A simple idea of relating the LQG and LQC degrees of freedom is discussed in context of toroidal Bianchi I model. The idea is an expansion of the construction originally introduced by Ashtekar and Wilson-Ewing and relies on explicit averaging of certain sub-class of spin-networks over the subgroup of the diffeomorphisms remaining after the gauge fixing used in homogeneous LQC. It is based on the set of clearly defined principles, thus is a convenient tool to control the emergence and behavior of the cosmological degrees of freedom in studies of dynamics in canonical LQG. Relating the proposed LQG-LQC interface with some results on black hole entropy suggests a modification to the area gap value currently used in LQC.

PACS numbers: 04.60.Pp, 98.80.Qc

I. INTRODUCTION

Loop Quantum Gravity (LQG) is one of the leading attempts to provide a solid framework unifying General Relativity with quantum aspects of reality. It has matured over the past years to the level, in which extracting the concrete dynamical predictions out of it became a technically feasible task. Despite this success the concrete results regarding the evolution of quantum spacetime in LQG are yet to appear. On the other hand, in past ten years a series of dynamical predictions have been made (with various level of rigor) within the symmetry reduced framework originating from LQG known as Loop Quantum Cosmology (LQC). There, the set of already known results ranges from establishing a singularity resolution through qualitative changes of the standard early universe dynamics picture (found on the genuine quantum level) to predictions of the behavior of cosmological perturbations and (in some cases) nonperturbative inhomogeneities.

The results obtained within LQC framework cannot be treated as final, as LQC was never derived from LQG in any systematic way. Instead, it is a stand-alone theory constructed by applying the methods of LQG to cosmological models further enhanced via parochuting some results and properties of LQG on the phenomenological level. Therefore, it is not a priori clear to what extent (if any) the predictions mentioned above reflect the true features of full LQG. Addressing this issues has brought a considerable interest within the loop community. Its pioneering studies were aimed towards controlling the so called inverse volume corrections in LQC and as tools to control the heuristic effective descriptions of inhomogeneous extensions of LQC. Presently, the attempts to provide a precise connection between LQG and LQC are directed in three main areas: (i) direct embedding of LQC framework within LQG, (ii) approximation of cosmological (symmetric) solutions in LQG, and (iii) emergence of cosmological (LQC) degrees of freedom in appropriate scenarios within LQG.

The approach (i), realized on the level of mathematical formalism, focuses on embedding the elements of the LQC formalism (for example straight holonomies) as a proper subclass of their analogs in LQG. So far however the most natural ways of constructing such embeddings have proved to lead to inconsistencies and resulted in several no-go statement. The main problem encountered in the attempts is the inherent diffeomorphism-invariance of LQG and the fact that the symmetries characterizing the cosmological solutions are a subgroup of the diffeomorphism group. In consequence the formalism of the theory is (by construction) insensitive to the very components distinguishing the cosmological spacetimes. On the other hand, recently, extending the standard LQC holonomy-flux algebra by all holonomies along piecewise analytic curves and imposing the symmetries on the classical level led to a viable embedding of LQC in LQG.

To overcome the problems of (i) another route was explored. There, instead of encoding the symmetries in the formalism one considered “the operational approach” – defining the symmetries through relations between spacetime quantities provided by the LQG observables. The idea has been realized by constructing the coherent states peaked about symmetric spacetimes (see for example). Provided the evolution of such coherent states in LQG can be controlled on the level of full theory this description can provide a definition of a (n effective)
reduced formalism on its own. Such formalism will have much stronger foundation in its LQG origin than LQC but applying it to probe for predictions is still a matter of the future. On the other hand, that formalism would lose its anchor in LQC, thus it would not be able to provide a solid connection with existing formulation of LQC or justification for the particular constructions implemented in it. As a consequence the utility of this approach as LQC test would be limited.

To mend this gap one needs to construct a precise dictionary between objects of LQG and LQC formalism, while keeping both the theories autonomous (the approach (iii)). Such dictionary should associate selected (possibly emergent) degrees of freedom of LQG with cosmological ones – working as fundamental for LQC. In that aspect it is not necessary to restrict to symmetric or near symmetric states in LQG. Building this dictionary would be just a process of extracting some (global) degrees of freedom and as such should be well defined for any quantum spacetime (or at least for a family of such which is sufficiently large to cover physically interesting scenarios). A good example of such procedure is a simple toy model used in [14] to fix certain ambiguities in quantizing the Bianchi I spacetimes in LQC. There, as an example of the spin network one considered a regular (cubic) lattice of which edges have been excited to the first state above the ground one. Much simpler models attempting to mimic cosmology, playing the role analogous to the one above, are also being constructed in Spin Foam formalism (often considered as the covariant formulation of LQG) [20].

It is also worth remembering, that outside of the top-down approaches presented above there is a considerable literature on bottom-up approaches, where the components of LQC are cast onto LQG structures. An example of such is the so-called lattice LQC [21], where the structure of degrees of freedom and elementary operators inherent to perturbative cosmology are defined on the regular lattice, again playing the role of an example of an LQG spin network. Another, bit more distant example is testing the BKL conjecture in context of LQC [22].

The construction presented here is strongly motivated by the Bianchi I toy model mentioned two paragraphs above. It shares with this model the choice of spin network topology. Outside of this initial choice however we will keep the construction as general as possible (avoiding the dependence on particular prescription in either LQG or LQC) at the same time keeping full control over the assumptions entering the construction. In particular, as the unreduced side of the interface we will use genuine LQG without any simplifications. The dictionary will be provided by objects having a precise physical interpretation and well defined in both theories.

To have such a simple tool available becomes recently more than just a convenience, as there is an increasing amount of effort towards making the preliminary dynamical predictions of either conservative LQG [23, 24], its modifications [25, 26] or simplifications (in particular the simplifications of SU(2) gauge to U(1)3 [27, 28]) via semiclassical approximations. One can thus confront the results of these projects with the predictions of LQC to either verify or falsify the latter.

Before proceeding with the construction of the LQG-LQC interface let us briefly recall those element of both theories which will be relevant for its construction.

II. ELEMENTS OF LQC AND LQC

Since for all practical purposes both LQG and LQC are independent theories, just sharing common quantization methodology [11, 16] we proceed with presenting them separately. Let us start with LQG.

A. Loop quantum gravity

LQG is a quantization of canonical general relativity, owing a lot of its mathematical components to Young-Mills theories on the lattice. It’s starting point is a 3 + 1 canonical splitting, with the phase space coordinatized by the Ashtekar-Barbero variables: su(2) valued connection Aa i and the densitized triad Eα i , where Aa i is a combination of the Levi-Civita connection and exterior curvature Aa i = Γa i + γKi a (with γ being the Barbero-Immirzi parameter) [30]. As any representation of GR it is a constrained theory with the algebra of constraints generated by: the Gauss constraint, the spatial diffeomorphisms and the Hamiltonian constraint. To deal with them the Dirac program is implemented: theory is quantized without constraints (so called kinematical level), which are next solved on the quantum level (with solutions forming the physical sector of the theory).

The basic objects of the theory are the holonomies of A along the piecewise analytic curves Uγ (A) = P exp( ∫γ Aa i τi dz α) and the fluxes of Eα across surfaces Kt = ∫K α Eα dσ α. Together they form the holonomy-flux algebra, which is the fundamental object in constructing quantum theory. An application of the GNS (Gelfand-Naimark-Segal) construction to this algebra leads to the kinematical Hilbert space Hkin LQG spanned by the cylindrical functions supported on the graphs embedded in 3-dimensional differential manifold. These functions are conveniently labeled by the su(2) representations (on each edge of the graph + the internal edges within the graph vertices) – enumerated
The properties of this operator are known in detail (see for example \cite{1, 3}). Its action on $\mathcal{H}_\text{diff}^{\text{LQG}}$ (i.e. on its spin network basis elements) is relatively simple \cite{30}: the area of chosen (arbitrary) 2-surface $S$ depends only on the $j$-labels of the edges of a spin-network reaching (or intersecting) this surface

$$\text{Ar}(S)\Psi[A] = 4\pi\gamma\ell_{\text{Pl}}^2 \left[ \sum_{e^+} \sqrt{j_{e^+}(j_{e^+}+1)} + \sum_{e^-} \sqrt{j_{e^-}(j_{e^-}+1)} \right],$$

(2.1)

where $e^+$ are the incoming edges of the graph supporting $\Psi[A]$ which terminate on the surface, $e^-$ are the ones starting at the surface. $j_{e}^\pm$ are their respective $su(2)$ representation labels. The edges intersecting (piercing) the surface are counted as both incoming and outgoing (i.e. in that case a trivial 2-valent node is temporarily introduced on the surface), thus their contribution is twice the terminating ones. The form of (2.1) immediately implies, that the spectrum of $\text{Ar}(S)$ is discrete. In particular the first non-zero value of the area is isolated from zero and equals

$$A_1 = 2\sqrt{3\pi\gamma}\ell_{\text{Pl}}^2.$$

(2.2)
2. The Euclidean part of Hamiltonian

Classically the Hamiltonian constraint (or, more precisely, its part corresponding to gravity) is of the form

\[ C = \int \mathrm{d}^3 x \sqrt{-q} C, \quad C = \frac{\gamma^2}{2\sqrt{\det E}} F_{ij}^a F_j^b [\epsilon^{ij} k F_{ab}^b + 2(1 - \gamma^2) K_{i[a}^k K_{b]}^j] \] (2.3)

where the field strength \( F_{ij}^a \) is the curvature of the connection \( A \). The first term in \( C \) is the so called Euclidean term of the Hamiltonian density. The loop quantization procedure, requires to express \( C \) in terms of the holonomies and fluxes – the process known as Thiemann regularization \[23]. In particular to express the curvature term \( F \) one implements the known classical identity

\[ F_{ab}^i X^a Y^b(x) = \lim_{\mathrm{Ar}(\triangle(x)) \to 0} \frac{U_{\triangle(x)} - 1}{\mathrm{Ar}(\triangle(x))} \] (2.4)

where \( \triangle(x) \) is the closed, piecewise analytic loop such that the vectors \( X, Y \) are tangent to it at the point \( x \), \( U_{\triangle(x)} \) is the holonomy along this loop and \( \mathrm{Ar}(\triangle(x)) \) is the physical area of the loop. The right-hand side of this identity can be quantized directly in LQG, as the operators corresponding to the holonomy and the area are well defined.

