Gravitational waves in Intrinsic Time Geometrodynamics

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Gravitational waves are investigated in Intrinsic Time Geometrodynamics. This theory has a non-vanishing physical Hamiltonian generating intrinsic time development in our expanding universe, and four-covariance is explicitly broken by higher spatial curvature terms. Linearization of Hamilton’s equations about the de Sitter solution produces transverse traceless excitations, with the physics of gravitational waves in Einstein’s General Relativity recovered in the low curvature low frequency limit. A noteworthy feature of this theory is that gravitational waves always carry positive energy density, even for compact spatial slicings without any energy contribution from boundary Hamiltonian. The perturbations are expanded in terms of eigenfunctions of the Laplacian operator; these modes have discrete eigenvalues, and the equation governing time evolution of the mode coefficients is derived.

INTRODUCTION

In Intrinsic Time Quantum Geometrodynamics (ITQG) the fundamental dynamical variables are the unimodular spatial 3-metric $\bar{q}_{ij}$ and the traceless momentric variable $\bar{\pi}^i_j$. They are related to the standard General Relativity(GR) phase space variables, the three metric and conjugate momentum $(q_{ij}, \pi^i_j)$ by

$$\bar{q}_{ij} = q^{-1/3} q_{ij}; \quad \bar{\pi}^i_j = q^{1/3} \bar{q}_{jm} \left( \bar{\pi}^i_m - \frac{1}{3} q^{im} \bar{\pi}_m \right).$$

(1)

$\bar{\pi}^i_j$ is the traceless part of the momentric variable (first introduced by Klauder[2]), and the fundamental commutation relations for ITQG expressed through these variables are[1][3]

$$[\bar{q}_{ij}(x), \bar{q}_{kl}(y)] = 0, \quad [\bar{q}_{ij}(x), \bar{\pi}^k_l(y)] = i\hbar E^k_i \delta(x-y), \quad [\bar{\pi}^i_j(x), \bar{\pi}^k_l(y)] = i\hbar \left( \delta^k_j \bar{\pi}^i_l - \delta^i_j \bar{\pi}^k_l \right) \delta(x-y);$$

(2)

wherein $E^j_{mn} = \frac{1}{2} (\delta^j_m \bar{q}_{jn} + \delta^j_n \bar{q}_{mj}) - \frac{1}{2} \delta^j_m \bar{q}_{mn}$ is a traceless projector which also plays the role of the vielbein for the $(+, +, +, +)$ supermetric $G_{mpq} = E^i_{jmn} E^j_{ipq}$. It is noteworthy that the commutation relations of the momentric variables $\bar{\pi}^i_j$ are in fact the $su(3)$ algebra. The physical Hamiltonian $H^\text{phys}$ which generates evolution with respect to intrinsic time $T$ is[1],

$$H^\text{phys} = \frac{1}{\beta} \int_{\Sigma} d^3 x' \tilde{H}(x'),$$

(3)

with a local Hamiltonian of density weight one[13]

$$\tilde{H} = \sqrt{\bar{\pi}^m \bar{\pi}_m} + \alpha q (R - 2\Lambda) + (g \bar{C}^m n + b \sqrt{q} \bar{R}_{mn}) (g \bar{C}_m n + b \sqrt{q} \bar{R}_{mn}),$$

(4)

