Stability Analysis of Normal DGP Brane-world Model with Agegraphic Dark Energy

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Abstract

The aim of this work is to apply the dynamical system approach to study the linear dynamics of normal DGP brane-world model with agegraphic dark energy as the dark energy component. The stability analysis of the model will be investigated and phase plane portrait will be shown. The nature of critical points will be analyzed by evaluating the eigenvalues of linearized Jacobi matrix. Also, statefinder diagnostic procedure will be applied to compare deviation from $\Lambda$CDM model. One of the most interesting results of this work is the great alleviation of the coincidence problem.

Keywords: DGP, agegraphic dark energy, dynamical system, stability, coincidence problem

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I. INTRODUCTION

Nowadays we know from observations that our Universe is experiencing an accelerated expansion phase [1], [2]. This acceleration can be described in two distinct scenarios. In one of them one can modify the left-hand side of Einstein’s equation in different ways where are called the *modified gravity* theories [3], [4] and in the other, one can add a component with negative pressure to the right-hand side of Einstein’s equation which is dubbed the *dark energy* (DE). A great variety of DE models have been proposed in the literature [5]-[11]. Among them holographic dark energy (HDE) [12], and agegraphic dark energy (ADE) [13] which has been based respectively on the holographic principle and the quantum fluctuations of space-time, are of particular interest because they contain some important features of a quantum gravity theory, though, a complete and comprehensive formulation of this theory has not yet been established.

In the ADE approach, Károlyházy and his collaborators showed that in Minkowskian space-time the distance $t$, can not be known to a better accuracy than

$$\delta t = \gamma t_p^{2/3} t^{1/3},$$

(1)

where $\gamma$, is a dimensionless constant of order unity and $t_p$, denotes the reduced Planck time. Then, the authors in [15], using the time-energy uncertainty relation estimated the quantum energy density of the metric fluctuations of Minkowskian space-time where can be viewed as the energy density of ADE as

$$\rho_{DE} \sim \frac{1}{t_p^2 t^2} \sim \frac{M_p^2}{t^2},$$

(2)

where $M_p$, represents the reduced Planck mass. Replacing the proper time scale $t$, in the above with the age of the Universe, $T = \int_0^a \frac{da}{H(a)}$, where $a$ and $H$, are respectively the scale factor and the Hubble parameter, Cai obtained the energy density of ADE as

$$\rho_{DE} = \frac{3n^2 M_p^2}{T^2}.$$  

(3)

Here, $3n^2$, is a numerical factor which parameterizes some uncertainties such as the effects of curved space-time [44], and also the species of quantum fields in the Universe.

On the other hand, the extra dimensional theories have been attracted a considerable amount of attention in the past two decades [16]-[21]. In these models, our four-dimensional
(4D) Universe is a brane embedded in a higher dimensional space-time in the sense that the standard model of particle physics is confined to the brane and only graviton can propagate into the bulk. The extra dimensions affect the Friedmann equations on the brane by inducing a few additional terms [22]-[24]. In particular, the DGP brane-world model in which the bulk is an infinite five-dimensional (5D) Minkowski space-time, proposed by Dvali, Gabadadze and Porrati, has remarkably been studied, recently [25]. There are two distinguished branches for this model, depending on how the brane can be embedded in the bulk. The self-accelerating branch where results the late time acceleration of the Universe without the help of any DE component and the normal branch in which one must consider DE to produce an accelerating expansion phase.

Independent of the above subjects dynamical system techniques have greatly been used in studying cosmological models [26],[27]. The main advantage of these techniques is the possibility of studying all solutions with admissible initial conditions. Since there are always some uncertainties in the initial conditions of a model, a physically meaningful mathematical model where present detailed information on the evolution of the deviations of the possible trajectories of the dynamical system from a given reference trajectory seems very useful. The key requirement in these mathematical models is the understanding of the local stability of the physical and cosmological processes.

