What is the natural scale for a Lévy process in modelling term structure of interest rates?

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Abstract
This paper gives examples of explicit arbitrage-free term structure models with Lévy jumps via state price density approach. By generalizing quadratic Gaussian models, it is found that the probability density function of a Lévy process is a "natural" scale for the process to be the state variable of a market.

Keywords. State price density approach, term structure models, Shirakawa model, Lévy process, Probability density.

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1 Introduction

1.1 Literature review
In the classical Black-Scholes economy, the exponential of a Brownian motion (with drift), i.e.,
\[ S_t = S_0 \exp(aB_t + bt), \]
is used to model stock prices. When one takes jumps into account, extending the Black-Scholes scale naturally leads to the modelling economic factors by exponential of a Lévy process, say:
\[ X_t = X_0 \exp Z_t, \]

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where $Z$ is a Lévy process. Such a model is often called "exponential-Lévy" type, and has been widely used in financial modelling since Merton’s jump diffusion model [Mer76] appeared. The variance gamma model by Madan and his co-authors (see e.g. [MCC98]), and the hyperbolic model by Eberlein and his authors (see e.g. [EP02]), are two major representatives, but there are still many others including [BN98] and [BNL01]. For further references, see [Miy01] or [CT04, Chapter 8–11].

In the context of interest rate models, the exponential scale is also common (though the scale is in general not time-homogeneous any more); e.g., bond prices of Vasicek’s model [Vas77] or Gaussian HJM [HJM92], LIBOR rates of the BGM’s LIBOR market model [BGM97], etc. There are also exponential-Lévy type interest rate models with jumps; e.g., Shirakawa’s extension of Gaussian HJM [Shi91], Chiarella-Nikitopoulos [CNS03], Eberlein-Raible’s Lévy term structure models [ER99], or Albeverio-Lytvynov-Mahnig [ALM04], etc, etc (see also section 2.2.3 in the present paper).

There are two major substitutes in the continuous-path cases; the affine class and the quadratic class. Roughly speaking, we say the model is “(exponential-)quadratic” (w.r.t. a state process $Z$) if the price at time $t$ of the bond with maturity $T$ is given by

$$P^T_t = \exp \left\{ \langle A_2(t, T)Z_t, Z_t \rangle + \langle A_1(t, T), Z_t \rangle + A_0(t, T) \right\},$$

where $A_2(t, T) \in \mathbb{R}^n \otimes \mathbb{R}^n$ is a non-zero symmetric matrix, $A_1(t, T)$ is in $\mathbb{R}^n$, and $A_0(t, T) \in \mathbb{R}$. If $A_2 \equiv 0$ then the model is called “(exponential-)affine”.

Affine class w.r.t. a Lévy process is nothing but the above-mentioned exponential-Lévy class. When we refer to affine class, the state process is usually a Markov process whose state space is a convex cone in $\mathbb{R}^n$ (and hence symmetric processes are excluded). The class contains Cox-Ingersoll-Ross model [CIR85] and Duffie-Kan’s multi-dimensional generalization [DK96]. The jump-diffusion extension of the affine class was fully done by Duffie-Filipović-Schachermayer [DFS03] by generalizing Kawazu-Watanabe theorem [KW71]. It is also shown by Filipović-Teichmann [FT04] that the ”finite-dimensional realization” is limited to the affine class.

The quadratic class, which can be embedded into the (degenerate) affine class, is gaining much popularity among economists and practitioners mainly due to its tractability, (see e.g. [ADG02]) or partially explained from a mathematical background; it has a rich mathematical structure (see [AH06]). Contrary to the affine case, however, no jumps are allowed in the quadratic class as Chen-Filipović-Poor showed in [CFP04] within Markovian framework (though Levendorskiǐ [Lev05] gives a jump-type extension of quadratic models “in the direction of affine class”).
The main aim of the present paper is to give a description of a Lévy extension of the quadratic models in a totally different way from [CFP04] and [Lev05].

1.2 Our result

The above story of the quadratic term structure models tells us that exponential-quadratic functions are "natural" scales (at least in modelling interest rates) for Brownian motions but not for pure-jump Lévy processes. Then what scale is natural for a Lévy process?

