On the wave representation of hyperbolic, elliptic, and parabolic Eisenstein series

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Abstract

We develop a unified approach to the construction of the hyperbolic and elliptic Eisenstein series on a finite volume hyperbolic Riemann surface. Specifically, we derive expressions for the hyperbolic and elliptic Eisenstein series as integral transforms of the kernel of a wave operator. Established results in the literature relate the wave kernel to the heat kernel, which admits explicit construction from various points of view. Therefore, we obtain a sequence of integral transforms which begins with the heat kernel, obtains a Poisson and wave kernel, and then yields the hyperbolic and elliptic Eisenstein series. In the case of a non-compact finite volume hyperbolic Riemann surface, we finally show how to express the parabolic Eisenstein series in terms of the integral transform of a wave operator.

1 Introduction

1.1. Non-holomorphic Eisenstein series. Let $\Gamma \subset \text{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind which acts on the hyperbolic space $\mathbb{H}$ by fractional linear transformations, and let $M = \Gamma \backslash \mathbb{H}$ be the finite volume quotient. One can view $M$ as a finite volume hyperbolic Riemann surface, possibly with $p_1$ cusps and $e_1$ elliptic fixed points. Classically, associated to any cusp $p_j$ ($j = 1, \ldots, p_1$) of $M$, there is a non-holomorphic parabolic Eisenstein series defined for $z \in M$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$ by

$$E_{p_j}^{\text{par}}(z, s) = \sum_{\eta \in \Gamma_{p_j} \setminus \Gamma} \text{Im}(\sigma_{p_j}^{-1}\eta z)^s,$$  \hspace{1cm} (1)

where $\Gamma_{p_j} := \text{Stab}_\Gamma(p_j) = \langle \gamma_{p_j} \rangle$ is the stabilizer subgroup generated by a primitive, parabolic element $\gamma_{p_j} \in \Gamma$ and $\sigma_{p_j} \in \text{PSL}_2(\mathbb{R})$ is the scaling matrix which satisfies $\sigma_{p_j}^{-1}\gamma_{p_j}\sigma_{p_j} = \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$. The parabolic Eisenstein series is an automorphic function on $M$ and admits a meromorphic continuation to the whole complex $s$-plane with no poles on the line $\text{Re}(s) = 1/2$ (see, for example, [CdV81], [He83], [Iwa02], or [Ku73]). The parabolic Eisenstein series play a central role in the spectral theory of automorphic functions on $M$ by contributing the eigenfunctions for the continuous spectrum of the hyperbolic Laplacian on $M$, and, subsequently, in the Selberg trace formula.

In [KM79], Kudla and Millson defined and studied a form-valued, non-holomorphic Eisenstein series $E_{\gamma,\text{KM}}^{\text{hyp}}(z, s)$ associated to any simple closed geodesic of $M$, or equivalently to any primitive, hyperbolic element $\gamma \in \Gamma$. In analogy with results for scalar-valued, parabolic, non-holomorphic Eisenstein series, the following results were proved in [KM79]. First, the Eisenstein series $E_{\gamma,\text{KM}}^{\text{hyp}}(z, s)$ admits a meromorphic continuation to all $s \in \mathbb{C}$. Second, the special value at

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s = 0 of the meromorphic continuation is a harmonic form which is the Poincaré dual to the geodesic corresponding to the hyperbolic element γ ∈ Γ.

A scalar-valued, non-holomorphic, hyperbolic Eisenstein series was defined in [JKvP10], and the authors proved that the series admits a meromorphic continuation to all s ∈ ℂ. Let γ ∈ Γ be a primitive, hyperbolic element with stabilizer Γγ. Let Lγ be the geodesic in ℍ which is invariant by the action of γ on ℍ. Then, for z ∈ M and s ∈ ℂ with Re(s) > 1, the hyperbolic Eisenstein series is defined by

\[ \mathcal{E}_\gamma^{hyp}(z, s) = \sum_{\eta \in \Gamma \setminus \Gamma} \cosh(d_{hyp}(\eta z, L_\gamma))^{-s}, \]

where \( d_{hyp}(\eta z, L_\gamma) \) denotes the hyperbolic distance from the point \( \eta z \) to the arc \( L_\gamma \). In [JKvP10], it was proved that \( \mathcal{E}_\gamma^{hyp}(z, s) \) admits an \( L^2 \)-spectral expansion, which was explicitly computed, and from which the meromorphic continuation of \( \mathcal{E}_\gamma^{hyp}(z, s) \) in s was derived.

In an unpublished paper from 2004, Jorgenson and Kramer defined an elliptic Eisenstein series \( \mathcal{E}_w^{ell}(z, s) \) attached to any point \( w \) on \( M \). The problem under study by these authors was to establish a sup-norm bound for the ratio of the so-called canonical and hyperbolic one-forms on \( \Gamma \); the completion of this work has been published in [JK11]. For any point \( w \in M \), let \( \Gamma_w \) denote its stabilizer, which may be trivial. Then, for \( z \in M, z \neq w, \) and \( s \in ℂ \) with \( \text{Re}(s) > 1 \), the elliptic Eisenstein series is defined by

\[ \mathcal{E}_w^{ell}(z, s) = \sum_{\eta \in \Gamma \setminus \Gamma} \sinh(d_{hyp}(\eta z, w))^{-s}. \]

In [vP10], the author studied elliptic Eisenstein series from the point of view of automorphic forms, ultimately proving the following three main points. First, the elliptic Eisenstein series \( \mathcal{E}_w^{ell}(z, s) \) admits a meromorphic continuation to all \( s \in ℂ \). Second, the special value at \( s = 0 \) can be expressed in terms of the norm of a holomorphic modular form which vanishes only at the point \( w \). Finally, although the elliptic Eisenstein series are not in \( L^2 \), there is a series expansion which expresses \( \mathcal{E}_w^{ell}(z, s) \) in a manner similar to the spectral decomposition of an \( L^2 \)-function. This expansion of \( \mathcal{E}_w^{ell}(z, s) \) was developed from the spectral decomposition of the automorphic kernel \( K_s(z, w) \) which is defined for \( z, w \in M \) and \( s \in ℂ \) with \( \text{Re}(s) > 1 \) by

\[ K_s(z, w) = \sum_{\eta \in \Gamma} \cosh(d_{hyp}(z, \eta w))^{-s}. \]

In further studies, we refer the interested reader to the articles [Fa07], [GvP09] and [GJM08] which prove convergence results amongst the various Eisenstein series when considering a degenerating sequence of hyperbolic Riemann surfaces.

1.2. Purpose of this article. One of the main goals of the present article is to understand and establish the results of [vP10] from the point of view of integral transforms of kernel functions, an approach that could be generalized to more general settings. Specifically, we prove precise relations, or rather integral formulas, which begin with the heat kernel, proceed to a Poisson kernel and wave kernel, and end with the hyperbolic and the elliptic Eisenstein series. In the case of hyperbolic Riemann surfaces, which we study in this article, we obtain a precise connection involving many basic special functions, and we will re-prove some of the theorems mentioned in 1.1. In addition, if \( M \) is non-compact, then we define an automorphic kernel, which is constructed using the wave kernel, whose zeroth Fourier coefficient is the parabolic Eisenstein series. As a result, we are able to express the hyperbolic, elliptic, and parabolic Eisenstein series as integral transforms of the wave kernel.

This article is organized as follows. In §2 we establish notation and provide various background information needed to make the article as self-contained as possible; in §3 we recall results involving the wave kernel and define an associated wave distribution on a certain class of test functions. In §4 we express the automorphic kernel \( K_s(z, w) \) in terms of the wave distribution. We will prove that
both hyperbolic and elliptic Eisenstein series can be expressed in terms of this automorphic kernel, thus justifying our assumption that $K_s(z, w)$ is indeed the main building block for hyperbolic and elliptic Eisenstein series. The connection between $K_s(z, w)$ and hyperbolic Eisenstein series is given in §5, and the connection between the automorphic kernel and the elliptic Eisenstein series is given in §6. Finally, in section 7, we express the hyperbolic Green’s function in terms of the of the wave distribution which yields wave representations of the the parabolic Eisenstein series.

Additionally, we prove that the elliptic Eisenstein series can be realized through the use of a specific test function in the wave distribution transform. As a result, we obtain an expression for the elliptic Eisenstein series which can be readily generalized to settings beyond the study of function theory on finite volume, hyperbolic Riemann surfaces. Moreover, we derive the meromorphic continuation of the automorphic kernel, the hyperbolic Eisenstein series, and the elliptic Eisenstein series using the spectral decomposition of the heat kernel at the end of the corresponding sections.

1.3. Concluding comments. In our definition of the wave distribution transform, we use the spectral expansion of $L^2$-functions on $M$ which requires the meromorphic continuation of the parabolic Eisenstein series. As a consequence, the logical development of the results here build upon the continuation of the parabolic Eisenstein series.

Another interesting point in the study of the Eisenstein series considered here is the various forms of the Kronecker limit formula. In the subsequent article [JvPS15], we study the Kronecker limit formula for elliptic Eisenstein series established in [vP10] and [vP15]. We prove, among other results, a factorization theorem for holomorphic forms in terms of elliptic Kronecker limit functions. From this, we obtain a number of explicit evaluations of elliptic Kronecker limit functions in terms of holomorphic Eisenstein series.

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Finally, the first (J. J.) and third (L. S.) named authors wish to emphasize the influence of the work from [vP10]. The present article developed from the goal of revisiting the results from this article in order to obtain generalizations in other settings. We thank the second named author (A. v.P.) for sharing her results from this paper. Going forward, it is important to note that the study of elliptic Eisenstein series began in [vP10], and the present article establishes another point of view.

2 Background material

2.1. Basic notation. As mentioned in the introduction, we let $\Gamma \subset \text{PSL}_2(\mathbb{R})$ denote a Fuchsian group of the first kind acting by fractional linear transformations on the hyperbolic upper half-plane $\mathbb{H} := \{ z = x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}; y > 0 \}$. We let $M := \Gamma \backslash \mathbb{H}$, which is a finite volume hyperbolic Riemann surface, and denote by $p : \mathbb{H} \rightarrow M$ the natural projection. We assume that $M$ has $e_\Gamma$ elliptic fixed points and $p_\Gamma$ cusps. We identify $M$ locally with its universal cover $\mathbb{H}$.

We let $\mu_{\text{hyp}}$ denote the hyperbolic metric on $M$, which is compatible with the complex structure of $M$, and has constant negative curvature equal to minus one. The hyperbolic line element $ds_{\text{hyp}}^2$, resp. the hyperbolic Laplacian $\Delta_{\text{hyp}}$, are given as

$$ds_{\text{hyp}}^2 := \frac{dx^2 + dy^2}{y^2}, \quad \text{resp.} \quad \Delta_{\text{hyp}} := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

By $d_{\text{hyp}}(z, w)$ we denote the hyperbolic distance from $z \in \mathbb{H}$ to $w \in \mathbb{H}$, which satisfies the relation

$$\cosh(d_{\text{hyp}}(z, w)) = 1 + 2u(z, w),$$  \quad (2)
where
\[ u(z, w) = \frac{|z-w|^2}{4\text{Im}(z)\text{Im}(w)}. \] (3)

Under the change of coordinates
\[ x := e^\rho \cos(\theta), \quad y := e^\rho \sin(\theta), \]
the hyperbolic line element, resp. the hyperbolic Laplacian are rewritten as
\[ ds^2_{\text{hyp}} = d\rho^2 + d\theta^2, \quad \text{resp.} \quad \Delta_{\text{hyp}} = -\sin^2(\theta) \left( \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \theta^2} \right). \]

For \( z = x + iy \in \mathbb{H} \), we define the hyperbolic polar coordinates \( \varrho = \varrho(z), \vartheta = \vartheta(z) \) centered at \( i \in \mathbb{H} \) by
\[ \varrho(z) := d_{\text{hyp}}(i, z), \quad \vartheta(z) := \angle(\mathcal{L}, T_z), \]
where \( \mathcal{L} := \{ z \in \mathbb{H} \mid x = \text{Re}(z) = 0 \} \) denotes the positive \( y \)-axis and \( T_z \) is the euclidean tangent at the unique geodesic passing through \( i \) and \( z \) at the point \( i \). In terms of the hyperbolic polar coordinates, the hyperbolic line element, resp. the hyperbolic Laplacian take the form
\[ ds^2_{\text{hyp}} = \sinh^2(\varrho)d\vartheta^2 + d\varrho^2, \quad \text{resp.} \quad \Delta_{\text{hyp}} = -\frac{\partial^2}{\partial \varrho^2} - \frac{1}{\tanh(\varrho)} \frac{\partial}{\partial \varrho} - \frac{1}{\sinh^2(\varrho)} \frac{\partial^2}{\partial \vartheta^2}. \]

### 2.2. Hyperbolic and elliptic Eisenstein series.

In this subsection, we recall the Eisenstein series, which are associated to any primitive, hyperbolic or elliptic element of \( \Gamma \), or equivalently to any closed geodesic or elliptic fixed point of \( M \).

