INVERSES OF STRUCTURED VECTOR BUNDLES

INDRANIL BISWAS AND VAMSI P. PINGALI

Abstract. Structured vector bundles were introduced by J. Simons and D. Sullivan in [SS-2010]. We prove that all structured vector bundles whose holonomies lie in GL($N$, $\mathbb{C}$), SO($N$, $\mathbb{C}$), or Sp($2N$, $\mathbb{C}$) have structured inverses. This generalizes a theorem of Simons and Sullivan proved in [SS-2010].

1. Introduction

Differential K-theory is an enhanced version of topological K-theory constructed by incorporating connections and differential forms. It was developed in order to refine the families of Atiyah-Singer index theorem and also to classify the Ramond-Ramond field strengths in string theory [BS-2012], [Fr-2000]. One model of the first group in differential K-theory $\tilde{K}^0(X)$ for a manifold $X$ is a Grothendieck group of vector bundles equipped with connections and odd differential forms [Ka-1987]. In [SS-2010], Simons and Sullivan constructed another model of $\tilde{K}^0(X)$ using just vector bundles and connections. They got rid of the differential form at the cost of introducing an equivalence relation among the connections on the vector bundles. More precisely, they constructed $\tilde{K}^0(X)$ as the Grothendieck group of “structured” vector bundles; the definition of a structured vector bundle is recalled in Section 2.

In addition to constructing a model of $\tilde{K}^0(X)$, Simons and Sullivan in [SS-2010] proved an interesting result about the existence of stable inverses of hermitian structured bundles. (It is Theorem 1.15 in [SS-2010] which is essentially Theorem 1.1 below for the unitary group.) However, their proof works only for the unitary group (and also to the more general case of compact Lie groups) because they used the existence of universal connections à la Narasimhan-Ramanan [NR-1961]. In [PT-2014] a slightly different proof of the theorem of Simons and Sullivan was given that did not involve universal connections. In fact, the proof in [PT-2014] is valid even for connections that are not compatible with the metric.

All the flat connections considered here will have trivial monodromy representation. Note that if $\nabla$ is a flat connection with trivial monodromy on a vector bundle $V$ over $X$, and $x_0$ is a point of $X$, then there is a unique isomorphism $f$ of $V$ with the trivial vector bundle $X \times \mathbb{C}^M$ over $X$, equipped with the trivial connection, such that $f$ is connection preserving and coincides with the identity map over $x_0$ (here $V_{x_0}$ denotes the fiber of $V$ over $x_0$).

In this paper we prove the following theorem.

**Theorem 1.1.** Let $G$ be one of the groups GL($N$, $\mathbb{C}$), SO($N$, $\mathbb{C}$), Sp($2N$, $\mathbb{C}$). Given a structured vector bundle $\mathcal{V} = [V, [\nabla]]$ on a smooth manifold $X$ such that the holonomy of some equivalent connection $\nabla$ is in $G$, there exists a structured inverse $\mathcal{W} = [W, [\tilde{\nabla}]]$ with the property that the holonomy of $\tilde{\nabla}$ is in the same $G$, satisfying

$$\mathcal{V} \oplus \mathcal{W} = [X \times \mathbb{C}^M, [\nabla_f]]$$

where $\nabla_f$ is a flat connection with trivial monodromy on the trivial vector bundle $X \times \mathbb{C}^M$ over $X$.

As an immediate consequence of Theorem 1.1 we have the following corollary.