The particular implementation of this identity (and the action of the resulting regularized \( F \) operator) depends on the specific construction (or prescription) of the Hamiltonian constraint. In the original construction by Thiemann \[23] the (Euclidean part of) Hamiltonian constraint added at the vertices the spin network the small triangular loops via introducing on the edges converging to it two 3-valent nodes connected by an edge labeled by fundamental \( su(2) \) representation \( (j = 1/2) \). Then the limit of shrinking this triangular loop to a point (at the node) was taken in the sense of the embedding, which nonetheless led to a diffeomorphism invariant result due to the nature of operator components in the approximation of \( F \) (for details, see \[24\]).

In the alternative construction (see for example \[25\]), where Hamiltonian constraint does not generate new edges, the loop has to be formed by existing edges of the spin network. It is defined by a requirement to form a plaquet – the minimal closed surface, of interior not intersected by any edges. In particular, when the spin networks are supported on the regular lattice, these loops are the minimal squares.

The particular construction of the Hamiltonian constraint also affects critically the structure of the physical Hilbert space. Below we will briefly discuss the issues related to it.

3. Physical Hilbert space structure

Since in the Master program \( \mathcal{H}_{\text{LQG}}^{\text{phy}} \) is defined only abstractly (formally) to probe its properties we will focus on the deparametrization picture. To start with, we note that the kinematical Hilbert space which arises from the GNS construction is non-separable. The reason behind it is, that the induced inner product on \( \mathcal{H}_{\text{kin}}^{\text{LQG}} \) makes the states supported on disjoint graphs orthogonal. Each subspace of states supported on chosen graph is separable, with a discrete inner product, however the complete \( \mathcal{H}_{\text{kin}}^{\text{LQG}} \) contains a continuum of (disjoint) graphs. Unfortunately, the gauge invariant and diffeomorphism invariant Hilbert spaces retain this property: the former puts only the restrictions on the labels of the graph without significantly decreasing the possible graph structures. The latter still allows for the continuum of distinct (orthogonal) graphs with a discrete inner product between them. Since in the deparametrization picture the space \( \mathcal{H}_{\text{diff}}^{\text{LQG}} \) becomes the physical one, this deficiency is transmitted directly to the physical sector. As the non-separability can affect significantly the construction and properties of the coherent states and the statistical ensembles (see \[27\] for discussion of these issues on a simple quantum-mechanical example), this problem requires certain amount of care.

The particular treatment depends on whether the action of the (true in the deparametrization picture) Hamiltonian changes the graph topology. If the graph is fixed (see for example \[26\]) the Hamiltonian distinguishes subspaces invariant with respect to its action (supported on an unchanging graph). If the relevant observable operators are defined carefully and also preserve these subspaces, they become the superselection sectors, each of them being separable (as supported on one specific graph). The standard treatment calls then for a restriction of the studies to just one such sector.

If the Hamiltonian is graph changing, this procedure becomes less straightforward, although often superselection sectors can be distinguished due to the fact, that (in specific prescriptions) the Hamiltonian changes the graph in a specific controlled way. This happens for example in case of the original construction of \[23\]. However, those superselection sectors can become already non-separable.

On the other hand, our experience from LQC shows (see the discussion in \[32\]), that for certain models such restriction might be insufficient to provide a sufficiently large semiclassical sector reproducing General Relativity.
dynamics in small gravitational field regimes. In that case an alternative construction may be needed. Such alternative is provided for example in [32]. There one makes use of available Lebesgue measure on the space of superselection sectors. Then the inner product is defined as the integral with respect to that measure of inner products $\langle \cdot | \cdot \rangle_\epsilon$ on the single superselection spaces $\mathcal{H}_\epsilon$ (with $\epsilon$ being an abstract superselection sector label)

$$\forall \psi, \phi \in \mathcal{H} : \langle \psi | \phi \rangle = \int d\mu(\epsilon) \langle \psi_\epsilon | \phi_\epsilon \rangle_\epsilon, \quad (2.5)$$

where $\psi_\epsilon, \phi_\epsilon \in \mathcal{H}_\epsilon$ are the restrictions (projections) of the states to the single sector $\epsilon$. Action of the operators preserving the sectors extends in a straightforward way. The integral Hilbert space $\mathcal{H}$ is again separable.

B. Loop quantum cosmology

LQC, even when applied to the description of the inhomogeneous spacetimes, always relies on the reorganization of the geometry and matter degrees of freedom onto the quasi-global ones, for example the Fourier or spherical harmonic modes of the inhomogeneities/gravitational waves/matter (see [8, 38]). As a consequence in this description there are always distinguished degrees of freedom corresponding to the “background” homogeneous spacetime. This distinction is achieved by partial gauge fixing, which is naturally distinguished in the cases of homogeneous spacetimes and in perturbative approaches. The remaining (inhomogeneous) degrees of freedom are then treated as the objects “living” on that homogeneous background on the equal footing with the matter fields. Thus, in all of the models a proper handling of the homogeneous spacetimes is an essential first step.

Here, we focus on the simplest model representing such spacetime – the model of Bianchi I universe. For the most of the paper we further fix the topology of its spatial slices to 3-torus. The precise mathematical formulation of the LQC quantization of this model has been presented in [11] (isotropic spacetimes) and [19] (actual quantization of the homogeneous background). It is performed via direct repetition of the procedure developed for LQG, although here the symmetries distinguish additional structure, which plays an essential role in the process.

First, the homogeneity distinguishes the natural partial gauge, in which the spacetime metric takes the form

$$g = -N^2(t)dt^2 + a_1^2(t)dx + a_2^2(t)dy + a_3^2(t)dz \quad (2.6)$$

where $N(t)$ is the lapse function, $(a_1, a_2, a_3)$ are the scale factors in three orthogonal directions (in which the metric is diagonal) and $\mathcal{H} = dx^2 + dy^2 + dz^2$ is the isotropic fiducial metric constant in comoving coordinates $(x, y, z)$. This choice fixes all the spacetime diffeomorphisms up to: a global time reparametrization, and the (rigid in $\mathcal{H}$ metric) global spatial translations.

Similarly to the general GR case, we select the Ashtekar-Barbero variables, although here the fiducial metric distinguishes the orthonormal triad $\gamma^a_i$ of vectors pointing in eigendirections of the physical metric and preserving the spatial symmetries of the system. That structure again allows to partially gauge-fix the variables through selecting

$$A^a_i = c^i(L_i)^{-1} \omega^a_i, \quad E^a_i = p_i L_i V_o^{-1} \sqrt{\gamma} \epsilon^a_o, \quad (2.7)$$

where $\lambda^a_i$ is a co-triad dual to $\gamma^a_i$, $V_o$ is the fiducial (with respect to $\gamma$) volume of the homogeneous spatial slices and $L_i$ are their (also fiducial) linear dimensions. The global coefficients $c^i$ and $p_i$ are the so called connection and triad coefficients. They form the canonical set with Poisson bracket $\{c^i, p_j\} = 8\pi G \gamma^i_j$.

In the next step one constructs the holonomy-flux algebra. Here however one notices, that upon the choice (2.7) one can just select the subalgebra of holonomies $U^{(\lambda)}_i$ along the straight edges in direction $\gamma_i$ and the fluxes $S_i$ along the unit squares orthogonal to $\gamma_i$ as they suffice to separate the phase space points. Further, the fluxes $S_i$ can be associated with the triad coefficients themselves as

$$S_i = p_i. \quad (2.8)$$

On such restricted (subalgebra of the) holonomy-flux algebra one implements the GNS construction, arriving to the unique quantum representation [32]. The kinematical Hilbert space resulting from this construction is a product of the square summable functions on the Bohr compactification of the real line

$$\mathcal{H}_{\text{kin}}^{\text{LQC}} = \left[ \Sigma^2 (\mathbb{T}_{\text{Bohr}}, d\mu_{\text{Bohr}}) \right]^3, \quad (2.9)$$

1 The construction there is presented on the example of the simple quantum mechanical system – an harmonic oscillator, however the applications to LQG are also discussed there.