First, let \( L_\gamma \) be the closed geodesic on \( M \) in the homotopy class determined by a hyperbolic element \( \gamma \in \Gamma \) with associated stabilizer subgroup \( \Gamma_\gamma = \langle \gamma \rangle \). There is a scaling-matrix \( \sigma_\gamma \in \text{PSL}_2(\mathbb{R}) \) such that
\[ \sigma^{-1}_\gamma \gamma \sigma_\gamma = \begin{pmatrix} e^{\ell_\gamma/2} & 0 \\ 0 & e^{-\ell_\gamma/2} \end{pmatrix}, \]
where \( \ell_\gamma \) denotes the hyperbolic length of \( L_\gamma \). We note that \( L_\gamma = p(\mathcal{L}_\gamma) \), where \( \mathcal{L}_\gamma := \sigma_\gamma \mathcal{L} \) with \( \mathcal{L} \) denoting the positive \( y \)-axis. Using the coordinates \( \rho = \rho(z) \) and \( \theta = \theta(z) \) introduced in subsection 2.1, the hyperbolic Eisenstein series associated to the closed geodesic \( L_\gamma \) is defined as a function of \( z \in M \) and \( s \in \mathbb{C} \) by the series
\[ E_{\gamma}^{\text{hyp}}(z, s) = \sum_{\eta \in \Gamma_\gamma \setminus \Gamma} \sin(\theta(\sigma^{-1}_\gamma \eta z))^s. \]

From elementary hyperbolic geometry, one can show that the hyperbolic distance \( d_{\text{hyp}}(z, \mathcal{L}) \) from \( z \) to the geodesic line \( \mathcal{L} \) satisfies the formula
\[ \sin(\theta(z)) \cosh(d_{\text{hyp}}(z, \mathcal{L})) = 1. \]

Therefore, we can re-write the series which defines the hyperbolic Eisenstein series by
\[ E_{\gamma}^{\text{hyp}}(z, s) = \sum_{\eta \in \Gamma_\gamma \setminus \Gamma} \cosh(d_{\text{hyp}}(\eta z, \mathcal{L}_\gamma))^{-s}. \] (4)

Referring to [Fa07], [GJM08], or [Ri04], e.g., where detailed proofs are provided, we recall that the series (4) converges absolutely and locally uniformly for any \( z \in \mathbb{H} \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \),
and that the series is invariant with respect to \( \Gamma \). A straightforward computation shows that the hyperbolic Eisenstein series satisfies the differential equation
\[
(\Delta_{\text{hyp}} - s(1-s)) \mathcal{E}_{\text{hyp}}^j(z, s) = s^2 \mathcal{E}_{\text{hyp}}^j(z, s + 2).
\]

Now, let \( w \in M \) be an arbitrary point with scaling matrix \( \sigma_w \in \text{PSL}_2(\mathbb{R}) \) and stabilizer subgroup \( \Gamma_w = \langle \gamma_w \rangle \). The group \( \Gamma_w \) is trivial unless \( w = e_j \), where \( e_j \) (\( j = 1, \ldots, e_1 \)) is an elliptic fixed point of \( M \), and, hence, \( \gamma_w \) is a primitive, elliptic element of \( \Gamma \). Note that
\[
\sigma_w^{-1} \gamma_w \sigma_w = \left( \begin{array}{cc} \cos(\pi/\text{ord}(w)) & \sin(\pi/\text{ord}(w)) \\ -\sin(\pi/\text{ord}(w)) & \cos(\pi/\text{ord}(w)) \end{array} \right),
\]
where \( \text{ord}(w) = \text{ord}(\Gamma_w) \) denotes the order of \( w \). Using the hyperbolic polar coordinates \( \varrho = \varrho(z) \) and \( \vartheta = \vartheta(z) \) introduced in subsection 2.1, the elliptic Eisenstein series \( \mathcal{E}_{w}^{\text{ell}}(z, s) \) associated to the point \( w \) is defined by
\[
\mathcal{E}_{w}^{\text{ell}}(z, s) = \sum_{\eta \in \Gamma_w \setminus \Gamma} \sinh((\sigma_w^{-1}\eta z))^{-s}.
\]

Referring to [vP10], where detailed proofs are provided, we recall that the series (5) converges absolutely and locally uniformly for \( z \in M \) with \( z \neq w \) and for \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), and that the series is invariant with respect to \( \Gamma \). A straightforward computation shows that the elliptic Eisenstein series satisfies the differential equation
\[
(\Delta_{\text{hyp}} - s(1-s)) \mathcal{E}_{w}^{\text{ell}}(z, s) = -s^2 \mathcal{E}_{w}^{\text{ell}}(z, s + 2).
\]

### 2.3. Spectral expansions.

Let \( 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \) denote the discrete eigenvalues of the hyperbolic Laplacian \( \Delta_{\text{hyp}} \) acting on smooth functions on \( M \); we write \( \lambda_j = 1/4 + t_j^2 = s_j(1-s_j) \), i.e., \( s_j = 1/2 + it_j \) with \( t_j \geq 0 \) or \( t_j \in (0, i/2) \). When considering sums over the set of discrete eigenvalues, we will index the sum by \( \lambda_j \geq 0 \), which allows for the possibility that the set is finite, as predicted by the Phillips-Sarnak philosophy, see [PS85]. Also, if \( M \) is compact, then \( p_\Gamma = 0 \), so sums over the space of parabolic Eisenstein series do not occur.

Under certain hypotheses on a function \( f \) on \( M \), which are defined carefully in numerous references such as [Iwa02], [He83], or [Ku73], there is a spectral expansion of \( f \) in terms of the eigenfunctions \( \{\psi_j(z)\}_{\lambda_j \geq 0} \) associated to the discrete eigenvalues \( \lambda_j \) of the hyperbolic Laplacian \( \Delta_{\text{hyp}} \) and the parabolic Eisenstein series \( \mathcal{E}_{p_k}^{\text{par}} \) \((k = 1, \ldots, p_\Gamma)\) associated to the cusps of \( M \); without loss of generality, we assume that all eigenfunctions of the Laplacian are real-valued. More precisely, a function \( f \) on \( M \), under certain assumptions, admits the expansion
\[
f(z) = \sum_{\lambda_j \geq 0} \langle f, \psi_j \rangle \psi_j(z) + \frac{1}{4\pi} \sum_{k=1}^{p_\Gamma} \int_{-\infty}^{\infty} \langle f, \mathcal{E}_{p_k}^{\text{par}}(\cdot, 1/2 + ir) \rangle \mathcal{E}_{p_k}^{\text{par}}(z, 1/2 + ir) \, dr.
\]

In particular, under certain hypotheses on the point-pair invariant function \( k(z, w) \) on \( M \times M \), the automorphic kernel
\[
K(z, w) = \sum_{\eta \in \Gamma} k(z, \eta w)
\]
adopts the spectral expansion (6) as a function of \( z \). More precisely, let us write \( k(z, w) = k(u(z, w)) = k(u) \) as a function of \( u \) with \( u(z, w) \) given by (3). Suppose that the Selberg/ Harish-Chandra transform \( h(r) \) of \( k(u) \) exists and satisfies the following conditions:

(S1) \( h(r) \) is an even function;

(S2) \( h(r) \) is holomorphic in the strip \( |\text{Im}(r)| \leq \frac{1}{2} + \epsilon \) for some \( \epsilon > 0 \).
(S3) \( h(r) \ll (1 + |r|)^{-2-\delta} \) for some fixed \( \delta > 0 \), as \( r \to \infty \) in the domain of condition (S2).

Then, the automorphic kernel (7) admits a spectral expansion of the form

\[
K(z, w) = \sum_{\lambda_j \geq 0} h(t_j) \psi_j(z) \psi_j(w) + \frac{1}{4\pi} \sum_{k=1}^{P_F} \int_{-\infty}^{\infty} h(r) \mathcal{E}_{pk}^{par}(z, 1/2 + ir) \mathcal{E}_{pk}^{par}(w, 1/2 + ir) dr,
\]

which converges absolutely and uniformly on compacta; here, we have \( h(t_j) \psi_j(w) = \langle K(\cdot, w), \psi_j \rangle \) and \( h(r) \mathcal{E}_{pk}^{par}(w, 1/2 + ir) = \langle K(\cdot, w), \mathcal{E}_{pk}^{par}(\cdot, 1/2 + ir) \rangle \) (see, e.g., [Iwa02], Theorem 7.4.). We recall that the Selberg/Harish-Chandra transform \( h(r) \) of \( k(u) \) can be computed in the following three steps:

\[
Q(v) = \int_{v}^{\infty} \frac{k(u) du}{\sqrt{u-v}}, \quad g(u) = 2Q \left( \sinh \left( \frac{u}{2} \right) \right), \quad h(r) = \int_{-\infty}^{\infty} g(u) e^{iru} du.
\]

If \( h(r) \) satisfies (S1), (S2), and (S3), the Selberg/Harish-Chandra can be inverted as follows:

\[
g(u) = \frac{1}{\pi} \int_{0}^{\infty} h(r) \cos(ur) dr, \quad Q(v) = \frac{1}{2} g(2 \sin^{-1} (\sqrt{v})), \quad k(u) = -\frac{1}{\pi} \int_{u}^{\infty} \frac{dQ(v)}{\sqrt{v-u}}.
\]

For later purpose, we note that, for any function \( h : \mathbb{R} \to \mathbb{C} \) satisfying (S1) and (S3), the series, resp. the integral

\[
\sum_{\lambda_j \geq 1/4} h(t_j) \psi_j(z) \psi_j(w), \quad \text{resp.} \quad \int_{-\infty}^{\infty} h(r) \mathcal{E}_{pk}^{par}(z, 1/2 + ir) \mathcal{E}_{pk}^{par}(w, 1/2 + ir) dr
\]

converges absolutely and uniformly on compacta (see, e.g., [He83], formula (4.1) on p. 303 with \( \chi \) trivial). Hence, if the function \( h : \mathbb{R} \to \mathbb{C} \) satisfies condition (S1), the following condition

(S2') \( h(r) \) is a well-defined and an even function for \( r \in \mathbb{R} \cup [-i/2, i/2] \),

together with condition (S3) in the domain of condition (S2'), that is as \( r \) tends to \( \pm \infty \), then the series and integrals on the right-hand side of (8) are well-defined and converge absolutely and uniformly on compacta. However, these conditions on \( h \) do not ensure that the right-hand side of (8) represents the spectral expansion of some automorphic kernel. Note that we will refer to these conditions simply by writing (S1), (S2'), (S3).

2.4. The heat kernel and the translated Poisson kernel on \( M \). The hyperbolic heat kernel \( K_{\mathbb{H}}(z, w; t) \) for \( z, w \in \mathbb{H} \) and \( t \in \mathbb{R}_{>0} \) is given by the formula

\[
K_{\mathbb{H}}(z, w; t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} r \tanh(\pi r) e^{-(r^2+1/4)t} P_{\frac{1}{2}+ir}(\cosh(d_{\text{hyp}}(z, w))) dr,
\]

where \( P_{\nu}(\cdot) \) denotes the Legendre function of the first kind; see, for example, page 246 of [Ch84]. The heat kernel is a fundamental solution associated to the differential operator \( \Delta_{\text{hyp}} - \partial_t \).

The translated by 1/4-Poisson kernel \( P_{\mathbb{H}, \frac{1}{4}}(z, w; u) \) for \( z, w \in \mathbb{H} \) and \( u \in \mathbb{C} \) with \( \text{Re}(u) > 0 \) is given by the \( G \)-transform of the integral kernel \( K_{\mathbb{H}}(z, w; t) e^{t/4} \), namely

\[
P_{\mathbb{H}, \frac{1}{4}}(z, w; u) = \frac{u}{\sqrt{4\pi}} \int_{0}^{\infty} K_{\mathbb{H}}(z, w; t) e^{t/4} e^{-u^2/4t} t^{-1/2} dt.
\]

The 1/4-Poisson kernel \( P_{\mathbb{H}, \frac{1}{4}}(z, w; u) \) is a fundamental solution associated to the differential operator \( \Delta_{\text{hyp}} - 1/4 - \partial_u^2 \). By Theorem 5.1 and Remark 5.3 of [JLa03], the 1/4-Poisson kernel
$P_{3,4}(z, w; u)$ admits a branched meromorphic continuation to all $u \in \mathbb{C}$ with singularities at $u = \pm i d_{\text{hyp}}(z, w)$.

The hyperbolic heat kernel $K_M(z, w; t)$ on $M$ is a function of $z, w \in M$ and $t \in \mathbb{R}_{>0}$, and can be obtained by averaging over the group $\Gamma$, namely

$$K_M(z, w; t) = \sum_{\eta \in \Gamma} K_{\eta}(z, \eta w; t).$$

The spectral expansion (6) of the heat kernel $K_M(z, w; t)$ is given by

$$K_M(z, w; t) = \sum_{\lambda_j \geq 0} e^{-\lambda_j t} \psi_j(z) \psi_j(w) + \frac{1}{4\pi} \sum_{k=1}^{p_f} \int_{-\infty}^{\infty} e^{-(r^2 + 1/4)t} \mathcal{E}_{pk}^{\text{par}}(z, 1/2 + ir)\mathcal{E}_{pk}^{\text{par}}(w, 1/2 + ir) dr.$$  

(9)

For $Z \in \mathbb{C}$ with $\text{Re}(Z) \geq 0$, the translated by $-Z$-Poisson kernel $P_{M, -Z}(z, w; u)$ for $z, w \in M$ and $u \in \mathbb{C}$ with $\text{Re}(u) \geq 0$ is defined by

$$P_{M, -Z}(z, w; u) = \frac{u}{\sqrt{4\pi}} \int_{0}^{\infty} K_M(z, w; t)e^{-Zt}e^{-u^2/4t}t^{-1/2} dt.$$  

The function $K_M(z, w; t)e^{-Zt}$ is a fundamental solution associated to the differential operator $\Delta_{\text{hyp}} + Z - \partial_t$. Analogously, the $-Z$-Poisson kernel $P_{M, -Z}(z, w; u)$ is a fundamental solution associated to the differential operator $\Delta_{\text{hyp}} + Z - \partial_u^2$.

As before, we write $\lambda_j = 1/4 + t_j^2$, and divide the sum in the spectral expansion (9) of the heat kernel on $M$ into the finite sum over $\lambda_j \leq 1/4$, so then $t_j \in (0, i/2)$, and the sum over $\lambda_j \geq 1/4$, so then $t_j \in \mathbb{R}_{\geq 0}$.

