On the geometry of the slice of trace-free $SL_2(\mathbb{C})$-characters of a knot group

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Abstract Let $K$ be a knot in an integral homology 3-sphere $\Sigma$ with exterior $E_K$, and let $B_2$ denote the two-fold branched cover of $\Sigma$ branched along $K$. We construct a map $\Phi$ from the slice of trace-free $SL_2(\mathbb{C})$-characters of $\pi_1(E_K)$ to the $SL_2(\mathbb{C})$-character variety of $\pi_1(B_2)$. When this map is surjective, it describes the slice as the two-fold branched cover over the $SL_2(\mathbb{C})$-character variety of $B_2$ with branched locus given by the abelian characters, whose preimage is precisely the set of metabelian characters. We show that each metabelian character can be represented as the character of a binary dihedral representation of $\pi_1(E_K)$. The map $\Phi$ is shown to be surjective for all 2-bridge knots and all pretzel knots of type $(p, q, r)$. An extension of this framework to $n$-fold branched covers is also described.

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1 Introduction

The purpose of this paper is to explore the relationship between $SL_2(\mathbb{C})$-representations for a homology knot exterior and those for a finite cyclic cover over an integral...
homology 3-sphere $\Sigma$ with branch set the knot. We define a correspondence from $\text{SL}_2(\mathbb{C})$-representations of the fundamental groups of a knot exterior with meridional trace zero to those of finite cyclic covers branched along the knot. Then we describe features of this correspondence for knots in $S^3$ and two-fold branched covers in terms of character varieties.

In the interaction between knot theory and three-dimensional topology, special values of polynomial invariants of knots often give good expressions for topological invariants of closed three-manifolds. Fox’s formula is a famous bridge between knot theory and three-dimensional topology. This formula says that the product of values of the Alexander polynomial of a knot at roots of unity gives the order of the first homology group of finite cyclic cover branched along the knot. Behind such a phenomenon, the representation space and the character variety of the fundamental group of a manifold has played an important role. The results in this paper were motivated by an attempt to understand a correspondence similar to Fox’s between $\text{SL}_2(\mathbb{C})$-characters of knot exteriors and those of finite cyclic covers branched along the knot.

The key to such a correspondence is choosing a subset of representation spaces and character varieties for knot exteriors. The $G$-character variety of a manifold $M$ can be considered as the set of characters of $G$-representations of the fundamental group $\pi_1(M)$ and admits the structure of an affine variety (for a precise definition for $G = \text{SL}_2(\mathbb{C})$, refer to Sect. 2.2 or [9]). Here we recall some beautiful results using the SU(2)-character varieties shown by Casson and Lin.

Casson introduced an invariant for an integral homology 3-sphere $\Sigma$, so-called the Casson invariant, originally by using the intersection between SU(2)-character varieties associated to a Heegaard splitting of $\Sigma$. More precisely, the character varieties of two handlebodies associated to a Heegaard splitting of $\Sigma$ are embedded in that of the common boundary surface of the handlebodies. Then the Casson invariant is defined as a half of the integer obtained by counting the algebraic intersection of the embedded character varieties of the handlebodies (see [1,23] for more details). Lin [16] defined an invariant for a knot $K$ in 3-sphere $S^3$, now known as the Casson–Lin invariant, by applying a similar idea to the variety of trace-free SU(2)-characters of the fundamental group of the knot exterior $E_K$. Here a trace-free character refers to any character of a representation sending the meridian to a matrix with trace zero.

Afterward, Herald generalized the Casson–Lin invariant by using gauge theoretic methods in [11] and Heusener and Kroll also have studied the same issue by topological methods in [12] to consider other trace condition of meridians indexed by $t \in (-2, 2)$. The knot invariants in Herald’s and Heusener–Kroll’s correspond to the equivariant signature of the knot. On the other hand, Collin and Saveliev [7] have studied SU(2)-character varieties of knot exteriors with other meridional trace conditions from the viewpoint of gauge theory with cyclic group actions. They considered finite cyclic branched covers over integral homology 3-spheres with branch set a knot and define a topological invariant, called the equivariant Casson invariant, for integral homology 3-spheres with finite cyclic group actions. Then it turned out that the equivariant Casson invariant can be identified with the equivariant knot signatures. Collin–Saveliev’s approach implies that the SU(2)-character variety of a knot exterior with trace conditions of meridians can be related to that of a finite cyclic branched cover.
In this perspective, for a knot $K$ in $\Sigma$, we construct a map $\Phi$ from $\text{SL}_2(\mathbb{C})$-representations of $\pi_1(E_K)$ with trace-free along meridians to those of the fundamental group of the two-fold branched cover $B_2$ over $\Sigma$ branched along $K$. The domain of this map $\Phi$ contains metabelian representations of the knot group $\pi_1(E_K)$ and $\Phi$ sends the metabelian representations to abelian representations of $\pi_1(B_2)$ (refer to Proposition 10). This map $\Phi$ naturally induces a map $\hat{\Phi}$ between characters of $\pi_1(E_K)$ and $\pi_1(B_2)$.

The domain of $\hat{\Phi}$ is the subset with trace-free along meridians, denoted by $S_0(E_K)$. Such a subset of $\text{SL}_2(\mathbb{C})$-characters with the trace of meridians fixed is called a slice since it is a level set of the function given by the evaluation of trace along meridians at the fixed values. In Theorem 1, Propositions 11 and 12, we will see some interesting properties of $\hat{\Phi}$ and $S_0(E_K)$ for 3-sphere $S^3$. In our description of $\hat{\Phi}$, the characters of metabelian representations have the significant feature and role. They form the fixed point set of an involution on $S_0(E_K)$ (refer to Proposition 4) and they are sent to the characters of abelian representation of $\pi_1(B_2)$ by $\hat{\Phi}$ one-to-one. We will also see that $\hat{\Phi}$ is surjective for all 2-bridge knots and all pretzel knots of type $(p, q, r)$ and then $S_0(E_K)$ is the two-fold branched cover over the character variety of $B_2$ via $\hat{\Phi}$.

The above framework for a two-fold branched cover $B_2$ are naturally generalized to the case of an $n$-fold cyclic branched cover $B_n$. Namely, we can also define a map from the subset $R_{2 \cos(\pi k/n)}(E_K)$ of $\text{SL}_2(\mathbb{C})$-representations of $\pi_1(E_K)$ with trace $2 \cos(\pi k/n)$ along meridians to $\text{SL}_2(\mathbb{C})$-representations $R(B_n)$ of $\pi_1(B_n)$. We can descend it to a map from the slice $S_{2 \cos(\pi k/n)}(E_K)$ of characters, associated with $R_{2 \cos(\pi k/n)}(E_K)$, to characters of $\pi_1(B_n)$. This extension of framework is another main result in this article.

**Organization** In Sect. 2, we review some notions about representations and the characters. Then we define notations used throughout this article. Section 3 gives a review of Lin presentation of a knot group and study its properties. In Sect. 4, we discuss the characters of metabelian representations and see important properties of binary dihedral representations in metabelian ones. In Sect. 5, we derive how to make an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(B_2)$ from that of $\pi_1(E_K)$. In Sect. 6, we focus on the structure of the slice $S_0(E_K)$ with trace-free along meridians for 3-sphere $S^3$ via the correspondence to the characters of $\pi_1(B_2)$. In the final section, we show several applications including the surjectivity of the map $\hat{\Phi}$ for two-bridge knots and pretzel knots of type $(p, q, r)$.

**2 Preliminaries**

2.1 General notations

We use the symbol $\Sigma$ to denote an integral homology 3-sphere and $K$ to denote a knot in $\Sigma$. Then $E_K$ stands for the knot exterior, i.e., it is obtained by removing an open tubular neighbourhood of $K$ from $\Sigma$. Let us denote by $B_n$ the $n$-fold cyclic branched cover of $\Sigma$ along the branched set $K$ and by $C_n$ the $n$-fold cyclic cover of $E_K$. The following diagram expresses the relationship among these spaces,
The right arrows are inclusions and the down arrows are projections. When we consider several fundamental groups \( \pi_1(\Sigma) \), \( \pi_1(E_K) \), \( \pi_1(C_n) \) and \( \pi_1(B_n) \), we assume that the base points of \( \Sigma \) and \( E_K \) are given by the same point \( b \) which lives in the boundary torus of \( E_K \) and those of \( C_n \) and \( B_n \) are given by a lift \( \hat{b} \) of \( b \). We will abbreviate \( \pi_1(\Sigma, b) \) to \( \pi_1(\Sigma) \) and so on. We denote by \( \mu \) a meridian of a knot \( K \), which is the element in \( \pi_1(E_K) \) represented by a simple closed curve in the boundary torus.

2.2 Review on the \( SL_2(\mathbb{C}) \)-character variety

We briefly review the \( SL_2(\mathbb{C}) \)-representation space and the character variety. We consider \( SL_2(\mathbb{C}) \)-representations of the fundamental group of a compact manifold \( M \), i.e., homomorphisms from \( \pi_1(M) \) into \( SL_2(\mathbb{C}) \). We let

\[
R(M) = \text{Hom}(\pi_1(M); SL_2(\mathbb{C}))
\]

denote the space of \( SL_2(\mathbb{C}) \)-representations for \( M \). We can see that \( R(M) \) is an affine algebraic set from a presentation of \( \pi_1(M) \), by viewing \( R(M) \) as solutions to polynomial equations in a product of copies of \( SL_2(\mathbb{C}) \) for each generator in \( \pi_1(M) \).

A representation \( \rho : \pi_1(M) \to SL_2(\mathbb{C}) \) is called abelian if the image \( \rho(\pi_1(M)) \) is an abelian subgroup in \( SL_2(\mathbb{C}) \). We say that \( \rho \) is metabelian if the commutator subgroup \( [\pi_1(M), \pi_1(M)] \) is sent to an abelian subgroup in \( SL_2(\mathbb{C}) \) by \( \rho \). Of course, all abelian representations are metabelian. It is well-known that there exist only the following two maximal abelian subgroups, up to conjugation, in \( SL_2(\mathbb{C}) \):

\[
\text{Hyp} := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \bigg| \lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \right\}, \quad \text{Para} := \left\{ \pm \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \bigg| \omega \in \mathbb{C} \right\}.
\]

Here Hyp means hyperbolic elements in \( SL_2(\mathbb{C}) \) and Para means parabolic elements in \( SL_2(\mathbb{C}) \). Hence if \( \rho \) is abelian (resp. metabelian), the image (resp. the image of \( [\pi_1(M), \pi_1(M)] \)) can be contained in Hyp or Para by taking conjugation.

An \( SL_2(\mathbb{C}) \)-representation \( \rho \) is called reducible if there exists an invariant line \( L \subset \mathbb{C}^2 \) such that \( \rho(g)(L) \subset L \), for all \( g \in \pi_1(M) \). Namely, there exists an element \( A \) of \( SL_2(\mathbb{C}) \) such that \( A\rho(g)A^{-1} \) is an upper triangular matrix for any \( g \in \pi_1(M) \). Of course, any abelian representation is reducible (while the converse is false in general). Moreover by Culler and Shalen [9, Lemma 1.2.1], all reducible representations send the commutator subgroup \( [\pi_1(M), \pi_1(M)] \) into the maximal abelian subgroup Para, up to conjugate. Hence reducible representations are metabelian. A representation is called irreducible if it is not reducible.

The group \( SL_2(\mathbb{C}) \) acts on the representation space \( R(M) \) by conjugation, however the naive quotient \( R(M)/SL_2(\mathbb{C}) \) is not Hausdorff in general. To construct
On the meridional trace-free slice of the character variety

an appropriate quotient, we take the affine algebraic set determined by the condition that its coordinate ring is isomorphic to the ring of invariant regular functions $\mathbb{C}[R(M)]^{\text{PSL}_2(\mathbb{C})}$. By Culler and Shalen [9], we can endow the set of characters of representations with such affine algebraic structure. Here the character associated to $\rho \in R(M)$ is the map $\chi_\rho : \pi_1(M) \to \mathbb{C}$, defined by $\chi_\rho(g) = \text{tr} \rho(g)$ where $\text{tr}$ denotes the trace of matrices. In this sense, the set of characters of $R(M)$ is called the character variety of $M$ and denoted by $X(M)$.

Let $R^\text{irr}(M)$ denote the subset of irreducible representations of $\pi_1(M)$, and $X^\text{irr}(M)$ denote its image under the map $t : R(M) \to X(M), t(\rho) = \chi_\rho$. Note that two irreducible representations of $\pi_1(M)$ in $\text{SL}_2(\mathbb{C})$ with the same character are conjugate by an element of $\text{SL}_2(\mathbb{C})$ (see [9, Proposition 1.5.2]). Similarly, we write $X^\text{red}(M)$ for the image of the set $R^\text{red}(M)$ of reducible representations under $t$. We also use $R^\text{ab}(M)$ (resp. $R^\text{meta}(M)$) for the set of abelian (resp. metabelian) representations and $X^\text{ab}(M)$ (resp. $X^\text{meta}(M)$) for the image by $t$. We refer the character of an irreducible representation to an irreducible character and similarly for reducible, abelian, metabelian character and so on.

3 Free Seifert surfaces and Lin presentations

Lin has introduced a new method to represent generators and relations of a knot group by using a free Seifert surface and the induced Heegaard splitting of $S^3$. We call this method Lin presentation. His approach also induces presentations of the fundamental groups of covering spaces and allows us to investigate character varieties of cyclic covers over a knot exterior. We will use these presentations as a main tool for our study in the rest of this article. In Sect. 3.1, we review free Seifert surfaces and Lin presentations. In Sects. 3.2 and 3.3, we show how a Lin presentation of $\pi_1(E_K)$ determines presentations of the fundamental groups $\pi_1(C_n)$ and $\pi_1(B_n)$ for the $n$-fold cyclic and branched covers.

3.1 Lin presentation for an integral homology sphere

Lin presentation was given for a knot in $S^3$, however, his construction can be extended for a knot in an integral homology 3-sphere $\Sigma$. To explain this extension, we review his construction.

