Bogoliubov spectrum and the dynamic structure factor in a quasi-two-dimensional spin-orbit coupled BEC

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Abstract

We compute the Bogoliubov–de Gennes excitation spectrum in a trapped two-component spin-orbit-coupled (SOC) Bose–Einstein condensate (BEC) in quasi-two-dimensions as a function of linear and angular momentum and analyze them. The excitation spectrum exhibits a minima-like feature at finite momentum for the immiscible SOC-BEC configuration, which implies dynamical instability. We augment these results by computing the dynamic structure factor in the density and pseudo-spin sector, and discuss its interesting features that can be experimentally measured through Bragg spectroscopy of such ultracold condensates.

Keywords: spin-orbit coupled BEC, Bogoliubov–de-Gennes excitation, dynamic structure factor, quasi two dimensional condensate

1. Introduction

The low energy excitation spectrum of an ultracold atomic Bose–Einstein condensate (BEC) [1, 2] reveals a plethora of information about the collective behavior of these macroscopic quantum systems. For example, the dispersion relation indicates the breakdown of superfluidity when an object moves through BECs [3–5], the nature of the quasiparticle modes plays a vital role in characterizing critical behavior near the phase transition [6], and most importantly it registers the response of such superfluids to external fluctuations [7] within the framework of perturbation theory. Realization of artificial light-induced spin–orbit coupling [8–10] in BECs has appended a new powerful tool to the simulation toolbox [11] for such ultracold atomic systems. Naturally, an extensive insight into the novel properties of such systems can be gained by looking into the behavior of their collective excitations [12–18].

Several studies investigating the effect of collective modes on various properties of such spin-orbit-coupled (SOC) bosonic superfluids in different dimensions have been carried out focusing on certain specific aspects. They include the study of mean-field dynamics of SOC-BEC in terms of their collective modes in one- and two-dimensions [14, 15], the role of dipole oscillation on the collective behavior of the system as it goes through a phase transition [16, 17], the change in the velocity of sound as a quasi-one dimensional SOC condensate goes through a second-order phase transition [19], study of collective modes under hydrodynamic approximation [18, 19], and the existence of mean-field ground state with exotic topology and its subsequent evolution [13]. For a SOC-BEC with one-dimensional spin–orbit coupling, the collective excitations have been experimentally observed using Bragg spectroscopy, both in homogeneous [20] and trapped configurations [21]. In a recent work [22], a system of binary BEC in...
quasi-two-dimensions with spin-orbit (SO) and Rabi coupling was also considered. By time evolving the system using the time-dependent Gross–Pitaevskii (GP) equation and analyzing the evolution of the stationary state solution in terms of plane wave excitations within the framework of Bogoliubov theory, the stability of the phases was examined. These theoretical and experimental studies clearly point to the necessity of a detailed investigation of the full spectrum of these SOC-BEC systems in realistic trapping geometry for other dimensions, particularly in two spatial dimensions, and the subsequent calculation of the dynamic structure factor using these excitations that can be experimentally measured using Bragg spectroscopy [23–27].

Accordingly, in this work we investigate the excitation spectrum of a trapped quasi-two-dimensional SOC-BEC for both miscible and immiscible configurations and present a systematic analysis of the nature of various quasiparticle excitation modes exploring a wide range of excitation spectra. In particular, we evaluate the excitation spectrum both as a function of linear momentum as well as angular momentum, which have not been explored in detail in earlier works. Through this analysis we demonstrate that the lowest-energy excitation in SOC-BEC has a finite rms value of angular momentum, which consequently leads to a violation of the irrotational condition on the velocity fields in the SOC-BEC superfluid [28]. This is in contrast to the cases of the single-component (scalar) BEC and two-component BEC without SOC. Furthermore, we observe a minima-like dip at finite wave vector in the excitation spectrum and observe smudged amplitudes of the quasi-particles near the minima in the immiscible case. Such minima in the excitation spectrum hold an important indication of the presence of roton-softening, which is an essential characteristic of supersolids [19, 20]. Subsequently, we study the consequences of the Bogoliubov–de Gennes (BdG) spectrum by computing the dynamic structure factor in the linear-response regime, a quantity that can be explored in experiments using Bragg spectroscopy.

Accordingly, the outline of the paper is as follows. In section 2.1, we discuss briefly the model system taken under consideration. In section 2.2, we discuss the BdG excitation spectrum for this configuration and characterize the various quasiparticle excitations for various ranges of momenta and energy. We also discuss the features of the low-lying excitations such as dipole and quadrupole modes in this system. In section 3, we compute the static and dynamic structure factor by which one can experimentally probe excitations in the trapped SOC-BEC.

2. Model Hamiltonian for SOC BEC and their low energy collective excitations

In this section we spell out the details of the methodology for computing the excitation spectrum of a SOC-BEC. In the first part section 2.1, we shall introduce the nature of spin–orbit coupling, the model Hamiltonian, and the corresponding GP equation. In section 2.2 we introduce the details of the Bogoliubov de Gennes formalism applied to such SOC BEC.

2.1. Synthetic spin–orbit coupling for an ultracold atom and the GP equation

We consider the spin-orbit coupled single-particle Hamiltonian, possessing a non-abelian gauge potential of the form $A : m(n\hat{\sigma}_y, \eta' \hat{\sigma}_z, 0)$ [29], given by:

$$\hat{h} = \frac{\hat{p}^2}{2m} - \eta \hat{p}_x \hat{\sigma}_y - \eta' \hat{p}_y \hat{\sigma}_z$$

(1)

where $\hat{p} = \{\hat{p}_x, \hat{p}_y, \hat{p}_z\}$; $\hat{\sigma}$ and $\hat{l}$ are Pauli and Identity matrices, respectively, and $\eta, \eta'$ are the strengths of spin–orbit coupling (SOC). The interaction effects are incorporated through the two-body mean field interaction term $1/2 \sum_{\kappa \kappa'} \int d\mathbf{r} V^{\kappa \kappa'}_{int} n_{\kappa} n_{\kappa'}$ where $n_{\kappa}$ is the density of the ‘$\kappa$’-component, $V^{\kappa \kappa'}_{int} = 4\pi \hbar^2 a_{\kappa \kappa'} / m$ correspond to the coupling constants between different spin channels and $a_{\kappa \kappa'}$ represents the $s$-wave scattering length.

Any combination of Rashba [30, 31] or linear Dresselhaus spin-orbit interaction [32] can be reproduced from the model Hamiltonian chosen in this work, equation (1). For instance, the spin–orbit coupling term of the considered Hamiltonian for isotropic SOC strengths $\eta = \eta'$, is equivalent to the Rashba SO coupling [30, 31]. Equation (1) reduces to the Rashba model through a rotation ‘exp(\text{i}\eta' \hat{\sigma}_y / 4)’ in the pseudo-spin space for $\eta' = \eta$. One can also examine ‘Ising-type’ SO coupling in atomic systems, by considering the SOC strengths as $\eta \neq \eta' = 0$ [33].