For $\text{Re}(u) > 0$ and $\text{Re}(u^2) > 0$, using formula (see [GR07], formula 3.471.9)

$$\frac{u}{\sqrt{4\pi}} \int_{0}^{\infty} e^{-(\lambda_j + Z)t} e^{-u^2/4t} t^{-1/2} \frac{dt}{t} = e^{-u \sqrt{\lambda_j + Z}},$$

where we take the principal branch of the square root, we get, for $Z \in \mathbb{C}$ with $\text{Re}(Z) \geq 0$, the spectral expansion

$$P_{M, -Z}(z, w; u) = \sum_{\lambda_j < 1/4} e^{-u \sqrt{\lambda_j + Z}} \psi_j(z) \psi_j(w) + \sum_{\lambda_j \geq 1/4} e^{-u \sqrt{\lambda_j + Z}} \psi_j(z) \psi_j(w)$$

$$+ \frac{1}{4\pi} \sum_{k=1}^{p_f} \int_{-\infty}^{\infty} e^{-u \sqrt{r^2 + 1/4 + Z}} \mathcal{E}_{pk}^{\text{par}}(z, 1/2 + ir)\mathcal{E}_{pk}^{\text{par}}(w, 1/2 + ir) dr.$$  

By Theorem 5.2 and Remark 5.3 of [JLa03], the $-Z$-Poisson kernel $P_{M, -Z}(z, w; u)$ admits an analytic continuation to $Z = -1/4$. Following the proof of Theorem 5.2 from [JLa03], we have that the continuation of $P_{M, -Z}(z, w; u)$ to $Z = -1/4$ with $\text{Re}(u) > 0$ and $\text{Re}(u^2) > 0$ is given by

$$P_{M, -Z}(z, w; u) = \sum_{\lambda_j < 1/4} e^{-u \sqrt{\lambda_j - 1/4}} \psi_j(z) \psi_j(w) + \sum_{\lambda_j \geq 1/4} e^{-ut_j} \psi_j(z) \psi_j(w)$$

$$+ \frac{1}{4\pi} \sum_{k=1}^{p_f} \int_{-\infty}^{u |r|} e^{-u |r|} \mathcal{E}_{pk}^{\text{par}}(z, 1/2 + ir)\mathcal{E}_{pk}^{\text{par}}(w, 1/2 + ir) dr,$$  

(10)

where $\sqrt{\lambda_j - 1/4} = t_j \in (0, i/2]$ is taken to be the branch of the square root obtained by analytic continuation through the upper half-plane. Since the function $h(r) = \exp(-u |r|)$ satisfies the
conditions (S1), (S2'), (S3), the series and the integral on the right hand side of (10) are convergent (see subsection 2.3).

By Theorem 5.2 of [JLa03], the $1/4$-Poisson kernel $P_{M,4}(z, w; u)$ admits a branched meromorphic continuation to all $u \in \mathbb{C}$ with branched singularities at all points of the form $u = \pm ip$ whenever $\rho$ is the length of a geodesic path from $z$ to $w$, and possibly at $u = 0$. This result is proved in greater generality using the language of pseudo-differential operators in [DG75].

3 The wave distribution

Choosing a branch of the meromorphic continuation of the $1/4$-Poisson kernel $P_{M,4}(z, w; u)$ to all $u \in \mathbb{C}$, the functions $P_{M,4}(z, w; \pm iu)$ represent a fundamental solution of the translated wave equation $\Delta_{\text{hyp}} - 1/4 + \partial_u^2$. For $z, w \in M$, $z \neq w$, and $u \in \mathbb{R}$, we define the translated by $1/4$ wave kernel on $M$ by

$$W_{M,4}(z, w; u) := P_{M,4}(z, w; iu) + P_{M,4}(z, w; -iu).$$

Because of convergence issues, the meromorphic continuation may not be obtained by simply replacing $u$ by $-iu$ (or $iu$) in the formula (10) for the Poisson kernel on $M$. Therefore, we will introduce the wave distribution $W_{M,4}(z, w)$ using the exact expression (10) for the analytic continuation of $P_{M,4}(z, w; u)$. Loosely speaking, we may say that the wave distribution is defined in such a way that in (10), the function $\exp(-ur)$ is replaced by the distribution associated to $\cos(ur)$, or equivalently, by the distribution associated to $\cos(ur)$ when restricting our attention to even functions of $r$.

Let $\mathbb{R}^+$ denote the set of non-negative real numbers. By $C^\infty(\mathbb{R}^+)$ we mean the class of all infinitely differentiable functions on $\mathbb{R}^+$ and by $C_0^\infty(\mathbb{R}^+) \subset C^\infty(\mathbb{R}^+)$ we denote the subclass of functions with compact support on $\mathbb{R}^+$. As usual, $g^{(j)}$ denotes the $j$-th derivative of the function $g$ ($j \geq 1$) and the Schwartz space of functions are those functions which are infinitely differentiable and rapidly decreasing.

**Definition 1.** For $a \in \mathbb{R}^+$, let $S'(\mathbb{R}^+, a)$ denote the subspace of functions $g$ in the Schwartz space on $\mathbb{R}^+$ with $g'(0) = 0$ and such that $|g(u)| \exp(ua)$ is dominated by an integrable function $G(u)$ on $\mathbb{R}^+$.

**Definition 2.** Let $a \in \mathbb{R}^+$ and $g \in S'(\mathbb{R}^+, a)$. For $r \in \mathbb{C}$, we formally define the function

$$H(r, g) := 2 \int_0^\infty \cos(ur)g(u)du.$$  \hspace{1cm} (11)

**Lemma 3.**

(i) Let $g \in S'(\mathbb{R}^+, 1/2)$ be such that the first three derivatives of $g$ are integrable and have a limit as $u \to \infty$. Then, the function $H(r, g)$ is well-defined and satisfies the conditions (S1), (S2'), and (S3) of subsection 2.3 with $\delta = 1$.

(ii) Let $\eta > 0$ and let $g \in S'(\mathbb{R}^+, 1/2 + \eta)$ be such that $g^{(j)}(u) \exp(u(1/2 + \eta))$ is bounded by some integrable function on $\mathbb{R}^+$ for $j = 1, 2, 3$. Then, the function $H(r, g)$ satisfies the conditions (S1), (S2), and (S3) of subsection 2.3 for any $0 < \epsilon < \eta$ and with $\delta = 1$.

**Proof.** Part (i): Trivially, for $u \in \mathbb{R}^+$ and $r \in \mathbb{C}$ with $|\text{Im}(r)| \leq 1/2$, the function $\cos(ur)$ is dominated by $\exp(u/2)$. Since $g \in S'(\mathbb{R}^+, 1/2)$ by assumption, the function $|g(u)| \exp(u/2)$ is dominated by an integrable function $G(u)$ on $\mathbb{R}^+$. Therefore, the integral (11) defining $H(r, g)$ exists and, hence, the function $H(r, g)$ is well-defined for $r \in \mathbb{C}$ with $|\text{Im}(r)| \leq 1/2$ and represents an even function of $r$. Further, when $r$ is real, integrating by parts three times, using the fact that $g'(0) = 0$ and that first three derivatives of $g$ are integrable and have a limit as $u \to \infty$, we get that $H(r, g)$ satisfies the asserted bound of condition (S3).
Part (i): For \( u \in \mathbb{R}^+ \), the function \( g(u) \cos(u r) \) is uniformly bounded in the strip \( |\text{Im}(r)| \leq 1/2 + \epsilon \) by \( G(u) \exp(-\eta - \epsilon u) \) for some integrable function \( G(u) \) on \( \mathbb{R}^+ \). Therefore, the integral (11) defining \( H(r, g) \) converges absolutely and uniformly in \( r \) on every compact subset of the strip \( |\text{Im}(r)| \leq 1/2 + \epsilon \), thus representing a holomorphic function that is obviously even. This proves that the function \( H(r, g) \) satisfies conditions (S1) and (S2). In order to prove (S3), we apply integration by parts three times, now using the fact that \( g'/(0) = 0 \) and that \( \cos(u r)g^{(j)}(u) \) and \( \sin(u r)g^{(j)}(u) \) are uniformly bounded in \( r \) by \( G_j(u) \exp(-\eta - \epsilon u) \) for some integrable functions \( G_j \) on \( \mathbb{R}^+ \) (\( j = 1, 2, 3 \)).

\[ \square \]

**Definition 4.** Let \( z, w \in M \). For \( g \in C^\infty_0(\mathbb{R}^+) \), the wave distribution \( \mathcal{W}_{M, \frac{1}{4}}(z, w)(g) \) applied to the function \( g \) is defined by

\[
\mathcal{W}_{M, \frac{1}{4}}(z, w)(g) = \sum_{\lambda_j \geq 0} H(t_j, g) \psi_j(z) \psi_j(w) + \frac{1}{4\pi} \sum_{k=1}^{p_1} \int \left[ H(r, g) \mathcal{E}^{\text{par}}_{p_k}(z, 1/2 + ir) \mathcal{E}^{\text{par}}_{p_k}(w, 1/2 + ir) dr \right],
\]

where \( \sqrt{\lambda_j - 1/4} = t_j \) (in the case when \( \lambda_j < 1/4 \) we take \( t_j \in (0, i/2] \)) and the coefficient \( H(\cdot, g) \) is given by (11).

**Proposition 5.** (i) Let \( z, w \in M \). Let \( g \in S'(\mathbb{R}^+, 1/2) \) be such that the first three derivatives of \( g \) are integrable and have a limit as \( u \to \infty \). Then, the wave distribution \( \mathcal{W}_{M, \frac{1}{4}}(z, w)(g) \) is well-defined, meaning the integral and series in (12) are convergent.

(ii) Let \( z, w \in M \). Let \( \eta > 0 \) and let \( g \in S'(\mathbb{R}^+, 1/2 + \eta) \) be such that \( g^{(j)}(u) \exp((1/2 + \eta) u) \) is bounded by some integrable function on \( \mathbb{R}^+ \) for \( j = 1, 2, 3 \). Then, the wave distribution represents the spectral expansion of an automorphic kernel, namely we have

\[
\mathcal{W}_{M, \frac{1}{4}}(z, w)(g) = \sum_{\eta \in \Gamma} k(z, \eta w),
\]

where \( k(z, w) = k(u(z, w)) = k(u) \) is the inverse of the Selberg/Harish-Chandra transform of \( H(r, g) \). Moreover, the series in (13) converges absolutely and uniformly on compacta.

**Proof.** Part (i): By part (i) of Lemma 3, \( H(r, g) \) is a well-defined and even function for \( r \in [-i/2, i/2] \). Hence, the finite sum

\[
\sum_{\lambda_j < 1/4} H(t_j, g) \psi_j(z) \psi_j(w)
\]

is well-defined. Further, for \( r \in \mathbb{R} \), the function \( H(r, g) \) satisfies property (S3) with \( \delta = 1 \). As recalled in subsection 2.3, this ensures that the series and the integral on right hand side of (12)

\[
\sum_{\lambda_j \geq 1/4} H(t_j, g) \psi_j(z) \psi_j(w), \quad \int_{-\infty}^{\infty} H(r, g) \mathcal{E}^{\text{par}}_{p_k}(z, 1/2 + ir) \mathcal{E}^{\text{par}}_{p_k}(w, 1/2 + ir) dr
\]

converge absolutely, which completes the proof of part (i).

Part (ii): By part (ii) of Lemma 3, \( H(r, g) \) satisfies the (S1), (S2), and (S3). As recalled in subsection 2.3, this ensures that the automorphic kernel

\[
\sum_{\eta \in \Gamma} k(z, \eta w)
\]

admits a spectral expansion of the form (8), which converges absolutely and uniformly on compacta and which equals the right hand side of (12), since \( h(r) = H(r, g) \) is the Selberg/Harish-Chandra transform of \( k(u) = k(u(z, w)) = k(z, w) \). This completes the proof. \[ \square \]
Proposition 6. Let \( z, w \in M \) with \( z \neq w \). Then, there exists a continuous, real-valued function \( F(z, w; u) \) of \( u \in \mathbb{R}^+ \) with the following properties:

(i) \( F(z, w; u) = - \sum_{\lambda_j < \frac{1}{4}} e^{u \sqrt{1/4 - \lambda_j}} \left( \sqrt{1/4 - \lambda_j} \right)^{-3} \psi_j(z) \psi_j(w) + O(u^3) \) as \( u \to \infty \).

(ii) \( F^{(j)}(z, w; u) = O(u^{3-j}) \) as \( u \to 0 \) \( (j = 0, 1, 2) \).

(iii) For any \( g \in S'((\mathbb{R}^+), 1/2) \) such that \( g^{(j)}(u) \exp(u/2) \) has a limit as \( u \to \infty \) and is bounded by some integrable function on \( \mathbb{R}^+ \) for \( j = 0, 1, 2, 3 \), we have

\[
W_{M, \frac{1}{4}}(z, w)(g) = \int_0^\infty F(z, w; u) g^{(3)}(u) du. \tag{14}
\]

Proof. Let \( z, w \in M \) with \( z \neq w \). For \( \zeta \in \mathbb{C} \) with \( \text{Re}(\zeta) \geq 0 \), we define

\[
\tilde{F}_{M, \frac{1}{4}}(z, w; \zeta) = \sum_{\lambda_j < \frac{1}{4}} f(t_j, \zeta) \psi_j(z) \psi_j(w) + \sum_{\lambda_j \geq \frac{1}{4}} f(t_j, \zeta) \psi_j(z) \psi_j(w)
+ \frac{1}{4\pi} \sum_{k=1}^{p_4} \int_{1-\infty}^\infty f(|r|, \zeta) \mathcal{C}_{p_k}(z, 1/2 + ir) \mathcal{C}_{p_k}(w, 1/2 + ir) dr, \tag{15}
\]

where \( \sqrt{\lambda_j - 1/4} = t_j \) (in the case when \( \lambda_j < 1/4 \) we take \( t_j \in (0, i/2] \)) and where we have set

\[
f(r, \zeta) = \frac{\exp(-\zeta r) - 1 + \zeta \sin(r) - (\zeta^2/2) \sin(r)^2}{(-r)^3}
\]

for \( r \neq 0 \) in the strip \( |\text{Im}(r)| \leq 1/2, \) and \( f(0, \zeta) = (\zeta^3 + \zeta)/6, \) by continuation. Observe that, for \( \text{Re}(\zeta) \geq 0 \) and \( r \in \mathbb{R}^+ \), we have

\[
f(r, \zeta) = O(\zeta) \text{ as } r \to 0 \quad \text{and} \quad f(r, \zeta) = O(\zeta^{-3}) \text{ as } r \to +\infty.
\]

Therefore, the spectral series (15) converges absolutely and uniformly, provided \( \text{Re}(\zeta) \geq 0 \), by the statements in subsection 2.3.

