsilon}_1((1, 0)) = \emptyset \text{ and } \hat{\varepsilon}_1((1, 1)) = \{0\},
\]

Then \( \hat{\varepsilon}_1 \) is a sup\(^+\) HFRI of \( A \) but not a sup\(^+\) HFLI of \( A \) because

\[
\hat{\varepsilon}_1((1, 0)(0, 0)) \subseteq_0^+ \hat{\varepsilon}_1((0, 0)).
\]
(2) A HFS \( \hat{\varepsilon}_2 \) of \( \mathcal{A} \) is defined by
\[
\hat{\varepsilon}_2((0, 0)) = \{0.2, 0.4, 0.5\}, \ \hat{\varepsilon}_2((0, 1)) = \{0.3, 0.7\}, \ \hat{\varepsilon}_2((1, 0)) = \{0, 0.5\} \text{ and } \hat{\varepsilon}_2((1, 1)) = \{0.7, 0.8, 0.9\}.
\]

Then \( \hat{\varepsilon}_2 \) is a sup\(^+_\alpha\)-HFLI of \( \mathcal{A} \) but not a sup\(^+_\alpha\)-HFRI of \( \mathcal{A} \) because
\[
\hat{\varepsilon}_2((1, 1)(1, 0)) \sqsubseteq \hat{\varepsilon}_2((1, 1)).
\]

**Example 3.2.** Let \( \mathcal{A} = \{p_1, p_2, p_3, p_4\} \) and define the binary operation “\( \cdot \)” on \( \mathcal{A} \) as follows:

\[
\begin{array}{c|cccc}
\cdot & p_1 & p_2 & p_3 & p_4 \\
\hline
p_1 & p_1 & p_1 & p_1 & p_1 \\
p_2 & p_1 & p_1 & p_1 & p_1 \\
p_3 & p_1 & p_1 & p_2 & p_1 \\
p_4 & p_1 & p_1 & p_2 & p_2 \\
\end{array}
\]

Then \( \mathcal{A} \) is a be the semigroup under the binary operation “\( \cdot \)” [30]. Now, define HFSs \( \hat{\varepsilon}_1 \) and \( \hat{\varepsilon}_2 \) on \( \mathcal{A} \) by
\[
\hat{\varepsilon}_1(p_1) = [0.3, 0.6], \ \hat{\varepsilon}_1(p_2) = \{0.3, 0.5\}, \ \hat{\varepsilon}_1(p_3) = \emptyset \text{ and } \hat{\varepsilon}_1(p_4) = \{1\},
\]
\[
\hat{\varepsilon}_2(p_1) = [0.3, 0.7], \ \hat{\varepsilon}_2(p_2) = \{0.3, 0.5, 0.8\}, \ \hat{\varepsilon}_2(p_3) = \emptyset \text{ and } \hat{\varepsilon}_2(p_4) = \{0.2, 0.5\}.
\]

Thus
\[
\begin{align*}
(1) & \ \hat{\varepsilon}_1 \text{ is a sup}^+\_0\_4\text{-HFLI but not a sup}^+\_0\_4\text{-HFI of } \mathcal{A} \text{ because } \hat{\varepsilon}_1(p_4p_4) \nsubseteq \hat{\varepsilon}_1(p_4), \\
(2) & \ \hat{\varepsilon}_2 \text{ is a sup}^+\_0\_3\text{-HFI of } \mathcal{A}.
\end{align*}
\]

**Proposition 3.1.** Every sup-HFRI (resp., sup-HFLI, sup-HFII, sup-HFI) of \( \mathcal{A} \) is a sup\(_\alpha\)-HFRI (resp., sup\(_\alpha\)-HFLI, sup\(_\alpha\)-HFII, sup\(_\alpha\)-HFI) of \( \mathcal{A} \) for all \( \alpha \in [0, 1] \).

**Proof.** Assume that \( \hat{\varepsilon} \) is a sup-HFRI of \( \mathcal{A}, \alpha \in [0, 1] \) and \( p, q \in \mathcal{A} \). Then \( \text{SUP}^+\hat{\varepsilon}(pq) \geq \text{SUP}^+\hat{\varepsilon}(p) \) and so
\[
\text{SUP}^+\alpha\hat{\varepsilon}(pq) = \min\{\text{SUP}^+\hat{\varepsilon}(pq) + \alpha, 1\} \geq \min\{\text{SUP}^+\hat{\varepsilon}(p) + \alpha, 1\} = \text{SUP}^+\alpha\hat{\varepsilon}(p).
\]

Hence \( \hat{\varepsilon}(p) \nsubseteq \hat{\varepsilon}(pq) \). Therefore, we obtain that \( \hat{\varepsilon} \) is a sup\(_\alpha\)-HFRI of \( \mathcal{A} \).

Similarly, we can prove the other results. \( \square \)

**Example 3.3.** Let \( \mathcal{A} = \{p_1, p_2, p_3, p_4\} \) be the semigroup defined in Example 3.2. We define a HFS \( \hat{\varepsilon} \) on \( \mathcal{A} \) by
\[
\hat{\varepsilon}(p_1) = \{0.1, 0.5, 0.8\}, \ \hat{\varepsilon}(p_2) = [0, 0.9], \ \hat{\varepsilon}(p_3) = [0, 0.5] \text{ and } \hat{\varepsilon}(p_4) = \emptyset.
\]

Then \( \hat{\varepsilon} \) is a sup\(_\alpha\)-HFRI (resp., sup\(_\alpha\)-HFLI, sup\(_\alpha\)-HFII, sup\(_\alpha\)-HFI) of \( \mathcal{A} \) for all \( \alpha \in [0.2, 1] \) but \( \hat{\varepsilon} \) is not a sup-HFRI (resp., sup-HFLI, sup-HFII, sup-HFI) of \( \mathcal{A} \). Indeed, \( \hat{\varepsilon} \) is not a sup-HFRI and sup-HFI of \( \mathcal{A} \) because \( \text{SUP}^+\hat{\varepsilon}(p_2p_1) < \text{SUP}^+\hat{\varepsilon}(p_2) \), \( \hat{\varepsilon} \) is not a sup-HFII of \( \mathcal{A} \) because \( \text{SUP}^+\hat{\varepsilon}(p_1p_2) < \text{SUP}^+\hat{\varepsilon}(p_2) \), and \( \hat{\varepsilon} \) is not a sup-HFII of \( \mathcal{A} \) because \( \text{SUP}^+\hat{\varepsilon}(p_1p_2p_3) < \text{SUP}^+\hat{\varepsilon}(p_2) \).
From Proposition 3.1 and Example 3.3, we have that the concept of a $\sup^+_{\alpha}$-HFRI (resp., $\sup^+_{\alpha}$-HFLI, $\sup^+_{\alpha}$-HFII, $\sup^+_{\alpha}$-HFI) of a semigroup $\mathcal{A}$ is a generalization of the concept of a $\sup^-$HFRI (resp., $\sup^-$HFLI, $\sup^-$HFII, $\sup^-$HFI) of $\mathcal{A}$.

**Proposition 3.2.** Let $\hat{\varepsilon}$ be a HFS on $\mathcal{A}$ and $k \in [0, 1]$. If $\hat{\varepsilon}$ is a $\sup^+_{\alpha}$-HFRI (resp., $\sup^+_{\alpha}$-HFLI, $\sup^+_{\alpha}$-HFII, $\sup^+_{\alpha}$-HFI) of $\mathcal{A}$ for all $\alpha \in [0, k]$, then $\hat{\varepsilon}$ is a $\sup^-$HFRI (resp., $\sup^-$HFLI, $\sup^-$HFII, $\sup^-$HFI) of $\mathcal{A}$.

**Proof.** Let $\hat{\varepsilon}$ be a $\sup^+_{\alpha}$-HFRI of $\mathcal{A}$ for all $\alpha \in [0, k]$. Suppose $\hat{\varepsilon}$ is not a $\sup^-$HFRI of $\mathcal{A}$, that is, there exist $p, q \in \mathcal{A}$ such that $\sup \hat{\varepsilon}(pq) < \sup \hat{\varepsilon}(p)$. Choose

$$\alpha = \min\left\{ \frac{\sup \hat{\varepsilon}(p) - \sup \hat{\varepsilon}(pq)}{2}, k \right\}.$$  

Then $\alpha \in [0, k]$ and

$$\begin{align*}
\sup \hat{\varepsilon}(pq) + \alpha &\leq \sup \hat{\varepsilon}(pq) + \left( \frac{\sup \hat{\varepsilon}(p) - \sup \hat{\varepsilon}(pq)}{2} \right) \\
&< \sup \hat{\varepsilon}(pq) + (\sup \hat{\varepsilon}(p) - \sup \hat{\varepsilon}(pq)) \\
&= \sup \hat{\varepsilon}(p) \\
&\leq 1.
\end{align*}$$  

Thus

$$\begin{align*}
\sup^+_{\alpha} \hat{\varepsilon}(p) &= \min\{ \sup \hat{\varepsilon}(p) + \alpha, 1 \} \\
&> \sup \hat{\varepsilon}(pq) + \alpha \\
&= \min\{ \sup \hat{\varepsilon}(pq) + \alpha, 1 \} \\
&= \sup^+_{\alpha} \hat{\varepsilon}(pq).
\end{align*}$$

Hence $\hat{\varepsilon}(pq) \sqsubseteq^+ \hat{\varepsilon}(p)$. Since $\hat{\varepsilon}$ is a $\sup^+_{\alpha}$-HFRI of $\mathcal{A}$, we have

$$\hat{\varepsilon}(pq) \sqsubseteq^+ \hat{\varepsilon}(p) \sqsubseteq^+ \hat{\varepsilon}(pq).$$

This is a contradiction. Therefore, $\hat{\varepsilon}$ is a $\sup^-$HFRI of $\mathcal{A}$.

Similarly, we can prove the other results. \(\square\)

**Proposition 3.3.** If $\hat{\varepsilon}$ is a $\sup^+_{\alpha}$-HFRI (resp., $\sup^+_{\alpha}$-HFLI, $\sup^+_{\alpha}$-HFII, $\sup^+_{\alpha}$-HFI) of $\mathcal{A}$, then $\hat{\varepsilon}$ is a $\sup^+_{k}$-HFRI (resp., $\sup^+_{k}$-HFLI, $\sup^+_{k}$-HFII, $\sup^+_{k}$-HFI) of $\mathcal{A}$ for all $k \in [\alpha, 1]$.

**Proof.** Assume that $\hat{\varepsilon}$ is a $\sup^+_{\alpha}$-HFRI of $\mathcal{A}$, $k \in [\alpha, 1]$ and $p, q \in \mathcal{A}$. Then $\hat{\varepsilon}(p) \sqsubseteq^+ \hat{\varepsilon}(pq)$, that is, $\min\{ \sup \hat{\varepsilon}(pq) + \alpha, 1 \} \geq \min\{ \sup \hat{\varepsilon}(p) + \alpha, 1 \}$. If $\sup \hat{\varepsilon}(pq) + \alpha \geq 1$, then

$$\sup \hat{\varepsilon}(pq) + k \geq \sup \hat{\varepsilon}(pq) + \alpha \geq 1 \geq \sup^+_{k} \hat{\varepsilon}(p)$$
and so $\bar{\varepsilon}(p) \supseteq_k^+ \bar{\varepsilon}(pq)$. On the other hand, suppose that $\sup \bar{\varepsilon}(pq) + \alpha \geq \sup \bar{\varepsilon}(p) + \alpha$. Then $\sup \bar{\varepsilon}(pq) \geq \sup \bar{\varepsilon}(p)$ and so

$$\sup \bar{\varepsilon}(pq) + k \geq \sup \bar{\varepsilon}(p) + k \geq \sup_k^+ \bar{\varepsilon}(p).$$

Thus $\bar{\varepsilon}(p) \supseteq_k^+ \bar{\varepsilon}(pq)$. Therefore, $\bar{\varepsilon}$ is a $\sup_k^+$-HFRI of $\mathcal{A}$.