The full Hamiltonian within the framework of GP theory, after projecting the single-particle Hamiltonian into the lower energy subspace, satisfies the following spinorial time-dependent GP equation (see [34] for details):

$$i\hbar \frac{\partial \psi_\kappa}{\partial t} = \left[ \frac{\hbar^2}{2m} \left( -i \hat{\sigma}_z - \kappa \hat{\sigma}_y \right)^2 - \frac{\hbar^2}{2m} \hat{\sigma}_y^2 V_{\text{Trap}} + \frac{g + g_-}{2} \psi^\dagger \psi \frac{g - g_+}{2} \left( \psi^\dagger \hat{\sigma}_z \psi \right) \right] \psi_\kappa$$

(2)

where $\psi = [\psi_+, \psi_-]^T$ and $\kappa = \pm$ labels the two components. Also, $m_s = \frac{m}{1 - (\frac{\hbar^2}{2m})^2}$ is the effective mass along the $y$-direction (with $\eta' < \eta$) and, $V_{\text{Trap}} = m (\omega_x^2 x^2 + \omega_y^2 y^2)/2$ is the external trapping potential with trapping frequencies $\omega_{x,y}$ along $x$ and $y$ directions, respectively. Here, we have considered tight confinement along the $z$-direction and thus frozen the dynamics along the $z$-axis [35]. We have considered the total number of atoms $N = 6 \times 10^3$ in our simulations with equal atom number in the two-components, and the harmonic trap parameters are $\omega_x = \omega_y = 2\pi \times 4.5$ Hz, $\omega_z = 2\pi \times 123$ Hz. The inter- and intraspecies interaction strengths for this quasi-two-dimensional condensate are denoted by $g_{+-}$.
and \( g_{\kappa \kappa'} \), respectively; \( g_{\kappa \kappa'} = \sqrt{\frac{m_{\gamma'}}{m_{\gamma}}} \) where \( a_\perp = \sqrt{\frac{\hbar}{m_{\gamma}}} \) is the transverse harmonic oscillator length.

In equation (2), we have considered the intraspecies interaction strengths to be equal \( g_{++} = g_{--} (= g) \), which is a good approximation to the experimental situation with \(^{87}\)Rb atoms [36] chosen in this work. In various places in the paper, we compare our results to a single-component BEC [1, 2, 37] and a two-component BEC [1, 2, 38–40] as limiting cases of equation (2). This corresponds to substituting the parameters \( g_{+-} = 0, m_1 = m_2 \) and \( \eta = 0 \) in equation (2), for the single-component BEC. For a two-component BEC we set in equation (2), \( g_{+-} \neq 0, m_1 = m_2 \) and \( \eta = 0 \). Additionally, we consider the miscible, i.e., \( g_{+-} < g \) and immiscible, i.e., \( g_{+-} > g \) [41, 42], configurations for a given spin–orbit coupling strength. The condensate components overlap with each other in the miscible configuration, whereas they are spatially separated in the immiscible configuration.

2.2. Bogoliubov–de Gennes formalism

We numerically propagate the GP equation (2) in imaginary time to obtain the two-component ground state wave function of the condensate, \( \psi_\kappa(x,y) = \{ \psi_{\kappa,0}^*(x,y) \psi_{\kappa,0}^-(x,y) \}^T \). The ground state solution obtained from the GPE in this way, for various cases, are shown in figures 16 and 17. The excitation spectrum and the nature of the quasiparticle amplitudes in such SOC-BEC are then obtained in the framework of the Bogoliubov theory [1, 2] by considering the fluctuations over the ground state wavefunction, of the GP equation (2) as:

\[
\begin{pmatrix}
\psi_+ \\
\psi_-
\end{pmatrix} = e^{-\frac{i}{\hbar} \int dx dy \left[ \begin{array}{c}
\psi_{+0} \\
\psi_{-0}
\end{array} \right] + \sum_j \left[ \begin{array}{c}
u_{+j} \\
\nu_{-j}
\end{array} \right] e^{-i \omega_j t} \left[ \begin{array}{c}
\phi_{+j} \\
\phi_{-j}
\end{array} \right] e^{i \omega_j t} \right)
\]

(3)

where \( \mu \) is the chemical potential of the condensate, and \( \omega_j = \epsilon_j / \hbar \) where \( \epsilon_j \) is the energy corresponding to each quasiparticle excitation.

The index ‘\( j \)’ represents the sequence of quasiparticle excitation. ‘\( u_{\kappa,j} \)’ and ‘\( v_{\kappa,j} \)’ are spatially dependent complex functions denoting the Bogoliubov quasiparticle amplitudes corresponding to the \( j \)th energy eigenstate and are normalized as,

\[
\int \int dx dy \sum_{\kappa = \pm} \left[ |u_{\kappa,j}(x,y)|^2 - |v_{\kappa,j}(x,y)|^2 \right] = 1.
\]

(4)

Inserting \( \psi_{\pm}(r,t) \) using equation (3) into GP equation (2) and retaining the fluctuations up to the linear order, we get the Bogoliubov–de-Genes (BdG) equations for the considered SOC-BEC system:

\[
\begin{pmatrix}
L_+ \\
L_-
\end{pmatrix} = \begin{pmatrix}
\frac{\hbar^2}{2m_{ \gamma}} \left[ -\frac{\partial^2}{\partial x^2} + \frac{m_{ \gamma}^2 \hbar^2}{2} \right] - \frac{\hbar^2}{2m_{ \gamma}} \partial^2 + V_{\text{Trap}} \\
\frac{\hbar^2}{2m_{ \gamma}} \left[ -\frac{\partial^2}{\partial x^2} + \frac{m_{ \gamma}^2 \hbar^2}{2} \right] - \frac{\hbar^2}{2m_{ \gamma}} \partial^2 + V_{\text{Trap}}
\end{pmatrix} \begin{pmatrix}
u_{+j} \\
v_{-j}
\end{pmatrix} = \epsilon_j \begin{pmatrix}
u_{+j} \\
v_{-j}
\end{pmatrix}
\]

(5)

where,

\[
L_+ = \frac{\hbar^2}{2m_{ \gamma}} \left[ -\frac{\partial^2}{\partial x^2} + \frac{m_{ \gamma}^2 \hbar^2}{2} \right] - \frac{\hbar^2}{2m_{ \gamma}} \partial^2 + V_{\text{Trap}}
\]

\[
L_- = \frac{\hbar^2}{2m_{ \gamma}} \left[ -\frac{\partial^2}{\partial x^2} + \frac{m_{ \gamma}^2 \hbar^2}{2} \right] - \frac{\hbar^2}{2m_{ \gamma}} \partial^2 + V_{\text{Trap}}
\]

We diagonalize the above BdG matrix (5) numerically to find quasiparticle excitation energy \( \epsilon_j \), as well as the Bogoliubov quasiparticle amplitudes \( \{ u_{\kappa,j} \} \) and \( \{ v_{\kappa,j} \} \). To this end, we expand \( u_{\kappa,j} \) and \( v_{\kappa,j} \) in terms of harmonic oscillator eigenfunction \( \phi_{\kappa}(x) \) (for such harmonically trapped SOC-BEC (details given in appendix A) as

\[
\begin{pmatrix}
u_{+j} \\
v_{-j}
\end{pmatrix} = \sum_{k,l=0}^{N_{ \gamma}} p_{jkl} \phi_{k}(x) \phi_{l}(y)
\]

\[
\begin{pmatrix}
u_{+j} \\
v_{-j}
\end{pmatrix} = \sum_{k,l=0}^{N_{ \gamma}} q_{jkl} \phi_{k}(x) \phi_{l}(y)
\]

\[
\begin{pmatrix}
u_{+j} \\
v_{-j}
\end{pmatrix} = \sum_{k,l=0}^{N_{ \gamma}} r_{jkl} \phi_{k}(x) \phi_{l}(y)
\]

\[
\begin{pmatrix}
u_{+j} \\
v_{-j}
\end{pmatrix} = \sum_{k,l=0}^{N_{ \gamma}} s_{jkl} \phi_{k}(x) \phi_{l}(y)
\]

(6)

where \( p_{jkl}, q_{jkl}, r_{jkl} \) and \( s_{jkl} \) are the coefficients of the linear combination, and \( \phi_\kappa(x) = \sqrt{\frac{m_{\gamma}}{\pi \hbar^2}} e^{-x^2/2\hbar^2} H_n(x/a_\gamma) \); where \( H_n \) is the \( n \)th order Hermite polynomial and \( a_\gamma = \sqrt{\frac{\hbar}{m_{\gamma}}} \). Substituting the above expression, after some straightforward algebra, the resulting equation can be written as
where each matrix $[.]$ is of the dimension $(N_{p} + 1)^2 \times (N_{p} + 1)^2$ and the elements of each of these matrices are evaluated using the integrals defined in appendix A. The resulting BdG matrix in equation (7) has $4(N_{p} + 1) \times 4(N_{p} + 1)$ dimensions for equal number of basis ($=N_{p}$) along $x$ and $y$ directions. The size of the BdG matrix increases rapidly due the $N_{p}^2$ scaling. We obtain the converged eigenvalues (up to four digits after decimal) for $N_{p} = 50$.

