ON GENERALIZED GAUSS MAPS OF MINIMAL SURFACES SHARING HYPERSURFACES IN A PROJECTIVE VARIETY

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Abstract. In this article, we study the uniqueness problem for the generalized gauss maps of minimal surfaces (with the same base) immersed in $\mathbb{R}^{n+1}$ which have the same inverse image of some hypersurfaces in a projective subvariety $V \subset \mathbb{P}^n(\mathbb{C})$. As we know, this is the first time the unicity of generalized gauss maps on minimal surfaces sharing hypersurfaces in a projective varieties is studied. Our results generalize and improve the previous results in this field.

1. Introduction and Main results

Let $x_1 : S_1 \to \mathbb{R}^{n+1}$ and $x_2 : S_2 \to \mathbb{R}^{n+1}$ be two oriented non-flat minimal surfaces immersed in $\mathbb{R}^{n+1}$ and let $G_1 : S_1 \to \mathbb{P}^n(\mathbb{C})$ and $G_2 : S_2 \to \mathbb{P}^n(\mathbb{C})$ be their generalized Gauss maps. Assume that there is a conformal diffeomorphism $\Phi$ of $S_1$ onto $S_2$ and the Gauss map of the minimal surface $x_2 \circ \Phi : S_1 \to \mathbb{P}^n(\mathbb{C})$ is given by $G_2 \circ \Phi$. Then $f^1 = G_1$, $f^2 = G_2 \circ \Phi$ are two nonconstant holomorphic maps from $S_1$ into $\mathbb{P}^n(\mathbb{C})$. In 1993, Fujimoto obtained the following result.

Theorem A (cf. [4, Theorem 1.2]). Under the notation be as above, let $H_1,\ldots,H_q$ be $q$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position such that

- $(f^1)^{-1}(H_j) = (f^2)^{-1}(H_j)$ for every $j$,
- $f^1 = f^2$ on $\bigcup_{j=1}^q (f^1)^{-1}(H_j) \setminus K$ for a compact subset $K$ of $S_1$.

Then we have necessarily $f^1 = f^2$

(1) if $q > (n+1)^2 + \frac{n(n+1)}{2}$ for the case where $S_1$ is complete and has infinite total curvature or
(2) if $q \geq (n+1)^2 + \frac{n(n+1)}{2}$ for the case where $K = \emptyset$ and $S_1$ and $S_2$ are both complete and have finite total curvature.

In 2017, J. Park and M. Ru [5] considered the case where $f^1$ and $f^2$ are linearly nondegenerate with an addition assumption that $\bigcap_{i=1}^k (f^1)^{-1}(H_{i_j}) = \emptyset$ for every $1 \leq i_1 < \cdots < i_k \leq q$ ($k \geq 2$).

Recently, in [11], the author initially studied the modified defect relation for the Gauss map of a minimal surface into a projective variety with hypersurfaces in subgeneral position. Motivated by the methods of [10, 11], in this paper, we will generalize the above

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mentioned results to the cases where gauss maps into a projective subvariety of \( \mathbb{P}^n(\mathbb{C}) \) have the same inverse image for some hypersurfaces in subgeneral position.

In order to state our results, we recall the following. Let \( S \) be an open complete Riemann surface in \( \mathbb{R}^{n+1} \). Let \( f \) be a holomorphic map from \( S \) into an \( \ell \)-dimension projective subvariety \( V \) of \( \mathbb{P}^n(\mathbb{C}) \) and let \( Q \) be a hypersurface in \( \mathbb{P}^n(\mathbb{C}) \) of degree \( d \). By \( \nu_Q(f) \) we denote the pull-back of the divisor \( Q \) by \( f \). Let \( F = (f_0, \ldots, f_n) \) be a reduced representation of \( f \). Assume that, the hypersurface \( Q \) has a defining polynomial, denoted again by the same notation \( Q \) (throughout this paper) if there is no confusion, given by

\[
Q(x_0, \ldots, x_n) = \sum_{I \in \mathcal{T}_d} a_I x^I,
\]

where \( \mathcal{T}_d = \{(i_0, \ldots, i_n) \in \mathbb{Z}_{\leq 0}^{n+1}; i_0 + \cdots + i_n = d\}, a_I \in \mathbb{C} \) are not all zero for \( I \in \mathcal{T}_d \) and \( x^I = x_0^{i_0} \cdots x_n^{i_n} \) for each \( i = (i_0, \ldots, i_n) \). We set

\[
Q(F) = \sum_{I \in \mathcal{T}_d} a_I f^I,
\]

where \( f^I = f_0^{i_0} \cdots f_n^{i_n} \) for each \( I \in \mathcal{T}_d \). Throughout this paper, for each given hypersurface \( Q \) we assume that \( ||Q|| = (\sum_{I \in \mathcal{T}_d} |a_I|^2)^{1/2} = 1 \).

Denote by \( I(V) \) the ideal of homogeneous polynomials in \( \mathbb{C}[x_0, \ldots, x_n] \) defining \( V \) and by \( \mathbb{C}[x_0, \ldots, x_n]_d \) the vector space of all homogeneous polynomials in \( \mathbb{C}[x_0, \ldots, x_n] \) of degree \( d \) including the zero polynomial. Define

\[
I_d(V) := \frac{\mathbb{C}[x_0, \ldots, x_n]_d}{I(V) \cap \mathbb{C}[x_0, \ldots, x_n]_d} \quad \text{and} \quad H_V(d) := \dim I_d(V).
\]

Denote by \([D]\) the equivalent class in \( I_d(V) \) of the element \( D \in \mathbb{C}[x_0, \ldots, x_n]_d \).

For the variety \( V \) of \( \mathbb{P}^n(\mathbb{C}) \) such that \( f(S) \subset V \), we say that \( f \) is nondegenerate over \( I_d(V) \) if there is no \([Q] \in I_d(V) \setminus \{0\}\) such that \( Q(F) \equiv 0 \).

Let \( Q_1, \ldots, Q_q \) \((q \geq N + 1)\) be \( q \) hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \). The hypersurfaces \( Q_1, \ldots, Q_q \) are said to be in \( N \)-subgeneral position with respect to \( V \) if

\[
V \cap \left( \bigcap_{j=1}^{N+1} Q_{i_j} \right) = \emptyset \quad \forall \quad 1 \leq i_1 < \cdots < i_{N+1} \leq q.
\]

Our first main result is stated as follows.

**Theorem 1.1.** Let \( V \) be an \( \ell \)-dimension projective subvariety of \( \mathbb{P}^n(\mathbb{C}) \). Let \( S_1, S_2 \) be non-flat minimal surfaces immersed in \( \mathbb{R}^{n+1} \) with the Gauss maps \( G_1, G_2 \) into \( V \), respectively. Assume that there are conformal diffeomorphisms \( \Phi_i \) of \( S_1 \) onto \( S_2 \). Let \( f^1 = G_1, f^2 = G_2 \circ \Phi \). Let \( Q_1, \ldots, Q_q \) be \( q \) hypersurfaces of \( \mathbb{P}^n(\mathbb{C}) \) in \( N \)-subgeneral position with respect to \( V \), \( d = \text{lcm} (\deg Q_1, \ldots, \deg Q_q) \) and let \( k \) be a positive integer such that:

(a) \((f^1)^{-1}(Q_j) = (f^2)^{-1}(Q_j)\) for every \( j \in \{1, \ldots, q\}\),
(b) \(\bigcap_{j=0}^{k-1}(f^1)^{-1}(Q_{i_j}) = \emptyset\) for every \( 1 \leq i_0 < \cdots < i_k \leq q \),
(c) \( f^1 = f^2 \) on \( \bigcup_{j=1}^{q-1}(f^1)^{-1}(Q_j) \).
Suppose that $f^1$ is linear nondegenerate over $I_d(V)$. If $S^1$ is complete and
\[
q > 2N - \ell + 1 \left( M + 1 + \frac{2(M + 1)}{2d} \right)
\]
where $M = H_d(V) - 1$, $\sigma_p = \frac{p(p+1)}{2}$ for every $p \geq 0$ and $\sigma_p = 0$ for every $p \leq 0$, then $f^1 \equiv f^2$.

Remark 1: If $V$ is the smallest linear subspace of $\mathbb{P}^n(\mathbb{C})$ containing $f^1(S)$ and $Q_1, \ldots, Q_q$ are hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position, then $V = \mathbb{P}^\ell(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$, $d = 1$, $N = n, M = \ell$. Therefore, from Theorem 1.1, $f^1 = f^2$ if
\[
q > \frac{2n - \ell + 1}{\ell + 1} \left( \ell + 1 + \frac{3\ell(\ell + 1)}{2} \right) = \frac{(2n - \ell + 1)(3\ell + 2)}{2}.
\]
This condition is always fulfilled if $q > \frac{(n+1)(3n+2)}{2} = (n + 1)^2 + \frac{n(n+1)}{2}$ (without any condition on $f^1(S)$). Then this theorem gives an improvement for Theorem A(1).

