SPACES OF MAPS INTO TOPOLOGICAL GROUP WITH THE
WHITNEY TOPOLOGY

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Abstract. Let $X$ be a locally compact Polish space and $G$ a non-discrete
Polish ANR group. By $C(X, G)$, we denote the topological group of all con-
tinuous maps $f : X \to G$ endowed with the Whitney (graph) topology and
by $C_c(X, G)$ the subgroup consisting of all maps with compact support. It
is known that if $X$ is compact and non-discrete then the space $C(X, G)$ is
an $l_2$-manifold. In this article we show that if $X$ is non-compact and not
end-discrete then $C_c(X, G)$ is an $(\mathbb{R}^\infty \times l_2)$-manifold, and moreover the pair
$(C(X, G), C_c(X, G))$ is locally homeomorphic to the pair of the box and the
small box powers of $l_2$.

1. Introduction

This paper is one of studies on local or global topological types of infi nite-
dimensional topological groups, which appear as function spaces over non-compact
spaces ([1, 2, 3, 4, 6]). For spaces $X$ and $Y$ let $C(X, Y)$ denote the space of all
continuous maps $f : X \to Y$ endowed with the Whitney (graph) topology. When
$X$ is compact, this topology coincides with the compact-open topology. In [16] the
third author showed that if $X$ is a non-discrete compact metrizable space and
$Y$ is a Polish ANR without isolated points, then the space $C(X, Y)$ is an $l_2$-manifold, i.e.,
a paracompact Hausdorff space which is locally homeomorphic to $(\approx_{t_2})$ the separa-
brable Hilbert space $l_2$. Here and below, a Polish space means a separable completely
metrizable space.

Suppose $X$ is a paracompact space and $G$ is a Hausdorff topological group with
the unit element $e$. Then, the space $C(X, G)$ is a topological group under the
pointwise multiplication. Let $C_c(X, G)$ denote the subgroup of $C(X, G)$ consisting
of maps $f : X \to G$ with compact support. Here, the support of $f \in C(X, G)$
is defined by $\text{supp}(f) = \text{cl}_X \{x \in X : f(x) \neq e\}$. When $X$ is a non-discrete
compact metrizable space and $G$ is a non-discrete Polish ANR group, from [10] or
the famous Dobrowolski- Toruńczyk’s theorem on $l_2$-manifold topological groups, it
follows that $C(X, G)$ is an $l_2$-manifold. In this paper we study topological types of
the spaces $C(X, G)$ and $C_c(X, G)$ in the case where $X$ is non-compact. The group
structure on these spaces is a key ingredient to our arguments.

A Frechet space is a completely metrizable locally convex topological linear space.
An LF-space is the direct limit of increasing sequence of Frechet spaces in the
category of (locally convex) topological linear spaces. In [13] it is shown that a

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manifold, function space, the Whitney (graph) topology, the box product, the small box product, end-discrete.

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separable LF-space is homeomorphic to (≈) either $l_2$, $\mathbb{R}^\infty$ or $\mathbb{R}^\infty \times l_2$, where $\mathbb{R}^\infty$ is the direct limit of the tower

$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots.$$ 

The spaces $\mathbb{R}^\infty$ and $\mathbb{R}^\infty \times l_2$ are homeomorphic to the countable small box powers $\square^n \mathbb{R}$ and $\square^n l_2$, respectively. The latter spaces are subspaces of the box powers $\square^n \mathbb{R}$ and $\square^n l_2$. See Section 2 for the definition of the (small) box powers (or products). For simplicity, the pair $(\square^n l_2, \square^n l_2)$ is denoted by $(\square, \square) l_2$. We say that a space $X$ is end-discrete if $X \setminus K$ is discrete for some compact subset $K$ of $X$. The following are the main results of this paper.

**Theorem 1.1.** Let $X$ be a non-compact locally compact Polish space and $G$ a non-discrete Polish AR (ANR) group.

1. If $X$ is not end-discrete or $G$ is not locally compact, then $$(C(X, G), C_c(X, G)) \approx_{(t)} (\square, \square) l_2.$$

2. Suppose $G$ is locally compact.
   (i) If $X$ is end-discrete and non-discrete, then $$(C(X, G), C_c(X, G)) \approx_{(t)} l_2 \times (\square, \square) \mathbb{R}.$$ 
   (ii) If $X$ is discrete (i.e., $X \approx N$), then $$(C(X, G), C_c(X, G)) \approx_{(t)} (\square, \square) \mathbb{R}.$$ 

**Theorem 1.2.** Let $X$ be a non-compact locally compact Polish space and $G$ a Hausdorff topological group.

1. If $G$ is (locally) contractible, then $C_c(X, G)$ is (locally) contractible.

2. Suppose $G$ is a non-discrete Polish ANR group.
   (i) If $X$ is non-discrete or $G$ is not locally compact, then the space $C_c(X, G)$ is an $(\mathbb{R}^\infty \times l_2)$-manifold. In addition, if $G$ is an AR, then $C_c(X, G) \approx \mathbb{R}^\infty \times l_2$.
   (ii) If $X$ is discrete and $G$ is locally compact, then the space $C_c(X, G)$ is an $\mathbb{R}^\infty$-manifold. In addition, if $G$ is an AR, then $C_c(X, G) \approx \mathbb{R}^\infty$.

