Abstract: Radio antennas use different frequency bands of Electromagnetic (EM) Spectrum for switching signals in the forms of radio waves. Regulatory authorities issue a unique number (unique identifying call sign) to each radio center, that must be used in all transmissions. Each radio center propagates channels to the two nearer radio centers so they must use distinctive numbers to avoid interruption. The task of effectively apportioning channels to transmitters is known as the Channel Assignment (CA) problem. CA Problem is discussed under the topic of graph coloring by mathematicians. The radio number of a graph can be used in many parts of the field communication. In this paper, we determined the radio number of zero-divisor graphs $\Gamma(Z_p^2 \times Z_q^2)$ for $p, q$ prime numbers.

Keywords: radio labeling; radio number; zero divisor graph

MSC: 05C78

1. Introduction

The antennas propagate electromagnetic waves which have different frequencies. These waves are known as Radio waves. A specific signal can be accessed by tuning the radio receiver to a particular frequency. Every radio station must be assigned distinct channels, located within a certain proximity of one another. The two radio stations are closer to each other, and then their assigned channels must have a greater difference. The task of efficiently allocating channels to transmitters is called the Channel Assignment (CA) problem.

Radio stations in the past have been used as a great source of communication but gradually the target audience is lowering. Network radio as an alternative can give news customized for a small population about local issues. But in 39 African nations, some charitable associations are using country radio stations because of specialized apparatuses for the provincial poor. In countries where access to the web is restricted and the education rate is very low, radio broadcasts are still useful. Even in advanced countries when web access is blocked, and telephone lines are cut due to some disasters people can, in any case, scan the wireless transmissions as dependable sources.

Numbers stations are shortwave radio broadcasts that are accepted to be routed to insight officials working in outside countries. Most recognized stations use discourse amalgamation to vocalize digital modes like phase-shift keying (PSK) and frequency-shift keying (FSK). Stations can have set frequencies in the HF band but some numbers stations are found by area, the innovation used to communicate the numbers has verifiably been clear-stock shortwave transmitters utilizing powers from 10 KW to 100 KW.

2. Motivation and Application

Graph labeling is useful in networks because each network node has a different transmission capacity for sending or receiving messages in wired or wireless links. For this purpose, the node is
labeled with a unique number. This number is attained automatically by subtracting the label of link and conversely, whereas the path label specifies the pair of interconnecting terminals. For the transmitting device, a positive integer is assigned to each channel to avoid interference. This interference is so high for short-distance communication like wireless networks so the difference in channel assignment should be greater.

By using radio labeling network protocols can efficiently determine the time on which sensors communicate, this phenomenon becomes a commodity for safe transmission in Wi-Fi and cellular networks. It is unpleasant being on the phone and facing interference that is caused by unconstrained simultaneous transmissions. Two closer or physically overlapped channels can interfere or resonate to damage communications and the solution to this problem is a suitable channel assignment.

3. Literature Review

In 1980, William Hale [1] presented a model of the CA problem. Generally, the CA problem has been displayed as a diagram coloring and labeling problem, where the transmitters are spoken to as the vertices of a diagram. If the transmitters are altogether near one another, at that point, two vertices are adjoining. The channels allotted to the transmitters are the labels to the vertices. For each pair of labels, there is a base satisfaction separation between two different vertices with assigned labels. The last point is to find optimal labeling with the span (range) of the channels utilized is minimal.

A graph \( G = (V, E) \) consists of \( V \), a nonempty set of vertices and \( E \), a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints [2]. The example of radio transmitters can be well mapped on a graph by considering the transmitters as vertices and channels as edges. Each transmitter is allocated a unique channel (a positive whole number) to avoid interference. The minimum distance between stations can increase interference. Thus the distinction in channel tasks must be more noteworthy. Every transmitter deals with one transmitter or pair of neighboring transmitters. The maximum distance between any two vertices in a simple and connected graph \( G \) is known as diameter of \( G \), it is denoted as \( \text{diam}(G) \).

