QUANTITATIVE SMALL SUBGRAPH CONDITIONING

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Abstract. We revisit the method of small subgraph conditioning, used to establish that random regular graphs are Hamiltonian a.a.s. We refine this method using new technical machinery for random $d$-regular graphs on $n$ vertices that hold not just asymptotically, but for any values of $d$ and $n$. This lets us estimate how quickly the probability of containing a Hamiltonian cycle converges to 1, and it produces quantitative contiguity results between different models of random regular graphs. These results hold with $d$ held fixed or growing to infinity with $n$. As additional applications, we establish the distributional convergence of the number of Hamiltonian cycles when $d$ grows slowly to infinity, and we prove that the number of Hamiltonian cycles can be approximately computed from the graph’s eigenvalues for almost all regular graphs.

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1. Introduction

The uniform model $G_{n,d}$ of random regular graph of degree $d$ on $n$ vertices is the setting for many celebrated theorems concerning discrete random structures, and much is known about it. For an excellent survey of random regular graphs, consider [Wor99]. In a line of work going back to Fenner and Frieze [FF84], Bollobás [Bol83], Frieze [Fri88], it was settled finally by Robinson and Wormald [RW92, RW94] that a uniformly chosen $d$-regular graph was a.a.s. Hamiltonian as $n \to \infty$. The techniques of Bollobás, Fenner and Frieze are algorithmic, while the work of Robinson and Wormald follows a second-moment method approach together with what is known as small subgraph conditioning. The combined efforts of [FJM+96] show that there are many Hamiltonian cycles a.a.s. and produce an algorithm that finds them a.a.s.
These results are concerned with holding \( d \) fixed and letting \( n \) tend to infinity. An alternative is to allow \( d = d(n) \) to vary with \( n \), possibly growing to infinity at some rate. Along this line of reasoning, it is shown in \cite{CFR02} that there is a constant \( c > 0 \) so that if \( d_0 \leq d(n) \leq cn \), the graph is Hamiltonian a.a.s. By a different approach, it is shown in \cite{KSVW01} that if \( d(n) \geq \sqrt{n} \log n \) then the graph is Hamiltonian a.a.s.

All of these techniques are ultimately asymptotic, in the sense that they show a graph feature holds with some probability tending to 1. In this paper, we will show how the small subgraph conditioning method can be used to produce estimates that do not just hold in the limit as \( n \to \infty \) but hold for all \( n \) and \( d \) simultaneously. In particular, we extend the method of small subgraph conditioning to the regime where \( d = d(n) \) may grow to infinity, and we are principally interested in the regime in which \( \log d / \log n \to 0 \).

As with much work on the uniform model, we actually work with the configuration model (or pairing model) \( \mathcal{P}_{n,d} \). In this model, \( nd \) balls are partitioned into \( n \) bins of equal size, noting this requires \( nd \) to be even. A matching of all the balls is chosen uniformly at random, and then the balls in each bin are identified to form vertices. All the edges are preserved in the identification to produce a \( d \)-regular pseudograph, which we call the projection of the pairing. For definiteness, we will refer to the balls as prevertices, which are partitioned into \( n \) vertex bins of size \( d \), and we will reserve typical graph nomenclature for the projected pseudograph.

One central object of study is \( H_n \), the number of Hamiltonian cycles in a pseudograph \( G \), which is a projection of a matching sampled from \( \mathcal{P}_{n,d} \). Because \( H_n \) is independent of the orderings of the prevertices within vertex bins, \( H_n \) is equivalently considered a function of the pairing or the regular pseudograph.

Our first theorem gives a bound on the probability that \( G \) contains no Hamiltonian cycles.

**Theorem 1.1.** Suppose that \( d = d(n) \geq 4 \) satisfies \( \log d / \log n \to 0 \). For every \( \epsilon > 0 \),

\[
\Pr_{\mathcal{P}_{n,d}}[H_n = 0] = O(n^{-1/3+\epsilon}).
\]

If \( d \geq 3 \) and \( d^2 / \log n \to 0 \), then for every \( \epsilon > 0 \),

\[
\Pr_{\mathcal{G}_{n,d}}[H_n = 0] = O(n^{-1/3+\epsilon}).
\]

Note that when \( d = 3 \), the theorem is not true for \( \mathcal{P}_{n,d} \). A self-loop anywhere in the graph obstructs the existence of Hamiltonian cycles, and the number of self-loops is asymptotically Poisson(1) (see Corollary 1.8). Whereas previous results show that these probabilities are \( o(1) \), the novelty here is the establishment of a rate. Previous work of \cite{CFR02} shows that for \( d \geq d_0 \) large, this probability is at most \( O(n^{-2}) \), but the approach taken in that paper is unlikely to extend to all \( d \geq 3 \). It remains an open question to determine the true rate, or even to determine if the rate decays as a power of \( n \).

### 1.1. Contiguity.

After the initial developments by Robinson and Wormald, Janson further developed small subgraph conditioning \cite{Jan95} by showing something known as contiguity. Two sequences of laws \( P_n \) and \( Q_n \) on a common measurable space are contiguous if for any sequence \( A_n \) of measurable events, \( P_n(A_n) \to 0 \iff Q_n(A_n) \to 0 \), which is a sort of qualitative asymptotic equivalence between the two models. Contiguity has proven useful in that it allows difficult estimates, such as Friedman’s second eigenvalue bounds \cite{Fri03}, to be made for a regular graph model of choice and then transferred to other models. Contiguity alone, however, gives only an asymptotic estimate for the probabilities in one model based on the probabilities in the other.

Beyond generalizing small subgraph conditioning to growing \( d \), we seek to understand how nearly estimates for a probability in one random regular graph model transfer to another. We will initiate this study for \( \mathcal{P}_{n,d} \) and a second graph model pertinent to studying \( H_n \) in \( \mathcal{P}_{n,d} \). We define the model \( \mathcal{F}_{n,d} \) that, as a pseudograph model, can be considered as a degree \( d - 2 \) regular pseudograph induced from \( \mathcal{P}_{n,d-2} \) with a superimposed, independent and uniformly chosen Hamiltonian cycle. At the pairing level, we define it by adding two prevertices to each bin of \( \mathcal{P}_{n,d-2} \), sampling a uniform matching of these new prevertices conditioned to project to a Hamiltonian cycle, and then randomizing the ordering the prevertices in each bin. Formally, we consider both \( \mathcal{F}_{n,d} \) and \( \mathcal{P}_{n,d} \) as laws of pairings on \( nd \) prevertices, and we refer to a pairing event as any set of these pairings. Further, the law of \( \mathcal{F}_{n,d} \) is absolutely continuous with respect to \( \mathcal{P}_{n,d} \), and the Radon-Nikodym derivative is precisely \( H_n/EH_n \).

Thus, small subgraph conditioning actually shows that \( \mathcal{F}_{n,d} \) and \( \mathcal{P}_{n,d} \) are contiguous. As \( \mathcal{F}_{n,d} \) always has a Hamiltonian cycle, the consequence that \( \mathcal{P}_{n,d} \) is Hamiltonian a.a.s. would now follow directly from the contiguity of the models, which we extend to the case of growing \( d \).
Theorem 1.2. Suppose that \( d = d(n) \) satisfies \( 4 \leq d \leq n^{\alpha_0 - \epsilon} \) where \( \alpha_0 = \frac{8}{3(8 + \sqrt{83})} \approx 0.283 \), for some \( \epsilon > 0 \). Then for any sequence of pairing events \( A_n \),

\[
\Pr_{\mathcal{F}_{n,d}} [A_n] \to 0 \iff \Pr_{\mathcal{G}_{n,d}} [A_n] \to 0.
\]

If in addition \( d \to \infty \), then \( dTV(\mathcal{F}_{n,d}, \mathcal{G}_{n,d}) \to 0 \).

By conditioning the pairings to project to simple graphs, these results can be transferred to the uniform model. This requires that we introduce \( \mathcal{F}_{n,d} \), which is \( \mathcal{F}_{n,d} \) conditioned to project to a simple graph. Note that on conditioning, \( H_n \) is still the Radon-Nikodym derivative between \( \mathcal{F}_{n,d}^* \) and \( \mathcal{G}_{n,d} \) up to renormalization. We are not able to show this same sort of general contiguity statement for \( \mathcal{F}_{n,d}^* \) and \( \mathcal{G}_{n,d} \) when \( d \to \infty \). However, we do show that a certain type of quantitative contiguity does transfer.

Theorem 1.3. Suppose that \( d \geq 4 \) and \( \log d / \log n \to 0 \), and suppose that \( A_n \) is some sequence of pairing events. Let \( \alpha > 0 \) be fixed. Then,

\[
\Pr_{\mathcal{F}_{n,d}} [A_n] = O(n^{-\alpha}) \implies \Pr_{\mathcal{G}_{n,d}} [A_n] = O(n^{-\beta + \epsilon}) \quad \forall \epsilon > 0,
\]

where \( \beta = \alpha \wedge \frac{1}{3} \). Likewise,

\[
\Pr_{\mathcal{F}_{n,d}} [A_n] = O(n^{-\alpha}) \implies \Pr_{\mathcal{G}_{n,d}} [A_n] = O(n^{-\beta + \epsilon}) \quad \forall \epsilon > 0,
\]

where \( \beta = \alpha \wedge \frac{1}{3} \). In particular

\[
\Pr_{\mathcal{F}_{n,d}} [A_n] = O(n^{-1/3 + \epsilon}) \quad \forall \epsilon > 0 \iff \Pr_{\mathcal{G}_{n,d}} [A_n] = O(n^{-1/3 + \epsilon}) \quad \forall \epsilon > 0.
\]

If we additionally assume that \( A_n \) consists only of pairings that project to simple graphs, then we may assume \( d \geq 3 \). The same results hold with \( \mathcal{F}_{n,d} \) replaced by \( \mathcal{F}_{n,d}^* \) and \( \mathcal{G}_{n,d} \) replaced by \( \mathcal{G}_{n,d} \) for \( 3 \leq d = o(\sqrt{\log n}) \).

Note that Theorem 1.1 is an immediate consequence of Theorem 1.3.

1.2. Other applications. The machinery developed here has further applications beyond the contiguity results. In [Jan95], the limiting distribution of \( H_n \) is derived for \( d \) fixed and \( n \to \infty \). We can derive the distributional convergence of \( H_n \) in the \( d \to \infty \) regime. As expected, its logarithm is asymptotically normal.

Theorem 1.4. If \( d \to \infty \) slowly enough that \( \log d / \log n \to 0 \), then with \( P \sim \mathcal{P}_{n,d} \),

\[
\frac{\log H_n(P) - \log \mathbb{E}H_n(P)}{\sqrt{2/d}} \Rightarrow N(0, 1).
\]

In fact, it can be seen that \( H_n / \mathbb{E}H_n \) is well-approximated by a multiple of the number of self-loops in the \( d \to \infty \) regime, by virtue of which the normality follows. Better approximations for \( H_n / \mathbb{E}H_n \) can be obtained by using more cycle information. Also, in a sufficiently sparse regime, cycle counts can be computed from the graph’s eigenvalues with high probability. This allows the Hamiltonian cycle count to be well-approximated by an easily computable trace, for almost all regular graphs:

Theorem 1.5. Suppose that \( 3 \leq d \leq n^{1/12} \). There is a polynomial \( \Pi_{d,n}(x) \), given in (70), such that

\[
\Pr_{\mathcal{P}_{n,d}} \left[ \left| \frac{H_n}{\mathbb{E}H_n} - \exp(\text{tr} \Pi_{d,n}(P)) \right| > n^{-1/12} \right] = O((\log n)^2 n^{-1/6}),
\]

where \( \text{tr} \Pi_{d,n}(P) = \sum_{i=1}^n \Pi_{d,n}(\lambda_i / \sqrt{d-1}) \), with \( \lambda_1, \ldots, \lambda_n \) the eigenvalues of the adjacency matrix of \( P \).

1.3. Small subgraph conditioning. To introduce the method, we will show how small subgraph conditioning can be used to estimate \( \Pr[H_n = 0] \). One possible first instinct is to apply the second moment method, but there is the unfortunate difficulty that \( \text{Var} H_n \) is the same order as \( (\mathbb{E}H_n)^2 \). The miracle is that most of the variance can be understood as arising from short cycles.

Let \( X_r \) be the number of cycles of length \( r \) in the induced pseudograph, so that

\[
\frac{f_{r,n}}{3} := \mathbb{E} \left[ \frac{H_n}{\mathbb{E}H_n} X_1, X_2, \ldots, X_r \right]
\]
is the Radon-Nikodym derivative of the cycle-count vector of $\mathcal{P}_{n,d}$ with respect to the cycle-count vector of $\mathcal{P}_{n,d}$. It can be shown that the cycle counts in both models are asymptotically vectors of independent Poissons. Thus from Fatou’s lemma, one can calculate

$$V_r := \lim_{n \to \infty} \text{Var} \left( \mathbb{E} \left[ \frac{H_n}{\mathbb{E}H_n} X_1, X_2, \ldots, X_r \right] \right)$$

solely using the limiting Poisson structure. On the other hand, an explicit variance calculation shows that

$$V_\infty := \lim_{n \to \infty} \frac{\text{Var} H_n}{(\mathbb{E}H_n)^2} = \lim_{r \to \infty} V_r,$$

which in a sense says that the two graph models $\mathcal{J}_{n,d}$ and $\mathcal{P}_{n,d}$ conditioned to have the same short cycle counts are asymptotically indistinguishable. Note that these limits being equal is not simply a question of reversing the order of the $r$ and $n$ limits; it asserts, moreover, that the cycle count $\sigma$-algebra asymptotically determines the Radon-Nikodym derivative.

Then, for any $r$ and $\epsilon > 0$ one has the bound

$$\Pr[ H_n = 0 ] \leq \Pr[f_{r,n} \leq \epsilon] + \Pr \left[ \left| \frac{H_n}{\mathbb{E}H_n} - f_{r,n} \right| \geq \epsilon \right]$$

$$\leq \Pr[f_{r,n} \leq \epsilon] + \frac{(\text{Var} H_n)/(\mathbb{E}H_n)^2 - \text{Var} f_{r,n}}{\epsilon^2},$$

where the bound follows from Chebyshev’s inequality and the properties of conditional expectation. Taking the limit supremum,

$$\limsup_{n \to \infty} \Pr[ H_n = 0 ] \leq \lim_{n \to \infty} \Pr[f_{r,n} \leq \epsilon] + \frac{V_\infty - V_r}{\epsilon^2}.$$

From the limiting Poisson structure of the cycle counts, this limiting probability has an explicit form, and it is now a calculation to choose $r$ and $\epsilon$ appropriately to make this bound as small as desired.

1.4. Quantitative estimate. We essentially follow the approach outlined above in the classic small subgraph conditioning method. The two innovations necessary to produce a rate in this argument are a variance estimate of $H_n$ that holds for a large range of $n$ and $d$ simultaneously and an estimate on how nearly Poisson are the cycle counts. The remainder of the work is to make estimates of the conditioned Radon-Nikodym derivative using the Poisson approximations.

Because of the nature of our Poisson approximation, we will also change the $\sigma$-algebra used in the conditioning. Roughly speaking, we will keep track of not only how many cycles appear but where they appear as well. As always, we work in the pairing model. By a cycle in a pairing $P$, we mean a collection of pairs that projects down to a cycle in the pseudograph.

Let $\mathcal{J}_k$ be the set of all possible cycles of length $k$ that could appear in an instance of $\mathcal{P}_{n,d}$. We note that $|\mathcal{J}_k| = \binom{n}{k} k! (d(k - 1))^{k}/2k$, and that this holds even for $k = 1$ and $k = 2$. For any $\alpha \in \mathcal{J}_k$, define $I_\alpha(P) = 1\{P \text{ contains } \alpha\}$. Further, let $L_\alpha = \prod_{k=1}^r (I_\alpha(P))$, $\alpha \in \mathcal{J}_k$. Our main approximation theorem says that $L_\alpha(P)$ is well-approximated by a vector of independent Poissons, with $P$ drawn from either $\mathcal{P}_{n,d}$ or $\mathcal{J}_{n,d}$. These Poisson vectors have slightly different means, and this difference will ultimately account for the dominant term in the variance of $H_n$. We will use $\lambda_\alpha$ and $\mu_\alpha$ to denote the approximate means of $I_\alpha(P)$ with $P$ drawn from $\mathcal{P}_{n,d}$ or $\mathcal{J}_{n,d}$ respectively. So, we define, for $\alpha \in \mathcal{J}_k$,

$$\lambda_\alpha := \frac{1}{(nd)^k}$$

$$\mu_\alpha := \frac{1}{(nd)^k} + \frac{(-1)^k - 1}{(nd(d - 1))^{k}}.$$

Let $Z = \prod_{k=1}^r (Z_\alpha, \alpha \in \mathcal{J}_k)$ be a vector whose coordinates are independent Poisson random variables with $\mathbb{E}Z_\alpha = \lambda_\alpha$ for $\alpha \in \mathcal{J}_k$. Likewise, let $\tilde{Z} = \prod_{k=1}^r (\tilde{Z}_\alpha, \alpha \in \mathcal{J}_k)$ be a vector whose coordinates are independent Poisson random variables with $\mathbb{E}\tilde{Z}_\alpha = \mu_\alpha$ for $\alpha \in \mathcal{J}_k$.

A typical distributional approximation theorem between $L_\alpha(P)$ and $Z$ might be given as a bound between their laws in some probability metric, such as the total variation distance. This is not quite enough for all of our purposes. Especially when it comes to estimating the variance of the conditional Radon-Nikodym
derivative (see Lemma [5.1]), we need better control over the point probabilities of the law of \( \mathbf{I}_r(P) \) for a pairing \( P \) from either \( \mathcal{P}_{n,d} \) or \( \mathcal{I}_{n,d} \). Thus by modifying the standard Stein’s method machinery, we seek to show that for a fixed \( \{0,1\} \)-vector \( x \),
\[
\frac{\Pr[\mathbf{I}_r(P) = x]}{\Pr[\mathbf{Z} = x]} \approx 1.
\]
We are not able to do this for all \( x \): indeed, \( \Pr[\mathbf{I}_r(P) = x] \) is zero for some choices of \( x \). We restrict ourselves to vectors that neither imply too many cycles nor imply cycles that overlap. Specifically, we estimate the ratio for cycle configurations that are strictly \( \lambda \)-neat, as defined below.

**Definition 1.6.** For some \( \lambda \geq 1 \), a \( \{0,1\} \)-vector \( x = (x_\alpha, \alpha \in J) \) is strictly \( \lambda \)-neat if the following hold:

i) If \( x_\alpha = x_\beta = 1 \), then \( \alpha \) and \( \beta \) do not share a vertex in the graph projection.

ii) The total number of prevertices contained in \( x \), given by
\[
\sum_{k=1}^r \sum_{\alpha \in J_k} 2k x_\alpha,
\]
is at most \( \lambda(d-1)^r \).

We now present our Poisson approximation theorem.

**Proposition 1.7.** There is an absolute constant \( C_1 \) such that for all strictly \((\log n)\)-neat \( x \) and all \( d \geq 3 \), \( r \geq 4 \), and \( n \) satisfying \( C_1(\log n)^2(d-1)^{2r-1} < n/2 \),
\[
\left| \frac{\Pr[\mathbf{I}_r(P) = x]}{\Pr[\mathbf{Z} = x]} - 1 \right| \leq \frac{C_1(\log n)^2(d-1)^{2r-1}}{n} \quad \text{for } P \sim \mathcal{P}_{n,d},
\]
and
\[
\left| \frac{\Pr[\mathbf{I}_r(\tilde{P}) = x]}{\Pr[\mathbf{Z} = x]} - 1 \right| \leq \frac{C_1(\log n)^2(d-1)^{2r-1}}{n} \quad \text{for } \tilde{P} \sim \mathcal{I}_{n,d}.
\]

To prove this proposition, we develop a variation on Stein’s method. The approach is similar to the method of size-bias couplings for Poisson approximation expounded in [BHJ92], and it also has much in common with the method of switchings used to derive point probability estimates in [MWW04] (see [Wor99, Section 2.4] for a good, general introduction to the method of switchings). We discuss our technique more in Section 3.4.

We also bound the probability that \( \mathbf{I}_r(P) \) or \( \mathbf{I}_r(\tilde{P}) \) is not strictly \((\log n)\)-neat to be of order \((d-1)^{2r}/n\) (see Proposition 3.8). As a consequence, we can derive total variation bounds.

**Corollary 1.8.** There is an absolute constant \( C_2 \) such that for all \( d \geq 3 \) and \( r \geq 4 \)
\[
d_{TV}(\mathbf{I}_r(P), \mathbf{Z}) \leq \frac{C_2(d + (\log n)^2)(d-1)^{2r-1}}{n} \quad \text{for } P \sim \mathcal{P}_{n,d},
\]
and
\[
d_{TV}(\mathbf{I}_r(\tilde{P}), \mathbf{Z}) \leq \frac{C_2(d + (\log n)^2)(d-1)^{2r-1}}{n} \quad \text{for } \tilde{P} \sim \mathcal{I}_{n,d}.
\]

To go with our quantitative Poisson approximations, we will need a quantitative estimate of the second moment of \( H_n \). Let \( f_n = \frac{H_n}{\mathbb{E} H_n} \) be the Radon-Nikodym derivative of the law of \( \mathcal{I}_{n,d} \) with respect to the law of \( \mathcal{P}_{n,d} \). We also define \( f_{r,n} \) to be the Radon-Nikodym derivative of the cycle processes, i.e. \( f_{r,n} := \mathbb{E}[f_n | \sigma(\mathbf{I}_r)] \). For any \( r \), define the limiting second moment expression
\[
\log V^{(r)} := \sum_{k=1}^r \frac{((-1)^k - 1)^2}{2k(d-1)^k}.
\]
Proposition 1.9. For any \( h \) (see Section A) using size-bias coupling. We then compare these expectations by a modification of Stein’s functional on the Markov chain. Thus, we are able to very precisely estimate its difference from a normal so that
\[
E \phi Z \text{ where } Z
\]
and we show that the second moment of \( f \) is approximately \( V(\infty) : \)

\[
\text{Proposition 1.9. For any } \alpha \text{ with } 1 \leq \alpha < 8/\sqrt{2}, \text{ there is a constant } M_\alpha \text{ so that for any } d \leq n^{1/2}/\log n
\]
\[
E_{\phi_n, a}[f_n^2] \leq V(\infty) + M_\alpha d^{2(1+1/\alpha)}/\sqrt{n}.
\]

This differs from previous work, such as in \( [FJM^*96] \), in that we develop a bound that works for a range of \( d \) and \( n \) simultaneously. Our methodology differs significantly from their work, and we develop a semiprobabilistic technique for making the comparison. We show that there is a law \( \phi \) of two-colorings of the edges of a cycle so that
\[
E_{\phi_n, a}[f_n^2] \approx E_{\phi} \sqrt{d/(d-2)} \exp(Z_n^2/d),
\]
where \( Z_n \) is a statistic of the two-coloring that is approximately standard normal. This \( \phi \) has the form of a factor model or graphical model (see \([DM10]\) for an overview of the general theory of these objects). We then show that \( \phi \) is very nearly that joint law on \( \{0,1\}^n \) that would come from a 2-state Markov chain \( \pi \), so that \( E_\phi \exp(Z_n^2/d) \approx E_\pi \exp(Z_n^2/d) \). Under this law, \( Z_n \) can be understood as a centered, scaled additive functional on the Markov chain. Thus, we are able to very precisely estimate its difference from a normal (see Section \( A \) using size-bias coupling. We then compare these expectations by a modification of Stein’s method suitable to comparing expectations of test functions of the form \( h(x) = e^{ax^2} \) for positive \( a \).

1.5. Organization. This paper is organized into four sections and one appendix. We begin in Section \( 2 \) with some preliminary calculations and lemmas that are useful throughout the paper. In Section \( 3 \) we prove the multiplicative Poisson bound Proposition \( 1.7 \) and estimates for the number of non \((\log n)\)-neat graphs. In Section \( 4 \) we prove the variance bound Proposition \( 1.9 \) we do not include the Markov chain estimates here. In Section \( 5 \) we prove the main theorems using the tools developed. Finally, we include Appendix \( A \) in which we prove precise estimates for a 2-state Markov chain.

1.6. Notation. Here and throughout the paper, we use the \( O(\cdot), \Omega(\cdot), \Theta(\cdot) \) to mean something stronger than their usual meaning. We always mean the implied constants are independent of \( d, n, \) and \( r \), and that these bounds hold for all \( d, n, \) and \( r \) in the ranges considered.

We will also make use of the following falling factorial notation. We let \([a]_b \) be the usual falling factorial \([a]_b = a(a-1) \cdots (a-b+1) \). We also use the double falling factorial \([a]_b \) (in analogy with double factorial), which is useful for describing combinatorial quantities arising from matchings. This is given by \([a]_b = (a-1)(a-3) \cdots (a-2b+1) \) with the caveat that \([2n-1]_n = [2n-1]_{n-1} = (2n)!! \). We additionally use the notation \([n] \) to mean the integers \( \{1, 2, \ldots, n\} \), noting that the falling factorial always has a subscript.

2. Supporting tools

Here we collect some important technical tools we will use throughout the paper. Frequently need to make calculations of statistics that are computable in terms of 2-colorings of a cycle. Thus, we find some explicit expressions for polynomials that can be used to do these calculations. Consider edge coloring a cycle \( C_k \) on \( k \) vertices by two colors \( \{R, B\} \). Choose an orientation for \( C_k \), and let \( r_1 \) denote the number of vertices with an incoming \( R \) edge and an outgoing \( B \) edge. Let \( r_2 \) denote the number of vertices with two incident \( R \) edges, and let \( b_2 \) denote the number of vertices with two incident \( B \) edges. Note that all of these statistics are independent of the orientation chosen. Define
\[
q_k(a, b, c) = \sum_{f: \mathcal{E}(C_k) \to \{R, B\}} a^{r_2} b^{r_1} c^{b_2},
\]
where $E$ denotes the edge set. Further, define

$$p_k(a, b) = q_k(a, b, 1) = \sum_{f : E(C_k) \to \{R, B\}} a^{r_2} b^{r_1}.$$  

Note that $k = r_2 + b_2 + 2r_1$, and hence $q_k = c^k p_k(a/c, b/c^2)$. Thus, it suffices to compute $p_k$.

To compute $p_k$, we will break the cyclic structure and instead work with analogous polynomials with respect to colorings of the directed path $P_{k+2}$ on $k + 2$ vertices. We will identify $E(P_{k+2})$ with $[k+1]$ in the natural way and define

$$p_k^R(a, b) = \sum_{f : E(P_{k+2}) \to \{R, B\}} a^{r_2} b^{r_1},$$

$$p_k^B(a, b) = \sum_{f : E(P_{k+2}) \to \{R, B\}} a^{r_2} b^{r_1},$$

By identifying the first and last edges of this path, we have that

$$p_k(a, b) = p_k^R(a, b) + p_k^B(a, b).$$

Beyond their use for computing $p_k(a, b)$, these polynomials are also needed in Section 3.

We use the method of the transfer matrix to find generating functions for these expressions. Consider a color pattern $f \in \{B, R\}^{k+1}$ as the walk on the following directed graph of length $k$ whose vertices spell out $f$:

![Diagram of directed graph]

The product of the edge weights is $a^{r_2} b^{r_1}$. Let $A = [\begin{smallmatrix} \frac{1}{a} & \frac{2}{b} \\ \frac{2}{a} & \frac{1}{b} \end{smallmatrix}]$. Then

$$A^k = \begin{bmatrix} p_k^R(a, b) & p_k^{RB}(a, b) \\ p_k^{BR}(a, b) & p_k^B(a, b) \end{bmatrix}$$

with $p_k^{BR}$ and $p_k^{RB}$ defined analogously to $p_k^R$ and $p_k^B$. By Theorem 4.7.2 in [Sta12], we find the following generating functions:

$$\sum_{k \geq 0} p_k^R(a, b) t^k = \frac{1 - t}{(a-b)t^2 - (a+1)t + 1},$$

$$\sum_{k \geq 0} p_k^B(a, b) t^k = \frac{1 - at}{(a-b)t^2 - (a+1)t + 1}.$$

Using partial fraction expansions, we arrive at

$$p_k^R(a, b) = \frac{t_+ t_-}{t_+ - t_-} \left( \frac{1 - t_-}{t_-^{k+1}} - \frac{1 - t_+}{t_+^{k+1}} \right),$$

$$p_k^B(a, b) = \frac{t_+ t_-}{t_+ - t_-} \left( \frac{1 - at_-}{t_-^{k+1}} - \frac{1 - at_+}{t_+^{k+1}} \right),$$

where

$$t_\pm = \frac{a + 1 \pm \sqrt{(a-1)^2 + 4b}}{2(a-b)}.$$  

It is now a simple exercise to produce an expression for $q_k$. Letting

$$\tau_\pm = \frac{c + a \pm \sqrt{(c-a)^2 + 4b}}{2},$$

we can produce an expression for $q_k$. Specifically, we have

$$q_k = \sum_{k \geq 0} q_k(a, b, 1) t^k.$$
we have that
\begin{equation}
\varrho_k(a, b, c) = \tau_+^k + \tau_-^k.
\end{equation}

**Poisson tails**

We will frequently require tail estimates of functions of Poisson fields. For this purpose, we use a bound that can be derived from modified log-Sobolev inequalities.

**Lemma 2.1.** Let \( \pi \) be a product measure of \( q \) Poisson laws, with means \( m_i \) for \( 1 \leq i \leq q \). Let \( F \) be a function from \( \mathbb{N}^q \to \mathbb{R} \), and define \( \nabla_i F(x) = F(x + e_i) - F(x) \) with \( e_i \) a standard basis vector. Further, let \( \|\nabla_i F\| = \sup_{x \in \mathbb{N}^q} |\nabla_i F(x)| \). If there are positive reals \( M_1 \) and \( M_2 \) so that
\[
\sum_{i=1}^q \|\nabla_i F\|^2 m_i \leq M_1 \quad \text{and} \quad \max_{i=1,\ldots,q} \|\nabla_i F\| \leq M_2,
\]
then for every \( r \geq 0 \),
\[
\pi(F \geq EF + r) \leq \exp\left(-\frac{r}{2M_2} \log \left(1 + \frac{M_2r}{M_1}\right)\right).
\]

**Proof.** This is a special case of the stronger theorem of Wu [Wu00], Proposition 3.1. \( \square \)

### 3. Poisson approximations

In this section, we will prove Proposition 1.7 establishing Poisson approximations for the cycle process \( I_r(P) \) when \( P \sim \mathcal{P}_{n,d} \) or \( P \sim \mathcal{F}_{n,d} \). Let \( H \) be a uniformly sampled Hamiltonian cycle on vertices \( \{1, \ldots, n\} \), and let \( Q \sim \mathcal{P}_{n,d-2} \). Recall that by representing \( H \) as a pairing, combining this pairing with \( P \), and randomly reordering the prevertices in each vertex bin, we obtain the model \( \mathcal{F}_{n,d} \). Our strategy will be to avoid this complication for as long as possible, and rather to work directly with \((H, Q)\). Let \( \mathcal{S}_{n,d} \) be the distribution of \((H, Q)\), which we call the unscrambled mixed model.