To answer this question we start from the observation that within the exponential-quadratic scale lies the probability density of the Gaussian distribution. This observation naturally leads to asking if its probability density is a natural scale for a general Lévy process or not. To be more precise, the question is: given a Lévy process $Z$, the market consists of

$$P^T_t = p(A(t, T), Z_t) \quad \text{(or something like this)}$$

where $p(t, x) = P(Z_t \in dx | Z_0 = 0) / dx$ can be consistent? or in particular, arbitrage-free?

An answer to this question is given as Theorem 3.1. It says that if the instantaneous forward rate $f(t, T) := -\partial_T \log P^T_t$ is given by

$$-\partial_T \log p(\lambda_T + T - t, Z_t)$$

for some continuous $\lambda : [0, \infty) \to [0, \infty)$, then the market is arbitrage-free (Theorem 3.1).

To construct arbitrage-free interest rate model, we rely on so-called state price density, or pricing kernel approach [BM01, pp371–], which was initiated by Constantinides [Con92] and later developed by Rogers [Rog97] and Hughston and his co-authors [FH92, BH04, HR05] (see also the textbook by Hunt and Kennedy [HK04]).

1.3 Organization of the present paper

We will start from a brief survey of the approach (section 2.1), and then give two important classes; Gaussian (section 2.5) and quadratic Gaussian (section 2.2.2). A direct jump-type extension which we call generalized Shirakawa model will be given in section 2.2.3. In section 3.1 Theorem 3.1 and its proof will be given. Examples based on our new framework will be presented in section 3.3.
2 The state price density approach to interest rate modelling

2.1 Review

In principle, a strictly positive process \( \{ \pi_t \} \) is a state price density with respect to a market on a filtered probability space \( (\Omega, \mathcal{F}, P, \{ \mathcal{F}_t \}) \) if for any asset indexed by \( i \in I \) that generates \( \{ D^i_s \} \) cash flows in the future, its market price at time \( t \) is given by

\[
S^i_t = \pi^{-1}_t E^P [ \int_{t+}^{\infty} \pi_s dD^i_s | \mathcal{F}_t ],
\]

or for any \( T > t \),

\[
S^i_t = \pi^{-1}_t E^P [ \pi_T S^i_T + \int_{t+}^{T-} \pi_s dD^i_s | \mathcal{F}_t ].
\]

In other words, \( \pi_T/\pi_t \times \text{probability density with respect to } P \) gives (random) discount factor of a (random) cash flow at time \( T \). In particular, if we denote by \( P^T_t \) the market value at time \( t \) of zero-coupon bond with maturity \( T \), we have

\[
P^T_t = \pi^{-1}_t E^P [ \pi_T | \mathcal{F}_t ].
\]

In the famous text book by Duffie \[Duf01\], an arbitrage-free market is characterized by the existence of a state price density. This duality, which comes from the Hahn-Banach theorem and its variants, is proven for fully discrete markets (finite dimensional cases) and for Brownian markets (Brownian filtration cases). In more general cases one can extend the duality though she/he needs to be careful about the exact meaning of the arbitrage. It depends on what kind of assets are traded and what kind of trading strategies are admissible in the market. To determine how far we can extend is, however, out of the scope of the present paper\[1\]. Here we just assume that the space of the value process of trading strategies is orthogonal to \( \pi \) in the \( L^2 \) space of stochastic processes, meaning that we presume that the existence of a state price density implies the market is arbitrage-free (though we leave the problem of market completeness behind). This assumption is robust because at least it is fulfilled if we admit simple strategies\[2\] only.

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\[1\] Extensive studies in this direction are found in \[DS94, DS99, DS06\], and for the cases of bond markets \[BMKR97, BKR97, DD04\] to name a few.

\[2\] by which we mean those strategies which remain constant for a short time interval.
From a perspective of modeling term structure of interest rates, the formula (2.2) says that, given a filtration, each strictly positive process \( \pi \) generates an arbitrage-free interest rate model. On the basis of this observation, we will construct arbitrage-free interest rate model.

\[ \text{(2.2)} \]

\[ \pi_t = P_0^t \exp \left\{ \int_0^t \langle h_s(t, s), dW_s \rangle_{\mathbb{R}^d} - \frac{1}{2} \int_0^t |h_s(t, s)|^2 ds \right\}. \]  

2.2 Examples

In practice, an explicit formula for the bond price (and hence interest rates) is desirable; it gives a parameterization of the entire term structure, and it becomes applicable to, say, duration-based hedging. The more we have explicit formulas for present values \( S^i \) for the cash flow \( D^i \) through (2.1) the better the model becomes.