**Theorem 1.2.** Let $V$ be an $\ell$–dimensional projective subvariety of $\mathbb{P}^n(\mathbb{C})$. Let $S_1, S_2$ be non-flat minimal surfaces in $\mathbb{R}^{n+1}$ with the Gauss maps $G_1, G_2$ into $V$, respectively. Assume that there are conformal diffeomorphisms $\Phi$ of $S_1$ onto $S_2$. Let $f^1 = G_1, f^2 = G_2 \circ \Phi$. Let $Q_1, \ldots, Q_q$ be hypersurfaces (not containing $V$) of $\mathbb{P}^n(\mathbb{C})$ in $N$–subgeneral position with respect to $V$, $d = \text{lcm(deg } Q_1, \ldots, \text{deg } Q_q)$ and let $k$ be a positive integer such that:

(a) $(f^1)^{-1}(Q_j) = (f^2)^{-1}(Q_j)$ for every $j \in \{1, \ldots, q\}$,
(b) $\bigcap_{j=0}^k (f^1)^{-1}(Q_{i_j}) = \emptyset$ for every $1 \leq i_0 < \cdots < i_k \leq q$.
(c) $f^1 = f^2$ on $\bigcup_{j=1}^k (f^1)^{-1}(Q_j)$.

If $f^1$ is nondegenerate over $I_d(V)$, $S^1$ is complete, $q \geq 2Mk + 2k$ and
\[
q > 2N - \ell + 1 \left( M + 1 + \frac{2Mk\sigma}{q + 2(M - 1)kd} + \frac{M(M + 1)}{2d} \right)
\]
then there are $\left[\frac{q}{2}\right]$ indices $i_1, \ldots, i_{[q/2]} \in \{1, \ldots, q\}$ such that
\[
\frac{Q_{i_1}(F^1)}{Q_{i_1}(F^2)} = \cdots = \frac{Q_{i_{[q/2]}}(F^1)}{Q_{i_{[q/2]}}(F^2)}
\]
for any two representations $F^1, F^2$ of $f^1, f^1$, respectively.

Remark 2: In the above theorem, suppose that $V = \mathbb{P}^n(\mathbb{C}), Q_1, \ldots, Q_q$ are hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position. Then $d = 1$, $M = N = \ell = n$. Therefore, from the above theorem, $f^1 = f^2$ if $q \geq 2nk + 2k$ and
\[
q > n + 1 + \frac{2nk\sigma}{q + 2nk - 2k} + \frac{n(n+1)}{2}.
\]
Therefore, this result implies the previous result of J. Park and M. Ru in [8].
2. Main lemmas

Let $V$ be $\ell$-dimension subvariety of $\mathbb{P}^n(\mathbb{C})$. Let $d$ be a positive integer. Throughout this section and Section 3, we fix a $\mathbb{C}$-ordered basis $\mathcal{V} = ([v_0], \ldots, [v_M])$ of $I_d(V)$, where $v_i \in H_d$ and $M = H_V(d) - 1$.

Let $S$ be an open Riemann surface and let $z$ be a conformal coordinate. Let $f$ be a holomorphic map of $S$ into $V$, which is nondegenerate over $I_d(V)$. Suppose that $F = (f_0, \ldots, f_n)$ is a reduced representation of $f$. We set

$$F = (v_0(F), \ldots, v_M(F))$$

and

$$F_p := F^{(0)} \wedge F^{(1)} \wedge \cdots \wedge F^{(p)} : S \to \bigwedge_{p+1}^{C+1}$$

for $0 \leq p \leq M$, where

- $F^{(0)} := F = (v_0(F), \ldots, v_M(F))$,
- $F^{(l)} = F^{(l)} := (v_0(F)^{(l)}, \ldots, v_M(F)^{(l)})$ for each $l = 0, 1, \ldots, p$,
- $v_i(F)^{(l)} (i = 0, \ldots, M)$ is the $l^{th}$- derivatives of $v_i(F)$ taken with respect to $z$.

The norm of $F_p$ is given by

$$|F_p| := \left( \sum_{0 \leq i_0 < i_1 < \cdots < i_p \leq M} |W(v_{i_0}(F), \ldots, v_{i_p}(F))|^2 \right)^{1/2},$$

where

$$W(v_{i_0}(F), \ldots, v_{i_p}(F)) := \det (v_{i_j}(F)^{(l)})_{0 \leq l, j < p}.$$

Denote by $\langle , \rangle$ the canonical hermitian product on $\bigwedge^{k+1}^{C+1}$ for $0 \leq k \leq M$. For two vectors $A \in \bigwedge^{k+1}^{C+1}$ and $B \in \bigwedge^{p+1}^{C+1}$, there is one and only one vector $C \in \bigwedge^{k-p}^{C+1}$ satisfying

$$\langle C, D \rangle = \langle A, B \wedge D \rangle \forall D \in \bigwedge^{k-p}^{C+1}.$$

The vector $C$ is called the interior product of $A$ and $B$, and denoted by $A \wedge B$.

Now, for a hypersurface $Q$ of degree $d$ in $\mathbb{P}^n(\mathbb{C})$, we have

$$[Q] = \sum_{i=0}^{M} a_i[v_i].$$

Hence, we associate $Q$ with the vector $(a_0, \ldots, a_M) \in \mathbb{C}^{M+1}$ and define $F_p(Q) = F_p \wedge H$. Then, we may see that

$$F_0(Q) = a_0v_0(F) + \cdots + a_Mv_M(F) = Q(F),$$

$$|F_p(Q)| = \left( \sum_{0 \leq i_1 < \cdots < i_p \leq M} \sum_{l \neq i_1, \ldots, i_p} a_l |W(v_l(F), v_{i_1}(F), \ldots, v_{i_p}(F))|^2 \right)^{1/2}.$$
For $0 \leq p \leq M$, the $p^{th}$-contact function of $f$ for $Q$ is defined by

$$\varphi_p(Q) := \frac{|F_p(Q)|^2}{|F_p|}.$$

**Lemma 2.1** (cf. [9 Lemma 3]). Let $Q_1, \ldots, Q_q$ be $q$ ($q > 2N - \ell + 1$) hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in $N$-subgeneral position with respect to $V$ of the same degree $d$. Then, there are positive rational constants $\omega_i$ ($1 \leq i \leq q$) satisfying the following:

i) $0 < \omega_i \leq 1 \forall i \in \{1, \ldots, q\}$,

ii) Setting $\tilde{\omega} = \max_{j \in Q} \omega_j$, one gets $\sum_{j=1}^q \omega_j = \tilde{\omega}(q - 2N + \ell - 1) + \ell + 1$.

iii) $2N - \ell + 1 \leq \tilde{\omega} \leq \frac{\ell + 1}{N}$.

iv) For each $R \subset \{1, \ldots, q\}$ with $\sharp R = N + 1$, then $\sum_{i \in R} \omega_i \leq \ell + 1$.

v) Let $E_i \geq 1$ ($1 \leq i \leq q$) be arbitrarily given numbers. For each $R \subset \{1, \ldots, q\}$ with $\sharp R = N + 1$, there is a subset $R^o \subset R$ such that $\sharp R^o = \text{rank}\{[Q_i]\}_{i \in R^o} = \ell + 1$ and

$$\prod_{i \in R} E_i^{\omega_i} \leq \prod_{i \in R^o} E_i.$$

The following theorem is due to the author in recent works [11] [12] [13].

