ON AN EXTENSION OF THE $H^k$ MEAN CURVATURE FLOW OF CLOSED CONVEX HYPERSURFACES

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Abstract. In this paper we prove that the $H^k$ ($k$ is odd and larger than 2) mean curvature flow of a closed convex hypersurface can be extended over the maximal time provided that the total $L^p$ integral of the mean curvature is finite for some $p$.

1. Introduction

Let $M$ be a compact $n$-dimensional hypersurface without boundary, which is smoothly embedded into the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ by the map

$$F_0 : M \rightarrow \mathbb{R}^{n+1}.$$ (1.1)

The $H^k$ mean curvature flow, an evolution equation of the mean curvature $H(\cdot, t)$, is a smooth family if immersions $F(\cdot, t) : M \rightarrow \mathbb{R}^{n+1}$ given by

$$\frac{\partial}{\partial t} F(\cdot, t) = -H^k(\cdot, t)\nu(\cdot, t), \quad F(\cdot, 0) = F_0(\cdot),$$ (1.2)

where $k$ is a positive integer and $\nu(\cdot, t)$ denotes the outer unit normal on $M_t := F(M, t)$ at $F(\cdot, t)$.

The short time existence of the $H^k$ mean curvature flow has been established in [3], i.e., there is a maximal time interval $[0, T_{\text{max}})$, $T_{\text{max}} < \infty$, on which the flow exists. In [2], we proved an extension theorem on the $H^k$ mean curvature flow under some curvature condition. In this paper, we give another extension theorem of the $H^k$ mean curvature flow for convex hypersurfaces.

Theorem 1.1. Suppose that the integers $n$ and $k$ are greater than or equal to 2, $k$ is odd, and $n+1 \geq k$. Suppose that $M$ is a compact $n$-dimensional hypersurface without boundary, smoothly embedded into $\mathbb{R}^{n+1}$ by a smooth function $F_0$. Consider the $H^k$ mean curvature flow on $M$,

$$\frac{\partial}{\partial t} F(\cdot, t) = -H^k(\cdot, t)\nu(\cdot, t), \quad F(\cdot, 0) = F_0(\cdot).$$

If

(a) $H(\cdot) > 0$ on $M$,
(b) for some $\alpha \geq n + k + 1$,

$$||H(\cdot, t)||_{L^\alpha(M \times [0, T_{\text{max}}])} := \left( \int_0^{T_{\text{max}}} \int_M |H(\cdot, t)|^\alpha g(\cdot, t) d\mu(t) dt \right)^{\frac{1}{\alpha}} < \infty,$$

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then the flow can be extended over the time $T_{\text{max}}$. Here $d\mu(t)$ denotes the induced metric on $M_t$.

2. Evolution equations for the $H^k$ mean curvature flow

Let $g = \{g_{ij}\}$ be the induced metric on $M$ obtained by the pullback of the standard metric $g_{\mathbb{R}^{n+1}}$ of $\mathbb{R}^{n+1}$. We denote by $A = \{h_{ij}\}$ the second fundamental form and $d\mu = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n$ the volume form on $M$, respectively, where $x^1, \cdots, x^n$ are local coordinates. The mean curvature can be expressed as

$$H = g^{ij}h_{ij}, \quad g_{ij}\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right)_{g_{\mathbb{R}^{n+1}}}$$

meanwhile the second fundamental forms are given by

$$h_{ij} = -\left(\nu, \frac{\partial^2 F}{\partial x^i \partial x^j}\right)_{g_{\mathbb{R}^{n+1}}}.$$

We write $g(t) = \{g_{ij}(t)\}, A(t) = \{h_{ij}(t)\}, \nu(t), H(t), d\mu(t), \nabla_t$, and $\Delta_t$ the corresponding induced metric, second fundamental form, outer unit normal vector, mean curvature, volume form, induced Levi-Civita connection, and induced Laplacian operator at time $t$. The position coordinates are not explicitly written in the above symbols if there is no confusion.