Similarly, we can prove the other results. \hfill $\Box$

**Proposition 3.4.** Every $\sup_\alpha^+$-HFI of $\mathcal{A}$ is a $\sup_\alpha^+$-HFII of $\mathcal{A}$.

**Proof.** Assume that $\bar{\varepsilon}$ is a $\sup_\alpha^+$-HFI of $\mathcal{A}$. Then $\bar{\varepsilon}(w) \supseteq_\alpha^+ \bar{\varepsilon}(wq) \supseteq_\alpha^+ \bar{\varepsilon}(pwq)$ for all $p, q, w \in \mathcal{A}$. Therefore, $\bar{\varepsilon}$ is a $\sup_\alpha^+$-HFII of $\mathcal{A}$. \hfill $\Box$

From Proposition 3.4 and Example 3.2, we have that the concept of a $\sup_\alpha^+$-HFII of a semigroup $\mathcal{A}$ is a generalization of the concept of a $\sup_\alpha^+$-HFI of $\mathcal{A}$.

### 3.2. $\sup_\beta^-$-hesitant fuzzy ideals

In this part, we introduce a $\sup_\beta^-$-hesitant fuzzy right ideal, a $\sup_\beta^-$-hesitant fuzzy left ideal, a $\sup_\beta^-$-hesitant fuzzy interior ideal and a $\sup_\beta^-$-hesitant fuzzy two-sided ideal of a semigroup, and investigate some of their properties. Moreover, it is shown that a $\sup_\beta^-$-hesitant fuzzy left (resp., right, interior, two-sided) ideal of a semigroup is a generalization of the concept of a $\sup$-hesitant fuzzy left (resp., right, interior, two-sided) ideal.

For a HFS $\bar{\varepsilon}$ on $\mathcal{A}$ and $\nabla, \Delta \in \wp([0, 1])$, we define

1. $\sup_\beta^- \nabla = \max\{\sup \nabla - \beta, 0\}$,
2. $\nabla \subseteq_\beta^- \Delta$ if and only if $\sup_\beta^- \nabla \subseteq \sup_\beta^- \Delta$,
3. $\nabla \subseteq_\beta \Delta$ if and only if $\sup_\beta^- \nabla < \sup_\beta^- \Delta$,
4. $\nabla \approx_\beta^- \Delta$ if and only if $\sup_\beta^- \nabla = \sup_\beta^- \Delta$.

Then we have

1. $\sup_0^- \nabla = \sup \nabla$,
2. $\nabla \subseteq \Delta$ if and only if $\nabla \subseteq_0^- \Delta$,
3. $\nabla \subseteq \Delta$ if and only if $\nabla \subseteq_0^- \Delta$,
4. $\nabla \approx \Delta$ if and only if $\nabla \approx_0^- \Delta$.

**Definition 3.2.** Let $\bar{\varepsilon}$ be a HFS on $\mathcal{A}$.

1. $\bar{\varepsilon}$ is called a $\sup_\beta^-$-hesitant fuzzy right ideal ($\sup_\beta^-$-HFRI) of $\mathcal{A}$ if $(\forall p, q \in \mathcal{A})(\bar{\varepsilon}(p) \subseteq_\beta^- \bar{\varepsilon}(pq))$.
2. $\bar{\varepsilon}$ is called a $\sup_\beta^-$-hesitant fuzzy left ideal ($\sup_\beta^-$-HFLI) of $\mathcal{A}$ if $(\forall p, q \in \mathcal{A})(\bar{\varepsilon}(q) \subseteq_\beta^- \bar{\varepsilon}(pq))$.
3. $\bar{\varepsilon}$ is called a $\sup_\beta^-$-hesitant fuzzy two-sided ideal (or a $\sup_\beta^-$-hesitant fuzzy ideal ($\sup_\beta^-$-HFI)) of $\mathcal{A}$ if it is both a $\sup_\beta^-$-HFRI and a $\sup_\beta^-$-HFII of $\mathcal{A}$.
4. $\bar{\varepsilon}$ is called a $\sup_\beta^-$-hesitant fuzzy interior ideal ($\sup_\beta^-$-HFII) of $\mathcal{A}$ if $(\forall p, q, w \in \mathcal{A})(\bar{\varepsilon}(w) \subseteq_\beta^- \bar{\varepsilon}(pwq))$.

**Example 3.4.** Let $\mathcal{A} = \{(1, 1), (0, 1), (0, 0), (1, 0)\}$ be a semigroup defined in Example 3.1.
Proposition 3.5. Every $\hat{\epsilon}$ is a sup$_{\beta}^{-}$-HFI of $A$ for all $\beta \in [0, 1]$ but not a sup$_{k}^{-}$-HFI of $A$ because $\hat{\epsilon}(p_4) \subset_k \hat{\epsilon}(p_4)$ for all $k \in [0, 0.7)$.

(2) $\hat{\epsilon}$ is a sup$_{\beta}^{-}$-HFI of $A$ for all $\beta \in [0.7, 1]$.

Example 3.5. Let $A = \{p_1, p_2, p_3, p_4\}$ be the semigroup defined in Example 3.2. We define a HFS $\hat{\epsilon}$ on $A$ by

$$\hat{\epsilon}(p_1) = [0.3, 0.7], \hat{\epsilon}(p_2) = [0.3, 0.5], \hat{\epsilon}(p_3) = \emptyset, \text{ and } \hat{\epsilon}(p_4) = (0, 0.7).$$

Thus

(1) $\hat{\epsilon}$ is a sup$_{\beta}^{-}$-HFI of $A$ for all $\beta \in [0, 1]$ but not a sup$_{k}^{-}$-HFI of $A$ for all $k \in [0, 0.7)$ because $\hat{\epsilon}(p_4) \subset_k \hat{\epsilon}(p_4)$ for all $k \in [0, 0.7)$.

(2) $\hat{\epsilon}$ is a sup$_{\beta}^{-}$-HFI of $A$ for all $\beta \in [0.7, 1]$.

Proposition 3.5. Every sup-HFRI (resp., sup-HFLI, sup-HFII, sup-HFI) of $A$ is a sup$_{\beta}^{-}$-HFI (resp., sup$_{\beta}^{-}$-HFLI, sup$_{\beta}^{-}$-HFII, sup$_{\beta}^{-}$-HFI) of $A$ for all $\beta \in [0, 1]$.

Proof. Assume that $\hat{\epsilon}$ is a sup-HFRI of $A$, $\beta \in [0, 1]$ and $p, q \in A$. Then $\sup \hat{\epsilon}(pq) \geq \sup \hat{\epsilon}(p)$ and thus

$$\sup_{\beta} \hat{\epsilon}(pq) = \max \{\sup \hat{\epsilon}(pq) - \beta, 0\} \geq \max \{\sup \hat{\epsilon}(p) - \beta, 0\} = \sup_{\beta} \hat{\epsilon}(p).$$

Hence $\hat{\epsilon}(p) \subset_{\beta} \hat{\epsilon}(pq)$. Therefore, $\hat{\epsilon}$ is a sup$_{\beta}^{-}$-HFI of $A$.

Similarly, we can prove the other results. 

Example 3.6. From Example 3.3, we get that the HFS $\hat{\epsilon}$ is a sup$_{\beta}^{-}$-HFI, sup$_{\beta}^{-}$-HFLI, sup$_{\beta}^{-}$-HFII and sup$_{\beta}^{-}$-HFI of $A$ for all $\beta \in [0.3, 1]$. However, $\hat{\epsilon}$ is not a sup-HFRI, sup-HFLI, sup-HFII and sup-HFI of $A$.

From Example 3.6 and Proposition 3.5, we obtain that the concept of a sup$_{\beta}^{-}$-HFI (resp., sup$_{\beta}^{-}$-HFLI, sup$_{\beta}^{-}$-HFII, sup$_{\beta}^{-}$-HFI) of a semigroup $A$ is a generalization of the concept of a sup-HFRI (resp., sup-HFLI, sup-HFII, sup-HFI) of $A$.

Proposition 3.6. Let $\hat{\epsilon}$ be a HFS on $A$ and $k \in [0, 1]$. If $\hat{\epsilon}$ is a sup$_{\beta}^{-}$-HFI (resp., sup$_{\beta}^{-}$-HFLI, sup$_{\beta}^{-}$-HFII, sup$_{\beta}^{-}$-HFI) of $A$ for all $\beta \in [0, k]$, then $\hat{\epsilon}$ is a sup-HFRI (resp., sup-HFLI, sup-HFII, sup-HFI) of $A$. 

Proof. Let \( \hat{\epsilon} \) be a sup\(^{-}\)-HFRI of \( \mathcal{A} \) for all \( \beta \in [0, k] \). Suppose that \( \text{SUP} \hat{\epsilon}(pq) < \text{SUP} \hat{\epsilon}(p) \) for some \( p, q \in \mathcal{A} \). Choose \( \beta \in [0, k] \) such that \( \text{SUP} \hat{\epsilon}(p) - \beta > 0 \). Then
\[
\text{SUP} \hat{\epsilon}(p) = \max\{\text{SUP} \hat{\epsilon}(p) - \beta, 0\} = \text{SUP} \hat{\epsilon}(p) - \beta.
\]
Since \( \hat{\epsilon} \) is a sup\(^{-}\)-HFRI of \( \mathcal{A} \), we have
\[
\text{SUP} \hat{\epsilon}(pq) \geq \text{SUP} \hat{\epsilon}(p) = \text{SUP} \hat{\epsilon}(p) - \beta > 0.
\]
Thus
\[
\text{SUP} \hat{\epsilon}(pq) = \text{SUP} \hat{\epsilon}(p) - \beta < \text{SUP} \hat{\epsilon}(p) - \beta = \text{SUP} \hat{\epsilon}(p) \leq \text{SUP} \hat{\epsilon}(pq),
\]
which is a contradiction. Hence \( \hat{\epsilon} \) is a sup-HFRI of \( \mathcal{A} \).

Similarly, we can prove the other results. \( \square \)

Proposition 3.7. If \( \hat{\epsilon} \) is a sup\(^{-}\)-HFRI (resp., sup\(^{-}\)-HFLI, sup\(^{-}\)-HFII, sup\(^{-}\)-HFI) of \( \mathcal{A} \), then \( \hat{\epsilon} \) is a sup\(^{-}\)-HFRI (resp., sup\(^{-}\)-HFLI, sup\(^{-}\)-HFII, sup\(^{-}\)-HFI) of \( \mathcal{A} \) for all \( k \in [\beta, 1] \).

Proof. Assume that \( \hat{\epsilon} \) is a sup\(^{-}\)-HFRI of \( \mathcal{A} \), \( k \in [\beta, 1] \) and \( p, q \in \mathcal{A} \). Then \( \hat{\epsilon}(p) \subseteq \hat{\epsilon}(pq) \), that is,
\[
\max\{\text{SUP} \hat{\epsilon}(pq) - \beta, 0\} \geq \max\{\text{SUP} \hat{\epsilon}(p) - \beta, 0\} \geq \text{SUP} \hat{\epsilon}(p) - \beta.
\]
If \( 0 \geq \text{SUP} \hat{\epsilon}(p) - \beta \), then
\[
\text{SUP} \hat{\epsilon}(pq) \geq 0 \geq \text{SUP} \hat{\epsilon}(p) - \beta \geq \text{SUP} \hat{\epsilon}(p) - k
\]
and so \( \text{SUP} \hat{\epsilon}(pq) \geq \text{SUP} \hat{\epsilon}(p) \), which implies that \( \hat{\epsilon}(p) \subseteq \hat{\epsilon}(pq) \). On the other hand, suppose that \( \text{SUP} \hat{\epsilon}(pq) - \beta \geq \text{SUP} \hat{\epsilon}(p) - \beta \). Then \( \text{SUP} \hat{\epsilon}(pq) \geq \text{SUP} \hat{\epsilon}(p) \). Thus \( \text{SUP} \hat{\epsilon}(pq) - k \geq \text{SUP} \hat{\epsilon}(p) - k \). Hence \( \text{SUP} \hat{\epsilon}(pq) \geq \text{SUP} \hat{\epsilon}(p) \), which implies that \( \hat{\epsilon}(p) \subseteq \hat{\epsilon}(pq) \). Therefore, \( \hat{\epsilon} \) is a sup\(^{-}\)-HFRI of \( \mathcal{A} \).

Similarly, we can prove the other results. \( \square \)

Proposition 3.8. Every sup\(^{-}\)-HFI of \( \mathcal{A} \) is a sup\(^{-}\)-HFII of \( \mathcal{A} \).

