ON THE STRUCTURE OF $K^*_G(T_G^*M)$

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Abstract. In this expository paper, we revisit the results of Atiyah-Singer and de Concini-Procesi-Vergne concerning the structure of the $K$-theory groups $K^*_G(T_G^*M)$.

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1. Introduction

When a compact Lie group $G$ acts on a compact manifold $M$, the $K$-theory group $K^0_G(T^*M)$ is the natural receptacle for the principal symbol of any $G$-invariant elliptic pseudo-differential operators on $M$. One important point of Atiyah-Singer’s Index Theory [1, 3, 5, 6] is that the equivariant index map $Index^G_M : K^0_G(T^*M) \to R(G)$ can be defined as the composition of a pushforward map $i_! : K^0_G(T^*M) \to K^0_G(T^*V)$ associated to an embedding $M \overset{i}{\hookrightarrow} V$ in a $G$-vector space, with the index map $Index^G_V : K^0_G(T^*V) \to R(G)$ which is the inverse of the Bott-Thom isomorphism [13].

In his Lecture Notes [1] describing joint work with I.M. Singer, Atiyah extends the index theory to the case of transversally elliptic operators. If we denote by $T^*_G M$ the closed subset of $T^*M$, union of the conormals to the $G$-orbits, Atiyah explains how the principal symbol of a pseudo-differential transversally elliptic operator on $M$ determines an element of the equivariant $K$-theory group $K^0_G(T^*_G M)$, and how the analytic index induces a map

$$\text{Index}^G_M : K^0_G(T^*_G M) \to R^{-\infty}(G),$$

where $R^{-\infty}(G) := \text{hom}(R(G), \mathbb{Z})$.

Like in the elliptic case the map [1] can be seen as the composition of a push-forward map $i_! : K^0_G(T^*_G M) \to K^0_G(T^*_G V)$ with the index map $Index^G_V : K^0_G(T^*_G V) \to R^{-\infty}(G)$. Hence the comprehension of the $R(G)$-module

$$K^0_G(T^*_G V)$$

is fundamental in this context. For example, in [3, 14] the authors gave a cohomological formula for the index and the knowledge of the generators of $K^0_{U(1)}(T^*_U(1) \mathbb{C})$ was used to establish the formula. In [11], de Concini-Procesi-Vergne proved a formula for the multiplicities of the index by checking it on the generators of [2].

When $G$ is abelian, Atiyah-Singer succeeded to find a set of generators for [2], and recently de Concini-Procesi-Vergne have shown that the index map identifies [2] with a generalized Dahmen-Michelli space [9, 10]. Let us explain their result.

Let $\hat{G}$ be the set of characters of the abelian compact Lie group $G$: for any $\chi \in \hat{G}$ we denote $\mathbb{C}_\chi$ the corresponding complex one dimensional representation of $G$. We associate to any element $\Phi := \sum_{\chi \in \hat{G}} m_\chi \mathbb{C}_\chi \in R^{-\infty}(G)$ its support $\text{Supp} (\Phi) = \{ \chi \mid m_\chi \neq 0 \} \subset \hat{G}$.

For any real $G$-module $V$, we denote $\Delta_G(V)$ the set formed by the infinitesimal stabilizer of points in $V$: we denote $\mathfrak{h}_{\text{min}}$ the minimal stabilizer. For any $\mathfrak{h} \in \Delta_G(V)$, we denote $H := \exp(\mathfrak{h})$ the corresponding torus and we denote $\pi_H : \hat{G} \to \hat{H}$ the restriction map.

We denote $R^{-\infty}(G/H) \subset R^{-\infty}(G)$ the subgroup formed by the elements $\Phi \in R^{-\infty}(G)$ such that $\pi_H(\text{Supp} (\Phi)) \subset \hat{H}$ is reduced to the trivial representation. Let

$$\langle R^{-\infty}(G/H) \rangle \subset R^{-\infty}(G)$$

be the $R(G)$-submodule generated by $R^{-\infty}(G/H)$. We have $\Phi \in \langle R^{-\infty}(G/H) \rangle$ if and only if $\pi_H(\text{Supp} (\Phi)) \subset \hat{H}$ is finite.

For any subspace $a \subset \mathfrak{g}$, we denote $V^a \subset V$ the subspace formed by the vectors fixed by the infinitesimal action of $a$. We fix an invariant complex structure on
V/V^g$, hence the vector space $V/V^g \subset V/V^h$ is equipped with a complex structure for any $h \in \Delta_G(V)$. Following [11], we introduce the following submodule of $R^{-\infty}(G)$: the Dahmen-Michelli submodule

$$DM_G(V) := \langle R^{-\infty}(G/H_{min}) \rangle \cap \left\{ \Phi \in R^{-\infty}(G) \mid \wedge^* \frac{V}{V^g} \otimes \Phi = 0, \forall h \neq h_{min} \in \Delta_G(V) \right\},$$

and the generalized Dahmen-Michelli submodule

$$F_G(V) := \left\{ \Phi \in R^{-\infty}(G) \mid \wedge^* \frac{V}{V^h} \otimes \Phi \in \langle R^{-\infty}(G/H) \rangle, \forall h \in \Delta_G(V) \right\}.$$

Note that the relation $\wedge^* \frac{V}{V^h} \otimes \Phi \in \langle R^{-\infty}(G/H) \rangle$ becomes $\Phi \in \langle R^{-\infty}(G/H_{min}) \rangle$ when $h = h_{min}$. Hence $DM_G(V)$ is contained in $F_G(V)$. We have the following remarkable result [11].

**Theorem 1.1** (de Concini-Procesi-Vergne). Let $G$ be an abelian compact Lie group, and let $V$ be a real $G$-module. Let $V^{gen} \subset V$ be its open subset formed by the $G$-orbits of maximal dimension. The index map defines

- an isomorphism between $K_G^0(T_G^*V)$ and $F_G(V)$,
- an isomorphism between $K^0_G(T_G^*V^{gen})$ and $DM_G(V)$.

The purpose of this note is to give a comprehensive account on the work of Atiyah-Singer and de Concini-Procesi-Vergne concerning the structure of (2) when $G$ is a compact abelian Lie group. We will explain in details the following facts:

- The decomposition of $K_G^0(T_G^*M)$ relatively to the stratification of the manifold $M$ relatively to the type of infinitesimal stabilizers.
- A set of generators of $K_G^0(T_G^*V)$.
- A set of generators of $K_G^0(T_G^*V^{gen})$.
- The injectivness of the index map $\text{Index}^G : K_G^0(T_G^*V) \to R^{-\infty}(G)$.
- The isomorphisms $K_G^0(T_G^*V) \simeq F_G(V)$ and $K_G^0(T_G^*V^{gen}) \simeq DM_G(V)$.

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## 2. Preliminary on $K$-theory

In this section, $G$ denotes a compact Lie group. Let $R(G)$ be the representation ring of $G$ and let $R^{-\infty}(G) = \text{hom}(R(G), \mathbb{Z})$.

### 2.1. Equivariant $K$-theory.

We briefly review the notations for $K$-theory that we will use, for a systematic treatment see Atiyah [2] and Segal [15].

Let $N$ be a locally compact topological space equipped with a continuous action of $G$. Let $E^\pm \to N$ be two $G$-equivariant complex vector bundles. An equivariant morphism $\sigma$ on $N$ is defined by a vector bundle map $\sigma \in \Gamma(N, \text{hom}(E^+, E^-))$, that we denote also $\sigma : E^+ \to E^-$. At each point $n \in N$, we have a linear map $\sigma(n) : E^+_{n} \to E^-_{n}$. The support of the morphism $\sigma$ is the closed set formed by the point $n \in N$ where $\sigma(n)$ is not an isomorphism. We denote it $\text{Support}(\sigma) \subset N$.

A morphism $\sigma$ is elliptic when its support is compact, and then it defines a class $[\sigma] \in K_G^0(N)$ in the equivariant $K$-group [15]. The group $K_G^1(N)$ is by definition the group $K_G^0(N \times \mathbb{R})$ where $G$ acts trivially on $\mathbb{R}$.
Let \( j : U \hookrightarrow N \) be an invariant open subset, and let us denote by \( r : N \setminus U \hookrightarrow N \) the inclusion of the closed complement. We have a push-forward morphism \( j_* : K_G^*(U) \to K_G^*(N) \) and a restriction morphism \( r^* : K_G^*(N) \to K_G^*(N \setminus U) \) that fit in a six terms exact sequence:

\[
\begin{array}{cccccc}
K_G^0(U) & \xrightarrow{j_*} & K_G^0(N) & \xrightarrow{r^*} & K_G^0(N \setminus U) & \\
\downarrow & & \downarrow & & \downarrow & \\
K_G^1(N \setminus U) & \xrightarrow{r^*} & K_G^1(N) & \xrightarrow{j_*} & K_G^1(U) & \\
\end{array}
\]

In the next sections we will use the following basic lemma which is a direct consequence of \([2]\).

**Lemma 2.1.** Suppose that we have a morphism \( S : K_G^*(N \setminus U) \to K_G^*(N) \) of \( R(G) \)-module such that \( r^* \circ S \) is the identity on \( K_G^*(N \setminus U) \). Then

\[
K_G^*(N) \cong K_G^*(U) \oplus K_G^*(N \setminus U)
\]
as \( R(G) \)-module.

We finish this section by considering the case of torus \( T \) belonging to the center of \( G \). Let \( i : T \hookrightarrow G \) be the inclusion map. We still denote \( i : \text{Lie}(T) \to \mathfrak{g} \) the map of Lie algebra, and \( i^* : \mathfrak{g}^* \to \text{Lie}(T)^* \) the dual map. Note that the restriction to \( T \) of an irreducible representation \( V^G_\lambda \) is isomorphic to \((\mathbb{C}_{\cdot}^*, \lambda)^p \) with \( p = \dim(V^G_\lambda) \). The representation ring \( R(G) \) contains as a subring \( R(G/T) \). At each character \( \mu \) of \( T \), we associate the \( R(G/T) \)-submodule of \( R(G) \) defined by

\[
R(G)_\mu = \sum_{i^* (\lambda) = \mu} Z V^G_\lambda.
\]

Note that \( R(G)_0 = R(G/T) \).

We have then a grading \( R(G) = \bigoplus_{\mu \in \hat{\mathbb{T}}} R(G)_\mu \) since \( R(G)_\mu \cdot R(G)_{\mu'} \subset R(G)_{\mu + \mu'} \).

If we work now with the \( R(G) \)-module \( R^{-\infty}(G) \), we have also a decomposition\(^1\)

\[
R^{-\infty}(G) = \bigoplus_{\mu \in \hat{\mathbb{T}}} R^{-\infty}(G)_{\mu} \quad \text{such that} \quad R(G)_\mu \cdot R^{-\infty}(G)_{\mu'} \subset R^{-\infty}(G)_{\mu + \mu'}.
\]

Let us consider now the case of a \( G \)-space \( N \), connected, such that the action of the subgroup \( T \) is trivial. Each \( G \)-equivariant complex vector bundle \( E \to N \) decomposes as a finite sum

\[
E = \bigoplus_{\mu \in A} E_\mu
\]

where \( E_\mu \cong \text{hom}_T(\mathbb{C}_\mu, E) \) is the \( G \)-sub-bundle where \( T \) acts through the character \( t \mapsto t^\mu \). Note that a \( G \)-equivariant morphism \( \sigma : E^+ \to E^- \) is equal to the sum of morphisms \( \sigma_\mu : E^+_\mu \to E^-_\mu \). Hence, at the level of \( K \)-theory we have also a decomposition

\[
K_G^*(N) = \bigoplus_{\mu \in \hat{\mathbb{T}}} K^*_G(N)_\mu
\]
such that \( R(G)_\mu \cdot K_G^*(N)_{\mu'} \subset K_G^*(N)_{\mu + \mu'} \).

\(^1\)The sign \( \bigoplus \) means that one can take infinite sum.
2.2. Index morphism: excision and free action. When $M$ is a compact $G$-manifold, an equivariant morphism $\sigma$ on the tangent bundle $T^*M$ is called a symbol on $M$. We denote by $T^*_G M$ the following subset of $T^* M$

\[ T^*_G M := \{ (m, \xi) \in T^* M \mid \langle \xi, X_M(m) \rangle = 0 \text{ for all } X \in \mathfrak{g} \}. \]

where $X_M(m) := \frac{d}{dt} e^{-tX} \cdot m |_{t=0}$ is the vector field generated by the infinitesimal action of $X \in \mathfrak{g}$. More generally, if $D \subset G$ is a distinguished subgroup, we can consider the $G$-invariant subset

\[ T^*_D M \supset T^*_G M. \]

An elliptic symbol $\sigma$ on $M$ defines an element of $K^0_G(T^*_G M)$, and the index of $\sigma$ is a virtual finite dimensional representation of $G$ that we denote $\text{Index}^G_M(\sigma) \equiv \mathbb{H}_3 \equiv \mathbb{H}_6$. An equivariant symbol $\sigma$ on $M$ is transversally elliptic when $\text{Support}(\sigma) \cap T^*_G M$ is compact; in this case Atiyah and Singer have shown that its index, still denoted $\text{Index}^G_M(\sigma)$, is well defined in $R^{-\infty}(G)$ and it depends only of the class $[\sigma] \in K^0_G(T_G M)$ (see [1] for the analytic index and [14] for the cohomological one). It is interesting to look at the index map as a pairing

\[ \text{Index}^G_M : K^0_G(T^*_G M) \times K^0_G(M) \rightarrow R^{-\infty}(G). \]

Let $\sigma$ be a $G_1 \times G_2$-equivariant symbol $\sigma$ on a manifold $M$. If $\sigma$ is $G_1$-transversally elliptic it defines a class

\[ [\sigma] \in K^0_{G_1 \times G_2}(T^*_G M), \]

and its index is smooth relatively to $G_2$. It means that $\text{Index}^{G_1 \times G_2}_M(\sigma) = \sum_{\mu \in G_1} \theta_\mu \otimes V_\mu^{G_1}$ where $\theta_\mu \in R(G_2)$ for any $\mu$. Hence

- the $G_1$-index $\text{Index}^{G_1}_M(\sigma) = \sum_{\mu \in G_1} \dim(\theta_\mu) \otimes V_\mu^{G_1}$ is equal to the restriction of $\text{Index}^{G_1 \times G_2}_M(\sigma)$ to $\mathfrak{g} = 1 \in G_2$.
- the product of $\text{Index}^{G_1 \times G_2}_M(\sigma)$ with any element $\Theta \in R^{-\infty}(G_1)$ is a well defined element $\Theta \cdot \text{Index}^{G_1 \times G_2}_M(\sigma)$ in $R^{-\infty}(G_1 \times G_2)$.

Remark 2.3. Suppose that a torus $T$ belonging to the center of $G$ acts trivially on the manifold $M$. Since the index map $\text{Index}^G_M$ is a morphism of $R(G)$-module, the pairing (7) specializes in a map from $K^0_G(T^*_G M) \mu \times K^0_G(M)_{\mu'}$ into $R^{-\infty}(G)_{\mu + \mu'}$. Hence on can extend the pairing (7) to

\[ \text{Index}^G_M : K^0_G(T^*_G M) \times \widehat{K^0_G}(M) \rightarrow R^{-\infty}(G). \]

See Definition 2.2 for the notation $\widehat{K^0_G}(M)$.

Let $U$ be a non-compact $K$-manifold. Lemma 3.6 of [1] tell us that, for any open $K$-embedding $j : U \hookrightarrow M$ into a compact manifold, we have a push-forward map $j_* : K^*_G(T_G U) \rightarrow K^*_G(T^*_G M)$.

Let us rephrase Theorem 3.7 of [1].
Theorem 2.4 (Excision property). The composition
\[ K^0_G(T_G^* U) \xrightarrow{j_*} K^0_G(T_G^* M) \xrightarrow{\text{index}_M^G} R^{-\infty}(G) \]
is independent of the choice of \( j : U \hookrightarrow M \): we denote this map \( \text{index}_M^G \).

Note that a relatively compact \( G \)-invariant open subset \( U \) of a \( G \)-manifold admits an open \( G \)-embedding \( j : U \hookrightarrow M \) into a compact \( G \)-manifold. So the index map \( \text{index}_M^G \) is defined in this case. Another important example is when \( U \to N \) is a \( G \)-equivariant vector bundle over a compact manifold \( N \): we can imbed \( U \) as an open subset of the real projective bundle \( \mathbb{P}(U \oplus \mathbb{R}) \).

Let \( K \) be another compact Lie group. Let \( P \) be a compact manifold provided with an action of \( K \times G \). We assume that the action of \( K \) is free. Then the manifold \( M := P/K \) is provided with an action of \( G \) and the quotient map \( q : P \to M \) is \( G \)-equivariant. Note that we have the natural identification of \( T^*_K P \) with \( q^* T^* M \), hence \( (T^*_K P)/K \simeq T^*_M \) and more generally
\[ (T^*_K \times_G P)/K \simeq T^*_G M. \]

This isomorphism induces an isomorphism
\[ Q^* : K^0_G(T^*_G M) \to K^0_G(T^*_K \times_G P). \]

The following theorem was obtained by Atiyah-Singer in \cite{AS}. For any \( \Theta \in R^{-\infty}(K \times G) \), we denote \([\Theta]^K \in R^{-\infty}(G)\) its \( K \)-invariant part.

Theorem 2.5 (Free action property). For any \([\sigma] \in K^0_G(T^*_G M)\), we have the following equality in \( R^{-\infty}(K)\):
\[ \left[ \text{index}^K_{\mathcal{P}}(Q^*[\sigma]) \right]^K = \text{index}_M^G([\sigma]). \]

2.3. Product. Suppose that we have two \( G \)-locally compact topological spaces \( N_k, k = 1, 2 \). For \( j \in \mathbb{Z}/2\mathbb{Z} \), we have a product
\[ (\Theta) \circ \text{ext} : K^0_G(N_1) \times K^0_G(N_2) \to K^0_G(N_1 \times N_2) \]
which is defined as follows \cite{AS}. Suppose first that \( * = 0 \). For \( k = 1, 2 \), let \( \sigma_k : E_k^+ \to E_k^- \) be a morphism on \( N_k \). Let \( E^\pm = E_1^\pm \oplus E_2^\pm \) and \( E^* = E_1^* \oplus E_2^* \). On \( N_1 \times N_2 \), the morphism \( \sigma_1 \circ \text{ext} \circ \sigma_2 : E^+ \to E^* \), is defined by the matrix
\[ \sigma_1 \circ \text{ext} \circ \sigma_2(a, b) = \begin{pmatrix} \sigma_1(a) \circ \text{Id} & -\text{Id} \circ \sigma_2(b) \\ \text{Id} \circ \sigma_2(b) & \sigma_1(a) \circ \text{Id} \end{pmatrix}, \]
for \((a, b) \in N_1 \times N_2\). Note that \( \text{Support}(\sigma_1 \circ \text{ext} \circ \sigma_2) = \text{Support}(\sigma_1) \times \text{Support}(\sigma_2) \). Hence the product \( \sigma_1 \circ \text{ext} \circ \sigma_2 \) is elliptic when each \( \sigma_k \) is elliptic, and the product \([\sigma_1] \circ \text{ext} \circ [\sigma_2] \) is defined as the class \([\sigma_1 \circ \text{ext} \circ \sigma_2]\). When \( * = 1 \), we make the same construction with the spaces \( N_1 \) and \( N_2 \times \mathbb{R} \).

