DILATION OF RITT OPERATORS ON $L^p$-SPACES

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Abstract. For any Ritt operator $T : L^p(\Omega) \to L^p(\Omega)$, for any positive real number $\alpha$, and for any $x \in L^p(\Omega)$, we consider $\|x\|_{T,\alpha} = \left\| \left( \sum_{k=1}^{\infty} k^{2\alpha-1} |T^k(I-T)\alpha x|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$. We show that if $T$ is actually an $R$-Ritt operator, then the square functions $\| \cdot \|_{T,\alpha}$ are pairwise equivalent. Then we show that $T$ and its adjoint $T^* : L^p(\Omega) \to L^p(\Omega)$ both satisfy uniform estimates $\|x\|_{T,1} \lesssim \|x\|_{L^p}$ and $\|y\|_{T^*,1} \lesssim \|y\|_{L^p'}$ for $x \in L^p(\Omega)$ and $y \in L^p'(\Omega)$ if and only if $T$ is $R$-Ritt and admits a dilation in the following sense: there exist a measure space $\tilde{\Omega}$, an isomorphism $U : L^p(\tilde{\Omega}) \to L^p(\tilde{\Omega})$ such that $\{U^n : n \in \mathbb{Z}\}$ is bounded, as well as two bounded maps $L^p(\Omega) \xrightarrow{J} L^p(\tilde{\Omega}) \xrightarrow{Q} L^p(\Omega)$ such that $T^n = QU^nJ$ for any $n \geq 0$. We also investigate functional calculus properties of Ritt operators and analogs of the above results on noncommutative $L^p$-spaces.

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1. Introduction.

Let $(\Omega, \mu)$ be a measure space and let $1 < p < \infty$. For any bounded operator $T : L^p(\Omega) \to L^p(\Omega)$, consider the ‘square function’

$$\|x\|_{T,1} = \left\| \left( \sum_{k=1}^{\infty} k |T^k(x) - T^{k-1}(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p},$$

(1.1)

defined for any $x \in L^p(\Omega)$. Such quantities frequently appear in the analysis of $L^p$-operators. They go back at least to [50], where they were used in connection with martingale square functions to study diffusion semigroups and their discrete counterparts. Similar square functions for continuous semigroups played a key role in the recent development of $H^\infty$-calculus and its applications. See in particular the fundamental paper [12], the survey [31] and the references therein.

It is shown in [34] that if $T$ is both a positive contraction and a Ritt operator, then it satisfies a uniform estimate $\|x\|_{T,1} \lesssim \|x\|_{L^p}$ for $x \in L^p(\Omega)$. This estimate and related ones lead to strong maximal inequalities for this class of operators (see also [35]). Next in the paper [30], the second named author studies the operators $T$ such that both $T : L^p(\Omega) \to L^p(\Omega)$ and its adjoint operator $T^* : L^p(\Omega) \to L^p(\Omega)$ satisfy uniform estimates

$$\|x\|_{T,1} \lesssim \|x\|_{L^p} \quad \text{and} \quad \|y\|_{T^*,1} \lesssim \|y\|_{L^p'}$$

(1.2)
for $x \in L^p(\Omega)$ and $y \in L^{p'}(\Omega)$. (Here $p' = \frac{p}{p-1}$ is the conjugate number of $p$.) It is shown that $L^p(\Omega)$ implies that $T$ is an $R$-Ritt operator (see Section 2 below for the definition) and that $L^p(\Omega)$ is equivalent to $T$ having a bounded $H^\infty$-calculus with respect to a Stolz domain of the unit disc with vertex at 1.

The present paper is a continuation of these investigations. Our main result is a characterization of $L^p(\Omega)$ in terms of dilations. We show that (1.2) holds true if and only if $T$ is an $R$-Ritt and there exist another measure space $(\tilde{\Omega}, \tilde{\mu})$, two bounded maps $J: L^p(\Omega) \to L^p(\tilde{\Omega})$ and $Q: L^p(\tilde{\Omega}) \to L^p(\Omega)$, as well as an isomorphism $U: L^p(\tilde{\Omega}) \to L^p(\tilde{\Omega})$ such that $\{U^n : n \in \mathbb{Z}\}$ is bounded and

$$T^n = QU^n J, \quad n \geq 0.$$  

This result will be established in Section 4. It should be regarded as a discrete analog of the main result of [19].

In Section 3, we consider variants of (1.1) as follows. Assume that $T: L^p(\Omega) \to L^p(\Omega)$ is a Ritt operator. Then $I - T$ is a sectorial operator and one can define its fractional power $(I - T)^\alpha$ for any $\alpha > 0$. Then we consider

\[ (1.3) \quad \|x\|_{T,\alpha} = \left( \sum_{k=1}^{\infty} k^{2\alpha - 1} |(I - T)^\alpha x|^2 \right)^{\frac{1}{2}} \]  

for any $x \in L^p(\Omega)$. Our second main result (Theorem [3.3] below) is that when $T$ is an $R$-Ritt operator, then the square functions $\|\|_{T,\alpha}$ are pairwise equivalent. This result of independent interest should be regarded as a discrete analog of [33] Thm. 1.1. We prove it here as it is a key step in our characterization of (1.2) in terms of dilations.

Section 2 mostly contains preliminary results. Section 5 is devoted to complements on $L^p$-operators and their functional calculus properties, in connection with $p$-completely bounded maps. Finally Section 6 contains generalizations to operators $T: X \to X$ on general Banach spaces $X$. We pay a special attention to noncommutative $L^p$-spaces, in the spirit of [22].

We end this introduction with a few notation. If $X$ is a Banach space, we let $B(X)$ denote the algebra of all bounded operators on $X$ and we let $I_X$ denote the identity operator on $X$ (or simply $I$ if there is no ambiguity on $X$). For any $T \in B(X)$, we let $\sigma(T)$ denote the spectrum of $T$. If $\lambda \in \mathbb{C} \setminus \sigma(T)$ (the resolvent set of $T$), we let $R(\lambda, T) = (\lambda I_X - T)^{-1}$ denote the corresponding resolvent operator. We refer the reader to [14] for general information on Banach space geometry. We will frequently use Bochner spaces $L^p(\Omega; X)$, for which we refer to [15].

For any $a \in \mathbb{C}$ and $r > 0$, we let $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ and we let $\mathbb{D} = D(0, 1)$ denote the open unit disc centered at 0. Also we let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote its boundary.

Whenever $\emptyset \subset \mathbb{C}$ is a non empty open set, we let $H^\infty(\emptyset)$ denote the space of all bounded holomorphic functions $f: \emptyset \to \mathbb{C}$. This is a Banach algebra for the norm

$$\|f\|_{H^\infty(\emptyset)} = \sup\{|f(z)| : z \in \emptyset\}.$$  

Also we let $\mathcal{P}$ denote the algebra of all complex polynomials.
In the above presentation and later on in the paper we will use \( \lesssim \) to indicate an inequality up to a constant which does not depend on the particular element to which it applies. Then \( A(x) \approx B(x) \) will mean that we both have \( A(x) \lesssim B(x) \) and \( B(x) \lesssim A(x) \).

2. Preliminaries on \( R \)-boundedness and Ritt operators.

This section is devoted to definitions and preliminary results involving \( R \)-boundedness (and the companion notion of \( \gamma \)-boundedness), matrix estimates and Ritt operators. We deal with operators acting on an arbitrary Banach space \( X \) (as opposed to the next two sections, where \( X \) will be an \( L^p \)-space).

Let \( (\varepsilon_k)_{k \geq 1} \) be a sequence of independent Rademacher variables on some probability space \( \Omega_0 \). We let \( \text{Rad}(X) \subset L^2(\Omega_0; X) \) be the closure of \( \text{Span}\{\varepsilon_k \otimes x : k \geq 1, \ x \in X\} \) in the Bochner space \( L^2(\Omega_0; X) \). Thus for any finite family \( x_1, \ldots, x_n \) in \( X \), we have

\[
\left\| \sum_{k=1}^n \varepsilon_k \otimes x_k \right\|_{\text{Rad}(X)} = \left( \int_{\Omega_0} \left\| \sum_{k=1}^n \varepsilon_k(\omega) x_k \right\|^2_X d\omega \right)^{\frac{1}{2}}.
\]

We say that a set \( F \subset B(X) \) is \( R \)-bounded provided that there is a constant \( C \geq 0 \) such that for any finite families \( T_1, \ldots, T_n \) in \( F \) and \( x_1, \ldots, x_n \) in \( X \), we have

\[
\left\| \sum_{k=1}^n \varepsilon_k \otimes T_k(x_k) \right\|_{\text{Rad}(X)} \leq C \left\| \sum_{k=1}^n \varepsilon_k \otimes x_k \right\|_{\text{Rad}(X)}.
\]

In this case we let \( R(F) \) denote the smallest possible \( C \), which is called the \( R \)-bound of \( F \).

Let \( (g_k)_{k \geq 1} \) denote a sequence of independent complex valued, standard Gaussian random variables on some probability space \( \Omega_1 \), and let \( \text{Gauss}(X) \subset L^2(\Omega_1; X) \) be the closure of \( \text{Span}\{g_k \otimes x : k \geq 1, \ x \in X\} \). Then replacing the \( \varepsilon_k \)'s and \( \text{Rad}(X) \) by the \( g_k \)'s and \( \text{Gauss}(X) \) in the above paragraph, we obtain the similar notion of \( \gamma \)-bounded set. The corresponding \( \gamma \)-bound of a set \( F \) is denoted by \( \gamma(F) \).

These two notions are very close to each other, however we need to work with both of them in this paper. Comparing them, we recall that any \( R \)-bounded set \( F \subset B(X) \) is automatically \( \gamma \)-bounded, with \( \gamma(F) \leq R(F) \). Moreover if \( X \) has a finite cotype, then the Rademacher averages and the Gaussian averages are equivalent on \( X \) (see e.g. [14, Prop. 12.11 and Thm. 12.27]), hence \( F \) is \( R \)-bounded if (and only if) it is \( \gamma \)-bounded.

\( R \)-boundedness was introduced in [5] and then developed in the fundamental paper [9]. We refer to the latter paper and to [26, Section 2] for a detailed presentation. We recall two facts which are highly relevant for our paper. First, the closure of the absolute convex hull of any \( R \)-bounded set is \( R \)-bounded [9, Lem. 3.2]. This implies the following.

**Lemma 2.1.** Let \( F \subset B(X) \) be an \( R \)-bounded set, let \( J \subset \mathbb{R} \) be an interval and let \( C \geq 0 \) be a constant. Then the set

\[
\left\{ \int_J a(t)V(t) \, dt \mid V : J \to F \text{ is continuous, } a \in L^1(J), \ \|a\|_{L^1(J)} \leq C \right\}
\]

is \( R \)-bounded.
Second, if \( X = L^p(\Omega) \) is an \( L^p \)-space with \( 1 \leq p < \infty \), then \( X \) has a finite cotype and we have an equivalence

\[
\left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(L^p(\Omega))} \approx \left\| \sum_k |x_k|^2 \right\|_{L^p(\Omega)}^{\frac{1}{2}}
\]

for finite families \((x_k)_k\) of \( L^p(\Omega) \). Consequently a set \( F \subset B(L^p(\Omega)) \) is \( R \)-bounded if and only if it is \( \gamma \)-bounded, if and only if there exists a constant \( C \geq 0 \) such that for any finite families \( T_1, \ldots, T_n \) in \( F \) and \( x_1, \ldots, x_n \) in \( L^p(\Omega) \), we have

\[
\left\| \left( \sum_{k=1}^n |T_k(x_k)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \leq C \left\| \left( \sum_{k=1}^n |x_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}.
\]

In the sequel we represent any element of \( B(\ell^2) \) by an infinite matrix \([c_{ij}]_{i,j \geq 1}\) in the usual way. Likewise for any integer \( n \geq 1 \), we identify the algebra \( M_n \) of all \( n \times n \) matrices with the space of linear maps \( \ell^2_n \to \ell^2_n \). Clearly an infinite matrix \([c_{ij}]_{i,j \geq 1}\) represents an element of \( B(\ell^2) \) (in the sense that it is the matrix associated to a bounded operator \( \ell^2 \to \ell^2 \)) if and only if

\[
\sup_{n \geq 1} \left\| [c_{ij}]_{1 \leq i,j \leq n} \right\|_{B(\ell^2_n)} < \infty.
\]

For any \([c_{ij}]_{1 \leq i,j \leq n}\) in \( M_n \), we set

\[
\left\| [c_{ij}] \right\|_{\text{reg}} = \left\| [c_{ij}] \right\|_{B(\ell^2_n)}.
\]

This is the so-called ‘regular norm’ of the operator \([c_{ij}] : \ell^2_n \to \ell^2_n\).

**Lemma 2.2.** For any matrix \([c_{ij}]\) in \( M_n \), the following assertions are equivalent.

(i) We have \( \left\| [c_{ij}] \right\|_{\text{reg}} \leq 1 \).

