A geometric approach to the equilibrium shapes of self-gravitating fluids

D. Peralta-Salas*

Departamento de Física Teórica II, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain

Abstract

The classification of the possible equilibrium shapes that a self-gravitating fluid can take in a Riemannian manifold is a classical problem in mathematical physics. In this paper it is proved that the equilibrium shapes are isoparametric submanifolds. Some geometric properties of the equilibrium shapes are also obtained, specifically the relationship with the isoperimetric problem and the group of isometries of the manifold. This work follows the new geometric approach to the problem developed by the author.

1 Introduction

Let \((M, g)\) be an analytic, complete and connected \(n\)-dimensional Riemannian manifold and \(\Omega\) an open connected subset of \(M\) (bounded or not) occupied by a mass of fluid. We say that a fluid is self-gravitating if the only significative forces are its interior pressure and its own gravitation. Depending on whether the gravitational field is modelled by Poisson or Einstein equations we say that the fluid is Newtonian or relativistic. An important problem in Fluid Mechanics consists in studying the shape that a self-gravitating fluid will take when it reaches the equilibrium state. By the term shape of a fluid I mean the topological and geometrical properties of the boundary \(\partial \Omega\). The mathematical description of this kind of fluids

*dperalta@fis.ucm.es
only involves three physical quantities, the gravitational potential, which is a function $f_1 : M \to \mathbb{R}$, and the density and pressure, which are two analytic ($C^\infty$) functions $f_2, f_3 : \Omega \to \mathbb{R}$. The set of partial differential equations that the functions $(f_1, f_2, f_3)$ must verify is of free-boundary type because the domain $\Omega$ is an unknown of the problem. In the relativistic case the metric tensor $g$ is also an unknown and it must satisfy the coupling condition

$$R_{ab} = f_1^{-1} f_{1;ab} + 4\pi (f_2 - f_3) g_{ab},$$

where $R_{ab}$ stands for the Ricci tensor and the semicolon standing for covariant derivative.

The standard approaches to the problem of classifying the equilibrium shapes of $\partial \Omega$ generally employ analytical techniques. In the Newtonian case maximum principles for elliptic equations are used in order to prove the existence of symmetries of the solutions $(f_1, f_2, f_3)$. Lichtenstein [2] and later on Lindblom [3] proved the existence of spherical symmetry (i.e. $\partial \Omega$ is a round sphere) when $M$ is the Euclidean 3-dimensional space, $\Omega$ is bounded and the functions $(f_1, f_2, f_3)$ satisfy some physical constraints. In the relativistic case arguments involving the positive mass theorem are used for obtaining the conformal flatness of the metric tensor. As a consequence of this technique Beig&Simon [4] and Lindblom&Masood-ul-Alam [5] proved again spherical symmetry when $\Omega$ is bounded and the solutions verify hard physical constraints. Despite of these important results many questions remain open: What about Newtonian fluids in Riemannian manifolds? What about unbounded domains $\Omega$? Do the same results hold if we drop the physical constraints?

In this work we follow the geometrical theory for studying equilibrium shapes developed by the author in [1]. The system of equations that we consider is the following, which we set up as problem ($P1$)

\[
\begin{align*}
\Delta f_1 &= F(f_1, f_2, f_3) \text{ in } \Omega \\
H(f_1) \nabla f_3 + G(f_2, f_3) \nabla f_1 &= 0 \text{ in } \Omega \\
f_1 &= c, \ c \in \mathbb{R}, \ \nabla f_1 \neq 0 \text{ and } f_1 \in C^2_t \text{ on } \partial \Omega \\
\Delta f_1 &= 0 \text{ in } M - \bar{\Omega}
\end{align*}
\]

$F$, $G$ and $H$ are analytic functions in their arguments and the symbol $C^2_t$ means that $f_1$ is $C^1$ on the boundary and its tangential second derivatives $f_{1,ij} t^j$ are continuous for any vector field $t = t^i \partial_i$ tangent to the boundary. Recall that $c$ is a constant not a priori prescribed. Note that this set
of equations is a generalization of the equations modelling Newtonian and relativistic fluids.

In section 2 we review some important theorems obtained in reference [1] and prove new results regarding the geometric properties of the equilibrium shapes, in particular that the level sets of the function \( f_1 \) are isoparametric submanifolds. In section 3 we obtain some sufficient conditions for the existence of equilibrium shapes on certain spaces and give some examples of manifolds for which isoparametric functions (and therefore equilibrium shapes) do not exist. Finally in section 4 the relationship between the equilibrium shapes and the Killing vector fields of \( M \) is studied. These results are related to some theorems recently proved by J. Szenthe [6]. We also point out an interesting property regarding isoperimetric domains. All these results are new.

