Stirling Posets

Mahir Bilen Can\(^1\) and Yonah Cherniavsky\(^2\)

\(^1\)Tulane University, New Orleans; mahirbilencan@gmail.com
\(^2\)Ariel University, Israel; yonahch@ariel.ac.il

June 12, 2018

Abstract

We define combinatorially a partial order on the set partitions and show that it is equivalent to the Bruhat-Chevalley-Renner order on the upper triangular matrices. By considering subposets consisting of set partitions with a fixed number of blocks, we introduce and investigate “Stirling posets.” As we show, the Stirling posets have a hierarchy and they glue together to give the whole set partition poset. Moreover, we show that they (Stirling posets) are graded and EL-shellable. We offer various reformulations of their length functions and determine the recurrences for their length generating series.

Keywords: Borel monoid, Stirling numbers.
MSC: 05A15, 14M15.

1 Introduction

Let \( n \) be a nonnegative integer. A collection \( S_i, \ldots, S_r \) of subsets of an \( n \)-element set \( S \) is said to be a set partition of \( S \) if \( S_i \)'s \((i = 1, \ldots, r)\) are mutually disjoint and \( \bigcup_{i=1}^{r} S_i = S \). In this case, \( S_i \)'s are called the blocks of the partition. If \( n > 0 \) and \( S = \{1, \ldots, n\} \), the collection of all set partitions of \( S \) is denoted by \( \Pi_n \). We will often drop set parentheses and commas and just put vertical bars between blocks. If \( B_1, \ldots, B_k \) are the blocks of a set partition \( \pi \) from \( \Pi_n \), then the standard form of \( \pi \) is defined as \( B_1 | B_2 | \cdots | B_k \), where we assume that \( \min B_1 < \cdots < \min B_k \) and the elements of each block are listed in increasing order. For example, \( \pi = 136|2459|78 \) is a set partition from \( \Pi_9 \).

The set \( \Pi_n \) is known to be a host to many interesting algebraic and combinatorial structures. Among these structures is the following well studied partial ordering: let \( A \) and \( A' \) be two set partitions of \( S \). \( A \) is said to refine \( A' \) if each block of \( A \) is contained in some block of \( A' \). This “refinement ordering” makes \( \Pi_n \) into a lattice, called the partition lattice, and by a result of Pudlak and Tuma (see \([12]\)) it is known that every lattice is isomorphic to a sublattice of \( \Pi_n \) for some \( n \).
A property that is shared by all partition lattices is that their order complexes have the homotopy type of a wedge of spheres. This important combinatorial topological property is seen by analyzing the labelings of the covering relations of the refinement ordering. Indeed, it follows as a consequence of the fact that the refinement ordering is an "edge lexicographically shellable" (EL-shellable for short) poset as shown by Gessel (mentioned in [1]) and by Wachs in [16]. We postpone the proper definition of EL-shellability to our preliminaries section but let us only mention very briefly that the property of EL-shellability of a graded poset is a way of linearly ordering of the maximal faces of the associated order complex, say $F_1, \cdots, F_m$, in such a way that $F_k \cap \left( \bigcup_{i=1}^{k-1} F_i \right)$ is a nonempty union of maximal proper faces of $F_k$ ($k = 2, \ldots, m$). Having this property immediately implies a plethora of results on the topology of the underlying poset, such as Cohen-Macaulayness. It is also helpful for better understanding the Möbius function of the poset. Our purpose in this paper is to present another natural partial ordering on $\Pi_n$ and to show that our poset is EL-shellable as well. To define our ordering we start with defining its most basic ingredient, namely the “arc-diagram.” It is customary to call a linearly ordered poset a chain. Here we will identify chains by their Hasse diagrams and draw them in an unorthodox way, horizontally, by placing the smallest entry on the left and connecting the vertices by arcs. In Figure 1.1 we depicted the chain on 9 vertices, where each arc represents a covering relation.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{chain.png}
\caption{A chain on 9 vertices.}
\end{figure}

Definition 1.1. By a labeled chain we mean a chain whose vertices are labeled by distinct numbers. An arc-diagram on $n$ vertices is a disjoint union of labeled chains where the labels are from $\{1, \ldots, n\}$ and each label $i \in \{1, \ldots, n\}$ is used exactly once.

See Figure 1.2 for an example.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{arc-diagram.png}
\caption{An arc-diagram on 9 vertices}
\end{figure}

It is easy to see that the arc-diagrams on $n$ vertices are in bijection with the elements of $\Pi_n$. Indeed, the map that is defined by grouping the labels of a chain into a set extends to define a bijection from arc-diagrams to the set partitions. For example, under this bijection, the arc-diagram in Figure 1.2 corresponds to the set partition 18|2569|37|4 in $\Pi_9$. In the light of this bijection, from now on, we will work with the arc-diagrams instead of set partitions.
Let us use the notation $A_n$ for denoting the set of all arc-diagrams on $n$ vertices. The goal of our article is to endow $A_n$ with a partial order and to use it to investigate certain subposets of $A_n$. In particular, we will focus on the subposets $A_{n,k} \subset A_n$, where the elements of $A_{n,k}$ have exactly $k$ chains. We will call these subposets as the title of our paper, namely, the Stirling posets.

Next we proceed to define the partial order that we will use throughout the paper. Let $A$ be an arc-diagram. We will identify the vertices of $A$ with their labels. An arc in $A$ is a covering relation in any of the labeled chains in $A$. If the arc denoted by $\alpha$ is a covering relation between the vertices $i$ and $j$, then we write $\alpha = \{i, j\}$. In practice (while drawing the diagrams) we will always think of an arc as the graph of a connected concave down path in $\mathbb{R}^2$. From this point of view, one of our most crucial conventions is that the arcs of $A$ do not intersect each other if they do not have to. We illustrate what we mean here in Figure 1.3. If there is no possibility of continuously deforming two arcs $\alpha_1$ and $\alpha_2$ so that they do not intersect in $\mathbb{R}^2$, then they are said to cross each other. Otherwise, we call them non-crossing arcs.

Before we proceed to explain our ordering on the arc-diagrams we will introduce a very useful function which will eventually lead us to a grading on our poset. This function is defined on all of the set of vertices, arcs, and chains of the arc-diagram. We will occasionally call a pair of non-crossing arcs nested if both of the starting and the ending vertices of one of the arcs stay below the other arc.

**Definition 1.2.** Let $A$ be an arc-diagram and let $\alpha$ be a vertex, or an arc, or a chain from $A$. The depth of $\alpha$, denoted by $\text{depth}(\alpha)$ is the total number of arcs “above” $\alpha$.

Let us be more specific about what we mean by the word “above” in Definition 1.2: If $\alpha$ is a chain where $i$ is its leftmost vertex and $j$ is its rightmost vertex, then an arc $\{r, s\}$ is said to be above $\alpha$ if $r < i$ and $s > j$. For an example, see Figure 1.4, where every arc is of depth 0 and the vertex 4 has depth 3. Obviously, for every arc-diagram the depths of the first and the last vertices are zero, that is

$$\text{depth}(1) = \text{depth}(n) = 0.$$  

Another simple observation that will be useful in the sequel is that if an arc-diagram $A$ on $n$ vertices has $k$ arcs, then $A$ has exactly $n - k$ chains. In this regard, let us point out that the
number of set partitions in $\Pi_n$ with $k$ blocks, hence the number of arc-diagrams in $A_n$ with $k$ chains, is given by the Stirling numbers of the second kind; it is easy to calculate them by using the simple recurrence

$$S(n, k) = S(n-1, k-1) + kS(n-1, k).$$

Let $A$ and $B$ be two arc-diagrams on $n$ vertices. $B$ is said to cover $A$, and denoted by $A \prec B$, if it is obtained from $A$ by one of the following three operations:

**Rule 1. The shortening of an arc of $A$.**

In this operation we move exactly one endpoint of an arc to another vertex so that the resulting arc is shortened as minimally as possible but the number of crossings does not change. For example, see Figure 1.5, where we depict two examples. In the bottom example, the left endpoint of the arc $\{1, 4\}$ is moved to the nearest available position, which is the vertex 3. Indeed, there is already an arc which emanates to the right from the vertex 2.

**Rule 2. Deleting a crossing.**

In this operation we interchange the rightmost endpoints of two crossing arcs so that they become a pair of non-crossing and nested arcs; we require in this operation that only one arc is deleted as a result of this operation. For example, in Figure 1.6, the endpoints of $\{1, 5\}$ and $\{2, 6\}$ are interchanged.

As a non-example, we consider $A = \{1, 4\}\{2, 5\}\{3, 6\}$, which has three crossings. The removal of the crossing between $\{1, 4\}$ and $\{3, 6\}$ according to the rule that we described in the previous paragraph gives $A' = \{1, 6\}\{2, 5\}\{3, 4\}$, which has no crossings.
Rule 3. *Adding a new arc.*

In this operation a new arc is introduced between two vertices in such a way that the new arc is not under any other (older) arcs and the endpoints of the new arc are as far from each other as possible. In Figure 1.7 we depict two examples. In the former one the new arc is \( \{1, 6\} \) and in the latter the new arc is \( \{3, 6\} \).

From now on we will call the set \( A_n \) together with the transitive closure of the covering relations we just defined the arc-diagram poset and denote it by \( (A_n, \prec) \).

Next, we define our first combinatorial statistic.

**Definition 1.3.** Let \( A \) be an arc-diagram on \( n \) vertices \( v_1, \ldots, v_n \) and with \( k \) arcs \( \alpha_1, \alpha_2, \ldots, \alpha_k \). We define the depth-index of \( A \), denoted by \( t(A) \) by the formula

\[
t(A) = \sum_{i=1}^{k} (n - i) - \sum_{j=1}^{n} \text{depth}(v_j) + \sum_{m=1}^{k} \text{depth}(\alpha_m).
\]

One of the main results of our paper is the following statement.

