GROUP ACTIONS ON SIMPLE TRACIALLY $\mathcal{Z}$-ABSORBING $C^*$-ALGEBRAS

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Abstract. We show that if $A$ is a simple (not necessarily unital) tracially $\mathcal{Z}$-absorbing $C^*$-algebra and $\alpha: G \to \text{Aut}(A)$ is an action of a finite group $G$ on $A$ with the weak tracial Rokhlin property, then the crossed product $C^*(G, A, \alpha)$ and the fixed point algebra $A^\alpha$ are simple and tracially $\mathcal{Z}$-absorbing, and they are $\mathcal{Z}$-stable if, in addition, $A$ is separable and nuclear. The same conclusion holds for all intermediate $C^*$-algebras of the inclusions $A^\alpha \subseteq A$ and $A \subseteq C^*(G, A, \alpha)$. We prove that if $A$ is a simple tracially $\mathcal{Z}$-absorbing $C^*$-algebra, then, under a finiteness condition, the permutation action of the symmetric group $S_m$ on the minimal $m$-fold tensor product of $A$ has the weak tracial Rokhlin property. We define the weak tracial Rokhlin property for automorphisms of simple $C^*$-algebras and we show that—under a mild assumption—(tracial) $\mathcal{Z}$-absorption is preserved under crossed products by such automorphisms.

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1. INTRODUCTION AND MAIN RESULTS

The study of group actions on $C^*$-algebras and the corresponding crossed products is at the core of the modern theory of operator algebras. With the near completion of the Elliott’s program to classify simple separable unital nuclear $\mathcal{Z}$-stable $C^*$-algebras (see W. Winter’s ICM talk in 2018 [44] for details), the classification of nonunital $C^*$-algebras is now of great interest [15, 16, 17, 11]. Among equivalent conditions which characterize classifiability, the notion of $\mathcal{Z}$-stability has a central role in the classification program.

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The property of tracial $\mathcal{Z}$-absorption, studied by Hirshberg and Orovitz for unital C*-algebras in [19], and by the authors for nonunital C*-algebras in [1], is a local version of $\mathcal{Z}$-stability, and is equivalent to that for simple separable nuclear C*-algebras [19, 8].

In this paper we investigate finite group actions on simple (not necessarily unital) tracially $\mathcal{Z}$-absorbing C*-algebras. In particular, we deal with the $\mathcal{Z}$-stability of the resulting crossed products. For actions on simple unital C*-algebras, there are already nice results in the literature (however, even in the unital case, some of our results are not covered by the existing known results). For instance, Matui and Sato showed that if $\alpha : \Gamma \to \text{Aut}(A)$ is a strongly outer action of an elementary amenable group $\Gamma$ on a simple unital separable nuclear stably finite infinite dimensional C*-algebra $A$ with finitely many extremal tracial states, then the $\mathcal{Z}$-stability of $A$ implies that of the crossed product $C^*(\Gamma, A, \alpha)$ (see [30, Corollary 4.11]). In [19, Theorem 5.6], Hirshberg and Orovitz showed that if $\alpha : G \to \text{Aut}(A)$ is an action of a finite group $G$ on a simple unital separable tracially $\mathcal{Z}$-absorbing C*-algebra $A$, then the crossed product $C^*(G, A, \alpha)$ is tracially $\mathcal{Z}$-absorbing, whenever $\alpha$ has the generalized tracial Rokhlin property.

There are a few results about the $\mathcal{Z}$-stability of the crossed products of actions on nonunital C*-algebras, but the assumptions on the actions are rather strong. For instance, if $\alpha : G \to \text{Aut}(A)$ is an action of a finite group $G$ on a separable C*-algebra $A$, then the $\mathcal{Z}$-stability of $A$ implies that of the crossed product, provided that $\alpha$ has the Rokhlin property [41, Theorem 3(v)]. More generally, the same conclusion holds even if $\alpha$ has finite Rokhlin dimension with commuting towers [20, Theorem 2.2]. However, the Rokhlin property is rather strong and imposes certain restrictions on K-theory of the C*-algebra. For instance, if $A$ is simple and separable such that either $K_0(A)$ or $K_1(A)$ is isomorphic to $\mathbb{Z}$, then there is no nontrivial finite group actions on $A$ with the Rokhlin property (see [31, Corollary 3.10]). Likewise, there is no action of a nontrivial finite group on $\mathcal{Z}$ or $\mathcal{O}_\infty$ which has finite Rokhlin dimension with commuting towers [20, Corollary 4.8] (cf. Theorem B and Corollary 3.12 below). For this reason, one would prefer instead to use the weak tracial Rokhlin property [13].

In the first main result we deal with the preservation of (tracial) $\mathcal{Z}$-absorption for finite groups actions on nonunital simple C*-algebras.

**Theorem A.** Let $A$ be a simple tracially $\mathcal{Z}$-absorbing C*-algebra and let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$ with the weak tracial Rokhlin property. Then

1. the crossed product $C^*(G, A, \alpha)$ and the fixed point algebra $A^\alpha$ are simple and tracially $\mathcal{Z}$-absorbing;
2. if $A$ is $\sigma$-unital then all intermediate C*-algebras $B$ and $D$ with $A^\alpha \subseteq B \subseteq A \subseteq D \subseteq C^*(G, A, \alpha)$ are simple and tracially $\mathcal{Z}$-absorbing;
(3) If $A$ is separable and nuclear then all intermediate $C^*$-algebras $B$ and $D$ as above are nuclear and $Z$-stable.

Part (1) extends [19, Theorem 5.6] to the nonunital case in which the fixed point algebra is not considered. (Note that, the assumption of separability is implicit in the statement of [19, Theorem 5.6], since the authors use their Lemma 5.5 in the proof.) To prove Part (1), we use ideas from [19] and the machinery of Cuntz subequivalence in our nonunital setting instead of using the tracial state space (see Theorem 2.5).

Part (2) provides a way to identify more tracially $Z$-absorbing $C^*$-algebras, by looking at intermediate $C^*$-algebras. Recently, the intermediate $C^*$-algebras (and their classification) have attracted some interest [9, 2]. We use Izumi’s result on the Galois correspondence for intermediate $C^*$-algebras in the setting of finite group actions [21, Corollary 6.6] (see Corollary 2.7 below). Part (3) follows from Part (2) and the equivalence of $Z$-stability and tracial $Z$-absorption for simple separable nuclear $C^*$-algebras [19, 8]. See Corollary 2.8 for the statement and proof of this part.

In [19], it is shown that the permutation action (see Section 3) of the symmetric group $S_m$ on $Z^\otimes m \cong Z$ has the generalized tracial Rokhlin property (and hence the weak tracial Rokhlin property, by Proposition 2.3). Our next result is about the weak tracial Rokhlin property of the permutation action of the symmetric group $S_m$ on the minimal tensor product of $m$ copies of a simple tracially $Z$-absorbing $C^*$-algebra. The combination of this result and Theorem A could be used to give new examples $Z$-stable $C^*$-algebras.

**Theorem B.** Let $A$ be a simple tracially $Z$-absorbing $C^*$-algebra and $A^\otimes m$ be finite for some $m \in \mathbb{N}$. Then the permutation action $\beta: S_m \to \text{Aut}(A^\otimes m)$ has the weak tracial Rokhlin property.

Here, finiteness of $A^\otimes m$ means that its unitization is finite in the usual sense, which is the case, for example, if $A^\otimes m$ is stably projectionless or if $A$ is simple, exact, and stably finite in the sense of [37].

Though the idea of the proof is rather simple, the proof is long and needs several technical lemmas (see Section 3). Also, we use a recent result which says that for actions on finite $C^*$-algebras, Condition (4) in the definition of the weak tracial Rokhlin property (Definition 2.2) follows from the other conditions [1, Proposition 7.12].

As an application of Theorems B, the permutation action of $S_m$ on the Razak-Jacelon algebra $W \cong W^\otimes m$ has the weak tracial Rokhlin property, since $W$ is simple, finite, and $Z$-stable. Also, Theorem A implies that $C^*(S_m, W, \beta)$ is $Z$-stable (since $W$ is nuclear). In [13, Example 3.12], various examples of actions with the weak tracial Rokhlin property are constructed using Theorem B. For instance, if $A$ is a simple nonelementary $C^*$-algebra with tracial rank zero (hence tracially $Z$-absorbing, by [1, Theorem A]), then $\beta: S_m \to \text{Aut}(A^\otimes m)$ has the weak tracial Rokhlin property and the
resulting crossed product is tracially $\mathcal{Z}$-absorbing, and if, in addition, $A$ is separable and nuclear, then the crossed product is $\mathcal{Z}$-stable.

Using a recent dichotomy for tracially approximately divisible C*-algebras [14] and Theorems A and B we obtain the following. See Corollary 3.14.

**Corollary C.** Let $A$ be a simple separable exact tracially $\mathcal{Z}$-absorbing C*-algebra. Let $m \in \mathbb{N}$ and consider the permutation action $\beta: S_m \to \text{Aut}(A^\otimes m)$. Then all intermediate C*-algebras of the inclusions $(A^\otimes m)^\beta \subseteq A^\otimes m$ and $A^\otimes m \subseteq C^*(S_m, A^\otimes m, \beta)$ are simple and tracially $\mathcal{Z}$-absorbing. If, in addition, $A$ is nuclear then all these intermediate C*-algebras are nuclear and $\mathcal{Z}$-stable.

The next main result is about $\mathcal{Z}$-actions and (tracial) $\mathcal{Z}$-absorption of the crossed product. This result is a nonunital version of [19, Theorem 6.7]. We weaken the assumption in [19, Theorem 6.7] that $\alpha^m$ acts trivially on $T(A)$ for some $m \in \mathbb{N}$, and we don’t need separability. We extend the definition of the (weak) tracial Rokhlin property for actions of $\mathbb{Z}$ to the nonunital case (cf. [32, Definition 1.1]). See Definitions 4.1 and 4.5 and Theorem 4.11.

**Theorem D.** Let $A$ be a simple tracially $\mathcal{Z}$-absorbing C*-algebra and let $\alpha \in \text{Aut}(A)$ have the controlled weak tracial Rokhlin property. Then the crossed product $C^*(\mathbb{Z}, A, \alpha)$ is simple and tracially $\mathcal{Z}$-absorbing. If moreover $A$ is separable and nuclear then $C^*(\mathbb{Z}, A, \alpha)$ is $\mathcal{Z}$-stable.

As an example, if $A$ is any simple $\mathcal{Z}$-stable C*-algebra, then, starting from an arbitrary automorphism of $A$, we obtain an automorphism $\alpha \in \text{Aut}(A)$ with the controlled weak tracial Rokhlin property (Example 4.15). The resulting crossed product is tracially $\mathcal{Z}$-absorbing, and it is $\mathcal{Z}$-stable if, in addition, $A$ is separable and nuclear.

Note that the controlled weak tracial Rokhlin property is equivalent to the weak tracial Rokhlin property for automorphisms of simple unital exact tracially $\mathcal{Z}$-absorbing C*-algebras with finitely many extremal traces (see Corollary 4.8).

The paper is organized as follows. In Section 2, we prove Theorem A. In Section 3, we prove that if $A$ is a simple tracially $\mathcal{Z}$-absorbing C*-algebra, then, under a finiteness condition, the permutation action of the symmetric group on the minimal tensor product of finitely many copies of $A$ has the weak tracial Rokhlin property. In Section 4, we first define the weak tracial Rokhlin property for automorphisms of simple (not necessarily unital) C*-algebras. Then we show—under a mild assumption—that (nonunital) tracial $\mathcal{Z}$-absorption passes to crossed products by automorphisms with the weak tracial Rokhlin property (Theorem 4.11).

We use the following (standard) notations. For a C*-algebra $A$, $A_+$ denotes the positive cone of $A$. Also, $A^+$ denotes the minimal unitization of $A$ (adding a new identity even if $A$ is unital), while $A^\sim = A$ if $A$ is unital and...
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$A^\sim = A^+$ if $A$ is nonunital. The notation $a \approx_\varepsilon b$ means $\|a - b\| < \varepsilon$. For $a, b \in A_+$, we use $a \precsim b$ to mean that $a$ is Cuntz subequivalent to $b$. (See, for example, [35, 1] for preliminaries on the Cuntz subequivalence.) We write $\mathcal{K} = \mathcal{K}(\ell^2)$ and $M_n = M_n(\mathbb{C})$. We take $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. (The $p$-adic integers will never appear.) We take $\mathbb{N} = \{1, 2, \ldots\}$. We abbreviate “completely positive contractive” to “c.p.c.”. If $\alpha: G \to \text{Aut}(A)$ is an action of a group $G$ on a C*-algebra $A$, then we denote the fixed point algebra by $A^\alpha$. Finally, for any C*-algebra $A$, we denote its tracial state space by $T(A)$ and by $A^{\otimes m}$ the minimal tensor product of $m$ copies of $A$.

2. Finite group actions

For actions of finite groups on not necessarily unital C*-algebras, at least five Rokhlin type properties have appeared in the literature. One is the multiplier Rokhlin property [33, Definition 2.15], which is defined using projections in the multiplier algebra. A second, given in [31, Definition 3.1] and just called the Rokhlin property, is defined for $\sigma$-unital C*-algebras using projections in the central sequence algebra of the given C*-algebra. A third, Definition 2 in [41, Section 3], is called there just the Rokhlin property, and is defined using positive contractions in the algebra instead of projections. In the unital case, it is equivalent to the usual Rokhlin property (Corollary 1 in [41, Section 3]), in the separable nonunital case it is equivalent to [31, Definition 3.1] (Corollary 2 in [41, Section 3]), and it is implied by the multiplier Rokhlin property (Corollary 3 in [41, Section 3]), although the converse is false (Example 1 in [41, Section 3]). A nonunital version of finite Rokhlin dimension with commuting towers is given in [20, Definition 1.14], and of course there is an analog without commuting towers. The fifth Rokhlin type property is a weak version (using positive elements instead of projections) of the tracial Rokhlin property for finite group actions on simple C*-algebras [13]. This is the one we use here. It is recalled in Definition 2.2 below. Even in the unital case, it is much weaker than the first three properties above, as discussed in [33, Example 3.12], and it also does not imply finite Rokhlin dimension with commuting towers, as follows from [20, Corollary 4.8]. It seems likely that finite Rokhlin dimension with commuting towers implies the property in Definition 2.2. As far as we know, however, this has never been checked.

In this section we prove (Theorem 2.5 below) that if $\alpha: G \to \text{Aut}(A)$ is an action of a finite group $G$ on a simple (not necessarily unital) tracially $\mathbb{Z}$-absorbing C*-algebra and $\alpha$ has the weak tracial Rokhlin property, then $C^*(G, A, \alpha)$ and $A^\alpha$ are also simple tracially $\mathbb{Z}$-absorbing C*-algebras. This is a generalization of Theorem 5.6 of [19] to the nonunital case. In [19] the fixed point algebra is not considered. We also remove the assumption of separability from Theorem 5.6 of [19]. (This assumption is implicit in the
statement of Theorem 5.6 of [19] because the authors use their Lemma 5.5 in the proof.)

First we recall the definition of tracial $\mathcal{Z}$-absorption introduced in [19] in the unital case and extended to the nonunital case in [1].

**Definition 2.1** ([1], Definition 3.6). We say that a simple C*-algebra $A$ is **tracially $\mathcal{Z}$-absorbing** if $A \not\cong \mathbb{C}$ and for every $x, a \in A_+$ with $a \neq 0$, every finite set $F \subseteq A$, every $\varepsilon > 0$, and every $n \in \mathbb{N}$, there is a c.p.c. order zero map $\varphi : M_n \to A$ such that:

1. $(x^2 - x\varphi(1)x - \varepsilon)_+ \preceq a$.
2. $|||\varphi(z), b|| < \varepsilon$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $b \in F$.

We recall the definition of the weak tracial Rokhlin property for finite group actions on simple (not necessarily unital) C*-algebras from [13].

**Definition 2.2** ([13], Definition 4.4). Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a simple C*-algebra $A$. Then $\alpha$ has the **weak tracial Rokhlin property** if for every $\varepsilon > 0$, every finite set $F \subseteq A$, and every $x, y \in A_+$ with $\|x\| = 1$, there exist orthogonal positive contractions $f_g \in A$ for $g \in G$ such that, with $f = \sum_{g \in G} f_g$, the following hold:

1. $\|f_g a - a f_h\| < \varepsilon$ for all $a \in F$ and all $g \in G$.
2. $\|\alpha_g(f_h) - f_{gh}\| < \varepsilon$ for all $g, h \in G$.
3. $(y^2 - y f y - \varepsilon)_+ \preceq x$.
4. $\|f x f\| > 1 - \varepsilon$.

We relate the weak tracial Rokhlin property to the generalized tracial Rokhlin property of [19].

**Proposition 2.3.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a simple unital C*-algebra $A$.

1. If $\alpha$ has the weak tracial Rokhlin property, then $\alpha$ has the generalized tracial Rokhlin property of [19, Definition 5.2].
2. If $A$ is finite and $\alpha$ has the generalized tracial Rokhlin property of [19, Definition 5.2], then $\alpha$ has the weak tracial Rokhlin property.

**Proof of Proposition 2.3.** Suppose $\alpha$ has the weak tracial Rokhlin property.

Define continuous functions $k, l : [0, 1] \to [0, 1]$ by

$$k(t) = \begin{cases} 3t & 0 \leq t \leq \frac{1}{3} \\ 1 & \frac{1}{3} < t \leq 1 \end{cases} \quad \text{and} \quad l(t) = \begin{cases} 0 & 0 \leq t \leq 2/3 \\ 3t - 2 & 2/3 < t \leq 1 \end{cases}.$$

To prove that $\alpha$ has the generalized tracial Rokhlin property, let $\varepsilon > 0$, let $F \subseteq A$ be finite, and let $a \in A_+ \setminus \{0\}$. Without loss of generality, $\|c\| \leq 1$ for all $c \in F$.