We will represent \((H, Q)\) as a pseudograph formed by superimposing their projections. Further, we color the edges from the configuration model red (denoted simply \( R \)), and edges from the Hamiltonian cycle blue (denoted simply \( B \)), which agrees with our terminology in other sections. We will label each endpoint of a red edge by the prevertex in \( Q \) from which it comes (see Figure 1). This provides the same information as \((H, Q)\), and we will go back and forth between the two views as the situation demands.

We define \( H_k \) to be the set of all possible cycles of length \( k \) in \((H, Q)\), in analogy with \( J_k \). As with the graph representation of \((H, Q)\), we represent these by \{\(R, B\)\}–edge–colored cycles, with prevertex labels on each red edge. We will refer to the color pattern of an element of \( H_k \) as a sequence of Rs and Bs of length \( k \) identified up to rotation and reversal, corresponding to the order in which the edge colors appear on the cycle. Define \( \mathcal{H} = \bigcup_{k=1}^\ell H_k \). Let \( I_\alpha \) be the indicator that \((H, Q)\) contains the cycle \( \alpha \), for any \( \alpha \in \mathcal{H} \), and
Corollary 3.3. Recall that of red and blue edges in these cycles:

All in all, this section is quite technical and delicate. For the reader who wants to skip to the chase, we recommend focusing on the arguments for $I_r(P)$, which typically use the same ideas as those for $I_r(H, Q)$ but have fewer technical details. The most important part of our argument is in Section 3.4, from Lemma 3.17 to Corollary 3.23.

In Section 3.1, we use the polynomials from Section 2 to compute the expected number of cycles of each size in $(H, Q)$, as well as a few related quantities. Section 3.2 is devoted to a bound on the probability that the cycles in $P$ or $(H, Q)$ are exceptional, in that they overlap each other or there are an unusually large number of them. In Section 3.3, we give a coupling of the model $\mathcal{P}_{n, d}$ with a conditioned version of itself, and we do the same thing for $\mathcal{F}_{n, d}$. Finally, in Section 3.4, we give the main argument and prove Proposition 1.17.

3.1. Expectations of cycle counts. Our first job is to use the polynomials from Section 2 to compute the expected number of cycles in $(H, Q)$. We will need the following facts about the asymptotics of $[n]_k$ and $\lceil n \rceil_k$, which are elementary to check.

**Lemma 3.1.** For all $k < n/2$,

$$n^k e^{-O(k^2/n)} \leq [n]_k \leq n^k,$$

and

$$n^k e^{-O(k^2/n)} \leq \lceil n \rceil_k \leq n^k.$$

**Lemma 3.2.** Suppose $\alpha$ is an $\{R, B\}$-edge-colored $k$-cycle. Let $r_1$ and $r_2$ be as in Section 2. Then for all $k < n/2$,

$$p_{\alpha} := EI_{\alpha} = \frac{2^{r_1}}{[n - 1]_{k - r_1}} \lceil n(d - 2) \rceil_{r_2 + r_1}$$

$$= \frac{2^{r_1}}{n^k (d - 2)^{r_2 + r_1}} \exp \left( O \left( \frac{(k - r_1 - r_2)^2}{n} + \frac{(r_1 + r_2)^2}{nd} \right) \right)$$

$$= \frac{2^{r_1}}{n^k (d - 2)^{r_2 + r_1}} e^{O(k^2/n)}.$$

**Proof.** The cycle $\alpha$ contains $r_1 + r_2$ red edges and $k - r_1 - r_2$ blue edges. The probability that $Q$ contains all of these red edges is $1/\lceil n(d - 2) \rceil_{r_1 + r_2}$. The blue edges form $r_1$ disjoint paths, and the probability that $H$ contains these paths is $2^{r_1}/[n - 1]_{k - r_1 - r_2}$. The approximations follow from Lemma 3.1.$\square$

Now, we compute the expected number of cycles of each length in $(H, Q)$, as well as the expected number of red and blue edges in these cycles:

**Corollary 3.3.** Recall that $I_{\alpha}$ is the indicator that $(H, Q)$ contains the cycle $\alpha$. Let $s_{\alpha}$ be the number of red edges and $t_{\alpha}$ be the number of blue edges in $\alpha$. Then

$$\mathbb{E} \sum_{\alpha \in H_k} s_{\alpha} I_{\alpha} = \frac{(d - 2)(d - 1)^k + 2(-1)^k}{2d} e^{O(k^2/n)},$$

$$\mathbb{E} \sum_{\alpha \in H_k} t_{\alpha} I_{\alpha} = \frac{2(d - 1)^k + (d - 2)(-1)^k - d}{2d} e^{O(k^2/n)},$$

$$\mathbb{E} \sum_{\alpha \in H_k} I_{\alpha} = \frac{(d - 1)^k + (-1)^k - 1}{2k} e^{O(k^2/n)}.$$

**Proof.** We begin by counting the expected number of cycles of a given length and color pattern. Fix an $\{R, B\}$-edge-colored, rooted oriented cycle $C_k$. There are $[n]_k$ different ways to choose the vertices of such a cycle, and there are $(d - 2)^{2r_1 + r_2}(d - 3)^{r_2}$ ways to assign edge labels to it. So long as the cycle is not all
blue, each such cycle has the probability given in Lemma 3.2 so the expected number of cycles with this color pattern is
\[
2^{r_1} \binom{n}{k} (d-2)^{2r_1+r_2}(d-3)^{r_2} = (2(d-2))^{r_1}(d-3)^{r_2}e^{O(k^2/n)}.
\]
Summing this over all possible edge colorings besides \(B^k\), the expected number of rooted, oriented cycles of length \(k\) is
\[
2kE \sum_{\alpha \in \mathcal{H}_k} I_\alpha = (p_k(d-3,2(d-2)) - 1)e^{O(k^2/n)},
\]
proving (12). If we repeat the same counting procedure, but only sum over edge colorings of \(C_k\) that color the first edge red, we count each cycle \(\alpha\) a total of \(2s_\alpha\) times, and so
\[
E \sum_{\alpha \in \mathcal{H}_k} 2s_\alpha I_\alpha = p_k(d-3,2(d-2))e^{O(k^2/n)},
\]
which proves (10). To show (11), we do the same thing and subtract off the term given by the all blue pattern:
\[
E \sum_{\alpha \in \mathcal{H}_k} 2t_\alpha I_\alpha = (p_k(d-3,2(d-2)) - 1)e^{O(k^2/n)}.
\]

3.2. Exceptional cycle counts. If \(x = (x_\alpha, \alpha \in \mathcal{H})\) or \(x = (x_\alpha, \alpha \in \mathcal{J})\) with \(x_\alpha\) equal to zero or one for each \(\alpha\), then we interpret \(x\) as a collection of cycles, and we will say that \(x\) contains \(\alpha\) to mean \(x_\alpha = 1\). Our estimates will fail for states \(x\) that contain too many cycles or overlapping cycles. The following definitions describe which states in the unscrambled mixed model \(\mathcal{S}_{n,d}\) and in the pairing model \(\mathcal{P}_{n,d}\) we will be able to analyze:

**Definition 3.4.** For some \(\lambda \geq 1\), a vector \(x = (x_\alpha, \alpha \in \mathcal{H})\) is \(\lambda\)-neat if the following hold:

i) The vector \(x\) does not contain any overlapping cycles; that is, if \(x_\alpha = x_\beta = 1\), then \(\alpha\) and \(\beta\) share no prevertices or Hamiltonian vertices.

ii) Let \(x\) contain a total of \(\Phi\) prevertices and \(\Psi\) Hamiltonian cycle vertices. These two counts satisfy
\[
\Phi \leq \lambda(d-1)^r, \quad \Psi \leq \lambda(d-1)^{r-1}.
\]

**Definition 3.5.** For some \(\lambda \geq 1\), a vector \(x = (x_\alpha, \alpha \in \mathcal{J})\) is \(\lambda\)-neat if the following hold:

i) The vector \(x\) does not contain any overlapping cycles; that is, if \(x_\alpha = x_\beta = 1\), then \(\alpha\) and \(\beta\) share no prevertices.

ii) The total number of prevertices contained in \(x\) is at most \(\lambda(d-1)^r\).

The point of this section is to show that when \(\lambda\) grows logarithmically, nearly all graphs have cycle counts satisfying these criteria.

**Proposition 3.6.** For \(d \geq 3\), and all \(r\) and \(n\),
\[
\Pr[I_r(P) \text{ is not } (\log n)\text{-neat}] \leq \frac{C_3(d-1)^{2r-1}}{n}, \quad P \sim \mathcal{P}_{n,d}.
\]

**Proposition 3.7.** For \(d \geq 3\), and all \(r\) and \(n\),
\[
\Pr[I_r(H, Q) \text{ is not } (\log n)\text{-neat}] \leq \frac{C_3(d-1)^{2r-1}}{n}.
\]

These two propositions have essentially the same proof, except that the details of the second one are somewhat trickier.

**Proof of Proposition 3.6.** Let \(\lambda = \log n\). We define two events whose probability we wish to bound:

- **OVERLAP** = \(\{P \text{ contains two cycles of length } r \text{ or less sharing an edge}\}\),

- **MANY** = \(\{I_r(P) \text{ contains more than } \lambda(d-1)^r \text{ prevertices}\} \cap \overline{\text{OVERLAP}}\).

Many
The cycle $\alpha$, with $\gamma$ dotted. The subgraph $\gamma$ has components $A_1, \ldots, A_p$. In this example, the number of components of $\gamma$ is $p = 3$, the size of $\alpha$ is $k = 11$, and the number of edges in $\gamma$ is $l = 4$. In this example, we will construct a cycle $\beta$ of length $j = 10$ that overlaps with $\alpha$ at $H$.

**Step 1.** We lay out the components $A_1, \ldots, A_p$. We can order and orient $A_2, \ldots, A_p$ however we would like, for a total of $(p-1)!^2$ choices. Here, we have ordered the components $A_1, A_3, A_2$, and we have reversed the orientation of $A_3$.

**Step 2.** Next, we choose how many edges will go in each gap between components. Each gap must contain at least one edge, and we must add a total of $(j-l-1)!^{p-1}$ choices. In this example, we have added one edge after $A_1$, three after $A_3$, and two after $A_2$.

**Step 3.** We can choose the new vertices in $[n-p-l]^{j-p-l}$ ways, and we can direct and give labels to the new edges in $(d-1)^{j-l+p}d^{j-l-p}$ ways.

**Figure 2.** Counting the cycles that overlap $\alpha$ at $\gamma$.

To bound the probability of OVERLAP, we bound $\sum_{\alpha, \beta} \mathbb{E}I_{\alpha}I_{\beta}$, where $\alpha$ and $\beta$ range over all pairs of overlapping cycles. For some $\alpha \in \mathcal{J}$, let $\mathcal{J}_l^{\alpha} \subseteq \mathcal{J}$ denote the set of cycles that share exactly $l$ pairs with $\alpha$, but otherwise do not share any prevertices. For any $\beta \in \mathcal{J}_l^{\alpha}$,

$$
\mathbb{E}[I_{\alpha}I_{\beta}] = \Pr[P \text{ contains } \alpha \text{ and } \beta] = \frac{1}{\|nd\|_{|\alpha|+|\beta|-l}}.
$$

Our plan is to bound the size of $\mathcal{J}_l^{\alpha}$. Fix some $\alpha \in \mathcal{J}_k$ and let $\gamma$ be some set of its edges of size $|\gamma| = l$ and with $p$ connected components (and thus $p + l$ vertices). We will show that the number of $j$-cycles that overlap with $\alpha$ at $\gamma$ is bounded by

$$
2^{p-l}(p-1)!\left(\frac{j-l-1}{p-1}\right)[n-p-l]^{j-p-l}(d-1)^{j-l+p}d^{j-l-p}
$$

Call the components of $\gamma A_1, \ldots, A_p$. We can construct any $\beta \in \mathcal{J}_j$ that overlaps with $\alpha$ at $\gamma$ by stringing together these components with other edges in between them. The components can appear in $\beta$ in any order, and each can appear with one of two orientations. Since the vertices in $\beta$ are only given up to
cyclic rotation, we can assume without loss of generality that component $A_1$ appears first, with some fixed orientation, followed by $A_2, \ldots, A_p$ in any order with any orientation, for a total of $2^{p-1}(p-1)!$ choices.

Now, imagine the components laid in a line, with gaps between them, and count the number of ways to fill the gaps. Each of the $p$ gaps must contain at least one edge, and the total number of edges in the gaps is $j - l$. Thus the total number of possible gap sizes is the number of compositions of $j - l$ into $p$ parts, \( \binom{j-l-1}{p-1} \).

Now that we have chosen the number of edges to appear in each gap, we choose the edges themselves. We can do this by giving an ordered list of $j - p - l$ vertices to go in the gaps, along with a label and an orientation for each of the $j - l$ new edges. There are $|n - p - l|_{j-p-l}$ ways to choose the vertices, and $(d-1)^{j-l+p}d^{j-l-p}$ ways to choose the labels. This establishes \( (13) \). It is a bound rather than an equality because some cycles constructed in this way might have additional overlap with $\alpha$.

Next, we count the number of ways to choose a subgraph $\gamma$ of $\alpha$ with $l$ edges and $p$ components. Let $s_1, \ldots, s_k$ be the vertices of $\alpha$, in order. Suppose that we have sequences of positive integers satisfying $a_1 + \cdots + a_p = l$ and $b_1 + \cdots + b_p = k - l$. Then we can obtain a subgraph of $\alpha$ with $l$ edges and $p$ components by starting at some vertex $s_i$ and including the next $a_j$ edges of $\alpha$ in $\gamma$, then excluding the next $b_j$, then including the next $s_k$, and so on. Every subgraph with $p$ components and $l$ edges is given in exactly $p$ ways, since $s_i$ can be at the beginning of any of the $p$ components. The number of ways to choose $i$ and the two sequences is $k(p-1)^{k-l-1}$, and so the total number of such subgraphs is this, divided by $p$.

All together, we have

$$\left| J^{l}_{\alpha} \cap J_{j} \right| \leq \sum_{p=1}^{l \wedge j-l} k \binom{l-1}{p-1} \binom{k-l-1}{p-1} 2^{p-1}(p-1)! \binom{j-l-1}{p-1} \times |n - p - l|_{j-p-l}(d-1)^{j-l+p}d^{j-l-p}$$

We apply the bounds

$$\binom{l-1}{p-1} \leq \frac{p^{p-1}}{(p-1)!},$$

$$\binom{k-l-1}{p-1}, \binom{j-l-1}{p-1} \leq (er/(p-1))^{p-1},$$

to get

$$\left| J^{l}_{\alpha} \cap J_{j} \right| \leq \sum_{p=1}^{l \wedge j-l} k \frac{2e^2p^3}{(p-1)^2} p^{-1} |n - p - l|_{j-p-l}(d-1)^{j-l+p}d^{j-l-p}$$

$$= k(d-1)^{j-l+1}d^{j-l-1} |n - l|_{j-l-1} \times \left( 1 + \sum_{p=2}^{l \wedge j-l} \frac{1}{p[n-l-1]_{p-1}} \left( \frac{2(d-1)e^2p^3}{d(p-1)^2} \right)^{p-1} \right).$$

We can assume without loss of generality that $r \leq n^{1/10}$, since the proposition holds for all $r > n^{1/10}$ just by choosing $C_3$ large enough to make $C_3(d-1)^{2r-1}/n \geq 1$ in this case. Thus the sum in the above equation is bounded by a universal constant, and we can compute

$$\sum_{\beta \in J^{l}_{\alpha}} E_{\alpha} I_{\beta} \leq \sum_{j=l+1}^{r} \sum_{\beta \in J^{l}_{\alpha} \cap J_{j}} \frac{1}{[nd]_{k+j-l}}$$

$$= \sum_{j=l+1}^{r} O \left( \frac{k(d-1)^{j-l+1}}{(nd)^{k+1}} \right) = O \left( \frac{k(d-1)^{r-l+1}}{(nd)^{k+1}} \right),$$
and

$$\Pr[\text{OVERLAP}] \leq \sum_{\alpha \in J} \sum_{i \geq 1} \sum_{\beta \in J_\alpha} \mathbb{E}I_\alpha I_\beta \leq \sum_{k=1}^r \frac{n_k d^k (d-1)^k}{2k} \sum_{l=1}^{k-1} O \left( \frac{(k(d-1)^{r-l+1})}{(nd)^{k+1}} \right)$$

$$= \sum_{k=1}^r \frac{n_k d^k (d-1)^k}{2k} \sum_{l=1}^{k-1} O \left( \frac{(k(2d-1)^{r-l-1})}{(nd)^{k+1}} \right)$$

$$= O \left( \frac{(d-1)^{2r-1}}{n} \right).$$

(14)

Now, we bound \( \Pr[\text{MANY}] \), using another union bound, which will reduce the problem to computing tail probabilities of a Poisson process. Let \( S \subseteq J \), and let \( |S| \) denote the total number of edges in all cycles in \( S \). We call \( S \) a bad set if \( |S| > \lambda (d-1)^r \) and its cycles do not overlap at any prevertices. If no proper subset of \( S \) contains more than \( \lambda (d-1)^r \) edges, we call \( S \) a minimal bad set. For any choice of \( m \) pairs of distinct prevertices out of \( nd \), the probability that \( P \) contains all of them is exactly \( 1/[[nd]]^m \). Thus by a union bound,

$$\Pr[\text{MANY}] \leq \sum_S \frac{1}{[[nd]]^{|S|}}$$

where \( S \) ranges over all minimal bad subsets of \( J \).

If \( S \) is a minimal bad set, then it contains at most \( \lambda (d-1)^r/2 + r \) edges. By Lemma 3.1

$$\frac{1}{[[nd]]^{|S|}} \leq (nd)^{|S|} \exp \left( O \left( \frac{|S|^2}{nd} \right) \right) \leq (nd)^{|S|} \exp \left( C_4 \lambda^2 (d-1)^{2r-1} \right)$$

for some absolute constant \( C_4 \). Now, we estimate

$$\sum_{S} \frac{1}{[[nd]]^{|S|}} \leq \exp \left( \frac{C_4 \lambda^2 (d-1)^{2r-1}}{n} \right) \sum_{S} \prod_{\alpha \in S} (nd)^{||\alpha||}$$

$$\leq \exp \left( \frac{C_4 \lambda^2 (d-1)^{2r-1}}{n} \right) e^\mu \sum_{S} \Pr[Z = 1\{S\}],$$

(15)

where we recall that \( Z = (Z_\alpha, \alpha \in J) \) has as its entries independent Poisson random variables with \( \mathbb{E}Z_\alpha = (nd)^{-|\alpha|} \), and \( \mu = \sum_\alpha \mathbb{E}Z_\alpha \). Define \( F(x) \) for \( x = (x_\alpha, \alpha \in J) \) by \( F(x) = \sum_\alpha 2 |\alpha| x_\alpha \), so that \( F(1\{S\}) \) is the number of prevertices in all cycles in \( S \). Now,

$$\sum_S \Pr[Z = 1\{S\}] \leq \Pr[F(Z) > \lambda (d-1)^r],$$

and we can bound this probability with the modified log-Sobolev inequalities. First, we compute

$$\mathbb{E}F(Z) = \sum_{k=1}^r \frac{2k |\mathcal{J}_k|}{(nd)^k} \leq C_5 (d-1)^r$$

for a constant \( C_5 \). In the notation of Lemma 2.1

$$\sum_{\alpha \in J} \|\nabla F\|^2 \mathbb{E}Z_\alpha = \sum_{\alpha \in J} (2 |\alpha|)^2 \mathbb{E}Z_\alpha \leq 2r \sum_{\alpha \in J} 2 |\alpha| \mathbb{E}Z_\alpha$$

$$= 2r \mathbb{E}F(Z) \leq 2C_5 r (d-1)^r,$$

and

$$\max_{\alpha \in J} \|\nabla \alpha F\| \leq 2r.$$

By Lemma 2.1

$$\Pr[F(Z) > \lambda (d-1)^r] \leq \exp \left( -\frac{(\lambda - C_5) (d-1)^r}{4r} \log \left( \frac{\lambda}{C_5} \right) \right).$$
Now, we substitute this back into (13). Making sure that we have chosen $C_5$ large enough, we have $\mu \leq C_5(d - 1)^r / r$, and we obtain

$$Pr[\text{MANY}] \leq \exp \left( - (d - 1)^r \left( \frac{\log n - C_5}{4r} \log \left( \frac{\log n}{C_5} \right) - \frac{C_5}{r} \right) \right).$$

(16)

For all $n$, $d$, $r$ such that $C_4(\log n)^2(d - 1)^{r-1} > n$, the proposition holds trivially by choosing $C_3$ sufficiently large. For the remaining values of $n$, $d$, and $r$, (16) shows that $Pr[\text{MANY}] = O(1/n)$, which with (14) completes the proof. \qed

**Proof of Proposition 3.7.** Define events OVERLAP and MANY as in the previous proposition. Let $H_\alpha$ consist of all cycles in $H$ that share an entire edge with $\alpha$ (and possibly other edges and prevertices as well), and let $H'_\alpha$ consist of all cycles that share no edges with $\alpha$ but do share Hamiltonian vertices.

First, we show that

$$\sum_{\alpha \in H} \sum_{\beta \in H_\alpha} \mathbb{E}I_\alpha I_\beta = O \left( \frac{(d - 1)^{2r-1}}{n} \right).$$

(17)

Let $\tilde{\alpha}$ and $\tilde{\beta}$ denote graph cycles without edge colors or labels. Let $\tilde{\gamma}$ be the subgraph made up of edges common to $\tilde{\alpha}$ and $\tilde{\beta}$. Suppose that $\tilde{\gamma}$ consists of $p$ paths, with a total of $l$ edges. First, we count how many possible $\tilde{\alpha}$ and $\tilde{\beta}$ can give rise to $\tilde{\gamma}$ with these properties.

Fix a choice of $\tilde{\alpha}$ and $\tilde{\gamma} \subseteq \tilde{\alpha}$, and we will determine how many possible $\tilde{\beta}$ there are. Let the components of $\tilde{\gamma}$ be $A_1, \ldots, A_p$, in the order they appear in $\tilde{\alpha}$. To construct $\tilde{\beta}$, imagine laying out these components, with $A_2, \ldots, A_p$ ordered and oriented any way, a total of $2^{p-1}(p - 1)!$ choices. Then, we will create $\tilde{\beta}$ by filling in the gaps between these components. Each gap between components must contain at least one edge, and there are a total of $j - l$ edges to add. So, the number of possible gap sizes is $\binom{j - l - 1}{p - 1}$, the number of compositions of $j - l$ into $p$ parts. This creates $j - p - l$ new vertices, and we have less than $n^{j-p-l}$ choices for these. Thus, for this fixed $\tilde{\alpha}$ and $\tilde{\gamma}$, there are at most

$$2^{p-1}(p - 1)! \binom{j - l - 1}{p - 1} n^{j-p-l}$$

choices of $\tilde{\beta}$.

We choose $\tilde{\alpha}$ from the $\binom{n}{k} / 2k < n^k / 2k$ possible $k$-cycles (without edge labels). To count how many $\tilde{\gamma} \subseteq \tilde{\alpha}$ we can form with $p$ components and $l$ total edges, fix a vertex in $\tilde{\alpha}$. Then, we can specify which edges to include in $\tilde{\gamma}$ by giving a sequence $a_1, b_1, \ldots, a_p, b_p$ instructing us to include in $\tilde{\gamma}$ the first $a_1$ edges after the vertex, then to exclude the next $b_1$, then to include the next $a_2$, and so on. Any sequence for which $a_i$ and $b_i$ are positive integers, $a_1 + \cdots + a_p = l$, and $b_1 + \cdots + b_p = k - l$ gives us a valid choice of $l$ edges of $\tilde{\alpha}$ making up $p$ components. This counts each subgraph $\tilde{\gamma}$ a total of $p$ times, since we could begin with any component of $\tilde{\gamma}$. Hence the number of subgraphs $\tilde{\gamma}$ with $l$ edges and $p$ components is $(k/p)\binom{l-1}{p-1}\binom{k-l-1}{p-1}$. In all, there are at most

$$2^{p-1}(p - 1)! \binom{j - l - 1}{p - 1} n^{j-p-l} \frac{n^k}{2k} \frac{(k/p)\binom{l-1}{p-1}\binom{k-l-1}{p-1}}{p-1} \frac{1}{2p} \left( \frac{2e^2}{(p-1)^2} \right)^{p-1} n^{k+j-p-l}$$

(18)

pairs of cycles $\alpha$ and $\beta$ such that their edges intersect to form $p$ paths with a total of $l$ edges.

Now, we bound all the ways to add edge colors and labels to $\tilde{\alpha}$ and $\tilde{\beta}$ to form $\alpha$ and $\beta$, and the probability that $\alpha \cup \beta$ appears in $(H, Q)$. Let $B_1, \ldots, B_p$ be the components of $\tilde{\beta}$ that do not overlap with $\tilde{\alpha}$, with $B_i$ appearing immediately after $A_i$. Let $B_i$ have $m_i$ edges, with $\sum m_i = j - l$. 
Every edge coloring of $\hat{\alpha} \cup \hat{\beta}$ gives rise to an $f \in \{B, R\}^k$, the edge coloring of $\hat{\alpha}$ and $\hat{f}_i \in \{B, R\}^m$, the edge coloring of $B_i$ for $1 \leq i \leq p$. Define $f_i$ from $\hat{f}_i$ by prepending and appending a $B$ to each side of $f_i$, so that $f_i \in B\{B, R\}^mB$. Any edge coloring of $\alpha \cup \beta$ is determined by the colorings $f, f_1, \ldots, f_p$.

Let $r_1$ and $r_2$ be the number of times the $RB$ and $RR$ patterns respectively occur in $f$. Let $r_1^{(i)}$ and $r_2^{(i)}$ be the same quantities for $f_i$. Let $R_1 = r_1 + \sum r_1^{(i)}$ and $R_2 = r_2 + \sum r_2^{(i)}$.

We will now turn this procedure around and edge-color $\alpha \cup \beta$ according to the colorings $f, f_1, \ldots, f_p$, as described above. This may not yield a possible coloring of $(H, Q)$. For example, it may be that $\alpha \cup \beta$ has three blue edges incident to a vertex. For any such nonsense coloring, we may simply take $EI_\alpha I_\beta = 0$.

Regardless, the subgraph $\alpha \cup \beta$ has a total of $R_1 + R_2$ red edges and $j + k - l - R_1 - R_2$ blue edges. It contains at most $R_1 + p$ disjoint blue paths. Thus for every coloring of $\alpha \cup \beta$ attained in this way, we have

$$EI_\alpha I_\beta \leq \frac{2^{R_1+p}}{n^{j+k-l}(d-2)^{R_1+R_2}} e^{O(r^2/n)}$$

(19)

by Lemma 3.1.

Next, we determine how many different ways we can add edge labels to $\alpha$ and $\beta$, given the edge colors. There will be a total of $2R_1 + 2R_2$ edge labels to assign, two for each red edge. Imagine walking around $\alpha \cup \beta$ assigning edge labels to its red edges. At each end of a red edges, we have $d - 2$ choices of edge labels if no adjacent red edge has had an edge label assigned yet, and we have $d - 3$ choices or fewer otherwise. For each $RR$ in $f, f_1, \ldots, f_p$, there is a vertex label with $d - 3$ or fewer choices. There are $R_2$ of these, so for at most $2R_1 + R_2$ of the edge labels do we have $d - 3$ choices. So, the number of ways to assign edge labels is at most $(d - 3)^{R_2} (d - 2)^{2R_1+R_2}$. (Note that this analysis works even in the $d = 3$ case, when containing the pattern $RR$ implies the $\alpha \cup \beta$ cannot occur.)

Multiplying this by the bound given in (19), the sum of $EI_\alpha I_\beta$ as $\alpha$ and $\beta$ range over all cycles formed by adding edge labels to $\hat{\alpha}$ and $\hat{\beta}$ is at most

$$\frac{2^p e^{O(r^2/n)}}{n^{j+k-l}} \sum_{f, f_1, \ldots, f_p} (2(d-2))^{R_1} (d-3)^{R_2}$$

$$= \frac{2^p e^{O(r^2/n)}}{n^{j+k-l}} \sum_{f, f_1, \ldots, f_p} p_k (d-3, 2(d-2)) \prod_{i=1}^{p} \left( d - 3, 2(d-2) \right)$$

$$= \frac{2^p e^{O(r^2/n)}}{n^{j+k-l}} O((d-1)^k) \prod_{i=1}^{p} \left( d - 1 \right)^{m_i}$$

$$= O(1) \left( \frac{d-1}{n} \right)^{j+k-l} 5^p e^{O(r^2/n)}.$$

We can assume without loss of generality that $r \leq n^{1/10}$ since the proposition holds for all $r > n^{1/10}$ by choosing $C_3$ sufficiently large. This means that the $e^{O(r^2/n)}$ factor can be absorbed into the $O(1)$ factor. Applying (18) and summing over all $k, j, p$, and $l$, we have

$$\sum_{\alpha \in H, \beta \in R_{\alpha}} EI_\alpha I_\beta$$

$$\leq \sum_{k, j=1}^{r} \sum_{l \geq 1} \sum_{p \geq 1} \frac{1}{2p} \left( \frac{2e^2 r^3}{(p-1)^2} \right)^{p-1} n^{k+j-l-p} O \left( 5^p \left( \frac{d-1}{n} \right)^{j+k-l} \right)$$

$$\leq \sum_{k, j=1}^{r} \sum_{l \geq 1} O \left( \frac{(d-1)^{j+k-l}}{n} \right) \sum_{p \geq 1} \frac{1}{2p} \left( \frac{10e^2 r^3}{(p-1)^2 n} \right)^{p-1}.$$

Our assumption that $r \leq n^{1/10}$ implies that the innermost sum is bounded by an absolute constant, proving (17).
Now, it only remains to bound $\sum_{\alpha \in \mathcal{H}} \sum_{\beta \in \mathcal{H}_\alpha} E I_{\alpha} I_{\beta}$. If $\beta \in \mathcal{H}_\alpha'$, then $E I_{\alpha} I_{\beta} \leq p_\alpha p_\beta$, so we will use this as a summand. We enumerate all $\beta \in \mathcal{H}_\alpha'$. Choose some Hamiltonian vertex $v$ of $\alpha$ and an orientation for $\alpha$. We will count all possible ways of constructing a rooted, oriented cycle $\beta$ that also contains $v$. Fix an edge coloring $f \in B\{B, R\}^{j-1}$ and let $r_1$ and $r_2$ be the values from Lemma 3.2 corresponding to $f$. There are at most $n^{j-1}(d-2)^{2r_1+r_2}(d-3)^{r_2}$ ways to fill in the remaining vertices and edge labels of $\beta$. Suppose that $\alpha$ contains $\phi$ prevertices. Then,

$$
\sum_{\beta \in \mathcal{H}_\alpha', \; |\beta|=j} p_\beta \leq (2k-\phi) \sum_{f \in B\{B, R\}^{j-1}} n^{j-1}(d-2)^{2r_1+r_2}(d-3)^{r_2} \frac{2^{r_1}}{n-1}_{j-r_2-r_1} \frac{n(d-2)}{n(d-2)}^{r_2+r_1} 
\leq (2k-\phi)p_j^B (d-3, 2(d-2))O(n^{-1})
= O\left(\frac{k(d-1)^{j-1}}{n}\right).
$$

Now we have

$$
\sum_{\alpha \in \mathcal{H}} \sum_{\beta \in \mathcal{H}_\alpha'} p_\alpha p_\beta \leq \sum_{\alpha \in \mathcal{H}} p_\alpha \sum_{j=1}^{r} O\left(\frac{|\alpha|}{n} (d-1)^{j-1}\right)
= \sum_{\alpha \in \mathcal{H}} p_\alpha O\left(\frac{|\alpha|}{n} (d-1)^{r-1}\right).
$$

(20)

This and (17) combine to show that $Pr[\text{OVERLAP}] = O((d-1)^{2r-1}/n)$.