We start with the definition of a free Seifert surface and existence of such a Seifert surface in $\Sigma$.

**Definition 1** A Seifert surface $S$ of a knot $K$ is called free if $\Sigma = N(S) \cup \Sigma \setminus N(S)$ is a Heegaard splitting of $\Sigma$. Here $N(S)$ is a closed tubular neighborhood of $S$ in $\Sigma$.

**Lemma 1** Let $K$ be a knot in $\Sigma$. Then there exists an embedded surface $S$ in $\Sigma$ such that $S \times [-1, 1] \cup \Sigma \setminus S \times [-1, 1]$ is a Heegaard splitting of $\Sigma$ satisfying $K = \partial S$.

For more information about the existence of a free Seifert surface and the induced Heegaard splitting of $\Sigma$, for example, see [23, Lemma 17.2].

A presentation of $\pi_1(E_K)$ is obtained from the Heegaard splitting associated to a free Seifert surface.
Lemma 2 (Lemma 2.1 in [17]) Assume that $S$ is a free Seifert surface for $K$ in $\Sigma$. Let $\Sigma = H_1 \cup H_2$ be the Heegaard splitting associated to $S$. For a basis $x_1, \ldots, x_{2g}$ of the free group $\pi_1(H_2)$, we have the following presentation of $\pi_1(E_K)$:

$$\pi_1(E_K) = \langle x_1, \ldots, x_{2g}, \mu \mid \mu \alpha_i \mu^{-1} = \beta_i, i = 1, \ldots, 2g \rangle,$$

where $\alpha_i, \beta_i$ are words in $x_1, \ldots, x_{2g}$, determined by the spine of $S$, and $g$ is the genus of $S$.

We call the presentation (1) a Lin presentation of $\pi_1(E_K)$.

Remark 1 Note that every $x_i$ is contained in the commutator subgroup of $\pi_1(E_K)$.

Proof of Lemma 2 This can be shown by van Kampen’s theorem. We begin by a decomposition of $E_K$ associated with a Heegaard splitting of $\Sigma$ by removing a solid torus inside single handlebody, which is isotopic to an open tubular neighbourhood of $K$. As in Lemma 1, the integral homology sphere $\Sigma$ is decomposed into two handlebodies $H_1$ and $H_2$ where $H_1 = S \times [-1, 1]$. By pushing $K$ into $H_1$ slightly, we have a simple closed curve $K'$ on $S \times \{0\}$, which is parallel to $\partial S = \partial \Sigma$. We identify $E_K$ with $E_{K'}$ and apply van Kampen’s theorem to the decomposition $E_K = (H_1 \setminus \text{int } N(K)) \cup H_2$.

Then van Kampen’s theorem says that the knot group

$$\pi_1(E_K) \simeq \pi_1(H_1 \setminus \text{int } N(K)) \ast_{\pi_1(\partial H_1)} \pi_1(H_2).$$

and by homotopy equivalence and van Kampen’s theorem again, we have

$$\pi_1(H_1 \setminus \text{int } N(K)) \simeq \pi_1(\partial N(K)) \ast_{\pi_1(\text{longitude})} \pi_1(S) = \langle \mu, a_1, \ldots, a_{2g} \mid \mu, \prod_{i=1}^{g} [a_{2i-1}, a_{2i}] = 1 \rangle$$

where $\mu$ is a meridian and $a_1, \ldots, a_{2g}$ are circles on $S$ and give a spine of $S$, which is a deformation retract of $S$ and $g$ is the genus of $S$.

Hence $\pi_1(E_K)$ is generated by $x_1, \ldots, x_{2g}, a_1, \ldots, a_{2g}$ and $\mu$. But since every $a_i$ is written by a word in $x_1, \ldots, x_{2g}$ in $\pi_1(E_K)$ by the homeomorphism between $\partial H_1$ and $\partial H_2$, we can reduce the generators to $x_1, \ldots, x_{2g}, \mu$. The relations of $\pi_1(E_K)$ are given by disks in $H_1 \setminus \text{int } N(K)$, whose boundaries belong to $\partial (H_1 \setminus \text{int } N(K)) \cup S$. We can assume that such a disk $D^2$ in $H_1 \setminus \text{int } N(K)$ intersects with $S$ transversally. Then the homotopy class of $\partial D^2$ is given by a word in $\mu a_i^\pm \mu^{-1} (a_i^-)^{-1}$ ($i = 1, \ldots, 2g$) where $a_i^\pm$ is $a_i \times \{\pm 1\}$, respectively. The relations of $\pi_1(E_K)$ are generated by $\mu a_i^\pm \mu^{-1} (a_i^-)^{-1}$ ($i = 1, \ldots, 2g$). So, by rewriting $a_i^\pm$ as words $\alpha_i$ and $\beta_i$ in $x_1, \ldots, x_{2g}$, we obtain the presentation in Lemma 2 (Fig. 1). \hfill \Box

Let us denote by $v_{i,j}$ the exponent sum of $x_j$ in $\alpha_i$ and $u_{i,j}$ the exponent sum of $x_i$ in $\beta_j$. We set two $(2g \times 2g)$-matrices $V := (v_{i,j})$ and $U := (u_{i,j})$ with integer entries.
On the meridional trace-free slice of the character variety

For a knot in $S^3$, Lin has introduced special free Seifert surface, called regular (see [17]). A free Seifert surface $S$ is called regular if it has a spine where embedding in $S^3$ induced by $S$ is isotopic to the standard embedding. By setting appropriate orientations on $a_i$ and $x_j$, we obtain the following standard properties of $V$ and $U$:

1. $U$ is so-called the Seifert matrix $Q = (Q_{i,j})$, $Q_{i,j} = lk(a_i, a_j^+)$ for $S$;
2. $V = tU$, where $tU$ is the transpose matrix of $U$.

Note that Lin used the matrix $(lk(a_i^+, a_j))_{i,j}$ as the Seifert matrix, which is different from ours $(lk(a_i, a_j^+))_{i,j}$. This gives rise to differences between the convention in [17] of matrices $U$ and $V$ and ours.

Unfortunately, regular Seifert surfaces are defined only for knots in $S^3$. However, $V$ and $U$ are expressed by using the Seifert matrix for a general free Seifert surface of a knot in an integral homology sphere $\Sigma$. To describe $V$ and $U$ for a free Seifert surface in $\Sigma$, we need one more $(2g \times 2g)$-matrix $T$ which represents a boundary operator in the Morse complex $\bigoplus_{i=0}^{2g} C^\text{Morse}_i(\Sigma; \mathbb{Z})$, where $C^\text{Morse}_i(\Sigma; \mathbb{Z})$ is generated by critical points with index $i$ of a Morse function associated to the Heegaard splitting as follows.

Let $S$ be a free Seifert surface for a knot $K$ in $\Sigma$. Set $\Sigma = H_1 \cup H_2$ as a Heegaard splitting associated to $S$. The set of cores of 1-handle in $H_1$ are given by $\{a_1, \ldots, a_{2g}\}$ in the spine $W_{2g}$. We denote by $D_i$ a meridian disk for $a_i$. The cores of 2-handles represent the generators $\{x_1, \ldots, x_{2g}\}$ in $\pi_1(H_2)$. We denote by $D'_i$ a meridian disk for $x_i$.

It is known that there exists a Morse function $f : Y \to [0, 3]$ compatible with the Heegaard splitting (see for instance [19]), i.e., $f$ is a self-indexing Morse function on $\Sigma$ with one minimum and one maximum and $f$ induces the Heegaard decomposition with surface $S = f^{-1}(3/2)$, $H_1 = f^{-1}([0, 3/2])$, $H_2 = f^{-1}([3/2, 3])$. The attaching circles $\partial D_i$ and $\partial D'_i$ are the intersections of $S$ with the ascending and descending gradient flows from the index one and two critical points, respectively.

The intersection point between the meridian disk $D_i$ and the core $a_i$ gives a critical point with index one. We denote by $q_i$ this critical point. The intersection $D'_i$ with $x_i$ gives a critical point with index two. We denote it by $p_i$. Each 2-handle is attached to $H_1$ along the image $\partial D'_i$ in $\partial H_1 = S$. We set an integer $t_{k,j}$ as the intersection number $\partial D_j$ with $\partial D'_k$ in $\partial H_1$. The matrix $(t_{k,j})$ gives the boundary operator from $C^\text{Morse}_2(\Sigma; \mathbb{Z})$ to $C^\text{Morse}_1(\Sigma; \mathbb{Z})$. The correspondence between two bases $(p_1, \ldots, p_{2g})$ and $(q_1, \ldots, q_{2g})$ is expressed as (Fig. 2)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{seifert_surface.png}
\caption{Seifert surface}
\end{figure}
\[ \partial(p_1, \ldots, p_{2g}) = (q_1, \ldots, q_{2g}) T, \]

where \( T \) is the \((2g \times 2g)\)-matrix \((t_{k,j})\).

**Remark 2** Since \( \Sigma \) is an integral homology 3-sphere, the representation matrix \( T \) is invertible. In particular the determinant \( \text{det}(T) \) is equal to \( \pm 1 \).

**Proposition 1** We suppose that \( S \) is a free Seifert surface of a knot \( K \). Let \( V \) and \( U \) be the matrices defined by the relations \( \alpha_i \) and \( \beta_i \) in Lin presentation (1) of the knot group. Then the matrices \( V \) and \( U \) are expressed as

\[ V = \text{'}QT, \quad U = QT, \]

where \( Q \) is the Seifert matrix for \( S \) and \( T \) is the matrix corresponding to the boundary operator from \( C_{\text{Morse}}^2(\Sigma; \mathbb{Z}) \) to \( C_{\text{Morse}}^1(\Sigma; \mathbb{Z}) \).

**Proof** From the definition, the integer \( v_{i,j} \) is given by the intersection number of \( a_i^+ \) with \( \partial D'_j \) in \( \partial H_1 \). Since \( \partial D'_j \) is homologue to \( \sum_{k=1}^{2g} t_{k,j} a_k \), by using linking number, the entry \( v_{i,j} \) in \( V \) is expressed as

\[ v_{i,j} = \text{lk}(a_i^+, \sum_{k=1}^{2g} t_{k,j} a_k), \]  

where \( t_{k,j} \) is the entries in the matrix \( T \). The right hand side of (2) turns into

\[ \text{lk}(a_i^+, \sum_{k=1}^{2g} t_{k,j} a_k) = \sum_{k=1}^{2g} t_{k,j} \text{lk}(a_i^+, a_k) \]

\[ = \sum_{k=1}^{2g} t_{k,j} Q_{k,i}. \]

This means that \( V = \text{'}QT \). Similarly we can prove that \( U = QT \).

**Remark 3** The Alexander polynomial \( \Delta_K(t) \) is given by \( \text{det}(tU - V) \).

**Remark 4** If \( K \) is a knot in \( S^3 \) and \( S \) is regular, then the matrix \( T \) is the identity matrix.
3.2 The induced presentation for the fundamental group of the \( n \)-fold cyclic cover

The main results in this subsection are Lemmas 3 and 4. Lemma 3 describes a presentation for \( \pi_1(C_n) \) induced by a choice of Lin presentation for \( \pi_1(E_K) \) and Lemma 4 establishes a useful relation in the homology \( H_1(C_n; \mathbb{Z}) \).

The \( n \)-fold cyclic cover \( C_n \) over \( E_K \) has the covering transformation, denoted by \( \sigma \). The covering transformations form a cyclic group with order \( n \). This action induces the automorphism \( \tau \) of \( \pi_1(C_n) \),

\[
\tau : \pi_1(C_n, \hat{b}) \to \pi_1(C_n, \hat{b}),
\]

\[ [\ell] \mapsto [\hat{m} \cdot \sigma(\ell) \cdot \hat{m}^{-1}], \]

where \( \hat{b} \) is a lift of the base point \( b \) of \( \pi_1(E_K) \), \( \ell \) denotes a closed loop in \( C_n \) and \( \hat{m} \) is a lift of the meridian \( \mu \) with the initial point \( \hat{b} \). Let \( p_* : \pi_1(C_n, \hat{b}) \to \pi_1(E_K, b) \) be the homomorphism induced by the projection from \( C_n \) to \( E_K \). Then we denote by \( \mu_n \) a meridian of \( C_n \) such that \( p_*(\mu_n) = \mu^n \). Note that the covering transformation does not preserve base point \( \hat{b} \).

**Remark 5** It follows from the definition of \( \tau \) that

1. for every element \( \gamma \) in \( \pi_1(C_n) \), the action of \( \tau^n \) sends \( \gamma \) to \( \mu_n \gamma \mu_n^{-1} \) where \( \tau^k \) denotes \( k \) times composition of \( \tau \); and
2. the homomorphism \( \tau \) sends \( \mu_n \) to itself in \( \pi_1(C_n) \).

Lin presentations are useful to describe the fundamental groups of the cyclic cover \( C_n \) over \( E_K \). Indeed, we can construct \( C_n \) by cutting \( E_K \) along a Seifert surface and gluing \( n \) copies. Then every closed loop on the Seifert surface can be lifted to \( C_n \).

**Lemma 3** Given a Lin presentation of \( \pi_1(E_K) \) of the form

\[
\pi_1(E_K) = \langle x_1, \ldots, x_{2g}, \mu \mid \mu \alpha_i \mu^{-1} = \beta_i, i = 1, \ldots, 2g \rangle,
\]

then \( \pi_1(C_n) \) admits a presentation of the form

\[
\pi_1(C_n) = \left\{ \tilde{x}_1, \ldots, \tilde{x}_{2g}, \right. \left. \tau^j \tilde{x}_1, \ldots, \tau^j \tilde{x}_{2g}, \mu_n \mid \mu_n \tilde{\alpha}_i(0) \mu_n^{-1} = \tilde{\beta}_i^{(n-1)}, \tilde{\alpha}_i^{(j)} = \tilde{\beta}_i^{(j-1)} \right\},
\]

where \( \tilde{x}_i \) is the lift of \( x_i \) to \( C_n \) and \( \tilde{\alpha}_i^{(j)} \), \( \tilde{\beta}_i^{(j)} \) denote the words obtained from \( \alpha_i \), \( \beta_i \) by replacing \( x_1, \ldots, x_{2g} \) with \( \tau^j \tilde{x}_1, \ldots, \tau^j \tilde{x}_{2g} \) for \( i = 1, \ldots, 2g \) and \( j = 0, \ldots, n - 1 \).

