NON-REMOVABILITY OF SIERPIŃSKI CARPETS

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ABSTRACT. We prove that all Sierpiński carpets in the plane are non-removable for (quasi)conformal maps. More precisely, we show that for any two Sierpiński carpets \( S, S' \subset \hat{\mathbb{C}} \) there exists a homeomorphism \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) that is conformal in \( \hat{\mathbb{C}} \setminus S \) and it maps \( S \) onto \( S' \). As a corollary, we obtain a partial answer to a question of Bishop [Bis94], whether any planar compact set with empty interior and positive measure can be mapped to a set of measure zero with an exceptional homeomorphism of the plane, conformal off that set.

1. INTRODUCTION

In recent work [Nta19, Theorem 1.8] the current author proved that the Sierpiński gasket, also called the Sierpiński triangle, is non-removable for conformal maps. In the same paper the author provided some evidence that all homeomorphic copies of the Sierpiński gasket should be non-removable for conformal maps; see [Nta19, Theorem 1.7]. This gave birth to the conception that “some sets should be non-removable for topological reasons”. Sierpiński carpets are topologically “larger” than gaskets and provide a perfect candidate to test this heuristic. In this work we prove that this is actually the case:

**Theorem 1.1.** All Sierpiński carpets \( S \subset \hat{\mathbb{C}} \) are non-removable for conformal maps.

This result resolves a conjecture of the author [Nta19, Conjecture 1]. Here, we pose another more general conjecture:

**Conjecture 1.1.** Every planar compact set containing a homeomorphic copy of \( C \times [0, 1] \) is non-removable for conformal maps, where \( C \) is the middle-thirds Cantor set.

We provide some necessary definitions before stating our other results.

**Definition 1.2.** Let \( K \subset \hat{\mathbb{C}} \) be a compact set. We say that \( K \) is conformally removable or removable for conformal maps if any homeomorphism \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) that is conformal in \( \hat{\mathbb{C}} \setminus K \) is actually conformal in \( \hat{\mathbb{C}} \) and thus, it is Möbius.

An equivalent notion is that of quasiconformal removability. We direct the reader to [You15], [Nta19], and the references therein for more background.

The *standard Sierpiński carpet* is constructed by subdividing the unit square \([0, 1]^2\) into nine squares of sidelength 1/3 and removing the middle square, and
then proceeding inductively in each of the remaining eight squares. It can be easily
proven that the standard Sierpiński carpet is non-removable for conformal maps;
see e.g. the discussion in [Nta19, Section 1.1].

In general, a Sierpiński carpet \( S \subset \hat{\mathbb{C}} \) is a set homeomorphic to the standard
carpet. It is a fundamental result of Whyburn [Why58] that a set \( S \) is a Sierpiński
carpet if and only if \( S \) has empty interior and \( S = \hat{\mathbb{C}} \setminus \bigcup_{i \in \mathbb{N}} Q_i \), where \( \{Q_i\}_{i \in \mathbb{N}} \) is a family of Jordan regions with disjoint closures and (spherical) diameters converging
to 0. The regions \( Q_i, i \in \mathbb{N}, \) are called the peripheral disks and the boundaries \( \partial Q_i \),
\( i \in \mathbb{N}, \) are called the peripheral circles of \( S \). Our main result is the following.

**Theorem 1.3.** Let \( S, S' \subset \hat{\mathbb{C}} \) be Sierpiński carpets. Then there exists a homeomor-
phism \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) with \( f(S) = S' \) such that \( f \) is conformal on \( \hat{\mathbb{C}} \setminus S \). Moreover, the
image of any finite collection of peripheral circles can be prescribed: if \( C_1, \ldots, C_N \) and \( C'_1, \ldots, C'_N \) are peripheral circles of \( S \) and \( S' \), respectively, then \( f \) can be chosen
such that \( f(C_i) = C'_i \) for all \( i = 1, \ldots, N \).

One can easily construct carpets \( S' \) with positive Lebesgue measure. The theorem,
combined with this fact, proves immediately Theorem 1.1, since a map \( f \) that
sends a carpet \( S \) of measure zero or positive measure to a carpet \( S' \) of positive
measure or measure zero, respectively, cannot be Möbius. Our proof of Theorem 1.3
is topological and utilizes the ideas of Whyburn [Why58].

