NONCOMMUTATIVE VARIATIONS ON LAPLACE’S EQUATION

JONATHAN ROSENBERG

Abstract. As a first step at developing a theory of noncommutative nonlinear elliptic partial differential equations, we analyze noncommutative analogues of Laplace’s equation and its variants (some of the them nonlinear) over noncommutative tori. Along the way we prove noncommutative analogues of many results in classical analysis, such as Wiener’s Theorem on functions with absolutely convergent Fourier series, and standard existence and non-existence theorems on elliptic functions. We show that many many classical methods, including the Maximum Principle, the direct method of the calculus of variations, the use of the Leray-Schauder Theorem, etc., have analogues in the noncommutative setting.

1. Introduction

Gelfand’s Theorem shows that $X \leadsto C_0(X)$ sets a contravariant equivalence of categories from the category of locally compact [Hausdorff] spaces and proper maps to the category of commutative $C^*$-algebras and $*$-homomorphisms. This observation is the key to the whole subject of noncommutative geometry, which is based on the following dictionary:

| Classical | Noncommutative |
|-----------|----------------|
| locally compact space | $C^*$-algebra |
| compact space | unital $C^*$-algebra |
| vector bundle | f. g. projective module |
| smooth manifold | $C^*$-algebra with “smooth subalgebra” |
| real-valued function | self-adjoint element |
| partial derivative | unbounded derivation |
| integral | tracial state |

The object of this paper is to begin to use this dictionary to set up a noncommutative theory of elliptic partial differential equations, both linear and nonlinear, along with corresponding aspects of the calculus of variations. Since the theory is still in its infancy, we begin with the very simplest case: Laplace’s equation and PDEs closely connected to it, and concentrate on the simplest nontrivial example of a noncommutative manifold, the irrational rotation algebra (or noncommutative 2-torus) $A_\theta$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$. A definition of elliptic partial differential operators, along with

2000 Mathematics Subject Classification. Primary 58B34; Secondary 58J05 35J05 35J20 30D30 46L87.

Partially supported by NSF Grants DMS-0504212 and DMS-0805003. This paper grew out of work on the paper [20]. Some of the results of this paper were presented in the Special Session on E-Theory, Extensions, and Elliptic Operators at the Joint Mathematics Meetings, San Diego, California, January 9, 2008.
the study of one example associated with the irrational rotation algebra, was given
in Connes’ fundamental paper [5], but there the emphasis was on pseudodifferential
calculus and index theory. Here we focus on several other things: variational meth-
ods, the Maximum Principle, an analogue of Wiener’s Theorem, tools for treating
nonlinear equations, the beginnings of a theory of harmonic unitaries, and some
aspects of noncommutative complex analysis.

What is the motivation for a noncommutative theory of elliptic PDE? For the
most part, it comes from physics. Many of the classical elliptic PDEs arise from
variational problems in Riemannian geometry, and are also the field equations of
physical theories. But the uncertainty principle forces quantum observables to be
noncommutative. There is also increasing evidence (e.g., [7, 4, 6, 34, 21, 22]) that
quantum field theories should allow for the possibility of noncommutative space-
times. Noncommutative sigma-models, for which the very earliest and simplest
investigations are in [9, 10], will require the noncommutative harmonic map equa-
tion, which generalizes the Laplace equation studied in this paper.

We use as our starting point the “noncommutative differential geometry” of Alain
Connes [5]. This theory only works well with “highly symmetric” noncommutative
spaces, as the “smooth” elements are taken to be the $C^\infty$ vectors for an action
of a Lie group on a $C^*$-algebra, but this theory is well adapted to the case of the
irrational rotation algebra, which carries an ergodic “gauge action” of the 2-torus
$T^2$.

The outline of this paper is as follows. We begin in Section 2 with the basic
properties of the Laplacian on $A_\theta$. Included are analogues of Wiener’s Theorem
(Theorem 2.8) and the Maximum Principle (Proposition 2.9). In Section 3, we take
up the basic properties of Sobolev spaces on $A_\theta$, which are needed for a deeper
analysis of some aspects of noncommutative PDEs. We should point out that some
of the material of this section has already appeared in [28, §3] and in [19]. The heart
of this paper is contained in Sections 4 and 5, which begin to develop a theory of
nonlinear elliptic partial differential equations, using methods analogous to those
traditional in the theory of nonlinear elliptic PDE. Finally, Section 6 deals with
noncommutative complex analysis.

We should mention that another example of noncommutative elliptic PDE and
an associated variational problem on noncommutative tori, namely, noncommu-
tative Yang-Mills theory, has already been studied by Connes and Rieffel [8, 32].
Furthermore, Theorem 2.8 was previously proved by Gröchenig and Leinert [12]
by another method, and variations on the Gröchenig-Leinert work can be found in
[19]. In their paper, Gröchenig and Leinert point out some applications to harmonic
analysis and wavelet theory, which go off in a somewhat different direction than the
applications to mathematical physics which we envisage, though obviously there is
some overlap between the two.

I would like the thank the referee for several useful comments and especially for
the reference to [12]. I would also like to thank Hanfeng Li for pointing out an error
in the original proof of Theorem 2.8.

2. The linear Laplacian

We will be studying the $C^*$-algebra $A_\theta$ generated by two unitaries $U, V$ satisfying
$UV = e^{2\pi i \theta} VU$. $A_\theta$ is simple with unique trace $\tau$ if $\theta \notin \mathbb{R} - \mathbb{Q}$. (See for example
The torus $G = \mathbb{T}^2$ acts by
\[ \alpha(z_1, z_2)U = z_1 U, \quad \alpha(z_1, z_2)V = z_2 V, \quad |z_1| = |z_2| = 1. \]
The space of $C^\infty$ vectors for the action $\alpha$ is the “smooth irrational rotation algebra”
\[ A^\infty_\theta = \left\{ \sum_{m,n} c_{m,n} U^m V^n : c_{m,n} \text{ rapidly decreasing} \right\}. \]
This should be viewed as a noncommutative deformation of the algebra $C^\infty(\mathbb{T}^2)$ of smooth functions on an ordinary 2-torus, and the decomposition of an element of this algebra in terms of multiples of $U^m V^n$ should be viewed as a sort of noncommutative Fourier series decomposition, with $c_{m,n}$ as a sort of Fourier coefficient. For $a \in A_\theta$ but not necessarily in $A^\infty_\theta$, the Fourier coefficients $c_{m,n}$ are well defined and satisfy $|c_{m,n}| \leq \|a\|$, since $c_{m,n} = \tau(V^{-n}U^{-m}a)$, but the Fourier series expansion of $a$ is only a formal expansion, and need not converge in the topology of $A_\theta$, just as one has functions in $C(\mathbb{T}^2)$ whose Fourier series do not converge absolutely or even pointwise.

We denote by $\delta_1$ and $\delta_2$ the infinitesimal generators of the actions of the two $T$ factors in $\mathbb{T}^2$ under $\alpha$. These are unbounded derivations on $A_\theta$, and map $A^\infty_\theta$ to itself. They are given by
\[ \delta_1(U) = 2\pi i U, \quad \delta_2(V) = 2\pi i V, \quad \delta_2(U) = \delta_1(V) = 0. \]
These derivations $\delta_j$ obviously commute with the adjoint operation $\ast$, and play the roles of the partial derivatives $\partial/\partial x_j$ in classical analysis on the 2-torus. Since the action $\alpha$ of $\mathbb{T}^2$ preserves the tracial state $\tau$, $\tau \circ \delta_j = 0$, $j = 1, 2$. This fact is the basis for the following Lemma, which we will use many times in the future.

**Lemma 2.1 (“Integration by Parts”).** If $a, b \in A^\infty_\theta$, then $\tau(\delta_j(ab)) = -\tau(\delta_j(b)a)$, $j = 1, 2$.

**Proof.** We have
\[ 0 = \tau(\delta_j(ab)) = \tau(\delta_j(ab)) + \tau(ab\delta_j(b)). \]
The result follows. \hfill \Box

**Definition 2.2.** In analogy with the usual notation in analysis, we let
\[ \Delta = \delta_1^2 + \delta_2^2. \]
This should be viewed as a “noncommutative elliptic partial differential operator.” (The notion of ellipticity was defined rigorously in [5] p. 602.) Clearly, $\Delta$ is a “negative” operator, and its spectrum consists of the numbers $-4\pi^2(m^2 + n^2)$, $m, n \in \mathbb{Z}$, with eigenfunctions $U^m V^n$. Via the noncommutative Fourier expansion discussed earlier, the pair $(A^\infty_\theta, \Delta)$ is isomorphic to $C^\infty(\mathbb{T}^2)$ with the usual Laplacian $\Delta$, provided one looks just at the linear structure and forgets the noncommutativity of the multiplication. (This was already observed in [5] p. 602.)

**Proposition 2.3.** For any $\lambda > 0$ (or not of the form $-4\pi^2n$ with $n \in \mathbb{N}$), $-\Delta + \lambda : A^\infty_\theta \to A^\infty_\theta$ is bijective.

**Proof.** We have
\[ (-\Delta + \lambda) \left( \sum_{m,n} c_{m,n} U^m V^n \right) = \sum_{m,n} \left( 4\pi^2(m^2 + n^2) + \lambda \right) c_{m,n} U^m V^n. \]
It is immediate that $-\Delta + \lambda$ has no kernel and has an inverse given by the formula
$$\sum_{m,n} c_{m,n} U^m V^n \mapsto \sum_{m,n} \frac{1}{4\pi^2 (m^2 + n^2) + \lambda} c_{m,n} U^m V^n,$$

since if $c_{m,n}$ is rapidly decreasing, so are the coefficients on the right. □

It is also easy to characterize the image of $\Delta$.

