Higher-dimensional attractors with absolutely continuous invariant probability

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Abstract
Consider a dynamical system $T : \mathbb{T}^n \times \mathbb{R}^d \to \mathbb{T}^n \times \mathbb{R}^d$ given by $T(x, y) = (E(x), C(y) + f(x))$, where $E$ is a linear expanding map of $\mathbb{T}^n$, $C$ is a linear contracting map of $\mathbb{R}^d$ and $f$ is in $C^2(\mathbb{T}^n, \mathbb{R}^d)$. We provide sufficient conditions for $E$ that imply the existence of an open set $\mathcal{U}$ of pairs $(C, f)$ for which the corresponding dynamic $T$ admits a unique absolutely continuous invariant probability.

A geometrical characteristic of transversality between self-intersections of images of $\mathbb{T}^n \times \{0\}$ is present in the dynamic of the maps in $\mathcal{U}$. In addition, we give a condition between $E$ and $C$ under which it is possible to perturb $f$ to obtain a pair $(C, \tilde{f})$ in $\mathcal{U}$.

Keywords: hyperbolic endomorphism, transversality method, absolutely continuous invariant probability, higher-dimensional attractors

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1. Introduction

In the study of ergodic theory, proper hyperbolic attractors for smooth invertible maps admit only singular invariant measures with respect to volume since they have zero Lebesgue measure [3, 7]. For non-invertible maps, absolutely continuous invariant measures do exist and are usually associated with the positivity of the Lyapunov exponents (see [2, 11, 18]).

There are attractors on surfaces with one negative Lyapunov exponent that admit absolutely continuous invariant probabilities. Fat backer maps [1] and fat solenoidal attractors [5, 19] are examples of such maps.
In the present work, we study higher-dimensional attractors with $d$ negative Lyapunov exponents that admit absolutely continuous invariant measures.

We consider volume expanding skew-products $T: \mathbb{T}^n \times \mathbb{R}^d \to \mathbb{T}^n \times \mathbb{R}^d$ given by

$$T(x, y) = (E(x), C(y) + f(x))$$

where $E: \mathbb{T}^n \to \mathbb{T}^n$ is an expanding map induced by a linear map $E: \mathbb{R}^n \to \mathbb{R}^n$ that preserves the lattice $\mathbb{Z}^n$, $C = \mathbb{R}^n/\mathbb{Z}^n$, $C$ a linear contracting map of $\mathbb{R}^d$ and $f$ is a $C^\infty$ function of $\mathbb{T}^n$ into $\mathbb{R}^d$.

Considering some constant $K > 0$ for which $M = \mathbb{T}^n \times [-K, K]^d$ satisfies $T(M) \subset M$ we obtain the attractor $\Lambda = \cap_{n \geq 0} T^n(M)$. Since the restriction of $T$ to $\Lambda$ is a transitive hyperbolic endomorphism, $T$ admits a unique SRB measure $\mu_T$ supported on $\Lambda$. The goal of this work is to provide conditions that guarantee the absolute continuity of $\mu_T$ with respect to the volume measure of $\mathbb{T}^n \times \mathbb{R}^d$.

Let $E(u)$ be the set of linear expanding maps of $\mathbb{T}^n$ and $C(d)$ the set of linear contractions of $\mathbb{R}^d$. Denoting $T = T(E, C, f)$, the first result in this work is:

**Theorem A.** Given integers $u \geq d$, if $E \in E(u)$ is induced by a diagonal matrix with at most $d$ distinct eigenvalues, then there exists a nonempty open subset $U$ of $C(d) \times C^2(\mathbb{T}^n, \mathbb{R}^d)$ such that the corresponding SRB measure $\mu_T$ of every map $T = T(E, C, f)$ for $(C, f) \in U$ is absolutely continuous with respect to the volume of $\mathbb{T}^n \times \mathbb{R}^d$.

Two features of non-invertible maps are important for the proof of the theorem in order to obtain absolute continuity of the invariant measure: volume-expansion of the dynamic and transversal overlaps between the images of the subsets of the domain (see theorem 2.9).

Given $E \in E(u)$, let us consider the following subset $C(d; E)$ of $C(d)$:

$$C(d; E) = \left\{ C \in C(d), \left| \det E \right| \det C |C|^{-1} > |C|^{-2d} \text{ and } \|C\| < \frac{\|E^{-1}\|^{-1}}{|\det E|^{\frac{1}{\|E\|}}} \right\}. \quad (2)$$

The condition $|\det E| |\det C|^{-1} > |C|^{-2d}$ implies that $T$ is volume-expanding ($|\det E \det C| > 1$). When $C$ is conformal, the condition $|\det E| |\det C|^{-1} > |C|^{-2d}$ is equivalent to the volume expansion of $T$.

When $C(d; E)$ is not empty, the next theorem shows that there exist open sets of pairs $(C, f)$ such that the measure $\mu_T$ is absolutely continuous for $T = T(E, C, f)$.

**Theorem B.** Given integers $u \geq d$, if $E \in E(u)$ is such that $C(d; E) \neq \emptyset$ then there exists a nonempty open subset $U$ of $C(d) \times C^2(\mathbb{T}^n, \mathbb{R}^d)$ such that the corresponding SRB measure $\mu_T$ of every map $T = T(E, C, f)$ for $(C, f) \in U$ is absolutely continuous with respect to the volume of $\mathbb{T}^n \times \mathbb{R}^d$.

**Theorem C.** Given integers $u \geq d$, $E \in E(u)$, $C \in C(d; E)$, there exist functions $\phi_k \in C^\infty(\mathbb{T}^n, \mathbb{R}^d)$, $k = 1, 2, \ldots, s$, such that for any $f \in C^2(\mathbb{T}^n, \mathbb{R}^d)$ the set of parameters...
Let us fix some notation involving the partition of the base space $\mathbb{T}^u$ that codifies the action of the expanding map $E$.

Given integers $u$ and $d$, we consider the dynamic $T = T(E, C, f) : \mathbb{T}^u \times \mathbb{R}^d \to \mathbb{T}^u \times \mathbb{R}^d$ given by

$$T(x, y) = (E(x), C(y) + f(x)), \quad (3)$$

where $E \in E(u)$ is a map whose lift $E : \mathbb{R}^u \to \mathbb{R}^u$ is a linear map with $\|E^{-1}\|^{-1} > 1$ that preserves the lattice $\mathbb{Z}^u$, $C \in C(d)$ is a linear contracting map, that is, $\|C\| < 1$ and $f \in C^2(\mathbb{T}^u, \mathbb{R}^d)$. 

A similar idea can be applied to partial hyperbolic dynamical systems in [19, 20]. A similar idea can be used to prove the existence and finiteness of physical measures [6, 20], which is conjectured to be valid for a typical dynamical system [15].

In section 2, we provide the definitions, including the transversality condition, and the statements of this work. In section 3, we prove that the transversality condition implies the absolute continuity of the SRB measure, giving estimates on the $L^2$-regularity of this measure. In section 4, we prove that the transversality condition is generic under the assumption that $C \in C(d; E)$.
We suppose in the whole text that $T$ is volume expanding. If $|\det E \det C| < 1$ then the attractor has zero volume and supports no absolute continuous invariant measure.

Let $R = \{ R(1), \cdots, R(r) \}$ be a fixed Markov partition for $E$, that is, $R(i)$ are disjoint open sets, the interior of each $R(i)$ coincides with $R(i)$, $E|_{R(i)}$ is one-to-one, $\bigcup_{i} R(i) = \mathbb{T}^u$ and $E(R(i)) \cap R(j) \neq \emptyset$ implies that $R(j) \subset E(R(i))$. It is a well-known fact that Markov partitions always exist for expanding maps (see, for example, [16]).

Given $z \in \mathbb{T}^u$ and $0 < r < 1/2$, we denote by $B(z, r)$ the ball centered in $z$ with radius $r$.

Let us suppose that $\text{diam}(R) < \gamma$, where $0 < \gamma < 1/2$ is a constant such that for every $x \in \mathbb{T}^u$ and $y \in E^{-1}(x)$ there exists a unique affine inverse branch $g_{y,x} : B(x, \gamma) \rightarrow B(y, \gamma)$ such that

$$g_{y,x}(x) = y \quad \text{and} \quad E(g_{y,x}(z)) = z$$

for every $z \in B(x, \gamma)$ [13, chapter 11].

Consider the set $\mathcal{L} = \{ 1, \cdots, r \}$ and $\mathcal{T}$ the set of words of length $n$ with letters in $\mathcal{L}$, $1 \leq n \leq \infty$. Denoting by $a = (a_j)_{j=1}^{n}$ a word in $\mathcal{T}$, define $\mathcal{P}$ the subset of words $a = (a_j)_{j=1}^{n}$ with the property that

$$E(\mathcal{R}(a_{i+1})) \cap \mathcal{R}(a_i) \neq \emptyset \quad \text{for every} \quad 0 \leq i \leq n - 1.$$

Consider the partition $\mathcal{R}^n := \cup_{n-1}^{0} E^{-1}(\mathcal{R})$ and, for every $a \in \mathcal{P}$, the sets $\mathcal{R}(a) = \bigcap_{n-1}^{0} E^{-1}(\mathcal{R}(a_{n-1}))$ in $\mathcal{R}^n$, which are nonempty if and only if $a \in \mathcal{P}$. The truncation of $a = (a_j)_{j=1}^{n}$ to length $1 \leq p \leq n$ is denoted by $[a]_p = (a_j)_{j=1}^{p}$.

For any $x \in \mathbb{T}^u$, let us fix some $\pi(x) \in \mathcal{L}$ such that $x \in \mathcal{R}(\pi(x))$ (it is unique for almost every $x \in \mathbb{T}^u$). For any $c \in \mathcal{P}$, $1 \leq p < \infty$, we consider $\mathcal{P}^c(c)$ the set of words $a \in \mathcal{P}$ such that $E^c(\mathcal{R}(a)) \cap \mathcal{R}(c) \neq \emptyset$. Define $\mathcal{P}^c(x) := \mathcal{P}^c(\pi(x))$ and, for $a \in \mathcal{P}^c(x)$, denote by $a(x)$ the point $y \in \mathcal{R}(a)$ that satisfies $E^c(y) = x$.

For any $a \in \mathcal{P}$ and $1 \leq n < \infty$ we consider the set $\mathcal{D}(a) := \{ x \in \mathbb{T}^u | a \in \mathcal{P}^c(x) \}$, which is a union of rectangles of the Markov partition. The image of $\mathcal{R}(a) \times \{ 0 \}$ by $\mathcal{T}$ is the graph of the function $S(\cdot, a) : \mathcal{D}(a) \rightarrow \mathbb{R}^d$ given by

$$S(x, a) := \sum_{i=1}^{n} C^{-1} f(E^{-i}(a(x))) = \sum_{i=1}^{n} C^{-1} f([a]_i(x)).$$

Consider the sets $I^\infty(x) = \{ a \in I^\infty \text{ such that } [a]_i \in \mathcal{P}^c \text{ for every } i \geq 1 \}$ and $\mathcal{D}(a) := \{ x \in \mathbb{T}^u | a \in I^\infty(x) \}$ for $a \in I^\infty(x)$. For any $p \geq 1$ and $c \in \mathcal{P}$, the restriction of $S(\cdot, a)$ to each atom $\mathcal{R}(c)$ of the partition $\mathcal{R}$ is uniformly bounded in the $C^2$-topology, so it can be extended to the closure as a $C^2$ function redefining it on the border, which we denote by $S_c(\cdot, a)$.

**Remark 2.1.** Note that, if $a \in \mathcal{P}^c(x)$, then $\pi(a(x)) = a_q$ and $\pi([a]_{q-j}(x)) = \pi(E^j(a(x))) = a_{q-j} - j$, $0 \leq j \leq q - 1$.

**Remark 2.2.** Let $N := |\det E|$ be the degree of $E$, we have that $\# \mathcal{P}(c) = N^q$ for every $c \in \mathcal{P}$. Actually, for almost every $x \in \mathbb{T}^u$ and $q \geq 1$, there are exactly $N^q$ rectangles $\mathcal{R}(a)$, $a \in \mathcal{P}$ such that $E^q(\mathcal{R}(a)) \cap \mathcal{R}(\pi(x)) \neq \emptyset$, corresponding to those that contain the $N^q$ preimages of $x$.

In the first part of the work, we define the transversality condition that implies the absolute continuity of the SRB measure of the dynamic $T$; this transversality will be defined in terms of the smallest singular value of the difference between two linear maps.

Given a linear map $A : \mathbb{R}^u \rightarrow \mathbb{R}^d$, $u \geq d$, denote by
\[
m(A) := \sup_{\dim W = d} \inf_{\|v\| = 1, v \in W} \|A(v)\|
\]

the smallest singular value of \(A\). Consider the constants
\[
\underline{\mu} = \|E^{-1}\|^{-1} \quad \underline{\pi} = \|E\|
\]
\[
\underline{\lambda} = \|C^{-1}\|^{-1} \quad \underline{\lambda} = \|C\|
\]

which are the minimum and maximum rates of expansion (or contraction) of \(E\) (or \(C\)). Consider also
\[
\alpha_0 = \frac{\|f\|_{C^2}}{1 - \|C\|} \quad \theta = \underline{\lambda}^{-1}
\]

and
\[
J = |\det E| |\det C|^{-1} ||C^{-1}||^{-2d}.
\]

**Remark 2.3.** Note that if \(C\) is conformal, then \(J\) is equal to \(|\det E \det C|\), corresponding to the Jacobian of \(T\).