To construct the dispersion relation of the Bogoliubov modes, we adopt the methodology used for the trapped configurations such as two-component BEC [43] and dipolar condensates [44–47]. Since there is no translational symmetry in such finite, trapped configuration, the linear momentum is not a good quantum number. Thus, each quasiparticle can be linked to an $rms$ value of the linear momentum. Additionally, the irrotationality condition that is obeyed in the single-component BEC-superfluid with no SOC coupling gets violated for strengths $\epsilon$ of both the linear and angular momenta is shown in equation (7) has $4(N_{p} + 1) \times 4(N_{p} + 1)$ dimensions for equal number of basis ($=N_{p}$) along $x$ and $y$ directions. The size of the BdG matrix increases rapidly due the $N_{p}^2$ scaling.

We start our discussion with the low-lying dipole and quadrupole excitations. Collective dipole oscillations represents the center-of-mass motion of all the atoms and are the excitations corresponding to the lowest finite energy modes [37]. In order to excite these modes, the trap can be suddenly displaced in such a way that the initial ground state of the BEC is now no longer the ground state of the displaced trap. This results in collective excitation of the atoms and thus can be easily excited experimentally [49]. For the first case shown in figure 1(a), the dipole mode occurs at energy $\epsilon = \hbar \omega_{\rho}$. This is also in consonance with the fact that for a scalar condensate, the frequency of the dipole oscillation is just the harmonic-trap frequency [37, 50]. However, for a SO-coupled condensate, the dipole-oscillation frequency deviates from the trap frequency [15–17, 28], which is also consistent with our simulations. For example, the energy at which the dipole mode occurs in SOC-BEC for strengths $\eta'/\eta = 0.78$ is $E = \epsilon = \hbar \omega_{\rho} = 0.58$, which is different from 1.

We next investigate the structure of the quasiparticle amplitudes to interpret the physics behind the dispersion curve. For comparison, the quasiparticle amplitudes $\{u_+, v_+\}$ of the dipole and quadrupole mode for the case of single-component BEC are shown in figures 2(a), (b) and (c), (d), respectively. Examining the dipole mode and quadrupole mode behavior provides a signature of the quantum phase transition and is studied in SOC-BEC experimentally [51]. We here show the quasiparticle amplitudes for both components labeled through $\pm$, $\{u_+, v_+\}$ for dipole and quadrupole modes, for one of the SOC strengths $\eta'/\eta = 0.78$ with non-zero intra-species interaction strength, in figures 3(a)–(d) and 4(a)–(d), respectively. In both these figures, (a) and (b) shows the quasiparticle amplitudes for the ‘+’ component and (c) and (d) contain the quasiparticle amplitudes for the ‘−’ component.

After discussing the low energy excitations, we will now examine the nature of the excitations for higher energy values. For illustration, we evaluate the quasiparticle amplitudes at dimensionless energies $\epsilon/(\hbar \omega_{\rho})$ near to 1 (low), 7 (intermediate) and 14 (high), marked with $\bullet$, $\triangle$, and $\Delta$, respectively. The three energy ranges are chosen to examine the whole range of energies in the dispersion (figure 1) exhibited by the cases under consideration.

For each energy range chosen, we now discuss the effect of different $k_{rms}$ values on the quasiparticle amplitudes. We choose a particular excitation, at a given energy, each from the rightmost and leftmost point from the dispersion curve in figure 1, and one in between these two. The leftmost excitation (with smaller $k_{rms}$) that lies nearly on the linear part of the dispersion, is phonon-like. The rightmost one (with larger $k_{rms}$) corresponds to the surface excitation [37], as we shall see the quasiparticle amplitude gets distributed over the boundary of the condensate only for such modes.

Again for comparison, we discuss the case of a scalar condensate, whose quasiparticle amplitudes $\{u_+\}$ are displayed, in figure 5. We analyze the radial and angular nodes in various examples of quasiparticle excitation. The radial nodes are the nodes existing in each quasiparticle excitation mode along any cross-section from the origin ($r = 0$), i.e., nodes
along $r = \sqrt{x^2 + y^2}$ counts $n_r$ except the zero at large $r$. The number of azimuthal nodes ($n_\phi$) is equal to twice the maxima (or minima) number along the outermost boundary of the excitation amplitude containing both. Figures 5(a) and (b) show two quasiparticles with energy, $\epsilon/(\hbar \omega_p) \approx 7$ for these illustrative values of $k_{rms}$; low (c), intermediate (d) and high (e) values. The excitation at lower value of $k_{rms}$ lies nearly on the linear part of the dispersion curve and has three radial nodes and 10 angular nodes. The intermediate $k_{rms}$ value at the same energy value has two radial nodes and 16 angular nodes, shown in figure 5(d), whereas larger $k_{rms}$ values at the same energy value have only one radial node and 22 angular nodes. Therefore, at roughly the same energy range, the quasiparticle amplitude gains azimuthal nodes at the cost of a loss of radial nodes. This is also evident from figure 1(a), where the quasiparticle excitation mode with smaller $k_{rms}$ has lower value of $(L_z)_{rms}$ and the quasiparticle excitation mode with larger $k_{rms}$ value has higher $(L_z)_{rms}$ value. The observation also holds for other considered cases.

Additionally, figure 1 illustrates that for a scalar BEC and a two-component BEC, the lowest quasiparticle excitation has a zero $rms$ value of angular momentum. However, for the SOC case, the gauge part of the $rms$ value of angular momentum $(L_g)$ results in a contribution of non-zero angular momentum to the lowest-energy excitation mode, shown in figure 1(d) (inset) for one of the SOC strengths. This occurs due to the modification of the effective momentum along a particular direction (equation (2)), subjected to variation in spin–orbit coupling parameters, which results in an anisotropic nature of the SOC-BEC velocity profile. In turn, this leads to a violation of the irrotationality condition on the velocity fields in the SOC-BEC superfluid, which is in contrast to the case of the scalar BEC superfluid [28]. Hence, the lowest energy excitations in

Figure 1. The discrete BdG quasiparticle excitation spectrum against their $rms$ linear $(k_{rms})$ and angular momentum $(L_z)_{rms}$ for: (a) trapped single-component BEC with $g_{+-} = 0$ (b) trapped two-component BEC with $g_{+-} = g$. (c) and (d) shows the dispersion relation for trapped SOC-BEC with SOC strength’s $\eta'/\eta = 0.4$ and 0.78 respectively, with $\eta = 0.6 \sqrt{\hbar \omega_p/m}$ and $g_{+-} = g$. Inset in (a) and (c) shows the zoom-in of the low-lying spectrum with respect to $k_{rms}$. Inset of (d) shows the canonical and gauge contribution to the $rms$ value of total angular momentum (see text).
Figure 2. Quasi-particle amplitudes $u_+^\pm$ and $v_+^\pm$ for (a), (b) dipole and (c), (d) quadrupole modes for the single-component BEC (with $g_{+-} = 0$). $u_+^\pm$ and $v_+^\pm$ are in units of $a_+^{-1}$ $\rho$. Each sub-figure is superimposed with condensate density contours (black lines).