**Proposition 2.4.** The image of $\Delta: A^\infty_\theta \to A^\infty_\theta$ is precisely $A^\infty_\theta \cap \ker \tau$, the smooth elements with zero trace.

**Proof.** We have $\Delta(\sum_{m,n} c_{m,n} U^m V^n) = -4\pi^2 \sum_{m,n} (m^2 + n^2) c_{m,n} U^m V^n$, and the factor $(m^2 + n^2)$ kills the term with $m = n = 0$. Thus the image of $\Delta$ is contained in the kernel of $\tau$. Conversely, suppose $a = \sum_{m,n} d_{m,n} U^m V^n$ is an arbitrary element of $A^\infty_\theta \cap \ker \tau$. That means $d_{m,n}$ is rapidly decreasing and $d_{0,0} = 0$. Then $d_{m,n}/(m^2 + n^2)$ is also rapidly decreasing, and
$$\sum_{m,n} \frac{-d_{m,n}}{4\pi^2 (m^2 + n^2)} U^m V^n,$$

where the $'$ indicates we omit the term with $m = n = 0$, converges to an element $b$ of $A^\infty_\theta$ with $\Delta b = a$. □

The following consequence is an analogue of a well-known fact about subharmonic functions on compact manifolds.

**Corollary 2.5.** If $a \in A^\infty_\theta$ is subharmonic (i.e., if $\Delta a \geq 0$), then $a$ is constant.

**Proof.** Suppose $a \in A^\infty_\theta$ and $\Delta a \geq 0$. By Proposition 2.4 $\tau(\Delta a) = 0$. But $\tau$ is a faithful trace, which means that if $b \geq 0$ and $\tau(b) = 0$, then $b = 0$. Apply this with $b = \Delta a$ and we see that $\Delta a = 0$. This implies $a$ is a scalar multiple of 1. □

For future use, we are also going to want to study other “function spaces” on the noncommutative torus. For example, we have the analogue of the Fourier algebra of functions with absolutely convergent Fourier series.

**Definition 2.6.** Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let
$$B_\theta = \left\{ \sum_{m,n} c_{m,n} U^m V^n : \sum_{m,n} |c_{m,n}| < \infty \right\}.$$

This is obviously a Banach subspace of $A_\theta$ with norm $\| \cdot \|_{\ell^1}$ given by the $\ell^1$ norm of the coefficients $c_{m,n}$. We also obviously have $\|a\|_{\ell^1} \geq \|a\|$ for $a \in B_\theta$. ($\| \cdot \|$ will for us always denote the $C^*$-algebra norm.)

The following lemma, related in spirit to the Sobolev Embedding Theorem [16, Theorem 1.1], relates the topology of $B_\theta$ to the subject of Propositions 2.3 and 2.4. More details of noncommutative Sobolev space theory will be taken up in Section 3 below.

**Lemma 2.7.** Let $f \in A^\infty_\theta$. Then there is a constant $C > 0$ such that (in the notation of Definition 2.6) $\|f\|_{\ell^1} \leq C \|(-\Delta + 1)f\|$. In particular, the domain of $\Delta$, as an unbounded operator on $A_\theta$, is contained in $B_\theta$. 
Proof. Suppose $f = \sum_{m,n} c_{m,n} U^m V^n \in A_\theta^\infty$. Then

$$\|f\|_{\ell^2} = \sum_{m,n} |c_{m,n}| = \sum_{m,n} \left( 1 + 4\pi^2(m^2 + n^2) \right) c_{m,n} \cdot \frac{a_{m,n}}{1 + 4\pi^2(m^2 + n^2)},$$

where $|a_{m,n}| = 1$. View this as an $\ell^2$ inner product and estimate it by Cauchy-Schwarz. We obtain

$$\|f\|_{\ell^2} \leq C \|(-1 + \Delta) f\|_{\ell^2},$$

where $\| \cdot \|_{\ell^2}$ is the $\ell^2$ norm of the sequence of Fourier coefficients (this can also be defined by $\|c\|_{\ell^2} = \tau(c^*c)^{\frac{1}{2}}$) and where

$$C = \left\| \left\{ (1 + 4\pi^2(m^2 + n^2))^{-1} \right\}_{m,n} \right\|_{\ell^2} = \left( \sum_{m,n} \frac{1}{(1 + 4\pi^2(m^2 + n^2))^2} \right)^{\frac{1}{2}} < \infty.$$

Since the $\ell^2$ norm is bounded by the $C^*$-algebra norm, as $\|c\|_{\ell^2} = \tau(c^*c)^{\frac{1}{2}} \leq \|c^*c\|^{\frac{1}{2}} = \|c\|$, the result follows. $\square$

The following result was proved several years ago by Gröchenig and Leinert [12], using the theory of symmetric $L^1$-algebras as developed by Leptin, Ludwig, Hulanicki, et al. We include a brief proof here for the sake of completeness.

**Theorem 2.8 (“Wiener’s Theorem”).** The Banach space $B_\theta$ is a Banach $*$-algebra and is closed under the holomorphic functional calculus of $A_\theta$. Thus if $a \in B_\theta$ and $a$ is invertible in $A_\theta$, $a^{-1} \in B_\theta$.

**Proof.** Suppose $a = \sum_{m,n} c_{m,n} U^m V^n$ with the sum absolutely convergent. Then

$$a^* = \sum_{m,n} c_{m,n} V^{-n} U^{-m} = \sum_{m,n} c_{m,n} e^{-2\pi imn\theta} U^{-m} V^{-n}$$

so $a^* \in B_\theta$. Similarly, if also $b = \sum_{m,n} d_{m,n} U^m V^n$ (absolutely convergent sum), then $ab$ has Fourier coefficients given by “twisted convolution” of the Fourier coefficients of $a$ and $b$, and since the twisting only involves scalars of absolute value 1, the Fourier coefficients of $ab$ are absolutely convergent. More precisely,

$$ab = \left( \sum_{m,n} c_{m,n} U^m V^n \right) \left( \sum_{k,l} d_{k,l} U^k V^l \right)$$

$$= \sum_{m,n,k,l} c_{m,n} d_{k,l} U^m V^n U^k V^l$$

$$= \sum_{m,n,k,l} c_{m,n} d_{k,l} e^{-2\pi ikn\theta} U^{m+k} V^{n+l}$$

$$= \sum_{p,q} f_{p,q} U^p V^q,$$

where

$$f_{p,q} = \sum_{m,n} c_{m,n} d_{p-m,q-n} e^{-2\pi ipm\theta},$$

so that

$$|f_{p,q}| \leq \sum_{m,n} |c_{m,n}| |d_{p-m,q-n}| \leq \|c\|_{\ell^2} \|d\|_{\ell^2}.$$

This confirms that $B_\theta$ is a Banach $*$-algebra and of course a $*$-subalgebra of $A_\theta$.

To prove the analogue of Wiener’s Theorem, we unfortunately cannot use the cute proof using the Gelfand transform, since $B_\theta$ is not commutative. We also
cannot use another very elementary proof from [24] since this also relies on commutativity. However, Newman’s proof is related to another well-known fact (implicit in [5], Lemma 1), that $A_\theta^\infty$ is closed under the holomorphic functional calculus of $A_\theta$. To prove this one has to show that if $b \in A_\theta^\infty$ with $b$ invertible in $A_\theta$, then $b^{-1}$ also lies in $A_\theta^\infty$. To prove this fact, iterate the identity $\delta_j(b^{-1}) = -b^{-1} \delta_j(b) b^{-1}$ to see that $b^{-1}$ lies in the domain of all monomials in $\delta_1$ and $\delta_2$. One might think that since $A_\theta^\infty$ is dense in $B_\theta$, this should be enough to prove Wiener’s Theorem for the latter, but this doesn’t work, since in general the spectrum and spectral radius functions are only upper semicontinuous, not continuous, on a noncommutative Banach algebra [23].

To prove the theorem, we rely on an observation of Hulanicki [15] Proposition 2.5] based on a theorem of Raikov [29] Theorem 5]: that if a Banach $*$-algebra $B$ (with isometric involution and a faithful $*$-representation on a Hilbert space) is embedded in its enveloping $C^*$-algebra $A$, then the spectra of self-adjoint elements of $B$ are the same whether computed in $B$ or in $A$ if and only if $B$ is symmetric (i.e., for $x \in B$, the spectrum in $B$ of $x^* x$ is contained in $[0, \infty)$). We will apply this with $B = B_\theta$ and $A = A_\theta$. Hulanicki also showed [14] that the $L^1$ algebras of discrete nilpotent groups are symmetric. In particular, the $L^1$ algebra of the discrete Heisenberg group $H$ (with generators $a, b, c$, where $c$ is central and $a b a^{-1} b^{-1} = c$) is symmetric. Thus $B_\theta$, which is the quotient of $L^1(H)$ by the (self-adjoint) ideal generated by $c - e^{2\pi i \theta}$, is also symmetric. (If $B$ is a symmetric Banach $*$-algebra and $J$ is a closed self-adjoint ideal, then $B/J$ is also symmetric, since if $x \in B/J$ is the image of $x \in B$, then the spectrum of $x^* x$ in $B/J$ is contained in the spectrum of $x^* x$ in $B$, hence is contained in $[0, \infty)$.) So for $x = x^* \in B_\theta$, by Hulanicki’s observation, if $x$ is invertible in $A_\theta$, $x^{-1} \in B_\theta$. Suppose $a \in B_\theta$ and $a$ is invertible in $A_\theta$. Then $a^*$ is also invertible in $A_\theta$, so $x = a^* a \in B_\theta$ and $x$ is invertible in $A_\theta$. Hence $x^{-1} = a^{-1} a^*-1 \in B_\theta$ and $a^{-1} = x^{-1} a^* \in B_\theta$. 