**Remark 2.4.** If \(C \in C(d; E)\), then \(J > 1\).

**Definition 2.5.** Given \(T = (E, C, f)\) as above, integers \(1 \leq p < \infty, c \in I^p\) and \(a, b \in I^p(c)\), we say that \(a\) and \(b\) are transversal on \(c\) if
\[
m(DS_c(x, a) - DS_c(y, b)) > 3\theta \alpha_0
\]
for every \(x, y \in \overline{R(c)}\).

Let us define a function that counts the maximum amount of pairs \(a\) and \(b\) that are not transversal on \(c\).

**Definition 2.6.** Given \(T\) as above, define the integers \(\tau(q, p)\) and \(\tau(q)\) by
\[
\tau(q, p) = \max_{c \in I^p} \max_{a \in I^p(c)} \# \{b \in I^p(c) | a \text{ is not transversal to } b \text{ on } c\}.
\]
\[
\tau(q) = \min_{p \geq 1} \tau(q, p).
\]

We say that it holds the transversality condition if for some integer \(q \in \mathbb{N}\) we have \(\tau(q) < J^q\).

**Remark 2.7.** Note that \(\tau(q, p)\) is monotone non-increasing in \(p\), so the minimum above is well defined, which implies that for every \(q \geq 1\), there exists an integer \(p_0(q)\) such that \(\tau(q) = \tau(q, p_0)\) for every \(p \geq p_0(q)\). If \(\tau(q, p) < J^q\) for some pair \((q, p)\), then \(\tau(q) \leq \tau(q, p) < J^q\) and the transversality condition are valid.

**Remark 2.8.** The transversality condition is open with respect to \((C, f) \in C(d) \times C^2(\mathbb{T}^n, \mathbb{R}^d)\) with respect to the product topology induced by the supremum norm in \(C(d)\) and by the \(C^2\)-norm in \(C^2(\mathbb{T}^n, \mathbb{R}^d)\). This is because \(J, \alpha_0\) and \(S_c(\cdot, a)\) depend continuously on \((C, f)\), and (12) is an open condition, which implies that \(\tau(q, p)\) is upper semi-continuous for fixed \(q, p\), following that the property \(\tau(q, p) < J^q\) is open with respect to \((C, f)\).
The transversality condition means that for each $q$ and $a \in I^\infty$ the number of $b$’s that are not transversal to $|a|_q$ may increase with $q$ at most at a rate smaller than $J^q$. This condition is used to estimate a regularity of the SRB measure, which will give its absolute continuity. The main step in the proof of theorem B corresponds to the following theorem.

**Theorem 2.9.** Given $(C,f) \in C(d) \times C^2(\mathbb{T}^u, \mathbb{R}^d)$ and $E \in \mathcal{E}(u)$ with $T = T(E,C,f)$ satisfying the transversality condition, there exists a neighborhood $U \subset C(d) \times C^2(\mathbb{T}^u, \mathbb{R}^d)$ of $(C,f)$ such that for every $(\tilde{C},\tilde{f}) \in U$ the corresponding SRB measure $\mu_T$ of $\tilde{T} = T(E,\tilde{C},\tilde{f})$ is absolutely continuous with respect to the volume of $\mathbb{T}^u \times \mathbb{R}^d$ and its respective density is in $L^2(\mathbb{T}^u \times \mathbb{R}^d)$.

To finish the proof of theorem B, we verify that there exists a dynamic $T = T(E,C,f)$ with $C \in C(d)$ that satisfies the transversality condition.

**Proposition 2.10.** Given $E \in \mathcal{E}(u)$ such that $C(d;E) \neq 0$, there exists a pair $(\tilde{C},\tilde{f}) \in C(d;E) \times C^2(\mathbb{T}^u, \mathbb{R}^d)$ such that $T = T(E,C,f)$ satisfies the transversality condition.

In the final part of the work, we deal with families of perturbations that give a dynamic satisfying the transversality condition.

Given $f_0 \in C^2(\mathbb{T}^u, \mathbb{R}^d)$ and functions $\phi_1, \cdots, \phi_s \in C^\infty(\mathbb{T}^u, \mathbb{R}^d)$, for any $s$-uple of parameters $t = (t_1, \cdots, t_s)$, we will consider the corresponding function $f_t(x) = f_0(x) + \sum_{k=1}^s t_k \phi_k(x)$ and the corresponding dynamic $T_t = T(E,C,f_t)$.

For fixed $E, C$, define the set

$$\mathcal{T} = \left\{ f \in C^2(\mathbb{T}^u, \mathbb{R}^d), \limsup_{q \to \infty} \frac{\log \tau(q)}{q} \geq \log J \right\}. \quad (15)$$

**Remark 2.11.** If $f$ is not in $\mathcal{T}$, then the dynamic $T = T(E,C,f)$ satisfies the transversality condition and there exists some integer $q_0$ such that $\tau(q) < J^q$ for every $q \geq q_0$. The condition of $f \notin \mathcal{T}$ is a stronger transversality condition.

In section 4, we will describe the construction of a family of functions $\phi_1, \cdots, \phi_s$ with the property that for every $f_0 \in C^2(\mathbb{T}^u, \mathbb{R}^d)$ the finite dimensional subspace $H := \text{span} \{ \phi_1, \cdots, \phi_s \} \subset C^2(\mathbb{T}^u, \mathbb{R}^d)$ satisfies the condition $f \notin \mathcal{T}$ which is valid for almost every $f \in f_0 + H$.

**Theorem 2.12.** Given $u \geq d$, $E \in \mathcal{E}(u)$ and $C \in C(d;E)$, there exist functions $\phi_k \in C^\infty(\mathbb{T}^u, \mathbb{R}^d)$, $1 \leq k \leq s$, such that for every $f_0 \in C^2(\mathbb{T}^u, \mathbb{R}^d)$ the set of parameters $t \in \mathbb{R}^d$ for which $f_t \in \mathcal{T}$ has zero Lebesgue measure.

### 3. Absolute continuity of the SRB measure

In this section, we will present some tools that will be used to prove theorem 2.9.

#### 3.1. The semi-norm

In order to obtain the absolute continuity of the measure $\mu_T$, we will use a semi-norm that measures the regularity of the measure related to the $L^2$-norm of the density function; this semi-norm is similar to that considered in [19].

Let us denote $m_d$ as the usual Lebesgue measure on $\mathbb{R}^d$, $m$ the normalized Lebesgue measure on $\mathbb{T}^u$, $\pi_1 : \mathbb{T}^u \times \mathbb{R}^d \to \mathbb{T}^u$ the projection into the first coordinate and for every
measure $\nu$ in $\mathbb{T}^n \times \mathbb{R}^d$ its push-forward $(\pi_1)_* \nu$ is the measure in $\mathbb{T}^n$ given by the relation $(\pi_1)_* \nu(E) := \nu(\pi_1^{-1}(E))$ for every $E \subset \mathbb{T}^n$.

**Definition 3.1.** Given finite measures $\mu_1$ and $\mu_2$ on $\mathbb{R}^d$ and $r > 0$, we define the bilinear form

$$\langle \mu_1, \mu_2 \rangle_r = \int_{\mathbb{R}^d} \mu_1(B(x,r)) \mu_2(B(x,r)) \, dm_d(x),$$

(16)

where $B(z,r) \subset \mathbb{R}^d$ is the ball of radius $r$ centered in $z$. The norm $\|\mu\|_r$ of the finite measure $\mu$ is defined by

$$\|\mu\|_r = \sqrt{\langle \mu, \mu \rangle_r}.$$

The value of $\|\mu\|_r$ when $r$ tends to 0 is related to the $L^2$-norm of the density of $\mu$, as the following lemma states.

**Lemma 3.2.** There exists a constant $C_d$ such that if a finite measure $\mu$ on $\mathbb{R}^d$ satisfies

$$\liminf_{r \to 0^+} \frac{\|x\|_r}{r} < \infty,$$

(17)

then $\mu$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, its density $\frac{d\mu}{dm_d}$ is in $L^2(\mathbb{R}^d)$ and satisfies $\|\frac{d\mu}{dm_d}\|_{L^2(\mathbb{R}^d)} \leq C_d \liminf_{r \to 0^+} \frac{\|\mu\|_r}{r^d}$.

**Proof.** Let $C_d$ be the constant such that $m_d(B(z,r)) = C_d^{-1} r^d$ for every $z \in \mathbb{R}^d$ and every $r > 0$.

Define the function $J_r(z) = \frac{\mu(B(z,r))}{m_d(B(z,r))}$ and note that $\|J_r\|_{L^2(\mathbb{R}^d)} = C_d \frac{\|\mu\|_r}{r^d}$. Then the condition $\liminf_{r \to 0^+} \frac{\|\mu\|_r}{r^d} < \infty$ implies that there exists a uniformly bounded subsequence of $J_r$ in $L^2(\mathbb{R}^d)$ with $r \to 0^+$. Thus we can consider a subsequence $r_n$ such that $J_{r_n}$ converges weakly to some $J_\infty \in L^2$, following that

$$\int_{\mathbb{R}^d} \phi J_\infty \, dm_d = \lim_{n \to \infty} \int_{\mathbb{R}^d} \phi J_{r_n} \, dm_d = \int_{\mathbb{R}^d} \phi \, d\mu$$

for every continuous function $\phi$ with compact support. So we have $\frac{d\mu}{dm_d}(z) = J_\infty(z)$ and

$$\frac{d\mu}{dm_d} \|_{L^2(\mathbb{R}^d)} \leq C_d \liminf_{r \to 0^+} \frac{\|\mu\|_r}{r^d}.$$

Given any finite measure $\mu$ on $\mathbb{T}^n \times \mathbb{R}^d$ satisfying $(\pi_1)_* \mu = m$, we consider $\mathcal{P}$ the partition of $\mathbb{T}^n \times \mathbb{R}^d$ into the sets $\{x\} \times \mathbb{R}^d$, $x \in \mathbb{T}^n$, and $\mu_x$ the conditional measures of $\mu$ relative to $\mathcal{P}$: that is, $\mu_x(\{x\} \times \mathbb{R}^d) = 1$ for $m$-almost every $x \in \mathbb{T}^n$ and $\mu(E) = \int_{\mathbb{T}^n} \mu_x(E) \, dm(x)$ for every measurable subset $E \subset \mathbb{T}^n$. The existence of the conditional measures is given by Rokhlin’s theorem [13, 17].

**Definition 3.3.** Given a finite measure $\mu$ on $\mathbb{T}^n \times \mathbb{R}^d$ and $r > 0$, we define the semi-norm $\|\|\mu\||_r$, by

$$\|\|\mu\||_r^2 = \int_{\mathbb{T}^n} \frac{\|\mu_x\|^2}{r^d} \, dm(x).$$

(18)
As a consequence of lemma 3.2, we have a criterion of absolute continuity for measures $\mu$ on $\mathbb{T}^n \times \mathbb{R}^d$, provided by the following:

**Corollary 3.4.** There exists a constant $C_d$ such that if a finite measure $\mu$ on $\mathbb{T}^n \times \mathbb{R}^d$ satisfies $(\pi_1)_* \mu = m$ and

$$
\lim_{r \to 0^+} \frac{\|\mu\|}{r^d} < \infty,
$$

then $\mu$ is absolutely continuous with respect to the volume $v = m \times m_d$ on $\mathbb{T}^n \times \mathbb{R}^d$, its density $\frac{\partial \mu}{\partial v}$ is in $L^2(\mathbb{T}^n \times \mathbb{R}^d)$ and satisfies $\|\frac{\partial \mu}{\partial v}\|_{L^2(\mathbb{T}^n \times \mathbb{R}^d)} \leq C_d \lim_{r \to 0^+} \frac{\|\mu\|}{r^d}$.

**Proof.** From Fatou’s inequality, we have

$$
\int_{\mathbb{T}^n} \liminf_{r \to 0} \frac{\|\mu_r\|^2}{r^{2d}} \, dm(x) \leq \liminf_{r \to 0} \int_{\mathbb{T}^n} \frac{\|\mu_r\|^2}{r^{2d}} \, dm(x) < \infty.
$$

This inequality and the previous lemma imply that, for Lebesgue, almost every $x \in \mathbb{T}^n$, $\mu_x$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and the density function $g_x$ for every $x \in \mathbb{T}^n$, $\mu_x$ satisfies $\|g_x\|_{L^2(\mathbb{R}^d)} \leq C_d \liminf_{r \to 0^+} \frac{\|\mu_r\|}{r^d}$.