Note that \( x_i \) in \( E_K \) does not intersect with the free Seifert surface. So we can take a closed loop in \( C_n \) as a lift.

**Proof** The presentation (3) follows from the construction of \( C_n \) by cutting \( E_K \) along the free Seifert surface \( S \) and gluing \( n \) copies of it along their boundary surfaces in an appropriate manner. First we separate \( \Sigma \) into two handlebodies \( H_1 = S \times [-1, 1] \) and \( H_2 \) as in Lemma 1 and take a simple closed curve \( K' \) on \( S \times \{0\} \) parallel to \( K \) by
pushing $K$ slightly into $H_1$. We denote by $S'$ a compact surface obtained by shrinking $S$ so that the boundary coincides with $K'$. Cutting $H_1$ along $S'$ and gluing $n$ copies, we have the cyclic cover $\tilde{H}_1$ of $H_1$. Since the cyclic cover $\tilde{H}_1$ is deformed to the union $T^2$ and $n$ copies of $S'$, the fundamental group $\pi_1(\tilde{H}_1)$ is generated by the homotopy classes of all lifts of $a_i (1 \leq i \leq 2g)$ and $\mu_n$ where $a_1, \ldots, a_{2g}$ are circles in a spine of $S'$. Next gluing $n$ copies of $H_2$ to $\tilde{H}_1$ as in Fig. 3, we obtain the $n$-fold cyclic cover of $E_K$ and the presentation (3) by van Kampen’s theorem.

The presentation of $\pi_1(C_n)$ in Lemma 3 induces a presentation of $H_1(C_n; \mathbb{Z})$ as follows. It is known that this presentation of $H_1(C_n; \mathbb{Z})$ can be also proved by using a Mayer–Vietoris argument and the Alexander duality (see [15, Chapter 9]).

**Lemma 4** We keep the notations in Proposition 1 and Lemma 3. Let $Z$ be a free abelian group generated by $2ng$ elements as follows:

$$Z := \left( \bigoplus_{i=1}^{2g} \mathbb{Z}[\overline{x}_i] \right) \oplus \left( \bigoplus_{i=1}^{2g} \mathbb{Z}[\tau(\overline{x}_i)] \right) \oplus \cdots \oplus \left( \bigoplus_{i=1}^{2g} \mathbb{Z}[\tau^{n-1}(\overline{x}_i)] \right).$$

Then the presentation of $H_1(C_n; \mathbb{Z})$ is given by

$$Z \xrightarrow{A} Z \oplus \mathbb{Z}[\mu_n] \rightarrow H_1(C_n; \mathbb{Z}),$$

where $A$ is the $(2ng \times 2ng)$-matrix given by

$$A = \begin{pmatrix}
V & -U & \cdots & 0 & 0 \\
0 & V & -U & \cdots & 0 \\
0 & 0 & V & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
-U & \cdots & 0 & 0 & V
\end{pmatrix}.$$ 

Moreover if we set $x_j = \ell([\tau^j(\overline{x}_1)], \ldots, [\tau^j(\overline{x}_{2g})])$, then we have the following relation in $H_1(C_n; \mathbb{Z})$:

$$x_0 + \cdots + x_{n-1} = 0.$$ 

**Proof** The above presentation is given by the abelianization of the presentation in Lemma 3. So we omit the details. Here we show the relation $x_0 + \cdots + x_{n-1} = 0$. 

\[ \text{Springer} \]
From the presentation of $H_1(C_n; \mathbb{Z})$, we have the following relation:

$$\begin{align*}
Vx_0 & - Ux_{n-1} = 0 \\
-Ux_0 + Vx_1 & = 0 \\
& \quad \vdots \\
-Ux_{n-2} + Vx_{n-1} = 0.
\end{align*}$$

By taking sum on the both side, we have

$$(V - U)(x_0 + \cdots + x_{n-1}) = 0.$$ 

By Proposition 1 (cf. Remark 3) and the fact that the Alexander polynomial of a knot at $t = 1$ always equals ±1, we see that $\det(V - U) = \Delta_K(1) = \pm 1$, which implies that $x_0 + \cdots + x_{n-1} = 0$. \qed

### 3.3 The induced presentation for the fundamental group of the $n$-fold branched cover

The main result in this subsection is Lemma 6, which describes the presentation for $\pi_1(B_n)$ induced by a choice of Lin presentation for $\pi_1(E_K)$.

We denote by $j_n$ the homomorphism from $\pi_1(C_n)$ to $\pi_1(B_n)$ induced by the inclusion from $C_n$ onto $B_n$. Now we have the following diagrams on the knot exterior and its covering spaces:

\[
\begin{array}{ccc}
C_n & \xrightarrow{j} & B_n \\
p & \downarrow & q \\
E_K & \xrightarrow{i} & \Sigma, \\
p_* & \downarrow & q_* \\
\pi_1(C_n) & \xrightarrow{j_*} & \pi_1(B_n) \\
\pi_1(E_K) & \xrightarrow{i_*} & \pi_1(\Sigma). \\
\end{array}
\]

Since $B_n$ is obtained from $C_n$ by gluing $D^2 \times S^1$ along their boundaries and the universality of the amalgamated product, we have the following diagram:

\[
\begin{array}{ccc}
\pi_1(D^2 \times S^1) & \xrightarrow{h} & \pi_1(B_n) \\
\pi_1(T^2) & \xrightarrow{\pi_1(T^2) \ast \pi_1(C_n)} & \pi_1(D^2 \times S^1) \\
\pi_1(C_n) & \xrightarrow{\pi_1(T^2)} & \pi_1(D^2 \times S^1) \\
\end{array}
\]

Van Kampen’s theorem implies that the above map $h$ is an isomorphism. The next result is a basic fact in group theory.
Lemma 5 Suppose $\varphi: H \to G_1$ is an epimorphism and the following diagram of groups is commutative:

$$
\begin{array}{ccc}
H & \xrightarrow{\psi} & G_1 \\
\uparrow & & \downarrow \iota_1 \\
\psi & & \iota_2 \\
\uparrow & & \downarrow \\
G_2 & \xrightarrow{G_1 \ast H G_2} & \end{array}
$$

Then $\iota_2$ is also surjective and the kernel of $\iota_2$ is given by $\langle \langle \psi(\ker \varphi) \rangle \rangle$ where $\langle T \rangle$ is the normal closure of a subset $T$. In particular, we have the isomorphism $G_2/\langle \langle \psi(\ker \varphi) \rangle \rangle \simeq G_1 \ast H G_2$.

Since the homomorphism $\pi_1(T^2) \to \pi_1(D^2 \times S^1)$ is surjective in the diagram (4) and the kernel is generated by a meridian, we have the isomorphism:

$$
\pi_1(C_n)/\langle \langle \mu \rangle \rangle \simeq \pi_1(B_n),
$$

where $\mu$ is the meridian such that $p_\ast(\mu) = \mu^n$. Hence we can regard the homomorphism $\hat{\pi}$ as the projection from $\pi_1(C_n)$ onto $\pi_1(C_n)/\langle \langle \mu \rangle \rangle$. Similarly, we have the following isomorphism for $\pi_1(\Sigma)$:

$$
\pi_1(E_K)/\langle \langle \mu \rangle \rangle \simeq \pi_1(\Sigma).
$$

From Lemmas 3 and 4 and the above relation between $\pi_1(C_n)$ and $\pi_1(B_n)$, we obtain the following presentations of $\pi_1(B_n)$ and $H_1(B_n; \mathbb{Z})$.

Lemma 6 Given a Lin presentation of $\pi_1(E_K)$ of the form

$$
\pi_1(E_K) = \langle x_1, \ldots, x_{2g}, \mu | \mu \alpha_i \mu^{-1} = \beta_i, i = 1, \ldots, 2g \rangle,
$$

then $\pi_1(B_n)$ admits a presentation of the form

$$
\pi_1(B_n) = \langle \tau^j \tilde{x}_1, \ldots, \tau^j \tilde{x}_{2g} | \tilde{\alpha}_i^{(j)} = \tilde{\beta}_i^{(j-1)}, 1 \leq i \leq 2g, 1 \leq j \leq n \rangle,
$$

where $\tilde{x}_i$ is the lift of $x_i$ to $B_n$ and $\tilde{\alpha}_i^{(j)}$, $\tilde{\beta}_i^{(j)}$ denote the words obtained from $\alpha_i$, $\beta_i$ by replacing $x_1, \ldots, x_{2g}$ with $\tau^j \tilde{x}_1, \ldots, \tau^j \tilde{x}_{2g}$ for $i = 1, \ldots, 2g$ and $j = 1, \ldots, n$. The homology group $H_1(B_n; \mathbb{Z})$ is expressed as

$$
\begin{array}{c}
\mathbb{Z} \xrightarrow{A} \mathbb{Z} \to H_1(B_n; \mathbb{Z}) \end{array}
$$

where $Z$ and $A$ are as in Lemma 4. Moreover for $x_j = ![\tau^j(\tilde{x}_1)], \ldots, [\tau^j(\tilde{x}_{2g})]$, the following relation holds in $H_1(C_n; \mathbb{Z})$:

$$
x_0 + \cdots + x_{n-1} = 0.
$$

Note that it holds that $\tau^n \tilde{x}_i = \mu_n \tilde{x}_i \mu_n^{-1} = \tilde{x}_i$ in $\pi_1(B_n)$.
Remark 6 Since the group $\pi_1(B_n)$ is just the quotient of $\pi_1(C_n)$ by the relation $\mu_n = 1$, we can view $R(B_n)$ as consisting of representations $\rho \in R(C_n)$ satisfying $\rho(\mu_n) = 1$.

4 On the characters of metabelian representations

The purpose of this section is to describe a feature of metabelian characters via $\mathbb{Z}_2$-action (involution) on character varieties. Sections 4.1 and 4.2 deal with reducible metabelian representations and irreducible ones separately. We will see that the metabelian characters gives the fixed points on the character variety via the involution in Sect. 4.3.

4.1 Reducible representations

From the definition, every reducible representation is conjugate to a representation whose image consists of upper triangular matrices in $\text{SL}_2(\mathbb{C})$.

Remark 7 If $A$ and $B$ are upper triangular $\text{SL}_2(\mathbb{C})$-elements, then the commutator $[A, B]$ lies in Para with eigenvalues 1. So we can see that the commutator subgroup of the image of any reducible representation is an abelian subgroup in $\text{SL}_2(\mathbb{C})$. Hence all reducible representations are metabelian. The set $R^\text{meta}(E_K)$ of metabelian representations can be decomposed as a union

$$\{\rho \in R^\text{irr}(E_K) \mid \rho : \text{metabelian}\} \cup R^\text{red}(E_K).$$

Furthermore we have two kinds of reducible representations, one is abelian and another is non-abelian. Every abelian representation factors through the abelianization $H_1(E_K; \mathbb{Z})$. Since the homology group $H_1(E_K; \mathbb{Z})$ is generated by the homology class of the meridian $\mu$, each abelian representation is determined by the image of the meridian $\mu$. For a given $\lambda \in \mathbb{C}^*$, we denote by $\varphi_\lambda$ the abelian representation given by

$$\varphi_\lambda : \pi_1(E_K) \rightarrow \text{SL}_2(\mathbb{C}), \quad \mu \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$  

By taking conjugation, we can assume that the image of any reducible representation consists of upper triangular $\text{SL}_2(\mathbb{C})$-elements. This means that for any reducible representation $\rho$, there is a complex number $\lambda$ such that the character of $\rho$ is the same as that of the abelian representation $\varphi_\lambda$. For non-abelian reducible representations, it is known that the following fact holds.

Lemma 7 [8,13] There exists a reducible non-abelian representation

$$\rho : \pi_1(E_K) \rightarrow \text{SL}_2(\mathbb{C})$$

such that $\chi_\rho = \chi_{\varphi_\lambda}$ if and only if $\Delta_K(\lambda^2) = 0$.

This is a well-known result of Burde [4] and de Rham [10] if $E_K$ is the knot exterior of a knot in $S^3$. 

$\square$ Springer
4.2 The characters of metabelian and binary dihedral representations

In this subsection, we focus on irreducible metabelian representations of a knot group. We begin with a review on binary dihedral representations of a knot group $\pi_1(E_K)$. As we will see, binary dihedral representations form a subset of metabelian representations and give the representatives of conjugacy classes for irreducible metabelian $SL_2(\mathbb{C})$-representations.

First we recall the binary dihedral group in $SU(2)$. Set

$$S_A^1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \mid a \in \mathbb{C}, |a| = 1 \right\} \quad \text{and} \quad S_B^1 = \left\{ \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \mid b \in \mathbb{C}, |b| = 1 \right\}$$

and define the binary dihedral group as $N = S_A^1 \cup S_B^1 \subset SU(2)$. It is easy to see $N$ is closed under multiplication, and a simple calculation reveals that for any $g, h \in N$, we have $[g, h] \in S_A^1$. An $SL_2(\mathbb{C})$-representation is said to be binary dihedral if the image is contained in $N$. It follows that $[N, N] = S_A^1$ is abelian, that $N$ is a metabelian group, and that every $SL_2(\mathbb{C})$-representation $\rho$ with image in $N$ is metabelian. This shows that every binary dihedral representation is metabelian, and the next result gives an equivalence between irreducible metabelian characters and irreducible binary dihedral characters.