**Theorem 2.2** (cf. [11 Theorem 3.3], [12 Theorem 3.5], [13 Theorem 2.7]). Let the notations be as above and let $\tilde{\omega}$ be the constant defined in the Lemma 2.1 with respect to the hypersurfaces $Q_1, \ldots, Q_q$. Then, for every $\epsilon > 0$, there exist a positive number $\delta$ ($> 1$) and $C$, depending only on $\epsilon$ and $Q_j$ such that

$$\operatorname{dd}^c \log \frac{\prod_{1 \leq j \leq q, 0 \leq p \leq M-1} |F_p|^{2\epsilon}}{\prod_{1 \leq j \leq q, 0 \leq p \leq M-1} \log^{2\omega_j} (\delta/\varphi_p(Q_j))} \geq C \left( \frac{|F_0|^{2(\tilde{\omega}(q - (2N - k + 1)) - M + k)} |F_M|^2}{\prod_{j=1}^q (|F_0(Q_j)|^2 \prod_{p=0}^{M-1} \log^{2\omega_j} (\delta/\varphi_p(Q_j)))^{\omega_j}} \right)^{\frac{2}{M(M+1)}} \operatorname{dd}^c |z|^2.$$

**Theorem 2.3** (cf. [5 Proposition 2.5.7]). Set $\tau_m = \sum_{p=1}^m \sigma_m$ for each integer $m$. We have

$$\operatorname{dd}^c \log(|F_0|^2 \cdots |F_{M-1}|^2) \geq \frac{\tau_m}{\sigma_m} \left( \frac{|F_0|^2 \cdots |F_M|^2}{|F_0|^2 \sigma_{M+1}} \right)^{1/\tau_m} \operatorname{dd}^c |z|^2.$$

**Theorem 2.4.** Let the notations be as above and let the assumption be as in Lemma 2.1. We have

$$\nu_{F_M^1} \geq \sum_{j=1}^q \omega_j \nu_{Q_j(F)} - (\sigma_M - \sigma_{M-\min\{k, \ell\}}) \nu_{\prod_{j=1}^q Q_j(F)}.$$

**Proof.** For a point $a \in \bigcup_{j=1}^q (f^1)^{-1}(Q_j)$, since $\{Q_j\}_{j=1}^q$ is in $N$-subgeneral position with respect to $V$, there are at most $N$ indices $j$ such that $Q_j(f^1)(a) = 0$. Then, there is a subset $R \subset \{1, \ldots, q\}$ with $\sharp R = N + 1$ such that $Q_j(f^1)(a) \neq 0 \forall j \notin R$. Applying Lemma 2.1, there exists a subset $R^o \subset R$ with $\sharp R^o = \ell + 1$ such that $\text{rank}_C\{[Q_j]; j \in R^o\} = \ell + 1$.
and
\[
\sum_{j=1}^{q} \omega_j \nu_{Q_j}(F)(a) = \sum_{j \in R} \omega_j \nu_{Q_j}(F^1)(a) \leq \sum_{j \in R^0} \nu_{Q_j}(F^1)(a).
\]
We set \(k' = \min\{k, \ell\}\). Since there are at most \(k'\) indices \(j \in R^0\) such that \(Q_j(F^1)(a) = 0\), we also may assume further that \(R^0 = \{1, \ldots, \ell + 1\}\), \(Q_j(F^1)(a) \neq 0\) for all \(j > k', j \in R^0\).

By the basis property of the wronskian, we have
\[
\nu_{F^1_M}(a) \geq \min_{\alpha} \left\{ \sum_{j=1}^{k'} \max\{0, \nu_{Q_j(F^1)}(a) - (M - \alpha(j))\} \right\} \geq \sum_{j=1}^{k'} \nu_{Q_j(F^1)}(a) - (\sigma_M - \sigma_{M-k'}),
\]
where the minimum is taken over all bijections \(\alpha : \{1, \ldots, k'\} \to \{0, \ldots, k' - 1\}\). Thus
\[
\nu_{F^1_M} \geq \sum_{j=1}^{q} \omega_j \nu_{Q_j}(F) - (\sigma_M - \sigma_{M-k'\ell}) \prod_{j=1}^{1} Q_j(F).
\]
The theorem is proved. \(\square\)

**Lemma 2.5** (Generalized Schwarz’s Lemma \[1\]). Let \(v\) be a non-negative real-valued continuous subharmonic function on \(\Delta(R) = \{z \in \mathbb{C}; |z| < R\}\). If \(v\) satisfies the inequality \(\Delta \log v \geq v^2\) in the sense of distribution, then
\[
v(z) \leq \frac{2R}{R^2 - |z|^2}.
\]

### 3. Holomorphic curves from complex discs into projective varieties

**Lemma 3.1.** Let \(V\) be an \(\ell\)-dimension projective subvariety of \(\mathbb{P}^n(\mathbb{C})\). Let \(Q_1, \ldots, Q_q\) be \(q\) hypersurfaces of \(\mathbb{P}^n(\mathbb{C})\) in \(N\)-subgeneral position with respect to \(V\) and let \(d\) be the least common multiple of \(\deg Q_1, \ldots, \deg Q_q\). Let \(f^1, \ldots, f^m\) be \(m\) holomorphic maps from \(\Delta(R)\) into \(V\) \((1 \leq m \leq n + 1)\), which are nondegenerate over \(I_d(V)\). Assume that there exists a holomorphic function \(h\) on \(\Delta(R)\) satisfying
\[
\lambda v_h + \sum_{i=1}^{m} \nu_{F^i_M} \geq \sum_{i=1}^{m} \sum_{j=1}^{q} \omega_j \nu_{Q_j(F^i)} \text{ and } |h| \leq \prod_{i=1}^{m} |F^i|^{\rho},
\]
where \(F^i = (F^i_0, \ldots, F^i_n)\) is a reduced representation of \(f^i\) \((1 \leq i \leq m)\), \(\lambda\) and \(\rho\) are non-negative numbers. Then for an arbitrarily given \(\epsilon\) satisfying
\[
\gamma = \sum_{j=1}^{q} \omega_j - M - 1 - \frac{\lambda \rho}{d} > \epsilon \left(\sigma_{M+1} + \frac{\rho}{d}\right),
\]
the pseudo-metric \(d\tau^2 = \eta^{2/m}|dz|^2\), where
\[
\eta = \left(\frac{|h|^\lambda \prod_{i=1}^{m} |F^i_0|^{\gamma - \epsilon(\sigma_{M+1} + \frac{\rho}{d})} |F^i_M| \prod_{p=0}^{M} |F^i_p|^{\epsilon} \prod_{j=1}^{q} (|Q_j(F^i)| \cdot \prod_{j=1}^{M-1} \log(\delta^j/\varphi^j_p(Q_j)))^{\omega_j}}{\gamma_{M+1} M^{M}}\right)^{\frac{1}{\gamma_{M+1} M}}
\]
and \(\delta^i\) is the number satisfying the conclusion of Theorem 2.2 with respect to the map \(f^i\), is continuous and has strictly negative curvature.
Here and throughout this paper, \( F^i_p \) and \( \varphi^i_p \) are defined with respect to the map \( f^i \). For simplicity, we sometimes write \( \prod_{i,j} \) and \( \prod_{j,p} \) for \( \prod_{i=1}^m \prod_{j=1}^q \) and \( \prod_{j=1}^q \prod_{p=1}^{M-1} \), respectively.

**Proof.** We see that the function \( \eta \) is continuous at every point \( z \) with \( \prod_{i,j} Q_j(F^i)(z) \neq 0 \). For a point \( z_0 \in \Delta(R) \) such that \( \prod_{i,j} Q_j(F^i)(z_0) = 0 \), we have

\[
\nu_\eta(z_0) \geq \frac{1}{\sigma_M + \epsilon \tau_M} \left( \lambda \nu_\eta(z_0) + \sum_{i=1}^m \nu_{F^i_M}(z_0) - \sum_{i=1}^m \sum_{j=1}^q \omega_j \nu_{Q_j(F^i)}(z_0) \right) \geq 0.
\]

This implies that \( d\tau^2 \) is a continuous pseudo-metric on \( \Delta(R) \).