Now, we bound $Pr[\text{MANY}]$. We say that $S \subseteq \mathcal{H}$ is a minimal bad set if it consists of non-overlapping cycles with a total of either more than $\lambda(d-1)^r$ prevertices or more than $\lambda(d-1)^{r-1}$ Hamiltonian vertices, and if no proper subset of $S$ has this property.

Let $r_1^{(\alpha)}$ and $r_2^{(\alpha)}$ be the number of $RB$s and $RR$s in the color pattern of $\alpha$, as in Lemma 3.2 and let

$$
r_1^{(S)} = \sum_{\alpha \in S} r_1^{(\alpha)}, \quad r_2^{(S)} = \sum_{\alpha \in S} r_2^{(\alpha)}.
$$

Let $|S|$ denote the total number of edges in $S$. Let $p_S$ denote the probability that $(H, Q)$ contains every cycle in $S$. The total number of prevertices in a minimally bad set is at most $\lambda(d-1)^r + 2r$, and the number of Hamiltonian vertices is at most $\lambda(d-1)^{r-1} + r$. Thus for a minimal bad set $S$, the total number of red edges, $r_1^{(S)} + r_2^{(S)}$, and the total number of blue edges, $|S| - r_1^{(S)} + r_2^{(S)}$, satisfy

$$
r_1^{(S)} + r_2^{(S)} \leq \frac{\lambda}{2} (d-1)^r + r,
|S| - r_1^{(S)} - r_2^{(S)} \leq \lambda(d-1)^{r-1} + r - 1.
$$

By Lemma 3.1

$$
p_S = \frac{2^{r_1^{(S)}}}{n^{|S|} (d-2)^{r_1^{(S)} + r_2^{(S)}}} \exp\left(\frac{\lambda^2 (d-1)^{2r-2}}{n} + O\left(\frac{\lambda^2 (d-1)^2r}{n(d-2)}\right)\right)
\leq \frac{2^{r_1^{(S)}}}{n^{|S|} (d-2)^{r_1^{(S)} + r_2^{(S)}}} \exp\left(C_6 \lambda^2 (d-1)^{2r-1}\right)
$$

for some absolute constant $C_6$. 

Our goal is to bound $\sum_S p_S$ as $S$ ranges over all minimal bad subsets of $H$. Let $Y = (Y_\alpha, \alpha \in H)$ be a vector of independent Poisson random variables with

$$\mathbb{E} Y_\alpha = \frac{2^{r_1^{(\alpha)}}}{n^{\alpha_1}(d - 2)^{r_1^{(\alpha)} + r_2^{(\alpha)}},}$$

and let $\mu = \sum_{\alpha \in H} \mathbb{E} Y_\alpha$.

$$\Pr[\text{Many}] \leq \sum_S p_S \leq \exp \left( \frac{C_6 \lambda^2 (d - 1)^{2r-1}}{n} \right) \sum_{\alpha \in S} \prod_{\alpha \in S} \frac{2^{r_1^{(\alpha)}}}{n^{\alpha_1}(d - 2)^{r_1^{(\alpha)} + r_2^{(\alpha)}}}$$

$$= \exp \left( \frac{C_6 \lambda^2 (d - 1)^{2r-1}}{n} \right) e^\mu \sum_{\alpha \in S} \Pr[Y = 1\{S\}].$$

(21)

For any cycle $\alpha \in H$, let $\phi_\alpha$ and $\psi_\alpha$ be the number of prevertices and Hamiltonian vertices, respectively, in $\alpha$. Let $s_\alpha$ and $t_\alpha$ be the number of red and blue edges, respectively, in the color pattern of $\alpha$, as in Section 3.1. Note that $\phi_\alpha = 2s_\alpha$ and $\psi_\alpha \leq 2t_\alpha$. So,

$$\sum_S \Pr[Y = 1\{S\}] \leq \Pr \left[ \sum_{\alpha \in H} \phi_\alpha Y_\alpha > \lambda (d - 1)^r \text{ or } \sum_{\alpha \in H} \psi_\alpha Y_\alpha > \lambda (d - 1)^{r-1} \right]$$

$$\leq \Pr \left[ \sum_{\alpha \in H} 2s_\alpha Y_\alpha > \lambda (d - 1)^r \text{ or } \sum_{\alpha \in H} 2t_\alpha Y_\alpha > \lambda (d - 1)^{r-1} \right].$$

Now, we use the modified log-Sobolev inequalities to get a tail estimate for each of these sums. Let $F(x) = \sum_\alpha 2s_\alpha x_\alpha$ for $x = (x_\alpha, \alpha \in H)$. By the same proof as for (10),

$$\mathbb{E} F(Y) = \sum_{\alpha \in H} 2s_\alpha \mathbb{E} Y_\alpha \leq C_7 (d - 1)^r$$

for some absolute constant $C_7$. In the notation of Lemma 2.1,

$$\sum_{\alpha \in H} \|\nabla F\|^2 \mathbb{E} Y_\alpha = \sum_{\alpha \in H} (2s_\alpha)^2 \mathbb{E} Y_\alpha \leq 2r \sum_{\alpha \in H} 2s_\alpha \mathbb{E} Y_\alpha$$

$$= 2r \mathbb{E} F(Y) \leq 2C_7 r (d - 1)^r,$$

and

$$\max_{\alpha \in H} \|\nabla F\| \leq 2r.$$

By Lemma 2.1

$$\Pr \left[ \sum_\alpha 2s_\alpha Y_\alpha > \lambda (d - 1)^r \right] \leq \exp \left( \frac{(\lambda - C_7)(d - 1)^r}{4r} \log \left( \frac{\lambda}{C_7} \right) \right).$$

(22)

In the same way, if $G(x) = \sum_\alpha 2t_\alpha x_\alpha$, then

$$\mathbb{E} G(Y) \leq C_7 (d - 1)^{r-1},$$

and

$$\sum_{\alpha \in H} \|\nabla G\|^2 \mathbb{E} Y_\alpha = \sum_{\alpha \in H} (2t_\alpha)^2 \mathbb{E} Y_\alpha \leq 2r \sum_{\alpha \in H} 2t_\alpha \mathbb{E} Y_\alpha$$

$$\leq 2r \mathbb{E} G(Y) \leq 2C_7 r (d - 1)^{r-1},$$

and

$$\max_{\alpha \in H} \|\nabla G\| \leq 2r.$$

Thus by Lemma 2.1

$$\Pr \left[ \sum_{\alpha \in H} 2t_\alpha Y_\alpha > \lambda (d - 1)^{r-1} \right] \leq \exp \left( \frac{(\lambda - C_7)(d - 1)^{r-1}}{4r} \log \left( \frac{\lambda}{C_7} \right) \right).$$

(23)
Making sure that we have chosen $C_7$ to be large enough, by the proof of (12), we have $\mu \leq C_7(d-1)^r/r$. We now sum (22) and (23) to show that

$$\Pr \left[ \sum_{\alpha \in H} \phi_{\alpha} Y_{\alpha} > \lambda(d-1)^r \text{ or } \sum_{\alpha \in H} \psi_{\alpha} Y_{\alpha} > \lambda(d-1)^{r-1} \right] \leq \exp \left( -\frac{(\lambda - C_7)d(d-1)^{r-1}}{4r} \log \left( \frac{\lambda}{C_7} \right) \right),$$

and then substitute this into (21) to get

$$\Pr[\text{OVERLAP}] \leq \exp \left[ -(d-1)^r \left( \frac{d(\lambda - C_7)}{4(d-1)^r} \log \left( \frac{\lambda}{C_7} \right) - \frac{C_7}{r} - \frac{C_6\lambda^2(d-1)^{r-1}}{n} \right) \right].$$

As with (16), this is $O(1/n)$ so long as $C_6\lambda^2(d-1)^{r-1} > n$, which finishes the proof, together with the bound on $\Pr[\text{OVERLAP}].$}

□

**Proposition 3.8.** For $d \geq 3$ and all $r$ and $n$,

$$\Pr[I_r(P)] \text{ is not strictly } (\log n)\text{-neat} \leq \frac{C_6(d-1)^{2r}}{n},$$

$$\Pr[I_r(\tilde{P})] \text{ is not strictly } (\log n)\text{-neat} \leq \Pr[I_r(H, Q) \text{ is not strictly } (\log n)\text{-neat}] \leq \frac{C_6(d-1)^{2r}}{n},$$

Proof. We only need to make minor changes to the previous proofs. To prove (25), define

$$\text{OVERLAP} = \{ P \text{ contains two cycles of length } r \text{ or less sharing a vertex} \}.$$

Again, we will bound $\sum_{\alpha, \beta} E[I_{\alpha} I_{\beta}]$ where $\alpha$ and $\beta$ range over all overlapping pairs of cycles. We have already shown in (14) that the sum over the cycles that overlap at an entire edge is $O((d-1)^{2r-1}/n)$. For any $k$-cycle $\alpha$, the number of $j$-cycles with a vertex in common with $\alpha$ is at most $kn^{j-1}(d(d-1)^{j})$. For any pair $\alpha$ and $\beta$ with a vertex in common but no edge in common,

$$E[I_{\alpha} I_{\beta}] = \frac{1}{\|nd\|_{j+k}}.$$

Thus the probability that $P$ contains some a cycle $\alpha$ and a cycle $\beta$ overlapping $\alpha$ at a vertex but at no edges is at most

$$\sum_{k=1}^{r} |J_k| \sum_{j=1}^{r} kn^{j-1}(d(d-1)^{j}) \frac{1}{\|nd\|_{j+k}} = \sum_{k=1}^{r} \frac{|n|_k (d(d-1))^{k}(2k-1)}{nd^{j+k-1}} \sum_{j=1}^{k} \frac{kn^{j-1}(d(d-1)^{j})}{\|nd\|_{j+k}} = O \left( \frac{(d-1)^{2r}}{n} \right),$$

proving that $\Pr[\text{OVERLAP}] = O((d-1)^{2r}/n)$. Combined with the bound on $\Pr[\text{OVERLAP}]$ from Proposition 3.6, this proves (25).

The equality in (26) holds because if $\tilde{P}$ is given by scrambling the prevertices in each bin of $(H, Q)$, then $\tilde{P}$ is strictly $\lambda$-neat if and only if $(H, Q)$ is. To adjust the proof of Proposition 3.7, we just need to change the definition of $\mathcal{H}'_\alpha$ to be all cycles that share no edges with $\alpha$ but do have a vertex in common, Hamiltonian or otherwise, and then do the computations leading up to (20) again. To enumerate all cycles in $\mathcal{H}'_\alpha$, first choose any vertex in $\alpha$. Let $\beta$ have color pattern $f$, and let $r_1$ and $r_2$ have their usual definitions of the number of RBs and RRs in $f$. The number of ways to fill in the remaining vertices and edge labels of $\beta$ is
at most \( n^{3-1(d-2)^{2r_1+rr_2}(d-3)^{2r_2}} \). Thus
\[
\sum_{\beta \in \mathcal{H}'_i} p_{\beta} \leq k \sum_{f \in (B,R)^i} \frac{n^{3-1(d-2)^{2r_1+rr_2}(d-3)^{2r_2}}}{(n-1)_{j-rr_2-r_1} \|n(d-2)\|_{rr_2+r_1}} \leq kp_j(d-3,2(d-2))O(n^{-1})
\]
\[
= O \left( \frac{k(d-1)^j}{n} \right).
\]
Now,
\[
\sum_{\alpha \in \mathcal{H}} \sum_{\beta \in \mathcal{H}'_i} p_{\alpha}p_{\beta} \leq \sum_{\alpha \in \mathcal{H}} p_{\alpha} \sum_{j=1}^{r} O \left( \frac{|\alpha|(d-1)^j}{n} \right)
\]
\[
= \sum_{\alpha \in \mathcal{H}} p_{\alpha} O \left( \frac{|\alpha|(d-1)^r}{n} \right)
\]
\[
= O \left( \frac{(d-1)^{2r}}{n} \right).
\]
The rest of Proposition 3.7 goes through as before. \(\square\)

3.3. Couplings. We will employ some variations of Stein’s method that use coupling techniques, so we will need to define couplings between conditioned pairings and Hamiltonian cycles and their unconditioned counterparts.

Suppose that a pairing contains the edges \( a \sim A \) and \( b \sim B \). We can delete these edges and replace them with \( a \sim b \) and \( A \sim B \) to get a new pairing. We call this switching the edges \( a \sim A \) and \( b \sim B \). This only makes sense if \( a \neq b \), but it is not a problem if \( A = b \) and \( B = a \), in which case the switching has no effect. We will use this operation to define couplings of pairings \( P \) and \( P' \):

**Coupling 3.9.** Let \( P \sim \mathcal{P}_{n,d} \), and fix distinct prevertices \( a_1, \ldots, a_k \) and \( b_1, \ldots, b_k \). Let \( A_1 \) and \( B_1 \) be the (random) prevertices paired with \( a_1 \) and \( b_1 \), respectively. Define \( P' \) to be the pairing obtained by the following procedure: Switch \( a_1 \sim A_1 \) and \( b_1 \sim B_1 \). Let \( A_2 \) and \( B_2 \) be the prevertices now paired with \( a_2 \) and \( b_2 \), and switch \( a_2 \sim A_2 \) and \( b_2 \sim B_2 \). Repeat for the remaining \( a_i \) and \( b_i \).

**Coupling 3.10.** Fix distinct prevertices \( a_1, \ldots, a_k \) and \( b_1, \ldots, b_k \), and let \( P' \) be distributed as \( \mathcal{P}_{n,d} \) conditioned to contain the pairs \( a_i \sim b_i \) for \( 1 \leq i \leq k \). Define \( P \) as follows: Sample \( A_k \) uniformly from all prevertices except \( a_1, \ldots, a_k \) and \( b_1, \ldots, b_k \), and let \( B_k \) denote the prevertex for which \( B_k \sim A_k \). Switch \( a_k \sim b_k \) and \( A_k \sim b_k \). Then sample \( A_{k-1} \) uniformly from all prevertices except \( a_1, \ldots, a_{k-1} \) and \( b_1, \ldots, b_{k-2} \), let \( B_{k-1} \sim A_{k-1} \), and switch \( a_{k-1} \sim b_{k-1} \) and \( A_{k-1} \sim B_{k-1} \). Repeat another \( k-2 \) times.

**Proposition 3.11.** In both couplings, \( P \sim \mathcal{P}_{n,d} \) and \( P' \) is distributed as \( P \) conditioned to contain \( a_i \sim b_i \) for \( 1 \leq i \leq k \). (In fact, these couplings are the same, though this is not important to us.)

**Proof.** Let \( \mathcal{P}_i \) be the set of all pairings on \( nd \) prevertices such that \( a_j \sim b_j \) for \( j \leq i \), with \( \mathcal{P}_0 \) the set of all pairings. Let \( p \in \mathcal{P}_{i-1} \), and let \( a' \sim a_i \) and \( b' \sim b_i \). Define \( p' \) from \( p \) by switching these two edges, as in Coupling 3.9. Let \( \varphi_i : \mathcal{P}_{i-1} \rightarrow \mathcal{P}_i \) be given by \( \varphi_i(p) = p' \). The elements of \( \varphi_i^{-1}(p') \) are all given by switching the edge \( a_i \sim b_i \) in \( p' \) with some edge other than \( a_j \sim b_j \) for \( j \leq i \), and other than \( b_j \sim a_j \) for \( j < i \). (Switching \( a_i \sim b_i \) with \( b_j \sim a_j \) does give an element of \( \varphi_i^{-1}(p') \), the pairing \( p' \) itself.) This demonstrates that \( \varphi_i^{-1}(p') \) has the same size regardless of \( p' \), namely \( nd - 2i + 1 \).

Step \( i \) of Coupling 3.9 can be interpreted as plugging the current random pairing into \( \varphi_i \). Similarly, step \( i \) (counting backward) of Coupling 3.10 can be interpreted as randomly choosing one of the preimages under \( \varphi_i \) of the current pairing. Because each preimage of \( \varphi_i \) is the same size, at step \( i \) of either coupling, the pairing is distributed uniformly on \( \mathcal{P}_i \). \(\square\)

We now consider similar couplings for random Hamiltonian cycles. It will make things simpler to work with Hamiltonian cycles that are oriented; that is, we identify two Hamiltonian cycles if they differ by a rotation, but not if they differ by a flip, so that there are \((n-1)!\) cycles on \( n \) vertices instead of the usual
When we refer to an edge $uv$ in a Hamiltonian cycle, it means an edge that goes from $u$ to $v$ in a canonical “forward" direction. To return to unoriented Hamiltonian cycles, take an unoriented cycle to have either orientation with probability $\frac{1}{2}$.

Suppose that $uU$ and $vV$ are edges in a Hamiltonian cycle. We define a switching of these edges as the action of deleting these edges and replacing them by $uv$ and $UV$, as in Figure 3. The orientation of these edges is important here: if $V$ were backwards instead of forwards from $v$, we would split the Hamiltonian cycle in two when we applied the switching. Also note that by definition, this switching yields a Hamiltonian cycle containing $uv$, not $vu$. We now demonstrate how we will condition a random Hamiltonian cycle to contain a collection of edges.

**Coupling 3.12.** Let $P_1,\ldots,P_p$ be a set of disjoint paths of vertices from $\{1,\ldots,n\}$, with $P_i = v_{i1} \cdots v_{ii}$. Let $H$ be a random Hamiltonian cycle. Let $v_{11}V_1$ and $v_{12}V'_1$ be edges in $H$, both with the forward orientation. Define $H'$ to be the Hamiltonian cycle obtained by switching $v_{11}V_1$ and $v_{12}V'_1$. Now, let $v_{12}V_2$ and $v_{13}V'_2$ be edges in the resulting cycle, and switch them. Repeat for the the rest of the edges in the path, and then for the other paths.

**Coupling 3.13.** Let $P_1,\ldots,P_p$ be a set of disjoint paths of vertices from $\{1,\ldots,n\}$, with $P_i = v_{i1} \cdots v_{il}$. Let $H'$ be distributed as a uniformly random Hamiltonian cycle conditioned to contain these paths with any orientation for $P_1,\ldots,P_{k-1}$ and with the specified orientation for $P_k$. Define $H$ by the following algorithm: Sample $UV$ uniformly from the edges of $H'$ not in any $P_1,\ldots,P_k$. Switch the last edge of $P_k$ and $UV$, and redefine $P_k$ by removing the last edge. Repeat this procedure by sampling a new edge $UV$ in the same way, switching it with the last edge of $P_k$ and redefining $P_k$ until $P_k$ is empty. Then reorient the cycle so that $P_{k-1}$ is oriented as specified. Repeat with the other paths.

**Proposition 3.14.** In both couplings, $H$ is a uniformly random Hamiltonian cycle, and $H'$ is distributed as $H$ conditioned to contain the paths $P_1,\ldots,P_{p-1}$ in any orientation, and $P_p$ in the given orientation.

**Proof.** Index the steps of Coupling 3.12 as $(i,j)$, where the first step is $(1,1)$, the next is $(1,2)$, and so on to the end of the first path, where it starts up again with $(2,1)$. Index the steps of Coupling 3.13 the same, starting at $(p,l_p)$ and counting down. Let $\mathcal{H}_{i,j}$ be the set of oriented Hamiltonian cycles that contain paths $P_l$ for $l < i$ in any orientation, and that contain the first $j$ edges of $P_i$ in the specified orientation.

Suppose that $h \in \mathcal{H}_{i,j-1}$, and let $v_{i,j}V$ and $v_{i,j+1}V'$ be edges in $h$. Define $h'$ by switching these two edges. (If $h$ already contains $v_{i,j}v_{i,j+1}$, this makes $h' = h$. If $h$ contains $v_{i,j+1}v_{i,j}$, then $h'$ is $h$ with its orientation switched.) Let $\varphi_{i,j} : \mathcal{H}_{i,j-1} \to \mathcal{H}_{i,j}$ be given by $\varphi_{i,j}(h) = h'$. As in Proposition 3.11, the elements of $\varphi_{i,j}^{-1}(h')$ are given by switching $v_{i,j}v_{i,j+1}$ with any edge except those in the paths $P_l$ for $l < i$ or in the first $j$ edges of $P_i$. Thus every fiber $\varphi_{i,j}^{-1}(h')$ is a set of the same size, regardless of the choice of $h'$.

Step $(i,j)$ of Coupling 3.12 can be viewed as plugging in a random Hamiltonian cycle into $\varphi_{i,j}$. Each step $(i,j)$ with $j > 1$ in Coupling 3.13 is equivalent to choosing a random element of the preimage of the current Hamiltonian cycle under $\varphi_{i,j}$. Because the preimages of $\varphi_{i,j}$ are all the same size, the uniform distribution is maintained at step $(i,j)$ with $j > 1$ in both couplings.

When $j = 1$, step $(i,j)$ in Coupling 3.13 has an added twist of reorienting the cycle to give path $P_{i-1}$ the specified direction. In Coupling 3.12, the step is equivalent to plugging in a random Hamiltonian cycle
Figure 4. Conditioning a graph formed by a random Hamiltonian cycle superimposed on the configuration model to contain a cycle $v_1 \cdots v_7$. Edges from the configuration model are colored red, and edges from the Hamiltonian cycle model are colored blue. The color pattern of the cycle to be created is red, red, blue, red, blue, blue, blue.

into $\varphi_{i,j}$, but the cycle is distributed as a random element of $H_{i-1,l_i}$ rather than $H_{i,0}$ (that is, it always has path $P_i$ oriented as specified). In both directions, it is not hard to see that the random Hamiltonian cycles remain uniformly distributed on $H_{i-1,l_i}$ and $H_{i,1}$.

The point of these couplings is to condition a graph to contain some given cycle. We will refer to the configuration model part and the Hamiltonian part of an element $\alpha \in J$, meaning the edges of $\alpha$ that come from each respective part of the model. We say that two cycles in $J$ overlap if their configuration model parts contain any prevertices in common, or if their Hamiltonian cycle parts contain any vertex in common.

For any cycle $\alpha \in H$, we define a pairing $Q_\alpha$ conditioned to contain the configuration model part of $\alpha$, coupled with $Q$ according to Coupling 3.9. We define a Hamiltonian cycle $H_\alpha$ conditioned to contain the Hamiltonian part of $\alpha$, coupled with $H$ according to Coupling 3.12 (recall that $H$ is given a random orientation so that Coupling 3.12 can be applied). Let $G$ and $G_\alpha$ be the $d$-regular pseudograph given by the projections of $(H,Q)$ and $(H_\alpha,Q_\alpha)$, respectively. See Figure 4 for an illustration. We state some properties of these couplings that are apparent from their construction. First, we define a subgraphs $K_i$ of $G$ for $1 \leq i \leq k$ in which $G$ and $G_\alpha$ might differ.

**Definition 3.15.** Fix some cycle $\alpha \in H$. Projected onto a pseudograph, let its vertices be $v_1, \ldots, v_k$. If $v_i$ lies between two red edges in $\alpha$, then there are two prevertices $a$ and $b$ used by $\alpha$ at $v_i$. Each is the endpoint of an edge in $Q$. Let $K_i$ be the projection of these two red edges and their endpoints. If $v_i$ lies between two blue edges in $\alpha$, then let $K_i$ be the two blue edges incident to $v_i$ and their endpoints. Finally, if $v_i$ lies between one blue edge and one red edge in $\alpha$, then let $K_i$ be the two blue edges incident to $v_i$, the red edge labeled by the prevertex used by $\alpha$ and their endpoints. These graphs are illustrated by the connected components of the top graph in Figure 4.

**Proposition 3.16.**

i) Suppose that $(H,Q)$ contains the cycle $\beta$, but $(H_\alpha,Q_\alpha)$ does not. Then $\beta$ and $\alpha$ overlap.

ii) Suppose that some edge is present in $(H_\alpha,Q_\alpha)$ but not in $(H,Q)$. Then this edge is either contained in $\alpha$, or its projection to $G$ is an edge between a vertex in $K_i$ and a vertex in $K_{i+1}$ for some $i$ (considering indices modulo $|\alpha|$).

**Proof.** For the first claim, note that the only edges that are destroyed by the coupling have some $v_i$ as an endpoint. For the second, the only edges that are created by the coupling appear between some $K_i$ and $K_{i+1}$ at the corresponding step in the coupling algorithm. □
3.4. Poisson approximation with multiplicative bounds. The usual goal in Poisson approximation is to bound the total variation distance between some distribution \( \mu \) and the Poisson distribution. This gives an estimate of the point probabilities \( \mu(k) \) with a uniform, additive error. We, on the other hand, want an approximation of these point probabilities in which the error term is relative to the size of \( \mu(k) \).

First, we present a framework for this form of approximation, echoing the one given in [BHJ92]. Let \( F = (F_\alpha, \alpha \in I) \) be a vector of Bernoulli random variables, and let \( p_\alpha = \mathbb{E}F_\alpha \). Suppose that we have a family of random vectors \( J_\alpha = (J_\beta, \beta \in I) \), each coupled with \( F \), such that \( J_\alpha \) is distributed as \( F \) conditioned on \( F_\alpha = 1 \). Let \( e_\alpha \) denote the standard basis vector equal to one at position \( \alpha \) and zero elsewhere. Define \( E_\alpha \) to be the event that \( J_\alpha = F + e_\alpha \). The idea is that bounds on conditional probabilities of \( E_\alpha \) can be turned into estimates on point probabilities relative to each other.

**Lemma 3.17.** Let \( x = (x_\beta, \beta \in I) \) be a vector of zeros and ones. For some \( \alpha \in I \) with \( x_\alpha = 1 \), suppose that \( J_\alpha \) is coupled with \( F \) and distributed as described above. If

\[
\begin{align*}
\Pr[E_\alpha \mid F = x - e_\alpha] &\geq 1 - \epsilon, \\
\Pr[E_\alpha \mid J_\alpha = x] &\geq 1 - \epsilon,
\end{align*}
\]

then

\[
(1 - \epsilon)p_\alpha \Pr[F = x - e_\alpha] \leq \Pr[F = x] \leq (1 - \epsilon)^{-1}p_\alpha \Pr[F = x - e_\alpha].
\]

**Proof.** These inequalities follows directly from the definitions:

\[
\begin{align*}
\Pr[F = x] &= \Pr[F_\alpha = 1 \text{ and } F = x] \\
&= p_\alpha \Pr[J_\alpha = x] \\
&\geq p_\alpha \Pr[F = x - e_\alpha \text{ and } E_\alpha] \\
&= p_\alpha \Pr[F = x - e_\alpha] \Pr[E_\alpha \mid F = x - e_\alpha] \geq (1 - \epsilon)p_\alpha \Pr[F = x - e_\alpha],
\end{align*}
\]

and

\[
\Pr[F = x - e_\alpha] \geq \Pr[J_\alpha = x \text{ and } E_\alpha] \\
= \Pr[J_\alpha = x] \Pr[E_\alpha \mid J_\alpha = x] \\
\geq (1 - \epsilon) \Pr[J_\alpha = x] = (1 - \epsilon)p_\alpha^{-1} \Pr[F = x].
\]

**Remark 3.18.** The approach to Poisson approximation in [BHJ92] is to show that

\[
\mathbb{E}\left| \sum_\beta F_\beta + 1 - \sum_\beta J_\alpha \beta \right|
\]

is always small (see [BHJ92] Theorem 1.B). This is quite similar to proving that \( \Pr[E_\alpha] \) is nearly one. The gist of our method is that by bounding conditional versions of this probability, we obtain more information.

This method was partially inspired by the use of *switchings* in random graph models, which also relate probabilities of slightly perturbed events. In [MWW03, Theorem 2], for example, the authors estimate the probability that a random regular graph has no cycles of size \( r \) or less, with a multiplicative error. Similar techniques are used in [Jan09]. See Remark 5.6 in that paper for an interpretation of switchings as approximate couplings.

Our goal now is to apply Lemma 3.17 to the cycle processes \( \mathbf{I}_r(P) \) and \( \mathbf{I}_r(H, Q) \) defined on p. 9. For some absolute constant \( C_9 \), the following two propositions hold:

**Proposition 3.19.** Fix some \( \alpha \in \mathcal{H} \), and let \( x \) be \( \lambda \)-neat with \( x_\alpha = 1 \). Let \( J_\alpha \) be distributed as \( \mathbf{I}_r(H, Q) \) conditioned on \( I_\alpha = 1 \). Then \( \mathbf{I}_r(H, Q) \) and \( J_\alpha \) can be coupled with

\[
\Pr[E_\alpha \mid \mathbf{I}_r(H, Q) = x - e_\alpha] \geq 1 - \frac{C_9 \lambda |\alpha| (d - 1)^r - 1}{n}.
\]

**Proposition 3.20.** Fix some \( \alpha \in \mathcal{H} \), and let \( x \) be \( \lambda \)-neat with \( x_\alpha = 1 \). Let \( J_\alpha \) be distributed as \( \mathbf{I}_r(H, Q) \) conditioned on \( I_\alpha = 1 \). If Then \( \mathbf{I}_r(H, Q) \) and \( J_\alpha \) can be coupled with

\[
\Pr[E_\alpha \mid J_\alpha = x] \geq 1 - \frac{C_9 \lambda |\alpha| (d - 1)^r - 1}{n}.
\]
Proof of Proposition 3.19. Take \((H, H_\alpha)\) from Coupling 3.12. We take \((Q, Q_\alpha)\) from Coupling 3.9 unless \(\alpha\) is a loop. In this case, we use a slight variation of Coupling 3.9 depicted in Figure 5. Let \(\alpha\) be made up of prevertices \(a \sim b\), with \(a\) and \(b\) belonging to the same vertex. If \(a \sim b\) already in \(Q\), then let \(Q_\alpha = Q\). Otherwise, suppose that \(A \sim a\) and \(B \sim b\) in \(Q\), and choose a prevertex \(A'\) uniformly out of the all prevertices other than \(a, b, A,\) and \(B\). Let \(B' \sim B\) in \(Q\). To form \(Q_\alpha\), delete \(a \sim A\), \(b \sim B\), and \(A' \sim B'\), and replace them with \(a \sim b\), \(A \sim A'\), and \(B \sim B'\). It is straightforward to check that \(Q_\alpha\) is distributed as \(Q\) conditioned on containing \(\alpha\).

Let \(k = |\alpha|\). Recall that the edge coloring of \(\alpha\) specifies which edges of \(\alpha\) come from the pairing model part of the graph, and which edges come from the Hamiltonian part. Suppose that \(\beta\) is some other \(k\)-cycle with the same color pattern\(^1\) as \(\alpha\), also disjoint from all cycles in \(x - e_\alpha\). By the symmetry of our model and our couplings,

\[
\Pr[E_\alpha \mid I_r(H, Q) = x - e_\alpha] = \Pr[E_\beta \mid I_r(H, Q) = x - e_\alpha].
\]

Taking this one step further, this statement still holds if \(\beta\) is chosen at random from all cycles with the same color pattern as \(\alpha\) that are disjoint from \(x - e_\alpha\).

