An asymptotic closed form solution for the distribution of combined wind speed and wind direction extremes

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Abstract. This paper presents an asymptotic expression for the joint distribution of the largest combined wind speed- and wind direction excursion, during a large but otherwise arbitrary recurrence period, based on a “mother” distribution that reflects the Exponential-like distribution behaviour of large wind speed excursions. The derived asymptotic joint distribution is shown to equal a joint Gumbel EV1 type distribution, and the associated distribution parameter is expressed as simple functions of basic parameters characterizing the stochastic longitudinal- and transversal wind speed processes in the atmospheric boundary layer and the recurrence time. Finally, predictions of the derived model are compared to results derived from full-scale measurements of wind speeds in the atmospheric boundary layer.

1. Introduction
The statistical distribution of extreme wind events, for a specified large recurrence period, is of crucial importance and of general interest in relation to design of wind sensitive structures. This is particularly true for wind turbine structures. For turbulence driven extreme wind events, which the present study is restricted to, a very versatile theory for consistent embedding of extreme gust events into a stochastic wind field has been developed [1], [2]. This “gust generator” is ideally suited for generation of synthetic extreme wind fields as input for aeroelastic design computations of wind turbines. In the framework of the “gust generator”, magnitudes of relevant gust events are required as input. The present work addresses an asymptotic closed form solution for the distribution of the magnitudes of a certain class of gust events characterized by synchronized wind speed increases and wind direction changes. The recurrence period for these events is assumed to be large, but is otherwise arbitrary.

The type of gust events considered here is also included the IEC 64100-1 standard [3] as an important extreme load case (ECD) for wind turbines, although in a simplified version where the gust description is stylized by adopting a deterministic course of the gust magnitude as function of time and further by being spatially coherent at any time instant. The stochastic character as well as the spatial limitation of a physically realistic gust event is thus disregarded in the code extreme load specification.

Basically, the proposed statistical model relies on two important observations that in a sense have formed the “boundary conditions” for the model development. The first observation is that the conventional Gaussian assumption for wind speed fluctuations is inadequate for the description of

1 The load case may e.g. be design driving for flap deflection of active stall controlled wind turbines.
events associated with large excursions from the mean [1], [4], [5]. Typically, the Gaussian assumption results in thinner distribution “tails” than observed in reality. This is illustrated in Figure 1 using measured full-scale longitudinal- and transversal turbulence components from the Oak Creek site in California [6]. The figure shows the (logarithmic of the) probability density functions (PDF’s) of the longitudinal- and transversal turbulence components, respectively, both normalized with respect to the mean wind speed. In the large excursion regime, the figure clearly demonstrates large deviations between the measured PDF’s and the respective Gaussian PDF’s fitted to the data. In general, the significantly more frequent occurrence of extreme turbulence excursions than predicted, as based on a Gaussian assumption, may result in under-prediction of the probability of large turbulence excursions by more than one decade. As a direct consequence, theoretical models of extreme distributions, if based on a Gaussian assumption, may result in distribution estimates exhibiting a dramatic bias as well as misleading variance properties – especially for long recurrence periods [9], [10].

![Figure 1](image-url)  

**Figure 1** Measured and Gaussian PDF fit compared for the normalized u- and v- turbulence components, respectively.

The second observation is that extremes, associated with turbulence driven full-scale events in the atmospheric boundary layer, usually seems to be well described by a Gumbel EV1 distribution [10], [11], [12]. Note in this respect that a Gaussian assumption, in case of a one-point wind speed gust event, will result in a theoretical extreme value distribution prediction different from the Gumbel EV1 distribution [13].

The two observations, as described above, have lead to the formulation of a model for the distribution of the largest simultaneously occurring wind speed and wind direction change based on a “mother” distribution that reflects the Exponential-like distribution behavior of large wind speed excursions in the atmospheric boundary layer. The model offers simple expressions for the extreme estimates that requires only a few, easy accessible, input parameters. Besides offering site-specific estimates, e.g. associated with application of the “gust generator” modeling tool, the model is also suitable for use in rational calibration of magnitudes of the ECD extreme event specified in the IEC design code.

The paper is organized in three sections – model derivation, results and conclusion.

2. Model

The basic vision behind the model is to derive an asymptotic closed form expression for the distribution of the “largest” joint event characterized by a synchronized wind speed increases (in the

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2 Exponential-like in the present context means that the tails of the PDF, in a logarithmic depiction, will be approximately linear.
mean wind direction) and wind direction changes in a single point in space. The lateral turbulence component is responsible for turbulence driven wind direction changes, and in the present context we more specifically define the “largest” joint event of the investigated type as the largest, simultaneous occurring, longitudinal- and transversal turbulence component within the considered long, but otherwise arbitrary, recurrence period.

The starting point is a “mother” distribution (i.e. a joint distribution of the turbulence excursions from the mean level for an assumed stationary process wind field) reflecting the observed Exponential-like distribution behaviour for large wind speed excursions. Guided by the results reported in [9], we postulate the following joint distribution of longitudinal- \((u_e)\) and lateral \((v_e)\) turbulence driven large excursions

\[
f(u_e, v_e) = \frac{1}{8\sqrt{C_u C_v} \pi \sigma_u \sigma_v} \sqrt{1 - \rho_{uv}^2} \sqrt{|u_e| |v_e|} \times \exp\left(-\frac{1}{2(1 - \rho_{uv}^2)} \left(\frac{|u_e|}{C_u \sigma_u} + \frac{|v_e|}{C_v \sigma_v} - \frac{2 \rho_{uv} \sign(u_e) \sign(v_e)}{\sqrt{C_u C_v} \sigma_u \sigma_v} \right)\right)
\]

where \(\sigma_u\) and \(\sigma_v\) are the standard deviations of the total data populations (i.e. including also small and medium range excursions) at the given altitude, \(C_u\) and \(C_v\) are dimensionless, but site- and height-dependent, positive constants, and \(\rho_{uv}\) is a parameter uniquely associated with the correlation between the \(u_e\) and \(v_e\) excursions, but not equal to the correlation coefficient of the \(u\)- and \(v\) excursions as might be expected (cf. Appendix A). An explicit dependence of the altitude, \(z\), of \(\sigma_u\), \(\sigma_v\), \(C_u\) and \(C_v\) in the expressions has been omitted for notational convenience. The proposed distribution of the tails of the total population of velocity fluctuations is seen to be a joint (double-sided) Gamma distribution with shape parameter ½. The motivation for this choice of “mother” distribution is that it will turn out to satisfy the two basic observations stated in the Introduction.

As an introduction to the next step, we note that applying (strictly) monotonic transformations on the physical variables \(u_e\) and \(v_e\) will transform local extremes in the physical domain into local extremes in the transformed domain. As a direct consequence, the number of local extremes (and their position on the time-axis) is invariant with respect to (strictly) monotonic transformations. Therefore, global extremes may be estimated/analyzed in the transformed domain and subsequently transformed back to the physical domain.