In the classical theory of the Laplacian, one of the most useful tools is the “Maximum Principle” (e.g., [19] p. 20]). The following is a noncommutative analogue.

**Proposition 2.9 (“Maximum Principle”).** Let $h = h^* \in A_\theta^\infty$, and let $[t_0, t_1]$ be the smallest closed interval containing the spectrum $\sigma(h)$ of $h$ (in $A_\theta$). In other words, let $t_1 = \max \{ t : t \in \sigma(h) \}$ and $t_0 = \min \{ t : t \in \sigma(h) \}$. Then there exists a state $\varphi$ of $A_\theta$ with $\varphi(h) = t_1$, and for such a state, $\varphi(\Delta h) \leq 0$. Similarly, there exists a state $\psi$ of $A_\theta$ with $\psi(h) = t_0$, and for such a state, $\psi(\Delta h) \geq 0$.

**Proof.** The commutative $C^*$-algebra $C^*(h)$ must have pure states $\varphi$ and $\tilde{\psi}$ with $\varphi(h) = t_1$, $\tilde{\psi}(h) = t_0$, since $t_0, t_1 \in \sigma(h)$. Extend these to states $\varphi, \tilde{\psi}$ of the larger $C^*$-algebra $A_\theta$. Then for $s \in G = T^2$, the functions $s \mapsto \varphi(\alpha_s(h))$ and $s \mapsto \tilde{\psi}(\alpha_s(h))$ must have a maximum (resp., minimum) at the identity element of $T^2$. (Recall that $\alpha$ is the gauge action by $*$-automorphisms.) Differentiate twice and the result follows by the “second derivative test.”

Just as in the classical setting, Laplace’s equation arises as the Euler-Lagrange equation of a variational problem.

**Definition 2.10.** For $a \in A_\theta^\infty$, let

$$E(a) = \frac{1}{2} \tau(\delta_1(a)^2 + \delta_2(a)^2).$$
This is clearly the noncommutative analogue of the classical energy functional

\[ f \mapsto \frac{1}{2} \int_M \| \nabla f \|^2 \, d\text{vol} \]

on a compact manifold \( M \).

**Proposition 2.11.** The Euler-Lagrange equation for critical points of the energy functional \( E \) of Definition 2.10, restricted to self-adjoint elements of \( A_\theta^\infty \), is just Laplace’s equation \( \Delta a = 0 \). Thus the only critical points are the scalar multiples of the identity, which are the points where \( E(a) = 0 \) and are strict minima for \( E \).

**Proof.** This works very much like the classical case. If \( a = a^* \) and \( h = h^* \), then

\[
\frac{d}{dt} \bigg|_{t=0} E(a + th) = \frac{1}{2} \tau (\delta_1(a) \delta_1(h) + \delta_1(h) \delta_1(a) + \delta_2(a) \delta_2(h) + \delta_2(h) \delta_2(a)) .
\]

Because of the trace property, we can write this as \( \tau (\delta_1(a) \delta_1(h) + \delta_2(a) \delta_2(h)) \). For \( a \) to be a critical point of \( E \), this must vanish for all choices of \( h \). “Integrating by parts” using Lemma 2.1, we obtain \( \tau (h \Delta(a)) = 0 \) for all \( h \), and since the trace pairing is nondegenerate, we get the Euler-Lagrange equation \( \Delta a = 0 \). Since \( \Delta \) has pure point spectrum with eigenvalues \(-4\pi^2(m^2 + n^2)\) and eigenfunctions \( U^m V^n \), the equation has the unique solution \( a = \lambda 1, \lambda \in \mathbb{R} \). These are also the points where \( E \) takes its minimum value of 0. \( \square \)

3. **Sobolev spaces**

In the treatment of Laplace’s equation above, we alluded to the theory of Sobolev spaces. One can develop this theory in the noncommutative setting in complete analogy with the classical case. To simplify the treatment, we deal here only with the \( L^2 \) theory, which gives rise to Hilbert spaces. These spaces are convenient for applications to nonlinear elliptic PDE, as we will see in the next section.

**Definition 3.1.** For \( a \in A_\theta \), we define its \( "L^2 norm" \)

\[ \| a \|_{L^2} = \tau (a^* a)^{\frac{1}{2}} . \]

We let \( L^2 \) or \( H^0 \) (this is the Sobolev space of “functions” with 0 derivatives in \( L^2 \)) be the completion of \( A_\theta \) in this norm. Obviously this is a Hilbert space, with inner product extending

\[ \langle a, b \rangle = \tau (b^* a) \]

\[ ^1 \text{This is really the norm for the Hilbert space of the II}_1 \text{ factor representation of } A_\theta \text{ determined by the trace } \tau. \]
on $A_\theta$. Also note that the norm of $L^2$ is simply the $\ell^2$ norm for the Fourier coefficients, since if $a \in A_\theta^\infty$ has the Fourier expansion $\sum_{m,n} c_{m,n} U^m V^n$, then

$$\|a\|_2^2 = \tau(a^* a) = \tau \left( \sum_{k,l,m,n} (c_{m,n} U^m V^n)^* c_{k,l} U^k V^l \right) = \tau \left( \sum_{k,l,m,n} \overline{c_{m,n}} c_{k,l} V^{-n} U^{-m} U^k V^l \right) = \sum_{m,n} |c_{m,n}|^2.$$

Now let $n \in \mathbb{N}$. We define the Sobolev space $H^n$ of “functions” with $n$ derivatives in $L^2$ to be the completion of $A_\theta^\infty$ in the norm

$$\|a\|_{H^n}^2 = \sum_{0 \leq |\beta| \leq n} \|\delta_\beta(a)\|_\ell^2.$$  

(These spaces are also defined, with slightly different notation, in [28, §3].) Here $\beta = \beta_1 \beta_2 \cdots \beta_{|\beta|}$ runs over sequences with $\beta_j = 1$ or 2 and $\delta_\beta$ means $\delta_{\beta_1} \cdots \delta_{\beta_{|\beta|}}$, a “partial derivative” of order $|\beta|$. For example,

$$\|a\|_{H^1}^2 = \|a\|_2^2 + \|\delta_1(a)\|_2^2 + \|\delta_2(a)\|_2^2.$$

The Sobolev space $H^n$ is clearly a Hilbert space, and we obviously have norm-decreasing inclusions $H^n \hookrightarrow H^{n-1}$. Furthermore, it is clear that the Sobolev norms are invariant under taking adjoints and can easily be expressed in terms of the Fourier coefficients; for example, if $a \in A_\theta^\infty$ has the Fourier expansion $\sum_{m,n} c_{m,n} U^m V^n$, then

$$\|a\|^2_{H^1} = \sum_{m,n} (1 + 4\pi^2 (m^2 + n^2)) |c_{m,n}|^2.$$

The following is the exact analogue of the classical Sobolev Embedding Theorem [16, Theorem 1.1] for $T^2$.

**Theorem 3.2 ("Sobolev Embedding").** The inclusion $H^n \hookrightarrow H^{n-1}$ is compact. The space $H^1$ is not contained in $A_\theta$, but $H^2$ has a compact inclusion into $B_\theta$ (and thus into $A_\theta$).

**Proof.** Since the Sobolev norms just depend on the decay of the Fourier coefficients, this follows immediately from the classical Sobolev Embedding Theorem in dimension 2. The inclusion of $H^2$ into $B_\theta$ also follows from the estimate

$$\|f\|_\ell^1 \leq C \|(-1 + \Delta)f\|_\ell^2,$$

in the proof of Lemma 2.7, with the compactness coming from the fact that we can approximate by the finite rank operators that truncate the Fourier series after finitely many terms.

---

2 Usually this would be called $H^{n,2}$, but we are trying to simplify notation.
4. **Nonlinear problems involving the Laplacian**

Somewhat more interesting, and certainly more difficult to treat than the situation of Proposition 2.11 are certain nonlinear problems involving the Laplacian, of the general form \( \Delta u = f(u) \). Such problems arise classically from the problem of prescribing the scalar curvature of a metric \( e^u g \) obtained by conformally deforming the original metric \( g \) on a Riemannian manifold \( M \) \([16] \) Chs. 5, 7]. For example, if \( g \) is the usual flat metric on \( \mathbb{T}^2 \), then the scalar curvature \( h \) of the pointwise conformal metric \( e^u g \) solves the equation \( \Delta u = -he^u \). (This equation is studied in detail in \([17] \) \$5\.) Because of the Gauss-Bonnet theorem on the torus, \( h \) must integrate out to 0, so there are no solutions with \( h \) a constant unless \( h = 0 \) and \( u \) is a constant. This fact has an exact analogue in our noncommutative setting.

**Proposition 4.1.** If \( \lambda \in \mathbb{R} \), the equation \( \Delta u = -\lambda e^u \) has no solution \( u = u^* \in A_\theta^\infty \) unless \( \lambda = 0 \) and \( u \) is a scalar multiple of 1.