Incorporation of DGP brane-world model and different kinds of DE components such as cosmological constant, quintessence, Chaplygin gas, HDE and ADE, has been discussed in literature [28]-[32]. In most of the cases, the dynamical system approach has been studied in details. In this manuscript we consider the normal branch of DGP model in the presence of ADE and investigate the stability analysis of the model. Our main motivation in this mixed model is the quantum nature of its components, in the sense that ADE comes out of a quantum gravity theory, like the extra dimensional gravity that results from string theory. The paper is organized as follows: In Sec.II we construct the model, introduce our new variables and write related ordinary differential equations. The critical points and related eigenvalues will be discussed in this section, as well. Sec.III deals with the statefinder diagnostic approach. Also, the coincidence problem is investigated in this section. Sec.IV includes a summary and conclusion of the article.
II. THE MODEL AND THE STABILITY ANALYSIS

In this section we first introduce normal DGP brane-world model in the presence of ADE and then utilize the dynamical system approach to investigate the model, carefully. We start with the cosmological equations of the model. Assuming the brane is spatially flat, homogeneous and isotropic, the Friedmann equation on the brane can be written as

$$H^2 + \frac{H}{r_c} = \frac{1}{3M_p^2} (\rho_m + \rho_{DE}) ,$$

(4)

where $\rho_m$, is the energy density of the matter of the Universe which is dominant by dark matter (DM), and $r_c$, is called the crossover distance which determines the transition from 4D to 5D regime. From conservation equations of DM and DE on the brane we have

$$\dot{\rho}_m + 3H\rho_m = 0 ,$$

(5)

$$\dot{\rho}_{DE} + 3H\rho_{DE}(1 + \omega_{DE}) = 0 .$$

(6)

Differentiating Eq.(3), we obtain

$$\dot{\rho}_{DE} = -2H\rho_{DE} \frac{\sqrt{\Omega_{DE}}}{n} ,$$

(7)

where

$$\Omega_{DE} = \frac{\rho_{DE}}{3M_p^2 H^2} = \frac{n^2}{H^2 T^2} .$$

(8)

Inserting Eq.(7), in Eq.(6), we find the equation of state (EoS) parameter of ADE, as

$$\omega_{DE} = -1 + \frac{2}{3n} \sqrt{\Omega_{DE}} ,$$

(9)

where indicates that EoS parameter of ADE, never crosses the phantom divide line. Also, the Raychaudhuri equation of our model can be obtained using Eqs.(4), (5) and (7), as

$$\dot{H} = \frac{-\rho_m - 2\omega_{DE} \sqrt{\Omega_{DE}}}{M_p^2 (2 + \frac{1}{H r_c})} .$$

(10)

The structure of the dynamical system can be studied via phase plane analysis. Generally, in order to perform the phase-space and stability analysis, one have to introduce some auxiliary variables to transform the cosmological equations of motion into a self-autonomous
dynamical system. Here, we introduce the following dimensionless phase variables:

\[ x = \sqrt{\frac{\rho_m}{3M_p^2(H^2 + \frac{H}{r_c})}}, \]
\[ y = \sqrt{\frac{\rho_{DE}}{3M_p^2(H^2 + \frac{H}{r_c})}}, \]
\[ z = \sqrt{1 + \frac{1}{Hr_c}} \]  \hspace{1cm} (11)

where if \( r_c \to \infty \), we get the 4D solutions. The new variable \( z \), satisfies \( z \geq 1 \), since \( H \) and \( r_c \), have positive values. Also, the Friedmann equation, Eq.(11), yields the constraint

\[ x^2 + y^2 = 1. \]  \hspace{1cm} (12)

With attention to this constraint and because our phase variables can not be negative, they have to satisfy \( 0 \leq x \leq 1 \), and \( 0 \leq y \leq 1 \).