2 The construction can also be extended to many cases with a singular measure.

3 This projections are known in the literature as the so called shadow states.
The basic operators – quantum counterparts of the holonomy-flux algebra elements – are the holonomy operators \( \hat{U}_i^{(\lambda)} \) and the unit flux operators \( \hat{\varphi}^i = \hat{S}^i \), the latter are known in the literature as the \( \text{LQC triad operators} \) due to the simple classical relation (2.8). We will implement the same naming policy here. One has to remember however, that these operators represent the \textit{fluxes}, not the triads. As in full LQG, in LQC the operators corresponding to the holonomies and triads themselves \textit{do not exist}. In fact, the relation between \( \varphi^i \) the scale factors \( a_i \) in (2.10) (see [10]) shows, that the operators \( \hat{\varphi}^i \) measure the area of the maximal surface orthogonal to \( \varepsilon p^i \).

The kinematical states \( |\psi\rangle \in \mathcal{H}_{\text{kin}}^{\text{LQC}} \) automatically satisfy the Gauss and diffeomorphism constraints. The only gauge transformations left after the partial gauge fixing – the global spatial translations act on the elements of \( \mathcal{H}_{\text{kin}}^{\text{LQC}} \) as an identity. The only nontrivial constraint remaining is the Hamiltonian one.

To construct the operator representing the (gravitational term of the) Hamiltonian constraint one repeats the Thiemann construction of LQG, partially discussed in sec. [II A 2]. For the Bianchi I model, the Lorentzian (exterior curvature) term is proportional to the Euclidean one, thus only the quantization of the latter is needed. For that one again has to deal with the field strength term, which is approximated by holonomies along a closed loop via (2.4). Since only the holonomies along the diagonal directions of \( q \) are available the loop is a rectangle oriented in directions of \( \varepsilon p^i \).

Here however we see an important difference with respect to LQG. In the full theory either (depending on the formulation) one could move (shrink) the loop in the embedding manifold and the transformation has not modified the physical area of the loop. In LQC, the presence of the \textit{background fiducial metric} \( q \) – an object responsible for the rigid relation between \( a_i \) and \( p_i \) – fixes a unique relation between the embedding (fiducial) area of the loop and its \textit{physical area}. For a loop spanned by (holonomies along) vectors \( \varepsilon e_j, \varepsilon e_k \) we have

\[
\text{Ar}(\square_{jk}) = \varepsilon^{ijk} p_i \lambda_j \lambda_k
\]

where \( \lambda_i \) are the fiducial lengths of the straight edges in direction \( \varepsilon e_i \) forming the loop.

This relation plays a crucial role in fixing the action of the Hamiltonian constraint in LQC [12]. In principle in LQC one can set the fiducial edge lengths \( \lambda_i \) freely which would allow to construct the loop or arbitrarily small areas. On the other hand in LQG the spectrum of the area operator is purely discrete and the first non-zero value of the area is determined by the theory. The goal of formulating LQC was the construction of the simplified settings approximating or mimicking the full theory as close as possible, this particular property (discreteness of the area) has been \textit{parachuted from LQG}. The fiducial lengths \( \lambda_i \) are fixed by the requirement that the physical area of the loop equals

\[
\text{Ar}(\square) =: \Delta = 2A_1,
\]

where \( A_1 \) is provided by (2.2). Loop of these dimensions is then considered as the minimal loop realized in LQC. The reason, why one takes as the minimal area \( 2A_1 \) instead of \( A_1 \) [10] is that it should correspond to the area of the \textit{surface pierced} by the edge, not just with the edge terminating on it. This requirement follows from the semi-heuristic lattice construction in [19] and will become apparent in the process of constructing the dictionary in further sections.

The particular way in which the requirement (2.11) fixes the lengths \( \lambda_i \) is construction dependent and in the past led to several distinct prescriptions in quantizing the Bianchi I model in LQC (see for example [11]). Subsequently one choice has been distinguished by the construction in [19] and by certain invariance requirements in application of the framework to the noncompact universes. At present the construction introduced in [19] is considered to be the unique consistent prescription.

With the lengths of holonomies fixed and the remaining components in the Hamiltonian constraint regularized via Thiemann construction, one arrives to the Hamiltonian constraint operator, which is a difference operator acting on the domain of elements of \( \mathcal{H}_{\text{kin}}^{\text{LQC}} \) supported on the finite number of points \( |p_1, p_2, p_3\rangle \) [see eq. (3.35)-(3.37) in [19]]. Given that, after coupling with appropriate matter fields the dynamical sector of the theory is determined by either group averaging [12] (see also [8, 43] for applications in context of LQC) and partial observable formalism or via deparametrization.

Here however we encounter the same problem as in full LQG: the kinematical Hilbert space \( \mathcal{H}_{\text{kin}}^{\text{LQC}} \) is non-separable (due to the discrete inner product). Since in the deparametrization picture it becomes the physical space, the latter is also non separable. In isotropic LQC the procedure of dealing with this problem makes use of the fact, that the Hamiltonian (or Hamiltonian constraint) distinguishes certain subsets ("lattices") invariant under its action. The

\[\text{For technical reasons (simplicity of the constraints) in [15] different labeling of the kinematical basis states is used.}\]
subspaces of states supported on those sets form then superselection sectors, each being separable. Subsequently one choses just one sector to describe the dynamics.

An extension of this approach to the anisotropic LQC is nontrivial, since for example in Bianchi I case the superselection sector “lattices” are formed out of families of sets dense on surfaces of codimension 1 in the configuration space [14]. Furthermore, it is not at all obvious, that a single sector would admit a proper semiclassical regime, where low energy dynamics conforms to GR.

In order to deal with such difficulties one can again implement the integral Hilbert space construction [32] following [22]. In the case of known isotropic models [8, 12] the results following from implementing this construction are (up to minor corrections) equivalent to the ones provided by treatment involving just one superselection sector.

### III. THE DICTIONARY

The review in the previous section shows clearly, that both LQG and LQC describe the geometry via very distinct sets of degrees of freedom. The principal difference between these frameworks is the presence of background (fiducial) geometry in LQC and absence of such in LQG. This LQC background structure is distinguished by the symmetries of the theory on the classical level. By its very construction, LQG does not “detect” these symmetries as there the symmetry transformations are just a specific class of finite diffeomorphisms, to which the framework is insensitive. As a consequence, building an LQG state representing the cosmological spacetime is a nontrivial task [18].

Our goal here is constructing the relation between the frameworks on the kinematical level, that is we do not address the matter of agreement of the dynamics between these two frameworks. Answering the question whether the LQC dynamics of a state representing a universe is a good approximation to the dynamics of the state representing the same universe in LQG, is beyond the scope of this article. Instead, we explore the interplay between the structures in both theories and the consequence the consistency requirements of one framework impose on the other. In the process we try to keep the full control over the initial assumptions entering the construction and to determine the freedom left after making these assumptions.

Our (principal) point of departure is the observation, that the basic quantities characterizing the state in LQC—the areas $p_i$ of maximal surfaces orthogonal to the basis triad vectors—correspond to observable quantities well defined for any physical LQG state. To see that, let us fix the type of spacetime represented and the approach to constructing the dynamical sector.

- Our objects of studies will be physical states corresponding to the (homogeneous but not isotropic) Bianchi I universe, of which the spatial slices have a 3-torus topology (although the results can be easily generalized to the noncompact flat case). Thus the embedding manifold for the kinematical spin-networks will be topologically $T^3$.

- Since the dictionary we are constructing involves the correspondence between the geometry degrees of freedom only and does not employ the dynamics of the system, we can safely assume, that it will not depend on the type of matter content coupled to gravity. Thus we can work with chosen particular type of matter and the results will automatically generalize to other types of matter. For that purpose we select the construction of the dynamical sector through the deparametrization with respect to the irrotational pressureless dust serving as the time frame in both LQG [8] and LQC [8].

While there is no consensus in the area as to whether the particular matter content selected above is well motivated physically, this choice provides simple and precise framework, circumventing the difficulties present in other approaches. In particular the diffeomorphism invariant Hilbert space in LQG and the kinematical Hilbert space of LQC automatically (and without any modifications or corrections) became the physical Hilbert spaces of their respective theories. Same applies to the area operators in LQG specified in sec. 1A1 and the “triad” operators $\hat{p}_i$ in LQC—they become physical observables. As a consequence they can be used directly, when comparing physical areas.

Given that one can associate with each physical state $|\Psi\rangle \in \mathcal{H}_{\text{phy}}^{\text{LQG}}$ (where we do not require this state to be supported on just one spin network) a cosmological state $|\Phi\rangle \in \mathcal{H}_{\text{phy}}^{\text{LQC}}$ such that the expectation values of the area operators along the “flat and orthogonal” maximal surfaces agree with the expectation values $\langle \Phi | \hat{p}_i | \Phi \rangle$. Despite lack of background metric, the notion of flatness and orthogonality can be made precise in terms of the expectation values. Below we present one of possible constructions. it will not be used in construction of LQG-LQC interface and it is

---

5 We use these terms since for homogeneous spacetimes (of diagonal spatial metric) they coincide with the standard meaning of flatness and orthogonality. This agreement may however not extend beyond that class of spacetimes.
presented solely as an example that making the abovementioned association between LQG and LQC states is possible. One of many ways to define such construction is:

1. First one chooses on the embedding manifold a point \( p \) and a triad of vectors anchored on it.

2. The angle operator in LQG is well defined \(^1\), thus one can distinguish (also in the embedding manifold) a triple of 2-surfaces of \( T^2 \) topology – sections of the embedding manifold – such that the respective pairs of distinguished triad vectors are tangent to them (at their intersection). Their relative orientation is then fixed by the requirement that the expectation value of the angle operator corresponds to normal angles.