Now, let \(\xi\) be chosen uniformly from the set of cycles in \(\mathcal{H}_k\) that share the color pattern of \(\alpha\), independent of all other random variables. A good way to think of \(\xi\) is as the cycle given by randomizing all the prevertex and vertex labels in \(\alpha\). Define the event

\[
F = \{\xi\text{ is disjoint from }x\text{ and }E_\xi\text{ holds}\}.
\]

Now \(\Pr[E_\alpha \mid I_r(H, Q) = x - e_\alpha] \geq \Pr[F \mid I_r(H, Q) = x - e_\alpha]\), and it suffices to bound this from below.

We break up the event \(F^c\) into three parts, with \(F^c \subseteq A_1 \cup A_2 \cup A_3\):

\[
\begin{align*}
A_1 & = \{\xi\text{ overlaps with a cycle in }x\}, \\
A_2 & = \{J_\beta\xi = 0 \text{ for some } \beta \neq \xi \text{ with } I_\beta = 1\}, \\
A_3 & = \{J_\beta\xi = 1 \text{ for some } \beta \neq \xi \text{ with } I_\beta = 0\}.
\end{align*}
\]

Let \(v_1, \ldots, v_k\) be the vertices of \(\xi\), with the starting vertex and orientation of the cycle arbitrarily fixed. By definition of \(\xi\), these vertices are randomly chosen without replacement from \(\{1, \ldots, n\}\). Let \(\phi\) and \(\psi\) be the number of prevertices and Hamiltonian vertices in \(\xi\), respectively.

To bound the probability of event \(A_1\), we observe that each prevertex in \(\xi\) is marginally uniform over all \(n(d-2)\) prevertices and each Hamiltonian vertex is marginally uniform over \(n\). Thus the chance that any particular prevertex in \(\xi\) matches one found in a cycle in \(x\) is at most \(\Phi/n(d-2)\), and the the chance that any particular Hamiltonian vertex in \(\xi\) matches one in \(x\) is at most \(\Psi/n\), where \(\Phi\) is the total number of prevertices and \(\Psi\) the total number of Hamiltonian vertices in \(x\), as in Definition 3.4. Applying this to all

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\(^1\)Recall two colored cycles have the same color pattern if and only if there is a graph isomorphism between them that preserves the edge coloring.
2k prevertices in $\xi$ with a union bound,
\[
\Pr[A_1 \mid I_r(H,Q) = x - e_\alpha] \leq \frac{\phi \Phi}{n(d-2)} + \frac{\psi \Psi}{n} \leq \frac{\phi \lambda(d-1)^r}{n(d-2)} + \frac{\psi \lambda(d-1)^{r-1}}{n} = O\left(\frac{k \lambda(d-1)^{r-1}}{n}\right).
\]

We now consider the event $A_2$. Assume first that $\xi$ is not a loop, and our usual coupling is in effect. Suppose that $I_\beta = 1$ and $J_\beta = 0$. Then by Proposition 3.14, the cycles $\beta$ and $\xi$ have a prevertex or a Hamiltonian vertex in common. Thus $A_2 \subseteq A_1$. If $\xi$ is a loop and our altered coupling of $(Q,Q_\xi)$ is in effect, then the situation is similar. Suppose that $I_\beta = 0$ but $J_\beta = 0$. Then either $\xi$ contains a prevertex in $\beta$, or the randomly chosen edge $A' \sim B'$ in $Q$ used to define $Q_\alpha$ contains a prevertex in $\beta$. In the first case, event $A_1$ holds. To bound the second case, we observe that both prevertices $A'$ and $B'$ are marginally distributed uniformly, and the probability that one of them matches a prevertex in $x$ is at most $2\Phi/n(d-2)$, which is $O(\lambda(d-1)^{r-1}/n)$ since $x$ is $\lambda$-near. In either case,
\[(28)\]
\[
\Pr[A_1 \cup A_2 \mid I_r(H,Q) = x - e_\alpha] = O\left(\frac{k \lambda(d-1)^{r-1}}{n}\right).
\]

In the final step, we will bound the event $A_3 \cap A_2^c$ using an approach similar to the switchings argument in [MWWM04]. Let $G$ and $G_\xi$ be the pseudographs defined by $(H,Q)$ and $(H_\xi,Q_\xi)$ respectively. We start with the case that $\xi$ is not a loop. Recall the subgraphs $K_1,\ldots,K_k$ from Definition 3.15. We claim that if $A_3 \cap A_2^c$ holds, then the following event holds:
\[
B = \left\{ \text{For some } 1 \leq i \leq k \text{ and } 1 \leq j \leq r/2, \text{ the distance in } G\text{ between } K_i \text{ and } K_{i+j} \text{ is less than or equal to } r-j \right\}.
\]
(Here and in the rest of the argument, we are considering indices modulo $k$.) Indeed, suppose $A_3 \cap A_2^c$ holds, and there exists some $\beta \neq \xi$ with $I_\beta = 0$ and $J_\beta = 1$. By Proposition 3.16, the only new edges in $G_\xi$ not found in $G$ are the ones in $\xi$, and an edge between $K_i$ and $K_{i+1}$ for each $i$. If $\beta$ is a loop, then it must consist of one of these edges between $K_i$ and $K_{i+1}$, in which case event $B$ holds because $K_i$ and $K_{i+1}$ have distance zero. Otherwise, $\beta$ must contain at least one path in $G \cap G_\xi$. Suppose it contains only one such path. The remainder of $\beta$ is either a single edge between some $K_i$ and $K_{i+j}$, or a portion of $\xi$. In both cases, the existence of this path implies event $B$. If instead $\beta$ contains more than one path in $G \cap G_\xi$, then one of them must have length strictly less than $r/2$. For some $i$ and some $0 \leq j \leq r/2$, this path goes between $K_i$ and $K_{i+j}$. If $j \geq 1$, then the path implies event $B$. If $j = 0$, then this path begins and ends at vertices in $K_i$. Along with either one or two edges present in $G$ by not in $G_\xi$, this forms a cycle in $G$. But then $A_2$ holds, contradicting our original assumption.

We now estimate the probability of that $B$ occurs. Let $d(K,K')$ denote the distance in $G$ between the two subgraphs $K$ and $K'$ (that is, the length of the shortest path between a vertex in one subgraph and a vertex in another).

**Claim 3.21.** For any $i \neq i'$,
\[
\Pr[d(K_i,K_{i'}) \leq D \mid I_r(H,Q) = x - e_\alpha] = O\left(\frac{(d-1)^D}{n}\right).
\]

**Proof:** Since $G$ is $d$-regular, the number of vertices within distance $D$ of $K_i$ is $O((d-1)^D)$. Even after conditioning on $H,Q$, and $K_i$, the vertex $v_i$ is a uniformly random choice out of all vertices except $v_i$. Thus the probability that it is within distance $D$ of $K_i$ is $O((d-1)^D/n)$, as is the probability that one of its (at most three) neighbors in $K_i$ are within $D$ of $K_i$.

By this claim,
\[
\Pr[B \mid I_r(H,Q) = x - e_\alpha] \leq \sum_{i=1}^k \sum_{j=1}^{r/2} O\left(\frac{(d-1)^{r-j}}{n}\right) = O\left(\frac{k(d-1)^{r-1}}{n}\right).
\]

(29)
It only remains to bound the probability of the event $A_3 \cap A_5' \cap B'$ when $\xi$ is a loop. Take $n \leq q$, $a$, $b$, $A_1, A_2, A'_1, A'_2$, and $B'$ as in the definition of the coupling on p. 23. Let $K$ be the subgraph of $G$ induced by the prevertices $A, a, B, b$, and let $K'$ be the subgraph induced by $A'$ and $B'$. We claim that if $A_3 \cap A_5'$ holds, then $d(K, K') \leq r - 1$. Indeed, suppose that $A_3$ holds and there exists some cycle other than $\xi$ in $G_\xi$ but not in $G$. This cycle must use one of the new edges $A \sim A'$ or $B \sim B'$. If it uses only one of them, then $G$ contains a path of length $r - 1$ or less either from $A$ to $A'$ or from $B$ to $B'$, and so $d(K, K') \leq r - 1$. If it uses both of them, then there are two possibilities: either $G$ contains a path of length $r - 1$ or less between $A$ and $B'$ or $B$ and $A'$, in which case $d(K, K') \leq r - 1$; or $G$ contains a cycle of length $r$ or less involving the edge $A' \sim B'$, in which case event $A_2$ holds.

By the same reasoning as in Claim 3.21,

$$\Pr[d(K, K') \leq r - 1 \mid I_r(H, Q) = x - c_\alpha] = O \left( \frac{(d - 1)^{r - 1}}{n} \right).$$

From this and (29), we have shown that in all cases

$$\Pr[A_3 \cap A_5' \mid I_r(H, Q) = x - c_\alpha] = O \left( \frac{k(d - 1)^{r - 1}}{n} \right).$$

Combining this with (28) completes the proof. \(\square\)

**Proof of Proposition 3.20** We take $(Q, Q_\beta)$ and $(H, H_\alpha)$ from Couplings 3.10 and 3.13 respectively. Let $k = |\alpha|$, and let $\phi$ and $\psi$ be the number of prevertices and Hamiltonian vertices, respectively, in $\alpha$. Let $G$ and $G_\alpha$ be the pseudographs given by $(H, Q)$ and $(H_\alpha, Q_\alpha)$.

Event $E_\alpha^c$ can happen in three ways: $G$ still contains $\alpha$, it is missing some cycle $\beta \neq \alpha$ present in $G_\alpha$, or it contains some cycle $\beta \neq \alpha$ not present in $G_\alpha$. We define three events $A_1, A_2, A_3$ based on this, with $E_\alpha^c \subseteq A_1 \cup A_2 \cup A_3$:

$$A_1 = \{Q \text{ or } H \text{ contains some edge of } \alpha\},$$

$$A_2 = \{I_\beta = 0 \text{ for some } \beta \neq \alpha \text{ with } J_{\beta\alpha} = 1\},$$

$$A_3 = \{I_\beta = 1 \text{ for some } \beta \neq \alpha \text{ with } J_{\beta\alpha} = 0\}.$$

Thus

$$\Pr[E_\alpha^c \mid J_{\bullet\alpha} = x] \leq \Pr[A_1 \mid J_{\bullet\alpha} = x] + \Pr[A_2 \mid J_{\bullet\alpha} = x] + \Pr[A_3 \mid J_{\bullet\alpha} = x].$$

We have made event $A_1$ broader than necessary; this will make it easier to bound the last term of this equation.

At each step of Coupling 3.10 an edge $a_i b_i$ in $\alpha$ is switched with a random edge. The edge is preserved only if $A_i = a_i$. Otherwise, no later switchings can cause it to return. Similarly, at each step of Coupling 3.13 an edge of $\alpha$ is switched with a random edge, and there are only two choices of this random edge that do not destroy the edge in $\alpha$. Thus

$$\Pr[A_1 \mid J_{\bullet\alpha} = x] \leq \frac{1}{n(d - 2) - \phi + 1} + \frac{1}{n(d - 2) - \phi + 3} + \cdots + \frac{1}{n(d - 2) - 1}$$

$$+ \frac{2}{n - \psi} + \frac{2}{n - \psi + 1} + \cdots + \frac{2}{n - 1}$$

$$= O \left( \frac{k}{n} \right).$$

Next, we consider event $A_2$. At each step of Couplings 3.10 and 3.13 an edge of $\alpha$ is switched with a random edge. If $J_{\beta\alpha} = 1$, then $I_\beta = 0$ only if one of these random edges contains a prevertex or a Hamiltonian vertex in $\beta$. This occurs for some prevertex contained in a cycle in $x$ with probability at most

$$\frac{2\Phi}{n(d - 2) - 2\phi + 1} + \frac{2\Phi}{n(d - 2) - 2\phi + 3} + \cdots + \frac{2\Phi}{n(d - 2) - 1} \leq \frac{2\Phi}{n(d - 2) - 2\phi + 1},$$

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and it occurs for some Hamiltonian vertex in a cycle in \( x \) with probability at most
\[
\frac{2\Psi}{n - \psi} + \frac{2\Psi}{n - \psi + 1} + \cdots + \frac{2\Psi}{n - 1} \leq \frac{2\psi\Psi}{n - \psi},
\]
where \( \Phi \) is the total number of prevertices and \( \Psi \) the total number of Hamiltonian vertices in \( x \), as in Definition 3.4. Since \( x \) is \( \lambda \)-neat, we can sum these to get
\[
\Pr[A_2 \mid J_{\omega} = x] = O\left(\frac{k\lambda(d - 1)^{r-1}}{n}\right).
\]

Last, we consider the event \( A_3 \cap A_1^c \cap A_2^c \). Consider Coupling 3.13 to take place after Coupling 3.10 so that we can say that there are \( k \) steps total to go from \( G_\alpha \) to \( G \) and number them from 1 to \( k \). Suppose we have just taken the \( i \)th step in the coupling process, switching two edges, whether in the pairing part of the graph or the Hamiltonian cycle part of the graph. Suppose that \( uv \) and \( UV \) are the edges deleted, with \( uv \) being part of \( \alpha \), and \( uU \) and \( vV \) are the edges created. We wish to show that it is unlikely that a new cycle has formed involving one of the new edges \( uU \) and \( vV \). More precisely, define \( C_i \) to be the event that a new cycle is formed in the \( i \)th step, and that it is the first new cycle formed by the coupling process. We will bound the probability of \( C_i \cap A_1^c \cap A_2^c \) under the assumption that \( J_{\omega} = x \).

The first thing to notice is that we cannot ignore the possibility of a new cycle forming involving both \( uU \) and \( vV \). Suppose that \( C_i \) holds, and that the new cycle formed uses both these edges. Then this cycle either contains paths between \( u \) and \( v \) and between \( U \) and \( V \), or paths between \( u \) and \( V \) and between \( U \) and \( v \). In the first case, \( uv \) is part of a cycle destroyed when the edge is switched with \( UV \). This cycle must have been present in \( G_\alpha \), since if \( C_i \) holds, then no new cycles have formed before step \( i \) in the coupling process. Thus event \( A_2 \) holds. In the second case, suppose that \( P(u, V) \) is the path from \( u \) to \( V \), and \( P(v, U) \) is the path from \( v \) to \( U \). The newly created cycle is \( UuP(u, V)vP(v, U) \). In the previous step, \( uvP(v, U)V P(V, u) \) is a cycle, and the switching deletes it. If this cycle is anything other than \( \alpha \), then event \( A_2 \) holds. If the cycle is \( \alpha \), then event \( A_1 \) holds, since this is the only way that \( UV \) can be part of \( \alpha \), assuming that \( J_{\omega} = x \) where \( x \) contains no cycles that overlap.

Thus we need only consider the possibility that \( C_i \cap A_1^c \cap A_2^c \) holds because a new cycle forms at step \( i \) involving only one of \( uU \) and \( vV \). Before step \( i \) in the coupling process, there are at most \( (d - 1)^{r-1} \) paths of length \( j - 1 \) starting from \( u \) whose first step is not \( uv \). The vertex \( U \) must be at the end of one of these paths if \( uU \) is to form a new cycle of length \( j \). This occurs with probability at most \( (d - 1)^j/(n(d - 2) - \phi + 1) \) if step \( i \) is part of the coupling process for \( (Q, Q_\alpha) \) and with probability at most \( (d - 1)^{r-1}/(n - \psi) \) if step \( i \) is part of the coupling process for \( (H, H_\alpha) \). The same is true for forming a new cycle involving \( vV \). Summing this bound over all \( j \) from 1 to \( r \),
\[
\Pr[C_i \cap A_1^c \cap A_2^c \mid J_{\omega} = x] = O\left(\frac{(d - 1)^{r-1}}{n}\right).
\]

Applying this for \( C_1, \ldots, C_k \), we have
\[
\Pr[A_3 \cap A_1^c \cap A_2^c \mid J_{\omega} = x] = O\left(\frac{k(d - 1)^{r-1}}{n}\right).
\]

Applying (31), (32), and (33) to (30) proves the proposition.

We will also need these results in \( \mathcal{P}_{n,d} \):

**Proposition 3.22.** Let \( P \sim \mathcal{P}_{n,d} \). Let \( J_{\omega} \) be distributed as \( I_r(P) \) conditioned on \( I_\alpha = 1 \). Let \( x \) be \( \lambda \)-neat, and let \( t_{\omega} = 1 \). Then \( I_r(P) \) and \( J_{\omega} \) can be coupled with
\[
\Pr[E_{\alpha} \mid I_r(P) = x - e_{\alpha}] \geq 1 - \frac{C_\beta \lambda \alpha}{n} (d - 1)^{r-1} - 1,
\]
\[
\Pr[E_{\alpha} \mid J_{\omega} = x] \geq 1 - \frac{C_\beta \lambda \alpha}{n} (d - 1)^{r-1} - 1.
\]

**Proof.** The proof of Propositions 3.19 and 3.20 go through exactly.

Propositions 3.19, 3.20, and 3.22 combine with Lemma 3.17 to give relative estimates on the point probabilities of \( I_r(H, Q) \) and \( I_r(P) \) with \( P \sim \mathcal{P}_{n,d} \).
Corollary 3.23. Either suppose that \( x = (x_\alpha, \alpha \in \mathcal{H}) \) and \( \mathbf{I} = \mathbf{I}_r(H,Q) \), or suppose that \( x = (x_\alpha, \alpha \in \mathcal{J}) \) and \( \mathbf{I} = \mathbf{I}_r(P) \) with \( P \sim \mathcal{P}_{n,d} \). In either case, suppose that \( x \) is \( \lambda \)-neat and \( C_9 \lambda |\alpha| (d-1)^{r-1} \leq n \). For any \( \alpha \) with \( x_\alpha = 1 \),

\[
(1 - \mathcal{E}(x, \alpha)) p_\alpha \Pr[ I = x - e_\alpha ] \leq \Pr[ I = x ] \leq (1 - \mathcal{E}(x, \alpha))^{-1} p_\alpha \Pr[ I = x + e_\alpha ] ,
\]

where \( p_\alpha = E I_\alpha \) and

\[
\mathcal{E}(x, \alpha) \leq \frac{C_9 \lambda |\alpha| (d-1)^{r-1}}{n}.
\]

By repeated application of this corollary, we can relate the probability of any \( \lambda \)-neat configuration of cycles to the probability that the graph contains no cycles at all of length \( r \) or less:

Proposition 3.24. Either suppose that \( x = (x_\alpha, \alpha \in \mathcal{H}) \) and \( \mathbf{I} = \mathbf{I}_r(H,Q) \), or suppose that \( x = (x_\alpha, \alpha \in \mathcal{J}) \) and \( \mathbf{I} = \mathbf{I}_r(P) \) with \( P \sim \mathcal{P}_{n,d} \). Suppose that \( C_9 \lambda r (d-1)^{r-1} \leq n/2 \). If \( x \) is \( \lambda \)-neat, then

\[
\exp \left( - \frac{C_{10} \lambda^2 (d-1)^{2r-1}}{n} \right) \prod_{\alpha : x_\alpha = 1} p_\alpha \leq \frac{\Pr[ I = x ]}{\Pr[ I = 0 ]} \leq \exp \left( \frac{C_{10} \lambda^2 (d-1)^{2r-1}}{n} \right) \prod_{\alpha : x_\alpha = 1} p_\alpha
\]

for some absolute constant \( C_{10} \).

Proof. Since \( C_9 \lambda r (d-1)^{r-1} \leq n/2 \),

\[
\left( 1 - \frac{C_9 \lambda k (d-1)^{r-1}}{n} \right)^{-1} = \exp \left( O \left( \frac{\lambda k (d-1)^{r-1}}{n} \right) \right)
\]

for any \( k \leq r \). Let \( c_k \) be the number of cycles of length \( k \) in \( x \). Let \( y = (y_\alpha, \alpha \in \mathcal{H}) \) or \( y = (y_\alpha, \alpha \in \mathcal{J}) \), as appropriate. If \( x \) is \( \lambda \)-neat and \( y_\alpha \leq x_\alpha \) for all \( \alpha \), then \( y \) is also \( \lambda \)-neat. Thus we can apply Corollary 3.23 repeatedly to get

\[
\frac{\Pr[ I = x ]}{\Pr[ I = 0 ]} \leq \prod_{k=1}^r \exp \left( O \left( \frac{\lambda k (d-1)^{r-1}}{n} \right) \right) c_k \prod_{\alpha : x_\alpha = 1} p_\alpha
\]

\[
= \exp \left( O \left( \frac{\lambda (d-1)^{r-1} \sum_{k=1}^r k c_k}{n} \right) \right) \prod_{\alpha : x_\alpha = 1} p_\alpha
\]

\[
\leq \exp \left( O \left( \frac{\lambda^2 (d-1)^{2r-1}}{n} \right) \right) \prod_{\alpha : x_\alpha = 1} p_\alpha.
\]

The lower bound has a nearly identical proof. \( \square \)

Proposition 3.25. Either suppose that \( \mathbf{I} = \mathbf{I}_r(H,Q) \) and \( \mu \) is the expected number of cycles of length \( r \) or less in \( (H,Q) \), or suppose that \( \mathbf{I} = \mathbf{I}_r(P) \) with \( P \sim \mathcal{P}_{n,d} \) and \( \mu \) is the expected number of cycles of length \( r \) or less in \( P \). In either case, for all \( d \geq 3 \) and \( r, n \) satisfying \( C_3(d-1)^{2r-1} \leq n/2 \),

\[
\Pr[ I = 0 ] = \exp \left( -\mu + O \left( \frac{(\log n)^2(d-1)^{2r-1}}{n} \right) \right).
\]

Proof. Let \( \mathbf{Y} = (Y_\alpha) \) be a vector of independent Poisson random variables with \( E Y_\alpha = E I_\alpha \), and with \( \alpha \) ranging over \( \mathcal{H} \) or \( \mathcal{J} \) as appropriate. Let \( \lambda = \log n \), and sum the upper bound from Proposition 3.24 over all \( \lambda \)-neat \( x \) to get

\[
\frac{\Pr[ I \text{ is } \lambda \text{-neat} ]}{\Pr[ I = 0 ]} \leq \exp \left( \frac{C_{10} \lambda^2 (d-1)^{2r-1}}{n} \right) \sum_{\lambda \text{-neat } x : x_\alpha = 1} \prod_{\alpha : x_\alpha = 1} p_\alpha
\]

\[
= \exp \left( \frac{C_{10} \lambda^2 (d-1)^{2r-1}}{n} \right) \sum_{\lambda \text{-neat } x} e^\mu \Pr[ \mathbf{Y} = x ]
\]

\[
\leq \exp \left( \frac{C_{10} \lambda^2 (d-1)^{2r-1}}{n} \right) e^\mu.
\]
By Proposition 3.6 or 3.7

\[ 1 - \frac{C_3(d - 1)^{2r-1}}{n} \leq \exp \left( \frac{C_1 \lambda^2 (d - 1)^{2r-1}}{n} \right) e^\mu \Pr[I = 0]. \]

Since \( C_3(d - 1)^{2r-1} < n/2, \)

\[ 1 - \frac{C_3(d - 1)^{2r-1}}{n} = \exp \left( -O \left( \frac{(d - 1)^{2r-1}}{n} \right) \right), \]

and so

\[ \Pr[I = 0] \geq \exp \left( -\mu + O \left( \frac{(\log n)^2 (d - 1)^{2r-1}}{n} \right) \right). \]

For the other direction, we use the lower bound from Proposition 3.24 to get

\[ \frac{1}{\Pr[I = 0]} \geq \frac{\Pr[I \text{ is } \lambda\text{-neat}]}{\Pr[I = 0]} \geq \exp \left( -\frac{C_1 \lambda^2 (d - 1)^{2r-1}}{n} \right) \sum_{\lambda\text{-neat } x} \Pr[Y = x] \]

\[ = \exp \left( -\frac{C_1 \lambda^2 (d - 1)^{2r-1}}{n} \right) e^\mu \Pr[Y \text{ is } \lambda\text{-neat}]. \]

We just need to bound \( \Pr[Y \text{ is } \lambda\text{-neat}] \). To handle the case where \( I = I_r(H,Q) \), see Proposition 3.7, where we considered a Poisson field \( Y \) with means differing very slightly from the \( Y \) in this proof. This makes no difference, and (24) applies and shows that the probability that \( Y \) fails to be \( \lambda\text{-neat} \) on account of containing too many prevertices or Hamiltonian vertices is easily \( O(n^{-1}) \). Similarly, the same argument used in (20) shows that the probability that \( Y \) contains overlapping cycles is \( O((d - 1)^{2r-1}/n) \). Taking the constant here to be \( C_3 \) (increasing it if necessary), it follows as with (35) that

\[ \Pr[I = 0] \leq \exp \left( -\mu + O \left( \frac{(\log n)^2 (d - 1)^{2r-1}}{n} \right) \right). \]

We now put all the pieces together and give the main result of this section.

Proof of Proposition 1.7. We start with \( I_r(P) \), proving (9). Let \( \lambda = \log n \). By Lemma 3.1

\[ \prod_{\alpha : x_{a} = 1} p_{\alpha} = \exp \left( O \left( \frac{\lambda^2 (d - 1)^{2r-1}}{n} \right) \right) \prod_{\alpha : x_{a} = 1} (nd)^{-|\alpha|}. \]

By Proposition 3.24

\[ \frac{\Pr[I_r(P) = x]}{\Pr[I_r(P) = 0]} = \exp \left( O \left( \frac{\lambda^2 (d - 1)^{2r-1}}{n} \right) \right) \prod_{\alpha : x_{a} = 1} (nd)^{-|\alpha|}. \]

We wish to replace \( \mu \) in (34) with \( \sum_{\alpha \in J} (nd)^{-|\alpha|} \). By Lemma 3.1

\[ \sum_{\alpha \in J} (nd)^{-|\alpha|} = \left( 1 + O \left( \frac{r^2}{n} \right) \right) \mu. \]

This together with Proposition 3.25 proves

\[ \Pr[I_r(P) = 0] = \exp \left( -\sum_{\alpha \in J} (nd)^{-|\alpha|} + O \left( \frac{(\log n)^2 (d - 1)^{2r-1}}{n} \right) \right). \]

Applying this to (36), we have shown that

\[ \Pr[I_r(P) = x] \leq \exp \left( \frac{\lambda C_1 (\log n)^2 (d - 1)^{2r-1}}{n} \right) \Pr[Z = x] \]

and

\[ \Pr[I_r(P) = x] \geq \exp \left( -\frac{\lambda C_1 (\log n)^2 (d - 1)^{2r-1}}{n} \right) \Pr[Z = x] \]
for some absolute constant $C_1$. For $|x| < 1/2$, $1 + x/2 \leq e^{x/2} \leq 1 + x$. Since $C_1 (\log n)^2 (d - 1)^{2r-1} < n/2$, this proves (3).

The proof of (4) is similar, but has a few more complications. The first is that we need to take into account the scrambling of the prevertices in each bin in $\mathcal{B}_{n,d}$. Suppose that $\ell$ is a coloring of the edges of cycles contained in $x$. Let $y_{\ell} = (y_{\alpha}, \alpha \in \mathcal{H})$ consist of the cycles in $x$, colored according to $\ell$. Let $Pr[\cdot | \ell]$ denote probability conditional on the prevertex scrambling inducing the coloring $\ell$ on the edges in cycles contained in $x$. Let $b$ be the total number of vertices in cycles in $y_{\ell}$ that are incident to either one or two blue edges in the cycle. Conditional on the coloring $\ell$, there are $2^b$ ways to assign prevertices for these Hamiltonian vertices, and thus

$$Pr[I_{r}(\tilde{P}) = x | \ell] = 2^{-b} Pr[I_{r}(H, Q) = y_{\ell}].$$

The probability that the cycles in $x$ get colored $\ell$ by the scrambling is

$$\prod_{\alpha : x_{\alpha} = 1} \left( \frac{2}{|d|_2} \right)^{|\alpha| - 2r_1 - r_2} \left( \frac{2(d - 2)}{|d|_2} \right)^{2r_1} \left( \frac{|d - 2|_2}{|d|_2} \right)^{r_2},$$

where $r_1$ and $r_2$ are as in Lemma 3.2 applied to the color pattern of $\alpha$. (This depends on the cycles in $x$ not overlapping even at a vertex.) Note that

$$b = \sum_{\alpha : x_{\alpha} = 1} (|\alpha| - r_2).$$

Summing over all possible $\ell$, we have

$$(38) \quad Pr[I_{r}(\tilde{P}) = x] = \sum_{\ell} \rho(\ell) Pr[I_{r}(H, Q) = y_{\ell}],$$

where

$$\rho(\ell) = 2^{-b} \prod_{\alpha : (y_{\ell})_{\alpha} = 1} \left( \frac{2}{|d|_2} \right)^{|\alpha| - 2r_1 - r_2} \left( \frac{2(d - 2)}{|d|_2} \right)^{2r_1} \left( \frac{|d - 2|_2}{|d|_2} \right)^{r_2}.$$

We would like to apply Proposition 3.24 to estimate $Pr[I_{r}(H, Q) = y_{\ell}]$, but there is a complication: just because $x$ is $\lambda$-neat does not necessarily mean that $y_{\ell}$ is, because it could contain more than $\lambda(d - 1)^{r-1}$ Hamiltonian vertices. The best we can say is that $y_{\ell}$ is $d\lambda$-neat, but using only this bound would introduce an extra factor of $d$ in the error term.