**Proof.** Suppose \( u = u^* \in A_\theta^\infty \). Then \( e^u \geq 0 \), so if \( \lambda \neq 0 \), either \( \lambda e^u \geq 0 \) or \( -\lambda e^u \geq 0 \). Thus if \( \Delta u = -\lambda e^u \), either \( u \) or \( -u \) is subharmonic. The result now follows from Corollary 2.5. \( \square \)

**Alternative Proof.** Use the Maximum Principle, Proposition 2.9. Let \( [a, b] \) be the smallest closed interval containing the spectrum of \( u \). Then for any state \( \varphi \) of \( A_\theta \), \( a \leq \varphi(u) \leq b \) and \( \varphi(e^u) \geq e^a \). If \( \Delta u = -\lambda e^u \) and \( \lambda > 0 \), then by Proposition 2.9 there is a state \( \varphi \) with \( \varphi(u) = a \) and \( \varphi(\Delta u) \geq 0 \), while \( \varphi(-\lambda e^u) < 0 \), a contradiction. Similarly, if \( \lambda < 0 \) and \( \Delta u = -\lambda e^u \), there is a state \( \varphi \) with \( \varphi(u) = b \) and \( \varphi(\Delta u) \leq 0 \), while \( \varphi(-\lambda e^u) > 0 \), a contradiction. \( \square \)

Proposition 4.1 suggests that we consider the equation \( \Delta u = -\frac{1}{2}(he^u + e^uh) \) with \( h = h^* \) not a scalar. (Note that we have symmetrized the right-hand side to make it self-adjoint, since \( u = u^* \) implies \( \Delta u \) is self-adjoint.) Once again, a slight variation on the argument of Proposition 4.1 shows that there is no solution if \( h \geq 0 \) or if \( h \leq 0 \); again this is not surprising since one gets the same result in the classical case as a consequence of Gauss-Bonnet.

**Proposition 4.2.** If \( h \geq 0 \) or \( h \leq 0 \) in \( A_\theta^\infty \), the equation \( \Delta u = -\frac{1}{2}(he^u + e^uh) \) has no solution \( u = u^* \in A_\theta^\infty \) unless \( h = 0 \) and \( u \) is a scalar multiple of 1.

**Proof.** This is just like the proof of Proposition 4.1. If \( h \geq 0 \) and \( \Delta u = -\frac{1}{2}(he^u + e^uh) \), then applying \( \tau \) to both sides, we get

\[
(4.1) \quad 0 = \tau(\Delta u) = -\tau(he^u) = -\tau(h^\frac{1}{2}e^uh^\frac{1}{2}).
\]

Since

\[
h^\frac{1}{2}e^uh^\frac{1}{2} = \left(e^\frac{1}{2}h^\frac{1}{2}\right)^* \left(e^\frac{1}{2}h^\frac{1}{2}\right) \geq 0
\]

and \( \tau \) is faithful, that implies \( e^\frac{1}{2}h^\frac{1}{2} = 0 \). Since \( e^\frac{1}{2} \) is invertible, it follows that \( h^\frac{1}{2} = 0 \) and \( h = 0 \). The case where \( h \leq 0 \) is almost identical; just replace \( h \) by \( -h \) and change the sign of the right-hand side of (4.1). \( \square \)

Unfortunately, the rest of the treatment in \([17] \) \$5\] doesn’t extend to our setting, since from the calculation

\[
\tau(h) = \frac{1}{2}\tau(e^{-u}he^u + h) = -\tau(e^{-u}\Delta u),
\]
Lemma 2.1) The left-hand side is \( \leq \) of (4.2) and we need to show we have a solution of (4.2). It remains to prove the uniqueness. Suppose \( (\Delta u) = 0, \) the right-hand side is \( \geq \) 0. Thus \( \vec{v} \) is not clear if \( \vec{v} < 0 \) follows. (The problem is that we can’t commute the various factors that arise from expanding \( \delta_j(e^{-u}) \) after “integration by parts.”) But since the main purpose of this section is just to test various techniques and see to what extent they apply to nonlinear noncommutative elliptic PDEs, we will consider instead the following more tractable equation from [16, Ch. 5]:

(4.2) \[ \Delta u = \mu e^u - \lambda, \quad \lambda, \mu \in \mathbb{R}, \lambda, \mu > 0. \]

Theorem 4.3. The equation (4.2) has the unique solution \( t_0 = \ln(\lambda/\mu) \) in \( (A_{\theta}^\infty)_{s.a.} \).

Proof. Let

\[ L(u) = E(u) + \tau(\mu e^u - \lambda u). \]

Note that for \( t \in \mathbb{R}, \mu e^t - \lambda t \) has an absolute minimum at \( t = t_0 \), so \( \mu e^{t_0} - \lambda t_0 \geq \lambda(1 - t_0) \) for \( u = u^* \) and so \( L(u) \geq \lambda(1 - t_0) \) for \( u = u^* \). Furthermore, the Euler-Lagrange equation for a critical point of \( L \) is precisely (4.2), since

\[ \frac{d}{dt} |_{t=0} L(u + th) = \tau(\delta_1(u)\delta_1(h) + \delta_2(u)\delta_2(h) - \lambda h) + \frac{d}{dt} |_{t=0} \tau(\mu e^{u+th}), \]

via the calculation in the proof of Proposition 2.11. Now

\[ \frac{d}{dt} |_{t=0} \tau(e^{u+th}) = \frac{d}{dt} |_{t=0} \sum_{n=0}^{\infty} \frac{1}{n!} \tau((u + th)^n) \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \tau(u^{n-1}h + u^{n-2}hu + \cdots + uh^{n-2} + hu^{n-1}) \]

\[ = \sum_{n=0}^{\infty} \frac{n}{n!} \tau(h^{n-1}) = \tau(he^u) \]

by invariance of the trace under cyclic permutations of the factors. So applying Lemma 2.1, we see that

\[ \frac{d}{dt} |_{t=0} L(u + th) = \tau(-h\Delta(u) - \lambda h + \mu he^u) = -\tau(h \cdot (\Delta u + \lambda - \mu e^u)). \]

So nondegeneracy of the trace pairing gives (4.2) as the Euler-Lagrange equation for a critical point of \( L \). It is also clear that \( t_0 \) is an absolute minimum for \( L \) and a solution of (4.2). It remains to prove the uniqueness. Suppose \( u \) is a solution of (4.2) and write \( u = t_0 + v \). Then \( v \) satisfies the equation \( \Delta v = \lambda(e^v - 1) \), and we need to show \( v = 0 \). Multiply both sides by \( v \) and apply \( \tau \). We obtain (using Lemma 2.1)

\[ -2E(v) = \tau(v\Delta v) = \lambda \tau(v(e^v - 1)). \]

The left-hand side is \( \leq 0 \), while since \( \lambda > 0 \) and \( t(e^t - 1) \geq 0 \) with equality only at \( t = 0 \), the right-hand side is \( \geq 0 \). Thus \( E(v) = 0 \), which implies \( v \) is a scalar with \( v(e^v - 1) = 0 \), i.e., \( v = 0 \).

With techniques reminiscent of [16, Ch. 5] we can study a slightly more complicated variant of (4.2).

Theorem 4.4. Let \( a \geq 0 \) be invertible in \( A_{\theta}^\infty \). Then the equation

(4.3) \[ \Delta u = \mu e^u - a, \quad \mu \in \mathbb{R}, \mu > 0 \]

has a solution \( u \in (A_{\theta}^\infty)_{s.a.} \).
We begin by using the Maximum Principle, Proposition 2.9, which implies an a priori bound on solutions of (4.4). (Compare the argument in [16, pp. 56–57].) Indeed, suppose a solution to (4.3), so far only known to be in $B$. We may choose a state $\varphi$ of $A_\theta$ with $\varphi(u) = d$, $\varphi(e^u) = e^d$, and by Proposition 2.9 $\varphi(\Delta u) \leq 0$. Since

$$\varphi((1 - t) u + t e^u - a) = (1 - t) d + t e^d - \varphi(a) \geq (1 - t) d + t e^d - \|a\|,$$

we get a contradiction if $(1 - t) d + t e^d - \|a\| > 0$, which is the case if $d > \|a\|$. So $d \leq \|a\|$. Similarly, we may choose a state $\psi$ of $A_\theta$ with $\psi(u) = c$, $\psi(e^u) = e^c$, and by Proposition 2.9 $\psi(\Delta u) \geq 0$. Since

$$\psi((1 - t) u + t e^u - a) = (1 - t) c + t e^c - \psi(a) \leq (1 - t) c + t e^c - \frac{1}{\|a^{-1}\|},$$

we get a contradiction if $e^c - \frac{1}{\|a^{-1}\|} < 0$. Thus $e^c - \frac{1}{\|a^{-1}\|} \geq 0$ and $c \geq - \ln \|a^{-1}\|$. In other words, we have shown that any solution of (4.4), for any $0 \leq t \leq 1$, satisfies the a priori estimate

$$- \ln \|a^{-1}\| \leq u \leq \|a\|.$$

Now rewrite (4.4) in the form

$$u = (-\Delta + 1)^{-1}(a + t u - t e^u).$$

The right-hand side is well-defined and continuous in the $C^*$-algebra norm topology for $u = (A_\theta)_{s.a.}$, since $(-\Delta + 1)^{-1}$ is bounded by Lemma 2.7. In fact, this Lemma also shows $(-\Delta + 1)^{-1}$ is bounded as a map $A_\theta \to B_\theta$, so as a map $A_\theta \to A_\theta$, it is a limit of operators of finite rank, namely the restrictions of the operator to the span of $\{U^{m}V^{n} : m^2 + n^2 \leq N\}$, as $N \to \infty$. Thus $(-\Delta + 1)^{-1}$ is not only bounded, but also compact. Together with the a priori estimate (4.5) and the fact that there is a solution for $t = 0$, this shows that (4.4) satisfies the hypotheses of the Leray-Schauder Theorem. Hence (4.4) has a solution for all $t \in [0, 1]$. Thus (4.3) (which is the special case of (4.4) for $t = 1$) has a solution in $\text{dom} \Delta \subseteq A_\theta$, and thus in $B_\theta$ by Lemma 2.7.