Use [29, Lemma 2.5.11(2)] to choose $\delta_1 > 0$ such that whenever $b, c \in A$ satisfy $0 \leq b, c \leq 1$ and $\|b - c\| < \delta_1$, then $\|k(b) - k(c)\| < \varepsilon$. Use [4, Lemma 2.5] to choose $\delta_2 > 0$ such that whenever $b \in A$ satisfies $0 \leq b \leq 1$ and $c \in F$ satisfies $\|b, c\| < \delta_2$, then $\|k(b), c\| < \varepsilon$. 
Define $\varepsilon_0 = \min(\delta_1, \delta_2, \frac{1}{3})$. Set $x = \|a\|^{-1}a$ and $y = 1$. Apply Definition 2.2 with $\varepsilon_0$ in place of $\varepsilon$, with $F$ as given, and these choices of $x$ and $y$, getting orthogonal positive contractions $f_g \in A$ for $g \in G$. Set $f = \sum_{g \in G} f_g$. Condition (4) in Definition 2.2 and $\varepsilon_0 \leq \frac{1}{3}$ imply $\|fxf\| > \frac{1}{3}$. In particular, $\|f\| > \left(\frac{5}{3}\right)^{1/2} > \frac{1}{3}$. So there is at least one $g_0 \in G$ such that $\|f_{g_0}\| > \frac{1}{3}$, and now Condition (2) in Definition 2.2 implies $\|f_g\| > \frac{1}{3}$ for all $g \in G$.

Define $e_g = k(f_g)$ for $g \in G$. The elements $e_g$ are orthogonal positive contractions, and the inequality $\|f_g\| > \frac{1}{3}$ ensures that $\|e_g\| = 1$ for all $g \in G$. In particular, we have (1) in [19, Definition 5.2].

By $\varepsilon_0 \leq \delta_1$ and the choice of $\delta_1$, we have $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$, which is (4) in [19, Definition 5.2]. Using $\delta_2$ in place of $\delta_1$, we similarly get $\|e_ga - ae_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$, which is (3) in [19, Definition 5.2].

Define $e = \sum_{g \in G} e_g$. It remains to show that $1 - e \precsim a$. One easily checks that $e = k(f)$. We have $1 - k(t) = l(1 - t)$ for all $t \in [0, 1]$, and for any $b \in A$ with $0 \leq b \leq 1$ we have $l(b) \sim (b - \frac{1}{2})_+$. Therefore

$$1 - e = 1 - k(f) = l(1 - f) \sim (1 - f - \frac{2}{3})_+ \leq (1 - f - \varepsilon)_+ \precsim x \sim a,$$

as desired.

Now suppose that $A$ is finite and $\alpha$ has the generalized tracial Rokhlin property. Let $\varepsilon, F, x,$ and $y$ be as in Definition 2.2. By [35, Lemma 2.9], there is $z \in (\overline{xAx})_+ \setminus \{0\}$ such that whenever $c \in A_+$ satisfies $0 \leq c \leq 1$ and $c \precsim z$, then $\|(1 - c)x(1 - c)\| > 1 - \varepsilon$. Apply [19, Definition 5.2], getting orthogonal positive contractions $f_g \in A$ for $g \in G$ such that, with $f = \sum_{g \in G} f_g$, the following hold:

1. $1 - f \precsim z$.
2. $\|f_ga - af_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$.
3. $\|\alpha_g(f_h) - f_{gh}\| < \varepsilon$ for all $g, h \in G$.

The last two conditions are (1) and (2) in Definition 2.2. For Condition (3) in Definition 2.2, we use (1) above at the fourth step and $z \in (\overline{xAx})_+$ at the fifth step, to get

$$(y^2 - yfy - \varepsilon)_+ \leq y(1 - f)y \sim (1 - f)^{1/2} y^2 (1 - f)^{1/2} \leq \|y\|^2(1 - f) \precsim z \precsim x.$$

Condition (4) in Definition 2.2 follows from (1) above and the choice of $z$. □

Lemma 2.4. Let $A$ be a simple tracially $\mathcal{Z}$-absorbing $C^*$-algebra and let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on $A$ with the weak tracial Rokhlin property. Then for every finite set $F \subseteq A$, every $\varepsilon > 0$, every nonzero positive element $a \in A$, every positive contraction $x \in A_+$, and every $n \in \mathbb{N}$, there is a c.p.c. order zero map $\psi: M_n \to A$ such that:

1. $\|x^2 - x\psi(1)x - \varepsilon\_+ \precsim a$.
2. $\|\psi(z), y\| < \varepsilon$ for any $z \in M_n$ with $\|z\| \leq 1$ and any $y \in F$. 

(3) \( \|a_g(\psi(z)) - \psi(z)\| < \varepsilon \) for any \( z \in M_n \) with \( \|z\| \leq 1 \) and any \( g \in G \).

**Proof.** Let \( F, \varepsilon, a, x, \) and \( n \) be given as in the statement. We may assume that \( F \subseteq A_+ \) and that \( \|y\| \leq 1 \) for all \( y \in F \). Choose \( \delta > 0 \) such that \( \delta < \varepsilon / [\|\text{card}(G)\|] \).

Use [27, Proposition 2.5] to choose \( \eta_0 > 0 \) such that whenever \( \varphi: M_n \to A \) is a c.p.c. map such that \( \|\varphi(y)\varphi(z)\| < \eta_0 \) for all \( y, z \in (M_n)_+ \) with \( yz = 0 \) and \( \|y\|, \|z\| \leq 1 \) (a c.p.c. \( \eta_0 \)-order zero map), then there is a c.p.c. order zero map \( \psi: M_n \to A \) such that \( \|\varphi(z) - \psi(z)\| < \delta \) for all \( z \in M_n \) with \( \|z\| \leq 1 \).

Use [4, Lemma 2.5] to choose \( \eta_1 > 0 \) such that whenever \( D \) is a C*-algebra and \( h, k \in D \) satisfy \( 0 \leq h, k \leq 1 \) and \( \|[h, k]\| < \eta_1 \), then \( \|[h^{1/2}, k]\| < \delta \). Now set \( \eta = \min(\eta_0, \eta_1, \delta) \).

We claim that there is \( b \in A_+ \setminus \{0\} \) such that:

(4) \( b \oplus \bigoplus_{g \in G} \alpha_g(b) \preceq a \).

To prove the claim, first set \( k = \text{card}(G) + 1 \) and use [35, Lemma 2.4] to find \( c \in A_+ \setminus \{0\} \) such that \( c \otimes 1_k \preceq a \) in \( M_\infty(A) \). Then [35, Lemma 2.6] implies that there is \( b \in A_+ \setminus \{0\} \) such that \( b \preceq \alpha_g^{-1}(c) \) for all \( g \in G \). Thus,

\[
b \oplus \bigoplus_{g \in G} \alpha_g(b) \preceq c \oplus \bigoplus_{g \in G} c = c \otimes 1_k \preceq a,
\]

which is (4).

Since \( \alpha \) has the weak tracial Rokhlin property, there are orthogonal positive contractions \( e_g \in A \) for \( g \in G \) such that, with \( e = \sum_{g \in G} e_g \), the following hold:

(5) \( \|e_g y - ye_g\| < \eta \) for all \( y \in F \cup \{x\} \) and all \( g \in G \).

(6) \( \|\alpha_g(e_h) - e_{gh}\| < \eta \) for all \( g, h \in G \).

(7) \( (x^2 - xe_x - \eta)_+ \preceq b \).

Set

\[
E = \{\alpha_g(y) : g \in G, y \in F\} \cup \{\alpha_h(e_g), \alpha_h(e_g)^{1/2} : g, h \in G\},
\]

which is a finite subset of \( A \). Since \( A \) is tracially \( \mathcal{Z} \)-absorbing, there is a c.p.c. order zero map \( \varphi: M_n \to A \) such that the following hold:

(8) \( (x^2 - x\varphi(1)x - \eta)_+ \preceq b \).

(9) \( \|\varphi(z)y\| < \eta \) for any \( z \in M_n \) with \( \|z\| \leq 1 \) and any \( y \in E \).

Define a c.p.c. map \( \tilde{\varphi}: M_n \to A \) by

\[
\tilde{\varphi}(z) = \sum_{g \in G} e_g^{1/2} \alpha_g(\varphi(z)) e_g^{1/2}
\]

for \( z \in M_n \). It follows from (9) that:

(10) \( \|\tilde{\varphi}(z) - \sum_{g \in G} e_g \alpha_g(\varphi(z))\| < \eta \text{card}(G) \) for any \( z \in M_n \) with \( \|z\| \leq 1 \).

For \( y, z \in (M_n)_+ \) with \( yz = 0 \) and \( \|y\|, \|z\| \leq 1 \), we use orthogonality of the elements \( e_g \) at the first two steps, \( \varphi(y) \varphi(z) = 0 \) at the third step, and (9)
at the fourth step, getting
\[
\| \tilde{\varphi}(y) \tilde{\varphi}(z) \| = \left\| \sum_{g \in G} e_{g}^{1/2} \alpha_{g}(\varphi(y)) e_{g}^{1/2} \cdot e_{g}^{1/2} \alpha_{g}(\varphi(z)) e_{g}^{1/2} \right\|
\leq \max_{g \in G} \left\| e_{g}^{1/2} \alpha_{g}(\varphi(y)) e_{g} \alpha_{g}(\varphi(z)) e_{g}^{1/2} \right\|
\leq \max_{g \in G} \left\| [\alpha_{g}^{-1}(e_{g}), \varphi(y)] \right\| < \eta \leq \eta_{0}.
\]
By the choice of \( \eta_{0} \), there is a c.p.c. order zero map \( \psi : M_{n} \to A \) such that:

(11) \( \| \tilde{\varphi}(z) - \psi(z) \| < \delta \) for any \( z \in M_{n} \) with \( \| z \| \leq 1 \).

We will show that \( \psi \) has the desired properties (1), (2), and (3) in the statement.

We prove (1). Using (7) and (8) at the first step and (4) at the third step, we get

\[
(2.1) \quad (x^{2} - xex - \eta)_{+} + \sum_{g \in G} e_{g}^{1/2} \alpha_{g}(\varphi^{x}(1)x - \eta)_{+}) e_{g}^{1/2} \approx b \oplus \bigoplus_{g \in G} e_{g}^{1/2} \alpha_{g}(b) e_{g}^{1/2} \approx b \oplus \bigoplus_{g \in G} \alpha_{g}(b) \approx a.
\]

On the other hand, using the assumption that \( x \in A^{\alpha} \) at the second step, using (5), \( \eta \leq \eta_{1} \), and the choice of \( \eta_{1} \) at the third step, and using (11) at the last step, we get

\[
(x^{2} - xex - \eta)_{+} + \sum_{g \in G} e_{g}^{1/2} \alpha_{g}(\varphi^{x}(1)x - \eta)_{+}) e_{g}^{1/2} \approx 2\eta x^{2} - xex + \sum_{g \in G} e_{g}^{1/2} \alpha_{g}(x^{2} - x\varphi^{x}(1)x) e_{g}^{1/2} = x^{2} - xex + \sum_{g \in G} e_{g}^{1/2} x e_{g}^{1/2} - \sum_{g \in G} e_{g}^{1/2} x \alpha_{g}(\varphi^{x}(1)) x e_{g}^{1/2} \approx 4\delta \text{card}(G) x^{2} - xex + \sum_{g \in G} x e_{g} x - \sum_{g \in G} x e_{g}^{1/2} \alpha_{g}(\varphi^{x}(1)) e_{g}^{1/2} x = x^{2} - x\tilde{\varphi^{x}(1)x} \approx \delta x^{2} - x\psi^{x}(1)x.
\]

So
\[
\left\| x^{2} - x\psi^{x}(1)x - \left[ (x^{2} - xex - \eta)_{+} + \sum_{g \in G} e_{g}^{1/2} \alpha_{g}(\varphi^{x}(1)x - \eta)_{+}) e_{g}^{1/2} \right] \right\| < 2\eta + 4\delta \text{card}(G) + \delta \leq 7\delta \text{card}(G) < \epsilon.
\]
Therefore, using \((2.1)\) at the second step, we have
\[
(x^2 - x\varphi(1)x - \varepsilon)_+ \precsim (x^2 - x\varepsilon(x - \eta))_+ + \sum_{g \in G} e_g^{1/2} \alpha_g((x^2 - x\varphi(1)x - \eta)_+)e_g^{1/2} \precsim a,
\]
as desired.

To prove \((2)\), let \(y \in F\) and let \(z \in M_n\) satisfy \(|z| \leq 1\). Using \((11)\) at the first step, \((10)\) at the second step, and \((5)\) and \((9)\) at the third step, we have
\[
|||\psi(z), y||| < 2\delta + ||[\tilde{\psi}(z), y]|| \\
\leq 2\delta + 2\eta \text{card}(G) + \left\| \sum_{g \in G} e_g \alpha_g(\varphi(z), y) \right\| \\
< 2\delta + 2\eta \text{card}(G) + 2\eta \text{card}(G) \leq 6\delta \text{card}(G) < \varepsilon.
\]

It remains to show \((3)\). Let \(g \in G\) and let \(z \in M_n\) satisfy \(|z| \leq 1\). Using \((11)\) at the first step, \((10)\) at the second step, and \((6)\) at the third step we have:
\[
||\alpha_g(\psi(z)) - \psi(z)|| \\
< 2\delta + ||\alpha_g(\tilde{\psi}(z)) - \tilde{\psi}(z)|| \\
\leq 2\delta + 2\eta \text{card}(G) + \left\| \sum_{h \in G} \alpha_g(e_h)\alpha_{gh}(\varphi(z)) - \sum_{h \in G} e_{gh}\alpha_{gh}(\varphi(z)) \right\| \\
\leq 2\delta + 2\eta \text{card}(G) + \eta \text{card}(G) \leq 5\delta \text{card}(G) < \varepsilon.
\]
This finishes the proof. \(\square\)

By averaging over the group and applying \([27, \text{Proposition 2.5}]\) again, we can improve the statement of Lemma 2.4 to require that \(\alpha_g(\psi(z)) = \psi(z)\) for all \(z \in M_n\) and all \(g \in G\). It seems simpler to do without this step. Moreover, the way we do it, the proof of Theorem 2.5 is a better model for the proof of Theorem 4.11, the case \(G = \mathbb{Z}\).

**Theorem 2.5.** Let \(\alpha : G \to \text{Aut}(A)\) be an action of a finite group \(G\) on a simple (not necessarily unital) tracially \(\mathbb{Z}\)-absorbing \(C^*\)-algebra \(A\). If \(\alpha\) has the weak tracial Rokhlin property, then \(C^*(G, A, \alpha)\) and \(A^\alpha\) are also simple tracially \(\mathbb{Z}\)-absorbing \(C^*\)-algebras.

**Proof.** By \([13, \text{Proposition 3.2}]\), \(\alpha\) is pointwise outer. So \([28, \text{Theorem 3.1}]\) implies that \(C^*(G, A, \alpha)\) is simple. Since \(A^\alpha\) is isomorphic to a hereditary \(C^*\)-subalgebra of \(C^*(G, A, \alpha)\) \([39]\), it follows that \(A^\alpha\) is also simple, and by \([1, \text{Theorem 4.1}]\), it also follows that it is enough to show that \(C^*(G, A, \alpha)\) is tracially \(\mathbb{Z}\)-absorbing.

To verify \([1, \text{Definition 3.6}]\) for \(C^*(G, A, \alpha)\), let \(F \subseteq C^*(G, A, \alpha)\) be a finite set, let \(x, a \in C^*(G, A, \alpha)_+\) with \(a \neq 0\), let \(\varepsilon > 0\), and let \(n \in \mathbb{N}\). The proof of \([19, \text{Lemma 5.1}]\) also works in the nonunital case, and we can apply this generalization of it to find \(a_0 \in A_+ \setminus \{0\}\) such that \(a_0 \precsim a\) in
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Next, let $(u_i)_{i \in I}$ be an approximate identity for $A$. For $i \in I$ set

$$v_i = \frac{1}{\text{card}(G)} \sum_{g \in G} \alpha_g(u_i) \in A^\alpha.$$ 

Then $(v_i)_{i \in I}$ is an approximate identity for $\mathcal{C}^*(G,A,\alpha)$ which is contained in $A^\alpha$. Therefore, by [1, Remark 3.8], we may assume that $x \in A^\alpha$.

For $g \in G$ let $u_g \in M(\mathcal{C}^*(G,A,\alpha))$ be the standard unitary associated with the crossed product. For $y \in F$ write $y = \sum_{g \in G} c_{y,g} u_g$ with $c_{y,g} \in A$ for $g \in G$. Define $E = \{c_{y,g} : y \in F \text{ and } g \in G\}$. Apply Lemma 2.4 with $E$ in place of $F$, with $a_0$ in place of $a$, with $\varepsilon/[2\text{card}(G)]$ in place of $\varepsilon$. We obtain a c.p.c. order zero map $\psi_0 : M_n \to A$. Let $\psi : A \to \mathcal{C}^*(G,A,\alpha)$ be its composition with the inclusion of $A$ in $\mathcal{C}^*(G,A,\alpha)$. We claim that $\psi$ satisfies the conditions of [1, Definition 3.6], for $\mathcal{C}^*(G,A,\alpha)$.

