SEMINORMED $\ast$-SUBALGEBRAS OF $\ell^\infty(X)$

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ABSTRACT. Arbitrary representations of a commutative unital ($\ast$-) $F$-algebra $A$ as a subalgebra of $F^X$ are considered, where $F = \mathbb{C}$ or $\mathbb{R}$ and $X \neq \emptyset$. The Gelfand spectrum of $A$ is explained as a topological extension of $X$ where a seminorm on the image of $A$ in $F^X$ is present. It is shown that among all seminorms, the sup-norm is of special importance which reduces $F^X$ to $\ell^\infty(X)$. The Banach subalgebra of $\ell^\infty(X)$ of all $\Sigma$-measurable bounded functions on $X$, $\text{Mb}(X, \Sigma)$, is studied for which $\Sigma$ is a $\sigma$-algebra of subsets of $X$. In particular, we study lifting of positive measures from $(X, \Sigma)$ to the Gelfand spectrum of $\text{Mb}(X, \Sigma)$ and observe an unexpected shift in the support of measures. In the case that $\Sigma$ is the Borel algebra of a topology, we study the relation of the underlying topology of $X$ and the one of the Gelfand spectrum of $\text{Mb}(X, \Sigma)$.

1. INTRODUCTION

It is common to look at rings and algebras as families of functions over a nonempty set with values in a suitable ring or field. This is specially helpful if one wants to study the ideal structure of a ring or algebra which naturally involves topological notions, mainly compactness.

In this article, we try to summarize some observations about topological algebras in an abstract manner. One motivation comes from [2] which attempts to represent positive linear functionals on a given commutative unital algebra as an integral with respect to a positive measure on the space of characters of the algebra. This is done by realizing the algebra as a subalgebra of continuous functions over the character space.

During the present article we always assume that $A$ is a commutative algebra over the field $F = \mathbb{R}$ or $\mathbb{C}$ equipped with a seminorm $\rho$. In section 2, first we provide a brief overview of the theory of seminormed algebras and their Gelfand spectrum. Then, we assume that $A$ can be embedded into $(F^X, \rho)$ for a nonemty set $X$ where $\rho$ defines a submultiplicative seminorm on a subalgebra of $F^X$ including the image of $A$. This induces a seminormed structure on $A$ as well. Theorem 2.5 gives a necessary and sufficient condition for $X$ to be dense in the Gelfand spectrum of $A$, that is, when the topology induced by the seminorm is equivalent to the topology induced by the norm infinity.

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Motivated by [2], where positive linear functionals on an algebra are presented as an integral with respect to a constructibly Radon measure, in section 3, we consider a measurable structure \( \Sigma \) on \( X \) and study the spectrum of the algebra of bounded measurable functions on \((X, \Sigma)\), denoted by \( M_b(X, \Sigma) \).

We prove that positive measures on \( X \) lift to positive measures over the spectrum of \( M_b(X, \Sigma) \), but this lifting shifts the support of the original measure out of \( X \) modulo at most a countable subset of \( X \) (propositions 3.8 and 3.9). Then we choose \( \Sigma \) to be \( B_\tau \), the Borel algebra of a topology \( \tau \) on \( X \), and observe some connections between \( \tau \) and the spectrum of \( M_b(X, B_\tau) \) (proposition 3.11 and theorem 3.12).

**Notations.** Let \( X \) be a non-empty set and \( \mathcal{S} \) be a structure on \( X \) which induces a topology on \( X \). We denote this topology by \( \tau(X, \mathcal{S}) \). For instance, let \( \mathcal{S} \) be a family of functions, defined on \( X \), with values in a topological space. Then \( \tau(X, \mathcal{S}) \) is the coarsest topology on \( X \) which makes every function in \( S \) continuous.

The \( \sigma \)-algebra of sets induced on \( X \) by a set \( \Lambda \subseteq 2^X \) is denoted by \( \sigma(\Lambda) \). In particular, if \( \tau \) is a topology on \( X \) then \( \sigma(\tau(X, \mathcal{S})) \) is the \( \sigma \)-algebra, denoted by \( B_\tau \), of all Borel subsets of \((X, \tau)\).

### 2. Involutive Subalgebras of \( \ell^\infty(X) \)

The set theory which is applied in this paper is ZFC. Throughout this article all algebras are assumed to be involutive and commutative over a field \( F \) (which is either \( \mathbb{R} \) or \( \mathbb{C} \) as specified). Subsequently, all \((\ast)\) algebra homomorphisms are also supposed to be \( F \)-module maps.

**Definition 2.1.** A function \( \rho : A \rightarrow [0, \infty] \) is called a quasi-norm on \( A \) if

1. \( \rho(a^*) = \rho(a) \) \( \forall a \in A \)
2. \( \rho(a + b) \leq \rho(a) + \rho(b) \) \( \forall a, b \in A \) (subadditive),
3. \( \rho(ra) = |r|\rho(a) \) \( \forall r \in F \) \( \forall a \in A \),

it is called submultiplicative, if

4. \( \forall a, b \in A \) \( \rho(a \cdot b) \leq \rho(a)\rho(b) \).