A function \( \phi : V(G) \rightarrow \mathbb{N} \) is called a radio labeling [3,4], if the following radio condition holds for any two distinct vertices \( x \) and \( y \):

\[
d(x, y) \geq 1 + \text{diam}(G) - |\phi(x) - \phi(y)|. \tag{1}\]

\( d(x, y) \) denotes the distance between two distinct vertices \( x, y \) of a connected graph \( G \). We denote by \( S(G, \phi) \) the set of consecutive integers \( \{a, a + 1, ..., A\} \), where \( a = \min_{x \in V(G)} \phi(x) \) and \( A = \max_{x \in V(G)} \phi(x) \) is the span of \( \phi \), denoted \( \text{span}(\phi) \) and \( \text{span}(\phi) = 1 + A - a \).

The minimum span of a radio labeling for \( G \) is called radio number of \( G \), denoted by \( rn(G) \). A radio labeling \( \phi \) of \( G \) with \( \text{span}(\phi) = rn(G) \) will be called optimal radio labeling for \( G \).

Other than its motivation by the channel problem, radio labeling itself is an appealing labeling problem and has been considered by various researchers. It is computationally extremely difficult to decide the radio number on an arbitrary diagram. The problem is known to be NP-hard for diagrams with a maximum distance of 2, yet the complexity, as a rule, is not known [5]. Thusly, the researchers start their investigation on a particular family of diagrams, for some fundamental classes of diagrams the problem showing to be difficult [4].

The radio numbers for paths and cycles were investigated in References [6-8], and were completely determined in Reference [4]. For all \( n \leq 20 \) with \( n \equiv 0 \) or 2 or 4 (mod 6) the radio number is studied for cube of \( C_n \) in Reference [9]. They also demonstrated the estimations of \( n \) for which this diagram is radio graceful. Ahmad and Marinescu-Ghemeci [10] investigated the radio numbers for some ladder-related diagrams. Marinescu-Ghemeci [11] determined the radio numbers for thorn stars and Saha et al. [12] investigated the radio numbers of toroidal grid. For additional detail, see References [13-17].
The idea of associating graphs with some algebraic objects was given by Beck [18], where he introduced a graph structure, called zero-divisor graph $\Gamma(R)$ of a commutative ring $R$ with unity. For $\Gamma(R)$, he let all the ring elements be vertices and two vertices $x$ and $y$ are adjacent if and only if $x \cdot y = 0$. After that, in 1999, Anderson and Livingston introduced a new method of associating a zero divisor graph to a commutative ring $R$ [19], where the vertices are all non-zero zero-divisors of $R$ and the condition for two vertices to be adjacent is the same as in Reference [18]. Ahmad and Haider [20] determined the radio number for $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$. Throughout the paper we use $\mathbb{Z}_{p^2}^* = \mathbb{Z}_{p^2} \setminus \{0\}$. For additional study of zero-divisor graphs, see References [21–23]. In the next section, we discuss the zero-divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q^2)$ and its radio number.

4. Zero-Divisor Graph

Let $p, q$ be two prime numbers and $\mathbb{Z}_{p^2} \times \mathbb{Z}_q^2$ be a commutative ring with unity. Consider a zero-divisor graph $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q^2)$, where the set of vertices consists of all non zero zero-divisors of $\mathbb{Z}_{p^2} \times \mathbb{Z}_q^2$. For $x \in \mathbb{Z}_{p^2}^*$ and $y \in \mathbb{Z}_q^*$, $(x, y) \in \mathbb{Z}_{p^2} \times \mathbb{Z}_q^2$ is a zero-divisor if and only if one of the following conditions holds:

1. at least one of $x$ and $y$ are zero-divisors
2. either $x = 0$ or $y = 0$.

According to condition 1, there are $(p - 1)(q^2 - q) + (q - 1)(p^2 - p) + (p - 1)(q - 1)$ number of elements that are zero-divisors for $\mathbb{Z}_{p^2} \times \mathbb{Z}_q^2$. According to condition 2, we have $p^2 - 1$ number of elements of the form $(x, 0)$ for $x \in \mathbb{Z}_{p^2}^*$ and $q^2 - 1$ number of elements of the form $(0, y)$ for $y \in \mathbb{Z}_q^*$. Thus the total number of zero-divisors are $(p - 1)(q^2 - q) + (q - 1)(p^2 - p) + (p - 1)(q - 1) + p^2 - 1 + q^2 - 1 = p^2q + pq^2 - pq - 1$. Hence, the order of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q^2)$ is

$$|V(\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q^2))| = p^2q + pq^2 - pq - 1.$$ 