The following evolution equations are obvious.

**Lemma 2.1.** For the $H^k$ mean curvature flow, we have

$$\frac{\partial}{\partial t}H(t) = kH^{k-1}(t)\Delta_t H(t) + H^k(t)|A(t)|^2 + k(k - 1)H^{k-2}(t)|\nabla_t H(t)|^2,$$

$$\frac{\partial}{\partial t}|A(t)|^2 = kH^{k-1}(t)\Delta_t |A(t)|^2 - 2kH^{k-1}(t)|\nabla_t A(t)|^2 + 2kH^{k-1}(t)|A(t)|^4 + 2k(k - 1)H^{k-2}(t)|\nabla_t H(t)|^2.$$

Here and henceforth, the norm $|\cdot|$ is respect to the induced metric $g(t)$.

**Corollary 2.2.** If $k$ is odd and larger than 2, then

$$H(t) \geq \min_M H(0)$$

along the $H^k$ mean curvature flow. In particular, $H(t) > 0$ is preserved by the $H^k$ mean curvature flow.

**Proof.** By Lemma 2.1, we have

$$\frac{\partial}{\partial t}H(t) = kH^{k-1}(t)\Delta_t H(t) + H^k(t)|A(t)|^2 + k(k - 1)H^{k-2}(t)|\nabla_t H(t)|^2$$

$$= kH^{k-1}(t)\Delta_t H(t) + \left(H^{k-1}(t)|A(t)|^2 + k(k - 1)H^{k-3}(t)|\nabla_t H(t)|^2\right)H(t).$$

Since $k \geq 2$ and $k$ is odd, it follows that

$$H^{k-1}(t)|A(t)|^2 + k(k - 1)H^{k-3}(t)|\nabla_t H(t)|^2$$

is nonnegative and then (2.3) follows from the maximum principle. \qed
Lemma 2.3. Suppose $k$ is odd and larger than 2, and $H > 0$. For the $H^k$ mean curvature flow and any positive integer $\ell$, we have

$$
\left( \frac{\partial}{\partial t} - kH^{k-1}(t) \Delta \right) \left( \frac{|A(t)|^2}{H^{\ell+1}(t)} \right)
= \frac{k(\ell + 1)}{k - 1} \left( \nabla_t H^{k-1}(t), \nabla_t \left( \frac{|A(t)|^2}{H^{\ell+1}(t)} \right) \right)
- \frac{2k}{H^{\ell+4-k}(t)} \left( H(t) \nabla_t A(t) - \frac{\ell + 1}{2} A(t) \nabla_t H(t) \right)^2 + \frac{2k(k - 1)}{H^{\ell+3-k}(t)} |\nabla_t H(t)|^2
+ \frac{2k}{H^{\ell+2-k}(t)} |A(t)|^4 - \frac{k(\ell + 1)(2k - \ell - 1)}{2H^{\ell+4-k}(t)} |A(t)|^2 |\nabla_t H(t)|^2.
$$

Proof. In the following computation, we will always omit time $t$ and write $\partial/\partial t$ as $\partial_t$. Then

$$
\partial_t H = kH^{k-1} \Delta H + H^k |A|^2 + k(k - 1)H^{k-2} |\nabla H|^2.
$$

By Corollary 2.2, $H(t) > 0$ along the $H^k$ mean curvature flow so that $|H(t)|^i = H^i(t)$ for each positive integer $i$. For any positive integer $\ell$, we have

$$
\partial_t |H|^\ell+1 = (\ell + 1)H^\ell \partial_t H
= (\ell + 1)H^\ell \left( kH^{k-1} \Delta H + H^k |A|^2 + k(k - 1)H^{k-2} |\nabla H|^2 \right)
= k(\ell + 1)H^{k+\ell-1} \Delta H + (\ell + 1)H^{k+\ell} |A|^2
+ k(k - 1)(\ell + 1)H^{k+\ell-2} |\nabla H|^2,
$$

$$
\Delta |H|^\ell+1 = \Delta H^{\ell+1} = (\ell + 1)\nabla \left( H^{\ell} \nabla H \right)
= (\ell + 1) \left( \ell H^{\ell-1} |\nabla H|^2 + H^\ell \Delta H \right)
= (\ell + 1)H^\ell \Delta H + (\ell + 1)H^{\ell-1} |\nabla H|^2.
$$

