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Stein’s method, logarithmic Sobolev and transport inequalities

Michel Ledoux∗  Ivan Nourdin†  Giovanni Peccati‡
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Abstract

We develop connections between Stein’s approximation method, logarithmic Sobolev and transport inequalities by introducing a new class of functional inequalities involving the relative entropy, the Stein (factor or) matrix, the relative Fisher information and the Wasserstein distance with respect to a given reference distribution on $\mathbb{R}^d$. For the Gaussian model, the results improve upon the classical logarithmic Sobolev inequality and the Talagrand quadratic transportation cost inequality. Further examples of illustrations include multidimensional gamma distributions, the uniform distribution on a compact interval, as well as families of log-concave densities. As a by-product, the new inequalities are relevant for entropic convergence expressed in terms of the Stein matrix. The tools rely on semigroup interpolation and bounds, in particular by means of the iterated gradients of the Markov generator with invariant measure the distribution under consideration. In a second part, motivated by the recent investigation by Nourdin, Peccati and Swan on Wiener chaoses, we address the issue of entropic bounds on multidimensional functionals $F$ with the Stein matrix via a set of data on $F$ and its gradients rather than on the Fisher information of the density. A natural framework for this investigation is given by the Markov Triple structure $(E, \mu, \Gamma)$ in which abstract Malliavin-type arguments may be developed and extend the Wiener chaos setting.

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1 Introduction

The classical logarithmic Sobolev inequality with respect to the standard Gaussian measure $d\gamma(x) = (2\pi)^{-d/2}e^{-|x|^2/2}dx$ on $\mathbb{R}^d$ indicates that for every probability $d\nu = hd\gamma$ with (smooth) density $h : \mathbb{R}^d \to \mathbb{R}_+$ with respect to $\gamma$,

$$H(\nu | \gamma) = \int_{\mathbb{R}^d} h \log h \, d\gamma \leq \frac{1}{2} \int_{\mathbb{R}^d} \frac{|
abla h|^2}{h} \, d\gamma = \frac{1}{2} I(\nu | \gamma)$$

where

$$H(\nu | \gamma) = \int_{\mathbb{R}^d} h \log h \, d\gamma = \text{Ent}_\gamma(h)$$

is the relative entropy of $d\nu = hd\gamma$ with respect to $\gamma$ and

$$I(\nu | \gamma) = \int_{\mathbb{R}^d} \frac{|
abla h|^2}{h} \, d\gamma = I_\gamma(h)$$

is the Fisher information of $\nu$ (or $h$) with respect to $\gamma$, see e.g. [B-G-L, Chapter II.5] for a general discussion. (Throughout this work, $| \cdot |$ denotes the Euclidean norm in $\mathbb{R}^d$.)

Inspired by the recent investigation [N-P-S1], this work puts forward a new form of the logarithmic Sobolev inequality (1.1) by considering a further ingredient, namely the Stein discrepancy given by the Stein matrix associated with $\nu$. A measurable matrix-valued map $\tau_\nu$ on $\mathbb{R}^d$ is said to be a Stein matrix for the (centered) probability $\nu$ if for every smooth test function $\varphi : \mathbb{R}^d \to \mathbb{R}$,

$$\int_{\mathbb{R}^d} x \cdot \nabla \varphi \, d\nu = \int_{\mathbb{R}^d} \langle \tau_\nu, \text{Hess}(\varphi) \rangle_{\text{HS}} \, d\nu$$

where, here and throughout, $\text{Hess}(\varphi)$ stands for the Hessian of $\varphi$, whereas $\langle \cdot, \cdot \rangle_{\text{HS}}$ and $\| \cdot \|_{\text{HS}}$ denote the usual Hilbert-Schmidt scalar product and norm, respectively. In dimension $d = 1$, Stein matrices are customarily called Stein factors. Note that, although these objects appear implicitly in the literature about Stein’s method (see e.g. [C1, C2, G-S1, G-S2]), the terms ‘Stein factor’ and ‘Stein matrix’ have been first introduced in [N-P-S1]. In recent years, Stein factors and matrices have gained momentum, specially in connection with probabilistic approximations involving random variables living on a Gaussian (Wiener) space (see the recent monograph [N-P2] for an overview of this emerging area).
According to the standard Gaussian integration by parts formula, the proximity of \( \tau_\nu \) with the identity matrix \( \text{Id} \) in \( \mathbb{R}^d \) indicates that \( \nu \) should be close to the Gaussian distribution \( \gamma \). Therefore, whenever such a Stein matrix \( \tau_\nu \) exists, the quantity, called \textit{Stein discrepancy} (of \( \nu \) with respect to \( \gamma \)),

\[
S(\nu \mid \gamma) = \| \tau_\nu - \text{Id} \|_{2,\nu} = \left( \int_{\mathbb{R}^d} \| \tau_\nu - \text{Id} \|_{\text{HS}}^2 \, d\nu \right)^{1/2}
\]

becomes relevant as a measure of the proximity of \( \nu \) and \( \gamma \). This quantity is actually at the root of the Stein method \([C-G-S, \, N-P2]\). For example, in dimension one, the classical Stein bound expresses that the total variation \( \text{TV}(\nu, \gamma) \) between a probability measure \( \nu \) and the standard Gaussian distribution \( \gamma \) is given by

\[
\text{TV}(\nu, \gamma) = \sup \left| \int_{\mathbb{R}} \varphi'(x) d\nu(x) - \int_{\mathbb{R}} x \varphi(x) d\nu(x) \right|
\]

where the supremum runs over all continuously differentiable functions \( \varphi : \mathbb{R} \to \mathbb{R} \) such that \( \| \varphi \|_\infty \leq \sqrt{\frac{2}{\pi}} \) and \( \| \varphi' \|_\infty \leq 2 \). In particular, by definition of \( \tau_\nu \) (and considering \( \varphi' \) instead of \( \varphi \)),

\[
\text{TV}(\nu, \gamma) \leq 2 \int_{\mathbb{R}} |\tau_\nu - 1| \, d\nu \leq 2 S(\nu \mid \gamma)
\]

justifying therefore the interest in the Stein discrepancy. It is actually a main challenge addressed in \([N-P-S1]\) and this work to investigate the multidimensional setting in which representations such as (1.2) are no more available.

With the Stein discrepancy \( S(\nu \mid \gamma) \), we emphasize here the inequality, for every probability \( d\nu = h d\gamma \),

\[
H(\nu \mid \gamma) \leq \frac{1}{2} S^2(\nu \mid \gamma) \log \left( 1 + \frac{I(\nu \mid \gamma)}{S^2(\nu \mid \gamma)} \right)
\]

as a new improved form of the logarithmic Sobolev inequality (1.1). In addition, this inequality (1.3) transforms bounds on the Stein discrepancy into entropic bounds, hence allowing for entropic approximations (under finiteness of the Fisher information).

The proof of (1.3) is achieved by the classical interpolation scheme along the Ornstein-Uhlenbeck semigroup \((P_t)_{t \geq 0}\) towards the logarithmic Sobolev inequality, but modified for time \( t \) away from 0 by a further integration by parts involving the Stein matrix. Indeed, while the exponential decay \( I_\gamma(P_t h) \leq e^{-2t} I\gamma(h) \) classically produces the logarithmic Sobolev inequality (1.1), the argument is supplemented by a different control of \( I_\gamma(P_t h) \) by the Stein discrepancy for \( t > 0 \). We actually show that the Stein discrepancy is itself exponentially decreasing along the semigroup flow in the sense that \( S(\nu_t \mid \gamma) \leq e^{-2t} S(\nu \mid \gamma) \) where \( d\nu_t = P_t h d\gamma \).

We call the inequality (1.3) \textit{HSI}, connecting entropy \( H \), Stein discrepancy \( S \) and Fisher information \( I \), by analogy with the celebrated Otto-Villani HWI inequality \([O-V]\).
relating entropy $H$, (quadratic) Wasserstein distance $W$ ($W_2$) and Fisher information $I$. We actually provide in Section 3 a comparison between the HWI and HSI inequalities (suggesting even an HWSI inequality). Moreover, based on the approach developed in [O-V], we prove that

$$W_2(\nu, \gamma) \leq S(\nu \mid \gamma) \arccos\left(e^{-\frac{H(\nu \mid \gamma)}{2s(\nu \mid \gamma)^2}}\right),$$

(1.4)
an inequality that improves upon the celebrated Talagrand quadratic transportation cost inequality [T]

$$W_2^2(\nu, \gamma) \leq 2H(\nu \mid \gamma)$$

(since $\arccos(e^{-r}) \leq \sqrt{2r}$ for every $r \geq 0$). We shall refer to (1.4) as the ‘WSH inequality’.

While put forward for the Gaussian measure $\gamma$, the question of the validity of (a form of) the HSI and WSH inequalities for other reference measures should be addressed. Natural examples exhibiting HSI inequalities may be described as invariant measures of second order differential operators (on $\mathbb{R}^d$) in order to run the semigroup interpolation scheme. The prototypical example is of course the Ornstein-Uhlenbeck operator with the standard Gaussian measure as invariant measure. But gamma or beta distributions associated to Laguerre or Jacobi operators may be covered in the same way, as well as families of log-concave measures. It should be mentioned that the definition of Stein factor or matrix has then to be adapted to the diffusion coefficient of the underlying differential operator. A convenient setting to work out this investigation is the one of Markov Triples $(E, \mu, \Gamma)$ and semigroups $(P_t)_{t \geq 0}$ as emphasized in [B-G-L] allowing for the $\Gamma$-calculus and the necessary heat kernel bounds in terms of the iterated gradients $\Gamma_n$. In particular, while the classical Bakry-Émery $\Gamma_2$ criterion [B-E] ensures the validity of the logarithmic Sobolev inequality in this context, it is worth mentioning that the analysis towards the HSI bound makes critical use of the associated $\Gamma_3$ operator, a rather new feature in the study of functional inequalities.

As alluded to above, the HSI inequality (1.3) is designed to yield entropic central limit theorems for sequences of probability measures of the form $d\nu_n = h_n d\gamma$, $n \geq 1$, such that $s_n = S(\nu_n \mid \gamma) \to 0$ and

$$\log \left(1 + \frac{I(\nu_n \mid \gamma)}{s_n^2}\right) = o(s_n^{-2}), \quad n \to \infty.$$  

This is achieved, for instance, when the sequence $I(\nu_n \mid \gamma)$, $n \geq 1$, is bounded. However, the principle behind the HSI inequality may actually be used to deduce entropic convergence (with explicit rates) in more delicate situations, including cases for which $I(\nu_n \mid \gamma) \to \infty$. Indeed, it was one main achievement of the work [N-P-S1] in the context of Wiener chaoses to set up bounds involving entropy and the Stein discrepancy without conditions on the Fisher information. Specifically, it was proved in [N-P-S1] that the entropy with respect to the Gaussian measure $\gamma$ of the distribution on $\mathbb{R}^d$ of a
vector $F = (F_1, \ldots, F_d)$ of Wiener chaoses may be controlled by the Stein discrepancy, providing the first multidimensional entropic approximation results in this context. The key feature underlying the HSI inequality is the control as $t \to 0$ of the Fisher information $I_t(P_t h)$ along the semigroup (where $h$ the density with respect to $\gamma$ of the law of $F$) by the Stein discrepancy. The arguments in [N-P-S1] actually provide the suitable small time behavior of $I_t(P_t h)$ relying on specific properties of the functionals (Wiener chaoses) under investigation and tools from Malliavin calculus.

In the second part of the work, we therefore develop a general approach to cover the results of [N-P-S1] and to include a number of further potential instances of interest. As before, the setting of a Markov Triple $(E, \mu, \Gamma)$ actually provides a convenient abstract framework to achieve this goal in which the $\Gamma$-calculus appears as a kind of substitute to the Malliavin calculus in this context. Let $\Psi$ be the function $1 + \log r$ on $\mathbb{R}_+$ but linearized by $r$ on $[0, 1]$, that is, $\Psi(r) = 1 + \log r$ if $r \geq 1$ and $\Psi(r) = r$ if $0 \leq r \leq 1$ (note that $\Psi(r) \leq r$ for every $r \in \mathbb{R}_+$). A typical conclusion is a bound of the type

$$H(\nu_F \mid \gamma) \leq C_F S^2(\nu_F \mid \gamma) \Psi \left( 1 + \frac{\tilde{C}_F}{S^2(\nu_F \mid \gamma)} \right)$$

(1.5)

of the relative entropy of the distribution $\nu_F$ of a vector $F = (F_1, \ldots, F_d)$ on $(E, \mu, \Gamma)$ with respect to $\gamma$ by the Stein discrepancy $S(\nu_F \mid \gamma)$, where $C_F, \tilde{C}_F > 0$ depend on integrability properties of $F$, the carré du champ operators $\Gamma(F_i, F_j), i, j = 1, \ldots, d$, and the inverse of the determinant of the matrix $(\Gamma(F_i, F_j))_{1 \leq i, j \leq d}$. In particular, $H(\nu_F \mid \gamma) \to 0$ as $S(\nu_F \mid \gamma) \to 0$ providing therefore entropic convergence under the Stein discrepancy. The general results obtained here cover not only normal approximation but also gamma approximation.

The inequality (1.5) thus transfers bounds on the Stein discrepancy to entropic bounds. The issue of controlling the Stein discrepancy $S(\nu_F \mid \gamma)$ itself (in terms of moment conditions for example) is not addressed here, and has been the subject of numerous recent studies around the so-called Nualart-Peccati fourth moment theorem (cf. [N-P2]). This investigation is in particular well adapted to functionals $F = (F_1, \ldots, F_d)$ whose coordinates are eigenfunctions of the underlying Markov generator. See [A-C-P, A-M-P, L2] for several results in this direction and [N-P2, Chapters 5-6] for a detailed discussion of estimates on $S(\nu_F \mid \gamma)$ that are available for random vectors $F$ living on the Wiener space.

The structure of the paper thus consists of two main parts, the first one devoted to the new HSI and WSH inequalities, the second one to an investigation of entropic bounds via the Stein discrepancy. Section 2 is devoted to the proof and discussions of the HSI inequality in the Gaussian case. In Section 3, we investigate connections between the Stein discrepancy, Wasserstein distances and transportation cost inequalities, in particular the HWI inequality, and establish the WSH inequality. Extensions of the HSI inequality to more general distributions arising as invariant probability measures of second order differential operators are addressed in Section 4. The second part consists
of Section 5 which develops a general methodology (in the context of Markov Triples) to reach entropic bounds on densities of families of functionals under conditions which do not necessarily involve the Fisher information.

2 Logarithmic Sobolev inequality and Stein discrepancy

Throughout this section, we fix an integer \( d \geq 1 \) and let \( \gamma = \gamma^d \) indicate the standard Gaussian measure on the Borel sets of \( \mathbb{R}^d \).

2.1 Stein matrix and discrepancy

Let \( \nu \) be a probability measure on the Borel sets of \( \mathbb{R}^d \). In view of the forthcoming definitions, we shall always assume (without loss of generality) that \( \nu \) is centered, that is, \( \int_{\mathbb{R}^d} x_j \, d\nu(x) = 0, \; j = 1, \ldots, d \).

As alluded to in the introduction, a measurable matrix-valued map on \( \mathbb{R}^d \):
\[
x \mapsto \tau_\nu(x) = \{ \tau_\nu^{ij}(x) : i, j = 1, \ldots, d \}
\]
is said to be a Stein matrix for \( \nu \) if \( \tau_\nu^{ij} \in L^1(\nu) \) for every \( i, j \) and, for every smooth \( \varphi : \mathbb{R}^d \to \mathbb{R} \),
\[
\int_{\mathbb{R}^d} x \cdot \nabla \varphi \, d\nu = \int_{\mathbb{R}^d} \langle \tau_\nu, \text{Hess}(\varphi) \rangle_{HS} \, d\nu. \tag{2.1}
\]
Observe from (2.1) that, without loss of generality, one may and will assume in the sequel that \( \tau_\nu^{ij}(x) = \tau_\nu^{ji}(x) \nu\text{-a.e.}, \; i, j = 1, \ldots, d \). Also, by choosing \( \varphi = x_i, \; i = 1, \ldots, d \), in (2.1) one sees that, if \( \nu \) admits a Stein matrix, then \( \nu \) is necessarily centered. Moreover, by selecting \( \varphi = x_i x_j, \; i, j = 1, \ldots, d \), and since \( \tau_\nu^{ij} = \tau_\nu^{ji} \),
\[
\int_{\mathbb{R}^d} x_i x_j \, d\nu = \int_{\mathbb{R}^d} \tau_\nu^{ij} \, d\nu, \; \; i, j = 1, \ldots, d
\]
(and in particular \( \nu \) has finite second moments). In dimension \( d = 1 \), Stein matrices are customarily called Stein factors.