With this in mind we introduce monotonic and memoryless (time independent) transformations, \(g_u(*)\) and \(g_v(*)\), defined by

\[
\tilde{u} = g_u(u_e) = \frac{\sigma_u}{C_u} \sign(u_e) \sqrt{|u_e|} ; \quad C_u > 0
\]

\[
\tilde{v} = g_v(v_e) = \frac{\sigma_v}{C_v} \sign(v_e) \sqrt{|v_e|} ; \quad C_v > 0
\]

along with their inverse transformations

\[
u_e = g_u^{-1}(\tilde{u}) = \frac{C_u}{\sigma_u} \sign(\tilde{u}) \tilde{u}^2 ; \quad C_u > 0
\]

\[
v_e = g_v^{-1}(\tilde{v}) = \frac{C_v}{\sigma_v} \sign(\tilde{v}) \tilde{v}^2 ; \quad C_v > 0
\]

\(3\) Note, that the Exponential-like behaviour, in the present case, is achieved at the expense of an acceptable distribution fit in the data population regime of small to medium excursions which, however, for an extreme investigation, associated with relatively large recurrence periods, is considered unimportant.
We note that due to the symmetry of the defined transformations, the mean value of the transformed processes is zero like the mean value of the physical processes. Formulated in terms of the joint probability density function introduced in (1), the joint probability density of the transformed variables is expressed as

$$f(\tilde{u}, \tilde{v}) = \frac{1}{J} \int f \left( \frac{C_u}{\sigma_u} \text{sign}(\tilde{u}) \tilde{u}^2, \frac{C_v}{\sigma_v} \text{sign}(\tilde{v}) \tilde{v}^2 \right),$$

where $J$ denotes the Jacobian, associated with the transformation, given by

$$J = \begin{vmatrix}
\frac{dg_u}{du_e} & \frac{dg_u}{dv_e} \\
\frac{dg_v}{du_e} & \frac{dg_v}{dv_e}
\end{vmatrix} = \begin{vmatrix}
\sqrt{\frac{\sigma_u \text{sign}(u_e)}{C_u}} \frac{1}{2\sqrt{|u_e|}} & 0 \\
0 & \sqrt{\frac{\sigma_v \text{sign}(v_e)}{C_v}} \frac{1}{2\sqrt{|v_e|}}
\end{vmatrix}
= \frac{\sigma_u \sigma_v}{4\sqrt{C_u C_v}} \text{sign}(u_e) \text{sign}(v_e) \frac{1}{\sqrt{|u_e||v_e|}}. $$

Introducing equations (7), (4) and (5) into equation (6) finally yields the following joint probability density function in the transformed variables

$$f(\tilde{u}, \tilde{v}) = \frac{1}{2\pi \sigma_u \sigma_v \sqrt{1 - \rho_{uv}^2}} \exp \left( -\frac{1}{2(1 - \rho_{uv}^2)} \left( \frac{\tilde{u}^2}{\sigma_u^2} + \frac{\tilde{v}^2}{\sigma_v^2} - 2\rho_{uv} \tilde{u} \tilde{v} \right) \right),$$

which is recognized as a 2D joint Gaussian PDF. Based on the previous considerations, the strategy is now to perform the extreme value analysis in the transformed domain, where the joint PDF between the transformed variables is known to be Gaussian and thus especially simple.

Before proceeding, we will simplify the problem further by presuming the longitudinal- and transversal turbulence components in a given spatial point to be uncorrelated. This assumption can be supported both theoretically and experimentally.

Neglecting the effect of Coriolis forces (caused by the rotation of the Earth) on the atmospheric boundary a symmetry argument will provide the theoretical basis for the assumed uncorrelated character/structure of the turbulence longitudinal- and transversal components as observed in one point in space. The argument goes as follows: The (atmospheric) flow over a plane area displays a left-right (lateral) symmetry with respect to the mean wind direction [8]. Since change of sign of the lateral turbulence component implies change of sign in the $uv$-cross correlation, this cross correlation must equal zero to maintain the stated field symmetry property.

In addition to the above theoretical considerations, the zero $uv$-correlation is in general also supported by full-scale experimental observations of the atmospheric boundary layer. However, deviations from this assumption have been observed, but thus mainly associated with offshore conditions with diverging wave and wind directions and/or with stable atmospheric conditions. As turbulence generated extreme gusts are intimately associated with turbulence intensity, and as turbulence intensity generally are somewhat smaller for the stable atmospheric regime, these exceptions are not a priori considered problematic extreme events.
For un-correlated processes with their joint PDF given by (1) it can be shown that $\rho_{uv} = 0$, and consequently that $u_e$ and $v_e$ are statistically independent (cf. Appendix A). Furthermore, it follows from (8) that also the introduced transformed variables are statistically independent under these circumstances. The latter result can also be obtained from a general result in [7] stating that if two random variables are independent, then the variables obtained by arbitrary individual transformations of each of those will again be statistically independent. This result may serve as a rudimentary check of the derivations leading to (8).

Introducing $\rho_{uv} = 0$ in (8), the joint PDF in the transformed variables is reduced to

$$f(\tilde{u}, \tilde{v}) = \frac{1}{2\pi \sigma_u \sigma_v} \exp \left( -\frac{1}{2} \left( \frac{\tilde{u}^2}{\sigma_u^2} + \frac{\tilde{v}^2}{\sigma_v^2} \right) \right) = \tilde{f}_u(\tilde{u}) \tilde{f}_v(\tilde{v}),$$  \hfill (9)

with one dimensional Gaussian distributions in the transformed variables introduced as

$$\tilde{f}_u(\tilde{u}) = \frac{1}{\sqrt{2\pi \sigma_u}} \exp \left( -\frac{\tilde{u}^2}{2\sigma_u^2} \right),$$  \hfill (10)

and

$$\tilde{f}_v(\tilde{v}) = \frac{1}{\sqrt{2\pi \sigma_v}} \exp \left( -\frac{\tilde{v}^2}{2\sigma_v^2} \right).$$  \hfill (11)

Note, that the introduced 1D Gaussian distributions of the transformed variables have the variances, $\sigma_u^2$ and $\sigma_v^2$, of the respective total physical $u$ and $v$ data populations.

With the joint PDF in the transformed variables reduced as specified in equation (9), we now turn to the problem of extreme identification. As $u_e$ and $v_e$, as well as their transformed analogous, are statistically independent, it follows directly that the extreme identification of these can be performed separately. We note the similarity between the two transformed random variables, and condense each of these into one normalized Gaussian distributed random variable $\eta$ with unit variance

$$f_\eta(\eta) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\eta^2}{2} \right),$$  \hfill (12)

by defining

$$\tilde{u} = \sigma_u \eta,$$  \hfill (13)

and

$$\tilde{v} = \sigma_v \eta.$$  \hfill (14)

With these transformations, evaluation of the distribution of the global extreme (the largest among the local extremes) of each of the transformed random variables is thus basically reduced to evaluation of the distribution of the global extreme associated with the “parent distribution” given by (12).