Defining $g : \mathbb{T}^n \times \mathbb{R}^d \to \mathbb{R}$ by $g(x,y) = g_x(y)$, we have $g \in L^2(\mathbb{T}^n \times \mathbb{R}^d)$ with $\|g\|_{L^2(\mathbb{T}^n \times \mathbb{R}^d)} \leq C_d \liminf_{r \to 0^+} \int_{\mathbb{T}^n} \frac{\|\mu_r\|}{r^d} \, dm(x)$ and

$$
\int_{\mathbb{T}^n \times \mathbb{R}^d} \phi(x,y) g(x,y) \, dm(x) \, dm_d(y) = \int_{\mathbb{T}^n} \left[ \int_{\mathbb{R}^d} \phi(x,y) g_x(y) \, dm_d(y) \right] \, dm(x)
= \int_{\mathbb{T}^n} \int_{\mathbb{R}^d} \phi(x,y) \, dm_d(y) \, dm(x)
= \int_{\mathbb{T}^n \times \mathbb{R}^d} \phi(x,y) \, dm(x)
$$

for every continuous function $\phi$ with compact support on $\mathbb{T}^n \times \mathbb{R}^d$. Above we used $(\pi_1)_* \mu = m$ in the disintegration of $\mu$ with respect to the partition into sets $\{x\} \times \mathbb{R}^d$. Then, $\mu$ is absolutely continuous with respect to the volume measure on $\mathbb{T}^n \times \mathbb{R}^d$ and $g = \frac{\partial \mu}{\partial v}$.

### 3.2. Useful lemmas on the transversality

In the following, we will state a quantitative lemma of change of variables by maps $g$ satisfying $m(Dg(x)) \geq \delta$.

**Lemma 3.5.** Given an open convex set $U \subset \mathbb{R}^n$, $g \in C^2(U, \mathbb{R}^d)$ with $\|g\|_{C^2} \leq 2\alpha_0$ and a real number $\delta > 0$ such that $\text{diam}(U) \leq \delta/4\alpha_0$ and $m(Dg(x)) \geq \delta$ for every $x \in U$, then there exists a constant $C_\delta > 0$ such that

$$
m_u(g^{-1} B(z,r)) < C_\delta r^d
$$

for every $z \in \mathbb{R}^d$ and every $r > 0$.

**Proof.** Let us first suppose that $u = d$. Using the convexity of $U$ and the Taylor formula, for $x \neq y \in U$ we have $g(y) = g(x) + Dg(x)(y - x) + R(y - x)$, where $\|R(y - x)\| \leq 2\alpha_0 \|y - x\|^2$. This implies that
\[
\|g(y) - g(x)\| \geq \delta \|y - x\| - 2\alpha_0 \|y - x\|^2 = \left(\delta - 2\alpha_0 \|y - x\|\right)\|y - x\| > 0.
\]

Hence \(g\) is injective, which implies that it is a diffeomorphism into the image. By a change of variables we have for every \(z \in \mathbb{R}^d\) and \(r > 0\) that:

\[
m_u(g^{-1}(B(z, r))) = \int_{g^{-1}(B(z, r)) \cap g(U)} 1 \, dm_u(x)
= \int_{B(z, r) \cap g(U)} |\det Dg(y)| \, dm_u(y)
\leq \delta^{-d} C_d^{-1} r^d.
\]

Now, suppose that \(u > d\). Fix a point \(a_0 \in U\), then there is a \(d\)-dimensional subspace \(W \subset \mathbb{R}^u\) such that \(m(Dg(a_0)|_W) \geq \delta\). As \(\text{diam}(U) < \delta/4\alpha_0\) and \(\|g\|_{C^2} \leq 2\alpha_0\), we have \(m(Dg(x)|_W) \geq \delta/2\) for every \(x \in U\).

For every \(a \in \mathbb{R}^u\), we denote \(W_a := a + W = \{a + x \in \mathbb{R}^u : x \in W\}\) and consider the usual orthogonal projection \(\pi_{W^\perp} : \mathbb{R}^u \to W^\perp\). For each \(y \in \pi_{W^\perp}(U)\) define \(U_y = W_y \cap U\) and \(h_y := g|_{U_y}\). Then \(h_y\) is a \(C^2\) transformation such that \(\|h_y\|_{C^2} \leq 2\alpha_0\) and \(m(Dh_y(x)) \geq \delta/2\) for every \(x \in U_y\). Applying the inequality above for \(g = h_y\) and \(U = U_y\), it follows that

\[
m_u(h_y^{-1}(B(z, r))) \leq \left(\frac{\delta}{2}\right)^{-d} C_d^{-1} r^d
\]

for every \(z \in \mathbb{R}^d\).

Finally, we use that \(\text{diam}(\pi_{W^\perp}(U)) < \delta/4\alpha_0\) and Fubini’s theorem:

\[
m_u(g^{-1}(B(z, r))) = \int_{\pi_{W^\perp}(U)} m_u(h_y^{-1}(B(z, r))) \, dm_{u-d}(y)
\leq m_{u-d}(\pi_{W^\perp}(U)) \left(\frac{\delta}{2}\right)^{-d} C_d^{-1} r^d
\leq C_{u-d}^{-1} \left(\frac{\delta}{8\alpha_0}\right)^{u-d} \left(\frac{\delta}{2}\right)^{-d} C_d^{-1} r^d = C \delta^{-d} r^d
\]

for every \(z \in \mathbb{R}^d\) and \(r > 0\).

\[\square\]

**Lemma 3.6.** \textit{Given} \(c \in \mathbb{R}\) \textit{and} \(a, b \in I(c)\), \textit{if} \(a\) \textit{and} \(b\) \textit{are transversal on} \(c\), \textit{then}
\[
m(DS_c(x, au) - DS_c(x, bv)) > \theta^q \alpha_0
\]
\textit{for every} \(x \in \overline{R(c)}\) \textit{and} \(u \in I^\infty(a)\) \textit{and} \(v \in I^\infty(b)\).

**Proof.** For every unitary vector \(v \in \mathbb{R}^u\), we have

\[
\|DS_c(x, au) - DS_c(x, a)\| v = \| \sum_{i=q+1}^\infty C^{i-q} Df([au]_i(x)) E^{-i} v \| \leq \| C \| q \| E^{-1} \| q \alpha_0.
\]

Analogously, we also have \(\|DS_c(x, bv) - DS_c(x, b)\| v \leq \| C \| q \| E^{-1} \| q \alpha_0\). By assumption, there is a \(d\)-dimensional subspace \(W \subset \mathbb{R}^d\) such that \(\|DS_c(x, au) - DS_c(x, bv)\| w \geq 3\theta^q \alpha_0 \|w\|\) for every \(w \in W\).
Recalling that $\theta = \|C\|E^{-1}$, for every vector $w \in \mathcal{W}$ we have
\[
\|(DS_c(x, au) - DS_c(x, bv))w\| \geq \|(DS_c(x, a) - DS_c(x, b))w\| \\
- \|(DS_c(x, au) - DS_c(x, a))w\| - \|(DS_c(x, bv) - DS_c(x, b))w\| \\
> 3\theta^2\alpha_0\|w\| - \theta^2\alpha_0\|w\| - \theta^2\alpha_0\|w\| = \theta^2\alpha_0\|w\|.
\]

What we want follows by taking the infimum over $w$. \qed

3.3. Symbolic description of $\mu_T$

Let us give a symbolic description of the dynamic $T$ and of the measure $\mu_T$. Consider $M = \bigsqcup_{i \in \mathcal{I}} \mathcal{R}(i) \times \mathcal{I}^\infty(\mathcal{R}(i)) \subset \mathcal{T}^u \times \mathcal{I}^\infty$ and $\hat{T}: M \to M$ given by
\[
\hat{T}(x, a) = (E(x), \pi(x)a).
\]
$\hat{T}$ is well defined because if $a \in \mathcal{I}^\infty(x)$ then $\pi(x)a \in \mathcal{I}^\infty(E(x))$. Moreover, for $a \in \mathcal{I}^\infty(x)$, we have the relation
\[
S(E(x), \pi(x)a) = f(x) + C(S(x, a)).
\]
This implies that $T \circ h = h \circ \hat{T}$, where $h: \hat{M} \to M$ is given by
\[
h(x, a) = (x, S(x, a)).
\]

**Remark 3.7.** Note that $\Lambda \subset \mathcal{h}(\hat{M})$. Actually, given $(x, y) \in \Lambda$, for every $n \in \mathbb{N}$, consider $a_n \in \mathcal{I}^n(x)$ such that $y = S(x, a_n) + C^n y_n$ for some $y_n \in [-K, K]^d$. Considering any $u_n \in \mathcal{I}^\infty(a_n(x))$, we have that $\|y - S(x, a_n u_n)\| \leq \|y - S(x, a_n)\| + \|S(x, a_n) - S(x, a_n u_n)\| \leq \|C^n K + \|C^n\|^{-1} \alpha_0$. Since $\mathcal{I}^\infty(x)$ is compact, we conclude that $y = S(x, b)$; that is, $(x, y) = h(x, b)$, where $b \in \mathcal{I}^\infty(x)$ is any limit point of the sequence $(a_n u_n)_{n \in \mathbb{N}}$.

Recalling that $N = \det E$ is the degree of $E$ and that for each $\mathcal{R}(i)$ there exists exactly $N$ rectangles $\mathcal{R}(j)$ such that $\mathcal{R}(i) \subset E(\mathcal{R}(j))$ (see remark 2.2), define the numbers $P_{ij} = \frac{1}{N}$ if $\mathcal{R}(i) \subset E(\mathcal{R}(j))$ and $P_{ij} = 0$ otherwise.

Consider the cylinders $\{m; a_m, \ldots, a_0\} = \{(u_1, u_2, \cdots) \in \mathcal{I}^\infty|u_i = a_m, m \leq i \leq n\}$ and $S$ the semi-ring formed by the sets $U \times C$, where $U \subset \mathcal{R}(i)$ for some $i$ and $C$ is a cylinder. Consider $\Sigma$ the $\sigma$-algebra in $\mathcal{T}^u \times \mathcal{I}^\infty$ generated by $S$. Since every cylinder is a disjoint union of cylinders of the form $[1; a_1, \ldots, a_n]$, there exists a probability $\hat{\mu}$ on $\mathcal{T}^u \times \mathcal{I}^\infty$ given by the relation
\[
\hat{\mu}(U \times V) := m(U)P_{a_1 a_2 \cdots a_{n-1} a_n} \quad (25)
\]
for any $U \subset \mathcal{R}(i)$ and any cylinder $V = [1; a_1, \ldots, a_n]$. We can extend (25) to $\Sigma$ by Carathéodory’s extension theorem [9, appendix A.1]. Note that $\hat{\mu}(\hat{M}) = 1$ and thus $h_*\hat{\mu}(\mathcal{T}^u \times \mathbb{R}^d) = 1$.

Recall that if $\mu$ is a probability measure supported in $\Lambda \subset \mathcal{T}^u \times \mathbb{R}^d$, the basin of $\mu$ is the set
\[
B(\mu) = \left\{ x \in \mathcal{T}^u \times \mathbb{R}^d \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(T^j(x)) \right. \left. = \int \phi d\mu \text{ for every } \phi \in C^0(\mathcal{T}^u \times \mathbb{R}^d, \mathbb{R}) \right\}.
\]
We say that $\mu$ is a physical measure if the volume of $B(\mu)$ in $\mathcal{T}^u \times \mathbb{R}^d$ is positive. In the present setting, a probability measure $\mu$ is an SRB measure if it is $T$-invariant and $(\pi)_* \mu \ll m$. 

\[
\text{2066}
\]
Remark 3.8. When $T$ is a hyperbolic endomorphism, a $T$-invariant measure $\mu$ is an SRB measure if its disintegration with respect to every measurable partition subordinate to the $W^u$ manifolds is absolutely continuous with respect to the Lebesgue measure on $W^u$. It is known that for hyperbolic transitive attractors for endomorphisms, there exists an invariant measure $\mu$ that is at the same time the unique SRB measure and the unique physical measure supported in the attractor. (We refer the reader to [21] and its references for more details.) In our setting, it is enough to check that this measure corresponds to $h_*\hat{\mu}$, given by the construction above.

Lemma 3.9. The measure $h_*\hat{\mu}$ is the unique SRB measure and is also the unique physical measure of $T$. In particular, $\mu_T = h_*\hat{\mu}$.

Proof. For every integer $n \geq 1$ and $k \geq 0$ we have:
\[ T^{-n-k}(U \times [1; a_1, \ldots, a_n]) = E^{-k}(E^{-n}(U) \cap E^{-n+1}(R(a_1)) \cap \cdots \cap R(a_n)) \times I^\infty. \]

It follows that $\hat{\mu}$ is an invariant probability for $\hat{T}$, because for any $U \subset R(i)$
\[ \hat{\mu}(\hat{T}^{-1}(U \times [1; a_1, \ldots, a_n])) = \hat{\mu}((R(a_1) \cap E^{-1}(U)) \times [1; a_2, a_3, \ldots, a_n]) = P_{a_1}m(U)P_{a_2} \cdots P_{a_{n-1}}a_n = \hat{\mu}(U \times [1; a_1, \ldots, a_n]). \]

To check that $\hat{\mu}$ mixes for $\hat{T}$, for any sets $A_1, A_2$ of the form $U \times [1; a_1, \ldots, a_n]$ with $U \subset R(j)$ we consider an integer $\ell \geq 1$ such that $\hat{T}^{-\ell}(A_1) = \hat{U}_A \times I^\infty$ for $i = 1, 2$. Since $m$ mixes for $E[13]$ and $\hat{\mu}(A_i) = \hat{\mu}(\hat{T}^{-\ell}(A_i)) = \hat{\mu}(\hat{U}_A \times I^\infty) = m(\hat{U}_A)$, we have:
\[ \hat{\mu}(\hat{T}^{-k}A_1 \cap A_2) = \hat{\mu}(\hat{T}^{-\ell-k}(A_1) \cap \hat{T}^{-\ell}(A_2)) = \hat{\mu}((E^{-k}\hat{U}_A \times I^\infty) \cap (\hat{U}_A \times I^\infty)) = m(E^{-k}\hat{U}_A \cap \hat{U}_A) \to m(\hat{U}_A)m(\hat{U}_A) = \hat{\mu}(A_1)\hat{\mu}(A_2). \]

For measurable subsets $A_1, A_2$, one proves that $\hat{\mu}(\hat{T}^{-k}A_1 \cap A_2) \to \hat{\mu}(A_1)\hat{\mu}(A_2)$ from standard arguments approximating $A_1, A_2$ by finite unions of the sets as above. So the measure $h_*\hat{\mu}$ is invariant and ergodic with respect to $T$.