For our convenience we can divide the vertex set of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_q^2)$ into the following seven disjoint sets:

$$A = \{(0, y) : y \in \mathbb{Z}_q^*, y \neq kq, 0 \leq k \leq q - 1\} = \{a_i : 1 \leq i \leq q^2 - q\}$$
$$B = \{(x, 0) : x = k_1p, 1 \leq k_1 \leq p - 1\} = \{b_i : 1 \leq i \leq p - 1\}$$
$$C = \{(x, y) : y \in \mathbb{Z}_q^*, y \neq kq, 0 \leq k \leq q - 1, x = k_1p, 1 \leq k_1 \leq p - 1\}$$
$$= \{c_i : 1 \leq i \leq (p - 1)(q^2 - q)\}$$
$$D = \{(x, y) : x \in \mathbb{Z}_{p^2}, x \neq k_1p, 0 \leq k_1 \leq p - 1, y = kq, 1 \leq k \leq q - 1\}$$
$$= \{d_i : 1 \leq i \leq (p^2 - p)(q - 1)\}$$
$$E = \{(x, y) : x = k_1p, 1 \leq k_1 \leq p - 1, y = kq, 1 \leq k \leq q - 1\}$$
$$= \{e_i : 1 \leq i \leq (p - 1)(q - 1)\}$$
$$G = \{(x, 0) : x \in \mathbb{Z}_{p^2}, x \neq k_1p, 0 \leq k_1 \leq p - 1\} = \{g_i : 1 \leq i \leq p^2 - p\}$$
$$F = \{(0, y) : y = kq, 1 \leq k \leq q - 1\} = \{f_i : 1 \leq i \leq q - 1\}.$$ 

From the above discussion we get $|A| = q^2 - q$, $|B| = p - 1$, $|C| = (p - 1)(q^2 - q)$, $|D| = (q - 1)(p^2 - p)$, $|E| = (p - 1)(q - 1)$, $|F| = q - 1$ and $|G| = p^2 - p$. The further detail about degree of vertices, distance between the above sets and proof of diameter is 3 can be seen in Reference [24].
5. Main Results

In the next theorem, we investigate the lower bound of radio number for zero-divisor graphs \( \Gamma(Z_{p^2} \times Z_{q^2}) \) for \( p, q \) prime numbers with \( p \geq 5 \) and \( q = 2 \).

**Theorem 1.** Let \( p, q \) be two prime numbers with \( p \geq 5 \) and \( q = 2 \), then \( \text{rn}(\Gamma(Z_{p^2} \times Z_{q^2})) \geq 4p^2 - 2 \).

**Proof.** If \( q = 2 \), then the cardinalities of the vertex sets are \( |A| = 2 \), \( |B| = p - 1 \), \( |C| = 2(p - 1) \), \( |D| = p^2 - p \), \( |E| = p - 1 \), \( |F| = 1 \) and \( |G| = p^2 - p \). It is easy to see that \( |D| = |G| = p^2 - p \), \( |B| = |E| = p - 1 \) and \( |C| = |B| + |E| \). Therefore, the total number of vertices are \( 2p^2 + 2p - 1 \).

Let \( d_S(x) \) denotes the degree of a vertex \( x \) in \( S \) and \( d(S_1, S_2) \) denotes the distance between the vertices of two sets \( S_1 \) and \( S_2 \). We know that the diameter is 3 i.e., \( \text{diam}(\Gamma(Z_{p^2} \times Z_{q^2})) = 3 \).

For any radio labeling \( \phi \) of a \( \Gamma(Z_{p^2} \times Z_{q^2}) \) must satisfy the following radio condition

\[
d(x, y) + |\phi(x) - \phi(y)| \geq \text{diam}(\Gamma(Z_{p^2} \times Z_{q^2})) + 1 = 4
\]

