Observer-based adaptive stabilization of a class of uncertain nonlinear systems

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In this paper, an adaptive output feedback stabilization method for a class of uncertain nonlinear systems is presented. Since this approach does not require any information about the bound of uncertainties, this information is not needed a priori and a mechanism for its estimation is exploited. The adaptation law is obtained using the Lyapunov direct method. Since all the states are not measurable, an observer is designed to estimate unmeasurable states for stabilization. Therefore, in the design procedure, first an observer is designed and then the control signal is constructed based on the estimated states and adaptation law with the σ-modification algorithm. The uniformly ultimately boundedness of all signals in the closed-loop system is analytically shown using the Lyapunov method. The effectiveness of the proposed scheme is shown by applying to a unified chaotic system.

Keywords: output feedback; adaptive control; uniformly ultimately boundedness; Lyapunov-based design

1. Introduction

The problem of the robust output feedback regulation of uncertain nonlinear systems is the design of a feedback control law for such systems in a way that the boundedness of signals in the closed-loop is guaranteed (Chen & Huang, 2005a, 2005b; Huang, & Chen, 2004). However, in many practical applications, measurement of all the states is not possible. Therefore, observer design is an essential step in this approach.

The design of observers has received a great deal of attention recently (Liu, 2009). However, there are two main restrictive conditions in the design of observer-based controllers. First, the nonlinearities are only functions of measurable signals, which is a common assumption in the literature. Moreover, it is assumed that the unknown non-linearity is bounded by output injection terms and this unknown nonlinearity satisfies a global Lipschitz condition, which is the second restriction (Liu, 2009). In practical cases, these conditions are not held.

In order to make the design procedure more practical, we should consider a mechanism to relax these conditions. These conditions were further relaxed recently in Liu (2009), Choi and Lim (2005), Alimhan and Inaba (2006) and Hou, Wu, and Duan (2009). In Liu and Zheng (2009), a Fuzzy logic system is employed to estimate the upper bound of nonlinear uncertainties. This procedure could be done by using other approximation tools such as neural networks (NNs) (Arefi & Jahed-Motlagh, 2011; Du & Chen, 2009).

However, the design of fuzzy logic system and NNs needs to incorporate the knowledge of an expert. In this paper, these limitations are thoroughly relaxed by the estimation of upper bound of uncertainties using an adaptive control strategy. Compared with Liu and Zheng (2009) and Du and Chen (2009), this method needs neither any information about the bound of uncertainties nor an experts or system specialist. Moreover, in the presented method in Liu and Zheng (2009) and Du and Chen (2009), semi-global results are obtained, while our proposed approach provides globally uniformly ultimately boundedness (UUB) for all the signals in the closed-loop system.

This paper is organized as follows: the problem formulation of output feedback stabilization of uncertain nonlinear systems is presented in the second section. Adaptive output feedback, observer design and the stability analysis of the algorithm are presented in Section 3. In Section 4, an uncertain unified chaotic system is adopted to evaluate the effectiveness of the proposed method. Finally, the conclusions are given in Section 5.

2. Problem formulation

Consider the following uncertain nonlinear system:

\[ \dot{x}(t) = Ax(t) + Bf(x) + Bu, \]

\[ y = C^T x, \]

Consider the following uncertain nonlinear system:
where \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) corresponds to the state vector of the system, \( y \in \mathbb{R}^m \) is the system output, \( u = [u_1, \ldots, u_m]^T \in \mathbb{R}^m \) is the input vector of the plant, \( f(x) = [f_1(x) \ldots f_m(x)]^T \in \mathbb{R}^m \) is the unknown nonlinear function vector, \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, \) and \( C \in \mathbb{R}^{p \times n} \) are constant matrices with appropriate dimensions. In the system (1), not all \( x_i \) are assumed to be measurable and only the system output \( y \) is assumed to be available for measurement. In the controller design, we need the following assumptions.

**Assumption 1** For the nonlinear function \( f(x) \), there exists a positive constant \( \beta \) such that
\[
\| f(x) \| \leq \beta.
\]

**Remark 1** Although Assumption 1 is restrictive, one can suppose that \( \beta \) is large enough so that this assumption is satisfied.

**Assumption 2** \((A, B)\) is controllable and \((A, C^T)\) is detectable.

The main goal is to design an output feedback controller such that the states of (1) in the presence of unknown nonlinear function are bounded.