and $\alpha = -\frac{1}{(2\pi\hbar)^2}$. The Cotton-York tensor density (of weight one) is denoted by $\bar{C}^m n$, $\bar{R}_{mn}$ is the traceless part of the spatial Ricci tensor, and $b$ (of dimension $\frac{1}{3}$) and $q$ (dimensionless) are coupling constants of the theory. It follows that $H^\text{phys}$ is invariant under spatial diffeomorphisms, and the super-momentum constraint, $H_i = 0$, can be added to the total Hamiltonian of the theory. $H^\text{phys}$ is not a local constraint, but a true non-vanishing Hamiltonian which generates physical evolution of the variables $(\bar{q}_{ij}, \bar{\pi}^i_j)$ with respect to the change $\delta T = \frac{\hbar}{\beta} \delta \ln V$, wherein $V$ is the spatial volume of our universe[1][4][7]. This is a very physical description of dynamics which resolves the ‘problem of time’ in GR and its extensions, and renders them amenable to the usual rules of classical and quantum dynamics. Unlike many Horava gravity theories[7] with an extra ambient time parameter, here $T$ is constructed from the intrinsic geometry of the 3-metric - a degree of freedom has been used, the determinant of the metric, $q$, obeys the Heisenberg
equation of motion $d\ln q^{1/3} = 1$, and the trace of the momentum, $\tilde{\pi}_i^i$, is totally absent in $H_{phys}$. Thus, despite not having a local Hamiltonian constraint (as in ‘projectable’ Horava gravity theories with an extra, and possibly pathological, mode), only two degrees of freedom (($\tilde{q}_{ij}, \tilde{\pi}_i^j$) with $H_t = 0$) are subject to fluctuations. An integrated Hamiltonian $H_{phys}$ rather than a local Hamiltonian constraint also ensures that addition of higher spatial curvature terms does not lead to intractable second class constraints and/or inconsistencies in the constraint algebra. Einstein’s GR is the limit $\beta^2 = 1/6$ and $b = g = 0$ i.e. when the potential term in $H$ reduces to just the spatial Ricci scalar and cosmological constant terms. Without 4-covariance and arbitrary a priori lapse function $N$, Einstein’s theory is recaptured in the sense that $H_{phys}$ produces an effective or emergent lapse while the EOM and constraints of GR lead to precisely this same a posteriori value of the lapse function [14]; and the square-root form of the Hamiltonian in [3] is needed for this agreement [3]. The presence of higher spatial curvature terms needed for improved ultraviolet convergence and completion of the theory signal the explicit loss of 4-covariance, and spatial compactness eliminates infrared divergence. A corresponding Lagrangian of the Baierlein-Sharp-Wheeler type can be found [4], but it is rather cumbersome to work with, and not really needed. The Hamiltonian description of mechanics is both complete and consistent.

**Hamilton equations**

In this work classical gravitational wave equations will be derived through Hamilton’s equations. The Heisenberg equations of motion for the unimodular metric variable and momentric,

$$\frac{\partial \tilde{q}_{ij}(x)}{\partial T} = \frac{1}{\hbar} [\tilde{q}_{ij}(x), H_{phys}], \quad \frac{\partial \tilde{\pi}_i^k(x)}{\partial T} = \frac{1}{\hbar} [\tilde{\pi}_i^k(x), H_{phys}],$$

lead [13], in the classical limit of setting $\hbar \to 0$, to

$$\frac{\partial \tilde{q}_{ij}(x)}{\partial T} = \frac{1}{\beta H(x)} \tilde{E}_{ij}(x) \tilde{\pi}_i^k(x),$$

and

$$\frac{\partial \tilde{\pi}_i^k(x)}{\partial T} = -\frac{1}{\beta} \tilde{E}_{ij}(x) \int_{\Sigma} d^3x' \frac{1}{H(x')} \left[ \alpha q(x') \delta R(x') + \frac{1}{2} \delta \tilde{\pi}_i^j(x') \right] \delta(x - x').$$

wherein $\delta C_{mm} := g \delta C_{mm} + b \sqrt{g} R_{mn}$. In the above, we have used the fact that traceless part of the momentric commutes with the kinetic operator, $\tilde{\pi}_m^m \tilde{\pi}_m^m$ of $H_{phys}$, which is a Casimir invariant of the $su(3)$ algebra generated by $\tilde{\pi}_j^i$. Formulas collected in the Appendix yield

$$\frac{\partial \tilde{\pi}_i^k(x)}{\partial T} = -\frac{\alpha}{2\beta} q^{1/3} \int_{\Sigma} d^3x' q(x') \tilde{E}_{ij}(x) \left[ -R^{ij}(x') + \nabla_i^j(x') - \tilde{\pi}_i^j(x') \nabla^j(x) \right] \delta(x - x') \nabla^j(x)$$

Besides projecting out the traceless part of the Ricci tensor, $\tilde{R}^k_i = R^k_i - \frac{1}{3} \delta^k_i R$, the trace-projector also annihilates the $\nabla^2$ term. Upon integration, the result is

$$\frac{\partial \tilde{\pi}_i^k(x)}{\partial T} = \frac{\alpha q}{2\beta H} \tilde{R}^k_i - \frac{\alpha}{2\beta} q^{1/3} \tilde{E}_{ij} \nabla^i \nabla^j \left( \frac{1}{H} - \frac{1}{\beta} \tilde{E}_{ij}(x) \right) \int_{\Sigma} d^3x' \frac{1}{H(x')} \delta C_{mm} \delta \tilde{q}_{ij}(x).$$