Condition (1) is clear. For Condition (2), let $z \in M_n$ satisfy $\|z\| \leq 1$ and let $y \in F$. Then

$$\| [\psi(z), y] \| \leq \sum_{g \in G} \| [\psi(z), c_{y,g} u_g] \|$$

$$\leq \sum_{g \in G} \| [\psi(z), c_{y,g}] \| + \sum_{g \in G} \| [\psi(z), u_g] \|$$

$$< \text{card}(G) \left( \frac{\varepsilon}{2\text{card}(G)} \right) + \text{card}(G) \left( \frac{\varepsilon}{2\text{card}(G)} \right) = \varepsilon.$$ 

This completes the proof. □

Remark 2.6. The proofs of Lemma 2.4 and Theorem 2.5 work if moreover $A$ is unital and $\alpha$ has the generalized tracial Rokhlin property ([19, Definition 5.2]) instead of having the weak tracial Rokhlin property. This is because Condition (4) in Definition 2.2 is not used in the proof of Lemma 2.4.

The point is that the definitions are written so that any purely infinite simple C*-algebra is tracially $\mathcal{Z}$-absorbing, but not every action of a finite group on a purely infinite simple C*-algebra has the weak tracial Rokhlin property. The generalized tracial Rokhlin property of [19] is strong enough to imply that the crossed product is simple. By [24, Theorem 4.5], the crossed product of a purely infinite simple unital C*-algebra by a pointwise outer action of a finite group is always purely infinite and simple. In fact, even if the action isn’t pointwise outer, the crossed product is a finite direct sum of purely infinite simple C*-algebras, so is purely infinite in the sense of [26, Definition 4.1].

We thus obtain an alternative proof for Theorem 5.6 of [19] which avoids dimension functions. In particular, the assumption of separability in Theorem 5.6 of [19] is unnecessary. (This assumption is implicit in the statement of that result because the authors use their Lemma 5.5.)

Corollary 2.7. Let $\alpha : G \to \text{Aut}(A)$ be an action of a finite group $G$ on a simple $\sigma$-unital tracially $\mathcal{Z}$-absorbing C*-algebra $A$. If $\alpha$ has the weak tracial
Rokhlin property then all intermediate $C^*$-algebras $B$ and $D$ with

$$A^\alpha \subseteq B \subseteq A \subseteq D \subseteq C^*(G, A, \alpha)$$

are simple and tracially $\mathcal{Z}$-absorbing.

**Proof.** Suppose that $B$ and $D$ are intermediate $C^*$-algebras as above. By [13, Proposition 3.2], $\alpha$ is pointwise outer. Then, [21, Corollary 6.6(1)] implies that there is a subgroup $H$ of $G$ such that $D = C^*(H, A, \beta)$ where $\beta: H \to \text{Aut}(A)$ is the restriction of $\alpha$ to $H$. By [13, Proposition 4.1], $\beta$ has the weak tracial Rokhlin property. Now, Theorem 2.5 yields that $D$ is simple and tracially $\mathcal{Z}$-absorbing. The corresponding result about $B$ follows similarly using the fact that $B$ is equal to the fixed point algebra of a restricted action $\beta: H \to \text{Aut}(A)$ as above [21, Corollary 6.6(2)].

**Corollary 2.8.** Let $\alpha: G \to \text{Aut}(A)$ be an action of a finite group $G$ on a simple separable nuclear $\mathcal{Z}$-stable $C^*$-algebra $A$. If $\alpha$ has the weak tracial Rokhlin property, then $C^*(G, A, \alpha)$ and $A^\alpha$ are simple $\mathcal{Z}$-stable $C^*$-algebras. Moreover, all intermediate $C^*$-algebras $B$ and $D$ as in Corollary 2.7 are simple, nuclear, and $\mathcal{Z}$-stable.

**Proof.** By Theorem 2.5, $C^*(G, A, \alpha)$ is simple and tracially $\mathcal{Z}$-absorbing. It is also nuclear since $A$ is nuclear and $G$ is amenable. By the main result of [8], we see that $C^*(G, A, \alpha)$ is $\mathcal{Z}$-stable. On the other hand, $A^\alpha$ is isomorphic to a full corner of $C^*(G, A, \alpha)$ [39], and by [43, Corollary 3.2], $\mathcal{Z}$-absorption is preserved under Morita equivalence in the class of separable $C^*$-algebras. Therefore $A^\alpha$ is also a simple $\mathcal{Z}$-stable $C^*$-algebra.

The second part about intermediate $C^*$-algebras follows from the first part and an argument similar to the proof of Corollary 2.7.

The following example can be thought of as the analog of [19, Example 5.10]. It is also a consequence of Theorem 3.10 below.

**Example 2.9.** Let $W$ be the Razak-Jacelon algebra [22], that is, the unique simple Razak algebra [36] which has a unique tracial state and no unbounded traces. Let $n \in \{2, 3, \ldots\}$, and let $A = W^\otimes n$, the tensor product of $n$ copies of $W$. We have $A \cong W$ by [15, Corollary 19.3] and $A \cong A \otimes \mathcal{Z}$ by [22, Corollary 6.3].

Let $\alpha$ be the action of the symmetric group $S_n$ on $A$ by permuting the tensor factors. We claim that $\alpha$ has the weak tracial Rokhlin property. First, let $\beta$ be the action of $S_n$ on $Z^\otimes n$ by permuting the tensor factors. By [19, Example 5.10], this action has the generalized Rokhlin property of [19, Definition 5.2]. By Proposition 2.3(2), it has the weak tracial Rokhlin property. Therefore the action $\alpha \otimes \beta$ of $S_n$ on $A \otimes Z^\otimes n = (W \otimes \mathcal{Z})^\otimes n$ has the weak tracial Rokhlin property by [13, Proposition 4.5]. Using an isomorphism $W \cong W \otimes \mathcal{Z}$ ([22, Corollary 6.3]), one sees that $\alpha$ is conjugate to $\alpha \otimes \beta$, and so has the weak tracial Rokhlin property. Alternatively, it follows from Theorem 3.10 below that $\alpha$ has the weak tracial Rokhlin property.

It follows from Theorem 2.5 that $C^*(S_n, W^\otimes n, \alpha)$ is tracially $\mathcal{Z}$-absorbing.
We don’t know whether the action in Example 2.9 has the Rokhlin property. The corresponding action on \( \mathbb{Z} \) does not have even any higher dimensional Rokhlin property with commuting towers, by [20, Corollary 4.8]. However, the proof relies, among other things, on nontriviality of \( K_0(\mathbb{Z}) \), while \( K_*(\mathcal{W}) = 0 \).

**Example 2.10.** Let \( A = \bigotimes_{k=1}^{\infty} M_3 \) be the UHF algebra of type \( 3^\infty \) and let \( B \) be a (not necessarily unital) simple C*-algebra. Then \( A \otimes B \) is \( \mathbb{Z} \)-stable, so it is tracially \( \mathbb{Z} \)-absorbing. Consider the action \( \alpha: \mathbb{Z}_2 \to \text{Aut}(A) \) generated by the automorphism

\[
\bigotimes_{k=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \text{Aut}(A),
\]

and let \( \beta: \mathbb{Z}_2 \to \text{Aut}(B) \) be an arbitrary action. The action \( \alpha \) has the tracial Rokhlin property (for unital C*-algebras and using projections) by the condition in [34, Proposition 2.5(3)] (but not the Rokhlin property, by the condition in [34, Proposition 2.4(3)]). By [13, Proposition 4.5], the action \( \alpha \otimes \beta: \mathbb{Z}_2 \to \text{Aut}(A \otimes B) \) has the weak tracial Rokhlin property. Now Theorem 2.5 implies that \( C^*(\mathbb{Z}_2, A \otimes B, \alpha \otimes \beta) \) is a simple tracially \( \mathbb{Z} \)-absorbing C*-algebra.

As an example for \( B \) and \( \beta \), let \( n \in \{2, 3, \ldots, \infty\} \), take \( B \) to be the minimal tensor product of two copies of a proper hereditary subalgebra of \( C^*_r(F_n) \), and take \( \beta \) to be generated by the (minimal) tensor flip.

A more interesting specific case would be gotten by taking \( B \) to be the reduced free product of two copies of the same nonunital C*-algebra, and taking \( \beta \) to be the free flip. Criteria for simplicity of reduced free products of unital C*-algebras are given in the corollary to [5, Proposition 3.1], but unfortunately we do not know criteria for simplicity of reduced free products of nonunital C*-algebras.

### 3. The permutation action on a finite tensor product

We will prove that if \( A \) is a simple tracially \( \mathbb{Z} \)-absorbing C*-algebra, then, under a finiteness condition, the permutation action of the symmetric group on the minimal tensor product of finitely many copies of \( A \) has the weak tracial Rokhlin property of Definition 2.2. The basic idea is simple. Fix \( m \in \mathbb{N} \), the number of tensor factors. Suppose \( n \in \mathbb{N} \) and \( \varphi: M_n \to A \) is a c.p.c. order zero map. Let \( (e_{j,k})_{j,k=1,2,\ldots,m} \) be the standard system of matrix units for \( M_n \). Then in the tensor product of \( m \) copies of \( A \), we can look at the elements

\[
\varphi(e_{r(1),r(1)}) \otimes \varphi(e_{r(2),r(2)}) \otimes \cdots \otimes \varphi(e_{r(m),r(m)})
\]

with

\[
r = (r(1), r(2), \ldots, r(m)) \in \{1, 2, \ldots, n\}^m.
\]
The permutation action on the tensor product translates into permutation of the indices in this expression. If $n$ is large, then most elements $r \in \{1, 2, \ldots, n\}^m$ have all coordinates distinct. The action is free on the set of such $r$, and adding up one of these from each orbit gives the elements required in the weak tracial Rokhlin property.

Some condition is presumably needed. The way the definitions are written, every purely infinite simple C*-algebra is tracially $\mathbb{Z}$-absorbing, but not every action on such a C*-algebra has the weak tracial Rokhlin property. The details are somewhat messy, so we start with some lemmas.

**Notation 3.1.** For $m \in \mathbb{N}$ let $S_m$ denote the corresponding symmetric group, the group of all permutations of $\{1, 2, \ldots, m\}$. Let $(e_{j,k})_{j,k=1,2,\ldots,m}$ be the standard system of matrix units for $M_m$. For a C*-algebra $A$, we let $A \otimes^m$ denote the minimal tensor product of $m$ copies of $A$, and for $x \in A$ we write $x \otimes^m$ for the $m$-fold tensor product $x \otimes x \otimes \cdots \otimes x \in A \otimes^m$. The permutation action $\beta$ of $S_m$ on $A \otimes^m$ is determined by

$$\beta_\sigma(a_1 \otimes a_2 \otimes \cdots \otimes a_m) = a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes \cdots \otimes a_{\sigma^{-1}(m)}$$

for $a_1, a_2, \ldots, a_m \in A$ and $\sigma \in S_m$. (We suppress $A$ and $m$ in the notation.) Finally, we recall [1, Notation 7.3]: the Pedersen ideal of $A$ is $\text{Ped}(A)$.

**Lemma 3.2.** Let $A$ be a C*-algebra, let $b \in A_+$, let $a \in (\text{Ab}A)_+$, and let $\varepsilon > 0$. Then there are $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in A$ such that $(a - \varepsilon)_+ = \sum_{j=1}^n x_j^*bx_j$.

**Proof.** Use [35, Lemma 1.13] to find $y_1, y_2, \ldots, y_n \in A$ such that

$$\left\| a - \sum_{j=1}^n y_j^*by_j \right\| < \varepsilon.$$

[1, Lemma 2.1], provides $d \in A$ such that

$$(a - \varepsilon)_+ = d^* \left( \sum_{j=1}^n y_j^*by_j \right) d.$$ 

Set $x_j = y_jd$ for $j = 1, 2, \ldots, n$. $\square$

**Lemma 3.3.** Let $A$ be a simple tracially $\mathbb{Z}$-absorbing C*-algebra, let $a \in A_+$, let $b \in A_+ \setminus \{0\}$, and let $\varepsilon > 0$. Then there is $n_0 \in \mathbb{N}$ such that whenever $\nu \in \mathbb{N}$ and $c \in A_+$ satisfy $n_0\nu(c) \leq \langle (a - \varepsilon)_+ \rangle$, then $\nu(c) \leq \langle b \rangle$.

One can prove this lemma using dimension functions: since $(a - \varepsilon)_+ \in \text{Ped}(A)$, the infimum of $d((a - \varepsilon)_+)$ over suitably normalized dimension functions will be strictly positive, and we can choose $\nu$ to be greater than the reciprocal of this infimum. However, the result can be gotten directly from almost unperforation.
Proof of Lemma 3.3. Lemma 3.2 provides \( n \in \mathbb{N} \) and \( x_1, x_2, \ldots, x_n \in A \) such that \((a - \varepsilon)_+ = \sum_{j=1}^{n} x_j^* b x_j \). Then, in \( M_n(A) \),

\[
(a - \varepsilon)_+ \preceq \text{diag}(x_1^* b x_1, x_2^* b x_2, \ldots, x_n^* b x_n)
\]

\[
\sim \text{diag}(b^{1/2} x_1^* b^{1/2}, b^{1/2} x_2^* b^{1/2}, \ldots, b^{1/2} x_n^* b^{1/2}) \preceq 1_n \otimes b.
\]

Set \( n_0 = n + 1 \). If \( n_0 \nu(c) \leq (a - \varepsilon)_+ \), then \((n + 1) \nu(c) \leq n(b)\), so \( \nu(c) \leq \langle b \rangle \) by [1, Theorem 6.4]. \( \square \)

Lemma 3.4. For every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that the following holds. Let \( n \in \mathbb{N} \), let \( A \) be a C*-algebra, let \( x \in A \) satisfy \( \| x \| \leq 1 \), and let \( \varphi : M_n \to A \) be a c.p.c. order zero map such that \( \| [\varphi(z), x] \| < \delta \) for any \( z \in M_n \) with \( \| z \| \leq 1 \). Then for every \( j, k \in \{1, 2, \ldots, n\} \) there is \( v \in A \) such that \( \| v \| \leq 1 \) and \( \| v^* x^* \varphi(e_{j,j}) x v - x^* \varphi(e_{k,k}) x \| < \varepsilon \).

Proof. Define \( f : [0, 1] \to [0, 1] \) by

\[
f(\lambda) = \begin{cases} 0 & 0 \leq \lambda \leq \frac{\varepsilon}{8} \\ 8\varepsilon^{-1} \lambda - 1 & \frac{\varepsilon}{8} < \lambda < \frac{\varepsilon}{4} \\ 1 & \frac{\varepsilon}{4} \leq \lambda \leq 1. \end{cases}
\]

Then \( f \in C_0((0, 1]) \). Further let \( t \in C_0((0, 1]) \) be the function \( t(\lambda) = \lambda \) for \( \lambda \in (0, 1] \). By [3, Lemma 2.5], the unitized cone \((CM_2)^+\) is generated by 1 and the elements \( t \otimes e_{j,k} \) for \( j, k = 1, 2 \). (There is a misprint in [3, Lemma 2.5]: the word “unital” is missing.) Therefore there is \( \delta > 0 \) such that whenever \( A \) is a C*-algebra, \( x \in A \) satisfies \( \| x \| \leq 1 \), \( \psi : (CM_2)^+ \to A^+ \) is a unital homomorphism, and \( \| [x, \psi(t \otimes e_{j,k})] \| < \delta \) for \( j, k = 1, 2 \), then \( \| [x, \psi(f \otimes e_{1,2})] \| < \frac{\varepsilon}{4} \).

Now let \( n, A, x, \varphi, j, \) and \( k \) be as in the hypotheses. There is nothing to prove if \( j = k \), so assume \( j \neq k \). Let \( \psi : CM_n \to A \) be the corresponding homomorphism ([45, Corollary 4.1]), satisfying \( \psi(t \otimes z) = \varphi(z) \) for \( z \in M_n \), and let \( \psi^+ : (CM_n)^+ \to A^+ \) be its unitization. Then \( \| [x, \psi^+(t \otimes e_{j,k})] \| < \delta \) for \( j, k = 1, 2, \ldots, n \). By considering the cone over the embedding of \( M_2 \) in \( M_n \) determined by

\[
e_{1,1} \mapsto e_{j,j}, \quad e_{1,2} \mapsto e_{j,k}, \quad e_{2,1} \mapsto e_{k,j}, \quad \text{and} \quad e_{2,2} \mapsto e_{k,k},
\]

we see that \( v = \psi^+(f \otimes e_{j,k}) \) satisfies \( \| [x, v] \| < \frac{\varepsilon}{4} \). Also clearly \( \| v \| \leq 1 \).

One checks that

\[
(f \otimes e_{j,k})^*(t \otimes e_{j,j})(f \otimes e_{j,k}) = tf^2 \otimes e_{k,k}
\]

and \( |\lambda f(\lambda) - \lambda| \leq \frac{\varepsilon}{2} \) for all \( \lambda \in [0, 1] \), so \( \| v^* \varphi(e_{j,j}) v - \varphi(e_{k,k}) \| \leq \frac{\varepsilon}{4} \). Now

\[
\| v^* x \varphi(e_{j,j}) x v - x^* \varphi(e_{k,k}) x \| \leq \| [v^*, x^*] \| \cdot \| \varphi(e_{j,j}) x v \| + \| [x^* v \varphi(e_{j,j})] \| \cdot \| [v, x] \|
\]

\[
< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon,
\]

as desired. \( \square \)
Lemma 3.5. Let $A$ be a simple tracially $\mathcal{Z}$-absorbing $C^*$-algebra. Then for every $x, a \in A_+$ with $a \neq 0$ and every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $\nu \in \mathbb{N}$, all $\rho > 0$, and every finite set $F \subseteq A$, there is a c.p.c. order zero map $\varphi : M_{n_0 \nu} \to A$ such that:

1. $\left( x^2 - x\varphi(1)x - \varepsilon \right)_+ \preceq a$.
2. $||[\varphi(z), b]|| < \rho$ for any $z \in M_{n_0 \nu}$ with $||z|| \leq 1$ and any $b \in F$.
3. $\nu \langle x\varphi(e_{1,1})x - \varepsilon \rangle_+ \leq \langle a \rangle$ in $W(A)$.

