The hexagon equations for dilogarithms and the Riemann-Hilbert problem

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Abstract

In this article we present the hexagon equations for dilogarithms which comes from the analytic continuation of the dilogarithm \( \text{Li}_2(z) \) to \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \). The hexagon equations are equivalent to the coboundary relations for a certain 1-cocycle of holomorphic functions on \( \mathbb{P}^1 \), and are solved by the Riemann-Hilbert problem of additive type. They uniquely characterize the dilogarithm under the normalization condition.

1 Introduction

Let \( D_\alpha \ (\alpha = 0, 1, \infty) \) be domains in the Riemann sphere \( \mathbb{P}^1 \) defined by

\[
D_0 = \mathbb{C} \setminus \{ z = x | 1 \leq x < +\infty \}, \quad D_1 = \mathbb{C} \setminus \{ z = x | -\infty < x \leq 0 \},
\]

\[
D_\infty = \mathbb{P}^1 \setminus \{ z = x | 0 \leq x \leq 1 \},
\]

which are open neighborhoods of the points \( 0, 1, \infty \), respectively. They give an open covering of the Riemann sphere \( \mathbb{P}^1 = D_0 \cup D_1 \cup D_\infty \), furthermore, satisfy \( D_0 \cap D_1 \cap D_\infty = H_+ \cup H_- \) where \( H_+ \) (\( H_- \)) is the upper (lower) half plane.

According to Theorem 2 in [4], the dilogarithms \( \text{Li}_2(z) \) and \( \text{Li}_2(1 - z) \) are characterized as the solutions \( f_0(z) \) and \( f_1(z) \) holomorphic in \( D_0 \) and \( D_1 \), respectively, to the following functional equation

\[
f_0(z) + \log z \log(1 - z) + f_1(z) = \zeta(2) \quad (z \in D_0 \cap D_1) \tag{1.1}
\]

under the asymptotic condition \( f'_\alpha(z) \to 0 \) (\( z \to \infty \), \( z \in D_\alpha, \alpha = 0, 1 \)) and the normalization condition \( f_0(0) = 0 \).
The equation (1.1) says that the function $-\log z \log(1 - z) + \zeta(2)$, which is holomorphic in $D_0 \cap D_1$, decomposes to the sum of $f_0(z)$ and $f_1(z)$ holomorphic in $D_0$ and $D_1$. That is to say, (1.1) is a Riemann-Hilbert problem (or, Plemelj-Birkhoff decomposition) of additive type. However, the previous asymptotic condition does not naturally come from the Riemann-Hilbert problem, is a rather technical one. So we would like to avoid that, and give more natural formulation of the Riemann-Hilbert problem.

2 The hexagon relations for dilogarithms

Let $\log(1 - z)$ be the principal value of the logarithm in $D_0$, namely, $\log 1 = 0$. Define a branch of the dilogarithm $\text{Li}_2(z)$ in $D_0$ by

$$\text{Li}_2(z) = -\int_0^z \frac{\log(1 - t)}{t} \, dt.$$  (2.1)

Then the Taylor expansion at $z = 0$ is

$$\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad (|z| < 1)$$  (2.2)

and by Abel’s continuous theorem, we have

$$\lim_{z \to 1, z \in D_0} \text{Li}_2(z) = \zeta(2).$$  (2.3)

It is well known that the dilogarithm satisfies the following functional relations:

$$\text{Li}_2(z) + \log z \log(1 - z) + \text{Li}_2(1 - z) = \zeta(2) \quad (z \in D_0 \cap D_1),$$  (2.4)

$$\text{Li}_2\left(\frac{z}{z - 1}\right) = -\text{Li}_2(z) - \frac{1}{2} \log^2(1 - z) \quad (z \in D_0).$$  (2.5)

The first formula describes the direct analytic continuation of $\text{Li}_2(z)$ along the path $01$, and the second one is regarded as the half-monodromy continuation of $\text{Li}_2(z)$ at $z = 0$. We call them Euler’s inversion formula and Landen’s inversion formula, respectively.

To consider the global analytic continuation of the dilogarithm, let us introduce automorphisms $t_i(z)$ of $\mathbb{P}^1$ as follows:

$$\begin{cases}
  t_0(z) = z_0 = z, & t_1(z) = z_1 = 1 - z, & t_2(z) = z_2 = \frac{z-1}{z}, \\
  t_3(z) = z_3 = \frac{1}{z}, & t_4(z) = z_4 = \frac{1}{1-z}, & t_5(z) = z_5 = \frac{z}{z-1}.
\end{cases}$$  (2.6)