2. The box topology and topological groups

In this preliminary section we recall basic facts on the box topology and topological groups (cf. [3] Section 2), [12, 17]). The box product $\square_{i \in N} X_i$ of a sequence of topological spaces $(X_i)_{i \in N}$ is the countable product $\prod_{i \in N} X_i$ endowed with the box topology. This topology is generated by the base consisting of boxes $\prod_{i \in N} U_i$, where $U_i$ is an open subset in $X_i$. The small box product $\square_{i \in N} X_i$ of a sequence of pointed spaces $(X_i, x_i)_{i \in N}$ is the subspace of $\square_{i \in N} X_i$ defined by

$$\square_{i \in N} X_i = \{ (x_i)_{i \in N} \in \square_{i \in N} X_i : \exists n \in \mathbb{N}, \forall i \geq n, x_i = x_i \}.$$ 

For notational simplicity, the pair $(\square_{i \in N} X_i, \square_{i \in N} X_i)$ is denoted by the symbol $(\square, \square) X_i$. In case $X_i = X$ for every $i \in \mathbb{N}$, $\square_{i \in N} X_i$ and $\square_{i \in N} X_i$ are respectively denoted by $\square^n X$ and $\square^n X$ and called the box power and the small box power, and moreover $(\square, \square) X_i$ is denoted by $(\square, \square)^n X$. 


Suppose $G$ is a topological group with the unit element $e \in G$. We always choose this unit element $e$ as the distinguished point of $G$ and its subgroups. A tower of closed subgroups in $G$ is a sequence $(G_i)_{i \in \mathbb{N}}$ of closed subgroups of $G$ such that

$$G_1 \subset G_2 \subset G_3 \subset \cdots \quad \text{and} \quad G = \bigcup_{i \in \mathbb{N}} G_i.$$ 

This tower yields a box product pair $(\square, \square)_{i \in \mathbb{N}}G_i$ and the multiplication map

$$p : \square_{i \in \mathbb{N}}G_i \to G, \quad p(x_1, x_2, \ldots, x_i) = x_1 x_2 \cdots x_i.$$ 

Note that $(\square, \square)_{i \in \mathbb{N}}G_i$ is a pair of a topological group and its subgroup with the unit element $e = (e, e, \ldots)$ and that the multiplication map $p$ is continuous ([3, Lemma 2.10]).

We say that (i) a map $f : X \to Y$ is open at a point $x \in X$ if for any neighborhood $U$ of $x$ in $X$ the image $f(U)$ is a neighborhood of $f(x)$ in $Y$ and (ii) $f : X \to Y$ has a local section at $y \in Y$ if there exist a neighborhood $V$ of $y$ in $Y$ and a map $s : V \to X$ such that $fs = \text{id}_V$.

Remark 2.1. (1) The map $p$ is open iff $p$ is open at $e$.

(2) If the map $p$ has a local section at $e$, then $p$ is open.

We say that a topological group $G$ is the direct limit of the tower $(G_i)_{i \in \mathbb{N}}$ in the category of topological groups if any group homomorphism $h : G \to H$ to an arbitrary topological group $H$ is continuous whenever its restrictions $h|_{G_i}$ are continuous for all $i \in \mathbb{N}$. Note that $G$ is the direct limit of $(G_i)_{i \in \mathbb{N}}$ if and only if $G$ carries the strongest group topology inducing the original topology on each group $G_i$. This description of the direct limit topology has the following consequence (cf. [4, Proposition 2.7], [5, Theorem 2.2]):

Proposition 2.2. If the map $p : \square_{i \in \mathbb{N}}G_i \to G$ is open, then $G$ is the direct limit of $(G_i)_{i \in \mathbb{N}}$ in the category of topological groups.

We conclude this section with some remarks on the (local) contractibility of the small box products. A space $X$ is called (i) strongly locally contractible at $x \in X$ if every neighborhood $U$ of $x$ contains a neighborhood $V$ of $x$ which contracts to the point $x$ in $U$ with keeping $x$ fixed (i.e., there is a contraction $h_1 : V \to U \ (t \in [0,1])$ of $V$ with $h_1(x) = x$), and (ii) strongly contractible at $x \in X$ if $X$ contracts to $x$ in itself with keeping $x$ fixed. Note that if a topological group $G$ is (locally) contractible then $G$ is strongly (locally) contractible at every $x \in G$ ([3, Remark 2.9]).

Proposition 2.3 ([3, Proposition 2.8]). The small box product $\square_{i \in \mathbb{N}}(X_i, x_i)$ is strongly (locally) contractible at the base point $x = (x_i)_{i \in \mathbb{N}}$ if and only if each space $X_i$ is strongly (locally) contractible at the base point $x_i$.

3. Topological Group $C(X, G)$

3.1. The graph topology. For spaces $X$ and $Y$ let $C(X, Y)$ denote the space of continuous maps $f : X \to Y$ endowed with the Whitney topology (or the graph topology). For $f \in C(X, Y)$ the symbol $\Gamma_f$ stands for the graph of $f$ (i.e., $\Gamma_f = \{ (x, f(x)) \in X \times Y : x \in X \}$). The graph topology on $C(X, Y)$ is generated by the base

$$\langle U \rangle = \{ f \in C(X, Y) : \Gamma_f \subset U \} \quad (U \text{ runs over the open sets in } X \times Y).$$
For pairs of spaces \((X, A)\) and \((Y, B)\) let \(C(X, A; Y, B)\) denote the subspace of \(C(X, Y)\) defined by
\[
C(X, A; Y, B) = \{ f \in C(X, Y) : f(A) \subset B \}.
\]

For a point \(y \in Y\) let \(\tilde{y} = \tilde{g}_x \in C(X, Y)\) denote the constant map onto \(y\).