Therefore

$$
\partial_t H^{\ell+1} = kH^{k-1} \Delta H^{\ell+1} - k(\ell + 1)H^{k+\ell-2} |\nabla H|^2
+ (\ell + 1)H^{k+\ell} |A|^2 + k(k - 1)(\ell + 1)H^{k+\ell-2} |\nabla H|^2.
$$

(2.4)

Recall from Lemma 2.1 that

$$
\partial_t |A|^2 = kH^{k-1} \Delta |A|^2 - 2kH^{k-1} |\nabla A|^2 + 2kH^{k-1} |A|^4 + 2k(k - 1)H^{k-2} |\nabla H|^2.
$$

Calculate, using (2.4),

$$
\partial_t \left( \frac{|A|^2}{|H|^\ell+1} \right)
= \frac{\partial_t |A|^2}{|H|^\ell+1} - \frac{|A|^2}{|H|^{2\ell+2}} \partial_t |H|^\ell+1
= \frac{kH^{k-1} \Delta |A|^2 - 2kH^{k-1} |\nabla A|^2 + 2kH^{k-1} |A|^4 + 2k(k - 1)H^{k-2} |\nabla H|^2}{|H|^{\ell+1}}
= \frac{kH^{k-1} |A|^2 \left[ kH^{k-1} \Delta H^{\ell+1} + (\ell + 1)H^{k+\ell} |A|^2 + k(k - \ell - 1)(\ell + 1)H^{k+\ell-2} |\nabla H|^2 \right]}{|H|^{2\ell+2}}
= \frac{kH^{k-1} |A|^2 \left[ \frac{\ell + 1}{H^{\ell+4-k}} |\nabla A|^2 + \frac{2k}{H^{\ell+4-k}} |A|^4 + \frac{2k(k - 1)}{H^{\ell+3-k}} |\nabla H|^2 \right]}{|H|^{\ell+2}}
- \frac{k|A|^2}{H^{2\ell+3-k}} \Delta H^{\ell+1} - \frac{\ell + 1}{H^{\ell+2-k}} |A|^4 - \frac{k(k - \ell - 1)(\ell + 1)}{H^{\ell+4-k}} |A|^2 |\nabla H|^2.
$$
Combining with all of them yields

\[
\Delta \left( \frac{|A|^2}{H^{\ell+1}} \right) = \frac{1}{H^{\ell+1}} \Delta |A|^2 + \Delta \left( \frac{1}{H^{\ell+1}} \right) |A|^2 + 2 \left< \nabla |A|^2, \nabla \left( \frac{1}{H^{\ell+1}} \right) \right>,
\]

\[
\nabla \left( \frac{1}{H^{\ell+1}} \right) = - \frac{(\ell + 1) H^\ell \nabla H}{H^{2\ell+2}} = - \frac{(\ell + 1) \nabla H}{H^{\ell+2}},
\]

\[
\Delta \left( \frac{1}{H^{\ell+1}} \right) = \nabla \left( - \frac{(\ell + 1) \nabla H}{H^{\ell+2}} \right).
\]