To deal with this, let $C_{\text{good}}$ be the set of colorings $\ell$ such that $y_{\ell}$ is $4\lambda$-neat, and let $C_{\text{bad}}$ be the remaining colorings. Let $E_{\ell}$ be defined by

$$\frac{Pr[I_{r}(H, Q) = y_{\ell}]}{Pr[I_{r}(H, Q) = 0]} = E_{\ell} \prod_{\alpha : (y_{\ell})_{\alpha} = 1} \frac{2^{r_1}}{n^{|\alpha|} (d - 2)^{r_1 + r_2}}.$$

If $\ell \in C_{\text{good}}$, then by Lemma 3.2

$$\prod_{\alpha : (y_{\ell})_{\alpha} = 1} p_{\alpha} = \exp \left( O \left( \frac{\lambda^2 (d - 1)^{2r-1}}{n} \right) \right) \prod_{\alpha : (y_{\ell})_{\alpha} = 1} \frac{2^{r_1}}{n^{|\alpha|} (d - 2)^{r_1 + r_2}},$$

and if $\ell \in C_{\text{bad}},$

$$\prod_{\alpha : (y_{\ell})_{\alpha} = 1} p_{\alpha} = \exp \left( O \left( \frac{\lambda^2 (d - 1)^{2r}}{n} \right) \right) \prod_{\alpha : (y_{\ell})_{\alpha} = 1} \frac{2^{r_1}}{n^{|\alpha|} (d - 2)^{r_1 + r_2}}.$$

By this and Proposition 3.24

$$E_{\ell} = \exp \left( O \left( \frac{\lambda^2 (d - 1)^{2r-1}}{n} \right) \right), \quad \ell \in C_{\text{good}},$$

$$E_{\ell} = \exp \left( O \left( \frac{\lambda^2 d^2 (d - 1)^{2r-1}}{n} \right) \right), \quad \ell \in C_{\text{bad}}.$$
For ease of presentation, we just show an upper bound on \( \Pr[I_r(\tilde{P}) = x] \). The lower bound has an identical proof. We first note that

\[
\rho(l) \prod_{\alpha : (y_t)_\alpha = 1} \frac{2r_1}{n|\alpha|} = \prod_{\alpha : (y_t)_\alpha = 1} \frac{(2(d-2))^{r_1}(d-3)^{r_2}}{(n[\ell])^{\alpha}},
\]

and that

\[
\sum_{\ell} \prod_{\alpha : (y_t)_\alpha = 1} \frac{(2(d-2))^{r_1}(d-3)^{r_2}}{(n[\ell])^{\alpha}} = \prod_{\alpha : x_\alpha = 1} \frac{p_{[\alpha]}(d-3, 2(d-2)) - 1}{(n[\ell])^{\alpha}} = \prod_{\alpha : x_\alpha = 1} \left( \frac{1}{(nd)^{\alpha}} + \frac{(-1)^{\alpha} - 1}{(n[\ell])^{\alpha}} \right) = e^\mu \Pr[\tilde{Z} = x],
\]

where \( \mu = \sum_{\alpha} \mathbb{E}_Z x_\alpha \). By the same reasoning as \( \text{(37)} \), Proposition \( \text{3.25} \) holds with its definition of \( \mu \) changed to this one. Applying all of this to \( \text{(38)} \),

\[
\Pr[I_r(\tilde{P}) = x] = \Pr[I_r(H, Q) = 0] \sum_{\ell} \mathcal{E}_d \rho(\ell) \prod_{\alpha : (y_t)_\alpha = 1} \frac{2r_1}{n|\alpha|(d-2)^{r_1+r_2}} \leq \exp\left( -\mu \frac{C_{11} \lambda^2 d^2(d-1)^{2r-1}}{n} \right) \sum_{\ell \in C_{\text{bad}}} \prod_{\alpha : (y_t)_\alpha = 1} \frac{(2(d-2))^{r_1}(d-3)^{r_2}}{(n[\ell])^{\alpha}}
\]

\[
+ \exp\left( -\mu \frac{C_{11} \lambda^2 d^2(d-1)^{2r-1}}{n} \right) \sum_{\ell \in C_{\text{bad}}} \prod_{\alpha : (y_t)_\alpha = 1} \frac{(2(d-2))^{r_1}(d-3)^{r_2}}{(n[\ell])^{\alpha}} \leq \exp\left( -\mu \frac{C_{11} \lambda^2 d^2(d-1)^{2r-1}}{n} \right) \sum_{\ell \in C_{\text{bad}}} \prod_{\alpha : (y_t)_\alpha = 1} \frac{(2(d-2))^{r_1}(d-3)^{r_2}}{(n[\ell])^{\alpha}}
\]

\[
+ \exp\left( C_{11} \lambda^2 d^2(d-1)^{2r-1} \right) \Pr[\tilde{Z} = x]
\]

for some absolute constant \( C_{11} \).

Thus, we need to show that the first term of \( \text{(39)} \) is negligible compared to the second one. Intuitively, this should hold because \( C_{\text{good}} \) contains the overwhelming majority of colorings. More precisely, we will show the following:

**Claim 3.26.**

\[
\sum_{\ell \in C_{\text{bad}}} \prod_{\alpha : (y_t)_\alpha = 1} (2(d-2))^{r_1}(d-3)^{r_2} \leq e^{-\lambda(d-1)^{r-2}} \sum_{\ell} \prod_{\alpha : (y_t)_\alpha = 1} (2(d-2))^{r_1}(d-3)^{r_2}.
\]

**Proof.** When \( d = 3, 4 \), the set \( C_{\text{bad}} \) is empty, since every coloring of a \( \lambda \)-neat \( x \) is \( d \lambda \)-neat, and \( C_{\text{bad}} \) consists of all colorings that fail to be \( 4 \lambda \)-neat. Thus we can assume that \( d \geq 5 \).

We will treat the sums probabilistically. Of course, each sum has a probabilistic interpretation in the first place, but we give a simpler one: For each edge \( e \) in a cycle in \( x \), interpret \( \omega_e = 1 \) to mean that \( e \) is colored blue, and \( \omega_e = 0 \) to mean that it is colored red. We will put a product measure on \( (\omega_e) \), assigning each edge blue with probability \( 3/d \) and red with probability \( 1 - 3/d \). (There is nothing special about these probabilities, and others would work as well.) Let \( R_l \) be the total number of \( RR \)s in the color patterns of all cycles in the coloring given by \( (\omega_e) \). Let \( R_2 \) be the total number of \( RR \)s in these patterns. Let \( m \) be the total number of edges in all cycles in \( x \). We define \( X \) to be zero if any cycle is colored all blue by \( (\omega_e) \);
With (41), this proves (40).

By Hoeffding’s inequality, otherwise,

\[ X := \left( \frac{6(d-2)}{d-3} \right)^{R_1} \left( \frac{d}{d-3} \right)^{R_2} \left( \frac{d}{3} \right)^m. \]

Since the total number of red edges is \( R_1 + R_2 \) and the total number of blue edges is \( m - R_1 - R_2 \), this makes

\[
\mathbb{E}X = \sum_{(\omega_e) \in \{0,1\}^m} \left( \frac{3}{d} \right)^{m-R_1-R_2} \left( \frac{d-3}{d} \right)^{R_1+R_2} \times \left( \frac{6(d-2)}{d-3} \right)^{R_1} \left( \frac{d}{3} \right)^m \mathbf{1}\{\text{no blue cycles}\}
\]

\[ = \sum_{\ell} \prod_{(y_e) : (y_e) \in \mathbb{R}_+} (2(d-2))^{r_1} (d-3)^{r_2}. \]

The number of Hamiltonian vertices in the random coloring is \( m - R_2 \). So, the claim takes on the form

(40) \[ \mathbb{E}[X \mathbf{1}\{m - R_2 > 4\lambda(d-1)^{r-1}\}] \leq e^{-\lambda(d-1)^{r-2}} \mathbb{E}X. \]

The random variable \( X \) is a decreasing function of \( (\omega_e) \): indeed, changing \( \omega_e \) from zero to one causes one of the following changes to \( R_1 \) and \( R_2 \), depending on the coloring of the neighbors of \( e \):

i) \( RRR \rightarrow RBR \): \( R_2 \) decreases by two, \( R_1 \) increases by one;

ii) \( RRB \rightarrow RBB \): \( R_2 \) decreases by one;

iii) \( BRB \rightarrow BBB \): \( R_1 \) decreases by one.

\( X \) decreases in all of these cases (we use the assumption that \( d \geq 5 \) in case [ii]). Changing \( \omega_e \) from zero to one might also cause a color to be colored all blue, in which case \( X \) decreases to zero. The random variable \( \mathbf{1}\{m - R_2 > 4\lambda(d-1)^{r-1}\} \) is an increasing function of \( (\omega_e) \). By the FKG inequality,

(41) \[ \mathbb{E}[X \mathbf{1}\{m - R_2 > 4\lambda(d-1)^{r-1}\}] \leq (\mathbb{E}X) \Pr[m - R_2 > 4\lambda(d-1)^{r-1}] \]

If \( m - R_2 > 4\lambda(d-1)^{r-1} \), then \( m - R_2 - R_1 > 2\lambda(d-1)^{r-1} \); this is because \( m - R_2 \) is the number of Hamiltonian vertices, and \( m - R_2 - R_1 \) is the number of blue edges in the coloring, and there are at most twice as many Hamiltonian vertices as blue edges. Thus

\[ \Pr[m - R_2 > 4\lambda(d-1)^{r-1}] \leq \Pr[m - R_2 - R_1 > 2\lambda(d-1)^{r-1}] \]

The number of blue edges, \( m - R_2 - R_1 \), is distributed as \( \text{Binom}(m, 3/d) \). Since \( x \) is \( \lambda \)-neat, the inequality \( m \leq \lambda(d-1)^r/2 \) holds. Thus

\[
\Pr[m - R_2 - R_1 > 2\lambda(d-1)^{r-1}]
= \Pr \left[ m - R_2 - R_1 - \mathbb{E}[m - R_2 - R_1] > 2\lambda(d-1)^{r-1} - \frac{3m}{d} \right]
\leq \Pr \left[ m - R_2 - R_1 - \mathbb{E}[m - R_2 - R_1] > \frac{\lambda}{2}(d-1)^{r-1} \right].
\]

By Hoeffding’s inequality,

\[
\Pr[m - R_2 - R_1 > 2\lambda(d-1)^{r-1}] \leq \exp \left( -\frac{\lambda^2(d-1)^{2r-2}}{2m} \right) \leq \exp \left( -\lambda(d-1)^{r-2} \right).
\]

With (41), this proves (40). \qed
Applying the claim to (39), we have shown that
\[
\Pr[\mathcal{L}(\tilde{P})] \leq \Pr[\tilde{Z} = x]\left( \exp \left( \frac{C_{11} \lambda^2 d^2 (d-1)^{2r-1}}{n} - \lambda (d-1)^{r-2} \right) + \exp \left( \frac{C_{11} \lambda^2 (d-1)^{2r-1}}{n} \right) \right)
\]
Using our assumptions that \( r \geq 4 \) and \( C_1 \lambda^2 (d-1)^{2r-1} < n/2 \), and assuming that we choose \( C_1 \) sufficiently larger than \( C_{11} \), we have
\[
\exp \left( \lambda (d-1)^{r-2} \left( \frac{C_{11} \lambda d^2 (d-1)^{r+1}}{n} - 1 \right) \right) \leq \exp \left( \frac{\lambda}{2} (d-1)^{r-2} \right) = n^{-(d-1)^{r-2}/2} = O(n^{-1}),
\]
and
\[
\exp \left( \frac{C_{11} \lambda^2 (d-1)^{2r-1}}{n} \right) = 1 + O \left( \frac{\lambda^2 (d-1)^{2r-1}}{n} \right).
\]
This and an identically derived lower bound complete the proof. \( \blacksquare \)

**Proof of Corollary 1.8** Suppose that \( \mu \) and \( \nu \) are probability measures on a discrete space \( \Omega \), and suppose that for some set \( A \subset \Omega \),
\[
\sum_{x \in A} |\mu(x) - \nu(x)| \leq \epsilon_1
\]
and \( \mu(A^c) \leq \epsilon_2 \). Then it is easily checked that \( d_{TV}(\mu, \nu) \leq \epsilon_1 + \epsilon_2 \). By virtue of Propositions 1.7 and 3.8, this is precisely the situation in which we are here. We note that we may assume that \( C_1 \left( (\log n)^2 \right) (d-1)^{2r-1} < n/2 \), for by adjusting \( C_2 \) to be sufficiently large, we may make the bound trivial. \( \blacksquare \)

4. **Variance calculation**

An alternative formulation of the second moment calculation that we need to make comes from the mixed model \( \mathcal{F}_{n,d} \). The quantity we need to estimate is \( \mathbb{E} H_n^2(P) \) with \( P \) drawn from the pairing model \( \mathcal{P}_{n,d} \). As \( H_n \) is the rescaled Radon-Nikodym derivative of \( \mathcal{F}_{n,d} \) with respect to \( \mathcal{P}_{n,d} \), it follows that
\[
\frac{\mathbb{E}_{\mathcal{P}_{n,d}}[H_n^2]}{\left( \mathbb{E}_{\mathcal{P}_{n,d}}[H_n] \right)^2} = \frac{\mathbb{E}_{\mathcal{P}_{n,d}}[H_n]}{\mathbb{E}_{\mathcal{P}_{n,d}}[H_n]}
\]
By the symmetry of both models, every fixed Hamiltonian cycle is equally probable in either \( \mathcal{P}_{n,d} \) or in \( \mathcal{F}_{n,d} \), and therefore, dividing through by the number of Hamiltonian cycles, it is equivalent to consider the ratio of probabilities of a fixed Hamiltonian cycle appearing. Thus, we fix distinct prevertices \( v_1, v_2, v_3, \ldots, v_{2n} \) where \( v_{2i}, v_{2i+1} \) come from vertex bin \( i \), and we consider the graph \( \Lambda \) on \( \{v_i\}_{i=1}^{2n} \) with edges \( \mathcal{E}(\Lambda) = \{v_{2i}v_{2i+1}\}_{i=1}^{n} \), where we let \( v_{2n+1} = v_1 \). Let \( E \) denote the event that a pairing contains \( \Lambda \) as a subgraph. By the note above, we have that
\[
\frac{\mathbb{E}_{\mathcal{P}_{n,d}}[H_n^2]}{\left( \mathbb{E}_{\mathcal{P}_{n,d}}[H_n] \right)^2} = \frac{\Pr_{\mathcal{F}_{n,d}}[E]}{\Pr_{\mathcal{P}_{n,d}}[E]}
\]
In \( \mathcal{F}_{n,d} \), the orderings of prevertices within each bin are uniformly and independently randomized. Thus, it can happen that one of \( \{v_2, v_3\} \) comes from the \( d-2 \) configuration graph, and the other comes from the superimposed Hamiltonian cycle. In this case, the event \( E \) cannot happen. Extending this idea, for \( E \) to happen it is necessary that all the edges of \( \Lambda \) have both endpoints coming from either the superimposed cycle or the underlying configuration graph. If this happens, \( \Lambda \) can have its edges labeled according to its endpoints. We color edges with endpoints in the configuration model \( \mathcal{R} \) and label them with \( R \), and we color edges coming from the Hamiltonian cycle \( \mathcal{B} \) and label them with \( B \). Formally, in this case there is a well-defined edge coloring \( \ell : \mathcal{E}(\Lambda) \to \{R, B\} \).
Define $V_n = \sum_{i=1}^{n} 1\{v_{2k}v_{2k+1} = R\}$. Conditioned on the edges of $\Lambda$ having the coloring $\ell$, it is straightforward to compute the conditional probability of $E$ in $\mathcal{G}_{n,d}$

$$
\Pr_{\mathcal{G}_{n,d}} [E \mid \ell] = \frac{1}{\| (d-2)n \|_{V_n} \| 2n-1 \|_{n-V_n}}.
$$

Meanwhile, it is possible to compute the exact probability under the law of $\mathcal{G}_{n,d}$ of a fixed edge coloring $\ell$ appearing. Let $b_2(\ell)$ be the number of vertex bins $i$ for which $\ell(\{v_{2i-1},v_{2i}\}) = B$ and $\ell(\{v_{2i-2},v_{2i+1}\}) = B$. Likewise, let $r_2(\ell)$ be the number of vertex bins for which $\ell(\{v_{2i},v_{2i+1}\}) = R$ and $\ell(\{v_{2i-1},v_{2i+1}\}) = R$. From the independence of the prevertex ordering,

$$
\Pr_{\mathcal{G}_{n,d}} [\ell] = \left( \frac{2}{|d|_2} \right)^{b_2(\ell)} \left( \frac{|d-2|_2}{|d|_2} \right)^{r_2(\ell)} \left( \frac{2(d-2)}{|d|_2} \right)^{n-b_2(\ell)-r_2(\ell)}.
$$

Combining (42) and (43), we have our first formula for $\Pr_{\mathcal{G}_{n,d}} (E)$, given by

$$
\Pr_{\mathcal{G}_{n,d}} [E] = \sum_{\ell} \left( \frac{2}{|d|_2} \right)^{b_2(\ell)} \left( \frac{|d-2|_2}{|d|_2} \right)^{r_2(\ell)} \left( \frac{2(d-2)}{|d|_2} \right)^{n-b_2(\ell)-r_2(\ell)} \Pr_{\mathcal{G}_{n,d}} [E \mid \ell],
$$

where the sum runs over all possible edge colorings $\ell$. However, this formula is ill-suited to asymptotic analysis, because exponentially rare $\ell$ contribute the majority of the sum. To rectify this, we define a new distribution on random colorings and use it to develop an alternate expression for $\Pr_{\mathcal{G}_{n,d}} (E)$. We will need to rescale $\Pr_{\mathcal{G}_{n,d}} [E \mid \ell]$ by $2^nV_{\ell}(d-2)V_n$. As $V_n(\ell)$ counts the total number of edges of the cycle colored $R$, we can express $V_n(\ell) = b_2(\ell) + (n - b_2(\ell) - r_2(\ell))/2$. Thus we define

$$
Z_\ell := \sum_{\ell} \left( \frac{1}{|d|_2} \right)^{b_2(\ell)} \left( \frac{|d-3|_2}{|d|_2} \right)^{r_2(\ell)} \left( \frac{\sqrt{2(d-2)}}{|d|_2} \right)^{n-b_2(\ell)-r_2(\ell)},
$$

again summing over all edge colorings.

For a cycle on vertices $[n]$ with nearest neighbor edges, define a law $\phi$ of a random edge-coloring $f$ on $\{R, B\}^n$ by

$$
\phi(\{f\}) := \frac{\left( \sqrt{2(d-2)} \right)^{n-b_2(f)-r_2(f)} (d-3)^{r_2(f)}}{Z_\phi},
$$

where $Z_\phi$ is a normalizing constant, $r_i(f)$ is the number of vertices with $i$ incident edges labeled $R$ and $b_i(f)$ is the number of vertices with $i$ incident edges labeled $B$.

Letting $V_n$ denote the number of $R$-labeled edges in a coloring sampled from $\{R, B\}^n$, this allows us to write

$$
\frac{\Pr_{\mathcal{G}_{n,d}} [E]}{Z_\ell} = \mathbb{E}_\phi \left( \frac{(d-2) V_n}{\| (d-2)n \|_{V_n} \| 2n-1 \|_{n-V_n}} \right),
$$

where we recall that $\| 2n-1 \|_{n-1} := \| 2n-1 \|_{n-V_n}$. It is fairly straightforward to calculate $Z_\ell$ exactly, as

$$
Z_\ell = \mathcal{O}_n \left( \frac{1}{|d|_2}, \frac{2(d-2)}{|d|_2}, \frac{d-3}{|d|_2} \right) = \left( \frac{1}{d} \right)^n + \left( -\frac{1}{|d|_2} \right)^n,
$$

by (9). Recalling that $\Pr_{\mathcal{G}_{n,d}} (E)$ is precisely $\| nd \|_n$, we can finally write

$$
\frac{\Pr_{\mathcal{G}_{n,d}} [E]}{\Pr_{\mathcal{G}_{n,d}} [E]} = \left( \mathbb{E}_\phi \left[ \frac{\| nd \|_n}{d^n} \frac{(d-2)^{V_n}}{\| (d-2)n \|_{V_n} \| 2n-1 \|_{n-V_n}} \right] \right) \left( 1 + \left( -\frac{1}{(d-1)} \right)^n \right),
$$

To estimate this expectation, we begin by approximating the integrand by something less complicated. This amounts to just applying Stirling’s approximation to each of the terms.

**Lemma 4.1.** Define $Z_n := \sqrt{\frac{d^3}{2n(d-2)^2}} (V_n - n d/2)$. Then,

$$
\frac{\| nd \|_n}{d^n} \frac{(d-2)^{V_n}}{\| (d-2)n \|_{V_n} \| 2n-1 \|_{n-V_n}} \leq \exp \left( \frac{Z_n^2}{d} + \xi_n \right) \sqrt{\frac{2n+1}{2V_n+1}},
$$

where

$$
\xi_n = \frac{d}{2} \left( \sqrt{\frac{d}{2n(d-2)^2}} + \frac{1}{2d} \right).
$$

This expression allows us to approximate $Z_n$ by $Z_n$, which is given by

$$
Z_n = \sqrt{\frac{d^3}{2n(d-2)^2}} (V_n - n d/2).
$$

Using this approximation, we can then write

$$
\frac{\Pr_{\mathcal{G}_{n,d}} [E]}{\Pr_{\mathcal{G}_{n,d}} [E]} \leq \exp \left( \frac{Z_n^2}{d} + \xi_n \right) \sqrt{\frac{2n+1}{2V_n+1}},
$$

which provides a useful bound on the probability of edge-coloring. This bound is particularly useful when $V_n$ is large and $d$ is small.
where \( \xi_n \) satisfies a bound of the form

\[
\xi_n \leq C_{12} \left( \frac{1}{n} + \frac{1}{V_n + 1} + \frac{1}{1 + n - V_n} \right),
\]

for some absolute constant \( C_{12} \).

**Proof.** By standard Stirling’s approximation, which we write in the form

\[
n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n e^{\lambda_n}
\]

for some \( \frac{1}{12n+1} \leq \lambda_n \leq \frac{1}{12n} \), we may approximate the \([a]_b\) terms. Specifically, we have

\[
\log \left( \frac{a}{e} \right)^b \left( \frac{a-2b}{a} \right)^{-\frac{2b}{2}} \leq \frac{C_{13}}{a} + \frac{C_{13}}{(2 + a - 2b)}
\]

for some absolute constant \( C_{13} \) and any \( a \geq 2 \) and \( b \geq 0 \) so that \( a - 2b \geq 0 \). We take the convention here that \( 0^0 = 1 \).

By applying this approximation, we get that

\[
\frac{[n d]_n}{d^n} \frac{(d-2)^V_n}{[(d-2)n]_{V_n}} \frac{2^{n-V_n}}{[2n]_{n-V_n}} = \\
\left( 1 + \frac{d}{d-2} \frac{V_n}{n} \right)^{(d-2)n + V_n} \left( 1 - \frac{2d}{(d-2)^2} \frac{V_n}{n} \right)^{(d-2)^2 + V_n} e^{\xi_n},
\]

where \( \bar{V}_n = V_n - \frac{d-2}{d} n \) and \( \xi_n \) is defined implicitly to make this an equality. Note that \( \xi_n \) satisfies the desired error bound by (48). Also note that the left hand side is not exactly the expression we need to approximate, as we have replaced \([2n-1]_{n-V_n}\) by \([2n]_{n-V_n}\).

By applying the bound \( 1 + a \leq e^a \) to (49) we get that

\[
\frac{[n d]_n}{d^n} \frac{(d-2)^V_n}{[(d-2)n]_{V_n}} \frac{2^{n-V_n}}{[2n]_{n-V_n}} \leq \frac{[2n]_{n-V_n}}{[2n-1]_{n-V_n}} \exp \left( \frac{2^2}{d} + \xi_n \right),
\]

and hence it suffices to show that there is some other error bound \( \xi_n' \) of the right form so that

\[
\frac{[2n]_{n-V_n}}{[2n-1]_{n-V_n}} \leq \sqrt{\frac{2n+1}{2V_n+1}} e^{\xi_n'}.
\]

For \( V_n \geq 1 \), we have that

\[
\frac{[2n]_{n-V_n}}{[2n-1]_{n-V_n}} = \left( \frac{2n}{2n-1} \right)^n \left( \frac{2V_n - 1}{2V_n} \right)^{V_n} \sqrt{\frac{2n-1}{2V_n-1}} e^{\xi_n'}.
\]

We bound the exponentials using \( 1 + a \leq e^a \). As for the radical, there is an absolute constant \( C_{14} \) so that for \( V_n \geq 1 \) we have

\[
\sqrt{\frac{2n-1}{2V_n-1}} \leq \sqrt{\frac{2n+1}{2V_n+1}} (1 + C_{14}(1/n + 1/V_n))
\]

Hence we get

\[
\frac{[2n]_{n-V_n}}{[2n-1]_{n-V_n}} \leq \sqrt{\frac{2n+1}{2V_n+1}} e^{\xi_n''},
\]

for some other error term \( \xi_n'' \) of the right form. In the case that \( V_n = 0 \), we have

\[
\frac{[2n]_{n-V_n}}{[2n-1]_{n-V_n}} = \left( \frac{2n}{n} \right)^2.
\]
which by direct approximation, is $2\sqrt{n}/\pi(1 + O(1/n))$. This is bounded by $\sqrt{2n}(1 + O(1/n))$, and by adjusting constants, we get that

$$\frac{\lfloor 2n \rfloor_{n-V_n}}{\lfloor 2n - 1 \rfloor_{n-V_n}} \leq \sqrt{\frac{2n + 1}{2V_n + 1}} \exp \left( C_{12}/n + C_{12}/(V_n + 1) \right)$$

for some absolute constant $C_{12}$. □

We will see that $Z_n$ is approximately standard normal and $\xi_n$ is negligible; making these replacements would give the desired $d/(d-2)$ in this expression. Executing the actual approximation is delicate, however, due to the Gaussian integral term; especially, we require a very strong Gaussian tail bound on $Z_n$. This rules out many available techniques for showing Gaussian concentration, as they do not provide sufficiently sharp constants. We prove a tail bound by a detailed analysis of the Laplace transform that is good enough for these purposes.

**Lemma 4.2.** For all $t \geq 0$

$$\Pr \left[ |V_n - \frac{d-2}{d} n| \geq t \right] \leq 8 \exp \left( -\frac{t^2}{2nc_d} \right),$$

where

$$c_d = \begin{cases} \frac{\sqrt{2}}{\sqrt{\pi}} & \text{if } d = 3, \\ \frac{2\sqrt{2}}{\sqrt{\pi}} & \text{if } d = 4, \\ \frac{\sqrt{2/(d-3)}}{\pi \sqrt{\sigma_d}} & \text{if } d \geq 5. \end{cases}$$

**Remark 4.3.** This tail bound is the principal reason that the error term in Proposition 1.9 has suboptimal $d$-dependence. The term $Z_n$ is chosen to have limiting variance 1, and thus $c_d$ would ideally behave more like $1/d$.

**Proof.** The key to computing the Laplace transform are the polynomials $\varrho_n(a, b, c)$ from Section 2. These polynomials give an explicit expression for the Laplace transform of $V_n$. In the notation of Section 2, $V_n$ can be written as $c_2 + c_1/2$, and so

$$E_\phi \exp(sV_n) = \frac{\varrho_n(a, be^s, ce^s)}{Z_\phi},$$

with $a = 1$, $b = 2(d-2)$ and $c = d-3$. Note that $Z_\phi = \varrho_n(a, b, c)$. In both cases, these polynomials can be written as $\tau_+^n + \tau_-^n$ for certain expressions in $a, b, c$. Explicitly, we recall (0):

$$\varrho_n(a, b, c) = \tau_+^n + \tau_-^n \quad \text{for } n \geq 1,$$

where $\tau_\pm = \frac{c + a \pm \sqrt{(c-a)^2 + 4b}}{2}$,

for all $a, b, c$. For all non-negative values of $a, b, c$, we have that $\tau_+ \geq |\tau_-|$. For these specific values: $a = 1$, $b = 2(d-2)$ and $c = d-3$, we have that

$$Z_\phi = \varrho_n(a, b, c) = (d-1)^n + (-1)^n \geq \frac{1}{d} (d-1)^n,$$

for $n \geq 1$ and $d \geq 3$. Combining these observations, we have

$$E_\phi \exp(sV_n) \leq 4(\tau_+^n)^n,$$

where $\tau_+ = \frac{ce^{s+a} + \sqrt{(ce^{s-a})^2 + 4be^s}}{2}$, with $a = \frac{1}{d-1}$, $b = \frac{2(d-2)}{(d-1)^2}$ and $c = \frac{d-3}{d-1}$.

We note that $\tau_+(0) = 1$ and that

$$\lim_{s \to 0} \frac{\log \tau_+(s)}{s} = \frac{d-2}{d}.$$

We proceed to estimating the derivative $(\log \tau_+(s)/s)'$, which we would like to bound by a constant. First, we note that we can pull out a factor of $e^{s/2}$ and keep the derivative the same, i.e.

$$\left( \frac{\log \tau_+(s)}{s} \right)' = \left( \frac{e^{s/2} \tau_+(s)}{s} \right)'.$$
So, we define \( q(s) = e^{-s/2} \tau_+(s) \). By doing integration by parts, we have that
\[
\log q(s) = s(\log q(s))' - s \int_0^s t (\log q(t))'' \, dt,
\]
and thus
\[
\left( \frac{\log q(s)}{s} \right)' = \frac{1}{s^2} \int_0^s t (\log q(t))'' \, dt.
\]
Therefore, it suffices to bound \((\log q(t))''\) above. Let \( f(s) = ce^{s/2} - ae^{-s/2} \), in terms of which we can write
\[
q(s) = \frac{2f'(s) + \sqrt{f(s)^2 + 4b}}{2}.
\]
Noting that \( f''(s) = \frac{1}{2}f(s) \), it is easily verified that
\[
(\log q(s))'' = \frac{2bf'(s)}{(f(s)^2 + 4b)^{3/2}}.
\]
This expression is \( C^1 \) for all \( s \in \mathbb{R} \). Further, it tends to 0 at both \( \pm \infty \), and so its maximum occurs at one of its critical points. By squaring and differentiating, it follows that its extrema occur at the roots of
\[
\frac{1}{2}f'(s)f(s)(f(s)^2 + 4b)^3 - 6(f'(s))^2(f(s)^2 + 4b)^2f'(s)f(s) = 0.
\]
When \( d \geq 4 \), there are three possible roots, given by the root of \( f(s) = 0 \) and possibly 2 roots of \( f(s)^2 + 4b - 12(f'(s))^2 = 0 \). These values are given by
\[
e^s = \frac{1}{d-3}, \quad \text{or} \quad e^s = \frac{4 \pm \sqrt{16 - 4(d-3)^2}}{2(d-3)}.
\]
Thus for \( d \geq 6 \), the maximum is given by the first root. For \( d = 5 \), the roots all coincide at \( e^s = \frac{1}{2} \). For \( d = 4 \), there are 3 distinct roots to check.

In the \( d = 3 \) case, it is no longer possible for \( f(s) = 0 \), but the equation \( f(s)^2 + 4b - 12(f'(s))^2 = 0 \) still has a root; however, the expression is no longer quadratic. We summarize the results of this calculus in the following table:

| \( d \) | Critical points | Maximizers | Maximum |
|--------|----------------|------------|---------|
| 3      | \( e^s = \frac{1}{2} \) | \( \frac{1}{3} \) | \( \frac{\sqrt{3}}{\sqrt{1}} \) |
| 4      | \( e^s = \frac{1}{d-3}, 2 \pm \sqrt{3} \) | \( 2 + \sqrt{3} \) | \( \frac{2\sqrt{3}}{\sqrt{3}} \) |
| 5      | \( e^s = \frac{1}{2} \) | \( \frac{1}{2} \) | \( \sqrt{2(d-3)} \) |
| \( d \geq 6 \) | \( e^s = \frac{1}{d-3} \) | \( \frac{1}{d-3} \) | \( \sqrt{\frac{d-3}{d-3}} \) |

All together this shows that, recalling equation \([50]\), that
\[
\left( \frac{\log \tau_+(s)}{s} \right)' \leq \frac{c_d}{2}.
\]
Integrating, we have that
\[
\tau_+(s) \leq \exp \left( \frac{d-2}{d} s + c_d s^2 \right),
\]
for all \( s \) and hence, by Markov’s inequality,
\[
\Pr \left[ |V_n - \frac{d-2}{d} n| \geq t \right] \leq 8 \exp \left( nc_d \frac{s^2}{2} - st \right),
\]
for all \( s \). Optimizing in \( s \) produces the stated bound. \( \square \)

As a consequence, we are able to estimate some small moments of \( \exp(Z_n^2/d) \) uniformly in \( d \) and \( n \).
Lemma 4.4. For every $\alpha$ with $1 \leq \alpha < 2/\sqrt{3}$, there is a constant $M_\alpha$ so that
\[ \mathbb{E}_\phi \exp(\alpha \frac{Z_n^2}{d}) \leq M_\alpha. \]
Further, for every $\alpha$ with $1 \leq \alpha < 8/\sqrt{2}$, there is a $d_0(\alpha)$ and a constant $M_\alpha$ so that
\[ \mathbb{E}_\phi \exp(\alpha \frac{Z_n^2}{d}) \leq M_\alpha \]
for all $d \geq d_0$.

