On a Theorem of Stelzer for Some Classes of Mixed Groups

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Abstract. We identify some classes $C$ of mixed groups such that if $G \in C$ has the cancellation property then the Walk-endomorphism ring of $G$ has the unit lifting property. In particular, if $G$ is a self-small group of torsion-free rank at most 4 with the cancellation property then it has a decomposition $G = F \oplus H$ such that $F$ is free and the Walk-endomorphism ring of $H$ has the unit lifting property.

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1. Introduction

In this paper all groups are abelian. If $G$ is a group then $T(G)$ represents the torsion part of $G$, if $p$ is a prime then $T_p(G)$ is the $p$-component of $G$, $r_0(G)$ is the torsion-free rank of $G$, and $\text{End}(G)$ is the endomorphism ring of $G$. If $G$ is not a torsion group, a subgroup $H \leq G$ is a full free subgroup if $H$ is free and $G/H$ is a torsion group. For other notions and notations, we refer to Ref. [17] and to Sect. 2.

A group $G$ has the cancellation property if whenever $H$ and $K$ are groups such that $G \oplus H \cong G \oplus K$ it follows that $H \cong K$. An important question is if we can find classes of groups which have the cancellation property and they can be characterized by using some properties of endomorphism rings. For instance, a group $G$ has the substitution property if and only if 1 is in the stable range of $\text{End}(G)$ [23].

For finite rank torsion-free groups, the cancellation property was characterized in [7, Theorem 21] by using some endomorphism rings: a torsion-free group $G$ of finite rank has the cancellation property if and only if

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$G = B \oplus C$ such that (a) $B$ is free (b) for every positive integer $n$ the units of $\text{End}(C)/n\text{End}(C)$ can be lifted to units of $\text{End}(C)$, and (c) the endomorphism rings of all quasi-direct summands of $C$ satisfy some technical conditions discovered by Eichler in the 1930s. If a ring $R$ has the property described in (b), i.e. for every positive integer $n$ all units of $R/nR$ can be lifted to units of $R$, then we will say that $R$ has the unit lifting property. The properties (a) and (b) for the necessity part of Blazhenov’s characterization were proved in [22, Theorem A]. In fact, Stelzer proved that if $G$ is a reduced torsion-free group of finite rank without free direct summands there exists a torsion-free group $H$ such that (i) $\text{Hom}(H, G) = 0 = \text{Hom}(G, H)$, (ii) $\text{End}(H) \cong \mathbb{Z}$, and (iii) for every positive integer $n$ there is an epimorphism $H \to G/nG$. Then the conclusion comes from a result of Fuchs [18], which states that if $G$ satisfies these three properties and has the cancellation property then $\text{End}(G)$ has the unit lifting property. Stelzer used in [22, Theorem B] this result to conclude that a strongly indecomposable torsion-free group $G$ has the cancellation property if and only if $G \cong \mathbb{Z}$ or 1 is in the stable range of $\text{End}(G)$. This extends the characterization of Fuchs and Loonstra [19], for torsion-free groups of rank 1 that have the cancellation property. It was proved in [21] that this characterization cannot be simplified.

Our main aim is to prove that a version of Stelzer’s Theorem [22, Theorem A], that uses the Walk-endomorphism ring instead of the classical endomorphism ring, is valid for some classes of mixed groups. In particular, it is proved that it works for self-small groups of torsion-free rank at most 4 (Theorem 4.5).

We denote by $S$ the class of infinite self-small groups of finite torsion-free rank. Some of the results about finite rank torsion-free groups and also some techniques used in the theory of finite rank torsion-free groups can be extended to some subclasses of $S$, e.g. [2,9,10]. However, this is not always valid. We refer to [2, Example 2.5] and [9, Example 3.6] for some examples. In [2, Section 2] some connections between substitution property and unit lifting property for groups in $S$ are presented. Mixed versions of [22, Theorem B] were proved for the class $QD$ of quotient-divisible groups (we remind that $QD \subseteq S$) and for self-small groups of torsion-free rank 1 in [2, Theorem 3.4 and Proposition 3.5]. A mixed version of Fuchs’s Lemma [18] was proved in [8], see Proposition 2.4.

For the reader’s convenience, we close this introduction with a sketch of the proof of this result. It is easy to see that every self-small group $G$ of finite torsion-free rank has a direct decomposition $G = F \oplus H$ such that $F$ is free and $H$ is self-small without free direct summands. Moreover, $G$ has the cancellation property if and only if $H$ has the cancellation property. Hence, it is enough to study self-small groups without free direct summands. Similar reasons allow us to restrict ourselves to the case of reduced groups. Let $S^*$ be the class of reduced infinite self-small groups of finite torsion-free rank without free direct summands. Our aim is to apply the mixed version of Fuchs’s Lemma, Proposition 2.4. Therefore, in Sect. 3, we will provide some sufficient conditions to conclude that a group $G \in S^*$ satisfies the hypothesis of this proposition. In the next section we will write every group $G \in S^*$ as a
sum $G = L + K$ where $L \cap K$ is a full free subgroup of $G$, $L$ is quotient-divisible and $K$ is torsion-free, where the Richman type of $K$ is reduced. Since all groups used here are of finite torsion-free rank, there exists a decomposition $L = F_1 \oplus L_1$ such that $F_1$ is free and $L_1$ has no free direct summands. The structure of the groups $G \in S^*$ for which the rank of $F_1$ or the torsion-free rank of $L_1$ are small enough is presented in Proposition 4.3. This result is used together with Lemma 4.1 to prove that if $G \in S$ is of torsion-free rank at most 4 and has the cancellation property then $G = F \oplus H$ such that $F$ is free and the Walk-endomorphism ring of $H$, $\text{End}_W(H) = \text{End}(H)/\text{Hom}(H, T(H))$, has the unit lifting property, Theorem 4.5. Consequently, a group $G \in S$ of rank at most 4 for which $Q\text{End}_W(G)$ is local has the cancellation property if and only if 1 is in the stable range of $\text{End}_W(G)$.