Proof. We may assume that $||x|| \leq 1$, since if $||x|| > 1$ we can replace $x$ with $x/||x||$ and $\varepsilon$ with $\varepsilon/||x||^2$ in (1) and (3), and then use $(Mw-M\varepsilon)_+ \sim (w-\varepsilon)_+$ for all $w \in A_+$ and all $M > 0$. Set $x_0 = (x - \frac{\varepsilon}{2})_+$. We require $x_0 \in F$.

Apply Lemma 3.3 with $x$ in place of $a$, with $a$ in place of $b$, and with $\frac{\varepsilon}{2}$ in place of $\varepsilon$, getting $n_0 \in \mathbb{N}$.

Let $\nu \in \mathbb{N}$. Apply Lemma 3.4 with $\frac{\varepsilon}{2}$ in place of $\varepsilon$, and call the resulting number $\rho_0$. Use [4, Lemma 2.5] to choose $\rho_1 > 0$ such that whenever $r, s \in A_+$ satisfy

$$||r|| \leq 1, \quad ||s|| \leq 1, \quad \text{and} \quad ||rs - sr|| < \rho_1,$$

then

$$||r^{1/2}s - sr^{1/2}|| < \frac{\varepsilon}{4n_0\nu}.$$ 

Set $\delta = \min(\rho, \rho_0, \rho_1)$.

Apply the definition of tracial $\mathcal{Z}$-absorption ([1, Definition 3.6]) to find a c.p.c. order zero map $\varphi : M_{n_0 \nu} \to A$ such that:

1. $\left( x_0^2 - x_0\varphi(1)x_0 - \frac{\varepsilon}{2} \right)_+ \preceq a$.
2. $||[\varphi(z), b]|| < \delta$ for any $z \in M_{n_0 \nu}$ with $||z|| \leq 1$ and any $b \in F$.

We have $||x_0 - x|| < \frac{\varepsilon}{2}$ and $||x||$, $||x_0|| \leq 1$, whence

$$||\left( x_0^2 - x_0\varphi(1)x_0 \right) - (x^2 - x\varphi(1)x) \rangle < \frac{\varepsilon}{2}.$$ 

Then [35, Corollary 1.6] implies

$$(x^2 - x\varphi(1)x - \varepsilon)_+ \preceq \left( x_0^2 - x_0\varphi(1)x_0 - \frac{\varepsilon}{2} \right)_+ \preceq a.$$ 

This is part (1) of the conclusion. Since $\delta \leq \rho$, part (2) of the conclusion is clear.

It remains to prove part (3). Since $x_0 \in F$, the choice of $\rho_0$ using Lemma 3.4 provides, for $k = 1, 2, \ldots, n_0 \nu$, an element $v_k \in A$ such that

$$||v_k x_0 \varphi(e_{k,k}) x_0 v_k - x_0 \varphi(e_{1,1}) x_0 || < \frac{\varepsilon}{2} \quad \text{and} \quad ||v_k|| \leq 1.$$ 

Then, using [35, Corollary 1.6] at the first step and [1, Lemma 2.3] and $||v_k|| \leq 1$ at the second step,

$$(3.1) \left( x_0 \varphi(e_{1,1}) x_0 - \varepsilon \right)_+ \preceq \left( v_k^* x_0 \varphi(e_{k,k}) x_0 v_k - \frac{\varepsilon}{2} \right)_+ \preceq \left( x_0 \varphi(e_{k,k}) x_0 - \frac{\varepsilon}{2} \right)_+.$$ 

Since $\delta \leq \rho_1$, the choice of $\rho_1$ implies that

$$||\varphi(e_{k,k})^{1/2} x_0 - x_0 \varphi(e_{k,k})^{1/2} || < \frac{\varepsilon}{4n_0\nu}.$$
for \( k = 1, 2, \ldots, n_0. \) So

\[
(3.2) \quad \left\| x_0 \varphi(1) x_0 - \sum_{k=1}^{n_0} \varphi(e_{k,k})^{1/2} x_0^2 \varphi(e_{k,k})^{1/2} \right\|
\]
\[
\leq \sum_{k=1}^{n_0} \left\| x_0 \varphi(e_{k,k}) x_0 - \varphi(e_{k,k})^{1/2} x_0^2 \varphi(e_{k,k})^{1/2} \right\|
\]
\[
\leq \sum_{k=1}^{n_0} \left( \| [x_0, \varphi(e_{k,k})^{1/2}] \| \cdot \| \varphi(e_{k,k})^{1/2} x_0 \|
+ \| \varphi(e_{k,k})^{1/2} x_0 \| \cdot \| [\varphi(e_{k,k})^{1/2}, x_0] \| \right)
\]
\[
< 2n_0 \varepsilon \left( \frac{\varepsilon}{4n_0} \right) = \varepsilon / 2.
\]

Since the elements \( \varphi(e_{k,k})^{1/2} x_0^2 \varphi(e_{k,k})^{1/2} \) are orthogonal, we have

\[
\left( \sum_{k=1}^{n_0} \varphi(e_{k,k})^{1/2} x_0^2 \varphi(e_{k,k})^{1/2} - \frac{\varepsilon}{2} \right) = \sum_{k=1}^{n_0} \left( \varphi(e_{k,k})^{1/2} x_0^2 \varphi(e_{k,k})^{1/2} - \frac{\varepsilon}{2} \right).
\]

Also by orthogonality, and by (3.2),

\[
\sum_{k=1}^{n_0} \langle (\varphi(e_{k,k})^{1/2} x_0^2 \varphi(e_{k,k})^{1/2} - \frac{\varepsilon}{2}) \rangle \leq \langle x_0 \varphi(1) x_0 \rangle \leq \langle x_0 \rangle.
\]

By [12, Proposition 2.3(ii)], we have

\[
(\varphi(e_{k,k})^{1/2} x_0^2 \varphi(e_{k,k})^{1/2} - \frac{\varepsilon}{2}) \sim (x_0 \varphi(e_{k,k}) x_0 - \frac{\varepsilon}{2}).
\]

Therefore, using (3.1), it follows that

\[
n_0 \nu \langle (x_0 \varphi(e_{1,1}) x_0 - \varepsilon) \rangle \leq \langle x_0 \rangle.
\]

Now the choice of \( n_0 \) using Lemma 3.3 implies \( \nu \langle (x_0 \varphi(e_{1,1}) x - \varepsilon) \rangle \leq \langle a \rangle \), as desired.

\[\square\]

**Lemma 3.6.** Let \( A \) and \( B \) be simple \( C^* \)-algebras, with \( B \) not of type I. Let \( a, x \in A_+ \setminus \{0\} \), and let \( b \in B_+ \setminus \{0\} \). Then for every \( \varepsilon > 0 \) there is \( y \in \overline{(bbb)}_+ \setminus \{0\} \) such that \( (a - \varepsilon) \otimes y \preceq x \otimes b \) in \( A \otimes_{\min} B \).

**Proof.** By [35, Lemma 1.13], there are \( n \in \mathbb{N} \) and \( c_1, c_2, \ldots, c_n \in A \) such that \( \| a - \sum_{j=1}^n c_j x c_j \| < \varepsilon \). Use [35, Lemma 2.1] to find orthogonal Cuntz equivalent elements \( y_1, y_2, \ldots, y_n \in \overline{(bbb)}_+ \setminus \{0\} \). Set \( y = y_1 \). Then in
\[ W(A \otimes \text{min } B) \] we have
\[ \langle (a - \varepsilon)_+ \otimes y \rangle \leq \left( \sum_{j=1}^{n} c_j^* x c_j \otimes y \right) \leq \sum_{j=1}^{n} \langle c_j^* x c_j \otimes y \rangle \]
\[ = \sum_{j=1}^{n} \langle x^{1/2} c_j^* x^{1/2} \otimes y \rangle \leq n \langle x \otimes y \rangle = \sum_{j=1}^{n} \langle x \otimes y_j \rangle \leq \langle x \otimes b \rangle. \]

This completes the proof. \( \square \)

**Lemma 3.7.** Let \( A \) be a simple \( C^* \)-algebra which is not of type I. Then for every \( m \in \mathbb{N} \), every \( b \in (A^{\otimes m})_+ \setminus \{0\} \), every \( x \in A_+ \) with \( \|x\| = 1 \), and for every \( \varepsilon > 0 \), there are \( \delta > 0 \) and \( z \in A_+ \setminus \{0\} \) such that, whenever \( a \in A_+ \) satisfies \( \|a\| \leq 1 \) and \( (x^2 - xax - \delta)_+ \precsim z \), then \( [(x^2)^{\otimes m} - (xax)^{\otimes m} - \varepsilon]_+ \precsim b \).

**Proof.** First suppose that there is \( y \in A_+ \setminus \{0\} \) such that \( b = y^{\otimes m} \). Set
\[ \delta = \frac{\varepsilon}{4m^2}. \]
By [35, Lemma 1.13], there are \( n \in \mathbb{N} \) and \( c_1, c_2, \ldots, c_n \in A \) such that
\[ \left\| x - \sum_{j=1}^{n} c_j^* y c_j \right\| < \delta. \]
Then, as in the proof of Lemma 3.3,
\[ (x - \delta)_+ \precsim 1_n \otimes y, \]
whence \( 1_n \otimes y \) in \( M_n(A) \). Therefore there is \( \delta_0 > 0 \) such that
\[ (x - 2\delta)_+ \precsim (1_n \otimes y - \delta_0)_+. \]
Use [35, Lemma 2.4] to find orthogonal Cuntz equivalent elements \( r_1, r_2, \ldots, r_n \in (y A y)_+ \setminus \{0\} \). Set \( r = r_1 \).

Apply Lemma 3.6 with \( y \) in place of both \( a \) and \( b \), with \( r \) in place of \( z \), and with \( \delta_0 \) in place of \( \varepsilon \), and call the resulting element \( s_0 \). Choose \( s \in A_+ \setminus \{0\} \) such that \( s \precsim s_0 \) and \( s \precsim r \). Then
\[ (y - \delta_0)_+ \otimes s \precsim r \otimes y. \]

Use [35, Lemma 2.4] to find orthogonal Cuntz equivalent elements \( z_{j_1, j_2, \ldots, j_{m-1}} \in (s A s)_+ \setminus \{0\} \) for \( j_1, j_2, \ldots, j_{m-1} = 1, 2, \ldots, n \). Set \( z = z_{1, 1, \ldots, 1} \).

Suppose \( (x^2 - xax - \delta)_+ \precsim z \). For \( k = 1, 2, \ldots, m \) define
\[ h_k = \left[ (x - 2\delta)_+ \otimes (x - 2\delta)_+ \right]^{\otimes (k-1)} \otimes (x^2 - xax - \delta)_+ \otimes \left[ ((x - 2\delta)_+) \right]^{2(k-m)} \]
and
\[ h_k(0) = (xax)^{\otimes (k-1)} \otimes (x^2 - xax) \otimes (x^2)^{\otimes (m-k)}. \]
Using \( \|x\| \leq 1 \) and \( ||a|| \leq 1 \), we get \( \|h_k - h_k(0)\| \leq (4m - 3)\delta. \) Since
\[ \sum_{k=1}^{m} h_k(0) = (x^2)^{\otimes m} - (xax)^{\otimes m}, \]
Therefore \( \varepsilon > 0 \) leads to the special case also work for \( \alpha \). Let \( y \) be an element. This completes the proof of the special case.

\[
(x^2)^m - (xax)^m - \sum_{k=1}^{m} h_k \leq m(4m - 3)\delta < \varepsilon. 
\]

We have, using (3.3) at the second step,

\[
h_1 = (x^2 - xax - \delta) + \big[ ((x - 2\delta)_+^2 \big]^{\otimes (m-1)} \notag
\]

\[
\lesssim z \otimes [1_n \otimes y]^{\otimes (m-1)} \notag
\]

\[
\sim \sum_{j_1, j_2, \ldots, j_m=1}^{n} z_{j_1, j_2, \ldots, j_m} \otimes y^{\otimes (m-1)} \notag
\]

\[
\lesssim [1_n \otimes (y - \delta_0)] + \otimes [1_n \otimes y]^{\otimes (k-1)} \notag
\]

\[
\sim \sum_{j_1, j_2, \ldots, j_m=1}^{n} (y - \delta_0) + \otimes y^{\otimes (k-2)} \otimes z_{j_1, j_2, \ldots, j_m} \otimes y^{\otimes (m-k)} \notag
\]

\[
\lesssim (y - \delta_0) + \otimes y^{\otimes (k-2)} \otimes s \otimes y^{\otimes (m-k)} \notag
\]

\[
\lesssim r \otimes y^{\otimes (k-2)} \otimes y \otimes y^{\otimes (m-k)} = r \otimes y^{\otimes (m-1)} \sim r_k \otimes y^{\otimes (m-1)}. 
\]

Therefore \( \sum_{k=1}^{m} h_k \lesssim \left( \sum_{k=1}^{m} h_k \right) \otimes y^{\otimes (m-1)} \lesssim y^{\otimes m}. \)

It now follows from (3.6) that

\[
[(x^2)^m - (xax)^m - \varepsilon]_+ \lesssim \sum_{k=1}^{m} h_k \lesssim y^{\otimes m}. 
\]

This completes the proof of the special case.

For the general case, use induction and Kirchberg's Slice Lemma ([37, Lemma 4.1.9]) to find \( y_1, y_2, \ldots, y_m \in A_+ \setminus \{0\} \) such that \( y_1 \otimes y_2 \otimes \cdots \otimes y_m \lesssim b \). Use [35, Lemma 2.6] to choose \( y \in A_+ \setminus \{0\} \) such that \( y \lesssim y_k \) for \( k = 1, 2, \ldots, m \). Then \( y^{\otimes m} \lesssim b \). The choices of \( \delta \) and \( z \) which work for \( y \) in the special case also work for \( b \).

**Lemma 3.8.** Let \( A \) be a simple \( C^* \)-algebra, let \( G \) be a finite group, and let \( \alpha : G \to \text{Aut}(A) \) be an action of \( G \) on \( A \). Let \( x \in A_+ \setminus \{0\} \). Suppose that an element \( y \in A_+ \) has the following property. For any finite set \( F \subseteq A \), any \( \varepsilon > 0 \), and any \( x \in A_+ \setminus \{0\} \), there exist orthogonal positive contractions \( f_g \in A \) for \( g \in G \) such that, with \( f = \sum_{g \in G} f_g \), the following hold:
Then every positive element \( z \in \overline{Ay} \) also has the same property.

Proof. The proof is essentially the same as that of [1, Lemma 3.5]. \( \square \)

Lemma 3.9. Let \( A \) be a finite simple \( C^* \)-algebra, let \( G \) be a finite group, and let \( \alpha : G \to \text{Aut}(A) \) be an action of \( G \) on \( A \). Suppose that for every \( \varepsilon > 0 \), every finite set \( F \subseteq A \), and every \( x, y \in A_{+} \) with \( x \neq 0 \), there exist orthogonal positive contractions \( f_g \in A \) for \( g \in G \) such that, with \( f = \sum_{g \in G} f_g \), the following hold:

1. \( \| f_g a - a f_g \| < \varepsilon \) for all \( a \in F \) and all \( g \in G \).
2. \( \| \alpha_g(f_h) - f_{gh} \| < \varepsilon \) for all \( g, h \in G \).
3. \( (y^2 - y f y - \varepsilon)_+ \asymp x \).

Then \( \alpha \) has the weak tracial Rokhlin property.

The conditions are the same as in Definition 2.2, except that we have omitted (4), the requirement that \( \| f x f \| > 1 - \varepsilon \).

Proof of Lemma 3.9. The proof is the same as that of [1, Proposition 7.12]. \( \square \)

Theorem 3.10. Let \( A \) be a simple tracially \( Z \)-absorbing \( C^* \)-algebra, let \( m \in \mathbb{N} \), and adopt Notation 3.1. Suppose that \( A^\otimes m \) is finite ([1, Definition 7.1]). Then the permutation action \( \beta : S_m \to \text{Aut}(A^\otimes m) \) has the weak tracial Rokhlin property.

Proof. We will make repeated use of the following estimate. Suppose \( a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in A \) satisfy \( \| a_j \| \leq 1 \) and \( \| b_j \| \leq 1 \) for \( j = 1, 2, \ldots, m \). Then

\[
(3.7) \quad \| a_1 \otimes a_2 \otimes \cdots \otimes a_m - b_1 \otimes b_2 \otimes \cdots \otimes b_m \| \leq \sum_{j=1}^{m} \| a_j - b_j \|.
\]

We verify the conditions in Lemma 3.9. Thus, let \( \varepsilon > 0 \), let \( F \subseteq A^\otimes m \) be finite, and let \( x, y \in (A^\otimes m)_+ \) with \( x \neq 0 \). We need to find orthogonal positive contractions \( f_{\sigma} \in A^\otimes m \) for \( \sigma \in S_m \) such that, with \( f = \sum_{\sigma \in S_m} f_{\sigma} \), the following hold:

1. \( \| f_{\sigma} a - a f_{\sigma} \| < \varepsilon \) for all \( a \in F \) and all \( \sigma \in S_m \).
2. \( \| \beta_{\sigma}(f_{\tau}) - f_{\sigma \tau} \| < \varepsilon \) for all \( \sigma, \tau \in S_m \).
3. \( (y^2 - y f y - \varepsilon)_+ \asymp x \).