3. Finally, the “flatness” of the surfaces is enforced by requirement that the surfaces minimize their physical area (again in terms of the expectation values of the area operator).

Then the area of the distinguished surfaces is associated with respective areas \( \langle \hat{p}_i \rangle \) in LQC.

This association between LQG and LQC states is far from unique: (i) it is fixed just by relation between expectation values of three observables, which is obviously insufficient to determine the state, (ii) the areas of the distinguished surfaces may depend on the point \( p \) and the chosen vector triad, and (iii) there may not be a global minimum of the areas in the point 3. of the above construction. At this moment however we do not look for the uniqueness of the association. We just want to show that it can be made. It may not be particularly useful for constructing the LQC limit of LQG, especially because so far we have not introduced any notion of symmetry. Literally, any (even very inhomogeneous) physical LQG state can be used in this construction.

On top of that deficiency, at present we are lacking any relation with the auxiliary structures in LQC, which is necessary to really understand the relation between the frameworks. Therefore in further studies we are going to restrict the space of possible physical states, by selecting off a very specific (yet sufficiently large to accommodate the physically interesting spacetimes) set of spin networks supporting the states.

A. The lattice spin-network

Our construction is heavily inspired by the semi-heuristic construction introduced in \(^{19}\). There, one equipped the embedding \( \mathbb{R}^3 \) manifold with a fiducial metric \( \xi \) and used it to define a regular lattice in it. This lattice has been next used to construct a specific spin-network by associating with each edge (link) of the lattice a \( j \)-label \((j = 1/2)\) corresponding to the fundamental \( su(2) \) representation – a minimal non-zero value allowed by the theory. Then, the gauge-invariant state has been distinguished as supported on this spin network only.

Given that state, one could introduce a so called fiducial cell – a compact region of space acting as the infrared regulator of the theory. It was chosen such that its edges were parallel (in the sense of \( \xi \)) to the edges of the lattice. Due to fixing of the \( j \)-labels the area of each face of the cell was then proportional to the number of the lattice edges piercing it. With these areas one subsequently associated the values of \( p_i \) in LQC.

Given that association, the regularity of the lattice allowed in turn to associate with each plaquet (minimal square loop of the lattice) a “physical” area. Finally the requirement that all these areas equal \( \Delta_{(2)} \) fixed the fiducial lengths of the edges of the plaquets, which in this construction are the curves along which the fundamental holonomies are taken.

In this article we expand on this idea, dropping however most of the assumptions made in \(^{19}\). To start with, we consider a single spin network, embedded in the \( T^3 \) manifold defined above. We further assume that our spin-network is topologically equivalent to the regular lattice\(^2\) or is a proper sub-graph of such. This graph is next equipped with the \( su(2) \) \( j \)-labels on the graph edges (and internal edges at each node) \textit{where in particular the value} \( j = 0 \text{ is allowed.} \)

If the original graph is the proper subgraph of the (topologically) regular lattice, it is completed to the lattice by adding appropriate edges with \( j = 0 \) and appropriate vertices. We further assume that the lattice is minimal: since two edges of \( j = 0 \) entering 2-valent vertex can be always replaced with one single edge, given a lattice spin-network we perform such reduction, whenever it does not destroy the regular lattice topology. Such construction of a spin-network, although abstract instead of embedded is used for example to formulate the \textit{Algebraic Quantum Gravity} \(^{23}\) framework.

To introduce the cosmological background structure we note, that a large class of spatial diffeomorphic gauge fixings can be implemented via equipping the embedding manifold with a metric tensor. For every spin network one can define a fiducial isotropic metric \( \tilde{\gamma} = dx^2 + dy^2 + dz^2 \) such that \( x_i := (x, y, z) \) are the functions defined along the

\(^6\) On the formal level this value cannot be associated with the expectation value of the LQG area operator since no edge intersect such plaquet. It can be however made precise with use of the so called \textit{dual graph} – a technique often applied in the spin-foam approaches.

\(^7\) To be mathematically precise, we define the graph which admits a set of discrete symmetries of the regular (closed) lattice on \( T^3 \).
edges of the graph, preserved by the discrete symmetries of the graph: for a cyclic permutation of the nodes along one lattice direction $\mathbf{3}$ the function $x_i(\vec{x})$ changes as follows

$$x_i \mapsto x_i + \lambda_i, \quad \lambda_i := 1/n_i$$

(3.1)

where $n_i$ is the number of graph edges forming a closed loop in direction $i$. The coordinates on the graph are next extended smoothly (non-uniquely) to the whole embedding manifold.

It is worth reiterating, that neither the partial gauge fixing introduced here nor the auxiliary structure play any role in describing the physics. All the physical geometry observables are insensitive to this choice, as their action depends only on the topology of the graph and its quantum labels.

Given the regular lattice defined above, one can precisely implement the construction of the LQG↔LQC dictionary specified at the beginning of sec. III. For that we choose the constancy surfaces $S_i$ of the coordinates $x_i$ (thus orthogonal to $p_i$). The areas of these surfaces (expectation value of the operator $\mathbf{2.1}$) are then associated with LQC “triad” or “flux” coefficients

$$\langle \text{Ar}(S_i) \rangle = p_i := \langle \hat{p}_i \rangle.$$  

(3.2)

The form of (2.1) implies immediately, that these areas do not depend on the way the coordinates $x_i$ have been completed between the elements of the graph. The values of $p_i$ depend however on the $j$-labels of the edges intersecting, terminating and contained within $S_i$ which may differ depending on the particular choice of the surface. Thus, this association is not unique. To emphasize this fact, we further denote these values as $p_{i,x_i}$ and the surfaces themselves as $S_{i,x_i}$. At this point we have to remember however, that there is some residual diffeomorphism freedom left in the system: the rigid (with respect to $p_i$) translations in $x_i$. We will exploit this freedom in the next subsection to complete the dictionary construction.

Before doing so however, we have to address one issue: since the dictionary will rely on the auxiliary structure, it is critical to check how it will be affected by the dynamics.

In case, the LQG Hamiltonian is graph-preserving (like for example in $\mathbf{25}$) there is no problem: the only elements affected are the spin labels. The graph itself does not change, thus its embedding in the manifold can be assumed to be constant in time. This in turn allows to keep the auxiliary structure constant.

The situation complicates a bit when the Hamiltonian is graph changing. There the preservation of the structure of the graph depends on particular form of that Hamiltonian. For example in original canonical LQG construction of $\mathbf{23}$ the Hamiltonian adds edges with $j = 1/2$ labels forming a triangular loops with existing ones. This can be easily implemented in the construction considered here, if instead of triangle we add a square loop with two new $j = 1/2$ edges. The new spin network can be then easily completed to a (topologically) regular lattice by adding $j = 0$ edges. The auxiliary structure can then be easily rebuilt, essentially in tow ways:

1. The coordinates $x_i$ and fiducial metric can be redefined so the new lattice becomes regular in them. This corresponds to discontinuously shifting the vertices of the graph to new positions on the embedding manifold (passive diffeomorphism). The discontinuity is however not a problem, as the Hamiltonian flow is not used in the construction of the dictionary and the auxiliary structure can be defined at each time slice independently.

2. The new nodes can be placed in the center (with respect to $p_i$ of existing plaquets. Then, the lattice would be a particular realization of the dense spin-network $\mathbf{13}$. It would however loose the regularity. The latter could in principle pose a problem as the fiducial length of the edges will be an essential component of the dictionary. In that case however one should average any quantity evaluated on the graph over the diffeomorphisms changing the fiducial lengths of the edges but preserving the directions (with respect to metric $\gamma$) of the edges of the graph. For quantities which are averages weighted by the fiducial lengths this averaging procedure yields exactly the same results as the “uniformization” defined in the point above (see Appendix $\mathbf{A}$).

It is important to note that the regularity assumption can be replaced with the average over (passive) diffeomorphisms preserving the directions of the edges (with respect to $\gamma$). Indeed one can represent the (relevant for the graph) passive diffeomorphisms as random distributions of the vertices coordinates over the interval $[0, 1]$, see Appendix $\mathbf{A}$.

---

$\mathbf{8}$ The direction is defined here by topology of the graph: each direction is the set of the classes of equivalence of minimal closed loops not shrinkable to a point on the embedding manifold.

$\mathbf{9}$ We remind that no restriction is made on the distribution of the values of $j$-labels on the graph.
At present a relevant difference remains between the LQG state constructed previously and its LQC analog. The LQC state is constructed with implicit assumption of representing the homogeneous spacetime, whereas the LQG one can a priori be highly inhomogeneous. This implies that the association of the values of \( p_i \) has to involve some kind of averaging (over the inhomogeneities) procedure. In LQG the lack of background structure makes the definition of such averaging difficult. Here however the choice of the spin network graph and partial gauge fixing allowed to construct the necessary background structure.

To employ it, we now consider the remaining rigid translations as the *active transformations* shifting the surfaces \( S_i \), along the graph and define the values \( p_i \) as the averages with respect to this translation group. In the mathematically precise sense the variables \( p_i \) are chosen to equal the expectation values of the area operator (of each surface \( S_i \)) *averaged over the rigid spatial translation group*. A simple calculation using (2.11) shows then, that

\[
p_i = \frac{8 \pi \gamma \ell_P^2}{n_i} \sum_{e \in \{e\}_i} \sqrt{j_e (j_e + 1)} = \frac{8 \pi \gamma \ell_P^2}{n_i} \Sigma_i,
\]

where \( \{e\}_i \) is the set of all the edges of the graph which point in direction of \( e_i \) and \( n_i := 1/\lambda_i \) is the number of graph edges pointing in direction of \( e_i \) and forming a close loop. Here, the edges terminating on the surface or contained within it do not contribute, as that would require tuning the translations and such translations are a zero measure set within the distinguished translation group.