Below we give two classical Term Structure Models (TSMs for short) which exhibit no jumps.

2.2.1 Gaussian TSMs

Let \( W_t \) be a standard Brownian motion taking values in \( \mathbb{R}^d \) and \( h(t, s) \) be an element of \( H \otimes H \) where

\[ H = \{ h : [0, \infty) \to \mathbb{R}^d, \text{absolutely continuous s.t. } \dot{h} \in L^2_{\text{loc}}[0, \infty) \}. \]

Given an initial data \( T \mapsto P^T_0 \), define

\[ \pi_t = P_0^t \exp \left\{ \int_0^t \langle h_s(t, s), dW_s \rangle_{\mathbb{R}^d} - \frac{1}{2} \int_0^t |h_s(t, s)|^2 ds \right\}. \]  

(2.3)

Then, by an easy manipulation we have

\[ P^T_t = \frac{P^T_0}{P_0} \exp \left\{ \int_0^t \langle h_s(T, s) - h_s(t, s), dW_s \rangle_{\mathbb{R}^d} \right. \]

\[ - \frac{1}{2} \int_0^t (|h_s(T, s)|^2 - |h_s(t, s)|^2) ds \right\}, \]  

(2.4)

where \( h_s \) stands for the partial derivative with respect to the latter variable.

Note that this class covers so-called Gaussian Heath-Jarrow-Morton models (see e.g. [MR97 319–]). In fact, we have

\[ f(t, T) := -\partial_T \log P^T_t \]

\[ = f(0, T) + \int_0^t \langle -h_{T,s}(T, s), dW_s \rangle_{\mathbb{R}^d} + \int_0^t \langle h_{T,s}(T, s), h_s(T, s) \rangle_{\mathbb{R}^d} ds \]

(2.5)

\[ = f(0, T) + \int_0^t \langle -h_{T,s}(T, s), dW_s - h_s(T, s) ds \rangle_{\mathbb{R}^d}. \]
The last expression shows that \( \{ f(t, T) \}_{t \leq T} \) is a martingale under the so-called forward measure \( P^T \) defined by

\[
dP^T/dP = \frac{P^T_{t}}{E[P^T_{t}]} = \exp \left\{ - \int_0^T \langle h_s(T, s), dW_s \rangle_{\mathbb{R}^d} - \frac{1}{2} \int_0^t |h_s(T, s)|^2 ds \right\}
\]

2.2.2 Quadratic Gaussian TSMs

Let \( A \) be a continuous map on \( \mathbb{R}_+ \) taking values in the set of all positive definite \( d \times d \)-symmetric matrices, and \( k : \mathbb{R}_+ \to \mathbb{R} \) be a continuous map. Define

\[
\pi_t(x) = \exp \left\{ -\langle A_t x, x \rangle_{\mathbb{R}^d} + k_t \right\}, \quad (x \in \mathbb{R}^d).
\]

Then we have, for a \( d \)-dimensional Wiener process \( W \) starting from the origin,

\[
P^T_t = E[\pi_T(W_T)|\sigma(W_s; s \leq t)]/\pi_t(W_t)
= \{\det(2(T - t)A_T + I)\}^{-1/2}. \cdot \exp \left\{ -\langle A_T - A_t - 2(T - t)(2(T - t)A_T + I)^{-1} \rangle W_t, W_t \right\} + (k_T - k_t)
\]

(2.6)

where \( I \) is the unit matrix.

For the derivation of (2.6), see Appendix.

Remark 2.1. There are several ways to introduce QTSMs. The above is just an illustration. For details see e.g. [CFP04] and [AH06].