Suppose \(G\) is a topological group with the unit element \(e\). Then, the space \(C(X, G)\) has the canonical group structure defined by the pointwise multiplication
\[
(fg)(x) = f(x)g(x) \quad (f, g \in C(X, G), \ x \in X).
\]

Then, the constant map \(\hat{e}\) is the unit element of \(C(X, G)\). The inverse \(f^{-1}\) of \(f \in C(X, G)\) is defined by
\[
f^{-1}(x) = (f(x))^{-1} \quad (x \in X).
\]

The support of \(f \in C(X, G)\) is defined by \(\text{supp}(f) = \text{cl}_X \{x \in X : f(x) \neq e\}\). We obtain a subgroup
\[
C_e(X, G) = \{ f \in C(X, G) : \text{supp}(f) \text{ is compact} \}.
\]

Now we shall show that when \(X\) is paracompact, \(G = C(X, G)\) is a topological group with respect to the graph topology. First we see that the inverse operator and the left/right multiplications are always continuous.

**Lemma 3.1.** The inverse operator \(\theta : G \to G, \theta(g) = g^{-1}\) and the left/right multiplications \(L_f, R_f : G \to G, L_f(g) = fg, R_f(g) = gf\) are continuous.

**Proof.** Since \(G\) is a topological group, we have the homeomorphisms:

1. \(\Theta : X \times G \approx X \times G, \quad \Theta(x, y) = (x, y^{-1})\)
2. \(\Phi_f, \Phi^f : X \times G \approx X \times G, \quad \Phi_f(x, y) = (x, f(x)y), \quad \Phi^f(x, y) = (x, yf(x))\).

Note that \(\Theta(\Gamma_g) = \Gamma_g^{-1}, \Phi_f(\Gamma_g) = \Gamma_gf\) and \(\Phi^f(\Gamma_g) = \Gamma_gf\) for any \(g \in C(X, G)\). Therefore, if \(g \in \langle U \rangle\), then \(g^{-1} \in \langle \Theta(U) \rangle\) and \(fg \in \langle \Phi_f(U) \rangle, gf \in \langle \Phi^f(U) \rangle\). This implies the assertions.

**Proposition 3.2.** For every paracompact space \(X\), the group \(C(X, G)\) is a topological group with respect to the graph topology.

By Lemma 3.1 the continuity of the multiplication of \(C(X, G)\) reduces to its continuity at \((\hat{e}, \hat{e})\). The latter is equivalent to the assertion that each neighborhood \(\langle U \rangle\) of \(\hat{e}\) admits a smaller neighborhood \(\langle V \rangle\) of \(\hat{e}\) with \(\langle V \rangle \circ \langle U \rangle \subset \langle U \rangle\). For \(U, V \subset X \times G\), we set \(U V = \{ (x, yz) \in X \times G : (x, y) \in U, (x, z) \in V \}\). Note that \(\Gamma_{fg} = \Gamma_f \Gamma_g\) and \(\langle U \rangle \langle V \rangle \subset \langle U V \rangle\). Therefore, Proposition 3.2 follows from the next lemma.

**Lemma 3.3.** If \(X\) is paracompact, then for any neighborhood \(U\) of \(\Gamma_{\hat{e}}\) there exists a neighborhood \(V\) of \(\Gamma_{\hat{e}}\) such that \(VV \subset U\).

**Proof.** For each \(x \in X\) there exist open neighborhoods \(U_x\) of \(x \in X\) and \(W_x\) of \(e \in G\) such that \(U_x \times W_x \subset U\). Since \(X\) is paracompact, the open covering \(\{U_x\}_{x \in X}\) admits a locally finite open refinement \(\{V_x\}_{x \in X}\) with \(V_x \subset U_x\) (\(x \in X\)). Each \(y \in X\) has an open neighborhood \(O_y\) which meets at most finitely many \(V_x\)'s. Then \(\tilde{W}_y = \bigcap \{W_x : O_y \cap V_x \neq \emptyset\}\) is an open neighborhood of \(e\) in \(G\). Finally, we define
\[
V = \bigcup_{y \in X} O_y \times \tilde{W}_y.
\]
It remains to show that $\mathcal{V} \subseteq \mathcal{U}$. Given any $(z, a), (z, b) \in \mathcal{V}$. There exist $y, y', x \in X$ such that $(z, a) \in O_y \times \mathcal{W}_y$, $(z, b) \in O_{y'} \times \mathcal{W}_{y'}$ and $z \in V_x$. Since $V_x$ meets both $O_y$ and $O_{y'}$, it follows that $\mathcal{W}_y, \mathcal{W}_{y'} \subseteq V_x$ and that $(z, ab) \in U_x \times \mathcal{W}_x \subseteq \mathcal{U}$. This means that $\mathcal{V} \subseteq \mathcal{U}$. □

### 3.2. Compact case.

When $X$ is compact, the graph topology on $C(X, Y)$ coincides with the compact-open topology. In this subsection we list some basic facts on the space $C(X, Y)$ for a compact metrizable space $X$. The proof of the main theorem in [16] can be modified to prove the following relative version:

**Theorem 3.4 ([16]).** Let $X$ be a compact metrizable space with a compact subset $K \subseteq X$ and $Y$ a Polish AR (ANR) without isolated points. If $X \setminus K$ is infinite or $Y \approx_l l_2$, then $C(X, K; Y) \approx_l l_2$ for any $y \in Y$.

When $Y$ is a topological group, we have a more precise conclusion.

**Theorem 3.5 ([11] [15], [7] Corollary 1).** Let $G$ be a Polish AR (ANR) group. Then,

1. $G \approx_l l_2$ if $G$ is non-locally compact, and
2. $G \approx_r \mathbb{R}^n$ for some $n \geq 0$ if $G$ is locally compact.