**Theorem 1.** For every positive integer \( n \), the arc-diagrams poset \( (A_n, \prec) \) is a bounded, graded, and an EL-shellable poset. The depth-index function is the grading of \( A_n \).

The proof of our theorem is at least as interesting as its statement. To explain it, we venture outside of combinatorics. Here we assume some familiarity with elementary algebraic geometry. Let \( \text{Mat}_n \) denote the linear algebraic monoid of \( n \times n \) matrices defined over \( \mathbb{C} \). The group of invertible elements, also called the **unit group**, of \( \text{Mat}_n \) is the general linear
group of invertible $n \times n$ matrices. The (standard) Borel subgroup of $\text{GL}_n$, denoted by $B_n$, is the subgroup $B_n \subset \text{GL}_n$ consisting of upper triangular matrices only. Then the doubled Borel group $B_n \times B_n$ acts on matrices via

$$(b_1, b_2) \cdot x = b_1 x b_2^{-1} \quad (b_1, b_2 \in B_n, \ x \in \text{Mat}_n)$$

(1.4)

Clearly, $\text{GL}_n$ is stable under this action. By the special case of an important result of Renner [14], it is known that the action (1.4) has finitely many orbits and moreover the orbits of the action are parametrized by a finite inverse semigroup:

$$\text{Mat}_n = \bigsqcup_{\sigma \in R_n} B_n \sigma B_n,$$

(1.5)

where $R_n$ is the finite monoid consisting of $n \times n$ 0/1 matrices with at most one 1 in each row and each column. The monoid $R_n$ is called the rook monoid; its elements are called rooks. (The nomenclature comes from the fact that the elements of $R_n$ are in bijection with the non-attacking rook placements on an $n \times n$ chessboard.) The Bruhat-Chevalley-Renner ordering on $R_n$ is the partial ordering that is defined by

$$\sigma \leq \tau \iff B_n \sigma B_n \subseteq B_n \tau B_n$$

(1.6)

for $\sigma, \tau \in R_n$. This poset structure on $R_n$ is well studied, [4]. It is known that $(R_n, \leq)$ is a graded, bounded, EL-shellable poset, see [3].

Towards a proof of Theorem 1, we make use of an important algebraic submonoid of $\text{Mat}_n$; it is the closure in Zariski topology of the Borel subgroup $B_n$ in $\text{Mat}_n$. We will call $\overline{B_n}$ the (standard) Borel submonoid. The first systematic study of the theory of Borel submonoids as a part of more general but interrelated theory of parabolic monoids is undertaken by Putcha in [13]. Here we are focusing on one extreme case only.

The Borel submonoid $\overline{B_n}$ consists of all upper triangular $n \times n$ matrices with complex entries. To see this, we use the standard (semidirect product) decomposition

$$B_n = T_n U_n,$$

where $T_n$ is the maximal torus consisting of invertible diagonal matrices and $U_n$ is the unipotent subgroup consisting of upper triangular unipotent matrices. It is easy to check that $U_n$ is already closed in $\text{Mat}_n$, therefore, the Borel submonoid is determined (generated) by its submonoids $\overline{T_n}$ and $U_n$. Here, $\overline{T_n}$ is the diagonal submonoid consisting of all diagonal matrices. Note that $\overline{T_n}$ is an affine toric variety and there is a one-to-one correspondence between the cones of its defining “fan” and its set of idempotents. (An idempotent in a monoid is an element $e$ such that $e^2 = id$.)

Let $M$ be a monoid and let $1_M$ denote its identity element. For us, a submonoid $N$ in a monoid $M$ is a subsemigroup $N \subseteq M$ such that $1_M \in N$. In particular, $1_M$ is the identity element in $N$. Now, $\overline{B_n}$ is a submonoid of $\text{Mat}_n$. Moreover, since it is closed under the two sided action of $B_n$, it has the induced Bruhat-Chevalley-Renner decomposition

$$\overline{B_n} = \bigsqcup_{\sigma \in B_n} B_n \sigma B_n.$$
Here, $B_n$ is the set of all $n \times n$ rooks which are upper triangular in shape. Note that $B_n$ is a submonoid of $R_n$ according to our definition. We call it the upper triangular rook monoid (on $n$ letters). In Figure 1.8 we depict the induced Bruhat-Chevalley-Renner ordering on $B_3$.

Another subsemigroup that is very useful for our purposes is the semigroup of all nilpotent rooks from $B_n$, which we call the standard nilpotent rook monoid and denote by $B_n^{nil}$. We should point out that the identity element of $B_n^{nil}$ is not the same as that of $B_n$. Nevertheless, $B_n^{nil}$ is a monoid. In fact, for $n > 0$, it is not difficult to see that $B_n^{nil}$ is isomorphic, as a monoid, to the upper triangular rook monoid $B_{n-1}$. By going through the same vein we observe that the semigroup of nilpotent elements in $B_n$ is isomorphic as a monoid to $B_{n-1}$. Moreover, this is an isomorphism of algebraic monoids.

The sets of idempotents of the monoids $B_n$ and $T_n$ are the same and it consists of $n \times n$ diagonal matrices with 0/1 entries. Let us denote this common set of idempotents by $E_n$.

It is not difficult to see that $E_n$ is a Boolean lattice with respect to the ordering

$$e \leq f \iff ef = fe = e \quad (e, f \in E_n),$$

In particular, $E_n$ has $2^n$ elements. We denote by $E_{n,k}$ the set of idempotents from $E_n$ whose matrix rank is $k$ and we define the following subvariety the Borel monoid:

$$B_{n,k} := \bigcup_{e \in E_{n,k}} \overline{B_n e B_n}. \quad (1.8)$$

Notice that except when $k \in \{0, n\}$, $B_{n,k}$ is not irreducible as an algebraic variety. Obviously, $B_{n,n}$ is equal to $\overline{B_n}$ and $B_{n,0} = B_n \cdot 0 \cdot B_n = \{0\}$.

The proofs of the following observations will be given in the sequel.

1. for $k = 0, \ldots, n$, the number of irreducible components of $B_{n,k}$ is $\binom{n}{k}$ and they are all equal dimensional.

2. $B_{n,k}$’s form a flag $\{0\} = B_{n,0} \subset B_{n,1} \subset \cdots \subset B_{n,n-1} \subset B_{n,n} = \overline{B_n}$.

3. each $B_{n,k}$ ($k = 0, \ldots, n$) has the structure of an algebraic semigroup.

4. each $B_{n,k}$ ($k = 0, \ldots, n$) has a Renner decomposition

$$B_{n,k} = \bigcup_{\sigma \in B_{n,k}} B_n \sigma B_n, \quad (1.9)$$

where $B_{n,k}$ is a finite subsemigroup of $B_n$ and it consists of rooks whose matrix rank is at most $k$. Moreover, with respect to induced Bruhat-Chevalley-Renner ordering the poset $(B_{n,k}, \leq)$ is a union of lower intervals of equal lengths in $B_n$.

5. The subsemigroups $B_{n,k} \subset B_n$ form a flag $\{0\} \subset B_{n,1} \subset \cdots \subset B_{n,n} = B_n$ and moreover the number of elements of $B_{n,k} - B_{n,k-1}$ is given by the Stirling number $S(n+1, n+1-k)$.
Figure 1.8: Bruhat-Chevalley-Renner order on $B_3$. 
6. The Bruhat-Chevalley-Renner ordering restricted to the subsets of the form $B_{n,k} - B_{n,k-1}$ (for $k = 1, \ldots, n$) is graded with a minimum and there are $\binom{n}{k}$ maximal elements. Each maximal interval in this poset is an interval in $B_n$, therefore, it is an EL-shellable poset.

As an application of our study of the Bruhat-Chevalley-Renner ordering on $B_{n,k}$’s we will prove the following theorem, which, in turn, will give us the proof of Theorem 1. Indeed, the poset $(B_{n,k}^{nil}, \leq)$ is a lower interval in the rook monoid, and $R_n$ is known to be an EL-shellable poset.

**Theorem 2.** The arc-diagram poset $(A_n, \prec)$ is isomorphic to $(B_{n,k}^{nil}, \leq)$.

Next, we show that the arc-diagram poset is a disjoint union of EL-shellable subposets, which are not necessarily intervals. The cardinalities of these subposets will be given by the Stirling numbers of the second kind.

**Theorem 3.** If $A_{n,k}$ denotes the set of arc-diagrams with $n - k$ chains, then $(A_{n,k}, \prec)$ is a graded EL-shellable poset with a unique minimum and $\binom{n}{k}$ maximum elements.

**Definition 1.10.** The $(n,k)$-th Stirling poset is the poset $(A_{n,k}, \prec)$. By abusing notation, we will denote it by $A_{n,k}$.

To contrast $A_{n,k}$ with the corresponding subposet in the refinement ordering on set partitions, let us mention that any two unequal set partitions of $\{1, \ldots, n\}$ with the same number of blocks are not comparable. In other words, the collection of arc-diagrams with the same number of chains do not form an interesting poset with respect to refinement ordering. On the other hand, similarly to the refinement ordering, in $(A_n, \prec)$, the Stirling subposets have a hierarchy in the sense that $A_{n,k}$ lies above $A_{n,k-1}$. Indeed, if $x$ and $y$ are two maximal elements from $A_{n,k}$ and $A_{n,k-1}$, respectively, then $t(x) - t(y) = n - k$. From a similar vein, if $x_0$ and $y_0$ denotes, respectively, the minimum elements of $A_{n,k}$ and $A_{n,k-1}$, then $t(x_0) - t(y_0) = k$.

It is not difficult to see that when $k = 1$, $A_{n,1}$ is the “fish net” as in Figure 1.9, hence every interval in $A_{n,1}$ is a lattice. As $k$ increases, $A_{n,k}$ becomes more complicated. Nevertheless, it is a pleasantly surprising fact that $A_{n,2}$ is a lattice as well. The smallest integer $n$ for which $A_{n,k}$ has a non-lattice subinterval is $n = 5$. See Figure 1.10.

