Holographic $SO(2, d)$ anomaly

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Abstract

In the $SO(2, d)$ gauge theory formalism of AdS gravity established in arXiv:1811.05286, the dynamics of bulk gravity is emergent from the vanishing of the boundary covariant anomaly for the $SO(2, d)$ conservation law. Parallel with the known results of chiral anomalies, we establish the descendent structure of the holographic $SO(2, d)$ anomaly. The corresponding anomaly characteristic class, bulk Chern-Simons like action as well as the boundary effective action are constructed systematically. The anomalous conservation law is presented both in terms of the covariant and consistent formalisms. Due to the existence of the ruler field, not only the Bardeen-Zumino polynomial, but also the covariant and consistent currents are explicitly constructed.
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1 Introduction

To understand the holographic emergence of bulk dynamics in AdS/CFT \cite{1,2,3}, various approaches have been suggested \cite{4,5,6,7,8,9}. In \cite{9}, after including the conformal transformation of the renormalization scale, it was shown that the bulk dynamics of a scalar field is highly constrained by the $SO(2,d)$ conformal symmetry of the dual CFT scalar operators. Furthermore, the holographic emergence of the dynamics for the bulk gravity itself is governed by the generic duality relation between the boundary global symmetry and the bulk gauge symmetry. Applying it to the $SO(2,d)$ conformal symmetry in CFT$_d$, the bulk AdS$_{d+1}$ gravity is reformulated as a $SO(2,d)$ gauge theory in \cite{10}. In this formalism, the pullback of the bulk Einstein equation on a codimension one hypersurface $\Sigma$ can be naturally related to the covariant anomaly of the CFT $SO(2,d)$ conservation law. Providing the $SO(2,d)$ conservation law is not anomalous for any local renormalization scale, the full bulk Einstein equation will be automatically implied.

In this paper, we study the formal mathematical structure of the holographic $SO(2,d)$ anomaly systematically. In Section 2.1, we briefly review the bulk $SO(2,d)$ covariant action \cite{10} for AdS gravity. The covariant formula of anomalous boundary conservation law is revealed during the Noether procedure in Section 2.2. In Section 3.1.1, the bulk Chern-Simons like action is constructed by the homotopic integration. To resolve the issue for large gauge transformation, a topological term is imposed in Section 3.1.2. In Section 3.2, the boundary effective action is also constructed via the homotopic integration method. Then in Section 3.3, the descendent structure of the holographic $SO(2,d)$ anomaly is established after obtaining the $d+2$ form anomaly characteristic class. In Section 4, we summarize our results and discuss possible generalizations.

2 AdS gravity as $SO(2,d)$ gauge theory

2.1 Bulk covariant action

By unifying the vielbein $e^a$ and the $SO(1,d)$ spin connection $\omega^{a\hat{b}}$ to the $SO(2,d)$ gauge field $A^{\hat{a}\hat{b}}$, the Palatini’s 1st order formalism for the AdS$_{d+1}$ gravity was reformulated as a $SO(2,d)$ gauge theory in \cite{10}. The action is

$$S_{(d+1)}^{\text{cov}} = \frac{1}{2\kappa^2} \int_M \left( \frac{2}{(d+1)!} \epsilon_{\hat{a}\hat{b}\hat{a}_1...\hat{a}_d} \left[ F^{\hat{a}\hat{b}} - \frac{2}{(d+1)!} \ell^2 Y^{\hat{a}} \right] \wedge \left( D Y^{\hat{a}_1} \wedge ... \wedge D Y^{\hat{a}_{d-1}} Y^{\hat{a}_d} \right) \right).$$

(1)

where $D = d + A$ is the $SO(2,d)$ gauge covariant derivative. We shall refer \cite{10} as the bulk covariant action since it is given by the sumamation of manifestly $SO(2,d)$ covariant terms. To figure out the $SO(2,d) \rightarrow SO(1,d)$ reduction manifestly, we have introduced the ruler field $Y^{\hat{a}}$ which satisfies the constraint

$$\eta_{\hat{a}\hat{b}} Y^{\hat{a}} Y^{\hat{b}} = -\ell^2.$$  

(2)
Obviously, $Y^\alpha$ is just an auxiliary field since it can be totally fixed via the $SO(2,d)$ gauge transformation. The physical degrees of freedom remains same as the original Palatini action.

