Holonomy on the principal $U(n)$ bundles over Grassmannian manifolds

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Abstract

Consider the principal $U(n)$ bundles over Grassmann manifolds $U(n) \rightarrow U(n + m)/U(m) \rightarrow G_{n,m}$. Let $S$ be a complete totally geodesic surface in the base space and $\gamma$ be a piecewise smooth, simple closed curve on $S$. Then the holonomy displacement along $\gamma$ is given by

$$V(\gamma) = e^{\lambda A(\gamma)i}$$

where $A(\gamma)$ is the area of the region on the surface $S$ surrounded by $\gamma$; $\lambda = \frac{1}{2}$ or 0 depending on whether $S$ is a complex submanifold or not.

In the process, we also characterize complete totally geodesic 2-dimensional submanifolds in Grassmanian manifolds $G_{n,m}$.

Keywords: principal $U(n)$ bundle, Grassmannian manifold, holonomy displacement, complete totally geodesic submanifold

2010 MSC: 53C29, 53C30, 53C35, 57N35

0. Introduction

Consider the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$. Let $\gamma$ be a simple closed curve on $S^2$. Pick a point in $S^3$ over $\gamma(0)$, and take the unique horizontal lift $\tilde{\gamma}$ of $\gamma$. Since $\gamma(1) = \gamma(0)$, $\tilde{\gamma}(1)$ lies in the same fiber as $\tilde{\gamma}(0)$ does. We are interested

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1This research was supported by National Research Foundation of Korea(NRF) grant funded by the Korean Government(2010-0019516).

2This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2013R1A1A2006926).
in understanding the difference between $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$. The following equality was already known [2]:

$$V(\gamma) = e^{\frac{1}{2} A(\gamma) i},$$

where $V(\gamma)$ is the holonomy displacement along $\gamma$, and $A(\gamma)$ is the area of the region surrounded by $\gamma$.

In this paper, we generalize this fact to the following higher dimensional Stiefel bundle over Grassmannian manifold:

$$U(n) \to U(n + m)/U(m) \xrightarrow{\pi} G_{n,m},$$

where $G_{n,m} = U(n+m)/(U(n) \times U(m))$. The main results are stated as follows:

Let $S$ be a complete totally geodesic surface in the base space. Let $\gamma$ be a piecewise smooth, simple closed curve on $S$ parametrized by $0 \leq t \leq 1$, and $\tilde{\gamma}$ its horizontal lift on the bundle $U(n) \to \pi^{-1}(S) \xrightarrow{\pi} S$, which is immersed in $U(n) \to U(n + m)/U(m) \xrightarrow{\pi} G_{n,m}$. Then

$$\tilde{\gamma}(1) = e^{\frac{1}{2} A(\gamma) i} \cdot \tilde{\gamma}(0) \quad \text{or} \quad \tilde{\gamma}(1) = \tilde{\gamma}(0),$$

where $A(\gamma)$ is the area of the region on the surface $S$ surrounded by $\gamma$, depending on whether $S$ has a complex submanifold or not. See Theorem 2.6.

We also characterize complete totally geodesic 2-dimensional submanifolds in Grassmannian manifolds $G_{n,m}$.

1. The bundle $U(1) \longrightarrow U(2)/U(1) \longrightarrow G_{1,1}$

First we study the case of $n = m = 1$ for the general principal bundle

$$U(n) \to U(n + m)/U(m) \to G_{n,m}.$$ 

We use $SU(2)$ rather than $U(2)$. Thus, our bundle is

$$U(1) \longrightarrow SU(2) \longrightarrow SU(2)/U(1).$$

Of course,

$$S^3 \cong SU(2) = \{ A \in \text{GL}(2, \mathbb{C}) : AA^* = I \text{ and } \det(A) = 1 \}.$$ 

From now on, we use the convention of $\mathfrak{gl}(k, \mathbb{C}) \subset \mathfrak{gl}(2k, \mathbb{R})$ by

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11} + iy_{11} & x_{12} + iy_{12} \\ x_{21} + iy_{21} & x_{22} + iy_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11} & -y_{11} & x_{12} & -y_{12} \\ y_{11} & x_{11} & y_{12} & x_{12} \\ x_{21} & -y_{21} & x_{22} & -y_{22} \\ y_{21} & x_{21} & y_{22} & x_{22} \end{bmatrix}. $$