**Theorem 4.** For all integers $n \geq 2$, the $(n,2)$-th Stirling poset $A_{n,2}$ is isomorphic to $B(n-1) - \{\{1, \ldots, n-1\}\}$, where $B(n-1)$ is the boolean lattice of all subsets of $\{1, \ldots, n-1\}$.

The arc-diagram poset $A_n$ contains many interesting (Stirling) posets. But it has more in it; we will justify our statement in our next result. Let us state the relevant terminology here. By a partial flag variety we mean a quotient variety of the form $GL_n/P$, where $P$ is a closed subgroup containing $B_n$. A Schubert variety is the Zariski closure of an orbit of $B_n$ on the partial flag variety. Note that $B_n$ acts on $GL_n/P$ via left multiplication.
Theorem 5. Let $X(n)$, $Y(n)$, and $Z(n)$, and be as in Figure 1.11. In addition, let $S_n$ denote the symmetric group on $\{1, \ldots, n\}$. Then the following statements hold true:

1. The interval $([Y(n), X(n)], \prec)$ in $\mathcal{A}_{2n}$ is isomorphic to $(S_n, \leq)$.
2. The interval $([Z(n), Y(n)], \prec)$ in $\mathcal{A}_{2n}$ and is isomorphic to $(B_n, \leq)$.
3. The interval $([Z(n), X(n)], \prec)$ in $\mathcal{A}_{2n}$ is isomorphic to $(R_n, \leq)$.
4. The interval $([Y(n), W(n)], \prec)$ in $\mathcal{A}_{2n}$ is isomorphic to the inclusion poset of Borel orbit closures in a Schubert variety.

Our final remark concerns the length generating function of the $(n,k)$-th Stirling poset. Let us denote by $t_k$ the length function on $\mathcal{A}_{n,k}$. Clearly, $t_k$ is equal to an appropriate shift of $t$. More precisely, let $A$ be an element from $\mathcal{A}_{n,k}$. If we view $A$ as an element of $\mathcal{A}_n$, then it is clear that $t(A) = t_k(A) + \binom{k}{2}$ since the unique minimum of $\mathcal{A}_{n,k}$ has depth-index $\binom{k}{2}$.

To be able to treat all length generating functions $t_k (k = 0, \ldots, n)$ together, we define

\[
\left[ \begin{array}{c} n \\ k \end{array} \right] := \sum_{A \in \mathcal{A}_{n,k}} q^{t(A)}. \tag{1.11}
\]

Obviously, (1.11) is a $q$-analog of the Stirling numbers of the second kind.

Theorem 6. For positive integers $n$ and $k$ such that $0 \leq k \leq n + 1$ the following recurrence holds true:

\[
\left[ \begin{array}{c} n + 1 \\ k \end{array} \right] = q^k \left[ \begin{array}{c} n \\ k \end{array} \right] + [n + 1 - k]_q q^k \left[ \begin{array}{c} n \\ k - 1 \end{array} \right],
\]

where $[k]_q$ is the polynomial $1 + q + \cdots + q^{k-1}$. The initial conditions are $\left[ \begin{array}{c} m \\ k \end{array} \right] = 1$ for all $m \in \mathbb{N}$. In addition, we assume that $\left[ \begin{array}{c} m \\ k \end{array} \right] = 0$ if $k < 0$ or $k > m$. 

10
Figure 1.10: A non-lattice maximal subinterval in $A_{5,3}$.

For various $(p, q)$-analogues of Stirling numbers of the second kind, see Wachs and White’s influential article [18]. Also, for many other poset theoretic properties of set-partitions (under refinement ordering) we recommend the excellent expository article [17] by Wachs.

We now describe the structure of our paper. We designed Section 2 so that it gives the necessary background for the subsequent sections. In particular, we review the concepts of EL-shellability, rook monoid, and recall some characterizations of the Bruhat-Chevalley-Renner ordering together with its length functions. The Section 3 is devoted to a proof of Theorem 2 and to a proof of the first part of Theorem 1. The second part of Theorem 1 is given in the subsequent Section 4, where we prove that the length function on $B_n$ is equivalent to the depth-index function $t$. In the same section, we introduce another statistic, denoted by $c$, and called the “crossing-index of an arc-diagram.” We prove that $c = t$. Section 5 is the most algebro-geometric section of our paper. We prove six properties that we mentioned above about the variety $\overline{B}_n$ and its subvarieties. The proof of Theorem 4 is recorded therein as well. In Section 6 we prove Theorem 5 which is a characterization of some special subintervals of $A_n$. Finally, in Section 7 we analyze the length generating function of the posets $A_{n,k}$ and prove Theorem 6.

2 Preliminaries

2.1 Set partitions

Although we do not use this fact in the sequel, let us mention that the number of set partitions from $\Pi_n$ is given by the $n$-th Bell number, which is denoted by $b_n$. The exponential generating
The series of $b_n$ is given by $e^{e^x - 1}$.

### 2.2 EL-shellable posets

A finite graded poset $P$ with a maximum and a minimum element is called **EL-shellable**, if there exists a map $f = f_{\Gamma} : C(P) \to \Gamma$ between the set of covering relations $C(P)$ of $P$ into a totally ordered set $\Gamma$ satisfying

1. in every interval $[x, y] \subseteq P$ of length $k > 0$ there exists a unique saturated chain $C : x_0 = x < x_1 < \cdots < x_{k-1} < x_k = y$ such that the entries of the sequence

$$f(C) = (f(x_0, x_1), f(x_1, x_2), \ldots, f(x_{k-1}, x_k))$$

is weakly increasing.

2. The sequence $f(C)$ of the unique chain $C$ from (1) is the lexicographically smallest among all sequences of the form $(f(x_0, x_1'), f(x_1', x_2'), \ldots, f(x_{k-1}', x_k))$, where $x_0 < x_1' < \cdots < x_{k-1}' < x_k$.

The **order complex** of a poset $P$ is the abstract simplicial complex $\Delta(P)$ whose simplices are the chains in $P$. For an EL-shellable poset the order complex is shellable, in particular it implies that $\Delta(P)$ is Cohen-Macaulay [1]. These, of course, are among the most desirable properties of a topological space.

**Remark 2.2.** In the sequel, specifically for the Stirling posets, we will relax the unique maximum element condition in the definition of EL-shellability.

**Remark 2.3.** There are various lexicographic shellability conditions in the literature and the EL-shellability defined here is among the stronger ones. See [2]
2.3 Algebraic monoids

In this section we provide the bare minimum background on reductive monoids to help the reader to understand the geometric/group theoretic angle of our work. We start with defining (more general) algebraic monoids.

Let \( k \) be an algebraically closed field and let \( M \) be an irreducible variety with a morphism \( a : M \times M \rightarrow M \) and an element \( e \in M \) such that

- \( a(x, a(y, z)) = a(a(x, y), z) \) for all \( x, y, z \) from \( M \);
- \( a(e, x) = a(x, e) = x \) for all \( x \) from \( M \).

Thus, \( M \) is an algebraic monoid. Let \( G \) denote the group of invertible elements in \( M \). If \( G \) is a reductive algebraic group, then \( M \) is called a reductive monoid.

Let \( E(M) \) denote the set of idempotents of \( M \). There is an important partial order on \( E(M) \) that is defined by

\[
ed \leq f \iff ef = e = fe.
\] (2.4)

For reductive monoids, there exists a finite sublattice \( \Lambda \subset E(M) \) such that

1. \( M = \bigsqcup_{e \in \Lambda} GeG \);
2. \( e \leq f \iff GeG \subset GfG \) for \( e, f \in \Lambda \).

In the second item, the bar stands for closure in Zariski topology. The lattice \( \Lambda \) (unique up to conjugation) is called the cross section lattice of \( M \) and it uniquely determines many important subgroups of \( G \). For example,

\[
B = \{ g \in G : ge = ege \text{ for all } e \in \Lambda \}
\] (2.5)

is a Borel subgroup and its opposite \( B^- \) is given by

\[
B^- = \{ g \in G : eg = ege \text{ for all } e \in \Lambda \}.
\] (2.6)

The maximal torus of \( B \) is

\[
T = \{ g \in G : ge = e \text{ for all } e \in \Lambda \}
\] (2.7)

Let \( N_G(T) \) denote the normalizer of \( T \) in \( G \) and let \( W \) denote \( N_G(T)/T \), the Weyl group of \( (G, T) \). Let \( S \subset W \) be a generating system consisting of simple reflections. It is well known that \( W \) is a graded poset with the rank function \( \ell : W \rightarrow \mathbb{Z} \) defined by

\[
\ell(w) = \text{dimension of the image of } BwB \text{ in } G/B .
\] (2.8)

The reductive monoid \( M \) has the Renner decomposition

\[
M = \bigsqcup_{\sigma \in \mathcal{R}(M)} B\sigma B
\]
where $R(M)$ is a finite inverse semigroup having $W$ as its unit group. In fact, $R(M) = N_G(T)/T$. We will call $R$ the Renner monoid of $(M, T)$. Extending the Bruhat-Chevalley ordering on the $W$, there is a natural graded partial order on the Renner monoid:

$$\sigma \leq \tau \iff B\sigma B \subseteq B\tau B \quad (2.9)$$

for $\sigma, \tau \in R(M)$. We will call (2.9) the Bruhat-Renner-Chevalley ordering.

**Remark 2.10.** If $\sigma$ and $\tau$ are two idempotents from $R(M)$, then $\sigma \leq \tau$ in (2.9) if and only if $\sigma \leq \tau$ in (2.4).

