Eigenmodes of aberrated systems: the tilted lens

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Abstract

When light propagates through aberrated optical systems, the resulting degradation in amplitude and phase has deleterious effects, for example, on resolution in imaging, spot sizes in focussing, and the beam quality factor of the output beam. Traditionally, this is either pre- or post-corrected by adaptive optics or phase conjugation. Here, we consider the medium as a complex channel and determine the corresponding eigenmodes which are impervious of the channel perturbation. We employ a quantum-inspired approach and apply it to the tilted lens as our example channel, a highly astigmatic system that is routinely used as a measure of orbital angular momentum. We find the eigenmodes analytically, show their robustness in a practical experiment, and outline how this approach may be extended to arbitrary astigmatic systems.

Supplementary material for this article is available online

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(Some figures may appear in colour only in the online journal)

1. Introduction

Transverse spatial modes that are solutions of the paraxial Helmholtz wave equation in vacuum are invariant to free-space propagation or media characterised by a uniform refractive index. Remarkably, orthogonality and linearity, which are key facets of vector spaces, guarantee a compact representation for spatial modes [1], akin to Jones matrix (vector) formalism for polarization but in higher dimensions [2]. Topical examples of mode families that inherit these characteristics, are the famous Laguerre–Gaussian (LG) and Hermite–Gaussian (HG) modes, forming solutions in cylindrical and Cartesian coordinates, respectively [3, 4]. For this reason, the modal description of light can be a powerful tool, including forming a measurement basis [5], for control of beam sizes [6, 7] to optical propagation description [8] and self-imaging selection rules [9]. Based on their constituting polynomials, it is possible to decompose some mode families into others and thus describe mode conversion in terms of constituting polynomials [10] but also using matrix formalism [11].

While structured light [12] has many applications, ranging from quantum [13] to classical [14] domains, concomitantly, they are fragile to inherent aberrations or imperfections in optical systems. In turn, the modal description of light has led to the search for the eigenmodes of the systems that light propagates through. Recent efforts have focused on developing methods for finding modes that are invariant to medium perturbations, with fruitful implementations comprising optical eigenmode imaging [15–17], super resolution [18] and in microcavities [19]. In many of these demonstrations the modal description of the eigenmodes using well known mode families (e.g. LG or HG modes) are often overlooked [20].

In this manuscript we use a quantum-inspired operator formalism, motivated by the linearity of optical transformation, to generate a matrix that describes a channel (equivalently medium) up until an arbitrary order. By way of example, we use the tilted lens aberration, associated with astigmatism, because of its extensive use in the sorting, detection and analysis of structure light carrying orbital angular momentum [3],
making it a highly topical example. From this matrix operator it is immediate to see that HG modes are the eigenmodes. We find a general framework for understanding the full eigenmode set, including superpositions grouped by eigenvalue class, and confirm the findings experimentally by creating these complex patterns of light and verifying their resilience through the channel. Although we treat the highly astigmatic tilted lens as our topical example, the framework outlined here will be of value for further studies of arbitrary aberrated optical channels.

2. Theory

2.1. The tilted lens

The tilted lens emerged as a useful orbital angular momentum (OAM) detector, converting OAM modes to HG modes in a single plane based on its rotation (tilting) [21]. It was a simplified version of an early approach based on cylindrical lenses [10, 22], now known as astigmatic mode converters, and can be described by connecting wave and ray optics [23, 24]. The evolution of LG beams under astigmatism has been investigated in [25–27] and [28] with the latter observing vortex splitting and elliptical intensity patterns similar to Ince–Gaussian modes [29]. Mode transformation between LG and HG modes has been analysed regarding OAM conservation [3], geometrical phase [30, 31], vector beam transformations [32] and as an OAM and radial mode detector [33–36]. It has also been performed holographically [37] and used to identify superpositions of transverse modes in combination with machine learning [38]. Henceforth we will refer to the tilted lens as an astigmatic mode converter.

