Abstract—In this paper, we present a coding-theoretic framework for message transmission over packet-switched networks. Network is modeled as a channel which can induce packet errors, deletions, insertions, and out of order delivery of packets. The proposed approach can be viewed as an extension of the one introduced by Kötter and Kschischang for networks based on random linear network coding. Namely, while their framework is based on subspace codes and designed for networks in which network nodes perform random linear combining of the packets, ours is based on the so-called subset codes, and is designed for networks employing routing in network nodes.

Index Terms—Subset codes, packet networks, routing, permutation channel, packet erasure codes, forward error correction.

I. INTRODUCTION

Packet-switched networks that employ routing as a means for transmitting packets between pairs of users are in widespread use in communications today [1]. We formulate here a framework for end-to-end forward error correction in such networks. We are motivated by the work of Kötter and Kschischang [2] in which the authors define so-called subspace codes and show that these codes, and particularly their constant-dimension versions, are adequate constructions for error and erasure recovery in networks employing random linear network coding (RLNC). The two frameworks turn out to be similar in many respects. Indeed, most concepts defined in our model have natural analogs in the subspace coding setting. On the other hand, there are some important differences between the two models, one of which will lead to a surprising conclusion that the codes for packet networks that are introduced here are equivalent (in a certain sense) to the classical binary codes in the Hamming space.

Let us now informally state the basic idea behind both approaches. Consider a network, abstracted as a communication channel, that acts on the transmitted packets by some randomized transformation (not including errors, erasures, etc.). In the case of RLNC networks, the channel transformation represents random linear combining of the source packets. In the case of networks based on routing, the transformation corresponds to permuting the packets in an unpredictable, and essentially random way. Namely, in such networks the packets with the same destination are frequently sent over different routes in the network and, as a consequence, they are received in practically arbitrary order (see the following section for a more detailed discussion of the channel model). The idea of sending information through such channels is very simple: Encode the information in an object that is invariant under the given transformation. This has led Kötter and Kschischang to the abstraction of the channel corresponding to RLNC networks (the operator channel) and the definition of codes for such a channel. In this case, the object invariant under random linear combinations of the packets is the vector space spanned by those packet[1]. Hence, the “codewords” are in this context taken to be subspaces of some ambient vector space [2].

In the case under consideration here, namely routed packet networks, we need an object that is invariant under random permutations of the packets. Such an object is a set. Therefore, a natural idea is to consider sets of packets as “codewords” in this context. If \( S \) is the set of all possible packets, the appropriate space in which such codes are to be defined is the set of all subsets of \( S \), denoted \( \mathcal{P}(S) \). In the following, we provide precise definitions and properties of the codes in \( \mathcal{P}(S) \), which are proposed as relevant to the problem of reliable data transmission over routed packet networks.

II. THE SYSTEM MODEL

Consider a packet-switched network in which a source node wishes to communicate with a destination node (or with multiple destination nodes). We assume that a message to be sent consists of a batch of packets (also called a generation) that are “simultaneously” injected into the network. Due to varying topology and load, the packets from the same batch can be sent over different routes in the network and their order is in general not preserved at the receiving side. This is especially true for, e.g., mobile ad-hoc networks where the topology is rapidly changing, and heavily loaded datagram-based networks in which the packets are frequently redirected in order to balance the load over different parts of the network. Hence, we will model networks as packet permutation channels which can deliver injected packets in an arbitrary order at the destination. Apart from permutations, there are various other unwanted effects the network can impose on the transmitted packets. We consider here three of them: errors, deletions, and insertions. Errors are random alterations of packet symbols caused by noise, malfunctioning of network equipment, etc. Packet deletions correspond to the fact that some packets can be “lost” in the channel, in which case the receiver is unaware of them being sent. They can occur for many reasons, finite buffering capabilities of routers, finite buffering capabilities of routers, etc.

