THE ÉTALE HOMOLOGY AND THE CYCLE MAPS IN ADIC COEFFICIENTS

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ABSTRACT. In this article, we define the ℓ-adic homology for a morphism of schemes satisfying certain finiteness conditions. This homology has these functors similar to the Chow groups: proper push-forward, flat pull-back, base change, cap-product, etc. In particular on singular varieties, this kind of ℓ-adic homology behaves much better than the classical ℓ-adic cohomology. As an application, we give an much easier approach to construct the cycle maps for arbitrary algebraic schemes over fields of finite cohomology dimension. And we prove these cycle maps kill the algebraic equivalences and commute with the Chern action of locally free sheaves.

INTRODUCTION

The étale cohomology, especially the ℓ-adic cohomology, is one of the most important tools of modern algebraic and arithmetic geometry, which allows us to construct a good cohomology theory for varieties over fields of arbitrary characteristic. More specifically, people use the ℓ-adic cohomology $H^*(X_{\text{ét}}, \mathbb{Z}_\ell)$ to substitute for singular cohomology on varieties of arbitrary characteristic. On a nonsingular varieties, the cohomology $H^*(X_{\text{ét}}, \mathbb{Z}_\ell)$ has very good properties and produces rich results. But on singular varieties or more generally on arbitrary schemes, the cohomology $H^*(X_{\text{ét}}, \mathbb{Z}_\ell)$ behave not so good, and many important constructions and results are not valid. So on singular varieties, the étale homology is more suitable than the étale cohomology.

In this paper, we generalize the étale homology defined in [11] in the following three facets. First, we define the étale homology in adic coefficients, which we call the ℓ-adic homology. Second, our theory of ℓ-adic homology is defined over schemes separated and of finite type over base schemes satisfying certain finiteness conditions, not just the algebraic schemes over separably closed fields as in [11]. In particular, algebraic schemes over fields which are not necessarily separably closed, are considered by us. Since our theory is based on the adic formalism created by Ekedahl [3], the ℓ-adic homology over base schemes of certain finiteness conditions shares almost the same good functorial properties, with that over separably closed base fields. Third, the ℓ-adic homology groups $H_n(X/Y, N)$ defined by us take value in arbitrary bounded complex $\mathcal{F}$, not just $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$ or $\mathbb{Z}/n\mathbb{Z}$ as in [11]. And almost all functors and properties are preserved when extending to complexes.

In §1, we briefly reiterate the category $D_c(X_{\text{ét}}, R\mathbb{A})$ together with the Grothendieck’s six operations in [3]. In §2, we recite the properties of the functor $Rf^!$ and use the language of [3] to rewrite the trace morphisms introduced in [6, XVIII] and [2, Cycle].

In §3, we define the ℓ-adic homology groups $H_n(X/Y, \mathcal{N})$ and $\mathbb{H}_n(X/Y, \mathcal{N})$ for a morphism $X \rightarrow Y$ of schemes satisfying certain finiteness conditions. These homology groups behave similarly in many facets to the bivariant Chow groups $A^{-n}(X \rightarrow Y)$ defined in [4, Ch. 17]. If $X$ is a $d$-dimensional nonsingular variety over a separably closed field $k$, then

$$\mathbb{H}_n(X/k, \mathcal{N}) = H^{2(d-n)}(X_{\text{ét}}, \mathcal{N}(d-n)).$$

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We define two maps: the push-forward maps $f_*$ and the pull-back maps $f^*$, for the $\ell$-adic homology groups, which correspond to these maps on Chow groups $\text{CH}_*(X)$ defined in [4, §1.4 & §1.7]. We prove that the two maps $f_*$ and $f^*$ commute (see Theorem 3.13), which is essential to the construct various cycle maps basing on $\ell$-adic homology. Moreover we define the base change maps on the $\ell$-adic homology.

In §4, we apply the $\ell$-adic homology in §3 to define the cycle map
\[ c_{1,\ell}: \text{CH}_*(X) \rightarrow H_*(X, \mathbb{Z}_\ell) \]
for arbitrary algebraic scheme $X$ over a field of finite cohomological dimension at $\ell$. We prove that the cycle map $c_{1,\ell}$ commutes with the push-forward map $f_*$ and the pull-back map $f^*$. And we prove the cycle maps kill the algebraic equivalence of algebraic cycles. In §5, we prove that the cycle map $c_{1,\ell}$ commutes with the Chow action $c_i(\mathcal{E}) \cap \bigcdot$ by locally free sheaves.

**Notation and Conventions.** A morphism $f: X \rightarrow Y$ of schemes is said to flat (resp. smooth) of relative dimension $n$ if $f$ is flat (resp. smooth) and all fibers of $f$ are $n$-equidimensional.

A morphism $f: X \rightarrow Y$ of Noetherian schemes is said to be compactifiable if it factors as $f = \bar{f} \circ j$ where $j: X \leftarrow \bar{X}$ is an open immersion, and $\bar{f}: \bar{X} \rightarrow Y$ is a proper morphism. By [?, Theorem 4.1], $f$ is compactifiable if and only if it is separated and of finite type.

An algebraic scheme over a field $k$ is a scheme separated, of finite type over $k$. A variety over $k$ is an integral algebraic scheme over $k$.

If $A$ is a Noetherian ring, we define $\text{D}_c(A)$ to be the full subcategory of $\text{D}(A)$ consisting of complexes cohomologically finitely generated.

If $\mathcal{F}$ is a complex of sheaves on $X_{\text{ét}}$, we write $\mathcal{F}^*(r) := \mathcal{F}^*[2r]$ for each $r \in \mathbb{Z}$.

The notation $\cong$ means being defined as; $\xrightarrow{\sim}$ means isomorphism; and the notation $\square$ in commutative diagrams means Cartesian square.

1. **The $\ell$-adic sheaves**

In this section, we briefly reiterate the theory of Ekedahl [3] about the category $\text{D}_c(X_{\text{ét}}, \mathcal{R}_*)$ together with the Grothendieck's six operations. See also [1] and [10].

Fix a prime number $\ell$, and let $R$ be the integral closure of $\mathbb{Z}_\ell$ in a finite extension field of $\mathbb{Q}_\ell$.

Let $X$ be a Noetherian scheme. We denote by $\text{S}(X_{\text{ét}}, \mathcal{R}_*)$ the abelian category of inverse systems
\[ \cdots \rightarrow \mathcal{F}_{n+1} \rightarrow \mathcal{F}_n \rightarrow \cdots \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_1 \]
such that each $\mathcal{F}_n$ is a sheaf of $R_n$-modules on $X_{\text{ét}}$. Set
\[ \text{D}(X_{\text{ét}}, \mathcal{R}_*) := \text{D}(\text{S}(X_{\text{ét}}, \mathcal{R}_*)) ; \]
and let $\text{D}_c(X_{\text{ét}}, \mathcal{R}_*)$ be the full subcategory of $\text{D}(X_{\text{ét}}, \mathcal{R}_*)$ consisting of complexes cohomologically AR-adic and constructible. Let $\text{D}_c(X_{\text{ét}}, \mathcal{R}_*)$ be the quotient of $\text{D}_c(X_{\text{ét}}, \mathcal{R}_*)$ by inverting AR-quasi-isomorphisms.

If $f: X \rightarrow Y$ is a morphism of Noetherian schemes, then we have a triangulated functor
\[ f^*: \text{D}_c(Y_{\text{ét}}, \mathcal{R}_*) \rightarrow \text{D}_c(X_{\text{ét}}, \mathcal{R}_*) . \]

As to other five operations, we must add some restrictions on the underlying schemes. We consider the following condition (†) relate to a scheme $X$:

(†) $X$ is Noetherian, quasi-excellent, of finite Krull dimension; $\ell$ is invertible on $X$ and $\text{cd}_\ell(X) < \infty$.

From the Gabber's finiteness theorem for étale cohomology in [5], we know the following facts
(1) If $X$ satisfies (†), then any scheme of finite type over $X$ satisfies (†).
(2) Let $R$ be a quasi-excellent, Henselian local ring with residue field $k$ such that $\text{cd}_\ell(k) < \infty$. Then $\text{Spec } R$ satisfies (†).
(3) If \( \ell \neq 2 \), then the affine scheme \( \text{Spec } \mathbb{Z}[1/\ell] \) satisfies (\( \dagger \)). (See [6, X, 6.1])
(4) If \( f: X \rightarrow Y \) is a compactifiable morphism of schemes satisfying (\( \dagger \)); then both \( Rf_* \) and \( Rf^! \) are of finite cohomological amplitude.