2000 Mathematics Subject Classification. 58A10, 53B15.

Key words and phrases. Structured vector bundle, connection, symplectic bundle, orthogonal bundle.
Corollary 1.2. Let \( \widetilde{K}_0^G(X) \) be the Grothendieck group of structured vector bundles \( \mathcal{V} = [\mathcal{V}, [\mathcal{V}]] \) satisfying the condition that the connection \( \nabla \) has holonomy in \( G \) (both \( G \) and \( X \) are as in Theorem 1.1). Let \( d \) denote the trivial flat connection on a trivial bundle \( X \times \mathbb{C}^k \). Then the following two hold:

1. Every element of \( \widetilde{K}_0^G(X) \) is of the form \( \mathcal{V} - [k] \) where \( [k] = [X \times \mathbb{C}^k, [d]] \) (so \( k = N \) if \( G = \text{GL}(N, \mathbb{C}) \) or \( SO(N, \mathbb{C}) \) and \( k = 2N \) if \( G = \text{Sp}(2N, \mathbb{C}) \)), and
2. \( \mathcal{V} = \mathcal{W} \) in \( \widetilde{K}_0^G(X) \) if and only if \( \mathcal{V} \oplus [\mathcal{N}] = \mathcal{W} \oplus [\mathcal{N}] \) as structured bundles for some flat bundle \( \mathcal{N} \) with trivial monodromy.

In [SS-2010], Simons and Sullivan proved what they called the Venice lemma which essentially says that every exact form arises out of Chern character forms of trivial bundles (see also [PT-2014]).

Using the same ideas as in the proof of Theorem 1.1 we give a proof of the following holonomy version of the Venice lemma.

Proposition 1.3. Fix \( G \) to be one of the groups \( \text{GL}(N, \mathbb{C}), \text{SO}(N, \mathbb{C}) \) and \( \text{Sp}(2N, \mathbb{C}) \). If \( \eta \) is any odd smooth form on \( X \), then there exists a trivial bundle \( T = X \times \mathbb{C}^k \) and a connection \( \nabla \) on it whose holonomy is in \( G \) such that

\[
\text{ch}(T, \nabla) - \text{ch}(T, d) = d\eta.
\] (1.1)

2. Preliminaries

As mentioned in the introduction, in order to define structured vector bundles we need to define an equivalence relation between connections on vector bundles on a smooth manifold. To do so, we recall the definition of the Chern-Simons forms. Throughout, \( V \) and \( W \) are smooth complex vector bundles on a smooth manifold \( X \). For a connection \( \nabla \) on a vector bundle \( V \), let \( F_V \in \mathcal{C}^\infty(X, \text{End}(V) \otimes \wedge^2 T^*X) \) be the curvature of \( \nabla \), and let

\[
\text{ch}(\nabla) = \text{Tr} \exp \left( \frac{\sqrt{-1}}{2\pi} F_\nabla \right)
\]

be the corresponding Chern character form on \( X \).

Definition 2.1. If \( \nabla_1 \) and \( \nabla_2 \) are smooth connections on \( V \), then the Chern-Simons form between them is defined as a sum of odd differential forms \( \text{CS}(\nabla_1, \nabla_2) \) modulo exact forms satisfying the following two conditions:

1. (Transgression) \( d\text{CS}(\nabla_1, \nabla_2) = \text{ch}(\nabla_1) - \text{ch}(\nabla_2) \).
2. (Functoriality) If \( f : Y \to X \) is a smooth map between smooth manifolds \( Y \) and \( X \), then \( \text{CS}(f^*\nabla_1, f^*\nabla_2) = f^*\text{CS}(\nabla_1, \nabla_2) \) modulo exact forms.

In [SS-2010] an equivalence relation between connections was defined, which we now recall.

Definition 2.2. If \( \nabla_1 \) and \( \nabla_2 \) are two smooth connections on a vector bundle \( V \) on \( X \), then

\[
\nabla_1 \sim \nabla_2
\]

if \( \text{CS}(\nabla_1, \nabla_2) = 0 \) modulo exact forms on \( X \). The equivalence class of \( \nabla \) is denoted by \([\nabla]\).