1 Strictly speaking, it is invariant only with high probability — if the transformation is full-rank.
2 In the networking literature, the term “erasures” is also used in this context. We will use the term “deletions” since it is more appropriate from the coding theory viewpoint.

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router/link failures, etc. Finally, packet insertions are a form of malicious behavior, where some user imitates the true source of the data, and wants the receiver to misinterpret the data.

In the following section we will introduce subset codes as adequate for the above-described network model, i.e., for the permutation channel with errors, deletions, and insertions. Given that these codes are, as already noted, defined on the power set of the set of all possible packets \( S \), we can also give a more formal definition of the considered channel: It is a discrete memoryless channel with input and output alphabets equal to \( \mathcal{P}(S) \). The channel is completely described by its transition probabilities (the probabilities of mapping the input subset \( X \) to the output subset \( Y \), for all \( X, Y \in \mathcal{P}(S) \)) which, on the other hand, are determined by the joint statistics of errors, deletions, and insertions of the elements of \( S \).

As a final remark in this section, we emphasize that this paper considers an end-to-end network transmission model. Hence, it is implicitly assumed that (subset) coding is done on the transport or application layer.

III. CODES FOR PACKET NETWORKS

A. Power sets and subset codes

Let \( S \) be a nonempty finite set, and let \( \mathcal{P}(S) \) denote the power set of \( S \), i.e., the set of all subsets of \( S \). A natural metric associated with this space is:

\[
d(X, Y) = |X \triangle Y|
\]

for \( X, Y \in \mathcal{P}(S) \), where \( \triangle \) denotes the symmetric difference of sets. It can also be written as \( d(X, Y) = |X \cup Y| - |X \cap Y| = |X| + |Y| - 2|X \cap Y| = 2|X \cup Y| - |X| - |Y| \). This distance is the length of the shortest path between \( X \) and \( Y \) in the Hasse diagram of the lattice of subsets of \( S \) ordered by inclusion. It is analogous to the subspace metric defined in [2]. This diagram plays a role similar to the Hamming hypercube for the classical codes in the Hamming metric (actually, it is isomorphic to the Hamming hypercube, see Section IV).

Another useful metric is given by:

\[
d'(X, Y) = \max\{|X \setminus Y|, |Y \setminus X|\}.
\]

It can also be written as \( d'(X, Y) = \max\{|X|, |Y|\} - |X \cap Y| = |X \cup Y| - \min\{|X|, |Y|\} \), and it is analogous to the injection metric for subspace codes [3]. Direct proofs that \( d \) and \( d' \) are indeed metrics are easy and very similar to the proofs for subspace and injection metrics, and we shall therefore omit them. In the following, we will only use distance \( d \) and refer to it as the subset metric.

One can define codes in the space \( \mathcal{P}(S) \) in the usual way. Namely, a subset code \( C \) is simply a nonempty subset of \( \mathcal{P}(S) \). Important parameters of such a code are its cardinality, \(|C|\), minimum distance:

\[
\min_{X,Y \in C, X \neq Y} d(X, Y),
\]

maximum cardinality of the codewords:

\[
\max_{X \in C} |X|,
\]

and the cardinality of the ambient set, \(|S|\). If \( C \subseteq \mathcal{P}(S) \) has minimum distance \( d \), and every codeword is of cardinality at most \( \ell \), we say that it is a code of type \( [\log |S|, \log |C|, d; \ell] \) (the base of the logarithm is generally arbitrary; we will assume that it is 2, and hence that the lengths of the messages are measured in bits). If all codewords of \( C \) are of cardinality \( \ell \), we say that it is a constant-cardinality code. A significant advantage of constant-cardinality codes is that the receiver knows in advance how many packets it needs to receive in order to initiate decoding, similarly to the constant-dimension codes in projective spaces [2]. The rate of an \([n, k, d; \ell] \) code is defined by:

\[
R = \frac{k}{n\ell}.
\]

In the intended application of subset codes, \( S \) will be the set of all possible packets, \( n = \log |S| \) the length of each packet, and \( \ell \) the number of packets one codeword contains. The source maps information sequence of length \( k \) bits to a codeword which is a set consisting of \( \ell \) packets of length \( n \) bits each, and sends these \( \ell \) packets through a channel. In the channel, these packets are permuted, some of them are deleted, some of them are received erroneously, and possibly some new packets are inserted by a malicious user. The receiver collects all these packets and attempts to reconstruct the codeword which was sent, i.e., the information sequence which corresponds to this codeword.