In particular if \( X \) is a scheme satisfying (\( \dagger \)), then \( X_{\text{et}} \) satisfies the condition A) in [3]; thus we have two bi-triangulated functors

\[
\bullet \otimes^L_R \bullet : D_c^-(X_{\text{et}}, R) \times D_c^-(X_{\text{et}}, R) \rightarrow D_c^-(X_{\text{et}}, R),
\]
\[
R\mathcal{H}om_R(\bullet, \bullet) : D_c^-(X_{\text{et}}, R)^{\text{opp}} \times D_c^+(X_{\text{et}}, R) \rightarrow D_c^+(X_{\text{et}}, R).
\]

And if \( f: X \rightarrow Y \) is a compactifiable morphism of schemes satisfying (\( \dagger \)), there are triangulated functors

\[
Rf_* : D_c(X_{\text{et}}, R) \rightarrow D_c(Y_{\text{et}}, R),
\]
\[
Rf_! : D_c(X_{\text{et}}, R) \rightarrow D_c(Y_{\text{et}}, R),
\]
\[
Rf^! : D_c(Y_{\text{et}}, R) \rightarrow D_c(X_{\text{et}}, R).
\]

For each scheme \( X \) satisfying (\( \dagger \)), each object \( \mathcal{F} \) in \( D_c(X_{\text{et}}, R) \), and each \( n \in \mathbb{N} \), we define

\[
H^n(X_{\text{et}}, \mathcal{F}) := \text{Hom}_{D_c(X_{\text{et}}, R)}(R_X, \mathcal{F}[n]).
\]

Note that this definition is compatible with the continuous étale cohomology \( H^n_{\text{cont}}(X_{\text{et}}, \mathcal{F}) \) defined in [2].

When we consider the schemes of finite type over a separably closed field, the following Theorem is essential.

**Theorem 1.1.** The right derived functors of \( (M_n) \mapsto \lim\nolimits_{\text{cont}} M_n \) and the left derived functors of \( M \mapsto (M \otimes_R R_n) \) define a natural equivalence of categories between \( D'_c(R) \) and \( D_{fg}(R) \).

**Proof.** See [1, Proposition 2.2.8]. \( \square \)

Now we fix a separably closed field \( k \). Note that \( D'_c((\text{Spec } k)_{\text{et}}, R) = D'_c(R) = D_{fg}(R) \).

**Notation 2.** Let \( X \) be an algebraic scheme over \( k, p: X \rightarrow \text{Spec } k \) the structural morphism. Put

\[
R\Gamma(X_{\text{et}}, \bullet) := Rp_* : D_c(X_{\text{et}}, R) \rightarrow D_{fg}(R),
\]
\[
R\Gamma_!(X_{\text{et}}, \bullet) := Rp_! : D_c(X_{\text{et}}, R) \rightarrow D_{fg}(R).
\]

Then for each \( q \in \mathbb{Z} \), we have \( H^q(X_{\text{et}}, \bullet) = H^q \circ R\Gamma(X_{\text{et}}, \bullet) \). And we define

\[
H^q_c(X_{\text{et}}, \bullet) := H^q \circ R\Gamma_!(X_{\text{et}}, \bullet).
\]

**Theorem 1.3 (The Künneth Formula).** Let \( X \) and \( Y \) be two algebraic schemes over \( k, Z := X \times_k Y \), \( f: Z \rightarrow X \) and \( g: Z \rightarrow Y \) the projections. Then for each \( \mathcal{F} \in D_{c}^*(X_{\text{et}}, R) \) and \( \mathcal{G} \in D_{c}^*(Y_{\text{et}}, R) \), there are two natural isomorphisms in \( D_{fg}(R) \):

\[
R\Gamma(X_{\text{et}}, \mathcal{F}) \otimes_R^L R\Gamma(Y_{\text{et}}, \mathcal{G}) \sim \rightarrow R\Gamma(Z_{\text{et}}, f^*\mathcal{F} \otimes_R^L g^*\mathcal{G}),
\]
\[
R\Gamma_!(X_{\text{et}}, \mathcal{F}) \otimes_R^L R\Gamma_!(Y_{\text{et}}, \mathcal{G}) \sim \rightarrow R\Gamma_!(Z_{\text{et}}, f^*\mathcal{F} \otimes_R^L g^*\mathcal{G}).
\]
Moreover there are two exact sequences of $R$-modules
\[
0 \to \bigoplus_{i+j=n} \text{H}^i(X_{\text{ét}}, \mathcal{F}) \otimes_R \text{H}^j(Y_{\text{ét}}, \mathcal{G}) \to \text{H}^n(Z_{\text{ét}}, f^* \mathcal{F} \otimes_R \mathcal{L} \cdot g^* \mathcal{G})
\]
\[
\to \bigoplus_{i+j=n+1} \text{Tor}_1^R(\text{H}^i(X_{\text{ét}}, \mathcal{F}), \text{H}^j(Y_{\text{ét}}, \mathcal{G})) \to 0,
\]
\[
0 \to \bigoplus_{i+j=n} \text{H}^i_c(X_{\text{ét}}, \mathcal{F}) \otimes_R \text{H}^j_c(Y_{\text{ét}}, \mathcal{G}) \to \text{H}^n_c(Z_{\text{ét}}, f^* \mathcal{F} \otimes_R \mathcal{L} \cdot g^* \mathcal{G})
\]
\[
\to \bigoplus_{i+j=n+1} \text{Tor}_1^R(\text{H}^i_c(X_{\text{ét}}, \mathcal{F}), \text{H}^j_c(Y_{\text{ét}}, \mathcal{G})) \to 0.
\]

2. The functor $Rf^!$ and the Trace Morphisms from SGA 4 & 4½

**Proposition 2.1.** Let $f : X \to Y$ be a compactifiable morphism of schemes satisfying (†) such that all fibers of $f$ are of dimensions $\leq d$. Then for each $a \in \mathbb{Z}$, $Rf^!$ sends $D_{c}^{\geq a}(X_{\text{ét}}, R^*)$ to $D_{c}^{\geq a-2d}(Y_{\text{ét}}, R^*)$.

**Proof.** See [6, XVIII, 3.1.7]. \qed

**Lemma 2.2.** Let $f : X \to Y$ be a compactifiable morphism of schemes satisfying (†). Then for every pair of objects $\mathcal{F}$ and $\mathcal{G}$ in $D_c(Y_{\text{ét}}, R^*)$, there is a natural morphism
\[
Rf^! \mathcal{F} \otimes_R^L f^* \mathcal{G} \to Rf^!(\mathcal{F} \otimes_R^L \mathcal{G})
\]
in $D_c(X_{\text{ét}}, R^*)$ which is functorial in $\mathcal{F}$ and $\mathcal{G}$.

**Proposition 2.3.** Let $f : X \to Y$ and $g : Y \to Z$ two compactifiable morphisms of schemes satisfying (†). For every pair of objects $\mathcal{F}$ and $\mathcal{G}$ in $D_c(Z_{\text{ét}}, R^*)$, there is a natural morphism
\[
Rf^! \circ g^! \mathcal{F} \otimes_R^L f^* \circ Rg^! \mathcal{G} \to R(g \circ f)^!(\mathcal{F} \otimes_R^L \mathcal{G})
\]
in $D_c(X_{\text{ét}}, R^*)$ which is functorial in $\mathcal{F}$ and $\mathcal{G}$.

**Proof.** We have
\[
Rf^! \circ g^! \mathcal{F} \otimes_R^L f^* \circ Rg^! \mathcal{G} \to Rf^!(g^! \mathcal{F} \otimes_R^L Rg^! \mathcal{G})
\]
\[
\to Rf^! \circ Rg^!(\mathcal{F} \otimes_R^L \mathcal{G})
\]
\[
\sim R(g \circ f)^!(\mathcal{F} \otimes_R^L \mathcal{G}). \quad \square
\]

**Proposition 2.4.** Let

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow p & & \downarrow q \\
X & \xrightarrow{f} & Y
\end{array}
\]

be a Cartesian square of schemes satisfying (†). Assume that $f$ is compactifiable.

1. For each object $\mathcal{F}$ in $D_c(X_{\text{ét}}, R^*)$, there is a natural morphism in $D_c(Y_{\text{ét}}, R^*)$
\[
q^* \circ Rf_* \mathcal{F} \to Rf_*^! \circ p^* \mathcal{F}.
\]

2. For each object $\mathcal{G}$ in $D_c(Y_{\text{ét}}, R^*)$, there is a natural morphism in $D_c(X_{\text{ét}}, R^*)$
\[
p^* \circ Rf^! \mathcal{G} \to Rf^! \circ q^* \mathcal{G}.
\]

3. Assume that $Y$ is an algebraic scheme over a field $k$, and there exists a $k$-scheme $T$ such that $Y' = Y \times_k T$. Then the morphisms in (1) and (2) are both isomorphisms.
(4) For each object $\mathcal{G}$ in $\mathcal{D}_c(Y_\text{â€¢}, R_\text{â€¢})$, there is a natural morphism in $\mathcal{D}_c(X_\text{â€¢}, R_\text{â€¢})$
\[ R_p^* \circ Rf^! \mathcal{G} \sim Rf^! \circ Rq_* \mathcal{G}. \]

Proof. (1) is induced by the classical base change morphisms.
(2) is from [6, XVIII, 3.1.14.2].
(3) is by [2, Th. finitude, 1.9].
(4) is by [6, XVIII, 3.1.12.3].