An isomorphism class \( \mathcal{V} = [\mathcal{V}, [\mathcal{V}]] \) as in Definition 2.2 is called a structured vector bundle. The direct sum of \( \mathcal{V} = [\mathcal{V}, [\mathcal{V}]] \) and \( \mathcal{W} = [\mathcal{W}, [\mathcal{W}]] \) is defined as

\[
\mathcal{V} \oplus \mathcal{W} = [\mathcal{V} \oplus \mathcal{W}, [\mathcal{V} \oplus \mathcal{W}]].
\]

A symplectic (respectively, orthogonal) bundle on \( X \) is a pair \( (E, \phi) \), where \( E \) is a \( \mathcal{C}^\infty \) vector bundle on \( X \) and \( \phi \) is a smooth section of \( E^* \otimes E^* \), such that

1. The bilinear form on \( E \) defined by \( \phi \) is anti-symmetric (respectively, symmetric), and
(2) the homomorphism

\[ E \rightarrow E^* \]  

defined by contraction of \( \varphi \) is an isomorphism, equivalently, the form \( \varphi \) is fiber-wise non-degenerate.

A connection on a vector bundle \( E \) induces a connection on \( E^* \otimes E^* \). A symplectic (respectively, orthogonal) connection on a symplectic (respectively, orthogonal) bundle \((E, \varphi)\) is a \( C^\infty \) connection \( \nabla \) on the vector bundle \( E \) such that the section \( \varphi \) is parallel with respect to the connection on \( E^* \otimes E^* \) induced by \( \nabla \).

If \((E, \varphi)\) is a symplectic (respectively, orthogonal) bundle, then the inverse of the isomorphism in (2.1) produces a symplectic (respectively, orthogonal) structure \( \varphi' \) on \( E^* \), because \((E^*, \varphi') = (E, \varphi)\). Let \( \nabla \) be a symplectic (respectively, orthogonal) connection on the symplectic (respectively, orthogonal) bundle \((E, \varphi)\). Then the connection \( \nabla' \) on \( E^* \) induced by \( \nabla \) is a symplectic (respectively, orthogonal) connection on \((E^*, \varphi')\), where \( \varphi' \) is defined above. We note that the isomorphism in (2.1) takes \( \varphi \) and \( \nabla \) to \( \varphi' \) and \( \nabla' \) respectively.

We define the holonomy version of differential K-theory next.

**Definition 2.3.** Let \( V \) be a vector bundle with a connection \( \nabla_V \) whose holonomy is in a group \( G \). Let \( \{\nabla_V\}_G \) denote the equivalence class of all such connections and denote the corresponding structured bundles by \( V \). Also, let \( T_G \) denote the free group of such structured bundles. The following group is the holonomy version of differential K-theory on a smooth manifold \( X \):

\[ \hat{K}^0_G(X) = \frac{T_G}{\mathcal{V} + \mathcal{W} - \mathcal{V} \oplus \mathcal{W}} \]  

(2.2)

**Remark 2.4.** Notice that we require all the equivalent connections in \( \{\nabla_V\}_G \) to have holonomy in \( G \) as a part of the definition of equivalence. In particular, equivalent connections in the sense of [SS-2010] do not necessarily have their holonomy in the same group.

From now onwards we drop the subscript \( G \) whenever it is clear from the context.

3. **Proof of Theorem 1.1**

We divide the proof of Theorem 1.1 into two cases.

3.1. **The case of \( G = \text{GL}(N, \mathbb{C}) \).**

This case has already been covered in [PT-2014]. However here we provide a different proof. Our approach relies on Lemma 3.1 proved below. We believe that Lemma 3.1 may be of interest in its own right. The geometrical content of the above mentioned lemma is that on \( \mathbb{R}^n \) every trivial bundle with a connection is a subbundle, equipped with the induced connection, of a trivial bundle equipped with a flat connection.

**Lemma 3.1.** Let \( V \) be a trivial complex vector bundle of rank \( r \) on \( \mathbb{R}^n \), and let \( A \) be a connection on \( V \). Then there exists an invertible, smooth \( (2n + 2)r \times (2n + 2)r \) complex matrix valued function \( g \) such that \( A_{ij} = [dgg^{-1}]_{ij} \), where \( 1 \leq i, j \leq r \).