If \( f(x) \) is known and the state vector \( x \) is available, the controller can be chosen as
\[
u = -f(x) - k_c^T x, \tag{2}\]
where \( k_c \) is the state feedback gain matrix. By substituting Equation (2) into Equation (1) we have
\[
\dot{x}(t) = (A - Bk_c^T)x(t). \tag{3}\]
Since the pair \((A, B)\) is controllable, the gain matrix \( k_c \) in Equation (3) can be chosen such that the characteristic polynomial of \( A - Bk_c^T \) is strictly Hurwitz. Then, it can be shown that \( \lim_{t \to \infty} \dot{x}(t) = 0 \). However, because \( f(x) \) is unknown and only the system output \( y \) is measurable, this controller cannot be implemented in practice. A solution is to estimate the upper bound of the known function using the adaptive control strategy and design a suitable observer to estimate the state vector \( x \) using the measurable output \( y \).

### 3. Adaptive output feedback controller and observer design

In the previous section we assumed that only the system output is measurable and other states cannot be used in the controller design. So, we need to design an observer to estimate the unmeasurable states. Suppose that \( \hat{x} \) is the estimation of state vector \( x \). The following observer is given as
\[
\hat{x}(t) = (A - Bk_c^T)x(t) + k_o(y - C^T \hat{x}), \tag{4}\]
\[
\hat{y} = C^T \hat{x}.
\]
Since the pair \((A, C^T)\) is observable, the gain matrix \( k_o \in \mathbb{R}^{n \times m} \) in Equation (4) can be chosen so that \( A - k_o C^T \) is strictly Hurwitz. Let the estimation error \( e = x - \hat{x} \). From Equations (1) and (4), we have
\[
\dot{e} = (A - k_o C^T)e + Bk_c^T \dot{x} + Bf(x) + Bu, \tag{5}\]
\[
\dot{\hat{e}} = e^T e. \tag{6}\]

**Assumption 3** There exist positive-definite matrices \( P \) and \( Q \) satisfying
\[
(A - k_o C^T)^TP + P(A - k_o C^T) + Q = 0, \tag{7}\]
\[
PB = C. \tag{8}\]

**Remark 2** If \( k_o \) can be chosen such that the triple \((A - k_o C^T, B, C)\) is strictly positive real (SPR), one can use the Kalman–Yakubovich–Popov lemma (Slotine & Li, 1991), which guarantees the existence of positive-definite symmetric matrices \( P \) and \( Q \) in Equation (6).

**Theorem** Consider the nonlinear system (1) and the observer given in Equation (4). Under Assumptions 1–3, construct the following adaptive controller:
\[
u = -k_c^T \dot{x} - \frac{\dot{\beta}}{\| \dot{\beta} \| + \epsilon}, \tag{9}\]
and the adaptation law is as follows:
\[
\dot{\beta} = \gamma \| \dot{\beta} \| - \sigma \dot{\beta}, \tag{10}\]
where \( \epsilon > 0, \gamma > 0, \dot{\beta}(t_0) > 0, \) and \( \sigma > 0 \) are the design parameters.

Then all the signals in the closed-loop system are UUB. Furthermore, the estimation error can approach an arbitrarily small value by choosing the design parameter appropriately.

**Proof** Choose the following continuously differentiable function as a Lyapunov candidate
\[
V = \frac{1}{2} \left[ e^T Pe + \frac{\dot{\beta}^2}{\gamma} \right], \tag{11}\]
where \( \dot{\beta} = \dot{\beta} - \dot{\beta} \in \mathbb{R} \) and \( \gamma \) is the adaptation gain given in Equation (10). The derivative of Equation (11), using Equation (5) is
\[
\dot{V} = -e^T (A_o^T P + P A_o)e + e^T PBk_c \dot{x} + e^T PBf(x)
\]
\[
+ e^T PBu + \frac{\dot{\beta}^2}{\gamma} \dot{\beta} \tag{12}\]
where \( A_o = A - k_o C^T \). According to Equation (6), we have
\[
e^T PB = e^T C = \ddot{e}^T. \tag{13}\]
By using Equations (6), (7), (10), and (11) we have
\[
\dot{V} = -\frac{1}{2} e^T Q e + e^T f - \frac{\| e \| \| \dot{\beta} \|}{\| \dot{\beta} \| + \epsilon} + \frac{1}{\gamma} \dot{\beta} \dot{\beta}. \tag{14}\]
Now, using the fact that $-e^T Q e \leq -\lambda_{\min}(Q) \| e \|^2$, where $\lambda_{\min}(Q)$ is the minimum eigenvalue of $Q$ and regarding to Assumption 1 we have

$$
\dot{V} \leq -\frac{1}{2} \lambda_{\min}(Q) \| e \|^2 + \| \hat{e} \| \beta - \| \hat{e} \| \dot{\beta}$

$$
\leq -\frac{1}{2} \lambda_{\min}(Q) \| e \|^2 + \beta - \frac{1}{\gamma} \dot{\beta} \beta.
$$

(13)

Furthermore, the following inequality is true for the third term of the right-hand side of inequality (13)

$$
-\| \hat{e} \| \dot{\beta} \leq -\| \hat{e} \| \dot{\beta} \left(1 + \frac{\epsilon}{\| \hat{e} \| \beta + \epsilon}\right)
$$

$$
\leq -\| \hat{e} \| \dot{\beta} + \epsilon.
$$

(14)

