MODELING RISK VIA REALIZED HYGARCH MODEL

El Hadji Mamadou Sall(1,∗), El Hadji Demé(1) and Abdou Ka Diongue (1)

(1) LERSTAD, UFR SAT Université Gaston Berger, BP 234 Saint-Louis, Sénégal
(∗) Corresponding autor: elhadjimadou.sall@yahoo.fr

Abstract.
In this paper, we propose the realized Hyperbolic GARCH model for the joint-dynamics of low-frequency returns and realized measures that generalizes the realized GARCH model of Hansen et al. in [16] as well as the FLoGARCH model introduced by Vander Elst in [21]. This model is sufficiently flexible to capture both long memory and asymmetries related to leverage effects. In addition, we will study the strictly and weak stationarity conditions of the model. To evaluate its performance, experimental simulations, using the Monte Carlo method, are made to forecast the Value at Risk (VaR) and the Expected Shortfall (ES). These simulation studies show that for ES and VaR forecasting, the realized Hyperbolic GARCH (RHYGARCH-GG) model with Gaussian-Gaussian errors provide more adequate estimates than the realized Hyperbolic GARCH model with student-Gaussian errors.

Keywords: Realized GARCH models, high-frequency data, long memory, realized measures, Value at Risk, Expected Shortfall

1 Introduction
Volatility forecast of asset returns is very important for option pricing as well as risk management. Since the Autoregressive Conditional Heteroskedasticity model (ARCH) introduced by Engle [12] and generalized by Bollerslev [6] are widely used to study the properties of volatility for economic and financial data. However, there are several shortcomings with using GARCH model for risk management or forecasting volatility. The major issue is the persistence of variance that evolves through time which the GARCH model fails to address. To overcome this problem, many models
are introduced in the literature. Among others, we can cite the IGARCH model of Engle and Bollerslev in [11], the FIGARCH model of Bollerslev and Baillie in [3], the FIIEGARCH model of Bollerslev and Baillie in [17], the HYGARCH model of Davidson in [8] where the conditional variance is a convex combination of the conditional variances of GARCH and FIGARCH models, the new HYGARCH of Li et al. in [18], and the S-HYGARCH model of Diongue and Guegan in [9]. However, as stated by Hansen et al. in [16] and discussed by Andersen et al. in [1], a single return only offers a weak signal about the current level of volatility. Therefore, the implication is that GARCH models are poorly suited for situations where volatility changes rapidly to a new level. Indeed, GARCH models are slow at catching up, and it will take many periods for the conditional variance to reach its new level. To alleviate this problem, researchers proposed to incorporate realized measures in the GARCH model.

Moreover, with the advent of high-frequency data, several measures have been developing in the literature, such as the Realized Variance and Realized Kernel, among many others Anderson and Bollerslev, Barndorff-Nielsen and Shephard, and Barndorff-Nielsen et al. in [2, 5, 4]. All of these measures provide more information on the current level of volatility compare to the square of returns. This aspect makes that realized measures have attracted recently the attentions of financial econometricians as an accurate estimator of volatility. For instance, Engle in [10] introduced the GARCH-X model by including realized measures in the GARCH equation. In [16], Hansen et al. proposed the Realized GARCH model by completing GARCH-X models with a measurement equation for the realized measure. Later, in [15], Hansen and Huang introduced the Realized EGARCH to capture the asymmetries related to leverage effects while Takahashi et al. [19] have extended the stochastic volatility model in the same direction. Watanabe in [22] used daily returns, realized volatility and realized kernel of the S&P 500 stock index to quantile forecasts and found that realized GARCH model with skewed Student’s t-distribution performs better than that with normal and Student’s t-distributions. In addition, Vander Elst [21] proposed FLoGARCH models (Fractionally integrated realized volatility GARCH) to capture also the property of long memory observed on the realized measure. He showed that, using the S&P 500 daily return, FLoGARCH models provide more accurate forecasts than realized GARCH models and FIGARCH models. However, FLoGARCH models used FIGARCH models, for which the existence of a stationary solution with infinite variance was not yet proved (see, Giraitis, Leipus and Surgailis in [14], Tayefi and Ramanathan in [20] among others), as GARCH equation.

To overcome the problem of infinite variance in FIGARCH models, in this work, we focus on another class of asymmetric long memory GARCH process that belong to the family of realized GARCH models introduced by Hansen et al. In particular, we introduce the realized Hyperbolic GARCH model (RHYGARCH) which extended the FLoGARCH model of Vander Elst [21]). Conditions
for strict and weak stationary of the model will be established. In addition, by considering two types of model (the realized HYGARCH model with Gaussian Gaussian error and the realized HYGARCH model with student-t Gaussian error), experimental simulations, using Monte Carlo method, are implemented for quantile forecasts (Value at Risk (VaR) and Expected Shortfall (ES)). For empirical studies, the model can be applied to the S&P 500 (SPY) stock index as in Vander Elst [21].