Similarly, the spectral series obtained by replacing \( f(r, \zeta) \) in (15) with any of its first three derivatives (with respect to the variable \( \zeta \)) is absolutely and uniformly convergent, provided \( \text{Re}(\zeta) > 0 \). Therefore, term by term differentiation is valid, and for \( \text{Re}(\zeta) > 0 \) one has, using \( d^3f(r, \zeta)/d\zeta^3 = \exp(-\zeta r) \) and by comparing with (10), the identity

\[
\frac{d^3}{d\zeta^3} \tilde{F}_{M, \frac{1}{4}}(z, w; \zeta) = P_{M, \frac{1}{4}}(z, w; \zeta). \tag{16}
\]

Now, let \( P_{M, \frac{1}{4}}^{(0)}(z, w; \zeta) = P_{M, \frac{1}{4}}(z, w; \zeta) \) and, for \( k \in \mathbb{N}, k \geq 1 \), define

\[
P_{M, \frac{1}{4}}^{(k)}(z, w; \zeta) = \int_0^\zeta P_{M, \frac{1}{4}}^{(k-1)}(z, w; \xi) d\xi,
\]

where the integral is taken along the ray \( [0, \zeta) \) inside the half plane \( \text{Re}(\zeta) > 0 \). With this notation, we have shown that

\[
\tilde{F}_{M, \frac{1}{4}}(z, w; \zeta) + q(z, w; \zeta) = P_{M, \frac{1}{4}}^{(3)}(z, w; \zeta), \tag{17}
\]

where \( q(z, w; \zeta) \) is a degree two polynomial in \( \zeta \) with coefficients which depend on \( z \) and \( w \). For \( z \neq w \), the function \( P_{M, \frac{1}{4}}^{(0)}(z, w; \zeta) \) has a limit as \( \zeta \) approaches zero; therefore, we get the bound

\[
P_{M, \frac{1}{4}}^{(k)}(z, w; \zeta) = O(\zeta^k) \text{ as } \zeta \to 0. \tag{18}
\]
With all this, we define, for $z, w \in M$ with $z \neq w$ and $u \in \mathbb{R}^+$, the function

$$F(z, w; u) = \frac{1}{i} \left( (\overline{F}_{M, \frac{1}{4}}(z, w; iu) + q(z, w; iu)) - (\overline{F}_{M, \frac{1}{4}}(z, w; -iu) + q(z, w; -iu)) \right).$$

The function $F(z, w; u)$ can be expressed using the spectral expansion (15), from which property (i) is proved, employing the equality

$$\frac{1}{i} (f(r, iu) - f(r, -iu)) = \frac{2}{r^3} (\sin(ur) - u\sin(r))$$

for $r \neq 0$ in the strip $|\text{Im}(r)| \leq 1/2$, and using that $q(z, w; iu) - q(z, w; -iu)$ is a degree one polynomial in $u$ with constant coefficient equal to zero. Assertion (ii) is proved directly from (17) using the bound (18).

Finally, one proves (iii) as follows. First, from integration by parts, we have that

$$\int_{0}^{\infty} (q(z, w; iu) - q(z, w; -iu)) g^{(3)}(u) du = 0.$$

Second, the spectral expansion (15) converges absolutely and uniformly for $\text{Re}(\zeta) = 0$, and the asymptotic expansion asserted in (i) holds. Therefore, we get

$$\frac{1}{i} \int_{0}^{\infty} \left( \overline{F}_{M, \frac{1}{4}}(z, w; iu) - \overline{F}_{M, \frac{1}{4}}(z, w; -iu) \right) g^{(3)}(u) du =$$

$$\sum_{\lambda_j \geq 0} G(t_j, g) \psi_j(z) \overline{\psi}_j(w) + \frac{1}{4\pi} \sum_{k=1}^{\infty} \int G(|r|, g) \mathcal{E}_{\text{par}}^{(z, 1/2 + ir)}(w, 1/2 + ir) dr,$$

where

$$G(r, g) = \frac{1}{i} \int_{0}^{\infty} (f(r, iu) - f(r, -iu)) g^{(3)}(u) du.$$

Integrating by parts three times, we finally derive $G(r, g) = H(r, g)$, which completes the proof.

4 The wave representation of an automorphic kernel

In this section, we study the following well-known automorphic kernel and we express it in terms of the wave distribution.

For $z, w \in M$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, we formally define the automorphic kernel $K_s(z, w)$ by the series

$$K_s(z, w) = \sum_{\eta \in \Gamma} \cosh(d_{\text{hyp}}(z, \eta w))^{-s}. \quad (19)$$

Elementary considerations show that the series which defines $K_s(z, w)$ converges absolutely for all $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, uniformly when $z$ and $w$ are restricted to any compact subset of $M$.

Furthermore, $K_s(z, w)$ is bounded on $M$ as a function of $z$ and, hence, belongs to $L^2(M)$ and admits a spectral expansion (see, e.g., [VP10], Proposition 5.1.1, for an explicit computation). In order to express the automorphic kernel in terms of the action of the wave distribution, we will outline a different proof of the spectral expansion following the ideas of Selberg.
Proposition 7. For $z, w \in M$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, the automorphic kernel $K_s(z, w)$ admits the spectral expansion

$$K_s(z, w) = \sum_{\lambda_j \geq 0} a_{1/2 + i\lambda_j} \psi_j(z)\psi_j(w) + \frac{1}{4\pi} \sum_{k=1}^{\infty} \int a_{1/2 + ir}(s) E^\text{par}_{p_k}(z, 1/2 + ir) E^\text{par}_{p_k}(w, 1/2 + ir) dr,$$

which converges absolutely and uniformly on compacta; here, we have set

$$a_{\nu}(s) := \frac{2^{\nu-1} \sqrt{\pi}}{\Gamma(s)} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s-1+\nu}{2}\right).$$

Proof. Let $z, w \in M$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$. First, we compute the Selberg/Harish-Chandra transform of the point-pair invariant function $k_s(z, w) := \cosh(d_{\text{hyp}}(z, w))^{-s}$ by applying the three steps recalled in subsection 2.3. To do this, we write $k_s(z, w)$ as a function of $u = u(z, w)$ using relation (2), namely $k_s(u(z, w)) = k_s(u) = (1 + 2u)^{-\nu}$. Hence, in the first step, we have to compute

$$Q_s(v) := \int_{0}^{\infty} \frac{k_s(u)}{\sqrt{u-v}} du = \int_{0}^{\infty} \frac{(1 + 2v + 2u)^{-s}}{\sqrt{u}} du = (1 + 2v)^{1/2-s} \frac{2^{-s/2}}{\sqrt{\pi}} \Gamma\left(\frac{s-1}{2}\right) \frac{2^{1/2}}{\Gamma(s)},$$

here, for the last equality we used [GR07], formula 3.251.11 with $\beta := 2/(1 + 2v)$, $\mu := 1/2$, $p = 1$ and $\nu := s$. In the second step, we get

$$g_s(u) := 2 Q_s\left(\sinh\left(\frac{u}{2}\right)^2\right) = \cosh(u)\left(1/2-s\right) 2^{1/2} \sqrt{\pi} \Gamma\left(\frac{s-1}{2}\right).$$

In the last step, we compute

$$h_s(r) := \int_{-\infty}^{\infty} g_s(u) e^{iru} du = \frac{2^{3/2}}{\sqrt{\pi}} \Gamma\left(s-\frac{1}{2}\right) \int_{0}^{\infty} \cos(ur) \cosh(u)^{-(s-1/2)} du = H(r, g_s),$$

where $H(r, g_s)$ is defined by (11). Now, in the case $r \in \mathbb{R}$, $r \neq 0$, the case $r \in [-i/2, i/2]$, $r \neq 0$, resp. the case $r = 0$, we have (see [GR07], formula 3.985.1, formula 3.512.1, formula 3.512.2, respectively)

$$\int_{0}^{\infty} \cos(ur) \cosh(u)^{-\nu} du = \frac{2^{\nu-2}}{\Gamma(\nu)} \Gamma\left(\frac{\nu - ir}{2}\right) \Gamma\left(\frac{\nu + ir}{2}\right),$$

where $\text{Re}(\nu) > 1/2$. Substituting (22) with $\nu := s - 1/2$ into (21), we immediately get for $r \in \mathbb{R}$ or $r \in [-i/2, i/2]$ the identity

$$h_s(r) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(s - \frac{1}{2} - ir\right) \Gamma\left(s - \frac{1}{2} + ir\right).$$

Now, let $\eta = (\text{Re}(s) - 1/2).$ Then, $1/2 + \eta = \text{Re}(s)/2$, and, obviously, $g_s(u) \in S'(\mathbb{R}^+, 1/2 + \eta)$. Moreover, $g_s^{(j)}(u) e^{u(1/2 + \eta)}$ is bounded by some integrable function on $[0, \infty)$ and obviously, $g_s'(u) \in S'(\mathbb{R}^+, 1/2 + \eta)$. Hence, by part (ii) of lemma 3, the function $H(r, g_s) = h_s(r)$ satisfies the conditions (S1), (S2), and (S3) of subsection 2.3 for any $0 < \epsilon < \eta$ and with $\delta = 1$. Therefore, $K_s(z, w)$ admits the spectral expansion

$$K_s(z, w) = \sum_{\lambda_j \geq 0} h_s(t_j) \psi_j(z)\psi_j(w) + \frac{1}{4\pi} \sum_{k=1}^{\infty} \int h_s(r) E^\text{par}_{p_k}(z, 1/2 + ir) E^\text{par}_{p_k}(w, 1/2 + ir) dr,$$

which converges absolutely and uniformly on compacta, as asserted. ■
A direct consequence of the proof of Proposition 7 is that the automorphic kernel \( K_s(z, w) \) may be represented via an action of the wave distribution.

**Theorem 8.** For \( z, w \in M \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), we have the representation

\[
K_s(z, w) = W_{M, \frac{1}{2}}(z, w)(g_s),
\]

which converges absolutely and uniformly on compacta; here, we have set

\[
g_s(u) = \frac{2^{1/2} \sqrt{\pi} \Gamma(s - \frac{1}{2})}{\Gamma(s)} \cosh(u)^{-s}, \quad u \in \mathbb{R}.
\]

**Proof.** Let \( z, w \in M \) and let \( \eta = (\text{Re}(s) - 1)/2 \). Then, we have \( g_s \in S'(\mathbb{R}^+, 1/2 + \eta) \) and \( g_s^{(j)}(u) \exp(u(1/2 + \eta)) \) is bounded by some integrable function on \( \mathbb{R}^+ \) for \( j = 1, 2, 3 \). Hence, by Proposition 5, we have

\[
W_{M, \frac{1}{2}}(z, w)(g_s) = \sum_{\eta \in \Gamma} k_s(z, \eta w) = K_s(z, w),
\]

which converges absolutely and uniformly on compacta; here, \( k_s(u) = k_s(u(z, w)) = k_s(z, w) = \cosh(d_{\text{hyp}}(z, w))^{-s} \) is the inverse of the Selberg/Harish-Chandra transform of \( H(r, g_s) \), as the proof of Proposition 7 shows.

We conclude this section by proving the meromorphic continuation of the automorphic kernel, which is known in the literature (see, for example, [P10]). However, the proof we present here is given within the framework of the wave distribution, hence provides a point of view which can be generalized to other settings.

**Lemma 9.** For \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), \( n \in \mathbb{N} \), and \( r \in \mathbb{R} \cup [-i/2, i/2] \), we have

\[
H(r, g_s) = \frac{2^{-2n}(s)_{2n}}{(\frac{s}{2} - \frac{1}{4} - \frac{ir}{2})n(\frac{s}{2} - \frac{1}{4} + \frac{ir}{2})n} H(r, g_{s+2n}),
\]

where \((\cdot)_n\) denotes the Pochhammer symbol, \( H(r, \cdot) \) is defined by (11), and \( g_s \) is given by (24).

**Proof.** Let \( n \in \mathbb{N} \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \). By definitions (11) and (24), we have

\[
H(r, g_s) = \frac{2^{3/2} \sqrt{\pi} \Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} \int_0^\infty \cos(ur) \cosh(u)^{-(\alpha - 1/2)} du.
\]

Hence, for \( r \in \mathbb{R} \cup [-i/2, i/2] \), using formula (22) with \( \nu := \alpha - 1/2 \) for \( \alpha = s \) and \( \alpha = s + 2n \), we derive the identity (25).