We now prove that \( d\tau^2 \) has strictly negative curvature on \( \Delta(R) \). Again, we have

\[
\sum_{i=1}^m \, \dd^c \log \frac{|F^i_M|^{1+\epsilon}}{\prod_{j=1}^q |Q_j(F^i)|^{\omega_j}} + (\lambda + \epsilon) \dd^c \log |h| \geq 0.
\]

Let \( \Omega \) be the Fubini-Study form of \( \mathbb{P}^n(\mathbb{C}) \) and denote by \( \Omega_{f^i} \) the pull-back of \( \Omega \) by the map \( f^i \) (\( 1 \leq i \leq m \)). By Theorems 2.2 and 2.3 we have

\[
\dd^c \log \eta^{1/m} \geq \frac{\gamma - \epsilon(\sigma_M+1+\frac{\epsilon}{\gamma})}{m(\sigma_M + \epsilon \tau_M)} \, \dd \sum_{i=1}^m \Omega_{f^i} + \frac{\epsilon}{4m(\sigma_M + \epsilon \tau_M)} \sum_{i=1}^m \, \dd^c \log (|F^i_0|^2 \cdots |F^i_{M-1}|^2)
\]

\[
+ \frac{1}{2m(\sigma_M + \epsilon \tau_M)} \sum_{i=1}^m \, \dd^c \log \prod_{p=0}^{M-1} |F^i_p|^{2(\omega_j)} \prod_{j=1}^q \, \dd M \log (\sigma_M + \epsilon \tau_M) |F^i_0|^2 \cdots |F^i_{M-1}|^2 \prod_{j=1}^q |Q_j(F^i)|^{\omega_j}
\]

\[
\geq \frac{\epsilon \tau_M}{4m \sigma_M (\sigma_M + \epsilon \tau_M)} \sum_{i=1}^m \left( \frac{|F^i_0|^2 \cdots |F^i_{M-1}|^2}{|F^i_0|^{2(\sigma_M+1)}} \right) \frac{1}{\tau} \, \dd^c |z|^2
\]

\[
+ C_0 \sum_{i=1}^m \left( \prod_{j=1}^q \frac{|Q_j(F^i)| |F^i_0|^2 \cdots |F^i_{M-1}|^2}{|F^i_0|^{2(\sigma_M+1)}} \right) \frac{1}{\tau M} \, \dd^c |z|^2
\]

\[
\geq \frac{\epsilon \tau_M}{4m \sigma_M (\sigma_M + \epsilon \tau_M)} \left( \frac{\prod_{i=1}^m |F^i_0|^2 \cdots |F^i_{M-1}|^2}{|F^i_0|^{2(\sigma_M+1)}} \right) \frac{1}{\tau M} \, \dd^c |z|^2
\]

\[
+ m C_0 \left( \prod_{i=1}^m \frac{|F^i_0|^2 \cdots |F^i_{M-1}|^2}{|F^i_0|^{2(\sigma_M+1)}} \right) \frac{1}{\tau M} \, \dd^c |z|^2
\]

\[
\geq C_1 \left( \frac{\prod_{i=1}^m |F^i_0|^2 \cdots |F^i_{M-1}|^2}{|F^i_0|^{2(\sigma_M+1)}} \right) \frac{1}{\tau M} \, \dd^c |z|^2
\]
for some positive constant $C_0, C_1$, where the last inequality comes from Hölder’s inequality. On the other hand, we have $|h| \leq \prod_{i=1}^{m} |F^i|^\rho \leq \prod_{i=1}^{m} |F^i_0|^\rho$ and
\[
\prod_{i=1}^{m} |F^i_0|^\omega (q-2N+k-1)-M+k-\epsilon \sigma_{M+1} \geq |h|^{\lambda + \epsilon} \prod_{i=1}^{m} |F^i_0|^\gamma - \epsilon (\sigma_{M+1} + \frac{\rho}{d}).
\]
This implies that $\Delta \log \eta^{2/m} \geq C_2 \eta^{2/m}$ for some positive constant $C_2$. Therefore, $d\tau^2$ has strictly negative curvature.

**Lemma 3.3.** Let the notations and the assumption as in Lemma 3.1. Then for an arbitrarily given $\epsilon$ satisfying
\[
\gamma = \sum_{j=1}^{q} \omega_j - M - 1 - \frac{\lambda \rho}{d} > \epsilon (\sigma_{M+1} + \frac{\rho}{d}),
\]
there exists a positive constant $C$, depending only on $\epsilon, Q_j$ ($1 \leq j \leq q$), such that
\[
\left( |h|^{\lambda + \epsilon} \prod_{i=1}^{m} |F^i_0|^\gamma - \epsilon (\sigma_{M+1} + \frac{\rho}{d}) |F^i_M| |F^i_0|^{\epsilon \rho} \prod_{j=1}^{q} |Q_j(F^i)|^{\omega_j} \right)^{1/m} \leq C \left( \frac{2R}{R^2 - |z|^2} \right)^{\sigma_{M+\epsilon \sigma_{M}}}
\]

**Proof.** As in the proof of Lemma 3.1, we have
\[
dd^c \log \eta^{1/m} \leq C_2 \eta^{2/m} \dd^c |z|^2.
\]
According to Lemma 2.5, this implies that
\[
\eta^{1/m} \leq C_3 \frac{2R}{R^2 - |z|^2},
\]
for some positive constant $C_3$. Then we have
\[
\left( |h|^{\lambda + \epsilon} \prod_{i=1}^{m} |F^i_0|^\gamma - \epsilon (\sigma_{M+1} + \frac{\rho}{d}) |F^i_M| |F^i_0|^{\epsilon \rho} \prod_{j=1}^{q} |Q_j(F^i)|^{\omega_j} \right)^{1/m(\sigma_{M+\epsilon \sigma_{M})} \leq C_3 \frac{2R}{R^2 - |z|^2}.
\]
It follows that
\[
\left( |h|^{\lambda + \epsilon} \prod_{i=1}^{m} |F^i_0|^\gamma - \epsilon (\sigma_{M+1} + \frac{\rho}{d}) |F^i_M| |F^i_0|^{\epsilon \rho} \prod_{j=1}^{q} |Q_j(F^i)|^{\omega_j} \right)^{1/m(\sigma_{M+\epsilon \sigma_{M})} \leq C_3 \frac{2R}{R^2 - |z|^2}.
\]
Note that the function $x^\omega \log^\omega \left( \frac{\delta}{x^2} \right)$ ($\omega > 0, 0 < x \leq 1$) is bounded. Then we have
\[
\left( |h|^{\lambda + \epsilon} \prod_{i=1}^{m} |F^i_0|^\gamma - \epsilon (\sigma_{M+1} + \frac{\rho}{d}) |F^i_M| |F^i_0|^{\epsilon \rho} \prod_{j=1}^{q} |Q_j(F^i)|^{\omega_j} \right)^{1/m(\sigma_{M+\epsilon \sigma_{M})} \leq C_4 \frac{2R}{R^2 - |z|^2},
\]
for a positive constant $C_4$. The lemma is proved.

**Lemma 3.4** (cf. Lemma 1.6.7). Let $d\sigma^2$ be a conformal flat metric on an open Riemann surface $S$. Then for every point $p \in S$, there is a holomorphic and locally biholomorphic map $\Phi$ of a disk $\Delta(R_0)$ onto an open neighborhood of $p$ with $\Phi(0) = p$ such that $\Phi$ is a local isometric, namely the pull-back $\Phi^*(d\sigma^2)$ is equal to the standard (flat)
metric on $\Delta(R_0)$, and for some point $a_0$ with $|a_0| = 1$, the curve $\Phi(0,R_0a_0)$ is divergent in $S$ (i.e., for any compact set $K \subset S$, there exists an $s_0 < R_0$ such that $\Phi(0,s_0a_0)$ does not intersect $K$).

**Theorem 3.5.** Let $S$ be an open Riemann surface and $V$ be an $\ell$-dimension projective subvariety of $\mathbb{P}^n(\mathbb{C})$. Let $f^1, \ldots, f^m$ be $m$ holomorphic curves from $S$ into $V$ ($1 \leq m \leq n$). Let $Q_1, \ldots, Q_q$ be $q$ hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in $N$-subgeneral position with respect to $V$ and $d = \text{lcm}(\deg Q_1, \ldots, \deg Q_q)$. Assume that each $f_i$ is nondegenerate over $I_d(V)$, there exists a holomorphic function $h$ on $S$ satisfying

$$\lambda h + \sum_{i=1}^{m} \nu_{F_i} \geq \sum_{i=1}^{q} \sum_{j=1}^{q} \omega_j \nu_{Q_j(F^i)} \text{ and } |h| \leq \prod_{i=1}^{m} \|F^i\|^{\rho},$$

where $F^i = (F^i_0, \ldots, F^i_n)$ is a reduced representation of $f^i$ ($1 \leq i \leq m$) and the metric

$$ds^2 = 2|\xi|^{2/m} \cdot \left( \prod_{i=1}^{m} \|F^i\| \right)^{2/m} |dz|^2,$$

where $\xi$ is a nowhere zero holomorphic function, is complete on $S$. Then we have

$$q \leq \frac{2N - \ell + 1}{\ell + 1} \left( M + 1 + \frac{\lambda}{d} + \frac{M(M+1)}{2d} \right).$$

**Proof.** If there are some hypersurfaces $Q_j$ such that $V \subset Q_j$, for instance they all are $Q_{q-r+1}, \ldots, Q_q$ ($0 \leq r \leq N - \ell + 1$), then by setting $N' = N - r, q' = q - r$ we have

$$\frac{2N - \ell + 1}{\ell + 1} \left( M + 1 + \frac{\lambda}{d} + \frac{M(M+1)}{2d} \right) - q \geq \frac{2N' - \ell + 1}{\ell + 1} \left( M + 1 + \frac{\lambda}{d} + \frac{M(M+1)}{2d} \right) - q'$$

and $Q_1, \ldots, Q_{q-r}$ are in $N'$-subgeneral position with respect to $V$. Then, without loss of generality, we may assume that $V \not\subset Q_j$ for all $j = 1, \ldots, q$.

