Using sophisticated string theory calculations, Maldacena and Susskind have intriguingly shown that near-extremal black holes are characterized by a finite mass gap above the corresponding zero-temperature (extremal) black-hole configuration. In the present compact paper we explicitly prove that the minimum energy gap $E_{\text{gap}} = \frac{\hbar^2}{M^3}$, which characterizes the mass spectra of near-extremal charged Reissner-Nordström black holes, can be inferred from a simple semi-classical analysis.

I. INTRODUCTION

Extremal black holes are of fundamental importance in candidate theories of quantum gravity since they mark the boundary between black-hole configurations and horizonless naked singularities [1–9]. These unique solutions of the Einstein field equations, which are characterized by the minimally allowed mass (radius) for a given amount of black-hole charge or angular momentum [10], have attracted much attention from physicists and mathematicians during the last five decades.

In particular, extremal black holes play a key role in various types of gedanken experiments that have been designed to challenge the Penrose cosmic censorship conjecture [1–9, 11, 12]. Likewise, extremal black holes play a fundamental role in various attempts to prove the validity of the intriguing weak gravity conjecture [13–21]. In addition, the thermodynamic and statistical properties of extremal black holes have been extensively studied by many physicists in order to gain some insights on the origin of the Bekenstein-Hawking entropy [22–28].

One of the most intriguing predictions of string theory is that near-extremal black holes are characterized by a finite (non-zero) energy gap of the first excited state above the extremal (zero-temperature) configuration [29]. In particular, it has been proved [30] that charged Reissner-Nordström (RN) black holes are characterized by the finite mass gap $E_{\text{gap}} = \frac{\hbar^2}{M^3}$; $M_{\text{extremal}} = Q$

of the first excited state above the extremal $M_{\text{extremal}} = Q$ black-hole configuration.

It is important to emphasize that the authors of the physically interesting work [30] have derived the suggested energy gap [11] using a physical argument which is based on holography without using explicit string theory calculations. In the present paper we shall argue that an energy gap of the form [11] arises naturally in a gedanken experiment that involves neither string theory calculations nor holography.

The main goal of the present compact paper is to gain some physical insights on the origins of the physically intriguing mass gap [11]. In particular, below we shall demonstrate that the interesting mass gap [11] can be inferred from a simple semi-classical gedanken experiment [we would like to stress the fact that we do not rule out the possible existence of an energy gap for extremal black holes which may be smaller than the suggested gap [11]]. To this end, we shall analyze the physical and mathematical properties of composed extremal-RN-black-hole-massive-shell configurations.

II. DESCRIPTION OF THE SYSTEM

We consider a physical system which is composed of a charged RN black hole of mass $M$ and electric charge $Q$ and a concentric spherically symmetric massive neutral shell of radius $R$ and energy (energy as measured by asymptotic observers) $E(R)$. According to the Birkhoff theorem, the spacetime inside the spherically symmetric shell is described by the curved RN line element [32, 34]

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$


In addition, according to the Birkhoff theorem, the curved spacetime outside the spherically symmetric shell of radius $R$ is described by the curved RN line element

$$ds^2 = -\left\{1 - \frac{2[M + E(R)]}{r} + \frac{Q^2}{r^2}\right\}dt^2 + \left\{1 - \frac{2[M + E(R)]}{r} + \frac{Q^2}{r^2}\right\}^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) .$$  \hspace{1cm} (3)

The (outer and inner) horizon radii of the central charged black hole are characterized by the simple relation

$$r_{\pm} = M \pm (M^2 - Q^2)^{1/2} .$$  \hspace{1cm} (4)

The Bekenstein-Hawking temperature of the black hole is given by the compact expression $^{[35, 36]}$

$$T_{\text{BH}} = \frac{(r_+ - r_-)\hbar}{2\pi^2 r_{\pm}^2} .$$  \hspace{1cm} (5)

In the present paper we shall focus our attention on extremal black holes. These zero-temperature ($T_{\text{BH}} = 0$) charged black-hole configurations, which mark the boundary between black-hole spacetimes and horizonless naked singularities, are characterized by the simple relation

$$Q = M .$$  \hspace{1cm} (6)

As explicitly shown in $^{[6]}$, the energy $E(R)$ of the massive shell as measured by far away asymptotic observers can be deduced by solving the equation of motion

$$\sqrt{g_{\text{in}}(r) + R^2} - \sqrt{g_{\text{out}}(r) + R^2} = -\frac{m}{r}$$  \hspace{1cm} (7)

of the shell in the curved black-hole spacetime, where [see Eqs. (2) and (3) with $Q = M$]

$$g_{\text{in}}(r) = \left(1 - \frac{M}{r}\right)^2 ,$$  \hspace{1cm} (8)

$$g_{\text{out}}(r) = 1 - \frac{2[M + E(R)]}{r} + \frac{M^2}{r^2} ,$$  \hspace{1cm} (9)

and $\dot{R} \equiv dR/d\tau$ $^{[37]}$.