(ii) There exist two matrices \([a_{ij}]\) and \([b_{ij}]\) in \( M_n \) such that \( c_{ij} = a_{ij}b_{ij} \) for any \( i, j = 1, \ldots, n \), and we both have

\[
\sup_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|^2 \leq 1 \quad \text{and} \quad \sup_{1 \leq j \leq n} \sum_{i=1}^n |b_{ij}|^2 \leq 1.
\]

The implication ‘(ii) ⇒ (i)’ is an easy application of the Cauchy-Schwarz inequality. The converse is due to Peller [42, Section 3] (see also [1]). We refer to [44] and [48, Sect. 1.4] for more about this result and complements on regular norms.

The following result extends the boundedness of the Hilbert matrix (which corresponds to the case \( \beta = \gamma = \frac{1}{2} \)). We thank Éric Ricard for his precious help in devising this proof.

**Proposition 2.3.** Let \( \beta, \gamma > 0 \) be two positive real numbers. Then the infinite matrix

\[
\begin{bmatrix}
\frac{i^{\beta-\frac{1}{2}}j^{\gamma-\frac{1}{2}}}{(i+j)^{\beta+\gamma}}
\end{bmatrix}_{i,j \geq 1}
\]

represents an element of \( B(\ell^2) \).
Proof. For any \( i, j \geq 1 \), set
\[
c_{ij} = \frac{i^\beta j^{-\frac{1}{2}} - \frac{1}{2} j^\gamma}{(i + j)^{3+\gamma}}, \quad a_{ij} = c_{ij} \left( \frac{i}{j} \right)^{\frac{1}{2}}, \quad \text{and} \quad b_{ij} = c_{ij} \left( \frac{j}{i} \right)^{\frac{1}{2}}.
\]
Then \( c_{ij} = a_{ij} b_{ij} \) for any \( i, j \geq 1 \), hence by the easy implication of Lemma 2.2 it suffices to show that
\[
(2.2) \quad \sup_{i \geq 1} \sum_{j=1}^{\infty} |a_{ij}|^2 < \infty \quad \text{and} \quad \sup_{j \geq 1} \sum_{i=1}^{\infty} |b_{ij}|^2 < \infty.
\]
Fix some \( i \geq 1 \). For any \( j \geq 1 \), we have
\[
|a_{ij}|^2 = c_{ij} \left( \frac{i}{j} \right)^{\frac{1}{2}} = \frac{i^\beta j^{-1}}{(i + j)^{3+\gamma}}.
\]
Hence
\[
\sum_{j=1}^{\infty} |a_{ij}|^2 = i^\beta \left( \frac{1}{(i + 1)^{3+\gamma}} + \sum_{j=2}^{\infty} \frac{1}{j^{1-\gamma}(i + j)^{3+\gamma}} \right).
\]
Looking at the variations of the function \( t \mapsto 1/(t^{1-\gamma}(i + t)^{3+\gamma}) \) on \((1, \infty)\), we immediately deduce that
\[
\sum_{j=1}^{\infty} |a_{ij}|^2 \leq 1 + 2 i^\beta \int_{1}^{\infty} \frac{1}{t^{1-\gamma}(i + t)^{3+\gamma}} \, dt.
\]
Changing \( t \) into \( it \) in the latter integral, we deduce that
\[
\sum_{j=1}^{\infty} |a_{ij}|^2 \leq 1 + 2 \int_{0}^{\infty} \frac{1}{t^{1-\gamma}(1 + t)^{3+\gamma}} \, dt.
\]
This upper bound is finite and does not depend on \( i \), which proves the first half of (2.2). The proof of the second half is identical. \( \square \)

We record the following elementary lemma for later use.

**Lemma 2.4.** Let \([c_{ij}]_{i,j \geq 1}\) and \([d_{ij}]_{i,j \geq 1}\) be infinite matrices of nonnegative real numbers, such that \( c_{ij} \leq d_{ij} \) for any \( i, j \geq 1 \). If the matrix \([d_{ij}]_{i,j \geq 1}\) represents an element of \( B(\ell^2) \), then the same holds for \([c_{ij}]_{i,j \geq 1}\).

We will need the following classical fact (see e.g. [14, Cor. 12.17]).

**Lemma 2.5.** Let \( X \) be a Banach space and let \([b_{ij}]_{1 \leq i,j \leq n}\) be an element of \( M_n \). Then for any \( x_1, \ldots, x_n \) in \( X \), we have
\[
\left\| \sum_{i,j=1}^{n} g_i \otimes b_{ij} x_j \right\|_{\text{Gauss}(X)} \leq \| [b_{ij}] \|_{B(\ell^2)} \left\| \sum_{j=1}^{n} g_j \otimes x_j \right\|_{\text{Gauss}(X)}.
\]
That result does not remain true if we replace Gaussian variables by Rademacher variables and this defect is the main reason why it is sometimes easier to deal with \( \gamma \)-boundedness than with \( R \)-boundedness.
Proposition 2.6. Let $X$ be a Banach space, let $F = \{T_{ij} : i, j \geq 1\}$ be a $\gamma$-bounded family of operators on $X$, let $n \geq 1$ be an integer and let $[c_{ij}]_{1 \leq i, j \leq n}$ be an element of $M_n$. Then for any $x_1, \ldots, x_n$ in $X$, we have

$$\left\| \sum_{i,j=1}^{n} g_i \otimes c_{ij} T_{ij}(x_j) \right\|_{\text{Gauss}(X)} \leq \gamma(F) \left\| [c_{ij}] \right\|_{\text{reg}} \left\| \sum_{j=1}^{n} g_j \otimes x_j \right\|_{\text{Gauss}(X)}.$$ 

Proof. We can assume that $\left\| [c_{ij}] \right\|_{\text{reg}} \leq 1$. By Lemma 2.2, we can write $c_{ij} = a_{ij} b_{ij}$ with

$$(2.3) \quad \sup_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|^2 \leq 1 \quad \text{and} \quad \sup_{1 \leq j \leq n} \sum_{i=1}^{n} |b_{ij}|^2 \leq 1.$$ 

Let $(g_{i,j})_{i,j \geq 1}$ be a doubly indexed family of independent Gaussian variables. For any integers $1 \leq i, j \leq n$, we define

$$A(i) = [ a_{i1} \ a_{i2} \ \ldots \ a_{in} ] \quad \text{and} \quad B(j) = [ b_{1j} \ b_{2j} \ \ldots \ b_{nj} ]^T.$$ 

Then we consider the two matrices

$$A = \text{Diag} \begin{pmatrix} A(1), \ldots, A(n) \end{pmatrix} \in M_{n,n^2} \quad \text{and} \quad B = \text{Diag} \begin{pmatrix} B(1), \ldots, B(n) \end{pmatrix} \in M_{n^2,n}.$$ 

Let $x_1, \ldots, x_n \in X$. Applying Lemma 2.5 successively to $A$ and $B$, we then have

$$\left\| \sum_{i,j=1}^{n} g_i \otimes c_{ij} T_{ij}(x_j) \right\|_{\text{Gauss}(X)} = \left\| \sum_{i,j=1}^{n} g_i \otimes a_{ij} b_{ij} T_{ij}(x_j) \right\|_{\text{Gauss}(X)}$$

$$\leq \|A\| \left\| \sum_{i,j=1}^{n} g_{ij} \otimes b_{ij} T_{ij}(x_j) \right\|_{\text{Gauss}(X)}$$

$$\leq \gamma(F) \|A\| \left\| \sum_{i,j=1}^{n} g_{ij} \otimes b_{ij} x_j \right\|_{\text{Gauss}(X)}$$

$$\leq \gamma(F) \|A\| \|B\| \left\| \sum_{j=1}^{n} g_j \otimes x_j \right\|_{\text{Gauss}(X)}.$$ 

We have

$$\|A\| = \sup_{1 \leq i \leq n} \|A(i)\|_{M_{1,n}} = \sup_{1 \leq i \leq n} \left( \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}},$$

hence $\|A\| \leq 1$ by (2.3). Likewise, we have $\|B\| \leq 1$ hence the above inequality yields the result. \ 

We now turn to Ritt operators, the key class of this paper, and recall some of their main features. Details and complements can be found in \cite{6,7,30,36,38,39,52}. We say that an operator $T \in B(X)$ is a Ritt operator if the two sets

$$(2.4) \quad \{T^n : n \geq 0\} \quad \text{and} \quad \{n(T^n - T^{n-1}) : n \geq 1\}$$
are bounded. This is equivalent to the spectral inclusion
\[ \sigma(T) \subset \overline{D} \]
and the boundedness of the set
\[ \{(\lambda - 1)R(\lambda, T) : |\lambda| > 1\} \]
This resolvent estimate outside the unit disc is called the ‘Ritt condition’.

Likewise we say that \( T \) is an \( R \)-Ritt operator if the two sets in (2.4) are \( R \)-bounded. This is equivalent to the inclusion (2.5) and the \( R \)-boundedness of the set (2.6).

For any angle \( \gamma \in (0, \frac{\pi}{2}) \), let \( B_\gamma \) be the interior of the convex hull of 1 and the disc \( D(0, \sin \gamma) \) (see Figure 1 below).

![Figure 1](image)

Then the Ritt condition and its \( R \)-bounded version can be strengthened as follows.

**Lemma 2.7.** Let \( T : X \to X \) be a Ritt operator (resp. an \( R \)-Ritt operator). There exists an angle \( \gamma \in (0, \frac{\pi}{2}) \) such that
\[ \sigma(T) \subset B_\gamma \cup \{1\} \]
and the set
\[ \{(\lambda - 1)R(\lambda, T) : \lambda \in \mathbb{C} \setminus B_\gamma, \lambda \neq 1\} \]
is bounded (resp. \( R \)-bounded).

This essentially goes back to [6], see [30] for details.

For any angle \( \theta \in (0, \pi) \), let
\[ \Sigma_\theta = \{z \in \mathbb{C} : |\text{Arg}(z)| < \theta\} \]
be the open sector of angle \( 2\theta \) around the positive real axis \((0, \infty)\). We say that a closed operator \( A : D(A) \to X \) with dense domain \( D(A) \) is sectorial if there exists \( \theta \in (0, \pi) \) such that \( \sigma(A) \subset \Sigma_\theta \) and the set
\[ \{zR(z, A) : z \in \mathbb{C} \setminus \Sigma_\theta\} \]
is bounded.

Let \( T \) be a Ritt operator and let \( \gamma \in (0, \pi) \) be such that the spectral inclusion (2.7) holds true and the set (2.8) is bounded. Then \( A = I - T \) is a sectorial operator. Indeed \( 1 - B_\gamma \subset \Sigma_\gamma \) and \( zR(z, A) = ((1 - z) - 1)R(1 - z, T) \) for any \( z \notin \Sigma_\gamma \). Hence for \( \theta = \gamma \), the set (2.10) is bounded. Thus for any \( \alpha > 0 \), one can consider the fractional power \( (I - T)^\alpha \). We refer e.g. to [20, Chap. 3] for various definitions of these (bounded) operators and their basic properties. Fractional powers of Ritt operators can be expressed by a natural Dunford-Riesz functional calculus formula. Indeed it was observed in [30] that for any polynomial \( \varphi \), we have

\[
\varphi(T)(I - T)^\alpha = \frac{1}{2\pi i} \int_{\partial B_\gamma} \varphi(\lambda)(1 - \lambda)^\alpha R(\lambda, T) d\lambda,
\]

where the countour \( \partial B_\gamma \) is oriented counterclockwise.

P. Vitse proved in [52] that if \( T : X \to X \) is a Ritt operator, then for any integer \( N \geq 0 \), the set \( \{n^N T^{n-1}(I - T)^N : n \geq 1\} \) is bounded. Our next statement is a continuation of these results.

**Proposition 2.8.** Let \( X \) be a Banach space and let \( T : X \to X \) be a Ritt operator (resp. an \( R \)-Ritt operator). For any \( \alpha > 0 \), the set

\[
\{n^\alpha (rT)^{n-1}(I - rT)^\alpha : n \geq 1, \ r \in (0, 1]\}
\]

is bounded (resp. \( R \)-bounded).

**Proof.** We will prove this result in the ‘\( R \)-Ritt case’ only. The ‘Ritt case’ is similar and simpler. Assume that \( T \) is \( R \)-Ritt. Applying Lemma 2.7 we let \( \gamma \in (0, \pi) \) be such that (2.7) holds true and the set (2.8) is \( R \)-bounded. Let \( r \in (0, 1] \) and let \( \lambda \in \mathbb{C} \setminus B_\gamma \), with \( \lambda \neq 1 \). Then \( \frac{\lambda}{r} \in \mathbb{C} \setminus B_\gamma \) hence \( \frac{\lambda}{r} \) belongs to the resolvent set of \( T \) and we have

\[
(\lambda - 1)R(\lambda, rT) = \frac{\lambda - 1}{\lambda - r} \left(\frac{\lambda}{r} - 1\right)R\left(\frac{\lambda}{r}, T\right).
\]

Since the set

\[
\left\{\frac{\lambda - 1}{\lambda - r} : \lambda \in \mathbb{C} \setminus B_\gamma, \ \lambda \neq 1, \ r \in (0, 1]\right\}
\]

is bounded, it follows from the above formula that the set

\[
(\lambda - 1)R(\lambda, rT) : \lambda \in \mathbb{C} \setminus B_\gamma, \ \lambda \neq 1, \ r \in (0, 1]\}
\]

is \( R \)-bounded.