The Renner monoid $R(M)$, is an inverse semigroup. This means that for each element $x$ of $R(M)$ there exists a corresponding $x^* \in R(M)$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An important commonality between all such monoids is that they admit a faithful linear semigroup representation. More precisely, let $R_n$ denote the Renner monoid $R(\text{Mat}_n)$. It is well known that $R_n$ is isomorphic to the inverse semigroup of all injective partial transformations on the set $\{1, \ldots, n\}$. Furthermore, if $R$ is an inverse semigroup, then for some $n$ there exists an injective semigroup homomorphism $\phi : R \to R_n$. Following our terminology from the introduction, will call $R_n$ the rook monoid since its elements can be viewed as rook placements on an $n \times n$ grid, where the nonzero entries of an element of $R_n$ are viewed as the non-attacking rook placements. It is also possible to represent the elements of $R_n$ in one-line notation and describe the covering relations of the Bruhat-Renner-Chevalley ordering in this context. We will briefly review this development.

Recall from [14] that the rank function on $R_n$ is given by

$$\ell(x) = \dim(BxB), \quad x \in R_n.$$  

There is a combinatorial formula for $\ell(x)$, $x \in R_n$. To explain we represent elements of $R_n$ by $n$-tuples. For $x = (x_{ij}) \in R_n$ we define the sequence $(a_1, \ldots, a_n)$ by

$$a_j = \begin{cases} 0 & \text{if the } j\text{-th column consists of zeros}, \\ i & \text{if } x_{ij} = 1. \end{cases} \quad (2.11)$$

By abuse of notation, we denote both the matrix and the sequence $(a_1, \ldots, a_n)$ by $x$. For example, the associated sequence of the partial permutation matrix

$$x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is $x = (3, 0, 4, 0)$.

Let $x = (a_1, \ldots, a_n) \in R_n$. A pair $(i, j)$ of indices $1 \leq i < j \leq n$ is called a *coinversion pair* for $x$, if $0 < a_i < a_j$. We denote the number of coinversion pairs of $x$ by $\text{coinv}(x)$.

**Example 2.12.** Let $x = (4, 0, 2, 3)$. Then, the only coinversion pair for $x$ is $(3, 4)$. Therefore, $\text{coinv}(x) = 1$.  

14
In [4], it is shown that the dimension $\ell(x) = \dim(BxB)$ of an orbit $BxB$, $x \in R_n$ is given by

\[ \ell(x) = (\sum_{i=1}^{n} a_i^*) - \text{coinv}(x), \text{ where } a_i^* = \begin{cases} a_i + n - i, & \text{if } a_i \neq 0 \\ 0, & \text{if } a_i = 0 \end{cases} \] 

Reformulating (2.13) gives

**Proposition 1.** Let $x = (a_1, \ldots, a_n) \in R_n$. Then

\[ \ell(x) = \sum a_i + \text{inv}(x), \]

where $\text{inv}(x) = |\{(i, j): 1 \leq i < j \leq n, a_i > a_j\}|$.

As a corollary of Proposition 1 we have

**Corollary 1.** Let $w = (a_1, \ldots, a_n) \in S_n$ be a permutation. Then $\ell(w) = \binom{n+1}{2} + \text{inv}(w)$.

First concrete description of the Bruhat-Chevalley-Renner ordering on $R_n$ is given in [11]:

**Theorem 7.** Let $x = (a_1, \ldots, a_n)$, $y = (b_1, \ldots, b_n) \in R_n$. The Bruhat-Renner-Chevalley ordering on $R_n$ is the smallest partial order on $R_n$ generated by declaring $x \leq y$ if either

1. there exists an $1 \leq i \leq n$ such that $b_i > a_i$ and $b_j = a_j$ for all $j \neq i$, or
2. there exist $1 \leq i < j \leq n$ such that $b_i = a_j$, $b_j = a_i$ with $b_i > b_j$, and for all $k \notin \{i, j\}$, $b_k = a_k$.

The covering relations of the order are analyzed in detail in [4], and the following two lemmas are found out to be very useful.

**Lemma 2.14.** Let $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ be elements of $R_n$. Suppose that $a_k = b_k$ for all $k \in \{1, \ldots, \hat{i}, \ldots, n\}$ and $a_i < b_i$. Then, $\ell(y) = \ell(x) + 1$ if and only if either

1. $0 = a_i$, $b_i = 1$ and $a_j = b_j > 0$ for all $j > i$, or
2. $0 < a_i$ and $b_i = a_i + 1$, or
3. there exists a sequence of indices $1 \leq j_1 < \cdots < j_s < i$ such that the set $\{a_{j_1}, \ldots, a_{j_s}\}$ is equal to $\{a_i + 1, \ldots, a_i + s\}$, and $b_i = a_i + s + 1$.

**Example 2.15.** Let $x = (4, 0, 5, 0, 3, 1)$, and let $y = (4, 0, 5, 0, 6, 1)$. Then $\ell(x) = 21$, and $\ell(y) = 22$. If $z = (4, 0, 5, 0, 3, 2)$, then $\ell(z) = 22$.

**Lemma 2.16.** Let $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ be two elements of $R_n$. Suppose that $a_i = b_j$ and $a_i = b_j$ for $i < j$. Furthermore, suppose that for all $k \in \{1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n\}$, $a_k = b_k$. Then, $\ell(y) = \ell(x) + 1$ if and only if for $s = i + 1, \ldots, j - 1$, either $a_j < a_s$, or $a_s < a_i$.

**Example 2.17.** Let $x = (2, 6, 5, 0, 4, 1, 7)$, and let $y = (4, 6, 5, 0, 2, 1, 7)$. Then $\ell(x) = 35$, and $\ell(y) = 36$. Let $z = (7, 6, 5, 0, 4, 1, 2)$. Then $\ell(z) = 42$. 

15
3 Proof of Theorem 2

Recall our notation that $B^\text{nil}_n$ denotes the strictly upper triangular elements of the Borel-Renner monoid $B_n$. Clearly, $B^\text{nil}_n$ is isomorphic to $B_{n-1}$ not only as a monoid but also as a poset.

Let $\sigma$ be an element from $B^\text{nil}_n$ and let $\sigma_1 \ldots \sigma_n$ be its one-line notation. We associate an arc-diagram $A = A(\sigma)$ to $\sigma$ as follows. If $i$ and $j$ are two positive integers such that $1 \leq i < j \leq n$, then $j$ covers $i$ in a chain of $A$ if and only if $\sigma_j = i$. Obviously, in this case, $\{i,j\}$ is an arc of $A$. This association is a version of a well-known bijection between set partitions $\Pi_n$ and the rook placements on an upper triangular board of base length $n - 1$.

Let us denote by $\varphi$ the bijection that is defined in the previous paragraph. Our goal in this section is to prove that

$$\varphi : (A_n, \prec) \rightarrow (B^\text{nil}_n, \leq) \quad (3.1)$$

is a poset isomorphism.

Let $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ be elements of $B^\text{nil}_n$ such that $y$ covers $x$. By Theorem 7 we know that either

1. there exists an $1 \leq i \leq n$ such that $b_i > a_i$ and $b_j = a_j$ for all $j \neq i$, or

2. there exist $1 \leq i < j \leq n$ such that $b_i = a_j$, $b_j = a_i$ with $b_i > b_j$, and for all $k \notin \{i,j\}$, $b_k = a_k$.

Let us proceed with the first case. Then by Lemma 2.14 we know that exactly one of the following statements hold true:

1.a $0 = a_i$, $b_i = 1$ and $a_j = b_j > 0$ for all $j > i$, or

1.b $0 < a_i$ and $b_i = a_i + 1$, or

1.c there exists a sequence of indices $1 \leq j_1 < \ldots < j_s < i$ such that the set $\{a_{j_1}, \ldots, a_{j_s}\}$ is equal to $\{a_i + 1, \ldots, a_i + s\}$, and $b_i = a_i + s + 1$.

In the case of 1.a we see that $\varphi^{-1}(x)$ has 1 as an isolated vertex and $\varphi^{-1}(y)$ has $\{1,j\}$ as an arc. Notice that no arc whose starting vertex is 1 lies under another arc. Moreover, since $a_j = b_j > 0$ for all $j > i$ by our hypothesis, the vertex $i$ has the biggest possible index that the arc starting at 1 can connect. Therefore, according to Rule 3, we have a covering relation $\varphi^{-1}(x) \prec \varphi^{-1}(y)$.

In the case of 1.b, $\{a_i, i\}$ is an arc in $\varphi^{-1}(x)$ and in $\varphi^{-1}(y)$ we have $\{a_i + 1, i\}$ as an arc. Therefore, an arc of $\varphi^{-1}(x)$ is shortened by 1, hence according to Rule 1. this is a covering relation.

The case of 1.c is similar to 1.a; it gives a covering relation by Rule 3.

Next, we look at the second type of covering relation as in Theorem 7. In this case, we look at the numbers $a_i, a_j, i$ and $j$ closely. By definition $\{a_i, i\}$ and $\{a_j, j\}$ are arcs in $\varphi^{-1}(x)$.
But both of the arcs \( \{a_j, i\} \) and \( \{a_i, j\} \) are contained in \( y \), therefore, \( a_i < a_j < i < j \). This means that \( \{a_i, i\} \) and \( \{a_j, j\} \) are crossing arcs in \( \varphi^{-1}(x) \). However, the arcs \( \{a_i, j\} \) and \( \{a_j, i\} \) are nested in \( \varphi^{-1}(y) \). By Rule 2., we see that \( \varphi^{-1}(y) \) covers \( \varphi^{-1}(x) \).

In summary, we showed that the map \( \varphi^{-1} \) is an order preserving bijection from \( (B_n^{nil}; \leq) \) to \( (A_n, \prec) \).