Because HG modes feature strongly in the application of such astigmatic mode converters [11, 21, 38], we use this as our basis for the modes and the operator. Let us consider the vector $|\Psi\rangle$ composed of every possible bi-dimensional Hermite–Gaussian transverse mode from the zeroth order up until the $N_{\text{max}}$th order as in

$$|\Psi\rangle = \sum_{o=0}^{N_{\text{max}}} \sum_{n=0}^{o} e_{o-n}^{\lambda} |HG_{o-n}\rangle,$$

with the spatial domain projection defined as

$$\langle x,y|HG_{m}^{n}(x,y) = HG_{m}^{n}(x,y) = H_m(\tilde{x})H_n(\tilde{y}) e^{i k z} e^{i \phi_{m,n}} e^{-i \left(\frac{\left|x^2+y^2\right|}{2}\right)} G(\tilde{x},\tilde{y}),$$

where $\tilde{x} = x/w(z)$, $\tilde{y} = y/w(z)$ the Gouy phase is $\phi_{m,n} = (m + n + 1) \arctan(z/z_R)$, $w(z) = w_0 \sqrt{1 + (z/z_R)^2}$ is the beam diameter for a $w_0$ beam waist, $z_R = \pi w_0^2 / \lambda$ is the Rayleigh length, $G(\tilde{x},\tilde{y})$ is the Gaussian envelope, $H_m(\tilde{x})$ is a Hermite polynomial of order $m$ and $R(z) = z \left[1 + (z/z_R)^2\right]$ is the radius of curvature of the wavefront. Because the HG basis forms a complete vector space, it can be used as a basis for transverse light fields. As such, the expansion in equation (1) represents a superposition of multiple scalar HG fields having corresponding normalised coefficients given by $c_{\lambda}^{\lambda}$. The dynamics of the optical field can be described using these coefficients [8] and this will become essential in expressing the action of the tilted lens on light beams.

Similar to the Jones formalism, in the modal space, we can describe the action of one or more optical elements as an operator $\mathcal{M}$ which operates in any state vector $|\Psi\rangle$ as in

$$|\Psi'\rangle = \mathcal{M}|\Psi\rangle \tag{3}$$

and if the vector $|\Psi'\rangle = \lambda|\Psi\rangle$ where $\lambda$ is a complex scalar, then $|\Psi'\rangle$ is an eigenvector of this operator. It has become more common in recent times to use quantum notation in optics because of the links between the two fields of study [2]. In the case of the aforementioned operator in terms of optical fields, this is equivalent to transforming a light beam having the profile $\Psi(x,y)$ to a new beam $\Psi'(x',y')$. The operator $\mathcal{M}$ can include phase perturbations, scattering effects or free space diffraction. Such optical channels can be represented either by transmission functions, integral transforms or combinations of both in the continuous basis. By using matrix formalism, we can collapse multiple operations into one operator that can be expressed in the complete modal basis.

Conveniently, in this work we use the HG basis for describing our operators. We will refer to [9] and rewrite it compactly as

$$\mathcal{M}_j = \sum_{o=0}^{N_{\text{max}}} \sum_{n=0}^{o} e^{i (n-2)-n} |HG_{o-n}\rangle\langle HG_{o-n}|.$$ \tag{4}

This operator defines the propagation of an optical mode to the tilted lens by a distance $d$ (where $d$ is the focal length of the untilted lens), the action of the lens itself, which is rotated around the $z$ axis by a small angle, and the subsequent propagation after the tilted lens by a distance $d$, to what we will call the conversion plane.

From the definition of $\mathcal{M}_j$ it is immediate to see that HG modes in the $x/y$ coordinates are all eigenvectors of this operator, independent of the order. However, since there are an infinite number of modes with discrete indices and their eigenvalues are given by a periodic function that increases with mode order, only a limited number of eigenvalues will be possible in this system. Therefore, regarding this transformation, an infinite number of HG modes can be grouped by a finite number of eigenvalues. This brings us to the often overlooked property of eigenvectors: any superposition of eigenvectors of the same eigenvalue is also an eigenvector.

We can grasp the concept in figure 1. The astigmatic mode converter can turn an LG beam into a HG mode in a propagation plane. This transformation does not change pure HG modes, but it can change an arbitrary superposition of them. However, by choosing a superposition of modes grouped by their eigenvalues, we generate states that are invariant through this astigmatic system.