The group $SU(2)$ has the following natural representation into $\text{GL}(4, \mathbb{R})$:

$$w = \begin{bmatrix} w_1 & w_2 & -w_3 & -w_4 \\ -w_2 & w_1 & w_4 & -w_3 \\ w_3 & -w_4 & w_1 & -w_2 \\ w_4 & w_3 & w_2 & w_1 \end{bmatrix}. $$
with the condition \( w_1^2 + w_2^2 + w_3^2 + w_4^2 = 1 \). In fact, the map

\[
  \begin{align*}
  w_1 + w_2 i + w_3 j + w_4 k & \mapsto w
  \end{align*}
\]

is a monomorphism from the unit quaternions into \( \text{GL}(4, \mathbb{R}) \). The circle group

\[
  \mathbb{S}^1 = \left\{ \begin{bmatrix} e^{-iz} & 0 \\ 0 & e^{iz} \end{bmatrix} : 0 \leq z \leq 2\pi \right\}
\]

is a subgroup of \( SU(2) \), and acts on \( SU(2) \) as right translations, freely with quotient \( \mathbb{CP}^1 = \mathbb{S}^2 \), the 2-sphere, giving rise to the fibration

\[
  \mathbb{S}^1 \rightarrow SU(2) \rightarrow \mathbb{CP}^1.
\]

Let \( \tilde{w} \) be the “\( i \)-conjugate” of \( w \) (replace \( w_2 \) by \( -w_2 \)). That is,

\[
  \tilde{w} = \begin{bmatrix}
    w_1 & -w_2 & -w_3 & -w_4 \\
    w_2 & w_1 & w_4 & -w_3 \\
    w_3 & -w_4 & w_1 & w_2 \\
    w_4 & w_3 & -w_2 & w_1
  \end{bmatrix}.
\]

Then,

\[
  w\tilde{w} = \begin{bmatrix}
    w_1^2 + w_2^2 - w_3^2 - w_4^2 \\
    2(w_1 w_3 + w_2 w_4) \\
    -2w_2 w_3 + 2w_1 w_4 \\
    2(w_1 w_3 + w_2 w_4)
  \end{bmatrix} \begin{bmatrix}
    w_1 + w_2 i + w_3 j + w_4 k
  \end{bmatrix}
\]

and

\[
  (w_1^2 + w_2^2 - w_3^2 - w_4^2)^2 + (2w_1 w_3 + 2w_2 w_4)^2 + (-2w_2 w_3 + 2w_1 w_4)^2 = 1.
\]

Clearly, \( \mathbb{CP}^1 \) can be identified with the following

\[
  \mathbb{CP}^1 = \left\{ \begin{bmatrix}
    x & 0 & -y & -z \\
    0 & x & z & -y \\
    y & -z & x & 0 \\
    z & y & 0 & x
  \end{bmatrix} : x^2 + y^2 + z^2 = 1 \right\}.
\]

Therefore, the map

\[
  p : SU(2) \rightarrow \mathbb{CP}^1
\]

defined by

\[
  p(w) = w\tilde{w}
\]

has the following properties

\[
  p(wv) = wp(v)\tilde{w} \quad \text{for all } w, v \in SU(2)
\]

\[
  p(wv) = p(w) \quad \text{if and only if } v \in \mathbb{S}^1.
\]

This shows that the map \( p \) is, indeed, the orbit map of the principal bundle \( \mathbb{S}^1 \rightarrow SU(2) \rightarrow \mathbb{CP}^1 \).
The Lie group $SU(2)$ will have a left-invariant Riemannian metric given by the following orthonormal basis on the Lie algebra $\mathfrak{su}(2)$

$$e_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Notice that $e_1$ and $e_2$ correspond to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ i \end{pmatrix}$ in $\mathfrak{gl}(2, \mathbb{C})$ and $[e_1, e_2] = 2e_3$. In order to understand the projection map better, consider the subset of $SU(2)$:

$$T = \left\{ \begin{bmatrix} \cos x & -(\sin x)e^{-iy} \\ (\sin x)e^{iy} & \cos x \end{bmatrix} : 0 \leq x \leq \pi, \ 0 \leq y \leq 2\pi \right\}$$

$$= \left\{ \begin{bmatrix} \cos x & 0 & -(\sin x)(\cos y) & -(\sin x)(\sin y) \\ 0 & \cos x & -(\sin x)(\sin y) & -(\sin x)(\cos y) \\ (\sin x)(\cos y) & -(\sin x)(\sin y) & \cos x & 0 \\ (\sin x)(\sin y) & -(\sin x)(\cos y) & 0 & \cos x \end{bmatrix} \right\}$$

which is the exponential image of

$$m = \left\{ \begin{bmatrix} 0 & -\xi^t \\ \xi & 0 \end{bmatrix} : \xi \in \mathbb{C} \right\}.$$