Next component needed in our dictionary are the (physical) areas of the minimal loops (plaquets) needed to approximate the field strength operator. To evaluate these areas we proceed exactly as in the case of the surfaces \( S \): we average the relevant area operators over the rigid (active) translations group. Again, a simple calculation yields (here we denote these areas by \( \sigma^i \))

\[
\sigma_i = \frac{8 \pi \gamma \ell_P^2}{n_1 n_2 n_3} \sum_{e \in \{e\}_i} \sqrt{j_e (j_e + 1)} = p_i \frac{\lambda_i}{\lambda_1 \lambda_2 \lambda_3}.
\]

As a consequence, the ratio \( \sigma_i/p_i \) (no summation over \( i \)) does not depend on the \( j \)-labels of the spin network. It depends *only on the number of edges of the graph* which is an expected result since (once the averaging is implemented) each surface \( S_i \) can be simply composed out of \( n_1 n_2 n_3/n_i \) plaquets. It is important to note, that, even though we are permitting the edges with \( j = 0 \), the numbers \( n_j \) are invariant due to gauge invariance and the requirement that the lattice is minimal.

The standard procedure implemented in LQC would call in the next step for an association \( \sigma_i = \Delta \), where \( \Delta \) is defined via (2.11). This would fix \( \lambda_i \) as the functions of the phase space, leading exactly to the dependence found in [19]. Here however we do not implement this step, expecting in turn that the value of \( \sigma_i \) should follow from the properties of the spin network. Thus, the only property at our disposal is the relation of \( \sigma_i \) with the average (over the graph) of \( j \)-labels associated with edges in direction \( e_i \)

\[
\sigma_i = 8 \pi \gamma \ell_P^2 \left[ \sqrt{j_e (j_e + 1)} \right]_{e_i} =: \Delta_i,
\]

where the symbol \( \left[ \right] \) denotes the average over the graph. For the models aimed to reproduce the cosmological spacetime via specific semiclassical states the value of such average depends on the details of the model and may in principle differ significantly from the value \( \Delta_i = \sqrt{3/2} \) consistent with \( \sigma_i = \Delta \) (see for example [28]). In such models, the values \( \sigma_i \) do not need to be fixed by any fundamental constant and a priori may depend on the state.

### 1. Alternative averaging procedure

The results of the averaging procedure implemented above can be easily understood on the intuitive level if we introduce a convenient decomposition of the group of rigid translations. First, one can introduce a discrete group \( \mathbb{Z}_{n_1,n_2,n_3} := \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3} \) of cyclic permutations of the graph vertices. The quotient of the group of translations over \( \mathbb{Z}_{n_1,n_2,n_3} \) is the group of translations over the distances \( \delta_i \in [0, \lambda_i) \). The averaging procedure can now be split onto two steps:

- Averaging over \( \mathbb{Z}_{n_1,n_2,n_3} \), which simply replaces the \( j \)-label of a single edge with the average \( \left[ \right]_{e_i} \) of all the edges parallel to it.
• Averaging over the quotient group. The result of this step follows directly from the observation that upon the action of the quotient group (with exception of the group neutral element which forms a zero measure set) each plaquet of the graph is intersected by exactly one edge (now carrying the averaged $j$) orthogonal to it.

The above procedure leads immediately to (3.5) and (3.4) and further, after reassembling the surfaces $S_i$ out of the plaquets, to (3.3).

The potential application of the studies performed here to the models, where the cosmological spacetime is defined by the semiclassical (often chosen to be coherent) state, lead to another complication. So far we have considered the single graph. Why such choice is perfectly fine to define a basis of a Hilbert (sub)space, it may be insufficient for such models. Therefore one needs to extend the dictionary to incorporate large number of such spin-network “superselection sectors”. We provide such extension below, using the integral Hilbert space construction presented in sec II A 3.

C. The integral extension

As in the case of a single lattice space, here we are going to define some subspace of $\mathcal{H}^{LQG}_{\text{phy}}$.

1. We start with a single lattice spin network defined in sec. II A (without introducing the background metric $\gamma$).

2. The plaquets of the spin network define three classes of surfaces of topology $T^2$ (maximal surfaces on the embedding manifold) such that within one class the surfaces do not intersect each other and all the intersections of the surfaces of distinct classes are of $S^1$ topology.

3. The discrete classes of surfaces are next completed to congruences of the embedding manifold (using the surfaces of the same topology) keeping the requirement, that intersections between representants of different classes are $S^1$. This (non-unique) extension always exists.

4. We extend the original lattice spin network to a class of disjoint spin networks of which edges are intervals of the intersections of surfaces defined above. This class is selected in such a way, that

   (a) each point of the embedding manifold is a node of exactly one spin network in the class, and

   (b) given two graphs of the set, the maximal $T^2$ surfaces of point 2 are interlaced, that is within each class of surfaces, between two surfaces of one graph there is exactly one surface of the other graph.

5. Each spin network is completed to a (topologically) regular lattice by adding edges with $j = 0$.

In its essence this method produces a continuum of (topologically) regular lattices of which edges are “parallel”. They define a distinguished coordinate system $x_i$ where the coordinates are functions constant on the $T^2$ surfaces from point 3. Given this coordinate system, one can equip the manifold with a background fiducial metric $\gamma = \sum_{ij=1}^{3}(dx_i)^2$. This construction is of course quite restrictive, however it allows to preserve the well defined notion of principal (diagonal) directions of the single lattice.

One way to produce the specific example of such set of spin-networks is to start with one regular lattice, equip the embedding manifold with the fiducial metric $\gamma$ (as in sec. II A) and then act with the active rigid translations defined in sec. II B. The family of possible continuous sets defined in points 1. − 5. is however much bigger. In particular, the lattices do not need to be regular with respect to the metric $\gamma$.

An important property of the selected set of spin networks is that it admits a well defined Lebesgue measure $d\sigma$ induced by the Lebesgue measure of a minimal cube of any (chosen arbitrarily) lattice within the set. This measure can now be used to construct the integral Hilbert space via (2.5) via setting $d\mu(\text{c}) = d\sigma$. Choosing different minimal cubes will lead to unitarily equivalent spaces.

---

10 One can introduce a notion of parallel edges terminating in a 6-valent node as the pair not being the edge of a single plaquet and next build the surface by selecting a plaquet and extending the surface by including plaquets whose least two edges are parallel to edges contained already by the surface.
Given the new (integral) Hilbert (sub)space we proceed with defining averaged quantities $p^i$, $\sigma$ exactly as in sec. III B. The only difference is an additional integration over the selected set of lattices. The calculations yield

$$p_i = \frac{8\pi \gamma \ell^2_p}{n_i} \int d\sigma(\vec{\varepsilon}) \sum_{e \in \{e\}_i} \sqrt{J_e(J_e + 1)} =: \frac{8\pi \gamma \ell^2_p}{n_i} \sum_{i},$$

$$\sigma_i = \frac{8\pi \gamma \ell^2_p}{n_1 n_2 n_3} \int d\sigma(\vec{\varepsilon}) \sum_{e \in \{e\}_i} \sqrt{J_e(J_e + 1)} = \sigma_1 \frac{\lambda_i}{\lambda_1 \lambda_2 \lambda_3} = 8\pi \gamma \ell^2_p [\sqrt{J(j + 1)}]_{e_i} =: \Delta_i,$$

where $\vec{e}$ labels the superselection sectors and $\Gamma(\vec{e})$ is the graph corresponding to superselection sector $\vec{e}$. We see, that the quantities $\lambda_i$, $\Delta_i$ are now defined by (3.6b) and do not necessary correspond to the average over fiducial lengths of the edges (which in turn may not be the constants of the graph). The average $j$-label, is however a proper average

$$\Delta_j = \int d\sigma(\vec{e}) \Delta_j(\vec{e}).$$

The above result has been found under assumption of a specific compact topology of the embedding manifold. However in LQC a consistency restrictions that actually do fix the theory originate in models of the noncompact universes, some of them already in isotropic sector. Therefore it is prudent to extend our dictionary to such case and to incorporate in it the notion of isotropy. For simplicity we will consider single lattice states only, although the generalization to the integral states of sec. III C is not difficult.

D. The noncompact extension and the isotropy

In order to keep the model as simple as possible we consider its extension to the flat space, assuming $R^3$ topology.

1. The extension to $R^3$

In LQC the standard method of dealing with infinities due to Cauchy slice noncompactness is selection of a compact region – the so called fiducial cell – which then becomes the infrared regulator of the model. The physical predictions are then extracted within the regulator removal limit. Well definiteness of that limit is the first consistency condition imposed on LQC and is precisely the origin of the so called improved dynamics prescription $^{12}$. Here we follow the same idea: We start with the construction of a single lattice spin network, as specified in sec. III A, although now the lattice is open and infinite. We then introduce the background structure exactly as in the compact case and distinguish the regulator – a rectangular cube of edges pointing in directions of $\vec{e}_i$. The expansion of the regulator is well defined in terms of the number of edges forming the interval of “straight lines” which is contained within the cube (or equivalently, the number of elementary cells stacked along the edge of the cube).