Two particular cases of this product are noteworthy:
- When \( N_1 = N_2 = N \), the inner product on \( K^0_G(N) \) is defined as \( a \circ b = \Delta^*(a \circ \text{ext} \circ b) \), where \( \Delta^* : K^0_G(N \times N) \to K^0_G(N) \) is the restriction morphism associated to the diagonal mapping \( \Delta : N \to N \times N \).
- The structure of \( R(G) \)-module of \( K^0_G(N) \) can be understood as a particular case of the exterior product, when \( N_1 \) is reduced to a point.
Let us recall the multiplicative property of the index for the product of manifolds. Consider a compact Lie group $G_2$ acting on two manifolds $M_1$ and $M_2$, and assume that another compact Lie group $G_1$ acts on $M_1$ commuting with the action of $G_2$. The external product of complexes on $T^*M_1$ and $T^*M_2$ induces a multiplication

\[
\odot_{\text{ext}} : K^0_{G_1 \times G_2}(T^*_G M_1) \times K^*_G(T^*_G M_2) \to K^0_{G_1 \times G_2}(T^*_G M_1 \times M_2).
\]

Since $T^*_G M_1 \times M_2 \neq T^*_G M_1 \times G^*_G M_2$ in general, the product $[\sigma_1] \odot_{\text{ext}} [\sigma_2]$ of transversally elliptic symbols need some care: we have to take representative $\sigma_2$ that are almost homogeneous (see Lemma 4.9 in [13]).

**Theorem 2.6 (Multiplicative property).** For any $[\sigma_1] \in K^0_{G_1 \times G_2}(T^*_G M_1)$ and any $[\sigma_2] \in K^0_{G_2}(T^*_G M_2)$ we have

\[
\text{index}^{G_1 \times G_2}_{M_1 \times M_2}([\sigma_1] \odot_{\text{ext}} [\sigma_2]) = \text{index}^{G_1}_{M_1}([\sigma_1]) \text{index}^{G_2}_{M_2}([\sigma_2]).
\]

In the last theorem, the product of $\text{index}^{G_1 \times G_2}_{M_1}(\sigma_1) \in R^{-\infty}(G \times G_2)$ and $\text{index}^{G_2}_{M_2}(\sigma_2) \in R^{-\infty}(G_2)$ is well defined since $\text{index}^{G_1 \times G_2}_{M_1}(\sigma_1)$ is smooth relatively to $G_2$ (see Section 2.2).

Suppose now that $G$ is abelian. For a generalized character $\Phi \in R^{-\infty}(G)$, we consider its support $\text{Sup}(\Phi) \subset G$ and the corresponding subset $\text{Sup}(\Phi) \subset g^*$ formed by the differentials.

Let $a \subset g$ a rational subspace, and let $\pi_a : g^* \to a^*$ be the projection. We will be interested to the $K$-groups $K^0_G(T^*_a M)$ associated to the $G$-spaces $T^*_a M := \{(m, \xi) \in T^* M \mid (\xi, X_M(m)) = 0 \text{ for all } X \in a\}$. We can prove that if $\sigma \in K^0_G(T^*_a M)$, then its index $\Phi := \text{Index}^{G}_a(\sigma) \in R^{-\infty}(G)$ has the following property: the projection $\pi_a$, when restricted to $\text{Sup}(\Phi)$, is proper (see [8]).

We have another version of Theorem 2.6.

**Theorem 2.7 (Multiplicative property - Abelian case).** Let $M_1$ and $M_2$ be two $G$-manifolds (with $G$ abelian), and let $a_1, a_2$ be two rational subspaces of $g$ such that $a_1 \cap a_2 = \{0\}$. If the infinitesimal action of $a_1$ is trivial on $M_2$, we have an external product

\[
\odot_{\text{ext}} : K^0_G(T^*_a M_1) \times K^*_G(T^*_a M_2) \to K^0_G(T^*_a M_1 \oplus a_2 M_2),
\]

and for any $[\sigma_k] \in K^0_G(T^*_a M_k)$ we have

\[
\text{index}^{G}_{M_1 \times M_2}([\sigma_1] \odot_{\text{ext}} [\sigma_2]) = \text{index}^{G}_{M_1}([\sigma_1]) \text{index}^{G}_{M_2}([\sigma_2]).
\]

Let us briefly explain why the product of the generalized characters $\Phi_k := \text{index}^{G}_{M_k}(\sigma_k) \in R^{-\infty}(G)$ is well-defined. We know that the projection $\pi_k : g^* \to a^*_k$ is proper when restricted to the infinitesimal support $\text{Sup}(\Phi_k) \subset g^*$. Since the infinitesimal action of $a_1$ is trivial on $M_2$, we know also that the image of $\text{Sup}(\Phi_2)$ by $\pi_1$ is finite (see Remark 2.3). These three facts insure that for any $\chi \in \tilde{G}$ the set $\{(\chi_1, \chi_2) \in \text{Sup}(\Phi_1) \times \text{Sup}(\Phi_2) \mid \chi_1 + \chi_2 = \chi\}$ is finite. Hence we can define the product $\Phi_1 \otimes \Phi_2$ as the restriction of $(\Phi_1, \Phi_2) \in R^{-\infty}(G \times G)$ to the diagonal.\(^2\)

\(^2\)A subspace $a \subset g$ is rational when it is the Lie algebra of a closed subgroup.
Remark 2.8. Consider an action of a compact abelian Lie group \( G \) on a manifold \( M \). Suppose that a torus subgroup \( H \subset G \) acts trivially on \( M \). Let \( H' \) be a closed subgroup of \( G \) such that \( G \simeq H \times H' \). In this case we have an isomorphism \( K_G^*(T_G^*M) \simeq R(H) \otimes K_H^*(T_H^*M) \) and we see that the index map sends \( K_G^*(T_G^*M) \) into \( R(H) \otimes R^{-\infty}(H') \simeq \langle R^{-\infty}(G/H) \rangle \). See the introduction where the submodule \( \langle R^{-\infty}(G/H) \rangle \) is defined without using a decomposition \( G \simeq H \times H' \).

2.4. Direct images and Bott symbols. Let \( \pi : \mathcal{E} \to N \) be a \( G \)-equivariant complex vector bundle. We define the Bott morphism on \( \mathcal{E} \)

\[
\text{Bott}(\mathcal{E}) : \wedge^+ \pi^* \mathcal{E} \to \wedge^- \pi^* \mathcal{E},
\]

by the relation \( \text{Bott}(\mathcal{E})(n,v) = \text{Cl}(v) : \wedge^+ \mathcal{E}_n \to \wedge^- \mathcal{E}_n \). Here the Clifford map is defined after the choice of a \( G \)-invariant Hermitian product on \( \mathcal{E} \).

Let \( s : N \to \mathcal{E} \) be the 0-section map. Since the support of \( \text{Bott}(\mathcal{E}) \) is the zero section, we have a push-forward morphism

\[
(10) \quad s_! : K^*_G(N) \to K^*_G(\mathcal{E}) \quad \sigma \quad \mapsto \quad \text{Bott}(\mathcal{E}) \circ_{\text{ext}} \pi^*(\sigma)
\]

which is bijective: it is the Bott-Thom isomorphism \([5]\).

Consider now an Euclidean vector space \( V \). Then its complexification \( V_C \) is an Hermitian vector space. The cotangent bundle \( T^*V \) is identified with \( V_C \): we associate to the covector \( \xi \in T^*_V \) the element \( v + i\xi \in V_C \), where \( \xi \in V^* \to \xi \in V \) is the identification given by the Euclidean structure.

Then \( \text{Bott}(V_C) \) defines an elliptic symbol on \( V \) which is equivariant relative to the action of the orthogonal group \( O(V) \). Its analytic index is computed in \([1]\). We have the equality

\[
(11) \quad \text{index}^{O(V)}_{V}(\text{Bott}(V_C)) = 1
\]

in \( R(O(V)) \).

Let \( \pi : V \to M \) be a \( G \)-equivariant real vector bundle over a compact manifold. We have the fundamental fact

**Proposition 2.9.** We have a push-forward morphism

\[
(12) \quad s_! : K^*_G(T^*_G M) \to K^*_G(T^*_G V)
\]

such that \( \text{index}^G_V \circ s_! = \text{index}^G_M \) on \( K^*_0(T^*_G M) \).

**Proof.** We fix a \( G \)-invariant euclidean structure on \( V \). Let \( n = \text{rank} \ V \). Let \( P \) be the associated orthogonal frame bundle. We have \( M = P/O \) and \( V = P \times_O V \) where \( V = \mathbb{R}^n \) and \( O \) is the orthogonal group of \( V \). For the cotangent bundle we have canonical isomorphisms

\[
T^*_G M \simeq T^*_{G \times O}(P/O) \quad \text{and} \quad T^*_G V \simeq T^*_{G \times O}(P \times V)/O
\]

which induces isomorphisms between \( K \)-groups

\[
Q_1 : K^*_G(T^*_G M) \to K^*_{G \times O}(T^*_{G \times O} P),
\]

\[
Q_2 : K^*_G(T^*_G V) \to K^*_{G \times O}(T^*_{G \times O}(P \times V)).
\]
Let us use the multiplicative property (see Section 2.3) with the groups $G_2 = G \times O, G_1 = \{1\}$ and the manifolds $M_1 = V, M_2 = P$. We have a map
\begin{equation}
(13) \quad s'_i : K^*_{G \times O}(T^*_{G \times O}P) \rightarrow K^*_{G \times O}(T^*_{G \times O}(P \times V))
\end{equation}
\[ \sigma \mapsto \text{Bott}(V \mathcal{C}) \odot \text{ext} \sigma \]

The map $s_i : K^*_{G}(T^*_G M) \rightarrow K^*_{G}(T^*_G V)$ is defined by the relation $s_i = Q^*_1 \circ s'_i \circ (Q^*_2)^{-1}$.

Thanks to Theorem 2.6 the relation (11) implies that $\text{index}_{P \times V}^G \circ s'_i = \text{index}_{P}^G$ on $K^*_{G \times O}(T^*_{G \times O}P)$. Thanks to Theorem 2.3 we have
\[
\text{index}_{V}^G(s_i(\sigma)) = \left[\text{index}_{P \times V}^G(s'_i \circ (Q^*_2)^{-1}(\sigma))\right]^O
= \left[\text{index}_{P}^G((Q^*_2)^{-1}(\sigma))\right]^O
= \text{index}_{M}^G(\sigma).
\]
for any $\sigma \in K^0_{G \times O}(T^*_{G \times O}P)$. □

We finish this section by considering the case of a $G$-equivariant embedding $i : Z \hookrightarrow M$ between $G$-manifolds.

**Proposition 2.10.** We have a push-forward morphism
\begin{equation}
(14) \quad i_* : K^*_G(T^*_G Z) \rightarrow K^*_G(T^*_G M)
\end{equation}
such that $\text{index}_{M}^G \circ i_* = \text{index}_{Z}^G$ on $K^0_G(T^*_G Z)$.

**Proof.** Let $\mathcal{N} = TM|_Z/TZ$ be the normal bundle. We know that an open $G$-invariant tubular neighborhood $U$ of $Z$ is equivariantly diffeomorphic with $\mathcal{N}$; let us denote by $\varphi : U \rightarrow \mathcal{N}$ this equivariant diffeomorphism. Let $j : U \hookrightarrow M$ be the inclusion. We consider the morphism $s_i : K^*_G(T^*_G Z) \rightarrow K^*_G(T^*_G \mathcal{N})$ defined in Proposition 2.9 the isomorphism $\varphi^* : K^*_G(T^*_G \mathcal{N}) \rightarrow K^*_G(T^*_G U)$ and the push-forward morphism $j_* : K^*_G(T^*_G U) \rightarrow K^*_G(T^*_G M)$. Thanks to Proposition 2.4 one sees that the composition $i_* = j_* \circ \varphi^* \circ s_i$ satisfies $\text{index}_{M}^G \circ i_* = \text{index}_{Z}^G$ on $K^0_G(T^*_G Z)$. □

### 2.5. Restriction: the vector bundle case.

Let $E \rightarrow M$ be a $G$-equivariant complex vector bundle. Let us introduce the invariant open subset $T^*_G(E \setminus \{0\})$ of $T^*_G E$ and its complement $T^*_G E|_{0\text{-section}} = T^*_G M \times E^*$. We denote
\begin{equation}
(15) \quad R : K^*_G(T^*_G E) \rightarrow K^*_G(T^*_G M)
\end{equation}
the composition of the restriction morphism $K^*_G(T^*_G E) \rightarrow K^*_G(T^*_G M \times E^*)$ with the Bott-Thom isomorphism $K^*_G(T^*_G M \times E^*) \simeq K^*_G(T^*_G M)$. Note that the morphism
\begin{equation}
(16) \quad R : K^*_G(T^*_D E) \rightarrow K^*_G(T^*_D M)
\end{equation}
is also defined when $D \subset G$ is a distinguished subgroup.

If $S = \{v \in E \mid ||v||^2 = 1\}$ is the sphere bundle, we have $E \setminus \{0\} \simeq S \times \mathbb{R}$ and then $T^*_G(E \setminus \{0\}) \simeq T^*_G S \times T^* \mathbb{R}$. Let $i : S \rightarrow E$ be the canonical immersion. The composition of the Bott-Thom isomorphism $K^*_G(T^*_G S) \simeq K^*_G(T^*_G (E \setminus \{0\}))$ with
the morphism \( j_s : K^*_G(T^*_G(E \setminus \{0\})) \to K^*_G(T^*_G(E)) \) correspond to the push-forward map \( i \) defined in Proposition \( 2.10 \). The six term exact sequence \( 3 \) becomes

\[
\begin{array}{ccc}
K^0_G(T^*_G\mathcal{S}) & \xrightarrow{i} & K^0_G(T^*_G\mathcal{E}) \\
\downarrow & & \downarrow \\
K^0_G(T^*_GM) & \xrightarrow{\delta} & K^1_G(T^*_GM) \\
\end{array}
\]

Let \( s_1 : K^*_G(T^*_G\mathcal{M}) \to K^*_G(T^*_G\mathcal{E}) \) be the push-forward morphism associated to the zero section \( s : M \to \mathcal{E} \) (see Proposition \( 2.10 \)). We have the fundamental

**Proposition 2.11.**

- The composition \( R \circ s_1 : K^*_G(T^*_G\mathcal{M}) \to K^*_G(T^*_G\mathcal{M}) \) is the map \( \sigma \mapsto \sigma \otimes \wedge^* \mathcal{E} \).
- The composition \( s_1 \circ R : K^*_G(T^*_G\mathcal{E}) \to K^*_G(T^*_G\mathcal{E}) \) is defined by \( \sigma \mapsto \sigma \otimes \wedge^* \mathcal{E} \).
- We have \( \text{index}_{\mathcal{M}}^G(R(\sigma)) = \text{index}^E_\mathcal{E}(\sigma \otimes \wedge^* \mathcal{E}) \) for any \( \sigma \in K^0_G(T^*_G\mathcal{E}) \).

**Proof.** The third point is a consequence of second point. Let us check the first two points.

We use the notations of the proof of proposition \( 2.9 \) we have a principal bundle \( P \to M = P/O \) and \( \mathcal{E} \) coincides as a real vector bundle with \( P \times O \). Since \( \mathcal{E} \) has an invariant complex structure, we can consider the frame bundle \( Q \subset P \) formed by the unitary basis of \( \mathcal{E} \). Here \( E = \mathbb{R}^{2n} = \mathbb{C}^n \). Let \( U \subset O \) be the unitary group of \( E \). Here the map \( s_1 \) and \( R \) can be defined with the reduced data \( (Q, U) \) through the maps

\[
s'_1 : K^*_G \times U(T^*_G \times UQ) \to K^*_G \times U(T^*_G \times U(Q \times E))
\]

and \( R' : K^*_G \times U(T^*_G \times U(Q \times E)) \to K^*_G \times U(T^*_G \times UQ) \). Since \( E \) admits a complex structure \( J \), the map \( w \oplus iv \mapsto (w + Jv, w - Jv) \) is an isomorphism between \( E \) and the orthogonal sum \( E \oplus \overline{E} \). Hence on \( E \) the Bott morphism \( \text{Cl}(w + iv) : \wedge^* E \to \wedge^* \overline{E} \) is equal to the product of the morphisms \( \text{Cl}(w + Jv) : \wedge^* E \to \wedge^* \overline{E} \) and \( \text{Cl}(w - Jv) : \wedge^* E \to \wedge^* \overline{E} \). When we restrict the Bott symbol \( \text{Bott}(E) \in K^0_U(T^*E) \) to the 0-section, we get

\[
(\wedge^* E \xrightarrow{\text{Cl}(w)} \wedge^* \overline{E}) \circ (\wedge^* E \xrightarrow{\text{Cl}(w)} \wedge^* \overline{E})
\]

which is equal to the class \( \text{Bott}(E) \otimes \wedge^* \mathcal{E} \) in \( K^0_G(E) \). Finally the composition \( R' \circ s'_1 : K^*_G \times U(T^*_G \times UQ) \to K^*_G \times U(T^*_G \times UQ) \) is equal to the map \( \sigma \mapsto \sigma \otimes \wedge^* \mathcal{E} \).

We get the first point through the isomorphism \( K^*_G \times U(T^*_G \times UQ) \cong K^*_G(T^*_G\mathcal{M}) \).