**GRAVITATIONAL WAVES ON DE SITTER BACKGROUND**

**Background solution and linearization**

We shall consider background solutions of the Hamilton equations with constant spatial 3-curvature geometries compatible with the Cosmological Principle. Explicitly, these Robertson-Walker 3-metrics are

$$dl^2 = a^2(T) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

(10)
with \( k = +1, 0, -1 \) respectively for the compact \( S^3 \), and non-compact \( \mathbb{R}^3 \) and \( H^3 \) intrinsic 3-geometries. The metric is Einstein, \( R_{ij} = \frac{1}{2} \tilde{q}_{ij} R \) with \( R = \frac{8 \pi \alpha}{q^2} \); hence \( \tilde{R}_{ij} \) and the Cotton-York tensor \( \tilde{C}_{ij} \) both vanish. In addition, \( \tilde{q}_{ij} = q^{-1/3} q_{ij} \) is independent of \( a \) (and thus of \( T \)). For the background extrinsic geometry, we set the traceless momentic variable \( \tilde{\pi}^j_i \) to zero. These considerations lead to spatially covariantly constant Hamiltonian density \( \tilde{H} = \sqrt{\alpha q (R - 2\Lambda)} \), and each term in \( \tilde{\pi}^j_i \) vanishes, resulting in \( \dot{\tilde{\pi}}^j_i = 0 \). Thus, the pair of Hamilton equations \( \{ \tilde{q}_{ij}, \tilde{\pi}^j_i \} \), \( \{ \tilde{\pi}^j_i, \tilde{H} \} \) are identically satisfied, the initial data is preserved; and the Robertson-Walker 3-geometry with vanishing momentic indeed constitutes a background solution of the theory, even when \( H_{phys} \) contains higher curvature terms \( \mathcal{C}^{mn} \mathcal{C}_{mn} \) in addition to the scalar potential of Einstein’s GR. The constant spatial curvature solution with vanishing momentic is also a saddle point of the exact vacuum solution in the Cotton-York era.[1]

It is noteworthy that \( \tilde{H} \) involves only the square of \( \mathcal{C}^{mn} \) (which is identically zero for any constant 3-curvature metric). Thus we may state a simple theorem: any spatially constant curvature solution of GR is also a solution of ITQC.[10]

In the Arnowitt-Deser-Misner(ADM) decomposition of any 4-dimensional classical solution with coordinate time variable \( t \), the lapse function takes the form \( N = \frac{\sqrt{\alpha q} \ln q^{1/3}}{4 \beta \sqrt{6 - k}} \) modulo spatial diffeomorphisms.[4, 6]. We can therefore recast the background solution into the usual 4-dimensional Robertson-Walker form by reparametrizing the cosmic time interval as \( dt' := N dt = \frac{\ln q^{1/3} dt}{\sqrt{6 - k}} \). To wit,

\[
ds^2 = -dt'^2 + a^2(t') \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],
\]

and the above relation between \( a \) and \( t' \) reduces to \( \frac{da}{dt} = \sqrt{\alpha \beta} \sqrt{\frac{\Delta \alpha}{6 - k}} \). This yields, for the GR value of \( \beta = \sqrt{\frac{1}{6}} \), the de Sitter solution with \( a(t') = \sqrt{\frac{3}{\Lambda}} \cosh \left[ \sqrt{\frac{3}{2 \Lambda}} (t' - t_0) \right] \), \( \Lambda c \sqrt{3/2 \Lambda} (t' - t_0) \) respectively for \( k = +1, 0, -1 \) spatial slicings.

We shall linearize the Hamilton equations about the de Sitter background \( \{ *\tilde{q}_{ij}, *\tilde{\pi}^j_i \} \) with \( S^3 \) slicings, and expand the variables as \( \tilde{q}_{ij} = *\tilde{q}_{ij} + \tilde{h}_{ij} \) and \( \tilde{\pi}^j_i = *\tilde{\pi}^j_i + \Delta \tilde{\pi}^j_i \). In addition, on account of spatial diffeomorphism symmetry, we require the physical fluctuations to be transverse i.e. \( * \nabla^i \tilde{h}_{ij} = 0 \); and \( \tilde{h}_{ij} \), being perturbations of the unimodular \( \tilde{q}_{ij} \), are traceless \( *\tilde{q}^{ij} \tilde{h}_{ij} = 0 \) as well.