**Proof.** Notice that \( A = \sum_{k=1}^n A_k dx^k \), where \( A_k \) are smooth \( r \times r \) complex matrix valued functions and \( x^k \) are coordinates on \( \mathbb{R}^n \). We may write \( A_k \) as

\[ A_k = 2I + A_k^r A_k + A_k - (2I + A_k^r A_k) \].
Using this it can be deduced that $A_k$ is a difference of two smooth functions with values in $r \times r$ positive definite matrices (an $r \times r$ matrix $B$ is called positive definite if $v^\dagger (B + B^\dagger)v > 0 \forall v \neq 0$). Indeed, we have

$$I + A_k^\dagger A_k + \frac{A_k + A_k^\dagger}{2} = \left(I + \frac{A_k}{2}\right)^\dagger \left(I + \frac{A_k}{2}\right) + \frac{3}{4} A_k^\dagger A_k \geq 0.$$ 

Also, $dx^k = e^{-x^k} d(e^{x^k})$ and $-dx^k = e^{x^k} d(e^{-x^k})$. Hence

$$A = \sum_{k=1}^{2n} f_k dh_k$$

where the $h_i$ are positive smooth functions and the $f_i$ are $r \times r$ positive-definite smooth matrix-valued functions.

We may attempt to find $g$ by forcing the first $r \times (2n + 2)r$ sub-matrix of $dg$ to be

$$\begin{bmatrix} dh_1 Id_{r \times r} & \ldots & dh_{2n} Id_{r \times r} & 0 & 0 \end{bmatrix}$$

and the first $(2n + 2)r \times r$ sub-matrix of $g^{-1}$ to be

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ -\sum h_k f_k \\ Id_{r \times r} \end{bmatrix}.$$

If we manage to find such a $g$, then $A_{ij} = [dg g^{-1}]_{ij}$.

Indeed, we claim that the matrix $g$ defined by

$$g = \begin{bmatrix} h_1 Id_{r \times r} & h_2 Id_{r \times r} & \ldots & h_{2n} Id_{r \times r} \\ Id_{r \times r} & 0 & \ldots & 0 \\ 0 & Id_{r \times r} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & Id_{r \times r} \\ 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} f_1 \left(\sum_k h_k f_k\right)^{-1} & 0 \\ 0 & f_2 \left(\sum_k h_k f_k\right)^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ f_{2n} \left(\sum_k h_k f_k\right)^{-1} & 0 \end{bmatrix}$$

does the job. (Here $Id_{r \times r}$ is the $r \times r$ identity matrix.) This can be verified by a straightforward computation. Note that $\sum_k h_k f_k$ is invertible because $h_k > 0$ and $f_k + f_k^\dagger > 0$. \hfill \Box

Since $\mathbb{R}^n$ is simply connected, any flat bundle on it has trivial monodromy. Lemma 3.1 implies the following generalization:

**Proposition 3.2.** Let $(V, \nabla = d + A)$ be a complex rank $r$ vector bundle equipped with a connection on a smooth manifold $X$ of dimension $n$. Then there exists a trivial complex vector bundle $T$ of rank $(4n + 8r + 2)(n + 2r)$ on $X$, and a smooth flat connection with trivial monodromy $\tilde{\nabla} = d + \tilde{A}$ on $T$, such that
Proof. Using the Whitney embedding theorem, there is an embedding of the total space of \( V \) in \( \mathbb{R}^{2n+4r} \). The zero section of \( V \) is diffeomorphic to \( X \) (and hence \( X \) also sits in \( \mathbb{R}^{2n+4r} \)). The tangent bundle \( TV \) of \( V \) is a subbundle of \( T\mathbb{R}^{2n+4r} |_V \).

