Quantum state tomography of the microwave field

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Abstract. We propose a new method for reconstructing the density matrix of a single-mode field of photons in the cavity using the discrete photodetection, which allows studying the quantum field in the microwave range. For reconstruction it is necessary to know the phase quadrature moments of all orders. A method of obtaining the first-order moment is discussed here.

1. Introduction
Recently the interest in quantum computation and quantum information has been increased, therefore much attention is given to quantum tomography, which allows reconstructing the density matrix of quantum field. In quantum mechanics it is impossible to develop a procedure that would allow to fully recover system state by several measurements, due to significant restrictions from the Heisenberg uncertainty principle and no-cloning theorem (except in the case when the quantum state coincides with the eigenstate of the measured value). Any measurement of a quantum system changes it, and the no-cloning theorem forbids the creation of an exact copy of the system without a priori knowledge of its condition. For recovering state of field the quantum tomography is used. The theory of quantum tomography of photon states in optical range is well developed in [1], [2], [3]. This theory is based on optical homodyne method. In the microwave range there are no photon counters. Therefore, the state of the microwave field in the cavity is measured by classic radio-frequency methods [4]. In [5] a quantum endoscopy method, which allows to obtain the wave function of a pure quantum state of the microwave field in the resonator is suggested.

In this paper we proposed a new method of quantum tomography of the photon field in the cavity using the discrete photodetection [6], an analogue of optical homodyne tomography [1], which in contrast to previous methods allows to recover the quantum field in the microwave range.

2. Procedure of tomography
Quantum tomography reconstructs the unknown density matrix \( \rho_f \) of field in the cavity [1]. For this purpose it is necessary to determine the k-order moments set \( \{ M_k(\phi) \} \) of phase quadrature operator \( K(\phi) \) of field

\[
M_k(\phi) = Tr_f \left[ (K(\phi))^k \rho_f \right]
\]  

Here \( \phi \) — operator's phase, \( Tr_f \) — a trace on the photon states,
where $a^*$, $a$ — the photon creation and annihilation operators. Characteristic function for phase quadrature operator [7], [8], [9] has the following form

$$
\chi(\eta, \phi) = \text{Tr}_f \left[ e^{i \eta K(\phi)} \rho_f \right] = \sum_k \frac{(i \eta)^k}{k!} \text{Tr}_f \left[ (K(\phi))^k \rho_f \right] = \sum_k \frac{(i \eta)^k}{k!} M_k(\phi).
$$

The probability $w(x, \phi)$ of finding the quadrature in eigenstate $|x\rangle$ at the measure is

$$
w(x, \phi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(\eta, \phi) e^{-i \eta x} d\eta = \text{Tr}_f \left[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i \eta x} e^{-i \eta K(\phi)} \rho_f, d\eta \right] = \langle x | \rho_f | x \rangle.
$$

This procedure reconstructs the diagonal elements of the density matrix $\rho_f$ in eigenbasis of the phase quadrature. Wigner function

$$
W(u, v, s) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} w(x, \phi) \exp\left( i \eta^2 / 8 + i \eta (x - u \cos \phi - v \sin \phi) \right) |\eta| dx d\eta d\phi,
$$

is related to the density matrix $\rho_f$, which can be restored by using the inverse Radon transform [7].

3. Process of detection

During the process of discrete detection [6], investigated cavity probed by single atoms-probes that passed through the cavity one at a time at a given speed. The probe has two energy states: the ground (unexcited) state $|0\rangle$ and the excited state $|1\rangle$. Interaction time of investigated field and the probe is stabilized using a velocity selector. It will vary according to initial velocity of the atom. After its escape from the cavity, the probe contains information about the quantum field mode inside the cavity. Further, laser pulse has an effect on atom within time $\mu$. Interaction time of atom and the laser is chosen so that the atom would come in a superposition of states [2] with a given phase $\phi$

$$
(\langle 0 |, |1 \rangle) \rightarrow \left\{ \frac{|0\rangle + i e^{-i\phi} |1\rangle}{\sqrt{2}}, \frac{|0\rangle + i e^{i\phi} |1\rangle}{\sqrt{2}} \right\}.
$$

Exposure to the laser pulse is equivalent to the influences of beam splitter in optical tomography with penetration/reflection probability is 50/50 percent [2]. After all interaction, the atom gets to the detectors (ionization chamber) which record its energy state. We will conduct a lot of these experiments on the detection of an atom by the known initial state, whereupon it is possible to determine the escape frequency of atom in the ground and excited states. This information will be used for calculation of quantum field quadrature in the cavity. The interaction between atom and facility for discrete photodetection is shown in figure 1.

Cavity for microwave fields has a high finesse. Therefore in this work we will not take into account the field relaxation in the cavity, we assume the efficiency of the detector being unitary. Atom is prepared in the excited state $|1\rangle$ beforehand, the transition frequency between the energy states of the atom is $\omega_a$. The eigenfrequency of the field in the cavity is $\omega = \omega_a$. The atom-cavity interaction time is $t$. Then the Hamiltonian of an atom-field system has the form

$$
H_{a\Phi} = \omega_a |1\rangle \langle 1| + \omega a^* a + \gamma (a^* |0\rangle \langle 1| + a |1\rangle \langle 0|),
$$
where $\gamma$ – the constant of atom-field interaction.

Figure 1. Facility for discrete photodetection
(a) – velocity selector, (b) – atom in excited state, (c) – cavity, (d) – laser beam, (e) – atom after the interaction with the cavity, (f) – selective detector on energy state.