for any distinct vertices \( x, y \in V(\Gamma(Z_{p^2} \times Z_{q^2})) \). Let \( \phi \) be an optimal radio labeling for \( \Gamma(Z_{p^2} \times Z_{q^2}) \). We count the number of values needed for labeling and add the minimum number of forbidden values for \( \phi \). It is possible to use consecutive labels between the vertices of sets that are at distance three. There are no forbidden values associated with vertices which are distance three apart. This implies that there is no forbidden values between the sets \( A \) and \( D \), \( C \) and \( D \), and \( G \) and \( C \) because these are at distance three. Since the vertices of sets \( G \) and \( D \) are at distance 2 from the vertices set \( B \). Therefore there are \( 2(p - 1) \) forbidden values between them. Similarly, the forbidden values between vertices of sets \( E \) and \( D \) are \( 2(p - 1) \). Since, \( |D| - |A| - |C| - |E| - |B| \) number of vertices will be left, the distance between the vertices of set \( D \) and \( G \) is 2, therefore the \( 2(p^2 - 5p + 1) \) forbidden values for the remaining vertices of set \( D \) and vertices of set \( G \). Now, the distance between the vertices of set \( G \) is also 2, therefore \( 4p - 2 \) forbidden values for the remaining vertices of set \( G \). Since the set \( F \) is at distance 1 from the vertices of sets \( G \) and \( D \) and the set \( F \) contains only one vertex for \( q = 2 \). Therefore, there are \( 3 \) forbidden values between the vertices of set \( G \) and the vertex of set \( F \).

Thus the total number of minimum forbidden values are \( 2(p - 1) + 2(p - 1) + 4p - 2 + 2(p^2 - 5p + 1) + 3 = 2p^2 - 2p + 1 \). Adding the forbidden values to the number of vertices, we get the radio number of the given graph. Hence \( \text{rn}(\Gamma(Z_{p^2} \times Z_{q^2})) \geq 2p^2 - 2p - 1 + 2p^2 + 2p - 1 = 4p^2 - 2 \), for \( p \geq 5 \) and \( q = 2 \). \( \square \)

**Theorem 2.** Let \( p, q \) be two prime numbers with \( p \geq 5 \) and \( q = 2 \), then \( \text{rn}(\Gamma(Z_{p^2} \times Z_{q^2})) \leq 4p^2 - 2 \).

**Proof.** We shall provide a radio labeling of \( \Gamma(Z_{p^2} \times Z_{q^2}) \) with span \( 4p^2 - 2 \), which implies \( \text{rn}(\Gamma(Z_{p^2} \times Z_{q^2})) \leq 4p^2 - 2 \). From Theorem 1 and our convenience we define the following:

The radio labeling \( \phi : V(\Gamma(Z_{p^2} \times Z_{q^2})) \to \mathbb{Z}^+ \) is defined the following:

\[
\phi(a_i) = 2i, \quad \text{for } 1 \leq i \leq 2; \quad \phi(c_i) = 2i + 4 \quad \text{for } 1 \leq i \leq 2(p - 1); \quad \phi(b_i) = 4(i + p - 1) + 3 \quad \text{for } 1 \leq i \leq p - 1; \quad \phi(e_i) = 4(i + 2p - 2) + 3 \quad \text{for } 1 \leq i \leq p - 1 \quad \text{and} \quad \phi(f_1) = 4p^2 - 2.
\]

\[
\phi(d_i) = \begin{cases} 2i - 1; & 1 \leq i \leq 2p + 1 \\ 4(i - p - 1) + 1; & 2p + 2 \leq i \leq p^2 - p. \end{cases}
\]

\[
\phi(g_i) = \begin{cases} 4(i + 3p - 3) + 3; & 1 \leq i \leq p^2 - 5p + 2 \\ 4(p^2 - 2p) + 2(i - p^2 + 5p - 3) + 1; & p^2 - 5p + 3 \leq i \leq p^2 - p. \end{cases}
\]

It is easy to see that the span of \( \phi \) is equal to \( 4p^2 - 2 \).
**Claim:** The labeling $\phi$ is a valid radio labeling.

We must show that the radio condition

$$d(x, y) + |\phi(x) - \phi(y)| \geq \text{diam}(\Gamma(Z_{p^2} \times Z_q^2)) + 1 = 4$$

holds for all pairs of vertices $x, y \in V(\Gamma(Z_{p^2} \times Z_q^2))$, where $x \neq y$.

1: Since the labels on the vertices of set $A$ are even integers and the distance between the vertices of set $A$ are also 2. Therefore, the radio condition (3) is satisfied for the vertices of set $A$.