2 Some geometric properties of the equilibrium shapes

Since \( \partial \Omega \) is a level set of \( f_1 \) the approach that we use to understand the equilibrium shapes consists in studying the geometrical properties of the fibres \( f_1^{-1}(c), c \in f_1(M) \). We assume that solutions to (P1) exist and characterize the structure of the level sets of these solutions. This is in strong contrast with the classical approaches where the problem of existence and uniqueness is firstly considered and then the geometrical restrictions arise.

The set formed by the union of all the connected components of the fibres of \( f_1 \) is called the partition of \( M \) induced by \( f_1 \) and is denoted by \( \beta_M(f_1) \). In general the dimension of the leaves of the partition is not constant because there can exist singular fibres (\( \nabla f_1 = 0 \)) and therefore \( \beta_M(f_1) \) is a singular foliation.

In [1] it is proved that the partitions induced by \( f_1, f_2 \) and \( f_3 \) agree fibrewise in \( \Omega \). It is also proved that \( f_1 \) is \( C^\omega \) in \( M - \partial \Omega \). As a consequence of this property the singular set of \( f_1 \) has null-Lebesgue measure and is topologically closed in \( M \).

The partition \( \beta_M(f_1) \) is not analytic across the free-boundary \( \partial \Omega \) but it has the remarkable property of being analytically representable, that is, there exists a function \( I : M \to \mathbb{R} \) analytic in the whole \( M \) such that \( \beta_M(f_1) = \beta_M(I) \). The idea of the proof, which appears in [1], is that the interior
symmetries propagate across the free-boundary and remain symmetries of the exterior solution. Then the partition is reconstructed from these symmetries in such a way that it turns out to be analytic. Note that it is not immediate that a partition which is analytic except for a fibre must be analytically representable, in fact many examples are known [1] in which the partition is not analytically representable across the pathological fibre.

The main theorem in [1], which provides a complete geometrical characterization of the leaves of $\beta_M(f_1)$, is the following:

**Theorem 1.** If $f_1$ is a solution of the problem (P1) then $\beta_M(f_1)$ is an equilibrium partition

The concept of equilibrium partition is introduced in [1]. We say that the analytic function $I : M \to \mathbb{R}$ is of equilibrium if $\Delta I$, $(\nabla I)^2$ and $I$ agree fibrewise in $M$. The partitions induced by equilibrium functions are called equilibrium partitions. Although this definition involves a particular function $I$ the concept of equilibrium partition is mainly geometrical, as the following proposition shows.

**Proposition 1.** Any analytic function representing an equilibrium partition $\Sigma$ is an equilibrium function.

**Proof.** By definition there exists an equilibrium function $I$ representing $\Sigma$. The lemma follows if we manage to prove that any analytic function $\hat{I}$ representing $\Sigma$ is of equilibrium. Suppose that $\nabla I$ does not vanish in certain open subset $U$ (it is always possible by the analyticity of $I$) and consider a local coordinate system $(x^1, \ldots, x^n)$. Assume, without loss of generality, that $I_{x^1} \neq 0$ in $U$, the subscript denoting partial differentiation. Then the implicit function theorem guarantees the following step: $x^1 = I^{-1}(I, x^2, \ldots, x^n) \implies \hat{I} = \hat{I}(I^{-1}(I, x^2, \ldots, x^n), x^2, \ldots, x^n) \equiv F(I, x^2, \ldots, x^n)$. It is easy to check that $F_{x^2} = \ldots = F_{x^n} = 0$. One only has to take into account the implicit function theorem and that $I, \hat{I}$ agree fibrewise. Hence in $U$ we get that $\hat{I} = F(I)$, where $F$ is an analytic function of its argument. Since $I$ is an equilibrium function we have that locally (by the same argument involving the implicit function theorem) $(\nabla I)^2$ and $\Delta I$ are functions of $I$. Now a straightforward computation yields that $(\nabla \hat{I})^2 = F'(I)^2(\nabla I)^2$ and $\Delta \hat{I} = F''(I)(\nabla I)^2 + F'(I)\Delta I$. This implies that locally $(\nabla \hat{I})^2$ and $\Delta \hat{I}$ are functions of $\hat{I}$, and therefore $\hat{I}$ is a local equilibrium function. The globalization of this property follows from an analytical continuation result for
analytic partitions, that is, if $f$ and $g$ are two analytic functions on $M$ which agree fibrewise in certain open subset $U$ then $\beta_M(f) = \beta_M(g)$. Indeed since $\beta_U(f) = \beta_U(g)$ we have that $\text{rank}(df, dg) \leq 1$ in $U$. Since $U$ is an open set and $f, g$ are analytic functions we have that this inequality is satisfied in the whole $M$, $\text{rank}(df, dg) \leq 1$ in $M$. This condition implies [7] that $f$ and $g$ are functionally dependent and so there exists an analytic function $Q : \mathbb{R}^2 \to \mathbb{R}$ such that $Q(f, g) = 0$ in $M$, which implies that the partitions of $f$ and $g$ agree. Application of this result to the triple $(\hat{I}, (\nabla \hat{I})^2, \Delta \hat{I})$ yields that $\hat{I}$ is an equilibrium function.