We next prove a simple, but basic fact about the correcting capabilities of subset codes.

Theorem 1: Assume that a code \( C \) of type \([n, k, d; \ell] \) is used for transmitting packets over a network. Then any pattern of \( t \) errors, \( \rho \) deletions, and \( s \) insertions can be corrected by the minimum distance decoder (with respect to the subset metric), as long as \( 2(\rho + 2t + s) < d \).

Proof: Let \( X \subseteq C \) be the set/codeword which is transmitted through a channel. Let \( Y \) be the received set. If \( \rho \) packets from \( X \) have been deleted, and \( s \) new packets have been inserted, then we easily deduce that \( |X \cap Y| \geq |X| - \rho \) and \( |Y| \leq |X| - \rho + s \). Observe further that errors can be regarded as combinations of deletions and insertions. Namely, an erroneous packet can be thought of as being inserted, while the original packet has been deleted. Therefore, the actual number of deletions and insertions is \( \rho + t \) and \( s + t \), respectively. We therefore conclude that \( |X \cap Y| \geq |X| - \rho - t \) and \( |Y| \leq |X| - \rho + s \), and so

\[
d(X, Y) = |X| + |Y| - 2|X \cap Y| \leq \rho + 2t + s.
\]

Now, if \( 2(\rho + 2t + s) < d \), then \( d(X, Y) \leq \lfloor \frac{d-1}{2} \rfloor \) and hence \( X \) can be recovered from \( Y \).

If only deletions can occur in the channel, we will have \( d(X, Y) = \rho \) and a sufficient condition for unique decodability will be \( \rho \leq \lfloor \frac{d-1}{2} \rfloor \).

As Theorem 1 establishes, large enough minimum distance \( d \) ensures that the sent codeword can be recovered for a certain level of channel impairments. Therefore, this parameter is determined by the channel statistics, i.e., probabilities of packet error/deletion/insertion, and packet delivery requirements (e.g., error probability). Other code parameters, \( \ell \) and \( n \), are also determined by certain delivery requirements, such as delay, and by the properties of the network, such as the maximal packet length. A general method for the construction...
of subset codes with specified parameters, which reduces to
the construction of binary codes, is described in Section [IV].

Another simple method, via packet-level block codes and
sequence numbers, is illustrated in the following subsection.

B. Examples of subset codes

In this subsection, we give a simple example of subset codes
to illustrate the above definitions.

How does one encode information in a set? One possible
solution (which is widely used in practice) is to add a sequence
number to every packet sent, thus achieving resilience to
arbitrary permutations. To illustrate this, assume that the
source has two packets to send, \( p_0 \) and \( p_1 \). Note that, from
the point of view of the receiver, the sequence \((p_0, p_1)\) is not
the same as the sequence \((p_1, p_0)\); these two sequences carry
different information. In the permutation channel, however,
either of these two sequences can be received when \((p_0, p_1)\) is
sent. The sender therefore sends \((q_0, q_1)\) instead, where
\( q_i = (i, p_i) \) is the new packet formed by prepending a
sequence number to the packet \( p_i \). Note that sequences \((q_0, q_1)\)
and \((q_1, q_0)\) are now identical to the receiver because in both
cases it will extract \((p_0, p_1)\) and further process these packets.
This means that the carrier of information is actually a set
\( \{q_0, q_1\} = \{(0, p_0), (1, p_1)\} \). This approach, combined with
some classical packet-level error-correcting code, provides an
easy example of subset codes that we describe next.