A quasi-norm \( \rho \) on \( A \) is called a seminorm if \( \rho(a) < \infty \) for every \( a \in A \).

Let \( A \) be a commutative algebra and let \( \rho \) be a quasi-norm on \( A \). The set of all elements of \( A \) with a finite quasi-norm \( \rho \) is denoted by \( B_\rho(A) \), i.e.,

\[ B_\rho(A) = \{ a \in A : \rho(a) < \infty \}. \]

If \( \rho \) is a submultiplicative quasi-norm \( \rho \), it is clear that \( B_\rho(A) \) is a \( \ast \)-subalgebra of \( A \) and the restriction of \( \rho \) to \( B_\rho(A) \) is a seminorm. An element \( a \in A \) is called symmetric if \( a^* = a \). The set of all symmetric elements of \( A \) are denoted by \( S(A) \). An algebra \( A \) with a seminorm \( \rho \) forms a seminormed algebra if \( \rho \) is submultiplicative. For a seminormed algebra \((A, \rho)\), the set of all non-zero \( \ast \)-algebra homomorphisms \( \alpha : A \rightarrow F \) is denoted by \( \mathcal{X}(A) \). The subset of \( \mathcal{X}(A) \) consisting of all \( \rho \)-continuous homomorphisms is called the Gelfand spectrum of \((A, \rho)\) and denoted by \( \text{sp}_\rho(A) \) which is a closed subspace of \( \mathcal{X}(A) \).
Note that $S(A)$ is always an $\mathbb{R}$-algebra and if $\mathbb{F} = \mathbb{C}$, then $A = S(A) \oplus iS(A)$. Moreover there is a bijective correspondence between $\mathcal{X}(S(A))$ and $\mathcal{X}(A)$: If $\alpha \in \mathcal{X}(A)$, then clearly $\alpha|_{S(A)}$ is real valued and hence $\alpha|_{S(A)} \in \mathcal{X}(S(A))$. Also for every $\alpha \in \mathcal{X}(S(A))$, its extension defined by $\tilde{\alpha}(a + ib) = \alpha(a) + i\alpha(b)$ is well-defined and $\tilde{\alpha}|_{S(A)} = \alpha$. Next, we give a characterization of all $\rho$-continuous $\mathbb{F}$-valued homomorphisms. The following lemma was proved as Lemma 3.2 in [3].

Lemma 2.2. Let $\alpha \in \mathcal{X}(A)$. Then $\alpha \in \text{sp}_\rho(A)$ if and only if $|\alpha(a)| \leq \rho(a)$, for all $a \in A$.

The Gelfand spectrum $\text{sp}_\rho(A)$ (as well as $\mathcal{X}(A)$) naturally carries a Hausdorff topology as a subspace of $\mathbb{F}^A$ with the product topology. For a real number $r > 0$, let $D_r := \{c \in \mathbb{F} : |c| \leq r\}$. According to Lemma 2.2, $\text{sp}_\rho(A) \subseteq \prod_{a \in A} D_{\rho(a)}$. One simple approximation argument implies that every element in the closure of $\text{sp}_\rho(A)$ is a $*$-algebra $\mathbb{F}$-homomorphism. But it also belongs to $\prod_{a \in A} D_{\rho(a)}$. Therefore, the closure of $\text{sp}_\rho(A)$ is a subset of $\text{sp}_\rho(A) \cup \{0\}$ where $0$ is the constant linear functional zero on $A$. From now on, we use $\text{sp}_\rho(A)$ to denote it as a topological subspace of $\prod_{a \in A} D_{\rho(a)}$.

Note that the difference between the following corollary and [3, Corollary 3.3] is due to the fact that we exclude zero in the definition of $\mathcal{X}(A)$.

Corollary 2.3. Let $(A, \rho)$ be a commutative seminormed algebra. If $A$ is unital then $\text{sp}_\rho(A)$ is compact. If $\text{sp}_\rho(A)$ is compact then there exists an element $a_0 \in A$ such that $|\alpha(a_0)| \geq 1$ for every $\alpha \in \text{sp}_\rho(A)$.

Proof. If $A$ is unital, one may use the identity element, $1$, (for which we have $\alpha(1) = 1$ for every $\alpha \in \text{sp}_\rho(A)$) to show that $0$ does not belong to the closure of $\text{sp}_\rho(A)$. Therefore, $\text{sp}_\rho(A)$ is indeed a closed set in $\prod_{a \in A} D_{\rho(a)}$. And subsequently, $\text{sp}_\rho(A)$ is compact.

Now suppose that $\text{sp}_\rho(A)$ is compact. Therefore, $\text{sp}_\rho(A)$ is a closed subset of $\prod_{a \in A} D_{\rho(a)}$, not containing $0$. So, there exists $a \in A$ and $\epsilon > 0$ such that the projection on $a^{th}$ component of the above product does not intersect with the neighbourhood $(-\epsilon, \epsilon)$ of $0$. Thus, this particular element $a$, satisfies $|\alpha(a)| \geq \epsilon$ for each $\alpha \in \text{sp}_\rho(A)$. Let $k = \inf\{|\alpha(a)| : \alpha \in \text{sp}_\rho(A)\} \geq \epsilon$ and $a_0 := a/k$. The claim follows for $a_0$. 