2: Consider the pairs $(b_i, b_j)$ with $i \neq j$, $1 \leq i, j \leq p - 1$, we have $d(b_i, b_j) = 1$ and the label difference for these pairs are $|\phi(b_i) - \phi(b_j)| = 4|\overline{i} - j| \geq 4$.

3: Consider the pairs $(c_i, c_j)$ with $i \neq j$, $1 \leq i, j \leq 2(p - 1)$, we have $d(c_i, c_j) = 2$ and $|\phi(c_i) - \phi(c_j)| = 2|\overline{i} - j| \geq 2$.

4: As $|\phi(d_i) - \phi(d_j)| = 2|\overline{i} - j| \geq 2$ or $|\phi(d_i) - \phi(d_j)| = 4|\overline{i} - j| \geq 4$ for pairs $(d_i, d_j)$ with $i \neq j$ and $d(d_i, d_j) = 2$.

5: As $|\phi(e_i) - \phi(e_j)| = 4|\overline{i} - j| \geq 4$ for pairs $(e_i, e_j)$ with $i \neq j$ and $d(e_i, e_j) = 1$.

6: As $|\phi(f_i) - \phi(f_j)| = 2|\overline{i} - j| \geq 2$ for pairs $(f_i, f_j)$ with $i \neq j$ and $d(f_i, f_j) = 2$.

7: Since the distance between the set $A$ with $D, C, E, F, B$ and $G$ is 3, 2, 2, 2, 1 and 1, respectively and $|\phi(a_i) - \phi(a_j)| \geq 1$, $|\phi(a_i) - \phi(c_j)| \geq 2$, $|\phi(a_i) - \phi(e_j)| \geq 2$, $|\phi(a_i) - \phi(f_j)| \geq 2$, $|\phi(a_i) - \phi(b_j)| \geq 3$, $|\phi(a_i) - \phi(g_j)| \geq 3$.

8: Since the distance between the set $B$ with $D, C, E, F, G$ and $E$ is 2, 2, 1, 1 and 1, respectively and $|\phi(b_i) - \phi(b_j)| \geq 2$, $|\phi(b_i) - \phi(g_j)| \geq 2$, $|\phi(b_i) - \phi(e_j)| \geq 3$, $|\phi(b_i) - \phi(f_j)| \geq 3$.

9: Since the distance between the set $C$ with $D, E, F$ and $G$ is 3, 2, 2 and 3, respectively and $|\phi(c_i) - \phi(c_j)| \geq 1$, $|\phi(c_i) - \phi(e_j)| \geq 2$, $|\phi(c_i) - \phi(f_j)| \geq 2$, $|\phi(c_i) - \phi(g_j)| \geq 1$.

10: Since the distance between the set $D$ with $E, F$ and $G$ is 2, 1 and 2, respectively and $|\phi(d_i) - \phi(f_j)| \geq 2$, $|\phi(d_i) - \phi(g_j)| \geq 2$.

11: Since the distance between the set $E$ with $F$ and $G$ is 1 and 2, respectively and $|\phi(e_i) - \phi(f_j)| \geq 3$, $|\phi(e_i) - \phi(g_j)| \geq 2$.

12: Since the distance between the set $F$ with $G$ is 1 and $|\phi(f_i) - \phi(g_j)| \geq 3$.

It follows that the radio condition (3) is for all pairs of vertices. Thus $\text{rn}(\Gamma(Z_{p^2} \times Z_q^2)) \leq \text{span}(\phi) \leq 4p^2 - 2$, for $p \geq 5$ and $q = 2$. This completes the proof. 

The labeling of the vertices $\Gamma(Z_{p^2} \times Z_q^2)$, for $p = 5$ and $q = 2$ is shown in Table 1.

**Table 1.** The vertex labels of $\Gamma(Z_{p^2} \times Z_q^2)$, for $p = 5$ and $q = 2$.

| Sets | The Labels of Vertices |
|------|------------------------|
| A    | 2, 4                   |
| B    | 23, 27, 31, 35         |
| C    | 6, 8, 10, 12           |
| D    | 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57 |
| E    | 39, 43, 47, 51         |
| F    | 98                     |
| G    | 55, 59, 61, 63, 65, 67, 69, 71, 73, 75, 77, 79, 81, 83, 85, 87, 89, 91, 93, 95 |

In Figure 1 the edge connectivity between sets show the complete bipartition of sets and circle around the sets representations the complete subgraphs.
Figure 1. Depicts one example of Theorem 2 for \( p = 5 \) and \( q = 2 \).