2.2. Conversion plane

We derived the distance $d$ (see supplementary information) from the lens to the conversion plane to be
of the beam, where relation defined in equation (5) is resilient to this aberration (forth row).

\[ d = \frac{d_a \pm \sqrt{7}d_b}{d_e} , \]  
(5a)

where

\[ d_a = f z_0^2 \left( \cos^2(\beta) \left(z_R^2 - 2fz_0\right) + \eta^2 \right) , \]  
(5b)

\[ d_b = -f^2 \cos^2(\beta) \left((f - z_0)^2 + z_R^2\right) \]  
\[ - 2fz_0 \cos^2(\beta) \left(f^2 - 2fz_0 + z_R^2\right) \]  
\[ + z_R^2 \left((f - z_0)^2 + z_R^2\right) \]  
(5c)

\[ d_e = 2z_R^2 \left(f \cos^2(\beta)(f - z_0) - fz_0 + z_R^2\right) . \]  
(5d)

Here, \( z_0 \) is the distance between the lens and the waist plane of the beam, \( z_R^2 = z_0^2 + z_R^2 \) and \( \eta \) is given by

\[ \eta = \sqrt{\frac{f^2 z_0^4 (\eta_a + \eta_b)}{\eta_c}} , \]  
(6)

\[ \eta_a = \cos^4(\beta) \left(z_R(z_R - 2f) + z_0^2\right) \times \left(z_R(2f + z_R) + z_0^2\right) , \]  
(7a)

\[ \eta_b = -2z_R^4 \cos^2(\beta) + \eta^2 , \]  
(7b)

\[ \eta_c = (f - z_0)^2 \left(f^2 \cos^2(\beta) - 2fz_0 \cos^2(\beta) + z_R^2\right)^2 . \]  
(7c)

It is important to notice that \( \eta \) can result in imaginary quantities, about which we interpret that no conversion happens. This sets an important requirement for the conversion to happen, being

\[ |\beta| > \arccos \left(\frac{\sqrt{z_0^2 + z_R^2}}{\sqrt{2z_0^2 + z_R^2}}\right) . \]  
(8)

To validate equation (5) we measured the conversion distance as a function of the tilting angle using different parameters. Figure 2 shows that the experimental data matches the theoretical predictions. More details on this derivation can be found in the supplementary information.

2.3. Generalization of eigenvalues

The eigenvalues in equation (4) are determined by the HG indices, constituting a complex argument for an exponential function, resulting in a \( 2\pi \) period for this function. Since the indices are all positive integers, the minimal difference between them is 1 and, consequently, the divisor factor 4 makes it so this period is divided into only 8 values. Given the form of the operator in equation (4), we can define \( m - n = j \in \{0, 7\} \). Due to the \( 2\pi \) periodicity of the function, there will be only eight possible eigenvalues for this operator, where \( \lambda_j = e^{i\pi j} \) is the \( j \)th eigenvalue.

Notably, \( j \) can assume negative values, but given the periodicity of the exponential function, we can use modular arithmetic to show that any negative values can be mapped to positive ones according to \( j_{pos} = 8\alpha + j_{neg} \), where \( \alpha \) is a positive integer. As an example, the value \( j = -1 \) would give the same eigenvalue as \( j = 7 \), depicted in figure 3, or a value of

![Figure 1. Laguerre-Gaussian modes are converted in HG modes when passed through a tilted lens (first row) after a propagation distance \( d \). HG modes in x/y directions are insensible to this transformation (second row) but an arbitrary combination of them is not (third row). A superposition of modes following the equivalence transformation (second row) but an arbitrary combination of them is not (third row).](image1)

![Figure 2. Conversion plane distance as a function of the tilting angle. System parameters used were \( w_0 = 0.5 \) mm, \( f = 25 \) cm and \( z_0 = 97 \) cm.](image2)
\[ |\Psi_j\rangle = \sum_{n=0}^{N_{\text{max}}-1} \sum_{\alpha=0}^{\frac{N_{\text{max}}-2n}{8}} c_{n}^{\alpha} \langle 8\alpha+j+n | H_{8\alpha+j+n}^{\alpha} \rangle + \sum_{n=0}^{N_{\text{max}}-1} \sum_{\alpha=1}^{\frac{N_{\text{max}}-2n}{8}} c_{n}^{8\alpha-n-j} \langle HG_{n}^{\alpha} + n-j | HG_{n}^{\alpha} + n-j \rangle \]  

where \( |\Psi_j\rangle \) is the linear combination of eigenvectors associated with the eigenvalue \( \lambda_j \) and \( N \) is the modal order up to which modes will be considered. The first summation includes modes with \( m > n \) where the second one includes modes with \( n > m \). The \( |\rangle \) notation represents the floor function. The summation over HG modes can be done up to an arbitrary number of modes given that established relations are followed. Truncating the sum up until an arbitrary maximum mode order is done in order to have an estimation of mode sizes and propagation dynamics, according to \([6, 7]\). From the Gouy phase, we know that the maximum mode order considered will dictate how ‘fast’ (in propagation) the intensity profile will vary and the lowest order included will determine how ‘fast’ the field will converge to its far-field pattern. Therefore, one can manipulate the propagation dynamics by excluding/including modes of given orders.