This paper is organized as follows. In Section 2, we present the realized HYGARCH model. In Section 3, we study the stationarity conditions, while in Section 4, we present the likelihood estimation functions. In Section 5, we present the forecasting method used to estimate the VaR and the ES. Section 6 investigates Monte Carlo simulation experiments in order to evaluate the finite sample properties of the model. Finally, Section 7 concludes.

2 The model

2.1 General structure of realized HYGARCH model

The general formula of the realized GARCH model is given by:

\[ r_t = h_t^{1/2} z_t, \]  
\[ h_t = u \left( h_{t-1}, \cdots, h_{t-p}, x_{t-1}, \cdots, x_{t-q} \right), \]  
\[ x_t = m \left( h_t, z_t, u_t \right), \]  

where \( r_t \) is the return, \( x_t \) a realized measure of volatility, \( (z_t)_t \) are identically independently distributed (i.i.d) with mean zero and variance one, \( (u_t)_t \) are also i.i.d with mean zero and variance \( \sigma_u^2 \). Here \( (z_t)_t \) and \( (u_t)_t \) are mutually independent. In addition, \( E(r_t | F_{t-1}) = 0 \) and \( E(r_t^2 | F_{t-1}) = h_t \), where \( F_{t-1} \) denotes the sigma field generated by the past information up to \( t - 1 \). More specifically \( F_t = \sigma(X_t, X_{t-1}, \cdots) \), with \( X_t = (r_t, x_t) \). We label equation (1) as return equation, equation (2) as the GARCH equation and equation (3) as the measurement equation.

Most (if not all) variants of ARCH and GARCH models are nested in the Realized HYGARCH framework. The nesting can be achieved by setting \( x_t = r_t \) or \( x_t = r_t^2 \), and the measurement equation is redundant for such models, because it is reduced to a simple identity. However, see Bollerslev in [7], the interesting case is when \( x_t \) is a high-frequency-based realized measure, or a vector containing several realized measures. Next we consider a particular variant of the Realized GARCH model, where an HYGARCH model is considered as GARCH equation.
2.2 Realized HYGARCH model

Recall that the log-linear realized GARCH\((p, q)\) by Hansen et al. in [16] is defined as follows:

\[
{r_t} = \frac{1}{2} h_t z_t, \\
\log (h_t) = \omega' + \beta (L) \log (h_t) + \alpha (L) \log (x_t), \\
\log (x_t) = \xi + \phi \log (h_t) + \tau (z_t) + u_t,
\]

where \(L\) denotes the lag or backshift operator, \(\beta (L) = \beta_1 L + \beta_2 L^2 + \cdots + \beta_p L^p\) and \(\alpha (L) = 1 + \alpha_1 L + \alpha_2 L^2 + \cdots + \alpha_q L^q\). The polynomial \(\tau (z) = \tau_1 z + \tau_2 (z^2 - 1)\) is called leverage function and facilitate a modeling of the dependence between return shocks and volatility shocks.

Remark that the GARCH\((p, q)\) process may be expressed as an ARMA\((m, p)\) process:

\[
[1 - \beta (L) - \alpha (L)] \log x_t = \omega' + [1 - \beta (L)] v_t,
\]

where \(m = \max (p, q)\) and \(v_t = \log (x_t) - \log (h_t)\). Thus, the process \((v_t)_t\) is interpreted as "innovations" for the conditional variance. When the polynomial \(1 - \beta (L) - \alpha (L)\) contains a unit root then \(\log (x_t)\) can be defined as an \(I(1)\) process which is written as

\[
\phi (L) (1 - L) \log (x_t) = \omega' + [1 - \beta (L)] v_t,
\]

where the polynomial \(\phi (L) = [1 - \beta (L) - \alpha (L)] (1 - L)^{-1}\) is of order \(m - 1\). Letting \(\gamma (L) = 1 - \beta (L) - \alpha (L)\), the model can be rearranged as:

\[
\log (h_t) = \omega + \left(1 - \frac{\gamma (L)}{1 - \beta (L)}\right) \log (x_t)
= \omega + \pi (L) \log (x_t)
\]

where \(\pi (L) = 1 - \frac{\gamma (L)}{1 - \beta (L)} = \omega = \frac{\omega'}{1 - \beta (L)}\). Notice that Vander Elst [21] replaces this expression with a fractional difference given by

\[
\pi (L) = 1 - \frac{\gamma (L) (1 - L)^d}{1 - \beta (L)},
\]

where \(\gamma (z) = 0\) has roots outside the unit circle, allows for long-range dependencies in \(\log (x_t)\). The model can be then written as:

\[
\log (h_t) = \omega + \left\{1 - \gamma (L) [1 - \beta (L)]^{-1} (1 - L)^d\right\} \log (x_t).
\]