Cartwright/Longuet-Higgins have treated this problem [13]. However, a direct application of the approach proposed by Cartwright/Longuet-Higgins is not possible. Basically, the Cartwright/Longuet-Higgins approach contains two steps – the first step being determination of the statistics of local extremes of a Gaussian process, and the second step is determination of the statistics of the largest of these local extremes within a given time horizon (recurrence period). We will follow the same two-step philosophy, however, with necessary adjustments of the second step to comply with the character of the present problem.
2.1. Distribution of the local extremes

The first step, dealing with establishing the statistics of the local extremes, has been solved by Rice [14]. Denoting local extremes by \( h_e \), the PDF of these may be expressed as

\[
f_\eta(h_e;\delta) = \frac{1}{\sqrt{2\pi}} \left[ \delta e^{-\frac{\eta^2}{2\delta^2}} + \eta e^{1-\delta^2} e^{-\frac{\eta^2}{2}} g(\eta_e,\delta) \right],
\]

with

\[
g(\eta_e,\delta) = \left[ \frac{\pi}{2^2} \right] \left[ 1 + \text{sign}(\eta_e) \text{Erf} \left( \frac{\eta_e \sqrt{1-\delta^2}}{\sqrt{2}\delta} \right) \right].
\]

where \( \text{Erf}(\ast) \) denotes the Error-function (which is only defined for positive or zero arguments), and \( \delta \) is the process bandwidth parameter. The bandwidth parameter expresses the relative width of the frequency spectrum of the process \( h(t) \) and can be expressed in terms of the process spectral moments, \( m_i \), as

\[
\delta = \sqrt{\frac{m_0 m_4 - m_2^2}{m_0 m_4}},
\]

where the i'th spectral moment for a single-sided power spectra \( S(f) \), with \( f \) being the frequency in Hz, is defined as

\[
m_i = \int_0^\infty df S(f) f^i.
\]

The distribution defined in (15) and (16) is denoted the Rice distribution in recognition of the pioneering work of Rice.

2.2. Distribution of the global extreme

Having determined the distribution of the individual local extremes, it remains to derive from those the distribution of the largest of these. As is traditionally done, we will assume the local extremes to be statistically independent. Based on this presumption, the resulting PDF of the largest extreme, \( h_{em} \), among \( N \) (independent) local extremes may be derived from (15) as

\[
f_{\max\eta}(\eta_{em};N,\delta) = \frac{d}{d\eta_{em}} \left[ 1 - \int_{\eta_{em}}^{\infty} d\eta_e f_\eta(\eta_e;\delta) \right]^{N-1} f_\eta(\eta_{em};\delta).
\]

We are aiming at an asymptotic expression for the PDF of the largest excursion associated with relatively large recurrence periods. We therefore consistently presume that \( \eta_{em} \) is large, and in this case the following asymptotic expression for the integral in (19) applies (for \( 0 \leq \delta < 1 \)) [13]

\[
\int_{\eta_{em}}^{\infty} d\eta_e f_\eta(\eta_e;\delta) \approx \sqrt{1-\delta^2} e^{-\frac{1}{2}\eta_{em}^2}.
\]
Introducing (20) into equation (19) yields the following asymptotic expression for the PDF of the largest excursion

\[ f_{\text{max}}(\eta_{\text{Em}}; N, \delta) = N \sqrt{1-\delta^2} \eta_{\text{Em}} \left[ 1 - \sqrt{1-\delta^2} e^{-\frac{1}{2} \eta_{\text{Em}}^2} \right]^{N-1} e^{-\frac{1}{2} \eta_{\text{Em}}^2}. \]  

(21)

The derived expression for the PDF of the largest excursion (21) can be further simplified if the number of local maxima, belonging to the extreme data segment, can be assumed large (i.e. if the recurrence time is large). This simplification is based on an asymptotic expansion of (21) for large values of the considered number of local extremes, \( N \). The asymptotic simplification takes advantage of the reformulation

\[ \left[ 1 - \sqrt{1-\delta^2} e^{-\frac{1}{2} \eta_{\text{Em}}^2} \right]^{N-1} = \text{Exp} \left\{ \ln \left[ 1 - \sqrt{1-\delta^2} e^{-\frac{1}{2} \eta_{\text{Em}}^2} \right]^{N-1} \right\}, \]

(22)

which, for large \( \eta_{\text{Em}} \), has the following asymptotic property

\[ \left[ 1 - \sqrt{1-\delta^2} e^{-\frac{1}{2} \eta_{\text{Em}}^2} \right]^{N-1} \rightarrow \text{Exp} \left[ N \sqrt{1-\delta^2} e^{-\frac{1}{2} \eta_{\text{Em}}^2} \right] \text{ for } N \rightarrow \infty. \]

(23)

Introducing (23) into (21) then yields the requested asymptotic expansion of the PDF of the largest excursion as

\[ f_{\text{max}}(\eta_{\text{Em}}; N, \delta) = N \sqrt{1-\delta^2} \eta_{\text{Em}} \text{Exp} \left[ -N \sqrt{1-\delta^2} e^{-\frac{1}{2} \eta_{\text{Em}}^2} \right] e^{-\frac{1}{2} \eta_{\text{Em}}^2}, \]

(24)

or by introducing the identity

\[ N \sqrt{1-\delta^2} e^{-\frac{1}{2} \eta_{\text{Em}}^2} = e^{\ln \left( N \sqrt{1-\delta^2} \right) - \frac{1}{2} \eta_{\text{Em}}^2} = e^{-\frac{1}{2} \eta_{\text{Em}}^2 + \ln \left( N \sqrt{1-\delta^2} \right)} \]

(25)

as

\[ f_{\text{max}}(\eta_{\text{Em}}; N, \delta) = \eta_{\text{Em}} \text{Exp} \left[ -e^{-\frac{1}{2} \eta_{\text{Em}}^2 + \ln \left( N \sqrt{1-\delta^2} \right)} \right] e^{-\frac{1}{2} \eta_{\text{Em}}^2 + \ln \left( N \sqrt{1-\delta^2} \right)}. \]

(26)

Thus, the PDF of the largest excursion is given by relation (21) which, for a large number of local extremes in the selected extreme data segment, degenerates to the expression given in (26). Note, that (26) resembles some of the functional characteristics of a Gumbel EV1 distribution, which we return to in the next Subsection.

2.3. Synthesis

Having obtained the PDF for the largest excursion from the mean value in the normalised transformed domain, as expressed in either (21) or (26), the final task is to transform this result to the physical domain. The first step is to recognize the statistical independence of the transformed variables as expressed in equation (9) and combine this result with equations (13), (14), and (26) to give the joint PDF of the largest excursions in the transformed domain.
\[
\tilde{f}_{\text{max}}(\bar{u}_m, \bar{v}_m; N_u, N_v, \delta_u, \delta_v) = \frac{\bar{u}_m}{\sigma_u} \exp \left[ -e^{-\frac{1}{2} \left( \frac{\bar{u}_m}{\sigma_u} \right)^2 + \ln\left(N_u \sqrt{1-\delta_u^2}\right)} \right] \times \frac{\bar{v}_m}{\sigma_v} \exp \left[ -e^{-\frac{1}{2} \left( \frac{\bar{v}_m}{\sigma_v} \right)^2 + \ln\left(N_v \sqrt{1-\delta_v^2}\right)} \right],
\]

(27)

with lower indices “\(u\)” and “\(v\)” indicating parameters associated with the \(u\)-variables and \(v\)-variables, respectively.