Remark 2.4. Every commutative seminormed algebra $(A, \rho)$ can be embedded into the unital algebra $A_1 := A \oplus \mathbb{F}$ with multiplication $(a, \lambda)(b, \gamma) = (ab + \gamma a + \lambda b, \lambda \gamma)$. Defining $\rho_1(a + \lambda) = \rho(a) + |\lambda|$ we also obtain a seminorm on $A_1$ which makes the natural embedding $a \mapsto (a, 0)$ continuous. For each $\alpha \in \mathcal{X}(A)$, define the extension $\alpha'(a, \lambda) = \alpha(a) + \lambda$ which is obviously an element in $\mathcal{X}(A_1)$. So one can regard $\mathcal{X}(A)$ as a subset of $\mathcal{X}(A_1)$. Regarding $\mathbb{F}$ as a commutative algebra, we know that $\mathcal{X}(\mathbb{F})$ has only one element which is the identity map. This leads to the fact that $\mathcal{X}(A_1) \setminus \mathcal{X}(A)$ consists of exactly one element which is zero on $A$ and maps $(a, \lambda)$ to $\lambda$ (denoted by $\infty$). Clearly, $\infty \in \text{sp}_\rho_1(A_1)$, therefore $A$ is a closed maximal ideal of $A_1$. Moreover, if $\text{sp}_\rho(A)$ is not compact, $\text{sp}_\rho_1(A_1)$ is the one-point compactification of $\text{sp}_\rho(A)$.
Every element \( a \in A \) induces a map \( \hat{a} : \mathcal{X}(A) \rightarrow \mathbb{F} \) defined by \( \hat{a}(\alpha) := \alpha(a) \) for each \( \alpha \in \mathcal{X}(A) \). Every \( * \)-algebra homomorphism \( \phi : A \rightarrow B \) induces a mapping \( \phi_* : \mathcal{X}(B) \rightarrow \mathcal{X}(A) \cup \{0\} \) defined by \( \phi_*(\beta) = \beta \circ \phi \) for each \( \beta \in \mathcal{X}(B) \).

Suppose that \( B \) is equipped with a seminorm \( \rho \). The homomorphism \( \phi \) induces a seminorm \( \rho_\phi \) on \( A \) defined by \( \rho_\phi(a) = \rho(\phi(a)) \). If \( \rho \) is submultiplicative, then so is \( \rho_\phi \). The map \( \phi \) as a homomorphism between seminormed algebras \((A, \rho_\phi)\) and \((B, \rho)\) is continuous. Therefore \( \phi_* \) maps \( \mathfrak{sp}_\rho(B) \) continuously into \( \mathfrak{sp}_{\rho_\phi}(A) \).

Here we are mainly interested in the case where \( B \) is a subalgebra of \( \mathbb{F}^X \) for a non-empty set \( X \). This generally enables us to realize \( \mathfrak{sp}(A) \) relative to \( X \).

Let \( \rho \) be a submultiplicative quasi-norm on \( \mathbb{F}^X \) with \( \rho(1) \geq 1 \). There is a natural map \( e : X \rightarrow \mathcal{X}(\mathbb{F}^X) \) which, to every \( x \in X \), assigns the evaluation map \( e_x : \mathbb{F}^X \rightarrow \mathbb{F} \), defined by \( e_x(f) := f(x) \). This is clear that \( e_x \in \mathcal{X}(\mathbb{F}^X) \). We denote the set of all \( \rho \)-continuous evaluations by \( X_\rho := eX \cap \mathfrak{sp}_\rho(X) \).

Let \( \iota : A \rightarrow B_\rho(\mathbb{F}^X) \) be an algebra homomorphism. By abuse of notation, we use \( \iota_* \) to denote the induced map \( \iota_*|_X : (X, \tau) \rightarrow \mathfrak{sp}_\rho(A) \).

**Theorem 2.5.** Let \((A, \rho)\) be a commutative seminormed algebra and let \( \iota : A \rightarrow B_\rho(\mathbb{F}^X) \) be a homomorphism. Define \( \rho_* \) on \( A \) as before. Then \( \iota_*|_X \) is dense in \( \mathfrak{sp}_{\rho_*}(A) \) if and only if there exists \( D > 0 \) such that

\[
\rho_* (a) \leq D \cdot \sup_{x \in X_\rho} |e_x(\iota a)|,
\]

for all \( a \in A \).

**Proof.** Let \( \mathfrak{C}(\mathfrak{sp}_{\rho_*}(A)) \) denote the space of all continuous functions on \( \mathfrak{sp}_{\rho_*}(A) \). The Gelfand map \( \hat{\cdot} : A \rightarrow \mathfrak{C}(\mathfrak{sp}_{\rho_*}(A)) \) is continuous and hence for some \( C > 0 \),

\[
\sup_{\beta \in \mathfrak{sp}_{\rho_*}(A)} |\hat{a}(\beta)| \leq C \cdot \rho_* (a), \quad \forall a \in A.
\]

(\( \Rightarrow \)) Since \( \iota_*|_X \) is dense in \( \mathfrak{sp}_{\rho_*}(A) \) we have

\[
\sup_{x \in X_\rho} |e_x(\iota a)| = \sup_{\beta \in \mathfrak{sp}_{\rho_*}(A)} |\beta(a)|,
\]

and it suffices to take \( D = C \).