Next, we will show that \( \varphi \) is an order preserving bijection from \( (A_n, \prec) \) to \( (B_n^{nil}; \leq) \). Let \( A \) and \( B \) be two arc-diagrams such that \( A \) is covered by \( B \) in \( (A_n, \prec) \). Let \( x \) and \( y \) denote, respectively, the images of \( A \) and \( B \) in \( B_n^{nil} \). (We continue to use the one-line notation for the elements of \( R_n \).) If the covering relation \( A \prec B \) is obtained from Rule 3., then \( y \) is obtained from \( x \) by inserting a nonzero entry to \( x \). But according to item 1. in Theorem 7, this is a covering relation in \( R_n \). If the covering relation \( A \prec B \) is obtained from Rule 2., then item 2. in Theorem 7 applies. Finally, if the covering relation \( A \prec B \) is obtained from Rule 1., then there are two possibilities. To describe, let \( a \) denote denote the arc \( a = \{v_i, v_j\} \) in \( A \) such that to obtain \( B \) from \( A \) we replace exactly one of the vertices \( v_i \) or \( v_j \) by another vertex \( v_k \) \( (i < k < j) \). In the first possible scenario, \( v_i \) is replaced by \( v_k \). This amounts to a covering relation as described in items 2 or 3 of Lemma 2.14. In the second possible scenario, \( v_j \) is replaced by \( v_k \). This amounts to the covering relation as in Lemma 2.16. Therefore, \( \varphi \) is order preserving as well, hence the proof of Theorem 2 is finished.

**Proof of the first claim of Theorem 1.** The poset \( (B_n^{nil}; \leq) \) is the interval \([0, \ldots, 0], (0, 1, \ldots, n-1)\) in \( (B_n; \leq) \), which, in turn, is the interval \([0, \ldots, 0], (1, \ldots, n)\) in \( (R_n, \leq) \). Therefore, by Theorem 2, the poset \( (A_n, \prec) \) is bounded, graded, and EL-shellable. \( \square \)

## 4 Statistics on arc-diagrams

In this section, to prove the second part of Theorem 1, we will show in Proposition 2 that the function \( t \) defined in the introduction section agrees with the length function on \( B_n \). Then we will give another combinatorial reformulation of \( t \).

**Proposition 2.** Let \( x \) be a partial permutation of the form \( x = (a_1, \ldots, a_n) \in R_n \) and let \( A_x \) be the arc-diagram on \( n \) vertices which corresponds to \( x \). Then

\[
\ell(x) = t(A_x). \tag{4.1}
\]

**Proof.** We will use induction on the number of vertices, \( n \). The base case of the induction is obvious. We assume that our claim (4.1) is true for all arc-diagrams with at most \( n \) vertices. We proceed to prove our claim for \( n + 1 \). Let \( x = (a_1, \ldots, a_{n+1}) \in R_{n+1} \) be a partial permutation with the corresponding arc-diagram \( A_x \) on \( n + 1 \) vertices and with \( k \) arcs.

Let \( s \) be a number such that \( 2 \leq s \leq n + 1 \). There are two cases to consider. First, if there exists an arc \( \{1, s\} \) in \( A_x \), then let \( \tilde{A} \) denote the arc-diagram that is obtained from \( A_x \) by removing \( \{1, s\} \), and let \( \tilde{x} \) be the partial permutation which corresponds to \( \tilde{A} \). Then \( k \) is the number of arcs in \( \tilde{A} \). Clearly, although \( \tilde{A} \) has \( n + 1 \) vertices, since its first vertex does not have any arcs emanating from it, \( \tilde{x} \) has a 0 in its first entry. Removing this entry from
\( \bar{x} \) and removing the first vertex from \( \bar{A} \) does not alter the difference between the length \( \ell \) and the statistics \( t \). Indeed, if \( \bar{x}' \) and \( \bar{A}' \) denotes the resulting partial permutation and the corresponding arc-diagram, then we see that
\[
\ell(\bar{x}) = \ell(\bar{x}') + k \quad \text{and} \quad t(A_{\bar{x}}) = t(\bar{A}') + k.
\]
Now by the induction hypothesis, we have \( \ell(\bar{x}) = t(\bar{A}) \). Secondly, if there is no arc of the form \( \{1, s\} \) in \( A_x \), then we repeat the previous argument.

Let us denote the \( i \)-th coordinates of \( x \) and \( \bar{x} \) by \( a_i(x) \) and \( a_i(\bar{x}) \), respectively, for \( i = 1, \ldots, n + 1 \). Notice that the \( s \)-th entry of \( \bar{x} \) is 0, and the \( s \)-th entry of \( x \) is 1. All other entries of \( \bar{x} \) and \( x \) coincide. So, we have
\[
\sum a_s(x) = \sum a_s(\bar{x}) + 1.
\]
Notice also that \( a_j(x) = 0 \) if and only if \( j \) is the starting point of a chain in \( A_x \), therefore,
\[
\text{inv}(x) = \text{inv}(\bar{x}) + 1 + r, \tag{4.2}
\]
where \( r \) is the number of chains in \( A_x \) that start at the \( j \)-th vertex with \( j > s \). Thus, \( \ell(x) = \ell(\bar{x}) + 1 + r \).

Next, we compare \( t(A_x) \) and \( t(\bar{A}) \). We have
\[
t(A) = t(\bar{A}) + n - k - 1 - (s - 2) + q, \tag{4.3}
\]
where \( q \) is the number of arcs under the arc \( \{1, s\} \). Let us explain the meanings of the summands on the right side of (4.3). The summand \( n - k - 1 \) appears since \( A \) has one more arc than that \( \bar{A} \) has; the contribution of the arcs in \( \bar{A} \) to \( t(A) \) is \( \sum_{i=1}^{k} (n - i) \) in \( t(\bar{A}) \) whereas the contribution of arcs of \( A_x \) to \( t(A_x) \) is \( \sum_{i=1}^{k+1} (n - i) \) in \( t(A) \). The summand \( -(s - 2) \) appears since the depths of each of the \( s - 2 \) vertices \( v_2, v_3, \ldots, v_{s-1} \) of \( \bar{A} \) increase by 1 when we include the arc \( \{1, s\} \). Finally, the summand \( q \) appears since the depths of each of the \( q \) arcs on the vertices \( v_2, v_3, \ldots, v_{s-1} \) of \( \bar{A} \) increase by 1 when we add include the \( \{1, s\} \).

Thus, in order to prove the equality \( \ell(x) = t(A) \) it suffices to show that \( r = n - k - s + q \), where \( r \) is as in (4.2). This equality holds in view of the following argument: \( n - k - r \) is the number of chains in \( \bar{A} \) starting at a vertex \( v_l \) with \( l \leq s \). If we add to this number the number of arcs on the vertices \( v_2, v_3, \ldots, v_{s-1} \), we get exactly \( s \). Consider the truncated sub-diagram of \( \bar{A} \) on the first \( s \) vertices. (The arcs \( \{i, j\} \) with \( i < s < j \) are deleted from \( \bar{A} \).) It is easy to see that, in any arc-diagram on \( s \) vertices, the number of arcs plus the number of chains equals to \( s \). Therefore the number of arcs in the truncated diagram is \( q \), and the number of chains therein is \( n - k - r \). So, \( s = n - k - r + q \), or, \( r = n - k - s + q \) is true. This finishes the proof of the equality \( \ell(x) = t(A) \).

Following the conventions that are set before Definition 1.2 on the crossings of arcs, we define the “crossing number” of an arc as follows.

**Definition 4.4.** Let \( \alpha \) be an arc in an arc-diagram \( A \). We denote by \( \text{cross}(\alpha) \) the total number of chains that \( \alpha \) crosses. Note that \( \alpha \) crosses a chain at most twice. In this case, we consider it as a single crossing.
Example 4.5. Let \( A \) be the arc-diagram in Figure 4.1. The crossing numbers of \( A \) are as follows: \( \text{cross}(\{1, 8\}) = \text{cross}(\{2, 5\}) = \text{cross}(\{3, 7\}) = 1 \), \( \text{cross}(\{5, 6\}) = 0 \), and \( \text{cross}(\{6, 9\}) = 2 \).

![Figure 4.1: Crossings.](image)

The following proposition shows the relation between \( \text{cross} \) and \( \text{depth} \).

Proposition 3. Let \( A \) be an arc-diagram on \( n \) vertices, denoted by \( v_1, \ldots, v_n \). Let \( \alpha_1, \ldots, \alpha_k \) denote its arcs, and let \( \beta_1, \ldots, \beta_{n-k} \) denote its chains. In this notation, the following equality holds true:

\[
\sum_{m=1}^{k} \text{cross}(\alpha_m) = \sum_{i=1}^{n} \text{depth}(v_i) - \sum_{m=1}^{k} \text{depth}(\alpha_m) - \sum_{j=1}^{n-k} \text{depth}(\beta_j). \tag{4.6}
\]

Proof. Once again, we use induction on \( n \). The base case is obvious, so we assume that our claim (4.6) holds true for arc-diagrams on \( m \leq n - 1 \) vertices and we will prove it for the arc-diagrams on \( n \) vertices.

Now, let \( A \) be an arc-diagram whose vertices, arcs, and chains are as in the hypothesis of the proposition. If there is no arc that emanates from the first vertex, then removal of the vertex does not alter neither the left hand side nor the right hand side of eqn. (4.6). So, in this case, by the induction hypothesis, we see that (4.6) holds true. Next, we will analyze how both sides of eqn. (4.6) changes if we add an arc \( \{1, t\} \) to \( A \). There are two cases. We abbreviate “left hand side of (4.6)” to “l.h.s.” and similarly we abbreviate “right hand side of (4.6)” to “r.h.s.”.