Proof. By scaling the tail bound in Lemma 4.2, we have
\[ \Pr \left[ \frac{Z_n}{\sqrt{d}} \geq t \right] \leq 8 \exp \left( -t^2 \beta \right), \]
where
\[ \beta := \frac{(d - 2)^2}{\alpha d^2 c_d}, \]
with equality when $d = 3$. Since, we now take
\[ \mathbb{E}_\phi \exp(\alpha \frac{Z_n^2}{d}) = \int_0^\infty e^t \Pr \left[ \frac{Z_n}{\sqrt{d}} \geq t \right] dt = \frac{8}{\beta - 1}, \]
provided $\beta > 1$. Thus it suffices to bound $\beta$ from below to control this constant. On the one hand, we have that for all $d \geq 3$, $\beta \geq \frac{8}{\sqrt{2} d^2}$, with equality when $d = 3$. On the other hand, we have that $\beta \to \frac{8}{\sqrt{2} \alpha}$ as $d \to \infty$, from which follows the second statement. \qed

Using this tail bound, we are able to estimate the contributions of the subexponential terms to the expectation, so that we have

Lemma 4.5. For $d \leq n^{1/2}/ \log n$,
\[ \mathbb{E}_\phi \left[ \frac{n d}{d^n} \left( \frac{(d - 2)^2}{d} \right)^{2n - \nu_{n}} \right] \leq \sqrt{\frac{d}{d - 2}} \mathbb{E}_\phi \exp \left( \frac{Z_n^2}{d} \right) + O \left( \frac{1}{\sqrt{n}} \right). \]

Proof. Our starting point is Lemma 4.1, we must bound
\[ \mathbb{E}_\phi \exp \left( \frac{Z_n^2}{d} + \xi_n \right) \sqrt{\frac{2n + 1}{2V_n + 1}}. \]
We first approximate this sum by replacing the $V_n$ in the square root by $\frac{d - 2}{d} n$. Thus, we seek to estimate
\[ E_1 := \mathbb{E}_\phi \exp \left( \frac{Z_n^2}{d} + \xi_n \right) \sqrt{\frac{2n + 1}{2V_n + 1} - \frac{2n + 1}{2(\frac{d - 2}{d} n + 1)}}, \]
from above. Let $f(x) = \sqrt{\frac{2x + 1}{2x + 1}}$. Note that there is a constant $C_{15}$ so that $\exp(\xi_n) \leq C_{15}$ with probability 1 for all $n$ and $d$, so that
\[ E_1 \leq C_{15} \mathbb{E}_\phi \exp \left( \frac{Z_n^2}{d} \right) \left| f(V_n) - f\left( \frac{d - 2}{d} n \right) \right|. \]
Fix some $\alpha$ with $1 < \alpha < 2/\sqrt{3}$ and apply Hölder’s inequality with exponent $\alpha$ and conjugate $\alpha^*$ to get
\[ E_1 \leq C_{15} \left( \mathbb{E}_\phi \exp \left( \alpha \frac{Z_n^2}{d} \right) \right)^{1/\alpha} \left( \mathbb{E}_\phi \left| f(V_n) - f\left( \frac{d - 2}{d} n \right) \right|^{\alpha^*} \right)^{1/\alpha^*}. \]
We note that $\frac{d - 2}{d} \geq \frac{1}{4}$ for all $d \geq 3$, and therefore by Lemma 4.2, there is some absolute constant $C_{16}$ so that
\[ \Pr \left[ V_n \leq \frac{1}{6} n \right] \leq \frac{1}{e} e^{-C_{16} n}. \]
The largest possible value of $f(V_n)$ is $\sqrt{2n + 1}$, and thus we have
\[ \mathbb{E}_\phi \left| f(V_n) - f\left( \frac{d - 2}{d} n \right) \right|^{\alpha^*} = O(n^{\alpha^*/2} e^{-C_{16} n}) + \mathbb{E}_\phi \left| f(V_n) - f\left( \frac{d - 2}{d} n \right) \right|^{\alpha^*} 1 \{ V_n \geq \frac{1}{6} n \}. \]
To estimate this other bit, we note that $|f'(x)/n|$ is bounded uniformly in $n$ for $x \geq \frac{1}{6} n$, and hence
\[
E_\phi \left| f(V_n) - f\left(\frac{d-2}{d} n \right) \right|^\alpha \mathbf{1}\{V_n \geq \frac{1}{6} n\} = O\left( E_\phi \left| V_n - \frac{d-2}{d} n \right|^{\alpha} \right) = O(n^{-\alpha/2}).
\]
Combining everything, we have that $E_1 = O(n^{-1/2})$, and we have therefore shown that
\[
E_\phi \exp \left( \frac{Z_n^2}{d} + \xi_n \right) \sqrt{\frac{2n+1}{2V_n+1}} \leq E_\phi \exp \left( \frac{Z_n^2}{d} + \xi_n \right) \sqrt{\frac{2n+1}{2d^2n+1}} + O(n^{-1/2}).
\]
Note that this radical is always less than $\sqrt{\frac{d}{d-2}}$, and so we turn to removing $\xi_n$; we must now bound
\[
E_2 := \sqrt{\frac{d}{d-2}} E_\phi \exp \left( \frac{Z_n^2}{d} \right) |e^{\xi_n} - 1|.
\]
Again, we apply Hölder’s inequality with the same $\alpha$ and in the same way as (51) to get
\[
E_2 = O\left( \left( E_\phi \left| e^{\xi_n} - 1 \right|^{\alpha} \right)^{1/\alpha} \right).
\]
Since $\xi_n$ is bounded uniformly in $n$, we have by Taylor approximation that
\[
|e^{\xi_n} - 1| \leq C_{17} \left( \frac{1}{V_n+1} + \frac{1}{1+n-V_n} \right)
\]
for some absolute constant $C_{17}$, which follows from Lemma 4.1. Thus, by the triangle inequality, it suffices to bound
\[
E_2 = O\left( \left( \frac{1}{V_n+1} \right)^{1/\alpha} + \left( \frac{1}{1+n-V_n} \right)^{1/\alpha} \right).
\]
For the first one, we have that by (52), the $\frac{1}{V_n+1}$ term is $O(n^{-1})$ except for with probability $O(e^{-C_{18}n})$. For the second, we note that $\frac{1}{1+n-V_n}$ has more complicated $d$ dependence, as when $d$ is large, the mean of $V_n$ is nearly $n$. That said, there is some absolute constant $C_{18} > 0$ so that
\[
\Pr \left[ V_n \geq \frac{d-1}{d} n \right] \leq \frac{1}{C_{18}} e^{-C_{18}n/d^2},
\]
which follows immediately by Lemma 4.2. Thus the $\frac{1}{1+n-V_n}$ term is $O(d/n)$ except for with probability $O(e^{-C_{18}n/d^2})$. By assumption that $d \leq \sqrt{n}/\log n$, this probability decays faster than any power of $n$, and certainly it is $O(1/\sqrt{n})$.

Combining these bounds, we get that
\[
\left( E_\phi \left| e^{\xi_n} - 1 \right|^{\alpha} \right)^{1/\alpha} = O\left( \frac{1}{\sqrt{n}} \right).
\]
Thus $E_1 = O(1/\sqrt{n})$ and $E_2 = O(1/\sqrt{n})$, which completes the proof. \qed

### 4.1. Markov chain approximation.
We will replace $\phi$ with a distribution that is amenable to easier analysis. Underlying this replacement is the idea that a random coloring $f$ drawn from $\phi$ produces a vector $(f(1), f(2), \ldots, f(n))$ that has nearly the same distribution as $(X_1, X_2, \ldots, X_n)$ where $X_k$ is the Markov chain on $\{R, B\}$ with transition probabilities $\Pr \left[ X_{k+1} = y \mid X_k = x \right] = p(x, y)$, and where $p(x, y)$ is given by
\[
p(R, R) = \frac{d-3}{d-1}, \quad p(R, B) = \frac{2}{d-1},
\]
\[
p(B, R) = \frac{d-2}{d-1}, \quad p(B, B) = \frac{1}{d-1}.
\]
This chain is easily checked to have stationary distribution that puts mass $(d-2)/d$ on $\{R\}$ and mass $2/d$ on $\{B\}$, and we will consider this chain started from stationarity.
This is a rapidly mixing chain, and its mixing properties can be controlled by the contraction coefficient \(\theta\), which for this chain is
\[
\theta := d_{TV}(\mathcal{L}(X_2 \mid X_1 = R), \mathcal{L}(X_2 \mid X_1 = B)) = \frac{1}{d-1},
\]
with \(\mathcal{L}\) denoting the law of a random variable. This gives a simple bound for the rate at which two Markov chains with the same transitions as \(X_k\) can be coupled. Suppose that \(\{X^1_k\}\) and \(\{X^2_k\}\) are two chains with the same transitions as \(X_k\) but with different starting states. There is a coupling of these two chains so that \(\tau = \inf \{k \geq 0 \mid X^1_k = X^2_k\}\) has \(\Pr[\tau > k] \leq \theta^k\).

The chain implicitly defines a distribution on edge colorings by simply defining a coloring \(f \in \{R, B\}^n\) by \(f(k) := X_k\). We will refer to the law on colorings defined in this way as \(\pi\). The precise relationship between \(\phi\) and \(\pi\) is that \(\phi\) is absolutely continuous with respect to \(\pi\), and the unscaled Radon-Nikodym derivative of \(\phi\) with respect to \(\pi\) is
\[
\rho(f) := \begin{cases} 
2(d - 3) & \text{if } f(1) = f(n) = R \\
2(d - 2) & \text{if } (f(n), f(1)) = (R, B) \\
2(d - 2) & \text{if } (f(n), f(1)) = (B, R) \\
(d - 2) & \text{if } f(1) = f(n) = B.
\end{cases}
\]

**Lemma 4.6.** With \(\pi\) as defined above,

\[
\frac{d\phi}{d\pi}(f) = \frac{\rho(f)}{E_{\pi}\rho(f)}.
\]

**Proof.** For an edge coloring \(f\) of the cycle, recall that \(r_i\) denotes the number of vertices with \(i\) neighboring edges colored \(R\) and \(b_i\) denotes the number of vertices with \(i\) neighboring edges colored \(B\). Likewise, let \(rb\) denote the number of vertices \(j\) with \(f(j - 1) = R\) and \(f(j) = B\), with the addition done mod \(n\). Similarly, let \(br\) denote the number of vertices \(j\) with \(f(j - 1) = B\) and \(f(j) = R\), with the addition done mod \(n\). Then, it follows that \(r_1 = b_1 = br + rb\), but also, because this a cycle, it must be that \(br = rb\).

For any coloring \(f\),
\[
\pi(\{f\})\rho(f) \propto (d - 3)^{r_2}2^{rb}(d - 2)^{br}1^{b_2}.
\]

On the other hand,
\[
\phi(\{f\}) \propto (d - 3)^{r_2}\sqrt{2(d - 2)^{f_1}}1^{b_2}.
\]

Using that \(r_1 = 2br = 2rb\), it now follows that \(\rho\) is the unscaled Radon-Nikodym derivative.

Using the Radon-Nikodym derivatives, we can transfer moment estimates from \(\phi\) to \(\pi\) with little effort.

**Lemma 4.7.** For every \(\alpha\) with \(1 \leq \alpha < 8/\sqrt{2}\), there is a constant \(M_\alpha\) and a constant \(d_0(\alpha)\) so that
\[
E_{\pi}\exp(\alpha \frac{Z_n^2}{d}) \leq M_\alpha d
\]
da\(n\) and all \(d \geq d_0\). If \(\alpha < 2/\sqrt{3}\), we can take \(d_0 = 3\). Furthermore, we have that for all \(t \geq 0\),
\[
\Pr[|Z_t| \geq t\sqrt{d}] \leq dM_\alpha \exp(-\alpha t^2).
\]

**Proof.** The second conclusion of the lemma follows immediately from the first by Markov’s inequality. As for the first, in the case that \(d \geq 4\), this is simply a consequence of Lemma 4.4 and the fact that \(\rho\) is bounded below by 1 \(\pi\)-almost surely; note
\[
M_\alpha \geq E_{\phi}\exp\left(\alpha \frac{Z_n^2}{d}\right) = E_{\pi}\rho(f)\exp\left(\alpha \frac{Z_n^2}{d}\right) \geq \frac{1}{E_{\pi}\rho(f)}E_{\pi}\exp\left(\alpha \frac{Z_n^2}{d}\right),
\]
so that rearranging,
\[
E_{\pi}\exp\left(\alpha \frac{Z_n^2}{d}\right) \leq E_{\pi}\rho(f)M_\alpha,
\]
and the result now follows from having \(E_{\pi}\rho(f) = O(d)\).
However, when $d = 3$, we require an additional argument, because $\rho$ can be 0. Consider the involution $\iota$ on colorings that swaps the color $f(n)$ between $R$ and $B$. Let $\iota^*(Z_n(f))$ denote the random variable $Z_n(\iota(f))$, so that we have

$$\exp\left(\frac{\alpha Z_n^2}{3}\right) \mathbf{1}\{f(n) = R\} = \exp\left(\frac{\alpha}{3}\iota^*(Z_n)^2 + \frac{2\iota^*(Z_n)q + q^2}{3}\right) \mathbf{1}\{\iota(f)(n) = B\},$$

where $q = \sqrt{\frac{3^3}{2n(3-2)^2}}$. For any coloring $f$ with $f(n) = R$, meanwhile, it must be that $f(n - 1) = B$ else $\pi(\{f\}) = 0$. Thus, for any coloring with $f(n - 1) = B$ and $f(n) = R$, we have that

$$\pi(\{f\}) = \pi(\{\iota(f)\}).$$

Thus, we can change the integration and get that

$$\mathbb{E}_\pi \exp\left(\frac{\alpha Z_n^2}{3}\right) \mathbf{1}\{f(n) = R\} = \mathbb{E}_\pi \exp\left(\frac{\alpha}{3}\iota^*(Z_n)^2 + \frac{2\iota^*(Z_n)q + q^2}{3}\right) \mathbf{1}\{f(n) = B\}.$$

This right-hand side can now be bounded in terms of $\phi$ by

$$\mathbb{E}_\pi \exp\left(\frac{\alpha Z_n^2}{3} + \frac{2Z_nq + q^2}{3}\right) \mathbf{1}\{f(n) = B\} \leq C_{19}\mathbb{E}_\phi \exp\left(\frac{\alpha Z_n^2}{3} + \frac{2Z_nq}{3}\right),$$

for some absolute constant $C_{19}$ as when $f(n) = B$, $\rho(f)$ is bounded below. Pick $\alpha'$ so that $\frac{2}{\sqrt{3}} > \alpha' > \alpha$. By Hölder’s inequality, we have that

$$\mathbb{E}_\phi \exp\left(\frac{\alpha Z_n^2}{3} + \frac{2Z_nq}{3}\right) \leq \left(\mathbb{E}_\phi \exp\left(\frac{\alpha Z_n^2}{3}\right)\right)^\frac{\alpha'}{\alpha}\left(\mathbb{E}_\phi \exp\left(\frac{\alpha'Z_n^2}{3}\right)\right)^\frac{\alpha - \alpha'}{\alpha'},$$

which is bounded uniformly in $n$ by Lemma 4.4.

The Radon-Nikodym derivative can be seen to be approximately independent of $Z_n$, as $Z_n$ is insensitive to a change of only 2 coordinates. For this reason, we can prove

**Lemma 4.8.**

$$\mathbb{E}_\phi \exp \left(\frac{Z_n^2}{d}\right) = \mathbb{E}_\pi \frac{\rho(f)}{\mathbb{E}_\pi \rho(f)} \exp \left(\frac{Z_n^2}{d}\right) = \mathbb{E}_\pi \exp \left(\frac{Z_n^2}{d}\right) + O\left(\sqrt{\frac{d}{n}}\right).$$

**Proof.** We need to prove that

$$\mathbb{E}_\pi \left[ \exp \left(\frac{Z_n^2}{d}\right) \mid X_1 = x, X_n = y \right] - \mathbb{E}_\pi \exp \left(\frac{Z_n^2}{d}\right)$$

is small, regardless of $x$ and $y$. To simplify notation, replace $B$ and $R$ with 0 and 1. Let $\{Y_1, \ldots, Y_n\}$ be a Markov chain with the same transition probabilities as $\{X_1, \ldots, X_n\}$, but started at $Y_1 = x$. We take the two chains to have the optimal Markovian coupling: conditional on $X_i$ and $Y_i$, the random variables $X_{i+1}$ and $Y_{i+1}$ are coupled by the optimal total variation coupling. Let $\tau$ be the first time that the two chains coincide (after which they stay together), or $\infty$ if they never do. For a chain on $\{0, 1\}$ with transition probability from 0 to 0 smaller than from 1 to 0, this coupling has the property that

$$(X_i, Y_i) = (1 - x, x) \quad \text{for all odd } i < \tau,$$

$$(X_i, Y_i) = (x, 1 - x) \quad \text{for all even } i < \tau.$$

Thus the sums of two chains differ by at most one, indicating that this statistic is quite insensitive to the starting point of the chain. We will write $\mathbb{E}_\pi[\cdot]$ with no subscript to indicate expectations with respect to this coupling, reserving the notation $\mathbb{E}_\pi[\cdot]$ for expectations that depend only on the first chain.
Let \( q = \sqrt{\frac{d^2}{2n(d-2)^2}} \). Let \( V'_n = \sum_{i=1}^{n} Y_i \), and let \( Z'_n = q(V'_n - dn/(d-2)) \). We rewrite the conditional expectation as

\[
\mathbb{E} \left[ \exp \left( \frac{Z_n^2}{d} \right) \mid X_1 = x, X_n = y \right] = \frac{\mathbb{E} \left[ \exp \left( \frac{Z_n^2}{d} \right) \mathbb{1}_{\{Y_n = y\}} \right]}{\mathbb{P}[Y_n = y]} \\
\leq \frac{\mathbb{E} \left[ \exp \left( \frac{Z_n^2}{d} \right) \mathbb{1}_{\{Y_n = y\}} \right]}{\mu(y) - (d-1)^{1-n}}.
\]

(55)

By the properties of the coupling mentioned above, \( Z'_n \leq Z_n + q \). So long as \( \tau \neq \infty \), we have \( X_n = Y_n \), and so

\[
\mathbb{E} \left[ \exp \left( \frac{Z_n^2}{d} \right) \mathbb{1}_{\{Y_n = y\}} \right] \leq \mathbb{E}_\pi \left[ \exp \left( \frac{(Z_n + q)^2}{d} \right) \mathbb{1}_{\{X_n = y\}} \right] \\
+ \mathbb{E}_\pi \left[ \exp \left( \frac{Z_n^2}{d} \right) \mathbb{1}_{\{\tau = \infty\}} \right].
\]

(56)

If \( \tau = \infty \) and \( n \) is even, then \( V'_n = n/2 \), and

\[
\frac{Z_n^2}{d} = \frac{(d-4)^2}{8(d-2)^2} \leq \frac{n}{8}.
\]

If \( \tau = \infty \) and \( n \) is odd, then \( V'_n = (n \pm 1)/2 \), and some algebra shows that \( Z_n^2/d \leq (n+1)/8 \). Thus

\[
\mathbb{E} \left[ \exp \left( \frac{Z_n^2}{d} \right) \mathbb{1}_{\{\tau = \infty\}} \right] \leq \mu(1-x)(d-1)^{1-n} \exp \left( \frac{n+11}{8} \right),
\]

which is easily \( O(1/n) \).

To deal with the first term of (56), we use the reversibility of the Markov chain to rewrite it as

\[
\mathbb{E}_\pi \left[ \exp \left( \frac{(Z_n + q)^2}{d} \right) \mathbb{1}_{\{X_n = y\}} \right] = \mu(y) \mathbb{E}_\pi \left[ \exp \left( \frac{(Z_n + q)^2}{d} \right) \mid X_1 = y \right].
\]

As before, there exists a coupling of \( Z_n \) with a random variable \( Z''_n \) such that \( Z''_n \) is distributed as \( Z_n \) conditioned on \( X_1 = y \), and \( Z''_n \leq Z_n + q \). Thus

\[
\mathbb{E}_\pi \left[ \exp \left( \frac{(Z_n + q)^2}{d} \right) \mathbb{1}_{\{X_n = y\}} \right] \leq \mu(y) \mathbb{E}_\pi \exp \left( \frac{(Z_n + 2q)^2}{d} \right).
\]

Fix some \( 1 < \alpha < 2/\sqrt{3} \) and apply Hölder’s inequality to get

\[
\mathbb{E}_\pi \exp \left( \frac{(Z_n + 2q)^2}{d} \right) - \mathbb{E}_\pi \exp \left( \frac{Z_n^2}{d} \right) = \mathbb{E}_\pi \left[ \exp \left( \frac{Z_n^2}{d} \right) \left( \exp \left( \frac{4qZ_n + 4q^2}{d} \right) - 1 \right) \right] \\
\leq (dM_\alpha)^{1/\alpha} \mathbb{E}_\pi \left[ \exp \left( \frac{4qZ_n + 4q^2}{d} \right) - 1 \right]^{1/\alpha^*}.
\]

By applying the bounds that \( |e^x - 1| \leq |x|e^{|x|} \) and that \( q^2/d = O(1/n) \), there is some absolute constant \( C_{20} \) so that

\[
\mathbb{E}_\pi \left| \exp \left( \frac{4qZ_n + 4q^2}{d} \right) - 1 \right|^{\alpha^*} \leq C_{20} \mathbb{E}_\pi \left| \frac{Z_n q}{d} \exp \left( \frac{C_{20} Z_n q}{d} \right) \right|^{\alpha^*}.
\]

Note that \( \frac{q}{2} = O(1/\sqrt{dn}) \) and hence by once again applying Hölder’s inequality and using the second part of Lemma [4.7] we conclude that

\[
\mathbb{E}_\pi \left| \frac{Z_n q}{d} \exp \left( \frac{C_{20} Z_n q}{d} \right) \right|^{\alpha^*} = O(1/n^{\alpha^*}).
\]

This shows that

\[
\mathbb{E}_\pi \exp \left( \frac{(Z_n + 2q)^2}{d} \right) - \mathbb{E}_\pi \exp \left( \frac{Z_n^2}{d} \right) \leq O \left( \frac{d}{n} \right).
\]
Applying (57) and (58) to (56) and substituting into (55),
\[
\mathbb{E}_x \left[ \exp \left( \frac{Z_n^2}{d} \right) \right | X_1 = x, X_n = y \leq \frac{\mu(y)}{\mu(y) - (d - 1)^{1-n}} \left( \mathbb{E}_\pi \exp \left( \frac{Z_n^2}{d} \right) + O \left( \frac{d}{n} \right) \right) \\
\leq \mathbb{E}_\pi \exp \left( \frac{Z_n^2}{d} \right) + C_{21} \frac{d}{n}.
\]
for some absolute constant \( C_{21} \), uniformly in \( x \) and \( y \). The conclusion of the lemma now follows by integrating
\[
\mathbb{E}_x \left[ \frac{\rho(f)}{\mathbb{E}_\pi \rho(f)} \exp \left( \frac{Z_n^2}{d} \right) \right] = \mathbb{E}_x \left[ \frac{\rho(f)}{\mathbb{E}_\pi \rho(f)} \mathbb{E}_\pi \exp \left( \frac{Z_n^2}{d} \right) | X_1, X_2 \right] \\
\leq \mathbb{E}_x \left[ \frac{\rho(f)}{\mathbb{E}_\pi \rho(f)} \left[ \mathbb{E}_\pi \exp \left( \frac{Z_n^2}{d} \right) + C_{21} \frac{d}{n} \right] \right] \\
= \mathbb{E}_\pi \exp \left( \frac{Z_n^2}{d} \right) + O(d/n). \quad \square
\]

4.2. Comparison with a standard normal by size-bias coupling. The remainder of the work is to compare these expectations in \( Z_n \) with that which we would get for a standard normal. For this task, we develop a modification of Stein’s method for normal approximation that allows us to directly compare these expectations. The basic outline of this approach follows the general method of size-bias couplings for normal approximation.\(^2\)

We define \( h(w) = \exp(w^2/d) \), and let \( \Phi(h) = \sqrt{\frac{d}{\pi^2}} \) denote the expectation of \( h \) applied to a standard normal. We let \( f_h \) be the solution to the differential equation
\[
(59) \quad f_h'(w) - w f_h(w) = h(w) - \Phi(h)
\]
that is given by the formulae
\[
f_h = \exp(w^2/2) \int_w^\infty \exp(-t^2/2) (\Phi(h) - h(t)) \ dt \\
= - \exp(w^2/2) \int_{-\infty}^w \exp(-t^2/2) (\Phi(h) - h(t)) \ dt.
\]

In the usual Stein’s method setup, the function \( h \) is bounded, from which it follows that \( f_h' \) and \( f_h'' \) are also bounded. This is not the case here, but it is easily verified that the growth rates of \( f_h \) and its derivatives are commensurate to the growth rate of \( h \).

**Lemma 4.9.** There is an absolute constant \( C_{22} \) so that
\[
|f_h(w)| \leq C_{22} \Phi(h)(1 + |w|)^{-1} h(w) \\
|f_h'(w)| \leq C_{22} \Phi(h) h(w) \\
|f_h''(w)| \leq C_{22} \Phi(h)(1 + |w|) h(w).
\]

**Proof.** We begin by noting that for all \( w \neq 0 \),
\[
\exp(\alpha w^2) \int_{|w|}^\infty \exp(-\alpha x^2) \ dx \leq \exp(\alpha w^2) \int_{|w|}^\infty \frac{x}{|w|} \exp(-\alpha x^2) \ dx \leq \frac{1}{2\alpha |w|}.
\]
From this, we observe that for all \( w \),
\[
\exp(\alpha w^2) \int_{|w|}^\infty \exp(-\alpha x^2) \ dx \leq \sqrt{\pi/4\alpha},
\]
as its derivative in \( w \) is negative for \( w > 0 \). It follows that there is an absolute constant \( C_{23} \) so that
\[
|f_h(w)| \leq C_{23} \Phi(h) h(w)(1 \wedge \frac{1}{|w|}).
\]

---

\(^2\) See Ross’s excellent survey [Ros11] for an overview; we will frequently reference general results surrounding Stein methodology from this source.
From the differential equation (59), we have that
\[ |f'_h(w)| \leq |w f_h| + h(w) + \Phi(h) \leq C_{24} \Phi(h) h(w) \]
for some larger absolute constant \(C_{24}\). By differentiating the Stein equation (59), we may also bound
\[ |f''_h(w)| \leq |f_h| + |w f'_h| + |h'(w)| \leq C_{25} \Phi(h)(1 + |w|)h(w) \]
for some other absolute constant \(C_{25}\).

Using the basic Stein’s method setup for size-bias coupling (see equation (3.25) of \([\text{Ros}11]\)), we have the following lemma, which refers to a size-bias coupling \((V^*_n, V_n)\) and an associated probability space constructed in the appendix.

**Lemma 4.10.** Let \(\mu = n(d-2)/d = \mathbb{E}_\pi V_n\) and \(\sigma^2 = \frac{2n(d-2)}{d}\). For any \(\sigma\)-algebra \(\mathcal{F}\) containing \(\sigma(Z_n)\),
\[
\mathbb{E}_\pi \left[ \left| f'_h(Z_n) \left( 1 - \frac{\mu}{\sigma^2} \mathbb{E} |V^*_n - V_n| \mathbb{F} \right) \right| \right] = \frac{1}{\sqrt{n}}
\]
where \(Z^*_n\) is in the interval with endpoints \(Z_n\) and \((V^*_n - \mu)/\sigma\).

Using this lemma, we finally estimate the difference in the expectations.

**Lemma 4.11.** For \(d \leq \sqrt{n}\), and for any \(\alpha < 8/\sqrt{2}\), we have that
\[
\mathbb{E}_\pi \left[ \frac{1}{d} \right] = \sqrt{\frac{d}{d^2 - 2}} + O \left( \frac{d^{\frac{3}{2}(1+1/\alpha)}}{\sqrt{n}} \right).
\]

**Proof.** We consider the size-bias coupling considered in the appendix, and the only probability space under consideration in this proof will be the one constructed there. We start from Lemma 4.10, by virtue of which we need only bound
\[
E_1 := \mathbb{E} \left[ f'_h(Z_n) \left( 1 - \frac{\mu}{\sigma^2} \mathbb{E} |V^*_n - V_n| \mathbb{F} \right) \right], \text{ where } \mathbb{F} := \sigma(X_1, \ldots, X_n),
\]
and
\[
E_2 := \frac{\mu}{2\sigma^3} \mathbb{E} \left[ f''_h(Z_n) |V^*_n - V_n|\right].
\]
For \(E_1\), it will turn out that the expectation of \(V^*_n - V_n\) is not exactly \(\sigma^2/\mu\). On the other hand, by Proposition A.6, we have an exact expression for \(\mathbb{E} |V^*_n - V_n|\). We note that, in the notation of that section, \(\lambda = -1/(d-1)\) and that \(p = (d-2)/d\). It follows that
\[
\mathbb{E} |V^*_n - V_n| = \frac{2(d-2)}{(d-1)^2} + 4(d-1) \frac{\sigma^2}{\mu} + O \left( \frac{1}{d^2} \right).
\]
From Lemmas 4.9 and 4.7 we have that
\[
\mathbb{E} |f'_h(Z_n)| = O \left( \mathbb{E} \left[ \exp(|Z_n^2|/d) \right] \right) = O(d).
\]
Applying this to \(E_1\), we conclude that
\[
E_1 = \frac{\mu}{\sigma^2} \mathbb{E} \left[ f'_h(Z_n) \left( \mathbb{E} |V^*_n - V_n| \mathbb{F} \right) \right] + O \left( \frac{1}{n} \right).
\]
From Corollary A.5, we have a uniform Gaussian tail bound on
\[
\mathbb{E} |V^*_n - V_n| - \mathbb{E} |V^*_n - V_n| \mathbb{F} |.
\]
In the notation of that corollary, we have \(\theta = \frac{1}{\sqrt{d}}\). If \(d = 3\), then \(\delta = 2\) and \(\gamma = \frac{2}{3}\), and if \(d > 3\), then \(\delta = 1\) and \(\gamma = \frac{2}{3}\). Thus the corollary implies that there is an absolute constant \(C_{26} > 0\) so that for any \(t \geq 0\),
\[
\text{Pr} \left( \left| \mathbb{E} |V^*_n - V_n| - \mathbb{E} |V^*_n - V_n| \mathbb{F} | \right| \geq \frac{t}{\sqrt{n}} \right) \leq 2 \exp \left( -C_{26} t^2 \right).
\]
In particular, this implies that for each fixed \(t > 0\),
\[
\mathbb{E} |\mathbb{E} |V^*_n - V_n| - \mathbb{E} |V^*_n - V_n| \mathbb{F}| | = O(n^{-t/2}).
\]
Thus for any \( \alpha < V \text{nonzero}, \) it has a subgeometric tail that is uniform in
\[ \text{where we let } Z = \frac{\zeta}{\sigma} \]
for \( 1 < \alpha 2/\sqrt{3} \), this only holds for \( d \geq d_0 \) for some \( d_0 \), while for \( \alpha < 2/\sqrt{3}, \) this holds for all \( d \geq 3 \). Thus for any \( \alpha < 8/\sqrt{2}, \) we may choose the implied constants sufficiently large that the inequality holds for all \( d \geq 3 \). Hence,
\[ E_1 = O \left( \frac{d^{1+1/\alpha}}{\sqrt{n}} \right). \]
We now turn to bounding \( E_2, \) which we recall is given by
\[ E_2 = \frac{\mu}{2\sigma^3} \mathbb{E} \left| f''_h (Z_n^*) (V_n^* - V_n)^2 \right|. \]

From Lemma 4.9 we have that \( f''_h (Z_n^*) = O((1 + |Z_n^*|) h(Z_n^*)) \). This is a monotone upper bound, and hence it suffices to bound
\[ E_3 := \frac{\mu}{2\sigma^3} \mathbb{E} \left| (1 + |Z_n|) h(Z_n)(V_n^* - V_n)^2 \right| \]
and
\[ E_4 := \frac{\mu}{2\sigma^3} \mathbb{E} \left| (1 + |Z_n^*|) h(Z_n^*)(V_n^* - V_n)^2 \right|, \]
where we let \( Z_n^* = (V_n^* - \mu)/\sigma \). In either case, we proceed along the usual line of applying Hölder’s inequality for \( 1 < \alpha < 8/\sqrt{2} \). We show the bound for \( E_4, \) as the bound for \( E_3 \) follows from a nearly identical argument. Thus we have
\[ E_4 \leq \frac{\mu}{2\sigma^3} \mathbb{E} [(1 + |Z_n^*|) h(Z_n^*))]^{1/\alpha} \left( \mathbb{E} |V_n^* - V_n|^{2/\alpha} \right)^{1/\alpha^*}. \]

By Proposition A.2 the variable is nonzero with probability at most \( O(1/d) \), and conditional on being nonzero, it has a subgeometric tail that is uniform in \( n \) and \( d \). Therefore, all the absolute moments of \( V_n^* - V_n \) are of order \( O(1/d) \). Meanwhile from the definition of the size-bias distribution, we have that
\[ \mathbb{E} [(1 + |Z_n^*|) h(Z_n^*))^{1/\alpha} \leq \mathbb{E} (1 + |Z_n^*|) [(1 + |Z_n|) h(Z_n)]^{\alpha} = O \left( \left(1 + \frac{2\alpha^*}{\alpha} \right) d^{1+1/2} \right). \]
Using that \( \mu/\sigma^3 = O(d^{3/2}/\sqrt{n}) \) and that \( \sqrt{d} \leq n \), we have that
\[ E_4 = O \left( \frac{d^{5/2+1/2-1/\alpha^*}}{\sqrt{n}} \right) \]
\[ = O \left( \frac{d^{2(1+1/\alpha)}}{\sqrt{n}} \right). \]

4.3. Summary. These lemmas taken together prove the needed variance bound. We will recapitulate them to prove Proposition 1.9.