They induce permutations $\sigma_{t_i}$ of the points $\{0, 1, \infty\},$

$$\sigma_{t_0} = \begin{pmatrix} 0 & 1 & \infty \\ 0 & 1 & \infty \end{pmatrix}, \quad \sigma_{t_1} = \begin{pmatrix} 0 & 1 & \infty \\ 1 & 0 & \infty \end{pmatrix}, \quad \sigma_{t_2} = \begin{pmatrix} 0 & 1 & \infty \\ \infty & 0 & 1 \end{pmatrix},$$

$$\sigma_{t_3} = \begin{pmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{pmatrix}, \quad \sigma_{t_4} = \begin{pmatrix} 0 & 1 & \infty \\ 1 & \infty & 0 \end{pmatrix}, \quad \sigma_{t_5} = \begin{pmatrix} 0 & 1 & \infty \\ 0 & \infty & 1 \end{pmatrix},$$
and give analytic isomorphisms $t_i : D_\alpha \to D_{t_i(\alpha)}$. Furthermore, we should note that

$$z_{j-1} = \frac{z_j}{z_j - 1}, \quad z_{j+1} = 1 - z_j, \quad (j = 0, 2, 4)$$

where the indices are defined modulo 6. Taking the inverse image of the relations (2.4) and (2.5) induced by $t_j$ ($j = 0, 2, 4$), we have the following proposition:

**Proposition 2.1.** The dilogarithm $\text{Li}_2(z)$ satisfies

$$\text{Li}_2(z_j) + \log z_j \log z_{j+1} + \text{Li}_2(z_{j+1}) = \zeta(2) \quad (z \in D_{t^{-1}_j(0)} \cap D_{t^{-1}_j(1)}), \quad (2.7)$$

$$\text{Li}_2(z_{j-1}) = -\text{Li}_2(z_j) - \frac{1}{2} \log^2 z_{j+1} \quad (z \in D_{t^{-1}_j(0)}), \quad (2.8)$$

where $j = 0, 2, 4$, and all the indices are defined modulo 6.

We refer to them as the hexagon relations for dilogarithms, which describe the analytic continuation of $\text{Li}_2(z)$ along the paths through the neighborhood of $0 \to 1 \to \infty \to 0$ in $H_\pm$ (see the figures below).

![Hexagon Diagram](image)

3 The hexagon equations for dilogarithms and a 1-cocycle of holomorphic functions on $\mathbb{P}^1$

In the sequel, the index $j$ denotes 0, 2, 4, and all the indices are defined modulo 6.

Let $f_j(z), f_{j-1}(z)$ be functions holomorphic in $D_{t^{-1}_j(0)}$, and consider the following functional equations for them:

$$f_j(z) + \log z_j \log z_{j+1} + f_{j+1}(z) = \zeta(2) \quad (z \in D_{t^{-1}_j(0)} \cap D_{t^{-1}_j(1)}), \quad (3.1)$$

$$f_{j-1}(z) = -f_j(z) - \frac{1}{2} \log^2 z_{j+1} \quad (z \in D_{t^{-1}_j(0)}). \quad (3.2)$$

We call them the hexagon equations for dilogarithms. We show that they are relevant to the cohomology theory on $\mathbb{P}^1$. 

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Proposition 3.1. To the covering \{D_0, D_1, D_\infty\} of \(\mathbb{P}^1\), we attach a 0-cochain 
\[ f = \{ f_{D_\alpha} \}_{\alpha=0,1,\infty} \] 
and a 1-cochain 
\[ F = \{ F_{D_\alpha \cap D_\beta} \}_{\alpha, \beta=0,1,\infty} \] 
where \(f_{D_\alpha}\) is a holomorphic function on \(D_\alpha\) and \(F_{D_\alpha \cap D_\beta}\) is a holomorphic function on \(D_\alpha \cap D_\beta\) defined by 
\[
f_{D_t^{-1}(0)}(z) = f_j(z), \quad F_{D_t^{-1}(0) \cap D_t^{-1}(1)}(z) = \frac{1}{2} \log^2 z_j - \log z_j \log z_{j+1} + \zeta(2),
\]
and \(F_{D_\beta \cap D_\alpha} = -F_{D_\alpha \cap D_\beta}\). Then we have the following:

1. \(F\) is a 1-cocycle. That is to say, \(F\) satisfies the cocycle condition 
\[
F_{D_1 \cap D_\infty}(z) - F_{D_0 \cap D_\infty}(z) + F_{D_0 \cap D_1}(z) = 0 \quad (z \in D_0 \cap D_1 \cap D_\infty).
\]

2. The condition (3.4) is equivalent to the Euler formula \(\zeta(2) = \frac{\pi^2}{6}\).

3. The hexagon equations (3.1) and (3.2) are equivalent to (3.2) and the 
coboundary relations 
\[
F_{D_t^{-1}(0) \cap D_t^{-1}(1)}(z) = f_{D_t^{-1}(0)}(z) - f_{D_t^{-1}(1)}(z) \quad (z \in D_t^{-1}(0) \cap D_t^{-1}(1)).
\]