As a consequence of theorem 1 the problem of classifying the partitions induced by the solutions of $(P1)$ is substituted by the problem of classifying the equilibrium partitions on different spaces. We have reduced the original problem involving a difficult system of PDE to a purely geometrical problem. In the next section we will show examples of manifolds which do not admit equilibrium functions. Thus in these manifolds we geometrically obtain an existence result: $(P1)$ cannot have solutions.

The following theorem characterizes the general properties that all the equilibrium partitions must possess on any Riemannian manifold [1].

**Theorem 2.** The partition induced by any equilibrium function $I$ on $M$ has a trivial fibre bundle local structure, each leaf has constant mean curvature and locally the leaves are geodesically parallel.

By the term trivial fibre bundle local structure I mean that $M$ is divided into a countable number of disconnected, $I$-partitioned (formed by level sets of $I$) open regions $M_i$ such that $M = \bigcup_i M_i \cup C(I)$, $C(I)$ standing for the critical set of $I$ (nowhere dense in $M$), and the equilibrium partition in each $M_i$ is a trivial fibre bundle. Note that each $M_i$ may be made up by several connected components $M_i'$. The interested reader can have a look at reference [11], where other geometrical properties of the equilibrium partitions are obtained in constant curvature, conformally flat and locally symmetric spaces. Note that the geodesical parallelism of equilibrium partitions implies that they are Riemannian (singular) foliations [9, 10], a property which will be important in the last section. In fact theorem 2 can be locally expressed as an equivalence.

**Proposition 2.** A Riemannian codimension 1 (singular) foliation whose non-singular leaves have constant mean curvature is locally an equilibrium partition.
Proof. Consider an open subset $U \subset M$ small enough so that the foliation in $U$ (which we assume regular) can be represented by a function $f$. If $M$ is compact, simply connected and the foliation has trivial holonomy then $f$ is defined on the whole $M$ [3]. If we assume that the foliation is locally trivial (this is the case when $M$ is compact and there are not dense leaves [9, 10]) then we can extend $U$ to a set $\Lambda$ formed by leaves of the foliation in such a way that the first integral $f$ is well defined in the whole $\Lambda$ (in general it will be defined on any globally trivial foliated set $\Lambda$, note that in this case the holonomy is trivial). Since the leaves are parallel then $f$ must satisfy that $(\nabla f)^2 = F(f)$ in $U$. Indeed if $g$ is the metric tensor and $D$ its associated covariant derivative it is immediate that $X((\nabla f)^2) = 2g(D_X\nabla f, \nabla f) = 2g(D_{\nabla f}\nabla f, X)$ for any vector field $X$ on $U$. Since the foliation is geodesically parallel then the gradient lines of $f$ are tangent to geodesics, that is $D_{\nabla f}\nabla f = \lambda \nabla f$ for certain real-valued function $\lambda$ on $U$. Identifying we get that $X((\nabla f)^2) = 2\lambda X(f)$ and therefore the symmetries of $f$ are also symmetries of $(\nabla f)^2$ which implies, via Frobenius theorem, that $(\nabla f)^2$ is a function of $f$. The constancy of the mean curvature $H$ on the leaves is expressed as $H = \text{div}(\frac{\nabla f}{||\nabla f||}) = G(f)$. After some computations, and taking into account that $(\nabla f)^2 = F(f)$, one readily gets that $\Delta f$ is also a function of $f$ in $U$ and hence we obtain the (local) equilibrium property. \[\square\]

Now we prove a result which relates the equilibrium condition to the well known isoparametric condition. Recall that a smooth function $f : M \rightarrow \mathbb{R}$ is called isoparametric if $(\nabla f)^2 = F(f)$ and $\Delta f = G(f)$ in $M$, $F$ and $G$ smooth functions of their argument. This concept was firstly introduced by Cartan [11, 12] and Segre [13] in a purely geometrical context. Two good surveys on this topic are the works of Nomizu [14] and Thorgbersson [15].