Proof. We start with [46].
\[ \frac{\mathbb{P}_{\Phi \cdot n \cdot d} [E]}{\mathbb{P}_{\Phi \cdot n \cdot d} [E]} = \left( \mathbb{E}_{\phi} [n d]_{\cdot} \frac{(d - 2)^V}{(d - 2)^V} \right) \left( \frac{2^n - V_n}{2^n - V_n} \right) \left( 1 + \left( \frac{1}{d} \right)^n \right). \]
We apply Stirling’s approximation and bound away the subexponential factors using Lemma 4.5 so that
\[ \frac{\mathbb{P}_{\Phi \cdot n \cdot d} [E]}{\mathbb{P}_{\Phi \cdot n \cdot d} [E]} \leq \sqrt{\frac{d}{d - 2}} \mathbb{E}_{\phi} \left[ \exp \left( \frac{Z_n^2}{d} \right) \right] + O \left( \frac{1}{\sqrt{n}} \right). \]
We then change the measure in the expectation from \( \phi \) to the Markov chain measure \( \pi \), using Lemma 4.8 to get
\[ \frac{\mathbb{P}_{\Phi \cdot n \cdot d} [E]}{\mathbb{P}_{\Phi \cdot n \cdot d} [E]} \leq \sqrt{\frac{d}{d - 2}} \mathbb{E}_{\pi} \left[ \exp \left( \frac{Z_n^2}{d} \right) \right] + O \left( \frac{1}{\sqrt{n}} \right). \]
Finally, we apply Stein’s method machinery to approximate the expectation by one with respect to Gaussian measure to conclude
\[
\frac{\Pr_{\mathcal{G}_{n,d}}[E]}{\Pr_{\mathcal{G}_{n,d}}[E]} \leq \frac{d}{d - 2} + O \left( \frac{d^{\frac{1}{2}} (1 + 1/n)}{\sqrt{n}} \right).
\]

5. MAIN RESULTS

We will now turn to proving our main results. Throughout this section, we let \( P \) denote a pairing sampled from \( \mathcal{G}_{n,d} \), and we let \( \mathcal{Y}_{r,n} \) refer to the Radon-Nikodym derivative between the Poisson laws of \( \mathcal{Z} \) and \( \mathcal{Z} \). This has an explicit form that we will need to use.

Let \( x = (x_\alpha, \alpha \in \mathcal{J}) \), and let \( c_k = \sum_{\alpha \in \mathcal{J}_k} x_\alpha \), the number of \( k \)-cycles represented by \( x \). Recalling \([1]\) and \([2]\),
\[
\mathcal{Y}_{r,n}(x) = \prod_{\alpha \in \mathcal{J}} e^{\lambda_\alpha - \mu_\alpha} \left( \frac{\mu_{\alpha}}{\lambda_\alpha} \right)^{x_\alpha}
\]
\[
= \prod_{1 \leq k \leq r, k \text{ odd}} \exp \left( \frac{[n]_k}{kn^k} \right) \left( 1 - \frac{2}{(d - 1)^k} \right)^{c_k}
\]
\[
= \prod_{1 \leq k \leq r, k \text{ odd}} \exp \left( \frac{1}{k} + O \left( \frac{k}{n} \right) \right) \left( 1 - \frac{2}{(d - 1)^k} \right)^{c_k}
\]
\[
= e^{O(\epsilon^2/n)} \prod_{1 \leq k \leq r, k \text{ odd}} e^{1/k} \left( 1 - \frac{2}{(d - 1)^k} \right)^{c_k}.
\]

Note that \( \mathcal{Y}_{r,n} \) is always positive for \( d \geq 4 \). For \( d = 3 \), we have \( \mathcal{Y}_{r,n} = 0 \) precisely when \( c_1 > 0 \). As we will need to condition on graphs being simple, we define the pairing event \( \text{Simple} = \{ P \text{ simple} \} \). Applying Proposition 3.25 with \( r = 2 \), we have that
\[
\Pr_{\mathcal{G}_{n,d}}[\text{Simple}] = \exp \left( -\frac{d - 1}{2} - \frac{(d - 1)^2}{4} + O \left( \frac{(\log n)^2 d^3}{n} \right) \right),
\]
\[
\Pr_{\mathcal{G}_{n,d}}[\text{Simple}] = \exp \left( -\frac{d - 3}{2} - \frac{(d - 1)^2}{4} + O \left( \frac{(\log n)^2 d^3}{n} \right) \right).
\]

Equation (61) is also obtained in \([MW91]\) without the \((\log n)^2 \) in the error term. We will start by estimating the second moment of the conditional Radon-Nikodym derivative \( f_{r,n} \).

**Lemma 5.1.** We set \( \epsilon \) to be
\[
\epsilon := \frac{d + C_2 (\log n)^2}{n} (d - 1)^{2r - 1}.
\]
There is an absolute constant \( C_{27} \) so that for \( r \geq 4 \)
\[
\mathbb{E}[f_{r,n}(P)^2] \geq V(r) (1 - C_{27}\epsilon).
\]

**Proof.** We may assume that \( \epsilon \leq \frac{1}{2} \), for by adjusting \( C_{27} \) to be at least 2, we may then make the bound trivial. Further we take \( \lambda = \log n \), so that for any strictly \( \lambda \)-neat cycle space point \( x \), Proposition 1.7 implies that
\[
f_{r,n}(x)^2 \Pr_{\mathcal{G}_{n,d}}[I_r = x] = \frac{\Pr_{\mathcal{G}_{n,d}}[I_r = x]^2}{\Pr_{\mathcal{G}_{n,d}}[I_r = x]} \geq \mathcal{Y}_{r,n}(x)^2 \Pr[Z = x] (1 - \epsilon)^2.
\]

By ignoring the non-neat cycle space points, we can immediately bound
\[
\mathbb{E}[f_{r,n}(P)^2] \geq \mathbb{E} \left[ \mathcal{Y}_{r,n}(Z)^2 \mathbf{1}[Z \text{ strictly } \lambda \text{-neat}] \right] (1 - O(\epsilon))
\]

To complete the lower bound, we need to estimate the contribution of the non-neat cycles to right hand side, and so we estimate \( \mathbb{E} \left[ \mathcal{Y}_{r,n}(Z)^2 \mathbf{1}[Z \text{ not strictly } \lambda \text{-neat}] \right] \) from above.
The key to making this estimate is to realize that \( \mathcal{Y}_{r,n}^2(x) \) is a rescaled Radon-Nikodym for yet another Poisson law. Let \( W = \prod_{k=1}^{2r}(W_{\alpha}, \alpha \in J_k) \) be a vector whose coordinates are independent Poisson random variables with \( \text{EW}_{\alpha} = \frac{\mu^2_{\alpha}}{\lambda_{\alpha}} \) for \( \alpha \in J_k \). It is easily checked that for any cycle space point \( x \)

\[
    \mathcal{Y}_{r,n}^2(x) \Pr[Z = x] = \exp \left( \sum_{\alpha \in J_k, k \leq r} \frac{(\mu_{\alpha} - \lambda_{\alpha})^2}{\lambda_{\alpha}} \right) \Pr[W = x].
\]

Further, we note that this renormalization constant is precisely

\[
    \mathbb{E} \left[ \mathcal{Y}_{r,n}^2(Z) \right] = \exp \left( r \sum_{k=1}^{2r} \frac{((-1)^k - 1)^2 [n]_k}{2k(d-1)^k n^k} \right)
    = V^r (1 - O(r^2/n)) = V^r (1 - O(\varepsilon)).
\]

Therefore, we have reduced the problem to estimating \( \Pr[W] \) not strictly \( \lambda \)-neat. We first apply Lemma 2.1 to bound the probability of \( W \) having too many cycles. Specifically, we define

\[
    F(x) := \sum_{\alpha \in J_k, k \leq r} |\alpha|W_{\alpha}.
\]

We note that \( ||\nabla F|| = |\alpha|, \) that \( \frac{\mu^2_{\alpha}}{\lambda_{\alpha}} \leq \lambda_{\alpha} \) and hence that

\[
    \sum_{\alpha \in J_k, k \leq r} |\alpha|^2 \frac{\mu^2_{\alpha}}{\lambda_{\alpha}} \leq \sum_{\alpha \in J_k, k \leq r} |\alpha|^2 \lambda_{\alpha}
    = \sum_{k=1}^{r} k^2 \frac{[n]_k (d)_2^k}{2k} \frac{1}{(nd)^k} = O(r(d - 1)^r).
\]

By applying Lemma 2.1 we conclude that for \( t > 0 \),

\[
    \Pr[F(W) \geq \mathbb{E}F(W) + t] \leq \exp \left( -\frac{t}{2r} \log \left( 1 + \frac{t}{C_{28}(d - 1)^r} \right) \right),
\]

for some absolute constant \( C_{28} \). Finally we bound the expectation of \( F(W) \) with

\[
    \mathbb{E}F(W) = \sum_{\alpha \in J_k, k \leq r} |\alpha|^2 \frac{\mu^2_{\alpha}}{\lambda_{\alpha}} \leq \sum_{k=1}^{r} \frac{k[n]_k [d^k_{\alpha}]}{2k} \frac{1}{(nd)^k} \leq (d - 1)^r.
\]

Thus we conclude that

\[
    \Pr[F(W) \geq \lambda(d - 1)^r] \leq \frac{1}{C_{29}} \exp \left( -C_{29} \left( \frac{(d - 1)^r}{r} \lambda \log \lambda \right) \right)
\]

for some absolute constant \( C_{29} \). It remains to estimate the probability under \( W \) that two cycles share a vertex. There are \( [n]_{k-1} [d^k_{\alpha}]/2k \) many \( \alpha \in J_k \) that use any given vertex. Thus, taking a union bound over all \( 1 \leq k \leq r \) and all \( 1 \leq l \leq r \),

\[
    \Pr[\exists \alpha \in J_k, \beta \in J_l \text{ sharing a vertex so that } W_{\alpha} = W_{\beta} = 1]
    = \sum_{1 \leq k, l \leq r} n [n]_{k-1} [d^k_{\alpha}] [n]_{l-1} [d^l_{\beta}] \frac{1}{2l} \frac{1}{(nd)^k (nd)^l} \leq \frac{(d - 1)^2r}{n},
\]

where we have used that \( \Pr[W_{\alpha} = 1] \leq \text{EW}_{\alpha} \leq \frac{1}{(nd)^{2\alpha}} \). By combining equations (65) and (66), we conclude that

\[
    \Pr[W \text{ not strictly } \lambda \text{-neat}] \leq \frac{(d - 1)^{2r}}{n} + \frac{1}{C_{29}} \exp \left( -C_{29} \left( \frac{(d - 1)^r}{r} \lambda \log \lambda \right) \right).
\]
By applying this bound, we conclude that
\[
\mathbb{E} \left[ Y_{r,n}^2(Z) 1\{Z \text{ not strictly } \lambda \text{-neat}\} \right] \\
= V^{(r)}(1 + O(\varepsilon)) \Pr[ W \text{ not strictly } \lambda \text{-neat}] \\
= V^{(r)} O(\varepsilon).
\]
(68)
We now combine (63), (64), and (68) to derive the lower bound
\[
\mathbb{E} [f_{r,n}^2(P)] \geq V^{(r)}(1 - O(\varepsilon)),
\]
which completes the proof. □

The lower bound on the conditional variance combined with the upper bound on the variance (Proposition 1.9) shows that \( f_n \) and \( f_{r,n} \) are close in \( L^2(\mathcal{P}_{n,d}) \).

**Lemma 5.2.** For every \( \alpha \) with \( 1 < \alpha < 8/\sqrt{2} \) there is a constant \( M_\alpha \) so that for all \( 3 \leq d \leq \sqrt{n}/\log n \) and all \( r \geq 4 \),
\[
\mathbb{E} |f_n(P) - f_{r,n}(P)|^2 \leq M_\alpha \left( (d - 1)^{-r-1} + \varepsilon + d^{2(1+1/\alpha)}/\sqrt{n} \right).
\]

**Proof.** By orthogonality, we have that \( \mathbb{E} |f_n(P) - f_{r,n}(P)|^2 = \mathbb{E} f_n^2(P) - \mathbb{E} f_{r,n}^2(P) \), and so by Proposition 1.9 and Lemma 5.1 we have that for any \( \alpha \) with \( 1 < \alpha < 8/\sqrt{2} \)
\[
\mathbb{E} |f_n(P) - f_{r,n}(P)|^2 = V^{(\infty)} - V^{(r)} + O(\varepsilon + d^{2(1+1/\alpha)}/\sqrt{n}).
\]
We note that there is some absolute constant \( C_{30} \) so that \( 0 \leq \log V^{(\infty)} - \log V^{(r)} \leq C_{30} (d - 1)^{-r-1} \). This in turn implies that \( V^{(\infty)} - V^{(r)} = O((d - 1)^{-r-1}) \), and hence we have completed the proof. □

We now develop estimates for \( f_{r,n} \) by comparing with the limiting Poisson structure.

**Lemma 5.3.** There is a constant \( C_{31} \) so that for \( \delta \geq \log r \geq \log 4, \varepsilon \leq \frac{1}{2} \) and \( d \geq 4 \),
\[
\Pr \left[ |\log f_{r,n}(P)| \geq \delta \land I_r(P) \text{ (log } n\text{-neat)} \right] \leq \frac{1}{C_{31}} \exp (-C_{31} d\delta \log \delta).
\]

In the case that \( d = 3 \),
\[
\Pr \left[ |\log f_{r,n}(P)| \geq \delta \land I_r(P) \text{ (log } n\text{-neat } \land \text{ P simple)} \right] \leq \frac{\Pr[P \text{ simple}]}{C_{31}} \exp (-C_{31} d\delta \log \delta).
\]

**Remark 5.4.** The same bound holds for \( \mathcal{F}_{n,d} \) as well, and the proof is identical, but we will not need it.

**Proof.** We will show the proof for \( d \geq 4 \). The proof for \( d = 3 \) follows by the same argument. We apply the multiplicative Poisson bound (Proposition 1.7) to get that for any strictly (log \( n \))-neat cycle space point \( x \),
\[
\mathcal{Y}_{r,n}(x)^{\frac{1+\varepsilon}{1-\varepsilon}} \geq f_{r,n}(x) \geq \mathcal{Y}_{r,n}(x)^{\frac{1-\varepsilon}{1+\varepsilon}}.
\]
We may therefore bound
\[
\Pr \left[ |\log f_{r,n}(P)| \geq \delta \land I_r(P) \text{ (strictly (log } n\text{-neat)} \right] \leq \Pr \left[ |\log \mathcal{Y}_{r,n}(P)| \geq \delta - \log \frac{1+\varepsilon}{1-\varepsilon} \land I_r \text{ strictly (log } n\text{-neat)} \right].
\]
As this probability is restricted to strictly (log \( n \))-neat \( x \), the multiplicative Poisson bound implies that
\[
\Pr \left[ |\log \mathcal{Y}_{r,n}(P)| \geq \delta + \log \frac{1+\varepsilon}{1-\varepsilon} \land I_r \text{ (strictly (log } n\text{-neat)} \right] \leq \Pr \left[ |\log \mathcal{Y}_{r,n}(Z)| \geq \delta + \log \frac{1+\varepsilon}{1-\varepsilon} \right] (1 + \varepsilon).
\]
As we have that \( \varepsilon \leq \frac{1}{2} \), it suffices to prove that there is an absolute constant \( C_{32} > 0 \) so that for all \( \delta \geq \log r \land 1 \)
\[
\Pr \left[ |\log \mathcal{Y}_{r,n}(Z)| \geq \delta \right] \leq \frac{1}{C_{32}} \exp (-C_{32} d\delta \log \delta)
\]
by adjusting constants.
For this purpose we note that the identity that for any cycle space point \( x \),

\[
- \log \lambda_{r,n}(x) = \sum_{\alpha \in J_k} \left[ -x_\alpha \log \frac{\mu_\alpha}{\lambda_\alpha} - \lambda_\alpha + \mu_\alpha \right].
\]

There is an absolute constant \( C_{33} > 0 \) so that for all \( \alpha, \frac{1}{\log (d-1)^{|\alpha|}} \leq - \log \frac{\mu_\alpha}{\lambda_\alpha} \leq C_{31} (d-1)^{-|\alpha|} \). Thus, we define

\[
F(x) := \sum_{\alpha \in J_k, 1 \leq k \leq r} \frac{x_\alpha}{(d-1)^{|\alpha|}},
\]

for cycle space point \( x \). Note that the added constant is

\[
0 \leq \sum_{\alpha \in J_k, 1 \leq k \leq r} [- \lambda_\alpha + \mu_\alpha] \leq \sum_{k=1}^{r} \sum_{\alpha \in J_k} \frac{[n]_k [d]_2^k}{n^k (d-1)^{2k} (nd)^k} = O(\log r).
\]

Also note that the expectation of \( F(Z) \) is

\[
\mathbb{E} F(Z) = \sum_{\alpha \in J_k, 1 \leq k \leq r} (d-1)^{-k} \lambda_\alpha \leq \sum_{k=1}^{r} \frac{[n]_k [d]_2^k}{(d-1)^{2k} (nd)^k} \leq O(\log r).
\]

Combining these observations, we note that it suffices to prove that there is an absolute constant \( C_{34} > 0 \) so that for all \( \delta \geq \log r \land 1 \),

\[
\Pr \left[ |F(Z) - \mathbb{E} F(Z)| \geq \delta \right] \leq \frac{1}{C_{34}^{d}} \exp (-C_{34} d \delta \log \delta),
\]

for by again adjusting constants, we may conclude the desired inequality.

This now follows from the modified log-Sobolev inequality bounds. We note that \( \| \nabla F \| = (d-1)^{-|\alpha|} \) and hence that

\[
\sum_{\alpha \in J_k, 1 \leq k \leq r} (d-1)^{-2k} \lambda_\alpha = \sum_{k=1}^{r} [n]_k [d]_2^k (d-1)^{2k} 2k \left( \frac{1}{(nd)^k} \right) = O \left( \frac{1}{d} \right).
\]

By applying Lemma 2.1, we conclude that for \( t > 0 \),

\[
\Pr \left[ |F(Z) - \mathbb{E} F(Z)| \geq t \right] \leq 2 \exp \left( - \frac{dt}{C_{35}} \log \left( 1 + \frac{t}{C_{35}} \right) \right),
\]

for some absolute constant \( C_{35} > 0 \).

\[\square\]

**Lemma 5.5.** There is an absolute constant \( C_{36} > 0 \) so that for any \( \delta \geq \log r \geq 4 \), any \( d \geq 4 \) and any pairing event \( A \),

\[
\Pr[\mathcal{G}_{n,d}[A] \leq C_{36} \left( \varepsilon + \varepsilon^{2\delta} \text{Var}_{\mathcal{G}_{n,d}}[f_n - f_{r,n}] + \exp (-C_{31} d \delta \log \delta) + \varepsilon^\delta \Pr[\mathcal{G}_{n,d}[A]] \right).
\]

If \( A \subseteq \text{SIMPLE} \) the same statement holds for \( d = 3 \).

**Proof.** We set \( \lambda = \log n \), and we bound

\[
\Pr[\mathcal{G}_{n,d}[A] \leq \Pr[\mathcal{G}_{n,d}[f_n \leq e^{-\delta}/2 \land \textbf{L}_r \text{ strictly } \lambda\text{-neat}]] + \Pr[\mathcal{G}_{n,d}[\textbf{L}_r \not\text{ strictly } \lambda\text{-neat}]] + \mathbb{E}[\mathcal{G}_{n,d}[2e^\delta f_n 1\{ A \}]].
\]

We note that the second line is \( O(\varepsilon) \) by Proposition 3.8. The third line is precisely \( 2e^\delta \Pr[\mathcal{G}_{n,d}[A]] \). To bound the first line, we write

\[
\Pr[\mathcal{G}_{n,d}[f_n \leq e^{-\delta}/2 \land \textbf{L}_r \text{ strictly } \lambda\text{-neat}]] \leq \Pr[\mathcal{G}_{n,d}[f_{r,n} \leq e^{-\delta} \land \textbf{L}_r \text{ strictly } \lambda\text{-neat}]] + \Pr[|f_n - f_{r,n}| \geq e^{-\delta}/2].
\]
The first of these we bound by Lemma 5.3 and the second we bound by Chebyshev’s inequality, completing the Lemma.

**Lemma 5.6.** There is an absolute constant $C_{37}$ so that for any pairing event $A$ and any $r \geq 4$,

$$\Pr_{\mathcal{F}_{n,d}}[A] \leq C_{37} \left( \varepsilon + \sqrt{\Var_{\mathcal{F}_{n,d}}[f_n - f_{r,n}]} \sqrt{\Pr_{\mathcal{F}_{n,d}}[A] + r \Pr_{\mathcal{F}_{n,d}}[A]} \right).$$

**Proof.** We may assume that $\varepsilon \leq \frac{1}{2}$, for by adjusting $C_{37} \geq 2$, we may make the bound trivial. Let $E$ be the event $E = \{I, \text{ strictly } \lambda\text{-neat }\}$,

$$\Pr_{\mathcal{F}_{n,d}}[A] \leq \Pr_{\mathcal{F}_{n,d}}[A \cap E] + \Pr_{\mathcal{F}_{n,d}}[E].$$

As we have that $\Pr_{\mathcal{F}_{n,d}}[E] = O(\varepsilon)$ from Proposition 3.8, it suffices to show the bound for $A \subseteq E$ by passing to $A \cap E$.

In this case, we have that for any strictly $\lambda$-neat cycle space point $x$,

$$f_{r,n}(x) \leq \frac{1+\varepsilon}{2} \mu_{r,n}(x) \leq 3 \prod_{\alpha \in \mathcal{J}_s, \ 1 \leq k \leq r} \exp(-\mu_\alpha + \lambda_\alpha) = O(r).$$

By applying Cauchy-Schwarz, we have that

$$\mathbb{E}_{\mathcal{F}_{n,d}} f_n \mathbb{1}[A] \leq \sqrt{\Var_{\mathcal{F}_{n,d}}[f_n - f_{r,n}]} \sqrt{\Pr_{\mathcal{F}_{n,d}}[A]} + \mathbb{E}_{\mathcal{F}_{n,d}} f_{r,n} \mathbb{1}[A],$$

and we conclude the lemma, noting that $f_{r,n} \mathbb{1}[A]$ can be bounded by $Cr \mathbb{1}[A]$ for an absolute constant $C$. \hfill \Box

We now turn to proving the main theorems.

**Proof of Theorem 1.3.** Fix a given sequence $D(n) \to \infty$ with $\log D(n)/\log n \to 0$. By passing to subsequences, it suffices to show the cases, where $d(n) \leq D(n)$ and where $d(n) \geq D(n)$. In the latter case, we need only prove the total variation bound. This, in turn follows from the simple inequality

$$\left| \Pr_{\mathcal{F}_{n,d}}(A) - \Pr_{\mathcal{F}_{n,d}}(A) \right| \leq \sqrt{\Var_{\mathcal{F}_{n,d}}[f_n]}.$$ 

From Proposition 1.9 we therefore have the bound that for any $1 < \alpha < 8/\sqrt{2}$

$$d_{TV}(\mathcal{P}_{n,d}, \mathcal{F}_{n,d}) = O \left( \frac{2}{d-2} + \frac{d^{\frac{3}{2}}(1+1/\alpha)}{\sqrt{n}} \right).$$

For $d(n) \leq n^{\alpha_0 - \varepsilon}$ where $\alpha_0 = \frac{8}{3(8+\sqrt{2})}$, we may therefore choose an $\alpha$ so that this tends to 0.

In the former case, we show the contiguity arguments one bound at a time. We start by assuming that $\Pr_{\mathcal{F}_{n,d}}[A_n] \to 0$. We then choose an integer sequence $r(n) \to \infty$ sufficiently slowly that $r(n) \Pr_{\mathcal{F}_{n,d}}[A_n] \to 0$ and $\varepsilon \to 0$. From Lemma 5.6 we have that

$$\Var_{\mathcal{F}_{n,d}}[f_n - f_{r,n}] \to 0,$$

and hence by Lemma 5.6 $\Pr_{\mathcal{F}_{n,d}}[A_n] \to 0$.

Suppose now that $\Pr_{\mathcal{F}_{n,d}}[A_n] \to 0$. We may choose $r(n)$ an integer sequence so that $r(n) \to \infty$, $r(n) \Pr_{\mathcal{F}_{n,d}}[A_n] \to 0$ and $r(n)^2 \varepsilon \to 0$. Apply Lemma 5.5 with $\delta = 2 \log r(n)$, and note that we have

$$e^{2\delta(n)} \Var_{\mathcal{F}_{n,d}}[f_n - f_{r,n}] \to 0,$$

so that $\Pr_{\mathcal{F}_{n,d}}[A_n] \to 0$. \hfill \Box

**Proof of Theorem 1.3.** The statements for $\mathcal{P}_{n,d}$ and $\mathcal{F}_{n,d}^*$ follow immediately from those for $\mathcal{P}_{n,d}$ and $\mathcal{F}_{n,d}$ together with the observation that for $d = o(\sqrt{\log n})$, both $\log \Pr_{\mathcal{P}_{n,d}}[\text{SIMPLE}] = o(\log n)$ and $\log \Pr_{\mathcal{F}_{n,d}}[\text{SIMPLE}] = o(\log n)$ (see [6.1] and [6.2]).

For the $\mathcal{P}_{n,d}$ case, we assume that $\log d(n)/\log n \to 0$, and we may choose $r(n) = \left\lfloor \frac{\log n}{\log (d(n)-1)} \right\rfloor$. 

\hfill \Box
Note that this implies that for all \( \epsilon > 0 \),
\[
\varepsilon = \frac{[d + C_1 \log n]^2}{n} (d - 1)^{2r-1} = O(n^{-1/3+\epsilon}).
\]
Likewise, by Lemma 5.2 we have that for all \( \epsilon > 0 \),
\[
\text{Var}_{\mathcal{G}_{n,d}}[f_n - f_{r,n}] = O((d - 1)^{-r-1} + \varepsilon + d^3/\sqrt{n}) = O(n^{-1/3+\epsilon}).
\]
Suppose that \( \text{Pr}_{\mathcal{G}_{n,d}}[A_n] = O(n^{-\alpha}) \). Then by taking \( \delta = \epsilon \log n \), we conclude by Lemma 5.5 that
\[
\text{Pr}_{\mathcal{G}_{n,d}}[A_n] = O(n^{-1/3+2\epsilon} + n^{-\alpha+2\epsilon}),
\]
completing the proof.
On the other hand, suppose that \( \text{Pr}_{\mathcal{G}_{n,d}}[A_n] = O(n^{-\alpha}) \). By Lemma 5.6 we conclude that
\[
\text{Pr}_{\mathcal{G}_{n,d}}[A_n] = O(n^{-\alpha/2 - 1/6 + \epsilon/2} + n^{-1/3+\epsilon} + n^{-\alpha+\epsilon}).
\]
Note that \( \frac{\alpha}{2} + \frac{1}{6} \geq \alpha \wedge \frac{1}{3} \), completing the proof. \( \square \)

**Lemma 5.7.** Let \( r = r(n) \geq 4 \) and \( d = d(n) \to \infty \) satisfy \( (d + (\log n)^2)(d - 1)^{2r-1} = o(n) \), then
\[
\frac{\log \mathcal{Y}_{r,n}(P)}{\sqrt{2/d}} \Rightarrow N(0,1),
\]
where \( P \sim \mathcal{G}_{n,d} \) and \( N(0,1) \) is a standard normal.