Let \( \sigma \in K^*_G \times U(T^*_G \times U(Q \times E)) \). For \( (x, \xi; v, w) \in T^*Q \times T^*E \), the transversally elliptic symbols

\[
\begin{align*}
\sigma(x, \xi; v, w) & \otimes \wedge^* \mathcal{E} \\
\sigma(x, \xi; v, w) & \otimes \text{Cl}(v) \\
\sigma(x, \xi; 0, w) & \otimes \text{Cl}(v) \\
R'(\sigma)(x, \xi) & \otimes \text{Cl}(w) \otimes \text{Cl}(v) \\
R'(\sigma)(x, \xi) & \otimes \text{Cl}(w + Jv) \otimes (w - Jv) \\
s'_1 \circ R'(\sigma)(x, \xi; v, w)
\end{align*}
\]
Theorem 3.4. Define the same class in $K^*_G \times U(T^*_G \times U(Q \times E))$. We have proved that $s'_i \circ R'(\sigma) = \sigma \otimes \wedge^* \overline{E}$, and we get the second point through the isomorphism $K^*_G \times U(T^*_G \times U(Q) \cong K^*_G(T^*_G \times M)$.

2.6. Restriction to a sub-manifold. Let $M$ be a $G$-manifold and let $Z$ be a closed $G$-invariant sub-manifold of $M$. Let us consider the open subset $T^*_G(M \setminus Z)$ of $T^*_G M$. Its complement is the closed subset $T^*_G M|_Z$. Let $N$ be the normal bundle of $Z$ in $M$. We have $T^*_M|_Z = T^*_Z \times N^*$ and then $T^*_G M|_Z = T^*_G Z \times N^*$.

We make the following hypothesis: the real vector bundle $N^* \to Z$ has a $G$-equivariant complex structure. Then we can define the map

$$R_Z : K^*_G(T^*_G M) \to K^*_G(T^*_G Z)$$

as the composition of the restriction $K^*_G(T^*_G M) \to K^*_G(T^*_G M|_Z) = K^*_G(T^*_G Z \times N^*)$ with the Bott-Thom isomorphism $K^*_G(T^*_G Z \times N^*) \to K^*_G(T^*_G Z)$.

3. Localization

In this section, $\beta \in \mathfrak{g}$ denotes a non-zero $G$-invariant element, and $\pi : E \to M$ is a $G$-equivariant hermitian vector bundle such that $\mathcal{E}^\beta = M$.

Remark 3.1. Note that (19) imposes the existence of a $G$-invariant complex structure on the fibers of $\mathcal{E}$. We take $J_\beta := \mathcal{L}(\beta)(-\mathcal{L}(\beta)^2) \mathcal{L}(\beta)$, where $\mathcal{L}(\beta)$ denotes the linear action on the fibers of $\mathcal{E}$.

The aim of this section is the following

Theorem 3.2. There exists a morphism $S_\beta : K^*_G(T^*_G M) \to K^*_G(T^*_G E)$ satisfying the following properties:

1. The composition $R \circ S_\beta$ is the identity on $K^*_G(T^*_G M)$.
2. For any $a \in K^*_G(T^*_G M)$, we have $S_\beta(a) \otimes \wedge^* \mathcal{E} = s(a)$.
3. For any $\sigma \in K^*_G(T^*_G M)$, we have the following equality

$$\text{Index}^G_\mathcal{E}(S_\beta(\sigma)) = \text{Index}^G_M(\sigma \otimes [\wedge^* \overline{\mathcal{E}}]^{-1})$$

in $R^{-\infty}(G)$, where $[\wedge^* \overline{\mathcal{E}}]^{-1}$ is a polarized inverse of $\wedge^* \overline{\mathcal{E}}$.

Remark 3.3. The maps $R$ and $S_\beta$ depend on the choice of the $G$-invariant complex structure on $\mathcal{E}$.

Theorem 3.2 tells us that (19) breaks in an exact sequence

$$0 \to K^*_G(T^*_G S) \to K^*_G(T^*_G E) \to K^*_G(T^*_G M) \to 0.$$ 

Since $R \circ S_\beta = R \circ S_\beta$ the image of the map $S_\beta - S_\beta : K^*_G(T^*_G M) \to K^*_G(T^*_G E)$ belongs to the image of the push-forward map $i_! : K^*_G(T^*_G S) \to K^*_G(T^*_G E)$.

Let us work now with the complex structure $J_\beta$ on $\mathcal{E}$. We denote $S^o_\beta$ the corresponding morphism. In Section 3.3.5.3 we will prove the following

Theorem 3.4. There exists a morphism $\theta_\beta : K^*_G(T^*_G M) \to K^*_G(T^*_G S)$ such that $S^o_\beta - S^o_\beta = i_! \circ \theta_\beta$.

---

3 Relatively to a $G$-invariant Euclidean metric on $\mathcal{E}$, the linear map $-\mathcal{L}(\beta)^2$ is positive definite, hence one can take its square root.
3.1. Atiyah-Singer pushed symbols. Let \( M \) be a \( G \)-manifold with an invariant almost complex structure \( J \). Then the cotangent bundle \( T^*M \) is canonically equipped with a complex structure, still denoted \( J \). The Bott morphism on \( T^*M \) associated to the complex vector bundle \((T^*M, J) \to M)\) is called the Thom symbol of \( M \), and is denoted \( \text{Thom}(M, J) \). Note that the product by the Thom symbol induces an isomorphism \( K^*_G(M) \cong K^*_G(T^*M) \).

For any \( X \in \mathfrak{g} \), we denote \( X_M(m) := \frac{d}{dt}|_{t=0} e^{-tx} \cdot m \) the corresponding vector field on \( M \). Thanks to an invariant Riemannian metric on \( M \), we define the 1-form \( \overline{X}_M(m) = (X_M(m), -) \).

From now on, we take \( X = \beta \) a non-zero \( G \)-invariant element. Then the corresponding 1-form \( \beta_M \) is \( G \)-invariant, and we define following Atiyah-Singer the equivariant morphism

\[
\text{Thom}_\beta(M, J)(m, \xi) := \text{Thom}(M, J)(m, \xi - \beta_M(m)), \quad (\xi, m) \in T^*M.
\]

We check easily that

\[
\text{Support}(\text{Thom}_\beta(M, J)) \cap T^*_\beta M = \{(m, 0); \ m \in M^\beta\}.
\]

where \( T^*_\beta = \exp(\mathbb{R}_\beta) \) is the torus generated by \( \beta \). In particular, we get a class

\[
(20) \ \ \text{Thom}_\beta(M, J) \in K^0_G(T^*_\beta M)
\]

when \( M^\beta \) is compact.

3.2. Atiyah-Singer pushed symbols: the linear case. Let us consider the case of a \( G \) -Hermitian vector space \( E \) such that \( E^\beta = \{0\} \).

Let \( \iota_1 : K^0_G(T^*_\beta S) \to K^0_G(T^*_\beta E) \) be the push-forward morphism associated to the inclusion \( i : S \to E \) of the sphere of radius one. Let \( R : K^0_G(T^*_\beta E) \to K^0_G(\{\bullet\}) \) be the restriction morphism. Since \( K^0_G(\{\bullet\}) = 0 \), the six term exact sequence \( \text{[17]} \) becomes

\[
(21) \quad 0 \to K^0_G(T^*_\beta S_E) \xrightarrow{\iota_1} K^0_G(T^*_\beta E) \xrightarrow{R} R(G).
\]

The pushed Thom symbol on \( E \) defines a class \( \text{Thom}_\beta(E) \in K^0_G(T^*_\beta E) \).

**Proposition 3.5.**

- We have \( R(\text{Thom}_\beta(E)) = 1 \) in \( R(G) \).
- The sequence \( \text{[21]} \) breaks down: we have a decomposition

\[
K^0_G(T^*_\beta E) = K^0_G(T^*_\beta S_E) \oplus \langle \text{Thom}_\beta(E) \rangle,
\]

where \( \langle \text{Thom}_\beta(E) \rangle \) denotes the free \( R(G) \)-module generated by \( \text{Thom}_\beta(E) \).

**Proof.** At \((x, \xi) \in T^*E\) the map \( \text{Thom}_\beta(E)(x, \xi) : \wedge^+ E \to \wedge^+ E \) is equal to \( \text{Cl}(\xi - \beta_E(x)) \), where \( \xi \in E^* \mapsto \xi \in E \) is the identification given by the Euclidean structure. We see that the restriction of \( \text{Thom}_\beta(E) \) to \( T^*_\beta E|_0 = E^* \) is equal to \( \text{Bott}(E^*) \) and then the first point follows. The second point is a direct consequence of the first one. \( \square \)

---

\(^4\)When the almost complex structure is understood, we will use the notation \( \text{Thom}(M) \).
Let $\hat{T}_\beta$ the group of characters of the torus $T_\beta$. The complex $G$-module $E$ decomposes into weight spaces $E = \sum_{\alpha \in \hat{T}_\beta} E_\alpha$ where each $E_\alpha = \{v \in E \mid t \cdot v = t^\alpha v\}$ are $G$-submodules. We define the $\beta$-positive and negative part of $E$,

$$E^{+,\beta} = \sum_{\alpha \in \hat{T}_\beta \ (\alpha, \beta) > 0} E_\alpha, \quad E^{-,\beta} = \sum_{\alpha \in \hat{T}_\beta \ (\alpha, \beta) < 0} E_\alpha$$

and the $\beta$-polarized module $|E|^\beta = E^{+,\beta} \oplus E^{-,\beta}$. It is important to note that the complex $G$-module $|E|^\beta$ is isomorphic to $(E, J_\beta)$, and so it does no depend on the initial complex structure of $E$.

Let $\hat{R}(G)$ be the $R(G)$-submodule of $R^{-\infty}(G)$ defined by the torus $T_\beta$ (see Definition 2.2). Since all the $\hat{T}_\beta$-weights in $|E|^\beta$ satisfy the condition $(\alpha, \beta) > 0$, the symmetric space $S^*\left(|E|^\beta\right)$ decomposes as a sum $\sum_{\mu \in \hat{T}_\beta} S^*\left(|E|^\beta\right)_\mu$ with $S^*\left(|E|^\beta\right)_\mu \in R(G)_\mu$. Hence $S^*\left(|E|^\beta\right)$ defines an element of $\hat{R}(G)$.

The following computation is done in [1] [Lecture 5] (see also [12] [Section 5.1]).

**Proposition 3.6.** We have the following equality in $R^{-\infty}(G)$:

$$\text{Index}_{\hat{G}}(\text{Thom}_\beta(E)) = (-1)^{\dim E^{+,\beta}} \det(E^{+,\beta}) \otimes S^*\left(|E|^\beta\right),$$

where $\det(E^{+,\beta})$ is a character of $G$.

**Example 3.7.** Let $V = \mathbb{C}$ with the canonical action of $G = S^1$. Let $\beta = \pm 1$ in $\text{Lie}(S^1) = \mathbb{R}$. The class $\text{Thom}_{\pm 1}(\mathbb{C}) \in K_G^0(T_{S^1} \mathbb{C})$ are represented by the symbols

$$\text{Cl}(\xi \pm ix) : \mathbb{C} \rightarrow \mathbb{C}, \quad (x, \xi) \in T^* \mathbb{C} \simeq \mathbb{C}^2.$$  

We have $\text{Index}_{\hat{C}}^C\left(\text{Cl}(\xi + ix)\right) = -\sum_{k \geq 1} t^k$, and $\text{Index}_{\hat{C}}^C\left(\text{Cl}(\xi - ix)\right) = \sum_{k \leq 0} t^k$ in $R^{-\infty}(S^1)$.

**Remark 3.8.** Let $J_k, k = 0, 1$ be two invariants complex structures on $E$, and let $\text{Thom}_\beta(E, J_k)$ be the corresponding pushed symbols. There exists an invertible element $\Phi \in R(G)$ such that

$$\text{Index}_{\hat{G}}^G(\text{Thom}_\beta(E, J_0)) = \Phi \cdot \text{Index}_{\hat{G}}^G(\text{Thom}_\beta(E, J_1)).$$

3.3. **Pushed symbols : functoriality.** Suppose now that we have a decomposition $V = W \oplus E$ of $G$-complex vector spaces such that $V^\beta \neq \{0\}$.

**Proposition 3.9.** In $K_G^0(T_G^*V)$, we have the equalities

$$\text{Thom}_\beta(V) \otimes \lambda_C V = \text{Bott}(V_C),$$

$$\text{Thom}_\beta(V) \otimes \lambda_C E = \text{Thom}_\beta(W) \otimes \text{Bott}(E_C).$$

**Proof.** Note that the first relation is a particular case of the second one when $W = 0$.

A covector $(x, \xi) \in T^*V$ decomposes in $x = x_W \oplus x_E$, and $\xi = \xi_W \oplus \xi_E$. The morphism $\sigma := \text{Thom}_\beta(W) \otimes \text{Bott}(E_C)$ defines at $(x, \xi)$ the map

$$\text{Cl}(\xi_W - \gamma_W(x_W)) \otimes \text{Cl}(x_E + i\xi_E)$$

from $(\wedge W \otimes \wedge E)^+$ to $(\wedge W \otimes \wedge E)^-$. 

\footnote{With $J_\beta = \mathcal{L}(\beta)(-\mathcal{L}(\beta))^{-1/2}$}

\footnote{These equalities holds also in $K_G^0(T_{\hat{T}_\beta}^*V)$.}
We have an isomorphism of complex $G$-modules: $E_C \simeq E \times \mathbb{E}$. We have two classes Bott($E$) and Bott($\mathbb{E}$) in $K^0_G(E)$ and Bott($E_C$) = Bott($E$) \circ Bott(\mathbb{E})$. At the level of endomorphism on $\wedge E_C \simeq \wedge E \otimes \mathbb{E}$, one has
\[(23)\quad \text{Cl}(x_E + iξE) = \text{Cl}(ξE - J_E x_E) \circ \text{Cl}(ξE + J_E x_E)\]
where $J_E$ is the complex structure on $E$. We consider the family of maps $σ_s(x, ξ): (\wedge W \otimes \wedge E \otimes \wedge \mathbb{E})^+ \rightarrow (\wedge W \otimes \wedge E \otimes \wedge \mathbb{E})^-$ defined by $Cl(ξW - βW(xW)) \circ Cl(ξE - θ_s(xE)) \circ Cl(ξE + J_E x_E)$ where $θ_s = (1 - s)J_E + sβ_E$. One checks easily that Support($σ_s$) $\cap T_G V = \{x_W = x_E = ξ_W = ξ_E = 0\}$ for any $s \in [0, 1]$. Hence $σ = σ_0$ is equal to $σ_1$ in $K^0(T_G^* E)$. Finally we check that $σ_1(x, ξ) = Cl(ξ - β_V(x)) \circ Cl(ξ_E + J_E x_E)$ can be deformed in $Cl(ξ - β_V(x)) \circ Cl(ξ_E + J_E x_E)$ can be deformed in
\[
\text{Cl}(ξ - β_V(x)) \circ \text{Cl}(0) = \text{Thom}_β(V) \otimes \wedge^\ast \mathbb{E},
\]
without changing its class in $K^0_G(T_G^* V)$. □

Since Index$^G_V$(Bott($V$)) = 1, the first relation of Proposition \ref{prop310} gives that
\[(24)\quad \text{Index}^G_V(\text{Thom}_β(V)) \cdot \wedge^\ast \mathbb{V} = 1\]
in $R^{-\infty}(G)$.

**Definition 3.10.** Let $V$ be a complex $G$-vector space such that $V^β = \{0\}$. We denote $[\wedge^\ast \mathbb{V}]^β_1 \in R^{-\infty}(G)$ the element $(-1)^{\dim V - β} \det(V^{-β}) \otimes S^\ast(\|V\|^β)$.

We come back to the morphism
\[(25)\quad R : K^0_G(T_G^* V) \rightarrow K^0_G(T_G^* W)\]
which is the composition of the restriction morphism $K^0_G(T_G^* V) \rightarrow K^0_G(T_G^* W \times E^*)$ with the Thom isomorphism $K^0_G(T_G^* W \times E^*) \simeq K^0_G(T_G^* W)$. We are interested by the image of the transversally elliptic symbols Thom$_β(V) \in K^0_G(T_G^* V)$ by the morphism $R$.

**Proposition 3.11.** We have the following equality in $K^0_G(T_G^* W)$
\[
R(\text{Thom}_β(V)) = \text{Thom}_β(W).
\]

**Proof.** The class Thom$_β(V)$ are defined by the symbols $Cl(ξ - β(x)) : ∧^+ V \rightarrow ∧^- V$, for $(x, ξ) \in TV$. Relatively to the decomposition $V = W \oplus E$, we write $x = x_W \oplus x_E$ and $ξ = ξ_W \oplus ξ_E$. If we restrict $Cl(ξ - β(x))$ to $T^* V|_W = T^* W \times E^*$ we get $Cl(ξ_W - β(x_W)) \circ Cl(ξ_E)$ acting from $(∧ W \otimes ∧ E)^+ \rightarrow (∧ W \otimes ∧ E)^-$. By definition of the map $R$ we find that $R(\text{Thom}_β(V)) = \text{Thom}_β(W)$. □

We consider now the case of a product of pushed symbols. Suppose that we have an invariant decomposition $E = E_1 \oplus E_2$ and invariant elements $β_1, β_2 \in \mathfrak{g}$ such that
- $E_1^{β_1} = E_2^{β_2} = \{0\}$,
- $β_2$ acts trivially on $E_1$.