The map $p$ restricted to $T$ is just the squaring map; that is,

$$p(w) = w^2, \quad w \in T.$$

**Theorem 1.1** ([2]). Let $S^1 \to SU(2) \to \mathbb{C}P^1$ be the natural fibration. Let $\gamma$ be a piecewise smooth, simple closed curve on $\mathbb{C}P^1$. Then the holonomy displacement along $\gamma$ is given by

$$V(\gamma) = e^{\frac{1}{2}A(\gamma)i} \in S^1$$

where $A(\gamma)$ is the area of the region in $\mathbb{C}P^1$ enclosed by $\gamma$.

**Proof.** Let $\gamma(t)$ be a closed loop on $\mathbb{C}P^1$ with $\gamma(0) = p(I_4)$. Therefore,

$$\gamma(t) = \begin{bmatrix} \cos 2x(t) & 0 & -\sin 2x(t) \cos y(t) & -\sin 2x(t) \sin y(t) \\ -\sin 2x(t) \cos y(t) & \cos 2x(t) & -\sin 2x(t) \sin y(t) & -\sin 2x(t) \cos y(t) \\ \sin 2x(t) \sin y(t) & -\sin 2x(t) \cos y(t) & \cos 2x(t) & 0 \\ \sin 2x(t) \cos y(t) & -\sin 2x(t) \sin y(t) & 0 & \cos 2x(t) \end{bmatrix}$$

Let

$$\tilde{\gamma}(t) = \begin{bmatrix} \cos x(t) & 0 & -\sin x(t) \cos y(t) & -\sin x(t) \sin y(t) \\ -\sin x(t) \cos y(t) & \cos x(t) & -\sin x(t) \sin y(t) & -\sin x(t) \cos y(t) \\ \sin x(t) \sin y(t) & -\sin x(t) \cos y(t) & \cos x(t) & 0 \\ \sin x(t) \cos y(t) & -\sin x(t) \sin y(t) & 0 & \cos x(t) \end{bmatrix}$$
with $0 \leq x(t) \leq \pi/2$ so that $p(\gamma(t)) = \gamma(t)$ (\gamma is a lift of \gamma), and let

$$\omega(t) = \begin{bmatrix}
\cos z(t) & -\sin z(t) & 0 & 0 \\
\sin z(t) & \cos z(t) & 0 & 0 \\
0 & 0 & \cos z(t) & \sin z(t) \\
0 & 0 & -\sin z(t) & \cos z(t)
\end{bmatrix}.$$ 

Put

$$\eta(t) = \tilde{\gamma}(t) \cdot \omega(t).$$

Then still $p(\eta(t)) = \gamma(t)$, and $\eta$ is another lift of $\gamma$. We wish $\eta$ to be the horizontal lift of $\gamma$. That is, we want $\eta'(t)$ to be orthogonal to the fiber at $\eta(t)$.

The condition is that $\langle \eta'(t), (\eta(t))_*(e_3) \rangle = 0$, or equivalently, $\langle (\eta(t)-1)_* \eta'(t), e_3 \rangle = 0$. That is,

$$\eta(t)^{-1} \cdot \eta'(t) = \alpha_1 e_1 + \alpha_2 e_2$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$. From this, we get the following equation:

$$z'(t) = \sin^2 x(t)y'(t). \quad (1-1)$$

Since any piecewise smooth curve can be approximated by a sequence of piecewise linear curves which are sums of boundaries of rectangular regions, it will be enough to prove the statement for a particular type of curves as follows [1]: Suppose we are given a rectangular region in the $xy$-plane

$$p \leq x \leq p + a, \quad q \leq y \leq q + b.$$ 

Consider the image $R$ of this rectangle in $\mathbb{CP}^1$ by the map

$$(x, y) \mapsto r(x, y) = (\cos 2x, (\sin 2x)(\cos y), (\sin 2x)(\sin y)).$$

Then $||r_x \times r_y|| = 2 \sin 2x$, (because $0 \leq x \leq \pi/2$). Thus, the area of $R$ is

$$\int_{q}^{q+b} \int_{p}^{p+a} 2 \sin 2x \, dx \, dy = 2b(\sin^2 p + a) - \sin^2(p).$$