(\( \Leftarrow \)) In contrary, suppose that \( \alpha \in \mathfrak{sp}_{\rho_*}(A) \setminus \iota_*|_X \) is not dense. There exists \( f \in \mathfrak{C}(\mathfrak{sp}_{\rho_*}(A)) \) such that \( f(\alpha) = 1 \) and \( f|_{\iota_*|_X} = 0 \). Since \( \hat{\cdot} \) separates points of \( \mathfrak{sp}_{\rho_*}(A) \), \( \hat{\alpha} \in \mathfrak{C}(\mathfrak{sp}_{\rho_*}(A)) \), and \( \mathfrak{sp}_{\rho_*}(A) \) is compact, by Stone-Weierstrass theorem, it is dense in \( \mathfrak{C}(\mathfrak{sp}_{\rho_*}(A)) \). Therefore, for \( \epsilon > 0 \), there is \( a_\epsilon \in A \) with \( \| f - \hat{a}_\epsilon \| < \epsilon \). Take an \( \epsilon > 0 \) such that \( \frac{1 - \epsilon}{\epsilon} > CD \). Then \( |f(\alpha) - \alpha(a_\epsilon)| = |1 - \alpha(a_\epsilon)| < \epsilon \) or \( 1 - \epsilon < |\alpha(a_\epsilon)| < 1 + \epsilon \). Also \( |f(\iota_* e_x) - e_x(\iota a_\epsilon)| = |0 - \iota a_\epsilon(x)| < \epsilon \) for all \( x \in X_\rho \). Now

\[
\sup_{\beta \in \mathfrak{sp}_{\rho_*}(A)} |\beta(a_\epsilon)| \leq C \rho_* (a_\epsilon) \leq CD \sup_{x \in X_\rho} |e_x(\iota a)| \leq CD \epsilon < 1 - \epsilon,
\]

and hence \( |\alpha(a_\epsilon)| < 1 - \epsilon \), a contradiction.
The clear implication of Theorem 2.5 is that if one is to realise a unital commutative algebra as a subalgebra of \((\mathbb{F}^X, \rho)\) the natural choice for \(\rho\) is the sup-norm over \(X\) which is defined by

\[(1) \quad \|f\|_X = \sup_{x \in X} |f(x)|.
\]

We denote \(B_{\|\cdot\|_X}(\mathbb{F}^X)\) by \(\ell^\infty(X)\). According to Theorem 2.5 the image of \(X\) under the map \(x \mapsto e_x\) is dense in \(sp_{\|\cdot\|_X}(\ell^\infty(X))\) and also for \(\iota : A \rightarrow \ell^\infty(X)\), we have

\[\ell_x(\mathbb{X}_{\|\cdot\|_X})_{\|\cdot\|_X} = sp_{\|\cdot\|_X}(A).
\]

Suppose that \(\tau\) is a topology on \(X\) and denote the set of all \(\tau\)-continuous bounded \(\mathbb{F}\)-valued functions on \(X\) by \(C_b(X, \tau)\) or simply by \(C_b(X)\), if there is no risk of confusion. Let \(\iota : A \rightarrow \ell^\infty(X)\) an algebra homomorphism. Then one can show that the induced map \(\iota_*|_X : (X, \tau) \rightarrow sp_{\|\cdot\|_X}(A)\) is continuous if and only if \(\iota A \subseteq C_b(X)\).

It is well known that if \((X, \tau)\) is a completely regular Hausdorff space, then \(sp_{\|\cdot\|_X}(C_b(X))\) is the Stone-Cech compactification of \((X, \tau)\). Moreover, every Hausdorff compactification of \((X, \tau)\) is homeomorphic to the spectrum of a unital subalgebra of \(C_b(X)\). In the next section we study the algebra of bounded measurable functions for a measurable structure on \(X\).

3. MEASURABLE STRUCTURES ON \(X\)

Let \(\Sigma\) be a \(\sigma\)-algebra of subsets of \(X\). Let \(M_b(X, \Sigma)\) be the algebra of all bounded \(\Sigma\)-measurable functions on \((X, \Sigma)\). Suppose that \(M_b(X, \Sigma)\) separates points of \(X\). Hence \(X\) sits inside \(sp_{\|\cdot\|_X}(M_b(X, \Sigma))\), injectively and its image is dense. Although we are assuming that \(M_b(X, \Sigma)\) separates points of \(X\), this does not imply that \(\Sigma\) contains singletons as we see in the following example.