2. Preliminaries and Known Results

2.1. Self-Small Groups

Self-small groups were introduced by Arnold and Murley in [6] as those groups $G$ such that for every set $I$ the canonical homomorphism $\text{Hom}(G, G(I)) \rightarrow \text{Hom}(G, G(I))$ is an isomorphism. The class of infinite self-small groups of finite torsion-free rank is denoted by $S$. The following characterizations for the groups from $S$ are presented in [1, Theorem 2.1]. For other characterizations, we refer to [1, Section 3] and [12, Theorem 3.1].

We denote by $\mathbb{P}$ the set of all prime integers.

**Theorem 2.1.** Let $G$ be an infinite group of finite torsion-free rank. The following are equivalent:

1. $G \in S$;
2. for all $p \in \mathbb{P}$ the $p$-components $T_p(G)$ are finite, and $\text{Hom}(G, T(G))$ is a torsion group;
3. for every $p \in \mathbb{P}$ the $p$-component $T_p(G)$ is finite and if $F_G \leq G$ is a full free subgroup of $G$ then $G/F_G$ is $p$-divisible for almost all $p \in \mathbb{P}$ such that $T_p(G) \neq 0$.

Recall that bounded pure subgroups are direct summands. Consequently, if $G \in S$ then for every finite set $P$ of primes the $P$-component $\oplus_{p \in P} T_p(G)$ is a finite direct summand of $G$. In this case we will fix a direct decomposition $G = (\oplus_{p \in P} T_p(G)) \oplus G(P)$. If $n > 1$ is an integer, we denote by $P_n$ the set of all prime divisors of $n$, and we will write $G = T_n(G) \oplus G(n)$, where $T_n(G) = \oplus_{p | P_n} T_p(G)$ and $G(n) = G(P_n)$.

There are two important classes of groups which are contained in $S$: the class $TF$ of finite rank torsion-free groups [6], and the class $QD$ of quotient-divisible groups [16]. We recall that a group of finite torsion-free rank $G$ is quotient-divisible if its torsion part is reduced and there exists a full free subgroup $F \leq G$ such that $G/F$ is divisible.

If $C \subseteq S$, we will denote by $QC$ the quasi-category associated to $C$. This is a category which has as objects the groups from $C$, and $\text{Hom}_{QC}(G, H) = \mathbb{Q} \otimes \text{Hom}(G, H)$ (this group is denoted by $\mathbb{Q}\text{Hom}(G, H)$) for all $G, H \in C$. 
The reader can find a general discussion about quasi-categories in [11]. Since for every \( G \in S \) the support group of \( \text{End}(G) \) is of finite torsion-free rank, the ring \( \mathbb{Q}\text{End}(G) \) is a finite dimensional \( \mathbb{Q} \)-algebra, and it follows that the quasi-category \( \mathbb{Q}S \) is Krull-Schmidt [10]. Moreover, a group \( G \in S \) is indecomposable in \( \mathbb{Q}S \) (such a group is called \textit{strongly indecomposable}) if and only if \( \mathbb{Q}\text{End}(G) \) is a local ring.

It was proved in [16] that there exists a duality \( d : \mathbb{QTF} \cong \mathbb{QTD} : d \). Moreover, the restriction of \( d \) to the subcategory \( \mathbb{QTFQD} \) of all torsion-free groups which are quotient-divisible induces a duality \( d : \mathbb{QTFQD} \to \mathbb{QTFQD} \) [5]. In the following we recall some properties associated to this duality. For the reader’s convenience, we point out that if \( X \) is in \( \mathbb{TF} \) or in \( \mathbb{QD} \) then \( d(X) \) is constructed in [16, Theorem 10] by using a full free subgroup \( F \leq X \). Therefore \( d(X) \) is unique up to a quasi-isomorphism.

**Lemma 2.2.** The following are true for a group \( X \) and a full free subgroup \( F \leq X \):

(a) if \( X \in \mathbb{TF} \) and \( p \) is a prime, the \( p \)-component of \( X/F \) is bounded if and only if \( d(X) = B \oplus Y \) with \( B \) a finite group and \( Y \) a \( p \)-divisible group;

(b) if \( X \in \mathbb{QD} \), \( X/F \) is divisible and \( p \) is a prime, we have \( (X/F)_p = 0 \) if and only if \( d(X) \) is \( p \)-divisible;

(c) \( d \) preserves the torsion-free rank.