**Theorem 3.** An equilibrium function is locally isoparametric.

**Proof.** Assume that the equilibrium function $f$ has $N$ different critical values (since $f$ is analytic the set of critical values is discrete in $\mathbb{R}$) and that $f(M) = (-\infty, +\infty)$. Otherwise the technique can be adapted without problem. In this case $M = \bigcup_{i=1}^{N+1} M_i \cup C(f)$, $C(f)$ standing for the critical set of $f$, and possibly $M_i$ being made up by several connected components $M^j_i$. In the open regions $M^j_i$ the equilibrium function is submersive and the partition is globally trivial, so we can take an open subset $\Lambda \subset M^j_i$ which is $f$-partitioned. An adaptation of an argument applied in the proof of proposition [1] yields that $(\nabla f)^2$ and $\Delta f$ are functions of $f$ in certain open subset.
of $\Lambda$. The globalization of this property to the whole $\Lambda$ stems from the existence of a local transversal (and analytic) curve to the fibres of $f$, which is a consequence of the triviality of the partition. In fact the isoparametric condition holds in each region $M_i^j$ because triviality implies the existence of a global transversal curve and therefore $f$, which is submersive, can be adapted to a global coordinate system in $M_i^j$ [16].

The following example illustrates the fact that the isoparametric character of an equilibrium partition is in general only local.

Example 1. The analytic function $f(x, y) = \cos \sqrt{x^2 + y^2}$ in $(\mathbb{R}^2, \delta)$ is of equilibrium type: the partition is formed by concentric circles and hence by proposition [11] it must be an equilibrium function. On the contrary it is not a global isoparametric function because $(\nabla f)^2 = 1 - f^2$ but $\Delta f$ cannot be globally expressed as a function of $f$ due to the existence of the critical fibres $r = i\pi, i \in \mathbb{N} \cup \{0\}$. Anyway, as proved in the theorem, $\Delta f$ is a well defined function of $f$ in the domains $M_i = \{i\pi < r < (i + 1)\pi\}$ and a straightforward computation yields $\Delta f = -f - \frac{(-1)^i \sqrt{1 - f^2}}{i\pi + \arccos((-1)^i f)}$.

The most remarkable feature of theorem [3] is that the idea of isoparametric submanifold, which was introduced in differential geometry many decades ago, naturally arises in a physical context. It is important to note that other authors have also employed the isoparametric condition in order to study the partitions induced by the solutions of certain PDE [17, 18, 19], but the techniques that we use are completely different to these authors’.

3 The existence problem of the equilibrium shapes

In general it is a difficult task to know whether an equilibrium (or isoparametric) function exists on a given Riemannian manifold. This problem is not only interesting from the mathematical viewpoint but also from the physical one. If certain space does not admit equilibrium partitions then a self-gravitating fluid will never reach static equilibrium, a non-existent result for the set $(P1)$ of PDE induced by the geometrical/topological properties of the manifold. It would be desirable to classify, in certain well defined sense, spaces admitting equilibrium or isoparametric functions, which would
be the suitable spaces for doing relevant physics. This restriction of physically admissible manifolds reminds us of the constraints imposed by general relativity: the geometry must be coupled with the matter. In fact, as we will show in this section, many of the spaces for which we manage to prove the existence of equilibrium functions possess geometric structures linked to the partitions, and without this link no isoparametric functions exist. It is surprising that this property arises without taking into account the coupling of general relativity but only the existence of equilibrium. As a remarkable consequence a Newtonian fluid, which a priori does not impose any constraint on the geometry, will not exist on certain spaces and the existence will be likely related to the coupling of certain geometric structures of \((M, g)\) with the equipotential hypersurfaces, as happens in the Einsteiniann theory. All the restrictions obtained in this section are of geometrical type and it remains open to ascertain whether topological restrictions also exist.

**Proposition 3.** An equilibrium partition with just one focal point (or caustic) \(P \in M\) exists on \((M, g)\) if and only if \(\det(g) = A(r)^2 B(\theta)^2\), where \(\det(g)\) is expressed in polar Riemann coordinates \((r, \theta)\) around \(p\). In this case the equilibrium partition is locally formed by geodesic spheres.