On the other hand, the change of $z(t)$ along the boundary $\gamma(t)$ of this region can be calculated using condition [1-1]. Let $\gamma(t)$ be represented by $(p + 4at, q)$ for $t \in [0, \frac{1}{4})$, $(p + a, q + b(4t - 1))$ for $t \in \left[\frac{1}{4}, \frac{1}{2}\right]$, $(p + a(3 - 4t), q + b)$ for $t \in \left[\frac{1}{2}, \frac{3}{4}\right]$, $(p, q + b(4 - 4t))$ for $t \in \left[\frac{3}{4}, 1\right]$. Then

$$z(1) - z(0) = \int_{0}^{1} z'(t) \, dt = b \cdot \sin^2(p + a) - b \cdot \sin^2(p).$$

Thus the total vertical change of $z$-values, $z(1) - z(0)$, along the perimeter of this rectangle is

$$b \cdot (\sin^2(p + a) - \sin^2(p))$$

which is $\frac{1}{2}$ times the area. Hence we get the conclusion.
2. The bundle $U(n) \rightarrow U(n+m)/U(m) \rightarrow G_{n,m}$

To deal with the bundle

$$U(n) \rightarrow U(n+m)/U(m) \rightarrow G_{n,m},$$

we investigate the bundle

$$U(n) \times U(m) \rightarrow U(n+m) \rightarrow G_{n,m}.$$

The Lie algebra of $U(n+m)$ is $u(n+m)$, the skew-Hermitian matrices, and has the following canonical decomposition:

$$g = h + m,$$

where

$$h = u(n) + u(m) = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} : A \in u(n), B \in u(m) \right\}$$

and

$$m = \left\{ \hat{X} := \begin{bmatrix} 0 & -X^* \\ X & 0 \end{bmatrix} : X \in M_{m,n}(\mathbb{C}) \right\}.$$

Define an Hermitian inner product $h : \mathbb{C}^m \rightarrow \mathbb{C}$ by

$$h(v, w) = v^* w,$$

where $v$ and $w$ are regarded as column vectors.

**Lemma 2.1.** Let

$$X = \left( a^r_k + ib^r_k \right), Y = \left( c^r_k + id^r_k \right) \in M_{m,n}(\mathbb{C})$$

for $r = 1, \cdots, m$, and $k = 1, \cdots, n$. Suppose that for their induced $\hat{X}, \hat{Y} \in m$,

$$[[\hat{X}, \hat{Y}], \hat{X}] = \hat{Z} \in m$$

for some $Z = \left( a^r_k \right) \in M_{m,n}(\mathbb{C})$ for $r = 1, \cdots, m$, and $k = 1, \cdots, n$. Then we have

$$\alpha_k^r = \sum_{j=1}^n (a^r_j + ib^r_j)(-2h(Y_j, X_k) + h(X_j, Y_k)) + \sum_{j=1}^n (c^r_j + id^r_j) h(X_j, X_k)$$

where $X_k$ and $Y_k$ are $k$-column vectors of $X$ and $Y$ for $k = 1, \cdots, n$.

**Proof.** It is easily obtained from

$$[[\hat{X}, \hat{Y}], \hat{X}] = \hat{X}(2\hat{Y} \hat{X} - \hat{X} \hat{Y}) - \hat{Y} \hat{X} \hat{X}.$$
Theorem 2.2. Let $U(n) \times U(m) \to U(n+m) \to G_{n,m}(\mathbb{C}), n \leq m$, be the natural fibration. Assume a $2$-dimensional subspace $m' = \text{Span}_{\mathbb{R}}\{\hat{X}, \hat{Y}\}$ of $m \subset u(n+m)$ satisfies
\[
X^* X = \lambda I_n, \quad X^* Y = \mu I_n, \quad \lambda \in \mathbb{R} - \{0\}, \mu \in \mathbb{C} \tag{2-1}
\]
for $X, Y \in M_{m,n}(\mathbb{C})$. Then $m'$ gives rise to a complete totally geodesic surface $S$ in $G_{n,m}(\mathbb{C})$ if and only if either

1. $[\hat{X}, \hat{Y}] \in u(m)$ and $Y^* Y = \eta I_n$ for some $\eta \in \mathbb{R}$ in case of $\text{Im} \mu = 0$,
2. $m'$ is $J$-invariant (i.e., has a complex structure) in case of $\text{Im} \mu \neq 0$.