2.2.3 Generalized Shirakawa TSMs

Let \( p \) be a stationary Poisson point process on a measurable space \( (E, \mathcal{B}_E) \). Then its counting measure defined by

\[
\mathcal{N}_p((s, t], A) := \sharp\{u \in (s, t]; p(u) \in A\}; \quad (s < t, A \in \mathcal{B}_E)
\]

is a stationary Poisson random measure; i.e. for mutually disjoint \( B_1, ..., B_n \in \mathcal{B}_{\mathbb{R}_+ \times E} \), the random variables \( \mathcal{N}_p(B_j) \)'s are mutually independent and Poisson distributed, and for \( A \in \mathcal{B}_E \) and \( s < t, \)

\[
P(\mathcal{N}_p((s, t], A) = n) = \{(t - s)\nu(A)\}^n e^{-(t-s)\nu(A)} \frac{n!}{n!},
\]

where \( \nu \) is a \( \sigma \)-finite measure on \( (E, \mathcal{B}_E) \). For a detailed instruction, see e.g. [IW89].

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Let $\delta$ be a Borel function from $\mathbb{R}_+ \times \mathbb{R}_+ \times E$ to $\mathbb{R}$ such that (the equivalence class of) $\delta(t, \cdot, \cdot)$ is in $\bigcap_{p \geq 1} L^p(dt \otimes \nu)$ and $\sup_{p \geq 1} \|\delta\|_{L^p}^p < \infty$.

For this $\delta$ and $h \in H \otimes H$, define

$$Z_t = \int_0^t h_s(t,s) dW_s - \frac{1}{2} \int_0^t |h_s(t,s)|^2 ds + \int_0^t \int_E \delta(t, s, x) \mathcal{N}_p(ds, dx),$$

where $H$, $h$, and $W$ are as in Example 2.2.1 and $W$ is independent of $p$ (or equivalently, $\mathcal{N}_p$).

By the assumption on $h$ and $\delta$, we have

$$E[e^{Z_t}] = \exp \left\{ \int_0^t \int_E \left( e^{\delta(t,s,x)} - 1 \right) \nu(dx) ds \right\} = \exp \left\{ \sum_{p=1}^{\infty} \int_0^t \int_E \{\delta(t, s, x)\}^p \nu(dx) ds \right\} < \infty.$$

Given an initial data $T \mapsto P^T_0$, define

$$\pi_t := P^T_0 e^{Z_t} / E[e^{Z_t}].$$

Then, denoting the bond price in Gaussian TSM (2.4) by $P^{T, \text{Gauss}}_t$, we have

$$P^T_t = E[\pi_T | \mathcal{F}^{p,W}_t] / \pi_t = P^{T, \text{Gauss}}_t \exp \left\{ \int_0^t \int_E \{\delta(T, s, x) - \delta(t, s, x)\} \mathcal{N}_p(ds, dx) \right\} \cdot \exp \left\{ - \int_0^t \int_E \left\{ (e^{\delta(T,s,x)} - 1) - (e^{\delta(t,s,x)} - 1) \right\} \nu(dx) ds \right\}.$$

Here the filtration $\{\mathcal{F}^{p,W}_t\}$ is the one generated by $p$ and $W$.

If we further assume that $\delta(t, \cdot, \cdot)$ is differentiable in $t$ and if its derivative $\partial_t \delta(t, \cdot, \cdot)$ is, say, uniformly bounded and if $\nu$ is finite, then denoting the forward rate in Gaussian TSM (2.5) by $f^{\text{Gauss}}(t,T)$, we have

$$f(t, T) := -\partial_T \log P^T_t = f^{\text{Gauss}}(t,T) + \int_0^t \int_E \delta_T(T, s, x) \{ \mathcal{N}_p(ds, dx) - e^{\delta(T,s,x)} \nu(dx) ds \}. \quad (2.7)$$

This expression can be regarded as an extension of Shirakawa’s model [Shi91], and as a special case of jump-diffusion or fairly general semi-martingale models, e.g. by [BKR97].
Remark 2.2. If $E$ is a finite set, then for each $t > 0$, $\omega \in \Omega$ we define the Poisson integral of $\delta$ as a random finite sum by
\[
\int_E \delta(x)N(t, dx)(\omega) = \sum_{x \in E} \delta(x)N(t, \{x\})(\omega),
\]
and in this case (2.7) becomes Shirakawa’s model.