Proof. Assume that \( \hat{\epsilon} \) is a sup\(^{-}\)-HFI of \( \mathcal{A} \). Then \( \hat{\epsilon}(w) \subseteq \hat{\epsilon}(wq) \subseteq \hat{\epsilon}(pwq) \) for all \( p, q, w \in \mathcal{A} \). Therefore, \( \hat{\epsilon} \) is a sup\(^{-}\)-HFII of \( \mathcal{A} \). \( \square \)

From Proposition 3.8 and Example 3.5, we have that the concept of a sup\(^{-}\)-HFII of a semigroup \( \mathcal{A} \) is a generalization of the concept of a sup\(^{-}\)-HFI of \( \mathcal{A} \).

Proposition 3.9. Let \( \hat{\epsilon} \) be a HFS on \( \mathcal{A} \). Then the followings are true:
Proof. It follows from Proposition 3.6.

**Proposition 3.10.** Let \( \hat{\varepsilon} \) be a HFS on \( A \). Then the followings are equivalent:

1. \( \hat{\varepsilon} \) is a sup-HFRI (resp., sup-HFLI, sup-HFII, sup-HFI) of \( A \).
2. \( \hat{\varepsilon} \) is a sup\( \beta \)-HFRI (resp., sup\( \beta \)-HFLI, sup\( \beta \)-HFII, sup\( \beta \)-HFI) of \( A \) for all \( \beta \in [0,1] \).
3. \( \hat{\varepsilon} \) is a sup\( \alpha \)-HFRI (resp., sup\( \alpha \)-HFLI, sup\( \alpha \)-HFII, sup\( \alpha \)-HFI) of \( A \) for all \( \alpha \in [0,1] \).

Proof. It follows from Propositions 3.1, 3.2, 3.5 and 3.6.

### 3.3. Fuzzy sets, \Lukasiewicz fuzzy sets, \Lukasiewicz anti-fuzzy sets and Pythagorean fuzzy sets

In this part, we characterize sup\( \alpha \)-HFRI s, sup\( \alpha \)-HFLI s, sup\( \alpha \)-HFII s, sup\( \alpha \)-HFI s, sup\( \beta \)-HFRI s, sup\( \beta \)-HFLI s, sup\( \beta \)-HFII s, and sup\( \beta \)-HFI s of semigroups in terms of FSs, PFSs, \Lukasiewicz fuzzy sets and \Lukasiewicz anti-fuzzy sets.

For a FS \( \xi \) of \( A \), consider the FS

\[
\xi^+_{\alpha} : A \rightarrow [0, 1], \quad p \mapsto \min\{\xi(p) + \alpha, 1\},
\]

which is called an \( \alpha \)-\Lukasiewicz anti-fuzzy set [22] of \( \xi \) in \( A \). In case that \( 0 \leq \alpha \leq 1 - \sup\{\xi(p) | p \in A\} \), the \Lukasiewicz anti-fuzzy set \( \xi^+_{\alpha} \) is called a fuzzy \( \alpha \)-translation [8] of \( \xi \) of type I.

For a FS \( \xi \) of \( A \), consider the FS

\[
\xi^-_{\beta} : A \rightarrow [0, 1], \quad p \mapsto \max\{\xi(p) - \beta, 0\}.
\]

Then \( \xi^-_{\beta}(p) = \max\{\xi(p) + (1 - \beta) - 1, 0\} \) for all \( p \in A \) and so \( \xi^-_{\beta} \) is called an \( 1 - \beta \)-\Lukasiewicz fuzzy set [23] of \( \xi \) in \( A \). In case that \( 0 \leq \beta \leq \inf\{\xi(p) | p \in A\} \), the \Lukasiewicz fuzzy set \( \xi^-_{\beta} \) is called a fuzzy \( \beta \)-translation [8] of \( \xi \) of type II. Then we have the following results:

1. \( \xi^-_{0} = \xi = \xi^+_{0} \).
2. \( \xi^-_{\beta} \leq \xi \leq \xi^+_{\alpha} \).
3. \( \xi^+_k \leq 1 \) for all \( k \in [\beta, 1] \).
4. \( \xi^-_k \leq \xi^+_{\alpha} \) for all \( k \in [0, \alpha] \).
5. the FS \( (f\hat{\varepsilon})^+_{\alpha} \) is an \( \alpha \)-\Lukasiewicz anti-fuzzy set of \( \hat{\varepsilon} \) in \( A \) and \( (f\hat{\varepsilon})^+_{\alpha}(p) = \sup\alpha\hat{\varepsilon}(p) \) for each HFS \( \hat{\varepsilon} \) on \( A \) and \( p \in A \).
6. the FS \( (f\hat{\varepsilon})^-_{\beta} \) is an \( 1 - \beta \)-\Lukasiewicz fuzzy set of \( \hat{\varepsilon} \) in \( A \) and \( (f\hat{\varepsilon})^-_{\beta}(p) = \sup\beta\hat{\varepsilon}(p) \) for each HFS \( \hat{\varepsilon} \) on \( A \) and \( p \in A \).

**Theorem 3.1.** For a HFS \( \hat{\varepsilon} \) on \( A \), the followings are equivalent:

1. \( \hat{\varepsilon} \) is a sup\( \alpha \)-HFRI (resp., sup\( \alpha \)-HFLI, sup\( \alpha \)-HFII, sup\( \alpha \)-HFI) of \( A \).
2. \( (f\hat{\varepsilon})^+_{\alpha} \) is a FRI (resp., FLI, FII, FI) of \( A \), and
Proof. (1) ⇒ (3). Assume that \( \bar{\varepsilon} \) is a sup\( _{\alpha} \)-HFRI of \( \mathcal{A} \), \( k \in [\alpha, 1] \) and \( p, q \in \mathcal{A} \). Then \( \bar{\varepsilon}(p) \leq_{\alpha} \bar{\varepsilon}(pq) \) and so \( \min\{\text{SUP}(\varepsilon(p) + \alpha, 1)\} \leq \text{SUP}(\varepsilon(p) + \alpha) \). If \( \min\{\text{SUP}(\varepsilon(p) + \alpha, 1)\} = \text{SUP}(\varepsilon(p) + \alpha) \), then \( \text{SUP}(\varepsilon(p)) \leq \text{SUP}(\varepsilon(pq)) \). Thus
\[
(f^\varepsilon)^+_k(pq) = \text{SUP}_{k}^+ \varepsilon(pq) \geq \text{SUP}_{k}^+ \varepsilon(p) = (f^\varepsilon)^+_k(p)
\]
and so \( (f^\varepsilon)^+_k(pq) \geq (f^\varepsilon)^+_k(p) \). On the other hand, suppose that \( \min\{\text{SUP}(\varepsilon(p) + \alpha, 1)\} = 1 \). Then
\[
\text{SUP}(\varepsilon(pq)) + k \geq \text{SUP}(\varepsilon(p) + \alpha, 1) \geq 1
\]
and so
\[
(f^\varepsilon)^+_k(pq) = \text{SUP}_{k}^+ \varepsilon(pq) = 1 \geq (f^\varepsilon)^+_k(p).
\]
Hence \( (f^\varepsilon)^+_k(pq) \geq (f^\varepsilon)^+_k(p) \). Therefore, \( (f^\varepsilon)^+_k \) is a FRI of \( \mathcal{A} \) for all \( k \in [\alpha, 1] \).

(3) ⇒ (2). It is directly obtained from taking \( k = \alpha \).

(2) ⇒ (1). Assume that \( (f^\varepsilon)^-_k \) is a FRI of \( \mathcal{A} \). Then \( (f^\varepsilon)^-_k(pq) \geq (f^\varepsilon)^-_k(p) \) for all \( p, q \in \mathcal{A} \). Thus
\[
\text{SUP}_{\alpha}^+ \varepsilon(pq) = (f^\varepsilon)^-_k(pq) \geq (f^\varepsilon)^-_k(p) = \text{SUP}_{\alpha}^+ \varepsilon(p)
\]
for all \( p, q \in \mathcal{A} \). Hence \( \bar{\varepsilon}(p) \leq_{\alpha} \bar{\varepsilon}(pq) \) for all \( p, q \in \mathcal{A} \), which implies that \( \bar{\varepsilon} \) is a sup\( _{\alpha} \)-HFRI of \( \mathcal{A} \). \( \square \)

**Theorem 3.2.** For a HFS \( \bar{\varepsilon} \) on \( \mathcal{A} \), the followings are equivalent:

1. \( \bar{\varepsilon} \) is a sup\( _{\beta} \)-HFRI (resp., sup\( _{\beta} \)-HFII, sup\( _{\beta} \)-HFII, sup\( _{\beta} \)-HFII) of \( \mathcal{A} \),
2. \( (f^\varepsilon)^-_\beta \) is a FRI (resp., FLI, FII, FI) of \( \mathcal{A} \), and
3. \( (f^\varepsilon)^-_k \) is a FRI (resp., FLI, FII, FI) of \( \mathcal{A} \) for all \( k \in [\beta, 1] \).

Proof. (1) ⇒ (3). Assume that \( \bar{\varepsilon} \) is a sup\( _{\beta} \)-HFRI of \( \mathcal{A} \), \( k \in [\beta, 1] \) and \( p, q \in \mathcal{A} \). If \( 0 \geq \text{SUP}(\varepsilon(p) - \beta) \), then \( 0 \geq \text{SUP}(\varepsilon(p) - k) \) and so
\[
(f^\varepsilon)^-_k(pq) \geq 0 = \text{SUP}_{k}^+ \varepsilon(p) = (f^\varepsilon)^-_k(p).
\]
Thus \( (f^\varepsilon)^-_k(pq) \geq (f^\varepsilon)^-_k(p) \). On the other hand, suppose that \( \text{SUP}(\varepsilon(p) - \beta) > 0 \). Since \( \bar{\varepsilon} \) is a sup\( _{\beta} \)-HFRI of \( \mathcal{A} \), we have \( \bar{\varepsilon}(p) \leq_{\beta} \bar{\varepsilon}(pq) \). Then \( \text{SUP}(\varepsilon(pq) - \beta) \geq \text{SUP}(\varepsilon(p) - \beta) \) and so \( \text{SUP}(\varepsilon(pq)) \geq \text{SUP}(\varepsilon(p)) \).

Thus
\[
(f^\varepsilon)^-_k(pq) = \text{SUP}_{k}^+ \varepsilon(pq) \geq \text{SUP}_{k}^+ \varepsilon(p) = (f^\varepsilon)^-_k(p).
\]
Hence \( (f^\varepsilon)^-_k(pq) \geq (f^\varepsilon)^-_k(p) \). Therefore, \( (f^\varepsilon)^-_k \) is a FRI of \( \mathcal{A} \) for all \( k \in [\beta, 1] \).

(3) ⇒ (2). It is directly obtained from taking \( k = \beta \).

(2) ⇒ (1). Assume that \( (f^\varepsilon)^-_\beta \) is a FRI of \( \mathcal{A} \) and \( p, q \in \mathcal{A} \). Then \( (f^\varepsilon)^-_\beta(pq) \geq (f^\varepsilon)^-_\beta(p) \) and so
\[
\text{SUP}_{\beta}^+ \varepsilon(pq) = (f^\varepsilon)^-_\beta(pq) \geq (f^\varepsilon)^-_\beta(p) = \text{SUP}_{\beta}^+ \varepsilon(p).
\]
Hence \( \bar{\varepsilon}(p) \leq_{\beta} \bar{\varepsilon}(pq) \). Therefore, \( \bar{\varepsilon} \) is a sup\( _{\beta} \)-HFRI of \( \mathcal{A} \). \( \square \)
Lemma 3.1. Let \( \hat{\varepsilon} \) be a HFS on \( \mathcal{A} \). Then \( \sup_k^+ \hat{\omega}(p) = 1 - \sup_k^- \hat{\varepsilon}(p) \) and \( \sup_k^- \hat{\omega}(p) = 1 - \sup_k^+ \hat{\varepsilon}(p) \) for all \( p \in \mathcal{A}, \hat{\omega} \in SC(\hat{\varepsilon}) \) and \( k \in [0, 1] \).