Given a single equivalence class \([j]\), if more than one mode is included, there is an infinite number of possible superpositions to be considered, since all \( c_{m}^{n} \) coefficients are continuous complex variables. Of course, the more HG modes are included in this summation, the more different intensity patterns look from each other. For simplicity, we will use all coefficients \( c_{m}^{n} = 1 \) since the resilience character of this modes is independent of an initial phase and relative non-zero weights. Since the auxiliary eigenvectors are composed of orthogonal modes and each mode can only be associated with only one auxiliary eigenvector, they are also orthogonal among themselves, not mattering the choice of relative weights and/or phases. The table 1 explicitly
Table 1. table depicting a superposition of modes grouped by eigenvalue up to a order of $N_{\text{max}} = 6$ with all coefficients set to 1.

| j | Eigenvalue | mode superposition |
|---|---|---|
| 0 | 1 | $|HG_{0}^{1}⟩ + |HG_{1}^{2}⟩ + |HG_{2}^{3}⟩ + |HG_{3}^{4}⟩$ |
| 1 | $\sqrt{-1}$ | $|HG_{0}^{2}⟩ + |HG_{1}^{3}⟩ + |HG_{2}^{4}⟩$ |
| 2 | $i$ | $|HG_{0}^{3}⟩ + |HG_{1}^{4}⟩ + |HG_{2}^{5}⟩ + |HG_{3}^{6}⟩$ |
| 3 | $(-1)^{3/4}$ | $|HG_{0}^{4}⟩ + |HG_{1}^{5}⟩ + |HG_{2}^{6}⟩$ |
| 4 | $-1$ | $|HG_{0}^{5}⟩ + |HG_{1}^{6}⟩ + |HG_{2}^{7}⟩ + |HG_{3}^{8}⟩$ |
| 5 | $-\sqrt{-1}$ | $|HG_{0}^{6}⟩ + |HG_{1}^{7}⟩ + |HG_{2}^{8}⟩$ |
| 6 | $-i$ | $|HG_{0}^{7}⟩ + |HG_{1}^{8}⟩ + |HG_{2}^{9}⟩ + |HG_{3}^{10}⟩$ |
| 7 | $(-1)^{3/4}$ | $|HG_{0}^{8}⟩ + |HG_{1}^{9}⟩ + |HG_{2}^{10}⟩ + |HG_{3}^{11}⟩$ |

Figure 5. The experimental setup consists of a mode being generated on the DMD and a imaging system. Initially, in situation A the imaging system faithfully reconstructs the generated field where we position a camera. In situation B we change to a scheme for astigmatic mode converters and capture the image at the conversion plane.

summarizes the superpositions considered for examples in this work, with corresponding equivalence classes and eigenvalues up until $N_{\text{max}} = 6$. The normalization factor was omitted.

3. Experimental setup

The experiment consists of a beam generation stage followed by a lens and a camera. The beam generation starts with the expansion and subsequent collimation of the beam, which impinges on a digital micromirror-device (DMD). In this device we encode binary holograms \([41, 42]\) which generates the desired field in a superposition of modes of a waist of $w_0 = 0.75$ mm. In order to confirm that we are generating the intended superposition of modes, we first build a 4\(f\) imaging system using a single lens, as depicted in figure 5 situation A. This arrangement images the auxiliary eigenvectors right after being shaped by the DMD right into the charged-coupled device (CCD) camera, where the pattern is confirmed to be the one encoded.

Afterwards, we know that the operation described by equation (4) only works in a specific plane in propagation after the lens, so it is necessary to make slight alterations and move from situation A to B, as depicted in figure 5. At a distance $f$ of the DMD screen we put a lens of focal length $f$, and the same distance away from the lens we place the camera. This arrangement, similar to the one depicted in figure 5 situation B, constitutes a 2\(f\) system in which the far-field distribution is obtained. It is used as an intermediate step when switching from situation A to B. In the case of an LG, both near-field and far-field are equivalent, so an LG mode is encoded on the DMD and confirmed to be observed. After this, we placed the camera in a distance $d$ and tilt the lens, finally achieving the setup depicted in figure 5 situation B.