Equation (7) is considered in the FLoGARCH model, introduced by Vander Elst [21], as the volatility equation. However, the FIGARCH\((p, d, q)\) model capture long-range dependence that possesses hyperbolic decay of ACF but has infinite variance which limits its application. Therefore, for the sake of generality, we write the volatility process as:

\[
\log (h_t) = \omega + \delta \left\{1 - \gamma (L) [1 - \beta (L)]^{-1} (1 - L)^d\right\} \log (x_t).
\]
with \( d \geq 0 \) and \( 0 \leq \delta \leq 1 \). Equation (8), which is the GARCH equation for the realized HYGARCH\((p,d,q)\) model, can be viewed as the HYGARCH\((p,d,q)\) model of Li et al. in [18] that has a form nearly the FIGARCH process while allowing the existence of finite variance. This model contains several other extensions among others, the FLoGARCH model when \( \delta = 1 \) and the realized GARCH model if \( d = 1 \) and \( \delta \leq 1 \). Equations (4), (6) and (8) define the realized HYGARCH model.

For the rest of the paper, the realized HYGARCH \((1,d,1)\) model is considered. It can be written as:

\[
\begin{align*}
    r_t &= h_t^{\frac{1}{2}} z_t \\
    \log h_t &= \omega + \delta \left[ 1 - \frac{1 - \gamma L}{1 - \beta L} (1 - L)^d \right] \log x_t \\
    \log x_t &= \epsilon + \phi \log h_t + \tau_1 z_t + \tau_2 (z_t^2 - 1) + \theta_u u_t,
\end{align*}
\]

with \( (1 - L)^d = \sum_{k=0}^{\infty} \frac{\Gamma (d + 1) (-L)^k}{\Gamma (k + 1) \Gamma (d - k + 1)} \).

\( r_t \) is the percentage log-return for day \( t \), \( z_t \ iid \sim D_1 (0,1) \), \( u_t \ iid \sim D_2 \left( 0, \sigma_u^2 \right) \) and \( x_t \) is the realized measure.

Hansen et al. in [16] used the RV (Realized Variance) and RK (Realized Kernel) as realized measures \( x_t \) in the realized GARCH model and considered Gaussian distributed errors \( D_1 (0,1) = N (0,1) \) and \( D_2 \left( 0, \sigma_u^2 \right) = N \left( 0, \sigma_u^2 \right) \). Gerlach and Chaowang in [13] used standardized student-t as \( D_1 (0,1) \) while Contino and Gerlach in [7] and Watanabe in [22] considered a Skew student-t as \( D_1 (0,1) \).

### 3 Existence of the second-order stationary solution

In this section, we study the existence conditions of stationary solution to equations (4), (6) and (8). More precisely, we investigate the strict and weak stationary conditions of the random sequence \((\log h_t, t \in \mathbb{Z})\). Thus, let \( \tilde{h}_t = \log h_t \) and \( \tilde{x}_t = \log x_t \) be related as

\[
\begin{align*}
    \tilde{h}_t &= \omega + \psi (L) \tilde{x}_t \\
    &= \omega + \sum_{i=1}^{\infty} \psi_i \tilde{x}_{t-i},
\end{align*}
\]

with

\[
\psi (L) = \delta \left\{ 1 - \phi (L) [1 - \alpha (L)]^{-1} (1 - L)^d \right\}.
\]
Lemma 1. Let \((\tilde{h}_t)\) be the process defined by (8), if the conditions

\[
|\phi \sum_{i=1}^{\infty} \psi_i| < 1 \quad \text{and} \quad \sum_{i=1}^{\infty} |\psi_i| < \infty
\]

are satisfied then

\[
\tilde{h}_t = \sum_{t=0}^{\infty} H_l(t),
\]

where

\[
H_l(t) = \sum_{i_1, i_2, \ldots, i_l=1}^{\infty} \phi^{l-1} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_l} (\omega \phi + \nu_{i_1-i_2-\cdots-i_l}) \quad \text{for} \quad l \geq 1;
\]

\[
H_0(t) = \omega.
\]

We resume in the following theorem, the necessary and sufficient conditions for the existence of a stationary solution for the process (4), (6) and (8).