The boundary \( \partial B_\gamma \) is the juxtaposition of the segment \( \Gamma_+ \) going from 1 to \( 1 - \cos(\gamma)e^{-i\gamma} \), of the segment \( \Gamma_- \) going from \( 1 - \cos(\gamma)e^{i\gamma} \) to 1 and of the curve \( \Gamma_0 \) going from \( 1 - \cos(\gamma)e^{i\gamma} \) to \( 1 - \cos(\gamma)e^{-i\gamma} \) counterclockwise along the circle of center 0 and radius \( \sin \gamma \).

Consider a fixed number \( \alpha > 0 \). For any integer \( n \geq 1 \) and any \( r \in (0, 1] \), we have

\[
(rT)^{n-1}(I - rT)^\alpha = \frac{1}{2\pi i} \int_{\partial B_\gamma} \lambda^{n-1}(1 - \lambda)^\alpha R(\lambda, rT) d\lambda
\]
by applying (2.11) to $rT$. Hence we may write
\[ n^\alpha (rT)^{n-1} (I - rT)^\alpha = \frac{-n^\alpha}{2\pi i} \int_{\partial B_r} \lambda^{n-1}(1 - \lambda)^{\alpha - 1}(\lambda - 1) R(\lambda, rT) \, d\lambda. \]

According to the $R$-boundedness of the set (2.12) and Lemma 2.1, it therefore suffices to show that the integrals
\[ I_n = n^\alpha \int_{\partial B_r} |\lambda|^n |1 - \lambda|^{\alpha - 1} |d\lambda| \]
are uniformly bounded (for $n$ varying in $\mathbb{N}$). Let us decompose each of these integrals as
\[ I_n = I_{n,0} + I_{n,+} + I_{n,-}, \]
with
\[ I_{n,0} = n^\alpha \int_{\Gamma_0} \cdot \cdot \cdot |d\lambda|, \quad I_{n,+} = n^\alpha \int_{\Gamma_+} \cdot \cdot \cdot |d\lambda|, \quad \text{and} \quad I_{n,-} = n^\alpha \int_{\Gamma_-} \cdot \cdot \cdot |d\lambda|. \]

For $\lambda \in \Gamma_0$, we both have
\[ \cos \gamma \leq |1 - \lambda| \leq 2 \quad \text{and} \quad |\lambda| = \sin \gamma. \]
Since the sequence $\left( n^\alpha (\sin \gamma)^n \right)_{n \geq 1}$ is bounded, this readily implies that the sequence $(I_{n,0})_{n \geq 1}$ is bounded.

Let us now estimate $I_{n,+}$. For any $t \in [0, \cos \gamma]$, we have $t^2 \leq t \cos \gamma$ hence
\[ |1 - te^{-i\gamma}|^2 = 1 + t^2 - 2t \cos \gamma \leq 1 - t \cos \gamma. \]
Hence
\[ I_{n,+} = n^\alpha \int_0^{\cos \gamma} |1 - te^{-i\gamma}|^n t^{\alpha - 1} \, dt \leq n^\alpha \int_0^{\cos \gamma} (1 - t \cos \gamma)^{\frac{\alpha}{2}} t^{\alpha - 1} \, dt. \]
Changing $t$ into $s = t \cos \gamma$ and using the inequality $1 - s \leq e^{-s}$, we deduce that
\[ I_{n,+} \leq \frac{n^\alpha}{(\cos \gamma)^\alpha} \int_0^{\cos^2 \gamma} s^{\alpha - 1} e^{-\frac{2s}{2}} \, ds. \]
This yields (changing $s$ into $u = \frac{sn}{2}$)
\[ I_{n,+} \leq \frac{2^\alpha}{(\cos \gamma)^\alpha} \int_0^{\infty} u^{\alpha - 1} e^{-u} \, du. \]
Thus the sequence $(I_{n,+})_{n \geq 1}$ is bounded. Since $I_{n,-} = I_{n,+}$, this completes the proof of the boundedness of $(I_n)_{n \geq 1}$. \[\square\]

### 3. Equivalence of square functions.

Throughout the next two sections, we fix a measure space $(\Omega, \mu)$ and a number $1 < p < \infty$. We shall deal with operators acting on the Banach space $X = L^p(\Omega)$. We start with a precise definition of (1.1) and (1.3) and a few comments.

Let $T : L^p(\Omega) \to L^p(\Omega)$ be a bounded operator and let $x \in L^p(\Omega)$. Let us consider
\[ x_k = k^{\frac{1}{2}}(T^k(x) - T^{k-1}(x)) \]
for any $k \geq 1$. If the sequence $(x_k)_{k \geq 1}$ belongs to the space $L^p(\Omega; \ell^2)$, then $\|x\|_{T,1}$ is defined as the norm of $(x_k)_{k \geq 1}$ in that space. Otherwise, we set $\|x\|_{T,1} = \infty$. If $T$ is a Ritt operator,
then the quantities $\|x\|_{T,\alpha}$ are defined in a similar manner for any $\alpha > 0$. In particular, $\|x\|_{T,\alpha}$ can be infinite.

These square functions are natural discrete analogs of the square functions associated to sectorial operators (see [12] and the survey paper [31]).

Assume that $T$ is a Ritt operator. Then $T$ is power bounded hence by the Mean Ergodic Theorem (see e.g. [25, Subsection 2.1.1]), we have a direct sum decomposition
\[(3.1)\]
\[L^p(\Omega) = \text{Ker}(I - T) \oplus \text{Ran}(I - T),\]
where Ker(·) and Ran(·) denote the kernel and the range, respectively. For any $\alpha > 0$, we have Ker\((I - T)^\alpha\) = Ker\((I - T)\). This implies that
\[(3.2)\]
\[\|x\|_{T,\alpha} = 0 \iff x \in \text{Ker}(I - T).\]

Given any $\alpha > 0$, a general question is to determine whether $\|x\|_{T,\alpha} < \infty$ for any $x$ in $L^p(\Omega)$. It is easy to check, using the Closed graph Theorem, that this finiteness property is equivalent to the existence of a constant $C \geq 0$ such that
\[(3.3)\]
\[\|x\|_{T,\alpha} \leq C\|x\|_{L^p}, \quad x \in L^p(\Omega).\]

In [30], the second named author established the following connection between the boundedness of discrete square functions and functional calculus properties.

**Theorem 3.1.** ([30]) Let $T: L^p(\Omega) \rightarrow L^p(\Omega)$ be a Ritt operator, with $1 < p < \infty$. The following assertions are equivalent.

(i) The operator $T$ and its adjoint $T^*: L^{p'}(\Omega) \rightarrow L^{p'}(\Omega)$ both satisfy uniform estimates
\[\|x\|_{T,1} \lesssim \|x\|_{L^p} \quad \text{and} \quad \|y\|_{T^*,1} \lesssim \|y\|_{L^{p'}}\]
for $x \in L^p(\Omega)$ and $y \in L^{p'}(\Omega)$.

(ii) There exists an angle $0 < \gamma < \frac{\pi}{2}$ and a constant $K \geq 0$ such that
\[\|\varphi(T)\| \leq K\|\varphi\|_{H^\infty(B_\gamma)}\]
for any $\varphi \in \mathcal{P}$.

(iii) The operator $T$ is R-Ritt and there exists an angle $0 < \theta < \pi$ such that $I - T$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus.

Besides [30], we refer to [12, 26, 32, 37] for general information on $H^\infty(\Sigma_\theta)$ functional calculus for sectorial operators.

The main purpose of this section is to show that if $T$ is R-Ritt, then the square functions $\|x\|_{T,\alpha}$ are pairwise equivalent. Thus the existence of an estimate (3.3) does not depend on $\alpha > 0$. This result (Theorem 3.3 below) is a discrete analog of the equivalence of square functions associated to $R$-sectorial operators, as established in [33].

We start with preliminary results which allow to estimate square functions $\|x\|_{T,\alpha}$ by means of approximation processes.

**Lemma 3.2.** Assume that $T: L^p(\Omega) \rightarrow L^p(\Omega)$ is a Ritt operator, and let $\alpha > 0$.

(1) For any operator $V: L^p(\Omega) \rightarrow L^p(\Omega)$ such that $VT = TV$ and any $x \in L^p(\Omega)$, we have
\[\|V(x)\|_{T,\alpha} \leq \|V\|\|x\|_{T,\alpha}.\]
(2) For any \( x \in \text{Ran}(I - T) \), we have \( \|x\|_{T, \alpha} < \infty \).

(3) Let \( \nu \geq \alpha + 1 \) be an integer and let \( x \in \text{Ran}((I - T)^\nu) \). Then

\[
\|x\|_{T, \alpha} = \lim_{r \to 1^-} \|x\|_{rT, \alpha}.
\]

**Proof.** (1): Consider \( V \in B(L^p(\Omega)) \). As is well-known, the tensor product \( V \otimes I_{\ell^2} \) extends to a bounded operator \( V \otimes I_{\ell^2} : L^p(\Omega; \ell^2) \to L^p(\Omega; \ell^2) \), with \( \|V \otimes I_{\ell^2}\| = \|V\| \). Assume that \( VT = TV \) and let \( x \) be such that \( \|x\|_{T, \alpha} < \infty \). Then we have

\[
(k^{\alpha - \frac{1}{2}}T^{k-1}(I - T)^\alpha(V(x)))_{k \geq 1} = V \otimes I_{\ell^2}\left[(k^{\alpha - \frac{1}{2}}T^{k-1}(I - T)^\alpha(x))_{k \geq 1}\right],
\]

and the result follows at once.

(2): Assume that \( x = (I - T)x' \) for some \( x' \in L^p(\Omega) \). By Proposition 2.8, there exists a constant \( C \) such that

\[
\sum_{k=1}^{\infty} \|k^{\alpha - \frac{1}{2}}T^{k-1}(I - T)^\alpha(x)\|_{L^p} = \sum_{k=1}^{\infty} k^{\alpha - \frac{1}{2}} \|T^{k-1}(I - T)^{\alpha+1}(x')\|_{L^p} \leq \|x'\|_{L^p} \sum_{k=1}^{\infty} k^{\alpha - \frac{1}{2}} \frac{C}{k^{\alpha+1}} \leq C \|x'\|_{L^p} \sum_{k=1}^{\infty} k^{-\frac{3}{2}} < \infty.
\]

This implies that \( (k^{\alpha - \frac{1}{2}}T^{k-1}(I - T)^\alpha(x))_{k \geq 1} \) belongs to \( L^p(\Omega; \ell^2) \).

(3): It is clear that \( (I - rT)^\alpha \to (I - T)^\alpha \) when \( r \to 1^- \). Assume that \( x \in \text{Ran}((I - T)^\nu) \). Arguing as in part (2) we find that the sequence \( (k^{\alpha - \frac{1}{2}}T^{k-1}(x))_{k \geq 1} \) belongs to \( L^p(\Omega; \ell^2) \). Then arguing as in part (1), we obtain that

\[
\left\|\left(k^{\alpha - \frac{1}{2}}T^{k-1}((I - rT)^\alpha - (I - T)^\alpha)(x)\right)_{k \geq 1}\right\|_{L^p(\ell^2)} \to 0
\]

when \( r \to 1^- \). This implies the convergence result. \( \Box \)

**Theorem 3.3.** Assume that \( T : L^p(\Omega) \to L^p(\Omega) \) is an R-Ritt operator. Then for any \( \alpha, \beta > 0 \), we have an equivalence

\[
\|x\|_{T, \alpha} \approx \|x\|_{T, \beta}, \quad x \in L^p(\Omega).
\]

**Proof.** We fix \( \gamma > 0 \) such that \( \alpha + \gamma \) is an integer \( N \geq 1 \). For any integer \( k \geq 1 \), we define the complex number

\[
c_k = \frac{k(k + 1) \cdots (k + N - 2)}{k^{\alpha - \frac{1}{2}}},
\]

with the convention that \( c_k = \frac{1}{k^{\alpha - \frac{1}{2}}} \) if \( N = 1 \). For any \( z \in \mathbb{D} \), we have

\[
\sum_{k=1}^{\infty} k(k + 1) \cdots (k + N - 2)z^{k-1} = \frac{(N - 1)!}{(1 - z)^N}.
\]
Hence
\[ \sum_{k=1}^{\infty} c_k k^{\frac{\alpha-1}{2}} z^{2k-2}(1 - z^2)^N = \sum_{k=1}^{\infty} k(k + 1) \cdots (k + N - 2)(z^2)^{k-1}(1 - z^2)^N = (N - 1)! \]

Since the operator $T$ is power bounded, we deduce that for every $r \in (0, 1)$ we have
\[ \sum_{k=1}^{\infty} c_k k^{\alpha-\frac{1}{2}} (rT)^{2k-2}(I - (rT)^2)^N = (N - 1)!I, \]
the series being absolutely convergent. Since $(I + rT)^N$ is invertible, this yields
\[ \sum_{k=1}^{\infty} c_k (rT)^{k-1}(I - rT)^\gamma k^{\alpha-\frac{1}{2}} (rT)^{k-1}(I - rT)^\alpha = (N - 1)!(I + rT)^{-N}. \]

Let $x \in L^p(\Omega)$. For any integer $m \geq 1$ and any $r \in (0, 1)$, we let
\[ y_m(r) = (N - 1)!(I + rT)^{-N} m^{\frac{\beta-1}{2}} (rT)^m (I - rT)^\alpha x. \]
Then it follows from the above identity that
\[ y_m(r) = \sum_{k=1}^{\infty} c_k m^{\frac{\beta-1}{2}} (rT)^{m+k-2}(I - rT)^{\beta+\gamma} \cdot k^{\alpha-\frac{1}{2}} (rT)^{k-1}(I - rT)^\alpha x. \]
For any $n \geq 1$, we consider the partial sum
\[ y_{m,n}(r) = \sum_{k=1}^{n} c_k m^{\frac{\beta-1}{2}} (rT)^{m+k-2}(I - rT)^{\beta+\gamma} \cdot k^{\alpha-\frac{1}{2}} (rT)^{k-1}(I - rT)^\alpha x, \]
and we have $y_{m,n}(r) \to y_m(r)$ when $n \to \infty$.