In this formalism, the spacetime metric is given by the gauge invariant quadratic form

$$g_{MN} = \eta_{\alpha\beta} D_M Y^{\alpha} D_N Y^{\beta}.$$  \hfill (3)

When the metric is not degenerate, $\{Y^\alpha, D_M Y^{\alpha}\}$ forms an intrinsic basis of the $SO(2,d)$ vector space. The spacetime geometric quantities can be recovered by expanding the $SO(2,d)$ covariant quantities on the intrinsic basis. For example, the $SO(2,d)$ field strength is expanded as

$$(F_{M_1 M_2})^{\alpha\beta} = \left( R^{N_1 N_2}_{M_1 M_2} + 2\epsilon^{-2}\delta_{[M_1}^{N_1} \delta_{M_2]}^{N_2} \right) D_{N_1} Y^{[\alpha} D_{N_2} Y^{\beta]} - 4\epsilon^{-2}\eta_{[M_1 M_2]} Y^{[\alpha} D_{N} Y^{\beta]},$$ \hfill (4)

where $\epsilon^N_{M_1 M_2}$ is the torsion and $\epsilon^{N_1 N_2}_{M_1 M_2}$ is the Riemann curvature for the torsional connection. We also notice that

$$\frac{1}{\ell!p!d(1-p)!} \epsilon_{\alpha_0 \cdots \alpha_{d+1}} \epsilon_{M_0 \cdots M_d} Y^{\alpha_0} \cdots D_{M_d} Y^{\alpha_d} Y^{\alpha_{d+1}} = D^{[M_0} Y^{[\alpha_0} \cdots D^{M_{p-1}} Y_{\alpha_{p-1}]},$$ \hfill (5)

where the spacetime indices are rising by $g^{MN}$ and

$$\epsilon^{01 \cdots d} = \frac{1}{\sqrt{g}}, \quad g = -\det(g_{MN}).$$ \hfill (6)

Combing (4) and (5), it is easy to show that the action (1) equals to the original Einstein-Hilbert action.

The variation of bulk covariant action on a manifold $M$ with boundary $\Sigma$ is given by

$$\delta S_{(d+1)}^{\text{cov}} = \int_M \left[ \delta A^{\alpha\beta} \wedge (J_{(d+1)})_{\alpha\beta} + \delta Y^{\alpha} (J_{(d+1)})_{\alpha} \right] + \int_{\Sigma} \left[ \delta A^{\alpha\beta} \wedge (K_{(d+1)})_{\alpha\beta} + \delta Y^{\alpha} (K_{(d+1)})_{\alpha} \right],$$ \hfill (7)

where $J_{(d+1)}$ are the bulk off-shell currents

$$(J_{(d+1)})_{\alpha\beta} = \frac{\partial L_{(d+1)}^{\text{cov}}}{\partial A^{\alpha\beta}} + d \left( \frac{\partial L_{(d+1)}^{\text{cov}}}{\partial (dA^{\alpha\beta})} \right) = \frac{(-1)^{d+1} \epsilon}{4\kappa^2 (d-2)!} \epsilon_{\alpha\beta\alpha_1 \cdots \alpha_d} F^{\alpha_1 \alpha_2} \wedge DY^{\alpha_3} \wedge \cdots \wedge DY^{\alpha_d},$$ \hfill (8)