The remaining spatial gauge freedom of the model is the same as in the compact case: the rigid translations. Now however we encounter a technical difficulty: the translation group is noncompact and does not preserve the regulator structure. We sidestep this problem, introducing the periodic boundary conditions on the faces of the regulator, thus restricting to certain compact “cyclic translation” group. The new “translations” are obviously not elements of the original translation group, however the proposed “trick” is well motivated on the heuristic level: as $(i)$ their action will equal that of the original translations group elements on those spin networks which are composed of the set of identical “copies” of the portion contained within the regulator cell, and $(ii)$ in the regulator removal limit we recover the original translation group.

On the technical level the above construction brings us exactly to a compact setting $T^3$ topology considered in previous subsections. Thus we proceed with construction of the dictionary exactly as before. The results (3.5)–(3.6) remain true here. In the regulator removal limit the values $p_i$ reach infinity, however the plaquets areas are well defined

$$\sigma_i = \lim_{n_1, n_2, n_3 \to \infty} \frac{8\pi \gamma \ell^2_p}{n_1 n_2 n_3} \sum_{e \in \{e\}_i} \sqrt{J_e(J_e + 1)} = 8\pi \gamma \ell^2_p [\sqrt{J(j + 1)}]_{e_i} =: \Delta_i,$$

provided that the limit on the right-hand side exists. It is however a reasonable expectation if we consider asymptotically homogeneous state. As a consequence the relation (3.3) extends to the noncompact case.
2. The isotropic sector

In comparison to the homogeneous non-isotropic spacetimes considered so far, the isotropic ones admit an additional symmetry class (subgroup) — the rotations.

Implementing these symmetries in the compact $T^3$ case is not possible, thus our starting point is the noncompact setting of the previous sub-section. Here we make one additional initial assumption: we restrict the auxiliary metric by requiring that the fiducial lengths of the edges in all three directions are the same. The flat metric $\gamma$ now defines the group of rigid rotations. As in the case of the translations, we consider them as active diffeomorphisms and average over them the observables used to define the dictionary.

Due to noncompactness, the only meaningful element of the dictionary is the average area of plaquets $\sigma_i$. The rotational transformation of the spin-network can be easily parametrized by the Euler angles. As in the case of translations, here we can distinguish a discrete group if the rotations by proper angles, which due to insensitivity of the area operator to the orientation of the edges can be replaced by a group $\Sigma^3$ of permutations of $\gamma e_i$. We can then distinguish a quotient group $SO(3)/\Sigma^3$.

Let us first consider the averaging over (the group of translations and) $\Sigma^3$. It follows immediately from (3.8) that

$$\sigma_\Sigma := \sigma_i = \lim_{n_1, n_2, n_3 \to \infty} \frac{8\pi\gamma_l^2}{n_1 n_2 n_3} \sum_{e \in E(\Gamma)} \sqrt{j_e(j_e + 1)} = 8\pi\gamma_l^2 E(\Gamma) \sum_{e \in E(\Gamma)} |q_{ab} n^a \gamma^e(e)| \sqrt{j_e(j_e + 1)} = \Delta_1 + \Delta_2 + \Delta_3 =: \Delta_* \quad (3.9)$$

where this time, we sum over all the edges of the spin network. This result is identical with the FRW limit of Bianchi I geometry (in LQC) studied in [19].

The averaging over (translations and) full $SO(3)$ is slightly more involved.

$$\sigma_R := \sigma_i = \lim_{n_1, n_2, n_3 \to \infty} \frac{8\pi\gamma_l^2}{n_1 n_2 n_3} \int_{SO(3)} d\sigma_{SO(3)} \sum_{e \in E(\Gamma)} |q_{ab} n^a \gamma^e(e)| \sqrt{j_e(j_e + 1)} \quad (3.10)$$

where $n^a$ is a unit (in $\gamma$) vector orthogonal to the plaquet and $\gamma^a e_i$ is the fiducial triad element tangent to the graph edge $e$. The factor $q_{ab} n^a \gamma^e(e)$ is a consequence of averaging over spatial translations as only the orthogonal (to $\gamma e_i$) “crossection” of the plaquet will contribute to the average over the translations along $\gamma e_i$. Using the known chart of $SO(3)$ defined by Euler angles (convention $Z(\alpha)X(\beta)Z(\gamma)$) [11] we get

$$\sigma_R = \lim_{n_1, n_2, n_3 \to \infty} \frac{8\pi\gamma_l^2}{n_1 n_2 n_3} \frac{4}{\pi^2} \int_{\alpha, \beta, \gamma} \sin(\beta) d\beta d\gamma \frac{1}{3} \left[ \sin(\beta)(\cos(\gamma) + \sin(\gamma)) + \cos(\beta) \right] \sum_{e \in E(\Gamma)} \sqrt{j_e(j_e + 1)} \quad (3.11)$$

$$= \frac{4\pi\gamma_l^2}{E(\Gamma)} \frac{3}{2} \Delta_*$$

where to write the first equality we split the rotation group onto $\Sigma^3$ and the quotient $SO(3)/\Sigma^3$.

As we can see, the contribution from the general angle rotations increases the physical plaquet area by a factor $3/2$. This discrepancy is unexpected, since the critical energy density (upper bound of the matter energy density operator spectrum) is a bijective function of the area gap and both Bianchi I and FRW spacetime models in LQC provide the same value of that quantity (see [40] versus [3]). To address this discrepancy we will investigate a bit closer the rotation group used in the averaging process: the rotations in fiducial metric $\gamma$.

To do so, let us consider a large (classical size) cube of fiducial size $L$. Denote its surface area (averaged over the translation group and $\Sigma^3$ to mimic an isotropic spacetime as close as possible without rotating the graph) in the case when its edges are oriented along the triad vectors $\gamma e_i$, by $A_{\gamma}$. By repeating the same calculation as in (3.11) one can show, that upon rotating this cube by Euler angles $(\alpha, \beta, \gamma)$ the surface area changes as follows:

$$A_{\gamma}(\alpha, \beta, \gamma) = A_{\gamma}[\sin(\beta)(\cos(\gamma) + \sin(\gamma)) + \cos(\beta)] \quad (3.12)$$

This implies in particular, that even if the $j$-label distributions in all the directions are the same this surface area is not invariant under the rotations. The direct consequence of it is the no-go statement: one can not build the isotropic spacetime using lattice spin network with the edges oriented in directions of one particular vector triad. For that, the large set of spin networks oriented in random directions would be needed. Such construction is quite easy, if for example one defines the integral Hilbert space structure (discussed in sec. [11A3] using the integral measure of $SO(3)$

---

11 Instead of rotating the plaquet, we keep it fixed at $z = 0$ and rotate the spin network.
In this case the averaging over rotations can be performed as averaging of the spatial oscillations of the electromagnetic field corresponding to the photon of certain energy (defined in turn the definitions of meter and second one sees, that the definition of a distance unit can be recast as a certain number proportional to the number of inhomogeneities given interval is able to accommodate, thus should be proportional to the number \( n_i \) of the spin-network edges.

\[
\sigma_i = \int_{SO(3)} d\sigma_{SO(3)} q_{ab} e_i^a e_i^b M(g)^{bc} e_i^c \lim_{n_1 n_2 n_3 \to \infty} \frac{8 \pi g^2 \ell^2_{Pl}}{n_1 n_2 n_3} \int d\sigma(\vec{\epsilon}) \sum_{e \in \{e\}_i(\vec{\epsilon}, g)} \sqrt{j_e(j_e + 1)}
\]

(3.13)

where \( g \) is the finite rotation (parametrized by Euler angles) and \( M(g) \) is the rotation matrix corresponding to it. The symbol \( \Gamma(\vec{\epsilon}, g) \) denotes here graphs oriented along the fiducial triad \( \vec{\epsilon}_i \) rotated by \( g \) (that is a support of a superselection sector of lattices oriented in the rotated triad) and belonging to the superselection sector labeled by \( \vec{\epsilon} \). Using again the \( SO(3) \) chart defined by Euler angles gives then

\[
\sigma_i = \frac{\gamma \ell^2_{Pl}}{\pi} \int_0^{2\pi} d\alpha \int_0^\pi \sin(\beta) d\beta \int_0^{2\pi} d\gamma \sum_{k=1}^{3} (e_i \cdot M(\alpha, \beta, \gamma) e_k) [\sqrt{j(j + 1)}]_{e_k, e_i(\alpha, \beta, \gamma)} =: \tilde{\Delta}_i,
\]

(3.14)

where the rotation matrix \( M \) are now expressed in terms of Euler angles and the averages over the translation superselection sectors of \( j \)-labels (originally defined in sec. [III C]) are now defined separately for each rotation superselection sector labeled by \( (\alpha, \beta, \gamma) \).