Second, and last, step is to derive the joint PDF for the largest excursions in the physical domain from (27) combined with transformations (4) and (5)

\[
f_{\text{max}}(u_{em}, v_{em}; N_u, N_v, \delta_u, \delta_v) = \frac{1}{|J'|} \tilde{f}_{\text{max}} \left( \frac{\sigma_u}{C_u} \right) \frac{\sigma_v}{C_v} \left( \frac{\bar{u}_m}{\sigma_u} \right) \frac{\bar{v}_m}{\sigma_v} \left( \sqrt{\bar{u}_m \sigma_u}, \sqrt{\bar{v}_m \sigma_v} \right), \quad (28)
\]

where \(J'\) denotes the Jacobian, associated with the relevant transformation, given by

\[
J' = \begin{vmatrix}
\frac{dg_{u,1}}{du} & \frac{dg_{u,1}}{dv} & 2\bar{u} & 0 \\
\frac{dg_{v,1}}{du} & \frac{dg_{v,1}}{dv} & 0 & 2\bar{v} \\
\frac{dg_{u,2}}{du} & \frac{dg_{u,2}}{dv} & 0 & 0 \\
\frac{dg_{v,2}}{du} & \frac{dg_{v,2}}{dv} & 0 & 0
\end{vmatrix} = 4 \frac{C_u C_v}{\sigma_u \sigma_v} \frac{\bar{u}_m}{\sigma_u} \frac{\bar{v}_m}{\sigma_v} \left( \sqrt{\bar{u}_m \sigma_u}, \sqrt{\bar{v}_m \sigma_v} \right)
\]

Thus, the following joint probability density function for the largest excursions in the physical domain is obtained

\[
f_{\text{max}}(u_{em}, v_{em}; N_u, N_v, \delta_u, \delta_v) = \frac{1}{2C_u \sigma_u} \left( \frac{\bar{u}_m}{\sigma_u} \right) \exp \left[ -e^{\frac{-1}{2 C_u \sigma_u} \left( \bar{u}_m \right)^2 + \ln\left(N_u \sqrt{1-\delta_u^2}\right)} \right] \times \frac{1}{2C_v \sigma_v} \left( \frac{\bar{v}_m}{\sigma_v} \right) \exp \left[ -e^{\frac{-1}{2 C_v \sigma_v} \left( \bar{v}_m \right)^2 + \ln\left(N_v \sqrt{1-\delta_v^2}\right)} \right].
\]

(30)

The asymptotic PDF, expressed in equation (30), is seen to be of the joint EV1 type in agreement with the requirements for the model stated in the Introduction. Since we are dealing with global maxima of zero mean processes, it is obvious that their sign is always positive, and the joint PDF of these is thus finally formulated as
In the present context we will define the requested (joint) extreme event from the extreme value distribution (31) as the most likely event; i.e. as associated with the mode(s), \( m[u_{em}] \) and \( m[v_{em}] \), of the joint distribution (31). The requested modes are determined from the equations

\[
\frac{\partial f_{\text{max}}(u_{em}, v_{em} ; N_u, N_v, \delta_u, \delta_v)}{\partial u_{em}} \bigg|_{u_{em} = m[u_{em}]} = 0 ,
\]

and

\[
\frac{\partial f_{\text{max}}(u_{em}, v_{em} ; N_u, N_v, \delta_u, \delta_v)}{\partial v_{em}} \bigg|_{v_{em} = m[v_{em}]} = 0 .
\]

The solution to equations (32) and (33) are

\[
m[u_{em}] = 2C_u\sigma_u \ln\left(N_u\sqrt{1 - \delta_u^2}\right) ,
\]

and

\[
m[v_{em}] = 2C_v\sigma_v \ln\left(N_v\sqrt{1 - \delta_v^2}\right) .
\]

Up to this point in the analysis of the extremes, the \( u_e \) and \( v_e \) processes have been “decoupled” due to the statistical independence of these. However, dealing with synchronized extreme \( u_e \) and \( v_e \) extreme events naturally requires a link between these to be established. This link is provided by the number of events. Thus, the number of synchronous events is equal in the expressions for the requested modes in equations (34) and (35). The number of these events is extensively treated in Appendix B.

The proposed asymptotic model is based on presumptions of large magnitude extremes as well as of a large number of local (joint) extremes; cf. Subsection 2.2. This is a delicate balance because a joint event of the treated type is a relatively rare event, especially if only joint events in the large excursion regime are counted, which is definitely the most logical choice with the present setout. For limited recurrence periods it is, however, suspected that it might be advantageous to count every occurring joint event, even though the restriction to events in the large excursion regime only then is violated. Therefore both cases are treated in Appendix B.

As for the expected number of joint events in the large excursion regime, within the time span \( T \), Appendix B gives the following expression (cf. (B.13))

\[
N = T \frac{m_{u_2}}{m_{u_0}} \frac{m_{v_2}}{m_{v_0}} \exp\left(\frac{k_u}{2C_u}\right) \exp\left(\frac{k_v}{2C_v}\right) ,
\]

where the constants, \( k_u \) and \( k_v \), defines the large excursion regime measured in terms of standard deviations of the total physical processes. Motivated by the investigation in [15], the extreme segment
of excursions is, in the present context, defined as defined as excursions larger than two times the respective standard deviations. With this choice, equation (36) simplify to
\[
N = T \sqrt{\frac{m_{u2}}{m_{u0}}} \sqrt{\frac{m_{v2}}{m_{v0}}} \exp \left( -\frac{1}{C_u} - \frac{1}{C_v} \right). 
\] (37)

As for the expected number of all joint events, within the time span \(T\), Appendix B gives the following expression (cf. (B.6))
\[
N = T \sqrt{\frac{m_{u4}}{m_{u2}^2 m_{v2}}}. 
\] (38)

Combining equations (34), (35), (17), and (37), we obtain the following joint extreme estimates, adopting the counting number associated with joint large excursion events only:
\[
m[u_{em}] = 2C_u \sigma_u \ln \left( T \sqrt{\frac{m_{u2}}{m_{u0}^2 m_{u4}}} \sqrt{\frac{m_{v2}}{m_{v0}}} \exp \left( -\frac{1}{C_u} - \frac{1}{C_v} \right) \right), 
\] (39)

and
\[
m[v_{em}] = 2C_v \sigma_v \ln \left( T \sqrt{\frac{m_{v2}}{m_{v0}^2 m_{v4}}} \sqrt{\frac{m_{u2}}{m_{u0}}} \exp \left( -\frac{1}{C_u} - \frac{1}{C_v} \right) \right). 
\] (40)

Combining equations (34), (35), (17), and (38), we obtain the following joint extreme estimates, adopting the counting number associated with all joint events only:
\[
m[u_{em}] = 2C_u \sigma_u \ln \left( T \sqrt{\frac{m_{u4} m_{u2}}{m_{u0}^2 m_{u4} m_{u2}}} \right), 
\] (41)

and
\[
m[v_{em}] = 2C_v \sigma_v \ln \left( T \sqrt{\frac{m_{v4} m_{v2}}{m_{v0}^2 m_{v4} m_{v2}}} \right). 
\] (42)

2.5. Parameter estimation

The resulting estimates of the investigated joint extreme event, as expressed in equations (39) – (42), are seen to depend only on certain spectral moments, the standard deviation of the driving processes, and two transformation constants, \(C_u\) and \(C_v\). Based on analysis of a huge number of full-scale wind speed data it has been demonstrated that, within certain generic terrain types, the \(C_u\)-dependence with the altitude is approximately the same [15]. In these analyses, the extreme segment of excursions was defined as excursions from the mean wind speed larger than two times the standard deviation, which is consistent with the present choice of \(k_u = k_v = 2\).