**Theorem 10.** For any \( z, w \in M \), the automorphic kernel \( K_s(z, w) \) admits a meromorphic continuation to the whole complex \( s \)-plane. The possible poles of the function \( \Gamma(s)\Gamma(s - 1/2)^{-1} K_s(z, w) \) are located at the points \( s = 1/2 \pm it_j - 2n \), where \( n \in \mathbb{N} \) and \( \lambda_j = 1/4 + t_j^2 \) is a discrete eigenvalue of \( \Delta_{\text{hyp}} \), at the points \( s = 1 - \rho - 2n \), where \( n \in \mathbb{N} \) and \( \rho \in (1/2, 1] \) is a pole of \( \mathcal{E}^\text{par}_{\rho_k}(z, s) \), and at the points \( s = \rho - 2n \), where \( n \in \mathbb{N} \) and \( \rho \) is a pole of \( \mathcal{E}^\text{par}_{\rho_k}(z, s) \) with \( \text{Re}(\rho) < 1/2 \).

**Proof.** We start by proving that \( K_s(z, w) \) has a meromorphic continuation to the halfplane \( \text{Re}(s) > 1 - 2n \) for any \( n \in \mathbb{N} \). Using the wave representation (23) of \( K_s(z, w) \) and substituting formula (25) for the coefficients \( H(t_j, g_s) \), we get for \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \) the identity

\[
\frac{2^{2n} \Gamma(s)}{\Gamma(s + 2n)} K_s(z, w) = \sum_{\lambda_j \geq 0} \frac{H(t_j, g_{s+2n})}{h_n(t_j, s)} \psi_j(z) \psi_j(w) + \frac{1}{4\pi} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \frac{H(r, g_{s+2n})}{h_n(r, s)} \mathcal{E}^\text{par}_{\rho_k}(z, 1/2 + ir) \mathcal{E}^\text{par}_{\rho_k}(w, 1/2 + ir) dr,
\]

(26)
where \( h_n(r, s) := \left(\frac{1}{4} - \frac{1}{2} - \frac{r^2}{4}\right) n \left(\frac{1}{4} + \frac{r^2}{4}\right) \) and \( \lambda_j = 1/4 + t_j^2 \). Since \( (a)_n = \prod_{j=0}^{n-1} (a + j) \), we have \( h_n(r, s) \sim r^{2n} \), as \( r \to \infty \). This proves that the series in (26) arising from the discrete spectrum is locally absolutely and uniformly convergent as a function of \( s \in \mathbb{C} \) for \( \Re(s) > 1 - 2n \) away from the poles of \( h_n(t_j, s)^{-1} \). The location of the poles arising from this part is now straightforward referring to the fact that zeros of \( h_n(t_j, s) \) for \( \Re(s) > 1 - 2n \) are at the points \( s = 1/2 \pm it_j - 2\ell \) for \( \ell = 0, \ldots, n - 1 \).

To prove the meromorphic continuation of the integral arising from the continuous spectrum in (26), we substitute \( r \mapsto 1/2 + ir \) and we observe that as a function of \( s \in \mathbb{C} \) the integral converges locally absolutely and uniformly for \( \Re(s) > 1 - 2n \) satisfying \( \Re(s) \neq 1/2 - 2\ell \) \( (\ell = 0, \ldots, n - 1) \). To obtain the meromorphic continuation across the lines \( \Re(s) = 1/2 - 2\ell \), we mimic the method used, e.g., in the proof of Theorem 2 of [JKvP10] or in [vP10]. Namely, for \( \ell = 0 \), we move the line of integration to \( \Re(s) = 1/2 + \varepsilon \) for some \( \varepsilon > 0 \) sufficiently small such that \( \mathcal{E}_{p_k}^\text{par} (z, s) \) has no poles in the strip \( 1/2 - \varepsilon < \Re(s) < 1/2 + \varepsilon \). Using twice the residue theorem and moving back the line of integration to \( \Re(s) = 1/2 \), we obtain the meromorphic continuation of the integral to the strip \( -3/2 < \Re(s) \leq 1/2 \). Repeating this process for \( \ell = 1, \ldots, n - 1 \), we obtain the desired meromorphic continuation. Multiplying the integral by \( \Gamma(s - 1/2)^{-1} \), the only poles arising in this process are at \( s = 1 - \rho - 2\ell \), where \( \rho \) is a pole of the Eisenstein series \( \mathcal{E}_{p_k}^\text{par} \) belonging to the line segment \( (1/2, 1] \) and at \( s = \rho - 2\ell \), where \( \rho \) is a pole of the Eisenstein series \( \mathcal{E}_{p_k}^\text{par} \) such that \( \Re(\rho) < 1/2 \) and \( \ell = 0, \ldots, n - 1 \).

Since \( n \in \mathbb{N} \) was chosen arbitrarily, this proves the meromorphic continuation of \( K_s(z, w) \) to the whole \( s \)-plane. \( \square \)

### 5 The wave representation of hyperbolic Eisenstein series

We now express the hyperbolic Eisenstein series recalled in subsection 2.2 in terms of the wave distribution. To do this, we first establish an integral representation for hyperbolic Eisenstein series using the automorphic kernel \( K_s(z, w) \) and then we apply Theorem 8.

**Proposition 11.** Let \( \gamma \in \Gamma \) be a primitive, hyperbolic element of \( \Gamma \) and let \( \mathcal{E}_{\text{hyp}}^\gamma(z, s) \) be the hyperbolic Eisenstein series associated to the closed geodesic \( L_\gamma \) on \( M \) in the homotopy class determined by \( \gamma \). Then, for \( z, w \in M \) and \( s \in \mathbb{C} \) with \( \Re(s) > 1 \), we have the identity

\[
\int_{L_\gamma} K_s(z, w) ds_{\text{hyp}}(w) = \frac{2^{s-1} \Gamma(\frac{s}{2})^2}{\Gamma(s)} \mathcal{E}_s^\text{hyp}(z, s).
\]

(27)

**Proof.** Let \( \gamma \in \Gamma \) be a primitive, hyperbolic element of \( \Gamma \) with scaling-matrix \( \sigma_\gamma \in \text{PSL}_2(\mathbb{R}) \), so then

\[
\sigma_\gamma^{-1} \gamma \sigma_\gamma = \begin{pmatrix} e^{t_\gamma / 2} & 0 \\ 0 & e^{-t_\gamma / 2} \end{pmatrix},
\]

where \( t_\gamma \) denotes the hyperbolic length of \( L_\gamma \). Without loss of generality, we may assume that \( \sigma_\gamma \) is the identity matrix, so then \( L_\gamma = p(L_\gamma) \), where \( L_\gamma = L \) is the positive \( y \)-axis. Let \( \Gamma_\gamma := \langle \gamma \rangle \) be the stabilizer subgroup of \( \gamma \) in \( \Gamma \); \( \Gamma_\gamma \) is isomorphic to \( \mathbb{Z} \) with generator \( \gamma \). Let us rewrite the automorphic kernel \( K_s(z, w) \) as

\[
K_s(z, w) = \sum_{\eta \in \Gamma} \cosh(d_{\text{hyp}}(\eta z, w))^{-s} = \sum_{\eta \in \Gamma_\gamma} \sum_{\eta' \in \Gamma_\gamma} \cosh(d_{\text{hyp}}(\eta' \eta z, w))^{-s} = \sum_{\eta \in \Gamma_\gamma \setminus \Gamma} \sum_{n \in \mathbb{Z}} \cosh(d_{\text{hyp}}(\eta z, e^{-n t_\gamma} w))^{-s}.
\]

(28)
Using elementary considerations involving counting functions, one has that the series in (28) converges absolutely and locally uniformly for \(z, w \in M\) and \(s \in \mathbb{C}\) with \(\text{Re}(s) > 1\); see [GJM08] for details. Therefore, we get

\[
\int_{L_{\gamma}} K_s(z, w) ds_{\text{hyp}}(w) = \sum_{\eta \in \Gamma \gamma, \Gamma} \sum_{n \in \mathbb{Z}} \int_{L_{\gamma}} \cosh(d_{\text{hyp}}(\eta z, e^{n\ell_{\gamma}} w))^{-s} ds_{\text{hyp}}(w). \tag{29}
\]

Set \(z' := \eta z\) and use the coordinates \(\rho = \rho(w)\) and \(\theta = \theta(w)\) from subsection 2.1. If we analyze each term in (29), we can write

\[
\int_{L_{\gamma}} \cosh(d_{\text{hyp}}(z', e^{n\ell_{\gamma}} w))^{-s} ds_{\text{hyp}}(w) = \int_{0}^{\ell_{\gamma}} \cosh(d_{\text{hyp}}(z', e^{n\ell_{\gamma} + \rho i}))^{-s} d\rho = \int_{n\ell_{\gamma}}^{(n+1)\ell_{\gamma}} \cosh(d_{\text{hyp}}(z', e^{\rho i}))^{-s} d\rho = \cosh(d_{\text{hyp}}(z', \mathcal{L}_{\gamma}))^{-s} \int_{n\ell_{\gamma}}^{(n+1)\ell_{\gamma}} \cosh(\rho - \text{log}(|z'|))^{-s} d\rho. \tag{30}
\]

All equalities in (30) are immediate except for the last one. For this, we use Theorem 7.11.1 from [Be95] which is an identity for right-angled hyperbolic triangles, namely

\[
cosh(d_{\text{hyp}}(z', e^{\rho i})) = \cosh(d_{\text{hyp}}(|z'| i, e^{\rho i}) \cosh(d_{\text{hyp}}(z', \mathcal{L})) = \cosh(\rho - \text{log}(|z'|)) \cosh(d_{\text{hyp}}(z', \mathcal{L}_{\gamma})).
\]

Substituting (30) into (29), we arrive at the formula

\[
\int_{L_{\gamma}} K_s(z, w) ds_{\text{hyp}}(w) = \sum_{\eta \in \Gamma \gamma, \Gamma} \cosh(d_{\text{hyp}}(\eta z, \mathcal{L}_{\gamma}))^{-s} \int_{n\ell_{\gamma}}^{(n+1)\ell_{\gamma}} \cosh(\rho - \text{log}(|\eta z|))^{-s} d\rho = \sum_{\eta \in \Gamma \gamma, \Gamma} \cosh(d_{\text{hyp}}(\eta z, \mathcal{L}_{\gamma}))^{-s} \int_{-\infty}^{\infty} \cosh(\rho - \text{log}(|\eta z|))^{-s} d\rho. \tag{31}
\]

Using (22) with \(\nu = s\) and \(r = 0\), we get that

\[
\int_{-\infty}^{\infty} \cosh(\rho - \text{log}(|\eta z|))^{-s} d\rho = 2 \int_{0}^{\infty} \cosh(\rho)^{-s} d\rho = \frac{2^{s-1} \Gamma(\frac{s}{2})^2}{\Gamma(s)}. \tag{32}
\]

Finally, substituting (32) into (31), we arrive at the formula

\[
\int_{L_{\gamma}} K_s(z, w) ds_{\text{hyp}}(w) = \frac{2^{s-1} \Gamma(\frac{s}{2})^2}{\Gamma(s)} \sum_{\eta \in \Gamma \gamma, \Gamma} \cosh(d_{\text{hyp}}(\eta z, \mathcal{L}_{\gamma}))^{-s} = \frac{2^{s-1} \Gamma(\frac{s}{2})^2}{\Gamma(s)} e_{\gamma}^{\text{hyp}}(z, s),
\]

which proves the assertion.

A direct consequence of Proposition 11 is that we can express the hyperbolic Eisenstein series \(E_{\gamma}^{\text{hyp}}(z, s)\) as an integral over \(L_{\gamma}\) of the wave distribution applied to \(g_s\), up to a multiplicative factor which is a universal function of \(s\).

**Theorem 12.** Let \(\gamma \in \Gamma\) be a primitive, hyperbolic element of \(\Gamma\) and let \(E^{\text{hyp}}(z, s)\) be the hyperbolic Eisenstein series associated to the closed geodesic \(L_{\gamma}\) on \(M\) in the homotopy class determined by \(\gamma\). Then, for \(z, w \in M\) and \(s \in \mathbb{C}\) with \(\text{Re}(s) > 1\), we have the representation

\[
E_{\gamma}^{\text{hyp}}(z, s) = \frac{2^{1-s} \Gamma(s)}{\Gamma(\frac{s}{2})^2} \int_{L_{\gamma}} \mathcal{W}_{M, \frac{1}{2}}(z, w)(g_s) ds_{\text{hyp}}(w),
\]

which converges absolutely and uniformly on compacta; here, the test function \(g_s\) is given by (24).
Proof. The assertion follows immediately by combining Proposition 11 with Theorem 8. \qed

Remark 13. The spectral expansion for hyperbolic Eisenstein series was established in [JKvP10]. By comparing this spectral expansion to the spectral expansion (20) of $K_s(z, w)$, one gets an alternative proof of Proposition 11. Conversely, substituting the spectral expansion (20) into (27), Proposition 11 gives rise to a new proof of the spectral expansion of hyperbolic Eisenstein series.

Remark 14. As proven in [JKvP10], for any $z \in M$, the hyperbolic Eisenstein series $\zeta_{hyp}^\gamma(z, s)$ admits a meromorphic continuation to the whole complex $s$-plane. Alternatively, this can be proved along the lines of the proof of Theorem 10 by starting with the identity

$$\frac{2^{s+2n-1} \Gamma(z)^2}{\Gamma(s+2n)} \zeta_{hyp}^\gamma(z, s) = \sum_{\lambda_j \geq 0} \frac{H(t_j, g_{s+2n})}{h_n(t_j, s)} \psi_j(z) \int_{L_\gamma} \psi_j(w) ds_{hyp}(w)$$

$$+ \frac{1}{4\pi} \sum_{k=1}^{\infty} \int_{\{-\infty\}}^{\infty} \frac{H(r, g_{s+2n})}{h_n(r, s)} \xi_{hyp}(z, 1/2 + ir) \int_{L_\gamma} \xi_{hyp}(w, 1/2 + ir) ds_{hyp}(w) dr,$$

which is deduced for $s \in \mathbb{C}$ with $\text{Re}(s) > 1$ by combining Proposition 11 with relation (26).