Let us write
(3.4)
\[ c_k m^{\frac{\beta-1}{2}} (rT)^{m+k-2}(I - rT)^{\beta+\gamma} = \frac{m^{\beta-\frac{1}{2}} c_k}{(m + k - 1)^{\gamma+\beta}} [(m + k - 1)^{\gamma+\beta} (rT)^{m+k-2}(I - rT)^{\beta+\gamma}] \]
for any $m, k \geq 1$. Since $c_k \sim_{+\infty} k^{\gamma-\frac{1}{2}}$, there exists a positive constant $K$ such that
\[ \frac{m^{\beta-\frac{1}{2}} c_k}{(m + k - 1)^{\gamma+\beta}} \leq K \frac{m^{\beta-\frac{1}{2}} k^{\gamma-\frac{1}{2}}}{(m + k)^{\gamma+\beta}} \]
for any $m, k \geq 1$. It therefore follows from Proposition 2.3 and Lemma 2.4 that the matrix
\[ \left[ \frac{m^{\beta-\frac{1}{2}} c_k}{(m + k - 1)^{\gamma+\beta}} \right]_{m,k \geq 1} \]
represents an element of $B(\ell^2)$. Moreover, by Proposition 2.8, the set
\[ F = \{ (m + k - 1)^{\gamma+\beta} (rT)^{m+k-2}(I - rT)^{\gamma+\beta} : m, k \geq 1, \ r \in (0, 1) \} \]
is $R$-bounded. Hence by (2.1), (3.4) and Proposition 2.6 we get to an estimate
\[ \left\| \left( \sum_{m=1}^{M} \left| y_{m,n}(r) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \left\| \left( \sum_{k=1}^{\infty} k^{2\alpha-1} |(rT)^{k-1}(I - rT)^\alpha x|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \]
for any integer $M \geq 1$. Passing to the limit, we deduce that
\[
\left\| \left( \sum_{m=1}^{\infty} |y_m(r)|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \left\| \left( \sum_{k=1}^{\infty} k^{2\alpha-1} |(rT)^k - (I-rT)^\alpha x|^2 \right)^{\frac{1}{2}} \right\|_{L^p}.
\]
Since the set $\{(N-1)!^{-1}(I+rT)^N : r \in (0,1)\}$ is bounded, we finally obtain that
\[
\|x\|_{rT,\beta} \lesssim \|x\|_{rT,\alpha}.
\]
It is crucial to note that in this estimate, the majorizing constant hidden in the symbol $\lesssim$ does not depend on $r \in (0,1)$.

Now let $\nu$ be an integer such that $\nu \geq \alpha + 1$ and $\nu \geq \beta + 1$. Applying Lemma 3.2 (3), we deduce a uniform estimate
\[
\|x\|_{rT,\beta} \lesssim \|x\|_{rT,\alpha}
\]
for $x \in \text{Ran}((I-T)^\nu)$. Next for any integer $m \geq 0$, set
\[
\Lambda_m = \frac{1}{m+1} \sum_{k=0}^{m} (I-T^k).
\]
It is clear that $\Lambda_m^\nu$ maps $X$ into $\text{Ran}((I-T)^\nu)$. Hence we actually have a uniform estimate
\[
\|\Lambda_m^\nu(x)\|_{T,\beta} \lesssim \|\Lambda_m^\nu(x)\|_{T,\alpha}, \quad x \in X, \ m \geq 1.
\]
Since $T$ is power bounded, the sequence $(\Lambda_m)_{m \geq 0}$ is bounded. Applying Lemma 3.2 (1), we deduce a further uniform estimate
\[
\|\Lambda_m^\nu(x)\|_{T,\beta} \lesssim \|x\|_{T,\alpha}, \quad x \in X, \ m \geq 1.
\]
Equivalently, we have
\[
\left\| \left( \sum_{k=1}^{l} k^{2\beta-1} |T^{k-1}(I-T)^\beta \Lambda_m^\nu(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|x\|_{T,\alpha}, \quad x \in X, \ m \geq 1, \ l \geq 1.
\]
For any $x \in \text{Ran}(I-T)$, $\Lambda_m(x) \to x$ and hence $\Lambda_m^\nu(x) \to x$ when $m \to \infty$. Hence passing to the limit in the above inequality, we obtain a uniform estimate $\|x\|_{T,\beta} \lesssim \|x\|_{T,\alpha}$ for $x$ in $\text{Ran}(I-T)$. Switching the roles of $\alpha$ and $\beta$, this shows that $\|\|_{T,\beta}$ and $\|\|_{T,\alpha}$ are equivalent on the space $\text{Ran}(I-T)$. Moreover $\|\|_{T,\beta}$ and $\|\|_{T,\alpha}$ vanish on $\text{Ker}(I-T)$ by (3.2). Appealing to the direct sum decomposition (3.1), we finally obtain that $\|\|_{T,\beta}$ and $\|\|_{T,\alpha}$ are equivalent on $L^p(\Omega)$.

The techniques developed so far in this paper allow us to prove the following proposition, which complements Theorem 3.1. For a Ritt operator $T$, we let $P_T$ denote the projection onto $\text{Ker}(I-T)$ which vanishes on $\text{Ran}(I-T)$ (recall (3.1)).

**Proposition 3.4.** Let $T : L^p(\Omega) \to L^p(\Omega)$ be a Ritt operator, with $1 < p < \infty$. Then the condition (i) in Theorem 3.1 is equivalent to:

(i) We have an equivalence
\[
\|x\|_{L^p} \approx \|P_T(x)\|_{L^p} + \|x\|_{T,1}
\]
for $x \in L^p(\Omega)$. 
Proof. That (i) implies (i)' was proved in [33], Rem. 3.4] in the case when $T$ is ‘contractively regular’. The proof in our present case is the same.

Assume (i)'. Let $y \in L^{p'}(\Omega)$. We consider a finite sequence $(x_k)_{k \geq 1}$ in $L^p(\Omega)$ and we set
\[
x = \sum_k k^{\frac{1}{2}} (T^*)^{k-1}(I - T)x_k.
\]
Then
\[
\left| \sum_k \langle k^{\frac{1}{2}} (T^*)^{k-1}(I - T^*)y, x_k \rangle \right| = |\langle y, x \rangle| \leq \|x\|_{L^p} \|y\|_{L^{p'}}.
\]
Moreover $x \in \text{Ran}(I - T)$ hence applying (i)', we deduce
\[
\left| \sum_k \langle k^{\frac{1}{2}} (T^*)^{k-1}(I - T^*)y, x_k \rangle \right| \lesssim \|y\|_{L^{p'}} \|x\|_{T, 1}.
\]
We will now show an estimate
\[
(3.5) \quad \|x\|_{T, 1} \lesssim \left\| \left( \sum_k |x_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p}.
\]
Then passing to the supremum over all finite sequences $(x_k)_{k \geq 1}$ in the unit ball of $L^p(\Omega; \ell^2)$, we deduce that $\|y\|_{T^*, 1} \lesssim \|y\|_{L^{p'}}$.

To show (3.5), first note that for any integer $m \geq 1$, we may write
\[
m^{\frac{1}{2}} T^{m-1}(I - T)x = \sum_k \frac{m^{\frac{1}{2}} k^{\frac{1}{2}}}{(m + k)^2} (m + k)^2 T^{m+k-2}(I - T)^2 x_k.
\]
Second according to [30], the assumption (i)' implies that $T$ is an $R$-Ritt operator. Hence by Proposition 2.8, the set \{(m + k)^2 T^{m+k-2}(I - T)^2 : m, k \geq 1\} is $R$-bounded. Therefore applying Propositions 2.3 and 2.6 we obtain (3.5). \qed

4. Loose dilations

We will focus on the following notion of dilation for $L^p$-operators.

\textbf{Definition 4.1.} Let $T : L^p(\Omega) \to L^p(\Omega)$ be a bounded operator. We say that it admits a loose dilation if there exist a measure space $(\widetilde{\Omega}, \widetilde{\mu})$, two bounded maps $J : L^p(\Omega) \to L^p(\widetilde{\Omega})$ and $Q : L^p(\widetilde{\Omega}) \to L^p(\Omega)$, as well as an isomorphism $U : L^p(\widetilde{\Omega}) \to L^p(\widetilde{\Omega})$ such that \{$U^n : n \in \mathbb{Z}$\} is bounded and
\[
T^n = QU^n J, \quad n \geq 0.
\]

That notion is strictly weaker than the following more classical one.

\textbf{Remark 4.2.} We say that a bounded operator $T : L^p(\Omega) \to L^p(\Omega)$ admits a strict dilation if there exist a measure space $(\widetilde{\Omega}, \widetilde{\mu})$, two contractions $J : L^p(\Omega) \to L^p(\widetilde{\Omega})$ and $Q : L^p(\widetilde{\Omega}) \to L^p(\Omega)$, as well as an isometric isomorphism $U : L^p(\widetilde{\Omega}) \to L^p(\widetilde{\Omega})$ such that $T^n = QU^n J$ for any $n \geq 0$.

This strict dilation property implies that $T$ is a contraction and that $J$ and $Q^*$ are both isometries.
Conversely in the case \( p = 2 \), Nagy’s dilation Theorem (see e.g. [51, Chapter 1]) ensures that any contraction \( L^2(\Omega) \rightarrow L^2(\Omega) \) admits a strict dilation.

Next, assume that \( 1 < p \neq 2 < \infty \). Then it follows from [1, 2, 10, 42] that \( T: L^p(\Omega) \rightarrow L^p(\Omega) \) admits a strict dilation if and only if there exists a positive contraction \( S: L^p(\Omega) \rightarrow L^p(\Omega) \) such that \( |T(x)| \leq S(|x|) \) for any \( x \in L^p(\Omega) \).

Except for \( p = 2 \) (see Remark 4.3 below), there is no similar description of operators admitting a loose dilation. The general issue behind our investigation is to try to characterize the \( L^p \)-operators which satisfy this property. Theorem 4.8 below gives a satisfactory answer for the class of Ritt operators.

**Remark 4.3.** Let \( H \) be a Hilbert space, let \( T: H \rightarrow H \) be a bounded operator and let us say that \( T \) admits a loose dilation if there exist a Hilbert space \( K \), two bounded maps \( J: H \rightarrow K \) and \( Q: K \rightarrow H \), and an isomorphism \( U: K \rightarrow K \) such that \( \{U^n : n \in \mathbb{Z}\} \) is bounded and \( T^n = QU^nJ \) for any \( n \geq 0 \). Then this property is equivalent to \( T \) being similar to a contraction.

Indeed assume that there exists an isomorphism \( V \in B(H) \) such that \( V^{-1}TV \) is a contraction. By Nagy’s dilation Theorem, that contraction admits a unitary dilation. In other words, there is a unitary \( U \) on a Hilbert space \( K \) containing \( H \), such that \( (V^{-1}TV)^n = qU^nJ \) for any \( n \geq 0 \), where \( j: H \rightarrow K \) is the canonical inclusion and \( q = j^* \) is the corresponding orthogonal projection. We obtain the loose dilation property of \( T \) by taking \( J = jV^{-1} \) and \( Q = Vq \).

The converse uses the notion of complete polynomial boundedness, for which we refer to [40, 41]. Assume that \( T \) admits a loose dilation. Using [41, Cor. 9.4] and elementary arguments, we obtain that \( T \) is completely polynomially bounded. Hence it is similar to a contraction by [40, Cor. 3.5].

According to the above result, the rest of this section is significant only in the case \( 1 < p \neq 2 < \infty \).

Let \( S: \ell^p_\mathbb{Z} \rightarrow \ell^p_\mathbb{Z} \) denote the natural shift operator given by \( S((t_k)_{k \in \mathbb{Z}}) = ((t_{k-1})_{k \in \mathbb{Z}}) \). For any \( \varphi \) in \( \mathcal{P} \) (the algebra of complex polynomials), we set

\[
\|\varphi\|_p = \|\varphi(S)\|_{B(\ell^p_\mathbb{Z})}.
\]

We recall that if \( \varphi \) is given by \( \varphi(z) = \sum_{k \geq 0} d_k z^k \), then \( \varphi(S) \) is the convolution operator (with respect to the group \( \mathbb{Z} \)) associated to the sequence \( (d_k)_{k \in \mathbb{Z}} \). Alternatively, \( \varphi(S) \) is the Fourier multiplier associated to the restriction of \( \varphi \) to the unitary group \( T \). We refer the reader to [17] for some elementary background on Fourier multiplier theory.