Differentiating with respect to \( T \), the Euler-Lagrange equation for the metric fluctuation,

\[
\frac{\partial^2 \tilde{q}_{ij}}{\partial T^2} = \frac{\partial}{\partial T} \left( \frac{1}{\beta H} \tilde{E}^l_{ki j} \right) \tilde{\pi}^k_l + \frac{1}{\beta H} \tilde{E}^l_{ki j} \frac{\partial \tilde{\pi}^k_l}{\partial T}, \tag{12}
\]

yields the linearized identity

\[
\frac{\partial^2 \tilde{h}_{ij}}{\partial T^2} = * \left( \frac{\partial}{\partial T} \left( \frac{1}{\beta H} \tilde{E}^l_{ki j} \right) \right) \Delta \tilde{\pi}^k_l + \Delta \left( \frac{\partial}{\partial T} \left( \frac{1}{\beta H} \tilde{E}^l_{ki j} \right) * \tilde{\pi}^k_l \right)
+ \Delta \left( \frac{1}{\beta H} \tilde{E}^l_{ki j} \right) * \left( \frac{\partial \tilde{\pi}^k_l}{\partial T} \right)
+ \left( \frac{1}{\beta H} \tilde{E}^l_{ki j} \right) \frac{\partial (\Delta \tilde{\pi}^k_l)}{\partial T}. \tag{13}
\]

On account of vanishing \( *\tilde{\pi}^j_i \), only the first and last terms remain. Linearization of \( \{ \tilde{q}_{ij}, \tilde{h}_{ij} \} \) yields

\[
(\beta^* \tilde{H}) \frac{\partial \tilde{h}_{ij}(x)}{\partial T} = * \tilde{E}^l_{ki j} \Delta \tilde{\pi}^k_l, \tag{14}
\]

and substituting for \( * \tilde{E}^l_{ki j} \Delta \tilde{\pi}^k_l \) in the first term of \( \{ \tilde{q}_{ij}, \tilde{h}_{ij} \} \) leads to the EOM for \( \tilde{h}_{ij} \) which is

\[
\frac{\partial^2 \tilde{h}_{ij}}{\partial T^2} = - \frac{\partial \ln \tilde{H}}{\partial T} \left( \frac{\partial \tilde{h}_{ij}}{\partial T} \right) + * \left( \frac{1}{\beta H} \tilde{E}^l_{ki j} \right) \frac{\partial (\Delta \tilde{\pi}^k_l)}{\partial T}. \tag{15}
\]

**Gravitational wave equation**

Explicit calculations lead to \( \frac{\partial \ln \beta \tilde{H}}{\partial T} = (\frac{R - 3\Lambda}{R - 2\Lambda}) \), and

\[
\frac{\partial (\Delta \tilde{\pi}^k_l)}{\partial T} = - \frac{\alpha q}{4 \beta^* \tilde{H}} q^{km} \left( \nabla^2 - \frac{1}{3} R \right) h_{ml} + \ldots . \tag{16}
\]
wherein by ... we mean the higher curvature contribution arising from $c_{mn}^n c_{mn}$. This shall be addressed later. The background Hamiltonian density $\hat{H}$ with zero momentic, $\hat{H}$, is covariantly constant, and $\delta \hat{H} = \frac{1}{R-2\Lambda}$ with $R = \frac{6k}{\sigma^2}$. Henceforth we drop the * label when there is no confusion, and it is understood that, apart from the perturbations $(\hat{h}_{ij}, \hat{\pi}_{ij})$, all other metric entities refer to the de Sitter background.

With the use of the above identities, (15) gives the resultant gravitational wave equation on de Sitter background, and expressed with respect to intrinsic time $T$, as

$$\frac{\partial^2 \hat{h}_{ij}}{\partial T^2} + \frac{(R - 3\Lambda)}{(R - 2\Lambda)} \frac{\partial \hat{h}_{ij}}{\partial T} + \frac{1}{4\beta^2(R - 2\Lambda)} (\nabla^2 - \frac{R}{3}) \hat{h}_{ij} + \cdots = 0$$