Since $\langle \pi_1 \rangle, h_*\hat{\mu} = m$ we get that $m(\pi_1(B(h_*\hat{\mu}))) = 1$, where $B(h_*\hat{\mu})$ is the basin of $h_*\hat{\mu}$. Finally, note that if $(x, y) \in B(h_*\hat{\mu})$ then the whole set $\{x\} \times \mathbb{R}^d$ is in $B(h_*\hat{\mu})$ because the vertical fibers are stable manifolds for $T$. Then $B(h_*\hat{\mu})$ has full volume in $T^d \times \mathbb{R}^d$, implying that $h_*\hat{\mu}$ is a physical measure. Since the basins of two different measures are disjoint, it follows that $h_*\hat{\mu}$ is the unique physical measure of $T$.

A similar argument shows that every ergodic SRB measure is a physical measure: if $\nu$ is an ergodic SRB measure, then $\langle \pi_1 \rangle, \nu \ll m$ due to the comment above remark 3.8, and it implies that $m(\pi_1(B(\nu))) > 0$. For every $(x, y) \in B(\nu)$, the set $\{x\} \times \mathbb{R}^d$ is contained in $B(\nu)$, thus $B(\nu)$ has positive volume and $\nu$ is a physical measure. Therefore, by the uniqueness of the physical measure $h_*\hat{\mu}$, every ergodic SRB measure must be equal to $h_*\hat{\mu}$. By the ergodic decomposition theorem, every SRB measure is equal to $h_*\hat{\mu}$, that is, $h_*\mu$ is the unique SRB measure of $T$. □

3.4. The main inequality

The transversality between $a$ and $b$ on $c$ provides an estimate on the integral of $\langle T^a_d\mu_{a(s)}, T^b_d\mu_{b(s)} \rangle$, over $R(c)$. 2067
In the continuation, let us simply write $\mu$ for the SRB measure $\mu_T$, which coincides with $h_*\hat{\mu}$, as described in the previous subsection. Consider $(\mu_x)_{x \in \mathcal{T}}$ the Rohlin disintegration of $\mu_T$ with respect to the partition $\mathcal{T} \times \mathbb{R}^d$, for each $x \in \mathcal{T}$. Consider an identification of $\{x\} \times \mathbb{R}^d \simeq \mathbb{R}^d$ and, for each $a \in I^p(x)$, we also identify the push forward of $\mu_{a(i)}$ by $T^a$, $T^a_{a(i)}$, with its restriction to $\{x\} \times \mathbb{R}^d$.

**Proposition 3.10.** Let $c \in I^p$ and $a, b \in I^p(c)$. If $a$ and $b$ are transversal on $c$, then there exists a constant $C_1 > 0$, depending on $q$, such that

$$
\int_{\mathcal{R}(c)} \langle T^a_{a(i)}\mu_{a(i)}, T^b_{a(i)}\mu_{b(i)} \rangle, \, dm(x) \leq C_1 r^{2d}
$$

for all $r > 0$.

**Proof.** Denote the indicator function of the ball $B(0, r) \subset \mathbb{R}^d$ by $\mathbb{I}_r$, and define $I := \int_{\mathcal{R}(c)} \langle T^a_{a(i)}\mu_{a(i)}, T^b_{a(i)}\mu_{b(i)} \rangle, \, dm(x)$. For $x \in \mathcal{R}(c)$, we have:

$$
\int_{\mathbb{R}^d} \frac{(T^a_{a(i)}\mu_{a(i)}), (T^b_{a(i)}\mu_{b(i)}))_r}{r^{2d}} = \int_{\mathbb{R}^d} \frac{\int_{\mathbb{R}^d} (T^a_{a(i)}\mu_{a(i)})(B(z, r)) \, T^b_{a(i)}(B(z, r)) \, dm(x)}{r^{2d}}(z)
$$

$$
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{I}_r(z_1 - z) \mathbb{I}_r(z_2 - z) \, d(T^a_{a(i)}\mu_{a(i)})(z_1) \times T^b_{a(i)}\mu_{b(i)}(z_2)) \, dm(z)
$$

$$
\leq C_1^{-1} r^{d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{I}_r(z_1 - z) \, d(T^a_{a(i)}\mu_{a(i)}) \times T^b_{a(i)}\mu_{b(i)}(z_1, z_2).
$$

The symbolic description for $\mu$ in section 3.3 can be extended in order to also describe the conditional measures $\mu_x$; $\mu_x$ is given by $\mu_x = (h_*\hat{\mu}_x)$, where $h_x(a) = (x, S(x, a))$ and $\hat{\mu}_x([1; a_1, \ldots, a_n]) = P_{\pi(x)a}P_{a_1} \cdots P_{a_{n-1}}u_a$. In particular, $x \mapsto \hat{\mu}_x$ is constant in each $\mathcal{R}(i)$.

Considering $\Psi_{a, x} : \mathbb{R}^d(\mathbb{a}(x)) \rightarrow (\{x\} \times \mathbb{R}^d) \cap \Lambda$ given by $\Psi_{a, x}(\mathbb{u}) = (x, S(x, \mathbb{a}(x)))$, it is valid that $\Psi_{a, x}(\mathbb{u}) = (T^a \circ h_x)(\mathbb{u}(x), \mathbb{u})$ for all $\mathbb{u} \in \mathbb{R}^d(\mathbb{a}(x))$, and so that $(\Psi_{a, x})_*\mu_x = T^a_{a(i)}\mu_{a(i)} = T^a_{b(i)} \mu_{b(i)}$. Analogously, we have that $(\Psi_{b, x})_*\mu_x = T^b_{a(i)}\mu_{a(i)}$, where $\Psi_{b, x}(\mathbb{u}(x), \mathbb{v}) = (x, S(x, \mathbb{b}(x)))$.

Notice that $T^a_{a(i)}\mu_{a(i)}$ is supported in $(\{x\} \times \mathbb{R}^d) \cap \Lambda$ and that

$$
\hat{\mu}_{a(i)}(I^\infty(a(x))) = \hat{\mu}_{a(i)}([1; k_1]) + \cdots + \hat{\mu}_{a(i)}([1; k_N]) = 1 + \cdots + \frac{1}{N} = 1,
$$

where $k_1, \ldots, k_N$ are the $N$ symbols such that $E^{-1}([a_1]_x) \cap \mathcal{R}(k_i) \neq \emptyset$. Similarly, $T^b_{b(i)}\mu_{b(i)}$ is supported in $(\{x\} \times \mathbb{R}^d) \cap \Lambda$ and $\hat{\mu}_{b(i)}(I^\infty(b(x))) = 1$.

Take $(x, z_1) = (\Psi_{a, x}, \Psi_{b, x})(\mathbb{u}, \mathbb{v})$ and denote $I_{a, b}(x) = I^\infty(a(x)) \times I^\infty(b(x))$, then $\langle T^a_{a(i)}\mu_{a(i)}, T^b_{b(i)}\mu_{b(i)} \rangle \mu_{b(i)}(\mathbb{u}, \mathbb{v})$ is at most

$$
C_1^{-1} r^{d} \int_{I_{a, b}(x)} \mathbb{I}_{2r}([S_{a}(x, \mathbb{u}) - S_{b}(x, \mathbb{b})]) \, d(\hat{\mu}_{a(i)} \times \hat{\mu}_{b(i)})(\mathbb{u}, \mathbb{v}).
$$

Since $I_{a, b}(x), \hat{\mu}_{a(i)}, \hat{\mu}_{b(i)}$ are constant in $\mathcal{R}(c)$, we can denote them as $I_{a, b}, \hat{\mu}_{a}, \hat{\mu}_{b}$. Integrating (28) on $\mathcal{R}(c)$, we have:
\[ I \leq C_d^{-1} r^d \int_{I_0} m(x) \leq \mathcal{R}(c), \| S(x, au) - S(x, bv) \| < 2r \) \ d(\hat{\mu}_a \times \hat{\mu}_b)(u, v). \]  
\hspace{2cm} (29)

To finish, it is enough to prove the following claim.

**Claim 3.11.** Given \( c \in I^p \) and \( a, b \in I^q(c) \), if \( a \) and \( b \) are transversal on \( c \), then there exists a constant \( C_2 > 0 \), depending on \( q \), such that

\[ m(x) \leq \mathcal{R}(c), \| S(x, au) - S(x, bv) \| < 2r \leq C_2 r^d \]

for every \( u \) and \( v \) such that \( au \in I^\infty(c) \) and \( bv \in I^\infty(c) \).

**Proof of claim 3.11.** For every \( x \in I^q(c) \), \( a \in I^q(c) \), \( 1 \leq n \leq \infty \) and \( 1 \leq j \leq n \) the maps

\[ h_{f, j}^a := g_{[a], [a]j-1}, \ldots \circ g_{[a]1, x} : f : B(x, \gamma) \to \mathbb{T}^m \]

are well defined by (4) and

\[ \hat{S}(\cdot, a(x)) = \sum_{j=1}^{n} C_j \circ f \circ h_{f, j}^a : B(x, \gamma) \to \mathbb{R}^d. \]  
\hspace{2cm} (30)

Fix some \( x_0 \in \mathcal{R}(c) \) and define \( g : B(x_0, \gamma) \ni \mathcal{R}(c) \to \mathbb{R}^d \) by

\[ g(x) := \hat{S}(x, au(x_0)) - \hat{S}(x, bv(x_0)). \]  
\hspace{2cm} (31)

Since \( \hat{S}(\cdot, au)|_{\mathcal{R}(c)} \) and \( S(\cdot, bv)|_{\mathcal{R}(c)} \) coincide with the restrictions of \( \hat{S}(\cdot, au(x_0)) \) and \( \hat{S}(\cdot, bv(x_0)) \) to \( \mathcal{R}(c) \), due to lemma 3.6, the transversality between \( a \) and \( b \) implies that

\[ m(Dg(x)) > \theta e \rho_0 \]  
\hspace{2cm} (32)

for every \( x \in \mathcal{R}(c) \).

We also have that \( ||g||_{C^1} \leq 2\rho_0 \). Defining \( \delta = \theta e \rho_0 \) and considering a covering of \( \mathcal{R}(c) \) with at most \( M = \lfloor (8\rho_0/\delta)^m \rfloor \) balls centered in points in \( \mathcal{R}(c) \) with radius at most \( \delta/4\rho_0 \), for each of such balls \( U \), we apply lemma 3.5 to \( g_{1_U} \), obtaining that \( m_h((g_{1_U})^{-1}B(0, 2r)) > C_\delta(2r)^d \).

Consequently, it follows that

\[ m(\{ x \in \mathcal{R}(c), \| S(x, au) - S(x, bv) \| < 2r \}) \leq 2^d C_\delta r^{d} = C_3 r^d. \]

Putting together claim 3.11 and (29), we have that

\[ I \leq C_3 r^d C_2 > 0 \]

The main step to proving theorem B is given by the following inequality. \( \square \)

**Proposition 3.12 (Main inequality).** For every integer \( 1 \leq q < \infty \), there exists a constant \( C_2(q) > 0 \) such that

\[ ||\mu||^2 \leq \frac{\tau(q)}{J_q} ||\mu||^2 \leq ||\mu||^2 + C_1(q) \]  
\hspace{2cm} (33)

for all \( r > 0 \).