Figure 3. Quasi-particle amplitudes $u_\pm^\pm$ and $v_\pm^\pm$ for the dipole mode in SOC-BEC with $\eta'/\eta = 0.78$ and $g_{+-} = g$. $u_\pm^\pm$ and $v_\pm^\pm$ are in units of $a_+^{-1}$. Each sub-figure is superimposed with condensate total density contours (black lines).

SOC-BEC, as a consequence, have a finite $rms$ value of angular momentum.

Also, in the same range of energy values, but with increasing value of $k_{rms}$ the quasiparticle amplitudes move toward the outermost boundary of the condensate and away from the center of the trap. To illustrate this, we have superposed the quasiparticle amplitude for a particular component $'u_+^\pm(x,y)'$ with the contour lines of the corresponding condensate density, i.e., for the $'+'$-component $|\psi_+|^2$. It is shown in figures 5(c)–(h). At relatively larger values of $k_{rms}$ shown in figures 5(e) and (h), the excitation amplitude lies at the outermost edge/contour of
the condensate and thus corresponds to the surface mode. It is shown that the coexistence of surface and phonon-like modes is observed in a similar range of energies. The observation is also consistent for the energy value $\sim 14$ at low, intermediate and high values of $k_{\text{rms}}$. (f) has 8 radial nodes and 4 angular nodes; (g) has 3 radial nodes and 28 angular nodes and (h) has only a single radial node but 36 angular nodes.

After discussing the nature of the quasiparticle amplitudes of the various excitations in a scalar condensate, we next analyze the nature of these in our SOC coupled configuration. Figures 6 and 7 show the quasiparticle amplitude $u_{+}(x, y)$ at different energy values for SOC strengths $\eta'/\eta = 0.4, 0.78$, respectively. Table 1 contains details about the number of radial nodes and angular nodes in the quasiparticle amplitude for the cases considered. Some points of differences and similarities between SOC and scalar condensate, based on table 1, are discussed in the following. Firstly, in a SOC-BEC, at low $k_{\text{rms}}$ and considered energy values, the quasiparticle excitations are aligned along a specific direction. It is along the $x$-direction in the case $\eta'/\eta = 0.78$, shown in figures 7(d) and (g), and along the $y$-direction in the case $\eta'/\eta = 0.4$, shown in figures 6(d) and (g). For the case of a scalar condensate, the distribution of amplitude is uniform and symmetric, as shown in figures 5(c) and (f). The anisotropy in quasiparticle amplitudes in SOC-BEC is present, although the trap is isotropic. This occurs due to the different effective masses and the effective momenta along the $x$ and $y$ directions in equation (2). Also, in a SOC-BEC, the number of nodes is direction-dependent and is not constant. As an example, we consider the case illustrated in figure 6(e) for one of the SOC-strengths $\epsilon/\hbar\omega_p = 7$. The number of radial nodes at an intermediate $k_{\text{rms}}$ consists of either 3 or 4 radial nodes depending on the direction chosen.

Furthermore, at high $k_{\text{rms}}$, the number of angular nodes for $\epsilon/\hbar\omega_p = 7, 14$ increases in the order: a single component, $\eta'/\eta = 0.78$, $\eta'/\eta = 0.4$. At the same value of $\epsilon/\hbar\omega_p = 7, 14$ but at low $k_{\text{rms}}$, the number of angular nodes is only 2 for both $\eta'/\eta = 0.78, 0.4$ and is largest for single-component BEC.
Also, the number of radial nodes for \( \epsilon/\hbar \omega_p = 1.7, 14 \), at high \( k_{\text{rms}} \), is mostly 1 for all cases. The number of radial nodes for \( \epsilon/\hbar \omega_p = 7, 14 \), at low \( k_{\text{rms}} \), is greater in SOC cases than in the single-component BEC.

Figures 8(a) and (b) show the dispersion relation in SOC-BEC for various SOC strengths \( \eta = 1.5, 1.8, 2 \) (in units of \( \sqrt{\hbar \omega_p/m} \)) with \( \eta'/\eta = 0.78 \) in miscible, i.e. \( g_+ < g \) and immiscible configurations, i.e. \( g_+ > g \) respectively.

The excitation spectrum of the immiscible system (figure 8(b)) contains a minima-like feature at finite \( k_{\text{rms}} \). A few outermost points (branches) of the dispersion, viewed against \( k_{\text{rms}} \), are marked by squares. The minima is right-shifted toward a higher value of \( k_{\text{rms}} \) with an increase in the magnitude of \( \eta \), as shown in figure 8(b1). We also show the magnified part of the dispersion from figure 8(b) concerning \( (L_z)_{\text{rms}} \) in figure 8(b2). The points corresponding to (b1) are also marked in (b2). Figure 8(b2) also contains minima at finite \( rms \) values of angular momentum, demonstrating that the lowest energy excitations have a non-zero, finite angular momentum, which increases with an increase in \( \eta \)'s magnitude.

The minima in the excitation spectrum illustrated in figure 8(b1) can be identified as a roton. Increasing the SOC strength \( \eta \), results in a decrease of the excitation energy, and is shown in figure 9. For \( \eta = 2\sqrt{\hbar \omega_p/m} \), the excitation energy touches zero at a finite \( k_{\text{rms}} \) and after that it becomes purely imaginary for \( \eta = 2.3, 2.4 \) (in units of \( \sqrt{\hbar \omega_p/m} \)). Therefore, we plot the real and imaginary parts of the excitation energy separately. For other cases mentioned earlier (and hereafter), we do not mention them separately because the imaginary part of the excitation energy is absent. Due to the presence of imaginary values in the excitation energy, dynamical instability will set-in and the corresponding modes will grow exponentially in time which may consequently, drive the system into some other phase (e.g. stripe phase [20]).

We next investigate the nature of the quasiparticle modes near the minima. For this purpose, we consider the dispersion relation from figure 8(b1) concerning \( k_{\text{rms}} \) only. It is shown in figure 10(a) for the parameters \( g < g_+ \) and \( \eta'/\eta = 0.78 \) with \( \eta = 1.8\sqrt{\hbar \omega_p/m} \). Figures 10(b)–(e) show the quasiparticle amplitude \( \psi_+(x,y) \) for the modes marked with ‘+’ in (a), for four illustrative modes labeled (i)–(iv). The mode marked (ii) identifies the minima location and has four angular nodes, Figure 10(b) with the label (i), has eight angular nodes. Both (d) and (e) marked with (iii) and (iv), respectively, have six angular nodes. Thus, the considered case for investigating quasiparticle modes near the minima has no radial nodes, only angular nodes. Here, the number of angular nodes is least at the minima location, with the largest on its left. Moreover, the effect of unbalanced interaction strengths, i.e. \( g_+ \neq g \) also impacts the spread of the quasiparticle amplitudes which become smeared spatially, as illustrated in figures 10(b)–(e).