Proof. Assume that $m'$ gives rise to a complete totally geodesic surface $S$ in $G_{n,m}(\mathbb{C})$. If $\text{Im} \mu = 0$, then $-X^* Y + Y^* X = -X^* Y + (X^* Y)^* = -2i\text{Im} \mu I_n = O_n$, so
\[
[\hat{X}, \hat{Y}] = \begin{bmatrix} O_n & 0 \\ 0 & -XY^* + YX^* \end{bmatrix} \in u(m) \subset u(n+m).
\]
Let $M = -XY^* + YX^*$. Then
\[
[\hat{X}, \hat{Y}] = \begin{bmatrix} O_n & 0 \\ 0 & M \end{bmatrix}
\]
and $[[\hat{Y}, \hat{X}], \hat{Y}] = -\hat{M} \hat{Y} \in m'$ from the hypothesis of the condition of totally geodesic. Note that
\[
-MY = XY^* Y - YX^* Y = XY^* Y - Y\mu I_n = XY^* Y - (\text{Re} Y)Y.
\]
Thus $XY^* Y = aX + bY$ for some $a, b \in \mathbb{R}$. Then $\lambda Y^* Y = X^* (XY^* Y) = X^* (aX + bY) = (a\lambda + b\text{Re} \mu) I_n$ and so
\[
Y^* Y = \frac{a\lambda + b\text{Re} \mu}{\lambda} I_n, \quad \frac{a\lambda + b\text{Re} \mu}{\lambda} \in \mathbb{R}.
\]
Suppose that $\text{Im} \mu \neq 0$. Let $e_k \in \mathbb{C}^m, k = 1, \ldots, m$, be an elementary vector which has all components 0 except for the $k$-component with 1. Then
\[
h(X_k, Y_j) = h(X e_k, Y e_j) = e_k^*(X^* Y) e_j.
\]
Then the condition $2.1$ is equivalent to
\[
h(X_k, Y_k) = \mu, \quad h(X_k, X_k) = \lambda, \quad h(X_k, X_j) = 0, \quad h(X_k, Y_j) = 0
\]
for $k \neq j$ in $\{1, \ldots, n\}$. From $h(X_k, Y_k) = \mu$, we obtain
\[
-2h(Y_k, X_k) + h(X_k, Y_k) = -\text{Re} \mu + 3i\text{Im} \mu
\]
Thus Lemma 2.1 says that
\[
[[\hat{X}, \hat{Y}], \hat{X}] = (-\text{Re} \mu + 3i\text{Im} \mu) \hat{X} + \lambda \hat{Y} = 3\text{Im} \mu (i\hat{X}) + (-\text{Re} \mu \hat{X} + \lambda \hat{Y}).
\]
From the hypothesis of the condition of totally geodesic, \([\hat{X}, \hat{Y}], \hat{X}] = a\hat{X} + b\hat{Y}\) for some \(a, b \in \mathbb{R}\). Since \(\text{Im} \mu \neq 0\), \(i\hat{X}\) will lie in \(\text{Span}_R \{\hat{X}, \hat{Y}\} = m'\), which implies that \(m'\) will be \(J\)-invariant.

Conversely, assume the necessary part holds. If the condition (1) holds, then \([[\hat{X}, \hat{Y}], \hat{X}] = M\hat{X}\) and \([[\hat{Y}, \hat{X}], \hat{Y}] = -MY\), where \(M = -XY^* + YX^*\). It suffices to show that \([[\hat{X}, \hat{Y}], \hat{X}] \in m'\) and \([[\hat{Y}, \hat{X}], \hat{Y}] \in m'\). Since

\[
M\hat{X} = -XY^*\hat{X} + YX^*\hat{X} = -X\hat{\mu}_n + Y\lambda_n = -\text{Re}uX + \lambda Y,
\]

we get \([[\hat{X}, \hat{Y}], \hat{X}] \in m'\). We also get \([[\hat{Y}, \hat{X}], \hat{Y}] \in m'\) since

\[
-MY = XY^*Y - YX^*Y = X\eta_n - Y\mu_n = \eta X - \text{Re}uY.
\]