**Remark 2.3.** The duality \( d \) preserves the quasi-exact sequences (i.e. exact sequences in \( \mathbb{QTF} \), respectively in \( \mathbb{QTD} \)) [15]. If \( 0 \to K \xrightarrow{f} L \xrightarrow{g} M \to 0 \) is a short exact sequence of torsion-free (resp. quotient-divisible) groups then for some suitable positive integers \( \ell \) and \( \ell \)-homomorphisms \( kd(f) \) and \( \ell d(g) \) can be viewed as group homomorphisms \( kd(f) : d(L) \to d(K) \) and \( \ell d(g) : d(M) \to d(L) \) such that \( kd(f)\ell d(g) = 0 \), cf. the proof of [16, Theorem 10]. It follows that \( \text{Ker}(\ell d(g)) \) is finite, \( \text{Im}(\ell d(g)) \) is a finite index subgroup of \( \text{Ker}(kd(f)) \), and \( \text{Im}(kd(f)) \) is a finite index subgroup of \( d(K) \).

### 2.2. The Cancellation Property

A group \( G \) has the \textit{cancellation property} if whenever \( H \) and \( K \) are groups such that \( G \oplus H \cong G \oplus K \) it follows that \( H \cong K \). The group \( G \) has the \textit{substitution property} if for every group \( A \) which has direct decompositions \( A = G_1 \oplus H = G_2 \oplus K \) such that \( G_1 \cong G_2 \cong G \) there exists \( G_0 \leq A \) such that \( G_0 \cong G \) and \( A = G_0 \oplus H = G_0 \oplus K \). It is easy to see that every group with the substitution property has the cancellation property. We refer to [17] for a survey on these properties. In particular, it was proved that there are strong connections between these properties and properties of the endomorphism ring of \( G \): the group \( G \) has the substitution property if and only if 1 is in the stable range of \( \text{End}(G) \) [23]; if \( G \in \mathbb{TF} \) has no free direct summands but has the cancellation property, then \( \text{End}(G) \) has the unit lifting property [22]; if \( R \) is a ring which is torsion-free of finite rank such that \( \mathbb{Q}R \) is local then \( R \) has the unit lifting property if and only if 1 is in the stable range of \( R \) [4].

For the case of (mixed) self-small groups, it was proved in [2] that it is useful to use the \textit{Walk-endomorphism ring} \( \text{End}_W(G) = \text{End}(G)/\text{Hom}(G,T(G)) \) instead of the classical endomorphism ring. A group \( G \in S \) has the
substitution property if and only if 1 is in the stable range of \(\text{End}_W(G)\) [2, Theorem 2.1]. Moreover, if \(G \in QD\) has no free direct summands but has the cancellation property then \(\text{End}_W(G)\) has the unit lifting property [8, Proposition 3.4].

The proofs for the unit lifting properties are based on the following result which was discovered by L. Fuchs in [18] for the torsion-free case. We will present here the mixed version proved in [8, Proposition 3.1]. An epimorphism \(\alpha : H \to U\) is called rigid if for every commutative diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\alpha} & U \\
\downarrow & & \downarrow \\
H & \xrightarrow{\alpha} & U,
\end{array}
\]

with \(\psi\) and \(\phi\) isomorphisms, we have \(\phi = \pm 1_U\).

**Proposition 2.4.** [8, Proposition 3.1] Suppose that \(G \in S\) and that for every positive integer \(n\) there exists a torsion-free group \(H\) such that the pair \((G, H)\) has the following properties

(I) \(\text{Hom}(G, H) = 0, \text{Hom}(H, G)\) is a torsion group, and

(II) there exists a rigid epimorphism \(\alpha : H \to G(n)/nG(n)\).

If \(G\) has the cancellation property then the Walk-endomorphism ring \(\text{End}_W(G) = \text{End}(G)/\text{Hom}(G, T(G))\) has the unit lifting property.

### 3. Groups that Verify the Hypothesis of Proposition 2.4

In this section, we will identify some classes of self-small groups which satisfy the hypothesis of Proposition 2.4.

#### 3.1. The Finite Rank Torsion-Free Case

In the case of torsion-free groups, Stelzer proved in [22, Theorem, p. 367] that every reduced finite rank torsion-free group without free direct summands satisfies the hypothesis of Proposition 2.4. For further use, we also record other results obtained in [22]. We recall that if \(P\) is a set of primes, a group \(K\) is \(P\)-divisible if for all \(p \in P\) and all \(a \in K\) the equation \(px = a\) has a solution in \(K\).

**Proposition 3.1.** [22] Let \(G \in S^*\) be torsion-free of rank \(k\). There exists a torsion-free group \(H\) of rank \(k + 1\) such that:

(i) \(\text{End}(H) \cong \mathbb{Z}\);

(ii) there exist a finite set of primes \(P\) and a free subgroup \(F_H \leq H\) such that the only subgroup of \(G\) which is \(P\)-divisible is 0 and \(H/F_H \cong \bigoplus_{p \in P} \mathbb{Z}(p^\infty)\);

(iii) for each positive integer \(n\) there is an epimorphism \(H \to (\mathbb{Z}/n\mathbb{Z})^k\);

(iv) all subgroups of \(H\) of rank \(\leq k\) are free;

(v) \(\text{Hom}(H, G) = 0\) and \(\text{Hom}(G, H) = 0\).

Consequently, there exists a group \(H\) such that for every positive integer \(n\), the pair \((G, H)\) satisfies Conditions (I) and (II)\(_n\).
The statement (ii) is extracted from the proof of [22, Theorem, p.367], while (iii) is a consequence of [22, Theorem, p.367].