Thus, we conclude that

\[
\partial_t - kH^{k-1} \Delta \left( \frac{|A|^2}{H^{\ell+1}} \right) = kH^{-\ell+2} \Delta |A|^2 - \frac{2k}{H^{\ell+2-k}} |\nabla A|^2 + \frac{2k}{H^{\ell+2-k}} |A|^4 + \frac{2k(k - 1)}{H^{\ell+3-k}} |\nabla H|^2
\]

\[
- \frac{k|A|^2}{H^{\ell+3-k}} \left[ (\ell + 1) H^\ell |\nabla H|^2 + (\ell + 1) H^\ell \Delta H \right] - \frac{\ell + 1}{H^{\ell+2-k}} |A|^4
\]

\[
- k(k - \ell - 1)(\ell + 1)|A|^2 |\nabla H|^2
\]

\[
- kH^{k-1} \left[ \frac{1}{H^{\ell+1}} \Delta |A|^2 - (\ell + 1) \frac{|A|^2 \Delta H}{H^{\ell+2}} + (\ell + 1)(\ell + 2) \frac{|A|^2 |\nabla H|^2}{H^{\ell+3}} \right]
\]

\[
- 2kH^{k-1} \left< \nabla |A|^2, \nabla \left( \frac{1}{H^{\ell+1}} \right) \right>
\]

\[
= - \frac{2k}{H^{\ell+2-k}} |\nabla A|^2 + \left( \frac{2k}{H^{\ell+2-k}} - \frac{\ell + 1}{H^{\ell+2-k}} \right) |A|^4 + \frac{2k(k - 1)}{H^{\ell+3-k}} |\nabla H|^2
\]

\[
- \frac{k(\ell + 1)(k + \ell + 1)|A|^2 |\nabla H|^2}{H^{\ell+4-k}} - 2kH^{k-1} \left< \nabla |A|^2, \nabla \left( \frac{1}{H^{\ell+1}} \right) \right>. 
\]

On the other hand,

\[
\left< \nabla |A|^2, \nabla \left( \frac{1}{H^{\ell+1}} \right) \right> = 2 \nabla A \cdot \nabla \frac{(\ell + 1) H^\ell \nabla H}{H^{2\ell+2}}
\]

\[
= \frac{-2(\ell + 1)}{H^{\ell+3}} H \cdot \nabla A \cdot A \cdot \nabla H.
\]

Thus, we conclude that

\[
\partial_t - kH^{k-1} \Delta \left( \frac{|A|^2}{H^{\ell+1}} \right) = - \frac{2k}{H^{\ell+2-k}} |\nabla A|^2 + \frac{2k - \ell - 1}{H^{\ell+2-k}} |A|^4 + \frac{2k(k - 1)}{H^{\ell+3-k}} |\nabla H|^2
\]

\[
- \frac{k(\ell + 1)(k + \ell + 1)|A|^2 |\nabla H|^2}{H^{\ell+4-k}} + 4k(\ell + 1)H^{\ell+4-k} \nabla A \cdot A \cdot \nabla H.
\]
Consider the function
\[
f := \frac{-2k}{H^{\ell+2-k}} |\nabla A|^2 - \frac{k(\ell + 1)(k + \ell + 1)|A|^2 |\nabla H|^2}{H^{\ell+4-k}} + \frac{4k(\ell + 1)}{H^{\ell+4-k}} H \cdot \nabla A \cdot \nabla H.
\]
Since
\[
\frac{2k(\ell + 1)}{H^{\ell+4-k}} H \cdot \nabla A \cdot \nabla H = \frac{k(\ell + 1)}{H^{\ell+3-k}} \nabla |A|^2 \cdot \nabla H,
\]
and
\[
\nabla \left( \frac{|A|^2}{H^{\ell+1}} \right) = \frac{\nabla |A|^2}{H^{\ell+1}} - \frac{(\ell + 1)|A|^2 \nabla H}{H^{\ell+2}},
\]
it follows that
\[
\frac{2k(\ell + 1)}{H^{\ell+4-k}} H \cdot \nabla A \cdot \nabla H = \frac{k(\ell + 1)}{k - 1} \left( \nabla H^{k-1} \cdot \nabla \left( \frac{|A|^2}{H^{\ell+1}} \right) \right) + \frac{k(\ell + 1)^2}{H^{\ell+4-k}} |A|^2 |\nabla H|^2.
\]
Consequently,
\[
f = \frac{-2k}{H^{\ell+2-k}} |\nabla A|^2 - \frac{k^2(\ell + 1)}{H^{\ell+4-k}} |A|^2 |\nabla H|^2
\]
\[
+ \frac{k(\ell + 1)}{k - 1} \left( \nabla H^{k-1} \cdot \nabla \left( \frac{|A|^2}{H^{\ell+1}} \right) \right) + \frac{2k(\ell + 1)}{H^{\ell+4-k}} H \cdot \nabla A \cdot \nabla H
\]
\[
= \frac{-2k}{H^{\ell+4-k}} \left[ \left( H \nabla A - \frac{\ell + 1}{2} A \cdot \nabla H \right)^2 \right]
\]
\[
- \frac{2k(\ell + 1)(2k - \ell - 1)}{4H^{\ell+4-k}} |A|^2 |\nabla H|^2 + \frac{k(\ell + 1)}{k - 1} \left( \nabla H^{k-1} \cdot \nabla \left( \frac{|A|^2}{H^{\ell+1}} \right) \right).
\]