**Proof.** Given \( q \), for every \( c \in I^p \), we denote by \( \Gamma(q, c) \) the set of pairs \( (a, b) \) with \( (a, b) \in I^q(c) \times I^q(c) \) such that \( a \) and \( b \) are not transversal on \( c \). Fix an integer \( p_0 = p_0(q) \) large enough such that
\[ \tau(q) = \max_{c \in \Gamma(q)} \max_{a \in \Gamma(q)} \# \{ b \in \Gamma(q) \mid (a, b) \in \Gamma(q, c) \}. \]

Consider also \( \Gamma(q) \) the set of triples \((a, b, c)\) with \( c \in I^0 \) and \((a, b) \in \Gamma(q, c)\). The invariance of \( \mu \) with respect to \( T \) and the uniqueness of the disintegration gives the relation

\[ \mu_x = N^{-q} \sum_{a \in \Gamma(c)} T^q_{\ast} \mu_{a(x)}. \]

Note that if \( c \in I^0 - I^0 \), then \( \mathcal{R}(c) = \emptyset \), therefore we may decompose \( |||\mu|||^2 \) into the following sum:

\[
|||\mu|||^2 = N^{-2q} \sum_{c \in I^0} \sum_{(a, b) \in \mathcal{H}(c) \times \mathcal{I}(c)} \int_{\mathcal{R}(c)} \frac{\langle T^q_{\ast} \mu_{a(x)}, T^q_{\ast} \mu_{b(x)} \rangle_r}{r^{2d}} \, dm(x)
\]

\[
= N^{-2q} \sum_{(a, b, c) \in \Gamma(q)} \int_{\mathcal{R}(c)} \frac{\langle T^q_{\ast} \mu_{a(x)}, T^q_{\ast} \mu_{b(x)} \rangle_r}{r^{2d}} \, dm(x)
\]

\[
+ N^{-2q} \sum_{(a, b, c) \in \Gamma(q)} \int_{\mathcal{R}(c)} \frac{\langle T^q_{\ast} \mu_{a(x)}, T^q_{\ast} \mu_{b(x)} \rangle_r}{r^{2d}} \, dm(x).
\]

Let us denote by \( S_1 \) the first sum above and \( S_2 \) the second sum. The transversality condition is used to bound \( S_2 \) from above: proposition 3.10 implies that

\[ \int_{\mathcal{R}(c)} \frac{\langle T^q_{\ast} \mu_{a(x)}, T^q_{\ast} \mu_{b(x)} \rangle_r}{r^{2d}} \, dm(x) \leq C_d r^{2d}. \]

Since \( \#(I^q \times I^q \times I^0) = N^{2q+\rho_0(q)} \) (see remark 2.2), it follows that

\[ S_2 \leq N^{2q+\rho_0(q)} N^{-2q} C_1(q). \]

In order to bound from above the first sum, we consider \( H = |\det C| \) and note that the restriction of \( T^q \) to each vertical fiber is an affine contraction with a linear part equal to \( C^{-q} \). Putting \( s = |||C^{-q}|||_r \), we have that

\[ |||T^q_{\ast}(\mu_{a(x)})|||_r^2 \leq H^q |||\mu_{a(x)}|||_r^2. \]

For \( a, b \in \Gamma(q) \), it holds:

\[ \langle T^q_{\ast} \mu_{a(x)}, T^q_{\ast} \mu_{b(x)} \rangle_r \leq \| T^q_{\ast} \mu_{a(x)} \|_r \| T^q_{\ast} \mu_{b(x)} \|_r \leq H^q \| \mu_{a(x)} \|_r^2 + \| \mu_{b(x)} \|_r^2. \]

Making a change of variables, we have

\[ \sum_{a \in \Gamma} \int_{\mathcal{T}_a} \frac{|||\mu_{a(x)}|||_r^2}{r^{2d}} \, dm(x) = \sum_{a \in \Gamma} \int_{\mathcal{R}(a)} \frac{|||\mu_{a(x)}|||_r^2}{r^{2d}} \, dm(y) = N^q \| \mu_{a(x)} \|_r^2. \]

Then:

\[ S_1 \leq r^{-2d} N^{-2q} H^q \tau(q) \sum_{a \in \Gamma} \int_{\mathcal{T}_a} \frac{|||\mu_{a(x)}|||_r^2}{r^{2d}} \, dm(x) \]

\[ \leq \frac{\tau(q)}{|\det E| \cdot |\det C|^{-1}} \| C^{-q} \|_{-2d} |||\mu|||_r^2. \]
Since \( \|C^{-q}\| \leq \|C^{-1}\|^q = \Lambda^{-q} \), we conclude that
\[
S_1 \leq \frac{\tau(q)}{||\det E|| \det C_1^{-1} \|C^{-1}\|^{-2d} q^2} \|\mu\|_C^{\epsilon_f},
\]
and what we want follows. \(\square\)

3.5. Absolute continuity of \(\mu_T\)

Finally, we are able to prove that the transversality condition implies the absolute continuity of \(\mu_T\) (theorem 2.9).

**Proof of theorem 2.9.** By the transversality condition of \(T\), there exists an integer \(q_0\) such that \(\tau(q_0) < J^{q_0}\). Consider \(\rho = \frac{T(q_0)}{J^{q_0}} < 1\) and take \(C_1(q_0)\) as in proposition 3.12 for this \(q_0\).

For \(\rho > 0\), we have
\[
\|\mu\|_C^2 \leq \rho \|\mu\|_C^{\epsilon_f} + C_1(q_0).
\]

Fixing some \(r_0 > 0\) and denoting \(\zeta = \|C^{-q_0}\|^{-1} < 1\), it follows that
\[
\|\mu\|_C^{r_0} \leq \rho \|\mu\|_{C^{r_0}} + C_1(q_0) \\
\leq \rho^2 \|\mu\|_C^{r_0} + C_1(q_0)(\rho + 1) \leq \ldots \\
\leq \rho^n \|\mu\|_C^{r_0} + C_1(q_0)(\rho^n + \ldots + \rho + 1) \leq \|\mu\|_C^{r_0} + \frac{C_1(q_0)}{1-\rho}.
\]

So we have \(\liminf_{r \to 0} \|\mu\|_C^r < \infty\), thus corollary 3.4 implies that the SRB measure is absolutely continuous and its density is in \(L^2(T^u \times \mathbb{R}^d)\).

The openness follows from an argument of semi-continuity. The transversality of a pair of functions \(S_{C}(\cdot, a)\) and \(S_{C}(\cdot, b)\) is open in \((C, f)\) because \(S_{C}(\cdot, a)\) is continuous on \(a\), on \(C\) and on \(f \in C^2(T^u, \mathbb{R})\). This implies that, for fixed \(q\), \(\tau(q)\) is upper semi-continuous and that the condition \(\tau(q) < (\|\det C|\|C^{-1}\|^{-2d} q^2)\) is open on \(f\). \(\square\)

3.6. Proof of theorem B

To conclude the proof of theorem B, we need proposition 2.10. To prove proposition 2.10, we will use theorem 2.12, which will be proved in section 4.

**Proof of proposition 2.10.** Given \(E\) such that \(C(d; E) \neq \emptyset\), \(C \in C(d; E)\) and any \(f_0 \in C^2(T^u, \mathbb{R}^d)\), theorem 2.12 implies that there exist functions \(\{\phi_k\}\) and \(a \in \mathbb{R}^d\) arbitrarily close to 0 such that \(f_i = f_0 + \sum \phi_k \not\in \mathcal{T}\). In particular, \(T = T(E, C, f_i)\) satisfies the transversality condition.

So we can prove theorem B.

**Proof of theorem B.** Consider the set \(\mathcal{U}\) formed by the pairs \((C, f) \in C(d) \times C^2(T^u, \mathbb{R}^d)\) such that the dynamic \(T = T(E, C, f)\) satisfies the transversality condition. Proposition 2.10 implies that \(\mathcal{U}\) is nonempty. Theorem 2.9 implies that \(\mathcal{U}\) is open and the absolute continuity of \(\mu_T\) for every \((C, f) \in \mathcal{U}\). \(\square\)
3.7. Proof of theorem A

Let us state a more precise version of theorem A.

**Proposition 3.13.** Given $u \geq d$, integers $\mu_1, \ldots, \mu_d \geq 2$, $u_1, \ldots, u_d \geq 1$ such that $u_1 + \cdots + u_d = u$ and a real number $\lambda \in (\max \{\mu_i^{-u_i} : i = 1, 2, \ldots, d\}, 1)$, consider $T^u = T^{u_1} \times \cdots \times T^{u_d}$, the expanding map $E : T^u \to T^u$, $E(x_1, \ldots, x_d) = (\mu_1 x_1, \ldots, \mu_d x_d) \mod \mathbb{Z}^u$, and the contracting map $C : \mathbb{R}^d \to \mathbb{R}^d$, $C(y_1, \ldots, y_d) = \lambda(y_1, \ldots, y_d)$. Then there exists a function $f \in C^2(T^u, \mathbb{R}^d)$ for which the corresponding map $T = T(E, C, f)$ satisfies the transversality condition.

**Proof of proposition 3.13.** Consider the maps $E_i : \mathbb{R}^{u_i} \to \mathbb{R}^{u_i}$ by $E_i(x_i) = \mu_i x_i$ and $C_i : \mathbb{R} \to \mathbb{R}$ by $C_i(y) = \lambda y$, for $i = 1, \ldots, d$. Note that $C_i \in C(1; E_i)$, for every $i = 1, \ldots, d$, then theorem 2.12 implies that there exist $f_i \in C^2(T^{u_i}, \mathbb{R})$ such that the map $T_i : T^{u_i} \times \mathbb{R} \to T^{u_i} \times \mathbb{R}$ given by $T_i(x, y) = (E_i(x), \lambda y + f_i(x))$ satisfies $\limsup_{q \to \infty} \frac{\log \lambda \mu_i^{u_i}}{q} < \log \lambda_i$, which means that $\tau(T_i, q) < J_i^q$ for every $q \geq q_i$.

Now, consider $f \in C^2(T^u, \mathbb{R}^d)$ given by $f(x_1, \ldots, x_d) = (f_1(x_1), \ldots, f_d(x_d))$, $x_i \in T^{u_i}$. The dynamic $T = T(E, C, f)$ has the form

$$T(x_1, \ldots, x_d, y_1, \ldots, y_d) = (\mu_1 x_1, \ldots, \mu_d x_d, \lambda y_1 + f_1(x_1), \ldots, \lambda y_d + f_d(x_d))$$

where $x_i \in T^{u_i}$ and $y_i \in \mathbb{R}$.

We claim that $T$ satisfies the transversality condition. To verify it, we consider the product alphabet $I_T = I_{T_1} \times \cdots \times I_{T_d}$, the product partition $R_T^u = R_{T_1}^{u_1} \times \cdots \times R_{T_d}^{u_d}$ and notice that if $a = (a_1, \ldots, a_d)$ is in $P = P_T^u$ then $S(x, a)$ is the product $S(x_1, \ldots, x_d, a_1, \ldots, a_d) = (S(x_1, a_1), \ldots, S(x_d, a_d))$, with $a_i \in I_{T_i}$. For $q \geq \max q_i$, we have $\tau(T, q) \leq J_1^q \cdots J_d^q = (\det C \det E)^q$, which means that $T$ satisfies the transversality condition, since $C$ is conformal.

**Proof of theorem A.** The dynamic of $E$ under the assumptions of theorem A corresponds to a map $E : T^u \to T^u$ given by $E(x_1, \ldots, x_d) = (\mu_1 x_1, \ldots, \mu_d x_d)$, where $\mu_1, \ldots, \mu_d$ are integers greater or equal than 2. So, theorem A is an immediate consequence of proposition 3.13 and theorem 2.9.

4. Genericity of the transversality condition

Recall that for $f_0 \in C^2(T^u, \mathbb{R}^d)$, $\phi_1, \ldots, \phi_d \in C^\infty(T^u, \mathbb{R}^d)$ and $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$, we define $f_t(x) = f_0(x) + \sum_{k=1}^d t_k \phi_k(x)$ and $T_t = T(E, C, f_t)$. For $a \in P(x)$, $1 \leq n \leq \infty$, we have the corresponding $S(x, a; t) = \sum_{i=1}^n C^{-i} f_t([a_i](x))$.

For a point $x \in T^u$ and a sequence $\sigma = (a_0, a_1, \ldots, a_k)$ of words in $I^\infty(x)$, we consider the affine map $\psi_{x, \sigma} : \mathbb{R}^d \to (M(d \times u))^k$ defined by

$$\psi_{x, \sigma}(t) = \left(DS(x, a_i; t) - DS(x, a_0; t)\right)_{i=1, \ldots, k}$$

where $M(d \times u)$ is the space of the matrices $d \times u$ with real entries.

Each entry of $\psi_{x, \sigma}$ is approximately the difference between the images of $T_t^u$ restricted to $R([a_i]_{\sigma}) \times \{0\}$ and to $R([a_0]_{\sigma}) \times \{0\}$.
Assuming that \( s \gg kdu \), let us denote by \( \text{Jac}\psi_{k,\sigma}(t) \) the supremum of the Jacobian of the restrictions of \( \psi_{k,\sigma} \) to \( k \)-dimensional subspaces:

\[
\text{Jac}(\psi_{k,\sigma})(t) = \sup_{\dim L = kdu} |\det D\psi_{k,\sigma}(t)|.
\]

(36)

As \( \psi_{k,\sigma} \) is an affine map, \( \text{Jac}(\psi_{k,\sigma})(t) \) does not depends on \( t \). Then in the sequel we write only \( \text{Jac}(\psi_{k,\sigma}) \).

The following definition is important in the proof of theorem C, because it corresponds to the kind of family \( T_i \) that we shall construct in order to prove that the transversality condition is generic.

**Definition 4.1.** For every integer \( n \geq 1 \), we say that the family \( T_i \) is \( n \)-generic on \( A \subset \mathbb{T}^n \) if for any \( x \in A \), every \( D \geq n^3 \) and any sequence \( (a_0, a_1, \ldots, a_D) \) of \( D + 1 \) words in \( P^\infty(x) \) such that \( [a_0], [a_1], \ldots, [a_D] \) are mutually distinct, taking \( \kappa = \left\lfloor \frac{D}{2n} \right\rfloor \), there exists a subsequence \( \sigma = (b_0, b_1, \ldots, b_s) \) of \( (a_0, a_1, \ldots, a_D) \) of length \( \kappa + 1 \) such that \( b_0 = a_0 \) and \( \text{Jac}(\psi_{k,\sigma}) > \frac{1}{2} \).

The proof of theorem C is divided into two parts: one corresponds to the construction of functions \( \phi_1, \ldots, \phi_s \) for which the family \( T_i \) is \( n \)-generic for some large value of \( n \) (proposition 4.2) and the other checks that the transversality condition is valid for almost every parameter \( t \) for a certain generic family (theorem 2.12).