**Lemma 1.** \( r_n(\Gamma(Z_{p^2} \times Z_{q^2}))) \leq 4p^2 - 2 = 34 \), for \( p = 3 \) and \( q = 2 \).

**Proof.** For \( p = 3 \) and \( q = 2 \), \( |D| = |G| \) and \( |D| = |A| + |C| \), Therefore the labeling of the vertices \( \Gamma(Z_{p^2} \times Z_{q^2}) \), for \( p = 3 \) and \( q = 2 \) is shown in Table 2 and Figure 2.

**Table 2.** The vertex labels of \( \Gamma(Z_{p^2} \times Z_{q^2}) \), for \( p = 3 \) and \( q = 2 \).

| Sets | The Labels of Vertices          |
|------|---------------------------------|
| A    | 2, 4                            |
| B    | 23, 27                          |
| C    | 6, 8, 10, 12                    |
| D    | 1, 3, 5, 7, 9, 11               |
| E    | 15, 19                          |
| F    | 34                              |
| G    | 13, 17, 21, 25, 29, 31          |

It is easy to see that the radio condition is satisfied. \( \square \)

By combining Lemma 1, Theorems 1 and 2, we get the following theorem.

**Theorem 3.** Let \( p, q \) be two prime numbers with \( p \geq 3 \) and \( q = 2 \), then \( r_n(\Gamma(Z_{p^2} \times Z_{q^2})) = 4p^2 - 2. \)

**Theorem 4.** Let \( p, q \) be two prime numbers with \( p > q \geq 3 \). Then \( r_n(\Gamma(Z_{p^2} \times Z_{q^2})) \geq 2p^2q + q - 3. \)

**Proof.** Since, \( |V(\Gamma(Z_{p^2} \times Z_{q^2}))| = p^2q + pq - p - 1 \) and \( |A| = q^2 - q, |B| = p - 1, |C| = (p - 1)(q^2 - q), |D| = (p^2 - p)(q - 1), |E| = (p - 1)(q - 1), |F| = q - 1 \) and \( |G| = p^2 - p \). For any radio labeling \( \phi \) of a \( \Gamma(Z_{p^2} \times Z_{q^2}) \) must satisfy the following the radio condition

\[
d(x, y) + |\phi(x) - \phi(y)| \geq diam(\Gamma(Z_{p^2} \times Z_{q^2})) + 1 = 4
\]

(4)

for any distinct vertices \( x, y \in V(\Gamma(Z_{p^2} \times Z_{q^2})) \). Let \( \phi \) be an optimal radio labeling for \( \Gamma(Z_{p^2} \times Z_{q^2}) \). We count the number of values needed for labeling and add the minimum number of forbidden
values for $\phi$. It is possible to use consecutive labels between the vertices of sets that are at distance three. There are no forbidden values associated with vertices which are distance three apart. Since the distance between the vertices of sets $A$ and $D$, $C$ and $D$, and $C$ and $G$ is 3 and among the vertices of sets is 2. Therefore, it is possible to use consecutive labels between the vertices of sets $A$ and $D$, $C$ and $D$, and $C$ and $G$. This means that there are no forbidden values associative with the vertices of sets $A$ and $C$. As, $|D| > |A| + |C|$ and $|D| - |A| - |C| = p^2q - p^2 - pq^2 + p$ number of elements will be left in set $D$.

The vertices of sets $B$ and $E$ are at distance 2 from the vertices of of the set $G$, i.e., $d(G, B) = d(G, E) = 2$ and $d(B, B) = d(E, E) = 1$. Therefore, there are $2(p - 1)$ and $2(p - 1)(q - 1)$ forbidden values between the vertices of $B$ and $E$, respectively. As, $|G| > |B| + |E|$ and $|G| - |B| - |E| = p^2 - pq - p + q$ number of elements will be left in set $G$. Hence, there are $p^2q - pq^2 - pq + q$ total number of elements are left in sets $D$ and $G$. Now, $d(F, F) = d(D, F) = d(G, F) = 1$ and $|F| = q - 1$. This implies that there are $3(q - 1)$ forbidden values between set $F$ with set $D$ or $G$. Hence, the remaining $p^2q - pq^2 - pq + q - (q - 1) = p^2q - pq^2 - pq + 1$ vertices will be left. Since, $d(D, D) = d(G, D) = d(D, G) = 2$, therefore the remaining vertices will be the forbidden values between them i.e., $p^2q - pq^2 - pq + 1$ forbidden values.