**Proof.** Recall that in polar Riemann coordinates centered at \(P \in M\) the metric tensor is expressed, in certain local neighborhood \(U\), as 
\[
ds^2 = dr^2 + G_{ij}(r, \theta)d\theta^i d\theta^j.
\]
The sufficiency condition stems from the fact that the function \(f = \frac{1}{2}r^2\) is of equilibrium. Indeed \((\nabla f)^2 = r^2 = 2f\) and \(\Delta f = \frac{\partial_r^2 (rA(r))}{A(r)}\)
which is an analytic function of \(r\) because \(A(r) = r^{\alpha-1} + O(r^\alpha)\). Therefore \(r^2\) induces a local equilibrium partition (the geodesic spheres) whose focal point is \(P\). Since polar Riemann coordinates are just local then this partition could not be globally defined except when the exponential map defines a global diffeomorphism from \(\mathbb{R}^n\) onto \(M\). This happens, for example, if the space is simply connected and the sectional curvature is non-positive (Cartan-Hadamard’s theorem). Conversely if one has an equilibrium partition with a caustic formed by the point \(P\) then the geodesical parallelism of the leaves implies that the partition must be formed by geodesic spheres centered at \(P\). This stems from the fact that the focal varieties of Riemannian (singular) submersions are smooth submanifolds of \(M\) and the regular leaves of the partition are tubes (constant distance) over either of the focal varieties \[20\]. On account of proposition \[1\]the function \(f = \frac{1}{2}r^2\) representing the same partition must be of equilibrium. The condition of \(\Delta f\) being a function of
r is expressed as $\partial_r \ln(r \sqrt{\det(g)}) = F(r)$ and a straightforward integration yields that $\det(g)$ must factorize in two functions of $r$ and $\theta$.

An important example in which $\det(g)$ factorizes like in proposition is when the manifold is rotationally symmetric around $P$ and therefore $\det(g) = A(r)^2 \Omega(\theta)^2$, where $\Omega(\theta)^2$ is the determinant of the metric of the Euclidean sphere $S^{n-1} \subset \mathbb{R}^n$ in its usual coordinates. Any constant curvature space satisfies this condition (with respect to any point) so the local equilibrium partitions with $P$ as focal point are the geodesic spheres centered at $P$. For the canonical constant curvature manifolds $S^n, \mathbb{R}^n$ and $\mathbb{H}^n$ the local partitions can be globalized (note that in $S^n$ there will appear a second focal point).

Another consequence is that spaces whose metric does not satisfy the assumption of factorization will not have (local) equilibrium partitions with just one focal point. This property is important from the physical viewpoint because a mass of fluid in general will enclose a contractible domain (for example think of a fluid-composed star) and hence in these spaces the fluid will not be able to reach static equilibrium.

The following proposition establishes a stronger result, the non-existence of equilibrium partitions, whatever the focal sets be, for certain 2-dimensional manifolds.

**Proposition 4.** Let $(\mathbb{R}^2, g)$ be a conformally flat 2-dimensional space. Then equilibrium partitions exist if and only if the conformal factor is an equilibrium function of $(\mathbb{R}^2, \delta)$ and they are the same as in the Euclidean case.

**Proof.** The conformally flat metric takes the canonical form $g = \exp(2\phi)\delta$. A straightforward computation yields that $(\nabla f)^2 = \exp(-2\phi)(\nabla Ef)^2$ and $\Delta f = \exp(-2\phi)\Delta Ef$, where the subscript $E$ means that the corresponding operation must be carried out in the Euclidean space. Locally the equilibrium function $f$ is isoparametric so $\exp(-2\phi)(\nabla Ef)^2 = F(f)$ and $\exp(-2\phi)\Delta Ef = G(f)$. Dividing both equations we get that $\frac{\Delta Ef}{(\nabla Ef)^2} = \frac{G(f)}{F(f)}$. This equation implies that the partition induced by $f$ is the same as the partition induced by certain harmonic function in $(\mathbb{R}^2, \delta)$. Indeed if $\Delta Ef = 0$ and we re-parametrize $\beta_{\mathbb{R}^2}(u)$ by another function $f$ such that $u = T(f)$ then $f$ will verify that $\frac{\Delta Ef}{(\nabla Ef)^2} = -\frac{T''(f)}{T(f)}$. Identifying we conclude that the re-parametrization is given by $-\frac{d}{df} \ln T''(f) = \frac{G(f)}{F(f)}$. The condition of inducing partitions given by harmonic functions is invariant under the change $f = R(h)$, for any $R$, and hence $\frac{\Delta Ef}{(\nabla Ef)^2} = M(h)$. Now the isoparametric
condition for $f$ implies that $R''(h)(\nabla E h)^2 + R'(h)\Delta E h = \hat{G}(h) \exp 2\phi$. Both equations are compatible if and only if $\phi$ is a function of $h$ and therefore the partitions will be isoparametric also in the Euclidean plane. The globalization is immediate for analytic functions.