Let \( \mathcal{A} \) be the set of all packets the source can possibly send.
Assume that \(|\mathcal{A}| = 2^m\), so that we can think of information
packets as having \( m \) bits. Assume further that the source
wishes to send \( k \) such packets, \( p_0, \ldots, p_{k-1} \) to a destination
over a network, i.e., over a permutation channel with errors,
deletions, and insertions. To protect the packets the source
defines some packet-level block code (see, e.g., [6]), and uses
the corresponding encoder to map these \( k \) packets to \( \ell > k \)
packets, \( q_0, \ldots, q_{\ell-1} \). To cope with the permutations in
the channel, the source further adds a sequence number of length
\( \log_2 \ell \) bits\(^3\) to every packet \( q_i \). This gives a subset code of
type \([m + \log_2 \ell, km, 2(\ell - k + 1); \ell] \), and rate:

\[
R = \frac{km}{\ell (m + \log_2 \ell)}. \tag{8}
\]

This code is a subset analog of the Kötter-Kschischang sub-
space code [2] designed for RLNC networks.

Even though RS codes are maximum distance separable
[7], the subset codes obtained in this way are not. Namely,
adding a sequence number is not an optimal way of encoding
information in a set (though this suboptimality is not a
concern in practice for sufficiently large packet lengths \( m \),
because sequence numbers only take a couple of bytes in the
packet header). The other reason for non-optimality is that
these codes are constant-cardinality codes; larger codes can
be obtained if one allows codewords of different cardinality.
This is analogous to the relation of general subspace codes
in projective spaces and constant-dimension codes [3]. In the
following section we discuss how one can construct optimal
(in any sense) subset codes.

As a final note here we point out that the codes constructed
in this way (via packet-level block codes and sequence num-
bers) are, to the best of our knowledge, the only type of error-
correcting codes for the permutation channel described in the
literature (see, for example, the construction of the “outer”
code in [3]). As established above, they are in fact only a
special case of subset codes.

IV. Subset codes as binary codes

Let \( S \) be a nonempty finite set with some implied ordering
of its elements, and observe the space \( \{0, 1\}^{|S|} \) of all binary se-
quencies of length \(|S|\) (denoted also \( 2^{|S|} \)). Each binary sequence
\( x \in 2^S \) defines a subset \( X \subseteq S \) containing elements defined
by the positions of ones in \( x \). As is well-known, this mapping
of subsets to binary sequences is an isomorphism between groups
\( (P(S), \triangle) \) and \( (2^S, \oplus) \), where \( \oplus \) denotes the XOR
operation (addition modulo 2). Furthermore, it is easy to show
that the Hamming distance between two sequences \( x, y \in 2^S \)
is precisely the subset distance between the corresponding
subsets \( X, Y \subseteq S \):

\[
d_h(x, y) = w_h(x \oplus y) = |X \triangle Y| = d(X, Y), \tag{9}
\]

where \( w_h \) denotes the Hamming weight of a sequence. In other
words, this mapping is also an isometry between metric spaces
\( (P(S), d) \) and \( (2^S, d_h) \). This means that the subset codes in
fact represent only another way to look at classical codes in
the binary Hamming space, and vice versa. In other words, the
study of subset codes and their properties reduces to the well-known theory of binary codes. Constant-cardinality codes are then equivalent to constant-weight binary codes. Finally, we note that the classical binary codes corresponding to \([n, k; d]\) subset codes have parameters \([2^n, k, d]\).

The above reasoning, though quite elementary, has an important implication. It shows that classical codes developed for binary channels (such as the Binary Symmetric Channel) define in a very natural way codes for correcting errors, deletions, and insertions in networks. Consequently, many familiar constructions of binary codes can be applied to subset codes. Namely, once the code parameters are determined from the given channel statistics and packet delivery requirements, the subset code with these parameters can be constructed via the corresponding binary code. For example, a constant-cardinality \([n, k; d]\) code could be designed as a constant-weight binary code with codeword weights \(\ell\) and parameters \([2^n, k, d]\), as explained above.