Remark 3.2. One of the main ingredients used in the proof of [22, Lemma 2] is that for every reduced torsion-free group of finite rank $G$ there exists a finite set of primes $P$ such that $G$ has no non-trivial $P$-divisible subgroups. This property is no longer true for the mixed case. For instance, if we consider a group $G$ of finite torsion-free rank with the properties

(i) $\bigoplus_{p \in P} \mathbb{Z}/p\mathbb{Z} \leq G \leq \prod_{p \in P} \mathbb{Z}/p\mathbb{Z}$,
(ii) $(\hat{1}_p)_{p \in P} \in G$ (here $\hat{1}_p$ denotes the unit of $\mathbb{Z}/p\mathbb{Z}$),
(iii) the factor group $G/(\bigoplus_{p \in P} \mathbb{Z}/p\mathbb{Z})$ is a quotient-divisible group which is not reduced,

then $\text{Hom}(G,T(G))$ is a torsion group, hence $G$ is self-small. However, for every finite set $P \subseteq \mathbb{P}$ the subgroup $H \leq G(P)$, defined by the properties $T(G(P)) \leq H \leq G(P)$ and $H/T(G(P))$ is the divisible part of $G(P)/T(G(P)) \cong G/T(G)$, is a non-trivial $P$-divisible subgroup of $G$.

3.2. The Case $G/T(G)$ is Reduced

We can apply Stelzer’s existence result to groups such that $G/T(G)$ is reduced.

Proposition 3.3. If $G \in S^*$ is a group such that $\overline{G} = G/T(G)$ is reduced then there exists a group $H$ such that the pair $(G,H)$ satisfies Conditions (I) and (II)$_n$ for every positive integer $n$.

Proof. Suppose that $r_0(G) = k$. Let $H$ be the group constructed via Proposition 3.1 such that $\text{Hom}(H,\overline{G}) = 0$ and $\text{Hom}(\overline{G},H) = 0$.

Since for every positive integer $n$ we have $G(n)/nG(n) \cong \overline{G}/n\overline{G}$ the pair $(G,H)$ satisfies the condition (II)$_n$.

The rank of $H$ is $k + 1$ and all subgroups of $H$ of rank $\leq k$ are free. Then $\text{Hom}(G,H) = 0$. If $f : H \to G$ is a homomorphism and $\pi_{T(G)} : G \to \overline{G}$ is the canonical surjection then $\pi_{T(G)}f = 0$. Then $\text{Im}(f)$ is a torsion group. Since $\text{Im}(f)$ is also reduced, it follows by using the condition (ii) from Proposition 3.1 that $\text{Im}(f)$ is finite. Then $(G,H)$ also satisfies the condition (I).

3.3. A Sufficient Condition

If $G/T(G)$ is not reduced, it is more difficult to construct a group $H$ such that the pair $(G,H)$ verifies Conditions (I) and (II)$_n$. In the case of quotient-divisible groups, this difficulty was passed by using an infinite family of groups. The proof of the following result can be extracted from [8, Proposition 3.4]. For the reader’s convenience, we give here some details.

Lemma 3.4. Suppose that for a fixed group $G \in S^*$ there exists an infinite family $W$ such that for all $W \in W$ we have

(a) $\text{Hom}(G,W) = 0$;
(b) $\text{Hom}(W,G)$ is a torsion group;
(c) $\text{End}(W) \cong \mathbb{Z}$;
(d) if $W_1, W_2 \in \mathcal{W}$ and $W_1 \neq W_2$ then $\text{Hom}(W_1, W_2) = 0$.

Then, for every positive integer $n$ there exists a group $H$ such that the pair $(G, H)$ verifies Conditions (I) and $(\text{II})_n$.

Proof. Let $n > 0$ be an integer. We consider a direct decomposition $G(n)/nG(n) = \bigoplus_{i=1}^{t} \langle u_i \rangle$. For every $i \in \{1, \ldots, t\}$ and for every $W \in \mathcal{W}$ there is an epimorphism $W \rightarrow \langle u_i \rangle$. Therefore, we can fix some epimorphisms $\alpha_i : W_i \rightarrow \langle u_i \rangle$, for all $i \in \{1, \ldots, t\}$ and $\alpha_{j\ell} : W_{j\ell} \rightarrow \langle u_j + u_\ell \rangle$ for all $j, \ell \in \{1, \ldots, t\}$ with $j < \ell$ such that the groups $W_1, W_2, W_1W_2, \ldots, W_{(t-1)t} \in \mathcal{W}$ are pairwise non-isomorphic. We denote $H = \langle \bigoplus_{i=1}^{t} W_i \rangle \oplus \langle \bigoplus_{1 \leq j < \ell \leq t} W_{j\ell} \rangle$, and we consider the epimorphism $\alpha : H \rightarrow G(n)/nG(n)$ induced by $\alpha_i$ and $\alpha_{j\ell}$. As in the proof of [8, Proposition 3.4], it can be proved by some direct computations that $\alpha$ is rigid, hence $H$ verifies $(\text{II})_n$. From a) and b) it is clear that $H$ verifies (I). \[\Box\]

The following result, which is useful in the study of the case when $G/T(G)$ is not reduced, was proved in [2, Proposition 3.3], for the particular case $m = k + 1$. In the proof of this result, the groups from the class $V$ are constructed in [20, Lemma 4.1]. It is easy to see that all details of this proof can be easily adapted for all integers $m > k$.