We fix a $\mathbb{C}$-ordered basis $\mathcal{V} = ([v_0], \ldots, [v_M])$ of $I_d(V)$ as in the Section 3. By replacing $Q_i$ with $Q_i^{d/\deg Q_i}$ ($1 \leq i \leq q$) if necessary, we may assume that all $Q_i$ ($1 \leq i \leq q$) are of the same degree $d$. Suppose that

$$[Q_j] = \sum_{i=0}^{M} a_{ji}[v_i],$$

where $\sum_{i=0}^{M} |a_{ji}|^2 = 1$.

Since $f^i$ ($1 \leq i \leq m$) is nondegenerate over $I_d(V)$, the contact functions satisfy

$$F^i_p(Q_j) \neq 0, \forall 1 \leq j \leq q, 0 \leq p \leq M.$$ 

Then, for each $j, p$ ($1 \leq j \leq q, 0 \leq p \leq M$), we may choose $i_1, \ldots, i_p$ with $0 \leq i_1 < \cdots < i_p \leq M$ such that

$$\psi(F^i)_{jp} = \sum_{s \neq i_1, \ldots, i_p} a_{js}w(v_s(F^i), v_{i_1}(F^i), \ldots, v_{i_p}(F^i)) \neq 0.$$
We note that $\psi(F^i)_{j0} = F^i_0(Q_j) = Q_j(F^i)$ and $\psi(F^i)_{jM} = F^i_M$.

Suppose contrarily that

$$q > \frac{2N - \ell + 1}{\ell + 1} \left( M + 1 + \frac{\lambda \rho}{d} + \frac{M(M + 1)}{2d} \right).$$

From Theorem 2.1 we have

$$(q - 2N + \ell - 1)\tilde{\omega} = \sum_{j=1}^{q} \omega_j - \ell - 1; \quad \tilde{\omega} \geq \omega_j > 0 \text{ and } \tilde{\omega} \geq \frac{\ell + 1}{2N - \ell + 1}.$$

Therefore,

$$\sum_{j=1}^{q} \omega_j - M - 1 - \frac{\lambda \rho}{d} \geq \tilde{\omega}(q - 2N + \ell - 1) - M + \ell - \frac{\lambda \rho}{d}$$

$$(3.6) \geq \frac{\ell + 1}{2N - \ell + 1} (q - 2N + \ell - 1) - M + \ell - \frac{\lambda \rho}{d}$$

$$= \frac{\ell + 1}{2N - \ell + 1} \left( q - \frac{(2N + \ell - 1)(M + 1 + \frac{\lambda \rho}{d})}{\ell + 1} \right)$$

$$> \frac{\ell + 1}{2N - \ell + 1} \cdot \frac{(2N + \ell - 1)M(M + 1)}{2d(\ell + 1)} = \frac{\sigma_M}{d}.$$

Then, we can choose a rational number $\epsilon (> 0)$ such that

$$\frac{d(\sum_{j=1}^{q} \omega_j - M - 1 - \frac{\lambda \rho}{d} - \sigma_M)}{d(\sigma_{M+1} + \frac{\rho}{d}) + \tau_M} > \epsilon > \frac{d(\sum_{j=1}^{q} \omega_j - M - 1 - \frac{\lambda \rho}{d} - \sigma_M)}{\frac{1}{mq} + d(\sigma_{M+1} + \frac{\rho}{d}) + \tau_M}.$$

We define the following numbers

$$\beta := d \left( \sum_{j=1}^{q} \omega_j - M - 1 - \frac{\lambda \rho}{d} - \epsilon \left( \sigma_{M+1} + \frac{\rho}{d} \right) \right) > \sigma_M + \epsilon \tau_M,$$

$$\rho := \frac{1}{\beta} (\sigma_M + \epsilon \tau_M),$$

$$\rho^\ast := \frac{1}{(1 - \rho)\beta} = \frac{1}{d(\sum_{j=1}^{q} \omega_j - M - 1 - \frac{\lambda \rho}{d}) - \sigma_M - \epsilon(d\sigma_{M+1} + \rho + \tau_M)}.$$

It is clear that $0 < \rho < 1$ and $\frac{\omega^\ast}{mq} > 1$.

We consider a set

$$S' = \{ a \in S; \psi(F^i)_{jp}(a) \neq 0, h(a) \neq 0 \forall 1 \leq i \leq m; j = 1, \ldots, q; p = 0, \ldots, M \}$$

and define a new pseudo-metric on $S'$ as follows

$$d\tau^2 = \left| \xi \right|^{2(1 + \beta + \omega^\ast)} \left( \frac{1}{\|h\|^{1+\epsilon}} \prod_{i=1}^{m} \frac{\prod_{j=1}^{q} |Q_j(F^i)|^{\omega_j}}{\prod_{j,p} |\psi(F^i)_{jp}|^{\omega_j}} \right)^{2\omega^\ast} |dz|^2.$$

Since $Q_j(F^i), F^i_M, \psi(F^i)_{jp}$ ($1 \leq j \leq q$) and $h$ are all holomorphic functions on $S'$, $d\tau^2$ is flat on $S'$. We now show that $d\tau^2$ is complete on $S'$. 
Indeed, suppose contrarily that $S'$ is not complete with $d\tau^2$, there is a divergent curve $\gamma : [0, 1) \to S'$ with finite length. Then, as $t \to 1$ there are only two cases: either $\gamma(t)$ tends to a point $a$ with

$$ (h \prod_{j=1}^{q} \prod_{p=0}^{M} \psi(F^j)_{jp})(a) = 0 $$

or else $\gamma(t)$ tends to the boundary of $S$.

For the first case, by Theorem 2.4, we have

$$ \nu_{d\tau}(a) \leq - \left( \sum_{i=1}^{m} \nu_{F^i}(a) - \sum_{i=1}^{m} \sum_{j=1}^{q} \omega_j \nu_{Q_j(F^i)}(a) + \lambda \nu_{h}(a) \right) $$

$$ + \left( \epsilon \sum_{i=1}^{m} \nu_{F^i}(a) + \epsilon \nu_{h}(a) + \frac{\epsilon}{q} \sum_{i=1}^{m} \sum_{j,p} \nu_{\psi(F^i)_{jp}}(a) \right) \rho^* $$

$$ \leq - \frac{\epsilon \rho^*}{m} \left( \sum_{i=1}^{m} \nu_{F^i}(a) + \nu_{h}(a) \right) - \frac{\epsilon \rho^*}{mq} \sum_{i=1}^{m} \sum_{j,p} \nu_{\psi(F^i)_{jp}}(a) \leq - \frac{\epsilon \rho^*}{mq} $$

Then, there is a positive constant $C$ such that

$$ |d\tau| \geq \frac{C}{|z-a|^{\frac{m}{mq}}} |dz| $$

in a neighborhood of $a$. Then we get

$$ L_{d\tau}(\gamma) = \int_{0}^{1} \|\gamma'(t)\|_{d\tau} dt = \int_{\gamma} d\tau \geq \int_{\gamma} \frac{C}{|z-a|^{\frac{m}{mq}}} |dz| = +\infty $$

($\gamma(t)$ tends to $a$ as $t \to 1$). This is a contradiction. Then, the second case must occur, i.e., $\gamma(t)$ tends to the boundary of $S$ as $t \to 1$.

Take a disk $\Delta$ (in the metric induced by $d\tau^2$) around $\gamma(0)$. Since $d\tau$ is flat, by Lemma 3.4, $\Delta$ is isometric to an ordinary disk in the plane. Let $\Phi : \Delta(r) = \{ |\omega| < r \} \to \Delta$ be this isometric with $\Phi(0) = \gamma(0)$. Extend $\Phi$ as a local isometric into $S'$ to the largest disk possible $\Delta(R) = \{ |\omega| < R \}$, and denoted again by $\Phi$ this extension (for simplicity, we may consider $\Phi$ as the exponential map). Since $\Phi$ cannot be extended to a larger disk, it must be hold that the image of $\Phi$ goes to the boundary of $S'$. But, this image cannot go to points $z_0$ of $S$ with $h(z_0) \prod_{i=1}^{m} \left( F^i_M(z_0) \prod_{j,p} \psi(F^i)_{jp}(z_0) \right) = 0$, since we have already shown that $\gamma(0)$ is infinitely far away in the metric $d\tau^2$ with respect to these points. Then the image of $\Phi$ must go to the boundary $S$. Hence, by again Lemma 3.4 there exists a point $w_0$ with $|w_0| = R$ so that $\Gamma = \Phi(0, w_0)$ is a divergent curve on $S$.