The above observation eases considerably the application of the proposed method for these terrain types, as only standard deviation and spectral moments then has to be determined directly from actual site measurements. Three generic terrain types have been investigated: offshore/coastal, flat homogeneous terrain and hilly scrub terrain. In the present context we further assume \(C_u\) and \(C_v\) to be identical (which will have to be verified at a later stage), and we thus suggest the following relationships
\[
C_u(z) = C_v(z) = az + b, 
\] (43)
with the $z$ denoting the height above terrain, and the values of $a$ and $b$ given in Table 1.

**Table 1** Estimated parameters defining $C_u$ and $C_v$ for various terrain types.

| Terrain type          | $a$      | $b$      |
|-----------------------|----------|----------|
| Offshore/coastal      | 0.0013   | 0.3026   |
| Flat, homogenous      | 0.0003   | 0.3011   |
| Hilly, scrub          | 0.0009   | 0.3581   |

In case time series data are not available for a particular site, the relevant spectral moments may have to be determined from generic wind spectra as specified in codes, thus leaving only the turbulence standard deviations to be assessed from site measurements. An often used reference for wind speed spectra is the empirical (single-sided) Kaimal spectrum [3] expressed by

$$S(f) = 4\sigma^2 \frac{L}{U} \left(1 + 6f_c \frac{L}{U}\right)^{-5/3},$$

(44)

where the frequency, $f_c$, are given in Hz, $\sigma$ is the standard deviation associated with the fundamental physical process, $\bar{U}$ is the mean wind speed, and $L$ is the integral scale parameter. With this spectral formulation, the requested second and forth order spectral moments can be expressed as [10]

$$m_2 = \int_0^\infty df \ f^2 \ S(f)$$

$$= \sigma^2 \frac{\bar{U}^2}{54L^3} \left[ \frac{3}{4} \left(1 + 6f_c \frac{L}{\bar{U}}\right)^4 - 6 \left(1 + 6f_c \frac{L}{\bar{U}}\right)^3 - \frac{2}{3} \left(1 + 6f_c \frac{L}{\bar{U}}\right)^3 + \frac{27}{4} \right],$$

(45)

and

$$m_4 = \int_0^{f_c} df \ f^4 S(f)$$

$$= \sigma^2 \frac{\bar{U}^4}{1944L^4} \left[ \frac{3}{10} \left(1 + 6f_c \frac{L}{\bar{U}}\right)^{10/3} - \frac{12}{7} \left(1 + 6f_c \frac{L}{\bar{U}}\right)^{7/3} + \frac{9}{2} \left(1 + 6f_c \frac{L}{\bar{U}}\right)^{4/3} \right]$$

$$+ \sigma^2 \frac{\bar{U}^4}{1944L^4} \left[ -12 \left(1 + 6f_c \frac{L}{\bar{U}}\right)^{1/3} - \frac{3}{2} \left(1 + 6f_c \frac{L}{\bar{U}}\right)^{-2/3} + \frac{729}{70} \right],$$

(46)

where $f_c$ denotes the cut-off frequency of the monitoring filter function.

### 3. Results

Predictions from the proposed model are compared with results from a direct extreme value analysis based on measured full-scale wind speed data reported in [16]. Emphasis has been on theoretical predictions based on the absolute minimum of input parameters only, meaning that generic wind spectra, as described in the previous section, have been applied. Thus, only site turbulence intensity is provided by the measurements.

4 The generic form of the spectrum can be used for both longitudinal- and transversal turbulence components, and therefore lower indices “$u$” and “$v$” are omitted.
As previously mentioned, gust events associated with synchronous wind speed- and wind direction changes are rare events. Even though a very large dataset from the Norwegian Skipheia site has been used, the experimental investigation in [16] suffers from the fact that very few fully synchronized wind speed- and wind direction gust events have been identified (i.e. only 65 synchronized extreme events out of a total population of N_t=43671 recordings). As a consequence, it has been necessary to pool data from the entire mean wind speed range (i.e. between 5m/s and 24m/s) in the experimental investigation, whereas the proposed theoretical model estimates joint extreme events conditioned on the mean wind speed. This discrepancy complicates a direct comparison. Referring to a recurrence period of 50 years, the result of the experimental analysis is an estimated most likely combined event with wind speed gust magnitude 14.2m/s and wind direction gust magnitude 27º.

Because the focus here is on large recurrence periods, the theoretical predictions are based on the estimates given in equations (39) and (40). The turbulence length scales entering the spectral moments as approximated by (45) and (46) are taken as specified in the IEC code [3]. The specified turbulence length scales, corresponding to the present measuring conditions (i.e. a measuring altitude equal to 101m), equals 340m and 113m. As for the involved cut-off frequency we have applied the Nyquist frequency associated with the sampling frequency. We have investigated three different recurrence periods – 1 month, 1 year, and 50 years. The associated model estimates for five different mean wind regimes are given in Table 2 and Table 3, respectively.

| Mean wind speed | 1 month | 1 year | 50 years |
|-----------------|---------|--------|----------|
| 8 m/s           | 1.0 m/s | 2.1 m/s| 3.8 m/s  |
| 12 m/s          | 2.1 m/s | 3.7 m/s| 6.2 m/s  |
| 16 m/s          | 3.3 m/s | 5.4 m/s| 8.8 m/s  |
| 20 m/s          | 4.6 m/s | 7.3 m/s| 11.5 m/s |
| 24 m/s          | 6.0 m/s | 9.2 m/s| 14.3 m/s |

Table 3: Estimated wind direction gust amplitudes for recurrence periods 1 month, 1 year, and 50 years.

| Mean wind speed | 1 month | 1 year | 50 years |
|-----------------|---------|--------|----------|
| 8 m/s           | 5.2º    | 9.5º   | 14.5º    |
| 12 m/s          | 6.7º    | 10.7º  | 15.3º    |
| 16 m/s          | 7.7º    | 11.4º  | 15.9º    |
| 20 m/s          | 8.5º    | 12.0º  | 16.3º    |
| 24 m/s          | 9.0º    | 12.5º  | 16.6º    |

The direction results, \( \theta_m \), in Table 2 are related to the turbulence gust estimates as

\[
\theta_m = \arctan\left(\frac{m[v_{em}]}{\overline{U} + m[u_{em}]}\right),
\]
present compound gust event is substantially less than the probability of the simpler isolated wind speed gust event.

As mentioned, the results in Tables 1 and 2 can not be directly compared to the experimental results in [16] because of data pooling from the whole available mean wind speed range. However, the resulting experimental estimates are expected to be somewhat dominated by the events associated with high mean wind speeds, because the turbulence standard deviation is roughly proportional to the mean wind speed. If we compare the model predictions associated with mean wind speeds in the high end of the experimental mean wind speed regime, say 20 m/s to 24 m/s, the predicted wind speed gust component is of the same magnitude as the experimentally determined one (14.2 m/s) for a 50 year recurrence period. This in contrary to the situation for the wind direction gust component, where we predicts only roughly 60% of the experimentally determined one. Note, however, in this respect that especially determination of the direction gust component in [16] is encumbered with uncertainty, because it is based on an assumed linear dependence between wind speed gust magnitude and wind direction gust magnitude, which in turn is encumbered with some scatter.