It is clearly sufficient (changing \( \varepsilon \)) to prove this for \( y \) in a dense subset of \( (A^\otimes m)_+ \). Combining this fact with Lemma 3.8, we see that we need only consider elements \( y \) in an approximate identity for \( A^\otimes m \). Thus, we may assume that there is \( y_0 \in A \) such that

\[
(3.8) \quad y_0 \geq 0, \quad \| y_0 \| = 1, \quad \text{and} \quad y = y_0^\otimes m.
\]
We may further assume that there is a finite set \( F_0 \subseteq A \) such that \( \|a\| \leq 1 \) for all \( a \in F_0 \) and

\[
F = \{a_1 \otimes a_2 \otimes \cdots \otimes a_m : a_1, a_2, \ldots, a_m \in F_0\}.
\]

Use \([35, \text{Lemma 2.4}]\) to choose nonzero orthogonal elements \( b, c \in (x(A^{\otimes m})x)_+ \).

Use induction and Kirchberg’s Slice Lemma \([37, \text{Lemma 4.1.9}]\) to find \( t_1, t_2, \ldots, t_m \in A_+ \setminus \{0\} \) such that \( t_1 \otimes t_2 \otimes \cdots \otimes t_m \preceq c \). Use \([35, \text{Lemma 2.6}]\) to choose \( c_0 \in A_+ \setminus \{0\} \) such that \( c_0 \preceq t_k \) for \( k = 1, 2, \ldots, m \). Then \( c_0^{\otimes m} \preceq c \). Without loss of generality, \( \|c_0\| = 1 \).

Define
\[
\epsilon_0 = \min\left(\frac{1}{2^t \frac{\epsilon}{32m}}\right) \quad \text{and} \quad y_1 = (y_0 - \epsilon_0)_+.
\]

Apply Lemma 3.3 with \( y_0 \) in place of \( a \), with \( c_0 \) in place of \( b \), and with \( \epsilon_0 \) in place of \( \epsilon \), getting \( n_0 \in \mathbb{N} \).

We have
\[
\lim_{t \to \infty} \left[n_0^m - (n_0 - \frac{m}{t})^m\right] = 0.
\]

Therefore there is \( n_1 \in \mathbb{N} \) with \( n_1 > m \) such that

\[
(3.10) \quad n_0^m - \left(n_0 - \frac{m}{n_1}\right)^m < 1.
\]

Define
\[
(3.11) \quad n = n_1n_0.
\]

Define
\[
\delta = \min\left(\frac{\epsilon_0}{n + 1}, \frac{\epsilon}{16mn^m}\right).
\]

Apply Lemma 3.4 with \( \delta \) in place of \( \epsilon \), getting a number \( \rho_1 > 0 \) \( (\text{called } \delta \text{ there}) \).

Use \([4, \text{Lemma 2.5}]\) to choose \( \rho_2 > 0 \) such that whenever \( s, t \in A_+ \) satisfy
\[
\|s\| \leq 1, \quad \|t\| \leq 1, \quad \text{and} \quad \|st - ts\| < \rho_2,
\]

then \( \|s^{1/2}t - ts^{1/2}\| < \delta \). Apply Lemma 3.7 with \( b \) and \( m \) as given, with \( \frac{\epsilon}{2} \) in place of \( \epsilon \), and with \( y_0 \) in place of \( x \), getting \( b_0 \in A_+ \setminus \{0\} \) (called \( z \) there) and \( \rho_3 > 0 \) \( (\text{called } \delta \text{ there}) \). Set
\[
\rho = \min\left(\rho_1, \rho_2, \rho_3, \frac{\epsilon}{n^m}\right).
\]

[1, Definition 3.6], now provides a c.p.c. order zero map \( \varphi : M_n \to A \) such that:

(4) \( (y_0^2 - y_0\varphi(1)y_0 - \rho)_+ \preceq b_0 \).

(5) \( \|\varphi(z), a\| < \rho \) for any \( z \in M_n \) with \( \|z\| \leq 1 \) and any \( a \in F_0 \cup \{y_0\} \).
Define $N = \{1, 2, \ldots, n\}$. Define
\[
R = \{r = (r(1), r(2), \ldots, r(m)) \in N^m : r(1) < r(2) < \cdots < r(m)\}.
\]
The group $S_m$ acts on $N^m$ by $(\sigma \cdot r)(j) = r(\sigma^{-1}(j))$ for $\sigma \in S_m$, $r = (r(1), r(2), \ldots, r(m)) \in N^m$, and $j = 1, 2, \ldots, m$. Using this action, we see that $S_m \cdot R$ is the set of all $r \in N^m$ such that the numbers $r(1), r(2), \ldots, r(m)$ are all distinct. Define $Q = N^m \setminus S_m \cdot R$. We have
\[
\text{card}(S_m \cdot R) = n(n - 1)(n - 2) \cdots (n - m + 1) \geq (n - m)^m,
\]
so
\[
\text{card}(Q) \leq n^m - (n - m)^m.
\]
To simplify notation, for $k \in N$ define $g_k = \varphi(e_{k,k})$, and for $r \in N^m$ define
\[
g_r = g_{r(1)} \otimes g_{r(2)} \otimes \cdots \otimes g_{r(m)}.
\]
One checks that $\beta_\sigma(g_r) = g_{\sigma \cdot r}$ for $\sigma \in S_m$ and $r \in N^m$. Also, if $\sigma \cdot r = \sigma' \cdot r'$ for some $\sigma, \sigma' \in S_m$ and $r, r' \in R$, then $\sigma = \sigma'$ and $r = r'$.

For $\sigma \in S_m$, define
\[
f_\sigma = \sum_{r \in R} g_{\sigma \cdot r}.
\]
Since $\varphi$ is a c.p.c. order zero map, $f_\sigma$ is a positive contraction. The sets $\sigma \cdot R$ are disjoint, so the elements $f_\sigma$ are orthogonal. Moreover, for $\sigma, \tau \in S_m$,
\[
\beta_\sigma(f_r) = \sum_{r \in R} \beta_\sigma(g_{\tau \cdot r}) = \sum_{r \in R} g_{\sigma \cdot \tau \cdot r} = f_{\sigma \cdot \tau}.
\]
So (2) holds.

We check (1). Let $a_1, a_2, \ldots, a_m \in F_0$ and set $a = a_1 \otimes a_2 \otimes \cdots \otimes a_m$. Let $r \in N^m$. Then one checks that
\[
[g_r, a] = \sum_{j=1}^m a_1 g_{r(1)} \otimes \cdots \otimes a_j - 1 g_{r(j-1)} \otimes [g_{r(j)}, a_j] \otimes g_{r(j+1)} a_{j+1} \otimes \cdots \otimes g_{r(m)} a_m.
\]
Since $\|a_j\| \leq 1$ and $\|g_{r(j)}\| \leq 1$ for $j = 1, 2, \ldots, m$, we have $\|[g_r, a]\| < m\rho$ by (5). Therefore, for $\sigma \in S_m$,
\[
\|[f_\sigma, a]\| \leq \sum_{r \in \sigma \cdot R} \|[g_r, a]\| < \text{card}(R)m\rho \leq n^m \rho \leq \varepsilon.
\]
This is (1).

It remains to prove (3), which requires considerable work. First, the relations $\rho \leq \rho_3$ and $(y_0^2 - y_0 \varphi(1)y_0 - \rho) + \succcurlyeq b_0$, together with the choice of $b_0$ and $\rho_3$ using Lemma 3.7, imply that
\[
(y^2 - y \varphi(1) \otimes \otimes_m y - \frac{\varepsilon}{2}) \succcurlyeq b.
\]
Next, for future reference, at the second step use $\rho \leq \rho_2$, the choice of $\rho_2$, and (5) to get, for any $k = 1, \ldots, n$,
\[
\|[y_0g_ky_0 - g_k^{1/2} y_0 g_k^{1/2}]\| \leq 2\|[y_0, g_k^{1/2}]\| < 2\delta.
\]
Applying (5) to \(y_0\), the inequality \(\rho \leq \rho_1\) and the choice of \(\rho_1\) using Lemma 3.4 provide, for \(k = 1, 2, \ldots, n\), elements \(v_k, w_k \in A\) such that 
\[
\|v_k\|, \|w_k\| \leq 1,
\]
(3.16) 
\[
\|v_k^* y_0 g_k y_0 v_k - y_0 g_1 y_0\| < \delta, \quad \text{and} \quad \|w_k^* y_0 g_1 y_0 w_k - y_0 g_k y_0\| < \delta.
\]

We claim that 
(3.17) 
\[
n_0 n_1 \langle (y_0 g_1 y_0 - 4\varepsilon_0)_+ \rangle \leq \langle y_1 \rangle.
\]
To prove the claim, use (3.15) at the first step and \(\varphi(1) = \sum_{k=1}^n g_k\) at the second step to get 
\[
\left\| y_1 \varphi(1) y_1 - \sum_{k=1}^n g_k^{1/2} y_0 g_k^{1/2} \right\| < \left\| y_1 \varphi(1) y_1 - \sum_{k=1}^n y_0 g_k y_0 \right\| + 2n \delta
\]
\[
\leq 2\|y_1 - y_0\| + 2n \delta \leq 2\varepsilon_0 + 2n \delta.
\]
Since \(g_1, g_2, \ldots, g_n\) are orthogonal, it follows that 
(3.18) 
\[
\sum_{k=1}^n \langle (g_k^{1/2} y_0 g_k^{1/2} - [2\varepsilon_0 + 2n \delta])_+ \rangle \leq \langle y_1 \varphi(1) y_1 \rangle \leq \langle y_1 \rangle.
\]
For \(k = 1, 2, \ldots, n\), using \((2n + 1)\delta < 2\varepsilon_0\) at the first step, using [35, Lemma 1.4(6)] at the second step, using (3.16), \(\|v_k\| \leq 1\), and [1, Lemma 2.3], at the third step, we get 
\[
(y_0 g_1 y_0 - 4\varepsilon_0)_+ \lesssim (y_0 g_1 y_0 - [(2n + 1)\delta + 2\varepsilon_0])_+
\]
\[
\sim (g_1^{1/2} y_0 g_1^{1/2} - [(2n + 1)\delta + 2\varepsilon_0])_+
\]
\[
\lesssim (g_k^{1/2} y_0 g_k^{1/2} - [2n \delta + 2\varepsilon_0])_+.
\]
Summing over \(k\) and combining this with (3.18) gives 
\[
n_1 \langle (y_0 g_1 y_0 - 4\varepsilon_0)_+ \rangle \leq \langle y_1 \rangle,
\]
so the claim follows from (3.11).

We next claim that 
(3.19) 
\[
[n^m - (n - m)^m] \langle [(y_0 g_1 y_0 - 4\varepsilon_0)_+]^{\otimes m} \rangle \leq \langle c \rangle.
\]
To prove this, use (3.17) and the choice of \(n_0\) using Lemma 3.3 to get 
\[
n_1 \langle (y_0 g_1 y_0 - 4\varepsilon_0)_+ \rangle \leq \langle c_0 \rangle.
\]
So 
\[
n_1 \langle [(y_0 g_1 y_0 - 4\varepsilon_0)_+]^{\otimes m} \rangle \leq \langle c_0^{\otimes m} \rangle \leq \langle c \rangle.
\]
The claim now follows by multiplying (3.10) by \(n_1^m\) and using (3.11) again.

Now we claim that for every \(r \in N^m\), we have 
(3.20) 
\[
(g_{r(1)}^{1/2} y_0 g_{r(1)}^{1/2} \otimes g_{r(2)}^{1/2} y_0 g_{r(2)}^{1/2} \otimes \cdots \otimes g_{r(m)}^{1/2} y_0 g_{r(m)}^{1/2}) = \frac{\varepsilon}{4}
\]
\[
\lesssim [(y_0 g_1 y_0 - 4\varepsilon_0)_+]^{\otimes m}.
\]
We prove the claim. Recalling the elements \(w_k\) in (3.16), for \(r \in N^m\) define 
\[
w_r = w_{r(1)} \otimes w_{r(2)} \otimes \cdots \otimes w_{r(m)} \in A^{\otimes m}.
\]
Then, using (3.7) at the first step and (3.15) and (3.16) at the second step,
\[
\left\| w_r^* \left[ (y_0 g_1 y_0 - 4 \varepsilon_0) + \right] \right\| \leq \sum_{j=1}^{m} \left[ \left\| w_{r(j)} \right\| (y_0 g_1 y_0 - 4 \varepsilon_0) + \right\} \right. - y_0 g_1 y_0 \right\| w_{r(j)} \|
\]
\[
< m(4 \varepsilon_0 + \delta + 2 \delta) = 4m \varepsilon_0 + 3m \delta \leq \varepsilon + \varepsilon = \frac{\varepsilon}{4}.
\]

The claim follows.

We now have, using (3.7) at the second step and (3.15) at the third step,
\[
\sum_{r \in Q} g_{r(1)}^{1/2} y_0 g_{r(1)}^{1/2} \otimes g_{r(2)}^{1/2} y_0 g_{r(2)}^{1/2} \otimes \cdots \otimes g_{r(m)}^{1/2} y_0 g_{r(m)}^{1/2} - \sum_{r \in Q} y_0^{\otimes m} g_r y_0^{\otimes m}
\]
\[
\leq \sum_{r \in Q} g_{r(1)}^{1/2} y_0 g_{r(1)}^{1/2} \otimes g_{r(2)}^{1/2} y_0 g_{r(2)}^{1/2} \otimes \cdots \otimes g_{r(m)}^{1/2} y_0 g_{r(m)}^{1/2} - y_0^{\otimes m} g_r y_0^{\otimes m}
\]
\[
\leq \sum_{r \in Q} \sum_{j=1}^{m} \left\| y_0 g_{r(j)} y_0 - g_{r(j)}^{1/2} y_0 g_{r(j)}^{1/2} \right\| < 2 \text{card}(Q) m \delta \leq 2mn^m \delta \leq \frac{\varepsilon}{4}.
\]

Therefore, using [35, Corollary 1.6] at the first step and orthogonality of \( g_1, g_2, \ldots, g_n \) at the second step,
\[
\left( \sum_{r \in Q} y_0^{\otimes m} g_r y_0^{\otimes m} - \frac{\varepsilon}{2} \right) +
\]
\[
\leq \sum_{r \in Q} \sum_{j=1}^{m} \left( g_{r(1)}^{1/2} y_0 g_{r(1)}^{1/2} \otimes g_{r(2)}^{1/2} y_0 g_{r(2)}^{1/2} \otimes \cdots \otimes g_{r(m)}^{1/2} y_0 g_{r(m)}^{1/2} - \frac{\varepsilon}{4} \right)
\]
\[
= \sum_{r \in Q} \sum_{j=1}^{m} \left( g_{r(1)}^{1/2} y_0 g_{r(1)}^{1/2} \otimes g_{r(2)}^{1/2} y_0 g_{r(2)}^{1/2} \otimes \cdots \otimes g_{r(m)}^{1/2} y_0 g_{r(m)}^{1/2} - \frac{\varepsilon}{4} \right).
\]

So, using (3.20) at the first step and (3.19) and (3.13) at the second step,
\[
\left\langle \left( \sum_{r \in Q} y_0^{\otimes m} g_r y_0^{\otimes m} - \frac{\varepsilon}{2} \right) \right\rangle \leq \text{card}(Q) \left\langle \left( [y_0 g_1 y_0 - 4 \varepsilon_0]^{\otimes m} \right) \right\rangle \leq \langle c \rangle.
\]

Finally, use [35, Lemma 1.5] to combine this last inequality with (3.14) and \( y \varphi(1) y - f = \sum_{r \in Q} y_0^{\otimes m} g_r y_0^{\otimes m} \) to get
\[
(y^2 - y f y - \varepsilon) + \leq (y^2 - y \varphi(1)^{\otimes m} y - \varepsilon) + \left( \sum_{r \in Q} y_0^{\otimes m} g_r y_0^{\otimes m} - \frac{\varepsilon}{2} \right) +
\]
\[
\leq b \otimes c \leq x.
\]
This is (3), and finishes the proof. \(\square\)

The following result follows from Theorems 3.10 and 2.5, and Corollary 2.7.

**Corollary 3.11.** Let \(A\) be a simple tracially \(\mathcal{Z}\)-absorbing \(C^*\)-algebra and let \(m \in \mathbb{N}\). Suppose that \(A^\otimes m\) is finite in the sense of [1, Definition 7.1]. Then the crossed product and the fixed point algebra are simple and purely infinite. If, in addition, \(A\) is \(\sigma\)-unital, then all intermediate \(C^*\)-algebras of the inclusions \((A^\otimes m)^\beta \subseteq A^\otimes m\) and \(A^\otimes m \subseteq C^*(S_m,A^\otimes m,\beta)\) are simple and tracially \(\mathcal{Z}\)-absorbing.

The following result follows from the fact that every finite group \(G\) embeds into some permutation group \(S_m\), Theorem 3.10, and [13, Proposition 4.1].

**Corollary 3.12.** For any finite group \(G\) and any finite, simple, self-absorbing, and tracially \(\mathcal{Z}\)-absorbing \(C^*\)-algebra \(A\), there is an action \(\alpha: G \to \text{Aut}(A)\) with the weak tracial Rokhlin property.