In the case, the averages of \( j \)-labels of edges in the same direction over the graph (within a single superselection sector) are equal, a simple calculation shows that

\[
\sigma_i = 4 \pi \gamma \ell^2_{Pl} \sum_{i=1}^{3} [\sqrt{j(j + 1)}]_{e_i, e_i} = \frac{1}{2} \sum_{i=1}^{3} \Delta_i,
\]

(3.15)

where \( \Delta_i \) is defined in (3.6b). This leads exactly to the correction of the Bianchi I plaquet area by a multiplicative factor \( 3/2 \) restoring the consistency with the isotropic limit.\footnote{In this case the averaging over rotations can be performed as averaging of \( j \)-labels between the rotation superselection sectors.}

### E. The physical consequences

For both construction of separable physical Hilbert space (the single superselection sector and the integral one) we reached the same conclusion. Given a “lattice” LQG state specified in sections [III A] or [III B] (as well as its extension discussed in previous subsection) the Bianchi I LQG state mimicking it has to have the single plaquet area is proportional to the average of \( j \)-labels (in appropriate direction) of the original LQG state. The time dependence of the latter depends in turn on (i) the choice of the (initial) LQG state and (ii) on the statistics of the particular Hamiltonian used to generate the time evolution. In particular, any model following from strictly graph preserving Hamiltonian will have

\[
\sigma_i \propto p_i
\]

(3.16)

which will lead to original Bojowald’s prescription in LQC. On the other hand, the studies of the noncompact model, show that in low energy limit \( \sigma_i \) should be constant. As a consequence, for the class of states considered in this article the average \( j \)-label, more precisely \( \Delta_j \) should approach constant in low curvature limit, thus the expansion of the spacetime in the process of dynamical evolution should follow from increasing the number of spin-network nodes rather than the \( j \)-labels.

This conclusion is supported by our intuitive understanding to the physical distance. Essentially, by examining the definitions of meter and second one sees, that the definition of a distance unit can be recast as a certain number of the spatial oscillations of the electromagnetic field corresponding to the photon of certain energy (defined in turn by particular spontaneous emission process). On the other hand, the coupling of matter to gravity in LQG leads to theory, where the matter degrees of freedom are represented by quantum labels living on the nodes (vertices) or the edges of the spin network (depending on the type of matter). This leads to an intuition, that the physical distance should be proportional to the “number of inhomogeneities” given interval is able to accommodate, thus should be proportional to the number \( n_i \) of the spin-network edges.
The meaning of orthogonality has been defined subsequently by no longer restrictive partial gauge fixing. Upon this fixing the spin network supporting the state become regular lattice. The choice of (metric) regularity of global orthogonal translations– which were considered as active transformations. In the process no restrictions regarding the distribution of the magnetic spins [48, 49, 58].

One of the ways of determining that value from genuine LQG is provided by the interface of Chern-Simons theory with LQG used to evaluate black hole entropy [48, 54]. Indeed, the comparison of two statistical calculations in [48] shows, that the edges with \( j > 1/2 \) provide significant contribution to the surface areas, thus the average “area gap” \( \Delta_i \) should be detectably larger than \( \Delta \). On the other hand, more detailed combinatorial analysis of the relevant statistics [55] shows that for small areas the BH entropy features a “stair-like” structure which on heuristic level can be interpreted as the existence of “quantum of the area”. The numerical simulations [50] determined it to approximately equal \( \Delta' \approx 7.565\ell_p^2 \), which would give the average \( \sqrt{j(j+1)} \approx 1.267 \) (roughly corresponding to average \( \bar{j} \approx 0.86 \)). The stair-like entropy structure dissipates for larger area due to dispersion in \( j \) and the complicated nature of the spectrum of area operator [56], however these are the low areas (small number of graph edges intersecting the area) where any effect of \( j \) distribution, especially any “peakness” of it, should be the most easy to observe.

While heuristic, the above argument provides a strong indication that, while the LQC area gap will not correspond exactly to \( \Delta \), it will remain of the same order. Given an interface constructed here it further suggests a specific correction to that area gap [13]. Its value can be determined more precisely via use of the same statistical methods originally applied to find the distribution of the magnetic spins [48, 49, 58]. This is however beyond the scope of this article.

IV. CONCLUSIONS

We have considered the relation between the loop quantum cosmology physical states and the physical states of full loop quantum gravity possibly representing homogeneous universe on the kinematical level (that is without controlling the consistency of dynamical predictions of LQG and LQC frameworks). The LQG has been applied at its genuine level, without any simplifications. Using the observable quantities well defined both in loop quantum gravity and cosmology we constructed a precise interface between these two frameworks for the class of Bianchi I spacetimes of toroidal spatial topology. To define this interface we used the specific subspace of genuine LQG states, distinguished by minimal selection criteria allowing to define the necessary components of the LQC auxiliary structure for these states and motivated by construction originally proposed in [19].

In the case the physical state is supported on one spin network we assumed that the spin networks supporting the state are topologically equivalent to a sub-networks of the regular lattices. When the state is composed out of a continuum of states on distinct spin networks we further provided a notion of congruence of the embedding manifold by parallel lattices. This criterion, being the sole restrictive condition on the LQG physical Hilbert space allowed to provide a precise notion of global orthogonal directions on the spatial manifold.

This structure has proven to be sufficient to define all the remaining auxiliary LQC structure necessary to construct a dictionary. The embedding manifold has been equipped with LQC fiducial background metric via partial gauge fixing. Upon this fixing the spin network supporting the state become regular lattice. The choice of (metric) regularity of the lattice was however not relevant in further construction of the dictionary, being instead a matter of convenience. Indeed, it was shown, that averaging over the diffeomorphisms preserving the global orthogonal directions leads to the same results.

The result of this identification associated the LQC area of each plaquet with the average of \( j \)-labels (orthogonal to the plaquet) of the LQG spin network. No restrictions stronger that this relation (orthogonal to the plaquet of the LQG spin network.

\[ \Delta_i \approx j \]

13 The idea that the value \( \Delta' \) should replace the LQC area gap has been originally suggested [50] by authors of [50]. It was however subsequently abandoned due to lack (at that time) of justification for such choice.

14 The meaning of orthogonality has been defined subsequently by no longer restrictive partial gauge fixing.
there we concluded, that the particular association of the values of these areas as the LQC phase space functions (known as LQC prescription choice) depend solely on the statistical properties of the Hamiltonian (constraint) generating the evolution of the spin network and are not restricted by the principles imposed in construction the the LQG-LQC interface. In particular both the Bojowald’s $\mu_n$ prescriptions and the so called improved dynamics can be a priori realized.

The results were further extended to the case of flat Bianchi I universe of topologically $\mathbb{R}^3$ spatial slices, where the relation found in $T^3$ case persisted unmodified \(48\). This relation and the consistency requirements on the LQC framework (existence of well defined infrared regulator removal limit) implies some restrictions on $j$-label statistics of LQG: for the class of states used to build the interface the averages of $j$-labels have to remain constant on low energy (gravitational field) limit.

The studies were further extended to the case of isotropic flat FRW universe, where the plaquet area operators were further averaged over additional symmetries admitted by these classes of spacetimes – the rotations. This process related the (now unique) plaquet area with the average of $j$-labels over all the edges of the spin network graph \(39\). Two levels of implementation of the symmetries were considered: averaging over a discrete group of rotations by proper angles and the full $SO(3)$. Studies of the former case lead to the FRW limit of Bianchi I cosmology consistent with the analogous limit found in \(19\). In the latter it was found, that due to contributions of all the graph $j$-labels in the case the rotation angles differ from (multiples of) proper ones the area of the plaquet is larger by a factor 3/2.

Due to apparent discrepancy of the above result with the FRW limit of Bianchi I found in \(19\) the changes of areas in considered model have been investigated up close. Consequently, it was found, that the set of lattices oriented in directions of one distinguished triad is insufficient to support the states accurately reproducing an isotropic spacetime. Consequently, an extension of the Hilbert space using the integral structure defined by the group of rotations was proposed. It was further shown that on the extended space the discrepancy is cured and the single plaquet area in the models of Bianchi I universes is also increased by a factor 3/2.

The found results have been finally confronted with the heuristic estimates of the $j$-statistics following from studies of black hole entropy in LQG. The dictionary constructed in this article indicates that associating the plaquet areas with the minimal nonzero LQG area is accurate only in cases when the spin network statistics makes $j > 1/2$ non-generic (zero measure contribution). On the other hand, the (known in literature) heuristic results following from numerical analysis of black hole entropy provide a natural (from the point of view of the constructed dictionary) estimate on the $j$-label averages. This estimate leads again to the constant area gap principle of improved dynamics. It indicates however a slightly different value of this area gap, corresponding to the LQC critical energy density $\rho_c \approx 0.19 \rho_{Pl}$. This value, while lower than the original LQC critical energy density ($\approx 0.41 \rho_{Pl}$) remains at the same level of magnitude.

The general purpose for constructing the above dictionary is to provide a viable tool of analyzing the cosmological limit of more advanced models aimed towards controlling or approximating the LQG dynamics (see of example \(21, 28\)). Since (i) the (restrictive) selection criteria are precisely controlled here and the formalism remains relatively general, and (ii) the formalism is adaptable to majority of prescriptions in defining the Hamiltonian (constraint) in LQG, it can be applied to a wide variety of models. It allows to extract the cosmological degrees of freedom out of such models in a precise way, further providing a tool for validating the initial assumptions selected in their construction (like for example the statistical averages of $j$ distributions). Through the consistency conditions on LQC in case of noncompact universes it also provides a tool for consistency control of the LQG models.