3. Structure factor

Using the excitation spectrum of a trapped SOC-BEC computed in the previous section, we can now calculate the dynamic structure factor (DSF), which can be measured using a robust experimental tool, Bragg spectroscopy [23–27]. The DSF yields information on the spectrum of collective

![Figure 8](https://example.com/image8.png)

Figure 8. Excitation spectrum of SOC-BEC in (a) miscible and, (b) immiscible configuration, for SOC strength's \( \eta = 1.5, 1.8, 2 \) (in units of \( \sqrt{\hbar \omega_p/m} \)) with \( \eta'/\eta = 0.78 \). (b1)–(b2), respectively, shows the magnified part of the dispersion, near the minima, in immiscible configuration concerning \( k_{\text{rms}} \) and \( (L_z)_{\text{rms}} \) separately. We mark the points on the outermost branch of the dispersion, viewed w.r.t \( k_{\text{rms}} \) in (b1) (using squares). The corresponding points are marked in (b2), and dashed lines in both (b1) and (b2) guide the eye joining these.
excitations, which can be explored at low momentum transfer [25]. It also provides knowledge about the momentum distribution through which the behavior of the system at high momentum transfer can be characterized, and where the response is dominated by single-particle effects [23]. Its investigation has provided a crucial understanding of physics in superfluid $^4$He [52]. In particular, it has facilitated the measurement of the roton spectrum [53], pair-distribution function and condensate fraction available from neutron scattering experiments [54], in this system.

In the experiment, the condensate is impinged by a Bragg pulse, with the help of two laser beams with wavevectors $k_1$ and $k_2$ and a frequency difference $\omega$. The following Hamiltonian describes the resulting external perturbation term:

$$\hat{H}_{\text{pen}}(t) = \lambda \hat{c}_k^\dagger e^{-i\omega t} e^{i\gamma t} + H.c. \quad (10)$$

where $\lambda$ is the strength of the Bragg potential, $k = k_1 - k_2$ is the momentum transferred by the Bragg pulse to the condensate and the factor $e^{i\gamma t}$ in equation (10), with $\gamma \to 0^+$, ensures that the system at initial time ($t = -\infty$) is governed by the unperturbed Hamiltonian. Directly after perturbing the system, the DSF relative to the operator $\hat{G}_k$ [1] can be probed:

$$S(k, \omega) = \sum_j Z_j \delta(h\omega - h\omega_j). \quad (11)$$

The quantity $Z_j = |\langle j|\hat{G}_k|0\rangle|^2$ is known as the strength of the operator $\hat{G}_k$ with respect to state $|j\rangle$ and $h\omega_j = \epsilon_j - \epsilon_0$. Here $|0\rangle$ corresponds to the ground (excited) state with the energy $\epsilon_0$ ($\epsilon_j$). For the two-component SOC-BEC, the DSF corresponding to the total density $\hat{\rho}(r) = \hat{\psi}^\dagger(r)\hat{\psi}(r)$ and spin density $\hat{s}_z(r) = \hat{\psi}^\dagger(r)\hat{s}_z\hat{\psi}(r)$ can be also calculated from the BdG excitations computed in the previous section, where $\hat{\psi}(r)$ is defined in equation (B5). Here, the operator $\hat{G}_k$, corresponding to the total density is $\hat{\rho}_k = \int dr \hat{\psi}^\dagger(r)\hat{\psi}(r) e^{ik \cdot r}$ and, the operator corresponding to the spin density is $\hat{s}_z = \int dr \hat{\psi}^\dagger(r)\hat{s}_z\hat{\psi}(r)e^{ik \cdot r}$, where $\hat{\psi}(r) = (\hat{\psi}_x(r) \hat{\psi}_y(r))^T$.

The strength $Z_j$ in equation (11) can be simplified and written in terms of the quasiparticle amplitudes (details in appendix B.2), resulting DSF corresponding to total density (spin density), labeled with subscript ‘$p$’ (‘$s$’), given as:

$$S_p(k, k_j, \omega) = \sum_j \int dxdy e^{i(kx_k + ky_k)t_0} \delta(h\omega - h\omega_j)$$

$$S_s(k, k_j, \omega) = \sum_j \int dxdy e^{i(kx_k + ky_k)t_0} \delta(h\omega - h\omega_j) . \quad (12)$$

The DSF integrated over $S(k_j, k_j, \omega)$ over frequency domain, gives the static structure factor,

$$S(k_j, k_j) = \frac{1}{N} \int d\omega S(k_j, k_j, \omega).$$

We start by discussing the analytical results for the structure factor in a uniform scalar and two-component BEC. The dispersions in these two cases are respectively given by

$$\epsilon_{s} = h \sqrt{c^2 k^2 + (h k^2 / 2m)^2}, \quad (13)$$

$$\epsilon_{D,S} = h \sqrt{c^2 k^2 + (h k^2 / 2m)^2}. \quad (14)$$

In a uniform scalar BEC, the sum in equation (11) is expended by only one mode. The static structure factor in this system is $S(k) = \epsilon_0(k) / \epsilon_B(k)$ [Feynman relation] [55], where $\epsilon_0(k)$ is the dispersion of non-interacting bosons. In the limit $k \to 0$, $S(k) = \hbar k / 2mc$ where $c = \sqrt{8\pi/m}$ is the Bogoliubov sound velocity. This can be generalized in a uniform two-component BEC using the dispersion relation [56] given in equation (14). The static factor in such configuration for $k \to 0$ gives $S_{D,S}(k) = \hbar k / 2mc_{D,S}$, where $c_{D,S}$ represents the sound velocity corresponding to density and spin-density, respectively. Thus, as $k \to 0$, the static structure factors in both configurations vary linearly with wavevector. However, the static structure factor approaches unity at large momentum [24, 25] in both systems, which is the value of the static structure factor for any momentum in uncorrelated non-interacting atoms.

To benchmark our computation for the trapped SOC-BEC, we now evaluate the dynamic and the static structure factor in the trapped single- and two-component BEC by considering a suitable limit of the excitation energy and the quasiparticle amplitudes determined in section 2.2. Substituting parameters $g = 0$, $m = m$, and $\eta = 0$ in equation (2) we get the results for trapped scalar BEC. The dynamic structure factor $S(k_f, k_j, \omega)$ for this case is shown figure 11(a). To gain a clear insight into its behavior, we integrate the dynamic structure factor $S(k_f, k_j, \omega)$ over the frequency domain and compute the static structure factor. The evaluated structure factor is illustrated, with a bold (blue) line in figures 13(a) and (b) (Here, $S_D = S_S = S$). The substitution of parameters $g \neq 0$, $m = m$, and $\eta = 0$ in equation (2) corresponds to the case of a trapped two-component BEC, discussed in [57].

**Figure 9.** Variation of the lowest lying excitation’s energy with SOC strength $\eta$, for immiscible SOC-BEC. Here, $\eta$ is in units of $\sqrt{\hbar\omega_d/m}$, and Re (Imag) refers to real (imaginary) part of the excitation energy.
Figure 10. Quasiparticle amplitudes for modes near the minima: (a) excitation spectrum for miscible SOC configuration with SOC strength’s $\eta'/\eta$ = 0.78 and $\eta$ = 1.8 (in units of $\sqrt{\hbar \omega_x / m}$) (see figure 8(b)). Inset shows the magnified view near the minima. (b)–(e) Quasiparticle amplitude $u_+(x,y)$ for modes labeled (i)–(iv) in (a) with marker ‘+’, respectively. The quasiparticle amplitudes are also superimposed with the condensate’s total density contours (black lines).