Finally, we note that both the model prediction of the direction gust and the direction gust estimated in [16] are well below the direction gust values specified in the IEC code [3]. Although the present approach is limited to gust events driven by turbulence, and thus consequently exclude e.g. gust contributions from frontal passages, we believe that the present results together with the experimental analysis performed in [16] indicate that the IEC specification of the extreme wind direction might be overestimated.

4. Conclusions

We have presented a closed form asymptotic expression for the joint distribution of the largest combined wind speed- and wind direction excursion, during a large but otherwise arbitrary recurrence period. The derived asymptotic distribution is shown to equal a joint Gumbel EV1 type distribution. The model is asymptotic in the sense that it assumes both large excursions and a large number of those. Especially the presumed large number of events is a strong requirement due to the relatively sparse occurrence the joint events in focus, which in turn most likely means that relatively large recurrence periods are required to satisfy the introduced asymptotic simplifications. However, as a 20 year design life time traditionally is adopted for wind turbine structures, it is believed that a sufficient number of such events will occur within the life time of a wind turbine structure, such that the asymptotic presumptions can be considered to be met in this case.

The model requires only a few, easy accessible, input parameters, and accounts approximately for variation of the magnitude of large excursions with height above terrain. The model is conditioned on the wind speed standard deviation and thus implicitly on the mean wind speed.

Besides offering site-specific estimates, e.g. associated with application of the “gust generator” modeling tool [1], [2], the model is also considered suitable for use in rational calibration of magnitudes of the ECD extreme event as specified in the IEC design code [3]. The result of the present investigation, together with another recent experimentally based result [16], might indicate that the wind direction specification in the IEC code [3] for the ECD extreme case is too conservative.

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5 For gust events defined by extreme excursions of just a single turbulence variable (as e.g. described in [9]) that occur more frequently than the present joint event, the resulting constraint on the size of the recurrence period is naturally relatively less restrictive.
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Appendix A

The present Appendix demonstrates that the large excursions $u_e$ and $v_e$ in (1) are uncorrelated if and only if the distribution parameter $\rho_{uv}$ is zero. The distribution parameter $\rho_{uv}$ being zero implies directly that the $u_e$ and $v_e$ excursions are statistically independent as is seen from (1).

By definition the cross correlation between $u_e$ and $v_e$, $E[u_e, v_e]$, is given by

$$E[u_e, v_e] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_e v_e f(u_e, v_e) \, du_e \, dv_e .$$  \hspace{1cm} \text{(A.1)}

Introducing the transformed variables defined by (2) and (3), as well as the joint PDF expressed in the transformed variables (8), equation (A.1) is reformulated as

$$E[u_e, v_e] = \frac{C_u C_v}{\sigma_u \sigma_v} \int \int_{-\infty}^{\infty} \text{sign}(u) \text{sign}(v) \tilde{u}^2 \tilde{v}^2 \tilde{f}(\tilde{u}, \tilde{v}) \, d\tilde{u} d\tilde{v} .$$  \hspace{1cm} \text{(A.2)}

First, it is shown that $\rho_{uv} = 0$ implies that $E[u_e, v_e] = 0$. For that purpose, we rewrite relation (A.2) as

$$E[u_e, v_e] = \frac{C_u C_v}{\sigma_u \sigma_v} \int_{0}^{\infty} \int_{0}^{\infty} \tilde{u}^2 \tilde{v}^2 \tilde{f}(\tilde{u}, \tilde{v}) \, d\tilde{u} d\tilde{v} - \frac{C_u C_v}{\sigma_u \sigma_v} \int_{0}^{\infty} \int_{0}^{\infty} \tilde{u}^2 \tilde{v}^2 \tilde{f}(\tilde{u}, \tilde{v}) \, d\tilde{u} d\tilde{v}$$  \hspace{1cm} \text{(A.3)}

For $\rho_{uv} = 0$ we note from (8) that the integral kernel $\tilde{u}^2 \tilde{v}^2 \tilde{f}(\tilde{u}, \tilde{v})$ is an even function in both $u_e$ and $v_e$. Therefore, the sign of the integral limits in (A.3) can be changed as follows

$$E[u_e, v_e] = \frac{C_u C_v}{\sigma_u \sigma_v} \int_{0}^{\infty} \int_{0}^{\infty} \tilde{u}^2 \tilde{v}^2 \tilde{f}(\tilde{u}, \tilde{v}) \, d\tilde{u} d\tilde{v} - \frac{C_u C_v}{\sigma_u \sigma_v} \int_{0}^{\infty} \int_{0}^{\infty} \tilde{u}^2 \tilde{v}^2 \tilde{f}(\tilde{u}, \tilde{v}) \, d\tilde{u} d\tilde{v}$$  \hspace{1cm} \text{(A.4)}

from which the requested implication is directly obtained

$$E[u_e, v_e] = 0 .$$  \hspace{1cm} \text{(A.5)}

Secondly and lastly, we show that if $E[u_e, v_e] = 0$ then $\rho_{uv} = 0$. This implication can be proved by direct integration of (1). However, a perhaps more elegant alternative is based on an application of Price’s theorem stating \cite{7}

$$\frac{\partial E[h(\tilde{u}, \tilde{v})]}{\partial \mu} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 h(\tilde{u}, \tilde{v})}{\partial \tilde{u} \partial \tilde{v}} \tilde{f}(\tilde{u}, \tilde{v}) \, d\tilde{u} d\tilde{v} .$$  \hspace{1cm} \text{(A.6)}

with $\tilde{f}(\tilde{u}, \tilde{v})$ being a joint Gaussian PDF, $h(\ast, \ast)$ being some arbitrary function, and

$$\mu = \rho_{uv} \sigma_u \sigma_v .$$  \hspace{1cm} \text{(A.7)}

From (A.7) and (A.6) follows directly that

$$\frac{\partial E[h(\tilde{u}, \tilde{v})]}{\partial \rho_{uv}} = \sigma_u \sigma_v \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 h(\tilde{u}, \tilde{v})}{\partial \tilde{u} \partial \tilde{v}} \tilde{f}(\tilde{u}, \tilde{v}) \, d\tilde{u} d\tilde{v} .$$  \hspace{1cm} \text{(A.8)}
We now define

$$E[u_e,v_e] = E[h(\tilde{u},\tilde{v})],$$

(A.9)

whereby, using (A.1), we obtain the following definition of $h(\tilde{u},\tilde{v})$

$$h(\tilde{u},\tilde{v}) = \frac{C_u C_v}{\sigma_u \sigma_v} \text{sign}(\tilde{u})\text{sign}(\tilde{v})\tilde{u}\tilde{v}^2.$$

(A.10)