Now we review the trace morphisms.

Definition 2.5. A morphism $f : X \to Y$ of schemes is said to be flat at dimension $d$ if there exists a nonempty open subset $U$ of $X$ satisfying the following conditions:
1. $f : U \to Y$ is flat;
2. for each point $y \in Y$, $U_y$ is either empty or $d$-dimensional;
3. every fiber of $X \setminus U \to Y$ is of dimension $< d$.

By [6, XVIII, 2.9], for every compactifiable morphism $f : X \to Y$ of schemes satisfying (†) which is flat at dimension $d$, and for every object $\mathcal{G}$ in $\mathcal{D}_c(Y_\text{â€¢}, R_\text{â€¢})$, we have a trace morphism:
\[ \text{Tr}_f : Rf^! \circ f^* \mathcal{G}(d) \to \mathcal{G}. \]

Since $Rf^!$ is right adjoint to $Rf_!$, the morphism $\text{Tr}_f$ induces a canonical morphism in $\mathcal{D}_c(X_\text{â€¢}, R_\text{â€¢})$:
\[ t_f : f^* \mathcal{G}(d) \to Rf^! \mathcal{G}. \]

Moreover we have a commutative diagram
\[ \begin{array}{ccc}
Rf_! \circ f^* \mathcal{G}(d) & \xrightarrow{Rf_!(t_f)} & Rf_! \circ Rf^! \mathcal{G} \\
\text{Tr}_f \downarrow & & \downarrow \text{Tr}_f \\
\mathcal{G} & & \mathcal{G}
\end{array} \]

(2.1)

By [6, XVIII, 3.2.5], we have

Proposition 2.6. Let $f : X \to Y$ be a compactifiable smooth morphism of relative dimension $d$ of schemes satisfying (†). Then for any object $\mathcal{G}$ in $\mathcal{D}_c(Y_\text{â€¢}, R_\text{â€¢})$, the canonical morphism
\[ t_f : f^* \mathcal{G}(d) \sim Rf^! \mathcal{G} \]
is an isomorphism in $\mathcal{D}_c(X_\text{â€¢}, R_\text{â€¢})$.

The following propositions 2.7-2.9 are deduced from [6, XVIII, 2.9].

Proposition 2.7. Let
\[ \begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
p \downarrow & \square & q \\
X & \xrightarrow{f} & Y
\end{array} \]
be a Cartesian square of schemes satisfying (†). Assume that $f$ is compactifiable and flat at dimension $d$. Then $f'$ is also flat at dimension $d$, and for each object $\mathcal{G}$ in $\mathcal{D}_c(Y_\text{â€¢}, R_\text{â€¢})$ we have
1. the composite morphism
\[ (Rf'_!) \circ f'^* \circ q^* \mathcal{G}(d) = (Rf_1^!) \circ p^* \circ f^* \mathcal{G}(d) \sim q^* \circ (Rf^!) \circ f^* \mathcal{G}(d) \xrightarrow{q^*(\text{Tr}_f)} q^* \mathcal{G} \]
is equal to $\text{Tr}_f$;
(2) the composite morphism

\[ f^* \circ q^* \mathcal{G}(d) = p^* \circ f^* \mathcal{G}(d) \xrightarrow{p^*(t_f)} p^* \circ Rf^! \mathcal{G} \to Rf^! \circ q^* \mathcal{G} \]

is equal to \( t_f \), where the last morphism is defined in Proposition 2.4 (2).

**Proposition 2.8.** Let \( f : X \to Y \) and \( g : Y \to Z \) two compactifiable morphisms of schemes satisfying (\( \dagger \)) which are flat at dimension \( d \) and \( e \) respectively, \( \mathcal{H} \) an object in \( D_c(Z_{\text{et}}, R_\bullet) \). Then we have

1. The composite morphism

\[
R(g \circ f_\dagger) \circ (g \circ f)^* \mathcal{H}(d + e) \xrightarrow{R(g_!)} (Rg_!) \circ (Rf_\dagger) \circ f^* \circ g^* \mathcal{H}(d + e)
\]

is equal to \( Tr_{g_0f} \).

2. The composite morphism

\[
f^*g^* \mathcal{H}(d + e) \xrightarrow{t_f} Rf^! \circ g^* \mathcal{H}(e) \xrightarrow{Rf^! (t_g)} Rf^! \circ Rg^! \mathcal{H} \xrightarrow{Tr} R(g \circ f)^! \mathcal{H}
\]

is equal to \( t_{g_0f} \).

**Proposition 2.9.** Let \( f : X \to Y \) be a finite morphism of schemes satisfying (\( \dagger \)) such that \( f_* \mathcal{O}_X \) is a locally free \( \mathcal{O}_Y \)-module of degree \( d \). Then for each object \( \mathcal{F} \) in \( D_c(Y_{\text{et}}, R_\bullet) \), the composite morphism

\[
\mathcal{F} \to f_* f^* \mathcal{F} \xrightarrow{Tr_f} \mathcal{F}
\]

is equal to the multiplication by \( n \).

The following proposition show that the trace morphism is essentially determined by the generic points. Let \( A \) be a Noetherian ring (in particular \( A = R_n \)).

**Proposition 2.10.** Let \( X \) be a \( n \)-dimensional algebraic scheme over \( k \), \( X_1, X_2, \ldots, X_r \) all irreducible components of dimension \( n \) of \( X \), \( F \) an \( A \)-module. For each \( i \), let \( Y_i \neq \emptyset \) be an open subset of \( X \) contained \( X_i \setminus \bigcup_{j \neq i} X_j \) and regard \( Y_i \) as a reduced subscheme of \( X \). For each \( i \), let \( x_i \) be the generic point of \( X_i \) and put \( a_i := \text{length}(\mathcal{O}_{X,x_i}) \). Then there is a canonical isomorphism \( \omega \) of \( A \)-modules which makes a commutative diagram.

\[
\begin{array}{c}
\bigoplus_{i=1}^r H_c^{2n}(Y_i_{\text{et}}, F(n)) \\
\xRightarrow{\omega} \\
\bigoplus_{i=1}^r a_i Tr_i
\end{array}
\]

3. \( \ell \)-ADIC HOMOLOGY FOR MORPHISMS OF ALGEBRAIC SCHEMES

**Notation 3.1.** Let \( f : X \to Y \) be a compactifiable morphism of schemes satisfying (\( \dagger \)). For each object \( \mathcal{N} \) in \( D_c(X_{\text{et}}, R_\bullet) \) and for each \( n \in \mathbb{Z} \), we define the \( n \)-th \( \ell \)-adic homology associated to \( f \) to be

\[
H_n(X \xrightarrow{f} Y, \mathcal{N}) := H^{-n}(X_{\text{et}}, Rf^! \mathcal{N})
\]

\[
= \text{Hom}_{D_c(X_{\text{et}}, R_\bullet)}(R_X, Rf^! \mathcal{N}[-n]),
\]

which is an \( R \)-module.
For convenient to define pull-backs along flat morphisms and cycle maps, we also define
\[ \mathbb{H}_n(X \xrightarrow{f} Y, \mathcal{N}) := H_{2n}(X \xrightarrow{f} Y, \mathcal{N}(-n)) = \text{Hom}_{D_c(X_{\text{et}}, R)}(R_X, Rf_! \mathcal{N}(-n)). \]

We set
\[ H_*(X \xrightarrow{f} Y, \mathcal{N}) := \bigoplus_{n \in \mathbb{Z}} \mathbb{H}_n(X \xrightarrow{f} Y, \mathcal{N}), \]
\[ \mathbb{H}_*(X \xrightarrow{f} Y, \mathcal{N}) := \bigoplus_{n \in \mathbb{Z}} H_n(X \xrightarrow{f} Y, \mathcal{N}). \]

We also use \( H_n(X/Y, \mathcal{N}) \) to denote \( H_*(X \xrightarrow{f} Y, \mathcal{N}) \) if no confusion arise. Similarly we may define \( \mathbb{H}_n(X/Y, \mathcal{N}) \).