6 The wave representation of elliptic Eisenstein series

We now express the elliptic Eisenstein series recalled in subsection 2.2 in terms of the wave distribution. We start by recalling the following relation of the elliptic Eisenstein series to the automorphic kernel $K_s(z, w)$, which was first given in [vP10].

Lemma 15. For $z, w \in M$ with $z \neq w$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, the elliptic Eisenstein series $\xi_{ell}^\gamma(z, s)$ associated to the point $w$ satisfies the relation

$$\xi_{ell}^\gamma(z, s) = \frac{1}{\text{ord}(w)} \sum_{k=0}^{\infty} \frac{(\frac{z}{w})^k}{k!} K_{s+2k}(z, w). \quad (33)$$

Proof. The proof of the assertion is immediate applying the general binomial theorem and is given in Lemma 3.3.8 of [vP10]. \qed

For technical reasons, see Remark 21, we will have to consider the following two test functions.

Definition 16. For $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, $\beta \in \mathbb{C}$ with $\text{Re}(\beta) > 0$, and $u \in \mathbb{R}^+$, we set

$$g_{s, \beta}(u) := g_s(u) \tanh(u)^\beta. \quad (34)$$

For $s, \beta \in \mathbb{C}$ with $\text{Re}(\beta) > \text{Re}(s-1) > 0$, and $u \in \mathbb{R}^+$, we set

$$G_{s, \beta}(u) := g_s(u) \tanh(u)^{\beta+1-s} F\left(\frac{1}{4}, \frac{3}{4}; \frac{s}{2}; \frac{1}{2}; \frac{1}{\cosh(u)^2}\right). \quad (35)$$

Here, $g_s(u)$ is given by (24) and $F(a; b; c; z)$ denotes the Gauss hypergeometric function.

Since $\text{Re}(1/4 + 3/4 - s/2 - 1/2) < 0$ for $\text{Re}(s) > 1$, the hypergeometric series in (35) converges uniformly for all $u \in \mathbb{R}$. Therefore, $G_{s, \beta}(u)$ is well-defined on $\mathbb{R}^+$ including $u = 0$ for the given range of complex parameters $s$ and $\beta$.

Lemma 17. Let $s \in \mathbb{C}$ with $\text{Re}(s) > 1$.

(i) For $\beta \in \mathbb{C}$ with $\text{Re}(\beta) > 8$, we have the bound

$$H(r, g_{s+2k, \beta}) = O(k^{-(\text{Re}(\beta)-8)/2} |r|^{-3}) \quad \text{as } r \to \pm \infty, k \to \infty. \quad (36)$$
(ii) For $\beta \in \mathbb{C}$ with $\text{Re}(\beta) > 0$, we have the bound

$$H(r, g_{s+2k}, \beta) = O(k^{-\text{Re}(\beta)/2}) \text{ as } k \to \infty,$$  \hspace{1cm} \text{(37)}

uniformly in $r$, for $r \in [-1, 1]$.

The implied constants in (i) and (ii) depend uniformly on $s$ and $\beta$.

Proof. Part (i): By definition, we have

$$H(r, g_{s+2k}, \beta) = \frac{2^{3/2} \sqrt{\pi} \Gamma(\alpha_k)}{\Gamma(\alpha_k + \frac{1}{2})} \int_0^\infty \cos(ur) \tanh(u)^\beta \cosh(u)^{-\alpha_k} du,$$  \hspace{1cm} \text{(38)}

where we have set $\alpha_k := s - 1/2 + 2k$. Integrating by parts three times, we find

$$H(r, g_{s+2k}, \beta) = \frac{2^{3/2} \sqrt{\pi} \Gamma(\alpha_k)}{\Gamma(\alpha_k + \frac{1}{2})} \frac{1}{r^3} \int_0^\infty \sin(u) \tanh(u)^\beta \cosh(u)^{-\alpha_k} S(\alpha_k, \beta; u) du$$

with

$$S(\alpha_k, \beta; u) := \sum_{j=0}^3 P_j(\alpha_k, \beta) \sinh(u)^{2j},$$

where $P_j(\alpha_k, \beta)$ is a polynomial in two variables $\alpha_k$ and $\beta$ of a degree equal to three; here, we used the fact that $g_{s+2k} \in S'(\mathbb{R}^+, 1/2 + \eta)$ for any $\eta \in (0, \text{Re}(s) - 1)$ assuming $\text{Re}(\beta) > 2$. By the bound

$$|S(\alpha_k, \beta; u)| \ll |\alpha_k|^3 \sum_{j=0}^3 \sinh(u)^{2j} \ll k^3 \sum_{j=0}^3 \sinh(u)^{2j},$$

we get

$$\left| \int_0^\infty \sin(u) \tanh(u)^\beta \cosh(u)^{-\alpha_k} S(\alpha_k, \beta; u) du \right| \ll k^3 \sum_{j=0}^3 \int_0^\infty \sinh(u)^{\mu_j} \cosh(u)^{-\nu} du = k^3 \sum_{j=0}^3 \frac{\Gamma(\nu+1) \Gamma(\mu-\nu)}{\Gamma(\mu+1) \Gamma(\nu+1)}$$

with $\mu_j := \text{Re}(\beta) - 3 + 2j$ and $\nu := \text{Re}(\alpha_k) + \text{Re}(\beta) + 3$; here, the last equality follows from formula 3.512.2 of [GR07] provided that $\text{Re}(\mu_j) > -1$ and $\text{Re}(\mu_j - \nu) < 0$.

An application of Stirling’s formula yields, for any fixed real $y$, the asymptotics $\Gamma(x)/\Gamma(x + y) \sim x^{-y}$, as $x \to \infty$. Hence, we have the bounds

$$\frac{\Gamma(\alpha_k)}{\Gamma(\alpha_k + 1)} = O(k^{-1/2}) \quad \text{and} \quad \frac{\Gamma(\nu+1) \Gamma(\mu-\nu)}{\Gamma(\mu+1) \Gamma(\nu+1)} = O(k^{-\text{Re}(\beta)/2 + 1 - j}),$$

as $k \to \infty$. Hence, summing up, we find

$$H(r, g_{s+2k}, \beta) = O(k^{-\text{Re}(\beta)/2} |r|^{-3}) \text{ as } |r|, k \to \infty,$$

where the implied constant depends uniformly on $s$ and $\beta$.

Part (ii): In the case when $r \in [-1, 1]$, using the trivial bound $|\cos(ur)| \leq 1$, we get immediately from (38) that

$$|H(r, g_{s+2k}, \beta)| \ll \frac{1}{k^{1/2}} \int_0^\infty \sinh(u)^{\text{Re}(\beta)} \cosh(u)^{-\text{Re}(\alpha_k) + \text{Re}(\beta)} du \text{ as } k \to \infty,$$

uniformly in $r$. The bound (37) is now obtained in the same way as above, using the representation of this integral in terms of Gamma factors and then applying Stirling’s formula.

This completes the proof. \hfill \Box
Lemma 18. For $s, \beta \in \mathbb{C}$ with $\text{Re}(s) > 1$ and $\text{Re}(\beta) > \text{Re}(s) + 8$, we have the relation

$$H(r, G_{s,\beta}) = \sum_{k=0}^{\infty} \frac{\left(\frac{s}{2}\right)_k}{k!} H(r, g_{s+2k,\beta}).$$

(39)

Proof. Let $s, \beta \in \mathbb{C}$ with $\text{Re}(\beta) > \text{Re}(s) + 8 > 9$ and $u \in \mathbb{R}^+$. We first prove the relation

$$G_{s,\beta}(u) = \sum_{k=0}^{\infty} \frac{\left(\frac{s}{2}\right)_k}{k!} g_{s+2k,\beta}(u).$$

(40)

The equality (40) obviously holds true for $u = 0$, since both sides of (40) are well-defined and equal to zero. Now, assume $u > 0$. Using formula $F(a; b; c; z) = (1 - z)^{c-a-b}F(c - a, c - b; c; z)$ (see [GR07], formula 9.131.1), we deduce

$$G_{s,\beta}(u) = g_s(u) \tanh(u)^\beta F\left(\frac{s}{2} - \frac{1}{4}; \frac{1}{2}; \frac{1}{2} + \frac{1}{2}; \frac{1}{2}; \frac{1}{2}: \frac{1}{\cosh(u)^2}\right).$$

(41)

Then, substituting the equality

$$g_{s,\beta}(u) = \frac{2^s \Gamma\left(\frac{s}{2} - \frac{1}{4}\right) \Gamma\left(\frac{s}{2} + \frac{3}{4}\right)}{\Gamma(s)} \tanh(u)^\beta \cosh(u)^{-\left(s-1/2\right)}$$

and the well-known series expansion of the hypergeometric function into (41), we get

$$G_{s,\beta}(u) = \tanh(u)^\beta \frac{2^s}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{s}{2} - \frac{1}{4} + k\right) \Gamma\left(\frac{s}{2} + \frac{1}{4} + k\right)}{k! \left(\frac{s}{2} + \frac{1}{2}\right)_k} \cosh(u)^{-\left(s+2k-1/2\right)}$$

$$= \sum_{k=0}^{\infty} \frac{2^{-2k} (s)_{2k}}{k! \left(\frac{s}{2} + \frac{1}{2}\right)_k} g_{s+2k,\beta}(u).$$

The relation (40) now follows from the dimidiation formula $2^{-2k} (a)_{2k} (a + 1/2)^{-1} = (a)_k$ for the Pochhammer symbol with $a := s/2$. Finally, using the relation (40), we get

$$H(r, G_{s,\beta}) = \frac{1}{2} \int_0^{\infty} \left(\sum_{k=0}^{\infty} \frac{\left(\frac{s}{2}\right)_k}{k!} g_{s+2k,\beta}(u)\right) \cos(ur) \, du = \sum_{k=0}^{\infty} \frac{\left(\frac{s}{2}\right)_k}{k!} H(r, g_{s+2k,\beta}).$$

The interchange of the sum and the integral defining $H(r, g_{s+2k,\beta})$ is justified by the bound

$$\left|\frac{\left(\frac{s}{2}\right)_k}{k!} H(r, g_{s+2k,\beta})\right| \ll k^{-\text{Re}(\beta)/2 - \text{Re}(s)/2 - 3} \ll k^{-(1+\varepsilon)},$$

for $\varepsilon > 0$ with $\text{Re}(\beta) = 2\varepsilon + \text{Re}(s) + 8$, which can be deduced from the bounds (36) and (37) together with

$$\left|\frac{\left(\frac{s}{2}\right)_k}{k!}\right| \ll k^{\text{Re}(s)/2 - 1}.$$

This completes the proof of the relation (39). \hfill \Box

Lemma 19. Let $s, \beta \in \mathbb{C}$ with $\text{Re}(\beta) > \text{Re}(s + 1) > 2$.

(i) For $z, w \in M$, the wave distribution $W_{M,\frac{1}{2}}(z, w)(g_{s,\beta})$ is well-defined and holomorphic as a function of $\beta$.

(ii) For $z, w \in M$ with $z \neq w$, the wave distribution $W_{M,\frac{1}{2}}(z, w)(g_{s,\beta})$ admits a holomorphic continuation in $\beta$ to $\text{Re}(\beta) > 0$. Further, for $\text{Re}(\beta) > 0$, assuming $\text{Re}(s) = \sigma \geq \sigma_0 > 1$ and $z \neq w$, there exists a constant $C > 1$ depending on $d_{\text{hyp}}(z, w)$ such that

$$W_{M,\frac{1}{2}}(z, w)(g_{s,\beta}) = O(C^{-(\sigma-\sigma_0)}) \quad \text{as } \sigma \to \infty.$$
Proof. Part (i): For $s, \beta \in \mathbb{C}$ with $\Re(\beta) > \Re(s + 1) > 2$, a straightforward computation shows that $g_{s, \beta}(u) \in S'(\mathbb{R}^+, 1/2)$, and that the first three derivatives of $g_{s, \beta}(u)$ are integrable and have a limit as $u \to \infty$. Hence, by part (i) of Proposition 5, the wave distribution $W_{M, \frac{1}{4}}(z, w)(g_{s, \beta})$ is well-defined and holomorphic as a function of $\beta$.

Part (ii): Moreover, for $\Re(\beta) > 2$ and $\Re(s) > 1$, the function $g_{s, \beta}(u)$ satisfies the assumptions of Proposition 6, part (iii). Hence, for $\delta > 0$, whose choice will be clarified during the course of the proof, we can use Proposition 6 to write

$$W_{M, \frac{1}{4}}(z, w)(g_{s, \beta}) = \int_0^\delta F(z, w; u) g_{s, \beta}^{(3)}(u) du + \int_\delta^\infty F(z, w; u) g_{s, \beta}^{(3)}(u) du. \quad (42)$$

Since the right hand side of (42) is meaningful for $\Re(\beta) > 0$, due to bound (ii) and (i) from Proposition 6, this provides the holomorphic continuation of $W_{M, \frac{1}{4}}(z, w)(g_{s, \beta})$ in the variable $\beta$ to the half-plane $\Re(\beta) > 0$, as asserted.