Theorem 1. Let \((\tilde{h}_t)_{t \in \mathbb{Z}}\) be the process defined by (8):

1. If the conditions

\[
|\phi \sum_{i=1}^{\infty} \psi_i| < 1 \quad \text{and} \quad \sum_{i=1}^{\infty} |\psi_i| < \infty
\]

are satisfied, where the weights \((\psi_i)_{i \in \mathbb{N}}\) are given in (14), then by restricting, the first moment of the bivariate process \((\tilde{h}_t, \tilde{x}_t)\) exists and is given by:

\[
E(\tilde{h}_t) = \frac{\omega + \xi \sum_{i=1}^{\infty} \psi_i}{1 - \phi \sum_{i=1}^{\infty} \psi_i} = \frac{\omega + \xi \psi(1)}{1 - \phi \psi(1)}
\]

and

\[
E(\tilde{x}_t) = \frac{\xi + \phi \omega}{1 - \phi \psi(1)}.
\]

2. If the conditions

\[
E(z_0^3) < \infty, \quad E(z_0^4) < \infty, \quad \omega = 0, \quad \phi > 0, \quad \psi_i \geq 0 \quad \forall \quad i \geq 1,
\]

and Lemma 1 are satisfied, then the second moment of the process \(\tilde{h}_t\) exists.

Under the conditions provided in Theorem 1, we investigate the strict and weak stationary solution for the realized HYGARCH model. The results concerning \(\tilde{h}_t\) are resumed in the following theorem.

Theorem 2. Let \((\tilde{h}_t)_{t \in \mathbb{Z}}\) be the process defined by (8):

1. Under conditions (14) and Lemma 1, (16) represents a unique strictly stationary solution for the process \((\tilde{h}_t)_{t \in \mathbb{Z}}\);

2. If, in addition, the condition (18) is verified then (16) is also a unique weakly stationary solution.
4 Likelihood Estimation

In this section, we tackle the problem of estimating the parameters of realized HYGARCH\((p, d, q)\) model. For this, the quasi-maximum likelihood method is used.

- Following Harry Vander Elst \cite{21} where \(D_1 = N (0, 1)\) and \(D_2 = N (0, \sigma_u^2)\), the log-likelihood function for the model is given by:

\[
l(r, x; \theta_G) = -\frac{1}{2} \sum_{t=1}^{n} \left( \log 2\pi + \log h_t + \frac{u_t^2}{h_t} \right) = -\frac{1}{2} \sum_{t=1}^{n} \left[ \log 2\pi + \log (\sigma_u^2) + \frac{u_t^2}{\sigma_u^2} \right], \tag{19}\]

where \(u_t = \log x_t - \epsilon - \phi \log h_t - \tau_1 z_t - \tau_2 (z_t^2 - 1)\). This model is denoted by RHYGARCH-GG (HYGARCH with Gaussian-Gaussian error). The parameters \(\theta_G = (w, d, \alpha, \beta, \epsilon, \phi, \tau_1, \tau_2, \sigma_u)\) with \(\alpha = (\alpha_1, \cdots, \alpha_q)\) and \(\beta = (\beta_1, \cdots, \beta_p)\).

- Under the choice \(D_1 = t^* (0, 1, \nu)\) and \(D_2 = N (0, \sigma_u^2)\) as in Gerlach and Chaowang \cite{13} and Contino and Gerlach \cite{7}, the log-likelihood function for this model is given by:

\[
l(r, x; \theta_\ell) = -\frac{1}{2} \sum_{t=1}^{n} \left[ \log 2\pi + \log (\sigma_u^2) + \frac{u_t^2}{\sigma_u^2} \right] = -\frac{1}{2} \sum_{t=1}^{n} \left( A(\nu) + \log [\pi (\nu - 2)] + \frac{\nu + 1}{2} \log \left( \frac{u_t^2}{h_t (\nu - 2)} \right) \right) \tag{20}\]

where \(u_t = \log x_t - \epsilon - \phi \log h_t - \tau_1 z_t - \tau_2 (z_t^2 - 1)\) and \(t^* (0, 1, \nu) = t (0, 1, \nu) \sqrt{\frac{\nu - 2}{2}}\), which is a student-\(t\) distribution with \(\nu\) degrees of freedom, scaled to have variance 1, and \(A(\nu) = \log \left[ \Gamma \left( \frac{\nu}{2} \right) \right] - \log \left[ \Gamma \left( \frac{\nu + 1}{2} \right) \right]\). The parameters \(\theta_\ell\) of this model, denoted by RHYGARCH-tG, is defined as \(\theta_\ell = (w, d, \delta, \alpha, \beta, \epsilon, \phi, \tau_1, \tau_2, \sigma_u, \nu)\) with \(\alpha = (\alpha_1, \cdots, \alpha_q)\) and \(\beta = (\beta_1, \cdots, \beta_p)\).