**Proposition 3.13.** Let \(A\) be a simple purely infinite \(C^*\)-algebra and let \(m \in \mathbb{N}\). Then the permutation action \(\beta: S_m \to \text{Aut}(A^\otimes m)\) is pointwise outer and the crossed product and the fixed point algebra are simple and purely infinite. If, in addition, \(A\) is \(\sigma\)-unital, then all intermediate \(C^*\)-algebras of the inclusions \((A^\otimes m)^\beta \subseteq A^\otimes m\) and \(A^\otimes m \subseteq C^*(S_m,A^\otimes m,\beta)\) are simple and purely infinite.

**Proof.** Suppose that there is \(\sigma \in S_m \setminus \{1\}\) and a unitary \(u\) in the multiplier algebra of \(A^\otimes m\) such that \(\beta_\sigma(c) = ucu^*\) for all \(c \in A^\otimes m\). Let \(q \in A\) be any nonzero projection and put \(p = q_{A^m}\) and \(B = qAq\). Then \(p\) is a \(\beta\)-invariant projection and so \(upu^* = \beta_\sigma(p) = p\). Thus \(pup = pu = up\). Put \(v = pup\) which is a unitary in \(B^\otimes m\) and \(\beta_\sigma(c) = vcv^*\) for all \(c \in B^\otimes m\).

Now, Sakai’s result [40, Theorem 7] implies that \(B\) is isomorphic to some full matrix algebra \(M_n\). (Note that in the Sakai’s result, it seems that the assumption of the unitality is implicit as \(K(H)\) also satisfies the conclusion of that result. However, his argument can be modified for the nonunital case to cover the algebra \(K(H)\).) This is a contradiction as \(B\) is purely infinite.

By [23, Theorem 3], pointwise outerness of \(\beta\) implies that the crossed product \(C^*(S_m,A^\otimes m,\beta)\) is simple and purely infinite. Since the fixed point algebra \((A^\otimes m)^\beta\) is isomorphic to a full corner of \(C^*(S_m,A^\otimes m,\beta)\), it is simple and purely infinite. If \(A\) is \(\sigma\)-unital, the same conclusion holds for all intermediate \(C^*\)-algebras of the inclusions \((A^\otimes m)^\beta \subseteq A^\otimes m\) and \(A^\otimes m \subseteq C^*(S_m,A^\otimes m,\beta)\), since, by [21, Corollary 6.6], they are, respectively, fixed point algebras and crossed products of the restriction of the action \(\beta: S_m \to \text{Aut}(A^\otimes m)\) to subgroups of \(S_m\). \(\square\)

**Corollary 3.14.** Let \(A\) be a simple separable exact tracially \(\mathcal{Z}\)-absorbing \(C^*\)-algebra. Let \(m \in \mathbb{N}\) and consider the permutation action \(\beta: S_m \to \text{Aut}(A^\otimes m)\). Then all intermediate \(C^*\)-algebras of the inclusions \((A^\otimes m)^\beta \subseteq A^\otimes m\) and \(A^\otimes m \subseteq C^*(S_m,A^\otimes m,\beta)\) are simple and tracially \(\mathcal{Z}\)-absorbing. If,
in addition, $A$ is nuclear then all these intermediate C*-algebras are nuclear and $\mathcal{Z}$-stable.

Proof. Since $A$ is simple, separable, and tracially $\mathcal{Z}$-absorbing, either $A$ is purely infinite or $sr(A) = 1$, by [14, Theorem 4.11 and Corollary 6.5]. If $A$ is purely infinite, the first part of the statement follows from Proposition 3.13. In the sequel, suppose that $A$ has stable rank one.

Since $A$ is tracially $\mathcal{Z}$-absorbing, so is $A^\otimes m$, by [1, Theorem 5.1]. Then, again by [14, Theorem 4.11 and Corollary 6.5], either $A^\otimes m$ is purely infinite or $sr(A^\otimes m) = 1$. If $sr(A^\otimes m) = 1$, then $A^\otimes m$ is finite in the sense of [1, Definition 7.1] (that is, its unitization is finite), and hence the first part of the statement follows from Corollary 3.11.

To complete the proof of the first part of the statement, it is enough to show that if $A$ has stable rank one, then $A^\otimes m$ cannot be purely infinite. Recall that a simple C*-algebra (unital or not) admits a densely defined quasitrace if and only if no matrix algebra over it contains an infinite projection [6, 7]. Since $A$ is simple with $sr(A) = 1$, this implies that $A$ has a densely defined quasitrace $\tau$, which is actually a densely defined trace, by the exactness of $A$. Now, $\tau^\otimes m$ is a densely defined trace on $A^\otimes m$ (whose domain is the algebraic tensor product of $m$ copies of $A$). So, $A^\otimes m$ does not have any infinite projection, and hence it is not purely infinite.

For the second part of the statement, the nuclearity of the intermediate C*-algebras follows from Izumi’s Galois correspondence for intermediate C*-algebras [21, Corollary 6.6], and their $\mathcal{Z}$-stability follows from [8, Theorem A]. □

4. Integer actions

In this section we show—under a mild additional assumption—that (nonunital) tracial $\mathcal{Z}$-absorption passes to crossed products by automorphisms with the weak tracial Rokhlin property, as in Definition 4.1 below. (See Theorem 4.11 below.) This result is the nonunital version of [19, Theorem 6.7]. We weaken the assumption in [19, Theorem 6.7] that $\alpha^m$ acts trivially on $T(A)$ for some $m \in \mathbb{N}$, and we don’t need separability.

We point out that the proof of [19, Theorem 6.7] isn’t valid without exactness, or at least without assuming that every quasitrace is a trace. Strict comparison is not defined in [19], so it isn’t clear whether the intention there is to use traces or quasitraces. The version using quasitraces is needed for the part of [19, Theorem 3.3] that says “and therefore $A$ has strict comparison”, and version using traces is needed at the end of the third last paragraph of the proof of [19, Lemma 6.6].

First we extend the definition of the (weak) tracial Rokhlin property for actions of $\mathbb{Z}$ to the nonunital case (cf. [32, Definition 1.1]). The analogous definition in [19], Definition 6.1 there, asks for orthogonal positive contractions $e_1, e_2, \ldots, e_n$, but we use the more conventional indexing $e_0, e_1, \ldots, e_n$. 
Definition 4.1. Let $A$ be a simple C*-algebra and let $\alpha \in \text{Aut}(A)$. We say that $\alpha$ has the weak tracial Rokhlin property if for every finite set $F \subseteq A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every $x, y \in A_+$ with $\|x\| = 1$, there exist orthogonal positive contractions $e_0, e_1, \ldots, e_n$ in $A$ such that, with $e = \sum_{j=0}^n e_j$, the following hold:

1. $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $j = 0, 1, \ldots, n - 1$.
2. $\|[e_j, b]\| \leq \varepsilon$ for $j = 0, 1, \ldots, n$ and all $b \in F$.
3. $(y^2 - y e y - \varepsilon)_+ \geq x$.
4. $\|e x e\| > 1 - \varepsilon$.

Proposition 4.2. Let $A$ be a simple unital C*-algebra and let $\alpha \in \text{Aut}(A)$.

1. If $\alpha$ has the weak tracial Rokhlin property, then $\alpha$ has the generalized tracial Rokhlin property of [19, Definition 6.1].
2. If $A$ is finite and $\alpha$ has the generalized tracial Rokhlin property of [19, Definition 6.1], then $\alpha$ has the weak tracial Rokhlin property.

Proof. The proof is similar to that of Proposition 2.3, using $\{1, 2, \ldots, n\}$ in place of $G$.

For (1), for given $\varepsilon > 0$, $n \in \mathbb{N}$, $F \subseteq A$ finite, and $a \in A_+ \setminus \{0\}$, we will ask for orthogonal positive contractions $e_0, e_1, \ldots, e_n \in A$ satisfying the conditions of [19, Definition 6.1], rather than $e_1, e_2, \ldots, e_n$; since $n$ is arbitrary, this is equivalent.

Choose $\delta_1, \delta_2 > 0$ as in the proof of Proposition 2.3(1). Set

$$\varepsilon_0 = \min\left(\delta_1, \delta_2, \frac{1}{2(n+1)}\right).$$

Define $x$ and $y$ as in the proof of Proposition 2.3(1). Apply Definition 4.1 with $\varepsilon_0$ in place of $\varepsilon$ and with $x, y, F, n$ as given, getting $f_0, f_1, \ldots, f_n$ (called $e_0, e_1, \ldots, e_n$ in Definition 4.1), and define $f = \sum_{j=0}^n f_j$. Then

$$\|f\| > \sqrt{1 - \varepsilon_0} \geq \sqrt{1 - \frac{1}{2(n+1)}} \geq 1 - \frac{1}{2(n+1)}.$$

Therefore there is $j_0 \in \{0, 1, \ldots, n\}$ such that

$$\|f_{j_0}\| > 1 - \frac{1}{2(n+1)}.$$  

Using Definition 4.1(1) and induction, for $j = 0, 1, \ldots, n$ we get

$$\|f_j\| > 1 - \frac{1}{2(n+1)} - \frac{|j - j_0|}{2(n+1)}.$$  

In particular, $\|f_j\| > \frac{1}{2} > \frac{1}{4}$ for all $j$. From here, finish as in the proof of Proposition 2.3(1).

The proof of (2) is the same as the proof of Proposition 2.3(2). □

Proposition 4.3. Let $A$ be a nonzero simple C*-algebra and let $\alpha \in \text{Aut}(A)$ have the weak tracial Rokhlin property. Then $\alpha^n$ is outer for every $n \in \mathbb{Z} \setminus \{0\}$.
Proof. It is enough to prove the case $n > 0$. So assume $n \in \mathbb{Z}$ and $n > 0$. Let $u \in M(A)$ be unitary, and assume that $\alpha^n = \text{Ad}(u)$; we derive a contradiction.

Define

$$\delta = \frac{1}{n^2 + 3n + 9}.$$ 

Use [4, Lemma 2.5] to choose $\delta_0 > 0$ such that whenever $D$ is a C*-algebra and $h, k \in D$ satisfy $0 \leq h, k \leq 1$ and $||[h, k]|| < \delta_0$, then $||[h^{1/2}, k]|| < \delta$. We also require $\delta_0 \leq \delta$. Choose $b_0 \in A_+$ such that $\|b_0\| = 1$. Set

$$F = \{\alpha^k(b_0) : k = -n, -n + 1, \ldots, n\} \cup \{u\alpha^{-k}(b_0^{1/2}) : k = 0, 1, \ldots, n\}.$$

Apply Definition 4.1 with this choice of $F$, with $\delta_0$ in place of $\varepsilon$, with $x = b_0$, and with $y = 0$. We get orthogonal positive contractions $f_0, f_1, \ldots, f_n \in A$ such that, with $f = \sum_{j=1}^nf_j$, the following hold:

1. $\|f_j\alpha^k(b_0) - \alpha^k(b_0)f_j\| < \delta_0$ for all $j \in \{0, 1, \ldots, n\}$ and all $k \in \{-n, -n + 1, \ldots, n\}$.
2. $\|f_ju\alpha^{-k}(b_0^{1/2}) - u\alpha^{-k}(b_0^{1/2})f_j\| < \delta_0$ for all $j, k \in \{0, 1, \ldots, n\}$.
3. $\|\alpha(f_j) - f_{j+1}\| < \delta_0$ for all $j \in \{0, 1, \ldots, n - 1\}$.
4. $\|f_0f\| > 1 - \delta_0$.

It follows that (using the choice of $\delta_0$ for (6) and (7)):

5. Whenever $j, k \in \{0, 1, \ldots, n\}$ then $\|\alpha^j\alpha^{-k}(f_k) - f_j\| < n\delta_0$.
6. $\|f_j^{1/2}\alpha^k(b_0) - \alpha^k(b_0)f_j^{1/2}\| < \delta$ for all $j \in \{0, 1, \ldots, n\}$ and all $k \in \{-n, -n + 1, \ldots, n\}$.
7. $\|f_j\alpha^k(b_0)^{1/2} - \alpha^k(b_0)^{1/2}f_j\| < \delta$ for all $j \in \{0, 1, \ldots, n\}$ and all $k \in \{-n, -n + 1, \ldots, n\}$.

We claim that there exists $l \in \{0, 1, \ldots, n\}$ such that

$$\|f_lf_0f_l\| > 1 - (n^2 + n + 1)\delta.$$ 

To prove the claim, first use (1) and orthogonality of $f_0, f_1, \ldots, f_n$ to see that if $j, k \in \{0, 1, \ldots, n\}$ with $j \neq k$, then $\|f_jb_0f_k\| < \delta_0 \leq \delta$, and

$$\left\| \sum_{j=0}^n f_jb_0f_j \right\| = \max_{0 \leq j \leq n} \|f_jb_0f_j\|.$$ 

Using these two facts at the third step, we get

$$1 - \delta < \|f_0f\| \leq \left\| \sum_{j=0}^n f_jb_0f_j \right\| + \left\| \sum_{j \neq k} f_jb_0f_k \right\| < \max_{0 \leq j \leq n} \|f_jb_0f_j\| + n(n + 1)\delta.$$ 

The claim follows.

Now let $l$ be as in the claim, and define $b = \alpha^{-l}(b_0)$. Then, using the claim and (5) at the third step,

$$\|f_0b\| = \|\alpha^l(f_0)b_0\| \geq \|f_ib_0\| - \|\alpha^l(f_0) - f_i\| \cdot \|b_0\| \geq \left[1 - (n^2 + n + 1)\delta\right] - n\delta = 1 - (n^2 + 2n + 1)\delta.$$
Define \( c = f_0^{1/2} \alpha^n(b) f_0^{1/2} \) and \( d = f_n^{1/2} \alpha^n(b) f_n^{1/2} \). Using (6) at the first and last steps, using (5) at the second step, using (7) at the fourth step, and using (2) at the fifth step, we get

\[
(4.2) \quad d \approx \delta f_n \alpha^n(b) \approx n \delta \alpha^n(b) = uf_0bu^* \\
\approx \delta ub^{1/2} f_0 b^{1/2} u^* \approx \delta_0 f_0 ub^{1/2} b^{1/2} u^* = f_0 \alpha^n(b) \approx \delta c.
\]

Using (4.1) and the appearance \( uf_0bu^* \) in the middle of (4.2), we get

\[ ||c|| > 1 - (n^2 + 2n + 1)\delta - (\delta_0 + 2\delta) \geq 1 - (n^2 + 2n + 4)\delta. \]

Also from (4.2), we get

\[ ||c - d|| < (n + 3)\delta + \delta_0 \leq (n + 4)\delta. \]

Therefore, at the second step using the fact that \( c \) and \( d \) are orthogonal positive elements,

\[ (n + 4)\delta > ||c - d|| = \max(||c||, ||d||) > 1 - (n^2 + 2n + 4)\delta, \]

which implies \( (n^2 + 3n + 8)\delta > 1 \). This inequality contradicts the choice of \( \delta \).

\[ \square \]

**Corollary 4.4.** Let \( A \) be a simple \( C^* \)-algebra and let \( \alpha \in \text{Aut}(A) \) have the weak tracial Rokhlin property. Then \( C^*(\mathbb{Z}, A, \alpha) \) is simple.

**Proof.** Combine Proposition 4.3 and [28, Theorem 3.1]. \( \square \)

The condition in the following definition substitutes for the hypothesis in [19, Lemma 6.6 and Theorem 6.7] that some power of the automorphism acts trivially on the tracial state space. The difference from Definition 4.1 is the addition of the last condition.

**Definition 4.5.** Let \( A \) be a simple \( C^* \)-algebra and let \( \alpha \in \text{Aut}(A) \). We say that \( \alpha \) has the controlled weak tracial Rokhlin property if for every \( z \in A_+ \setminus \{0\} \) there exists \( n_0 \in \mathbb{N} \) such that for every \( n \in \mathbb{N} \) with \( n \geq n_0 \), every finite set \( F \subseteq A \), every \( \varepsilon > 0 \), and every \( x, y \in A_+ \) with \( ||x|| = 1 \), there exist orthogonal positive contractions \( e_0, e_1, \ldots, e_n \) in \( A \) such that, with \( e = \sum_{j=0}^n e_j \), the following hold:

1. \( ||\alpha(e_j) - e_{j+1}|| < \varepsilon \) for \( j = 0, 1, \ldots, n - 1 \).
2. \( ||(e_j, b)|| < \varepsilon \) for \( j = 0, 1, \ldots, n \) and all \( b \in F \).
3. \( (y^2 - yey - \varepsilon)_+ \lesssim x \).
4. \( ||xe|| > 1 - \varepsilon \).
5. \( e_j \lesssim z \) for \( j = 0, 1, \ldots, n \).

We do not know any example of an automorphism of a simple \( C^* \)-algebra which has the weak tracial Rokhlin property but not the controlled weak tracial Rokhlin property, although we suspect such examples exist. It seems less likely that such examples can exist on a simple tracially \( \mathcal{Z} \)-absorbing \( C^* \)-algebra.

Note that if \( \alpha \in \text{Aut}(A) \) has the controlled weak tracial Rokhlin property then it has the weak tracial Rokhlin property, since by taking a fixed \( z_0 \in \mathcal{Z} \)
implies that there is \( n_0 \in \mathbb{N} \) such that Definition 4.1 holds for any \( n \geq n_0 \). Now if we are given \( F, \varepsilon, n, x, y \) as in Definition 4.1, then we can apply Definition 4.1 with \( m = (n + 1)n_0 - 1 \) in place of \( n \) and with \( \varepsilon/n_0 \) in place of \( \varepsilon \) to get \( e_0, \ldots, e_m \). Set \( f_j = \sum_{k=0}^{n_0-1} e_{j+k(n+1)} \), for \( 0 \leq j \leq n \). Then \( f_j \)'s satisfy Definition 4.1.