(17)

for physical, transverse traceless perturbations $\hat{h}_{ij}$. Without the higher order curvature terms, this reproduces the gravitational wave equation for GR on a de Sitter background for $\beta^2 = 1/6$. The factor $(R - 2\Lambda) = 6(\frac{k}{\sigma^2} - \frac{\Lambda}{3})$ vanishes only at the de Sitter “throat” at $t' = t_0'$ for $k = +1$, but this factor is otherwise always negative regardless of whether $k = +1, 0, -1$. Likewise, the coefficient $(R - 3\Lambda)$ is positive definite, and the $\frac{\partial h_{ij}}{\partial T}$ term plays the role of frictional force, tempered by the expansion of the universe. Bearing in mind $\frac{da}{dt} = \sqrt{6\beta} \sqrt{\frac{\Lambda}{3} a^2 - k}$, with $R = \frac{6k}{\sigma^2}$, and $dt = 2d\ln a(t')$, conversion of Eq. (17) into variation w.r.t. $t'$ yields $\left(\frac{1}{24\beta^2(1 - \frac{1}{3\sigma^2})} \frac{d^2}{dt^2} - \frac{1}{24\beta^2(1 - \frac{1}{3\sigma^2})} \nabla^2\right) \hat{h}_{ij} + ... = 0$

which implies the speed of the GR wave is 1 (in units $c = 1$ since we have previously used the notation $-d\tau^2$ instead of $-c^2d\tau^2$ in the de Sitter metric).

**Higher curvature contribution**

The higher curvature contribution to the wave equation which arises from (19) is

$$-\frac{1}{\beta^2} \frac{1}{\hat{H}(x')} \frac{\partial}{\partial x'} (E_{kij} E_{tuv}) (x) \int_{\Sigma} d^3x' \frac{1}{\hat{H}(x')} (\Delta C_{mn}(x') \frac{\delta C_{mn}(x)}{\delta q_{uv}(x)}),$$

(18)

and it can be explicitly computed in the Appendix. This may in turn be expressed as

$$- \int_{\Sigma} d^3x' \frac{q_{ij} + \frac{6}{\beta H(x')}}{2\beta H(x')} \Delta C_{mn}(x') \Omega_{mn} \delta(x - x'),$$

(19)

with $\Omega_{ijkl} := -\frac{1}{\beta H(x)} \left[ 4(q^{ik} e^{ilm} + q^{jk} e^{ilm} + q^{il} e^{jkm} + q^{il} e^{km}) \nabla_m + \frac{h}{4} \sqrt{q} (q^{ik} q^{jl} + q^{jk} q^{il}) (\nabla^2 - \frac{R}{3}) \right]$ for the background.

Upon integration, the equation with higher curvature contribution is,

$$\frac{\partial^2 \hat{h}_{ij}}{\partial T^2} + \frac{(R - 3\Lambda)}{(R - 2\Lambda)} \frac{\partial \hat{h}_{ij}}{\partial T} + \frac{1}{4\beta^2(R - 2\Lambda)} (\nabla^2 - \frac{R}{3}) \hat{h}_{ij} + \Omega_{ijkl} \Omega_{mn} \frac{kl}{6} \hat{h}_{kl} = 0,$$

(20)

wherein explicit the higher curvature term is $\Omega^{ijkl} \Omega_{mn} \hat{h}_{kl} = \frac{q_{ij}}{2\beta H(x)} \left( - \frac{a^2}{4} (\nabla^2 - \frac{R}{2}) + \frac{a^2}{4} \right) (\nabla^2 - \frac{R}{3}) \hat{h}_{ij}$. At this level of approximation, we may assume $d\tau = \frac{1}{3} d\ln V = 2d\ln a = -d\ln R$, or $e^{T - T_{non}} = \left( \frac{a}{a_{non}} \right)^2 = (1 + z)^{-2} = \frac{R_{non}}{R}$. Thus the equation may be expressed entirely in terms of $T$, $a$ or $z$-development. While all $k = 0, \pm 1$ are valid descriptives in this work[3] in which $d\tau = 2d\ln a$, in general ITQG uses $d\tau = \frac{1}{3} d\ln V$, thus favoring compact manifolds with finite spatial volumes.