The generalized Shirakawa model we have presented is basically a exponential-Lévy model, and therefore we cannot use those Lévy process without finite moments. For example, we miss the symmetric $\alpha$-stable processes, which is characterized by (constant times) $|\xi|^\alpha (\alpha \in (0, 2))$ as its Lévy symbol, i.e.; the Lévy process $Z$ with the property
\[
E[\exp\{i\langle \xi, Z_t - Z_s \rangle\}] = \exp\{- (t - s)\theta|\xi|^\alpha\}, \ \theta > 0, \xi \in \mathbb{R}^d, 0 < s \leq t. \ (2.8)
\]
In fact its $\alpha$-th moment explode. To construct an interest rate model driven by stable processes, we might use an approach given in the next section.

3 Lévy Density TSMs: A Generalization of QTSMs

3.1 Main result

Let $Z$ be a Lévy process in $\mathbb{R}^d$ starting from the origin, adapted to a given filtration $\{\mathcal{F}_t\}$. We assume that it has the probability density $p(t, x)$ with respect to the Lebesgue measure of $\mathbb{R}^d$:
\[
P(Z_t \in dx) = p(t, x)dx. \ (3.1)
\]
Here we assume that
\[
P(p(s, Z_t) > 0, \forall s > 0, \forall t > 0) = 1, \ (3.2)
\]
and
\[
p(t, \cdot)p(s, \cdot) \text{ is integrable w.r.t. the Lebesgue measure for all } t, s > 0. \ (3.3)
\]
Then we have the following.

Theorem 3.1. Under (3.2) and (3.3), the term structure model given by
\[
P^T_t = p(\lambda_T + T - t, Z_t)/p(\lambda_t, Z_t), \ 0 \leq t \leq T < \infty, \ (3.4)
\]
where $\lambda : [0, \infty) \rightarrow [0, \infty)$ is a continuous function, is arbitrage-free in the sense of (2.7) by putting $\pi_t = p(\lambda_t, Z_t)$. 
Proof. By the Markov property of $Z$, we have

$$E[p(\lambda_T, Z_T) | F_t] = E[p(\lambda_T, Z_T) | \sigma(Z_t)]$$

$$= \int_{R^d} p(\lambda_T, x + Z_t) P(Z_{T-t} \in dx)$$

$$= \int_{R^d} p(\lambda_T, x + Z_t) p(T-t, x) dx.$$  \hspace{2cm} (3.5)

Denote Lévy symbol of $Z$ by $\psi$:

$$\mathcal{F}[p(\lambda_T, \cdot)](\xi) \equiv \int_{R^d} e^{i\langle \xi, x \rangle} p(\lambda_T, x) dx = e^{-\lambda_T \psi(\xi)}, \quad (\xi \in R^d).$$

Then we have

$$p(T-t, x) = \mathcal{F}^*[e^{-(T-t)\psi(\cdot)}](x) \left( \equiv (2\pi)^{-d} \int_{R^d} e^{-i\langle \xi, x \rangle} e^{-(T-t)\psi(\xi)} d\xi \right).$$

Thus we can rewrite (3.5) as

$$E[p(\lambda_T, Z_T) | F_t] = \langle p(\lambda_T, \cdot + Z_t), \mathcal{F}^*[e^{-(T-t)\psi(\cdot)}] \rangle_{L^2}.$$  \hspace{2cm} (3.6)

Since $\mathcal{F} \circ \mathcal{F}^* = \text{id}$ on the space of density functions, we have

$$\langle p(\lambda_T, \cdot + Z_t), \mathcal{F}^*[e^{-(T-t)\psi(\cdot)}] \rangle_{L^2} = (2\pi)^{-d} \left( \mathcal{F}[p(\lambda_T, \cdot + Z_t)], e^{-(T-t)\psi(\cdot)} \right)_{L^2}. $$