Let us decompose \( (0, \pi) \) dyadically into the following family \( (I_j)_{j \in \mathbb{Z}} \) of intervals:

\[
I_j = \left\{ \left[ \frac{\pi}{2^j}, \frac{\pi}{2^{j+1}} \right) \right. \text{ if } j \geq 0 \right.
\left. \left[ 2^{j-1} \pi, 2^j \pi \right) \right. \text{ if } j < 0. \right. \]

Then we denote by \( \Delta_j \) the corresponding arcs of \( T \):

\[
\Delta_j = \{ e^{it} : t \in -I_j \cup I_j \}.
\]

We will use the following version of the Marcinkiewicz multiplier theorem (see [3] Thm. 4.3] and also [17]).
Theorem 4.4. Let $1 < p < \infty$. Let $\phi \in L^\infty(\mathbb{T})$ and assume that $\phi$ has uniformly bounded variations over the $(\Delta_j)_{j \in \mathbb{Z}}$. Then $\phi$ induces a bounded Fourier multiplier $M_\phi : \ell^p_\mathbb{Z} \to \ell^p_\mathbb{Z}$ and we have

$$
\|M_\phi\|_{B(\ell^p_\mathbb{Z})} \leq C_p \left( \|\phi\|_{L^\infty(\mathbb{T})} + \sup \{ \text{var} (\phi, \Delta_j) : j \in \mathbb{Z}\} \right),
$$

where $\text{var} (\phi, \Delta_j)$ is the usual variation of $\phi$ over $\Delta_j$ and the constant $C_p$ only depends on $p$.

For convenience, Definition 4.5 and Proposition 4.7 below are given for an arbitrary Banach space $X$, although we are mostly interested in the case when $X$ is an $L^p$-space.

Definition 4.5. We say that a bounded operator $T : X \to X$ is $p$-polynomially bounded is there exists a constant $C \geq 1$ such that

$$
\|\varphi(T)\| \leq C\|\varphi\|_p
$$

for any complex polynomial $\varphi$.

The following connection with dilations is well-known.

Proposition 4.6. If $T : L^p(\Omega) \to L^p(\Omega)$ admits a loose dilation, then it is $p$-polynomially bounded.

Proof. Assume that $T$ satisfies the dilation property given by Definition 4.1. Then for any $\varphi \in \mathcal{P}$, we have $\varphi(T) = Q\varphi(U)J$, hence

$$
\|\varphi(T)\| \leq \|Q\|\|J\|\|\varphi(U)\|.
$$

Moreover by the transference principle (see [11, Thm. 2.4]), $\|\varphi(U)\| \leq K^2\|\varphi\|_p$, where $K \geq 1$ is any constant such that $\|U^n\| \leq K$ for any integer $n$. This yields the result. \(\square\)

We will see in Section 5 that the converse of that proposition does not hold true.

The above proof shows that if $T : L^p(\Omega) \to L^p(\Omega)$ admits a strict dilation, then $\|\varphi(T)\| \leq \|\varphi\|_p$ for any $\varphi \in \mathcal{P}$, a very classical fact. The famous Matsaev Conjecture asks whether this inequality holds for any $L^p$-contraction $T$ (even those with no strict dilation). This was disproved very recently by Drury in the case $p = 4$ [16]. It is unclear whether there exists an $L^p$-contraction $T$ satisfying $\|\varphi(T)\| \leq \|\varphi\|_p$ for any $\varphi \in \mathcal{P}$, without admitting a strict dilation.

Proposition 4.7. Let $T : X \to X$ be a $p$-polynomially bounded operator. Then $I - T$ is sectorial and for any $\theta \in (\frac{\pi}{2}, \pi)$, it admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus.

Proof. Since $T$ is $p$-polynomially bounded, it is power bounded hence $\sigma(T) \subset \mathbb{D}$. We can thus define $\varphi(T)$ for any rational function with poles outside $\mathbb{D}$. Furthermore (4.2) holds as well for such functions, by approximation.

We fix two numbers $\frac{\pi}{2} < \theta < \theta' < \pi$ and we let (see Figure 2):

$$
\mathbb{D}_\theta = D(-i\cot(\theta), \frac{1}{\sin(\theta)}) \cup D(i\cot(\theta), \frac{1}{\sin(\theta)}).
$$
Clearly $\mathbb{D}_\theta$ contains $\mathbb{D}$. For any $t \in (-\pi, 0) \cup (0, \pi)$, let $r(t)$ denote the radius of the largest open disc centered at $e^{it}$ and included in $\mathbb{D}_\theta$. If $t$ is positive and small enough, we have

$$r(t) = \frac{1}{\sin(\theta)} - |e^{it} + i \cot(\theta)|$$

$$= \frac{1}{\sin(\theta)} - \sqrt{\cos^2(t) + (\sin(t) + \cot(\theta))^2}$$

$$= \frac{1}{\sin(\theta)} \left(1 - \sqrt{1 + 2 \sin(t) \sin(\theta) \cos(\theta)}\right)$$

$$= - \cos(\theta) t + \frac{1}{2} \left(\sin(\theta) \cos^2(\theta)\right) t^2 + O(t^3).$$

Consequently, we have $r(t) > - \cos(\theta) t$ for $t > 0$ small enough. We deduce that if $j < 0$ with $|j|$ large enough and $t \in I_j$, we have

$$D\left(e^{it}, - \cos(\theta) \frac{\pi}{2|j|+1}\right) \subset \mathbb{D}_\theta.$$ 

The same holds for $t \in -I_j$. Moreover the intervals $I_j$ and $I_{-j}$ of the dyadic decomposition have length equal to $\frac{\pi}{2|j|+1}$. Hence for any rational function $\varphi$ with poles outside $\overline{\mathbb{D}}_\theta$ and any $j < 0$ with $|j|$ large enough, we obtain that

$$\text{var}\left(\varphi|_{\Delta}, \Delta_j\right) = \int_{-I_j \cup I_j} |\varphi'(e^{it})| \, dt$$

$$\leq \int_{-I_j \cup I_j} \|\varphi\|_{H^\infty(\mathbb{D}_\theta)} \frac{\pi}{2|j|+1} \, dt \quad \text{by (4.3) and Cauchy’s inequalities,}$$

$$\leq \frac{\pi}{2|j|} \cdot \frac{\|\varphi\|_{H^\infty(\mathbb{D}_\theta)}}{- \cos(\theta) \frac{\pi}{2|j|+1}} = \frac{2\pi \|\varphi\|_{H^\infty(\mathbb{D}_\theta)}}{- \cos(\theta)}.$$
We have a similar result if \( j \geq 0 \) and large enough. Applying Theorem 4.4, we deduce a uniform estimate
\[
\| \varphi(S) \|_{B(\ell^p_2)} \lesssim \| \varphi \|_{H^\infty(\mathbb{D}_\theta)}.
\]
Combining with (4.2) -as explained at the beginning of this proof- we obtain the existence of a constant \( K \geq 0 \) such that for any rational function \( \varphi \) with poles outside \( \overline{\mathbb{D}_\theta} \),
\[
\| \varphi(T) \|_{B(X)} \leq K \| \varphi \|_{H^\infty(\mathbb{D}_\theta)}.
\]
Note that we have the following inclusion:
\[
1 - \mathbb{D}_\theta \subset \Sigma_\theta.
\]
Then let \( \mathcal{R}_{\theta'} \) be the algebra of all rational functions with poles outside \( \overline{\Sigma_{\theta'}} \) and with a nonpositive degree. We deduce from above that for any \( f \in \mathcal{R}_{\theta'} \),
\[
\| f(I - T) \| \leq K \| f(1 - \cdot) \|_{H^\infty(\mathbb{D}_\theta)} \leq K \| f \|_{H^\infty(\Sigma_\theta)}.
\]
According to [32, Prop. 2.10], this readily implies that \( I - T \) is sectorial and admits a bounded \( H^\infty(\Sigma_{\theta'}) \) functional calculus. \( \square \)

**Theorem 4.8.** Let \( T : L^p(\Omega) \to L^p(\Omega) \) be a bounded operator, with \( 1 < p < \infty \). The following assertions are equivalent.

(i) The operator \( T \) and its adjoint \( T^* : L^p(\Omega) \to L^{p'}(\Omega) \) both satisfy uniform estimates
\[
\| x \|_{T,1} \lesssim \| x \|_{L^p} \quad \text{and} \quad \| y \|_{T^*,1} \lesssim \| y \|_{L^{p'}}
\]
for \( x \in L^p(\Omega) \) and \( y \in L^{p'}(\Omega) \).

(ii) The operator \( T \) is \( R \)-Ritt and admits a loose dilation.

(iii) The operator \( T \) is \( R \)-Ritt and \( p \)-polynomially bounded.

**Proof.** That (ii) implies (iii) follows from Proposition 4.6.

Assume (iii). By Proposition 4.7, \( I - T \) admits a bounded \( H^\infty(\Sigma_\theta) \) functional calculus for any \( \theta > \frac{\pi}{2} \). Since \( T \) is \( R \)-Ritt, this implies (i) by Theorem 3.1.

Assume (i). It follows from [30] (see Theorem 3.1) that \( T \) is an \( R \)-Ritt operator. Thus we only need to establish the dilation property of \( T \). Since \( T \) is \( R \)-Ritt, Theorem 3.3 ensures that the square functions \( \| \cdot \|_{T,1} \) and \( \| \cdot \|_{T^{-1}} \) are equivalent on \( L^p(\Omega) \). Likewise, \( \| \cdot \|_{T^*,1} \) and \( \| \cdot \|_{T^{-1}} \) are equivalent on \( L^{p'}(\Omega) \). Consequently the assumption (i) implies the existence of a constant \( C \geq 1 \) such that
\[
\| x \|_{T,1} \leq C \| x \|_{L^p} \quad \text{and} \quad \| y \|_{T^*,1} \leq C \| y \|_{L^{p'}}
\]
for any \( x \in L^p(\Omega) \) and any \( y \in L^{p'}(\Omega) \).

We will use the direct sum decomposition (3.1), as well as the analogous decompositions of \( L^{p'}(\Omega) \) corresponding to \( T^* \). According to the above estimates, we may define two bounded maps
\[
j_1 : \text{Ran}(I - T) \to L^p(\Omega; \ell^2_\Omega) \quad \text{and} \quad j_2 : \text{Ran}(I - T^*) \to L^{p'}(\Omega; \ell^2_\Omega)
\]
as follows. For any \( x \in \text{Ran}(I - T) \) and any \( y \in \text{Ran}(I - T^*) \), we set
\[
x_k = T^{k-1}(I - T)^{\frac{1}{2}}x \quad \text{and} \quad y_k = (T^*)^{k-1}(I - T^*)^{\frac{1}{2}}y
\]
if $k \geq 0$, we set $x_k = 0$ and $y_k = 0$ if $k < 0$. Then we set
\[ j_1(x) = (x_k)_{k \in \mathbb{Z}} \quad \text{and} \quad j_2(y) = (y_k)_{k \in \mathbb{Z}}. \]

Then we let $J_1 : L^p(\Omega) \to L^p(\Omega) \oplus L^p(\Omega; \ell_2^2)$ be the linear map taking any $x \in \text{Ker}(I - T)$ to $(x, 0)$ and any $x \in \text{Ran}(I - T)$ to $(0, j_1(x))$. We define $J_2 : L^p(\Omega) \to L^p(\Omega) \oplus L^p(\Omega; \ell_2^2)$ in a similar way.