$$(J_{(d+1)})_{\alpha} = \frac{\partial L_{(d+1)}^{\text{cov}}}{\partial Y^{\alpha}} + d \left( \frac{\partial L_{(d+1)}^{\text{cov}}}{\partial (dY^{\alpha})} \right) = \frac{(-1)^{d+1} \epsilon}{2\kappa^2 \ell (d-2)!} \epsilon_{\alpha\beta\alpha_1 \cdots \alpha_d} Y^\beta D(F^{\alpha_1 \alpha_2} \wedge DY^{\alpha_3} \wedge \cdots \wedge DY^{\alpha_d}),$$ \hfill (9)

and $K_{(d+1)}^{\text{cov}}$ are the bulk off-shell Noether potentials [11]

$$(K_{(d+1)}^{\text{cov}})_{\alpha\beta} = \frac{\partial L_{(d+1)}^{\text{cov}}}{\partial (dA^{\alpha\beta})} = \frac{(-1)^{d+1}}{2\kappa^2 \ell (d-2)!} \epsilon_{\alpha\beta\alpha_1 \cdots \alpha_d} Y^{\alpha_1 \alpha_2} \wedge DY^{\alpha_3} \wedge \cdots \wedge DY^{\alpha_d},$$ \hfill (10)

$$(K_{(d+1)}^{\text{cov}})_{\alpha} = \frac{\partial L_{(d+1)}^{\text{cov}}}{\partial (dY^{\alpha})}$$

$$= \frac{(-1)^d}{2\kappa^2 \ell (d-2)!} \epsilon_{\alpha\beta\alpha_1 \cdots \alpha_d} Y^\beta \left[ F^{\alpha_1 \alpha_2} \wedge DY^{\alpha_3} \wedge \cdots \wedge DY^{\alpha_d} - \frac{2}{\ell^2 (d-1)} DY^{\alpha_1} \wedge \cdots \wedge DY^{\alpha_d} \right].$$ \hfill (11)

As analysed in [10], the bulk equation of motions for $\delta A$

$$(J_{(d+1)})_{\alpha\beta} = \frac{(-1)^{d+1} \epsilon}{4\kappa^2 (d-2)!} \epsilon_{\alpha\beta\alpha_1 \cdots \alpha_d} F^{\alpha_1 \alpha_2} \wedge DY^{\alpha_3} \wedge \cdots \wedge DY^{\alpha_d} = 0$$ \hfill (12)
is equivalent to the Einstein equation plus the torsion free condition. Providing \[12\], the bulk equation of motions for \(\delta Y\)

\[
(\mathcal{J}_{(d+1)})_{\hat{a}} = \frac{(-1)^{d+1}}{2\kappa^2 f(d-2)!} \mathcal{C}^{\hat{a}\hat{b}\hat{a}_1\ldots\hat{a}_d} Y^{\hat{b}} D(F^{\hat{a}_1\hat{a}_2} \wedge DY^{\hat{a}_3} \wedge \cdots \wedge DY^{\hat{a}_d}) = 0 \tag{13}
\]

is automatically satisfied. Therefore, the introducing of \(Y^{\hat{a}}\) does not impose any additional constraints other than the original Palatini equations.

\subsection*{2.2 The off-shell conservation laws}

Substituting the infinitesimal \(SO(2, d)\) gauge transformation

\[
\delta_u Y^{\hat{a}} = u^{\hat{a}}_{\hat{\beta}} Y^{\hat{\beta}}, \quad \delta_u A^{\hat{a}\hat{b}} = -D u^{\hat{a}\hat{\beta}}, \tag{14}
\]

into the general variation \[7\], the gauge invariance of the action \(\delta_u S_{(d+1)}^{\text{cov}} = 0\) implies the off-shell bulk conservation law

\[
D(\mathcal{J}_{(d+1)})_{\hat{a}\hat{\beta}} + Y^{\hat{\beta}} (\mathcal{J}_{(d+1)})(\hat{a}) = 0. \tag{15}
\]