In Theorem 3.3, if $G$ is an AR, then $G$ is an $l_2$-manifold in the case (1) and a Lie group in the case (2) ([7] Corollary 1]). For the AR case, note that (1) any contractible $l_2$-manifold is homeomorphic to $l_2$ itself and (2) any contractible $n$-dimensional Lie group is homeomorphic to $\mathbb{R}^n$. In fact, by Cartan-Malcev-Iwasawa’s polar decomposition theorem [15] any connected Lie group $G$ admits a factorization $G \approx K \times \mathbb{R}^n$, where $K$ is any maximal compact subgroup of $G$. If $G$ is contractible, then the closed manifold $K$ is also contractible and hence consists of a single point.

**Corollary 3.6.** Let $X$ be a compact metrizable space with a compact subset $K \subseteq X$ and $G$ a non-discrete Polish AR (ANR) group.

1. If $X \setminus K$ is infinite or $G$ is non-locally compact, then $C(X, K; G, e) \approx_l l_2$.
2. If $X \setminus K$ is finite and $G$ is locally compact, then $C(X, K; G, e) \approx_r \mathbb{R}^n$ for some $n \geq 1$.

We also need some extension/deformation lemmas for maps.

**Lemma 3.7.** Let $X$ be a compact metrizable space with $K, L \subseteq X$ disjoint compact subsets.

1. If $Y$ is an AR, then for any map $f : X \to Y$ there exist a neighborhood $U$ of $f|_K$ in $C(K, Y)$ and a map $s : U \to C(X, Y)$ such that $s(g)|_K = g$, $s(g)|_L = f|_L$ ($g \in U$) and $s(f|_K) = f$.
2. If $Y$ is an AR, then we can take $U = C(K, Y)$ in (1).

**Proof.** (1) We define a closed subset $H$ of the space $C(K, Y) \times X$ and a map $\varphi : H \to Y$ by

$$H = (C(K, Y) \times (K \cup L)) \cup \{f|_K \times X\} \quad \text{and}$$

$$\varphi(g, x) = \begin{cases} g(x) & \text{if } x \in K, \\ f(x) & \text{if } x \in L \text{ or } g = f|_K. \end{cases}$$

Since $Y$ is an ANR and $C(K, Y) \times X$ is metrizable, the map $\varphi$ extends to a map $\hat{\varphi} : \mathcal{V} \to Y$ over a neighborhood $\mathcal{V}$ of $H$ in $C(K, Y) \times X$. Using compactness of $X$,
Lemma 3.8. Suppose $X$ is a compact metrizable space and $K,L$ are disjoint compact subsets of $X$.

(1) If $Y$ is strongly locally contractible at a point $y \in Y$, then the following hold:

(i) For any neighborhood $V$ of $\tilde{y}_X$ in $C(X,Y)$ there exist a neighborhood $\mathcal{U}$ of $\tilde{y}_X$ in $C(X,Y)$ and a homotopy $s_t : \mathcal{U} \to V$ ($t \in [0,1]$) such that for each $f \in \mathcal{U}$

(a) $s_0(f) = f$, 
(b) $s_1(f) \in C(X,L;Y,y)$,
(c) $s_t(f)|_K = f|_K$ ($t \in [0,1]$),
(d) if $x \in X$ and $f(x) = y$, then $s_t(f)(x) = y$ ($t \in [0,1]$) (in particular, $s_1(\tilde{y}_X) = \tilde{y}_X$).

(ii) For any subset $F \subset X$, the subspace $C(X,F;Y,y)$ is strongly locally contractible at $\tilde{y}_X$.

(2) If $Y$ is strongly contractible at $y$, then the following hold:

(i) There exists a homotopy $s_t : C(X,Y) \to C(X,Y)$ ($t \in [0,1]$) which satisfies the consitions (1)(i)(a)- (d) for each $f \in C(X,Y)$.

(ii) For any subset $F \subset X$, the subspace $C(X,F;Y,y)$ is strongly contractible at $\tilde{y}_X$.

Proof. (1) (i) Take a map $\lambda : X \to [0,1]$ with $\lambda(K) = 0$ and $\lambda(L) = 1$. There exists a neighborhood $V$ of $y$ in $Y$ with $C(X,V) \subset V$. By the assumption, there is a neighborhood $U$ of $y$ in $V$ and a homotopy $h_t : U \to V$ ($t \in [0,1]$) such that $h_0 = \text{id}_U$, $h_1 = \tilde{y}_U$ and $h_t(y) = y$. Then $\mathcal{U} = C(X,U)$ is a neighborhood of $\tilde{y}_X$ and the required homotopy $s_t : \mathcal{U} \to V$ is defined by $s_t(f)(x) = h_{\lambda(x)}(f(x))$.

(ii) We can apply (1)(i) to $K = \emptyset$ and $L = X$ so to obtain strong local contractions $s_t : \mathcal{U} \to V$ at $\tilde{y}_X$ in $C(X,Y)$. The condition (d) implies that $s_t(\mathcal{U} \cap C(X,F;Y,y)) \subset V \cap C(X,F;Y,y)$.

(2) (i) The homotopy $s_t$ is defined in the same way as in (1)(i) by using a strong contraction $h_t$ of $Y$ at $y$.