Our goal now is to show that $\Gamma$ has finite length in the original metric $ds^2$ on $S$, contradicting the completeness of $S$. Let $g^i := f^i \circ \Phi : \Delta(R) \to V \subset \mathbb{P}^n(C)$ be a holomorphic map which is nondegenerate over $I_d(V)$. Then $g^i$ have a reduced representation

$$ G^i = (g^i_0, \ldots, g^i_n), $$
where \( g_i^j = f_i^j \circ \Phi \) (\( 1 \leq i \leq m, 0 \leq j \leq n \)). Hence, we have:

\[
\Phi^* ds^2 = 2 |\xi \circ \Phi|^{2/m} \prod_{i=1}^{m} \left\| F^i \circ \Phi \right\|^{2/m} |\Phi^* dz|^2 = 2 |\xi \circ \Phi|^{2/m} \left( \prod_{i=1}^{m} \left\| G^i \right\|^{2/m} \right) \left| \frac{d(z \circ \Phi)}{dw} \right| |dw|^2,
\]

\[
G_M^i = (F^i \circ \Phi)_M = F_M^i \circ \Phi \cdot \left( \frac{d(z \circ \Phi)}{dw} \right)^{\sigma_M},
\]

\[
\psi(G^i)_{jp} = \psi(F^i \circ \Phi)_{jp} = \psi(F^i)_{jp} \cdot \left( \frac{d(z \circ \Phi)}{dw} \right)^{\sigma_p}, (0 \leq p \leq M).
\]

On the other hand, since \( \Phi \) is locally isometric,

\[
|dw| = |\Phi^* d\tau|
\]

\[
= |\xi \circ \Phi|^{\frac{1+\beta\rho\rho^*}{m}} \left( \frac{1}{|h \circ \Phi|^{\lambda+\epsilon}} \prod_{i=1}^{m} \frac{|G_M^i|^{1+\epsilon}}{F_M^i \circ \Phi} \prod_{j=1}^{q} |Q_j(F^i)\circ\Phi|^{\omega_j} \right) \left( \frac{d(z \circ \Phi)}{dw} \right)^{\frac{\rho^*/m}{|1+\beta\rho\rho^*|}} \cdot |dw|
\]

(because \( 1 + \rho^* (\sigma_M + \epsilon \tau_M) = 1 + \beta \rho \rho^* \)). This implies that

\[
\left| \frac{d(z \circ \Phi)}{dw} \right| = |\xi \circ \Phi|^{-\frac{1}{m}} \left( \frac{1}{|h \circ \Phi|^{\lambda+\epsilon}} \prod_{i=1}^{m} \frac{|G_M^i|^{1+\epsilon}}{F_M^i \circ \Phi} \prod_{j=1}^{q} |Q_j(F^i)\circ\Phi|^{\omega_j} \right) \left( \frac{d(z \circ \Phi)}{dw} \right)^{\frac{\rho^*/m}{|1+\beta\rho\rho^*|}} \cdot |\xi \circ \Phi|^{\frac{1}{m}}
\]

\[
\leq |\xi \circ \Phi|^{-\frac{1}{m}} \left( \frac{1}{|h \circ \Phi|^{\lambda+\epsilon}} \prod_{i=1}^{m} \frac{|G_M^i|^{1+\epsilon}}{F_M^i \circ \Phi} \prod_{j=1}^{q} |Q_j(F^i)\circ\Phi|^{\omega_j} \right) \left( \frac{d(z \circ \Phi)}{dw} \right)^{\frac{\rho^*/m}{|1+\beta\rho\rho^*|}} \cdot |\xi \circ \Phi|^{\frac{1}{m}}
\]

Hence, we have

\[
\Phi^* ds \leq \sqrt{2} \prod_{i=1}^{m} \left\| G^i \right\|^{\frac{1}{m}} \left( |h \circ \Phi|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|G_M^i|^{1+\epsilon}}{F_M^i \circ \Phi} \prod_{j=1}^{q} |Q_j(F^i)\circ\Phi|^{\omega_j} \right)^{\frac{1}{m\beta}} |dw|
\]

\[
= \sqrt{2} \left( |h \circ \Phi|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|G_M^i|^{1+\epsilon}}{F_M^i \circ \Phi} \prod_{j=1}^{q} |Q_j(F^i)\circ\Phi|^{\omega_j} \right)^{\frac{1}{m\beta}} |dw|
\]

\[
\leq C_1 \left( |h \circ \Phi|^{\lambda+\epsilon} \prod_{i=1}^{m} \frac{|G_M^i|^{1+\epsilon}}{F_M^i \circ \Phi} \prod_{j=1}^{q} |Q_j(F^i)\circ\Phi|^{\omega_j} \right)^{\frac{1}{m\beta}} |dw|.
\]

with a positive constant \( C_1 \). We note that

\[
\sigma_M = \sum_{j=1}^{q} \omega_j - M - 1 - \frac{\lambda \omega - \epsilon}{\sigma_M + 1 + \frac{\lambda \rho}{M}}.
\]

Then the inequality (3.6) yields that the conditions of Lemma 2.5 are satisfied. Then, by
applying Lemma 2.5, we have
\[ \Phi^* ds \leq C_2 \left( \frac{2R}{R^2 - |w|^2} \right)^\rho |dw| \]
for some positive constant \( C_2 \). Also, we have that \( 0 < \rho < 1 \). Then
\[ L_{ds^2}(\Gamma) \leq \int_\Gamma ds = \int_{0, w_0} \Phi^* ds \leq C_2 \cdot \int_0^R \left( \frac{2R}{R^2 - |w|^2} \right)^\rho |dw| < +\infty. \]
This contradicts the assumption of completeness of \( S \) with respect to \( ds^2 \). Thus, \( ds^2 \) is complete on \( S' \).

Then, we note that the metric \( d\tau^2 \) on \( S' \) is flat. Then by a theorem of Huber (cf. [2 Theorem 13, p.61]), the fact that \( S' \) has finite total curvature (with respect to \( d\tau^2 \)) implies that \( S' \) is finitely connected. This means that there is a compact subregion of \( S' \) whose complement is the union of a finite number of doubly-connected regions. Therefore, the functions \( |\psi_i|^0 \prod_{j=1}^q |\psi_j(G)_{ij}| \) must have only a finite number of zeros, and the original surface \( S \) is finitely connected. Due to Osserman (cf. [7, Theorem 2.1]), each annular ends of \( S' \), and hence of \( S \), is conformally equivalent to a punctured disk. Thus, the Riemann surface \( S \) must be conformally equivalent to a compact surface \( \bar{S} \) punctured at a finite number of points \( P_1, \ldots, P_r \). Then, there are disjoint neighborhoods \( U_i \) of \( P_i \) \( (1 \leq i \leq r) \) in \( \bar{S} \) and biholomorphic maps \( \phi_i : U_i \rightarrow \Delta \) with \( \phi_i(P_i) = 0 \). Note that, the Poincare-metric on \( \Delta^* = \Delta \setminus \{0\} \) is given by \( ds^2_{\Delta^*} = \frac{4|dw|^2}{|w|^2 \log^2 |w|^2} \), where \( w \) is the complex coordinate on \( \Delta \).