Finally, the total number of minimum forbidden values will be the sum of all forbidden values which is equal to $2(p - 1) + 2(p - 1)(q - 1) + 3(q - 1) + p^2q - pq^2 - pq + 1 = p^2q - pq^2 + pq + q - 2$. By adding the forbidden values and the number of vertices to label provide a total of $2p^2q + q - 3$ labels, hence $rn(\Gamma(Z_{p^3} \times Z_{q^p})) \geq 2p^2q + q - 3$, for $p > q \geq 3$. □

![Figure 2](image-url)  
*Figure 2. The vertex labels of $\Gamma(Z_{p^3} \times Z_{q^p})$, for $p = 3$ and $q = 2$.***

**Theorem 5.** Let $p, q$ be two prime numbers with $p > q \geq 3$. Then $rn(\Gamma(Z_{p^3} \times Z_{q^p})) \leq 2p^2q + q - 3$.

**Proof.** We shall provide a radio labeling of $\Gamma(Z_{p^3} \times Z_{q^p})$ with span $2p^2q + q - 3$, which implies $rn(\Gamma(Z_{p^3} \times Z_{q^p})) \leq 2p^2q + q - 3$. The radio labeling $\phi : V(\Gamma(Z_{p^3} \times Z_{q^p})) \rightarrow \mathbb{Z}^+$ is defined the following:

- Let $a = q(p - 1) = 1$, $\phi(a_i) = 2i - 1$, for $1 \leq i \leq q^2 - q$.
- $\phi(d_i) = 2i$, for $1 \leq i \leq (p^2 - p)(q - 1)$.
- $\phi(c_i) = 2(q^2 - q) + 2i - 1$, for $1 \leq i \leq (q^2 - q)(p - 1)$.
- $\phi(e_i) = 2(p^2 - p)(q - 1) + 4i$, for $1 \leq i \leq (q - 1)(p - 1)$.
\( \phi(b_i) = 2(p^2 - p)(q - 1) + 4(p - 1)(q - 1) + 4i, \) for \( 1 \leq i \leq p - 1. \)
\( \phi(f_i) = 2(p^2 - p)(q - 1) + 4a + 2(p^2 - p - a - 1) + 3i, \) for \( 1 \leq i \leq q - 1. \)

\[
\phi(g_i) = \begin{cases} 
2(p^2 - p)(q - 1) + 2 + 4(i - 1); & 1 \leq i \leq a \\
2(p^2 - p)(q - 1) + 4a + 2(i - a - 1); & a + 1 \leq i \leq p^2 - p.
\end{cases}
\]

It is easy to see that the span of \( \phi \) is equal to \( 2p^2q + q - 3. \)

**Claim:** The labeling \( \phi \) is a valid radio labeling.

We must show that the radio condition

\[
d(x,y) + |\phi(x) - \phi(y)| \geq \text{diam}(\Gamma(Z_{p^2} \times Z_{q^2})) + 1 = 4
\] (5)

holds for all pairs of vertices \( x,y \in V(\Gamma(Z_{p^2} \times Z_{q^2})), \) where \( x \neq y. \)

1: Since the labels on the vertices of set \( A \) are odd integers and the distance between the vertices of set \( A \) are also 2. Therefore, the radio condition (5) is satisfied for the vertices of set \( A. \)

2: Consider the pairs \( (b_i, b_j) \) with \( i \neq j, 1 \leq i,j \leq p - 1, \) we have \( d(b_i, b_j) = 1 \) and the label difference for these pairs are \( |\phi(b_i) - \phi(b_j)| = 4|i - j| \geq 4. \)

3: Consider the pairs \( (c_i, c_j) \) with \( i \neq j, 1 \leq i,j \leq (p - 1)(q^2 - q), \) we have \( d(c_i, c_j) = 2 \) and
\[ |\phi(c_1) - \phi(c_2)| = 2|i - j| \geq 2. \]