The implication of this result is remarkable, in 2-dimensional conformally flat spaces there not exist solutions to problem (P1) unless the conformal factor be an Euclidean isoparametric function. In this case the equilibrium partitions are those of the conformal factor and are given, up to rigid motions, by $\beta_{\mathbb{R}^2}(x^2+y^2)$ and $\beta_{\mathbb{R}^2}(x)$. We have obtained this coupling condition without taking into account the Einsteinian coupling between matter and geometry, in particular it holds for Newtonian fluids. In fact the assumption that the conformal factor $\phi$ is a function of the gravitatory potential is very common in the physics literature (see for example [21]). We have hence justified this hypothesis in the 2-dimensional case: it is a geometrical constraint in order that equilibrium solutions exist. Propositions 3 and 4 show that physically relevant manifolds will have to satisfy hard constraints.

For instance in the Riemannian manifold $(\mathbb{R}^2, \exp(y - x^2)(dx^2 + dy^2))$ there not exist equilibrium partitions and therefore a self-gravitating fluid would never reach the static equilibrium, fluid-composed stars would not be possible. It is remarkable to note that this space does not possess (global) Killing vector fields, a fact which is very related to the existence of equilibrium functions, as we will show.

4 Equilibrium shapes, isoperimetric domains and isometries

The results of the preceding section suggest that equilibrium partitions are always linked to certain geometric structures of the manifold. If these geometric structures fail to exist then equilibrium partitions do not exist. For example consider the Euclidean space $(\mathbb{R}^n, \delta)$. It can be proved [2] that the equilibrium partitions of this manifold are geometrically trivial in the sense that all the leaves are globally isometric to $bS^p \times \mathbb{R}^q$, $p$ and $q$ fixed natural numbers such that $p+q = n-1$, $b \geq 0$. These partitions have the remarkable property of being generated by isometries of $(\mathbb{R}^n, \delta)$. In general, as a consequence of proposition 3, the equilibrium partitions with $P$ as focal point are induced by isometric group actions on $M$ whenever the space is rotationally
symmetric around \( P \). This fact suggests that the isometries of the manifold are somehow related to the equilibrium partitions. The following proposition establishes the equivalence between both concepts for 2-dimensional manifolds.

**Proposition 5.** Let \((M,g)\) be a 2-dimensional Riemannian space. Then the equilibrium partitions are 1-dimensional (singular) foliations generated by Killing vector fields of \((M,g)\).

**Proof.** The equilibrium partitions of \( M \) are formed by leaves of dimension 1, except for a nowhere dense and closed subset of singular points. This defines a 1-dimensional (singular) foliation. In dimension 1 mean curvature and Gauss curvature agree and therefore the orbits possess constant Gauss curvature. As a consequence of Gauss theorem we have that the sectional curvature of \((M,g)\) restricted to each leaf is also constant (the intrinsic sectional curvature of a 1-dimensional manifold is always trivial). This implies that the fibres are transitivity lines of an isometric group action on the space and therefore correspond to the orbits of a Killing vector field.

This proposition proves analogous results appearing in proposition 2 and theorem 2, reference [6], in the Riemannian case and without assuming that \( M \) is simply connected. Indeed if \( f \) is a submersive equilibrium function you can adapt it to a (local) coordinate system \((f,g)\). Since \( \partial_g \), tangent to the levels of \( f \), is a Killing then the metric can be locally expressed as a warped product, \( ds^2 = A(f)(df^2 + B(f)dg^2) \). In fact since \( f \) induces a Riemannian (non-singular) foliation with a global transversal curve \( \Sigma \) (diffeomorphic to \( S^1 \) or \( \mathbb{R} \)) then \( M \) can be globally expressed as a product \( \{f = 0\} \times \Sigma \), which just leaves the possibilities \( M \cong \mathbb{R}^2, \mathbb{R} \times S^1 \) and \( S^1 \times S^1 \), and therefore the warped product expression globalizes.

The converse of proposition 5 holds in very general situations, as we prove in the next theorem. Note that for 2-dimensional manifolds which do not admit Killing vector fields equilibrium partitions do not exist. This is directly related to proposition 4 and implies that the spaces \((\mathbb{R}^2, \exp(2\phi)\delta)\) for which \( \phi \) is not an equilibrium function of \((\mathbb{R}^2, \delta)\) do not possess Killing vector fields.