Using Theorem 1.3 we obtain a partial answer to a question raised by Bishop
[Bis94, Question 3]. He asked whether any compact set \( K \subset \mathbb{C} \) with empty interior and
positive area can be mapped to a set of measure zero with a homeomorphism
\( f : \mathbb{C} \to \mathbb{C} \) that is conformal on \( \mathbb{C} \setminus K \). A partial answer to that question was given
by Kaufman and Wu [KW96, Theorem 3], where the authors proved that there
exists a subset of \( K \) with positive, strictly smaller measure such that the answer
to the question is positive for that subset. We prove that one can “enlarge” \( K \)
to a set \( L \) by attaching to it a small set of measure zero and Hausdorff dimension
arbitrarily close to 1, so that the answer is positive for \( L \):

**Proposition 1.4.** Let \( K \subset \mathbb{C} \) be a compact set with empty interior. Then for each \( \varepsilon > 0 \)
there exists a compact set \( L \supset K \) such that the Hausdorff dimension of \( L \setminus K \)
is at most \( 1 + \varepsilon \) and the following holds: there exists a homeomorphism \( f : \mathbb{C} \to \mathbb{C} \)
that is conformal on \( \mathbb{C} \setminus K \) such that \( f(L) \) has Hausdorff dimension at most \( 1 + \varepsilon \)
and in particular Lebesgue measure zero.

In fact \( L \) can be taken to lie in the \( \varepsilon \)-neighborhood of \( K \) if we allow \( f \) to be
quasiconformal, but we will not go into details for the sake of brevity.

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2. Proof of Theorem 1.3

In the entire section we only use the spherical metric of \( \hat{\mathbb{C}} \). We start with the
following definition adapted from [Why58].
Definition 2.1. Let \( \varepsilon > 0 \) and \( Y \) be a closed subset of \( \hat{C} \). A partition \( \{Y_\alpha\} \) of \( Y \) is called an \( \varepsilon \)-subdivision if the collection \( \{Y_\alpha\} \) consists of finitely many closed Jordan regions, also called 2-cells, with disjoint interiors and diameter less than \( \varepsilon \).

A partition \( \{S_\alpha\} \) of a Sierpiński carpet \( S \subset \hat{C} \) into Sierpiński carpets is an \( \varepsilon \)-subdivision (rel. \( Q_1, \ldots, Q_N \)) if there exist peripheral disks \( Q_1, \ldots, Q_N \) of \( S \) and an \( \varepsilon \)-subdivision \( \{Y_\alpha\} \) of the closed domain \( Y := \hat{C} \setminus \bigcup_{i=1}^{\infty} Q_i \), for which

1. the boundaries of the 2-cells \( Y_\alpha \) are contained in \( \hat{C} \setminus \bigcup_{i=1}^{\infty} \partial Q_i \), where \( Q_i, i \geq N + 1 \), are the remaining peripheral disks of \( S \), and
2. \( \{S_\alpha\} = \{S \cap Y_\alpha\} \), i.e., the Sierpiński carpets in the subdivision \( \{S_\alpha\} \) arise as the intersections of the 2-cells in the subdivision \( \{Y_\alpha\} \) with the original Sierpiński carpet \( S \).

We remark that each peripheral disk of \( S \) is either one of the disks \( Q_1, \ldots, Q_N \), or it is a peripheral disk of one of the carpets \( S_\alpha \) in the \( \varepsilon \)-subdivision of \( S \).

Lemma 2.2. Let \( S, S' \subset \hat{C} \) be Sierpiński carpets. Fix peripheral disks \( Q_1, \ldots, Q_N \) and \( Q'_1, \ldots, Q'_M \) of \( S \) and \( S' \), respectively. Then for each \( \varepsilon > 0 \) there exists \( M > N \) and peripheral disks \( Q_{N+1}, \ldots, Q_M \) and \( Q'_{N+1}, \ldots, Q'_M \) of \( S \) and \( S' \), respectively, such that there exists an \( \varepsilon \)-subdivision of \( S \) rel. \( Q_1, \ldots, Q_M \) and an \( \varepsilon \)-subdivision of \( S' \) rel. \( Q'_1, \ldots, Q'_M \). Moreover, for each family of orientation-preserving homeomorphisms \( g_i: \overline{Q}_i \to \overline{Q}'_i \), \( i \in \{1, \ldots, M\} \), the \( \varepsilon \)-subdivisions can be taken in such a way that they correspond to each other under a homeomorphism \( g: \hat{C} \to \hat{C} \) that extends \( g_i, i \in \{1, \ldots, M\} \).