Proof. Let \( p \in \mathcal{A}, \hat{\omega} \in SC(\hat{\varepsilon}) \) and \( k \in [0, 1] \). Then

\[
\begin{align*}
\sup_k^+ \hat{\omega}(p) &= \min \{ \sup \hat{\omega}(p) + k, 1 \} \\
&= \min \{ (1 - \sup \hat{\varepsilon}(p)) + k, 1 \} \\
&= \min \{ 1 - (\sup \hat{\varepsilon}(p) - k), 1 \} \\
&= 1 - \max \{ \sup \hat{\varepsilon}(p) - k, 0 \} \\
&= 1 - \sup_k^- \hat{\varepsilon}(p)
\end{align*}
\]

and

\[
\begin{align*}
\sup_k^- \hat{\omega}(p) &= \max \{ \sup \hat{\omega}(p) - k, 0 \} \\
&= \max \{ (1 - \sup \hat{\varepsilon}(p)) - k, 1 - 1 \} \\
&= \max \{ 1 - (\sup \hat{\varepsilon}(p) + k), 1 - 1 \} \\
&= 1 - \min \{ \sup \hat{\varepsilon}(p) + k, 1 \} \\
&= 1 - \sup_k^+ \hat{\varepsilon}(p)
\end{align*}
\]

Therefore, \( \sup_k^+ \hat{\omega}(p) = 1 - \sup_k^- \hat{\varepsilon}(p) \) and \( \sup_k^- \hat{\omega}(p) = 1 - \sup_k^+ \hat{\varepsilon}(p) \). \qed

Theorem 3.3. For a HFS \( \hat{\varepsilon} \) on \( \mathcal{A} \), the followings are equivalent:

1. \( \hat{\varepsilon} \) is a sup\(^+\) -HFRI (resp., sup\(^+\) -HFLI, sup\(^+\) -HFII, sup\(^+\) -HFI) of \( \mathcal{A} \),
2. \( (f\hat{\varepsilon})^-_\alpha \) is an AFRI (resp., AFLI, AFII, AFI) of \( \mathcal{A} \),
3. \( (f\hat{\omega})^-_\alpha \) is an AFRI (resp., AFLI, AFII, AFI) of \( \mathcal{A} \) for all \( \hat{\omega} \in SC(\hat{\varepsilon}) \), and
4. \( (f\hat{\omega})^-_k \) is an AFRI (resp., AFLI, AFII, AFI) of \( \mathcal{A} \) for all \( \hat{\omega} \in SC(\hat{\varepsilon}) \) and \( k \in [\alpha, 1] \).

Proof. (1) \( \Rightarrow \) (4). Assume that \( \hat{\varepsilon} \) is a sup\(^+\) -HFRI of \( \mathcal{A} \). By Proposition 3.3, we have that \( \hat{\varepsilon} \) is a sup\(^+\) -HFRI of \( \mathcal{A} \) for all \( k \in [\alpha, 1] \). By Lemma 3.1, we get

\[
(f\hat{\omega})^-_k(p) = \sup_k^- \hat{\omega}(p) = 1 - \sup_k^+ \hat{\varepsilon}(p) \geq 1 - \sup_k^+ \hat{\varepsilon}(pq) = \sup_k^- \hat{\omega}(pq) = (f\hat{\omega})^-_k(pq)
\]

for all \( \hat{\omega} \in SC(\hat{\varepsilon}) \), \( k \in [\alpha, 1] \) and \( p, q \in \mathcal{A} \). Hence \( (f\hat{\omega})^-_k \) is an AFRI of \( \mathcal{A} \) for all \( \hat{\omega} \in SC(\hat{\varepsilon}) \) and \( k \in [\alpha, 1] \).

(4) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (2). They are clear.
(2) ⇒ (1). Assume that \((\hat{f}^\varepsilon)_{\alpha}\) is an AFRI of \(\mathcal{A}\) and \(p, q \in \mathcal{A}\). Then \((\hat{f}^\varepsilon)_{\alpha}(p) \geq (\hat{f}^\varepsilon)_{\alpha}(pq)\) and by Lemma 3.1, we have

\[
\text{SUP}_{\alpha}^+ \hat{\varepsilon}(p) = 1 - \text{SUP}_{\alpha}^- \hat{\varepsilon}^+(p) \\
= 1 - (\hat{f}^\varepsilon)_{\alpha}(p) \\
\leq 1 - (\hat{f}^\varepsilon)_{\alpha}(pq) \\
= 1 - \text{SUP}_{\alpha}^- \hat{\varepsilon}^+(pq) \\
= \text{SUP}_{\alpha}^+ \hat{\varepsilon}(pq).
\]

Hence \(\hat{\varepsilon}(p) \sqsubseteq_{\alpha} \hat{\varepsilon}(pq)\). Therefore, \(\hat{\varepsilon}\) is a sup_{\alpha}^+\text{-HFRI of } \mathcal{A}.

**Theorem 3.4.** For a HFS \(\hat{\varepsilon}\) on \(\mathcal{A}\), the followings are equivalent:

1. \(\hat{\varepsilon}\) is a sup_{\beta}^-\text{-HFRI (resp., sup_{\beta}^-\text{-HFLI, sup_{\beta}^-\text{-HFII, sup_{\beta}^-\text{-HFI}) of } \mathcal{A}}\),
2. \(\hat{f}^\varepsilon_{\beta}\) is an AFRI (resp., AFLI, AFII, AFI) of \(\mathcal{A}\),
3. \(\hat{f}^\varepsilon_{\beta}^\omega\) is an AFRI (resp., AFLI, AFII, AFI) of \(\mathcal{A}\) for all \(\hat{\omega} \in \mathcal{SC}(\hat{\varepsilon})\), and
4. \(\hat{f}^\varepsilon_{\beta}^k\) is an AFRI (resp., AFLI, AFII, AFI) of \(\mathcal{A}\) for all \(\hat{\omega} \in SC(\hat{\varepsilon})\) and for all \(k \in [\beta, 1]\).

**Proof.** (1) ⇒ (4). Assume that \(\hat{\varepsilon}\) is a sup_{\beta}^-\text{-HFRI of } \mathcal{A}. By Proposition 3.7 and Lemma 3.1, we get

\[
(f^\hat{\varepsilon}_{\beta}^+)(pq) = \text{SUP}_{\hat{\omega}}^+ \hat{\varepsilon}(pq) = 1 - \text{SUP}_{\hat{\omega}}^- \hat{\varepsilon}(pq) = 1 - \text{SUP}_{\hat{\omega}}^- \hat{\varepsilon}(p) = \text{SUP}_{\hat{\omega}}^+ \hat{\varepsilon}(p) = (f^\hat{\varepsilon}_{\beta}^+)(p)
\]

for all \(\hat{\omega} \in \mathcal{SC}(\hat{\varepsilon})\), \(k \in [\beta, 1]\) and \(p, q \in \mathcal{A}\). Hence \(f^\hat{\varepsilon}_{\beta}^+\) is an AFRI of \(\mathcal{A}\) for all \(\hat{\omega} \in SC(\hat{\varepsilon})\) and \(k \in [\beta, 1]\).

(4) ⇒ (3) and (3) ⇒ (2). They are clear.

(2) ⇒ (1). Assume that \(\hat{f}^\varepsilon_{\beta}\) is an AFRI of \(\mathcal{A}\). By Lemma 3.1, we get

\[
\text{SUP}_{\beta}^- \hat{\varepsilon}(pq) = 1 - \text{SUP}_{\beta}^+ \hat{\varepsilon}^+(pq) \\
= 1 - (f^\varepsilon_{\beta}^+)(pq) \\
\geq 1 - (f^\varepsilon_{\beta}^+)(p) \\
= 1 - \text{SUP}_{\beta}^+ \hat{\varepsilon}^+(p) \\
= \text{SUP}_{\beta}^- \hat{\varepsilon}(p)
\]

for all \(p, q \in \mathcal{A}\). Thus \(\hat{\varepsilon}(p) \sqsubseteq_{\beta} \hat{\varepsilon}(pq)\) for all \(p, q \in \mathcal{A}\), which implies that \(\hat{\varepsilon}\) is a sup_{\beta}^-\text{-HFRI of } \mathcal{A}.

**Theorem 3.5.** For a HFS \(\hat{\varepsilon}\) on \(\mathcal{A}\), the followings are equivalent:

1. \(\hat{\varepsilon}\) is a sup_{\alpha}^+\text{-HFRI (resp., sup_{\alpha}^+\text{-HFLI, sup_{\alpha}^+\text{-HFII, sup_{\alpha}^+\text{-HFI}) of } \mathcal{A}}\),
2. \((f^\hat{\varepsilon}^+_{\alpha}, f^\varepsilon_{\alpha})\) is a PFRI (resp., PFII, PFI) of \(\mathcal{A}\),
3. \((f^\hat{\varepsilon}^+_{\alpha}, f^\varepsilon_{\alpha})\) is a PFRI (resp., PFII, PFI) of \(\mathcal{A}\) for all \(\hat{\omega} \in \mathcal{SC}(\hat{\varepsilon})\), and
4. \((f^\hat{\varepsilon}^+_{k}, f^\varepsilon_{k})\) is a PFRI (resp., PFII, PFI) of \(\mathcal{A}\) for all \(\hat{\omega} \in SC(\hat{\varepsilon})\) and \(k \in [\alpha, 1]\).
Proof. It follows from Theorems 3.1 and 3.3.

\[ \square \]

**Theorem 3.6.** For a HFS \( \hat{\varepsilon} \) on \( \mathcal{A} \), the followings are equivalent:

1. \( \hat{\varepsilon} \) is a sup\( _{\beta} \)-HFRI (resp., sup\( _{\beta} \)-HFLI, sup\( _{\beta} \)-HFII, sup\( _{\beta} \)-HFI) of \( \mathcal{A} \),
2. \( [(f^{\hat{\varepsilon}})_{\beta}, (f^{\hat{\varepsilon}})^{+}_{\beta}] \) is a PFRI (resp., PFLI, PFII, PFI) of \( \mathcal{A} \),
3. \( [(f^{\hat{\varepsilon}})_{\beta}, (f^{\hat{\hat{\varepsilon}}}^{+})_{\beta}] \) is a PFRI (resp., PFLI, PFII, PFI) of \( \mathcal{A} \) for all \( \hat{\omega} \in SC(\hat{\varepsilon}) \), and
4. \( [(f^{\hat{\varepsilon}})_{\alpha}, (f^{\hat{\hat{\varepsilon}}}^{+})_{\alpha}] \) is a PFRI (resp., PFLI, PFII, PFI) of \( \mathcal{A} \) for all \( \hat{\omega} \in SC(\hat{\varepsilon}) \) and \( k \in [\beta, 1] \).

Proof. It follows from Theorems 3.2 and 3.4.

\[ \square \]

3.4. *Hesitant fuzzy sets and hybrid sets.* In this part, we characterize sup\( _{\alpha} \)-HFRI, sup\( _{\alpha} \)-HFLI, sup\( _{\alpha} \)-HFII, sup\( _{\alpha} \)-HFI, sup\( _{\beta} \)-HFRI, sup\( _{\beta} \)-HFLI, sup\( _{\beta} \)-HFII, sup\( _{\beta} \)-HFI of semigroups in terms of HFSs and hybrid sets.

**Theorem 3.7.** For a HFS \( \hat{\varepsilon} \) on \( \mathcal{A} \), the followings are equivalent:

1. \( \hat{\varepsilon} \) is a sup\( _{\alpha} \)-HFRI (resp., sup\( _{\alpha} \)-HFLI, sup\( _{\alpha} \)-HFII, sup\( _{\alpha} \)-HFI) of \( \mathcal{A} \), and
2. \( H_{\hat{\varepsilon}}^{(\hat{\varepsilon}, \bullet)} \) is a HFRI (resp., HFLI, HFII, HFI) of \( \mathcal{A} \) for all \( \bullet \in \mathcal{P}(0, 1-\alpha) \).