**Proposition 3.5.** Let $L \in \mathcal{S}^*$ be a quotient-divisible group with $r_0(L) = k \geq 1$. If $m > k$ is an integer, there exists an uncountable family $\mathcal{W}$ of pairwise non-isomorphic torsion-free groups of rank $m$ such that for all $W \in \mathcal{W}$ we have

(i) $\text{End}(W) \cong \mathbb{Z}$;
(ii) $\text{Hom}(L, W) = 0$;
(iii) all proper torsion-free quotient groups of $W$ are divisible;
(iv) $\text{Hom}(W, L)$ is a torsion group;
(v) all rank $1$ subgroups of $W$ are free and there exists a free subgroup $F_W \leq W$ such that $W/F_W \cong (\mathbb{Q}/\mathbb{Z})^{m-1}$.

**Remark 3.6.** Suppose that $W_1, W_2 \in \mathcal{W}$ and $W_1 \neq W_2$. If $f : W_1 \rightarrow W_2$ is a homomorphism then $f(W_1) \neq W_2$ (otherwise $W_1$ and $W_2$ would be isomorphic since they are torsion-free of the same rank). Then $f(W_1)$ is a free group, and this implies that $f(W_1) = 0$ since $W_1$ is indecomposable. It follows that $\text{Hom}(W_1, W_2) = 0$.

**Corollary 3.7.** [8, Proposition 3.4] If $G \in \mathcal{S}^*$ is quotient-divisible then for every positive integer $n$ there exists a group $H$ such that (I) and $(\text{II})_n$ are true for the pair $(G, H)$.

We will use the family $\mathcal{W}$ from Proposition 3.5 to prove a modified version of Proposition 3.1.

**Proposition 3.8.** Let $K \in \mathcal{S}^*$ be torsion-free of rank $k$. Then for every $m > k$ there exists an uncountable family $\mathcal{V}$ of torsion-free quotient-divisible groups of rank $m$ such that for all $V \in \mathcal{V}$ we have

(i) $\text{End}(V) \cong \mathbb{Z}$;
(ii) $\text{Hom}(V, K) = 0$;
(iii) all proper pure subgroups of $V$ are free (hence $\text{Hom}(K, V) = 0$);
Moreover, if $V_1, V_2 \in \mathcal{V}$ and $V_1 \neq V_2$ then $\text{Hom}(V_1, V_2) = 0$.

**Proof.** Let $d : \mathbb{Q}\mathcal{T} \mathcal{F} \cong \mathbb{Q}\mathcal{D} : d$ be the duality presented in Sect. 2.1. Since $d(\mathbb{Z}) = \mathbb{Q}$ and $d(\mathbb{Q}) = \mathbb{Z}$, it is easy to observe that the quotient-divisible group $d(K)$ belongs to $\mathcal{S}^*$. Then there exists a family $\mathcal{W}$ as in Proposition 3.5 which corresponds to $d(K)$. Let $\mathcal{V} = \{d(W) \mid W \in \mathcal{W}\}$. Since the restriction of $d$ to the class of quotient-divisible groups coincides with Arnold’s duality described in [5] and all $W$ are torsion-free quotient-divisible, the groups $d(W)$ are torsion-free and quotient-divisible.

(i) Since $d$ is a duality, for every $W \in \mathcal{W}$ we have $\mathbb{Q}\text{End}(d(W)) \cong \mathbb{Q}\text{End}(W) \cong \mathbb{Q}$. Then $\text{End}(d(W))$ is isomorphic to a unital subring of $\mathbb{Q}$. It follows that $\text{End}(d(W)) \cong \mathbb{Z}$ if and only if there exists a prime $p$ such that $\text{End}(d(W))$ is $p$-divisible.

Suppose that there exists such a prime $p$. Then $d(W)$ is $p$-divisible. Since there exists a full free subgroup $F_W \leq W$ such that $W/F_W \cong (\mathbb{Q}/\mathbb{Z})^{m-1}$, we can apply the statement 1) of Lemma 2.2 to obtain a contradiction. Therefore, for every $W \in \mathcal{W}$ we have $\text{End}(d(W)) \cong \mathbb{Z}$.

(ii) We have $\mathbb{Q}\text{Hom}(V, K) = \mathbb{Q}\text{Hom}(d(W), K) \cong \mathbb{Q}\text{Hom}(d(K), W) = 0$, hence $\text{Hom}(V, K) = 0$ since it is torsion-free.

(iii) and (iv) Let $U$ be a proper pure subgroup of $V = d(W)$. We apply $d$ to the exact sequence

$$0 \to U \overset{i}{\to} V \overset{\pi}{\to} V/U \to 0,$$

where $i$ is the inclusion map, and $\pi$ is the canonical surjection. By Remark 2.3 it follows that we can assume w.l.o.g. that $\text{Im}(d(\pi))$ is a finite index subgroup of $\text{Ker}(d(\iota))$ and that $\text{Im}(d(\iota))$ is a finite index subgroup of $d(U)$. However, all proper torsion-free quotient groups of $d(V)$ are divisible [20, Theorem 2.1]. Since $d$ preserves the rank, it follows that $\text{Im}(d(\iota))$ is divisible, and this implies that $d(U)$ is divisible. Then $U$ is free.

Similarly, if $U$ is of rank $m - 1$ then $d(V/U)$ is isomorphic to a (finite index) subgroup of the rank 1 subgroup $\text{Ker}(d(\iota))$. It follows that $d(V/U) \cong \mathbb{Z}$ since all rank 1 subgroups of $d(V)$ are isomorphic to $\mathbb{Z}$. Then $V/U \cong \mathbb{Q}$. □

### 3.4. When the Divisible Part of $G/T(G)$ Has Rank 1

**Proposition 3.9.** Suppose that $G \in \mathcal{S}^*$ is a group such that the divisible part of $G/T(G)$ has rank 1. Then for every positive integer $n$ there exists a group $H$ such that the pair $(G, H)$ verifies Conditions (I) and (II)$_n$.