After atom passed through the cavity, the classical laser pulse with a frequency $\omega_l (\omega = \omega_l)$ over time $\mu$ affects it. The interaction Hamiltonian of the atom and the laser is as follows

$$H_{al} = \omega_0 |1\rangle\langle 1| + \delta (e^{i\phi} |0\rangle\langle 1| + |0\rangle\langle e^{-i\phi} |) \cos(\omega, \mu)$$

$\delta$ – the interaction constant between the atom-probe with a laser, $\phi$ – the laser impulse phase. The laser impulse converts the atomic basis in superposition (3). After two interactions, full evolution operator of the atom-field density matrix has the form (we consider the non-dimensional time $\tau = l \gamma$)

$$R(\tau, \phi) = \frac{1}{\sqrt{2}} \begin{bmatrix}
\cos(\sqrt{\alpha a}) - e^{i\phi} \frac{\sin(\sqrt{\alpha a})}{\sqrt{\alpha a}} a & -ia^- \frac{\sin(\sqrt{\alpha a})}{\sqrt{\alpha a}} -ie^{i\phi} \cos(\sqrt{\alpha a}) \\
-i e^{-i\phi} \cos(\sqrt{\alpha a}) - i\frac{\sin(\sqrt{\alpha a})}{\sqrt{\alpha a}} a & -ie^{-i\phi} a^+ \frac{\sin(\sqrt{\alpha a})}{\sqrt{\alpha a}} + \cos(\sqrt{\alpha a})
\end{bmatrix},$$

The equation for the probability of the atoms detection in the state $|\beta\rangle$, $\beta = 0,1$, when the atom was conditionally prepared in the state $|\alpha\rangle$, $\alpha = 1$, are

$$P_{j0}(\tau, \phi) = Tr_j \left[ |\beta\rangle R(\tau, \phi)|\alpha\rangle \rho_j \langle \alpha | R^*(\tau, \phi)|\beta\rangle \right]$$

When

$$\tau \sqrt{\alpha a^+} \ll 1$$

the probability of the atoms detection (4) transforms to the form

$$P_{01}(\tau, \phi) \approx Tr_j \left[ \left( \frac{1 + ae^{i\phi} + a^+ e^{-i\phi}}{2} \cdot \tau \right) \rho_j \right], P_{10}(\tau, \phi) \approx Tr_j \left[ \left( \frac{1 - ae^{i\phi} + a^+ e^{-i\phi}}{2} \cdot \tau \right) \rho_j \right]$$

From (6) by the fixed phase $\phi$

$$\frac{\Delta DP(\tau, \phi)}{\sqrt{2} \Delta \tau} = tg \theta \approx Tr_j \left[ K(\phi) \rho_j \right] = M(\phi)$$

where $\theta$ – the slope angle of dependence $DP(\tau, \phi) = P_{01}(\tau, \phi) - P_{10}(\tau, \phi)$ from $\tau$ the interaction time of the atom with the cavity. We obtained connection of the first moment of quadrature with the
detection probability an atom in the ground and excited states by the fixed phase $\phi$. For the full procedure of tomography we must know the k-th order moments set.

4. Simulation of experiment
Simulation of experiment was conducted to verify the connection by the equation (7) of the first moment of the phase quadrature with the probability of atom detection in each state. According to the formula (1) by the condition (5) and the known $\rho_f$ density matrix of the field in the cavity, we evaluated the set $M_{\text{teor}}(\phi)$ of theoretical quadrature moments for the different phases $\phi$. The density matrix is generated by a Poisson distribution

$$\rho_f = e^{\xi^2} \sum_{i,j} \frac{\xi^i \xi^*^j}{\sqrt{i!j!}} |i\rangle \langle j|$$

The density matrix is the infinite size, therefore, the coherent state amplitude $\xi$ is specially selected, so that the elements of the density matrix $\rho_f$ is quickly decreases with increasing the number of photons $i$, $j$. The escape probability of atom $P_{0i}$ in the ground and $P_{1j}$ excited states is determined by the formula (4).

The first experimental moment $M_{\text{ex}}(\phi)$ can be found by the equation (7). For this we calculate $P0(\tau,\phi)$ the escape frequency of atom in the ground state and $P1(\tau,\phi)$ – in excited state by the fixed phase $\phi$. We assume that these frequencies are probabilities $P0(\tau,\phi) \approx P_{0i}(\tau,\phi)$, $P1(\tau,\phi) \approx P_{1j}(\tau,\phi)$.

![Figure 2](image-url)

**Figure 2.** Dependence of $DP(\tau)$ from $\tau$ by the fixed phase $\phi = 0.064$ and number of experiments $N = 500$ и $N = 30000$

We can derive the set of the theoretical and experimental moments of the phase quadrature of field in the cavity by means of varying the phase $\phi$ from 0 to $\pi$. On figure 3 it exhibit that the experimental moment has been calculated by the assumption from this paper near to the theoretical moment.
Figure 3. Dependence of theoretical $M_{\text{teor}}(\phi)$ and experimental first quadrature moment $M_{\text{ex}}(\phi)$ on phase $\phi$. For receipt of every quadrature moment at fixed phase was considered 10 time points (time interaction between the atom and the cavity). The number of phase points is 50. The number of experiments to determine the frequency of atom detection in each state is 500 and 30000.

5. Conclusion
In this paper we research the technique that allows us to study the microwave field by the method which is equivalent to the optical homodyne tomography. The relation between the first phase quadrature moments of the quantum field with the frequency of detection atom in the ground and excited state was received. The check that the proposed method can recover the first moment of the phase quadrature of the field was performed. For a complete tomography procedure it is necessary to know all the moments of the quadrature. They can be defined similarly.

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