Case 1. We assume that there is no arc of \( A \) which is of the form \( \{t, r\} \) with \( r > t \). In this case, let us denote by \( p \) the number of arcs which cross \( \{1, t\} \). In other words, the number of arcs \( \{a, b\} \) such that \( a < t \) and \( b > t \) is \( p \). If we add \( \{1, t\} \) back to \( A \), then \( \sum \text{cross}(\alpha_k) \) increases by \( 2p \). Let us look at the r.h.s. The sum of depths of vertices, \( \sum \text{depth}(v_i) \), increases by \( t - 2 \) since \( v_2, \ldots, v_{t-1} \) are now below the arc \( \{1, t\} \). The sum of the depths of arcs, \( \sum \text{depth}(\alpha_m) \), increases by \( q \), where \( q \) is the number of arcs under \( \{1, t\} \). (In other words, \( q \) is the number of arcs \( \{i, j\} \) such that \( 2 \leq i < j \leq t - 1 \).) By adding \( \{1, t\} \), we see that the sum of depths of chains, \( \sum \text{depth}(\beta_j) \), increases by \( s \), where \( s \) is the number of chains under the arc \( \{1, t\} \); but also it decreases by \( p \), since in \( A \) the vertex \( v_t \) was a chain by itself and there were \( p \) arcs above it. In conclusion, the l.h.s. increases by \( 2p \), while the r.h.s. increases by \( t - 2 - q - s + p \). Notice also the equality \( p + q + s = t - 2 \) which follows
from the fact that in any arc-diagram on \( n \) vertices the number of arcs plus the number of chains equals to \( n \). Now, since \( p + q + s = t - 2 \) is true, the r.h.s. and the l.h.s. are still equal after the arc \( \{1, t\} \) is added to \( A \). This finishes the proof of the first case.

**Case 2.** Assume that \( A \) has an arc of the form \( \{t, r\} \) with \( r > t \). Then, by adding \( \{1, t\} \) to \( A \), the l.h.s. increases by \( 2p - u \), where \( p \) is the total number of arcs of \( A \) that are of the form \( \{a, b\} \) with \( a < t \) and \( b > t \), and \( u \) is the number of arcs \( \{a, b\} \) with \( a < t \) and \( b > t \) that cross the chain \( \{t, r, \ldots\} \) in \( A \). Let us look at the r.h.s.. As in Case 1., the sum of depths of vertices, \( \sum \text{depth}(i) \), increases by \( t - 2 \); the sum of the depths of arcs, \( \sum \text{depth}(\alpha_m) \), increases by \( q \), where \( q \) is the number of arcs under the arc \( \{1, t\} \). Finally, the sum of the depths of chains, \( \sum \text{depth}(\beta_j) \), changes as follows: it increases by \( s \), where \( s \) is the number of chains under the arc \( \{1, t\} \), and it decreases by \( p - u \), where \( p \) and \( u \) are as before. In summary, the l.h.s increases by \( 2p - u \), while the r.h.s. increases by \( t - 2 - q - s + p - u \). Therefore, the l.h.s. and the r.h.s are equal in view of the equality \( p + q + s = t - 2 \) which is seen as in Case 1.. This finishes the proof of our claim.

\[\square\]

**Definition 4.7.** Let \( A \) be an arc-diagram on \( n \) vertices with \( k \) arcs denoted \( \alpha_1, \alpha_2, \ldots, \alpha_k \) and \( n - k \) chains denoted \( \beta_1, \beta_2, \ldots, \beta_{n-k} \). We define the crossing-index of \( A \) by the formula

\[
c(A) = \sum_{i=1}^{k} (n - i) - \sum_{j=1}^{n-k} \text{depth}(\beta_j) - \sum_{m=1}^{k} \text{cross}(\alpha_m).
\]

**Example 4.8.** We continue with Example 4.5. The arc-diagram \( A \) consists of four chains, \( \{1, 8\}, \{2, 5, 6, 9\}, \{3, 7\}, \{4\} \); it has five arcs, \( \{1, 8\}, \{2, 5\}, \{3, 7\}, \{5, 6\}, \{6, 9\} \). The depths of the vertices are \( \text{depth}(1) = \text{depth}(9) = 0, \text{depth}(2) = \text{depth}(8) = 1, \text{depth}(3) = \text{depth}(5) = \text{depth}(6) = \text{depth}(7) = 2, \text{and depth}(4) = 3 \). The depths of arcs are given by \( \text{depth}{}(\{1, 8\}) = \text{depth}(\{6, 9\}) = 0, \text{depth}(\{2, 5\}) = \text{depth}(\{3, 7\}) = 1, \text{depth}(\{5, 6\}) = 2 \). Therefore, the depth-index of \( A \) is given by

\[
t(A) = 8 + 7 + 6 + 5 + 4 - (0 + 1 + 2 + 3 + 2 + 2 + 2 + 1 + 0) + (0 + 1 + 1 + 2 + 0) = 21.
\]

Next, we will compute the crossing-index \( c(A) \). The depths of chains are given by \( \text{depth}(\{1, 8\}) = \text{depth}(\{2, 5, 6, 9\}) = 0, \text{depth}(\{3, 7\}) = 1, \text{depth}(\{4\}) = 3 \). The crossing numbers of \( A \) are as follows: \( \text{cross}(1, 8) = \text{cross}(2, 5) = \text{cross}(3, 7) = 1, \text{cross}(5, 6) = 0, \text{cross}(6, 9) = 2 \). In summary we have

\[
c(A) = 8 + 7 + 6 + 5 + 4 - (0 + 0 + 1 + 3) - (1 + 1 + 1 + 0 + 2) = 21.
\]

The equality of the depth-index and the crossing-index holds true for all arc-diagrams. The proof of this fact follows from Proposition 3 and the definitions, so we omit it.

**Proposition 4.** Let \( A \) be an arc-diagram. Then

\[
t(A) = c(A).
\]
5 Stirling posets

Recall that $E_{n,k}$ denotes the set of diagonal matrices $A$ such that $A$ is a diagonal matrix of rank $k$ and it has only 0's and 1's in its entries. Recall also that we defined the variety $B_{n,k}$ as the union $\bigcup_{e \in E_{n,k}} B_n e B_n$. In this section we will further explain and prove the 6 properties about $B_{n,k}$'s that we listed in Introduction.

We start with proving the following lemma.

**Lemma 5.1.** that the number of components of $B_{n,k}$ is $\binom{n}{k}$ and they are all equal dimensional.

**Proof.** We will show that the elements of $E_{n,k}$ are incomparable in BCR ordering and furthermore

$$e \in E_{n,k} \implies \ell(e) = \frac{k(2n-k+1)}{2}. \quad (5.2)$$

To this end, let $e$ and $f$ be two diagonal idempotents from $B_n$. By Remark 2.10 we know that

$$B_n e B_n \subset B_n f B_n \iff e f = e.$$

But for two diagonal matrices $e$ and $f$ with 0/1 entries and which are of the same rank, the equality $e = f e$ holds true if and only if $e = f$. Since each Borel orbit closure is an irreducible variety, it follows that each closed subset $B_n f B_n$ ($f \in E_{n,k}$) of $B_{n,k}$ is irreducible, and these are precisely the irreducible components of $B_{n,k}$. In particular, there are $\binom{n}{k}$ of them.

Next, we prove the length formula (5.2). We will accomplish this by inducting on $k$. We start with the base case $k = 1$. Let

$$e = (a_1, a_2, \ldots, a_n) \in E_{n,k}$$

be an idempotent that is given in one-line notation (2.11). Since $k = 1$, there exists a unique index $i$ ($1 \leq i \leq n$) such that $a_i = 1$ and $a_j = 0$ if $j \neq i$. Then by Proposition 1 we know that $\ell(e) = \sum a_i + \text{inv}(e) = i + (n - i) = n$ which agrees with (5.2). Now assume that our claim holds true for all idempotents of rank $k - 1$, we proceed to show that it is true for $e = (a_1, a_2, \ldots, a_n) \in E_{n,k}$. Let $m$ denote the largest index such that $a_m = m$. In this case, replacing this 1 with 0 gives us an element

$$e' = (a_1, \ldots, a_{m-1}, 0, a_{m+1}, \ldots, a_n) \in E_{n,k-1}$$
hence $\ell(e') = \frac{(k-1)(2n-k+2)}{2}$ by our induction assumption. By Proposition 1, we see that

\[
\ell(e') = \sum_{j \neq m} a_j + \text{inv}(e')
\]

\[
= \left( \sum_j a_j + m \right) + (\text{inv}(e) - (k - 1) + (n - m))
\]

\[
= \left( \sum_j a_j + \text{inv}(e) \right) + (m - (k - 1) + (n - m))
\]

\[
= \ell(e) + (n - k + 1)
\]

from which our claim follows. Hence, the proof is finished.

\[\square\]

**Remark 5.3.** A simple but useful fact regarding Bruhat-Renner-Chevalley order on $R_n$ is that if $\sigma \leq \tau$ for two elements $\sigma, \tau \in R_n$, then their matrix ranks satisfy $\text{rank}(\sigma) \leq \text{rank}(\tau)$.

**Lemma 5.4.** For every $k$ in the range $0, \ldots, n$, $B_{n,k}$ has a Renner decomposition of the form

\[
B_{n,k} = \bigsqcup_{\sigma \in B_{n,k}} B_n \sigma B_n,
\]

where $B_{n,k}$ is a finite subsemigroup of $B_n$ and it consists of rooks from $B_n$ whose matrix rank is at most $k$.

**Proof.** This follows from the definition of Bruhat-Chevalley Renner ordering and Remark 5.3.

\[\square\]

**Lemma 5.5.** If $n$ is a nonnegative integer $n$, then there is a filtration

\[
\{0\} = B_{n,0} \subset B_{n,1} \subset \cdots \subset B_{n,n-1} \subset B_{n,n} = \overline{B_n}.
\]

**Proof.** This from the fact that

\[
\overline{B_n} = \bigcup_{f \leq e} B_n f B_n
\]

and that $f \leq e \Rightarrow \text{rank}(f) \leq \text{rank}(e)$, where $e$ and $f$ are from $R_n$.

\[\square\]

**Corollary 2.** If $n$ and $k$ are two nonnegative integers such that $0 \leq k \leq n$, then $B_{n,k}$, hence $B_{n,k}$ (for all $k = 0, \ldots, n$) have the structure of an algebraic semigroup.