Here the correspondence of the subdivisions under \( g \) is understood in the sense that each 2-cell \( Y_\alpha \) of the subdivision of \( S \) is mapped by \( g \) onto a 2-cell \( Y'_\alpha \) of the subdivision of \( S' \).

A weaker version of this lemma is used by Whyburn in his topological characterization of the carpet; see [Why58, Lemma 1]. For the sake of completeness and to make this work self-contained, we include an outline of the proof of Lemma 2.2 later in Section 4.

Proof of Theorem 1.3. Let \( \varepsilon_k = 1/k \). We fix peripheral circles \( C_1, \ldots, C_N \) and \( C'_1, \ldots, C'_N \) of \( S \) and \( S' \), respectively, as in the statement of the theorem, and we consider conformal maps \( g_i, i \in \{1, \ldots, N\} \), from the peripheral disk \( Q_i \) of \( S \) bounded by \( C_i \) onto the peripheral disk \( Q'_i \) of \( S' \) bounded by \( C'_i \). These conformal maps extend to the boundaries by Carathéodory’s theorem and provide orientation-preserving homeomorphisms \( g_i: \overline{Q}_i \to \overline{Q}'_i \). By Lemma 2.2 we can find peripheral disks \( Q_{N+1}, \ldots, Q_M \) and \( Q'_{N+1}, \ldots, Q'_M \) of \( S \) and \( S' \), respectively, with conformal maps \( g_i: \overline{Q}_i \to \overline{Q}'_i \), \( i \in \{N+1, \ldots, M\} \), and we can find \( \varepsilon_1 \)-subdivisions of \( S \) and \( S' \) that correspond to each other under a global homeomorphic extension \( f_1: \hat{C} \to \hat{C} \) of the maps \( g_i: \overline{Q}_i \to \overline{Q}'_i \). Observe that \( \text{dist}(f_1(S), S') \leq \varepsilon_1 \) and \( \text{dist}(S, f_1^{-1}(S')) \leq \varepsilon_1 \).

Next, we fix one of the Sierpiński carpets \( S_\alpha \) in the \( \varepsilon_1 \)-subdivision of \( S \) that corresponds to a carpet \( S'_\alpha \) in the \( \varepsilon_1 \)-subdivision of \( S' \); that is, the corresponding 2-cells are mapped to each other under \( f_1 \) (however, we do not necessarily have \( f_1(S_\alpha) = S'_\alpha \)). Note that the peripheral circles of \( S_\alpha \) and \( S'_\alpha \) necessarily have diameters bounded above by \( \varepsilon_1 \).

There is a distinguished peripheral circle of \( S_\alpha \) \((S'_\alpha)\) that separates it from the other carpets in the subdivision of \( S \) \((S') \). We call \( R_1 \) \((R'_1) \) the peripheral disk of \( S_\alpha \) \((S'_\alpha) \) bounded by that peripheral circle. Consider the orientation-preserving
homeomorphism \( h_1 := f_1|_{R_1} : R_1 \rightarrow R_1' \). Now we apply Lemma 2.2 on \( S_\alpha \) and \( S'_\alpha \), to obtain peripheral disks \( R_2, \ldots, R_K \) and \( R'_2, \ldots, R'_K \) of \( S_\alpha \) and \( S'_\alpha \), respectively, together with conformal maps \( h_i : R_i \rightarrow R_i' \), \( i \in \{2, \ldots, K\} \), and we can find \( \varepsilon \)-subdivisions of \( S_\alpha \) and \( S'_\alpha \) (rel. \( R_i \) and \( R'_i \), \( i = 1, \ldots, K \), respectively) that correspond to each other under a homeomorphic extension \( f_{\alpha,2} : \hat{C} \setminus R_1 \rightarrow \hat{C} \setminus R_1' \) of the maps \( h_i, i \in \{2, \ldots, K\} \).

We repeat this procedure for each of the carpets \( S_\alpha \) in the \( \varepsilon \)-subdivision of \( S \).