The last step of the proof is “elliptic regularity.” In other words, we need to show that a solution to (4.3), so far only known to be in $B_\theta$, lies in $A_\theta$. Since $a \in A_\theta^\infty$ and $B_\theta$ is closed under holomorphic functional calculus (by Theorem 2.8), the right-hand side of (4.3) lies in $B_\theta$, i.e., has absolutely summable Fourier coefficients. Then
(4.3) implies that the Fourier coefficients $c_{m,n}$ of $u$ have even faster decay, namely,

$$\sum_{m,n} (1 + m^2 + n^2) |c_{m,n}| < \infty.$$ 

Now one can iterate this argument. This is a bit tricky, as at each step one needs a new Banach subalgebra of $A_\theta$ to replace $\mathcal{B}$ (we drop the subscript $\theta$ for simplicity of notation), so we indicate how this works at the next step, and then sketch how to proceed further. For $u \in \mathcal{B}$ with Fourier coefficients $c_{m,n}$, let

$$\|u\|_1 = \sum_{m,n} (2 + m^2 + n^2) |c_{m,n}|,$$

assuming this converges. We have seen that we know $\|u\|_1 < \infty$. We claim that $\| \cdot \|_1$ is a Banach $*$-algebra norm. This will follow by the argument in the proof of Theorem 2.8 if we can show that

$$\sum_{p,q} (2 + p^2 + q^2) \sum_{m,n} |c_{m,n}| |d_{p-m,q-n}| \leq \left( \sum_{m,n} (2 + m^2 + n^2) |c_{m,n}| \right) \left( \sum_{l,k} (2 + l^2 + k^2) |d_{l,k}| \right).$$

Comparing the two sides of this inequality, one sees it is equivalent to proving that

$$(2 + p^2 + q^2) \leq (2 + m^2 + n^2)(2 + (p - m)^2 + (q - n)^2),$$

or with $\vec{v} = (m,n)$ and $\vec{w} = (p - m, q - n)$ vectors in Euclidean 2-space, that

$$(2 + \|\vec{v} + \vec{w}\|^2) \leq (2 + \|\vec{v}\|^2)(2 + \|\vec{w}\|^2).$$

This inequality in turn follows from the standard inequality

$$\|\vec{v} + \vec{w}\|^2 \leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2 \|\vec{v}\| \cdot \|\vec{w}\| \leq 2(\|\vec{v}\|^2 + \|\vec{w}\|^2).$$

This shows the completion of $A_\infty^\phi$ in the norm $\| \cdot \|_1$ is a Banach $*$-algebra $\mathcal{B}_1$. Since $u$ and $a$ are in $\mathcal{B}_1$, so is $e^{u-a}$. By (4.3) again, $u$ has still more rapid decay; its Fourier coefficients satisfy

$$\sum_{m,n} (m^2 + n^2)^2 |c_{m,n}| < \infty.$$ 

Now we iterate again using still another Banach $*$-algebra $\mathcal{B}_2$ with the norm

$$\|u\|_2 = \sum_{m,n} (8 + (m^2 + n^2)^2) |c_{m,n}|.$$ 

Again one has to check that this is a Banach algebra norm, which will follow from the inequalities

$$8 + \|\vec{v} + \vec{w}\|^4 = 8 + (\|\vec{v}\|^2 + \|\vec{w}\|^2)^2 \leq 8 + 2(\|\vec{v}\|^2 + \|\vec{w}\|^2)^2 \leq 8 + 4(\|\vec{v}\|^4 + \|\vec{w}\|^4 + 2\|\vec{v}\|^2 \cdot \|\vec{w}\|^2) \leq 8 + 4(\|\vec{v}\|^4 + \|\vec{w}\|^4) \leq (8 + \|\vec{v}\|^4)(8 + \|\vec{w}\|^4).$$
Thus $B_2$ is a Banach algebra and $e^u - a \in B_2$, so that $\Delta u \in B_2$ and the Fourier coefficients of $u$ decay faster than $(m^2 + n^2)^3$, etc. Repeating in this way, we show by induction that $c_{m,n}$ is rapidly decreasing, and thus that $u \in A_0^\infty$.

**Sketch of a second proof.** One could also approach this problem using “variational methods.” By the argument at the beginning of the proof of Theorem 3.2, strongly in $H^2$, we can assume that $\{u_n\}$ converges weakly in the Hilbert space $H^1$, and by Theorem 5.2 strongly in $H^0 = L^2$, to some $u \in H^1$ which is a minimizer for $\mathcal{L}$. (Compare the argument in 

Thus we can assume our minimizing sequence $u_n$ is bounded in the Sobolev space $H^1$. Since the unit ball of a Hilbert space is weakly compact, after passing to a subsequence, we can assume that $u_n$ converges weakly in the Hilbert space $H^1$, and by Theorem 5.2 strongly in $H^0 = L^2$, to some $u \in H^1$ which is a minimizer for $\mathcal{L}$. (Compare the argument in 

So we will show that $\mathcal{L}$ must have a minimum point, which will be a solution of (4.3).

Choose $u_n = u_n \in A_0^\infty$ with $\mathcal{L}(u_n)$ decreasing to inf $\{\mathcal{L}(u) : u \in (A_0^\infty)_{s.a.}\}$. Since $E$ and $\tau(e^u - a e^u)$ are separately bounded below, $E(u_n)$ must remain bounded. That means that $\|\delta_j(u_n)\|_{L^2}$ remains bounded for $j = 1, 2$.

We can also assume that $\|u_n\|_{L^2}$ remains bounded. To see this, it is easiest to use a trick (cf. [10], pp. 56–57). Because of the a priori bound on solutions of (4.3), coming from the Maximum Principle (see the first proof above), we can modify the function $e^u$ on the right-hand side of the equation and replace it by some $C^\infty$ function that grows linearly for $u \geq \|a\| + 1$ and decays linearly for $u \leq -1 - \ln \|a\|$. (This does not affect the Maximum Principle argument, so the solutions of the modified equation are the same as for the original one.) This has the effect of changing the term $\tau(e^u)$ in the formula for $\mathcal{L}$ to something that outside of a finite interval behaves like a constant times $\tau(u^2)$, which is $\|u\|_{L^2}^2$.

Thus we can assume our minimizing sequence $u_n$ is bounded in the Sobolev space $H^1$. Since the unit ball of a Hilbert space is weakly compact, after passing to a subsequence, we can assume that $u_n$ converges weakly in the Hilbert space $H^1$, and by Theorem 5.2 strongly in $H^0 = L^2$, to some $u \in H^1$ which is a minimizer for $\mathcal{L}$. (Compare the argument in [10, Theorem 5.2].) This $u$ is a “weak solution” of our equation and we just need to show it is smooth, i.e., corresponds to a genuine element of $A_0^\infty$. This requires an “elliptic regularity” argument similar to the one in the first proof. □
5. Harmonic Unitaries

In this section, we discuss the noncommutative analogue of the classical problem of studying harmonic maps \( M \to S^1 \), where \( M \) is a compact Riemannian manifold and \( S^1 \) is given its usual metric. This problem was studied and solved in [11] pp. 128–129. The homotopy classes of maps \( M \to S^1 \) are classified by \( H^1(M, \mathbb{Z}) \). For each homotopy class in \( H^1(M, \mathbb{Z}) \), we can think of it as an integral class in \( H^1(M, \mathbb{R}) \), and represent it (by the de Rham and Hodge Theorems) by a unique harmonic 1-form with integral periods. Integrating this 1-form gives a harmonic map \( M \to S^1 \) in the given homotopy class. This map is not quite unique since we can compose with an isometry (rotation) of the circle, but except for this we have uniqueness. (This follows from [11] Proposition, p. 123.)

If we dualize a map \( M \to S^1 \), we obtain a unital \(*\)-homomorphism \( C(S^1) \to C(M) \), which since \( C(S^1) \) is the universal \(*\)-algebra on a single unitary generator, is basically the same as a choice of a unitary element \( u \in C(M) \). This analysis suggests that the noncommutative analogue of a harmonic map to \( S^1 \) should be a “harmonic” unitary in a noncommutative \(*\)-algebra \( A \). Each unitary in \( A \) defines a class in the topological \( K \)-theory group \( K_1(A) \), and for a unital \(*\)-algebra, every \( K_1 \) class is represented by a unitary in \( M_n(A) \) for some \( n \), so since we can replace \( A \) by \( M_n(A) \), the natural problem is to search for a harmonic representative in a given connected component of \( U(A) \) (or, passing to the stable limit, in a given \( K_1 \) class).