5 Forecasting Method

5.1 Value-at-Risk Forecasts

In order to forecast tail risk in a parametric realized HYGARCH setting, the model is used to estimate a one-ahead volatility forecast at both the 95% and 99% confidence level \(\alpha\). The conditional one-period-ahead VaR forecast is defined as:

\[
\alpha = P \left( r_{t+1} < VaR_{\alpha} \mid F_t \right),
\]

where \(r_{t+1}\) is the one-period return from time \(t\) to time \(t+1\), \(\alpha\) is the quantile level and \(F_t\) is the informative set at time \(t\). For a normal distribution, VaR is calculated via the inverse standard
Gaussian CDF, denoted by $\Phi^{-1}(\alpha)$:

$$VaR_\alpha = \sqrt{h_{t+1}} \Phi^{-1}(\alpha),$$

where $\Phi^{-1}(\alpha)$ is the inverse standard Gaussian. And similarly for a student-t distribution:

$$VaR_\alpha = \sqrt{h_{t+1}} T_{\nu}^{-1}(\alpha),$$

where $T_{\nu}^{-1}(\alpha)$ is the inverse standardized student-t CDF.

### 5.2 Conditional value-at-risk forecasts

The Conditional Value-at-Risk forecasts (CVaR) or Expected Shortfall (ES) is used to estimate a one-ahead volatility forecast, as it has become preferred to the VaR due to the latter’s shortcomings. The Expected Shortfall is defined as:

$$ES_\alpha = E (r_{t+1} \mid r_{t+1} \geq VaR_\alpha, F_t).$$

For a normal distribution, the Expected shortfall is calculated via that expression:

$$ES_\alpha = \sqrt{h_{t+1}} \frac{\phi (\Phi^{-1}(\alpha))}{1 - \alpha},$$

where $\phi(x)$ and $\Phi^{-1}(\alpha)$ are the normal probability density function and inverse distribution function respectively. For a student-t distribution, we can derive the Expected shortfall:

$$ES_\alpha = \sqrt{h_{t+1}} t_{\nu}[T_{\nu}^{-1}(\alpha)] \left[ \frac{\nu + (T_{\nu}^{-1}(\alpha))^2}{\nu - 1} \right],$$

where $\nu$ is the estimated degrees of freedom, $t_{\nu}(x)$ and $T_{\nu}^{-1}(\alpha)$ are the student-t probability density and inverse cumulative distribution function.

### 6 Simulation study

In this section, we have designed and executed Monte Carlo simulation with the aim of analyzing the sampling properties of the MLE estimators for the realized HYGARCH(1, d, 1) model with Gaussian Gaussian error and with student-t Gaussian error. Across M=1000 Monte Carlo replications and sample size T=1000 and T=3000, two specific models are considered:

1. Model 1

   $$r_t = \sqrt{h_t} z_t, \quad z_t \sim N(0, 1),$$
   $$\log h_t = 0.1 + 0.4 \left[ 1 - \frac{1 - 0.1L}{1 - 0.4L} (1 - L)^{0.4} \right] \log x_t,$$
   $$\log x_t = -0.1 + 1 \log h_t - 0.08 z_t + 0.06 (z_t^2 - 1) + u_t, \quad u_t \sim N(0, 0.4).$$
Table 1: Summary statistics for the estimator of the RHYGARCH-GG model, data simulated from Model 1

| Parameter | True | MSE   | Mean  | True | MSE   | Mean  |
|-----------|------|-------|-------|------|-------|-------|
| T=1000    |      |       |       |      |       |       |
| ω         | 0.1  | 0.0030| 0.1178| 0.1  | 0.0010| 0.1062|
| γ         | 0.1  | 0.0334| 0.1268| 0.1  | 0.0038| 0.0887|
| β         | 0.4  | 0.0327| 0.3710| 0.4  | 0.0122| 0.3683|
| δ         | 0.4  | 0.0546| 0.4113| 0.4  | 0.0207| 0.4218|
| d         | 0.4  | 0.0573| 0.4511| 0.4  | 0.0150| 0.39809|
| ξ         | -0.00| 0.0036| 0.0533| -0.00| 0.0011| -0.0030|
| φ         | 1    | 0.0433| 0.9688| 1    | 0.0145| 1.0204|
| τ₁        | -0.08| 0.0001| -0.0801| -0.08| 0.000059| -0.0802|
| τ₂        | 0.06 | 0.000103| 0.0612| 0.06 | 0.00003| 0.0595|
| σ_u       | 0.4  | 0.00139| 0.3667| 0.4  | 0.00103| 0.3681|
| T=3000    |      |       |       |      |       |       |
| 5%VaR     | -1.8547| 0.0051| -1.8738| -1.8567| 0.0024| -1.8684|
| 1%VaR     | -2.6834| 0.0089| -2.6402| -2.6524| 0.00572| -2.6433|
| 5%ES      | 0.1224| 0.000022| 0.1236| 0.1225| 0.00001| 0.1233|
| 1%ES      | 0.0310| 0.000001| 0.0306| 0.0307| 0.000007| 0.0305|