On the other hand, we can recast the Raychaudhury equation in terms of new variables as

\[ \frac{\dot{H}}{H^2} = -\frac{3z^2(x^2 + \frac{2}{3n}y^3z)}{z^2 + 1}. \]  \hspace{1cm} (13)

In addition, the EoS parameter of ADE, Eq.(11) and also the total EoS parameter of the Universe can be written as

\[ \omega_{DE} = -1 + \frac{2}{3n}y^z, \]  \hspace{1cm} (14)

and

\[ \omega_{tot} = -1 - \frac{2\dot{H}}{3H^2} = -1 + \frac{2z^2(x^2 + \frac{2}{3n}y^3z)}{z^2 + 1}, \]  \hspace{1cm} (15)

respectively. With attention to the phase-space variables, Eq.(11), and the Friedmann constraint, Eq.(12), and also Eq.(13), we reach to the following autonomous system of ordinary differential equations in \( (y, z) \) space

\[ y' = -\frac{y^2z}{n} + \frac{y}{2} \left( 3(1 - y^2) + \frac{2}{n}y^3z \right), \]  \hspace{1cm} (16)
\[ z' = \frac{z(z^2 - 1)}{2(z^2 + 1)} \left( 3(1 - y^2) + \frac{2}{n}y^3z \right), \]  \hspace{1cm} (17)

where prime means derivative with respect to \( \ln a \). These equations interpret the evolution of our model of non-interacting DM and ADE in the context of DGP brane-world model.

Stability analysis is to study the behavior of a system in the vicinity of its critical or fixed points. To determine the critical points of the dynamical system above we must impose the
TABLE I: The fixed points found for the normal DGP model with ADE term as dark energy.

| (z, y)   | eigenvalues | ω_{tot} | description                  | stability  |
|----------|-------------|---------|------------------------------|------------|
| A (1, 0) | (3/2, 3/2)  | 0       | DM domination                | unstable   |
| B (1, 1) | (1/n, (2-3n)/n) | -1 + 2/(3n) | DE domination                | saddle     |

conditions $y' = 0$, and $z' = 0$, simultaneously. The admissible results that satisfy the limitations on $y$, and $z$, have been shown in Table I.

Point $A$, relates to the matter dominated era because with attention to the Friedmann constraint, $y = 0$, is equivalent with $x = 1$. On the other hand, using Eq. (11), we have $x = \sqrt{\Omega_m}/z$, where $\Omega_m = \rho_m/3M_p^2H^2$, and since at point $A$, we have $z = 1$, so this point shows a matter dominated regime. In the same manner, point $B$, can be related to the DE dominated era. Fig. 1 represents the phase portrait of our dynamical system in $yz$-plane for $n = 2$. We should note here that if we evaluate Eq. (15) at point B, we find a necessary condition for the parameter $n$ as $n > 1$ to guarantee the late time acceleration.

FIG. 1: The 2D phase plane corresponding to the critical points.

We should note some important results here. One can rewrite Friedmann equation (see
Eq. (14)) as
\[ \Omega_m + \Omega_{DE} = z^2 = 1 + \frac{1}{Hr_c}. \] (18)
On the other hand from Table I one can see that at both the critical points we have \( z = 1 \), which is related to the formal limit \( r_c \to \infty \), that indicates these points correspond to standard 4D behavior in which \( \Omega_m + \Omega_{DE} = 1 \). Therefore the line \( z = 1 \) in Fig. I is the trajectory of a pure 4D cosmology which yields from any initial condition as \((y, z = 1)\). It shows that the 4D Universe starts from a matter dominated epoch and approaches a DE dominated era, as we expect in a standard 4D cosmology. All other phase trajectories in Fig. I which leave the phase line \( z = 1 \) and probe the phase plane \((y, z)\), arise from any other initial condition with \( z \neq 1 \) and show the effect of extra dimension in our model. In all these cases after the starting from a matter dominated era the Universe no longer experiences a 4D DE dominated period because with attention to Eq. (18) another term will be dominated that we can call it \( \Omega_{DGP} = \Omega_{DE} - 1/Hr_c \). Although one can consider it as an effective DE term but clearly it differs from the one related to the saddle point \( B \). It can be seen in Fig. I that all these trajectories are repelled from the line \( y = 1 \).

III. FEATURES OF THE MODEL

A. Statefinder Diagnostic

Statefinder diagnostic is a reliable approach in distinguishing various DE models. It is based on a pair of new geometrical variables which are related to the third derivative of scale factor with respect to time. In a flat Universe the statefinder parameters are defined as \[ r = \frac{\ddot{a}}{aH^3}, \quad s = \frac{r - 1}{3(q - \frac{1}{2})}, \] (19)
where \( q \) is the deceleration parameter. To classify different DE scenarios one can compare related trajectories in \( rs \)-plane. Also, the deviation from the \( \Lambda \)CDM model which is expressed by the point \((r = 1, s = 0)\), can be studied in this way (See Fig. II).