If \( X \) is an algebraic schemes over a separably closed field \( k \) and \( N \) is an object in \( D_c(R) \), we write
\[ H_n(X, N) := H_n(X \to \text{Spec} k, N), \quad \mathbb{H}_n(X, N) := H_n(X \to \text{Spec} k, N). \]

By Proposition 2.6 we have

**Lemma 3.2.** Let \( f: X \to Y \) be a compactifiable smooth morphism of relative dimension \( d \) of schemes satisfying (\( \dagger \)). Then for each object \( \mathcal{N} \) in \( D_c(Y_{\text{et}}, R) \) and for \( n \in \mathbb{Z} \), the morphism \( f_! \) induces a canonical isomorphism of \( R \)-modules:
\[ \mathbb{H}^{d-n}(X, f^* \mathcal{N}) \cong H_n(X \xrightarrow{f} Y, \mathcal{N}). \]

**Proposition 3.3.** Let \( f: X 
\to S \) be a compactifiable morphism of schemes satisfying (\( \dagger \)), \( Y \) a closed subscheme of \( X \) and \( U := X \setminus Y \). Then we have a long exact sequence
\[ \cdots \to H_n(Y/S, \mathcal{N}) \to H_n(X/S, \mathcal{N}) \to H_n(U/S, \mathcal{N}) \to H_{n-1}(Y/S, \mathcal{N}) \to \cdots . \]

**Proof.** Put \( \mathcal{M} := Rf^! \mathcal{N} \). Then the proposition follows from the distinguished triangle
\[ i_*j^! \mathcal{M} \to \mathcal{M} \to j_*j^* \mathcal{M} \to i_*j^! \mathcal{M}[1], \]
where \( i: Y \leftarrow X \) and \( j: U \leftarrow X \) are the inclusions. \( \square \)

**Proposition 3.4** (Mayer-Vietoris Sequence). Let \( f: X \to S \) be a compactifiable morphism of schemes satisfying (\( \dagger \)), \( X_1 \) and \( X_2 \) two closed subschemes of \( X \) such that \( X = X_1 \cup X_2 \) (as sets). Then we have a long exact sequence
\[ \cdots \to H_n((X_1 \cap X_2)/S, \mathcal{N}) \to H_n(X_1/S, \mathcal{N}) \oplus H_n(X_2/S, \mathcal{N}) \to H_n(X/S, \mathcal{N}) \]
\[ \to H_{n-1}((X_1 \cap X_2)/S, \mathcal{N}) \to \cdots . \]

**Proof.** Put \( \mathcal{M} := Rf^! \mathcal{N} \). Then the proposition follows from the distinguished triangle
\[ i_* \circ Rf^! \mathcal{M} \to i_{1*} \circ Rf_1^! \mathcal{M} \oplus i_{2*} \circ Rf_2^! \mathcal{M} \to \mathcal{M} \to i_* \circ Rf^! \mathcal{M}[1], \]
where \( i: X_1 \cap X_2 \leftarrow X, i_1: X_1 \leftarrow X, i_2: X_2 \leftarrow X \) are the inclusions. \( \square \)

**Proposition 3.5** (Vanishing). Let \( f: X \to Y \) be a compactifiable morphism of schemes satisfying (\( \dagger \)) such that all fibers of \( f \) are of dimensions \( \leq d \), \( \mathcal{N} \) an object in \( D^b_c(Y_{\text{et}}, R) \). Then \( H_n(X/Y, \mathcal{N}) = 0 \) whenever \( n > 2d - a \).

**Proof.** By Proposition 2.1, \( Rf^! \mathcal{N}[-n] \in D^{>0}_{c}(X_{\text{et}}, R) \). Thus if \( a - 2d + n > 0 \), then
\[ H_n(X \xrightarrow{f} Y, \mathcal{N}) = \text{Hom}_{D_c(X_{\text{et}}, R)}(R_X, Rf_! \mathcal{N}[-n]) = 0 . \]
Proposition 3.6. Let \( f : X \to S \) be a compactifiable morphism of schemes satisfying (†). \( Y \) a closed subscheme of \( X \) such that \( \dim Y_s \leq d \) for all \( s \in S \), \( X' := X \setminus Y \), \( \mathcal{N} \) an object in \( D^b_c(S_{\text{et}}, R_*) \). Then for each integer \( n > 2d + 1 - a \), there is a canonical isomorphism of \( R \)-modules \[
abla_n(X/S, \mathcal{N}) \simto \nabla_n(X'/S, \mathcal{N}).
\]

Proof. Apply Proposition 3.3 and Proposition 3.5.

Notation 3.7. Let \( f : X \to Y \) be a compactifiable morphism of schemes satisfying (†). For each object \( \mathcal{G} \) in \( D_c(Y_{\text{et}}, R_*) \), we define \[
\delta_f : \mathcal{G} \to \mathcal{R}f_* \circ f^* \mathcal{G} \quad \text{and} \quad \theta_f : \mathcal{R}f_! \circ \mathcal{R}f_!^! \mathcal{G} \to \mathcal{G}
\]
to be the canonical morphisms induced by the adjunctions \( f^* \dashv \mathcal{R}f_* \) and \( \mathcal{R}f_! \dashv \mathcal{R}f_!^! \) respectively.

The following map is a kind of variant of the Gysin homomorphism.

Definition 3.8 (Push-forward). Let \( p : X \to S \) and \( q : Y \to S \) be two compactifiable morphisms of schemes satisfying (†), \( f : X \to Y \) a proper \( S \)-morphism. For every object \( \mathcal{N} \) in \( D_c(S_{\text{et}}, R_*) \) and for every \( n \in \mathbb{Z} \), we define a homomorphism of \( R \)-modules \[
f_* : \nabla_n(X/S, \mathcal{N}) \to \nabla_n(Y/S, \mathcal{N})
\]
as follows. For each \( \alpha \in \nabla_n(X/S, \mathcal{N}) \), \( f_*(\alpha) \) is defined to be the composition \[
R_Y \xrightarrow{\delta_f} \mathcal{R}f_* R_X \xrightarrow{\mathcal{R}f_*(\alpha)} \mathcal{R}f_* \circ \mathcal{R}p_! \mathcal{N}[-n] \xrightarrow{\sim} \mathcal{R}f_* \circ \mathcal{R}f^! \circ \mathcal{R}q_! \mathcal{N}[-n] \xrightarrow{\theta_f} \mathcal{R}q_! \mathcal{N}[-n].
\]

Proposition 3.9. Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} S \) be a sequence of morphisms of schemes satisfying (†) such that \( f \) and \( g \) are proper, and \( h \) is compactifiable. Then for all \( \mathcal{N} \in D_c(S_{\text{et}}, R_*) \) and \( n \in \mathbb{Z} \), we have \( (g \circ f)_* = g_* \circ f_* : \nabla_n(X/S, \mathcal{N}) \to \nabla_n(Z/S, \mathcal{N}) \).

Proof. This is by the following simple lemma.

Lemma 3.10. Let \( f : X \to Y \) and \( g : Y \to Z \) be two compactifiable morphisms of schemes satisfying (†), \( \mathcal{H} \) an object in \( D_c(Z_{\text{et}}, R_*) \). Then we have

1. The following composition is equal to \( \delta_{g \circ f} \)
   \[
   \mathcal{H} \xrightarrow{\delta_g} \mathcal{R}g_* \circ g^* \mathcal{H} \xrightarrow{\mathcal{R}g_*(\delta_f)} \mathcal{R}g_* \circ \mathcal{R}f_* \circ f^* \circ g^* \mathcal{H} \xrightarrow{\sim} \mathcal{R}(g \circ f)_* \circ (g \circ f)^* \mathcal{H}.
   \]

2. The following composition is equal to \( \theta_{g \circ f} \)
   \[
   \mathcal{R}(g \circ f)_! \circ \mathcal{R}(g \circ f)^! \mathcal{H} \xrightarrow{\sim} \mathcal{R}g_! \circ \mathcal{R}f_! \circ \mathcal{R}g_! \circ \mathcal{R}f_!^! \mathcal{H} \xrightarrow{\mathcal{R}g_!(\theta_f)} \mathcal{R}g_! \circ \mathcal{R}g_!^! \mathcal{H} \xrightarrow{\theta_g} \mathcal{H}.
   \]

Definition 3.11 (Pull-back). Let \( p : X \to S \) and \( q : Y \to S \) be two compactifiable morphisms of schemes satisfying (†), \( f : X \to Y \) an \( S \)-morphism which is flat at dimension \( d \). For every object \( \mathcal{N} \) in \( D_c(S_{\text{et}}, R_*) \) and for every \( n \in \mathbb{Z} \), we define a homomorphism of \( R \)-modules \( f^* : \nabla_n(Y/S, \mathcal{N}) \to \nabla_{n+d}(X/S, \mathcal{N}) \) as follows. For each \( \beta \in \nabla_n(Y/S, \mathcal{N}) \), \( f^*(\beta) \) is defined to be the composition \[
R_X \xrightarrow{f!} \mathcal{R}f! R_Y(-d) \xrightarrow{\mathcal{R}f_!(\beta)} \mathcal{R}f! \circ \mathcal{R}q_! \mathcal{N}(-n + d) \xrightarrow{\sim} \mathcal{R}p^! \mathcal{N}(-n + d).
\]

Proposition 3.12. Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \to S \) be a sequence of compactifiable morphisms of schemes satisfying (†) such that \( f \) and \( g \) are flat at dimension \( d \) and \( e \) respectively. Then for all \( \mathcal{N} \in D_c(S_{\text{et}}, R_*) \) and \( n \in \mathbb{Z} \), we have \( (g \circ f)^* = f^* \circ g^* : \nabla_n(X/S, \mathcal{N}) \to \nabla_{n+d+e}(Z/S, \mathcal{N}) \).