(ii) The condition (d) assures that $s_t(C(X,F;Y,y)) \subset C(X,F;Y,y)$. \qed

4. Topological Type of $(C(X,G), C_c(X,G))$

In this section we prove Theorem 3.1. To simplify the arguments, we use the following terminology: For pairs of spaces $(X,A)$ and $(Y,B)$, (a) we set $(X,A) \times (Y,B) = (X \times Y, A \times B)$, (b) we write $(X,A) \approx_f (Y,B)$ if for each point $a \in A$ there exist an open neighborhood $U$ of $a$ in $X$ and an open subset $V$ of $Y$ which admit a homeomorphism of pairs of spaces $(U,U \cap A) \approx (V,V \cap B)$, and (c) we say that a map $p : (X,A) \to (Y,B)$ has a local section at a point $b \in B$, if there exist an open neighborhood $V$ of $b$ in $Y$ and a map of pairs $s : (V,V \cap B) \to (X,A)$ such that $ps = \text{id}_V$.

Suppose $G$ is a Hausdorff topological group and $X$ is a locally compact Polish space. Let $\mathcal{G} = C(X,G)$ and $\mathcal{G}_c = C_c(X,G)$. For subsets $K,N \subset X$, let

$\mathcal{G}_K = C(X,K;G,e)$, $\mathcal{G}(N) = \mathcal{G}_{X\setminus N}$,
$\mathcal{G}_K(N) = \mathcal{G}_K \cap \mathcal{G}(N)$ and $\mathcal{G}_c(N) = \mathcal{G}(N) \cap \mathcal{G}_c$. 

we can find a neighborhood $\mathcal{U}$ of $f|_K \in C(K,Y)$ such that $\mathcal{U} \times X \subset \mathcal{V}$. Then, the desired map $s : \mathcal{U} \to C(X,Y)$ is defined by $s(g)(x) = \tilde{\varphi}(g,x)$. 

(2) If $Y$ is an AR, then we can take $\mathcal{V} = C(K,Y) \times X$ and $\mathcal{U} = C(K,Y)$.
Every discrete family $\mathcal{L} = (L_i)_{i \in \mathbb{N}}$ of compact subsets of $X$ induces two maps

$$r_{\mathcal{L}} : (G, G_e) \to (\square_{i \in \mathbb{N}} C(L_i, G), \quad r_{\mathcal{L}}(f) = (f|_{L_i})_{i \in \mathbb{N}}, \quad \lambda_{\mathcal{L} : (\square_{i \in \mathbb{N}} G(L_i)) \to (G(L), G_e(L)), \quad \lambda_{\mathcal{L}}(f_i)_{i \in \mathbb{N}}|_{L_i} = f_i|_{L_i},$$

where $L = \bigcup_{i \in \mathbb{N}} L_i$. Note that the map $\lambda_{\mathcal{L}}$ is a homeomorphism.

Let $\mathcal{L} = (L_i)_{i \in \mathbb{N}}, \mathcal{N} = (N_i)_{i \in \mathbb{N}}$ and $\mathcal{K} = (K_i)_{i \in \mathbb{N}}$ be discrete families of compact subsets of $X$ such that $L_i \subset \text{Int} N_i (i \in \mathbb{N})$ and $X = L \cup K$, where $L = \bigcup_{i \in \mathbb{N}} L_i, K = \bigcup_{i \in \mathbb{N}} K_i$ and $N = \bigcup_{i \in \mathbb{N}} N_i$. The families $\mathcal{N}$ and $\mathcal{K}$ induces the homeomorphisms $\lambda_{\mathcal{N}} : (\square_{i \in \mathbb{N}} G(N_i)) \to (G(N), G_e(N))$ and $\lambda_{\mathcal{K}} : (\square_{i \in \mathbb{N}} G(K_i)) \to (G(K), G_e(K))$, and we obtain the map

$$\rho : (\square_{i \in \mathbb{N}} G(N_i) \times (\square_{i \in \mathbb{N}} G(K_i) \to (G, G_e), \quad \rho((g_i)_{i \in \mathbb{N}}, (h_i)_{i \in \mathbb{N}}) = \lambda_{\mathcal{N}}((g_i)_{i \in \mathbb{N}}) \cdot \lambda_{\mathcal{K}}((h_i)_{i \in \mathbb{N}}).$$

**Lemma 4.1.**

1. If $G$ is an AR (ANR), then

$$(G, G_e) \approx (\square_{i \in \mathbb{N}} C(L_i, G) \times (\square_{i \in \mathbb{N}} G(L_i).$$

2. If $G$ is (locally) contractible, then the map $\rho$ has a (local) section at $\hat{e}_X$.

**Proof.** (1) For each $i \in \mathbb{N}$, since $L_i$ and $F_i = \text{Fr}_X N_i$ are disjoint compact sets in $N_i$, by Lemma 3.7 (1), there exist an open neighborhood $\mathcal{V}_i$ of $\hat{e}_X \in C(L_i, G)$ and a map $s_i : \mathcal{V}_i \to C(N_i, F_i G, e) \approx G(N_i)$ such that $s_i(f)|_{L_i} = f (f \in \mathcal{V}_i)$ and $s_i(\hat{e}_X) = \hat{e}_X$. The maps $s_i (i \in \mathbb{N})$ determine a continuous map

$s : \square_{i \in \mathbb{N}} \mathcal{V}_i \to \square_{i \in \mathbb{N}} G(N_i), \quad s((f_i)_{i \in \mathbb{N}}) = (s_i(f_i))_{i \in \mathbb{N}}.$