As we known that the universal covering surface of \( S \) is biholomorphic to \( \mathbb{C} \) or a disk in \( \mathbb{C} \). If the universal covering of \( S \) is biholomorphic to \( \mathbb{C} \) (denote by \( \tilde{\Phi} : \mathbb{C} \rightarrow S \) this universal covering mapping), then from the assumption that
\[ \lambda \nu_h + \sum_{i=1}^m \nu_{F_i} \geq \sum_{i=1}^m \sum_{j=1}^q \omega_j \nu_{Q_i(F^i)} \text{ and } |h| \leq \prod_{i=1}^m ||F^i||^\rho, \]
we have
\[ \lambda \rho \sum_{i=1}^m T_{f^i \circ \Phi} \geq \sum_{j=1}^q \sum_{i=1}^m \left( \sum_{j=1}^q \omega_j N(r, \nu_{Q_i(F^i \circ \Phi)}) - N(r, \nu_{F^i \circ \Phi}) \right), \]
where \( T_f(r) \) is the characteristic function of the mapping \( f : \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C}) \) and \( N(r, \nu) \) is the counting function of the divisor \( \nu \) on \( \mathbb{C} \) (see [9] for the definitions). Using the second main theorem (Theorem 1.1 in [9]), we have
\[ \lambda \rho \sum_{i=1}^m T_{f^i \circ \Phi} \geq \sum_{i=1}^m \sum_{j=1}^q N^{[M]}(r, \nu_{Q_i(F^i \circ \Phi)}) \geq \left( q - \frac{(2N - \ell + 1)(M + 1)}{\ell + 1} \right) \sum_{i=1}^m T_{f^i \circ \Phi}. \]
Here, the symbol “\( \| \| \)” means the inequalities hold for all \( r \in [1, +\infty) \) outside a finite Borel measure set \( E \). Letting \( r \rightarrow +\infty \) \( (r \notin E) \), we get
\[ \lambda \rho \geq q - \frac{(2N - \ell + 1)(M + 1)}{\ell + 1} \]
and arrive at a contradiction.
Then, we only consider the case where the universal covering surface of \( S \) is biholomorphic to the unit disk \( \Delta \) in \( \mathbb{C} \). Let \( \Phi : \Delta \to S \) be this holomorphic universal covering. Consider the following metric
\[
d^2 = \eta^2 |dz|^2,
\]
where
\[
\eta = \left( |h|^\gamma + \prod_{i=1}^m \left| \frac{F_0^{\frac{1}{2}} | \frac{\gamma - \epsilon (\sigma_{M+1} + \frac{2}{\sigma}) | F_{M}^{\frac{1}{2}} | \prod_{j=1}^q | Q_j (F_i) |^{\gamma} }{m (\sigma_M + \epsilon_\sigma)} \right| \right)^{-\frac{1}{m (\sigma_M + \epsilon_\sigma)}}.
\]
It is obvious that \( d^2 \) is continuous on \( S \setminus \bigcup_{j=1}^q (f_i)^{-1}(Q_j) \). Take a point \( a \) such that \( \prod_{j=1}^q Q_j(F_i)(a) = 0 \). From the assumption, we have
\[
d^2(a) \geq \frac{1}{m (\sigma_M + \epsilon_\sigma)} \left( \lambda v_h(a) + \sum_{i=1}^m v_{F_i}(a) - \sum_{i=1}^m \sum_{j=1}^q \omega_j v_i Q_j(F_i)(a) \right) \geq 0.
\]
Therefore \( d^2 \) is continuous at \( a \). This yields that \( d^2 \) is a continuous pseudo-metric on \( S \).

Now, from Lemma 3.1, we see that \( d^2 \) has strictly negative curvature on \( S \). Hence, by the decreasing distance property, we have
\[
\Phi^* d^2 \leq d\sigma^2_\Delta \leq C_3 \cdot (\Phi \circ \phi_i^{-1})^* d\sigma^2_\Delta \quad (1 \leq i \leq r)
\]
for some positive constant \( C_3 \). This implies that
\[
\int_{U_i} \Omega_{d^2} \leq \int_{\Phi^{-1}(U_i)} \Phi^* \Omega_{d\sigma^2_\Delta} \leq l_0 C_3 \int_{\Delta^*} \Omega_{d\sigma^2_\Delta} < \infty.
\]
where \( l_0 \) is the number of the sheets of the covering \( \Phi \). Then, it yields that
\[
\int_S \Omega_{d^2} \leq \int_{S \setminus \bigcup_{i=1}^r U_i} \Omega_{d^2} + l_0 C_3 r \int_{\Delta^*} \Omega_{d\sigma^2_\Delta} < \infty.
\]

Now, denote by \( ds^2 \) the original metric on \( S \). Similar as (3.2), we have
\[
\text{dd}^c \log \eta \geq \frac{\gamma - \epsilon (\sigma_{M+1} + \frac{2}{\sigma})}{\sigma_M + \epsilon_\sigma} \int m \sum_{i=1}^m \Omega_i.
\]
Then there is a subharmonic function \( \nu \) such that
\[
\nu^2 |dz|^2 = e^\nu |\frac{\rho}{\sigma} \left( \prod_{i=1}^m \| F_i \|^2 \frac{\gamma - \epsilon (\sigma_{M+1} + \frac{2}{\sigma})}{m (\sigma_M + \epsilon_\sigma)} \right) |d\nu|^2
\]
\[
= e^\nu \left( \prod_{i=1}^m \| F_i \|^2 \frac{\gamma - \epsilon M + \frac{2}{\sigma M + \epsilon_\sigma}}{m (\sigma_M + \epsilon_\sigma)} \right) |d\nu|^2
\]
for a subharmonic function \( w \) on \( S \). Since \( S \) is complete with respect to \( ds^2 \), applying a result of S. T. Yau \([14]\) and L. Karp \([6]\) we have
\[
\int_S \Omega_{d^2} = \int_S e^w \Omega_{ds^2} = +\infty.
\]
This contradiction completes the proof of the theorem. \( \square \)
4. Uniqueness theorems for Gauss maps

In this section, we will prove main theorems of this paper. Firstly, we prove the following.

**Lemma 4.1.** Let $S$ be an open Riemann surface and $V$ be a $\ell$-dimension projective subvariety of $\mathbb{P}^n(\mathbb{C})$. Let $Q_1, \ldots, Q_q$ be $q$ hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in $N-$subgeneral position with respect to $V$ and $d = \text{lcm}(\deg Q_1, \ldots, \deg Q_q)$. Let $f^1, f^2$ be holomorphic maps from $S$ into $V$ such that

1. $\bigcap_{j=0}^k (f^1)^{-1}(Q_{j}) = \emptyset$ for every $1 \leq j_0 < \cdots < j_k \leq q$,
2. $f^1 = f^2$ on $\bigcup_{j=1}^q ((f^1)^{-1}(Q_j) \cup (f^2)^{-1}(Q_j))$.

If $f^1$ is nondegenerate over $I_d(V)$, $S$ is complete with a metric $ds^2 = |\xi|^2 F^1_0|dz^2|$, where $\xi$ is a non-vanishing holomorphic function, $z$ is a conformal coordinate on $S$, $F^1$ is a reduced presentation of $f^1$, and

$$q > \frac{2N - \ell + 1}{\ell + 1} \left( M + 1 + \sigma_M - \sigma_{M-\min(k,\ell)} + \frac{M(M+1)}{2d} \right)$$

then $f^2$ is nondegenerate over $I_d(V)$.

**Proof.** Let $F^i = (f^i_0, \ldots, f^i_n)$ be reduce representations of $f^i$ ($i = 1, 2$). Suppose contrarily that $f^2$ is degenerate over $I_d(V)$. Then there exists a hypersurface $Q$ of degree $d$ such that $V \not\subset Q, Q(F^2) \equiv 0$. By the assumption that $f^1$ is nondegenerate over $I_d(V)$, we have $Q(F^1) \not\equiv 0$. Since $f^1 = f^2$ on $\bigcup_{i=1}^q Q_i$, we have $Q(F^1) = 0$ on $\bigcup_{i=1}^q (f^1)^{-1}Q_i$. Therefore, setting $k' = \min\{k, \ell\}$ and $h = Q(F^1)$, from Theorem 2.4 we have

$$(\sigma_M - \sigma_{M-k'}) \nu_h(a) + \nu_{F^1_0}(a) \geq \sum_{j=1}^{k'} \nu_{Q_j(F^1)}(a) \geq \sum_{j=1}^q \omega_j \nu_{Q_j(F^1)}(a).$$

Also, it is clear that $|h| \leq \|F^1\|^d$. Applying Theorem 3.5, we have

$$q \geq \frac{2N - \ell + 1}{\ell + 1} \left( M + 1 + \sigma_M - \sigma_{M-k'} + \frac{M(M+1)}{2d} \right).$$

This contradiction completes the proof of the lemma. □

**Proof of Theorem 4.7.** Without loss of generality, we may assume that $\deg Q_j = d$ for all $1 \leq j \leq q$. We may suppose that $f^1(S) \not\subset Q_j$ for all $j \in \{1, \ldots, q\}$ (otherwise $f^1 = f^2$). Let $z$ be a conformal coordinate on $S^1$ and $F^1 = (f^1_0, \ldots, f^1_n)$ be the reduce representation of $f^1$ for each $i \in \{1, 2\}$. Since $\Phi$ is a conformal diffeomorphism, there exists a non-vanishing holomorphic function $\xi$ such that $ds^2 = \|F^1\|^2|dz^2| = |\xi|^2 \|F^2\|^2|dz^2|$. We have $ds^2$ is complete on $S^1$. Also, from the proof of Lemma 4.1 we see that $f^2(S^1) \subset V$ and $f^2$ is nondegenerate over $I_d(V)$.