Consequently, if $G$ has the cancellation property then $\text{End}_W(G)$ has the unit lifting property.

**Proof.** We have $G/T(G) = \overline{Q} \oplus \overline{M}$ such that $\overline{Q} \cong \mathbb{Q}$ and $\overline{M}$ is a reduced torsion-free group. Let $T(G) \leq Q, M \leq G$ such that $Q/T(G) = \overline{Q}$ and $M/T(G) = \overline{M}$. Then $Q$ and $M$ are pure subgroups of $G$, [13, Ex. S.3.26]. Moreover, $G/Q \cong \overline{M} \in \mathcal{S}^*$ is torsion-free, hence we can apply Proposition 3.8.

Let $\mathcal{V}$ be an infinite class of groups constructed as in Proposition 3.8 such that for all $V \in \mathcal{V}$ we have $r(V) = r_0(G) + 1$ and $\text{Hom}(V, \overline{M}) = 0$. Then $\text{Hom}(G, V) = 0$ for all $V \in \mathcal{V}$. 

Let $f : V \to G$ be a morphism, where $V \in \mathcal{V}$. Suppose that $f(V)$ is not a torsion group. Let $\pi : G \to G/T(G)$ and $\pi_M : G/T(G) \to \overline{M}$ be the canonical projections. It follows that $\pi_M \pi f = 0$, hence $\pi f(V) \leq \overline{Q}$ (note that $\overline{Q}$ is the unique direct complement of $\overline{M}$). Then the image $f(V)$ has the torsion-free rank 1 . Let $K$ be the kernel of $f$. Then the pure envelope $K \ast$ of $K$ is a proper pure subgroup of $V$, hence it is free, and it follows that $K \ast /K$ is finite. Since $V/K \ast \cong Q$, it follows that $f(V) \cong V/K \cong B \oplus Q$, hence $G$ is not reduced, a contradiction. It follows that $f(V)$ is a reduced torsion group. However, $V$ is quotient-divisible, and this implies that the reduced parts of all its torsion quotients are bounded. Then $f(V)$ is bounded. Then there exists a positive integer $m$ such that $mf = 0$. It follows that for all $V \in \mathcal{V}$ the group $\text{Hom}(V,G)$ is a torsion group. By Lemma 3.4 we conclude that for every positive integer $n$ there exists $H$ such that the pair $(G,H)$ verifies Conditions (I) and (II)$_n$. □

4. Self-Small Groups that Verify Stelzer’s Theorem

The main aim of this section is to determine the groups $G \in S^*$ that satisfy the mixed version of Stelzer’s Theorem, i.e., if $G$ has the cancellation property then $\text{End}_W(G)$ has the unit lifting property.

4.1. A Reduction Lemma

As a first step, we will prove a result which reduces our study to groups which have no mixed direct summands from the class $G$. We recall that a group $G \in S$ is in the class $G$ if $G/T(G)$ is divisible, [3]. It was proved in [14] that all groups from $G$ have the cancellation property. Moreover, for every $G \in G$ the Walk-endomorphism ring $\text{End}_W(G)$ is torsion-free and divisible, hence $G$ has the unit lifting property.

Lemma 4.1. Let $G = L \oplus K \in S$ such that $L \in G$. Then $\text{End}_W(G)$ has the unit lifting property if and only if $\text{End}_W(K)$ has the unit lifting property.

Proof. We write $\text{End}(G)$ as a matrix ring

$$\text{End}(G) = \left( \begin{array}{cc} \text{End}(L) & \text{Hom}(K,L) \\ \text{Hom}(L,K) & \text{End}(K) \end{array} \right).$$

Since $G$ is self-small, it follows that if $X,Y \in \{L,K\}$ then $T(\text{Hom}(X,Y)) = \text{Hom}(X,T(Y))$. Moreover, from the exact sequence

$$0 \to T(Y) \to Y \to Y/T(Y) \to 0,$$

we obtain an injective map

$$\text{Hom}(X,Y)/\text{Hom}(X,T(Y)) \hookrightarrow \text{Hom}(X,Y/T(Y)) \hookrightarrow \text{Hom}(X/T(X),Y/T(Y)).$$

The image of this map is pure [9, Lemma 2.6].

In the matrix representation

$$\text{End}_W(G) = \left( \begin{array}{cc} \text{End}_W(L) & \text{Hom}(K,L)/\text{Hom}(K,T(L)) \\ \text{Hom}(L,K)/\text{Hom}(L,T(K)) & \text{End}_W(K) \end{array} \right),$$
the additive groups

\[ \text{End}_W(L), \quad \text{Hom}(K, L)/\text{Hom}(K, T(L)), \quad \text{and} \quad \text{Hom}(L, K)/\text{Hom}(L, T(K)) \]

are torsion-free and divisible. Hence, for every positive integer \( n \) the factor ring \( \text{End}_W(G)/n\text{End}_W(G) \) can be identified with \( \text{End}_W(K)/n\text{End}_W(K) \). The conclusion is now obvious. \( \square \)

Consequently, to find classes of groups which satisfy the mixed version of Stelzer’s theorem, it is enough to identify self-small mixed groups which are direct sums of a group from \( G \) and a group which verifies the hypothesis of Proposition 2.4.