Remark 2.1. (a) Let \( d = 1 \) and assume that \( \nu \) has a density \( p \) with respect to the Lebesgue measure on \( \mathbb{R} \). In this case, it is easily seen that, whenever it exists, the Stein factor \( \tau_\nu \) is uniquely determined (up to sets of zero Lebesgue measure). Moreover, under standard regularity assumptions on \( p \), one deduces from a standard integration by parts that a version of \( \tau_\nu \) is given by \( \tau_\nu(x) = p(x)^{-1} \int_x^\infty y p(y) \, dy \) for \( x \) inside the support of \( p \).

(b) In dimension \( d \geq 2 \), a Stein matrix \( \tau_\nu \) may not be unique – see [N-P-S2, Appendix A].
(c) It is important to notice that, in dimension \( d \geq 2 \), the definition (2.1) of Stein matrix is actually weaker than the one used in [N-P-S1, N-P-S2]. Indeed, in those references a Stein matrix \( \tau_\nu \) is required to satisfy the stronger ‘vector’ (as opposed to the trace identity (2.1)) relation

\[
\int_{\mathbb{R}^d} x \cdot \nabla \varphi \, d\nu = \int_{\mathbb{R}^d} \tau_\nu \nabla \varphi \, d\nu
\]  

(2.2)

for every smooth test function \( \varphi : \mathbb{R}^d \to \mathbb{R} \). The definition (2.1) of a Stein matrix adopted in the present paper allows one to establish more transparent connections between normal and non-normal approximations, such as the ones explored in Section 4. Observe that we will need to use Stein matrices in the strong sense of [N-P-S1, N-P-S2] when dealing with Wasserstein distances of order \( \neq 2 \) in Section 3.2.

Definition (2.1) is directly inspired by the Gaussian integration by parts formula according to which

\[
\int_{\mathbb{R}^d} x \cdot \nabla \varphi \, d\gamma = \int_{\mathbb{R}^d} \Delta \varphi \, d\gamma = \int_{\mathbb{R}^d} \langle \text{Id}, \text{Hess}(\varphi) \rangle_{\text{HS}} \, d\nu
\]  

(2.3)

so that the proximity of \( \tau_\nu \) with the identity matrix \( \text{Id} \) indicates that \( \nu \) should be close to \( \gamma \). In particular, it should be clear that the notion of Stein matrix in the sense of (2.1) is motivated by normal approximation. Section 4 will introduce analogous definitions adapted to the target measure. Whenever a Stein matrix exists, we consider to this task the quantity, called Stein discrepancy of \( \nu \) with respect to \( \gamma \) in the introduction,

\[
S(\nu \mid \gamma) = \| \tau_\nu - \text{Id} \|_{2,\nu} = \left( \int_{\mathbb{R}^d} \| \tau_\nu - \text{Id} \|_{\text{HS}}^2 \, d\nu \right)^{1/2}.
\]

(Note that \( S(\nu \mid \gamma) \) may be infinite if one of the \( \tau_\nu^{ij} \)'s is not in \( L^2(\nu) \).) Whenever \( S(\nu \mid \gamma) = 0 \), then \( \nu = \gamma \) since \( \tau_\nu \) is the identity matrix (see e.g. [N-P2, Lemma 4.1.3]). Observe also that if \( C \) denotes the covariance matrix of \( \nu \), then

\[
S^2(\nu \mid \gamma) = \sum_{i,j=1}^{d} \text{Var}_\nu(\tau_\nu^{ij}) + \| C - \text{Id} \|_{\text{HS}}^2,
\]  

(2.4)

where \( \text{Var}_\nu \) indicates the variance under the probability measure \( \nu \).

### 2.2 The Gaussian HSI inequality

As before, write \( d\nu = hd\gamma \) to indicate a centered probability measure on \( \mathbb{R}^d \) which is absolutely continuous with respect to the standard Gaussian distribution \( \gamma \). We assume that there exists a Stein matrix \( \tau_\nu \) associated with \( \nu \) in the preceding sense (see (2.1)).
The following result emphasizes the Gaussian HSI inequality connecting entropy $H$, Stein discrepancy $S$ and Fisher information $I$. In the statement, we use the conventions $0 \log(1 + \frac{s}{\theta}) = 0$ and $\infty \log(1 + \frac{\infty}{\theta}) = \infty$ for every $s \in [0, \infty]$, and $r \log(1 + \frac{r}{\theta}) = \infty$ for every $r \in (0, \infty)$. Since $r \log(1 + \frac{s}{r}) \leq s$ for every $r > 0, s \geq 0$, the HSI inequality (2.5) improves upon the standard logarithmic Sobolev inequality (1.1).

**Theorem 2.2** (Gaussian HSI inequality). For any centered probability measure $d\nu = hd\gamma$ on $\mathbb{R}^d$ with smooth density $h$ with respect to $\gamma$,

$$H(\nu \mid \gamma) \leq \frac{1}{2} S^2(\nu \mid \gamma) \log \left( 1 + \frac{I(\nu \mid \gamma)}{S^2(\nu \mid \gamma)} \right).$$

(2.5)

The HSI inequality (2.5) may be extended to the case of a centered Gaussian distribution on $\mathbb{R}^d$ with a general non-degenerate covariance matrix $C$. We denote such a measure by $\gamma_C$, so that $\gamma = \gamma_{\text{Id}}$. We also denote by $\|C\|_{\text{op}}$ the operator norm of $C$, that is, $\|C\|_{\text{op}}$ is the largest eigenvalue of $C$.

**Corollary 2.3** (Gaussian HSI inequality, general covariance). Let $\gamma_C$ be as above (with $C$ non-singular), and let $d\nu = hd\gamma_C$ where $h$ is a centered smooth probability density with respect to $\gamma_C$. Assume that $\nu$ admits a Stein matrix $\tau_\nu$ in the sense of (2.1). Then,

$$H(\nu \mid \gamma_C) \leq \frac{1}{2} \left\| C^{-\frac{1}{2}} \tau_\nu C^{-\frac{1}{2}} - \text{Id} \right\|_{2,\nu}^2 \log \left( 1 + \frac{\|C\|_{\text{op}} I(\nu \mid \gamma_C)}{\|C^{-\frac{1}{2}} \tau_\nu C^{-\frac{1}{2}} - \text{Id}\|_{2,\nu}^2} \right),$$

where $C^{-\frac{1}{2}}$ denotes the unique symmetric non-singular matrix such that $(C^{-\frac{1}{2}})^2 = C^{-1}$.

Corollary 2.3 is easily deduced from Theorem 2.2 and details are left to the reader. The argument simply uses that if $M$ is the unique non-singular symmetric matrix such that $C = M^2$, then $H(\nu \mid \gamma_C) = H(\nu_0 \mid \gamma)$ where $d\nu_0(x) = h(Mx)d\gamma(x)$.

### 2.3 Proof of the Gaussian HSI inequality

According to our conventions, if either $S(\nu \mid \gamma)$ or $I(\nu \mid \gamma)$ is infinite, then (2.5) coincides with the logarithmic Sobolev inequality (1.1). On the other hand, if $S(\nu \mid \gamma)$ or $I(\nu \mid \gamma)$ equals zero, then $\nu = \gamma$, and therefore $H(\nu \mid \gamma) = 0$. It follows that, in order to prove (2.5), we can assume without loss of generality that $S(\nu \mid \gamma)$ and $I(\nu \mid \gamma)$ are both non-zero and finite.

The proof of Theorem 2.2 is based on the heat flow interpolation along the Ornstein-Uhlenbeck semigroup. We recall a few basic facts in this regard, and refer the reader to e.g. [B-G-L, Section 2.7.1] for any unexplained definition or result. Let thus $(P_t)_{t \geq 0}$ be the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^d$ with infinitesimal generator

$$\mathcal{L}f = \Delta f - x \cdot \nabla f = \sum_{i=1}^d \frac{\partial^2 f}{\partial x_i^2} - \sum_{i=1}^d x_i \frac{\partial f}{\partial x_i}$$

(2.6)
(acting on smooth functions \( f \)), invariant and symmetric with respect to \( \gamma \). We shall often use the fact that the action of \( P_t \) on smooth functions \( f : \mathbb{R}^d \to \mathbb{R} \) admits the integral representation (sometimes called \textit{Mehler’s formula})

\[
P_t f(x) = \int_{\mathbb{R}^d} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) d\gamma(y), \quad t \geq 0, \ x \in \mathbb{R}.
\]

The semigroup is trivially extended to vector-valued functions \( f : \mathbb{R}^d \to \mathbb{R}^d \). In particular, if \( f : \mathbb{R}^d \to \mathbb{R} \) is smooth enough,

\[
\nabla P_t f = e^{-t} P_t (\nabla f). \quad (2.7)
\]

One technical important property (part of the much more general Bismut formulas in a geometric context [Bi, B-G-L]) is the identity, between vectors in \( \mathbb{R}^d \),

\[
P_t (\nabla f)(x) = \frac{1}{\sqrt{1 - e^{-2t}}} \int_{\mathbb{R}^d} y f(e^{-t}x + \sqrt{1 - e^{-2t}}y) d\gamma(y), \quad (2.8)
\]

owing to a standard integration by parts of the Gaussian density.

The generator \( L \) is a diffusion and satisfies the integration by parts formula

\[
\int_{\mathbb{R}^d} f \ L g \ d\gamma = - \int_{\mathbb{R}^d} \nabla f \cdot \nabla g \ d\gamma \quad (2.9)
\]

on smooth functions \( f, g : \mathbb{R}^d \to \mathbb{R} \). In particular, given the smooth probability density \( h \) with respect to \( \gamma \),

\[
I_\gamma(h) = \int_{\mathbb{R}^d} \frac{|\nabla h|^2}{h} d\gamma = \int_{\mathbb{R}^d} |\nabla (\log h)|^2 h d\gamma = - \int_{\mathbb{R}^d} L (\log h) h d\gamma.
\]

As \( d\nu = h d\gamma \), setting \( v = \log h \),

\[
I(\nu | \gamma) = I_\gamma(h) = \int_{\mathbb{R}^d} |\nabla v|^2 d\nu = - \int_{\mathbb{R}^d} L v d\nu. \quad (2.10)
\]

(These expressions should actually be considered for \( h + \varepsilon \) as \( \varepsilon \to 0 \).) Finally, using \( P_t h \) instead of \( h \) in the previous relations and writing \( v_t = \log P_t h \), one deduces from the symmetry of \( P_t \) that

\[
I(\nu_t | \gamma) = I_\gamma(P_t h) = \int_E |\nabla v_t|^2 P_t h d\gamma = - \int_{\mathbb{R}^d} L P_t v_t h d\gamma = - \int_{\mathbb{R}^d} L P_t v_t d\nu. \quad (2.11)
\]

Recall finally that if \( d\nu_t = P_t h d\gamma \), \( t \geq 0 \) (with \( \nu_0 = \nu \) and \( \nu_t \to \gamma \)), the classical \textit{de Bruijn’s formula} (see e.g. [B-G-L, Proposition 5.2.2]) indicates that

\[
\frac{d}{dt} H(\nu_t | \gamma) = - I(\nu_t | \gamma). \quad (2.12)
\]
Theorem 2.2 will follow from the next Proposition 2.4. In this proposition, \( (i) \) corresponds to the integral version of (2.12) whereas \( (ii) \) describes the well-known exponential decay of the Fisher information along the Ornstein-Uhlenbeck semigroup. This decay actually yields the logarithmic Sobolev inequality (1.1), see [B-G-L, Section 5.7]. The new third point \( (iii) \) is a reformulation of [N-P-S1, Theorem 2.1] for which we provide a self-contained proof. It describes an alternate bound on the Fisher information along the semigroup in terms of the Stein discrepancy for values of \( t > 0 \) away from 0. It is the combination of \( (ii) \) and \( (iii) \) which will produce the HSI inequality. Point \( (iv) \) will be needed in the forthcoming proof of the WSH inequality (1.4), as well as in the proof of Proposition 3.2 providing a direct bound of the Wasserstein distance \( W_2 \) by the Stein discrepancy.

**Proposition 2.4.** Under the above notation and assumptions, denote by \( \tau_\nu \) a Stein matrix associated with \( d\nu = h d\gamma \). For every \( t > 0 \), recall \( d\nu_t = P_t h d\gamma \), and write \( v_t = \log P_t h \). Then,

- \( (i) \) (Integrated de Bruijn’s formula)
  \[
  H(\nu | \gamma) = \text{Ent}_\gamma(h) = \int_0^\infty I_\gamma(P_t h) dt. \tag{2.13}
  \]

- \( (ii) \) (Exponential decay of Fisher information) For every \( t \geq 0 \),
  \[
  I(\nu_t | \gamma) = I_\gamma(P_t h) \leq e^{-2t} I_\gamma(h) = e^{-2t} I(\nu | \gamma). \tag{2.14}
  \]

- \( (iii) \) For every \( t > 0 \),
  \[
  I_\gamma(P_t h) = \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ (\tau_\nu(x) - \text{Id})y \cdot \nabla v_t(e^{-t}x + \sqrt{1 - e^{-2t}} y) \right] d\nu(x) d\gamma(y). \tag{2.15}
  \]
  As a consequence, for every \( t > 0 \),
  \[
  I(\nu_t | \gamma) = I_\gamma(P_t h) = \int_{\mathbb{R}^d} |\nabla v_t|^2 d\nu
  \leq \frac{e^{-4t}}{1 - e^{-2t}} \|\tau_\nu - \text{Id}\|^2_{2,\nu} = \frac{e^{-4t}}{1 - e^{-2t}} S^2(\nu | \gamma). \tag{2.16}
  \]

- \( (iv) \) (Exponential decay of Stein discrepancy) For every \( t \geq 0 \),
  \[
  S(\nu_t | \gamma) \leq e^{-2t} S(\nu | \gamma). \tag{2.17}
  \]

**Proof.** In view of the preceding discussion, only the proofs of \( (iii) \) and \( (iv) \) need to be detailed. Throughout the various analytical arguments below, it may be assumed that
the density $h$ is regular enough, the final conclusions being then reached by approximation arguments as e.g. in [O-V, B-G-L]. Starting with (iii), use (2.11) and the definition (2.1) of $\tau_\nu$ to write, for any $t > 0$,

$$I_t(P_t h) = -\int_{\mathbb{R}^d} \mathcal{L} P_t v_t \, dv = -\int_{\mathbb{R}^d} [\Delta P_t v_t - x \cdot \nabla P_t v_t] \, dv = \int_{\mathbb{R}^d} \langle \tau_\nu - \text{Id}, \text{Hess}(P_t v_t) \rangle_{HS} \, dv. \quad (2.18)$$

Now, for all $i, j = 1, \ldots, d$, by (2.7) and (2.8),

$$\partial_{ij} P_t v_t(x) = e^{-2t} P_t (\partial_{ij} v_t)(x) = \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \int_{\mathbb{R}^d} y_i \partial_{x_j} (e^{-t} x + \sqrt{1 - e^{-2t}} y) d\gamma(y).$$

Hence

$$\int_{\mathbb{R}^d} \langle \tau_\nu - \text{Id}, \text{Hess}(P_t v_t) \rangle_{HS} \, dv = \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[ (\tau_\nu(x) - \text{Id}) y \cdot \nabla v_t(e^{-t} x + \sqrt{1 - e^{-2t}} y) \right] d\nu(x) d\gamma(y)$$

which is (2.15). To deduce the estimate (2.16), it suffices to apply (twice) the Cauchy-Schwarz inequality to the right-hand side of (2.15) in such a way that, by integrating out the $y$ variable,

$$I_t(P_t h) \leq \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(\tau_\nu(x) - \text{Id}) y| |\nabla v_t(e^{-t} x + \sqrt{1 - e^{-2t}} y)| d\nu(x) d\gamma(y)$$

$$\leq \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \left( \int_{\mathbb{R}^d} \|\tau_\nu - \text{Id}\|_{HS}^2 \, d\nu \right)^{1/2} \left( \int_{\mathbb{R}^d} P_t(|\nabla v_t|^2) \, d\nu \right)^{1/2}.$$

Since

$$\int_{\mathbb{R}^d} P_t(|\nabla v_t|^2) \, d\nu = \int_{\mathbb{R}^d} P_t(|\nabla v_t|^2) h d\gamma = \int_{\mathbb{R}^d} |\nabla v_t|^2 P_t h d\gamma = I_t(P_t h)$$

by symmetry of $P_t$, the proof of (2.16) is complete.