2. Model 2

\[ r_t = \sqrt{h_t}z_t, \quad z_t \sim t^*_\nu (0, 1), \]

\[ \log h_t = 0.1 + 0.4 \left[ 1 - \frac{1 - 0.1L}{1 - 0.4L} (1 - L)^{0.4} \right] \log x_t, \]

\[ \log x_t = -0.1 + 1 \log h_t - 0.08z_t + 0.06 (z_t^2 - 1) + u_t, \quad u_t \sim N(0, 0.4). \]

Where \( r_t \) is the daily log-return, \( x_t \) the daily realized measure and \( t^* \) represents the Student-\( t \) distribution standardized to have variance 1. Notice that the chosen parameters verify the stationary conditions. For other parameters choice, one can refer to Contino and Gerlach in [7], and Li et al. in [18] for the parameter \( \nu \).

Estimation results are summarized in Tables 1 and 2. Inspection of these tables reveals that, for all sample sizes, the QMLE procedure performs relatively well. Particularly, the MSE for the parameters \( \epsilon, \tau_1, \tau_2 \) and \( \sigma_u \) are very small indicating that estimators are consistent. Moreover, we notice that the bias as well as the MSE decreases when the sample size increases. We observe also that parameter estimates are more precise in the measure equation than in the GARCH equation.

Another finding is that the ES has lowest bias estimation under \( t \)-student distribution when the sample size increases and inversely under the Gaussian distribution. In addition, the results for ES and VaR of the RHYGARCH-GG model deliver more satisfactory estimates than the
Table 2: Summary statistics for the estimator of the RHYGARCH-tG model, data simulated from Model 2

| Parameter | True | MSE | Mean | True | MSE | Mean |
|-----------|------|-----|------|------|-----|------|
| $\omega$  | 0.1  | 0.0022 | 0.0911 | 0.1  | 0.0020 | 0.084 |
| $\gamma$  | 0.1  | 0.0208 | 0.1931 | 0.1  | 0.0110 | 0.1770 |
| $\beta$   | 0.4  | 0.0244 | 0.3316 | 0.4  | 0.0143 | 0.3592 |
| $\delta$  | 0.4  | 0.0352 | 0.3204 | 0.4  | 0.0216 | 0.32533 |
| $d$       | 0.4  | 0.0167 | 0.3779 | 0.4  | 0.0070 | 0.3762 |
| $\nu$     | 3    | 0.0572 | 3.1787 | 3    | 0.0455 | 3.1712 |
| $\xi$     | -0.00 | 0.0066 | 0.069  | -0.00 | 0.0069 | 0.068  |
| $\phi$    | 1    | 0.0138 | 0.9218 | 1    | 0.0194 | 0.9304 |
| $\tau_1$  | -0.08 | 0.0022 | -0.0760 | -0.08 | 0.0009 | -0.07602 |
| $\tau_2$  | 0.06  | 0.00038 | 0.0566 | 0.06  | 0.00097 | 0.0569 |
| $\sigma_u$| 0.4  | 0.0011 | 0.3708 | 0.4  | 0.0010 | 0.3725 |

|  | 5%VaR | 1%VaR | 5%ES | 1%ES |
|---|--------|--------|------|------|
| MSE | 0.0376 | -2.5672 | 0.0262 | -2.5652 |
| Mean | -2.625 | -5.9864 | 0.0776 | 0.0776 |
| Mean | -2.668 | -4.836 | 0.0736 | 0.0736 |

RHYGARCH-tG model. Suggesting that the RHYGARCH-GG model can be used to forecast the ES and VaR. The findings of this research are consistent with those from Gerlach and Chaowang [13].

### 7 Conclusion

In this work, the realized HYGARCH process is studied which generalizes the realized GARCH model of Hansen et al. in [16] and the FLoGARCH model introduced by Vander Elst in [21]. Under some assumptions, the model shows to be strictly and weak stationary. The parameter estimation problem is addressed using the quasi-maximum likelihood procedure. Finite sample behaviors of this method were studied using Monte Carlo simulations. It indicates that the approach can yield asymptotic efficient estimates. The simulation shows that the RHYGARCH-tG model deliver more adequate estimates than the RHYGARCH-GG model for forecasting ES. Nevertheless, it shows that the RHYGARCH-GG model has more precise estimates than the RHYGARCH-tG for forecasting the VaR. Since the results from the estimation methodology are encouraging, it will be interesting to examine, in a future work, the empirical application of the realized HYGARCH...
model in financial data.