Observing that

$$\mathcal{F}[p(\lambda_T, \cdot + Z_t)](x) = e^{-i\langle x, Z_t \rangle} \mathcal{F}[p(\lambda_T, \cdot)](x), \quad (x \in R^d),$$

we have

$$E[p(\lambda_T, Z_T) | F_t] = (2\pi)^{-d} \left( e^{-i\langle \cdot, Z_t \rangle} e^{-\lambda_T \psi(\cdot)}, e^{-(T-t)\psi(\cdot)} \right)_{L^2}$$

$$= (2\pi)^{-d} \int_{R^d} e^{-i\langle x, Z_t \rangle} e^{-(\lambda_T + T-t)\psi(x)} dx$$

$$= \mathcal{F}^*[e^{-(\lambda_T + T-t)\psi(\cdot)}](Z_t)$$

$$= p(\lambda_T + T-t, Z_t).$$

Hence we have

$$E[p(\lambda_T, Z_T) | F_t] / p(\lambda_T, Z_t) = \text{right-hand-side of (3.4)}.$$

This proves the assertion. \qed
3.2 Minor extensions

3.2.1 Translation

The assumption that $Z$ is starting from 0 is just a convention. The result is stable under the change of starting point. More precisely, for any $z \in \mathbb{R}^d$, substituting $Z_t + z$ for $Z_t$ in (3.2) and (3.4) does not cause any problem.

3.2.2 Product

Let $Z_1, \ldots, Z_k$ be mutually independent Lévy processes starting from the origin, $\lambda^l : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous maps, and $p_l(t, x)$, $l = 1, \ldots, k$ be their probability density functions satisfying (3.2) and (3.3). Then from Theorem 3.1, it is clear that the bond market given by

$$P^T_t = \prod_{l=1}^k p_l(\lambda^l T + T - t, Z^l_t)/p_l(t, Z^l_t)$$

is arbitrage-free. In fact, one can take $\prod_{l=1}^k p_l(t, Z^l_t)$ to be a state price density.

3.3 Examples of LDTSMs

3.3.1 Quadratic Gaussian TSMs as LDTSMs

Let $Z$ be a $d$-dimensional Gaussian process such that $Z_{t+\Delta t} - Z_t \sim N(0, \Sigma \Delta t)$, where $\Sigma$ is a positive definite matrix. Then its probability density is given by

$$p(t, x) := (2\pi t)^{-d/2} (\det \Sigma)^{-1/2} \exp\{-\frac{1}{2t} (\Sigma^{-1} x, x)\}.$$ 

Let $\lambda_t : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing continuous function. Using the formula (3.4), we have an LDTSM generated by $Z$ as

$$P^T_t = p(\lambda_T + T - t, Z_t)/p(\lambda_t, Z_t)$$

$$= \left(\frac{\lambda_t}{\lambda_T + T - t}\right)^{d/2} \exp\{-\frac{1}{2} ((\lambda_T + T - t)^{-1} - \lambda_t^{-1}) (\Sigma^{-1} Z_t, Z_t)\}. \quad (3.7)$$

Since $W_t := \Sigma^{-1/2} Z$ is a Brownian motion, (3.7) is also represented as

$$P^T_t = \left(\frac{\lambda_t}{\lambda_T + T - t}\right)^{d/2} \exp\{-\frac{1}{2} ((\lambda_T + T - t)^{-1} - \lambda_t^{-1}) |W_t|^2\}. \quad (3.8)$$

Note that $X_t := |W_t|^2$ is a $d$-dimensional squared Bessel process, and so (3.8) is a 1-dimensional affine TSM.
Set
\[ A_t := \frac{1}{2\lambda_t} I \quad \text{and} \quad k_t := -\frac{1}{2} \log(2\pi\lambda_t)^d. \]

Then we have
\[ p(\lambda_t, Z_t) = \exp\{-\langle A_t W_t, W_t \rangle_{\mathbb{R}^d} + k_t \}. \]

Therefore, we could also obtain (3.8) using the formula (2.6).

It should be noted that a more general QTSM like (2.6) is within our LDTSMS; it is shown via the ”product” construction of section 3.2.2.