Introducing (A.9) and (A.10) into (A.8) yields

$$\frac{\partial E[u_e,v_e]}{\partial \rho_{uv}} = 4C_u C_v \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{u} \tilde{v} f(\tilde{u},\tilde{v})d\tilde{u}d\tilde{v} = 4C_u C_v E[\tilde{u}\tilde{v}].$$

(A.11)

The expected value of the absolute value of the product of two joint Gaussian distributed random variables is a standard result from mathematical statistics and may be formulated as [7]

$$|E[u_e,v_e]| = \frac{2\sigma_u \sigma_v}{\pi} (\cos \alpha + \alpha \sin \alpha).$$

(A.12)

with

$$\rho_{uv} = \sin \alpha ; \quad -\pi/2 < \alpha \leq \pi/2.$$  

(A.13)

Introducing (A.12) into (A.11) yields the following relation

$$\frac{\partial E[u_e,v_e]}{\partial \rho_{uv}} = \frac{8}{\pi} C_u C_v \sigma_u \sigma_v (\cos \alpha + \alpha \sin \alpha).$$

(A.14)

Note from (A.14) that $\frac{\partial E[u_e,v_e]}{\partial \rho_{uv}}$ is an even function of $\alpha$. We have previously shown that $\rho_{uv} = 0$ implies that $E[u_e,v_e] = 0$. Using the this result, combined with (A.13) and (A.14) yields

$$E[u_e,v_e] = E[u_e,v_e]_{\rho_{uv}=0} + \int_0^{\rho_{uv}} \frac{\partial E[u_e,v_e]}{\partial \rho_{uv}} d\rho_{uv}$$

$$= \frac{8}{\pi} C_u C_v \sigma_u \sigma_v \int_0^{\text{Arc}\sin \rho_{uv}} (\cos^2 \alpha + \alpha \sin \alpha \cos \alpha) d\alpha.$$  

(A.15)

Integrating by parts, the integral in (A.15) is finally reduced to

$$E[u_e,v_e] = \frac{2}{\pi} C_u C_v \sigma_u \sigma_v \left[ 3\rho_{uv}\sqrt{1-\rho_{uv}^2} + (1 + 2\rho_{uv}^2)\text{Arc}\sin \rho_{uv} \right].$$

(A.16)

Equation (A.16) expresses the relationship between the $(u_e,v_e)$ cross correlation and the distribution parameter $\rho_{uv}$. What remains now is to prove that $E[u_e,v_e]$ is a strictly monotonic function of $\rho_{uv}$ within its definition interval $\rho_{uv} \in [-1;1]$. For this purpose we derive the gradient of $E[u_e,v_e]$ with respect to $\rho_{uv}$. Based on equation (A.16) we find

$$\frac{\partial E[u_e,v_e]}{\partial \rho_{uv}} = \frac{2}{\pi} C_u C_v \sigma_u \sigma_v \left[ \frac{5-4\rho_{uv}^2}{\sqrt{1-\rho_{uv}^2}} + 4\rho_{uv} \text{Arc}\sin \rho_{uv} \right],$$

(A.17)

whereby
\[
\frac{\partial E[u_e,v_e]}{\partial \rho_{uv}} = 0
\]  
(A.18)

which, since neither of \(\sigma_u, \sigma_v, C_u\) and \(C_v\) is zero for relevant cases, implies that

\[
\frac{5 - 4 \rho_{uv}^2}{\sqrt{1 - \rho_{uv}^2}} = -4 \rho_{uv} \arcsin \rho_{uv}. 
\]  
(A.19)

Equation (A.19) has no solutions, which is seen as follows: For \(\rho_{uv} \in ]0;1[ \lor \rho_{uv} \in ]-1;0[\) the left hand side of (A.19) is positive, whereas the right hand side is negative. Further, \(\rho_{uv} = 0\) is surely not a root. As a consequence, the continuous expression for the gradient must have a constant sign, whereby it is shown that \(E[u_e,v_e]\) is a strictly monotonic function of \(\rho_{uv}\).
Appendix B
This Appendix addresses the number of joint (i.e., simultaneously occurring) maxima of two stationary stochastic processes, \( \hat{u}(t) \) and \( \hat{v}(t) \), within a specified span of time \( T \). The parameter \( t \) is the time coordinate.

We now consider the joint PDF of the six stochastic processes \( \hat{u}(t) \), \( \hat{v}(t) \), \( \hat{u}(t) \), \( \hat{v}(t) \), \( \hat{v}(t) \), and \( \hat{v}(t) \), where an “upper dot” symbolizes differentiation with respect to the time coordinate. We denote this PDF by \( f_{\hat{u},\hat{v}}(\hat{u},\hat{v}) \). The conditions for \( \hat{u}(t) \) having a maximum within the infinitesimal time interval \([t';t'+dt']\), and \( \hat{v}(t) \) having a maximum within the infinitesimal time interval \([t^*;t^*+dt^*] \) is that the respective gradients, \( \hat{u}(t) \) and \( \hat{v}(t) \), change sign (from positive to negative) within the specified intervals. This is achieved if the following four conditions are met: 1) \( \hat{u}(t') < 0 \); 2) \( \hat{v}(t^*) < 0 \); 3) \( 0 \leq \hat{u}(t') \leq \hat{u}(t') dt' \); and 4) \( 0 \leq \hat{v}(t^*) \leq \hat{v}(t^*) dt^* \). The probability, \( P(t',t^*) \), for occurrence of arbitrary maxima of the two processes, within the specified infinitesimal intervals, is thus

\[
P(t',t^*) = \int_{t=-\infty}^{t=0} \int_{t'=-\infty}^{t'=0} \int_{u=-\infty}^{u=\infty} \int_{v=-\infty}^{v=\infty} d\hat{u} d\hat{v} d\hat{u} d\hat{v} f_{\hat{u},\hat{v}}(\hat{u},\hat{v})
\approx \int_{t=-\infty}^{t=0} \int_{t'=-\infty}^{t'=0} \int_{u=-\infty}^{u=\infty} \int_{v=-\infty}^{v=\infty} d\hat{u} d\hat{v} d\hat{u} d\hat{v} \int_{t'=t}^{t'=t+dt'} d\hat{t} \int_{t^*=t}^{t^*=t+dt^*} d\hat{t}^* f_{\hat{u},\hat{v}}(\hat{u},\hat{v})
\]
(B.1)

where the indicated, however very good, approximation arises from the fact that that the gradient, \( \hat{u}(t) \), is not necessarily zero at \( t = t' \) but rather “somewhere” in the infinitesimal range \([t';t'+dt']\) and analogues for \( \hat{v}(t) \).