We consider then $β^t = tβ_1 + β_2$ with $t > 0$. We have $V_1^{β^t} = \{0\}$ for any $t > 0$ and $V_2^{β^t} = \{0\}$ if $t > 0$ is small enough.
Lemma 3.12. Let $J = J_1 \oplus J_2$ be an invariant complex structure on $V = V_1 \oplus V_2$. Then if $t > 0$ is small enough, we have the following equality in $K_G^0(T_G^* V)$:

$$\Thom_{\beta'}(V, J) = \Thom_{\beta_1}(V_1, J_1) \odot \Thom_{\beta_2}(V_2, J_2).$$

Proof. Both symbols are maps from $(\wedge V_1 \otimes \wedge V_2)^{+}$ into $(\wedge V_1 \otimes \wedge V_2)^{-}$. We write a tangent vector $(\xi, x) \in TV$ as $\xi = \xi_1 \oplus \xi_2$ and $x = x_1 \oplus x_2$. The symbol $\Thom_{\beta'}(V, J)$ is equal to

$$\Cl(\xi_1 + \beta_1(x_1)) \odot \Cl(\xi_2 + \beta_2(x_2)) = \Cl(\xi_1 + t\tilde{\beta}_1(x_1)) \odot \Cl(\xi_2 + (t\tilde{\beta}_1 + \tilde{\beta}_2)(x_2))$$

Note that $\tilde{\beta}_2 : V_2 \to \wedge V_2$ is invertible, so there exist $c > 0$ such that $t\tilde{\beta}_1 + \tilde{\beta}_2$ is invertible for any $t \in [0, c]$. Hence $\Thom_{\beta'}(V, J)$ is transversally elliptic for $0 < t \leq c$. We consider the deformation

$$\sigma_s = \Cl(\xi_1 + (st + (1 - s))\tilde{\beta}_1(x_1)) \odot \Cl(\xi_2 + (st\tilde{\beta}_1 + \tilde{\beta}_2)(x_2))$$

for $s \in [0, 1]$. We check easily that $\text{Support}(\sigma_s) \cap T_G V = \{(0, 0)\}$ for any $s \in [0, 1]$. Hence $\sigma_0 = \Thom_{\beta'}(V_1, J)$ and $\sigma_0 = \Thom_{\beta_1}(V_1, J_1) \odot \Thom_{\beta_2}(V_2, J_2)$ defines the same class in $K_G^0(T_G^* V)$.

\[\square\]

3.4. The map $S_{\beta}$. We come back to the situation of a $G$-equivariant complex vector bundle $\pi : E \to M$ such that $E^\beta = M$. Since the torus $T_\beta$ acts trivially on $M$, we have a decomposition $E = \oplus_{\alpha \in X} E_\alpha$ where $X$ is a finite set of character of $T_\beta$, and $E_\alpha$ is the complex sub-bundle of $E$ where $T_\beta$ acts trough the character $t \mapsto t^\alpha$. Definition 3.10 can be extended as follows. We denote

$$[\wedge^\bullet E]^\beta = (-1)^{\dim c} \det(\wedge^\bullet \beta) \otimes S^\bullet(|\wedge E|^\beta)$$

where $E^\pm,\beta = \sum_{(\alpha, \beta) > 0} E_\alpha$ and $|\wedge E|^\beta = E^{+, \beta} \oplus E^{-, \beta}$. Note that $[\wedge^\bullet E]^\beta = \wedge^\bullet (C\alpha)^n_\alpha$, which is equipped with the standard Hermitian structure.

Let $U$ be the unitary group of $E$, and let $U'$ be the subgroup of elements that commute with the action of $T_\beta$: we have $U' \simeq \Pi_{\alpha \in X} U(C^n_\alpha)$. Let $P' \to M$ be the $U'$-principal bundle defined as follows: for $m \in M$, the fiber $P'_m$ is defined as the set of maps $f : E \to E_\alpha$ preserving the Hermitian structures and which are $T_\beta$-equivariant. By definition, the bundle $P' \to M$ is $G$-equivariant. We consider the following groups action:

- $G \times U'$ acts on $P'$,
- $U' \times T_\beta$ acts on $E$,
- $T_\beta$ and $G$ acts trivially respectively on $P'$ and on $E$.

Let us use the multiplicative property (see Section 2.3) with the groups $G_2 = G \times U'$, $G_1 = T_\beta$ and the manifolds $M_1 = E$, $M_2 = P'$. We have a product

$$K_{T_\beta \times G \times U'}^0(T_{T_\beta}^* E) \times K_{G \times U'}^0(T_{G \times U'}^* P') \to K_{T_\beta \times G \times U'}^0(T_{T_\beta \times G \times U'}^* (P' \times E)),$$
and the Thom class Thom_β(E) ∈ K^0_{T^*_β × U'(T^*_β × E)} induces the map

\[(27) \quad S'_β : K^*_G(T^*_β × E) \rightarrow K^*_G(T^*_β × U'(P × E)) \]

\[\sigma \mapsto \text{Thom}_β(E) \circ \text{ext} \sigma\]

After taking the quotient by \( U' \), we get a map

\[S'_β : K^*_G(T^*_β M) \rightarrow K^*_G(T^*_β × G E)\]

Finally, since \( T^*_β × G E = T^*_G E \), we can compose \( S'_β \) with the forgetful map \( K^*_β × G E \rightarrow K^*_β \) to get

\[S_β : K^*_G(T^*_β M) \rightarrow K^*_G(T^*_G E)\]

Now we see that in Theorem 3.2:

1. The relation \( R \circ S_β = \text{Id} \) is induced by the relation \( R(\text{Thom}_β(E)) = 1 \), where \( R : K^*_β × U'(T^*_β E) \rightarrow R(T^*_β × U') \) (see Proposition 3.3).
2. The relation \( S_β(a) \otimes \wedge^* E = s_β(a) \) is induced by the relation \( \text{Thom}_β(E) \otimes \wedge^* E = \text{Bott}(E_G) \) proved in Proposition 3.3.

Let us prove the last point of Theorem 3.2. Let \( σ ∈ K^0_G(T^*_β M) \) and let \( \tilde{σ} \) be the corresponding element in \( K^0_{T^*_β × U'(T^*_β × E)} \). The index \( \text{Index}^{G \times T^*_β}_E(S_β(σ)) \) is equal to the restriction of \( \text{Index}^{G \times T^*_β}_E(S_β(σ)) \) to \( R^{-∞}(G × T_β) \) (see Section 2.2). By definition we have the following equalities in \( R^{-∞}(G × T_β) \)

\[\text{Index}^{G \times T^*_β}_E(S_β(σ)) = \left[\text{Index}^{U' × G \times T^*_β}_E(S'_β(\tilde{σ}))\right]^{U'} = \left[\text{Index}^{U' × G}_E(\tilde{σ}) \cdot \text{Index}^{U' × T^*_β}_E(\text{Thom}_β(E))\right]^{U'} = \sum_{μ ∈ E_β} \text{Index}^G_M(σ \otimes W_μ) \otimes C_μ\]

where \( \text{Index}^{U' × T^*_β}_E(\text{Thom}_β(E)) = [\wedge^* E]^{-1}_β = \sum_{μ ∈ E_β} W_μ \otimes C_μ \) with \( W_μ ∈ R(U') \). We denote \( W_μ = P^* × U' W_μ \) the corresponding element in \( K^0_M(M) \). Finally we get

\[\text{Index}^G_E(S_β(σ)) = \sum_{μ ∈ E_β} \text{Index}^G_M(σ \otimes W_μ) = \text{Index}^G_M(σ \otimes [\wedge^* E]^{-1}_β)\]

where \( [\wedge^* E]^{-1}_β = \sum_{μ ∈ T} W_μ \in K^0_G(M) \).

3.5. The map \( θ_β \). We keep the same notation than the previous section: \( π : E → M \) is a \( G \)-equivariant complex vector bundle such that \( E^β = M \), but here we work with the complex structure \( J_β \) on \( E \). Since the map \( S^o_β \) are defined through the pushed Thom classes \( \text{Thom}_β(E) ∈ K^0_G(T^*_β × E) \) (see (27)), we have to study the class \( \text{Thom}_β(E) - \text{Thom}_β(E) \) in order to understand how the map \( S^o_β - S^o_β : K^*_G(T^*_β × E) \rightarrow K^*_G(T^*_β × G E) \) factorizes through the push-forward morphism \( i_1 : K^*_G(T^*_G E) → K^*_G(E) \).
3.1. The tangential Cauchy Riemann operator. Let $E$ be a Euclidean $G$-module such that $E^0 = \{0\}$. We equipped $E$ with the invariant complex structure $J_\beta$ (see Remark 3.11). Let $S \subset E$ be the sphere of radius one. Let us defined the tangential Cauchy Riemann operator on $S$. For $y \in S$, we have

$$T_y S = \{ \xi \mid (\xi, y) = 0 \}$$

$$= \mathcal{H}_y \oplus \mathbb{R}J_\beta y,$$

where $\mathcal{H}_y = (C_y)^\perp$ is a complex invariant subspace of $(E, J_\beta)$. Let $\mathcal{H} \to S$ be the corresponding Hermitian vector bundle. For $\xi \in T_y S$, we denote $\xi'$ its component in $\mathcal{H}_y$. Since $(\beta E(y), J_\beta y) \neq 0$ for $y \neq 0$, we see that for $\xi \in T_S S$, we have $\xi' = 0 \iff \xi = 0$.

**Definition 3.13.** The Cauchy Riemann symbol\(^7\) $\sigma_\beta^E: \wedge^+ \mathcal{H} \to \wedge^- \mathcal{H}$ is defined by $\sigma_\beta^E(y, \xi) = \text{Cl}(\xi') : \wedge^+ \mathcal{H}_y \to \wedge^- \mathcal{H}_y$. It defines\(^8\) a class $\sigma_\beta^E \in K^0_G(T^*_G S)$.

The Thom isomorphism tells us that $K^0_G(T^*_G S) \cong K^0_G(T^*_G (E \setminus \{0\}))$ and we know that $i: K^0_G(T^*_G S) \to K^0_G(T^*_G E)$ is injective. Hence, it will be convenient to use the same notations for $\sigma_\beta^E$ and $i(\sigma_\beta^E)$ and to consider them as a class in $K^0_G(T^*_G (E \setminus \{0\}))$ or in $K^0_G(T^*_G E)$.

**Example 3.14.** Consider the Cauchy Riemann symbol $\sigma_\beta^C \in K^0_G(T^*_G \mathbb{C})$ associated to the one dimensional representation $\mathbb{C}$ of $G$. We check that $\sigma_\beta^C$ is represented by the map $\rho: T^* \mathbb{C} \to \mathbb{C}$ defined by $\rho(w, z) = \Re(wz) + i(|z|^2 - 1)$.

We come back to the setting of Section 3.1. We have an exact sequence $0 \to K^0_G(T^*_G S) \xrightarrow{i} K^0_G(T^*_G E) \xrightarrow{\text{R}} R(G) \to 0$, and we know that $R(\text{Thom}_{\beta}(E)) = 1$. Then $\text{Thom}_{\beta}(E) - \text{Thom}_{-\beta}(E)$ belongs to $\ker(\text{R}) = \text{Im}(i)$.

The following result is due to Atiyah-Singer when $G$ is the circle group (see [1][Lemma 6.3]). The proof in the general case is given in Appendix B.

**Proposition 3.15.** Let $E$ be a $G$-module equipped with the invariant complex structure $J_\beta$. We have the following equality

$$\text{Thom}_{-\beta}(E) - \text{Thom}_{\beta}(E) = i(\sigma_\beta^E).$$

in $K^0_G(T^*_G E)$.

3.2. Functoriality. Suppose that $V = W \oplus E$ with $W^\beta = E^\beta = \{0\}$. We equipped $V, W, E$ by the invariant complex structures defined by $\beta$. Let $\sigma_\beta^W \in K^0_G(T^*_G (V \setminus \{0\}))$, $\sigma_\beta^V \in K^0_G(T^*_G (W \setminus \{0\}))$ be the corresponding Cauchy Riemann classes. We have a natural product

$$K^0_G(T^*_G (W \setminus \{0\})) \times K^0_G(T^*_G E) \to K^0_G(T^*_G (V \setminus \{0\})).$$

and a restriction morphism $R: K^0_G(T^*_G V) \to K^0_G(T^*_G W)$ (see [20]).

**Proposition 3.16.** We have

- $\sigma_\beta^W \otimes \text{Bott}(E) = \sigma_\beta^V \otimes \wedge^E$ in $K^0_G(T^*_G (V \setminus \{0\}))$,
- $R(\sigma_\beta^W) = \sigma_\beta^V$ in $K^0_G(T^*_G W)$.

\(^7\)Here we use an identification $T^* S \cong TS$ given by the Euclidean structure.

\(^8\)Note that $\sigma_\beta^E$ defines also a class in $K^0_G(T^*_G S)$.
we can define $E \to \text{complex vector bundle } G \to R \to S$.

After taking the quotient by $U$. We can precise the last statement of Proposition 3.16, by saying that the identity Thom $\cdots$ holds in $K_G(T_G(W \setminus \{0\}))$.

3.5.3. Definition of the map $\theta_\beta$. We come back to the setting of Section 3.4. The complex vector bundle $E \to M$ corresponds to $P' \times U', E \to P'/U'$, and the sphere bundle is $S = P' \times U' S_E$.

Let us use the multiplicative property (see Section 2.3) with the groups $G_2 = G \times U'$, $G_1 = T_\beta$ and the manifolds $M_1 = S_E, M_2 = P'$. Thanks to the product $K^0_{T_\beta \times G \times U'}(T^*_{G \times U'} S_E) \times K^*_G(T^*_{G \times U'}(P' \times S_E))$ we can define

$$
\theta'_\beta : K^*_G(T^*_{G \times U'}(P')) \to K^*_{T_\beta \times G \times U'}(T^*_{T_\beta \times G \times U'}(P' \times S_E))
\sigma \mapsto \sigma_{E, \beta}^{E, \beta} \circ \sigma.
$$

After taking the quotient by $U'$, we get a map

$$
\theta'_\beta : K_G(T_G M) \to K_{T_\beta \times G \times U'}(T^*_{T_\beta \times G \times U'}(S_E))
$$

Finally, since $T^*_{T_\beta \times G S} = T^*_{G S}$, we can compose $\theta'_\beta$ with the forgetful map $K^*_{T_\beta \times G \times U'} \to K_G(T_G S)$ to get $\theta_\beta : K^*_G(T_G M) \to K^*_G(T_G S)$.

The identity Thom $\cdots$ shows that we have a commutative diagram

$$
\begin{array}{ccc}
K^*_G(T^*_{G \times U'}(P')) & \xrightarrow{\theta'_\beta} & K^*_{T_\beta \times G \times U'}(T^*_{T_\beta \times G \times U'}(P' \times S_E)) \\
\downarrow & & \downarrow \\
\sigma'_{E, \beta} - \sigma'_{\beta} & & \eta \\
\end{array}
$$

After taking the quotient by $U'$, we get the commutative diagram

$$
\begin{array}{ccc}
K^*_G(T_G^* M) & \xrightarrow{\theta_\beta} & K^*_G(T_G S) \\
\downarrow & & \downarrow \\
\sigma'^{-1}_\beta - \sigma'^{-1}_\beta & & \mathbb{K}^*_E(T_G^* E)
\end{array}
$$

which is the content of Theorem 3.4.

3.6. Restriction to a fixed point sub-manifold. Let $M$ be a $G$-manifold and let $\beta \in g$ be a $G$-invariant element. Let $Z$ be a connected component of the fixed point set $M^\beta$. Note that $\beta$ defines a complex structure $J_\beta$ on the normal bundle of
Z in M. Following Section 2.6 we have a restriction morphism $R_Z$ that fits in the six term exact sequence

$$
\begin{align*}
\mathbf{K}_G^0(T_G^*(M \setminus Z)) & \xrightarrow{j_*} \mathbf{K}_G^0(T_G^*M) \xrightarrow{R_Z} \mathbf{K}_G^0(T_G^*Z) \\
\mathbf{K}_G^1(T_G^*Z) & \xleftarrow{R_Z} \mathbf{K}_G^1(T_G^*M) \xleftarrow{j_*} \mathbf{K}_G^1(T_G^*(M \setminus Z)).
\end{align*}
$$

Proposition 3.17.  
- There exists a morphism $S_{\beta,Z} : \mathbf{K}_G^*(T_G^*Z) \to \mathbf{K}_G^*(T_G^*M)$ such that $R_Z \circ S_{\beta,Z}$ is the identity on $\mathbf{K}_G^*(T_G^*Z)$.
- We have an isomorphism of $R(G)$-modules :

$$
\mathbf{K}_G^*(T_G^*M) \simeq \mathbf{K}_G^*(T_G^*Z) \oplus \mathbf{K}_G^*(T_G^*(M \setminus Z)).
$$

Proof. Let $\mathcal{N}$ be the normal bundle of $Z$ in $M$. Let $U$ be an invariant tubular neighborhood of $Z$, which is small enough so that we have an equivariant diffeomorphism $\phi : U \to \mathcal{N}$ which is the identity on $Z$. Let $S_{\beta,N} : \mathbf{K}_G^*(T_G^*Z) \to \mathbf{K}_G^*(T_G^*\mathcal{N})$ the map that we have constructed in Section 2.6. Let $j_* : \mathbf{K}_G^*(T_G^*U) \to \mathbf{K}_G^*(T_G^*M)$ be the push-forward map associated to the inclusion $j : U \hookrightarrow M$. Let $\phi^* : \mathbf{K}_G^*(T_G^*\mathcal{N}) \to \mathbf{K}_G^*(T_G^*U)$ be the isomorphism associated to $\phi$. We can consider the composition

$$S_{\beta,Z} := j_* \circ \phi^* \circ S_{\beta,N},$$

and we leave to the reader the verification that $R_Z \circ S_{\beta,Z} = \text{Id}$. The last point is a direct consequence of the first one. \qed

4. Decomposition of $\mathbf{K}_G^*(T_G^*M)$ when $G$ is abelian

In this section $G$ denotes a compact abelian Lie group, with Lie algebra $\mathfrak{g}$. Let $M$ be a (connected) manifold equipped with an action of $G$. For any $m \in M$, we denote $\mathfrak{g}_m \subset \mathfrak{g}$ its infinitesimal stabilizer.

Let $\Delta_G(M)$ be the set formed by the infinitesimal stabilizer of points in $M$. During this section, we suppose that $\Delta_G(M)$ is finite: it is the case if $M$ is compact or when $M$ is embedded equivariantly in a $G$-module. We have a partition

$$M = \bigsqcup_{b \in \Delta_G(M)} M_b$$

where $M_b := \{m \in M \mid \mathfrak{h} = \mathfrak{g}_m\}$ is an invariant open subset of the smooth submanifold $M_b := \{m \in M \mid \mathfrak{h} \subset \mathfrak{g}_m\}$.

On the other hand, we consider for $0 \leq k \leq s = \dim G$ the closed subset

$$M^{\leq k} \subset M$$

formed by the points $m \in M$ such that $\dim(G \cdot m) = \text{codim}(\mathfrak{g}_m) \leq k$. We have

$$M^{\leq k} = \bigsqcup_{\text{codim} h \leq k} M_b = \bigsqcup_{\text{codim} h \leq k} M^h$$

Let $M^{=k} = M^{\leq k} \setminus M^{\leq k-1}$ and $M^{>k} = M \setminus M^{\leq k-1}$. We note that

$$M^{=k} = \bigsqcup_{\text{codim} h = k} M_b.$$
Lemma 4.1. Let \( \gamma \) be the fixed point set \( \gamma \) of the open subset \( M \). We will use the increasing sequence of invariant open subsets

\[
M^{>s_0-1} \subset \cdots \subset M^{s_0} \subset M^{>0} \subset M.
\]

Here \( M^{>0} = M \setminus M^0 \), and \( M^{>s_0-1} = M^{gen} \) is the dense open subset formed by the \( G \)-orbits of maximal dimension. Note also that \( M^{gen} \) corresponds to \( M_{h_{min}} \) where \( h_{min} \) is the minimal stabilizer.