At this point it is necessary to remember, that the interface relies on quite strong restriction of the spin-network graph topology. In principle, no such restrictions should be made in order for the results to be completely robust. Any generalization however, for example using the random graphs \(59\) is extremely difficult, as in such cases the LQC auxiliary structure (being a relevant part of the interface) has to emerge on the physical level (via observables) and may strongly depend on the $j$-label statistics of the physical states, which in turn is decided by the details in Hamiltonian (constraint) construction.

**ACKNOWLEDGMENTS**

Author thanks Edward Wilson-Ewing for discussions and helpful comments and especially to Emanuele Alesci for extensive discussions and the encouragement to write this article. This work has been supported in parts by the Chilean FONDECYT organization under regular project 1140335 and the National Center for Science (NCN) of Poland research grant 2012/05/E/ST2/03308 as well as by UNAB via internal project DI-562-14/R.
Appendix A: Averaging over diffeomorphisms

Consider a 1-dimensional lattice \((n \text{ edges})\) spanned across the interval \([0,1]\) with uniform random distribution of the vertices and each edge equipped with value \(x_i\). Consider further the “average” of some function \(f(x_i)\) weighted by the length of each edge

\[
\mathcal{T} = \sum_{i=1}^{n} f(x_i)l_i
\]  

(A1)

The probabilistic space of the edge length distribution is the \(n\) dimensional romboid

\[
\sum_{i=1}^{n} l_i = 1, \quad \forall i \in \{1, \ldots, n\} : l_i > 0,
\]  

(A2)

with the measure \(dx = dl_1 \ldots dl_n\). The volume of this romboid is \(V_n = 1/n!\). The average value \(\langle l_i \rangle\) of \(l_i\) is the ratio of the volume of the romboid over the \(n-1\) dimensional volume of its base, which is \(V_{n-1}\). As a consequence we have

\[
\langle \mathcal{T} \rangle = \sum_{i=1}^{n} f(x_i)\langle l_i \rangle = \frac{1}{n} \sum_{i=1}^{n} f(x_i),
\]  

(A3)

which corresponds precisely to the case, where the vertices of the lattice are distributed uniformly.
[26] K. Giesel and T. Thiemann, *Algebraic Quantum Gravity (AQG). II. Semiclassical Analysis*, Class. Quant. Grav. 24 (2007) 2499–2564, [arXiv:0707.1000](http://arxiv.org/abs/0707.1000).

[27] E. Alesci, F. Cianfrani, and C. Rovelli, *Quantum-Reduced Loop-Gravity: Relation with the Full Theory*, Phys. Rev. D88 (2013) 104001, [arXiv:1309.6304](http://arxiv.org/abs/1309.6304).

[28] E. Alesci and F. Cianfrani, *Quantum Reduced Loop Gravity: Semiclassical limit*, arXiv:1402.3155.

[29] N. Bodendorfer, *A quantum reduction to Bianchi I models in loop quantum gravity*, arXiv:1410.5608.

[30] J. F. Barbero G., *Real Ashtekar variables for Lorentzian signature spacetime times*, Phys. Rev. D51 (1995) 5507–5510, [gr-qc/9410014](http://arxiv.org/abs/gr-qc/9410014).

[31] J. Lewandowski, A. Okolow, H. Sahlmann, and T. Thiemann, *Uniqueness of diffeomorphism invariant states on holonomy-flux algebras*, Commun. Math. Phys. 267 (2006) 703–733, [arXiv:0504147](http://arxiv.org/abs/0504147).

[32] B. Dittrich, *Partial and Complete Observables for Canonical General Relativity*, Class. Quant. Grav. 23 (2006) 6155–6184, [gr-qc/0507106](http://arxiv.org/abs/gr-qc/0507106).

[33] B. Dittrich, *Partial observables*, Phys. Rev. D65 (2002) 124013, [gr-qc/0110035](http://arxiv.org/abs/gr-qc/0110035).

[34] B. Dittrich and T. Thiemann, *Testing the master constraint programme for loop quantum gravity. I. General framework*, Class. Quant. Grav. 23 (2006) 1025–1066, [gr-qc/0411138](http://arxiv.org/abs/gr-qc/0411138).

[35] B. Dittrich and T. Thiemann, *Testing the master constraint programme for loop quantum gravity. II. Finite dimensional systems*, Class. Quant. Grav. 23 (2006) 1067–1088, [gr-qc/0411139](http://arxiv.org/abs/gr-qc/0411139).

[36] B. Dittrich and T. Thiemann, *Testing the master constraint programme for loop quantum gravity. III. SL(2,R) models*, Class. Quant. Grav. 23 (2006) 1089–1120, [gr-qc/0411140](http://arxiv.org/abs/gr-qc/0411140).

[37] B. Dittrich and T. Thiemann, *Testing the master constraint programme for loop quantum gravity. IV. Free field theories*, Class. Quant. Grav. 23 (2006) 1121–1142, [gr-qc/0411141](http://arxiv.org/abs/gr-qc/0411141).

[38] B. Dittrich and T. Thiemann, *Testing the master constraint programme for loop quantum gravity. V. Interacting field theories*, Class. Quant. Grav. 23 (2006) 1143–1162, [gr-qc/0411142](http://arxiv.org/abs/gr-qc/0411142).

[39] M. Domagala, K. Giesel, W. Kamiński, and J. Lewandowski, *Gravity quantized*, Phys. Rev. D82 (2010) 044024, [arXiv:0905.4949](http://arxiv.org/abs/0905.4949).

[40] A. Ashtekar and E. Wilson-Ewing, *A Geometric perspective on singularity resolution and uniqueness in loop quantum cosmology*, Phys. Rev. D80 (2009) 044024, [arXiv:0905.4949](http://arxiv.org/abs/0905.4949).
[47] M. Bojowald, *Homogeneous loop quantum cosmology*, Class.Quant.Grav. **20** (2003) 2595–2615, [gr-qc/0303073](http://arxiv.org/abs/gr-qc/0303073).

M. Bojowald, G. Date, and K. Vandersloot, *Homogeneous loop quantum cosmology: The Role of the spin connection*, Class.Quant.Grav. **21** (2004) 1253–1278, [gr-qc/0311004](http://arxiv.org/abs/gr-qc/0311004).

[48] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, *Quantum geometry and black hole entropy*, Phys.Rev.Lett. **80** (1998) 904–907, [gr-qc/9710007](http://arxiv.org/abs/gr-qc/9710007).

A. Ashtekar, J. C. Baez, and K. Krasnov, *Quantum geometry of isolated horizons and black hole entropy*, Adv.Theor.Math.Phys. **4** (2000) 1–94, [gr-qc/0005126](http://arxiv.org/abs/gr-qc/0005126).

[49] M. Domagała and J. Lewandowski, *Black hole entropy from quantum geometry*, Class.Quant.Grav. **21** (2004) 5233–5244, [gr-qc/0407051](http://arxiv.org/abs/gr-qc/0407051).

A. Corichi, J. Diaz-Polo, and E. Fernandez-Borja, *Black hole entropy quantization*, Phys.Rev.Lett. **98** (2007) 181301, [gr-qc/0609122](http://arxiv.org/abs/gr-qc/0609122).

A. Corichi, J. Diaz-Polo, and E. Fernandez-Borja, *Quantum geometry and microscopic black hole entropy*, Class.Quant.Grav. **24** (2007) 243–251, [gr-qc/0605014](http://arxiv.org/abs/gr-qc/0605014).

[51] I. Agullo, J. F. Barbero G., J. Diaz-Polo, E. Fernandez-Borja, and E. J. Villaseñor, *Black hole state counting in LQG: A Number theoretical approach*, Phys.Rev.Lett. **100** (2008) 211301, [arXiv:0802.4077](http://arxiv.org/abs/0802.4077).

[52] J. F. Barbero G. and E. J. Villaseñor, *On the computation of black hole entropy in loop quantum gravity*, Class.Quant.Grav. **26** (2009) 035017, [arXiv:0810.1599](http://arxiv.org/abs/0810.1599).

[53] J. Engle, A. Perez, and K. Noui, *Black hole entropy and SU(2) Chern-Simons theory*, Phys.Rev.Lett. **105** (2010) 031302, [arXiv:0905.3168](http://arxiv.org/abs/0905.3168).

[54] G. Fernando Barbero, J. Lewandowski, and E. J. Villaseñor, *Flux-area operator and black hole entropy*, Phys.Rev. **D80** (2009) 044016, [arXiv:0905.3465](http://arxiv.org/abs/0905.3465).

[55] I. Agullo, J. Fernando Barbero, E. F. Borja, J. Diaz-Polo, and E. J. Villaseñor, *Detailed black hole state counting in loop quantum gravity*, Phys.Rev. **D82** (2010) 084029, [arXiv:1101.3660](http://arxiv.org/abs/1101.3660).

[56] G. Fernando Barbero and E. J. Villaseñor, *Statistical description of the black hole degeneracy spectrum*, Phys.Rev. **D83** (2011) 104013, [arXiv:1101.3662](http://arxiv.org/abs/1101.3662).

[57] J. Diaz-Polo and E. Fernandez-Borja oral communication (IGPG seminar, PennState), 2007.

[58] K. A. Meissner, *Black hole entropy in loop quantum gravity*, Class.Quant.Grav. **21** (2004) 5245–5252, [gr-qc/0407052](http://arxiv.org/abs/gr-qc/0407052).

[59] H. Sahlmann, *Wave propagation on a random lattice*, Phys.Rev. **D82** (2010) 064018, [arXiv:0911.4180](http://arxiv.org/abs/0911.4180).