In fact $\hat{h}_{ij}$ can be explicitly expanded in terms of $k = \pm 1$ or $S^3$ tensor (density)[17] harmonics of Ref.[10], $\hat{Y}_{(4,5)}^{Klm}_{ij}$, with $K \geq 2$. These are the two orthogonal transverse traceless eigenfunctions of Laplacian operator, $\nabla^2$, with (negative) eigenvalues $E_K = \frac{2 - K(K+2)}{2} = \frac{K(2-K)(K+2)}{6}$. A similar expansion can be done for the transverse traceless $\Delta \hat{h}_{ij}$. Through $h_{ab} = \sum_{l=4,5} C_{Klm}^{(l)} \hat{Y}_{Klm}^{(l)}(ab)$ and orthogonality of the eigenfunctions, (20) reduces to an equation for intrinsic time-dependence of the mode coefficients which carry discrete eigenvalues $\{Klm\}$. The resultant equation which encodes full-fledged information of all time dependence of the physical modes arising from gravitational perturbations during different epochs of the expanding de Sitter universe is

$$\ddot{C}_{(K)} + \frac{(R - 3\Lambda)}{(R - 2\Lambda)} \dot{C}_{(K)} + \frac{E_K}{4\beta^2(R - 2\Lambda)} C_{(K)} + \frac{b^2 - q^2}{4\beta^2\alpha(R - 2\Lambda)} E_K^2 C_{(K)} = 0;$$

(21)

wherein, for simplicity, we have denoted $C_{(K)} := C_{Klm}^{(l)}$ and defined $E_K := E_K - \frac{R}{3} = -\frac{K(K+2)R}{6}$ which is the eigenvalue of $\nabla^2 - \frac{R}{3}$. 
ENERGY OF GRAVITATIONAL PERTURBATIONS, AND FURTHER REMARKS

ITQG, as in Horava-type gravity theories, introduces only higher order spatial, but not time, derivatives into the wave equation through higher order spatial curvature terms which improve the ultra-violet convergence of the theory without compromising unitarity; yet the theory captures the physics of Einstein’s GR in long wavelength low curvature circumstances. From the wave equation, we can also see that the propagator for flat background will contain additional terms (up to the highest order of $1/p^6$ from the square of Cotton-York tensor in $H_{\text{phys}}$), but there will be no additional poles for $p_0$ in the absence of higher time derivatives. In the modified wave equation, the ratio of the contribution from higher curvature terms to that of Einstein’s GR is explicitly 

$$\frac{[16\pi]^2 b^2 g^2 (E K - \frac{1}{2})]}{\alpha} \approx 6 [b^2 a^2 + g^2 (K + 1)] K (K + 2)(\frac{\Gamma_{\text{planck}}}{a})^4,$$

which can be computed for any given set of $K, R, b$ and $g$. In the current epoch, this ratio too small to be of significance at LIGO’s characteristic detection wavelengths. However, departures from Einstein’s theory can become significant in the regime of large curvatures in the early universe and/or for large values of $K$. In the era of $a \to 0$, all physics is dominated by the Cotton-York term, the de Sitter solution is a saddle point of the exact vacuum state $[1]$; and, instead of Einstein’s GR, Eq. (21) will be dominated by the last term associated with linearized excitations in the Cotton-York era.

There is no contradiction in having both physical local energy density and spatial diffeomorphism invariance. Without the paradigm of 4-covariance, the total Hamiltonian density is not required to vanish; at each point, 2 of the d.o.f. remain even after spatial diffeomorphisms are taken into account. In perturbative excitations, the remaining physical d.o.f. are precisely the transverse traceless $(\bar{h}_{ij}, \Delta \bar{\pi}_{ij})$ modes. A noteworthy feature of ITQG in which time change is identified with variation in the logarithm of (finite) spatial volume is that gravitational waves always carry physical positive energy density, even for compact spatial slicings without any energy contribution from boundary Hamiltonian. The energy for the gravitational wave excitation is

$$H_{\text{phys}}[\bar{\pi}_{ij}, q_{ij}] - H_{\text{phys}}[\bar{\pi}_{ij}^*, q_{ij}] \approx \int \frac{1}{2\beta^3 R^6} \left[ \Delta \pi^m_n \Delta \pi^m_m + \frac{1}{4} \pi^{ij} \pi^{ij} (\nabla^2 - \frac{\delta}{\partial x^2}) \bar{h}_{ij} + (\beta \bar{H})^2 (q^{ij} O_{mn} \bar{h}_{ij}) (q^{ij} O_{mn} \bar{h}_{ij}) \right] d^3x.$$

(22)

The expression is positive-definite for $k = 0$ and +1 since $\alpha < 0$, but the $R = 6k/a^2$ term is negative for $k = -1$. Again by expanding in eigenmodes of the Laplacian operator, the expression of the Hamiltonian density can be computed explicitly in terms of the mode coefficients. In the classical theory, the momentic related to the time change of the metric via Eq. (13).