Let us consider the related sequences of open subspaces

\[
T^*_G M^{>s_0-1} \subset \cdots \subset T^*_G M^{s_0} \subset T^*_G M^{>0} \subset T^*_G M.
\]

At level of \( K \)-theory the inclusion \( j_k : M^{s_0} \hookrightarrow M^{s_0-1} \) gives rise to the map

\[
(j_k)_* : K^*_G(T_G M^{s_0}) \longrightarrow K^*_G(T_G M^{s_0-1}).
\]

Let \( 0 \leq k \leq s_0 - 1 \). We have the decomposition

\[
T^*_G M^{s_0-k-1} = T^*_G M^{s_0-k} \bigcup \bigcup_{\text{codim} h = k} T^*_G M^{s_0-k-1}(h)
\]

\[
= T^*_G M^{s_0-k} \bigcup \bigcup_{\text{codim} h = k} T^*_G M^{s_0-k-1}(h) \times N_h^*
\]

where \( N_h \) is the normal bundle of \( M_h \) in \( M \). Note that \( M_h \) is a closed sub-manifold of the open subset \( M^{s_0-k-1} \), when \( \text{codim} h = k \).

Lemma 4.1. Let \( \gamma \in \Delta_G(M) \) with \( \text{codim} \gamma = k \). There exists \( \gamma_h \in \gamma \) so that \( M_h \) is equal to the fixed point set \( (M^{>k-1})^\gamma_h := \{ m \in M^{>k-1} \mid \gamma_h \in \gamma_m \} \). The element \( \gamma_h \) defines then a complex structure \( J_{\gamma_h} \) on the normal bundle \( N_h \).

Proof. Let \( H \) be the closed connected subgroup of \( G \) with Lie algebra \( \gamma \). Let \( \gamma_h \in \gamma \) generic so that the closure of \( \{ \exp(t\gamma_h), t \in \mathbb{R} \} \) is equal to \( H \). Then for any \( m \in M \),

\[
\{ m \in M^{>k-1} \mid \gamma_h \in \gamma_m \} = \{ m \in M^{>k-1} \mid h \subset \gamma_m \} = \{ m \in M \mid h = \gamma_m = M_h \}.
\]

Thanks to Lemma 4.1, we can exploit Section 3.6. For any \( \gamma \in \Delta_G(M) \) of codimension \( k \), we have a restriction morphism

\[
R_h : K^*_G(T^*_G M^{>k-1}) \longrightarrow K^*_G(T^*_G M_h)
\]

and a section

\[
S_h := S_{\gamma_h, M_h} : K^*_G(T^*_G M_h) \longrightarrow K^*_G(T^*_G M^{>k-1})
\]

such that \( R_h \circ S_h \) is the identity on \( K^*_G(T^*_G M_h) \).

We have also a long exact sequence

\[
K^0_\gamma(T^*_G M^{>k}) \xrightarrow{(j_k)_*} K^0_\gamma(T^*_G M^{>k-1}) \xrightarrow{R_k} \bigoplus_{\text{codim} h = k} K^0_\gamma(T^*_G M_h)
\]

\[
\bigoplus_{\text{codim} h = k} K^1_\gamma(T^*_G M_h) \xrightarrow{R_k} K^1_\gamma(T^*_G M^{>k-1}) \xrightarrow{(j_k)_*} K^1_\gamma(T^*_G M^{>k}).
\]
Theorem 4.4
in a less precise version in [1][Theorem 8.4].

is injective, since

Proof. The last point is a direct consequence of the firsts one. The first point is known, and the second assertion is due to the fact that \( M_a \cap M_b = \emptyset \) when \( a \neq b \).

The previous lemma shows that the map

\[
(30) \quad (j_k)_*: S_k : K^*_G(T^*_GM^{>k}) \times \oplus_{\text{codim} h = k} K^*_G(T^*_GM_h) \to K^*_G(T^*_GM^{>k-1})
\]
is an isomorphism of \( R(G) \)-module. In particular the maps \((j_k)_*\) are injective.

Remark 4.3. If we consider the open subset \( j: M^{\text{gen}} \to M \) formed by the \( G \)-orbits of maximal dimension, we know then that

\[ j_* : K^*_G(T^*_GM^{\text{gen}}) \to K^*_G(T^*_GM) \]
is injective, since \( j \) is the composition of all the \( j_k \).

The isomorphisms \([39]\) all together give the following Theorem (which was given in a less precise version in \([1]\) [Theorem 8.4]).

Theorem 4.4 (Atiyah-Singer). Let \( \gamma := \{ \gamma_h, h \in \Delta_G(M) \} \) such that \( M_h = \{ m \in M^{>\text{codim} h - 1} | \gamma_h \in \mathfrak{g}_m \} \). We have an isomorphism

\[
(31) \quad \Phi_{\gamma} : \bigoplus_{h \in \Delta_G(M)} K^*_G(T^*_GM_h) \to K^*_G(T^*_GM)
\]
of \( R(G) \)-module such that

\[
\text{Index}^G_M(\Phi_{\gamma} \oplus_{h} \sigma_h)) = \sum_{h \in \Delta} \text{Index}^G_M(\sigma_h \otimes S^*(N_h))
\]
for any \( \oplus_{h} \sigma_h \in \bigoplus_{h \in \Delta} K^0_G(T^*_GM_h) \). Here \( N_h \) is the normal bundle of \( M_h \) in \( M \) which is equipped with the complex structure defined by \(-\gamma_h\).

For any \( h \in \Delta_G(M) \) we denote \( H \subset G \) the closed connected subgroup with Lie algebra \( h \). Let us denote \( H' \subset G \) be a Lie subgroup such that \( G \simeq H \times H' \). Then the \( R(G) \)-module \( K^*_G(T^*_GM_h) \) is equal to

\[
K^*_H(T^*_H,M_h) \otimes R(H).
\]
Thus \( [4.4] \) says that \( K^*_G(T^*_GM) \) is isomorphic to

\[
\bigoplus_{h \in \Delta_G(M)} K^*_H(T^*_H,M_h) \otimes R(H).
\]

Note that the action of \( H' \) on \( M_h \) has finite stabilizers, hence the group \( K^*_H(T^*_H,M_h) \) is equal to \( K^*_\text{orb}(T^*M_h) \), where \( M_h = M_h/H' \) is an orbifold.
5. THE LINEAR CASE

In this section, the group $G$ is a compact abelian Lie group. Let $V$ be a real $G$-module. Let $V^{gen}$ be the open subset formed by the $G$-orbits of maximal dimension. We equip $V/V^\theta$ with an invariant complex structure. For any $\gamma \in g$, such that $V^\gamma = V^\theta$ we associate the class

$$\text{Thom}_G(V/V^\theta) \otimes \text{Bott}(V^\theta) \in K_G^0(T^*_G V).$$

Let $H_{min} \subset G$ be the minimal stabilizer for the $G$-action on $V$. Let $s := \dim G - \dim H_{min}$.

**Definition 5.1.** A $(G,V)$-flag $\varphi$ corresponds to a decomposition $V/V^\theta = V^\varphi_1 \oplus \cdots \oplus V^\varphi_s$ in complex $G$-subspaces, and a decomposition $g = h_{min} \oplus \mathbb{R} \beta^1_\varphi + \cdots + \mathbb{R} \beta^s_\varphi$ such that for any $1 \leq k \leq s$

1. $\beta^1_\varphi$ acts trivially on $V^\varphi_j$ when $j < k$,
2. $\beta^1_\varphi$ acts bijectively on $V^\varphi_k$.

We can associate to the data $\varphi$ above, the flags $V^\varphi = V^{[0],\varphi} \subset V^{[1],\varphi} \subset \cdots \subset V^{[s],\varphi} = V$ and $h_{min} = g^{[0],\varphi} \subset g^{[1],\varphi} \subset \cdots \subset g^{[s],\varphi} = g$ where

$$V^{[j],\varphi} = V^\theta \oplus \sum_{1 \leq k \leq j} V^\varphi_k,$$

and $g^{[j],\varphi} = h_{min} \oplus \mathbb{R} \beta^1_{\varphi_j} \oplus \cdots \oplus \mathbb{R} \beta^s_{\varphi_j}$.

We see that conditions c1 and c2 are equivalent to saying that the generic infinitesimal stabilizer of the $G$-action on the vector space $V^{[j],\varphi}$ is equal to $g^{[j],\varphi}$.

Thanks to c2, the Cauchy-Riemann symbol

$$\sigma^\varphi_{\beta^1_\varphi} \in K_G^0(T^*_G (V^\varphi_1 \setminus \{0\})),
$$

is well defined. Conditions c1 and c2 tell us also that $(V^{[i],\varphi}_i \setminus \{0\}) \times \cdots \times (V^{[s],\varphi}_s \setminus \{0\})$ is an open subset of $(V/V^\theta)^{gen}$, and thanks to Theorem 2.7 we know that the following product

$$\sigma^\varphi_{\beta^1_\varphi} \cdots \sigma^\varphi_{\beta^s_\varphi}$$

is a well defined class in $K_G^0(T^*_G (V/V^\theta)^{gen})$.

We need the following submodule of $R^{-\infty}(G)$ defined by the relations

$$\Phi \in \mathcal{F}_G(V) \iff \wedge^* V/V^\theta \otimes \Phi \in (R^{-\infty}(G/H), \forall h \in \Delta_G(V),$$

$$\Phi \in \mathcal{DM}_G(V) \iff \wedge^* V/V^\theta \otimes \Phi = 0, \forall h \neq h_{min}$$

and $\Phi \in (R^{-\infty}(G/H_{min}))$.

The purpose of this section is to give a detailed proof of the following theorem.

**Theorem 5.2.** Let $G$ a compact abelian Lie group and let $V$ be a real $G$-module. We have

1. $K_G(T^*_G V) = K_G(T^*_G V^{gen}) = 0$
2. The index map $\text{Index}_G^V : K_G^0(T^*_G V) \to R^{-\infty}(G)$ is one to one.
3. The elements $\text{Bott}(V^\theta_\gamma) \otimes \text{Thom}_G(V/V^\theta)$ generate $K_G^0(T^*_G V)$, when $\gamma$ runs over the elements such that $V^\gamma = V^\theta$.
4. The elements $\text{Bott}(V^\theta_\gamma) \otimes \sigma^\varphi_{\beta^1_\varphi}$ generate $K_G^0(T^*_G V^{gen})$, when $\varphi$ runs over the $(G,V)$-flag.
5. The image $K_G^0(T^*_G V)$ by $\text{Index}_G^V$ is equal to $\mathcal{F}_G(V)$.
6. The image $K_G^0(T^*_G V^{gen})$ by $\text{Index}_G^V$ is equal to $\mathcal{DM}_G(V)$.  

$^9\beta$ acts bijectively on a vector space $V$ if $V^\beta = \{0\}$.
Hence b., e. and f. say that the \( R(G) \)-modules \( K^*_G(T^*_G V) \) and \( K^*_G(T^*_G V^\text{gen}) \) are respectively isomorphic to \( F_G(V) \) and \( DM_G(V) \).

Note that, when \( \dim V/V^g = 0 \), we have \( T^*_G V = T^*_G V^\text{gen} = T^* V \) and all the points are direct consequences of the Bott isomorphism. Point d. is proved in [1], and points a., e. and f. are due to de Concini-Procesi-Vergne [9] [10]. Point b. is proved in [1] for the circle group, and in [9] [10] for the general case. In [9] [10], c. is obtained as a consequence of d. together with the decomposition formula [31].

We will give a proof by induction on \( \dim V/V^g \) that is based on the work of [9] [10]. But here our treatment differs from those of [1] [9] [10], since the proof of all points of Theorem 5.2 follows directly by a careful analysis of the exact sequence

\[
0 \longrightarrow K^*_G(T^*_G W) \xrightarrow{J} K^0_G(T^*_G V) \xrightarrow{R} K^0_G(T^*_G W) \longrightarrow 0.
\]

associated to an invariant decomposition \( V = \mathbb{C}_x \oplus W \).

### 5.1. Restriction to a subspace

Suppose that \( V \neq V^g \). Then \( V \) contains a complex representation \( \mathbb{C}_x \) attached to a surjective character \( \chi : G \rightarrow S^1 \). Let \( G_{\chi} = \ker(\chi) \) with Lie algebra \( \mathfrak{g}_x \). The differential of \( \chi \) is \( i\bar{\chi} \) with \( \bar{\chi} \in \mathfrak{g}^* \). Here \( \mathbb{C}_x \cap V^g = \{0\} \) since \( \bar{\chi} \neq 0 \).

Let us consider an invariant decomposition \( V = W \oplus \mathbb{C}_x \).

**Remark 5.3.** We check that \( \dim W/W^g = \dim V/V^g - 1 \), and \( \dim W/W^g \leq \dim V/V^g - 1 \).

We look at the open subset \( j : T^*_G(W \times \mathbb{C}_x \setminus \{0\}) \rightarrow T^*_G V \). Its complement is the closed subset \( T^*_G V|_{W\times\{0\}} \cong T^*_G W \times \mathbb{C}_x \). We have the six term exact sequence (32)

\[
\begin{array}{c}
K^1_G(T^*_G(W \times \mathbb{C}_x \setminus \{0\})) \xrightarrow{J} K^0_G(T^*_G V) \xrightarrow{R} K^0_G(T^*_G W \times \mathbb{C}_x) \\
\delta \downarrow & & \delta \downarrow \\
K^1_G(T^*_G W \times \mathbb{C}_x) \xleftarrow{r} K^0_G(T^*_G V) \xleftarrow{J} K^0_G(T^*_G (W \times \mathbb{C}_x \setminus \{0\})).
\end{array}
\]

Let \( R : K^*_G(T^*_G V) \rightarrow K^*_G(T^*_G W) \) be the composition of the map \( r \) with the Bott isomorphism \( K^*_G(T^*_G W \times \mathbb{C}_x) \rightarrow K^*_G(T^*_G W) \). Note that \( R \) depends of the choice of the canonical complex structure on \( \mathbb{C}_x \).

The open subset \( \mathbb{C}_x \setminus \{0\} \) with the \( G \)-action is isomorphic to \( G/G_{\chi} \times \mathbb{R} \). Hence \( T^*_G(W \times \mathbb{C}_x \setminus \{0\}) \cong T^*_G(W \times G/G_{\chi}) \times \mathbb{R} \). Since the \( G \)-manifold \( W \times G/G_{\chi} \) is isomorphic to \( G \times_{G_{\chi}} W \), we get finally

\[
K^*_G(T^*_G(W \times \mathbb{C}_x \setminus \{0\})) = K^*_G(T^*_G(W \times G/G_{\chi}) \times \mathbb{R}) \\
\cong K^*_G(T^*_G(W \times G/G_{\chi})) \\
\cong K^*_G(T^*_G(G \times_{G_{\chi}} W)) \\
\cong K^*_G(T^*_G(W \times \mathbb{C}_x \setminus \{0\})).
\]

Let \( J : K^*_G(T^*_G W) \rightarrow K^*_G(T^*_G V) \) be the composition of the map \( J \) with the previous isomorphism \( K^*_G(T^*_G W \times \mathbb{C}_x) \cong K^*_G(T^*_G(W \times \mathbb{C}_x \setminus \{0\})) \). The sequence (32)
becomes
\[
\begin{array}{ccc}
\mathbf{K}^0_{G,\chi}(T_{G,\chi}W) & \xrightarrow{J} & \mathbf{K}^0_G(T_G^*V) \\
\downarrow{\delta} & & \downarrow{\delta} \\
\mathbf{K}^1_{G,\chi}(T_{G,\chi}W) & \xleftarrow{R} & \mathbf{K}^1_G(T_G^*V) \\
\end{array}
\]

The following description of the morphism \(J\) will be used in the next sections.

Let \(\beta \in \mathfrak{g}\) such that \(\mathfrak{g} = \mathfrak{g}_\chi \oplus \mathbb{R} \beta\). Since the action of \(G\) is trivial on \(\mathbb{C}_\chi\), the product

\[
\mathbf{K}^0_G(T^*_G W) \times \mathbf{K}^0_{G,\beta}(T^*_{G,\beta} \mathbb{C}_\chi) \xrightarrow{\otimes} \mathbf{K}^0_G(T^*_G V)
\]

is well defined. Let \(\sigma^G_{\mathbb{C}_\chi} \in \mathbf{K}^0_G(T^*_G \mathbb{C}_\chi)\) be the Cauchy-Riemann class.

**Lemma 5.4.** Let \([\sigma] \in \mathbf{K}^0_{G,\chi}(T^*_{G,\chi} W)\) be a class that is represented by a \(G\)-equivariant, \(G\)-\(\chi\)-transversally elliptic morphism \(\sigma\). Then the product \(\sigma \otimes \sigma^G_{\mathbb{C}_\chi}\) is \(G\)-transversally elliptic and \(J([\sigma]) = \left[\sigma \otimes \sigma^G_{\mathbb{C}_\chi}\right]\) in \(\mathbf{K}^0_G(T^*_G V)\).

**Proof.** The character \(\chi\) defines the inclusion \(i : G/G_\chi \to \mathbb{C}_\chi, g \mapsto \chi(g)\). Let

\[
i : \mathbf{K}^0_G(T^*_G(G/G_\chi \times W)) \to \mathbf{K}^0_G(T^*_G V)
\]

be the push-forward morphism.

The manifold \(G \times W\) is equipped with two \(G \times G_\chi\)-actions: \((g,h) \cdot \chi((x,w)) = (gxh^{-1}, h \cdot w)\) and \((g,h) \cdot \pi((x,w)) = (gxh^{-1}, g \cdot w)\). The map \(\theta(x,w) = (x, x^{-1} \cdot w)\) is an isomorphism between \(G \times \mathbb{R} W\) and \(G \times_1 W\). The quotients by \(G\) and \(G_\chi\) give us the maps \(\pi_G : G \times_1 W \to W\), and \(\pi_{G,\chi} : G \times_2 W \to G/G_\chi \times W\).

We have

\[
J = i \circ (\pi^*_G)^{-1} \circ \theta^* \circ \pi^*_G
\]

where \((\pi^*_G)^{-1} \circ \theta^* \circ \pi^*_G : \mathbf{K}^0_{G,\chi}(T^*_{G,\chi} W) \to \mathbf{K}^0_G(T^*_G(G/G_\chi \times W))\) is an isomorphism.

It is an easy matter to check that if the class \([\sigma] \in \mathbf{K}^0_{G,\chi}(T^*_{G,\chi} W)\) is represented by a \(G\)-equivariant, \(G\)-\(\chi\)-transversally elliptic morphism \(\sigma\), then \((\pi^*_G)^{-1} \circ \theta^* \circ \pi^*_G([\sigma]) = [\sigma \otimes [0]]\) where \([0] : \mathbb{C} \to \{0\}\) is the zero symbol on \(G/G_\chi\). Finally \(J([\sigma]) = i([\sigma \otimes [0]]) = [\sigma \otimes \sigma^G_{\mathbb{C}_\chi}]\).