Suppose contrarily that $f^1 \neq f^2$. Then there exists $0 \leq i < j \leq n$ such that

$$h := f^1 f^2_j - f^1_i f^2_i \neq 0.$$

It is clear that $\nu_h \geq \nu_{[I_{i=1}^q Q_j(F^i)]}$ for each $i \in \{1, 2\}$. 


From Theorem 2.4 we have
\[ 2(\sigma_M - \sigma_M - \min\{k, \ell\})\nu_h + \sum_{i=1}^{2} \nu_{F_M^i} \geq \sum_{i=1}^{2} \sum_{j=1}^{q} \nu_{Q_j(F^i)}. \]

Note that \(|h| \leq \|F^1\| \cdot \|F^2\|\). By applying Theorem 3.5, we have
\[ q \leq \frac{2N - \ell + 1}{\ell + 1} \left( M + 1 + \frac{2(\sigma_M - \sigma_M - \min\{k, \ell\})}{d} + \frac{M(M + 1)}{2d} \right). \]

This contradiction completes the proof of the theorem. \( \square \)

**Proof of Theorem 1.2.** Let \( z \) be a conformal coordinate on \( S^1 \) and \( F^i \) be the reduced representation of \( f^i \) for each \( i \in \{1, 2\} \). Since \( \Phi \) is a conformal diffeomorphism, there exists a non-vanishing holomorphic function \( \xi \) such that \( ds^2 = \|F^1\|^2|dz|^2 = \|F^2\|^2|dz|^2 = |\xi| \cdot \|F^1\| \cdot \|F^2\||dz|^2 \). Note that, \( ds^2 \) is complete on \( S^1 \).

Suppose contrarily that the theorem does not hold. Consider the simple graph \( G \) with the set of vertices \( \{1, \ldots, q\} \) and the set of edges consisting of all pairs \( i, j \) such that \( Q_i(F^1)Q_j(F^2) - Q_j(F^1)Q_i(F^2) \neq 0 \). The supposition implies that the order of each vertex does not exceed \( [q/2] \). Then, by Dirac’s theorem, there is a Hamilton cycle \( i_1, \ldots, i_q, i_{q+1}, \) where \( i_{q+1} = i_1 \). We set \( u_j := i_j + 1 \) if \( j < q \) and \( u_q := i_1 \). Then we have
\[ h := \prod_{j=1}^{q}(Q_{i_j}(F^1)Q_{u_j}(F^2) - Q_{u_j}(F^1)Q_{i_j}(F^2)) \neq 0. \]

It is clear that \(|h| \leq \|F^1\|^d q \|F^2\|^d q \).

For each point \( a \in \bigcup_{j=1}^{q} (f^j)^{-1}(Q_j) \), take a subset \( I_1 \subset \{1, \ldots, q\} \) such that \( \sharp I = N + 1 \) and \( Q_j(F^1)(a) \neq 0 \) for every \( j \notin I_1 \). Then there is a subset \( I_2 \subset I_1 \) such that \( \sharp I_2 = \text{rank}_\mathbb{C}\{[Q_j]; j \in I_2\} = \ell + 1 \) and
\[ \sum_{i \in I_2} (\nu_{Q_i(F^1)}(a) + \nu_{Q_i(F^2)}(a)) \geq \sum_{i \in I_1} \omega_i(\nu_{Q_i(F^1)}(a) + \nu_{Q_i(F^2)}(a)). \]

Then, there exists a subset \( I \subset I_2 \) such that \( \sharp I = t \) and \( Q_j(F^1)(a) \neq 0 \) for every \( j \in I_2 \setminus I \). Hence, we have
\[ \sum_{i \in I} (\nu_{Q_i(F^1)}(a) + \nu_{Q_i(F^2)}(a)) \geq \sum_{i=1}^{q} \omega_i(\nu_{Q_i(F^1)}(a) + \nu_{Q_i(F^2)}(a)). \]
Then, we have
\[
\nu_h(a) \geq 2 \sum_{i=1}^{t} \min \{\nu_{Q_i(F^1)}(a), \nu_{Q_i(F^2)}(a)\} + (q - 2t)
\]
\[
\geq 2 \sum_{i=1}^{t} \left( \min \{\nu_{Q_i(F^1)}(a), M\} + \min \{\nu_{Q_i(F^2)}(a), M\} - M \right) + (q - 2t)
\]
\[
= 2 \sum_{i=1}^{t} \left( \min \{\nu_{Q_i(F^1)}(a), M\} + \min \{\nu_{Q_i(F^2)}(a), M\} \right) + (q - 2(M + 1)t)
\]
\[
\geq \frac{q + 2(M - 1)t}{2Mt} \sum_{i=1}^{t} \left( \min \{\nu_{Q_i(F^1)}(a), M\} + \min \{\nu_{Q_i(F^2)}(a), M\} \right).
\]

Also, by usual arguments we have
\[
\nu_{F^1_{M\ell}F^2_{M\ell}}(a) \geq \sum_{j=1}^{2} \sum_{i=1}^{t} \max \{0, \nu_{Q_i(F^j)}(a) - M\}.
\]

Therefore, we have the following estimate:
\[
\frac{2Mt}{q + 2(M - 1)t} \nu_h(a) + \nu_{F^1_{M\ell}F^2_{M\ell}}(a) \geq \sum_{i=1}^{t} (\nu_{Q_i(F^1)}(a) + \nu_{Q_i(F^2)}(a))
\]
\[
\geq \sum_{i=1}^{q} \omega_i(\nu_{Q_i(F^1)}(a) + \nu_{Q_i(F^2)}(a))
\]

By Theorem 3.5, we have
\[
q \leq \frac{2N - \ell + 1}{\ell + 1} \left( M + 1 + \frac{2Mtq}{(q + 2(M - 1)t)d} + \frac{M(M + 1)}{2d} \right)
\]
\[
= \frac{2N - \ell + 1}{\ell + 1} \left( M + 1 + \frac{2Mkq}{(q + 2(M - 1)k)d} + \frac{M(M + 1)}{2d} \right)
\]
and arrive at a contradiction. This completes the proof of the theorem. \(\square\)

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No potential conflict of interest was reported by the authors.

**References**

[1] L. V. Ahlfors, *An extension of Schwarz’s lemma*, Trans. Am. Math. Soc. **43** (1938) 359–364.

[2] A. Huber, *On subharmonic functions and differential geometry in large*, Comment. Math. Helv., **32** (1961), 13–72.

[3] H. Fujimoto, *Unicity theorems for the Gauss maps of complete minimal surface*, J. Math. Soc. Jpn., **45** (1998), 481–487.

[4] H. Fujimoto, *Unicity theorems for the Gauss maps of complete minimal Surfaces, II*, Kodai Math. J., **16** (1993), 335–354.

[5] H. Fujimoto, *Value Distribution Theory of the Gauss map of Minimal Surfaces in \(R^m\)*, Aspect of Math., Vol. E21, Vieweg, Wiesbaden (1993).

[6] L. Karp, *Subharmonic functions on real and complex manifolds*, Math. Z., **179** (1982), 535–554.
[7] R. Osserman, *On complete minimal surfaces*, Arch. Rational Mech. Anal., 13 (1963), 392–404.
[8] J. Park and M. Ru, *Unicity results for Gauss maps of minimal surfaces immersed in $\mathbb{R}^m$*, J. Geom., 108 (2017), 481–499.
[9] S. D. Quang and D. P. An, *Second main theorem and unicity of meromorphic mappings for hypersurfaces in projective varieties*, Acta Math. Vietnam, 42 (2017), no. 3, 455–470.
[10] S. D. Quang, *Generalization of uniqueness theorem for meromorphic mappings sharing hyperplanes*, Internat. J. Math., 30 (2019), no. 1, 1950011, 16 pp.
[11] S. D. Quang, *Modified defect relation for Gauss maps of minimal surfaces with hypersurfaces of projective varieties in subgeneral position*, arXiv:2107.08986 [math.DG].
[12] S. D. Quang, *Curvature estimate and the ramification of the holomorphic maps over hypersurfaces on Riemann surfaces*, Bull. Soc. Math. France 151 (1) (2023), 91-115.
[13] S. D. Quang, *Modified defect relation of Gauss maps on annular ends of minimal surfaces for hypersurfaces of projective varieties in subgeneral position*, J. Math. Anal. Appl. 23 (1) (2024), 127806.
[14] S.T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana U. Math. J., 25 (1976), 659–670.

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