Considering inequality (14) we have

$$
\dot{V} \leq -\frac{1}{2} \lambda_{\min}(Q) \| e \|^2 - \| \hat{e} \| \beta + \epsilon + \frac{1}{\gamma} \dot{\beta} \beta.
$$

(15)

Finally, by substituting the adaptation law (8) into Equation (15), we obtain

$$
\dot{V} \leq -\frac{1}{2} \lambda_{\min}(Q) \| e \|^2 + \epsilon - \sigma \dot{\beta} \beta,
$$

(16)

$$
\dot{V} \leq -\frac{1}{2} \lambda_{\min}(Q) \| e \|^2 + \epsilon - \sigma \beta^2 + \sigma |\beta| |\beta|.
$$

(17)

From Equations (9) and (17) and $\lambda_{\min}(P) \| e \|^2 \leq e^T Pe \leq \lambda_{\max}(P) \| e \|^2$ we have

$$
\dot{V} \leq -cV + \epsilon + \sigma |\beta|^2/2.
$$

(18)

where

$$
c = \min \left\{ \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}, \sigma \gamma \right\}.
$$

(19)

Solving inequality (18) gives

$$
0 \leq V(t) \leq \frac{\epsilon + \sigma |\beta|^2/2}{c} + \left( V(t_0) - \frac{\epsilon + \sigma |\beta|^2/2}{c} \right) e^{-ct} \quad \forall t \geq 0.
$$

(20)

Thus, $V(t) \leq \max\{V(t_0), (\epsilon + \sigma (|\beta|^2/2))/c\}, \forall t \geq 0$. From the definition of $V(t)$ in Equation (9), the error vector $e(t)$, is bounded by

$$
\| e(t) \| \leq \sqrt{\max\{V(t_0), (\epsilon + \sigma (|\beta|^2/2))/c\} / \lambda_{\min}(P)}.
$$

(21)

Equation (17) implies that $\dot{V} < 0$ when $\lambda_{\min}(Q) \| e \|^2 \geq 2\epsilon + \sigma |\beta|^2$, this shows that the system is UUB i.e. $e(t)$ converges to compact set $\Omega_\epsilon$ in finite time

$$
\Omega_\epsilon = \left\{ e(t) \mid \| e(t) \| \leq \sqrt{\frac{2\epsilon + \sigma |\beta|^2}{\lambda_{\min}(Q)}} \right\}.
$$

(22)

As a result, there exists a constant $T$, such that all trajectories will converge to $\Omega_\epsilon$ and remain in $\Omega_\epsilon$ for all time $t > T$. Since the characteristic polynomial of $A - Bk^e_\epsilon$ is strictly Hurwitz, it can be concluded from Equation (4) that $\dot{x}$ is bounded. Then according to $e = x - \hat{x}$, we can also conclude that $x$ is also bounded. In addition, based on the definition of $u$ in Equation (7), $u$ is also bounded. This completes the proof.

4. Simulation results

To show the proficiency of the presented algorithm, the simulation results are presented in this section. A class of more general nonlinear systems is studied in this section. For example, the following unified chaotic system is considered (Liu & Zheng, 2009):

$$
\dot{x}_1 = (25\alpha + 10)(x_2 - x_1),
$$

$$
\dot{x}_2 = (28 - 35\alpha)x_1 - x_1x_3 + (29\alpha - 1)x_2 + u_2,
$$

$$
\dot{x}_3 = x_1x_2 - \alpha + \frac{8}{3}x_3 + u_3,
$$

(23)

where $\alpha \in [0, 1]$. When $\alpha \in [0, 0.8]$, it is a Lorenz chaotic system, $\alpha = 0.8$ is the Lu chaotic system, and $\alpha \in (0.8, 1]$ is Chen’s chaotic system. The system can be easily transformed into the canonical form of Equation (1) with the following parameters:

$$
A = \begin{bmatrix}
-25\alpha & 25\alpha + 10 & 0 \\
28 - 35\alpha & 29\alpha - 1 & 0 \\
0 & 0 & -\alpha + \frac{8}{3}
\end{bmatrix},
$$

$$
B = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix},
$$

$$
f(x) = \begin{bmatrix}
-x_1x_3 \\
x_1x_2
\end{bmatrix},
$$

$$
C = \begin{bmatrix}
0 & 1 & 0
\end{bmatrix}^T.
$$

(24)