**Proposition 2** Every irreducible metabelian $SL_2(\mathbb{C})$-representation of $\pi_1(E_K)$ is conjugate to an irreducible binary dihedral representation. Hence we have the following equation:

$$\{ \chi_\rho \mid \rho : \text{irreducible metabelian} \} = \{ \chi_\rho \mid \rho : \text{irreducible binary dihedral} \}.$$ 

Moreover the number of conjugacy classes of irreducible metabelian representations is given by

$$\frac{|\Delta_K(-1)| - 1}{2},$$

where $\Delta_K(t)$ is the Alexander polynomial of $K$.

For a knot in $S^3$, we can find the formula for the number of conjugacy classes of irreducible metabelian representations in [18, Proposition 1.1 and Theorem 1.2]. Klassen [14, Theorem 10] also proved the same formula for binary dihedral representations. Hence Proposition 2 for a knot in $S^3$ can be deduced by combining these two results.

**Proof of Proposition 2** Let $\rho$ be an irreducible and metabelian $SL_2(\mathbb{C})$-representation of $\pi_1(E_K)$. Then for the generators in a Lin presentation (1) of a knot in an integral homology 3-sphere $\Sigma$, by taking conjugation, we have the following form of binary dihedral representations:

$$\rho(x_i) = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix}, \quad \rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ \(\text{(8)}\)
Then we can apply the same argument as in the proof of [18, Proposition 1.1 and Theorem 1.2] to deduce the above form and counting the conjugacy classes. □

Remark 8 We can find a higher rank analogy of Proposition 2 for knots in $S^3$ in [2].

4.3 An involution on the $\text{SL}_2(\mathbb{C})$-character variety

In this subsection, we introduce involutions $\iota$ and $\hat{\iota}$ on $R(E_K)$ and $X(E_K)$ and identify the fixed point set of $\hat{\iota}$ on $X^{\text{irr}}(E_K)$ with irreducible metabelian characters. For $g \in \pi_1(\Sigma)$, we denote by $[g] \in H_1(E_K; \mathbb{Z}) \simeq \mathbb{Z}$ the associated element in the homology group, which we identify with $\mathbb{Z}$ by the isomorphism that sends the meridian $\mu$ to the generator 1. Next, we define an involution $\iota$ on $R(E_K)$ as follows. Given $\rho \in R(E_K)$, let $\iota(\rho)$ be the representation such that $\iota(\rho)(g) = (-1)^{|g|} \rho(g)$ for all $g \in \pi_1(E_K)$. It is immediate that $\iota$ is an involution on $R(E_K)$, and it induces an involution $\hat{\iota}$ on the character variety $X(E_K)$ defined as follows. Given $\chi_\rho \in X(E_K)$, let $\hat{\iota}(\chi_\rho)$ be the character such that $\hat{\iota}(\chi_\rho) = (-1)^{|g|} \chi_\rho(g)$ for any $g \in \pi_1(E_K)$. It is easy to check that $\hat{\iota}$ is an involution on $X(E_K)$ and that $\hat{\iota}(\chi_\rho) = \chi_{\iota(\rho)}$ from the commutativity of trace with the scalar multiplication.

The involution $\hat{\iota}$ has the following fixed point set.

**Proposition 3** The fixed point set of $\hat{\iota}$ in $X^{\text{irr}}(E_K)$ is the set of irreducible metabelian characters, i.e.,

$$X^{\text{irr}}(E_K)^{\hat{\iota}} = \{ \chi_\rho \mid \rho : \text{irreducible metabelian} \}$$

We apply the following elementary lemma to prove Proposition 3.

**Lemma 8** We set $A \in M_2(\mathbb{C})$ and $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then we have

$$\text{tr} \ AC = 0 \Leftrightarrow A = ^tA.$$  

**Proof of Proposition 3** We choose a Lin presentation for the knot group:

$$\pi_1(E_K) = \langle x_1, \ldots, x_{2g}, \mu \mid \mu \alpha_i\mu^{-1} = \beta_i, \ i = 1, \ldots, 2g \rangle.$$  

First we show that for every irreducible metabelian $\text{SL}_2(\mathbb{C})$-representation $\rho$ the image $\iota(\rho)$ is conjugate to $\rho$. It follows from Proposition 2 that after taking suitable conjugation, the generators in the presentation (9) are sent to the following matrices by $\rho$:

$$\rho(x_i) = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix}, \quad \rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

$\square$ Springer
Since every $x_i$ is contained in the commutator subgroup of $\pi_1(E_K)$, the images of these generators by $\iota(\rho)$ are given by

$$\iota(\rho)(x_i) = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix}, \quad \iota(\rho)(\mu) = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

Hence we have

$$\iota(\rho) = P \rho P^{-1}, \quad P = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}. $$

In particular, we obtain the equation $\chi_\rho = \chi_{\iota(\rho)}$.

Next we show that an irreducible $\text{SL}_2(\mathbb{C})$-representation $\rho$ satisfying $\chi_\rho = \chi_{\iota(\rho)}$ is metabelian. From the equations $\chi_{\iota(\rho)}(\mu) = -\chi_\rho(\mu)$ and $\chi_\rho = \chi_{\iota(\rho)}$, we have $\text{tr} \rho(\mu) = 0$. Therefore by taking conjugation we can assume that $\rho(\mu)$ is given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We choose any two elements $\gamma_1, \gamma_2$ from $[\pi_1(E_K), \pi_1(E_K)]$. Since $[\gamma_1, \mu] = 1$, we have that $\chi_{\iota(\rho)}(\gamma_1, \mu) = -\chi_\rho(\gamma_1, \mu)$. From the assumption $\chi_\rho = \chi_{\iota(\rho)}$, we obtain $\text{tr} \rho(\gamma_1, \mu) = 0$. By Lemma 8, $\rho(\gamma_1) = t \rho(\gamma_1)$ holds for $i = 1, 2$. Since $\gamma_1 \gamma_2 \in [\pi_1(E_K), \pi_1(E_K)]$, we also have $\rho(\gamma_1 \gamma_2) = t \rho(\gamma_1) \rho(\gamma_2)$ Then we have the following equalities of matrices:

$$\rho(\gamma_1 \gamma_2) = t \rho(\gamma_1) \rho(\gamma_2) = \rho(\gamma_2) t \rho(\gamma_1) = \rho(\gamma_2 \gamma_1).$$

Therefore the representation $\rho$ is metabelian.

By Propositions 2 and 3, we can conclude the following corollary.

**Corollary 1**

$$X^\text{irr}(E_K) = \{ \chi_\rho | \rho: \text{irreducible binary dihedral} \}.$$

The fixed point set $X(E_K)^{\hat{\iota}}$ could also contain reducible characters. Actually, as we shall see, this fixed point set $X(E_K)^{\hat{\iota}}$ contains only one character coming from reducible representations.

Before giving this proof, we define subsets in the character variety where the fixed point set $X(E_K)^{\hat{\iota}}$ lives. Every character in such a subset sends the meridian $\mu$ to the same value. We define such subsets as a level set of a function on the character variety.

For any given $\gamma \in \pi_1(E_K)$, we denote by $I_\gamma$ a function on $X(E_K)$ defined as $I_\gamma: X(E_K) \to \mathbb{C}, \chi \mapsto \chi(\gamma)$. This function is called the trace function of $\gamma$.

**Definition 2** We define the subset $S_c(E_K)$ in $X(E_K)$ as the level set of $I_\mu$ at $c$, i.e.,

$$S_c(E_K) = I_\mu^{-1}(c).$$

This subset is called the slice with the trace of meridian fixed by $c$.

We also set $R_c(E_K)$ as $R_c(E_K) := \{ \rho \in R(E_K) | \text{tr} \rho(\mu) = c \}.$
Lemma 9 The fixed point set $X(E_K)^\hat{\iota}$ is contained in $S_0(E_K)$ and the restriction of $\hat{\iota}$ to $S_0(E_K)$ gives an involution on it.

Proof We have seen that every character $\chi$ in $X(E_K)^\hat{\iota}$ satisfies that $\chi(\mu) = 0$ in the proof of Proposition 3. For any element $\chi$ in $S_0(E_K)$, we have

$$\hat{\iota}(\chi)(\mu) = -\chi(\mu) = 0.$$ 

The image of $S_0(E_K)$ by $\hat{\iota}$ is contained in $S_0(E_K)$ itself.

Remark 9 The involution $\iota$ interchanges $R_c(E_K)$ and $R_{-c}(E_K)$, and likewise the involution $\hat{\iota}$ interchanges the $S_c(E_K)$ and $S_{-c}(E_K)$.

Every representation in $R_0(E_K)$ maps the meridian $\mu$ to matrices whose eigenvalues are $\pm\sqrt{-1}$. By Lemma 7 and $\Delta_K(-1) \neq 0$, we have no non-abelian reducible representation in $R_0(E_K)$.

Remark 10 Every reducible representation in $R_0(E_K)$ is abelian.

Therefore, by Lemma 9, the set of reducible characters in $X(E_K)^\hat{\iota}$ consists of only one character of the abelian representation $\varphi_{\sqrt{-1}}$. Summarizing, we have the following proposition.

Proposition 4 The fixed point set $X(E_K)^\hat{\iota}$ is expressed as

$$X(E_K)^\hat{\iota} = S_0(E_K)^\hat{\iota} = \{\chi_\rho \mid \rho : \text{irreducible binary dihedral}\} \cup \{\chi_{\varphi_{\sqrt{-1}}}\}.$$ 

5 Maps between representation spaces

In this section, we derive how to make an $SL_2(\mathbb{C})$-representation of $\pi_1(B_n)$ from that of $\pi_1(E_K)$ via the maps between representation spaces defined by the pullbacks of the homomorphisms $p_\ast, q_\ast, i_\ast$ and $j_\ast$ between the fundamental groups. These maps satisfy the following diagram among the $SL_2(\mathbb{C})$-representation spaces of $\pi_1(E_K), \pi_1(C_n), \pi_1(\Sigma)$ and $\pi_1(B_n)$:

\[
\begin{array}{cccc}
\pi_1(C_n) & j_\ast & \pi_1(B_n) & R(C_n) \\
p_\ast & \downarrow & q_\ast & p^\ast \\
\pi_1(E_K) & i_\ast & \pi_1(\Sigma), & R(E_K) \\
& & & R(\Sigma).
\end{array}
\]

Note that $j^\ast$ is injective since $j_\ast$ is surjective. As in Remark 6, we can regard $j^\ast(R(B_n))$ as the subset in $R(C_n)$ consisting of $\rho \in R(C_n)$ which sends $\mu_n$ to the identity matrix. Hereafter we simply identify $R(B_n)$ with its image under $j^\ast$, and likewise regard $R(\Sigma)$ as a subset of $R(E_K)$.

In this setting, $SL_2(\mathbb{C})$-representations of $\pi_1(B_n)$ is constructed from the following subset of those of $\pi_1(E_K)$:

$$\{\rho \in R(E_K) \mid \rho(\mu)^n = 1\} \cup \{\rho \in R(E_K) \mid \rho(\mu)^n = -1\}. \quad (11)$$
In fact, our interest lies in the case of $n = 2$ because this case has distinct features from the other cases (see Sect. 6). To compare their features, we deal with general definitions including the cases other than $n = 2$ in this section.

We start with the observation that the pull-back of $\rho$ in $R(E_K)$ satisfying $\rho(\mu)^n = 1$ defines an $SL_2(\mathbb{C})$-representation of $\pi_1(B_n)$ however this correspondence covers very few range of $R(B_n)$. To deal with more range, we consider the pair of subset as in (11). We first observe the intersection $p^*(R(E_K)) \cap R(B_n)$ and the induced correspondence from the following subset of $R(E_K)$.

Lemma 10

$$(p^*)^{-1}(R(B_n)) = \{ \rho \in R(E_K) \mid \rho(\mu)^n = 1 \}.$$  \hfill (12)

Proof It follows that $\rho \in (p^*)^{-1}(R(B_n)) \iff p^*\rho(\mu) = 1 \iff \rho(\mu)^n = 1$. \hfill $\square$

From the intersection (12) in Lemma 10, we can make a correspondence from the subset $\{ \rho \in R(E_K) \mid \rho(\mu)^n = 1 \}$ into $R(B_n)$ by the pull-back $p^*$, i.e.,

$$p^*: \{ \rho \in R(E_K) \mid \rho(\mu)^n = 1 \} \rightarrow R(B_n) \subset R(C_n).$$

Remark 11 In the special case that $n = 2$, the subset $(p^*)^{-1}(R(B_2))$ turns into

$$\{ \rho \in R(E_K) \mid \rho: \pi_1(E_K) \rightarrow \{\pm 1\} \}$$

consisting of only two conjugacy classes, both of whose restrictions to $\pi_1(B_2)$ are trivial. Hence, in the construction of an $SL_2(\mathbb{C})$-representation of $\pi_1(B_2)$ via the map $p^*$, we can only deal with the slices $R_{\pm 2}(E_K)$ at $\pm 2$, which correspond to the second roots of unity.

Next we see that the pull-back by $p^*$ of the subset (11) defines a $PSL_2(\mathbb{C})$-representation of $\pi_1(B_n)$. This allows us to consider much more elements of $R(E_K)$.

Lemma 11 If $\rho$ is an element in the subset (11), then the pull-back $p^*\rho$ defines $PSL_2(\mathbb{C})$-representation of $\pi_1(B_n)$.

Proof We can regard $\pi_1(B_n)$ as the quotient $\pi_1(C_n)/\langle \langle \mu_n \rangle \rangle$. It is enough to prove that $p^*\rho(\mu_n) = \pm 1$. This follows from direct calculations. \hfill $\square$

We remark several results on the lifting problem without a sign refinement. For example, Burde [5] has shown that binary dihedral representations form the branched point set of covering map from $SU(2)$-representations to $SO(3)$-representations for a two-bridge knot. Boyer and Zhang [3, Section 3] and Porti and Heusener [20, Section 4] deal with the lifting problem for $PSL_2(\mathbb{C})$-representations of a finitely generated and presented group in context of the character variety.