If the condition (2) holds, then \(m' = \text{Span}_R \{\hat{X}, i\hat{X}\}\), and

\[
[\hat{X}, i\hat{X}] = \begin{bmatrix}
-2i\lambda_n & 0 \\
0 & 2iXX^*
\end{bmatrix}.
\]

It suffices to show that \([[\hat{X}, i\hat{X}], \hat{X}] \in m'\) and \([[\hat{X}, i\hat{X}], i\hat{X}] \in m'\). Since \([\hat{X}, i\hat{X}] \in \mathfrak{u}(n + m)\), \(XX^*\) will be an element in \(\mathfrak{u}(m)\), so \(O_m\). Thus, \([\hat{X}, i\hat{X}] = -2i\lambda \begin{bmatrix}
I_n & 0 \\
0 & O_m
\end{bmatrix}\), and so \([[\hat{X}, i\hat{X}], \hat{X}] = 2\lambda i\hat{X}\) and \([[\hat{X}, i\hat{X}], i\hat{X}] = -2\lambda i\hat{X}\). Hence we get the conclusion.

**Remark 2.3.** The condition of \(X\) in Theorem 2.2 says \(X : \mathbb{C}^n \to \mathbb{C}^m\) is a conformal one-one linear map. In view of \(\hat{X} \in \mathfrak{u}(n + m) \subset \text{End}(\mathbb{C}^{n+m})\), \(\hat{X}\) sends the subspace \(\mathbb{C}^n\) to its orthogonal subspace \(\mathbb{C}^m\) conformally. And the condition of the relation between \(X\) and \(Y\) says that

\[
h_{\mathbb{C}^m}(Xv, Yw) = \mu h_{\mathbb{C}^k}(v, w) \quad \text{for } v, w \in \mathbb{C}^n,
\]

where \(h_{\mathbb{C}^k}\) is an Hermitian on \(\mathbb{C}^k\), \(k = 1, 2, \ldots\), given by

\[
h_{\mathbb{C}^k}(u_1, u_2) = u_1^*u_2 \quad \text{for } u_1, u_2 \in \mathbb{C}^k.
\]

When \(n = 1\), the condition (2.1) is satisfied automatically for any two vectors in \(\mathbb{C}^m\) by identifying \(M_{m,1}(\mathbb{C})\) with \(\mathbb{C}^m\). So we get

**Corollary 2.4.** A 2-dimensional subspace \(m'\) of \(\mathfrak{u}(m+1)\) gives rise to a complete totally geodesic submanifold of \(\mathbb{C}P^m\) if and only if either

1. \(m'\) is \(J\)-invariant (i.e., has a complex structure), or
2. \(m'\) has tangent vectors \(v\) and \(w\) such that \(\text{Im}h_{\mathbb{C}^m}(v, w) = 0\).

We return to the bundle \(U(n) \to U(n+m)/U(m) \xrightarrow{\pi} G_{n,m}\). Any subset \(A \subset G_{n,m}\) induces a bundle \(U(n) \to \pi^{-1}(A) \to A\), which is immersed in the original bundle and diffeomorphic to the pullback bundle with respect the inclusion of \(A\) into \(G_{n,m}\). In fact, in the bundle \(U(n) \times U(m) \to U(n+m) \xrightarrow{\pi} G_{n,m}\), the induced distribution in \(\pi^{-1}(A)\) from \(U(m)\) in \(U(n+m)\) is integrable, so this induces the bundle \(U(n) \to \pi^{-1}(A) \to A\).
Theorem 2.5. Assume the same condition for a complete totally geodesic surface $S$ of Theorem 2.2. Then, in the bundle $U(n) \to \pi^{-1}(S) \to S$, which is immersed in the original bundle $U(n) \to U(n+m)/U(m) \to G_{n,m}$, either

1. it is flat in case of $\text{Im} \mu = 0$, or
2. there exist a subbundle of rank 1, which is isomorphic to the Hopf bundle $S^1 \to S^3 \to S^2$ in case of $\text{Im} \mu \neq 0$.