**Theorem 4.** Let \( \Xi = \{\xi_1, \ldots, \xi_p\}, p \geq n - 1 \), be a Lie algebra of Killing vector fields of \((M,g)\). \( \Xi \) satisfies that rank\((\xi_1, \ldots, \xi_p)\) = \( n - 1 \) in \( M \), up to a null measure set, and it generates a closed subgroup of the group of isometries. Then the (singular) foliation induced by \( \Xi \) is an equilibrium partition.
Proof. \( \Xi \) generates an isometric group action \( G \) on \( M \). \( G \) is connected, simply connected (take the universal covering) and closed in the group of isometries (by assumption). This defines a proper group action on the manifold and therefore \( M \) can be divided into two components \([22]\), the principal part \( M^* \), which is open and dense in \( M \), and the singular part, which is formed by totally geodesic submanifolds. \( M^* \) is foliated by codimension 1 closed submanifolds of \( M \), in fact this foliation is a Riemannian submersion from \( M \) to \( M/G \) \([10]\). Note that \( M/G \) is a differential Hausdorff 1-manifold, and therefore diffeomorphic to \( \mathbb{R} \) or \( S^1 \). The submersion is analytic because we always assume in this paper that \((M,g)\) is analytic, and therefore also the Killing vector fields. Call \( f \) the function representing the foliation in \( M^* \); since it is Riemannian then \( f \) will satisfy that \((\nabla f)^2 = F(f)\), as proved in proposition \([2]\). Since the foliation is globally trivial (there exists a global transversal curve and the leaves are all diffeomorphic among them) this condition holds in the whole \( M^* \). Since the action of \( G \) is transitive on each leaf (the leaves are extrinsically homogeneous, that is homogeneous by isometries of the ambient space) then the mean curvature must be constant at all point of the leaf \([23]\). This follows from the fact that the second fundamental forms at two different points connected by an isometry correspond through this isometry. In terms of \( f \) this condition is expressed as \( H = \text{div}(\frac{\nabla f}{\|\nabla f\|}) = H(f) \). The following computation \( \text{div}(\frac{\nabla f}{\|\nabla f\|}) = \frac{\Delta f}{\|\nabla f\|} - \frac{\nabla f \nabla (\|\nabla f\|)}{(\|\nabla f\|)^2} \) readily implies that \( \Delta f = G(f) \). Since the non-principal set is nowhere dense the isoparametric condition extends to the whole \( M \) and therefore the foliation is of equilibrium. Note that the extended \( f \) could fail to be analytic in the singular set. \( \square \)

Remark 1. In general it is necessary to require that \( G \) be closed in the group of isometries \([23]\). For example, take the flat 2-torus \( S^1 \times S^1 \) and consider the action by the real line which is given by an irrational translation. This induces a Killing vector field, 1-dimensional, but the group generated is not closed in the isometry group of the torus, which we know is compact (it is \( O(2) \times O(2) \)). In fact this action is not proper since orbits are not embedded. Similar examples can be constructed in greater dimension.

Note that theorem \([4]\) generalizes theorem 1 in \([6]\) to arbitrary dimension in the Riemannian setting. In general the converse of this theorem is true only for 2-dimensional manifolds (proposition \([5]\)). Indeed consider a manifold which is not rotationally symmetric with respect to the point \( P \) but the determinant of the metric in polar Riemann coordinates factorizes as in

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proposition \[3\] Then the geodesic spheres around \( P \) are equilibrium submanifolds but they are not induced by an isometric group action. This shows that in general the converse theorem does not hold. It would be interesting to find conditions in order that the equilibrium partitions of a manifold be (singular) foliations induced by isometric group actions.

All these results show the deep relationship between isometries and equilibrium and suggest that physically relevant spaces should possess enough Killing vector fields. Consequently an effective procedure in order to obtain equilibrium partitions, and hence equilibrium configurations of self-gravitating fluids, is to compute the Killing vector fields of the space. It is likely that spaces which just admit a few isometries (or even no one) do not admit equilibrium functions either, lacking static configurations. Let us illustrate now theorem 4 with an example.

**Example 2.** Consider the space \( \mathbb{H}^2 \times \mathbb{R} \) endowed with the metric \( ds^2 = \frac{dx^2 + dy^2}{F^2} + dz^2 \), where \( F = \frac{2-x^2-y^2}{2} \) and \( \mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 2\} \). The Killing vector fields of this manifold are \[24\]: \( X_1 = (F + y^2)\partial_x - xy\partial_y, \quad X_2 = -xy\partial_x + (F + x^2)\partial_y, \quad X_3 = -y\partial_x + x\partial_y \) and \( X_4 = \partial_z \). A straightforward computation yields that \((\nabla f)^2 = F^2 (f_x^2 + f_y^2) + f_z^2 \) and \( \Delta f = F^2 (f_{xx} + f_{yy}) + f_{zz} \).