We will lift $PSL_2(\mathbb{C})$-representations of $\pi_1(B_n)$ given in Lemma 11 to $SL_2(\mathbb{C})$ by using a sign refinement in $R(C_n)$. We define an involution $\iota_n$ on $R(C_n)$ for our sign refinement in Definition 3.
**Definition 3** Set $\xi_{2n} = \exp(\pi \sqrt{-1}/n)$. Given $\rho \in R(C_n)$, let $\iota_n(\rho)$ be the representation such that

$$\iota_n(\rho)(g) = \left(\xi_{2n}\right)^{p_*(g)}\rho(g)$$

for all $g \in \pi_1(C_n)$, where $p_* : H_1(C_n; \mathbb{Z}) \to H_1(E_K) \cong \mathbb{Z}$ whose image is $n\mathbb{Z}$.

It is immediate that $\iota_n$ is an involution on $R(E_K)$, and it induces an involution $\widehat{\iota}_n$ on the character variety $X(C_n)$ defined as follows.

**Definition 4** Given $\chi_\rho \in X(C_n)$, let $\widehat{\iota}_n(\chi_\rho)$ be the character such that

$$\widehat{\iota}_n(\chi_\rho)(g) = \left(\xi_{2n}\right)^{p_*(g)}\chi_\rho(g).$$

It is easy to check that $\widehat{\iota}_n$ is an involution on $X(C_n)$ and that $\widehat{\iota}_n(\chi_\rho) = \chi_{\iota_n(\rho)}$ from the commutativity of trace with the scalar multiplication.

Now we construct a map

$$\Phi : \{\rho \in R(E_K) \mid \rho(\mu)^n = \pm 1\} \to R(B_n)$$

as follows. Here the domain of $\Phi$ is the subset (11), denoted by $D(\Phi)$. Since any element of $D(\Phi)$ is contained in the slice $R_{\xi(k/n)}(E_K)$ at some $\xi(k/n) = 2 \cos(k\pi/n)$, we focus on the slice $R_{2\cos(k\pi/n)}(E_K)$. Let $\rho$ be an element of $R_{\xi(k/n)}(E_K)$ such that $\rho(\mu)^n = (-1)^k$. Then we define an $SL_2(\mathbb{C})$-representation $\Phi(\rho)$ of $\pi_1(C_n)$ by sending each element $\gamma \in \pi_1(C_n)$ to the following matrix:

$$\Phi(\rho)(\gamma) = (\iota_n)^k(p^*\rho)(\gamma) = ((\xi_{2n})^{p_*[\gamma]}\rho(\gamma),$$

where $(\iota_n)^k$ is $k$ times composition of $\iota_n$. The well-definedness of $\Phi$ follows from the next proposition.

**Proposition 5** For $\rho \in R_{\xi(k/n)}(E_K)$ satisfying $\rho(\mu)^n = (-1)^k$, we have the equality $\Phi(\rho)(\mu) = 1$. In particular, this homomorphism induces a homomorphism from $\pi_1(C_n)/\langle\langle \mu \rangle\rangle$ into $SL_2(\mathbb{C})$.

**Proof** This proposition follows from the following direct calculation. Let $\rho \in R_{\xi(k/n)}(E_K)$ satisfy $\rho(\mu)^n = (-1)^k$.

$$\Phi(\rho)(\mu) = ((\xi_{2n})^{p_*[\mu]}\rho(\mu)$$

$$= ((\xi_{2n})^{kn} \cdot \rho(\mu)^n$$

$$= (-1)^k(-1)^k$$

$$= 1.$$
We note that the $\text{SL}_2(\mathbb{C})$-representation $\Phi(\rho)$ is a lift of the $\text{PSL}_2(\mathbb{C})$-representation $p^*\rho$ of $\pi_1(B_n)$:

$$
\begin{array}{ccc}
\text{SL}_2(\mathbb{C}) & \xrightarrow{\Phi(\rho)} & \pi_1(B_n) \\
\\downarrow & & \downarrow p^* \\
& \xrightarrow{\rho} & \text{PSL}_2(\mathbb{C}).
\end{array}
$$

The domain $D(\Phi)$ is decomposed as

$$
D(\Phi) = \{ \rho \in R(E_K) \mid \rho(\mu)^n = \pm 1 \}
= \bigcup_{k=1}^{n-1} \{ \rho \in R_{\xi(\frac{k}{n})}(E_K) \mid \rho(\mu)^n = \pm 1 \} \cup \{ \rho \in R(E_K) \mid \rho(\mu) = \pm 1 \}. \quad (13)
$$

**Remark 12** The set $\{ \rho \in R(E_K) \mid \rho(\mu) = 1 \}$ coincides with $i^*(R(\Sigma))$. Since the pull-back $i^*$ is injective, we identify $i^*(R(\Sigma))$ with $R(\Sigma)$. The involution $\iota$ interchanges the two sets $\{ \rho \in R(E_K) \mid \rho(\mu) = 1 \}$ and $\{ \rho \in R(E_K) \mid \rho(\mu) = -1 \}$ with each other. So we can rewrite the domain $D(\Phi)$ in (13) as

$$
\bigcup_{k=1}^{n-1} \{ \rho \in R_{\xi(\frac{k}{n})}(E_K) \mid \rho(\mu)^n = \pm 1 \} \cup R(\Sigma) \cup \iota(R(\Sigma)).
$$

The correspondence $\Phi$ induces a map $\hat{\Phi}$ on the subset $t(D(\Phi))$ in $X(E_K)$

$$
\hat{\Phi} : t(D(\Phi)) \to X(B_n)
\quad \chi_{\rho} \mapsto \chi_{\Phi(\rho)}
$$

It is easy to check the well-definedness from the construction of $\Phi$ and the commutativity of trace with the scalar multiplication. The domain $t(D(\Phi))$ of $\hat{\Phi}$ is decomposed as

$$
t(D(\Phi)) = \bigcup_{k=1}^{n-1} S_{\xi(\frac{k}{n})}(E_K) \cup X(\Sigma) \cup \tau(X(\Sigma)) = \bigcup_{k=0}^{n-1} S_{\xi(\frac{k}{n})}(E_K), \quad (14)
$$

where we identify $i^*(X(\Sigma))$ with $X(\Sigma)$. Hence $\hat{\Phi}$ is defined on $S_{\xi(\frac{k}{n})}(E_K)$ as

$$
\hat{\Phi} : S_{\xi(\frac{k}{n})}(E_K) \to X(B_n), \quad \chi_{\rho} \mapsto \chi_{(\iota_\Sigma)^k(p^*\rho)}. \quad (15)
$$

By the definition of $\Phi$, we see the image of any abelian representation by $\Phi$ as follows.

**Lemma 12** For every abelian representation $\rho$ in $D(\Phi)$, the image $\Phi(\rho)$ is the trivial representation of $\pi_1(B_n)$. 
Proof Since $\rho$ is abelian, it sends all elements in the commutator subgroup of $\pi_1(E_K)$ to the identity matrix. For each generator $\tau^j(\tilde{x}_i)$ in the presentation of Lemma 6, the image by $\Phi(\rho)$ is expressed as $\Phi(\rho)(\tau^j\tilde{x}_i) = \rho(\mu^j x_i \mu^{-j})$. Since all $x_i$ are contained in the commutator subgroup of $\pi_1(E_K)$, $\mu^j x_i \mu^{-j}$ is also an element in the commutator subgroup. Hence every $\mu^j x_i \mu^{-j}$ is sent to the identity matrix by $\rho$. Therefore $\Phi(\rho)$ is trivial. 

Remark 13 The involutions $\iota$ and $\hat{\iota}$ interchange slices as follows:

$$
\iota(R_{\xi(k/n)}(E_K)) = R_{\xi(1-k/n)}(E_K), \quad \hat{\iota}(S_{\xi(k/n)}(E_K)) = S_{\xi(1-k/n)}(E_K).
$$

We investigate the properties of the map $\Phi$ in the rest of this section. To consider the image of $\Phi$, we introduce the following notion and notation concerning the action $\tau$ on $\pi_1(C_n)$.

Definition 5 An $SL_2(\mathbb{C})$-representation $\rho$ of $\pi_1(C_n)$ is $\tau$-equivariant if there exists an element $C$ in $SL_2(\mathbb{C})$ such that the following diagram is commutative:

$$
\pi_1(C_n) \xrightarrow{\tau} \pi_1(C_n) \\
\downarrow \rho \quad \quad \quad \quad \quad \quad \downarrow \rho \\
SL_2(\mathbb{C}) \xrightarrow{AdC} SL_2(\mathbb{C}),
$$

where $AdC(X) := CXC^{-1}$ for elements $C$ and $X$ in $SL_2(\mathbb{C})$. We denote by $R^\tau(C_n)$ the set of $\tau$-equivariant $SL_2(\mathbb{C})$-representations.

The $\tau$-equivariance of $\rho$ means that the pull-back of $\rho$ by $\tau$ is expressed as the conjugation of an $SL_2(\mathbb{C})$-element.

Lemma 13 If $\rho$ is irreducible and $\tau$-equivariant, then a matrix $C$ satisfying the diagram (16) is uniquely determined up to sign.

Proof It follows from Schur’s lemma that for $\rho$ and $\tau^* \rho$ the scalar matrices in $SL_2(\mathbb{C})$ are only $\pm 1$. 

Similarly, we can also define $\tau$-equivariant representations for $\pi_1(B_n)$. We denote by $R^\tau(B_n)$ the set of $\tau$-equivariant representations of $\pi_1(B_n)$. We use the same symbol $\tau$ for the induced homomorphism on $\pi_1(B_n)$ by the next lemma.

Lemma 14 The following diagram is commutative:

$$
\pi_1(C_n) \xrightarrow{\tau} \pi_1(C_n) \\
\downarrow j_* \quad \quad \quad \quad \quad \quad \downarrow j_* \\
\pi_1(B_n) \xrightarrow{} \pi_1(B_n).
$$
The fundamental group $\pi_1(B_n)$ is expressed as $\pi_1(C_n)/\langle \langle \mu_n \rangle \rangle$. Since $\tau$ is automorphism and $\tau(\mu_n) = \mu_n$ in $\pi_1(C_n)$. Note that $\langle \langle \mu_n \rangle \rangle$ is expressed as $\langle (g_1^{-1}\mu_n g_1) \cdots (g_k^{-1}\mu_n g_k) | k \geq 0, g_i \in \pi_1(C_n), \epsilon_i = \pm 1 (1 \leq i \leq k) \rangle$. We have that $\tau(\langle \langle \mu_n \rangle \rangle) = \langle \langle \mu_n \rangle \rangle$. Hence $\tau$ induces an automorphism on the quotient group $\pi_1(C_n)/\langle \langle \mu_n \rangle \rangle$

Moreover we can identify the set $R^\tau(B_n)$ with $\{ \rho \in R^\tau(C_n) | \rho(\mu_n) = 1 \}$ by the following Lemma 15.

**Lemma 15** The subset $j^*(R^\tau(B_n))$ coincides with $\{ \rho \in R^\tau(C_n) | \rho(\mu_n) = 1 \}$.

**Proof** This follows from the commutative diagram (17) in Lemma 14.

The conjugation on representation spaces has the following property.

**Lemma 16** The conjugation for representations preserves $\tau$-equivariance.

**Proof** Let $\rho$ be a $\tau$-equivariant representation of $\pi_1(C_n)$. There exists an element $C$ in $\text{SL}_2(\mathbb{C})$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
\pi_1(C_n) & \xrightarrow{\tau} & \pi_1(C_n) \\
\rho \downarrow & & \rho \downarrow \\
\text{SL}_2(\mathbb{C}) & \xrightarrow{\text{Ad}_C} & \text{SL}_2(\mathbb{C}).
\end{array}
\]

Then for any $P$ in $\text{SL}_2(\mathbb{C})$, the representation $P\rho P^{-1}$ satisfies that

\[
\begin{array}{ccc}
\pi_1(C_n) & \xrightarrow{\tau} & \pi_1(C_n) \\
P\rho P^{-1} \downarrow & & P\rho P^{-1} \downarrow \\
\text{SL}_2(\mathbb{C}) & \xrightarrow{\text{Ad}_{PC^{-1}}} & \text{SL}_2(\mathbb{C}).
\end{array}
\]

Hence $P\rho P^{-1}$ is $\tau$-equivariant.

We will prove that $R^\tau(B_n)$ coincides with the image of $\Phi$. To prove this, we use the identification $R^\tau(B_n) = \{ \rho \in R^\tau(C_n) | \rho(\mu_n) = 1 \}$ in Lemma 15 and the following lemma.

**Lemma 17** Suppose that $\rho \in R^\tau(B_n)$ and $C$ is a matrix as in the commutative diagram (16). The adjoint action $\text{Ad}_{C^n}$ acts trivially on $\text{Im} \rho$.

**Proof** The $\tau$-equivariance of $\rho$ gives us the equality that $\rho(\tau^i(\gamma)) = \text{Ad}_{C^n}(\rho(\gamma))$ for any $\gamma$ in $\pi_1(C_n)$. For the case of $i = n$, since $\tau^n(\gamma) = \mu_n \gamma \mu_n^{-1}$ and $\rho(\mu_n) = 1$, we have that $\text{Ad}_{C^n}(\rho(\gamma)) = \rho(\gamma)$ for any $\gamma$ in $\pi_1(C_n)$, i.e, $\text{Ad}_{C^n}$ acts trivially on $\text{Im} \rho$.

We next show the following property on $p^*$ to study the image of $\Phi$.  

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Lemma 18  For every $\rho$ in $R(E_K)$, the image of $p^*: R(E_K) \to R(C_n)$ is contained in $R^\tau(C_n)$.

Proof Let $\gamma$ be an element in $\pi_1(C_n)$. By the following direct calculation, it follows that $p^* \rho$ is $\tau$-equivariant:

$$p^* \rho(\tau(\gamma)) = \rho(p_*(\tau(\gamma)))$$
$$= \rho(\mu \cdot p_*(\gamma) \cdot \mu^{-1})$$
$$= \text{Ad}_{\rho(\mu)}(p^* \rho(\gamma)).$$

$\blacksquare$

In fact, the map $p^*$ gives the following equality as sets:

Proposition 6  $\text{Im } \Phi = R^\tau(B_n)$.