Figure 11. (a) Dynamic structure factor for single-component BEC with respect to $k_x = k$ and $\omega$. (b), (c) Density $S_D(k_x, k_y = 0, \omega)$ and spin $S_S(k_x, k_y = 0, \omega)$ Dynamic structure factor in two-component BEC. (d) and (e) shows these for SOC case for strengths $\eta'/\eta = 0.78$ with $\eta = 0.6 \sqrt{\hbar \omega_x / m}$.

In the case of a two-component BEC, the density and spin dynamic structure factor are represented in figures 11(b) and (c), respectively. The density static structure factor’s general behavior is identical to that of the single-component BEC, illustrated in figure 13(a).

The magnitude of $S_D(k_x, k_y)$ in two-component BEC is relatively lower than that of the single-component BEC at most $k_x$ values for the uniform and trapped case. Similarly $S_S(k_x, k_y)$ has a higher magnitude relative to the corresponding single-component static structure factor. It reaches the plateau of the unit static structure factor rapidly, shown in figure 13(b).

Thus, the finite interspecies interaction strength diminishes the density static structure factor in the two-component BEC, whereas it intensifies the spin structure factor, in agreement with results of [57]. Again this can be understood from $S(k) = \epsilon_0(k) / \epsilon(k)$. For a finite $g_{+-}$, the effective interaction strength $(g + g_{+-})$ increases in the total density part (refer to the first term in the second line of equation (2)), leading to an increment in the energy of the excitation relative to the single-component case with $g_{+-} = 0$ and thus decreasing the static structure factor. Simultaneously, the effective interaction strength $(g - g_{+-})$ is reduced in the spin-density term (refer to the second term in the second line of equation (2)), resulting in a lower energy and an increase of the static structure factor relative to $S_D$ in two-component BEC.

However, the Feynman relation discussed above is not generally applicable to uniform BECs with spin–orbit coupling [58]. The relation is not satisfied in the whole momentum space, and its applicability depends on the type of ground state of SOC-BEC [59]. Therefore, we utilize equation (12) only to compute and discuss the dynamic structure factor features in trapped SOC-BEC.

Coming to the SOC-BEC with all interaction strengths equal, namely $g_{+-} = g$, the $S_D(k_x, k_y = 0, \omega)$ is shown for $\eta'/\eta = 0.78$ with $\eta = 0.6 \sqrt{\hbar \omega_x / m}$ in figures 11(d) and (e), respectively. The figures illustrate a relatively broader non-vanishing frequency response at a given wave vector and...
The density static structure factor’s maxima coincides approximately with the location of the fluctuation in the density relative to the total density, in the momentum space, i.e. \( \delta \rho(k_x, k_y = 0) / \rho(k_x, k_y = 0) \), illustrated in figure 15(a) corresponding to the total density. Figure 15(b) shows the Fourier transform of the fluctuations, \( (\delta x) \), in the ground state corresponding to the spin-density along with \( k_y = 0 \). It does not have clear maxima/minima as seen in figure 15(a). However, \( S_S(k_x, k_y) \) shown in figure 13(b) for \( \eta / \tilde{\eta} = 0.78 \) has a distinct peak whose location coincides with the peak in figure 15(b) for the corresponding case.

The spin–orbit coupling further enhances the structure factor amplitude for both the density and spin static structure factor, shown in figures 13(a) and (b), respectively. In addition, the peak amplitude at non-zero \( k_x \) is relatively large in the \( S_S(k_x, k_y) \) than in \( S_D(k_x, k_y) \). Similar to the case without SOC, the effective interaction strength gets reduced in the spin-density term, which further leads to lower excitation energy and enhanced spin static structure factor compared to the density structure factor.

The above discussion is confined to the balanced inter-component and intra-component interaction strengths, i.e.
For both the total density and spin-density are symmetric about $g_{\pm} = g$. Based on the interaction strengths, the characterization of the ground state in an interacting gas depends on the minima in the single-particle dispersion \cite{29, 34}, which is either a single-well (miscible, $g > g_{\pm}$) or double-well (immiscible, $g < g_{\pm}$) type. The ground state of our system has two possible phases: a zero-momentum phase and plane-wave phase \cite{60, 61}. In the miscible regime, the SOC-BEC condenses in the zero-momentum state and has an extremely small value ($\sim 0$) of spin-density, apparent from the magnitude of the randomly fluctuating spin-density, shown in figures 16 and 17(b). Whereas in the immiscible regime, the SOC-BEC chooses one of the two-wells as the ground state and is in the plane-wave phase. In this regime, the spin-density has a finite-value\cite{61}, illustrated in figures 16 and 17(f). Another typical ground state phase in the SOC-BEC literature is the stripe phase \cite{60}, where the BEC remains in a superposition state of both wells. However, the stripe phase is not achievable in our system because there is no off-diagonal coupling term in equation (2). We next discuss the density and spin dynamic structure factor and the corresponding static structure factors in SOC-BEC, in such miscible and immiscible configurations. For both these configurations, we have shown the total density and spin-density in the ground state with the respective dynamic factors in figures 16 and 17, for $\eta' / \eta = 0.78$ with $\eta = 0.6, 1.8$ (units of $\sqrt{\hbar \omega_p}/m$), respectively.

In the case of $g_{\pm} < g$, the density and spin dynamic structure factor $S_{D,S}(k_x, k_y = 0, \omega)$ for SOC strengths $\eta' / \eta = 0.78$ with $\eta = 0.6 \sqrt{\hbar \omega_p}/m$ is shown in figures 16(c) and (d) respectively, and for $g_{\pm} > g$ in figures 16(g) and (h) respectively. For a higher value of the SOC strength $\eta = 2 \sqrt{\hbar \omega_p}/m$ with $\eta' / \eta = 0.78$, the corresponding results appear in figure 17. At a given SOC-strength and $g_{\pm} < g$, the DSF for both the total density and spin-density are symmetric about $k_z = 0$, i.e. $S_{D,S}(k_x, k_y = 0, \omega) = S_{D,S}(-k_x, k_y = 0, \omega)$. The DSF is calculated with the expression given in equation (12) and depends on the details of quasiparticle amplitudes (see appendix B.2 for detail). In the plot of figures 17(c) and (g), we therefore superposed the plot of quasiparticle dispersion on the plot of $S_{D}(k_x, k_y = 0, \omega)$. The quasiparticle dispersion is plotted as a function of $k_{rms}a_{\rho}$ which is indicated along the horizontal axis on the upper part of the plot. The detailed structure of the $S_{D,S}(k_x, k_y = 0, \omega)$ at a given value of $\omega$ shows interesting variation as a function of $k_{rms}a_{\rho}$ which depends on the quasiparticle amplitude, and not amenable to a simple explanation. In the opposite case, $g_{\pm} > g$, the energy spectrum possesses a minima-like feature, as discussed earlier. In addition, the SOC-BEC chooses either of the two wells as the ground state through a spontaneous symmetry breaking mechanism, which
Figure 16. (a), (b) Total density $\rho(r) = \psi^\dagger(r) \psi(r)$ and spin-density $s_z(r) = \psi^\dagger(r) \sigma_z \psi(r)$ of the ground state, respectively, in miscible SOC configuration for $\eta'/\eta = 0.78$ with $\eta = 0.6 \sqrt{\hbar \omega_p/m}$. (c), (d) Dynamic structure factor for density $S_D(k_x,k_y=0,\omega)$ and spin-density $S_{S}(k_x,k_y=0,\omega)$, respectively, for the aforementioned miscible case. (e)–(h) shows the corresponding results for same SOC strength in the immiscible configuration with $g = 0.3 g_{+...}$. Solid (green) line in (c) and (g) are guide to the eyes indicating the excitation branch in the miscible and the immiscible case, respectively, illustrating the ‘roton’ minima in the latter.