\(\square\)

**Remark 5.5.** In the next sections, we will use the exact sequence \((33)\), when \(V\) is replaced by an invariant open subset \(U_V\). Suppose that there exist invariant open subsets \(U_W^1, U_W^2 \subset W\) such that \(U_V = U_W^1 \cup U_W^2 \times \mathbb{C}_\chi \setminus \{0\}\). Then \((33)\) becomes

\[
\begin{array}{ccc}
\mathbf{K}^0_{G,\chi}(T^*_G \mathcal{U}_W^1) & \xrightarrow{J} & \mathbf{K}^0_G(T^*_G \mathcal{U}_V) \\
\downarrow{\delta} & & \downarrow{\delta} \\
\mathbf{K}^1_{G,\chi}(T^*_G \mathcal{U}_W^1) & \xleftarrow{R} & \mathbf{K}^1_G(T^*_G \mathcal{U}_V) \\
\end{array}
\]

For example, if \(U_V = V^{gen}\), we take \(U_W^1 = W \cap V^{gen}\) and \(U_W^2 = W^{gen,G_\chi}\).
5.2. The index map is injective. Let us prove by induction on \( n \geq 0 \) the following fact

\[(H_n) \quad \text{Index}^G_v : \mathbb{K}^0_G(T_G^*V) \rightarrow R^{-\infty}(G) \quad \text{is one to one if} \quad \dim V/V^g \leq n.\]

If \( \dim V/V^g = 0 \), we have \( T_G^*V = T^*V \) and the index map \( \mathbb{K}^0_G(T^*V) \rightarrow R(G) \) is the inverse of the Bott isomorphism.

Suppose now that \((H_n)\) is true, and consider \( G \circ V \) such that \( \dim V/V^g = n+1 \). We start with a decomposition \( V = W \oplus C \chi \) and the exact sequence \( (33) \). The induction map \( \text{Ind}_G^G : R^{-\infty}(G) \rightarrow R^{-\infty}(G) \) is defined by the relation \( \text{Ind}_G^G(E) = [L^2(G) \otimes E]^G_x \). We denote \( \Lambda^* \mathbb{C}_\chi : R^{-\infty}(G) \rightarrow R^{-\infty}(G) \) the product by \( 1 - \mathbb{C}_\chi \).

**Proposition 5.6.** The following diagram is commutative

\[
\begin{array}{ccc}
\mathbb{K}^0_{G_x}(T_{G_x}^*W) & \xrightarrow{J} & \mathbb{K}^0_G(T_G^*V) \\
\downarrow{\text{Index}}^G_{W_x} & & \downarrow{\text{Index}}^G_v \\
R^{-\infty}(G_x) & \xrightarrow{\text{Ind}_G^G} & R^{-\infty}(G) \\
\end{array}
\]

**Proof.** Let \( \sigma \in \mathbb{K}^0_{G_x}(T_{G_x}^*W) \). We have \( \pi_G^x(\sigma) = \sigma \circ [0] \) where \( [0] : C \rightarrow \{0\} \) is the zero symbol on \( G \). Then the product formula says that \( \text{Index}^G_{G_x \times W^*}(\pi_G^x(\sigma)) = \text{Index}^G_{W^*}(\sigma) \otimes L^2(G) \) and thanks to \( (34) \), we see that

\[
\text{Index}^G_v(J(\sigma)) = \left[ \text{Index}^G_{G_x \times W^*}(\pi_G^x(\sigma)) \right] + \theta^* \circ \pi_G^x(\sigma)
\]

This proved the commutativity of the left part of the diagram, and the commutativity of the right part of the diagram is a particular case of Proposition \( 2.11 \). \( \square \)

We need now the following result that will be proved in Appendix A

**Lemma 5.7.** The sequence

\[
0 \rightarrow R^{-\infty}(G_x) \xrightarrow{\text{Ind}_G^G} R^{-\infty}(G) \xrightarrow{\Lambda^* \mathbb{C}_\chi} R^{-\infty}(G)
\]

is exact.

Lemma 5.7 tells us in particular that \( \text{Ind}_G^G \) is one to one. We can now finish the proof of the induction. In the commutative diagram \( (36) \), the maps \( \text{Index}^G_{W^*}, \text{Index}^G_{W^*} \) and \( \text{Ind}_G^G \) are one to one. It is an easy matter to deduces that \( \text{Index}^G_v \) is one to one.

We end up this section with the following statement which is the direct consequence of the injectivity of \( \text{Index}^G_v \) (see Remark \( 5.8 \)).

**Remark 5.8.** Let \( J_k, k = 0, 1 \) be two invariants complex structures on \( V \), and let \( \text{Thom}_\beta(V, J_k) \) be the corresponding pushed symbols attached to an element \( \beta \) satisfying \( V^\beta = \{0\} \). There exists an invertible element \( \Phi \in R(G) \) such that

\[
\text{Thom}_\beta(V, J_0) = \Phi \cdot \text{Thom}_\beta(V, J_1)
\]

in \( \mathbb{K}^0_G(T_G^*V) \).
5.3. Generators of $K_G^*(T_G^*V)$. Let $V$ be a real $G$-module: we equip $V/V^\theta$ with an invariant complex structure. Let $A_G(V) \subset K_G^0(T_G^*V)$ be the submodule generated by the family $\text{Bott}(V_C^\theta) \odot \text{Thom}_\gamma(V/V^\theta)$, where $\gamma$ runs over the element of $g$ satisfying $V^\gamma = V^\theta$. Remark 5.8 tells us that $A_G(V)$ is independent of the choice of the complex structure on $V/V^\theta$.

In this section we will prove by induction on $n \geq 0$ the following fact

$$(H_n) \quad K_G^1(T_G^*V) = 0 \quad \text{and} \quad K_G^0(T_G^*V) = A_G(V) \quad \text{if} \quad \dim V/V^\theta \leq n.$$

If $\dim V/V^\theta = 0$, we have $T_G^*V = T^*V$ and assertion $(H_0)$ is a direct consequence of the Bott isomorphism.

Suppose now that $(H_n)$ and is true, and consider $G \circ V$ such that $\dim V/V^\theta = n + 1$. We have a decomposition $V = W \oplus \tilde{C}_\chi$ with $\tilde{\chi} \neq 0$. If we apply $(H_n)$ to $G \circ W$ and $G_\chi \circ W$, we get first that $K_G^1(T_G^*W) = 0$ and $K_G^1(T_G^*\chi) = 0$.

The long exact sequence $(33)$ implies then that $K_G^0(T_G^*V) = 0$, and induces the short exact sequence

$$(38) \quad 0 \longrightarrow K_G^0(T_G^*\chi) \longrightarrow J K_G^0(T_G^*V) \longrightarrow R K_G^0(T_G^*W) \longrightarrow 0.$$
5.4. Generators of $\mathbf{K}^*_G(\mathbf{T}^*_G V^{\text{gen}})$. Let $B_G(V)$ be the submodule of $\mathbf{K}^*_G(\mathbf{T}^*_G V^{\text{gen}})$ generated by the family Bott($V^g$) $\cap \sigma^V_{\mathbf{F}}$ where $\varphi$ runs over the $(G, V)$-flag.

In this section we will prove by induction on $n \geq 0$ the following fact

\[(H'_n) \quad K^1_G(\mathbf{T}^*_G V^{\text{gen}}) = 0 \quad \text{and} \quad K^0_G(\mathbf{T}^*_G V^{\text{gen}}) = B_G(V) \quad \text{if} \quad \dim V/V^g \leq n.\]

If $\dim V/V^g = 0$, we have $\mathbf{T}^*_G V^{\text{gen}} = \mathbf{T}^* V$ and $(H'_0)$ is a direct consequence of the Bott isomorphism. Suppose now that $(H'_n)$ is true, and consider $G \circ V$ such that $\dim V/V^g = n + 1$. We have an invariant decomposition $V = W \oplus C_\chi$, with $\chi \neq 0$, and

$$V^{\text{gen}} = V^{\text{gen}} \cap W \bigcup W^{\text{gen}, G_x} \times C_\chi \setminus \{0\}.$$  

Note that $V^{\text{gen}} \cap W$ is either equal to $W^{\text{gen}}$ (if the $G$-orbits in $V$ and $W$ have the same maximal dimension) or is empty. Following Remark 5.5, we have the exact sequence

\[(39) \quad K^0_{G_x}(\mathbf{T}^*_G W^{\text{gen}, G_x}) \xrightarrow{J} K^0_G(\mathbf{T}^*_G V^{\text{gen}}) \xrightarrow{R} K^0_G(\mathbf{T}^*_G W^{\text{gen}}) \xrightarrow{\delta} K^1_G(\mathbf{T}^*_G V^{\text{gen}}) \xrightarrow{J} K^1_{G_x}(\mathbf{T}^*_G W^{\text{gen}, G_x})\]

when $V^{\text{gen}} \cap W \neq \emptyset$. On the other hand, when $V^{\text{gen}} \cap W = \emptyset$, we have an isomorphism

\[(40) \quad J : K^0_{G_x}(\mathbf{T}^*_G W^{\text{gen}, G_x}) \rightarrow K^0_G(\mathbf{T}^*_G V^{\text{gen}}).\]

If we apply $(H'_n)$ to $G \circ W$ and $G_x \circ W$, we get first $K^1_G(\mathbf{T}^*_G W^{\text{gen}}) = 0$ and $K^1_{G_x}(\mathbf{T}^*_G W^{\text{gen}, G_x}) = 0$. Using the bottom of the diagram (39) and the isomorphism (40), we get $K^1_G(\mathbf{T}^*_G V) = 0$. Moreover, the long exact sequence (29) induces the short exact sequence

$$0 \rightarrow K^0_{G_x}(\mathbf{T}^*_G W^{\text{gen}, G_x}) \xrightarrow{J} K^0_G(\mathbf{T}^*_G V^{\text{gen}}) \xrightarrow{R} K^0_G(\mathbf{T}^*_G (V^{\text{gen}} \cap W)) \rightarrow 0.$$  

Since the assertion $(H'_n)$ gives also

$K^0_G(\mathbf{T}^*_G W^{\text{gen}}) = B_G(W) \quad \text{and} \quad K^0_{G_x}(\mathbf{T}^*_G W^{\text{gen}, G_x}) = B_{G_x}(W),$  

the equality $K^0_G(\mathbf{T}^*_G V^{\text{gen}}) = A_G(V)$ will follow from following Lemma.

**Lemma 5.10.** We have

- $J(B_{G_x}(W)) \subset B_G(V),$
- $B_G(W) \subset R(B_G(V)), \text{ when } V^{\text{gen}} \cap W \neq \emptyset.$

**Proof.** Let $\beta \in \mathfrak{g}$ such that $\langle \chi, \beta \rangle > 0$: we have $\mathfrak{g} = \mathfrak{g}_\chi \oplus \mathbb{R} \beta$. For any $(W, G_x)$-flag $\varphi$, we consider the element

$$\alpha := \text{Bott}(W^{\mathfrak{g}_x}) \odot \sigma^W_{\mathbf{F}} \in K^0_{G_x}(\mathbf{T}^*_G W^{\text{gen}})$$

and we want to compute its image by $J$.

We note that the minimal stabilizer $H_{\min} \subset G$ for the $G$-action on $V$ is equal to the minimal stabilizer for the $G_x$-action on $W$. Let $s := \dim G_x - \dim H_{\min}$. A $(G_x, W)$-flag $\varphi$ corresponds to

- a decomposition $W/W^{\mathfrak{g}_x} = W^1 \oplus \cdots \oplus W^s$ in complex $G$-subspaces
Let $V$ be the corresponding decompositions. The hypothesis $h$ is the minimal stabilizer

\[
\text{Dahmen-Michelli submodule defined in the introduction. We start with the follow-}
\]

\[
g\text{to the restriction of the product of } \sigma.
\]

\[
\text{Suppose now that } V \text{ is contained in } \chi \leq G. \text{ Finally, thanks to Lemma 5.4 we have}
\]

\[
\text{we have proved that } J(\alpha) = \text{onto, we have proved that } J(\alpha) = \text{onto.}
\]

\[
\text{Here we use the identity } \sigma^\chi \circ \text{Bott}(W^g/W^g) = \Lambda^* W^g/W^g \otimes \sigma^w_1, \text{ valid in } K_0^0(T_{R,\beta}^* W_k^g), \text{ which is proved in Proposition 3.16. Since } R(G) \to R(G^\chi) \text{ is onto, we have proved that } J(A_G^\chi(W)) \subset A_G(W).
\]

\[
\text{Suppose now that } V^\text{gen} \cap W \neq \emptyset, \text{ and let us prove now that } A_G(W) \subset R(A_G(W)). \text{ Let } \varphi \text{ be a } (G, W)\text{-flag: let } W/W^g = W^\varphi_1 \oplus \cdots \oplus W^\varphi_s \text{ and } g = h_{\text{min}} \oplus R\beta^\varphi_1 \oplus \cdots \oplus R\beta^\varphi_s \text{ be the corresponding decompositions. The hypothesis } V^\text{gen} \cap W \neq \emptyset \text{ means that the minimal stabilizer } h_{\text{min}} \text{ for the } g\text{-action in } W \text{ is contained in } g^\chi. \text{ Hence } \chi \text{ does not belongs to } (R\beta^\varphi_1 \oplus \cdots \oplus R\beta^\varphi_s)^{-1}. \text{ Let}
\]

\[
k = \max\{i \mid \langle \chi, \beta^\varphi_i \rangle \neq 0\}.
\]

\[
\text{Let } \psi \text{ be the } (G, V)\text{-flag defined as follows:}
\]

\[
\bullet \text{ } V_i^\psi = W^\varphi_i \text{ is } i \neq k, \text{ and } V_k^\psi = W^\varphi_k \oplus \mathbb{C}_x,
\]

\[
\bullet \beta^\psi_k = \beta^\varphi_k \text{ for } 1 \leq k \leq s.
\]

\[
\text{Then we have}
\]

\[
R(\text{Bott}(V^g) \circ \sigma^{V/W^g,\psi}) = \text{Bott}(W^g) \circ \sigma^{\varphi_1}_\sigma \cdots \text{R}(\sigma^{\varphi_k}_\sigma) \cdots \sigma^{\varphi_s}_\sigma = \text{Bott}(W^g) \circ \sigma^{\varphi_1}_\sigma \cdots \sigma^{\varphi_k}_\sigma \cdots \sigma^{\varphi_s}_\sigma = \text{Bott}(W^g) \circ \sigma^{W/W^g,\varphi}_\sigma
\]

\[
\text{We use here the relation } R(\sigma^{\varphi}_\sigma) = \sigma^{\varphi}_\sigma \text{ (see Proposition 3.11). It proves that}
\]

\[
A_G(W) \subset R(A_G(W)). \quad \square
\]

5.5. $K_0^0(T^* G V) = \text{isomorphic to } F_G(V)$. For any $G$-module $V$, we denote $F_G(V)$ the image of $K_0^0(T^* G V)$ by Index$G$. We know from Section 5.2 that the index map Index$G^$ is injective, hence $F_G(V) = K_0^0(T^* G V)$. Let $F_G(V)$ be the generalized Dahmen-Michelli submodule defined in the introduction. We start with the following

\[
\text{Lemma 5.11. We have } F_G(V) \subset F_G(V).
\]
Proof. Let \( \sigma \in K^G_b(T^*_G V) \) and let \( \mathfrak{h} \in \Delta_G(V) \). Since the vector space \( V/V^h \) carries an invariant complex structure we have a restriction morphism \( R^h : K^G_b(T^*_G V) \rightarrow K^G_b(T^*_G V^h) \). Let \( i_t : K^G_b(T^*_G V^h) \rightarrow K^G_b(T^*_G V) \) be the push-forward morphism associated to the inclusion \( V^h \hookrightarrow V \). Thanks to Proposition \( \ref{prop:push-forward} \) we know that \( i_t \circ R^h(\sigma) = \sigma \otimes \wedge^* V/V^h \), and then

\[
\wedge^* \overline{\delta V/V^h} \otimes \text{Index}^G_{V^h}(\sigma) = \text{Index}^G_{V^h}(R^h(\sigma)).
\]

But since the action of \( H \) is trivial on \( V^h \), we know that \( \text{Index}^G_{V^h}(R^h(\sigma)) \in \langle R^{-\infty}(G/H) \rangle \) (see Remark \( \ref{rem:index} \)). The inclusion \( F_G(V) \subset F_G(V) \) is proved. \( \square \)

We will now prove by induction on \( n \geq 0 \) the following fact

\[
(H_n) \quad F_G(V) = F_G(V) \quad \text{if} \quad \dim V/V^g \leq n.
\]

If \( \dim V/V^g = 0 \), we have \( T^*_G V = T^* V \) and \( \Delta_G(V) = \{ g \} \). In this situation, \( \mathfrak{h}_{\text{min}} = g \) and \( \langle R^{-\infty}(G/H) \rangle = R(G) \). We have then \( F_G(V) = R(G) \), and \( (H_0) \) is a direct consequence of the Bott isomorphism.

Suppose now that \( (H_n) \) and is true, and consider \( G \subset V \) such that \( \dim V/V^g = n + 1 \). We have a decomposition \( V = W \oplus \mathbb{C} \chi \) with \( \chi \neq 0 \). If we apply \( (H_n) \) to \( G \subset W \) and \( G \subset W \), we get \( F_G(W)^T = F_G(W) \) and \( F_{G^\chi}(W)^T = F_{G^\chi}(W) \). The following Lemma will be the key point of our induction.