Proof For any representation $\rho$ in $D(\Phi) \cap R_{\xi(k/n)}(E_K)$ and each $\gamma$ in $\pi_1(C_n)$, the matrix $\Phi(\rho)(\tau(\gamma))$ is given by

$$\Phi(\rho)(\tau(\gamma)) = (((\xi_2n)^{p_*(\tau(\gamma))})^k \cdot p^* \rho(\tau(\gamma))$$
$$= (((\xi_2n)^{p_*(\tau(\gamma))})^k \cdot \text{Ad}_{\rho(\mu)}(p^* \rho(\gamma)) \quad \text{(by Lemma 18)}$$
$$= \text{Ad}_{\rho(\mu)}(\Phi(\rho)(\gamma)).$$

Hence the $\text{SL}_2(\mathbb{C})$-representation $\Phi(\rho)$ is $\tau$-equivalent. It is left to show the inclusion $\text{Im } \Phi \supset R^\tau(B_n)$. We take $\rho$ in $R^\tau(B_n)$ and let $C$ be a matrix as in the commutative diagram (16). We use a presentation of $\pi_1(C_n)$ induced from a Lin presentation of $\pi_1(E_K)$ as in Lemma 3. The relations of $\pi_1(C_n)$ give the equalities:

$$\rho(\mu_n \tilde{\alpha}_i^{(0)} \mu_n^{-1}) = \rho(\tilde{\beta}_i^{(n-1)}).$$

The left hand side turns into $\rho(\tilde{\alpha}_i^{(0)})$. Since the element $\tilde{\beta}_i^{(n-1)}$ can be written as $\tau^{n-1}(\tilde{\beta}_i^{(0)})$, the right hand side $\rho(\tilde{\beta}_i^{(n-1)})$ turns into $\text{Ad}_{C^{n-1}}(\rho(\tilde{\beta}_i^{(0)}))$. By Lemma 17, $\text{Ad}_C$ acts trivially on $\text{Im } \rho$. Hence we have the equalities:

$$C \rho(\tilde{\alpha}_i^{(0)}) C^{-1} = \rho(\tilde{\beta}_i^{(0)}).$$

To make a representation $\rho_0$ of $\pi_1(E_K)$ satisfying $\Phi(\rho_0) = \rho$, we set $\rho_0(\mu) = C$ and $\rho_0(x_i) = \rho(\tilde{x}_i)$. The relations $\rho_0(\mu \alpha_i \mu^{-1}) = C \rho(\tilde{\alpha}_i^{(0)}) C^{-1}$ and $\rho_0(\beta_i) = \rho(\tilde{\beta}_i^{(0)})$ show that

$$\rho_0(\mu \alpha_i \mu^{-1}) = \rho_0(\beta_i)$$

for all $i$. Hence $\rho_0$ is an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(E_K)$. We must show that $\rho_0$ is contained in $D(\Phi)$. By the following Lemma 19, there exists an integer $k$ such that $\rho_0$ is contained in $R_{\xi(k/n)}(E_K)$.

$\blacksquare$
Lemma 19 If \( \rho \) is a non-trivial representation in \( R^\tau(B_n) \) and \( C \) is an \( SL_2(\mathbb{C}) \) element as in the diagram (16), then \( C^n = \pm 1 \).

Proof We regard \( \rho \) as an element in \( R^\tau(C_n) \) satisfying \( \rho(\mu_n) = 1 \). By Lemma 17, the adjoint action \( Ad_{C_n} \) acts trivially on Im \( \rho \), i.e., \( C^n \) commutes with all elements in Im \( \rho \subset SL_2(\mathbb{C}) \). We suppose that \( C^n \) is not contained in the center \( \{ \pm 1 \} \) in \( SL_2(\mathbb{C}) \). This says that \( C \) and Im \( \rho \) are contained in the same maximal abelian subgroup in SL\(_2(\mathbb{C})\). Since \( \rho \) is \( \tau \)-equivariant and \( C \) commutes with all elements of Im \( \rho \), we have that \( \rho(\tau(\gamma)) = Ad_C(\rho(\gamma)) = \rho(\gamma) \) for any \( \gamma \) in \( \pi_1(C_n) \). Then \( \rho \) turns into the pull-back of abelian representation of \( \pi_1(E_K) \) given by the following corresponding:

\[
\mu \mapsto C, \quad x_i \mapsto \rho(\tilde{x}_i).
\]

for Lin presentations of \( \pi_1(E_K) \) and \( \pi_1(C_n) \). However every abelian representation of \( \pi_1(E_K) \) sends \( x_i \) to 1 since \( x_i \) is an element in the commutator subgroup. Thus \( \rho \) is trivial but this contradicts to our assumption that \( \rho \) is non-trivial.

Now, Proposition 6 can be naturally extended to the set of characters as follows:

Proposition 7 The image of \( \hat{\Phi} \) coincides with the fixed point set \( X(B_n)^\tau \) of the action \( \tau^* \) on \( X(B_n) \).

Proof It follows by definition that the image Im \( \hat{\Phi} = t(R^\tau(B_n)) \) is contained in \( X(B_n)^\tau \). We need to prove that every fixed point of \( \tau^* \) in \( X(B_n) \) is given by the character of a \( \tau \)-equivariant representation.

First we consider the character \( \chi_\rho \in X(B_n)^\tau \) associated to an irreducible representation \( \rho \) such that it is fixed by \( \tau^* \), i.e., \( \tau^*(\chi_\rho) = \chi_\tau \rho = \chi_\rho \). Since \( \rho \) is irreducible, the representation \( \rho \) is conjugate to \( \tau^* \rho \). This says that \( \rho \) is \( \tau \)-equivariant.

Next we consider the case that \( \chi_\rho \in X(B_n)^\tau \) is a reducible character such that \( \tau^*(\chi_\rho) = \chi_\rho \). For every reducible representation there exists an abelian representation of \( \pi_1(B_n) \) such that it has the same character as that of the reducible one and the image is contained in the maximal abelian subgroup Hyp. So we can assume without loss of generality that Im \( \rho \) is contained in Hyp. By using a presentation as in Lemma 6 and \( \tau^*(\chi_\rho) = \chi_\rho \), we have the equality \( tr \rho(\tau^j(\tilde{x}_i)) = tr \rho(\tilde{x}_i) \). Hence \( \rho(\tau^j(\tilde{x}_i)) \) is either \( \rho(\tilde{x}_i) \) or \( \rho(\tilde{x}_i)^{-1} \). Set \( \rho(\tilde{x}_i) \) and \( \rho(\tau^j(\tilde{x}_i)) \) as

\[
\rho(\tilde{x}_i) = \begin{pmatrix} r_i e^{\sqrt{-1} \theta_i} & 0 \\ 0 & r_i^{-1} e^{-\sqrt{-1} \theta_i} \end{pmatrix}, \quad \rho(\tau^j(\tilde{x}_i)) = \rho(\tilde{x}_i)^{\epsilon_j} \quad (\epsilon_j = \pm 1).
\]

Since \( \rho \) factors through \( H_1(B_n; \mathbb{Z}) \), the relation \(-Ux_j + Vx_{j+1} \) in \( H_1(B_n; \mathbb{Z}) \) gives us the following relation:

\[
\begin{pmatrix} \log r_1 \\ \vdots \\ \log r_{2g} \end{pmatrix} = 0, \quad \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_{2g} \end{pmatrix} \equiv 0 \pmod{2\pi \mathbb{Z}}.
\]

(18)
If there exists some $j$ such that $\epsilon_{j+1} = \epsilon_j$, then it follows by the relation (18) and $\det(-U + V) = \Delta_K(1) = \pm 1$ that $\rho(\tilde{x}_i) = 1$ for all $i$, i.e., $\rho$ is trivial. If the equality $\epsilon_{j+1} = -\epsilon_j$ holds for all $j$, then we have $\tau^* \rho = C \rho C^{-1}$ where $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus $\rho$ is $\tau$-equivariant in the both case. Therefore $\operatorname{Im} \hat{\Phi}$ coincides with the fixed point set $X(B_n)^\tau$. \hfill \Box

In the case of SU(2)-representations, such a correspondence like $\Phi$ has been considered by Collin and Saveliev [7]. They deal with irreducible SU(2)-representations mainly. Here we also draw attention to reducible representations of $\pi_1(B_n)$. Lin presentations lead us to the property of $\Phi$ for reducible representations.

**Proposition 8** In the image of $\Phi$, all reducible representations are abelian, i.e.,

$$\operatorname{Im} \Phi \cap \operatorname{R}^{\text{red}}(B_n) = \operatorname{Im} \Phi \cap \operatorname{R}^{\text{ab}}(B_n).$$

**Proof** It is obvious that any abelian representation is a reducible one. We must show that the left hand side of Eq. (19) is contained in the right hand side.

By Proposition 6, $\operatorname{Im} \Phi \cap \operatorname{R}^{\text{red}}(B_n)$ coincides with $\operatorname{R}^\tau(B_n) \cap \operatorname{R}^{\text{red}}(C_n)$ in $\operatorname{R}(C_n)$. So we can assume that $\rho$ is an element in $\operatorname{R}^{\text{red}}(C_n)$ such that $\rho(\mu_n) = I$ and $C$ is an SL$_2(\mathbb{C})$-element satisfying $\tau^* \rho = A \rho C$. We use the presentation of $\pi_1(C_n)$ as in Lemma 3. Since $\rho$ is reducible, by taking conjugation, we can assume that

$$\rho(\tau^j(\tilde{x}_i)) = \begin{pmatrix} \lambda_{i,j} & \kappa_{i,j} \\ 0 & \lambda_{i,j}^{-1} \end{pmatrix}$$

for all $i$ and $j$. If $\lambda_{i,j}$ is $\pm 1$ for all $i$ and $j$, then $\operatorname{Im} \rho$ is contained in the maximal abelian subgroup Para. Hence $\rho$ is abelian.

We consider the case that there exists a pair $(i, j)$ with $\lambda_{i,j} \neq \pm 1$. We can assume that $\rho(\tau^j(\tilde{x}_i))$ is a diagonal matrix by taking conjugation of an upper triangular matrix. Since $\rho$ is $\tau$-equivariant and reducible, $\rho(\tau^j(\tilde{x}_i)) = C \rho(\tau^j(\tilde{x}_i)) C^{-1}$ is also an upper triangular matrix. Writing $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have that $cd = 0$. If $c$ were zero, then we would have the following contradiction. When we set $\rho_0(\mu) = C$ and $\rho_0(x_i) = \rho(\tilde{x}_i)$, the correspondence $\rho_0$ defines an SL$_2(\mathbb{C})$-representation of $\pi_1(E_K)$. Moreover since the image of $\rho_0$ consists of upper triangular matrices, the representation $\rho_0$ is reducible. Hence for every commutator $\gamma$ in $\pi_1(E_K)$, the trace of $\rho_0(\gamma)$ is $\pm 2$. Applying this to the commutator $x_i$, we have that $\operatorname{tr} \rho_0(x_i) = \lambda_{i,j} + \lambda_{i,j}^{-1} = \pm 2$. This is a contradiction to our assumption that $\lambda_{i,j} \neq \pm 1$.

Now we have $d = 0$. Furthermore since $\rho(\tau^{j+h}(\tilde{x}_i)) = C^h \rho(\tau^j(\tilde{x}_i)) C^{-h}$ must be upper triangular, it follows that the matrix $C$ turns into $\begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$. By reducibility of $\rho$, the matrix $C \rho(\tau^j(\tilde{x}_k)) C^{-1}$ is also an upper triangular matrix. Therefore every $\kappa_{k,l}$ must be zero. Then the image of $\rho$ is contained in the maximal abelian subgroup Hyp, i.e., $\rho$ is also abelian in this case. This completes the proof. \hfill \Box
Proposition 9  The preimage $\Phi^{-1}(R_{\text{red}}(B_n))$ is expressed as

$$\Phi^{-1}(R_{\text{red}}(B_n)) = R_{\text{meta}}^{\text{meta}}(E_K) \cap D(\Phi).$$

Proof  By Proposition 8, it is enough to show that the preimage $\Phi^{-1}(R_{\text{ab}}(B_n))$ coincides with $R_{\text{meta}}^{\text{meta}}(E_K) \cap D(\Phi)$. Let $\rho$ be an element in $\Phi^{-1}(R_{\text{ab}}(B_n))$. By taking conjugation, we can assume that the image $\Phi(\rho)$ is contained in either the maximal abelian subgroups Hyp or Para. We choose a Lin presentation of $\pi_1(E_K)$ as in Lemma 2. In the case that $\text{Im} \Phi(\rho)$ is contained in Hyp, for all $i$ the matrix $\rho(x_i)$ is expressed as

$$\rho(x_i) = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix}.$$ 

If all $\lambda_i$ are $\pm 1$, then $\rho$ is abelian. We consider the case that there exist some $\lambda_i$ such that $\lambda_i \neq \pm 1$. Since $\Phi(\rho)(\tau(\tilde{x}_i)) = \pm \rho(\mu) \rho(x_i) \rho(\mu)^{-1}$ and $\Phi(\rho)(\tau(\tilde{x}_i))$ is also upper triangular, the matrix $\rho(\mu)$ is expressed as either

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}.$$ 

In both cases, the representation $\rho$ turns into metabelian (refer to Eq. (8)).

In the case that $\text{Im} \Phi(\rho)$ is contained in Para, for all $i$, $\rho(x_i)$ is expressed as

$$\rho(x_i) = \pm \begin{pmatrix} 1 & \kappa_i \\ 0 & 1 \end{pmatrix}.$$ 

If all $\kappa_i$ are zero, then $\rho$ is abelian. We consider the case that there exists some non-zero $\kappa_i$. Since $\Phi(\rho)(\tau(\tilde{x}_i)) = \pm \rho(\mu) \rho(x_i) \rho(\mu)$ is also a parabolic element, the matrix $\rho(\mu)$ must be upper triangular. Therefore $\rho$ is reducible, in particular, it is metabelian.