Proof. (1) \( \Rightarrow \) (2). Assume that \( \hat{\varepsilon} \) is a sup\( _{\alpha} \)-HFRI of \( \mathcal{A} \). Let \( \bullet \in \mathcal{P}(0, 1-\alpha) \), \( p, q \in \mathcal{A} \) and \( k \in H_{\hat{\varepsilon}}^{(\hat{\varepsilon}, \bullet)}(p) \). Then \( \hat{\varepsilon}(p) \subseteq \hat{\varepsilon}(pq) \) and \( \text{SUP}(\hat{\varepsilon}) \geq k \in \bullet \). Thus

\[
\text{SUP}(\hat{\varepsilon}(pq)) = (\text{SUP}(\hat{\varepsilon}(pq)) + \alpha) - \alpha
\geq \text{SUP}^{+}_{\alpha}\hat{\varepsilon}(pq) - \alpha
\geq \text{SUP}^{+}_{\alpha}\hat{\varepsilon}(p) - \alpha
= \min\{\text{SUP}(\hat{\varepsilon}(p)), 1-\alpha\}
\geq k
\]

which implies that \( k \in H_{\hat{\varepsilon}}^{(\hat{\varepsilon}, \bullet)}(pq) \). Hence \( H_{\hat{\varepsilon}}^{(\hat{\varepsilon}, \bullet)}(p) \subseteq H_{\hat{\varepsilon}}^{(\hat{\varepsilon}, \bullet)}(pq) \). Therefore, \( H_{\hat{\varepsilon}}^{(\hat{\varepsilon}, \bullet)} \) is a HFRI of \( \mathcal{A} \) for all \( \bullet \in \mathcal{P}(0, 1-\alpha) \).

(2) \( \Rightarrow \) (1). Assume that \( H_{\hat{\varepsilon}}^{(\hat{\varepsilon}, \bullet)} \) is a HFRI of \( \mathcal{A} \) for all \( \bullet \in \mathcal{P}(0, 1-\alpha) \). Let \( p, q \in \mathcal{A} \) and \( \bullet = [0, 1-\alpha] \). Then

\[
\text{SUP}^{+}_{\alpha}\hat{\varepsilon}(p) - \alpha = \min\{\text{SUP}(\hat{\varepsilon}(p)), 1-\alpha\} \in H_{\hat{\varepsilon}}^{(\hat{\varepsilon}, \bullet)}(p) \subseteq H_{\hat{\varepsilon}}^{(\hat{\varepsilon}, \bullet)}(pq).
\]

Thus

\[
\text{SUP}^{+}_{\alpha}\hat{\varepsilon}(pq) - \alpha = \min\{\text{SUP}(\hat{\varepsilon}(pq)), 1-\alpha\} \geq \text{SUP}^{+}_{\alpha}\hat{\varepsilon}(p) - \alpha.
\]

Hence \( \text{SUP}^{+}_{\alpha}\hat{\varepsilon}(pq) \geq \text{SUP}^{+}_{\alpha}\hat{\varepsilon}(p) \) which implies that \( \hat{\varepsilon}(p) \subseteq \hat{\varepsilon}(pq) \). Therefore, \( \hat{\varepsilon} \) is a sup\( _{\alpha} \)-HFRI of \( \mathcal{A} \).

\[ \square \]

**Theorem 3.8.** For a HFS \( \hat{\varepsilon} \) on \( \mathcal{A} \), the followings are equivalent:

1. \( \hat{\varepsilon} \) is a sup\( _{\beta} \)-HFRI (resp., sup\( _{\beta} \)-HFLI, sup\( _{\beta} \)-HFII, sup\( _{\beta} \)-HFI) of \( \mathcal{A} \),
(2) $H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}$ is a HFRI (resp., HFLI, HFII, HFI) of $\mathcal{A}$ for all $\nabla \in \varphi((\beta, 1])$.

Proof. (1) $\Rightarrow$ (2). Assume that $\hat{e}$ is a sup$^{-}\hat{\beta}$-HFRI of $\mathcal{A}$. Let $\nabla \in \varphi((\beta, 1])$, $p, q \in \mathcal{A}$ and $k \in H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}(p)$. Then $\text{SUP}\hat{e}(p) \geq k > \beta$, $k \in \nabla$ and $\hat{e}(p) \sqsubseteq_{\hat{\beta}} \hat{e}(pq)$. Thus

$$\max\{\text{SUP}\hat{e}(pq), \beta\} = \text{SUP}_{\hat{\beta}}\hat{e}(pq) + \beta$$
$$\geq \text{SUP}_{\hat{\beta}}\hat{e}(p) + \beta$$
$$= \max\{\text{SUP}\hat{e}(p), \beta\}$$
$$\geq k$$
$$> \beta,$$

that is, $\text{SUP}\hat{e}(pq) \geq k$. Hence $k \in H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}(pq)$ and so $H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}(p) \subseteq H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}(pq)$. Therefore, $H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}$ is a HFRI of $\mathcal{A}$ for all $\nabla \in \varphi((\beta, 1])$.

(2) $\Rightarrow$ (1). Assume that $H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}$ is a HFRI of $\mathcal{A}$ for all $\nabla \in \varphi((\beta, 1])$ and $p, q \in \mathcal{A}$. If $\text{SUP}\hat{e}(p) \leq \beta$, then

$$\text{SUP}_{\hat{\beta}}\hat{e}(p) = 0 \leq \text{SUP}_{\hat{\beta}}\hat{e}(pq)$$

and so $\hat{e}(p) \sqsubseteq_{\hat{\beta}} \hat{e}(pq)$. On the other hand, suppose that $\text{SUP}\hat{e}(p) > \beta$. Let $\nabla = (\beta, 1]$. Then

$$\text{SUP}_{\hat{\beta}}\hat{e}(p) + \beta = \max\{\text{SUP}\hat{e}(p), \beta\} = \text{SUP}\hat{e}(p) \in H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}(p) \subseteq H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}(pq).$$

Thus

$$\text{SUP}_{\hat{\beta}}\hat{e}(pq) + \beta \geq (\text{SUP}\hat{e}(pq) - \beta) + \beta = \text{SUP}\hat{e}(pq) \geq \text{SUP}_{\hat{\beta}}\hat{e}(p) + \beta.$$ 

Hence $\text{SUP}_{\hat{\beta}}\hat{e}(p) \leq \text{SUP}_{\hat{\beta}}\hat{e}(pq)$ and so $\hat{e}(p) \sqsubseteq_{\hat{\beta}} \hat{e}(pq)$. Therefore, $\hat{e}$ is a sup$^{-}\hat{\beta}$-HFRI of $\mathcal{A}$. \qed

**Theorem 3.9.** For a HFS $\hat{e}$ on $\mathcal{A}$, the followings are equivalent:

(1) $\hat{e}$ is a sup$^{+}_{\alpha}$-HFRI (resp., sup$^{+}_{\alpha}$-HFLI, sup$^{+}_{\alpha}$-HFII, sup$^{+}_{\alpha}$-HFI) of $\mathcal{A}$,

(2) $(H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}, (f^{\hat{\omega}})_{\alpha})$ is a HyRI (resp., HyLI, HyII, HyII) of $\mathcal{A}$ over $[0, 1]$ for all $\nabla \in \varphi([0, 1 - \alpha])$,

(3) $(H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}, (f^{0})_{\alpha})$ is a HyRI (resp., HyLI, HyII, HyII) of $\mathcal{A}$ over $[0, 1]$ for all $\hat{\omega} \in \mathcal{SC}(\hat{e})$ and $\nabla \in \varphi([0, 1 - \alpha])$, and

(4) $(H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}, (f^{0})_{k})$ is a HyRI (resp., HyLI, HyII, HyII) of $\mathcal{A}$ over $[0, 1]$ for all $\hat{\omega} \in \mathcal{SC}(\hat{e})$, $k \in [\alpha, 1]$ and $\nabla \in \varphi([0, 1 - \alpha])$.

Proof. It follows from Theorems 3.3 and 3.7. \qed

**Theorem 3.10.** For a HFS $\hat{e}$ on $\mathcal{A}$, the followings are equivalent:

(1) $\hat{e}$ is a sup$^{-}_{\beta}$-HFRI (resp., sup$^{-}_{\beta}$-HFLI, sup$^{-}_{\beta}$-HFII, sup$^{-}_{\beta}$-HFI) of $\mathcal{A}$,

(2) $(H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}, (f^{\hat{\omega}})_{\beta})$ is a HyRI (resp., HyLI, HyII, HyII) of $\mathcal{A}$ over $[0, 1]$ for all $\nabla \in \varphi((\beta, 1])$,

(3) $(H_{\mathcal{H}_{\text{SUP}}}^{(\hat{e}, \nabla)}, (f^{0})_{\beta})$ is a HyRI (resp., HyLI, HyII, HyII) of $\mathcal{A}$ over $[0, 1]$ for all $\hat{\omega} \in \mathcal{SC}(\hat{e})$ and $\nabla \in \varphi((\beta, 1])$, and
Proof. It follows from Theorems 3.4 and 3.8.

\[ \square \]

3.5. Interval-valued fuzzy sets and cubic sets. In this part, we characterize sup\(^+\)HFRI, sup\(^+\)HFII, sup\(^+\)HFIs, sup\(^+\)HFIs, sup\(^-\)HFRI, sup\(^-\)HFII, sup\(^-\)HFIs, and sup\(^-\)HFIs of semigroups in terms of ivFSs and cubic sets.

Let \( \xi \) and \( \eta \) be FSs of \( \mathcal{A} \) such that \( \xi \leq \eta \), the followings are true.

1. If \( \xi, \eta \) are ivFSs on \( \mathcal{A} \), then \( \xi, \eta \) are ivFSs on \( \mathcal{A} \).
2. If \( \alpha \geq \beta \), then \( [\xi^-_\beta, \eta^-_\beta] \) is an ivFS on \( \mathcal{A} \).
3. If \( \alpha \leq \beta \), then \( [\xi^-_\alpha, \eta^-_\beta] \) is an ivFS on \( \mathcal{A} \).
4. If \( \bar{\lambda} \) is an ivFS on \( \mathcal{A} \) and \( \bar{\lambda} = [\xi, \eta] \), then \( \bar{\lambda}^L = \xi \) and \( \bar{\lambda}^U = \eta \).

**Theorem 3.11.** For a HFS \( \hat{\epsilon} \) on \( \mathcal{A} \), the followings are equivalent:

1. \( \hat{\epsilon} \) is a sup\(^+\)HFRI (resp., sup\(^+\)HFII, sup\(^+\)HFIs, sup\(^+\)HFIs) of \( \mathcal{A} \).
2. \( [(f\hat{\epsilon})^+_(k), (f\hat{\epsilon})^-_(k)] \) is an ivFRI (resp., ivFLI, ivFII, ivFI) of \( \mathcal{A} \) for all \( k \in [\alpha, 1] \).
3. \( [(f\hat{\epsilon})^+_(k), (f\hat{\epsilon})^-_(k)] \) is an ivFRI (resp., ivFLI, ivFII, ivFI) of \( \mathcal{A} \) for all \( k, \beta \in [\alpha, 1] \) with \( k \leq \beta \).
4. \( H^{(\hat{\epsilon}, [0, 1-\alpha])}_{\text{sup}}(\mathcal{A}) \) is an ivFRI (resp., ivFLI, ivFII, ivFI) of \( \mathcal{A} \).

Proof. (1) \( \Leftrightarrow \) (2) and (1) \( \Leftrightarrow \) (3). It follows from Theorem 3.1.

(1) \( \Rightarrow \) (4). Assume that \( \hat{\epsilon} \) is a sup\(^+\)HFRI of \( \mathcal{A} \) and \( p, q \in \mathcal{A} \). Then \( \hat{\epsilon}(p) \sqsupseteq \hat{\epsilon}(pq) \) and so

\[
\text{SUP}^{+}\hat{\epsilon}(p) \leq \text{SUP}^{+}\hat{\epsilon}(pq).
\]

Thus

\[
H^{(\hat{\epsilon}, [0, 1-\alpha])}_{\text{sup}}(\mathcal{A})(p) = [0, \text{SUP}^{\alpha}_{\text{sup}}(\hat{\epsilon}(p) - \alpha)] \supseteq [0, \text{SUP}^{\alpha}_{\text{sup}}(\hat{\epsilon}(pq) - \alpha)] = H^{(\hat{\epsilon}, [0, 1-\alpha])}_{\text{sup}}(pq).
\]

Therefore, \( H^{(\hat{\epsilon}, [0, 1-\alpha])}_{\text{sup}} \) is an ivFRI of \( \mathcal{A} \).