For any $x \in \text{Ran}(I - T)$ and $y \in \text{Ran}(I - T^*)$, we have
\[
\langle J_1 x, J_2 y \rangle = \sum_{k=1}^{\infty} \langle T^{k-1}(I - T)^{\frac{1}{2}} x, (T^*)^{k-1}(I - T^*)^{\frac{1}{2}} y \rangle \\
= \sum_{k=1}^{\infty} \langle T^{2(k-1)}(I - T) x, y \rangle \\
= \sum_{k=1}^{\infty} \langle T^{2(k-1)}(I - T^2)(I + T)^{-1} x, y \rangle.
\]

For any integer $N \geq 1$
\[
\sum_{k=1}^{N} T^{2(k-1)}(I - T^2) = I - T^{2N}.
\]

Furthermore, $(I + T)^{-1} x$ belongs to $\text{Ran}(I - T)$ and the sequence $(T^n)_{n \geq 0}$ strongly converges to 0 on that subspace of $L^p(\Omega)$. Hence
\[
\langle J_1 x, J_2 y \rangle = \langle (I + T)^{-1} x, y \rangle.
\]

Let $\Theta : L^p(\Omega) \to L^p(\Omega)$ be the linear map taking any $x \in \text{Ran}(I - T)$ to $(I + T)x$ and any $x \in \text{Ker}(I - T)$ to itself. Then it follows from the above calculation that
\begin{equation}
\Theta J_2 J_1 = I_{L^p(\Omega)}.
\end{equation}

Let
\[
Z = L^p(\Omega) \oplus L^p(\Omega; \ell_2^2),
\]
and let $U : Z \to Z$ be the linear map which takes any $x \in L^p(\Omega)$ to itself and any sequence $(x_k)_{k \in \mathbb{Z}}$ in $L^p(\Omega; \ell_2^2)$ to the shifted sequence $(x_{k+1})_{k \in \mathbb{Z}}$. Next let $P : Z \to Z$ be the linear map which takes any $x \in L^p(\Omega)$ to itself and any sequence $(x_k)_{k \in \mathbb{Z}}$ in $L^p(\Omega; \ell_2^2)$ to the truncated sequence $(\ldots, 0, \ldots, 0, x_0, x_1, \ldots, x_k, \ldots)$. By construction, we have
\begin{equation}
PU^n J_1 = J_1 T^n, \quad n \geq 0.
\end{equation}

We also have $J_2^n P = J_2^n$ hence setting $J = J_1 : L^p(\Omega) \to Z$ and $Q = \Theta J_2^n : Z \to L^p(\Omega)$, we deduce from (4.4) and (4.5) that $T^n = QU^n J$ for any $n \geq 0$. Furthermore, $U$ is an isometric isomorphism on $Z$. Thus we have established that $T$ satisfies the dilation property stated in Definition 4.1, except that the dilation space is $Z$ instead of being an $L^p$-space.

It is easy to modify the construction to obtain a dilation through an $L^p$-space, as follows. First recall that using for example Gaussian variables, one can isometrically represent $\ell_2^2$ as
a complemented subspace of an $L^p$-space (see e.g. [43, Chapter 5]). The space $Z$ can be therefore represented as well as a complemented subspace of an $L^p$-space. Thus we have

$$Z \oplus W = L^p(\tilde{\Omega})$$

for an appropriate measure space $(\tilde{\Omega}, \tilde{\mu})$ and some Banach space $W$. Let $J': L^p(\Omega) \to L^p(\tilde{\Omega})$ be defined by $J'(x) = (J(x), 0)$, let $U': L^p(\tilde{\Omega}) \to L^p(\tilde{\Omega})$ be defined by $U'(z, w) = (Uz, w)$ and let $Q': L^p(\tilde{\Omega}) \to L^p(\tilde{\Omega})$ be defined by $Q'(z, w) = Q(z)$. Then $U'$ is an isomorphism, $(U'^n)_{n \in \mathbb{Z}}$ is bounded and $Q'U^mJ' = T^n$ for any $n \geq 0$.

5. Comparing $p$-boundedness properties

In this section we will consider an $L^p$-analog of complete polynomial boundedness going back to [45] (see also [47, Chap. 8]) and give complements to the results obtained in the previous section. In particular we will show the existence of $p$-polynomially bounded operators $L^p \to L^p$ without any loose dilatation.

In the sequel we assume that $1 \leq p < \infty$. Let $n \geq 1$ be an integer. For any vector space $V$, we let $M_n(V)$ denote the space of $n \times n$ matrices with entries in $V$. When $V = B(X)$ for some Banach space $X$, we equip this space with a specific norm, as follows. For any $[T_{ij}]_{1 \leq i,j \leq n}$ in $M_n(B(X))$, we set

$$\| [T_{ij}] \|_{p, M_n(B(X))} = \sup \left\{ \left( \sum_{i=1}^n \| \sum_{j=1}^n T_{ij}(x_j) \|_X^p \right)^{\frac{1}{p}} : x_1, \ldots, x_n \in X, \sum_{j=1}^n \| x_j \|_X^p \leq 1 \right\}. \tag{5.1}$$

In other words, we regard $[T_{ij}]$ as an operator $\ell^p_n(X) \to \ell^p_n(X)$ in a natural way and the norm of the matrix is defined as the corresponding operator norm.

Let $X, Y$ be two Banach spaces, let $V \subset B(X)$ be a subspace and let $u: V \to B(Y)$ be a linear mapping. We say that $u$ is $p$-completely bounded if there exists a constant $C \geq 0$ such that

$$\| [u(T_{ij})] \|_{p, M_n(B(Y))} \leq C \| [T_{ij}] \|_{p, M_n(B(X))}$$

for any $n \geq 1$ and any matrix $[T_{ij}]$ in $M_n(V)$. In this case, we let $\| u \|_{pcb}$ denote the smallest possible $C$.

Let us regard the vector space $\mathcal{P}$ of all complex polynomials as a subspace of $B(\ell^p_\mathbb{Z})$, by identifying any $\varphi \in \mathcal{P}$ with the operator $\varphi(S)$. Accordingly for any $[\varphi_{ij}]$ in $M_n(\mathcal{P})$, we set

$$\| [\varphi_{ij}] \|_p = \| [\varphi_{ij}(S)] \|_{p, M_n(B(\ell^p_\mathbb{Z}))}.$$  

This extends [41] to matrices. We say that a bounded operator $T: Y \to Y$ is $p$-completely polynomially bounded if the natural mapping $u: \mathcal{P} \to B(Y)$ given by $u(\varphi) = \varphi(T)$ is $p$-completely bounded. This is equivalent to the existence of a constant $C \geq 1$ such that

$$\| [\varphi_{ij}(T)] \|_{p, M_n(B(Y))} \leq C \| [\varphi_{ij}] \|_p$$

for any matrix $[\varphi_{ij}]$ of complex polynomials.

When $p = 2$ and $Y$ is a Hilbert space, the notions of 2-polynomial boundedness and 2-complete polynomial boundedness correspond to the usual notions of polynomial boundedness and complete polynomial boundedness from [40, 41]. See [41] for the rich connections.
with operator space theory. The existence of a polynomially bounded operator on Hilbert space which is not completely polynomially bounded is a major result due to Pisier. Indeed this is the heart of his negative solution to the Halmos problem \[16, 47\]. We will show that Pisier’s construction can be transferred to our $L^p$-setting.

We start with an elementary result which is obvious when $p = 2$ but requires attention when $p \neq 2$.

**Lemma 5.1.** Let $N \geq 1$ be an integer, let $H$ be a Hilbert space and let $\pi : B(\ell^2_N) \to B(H)$ be a unital $*$-representation. Then for any $n \geq 1$ and any matrix $[T_{ij}]$ in $M_n(B(\ell^2_N))$, we have

$$\| [T_{ij}] \|_{p,M_n(B(\ell^2_N))} \leq \| [\pi(T_{ij})] \|_{p,M_n(B(H))}.$$ 

**Proof.** As is well-known, there is a Hilbert space $K$ such that

$$H \cong \ell^2_N(K), \quad B(H) \cong B(\ell^2_N) \otimes B(K),$$

and $\pi(T) = T \otimes I_K$ for any $T \in B(\ell^2_N)$ (see e.g. \[13, Cor. III.1.7\]). Consider $[T_{ij}]$ in $M_n(B(\ell^2_N))$ and $x_1, \ldots, x_n$ in $\ell^2_N$. Fix some $e \in K$ with $\|e\| = 1$. Then

$$\sum_i \left\| \sum_j T_{ij}(x_j) \right\|_{\ell^2_N}^p = \sum_i \left\| \sum_j T_{ij}(x_j) \otimes e \right\|_{\ell^2_N(K)}^p$$

$$= \sum_i \left\| \sum_j [\pi(T_{ij})](x_j \otimes e) \right\|_{\ell^2_N(K)}^p$$

$$\leq \left\| [\pi(T_{ij})] \right\|_{p,M_n(B(H))}^p \sum_j \left\| x_j \otimes e \right\|_{\ell^2_N(K)}^p$$

$$\leq \left\| [\pi(T_{ij})] \right\|_{p,M_n(B(H))}^p \sum_j \left\| x_j \right\|_{\ell^2_N}^p,$$

and the result follows at once. \qed

**Proposition 5.2.** Suppose that $1 < p < \infty$. There exists a $p$-polynomially bounded operator $T : L^p([0, 1]) \to L^p([0, 1])$ which is not $p$-completely polynomially bounded.

**Proof.** We need some background on Pisier’s counterexample. We refer to \[47, Chap. 9\] and \[11, Chap. 10\] for a detailed exposition of this example and also to the necessary background on Hankel operators on $B(\ell^2(H))$ and their $B(H)$-valued symbols.

We start with a concrete description of a sequence of operators satisfying the so-called canonical anticommutation relations. Let $I_2$ denote the identity matrix on $M_2$. For any $k \geq 1$, consider the unital embedding $M_{2^k} \hookrightarrow M_{2^k+1} \cong M_{2^k} \otimes M_2$ given by $A \mapsto A \otimes I_2$. The closure of the union of the resulting increasing sequence $(M_{2^k})_{k \geq 1}$ is a $C^*$-algebra. Representing it as an algebra of operators, we obtain a Hilbert space $H$ and an embedding

$$\bigcup_{k \geq 1} \uparrow M_{2^k} \subset B(H)$$

whose restriction to each $M_{2^k}$ is a unital $*$-representation.
Consider the $2 \times 2$ matrices

\[ D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]

For any $k \geq 1$, we set

\[ C_k = E^\otimes (k-1) \otimes D \in M_{2^k}, \]

where $E^\otimes (k-1)$ denotes the tensor product of $E$ with itself $(k-1)$ times. Then following (5.2) we let $\tilde{C}_k$ denote this operator regarded as an element of $B(H)$. The distinction between $C_k$ and $\tilde{C}_k$ may look superfluous. The reason why we need this is that the inclusion providing the identification between $C_k$ and $\tilde{C}_k$ is a $\ast$-representation and a priori, $\ast$-representations are not $p$-complete isometries (i.e. they do not preserve $p$-matrix norms). However using Lemma 5.1, we see that for any $m \geq 1$, for any $n \geq 1$ and for any $a_1, \ldots, a_m \in M_n$,

\begin{equation}
\left\| \sum_{k=1}^m a_k \otimes C_k \otimes I_2^\otimes (m-k) \right\|_{p,M_n(B(\ell^2_m))} \leq \left\| \sum_{k=1}^m a_k \otimes \tilde{C}_k \right\|_{p,M_n(B(H))}.
\end{equation}

The above sequence of matrices has the following remarkable property (see [47, p. 70]): for any complex numbers $\alpha_1, \ldots, \alpha_m$,

\begin{equation}
\left\| \sum_{k=1}^m \alpha_k C_k \otimes I_2^\otimes (m-k) \right\|_{B(\ell^2_m)} = \left( \sum_{k=1}^m |\alpha_k|^2 \right)^{\frac{1}{2}}.
\end{equation}

Let $H = \ell^2(H) \oplus \ell^2(H)$, let $\sigma: \ell^2(H) \to \ell^2(H)$ denote the shift operator, let $\Gamma: \ell^2(H) \to \ell^2(H)$ be the Hankel operator associated to the $B(H)$-valued function $F$ given by

\[ F(t) = \sum_{k=1}^\infty \frac{\tilde{C}_k}{2^k} e^{-i(2^k-1)t}, \]

and let $T \in B(H)$ be the operator given by

\[ T = \begin{bmatrix} \sigma^* & \Gamma \\ 0 & \sigma \end{bmatrix}. \]

Pisier proved that this operator is polynomially bounded without being completely polynomially bounded. Since $\| \|_2 \leq \| \|_p$ on $\mathcal{P}$, the linear mapping

\[ u: (\mathcal{P}, \| \|_p) \longrightarrow B(H), \quad u(\varphi) = \varphi(T), \]

is therefore bounded. Our aim is now to show that $u$ is not $p$-completely bounded. We consider the auxiliary mapping $w: \mathcal{P} \to B(H)$ defined by letting

\[ w \left( \sum_{k \geq 0} d_k z^k \right) = \sum_{k \geq 0} d_k \tilde{C}_k \]

for any finite sequence $(d_k)_{k \geq 0}$ of complex numbers.

Let $j: H \to \ell^2(H)$ be the isometric embedding given by $j(x) = (x, 0, \ldots, 0, \ldots)$. Then let $v: B(\ell^2(H)) \to B(H)$ be defined by letting $v(R) = j^* R j$ for any $R \in B(H)$. It is easy to
By interpolation, we deduce that check that $v$ is $p$-completely bounded, with $\|v\|_{\text{pcb}} = 1$. On the other hand, for any $\varphi \in \mathcal{P}$, we have

$$\varphi(T) = \begin{bmatrix} \varphi(\sigma^{*}) & \Gamma \varphi'(\sigma) \\ 0 & \varphi(\sigma) \end{bmatrix},$$

see [47, (9.7)]. Let $\tilde{u} : \mathcal{P} \to \mathcal{B}(\ell^2(H))$ be defined by $\tilde{u}(\varphi) = \Gamma \varphi'(\sigma)$. Then the argument in the proof of [47, Thm. 9.7] shows that $w = v\tilde{u}$. Thus if $u$ were $p$-completely bounded, then $w$ would be $p$-completely bounded as well. Let us show that this does not hold true.