If we collect the \( \varepsilon \)-subdivisions of the carpets of the form \( S_\alpha \subset S \) and \( S'_\alpha \subset S' \), we obtain \( \varepsilon \)-subdivisions of the carpets \( S \) and \( S' \). Patching together the resulting maps of the form \( f_{\alpha,2} \) yields a homeomorphism \( f_{2} : \hat{C} \rightarrow \hat{C} \) under which the \( \varepsilon_1 \)- and \( \varepsilon_2 \)-subdivisions of \( S, S' \) are in correspondence. The map \( f_{2} \) agrees with \( f_1 \) on \( \bigcup_{i=1}^{M} \overline{Q_i} \) and maps conformally the peripheral disks of \( S \) having diameter larger than \( \varepsilon_2 \) onto some peripheral disks of \( S' \). Note that by construction \( f_{2}^{-1} \) maps conformally the peripheral disks of \( S' \) having diameter larger than \( \varepsilon_2 \) onto some peripheral disks of \( S \). Moreover, the \( L^\infty \)-distance of \( f_1 \) and \( f_2 \) is bounded by \( \varepsilon_1 \); the same statement holds for the inverses of the maps. Also, note that \( \text{dist}(f_{2}(S), S') \leq \varepsilon_2 \) and \( \text{dist}(S, f_{2}^{-1}(S')) \leq \varepsilon_2 \).

We repeat the procedure of the last two paragraphs for each carpet in the \( \varepsilon_2 \)-subdivision of \( S, S' \). Inductively, for each \( k \in \mathbb{N} \) we obtain a homeomorphism \( f_k \) of \( \hat{C} \) with the following properties:

1. The \( L^\infty \)-distance of \( f_k, f_m \) and the \( L^\infty \)-distance of \( f_k^{-1}, f_m^{-1} \) are bounded by \( \varepsilon_k \) for \( m \geq k \).
2. \( \text{dist}(f_k(S), S') \leq \varepsilon_k \) and \( \text{dist}(S, f_k^{-1}(S')) \leq \varepsilon_k \) for all \( k \in \mathbb{N} \).
3. For each \( k \in \mathbb{N} \) the map \( f_k \) maps conformally peripheral disks \( Q_{i_1}, \ldots, Q_{i_k} \)
   of \( S \) onto corresponding peripheral disks \( Q'_{i_1}, \ldots, Q'_{i_k} \) of \( S' \). In fact, for
   each \( l \in \mathbb{N} \) the sequence \( \{f_k\}_{k \in \mathbb{N}} \) is eventually constant on \( Q_{i_l} \).
4. The sequences of peripheral disks \( Q_{i_k}, Q'_{i_k}, k \in \mathbb{N} \), exhaust the sequences
   of peripheral disks of \( S, S' \), respectively.

Since \( \varepsilon_k \to 0 \), (1) implies that \( f_k \) converges uniformly in \( \hat{C} \) to a homeomorphism \( f \) of \( \hat{C} \). By (2), we have \( f(S) = S' \). Finally, (3) and (4) imply that each peripheral
disk of \( S \) is mapped conformally onto a peripheral disk of \( S' \). In particular, \( f \)
is conformal on \( \hat{C} \setminus S \).

3. PROOF OF PROPOSITION 1.4

An easily verifiable fact that we will use here is that for each \( \varepsilon > 0 \) there exists a
self-similar square carpet \( S \subset \mathbb{C} \) with Hausdorff dimension bounded above by \( 1 + \varepsilon \).

We fix \( \varepsilon > 0 \). Suppose that \( K \subset \mathbb{C} \) is a compact set with empty interior and
let \( B \) be a large ball containing \( K \). We consider a dyadic square decomposition of
the open set \( B \setminus K \). In each dyadic square \( Q \subset B \setminus K \) in this decomposition we
consider a square carpet \( S_Q \) with Hausdorff dimension bounded by \( 1 + \varepsilon \), scaled so that it “fits” in the square \( S \); that is to say that the boundary of the unbounded
peripheral disk of \( S_Q \) is precisely the boundary of \( Q \). Then we take \( L \) to be the
closure of the union of all carpets \( S_Q \).

Note that \( L \supset K \), since \( \text{int}(K) = \emptyset \), and that \( L \) is a Sierpiński carpet, as one can see by
Wójcik’s criterion (see the Introduction): \( \text{int}(L) = \emptyset \) (since \( \text{int}(K) = \emptyset \)) and
the complement of \( L \) in \( \hat{C} \) consists of countably many Jordan regions with
disjoint closures and diameters shrinking to 0. Moreover, \( L \setminus K \) has Hausdorff
dimension at most $1 + \varepsilon$, since it is the union of the circle $\partial B$ with countably many carpets $S_Q$, each having Hausdorff dimension at most $1 + \varepsilon$.