Finally, we complete the proof. \( \square \)

**Corollary 2.4.** Suppose \( k \) is odd and larger than 2, and \( H > 0 \). For the \( H^k \) mean curvature flow, we have
\[
\left( \frac{\partial}{\partial t} - k H^{k-1}(t) \Delta t \right) \left( \frac{|A(t)|^2}{H^{2k}(t)} \right)
\]
\[
= \frac{2k^2}{k - 1} \left( \nabla t H^{k-1}(t), \nabla t \left( \frac{|A(t)|^2}{H^{2k}(t)} \right) \right) + \frac{2k(k - 1)}{H^{k+2}(t)} |\nabla t H(t)|^2
\]
\[
- \frac{2k}{H^{k+3}(t)} [H(t) \cdot \nabla t A(t) - kA(t) \cdot \nabla t H(t)]^2.
\]

3. PROOF OF THE MAIN THEOREM

In this section we give a proof of theorem 2.1. For any positive constant \( C_0 \), consider the quantity
\[
Q(t) := \frac{|A(t)|^2}{H^{2k}(t)} + C_0 H^\ell(t),
\]
(3.1)
where the integer $\ell$ is determined later. By (2.4) and Corollary 2.4 we have

$$
\left( \frac{\partial}{\partial t} - k H^{k-1}(t) \Delta_t \right) Q(t)
\leq \frac{2k^2}{k - 1} \left( \nabla H^{k-1}(t), \nabla_{t} Q(t) - C_0 \nabla_{t} H^{\ell+1}(t) \right) + \frac{2k(k - 1)}{H^{k+2}(t)} \left| \nabla_{t} H(t) \right|^2 \\
+ C_0 \left[ (\ell + 1) H^{k+\ell}(t), A(t) \right]^2 + k(k - \ell - 1)(\ell + 1) H^{k+\ell-2}(t) \left| \nabla_{t} H(t) \right|^2
$$

$$
= \frac{2k^2}{k - 1} \left( \nabla H^{k-1}(t), \nabla_{t} Q(t) \right) - \frac{2k^2}{k - 1} C_0 (k - 1)(\ell + 1) H^{k+\ell-2}(t) \left| \nabla_{t} H(t) \right|^2 \\
+ \frac{2k(k - 1)}{H^{k+2}(t)} \left| \nabla_{t} H(t) \right|^2 + C_0 k(k - \ell - 1)(\ell + 1) H^{k+\ell-2}(t) \left| \nabla_{t} H(t) \right|^2 \\
+ C_0 (\ell + 1) H^{k+\ell}(t) \left[ Q(t) - C_0 H^{\ell+1}(t) \right] H^{2k}(t)
$$

$$
= \frac{2k^2}{k - 1} \left( \nabla H^{k-1}(t), \nabla_{t} Q(t) \right) \\
+ \left| \nabla_{t} H(t) \right|^2 \left[ \frac{2k(k - 1)}{H^{k+2}(t)} - C_0 (k + 1) \right] \\
+ C_0 (\ell + 1) H^{3k+\ell}(t) Q(t) - C_0^2 (\ell + 1) H^{3k+2\ell+1}(t).
$$