4: As
\[ |\phi(d_i) - \phi(d_j)| = 2|i - j| \geq 2 \text{ for pairs } (d_i, d_j) \text{ with } i \neq j \text{ and } d(d_i, d_j) = 2. \]

5: As
\[ |\phi(e_1) - \phi(e_2)| = 4|i - j| \geq 4 \text{ for pairs } (e_i, e_j) \text{ with } i \neq j \text{ and } d(e_i, e_j) = 1. \]

6: As
\[ |\phi(f_1) - \phi(f_2)| = 2|i - j| \geq 2 \text{ for pairs } (f_i, f_j) \text{ with } i \neq j \text{ and } d(f_i, f_j) = 2. \]

7: As
\[ |\phi(g_1) - \phi(g_2)| = 1 \text{ and } |\phi(g_1) - \phi(g_2)| = 1, \]

8: Since the distance between the set \( A \) with \( D, C, E, F, B \) and \( G \) is 3, 2, 2, 2, 2 and 1, respectively and
\[ |\phi(a_1) - \phi(a_2)| \geq 1, |\phi(a_1) - \phi(a_2)| \geq 2, |\phi(a_1) - \phi(a_2)| \geq 2, |\phi(a_1) - \phi(a_2)| \geq 3, \]

9: Since the distance between the set \( B \) with \( D, G, C, E \) and \( F \) is 2, 2, 1, 1 and 1, respectively and
\[ |\phi(b_1) - \phi(b_2)| \geq 2, |\phi(b_1) - \phi(b_2)| \geq 2, |\phi(b_1) - \phi(b_2)| \geq 3, |\phi(b_1) - \phi(b_2)| \geq 3. \]

10: Since the distance between the set \( C \) with \( D, E, F \) and \( G \) is 3, 2, 2 and 3, respectively and
\[ |\phi(c_1) - \phi(c_2)| \geq 1, |\phi(c_1) - \phi(c_2)| \geq 2, |\phi(c_1) - \phi(c_2)| \geq 2, |\phi(c_1) - \phi(c_2)| \geq 1. \]

11: Since the distance between the set \( D \) with \( E, F \) and \( G \) is 2, 1 and 2, respectively and
\[ |\phi(d_1) - \phi(d_2)| \geq 3, |\phi(d_1) - \phi(d_2)| \geq 3. \]

12: Since the distance between the set \( E \) with \( F \) and \( G \) is 1 and 2, respectively and
\[ |\phi(e_1) - \phi(e_2)| \geq 3, |\phi(e_1) - \phi(e_2)| \geq 3. \]

13: Since the distance between the set \( F \) with \( G \) is 1 and
\[ |\phi(f_1) - \phi(f_2)| \geq 3. \]

It follows that the radio condition (5) is for all pairs of vertices. Thus
\[ \text{rn}(\Gamma(Z_{p^2} \times Z_{q^2})) \leq \\
\text{span}(\phi) \leq 2p^2q + q - 3, \text{ for } p > q \geq 3. \]

This completes the proof. \( \Box \)

Combining the Theorems 4 and 5, we get the following theorem:

**Theorem 6.** Let \( p, q \) be two prime numbers with \( p > q \geq 3. \) Then
\[ \text{rn}(\Gamma(Z_{p^2} \times Z_{q^2})) = 2p^2q + q - 3. \]

### 6. Open Problems

Since we have found the radio number of zero divisor graph for a class of commutative ring \( Z_{p^2} \times Z_{q^2}. \) The following remains open to explore for the readers.

**Question:** What is the radio number associated to zero divisor graph of commutative rings, \( Z_n \) and \( Z_m \times Z_n, \) in general or even for appealing cases of integers \( m \) and \( n. \)
7. Conclusions

This article underlines the calculation of radio number for zero divisor graphs comprised of commutative rings \( \mathbb{Z}_p^2 \times \mathbb{Z}_q^2 \). The theoretical results obtained in this article have potential to be useful in communication networks, circuit design, and sensor networks. In a particular context, they are useful in radio engineering where frequency bands allowed individuals to transmit voice, text, image, and data communications in local and global span even into space.

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