The preimage $\mathcal{V} = r_{\mathcal{L}}^{-1}(\square_{i \in \mathbb{N}} \mathcal{V}_i)$ is an open neighborhood of $\hat{e}_X$ in $G$. Consider the map

$$\eta = \lambda_{\mathcal{N}} \circ s \circ r_{\mathcal{L}} : \mathcal{V} \to G(N).$$

Since $\eta(g)|_{L_i} = g|_{L_i}$, we have $\eta(g)^{-1}g \in G_L$. Hence, we obtain two maps

$$\phi : \mathcal{V} \to \square_{i \in \mathbb{N}} \mathcal{V}_i \times G_L, \quad \phi(g) = (r_{\mathcal{L}}(g), \eta(g)^{-1} \cdot g) \quad \text{and} \quad \theta : \square_{i \in \mathbb{N}} G(N_i) \times G_L \to G, \quad \theta((g_i)_{i \in \mathbb{N}}, h) = \lambda_{\mathcal{N}}((g_i)_{i \in \mathbb{N}}) \cdot h.$$

It follows that

$$\theta \circ (s \times \text{id}) \circ \phi = \text{id}_\mathcal{V}.$$

Indeed, we have

$$\theta \circ (s \times \text{id}) \circ \phi(g) = (\lambda_{\mathcal{N}} \circ s \circ r_{\mathcal{L}})(g) \cdot \eta(g)^{-1} \cdot g = g \quad (g \in \mathcal{V}).$$

Next, consider the map $\psi_0 = \theta \circ (s \times \text{id}) : \square_{i \in \mathbb{N}} \mathcal{V}_i \times G_L \to G$. For any $((f_i)_{i \in \mathbb{N}}, h) \in \square_{i \in \mathbb{N}} \mathcal{V}_i \times G_L$, we have

$$\psi_0((f_i)_{i \in \mathbb{N}}, h)|_{L_i} = \theta((s_i(f_i))_{i \in \mathbb{N}}, h)|_{L_i} = \lambda_{\mathcal{N}}((s_i(f_i))_{i \in \mathbb{N}})|_{L_i} \cdot h|_{L_i},$$

$$= \lambda_{\mathcal{N}}((s_i(f_i))_{i \in \mathbb{N}})|_{L_i} = f_i \in \mathcal{V}_i.$$

This means that $(r_{\mathcal{L}} \circ \psi_0)((f_i)_{i \in \mathbb{N}}, h) = (f_i)_{i \in \mathbb{N}}$ and $\psi_0((f_i)_{i \in \mathbb{N}}, h) \in r_{\mathcal{L}}^{-1}(\square_{i \in \mathbb{N}} \mathcal{V}_i) = \mathcal{V}$. Thus, we obtain the map

$$\psi = \theta \circ (s \times \text{id}) : \square_{i \in \mathbb{N}} \mathcal{V}_i \times G_L \to \mathcal{V}.$$

Now we shall show that the maps $\phi$ and $\psi$ are reciprocal homeomorphisms. It remains to show that $\phi \circ \psi = \text{id}$. For any $((f_i)_{i \in \mathbb{N}}, h) \in \square_{i \in \mathbb{N}} \mathcal{V}_i \times G_L$, if we put $g = \psi((f_i)_{i \in \mathbb{N}}, h)$, then we have (a) $r_{\mathcal{L}}(g) = (f_i)_{i \in \mathbb{N}}$, (b) $\eta(g) = (\lambda_{\mathcal{N}} \circ s \circ r_{\mathcal{L}})(g) = (\lambda_{\mathcal{N}} \circ s)((f_i)_{i \in \mathbb{N}})$ and (c) $g = \psi((f_i)_{i \in \mathbb{N}}, h) = \theta(s((f_i)_{i \in \mathbb{N}}), h) = (\lambda_{\mathcal{N}} \circ s)((f_i)_{i \in \mathbb{N}}) \cdot h$. Therefore, it follows that $(\phi \circ \psi)((f_i)_{i \in \mathbb{N}}, h) = \phi(g) = (r_{\mathcal{L}}(g), \eta(g)^{-1}g) = ((f_i)_{i \in \mathbb{N}}, h).$
Thus, we have \( (V, V \cap G_c) \approx (\emptyset, \emptyset) \) and \( (\emptyset, \emptyset) \approx (G_L, G_{L,c}) \).

Since \( X = L \cup K \), we have \( G_L \subset G(K) \). Thus, \( \lambda_K \) restricts to the homeomorphism
\[
\lambda_K : (\emptyset, \emptyset) \ni G_L(K_i) \approx (G_L, G_{L,c}).
\]

Thus, we have \( (V, V \cap G_c) \approx (\emptyset, \emptyset) \) and \( (\emptyset, \emptyset) \approx (G_L, G_{L,c}) \).

If \( G \) is an AR, then we can take \( V_i = C(L_i,G) \) and \( V = G \). Then, we have
\[
(G, G_c) \approx (\emptyset, \emptyset) \ni C(L_i, G) \times (\emptyset, \emptyset) \ni G_L(K_i).
\]

(2) The argument proceeds along the same line as the first half part of (1). The family \( \mathcal{N} \) induces the map
\[
r_{\mathcal{N}} : G \to \square_{i \in \mathbb{N}} C(N_i, G), \quad r_{\mathcal{N}}(f) = (f|_{N_i})_{i \in \mathbb{N}}.
\]