Figure 17. (a), (b) Total density and spin-density of the ground state, respectively, in miscible SOC configuration for $\eta'/\eta = 0.78$ with $\eta = 2 \sqrt{\hbar \omega_p/m}$. (c), (d) Dynamic structure factor for density $S_D(k_x,k_y=0,\omega)$ and spin-density $S_{S}(k_x,k_y=0,\omega)$, respectively, for aforementioned miscible case. (e)–(h) shows the corresponding results for same SOC strength’s in immiscible configuration with $g = 0.3 g_{+...}$. (c) and (g) contains the excitation spectrum w.r.t. $k_{rms}$ superposed on the DSF in green (light).

correspondingly affects the nature of the DSF [19]. The symmetry under the exchange of $k_x$ to $-k_x$ is no longer respected. Again, the detailed structure of $S_{D,S}(k_x,k_y=0,\omega)$ at a given value of $\omega$ shows variation which is now different as compared to the features in figures 17(c) and (d). The DSF exhibits an asymmetric character of about $k_x = 0$ in both total density and spin-density, i.e. $S_{D,S}(k_x,k_y=0,\omega) \neq S_{D,S}(-k_x,k_y=0,\omega)$, illustrated respectively in figure 16, 17(g) and (h).

Figure 18 displays the static structure factor in both miscible and immiscible SOC-BEC. In the miscible case, for a ratio of SOC strengths $\eta'/\eta = 0.78$ with values of $\eta$ listed in figure 18, the density and spin-density static structure factors are shown in (a) and (b), respectively; (c) and (d) show the corresponding results for the immiscible case. The features discussed previously for balanced interaction strengths, e.g. the non-zero static structure factor as $k_x \to 0$ and the magnitude of the static structure factor approaches unity at large wave vectors, remains valid.

The static structure factors in figure 18 are normalized by their maximum value, i.e. $S_{D,S}/\max\{S_{D,S}\}$, to show general features such as peaks, maxima/minima, and not magnitude. The behavior of both the density and spin static structure
Einstein–de Haas effect. This non-zero angular momentum may lead to the study of the angular momentum for the lowest-energy excitations in SOC-BEC. How the spin-dependent gauge part leads to a non-zero angular form and spin-dependent gauge form, we demonstrate by writing the angular momentum operator as a sum of its components even in an isotropic trapping potential. By writing the angular momentum operator as a sum of its components even in an isotropic trapping potential.

In this work, within the BdG approximation, we analyze the excitation spectrum of a trapped quasi-two-dimensional two-component SOC-BEC for both balanced and unbalanced interaction strengths. In the initial part of the work, we computed the excitation spectrum and compared the excitation spectrum with the corresponding excitation spectrum in the single-component and two-component BEC without SOC. In this analysis, we identify the low-energy excitations such as dipole and quadrupole modes as well as various other excitation modes probing a range of energy spectrum in SOC-BEC, and compare them with the corresponding excitation modes in single- and two-component BEC without SOC. We demonstrate how the SOC leads to anisotropic quasiparticle modes in single- and two-component BEC without SOC. In this work, within the BdG approximation, we analyze the excitation spectrum of a trapped quasi-two-dimensional two-component SOC-BEC for both balanced and unbalanced interaction strengths. The results in section 3 display the strong impact of SOC strengths and the interaction strengths on the dynamic and static structure factors. The results presented can be used for beyond mean-field studies that take into account quantum fluctuations, such as one done under the truncated Wigner approximation. This also enables the evaluation of the structure factor in SOC-BEC at finite temperatures. DSF can be directly probed by experimental techniques such as Bragg spectroscopy that allow one to tune the momentum transfer over a wide range, and various properties, e.g. coherence [23], and vortices [65, 66] can be probed in such a configuration.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Details of basis chosen and integrals used in section 2.2

Consider the following part of $\hat{L}_+$ from equation (5),

$$\left[ \frac{1}{2m} (p_x + m n)^2 + \frac{1}{2} m \omega^2 x^2 \right] + \left[ \frac{1}{2m} p_y^2 + \frac{1}{2} m \omega^2 y^2 \right].$$

The above can be rewritten as:

$$[P_1^2 + X_1^2] + [P_2^2 + X_2^2].$$

where, $P_1 = \frac{1}{\sqrt{2m}} (p_x + m n)$, $P_2 = \frac{1}{\sqrt{2m}}$, $X_1 = \sqrt{\frac{\hbar}{2}} \omega x$ and $X_2 = \sqrt{\frac{\hbar}{2}} \omega y$. Defining the operators:

$$a_1' = \frac{1}{\sqrt{\hbar \omega}} (X_1 + iP_1), \quad a_2' = \frac{1}{\sqrt{\hbar \omega}} \left( \frac{m}{m_s} \right)^{1/4} (X_2 + iP_2).$$

$$\psi(x) = \frac{1}{\sqrt{\sqrt{2\pi}\hbar}} \int \frac{dp_x}{2\pi}\int \frac{dp_y}{2\pi} (a_1 + a_2') \phi(p_x, p_y).$$
We get,
\[
X_1^2 + P_1^2 = a_1^1 a_1^1 + \frac{1}{2} \hbar \omega, \\
X_2^2 + P_2^2 = a_2^1 a_2^1 + \frac{1}{2} \hbar \omega \sqrt{\frac{m}{m_y}}.
\]

Using these operators, first part of \( L_+ \) (or \( \hat{L}_+ \)) becomes diagonal in the chosen basis with energy
\[
E(K, L) = \left( K + m_R L + \frac{1}{2} [1 + m_R] \right) \hbar \omega
\]
where \( m_R = \sqrt{m/m_y} \) and \( K, L = 0, 1, \ldots, N_R \).

We next provide the explicit form of the integrals used for computing the matrix elements of equation (7) in the following:

\[
A_{k'\nu'\nu} = \int dx dy \phi_k^* \phi_{k'} \hat{L}_+ \phi_k^* \phi_{k'} \\
B_{k'\nu'\nu} = \int dx dy \phi_k^* \phi_{k'} \left[ g_{2D} \psi_{\nu'0}^2 \right] \phi_k^* \phi_{k'} \\
C_{k'\nu'\nu} = \int dx dy \phi_k^* \phi_{k'} \left[ g_{2D} \psi_{\nu'0}^2 \right] \phi_k^* \phi_{k'} \\
D_{k'\nu'\nu} = \int dx dy \phi_k^* \phi_{k'} \left[ g_{2D} \psi_{\nu'0}^2 \right] \phi_k^* \phi_{k'} \\
E_{k'\nu'\nu} = \int dx dy \phi_k^* \phi_{k'} \left[ g_{2D} \psi_{\nu'0}^2 \right] \phi_k^* \phi_{k'} \\
G_{k'\nu'\nu} = \int dx dy \phi_k^* \phi_{k'} \left[ g_{2D} \psi_{\nu'0}^2 \right] \phi_k^* \phi_{k'} \\
H_{k'\nu'\nu} = \int dx dy \phi_k^* \phi_{k'} \left[ g_{2D} \psi_{\nu'0}^2 \right] \phi_k^* \phi_{k'} \\
K_{k'\nu'\nu} = \int dx dy \phi_k^* \phi_{k'} \left[ \hat{L}_- \right] \phi_k^* \phi_{k'} \\
L_{k'\nu'\nu} = \int dx dy \phi_k^* \phi_{k'} \left[ g_{2D} \psi_{\nu'0}^2 \right] \phi_k^* \phi_{k'} \\
O_{k'\nu'\nu} = \int dx dy \phi_k^* \phi_{k'} \left[ g_{2D} \psi_{\nu'0}^2 \right] \phi_k^* \phi_{k'} \\
F_{k'\nu'\nu} = -A_{k'\nu'\nu}, \quad J_{k'\nu'\nu} = -H_{k'\nu'\nu}, \quad I_{k'\nu'\nu} = D_{k'\nu'\nu} \\
M_{k'\nu'\nu} = G_{k'\nu'\nu}, \quad N_{k'\nu'\nu} = -C_{k'\nu'\nu}, \quad P_{k'\nu'\nu} = -K_{k'\nu'\nu}.
\]