Now, let $\Re(\beta) > 0$ and assume $\Re(s) = \sigma > \sigma_0 > 1$. The bound (i) from Proposition 6 together with the inequality $\tanh(u) \leq 1$ implies the bound

$$\left| \int_\delta^\infty F(z, w; u) g_{s, \beta}^{(3)}(u) du \right| = O(C_{\delta}^{-\sigma + 1}) \quad (43)$$

for some constant $C_{\delta} > 1$, as $\sigma \to \infty$. Further, using the notation and results of Proposition 6, we can integrate by parts to write

$$\int_0^\delta F(z, w; u) g_{s, \beta}^{(3)}(u) du = \int_0^\delta \left( P_{M, \frac{1}{4}}(z, w; iu) + P_{M, \frac{1}{4}}(z, w; -iu) \right) g_{s, \beta}(u) du + O(C_{\delta}^{-\sigma + 1})$$

for some constant $\tilde{C}_{\delta} > 1$, as $\sigma \to \infty$; in particular, we used identity (16) which holds by analytic continuation for $\zeta = iu$ with $0 < u \leq \delta$ and $\delta$ chosen as below. Since $z \neq w$, the function

$$\tilde{P}(z, w; u) := P_{M, \frac{1}{4}}(z, w; iu) + P_{M, \frac{1}{4}}(z, w; -iu)$$

is real valued and continuous as $u$ approaches zero. Therefore, we can choose $\delta > 0$ sufficiently small so that the function $\tilde{P}(z, w; u)$ has one sign when $u \in (0, \delta)$, hence

$$\left| \int_0^\delta \tilde{P}(z, w; u) g_{s, \beta}(u) du \right| \leq \int_0^\delta \left| \tilde{P}(z, w; u) \right| g_{\sigma}(u) du = \int_0^\delta \tilde{P}(z, w; u) g_{\sigma}(u) du,$$

where the inequality follows, since $\tanh(u) \leq 1$, and the last equality follows, since the function $\tilde{P}(z, w; u)$ has one sign and $g_{\sigma}(u)$ is positive. Integrating by parts once again, we get

$$\int_0^\delta \tilde{P}(z, w; u) g_{\sigma}(u) du = \int_0^\delta F(z, w; u) g_{\sigma}^{(3)}(u) du = W_{M, \frac{1}{4}}(z, w)(g_{\sigma}) - \int_\delta^\infty F(z, w; u) g_{\sigma}^{(3)}(u) du$$

as $\sigma \to \infty$, by computations similar to the ones used to obtain the bound (43). Now, Theorem 8 yields $W_{M, \frac{1}{4}}(z, w)(g_{\sigma}) = K_{\sigma}(z, w)$, hence, from the series definition (19), we derive the bound

$$|W_{M, \frac{1}{4}}(z, w)(g_{\sigma})| \leq \tilde{C}^{-(\sigma - \sigma_0)} K_{\sigma_0}(z, w) \ll \tilde{C}^{-(\sigma - \sigma_0)}$$

for a constant $\tilde{C} > 1$ such that $\tilde{C} < \cosh(d_{\text{hyp}}(z, \eta w))$ for any $\eta \in \Gamma$. The proof is complete by substituting the last bound together with the bounds (43) and (44) into equality (42) and letting $C = \min\{C_{\delta}, \tilde{C}_{\delta}, \tilde{C}\} > 1$. □
Theorem 20. Let $s, \beta \in \mathbb{C}$ with $\text{Re}(\beta) > \text{Re}(s + 1) > 2$.

(i) For $z, w \in M$, the wave distribution $W_{M, \frac{1}{2}}(z, w)(G_{s, \beta})$ is well-defined and holomorphic as a function of $\beta$.

(ii) For $z, w \in M$ with $z \neq w$, the wave distribution $W_{M, \frac{1}{2}}(z, w)(G_{s, \beta})$ admits a holomorphic continuation in $\beta$ to $\{ \beta \in \mathbb{C} \mid \text{Re}(\beta) > 0 \} \cup \{ 0 \}$, and we have the representation

$$\text{ord}(w)E_{w}^{\text{ad}}(z, s) = W_{M, \frac{1}{2}}(z, w)(G_{s, \beta}) \big\rvert_{\beta = 0}^-.$$

Here, the test function $G_{s, \beta}$ is given by (35).

Proof. Part (i): Let $s, \beta \in \mathbb{C}$ with $\text{Re}(\beta) > \text{Re}(s + 1) > 2$. A straightforward computation shows that, for any $\eta \in (0, \text{Re}(s) - 1)$, we have $G_{s, \beta} \in S'(\mathbb{R}^+, 1/2 + \eta)$ and $G_{s, \beta}^{(j)}(w) \exp(u(1/2 + \eta))$ is bounded by some integrable function on $\mathbb{R}^+$ for $j = 1, 2, 3$. Hence, by Proposition 5, $W_{M, \frac{1}{2}}(z, w)(G_{s, \beta})$ is well-defined and a holomorphic function in $\beta$.

Part (ii): Let $s \in \mathbb{C}$ with $\text{Re}(s) \geq \sigma_0 > 1$. Then, for $z, w \in M$ with $z \neq w$, the wave distribution $W_{M, \frac{1}{2}}(z, w)(g_{s + 2k, \beta})$ is holomorphic in $\beta$ for any $k \in \mathbb{N}$. Further, by part (ii) of Lemma 19, we get the bound

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})^k}{k!} W_{M, \frac{1}{2}}(z, w)(g_{s + 2k, \beta}) \ll \sum_{k=0}^{\infty} \frac{(\text{Re}(s))^k}{k!} C^{-2k} < \infty,$$

uniformly for any $\beta \in \mathbb{C}$ with $\text{Re}(\beta) > 0$. Hence, the series

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})^k}{k!} W_{M, \frac{1}{2}}(z, w)(g_{s + 2k, \beta})$$

converges absolutely and uniformly, and hence represents a holomorphic function for $\text{Re}(\beta) > 0$.

Further, for $\text{Re}(\beta)$ large enough (e.g. $\text{Re}(\beta) > \text{Re}(s) + 8 > 9$), by the bounds (36) and (37) (both bounds are needed in the case of the integral), we are allowed to interchange the sum over $k$ with the sum, resp. the integral appearing in the definition of the wave distribution $W_{M, \frac{1}{2}}(z, w)(g_{s + 2k, \beta})$, and we obtain

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})^k}{k!} W_{M, \frac{1}{2}}(z, w)(g_{s + 2k, \beta}) =$$

$$\sum_{\lambda_j \geq 0} H(t_j, G_{s, \beta}) \psi_j(z) \psi_j(w) + \frac{1}{4\pi} \sum_{j=1}^{\infty} \int H(r, G_{s, \beta}) E_{p_j}^{\text{par}}(z, 1/2 + ir) E_{p_j}^{\text{par}}(w, 1/2 + ir) dr,$$  

employing the relation (39). Thus, for $\text{Re}(\beta) \gg 0$, we have

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})^k}{k!} W_{M, \frac{1}{2}}(z, w)(g_{s + 2k, \beta}) = W_{M, \frac{1}{2}}(z, w)(G_{s, \beta}). \tag{45}$$

Since the left hand side of (45) is holomorphic for $\text{Re}(\beta) > 0$, this identity provides the holomorphic continuation of $W_{M, \frac{1}{2}}(z, w)(G_{s, \beta})$ to the half-plane $\text{Re}(\beta) > 0$. Moreover, letting $\beta = 0$ on the left hand side of (45) and combining relation (33) with Theorem 8, we deduce

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})^k}{k!} W_{M, \frac{1}{2}}(z, w)(g_{s + 2k}) = \text{ord}(w)E_{w}^{\text{ad}}(z, s),$$

Hence, by the principle of analytic continuation, we obtain the identity

$$\text{ord}(w)E_{w}^{\text{ad}}(z, s) = W_{M, \frac{1}{2}}(z, w)(G_{s, \beta}) \big\rvert_{\beta = 0}^-,$$  

thereby completing the proof of the theorem.
Remark 21. The elliptic Eisenstein series can be viewed as the automorphic kernel associated to the point-pair invariant

\[ k_{\text{ell}}^G(z, w) := \sinh(d_{\text{hyp}}(z, w))^{-s}. \]  

Having in mind relation (13), one may ask whether there is a more direct approach to the wave representation of the elliptic Eisenstein series by computing the Selberg/Harish-Chandra transform of (46) (see subsection 2.3). To do this, we first write \( k_{\text{ell}}^G(\alpha(z, w)) = k_{\text{ell}}^G(\alpha) = 2^{-s}u^{-s/2}(u+1)^{-s/2}, \) where \( u = u(z, w) \) is defined by (3), and get

\[ Q^G_s(v) := \int_{\nu} k_{\text{ell}}^G(\alpha) \text{d}u = 2^{-s} \int_{0}^{\nu} u^{-\frac{s}{2}}(u+v)^{-s/2}(u+v+1)^{-s/2} \text{d}u \]

\[ = \frac{2^{-s} \sqrt{\pi} \Gamma(s - \frac{1}{2}) (v+1)^{1/2-s/2}}{\Gamma(s) v^{s/2}} F \left( \frac{1}{2}, \frac{s}{2}; s; \frac{1}{v} \right); \]

here, for the last equality we used formula 3.197.1. of [GR07] with \( \beta = v, \gamma = v+1, \nu = 1/2, \mu = \rho = s/2 \) keeping in mind that \( \text{Re}(s)/2 > \text{Re}(1/2 - s/2). \) Finally, we get

\[ g^G_s(u) := 2 Q^G_s(\sinh \left( \frac{u}{2} \right)^2) = \frac{2^{-s} \sqrt{\pi} \Gamma(s - \frac{1}{2}) \cosh \left( \frac{u}{2} \right)^{1-s}}{\sinh \left( \frac{u}{2} \right)^s} F \left( \frac{1}{2}, \frac{s}{2}; s; \frac{1}{\cosh \left( \frac{u}{2} \right)^2} \right) \]

where the last formula follows from formula 9.134.1. of [GR07] with \( \alpha = 1/2, \beta = s/2 \) and \( z = -1/(\sinh^2(u/2)). \) Thus, for \( u \in \mathbb{R}^+ \) with \( u > 0, \) we have \( g^G_s(u) = G_{s,0}(u). \) However, the functions \( G_{s,0}(u) \) and \( g^G_s(u) \) are not defined for \( u = 0 \) for \( \text{Re}(s) > 1. \) Furthermore, due to the growth of \( \tanh^{-s}(u) \) as \( u \downarrow 0, \) the Selberg/Harish-Chandra transform of (46) does not exist for \( \text{Re}(s) > 2. \) Therefore, it is not possible to provide the above mentioned direct approach.

Remark 22. As proved in [vP10], for any \( z, w \in M \) with \( z \neq w, \) the elliptic Eisenstein series \( E_{w}^G(z, s) \) admits a meromorphic continuation to the whole complex \( s \)-plane. The main step of this proof consists of using relation (33) and writing, now in terms of the wave representation,

\[ \text{ord}(w) E_{w}^G(z, s) = \sum_{k=0}^{n} \left( \frac{z}{k!} \right) W_{M, \frac{1}{2}}(z, w)(g_{s+2k}) + \sum_{k=n+1}^{\infty} \left( \frac{z}{k!} \right) W_{M, \frac{1}{2}}(z, w)(g_{s+2k}) \]

for any positive integer \( n. \) The second series in (47) represents a holomorphic function for \( \text{Re}(s) > 1 - 2n; \) the first sum in (47) admits a meromorphic continuation to the half-plane \( \text{Re}(s) > 1 - 2n \) as a consequence of Theorem 10.

7 The wave representation of parabolic Eisenstein series

We now develop a representation of the parabolic Eisenstein series (1) in terms of the wave distribution. We start by recalling that the parabolic Eisenstein arises in the zeroth coefficient of the Fourier expansion of the following automorphic kernel (see, e.g., [He83], [Iwa02]).

Definition 23. The hyperbolic Green’s function \( G_s(z, w) \) is defined for \( z, w \in M \) with \( z \neq w, \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1, \) by the following series

\[ G_s(z, w) := \frac{1}{2\pi} \sum_{\eta \in \mathcal{F}} Q_{s-1}(1 + 2u(z, \eta w)); \]

where \( Q_{\nu}(\cdot) \) denotes the Legendre function of the second kind and \( u(z, w) \) given by (3).

We note that \( G_s(z, w) \) as a function of \( z \) (with \( w \) fixed), or as a function of \( w \) (with \( z \) fixed), is not a \( L^2 \)-function on \( M. \)
**Proposition 24.** Let $E_{p_j}^\text{par}(z,s)$ be the parabolic Eisenstein series associated to the cusp $p_j$ with scaling matrix $\sigma_{p_j} \in \text{PSL}_2(\mathbb{R})$ ($j = 1, \ldots, p_f$). Furthermore, let $L_{p_j}:= \sigma_{p_j}L_a$, where $L_a$ is the horocycle given by $L_a := \{z \in \mathbb{H} \mid \text{Re}(z) \in [0,1], \text{Im}(z) = a\}$. Then, for $z \in M$, $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, and $a \in \mathbb{R}$ with $a > \text{Im}(\eta z)$ for any $\eta \in \Gamma$, we have the identity

$$E_{p_j}^\text{par}(z,s) = (2s-1)\text{Im}(w)^{s-1} \int_{L_{p_j}} G_s(z,w)ds_{\text{hyp}}(w).$$

**Proof.** From the Fourier expansion of the hyperbolic Green’s function with respect to the variable $w$ (see, e.g., [Iwa02], Theorem 5.3.), we deduce under the above assumptions

$$\int_0^1 G_s(z,\sigma_p(x + iy'))dx' = (2s-1)^{-1/2} \zeta(1-s) \sum_{k \geq 1} \frac{1}{k^s} \text{Re}(z)^{s-1} \frac{1}{\Gamma(s)},$$

which after the change of coordinates $w = x' + iy' \rightarrow \sigma_p w$ yields the assertion. \(\Box\)

From Proposition 24, it suffices to represent the automorphic kernel $G_s(z,w)$ in terms of the wave distribution. To do so, we first derive the following general statement.