While it is true that adopting a d.o.f. as ‘clock’ can yield a non-vanishing local Hamiltonian even when 4-covariance is maintained, multi-fingered ‘time’ suffers from ordering problems and clock-dependent alternative histories. ITQG, or in this regard, Horava gravity theories, are not gauge-fixed versions of Einstein’s GR; they have true Hamiltonians, global time evolutions and ‘preferred slicings’. Positive-definite spatial metric bequeaths space-like separation, “a notion of ‘simultaneity’ and a common moment of a rudimentary ‘time’” $[1]$. Dynamical fields evolve, expansion of our universe is a ‘time’ change, and energy associated with that generator $H_{\text{phys}}$ is physical. “The universe allows no, and needs no, external clock” $[12]$, it and only it is all-encompassing and robust enough to be the universal and ultimate clock.

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APPENDIX

In three dimensions the Weyl curvature is zero, and the Riemann curvature tensor $R^n_{mjk} = \partial_j \Gamma^n_{mk} - \partial_k \Gamma^n_{mj} + \Gamma^u_{jse} \Gamma^n_{uk} - \Gamma^u_{kse} \Gamma^n_{mj}$ can be written completely in terms of the Ricci curvature through

$$R^n_{mjk} = q_{mk} R^n_j + \delta^n_j R_{mk} - q_{mj} R^n_k - \delta^n_k R_{mj} + \frac{R}{2} (\delta^n_k q_{mj} - \delta^n_j q_{mk}).$$

(23)
Variation of the connection is given by $\delta \Gamma_{in}^m = \frac{1}{2} q^{mr} \left( \nabla_i \delta q_{rn} + \nabla_n \delta q_{ri} - \nabla_r \delta q_{in} \right)$, while variation of the Ricci tensor and the curvature scalar result in

$$\delta R_{ij} = \frac{1}{2} \left( \nabla^a \nabla_i \delta q_{nj} + \nabla^a \nabla_j \delta q_{in} - \nabla^2 \delta q_{ij} - q^{kl} \nabla_i \nabla_j \delta q_{kl} \right);$$

$$\delta R = \delta(q^{ij} R_{ij}) = q^{ij} \delta R_{ij} - R^{ij} \delta q_{ij} = -R^{ij} \delta q_{ij} + \nabla^i \nabla^j \delta q_{ij} - q^{ij} \nabla^2 \delta q_{ij}. \tag{24}$$

Another useful result from commuting covariant derivatives involving a covariant divergence is

$$\nabla^2 \delta q_{ij} = \nabla_i \nabla^j \delta q - \frac{R}{2} \delta q_{ij} - q^{mn} R_{ij}^l \delta q_{nl};$$

$$\nabla_i \nabla^j \delta q - \frac{R}{2} \delta q_{ij} - q^{mn} R_{ij}^l \delta q_{nl} + (2 R^n \delta q_{nj} + R^n \delta q_{ni}) - (R_{ij} - \frac{1}{2} q_{ij} R) \delta q_n,$$}

where we have used \[28\]. Putting these facts together, we have, for transverse-traceless variations with respect to the $S^3$ background that

$$(\delta R_{ij})_{S^3} = \frac{1}{2} \left( \frac{R}{2} \delta q_{ij} + \frac{R}{2} \delta q_{ij} - \nabla^2 \delta q_{ij} - \nabla_i \nabla_j (q^{kl} \delta q_{kl}) \right) = -\frac{1}{2}(\nabla^2 - R) \delta q_{ij}, \tag{26}$$

and $(\nabla^a \delta q_{nj})_{S^3} = \frac{1}{2} R \delta q_{ij}$.

The Cotton-York tensor density of weight one $\tilde{C}^{ij}$ is the functional derivative of the Chern-Simons functional

$$W_{CS} = \frac{1}{4} \int \epsilon^{ikj} \left( \Gamma_{in}^m \partial_j \Gamma_{km}^n + \frac{2}{3} \Gamma_{in}^m \Gamma_{j}^s \Gamma_{km}^r \right) d^3 x,$$}

i.e.

$$\frac{\delta W_{CS}}{\delta q_{ij}} = \tilde{C}_{ij} = \epsilon^{imn} \nabla_m (R_{ij}^l - \frac{1}{4} R \delta_{ij}^l) = \frac{1}{2} (\epsilon^{imn} \nabla_m R_{ij}^l + \epsilon^{imn} \nabla_m R_{il}^j). \tag{27}$$