(4) \( \Rightarrow \) (1). Assume that \( H^{(\hat{\epsilon}, [0, 1-\alpha])}_{\text{sup}} \) is an ivFRI of \( \mathcal{A} \) and \( p, q \in \mathcal{A} \). Then

\[
[0, \text{SUP}^{\alpha}_{\text{sup}}(\hat{\epsilon}(p) - \alpha)] = H^{(\hat{\epsilon}, [0, 1-\alpha])}_{\text{sup}}(p) \supseteq H^{(\hat{\epsilon}, [0, 1-\alpha])}_{\text{sup}}(pq) = [0, \text{SUP}^{\alpha}_{\text{sup}}(\hat{\epsilon}(pq) - \alpha)].
\]

Thus

\[
\text{SUP}^{\alpha}_{\text{sup}}(\hat{\epsilon}(p)) = (\text{SUP}^{\alpha}_{\text{sup}}(\hat{\epsilon}(p) - \alpha) + \alpha) \leq (\text{SUP}^{\alpha}_{\text{sup}}(\hat{\epsilon}(pq) - \alpha) + \alpha) = \text{SUP}^{\alpha}_{\text{sup}}(\hat{\epsilon}(pq)).
\]

Hence \( \hat{\epsilon}(p) \sqsupseteq \hat{\epsilon}(pq) \). Therefore, \( \hat{\epsilon} \) is a sup\(^+\)HFRI of \( \mathcal{A} \).

\[ \square \]

**Theorem 3.12.** For a HFS \( \hat{\epsilon} \) on \( \mathcal{A} \), the following are equivalent:

1. \( \hat{\epsilon} \) is a sup\(^-\)HFRI (resp., sup\(^-\)HFII, sup\(^-\)HFIs, sup\(^-\)HFIs) of \( \mathcal{A} \).
2. \( [(f\hat{\epsilon})^+_(k), (f\hat{\epsilon})^-_(k)] \) is an ivFRI (resp., ivFLI, ivFII, ivFI) of \( \mathcal{A} \) for all \( k \in [\beta, 1] \), and
3. \( [(f\hat{\epsilon})^+_(k), (f\hat{\epsilon})^-_(k)] \) is an ivFRI (resp., ivFLI, ivFII, ivFI) of \( \mathcal{A} \) for all \( k, \alpha \in [\beta, 1] \) with \( \alpha \geq k \).

Proof. It follows from Theorem 3.2.
Theorem 3.13. Let \( \hat{\varepsilon} \) be a HFS on \( \mathcal{A} \) and \( \hat{\lambda} = \mathcal{H}_{\sup_{\alpha}}^{(\hat{\varepsilon};[0,1-\alpha])} \). Then the followings are equivalent:

1. \( \hat{\varepsilon} \) is a \( \sup_{\alpha}^+ \)-HFRI (resp., \( \sup_{\alpha}^- \)-HFII, \( \sup_{\alpha}^+ \)-HFII, \( \sup_{\alpha}^- \)-HFII) of \( \mathcal{A} \),
2. \( (\hat{\lambda},(f^{\hat{\varepsilon}})_\alpha^+) \) is a CuRI (resp., CuLI, CuII, CuI) of \( \mathcal{A} \),
3. \( (\hat{\lambda},(f^{\hat{\varepsilon}})_\alpha^-) \) is a CuRI (resp., CuLI, CuII, CuI) of \( \mathcal{A} \) for all \( \hat{\omega} \in SC(\hat{\varepsilon}) \),
4. \( (\hat{\lambda},(f^{\hat{\varepsilon}})_\lambda^+) \) is a CuRI (resp., CuLI, CuII, CuI) of \( \mathcal{A} \) for all \( \hat{\omega} \in SC(\hat{\varepsilon}) \) and \( k \in [\alpha, 1] \).

Proof. It follows from Theorems 3.3 and 3.11. \( \square \)

Theorem 3.14. Let \( \hat{\varepsilon} \) be a HFS on \( \mathcal{A} \), \( k \in [\beta, 1] \) and \( \hat{\lambda} = [(f^{\hat{\varepsilon}})_k^-, (f^{\hat{\varepsilon}})_\beta^-] \). Then the followings are equivalent:

1. \( \hat{\varepsilon} \) is a \( \sup_{\beta}^- \)-HFRI (resp., \( \sup_{\beta}^- \)-HFII, \( \sup_{\beta}^- \)-HFII, \( \sup_{\beta}^- \)-HFII) of \( \mathcal{A} \),
2. \( (\hat{\lambda},(f^{\hat{\varepsilon}})_\beta^+) \) is a CuRI (resp., CuLI, CuII, CuI) of \( \mathcal{A} \),
3. \( (\hat{\lambda},(f^{\hat{\varepsilon}})_\beta^-) \) is a CuRI (resp., CuLI, CuII, CuI) of \( \mathcal{A} \) for all \( \hat{\omega} \in SC(\hat{\varepsilon}) \).

Proof. It follows from Theorems 3.4 and 3.12. \( \square \)

4. Characterizing semigroups by \( \sup_{\alpha}^- \)-type and \( \sup_{\beta}^- \)-type of HFSs

In this section, we characterize intra-regular, completely regular, left (right) regular, left (right) simple and simple semigroups and groups in terms of \( \sup_{\alpha}^- \)-type and \( \sup_{\beta}^- \)-type of HFSs.

A semigroup \( \mathcal{A} \) is called

1. intra-regular if for each \( w \in \mathcal{A} \), there exist \( p, q \in \mathcal{A} \) such that \( w = pw^2q \),
2. completely regular if for each \( p \in \mathcal{A} \) there exists \( q \in \mathcal{A} \) such that \( p = qpq \) and \( pq = qp \),
3. left regular if for each \( p \in \mathcal{A} \) there exists \( q \in \mathcal{A} \) such that \( p = qp^2 \),
4. right regular if for each \( p \in \mathcal{A} \) there exists \( q \in \mathcal{A} \) such that \( p = p^2q \),
5. left simple if \( \mathcal{A} = B \) for each left ideal \( B \) of \( \mathcal{A} \),
6. right simple if \( \mathcal{A} = B \) for each right ideal \( B \) of \( \mathcal{A} \),
7. simple if \( \mathcal{A} = B \) for each ideal \( B \) of \( \mathcal{A} \),
8. group if it is both left simple and right simple.

It is well-known that \( \mathcal{A} \) is completely regular if and only if it is both left and right regular.

Proposition 4.1. Let \( \hat{\varepsilon} \) be a HFS on an intra-regular semigroup \( \mathcal{A} \). Then \( \hat{\varepsilon} \) is a \( \sup_{\alpha}^+ \)-HFII (resp., \( \sup_{\beta}^- \)-HFII) of \( \mathcal{A} \) if and only if \( \hat{\varepsilon} \) is a \( \sup_{\alpha}^- \)-HFI (resp., \( \sup_{\beta}^- \)-HFI) of \( \mathcal{A} \).

Proof. \((\Rightarrow)\). Assume that \( \hat{\varepsilon} \) is a \( \sup_{\alpha}^+ \)-HFII of \( \mathcal{A} \) and \( p, q \in \mathcal{A} \). Then there exist \( w_1, w_2, w_3, w_4 \in \mathcal{A} \) such that \( p = w_1p^2w_2 \) and \( q = w_3q^2w_4 \). Thus \( \hat{\varepsilon}(p) \subseteq_{\alpha}^+ \hat{\varepsilon}(w_1p^2w_2q) = \hat{\varepsilon}(pq) \) and \( \hat{\varepsilon}(q) \subseteq_{\alpha}^+ \hat{\varepsilon}(pw_3q^2w_4) = \hat{\varepsilon}(pq) \). Therefore, \( \hat{\varepsilon} \) is a \( \sup_{\alpha}^+ \)-HFI of \( \mathcal{A} \).

\((\Leftarrow)\). It follows from Proposition 3.4. \( \square \)

Theorem 4.1. Let \( \mathcal{A} \) be a semigroup. The followings are equivalent:
(1) \( A \) is intra-regular,
(2) \( \hat{\varepsilon}(p) \cong k^+ \hat{\varepsilon}(p^2) \) for each \( k \in [0,1] \), \( \sup_k^+ \)-HFI \( \hat{\varepsilon} \) of \( A \) and \( p \in A \),
(3) \( \hat{\varepsilon}(p) \cong k^+ \hat{\varepsilon}(p^2) \) for each \( k \in [0,1] \), \( \sup_k^+ \)-HFI \( \hat{\varepsilon} \) of \( A \) and \( p \in A \),
(4) \( \hat{\varepsilon}(p) \cong k^+ \hat{\varepsilon}(p^2) \) for each \( k \in [0,1] \), \( \sup_k^- \)-HFI \( \hat{\varepsilon} \) of \( A \) and \( p \in A \),
(5) \( \hat{\varepsilon}(p) \cong k^+ \hat{\varepsilon}(p^2) \) for each \( k \in [0,1] \), \( \sup_k^- \)-HFI \( \hat{\varepsilon} \) of \( A \) and \( p \in A \),
(6) \( \hat{\varepsilon}(p) \cong \hat{\varepsilon}(p^2) \) for each \( \sup \)-HFI \( \hat{\varepsilon} \) of \( A \) and \( p \in A \),
(7) \( \hat{\varepsilon}(p) \cong \hat{\varepsilon}(p^2) \) for each \( \sup \)-HFI \( \hat{\varepsilon} \) of \( A \) and \( p \in A \).

Proof. (7) \( \Rightarrow \) (6). It follows from Proposition 3.4.

(2) \( \Leftrightarrow (4) \Leftrightarrow (6) \) and (3) \( \Leftrightarrow (5) \Leftrightarrow (7) \). They follow from Proposition 3.10.

(1) \( \Rightarrow \) (2). Assume that (1) holds, \( k \in [0,1] \), \( \hat{\varepsilon} \) is a \( \sup_k^+ \)-HFI of \( A \) and \( p \in A \). There exist \( q, w \in A \) such that \( p = qp^2w \). Thus

\[
\hat{\varepsilon}(p) \subseteq k^+ \hat{\varepsilon}(p^2) \subseteq k^+ \hat{\varepsilon}(p^2w) \subseteq k^+ \hat{\varepsilon}(qp^2w) = \hat{\varepsilon}(p).
\]

Hence \( \hat{\varepsilon}(p) \cong k^+ \hat{\varepsilon}(p^2) \). Therefore, \( \hat{\varepsilon}(p) \cong k^+ \hat{\varepsilon}(p^2) \) for each \( k \in [0,1] \), \( \sup_k^+ \)-HFI \( \hat{\varepsilon} \) of \( A \) and \( p \in A \).

(1) \( \Rightarrow \) (3). It is similar to prove (1) \( \Rightarrow \) (2).

(7) \( \Rightarrow \) (1). Assume that (7) holds and \( p \in A \). Then \( J[p^2] = \{p^2\} \cup Ap^2 \cup p^2A \cup Ap^2A \) is an interior ideal of \( A \) and by Theorem 2.9, \( \hat{\varepsilon}_{J[p^2]} \) is a \( \sup \)-HFI of \( A \). Since \( p^2 \in J[p^2] \), we get

\( \hat{\varepsilon}_{J[p^2]}(p) \cong \hat{\varepsilon}_{J[p^2]}(p^2) = [0,1] \) which implies that \( \hat{\varepsilon}_{J[p^2]}(p) = [0,1] \). Thus \( p \in J[p^2] \) and so \( p \in Ap^2A \). Hence \( A \) is intra-regular.

(6) \( \Rightarrow \) (1). It is similar to prove (7) \( \Rightarrow \) (1). \( \square \)

Proposition 4.2. Let \( \hat{\varepsilon} \) be a HFS of an intra-regular semigroup \( A \). Then the followings are true:

(1) \( \hat{\varepsilon}(pq) \cong_A \hat{\varepsilon}(qp) \) for each \( \sup_A^+ \)-HFI of \( A \) and \( p, q \in A \),
(2) \( \hat{\varepsilon}(pq) \cong_A \hat{\varepsilon}(qp) \) for each \( \sup_A^+ \)-HFI of \( A \) and \( p, q \in A \).