### 3.3.2 Cauchy TSMs

Let \( Z \) be a Cauchy process in \( \mathbb{R}^d \) starting from 0, whose probability density is given by
\[ p(t, x) := P(Z_t \in dx) = \frac{\Gamma((d + 1)/2)\theta t}{(\pi\{(\theta t)^2 + |x - t\gamma|^2\})^{(d+1)/2}}, \tag{3.9} \]

where \( \theta > 0 \) and \( \gamma \in \mathbb{R}^d \). Note that Cauchy processes are strictly stable (or self-similar) process with parameter 1; namely,
\[ Z_{ct} \overset{d}{=} cZ_t, \quad c > 0, t > 0. \]

This property is seen from their Lévy symbol: actually we have
\[ E[e^{i\langle \xi, Z_t \rangle}] = e^{-t|\xi|^2 - i\langle \xi, \gamma \rangle}. \]

Apparently \( p(t, x) \) in (3.9) satisfies the assumptions (3.2) and (3.3). Thus we can obtain a closed form expression for an arbitrage-free TSM driven by a Cauchy process as follows
\[ P^T_t = \frac{\lambda_T + T - t}{\lambda_t} \left( \frac{\theta^2\lambda^2 + |Z_t - t\gamma|^2}{\theta^2(\lambda_T + T - t)^2 + |Z_t - t\gamma|^2} \right)^{(d+1)/2}. \]

### 3.3.3 Gamma TSMs

Let \( Z \) be a one dimensional gamma process with parameters \( a, b > 0 \), so that each \( Z_t \) has density
\[ p(t, x) = \frac{t^a}{\Gamma(at)} x^{a-1} e^{-bx}, \]

About the stable distributions, [Sat99] is a good reference.
for fixed $x \geq 0$. It is easy to check that (3.2) and (3.3) are satisfied. The market value at time $t$ is given by

$$
\tilde{P}_t^T = b^{\lambda_T - \lambda_t + T - t} \frac{\Gamma(a \lambda_t)}{\Gamma(a (\lambda_T + T - t))} Z_t^{a (\lambda_T - \lambda_t + T - t)}. \quad (3.10)
$$

Here we may think $Z_0 > 0$ (see section 3.2.1).

The expression $\tilde{P}_t^T$ jumps only upwards, which is unrealistic for the bond price movement. But as we remarked in section 3.2.2 we can consider $\tilde{P}_t^T$ in (3.10) as a factor of the bond price.

**Remark 3.1.** We can further generalize the Gamma TSMs to include CME (convolutions of mixtures of exponential distributions) and others appearing in [Yam06].

**Appendix: A derivation of (2.6)**

We let $k_t \equiv 0$ for the moment. By the Markov property of $W$, we have

$$
E[\pi_T(W_T)|\sigma(W_s; s \leq t)] = E[\pi_T(W_T)|\sigma(W_t)]
$$

$$
= \int_{\mathbb{R}^d} \pi_T(x) P(W_T \in dx|W_t)
$$

$$
= \int_{\mathbb{R}^d} \pi_T(x + W_t) P(W_{T-t} \in dx)
$$

$$
= \left(\frac{1}{2\pi(T-t)}\right)^{d/2} \int_{\mathbb{R}^d} \pi_T(x + W_t) \exp\left(-\frac{|x|^2}{2(T-t)}\right) dx
$$

$$
= \left(\frac{1}{2\pi(T-t)}\right)^{d/2} \int_{\mathbb{R}^d} e^{-\langle A_T W_t, x \rangle} \int_{\mathbb{R}^d} e^{-\langle A_T W_t, x \rangle} \frac{|x|^2}{2(T-t)} dx.
$$

The last equality comes from the following:

$$
\langle A_T(x + W_t), (x + W_t) \rangle_{\mathbb{R}^d} = \langle A_T W_t, W_t \rangle + 2 \langle A_T x, W_t \rangle + \langle A_T x, x \rangle_{\mathbb{R}^d}.
$$

Let $U_T$ be an orthogonal matrix such that $A_T = U_T \text{diag}[\epsilon_T^1, ..., \epsilon_T^d] U_T$. Here, of course, $\epsilon_T^1, ..., \epsilon_T^d$ are eigenvalues of $A_T$. Then we have

$$
2\langle A_T x, W_t \rangle + \langle A_T x, x \rangle_{\mathbb{R}^d} = 2\langle \text{diag}[\epsilon_T^1, ..., \epsilon_T^d] U_T x, U_T W_t \rangle_{\mathbb{R}^d} + \langle \text{diag}[\epsilon_T^1, ..., \epsilon_T^d] U_T x, U_T x \rangle_{\mathbb{R}^d}.
$$