Let us now turn to the proof of (2.17). For any smooth test function $\varphi$ on $\mathbb{R}^d$, by symmetry of $(P_t)_{t \geq 0}$, for any $t \geq 0$,

$$\int_{\mathbb{R}^d} x \cdot \nabla \varphi \, dv_t = \int_{\mathbb{R}^d} x \cdot \nabla \varphi \, P_t h d\gamma = \int_{\mathbb{R}^d} P_t (x \cdot \nabla \varphi) h d\gamma = \int_{\mathbb{R}^d} P_t (x \cdot \nabla \varphi) d\nu.$$

By the integral representation of $P_t$,

$$\int_{\mathbb{R}^d} P_t (x \cdot \nabla \varphi) d\nu = e^{-t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} x \cdot \nabla \varphi(e^{-t} x + \sqrt{1 - e^{-2t}} y) d\nu(x) d\gamma(y)$$

$$+ \sqrt{1 - e^{-2t}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} y \cdot \nabla \varphi(e^{-t} x + \sqrt{1 - e^{-2t}} y) d\nu(x) d\gamma(y).$$
Use now the definition of $\tau_\nu$ in the $x$ variable and integration by parts in the $y$ variable to get that
\[
\int_{\mathbb{R}^d} P_t(x \cdot \nabla \varphi) \, d\nu = e^{-2t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left< \tau_\nu(x), (\text{Hess}(\varphi))(e^{-t}x + \sqrt{1-e^{-2t}}y) \right>_{\text{HS}} \, d\nu(x) \, d\gamma(y)
\]
\[
+ (1 - e^{-2t}) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta \varphi (e^{-t}x + \sqrt{1-e^{-2t}}y) \, d\nu(x) \, d\gamma(y)
\]
\[
= e^{-2t} \int_{\mathbb{R}^d} \left< \tau_\nu, P_t(\text{Hess}(\varphi)) \right>_{\text{HS}} \, d\nu + (1 - e^{-2t}) \int_{\mathbb{R}^d} P_t(\Delta \varphi) \, d\nu
\]
\[
= e^{-2t} \int_{\mathbb{R}^d} \left< P_t(h\tau_\nu), \text{Hess}(\varphi) \right>_{\text{HS}} \, d\gamma + (1 - e^{-2t}) \int_{\mathbb{R}^d} \Delta \varphi \, P_t h \, d\gamma.
\]
As a consequence, a Stein matrix for $\nu_t$ is
\[
\tau_{\nu_t} = e^{-2t} \frac{P_t(h\tau_\nu)}{P_t h} + (1 - e^{-2t}) \text{Id}.
\]
Therefore,
\[
\int_{\mathbb{R}^d} \| \tau_{\nu_t} - \text{Id} \|_{\text{HS}}^2 \, d\nu_t = e^{-4t} \int_{\mathbb{R}^d} \| P_t(h(\tau_\nu - \text{Id})) \|_{\text{HS}}^2 \, d\gamma.
\]
By the Cauchy-Schwarz inequality along $P_t$,
\[
\| P_t(h(\tau_\nu - \text{Id})) \|_{\text{HS}}^2 \leq P_t \| \tau_\nu - \text{Id} \|_{\text{HS}}^2 P_t h.
\]
Hence,
\[
\int_{\mathbb{R}^d} \| \tau_{\nu_t} - \text{Id} \|_{\text{HS}}^2 \, d\nu_t \leq e^{-4t} \int_{\mathbb{R}^d} P_t(\| \tau_\nu - \text{Id} \|_{\text{HS}}^2) \, d\gamma
\]
\[
= e^{-4t} \int_{\mathbb{R}^d} \| \tau_\nu - \text{Id} \|_{\text{HS}}^2 \, h \, d\gamma = e^{-4t} \int_{\mathbb{R}^d} \| \tau_\nu - \text{Id} \|_{\text{HS}}^2 \, d\nu,
\]
that is the announced result $(iv)$. Proposition 2.4 is established. 

\[\square\]

**Remark 2.5.** For every $t > 0$, it is easily checked that the mapping $x \mapsto \tau_{\nu_t}(x)$ appearing in (2.19) admits the probabilistic representation
\[
\tau_{\nu_t}(x) = \mathbb{E} \left[ e^{-2t} \tau_\nu(F) + (1 - e^{-2t}) \text{Id} \mid F_t = x \right] \, d\nu_t(x) - \text{a.e.},
\]
where, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $F$ has distribution $\nu$ and $F_t = e^{-t}F + \sqrt{1-e^{-2t}}Z$, with $Z$ a $d$-dimensional vector with distribution $\gamma$, independent of $F$.

We are now in a position to prove Theorem 2.2.
Proof of Theorem 2.2. As announced, on the basis of the interpolation (2.13), we apply (2.14) and (2.16) respectively to bound the Fisher information $I_{\gamma}(P, t)$ for $t$ around 0 and away from 0. We thus get, for every $u > 0$, 

\[
H(\nu | \gamma) = \int_0^u I_{\gamma}(P, t) dt + \int_u^\infty I_{\gamma}(P, t) dt 
\leq I(\nu | \gamma) \int_0^u e^{-2t} dt + S^2(\nu | \gamma) \int_u^\infty \frac{e^{-4t}}{1 - e^{-2t}} dt 
\leq \frac{1}{2} I(\nu | \gamma)(1 - e^{-2u}) + \frac{1}{2} S^2(\nu | \gamma)(e^{-2u} - 1 - e^{-2u}).
\]

Optimizing in $u$ (set $1 - e^{-2u} = r \in (0, 1)$) concludes the proof. □

Remark 2.6. It is worth mentioning that a slight modification of the proof of (iii) in Proposition 2.4 leads to the improved form of the exponential decay (2.14) of the Fisher information

\[
I(\nu | \gamma) \leq \frac{e^{-2t} S^2(\nu | \gamma) I(\nu | \gamma)}{S^2(\nu | \gamma) + (e^{2t} - 1) I(\nu | \gamma)}.
\]

(2.21)

As for the classical logarithmic Sobolev inequality, this inequality may be integrated along de Bruijn’s formula (2.12) towards the better, although less tractable, HSI inequality

\[
H \leq \frac{S^2 I}{2(S^2 - 1)} \left(1 + \frac{1}{S^2 - 1} \log \left(\frac{I}{S^2}\right)\right)
\]

(understood in the limit as $S^2 = 1$), where $H = H(\nu | \gamma)$, $S = S(\nu | \gamma)$ and $I = I(\nu | \gamma)$.

2.4 Total variation and Stein discrepancy

In the last part of this section, we discuss a further result involving the Stein discrepancy, this time in the context of the classical Pinsker-Csiszár-Kullback inequality (see, e.g. [V, Remark 22.12]). Denote by $TV(\nu, \mu)$ the total variation distance between two probability measures $\mu, \nu$ on $\mathbb{R}^d$. While the classical Pinsker-Csiszár-Kullback inequality indicates that

\[
TV(\nu, \mu) \leq \sqrt{\frac{1}{2} H(\nu | \mu)},
\]

we mention here a bound (however only for the Gaussian measure $\mu = \gamma$) involving the Stein discrepancy. Recall that $\Psi$ on $\mathbb{R}_+$ is given by $\Psi(r) = 1 + \log r$ if $r \geq 1$ and $\Psi(r) = r$ if $0 \leq r \leq 1$.

Proposition 2.7 (Total variation and Stein discrepancy). For any centered probability measure $d\nu = hd\gamma$ on $\mathbb{R}^d$ with smooth density $h$ with respect to $\gamma$,

\[
TV(\nu, \gamma) \leq \sqrt{2} S(\nu | \gamma) \Psi \left(\frac{\sqrt{2} C \int_{\mathbb{R}^d} |\nabla h| d\gamma}{S(\nu | \gamma)}\right)
\]

where $C = \int_{\mathbb{R}^d} |y| d\gamma \leq \sqrt{d}$. 

**Proof.** We only sketch the argument. To control the total variation distance $\text{TV}(\nu, \gamma)$, we examine $\int_{\mathbb{R}^d} g d\nu - \int_{\mathbb{R}^d} g d\gamma$ for every bounded (by 1) measurable functions $g : \mathbb{R}^d \to \mathbb{R}$. To this task, we again use interpolation along the Ornstein-Uhlenbeck semigroup $(P_t)_{t \geq 0}$ to get

$$\int_{\mathbb{R}^d} g d\nu - \int_{\mathbb{R}^d} g d\gamma = \int_0^\infty \left( \int_{\mathbb{R}^d} L P_t g d\nu \right) dt.$$  

As for the proof of Theorem 2.2, we distinguish between large $t > 0$ and small $t > 0$. For large $t$'s, we may write similarly by definition of the Stein matrix that

$$\int_{\mathbb{R}^d} L P_t g d\nu = \int_{\mathbb{R}^d} [\Delta P_t g - x \cdot \nabla P_t g] d\nu = \int_{\mathbb{R}^d} \langle \tau \nu - \text{Id}, \text{Hess}(P_t \nu) \rangle_{\text{HS}} d\nu.$$  

By the Mehler integral representation of $P_t g$ and integration by parts (similar to (2.8)), for every $i, j = 1, \ldots, d$ and $t > 0$,

$$\partial_{ij} P_t g(x) = \frac{e^{-2t}}{1 - e^{-2t}} \int_{\mathbb{R}^d} (y_i y_j - \delta_{ij})g(e^{-t}x + \sqrt{1 - e^{-2t}} y) d\gamma(y).$$  

Hence, for every $g$ with $\|g\|_{\infty} \leq 1$,

$$\left| \int_{\mathbb{R}^d} L P_t g d\nu \right| \leq \frac{e^{-2t}}{1 - e^{-2t}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{i,j=1}^d (\tau_{ij}(x) - \delta_{ij})(y_i y_j - \delta_{ij}) d\nu(x) d\gamma(y).$$  

Bounding the first moment by the second one and integrating in the $y$ variable leads to

$$\left| \int_{\mathbb{R}^d} L P_t g d\nu \right| \leq \frac{\sqrt{2} e^{-2t}}{1 - e^{-2t}} \|\tau - \text{Id}\|_{2,\nu}.$$  

To handle the small values of $t > 0$, by integration by parts (2.9),

$$\int_{\mathbb{R}^d} L P_t g d\nu = \int_{\mathbb{R}^d} L P_t g h d\gamma = -\int_{\mathbb{R}^d} \nabla P_t g \cdot \nabla h d\gamma.$$  

so that, by (2.7) and (2.8), and since $\|g\|_{\infty} \leq 1$,

$$\left| \int_{\mathbb{R}^d} L P_t g d\nu \right| \leq \frac{C e^{-t}}{\sqrt{1 - e^{-2t}}} \int_{\mathbb{R}^d} |\nabla h| d\gamma.$$  

As a conclusion, for every $u > 0$,

$$\text{TV}(\nu, \gamma) \leq C \int_{\mathbb{R}^d} |\nabla h| d\gamma \int_0^u \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} dt + \sqrt{2} S(\nu | \gamma) \int_u^\infty \frac{e^{-2t}}{1 - e^{-2t}} dt.$$  

Choosing $u$ such that

$$e^{-2u} = \frac{\left( \int_{\mathbb{R}^d} |\nabla h| d\gamma \right)^2}{\left( \int_{\mathbb{R}^d} |\nabla h| d\gamma \right)^2 + 2 S^2(\nu | \gamma)}$$
leads to the best optimization. In order however to describe a simple meaningful upper bound, write

\[
TV(\nu, \gamma) \leq C \int_{\mathbb{R}^d} |\nabla h| d\gamma \int_0^u \frac{e^{-t}}{\sqrt{1 - e^{-t}}} dt - \frac{1}{\sqrt{2}} S(\nu | \gamma) \log(1 - e^{-2u})
\]

\[
\leq 2C \int_{\mathbb{R}^d} |\nabla h| d\gamma \sqrt{1 - e^{-u}} - \frac{1}{\sqrt{2}} S(\nu | \gamma) \log(1 - e^{-u}).
\]

Optimizing then in \( \sqrt{1 - e^{-u}} = r \in (0, 1) \) yields the claim of the statement.

\[\square\]

3 Transport distances and Stein discrepancy

In this section, we develop further inequalities involving the Stein discrepancy, this time in relation with Wasserstein distances. A new improved form of the Talagrand quadratic transportation cost inequality, called WSH, is emphasized, and comparison between the HSI inequality and the Talagrand and Otto-Villani HWI inequalities is provided. Let again \( \gamma = \gamma^d \) denote the standard Gaussian measure on \( \mathbb{R}^d \).

**Definition 3.1** (Wasserstein distance). Fix \( p \geq 1 \). Given two probability measures \( \nu \) and \( \mu \) on the Borel sets of \( \mathbb{R}^d \) whose marginals have finite absolute moments of order \( p \), define the Wasserstein distance (of order \( p \)) between \( \nu \) and \( \mu \) as the quantity

\[
W_p(\nu, \mu) = \inf \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y) \right)^{1/p}
\]

where the infimum runs over all probability measures \( \pi \) on \( \mathbb{R}^d \times \mathbb{R}^d \) with marginals \( \nu \) and \( \mu \).

Relevant information about Wasserstein distances can be found, e.g. in [V, Section I.6].

We shall subdivide the analysis into two parts. In Section 3.1, we deal with the special case of the quadratic Wasserstein distance \( W_2 \), for which we use the definition (2.1) of a Stein matrix. In Section 3.2, we deal with general Wasserstein distances \( W_p \) possibly of order \( p \neq 2 \), for which it seems necessary to use the stronger definition (2.2) adopted in [N-P-S1, N-P-S2].

3.1 The case of the Wasserstein distance \( W_2 \)

We provide here a dimension-free estimate on the Wasserstein \( W_2 \) distance expressed in terms of the Stein discrepancy. In the forthcoming statement, denote by \( \nu \) a centered probability measure on \( \mathbb{R}^d \) admitting a Stein matrix \( \tau_\nu \) (that is, \( \tau_\nu \) verifies (2.1) for every smooth test function \( \varphi \)). It is not assumed that \( \nu \) admits a density with respect to the Lebesgue measure on \( \mathbb{R}^d \) (in particular, \( \nu \) can have atoms). As already observed, the existence of a Stein matrix for \( \nu \) implies that \( \nu \) has finite moments of order 2.
Proposition 3.2 (W₂ distance and Stein discrepancy). For every centered probability measure ν on \( \mathbb{R}^d \),
\[
W_2(\nu, \gamma) \leq S(\nu \mid \gamma).
\] (3.1)

Proof. Assume first that \( d\nu = hd\gamma \) where \( h \) is a smooth density with respect to the standard Gaussian measure \( \gamma \) on \( \mathbb{R}^d \). As before, write \( v_t = \log P_t h \) and \( d\nu_t = P_t hd\gamma \). We shall rely on the estimate, borrowed from [O-V, Lemma 2] (cf. also [V, Theorem 24.2(iv)]),
\[
\frac{d}{dt} W_2(\nu, \nu_t) \leq \left( \int_{\mathbb{R}^d} |\nabla v_t|^2 d\nu_t \right)^{1/2}.
\] (3.2)

Note that (3.2) is actually the central argument in the Otto-Villani theorem [O-V] asserting that a logarithmic Sobolev inequality implies a Talagrand transport inequality. Here, by making use of (3.2) and then (2.16) we get
\[
W_2(\nu, \gamma) \leq \int_0^\infty \left( \int_{\mathbb{R}^d} |\nabla v_t|^2 d\nu_t \right)^{1/2} dt \leq S(\nu \mid \gamma) \int_0^\infty e^{-2t} \sqrt{1 - e^{-2t}} dt
\]
which is the result in this case.