The following toy example illustrates the above notions.

**Example 1:** Let \(S = \{a, b, c, d\}\). Any subset of \(S\) can be identified by a binary sequence of length 4; for example \(\{a, b\} \leftrightarrow 1100, \{b, d\} \leftrightarrow 0101, \text{etc.}\). Consider now some code in \([0, 1]^4\), e.g., \(C = \{1100, 1010, 0110, 0011\}\). The subset counterpart of this code is then \(C_S = \{(a, b), (a, c), (b, c), (c, d)\}\). The distance between two subsets of \(S\) is the Hamming distance between the corresponding binary sequences, for example:

\[
d((a, b), (a, c)) = |\{b, c\}| = 2 = d_0(1100, 1010)
\]

so that all properties of \(C\) directly translate into equivalent properties of the subset code \(C_S\). The code \(C_S\) is a constant-cardinality code of type \([2, 2, 2, 2]\).

The above example can be extended to arbitrary sets \(S\) and binary codes \(C \subseteq 2^S\).

V. SOME PRACTICAL CONSIDERATIONS

In this section, we give some comments on subset codes and the channel model that could be relevant for their analysis in practical scenarios.

**Comments on binary codes:** One constraint on the binary codes corresponding to \([n, k; d; \ell]\) subset codes should be pointed out. Namely, “practical” subset codes will certainly require that \(\ell \ll 2^n\), i.e., that the number of packets in one codeword is much smaller than the number of all possible packets. This means that binary codes corresponding to (practically feasible) subset codes will only have small weight codewords. Moreover, the fact that binary codes corresponding to \([n, k; d; \ell]\) subset codes have exponential length \((2^n)\) places additional complexity constraints on the code design.

**Comments on the channel model:** The links in networks can generally be unreliable. For example, if a large packet is sent over a wireless link, it is highly probable that it will be hit by an error, i.e., that at least one of its bits/symbols will be received incorrectly. Furthermore, this error probability increases with the packet length \(n\). In such a scenario it can happen (with fairly high probability) that all of the packets from the sent codeword are erroneous, in which case \(X \cap Y = \emptyset\) and reliable recovery is impossible. Subset codes alone do not provide a good protection from errors in such cases. One way to solve this problem is to additionally protect each packet with its own error correcting code. This solution is in agreement with current networking practice. Namely, as already noted, we treat here an end-to-end network model and hence assume that (subset) coding is done on the transport or application layer. In most networks, packets on lower layers (e.g., link and physical layer) include some error correcting/error detecting codes (such as LDPC codes for error correction combined with CRC codes for error detection). These codes effectively create a channel that we treat here, namely, they keep the link-layer packet error probability at a “reasonable” level.

Packet insertions also deserve a comment regarding possible practical applications of subset codes. In general, by inserting enough packets an adversary can always prevent the receiver from correctly decoding the received set. Thus we also assume in our model that the number of insertions is relatively small, or at least that it behaves as a random variable whose parameters we can estimate and then design the code with respect to this estimated channel statistics. This may not be the case in practice because insertions inherently represent deliberate interference, but our assumption can certainly be achieved by a proper authentication protocol; that way the receiver will recognize and disregard (most of) the inserted packets. That is to say that subset codes do not provide any cryptographic protection; insertions are treated here because they naturally fit in the model, along with deletions and errors.

We note that the above comments on errors and insertions are also valid for subspace codes in network coded networks.

VI. CONCLUSION

We have presented a different view on the problem of forward error correction in the packet permutation channels. One advantage of the presented approach is that it unifies to some extent coding for RLNC networks and routed packet networks. We have introduced subset codes as appropriate constructs for these channels. Some basic properties of subset codes have been established, the most interesting of which is their equivalence to the classical binary codes.

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