**Lemma 5.12.** Let \( H \subset G \) be a closed subgroup \( (G \) is abelian). For any \( \Phi \in R^{-\infty}(G \chi) \), we have the equivalences

\[
(\Phi) \quad \Phi \in \langle R^{-\infty}(G \chi/H) \rangle \iff \text{Ind}^G_{G \chi}(\Phi) \in \langle R^{-\infty}(G/H) \rangle,
\]

\[
(\Phi) \quad \Phi \in F_{G \chi}(W) \iff \text{Ind}^G_{G \chi}(\Phi) \in F_G(V).
\]

The exact sequence \( \ref{eq:exact_sequence} \) specializes in the exact sequence

\[
0 \rightarrow F_{G \chi}(W) \xrightarrow{\text{Ind}^G_{G \chi}} F_G(V) \xrightarrow{\wedge^* \overline{\delta V/V^h}} F_G(W).
\]

**Proof.** Let us consider the first point. For \( \Phi := \sum_{\mu \in \widehat{\mathbb{C} \chi}} m(\mu) \mathbb{C} \), \( \mu \in R^{-\infty}(G \chi) \), we have \( \text{Ind}^G_{G \chi}(\Phi) = \sum_{\varphi \in \hat{G}} \pi_{\varphi}(\mu) \mathbb{C} \), \( \varphi \in \hat{G} \). We see then that \( \text{Supp}(\text{Ind}^G_{G \chi}(\Phi)) = \pi_{\text{Ind}^{-1}}(\text{Supp}(\Phi)) \). If \( \pi_H : \hat{G} \rightarrow \hat{H} \) and \( \pi_{H'} : \hat{G} \rightarrow \hat{H} \) denote the projections, we have then the following relation

\[
\pi_H \left( \text{Supp} \left( \text{Ind}^G_{G \chi}(\Phi) \right) \right) = \pi_{H'} \left( \text{Supp}(\Phi) \right)
\]

that induces \( \ref{eq:exact_sequence} \).

For any \( \Phi \in R^{-\infty}(G \chi) \) and any subspace \( \mathfrak{h} \in \Delta_G(V) \), we consider the expression

\[
\Omega := \wedge^* \overline{V/V^h} \otimes \text{Ind}^G_{G \chi}(\Phi).
\]

We have two cases:

- Either \( \mathfrak{h} \not\subset \mathfrak{g}_\chi \) : here \( \mathbb{C} \chi \subset V/V^h \) and \( \wedge^* \overline{V/V^h} = \wedge^* \overline{\mathbb{C} \chi} \otimes \delta \). In this case, \( \Omega = 0 \) because \( \wedge^* \overline{\mathbb{C} \chi} \circ \text{Ind}^G_{G \chi} = 0 \).

- Or \( \mathfrak{h} \subset \mathfrak{g}_\chi \) : here \( \mathfrak{h} \in \Delta_G(W) \) and \( V/V^h = W/W^h \). In this case \( \Omega = \text{Ind}^G_{G \chi}(\wedge^* \overline{W/W^h} \otimes \Phi) \).
It is then immediate that the equivalence \(\text{13}\) follows from \(\text{12}\).

Thanks to \(\text{13}\), it is an easy matter to check that the sequence \(\text{14}\) is exact at \(\mathcal{F}_G(V)\). We leave to the reader the checking that \(\wedge^\bullet \psi : \mathcal{F}_G(V) \subset \mathcal{F}_G(W)\). So the second point is proved.

Let \(I_F : \mathcal{F}_G(V)' \rightarrow \mathcal{F}_G(V)\) be the inclusion. Finally, we have the following commutative diagram, where all the horizontal sequences are exact:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K^0_{G, \chi}(T^*_G V) & \longrightarrow & K^0_{G}(T^*_G V) & \longrightarrow & R & \longrightarrow & K^0_{G}(T^*_G W) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}_{G, \chi}(W)' & \longrightarrow & \mathcal{F}_G(V)' & \longrightarrow & I & \longrightarrow & \mathcal{F}_G(W) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \mathcal{F}_{G, \chi}(W) & \longrightarrow & \mathcal{F}_G(V) & \longrightarrow & \wedge^\bullet \psi & \longrightarrow & \mathcal{F}_G(W). & & \\
\end{array}
\]

Except for \(I_F\), we know that all the vertical arrows are isomorphism. It is an easy exercise to check that \(I_F\) must be an isomorphism.

5.6. \(K^0_G(T^*_G V^{gen})\) is isomorphic to \(\text{DM}_G(V)\). For any \(G\)-module \(V\), we denote \(\text{DM}_G(V)'\) the image of \(K^0_G(T^*_G V^{gen})\) by \(\text{Index}^G\). Since the maps \(j_* : K^0_G(T^*_G V^{gen}) \rightarrow K^0_G(T^*_G V)\) and \(\text{Index}^G\) are injective (see Remark \(\text{4.3}\) and Section \(\text{5.2}\)), we have

\(\text{DM}_G(V)' \simeq K^0_G(T^*_G V^{gen})\)

for any \(G\)-module. Let \(\text{DM}_G(V)\) be the generalized Dahmen-Michelli submodules defined in the introduction. We start with the following

Lemma 5.13. We have \(\text{DM}_G(V)' \subset \text{DM}_G(V)\).

Proof. Let \(\tau \in K^0_G(T^*_G V^{gen})\) and \(j_*(\tau) \in K^0_G(T^*_G V)\). First we remark that \(\text{Index}^G_\chi(\tau) \in (R^{-\infty}(G/H_{min}))\) since \(H_{min}\) acts trivially on \(V\) (see Remark \(\text{4.3}\)).

Let \(\mathfrak{h} \neq h_{min}\) be a stabilizer in \(\Delta G(V)\). Since \(V^\mathfrak{h} \cap V^{gen} = \emptyset\) the composition \(R^\mathfrak{h} \circ j_*\) is the zero map, and \(\text{Index}^G_\chi(\tau) = 0\).

Since by definition \(\text{Index}^G_\chi(\tau) = \text{Index}^G_\chi(j_*(\tau))\), the inclusion \(\text{DM}_G(V)' \subset \text{DM}_G(V)\) is proved.

We will now prove by induction on \(n \geq 0\) the following fact

\((H''^n)_{n \geq 0} \quad \text{DM}_G(V)' = \text{DM}_G(V) \quad \text{if} \quad \dim V/V^\mathfrak{h} \leq n.\)

If \(\dim V/V^\mathfrak{h} = 0\), we have \(T^*_G V = T^*_G V, V^{gen} = V\) and \(\Delta G(V) = \{\mathfrak{g}\}\). In this situation, \(h_{min} = \mathfrak{g}\) and \(R^{-\infty}(G/H_{min}) = R(G)\). We have then \(\text{DM}_G(V) = R(G)\), and assertion \((H''^n)\) is a direct consequence of the Bott isomorphism.

Suppose now that \((H''^n)\) and is true, and consider \(G \subset V\) such that \(\dim V/V^\mathfrak{h} = n + 1\). We have a decomposition \(V = W \oplus \mathbb{C}_\chi\) with \(\bar{\chi} \neq 0\). If we apply \((H''^n)\) to \(G \subset W\) and \(G\chi \subset W\), we get \(\text{DM}_G(W)' = \text{DM}_G(W)\) and \(\text{DM}_G(W)' = \text{DM}_G(W)\).

It works like in the previous section, apart for the dichotomy concerning \(V^{gen} \cap W\). We have the following
Lemma 5.14. Let $H \subset G_\chi$ be a closed subgroup ($G$ is abelian). For any $\Phi \in R^{-\infty}(G_\chi)$, we have the equivalences

\begin{equation}
\Phi \in DM_{G_\chi}(W) \iff \text{Ind}_{G_\chi}^G(\Phi) \in DM_G(V).
\end{equation}

\begin{itemize}
\item If $V^{gen} \cap W \neq \emptyset$, the exact sequence (37) specializes in the exact sequence

\begin{equation}
0 \longrightarrow DM_{G_\chi}(W) \overset{\text{Ind}_{G_\chi}^G}{\longrightarrow} DM_G(V) \overset{\wedge V/W^b}{\longrightarrow} DM_G(W).
\end{equation}

\item If $V^{gen} \cap W = \emptyset$, the exact sequence (37) induces the isomorphism

\begin{equation}
\text{Ind}_{G_\chi}^G : DM_{G_\chi}(W) \overset{\sim}{\longrightarrow} DM_G(V).
\end{equation}
\end{itemize}

\textbf{Proof.} Let $\Phi \in R^{-\infty}(G_\chi)$ and $h \in \Delta_G(V)$. We consider the term $\Omega := \wedge^*V/W^b \otimes \text{Ind}_{G_\chi}^G(\Phi)$. Like in the proof of lemma 5.12 we have two cases:

\begin{itemize}
\item Either $h \notin g_\chi$: in this case $\Omega = 0$.
\item Or $h \subset g_\chi$: here $h \in \Delta_{G_\chi}(W)$ and $V/W^b = W/W^b$. In this case $\Omega = \text{Ind}_{G_\chi}^G(\eta)$ with $\eta = \wedge^*W/W^b \otimes \Phi$.
\end{itemize}

Since the minimal stabilizer for the $G_\chi$ action on $W$ coincides with the minimal stabilizer for the $G$ action on $V$, the relation (47) induces the equivalence $\Phi \in \langle R^{-\infty}(G_\chi/H_{min}) \iff \text{Ind}_{G_\chi}^G(\Phi) \in \langle R^{-\infty}(G/H_{min})\rangle$. For the stabilizers $h_{min} \subset h \subset g_\chi$, using the fact that $\text{Ind}_{G_\chi}^G$ is injective, we see that $\wedge^*V/W^b \otimes \text{Ind}_{G_\chi}^G(\Phi) = 0$ if and only if $\wedge^*W/W^b \otimes \Phi = 0$. The first point follows.

Thanks to (45) it is an easy matter to check that the sequence (37) specializes in the exact sequence $0 \rightarrow DM_{G_\chi}(W) \overset{\alpha}{\longrightarrow} DM_G(V) \overset{\beta}{\longrightarrow} R^{-\infty}(G)$, where $\alpha = \text{Ind}_{G_\chi}^G$ and $\beta = \wedge^*C_\chi$. We can precise this sequence as follows.

Let $h_{min}(W), h_{min}(V)$ be respectively the minimal infinitesimal stabilizer for the $G$-action on $W$ and $V$. We note that $V^{gen} \cap W \neq \emptyset \iff h_{min}(W) \subset g_\chi \iff h_{min}(W) = h_{min}(V)$.

Suppose that $V^{gen} \cap W \neq \emptyset$, and let us check that the image of $\beta$ is contained in $DM_G(W)$. Take $\Phi \in DM_G(V)$ and $h \in \Delta_G(W)$. Let $\beta(\Phi) = \wedge^*C_\chi \otimes \Phi$. We have to consider three cases:

\begin{enumerate}
\item If $h = h_{min}(W)$, then $\Phi$ and $\beta(\Phi)$ belong to $\langle R^{-\infty}(G/H_{min}(W)) \rangle = \langle R^{-\infty}(G/H_{min}(W)) \rangle$.
\item If $h_{min}(W) \subset h \subset g_\chi$, then $\wedge^*V/W^b = \wedge^*W/W^b$ and $\wedge^*W/W^b \otimes \beta(\Phi) = \beta(\wedge^*V/W^b \otimes \Phi) = 0$.
\item If $h \not\subset g_\chi$, then $V/W^b = W/W^b \oplus C_\chi$. We get then $\wedge^*W/W^b \otimes \beta(\Phi) = \wedge^*V/W^b \otimes \Phi = 0$.
\end{enumerate}

We have proved that $\beta(\Phi) \in DM_G(W)$.

Suppose now that $V^{gen} \cap W = \emptyset$, and let us check that $\beta$ is the zero map. Let $h := h_{min}(W) \in \Delta_G(V)$. We have $V/W^b = C_\chi$ since $h \not\subset g_\chi$, and by definition we have $\beta(\Phi) = \wedge^*C_\chi \otimes \Phi = \wedge^*V/W^b \otimes \Phi = 0$ for any $\Phi \in DM_G(V)$.

\[\text{\textsuperscript{11} Says } h_{min} \text{ with corresponding group } H_{min} = \exp(h_{min}).\]
Let \( I_{DM} : \text{DM}_G(V)' \hookrightarrow \text{DM}_G(V) \) be the inclusion. If \( V^{gen} \cap W \neq \emptyset \), we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K^0_{G,x}(T^*_{\mathfrak{g}_x}W^{gen,G_x}) \quad \longrightarrow & K^0_G(T^*_G V^{gen}) \quad \longrightarrow & K^0_G(T^*_G W^{gen}) \quad \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{DM}_G(x)(W)' \quad \longrightarrow & \text{DM}_G(V)' \quad \longrightarrow & \text{DM}_G(W)' \quad \longrightarrow & 0 \\
0 & \longrightarrow & \text{DM}_G(x)(W) \quad \longrightarrow & \text{DM}_G(G)(W) \quad \longrightarrow & \text{DM}_G(W) \quad \longrightarrow & 0 \\
\end{array}
\]

and if \( V^{gen} \cap W = \emptyset \), we have the other commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K^0_{G,x}(T^*_{\mathfrak{g}_x}W^{gen,G_x}) \quad \longrightarrow & K^0_G(T^*_G V^{gen}) \quad \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{DM}_G(x)(W)' \quad \longrightarrow & \text{DM}_G(V)' \quad \longrightarrow & 0 \\
0 & \longrightarrow & \text{DM}_G(x)(W) \quad \longrightarrow & \text{DM}_G(V) \quad \longrightarrow & 0 \\
\end{array}
\]

In both diagrams, all the horizontal sequences are exact, and except for \( I_{DM} \), we know that all the vertical arrows are isomorphisms. It is an easy exercise to check that in both cases \( I_{DM} \) must be an isomorphism.

5.7. **Decomposition of** \( K^0_G(T^*_G V) \simeq \text{DM}_G(V) \). Let \( V \) be a real \( G \)-module such that \( V^0 = \{0\} \). Let \( J \) be an invariant complex structure on \( V \). Let \( \mathcal{W} \subset \mathcal{G} \) be the set of weights: \( \chi \in \mathcal{W} \) if \( V_\chi := \{ v \in V \mid g \cdot v = \eta(g) \cdot v \} \neq \{0\} \). The differential of \( \eta \) is denoted \( \tilde{\eta} \) with \( \tilde{\eta} \in \mathfrak{g}^* \). Let \( \mathcal{W} := \{ \tilde{\eta} \mid \eta \in \mathcal{W} \} : \) it is the set of infinitesimal for the action of \( \mathfrak{g} \) on \( V \).

Let \( \Delta_G(V) \) be the finite set formed by the infinitesimal stabilizer of points in \( V \). For a subspace \( \mathfrak{h} \subset \mathfrak{g} \), we see that \( \mathfrak{h} \in \Delta_G(V) \) if and only if \( \mathfrak{h}^\perp \subset \mathfrak{g}^* \) is generated by \( \mathfrak{h} \cap \mathcal{W} \).

Any vector \( v \in V \) decomposes as \( v = \sum_\eta v_\eta \) with \( v_\eta \in V_\eta \). The subalgebra \( g_v \) that stabilizes \( v \) is equal to \( \cap_\eta \neq 0 \ker(\tilde{\eta}) = (\sum_\eta \neq 0 \mathbb{R} \tilde{\eta})^\perp \). For a subspace \( \mathfrak{h} \subset \Delta_G(V) \), we see that the subspace \( V^\mathfrak{h} := \{ v \mid \mathfrak{h} \subset g_v \} \) is equal to \( \bigoplus_{\tilde{\eta} \in \mathfrak{h}^\perp} V_\eta \) and \( V^\mathfrak{h} := \{ v \mid \mathfrak{h} = \mathfrak{g}_v \} \) is the subspace \( (V^\mathfrak{h})^{gen} \) formed by the vectors \( v := \sum_{\tilde{\eta} \notin \mathfrak{h}^\perp} v_\eta \) such that \( \sum_\eta \neq 0 \mathbb{R} \tilde{\eta} = \mathfrak{h}^\perp \).

We have \( V/V^\mathfrak{h} \simeq \sum_{\tilde{\eta} \notin \mathfrak{h}^\perp} V_\eta \). Following Section 4, we consider a collection \( \gamma := \{ \gamma_\mathfrak{h} \in \mathfrak{h}, \mathfrak{h} \in \Delta_G(V) \} \) such that \( (V/V^\mathfrak{h})^{\gamma} = \{0\} \). We look at the \( H \)-transversally elliptic symbol \( \text{Thom}_{\gamma^\mathfrak{h}}(V/V^\mathfrak{h}) \) on \( V/V^\mathfrak{h} \). Since the action of \( H \) is trivial on \( V^\mathfrak{h} \), the following map

\[
K^0_G(T^*_G(V^\mathfrak{h})^{gen}) \longrightarrow K^0_G(T^*_G((V^\mathfrak{h})^{gen} \times V/V^\mathfrak{h}))
\]

is well defined. We can compose the previous map with the push-forward morphism \( K^0_G(T^*_G((V^\mathfrak{h})^{gen} \times V/V^\mathfrak{h})) \to K^0_G(T^*_GV) : \) let us denote \( S^\mathfrak{h} \) the resulting map.
We can now state Theorem 5.15 in our linear setting.

**Theorem 5.15.** The map

\[ S_\gamma := \bigoplus_{\mathfrak{h} \in \Delta_G(V)} \mathbb{K}^b_G(T_G(V^b)) \rightarrow \mathbb{K}^b_G(T_G(V)) \]

is an isomorphism of $R(G)$-modules.

Now we can translate the previous decomposition through the index map. For $\mathfrak{h} \in \Delta_G(V)$, we consider the element $[\wedge^* V/V^b]^{-1}_{\gamma_\mathfrak{h}} \in R^{-\infty}(G)$ which is due to the isomorphisms $h \in \Delta_G(V)$ (see Definition 5.10 and Proposition 5.11).

We need first the following

**Lemma 5.16.** The product by $[\wedge^* V/V^b]^{-1}_{\gamma_\mathfrak{h}}$ defines a map from $DM_G(V^b)$ into $\mathcal{F}_G(V)$.

**Proof.** Since the symbol $Thom_{\gamma_\mathfrak{h}}(V/V^b)$ is $H$-transversally elliptic, the projection $\pi_\mathfrak{h} : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$ is proper when restricted to the infinitesimal support $\text{Supp}(\Omega)$ of $\Omega := [\wedge^* V/V^b]^{-1}_{\gamma_\mathfrak{h}}$. Let $\Phi \in (R^{-\infty}(G/H))$: the image of $\text{Supp}(\Phi)$ by $\pi_\mathfrak{h}$ is finite. It is now easy to check that for any $\chi \in \widehat{G}$ the set $\{(\chi_1, \chi_2) \in \text{Supp}(\Omega) \times \text{Supp}(\Phi) \mid \chi_1 + \chi_2 = \chi\}$ is finite: the product $[\wedge^* V/V^b]^{-1}_{\gamma_\mathfrak{h}} \otimes \Phi$ is well-defined.