**Proof.** Since the rank of the product of two matrices is bounded by the ranks of the multiplicands, our claim follows from Lemma 5.5.

\[\square\]

**Lemma 5.6.** The subsemigroups $B_{n,k} \subset B_n$ form a flag $\{0\} \subset B_{n,1} \subset \cdots \subset B_{n,n} = B_n$ and moreover the number of elements of the difference $B_{n,k} - B_{n,k-1}$ is given by the Stirling number $S(n+1, n+1 - k)$.
Proof. The first claim follows from Lemmas 5.4 and 5.5. The second claim follows from the fact that \( B_{n,k} - B_{n,k-1} \) consists of upper triangular rooks whose matrix rank is \( k \) and that the set partitions of \( \{1, \ldots, n+1\} \) with \( k \) blocks is in bijection with upper triangular \( n \times n \) rook matrices of rank \( k \).

Proposition 5. The Bruhat-Chevalley-Renner ordering restricted to the subsets of the form \( B_{n,k} - B_{n,k-1} \) (for \( k = 1, \ldots, n \)) is a graded poset with a minimum element. It has \( \binom{n}{k} \) maximal elements. Each maximal interval in this poset is an interval in \( B_n \), therefore, it is an EL-shellable poset.

Proof. Let \( P_{n,k} \) denote \( (B_{n,k} - B_{n,k-1}, \leq) \), where \( k \in \{1, \ldots, n\} \). The poset \( (P_{n,k}, \leq) \), where \( \leq \) is the Bruhat-Chevalley-Renner ordering has \( \binom{n}{k} \) maximal elements which are given by the incomparable diagonal \( n \times n \) rook matrices of rank \( k \). It is easy to check that the rook matrix (given in one-line notation)

\[
e_0 := (0, \ldots, 0, 1, 2, \ldots, k) \in B_n
\]

is the smallest element of \( (P_{n,k}, \leq) \). Therefore, \( P_{n,k} \) is a union of \( \binom{n}{k} \) maximal intervals all of which has the same poset rank. It is clear that these maximal intervals are intervals in \( B_n \), hence in \( R_n \) as well. In particular we see that \( P_{n,k} \) is an EL-shellable poset.

Now we are ready to prove Theorem 3 which states that for every pair of nonnegative integers \( (n, k) \) such that \( 0 \leq k \leq n \) the Stirling poset \( A_{n,k} \) is a graded and EL-shellable poset. Furthermore, the cardinality of \( A_{n,k} \) is given by \( S(n, n-k) \).

Proof of Theorem 3. The second claim of the theorem is straightforward to prove. To prove the first claim, following the notation in the proof of Proposition 5, we denote the poset \( (B_{n,k} - B_{n,k-1}, \leq) \) by \( P_{n,k} \). Recall from Section 3 that there is a poset isomorphism \( \varphi : (A_n, \prec) \to (B_{n-1}, \leq) \) that is defined by \( \varphi(A) = \sigma \) whenever \( A \) and \( \sigma \) are related as follows: \( \{i, j\} \) is an arc in \( A \) if and only if \( \sigma_j = i \). In particular, if \( A \) is an arc-diagram with \( k \) chains, then \( \sigma \) is a rook matrix of rank \( n-k \). Therefore, we see that \( A_{n,k} \) is isomorphic to \( P_{n-1,n-k} \), hence \( A_{n,k} \) is graded and EL-shellable.

Convention 5.7. In the light of Theorem 3, whenever it is more convenient, we will use the poset \( (B_{n-1,n-k} - B_{n-1,n-k+1}, \leq) \), which is abbreviated to \( P_{n-1,n-k} \), in place of \( (A_{n,k}, \prec) \).

We proceed to prove Theorem 4 which states that \( A_{n,2} \) is the boolean lattice \( B(n-1) - \{1, \ldots, n-1\} \).

Proof of Theorem 4. Following Convention 5.7, we will identify \( P_{n-1,n-2} \) with \( (A_{n,2}, \prec) \). A rook matrix \( x \in B_n \) is an element of \( P_{n-1,n-2} \) if its one-line notation \( x = (a_1, \ldots, a_{n-1}) \) satisfies the following properties:

- the cardinality of \( \{a_1, \ldots, a_{n-1}\} \) is \( n-1 \). This means that the entries of \( x \) are mutual different.
For $i = 1, \ldots, n - 1$, $0 \leq a_i \leq n - 1$.

Clearly, the smallest element of $P_{n-1,n-2}$ is $x_0 = (0, 1, 2, \ldots, n - 2)$. Indeed, $x_0$ has no inversions, and the sum of entries of $x_0$ is the unique minimum of the function

$$x = (a_1, \ldots, a_{n-1}) \mapsto \sum_{i=1}^{n-1} a_i$$
on $P_{n-1,n-2}$.

Next, we define the map $\psi : P_{n-1,n-2} \to B(n - 1) - \{\{1, \ldots, n - 1\}\}$ by

$$\psi(x) = \{a_i : a_i = i, \text{ where } i \in \{1, \ldots, n - 1\}\}.$$

Our goal is to prove that $\psi$ is a poset isomorphism by showing that for every $x = (a_1, \ldots, a_{n-1})$ from $P_{n-1,n-2}$, the interval $[x_0, x]$ is isomorphic to $B(r)$, where $r$ is the number of indices $i = 1, \ldots, n - 1$ such that $a_i = i$. To this end, we proceed by induction on $n$. The base case is when $n = 2$ and in this case $P_{2,1}$ is isomorphic to a fish net poset with 3 elements, so our claim holds true. (In a similar manner, $P_{3,2}$ case be checked by hand.)

Now, let $x = (a_1, \ldots, a_{n-1})$ be an element with $r$ fixed points, that is to say the cardinality of $\{a_i : a_i = i \text{ where } i \in \{0, 1, \ldots, n - 1\}\}$ is $r$. We notice that the non-fixed entries of $x$ appear in an increasing order. In other words, if $a_{i_1}, \ldots, a_{i_{n-1-r}}$ are the entries in $x$ such that $0 \leq a_{i_j} < i_j$ ($j = 1, \ldots, n-1-r$), then $a_{i_1} < \cdots < a_{i_{n-1-r}}$. Next, we observe that if $x$ covers $y$ in $P_{n-1,n-2}$, then $y$ is obtained from $x$ by interchanging exactly one of the fixed entries $a_i (= i)$ with a non-fixed entry $a_j (\neq j)$. In this case, by our induction hypothesis $[x_0, y]$ is isomorphic to the Boolean lattice $B(r - 1)$. Since there are exactly $r$ such subintervals in $[x_0, x]$, we see that $[x_0, x]$ is isomorphic to $B(r)$, hence the proof is finished.

\[\square\]

6 Intervals in $A_n$

From an algebraic point of view, the Borel submonoid $B_n$ may look much simpler compared to its ambient monoid $\text{Mat}_n$. Our goal in this section is to show that, once its size is doubled, the Borel monoid $B_{2n}$ packs at least the same amount of combinatorial information as $\text{Mat}_n$ does.

We start with proving Theorem 5. Its first three items states the following:

1. The interval $[\{Y(n), X(n)\}, \prec]$ in $A_{2n}$ is isomorphic to $\{S_n, \leq\}$.
2. The interval $[\{Z(n), Y(n)\}, \prec]$ in $A_{2n}$ and is isomorphic to $\{B_n, \leq\}$.
3. The interval $[\{Z(n), X(n)\}, \prec]$ in $A_{2n}$ is isomorphic to $\{R_n, \leq\}$.

where $X(n), Y(n), Z(n)$ and $W(n)$ are as in Figure 1.11.
**Proof of Theorem 5.** For the proofs of these statements, once again, we will use the poset isomorphism (3.1) between $A_{2n}$ and $B_{2n-1}$. In particular, the one-line notation for the rook matrices $\varphi(W(n)), \varphi(Z(n)), \varphi(X(n))$, and $\varphi(Y(n))$ are given by

$$\begin{align*}
\varphi(W(n)) &= (0, 1, 2, \ldots, n, 0, \ldots, 0), \\
\varphi(Z(n)) &= (0, \ldots, 0), \\
\varphi(Y(n)) &= (0, \ldots, 0, 1, 2, \ldots, n), \\
\varphi(X(n)) &= (0, \ldots, 0, n, n-1, \ldots, 1).
\end{align*}$$

It is easy to see, by using Theorem 7, that if two rooks $x = (a_1, \ldots, a_{2n})$ and $y = (b_1, \ldots, b_{2n})$ have their first $n$ entries the same, that is $a_i = b_i$ for $i = 1, \ldots, n$, then $x \leq y$ if and only if $(a_{n+1}, \ldots, a_{2n}) \leq (b_{n+1}, \ldots, b_{2n})$. The proofs of our claims 1., 2., and 3. follow easily from this simple observation.