The general case is obtained by a simple regularization procedure which is best presented in probabilistic terms. Fix \( \varepsilon > 0 \) and introduce the auxiliary random variable \( F_\varepsilon = e^{-\varepsilon} F + \sqrt{1 - e^{-2\varepsilon}} Z \) where \( F \) and \( Z \) are independent with respective laws \( \nu \) and \( \gamma \). It is immediately checked that: (a) the distribution of \( F_\varepsilon \), denoted by \( \nu_\varepsilon \), admits a smooth density \( h_\varepsilon \) with respect to \( \gamma \) (of course, this density coincides with \( P_\varepsilon h \) whenever the distribution of \( F \) admits a density \( h \) with respect to \( \gamma \) as in the first part of the proof); (b) a Stein matrix for \( \nu_\varepsilon \) is given by
\[
\tau_{\nu_\varepsilon}(x) = \mathbb{E}[e^{-2\varepsilon} \tau_\nu(F) + (1 - e^{-2\varepsilon}) \text{Id} \mid F_\varepsilon = x] \quad d\nu_\varepsilon(x) - \text{a.e.}
\]
(consistent with (2.20)); (c) \( S(\nu_\varepsilon \mid \gamma) \leq e^{-2\varepsilon} S(\nu \mid \gamma) \); (d) as \( \varepsilon \to 0 \), \( F_\varepsilon \) converges to \( F \) in \( L^2 \), so that, in particular, \( W_2(\nu_\varepsilon, \gamma) \to W_2(\nu, \gamma) \). One therefore infers that
\[
W_2(\nu, \gamma) = \lim_{\varepsilon \to 0} W_2(\nu_\varepsilon, \gamma) \leq \limsup_{\varepsilon \to 0} S(\nu_\varepsilon \mid \gamma) \leq S(\nu \mid \gamma),
\]
and the proof is concluded.

The inequality (3.1) may of course be compared to the Talagrand quadratic transportation cost inequality [T, V, B-G-L]
\[
W_2^2(\nu, \gamma) \leq 2H(\nu \mid \gamma).
\] (3.3)

As announced in the introduction, one can actually further refine (3.1) in order to deduce an improvement of (3.3) in the form of a WSH inequality. The refinement relies on the HSI inequality itself.
Theorem 3.3 (Gaussian WSH inequality). Let \( d\nu = hd\gamma \) be a centered probability measure on \( \mathbb{R}^d \) with smooth density \( h \) with respect to \( \gamma \). Assume further that \( S(\nu \mid \gamma) \) and \( H(\nu \mid \gamma) \) are both positive and finite. Then

\[
W_2(\nu, \gamma) \leq S(\nu \mid \gamma) \arccos \left( e^{\frac{-H(\nu \mid \gamma)}{S^2(\nu \mid \gamma)}} \right).
\]

Proof. For any \( t \geq 0 \), recall \( d\nu_t = P_t h d\gamma \) (in particular, \( \nu_0 = \nu \) and \( \nu_t \to \gamma \) as \( t \to \infty \)). The HSI inequality (2.5) applied to \( \nu_t \) yields that

\[
H(\nu_t \mid \gamma) \leq \frac{1}{2} S^2(\nu_t \mid \gamma) \log \left( 1 + \frac{I(\nu_t \mid \gamma)}{S^2(\nu_t \mid \gamma)} \right).
\]

Now, \( S^2(\nu_t \mid \gamma) \leq S^2(\nu \mid \gamma) \) by (2.17) and \( r \mapsto r \log (1 + \frac{s}{r}) \) is increasing for any fixed \( s \) from which it follows that

\[
H(\nu_t \mid \gamma) \leq \frac{1}{2} S^2(\nu \mid \gamma) \log \left( 1 + \frac{I(\nu_t \mid \gamma)}{S^2(\nu \mid \gamma)} \right).
\]

By exponentiating both sides, this inequality is equivalent to

\[
\sqrt{I(\nu_t \mid \gamma)} \leq \frac{I(\nu_t \mid \gamma)}{S(\nu \mid \gamma) \sqrt{e^{\frac{2H(\nu_t \mid \gamma)}{S^2(\nu_t \mid \gamma)}} - 1}}.
\]

Combining with (3.2) and recalling (2.12) leads to

\[
\frac{d}{dt} W_2(\nu, \nu_t) \leq \sqrt{I(\nu_t \mid \gamma)} \leq -\frac{d}{dt} H(\nu_t \mid \gamma)
\]

\[
= -\frac{d}{dt} \left( S(\nu \mid \gamma) \arccos \left( e^{\frac{-H(\nu \mid \gamma)}{S^2(\nu \mid \gamma)}} \right) \right).
\]

In other words,

\[
\frac{d}{dt} \left( W_2(\nu, \nu_t) + S(\nu \mid \gamma) \arccos \left( e^{\frac{-H(\nu \mid \gamma)}{S^2(\nu \mid \gamma)}} \right) \right) \leq 0.
\]

The desired conclusion is achieved by integrating between \( t = 0 \) and \( t = \infty \). The proof of Theorem 3.3 is complete.

Proposition 3.2 raises a number of observations.

Remark 3.4. (a) The Talagrand inequality may combined with the HSI inequality of Theorem 2.2 to yield the bound

\[
W_2^2(\nu, \gamma) \leq S^2(\nu \mid \gamma) \log \left( 1 + \frac{I(\nu \mid \gamma)}{S^2(\nu \mid \gamma)} \right). \tag{3.4}
\]
(b) (HWI inequality). As described in the introduction, a fundamental estimate connecting entropy $H$, Wassertein distance $W_2$ and Fisher information $I$ is the so-called HWI inequality of Otto and Villani [O-V] stating that, for all $d\nu =hd\gamma$ with density $h$ with respect to $\gamma$,

$$H(\nu|\gamma) \leq W_2(\nu, \gamma) \sqrt{I(\nu|\gamma)} - \frac{1}{2} W_2^2(\nu, \gamma)$$

(3.5)

(see, e.g. [V, pp. 529-542] or [B-G-L, Section 9.3.1] for a general discussion). Recall that the HWI inequality (3.5) improves upon both the logarithmic Sobolev inequality (1.1) and the Talagrand inequality (3.3). It is natural to look for a more general inequality, involving all four quantities $H$, $W_2$, $I$ and the Stein discrepancy $S$, and improving both the HSI and HWI inequalities. One strategy towards this task would be to follow again the heat flow approach of the proof of Theorem 2.2 and write, for $0 < u \leq t$,

$$\text{Ent}_{\gamma}(h) = \int_0^t I_{\gamma}(P_s h) ds + \text{Ent}_{\gamma}(P_t h)$$

$$\leq I_{\gamma}(h) \int_0^u e^{-2s} ds + S^2(\nu|\gamma) \int_u^t \frac{e^{-4s}}{1 - e^{-2s}} ds + \frac{e^{-2t}}{2(1 - e^{-2t})} W_2^2(\nu, \gamma).$$

Here, we used (2.14) and (2.16), as well as the known reverse Talagrand inequality along the semigroup given by

$$\text{Ent}_{\gamma}(P_t h) \leq \frac{e^{-2t}}{2(1 - e^{-2t})} W_2^2(\nu, \gamma)$$

(cf. e.g. [B-G-L, p. 446]). Setting $\alpha = 1 - e^{-2u} \leq 1 - e^{-2t} = \beta$, the preceding estimate yields

$$H(\nu|\gamma) \leq \inf_{0 < \alpha \leq \beta \leq 1} \Phi(\alpha, \beta)$$

where

$$\Phi(\alpha, \beta) = \alpha I(\nu|\gamma) + (\alpha - \log \alpha) S^2(\nu|\gamma) + \frac{1 - \beta}{\beta} W_2^2(\nu, \gamma) + (\log \beta - \beta) S^2(\nu|\gamma).$$

However, elementary computations show that, unless the rather unnatural inequality $2W_2(\nu, \gamma) \leq S(\nu|\gamma)$ is verified, the minimum in the above expression is attained at a point $(\alpha, \beta)$ such that either $\alpha = \beta$ (and in this case one recovers HWI) or $\beta = 1$ (yielding HSI). Hence, at this stage, it seems difficult to outperform both HWI and HSI estimates with a single ‘HWIS’ inequality. In the subsequent point (c), we provide an elementary explicit example in which the HSI estimate perform better than the HWI inequality.
(c) In this item, we thus compare the HSI and HWI inequalities on a specific example in dimension $d = 1$. For every $n \geq 1$, consider the probability measure $d\nu_n(x) = p_n(x)dx$ with density

$$p_n(x) = \frac{1}{\sqrt{2\pi}} [(1 - a_n)e^{-x^2/2} + na_n e^{-n^2x^2/2}], \quad x \in \mathbb{R},$$

where $(a_n)_{n\geq 1}$ is such that $a_n \in [0, 1]$ for every $n \geq 1$, $a_n = o\left(\frac{1}{\log n}\right)$ and $n^{2/3}a_n \to \infty$. A direct computation easily shows that $H(\nu_n | \gamma) \to 0$. Also, since

$$p_n'(x) = -\frac{x}{\sqrt{2\pi}} [(1 - a_n)e^{-x^2/2} + a_n e^{-n^2x^2/2}],$$

one may show after simple (but a bit lengthy) computations that

$$I(\nu_n | \gamma) = \int_{\mathbb{R}} \frac{p_n'(x)^2}{p_n(x)}dx - 1 \sim n^2 a_n \quad \text{as} \quad n \to \infty.$$

We next examine the Stein discrepancy $S(\nu_n | \gamma)$ and Wasserstein distance $W_2(\nu_n, \gamma)$. Since a Stein factor $\tau_n$ of $\nu_n$ is given by

$$\tau_n(x) = \frac{1}{\sqrt{2\pi} p_n} [(1 - a_n)e^{-x^2/2} + \frac{a_n}{n} e^{-n^2x^2/2}],$$

it is easily seen that

$$S^2(\nu_n | \gamma) = \int_{\mathbb{R}} (\tau_n(x) - 1)^2 p_n(x)dx \leq a_n \to 0.$$

Concerning the Wasserstein distance, from the inequality (3.1), we deduce that $W_2(\nu_n, \gamma) \leq \sqrt{a_n}$. On the other hand, by the Lipschitz characterization of $W_1$ (specializing to the Lipschitz function $x \mapsto |\cos(x)|$), cf. e.g. [V, Remark 6.5]),

$$W_2(\nu_n, \gamma) \geq W_1(\nu_n, \gamma) \geq \left| \int_{\mathbb{R}} |\cos(x)| d\nu_n(x) - \int_{\mathbb{R}} |\cos(x)| d\gamma(x) \right|.$$

Now, the right-hand side of this inequality multiplied by $\frac{1}{a_n}$ is equal to

$$\left| n \int_{\mathbb{R}} |\cos(x)| e^{-n^2x^2/2} \frac{dx}{\sqrt{2\pi}} - \int_{\mathbb{R}} |\cos(x)| d\gamma(x) \right|$$

$$= \left| \int_{\mathbb{R}} \left[ |\cos(x)| - |\cos(x)| \right] d\gamma(x) \right|$$

which, by dominated convergence, converges to a non-zero limit. As a consequence, there exists $c > 0$ such that, for $n$ large enough, $W_2(\nu_n, \gamma) \geq c a_n$. 
Summarizing the conclusions, the quantity
\[ W_2(\nu_n, \gamma) \sqrt{I(\nu_n \mid \gamma)} - \frac{1}{2} W_2^2(\nu_n, \gamma) \]
is bigger than a sequence of the order of \( n a_n^{3/2} = (n^{2/3} a_n)^{3/2} \), which (by construction) diverges to infinity as \( n \to \infty \). This fact implies that, in this specific case, the bound in the HWI inequality diverges to infinity, whereas \( H(\nu_n \mid \gamma) \to 0 \). On the other hand, the HSI bound converges to zero, since
\[ S^2(\nu_n \mid \gamma) \log \left( 1 + \frac{I(\nu_n \mid \gamma)}{S^2(\nu_n \mid \gamma)} \right) \leq a_n \log(1 + n^2) \sim 2a_n \log n \to 0. \]

### 3.2 General Wasserstein distances under a stronger notion of Stein matrix

In this part, we obtain bounds in terms of Stein discrepancies on the Wasserstein distance \( W_p \) of any order \( p \) between a centered probability measure \( \nu \) on \( \mathbb{R}^d \) and the standard Gaussian distribution \( \gamma \). As in Proposition 3.2, we shall consider probabilities \( \nu \) not necessarily admitting a density with respect to \( \gamma \). However, it will be assumed that \( \nu \) has a Stein matrix \( \tau_\nu \) verifying the stronger ‘vector’ relation (2.2). The reason for this is that, in order to deal with Wasserstein distances of the type \( W_p, p \neq 2 \), one needs to have access to the explicit expression of the score function \( \nabla \log P_t h \) along the Ornstein-Uhlenbeck semigroup, as proved in [N-P-S1, Lemma 2.9] in the framework of Stein matrices verifying (2.2). Recall that the existence of \( \tau_\nu \) implies that \( \nu \) has finite moments of order 2.

**Proposition 3.5 (\( W_p \) distance and Stein discrepancy).** Let \( \nu \) be a centered probability measure on \( \mathbb{R}^d \) with Stein matrix \( \tau_\nu \) in the sense of (2.2). For every \( p \geq 1 \), set
\[ \| \tau_\nu - \text{Id} \|_{p,\nu} = \left( \sum_{i,j=1}^d \int_{\mathbb{R}^d} |\tau^{ij}_\nu - \delta_{ij}|^p d\nu \right)^{1/p} \]
(where \( \delta_{ij} = 1 \) if \( i = j \) and 0 if not), possibly infinite if \( \tau^{ij}_\nu \notin L^p(\nu) \). In particular, \( \| \tau_\nu - \text{Id} \|_{2,\nu} = S(\nu \mid \gamma) \).

(i) Let \( p \in [1, 2) \). Then,
\[ W_p(\nu, \gamma) \leq C_p d^{1-1/p} \| \tau_\nu - \text{Id} \|_{p,\nu} \] (3.6)
where \( C_p = \int_{\mathbb{R}} |x|^p d\gamma^1(x) \).

(ii) Let \( p \in [2, \infty) \). If \( \nu \) has finite moments of order \( p \), then (with the same \( C_p \) as in (i))
\[ W_p(\nu, \gamma) \leq C_p d^{3-2/p} \| \tau_\nu - \text{Id} \|_{p,\nu}. \] (3.7)
In particular, for \( p = 2 \) we recover (3.1).
Proof. Owing to an approximation argument analogous to the one rehearsed at end of the proof of Proposition 3.2, it is sufficient to consider the case \( d\nu = h\,d\gamma \) where \( h \) is a smooth density. Write as before \( v_t = \log P_th \) and \( d\nu_t = P_th d\gamma \). By virtue of [N-P-S1, Lemma 2.9], under thus the strengthened assumption (2.2), a version of \( \nabla v_t, t > 0 \), is given by

\[
x \mapsto \nabla v_t(x) = \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \mathbb{E}[ (\tau_{\nu}(F) - \text{Id}) Z \mid F_t = x], \quad x \in \mathbb{R}^d,
\]

where, as in Remark 2.5, yielding (ii) the proof of Proposition 3.2, it is sufficient to consider the case \( d\nu_t = \nu, \gamma \) (cf. also [V, Theorem 24.2(iv)]) in order to obtain the general estimate

\[
\frac{dt}{d} W_p(\nu, \nu_t) \leq \left( \int_{\mathbb{R}^d} |\nabla v_t|^p \, d\nu_t \right)^{1/p}.
\]

It follows that

\[
W_p(\nu, \gamma) \leq \int_0^\infty \left( \int_{\mathbb{R}^d} |\nabla v_t|^p \, d\nu_t \right)^{1/p} \, dt
= \int_0^\infty \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \mathbb{E} \left[ \left( \sum_{i=1}^d \mathbb{E} \left[ \sum_{j=1}^d (\tau_{\nu}^{ij}(F) - \delta_{ij}) Z_j \middle| F_1 \right] \right)^p \right]^{1/p} \, dt.
\]

Now, if \( 1 \leq p < 2 \),

\[
W_p(\nu, \gamma) \leq \int_0^\infty \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \, dt \left( \sum_{i=1}^d \mathbb{E} \left[ \left( \sum_{j=1}^d (\tau_{\nu}^{ij}(F) - \delta_{ij}) Z_j \right)^p \right] \right)^{1/p}
\leq C_p \, d^{1/2 - 1/p} \left( \sum_{i,j=1}^d \mathbb{E} \left[ |\tau_{\nu}^{ij}(F) - \delta_{ij}|^p \right] \right)^{1/p}
\]

yielding (i). On the other hand, if \( p \geq 2 \), then

\[
W_p(\nu, \gamma) \leq \int_0^\infty \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \mathbb{E} \left[ \left( \sum_{i=1}^d \mathbb{E} \left[ \left( \sum_{j=1}^d (\tau_{\nu}^{ij}(F) - \delta_{ij}) Z_j \right)^2 \right] \right)^{p/2} \right]^{1/p} \, dt
\leq d^{1/2 - 1/p} \left( \sum_{i=1}^d \mathbb{E} \left[ \left( \sum_{j=1}^d (\tau_{\nu}^{ij}(F) - \delta_{ij}) Z_j \right)^2 \right] \right)^{1/p}
= C_p \, d^{1/2 - 1/p} \left( \sum_{i,j=1}^d \mathbb{E} \left[ |\tau_{\nu}^{ij}(F) - \delta_{ij}|^2 \right] \right)^{1/p}
\leq C_p \, d^{1-2/p} \left( \sum_{i,j=1}^d \mathbb{E} \left[ |\tau_{\nu}^{ij}(F) - \delta_{ij}|^p \right] \right)^{1/p}
\]

which immediately yields (ii). The proof of Proposition 3.5 is complete. \( \square \)
Remark 3.6. Specializing (3.6) to the case $p = 1$ yields the estimate

$$W_1(\nu, \gamma) \leq \sqrt{\frac{2}{\pi}} \|\tau_\nu - \text{Id}\|_{1,\nu}$$

which improves previous dimensional bounds obtained by an application of the multidimensional Stein method (cf. the proof of [N-P2, Theorem 6.1.1]). It is important to note that, apart from the results obtained in the present paper, there is no other version of Stein’s method allowing one to deal with Wasserstein distances of order $p > 1$. Observe that coupling results from [C2] (that are based on completely different methods) may be used to deduce analogous estimates in the case when $d = 1$ and the Stein factor $\tau_\nu$ is bounded.