**Example 3.1.** Recall that a topological space \((X, \tau)\) is called a \(T_0\) space, if for every \(x \neq y \in X\), either \(x \notin \{y\}^\tau\) or \(y \notin \{x\}^\tau\). Then characteristic functions of open sets clearly separate points of \(X\). Let \(\omega_1\) be the first uncountable ordinal and \(X = \omega_1 + 1\). The family of sets \(R_a := \{x \in X : x > a\}\) form a basis for a topology on \(X\). This topology is evidently \(T_0\) and satisfies first axiom of countability at every point except \(\omega_1\). Although \(\{\omega_1\} = \bigcap_{a > \omega_1} R_a\), every countable intersection of sets \(R_a\) for \(a < \omega_1\) contains ordinals smaller than \(\omega_1\). Thus \(\{\omega_1\}\) does not belong to the \(\sigma\)-algebra generated by \(\{R_a : a \in \mathbb{X}\}\), denoted by \(\Sigma_r\), while \(M_b(X, \Sigma_r)\) separates points of \(X\). Note that the topology of \(\omega_1 + 1\) in this case is not first countable. Singletons always belong to the \(\sigma\)-algebra of Borel subsets of first countable spaces.

We denote \(sp_{\|\cdot\|_X}(M_b(X, \Sigma))\) by \(\xi_\Sigma X\). Note that since \(M_b(X, \Sigma)\) separates the points of \(X\), we can think of \(\xi_\Sigma X\) as a compactification of \(X\), i.e., it is a compact Hausdorff space such that there is an injection \(\psi : X \rightarrow \xi_\Sigma X\) such that \(\psi(X)\) is a dense subspace of \(\xi_\Sigma X\). Further, for every bounded \(\Sigma\)-measurable function \(f\) on \(X\), the function \(f \circ \psi^{-1}\) is continuously extendible over \(\xi_\Sigma X\). Also, \(\xi_\Sigma X\) is unique (up to a homeomorphism) with this property in this sense that for every other
compactification of $X$, say $\gamma X$, on which elements of $M_0(X, \Sigma)$ are continuously extendible, there is a continuous map $\iota: \gamma X \to \xi X$ agreeing on the images of $X$ in $\xi X$ and $\gamma X$.

Let $\chi_E$ be the characteristic function of $E$ as on $X$ for $E \in \Sigma$. Denoting its continuous extension to $\xi X$ with $\tilde{\chi}_E$ we have:

$$\tilde{\chi}_E^2 = \chi_E^2 = \tilde{\chi}_E;$$

thus it ranges over $\{0, 1\}$, which implies that $\tilde{\chi}_E$ itself must be the characteristic function of a set, say $\tilde{E}$ in $\xi X$.

**Lemma 3.2.** Let $E \in \Sigma$. Then $\overline{E} = \tilde{E}$ where $\overline{E}$ is the closure of $E$ in $\xi X$.

**Proof.** It is clear that $\tilde{E} = \tilde{\chi}_E^{-1}(\{1\})$ is closed and $E \subseteq \tilde{E}$. Thus $\overline{E} \subseteq \tilde{E}$. If $z \notin \overline{E}$, then for an open neighbourhood $U$ of $z$ we have $U \cap E = \emptyset$. Therefore there is a function $f \in M_0(X, \Sigma)$ and an open interval $I$ in $\mathbb{R}$ such that $z \in f^{-1}(I) \subseteq U$.

Let $F = f^{-1}(I) \in \Sigma$, then $E \cap F = \emptyset$, so $\chi_E \cdot \chi_F = 0$ and $\tilde{\chi}_E \cdot \tilde{\chi}_F = 0$. Since $\tilde{\chi}_F(z) = 1$ the later equation implies $\tilde{\chi}_E(z) = 0$. This contradicts assumption $z \in \tilde{E}$, therefore $\tilde{E} = \overline{E}$. \qed

Using the above lemma, we investigate some properties of $X$ as a subspace of $\xi X$.

**Corollary 3.3.**

1. $E$ is a clopen subset of $\xi X$ for every $E \in \Sigma$;
2. $\tilde{\Sigma} := \{\tilde{E} : E \in \Sigma\}$ forms a basis for the topology of $\xi X$;
3. $X$ is an open dense subspace of $\xi X$ whose subspace topology is discrete;
4. For a subset $Y \subset X$, $\overline{\chi} = \overline{Y}$ if and only if $Y$ is finite.

**Proof.** (1) Since $E = \tilde{E} = \tilde{\chi}_E^{-1}(\{1\}) = \tilde{\chi}_E^{-1}(\frac{1}{2}, \infty)$ and $\tilde{\chi}_E$ is continuous, we conclude that $\tilde{E}$ is clopen.

(2) The family $\{f^{-1}[0, 1] : f \in M_0(X, \Sigma)\}$ forms a basis for the closed sets of $\xi X$. Note that $E = f^{-1}[0, 1] \in \Sigma$ and $\overline{E} = \overline{f^{-1}[0, 1]}$ which is clopen by (1) and the conclusion follows.

(3) By (1), the closure of every element of $\Sigma$ is open in $\xi X$. Since the topology of $\xi X$ is Hausdorff and $\Sigma$ contains all singletons, singletons are closed. Therefore $\{x\}$ is a clopen for every $x \in X$ and hence $X$ is open in $\xi X$, dense by Theorem 2.5 and the subspace topology on $X$ is discrete.