Now, we apply Theorem 4.1, which provides us with a homeomorphism $f : \hat{C} \to \hat{C}$ that maps the carpet $L$ onto a fixed square carpet with Hausdorff dimension at most $1 + \varepsilon$. In particular, $f(L)$ has Lebesgue measure zero. By post-composing with a Möbius transformation of $\hat{C}$, which does not affect the Hausdorff dimension, we may assume that $f(\infty) = \infty$ and thus, we have a homeomorphism $f : C \to \hat{C}$ that is conformal in $C \setminus L$, as desired. □

4. OUTLINE OF PROOF OF LEMMA 2.2

The proof relies on the following version of Moore’s theorem [Moo25]:

**Theorem 4.1.** [Dav86] Corollary 6A, p. 56] Let $\{R_i\}_{i \in \mathbb{N}}$ be a sequence of Jordan regions in the sphere $\hat{C}$ with mutually disjoint closures and diameters converging to 0, and consider an open set $\Omega \supset \bigcup_{i \in \mathbb{N}} \overline{R}_i$. Then there exists a continuous, surjective map $F : \hat{C} \to S^2$ that is the identity outside $\Omega$ and it induces the decomposition of $\hat{C}$ into the sets $\{\overline{R}_i\}_{i \in \mathbb{N}}$ and points. In other words, there are countably many distinct points $p_i \in S^2$, $i \in \mathbb{N}$, such that $F^{-1}(p_i) = \overline{R}_i$ for $i \in \mathbb{N}$ and $F$ is injective on $\hat{C} \setminus \bigcup_{i \in \mathbb{N}} \overline{R}_i$ with $F(\hat{C} \setminus \bigcup_{i \in \mathbb{N}} \overline{R}_i) = S^2 \setminus \{p_i : i \in \mathbb{N}\}$.

Here $S^2$ is identical to $\hat{C}$, but it will be more convenient to have different notation for the target and view $S^2$ mostly from a topological point of view.

First, we consider the peripheral disks $Q_1, \ldots, Q_N$ of $S$ and $Q'_1, \ldots, Q'_N$ of $S'$ given in the statement of Lemma 2.2 and fix $\varepsilon > 0$. We append to these collections some peripheral disks $Q_{N+1}, \ldots, Q_M$ of $S$ and $Q'_{N+1}, \ldots, Q'_M$ of $S'$ so that the remaining peripheral disks of $S$ and $S'$ have diameters less than $\varepsilon$. Now, we apply Theorem 4.1 to the region $\Omega = \hat{C} \setminus \bigcup_{i=1}^M \overline{Q}_i$, and to the remaining peripheral disks $\{R_i\}_{i \in \mathbb{N}}$ of $S$ contained in $\Omega$, all of which have diameter less than $\varepsilon$. This yields a “collapsing” map $F : \hat{C} \to S^2$ that collapses each $R_i$ to a point $p_i$, but is injective otherwise. We apply analogously Theorem 4.1 to the carpet $S'$ and obtain a map $F'$ that collapses peripheral disks $R'_i$ to points $p'_i$.

Now, we consider given orientation-preserving homeomorphisms $g_i : \overline{Q}_i \to \overline{Q}'_i$, $i \in \{1, \ldots, M\}$, as in the statement of Lemma 2.2. These homeomorphisms “project” down to homeomorphisms

$$\overline{g}_i := F' \circ g_i \circ (F|_{\overline{Q}_i})^{-1} : F(\overline{Q}_i) \to F'(\overline{Q}'_i), \quad 1 \leq i \leq M.$$  

These are orientation-preserving homeomorphisms between finitely many disjoint Jordan regions in $S^2$. This implies that there exists a homeomorphic extension $\overline{g} : S^2 \to S^2$ of $\overline{g}_i$, $i \in \{1, \ldots, M\}$.

Consider the countable set $A' = \{\overline{g}(p_i) : i \in \mathbb{N}\} \cup \{p'_i : i \in \mathbb{N}\} \subset S^2$. This is the union of the set of points $p'_i$ arising from collapsing the peripheral disks of $S'$ (under $F'$) that are different from $Q'_1, \ldots, Q'_M$, together with the images $\overline{g}(p_i)$ of the corresponding points $p_i$ arising from collapsing the peripheral disks of $S$ (under $F$). We fix a $\delta > 0$ (to be specified) and a $\delta$-subdivision $\{Y'_\alpha\}$ of the closed domain $Y' = S^2 \setminus \bigcup_{i=1}^M F'(Q'_i)$ so that the boundaries $\partial Y'_\alpha$ avoid the countable set $A'$.