III. THE MINIMAL MASS GAP $\mathcal{E}_{\text{gap}} \equiv M - Q$ OF NEAR-EXTREMAL BLACK HOLES

In the present section we shall use analytical techniques in order to provide supporting evidence for the existence of a finite mass gap,

$$\mathcal{E}_{\text{gap}} \equiv M - Q ,$$  \hspace{1cm} (10)

for a near-extremal RN black hole above the extremal limit $^{[6]}$. To this end, we shall analyze a gedanken experiment in which a shell of proper mass $m$, which is concentric with an extremal charged RN black hole, is lowered towards the central black hole. At some point, the shell would merge with the original extremal black hole to form a larger black hole [see Eq. (14) below], thus increasing the mass (energy) of the central black hole. Below we shall try to minimize the energy increase $^{[10]}$.

In order to determine the minimal mass gap of near-extremal black holes above the extremal limit $^{[6]}$, one should try to deliver to the central extremal black hole the smallest possible (non-zero) amount of energy. We shall therefore lower the massive shell slowly towards the central black hole with an infinitesimally small radial velocity

$$\dot{R} \to 0 .$$  \hspace{1cm} (11)

As we shall explicitly show below, most (but not all) of the mass-energy of the shell as measured by asymptotic observers would be red-shifted by the gravitational field of the central black hole during this adiabatic lowering process. It is important to emphasize that the physical apparatus which is required in order to perform the adiabatic lowering process of the shell towards the black hole would probably need to withstand extremely high tensions $^{[38]}$. 


Thus, it should be realized that the present gedanken experiment is a highly idealized one and technical problems may prevent the actual performance of the experiment.

Taking cognizance of Eqs. (7), (8), (9), and (11), one finds that the total energy (as measured by far away asymptotic observers) of the massive shell in the curved black-hole spacetime is given by the simple functional expression

$$E(R) = m \cdot \left(1 - \frac{M}{R}\right) - \frac{m^2}{2R},$$  \hspace{1cm} (12)

where $R$ is the radius of the shell. The first term on the r.h.s of (12) represents the red-shifted mass-energy of the shell in the curved black-hole spacetime. The second term on the r.h.s of (12) represents the gravitational self-energy (an energy term proportional to $m^2$) of the massive shell.

It is worth noting that if the massive shell could be lowered slowly all the way down to the horizon ($r_H = M$) of the central extremal RN black hole, then the mass term $m \cdot (1 - M/R)$ in (12) would be completely red-shifted in this process [see, in particular, the energy expression (12) with $R = r_H = M$]. However, as we shall now prove explicitly, the adiabatic lowering process of the massive shell towards the central black hole must stop in the regime $R > M$ before the shell reaches the original horizon $r_H = M$ of the central extremal black hole.

In particular, it is easy to verify that the lowering process of the shell towards the central black hole produces a new (and larger) horizon, which is characterized by the relation $r_{NH} > M$, before the massive shell reaches the radius $r_H = M$ of the original black-hole horizon. Taking cognizance of the curved line element (3), one finds that the physical condition for the formation of a new engulfing horizon in the composed black-hole-massive-shell system is given by the characteristic functional expression

$$1 - \frac{2(M + E(R))}{R} + \frac{Q^2}{R^2} = 0 \quad \text{with} \quad Q = M.$$  \hspace{1cm} (13)

Substituting the $R$-dependent energy (12) of the shell into the characteristic black-hole relation (13), one deduces that a new and larger horizon, which engulfs both the original central black hole and the massive shell, is formed in the composed spherically symmetric black-hole-massive-shell system when the radius of the shell reaches the critical value

$$R \rightarrow r_{NH} = M + m.$$  \hspace{1cm} (14)

The radius (14) of the new horizon in the composed black-hole-shell system is larger than the original radius $r_H = M$ of the central extremal black hole.

Before proceeding, it is worth stressing the fact that the analytically derived expression (14) provides the minimum possible radius of the new engulfing horizon that results when a shell of mass $m$ is added to the central extremal RN black hole. In particular, the total energy of a massive shell which has a non-vanishing kinetic energy (a non-vanishing radial velocity) is larger than the one given by the energy expression (12). Hence, a massive shell which falls towards the central extremal black hole with a non-vanishing radial velocity would form an engulfing new horizon which is larger than the one given by the relation (14) [3].