It is worth mentioning that Assumption 1, which imposes an upper bound for $f(x)$, is simply satisfied in chaotic systems due to the boundedness of trajectories in these systems (Arefi & Jahed-Motalgh, 2012). The simulation is carried out with initial conditions $x_0 = [2, 1, -1]^T$, $\dot{x}_0 = [3, 2, 1]^T$. It is straightforward to verify that the triple $(A - K_\epsilon C, B, C)$ can be made SPR by the choice of the observer gain.
Besides, the observer and state feedback gain are selected as
\[ k_c = \begin{bmatrix} 0 & 200 & 50 \\ 50 & 10 & 100 \end{bmatrix}^T, \quad k_o = \begin{bmatrix} 60 & 10 & 15 \\ 30 & 100 & 15 \end{bmatrix}^T. \]
Furthermore, the parameters of the controller and adaptation law are as follows:
\[ \varepsilon = 0.01, \quad \gamma = 0.5, \quad \sigma = 0.001, \quad \hat{\beta}(0) = 0.1. \]
Figure 1 shows the chaotic behavior of system (23) with \( u = 0 \) and \( \alpha = 0.5 \).

![Figure 1](image)

Figure 1. The chaotic behavior of system with \( u = 0 \) where \( \alpha \) is chosen as \( \alpha = 0.5 \).

The state trajectories of the system by applying the controller (7) with \( \alpha = 0.5 \) are shown in Figure 2. It can be seen from the simulations that the adaptive output feedback controller (7) makes the state estimations tend the actual states precisely. As this figure shows, the proposed controller stabilizes the system in the presence of unknown nonlinear uncertainties.

The time responses of the control input and adaptation parameter \( \hat{\beta} \) are shown in Figures 3 and 4, respectively.

![Figure 2](image)

Figure 2. (a) \( x_1 \) (solid line) and \( \hat{x}_1 \) (dashed line); (b) \( x_2 \) (solid line) and \( \hat{x}_2 \) (dashed line); (c) \( x_3 \) (solid line) and \( \hat{x}_3 \) (dashed line).
Figure 3. Control inputs of the system (a) $u_1$ and (b) $u_2$.

Figure 4. Adaptation parameter $\dot{\hat{\beta}}$.

5. Conclusion

In this paper, an adaptive output feedback stabilization strategy for a class of uncertain nonlinear systems was proposed. Since all the states are not measurable, an observer was presented to estimate unmeasurable states. The design method is based on the Lyapunov stability Theorem, and it was shown that all signals in the closed-loop system are UUB. Additionally, in this method, the mere knowledge of boundedness of uncertain term is sufficient.

Simulation results for the stabilization of a unified chaotic system show that the proposed approach has a fast response in stabilization. Moreover, the norm of the estimation errors is bounded, while the control signal is completely smooth.

References

Alimhan, K., & Inaba, H. (2006). Output feedback control for a class of nonlinear systems. International Journal of Automation and Computation, 3(3), 215–221.

Arefi, M. M., & Jahed-Motlagh, M. R. (2011). Observer-based adaptive neural control for a class of nonlinear non-affine systems with unknown gain sign, 18(4)(pp. 2644–2649). 18th IFAC world congress, Milano.

Arefi, M. M., & Jahed-Motalgh, M. R. (2012). Robust synchronization of Rossler systems with mismatched time-varying parameters. Nonlinear Dynamics, 67(2), 1233–1245.

Chen, Z., & Huang, J. (2005a). Global robust output regulation for output feedback systems. IEEE Transactions on Automatic Control, 50, 117–121.

Chen, Z., & Huang, J. (2005b). A general formulation and solvability of the global robust output regulation problem. IEEE Transactions on Automatic Control, 50, 448–462.
Choi, H. L., & Lim, J. T. (2005). Global exponential stabilization of a class of nonlinear systems by output feedback. *IEEE Transaction on Automatic Control, 50*(2), 255–257.

Du, H., & Chen, X. (2009). NN-based output feedback adaptive variable structure control for a class of non-affine nonlinear systems: A nonseparation principle design. *Neurocomputing, 72*(7–9), 2009–2016.

Hou, M., Wu, A., & Duan, G. (2009). *Adaptive control of a class of nonlinear systems by output feedback* (pp. 238–243). Proceedings of the 7th Asian control conference, Hong Kong, China.

Huang, J., & Chen, Z. (2004). A general framework for tackling the output regulation problem. *IEEE Transactions on Automatic Control, 49*, 2203–2218.

Khalil, H. K. (2002). *Nonlinear systems* (3rd ed.). New Jersey, NJ: Prentice-Hall.

Liu, Y. (2009). Robust adaptive observer for nonlinear systems with unmodeled dynamics. *Automatica, 45*(8), 1891–1895.

Liu, Y. J., & Zheng, Y. Q. (2009). Adaptive robust fuzzy control for a class of uncertain chaotic systems. *Nonlinear Dynamics, 57*, 431–439.

Slotine, J. J., & Li, W. (1991). *Applied nonlinear control*. Englewood Cliffs, NJ: Prentice-Hall.