Proof. (1). Let \( \hat{\varepsilon} \) be a \( \sup_A^+ \)-HFI of \( A \) and \( p, q \in A \). By Theorem 4.1, we have

\[
\hat{\varepsilon}(pq) \subseteq_A \hat{\varepsilon}(p(sp)q) = \hat{\varepsilon}((pq)^2) \cong_A \hat{\varepsilon}(pq) \subseteq_A \hat{\varepsilon}(qs(sp)q) = \hat{\varepsilon}((pq)^2) \cong_A \hat{\varepsilon}(pq).
\]

Thus \( \hat{\varepsilon}(pq) \cong_A \hat{\varepsilon}(pq) \).

(2). It follows from (1) and Proposition 4.1. \( \square \)

Similarly we can prove the following theorem.

Proposition 4.3. Let \( \hat{\varepsilon} \) be a HFS of an intra-regular semigroup \( A \). Then the followings are true:

(1) \( \hat{\varepsilon}(pq) \cong_B \hat{\varepsilon}(qp) \) for each \( \sup_B^- \)-HFI of \( A \) and \( p, q \in A \),
(2) \( \hat{\varepsilon}(pq) \cong_B \hat{\varepsilon}(qp) \) for each \( \sup_B^- \)-HFI of \( A \) and \( p, q \in A \).

Theorem 4.2. Let \( A \) be a semigroup. The followings are equivalent:

(1) \( A \) is left regular,
Then it can be easily seen the following conditions:

\[ (2) \, \bar{\varepsilon}(p) \cong_k^+ \bar{\varepsilon}(p^2) \text{ for each } k \in [0, 1], \sup_k^+\text{-HFLI of } \mathcal{A} \text{ and } p \in \mathcal{A}, \]

\[ (3) \, \bar{\varepsilon}(p) \cong_k^- \bar{\varepsilon}(p^2) \text{ for each } k \in [0, 1], \sup_k^-\text{-HFLI of } \mathcal{A} \text{ and } p \in \mathcal{A}, \]

\[ (4) \, \bar{\varepsilon}(p) \cong \bar{\varepsilon}(p^2) \text{ for each sup-HFLI of } \mathcal{A} \text{ and } p \in \mathcal{A}. \]

**Proof.** (2) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (4). It follows from Proposition 3.10.

(1) \( \Rightarrow \) (2). Assume that (1) holds, \( k \in [0, 1], \bar{\varepsilon} \) is a \( \sup_k^+\text{-HFLI of } \mathcal{A} \) and \( p \in \mathcal{A} \). Since \( \mathcal{A} \) is left regular, there exists \( q \in \mathcal{A} \) such that \( p = qp^2 \). Thus, by \( \bar{\varepsilon} \) is a \( \sup_k^+\text{-HFLI of } \mathcal{A} \), we obtain

\[ \bar{\varepsilon}(p) \sqsubseteq_k^+ \bar{\varepsilon}(p^2) \sqsubseteq_k^+ \bar{\varepsilon}(qp^2) = \bar{\varepsilon}(p). \]

Hence \( \bar{\varepsilon}(p) \cong_k^+ \bar{\varepsilon}(p^2) \). Therefore \( \bar{\varepsilon}(p) \cong_k^+ \bar{\varepsilon}(p^2) \) for each \( k \in [0, 1], \sup_k^+\text{-HFLI of } \mathcal{A} \) and \( p \in \mathcal{A} \).

(4) \( \Rightarrow \) (1). Assume that (4) holds and \( p \in \mathcal{A} \). Then \( L[p^2] = \{ p^2 \} \cup Ap^2 \) is a left ideal of \( \mathcal{A} \) and by Theorem 2.11, we get that \( \hat{\mathcal{X}}_{L[p^2]} \) is a sup-HFLI of \( \mathcal{A} \). Since \( p^2 \in L[p^2] \), we have \( \hat{\mathcal{X}}_{L[p^2]}(p) \cong \hat{\mathcal{X}}_{L[p^2]}(p^2) = [0, 1] \) and so \( \hat{\mathcal{X}}_{L[p^2]}(p) = [0, 1] \). Thus \( p \in L[p^2] \) which implies that \( p \in Ap^2 \). Therefore, \( \mathcal{A} \) is left regular. \( \square \)

The left-right dual of Theorem 4.2 reads as follows:

**Theorem 4.3.** Let \( \mathcal{A} \) be a semigroup. The followings are equivalent:

1. \( \mathcal{A} \) is right regular,
2. \( \bar{\varepsilon}(p) \cong_k^+ \bar{\varepsilon}(p^2) \) for each \( k \in [0, 1], \sup_k^+\text{-HFRI } \bar{\varepsilon} \) of \( \mathcal{A} \) and \( p \in \mathcal{A} \),
3. \( \bar{\varepsilon}(p) \cong_k^- \bar{\varepsilon}(p^2) \) for each \( k \in [0, 1], \sup_k^-\text{-HFRI } \bar{\varepsilon} \) of \( \mathcal{A} \) and \( p \in \mathcal{A} \),
4. \( \bar{\varepsilon}(p) \cong \bar{\varepsilon}(p^2) \) for each sup-HFRI \( \bar{\varepsilon} \) of \( \mathcal{A} \) and \( p \in \mathcal{A} \).

**Theorem 4.4.** Let \( \mathcal{A} \) be a semigroup. The followings are equivalent:

1. \( \mathcal{A} \) is completely regular,
2. \( \bar{\varepsilon}(p) \cong_k^+ \bar{\varepsilon}(p^2) \) and \( \hat{\omega}(p) \cong_k^+ \hat{\omega}(p^2) \) for each \( k \in [0, 1], \sup_k^+\text{-HFRI } \bar{\varepsilon}, \sup_k^+\text{-HFLI } \hat{\omega} \) of \( \mathcal{A} \) and \( p \in \mathcal{A} \),
3. \( \bar{\varepsilon}(p) \cong_k^- \bar{\varepsilon}(p^2) \) and \( \hat{\omega}(p) \cong_k^- \hat{\omega}(p^2) \) for each \( k \in [0, 1], \sup_k^-\text{-HFRI } \bar{\varepsilon}, \sup_k^-\text{-HFLI } \hat{\omega} \) of \( \mathcal{A} \) and \( p \in \mathcal{A} \),
4. \( \bar{\varepsilon}(p) \cong \bar{\varepsilon}(p^2) \) and \( \hat{\omega}(p) \cong \hat{\omega}(p^2) \) for each sup-HFRI \( \bar{\varepsilon}, \sup\text{-HFLI } \hat{\omega} \) of \( \mathcal{A} \) and \( p \in \mathcal{A} \).

**Proof.** It follows from Theorems 4.2 and 4.3. \( \square \)

A HFS \( \bar{\varepsilon} \) on \( \mathcal{A} \) is called

1. constant if \( \bar{\varepsilon}(p) = \bar{\varepsilon}(q) \) for all \( p, q \in \mathcal{A} \),
2. \( \sup_k^+\text{-constant if } \bar{\varepsilon}(p) \cong_k^+ \bar{\varepsilon}(q) \) for all \( p, q \in \mathcal{A} \),
3. \( \sup_k^-\text{-constant if } \bar{\varepsilon}(p) \cong_k^- \bar{\varepsilon}(q) \) for all \( p, q \in \mathcal{A} \), and
4. sup-constant if \( \bar{\varepsilon}(p) \cong \bar{\varepsilon}(q) \) for all \( p, q \in \mathcal{A} \).

Then it can be easily seen the following conditions:

1. if \( \bar{\varepsilon} \) is constant, then \( \bar{\varepsilon} \) is sup-constant,
(2) if $\bar{e}$ is sup-constant, then $\bar{e}$ is both $\sup_k^+$-constant and $\sup_k^-$-constant.

**Theorem 4.5.** Let $\mathcal{A}$ be a semigroup. The followings are equivalent:

1. $\mathcal{A}$ is left simple,
2. $\bar{e}$ is $\sup_k^+$-constant for every $k \in [0, 1]$ and $\sup_k^+$-HFLI $\bar{e}$ of $\mathcal{A}$,
3. $\bar{e}$ is $\sup_k^-$-constant for every $k \in [0, 1]$ and $\sup_k^-$-HFLI $\bar{e}$ of $\mathcal{A}$,
4. $\bar{e}$ is sup-constant for every sup-HFLI $\bar{e}$ of $\mathcal{A}$.

**Proof.** $(2) \Leftrightarrow (3) \Leftrightarrow (4)$. It follows from Proposition 3.10.

$(1) \Rightarrow (2)$. Assume that $(1)$ holds, $k \in [0, 1]$ and $\bar{e}$ is a $\sup_k^+$-HFLI of $\mathcal{A}$. Let $p, q \in \mathcal{A}$. Since $\mathcal{A}$ is left simple, we have $p \in \mathcal{A} = Ap$ and $q \in \mathcal{A} = Ap$. Thus $p = w_1 q$ and $q = w_2 p$ for some $p, q \in \mathcal{A}$. Since $\bar{e}$ is a $\sup_k^+$-HFLI of $\mathcal{A}$, we get

$$\bar{e}(p) \subseteq^+ \bar{e}(w_2 p) = \bar{e}(q) \subseteq^+ \bar{e}(w_1 q) = \bar{e}(p).$$

Then $\bar{e}(p) \cong^+ \bar{e}(q)$. Hence $\bar{e}$ is $\sup_k^+$-constant. Therefore, we obtain that $\bar{e}$ is $\sup_k^+$-constant for every $k \in [0, 1]$ and $\sup_k^+$-HFLI $\bar{e}$ of $\mathcal{A}$.

$(4) \Rightarrow (1)$. Assume that $(4)$ holds. Let $L$ be a left ideal of $\mathcal{A}$ and $w \in L$. Then, by Theorem 2.11, we have $\hat{\chi}_L$ is sup-HFLI of $\mathcal{A}$. By assumption $(4)$, we get that $\hat{\chi}_L$ is sup-constant. Thus $\hat{\chi}_L(p) \cong \hat{\chi}_L(w) = [0, 1]$ for all $p \in \mathcal{A}$ which implies that $\hat{\chi}_L(p) = [0, 1]$ for all $p \in \mathcal{A}$. Hence $\mathcal{A} = L$. Therefore, $\mathcal{A}$ is left simple.

The left-right dual of Theorem 4.5 reads as follows:

**Theorem 4.6.** Let $\mathcal{A}$ be a semigroup. The followings are equivalent:

1. $\mathcal{A}$ is right simple,
2. $\bar{e}$ is $\sup_k^+$-constant for every $k \in [0, 1]$ and $\sup_k^+$-HFRI $\bar{e}$ of $\mathcal{A}$,
3. $\bar{e}$ is $\sup_k^-$-constant for every $k \in [0, 1]$ and $\sup_k^-$-HFRI $\bar{e}$ of $\mathcal{A}$,
4. $\bar{e}$ is sup-constant for every sup-HFRI $\bar{e}$ of $\mathcal{A}$.

The following theorem can be seen in a similar way as in the proof of Theorem 4.5.

**Theorem 4.7.** Let $\mathcal{A}$ be a semigroup. The followings are equivalent:

1. $\mathcal{A}$ is simple,
2. $\bar{e}$ is $\sup_k^+$-constant for every $k \in [0, 1]$ and $\sup_k^+$-HFLI $\bar{e}$ of $\mathcal{A}$,
3. $\bar{e}$ is $\sup_k^-$-constant for every $k \in [0, 1]$ and $\sup_k^-$-HFLI $\bar{e}$ of $\mathcal{A}$,
4. $\bar{e}$ is sup-constant for every sup-HFLI $\bar{e}$ of $\mathcal{A}$.

From Theorems 4.5 and 4.6, we have the following theorem.

**Theorem 4.8.** Let $\mathcal{A}$ be a semigroup. The followings are equivalent:

1. $\mathcal{A}$ is group,
(2) $\hat{\epsilon}$ and $\hat{\omega}$ are $\sup_k^+$-constant for every $k \in [0,1]$, $\sup_k^+$-HFLI $\hat{\epsilon}$ and $\sup_k^+$-HFRI $\hat{\omega}$ of $A$.

(3) $\hat{\epsilon}$ and $\hat{\omega}$ are $\sup_k^-$-constant for every $k \in [0,1]$, $\sup_k^-$-HFLI $\hat{\epsilon}$ and $\sup_k^-$-HFRI $\hat{\omega}$ of $A$.

(4) $\hat{\epsilon}$ and $\hat{\omega}$ are $\sup$-constant for every $\sup$-HFLI $\hat{\epsilon}$ and $\sup$-HFRI $\hat{\omega}$ of $A$.