The next level of complexity up from the case where \( A = C(M) \) is commutative is the case where \( A = C(M, M_n(\mathbb{C})) \) for some \( n \). In this case, a unitary in \( U(A) \) is the same thing as a map \( M \to U(n) \), and a harmonic unitary should be the same thing as a harmonic map \( M \to U(n) \). For example, suppose \( M = S^3 \) and \( n = 2 \). Since there are no maps \( M \to S^1 \) which are not homotopic to a constant, it is natural to look first at smooth maps \( f : S^3 \to U(2) \) with \( \det f : S^3 \to \mathbb{T} \) identically equal to 1, i.e., to look at maps \( f : S^3 \to SU(2) = S^3 \), with both copies of \( S^3 \) equipped with the standard “round” metric. This problem is treated in [11] Proposition, pp. 129–131. For example, the identity map \( S^3 \to S^3 = SU(2) \hookrightarrow U(2) \) is a harmonic map representing the generator of \( K_1(A) = K^{-1}(S^3) \). The study of harmonic maps in other homotopy classes, even just in the simple case of \( S^3 \to S^3 \), is a complicated issue (see, e.g., [11] Proposition, pp. 129–131 and [33]: however, this is quite tangential to the main theme of this article, so we won’t consider it further.

Instead, we consider now the notion of harmonic unitaries in the case of \( A_\theta \). Recall first that \( K_1(A_\theta) \cong \mathbb{Z}^2 \), with \( U \) and \( V \) as generators [25] Corollary 2.5], and that the canonical map \( U(A_\theta)/U(A_\theta)_0 \to K_1(A_\theta) \) is an isomorphism [31].

**Definition 5.1.** If \( u \in A_\theta^\infty \) is unitary, we define the **energy** of \( u \) to be

\[
E(u) = \frac{1}{2} \tau((\delta_1(u))^* \delta_1(u) + (\delta_2(u))^* \delta_2(u)).
\]

Obviously this is constructed so as to be \( \geq 0 \). This definition also coincides with the energy defined in Definition 2.10 provided we insert the appropriate \(*\)'s in the latter (which we can do without changing anything since there we were taking \( u \) to be self-adjoint). The unitary \( u \) is called harmonic if it is a critical point for \( E : U(A_\theta^\infty) \to [0, \infty) \). By the discussion above, a harmonic unitary is the noncommutative analogue of a harmonic circle-valued function on a manifold.
Remark 5.2. Note that in Definition 5.1, \( E(u) \) is invariant under multiplication of \( u \) by a scalar \( \lambda \in \mathbb{T} \). Thus \( E \) descends to a functional on the projective unitary group \( PU(A_\infty^\theta) \) and any sort of uniqueness result for harmonic unitaries can only be up to multiplication of \( u \) by a scalar \( \lambda \in \mathbb{T} \). This is analogous to what happens in the case of harmonic maps \( M \to \mathbb{T} \), where the associated harmonic 1-form is unique but the map itself is only defined up to a “constant of integration.”

Theorem 5.3. If \( u \in A_0^\omega \) is unitary, then \( u \) is harmonic if and only if it satisfies the Euler-Lagrange equation

\[
\tag{5.1} \quad u^*(\Delta u) + (\delta_1(u))^* \delta_1(u) + (\delta_2(u))^* \delta_2(u) = 0.
\]

Note that this equation is “elliptic” (if we drop lower-order terms, it reduces to Laplace’s equation \( \Delta u = 0 \)), but highly nonlinear.

Proof. First note that for \( u \) unitary, since \( uu^* = u^*u = 1 \), we have

\[
\delta_j(u) u^* + u (\delta_j(u))^* = (\delta_j(u))^* u + u^* \delta_j(u) = 0,
\]

\( j = 1, 2 \). If \( u \) is unitary, then any nearby unitary is of the form \( ue^{ith}, h = h^* \), and

\[
\left. \frac{d}{dt} \right|_{t=0} E(ue^{ith}) = \frac{1}{2} \tau \left( -i \delta_1(h)u^* \delta_1(u) + i \delta_1(u)^* u \delta_1(h) \right.
\]

\[
+ \text{similar expression with } \delta_2 \Big).
\]

We can use the trace property to move all the \( \delta_j(h) \)'s to the front. So \( u \) is a critical point if and only if for all \( h = h^* \),

\[
\tag{5.2} \quad \tau \left( \delta_1(h) \text{Im}(u^* \delta_1(u)) + \delta_2(h) \text{Im}(u^* \delta_2(u)) \right) = 0.
\]

In (5.2), the Im’s can be omitted since we have seen that \( u \) unitary \( \Rightarrow \delta_j(u)^* u \) skew-adjoint. Thus \( u \) is harmonic if and only if

\[
\tau \left( \delta_1(h) \left( u^* \delta_1(u) \right) + \delta_2(h) \left( u^* \delta_2(u) \right) \right) = 0
\]

for all \( h = h^* \in A_\infty^\omega \). Now apply integration by parts (Lemma 2.1). We see that \( u \) is harmonic if and only if

\[
\tau \left( h \delta_1 \left( u^* \delta_1(u) \right) + h \delta_2 \left( u^* \delta_2(u) \right) \right) = 0
\]

for all \( h = h^* \in A_\infty^\omega \). Since the trace pairing is nondegenerate, the Theorem follows. \( \square \)

It seems natural to make the following conjecture:

Conjecture 5.4. In each connected component of \( PU(A_0^\omega) \), the functional \( E \) has a unique minimum, given by scalar multiples of \( U^nV^m \). These are the only harmonic unitaries in this component.

Unfortunately, because of the complicated nonlinearity in (5.1), plus complications coming from noncommutativity, we have not been able to prove the Conjecture 5.4. However, we have the following partial result. In particular, we see that every connected component in \( U(A_0^\omega) \) contains a harmonic unitary which is energy-minimizing.
Theorem 5.5. The scalar multiples of $U^m V^n$ are harmonic and are strict local minima for $E$. Any harmonic unitary $u$ depending on $U$ alone is a scalar multiple of a power of $U$. Similarly, any harmonic unitary $u$ depending on $V$ alone is a scalar multiple of a power of $V$.

Proof. First suppose $u$ depends on $U$ alone. Then $\delta_1(u) = 0$. So by the proof of Theorem 5.3 if $u$ is harmonic, then $\tau \left( \delta_1(h) \cdot \delta_1(u)^* u \right) = 0 \forall h = h^*$. This must also hold for general $h$ (not necessarily self-adjoint) since we can split $h$ into its self-adjoint and skew-adjoint parts. Since the range of $\delta_1$ contains $U^m$ unless $m = 0$, $\tau(\delta_1(u)^* u U^m) = 0$ for $m \neq 0$, which means (since $\delta_1(u)^* u$ depends only on $U$) that $\delta_1(u)^* u$ is a scalar. Thus $u$ is an eigenfunction for $\delta_1$ and so $u = e^{i\lambda} U^m$ for some $m$. The case where $u$ depends on $V$ alone is obviously similar.

Next let's examine $u = U^m V^n$. Since $E(U^m V^n) = 2\pi^2 (m^2 + n^2)$ while

$$(U^m V^n)^* \Delta(U^m V^n) = -4\pi^2 (m^2 + n^2),$$

$u$ satisfies (5.1) and is therefore harmonic. We show it is a local minimum for $E$; in fact, the minimum is strict once we pass to $PU(A^\infty_0)$. We expand $\delta_j(u e^{i\lambda t})$, with $h = h^*$, out to second order in $t$. Note that with $\delta = \delta_1$ or $\delta_2$,

$$\delta(u e^{i\lambda t}) = \delta(u) + it \left[ \delta(u) h + u \delta(h) \right] - \frac{t^2}{2} \left[ \delta(u) h^2 + u \delta(h) h + u h \delta(h) \right] + O(t^3).$$

We substitute this into the formula for $E(u e^{i\lambda t})$. The terms linear in $t$ cancel since $u$ is harmonic, and we find that

$$E(u e^{i\lambda t}) = 2\pi^2 (m^2 + n^2) + t^2 \tau \left[ (\delta_1(u) h + u \delta_1(h))^* (\delta_1(u) h + u \delta_1(h)) \right]$$

$$- \frac{1}{2} (\delta_1(u)^* (\delta_1(u) h^2 + u \delta_1(h) h + u h \delta_1(h))$$

$$- \frac{1}{2} (h^2 \delta_1(u)^* + h \delta_1(h) u^* + \delta_1(h) h u^*) \delta_1(u)$$

$$+ \text{similar expressions with } \delta_2 \right] + O(t^3).$$

This actually simplifies considerably since $u$ is an eigenvector for both $\delta_1$ and $\delta_2$, so that $\delta_j(u)^* \delta_j(u)$, $\delta_j(u)^* u$, and $u^* \delta_j(u)$ are all scalars. It turns out that almost everything cancels and one gets

$$E(u e^{i\lambda t}) = 2\pi^2 (m^2 + n^2) + \frac{1}{2} t^2 \tau \left( \delta_1(h)^2 + \delta_2(h)^2 \right) + O(t^3)$$

$$= 2\pi^2 (m^2 + n^2) + t^2 \ E(h) + O(t^3).$$

By Proposition 2.11 the term in $t^2$ vanishes exactly when $h$ is a constant, and in that case $E(u e^{i\lambda t}) = E(u) = 2\pi^2 (m^2 + n^2)$ (exactly). Otherwise, the coefficient of $t^2$ is strictly positive and $E(u e^{i\lambda t})$ has a strict local minimum at $t = 0$. \qed

6. THE LAPLACIAN AND HOLOMORPHIC GEOMETRY

As we have seen, $\Delta$ on $A_0$ behaves very much like the classical Laplacian on $\mathbb{T}^2$. But the Laplacian in (real) dimension 2 is very closely related to holomorphic geometry in complex dimension 1. That suggests that the theory we have developed should be closely related to the Cauchy-Riemann operators $\partial$ and $\bar{\partial}$ on “noncommutative elliptic curves,” as developed in references like [27] [26].
In classical analysis (in one complex variable), one usually sets \( \overline{\partial} = \frac{1}{2}(\overline{\partial_x} + i \overline{\partial_y}) \), the Cauchy-Riemann operator, with \( \partial \) its complex conjugate. Then \( \Delta = 4\overline{\partial}\partial \). In our situation, the obvious analogue is to set \( \overline{\partial} = \frac{1}{2}(\delta_1 + i \delta_2) \). Comparable to Proposition 2.4 is:

**Proposition 6.1.** The operator \( \overline{\partial} : A^\infty_b \to A^\infty_b \) has kernel given by scalar multiples of the identity, and restricts to a bijection on \( \ker \tau \).