B. Coincidence Problem

Coincidence problem where is a fine-tuning cosmological problem appears because the energy densities of DM and DE are of the same order around the present time. Too many
FIG. 2: The plot of statefinder parameters \( \{r, z\} \), \( \{s, z\} \) and \( \{r, s\} \). We have used the initial conditions \( H_0 = 0.67 \) and \( \Omega_{m0} = 0.31 \) from Planck results \[43\], and also \( n = 2 \) and \( \Omega_{rc} = 1/4r_c^2H_0^2 = 0.002 \).

Some of the authors have used different approaches to solve or at least alleviate this problem \[35\]-\[41\]. Some of them come on the attractor solution and try to show that the ratio between the dark sectors, \( R \equiv \rho_m/\rho_{DE} \), is independent of the initial conditions. Another group demonstrate that \( R \), does not vary so much during the evolution of the Universe and some others indicate that \( R \), approaches a constant value at late times or changes slower than the scale factor today. Some of them utilize the dynamical DE models, but others consider the models in which an interaction between the dark sectors of the Universe is taking into account.

In an earlier work we tried to show the role of extra dimension in resolving this problem.
in a normal DGP scenario in the presence of $\Lambda$, as the DE component \[42.\] Here, we will represent the effect of normal DGP model with ADE, in ameliorating the coincidence problem. Comparing Eq.(4), with the standard Friedmann equation, we can introduce an effective DE component in our model with the energy density $\rho_{\text{eff}}$, as
\[\rho_{\text{eff}} = \rho_{\text{de}} - \frac{3M_p^2 H}{r_c}.\] (20)

Then, we can represent the above ratio in our model as $R \equiv \rho_m/\rho_{\text{eff}}$.

Fig.3 illustrates the behavior of $R$, in our model and compares it with the one of the standard $\Lambda$CDM model and also the one in ADGP model. One of the initial values we have used to plot Fig.3 is $\Omega_{rc} = 1/4r_c^2 H_0^2 = 0.002$. Obviously, with this choice, $r_c$ has a finite value and so $z \neq 1$. Therefore the effect of extra dimension has surely been imported in this figure. Although this figure shows that extra dimension can solely alleviate the coincidence problem, but it is obvious that the type of the DE component under consideration is very important, too. Clearly, the combination of DGP and ADE has much more influence in ameliorating the coincidence problem than the ADGP model. We should note here that with attention to Eq.(9), in the limit $n \to \infty$, our model approaches a ADGP model and if besides that $\Omega_{rc} \to 0$ which means $r_c \to \infty$, we then reach to the standard $\Lambda$CDM model.

FIG. 3: Coincidence problem is alleviated in ADE+DGP model, significantly. We have used the initial conditions $H_0 = 0.67$ and $\Omega_m = 0.31$ from Planck results \[43\], and also $n = 2$ and $\Omega_{rc} = 1/4r_c^2 H_0^2 = 0.002$. $z$ in $\ln(1 + z)$, is the redshift.
IV. CONCLUSION

In this article we considered the normal branch of DGP brane-world model which needs a DE component to produce late-time acceleration of the Universe. Among different DE scenarios we assumed ADE component. On the other hand, investigating a cosmological model in the context of dynamical system analysis can be very useful and interesting. After introducing a set of new variables we transformed our cosmological equations to an autonomous dynamical system and found the critical points related to two important cosmological epochs. Although 5D behavior related to dynamical screening does actually arise in our model but the critical points does not differ from the standard ones of 4D Einstein-Hilbert gravity. Afterwards, a quantitative analysis of some important physical parameters, i.e., statefinder parameters in the theory was presented. We found that the Universe leaves an unstable state in the past, passes the ΛCDM state and finally approaches the saddle state in the future. Moreover, this model is in good agreement with ΛCDM model. Finally, we investigated the coincidence problem and found that the normal DGP with ADE could significantly alleviate this problem.

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Because the energy density is derived for Minkowskian space-time