Let $\Phi \in (R^{-\infty}(G/H))$. For any $\mathfrak{a} \in \Delta_G(V)$ we have the ‘mother’ formula\(^{12}\)

\[ (48) \quad \wedge^* V/V^\mathfrak{a} \otimes [\wedge^* V/V^b]^{-1}_{\gamma_\mathfrak{h}} \otimes \Phi = \wedge^* V^h/V^{h+a} \otimes [\wedge^* V^a/V^{h+a}]^{-1}_{\gamma_\mathfrak{h}} \otimes \Phi \]

which is due to the isomorphisms $V^\mathfrak{a} \simeq V/(V^h + V^a) \oplus V^h/V^{h+a}, V^\mathfrak{a} \simeq V/(V^h + V^a) \oplus V^a/V^{h+a}$, and the relation

\[ \wedge^* W \otimes [\wedge^* W]^{-1} = 1 \]

that holds for any $G$-module such that $W^\gamma = \{0\}$.

Note that for any $\mathfrak{a}, \mathfrak{h} \in \Delta_G(V)$ we have the equivalence $V^{h+a} = V^\mathfrak{h} \iff \mathfrak{a} \subseteq \mathfrak{h}$. Suppose now that $\Phi \in DM_G(V^b)$ and consider the product $\Omega := [\wedge^* V/V^b]^{-1}_{\gamma_\mathfrak{h}} \otimes \Phi \in R^{-\infty}(G)$. If $\mathfrak{a} \subseteq \mathfrak{h}$, we have

\[ \wedge^* V/V^\mathfrak{a} \otimes \Omega = [\wedge^* V^a/V^h]^{-1}_{\gamma_\mathfrak{h}} \otimes \Phi \in (R^{-\infty}(G/A)) \]

since $[\wedge^* V^a/V^h]^{-1}_{\gamma_\mathfrak{h}} \in (R^{-\infty}(G/A))$ and $\Phi \in (R^{-\infty}(G/H)) \subset (R^{-\infty}(G/A))$. In the other hand, if $\mathfrak{a} \not\subseteq \mathfrak{h}$, we have $\wedge^* V/V^\mathfrak{a} \otimes \Omega = 0$ since $[\wedge^* V^h/V^{h+a}]^{-1}_{\gamma_\mathfrak{h}} \otimes \Phi = 0$.

We have proved that $\Omega = [\wedge^* V/V^b]^{-1}_{\gamma_\mathfrak{h}} \otimes \Phi$ belongs to $\mathcal{F}_G(V)$. \hfill \box

The map

\[ S_\gamma := \bigoplus_{\mathfrak{h} \in \Delta_G(V)} DM_G(V^h) \rightarrow \mathcal{F}_G(V) \]

\(^{12}\)See formula (2) in \([9]\).
defined by $S_* : \bigoplus_b \Phi_b := \sum_{b \in D_G(V)}[^\wedge \bullet V/V]^b \otimes \Phi_b$ satisfies the following commutative diagram

$$
\begin{array}{ccc}
\bigoplus_b K_G^0(T^*_G(V^b)_{\text{gen}}) & \xrightarrow{S_*} & K_G^0(T^*_G V) \\
\downarrow \otimes \text{Index}_{V_b}^G & & \downarrow \text{Index}_{V}^G \\
\bigoplus_b \text{DM}_G(V^b) & \xrightarrow{S_*} & \mathcal{F}_G(V).
\end{array}
$$

Since $S_*$ and the index maps $\text{Index}_{V_b}^G, \text{Index}_{V}^G$ are isomorphisms we recover the following theorem of de Concini-Procesi-Vergne [9].

**Theorem 5.17.** The map $S_\gamma$ is an isomorphism of $R(G)$-modules.

6. Appendix

6.1. **Appendix A.** Let $G$ be a compact abelian Lie group, and let $\chi : G \to U(1)$ be a surjective morphism. We want to prove that the sequence

$$(49) \quad 0 \to R^{-\infty}(G_\chi) \xrightarrow{\text{Ind}_{G_\chi}^G} R^{-\infty}(G) \xrightarrow{\wedge \chi} R^{-\infty}(G)$$

is exact. Note that the induction map $\text{Ind}_{G_\chi}^G : R^{-\infty}(G_\chi) \to R^{-\infty}(G)$ is the dual of the restriction morphism $R(G) \to R(G_\chi)$. Hence the injectivity of $\text{Ind}_{G_\chi}^G$ will follows from the classical

**Lemma 6.1.** Let $H$ be a closed subgroup of a compact abelian Lie group $G$. The restriction $R(G) \to R(H)$ is onto.

**Proof.** Let $\theta$ be a character of $H$. For any $L^1$-function $\phi : G \to \mathbb{C}$, we consider the average $\bar{\phi}(g) = \int_H \phi(gh)\theta(h)^{-1}dh$ : we have then

$$(50) \quad \bar{\phi}(gh) = \bar{\phi}(g)\theta(h) \quad \text{for any} \quad (g, h) \in G \times H.$$ 

Let us choose $\phi$ such that $\bar{\phi} \neq 0$. For any character $\chi : G \to \mathbb{C}$, we consider the function

$$\tilde{\phi}_\chi(t) := \int_G \tilde{\phi}(tg)\chi(g)^{-1}dg.$$ 

We have $\tilde{\phi}_\chi = (\tilde{\phi}, \chi)\chi$ where $(\tilde{\phi}, \chi) = \int_G \tilde{\phi}(g)\chi(g)^{-1}dg \in \mathbb{C}$. It is immediate that $(50)$ gives that $\tilde{\phi}_\chi(h) = (\tilde{\phi}, \chi)\theta(h)$ for $h \in H$. Hence the restriction of $\chi$ to $H$ is equal to $\theta$ when $(\tilde{\phi}, \chi) \neq 0$. By a density argument, we know that such $\chi$ exists. \hfill $\Box$

Now we want to prove that $\text{Image}(\text{Ind}_{G_\chi}^G) = \ker(\wedge \chi).$ The inclusion $\text{Image}(\text{Ind}_{G_\chi}^G) \subset \ker(\wedge \chi)$ comes from the fact that $\wedge \chi = 0$ in $R(G_\chi)$.

For the other inclusion, we consider $\Phi := \sum_{\mu \in G} m(\mu)\mathbb{C}_\mu \in \ker(\wedge \chi)$. We have the relation $\Phi \otimes \mathbb{C} = \Phi$, which means that $m(\mu + \chi) = m(\mu)$ for all $\mu \in \widehat{G}$. Let $\pi : \widehat{G} \to \widehat{G}_\chi$ be the restriction morphism. Thanks to Lemma 6.1, we know that $\pi$ is surjective, and we see that for $\theta \in \widehat{G}_\chi$, $\pi^{-1}(\theta)$ is of the form $\{k\chi + \theta', k \in \mathbb{Z}\}$. 


For $\theta \in \hat{G}_x$, we denote $n(\theta) \in \mathbb{Z}$ the integer $m(\mu)$ for $\mu \in \pi^{-1}(\theta)$. We have then

$$\Phi = \sum_{\mu \in \hat{G}} m(\mu)C_{\mu} = \sum_{\theta \in \hat{G}_x} \sum_{\mu \in \pi^{-1}(\theta)} m(\mu)C_{\mu} = \sum_{\theta \in \hat{G}_x} n_{\theta} \sum_{k \in \mathbb{Z}} C_{k\chi+\theta}. $$

$$= \text{Ind}_{\hat{G}_x}^G \left( \sum_{\theta \in \hat{G}_x} n_{\theta} C_{\theta} \right).$$

### 6.2. Appendix B

This section is devoted to the proof of Proposition [3.15](#). Let $V$ be equipped with the complex structure $J := J_\beta$. The class $\text{Thom}_{\pm \beta}(V) \in K^0_G(T_G^*V)$ are represented by the symbols $\text{Cl}(\xi \pm \beta(x)) : \wedge^+ V \rightarrow \wedge^- V$. Since $-\text{Thom}_{-\beta}(V)$ is represented by $-\text{Cl}(\xi + \beta(x)) : \wedge^- V \rightarrow \wedge^+ V$, the class $\text{Thom}_{-\beta}(V)$ is represented by the symbol

$$\tau(x, \xi) : \wedge^* V \rightarrow \wedge^* V$$

defined by $\tau(x, \xi) = \text{Cl}(\xi) \circ \epsilon - \text{Cl}(\beta(x))$, where $\epsilon(w) = (-1)^{|w|}w$. We consider the family $\tau_s(x, \xi) = (s \text{Id} + \text{Cl}(\xi)) \circ \epsilon - \text{Cl}(\beta^s(x))$, $s \in [0, 1]$, where $\beta^s = sJ + (1 - s)\beta$. Note that $\beta^s$ is invertible for any $s \in [0, 1]$.

**Lemma 6.2.** The family $\tau_s, s \in [0, 1]$ is an homotopy of transversally elliptic symbols.

Thanks to the last lemma, we know that $\tau = \tau_1$ in $K^0_G(T_G^*V)$. Since $\text{Support}(\tau_1) \cap T^*_G V \subset T^*_G(V \setminus \{0\})$, the restriction $\tau' := \tau_1|V \setminus \{0\}$ is a $G$-transversally elliptic symbol on $V \setminus \{0\}$, and the excision property tells us that $j_i(\tau') = \tau_1 = \tau$ in $K^0_G(T_G^*V)$.

For $(x, \xi) \in T^*(V \setminus \{0\})$, the map $\tau'(x, \xi) : \wedge^* V \rightarrow \wedge^* V$ is given by

$$\tau'(x, \xi) = (\text{Id} + \text{Cl}(\xi)) \circ \epsilon - \text{Cl}(Jx).$$

Let $S$ be the sphere of radius one of $V$. We work with the isomorphism $S \times \mathbb{R} \simeq V \setminus \{0\}, (y, t) \mapsto e^t y$. Let $S \subset S \times \mathbb{R} \times \mathbb{C}$ be the trivial complex vector bundle. Let $H: S \times \mathbb{R}$ be the bundle defined by $H_{(y, t)} := (\mathbb{C}y) \subset T_y S$. We use the isomorphism of vector bundle

$$\phi : H \oplus \mathbb{C} \rightarrow T(S \times \mathbb{R})$$

defined by $\phi_{(y, t)}(\xi' + a + ib) = (\xi' + bJ(y), a) \in T_y S \times T_y \mathbb{R}$. Through $\phi$ the bundle map $\text{Cl}(\xi) : \wedge^+ V \rightarrow \wedge^- V$ for $\xi \in T_y V$ becomes

$$\text{Cl}_{(y, t)}(\xi' + z) : (\wedge H_y \otimes \mathbb{C})^+ \rightarrow (\wedge H_y \otimes \mathbb{C})^-$$

Through $\phi$, the vector field $x \mapsto J_x x$ becomes the section of $\mathbb{C}$ given by $(y, t) \mapsto e^t i$, and the morphism $\tau'$ is defined as follows: for $(y, t) \in S \times \mathbb{R}$, and $\xi' + z \in H_y \oplus \mathbb{C}$, the map $\tau'_{(y, t)}(\xi' + z) : \wedge H_y \otimes \mathbb{C} \rightarrow \wedge H_y \otimes \mathbb{C}$ is defined by

$$\tau'_{(y, t)}(\xi' + z) = (\text{Id} + \text{Cl}(\xi' + z)) \circ \epsilon - e^t \text{Cl}(i).$$

Let $A_{\xi', \xi} = \text{Cl}(\xi' + z)$ and $B = \text{Cl}(i)$ be the maps from $(\wedge H_y \otimes \mathbb{C})^+$ into $(\wedge H_y \otimes \mathbb{C})^-$. The matrix of $\tau'_{(y, t)}(\xi' + z)$ relatively to the grading of $\wedge H_y \otimes \mathbb{C}$ is

$$\begin{pmatrix} \text{Id} & A_{\xi', \xi} + e^t B^* \\ A_{\xi', \xi} + e^t B & -\text{Id} \end{pmatrix}. $$
Let us consider the deformation of $\tau'$ in a family

$$\sigma_s := \begin{pmatrix} \Id & e^t B^* \\ A_{z,\xi'} - e^t B & -\Id \end{pmatrix}, \quad s \in [0, 1].$$

**Lemma 6.3.** The family $\sigma_s, s \in [0, 1]$ is an homotopy of transversally elliptic symbols.

The symbol $\sigma_0 := \begin{pmatrix} \Id & e^t B^* \\ A_{z,\xi'} - e^t B & -\Id \end{pmatrix}$ is clearly homotopic to $\sigma_2 := \begin{pmatrix} \Id & 0 \\ e^{-t} B & \Id \end{pmatrix} \sigma_0 \begin{pmatrix} \Id & 0 \\ -e^{-t} B & \Id \end{pmatrix}

= \begin{pmatrix} \Id & (e^t - e^{-t}) B \\ A_{z,\xi'} - (e^t - e^{-t}) B & 0 \end{pmatrix}$

Since the morphism $e^t B : (\wedge \mathcal{H} \otimes \wedge \mathcal{C}^-) \to (\wedge \mathcal{H} \otimes \wedge \mathcal{C}^+) = 0$ is always invertible, its class vanishes. Hence we have $[\tau'] = [\sigma_0] = [\sigma_2] = [A_{z,\xi'} - (e^t - e^{-t}) B]$ in $K^0_G(T^*_G(S \times \mathbb{R}))$.

We are now working with the morphism $\sigma_3 : (\wedge \mathcal{H} \otimes \wedge \mathcal{C}^+) \to (\wedge \mathcal{H} \otimes \wedge \mathcal{C}^-)$ defined by

$$\sigma_3(\xi' \oplus z) = \Cl(\xi') + \Cl(z - (e^t - e^{-t}) i).$$

Since $\frac{e^t - e^{-t}}{t^2} > 0$ on $\mathbb{R}$, we can deform the term $z - (e^t - e^{-t}) i$ in $t + i \Re(z)$ without changing the intersection of the support with $T^*_G(S \times \mathbb{R})$.

Finally we have proved that $\Thom_{\beta} \circ \mathcal{V}(V) - \Thom_{\beta} \circ \mathcal{V}(V)$ is represented on $S \times \mathbb{R}$ by the morphism $\Cl(\xi') + \Cl(t + i \Re(z)) : (\wedge \mathcal{H} \otimes \wedge \mathcal{C}^+) \to (\wedge \mathcal{H} \otimes \wedge \mathcal{C}^-)$ which is by definition equal to $\sigma_3^V \circ \text{Bott}(T\mathbb{R}) = i_!(\sigma_3^V)$.

We finish this section with the proofs of the deformation Lemmas. For the family $\tau_s(x, \xi) = (s \Id + \Cl(\xi)) \circ \epsilon - \Cl(\beta_s(x))$, we have

$$(\tau_s(x, \xi))^* \tau_s(x, \xi) = \begin{pmatrix} s^2 + \|\xi - \beta_s(x)\|^2 & -2s \Cl(\beta_s(x)) \\ 2s \Cl(\beta_s(x)) & s^2 + \|\xi + \beta_s(x)\|^2 \end{pmatrix}$$

Then $\det(\tau_s(x, \xi)) = 0$ if and only if

$$(s^2 + \|\xi - \beta_s(x)\|^2)(s^2 + \|\xi + \beta_s(x)\|^2) = 4s^2 \|\beta_s(x)\|^2$$

which is equivalent to the equality $(s^2 + \|\xi\|^2 + \|\beta_s(x)\|^2)^2 = 4s^2 \|\beta_s(x)\|^2 + 4\xi, \beta_s(x))^2$.

If $\xi \notin \mathbb{R} \beta_s(x)$, we have $(\xi, \beta_s(x))^2 < \|\xi\|^2 \|\beta_s(x)\|^2$, and then

$$(s^2 + \|\xi\|^2 + \|\beta_s(x)\|^2)^2 < 4(s^2 + \|\xi\|^2) \|\beta_s(x)\|^2$$

which gives $(s^2 + \|\xi\|^2 - \|\beta_s(x)\|^2)^2 < 0$ which is contradictory. Then $\det(\tau_s(x, \xi)) = 0$ if and only if $\xi \in \mathbb{R} \beta_s(x)$ and $s^2 + \|\xi\|^2 - \|\beta_s(x)\|^2 = 0$. If furthermore $\xi \in T^*_G \mathcal{V}|_x$, then $\xi = 0$. We have proved that $\text{Support}(\tau_s) \cap T^*_G \mathcal{V}$ is equal to the compact set $\{(x, \xi) \mid \xi = 0 \text{ and } s^2 - \|\beta_s(x)\|^2 = 0\}$. So $s \in [0, 1] \to \tau_s$ is an homotopy of transversally elliptic symbols.

\[\text{It is due to the fact that } (\beta_s(x), \beta(x)) > 0 \text{ when } x \neq 0.\]
For the family $\sigma_s := \begin{pmatrix} \text{Id} & sA^* + e^tB^* \\ A - e^tB & -\text{Id} \end{pmatrix}$, we have

$$(\sigma_s)^*\sigma_s = \begin{pmatrix} \rho^+ \text{Id} & (1 - s)A^* + 2e^tB^* \\ (1 - s)A + 2e^tB & \rho^-(s)\text{Id} \end{pmatrix}$$

with $\rho^+ = 1 + \|\xi\|^2 + \|z - e^ti\|^2$ and $\rho^- (s) = 1 + \|s\xi\|^2 + \|sz + e^ti\|^2$. We check easily that $((1 - s)A^* + 2e^tB^*)((1 - s)A + 2e^tB) = \rho(s)\text{Id}$ with

$$\rho(s) = \|(s - 1)\xi\|^2 + \|(s - 1)z + 2e^ti\|^2.$$ 

Finally $\det(\sigma_s) = 0$ if and only if $\rho(s) = \rho^-(s)\rho^+$. In other words, $(y, t; \xi' \oplus z)$ belongs to the support of $\sigma_s$ if and only if

$$\|(s-1)\xi\|^2 + \|(s-1)z + 2e^ti\|^2 = (1 + \|s\xi\|^2 + \|sz + e^ti\|^2) \left(1 + \|\xi\|^2 + \|z - e^ti\|^2\right).$$

Let us suppose now that $\xi' \oplus z \in T^*_G(S \times \mathbb{R})$. It imposes $\text{Im}(z) = 0$, and the last relation becomes $(s - 1)^2\Theta + 4e^{2ti} = (1 + e^{2ti} + s^2\Theta)(1 + e^{2ti} + \Theta)$ with $\Theta = [\|\xi\|^2 + \|z\|^2]$. It is easy to see that the last relation holds if and only if $t = \Theta = 0$. Finally we have proved that

$$\text{Support}(\sigma_s) \cap T^*_G(S \times \mathbb{R}) = \{(y, t; \xi' \oplus z) \mid t = 0, \ \xi' = 0, \ z = 0\},$$

and then $\sigma_s, s \in [0, 1]$ defines an homotopy of transversally elliptic symbols.

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14 We write $A$ for $A_{s,t'}$. 
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