Proof. This follows from Proposition 2.8 (2).
Theorem 3.13. Let $S$ be a scheme satisfying (†), $r : Y \to S$ a compactifiable morphism. Let

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow p & \square & \downarrow q \\
X & \xrightarrow{f} & Y
\end{array}
\]

be a Cartesian square of schemes such that $f$ is proper and $q$ is compactifiable and flat at dimension $d$; $\mathcal{N}$ an object in $D_c(S_{\text{et}}, R_\bullet)$ and $n \in \mathbb{Z}$. Then we have

\[
q^* \circ f_* = f'_* \circ p^*: \mathbb{H}_n(X/S, \mathcal{N}) \to \mathbb{H}_{n+d}(Y'/S, \mathcal{N}).
\]

Proof. Put $\mathcal{M} := R^1 f_! \mathcal{N}$. Let $\alpha \in \mathbb{H}_n(X/S, \mathcal{N})$. Then $q^* \circ f_*(\alpha)$ is equal to the composition

\[
R_Y \xrightarrow{t_q} R^q f^! R_Y (-d) \xrightarrow{R_q(\delta_f)} R^q f_*^! R_X (-d) \xrightarrow{R^q f^! R f_* (\alpha)} R^q f_*^! R f_*^! R f_Y (-d) \xrightarrow{R^q f_*^! R f'_* (\alpha)} R^q f_*^! R f'_*^! R f_\mathcal{M} (-n + d) \xrightarrow{R^q(\theta_f)} R^q f'_! \mathcal{M} (-n + d);
\]

and $f'_* \circ p^*(\alpha)$ is equal to the composition

\[
R_Y \xrightarrow{\delta_f} R f'_* \circ f'^* R_Y = R f'_* \circ f'^* \circ q^* R_Y \xrightarrow{R f'_* \circ f'^* (\alpha)} R f'_* \circ f'^* \circ q^! R_Y (-d)
\]

After applying Proposition 2.7 (2) to $t_p$, we obtain that the morphism $f'_* \circ p^*(\alpha)$ is equal to the composition

\[
\text{Consider the following diagram}
\]

where $\bigcirc$ means commutative square. The commutativity of (a) and (b) are by the following simple Lemma 3.14. So the whole diagram is commutative. Note that the composition along the direction $\xrightarrow{\delta_f}$ in above diagram is equal to $q^* \circ f_*(\alpha)$; and the composition along $\xrightarrow{\delta_f}$ is equal to $f'_* \circ p^*(\alpha)$. Thus $q^* \circ f_*(\alpha) = f'_* \circ p^*(\alpha)$. \qed
Lemma 3.14. Let

\[ \begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
p \downarrow & & \downarrow q \\
X & \xrightarrow{f} & Y
\end{array} \]

be a Cartesian square of schemes satisfying (†) with all morphisms compactifiable. Then we have

1. For each object \( \mathcal{G} \) in \( D_c(Y_{\acute{e}t}, R_* \) ), the diagram

\[ \begin{array}{ccc}
Rq_{Y'}^! \mathcal{G} & \xrightarrow{Rq_{Y'}^!(\delta_f)} & Rq_Y^! \circ Rf_* \circ f'^* \mathcal{G} \\
\delta_{f'} \downarrow & \cong & \downarrow \varphi \\
Rf'_* \circ f'^* \circ Rq_{Y'}^! \mathcal{G} & \xrightarrow{Rf'_* (\psi)} & Rf'_* \circ Rp_* \circ f'^* \mathcal{G}
\end{array} \]

is commutative in \( D_c(Y_{\acute{e}t}', R_* \), where \( \varphi \) is defined in Proposition 2.4 (4) and \( \psi \) is defined in Proposition 2.4 (2).

2. Assume that \( f \) is proper. Then for each object \( \mathcal{G} \) in \( D_c(Y_{\acute{e}t}, R_* \) ), the diagram

\[ \begin{array}{ccc}
Rf'_* \circ Rp_* \circ f'^* \mathcal{G} & \xrightarrow{\alpha} & Rf_* \circ Rf' ! \mathcal{G} \\
\cong & \downarrow \cong & \downarrow \theta f' \\
Rf'_* \circ Rf'^* \circ Rq_{Y'}^! \mathcal{G} & \longleftarrow \theta f' & Rq_Y^! \mathcal{G}
\end{array} \]

is commutative in \( D_c(Y_{\acute{e}t}', R_* \), where \( \alpha \) is defined in Proposition 2.4 (4) and \( \beta \) is induced by the composition

\[ Rf'_* \circ Rp_* \circ f'^* \mathcal{G} \cong R(f \circ p)'! = R(q \circ f')'! \cong R f'^* \circ Rq_* \mathcal{G} \]

\[ (\beta) \]

Definition 3.15 (Base Change). Let

\[ \begin{array}{ccc}
X' & \xrightarrow{f'} & S' \\
p \downarrow & & \downarrow u \\
X & \xrightarrow{f} & S
\end{array} \]

be a Cartesian square of schemes satisfying (†) with \( f \) compactifiable. For every object \( \mathcal{N} \) in \( D_c(S_{\acute{e}t}, R_* \) ) and for every \( n \in \mathbb{Z} \), we define a homomorphism of \( R \)-modules

\[ u^* : H_n(X/S, \mathcal{N}) \to H_n(X'/S', u^* \mathcal{N}) \]

as follows. For each \( \alpha \in H_n(X/S, \mathcal{N}) \), \( u^*(\alpha) \) is defined to be the composition

\[ R_{X'} = p^* R_X \xrightarrow{p^*(\alpha)} p^* \circ Rf' ! \mathcal{N}'[-n] \xrightarrow{\varphi} R f'^* \circ u^* \mathcal{N}'[-n] \]

where \( \varphi \) is defined in Proposition 2.4 (2).

We have the following three obvious propositions about the base change homomorphisms.

Proposition 3.16. Let \( k \subseteq K \) be two separably closed fields, \( f : X \to S \) a morphism of algebraic schemes over \( k \), \( u : S_K \to S \) the projection. Then for each object \( \mathcal{N} \) in \( D_c(S_{\acute{e}t}, R_* \) ) and for each \( n \in \mathbb{Z} \), the homomorphism

\[ u^* : H_n(X/S, \mathcal{N}) \xrightarrow{\sim} H_n(X_K/S_K, u^* \mathcal{N}) \]

is an isomorphism.
Proof. It follows from Proposition 2.4 (3) and Theorem 1.1.

Proposition 3.17. Let

\[
\begin{array}{ccc}
X'' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
S'' & \longrightarrow & S'
\end{array}
\]

be a commutative diagram of schemes satisfying (1) with both squares Cartesian, and all three vertical arrows being compactifiable. Then for all \( N \in \mathcal{D}_c(S_\text{ét}, R_*) \) and \( n \in \mathbb{Z} \), we have

\[(u \circ v)^* = v^* \circ u^* : H_n(X/S, N) \rightarrow H_n(X''/S'', (u \circ v)^* N) .\]

Proposition 3.18. Let

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
S' & \longrightarrow & S
\end{array}
\]

be a commutative diagram of schemes satisfying (1) with both squares Cartesian, and all level arrows being compactifiable. Let \( N \) be an object in \( \mathcal{D}_c(S_\text{ét}, R_*) \) and \( n \in \mathbb{Z} \). Then we have

1. If \( f \) is proper, then
   \[ u^* \circ f_* = f'_* \circ u^* : H_n(X/S, N) \rightarrow H_n(Y'/S', u^* N) .\]

2. If \( f \) is flat at dimension \( d \), then
   \[ u^* \circ f^* = f'^* \circ u^* : \mathbb{H}_n(Y/S, N) \rightarrow \mathbb{H}_{n+d}(X'/S', u^* N) .\]

Definition 3.19 (Galois action). Let \( k_0 \) be a field, \( k \) the separably closed field of \( k_0, G := \text{Gal}(k/k_0) \), \( X \) an algebraic scheme over \( k, Y_0 \) an algebraic scheme over \( k_0, Y := Y_0 \otimes_{k_0} k \), \( \mathcal{N}_0 \) an object in \( \mathcal{D}_c(Y_0_\text{ét}, R_*) \) and \( \mathcal{N} \) the pull-back of \( \mathcal{N}_0 \) on \( Y \). Then there is an action of \( G \) on \( H_n(X/Y, \mathcal{N}) \) defined by

\[(g, \alpha) \mapsto (\text{id}_{Y_0} \otimes g)^* (\alpha), \quad g \in G, \alpha \in H_n(X/Y, \mathcal{N}) .\]

In particular if \( N \in \mathcal{D}_R(R) \) and \( n \in \mathbb{Z} \), then there is a Galois action of \( G \) on \( H_n(X, N) \).