On the other hand, by the proof of Proposition 2 and Remark 7, every metabelian representation is conjugate to either

$$\rho(x_i) = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix}, \quad \rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

or

$$\rho(x_i) = \begin{pmatrix} 1 & \omega_i \\ 0 & 1 \end{pmatrix}, \quad \rho(\mu) = \begin{pmatrix} \lambda & \kappa \\ 0 & \lambda^{-1} \end{pmatrix}.$$ 

It follows from the definition of $\Phi$ that all metabelian representations in $D(\Phi)$ are sent to abelian ones of $\pi_1(B_n)$.

\[\square\]
6 On the geometry of the slice $S_0(E_K)$

In the subsequent section, we focus on knots in $S^3$ mainly to see the interesting structure of the slice $S_0(E_K)$ through the map $\hat{\Phi}$. As described below, when $\hat{\Phi}$ is surjective, it describes the slice $S_0(E_K)$ as the two-fold branched cover over the $\text{SL}_2(\mathbb{C})$-character variety of $B_2$ with branched locus given by the abelian characters, whose preimage is precisely the set of metabelian characters. Note that the results in this section can be naturally extended to the case of an integral homology 3-sphere with some modifications.

First we describe the representation spaces $R_0(E_K)$ and $R^\tau(B_2)$ in detail.

**Lemma 20** If $\rho \in R_0(E_K)$, then $\rho(\mu)^2 = -1$. Therefore the domain of $\Phi$ is given by the union $R_0(E_K) \cup R(S^3) \cup \iota(R(S^3)) = R_0(E_K) \cup \{\text{trivial rep.}\} \cup \{\iota(\text{trivial rep.})\}$. Moreover the image of $\Phi$ coincides with that of the restriction on $R_0(E_K)$.

**Proof** This follows from the Cayley–Hamilton identity and Remark 12. The equality $\text{Im } \Phi = \text{Im } \Phi|_{R_0(E_K)}$ can be deduced from the fact that $R(S^3)$ and $\iota(R(S^3))$ consist of only the trivial representation and its image by $\iota$, respectively. It follows from Lemma 12 that they are sent to the trivial representation of $\pi_1(B_2)$ by $\Phi$ and all abelian representations in $R_0(E_K)$ are also sent to the trivial one of $\pi_1(B_2)$. This shows that $\text{Im } \Phi = \text{Im } \Phi|_{R_0(E_K)}$. \qed

By Lemma 20, in the case of knots in $S^3$, the domain $D(\Phi)$ is reduced to the slice $R_0(E_K)$. This makes the properties on $\Phi$ and $R_0(E_K)$ much clearer. For example, we obtain the next proposition by Propositions 6, 8, 9 and Lemma 12.

**Proposition 10** The map $\Phi$ from $R_0(E_K)$ onto $R^\tau(B_2)$ makes the following correspondence. For an element $\rho$ in $R_0(E_K)$,

1. If $\rho$ is abelian, then $\Phi(\rho)$ is the trivial representation of $\pi_1(B_2)$;
2. If $\rho$ is irreducible and metabelian, then $\Phi(\rho)$ is a non-trivial abelian representation in $R^\tau(B_2)$; and
3. If $\rho$ is non-metabelian, in particular irreducible, then $\Phi(\rho)$ is an irreducible representation in $R^\tau(B_2)$.

Recall that the map $\Phi$ induces the map $\hat{\Phi} : S_0(E_K) \to X(B_2)$ defined by $\hat{\Phi}(\chi_\rho) := \chi_{\Phi(\rho)}$ (see also Eq. (15)). This gives us the following theorem, which is one of the main results in this article.

**Theorem 1** The image of $S_0(E_K)$ by $\hat{\Phi}$ coincides with the subset $X(B_2)^\tau$ defined by $\chi_{\Phi(\rho)} \in X(B_2) | \rho : \tau$-equivariant}. The map $\hat{\Phi} : S_0(E_K) \to X(B_2)^\tau$ is one-to-one correspondence on the fixed point set $S_0(E_K)^\tau$ and two-to-one correspondence on $S_0(E_K) \setminus S_0(E_K)^\tau$. Moreover two points in the preimage of a point in $X(B_2)^\tau$ are interchanged by the involution $\hat{\tau}$ on $S_0(E_K)$.

The rest of this section is devoted to proving Theorem 1.

**Lemma 21** If $\rho$ is an element in $D(\Phi)$, then the map $\Phi$ sends both $\rho$ and $\iota(\rho)$ to the same $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(B_2)$, i.e., $\Phi(\rho) = \Phi(\iota(\rho))$. Moreover, this equality gives $\hat{\Phi}(\chi_\rho) = \hat{\Phi}(\chi_{\iota(\rho)})$. 

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Proof The map $\Phi$ is constructed by the action $\iota$ on $R(C_2)$ and the pull-back $p^*$. For each element $\rho$ in $R_0(E_K)$, the $\text{SL}_2(\mathbb{C})$-representation $\Phi(\rho)$ is given by $\iota(p^*\rho)$ and $\Phi(\iota(\rho))$ by $\iota(p^*\iota(\rho))$. Moreover it follows from $p_*(H_1(C_2;\mathbb{Z})) \simeq 2\mathbb{Z}$ in $H_1(E_K;\mathbb{Z}) \simeq \mathbb{Z}$ that for any $g$ in $\pi_1(C_2)$
\[ p^*\iota(\rho)(g) = (-1)^{p_*(|g|)} p^*\rho(g) = p^*\rho(g). \]
Hence $\Phi(\iota(\rho))$ coincides with $\Phi(\rho)$. This equality also holds for the induced map $\widehat{\Phi}$. 

By Proposition 3, the fixed point set $S_0(E_K)\hat{\iota}$ in $S_0(E_K)$ consists entirely of metabelian characters, which are in $R_0(E_K)$.

**Proposition 11** The restriction of $\widehat{\Phi}$ on $S_0(E_K)\hat{\iota}$ gives a bijection between $S_0(E_K)\hat{\iota}$ and all abelian characters of $\pi_1(B_2)$.

**Proof** By Proposition 3 and Lemma 9, we focus on metabelian characters. By Propositions 8 and 9, the map $\hat{\iota}$ sends the metabelian characters of $\pi_1(E_K)$ onto the abelian characters of $\pi_1(B_2)$. By Propositions 2 and 4, the number of metabelian characters is equal to $(|\Delta_K(-1)| - 1)/2 + 1$ where $\Delta_K(t)$ is the Alexander polynomial of $K$. On the other hand, we see that the number of abelian characters of $\pi_1(B_2)$ is also equal to $(|\Delta_K(-1)| - 1)/2 + 1$ as follows.

Every abelian representation factors through the abelianization $H_1(B_2;\mathbb{Z})$. Since the order of $H_1(B_2;\mathbb{Z})$ is finite (an odd integer $|\Delta_K(-1)|$), $H_1(B_2;\mathbb{Z})$ is decomposed into the direct sum of some finite cyclic groups. The character of an abelian representation is determined by the traces for generators of these cyclic groups. By conjugation, the images of generators of cyclic groups are given by the diagonal matrices whose diagonal components are roots of unity (because the order of each cyclic group is finite). Combining these facts, we can show that the number of abelian characters is given by $(|\Delta_K(-1)| - 1)/2 + 1$. Therefore the restriction of $\widehat{\Phi}$ on $S_0(E_K)\hat{\iota}$ gives a one-to-one correspondence. 

**Proposition 12** The restriction of $\widehat{\Phi}$ to $S_0(E_K) \setminus S_0(E_K)\hat{\iota}$ gives a two-to-one correspondence.

**Proof** It is sufficient to show that if $\rho$ and $\rho'$ are two non-metabelian representations of $\pi_1(E_K)$ satisfying $\hat{\Phi}(\chi_\rho) = \hat{\Phi}(\chi_{\rho'})$, then $\chi_\rho$ coincides with either $\chi_{\rho'}$ or $\hat{\iota}(\chi_\rho)$. By Proposition 10, the representations $\Phi(\rho)$ and $\Phi(\rho')$ are irreducible. Since $\Phi(\rho)$ and $\Phi(\rho')$ have the same characters, by Culler and Shalen [9, Proposition 1.5.2], these two representations are conjugate, i.e., there exists an $\text{SL}_2(\mathbb{C})$-element $P$ such that
\[ \Phi(\rho') = P\Phi(\rho)P^{-1}. \] (20)

We use the presentations of $\pi_1(E_K)$ and $\pi_1(B_2)$ as in Lemma 6. By the construction of $\Phi$, the matrix $\Phi(\rho)(\overline{\chi}_i)$ is expressed as
\[ \Phi(\rho)(\overline{\chi}_i) = \rho(\chi_i). \] (21)
Combining Eq. (21) with (20), we have the equality

$$\rho'(x_i) = P \rho(x_i) P^{-1}. \quad (22)$$

Similarly, the matrix $\Phi(\rho)(\tau(\tilde{x}_i))$ is expressed as

$$\Phi(\rho)(\tau(\tilde{x}_i)) = \rho(\mu) \rho(x_i) \rho(\mu)^{-1} = Ad_{\rho(\mu)}(\Phi(\rho)(\tilde{x}_i)). \quad (23)$$

Combining Eq. (23) with (20), we have the equality

$$\Phi(\rho')(\tau(\tilde{x}_i)) = Ad_{P \rho(\mu) P^{-1}}(\Phi(\rho')(\tilde{x}_i)).$$

The matrix $\Phi(\rho)(\tau(\tilde{x}_i))$ is also expressed as $Ad_{\rho'(\mu)}(\Phi(\rho')(\tilde{x}_i))$. By Lemma 13, we obtain the equality

$$\rho'(\mu) = \pm P \rho(\mu) P^{-1}. \quad (24)$$

From the relations that $\iota(\rho)(x_i) = \rho(x_i)$ and $\iota(\rho)(\mu) = -\rho(\mu)$ and Eqs. (22) and (24), we can conclude that $\rho'$ is conjugate to either $\rho$ or $\iota(\rho)$ by $P$.

**Proof of Theorem 1** Combining Lemma 21, Propositions 11 and 12, we obtain Theorem 1.

Moreover $\tau$-equivariant representations subspace $R^\tau(B_2)$ have the following distinct property from the cases other than $n = 2$ as mentioned at the beginning of Sect. 5 (compare to Proposition 13).

**Lemma 22** The set $R^\tau(B_2)$ contains $R^{ab}(B_2)$.

**Proof** Let $\rho$ be an abelian $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(B_2)$. Since this representation factors through $H_1(B_2; \mathbb{Z})$ which is a finite abelian group with the order $|\Delta_K(-1)|$. By taking conjugation, the image Im $\rho$ is contained in the maximal abelian subgroup Hyp. So we can set $\rho(\tilde{x}_i)$ as

$$\rho(\tilde{x}_i) = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix}.$$  

By Lemma 6, we have $[\tilde{x}_i] + [\tau(\tilde{x}_i)] = 0$ in $H_1(B_2; \mathbb{Z})$ for all $i$. Hence $\rho(\tau(\tilde{x}_i))$ is given by $\rho(\tilde{x}_i)^{-1}$.

If we set $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then the diagram (16) becomes commutative. Therefore $\rho$ is $\tau$-equivariant.

**Remark 14** Lemma 22 does not hold for three-fold branched covers of $S^3$. As shown below, for the trefoil knot in $S^3$ there exists an abelian representation of $\pi_1(\Sigma_3)$ which is not $\tau$-equivariant.
Proposition 13 Let \( K \) be the left-handed trefoil knot in \( S^3 \). The set \( R^{ab}(\Sigma_3) \) is not contained in \( R^{\tau}(\Sigma_3) \).

To show Proposition 13, we consider regular Seifert surface \( S \) of the left-handed trefoil depicted in Fig. 4. We fix a spine of \( S \) as in Fig. 4. Then the Seifert matrix is given by the following matrix

\[
Q = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.
\]

Now we consider the Heegaard splitting \( S^3 \setminus N(S) \cup N(S) \) associated to \( S \) as in Fig. 5. Each attaching circle of 2-handles intersects with the boundaries of meridian disks of 1-handles at only one point in this Heegaard splitting. In this setting, the boundary operator from \( C^\text{Morse}_2(S^3, \mathbb{Z}) \) to \( C^\text{Morse}_1(S^3, \mathbb{Z}) \) is expressed as the identity matrix (see Remark 4). The relations in the Lin presentation for \( S \) are given by

\[
\alpha_1 = x_1x_2^{-1}, \quad \alpha_2 = x_2, \quad \beta_1 = x_1, \quad \beta_2 = x_1^{-1}x_2.
\]

Hence, by Lemma 6, a presentation of \( \pi_1(\Sigma_3) \) is given by

\[
(\bar{x}_1, \bar{x}_2, \tau(\bar{x}_1), \tau(\bar{x}_2), \tau^2(\bar{x}_1), \tau^2(\bar{x}_2) | \bar{\alpha}_1^{(k)} = \bar{\beta}_1^{(k-1)}, \bar{\alpha}_2^{(k)} = \bar{\beta}_2^{(k-1)} (k \text{ mod } 3))
\]

This presentation is reduced as

\[
(\bar{x}_1, \bar{x}_2 | \bar{x}_1\bar{x}_2^{-1} = \bar{x}_2\bar{x}_1, \bar{x}_2\bar{x}_1 = \bar{x}_1^{-1}\bar{x}_2).
\]

About another method to obtain this presentation, for example see [22, Chapter 10].
Proof of Proposition 13 By Lemma 6, the homology group $H_1(\Sigma_3; \mathbb{Z})$ is presented as

$$\mathbb{Z}^6 \xrightarrow{A} \mathbb{Z}^6 \rightarrow H_1(\Sigma_3; \mathbb{Z}),$$

where

$$A = \begin{pmatrix} tQ & -Q & 0 \\ 0 & tQ & -Q \\ -Q & 0 & tQ \end{pmatrix}.$$

If we set

$$\begin{cases} 
\rho(\tilde{x}_1) = 1, & \rho(\tau(\tilde{x}_1)) = -1, & \rho(\tau^2(\tilde{x}_1)) = -1, \\
\rho(\tilde{x}_2) = -1, & \rho(\tau(\tilde{x}_2)) = 1, & \rho(\tau^2(\tilde{x}_2)) = -1,
\end{cases}$$

then this correspondence defines an abelian representation of $\pi_1(\Sigma_3)$. Since the trace of $\rho(\tilde{x}_1)$ is not equal to that of $\rho(\tau(\tilde{x}_1))$, there does not exist any $SL_2(\mathbb{C})$-element such that the diagram (16) commutes. \qed

7 Applications

In the previous section, we have investigated the correspondence $\hat{\Phi}$ between $S_0(E_K)$ and $X(B_2)$. In particular, Lemma 22 shows that $R^{ab}(B_2) \subset R^\tau(B_2)$. In this section, we look into surjectivity of $\hat{\Phi}: S_0(E_K) \rightarrow X(B_2)$ for two-bridge knots and pretzel knots of type $(p, q, r)$. Namely, we show that $R^\tau(B_2)$ contains all $SL_2(\mathbb{C})$-representations $R(B_2)$.