**Proof.** Immediate from the fact that if \( a = \sum_{m,n} c_{m,n} U^m V^n \), then
\[
\overline{\partial}a = \pi i \sum_{m,n} (m + in) c_{m,n} U^m V^n ,
\]
together with the characterization of elements of \( A^\infty_b \) in terms of rapidly decreasing Fourier series. \( \square \)

Thus the noncommutative torus admits no nontrivial global “holomorphic functions.” This is not surprising since a compact complex manifold admits no nonconstant global holomorphic functions. However, assuming \( \tau(f) = 0 \), we can solve the inhomogeneous Cauchy-Riemann equation \( \overline{\partial}u = f \), which in the classical case is related to the proof of the Mittag-Leffler Theorem (see, for example, [13, Ch. 1]).

In some situations, one is led to the more complicated equation (similar to the above but with \( \overline{\partial} \) replaced by the “logarithmic Cauchy-Riemann operator”) \( (\overline{\partial}u) u^{-1} = f \), which we can rewrite as \( \overline{\partial}u = fu \). This equation was already studied (modulo a change of conventions about whether one should multiply on the left or the right) in a (different) noncommutative context in [3], and then in [28].

**Theorem 6.2** (Polishchuk, [28]). Let \( f \in A_b \). Then the equation \( \overline{\partial}u = fu \) has a nonzero solution if and only if \( \tau(f) \in \pi i(\mathbb{Z} + i\mathbb{Z}) \).

[Comment: Polishchuk and Schwarz in [27] [28] use a slightly different convention. They take \( \overline{\partial} \) to be \( (x + iy)\delta_1 + \delta_2 \), with \( y < 0 \); when \( x = 0 \) and \( y = -1 \), this is what we have here, up to a constant factor of \(-2i\). This constant explains why the result looks different. With our convention, \( u = U^m V^n \) solves \( \overline{\partial}u = fu \) with \( f = \pi i(m + in) \).

The relevance of this result concerns the theory of noncommutative meromorphic functions. While a compact complex manifold admits no nonconstant global holomorphic functions, it can admit nonconstant meromorphic functions, such as (in the case of an elliptic curve) elliptic functions like the Weierstraß \( \wp \) function. There are two ways we can view meromorphic functions on a Riemann surface \( M \). On the one hand, they can be considered as ratios of holomorphic sections of holomorphic line bundles \( \mathcal{L} \) of \( M \). On the other hand, they can be considered as formal quotients of functions that satisfy the Cauchy-Riemann equation. These points of view, applied to a noncommutative torus, are equivalent via the following reasoning. A holomorphic vector bundle is defined via its module of (smooth) sections, which is a finitely generated projective (right) \( A^\infty_b \)-module. This module must be equipped with an operator \( \nabla \) satisfying the basic axiom
\[
\nabla(s \cdot a) = \nabla(s) \cdot a + s \cdot \overline{\partial}(a).
\]

We could also study different conformal structures on the torus, by changing the \( i \) here to another complex number in the upper half-plane, but for the problems we will study here, this makes no essential difference.
If we assume the module is $A_\theta^\infty$ itself (i.e., the vector bundle is of “dimension 1,” i.e., is a line bundle), then this operator is determined by $f = \nabla(1)$, in that for any $s$,

$$\nabla(s) = \nabla(1 \cdot s) = f \cdot s + 1 \overline{\partial}(s) = \overline{\partial}(s) + fs.$$  

A “holomorphic section” of the bundle is then a solution $s$ of $\overline{\partial}(s) + fs = 0$.

On the other hand, the natural definition of “meromorphic functions” is the following.

**Definition 6.3.** A meromorphic function on the noncommutative torus $A_\theta$ is a formal quotient $u^{-1}v$, with $u, v \in \text{dom}(\overline{\partial}) \subset A_\theta$, satisfying the Cauchy-Riemann equation (in the sense to be made precise below). Here we don’t want to require that $u$ be invertible in $A_\theta$ (otherwise $u^{-1}v$ would be holomorphic, hence constant), so we simply want $u$ to be “regular” (in the sense of not being either a left or right zero divisor), and the inverse is to be interpreted in a formal sense (or in the maximal ring of quotients $[2]$, the algebra of unbounded operators affiliated to the hyperfinite $\text{II}_1$ factor obtained by completing $A_\theta$ in its trace representation). Then the condition that $u^{-1}v$ be meromorphic is that

$$0 = \overline{\partial}(u^{-1}v) = \overline{\partial}(u^{-1})v + u^{-1}\overline{\partial}v = -u^{-1}\overline{\partial}(u)u^{-1}v + u^{-1}\overline{\partial}v,$$

or (via multiplication by $u$ on the left) that

$$(6.1) \quad \overline{\partial}v = fv, \quad \overline{\partial}u = fu,$$

which says precisely that our meromorphic function is a quotient of two holomorphic sections of a holomorphic line bundle with $\nabla = \overline{\partial} + f$. In the other direction, if $u$ and $v$ satisfy (6.1) and $u$ is regular, so that the formal expression $u^{-1}v$ makes sense, then we formally have

$$\overline{\partial}(u^{-1}v) = \overline{\partial}(u^{-1})v + u^{-1}\overline{\partial}v = -u^{-1}\overline{\partial}(u)u^{-1}v + u^{-1}\overline{\partial}v$$

$$= -u^{-1}fuu^{-1}v + u^{-1}fv = -u^{-1}fv + u^{-1}fv = 0,$$

and $u^{-1}v$ is meromorphic.

In accordance with the classical existence theorem of Weierstraß for elliptic functions, we have:

**Proposition 6.4.** There exist nonconstant meromorphic functions on the noncommutative torus $A_\theta$, in the sense of Definition 6.3.

**Proof.** This follows immediately from the discussion in [28 §3], which shows that there are choices for $f$ for which the holomorphic connection $\nabla$ is reducible, with a space of holomorphic sections of dimension bigger than 1, and thus there are solutions of (6.1) with $u$ and $v$ not linearly dependent. Note that if this is the case, $u$ cannot be invertible ([28 Lemma 3.14]—we also know this independently from Proposition 6.1). But we do require $u$ to be regular, so we need to check that this can be achieved. For example, suppose $e$ is a proper projection in $A_\theta^\infty$ (“proper” means $0 < \tau(e) < 1$) of trace $m + n\theta$ with $n$ relatively prime to both $m$ and $1 - m$. The trivial rank-one right $A_\theta^\infty$ module splits as $eA_\theta^\infty \oplus (1 - e)A_\theta^\infty$, and we can arrange to choose a holomorphic connection on $A_\theta^\infty$ that is reducible in a way compatible with this splitting, so that there are 1-dimensional spaces of holomorphic sections on each of $eA_\theta^\infty$ and $(1 - e)A_\theta^\infty$. By the explicit formulas in [27 Proposition 2.5], these come from real-analytic functions in $S(\mathbb{R})$, and so it’s
evident that the \( u \) that results from putting these together is regular, as by \([2]\), it’s enough to observe that its left and right support projections are equal to 1. □

On the other hand, there is also a non-existence result for meromorphic functions on the [classical] torus: no such nonconstant function exists with only a single simple pole \([1, \text{Corollary to Theorem 4, p. 271}]\). We can find an analogue of this in the noncommutative situation also. To explain it, first note that in the sense of distributions on the complex plane, \( \overline{\partial}(\frac{1}{z}) \) is not zero (if it were, \( \frac{1}{z} \) would have a removable singularity, by elliptic regularity), but rather is equal to \( \pi \delta \), where \( \delta \) is the Dirac \( \delta \)-distribution at 0. Suppose there were a meromorphic function \( f \) on \( T^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \) with at worst one simple pole and no other poles. Then \( f \) would be locally integrable and, after translation to move the pole to 0, would define a distribution on \( T^2 \) with \( \overline{\partial}(f) \) a multiple of \( \delta \). Thus the Fourier series of \( \overline{\partial}(f) \) would be a multiple of the Fourier series of \( \delta \), which is \( \sum_{m,n} U^m V^n \). And in fact Fourier analysis gives another proof of the nonexistence theorem not using residue calculus. Suppose \( f \) were nonconstant. Since a compact complex manifold admits no nonconstant holomorphic functions, \( f \) cannot be holomorphic, which means that \( \overline{\partial}f \) must be non-zero in the sense of distributions. Since \( \overline{\partial}(f) \) is a multiple of \( \sum_{m,n} U^m V^n \), the proportionality constant, which is also the \((0,0)\) Fourier coefficient of \( \overline{\partial}f \), must be non-zero. But this is impossible since the Fourier series of any distribution in the image on \( \overline{\partial} \) must have zero constant term. The noncommutative analogue of all this is the following:

**Proposition 6.5.** Let \( f \) be a distribution in the dual of \( A^\infty_\theta \). (The distributions consist of formal Fourier series \( \sum_{m,n} c_{m,n} U^m V^n \) with \( \{c_{m,n}\} \) of tempered growth.) Suppose \( \overline{\partial}f \) is a multiple of \( \sum_{m,n} U^m V^n \). Then \( f \) is a constant.