We fix a Jordan curve $Y'_\alpha$ and consider the preimage $(F')^{-1}(\partial Y'_\alpha)$, which is a Jordan curve since $F'$ is injective on it. Let $X'_\alpha \subset \hat{C}$ be the Jordan region bounded by that Jordan curve with the property that $(F')^{-1}(p'_i) \subset X'_\alpha$ for some $p'_i \in Y'_\alpha$;
in fact, this will hold for all \( p'_i \in Y'_\alpha \), as one can see using homotopy arguments. Then the Jordan regions \( \{ X'_\alpha \} \) give a subdivision of the closed domain \( \overline{C \setminus \bigcup_{i=1}^{\infty} \partial Q'_i} \), in the sense of Definition 2.1, and the boundaries of the 2-cells \( X'_\alpha \) are contained in \( S' \setminus \bigcup_{i=1}^{\infty} \partial Q'_i \) (here \( Q'_i, \ i \geq M + 1 \), are the remaining peripheral disks of \( S' \)). Finally, the sets \( S'_\alpha := S' \cap Y'_\alpha \) are Sierpiński carpets by Whyburn’s criterion (stated before Theorem 1.3), since \( S'_\alpha \subset S' \) has empty interior and the complement of \( S'_\alpha \) consists of Jordan regions with disjoint closures, whose diameters converge to 0; what is important here is that \( \partial S'_\alpha = (F')^{-1}(\partial Y'_\alpha) \) does not intersect any peripheral circle \( \partial Q'_i, \ i \geq M + 1 \). Therefore, the collection \( \{ S'_\alpha \} \) is a subdivision of the carpet \( S' \), as in Definition 2.1. One can ensure that this subdivision is an \( \varepsilon \)-subdivision by choosing a sufficiently small \( \delta > 0 \) in the previous paragraph and using the following elementary lemma:

**Lemma 4.2.** Suppose that \( X \) and \( Y \) are compact metric spaces and \( h: X \to Y \) is a continuous, surjective map with the property that the preimage of each point has diameter at most a given number \( \varepsilon > 0 \). Then there exists a \( \delta > 0 \) such that for each set \( B \subset Y \) with \( \text{diam}(B) < \delta \) we have \( \text{diam}(h^{-1}(B)) < \varepsilon \).

Now, we consider the preimages \( Y_{\alpha} \) under \( \tilde{g} \) of the Jordan regions \( Y'_\alpha \) lying in the partition of \( Y' = S^2 \setminus \bigcup_{i=1}^{M} F(Q'_i) \). The sets \( Y_{\alpha} \) provide a partition of \( Y = S^2 \setminus \bigcup_{i=1}^{M} F(Q_i) \) into Jordan regions and the boundaries \( \partial Y_{\alpha} \) of the Jordan regions in the partition avoid the countable set \( A = \{ p_i : i \in \mathbb{N} \} \). One now takes preimages under \( F \) of the Jordan curves \( \partial Y_{\alpha} \); these provide a subdivision of \( S \) as in the previous paragraph. We denote by \( X_{\alpha} \subset \overline{C} \) the Jordan region bounded by \( F^{-1}(\partial Y_{\alpha}) \) that contains a carpet \( S_{\alpha} \) in the subdivision of \( S \). Again, if \( \delta \) is chosen to be sufficiently small, then by continuity the regions \( Y_{\alpha} \) will be small enough, so that the preceding lemma can guarantee that the sets \( X_{\alpha} \) yield an \( \varepsilon \)-subdivision of \( S \).

The map \( \tilde{g} \) (composed appropriately with \( F \) and \( (F')^{-1} \)) provides a homeomorphism \( g \) from the union of the Jordan curves of the form \( \partial X_{\alpha} \) onto the union of the Jordan curves of the form \( \partial X'_\alpha \). This homeomorphism extends the maps \( g_i: \partial Q_i \to \partial Q'_i, \ i \in \{1, \ldots, M\} \). The union of the Jordan curves \( \partial X_{\alpha} \) is the 1-skeleton of a 2-cell decomposition of \( \overline{C} \), in which the 2-cells are the sets \( X_{\alpha} \), together with the sets \( \overline{Q}_i, \ i \in \{1, \ldots, M\} \). The analogous statement holds for \( X'_\alpha \) and \( \overline{Q'}_i, \ i \in \{1, \ldots, M\} \). Now, one can extend the homeomorphism \( g \) to a homeomorphism of \( \overline{C} \); see e.g. [BM17, Chapter 5.2]. By construction, the \( \varepsilon \)-subdivisions of \( S \) and \( S' \) are in correspondence under \( g \), in the sense of the comments following the statement of Lemma 2.2. \( \square \)

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