By Lemma \( \text{3.8} \) (1), for each \( i \in \mathbb{N} \), there exist an open neighborhood \( V_i \) of \( \hat{e}_X \) in \( C(N_i, G) \) and a map \( s_i^{\prime} : V_i \to C(N_i, F_i ; G,c) \approx (G(N_i)) \) such that \( s_i^{\prime}(f)|_{L_i} = f|_{L_i} \) for \( f \in V_i \) and \( s_i^{\prime}(\hat{e}_N) = \hat{e}_N \). The maps \( (s_i^{\prime})_{i \in \mathbb{N}} \) determine a continuous map
\[
s' : \square_{i \in \mathbb{N}} V_i \to \square_{i \in \mathbb{N}} G(N_i), \quad s'((f_i)_{i \in \mathbb{N}}) = (s_i^{\prime}(f_i))_{i \in \mathbb{N}}.
\]

As before, \( V' = r_{\mathcal{N}}^2(\square_{i \in \mathbb{N}} V_i) \) is an open neighborhood of \( \hat{e}_X \) in \( G \) and we have maps
\[
\eta' = \lambda_{\mathcal{N}} \circ s' \circ r_{\mathcal{N}} : V' \to G(N) \quad \text{and} \quad \phi' : V' \to \square_{i \in \mathbb{N}} V_i \times G_L. \quad \phi'(g) = (r_{\mathcal{N}}(g), \eta'(g)^{-1}g).
\]

Note that the map \( \rho \) has the factorization
\[
\rho : \square_{i \in \mathbb{N}} G(N_i) \times \square_{i \in \mathbb{N}} G_L(K_i) \approx \square_{i \in \mathbb{N}} G(N_i) \times G_L \to G.
\]

Again, it is seen that \( \theta \circ (s' \times \text{id}) \circ \phi' = \text{id}_{V'} \), and hence the map \( \theta \) has a local section at \( \hat{e}_X : (s' \times \text{id})\phi' : (V', V' \cap G_c) \to (\emptyset, \emptyset) \ni G(N_i) \times (G_L, G_{L,c}). \)

Thus, the map \( \rho \) also has a local section at \( \hat{e}_X \).

If \( G \) is contractible, then we can take \( V_i^c = C(N_i, G) \) and \( V' = G \). Hence, \( \rho \) has a global section. This completes the proof. \( \square \)

A subset \( A \) of \( M \) is regular closed if \( A = \text{cl}_M(\text{int}_M A) \). Since \( X \) is locally compact and \( \sigma \)-compact, there exists a sequence \( (X_i)_{i \in \mathbb{N}} \) of compact regular closed subsets of \( X \) such that \( X_i \subset \text{int}_X X_{i+1} \) ( \( i \in \mathbb{N} \) ) and \( X = \bigcup_{i \in \mathbb{N}} X_i \). It induces the tower \( (G(X_i))_{i \in \mathbb{N}} \) of closed subgroups of \( G_c \) and the multiplication map
\[
p : \square_{i \in \mathbb{N}} G(X_i) \to G_c, \quad p(h_1, \ldots, h_n) = h_1 \cdots h_n.
\]

Let \( L_i = X_i \setminus \text{int}_X X_{i-1} \) for each \( i \in \mathbb{N} \), where \( X_0 = \emptyset \). Since each \( X_i \) is regular closed, \( L_i \) is also regular closed. We call the sequence \( (X_i, L_i)_{i \in \mathbb{N}} \) an exhausting sequence for \( X \).

**Proposition 4.2.** Suppose \((X_i, L_i)_{i \in \mathbb{N}}\) is an exhausting sequence for \( X \).

1. If \( G \) is an AR (ANR), then
\[
(G, G_c) \approx (\emptyset, \emptyset) \ni C(L_{2i}, G) \times (\emptyset, \emptyset) \ni G(L_{2i-1}).
\]

2. If \( G \) is (locally) contractible, then
   (i) the map \( p : \square_{i \in \mathbb{N}} G(X_i) \to G_c \) has a (local) section \( s \) at \( \hat{e}_X \) such that
   \[
s(\hat{e}_X) = (\hat{e}_X, \hat{e}_X, \ldots),
\]
Since $G_X \geq n$ assumption, we can find a compact open subset $\square$ portion 4.2 (2) the map $\rho : (\square, \square) \in \mathcal{G}(N_2i) \times (\square, \square) \in \mathcal{G}(L_{2i-1}) \rightarrow (\mathcal{G}, \mathcal{G}_c)$.

Proof of Theorem 1.2. By Lemma 4.1 (2) the map

$$\rho : (\square, \square) \in \mathcal{G}(N_2i) \times (\square, \square) \in \mathcal{G}(L_{2i-1}) \rightarrow (\mathcal{G}, \mathcal{G}_c)$$

has a local section $S : (\mathcal{V}, \mathcal{V} \cap \mathcal{G}_c) \rightarrow (\square, \square) \in \mathcal{G}(N_2i) \times (\square, \square) \in \mathcal{G}(L_{2i-1})$ at $\hat{e}_X$ such that

$$S(\hat{e}_X) = ((\hat{e}_X, \hat{e}_X, \ldots), (\hat{e}_X, \hat{e}_X, \ldots))$$.

Therefore, the required local section of the map $s : \mathcal{V} \rightarrow \square \in \mathcal{G}(X_i) \rightarrow \mathcal{G}_c$ is defined by

$$s : \mathcal{V} \rightarrow \square \in \mathcal{G}(X_i), \quad s(h) = (\hat{e}_X, \hat{e}_X, f_1, g_1, f_2, g_2, \ldots).$$

If $G$ is contractible, then we can take $\mathcal{V} = \mathcal{G}$ and $\mathcal{W} = \mathcal{G}_c$.

(ii) From (i) and Remark 4.1, it follows that $p$ is open at $e = (\hat{e}_X, \hat{e}_X, \ldots)$. Thus, the assertion follows from Proposition 2.2. This completes the proof.