**Remark 4.6.** Definition 4.5 is equivalent to Definition 4.1 if \( A \) is purely infinite, since in this case, \( a \precsim z \) for all \( a, z \in A_+ \setminus \{0\} \).

**Proposition 4.7.** Let \( A \) be a simple unital exact tracially \( \mathcal{Z} \)-absorbing \( C^* \)-algebra and let \( \alpha \in \text{Aut}(A) \) have the generalized tracial Rokhlin property of [19, Definition 6.1]. Suppose that there is \( m > 0 \) such that \( \alpha^m \) acts trivially on the tracial state space \( T(A) \). Then \( \alpha \) has the controlled weak tracial Rokhlin property.

**Proof.** If \( A \) is traceless then by [38, Corollary 5.1], \( A \) is purely infinite. Then by Remark 4.6 we are done. In the sequel, assume that \( T(A) \neq \emptyset \).

We first prove the version without the condition \( \|exe\| > 1 - \varepsilon \); see (1), (2), (3), (4), and (6) below.

Let \( z \in A_+ \setminus \{0\} \). Set \( r = \inf_{\tau \in T(A)} d_{\tau}(z) \). Choose some \( r \in (\mathbb{Z}+z)_+ \) such that \( |r| = 1 \). Then \( r \geq \inf_{\tau \in T(A)} \tau(r) > 0 \).

By [35, Lemma 2.4], for every \( k \in \mathbb{N} \) there is \( b \in A_+ \setminus \{0\} \) such that \( k\langle b \rangle \leq \langle 1 \rangle \) in \( W(A) \), and by [35, Lemma 2.4] for every \( x \in A_+ \setminus \{0\} \) there is \( c \in A_+ \setminus \{0\} \) such that \( c \precsim x \) and \( c \precsim z \). Using these two facts and [19, Lemma 6.5], we see that there is \( n_0 \in \mathbb{N} \) such that for every \( n \in \mathbb{N} \) with \( n \geq n_0 \), every finite set \( F \subseteq A \), every \( \varepsilon > 0 \), and every \( x \in A_+ \setminus \{0\} \), there exist orthogonal positive contractions \( e_0, e_1, \ldots, e_n \) in \( A \) such that, with \( e = \sum_{j=0}^n e_j \), the following hold:

1. \( \|e_j\| = 1 \) for \( j = 0, 1, \ldots, n \).
2. \( \|\alpha(e_j) - e_{j+1}\| < \varepsilon \) for \( j = 0, 1, \ldots, n - 1 \).
3. \( \|\langle e_j, b \rangle \| \leq \varepsilon \) for \( j = 0, 1, \ldots, n \) and all \( b \in F \).
4. \( 1 - \sum_{j=0}^n e_j \precsim x \).
5. \( d_{\tau}(e_j) < \rho \) for \( j = 0, 1, \ldots, n \).

Since by [19, Theorem 3.3] \( A \) has strict comparison and, by [18, Theorem 5.11], all quasi-traces on \( A \) are traces, so the last condition implies that:

6. \( e_j \precsim z \) for \( j = 0, 1, \ldots, n \).

The rest of the proof is now the same as the proof of Proposition 2.3(2). \( \square \)

Combing Proposition 2.3(1), the previous propostion, and [19, Remark 6.8], we obtain the following:

**Corollary 4.8.** Let \( A \) be a simple unital exact tracially \( \mathcal{Z} \)-absorbing \( C^* \)-algebra such that \( T(A) \) has finitely many extremal traces. If \( \alpha \in \text{Aut}(A) \) has the weak tracial Rokhlin property, then it has the controlled weak tracial Rokhlin property.
Lemma 4.9. Let $A$ be a stably finite $C^*$-algebra, let $G$ be a discrete group, and let $\alpha: G \to \text{Aut}(A)$ be an action of $G$ on $A$. Let $F \subseteq A$ and $S \subseteq G$ be finite. Then for every $\varepsilon > 0$, there is $e \in A$ such that $0 \leq e \leq 1$, $\|ea - a\| < \varepsilon$ for all $a \in F$, and $\|\alpha_g(e) - e\| < \varepsilon$ for all $g \in S$.

The group $G$ need not be amenable, and there are no conditions at all on the action. The methods are taken from the proof of [4, Theorem 4.6]. The situation there, however, is much more complicated, and it seems simplest to give a full proof here, relying on several lemmas from [4].

Proof of Lemma 4.9. Choose an integer $n \geq 2$ such that $n > \frac{4}{\varepsilon}$. Define subsets of $G$ by

$$S_0 = \{1\}, \quad S_1 = S \cup S^{-1} \cup \{1\}, \quad S_2 = S_1 S_1,$$

$$S_3 = S_1 S_2, \quad \ldots, \quad S_n = S_1 S_{n-1}.$$

Use [4, Lemma 4.4] to choose $\rho > 0$ such that whenever $e, x \in A$ satisfy

$$0 \leq e \leq 1, \quad 0 \leq x \leq 1, \quad \text{and} \quad \|ex - x\| < \rho,$$

then $\|e^{1/2}x - x\| < \frac{\varepsilon}{4}$.

For $k = 0, 1, \ldots, n$, let $D_k \subseteq A$ be the hereditary subalgebra of $A$ generated by all $\alpha_g(a)$ for $g \in S_k$ and $a \in F$. Then $\alpha_g(D_k) \subseteq D_{k+1}$ for $g \in S_1$ and $k = 0, 1, \ldots, n - 1$. Choose $c_0 \in D_0$ such that $0 \leq c_0 \leq 1$ and

$$\|c_0a - a\| < \rho$$

for $a \in F$.

By induction, choose $c_k \in D_k$ for $k = 1, 2, \ldots, n$ such that $0 \leq c_k \leq 1$,

$$\|c_k \alpha_g(c_l) - \alpha_g(c_l)\| < \rho$$

for

$$k = 1, 2, \ldots, n, \quad g \in S_1, \quad \text{and} \quad l = 0, 1, \ldots, k - 1,$$

and

$$\|c_k a - a\| < \varepsilon$$

for $k = 0, 1, \ldots, n$ and $a \in F$.

Now define $e = \frac{1}{n} \sum_{k=0}^{n-1} c_k$. Clearly $0 \leq e \leq 1$.

For $a \in F$ we use (4.5) to get

$$\|ea - a\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|c_k a - a\| < \left(\frac{1}{n}\right) n \varepsilon = \varepsilon.$$

It remains to prove that $\|\alpha_g(e) - e\| < \varepsilon$ for all $g \in S$. So let $g \in S$.

We will apply [4, Lemma 4.3]. We define $c_{-1} = 0$, and take the selfadjoint elements $a, b, d, r, x, y \in A$ to be

$$a = \frac{1}{n} \sum_{k=0}^{n-2} c_k, \quad b = \frac{1}{n} \sum_{k=0}^{n} c_k, \quad d = e, \quad r = \alpha_g(e),$$
\[ x = \frac{1}{n} \sum_{k=0}^{n-1} \alpha_g(c_k^{1/2})c_{k-1} \alpha_g(c_k^{1/2}), \quad \text{and} \quad y = \frac{1}{n} \sum_{k=0}^{n-1} c_{k+1}^{1/2} \alpha_g(c_k) c_{k+1}^{1/2}. \]

To apply [4, Lemma 4.3], in place of \( \varepsilon \) we use \( \frac{2}{n} \), and in place of \( \rho \) we use \( \frac{\varepsilon}{2} \).

We verify the hypotheses of [4, Lemma 4.3]. That \( a \leq d \leq b \) is clear. The relations \( x \leq r \) and \( y \leq b \) follow from

\[ \alpha_g(c_k^{1/2})c_{k-1} \alpha_g(c_k^{1/2}) \leq \alpha_g(c_k) \quad \text{and} \quad c_{k+1}^{1/2} \alpha_g(c_k) c_{k+1}^{1/2} \leq c_{k+1} \]

for \( k = 0, 1, \ldots, n - 1 \). We also have

\[ \|b - a\| = \left\| \frac{1}{n} \sum_{k=0}^{n} c_k - \frac{1}{n} \sum_{k=0}^{n-2} c_k \right\| \leq \frac{1}{n} (\|c_{n-1}\| + \|c_n\|) \leq \frac{2}{n}. \]

It remain to prove that \( \|a - x\| < \frac{\varepsilon}{2} \) and \( \|r - y\| < \frac{\varepsilon}{2} \).

The choice of \( \rho \) and the containment \( S^{-1} \subseteq S_1 \) imply that for \( k = 0, 1, \ldots, n - 1 \) we have

\[ \|c_k^{1/2} \alpha_g(c_{k+1}) - \alpha_g(c_k)\| < \frac{\varepsilon}{4}, \quad \|\alpha_g(c_{k+1})^{1/2} - \alpha_g(c_k)\| < \frac{\varepsilon}{4}, \quad \|\alpha_g(c_{k+1}^{1/2}) c_k - c_k\| < \frac{\varepsilon}{4}, \quad \text{and} \quad \|c_k c_{k+1}^{1/2} - c_k\| < \frac{\varepsilon}{4}. \]

These relations imply that

\[ \|a - x\| < \left( \frac{1}{n} \right)(n - 1) \cdot 2 \left( \frac{\varepsilon}{4} \right) < \frac{\varepsilon}{2} \quad \text{and} \quad \|r - y\| < \left( \frac{1}{n} \right) n \cdot 2 \left( \frac{\varepsilon}{4} \right) = \frac{\varepsilon}{2}. \]

The conclusion of [4, Lemma 4.3] now tells us that

\[ \|\alpha_g(e) - e\| \leq \frac{2}{n} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \]

as desired. \( \square \)

The next lemma is the analog of [19, Lemma 6.6]. There are enough additional wrinkles that we give the proof in full.

**Lemma 4.10.** Let \( A \) be a simple tracially Z-absorbing C*-algebra and let \( \alpha \in \text{Aut}(A) \) have the controlled weak tracial Rokhlin property. Then for every finite set \( F \subseteq A \), every \( \varepsilon > 0 \), every \( a \in A_+ \setminus \{0\} \), every \( n \in \mathbb{N} \), and every positive contraction \( x \in A \), there exists a c.p.c. order zero map \( \psi : M_n \to A \) such that:

1. \( (x^2 - x \psi(1)x - \varepsilon) + \lhd a. \)
2. \( \|\psi(z)y\| < \varepsilon \) for all \( y \in F \) and all \( z \in M_n \) with \( \|z\| \leq 1. \)
3. \( \|\alpha(\psi(z)) - \psi(z)\| < \varepsilon \) for all \( z \in M_n \) with \( \|z\| \leq 1. \)

**Proof.** Let \( F, \varepsilon, a, n \), and \( x \) be as in the statement. We may assume that \( \|b\| \leq 1 \) for all \( b \in F \) and \( \varepsilon < 1 \). Choose \( M \in \mathbb{N} \) such that \( \frac{1}{M} < \frac{\varepsilon^2}{256} \). By [35, Lemma 2.1], there is a nonzero positive element \( a_0 \in A \) with

\[ a_0 \otimes 1_{2M+2} \lhd a \]
in $M_\infty(A)$. Choose $n_0 \in \mathbb{N}$ for $z = a_0$ according to Definition 4.5. Choose $N \in \mathbb{N}$ such that $N > \max(2M,n_0)$. Choose $\delta > 0$ such that

$$\delta \leq \min \left(1, \frac{\varepsilon}{12N + 22}, \frac{\varepsilon^2}{64N^2} \right)$$

Use [27, Proposition 2.5]) to choose $\eta_0 > 0$ such that whenever $\varphi : M_n \to A$ is a c.p.c. map such that $\|\varphi(y)\varphi(z)\| < \eta_0$ for all $y, z \in (M_n)_+$ with $yz = 0$ and $\|y\|, \|z\| \leq 1$ (a c.p.c. $\eta_0$-order zero map), then there is a c.p.c. order zero map $\psi : M_n \to A$ such that $\|\varphi(z) - \psi(z)\| < \delta$ for all $z \in M_n$ with $\|z\| \leq 1$.

Use [4, Lemma 2.5] to choose $\eta_1 > 0$ such that whenever $D$ is a C*-algebra and $h, k \in D$ satisfy $0 \leq h, k \leq 1$ and $\|[h,k]\| < \eta_1$, then $\|[h^{1/2},k]\| < \delta$. Now set $\eta = \min(\eta_0, \eta_1, \delta, \eta_1, \delta, \eta_1, \delta, \eta_1, \delta, \eta_1, \delta, \eta_1, \delta, \eta_1, \delta, \eta_1, \delta, \eta_1, \delta)$.

We claim that there is $b \in A_+ \setminus \{0\}$ such that:

$$\bigoplus_{j=0}^{N} \alpha^j(b) \precsim a_0 \text{ in } M_\infty(A).$$

To prove the claim, since $A$ is not type I, we can use [35, Lemma 2.1] to find $c \in A_+ \setminus \{0\}$ such that $c \otimes 1_{N+1} \precsim a$ in $M_\infty(A)$. Then [35, Lemma 2.4] implies that there is $b \in A_+ \setminus \{0\}$ such that $b \precsim \alpha^{-j}(c)$ for $j = 0,1,\ldots,N$. Thus

$$\bigoplus_{j=0}^{N} \alpha^j(b) \precsim \bigoplus_{j=0}^{N} c = c \otimes 1_N \precsim a_0.$$  

The claim is proved.

Use Lemma 4.9 to choose $h \in A$ such that $0 \leq h \leq 1$, $\|hx - x\| < \eta$, and $\|\alpha^k(h) - h\| < \eta$ for $k = 1,2,\ldots,N$.

Since $\alpha$ has the controlled weak tracial Rokhlin property, there are orthogonal positive contractions $e_0,e_1,\ldots,e_N \in A$ such that, with $e = e_0 + e_1 + \cdots + e_N$, the following hold:

1. $\|e_jy\| < \eta$ for $j = 0,1,\ldots,N$ and all $y \in F \cup \{h\}$.
2. $\|\alpha(e_j) - e_{j+1}\| < \eta$ for $j = 0,1,\ldots,N-1$.
3. $\|h^2 - heh - \eta\| < \eta$.
4. $e_j \precsim b \precsim a_0$ for $j = 0,1,\ldots,N$.

Set

$$E = \{ \alpha^{-j}(e_k), \alpha^{-j}(e_k^{1/2}), \alpha^{-j}(y) : 0 \leq j, k \leq N \text{ and } y \in F \cup \{h\} \},$$

which is a finite subset of $A$. Since $A$ is tracially $\mathcal{Z}$-absorbing there is a c.p.c. order zero map $\varphi : M_n \to A$ such that the following hold:

1. $\|h^{1/2} - \varphi(1)h - \eta\| < \eta$
2. $\|\varphi(z)\| < \frac{\eta}{\sqrt{N}}$ for all $z \in M_n$ with $\|z\| \leq 1$ and all $y \in E$. 


For \( j \in \mathbb{Z} \) define
\[
f_j = \begin{cases} 
0 & j \leq 0 \\
(j/M)e_j & 1 \leq j \leq M - 1 \\
e_j & M \leq j \leq N - M \\
((N - j)/M)e_j & N - M + 1 \leq j \leq N - 1 \\
0 & N \leq j.
\end{cases}
\]
Using (6), we get:
\[
\|\alpha(f_j) - f_{j+1}\| < \frac{\eta}{M} \quad \text{for} \quad j = 0, 1, \ldots, M - 1 \quad \text{and} \quad j = N - M, \ldots, N - 1, \quad \text{and} \quad \|\alpha(f_j) - f_{j+1}\| < \eta \quad \text{for all other} \quad j \in \mathbb{Z}.
\]
Define a c.p.c. map \( \tilde{\varphi}: \mathbb{M}_n \to A \) by
\[
\tilde{\varphi}(z) = \sum_{j=0}^{N} f_j^{1/2} \alpha^j(\varphi(z))f_j^{1/2}
\]
for \( z \in \mathbb{M}_n \). By (10) and the definition of \( f_j \), for \( j = 0, 1, \ldots, N \) and \( z \in \mathbb{M}_n \) with \( \|z\| \leq 1 \), we have
\[
(4.7) \quad \|[f_j, \alpha^j(\varphi(z))]\| \leq \|[e_j, \alpha^j(\varphi(z))]\| = \|[\alpha^{-j}(e_j), \varphi(z)]\| < \frac{\eta}{N}
\]
and similarly
\[
(4.8) \quad \|[f_j^{1/2}, \alpha^j(\varphi(z))]\| < \frac{\eta}{N}.
\]
It follows from (4.8) that:
\[
(12) \quad \|\tilde{\varphi}(z) - \sum_{j=1}^{N} f_j \alpha^j(\varphi(z))\| < \eta \quad \text{for any} \quad z \in \mathbb{M}_n \quad \text{with} \quad \|z\| \leq 1.
\]
Together with orthogonality of the elements \( f_j \) and the fact that \( \varphi \) has order zero, (4.7) implies that for \( y, z \in (\mathbb{M}_n)_+ \) with \( \|y\|, \|z\| \leq 1 \) and \( yz = 0 \), we have \( \|\tilde{\varphi}(y)\tilde{\varphi}(z)\| < \eta \leq \eta_0 \). By the choice of \( \eta_0 \), there is a c.p.c. order zero map \( \psi: \mathbb{M}_n \to A \) such that:
\[
(13) \quad \|\tilde{\varphi}(z) - \psi(z)\| < \delta \quad \text{for any} \quad z \in \mathbb{M}_n \quad \text{with} \quad \|z\| \leq 1.
\]
We will show that \( \psi \) has properties (1), (2), and (3) in the statement.