For the manifold with boundary \(\Sigma\)

\[
\mathcal{Y}^{\hat{a}} = f^*_\Sigma [Y^{\hat{a}}], \quad \mathcal{A}^{\hat{a}\hat{\beta}} = f^*_\Sigma [A^{\hat{a}\hat{\beta}}], \tag{16}
\]

the gauge invariance also implies the boundary off-shell anomalous conservation law

\[
D(\mathcal{J}_{(d)})_{\hat{a}\hat{\beta}} + \mathcal{Y}^{\hat{\beta}} (\mathcal{J}_{(d)})^{\hat{a}} = (\mathcal{A}_{(d)}^{\text{cov}})_{\hat{a}\hat{\beta}}, \tag{17}
\]

where the boundary currents and anomaly are given by pullback of the bulk Noether potential and current on \(\Sigma\) respectively

\[
(\mathcal{J}_{(d)}^{\text{cov}})_{\hat{a}\hat{\beta}} = -f^*_\Sigma [(\mathcal{A}_{(d+1)}^{\text{cov}})_{\hat{a}\hat{\beta}}], \quad (\mathcal{J}_{(d)}^{\text{cov}})^{\hat{a}} = -f^*_\Sigma [((\mathcal{C}_{(d+1)}^{\text{cov}}))_{\hat{a}}], \quad (\mathcal{A}_{(d)}^{\text{cov}})^{\hat{a}\hat{\beta}} = -f^*_\Sigma [((\mathcal{J}_{(d+1)})_{\hat{a}\hat{\beta}}]. \tag{18}
\]

The same results were obtained in the Hamiltonian analysis in \[10\]. The boundary currents and anomaly are manifestly covariant under the local \(SO(2, d)\) transformation. From the dual CFT point of view, \[18\] give rise to the covariant formalism of the CFT currents

\[
(\mathcal{J}_{(d)}^{\text{cov}})_{\hat{a}\hat{\beta}} = \left(\frac{\delta S_{\text{CFT}}}{\delta \mathcal{A}^{\hat{a}\hat{\beta}}} \right), \quad (\mathcal{J}_{(d)}^{\text{cov}})^{\hat{a}} = \left(\frac{\delta S_{\text{CFT}}}{\delta \mathcal{Y}^{\hat{a}}} \right), \tag{19}
\]

as well as the corresponding covariant \(SO(2, d)\) anomaly. When the pullback of the bulk Einstein equations on the hypersurface \(z = \zeta(x)\) is satisfied, the covariant anomaly vanishes for the CFT with the local renormalization scale \(\mu = \zeta^{-1}\). Inversely, if we require that the CFT \(SO(2, d)\) covariant conservation law is not anomalous for any local renormalization scale, the full bulk Einstein equations are automatically implied.
3 The descendent structure of $SO(2, d)$ anomaly

3.1 Bulk consistent action

3.1.1 Chern-Simons like action

To describe the descendent structure of $SO(2, d)$ anomaly, we need to introduce the homotopic quantities

$$D(s)Y = dY + sAY, \quad F(s) = dA(s) + A(s) \wedge A(s) = sF - s(1 - s)A \wedge A$$ \hspace{1cm} (20)

by the replacement

$$A \rightarrow A(s) = sA.$$ \hspace{1cm} (21)

As in the case of chiral anomalies \cite{12}, the bulk Chern-Simons like action can be constructed in terms of the homotopic integration over the boundary covariant anomaly. That is

$$S_{(d+1)}^{CS} = \int_M \mathcal{L}_{(d+1)}^{CS} = - \int_M \int_0^1 ds \, A^{a\beta} \wedge \mathcal{A}_{(d)}^{\alpha\beta}(A(s), Y) = \int_M \int_0^1 ds \, A^{a\beta} \wedge (\mathcal{J}_{(d+1)})_{\alpha\beta}(A(s), Y)$$

$$= \frac{(-1)^{d+1} \ell}{4\kappa^2 (d-2)!} \int_M \int_0^1 ds \, \epsilon_{a\beta a_1 \cdots a_d} A^{a_1 a_2} \wedge F^{a_3 \cdots a_d}(s) \wedge D(s)Y^{a_3} \wedge \cdots \wedge D(s)Y^{a_d}. \hspace{1cm} (22)$$