Note that for any Banach space $X$, for any integer $N \geq 1$, for any $T \in \mathcal{B}(X)$ and for any $A \in \mathcal{B}(\ell^1_N)$, we have

$$\|A \otimes T : \ell^1_N(X) \to \ell^1_N(X)\| = \|A\|_{\mathcal{B}(\ell^1_N)} \|T\|_{\mathcal{B}(X)}.$$ 

This can be be seen as a consequence of the fact that $\ell^1_N(X)$ is the projective tensor product of $\ell^1_N$ and $X$, see [15, Chapter VIII], however an elementary proof is also possible (we leave this to the reader).

Let $m \geq 1$. Clearly $\|E\|_{\mathcal{B}(\ell^1_2)} = \|D\|_{\mathcal{B}(\ell^1_2)} = 1$. Hence applying the above property we have $\|C_k\|_{\mathcal{B}(\ell^1_k)} = \|E\|_{\mathcal{B}(\ell^1_2)} = 1$ and hence

$$\|C_k \otimes I_2^{\otimes (m-k)} \otimes S^{2k}\|_{\mathcal{B}(\ell^1_{m}(\ell^1_2))} = 1, \quad k = 1, \ldots, n. \tag{5.5}$$

Let $\varphi_m \in M_{2^m} \otimes \mathcal{P}$ be given by

$$\varphi_m(z) = \sum_{k=1}^{m} C_k \otimes I_2^{\otimes (m-k)} z^{2k}.$$ 

By (5.4), we have

$$\|\varphi_m\|_2 = \sup_{|z|=1} \|\varphi_m(z)\|_{\mathcal{B}(\ell^1_{2^m})} = \sup_{|z|=1} \left\| \sum_{k=1}^{m} z^{2k} C_k \otimes I_2^{\otimes (m-k)} \right\|_{\mathcal{B}(\ell^1_{2^m})} = \sqrt{m}.$$ 

On the other hand, applying (5.3) we have

$$\|\varphi_m\|_1 = \left\| \sum_{k=1}^{m} C_k \otimes I_2^{\otimes (m-k)} \otimes S^{2k}\right\|_{\mathcal{B}(\ell^1_{2^m}(\ell^1_2))} \leq \sum_{k=1}^{m} \|C_k \otimes I_2^{\otimes (m-k)} \otimes S^{2k}\|_{\mathcal{B}(\ell^1_{2^m}(\ell^1_2))} = m.$$ 

By interpolation, we deduce that

$$\|\varphi_m\|_p \leq m^{\frac{1}{p}}. \tag{5.6}$$

Next we have

$$\left(I_{\mathcal{B}(\ell^1_{2^m})} \otimes w\right)(\varphi_m) = \sum_{k=1}^{m} C_k \otimes I_2^{\otimes (m-k)} \otimes \tilde{C}_k. \tag{5.7}$$
Let us estimate the norm of this tensor product in $B(\ell^p_{2m}(H))$. Let $(e_1, e_2)$ denote the canonical basis of $\mathbb{C}^2$. For any $k = 1, \ldots, m$ and any $i_1, \ldots, i_m, j_1, \ldots, j_m$ in $\{1, 2\}$,

$\left\langle \left( \sum_{k=1}^m \left( C_k \otimes I_2^\otimes(m-k) \right) \right) \left( e_{j_1} \otimes \cdots \otimes e_{j_m}, e_{i_1} \otimes \cdots \otimes e_{i_m} \right) \right\rangle$

$= \left\langle E e_{j_1} \otimes \cdots \otimes E e_{j_{k-1}} \otimes D e_{j_k} \otimes e_{j_{k+1}} \otimes \cdots \otimes e_{j_m}, e_{i_1} \otimes \cdots \otimes e_{i_m} \right\rangle$

$= \left\langle (-1)^{\delta_{j_1,2}} e_{j_1} \otimes \cdots \otimes (-1)^{\delta_{j_{k-1},2}} e_{j_{k-1}} \otimes \delta_{j_k,2} e_1 \otimes e_{j_{k+1}} \otimes \cdots \otimes e_{j_m}, e_{i_1} \otimes \cdots \otimes e_{i_m} \right\rangle$

$= \left\langle (-1)^{\delta_{j_1,2}} \cdots (-1)^{\delta_{j_{k-1},2}} \delta_{j_k,2} \langle e_{j_1}, e_{i_1} \rangle \cdots \langle e_{j_{k-1}}, e_{i_{k-1}} \rangle \langle e_1, e_{i_k} \rangle \langle e_{j_{k+1}}, e_{i_{k+1}} \rangle \cdots \langle e_{j_m}, e_{i_m} \rangle \right\rangle$

$= \left\langle (-1)^{\delta_{j_1,2}} \cdots (-1)^{\delta_{j_{k-1},2}} \delta_{j_k,2} \delta_{j_{k+1},i_{k+1}} \cdots \delta_{j_{m-1},i_{m-1}} \delta_{j_m,i_m} \right\rangle$

Hence

$\left\langle \left( \sum_{k=1}^m \left( C_k \otimes I_2^\otimes(m-k) \right) \right) \left( \sum_{j_1, \ldots, j_m=1}^2 e_{j_1} \otimes \cdots \otimes e_{j_m}, e_{i_1} \otimes \cdots \otimes e_{i_m} \right) \right\rangle$

is equal to

$\sum_{k=1}^m \sum_{j_1, \ldots, j_m, i_1, \ldots, i_m=1}^2 \left\langle \left( C_k \otimes I_2^\otimes(m-k) \right) \left( e_{j_1} \otimes \cdots \otimes e_{j_m}, e_{i_1} \otimes \cdots \otimes e_{i_m} \right) \right\rangle^2$

$= \sum_{k=1}^m \sum_{j_1, \ldots, j_m, i_1, \ldots, i_m=1}^2 \left( \delta_{j_k,2} \delta_{i_k} \delta_{j_1,i_1} \cdots \delta_{j_{k-1},i_{k-1}} \delta_{j_{k+1},i_{k+1}} \cdots \delta_{j_{m-1},i_{m-1}} \delta_{j_m,i_m} \right)^2$

$= \sum_{k=1}^m 2^{m-1} = m 2^{m-1}.$

Since the norm of

$\sum_{i_1, \ldots, i_m=1}^2 e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e_{i_1} \otimes \cdots \otimes e_{i_m}$

in $\ell^p_{2m}(\ell^2_{2m})$ (resp. in $\ell^p_{2m}(\ell^2_{2m})$) is equal to

$\left( \sum_{i_1, \ldots, i_m=1}^2 \| e_{i_1} \otimes \cdots \otimes e_{i_m} \|_{\ell^2_{2m}}^p \right)^{\frac{1}{p}} = 2^m$

(resp. $2^m$), we deduce that

$\left\| \sum_{k=1}^m \left( C_k \otimes I_2^\otimes(m-k) \right) \left( C_k \otimes I_2^\otimes(m-k) \right) \right\|_{B(\ell^p_{2m}(\ell^2_{2m}))} \geq \frac{m}{2}.$

Combining with (5.3) and (5.7) we obtain that

$\left\| \left( I_B(\ell^p_{2m}) \otimes w \right)(\varphi_m) \right\| \geq \frac{m}{2}.$
Together with (5.6), this implies that \(w\) is not \(p\)-completely bounded. Thus \(T\) is not \(p\)-completely polynomially bounded.

So we are done except that \(T\) acts on the Hilbert space \(\mathcal{H}\) and not on \(L^p([0,1])\). However arguing as in the last part of the proof of Theorem 4.8 it is easy to pass from \(\mathcal{H}\) to the space \(L^p([0,1])\). □

The proof of Proposition 4.6 actually yields the following stronger result: if an operator \(T: L^p(\Omega) \to L^p(\Omega)\) admits a loose dilation, then it is \(p\)-completely polynomially bounded (details are left to the reader). Hence the above proposition yields the following.

**Corollary 5.3.** There exists a \(p\)-polynomially bounded operator \(T: L^p([0,1]) \to L^p([0,1])\) which does not admit any loose dilation.

Note also that according to Theorem 4.8 and the above observation, no \(R\)-Ritt operator can satisfy Proposition 5.2. Namely, if \(T: L^p(\Omega) \to L^p(\Omega)\) is an \(R\)-Ritt operator and is \(p\)-polynomially bounded, then it is \(p\)-completely polynomially bounded.

Remark 4.3 and the above investigations lead to the following open problem (for \(p \neq 2\)): does any \(p\)-completely polynomially bounded operator \(T: L^p(\Omega) \to L^p(\Omega)\) admit a loose dilation?

In the last part of this section we are going to consider another type of counterexamples. Clearly any \(p\)-polynomially bounded \(T: L^p(\Omega) \to L^p(\Omega)\) is automatically power bounded, that is,

\[\sup_{n \geq 0} \|T^n\| < \infty.\]

The existence of a power bounded operator on Hilbert space which is not polynomially bounded is an old result of Foguel [18]. Our aim is to prove an \(L^p\)-analog of that result. We will actually show a stronger form: there exists a Ritt operator which is not polynomially bounded. To achieve this, we will adapt a technique from [24].

We need some background on Schauder bases and their multipliers that we briefly recall. We let \(v_1\) denote the set of all sequences \((c_n)_{n \geq 0}\) of complex numbers whose variation \(\sum_{n=1}^{\infty} |c_n - c_{n-1}|\) is finite. Any such sequence is bounded and \(v_1\) is a Banach space for the norm

\[\|(c_n)_{n \geq 0}\|_{v_1} = |c_0| + \sum_{n=1}^{\infty} |c_n - c_{n-1}|.\]

Let \((e_n)_{n \geq 0}\) be a Schauder basis on some Banach space \(X\). For any \(n \geq 0\), let \(Q_n: X \to X\) be the projection defined by

\[Q_n \left( \sum_{k=0}^{\infty} a_k e_k \right) = \sum_{k=n}^{\infty} a_k e_k\]

for any converging sequence \(\sum_k a_k e_k\). The sequence \((Q_n)_{n \geq 0}\) is bounded and by a standard Abel summation argument, we have the following.
Lemma 5.4. For any $c = (c_n)_{n \geq 0}$ in $v_1$, there exists a (necessarily unique) bounded operator $T_c : X \to X$ such that

$$T_c \left( \sum_{n=0}^{\infty} a_n e_n \right) = \sum_{n=0}^{\infty} c_n a_n e_n$$

for any converging sequence $\sum_n a_n e_n$. Furthermore,

$$\|T_c\| \leq \left( \sup_{n \geq 0} \|Q_n\| \right) \| (c_n)_{n \geq 1} \|_{v_1}.$$ 

The above operator $T_c$ is called the multiplier associated to the sequence $c$.

Proposition 5.5. Let $1 < p < \infty$.

(1) There exists a Ritt (hence a power bounded) operator on $\ell^2$ which is not $p$-polynomially bounded.

(2) There exists an $R$-Ritt operator on $L^p([0, 1])$ which is not $p$-polynomially bounded.

Proof. (1): We let $(e_n)_{n \geq 0}$ be a Schauder basis of $H = \ell^2$. It is clear that the sequence $(1 - \frac{1}{2^n})_{n \geq 0}$ has a finite variation. According to the above discussion, we let $T : H \to H$ denote the multiplier associated to that sequence.

For any $\theta \in (-\pi, 0) \cup (0, \pi]$, set

$$c(\theta)_n = \frac{1}{e^{i\theta} - (1 - \frac{1}{2^n})}, \quad n \geq 0.$$ 

We have

$$\sum_{n=1}^{+\infty} |c(\theta)_n - c(\theta)_{n-1}| = \sum_{n=1}^{+\infty} \left| \int_{1 - \frac{1}{2^{n+1}}}^{1 - \frac{1}{2^n}} \frac{dt}{(e^{i\theta} - t)^2} \right| 
\leq \sum_{n=1}^{+\infty} \int_{1 - \frac{1}{2^{n+1}}}^{1 - \frac{1}{2^n}} \frac{dt}{|e^{i\theta} - t|^2} 
\leq \int_{0}^{1} \frac{dt}{|e^{i\theta} - t|^2}.$$ 

Let $I(\theta)$ denote the latter integral. It is finite hence $c(\theta) = (c(\theta)_n)_{n \geq 0}$ belongs to $v_1$. It is easy to deduce that $e^{i\theta} - T$ is invertible, the operator $R(e^{i\theta}, T)$ being the multiplier associated to the sequence $c(\theta)$.

For $\theta \neq \pi$, elementary computations yield

$$I(\theta) = \int_{0}^{1} \frac{dt}{(t - \cos(\theta))^2 + \sin^2(\theta)} 
= \frac{1}{\sin(\theta)} \left[ \int_{\frac{-\cos(\theta)}{\sin(\theta)}}^{\frac{1-\cos(\theta)}{\sin(\theta)}} \frac{du}{1+u^2} \right] 
= \frac{\pi - \theta}{2\sin(\theta)}.$$ 

Moreover \(|e^{i\theta} - 1| = 2 \sin\left(\frac{\theta}{2}\right)\), hence

\[
|e^{i\theta} - 1|/I(\theta) = (\pi - \theta) \frac{\sin\left(\frac{\theta}{2}\right)}{\sin(\theta/2)} = \frac{\pi - \theta}{2 \cos\left(\frac{\theta}{2}\right)}.
\]

This is bounded for \(\theta\) varying in \((-\pi, 0) \cup (0, \pi)\). According to Lemma 5.4 this shows that \(\sigma(T) \subset \mathbb{D} \cup \{1\}\) and \(\{(\lambda - 1)R(\lambda, T) : \lambda \in T \setminus \{1\}\}\) is bounded.