Now we choose $\ell$ so that the following constraints

$$
\ell + 1 \leq 0, \quad k + \ell + 1 \leq 0, \quad 3k + 2\ell + 1 \geq 0
$$

are satisfied; that is

$$
(3.2) \quad -\frac{1}{2} - \frac{3}{2} k \leq \ell \leq -1 - k.
$$

In particular, we can take

$$
\ell := -2 - k.
$$

By our assumption on $k$, we have $k \geq 3$ and hence (3.3) implies (3.2). Plugging (3.3) into the above inequality yields

$$
\left( \frac{\partial}{\partial t} - k H^{k-1}(t) \Delta_t \right) Q(t)
\leq \frac{2k^2}{k - 1} \left( \nabla H^{k-1}(t), \nabla_{t} Q(t) \right) + \left| \nabla_{t} H(t) \right|^2 \left[ \frac{2k(k - 1)}{H^{k+2}(t)} - C_0 k(k + 1) \right] \\
- C_0 (1 + k) H^{2k-2}(t) Q(t) + C_0^2 (1 + k) H^{k-3}(t).
$$

Choosing

$$
C_0 := \frac{1}{2} \frac{k + 1}{k - 1} H^{k+2}_{\text{min}},
$$

where $H_{\text{min}} := \min_M H = \min_M H(0)$, we arrive at

$$
\frac{2k(k - 1)}{C_0 k(k + 1)} \leq H^{k+2}_{\text{min}} \leq H^{k+2}(0) \leq H^{k+2}(t)
$$

according to (2.3). Consequently,

$$
\left( \frac{\partial}{\partial t} - k H^{k-1}(t) \Delta_t \right) Q(t) \leq \frac{2k^2}{k - 1} \left( \nabla H^{k-1}(t), \nabla_{t} Q(t) \right) \\
- C_1 H^{2k-2}(t) Q(t) + C_2 H^{k-3}(t),
$$

(3.6)
for $C_1 := C_0(1 + k)$ and $C_2 := C_0^2(1 + k)$.

**Lemma 3.1.** If the solution can not be extended over $T_{\text{max}}$, then $H(t)$ is unbounded.

**Proof.** By the assumption, we know that $|A(t)|$ is unbounded as $t \to T_{\text{max}}$. We now claim that $H(t)$ is also unbounded. Otherwise, $0 < H_{\text{min}} \leq H(t) \leq C$ for some uniform constant $C$. If we set

$$C_3 := C_1 H_{\text{min}}^{2k-2}, \quad C_4 := C_2 C^{k-3},$$

then (3.6) implies that

$$\left(\frac{\partial}{\partial t} - k H^{k-1}(t) \Delta_t\right) Q(t) \leq \frac{2k^2}{k-1} \langle \nabla_t H^{k-1}(t), \nabla_t Q(t) \rangle - C_3 Q(t) + C_4.$$

By the maximum principle, we have

$$Q'(t) \leq -C_5 Q(t) + C_4$$

where

$$Q(t) := \max_M Q(t).$$

Solving (3.8) we find that

$$Q(t) \leq \frac{C_4}{C_5} + \left( Q(0) - \frac{C_4}{C_5} \right) e^{-C_5 t}.$$

Thus $Q(t) \leq C_5$ for some uniform constant $C_5$. By the definition (3.1) and the assumption $H(t) \leq C$, we conclude that $|A(t)| \leq C_6$ for some uniform constant $C_6$, which is a contradiction. 

The rest proof is similar to [2, 4]. Using Lemma 3.1 and the argument in [2] or in [4], we get a contradiction and then the solution of the $H^k$ mean curvature flow can be extended over $T_{\text{max}}$.

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