**Proof of Lemma 5.7.** We once again write
\[
\varepsilon = \frac{[d + C_1 \log n]^2}{n} (d - 1)^{2r-1}.
\]
From the total variation bound Corollary 1.8 we can construct a probability space on which \( \mathbf{I}_r(P) \) and \( Z \)
are defined and satisfy
\[
\text{Pr}[\mathbf{I}_r(P) \neq Z] = d_{TV}(\mathbf{I}_r(P), Z) = O(\varepsilon).
\]
On this space we have that
\[
\text{Pr}[\log \mathcal{Y}_{r,n}(\mathbf{I}_r(P)) \neq \log \mathcal{Y}_{r,n}(Z)] = O(\varepsilon).
\]
Writing \( Z_k = \sum_{\alpha \in \mathcal{J}_k} Z_\alpha \), we have by (60) that
\[
\log \mathcal{Y}_{r,n}(Z) = O(r^2/n) + \sum_{1 \leq k \leq r, k \text{ odd}} \left[ \frac{1}{k} + Z_k \log \left(1 - \frac{2}{(d - 1)^k}\right)\right].
\]
Note that \( Z_k \sim \text{Poisson} \left( \frac{|n|_k (d-1)^k}{2kn^k} \right) \), and hence
\[
\mathbb{E} \left| \frac{1}{k} + Z_k \log \left(1 - \frac{2}{(d - 1)^k}\right)\right| \leq \left| \frac{1}{k} + \frac{|n|_k (d-1)^k}{2kn^k} \log \left(1 - \frac{2}{(d - 1)^k}\right)\right| + \left| \log \left(1 - \frac{2}{(d - 1)^k}\right)\right| \sqrt{\text{Var} Z_k}.
\]
As the log terms can be approximated by \( -\frac{2}{(d-1)^k} \) uniformly in \( k \) and \( d \), it follows that
\[
\mathbb{E} \left| \log \mathcal{Y}_{r,n}(Z) - 1 + \frac{2}{d-1} Z_1 \right| = O(r^3/n + 1/d).
\]
Now from the classical central limit theorem, we have that
\[
\frac{d-1}{\sqrt{d/2}} - Z_1 \Rightarrow N(0,1),
\]
which completes the proof. \( \square \)
Proof of Theorem 1.4. Take \( r(n) = \left\lfloor \frac{\log n}{3\log(d(n)-1)} \right\rfloor \). By Lemma 5.7, it suffices to show that
\[
\frac{\log f_n(P) - \log \mathcal{Y}_{r,n}(I_r(P))}{\sqrt{2/d}} \xrightarrow{p} 0.
\]
On the one hand, we have that
\[
\frac{\log f_{r,n}(P) - \log \mathcal{Y}_{r,n}(I_r(P))}{\sqrt{2/d}} \xrightarrow{p} 0,
\]
as for a cycle space point \( x \), that is \( (\log n) \)-neat, we have by Proposition 1.7
\[
\mathcal{Y}_{r,n}(x)^{1+\varepsilon} \geq f_{r,n}(x) \geq \mathcal{Y}_{r,n}(x)^{1-\varepsilon}.
\]
As \( I_r(P) \) is \( (\log n) \)-neat with probability going to 1 and \( \varepsilon d / 2 \to 0 \), the desired statement on the logarithms follow. This reduces the problem to showing that
\[
\frac{\log f_n(P) - \log f_{r,n}(P)}{\sqrt{2/d}} \xrightarrow{p} 0.
\]
We note that we have
\[
\Pr_{\mathcal{G}_{n,d}} \left[ |f_n - 1| \geq \frac{1}{2} \right] \leq 4 \Var_{\mathcal{G}_{n,d}}[f_n] \to 0
\]
by Proposition 1.9. Likewise, we get
\[
\Pr_{\mathcal{G}_{n,d}} \left[ |f_{r,n} - 1| \geq \frac{1}{2} \right] \leq 4 \Var_{\mathcal{G}_{n,d}}[f_{r,n}] \leq 4 \Var_{\mathcal{G}_{n,d}}[f_n] \to 0,
\]
by the contraction properties of the conditional expectation. By the mean value theorem, we have that
\[
\Pr_{\mathcal{G}_{n,d}} \left[ |\log f_n - \log f_{r,n}| > t \sqrt{2/d} \right] \leq \Pr_{\mathcal{G}_{n,d}} \left[ 2 |f_n - f_{r,n}| > t \sqrt{2/d} \right] + 8 \Var_{\mathcal{G}_{n,d}}[f_n].
\]
Hence, by Lemma 5.2, we have that for all \( k \), \( \Var_{\mathcal{G}_{n,d}}[f_n - f_{r,n}] = o(d^{-k}) \), and hence this term goes to 0. \( \square \)

It remains to prove Theorem 1.5 on approximating the number of Hamiltonian cycles by a graph’s eigenvalues. We give some combinatorial definitions. A closed non-backtracking walk on a graph is a walk that begins and ends at the same vertex, and that never follows an edge and immediately follows that same edge backwards. If the last step of a closed non-backtracking walk is anything other than the reverse of the first step, we say that the walk is cyclically non-backtracking. Let \( P \sim \mathcal{G}_{n,d} \) and let \( \text{CNBW}^{(n)}_k \) denote the number of closed cyclically non-backtracking walks of length \( k \) on the pseudograph given by \( P \).

Let \( T_i(x) \) be the Chebyshev polynomial of the first kind of degree \( i \) on the interval \([-1, 1]\). We define a set of polynomials
\[
\Gamma_0(x) = 1,
\]
\[
\Gamma_{2i}(x) = 2T_{2i}(\frac{x}{2}) + \frac{d-2}{(d-1)^i} \quad \text{for } i \geq 1,
\]
\[
\Gamma_{2i+1}(x) = 2T_{2i+1}(\frac{x}{2}) \quad \text{for } i \geq 0.
\]
These polynomials allow us to count cyclically non-backtracking walks from a graph’s eigenvalues. Let \( \text{tr} f(P) = \sum_{i=1}^{n} f(\lambda_i/\sqrt{d-1}) \), where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of the adjacency matrix of \( P \).

**Proposition 5.8** (Proposition 32 in [DJPP13]).
\[
\text{tr} \Gamma_k(P) = (d-1)^{-k/2} \text{CNBW}^{(n)}_k.
\]

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We will define another set of polynomials whose traces give the cycle counts of $P$, with high probability. Let $\mu$ be the M"obius function, given by
\[
\mu(n) = \begin{cases} 
1 & \text{if } n = 1, \\
(-1)^{a} & \text{if } n \text{ is the square-free product of } a \text{ primes}, \\
0 & \text{otherwise}.
\end{cases}
\]
For $k \geq 1$, define
\[
(69) \Xi_k(x) := \frac{1}{2k} \sum_{j|k} \mu \left( \frac{k}{j} \right) (d - 1)^{j/2} \Gamma_j(x).
\]

**Proposition 5.9.** With probability at least $1 - O(r(d - 1)^r/n)$, the number of cycles of length $k$ in $P$ is $\text{tr} \Xi_k(P)$ for all $1 \leq k \leq r$.

**Proof.** We will show that with high probability, all cyclically non-backtracking walks in $P$ are repeated walks around cycles. For this to fail, $P$ must contain cycles of length $k$ and $j$ at distance $l$ (possibly zero) from each other, with $k + j + 2l \leq r$.

By a slight variation of (14) and (27), the probability that $P$ contains cycles of length $j$ and $k$ with $k + j \leq r$ that overlap is $O(r(d - 1)^r/n)$. The number of possible edge-labeled subgraphs consisting of cycles of length $k$ and $j$ with a path of length $l \geq 1$ between them is at most
\[
[n]_{j+k+l-1} (d - 1)^{j+k+l+1} d^{j+k+l-1},
\]
and each subgraph is contained in $P$ with probability $1/[[d]_{j+k+l}]$. By a union bound, $P$ contains some such subgraph with probability $O((d - 1)^{j+k+l}/n)$. The sum of this over all $1 \leq j, k \leq r$ and $l \geq 1$ satisfying $j + k + 2l \leq r$ is $O(r(d - 1)^{r-1}/n)$.

Let $C_k^{(n)}$ denote the number of cycles of length $k$ in $P$. If all cyclically non-backtracking walks are repeated walks around cycles, then
\[
\text{CNBW}_k^{(n)} = \sum_{j|k} 2j C_k^{(n)}.
\]

The proposition follows by applying the M"obius inversion formula to write $C_k^{(n)}$ in terms of $\text{CNBW}_j^{(n)}$ for $j \mid k$, and then applying Proposition 5.8. \hfill \Box

We define one last polynomial:
\[
(70) \Pi_{d,n}(x) = \sum_k \left[ \frac{1}{kn} + \log \left( 1 - \frac{2}{(d - 1)^k} \right) \Xi_k(x) \right],
\]
where the sum ranges over all odd values from 1 to $\lceil \log n/3 \log(d - 1) \rceil$. In $\Pi_{d,n}(x)$, the coefficient of $\Xi_1(x)$ is log 0. We interpret this as $-\infty$, and we say that $\text{tr} \Pi_{d,n}(P) = -\infty$ unless $\text{tr} \Xi_1(P) = 0$.

**Lemma 5.10.** Let $r = \lceil \log n/3 \log(d - 1) \rceil$. Suppose that $d - 1 \leq n^{1/3}$, so that $r \geq 1$ and $\Pi_{d,n}$ is nonzero. For some absolute constants $C_{38}$ and $C_{39}$,
\[
\Pr \left[ \left| f_{r,n}(P) - \exp(\text{tr} \Pi_{d,n}(P)) \right| > C_{38} \log n^{5/2} n^{-1/3} \right] < C_{39} n^{-1/3}.
\]

**Proof.** Suppose that the event of Proposition 5.9 holds, which occurs with probability $1 - O(r(d - 1)^r/n)$, and that $I_r(P)$ is strictly $(\log n)$-neat, which occurs with probability $1 - O((d - 1)^{2^r}/n)$ by Proposition 3.8. On this event, by (60),
\[
\mathcal{Y}_{r,n}(I_r(P)) = (1 + O((\log n)^2 n^{-1/3})) \exp(\text{tr} \Pi_{d,n}(P)).
\]
By Proposition 1.7,
\[
f_{r,n}(I_r(P)) = (1 + O((\log n)^2 n^{-1/3})) \mathcal{Y}_{r,n}(I_r(P)).
\]
Now,
\[ |f_{r,n}(I_r(P)) - \exp(\text{tr} \Pi_{d,n}(P))| = |f_{r,n}(I_r(P)) - (1 + O(\log^2 n/n))Y_{r,n}(I_r(P))| = Y_{r,n}(I_r(P))O((\log n)^2n^{-1/3}). \]

As
\[ Y_{r,n}(I_r(P)) \leq \prod_{1 \leq k \leq r, k \text{ odd}} e^{1/k} \leq \exp \left( \frac{\log r}{2} + O(1) \right) = O(\sqrt{\log n}), \]

it holds that
\[ |f_{r,n}(I_r(P)) - \exp(\text{tr} \Pi_{d,n}(P))| = O((\log n)^{5/2}n^{-1/3}) \]
on an event which occurs with probability \( 1 - O((d - 1)^{2r}/n) = 1 - O(n^{-1/3}). \)

**Proof of Theorem 1.5.** Let \( r = \left\lfloor \frac{\log n}{3\log(d-1)} \right\rfloor. \) Since \( r \geq 4, \) we can apply Lemma 5.2 with \( \alpha = 5 \) to get
\[ \mathbb{E} |f_n(P) - f_{r,n}(P)|^2 = O(n^{-1/3} + (\log n)^2n^{-1/3} + n^{-7/20}) = O((\log n)^2n^{-1/3}). \]

By Chebyshev’s inequality,
\[ \Pr \left[ |f_n(P) - f_{r,n}(P)| > n^{-1/12} - C_{38}(\log n)^{5/2}n^{-1/3} \right] = O((\log n)^2n^{-1/6}). \]

This and Lemma 5.10 combine to prove the theorem.

**Appendix A. Size-bias coupling of a 2-state Markov chain**

Let \( X_1, X_2, \ldots, X_n \) be \( n \) steps of a stationary, reversible Markov chain on two states \( \{0, 1\}, \) and let \( \mu \) denote its stationary measure. We let \( p = \mu(\{1\}), \) and we let \( \theta \) be the contraction coefficient of the chain, which as this is a 2-element space, is simply
\[ \theta = d_{TV}(\mathcal{L}(X_2 | X_1 = 1), \mathcal{L}(X_2 | X_1 = 0)), \]

where \( \mathcal{L} \) denotes the law of a variable. This regulates the optimal rate at which two chains with the same transition rule can be coupled in a Markovian fashion. Let \( Y_1, Y_2, \ldots, Y_n \) be an independent copy of \( X_1, X_2, \ldots, X_n. \) Then we define
\[ \beta = \Pr [X_2 \neq Y_2 | X_1 = 1, Y_1 = 0]. \]

This, in a sense, governs the coupling rate of the slowest possible Markovian coupling. It is always the case that \( \theta \leq \beta, \) and we will work in the case that \( \beta < 1. \)

Let \( V_n \) denote the number of \( X_i \) that are 1, so that \( V_n = \sum_{i=1}^{n} X_i. \) We will show a general construction for a size-bias coupling for \( V_n \) and show some estimates for this coupling that can be used for normal approximation of \( V_n. \) While we will not directly need this normal approximation, it follows immediately from the estimates that we do need, and so we state it as a result of possibly independent interest.

**Proposition A.1.** With \( V_n \) as above, with \( 0 < p < 1, \) and with \( \beta < 1, \) there is a constant \( C > 0, \) depending on the law of the Markov chain, so that
\[ d_W \left( \frac{V_n - \mathbb{E}V_n}{\sqrt{\text{Var}V_n}}, Z \right) \leq \frac{C}{\sqrt{n}}, \]

where \( d_W \) denotes the Wasserstein 1-distance (see Section 1.1.1 of [Ros11]) and where \( Z \) is a standard normal.
A.1. Construction of the coupling. Since \( V_n \) is a sum of indicators with equal means, its size-bias distribution \( V_n^s \) can be realized by choosing \( I \sim \text{Unif}([n]) \) independently of the chain and defining

\[
V_n^s = 1 + \sum_{i \neq I} Y_i,
\]

where the collection \( \{Y_i\}_{i=1}^n \) has the distribution of \( \{X_i\}_{i=1}^n \) conditioned on \( X_I = 1 \) (this follows directly from [Ros11, Corollary 3.24]). This conditioning can be accomplished by defining \( \{Z_k\}_{k=-\infty}^\infty \) to be a Markov chain independent of \( \{X_i\} \) and with with its same transition probabilities, but started at \( Z_0 = 1 \). As the chain is reversible, we can shift coordinates to obtain a chain \( \{Z_{i-1}\}_{i=1}^n \) with same distribution as the original chain \( \{X_i\} \) conditioned on \( X_I = 1 \).

To couple the conditioned chain \( \{Z_{i-1}\} \) back to \( \{X_i\} \), we have it join back up with \( X_i \) at the first time \( k \) before and after time \( I \) that \( X_k = Z_{k-I} \), taking advantage of the reversibility of our chains. Formally, let

\[
\tau_+ = \inf\{k \geq 0: X_{I+k} = Z_k\} \land (n-I+1),
\]
\[
\tau_- = \inf\{k \geq 0: X_{I-k} = Z_k\} \land I,
\]

and define

\[
Y_k = \begin{cases} 
Z_{k-I} & \text{if } \tau_- \leq k-I \leq \tau_+, \\
X_k & \text{otherwise.}
\end{cases}
\]

The pair \( \{(Y_i)_{i=1}^n, (X_i)_{i=1}^n\} \) is the desired coupling of the underlying state space, and setting \( V_n^s = \sum_{i=1}^n Y_i \), we obtain a size-bias coupling \( (V_n^s, V_n) \).

A.2. Estimates. For application of Stein’s method, there are two quantities that need to be controlled. Roughly, \( V_n^s - V_n \) needs to be at constant order and \( \mathbb{E}[V_n^s - V_n \mid V_n] \) needs to be shrinking. Bounding the first of the two is the more straightforward. We will show that

**Proposition A.2.** For any \( k \geq 1 \),

\[
\Pr[|V_n^s - V_n| \geq k] \leq (1-p)k\beta^{k-1},
\]

and

\[
\Pr[V_n^s = V_n] \geq p.
\]

**Proof.** Recalling the coupling times \( \tau_+ \) and \( \tau_- \) defined in (71), the chains \( X_i \) and \( Y_i \) differ only for \( I - \tau_- < i < I + \tau_+ \). Thus

\[
|V_n^s - V_n| \leq (\tau_+ + \tau_- - 1) \lor 0.
\]

Each of these stopping times is 0 if and only if \( X_I = 1 \), and thus we have that

\[
\Pr[V_n^s = V_n] \geq \Pr[X_I = 1] = p
\]

by stationarity.

Regardless of \( I \), the tails of each of these stopping times can be controlled by the constant \( \beta \), as for any non-negative integer \( k \),

\[
\Pr[\tau_+ > k \mid X_I = 0] \leq \beta^k.
\]

More generally, as this is simple worst-case behavior, the same bound holds for \( \tau_- \) and \( \tau_+ \) jointly in that for any non-negative integers \( k \) and \( l \),

\[
\Pr[\tau_+ > k \text{ and } \tau_- > l \mid X_I = 0] \leq \beta^{k+l}.
\]

We can now sum this bound to conclude that

\[
\Pr[|V_n^s - V_n| \geq k \mid X_I = 0] \leq \Pr[\tau_+ + \tau_- - 1 \geq k]
\]
\[
\leq \sum_{l=0}^{k-1} \Pr[\tau_+ > k-l-1 \text{ and } \tau_- > l \mid X_I = 0]
\]
\[
\leq k\beta^{k-1}.
\]
It remains to control the conditional expectation
\[ \mathbb{E} [V_n^* - V_n \mid V_n] . \]
An estimate of the variance of this expression would suffice for the usual application of Stein's method (see [Ros11, Theorem 3.20]), but this will not quite be sufficient for our purposes, as we will need some higher moments. We will use a functional equality to control the deviations of this expression. We recall that the Hamming distance on \( \{0,1\}^n \) is given by the minimum number of coordinate changes required to change one string into another; in this case, it coincides with the \( L^1 \) distance. A function \( f: \{0,1\}^n \to \mathbb{R} \) is called \( K \)-Lipschitz if
\[ |f(x) - f(y)| \leq K \|x - y\|_1 \]
for all \( x \) and \( y \) in \( \{0,1\}^n \).
We define
\[ F(x_1, \ldots, x_n) := \mathbb{E}[V_n^* - V_n \mid X_1 = x_1, \ldots, X_n = x_n], \]
so that
\[ \mathbb{E}[V_n^* - V_n \mid X_1 = x_1, \ldots, X_n = x_n] = F(x_1, \ldots, x_n), \]
and we turn to estimating the Lipschitz constant of \( F \). Let \( P \) be the transition matrix of the chain \( X_1, \ldots, X_n \). Our assumptions that the chain is reversible and that \( \beta < 1 \) imply that either all entries of \( P \) are less than one, or all entries of \( P^2 \) are. Let \( \delta \) equal 1 in the first case and 2 in the second, and let \( \gamma \) be the maximum entry of \( P^\delta \).

**Proposition A.3.** The function \( F \) is Lipschitz with constant \( (1 - \gamma + 2\delta)\delta/(1 - \gamma)^2 n \).

**Proof.** Fix some \( x_1, \ldots, x_n \) and some \( 1 \leq j \leq n \), and let \( x'_j = 1 - x_j \). To simplify notation, we assume for this proof that all random variables defined previously, such as \( Y_1, \ldots, Y_n, V_n^*, V_n \), are distributed conditional on \( X_1 = x_1, \ldots, X_n = x_n \). Define \( Y'_1, \ldots, Y'_n \) as in (72), using the same random index \( I \) and the same Markov chain \( \{Z_i\}_{i=1}^\infty \), but conditional on
\[ X_j = x'_j, \]
\[ X_i = x_i \quad \text{for} \ i \neq j. \]
This defines a coupling of the two conditional expectations \( F(x_1, \ldots, x_n) \) and \( F(x_1, \ldots, x_{j-1}, 1-x_j, x_{j+1}, \ldots, x_n) \). Define
\[ W_n^* = \sum_{i=1}^n Y'_i, \quad W_n = \sum_{i=1}^n x_i - x_j + x'_j. \]
We now have
\[ F(x_1, \ldots, x_n) - F(x_1, \ldots, x_{j-1}, x'_j, x_{j+1}, \ldots, x_n) = \mathbb{E}[V_n^* - V_n - (W_n^* - W_n)]. \]
Define \( \tau'_+ \) and \( \tau'_- \) analogously to \( \tau_+ \) and \( \tau_- \). The process \( Y'_* \) matches up with \( Z_{\tau'-1} \) from time \( I - \tau_- \) to \( I + \tau_+ \) and with its base sequence \( x_1, \ldots, x_n \) outside of those times. Similarly, \( Y'_* \) matches up with \( Z_{\tau'-1} \) from time \( I - \tau_- \) to \( I + \tau_+ \), and with its base sequence \( x_1, \ldots, x'_j, \ldots, x_n \) outside of those times. If \( j \notin \{I - \tau_- \ldots, I + \tau_+ \} \), then \( \tau'_\pm = \tau_\pm \), and \( Y'_* \) and \( Y'_* \) are equal to each other from \( I - \tau_- \) to \( I + \tau_+ \) and to their base sequences outside of those times. It thus holds that \( V_n^* - V_n = W_n^* - W_n \). Furthermore, this what usually occurs, as by direct calculation,
\[ \Pr[j \in \{I - \tau_- \ldots, I + \tau_+ \}] = \frac{1}{n} \sum_{i=1}^n \Pr[j \in \{I - \tau_- \ldots, I + \tau_+ \} \mid I = i] \]
\[ \leq \frac{1}{n} \sum_{i=1}^n \left( \gamma^{|j-i| - 1/\delta} \wedge 1 \right) \]
\[ \leq \frac{1}{n} \left( 1 + 2 \sum_{i=0}^{\infty} \gamma^{1/\delta} \right) = \frac{1 - \gamma + 2\delta}{(1 - \gamma)n}, \]

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If \( j \in \{I - \tau_-, \ldots, I + \tau_+\} \), then one of the two chains \( Y_\bullet \) and \( Y'_\bullet \) rejoins its base sequence at time \( j \), and the other does not. Let \( v \) be the number of extra steps it takes for this chain to rejoin its base sequence. Then \( V_n^s - V_n \) and \( W_n^s - W_n \) differ by at most \( v \). We bound the probability that \( v \) is large:

\[
\Pr [v > k | j \in \{I - \tau_-, \ldots, I + \tau_+\}] \leq \gamma^{[k/\delta]}.
\]

It follows that

\[
\mathbb{E} [v | j \in \{I - \tau_-, \ldots, I + \tau_+\}] = \sum_{k=0}^{\infty} \Pr [v > k | j \in \{I - \tau_-, \ldots, I + \tau_+\}] \leq \sum_{k=0}^{\infty} \gamma^{[k/\delta]} = \frac{\delta}{1 - \gamma}.
\]

Thus,

\[
|\mathbb{E} [V_n^s - V_n - (W_n^s - W_n)]| \\
\leq \frac{1 - \gamma + 2\delta}{(1 - \gamma) n} \mathbb{E} [V_n^s - V_n - (W_n^s - W_n) | j \in \{I - \tau_-, \ldots, I + \tau_+\}] \\
\leq \frac{(1 - \gamma + 2\delta) \delta}{(1 - \gamma)^2 n},
\]

thus showing that \( F \) is Lipschitz with the above constant.

The advantage of knowing that this function is Lipschitz is that we immediately get strong concentration in terms of the contraction coefficient of the chain.

**Proposition A.4.** For a \( K \)-Lipschitz function \( F \) of \( \{0, 1\}^n \)

\[
\Pr (|F(X) - \mathbb{E} F(X)| \geq t) \leq 2 \exp \left( -\frac{t^2(1 - \theta)^2}{2nK^2} \right),
\]

where \( X = (X_1, X_2, \ldots, X_n) \).

**Proof.** See Theorem 1.2 of [KR08] and the paragraph following it.

**Corollary A.5.** For any \( t \geq 0 \),

\[
\Pr \left( |\mathbb{E} [V_n^s - V_n] - \mathbb{E} [V_n^s - V_n | X] | \geq \frac{t}{\sqrt{n}} \right) \leq 2 \exp \left( -\frac{t^2(1 - \gamma)^2(1 - \theta)^2}{2(1 - \gamma + 2\delta)^2 \delta^2} \right).
\]

**Proof.** This is simply Proposition A.3 Proposition A.4 and a change of variables.

**Proposition A.6.** Let \( \lambda \) be the second eigenvalue of the Markov kernel, which is given by

\[
(73) \quad \lambda = \Pr [X_1 = 1 | X_0 = 1] - \Pr [X_1 = 1 | X_0 = 0].
\]

(\textit{Note that }|\lambda| = \theta.) Then

\[
\mathbb{E} [V_n^s - V_n] = \frac{(1 - p)(1 + \lambda)}{1 - \lambda} - \frac{2(1 - p)\lambda(1 - \lambda^n)}{(1 - \lambda)^2 n}.
\]

**Proof.** Let \( \hat{\tau}_+ = \tau_+ \wedge (n - I) \) and \( \hat{\tau}_- = \tau_- \wedge (I - 1) \). This only makes a difference when \( Z_{\bullet - I} \) never rejoins \( X_\bullet \), and \( \tau_+ = n - I + 1 \) or \( \tau_- = I \). We first use symmetry so that we can ignore one of \( \hat{\tau}_+ \) and \( \hat{\tau}_- \):

\[
\mathbb{E} [V_n^s - V_n] = \mathbb{E} \left[ \sum_{i=0}^{\hat{\tau}_+} (Z_i - X_{I+i}) \right] \\
= \mathbb{E} \left[ \sum_{i=0}^{\hat{\tau}_+} (Z_i - X_{I+i}) + \sum_{i=0}^{\hat{\tau}_-} (Z_{I-i} - X_{I-i}) - Z_0 + X_I \right] \\
= 2 \mathbb{E} \left[ \sum_{i=0}^{\hat{\tau}_+} (Z_i - X_{I+i}) \right] - 1 + p.
\]

(74)
Define
\[
M_i := \sum_{j=0}^{i} (Z_j - X_{I+j}) - \sum_{j=0}^{i-1} E[Z_{j+1} - X_{I+j+1} \mid \mathcal{F}_j],
\]
where \( \mathcal{F}_j = \sigma(I, X_I, \ldots, X_{I+j}, Z_0, \ldots, Z_j) \). The process \((M_i, i \geq 0)\) is a martingale with respect to the filtration \((\mathcal{F}_i, i \geq 0)\). By explicitly computing this conditional expectation, we see that
\[
M_i = (1 - \lambda) \sum_{j=0}^{i-1} (Z_j - X_{I+j}) + Z_i - X_{I+i}.
\]

Since \( \hat{\tau}_+ \) is a bounded stopping time with respect to this filtration,
\[
1 - p = EM_0 = EM_{\hat{\tau}_+} = (1 - \lambda) E \left[ \sum_{i=0}^{\hat{\tau}_+ - 1} (Z_i - X_{I+i}) \right] + E [Z_{\hat{\tau}_+} - X_{I+\hat{\tau}_+}]
\]
\[
= (1 - \lambda) E \left[ \sum_{i=0}^{\hat{\tau}_+} (Z_i - X_{I+i}) \right] + \lambda E [Z_{\hat{\tau}_+} - X_{I+\hat{\tau}_+}]
\]
by the optional stopping theorem. Thus,
\[
(75) \quad E \left[ \sum_{i=0}^{\hat{\tau}_+} (Z_i - X_{I+i}) \right] = \frac{1 - p - \lambda E [Z_{\hat{\tau}_+} - X_{I+\hat{\tau}_+}]}{1 - \lambda}.
\]

All that remains is to determine \( E[Z_{\hat{\tau}_+} - X_{I+\hat{\tau}_+}] \). The expression \( Z_{\hat{\tau}_+} - X_{I+\hat{\tau}_+} \) is zero unless the Markov chains \( Z_\bullet \) and \( X_{I+\bullet} \) never match up. So,
\[
E[Z_{\hat{\tau}_+} - X_{I+\hat{\tau}_+}] = E \left[ \mathbf{1}_{\{Z_0 \neq X_I, \ldots, Z_{n-I} \neq X_n\}} (Z_{n-I} - X_n) \right].
\]

Averaging over the possible choices of \( I \) gives
\[
E[Z_{\hat{\tau}_+} - X_{I+\hat{\tau}_+}] = \frac{1}{n} \sum_{i=1}^{n} E \left[ \mathbf{1}_{\{Z_0 \neq X_I, \ldots, Z_{n-I} \neq X_n\}} (Z_{n-I} - X_n) \right]
\]
\[
= \frac{1}{n} \sum_{i=0}^{n-1} E \left[ \mathbf{1}_{\{Z_0 \neq X_I, \ldots, Z_{n-I} \neq X_n\}} (Z_{n-I} - X_n) \right].
\]

In the last step, we shifted \( X_\bullet \) and replaced \( i \) with \( n-i \) to make the indices easier to deal with; the step is justified because \( X_\bullet \) is stationary and independent of \( Z_\bullet \). Next, we take an inductive approach. By the Markov property,
\[
E \left[ \mathbf{1}_{\{Z_0 \neq X_I, \ldots, Z_{n-I} \neq X_n\}} (Z_i - X_i) \mid X_0, \ldots, X_{i-1}, Z_0, \ldots, Z_{i-1} \right]
\]
\[
= \mathbf{1}_{\{Z_0 \neq X_I, \ldots, Z_{i-1} \neq X_{i-1}\}} E[Z_i - X_i \mid X_{i-1}, Z_{i-1}]
\]
\[
= \mathbf{1}_{\{Z_0 \neq X_I, \ldots, Z_{i-1} \neq X_{i-1}\}} \lambda (Z_{i-1} - X_{i-1}).
\]

Taking expectations,
\[
E \left[ \mathbf{1}_{\{Z_0 \neq X_I, \ldots, Z_{n-I} \neq X_n\}} (Z_i - X_i) \right]
\]
\[
= \lambda E \left[ \mathbf{1}_{\{Z_0 \neq X_I, \ldots, Z_{n-I} \neq X_{n-1}\}} (Z_{n-I} - X_{n-1}) \right]
\]
\[
= \lambda \lambda^{n-I} (1 - p).
\]

Thus
\[
E[Z_{\hat{\tau}_+} - X_{I+\hat{\tau}_+}] = \frac{1 - p}{n} \sum_{i=0}^{n-1} \lambda^i
\]
\[
= \frac{(1 - p)(1 - \lambda^n)}{n(1 - \lambda)}.
\]

Substituting this into (74) and (75) completes the proof. □
Finally, we prove the quantitative Markov central limit theorem by combining these facts, which we emphasize is not needed for the main results of this paper.

Proof of Proposition A.7. Let $W = (V_n - \mathbb{E} V_n)/\sqrt{\text{Var} V_n}$ and let $Z$ be a standard normal. Our starting point is the standard Stein’s method through size-bias coupling lemma (see Theorem 3.20 of [Ros11]), which states that

$$d_{W}(W, Z) \leq \frac{\mathbb{E} V_n}{\sqrt{\text{Var} V_n}} \sqrt{\frac{2}{\pi}} \sqrt{\text{Var} (V_n^* - V_n | V_n)} + \frac{(\mathbb{E} V_n)^{3/2}}{(\text{Var} V_n)^{3/2}} \mathbb{E} \left[ (V_n^* - V_n)^2 \right].$$

We will need to know that the standard deviation of $V_n$ is $\Theta(\sqrt{n})$. (In this proof, the implicit constants in asymptotic expressions should be understood to depend on the law of the Markov chain.) On the one hand, we know that $\mathbb{E} V_n = np = \Theta(n)$ by stationarity. On the other hand, from the definition of the size-bias distribution that

$$\mathbb{E} \left[ (V_n^* - V_n)^2 \right] = \frac{\text{Var} V_n}{\mathbb{E} V_n}.$$

By hypothesis on $\mu$ and $\beta$, it follows that $\text{Var} V_n = \Theta(1)$, and hence $\text{Var} V_n = \Theta(n)$.

Meanwhile, by Jensen’s inequality and Corollary A.5,

$$\text{Var}(\mathbb{E} (V_n^* - V_n | V_n)) \leq \text{Var}(\mathbb{E} (V_n^* - V_n | X)) = O(1/n),$$

and by Proposition A.2

$$\mathbb{E} \left[ (V_n^* - V_n)^2 \right] = O(1).$$

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