Common factors in automatic and Sturmian sequences

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Abstract
In this short note we show that a $k$-automatic sequence and a Sturmian sequence cannot have arbitrarily large factors in common.

1 Introduction

Sturmian sequences are those given by the first differences of sequences of the form

$$(\lfloor na + \beta \rfloor)_{n \geq 1},$$

where $0 \leq a, \beta < 1$ and $a$ is irrational \[1\]. It is well-known that a Sturmian sequence cannot be $k$-automatic; that is, it cannot be generated by a finite automaton reading $n$ expressed in an integer base $k \geq 2$. This follows from the fact that the limiting frequency of 1’s in a Sturmian sequence is $\alpha$, whereas if a letter in a $k$-automatic sequence has a limiting frequency, that frequency must be rational \[3, Thm. 6, p. 180 \].

Recall that by factor of a word or sequence $s$, we mean a contiguous block of symbols $x$ inside $s$. Then a natural question is, can a Sturmian sequence and a $k$-automatic sequence have arbitrarily large finite factors in common? This question is related to a problem recently studied by Byszewski and Konieczny \[2\]: they wish to determine which generalized polynomial functions (these are sequences defined by expressions involving algebraic operations along with the floor function) can be $k$-automatic\[3\].

\[1\]We obtained these results in July 2016. The result was also mentioned at the Bridges between Automatic Sequences, Algebra, and Number Theory School held at the CRM in Montreal in April 2017, where Jakub Byszewski pointed out the connections to his work. It is for this reason that we are posting the proof of this result.
We also mention the work of Tapsoba [5]. Recall that the complexity of a word $s$ is the function counting the number of distinct factors of length $n$ in $s$. It is also well-known that Sturmian words have the minimum possible complexity $n + 1$ achievable by an aperiodic infinite word. Tapsoba shows another distinction between automatic sequences and Sturmian words by giving a formula for the minimal complexity function of the fixed point of an injective $k$-uniform binary morphism and comparing this to the complexity function of Sturmian words.

Our main result is the following:

**Theorem 1.** Let $x$ be a $k$-automatic sequence and let $a$ be a Sturmian sequence. There exists a constant $C$ (depending on $x$ and $a$) such that if $x$ and $a$ have a factor in common of length $n$, then $n \leq C$.

Note that this result would follow fairly easily from the frequency results mentioned previously, if $x$ is uniformly recurrent (meaning that for every factor $z$ of $x$ occurs infinitely often, and with bounded gap size between two consecutive occurrences). However, unlike Sturmian sequences, automatic sequences need not be uniformly recurrent: consider, for example, the 2-automatic sequence that is the characteristic sequence of the powers of 2. Our proof is therefore based on the finiteness of the $k$-kernel of $x$, along with the uniform distribution property of Sturmian sequences (similar arguments have previously been used by the second author [4]).

## 2 Proof of Theorem 1

**Proof.** Let $x = x_0x_1\cdots$ and $a = a_0a_1\cdots$. Since the factors of a Sturmian word do not depend on $\beta$, without loss of generality, we may suppose that $\beta = 0$ (or, in other words, that $a$ is a characteristic word). Then there exists an irrational number $\alpha$ such that $a$ is defined by the following rule:

$$a_n = \begin{cases} 1, & \text{if } ((n+1)\alpha) < \alpha; \\ 0, & \text{otherwise}. \end{cases}$$

Here $\{\cdot\}$ denotes the fractional part of a real number.

Suppose that for some $L$, the words $x$ and $a$ have a factor of length $L$ in common: i.e., for some $i \leq j$

$$x_i \cdots x_{i+L-1} = a_j \cdots a_{j+L-1}.$$

(We may assume that $i \leq j$ since $a$ is recurrent, but this is not important for what follows.)

Suppose that the $k$-kernel of $x$,

$$\{(x_{nk^r+s})_{n \geq 0} : r \geq 0 \text{ and } 0 \leq s < k^r\},$$

has $Q$ distinct elements. Let $r$ satisfy $k^r > Q$. There there exist integers $s_1, s_2$ with $0 \leq s_1 < s_2 < k^r$ such that

$$\begin{align*}
(x_{nk^r+s_1})_{n \geq 0} &= (x_{nk^r+s_2})_{n \geq 0}.
\end{align*}$$
Define
\[ d_1 := s_1 + j - i + 1 \]
\[ d_2 := s_2 + j - i + 1. \]

For all \( n \) satisfying \( i \leq nk^r + s_1 \) and \( nk^r + s_2 \leq i + L - 1 \) we have \( x_{nk^r+s_1} = a_{nk^r+d_1-1} \) and \( x_{nk^r+s_2} = a_{nk^r+d_2-1} \). Since \( x_{nk^r+s_1} = x_{nk^r+s_2} \), we have \( a_{nk^r+d_1-1} = a_{nk^r+d_2-1} \). This means that either the inequalities
\[ ((nk^r + d_1)a) < \alpha \text{ and } ((nk^r + d_2)a) < \alpha \] (1)
both hold, or the inequalities
\[ ((nk^r + d_1)a) \geq \alpha \text{ and } ((nk^r + d_2)a) \geq \alpha \] (2)
both hold.

If \( L \) is arbitrarily large, then there exist arbitrarily large sets \( I \) of consecutive positive integers such that every \( n \in I \) satisfies either (1) or (2). Without loss of generality, suppose that \( \{d_2\alpha\} > \{d_1\alpha\} \). Choose \( \epsilon > 0 \) such that \( \epsilon < (d_2\alpha) - (d_1\alpha) \). Note that \( d_2 - d_1 = s_2 - s_1 \), so \( \epsilon \) does not depend on \( L \) (or \( I \)). Since \( k^r\alpha \) is irrational, if \( I \) is sufficiently large then by Kronecker’s theorem (which asserts that the set of points \( \{na\} \) is dense in \((0,1)\)) there exists \( N \in I \) such that
\[ \{N(k^r\alpha) + d_2\alpha\} \in [\alpha, \alpha + \epsilon]. \]
By the choice of \( \epsilon \), this implies that
\[ \{N(k^r\alpha) + d_2\alpha\} \geq \alpha \text{ and } \{N(k^r\alpha) + d_1\alpha\} < \alpha, \]
contradicting the assumption that \( N \) satisfies one of (1) or (2). The contradiction means that \( L \) must be bounded by some constant \( C \), which proves the theorem. \( \square \)

**Example 2.** Consider the Thue-Morse word \( t = 01101001 \cdots \) given by the fixed point of the morphism \( 0 \rightarrow 01 \) and \( 1 \rightarrow 10 \), and the Fibonacci word \( f = 01001010 \cdots \) given by the fixed point of \( 0 \rightarrow 01 \) and \( 1 \rightarrow 0 \). The latter is Sturmian. The set of common factors is
\[ \{e, 0, 1, 00, 01, 10, 001, 010, 100, 101, 0010, 0100, 0101, 1001, 1010, 00101, 01001, 01010, 010010, 010100, 0100101, 1010010, 10100101, 0101001, 1010011, 0101101, 0101100, 1011001, 1010011, 0101001, 1010010, 1010011, 0101001, 1010010, 1010011\}, \]
so \( C = 8 \).

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