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A Appendix section

Proof. Denote by $v_t = \xi + \tau(z_t) + u_t$. By (6) and (12), we have

$$\tilde{h}_t = \omega + \sum_{i_1=1}^{\infty} \psi_{i_1} \left( \phi \tilde{h}_{t-i_1} + v_{t-i_1} \right)$$

$$= \omega + \sum_{i_1=1}^{\infty} \psi_{i_1} v_{t-i_1} + \phi \sum_{i_2=1}^{\infty} \psi_{i_2} \tilde{h}_{t-i_2}$$

$$= \omega + \sum_{i_1=1}^{\infty} \psi_{i_1} v_{t-i_1} + \phi \sum_{i_2=1}^{\infty} \psi_{i_1} \psi_{i_2} v_{t-i_1-i_2} + \phi^2 \sum_{i_1,i_2=1}^{\infty} \psi_{i_1} \psi_{i_2} \tilde{h}_{t-i_1-i_2}$$

$$= \omega + \sum_{i_1=1}^{\infty} \psi_{i_1} v_{t-i_1} + \phi \sum_{i_1,i_2=1}^{\infty} \psi_{i_1} \psi_{i_2} v_{t-i_1-i_2}$$

$$= \omega + \sum_{i_1=1}^{\infty} \psi_{i_1} v_{t-i_1} + \phi \sum_{i_1,i_2=1}^{\infty} \psi_{i_1} \psi_{i_2} \tilde{h}_{t-i_1-i_2} + \phi^2 \sum_{i_1,i_2,i_3=1}^{\infty} \psi_{i_1} \psi_{i_2} \psi_{i_3} \tilde{h}_{t-i_1-i_2-i_3}$$

for $m$ step we have

$$\tilde{h}_t = \sum_{l=0}^{m} \sum_{i_1,\ldots,i_l}^{\infty} \phi^{l-1} \psi_{i_1} \cdots \psi_{i_l} \left( \omega \phi + v_{t-i_1-\cdots-i_l} \right)$$

$$+ \phi^m \sum_{i_1,\ldots,i_m+1}^{\infty} \psi_{i_1} \cdots \psi_{i_m+1} \tilde{x}_{t-i_1-\cdots-i_m+1}$$

$$\tilde{h}_t = \sum_{l=0}^{m} \sum_{i_1,\ldots,i_l}^{\infty} \phi^{l-1} \psi_{i_1} \cdots \psi_{i_l} \left( \omega \phi + v_{t-i_1-\cdots-i_l} \right)$$

$$+ \sum_{i_1}^{\infty} \phi \psi_{i_1} \sum_{i_{m+1}}^{\infty} \psi_{i_{m+1}} \tilde{x}_{t-i_1-\cdots-i_{m+1}}.$$
Taking the expectations in (6) and (12) and solving the linear system gives that if the bivariate process \((\tilde{h}_t, \tilde{x}_t)\) is mean-stationary, then

$$E(\tilde{h}_t) = \frac{\omega + \xi \sum_{i=1}^{\infty} \psi_i}{1 - \phi \sum_{i=1}^{\infty} \psi_i} = \frac{\omega + \xi \psi(1)}{1 - \phi \psi(1)}$$

and

$$E(\tilde{x}_t) = -\frac{\xi + \phi \omega}{1 - \phi \psi(1)}.$$

Let’s now proved the sufficient condition for the existence of the second moment of the process \(\tilde{h}_t\). Applying the Minkowski inequality norm to (16) in conditions (18), we get:

$$E(\tilde{h}_t^2) \leq \sum_{l=0}^{\infty} \sum_{i_1, i_2, \ldots, i_l} \phi^{l-1} \psi_{i_1} \cdots \psi_{i_l} \left[ E(\omega \psi + v_{t-i_1-i_2-\ldots-i_l})^2 \right]^{\frac{1}{2}}.$$

Let define and denote by \(B\) the above equality:

$$B = \sum_{l=0}^{\infty} \sum_{i_1, i_2, \ldots, i_l} \phi^{l-1} \psi_{i_1} \cdots \psi_{i_l} \left[ \omega^2 \phi^2 + 2 \xi \omega \phi + E(v_{t-i_1-i_2-\ldots-i_l})^2 \right]^{\frac{1}{2}}.$$