By endowing \( V \) with the metric induced from the Euclidean metric on \( \mathbb{R}^{2n+4r} \), we may find the orthogonal complement \( TV \) of \( TV \). It satisfies
\[
TV|_X \oplus TV^\perp = T\mathbb{R}^{2(n+2r)}|_X.
\]
The vector bundle \( V \) itself may be identified with a subbundle of \( TV \). Using the induced metric we may find the orthogonal complement \( V^\perp \) of \( V \) in \( TV \). Therefore, there exists a vector bundle \( U = V^\perp \oplus TV^\perp \) on \( X \) such that \( V \oplus U = X \times \mathbb{C}^{n+2r} = Q \).

We may endow \( U \) with some arbitrary connection \( \nabla_U \). This induces the connection \( \nabla_Q = \nabla \oplus \nabla_U \) on the bundle \( Q \). Using a tubular neighborhood and a partition of unity we may extend \( \nabla_Q \) from \( X \) to a connection \( \nabla_Q \) on the trivial vector bundle of rank \((n+2r)\) defined on all of \( \mathbb{R}^{2n+4r} \). Now we may use Lemma \[SS-2010, Lemma 1.16\] to come up with a vector bundle \( \widetilde{T} \) of rank \((4n+8r+2)(n+2r)\) on \( \mathbb{R}^{2n+4r} \), equipped with a flat connection \( \nabla_{\widetilde{T}} \), such that \( \nabla_Q \) is induced from it. Restricting our attention to \( X \) we see that the vector bundle \( T = \widetilde{T}|_X \) equipped with the connection \( \nabla_{\widetilde{T}|X} \) satisfies the conditions in the proposition. \( \square \)

Proposition \[SS-2010\] may be viewed as a vector bundle version of the Nash embedding theorem because it states that every connection arises out of a flat connection with trivial monodromy. We now state a useful lemma \[SS-2010\] Lemma 1.16.

Lemma 3.3 (Simons-Sullivan). Let \( V \) and \( W \) be smooth vector bundles on a smooth manifold \( X \). Let \( \nabla \) be a smooth connection on the direct sum \( V \oplus W \) with curvature \( R \). Let \( \nabla_V \) and \( \nabla_W \) be the connections on \( V \) and \( W \) respectively constructed from \( \nabla \) using the decompositions \( V \oplus W \). Suppose that \( R_{r,s}(V) \subseteq V \) and \( R_{r,s}(W) \subseteq W \) for all tangent vectors \( r,s \) at any point of \( X \). Then
\[
\text{CS}(\nabla_V \oplus \nabla_W, \nabla) = 0 \text{ modulo exact forms on } X.
\]

Lemma 3.3 in conjunction with Lemma \[SS-2010\] implies Theorem \[SS-2010\] in the case \( G = \text{GL}(N,\mathbb{C}) \). Indeed, given a structured bundle \( \mathcal{V} = [V, \nabla_V] \), lemma \[SS-2010\] furnishes a flat bundle \( \mathcal{T} = [T, \nabla_T] \) such that \( \mathcal{V} \) is a subbundle of \( \mathcal{T} \) with the connection \( \nabla_V \) being induced from \( \nabla_T \). Using the natural metric on \( T \) we may find an orthogonal complement \( W \) to \( V \) so that \( V \oplus W = T \). Endowing \( W \) with the induced connection \( \nabla_W \) from \( \nabla_T \) (which is flat), it is straight-forward to check that the conditions of lemma \[SS-2010\] are satisfied. Let \( \mathcal{W} = [W, \nabla_W] \).

Using lemma \[SS-2010\] we see that
\[
[T, \{\nabla_T\}] = [V \oplus W, \{\nabla_T\}] = [V \oplus W, \{\nabla_V \oplus \nabla_W\}] = \mathcal{V} \oplus \mathcal{W}.
\]

3.2. \( G = \text{Sp}(2N,\mathbb{C}) \) or \( G = \text{SO}(N,\mathbb{C}) \).