**Proposition 4.2.** Given \( u \geq d \), \( E \in E(u) \), \( C \in C(d) \), there exists an integer \( n_0 \) such that for every \( n \geq n_0 \) there exist functions \( \phi_k \in C^\infty(\mathbb{T}^u, \mathbb{R}^d) \), \( 1 \leq k \leq s \), such that for every \( f_0 \in C^2(\mathbb{T}^u, \mathbb{R}^d) \) the corresponding family \( T_i \) is \( n \)-generic on \( U_x \).

### 4.1. Construction of generic families

The construction of generic families in proposition 4.2 can be reduced to a local version of itself.

**Proposition 4.3.** [Local version of proposition 4.2] Given \( u \geq d \), \( E \in E(u) \), \( C \in C(d) \), for every point \( x \in \mathbb{T}^u \) there exists an integer \( n_0 \) such that for every \( n \geq n_0 \) there exists a neighborhood \( U_x \) of \( x \) and functions \( \phi_k \in C^\infty(\mathbb{T}^u, \mathbb{R}^d) \), \( 1 \leq k \leq s \), such that for every \( f_0 \in C^2(\mathbb{T}^u, \mathbb{R}^d) \) the corresponding family \( T_i \) is \( n \)-generic on \( U_x \).

Assuming proposition 4.3 we can prove proposition 4.2.

**Proof of proposition 4.2.** Consider \( n_0 \) as given by proposition 4.3: for every \( n \geq n_0 \) there exists a finite set of points \( x_k \), \( 1 \leq k \leq m \), such that the neighborhoods \( U_{x_k} \) given by proposition 4.3 cover \( \mathbb{T}^u \). Associated with each \( x_k \) and \( U_{x_k} \) we have the \( C^\infty \) functions \( \phi_1^{(k)}, \ldots, \phi_s^{(k)} \).

Denote by \( \{\phi_i\}_{i=1}^{m} \) the union of these functions and \( t_i \) the parameter corresponding to the function \( \phi_i \). For every \( x \in \mathbb{T}^u \), take \( k_0 \) such that \( x \in U_{x_{k_0}} \) the Jacobian of \( \psi_{k,\sigma} \) is greater than the Jacobian of \( \psi_{k,\sigma} \) restricted to the subspace \( W \) generated by \( \{t_i\}_{i=1}^{m(k_0)} \), which satisfies \( \text{Jac}(\psi_{k,\sigma}^w) > \frac{1}{2} \).

Given \( x \in \mathbb{T}^u \) and \( n \in \mathbb{N} \), we will consider a family of functions \( \phi_i^n \), \( 1 \leq k \leq ud \), \( a \in I^u(x) \), that are supported on a neighborhood of the pre-image \( a(x) \). Deformations of \( f \) along \( \phi_i^n \) will barely be relevant in the expression of \( S(x, b, t) \) for \( b \in I^\infty(x) \) if \( [b_i](x) \) is far from \( a(x) \) for small values of \( i \).
Proof of proposition 4.3. For $x \in \mathbb{T}^n$, $n \in \mathbb{N}$ and $b \in P^n(x)$, define $E(b, x) = \{a \in P^n(x) \mid E'(b(x)) = a(x) \text{ for some } i \geq 0\}$. The points $a(x)$, for $a \in E(b, x)$, are in the forward orbit of $b(x)$.

Fix some positive integer $k_i$ such that
\[
4 \delta dN^2(1 - \bar{\lambda}_x\mu^{-1})^{-1} \lambda^{-1}\pi(\bar{\lambda}_x\mu^{-1})^{k_i} < 1, \tag{37}
\]
and define the numbers
\[
\nu(n) = k_i n \quad \text{and} \quad \epsilon(n) = 4 \bar{\lambda}(1 - \bar{\lambda}_x\mu^{-1})^{-1}[\lambda^{-1}\pi(\bar{\lambda}_x\mu^{-1})^{k_i+1}]^n, \tag{38}
\]
These choices ensure that there is an integer $n_0$ such that
\[
(1 - 4\delta dN^n \epsilon(n))^N > \frac{1}{2}, \tag{39}
\]
and
\[
n + \nu(n) + 2 < \frac{n^2}{3} < \left[ \frac{n^3}{n + 1} \right] - \frac{n^3}{2n} < \left[ \frac{D}{n + 1} \right] - \frac{D}{2n}, \tag{40}
\]
for every $n \geq n_0$ and every $D \geq n^3$. Inequality (39) is used in (49) and (40) is used in lemma 4.5. We also suppose that $n_0$ is large enough such that $\|E\|^n > 6$.

For this value of $n = \nu(n)$, with $n \geq n_0$, let us consider $\epsilon_0 = \epsilon_0(n) > 0$ small such that $\epsilon_0 < \gamma$ and
\[
E'(B(b(x), \epsilon_0)) \cap B(a(x), \|E\|^n + \nu \epsilon_0) \neq \emptyset \text{ only if } E'(b(x)) = a(x) \tag{41}
\]
for every $0 \leq i \leq n + \nu$ and $a, b \in P^n(x)$.

Remark 4.4. If $a(x) \neq b(x)$ for $a, b \in P^n(x)$, then, in particular:
\[
B(b(x), \epsilon_0) \cap B(a(x), \|E\|^n + \nu \epsilon_0) = \emptyset. \tag{42}
\]
By the choice of $\epsilon_0$, the distance between $B(b(x), \epsilon_0)$ and $B(a(x), \epsilon_0)$ is at least $(\|E\|^n + 2\epsilon_0) > 4\epsilon_0$.

Denote by $E_{j'}^i$ the matrix that has entry 1 in the intersection of the $i'$th line row of the $j'$th column and with all the other entries equal to 0. For each $a \in P^n(x)$, $1 \leq i' \leq d, 1 \leq j' \leq u$, we consider a C$^\infty$ function $\phi_{j',i'}^a : \mathbb{T}^n \to \mathbb{R}^d$ such that:

- $\phi_{j',i'}^a$ is supported in $B(a(x), \pi^{-n}\epsilon_0)$;
- $D\phi_{j',i'}^a(y) = C^{-n+1}E_{j',i'}^a E^n$ for every $y \in B(a(x), \pi^{-n}\epsilon_0/3)$;
- $\|D\phi_{j',i'}^a(y)\| < 2\pi^{-n+1}\pi^n$ for every $y \in B(a(x), \pi^{-n}\epsilon_0)$.

Define $U_x := B(x, (\mu^{-1})^{-n}\epsilon_0/3)$. For every $y \in U_x$, there exists a bijection $\Phi_{j',i'} : I^\infty(x) \to I^\infty(y)$ given by $\Phi_{j',i'}(a) = (\hat{a}_j)_{j=1}^\infty$ such that $\hat{a}_j = \pi(h_{j'}^a(y))$, where $h_{j'}^a$ is the inverse branch of $E'$ defined in (30). Note that
\[
d([a], [\Phi_{j',i'}^{-1}(a)], (x)) \leq \|E^{-1}\| \|d(x, y)\| \tag{42}
\]
for every \(a \in I^\infty(y)\) and every \(i \geq 0\). If \(\pi(x) = \pi(y)\), then \(\Phi_{xy}\) is simply the identity. Also, note that by the definition of \(U_\varepsilon\) and (42), \([a]|_n(y) \in B([\Phi_{xy}^{-1}(a)]|_n(x), \mu^{-n} \varepsilon_0/3)\).

**Lemma 4.5.** Given a sequence \(\sigma = (a_0, a_1, \cdots, a_D)\) in \(I^\infty(x)\) with \([a_0]|_n, \cdots, [a_D]|_n\) distinct and \(\kappa = \lfloor \frac{D}{2\varepsilon} \rfloor\) there exists a subsequence \(\hat{\sigma} = (b_0, b_1, \cdots, b_n)\) in \(I^\infty(x)\), with \(b_0 = a_0\), such that

\[
\phi^{[b|_y]}([\Phi_{xy}(b)]|_y(y)) = 0
\]

(43)

for every \(y \in B(x, \varepsilon_0)\), \(1 \leq l \leq \kappa\), \(1 \leq i' \leq d\), \(0 \leq l \leq \kappa\), \(l \neq i'\) and \(i = 0, 1, \cdots, n + \nu\). Moreover, for \(l = l'\), (43) holds if \(i \neq n\).

**Proof of lemma 4.5.** Let us first state two claims corresponding to consequences of (41) that shall be used in the proof of lemma 4.5.

**Claim 4.6.** For every \(a, b \in I^\infty(x)\) and \(0 \leq p \leq 2n + \nu\), we have the following:

(a) If \([a]|_n \not\in \mathcal{E}([b]|_n, x)\) and \([b]|_n \not\in \mathcal{E}([a]|_n, x)\), then

\[
B([a]|_n(x), \varepsilon_0) \cap B([b]|_p(x), \varepsilon_0) = \emptyset.
\]

(b) If \([a]|_n(x)\) is not \(E\)-periodic and \(p \neq n\), then

\[
B([a]|_n(x), \varepsilon_0) \cap B([a]|_p(x), \varepsilon_0) = \emptyset.
\]

**Proof.**

(a) For \(0 \leq p \leq n\), since \(E^{n-p}(b|_n(x)) \neq [a]|_n(x)\), (41) implies that

\[
E^{n-p}(B([b]|_n(x), \varepsilon_0)) \cap B([a]|_n(x), \varepsilon_0) = \emptyset.
\]

The inclusion \(B([b]|_p(x), \varepsilon_0) \subseteq E^{n-p}(B([b]|_n(x), \varepsilon_0))\) implies (44) in this case.

For \(n < p \leq 2n + \nu\), since \(E^{p-n}[a]|_n(x) \neq [b]|_n(x)\), (41) implies that

\[
E^{p-n}(B([a]|_n(x), \varepsilon_0)) \cap B([b]|_n(x), \|E\|^{n+p} \varepsilon_0) = \emptyset.
\]

It follows (44) by taking preimages with respect to \(E^{-(p-n)}\) and noticing that

\[
B([b]|_p(x), \varepsilon_0) \subseteq E^{-(p-n)}(B([b]|_n(x), \|E\|^{n+p} \varepsilon_0)).
\]

(b) For \(0 \leq p < n\), by \([a]|_n(x) \neq E^{n-p}([a]|_n(x))\) and (41), we have:

\[
B([a]|_n(x), \|E\|^{n+p} \varepsilon_0) \cap E^{n-p}(B([a]|_n(x), \varepsilon_0)) = \emptyset.
\]

So (45) follows in this case since \(B([a]|_p(x), \varepsilon_0) \subseteq E^{n-p}(B([a]|_n(x), \varepsilon_0))\).

For \(n < p \leq 2n + \nu\), the fact that \([a]|_n(x) \neq E^{p-n}([a]|_n(x))\) together with (41) implies that

\[
B([a]|_n(x), \|E\|^{n+p} \varepsilon_0) \cap E^{p-n}(B([a]|_n(x), \varepsilon_0)) = \emptyset.
\]

Once again, (45) follows by taking the preimages with respect to \(E^{-(p-n)}\) and using that \(B([a]|_p(x), \varepsilon_0) \subseteq E^{-(p-n)}(B([a]|_n(x), \|E\|^{n+p} \varepsilon_0))\).

Now we proceed to the proof of lemma 4.5. First, let us note that the cardinality of each \(\mathcal{E}([a]|_n, x)\) is at most \(n + 1\). Actually, if some \(a \in I^p(x)\) is in \(\mathcal{E}([a]|_n, x)\) and \(a \neq [a]|_n\), then \(x\) is periodic.
and its period is at most \( n \), because in this case there exists \( k \geq 0 \), such that \( E^k(\{a\}_n(x)) = a(x) \) and therefore \( E^p(x) = E^p(E^k(\{a\}_n(x))) = E^p(a(x)) = x \). This implies that there must exist at most one \( a \in \mathcal{P}(x) \) such that \( E^i(x) = a(x) \) for some \( i \geq 0 \), because if \( E^p(x) = a(x) \) for another \( \bar{a} \in \mathcal{P}(x) \) and \( p > i \) then \( i + n \) and \( p + n \) are multiples of the period of \( x \). Consequently, \( p - i \) is a multiple of the period of \( x \) and \( a(x) = E^i(x) = E^p(E^{p-i}(x)) = E^p(x) = \bar{a}(x) \), that is, \( a = \bar{a} \). So, for each \( 0 \leq i < n \) there is at most one \( a \in \mathcal{P}(x) \) such that \( a(x) = E^i(\{a\}_n(x)) \), and there also exists at most one \( a \in \mathcal{P}(x) \) such that \( a(x) = E^i(\{a\}_n(x)) \) for some \( i \geq n \).

As a consequence, there are at least \( \lfloor \frac{D}{\# \mathcal{F} + 1} \rfloor \) elements \( \{a_{\xi(j)}\} \) such that \( a_{\xi(j)} \in \mathcal{E}(\{a_{\xi(j)}\}_n, x) \) for \( j \neq j' \).