Applying the maximum principle to the function \(z \mapsto (1 - z)(I_H - zT)^{-1}\), we deduce that the set \(\{(\lambda - 1)R(\lambda, T) : |\lambda| > 1\}\) is bounded as well, and hence \(T\) is a Ritt operator.

(2): Since all bounded subsets of \(B(\ell^2)\) are \(R\)-bounded, the operator considered in part (1) is automatically an \(R\)-Ritt operator. Then arguing again as in the proof of Theorem 4.8 it is easy to pass from an \(\ell^2\)-operator to an \(L^p([0, 1])\)-operator which is not \(p\)-polynomially bounded although being an \(R\)-Ritt operator.

\[\square\]

6. Generalizations to general Banach spaces

Up to now we have mostly dealt with operators acting on (commutative) \(L^p\)-spaces. In this last section, we shall consider more general Banach spaces, in particular noncommutative \(L^p\)-spaces. We aim at extending our main results from Sections 3 and 4 to this broader context.

We will use classical notions from Banach space theory such as cotype, \(K\)-convexity and the UMD property. We refer the reader to [8, 14, 43] for background.

In accordance with (2.1), we are going to extend the definitions (1.1) and (1.3) to arbitrary Banach spaces using Rademacher averages. Recall Section 2 for notation. The use of such averages as a substitute of square functions on abstract Banach spaces is a classical and fruitful principle. See e.g. [22, 23, 30].

Let \(X\) be a Banach space, let \(T : X \to X\) be any bounded operator and let \(x \in X\). Consider the element \(x_k = k^{\frac{1}{2}}(T^k(x) - T^{k-1}(x))\) for any \(k \geq 1\). If the series \(\sum_k \varepsilon_k \otimes x_k\) converges in \(L^2(\Omega_0; X)\) then we set

\[
\|x\|_{T, 1} = \left\| \sum_{k=1}^{\infty} k^{\frac{1}{2}} \varepsilon_k \otimes (T^k(x) - T^{k-1}(x)) \right\|_{\text{Rad}(X)}.
\]

We set \(\|x\|_{T, 1} = \infty\) otherwise. Likewise, if \(T\) is a Ritt operator and \(\alpha > 0\) is a positive real number, then we set

\[
\|x\|_{T, \alpha} = \left\| \sum_{k=1}^{\infty} k^{\alpha - \frac{1}{2}} \varepsilon_k \otimes T^{k-1}(I - T)^\alpha x \right\|_{\text{Rad}(X)}.
\]
if the corresponding series converges in $L^2(\Omega_0; X)$, and $\|x\|_{T,\alpha} = \infty$ otherwise. The following extends Theorem \[3.3\]

**Theorem 6.1.** Assume that $X$ is reflexive and has a finite cotype. Let $T : X \to X$ be an $R$-Ritt operator. Then for any $\alpha > 0$ and $\beta > 0$, we have an equivalence

$$\|x\|_{T,\alpha} \approx \|x\|_{T,\beta}, \quad x \in X.$$ 

**Proof.** We noticed in Section 2 that if $X$ has a finite cotype, then Rademacher averages and Gaussian averages are equivalent on $X$.

Furthermore, the reflexivity of $X$ ensures that it satisfies the Mean Ergodic Theorem. We thus have

$$X = \text{Ker}(I - T) \oplus \overline{\text{Ran}(I - T)}.$$

Lastly, since $X$ has a finite cotype, it cannot contain $c_0$ (as an isomorphic subspace). Hence by \[27\], a series $\sum_k \varepsilon_k \otimes x_k$ converges in $L^2(\Omega_0; X)$ if (and only if) its partial sums are uniformly bounded, that is, there is a constant $K \geq 0$ such that

$$\left\| \sum_{k=1}^N \varepsilon_k \otimes x_k \right\|_{\text{Rad}(X)} \leq K, \quad N \geq 1.$$

With these three properties in hand, it is easy to see that our proof of Theorem \[3.3\] extends verbatim to the general case. \hfill \square

In the rest of this section we are going to focus on noncommutative $L^p$-spaces. We let $M$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace and for any $1 \leq p < \infty$, we let $L^p(M)$ denote the associated (noncommutative) $L^p$-space. We refer to \[49\] for background and information on these spaces. Any element of $L^p(M)$ is a (possibly unbounded) operator and for any such $x$, we set

$$|x| = (x^*x)^{\frac{1}{2}}.$$

We recall the noncommutative analog of (2.1) from \[28\] (see also \[29\]). For finite families $(x_k)_k$ of $L^p(M)$, we have the following equivalences. If $2 \leq p < \infty$, then

\begin{equation}
(6.1) \quad \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(L^p(M))} \approx \max \left\{ \left\| \left( \sum_k |x_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)}, \left\| \left( \sum_k |x_k^*|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \right\}.
\end{equation}

If $1 < p \leq 2$, then

\begin{equation}
(6.2) \quad \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(L^p(M))} \approx \inf \left\{ \left\| \left( \sum_k |u_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} + \left\| \left( \sum_k |v_k^*|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \right\},
\end{equation}

where the infimum runs over all possible decompositions $x_k = u_k + v_k$ in $L^p(M)$.

Let $T : L^p(M) \to L^p(M)$ be a bounded operator. We say that $T$ admits a noncommutative loose dilation if there exist a von Neumann algebra $\tilde{M}$, an isomorphism $U : L^p(\tilde{M}) \to L^p(\tilde{M})$ such that the set $\{U^n : n \in \mathbb{Z}\}$ is bounded and two bounded maps $L^p(M) \xrightarrow{J} L^p(\tilde{M})$ and $L^p(\tilde{M}) \xrightarrow{Q} L^p(M)$ such that $T^n = QU^nJ$ for any integer $n \geq 0$. We say that $T$ admits a noncommutative strict dilation if this holds true for an isometric isomorphism $U$ and two
contractions $J$ and $Q$. As opposed to the commutative case (see Remark 4.2), there is no characterization of contractions $T: L^p(M) \to L^p(M)$ which admit a noncommutative strict dilation. The gap with the commutative situation is illustrated by the following result [21 Thm 5.1]: for any $p \neq 2$, there exist a completely positive contraction on some finite dimensional noncommutative $L^p$ which does not admit any noncommutative strict dilation.

We now turn to loose dilations. In the commutative setting, the following proposition is a combination of Propositions 4.6 and 4.7.

Proposition 6.2. Let $T: L^p(M) \to L^p(M)$, with $1 < p < \infty$. If $T$ admits a noncommutative loose dilation, then $I - T$ is sectorial and admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta \in \left(\frac{\pi}{2}, \pi\right)$.

Proof. Let us explain how to adapt the ‘commutative’ proof to the present setting.

First we extend the definition (4.1) as follows. For any Banach space $X$, let $S_X: \ell_\infty(Z) \to \ell_\infty(Z)$ denote the shift operator. Then for any $\phi \in \mathcal{P}$, we set

$$\|\phi\|_{p,X} = \left\| \phi(S_X) \right\|_{B(\ell_\infty(Z))}.$$ 

It follows from [6, Thm. 4.3] that if $X$ is UMD, then Theorem 4.4 holds as well for scalar valued Fourier multipliers on $\ell_\infty(Z)$. In this case, the argument in the proof of Proposition 4.7 leads to the following: for any $\theta \in \left(\frac{\pi}{2}, \pi\right)$, there is an estimate

$$\|\phi\|_{p,X} \lesssim \|\phi\|_{H^\infty(D_\theta)}$$

for rational functions $\phi$ with poles outside $D_\theta$.

Second we note that if $U: L^p(\hat{\tilde{M}}) \to L^p(\hat{\tilde{M}})$ is an isomorphism such that $K = \sup \{\|U^n\| : n \in \mathbb{Z}\} < \infty$, then the vectorial version of the transference principle (see [4, Thm. 2.8]) ensures that for any $\phi$ as above, we have

$$\|\phi(U)\| \leq K^2 \|\phi\|_{p,L^p(\hat{\tilde{M}})}.$$ 

Assume now that $T: L^p(M) \to L^p(M)$ admits a noncommutative loose dilation. Noncommutative $L^p$-spaces are UMD hence property (6.3) applies to them. Hence arguing as in proof of Proposition 4.6 we find an estimate

$$\|\phi(T)\| \lesssim \|\phi\|_{H^\infty(\Sigma_\theta)}$$

for rational functions $\phi$ with poles outside $\overline{D_\theta}$. Finally the argument at the end of the proof of Proposition 4.7 yields that $I - T$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \frac{\pi}{2}$. We skip the details. \qed

We are now ready to give the noncommutative analog of Theorem 4.8.

Theorem 6.3. Let $T: L^p(M) \to L^p(M)$ be a bounded operator, with $1 < p < \infty$.

1. The following assertions are equivalent.
   (i) The operator $T$ is R-Ritt and admits a noncommutative loose dilation.
   (ii) The operator $T$ and its adjoint $T^*: L^p'(M) \to L^p'(M)$ both satisfy uniform estimates

$$\|x\|_{T^*,1} \lesssim \|x\|_{L^p(M)} \quad \text{and} \quad \|y\|_{T,1} \lesssim \|y\|_{L^p'(M)}$$

for $x \in L^p(M)$ and $y \in L^p'(M)$. 

(2) Assume that \( p \geq 2 \). Then the above conditions are equivalent to the existence of a constant \( C \geq 1 \) for which the following two properties hold.

(iii) For any \( x \in L^p(M) \),
\[
\left\| \left( \sum_{k=1}^{\infty} k |T^k(x) - T^{k-1}(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \leq C \|x\|_{L^p(M)}
\]
and
\[
\left\| \left( \sum_{k=1}^{\infty} k |(T^k(x) - T^{k-1}(x))^*|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \leq C \|x\|_{L^p(M)}.
\]

(iii)* For any \( y \in L^{p'}(M) \), there exist two sequences \((u_k)_{k \geq 1}\) and \((v_k)_{k \geq 1}\) of \( L^p(M) \) such that
\[
\left\| \left( \sum_{k=1}^{\infty} |u_k|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(M)} \leq C \|y\|_{L^{p'}(M)}, \quad \left\| \left( \sum_{k=1}^{\infty} |v_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \leq C \|y\|_{L^{p'}(M)},
\]
and
\[
u_k + v_k = k^{\frac{1}{2}}(T^*(y) - T^{*(k-1)}(y)) \quad \text{for any } k \geq 1.
\]

Proof. Theorem 5.1 holds as well on noncommutative \( L^p \)-spaces, by [30]. Combining that result with Proposition 6.2, we obtain that (i) implies (ii).

Assume (ii) and suppose for simplicity that \( I - T \) is 1-1 (the changes to treat the general case are minor ones). By Theorem 6.1 we have uniform estimates
\[
\|x\|_{T^*\frac{1}{2}} \lesssim \|x\|_{L^p(M)} \quad \text{and} \quad \|y\|_{T^{*\frac{1}{2}}} \lesssim \|y\|_{L^{p'}(M)}
\]
for \( x \in L^p(M) \) and \( y \in L^{p'}(M) \). As in the proof of Theorem 4.8, we may therefore define \( J_1 : L^p(M) \to \text{Rad}(L^p(M)) \) and \( J_2 : L^{p'}(M) \to \text{Rad}(L^{p'}(M)) \) by setting
\[
J_1(x) = \sum_{k=1}^{\infty} \varepsilon_k \otimes T^{k-1}(I - T)\frac{1}{2} x \quad \text{and} \quad J_2(y) = \sum_{k=1}^{\infty} \varepsilon_k \otimes T^{*(k-1)}(I - T^*)\frac{1}{2} y
\]
for any \( x \in L^p(M) \) and any \( y \in L^{p'}(M) \). Since \( L^p(M) \) is \( K \)-convex, we have a natural isomorphism
\[
(\text{Rad}(L^p(M)))^* \approx \text{Rad}(L^{p'}(M)).
\]
Hence one can consider the composition \( J_2^* J_1 \), it is equal to \((I + T)^{-1}\) and one obtains (i) by simply adapting the proof of Theorem 4.8.

Finally the equivalence between (ii) and (iii)+(iii)* follows from (6.1) and (6.2).

Switching (iii) and (iii)*, we find a version of (2) for the case \( p \leq 2 \).

Remark 6.4.

(1) Let \( T : L^p(\Omega) \to L^p(\Omega) \) be an \( R \)-Ritt operator on some commutative \( L^p \)-space. Combining Theorems 6.3 and 4.8 we find that \( T \) admits a noncommutative loose dilation (if and only if) it admits a commutative one.

(2) Proposition 3.4 holds true on noncommutative \( L^p \)-spaces. The proof is similar, using (6.4) instead of the duality \( L^p(\Omega; \ell^2)^* = L^{p'}(\Omega; \ell^2) \).
We refer the reader to the forthcoming paper [3] for more about square functions associated to Ritt operators on noncommutative $L^p$-spaces.

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