(4) If $Y$ is finite, then since the topology of $\xi X$ is Hausdorff, it is also closed. Let $Y$ be an arbitrary subset of $X$. The set $\overline{Y} \subseteq \xi X$ is compact. If $\overline{Y} = \overline{Y}$, then $\{\{x\} : x \in Y\}$ is an open cover of $Y$ which will not have a finite subcover, if $Y$ is infinite. \qed

**Remark 3.4.** Let $\Sigma$ be a $\sigma$-algebra of subsets of an infinite set $X$. If there are infinitely many disjoint sets in $\Sigma$, then $M_0(X, \Sigma)$ is not separable. The proof is similar to the classical proof of the fact that $\ell^\infty(\mathbb{N})$ is not separable. Hence, in this case $\xi X$ is not metrizable. (It is classically known that for a compact space $X$, $C_0(X)$ is separable if and only if $X$ is metrizable ([1, Theorem 2.4]).)
By Lemma 3.3(2), $\xi_\Sigma X$ is totally disconnected. One can prove that $\{\hat{E} : E \in \Sigma\}$ contains all clopen subsets of $\xi_\Sigma X$. To see this suppose that $Y \subseteq \xi_\Sigma X$ is clopen. Since $\xi_\Sigma X$ is compact, so is $Y$. By 3.3(2), $Y = \bigcup_{i \in I} \hat{E}_i$ as an open set, for a family $\{E_i\}_{i \in I} \subseteq \Sigma$. Therefore $Y = \hat{E}_{i_1} \cup \cdots \cup \hat{E}_{i_n}$ for $i_1, \ldots, i_n \in I$, which belongs to $\tilde{\Sigma}$.

A topological space is called extremally disconnected if the closure of every open set is open. The following shows when $\xi_\Sigma X$ is extremally disconnected. For the relation between Boolean algebras and extremely disconnected spaces see [6, §3.5] or [7, 22.4]. Commutative algebras with extremely disconnected Gelfand spectrum are forming the commutative class of AW*-algebras where $F = C$.

An algebra of sets is said to be complete if it is closed under arbitrary union and hence intersection

**Proposition 3.5.** If $\xi_\Sigma X$ is extremely disconnected, then $\tilde{\Sigma}$ is complete. Conversely, if $\Sigma$ is complete, then $\xi_\Sigma X$ is extremely disconnected.

**Proof.** Suppose that $\xi_\Sigma X$ is extremely disconnected and let $\Delta \subseteq \tilde{\Sigma}$. Then $U = \cup \Delta$ is open and hence $\overline{U}$ is also open, thus belongs to $\tilde{\Sigma}$, say $\overline{U} = \hat{E}$. If $\hat{E} \setminus \cup \Delta \neq \emptyset$, then there exists is a clopen set $\emptyset \neq \hat{F} \subseteq \hat{E} \setminus \cup \Delta$. Therefore $\overline{U} \subseteq \hat{E} \setminus \hat{F} \subseteq \hat{E}$, a contradiction.

Now, suppose that $\Sigma$ is complete and let $U$ be an open set in $\xi_\Sigma X$. Take $\Delta \subset \Sigma$ such that $U = \cup \Delta$. Since $\Sigma$ is complete, $E = \cup \Delta \in \Sigma$ and $\overline{U} \subseteq \overline{E} = \hat{E}$. If $\hat{E} \setminus \overline{U} \neq \emptyset$ (so open) then it contains a nonempty clopen $\hat{F} \in \tilde{\Sigma}$. Now $\hat{E} \setminus \hat{F}$ is a clopen containing $\cup \Delta$ and strictly contained in $\hat{E}$, a contradiction. Thus $U = \hat{E}$ is clopen and hence $\xi_\Sigma X$ is extremely disconnected.

To prove the last proposition of this subsection, we need the following well-known lemma. The proof is straightforward, so we omit it here.

**Lemma 3.6.** Let $(X, \tau)$ be a second countable topological space and let $B_\tau$ be the $\sigma$-algebra of Borel subsets of $X$. Then $B_\tau$ is countably generated.

**Proposition 3.7.** Suppose that $\Sigma$ is a countably generated $\sigma$-algebra on $X \neq \emptyset$ such that every open subset of $\xi_\Sigma X$ belongs to $\sigma(\tilde{\Sigma})$. Let $\iota : A \rightarrow \ell^\infty(X)$ be an algebra homomorphism. Then the induced map $\iota_*|X : (X, \Sigma) \rightarrow sp_{||\cdot||_X}(A)$ is $\Sigma$-measurable if and only if $\iota \in M_b(X, \Sigma)$.