4 HSI inequalities for further distributions

On the basis of the Gaussian example of Section 2, we next address the issue of HSI inequalities for distributions on $\mathbb{R}^d$, $d \geq 1$, that are not necessarily Gaussian. In order to reach the basic semigroup ingredients towards such HSI inequalities put forward in Proposition 2.4, a convenient family of measures to deal with is the family of invariant measures of second order differential operators. These include gamma and beta distributions, as well as families of log-concave measures as illustrations. We present them here in the framework of Markov Triples as developed in [B-G-L] and, for simplicity, only consider operators and measures on $\mathbb{R}^d$.

4.1 A general statement

Let $E$ be a domain of $\mathbb{R}^d$ and consider a family of real-valued $C^\infty$-functions $a^{ij}(x)$ and $b^i(x)$, $i, j = 1, \ldots, d$, defined on $E$. We assume that the matrix $a(x) = (a^{ij}(x))_{1 \leq i, j \leq d}$ is symmetric and positive definite for any $x \in E$. For every $x \in E$, we let $a^2(x)$ be the unique symmetric non-singular matrix such that $(a^{ij}(x))^2 = a(x)$. Let $\mathcal{A}$ denote the algebra of $C^\infty$-functions on $E$ and $\mathcal{L}$ be the second order differential operator given on functions $f \in \mathcal{A}$ by

$$\mathcal{L}f = \langle a, \text{Hess}(f) \rangle_{\text{HS}} + b \cdot \nabla f = \sum_{i,j=1}^d a^{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^d b^i \frac{\partial f}{\partial x_i}. \quad (4.1)$$

The operator $\mathcal{L}$ satisfies the chain rule formula and defines a diffusion operator. We assume that $\mathcal{L}$ is the generator of a symmetric Markov semigroup $(P_t)_{t \geq 0}$, where the symmetry is with respect to an invariant probability measure $\mu$.

A central object of interest in this context is the carré du champ operator $\Gamma$ defined from the generator $\mathcal{L}$ by

$$\Gamma(f, g) = \frac{1}{2} \left[ \mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f \right] = \sum_{i,j=1}^d a^{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$
for all \((f, g) \in \mathcal{A} \times \mathcal{A}\). Note that \(\Gamma\) is bilinear and symmetric and \(\Gamma(f, f) \geq 0\). Moreover, the integration by parts property for \(\mathcal{L}\) with respect to the invariant measure \(\mu\) is expressed by the fact that, for functions \(f, g \in \mathcal{A}\),

\[
\int_E f \mathcal{L} g \, d\mu = - \int_E \Gamma(f, g) \, d\mu.
\]

The structure \((E, \mu, \Gamma)\) then defines a Markov Triple in the sense of [B-G-L] to which we refer for the necessary background.

The requested semigroup analysis toward HSI inequalities will actually involve in addition the iterated gradient operators \(\Gamma_n, n \geq 1\), defined inductively for \((f, g) \in \mathcal{A} \times \mathcal{A}\) via the relations \(\Gamma_0(f, g) = fg\) and

\[
\Gamma_n(f, g) = \frac{1}{2} \left[ \mathcal{L} \Gamma_{n-1}(f, g) - \Gamma_{n-1}(f, \mathcal{L} g) - \Gamma_{n-1}(g, \mathcal{L} f) \right], \quad n \geq 1.
\]

In particular \(\Gamma_1 = \Gamma\) and the operators \(\Gamma_n, n \geq 1\), are similarly symmetric and bilinear. In what follows, we shall often adopt the shorthand notation \(\Gamma_n(f)\) instead of \(\Gamma_n(f, f)\).

The \(\Gamma_2\) operator is part of the famous Bakry-Émery criterion for logarithmic Sobolev inequalities [B-E], [B-G-L, Section 5.7]. As a new feature of the analysis here, the iterated gradient \(\Gamma_3\) will turn essential towards a suitable analogue of \((iii)\) in Proposition 2.4.

A prototypical example of this setting is of course the example of the Ornstein-Uhlenbeck operator \(\mathcal{L} = \Delta - x \cdot \nabla\) on \(\mathbb{R}^d\) considered earlier, with the standard Gaussian measure \(\gamma\) as symmetric and invariant measure. In this case, the carré du champ operator is simply given by \(\Gamma(f) = |\nabla f|^2\) on smooth functions \(f\). It is easily seen that, for example (cf. [L1]),

\[
\Gamma_2(f) = \sum_{i,j=1}^d \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 + \Gamma(f)
\]

and

\[
\Gamma_3(f) = \sum_{i,j,k=1}^d \left( \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \right)^2 + 3 \Gamma_2(f) - 2 \Gamma(f).
\]

Given thus the preceding Markov Triple \((E, \mu, \Gamma)\) associated to the second order differential operator \(\mathcal{L}\) of (4.1), let \(d\nu = h \, d\mu\) where \(h\) is a smooth probability density with respect to \(\mu\). As in the Gaussian case, the relative entropy of \(\nu\) with respect to \(\mu\) is the quantity

\[
H(\nu | \mu) = \text{Ent}_\mu(h) = \int_E h \log h \, d\mu.
\]

Similarly, the Fisher information of \(\nu\) (or \(h\)) with respect to \(\mu\) is defined as

\[
I(\nu | \mu) = I_\mu(h) = \int_E \frac{\Gamma(h)}{h} \, d\mu = \int_E \Gamma(\log h) \, h \, d\mu = - \int_E \mathcal{L}(\log h) \, d\nu.
\]

(4.2)
The (integrated) de Bruijn’s identity (cf. Proposition 5.2.2 in [B-G-L]) reads as in (i) of Proposition 2.4,
\[ \mathcal{H}(\nu \mid \mu) = \int_0^\infty I_\mu(P_th) d\mu. \]

Let \( \mathcal{M}_{d \times d} \) denote the class of \( d \times d \) matrices with real entries. Analogously to the definition of Stein matrix of Section 2.1, we shall say that a matrix-valued mapping \( \tau_\nu : \mathbb{R}^d \to \mathcal{M}_{d \times d} \) satisfying \( \tau_\nu^{ij} \in L^1(\nu) \) for every \( i, j = 1, \ldots, d \), and
\[ -\int_E b \cdot \nabla f \, d\nu = \int_E \langle \tau_\nu, \text{Hess}(f) \rangle_{\text{HS}} \, d\nu, \quad f \in \mathcal{A}, \quad (4.3) \]
is a Stein matrix for the probability \( \nu \) on \( E \) with respect to the generator of \( L \) of (4.1), where \( b = (b_i(x))_{1 \leq i \leq d} \) is part of the definition of \( \mathcal{L} \). For the Ornstein-Uhlenbeck operator \( \mathcal{L} = \Delta - x \cdot \nabla \), the definition corresponds to (2.1). Since \( \int_E \mathcal{L} f \, d\mu = 0 \), observe that \( a \) is a Stein matrix for \( \mu \). The main result in this section is an HSI inequality that relates \( \mathcal{H}(\nu \mid \mu) \), \( \mathcal{I}(\nu \mid \mu) \) and the Stein discrepancy of \( \nu \) with respect to \( \mu \)
\[ S(\nu \mid \mu) = \left( \int_E \| a^{-\frac{1}{2}} \tau_\nu a^{-\frac{1}{2}} - \text{Id} \|_{\text{HS}}^2 \, d\nu \right)^{1/2} \quad (4.4) \]
that we regard, as in the Gaussian case of Section 2, as a measure of the distance between \( \nu \) and \( \mu \) (since \( \tau_\mu = a \)). Note that choosing \( a = C \) in (4.4), with \( C \) non-singular, yields the quantity arising in Corollary 2.3. It should also be mentioned that the Stein discrepancy (4.4) is somewhat in contrast with the bounds one customarily obtains when applying Stein’s method (see e.g. [N-P1] for the specific example of the one-dimensional Gamma distribution, or [R] for a general reference), which typically involve quantities of the type \( \int_E \| \tau_\nu - a \|_{\text{HS}}^2 \, d\nu \). The appearance of the inverse matrices \( a^{-\frac{1}{2}} \) seems to be inextricably connected with the fact that we deal with information-theoretical functionals.

The following general statement collects the necessary assumptions on the iterated gradients \( \Gamma, \Gamma_2 \) and \( \Gamma_3 \) to achieve the expected HSI inequality by the semigroup interpolation scheme. The next paragraphs will provide illustrations in various concrete instances of interest. In Theorem 4.1 below, (i) amounts to the Bakry-Émery \( \Gamma_2 \) criterion to ensure the logarithmic Sobolev inequality. Condition (ii) linking the \( \Gamma_2 \) and \( \Gamma_3 \) operators will provide (together with (iii)) the suitable semigroup bound for the time control of \( \mathcal{I}(P_th) \) away from 0. Recall \( \Psi(r) = 1 + \log r \) if \( r \geq 1 \) and \( \Psi(r) = r \) if \( 0 \leq r \leq 1 \).

**Theorem 4.1 (Abstract HSI inequality).** In the preceding context, let \( d\nu = hd\mu \) where \( h \) be a smooth probability density with respect to \( \mu \). Assume that there exists \( \rho, \kappa, \sigma > 0 \) such that, for any \( f \in \mathcal{A} \),
\[ (i) \ \Gamma_2(f) \geq \rho \Gamma(f); \]
\[ (ii) \ \Gamma_3(f) \geq \kappa \Gamma_2(f); \]
(iii) \( \Gamma_2(f) \geq \sigma \| a^{\frac{1}{2}} \Hess(f) a^{\frac{1}{2}} \|_{HS}^2 \) (with \( a \) as in (4.1)).

Then,
\[
H(\nu | \mu) \leq \frac{1}{2\sigma} S^2(\nu | \mu) \Psi \left( \frac{\sigma \max(\rho, \kappa) I(\nu | \mu)}{\rho \kappa S^2(\nu | \mu)} \right).
\]

Note that in the Ornstein-Uhlenbeck example, \( \rho = \kappa = \sigma = 1 \) from which we recover the HSI inequality (2.5), however in a slightly weaker formulation.

**Proof.** It is therefore a classical fact (see e.g. [B-G-L, (5.7.4)]) that (i) ensures the exponential decay of the Fisher information along the semigroup
\[
I_\mu(P_t h) \leq e^{-2\rho t} I_\mu(h) = e^{-2\rho t} I(\mu | \nu)
\]
for every \( t \geq 0 \) (and then yields a logarithmic Sobolev inequality for \( \mu \).) Now, fix \( t > 0 \) and let \( f \in A \). The \( \Gamma \)-calculus as developed in [B-G-L], but at the level of the \( \Gamma_2 \) and \( \Gamma_3 \) operators, yields on \([0, t]\) (by the very definition of \( \Gamma_3 \) from \( \Gamma_2 \)),
\[
\frac{d}{ds} \left( P_s(\Gamma_2(P_t-f)) e^{-2\kappa s} \right) = 2e^{-2\kappa s} \left( P_s(\Gamma_3(P_t-f)) - \kappa P_s(\Gamma_2(P_t-f)) \right)
\]
\[
= 2e^{-2\kappa s} P_s((\Gamma_3 - \kappa \Gamma_2)(P_t-f)).
\]
By (ii), the latter is non-negative so that the map \( s \mapsto P_s(\Gamma_2(P_t-f)) e^{-2\kappa s} \) is increasing on \([0, t]\), and thus
\[
P_t(\Gamma(f)) - \Gamma(P_t(f)) = 2 \int_0^t P_s(\Gamma_2(P_{t-s}-f)) ds
\]
\[
\geq 2 \Gamma_2(P_t f) \int_0^t e^{2\kappa s} ds = \frac{1}{\kappa} (e^{2\kappa t} - 1) \Gamma_2(P_t f).
\]
Together with (iii), it then follows that
\[
P_t(\Gamma(f)) \geq P_t(\Gamma(f)) - \Gamma(P_t(f)) \geq \frac{\sigma}{\kappa} (e^{2\kappa t} - 1) \| a^{\frac{1}{2}} \Hess(P_t f) a^{\frac{1}{2}} \|_{HS}^2.
\]

We shall apply (4.6) to \( v_t = \log P_t h \) (with \( h \) regular enough). First, by symmetry of \( \mu \) with respect to \( (P_t)_{t \geq 0} \),
\[
I_\mu(P_t h) = - \int_E \mathcal{L} v_t P_t h d\mu = - \int_E \mathcal{L} P_t v_t h d\mu = - \int_E \mathcal{L} P_t v_t d\nu.
\]
Hence, by (4.1) and (4.3),
\[
I_\mu(P_t h) = - \int_E \langle a, \Hess(P_t v_t) \rangle_{HS} d\nu - \int_E b \cdot \nabla P_t v_t d\nu
\]
\[
= \int_E \langle \tau_{\nu} - a, \Hess(P_t v_t) \rangle_{HS} d\nu.
\]
Now, by the Cauchy-Schwarz inequality,

\[
I_\mu(P_t h) = \int_E \langle a^{-\frac{1}{2}} \tau v, a^{-\frac{1}{2}} - 1, a^{\frac{1}{2}} \text{Hess}(P_t v) a^{\frac{1}{2}} \rangle_{\text{HS}} d\nu
\]

\[
\leq \left( \int_E \| a^{-\frac{1}{2}} \tau v, a^{-\frac{1}{2}} - 1 \|_{\text{HS}}^2 d\nu \right)^{1/2} \left( \int_E \| a^{\frac{1}{2}} \text{Hess}(P_t v) a^{\frac{1}{2}} \|_{\text{HS}}^2 d\nu \right)^{1/2}
\]

\[
\leq S(\nu \mid \mu) \left( \frac{\kappa}{\sigma(e^{2\kappa t} - 1)} \int_E P_t(\Gamma(v_t)) d\nu \right)^{1/2}
\]

where the last step follows from (4.6). Since

\[
\int_E P_t(\Gamma(v_t)) d\nu = \int_E P_t(\Gamma(v_t)) h d\mu = \int_E \Gamma(v_t) P_t h d\mu = I_\mu(P_t h),
\]

it follows that

\[
I_\mu(P_t h) \leq \frac{\kappa}{\sigma(e^{2\kappa t} - 1)} S^2(\nu \mid \mu). \tag{4.8}
\]

Finally, using (4.5) for small \( t \) and (4.8) for large \( t \), one deduces that, for every \( u > 0 \),

\[
H(\nu \mid \mu) \leq I(\nu \mid \mu) \int_0^u e^{-2\rho t} dt + S^2(\nu \mid \mu) \int_u^\infty \frac{\kappa}{\sigma(e^{2\kappa t} - 1)} dt
\]

\[
= \frac{I(\nu \mid \mu)}{2\rho} (1 - e^{-2\rho u}) - \frac{S^2(\nu \mid \mu)}{2\sigma} \log(1 - e^{-2\kappa u}).
\]

Setting \( r = e^{-2u} \),

\[
H(\nu \mid \mu) \leq \inf_{0 < r < 1} \left\{ \frac{I(\nu \mid \mu)}{2\rho} (1 - r^\rho) - \frac{S^2(\nu \mid \mu)}{2\sigma} \log(1 - r^\kappa) \right\}.
\]

Now, using that \( 1 - r^\rho \leq \max(1, \frac{e}{r})(1 - r^\kappa) \) for \( r \in (0, 1) \), a simple (non-optimal) optimization yields the desired conclusion. The proof of Theorem 4.1 is complete.

\[\square\]

**Remark 4.2.** It should be pointed out that, on the basis of (4.8), transport inequalities as studied in Section 3 may be investigated similarly in the preceding general context, and with similar illustrations as developed below. For example, as an analogue of (3.1),

\[
W_2(\nu, \mu) \leq \frac{2}{\sqrt{\kappa \sigma}} S(\nu \mid \mu).
\]

In order not to expand too much the exposition, we leave the details to the reader.