Proof. From (3.3), it follows that 
\[
F_{D_1 \cap D_\infty}(z) - F_{D_0 \cap D_\infty}(z) + F_{D_0 \cap D_1}(z) = 3\zeta(2) + \frac{1}{2} \left( \log z + \log \left( \frac{z-1}{z} \right) + \log \left( \frac{1}{1-z} \right) \right)^2.
\]
As the logarithms satisfy the half-monodromy relations 
\[
\log z + \log \left( \frac{z-1}{z} \right) + \log \left( \frac{1}{1-z} \right) = \pm \pi i \quad (z \in H_\pm),
\]
we have \(F_{D_1 \cap D_\infty}(z) - F_{D_0 \cap D_\infty}(z) + F_{D_0 \cap D_1}(z) = 3\zeta(2) - \frac{\pi^2}{2}\). Thus the claims (i) and (ii) are proved.

The claim (iii) immediately follows from \(t_j^{-1}(0) = t_j^{-1}(1)\). 

4 Main theorem

From Proposition 3.1 and the results on the cohomology theory that \(H^1(\mathbb{P}^1, O) = 0\), \(H^0(\mathbb{P}^1, O) = \mathbb{C}\), where \(O\) denotes the sheaf of holomorphic functions on \(\mathbb{P}^1\), one can deduce that the solutions to the hexagon equations exist uniquely under the normalization condition \(f_0(0) = 1\). However one cannot know how the solutions are relevant to dilogarithms. So let us solve the hexagon equation by the Riemann-Hilbert problem of additive type.
Theorem 4.1. The hexagon equations for dilogarithms have a unique solution under the normalization condition \( f_0(0) = 0 \), and the solutions are expressed as

\[
f_i(z) = \text{Li}_2(z_i) \quad (i = 0, 1, \ldots, 5).
\]  

(4.1)

Proof. Differentiating the equations (3.1) and (3.2) with respect to the variable \( z = z_0 \), we have the following:

\[
f_j'(z) + \log z_j \left( \log z_j \right)' = -f_{j+1}'(z) - \log z_j \left( \log z_{j+1} \right)', \quad j = 0, 2, 4. \]  

(4.2)

\[
f_{j+1}'(z) + \log z_j \left( \log z_{j+1} \right)' = -f_{j+2}'(z) - \log z_{j+3} \left( \log z_{j+2} \right)', \quad j = 0, 2, 4. \]  

(4.3)

where \( j = 0, 2, 4 \). Put

\[
f(z) = f_0'(z) + \log z_1 \left( \log z_0 \right)' = f'_0(z) + \frac{\log(1 - z)}{z}.
\]

This is a function holomorphic in \( D_0 \). From (4.2), it is analytically continued to \( g(z) \) and \( h(z) \)

\[
g(z) = -f_1'(z) - \log z_0 \left( \log z_1 \right)' = -f_1'(z) + \frac{\log z}{1 - z},
\]

\[
h(z) = -f_3'(z) - \log z_2 \left( \log z_3 \right)' = -f_3'(z) + \frac{\log \left( \frac{z - 1}{z} \right)}{z},
\]

which are holomorphic in \( D_1 \) and \( D_\infty \), respectively. Hence \( f(z) \) is holomorphic on \( P^1 \) so that \( f(z) = g(z) = h(z) = A \) where \( A \) is a constant. Substitute this into (4.2) and (4.3), and integrate them by using

\[
\log z_{j+1} \left( \log z_j \right)' = -\left( \text{Li}_2(z_j) \right)' , \quad \log z_j \left( \log z_{j+1} \right)' = -\left( \text{Li}_2(z_{j+1}) \right)'.
\]

Then we obtain

\[
f_i(z) = \text{Li}_2(z_i) + (-1)^i A z + b_i \quad (i = 0, 1, \ldots, 5).
\]

Here \( b_i \) are integral constants. From the normalization condition \( f_0(0) = 0 \), it is clear that \( b_0 = 0 \). Hence from (3.1), we have

\[
\text{Li}_2(z) + \log z \log(1 - z) + \text{Li}_2(1 - z) + b_1 = \zeta(2).
\]

Taking the limit as \( z \to 0 \) (\( z \in D_0 \cap D_1 \)) in this equation, we have \( b_1 = 0 \) and \( f_1(z) = \text{Li}_2(1 - z) - A z \). In a similar way, we show that all the constants \( b_i \) are 0. Furthermore, since \( f_3(z) \), which is expressed as

\[
f_3(z) = \text{Li}_2 \left( \frac{1}{z} \right) + A z,
\]

should be holomorphic at \( z = \infty \), we have \( A = 0 \). Thus the proof is completed.
5 Discussions

In [5], the Riemann-Hilbert approach generalizes to the case of the fundamental solution normalized at the origin of the KZ equation of one variable. So it will be worth trying to generalize the hexagon equations approach to the case of the fundamental solutions of the KZ equation of one variable (cf. [7]), and further to the case of the KZ equation of two variables (cf. [6]).

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