The following theorem is used to prove that cycle maps eliminate algebraic equivalent classes.

Theorem 3.20. Let \( f : X \rightarrow Y \) be a morphism of algebraic schemes over a separably closed field \( k \), \( Z \) a nonsingular variety over \( k \), \( \mathcal{N} \) an object in \( \mathcal{D}_c(Y_\text{ét}, R_*) \),

\[ \alpha \in H_n((X \times_k Z)/(Y \times_k Z), \text{pr}_1^* \mathcal{N}) .\]

For each \( z \in Z(k) \), put

\[ j_z := \text{id}_Y \times z : Y \rightarrow Y \times_k Z .\]

Then \( z \mapsto j_z^* (\alpha) \) is a constant map from \( Z(k) \) to \( H_n(X/Y, \mathcal{N}) \).

Proof. By Proposition 3.16 we may assume that \( k \) is algebraically closed. Since every two rational points of \( Z \) can be joined by a series of nonsingular curves, we may further assume that \( Z \) is a complete nonsingular curve. First we have a commutative diagram with both squares Cartesian.

\[
\begin{array}{ccc}
X \times_k Z & \xrightarrow{f'} & Y \times_k Z \\
\downarrow p & & \downarrow q \\
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
& & \text{Spec } k
\end{array}
\]
By Proposition 2.4, we have
\[ Rf^! \circ u^* \mathcal{M}[-n] \overset{\sim}{\longrightarrow} p^* Rf^! \mathcal{M}[-n] = p^* Rf^! \mathcal{M}[-n] \otimes_R (v \circ f')^* RZ. \]
Since \(Z\) is a complete nonsingular curve over \(k\), we have \(H^0(Z_{\text{et}}, R) \cong R, H^1(Z_{\text{et}}, R) \cong R^{g}\) (where \(g\) is the genus of \(Z\)), and \(H^2(Z_{\text{et}}, R) \overset{\sim}{\longrightarrow} R\) are all free \(R\)-modules. Now we apply Theorem \ref{thm1} to obtain an isomorphism:

\[ H_n((X \times_k Z)/(Y \times_k Z), pr_1^* \mathcal{M}) \overset{\sim}{\longrightarrow} H^0(X \times_k Z, p^* Rf^! \mathcal{M}[-n] \otimes_R (v \circ f')^* RZ) \]
\[ \overset{\sim}{\longrightarrow} H_n(X/Y, \mathcal{M}) \oplus (H_{n+1}(X/Y, \mathcal{M}) \otimes_R H^1(Z_{\text{et}}, R)) \oplus (H_{n+2}(X/Y, \mathcal{M}) \otimes_R H^2(Z_{\text{et}}, R)). \]
Let \(\beta \in H_n(X/Y, \mathcal{M})\) be the image of \(\alpha\) induced by above isomorphism. Then \(j^*_z(\alpha) = \beta\) for all \(z \in Z(k)\).

4. The Cycle Maps for Chow Groups

In this section, we construct the cycle maps for arbitrary algebraic schemes over \(k\), where \(k\) is a field such that \(\text{char } k \neq \ell\) and \(c_{d\ell}(k) < \infty\).

**Notation 4.1.** Let \(f : X \to Y\) be a compactifiable morphism of schemes satisfying (†) which is flat at dimension \(d\). We define
\[ c_\ell(X/Y) := t f : \mathbb{Z}_{\ell,X} \to Rf^! \mathbb{Z}_{\ell,Y}(-d) \]
in \(D^b(X_{\text{et}}, \mathbb{Z}_{\ell,\bullet})\), i.e., \(c_\ell(X/Y) \in \mathbb{H}_d(X/Y, \mathbb{Z}_\ell)\).

**Proposition 4.2.** Let \(X \to S\) and \(Y \to S\) be two compactifiable morphisms of schemes satisfying (†), \(f : X \to Y\) a morphism of \(S\)-schemes. Assume that \(Y \to S\) and \(f : X \to Y\) are flat at dimension \(n\) and \(d\) respectively. Then we have
\[ f^* c_\ell(Y/S) = c_\ell(X/S) \in \mathbb{H}_{d+n}(X/S, \mathbb{Z}_\ell). \]

**Proof.** This follows from Proposition 2.8 (2).

**Proposition 4.3.** Let \(p : X \to S\) and \(q : Y \to S\) be two morphisms of schemes satisfying (†) both of which are compactifiable and flat at dimension \(d\), \(f : X \to Y\) a finite \(S\)-morphism such that \(f_* f_X\) is a locally free \(\mathcal{O}_Y\)-module of degree \(n\). Then we have
\[ f_* c_\ell(X/S) = n \cdot c_\ell(Y/S) \in \mathbb{H}_{d}(Y, \mathbb{Z}_\ell). \]

**Proof.** By the definition of \(f_*\) and Proposition 2.8 (2), the element \(f_* c_\ell(X/S)\) is equal to the composite morphism
\[ Z_{\ell,Y} \overset{\delta f}{\longrightarrow} f_* Z_{\ell,X} \overset{f_* (t_f)}{\longrightarrow} f_* \circ Rf^! Z_{\ell,Y} \overset{f_* \circ Rf^! (t_q)}{\longrightarrow} f_* \circ Rf^! \circ Rq^! Z_{\ell,Y} \overset{\theta_f}{\longrightarrow} Rq^! Z_{\ell,Y}. \]
By Diagram (2.1) and Proposition 2.9, we have a commutative diagram
\[ Z_{\ell,Y} \overset{\delta f}{\longrightarrow} f_* Z_{\ell,X} \overset{f_* (t_f)}{\longrightarrow} f_* \circ Rf^! Z_{\ell,Y} \overset{f_* \circ Rf^! (t_q)}{\longrightarrow} f_* \circ Rf^! \circ Rq^! Z_{\ell,Y} \]
\[ \overset{\theta_f}{\longrightarrow} Rq^! Z_{\ell,Y}. \]
Thus we get the proof.

**Notation 4.4.** Let \(X \to S\) be a compactifiable morphism of schemes satisfying (†), \(i : Y \hookrightarrow X\) a closed immersion. Assume that the morphism \(Y \to S\) is flat at dimension \(d\). Then we define
\[ \overline{c}_X(S,\ell)(Y) := i_* c_\ell(Y/S) \in \mathbb{H}_d(X/S, \mathbb{Z}_\ell). \]
**Notation 4.5.** Let $X$ be an algebraic scheme over $k$. Then for each $n \in \mathbb{Z}$, there is a canonical homomorphism of groups
\[
\tilde{c}_X/\ell: \mathbb{Z}_n(X) \to \mathbb{H}_n(X, \mathbb{Z}_\ell), \quad \sum a_i \cdot [Y_i] \mapsto \sum a_i \cdot \tilde{c}_{X/k,\ell}(Y_i).
\]

**Proposition 4.6.** Let $f: X \to Y$ be a proper morphism of algebraic schemes over $k$. Then for every $n \in \mathbb{N}$, we have a commutative diagram
\[
\begin{array}{ccc}
Z_n(X) & \xrightarrow{\tilde{c}_{X/\ell}} & \mathbb{H}_n(X, \mathbb{Z}_\ell) \\
f_* & & f_* \\
Z_n(Y) & \xrightarrow{\tilde{c}_{Y/\ell}} & \mathbb{H}_n(Y, \mathbb{Z}_\ell)
\end{array}
\]

**Proof.** Let $X'$ be a $n$-dimensional subvariety of $X$, $Y' := f(X')$, $i: X' \hookrightarrow X$ and $j: Y' \hookrightarrow Y$ the inclusion, $g: X' \to Y'$ the induced morphism. By Proposition 3.9, we have
\[
f_* \circ \tilde{c}_{X/\ell}([X']) = f_* \circ i_* c_\ell(X'/k) = j_* \circ g_* c_\ell(X'/k) \in \mathbb{H}_n(X, \mathbb{Z}_\ell).
\]

Since $f_*[X'] = \deg(X'/Y')[Y']$ (see [4, 1.4]), we have only to prove that
\[
g_* c_\ell(X'/k) = \deg(X'/Y') \cdot c_\ell(Y'/k) \in \mathbb{H}_n(Y', \mathbb{Z}_\ell).
\]

Case 1. $\dim Y' < n$. Then $\deg(X'/Y') = 0$. And by Proposition 3.5, $\mathbb{H}_n(Y', \mathbb{Z}_\ell) = 0$.