The next paragraphs present various illustrations of Theorem 4.1.
4.2 Multivariate gamma distribution

As a first example of illustration of the preceding general result, we consider the case of the multidimensional Laguerre operator, which is the product on $\mathbb{R}^d_+$ of one-dimensional Laguerre operators of parameters $p_i > 0$, $i = 1, \ldots, d$, that is,

$$\mathcal{L}f = \sum_{i=1}^d x_i \frac{\partial^2 f}{\partial x_i^2} + \sum_{i=1}^d (p_i - x_i) \frac{\partial f}{\partial x_i}.$$  

In particular, \(a(x) = (x_i\delta_{ij})_{1 \leq i,j \leq d}\) in (4.1). It is a standard fact that the invariant measure $\mu$ associated with $\mathcal{L}$ has a density with respect to the Lebesgue measure given by the tensor product of $d$ gamma densities of the type $f_i(x_i) = \Gamma(p_i)^{-1} x_i^{p_i-1} e^{-x_i}$, $x_i \in \mathbb{R}_+$, $i = 1, \ldots, d$. For reasons that will become clear later on, we assume that $p_i \geq \frac{3}{2}$, $i = 1, \ldots, d$.

After some easy but cumbersome calculations, it may be checked that, along suitable smooth functions $f$,

$$\Gamma(f) = \sum_{i=1}^d x_i \left(\frac{\partial f}{\partial x_i}\right)^2$$

$$\Gamma_2(f) = \sum_{i,j=1}^d x_i x_j \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2 + \sum_{i=1}^d x_i \frac{\partial f}{\partial x_i} \frac{\partial^2 f}{\partial x_i^2} + \frac{1}{2} \sum_{i=1}^d (p_i + x_i) \left(\frac{\partial f}{\partial x_i}\right)^2$$

$$\Gamma_3(f) = \sum_{i,j,k=1}^d x_i x_j x_k \left(\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}\right)^2 + 3 \sum_{i,j=1}^d x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial^2 f}{\partial x_i^2} + \frac{3}{2} \sum_{i=1}^d x_i \left(\frac{\partial^2 f}{\partial x_i^2}\right)^2$$

$$+ \frac{3}{2} \sum_{i,j=1}^d (p_i + x_i) x_j \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2 + \frac{3}{2} \sum_{i=1}^d x_i \left(\frac{\partial^2 f}{\partial x_i^2}\right)^2$$

$$+ \frac{1}{4} \sum_{i=1}^d (3p_i + x_i) \left(\frac{\partial f}{\partial x_i}\right)^2.$$  

Note that (recall $x_i, x_j, x_k \geq 0$)

$$\sum_{i,j,k=1}^d x_i x_j x_k \left(\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}\right)^2 + 3 \sum_{i,j=1}^d x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial^2 f}{\partial x_i^2} \frac{\partial^2 f}{\partial x_j^2}$$

$$\geq \sum_{i,j=1}^d x_i^2 x_j \left(\frac{\partial^3 f}{\partial x_i^2 \partial x_j}\right)^2 + 3 \sum_{i,j=1}^d x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{\partial^2 f}{\partial x_i^2} \frac{\partial^2 f}{\partial x_j^2}$$

$$\geq -\frac{9}{4} \sum_{i,j=1}^d x_j \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2.$$
Therefore

\[
\Gamma_3(f) \geq 3 \sum_{i,j=1}^{d} (p_i - \frac{3}{2} + x_i) x_j \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 + 3 \sum_{i=1}^{d} x_i \left( \frac{\partial^2 f}{\partial x_i^2} \right)^2 \\
+ \frac{3}{2} \sum_{i=1}^{d} x_i \frac{\partial f}{\partial x_i} \frac{\partial^2 f}{\partial x_i^2} + \frac{1}{4} \sum_{i=1}^{d} (3p_i + x_i) \left( \frac{\partial f}{\partial x_i} \right)^2.
\]

Since \(p_i \geq \frac{3}{2}\), it follows at once that \(\Gamma_3(f) \geq \frac{1}{2} \Gamma_2(f)\). Analogous computations lead to

\[
\sum_{i=1}^{d} x_i \frac{\partial f}{\partial x_i} \frac{\partial^2 f}{\partial x_i^2} \geq -\frac{1}{2} \sum_{i=1}^{d} x_i^2 \left( \frac{\partial^2 f}{\partial x_i^2} \right)^2 - \frac{1}{2} \sum_{i=1}^{d} \left( \frac{\partial f}{\partial x_i} \right)^2,
\]

implying that

\[
\Gamma_2(f) \geq \frac{1}{2} \sum_{i=1}^{d} x_i^2 \left( \frac{\partial^2 f}{\partial x_i^2} \right)^2 + \frac{1}{2} \sum_{i=1}^{d} (p_i - 1 + x_i) \left( \frac{\partial f}{\partial x_i} \right)^2 \geq \frac{1}{2} \Gamma(f).
\]

Finally, one has

\[
\frac{1}{2} \left\| \sqrt{a} \text{Hess}(f) \sqrt{a} \right\|_{\text{HS}}^2 = \frac{1}{2} \sum_{i,j=1}^{d} x_i x_j \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 \\
\leq \frac{1}{2} \sum_{i,j=1}^{d} x_i x_j \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 + \frac{1}{2} \sum_{i=1}^{d} \left( x_i \frac{\partial^2 f}{\partial x_i^2} + \frac{\partial f}{\partial x_i} \right)^2 \\
\leq \Gamma_2(f).
\]

As a consequence, Theorem 4.1 applies with \(\rho = \kappa = \sigma = \frac{1}{2}\) to yield the following result (the numerical constants there are not sharp). It is not difficult to see from the preceding computations that in the one-dimensional case \(d = 1\), it is actually enough to assume that \(p \geq \frac{3}{2}\).

**Proposition 4.3** (HSI inequality for gamma distribution). Let \(\mu\) be the product measure of gamma distributions \(\Gamma(p_i)^{-1} x_i^{p_i-1} e^{-x_i} dx_i\) on \(\mathbb{R}_+^d\) with \(p_i \geq \frac{3}{2}\), \(i = 1, \ldots, d\). Then, for any \(d\nu = hd\mu\) where \(h\) is a smooth probability density,

\[
\mathcal{H}(\nu | \mu) \leq S^2(\nu | \mu) \Psi \left( \frac{I(\nu | \mu)}{S^2(\nu | \mu)} \right).
\]

### 4.3 One-dimensional uniform distribution on \([-1, 1]\)

In this section, we examine the case of the one-dimensional Jacobi operator of parameters \(\alpha = \beta = 1\), that is,

\[
\mathcal{L} f = (1 - x^2) f'' - 2xf',
\]
whose associated invariant measure $\mu$ is uniform distribution on $[-1, +1]$. The general family of parameters with the beta distributions as invariant measures (cf. [B-G-L, Section 2.7.4]) may be considered similarly, at the expense however of tedious computations, as well as multivariate (product) versions. For simplicity, we only detail this case to better illustrate the conclusion.

Easy calculations lead to, for a smooth function $f$ (with compact support in $(-1, +1)$),

$$\Gamma(f) = (1 - x^2) f''^2$$
$$\Gamma_2(f) = (1 + x^2) f''^2 + (1 - x^2)^2 f'''^2 - 2x(1 - x^2) f' f''$$
$$\Gamma_3(f) = (1 - x^2)^3 f''''^2 - 6x(1 - x^2)^2 f''' f'' - 2(1 - x^2)^2 f' f''''$$
$$+ 3(1 - x^2)(1 + 3x^2)f''^2 + 6x(1 - x^2)f' f'' + (3 - x^2)f'^2.$$

Observe that

$$\Gamma_2(f) = f''^2 + (xf' - (1 - x^2)f''')^2 \geq \Gamma(f).$$

Furthermore,

$$\Gamma_3(f) - \Gamma_2(f) = (1 - x^2) \left[ (1 - x^2)^2 f''''^2 - 6x(1 - x^2)f''' f''ight.$$  
$$- 2(1 - x^2)f' f'' + 2(1 + 5x^2)f''^2 + 8xf' f'' + 2f'^2 \right]$$
$$= (1 - x^2) \left[ ((1 - x^2)f'''' - 3x f''' - f'')^2 + (f' + xf'')^2 + 2f''^2 \right] \geq 0$$

so that $\Gamma_3(f) \geq \Gamma_2(f)$. Also,

$$\Gamma_2(f) \geq (1 + x^2) f''^2 + (1 - x^2)^2 f''^2 - 2x^2 f'^2 - \frac{1}{2}(1 - x^2)^2 f'''^2$$
$$= (1 - x^2)f''^2 + \frac{1}{2}(1 - x^2)^2 f''^2$$
$$\geq \frac{1}{2}(1 - x^2)^2 f''^2.$$

Hence, Theorem 4.1 applies with $\rho = \kappa = 1$ and $\sigma = \frac{1}{2}$ (note that $a(x) = 1 - x^2$) to yield the following conclusion. Again, the numerical constants are not sharp.

**Proposition 4.4** (HSI inequality for the uniform distribution). Let $\mu$ be uniform probability measure on $[-1, +1]$. Then, for any $d\nu = h d\mu$ where $h$ is a smooth probability density,

$$\mathcal{H}(\nu | \mu) \leq S^2(\nu | \mu) \Psi\left(\frac{I(\nu | \mu)}{2 S^2(\nu | \mu)}\right)$$

### 4.4 Families of log-concave distributions

We consider here a diffusion operator on the line of the type

$$\mathcal{L} f = f'' - u' f'$$
associated with a symmetric invariant probability measure \( d\mu = e^{-u}dx \), where \( u \) is a smooth potential on \( \mathbb{R} \). The Gaussian model corresponds to the quadratic potential 
\[ u(x) = \frac{x^2}{2}. \]

We have, for smooth functions \( f \),
\[
\Gamma(f) = f'^2,
\Gamma_2(f) = f''^2 + u''f'^2,
\Gamma_3(f) = f'''^2 + 3u'''f'f'' + 3u''f''^2 + \frac{1}{2} \left( (u^{(4)} - u'u''' + 2u''^2)f'^2 \right).
\]

Assume that there exists \( c > 0 \) such that, uniformly, \( u'' \geq c \),
\[
(4.9) \quad u^{(4)} - u'u''' + 2u''^2 - 6cu'' \geq 0
\]
and
\[
(4.10) \quad 3u'''^2 \leq 2(u'' - c)\left( (u^{(4)} - u'u''' + 2u''^2 - 6cu'') \right).
\]

Then \( \Gamma_2(f) \geq c \Gamma(f) \), \( \Gamma_2(f) \geq f'''^2 \) and \( \Gamma_3(f) \geq 3c \Gamma_2(f) \) for every \( f \). Hence, Theorem 4.1 applies with \( \rho = c \), \( \kappa = 3c \) and \( \sigma = 1 \).

**Proposition 4.5** (HSI inequality for log-concave distribution). Let \( d\mu = e^{-u}dx \) on \( \mathbb{R} \) where \( u \) is a smooth potential on \( \mathbb{R} \) such that for some \( c > 0 \), \( u'' \geq c \) and \( (4.9) \) and \( (4.10) \) hold. Then, for any \( d\nu = h d\mu \) where \( h \) is a smooth probability density,
\[
H(\nu \mid \mu) \leq \frac{1}{2} S^2(\nu \mid \mu) \Psi\left( \frac{I(\nu \mid \mu)}{c S^2(\nu \mid \mu)} \right).
\]

Recall that in this context, the only condition \( u'' \geq c > 0 \) ensures the logarithmic Sobolev inequality for \( \mu \) [B-G-L, Corollary 5.7.2]. It is not difficult to find (simple) examples outside the Gaussian model (corresponding to \( c = \frac{1}{2} \)) such that conditions \( (4.9) \) and \( (4.10) \) are fulfilled. For example, if \( u(x) = \frac{x^2}{2} + \epsilon x^4 \), it is easily seen that these hold for \( c = \frac{1}{4} \) and \( \epsilon = \frac{1}{12} \) (for instance). In the Gaussian case, the estimate obtained in this proposition is somewhat worse than the HSI inequality of Theorem 2.2. At the expenses of more involved conditions \( (4.9) \) and \( (4.10) \), multidimensional versions may be considered similarly.

## 5 Entropy bounds on laws of functionals

As emphasized in the introduction, the new HSI inequalities described in the preceding sections provide entropic bounds on probability measures \( \nu \) which may be used towards convergence in entropy via the Stein discrepancy \( S(\nu \mid \mu) \). Now, these bounds assume that the Fisher information \( I_\mu(h) \) of the density \( h \) of \( \nu \) with respect to \( \mu \) is finite (in order to control \( I_\mu(P_th) \) in small time), which may not hold in specific illustrations. The goal pursued in the second part of this work is actually to overcome this difficult
and to describe conditions (integrability and tail behavior) on the initial data itself of a multidimensional functional $F = (F_1, \ldots, F_d)$ with distribution $\nu = \nu_F$ (on $\mathbb{R}^d$) in order to control the Fisher information $I_{\mu}(P_th)$ in small time. This investigation was initiated in [N-P-S1] in Wiener space towards the first normal approximation results in entropy for Wiener chaos distributions. Here, we consider distributions of functionals on a Markov Triple structure $(E, \mu, \Gamma)$ already put forward in the preceding section, and describe how the associated $\Gamma$-calculus may be developed towards normal (as well as gamma) approximations in the entropic sense.

Referring as before to [B-G-L] for a complete account, we thus deal with a Markov Triple $(E, \mu, \Gamma)$ on a probability space $(E, \mathcal{E}, \mu)$, with Markov semigroup $(P_t)_{t \geq 0}$ with symmetric and invariant probability measure $\mu$, infinitesimal generator $L$, associated carré du champ operator $\Gamma$ and underlying algebra of (smooth) functions $\mathcal{A}$. Integration by parts expresses that

$$\int_E f L g \, d\mu = -\int_E \Gamma(f, g) \, d\mu$$

for every $f, g \in \mathcal{A}$.

The second order differential operators of Section 4 provide instances of this general framework. Gaussian and Wiener spaces with associated Ornstein-Uhlenbeck semigroup and generator are a prototypical example for the illustrations. Note in particular that Wiener chaoses as investigated in [N-P-S1] are eigenfunctions of the Ornstein-Uhlenbeck generator. Eigenfunctions of the underlying operator $L$ are actually of special interest in the context of the Stein method as illustrated in Section 5.1.

For $d \geq 1$, let $F = (F_1, \ldots, F_d)$ be a vector defined on $(E, \mathcal{E}, \mu)$, where each $F_i$ is centered and square-integrable, and denote by $\nu_F$ the law of $F$. Common to the three Sections 5.1–5.3, assume that the distribution $\nu_F$ of $F$ admits a density $h$ with respect to the standard Gaussian distribution $\gamma$ on $\mathbb{R}^d$ (in particular, $\nu_F$ is absolutely continuous with respect to the Lebesgue measure). In the first part, we describe the Stein matrix and discrepancy for vectors of eigenfunctions of $L$. Next, we address some direct bounds on the Fisher information $I_{\gamma}(h)$ in terms of the data of the functional $F$ and its gradients. Then, we develop the results on entropic normal approximations, extending the conclusions in [N-P-S1], by an analysis of the small time behavior of $I_{\gamma}(P_th)$. Finally, we address similar issues in the context of one-dimensional gamma approximation.

5.1 Stein matrix and discrepancy for eigenfunctions

The first statement shows that, whenever the vector $F$ is composed of eigenfunctions of $L$, then a Stein matrix $\tau_{\nu_F}$ of $\nu_F$ with respect to $\gamma$ as defined in (2.1) can be expressed in terms of the carré du champ operator $\Gamma$.