From Section \[SS-2010\] we know that there is a structured inverse \( \mathcal{W} = [W, \{\nabla\}] \) of \( (E, \nabla) \). We clarify that \( W \) does not necessarily have a \( G \)-structure. Let \( \nabla' \) denote the connection on \( W^* \) induced by \( \nabla \). Using the natural pairing of \( W \) with \( W^* \), the vector bundle \( W \oplus W^* \) has a canonical \( G \)-structure \( \phi_0 \).

We note that \( \nabla^* \oplus \nabla' \) is a \( G \)-connection on \( (W \oplus W^*, \phi_0) \).
Clearly, \((W^*, \tilde{V}')\) is a structured inverse of \((E^*, V')\). Therefore,

\[(E^* \oplus W \oplus W^*, V' \oplus \tilde{V} \oplus \tilde{V}')\]

is a structured inverse of \((E, V)\). The connection \(V' \oplus \tilde{V} \oplus \tilde{V}'\) preserves the \(G\)-structure \(\varphi' \oplus \varphi_0\) on the vector bundle \(E^* \oplus W \oplus W^*\).

4. Applications

In this section we prove Corollary 1.2 and Proposition 1.3.

4.1. Proof of Corollary 1.2

(1) Any element of \(\tilde{K}_0^G(X)\) is of the form \([V] - [W]\) by definition. Since there exists an inverse \(Q\) to \(W\) such that \(Q \oplus W = [k]\), where \([k]\) is flat with trivial monodromy, we see that \([V] - [W] = [V \oplus Q] - [k]\).

(2) If \([V] = [W]\) in \(\tilde{K}_0^G(X)\), then \(V \oplus P = W \oplus P\) for some structured bundle \(P\). Let \(E\) be an inverse of \(P\). Adding \(E\) to both sides we see that \(V \oplus [N] = W \oplus [N]\) for some flat vector bundle \(N\) with trivial monodromy.

4.2. Proof of Proposition 1.3. Using the Venice lemma in [PT-2014] we see that there exists a trivial bundle \(\tilde{T}\) with a connection \(\tilde{\nabla}_\tilde{T} = d + A\) such that

\[
\frac{d\eta}{2} = \text{ch}(\tilde{T}, \nabla_{\tilde{T}}) - \text{ch}(\tilde{T}, d).
\]

It is not necessarily the case that the holonomy of \(\nabla_{\tilde{T}}\) lies in \(G\). However, we know that

\[
\frac{d\eta}{2} = \text{ch}(\tilde{T}^*, \nabla_{\tilde{T}^*}) - \text{ch}(\tilde{T}^*, d)
\]

because \(d\eta\) is an even form. Therefore,

\[
d\eta = \text{ch}(\tilde{T}^* \oplus \tilde{T}, \nabla_{\tilde{T}^*} \oplus \nabla_{\tilde{T}}) - \text{ch}(\tilde{T}^* \oplus \tilde{T}, d).
\]

Using the same reasoning as in Section 3.2 we obtain the desired result.

Acknowledgements

We are grateful to the referee for detailed comments to improve the exposition. The first–named author acknowledges the support of a J. C. Bose Fellowship.

References

[BS-2012] U. Bunke and T. Schick. Differential K-theory: a survey. Global differential geometry. Springer Berlin Heidelberg, 2012. 303–357.

[Fr-2000] D. Freed. Dirac charge quantization and generalized differential cohomology. Surv. Diff. Geom. 7 (2000), 129–194.

[Ka-1987] M. Karoubi. Homologie cyclique et K-théorie. Asterisque (149) : 147, (1987).

[NR-1961] M. S. Narasimhan and S. Ramanan. Existence of universal connections. Am. Jour. Math. 83 (1961), 563–572.

[PT-2014] V. Pingali and L. Takhtajan. On Bott Chern forms and their applications. Math. Ann. 360 (2014), 519–546.

[SS-2010] J. Simmons and D. Sullivan. Structured vector bundles define differential K-theory. Quanta of maths, Clay Math. Proc., Vol. 11, Amer. Math. Soc., Providence, RI, 579–599 (2010).