Thus, putting $y = (y_t^1, ..., y_t^d) = U_t x$,

$$
2\langle A_T x, W_t \rangle + \langle A_T x, x \rangle_{\mathbb{R}^d} = 2\langle \text{diag}[\epsilon_T^1, ..., \epsilon_T^d] y, U_T W_t \rangle_{\mathbb{R}^d} + \langle \text{diag}[\epsilon_T^1, ..., \epsilon_T^d] y, y \rangle_{\mathbb{R}^d} + \frac{|y|^2}{2(T-t)}.
$$
and denoting $c_i \equiv c_i(t, T) := c_i^T + \frac{1}{2(T-t)}$,

$$
= \sum_{i=1}^{d} \left( c_i(y_i^T)^2 + 2y_i^T(U_TW_t)_i \right)
= \sum_{i=1}^{d} c_i \left\{ (y_i^2)^2 + 2 \frac{(U_TW_t)_i^2}{c_i} + \left( \frac{U_TW_t}_i^2 \right)^2 \right\} - \sum_{i=1}^{d} \frac{(U_TW_t)_i^2}{c_i}.
$$

We claim here that

$$
\sum_{i=1}^{d} \frac{(U_TW_t)_i^2}{c_i} = 2(T-t)\langle(2(T-t)A_T + I)^{-1}W_t, W_t\rangle, \quad (3.12)
$$

where $I$ is the unit matrix.

Assuming (3.12) for the moment, we have

(3.11)

$$
= \left( \frac{1}{2\pi(T-t)} \right)^{d/2} e^{-\left\{ (A_T - 2(T-t)(2(T-t)A_T + I)^{-1})W_t, W_t \right\}} 
\cdot \int_{\mathbb{R}^d} e^{-\sum_{i=1}^{d} c_i \left\{ (y_i^2)^2 + \frac{(U_TW_t)_i^2}{c_i} \right\}^2} dy^1 \cdots dy^d 
\cdot \int_{\mathbb{R}^d} e^{-\left\{ (A_T - 2(T-t)(2(T-t)A_T + I)^{-1})W_t, W_t \right\}} (c_1 \cdots c_d)^{-1/2} 
= \left( \frac{1}{2(T-t)} \right)^{d/2} e^{-\left\{ (A_T - 2(T-t)(2(T-t)A_T + I)^{-1})W_t, W_t \right\}} (c_1 \cdots c_d)^{-1/2} 
\cdot \{\text{det}(2(T-t)A_T + I)\}^{-1/2} e^{-\left\{ (A_T - 2(T-t)(2(T-t)A_T + I)^{-1})W_t, W_t \right\}}.
$$

Thus we have (2.8).

Now we prove (3.12). First observation is the following:

$$
\text{diag}[c_1, \ldots, c_d] = \text{diag}[c_1^T, \ldots, c_d^T] + \frac{1}{2(T-t)}I 
= U_T A_T U_T^* + \frac{1}{2(T-t)}I = U_T (A_T + \frac{1}{2(T-t)}I) U_T^*.
$$

We then notice that

$$
\text{diag}\left[\frac{1}{c_1}, \ldots, \frac{1}{c_d}\right] = U_T (A_T + \frac{1}{2(T-t)}I)^{-1} U_T^*.
$$
Therefore,

\[
\sum_{i=1}^{d} \left( \frac{\langle \text{diag}\left[ \frac{1}{c_1}, ..., \frac{1}{c_d} \right] U_T W_t, U_T W_t \rangle}{c_i} \right) = \langle \text{diag}\left[ \frac{1}{c_1}, ..., \frac{1}{c_d} \right] U_T W_t, W_t \rangle \\
= \langle (A_T + \frac{1}{2(T-t)}I)^{-1} W_t, W_t \rangle \\
= 2(T-t)\langle (2(T-t)A_T + I)^{-1} W_t, W_t \rangle.
\]

This completes the proof.

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