**Proposition 5.1** (Stein matrix for eigenfunctions). Let $F = (F_1, \ldots, F_d)$ on $(E, \mathcal{E}, \mu)$ such that, for every $i = 1, \ldots, d$, the random variable $F_i$ is an eigenfunction of $-L$, with
eigenvalue $\lambda_i > 0$. Assume moreover that $\Gamma(F_i, F_j) \in L^1(\mu)$ for every $i, j = 1, \ldots, d$. Then, the matrix-valued map $\tau_{\nu_F}$ defined as

$$
\tau_{\nu_F}^{ij}(x_1, \ldots, x_d) = \frac{1}{\lambda_i} E_{\mu} \left[ \Gamma(F_i, F_j) \bigg| F = (x_1, \ldots, x_d) \right], \quad i, j = 1, \ldots, d, \tag{5.2}
$$

is a Stein matrix for $\nu_F$, that is, it satisfies (2.1). (The right-hand side of (5.2) indicates a version of the conditional expectation of $\Gamma(F_i, F_j)$ with respect to $F$ under the probability measure $\mu$.)

Proof. Use integration by parts with respect to $L$ to get that, for every smooth test function $\varphi$ on $\mathbb{R}^d$ and every $i = 1, \ldots, d$,

$$
\lambda_i \int_E F_i \varphi(F) d\mu = - \int_E LF_i \varphi(F) d\mu = \sum_{j=1}^d \int_E \Gamma(F_i, F_j) \frac{\partial \varphi}{\partial x_j}(F) d\mu.
$$

The proof is concluded by taking conditional expectations. \qed

As a consequence, together with (2.4) and Jensen’s inequality,

$$
S^2(\nu_F \mid \gamma) \leq \sum_{i,j=1}^d \frac{1}{\lambda_i^2} \text{Var}_\mu(\Gamma(F_i, F_j)) + \|C - \text{Id}\|^2_{\text{HS}} = V^2 \tag{5.3}
$$

where $C$ denotes the covariance matrix of $\nu_F$, providing therefore a tractable way to control the Stein discrepancy in this case. In addition, combining with Theorem 2.2 immediately yields the following statement.

**Corollary 5.2.** Under the assumptions and notation of Proposition 5.1,

$$
H(\nu_F \mid \gamma) \leq V^2 \log \left( 1 + \frac{1(\nu_F \mid \gamma)}{V^2} \right). \tag{5.4}
$$

In particular, if $d = 1$ and $C = 1$, $H(\nu_F \mid \gamma) \to 0$ whenever $\text{Var}(\Gamma(F)) \to 0$ (cf. [N-P2, L2]).

**Example 5.3.** A typical example of a Markov Triple for which the quantity $V^2$ appearing in the above bound can be estimated explicitly corresponds to the case where $(E, \mathcal{E}, \mu)$ is a probability space supporting an isonormal Gaussian process $X = \{X(h) : h \in \mathcal{H}\}$ over some real separable Hilbert space $\mathcal{H}$, and $L$ is the generator of the associated Ornstein-Uhlenbeck semigroup. In this case, $\Gamma(F, G) = \langle DF, DG \rangle_{\mathcal{H}}$ for smooth functionals $F$ and $G$, where $D$ stands for the Malliavin derivative operator, and the eigenspaces of $-L$ are the so-called Wiener chaoses $\{C_k : k \geq 0\}$ of $X$. For $k = 0, 1, 2, \ldots$, the eigenvalue of $C_k$ is given by $k$. A detailed discussion about how to bound a quantity such as $V^2$, in the case of random vectors with components inside a Wiener chaos, can
be found in [N-P2, Chapter 6]. In particular, if $d = 1$ and $F$ belongs to $C_k$, then $V^2$ can be controlled by the second and fourth moments of $F$ as

$$V^2 = (\mathbb{E}[F^2] - 1)^2 + \frac{1}{k^2} \text{Var}(\|DF\|^2)$$

$$\leq (\mathbb{E}[F^2] - 1)^2 + \frac{k - 1}{3k} (\mathbb{E}[F^4] - 3 \mathbb{E}[F^2]^2).$$

In particular, such an estimate provides a proof of the famous ‘fourth moment theorem’ for chaotic random variables, cf. [N-P2, Theorem 5.2.7].

**Remark 5.4.** While eigenfunctions appear as functionals of particular interest, the $\Gamma$-calculus actually provides a formal description of Stein factors or matrices of a given functional $F$ on $(E, \mu, \Gamma)$ (in dimension one for simplicity) as the conditional expectation with respect to $F$ of $\Gamma(F, L^{-1}F)$ (where $L^{-1}F = \int_0^\infty P_t F dt$). This observation further expands on the preceding example.

### 5.2 Bounds on the Fisher information

When dealing with the upper-bound (5.4), the Fisher information $I(\nu_F | \gamma) = I_\gamma(h)$ of the density $h$ of the law $\nu_F$ of $F$ cannot always be explicitly deduced from the data concerning the random vector $F$. We show here how to deduce some useful bounds on $I(\nu_F | \gamma)$ in terms of $F$ and its gradients.

Let $F = (F_1, \ldots, F_d)$ be general vector of centered and square-integrable random variables (that need not necessarily be eigenfunctions of $-L$). Recall that the distribution $\nu_F$ of $F$ admits a (smooth) density $h$ with respect to the standard Gaussian distribution $\gamma$ on $\mathbb{R}^d$. It is implicitly assumed that all the $F_i$’s are in $\mathcal{A}$ (or some extended algebra in the sense of [B-G-L]) allowing for the formal computations developed next. These assumptions should then be verified on the concrete examples of interest (such as Wiener chaoses).

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be smooth enough. By integration by parts (5.1) with respect to $L$, for every $w \in \mathcal{A}$, and every $i, j = 1, \ldots, d$,

$$\sum_{k=1}^d \int_E w \Gamma(F_i, F_k) \frac{\partial^2 \phi}{\partial x_k \partial x_j}(F)d\mu = -\int_E LF_i w \frac{\partial \phi}{\partial x_j}(F)d\mu - \int_E \Gamma(F_i, w) \frac{\partial \phi}{\partial x_j}(F)d\mu.$$

Let $\tilde{\Gamma}$ be the symmetric matrix with entries $\Gamma(F_i, F_j)$, $i, j = 1, \ldots, d$. Applying the latter to $w = w_{ij}$, symmetric in $i, j$, yields

$$\int_E \text{Tr}(W \tilde{\Gamma} \text{Hess}(\phi)(F))d\mu$$

$$= -\sum_{i,j=1}^d \int_E LF_i w_{ij} \frac{\partial \phi}{\partial x_j}(F)d\mu - \sum_{i,j=1}^d \int_E \Gamma(F_i, w_{ij}) \frac{\partial \phi}{\partial x_j}(F)d\mu$$

(5.5)
where $W = (w_{ij})_{1 \leq i, j \leq d}$. Provided it exists, set $W = \tilde{\Gamma}^{-1}$, so that the left-hand side in the previous identity is just $\int_E \Delta \phi(F) d\mu$. Recalling from (2.6) the Ornstein-Uhlenbeck generator $L = \Delta - x \cdot \nabla$ associated with the standard Gaussian distribution $\gamma$ on $\mathbb{R}^d$, it follows that

$$- \int_E L\phi(F) d\mu = \sum_{i,j=1}^d \int_E L F_i (\tilde{\Gamma}^{-1})_{ij} \frac{\partial \phi}{\partial x_j}(F) d\mu + \sum_{i,j=1}^d \int_E \Gamma(F_i, (\tilde{\Gamma}^{-1})_{ij}) \frac{\partial \phi}{\partial x_j}(F) d\mu + \sum_{i=1}^d \int_E F_i \frac{\partial \phi}{\partial x_i}(F) d\mu.$$ 

In more compact notation, if

$$V = \left( \sum_{i=1}^d \Gamma(F_i, (\tilde{\Gamma}^{-1})_{ij}) \right)_{1 \leq j \leq d} \quad \text{and} \quad U = \tilde{\Gamma}^{-1}LF + V + F,$$

then

$$- \int_E L\phi(F) d\mu = \int_E U \cdot \nabla \phi(F) d\mu.$$ 

Applied to $\phi = v = \log h$, by the Cauchy-Schwarz inequality and (4.2),

$$I_\gamma(h) \leq \int_E |U|^2 d\mu. (5.6)$$

The consequences of the previous computations are gathered together in the next statement, where we point out a set of sufficient conditions on $F$ and its gradients $\Gamma(F_i, F_j)$ ensuring that the random variable $U$ is indeed square-integrable.

**Proposition 5.5** (Bound on the Fisher information). Let $F = (F_1, \ldots, F_d)$ be a vector of elements of $A$ on $(E, \mu, \Gamma)$. Assume that all the $F_i$, $LF_i$, $\Gamma(F_i, F_j)$, $i, j = 1, \ldots, d$, and $\frac{1}{\det(\Gamma)}$ are in $L^p(\mu)$ for every $p \geq 1$. Then, $\int_E |U|^2 d\mu < \infty$ and

$$I(\nu_F \gamma) \leq \int_E |U|^2 d\mu. \quad (5.6)$$

The condition on $\frac{1}{\det(\Gamma)}$ in Proposition 5.5 has some similarity with basic assumptions in Malliavin calculus (cf. [N, N-P2]).

**Example 5.6.** One may of course wonder whether the bound (5.6) is of any interest. Here is a simple example showing that there are instances where $I_\gamma(h)$ might be quite intricate to handle directly on the density $h$ of the distribution of $F$ while $U$ has a clearly description. On $E = \mathbb{R}^{2n}$ with the standard Gaussian measure $\mu = \gamma$ and $\Gamma(f) = |\nabla f|^2$ the standard carré du champ operator, let

$$F(x) = x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n}, \quad x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}.$$
It is classical that the distribution of the product of two independent standard normal has a density (with respect to the Lebesgue measure on \( \mathbb{R} \)) given by a Bessel function. The density \( h \) of the distribution of \( F \) is thus rather involved. On the other hand, it is easily seen that

\[
L F = -2F \quad \text{and} \quad \Gamma(F) = x_1^2 + \cdots + x_{2n}^2 = \rho^2
\]

so that

\[
U = F \left( -\frac{2}{\rho^2} - \frac{4}{\rho^4} + 1 \right).
\]

By using polar coordinates, it is immediately seen that \( \int_{\mathbb{R}^{2n}} U^2 d\mu < \infty \) as soon as \( n \geq 5 \).

### 5.3 Fisher information growth and normal approximation

One evident drawback of the results of the previous paragraph is that, since the quantity \( |U| \) is singular as the determinant of \( \overline{\Gamma} \) is close to 0, one is forced to assume that \( \frac{1}{\det(\overline{\Gamma})} \) is in all \( L^p(\mu) \) spaces (or at least for some \( p \) large enough depending on \( d \)). This assumption is in general too strong, and very difficult to check in concrete situations. The idea developed in this section (which generalizes the approach initiated in [N-P-S1]) is that, under weaker moment assumptions, while the Fisher information \( I(\gamma) \) might be infinite, it is nevertheless possible to control the growth as \( t \to 0 \) of \( I(t,h) \). Together with the control in terms of the Stein discrepancy for large time achieved in Section 2, one may then reach entropic bounds which can be handled in concrete examples (such as those of random vectors whose components belong to some Wiener chaos).

As before, let \( F = (F_1, \ldots, F_d) \) be a general vector of centered and square-integrable random variables (in the algebra \( \mathcal{A} \) or some natural extension), with distribution \( dv_F = hd\gamma \). As a crucial assumption, \( \nu_F \) has a Stein matrix \( \tau_{\nu_F} \) with respect to \( \gamma \) as defined in (2.1) (see also Proposition 5.1 and Remark (5.4)). Recall the matrix \( \overline{\Gamma} \) with entries \( \Gamma(F_i, F_j), i, j = 1, \ldots, d \). Also, in what follows we use the convention that, if \( \overline{\Gamma} \) is singular, then the matrix \( \det(\overline{\Gamma})^{-1} \) must be understood as the transpose of usual adjugate matrix operator of \( \overline{\Gamma} \) (both quantities being of course equal for non-singular matrices).

With the notation of the preceding section, given \( \varepsilon > 0 \), write first, again for a smooth function \( \phi \) on \( \mathbb{R}^d \) and \( L \) the Ornstein-Uhlenbeck operator in \( \mathbb{R}^d \),

\[
\int_E L \phi(F) d\mu = \int_E \Delta \phi(F) d\mu - \int_E F \cdot \nabla \phi(F) d\mu
\]

\[
= \int_E \frac{\det(\overline{\Gamma})}{\det(\Gamma)} \Delta \phi(F) d\mu + \int_E \frac{\varepsilon}{\det(\Gamma)} \Delta \phi(F) d\mu
\]

\[
- \int_E F \cdot \nabla \phi(F) d\mu.
\]
Choose $W = \frac{\det(\tilde{\Gamma})}{\det(\Gamma) + \varepsilon}$ in (5.5), so that
\[
\int_E \frac{\det(\tilde{\Gamma})}{\det(\Gamma) + \varepsilon} \Delta\phi(F) d\mu = -\int_E \left( \frac{\det(\tilde{\Gamma}) \tilde{\Gamma}^{-1} LF + V_1}{\det(\Gamma) + \varepsilon} - \frac{V_2}{(\det(\Gamma) + \varepsilon)^2} \right) \cdot \nabla\phi(F) d\mu
\]
where
\[
V_1 = \left( \sum_{i=1}^d \Gamma(F_i, \det(\tilde{\Gamma})(\tilde{\Gamma}^{-1})_{ij}) \right)_{1 \leq j \leq d}
\]
and
\[
V_2 = \left( \sum_{i=1}^d \det(\tilde{\Gamma})(\tilde{\Gamma}^{-1})_{ij} \Gamma(F_i, \det(\tilde{\Gamma})) \right)_{1 \leq j \leq d}
\]
Apply now the preceding to $\phi = P_t v_t$, $v_t = \log P_t h$, $t > 0$. Since $\nabla P_t v_t(F) = e^{-t} P_t(\nabla v_t)$ and
\[
I_\gamma(P_t h) = \int_E P_t(|\nabla v_t|^2)(F) d\mu,
\]
by the Cauchy-Schwarz inequality, assuming for simplicity that $0 < \varepsilon \leq 1$,
\[
\left| \int_E \frac{\det(\tilde{\Gamma})}{\det(\Gamma) + \varepsilon} \Delta P_t v_t(F) d\mu - \int_E F \cdot \nabla P_t v_t(F) d\mu \right| \leq \frac{e^{-t}}{\varepsilon^2} \left( \int_E \left[ |\det(\tilde{\Gamma})\tilde{\Gamma}^{-1} LF| + |V_1| + |V_2| + |F| \right]^2 d\mu \right)^{1/2} I_\gamma(P_t h)^{1/2}
\]
On the other hand, using the same semigroup computations as in Section 2,
\[
\Delta P_t v_t(F) = \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \int_{R^d} y \cdot \nabla v_t(e^{-t} F + \sqrt{1 - e^{-2t}} y) d\gamma(y)
\]
so that
\[
\left| \int_E \frac{\varepsilon}{\det(\tilde{\Gamma}) + \varepsilon} \Delta P_t v_t(F) d\mu \right| \leq \frac{\sqrt{d} \varepsilon e^{-2t}}{\sqrt{1 - e^{-2t}}} \left( \int_E \frac{1}{(\det(\tilde{\Gamma}) + \varepsilon)^2} d\mu \right)^{1/2} I_\gamma(P_t h)^{1/2}.
\]
Assume now that
\[
\int_E \left[ |\det(\tilde{\Gamma})\tilde{\Gamma}^{-1} LF| + |V_1| + |V_2| + |F| \right]^2 d\mu = A_F < \infty \quad (5.7)
\]
and that
\[
\int_E \frac{1}{(\det(\tilde{\Gamma}) + \varepsilon)^2} d\mu \leq \delta(\varepsilon). \quad (5.8)
\]
Collecting the preceding bounds and recalling from (4.7) that
\[
\int_E \mathcal{L} P_t v_t(F) d\mu = \int_E \mathcal{L} P_t v_t h d\mu = -I_\gamma(P_t h)
\]
yields that, for \( t > 0 \) and \( 0 < \varepsilon \leq 1 \),
\[
I_\gamma(P_t h) \leq 2e^{-2t}\left(\frac{A_F}{\varepsilon^4} + \frac{d \varepsilon^2 \delta(\varepsilon)}{1 - e^{-2t}}\right).
\] (5.9)

In the following statement, we determine a handy set of sufficient conditions on \( F \) and its gradients ensuring that, for some choice of \( \varepsilon = \varepsilon(t) > 0 \), the function on the right-hand side of (5.9) is integrable for the small values of \( t > 0 \). Combined with (2.18) for the large values of \( t > 0 \), a control of the entropy of \( \nu_F \) in terms of the Stein discrepancy \( S(\nu_F \mid \gamma) \) may then be produced. Recall the function \( \Psi \) on \( \mathbb{R}_+ \) given by \( \Psi(r) = 1 + \log r \) if \( r \geq 1 \) and \( \Psi(r) = r \) if \( 0 \leq r \leq 1 \).