4.2. Self-Small Groups as Pushouts

We will describe (up to a finite summand) the groups in \( S^* \) in a manner similar to that presented in [12, Proposition 3.2], i.e., as pushouts of quotient-divisible groups and torsion-free groups.

**Lemma 4.2.** Let \( G \in S^* \) be of torsion-free rank \( k \). There exists a direct decomposition \( G = V \oplus G' \) such that \( V \) is finite and \( G' \) can be embedded in a pushout diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & F & \rightarrow & K & \rightarrow & K/F & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & L & \rightarrow & G' & \rightarrow & K/F & \rightarrow & 0
\end{array}
\]

such that

(1) \( F \) is a full free subgroup \( F \leq G' \), and \( G'/F \) is \( p \)-divisible for all primes \( p \) with \( T_p(G') \neq 0 \);
(2) \( L \) is quotient-divisible such that \( T(G') \leq L \) and \( L/F \) is divisible;
(3) \( K \) is torsion-free and \( K/F \) is reduced;
(4) for every prime \( p \) we have \( (G'/L)_p = 0 \) or \( T_p(G') = 0 \).

**Proof.** Let \( F_0 \) be a full free subgroup of \( G \). It follows that the set

\[
U = \{ p \in \mathbb{P} \mid (G/F_0)_p \neq p(G/F_0)_p \quad \text{and} \quad T_p(G) \neq 0 \}
\]

is finite. We consider the subgroup \( V = \bigoplus_{p \in U} T_p(G) \) and we fix a direct decomposition \( G = V \oplus G' \). Then \( F = F_0 \cap G' \) is a full free subgroup of \( G' \), and \( G'/F \) is isomorphic to a finite index subgroup of \( G/F_0 \). Suppose that there exists a prime \( p \) such that \( G'/F \) is not \( p \)-divisible and \( T_p(G') \neq 0 \). Then \( G/F_0 \) is not \( p \)-divisible and \( T_p(G) \neq 0 \), hence \( p \in U \). This is not possible since \( T_p(G') = 0 \) for all \( p \in U \). It follows that for every prime \( p \) such that \( T_p(G') \neq 0 \) the group \( G'/F \) is \( p \)-divisible.

We write \( G'/F = \widehat{L} \oplus \widehat{K} \) such that \( \widehat{L} \) is divisible and \( \widehat{K} \) is reduced. We denote by \( L \) and \( K \) the subgroups of \( G' \) such that \( F \leq L \cap K \), \( L/F = \widehat{L} \), and \( K/F = \widehat{K} \). We observe that \( K \) is torsion-free and \( T(G') \leq L \). Moreover, for every prime \( p \) we have \( (G'/L)_p = 0 \) or \( T_p(G') = 0 \).

To complete the proof, it is enough to apply [12, Lemma 3.1] to conclude that all these data can be included in a pushout diagram. \( \square \)
**Proposition 4.3.** Suppose that $G \in \mathcal{S}^{\ast}$ is a group of torsion-free rank $k$ that can be embedded in a pushout diagram as in Lemma 4.2. Moreover, we fix a decomposition $L = F_1 \oplus L_1$ such that $F_1$ is free and $L_1$ has no free direct summands.

(a) If $K$ is free then $G$ is quotient-divisible.
(b) If $L$ is free then $G$ is torsion-free.
(c) If $L_1 \in \mathcal{G}$ then $G = L_1 \oplus K_1$, where $K_1$ is torsion-free.
(d) If $r(F_1) \leq 1$ then for every positive integer $n$ there exists a group $H$ such that the pair $(G, H)$ verifies Conditions (I) and (II)$_n$.
(e) If $r_0(L_1) = 1$ then one of the following properties holds true:
   
   (i) $G/T(G)$ is reduced, or 
   (ii) $G = L_1 \oplus K_1$, where $L_1 \in \mathcal{G}$ and $K_1$ is torsion-free.

(f) If $r_0(L_1) = 2$ then one of the following properties holds true:
   
   (i) $G/T(G)$ is reduced;
   (ii) for every positive integer $n$ there exists a group $H$ such that the pair $(G, H)$ verifies Conditions (I) and (II)$_n$;
   (iii) $G = L_1 \oplus K_1$, where $L_1 \in \mathcal{G}$ and $K_1$ is torsion-free;

**Proof.** (a) This follows from the isomorphism $G/K \cong L/F$.

(b) We have $T(G) \leq L$. Since $L$ is free, it follows that $T(G) = 0$.

(c) For every prime $p$ we have $(G/L)_p = 0$ or $T_p(G) = 0$. This implies that $\text{Ext}(G/L, T(G)) = 0$.

Moreover, $T(G) \leq L_1$ and $L_1/T(G)$ is divisible. Applying the covariant functor $\text{Ext}(G/L, -)$ to the exact sequence $0 \to T(G) \to L_1 \to L_1/T(G) \to 0$, we obtain $\text{Ext}(G/L, L_1) = 0$. Therefore, in the exact sequence

$$\text{Ext}(G/L, L_1) \to \text{Ext}(G/L_1, L_1) \to \text{Ext}(L/L_1, L_1)$$

the first group is 0. But $\text{Ext}(L/L_1, L_1) = 0$ because the group $L/L_1$ is free, hence $\text{Ext}(G/L_1, L_1) = 0$. It follows that $L_1$ is a direct summand of $G$. Then $G = L_1 \oplus K_1$, where $K_1$ is torsion-free.