The expected number of (arbitrary) maxima, \( N \), of the two processes during a time span, \( T \), is consequently determined from (B.1) as

\[
N = T^{2} \int_{t=-\infty}^{t=0} \int_{t'=-\infty}^{t'=0} \int_{u=-\infty}^{u=\infty} \int_{v=-\infty}^{v=\infty} d\hat{u} d\hat{v} d\hat{u} d\hat{v} \int_{t'=t}^{t'=t+dt'} d\hat{t} \int_{t^*=t}^{t^*=t+dt^*} d\hat{t}^* f_{\hat{u},\hat{v}}(\hat{u},\hat{v})
\]
(B.2)

If we now restrict the search to arbitrary size maxima occurring simultaneously, the expected number of such events, \( N_0 \), is derived from (B.1) with the constraint that \( t' = t^* \) as

\[
N = T^{2} \int_{t=-\infty}^{t=0} \int_{u=-\infty}^{u=\infty} \int_{v=-\infty}^{v=\infty} d\hat{u} d\hat{v} \int_{t'=t}^{t'=t+dt'} d\hat{t} \int_{t^*=t}^{t^*=t+dt^*} d\hat{t}^* f_{\hat{u},\hat{v}}(\hat{u},\hat{v})
\]
(B.3)

where \( \delta(\cdot) \) denotes the Dirac function.
In line with the considerations in Section 2, we now restrict the investigation to Gaussian processes. For Gaussian processes, the derivatives are known to be Gaussian also, and consequently becomes a 6 variate Gaussian PDF. A further simplification is achieved by restricting to statistically independent joint Gaussian distributions of the form given by equation (9) in Section 2. In this case \( f_i(\dot{u}, \ddot{u}, \tilde{u}, \dot{v}, \ddot{v}, \tilde{v}) \) reduces to a joint Gaussian distribution with the co-variance matrix, \( R \), given by

\[
R = \begin{pmatrix}
0 & 0 & m_{u2} & 0 & 0 & 0 \\
0 & -m_{v2} & 0 & 0 & 0 & 0 \\
m_{u2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
m_{v2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]  

with \( m_{ui} \) and \( m_{vi} \) being the spectral moments associated with the \( \dot{u}(t) \)- and the \( \dot{v}(t) \) process, respectively. For a definition of spectral moments we refer to equation (18) in Section 2. The block matrix structure of \( R \) reflects the presumed statistical independence between \( (\dot{u}, \ddot{u}, \tilde{u}) \) and \( (\dot{v}, \ddot{v}, \tilde{v}) \), and the two diagonal blocks are obtained directly from Rice’s analysis [14] of distribution of maxima in a single Gaussian process.

Utilizing the introduced simplifications, equation (B.3) reduces to

\[
N = T \int_{-\infty}^{0} \int_{-\infty}^{0} \int_{-\infty}^{0} d\tilde{u} d\tilde{v} d\tilde{u} d\tilde{v} f_i(\ddot{u}, 0, \ddot{u}, 0, \ddot{v}, 0) \\
= T \int_{-\infty}^{0} \int_{-\infty}^{0} d\tilde{u} d\tilde{u} f_{u0}(\tilde{u}, 0, \ddot{u}) \times \int_{-\infty}^{0} \int_{-\infty}^{0} d\tilde{v} d\tilde{v} f_{v0}(\ddot{v}, 0, \ddot{v}),
\]

where \( f_{u0}(\tilde{u}, 0, \ddot{u}) \) and \( f_{v0}(\ddot{v}, 0, \ddot{v}) \) denote three variate joint Gaussian PDF’s with covariance matrices equal to the upper and lower diagonal blocks in (B.4), respectively. The two double integrals in (B.5) is of the type considered by Rice for a single stochastic process [14], and using this result equation (B.5) is finally expressed as

\[
N = T \frac{m_{u4}m_{v4}}{m_{u2}m_{v2}}.
\]

With the approach leading to (B.6) all local maxima are in principle accounted for no matter their magnitude. However, since the present investigation focuses on the extreme segment of excursions (cf. Section 2), only excursions belonging to this segment should logically contribute to the number of (large) local extremes, \( N_e \), appearing in equation (19). Therefore, only \( \ddot{u} \)- excursions above a certain threshold, \( \tilde{U}_0 \), and only \( \ddot{v} \)- excursions above a certain threshold, \( \tilde{V}_0 \), should in principle be accounted for. Accounting for only large maxima exceeding the two thresholds equation (B.5) is modified to

\[
N_e = T \int_{-\infty}^{0} \int_{-\infty}^{0} d\tilde{u} d\tilde{u} f_{u0}(\ddot{u}, 0, \ddot{u}) \times \int_{-\infty}^{0} \int_{-\infty}^{0} d\tilde{v} d\tilde{v} f_{v0}(\ddot{v}, 0, \ddot{v}).
\]
In the reduction of (B.7), the decomposition in two statistically independent terms again prove useful, as Rice [14] offers an asymptotic approximation to the two involved integral kernels when both $\bar{U}_0$ and $\bar{V}_0$ are large. Retaining only the important terms we obtain

$$
N_e = T \int_{\bar{u}=\bar{U}_0}^\infty du \int_{\bar{v}=\bar{V}_0}^\infty dv \frac{m_u}{m_{u_0}} \frac{\bar{u}^2}{2m_{u_0}} \text{Exp}\left(-\frac{\bar{u}^2}{2m_{u_0}}\right) \times \frac{m_v}{m_{v_0}} \frac{\bar{v}^2}{2m_{v_0}} \text{Exp}\left(-\frac{\bar{v}^2}{2m_{v_0}}\right)
$$

$$
= T \sqrt{\frac{m_{u_0}}{m_u}} \sqrt{\frac{m_{v_0}}{m_v}} \text{Exp}\left(-\frac{\bar{U}_0^2}{2m_{u_0}}\right) \text{Exp}\left(-\frac{\bar{V}_0^2}{2m_{v_0}}\right). 
$$

(B.8)

Definition of the extreme data segments will, for a given site, depend on the degree of agreement between the measured data and a Gaussian fit. Let the excursion magnitudes (in the physical domain), where the deviations between data material and the Gaussian description “start” (cf. Figure 1), be denoted as $U_0$ and $V_0$, respectively. Normalizing these magnitudes with the respective standard deviations, we have

$$
k_u = \frac{U_0}{\sigma_u}, 
$$

(B.9)

and

$$
k_v = \frac{V_0}{\sigma_v}. 
$$

(B.10)

To facilitate application of the derived formalism, the thresholds defined in the physical data domain have to transformed to the corresponding/associated thresholds in the transformed data domain, $\bar{U}_0$ and $\bar{V}_0$. Using equations (2) and (3) we find the following relationships

$$
\bar{U}_0 = g_\alpha(U_0) = \sqrt{\frac{\sigma_u}{C_u}} U_0 = \sqrt{k_u \frac{\sigma_u}{\sqrt{C_u}}}; \quad C_u > 0 , 
$$

(B.11)

and

$$
\bar{V}_0 = g_\gamma(V_0) = \sqrt{\frac{\sigma_v}{C_v}} V_0 = \sqrt{k_v \frac{\sigma_v}{\sqrt{C_v}}}; \quad C_v > 0 . 
$$

(B.12)

Finally, introducing (B.11) and (B.12) into (B.8), we have the following expression for the expected number of synchronous maxima above the respective extreme segment thresholds

$$
N_e = T \sqrt{\frac{m_{u_0}}{m_u}} \sqrt{\frac{m_{v_0}}{m_v}} \text{Exp}\left(-\frac{k_u}{2C_u}\right) \text{Exp}\left(-\frac{k_v}{2C_v}\right). 
$$

(B.13)

Note, that relation (B.13) is derived under the assumption of a large threshold, and therefore $N_e$, as defined by (B.13), does not degenerate to the derived expression for the number of all zero’s (B.6) for the threshold put equal to zero.