We have

$$E(v_{t-i_1-i_2-\ldots-i_l})^2 = E[\xi + \tau_1 z_{t-i_1-i_2-\ldots-i_l} + \tau_2 (z_{t-i_1-i_2-\ldots-i_l}^2 - 1) + u_{t-i_1-i_2-\ldots-i_l}]^2,$$

by developing this expression and using the fact that \(u_t\) and \(z_t\) are mutually independent we get:

$$E(v_{t-i_1-i_2-\ldots-i_l})^2 = \xi^2 + \tau_1^2 + \tau_2^2 + 2 \tau_1 \tau_2 E(z_0^3) + \tau_2^2 E(z_0^4),$$

so we have:

$$B = \left[ k + 2 \tau_1 \tau_2 E(z_0^3) + \tau_2^2 E(z_0^4) \right] \sum_{l=0}^{\infty} \left( \sum_{i=1}^{\infty} \psi_i \right)^l,$$

where \(k = \omega + \frac{2 \xi \omega}{\phi} + \xi^2 \phi^{-2} + \tau_1^2 \phi^{-2} + \tau_2^2 \phi^{-2} - \frac{\tau_2^2}{\phi^2}.\) If the conditions (18) and Lemma 1 are satisfied, the second moment of the process \((\tilde{h}_t)\) exists.

\(\square\)

**Proof.** Theorem 2
1. Since it is very easy to verify that $(16)$ is a stationary solution, therefore we show that it is the unique strictly stationary solution. Assume that $y_t$ is any strictly stationary solution with finite first moment $E(y_0)$. Then, applying relations $(12)$ and $(6)$ after $m$ steps we obtain.

$$
y_t = \sum_{l=0}^{m} \sum_{i_1, \ldots, i_l} \phi^{l-1} \psi_{i_1} \cdots \psi_{i_l} (\omega \phi + v_{t-i_1-\cdots-i_l}) + \sum_{i_1, \ldots, i_{m+1}} \phi^m \psi_{i_1} \cdots \psi_{i_{m+1}} (\phi y_{t-i_1-\cdots-i_{m+1}} + v_{t-i_1-\cdots-i_{m+1}}).
$$

Therefore, we have

$$
y_t - h_t = \sum_{i_1, \ldots, i_{m+1}} \phi^m \psi_{i_1} \cdots \psi_{i_{m+1}} (\phi y_{t-i_1-\cdots-i_{m+1}} + v_{t-i_1-\cdots-i_{m+1}})
- \sum_{l=m+1}^{\infty} \sum_{i_1, \ldots, i_{l}} \phi^{l-1} \psi_{i_1} \cdots \psi_{i_{l}} (\omega \phi + v_{t-i_1-\cdots-i_{l}}).$$

(21)

Applying Chebychev’s inequality to the first term on the right-hand side of (21), we obtain

$$
\varepsilon P \left[ \sum_{i_1, \ldots, i_{m+1}} \phi^m \prod_{j=1}^{m+1} \psi_j (\phi y_{t-i_1-\cdots-i_{m+1}} + v_{t-i_1-\cdots-i_{m+1}}) > \varepsilon \right] \leq \left( E(y_0) + \frac{\xi}{\phi} \right) \times \left( \phi \sum_{i=1}^{\infty} \psi_i \right)^{m+1}.
$$

By $(15)$ and the Borel-Cantelli lemma, this implies almost sure convergence to zero as $m \to \infty$. We have $\sum_{t=0}^{\infty} H_t(t) < \infty$, choosing $m$ large enough, the second term on the right-hand side of (21) can be made small with probability 1. Thus $h_t = y_t$ a.s.

2. According to condition $(18)$ and lemma $(10)$ the second moment of the process exists. To verify that the sequence $\tilde{h}_t$ defined by $(16)$ is weakly stationary, observe that:

$$
E(\tilde{h}_t) = \frac{\omega + \xi \sum_{i=1}^{\infty} \psi_i}{1 - \phi \sum_{i=1}^{\infty} \psi_i} = \frac{\omega + \xi \psi(1)}{1 - \phi \psi(1)}
$$

$$
Cov(\tilde{h}_t, \tilde{h}_{t+k}) = E(\tilde{h}_t, \tilde{h}_{t+k}) - E(\tilde{h}_t) E(\tilde{h}_{t+k})
= \sum_{l,k=1}^{\infty} \sum_{i_1, i_2, \ldots, i_l, j_1, j_2, \ldots, j_k=1}^{\infty} \phi^{l+k-2} \psi_{i_1} \psi_{j_1} \cdots \psi_{i_l} \psi_{j_2} \cdots \psi_{j_k}
Cov(v_{t-i_1-\cdots-i_l}, v_{t-j_1-\cdots-j_k}) - \left( \frac{\omega + \xi \psi(1)}{1 - \phi \psi(1)} \right)^2
= Cov(\tilde{h}_0, \tilde{h}_t).
$$

Unicity results is obtained using the same lines as in 1. 

\[ \square \]