Proof of Theorem 1.1. By Proposition 1.2(1) it only remains to determine the topological type of the spaces $C(L_{2i}, G)$ and $\mathcal{G}(L_{2i-1}) \approx C(L_{2i-1}, \text{Fr}_X L_{2i-1}; G, e)$ for a suitable exhausting sequence $(X_i, L_i)_{i \in \mathbb{N}}$ for $X$.

(1) If $G$ is non-locally compact, we choose any exhausting sequence $(X_i, L_i)_{i \in \mathbb{N}}$. If $X$ is not end-discrete, we can find an exhausting sequence $(X_i, L_i)_{i \in \mathbb{N}}$ such that $\text{Int}_X L_i$ is infinite for each $i \geq 1$. In each case we have $C(L_{2i}, G) \approx (\rho_{(i)}) l_2$ and $\mathcal{G}(L_{2i-1}) \approx (\rho_{(i)}) l_2 (i \geq 1)$ by Corollary 3.6(1).

(2) Since $G$ is locally contractible, it follows that $G$ is a Lie group of dimension $n \geq 1$ (so that $G \approx \mathbb{R}^n$) and that $G \approx \mathbb{R}^n$ if $G$ is contractible. (i) By the assumption, we can find a compact open subset $X_1$ of $X$ such that $X_1$ is infinite and $X - X_1$ is discrete. Thus we have an exhausting sequence $(X_i, L_i)_{i \in \mathbb{N}}$ such that $X_i$ is infinite and (b) $L_i$ is finite for each $i \geq 1$. Then, $\mathcal{G}(L_1) \approx (\rho_{(i)}) l_2$ and $C(L_{2i}, G) \approx (\rho_{(i)}) l_2 (i \geq 1)$.

(ii) Choose an exhausting sequence $(X_i, L_i)_{i \in \mathbb{N}}$ such that each $L_i$ is a one-point set. Then, we have $\mathcal{G}(L_{2i-1}) \approx C(L_{2i}, G) \approx (\rho_{(i)}) l_2 (i \geq 1)$.

Proof of Theorem 1.2. (1) Choose any exhausting sequence $(X_i, L_i)_{i \in \mathbb{N}}$ for $X$. Since $G$ is (locally) contractible, so is each $\mathcal{G}(X_i) \approx C(X_i, \text{Fr}_X X_i; G, e)$ by Lemma 3.8. Hence, $\square_{i \in \mathbb{N}} \mathcal{G}(X_i)$ is (locally) contractible from Proposition 2.2. By Proposition 4.2(2) the map $p : \square_{i \in \mathbb{N}} \mathcal{G}(X_i) \rightarrow \mathcal{G}_c$ admits a (local) section $s$ at $\hat{e}_X$ such
that \( s(\hat{e}_X) = (\hat{e}_X, \hat{e}_X, \ldots) \). Thus, the group \( \mathcal{G}_e \) is also seen to be (locally) contractible.

(2) The paracompactness of the space \( C_c(X, G) \) follows from Proposition 4.4 below. Thus, the assertions follow from Theorem 1.1. □

Let \( V \) be a topological linear space and \( (V_i)_{i \in \mathbb{N}} \) a tower of closed linear subspaces of \( V \). Then, they are abelian groups and the direct limit of the tower \( (V_i)_{i \in \mathbb{N}} \) in the category of topological groups is also a topological linear space. Thus, we have the following as a corollary of Proposition 4.2 (2)(ii).

**Corollary 4.3.** Suppose \( X \) is a locally compact Polish space and \( (X_i)_{i \in \mathbb{N}} \) is a sequence of compact subsets of \( X \) such that \( X_i \subset \text{int}_X X_{i+1} \) (\( i \in \mathbb{N} \)) and \( X = \bigcup_{i \in \mathbb{N}} X_i \). Then, for any Fréchet space \( F \) the space \( C_c(X, F) \) is the LF-space with respect to the tower of Fréchet spaces \( (C(X_i, \text{Fr}_X X_i; F, 0))_{i \in \mathbb{N}} \).

Let \((Y, *)\) be a pointed space. Similarly to the above, the support of \( f \in C(X, Y) \) is defined by \( \text{supp}(f) = \text{cl}_X \{x \in X : f(x) \neq *\} \) and we define a space \( C_c(X, Y) \) as follows:

\[ C_c(X, Y) = \{ f \in C(X, G) : \text{supp}(f) \text{ is compact} \}. \]

A space is said to be **perfectly paracompact** if it is paracompact and each open subset is of type \( F_{\sigma} \). It should be noticed that any subspace of a perfectly paracompact space is perfectly paracompact (cf. [10, Theorem 5.1.28], [9, Chapter V, II, Section 2, 2.5]) From [8, Section 4] it follows that the small box product of metrizable spaces is perfectly paracompact. Since any LF-space is homeomorphic to a small box product of Fréchet spaces [14], it follows that every LF-space is perfectly paracompact. Hence, Corollary 4.3 has the following consequence.

**Proposition 4.4.** Suppose \( X \) is a locally compact Polish space and \((Y, *)\) is a pointed metrizable space. Then, \( C_c(X, Y) \) is perfectly paracompact.

**Proof.** Let \((N, 0)\) be a Banach space which contains \((Y, *)\) as a subspace. By Corollary 4.3 the space \( C_c(X, N) \) is an LF-space, and hence it is perfectly paracompact. Since \( C_c(X, Y) \) is a subspace of \( C_c(X, N) \), we have the conclusion. □

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