**Proof.** Note that by Corollary 3.3, $\tilde{\Sigma}$ forms a basis for the topology of $\xi_\Sigma X$. But since $\Sigma$ and subsequently $\tilde{\Sigma}$ are countably generated, every Borel subset of $\xi_\Sigma X$ belongs to $\tilde{\Sigma}$, by Lemma 3.6. A basic open set of $sp_{||\cdot||_X}(A)$ is of the form $\hat{a}^{-1}(O)$ where $O \subseteq F$ is open. Computing the inverse image of $\hat{a}^{-1}(O)$ under $\iota_*$ we have:

$$\iota_*|X^{-1}\hat{a}^{-1}(O) = \hat{\iota a}^{-1}(O) \cap X$$

$(\Rightarrow)$ Suppose that $\iota_*$ is $\Sigma$-measurable. If in contrary $\hat{\iota a} \notin M_b(X, \tau)$ for some $a \in A$, then there exists a set $O \subseteq F$, such that $\hat{\iota a}^{-1}(O) \cap X$ is not $\Sigma$-measurable and hence by (2), $\iota_*|X$ can not be $\Sigma$-measurable, a contradiction.

$(\Leftarrow)$ If each $\hat{\iota a}$ is $\Sigma$-measurable, then $\hat{\iota a}^{-1}(O)$ is $\Sigma$-measurable for any open $O \subseteq F$ and again by (2), $\iota_*|X$ is $\Sigma$-measurable.
3.1. **Measures on** $(X, \Sigma)$ **and** $\xi_\Sigma X$. Starting with a measurable structure $(X, \Sigma)$, we identified $X$ as an open dense subset of a totally disconnected compact space $\xi_\Sigma X$ where every bounded $\Sigma$-measurable function on $X$ extends continuously to $\xi_\Sigma X$. This naturally leads one to ask about the relation between measures on $(X, \Sigma)$ and $\xi_\Sigma X$.

**Proposition 3.8.** Let $\mu$ be a positive measure on $(X, \Sigma)$. Then $\mu$ extends to a Borel measure $*\mu$ on $\xi_\Sigma X$, satisfying

$$\forall E \in \Sigma \quad *\mu(\tilde{E}) = \mu(E).$$

**Proof.** Define a linear functional $L : C(\xi_\Sigma X) \to \mathbb{R}$ by

$$L(f) = \int_X f|_X \, d\mu, \quad \forall f \in C(\xi_\Sigma X).$$

Clearly $L$ is positive and hence by Riesz representation theorem, there exists a Borel measure $*\mu$ on $\xi_\Sigma X$ such that

$$L(f) = \int_{\xi_\Sigma X} f \, d*\mu, \quad \forall f \in C(\xi_\Sigma X).$$

Note that for every $E \in \Sigma$, $*\mu(\tilde{E}) = \int \tilde{\chi}_E \, d*\mu = L(\tilde{\chi}_E) = \int \chi_E \, d\mu = \mu(E)$. \(\square\)

Although the measure $*\mu$ obtained in Proposition 3.8 seems to be mainly supported on $X$, but in fact, the size of $X \cap \text{supp}(*\mu)$ is rather small as it is pointed out in the following proposition.

**Proposition 3.9.** Let $\mu$ be a finite Borel measure on $\xi_\Sigma X$ and $\Sigma$ contains all singletons. Then $X \cap \text{supp}(\mu)$ is at most countable.

**Proof.** By definition, a point $x \in \xi_\Sigma X$ belongs to $\text{supp}(\mu)$ if and only if for every neighbourhood $U$ of $x$, $\mu(U) > 0$. Every singleton $\{z\}$ for $z \in X$ is open in $\xi_\Sigma X$, thus for every $z \in X \cap \text{supp}(\mu)$, $\mu(\{z\}) > 0$. Since $\mu(\xi_\Sigma X) < \infty$, $X \cap \text{supp}(\mu)$ can not be uncountable. \(\square\)

**Corollary 3.10.** Let $\mu$ be a positive measure on $(X, \Sigma)$. If $\mu(\{x\}) = 0$, for some $x \in X$, then $x \notin \text{supp}(\mu)$.

**Proof.** Since $\{x\} \in \Sigma$ and $\mu(\{x\}) = 0$, $\chi_x \in M_0(X, \Sigma)$ and $\int_X \chi_x \, d\mu = 0$. Thus $*\mu(\{x\}) = \int_{\xi_\Sigma X} \tilde{\chi}_x \, d*\mu = 0$. But $\{x\}$ is open and hence $x \notin \text{supp}(\mu)$. \(\square\)

3.2. **Borel Algebra of a topology.** Let $(X, \tau)$ be a $T_1$ topological space and denote by $\mathcal{B}_\tau$ its Borel algebra. Since the topology is $T_1$, singletons are Borel and hence $M_0(X, \mathcal{B}_\tau)$ separates points of $X$. Clearly the inclusion $\iota : \mathcal{C}_b(X, \tau) \to M_0(X, \mathcal{B}_\tau)$ is continuous and hence $\iota_* : \xi_\mathcal{B}_\tau X \to \mathcal{S}(\mathcal{C}_b(X, \tau))$ is onto. Consequently, if $\tau$ is completely regular, then $\beta X$ is a continuous image of $\xi_\mathcal{B}_\tau X$ where $\beta X$ is the Stone-Čech compactification of $X$ (look at [4, 6.5]). But $\iota_*$ cannot be injective unless $\tau$ is extremely disconnected, in which case $\mathcal{B}_\tau = \tau$ and hence $\xi_\mathcal{B}_\tau$ and $\beta$ are identical. It is natural to ask if there is any relation between topological structure of $(X, \tau)$ and $\xi_\mathcal{B}_\tau X$. 