To explain and prove the last item, we briefly review “Ding’s Schubert varieties.” Let $\text{Mat}_{n,m}$ denote the set of all $n \times m$ matrices of rank $n$, hence we implicitly assume that $m \geq n$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with $\lambda_i \geq \lambda_{i+1}$ for $1 \leq i \leq n-1$ and $\lambda_1 = m$. For us, a Ferrers board $F_\lambda$ is a top-right justified subarray in an $n \times m$ matrix such that the length of the $i$-th row is $\lambda_i$. For example, if $\lambda = (6, 3, 1)$ (hence $n = 3, m = 6$), then the corresponding Ferrers board is of the form

$$\begin{bmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
\end{bmatrix},$$

We denote by $M_\lambda$ the set $M_\lambda = \{(a_{i,j}) \in \text{Mat}_{n,m} : a_{i,j} = 0 \text{ if } (i,j) \notin F_\lambda\}$. As it is shown in [7], the quotient space $B_n \setminus M_\lambda$, which is denoted by $X_\lambda$ has the structure of a smooth projective variety and it is noticed by Develin, Martin, and Reiner that $X_\lambda$ is actually isomorphic to a Schubert variety $X_w$ in $\text{GL}_m/P_\lambda$, where $P_\lambda$ is the parabolic subgroup of matrices of the form

$$\begin{pmatrix}
A_1 & * \\
0 & A_2
\end{pmatrix},$$

where $A_1$ is an upper triangular invertible $n \times n$ matrix, and $A_2$ is an $m-n \times m-n$ invertible matrix. (See Section 2 of [6]. Note that in the cited reference the authors use flags rather than matrices to describe the partial flag varieties and their Schubert varieties.)

Now, we specialize to our situation by choosing $\lambda = (2n - 1, 2n - 2, \ldots, n-1)$. Thus, we have $m = 2n - 1$. However, we will view $F_\lambda$ as a top-right justified subarray of a $2n \times 2n$ matrix, hence $M_\lambda$ is contained in $\text{Mat}_{2n}$ as an affine subvariety. It is not difficult to check that $M_\lambda$ is closed under the two sided action of $B_{2n} \times B_{2n}$. Consider the following subgroup of $B_{2n}$:

$$H := \left\{ \begin{bmatrix} A & 0 \\ 0 & \text{id}_n \end{bmatrix} : A \in B_n \right\}.$$ 

Clearly, $H$ is isomorphic to $B_n$ and the left multiplication action of $B_{2n}$ on $M_\lambda$ is equivalent to the left multiplication action of $B_n$. Therefore, the isomorphism $B_n \setminus M_\lambda \rightarrow X_\lambda$ is $B_{2n}$-equivariant, where the $B_{2n}$ action on $M_\lambda$ is on the right and $B_{2n}$ action on $X_\lambda$ is on the left.
Now we need a general fact about the topology of Schubert varieties. Let $G$ be a reductive algebraic group, $P \subset G$ be a parabolic subgroup containing a Borel subgroup $B$. It is a well known fact that for every Schubert variety $X$ in $G/P$, the orbits of $B$ in $X$ are affine spaces and furthermore these affine spaces give a cell decomposition. Therefore, in our case, the cells of $X_\lambda$ are given by the $B_{2n}$ orbits in $X_\lambda$. Said differently, each $B_{2n} \times B_{2n}$ orbit in $M_\lambda$ corresponds to an affine cell in $X_\lambda$. Since these orbits are the same as the orbits of $B_{2n} \times B_{2n}$, we see that the partial order $\leq$ restricted to the rook matrices in $M_\lambda$ describes the inclusion poset on the cells of the Schubert variety $X_\lambda$. It is clear that $\varphi(W(n)) = (0, 1, 2, \ldots, n, 0, \ldots, 0)$ corresponds to the maximal dimensional cell and $\varphi(Y(n)) = (0, \ldots, 0, 1, 2, \ldots, n)$ corresponds to the smallest dimensional cell, hence our proof is finished.

7 A recurrence

We already pointed out in the introductory section that the generating function $[2]$ of $A_{n,k}$ is a $q$-analog of the Stirling numbers of second kind. Another closely related $q$-analog, which is introduced by Garsia $[8]$ and studied in $[10, 9]$ is as follows. For $k = 0, \ldots, n$, $S_{n,k}(q)$ is defined as the polynomial that solves the recurrences

$$S_{n+1,k}(q) = q^{k-1}S_{n,k-1}(q) + [k]_q S_{n,k}(q)$$

(7.1)

with initial conditions $S_{0,0} = 1$ and conventions $S_{n,k} = 0$ whenever $0 > k$ or $k > n$. It is shown by Garsia and Remmel in $[9]$ that

$$S_{n,k}(q) = R_{n-k}(\delta_n, q),$$

(7.2)

where $R_k(\delta_n, q)$ (for $k = 0, \ldots, n$) is the combinatorially defined function

$$R_k(\delta_n, q) = \sum_{r \in C_k(\delta_n)} q^{\text{stat}(r)}.$$  

(7.3)

Here $\delta_n$ is the staircase board, namely the bottom-right justified arrangement of boxes with $i-1$ boxes in the $i$-th row, $C_k(\delta_n)$ is the set of all placements of $k$ non-attacking rooks in $\delta_n$. Clearly, $k$-rook placement can be thought of as an element of $P_{n,k}$ by turning the staircase board up-side-down and then completing it to a square $n \times n$ matrix with 0’s and 1’s where 1’s represent the placements of non-attacking rooks. Finally, the statistics in (7.3) is the inversion statistics of the rook placements. Rather than defining this combinatorial statistic on rook placements, which is somewhat lengthy, we will mention a useful result that gives us an equivalent form. The following observation is recorded in $[5, \text{Lemma 5.3}]$ in a slightly different terminology.

Lemma 7.4. Let $\sigma$ be a rook matrix from $P_{n-1,k}$ and let $r = r(\sigma)$ denote the corresponding non-attacking $k$-rook placement in $\delta_n$. In this case, the following equality holds true

$$\dim(B_n\sigma B_n) = \binom{n}{2} - \text{stat}(r).$$
As a consequence of Lemma 7.4 and definitions,

\[
\left[ \frac{n+1}{k} \right] = q^{\binom{n+1}{2}} R_k \left( \delta_{n+1, \frac{1}{q}} \right). \tag{7.5}
\]

Now we are ready to finish our paper by proving Theorem 6.

**Proof of Theorem 6.** By (7.2) and the recurrence relation in (7.1), we have

\[
q^{\binom{n+1}{2}} R_k \left( \delta_{n+1, \frac{1}{q}} \right) = q^{\binom{n+1}{2}} S_{n+1, n+1-k} \left( \frac{1}{q} \right)
\]

\[
= q^{\binom{n+1}{2}} q^{-n-k} S_{n, n-k} \left( \frac{1}{q} \right) + q^{\binom{n+1}{2}} [n+1-k] \frac{1}{q} S_{n+1, n+1-k} \left( \frac{1}{q} \right)
\]

\[
= q^{\binom{n}{2} + k} R_k \left( \delta_n, \frac{1}{q} \right) + q^{\binom{n+1}{2} - (n-k)} [n+1-k] \frac{1}{q} R_{k-1} \left( \delta_n, \frac{1}{q} \right). \tag{7.6}
\]

It follows from (7.5) and (7.6) that

\[
\left[ \frac{n+1}{k} \right] = q^k \left[ \frac{n}{k} \right] + [n+1-k] q^k \left[ \frac{n}{k} \right]. \tag{7.7}
\]

This finishes the proof of our theorem.

\[\square\]

**References**

[1] Björner, A. *Shellable and Cohen-Macaulay partially ordered sets*. Trans. Amer. Math. Soc. 260 (1980), no. 1, 159–183.

[2] Björner, A. and Wachs M. *Bruhat order of Coxeter groups and shellability*. Adv. in Math., 43(1):87–100, 1982.

[3] Can, M.B. *Rook Monoid is Lexicographically Shellable*. https://arxiv.org/abs/1001.5104

[4] Can, M.B., Renner, L.E. *Bruhat-Chevalley order on the rook monoid*. Turkish J. Math. 36 (2012), no. 4, 499–519.

[5] Can, M.B., Renner, L.E. *H-polynomials and rook polynomials*. Internat. J. Algebra Comput. 18 (2008), no. 5, 935–949.

[6] Develin, M., Martin, J.L., Reiner, V. *Classification of Ding’s Schubert varieties: finer rook equivalence*. Canad. J. Math. 59 (2007), no. 1, 36–62.

[7] Ding, K. *Rook placements and cellular decomposition of partition varieties*. Discrete Math. 170 (1997), no. 1-3, 107–151.

27
[8] Garsia, A.M. *On the ”maj” and ”inv” q-analogues of Eulerian polynomials*. Linear and Multilinear Algebra 8 (1979/80), no. 1, 21–34.

[9] Garsia, A.M., Remmel, J.B. *Q-counting rook configurations and a formula of Frobenius*. J. Combin. Theory Ser. A 41 (1986), no. 2, 246–275.

[10] Milne, S.C. *Restricted growth functions, rank row matchings of partition lattices, and q-Stirling numbers*. Adv. in Math. 43 (1982), no. 2, 173–196.

[11] Pennell, E.A., Putcha, M.S., Renner, L.E. *Analogue of the Bruhat-Chevalley order for reductive monoids*. J. Algebra 196 (1997), no. 2, 339–368.

[12] Pudlak, P., Tuma, J. *Every finite lattice can be embedded in a finite partition lattice*. Algebra Universalis 10 (1980), no. 1, 74–95.

[13] Putcha, M.S. *Parabolic monoids. I. Structure*. Internat. J. Algebra Comput. 16 (2006), no. 6, 1109–1129.

[14] Renner, L.E. *Analogue of the Bruhat decomposition for algebraic monoids*. J. Algebra 101 (1986), no. 2, 303–338.

[15] Renner, L.E. *Linear algebraic monoids*. Encyclopaedia of Mathematical Sciences, 134. Invariant Theory and Algebraic Transformation Groups, V. Springer-Verlag, Berlin, 2005.

[16] Wachs, M. *A basis for the homology of the d-divisible partition lattice*. Adv. Math. 117 (1996), no. 2, 294–318.

[17] Wachs, M. *Poset topology: tools and applications*. In Geometric combinatorics, volume 13 of IAS/Park City Math. Ser., pages 497–615. Amer. Math. Soc., Providence, RI, 2007.

[18] Wachs, M., White, D. *p,q-Stirling numbers and set partition statistics*. J. Combin. Theory Ser. A 21 56 (1991) 27–46.