Some easy, although long, computations show that the codimension 1 partitions (up to null measure set) induced by the Killings vector fields are:

- \( \{X_3, X_4\} \Rightarrow f = x^2 + y^2 \), which is an equilibrium function.
- \( \{X_i, X_j\}, i \neq j = 1, 2, 3 \Rightarrow f = z \), which is an equilibrium function.
- \( \{X_1, X_4\} \Rightarrow f = \frac{x^2 + y^2 - 2}{y} \), which is an equilibrium function (with a singular set).
- \( \{X_2, X_4\} \Rightarrow f = \frac{x^2 + y^2 - 2}{x} \), which is an equilibrium function (with a singular set).

From the physical viewpoint it is reasonable to compare the shapes of a compact self-gravitating fluid with the isoperimetric domains. By the term isoperimetric I mean the sets which minimize the area for variations which leave fixed the volume. In the Euclidean space the only compact equilibrium submanifold is the round sphere, which is exactly the solution to the isoperimetric problem. The physical meaning is clear: fluid-composed stars would minimize their surfaces in order to achieve equilibrium. Regretfully for general Riemannian manifolds an equilibrium submanifold does not solve
the isoperimetric problem. The most general result that can be proved is the following.

**Proposition 6.** Let $S$ be a compact equilibrium codimension 1 submanifold. Then $S$ is a critical point of the $(n-1)$-area $A(t)$ for all variations $S_t$ that leave constant the $n$-volume $V(t)$ enclosed by $S$.

**Proof.** $S$ is the level set of an analytic function and therefore it has no boundary. Since it is compact it encloses a finite volume. The equilibrium condition implies that the mean curvature is a constant $H$. Let $S_t$, $t \in (-\epsilon, \epsilon)$ and $S_0 = S$, be a variation of $S$. The first variation of the area at $t = 0$ is given by \[ A'(0) = -(n-1)H \int_S f dS, \] where $f$ is the normal component of the variation vector of $S_t$ and $dS$ is the $(n-1)$-area element of $S$. Since the variation is volume preserving then $V'(0) = \int_S f dS = 0$ and therefore we get that $A'(0) = 0$.

This result cannot be improved in general. We can find manifolds for which equilibrium shapes are minimizers of the area and other manifolds for which they are maximizers or saddle points. Even the weaker condition of being stable, that is $A''(0) \geq 0$, is not generally verified. It would be interesting to classify all the spaces whose compact equilibrium submanifolds are stable. The following list gives some of them:

- Constant curvature simply connected manifolds. The geodesic spheres are the only stable submanifolds \[25]. They are also of equilibrium on account of proposition \[3\].

- Rotationally symmetric planes with decreasing curvature from the origin. The geodesic circles are stable and enclose isoparametric domains \[26\], they are also of equilibrium.

- Rotationally symmetric spheres with curvature increasing from the equator and equatorial symmetry. The geodesic circles are stable and enclose isoperimetric domains \[26\], they are also of equilibrium.

- Rotationally symmetric cylinders with decreasing curvature from one end and finite area. The circles of revolution are stable, enclose isoperimetric domains \[26\] and a straightforward computation yields that they are also of equilibrium.
It is not difficult to construct examples of manifolds with equilibrium partitions whose leaves are not stable. For instance consider the plane with the following metric tensor in polar coordinates $ds^2 = dr^2 + r^2(1 + r^2)^2 d\theta^2$. The function $I = \frac{1}{2} r^2$ is of equilibrium, it induces the equilibrium partition given by the geodesic circles. Now, if you set $f(r) = r(1 + r^2)$, the expression $f'^2 - ff'' = 1 + 3r^4$ is greater than 1 when $r > 0$. This implies that no stable curves exist. Other similar examples in dimension 2 can be found in the work of Ritore. Other interesting example is given by the symmetric spaces of rank 1. The geodesic spheres are transitivity hypersurfaces of the group of isometries and therefore they are equilibrium submanifolds (theorem 4). However not all the geodesic spheres are stable.

The most remarkable physical conclusion of this work is that if we want to recover the physical intuition that we have in the Euclidean space, such as the existence of contractible fluid domains in equilibrium, the coupling between symmetries of the space and the shapes of the fluid or the stability of the fluid regions we will have to restrict the base manifold. It would be no surprising that only a few base spaces were able to give physically relevant results.

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