Some slight compensation of the propagation direction is necessary, as the tilting of the lens can redirect the beam, similarly to the action of a prism. We subsequently move the camera until an exact image of a HG mode is observed. With this we confirm that the observed plane is indeed the proper conversion plane. The calculated distance $d = 27.9$ cm was measured between the lens and the camera when used a lens of $f = 30$ cm with a measured inclination of $\beta \approx 24^\circ$. We then proceed by changing the encoded modes on the DMD to those of equation (10). We demonstrate this by setting $N_{\text{max}} = 6$ and generating the auxiliary eigenvectors according to table 1. This is illustrated in figure 5 situation B.
3.1. Propagation

While variant in free-space propagation, the dynamics of a superposition of modes is trivial: besides resizing, each mode picks up a phase according to their Gouy phases, which is proportional to the mode order. In the auxiliary eigenvectors intensity pattern this is translated into the interesting behaviour of the far-field being the mirror image of the near-field, a flip in the $x$ direction. This is illustrated with experimental data in figure 6. It is also interesting to point out that, since the propagation phase is manifested as intramodal phases, the field in any point in propagation is still an auxiliary eigenvector. The operator of the tilted lens $M_t$ describes three processes: propagation to the lens, application of an astigmatic phase given by the tilting of the lens and the propagation to the conversion plane, as illustrated in figure 1. When considering single modes or superpositions of modes of the same order, the first propagation stage is negligible, as it only rescales the modes. In the case of a superposition of modes of different orders, the field that reaches the lens is not the original encoded near-field since the propagation phase that each constituting mode picks up will be different. Therefore, when dealing with auxiliary eigenvectors, the initial propagation has to be taken into account.

We dealt with this in two different ways: the first one is that, since we are dealing with paraxial modes, any phase propagation is just a relative phase between modes of different orders. We can compensate this by adjusting the encoded hologram so that we change the initial field in order to obtain a desired field after the tilted lens. The second way is to use a digital propagation technique [43] whereby encoding a lens function in the phase of the encoded mode we shift the Fourier plane. Both ways showed identical results in the conversion plane.

This claim is supported by the fact that in [21] the astigmatic mode conversion happens when the phase is applied to the field that reaches the lens. For a field consisting of a single mode, the near-field and the one that reaches the lens are essentially the same, but this is not true for superpositions.

**Figure 6.** Propagation dynamics of auxiliary eigenvectors of $N_{\text{max}} = 6$. 
Figure 7. Experimental results for the modes conversion of modes grouped by $j$ value up to the order $N_{\text{max}} = 6$. First column shows theoretical simulations of the intensity profile. Second column shows the untitled image and third image shows the mode unaffected by the conversion process.

of modes that propagate a non-negligible amount compared to the diffraction length. Both methods were shown to compensate for the initial propagation phase. While the second one maintains the encoded intensity profile both when imaging the near-field and when looking at the conversion plane, the distance it would be able to compensate for is dependent on the resolution of the spatial light modulator, since the applied phase may vary very rapidly in the radial direction. The first method does not have this problem, as no extra phase is applied but instead the near-field is different from the field in the conversion plane.

To further support our claims, we adapt our setup to that of a digitally applied astigmatic phase, by first setting $\beta = 0^\circ$ (untilting the lens) and applying a phase proportional to the Zernike polynomial $Z_2^2$ associated with horizontal astigmatism. In this way, only the evolution under astigmatism is considered. The lens of focal distance $f$ in a distance $f$ after the DMD then creates a system that propagates the beam to its far-field at a distance $f$, where the camera is placed. This situation is analogous to the field reaching the tilted lens without any propagation effects and then being propagated to the far field.

4. Results

In figure 7 we can see results for a few modes of $N_{\text{max}} = 6$. In the first column we show theoretical simulations of the encoded modes. The second column (Untilted) shows the near-field images obtained with the imaging setup described in figure 5 situation A. The third column (Tilted) shows the modes after the tilted lens in the conversion plane, regarding figure 5 situation B. It is possible to see remarkable agreement of simulations with experimental data: the originally encoded auxiliary eigenvectors are seen unaltered at the conversion plane of the tilted lens.