Given \( y \in \mathcal{B}(x, \varepsilon_0) \), equation (42) implies that \([\Phi_{x,0}(a_{\xi(j)}), y]\) is contained in \( \mathcal{B}(\{a_{\xi(j)}\}_n, x) \). Since \( B(\{a_{\xi(j)}\}_n, x, \varepsilon_0) \cap B(\{a_{\xi(j)}\}_n, x, \varepsilon_0) = \emptyset \) (by claim 4.6) and \( \Phi_{x,0}(\{a_{\xi(j)}\}_n) \) is supported in \( B(\{a_{\xi(j)}\}_n, x, \varepsilon_0) \), we get that
\[
\phi_{x,j}^{\{a_{\xi(j)}\}_n}(\Phi_{x,0}(\{a_{\xi(j)}\}_n)) = 0
\]
for every \( i = 0, \ldots, 2n + \nu \) and every \( y \in \mathcal{B}(x, \varepsilon_0) \), if \( j \neq j' \).

Since all the points \( \{a_{\xi(j)}\}_n \) are distinct, by remark 4.4, for each \( i = 1, \ldots, n + \nu \) there is at most one \( l = (i) \) such that \( B(\{a_{\xi(j)}\}_n, x, \varepsilon_0) \cap B(\{a_{\xi(j)}\}_n, x, \varepsilon_0) = \emptyset \). Denote by \( \mathcal{K} = \{a_{\xi(j)}\}_n \) and consider \( \mathcal{J} = \{a \in \mathcal{K} | a \neq 0 \} \) and \( a \neq a_{\xi(j)} \) for \( i = 1, \ldots, n + \nu(n) \), then \( \mathcal{J} \) has at least \( \lfloor \frac{D}{\# \mathcal{F} + 1} \rfloor - n - \nu(n) - 2 \) elements. Since \( \kappa = \lfloor \frac{D}{\# \mathcal{F} + 1} \rfloor \) (by inequality (40)), we can consider \( \mathcal{B}_1, \ldots, \mathcal{B}_k \), \( J \).

Consider also \( \mathcal{B}_0 = \mathcal{B}_0 \) and the subsequence \( \sigma = (\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_k) \); this choice implies that \( \phi_{x,j}^{\{a_{\xi(j)}\}_n}(\Phi_{x,0}(\{a_{\xi(j)}\}_n)) = 0 \) for every \( y \in \mathcal{B}(x, \varepsilon_0) \), for \( l = 1, \ldots, \kappa \) and \( 0 \leq i \leq n + \nu \).

Finally, for fixed \( 1 \leq \ell \leq \kappa \), \( \{a_{\xi(j)}\}_n \) is not periodic, because otherwise \( \{a_{\xi(j)}\}_n \in \mathcal{E}(\{a_{\xi(j)}\}_k, x) \) for all \( k \neq I \), which is not the case. Then, claim 4.6 implies that
\[
\mathcal{B}(\{a_{\xi(j)}\}_n, x, \varepsilon_0) \cap \mathcal{B}(\{a_{\xi(j)}\}_n, x, \varepsilon_0) = \emptyset
\]
for every \( i = 0, 1, \ldots, 2n + \nu \setminus \{n\} \). Thus, for \( l = I', (43) \) holds if \( i \neq n \).

Let us continue the proof of proposition 4.3.

Given any sequence \( (a_0, a_1, \ldots, a_d) \) in \( \mathcal{F}^\infty(y) \) with \( \{a_i\}_n \) distinct, we consider \( \bar{a} := \Phi_{x,d}^{-1}(a_i) \) and the sequence \( \bar{a}_0 := (a_0, a_1, \ldots, a_0) \) in \( \mathcal{F}^\infty(x) \), which also has \( \{a_i\}_n \) distinct, and consider \( \bar{a}_0 = (\bar{a}_0, a_1, \ldots, a_0) \); the subsequence of \( \bar{a}_0 \) given by lemma 4.5. Denote \( b_i = \Phi_{x,d}(b_i) \) in \( \mathcal{F}^\infty(y) \) for \( l = 0, 1, \ldots, \kappa \) and consider \( \bar{a} = (b_0, b_1, \ldots, b_k) \) a subsequence of \( (a_0, a_1, \ldots, a_d) \). We will prove in the following that \( \text{Jac}(\psi_{x,\sigma}) > \frac{1}{\ell} \).

Identifying a point \( t \in \mathbb{R}^\ell = \mathbb{R}^{d \# \mathcal{F}(y)} \) with \( (T^y)_{a \in \mathcal{F}(y)} \), where \( T^y = [\phi_{x,j}^{\mathcal{F}}] \) is a \( d \times u \) matrix, we consider \( f_i(y) = f(y) + \sum_{a \in \mathcal{F}} \phi_{x,j}^{\mathcal{F}}(y), a \in \mathcal{F}(y), 1 \leq i \leq \ell \) \leq d, \( 1 \leq \ell' \leq u \), where \( \bar{a} = (\Phi_{x,d}^{-1}(\bar{a}))_n \) in \( \mathcal{F}(x) \), for some \( \bar{a} \in \mathcal{F}(y) \) such that \( \{a\}_n = a \). The map \( \psi_{x,\sigma} : \mathbb{R}^\ell \rightarrow (M(d \times u))^\infty \) is an affine map of the form
\[
\psi_{x,\sigma}(t) = A_j(y) + \sum_{a \in \mathcal{F}} \phi_{x,j}^{\mathcal{F}}(y)
\]
(46)
where the \( \ell \)th coordinate of the linear term \( L_{x,j}(y) \), \( 1 \leq \ell \leq \kappa \), is given by the series
\[
[L_{x,j}(y)]_i = \sum_{i=1}^{\infty} C^{-1}(D\phi_{x,j}^{\mathcal{F}}([b]_n)) - D\phi_{x,j}^{\mathcal{F}}([b]_n) \mathcal{E}^{-1}.
\]
(47)
Let us write \( L_{\bar{a},j}(y) = L_{\bar{a},j}(y) + L_{\bar{a},j}(y) \), where
\[ [L^1_{a_{l'}}^l(y)]_j = \sum_{i=1}^{n+\nu} C^{-1}\left(D\phi^l_{l'}([b]_i)(y) - D\phi^l_{l'}([b]_0)(y)\right)E^{-1}, \]  

and consider \( W_0 = \{(T^a)_{a \in P_v(y)} | T^a = 0 \text{ if } a \neq [b]_a \text{ for every } 1 \leq l \leq \kappa \} \) a \( \kappa \text{-ud} \)-dimensional subspace of \( \mathbb{R}^{\nu} \). We identify \( W_0 \) with \((M(d \times u))^\kappa\) and we take the canonical base of \((M(d \times u))^\kappa\) formed by the vectors \( V_{ij}^b \) defined by

\[ V_{ij}^b = (M_i, M_2, \ldots, M_n) \]

where \( M_l = 0 \) if \( l' \neq l \) and \( M_l = E_{l,j} = (a_{l,j}) \) with \( a_{ij} = 1 \) and all the other entries are zero.

From lemma 4.5, for every \( 0 \leq l \leq \kappa, 1 \leq l' \leq \kappa \) and \( i = 0, 1, \ldots, n+\nu \), the value of \( \phi^l_{l'}([b]_j)(y) \) is non-zero only if \( l = l' \) and \( i = n \). Moreover, since \( y \in U_v \), the value of \( D\phi^l_{l'}([b]_j)(y) \) is equal to \( C_{n+1}E_{l,j}E^q \). Then, from (48), \( [L^1_{b_l,l'}^l(y)]_j = 0 \), if \( l' \neq l \) and \( [L^1_{b_l,l'}^l(y)]_j = E_{l,j} \). Therefore, if we denote by \( G_{l,j}(\psi_{l,j}) \) the linear part of \((\psi_{l,j})w_0\), we have

\[ G_{l,j}(V_{ij}^b) = L_{b_l,l'}(y) = V_{ij}^b + L_{b_l,l'}^2(y). \]

Since \( \|C^{-1}\| \leq \mathcal{X}^{-1}, \|E^{-1}\| \leq \mathcal{L}^{-r} \) and \( \|D\phi^l_{l'}([b]_j)(y)\| \leq 2\lambda^{-n+1}p^n \), the norm of the supremum of \( L_{b_l,l}^2(y) \) is bounded by:

\[ \sum_{j=0}^{\nu} 4\lambda^{-n+1}\mathcal{X}^{-1}\mathcal{L}^{-r} \leq 4\lambda^{-n+1}\mathcal{X}^{\nu}\mathcal{L}^{-n-\nu}(1 - \mathcal{L}^{-1})^{-1} = \epsilon(n), \]

where \( \nu = \nu(n) \) and \( \epsilon(n) \) were defined in (38).

Therefore, \( G_{l,j}(\psi_{l,j}) \) is represented in the canonical base by a \( \kappa \text{-ud} \times \kappa \text{-ud} \) matrix \( [G_{l,j}^\kappa] = \text{Id} + R \), where \( \text{Id} \) is the identity matrix and \( R \) has all entries bounded by \( \epsilon(n) \). By \( [14] \) and \( \kappa \leq N^n = |\det E|^n \), the determinant of \( G_{l,j}(\psi_{l,j}) \) is bounded below by

\[ (1 - \kappa du \epsilon(n))^{'\text{ud}} \geq \left(1 - 4\kappa du N^n \epsilon(n)\right)^{'\text{ud}} > \frac{1}{2}. \]

Following that \( \text{Jac}(\psi_{l,j}) \geq \text{Jac}(\psi_{l,j}|w_0) > \frac{1}{2} \). \( \square \)

4.2. The amount of non-transversality is not too large in generic families

In order to prove theorem 2.12, we will first see that if \( T \) does not satisfy the transversality conditions then there must be a large amount of words \( a, \in I^\infty \) that are non-transversal in some \( c \) with distinct truncations \( [a]_a \).

For each \( q \geq 0 \), we consider an integer \( p = p(q) := \left\lceil \frac{-\log \log \sqrt{q}}{\log \sqrt{q}} \right\rceil + 1 \) (which satisfies \( \sqrt{q}^{\mathcal{L}^{-p}} < \theta^q \)) and we consider the constant \( B = \lim_{q \to \infty} \frac{p(q)}{q} \). Fix one point \( x_\in \mathcal{R}(c) \) for each \( c \in I^{\mathcal{R}(q)} \). Let us fix integers \( n_0, D_0, \kappa_0 \geq 2 \) such that

\[ (D_0 + 1)\mathcal{L}^{-n_0} < \frac{1}{2} \tag{50} \]

\[ N^\kappa_0 + B + 1 \theta^{n - d + 1} \kappa_0 < 1 \tag{51} \]

\[ \kappa_0 + 1 < \frac{D_0}{2n_0} \tag{52} \]

\[ n_0^3 < D_0. \tag{53} \]
There exist such integers above because $J > 1$ and because the map $C$ is in $C(d; E)$. The choice of each one will be used in the continuation: (50) is used in (54) and (51) is used in (61), (52) and (53) are used in the proof of theorem 2.12 to obtain $κ = κ₀$ for $D = D₀$ in the definition of $r₀$-generic family.

**Lemma 4.7.** If $f ∈ T$, then for every $q_0 ≥ 1$ there exists $q > q₀$ such that there exists a word $c ∈ I^{p(q)}$ and $I + D₀$ words $a_i ∈ I^p(c)$ with $|a_i|_{b_i}$ distinct such that for any $1 < i < D₀$
\[ m(DS(x_0, a_i) - DS(x_0, a_0)) ≤ 6θ_i α₀ \]

**Proof.** Given $q₀$, we consider $q₀$ such that $q₀ > q₀ > q_0$. Since $f ∈ T$, we can take $q > q₀$ large enough that there exists a word $c ∈ I^{p(q)}$, a subset $L ⊂ I^q(c)$ and some $u₀ ∈ L$ such that $♯L ≥ J^q$ and for every $u ∈ L$ there exist points $x_u$ and $y_u$ in $R(c)$ satisfying:
\[ m(DS(x_u, u) - DS(y_u, u)) ≤ 3θ_i α₀ \]

Fixing some $x ∈ R(c)$, for each $d$-dimensional subspace $W$ there exists a unitary vector $v_W ∈ W$ such that $\|DS(x_u, u) - DS(y_u, u)\|v_W\| ≤ 3θ_i α₀$ (recall the definition (7)) which implies that the value of $\|DS(x, u) - DS(x, u)\|v_W\|$ is at most:
\[ \|DS(x, u)v_W - DS(x, u)v_W\| + \|DS(x, u)v_W - DS(x, u)v_W\| + \|DS(x, u)v_W - DS(x, u)v_W\| ≤ \sqrt{dJ}^{-p(\tilde{q})} α₀ + 3θ_i α₀ + \sqrt{dJ}^{-p(\tilde{q})} α₀ ≤ 5θ_i α₀. \]

In the last estimates we use $\|DS(., a)\| ≤ α₀$ for all $a ∈ I^q(., a)$ and the diameter of $R(c)$ is bounded by $\sqrt{dJ}^{-p(\tilde{q})}$.

Taking the supremum on $W$, we have
\[ m(DS(x, u) - DS(x, u)) ≤ 5θ_i α₀. \]

To obtain word $a_i$ with distinct truncations $|a_i|_{b_i}$ for each $0 ≤ j ≤ |\tilde{q}/n_0|$ and $a ∈ L$, we define
\[ L(j, a) = \{ b ∈ L, |a|_{b_i} = |b|_{b_i} \} \]
if $j ≥ 1$ and $L(0, a) = L$.