**Theorem 5.7** (normal entropic approximation via Stein discrepancy). Let \( F = (F_1, \ldots, F_d) \) be a vector of centered elements of \( A \) on \( (E, \mu, \Gamma) \). Assume that all the \( F_i, L F_i, \Gamma(F_i, F_j) \), \( i, j = 1, \ldots, d \), are in \( L^p(\mu) \) for every \( p \geq 1 \), and that
\[
B_F = \int_E \frac{1}{\det(\tilde{\Gamma})^\alpha} d\mu < \infty
\] (5.10)
for some \( \alpha > 0 \). Then, \( A_F < \infty \) (as defined in (5.7)) and
\[
H(\nu_F \mid \gamma) \leq \frac{S^2(\nu_F \mid \gamma)}{2(1 - 4\kappa)} \Psi\left(\frac{2(A_F + d(B_F + 1))}{S^2(\nu_F \mid \gamma)}\right)
\] (5.11)
where \( \kappa = \frac{2+\alpha}{2(4+3\alpha)} \left( < \frac{1}{4} \right) \). In particular, under the assumptions on \( F \), \( H(\nu_F \mid \gamma) \to 0 \) as \( S(\nu_F \mid \gamma) \to 0 \).

**Proof.** First of all, we have that the parameter \( A_F \) is finite, since the expressions of \( \det(\tilde{\Gamma})\tilde{\Gamma}^{-1}LF_i, V_1 \) and \( V_2 \) only involve products of \( F_i, LF_i \) and \( \Gamma(F_i, F_j) \), \( i, j = 1, \ldots, d \).

Now, for every \( \varepsilon > 0 \) and \( \lambda > 0 \),
\[
\int_E \frac{1}{(\det(\tilde{\Gamma}) + \varepsilon)^2} d\mu \leq \frac{1}{\varepsilon^2} \mu(\det(\tilde{\Gamma}) \leq \lambda) + \frac{1}{\lambda^2} \leq \frac{B_F \lambda^\alpha}{\varepsilon^2} + \frac{1}{\lambda^2}.
\] (5.12)

The choice of \( \lambda = \varepsilon^{\frac{2}{4+3\alpha}} \) yields (5.8) with \( \delta(\varepsilon) = (B_F + 1)\varepsilon^{-\frac{4}{4+3\alpha}} \). Let then \( \varepsilon = \varepsilon(t) = (1 - e^{-2t})^\kappa, t \geq 0, \) for \( \kappa = \frac{2+\alpha}{2(4+3\alpha)} \left( < \frac{1}{4} \right) \). Then
\[
\frac{A_F}{\varepsilon^4} + \frac{d \varepsilon^2 \delta(\varepsilon)}{1 - e^{-2t}} \leq \frac{A_F + d(B_F + 1)}{(1 - e^{-2t})^{4\kappa}}
\]
from which, as a consequence of (5.9), for every \( t > 0 \),
\[
I_\gamma(P_t h) \leq 2\left[A_F + d(B_F + 1)\right] \frac{e^{-2t}}{(1 - e^{-2t})^{4\kappa}}.
\] (5.13)
To conclude, recall, as in the proof of Theorem 2.2, the decomposition for every \( u > 0 \),

\[
H(\nu_F | \gamma) \leq \int_0^u \mathbb{I}_\gamma(P_t h) dt + S^2(\nu_F | \gamma) \int_u^\infty \frac{e^{-4t}}{1 - e^{-2t}} dt.
\]

Therefore, by (5.13),

\[
H(\nu_F | \gamma) \leq \frac{A_F + d(B_F + 1)}{1 - 4\kappa} (1 - e^{-2u})^{1 - 4\kappa} \\
+ \frac{1}{2} S^2(\nu_F | \gamma)( - e^{-2u} - \log(1 - e^{-2u})) \\
\leq \frac{A_F + d(B_F + 1)}{1 - 4\kappa} (1 - e^{-2u})^{1 - 4\kappa} - \frac{1}{2} S^2(\nu_F | \gamma) \log(1 - e^{-2u}),
\]

and the bound (5.11) in the statement follows by optimizing in \( u > 0 \) (set \( (1 - e^{-2u})^{1 - 4\kappa} = r \in (0, 1) \)). Theorem 5.7 is established.

Since \( \Psi(r) \leq r \) for every \( r \in \mathbb{R}_+ \), observe from (5.11) that

\[
H(\nu_F | \gamma) \leq \frac{A_F + d(B_F + 1)}{(1 - 4\kappa)}
\]

so that, under the assumptions of Theorem 5.7, one also has that \( H(\nu_F | \gamma) < \infty \), a conclusion of independent interest.

The quantity \( A_F \) of (5.7) involves integrability conditions on \( F \) and its gradients (they may actually be weakened according to the precise expression of \( A_F \)). On the other hand, \( B_F \) of (5.10) is rather concerned with a small ball behavior. For a vector \( F = (F_1, \ldots, F_d) \) of eigenvectors of the underlying Markov generator \( L \), Theorem 5.7 may be combined with (5.3) to fully control the relative entropy in terms of \( F \) and its gradients as now illustrated in some instances.

**Example 5.8.** We describe, in part following [N-P-S1], how the preceding developments may be applied to concrete examples of interest.

(a) As already mentioned in Example 5.3, one such model is the case of a Gaussian vector chaos \( F = (F_1, \ldots, F_d) \), each \( F_i \) being a chaos on Wiener space, in which case (see Example 5.3) \( \Gamma(F_i, F_j) = \langle DF_i, DF_j \rangle_\mathcal{H} \). As put forward in [N-P-S], the first part of the hypotheses in Theorem 5.7 is fulfilled by the integrability of Wiener chaoses and of their derivatives. Concerning the second part of the hypotheses, the relevant property emphasized in [N-P-S] is that whenever the law of \( F \) has a density with respect to the Lebesgue measure, which amounts to the fact that \( \mathbb{E}(\det(\widetilde{\Gamma})) > 0 \), then for some universal constant \( c > 0 \),

\[
\mathbb{P}(\det(\widetilde{\Gamma}) \leq \lambda) \leq cN \lambda^{1/N} \mathbb{E}(\det(\widetilde{\Gamma}))^{-1/N}
\]

(5.14)
for every $\lambda > 0$, where $N \geq 1$ is an integer related to the degrees of the $F_i$’s. Under (5.14), the second hypothesis of Theorem 5.7 clearly holds for any $\alpha < \frac{1}{N}$ (cf. (5.12)). The latter then applies to basically recover the main conclusion of [N-P-S].

(b) It may be observed that the same conclusion (5.14) holds true when the $F_i$’s are polynomials under a log-concave measure $d\mu = e^{-u}dx$ on $\mathbb{R}^n$, at least when $u$ is a polynomial or such that $|\nabla u| \in L^p(\mu)$ for every $p \geq 1$. Indeed, the determinant $\det(\Gamma)$ is then also of this form, and the seminal result from [C-W] used in [N-P-S] applies similarly.

### 5.4 Fisher information growth and gamma approximation

This final section develops the analogous investigation towards gamma approximation, for simplicity one-dimensional. Denote by $\gamma_p$ the gamma distribution (on the positive real line) with parameter $p > 0$, invariant measure of the Laguerre operator

$$\mathcal{L}_p f = xf'' + (p - x)f'.$$

Consider a random variable $F \geq 0$ with law $d\nu_F = h d\gamma_p$ absolutely continuous with respect to $\gamma_p$. Assume that $\nu_F$ admits a Stein factor $\tau_{\nu_F}$ with respect to $\gamma_p$, that is, according to (4.3) (taking into account the diffusion coefficient $a(x) = x$ in (5.15)), $\tau_{\nu_F}$ is a mapping on $\mathbb{R}_+$ verifying

$$\int_{\mathbb{R}_+} (x - p) \varphi \, d\nu_F = \int_{\mathbb{R}_+} \tau_{\nu_F} \varphi' \, d\nu_F$$

for every smooth test function $\varphi$. In particular, $\int_{\mathbb{E}} F \, d\mu = p$. Note that, in this case,

$$S^2(\nu_F \mid \gamma_p) = \int_{\mathbb{E}} \left( \frac{\tau_{\nu_F}(F)}{F} - 1 \right)^2 \, d\mu.$$

From the study of Gaussian chaoses for example, and as already mentioned earlier, it appears that the latter $S(\nu_F \mid \gamma_p)$ might not always be the relevant quantity of interest (cf. [N-P1, R]). Indeed, for an eigenfunction $F$ with eigenvalue $-\lambda$, $\lambda > 0$, the Stein factor $\tau_{\nu_F}(F)$ may be identified with the conditional expectation of $\lambda^{-1} \Gamma(F)$ knowing $F$. Now, for such a functional, moment conditions on $F$ may be used to rather control the variance of $\lambda^{-1} \Gamma(F) - F$, and similarly higher moments (cf. [A-C-P, A-M-P, L2]). Of course, by Hölder’s inequality,

$$\left( \int_{\mathbb{E}} \left( \frac{\Gamma(F)}{\lambda F} - 1 \right)^2 \, d\mu \right)^{1/2} \leq \left( \int_{\mathbb{E}} F^{-2r} \, d\mu \right)^{1/r} \left( \int_{\mathbb{E}} \left| \frac{\Gamma(F)}{\lambda} - F \right|^{2s} \, d\mu \right)^{1/s}$$

for $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$. Provided it may be ensured that $\int_{\mathbb{E}} F^{-2r} \, d\mu < \infty$ for some $r > 1$, the results here are nevertheless still of interest.
We assume below that \( p \geq \frac{1}{6} \) so that the estimates (4.6) and (4.8) are verified, with the choice of parameters \( d = 1 \) and \( \rho = \kappa = \sigma = \frac{1}{2} \) (see the comment preceding Proposition 4.3). The proof of the following statement will follow the one developed for Theorem 5.7.

**Theorem 5.9** (Gamma entropic approximation via Stein discrepancy). On \((E, \mu, \Gamma)\), let \( F \geq 0 \) in \( A \). Assume that \( F, L F, \Gamma(F) \) and \( \Gamma(F, \Gamma(F)) \) are in \( L^q(\mu) \) for every \( q \geq 1 \) and that

\[
B_F = \int_E \frac{1}{\Gamma(F)^\alpha} \, d\mu < \infty
\]

for some \( \alpha > 0 \). Then

\[
A_F = \int_E \frac{1}{F} \left[ (F|L F| + \Gamma(F) + F|\Gamma(F, \Gamma(F))|) + p + F \right]^2 \, d\mu < \infty
\]

and

\[
H(\nu_F \mid \gamma_p) \leq \frac{S^2(\nu_F \mid \gamma_p)}{2(1 - 4\kappa)} \Psi \left( \frac{2(A_F + B_F + 1)}{S^2(\nu_F \mid \gamma_p)} \right)
\]

where \( \kappa = \frac{2 + \sigma}{2(4 + 3\sigma)} \) \((< \frac{1}{4})\). In particular, under the assumptions on \( F \), \( H(\nu_F \mid \gamma_p) \to 0 \) as \( S(\nu_F \mid \gamma_p) \to 0 \).

**Proof.** Denoting by \((P_t)_{t \geq 0}\) the semigroup with infinitesimal generator \( \mathcal{L}_p \), we have as in (4.7),

\[
I_{\gamma_p}(P_t h) = - \int_{\mathbb{R}^+} \mathcal{L}_p P_t v_t \, h \, d\gamma_p = - \int_E \mathcal{L}_p P_t v_t(F) \, d\mu
\]

where \( v_t = \log P_t h \). Now, for every \( \varepsilon > 0 \),

\[
\int_E \mathcal{L}_p P_t v_t(F) \, d\mu = \int_E F(P_t v_t)'(F) \, d\mu + \int_E (p - F)(P_t v_t)'(F) \, d\mu
\]

\[
= \int_E F(P_t v_t)'(F) \frac{\Gamma(F)}{\Gamma(F) + \varepsilon} \, d\mu + \int_E F(P_t v_t)'(F) \frac{\varepsilon}{\Gamma(F) + \varepsilon} \, d\mu
\]

\[
+ \int_E (p - F)(P_t v_t)'(F) \, d\gamma_p.
\]

By integration by parts,

\[
\int_E F(P_t v_t)'(F) \frac{\Gamma(F)}{\Gamma(F) + \varepsilon} \, d\mu = \int_E (P_t v_t)'(F) \left[ \frac{F(-L F)}{\Gamma(F) + \varepsilon} - \Gamma(F, \frac{F}{\Gamma(F)} + \varepsilon) \right] \, d\mu.
\]

Using that

\[
\Gamma(F, \frac{F}{\varepsilon + \Gamma(F)}) = \frac{\Gamma(F)}{\Gamma(F) + \varepsilon} - \frac{F \Gamma(F, \Gamma(F))}{(\Gamma(F) + \varepsilon)^2}
\]

it follows that

\[
\int_E \mathcal{L}_p P_t v_t(F) \, d\mu = \int_E \sqrt{F}(P_t v_t)'(F) W_z(F) \, d\mu + \int_E F(P_t v_t)'(F) \frac{\varepsilon}{\Gamma(F) + \varepsilon} \, d\mu
\]

\[
+ \int_E (p - F)(P_t v_t)'(F) \, d\gamma_p.
\]
with
\[
W_\varepsilon(F) = \frac{\sqrt{F}(-LF)}{\Gamma(F) + \varepsilon} - \frac{\Gamma(F)}{\sqrt{F}(\Gamma(F) + \varepsilon)} + \frac{\sqrt{F}}{\Gamma(F) + \varepsilon} \left( \frac{\Gamma(F, \Gamma(F))}{\Gamma(F)} + \frac{p}{\sqrt{F}} - \sqrt{F} \right).
\]

Now, for every \(0 < \varepsilon \leq 1\),
\[
|W_\varepsilon(F)| \leq \frac{1}{\varepsilon^2 \sqrt{F}} \left[ F|LF| + \Gamma(F) + F\Gamma(F, \Gamma(F)) + p + F \right].
\]

As a consequence, with the notation introduced in the statement,
\[
\int_E W_\varepsilon^2(F) d\mu \leq A_F \varepsilon^4.
\]

By the Cauchy-Schwarz inequality,
\[
\left| \int_E \sqrt{F}(P_tv_t)'(F)W_\varepsilon(F) d\gamma_p \right| \leq \left( \int_E W_\varepsilon^2(F) d\mu \right)^{1/2} \left( \int_E F(P_tv_t)^2(F) d\mu \right)^{1/2} = \left( \int_E W_\varepsilon^2(F) d\mu \right)^{1/2} \left( \int_{R_+} \Gamma(P_tv_t)hd\gamma_p \right)^{1/2}.
\]

Since \(\Gamma(P_tv_t) \leq e^{-t} P_t(\Gamma(v_t))\) (Theorem 3.2.4 in [B-G-L]),
\[
\left| \int_E \sqrt{F}(P_tv_t)'(F)W_\varepsilon(F) d\gamma_p \right| \leq \frac{e^{-t/2}}{\varepsilon^2} A_F^{1/2} I_{v_p}(P_t h)^{1/2}.
\]

On the other hand, the estimate (4.6) yields the bound
\[
\int_E F^2(P_tv_t)''(F)^2 d\mu = \int_{R_+} x^2(P_tv_t)^2hd\gamma_p
\]
\[
\leq \frac{1}{e^t - 1} \int_{R_+} P_t(\Gamma(v_t)) hd\gamma_p = \frac{1}{e^t - 1} I_{v_p}(P_t h) = \frac{1}{e^t - 1} I_{v_p}(P_t h).
\]

This in turn implies that
\[
\left| \int_E F(P_tv_t)''(F) \frac{\varepsilon}{\Gamma(F) + \varepsilon} d\mu \right| \leq \frac{1}{\sqrt{e^t - 1}} I_{v_p}(P_t h)^{1/2} \left( \int_E \left( \frac{\varepsilon}{\Gamma(F) + \varepsilon} \right)^2 d\mu \right)^{1/2}.
\]

Gathering together all the previous estimates, we deduce that, for every \(0 < \varepsilon \leq 1\) and \(t > 0\),
\[
I_{v_p}(P_t h) \leq \frac{2e^{-t} A_F}{\varepsilon^4} + \frac{2}{e^t - 1} \int_E \left( \frac{\varepsilon}{\Gamma(F) + \varepsilon} \right)^2 d\mu.
\]

On the basis of this estimate, we then conclude exactly as in the proof of Theorem 5.7. □
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