Case 2. $\dim Y' = n$. We apply the result in [8, Ex. 3.7]. Since the morphism $g$ is generically finite and $Y'$ is an integral scheme, there exists an nonempty subscheme $V$ of $Y'$ such that $g: g^{-1}(V) \to V$ is a finite morphism and $g_* O_{X'/V}$ is a locally free $O_V$-module. Now the proposition follows from Proposition 3.6 and Proposition 4.3. 

**Proposition 4.7.** Let $X$ be an algebraic scheme over $k$, $Y$ a $n$-equidimensional closed subscheme of $X$. Then we have
\[
\tilde{c}_{X/k,\ell}(Y) = \tilde{c}_{X/\ell}([Y]) \in \mathbb{H}_n(X, \mathbb{Z}_\ell).
\]

**Proof.** This is easily deduced from Proposition 2.10. 

**Proposition 4.8.** Let $f: X \to Y$ be a flat morphism of relative dimension $d$ of algebraic schemes over $k$. Then for every $n \in \mathbb{N}$, we have a commutative diagram
\[
\begin{array}{ccc}
Z_n(Y) & \xrightarrow{\tilde{c}_{Y/\ell}} & \mathbb{H}_n(Y, \mathbb{Z}_\ell) \\
f_* & & f_* \\
Z_{n+d}(X) & \xrightarrow{\tilde{c}_{X/\ell}} & \mathbb{H}_{n+d}(X, \mathbb{Z}_\ell)
\end{array}
\]

**Proof.** Let $\alpha \in Z_n(Y)$. We may assume that $Y$ is a variety of dimension $n$ and $\alpha = [Y]$. Then we have only to apply Proposition 4.2. 

Now we could prove that $\tilde{c}$ annihilates the rational equivalence.

**Lemma 4.9.** Let $X$ be a nonsingular variety of dimension $n$ over $k$, $D$ an effective divisor on $X$. Then
\[
\tilde{c}_{X/S,\ell}(D) = c_1(O(D)) \in \mathbb{H}_{-1}(X, \mathbb{Z}_\ell) = \mathbb{H}^1(X, \mathbb{Z}_\ell).
\]

**Proof.** See [9, (3.26)].
**Theorem 4.10.** Let $X$ be an algebraic scheme over $k$. Then for each $n \in \mathbb{N}$,
\[
\text{Rat}_n(X) \subseteq \ker\left(\widetilde{c}_{X,\ell} : \mathbb{Z}_n(\ell) \to \mathbb{H}_n(X,\mathbb{Z}_\ell)\right),
\]
i.e., the homomorphism $\widetilde{c}_{X,\ell}$ factors through $\mathbb{C}H_n(X)$. We use $c_{X,\ell}$ or $c_{\ell}$ or $c_X$ to denote the induced homomorphism $\mathbb{C}H_n(X) \to \mathbb{H}_n(X,\mathbb{Z}_\ell)$.

**Proof.** After applying [4, Proposition 1.6] together with Proposition 4.6 and Proposition 4.8, we have only to prove that
\[
\widetilde{c}_{P_1,\ell}(0) = \widetilde{c}_{P_1,\ell}(\infty) \in \mathbb{H}_0(P_1,\mathbb{Z}_\ell).
\]
This is by Lemma 4.9. \hfill \Box

In the following, we define the degree map for the homology of degree zero. Note that $\mathbb{H}_0(\text{Spec } k,\mathbb{Z}_\ell) = \mathbb{Z}_\ell$. So we have

**Definition 4.11.** For any proper algebraic scheme $X$ over $k$, we define **degree map** $\text{deg}_\ell$ to be the homomorphism
\[
\mathbb{H}_0(X,\mathbb{Z}_\ell) \xrightarrow{p^*} \mathbb{H}_0(\text{Spec } k,\mathbb{Z}_\ell) = \mathbb{Z}_\ell,
\]
where $p : X \to \text{Spec } k$ is the structural morphism.

**Lemma 4.12.** Let $X$ be a $n$-dimensional proper algebraic scheme over $k$.

1. We have a commutative diagram.

\[
\begin{array}{ccc}
\mathbb{H}^{2n}(X_{\text{ét}},\mathbb{Z}_\ell(n)) & \xrightarrow{(t_X)_*} & \mathbb{H}_0(X,\mathbb{Z}_\ell) \\
\downarrow \text{Tr} & & \downarrow \text{deg}_\ell \\
\mathbb{Z}_\ell & & \mathbb{Z}_\ell
\end{array}
\]

2. We have a commutative diagram.

\[
\begin{array}{ccc}
\mathbb{C}H_0(X) & \xrightarrow{c_{X,\ell}} & \mathbb{H}_0(X,\mathbb{Z}_\ell) \\
\downarrow \text{deg} & & \downarrow \text{deg}_\ell \\
\mathbb{Z} & \xrightarrow{c_{X,\ell}} & \mathbb{Z}_\ell
\end{array}
\]

**Proof.** (1) is by the commutative diagram 2.1. (2) is by Proposition 4.6. \hfill \Box

**Proposition 4.13.** Assume that $k$ is separably closed and let $X$ be a nonsingular complete variety over $k$. Then $\text{deg}_\ell : \mathbb{H}_0(X,\mathbb{Z}_\ell) \xrightarrow{\sim} \mathbb{Z}_\ell$ is an isomorphism.

**Proof.** Put $\dim X = n$. By Proposition 2.6, we have only to prove that $\text{Tr}_X : \mathbb{H}^{2n}(X_{\text{ét}},\mathbb{Z}_\ell(n)) \to \mathbb{Z}_\ell$ is an isomorphism. This is by [12, VI, 11.1 (a)]. \hfill \Box

The following theorem shows that the cycle map $c_{X,\ell}$ annihilate algebraic equivalence of cycles.

**Theorem 4.14.** Assume that $k$ is separably closed and let $X$ be an algebraic scheme over $k$. Then for each $n \in \mathbb{N}$, the cycles in $\mathbb{C}H_n(X)$ which are algebraically equivalent to zero (in the sense of [4, 10.3]), are contained in $\ker(c_{X,\ell})$.

**Proof.** By Proposition 3.16 we may assume that $k$ is algebraically closed. Let $c_1, c_2 \in \mathbb{C}H_n(X)$ such that $c_1 \sim_a c_2$; and let $T$ be a nonsingular curve over $k$. $t_1, t_2 \in T(k)$, $c \in \mathbb{C}H_{n+1}(X \times_k T)$ such that $c_{t_i} = c_i$ for $i = 1, 2$. Obviously we may assume that $c = [Y]$, where $Y$ is a $(n + 1)$-dimensional subvariety of $X \times_k T$ such that for all $t \in T(k)$, $Y$ is not contained in $\text{id} \times t_i : X \hookrightarrow X \times_k T$. 


Obviously the induced morphism $Y \to T$ is dominant and flat. Put
$$
\alpha := c(X \times_k T)/T, \tau(Y) \in \mathbb{H}_n((X \times_k T)/T, \mathbb{Z}_\ell).
$$
By Proposition 3.18 and Proposition 2.7 (2), we have
$$
t^\tau_\ell(\alpha) = c_{X, \ell}(Y_{t_\ell}) = c_{X, \ell}(c_{t_\ell}).
$$
So we have only to apply Proposition 3.20. □

5. CAP-PRODUCTS AND COMPATIBILITY WITH CHERN CLASSES

First we define the cap-products for the ℓ-adic homology.

**Definition 5.1** (Cap-Product). Let $f: X \to Y$ and $g: Y \to Z$ be compactifiable morphisms of schemes satisfying ( AttributeSet), $\mathcal{M}$ and $\mathcal{N}$ two objects in $D'_c(Z_{\text{ét}}, R_*)$.