Proof. Assume that $\text{Im} \mu = 0$. Consider the bundle $U(n) \times U(m) \to U(n+m) \to G_{n,m}$. Then $S$ induces a bundle $U(n) \times U(m) \to p^{-1}(S) \to S$. Totally geodesic condition says that the distribution induced from $\text{Span}_R \{X, Y, [X, Y]\}$ is integrable. Since $[X, Y]$ is contained in the Lie algebra $u(m)$ of $U(m)$, (1) is obtained.

Assume that $\text{Im} \mu \neq 0$. Consider the following three elements in $\mathfrak{su}(1+1)$:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}.$$ 

Since $m'$ is $J$-invariant, there is a Lie algebra monomorphism $f : \mathfrak{su}(1+1) \to u(n+m)$, given by

$$f(aA + bB + cC) = a\hat{X} + bi\hat{X} + cK,$$

where $K = \begin{bmatrix} -\lambda I_n & 0 \\ 0 & O_m \end{bmatrix} \in u(n)$. In fact,

$$[A, B] = 2C, \quad [C, A] = 2B, \quad [C, B] = -2A$$

and

$$[\hat{X}, i\hat{X}] = 2K, \quad [K, \hat{X}] = 2i\hat{X}, \quad [K, i\hat{X}] = -2\hat{X}.$$ 

Thus $f$ will induce a Lie group monomorphism $\tilde{f} : SU(1+1) \to U(n+m)$ with $\tilde{f}(S(U(1) \times U(1))) \subset U(n) \times U(m)$ since $SU(2)$ is simply connected and $S(U(1) \times U(1))$ is connected. Furthermore, it is the bundle map from

$$S(U(1) \times U(1)) \to SU(1+1) \to G_{1,1} = SU(1+1)/S(U(1) \times U(1))$$

to

$$U(n) \times U(m) \to U(n+m) \to G_{n,m}.$$ 

Since $\text{Span}_R \{\hat{X}, i\hat{X}, K\} \perp u(m)$, the linearity and the left invariance of vector fields will induce the bundle map from

$$S(U(1) \times U(1)) \to SU(1+1) \to G_{1,1} = SU(1+1)/S(U(1) \times U(1))$$

to

$$U(n) \to U(n+m)/U(m) \to G_{n,m}.$$
thorough the immersed bundle \( U(n) \to p^{-1}(S) \to S \) of \( U(n) \times U(m) \to U(n + m) \xrightarrow{\pi_2} G_{n,m} \). Then the following three different expressions of the bundle equivalences of the Hopf bundles

\[
S^1 \to S^3 \to S^2,
\]

\[
U(1) \to SU(2) \to SU(2)/U(1),
\]

and

\[
S(U(1) \times U(1)) \to SU(1 + 1) \to G_{1,1} = SU(1 + 1)/S(U(1) \times U(1))
\]

displays (2).

By combining Theorems 1.1 and 2.5, we have now

**Theorem 2.6.** Let \( U(n) \to U(n + m)/U(m) \xrightarrow{\pi} G_{n,m} \) be the natural fibration. Assume the same condition for a complete totally geodesic surface \( S \) of Theorem 2.2 and consider the bundle \( U(n) \to \pi^{-1}(S) \xrightarrow{\pi} S \). Let \( \gamma \) be a piecewise smooth, simple closed curve on \( S \). Then the holonomy displacement along \( \gamma \) is given by

\[
V(\gamma) = e^{\frac{A(\gamma)}{2}i} \quad \text{or} \quad e^{0i} \in S^1
\]

where \( A(\gamma) \) is the area of the region on the surface \( S \) surrounded by \( \gamma \), depending on whether \( S \) is a complex submanifold or not.

**Remark 2.7.** For \( n = 1 \), we have the following natural bundle \( S^1 \to S^{2m+1} \to \mathbb{C}P^m \). Let \( S \) be a complete totally geodesic surface in \( \mathbb{C}P^m \) and \( \gamma \) be a piecewise smooth, simple closed curve on \( S \). Then the holonomy displacement along \( \gamma \) is given by

\[
V(\gamma) = e^{\frac{A(\gamma)}{2}i} \quad \text{or} \quad e^{0i} \in S^1
\]

where \( A(\gamma) \) is the area of the region on the surface \( S \) surrounded by \( \gamma \), depending on whether \( S \) is a complex submanifold or not. See Corollary 2.4.

**References**

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