5. Conclusions and Future Works

In present paper, we have introduced the concepts of $\sup_\alpha^+$-HFRIs (resp., $\sup_\alpha^+$-HFLIs, $\sup_\alpha^+$-HFIs) and $\sup_\beta^-$-HFRIs (resp., $\sup_\beta^-$-HFLIs, $\sup_\beta^-$-HFIs) which are generalizations of the concepts of $\sup$-HFRIs (resp., $\sup$-HFLIs, $\sup$-HFIs) of semigroups, and discussed their some properties. Furthermore, the concepts have been established by FSs, Łukasiewicz fuzzy sets, Łukasiewicz anti-fuzzy sets, PFSs, HFSs, hybrid sets, IvFSs and cubic sets. Finally, we have characterized intra-regular, left (right) regular, completely regular, left (right) simple and simple semigroups in terms of $\sup_\alpha^+$-type and $\sup_\beta^-$-type of hesitant fuzzy sets.

The following are objectives for study and research in semigroups and other algebras:

- to introduce and study $\sup_\alpha^+$-type and $\sup_\beta^-$-type of HFSs based on bi-ideals of semigroups,
- to introduce and study $\sup_\alpha^+$-type and $\sup_\beta^-$-type of HFSs based on ideal theory in BCK/BCI-algebras, ternary semigroups, $\Gamma$-semigroups and LA-semigroups,
- to introduce and study $\sup_\alpha^+$-type and $\sup_\beta^-$-type of HFSs based on substructures of GE-algebras, BRK-algebras, BE-algebras and IUP-algebras [5, 6, 11, 26],
- to apply this study to the concept of rough sets according to Ansari’s study [2, 3].

Acknowledgment: This work was supported by Research Development and Research Management Fund of Pibulsongkram Rajabhat University for 2022 [Grant number RDI-2-65-19]. The authors wish to express their sincere thanks to the referees for the valuable suggestions which lead to an improvement of this paper.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. Anis, M. Khan, Y.B. Jun, Hybrid Ideals in Semigroups, Cogent Math. 4 (2017), Art. ID 1352117. https://doi.org/10.1080/23311835.2017.1352117.

[2] M.A. Ansari, Rough Set Theory Applied to JU-Algebras, Int. J. Math. Comput. Sci. 16 (2021), 1371–1384.

[3] M.A. Ansari, A. Iampan, Generalized Rough $(m, n)$ Bi-$\Gamma$-Ideals in Ordered LA-$\Gamma$-Semigroups, Commun. Math. Appl. 12 (2021), 545–557.

[4] K.T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 20 (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3.

[5] R.K. Bandaru, On BRK-Algebras, Int. J. Math. Math. Sci. 2012 (2012), Art. ID 952654. https://doi.org/10.1155/2012/952654.

[6] R.K. Bandaru, A.B. Saeid, Y.B. Jun, On GE-Algebras, Bull. Sect. Logic Univ. Łódź 50 (2021), 81–96. https://doi.org/10.18775/0138-0680.2020.20.
[7] V. Chinnadurai, A. Arulselvam, On Pythagorean Fuzzy Ideals in Semigroups, J. Xi’an Univ. Archit. Technol. 12 (2020), 1005–1012.

[8] T. Guntasow, S. Sajak, A. Jomkham, A. Iampan, Fuzzy Translations of a Fuzzy Set in UP-Algebras, J. Indones. Math. Soc. 23 (2017), 1–19. https://doi.org/10.22232/jims.23.2.371.1-19.

[9] B. Elavarasan, K. Porsevli, Y.B. Jun, Hybrid Generalized Bi-Ideals in Semigroups, Int. J. Math. Comput. Sci. 14 (2019), 601–612.

[10] H. Harizavi, Y.B. Jun, Sup-Hesitant Fuzzy Quasi-Associative Ideals of BCI-Algebras, Filomat. 34 (2020), 4189–4197. https://doi.org/10.2298/FIL2012189H.

[11] A. Iampan, P. Julatha, P. Khamrot, et al. Independent UP-Algebras, J. Math. Comput. Sci. 27 (2022), 65–76. https://doi.org/10.22436/jmcs.027.01.06.

[12] U. Jittburus, P. Julatha, New Generalizations of Hesitant and Interval-Valued Fuzzy Ideals of Semigroups, Adv. Math., Sci. J. 10 (2021), 2199–2212. https://doi.org/10.37418/amssj.10.4.34.

[13] P. Julatha, T. Gaketem, P. Khamrot, et al. Sup-Hesitant Fuzzy Ideals and Bi-Ideals of Semigroups, Submitted.

[14] P. Julatha, A. Iampan, A New Generalization of Hesitant and Interval-Valued Fuzzy Ideals of Ternary Semigroups, Int. J. Fuzzy Log. Intell. Syst. 21 (2021), 169–175. https://doi.org/10.5391/IJFIS.2021.21.2.169.

[15] P. Julatha, A. Iampan, On inf-Hesitant Fuzzy Γ-Ideals of Γ-Semigroups, Adv. Fuzzy Syst. 2022 (2022), 9755894. https://doi.org/10.1155/2022/9755894.

[16] P. Julatha, A. Iampan, SUP-Hesitant Fuzzy Ideals of Γ-Semigroups, J. Math. Comput. Sci. 26 (2022), 148–161. https://doi.org/10.22436/jmcs.026.02.05.

[17] P. Julatha, M. Siripitukdet, Some Characterizations of Anti-Fuzzy (Generalized) Bi-Ideals of Semigroups, Thai J. Math. 16 (2018), 335–346.

[18] Y.B. Jun, M.S. Kang, C.H. Park, N-Subalgebras in BCK/BCI-Algebras Based on Point N-Structures, Int. J. Math. Math. Sci. 2010 (2010), 303412. https://doi.org/10.1155/2010/303412.

[19] Y.B. Jun, C.S. Kim, M.S. Kang, Cubic Subalgebras and Ideals of BCK/BCI-Algebras, Far East J. Math. Sci. 44 (2010), 239–250.

[20] Y.B. Jun, A. Khan, Cubic Ideals in Semigroups, Honam Math. J. 35 (2013), 607–623. https://doi.org/10.5831/HMJ.2013.35.4.607.

[21] Y.B. Jun, K.J. Lee, S.Z. Song, Hesitant Fuzzy Bi-Ideals in Semigroups, Commun. Korean Math. Soc. 30 (2015), 143–154. https://doi.org/10.4134/CKMS.2015.30.3.143.

[22] Y.B. Jun, Łukasiewicz Anti Fuzzy Set and Its Application in BE-Algebras, Trans. Fuzzy Sets Syst. In Press. http://doi.org/10.30495/TFSS.2022.1960391.1037.

[23] Y.B. Jun, Łukasiewicz Fuzzy Subalgebras in BCK-Algebras and BCI-Algebras, Ann. Fuzzy Math. Inform. 23 (2022), 213–223. http://doi.org/10.30948/afmi.2022.23.2.213.

[24] Y.B. Jun, S.Z. Song, G. Muniuddin, Hybrid Structures and Applications, Ann. Commun. Math. 1 (2018), 11–25.

[25] M. Khan, T. Asif, Characterizations of Semigroups by Their Anti-Fuzzy Ideals, J. Math. Res. 2 (2010), 134–143.

[26] H.S. Kim, Y.H. Kim, On BE-Algebras, Sci. Math. Jpn. 66 (2007), 113–116. https://doi.org/10.32219/iamsm.66.1_113.

[27] N. Kuroki, Fuzzy Bi-Ideals in Semigroups, Comment. Math. Univ. St. Pauli. 28 (1979), 17–21. https://doi.org/10.14992/00010265.

[28] N. Kuroki, On Fuzzy Ideals and Fuzzy Bi-Ideals in Semigroups, Fuzzy sets Syst. 5 (1981), 203–215. https://doi.org/10.1016/0165-0114(81)90018-X.

[29] K.J. Lee, Bipolar Fuzzy Subalgebras and Bipolar Fuzzy Ideals of BCK/BCI-Algebras, Bull. Malays. Math. Sci. Soc. 32 (2009), 361–373.

[30] J.N. Mordeson, D.S. Malik, N. Kuroki, Fuzzy Semigroups, Springer, Berlin, (2012).
[31] G. Muhiuddin, Cubic Interior Ideals in Semigroups, Appl. Appl. Math. 14 (2019), 463–474. https://digitalcommons.pvamu.edu/aam/vol14/iss1/32.

[32] G. Muhiuddin, Y.B. Jun, Sup-Hesitant Fuzzy Subalgebras and Its Translations and Extensions, Ann. Commun. Math. 2 (2019), 48–56.

[33] A. L. Narayanan, T. Manikantan, Interval-Valued Fuzzy Ideals Generated by an Interval-Valued Fuzzy Subset in Semigroups, J. Appl. Math. Comput. 20 (2006), 455–464. https://doi.org/10.1007/BF02831952.

[34] Z. Pawlak, Rough Sets, Int. J. Inform. Comput. Sci. 11 (1982), 341–356. https://doi.org/10.1007/BF01001956.

[35] P. Phummee, S. Papan, C. Noyoampaeng, et al. sup-Hesitant Fuzzy Interior Ideals of Semigroups and Their sup-Hesitant Fuzzy Translations, Int. J. Innov. Comput. Inform. Control. 18 (2022), 121–132. https://doi.org/10.24507/ijicic.18.01.121.

[36] N. Ratchakhwan, P. Julatha, T. Gaketem, et al. (inf, sup)-Hesitant Fuzzy Ideals of BCK/BCI-Algebras, Int. J. Anal. Appl. 20 (2022), 34. https://doi.org/10.28924/2291-8639-20-2022-34.

[37] M. Shabir, Y. Nawaz, Semigroups Characterized by the Properties of Their Anti-Fuzzy Ideals, J. Adv. Res. Pure Math. 1 (2009), 42–59.

[38] M.M.Takallo, R.A. Borzooei, Y.B. Jun, Sup-Hesitant Fuzzy p-Ideals of BCI-Algebras, Fuzzy Inform. Eng. 13 (2021), 460–469. https://doi.org/10.1080/16168658.2021.1993668.

[39] A.F. Talee, M.Y. Abassi, S.A. Khan, Hesitant Fuzzy Ideals in Semigroups With a Frontier, Arya Bhatta J. Math. Inform. 9 (2017), 163–170.

[40] N. Thillaigovindan, V. Chinnadurai, On Interval Valued Fuzzy Quasi-Ideals of Semigroups, East Asian Math. J. 25 (2009), 449–467.

[41] V. Torra, Hesitant Fuzzy Sets, Int. J. Intell. Syst. 25 (2010), 529–539. https://doi.org/10.1002/int.20418.

[42] V. Torra, Y. Narukawa, On hesitant fuzzy sets and decision, in: 2009 IEEE International Conference on Fuzzy Systems, IEEE, Jeju Island, South Korea, 2009: pp. 1378–1382. https://doi.org/10.1109/FUZZY.2009.5276884.

[43] S. Umar, A. Hadi, A. Kham, On Prime Cubic Bi-Ideals of Semigroups, Ann. Fuzzy Math. Inform. 9 (2015), 957–974.

[44] R. R. Yager, A. M. Abbasov, Pythagorean Membership Grades, Complex Numbers, and Decision Making, Int. J. Intell. Syst. 28 (2013), 436–452. https://doi.org/10.1002/int.21584.

[45] R.R. Yager, Pythagorean Fuzzy Subsets, in: 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS). IEEE, Edmonton, AB, Canada, 2013: pp. 57–61. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375.

[46] L. A. Zadeh, Fuzzy Sets, Inform. Control. 8 (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[47] L. A. Zadeh, The Concept of a Linguistic Variable and Its Application to Approximate Reasoning I, Inform. Sci. 8 (1975), 199–249. https://doi.org/10.1016/0020-0255(75)90036-5.

[48] W.R. Zhang, Bipolar Fuzzy Sets and Relations: A Computational Framework for Cognitive Modeling and Multiagent Decision Analysis, in: NAFIPS/IFIS/NASA ’94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intelligent Systems Conference, and the NASA Joint Technology Wo, IEEE, San Antonio, TX, USA, 1994: pp. 305–309. https://doi.org/10.1109/IJCF.1994.375115.