For every $m, n \in \mathbb{Z}$, there is a cap-product
$$
H_m(X \xrightarrow{f} Y, g^* \mathcal{M}) \times H_n(Y \xrightarrow{g} Z, \mathcal{N}) \xrightarrow{\cup} H_{m+n}(X \xrightarrow{g \circ f} Z, \mathcal{M} \otimes_R^L \mathcal{N}),
$$
defined as follows. Let $\alpha \in H_m(X \xrightarrow{f} Y, g^* \mathcal{M})$ and $\beta \in H_n(Y \xrightarrow{g} Z, \mathcal{N})$, then we define $\alpha \cap \beta$ to be the composite morphism
$$
R_X \xrightarrow{\alpha \otimes_L^R f^* \beta} R f^! \circ g^* \mathcal{M}[-m] \otimes_R^L g^* \mathcal{N}[-n] \otimes R (g \circ f)^!(\mathcal{M} \otimes_R^L \mathcal{N})[-(m+n)].
$$
where $\varphi$ is defined in Proposition 2.23.

Similarly we may define the cap-product for $\mathbb{H}_*$ as follows:
$$
\mathbb{H}_m(X \xrightarrow{f} Y, g^* \mathcal{M}) \times \mathbb{H}_n(Y \xrightarrow{g} Z, \mathcal{N}) \xrightarrow{\cup} \mathbb{H}_{m+n}(X \xrightarrow{g \circ f} Z, \mathcal{M} \otimes_R^L \mathcal{N}),
$$
In particular if $X \to S$ is a compactifiable morphisms of schemes satisfying ( AttributeSet), and $\mathcal{N}$ an object in $D'_c(S_{\text{ét}}, R_*)$, then for every $m, n \in \mathbb{Z}$, there are cap-products
$$
H^m(X, R) \times H_n(X/S, \mathcal{N}) \xrightarrow{\cup} H_{m-n}(X/S, \mathcal{N}),
$$
$$
\mathbb{H}^m(X, R) \times \mathbb{H}_n(X/S, \mathcal{N}) \xrightarrow{\cup} \mathbb{H}_{m-n}(X/S, \mathcal{N}).
$$

The following Proposition can be directly calculated.

**Proposition 5.2** (Projection Formula). Let $f: X \to Y$ and $g: Y \to S$ be morphisms of schemes satisfying ( AttributeSet) with $f$ proper and $g$ compactifiable, $\mathcal{N}$ an object in $D'_c(Y_{\text{ét}}, R_*)$. Then we have

1. For every $\alpha \in H^r(Y, R)$ and $\beta \in H_n(X/S, \mathcal{N})$, we have
$$
\alpha \cap f_* \beta = f_* (f^* \alpha \cap \beta) \in H_{n-r}(X/S, \mathcal{N}).
$$

2. For every $\alpha \in \mathbb{H}^r(Y, R)$ and $\beta \in \mathbb{H}_n(X/S, \mathcal{N})$, we have
$$
\alpha \cap f_* \beta = f_* (f^* \alpha \cap \beta) \in \mathbb{H}_{n-r}(Y/S, \mathcal{N}).
$$

It may be further showed that the cup-product defined in Definition 5.1 has many similar properties with bivariant intersection theory defined in [4, Ch. 17], i.e., has associativity and is compatible with the Pull-back functor $f_*$, the push-out functor $f^*$ and the base change functor $u^*$. Since we need not them here, so we leave it to the readers.

Next, we review the cycle maps for locally free sheaves. First by [9, (3.26) a)], we have a homomorphism of groups
$$
c^\ell_1: \text{Pic} X \to \mathbb{H}^1(X, \mathbb{Z}_\ell)
$$
for every scheme $X$ satisfying ( AttributeSet). The following two propositions depict the cycle maps for locally free sheaves.
Moreover we may assume that $\mathcal{O}_X$ is an invertible $\mathcal{O}_S$-module of constant rank $r + 1$. Then we have

$$\mathbb{H}^*(S, \mathbb{Z}[T])/(T^{r+1}) \sim \mathbb{H}^*(P, \mathbb{Z}), \quad \mathbb{T} \mapsto c_f^i(\mathcal{O}_P(1)).$$

Proof. See [9, (6.13)] or [7, VII, 2.2.6].

Proposition 5.4. Let $X$ be a scheme satisfying (†), $\mathcal{E}$ a locally free $\mathcal{O}_S$-module of constant rank $m$, $P := \mathbb{P}(\mathcal{E})$, $p : P \to X$ the projection, $\xi := c_f^i(\mathcal{O}_P(1))$. Then there exists a unique element $c_f^i(\mathcal{E}) \in \mathbb{H}^r(X, \mathbb{Z})$ for each $r \in \mathbb{N}$, such that

$$\begin{cases}
\sum_{i=0}^m c_f^i(\mathcal{E})\xi^{m-i} = 0, \\
c_f^0(\mathcal{E}) = 1, \\
c_f^i(\mathcal{E}) = 0 \text{ for } r > m.
\end{cases}$$

Now we define the trace morphisms for regular immersions of codimension 1. Let $X$ be a scheme satisfying (†) and $i : D \hookrightarrow X$ a regular closed immersion of codimension 1. By [9, (3.26) and the proof], $i : D \hookrightarrow X$ determines an element

$$t_i \in \mathbb{H}_D^{2\text{cont}}(X_{et}, \mathbb{Z}_l(1)) = \text{Hom}_{D_{et}(\mathcal{D}_l, \mathbb{Z}_l \otimes)}(\mathbb{Z}_l, \mathbf{R}t^1_i \mathbb{Z}_l(-1)).$$

Similar to [2, (cycle) 2.3.1], we have

Proposition 5.5. Let $S$ be a scheme satisfying (†), $f : X \to S$ and $g : Y \to S$ two compactifiable morphisms which are flat at dimension $n$ and $n - 1$ respectively, $i : Y \hookrightarrow X$ a regular closed immersion of codimension 1 such that $f \circ i = g$. Then we have

1. The composite morphism

$$\mathbb{R}g_{\mathbb{Z}_l} \mathbb{R}f_i(t_i) = \mathbb{R}f_i \circ \mathbb{R}t_{\mathbb{Z}_l}(1) = \mathbb{R}f_i \circ \mathbb{R}t_{\mathbb{Z}_l}(1) \xrightarrow{\mathbb{R}f_i(\theta_i)} \mathbb{R}f_i \mathbb{Z}_l(1) \xrightarrow{T_{r,f}} \mathbb{Z}_l(-n-1)$$

is equal to $T_g$.

2. The composite morphism

$$\mathbb{R}t_{\mathbb{Z}_l}(1) \xrightarrow{\mathbb{R}t_{\mathbb{Z}_l}(f)} \mathbb{R}t_{\mathbb{Z}_l} \mathbb{Z}_l(-n-1) = \mathbb{R}g_{\mathbb{Z}_l} \mathbb{Z}_l(-n-1)$$

is equal to $t_g$.

Finally we could prove that the cycle maps are compatible with Chern classes. According to [4, Ch. 3], if $X$ is an algebraic scheme over $k$ and $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module, then there is an operation of Chern classes on each Chow group

$$\text{CH}_* (X) \to \text{CH}_{*-i} (X), \quad \alpha \mapsto c_i(\mathcal{E}) \cap \alpha.$$  

Theorem 5.6. Let $X$ be an algebraic scheme over $k$, $\mathcal{E}$ a locally free $\mathcal{O}_X$-module, $\alpha \in \text{CH}_{r}(X)$. Then we have

$$c_i^1(\mathcal{E}) \cap c_{X,E}(\alpha) = c_{X,E}(c_i(\mathcal{E}) \cap \alpha) \in \text{HH}_{r-1}(X, \mathbb{Z}).$$

Proof. By the projection formulas (Proposition 5.2 and [4, Theorem 3.2 (c)]), we obtain that if $f : X' \to X$ is a proper morphism and $\alpha' \in \text{CH}_r(X')$ such that $f_*(\alpha') = \alpha$ and the pair $(f^*\mathcal{E}, \alpha')$ satisfies (5.2), then the pair $(\mathcal{E}, \alpha)$ also satisfies (5.2). Thus by the splitting construction (see [4, §3.2]), we may assume that $\mathcal{E} = \mathcal{L}$ is an invertible $\mathcal{O}_X$-module and have only to prove that

$$c_f^i(\mathcal{L}) \cap c_{X,E}(\alpha) = c_{X,E}(c_1(\mathcal{L}) \cap \alpha) \in \text{HH}_{r-1}(X, \mathbb{Z}).$$

Moreover we may assume that $X$ is a variety of dimension $r$ and $\alpha = [X]$. After replacing $X$ with its normalization, we may assume that $X$ is normal. Then we may assume that $\mathcal{L} = \mathcal{O}(Y)$ where $Y \hookrightarrow X$ is a regular closed immersion of codimension 1. Then we have only to apply Proposition 5.5. $\square$
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