That $\tilde{C}^{ij}$ is symmetric and the last equality above can be established through the Bianchi identity $\nabla_m (R_{mn}^i - \frac{3}{2} R \delta_{mn}^i) = 0$. The corresponding functional variation of $\tilde{C}^{ij}$ is

$$2 \delta \tilde{C}^{ij} = (\epsilon^{imn} \nabla_m \delta R_{ij}^l + \epsilon^{imn} ((\delta \Gamma_{ij}^m) R_m^n - (\delta \Gamma_{mn}^i) R_j^n) + i \leftrightarrow j)$$

$$= \epsilon^{imn} \nabla_m \delta R_{ij}^l + \frac{1}{2} \epsilon^{imn} \left( \frac{q^{jr}}{\nabla_m \delta q_{rn} + \nabla_s \delta q_{rm} - \nabla_r \delta q_{ms}} \right) \left( R_{ij}^s - q^{sr} \left( \nabla_m \delta q_{rn} + \nabla_n \delta q_{rm} - \nabla_r \delta q_{mn} \right) R_{ij}^s \right) + i \leftrightarrow j. \tag{29}$$

Taking into account the symmetrization of the indices $i, j$; and $k, l$; we arrive at

$$\frac{\delta \tilde{C}^{ij}(x')}{\delta q_{kl}(x)} = \frac{1}{8} \left( q^{ik} \epsilon_{jlm} + q^{jk} \epsilon_{ilm} + q^{jl} \epsilon_{ikm} + q^{il} \epsilon_{jkm} \right) \nabla_m \left( \nabla^2 - \frac{R}{3} \right) \delta(x - x'). \tag{30}$$

Thus,

$$\frac{\delta \tilde{C}^{ij}(x')}{\delta q_{kl}(x)} = q^{ik} \frac{1}{2} \frac{\delta \tilde{C}^{ij}(x')}{\delta q_{kl}(x')} \delta(x - x') = -\frac{1}{2} \frac{\delta \tilde{C}^{ij}(x')}{\delta q_{kl}(x')} \delta(x - x'); \tag{31}$$

and $\Delta \tilde{C}^{ij}(x) = \int \frac{\delta \tilde{C}^{ij}(x')}{\delta q_{kl}(x')} \tilde{h}_{kl}(x') d^3 x'$. By explicit computations, it follows that on $S^3$,

$$O^1_{mn} \partial^m_{mn} q_i R_{ij}^l = \frac{q^{ik} \epsilon_{jlm}}{2 \beta^2 H^2} \left( -\frac{q^{ij} \epsilon_{jlm}}{4} (\nabla^2 - \frac{R}{2}) + \frac{b^2}{4} \right) (\nabla^2 - \frac{R}{3})^{2} \tilde{h}_{ij}. \tag{32}$$

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Higher curvature contributions to gravitational waves in other modified gravity theories have also been discussed in A. De Felice and S. Tsujikawa, Phys. Rev. D 91, 103506 (2015).

The case of $k = -1$ for GR has been considered previously in S. W. Hawking, T. Hertog and N. Turok, Phys. Rev. D 62, 063502 (2000).

Lee Smolin, remarks in Time in Cosmology Conference, Perimeter Institute (June 27-30, 2016).

As shown in [1], the Hamiltonian density is the square root of a positive semi-definite, self-adjoint operator that governs evolution with respect to intrinsic time.

For any ADM decomposition of the metric, $\mathcal{N} = \sqrt{\left(\partial_t \ln q^{1/3} - \frac{2}{3} \nabla_i N^i\right)^2}$. This in fact ensures the Hamiltonian constraint is satisfied classically in the form $\left(T^\tau \mathcal{K}\right)^2 = \frac{4\kappa^2}{\mathcal{N}} \dot{\mathcal{H}}^2$. Further details on the equivalence between GR and $\mathcal{H}_{\text{ADM}}$ can be found in [4–6].

An explicit metric representation of the momentric operator is $\pi^j_i = \hbar E^i_{jm} \nabla_m q^j$. See also Huei-Chen Lin and Chopin Soo, Chin. J. Phys. 53, 110106-1 (2015).

The eigenvalues of the Laplacian operator are not affected by multiplication with any power of $q$ which is covariantly constant.