As regards two-bridge knots, it is well-known that the two-fold branched cover $B_2$ along a two-bridge knot is a three-dimensional lens space. Since the fundamental group of a lens space is cyclic, the $SL_2(\mathbb{C})$-representation space for a lens space consists entirely of abelian representations. By Theorem 1 and Proposition 10, the slice $S_0(E_K)$ consists of metabelian characters of $\pi_1(E_K)$. It is known that the order of $H_1(B_2; \mathbb{Z})$ is given by $|\Delta_K(-1)|$. Thus we have proved the following lemma, which follows originally from the proof of Theorem 1.3 in [18].

**Lemma 23** For a two-bridge knot $K$, the slice $S_0(E_K)$ coincides with the fixed point set $S_0(E_K)^\hat{\Phi}$ and it can be identified with the set of $((|\Delta_K(-1)| - 1)/2 + 1)$ conjugacy classes of the $SL_2(\mathbb{C})$-metabelian representations of $\pi_1(E_K)$.

**Remark 15** We do not have to consider reducible and non-abelian representations in Lemma 23 since there exist no such representations in $R_0(E_K)$. This is due to Remark 10.

Another way of stating Lemma 23 is to say that if $B_2$ is a lens space, then the slice $S_0(E_K)$ coincides with the fixed point set $S_0(E_K)^\hat{\Phi}$. This means that the difference $S_0(E_K) \setminus S_0(E_K)^\hat{\Phi}$ is an obstruction for $B_2$ to be a lens space and thus for $K$ to be a two-bridge knot.
Theorem 2  If $S_0(E_K) \setminus S_0(E_K)^\mathbb{Z} \neq \emptyset$, then $B_2$ is not a lens space. In particular, $K$ is not a two-bridge knot.

Remark 16  The similar statement also holds for a knot in an integral homology 3-sphere.

Here we give explicit representatives in conjugacy classes of metabelian representations by using Riley’s construction \[21\] and the correspondence $\Phi$. We choose the following presentation of a two-bridge knot group as

$$\langle x, y \mid wx = yw \rangle,$$

where the meridians $x$ and $y$ are depicted in Fig. 6 and $w$ is a word in $x$ and $y$. We will show the following theorem.

Theorem 3  Let $K$ be a two-bridge knot and $p$ the determinant of $K$, i.e., $p = |\Delta_K(-1)|$. Then the slice $S_0(E_K)$ is identified with the following set of conjugacy classes of $SL_2(\mathbb{C})$-representations:

$$\{[\rho_k] \mid k = 1, \ldots, (p - 1)/2\} \cup \{[\varphi_{\sqrt{-1}}]\},$$

where the representation $\rho_k$ is given by

$$x \mapsto \begin{pmatrix} \sqrt{-1} & -\sqrt{-1} \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad y \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ -\sqrt{-1}(e^{k\pi\sqrt{-1}/p} - e^{-k\pi\sqrt{-1}/p})^2 & -\sqrt{-1} \end{pmatrix},$$

and $\varphi_{\sqrt{-1}}$ is the abelian representation given by

$$\varphi_{\sqrt{-1}}(\mu) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

Riley had shown a construction of non-abelian $SL_2(\mathbb{C})$-representations of two-bridge knot groups in \[21\] by using a polynomial equation $\phi_K(t, u) = 0$. His construction gives every representative in each conjugacy class of a non-abelian representation. Note that the irreducible metabelian representations are given by using roots of $\phi_K(-1, u) = 0$. We also obtain the next statement about $\phi_K(-1, u)$.
**Theorem 4** We keep notations in Theorem 3. The polynomial \( \phi_K(-1, u) \) has distinct \((p - 1)/2 \) roots \( \{(e^{k\pi \sqrt{-1}/p} - e^{-k\pi \sqrt{-1}/p})^2 \mid k = 1, \ldots, (p - 1)/2 \} \). Namely, \( \phi_K(-1, u) \) is expressed as

\[
\phi_K(-1, u) = (-1)^{(p-1)/2} \prod_{k=1}^{(p-1)/2} \left\{ u - (e^{k\pi \sqrt{-1}/p} - e^{-k\pi \sqrt{-1}/p})^2 \right\}.
\]

Note that a generator of \( \pi_1(B_2) \) in \( S^3 \) is illustrated as in Fig. 6 (for details about the generator, see [6, Chapter 12]).

**Proof of Theorem 3** We focus on the non-abelian part in \( S_0(E_K) \). By Theorem 1 in [21], every conjugacy class of non-abelian \( \text{SL}_2(\mathbb{C}) \)-representations has a representative such that:

\[
\rho(x) = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \sqrt{1/t} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \sqrt{t} & 0 \\ u\sqrt{t} & 1/\sqrt{t} \end{pmatrix},
\]

where \( t \) and \( u \) satisfy the equation that \( \phi_K(t, u) = 0 \). We now consider the case that \( t = -1 \) to describe elements in \( S_0(E_K) \). We can take a generator \( \gamma \) of the cyclic group \( \pi_1(B_2) \) so that \( p_*(\gamma) = xy^{-1} \) holds. The image of the homology class \( [\gamma] \) by \( p_* \) is null-homologous. Then \( \Phi(\rho)(\gamma) \) is expressed as follows:

\[
\Phi(\rho)(\gamma) = \left( \sqrt{-1} \right)^{p_*[\gamma]} \cdot \rho(p_*(\gamma)) = \rho(xy^{-1}) = \begin{pmatrix} u + 1 & 1 \\ u & 1 \end{pmatrix}.
\]

By Theorem 1, the representation \( \Phi(\rho) \) is abelian. Every abelian representation of \( \pi_1(B_2) \) is determined by the eigenvalues for the generator \( \gamma \). Since \( \pi_1(B_2) \) is the cyclic group with order \( p \), the matrix \( \Phi(\rho)(\gamma) \) is conjugate to

\[
\begin{pmatrix}
e^{2\pi \sqrt{-1}/p} & 0 \\
0 & e^{-2\pi \sqrt{-1}/p} \end{pmatrix}.
\]

Comparing the traces of \( \Phi(\rho)(\gamma) \) and the above diagonal matrix, we have that

\[
u \equiv \left( e^{\pi \sqrt{-1}/p} - e^{-\pi \sqrt{-1}/p} \right)^2.
\]

Since \( \Phi \) gives a one-to-one correspondence on metabelian characters, we can conclude the statement. \( \square \)

**Proof of Theorem 4** If we fix a choice of square root of \(-1\), the roots of \( \phi_K(-1, u) \) correspond to the representatives of non-abelian part in \( S_0(E_K) \) by one-to-one.
From [21], the highest degree of $u$ in $\phi_K(t, u)$ is given by the determinant of $K$, i.e., $p = |\Delta_K(-1)|$. Moreover it follows from the proof of [21, Lemma 2] that the coefficient of leading term on $u$ of $\phi_K(t, u)$ is equal to $(-1)^{(p-1)/2}$. By the degree of $u$ and Theorem 3, all roots of $\phi_K(-1, u)$ are given precisely by $(p - 1)/2$ distinct real numbers:

$$\{(e^{k\pi\sqrt{-1}/p} - e^{-k\pi\sqrt{-1}/p})^2 \mid k = 1, \ldots, (p - 1)/2\}.$$ 

This completes the proof. \hfill \Box

**Example** Let $K$ be the trefoil knot. Then $\phi_K(t, u)$ is given by $u - t - t^{-1} + 1$. The root of $\phi_K(-1, u)$ is $u = -3$. The corresponding representation $\rho$ is expressed as

$$\rho(x) = \begin{pmatrix} \sqrt{-1} & -\sqrt{-1} \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \sqrt{-1} & 0 \\ 3\sqrt{-1} & -\sqrt{-1} \end{pmatrix}.$$ 

Then the matrix $\Phi(\rho)(xy^{-1})$ is given by $\begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix}$. Direct calculations show that $\Phi(\rho)(xy^{-1})^3 = 1$. Since $|\Delta_K(-1)| = 3$, the formula in Theorem 4 turns into

$$u - \left( e^{\pi\sqrt{-1}/3} - e^{-\pi\sqrt{-1}/3} \right)^2 = u + 3.$$ 

This polynomial coincides with $\phi_K(-1, u)$.

Lemma 23 and Theorem 1 show that the map $\hat{\Phi}$ is bijective for all two-bridge knots. Moreover the following holds on the surjectivity of $\hat{\Phi}$.

**Proposition 14** If $K$ is a pretzel knot of type $(p, q, r)$, then the map $\hat{\Phi}$ from $S_0(E_K)$ to $X(B_2)$ is surjective.

The key to Proposition 14 is that the two-fold branched cover $B_2$ along a pretzel knot of type $(p, q, r)$ is the Brieskorn manifold of type $(p, q, r)$. The fundamental group of this Brieskorn manifold has a presentation with four generators $s_1, s_2, s_3$ and $h$ (a central element) as that of a Seifert manifold (for more details, see [6, Chapter 12] and [24]). In fact, it is more convenient to work with another set of generators $t_1 = s_1, t_2 = s_1s_2$ and $h$ when we consider the induced action on the fundamental group by the covering transformation. We use this idea to prove Proposition 14.

**Proof** The character variety $X(B_2)$ is expressed as the union $X^{ab}(B_2) \cup X^{irr}(B_2)$. We will show that each part is contained in $\text{Im} \hat{\Phi}$. By Proposition 11, $X^{ab}(B_2)$ is contained in $\text{Im} \hat{\Phi}$. Hence it suffices to show that $X^{irr}(B_2)$ is contained in $\text{Im} \hat{\Phi}$. Since $\text{Im} \hat{\Phi}$ is $X(B_2)^r$, we check the inclusion $X^{irr}(B_2) \subset X(B_2)^r$. For $\chi_\rho \in X^{irr}(B_2)$, we have the following equivalence relations:

$$\tau^*(\chi_\rho) = \chi_\rho \iff \chi_{\tau^*\rho} = \chi_\rho$$

$$\iff \rho \overset{\text{conj}}{\sim} \tau^* \rho \iff \rho \in R^r(B_2).$$
So, to complete the proof, it is enough to show that every irreducible representation is \( \tau \)-equivariant. The fundamental group \( \pi_1(B_2) \) has the following presentation:

\[
\langle s_1, s_2, s_3, h \mid s_1^p h = 1, s_2^q h = 1, s_3^r h = 1, [s_i, h] = 1 \ (1 \leq i \leq 3), s_1 s_2 s_3 = 1 \rangle.
\]

The action of \( \tau \) on the generators \( s_1, s_2, s_3, h \) is expressed as follows:

\[
\tau : h \mapsto h, \quad s_1 \mapsto s_1^{-1}, \quad s_2 \mapsto s_1 s_2^{-1} s_1^{-1}, \quad s_3 \mapsto s_1 s_2 s_3^{-1} s_1^{-1} s_1^{-1}.
\]

We set \( t_1 \) and \( t_2 \) as \( t_1 = s_1 \) and \( t_2 = s_1 s_2 \). Then three elements \( h, t_1 \) and \( t_2 \) generate the group \( \pi_1(B_2) \). For the new generators, the action of \( \tau \) is expressed as

\[
\tau : h \mapsto h, \quad t_1 \mapsto t_1^{-1}, \quad t_2 \mapsto t_2^{-1}.
\]

Since \( \rho \) is irreducible, the image \( \text{Im} \rho \) is a non-abelian subgroup in \( SL_2(\mathbb{C}) \). Hence the central element \( \rho(h) \) in \( \text{Im} \rho \) is \( \pm 1 \). By the relation \( t_1^p h = 1 \), the matrix \( \rho(t_1) \) has a finite order, in particular it is hyperbolic. By taking conjugation, we can suppose that the irreducible representation \( \rho \) is given by

\[
\rho(h) = \pm 1, \quad \rho(t_1) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad \rho(t_2) = \begin{pmatrix} s & t \\ u & v \end{pmatrix},
\]

where \( u t \neq 0 \). We set a complex number \( \delta \) as \( \delta^2 = \sqrt{-u/t} \). Moreover, taking a conjugate by the matrix \( \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} \), we can assume that \( \rho(t_2) \) is given by \( \begin{pmatrix} s & t \\ -t & v \end{pmatrix} \) at the beginning.

For a unit quaternion \( k = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \), we have

\[
k \rho(h) k^{-1} = \rho(h), \quad k \rho(t_1) k^{-1} = \rho(t_1)^{-1}, \quad k \rho(t_2) k^{-1} = \rho(t_2)^{-1}.
\]

Therefore it follows that \( \tau^* \rho \) coincides with \( k \rho k^{-1} \), i.e., \( \rho \) is \( \tau \)-equivariant. This completes the proof. \( \square \)

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