(d) We will prove that $G$ verifies the hypothesis of Lemma 3.4.

We have to study the cases when $L$ has no free direct summands or when $L = \langle x \rangle \oplus L_1$, where $L_1$ has no free direct summands. We will present the proof in the second case. The proof in the first case can be done in a similar way.

Let $W$ be a class associated to $L_1$ as in Proposition 3.5 such that all groups $W \in \mathcal{W}$ have the rank $k + 1$. We have to prove that for every $W \in \mathcal{W}$ we have $\text{Hom}(G, W) = 0$ and $\text{Hom}(W, G)$ is a torsion group.

Let $f : W \to G$ be a homomorphism. The rank of $W$ is $k + 1$, hence $\text{Ker}(f) \neq 0$. Let $x \in \text{Ker}(f)$. If $U$ is the pure subgroup generated by $x$ then it is isomorphic to $\mathbb{Z}$ and $W/U$ is divisible. Since $U/\langle x \rangle$ is finite, it follows that $W/\langle x \rangle = B \oplus D$ with $B$ a finite group and $D$ a torsion-free divisible group. Since $\text{Im}(f)$ is a quotient of $W/\langle x \rangle$, it follows that $\text{Im}(f)$ is a direct sum of a divisible group and a finite group. However, $G$ is reduced, hence $\text{Im}(f)$ is finite.

Suppose that there exists $f : G \to W$ a nonzero morphism. Since $f_{|L_1} = 0$, we obtain a morphism $\bar{f} : G/L_1 \to W$, where the torsion-free rank of
$G/L_1$ is 1. Suppose that $\overline{f} \neq 0$. Then $\text{Im}(\overline{f})$ is a rank 1 subgroup of $W$, hence $\text{Im}(\overline{f}) \cong \mathbb{Z}$. This implies that $G$ has a direct summand isomorphic to $\mathbb{Z}$, a contradiction.

(e) Suppose that $G/T(G)$ is not reduced. For every subgroup $X \leq G$ we denote $\overline{X} = (X + T(G))/T(G)$. Then $G/T(G) = \overline{L} + \overline{K}$ and $\overline{K} \cong K_1$. From Lemma 4.2, it follows that $G/L \cong K/(L \cap K)$ is reduced. Then the divisible part of $G/T(G)$ is contained in $\overline{L} = L/T(G)$ is not reduced. Using a similar argument it follows that the divisible part of $G/T(G)$ is included in $L_1$. Since the torsion-free rank of $L_1$ is 1, we obtain $L_1/T(G) \cong \mathbb{Q}$, hence $L_1 \in G$. The conclusion follows from the case c).

(f) As in the previous case, we can assume that $G/T(G)$ is not reduced. From the argument presented in the case e) it follows that the divisible part of $G/T(G)$ is included in $L_1$. If $L_1$ is divisible, we can apply case c). Suppose that $L_1$ is not divisible. Since $L_1/T(G)$ is of rank 1, it follows that the divisible part of $G/T(G)$ has rank 1, and the conclusion follows from Proposition 3.9.

Corollary 4.4. Suppose that $G \in S^*$ has the cancellation property, and has a decomposition as in Lemma 4.2 such that $G'$ verifies one of the hypotheses listed in Proposition 4.3. Then $\text{End}_W(G)$ has the unit lifting property.

Proof. Observe that $G'$ has the cancellation property. Moreover, $\text{End}_W(G) \cong \text{End}_W(G')$, so the conclusion follows from Lemma 4.1 and Proposition 2.4.

4.3. Groups of Torsion-Free Rank at Most 4

We are ready to prove that Stelzer’s Theorem is true for self-small groups of torsion-free rank at most 4. A proof for groups of torsion-free rank 1 is presented in [2, Proposition 3.5 and Corollary 3.6].

Theorem 4.5. Suppose that $G$ is a self-small group of torsion-free rank at most 4. If $G$ has the cancellation property then $G = F \oplus H$ such that $F$ is a free group and $\text{End}_W(H)$ has the unit lifting property.

Proof. We write $G = F \oplus K \oplus Q$, where $F$ is free, $Q$ is divisible, and $K$ is finite or $K \in S^*$. Since $F$ and $Q$ have the cancellation property, it follows that $G$ has the cancellation property if and only if $K$ has the cancellation property. We recall that $Q$ is torsion-free [6]. We denote $H = K \oplus Q$, and we observe that it is enough to prove that the property stated in our theorem is valid for the groups from $S^*$.

Let $G \in S^*$ be a group of torsion-free rank at most 4 such that $G$ has the cancellation property. Consider a pushout diagram as in Lemma 4.2, and write $L = F_1 \oplus L_1$ such that $F_1$ is free and $L_1$ has no free direct summands. It is easy to see that if $r_0(F_1) \geq 2$ then $r_0(L_1) \leq 2$. Hence we can apply Corollary 4.4, and the proof is complete.

We recall from [10] that a group $G \in S$ is strongly indecomposable if the $\mathbb{Q}$-algebra $\mathbb{Q}\text{End}_W(G)$ is local. From [2, Theorem 2.2] and [8, Proposition 3.1] we obtain the following
Corollary 4.6. A strongly indecomposable self-small reduced group $G$ of torsion-free rank at most 4 has the cancellation property if and only if there exists a finite group $B$ such that $G \cong B \oplus \mathbb{Z}$ or 1 is in the stable range of $\text{End}_W(G)$.

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