Let \( x \in X \) and \( \mathcal{N}_\tau(x) \) be the family of open neighbourhoods of \( x \) in \( \tau \) and \( \mathcal{N}_\tau(x) = \{ \tilde{U} : U \in \mathcal{N}_\tau(x) \} \). Define the halo of \( x \) in \( \xi_B \), \( X \) as

\[
h(x) := \bigcap \mathcal{N}_\tau(x).
\]

The set \( h(x) \) is compact and contains all points of \( \xi_B \), \( X \) that can not be distinguished from \( x \) via the image of \( \tau \). If \( \tau \) is Hausdorff, then for every \( x \neq y \in X \), there are open sets \( U_x, U_y \in \tau \) with \( U_x \cap U_y = \emptyset \). Thus \( U_x \cap \tilde{U}_y = \emptyset \), and therefore \( h(x) \cap h(y) = \emptyset \).

**Proposition 3.11.** If \( \tau \) is Hausdorff, then \( h(x) \) is open if and only if \( \{ x \} \) is open in \( (X, \tau) \).

**Proof.** If \( \{ x \} \) is open, then \( \{ x \} \in \mathcal{N}_\tau(x) \). Since \( \tilde{\mathcal{N}} \) is a basis for \( \xi_B \), \( X \), clearly, \( x \in h(x) \subseteq \{ x \} \). Conversely, if \( h(x) \) is open, then it is clopen and hence \( h(x) = E \in \mathcal{B}_\tau \). If \( E \neq \{ x \} \), then \( E \) contains another point \( y \in X, y \neq x \). Thus \( y \in h(x) \) which implies that \( h(x) \cap h(y) \neq \emptyset \), contradicting the above argument. \( \square \)

Proposition 3.11 can be read as \( h(x) = \{ x \} \) if and only if \( \{ x \} \) is open in \( (X, \tau) \). The following shows how the compactness of a Borel subset of \( (X, \tau) \) is reflected in \( \xi_B \), \( X \).

**Theorem 3.12.** Let \( Y \subseteq (X, \tau) \) be a Borel subspace. Then \( Y \) is compact if and only if \( \bar{Y} \subseteq \bigcup_{y \in Y} h(y) \).

**Proof.** \( \Rightarrow \) Suppose that \( Y \) is compact and let \( z \in \xi_B \), \( X \setminus \bigcup_{y \in Y} h(y) \). We show \( z \not\in \bar{Y} \). Since \( z \not\in \bigcup_{y \in Y} h(y) \), for each \( y \in Y \), there exists \( O_y \in \mathcal{N}_\tau(y) \) such that \( z \not\in \tilde{O}_y \). Now \( \{ O_y : y \in Y \} \) is an open cover of the compact set \( Y \). Let \( \{ O_{y_1}, \ldots, O_{y_k} \} \) be such that \( Y \subseteq \bigcup_{i=1}^k O_{y_i} \), then \( \bar{Y} \subseteq \bigcup_{i=1}^k \tilde{O}_{y_i} \), which proves \( z \not\in \bar{Y} \), and hence \( Y \subseteq \bigcup_{y \in Y} h(y) \).

\( \Leftarrow \) Suppose that \( \bar{Y} \subseteq \bigcup_{y \in Y} h(y) \), but \( Y \) is not compact. Then there exists an open cover \( \{ O_i \}_{i \in I} \) of \( Y \) with no finite subcover. So, for every finite subset \( \{ i_1, \ldots, i_n \} \) of \( I \),

\[
Y \cap \left( \bigcap_{k=1}^n O_{i_k} \right) \neq \emptyset.
\]

Since \( Y \) is Borel, \( \bar{Y} \) is compact and hence \( \tilde{\{ O_i \}}_{i \in I} \) forms a basis for an ultrafilter \( \mathcal{F} \) in \( \xi_B \), \( X \). Clearly \( \bar{Y} \in \mathcal{F} \) and hence \( z = \lim \mathcal{F} \in \bar{Y} \) (for more detail on filters see [8, §12]). For every \( y \in Y \), there exists \( i \in I \) such that \( O_i \in \mathcal{N}_\tau(y) \) and hence \( z \not\in \tilde{O}_i \). Thus \( z \not\in h(y) \subseteq \tilde{O}_i \). This proves

\[
z \in \bar{Y} \setminus \bigcup_{y \in Y} h(y),
\]
as desired. \( \square \)

It is worth mentioning that the results of 3.2 resemble significant similarities between properties of \( \xi_B \), \( X \) and nonstandard extensions of \( (X, \tau) \). We can consider \( \xi_B \), \( X \) as a nonstandard model of \( (X, \tau) \) and characterize halos as analogue to monads and so on. In this scope Theorem 3.12 is the analogue of Robinson’s theorem [5, Theorem III.2.2] about nonstandard extension of compact spaces.
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