Substituting Eq. (14) into Eq. (12) one finds that, for a given mass $M$ of the original central black hole, the energy gap $\mathcal{E}_{\text{gap}}$ (the minimum energy which is delivered to the extremal black hole due to the assimilation of the massive shell) is given by the remarkably simple functional expression

$$\mathcal{E}_{\text{gap}} = E_{\text{min}}(m; M) = \frac{m^2}{2(M + m)}.$$  \hspace{1cm} (15)

The energy expression (15) implies that the minimum energy $E_{\text{min}}(m; M)$ which is delivered to the extremal RN black hole in the present gedanken experiment can be minimized by minimizing the mass $m$ of the shell. We therefore raise here the following physically important question: How small can the mass $m$ of the shell be?

In order to address this physically interesting question, we can think of the shell as representing a spherically symmetric localized wave packet of a massive particle. It is well known that, according to quantum theory and special relativity, a wave packet representing a particle of mass $m$ cannot be localized to better than its Compton length: $R \geq h/2m$ [10, 11]. This relation implies that the circumference $C = 2\pi R$ of the spherically symmetric wave packet is bounded from below by the relation

$$2\pi R \geq \frac{h}{2m}.$$  \hspace{1cm} (16)

One therefore concludes that a spherically symmetric compact wave packet of circumference radius $R$ representing a particle of mass $m$ is characterized by the lower bound

$$m \geq \frac{h}{2R}.$$  \hspace{1cm} (17)
Since in our case the radius of the shell at the point of assimilation is \( R_{\text{min}} = M + m \) [see Eq. (14)], one finds the simple expression

\[
m_{\text{min}} = \frac{\hbar}{2M} \cdot [1 + O(\hbar/M^2)]
\]  

(18)

for the minimally allowed mass of the shell which is consistent with quantum theory and special relativity. Substituting (18) into (15), one finds the expression

\[
\mathcal{E}_{\text{gap}} = E_{\text{min}} = \frac{\hbar^2}{8M^3}
\]  

(19)

for the energy gap of the central extremal black hole in the regime \( M^2 \gg \hbar \) of large black holes. It is important to emphasize that we cannot rule out the possible existence of a different physical process that may yield an energy gap which may be smaller than the analytically derived energy gap (19).

IV. SUMMARY

Using a sophisticated string theory analysis, Maldacena and Susskind [29] have nicely shown that near-extremal black holes are characterized by a finite mass gap of the first excited state above the extremal (zero-temperature) black-hole configuration. As explicitly proved in the physically interesting work [30], the same energy gap can be inferred using a physical argument which is based on holography without using explicit string theory calculations.

The main goal of the present compact paper was to gain some physical insights on the origin of the intriguing mass gap (1). In particular, we have demonstrated that the suggested finite mass gap (1) can be deduced from a simple semi-classical gedanken experiment in which a spherically symmetric massive shell is lowered towards a central extremal Reissner-Nordström black hole which eventually absorbs the shell. In order to determine the minimum mass gap of the system above the extremal black-hole limit (6), we have tried to minimize the energy which is delivered to the black hole by the shell. We have therefore assumed that the massive shell is lowered adiabatically (slowly) towards the central black hole.

The main results derived in the present paper and their physical implications are as follows:

1. It has been shown that during the lowering process of the shell, most (but not all) of its mass-energy as measured by far away asymptotic observers is red-shifted by the gravitational field of the central black hole. Intriguingly, however, we have explicitly proved that at some point during the lowering process of the shell, a new and larger horizon, which engulfs both the original black hole and the massive shell, is formed. In particular, the engulfing new horizon with \( r_{\text{NH}} > M \) [see Eq. (14)] always forms before the massive shell reaches the horizon \( r_H = M \) of the original extremal black hole.

2. The formation of the new and larger horizon prevents one from totally red-shifting the mass-energy of the lowered shell. Thus, for a given proper mass \( m \) of the shell, there is always a finite (non-zero) lower bound \( E_{\text{min}}(m; M) = m^2/[2(M + m)] \) [see Eq. (15)] on the energy which is delivered to the central extremal black hole.

3. Taking cognizance of the fact that a combination of quantum theory and special relativity sets the lower bound \( R \geq \hbar/2m \) on the proper circumference radius of a shell of proper mass \( m \), we have derived the semi-classical expression

\[
\mathcal{E}_{\text{gap}} = M - Q = \frac{\hbar^2}{8M^3}
\]  

(20)

for the minimum energy which is delivered to the central extremal black hole in the present gedanken experiment.

Interestingly, the results of the present semi-classical gedanken experiment support the validity of the previously derived [29, 20] quantum energy gap (1) of extremal black holes in a wider context than string theory and holography. As emphasized above, our analysis cannot rule out the possible existence of another physical process that may yield an energy gap for extremal black holes which may be smaller than the characteristic gap (20).

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