### Appendix B. Details of derivation from section 3

#### B.1. Dynamic structure factor in Lehmann representation

The dynamic correlation of a density fluctuation at time \( t = 0 \) and one at time ‘\( t \)’ is given as
\[
S_D(k, t) = \langle 0 | \hat{\rho}_k(t) \hat{\rho}_k(0) | 0 \rangle. \tag{B1}
\]

Introducing a complete set of eigenstates of the Hamiltonian \( \hat{\rho}_k \) and using the time evolution of the density fluctuation i.e. \( \hat{\rho}_k(t) | 0 \rangle = e^{iH_{\text{eff}}t} \hat{\rho}_k e^{-iH_{\text{eff}}t} | 0 \rangle \), equation (B1) becomes
\[
S_D(k, t) = \sum_j \langle 0 | \hat{\rho}_k | j \rangle \langle j | \hat{\rho}_k | 0 \rangle e^{-i(\epsilon_j - \epsilon_0)t} \tag{B2}
\]

where \( H_{\text{eff}} \) is the effective Hamiltonian used for obtaining equation (2) and is given as
\[
H_{\text{eff}} = \frac{1}{2m} \left( \hat{p}^2 - \sigma_n | m \rangle \langle m | \right) + \frac{\hat{p}^2}{2m} I \\
+ \left[ V_{\text{Trap}} + g_{2D} \left( | \psi_+ \rangle \langle \psi_+ | + | \psi_- \rangle \langle \psi_- | \right) \right] I. \tag{B3}
\]

Taking the Fourier transform of equation (B2), we get
\[
S_D(k, \omega) = \sum_j \langle 0 | \hat{\rho}_k | j \rangle | j \rangle \langle j | \hat{\rho}_k | 0 \rangle \delta (\omega - (\epsilon_j - \epsilon_0)/\hbar). \tag{B4}
\]

The above expression is written in Lehmann representation [67]. Similarly, the DSF for spin-density can be computed.

#### B.2. Details of matrix element in equation (12)

In this part of appendix, we will compute the matrix element \( \langle 0 | \hat{\rho}(\hat{r}) | j \rangle \). To this purpose, we define the field operators corresponding to each component as,
\[
\hat{\psi}(\hat{r}) = e^{\frac{-i}{\hbar}} \left( \hat{\psi}_0(\hat{r}) + \sum_j \left\{ u_j(\hat{r}) \hat{b}_j e^{-i\omega_j t} + v_j(\hat{r}) \hat{b}_j^\dagger e^{i\omega_j t} \right\} \right), \tag{B5}
\]

where, \( \hat{\psi}(\hat{r}) = [\hat{\psi}_+(\hat{r}) \hat{\psi}_-(\hat{r})]^T \), \( \hat{\psi}_0(\hat{r}) = [\hat{\psi}_{+0}(\hat{r}) \hat{\psi}_{-0}(\hat{r})]^T \), \( u_j(\hat{r}) = [u_{+j}(\hat{r}) u_{-j}(\hat{r})]^T \) and, \( v_j(\hat{r}) = [v_{+j}(\hat{r}) v_{-j}(\hat{r})]^T \).

The density operator corresponding to each component is given as,
\[
\hat{\rho}_+(\hat{r}) = | \hat{\psi}_+(\hat{r}) \rangle \langle \hat{\psi}_+(\hat{r}) | = \sum_j \hat{b}_j e^{-i\omega_j t} \times \left[ u_{+j}(\hat{r}) \hat{\psi}_{+0}(\hat{r})^* \hat{\psi}_{+0}(\hat{r}) + v_{+j}(\hat{r}) \hat{\psi}_{-0}(\hat{r})^* \hat{\psi}_{-0}(\hat{r}) \right] + c.c. \\
| \hat{\psi}_-(\hat{r}) \rangle \langle \hat{\psi}_-(\hat{r}) | = \sum_j \hat{b}_j e^{-i\omega_j t} \times \left[ u_{-j}(\hat{r}) \hat{\psi}_{-0}(\hat{r})^* \hat{\psi}_{-0}(\hat{r}) + v_{-j}(\hat{r}) \hat{\psi}_{+0}(\hat{r})^* \hat{\psi}_{+0}(\hat{r}) \right] + c.c. \tag{B6}
\]

The matrix element \( \langle 0 | \hat{\rho}(\hat{r}) | j \rangle = \langle 0 | \hat{\rho}(\hat{r}) | \hat{b}_j \rangle | 0 \rangle \) as \( \hat{b}_j | 0 \rangle = | j \rangle \). Therefore, the matrix element corresponding to operators for total density \( \hat{\rho}(\hat{r}) = | \hat{\psi}_+(\hat{r}) \rangle \langle \hat{\psi}_+(\hat{r}) | + | \hat{\psi}_-(\hat{r}) \rangle \langle \hat{\psi}_-(\hat{r}) | \) and spin-density \( \hat{s}_z(\hat{r}) = | \hat{\psi}_+(\hat{r}) \rangle \langle \hat{\psi}_+(\hat{r}) | - | \hat{\psi}_-(\hat{r}) \rangle \langle \hat{\psi}_-(\hat{r}) | \) can be expressed as,

\[
\langle 0 | \hat{\rho}(\hat{r}) | j \rangle = e^{-i\omega_j t} \left\{ \left[ u_{+j}(\hat{r}) \hat{\psi}_{+0}(\hat{r})^* \hat{\psi}_{+0}(\hat{r}) + v_{+j}(\hat{r}) \hat{\psi}_{+0}(\hat{r})^* \hat{\psi}_{+0}(\hat{r}) \right] \\
+ \left[ u_{-j}(\hat{r}) \hat{\psi}_{-0}(\hat{r})^* \hat{\psi}_{-0}(\hat{r}) + v_{-j}(\hat{r}) \hat{\psi}_{-0}(\hat{r})^* \hat{\psi}_{-0}(\hat{r}) \right] \right\} \\
\langle 0 | \hat{s}_z(\hat{r}) | j \rangle = e^{-i\omega_j t} \left\{ \left[ u_{+j}(\hat{r}) \hat{\psi}_{+0}(\hat{r})^* \hat{\psi}_{+0}(\hat{r}) + v_{+j}(\hat{r}) \hat{\psi}_{+0}(\hat{r})^* \hat{\psi}_{+0}(\hat{r}) \right] \\
- \left[ u_{-j}(\hat{r}) \hat{\psi}_{-0}(\hat{r})^* \hat{\psi}_{-0}(\hat{r}) + v_{-j}(\hat{r}) \hat{\psi}_{-0}(\hat{r})^* \hat{\psi}_{-0}(\hat{r}) \right] \right\} .
\]
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