Notably, there is a lack of trivial symmetry. Modes with $[j] \neq [0],[4]$ are not symmetric in either horizontal, vertical or $\pm 45^\circ$ directions, while for $[0],[4]$ there is inversion symmetry
in ±45°. This can be explained by the fact that all constituent modes have different symmetries, which ultimately lead to breaking one another. For the case of [0] one might observe that, since it follows that \( m - n = 0 \), all modes will be symmetric in both \( x, y \) directions. This auxiliary eigenvector however can also include \( |HG_{m,n}^{6}\rangle \) and \( |HG_{m,n}^{7}\rangle \) which breaks the \( x, y \) symmetry but creates one at 45°. The same can be said for [4] which for every \( |HG_{m,n}^{n}\rangle \) also contains \( |HG_{m,n}^{m}\rangle \) and these two are the same, except for a 90° rotation. One might also notice that they have most of their intensities distributed along the 45° line. This, however, is not related to astigmatic resilience, as we also know from [11] that a HG mode at 45° is very similar but is not an eigenmode and gets converted into an LG mode.

Upon moving to the scheme where we can carefully increase the astigmatism of the system, we see the results in figure 8. There is no effect in a HG mode with \( m,n = 4, 4 \). Then we apply the astigmatism to a LG mode of \( l, p = 5, -4 \). We can see that this mode is converted into a HG oriented at −45° confirming the sign of the topological charge, as well as the indices \( m, n \) confirming the \( p, l \) indices according to [10]. We then encode an auxiliary eigenvectors of \( N_{max} = 6 \) and \( j = 3 \) and show that, despite seeing changes when astigmatism is applied, the original profile is recovered for a level of astigmatism correspondent to full conversion.

5. Discussion and conclusion

Previous works have tackled the problem of generating beams with optical vortices resilient to astigmatism [27, 44]. Notably, the results in the second row of figure 8 are in good agreement with those of [44]. However, differently from those works, we do not focus on optical vortices and provide a solution compatible with existing techniques using HG modes.

Our work addresses the limited extension of the modal description of light, to light propagation through aberrated channels. We have contributed a study that focuses on a simple and commonly encountered aberration, i.e. an astigmatic tilted lens, defined by an operator with eigenvalues that have a given periodicity in the mode space.

To achieve this, we first characterized this aberrated channel according to a new criteria of conversion and optical element modelling, and established for the first time in the literature, a clear analytical relation between an arbitrary position for the lens, the lens astigmatism and the conversion plane. We subsequently found the full set of eigenvectors of a highly aberrated channel, including the superpositions of eigenvectors, which can be grouped according to equivalence classes by a congruence relation to create new auxiliary eigenvectors. By truncating up to a maximum mode order we ensured that it is possible to control the physical dimensions.
and propagation dynamics. In this work we set the coefficients of all modes to be 1 for visualization, but this is not a requirement. Although these modes are not stable under propagation, they all show the behaviour of the far-field being the flipped image of the near-field in the x axis. Interestingly, propagation for these modes is only an intra-modal phase given by the Gouy phase, which in turn does not change the validity of the eigenmode superposition still being an eigenmode. Apart from this interesting physics, the practical implication is that we can find exact eigenmodes with some propagation distance included. In other words, they hold valid for a user-defined plane after the aberration, as is commonly done in applications such as laser materials processing. This implies that every intensity form seen in figure 6 is also an auxiliary eigenmode which propagate independently of the astigmatism. However, we also showed that the level of astigmatism is an important condition to the full recovery of the original profile.

We believe that our approach can be extended to any aberrated optical system, given that it is unitary. By probing a system with a basis and generating a matrix in the modal space, it is not only possible to obtain eigenmodes of the system, but any periodicity of the eigenvalues would be detected if probed with a sufficient number of modes. Those could be grouped to form intricate patterns that are unaltered by this system and even exhibit properties beyond the original eigenmodes while retaining the resilience property, such as the x flip behaviour and symmetry manipulation.

In this system, the equivalence classes can have beams of many different orders. Of course, the size and type of equivalence classes depends heavily on the optical system, but this separation can be very broad and mutually non-exclusive with recent applications based on Gouy phase effects, such as [35, 36].

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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