**Proposition 25.** Let $k(z,w)$ be a point-pair invariant function on $\mathbb{M} \times \mathbb{M}$ and let us write $k(z,w) = k(u(z,w)) = k(u)$ as a function of $u$ with $u(z,w)$ given by (3). Suppose that the Selberg/Harish-Chandra transform $h(r)$ of $k(u)$ exists and satisfies conditions (S1) and (S2) of subsection 2.3 together with the bound $h(r) = O \left( (1 + |r|)^{-2} \right)$ as $\text{Im}(r) \rightarrow \pm \infty$ in the domain of condition (S2). Furthermore, assume that the automorphic kernel $K(z,w)$ associated to $k(u)$ can be realized as

$$K(z,w) = \frac{1}{4\pi} \sum_{\eta \in \Gamma} \frac{1}{|\eta|^2} \int_{\text{Re}(\nu) = 1+\delta} \text{Re}(\nu)^{1/2} \Gamma(\nu/2) h(i(1/2 - \nu)) \mathcal{K}_\nu(z,w) d\nu,$$

where $P_{-1/2+ir}(\cdot)$ denotes the Legendre function of the first kind. Then, for some $0 < \delta < 1/2$, and $z,w \in \mathbb{M}$ with $z \neq w$, we have the representation

$$K(z,w) = \frac{1}{2\pi^{3/2} i} \int \frac{2^{-\nu} \Gamma(\nu/2)}{\Gamma(\nu - 1/2)} h \left( i \left( \frac{1}{2} - \nu \right) \right) K_{\nu+2k}(z,w) d\nu,$$

where we have set

$$K_{\nu+2k}(z,w) := \sum_{k=0}^{\infty} \frac{b_k(\nu)}{k!} K_{\nu+2k}(z,w)$$

with $b_k(\nu) := \left( \frac{\nu}{2} \right)_k \left( \frac{\nu+1}{2} \right)_k / \left( \frac{1}{2} \right)_k$.

**Proof.** To begin, we note that

$$P_{-1/2+ir}(\cosh(d_{\text{hyp}}(z,\eta w))) = P_{-1/2+ir}(1 + 2u(z,\eta w)).$$

Set $u := u(z,\eta w) > 0$. Using formula 8.820.4 from [GR07], we can represent the Legendre function as

$$P_{-1/2+ir}(1 + 2u) = (1 + u)^{-1/2+ir} F \left( \frac{1}{2} - ir, \frac{1}{2} - ir; 1; \frac{u}{u+1} \right).$$

The application of formula 9.134.2 from [GR07] then yields

$$P_{-1/2+ir}(1 + 2u) = (1 + 2u)^{-1/2+ir} F \left( \frac{1}{4} - \frac{ir}{2}, \frac{3}{4} - \frac{ir}{2}; 1; \frac{(2u+1)^2 - 1}{(2u+1)^2} \right).$$
Formula 9.131.2 from [GR07], together with the duplication formula for the $\Gamma$-function, then implies that the function $P_{\frac{1}{2} + ir}(1 + 2u)$ can be represented as the sum of two hypergeometric functions, namely, we have that

$$P_{\frac{1}{2} + ir}(1 + 2u) = H\left(\frac{1}{2} + ir, u\right) + H\left(\frac{1}{2} - ir, u\right)$$  \hspace{1cm} (49)

with

$$H(\nu, u) := \frac{2^\nu \Gamma(1 - 2\nu)}{\Gamma(\nu - \nu^2)} (1 + 2u)^{-\nu} F\left(\nu, \frac{\nu + 1}{2}; \nu + 1; \frac{1}{(1 + 2u)^2}\right).$$

Hence, substituting (49) with $u = u(z, \eta w)$ into (48) and observing that the function $r \tanh(\pi r) h(r)$ is even in $r$, we derive the representation

$$K(z, w) = \frac{1}{2\pi} \sum_{\eta \in \Gamma} \int_{-\infty}^{\infty} r \tanh(\pi r) h(r) H\left(\frac{1}{2} + ir, u(z, \eta w)\right) dr.$$  \hspace{1cm} (50)

By combining formulas 8.332.1 and 8.332.2 from [GR07], we obtain the identity

$$r \tanh(\pi r) = \frac{\Gamma\left(\frac{1}{2} + ir\right) \Gamma\left(\frac{1}{2} - ir\right)}{\Gamma(\pi r) \Gamma(-ir)}.$$  

Using this formula, together with the duplication formula for the $\Gamma$-function, we obtain the equality

$$r \tanh(\pi r) H\left(\frac{1}{2} + ir, u\right) = \frac{2^{-1/2 - \nu} \Gamma\left(\frac{1}{2} + ir\right)}{\sqrt{\pi} \Gamma(\pi r)} (1 + 2u)^{-\frac{1}{2} - ir} F\left(\frac{1}{4} + \frac{ir}{2}, \frac{1}{2}; 1 + ir; \frac{1}{(1 + 2u)^2}\right).$$

Substituting $\nu := 1/2 + ir$ in (50) therefore gives

$$K(z, w) = \sum_{\eta \in \Gamma} \int_{\text{Re}(\nu) = 1/2} c(\nu) \frac{h(i\left(\frac{1}{2} - \nu\right))}{(1 + 2u(z, \eta w))^\nu} F\left(\nu, \frac{\nu + 1}{2}; \nu + 1; \frac{1}{(1 + 2u(z, \eta w))^2}\right) d\nu.$$  \hspace{1cm} (51)

with

$$c(\nu) := \frac{1}{2^{1/2} \pi^{\nu/2} \Gamma(\nu - 1/2)}.$$

It is immediate that the integrand in (51) is a holomorphic function for $1/2 \leq \text{Re}(w) \leq 1 + \delta < 3/2$. The bound on the function $h$ and the Stirling formula for the $\Gamma$-function imply that we may apply Cauchy’s formula and move the line of integration in (51) to the line $\text{Res} = 1 + \delta$, thus obtaining that

$$K(z, w) = \sum_{\eta \in \Gamma} \int_{\text{Re}(\nu) = 1 + \delta} c(\nu) \frac{h(i\left(\frac{1}{2} - \nu\right))}{\cosh(d_{\text{hyp}}(z, \eta w))^\nu} F\left(\nu, \frac{\nu + 1}{2}; \nu + 1; \frac{1}{\cosh(d_{\text{hyp}}(z, \eta w))^2}\right) d\nu.$$  \hspace{1cm} (52)

Let $\tilde{C} > 1$ be a constant such that $\tilde{C} < \cosh(d_{\text{hyp}}(z, \eta w))$ for any $\eta \in \Gamma$, as in the proof of Lemma 19. Since $\text{Re}\left(\nu + \frac{\nu}{2} + \frac{1}{2} - \nu - \frac{1}{2}\right) = 0$ and $\cosh(d_{\text{hyp}}(z, \eta w))^{-2} < \tilde{C}^{-2} < 1$, the hypergeometric function in (52) converges uniformly in $\nu$ and is bounded as a function of $\nu$ and $\eta \in \Gamma$. Therefore, the Stirling formula for the gamma function and the bound for the function $h$ imply that the integrand in (52) is uniformly bounded by

$$\frac{1}{(1 + |\nu|)^{1/2}} \cosh(d_{\text{hyp}}(z, \eta w))^{-1 - \delta}.$$  \hspace{1cm} (53)

Therefore, the series in (52) is majorized by the automorphic kernel $K_{1+\delta}(z, w)$, which allows us to interchange the sum and the integral in (52).

Finally, arguing in the same manner, we deduce that

$$\sum_{\eta \in \Gamma} \cosh(d_{\text{hyp}}(z, \eta w))^{-\nu} F\left(\nu, \frac{\nu + 1}{2}; \nu + 1; \frac{1}{\cosh(d_{\text{hyp}}(z, \eta w))^2}\right) = \mathcal{K}_\nu(z, w),$$

for $\text{Re}(\nu) = 1 + \delta > 1$. This completes the proof. \hfill \Box
Corollary 26. Let $\delta \in (0, 1/2)$. Then, for $z, w \in M$ with $z \neq w$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, we have the representation

$$G_s(z, w) = \int_{\text{Re}(\nu) = 1 + \delta} \frac{c(\nu)}{(s - \frac{1}{2})^2 - (\nu - \frac{1}{2})^2} \mathcal{K}_\nu(z, w) d\nu,$$

where $b_k(\nu) := \left(\frac{\nu}{2}\right)_k \left(\frac{\nu + 1}{2}\right)_k / (\nu + \frac{1}{2})_k$ and with $c(\nu) := 2^{-\nu-1}\Gamma(\nu) / (\Gamma(\nu - \frac{1}{2}) \pi^{3/2} i)$.

Proof. First, we note that $Q_{s-1}(1 + 2u(z, w)) = Q_{s-1}(\cosh(d_{hyp}(z, w)))$. Hence, by formula 7.213 from [GR07] with $a = s - 1/2$ and $b = d_{hyp}(z, w)$, we have, for $\text{Re}(s) > 1/2$, the equality

$$Q_{s-1}(1 + 2u(z, w)) = \frac{1}{2} \int_{-\infty}^{\infty} r \tanh(\pi r) h(r) P_{-\frac{1}{4} + i r} (\cosh(d_{hyp}(z, w))) dr$$

with

$$h(r) := \frac{1}{(s - \frac{1}{2})^2 + r^2}.$$ 

The function $h(r)$ is obviously even and holomorphic in the strip $|\text{Im}(r)| \leq \frac{1}{2} + \epsilon$, where $\epsilon > 0$ is such that $\epsilon < \text{Re}(s) - 1/2$. Therefore, $h(r)$ satisfies conditions (S1) and (S2) of subsection 2.3. Moreover, we have the bound $h(r) = O((1 + |r|)^{-2})$, as $\text{Im}(r) \to \pm \infty$ in the domain of condition (S2). With all this, the conditions of Proposition 25 are satisfied and the asserted representation can be immediately derived.

□

Definition 27. For $\nu, \beta \in \mathbb{C}$ with $\text{Re}(\beta) > \text{Re}(\nu) - 1 > 0$, and $u \in \mathbb{R}^+$, we set

$$\tilde{G}_{\nu, \beta}(u) := g_\nu(u) \tanh(u)^\beta F\left(\frac{\nu}{2}, \frac{\nu}{2} + \frac{1}{2}; \frac{1}{2}; \frac{1}{\cosh(u)^2}\right),$$

where $g_\nu(u)$ is given by (24).

Theorem 28. Let $\nu, \beta \in \mathbb{C}$ with $\text{Re}(\beta) > \text{Re}(\nu + 1) > 2$.

(i) For $z, w \in M$, the wave distribution $\mathcal{W}_{M, 1/4}(z, w)(\tilde{G}_{\nu, \beta})$ is well-defined and holomorphic as a function of $\beta$.

(ii) For $z, w \in M$ with $z \neq w$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, the wave distribution $\mathcal{W}_{M, 1/4}(z, w)(\tilde{G}_{\nu, \beta})$ admits a holomorphic continuation in $\beta$ to $\{\beta \in \mathbb{C} | \text{Re}(\beta) > 0\} \cup \{0\}$, we have the representation

$$G_s(z, w) = \int_{\text{Re}(\nu) = 1 + \delta} \left. \frac{c(\nu)}{(s - \frac{1}{2})^2 - (\nu - \frac{1}{2})^2} \mathcal{W}_{M, 1/4}(z, w)(\tilde{G}_{\nu, \beta}) \right|_{\beta = 0} d\nu,$$

for some $\delta > 0$ sufficiently small, and with $c(\nu)$ defined in Corollary 26.

Proof. The proof is analogous to the method of proof of Theorem 20 in section 6.

Part (i): We first employ the equality

$$\tilde{G}_{\nu, \beta}(u) = \sum_{k=0}^{\infty} \left(\frac{u}{2}\right)_k \left(\frac{u + 1}{2}\right)_k k! \left(\nu + \frac{1}{2}\right)_k g_{\nu + 2k, \beta}(u),$$

where $g_{\nu, \beta}(u)$ is given by (34). Along the lines of the proof of Lemma 17 and Lemma 18, we then derive, for $\nu, \beta \in \mathbb{C}$ with $\text{Re}(\nu) > 1$ and $\text{Re}(\beta) > \text{Re}(\nu) + 8$, the relation

$$H(r, \tilde{G}_{\nu, \beta}) = \sum_{k=0}^{\infty} \left(\frac{u}{2}\right)_k \left(\frac{u + 1}{2}\right)_k k! \left(\nu + \frac{1}{2}\right)_k H(r, g_{\nu + 2k, \beta});$$
here, for $\Re(\nu) = 1 + \delta$, we employed the bound
\[
\left| \left( \frac{\nu}{k} \right)_k \left( \frac{\nu + \frac{1}{2}}{k} \right)_k \right| \leq \left| \left( \frac{\nu}{k} \right)_k \right| \frac{1}{k!} \lesssim \frac{1}{k!},
\]
as $k \to \infty$. Finally, applying Lemma 19 and following the lines of the proof of Theorem 20, we deduce the statement (i).

Part (ii): Again, following lines of the proof of Theorem 20 (ii), we deduce that, for $z, w \in M$ with $z \neq w$, the wave distribution $W_{M, \frac{1}{4}}(z, w)(\widehat{G}_{\nu, \beta})$ admits a holomorphic continuation in $\beta$ to \( \{ \beta \in \mathbb{C} \mid \Re(\beta) > 0 \} \cup \{0\} \) such that
\[
W_{M, \frac{1}{4}}(z, w)(\widehat{G}_{\nu, \beta}) |_{\beta = 0} = \sum_{k=0}^{\infty} \frac{b_k(\nu)}{k!} K_{\nu+2k}(z, w) = K_{\nu}(z, w).
\]
Hence, the statement of Corollary 26 completes the proof of the theorem.

Combining Theorem 28 with Proposition 24 yields the wave representation of the parabolic Eisenstein series.

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