**Proof.** This follows exactly the lines as the argument above for the classical theorem. If \( \overline{\partial}f \) has formal Fourier expansion \( c \sum_{m,n} U^m V^n \), then the \((m,n)\) coefficient, \( c \), must be divisible by \( m + in \) for all \((m,n)\). Because of the \((0,0)\) coefficient, this is only possible if \( c = 0 \). But if \( c = 0 \), then \( f \) is in the distributional kernel of \( \overline{\partial} \), which forces all the Fourier coefficients of \( f \) to vanish except for the constant term. □

In fact, essentially the same proof proves a slightly more general statement, which in the classical case is equivalent to \([1, \text{Theorem 4, p. 271}]\). For the analysis above shows that the “sum of the residues” of a meromorphic function \( f \) on \( T^2 \), when the function is considered as a distribution\(^4\) is precisely the constant term in the Fourier series of \( \overline{\partial}f \), up to a factor of \( \pi \). The analogue of the “sum of the residues” theorem in the noncommutative world is:

**Proposition 6.6.** Let \( f \) be a distribution in the dual of \( A^\infty_\theta \). Then the constant term in the (formal) Fourier series of \( \overline{\partial}f \) is zero.

**Proof.** Essentially the same as before. □

The connection with the main subject of this paper is of course that “meromorphic functions” \( w \) as studied in this section are “singular” solutions of Laplace’s

\(^4\)This requires a comment. A meromorphic function with simple poles is locally integrable, thus defines a distribution in the obvious way. A meromorphic function with higher-order poles is not locally integrable, but can be made into a distribution of “principal value integral” type. This distribution is not a measure.
equation $\Delta w = 0$, since $\Delta = 4 \partial \partial$. More precisely, “singular solution” means classically that as a distribution, $\Delta w$ is not necessarily 0, but has countable support. In the noncommutative setting, we do not have a notion of support for a distribution, but the same basic idea applies.

References

1. Lars V. Ahlfors, *Complex analysis*, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics. MR510197 (80c:30001)
2. S. K. Berberian, *The maximal ring of quotients of a finite von Neumann algebra*, Rocky Mountain J. Math. **12** (1982), no. 1, 149–164. MR649748 (83i:16005)
3. J.-B. Bost, *Principe d’Oka, K-théorie et systèmes dynamiques non commutatifs*, Invent. Math. **101** (1990), no. 2, 261–333. MR1062964 (92j:46126)
4. Ali H. Chamseddine and Alain Connes, *The spectral action principle*, Comm. Math. Phys. **186** (1997), no. 3, 731–750. MR1463819 (99c:58010)
5. Alain Connes, *C∗ algebres et géométrie différentielle*, C. R. Acad. Sci. Paris Sér. A-B **290** (1980), no. 13, A599–A604, available now at http://www.alainconnes.org/. MR572645 (81c:46053)
6. Alain Connes, Michael R. Douglas, and Albert Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, J. High Energy Phys. (1998), no. 2, Paper 3, 35 pp. (electronic). MR1613978 (99b:58023)
7. Alain Connes and John Lott, *Particle models and noncommutative geometry*, Nuclear Phys. B Proc. Suppl. **18B** (1990), 29–47 (1991), Recent advances in field theory (Annecy-le-Vieux, 1990). MR1128127 (93a:58015)
8. Alain Connes and Marc A. Rieffel, *Yang-Mills for noncommutative two-tori*, Operator algebras and mathematical physics (Iowa City, Iowa, 1985), Contemp. Math., vol. 62, Amer. Math. Soc., Providence, RI, 1987, pp. 237–266. MR878383 (88b:58023)
9. Ludwik Dąbrowski, Thomas Krajewski, and Giovanni Landi, *Some properties of nonlinear σ-models in noncommutative geometry*, Internat. J. Modern Phys. B **14** (2000), no. 22-23, 2367–2382, Proceedings of the 1999 Euroconference: On Non-commutative Geometry and Hopf Algebras in Field Theory and Particle Physics (Torino). MR1818819 (2002a:81295)
10. , *Non-linear σ-models in noncommutative geometry: fields with values in finite spaces*, Modern Phys. Lett. A **18** (2003), no. 33-35, 2371–2379, Spacetime and fundamental interactions (Vietri sul Mare, 2003). MR2035840 (2004k:81186)
11. James Eells, Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160. MR0164306 (29 #1603)
12. Karlheinz Gröchenig and Michael Leinert, *Wiener’s lemma for twisted convolution and Gabor frames*, J. Amer. Math. Soc. **17** (2004), no. 1, 1–18 (electronic). MR2015328 (2004m:42037)
13. Lars Hörmander, *An introduction to complex analysis in several variables*, third ed., North-Holland Mathematical Library, vol. 7, North-Holland Publishing Co., Amsterdam, 1990. MR1045639 (91a:32001)
14. A. Hulanicki, *On symmetry of group algebras of discrete nilpotent groups*, Studia Math. **35** (1970), 207–219 (errata insert). MR0278082 (43 #3814)
15. , *On the spectrum of convolution operators on groups with polynomial growth*, Invent. Math. **17** (1972), 135–142. MR0323951 (48 #2304)
16. Jerry L. Kazdan, *Some applications of partial differential equations to problems in geometry*, Tech. report, Tokyo University, 1983, originally part of the Surveys in Geometry series, available now in a revised edition, 1993, at http://www.math.upenn.edu/~kazdan/japan/japan.pdf.
17. Jerry L. Kazdan and F. W. Warner, *Curvature functions for compact 2-manifolds*, Ann. of Math. (2) **99** (1974), 14–47. MR0343205 (49 #7949)
18. Jean Leray and Jules Schauder, *Topologie et équations fonctionnelles*, Ann. Sci. École Norm. Sup. (3) **51** (1934), 45–78, available now at http://www.numdam.org/item?id=ASENS_1934_3_51_45_0. MR1509338
19. Franz Luef, *On spectral invariance of non-commutative tori*, Operator theory, operator algebras, and applications, Contemp. Math., vol. 414, Amer. Math. Soc., Providence, RI, 2006, pp. 131–146. MR2277208
20. Varghese Mathai and Jonathan Rosenberg, *A noncommutative sigma-model*, In preparation.
21. , *T-duality for torus bundles with H-fluxes via noncommutative topology*, Comm. Math. Phys. 253 (2005), no. 3, 705–721. MR2116734 (2006h:58008)
22. , *T-duality for torus bundles with H-fluxes via noncommutative topology. II. The high-dimensional case and the T-duality group*, Adv. Theor. Math. Phys. 10 (2006), no. 1, 123–158. MR2222224 (2007m:58009)
23. J. D. Newburgh, *The variation of spectra*, Duke Math. J. 18 (1951), 165–176. MR0051441 (14,481b)
24. D. J. Newman, *A simple proof of Wiener’s 1/f theorem*, Proc. Amer. Math. Soc. 48 (1975), 264–265. MR0365002 (51 #1255)
25. M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain cross-product C*-algebras*, J. Operator Theory 4 (1980), no. 1, 93–118. MR587369 (82c:46074)
26. A. Polishchuk, *Classification of holomorphic vector bundles on noncommutative two-tori*, Doc. Math. 9 (2004), 163–181 (electronic). MR2054986 (2005c:58011)
27. A. Polishchuk and A. Schwarz, *Categories of holomorphic vector bundles on noncommutative two-tori*, Comm. Math. Phys. 236 (2003), no. 1, 135–159. MR1977884 (2004k:58011)
28. Alexander Polishchuk, *Analogues of the exponential map associated with complex structures on noncommutative two-tori*, Pacific J. Math. 226 (2006), no. 1, 153–178. MR2247860 (2007k:58011)
29. D. A. Raikov, *To [sic] the theory of normed rings with involution*, C. R. (Doklady) Acad. Sci. URSS (N.S.) 54 (1946), no. 5, 387–390. MR0019845 (8,469d)
30. Marc A. Rieffel, *C*-algebras associated with irrational rotations, Pacific J. Math. 93 (1981), no. 2, 415–429. MR623572 (83b:46087)
31. , *The homotopy groups of the unitary groups of noncommutative tori*, J. Operator Theory 17 (1987), no. 2, 237–254. MR887221 (88f:22018)
32. , *Critical points of Yang-Mills for noncommutative two-tori*, J. Differential Geom. 31 (1990), no. 2, 535–546. MR1037414 (91b:58014)
33. Richard Schoen and Karen Uhlenbeck, *Regularity of minimizing harmonic maps into the sphere*, Invent. Math. 78 (1984), no. 1, 89–100. MR762354 (86a:58024)
34. Nathan Seiberg and Edward Witten, *String theory and noncommutative geometry*, J. High Energy Phys. (1999), no. 9, Paper 32, 93 pp. (electronic). MR1720697 (2001i:81237)

Department of Mathematics, University of Maryland, College Park, MD 20742, USA
E-mail address: jmr@math.umd.edu
URL: http://www.math.umd.edu/~jmr