We prove (1). Define
\[
w = \sum_{j=0}^{M-1} \left(1 - \frac{j}{M}\right) e_j^{1/2} h^2 e_j^{1/2} + \sum_{j=N-M+1}^{N} \left(1 - \frac{N-j}{M}\right) e_j^{1/2} h^2 e_j^{1/2}.
\]
Then by (8) and (4) we have
\[
(4.9) \quad w \preceq a_0 \otimes 1_{2M},
\]
and, using (5), \( \eta \leq \eta_1 \), and the choice of \( \eta_1 \) at the third step, we have

\[
(4.10) \quad \left\| heh - \sum_{j=0}^{N} f_j^{1/2} h f_j^{1/2} - w \right\| = \left\| \sum_{j=0}^{N} he_j h - \sum_{j=0}^{N} e_j^{1/2} h^2 e_j^{1/2} \right\|
\leq \sum_{j=0}^{N} 2\left\| [e_j^{1/2}, h] \right\| < 2(N + 1)\delta.
\]

Furthermore, for \( j = 1, 2, \ldots, N \) we have

\[
(4.11) \quad \left\| \alpha^j (h^2 - h\varphi(1)h) - [h^2 - h\alpha^j(\varphi(1))h] \right\| \leq 4\|\alpha^j(h) - h\| < 4\eta.
\]

Now, using (4.11) at the second step, using (4.10) at the third step, using (5), \( \eta \leq \eta_1 \), the choice of \( \eta_1 \), and the definition of \( f_j \) at the fourth step, and using (13) at the fifth step, we have

\[
\begin{align*}
(h^2 - heh - \eta)_+ + \sum_{j=0}^{N} f_j^{1/2} \alpha^j ((h^2 - h\varphi(1)h - \eta)_+) f_j^{1/2} &
\approx_{(N+2)\eta} (h^2 - heh) + \sum_{j=0}^{N} f_j^{1/2} \alpha^j (h^2 - h\varphi(1)h) f_j^{1/2} \\
&\approx_{4(N+1)\eta} (h^2 - heh) + \sum_{j=0}^{N} f_j^{1/2} h^2 f_j^{1/2} - \sum_{j=0}^{N} f_j^{1/2} h\alpha^j(\varphi(1))h f_j^{1/2} \\
&\approx_{2(N+1)\delta} h^2 - w - \sum_{j=0}^{N} f_j^{1/2} h\alpha^j(\varphi(1))h f_j^{1/2} \\
&\approx_{2(N+1)\delta} h^2 - w - \sum_{j=0}^{N} h f_j^{1/2} \alpha^j(\varphi(1)) f_j^{1/2} h \\
&\approx_{\delta} h^2 - h\psi(1)h - w.
\end{align*}
\]

Thus we get

\[
\left\| (h^2 - h\psi(1)h) - \left[ w + (h^2 - heh - \eta)_+ + \sum_{j=0}^{N} f_j^{1/2} \alpha^j ((h^2 - h\varphi(1)h - \eta)_+) f_j^{1/2} \right] \right\|
\leq (N + 2)\eta + 4(N + 1)\eta + 2(N + 1)\delta + 2(N + 1)\delta + \delta \\
\leq (6N + 11)\delta \leq \frac{\varepsilon}{2}.
\]

Then, using [1, Lemma 2.3], at the first step, using (4.9), (7), and (9) at the third step, using (4) at the fourth step, and using (4.6) at the fifth step, we
have
\[ [x(h^2 - h\psi(1)h)x - \frac{\varepsilon}{2}] + \lesssim (h^2 - h\psi(1)h - \frac{\varepsilon}{2})_+ \lesssim w + (h^2 - h\varepsilon - \eta)_+ \]
\[ + \sum_{j=0}^N f_j^{1/2} \alpha^j ((h^2 - h\psi(1)h - \eta)_+) f_j^{1/2} \]
\[ \lesssim (a_0 \otimes 1_{2M}) \oplus b \oplus \bigoplus_{j=1}^N \alpha^j (b) \]
\[ \lesssim (a_0 \otimes 1_{2M+2}) \]
\[ \lesssim a. \]

Using the choice of \( h \) and \( \eta \leq \frac{\varepsilon}{8} \), we have
\[ \|x(h^2 - h\psi(1)h)x - [x^2 - x\psi(1)x]\| < 4\eta \leq \frac{\varepsilon}{2}. \]

So [35, Corollary 1.6] implies
\[ (x^2 - x\psi(1)x - \varepsilon)_+ \lesssim [x(h^2 - h\psi(1)h)x - \frac{\varepsilon}{2}]_+ \lesssim a, \]
as desired.

To prove (2), let \( y \in F \) and let \( z \in M_n \) satisfy \( \|z\| \leq 1 \). Using (13) at the first step, using (12) and \( f_0 = 0 \) at the second step, using (10), (5), and the definition of \( f_j \) at the third step, and using \( \eta \leq \delta \) at the fourth step, we get
\[ \|\psi(z), y\| < 2\delta + \|\tilde{\psi}(z), y\| \]
\[ \leq 2\delta + 2\eta + \left\| \sum_{j=1}^N [f_j \alpha^j(\varphi(z)), y] \right\| \]
\[ < 2\delta + 2\eta + \eta + N\eta < 6N\delta < \varepsilon, \]
as desired.

Finally, we prove (3). In preparation for the estimate (4.12) below, we estimate the expressions
\[ \| (\alpha(f_j) - f_{j+1})(\alpha(f_k) - f_{k+1}) \| \]
for \( j, k = 0, 1, \ldots, N - 1 \). If \( j \neq k \), then, using (6) and orthogonality of \( e_0, e_1, \ldots, e_N \) at the fourth step,
\[ \| (\alpha(f_j) - f_{j+1})(\alpha(f_k) - f_{k+1}) \| = \| - \alpha(f_j)f_{k+1} - f_{j+1}\alpha(f_k) \| \]
\[ \leq \| \alpha(f_j)f_{k+1} \| + \| f_{j+1}\alpha(f_k) \| \]
\[ \leq \| \alpha(e_j)e_{k+1} \| + \| e_{j+1}\alpha(e_k) \| \]
\[ < 2\eta \sim 2\delta. \]

If \( j = k \), we use (11). If \( 0 \leq j \leq M - 1 \) or \( N - M \leq j \leq N - 1 \), then
\[ \| (\alpha(f_j) - f_{j+1})^2 \| < \left( \frac{1}{M} + \eta \right)^2 \leq \frac{4}{M^2}, \]
while otherwise
\[ \|(\alpha(f_j) - f_{j+1})^2\| < \eta^2 \leq \delta^2 \leq \delta. \]

Next, let \( z \in M_n \) satisfy \( \|z\| \leq 1 \). Then
\[ (4.12) \]
\[ \left\| \sum_{j=0}^{N-1} \alpha(f_j)\alpha_j^{j+1}(\varphi(z)) - \sum_{j=0}^{N-1} f_{j+1}\alpha_j^{j+1}(\varphi(z)) \right\|^2 \]
\[ = \left\| \sum_{j,k=0}^{N-1} \alpha^{j+1}(\varphi(z))\alpha(f_j) - f_{j+1})(\alpha(f_k) - f_{k+1})\alpha^{k+1}(\varphi(z)) \right\| \]
\[ \leq \sum_{j,k=0}^{N-1} \|\alpha^{j+1}(\varphi(z))\|\|\alpha(f_j) - f_{j+1})\|\|\alpha^{k+1}(\varphi(z))\| \]
\[ < 2N(N-1)\delta + 2M \left( \frac{4}{M^2} \right) + (N-2M)\delta \]
\[ \leq 2N^2\delta + \frac{8}{M} \leq \frac{\varepsilon^2}{32} + \frac{\varepsilon^2}{32} = \frac{\varepsilon^2}{16}. \]
Therefore
\[ \left\| \alpha\left( \sum_{j=0}^{N-1} f_j\alpha_j^{j}(\varphi(z)) \right) - \sum_{j=0}^{N-1} f_{j+1}\alpha_j^{j+1}(\varphi(z)) \right\| < \frac{\varepsilon}{4}. \]

Combining this estimate with two applications each of (12) and (13), and using \( f_0 = f_N = 1 \), we get
\[ \|\alpha(\psi(z)) - \psi(z)\| < 2\delta + 2\eta + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon. \]

This finishes the proof. \( \square \)

**Theorem 4.11.** Let \( A \) be a simple tracially \( \mathcal{Z} \)-absorbing \( C^* \)-algebra and let \( \alpha \in \text{Aut}(A) \) have the controlled weak tracial Rokhlin property. Then \( C^*(\mathbb{Z}, A, \alpha) \) is also a simple tracially \( \mathcal{Z} \)-absorbing \( C^* \)-algebra. If moreover \( A \) is separable and nuclear then \( C^*(\mathbb{Z}, A, \alpha) \) is \( \mathcal{Z} \)-stable.

The proof is similar to that of Theorem 2.5, but requires several more steps.

**Proof of Theorem 4.11.** Proposition 4.3 and [28, Theorem 3.1] imply that \( C^*(\mathbb{Z}, A, \alpha) \) is simple.

To verify [1, Definition 3.6], for \( C^*(\mathbb{Z}, A, \alpha) \), let \( F \subseteq C^*(\mathbb{Z}, A, \alpha) \) be a finite set, let \( x, a \in C^*(\mathbb{Z}, A, \alpha)_+ \) with \( a \neq 0 \), let \( \varepsilon > 0 \), and let \( n \in \mathbb{N} \). Without loss of generality \( \varepsilon < 1 \). The proof of [19, Lemma 5.1] also works in the nonunital case, and we can apply this generalization of it to find \( a_0 \in A_+ \setminus \{0\} \) such that \( a_0 \geq a \) in \( C^*(\mathbb{Z}, A, \alpha) \). Next, let \( (u_i)_{i \in I} \) be an approximate identity for \( A \). Then \( (u_i)_{i \in I} \) is also an approximate identity for \( C^*(\mathbb{Z}, A, \alpha) \). Therefore, by [1, Remark 3.8], we may assume that \( x \in A \).
Let $u \in M(C^*(\mathbb{Z}, A, \alpha))$ be the standard unitary associated with the crossed product. Choose $M \in \mathbb{N}$ such that for $y \in F$ there are $c_{y, -M}, c_{y, -M+1}, \ldots, c_{y, M} \in A$ satisfying

$$\left\| y - \sum_{m=-M}^{M} c_{y, m} u^m \right\| \leq \frac{\varepsilon}{4}.$$ 

Define

$$E = \{ c_{y, m} : y \in F \text{ and } -M \leq m \leq M \} \quad \text{and} \quad R = \sup_{c \in E} \| c \|.$$

Set

$$\varepsilon_0 = \frac{\varepsilon}{2(RM + 1)(2M + 1)}.$$

Apply Lemma 4.10 with $E$ in place of $F$, with $a_0$ in place of $a$, with $x$ as given, and with $\varepsilon_0$ in place of $\varepsilon$. We obtain a c.p.c. order zero map $\psi_0 : M_n \to A$. Let $\psi : A \to C^*(\mathbb{Z}, A, \alpha)$ be its composition with the inclusion of $A$ in $C^*(\mathbb{Z}, A, \alpha)$. We claim that $\psi$ satisfies the conditions of [1, Definition 3.6], for $C^*(\mathbb{Z}, A, \alpha)$. Condition (1) is clear. For condition (2), let $z \in M_n$ satisfy $\| z \| \leq 1$ and let $y \in F$. Then for all $m \in \mathbb{Z}$ we have $\| \alpha^m(\psi(z)) - \psi(z) \| \leq |m| \varepsilon_0$. Therefore

$$\|[\psi(z), y]\| \leq 2 \left\| y - \sum_{m=-M}^{M} c_{y, m} u^m \right\| + \sum_{m=-M}^{M} \|[\psi(z), c_{y, m} u^m]\|$$

$$\leq \frac{\varepsilon}{2} + \sum_{m=-M}^{M} \|[\psi(z), c_{y, m}]\| \cdot \| u^m \| + \sum_{m=-M}^{M} \| c_{y, m} \| \cdot \|[\psi(z), u^m]\|$$

$$\leq \frac{\varepsilon}{2} + (2M + 1) \varepsilon_0 + R \sum_{m=-M}^{M} |m| \varepsilon_0$$

$$\leq \frac{\varepsilon}{2} + (2M + 1) \varepsilon_0 + R(2M + 1) M \varepsilon_0 = \varepsilon.$$ 

This completes the proof first part of the statement. The second part follows from the first part and [8, Theorem A].

**Remark 4.12.** As discussed in the introduction to this section, exactness, or at least the assumption that every quasitrace is a trace, is needed for the proof of [19, Theorem 6.7] to be valid. With this correction, [19, Theorem 6.7], at least for the stably finite case, is a corollary of Theorem 4.11, by Proposition 4.7. Moreover, we don’t need the separability assumption in [19, Theorem 6.7].

In fact, one can say a little more.

**Remark 4.13.** Let $A$ be a simple unital $C^*$-algebra and let $\alpha \in \text{Aut}(A)$. Define the *controlled generalized tracial Rokhlin property* by modifying [19, Definition 6.1] in the same way that Definition 4.5 modifies Definition 4.1. If
A is tracially $Z$-absorbing and $\alpha$ has this property, then $C^*(Z, A, \alpha)$ is also a simple tracially $Z$-absorbing C*-algebra. To prove this, let $\alpha \in \Aut(A)$ have the controlled generalized tracial Rokhlin property. By [19, Proposition 6.3], $\alpha^m$ is outer for all $m \in \mathbb{Z} \setminus \{0\}$, and hence $C^*(Z, A, \alpha)$ is simple by [28, Theorem 3.1]. Note that $\alpha$ satisfies Conditions (1), (2), and (3) in Definition 4.1 (but not necessarily Condition (4) there). However, the proof of Lemma 4.10 works for $\alpha$ (with $x = 1$), since Condition (4) of Definition 4.1 is not used in the proof of that lemma. Using [1, Lemma 3.3], the proof of Theorem 4.11 again works in this setting.

The proof of the following proposition is similar to that of [13, Theorem 4.5] except that we need Lemma 4.9 instead of using the finiteness of the group.

**Proposition 4.14.** Let $A$ and $B$ be simple C*-algebras and let $\alpha \in \Aut(A)$ and $\beta \in \Aut(B)$. If $\alpha$ has the (controlled) weak tracial Rokhlin property, then so does the automorphism $\alpha \otimes \beta$ of $A \otimes_{\min} B$.

**Proof.** First suppose that $\alpha$ has the weak tracial Rokhlin property. We proceed as in the proof of [13, Theorem 4.5] except that we use Lemma 4.9 to choose a suitable positive element $s \in B_+$ in item (12) of the proof of [13, Theorem 4.5]. We put $f_j = r_j \otimes s$ for $0 \leq j \leq n$. The rest of that proof works and implies that $f_0, f_1, \ldots, f_n$ satisfy Definition 4.1. Thus $\alpha \otimes \beta$ has the weak tracial Rokhlin property.

Now, suppose that $\alpha$ has the controlled weak tracial Rokhlin property. To verify Definition 4.5 for $\alpha \otimes \beta$, let $z_0 \in A \otimes_{\min} B$ be positive and nonzero. By Kirchberg’s Slice Lemma ([37, Lemma 4.1.9]), there are nonzero positive elements $z_1 \in A$ and $z_2 \in B$ such that $z_1 \otimes z_2 \lesssim z_0$. We proceed as the previous paragraph to obtain $s \in B_+$ satisfying item (12) of the proof of [13, Theorem 4.5]. Since $\lim_{\xi \to 0} (s - \xi)_+ = s$, there is $\xi > 0$ such that item (12) of the proof of [13, Theorem 4.5] holds for $(s - \xi)_+$ in place of $s$. By [35, Proposition 1.14], there is $m \in \mathbb{N}$ such that $(s - \xi)_+ \lesssim 1_m \otimes z_2$. By Proposition 4.3, $A$ is non-Type I and hence [35, Lemma 2.4] provides a nonzero positive element $z_3 \in A$ such that $z_3 \otimes 1_m \lesssim z_1$. Choose $n_0$ as in Definition 4.5 for the action $\alpha$ with $z_3$ in place of $z$, getting $r_0, r_1, \ldots, r_n$ for $n \geq n_0$. We only need to deal with Part (5) of Definition 4.5. Put $f_j = r_j \otimes (s - \xi)_+$ for $0 \leq j \leq n$. Then

$$f_j \lesssim z_3 \otimes (s - \xi)_+ \lesssim z_3 \otimes (1_m \otimes z_2) \lesssim z_1 \otimes z_2 \lesssim z_0.$$ 

Therefore, $\alpha \otimes \beta$ has the controlled weak tracial Rokhlin property. \qed

**Example 4.15.** Let $A$ be a simple $Z$-stable C*-algebra and let $\beta$ be an arbitrary automorphism of $A$. Let $\gamma$ be the bilateral tensor shift on $Z^{\otimes \infty} \cong Z$ (see [42]). Consider the automorphism $\alpha = \beta \otimes \gamma$ on $A \otimes Z$. Then $\alpha$ has the controlled weak Rokhlin property. To see this, first, by [19, Example 6.9] and Proposition 4.2(2), $\gamma$ has the weak tracial Rokhlin property. Since $Z$ has a unique tracial state, Corollary 4.8 implies that $\gamma$ has
the controlled weak tracial Rokhlin property. Now, Proposition 4.14 implies that $\alpha$ has the controlled weak tracial Rokhlin property. Hence the crossed product $C^*(\mathbb{Z}, A, \alpha)$ is simple and tracially $\mathbb{Z}$-absorbing (by Theorem 4.11). If moreover, $A$ is separable and nuclear then the crossed product is nuclear and $\mathbb{Z}$-stable (by [8, Theorem A]).

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