Note that $L(j_1, a) ⊂ L(j_2, a)$ if $j_1 < j_2$ and that $L(j, a₁) ∩ L(j, a₂) = \emptyset$ if $|a₁|_{b_i} ≠ |a₂|_{b_i}$. Define
\[ H(j, a) = \{ b ∈ L | b ∈ L(j, a) ∩ L(j - 1, aᵢ) \} \text{ and } h_j := \max_{a_i ∈ L} \#H(j, a) \]
for $j ≥ 1$ and $h₀ = L(0, a) = J^q$.

Since $L(j, a)$ has the same cardinality as $I^{p(q₀/|a|_{b_i})}$, remark 2.2 implies that $h_j ≤ N^{p(\tilde{q}/n_0)}$, so there exists $j ≤ |\tilde{q}/n_0|$ such that $h_{j+1} < J^{-m_2/2} j$. Let $j₀$ be the minimum of such integers $j$ and put $q = \tilde{q} - m_j$. By the minimality of $j$, we have that $h_{j₀} ≥ J^q$. We also have that $q ≥ q₀ > q₀$. 

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Considering $v_0$ such that $h_{j_e} = \#H(j_e, v_0)$, the set $H(j_e, v_0)$ contains at least $1 + D_0$ sets $H(j_e + 1, 0)$, because

$$h_{j_e} - (D_0 + 1)h_{j_e+1} > h_{j_e} - (D_0 + 1)J^{-\nu_0/2}h_{j_e} > 0, \quad (54)$$

due to (50).

We consider the disjoint sets $H(j_e + 1, u_0), H(j_e + 1, u_1), \cdots, H(j_e + 1, u_{D_0})$. In particular, $[u_i]([j_e + 1]_s)$ are distinct for $i' = 0, 1, \cdots, D_0$.

So, there exist $b \in \mathcal{I}^{l-q}$ and $a_i \in \mathcal{I}^s$, $0 \leq i \leq D_0$ such that $b a_i = u_i \in H(j_e, v_0)$ for $0 \leq i \leq D_0$ and that $[a_i]_{|a_i|} \neq [a_j]_{|a_j|}$ if $i \neq j$.

Finally, from

$$DS(x, ba_i) = DS(x, b) + C^{l-d}DS(b(x), a_i)E^{-\tilde{q}+q} \quad (55)$$

we have that:

$$m(DS(b(x), a_i) - DS(b(x), a_0)) \leq 50^q \alpha_0.$$

Taking $c \in I^{l(q)}$ such that $b(x) \in R(c)$, we have $\|b(x) - x_c\| \leq \sqrt{\tilde{d}} \|x_c\| \leq \theta^q$, which implies the lemma. \hfill \Box

To finish the proof of theorem 2.12, we will use two lemmas of linear algebra.

**Lemma 4.8.** Given integers $s$ and $k$, there exists a constant $C_3 > 0$ such that if $G : \mathbb{R}^s \rightarrow \mathbb{R}^k$ is an affine function with $\text{Jac}(G) > \delta$, then

$$m_0(G^{-1}(Y) \cap (-1, 1)^s) \leq C_3 \frac{m_k(Y)}{\delta} \quad (56)$$

for every measurable set $Y \subset \mathbb{R}^k$.

**Proof.** Immediate from the Jacobian of an affine map. \hfill \Box

In the next lemma, we use the canonical identification of $M(d \times u)$ with $\mathbb{R}^{du}$ and we identify the volume in the space $M(d \times u)$ as the Lebesgue measure $m_{du}$ in $\mathbb{R}^{du}$.

**Lemma 4.9.** For every $u \geq d$, there exists a constant $C_4 > 0$ such that the set

$$X(r) := \{M \in M(d \times u), \|M\| \leq 2\alpha_0, m(M) < r\}$$

has volume bounded by $C_4 r^{u-1}$ for every $r > 0^3$.

**Proof.** Let $r(i)$ be the $i$th row of the matrix $M$. First, we claim that $m(M)$ is equal, up to a bounded factor, to

min$_i d\left(r(i), \text{span}_{j \neq i}(r(j)) \right)$.

**Claim 4.10.** For every $M \in M(d \times u)$, it is valid that:

$$m(M) \leq \min_i d\left(r(i), \text{span}_{j \neq i}(r(j)) \right) \leq \sqrt{d} m(M).$$

\footnote{The authors are grateful to Quas for pointing this out.}
Proof claim 4.10. Consider $D := \min_i d(i, \text{span}_{j \neq i}(r(j)))$. Recall that $m(M) = m(M^T)$ and that $r(i) = M^T(e_i)$, where $M^T$ is the transpose matrix of $M$ and $\{e_i\}_{i=1}^d$ the canonical basis of $\mathbb{R}^d$. Considering $v = (x_1, \ldots, x_d) \in \mathbb{R}^d$ a unitary vector such that $\|M^T(v)\| = m(M^T) = m(M)$ and $i_1$ such that $|x_{i_1}| = \max_j |x_j|$, we have that $|x_{i_1}| \geq \frac{1}{\sqrt{d}}$. Then

$$\sqrt{d}m(M) \geq \frac{\|M^T(v)\|}{|x_{i_1}|} = \left\| \frac{x_{i_1}}{|x_{i_1}|} r(i_1) + \sum_{j \neq i_1} \frac{x_j}{|x_{i_1}|} r(j) \right\| \geq d \left(r(i_1), \text{span}(r(j)) \right) \geq D.$$

On the other hand, let $i_2$ be such that $d \left(r(i_2), \text{span}_{j \neq i_2}(r(j)) \right) = D$. There exist real numbers $b_j, j \neq i_2$, such that $D = \|r(i_2) + \sum_{j \neq i_2} b_j r(j)\|$. Then

$$D = \|M^T(e_{i_2}) + \sum_{j \neq i_2} b_j e_j\| \geq m(M^T)\|e_{i_2} + \sum_{j \neq i_2} b_j e_j\| \geq m(M).$$

Denote a matrix $M \in M(d \times u)$ by $M = [r(1), \ldots, r(d)]^T$, where $r(j) \in \mathbb{R}^u$ is its $j$th row. It follows from claim 4.10 that the set $\mathcal{X}(r)$ is contained in $\bigcup_{i=1}^u \mathcal{M}_i(r)$, where $\mathcal{M}_i(r) \subset M(d \times u)$ is the set of matrices $M = [r(1), \ldots, r(d)]^T$ such that $\|r(k)\| \leq 2\alpha_0$ for every $k = 1, 2, \ldots, d$ and $d(r(i), \text{span}_{j \neq i}(r(j))) < \sqrt{d}r$. Consider that $m_i$ is the Lebesgue measure in $\mathbb{R}^u$, then the volume in $M(d \times u)$ is the product measure $m_1 \times \cdots \times m_d$.

For every choice of $d - 1$ rows of $M$, their span in $\mathbb{R}^d$ is a $(d - 1)$-dimensional subspace $H$ and the intersection of the neighborhood of radius $\sqrt{d}r$ of $H$ with the ball $B(0, 2\alpha_0)$ of $\mathbb{R}^u$ has volume bounded from above by $(4\alpha_0)^{d-1}(\sqrt{d})^{u-d+1}r^{u-(d-1)}$.

So, for fixed $r_j \in B(0, 2\alpha_0) \subset \mathbb{R}^u, j \in \{1, \ldots, d\} - \{i\}$, if we denote

$$W_{r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_d} := \{r_i \in B(0, 2\alpha_0) | r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_d \} \in \mathcal{M}_i(r),$$

we have

$$m_i(W_{r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_d}) \leq (4\alpha_0)^{d-1}(\sqrt{d})^{u-d+1}r^{u-(d-1)}. \quad (57)$$

Consider $B = B(0, 2\alpha_0) \subset \mathbb{R}^u$, and $m_i = m_1 \times \cdots \times m_{i-1} \times m_{i+1} \cdots \times m_d$ the product of the measures $m_j$ except $m_i$. By Fubini’s theorem,

$$\text{vol}(\mathcal{M}_i(r)) \leq \int_B \cdots \int_B \mathbb{1}_{\mathcal{M}_i(r)}([r(1), \ldots, r(d)])^T \, dm_1 \cdots dm_d \leq \int_B m_i(W_{r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_d}) \, dm_i \leq \int_B (4\alpha_0)^{d-1}(\sqrt{d})^{u-d+1}r^{u-d} \, dm_i \leq \int_B (4\alpha_0)^{d-1}(\sqrt{d})^{u-d+1}r^{u-d+1} \, dm_i \leq (4\alpha_0)^{d-1}(\sqrt{d})^{u-d+1}r^{u-(d-1)} \leq (4\alpha_0)^{d-1}(\sqrt{d})^{u-d+1}r^{u-d+1} \leq (4\alpha_0)^{d-1+1}(\sqrt{d})^{u-d+1}r^{u-d+1},$$

following that the volume of $\mathcal{X}(r)$ is bounded above by $C\alpha_0^{u-d+1}$. \hfill \Box

**Proof of theorem 2.12.** We consider $\{\phi_i\}_{i=1}^\infty$ the functions given by proposition 4.2 for $E$, $C$ and $n = n_0$.

Fixing words of infinite length $a^1, \ldots, a^\infty \in I^\infty$ with $[a^i]_1 = i$, we associate for every word $a$ of finite length a word $\hat{a} = a a^\infty \in I^\infty$. For any sequence $\sigma = (b_i)_{i=0}^{n_0}$ in $(P^\infty)^{1+n_0}$, denote $\hat{\sigma} = (b_i)_{i=0}^{n_0} \in (P^\infty)^{1+n_0}$. 

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Let us consider $T_0 := \{ t \in \mathbb{R}^4 | t_i \in T \}$. If $t \in T_0$, lemma 4.7 implies that for every integer $q_0$ there exists an integer $q \geq q_0$, a word $c \in \mathcal{P}_{(q)}$ and $1 + D_0$ words $a_i \in \mathcal{P}(c)$ with $[a_i]_{n_0}$ distinct such that $m(DS(x_c, a_i) - DS(x_c, a_0)) \leq 60^q \alpha_0$ for any $i = 1, \ldots, D_0$. By the triangular inequality, it holds that

$$m(DS(x_c, \hat{a}) - DS(x_c, \hat{a}_0)) \leq 80^q \alpha_0.$$  \hfill (58)

We consider also the sets $B^q = \{(\sigma, c) \in (\mathcal{P})^{1+q_0} \times \mathcal{P}(q) | \text{Jac}(\psi_{x_c, \hat{a}}) > \frac{1}{2} \}$ and $T(q) := \cup_{(\sigma, c) \in B^q} \mathcal{A}((80^q \alpha_0)^{n_0})$.

Given $x_c$ and the sequence $(\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_n)$, the fact that the family $T^I_{n_0}$ is $m_0$-generic, that $\frac{n_0}{\kappa_0} > \kappa_0$ and $D_0 > n_0^3$ imply that there exists a subsequence $\sigma = (b_0, b_1, \ldots, b_{n_0}) \in (\mathcal{P})^{1+q_0}$ such that each entry of $\psi_{x_c, \hat{a}}(t)$, for $t \in T_0$, is in the set

$$\mathcal{A}(80^q \alpha_0) = \{ M \in M(d \times u), \| M \| \leq 2 \alpha_0 \text{ and } m(M) < 80^q \alpha_0 \},$$

which means that

$$T_0 \subset \liminf_{q \to +\infty} T(q).$$  \hfill (59)

Since for every $i \in \mathcal{T}$ there are exactly $N$ sets $\mathcal{R}(j)$ such that $E(\mathcal{R}(j)) \cap \mathcal{R}(i) \neq \emptyset$, we get that $\#B^q = rN^{p-1}$ and that

$$\#B^q \leq (rN^{q-1})^{|N|+1} rN^{p(q)-1}.$$  \hfill (60)

Putting together lemmas 4.8, 4.9 and (60), we get that the estimate

$$m_s(T(q)) \leq 2^{|N|} rN^{q_0 + q_0 + p(q)-2 + \kappa_0} (2C_4 \alpha_0)^{n-d+1}$$

is valid for infinitely many $q$'s.

So, there is a constant $C_5 > 0$ such that the term in (61) is bounded above by

$$C_5 (N^{n-d+1} r)^q$$

when $q$ is sufficiently large. By the choice of $\kappa_0$, $m_s(T(q))$ converges to zero exponentially fast when $q \to +\infty$, implying, together with (59) and the Borel–Cantelli lemma, $m_s(T_0) = 0$; that is, the conclusion of theorem 2.12.

4.3. Proofs of theorem C and corollary D

Finally, we are able to prove theorem C and corollary D.

Proof of theorem C. Given the map $T = T(E, C, \mathcal{F})$ satisfying the assumptions of theorem C, we consider the functions $\{ \phi_k \}_{k=1}^n \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ given by theorem 2.12 for some $n$ sufficiently large. Then the set of parameters $t = (t_1, \ldots, t_n)$ for which the corresponding map $T_t$ satisfies the transversality condition has full Lebesgue measure. Theorem 2.9 implies that for such maps the SRB measure $\mu_{T_t}$ is absolutely continuous.

Proof of corollary D. When $d = 1$, the map $C$ is just a multiplication by a factor $\lambda \in \mathbb{R}$ and if $E = \mu I$ for some integer $\mu$ $\geq 2$, then the relations in the definition of $C(d; E)$ become $\lambda \in \{ \frac{1}{\mu}, 1 \}$. So we are under the assumptions of theorem C.
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