The variation of $S_{(d+1)}^{CS}$ is given by

$$\delta S_{(d+1)}^{CS} = \int_M \left[ \delta A^{a\beta} \wedge (\mathcal{J}_{(d+1)})_{a\beta} + \delta Y^{a}(\mathcal{J}_{(d+1)})_{a} \right] + \int_M \left[ \delta A^{a\beta} \wedge (K_{(d+1)})_{a\beta} + \delta Y^{a}(K_{(d+1)})_{a} \right], \hspace{1cm} (23)$$

where the Noether potentials are

$$(K_{(d+1)})_{a} = \frac{(-1)^{d+1} \ell}{4\kappa^2 (d-2)!} \int_0^1 ds \, \epsilon_{a\beta a_1 \cdots a_d} A^{a_1 a_2} \wedge D(s)Y^{a_3} \wedge \cdots \wedge D(s)Y^{a_d}, \hspace{1cm} (24)$$

$$(K_{(d+1)})_{a} = \frac{(-1)^d \ell}{2\kappa^2 (d-2)!} \epsilon_{a\beta a_1 \cdots a_d} Y^{a_2} \left[ F^{a_1 a_2} \wedge D(s)Y^{a_3} \wedge \cdots \wedge D(s)Y^{a_d} \right. \nonumber$$

$$\left. - \int_0^1 ds \, D(s) \left( A^{a_1 a_2} \wedge D(s)Y^{a_3} \wedge \cdots \wedge D(s)Y^{a_d} \right) \right]. \hspace{1cm} (25)$$

We notice that $\delta S_{(d+1)}^{CS}$ gives rise to the same bulk terms as $\delta S_{(d+1)}^{\text{cov}}$. Thus these two Lagrangians are different only by a closed term, and both of them can be viewed as the $SO(2, d)$ uplifting of the bulk Einstein gravity. For $d = 2$, $S_{(d+1)}^{CS}$ comes back to the $SO(2, 2) = SL(2, R) \times SL(2, R)$ Chern-Simons gauge theory \cite{13}.

As in the chiral anomaly case, the Chern-Simons like action \cite{22} is invariant under the perturbative gauge transformations, but not invariant under the large gauge transformations. Let us consider the finite $SO(2, d)$ gauge transformation

$$Y \rightarrow \hat{Y} = UY, \quad A \rightarrow \hat{A} = UAU^{-1} - dUU^{-1} = U(A - A)U^{-1},$$ \hspace{1cm} (26)

where we denote $\hat{A} = U^{-1}dU$. The corresponding homotopic quantities are

$$\hat{A}(s) = s\hat{A} = sU^{-1}dU, \quad \hat{D}(s)Y = dY + s\hat{A}Y = dY + sU^{-1}dUY,$$
\[
\tilde{F}(\hat{s}) = d\tilde{A}(\hat{s}) + \tilde{A}(\hat{s}) \wedge \hat{A}(\hat{s}) = 0 \Rightarrow \hat{F}(1 - \hat{s}) = \hat{F}(1 - \hat{s}),
\]
\[
A(s, \hat{s}) = sA + \hat{s}A,
\]
\[
D(s, \hat{s})Y = dY + sAY + \hat{s}\hat{A}Y,
\]
\[
F(s, \hat{s}) = dA(s, \hat{s}) + A(s, \hat{s}) \wedge A(s, \hat{s}) = F(s) + \tilde{F}(\hat{s}) + s\hat{s}(A \wedge \hat{A} + \hat{A} \wedge A). \tag{27}
\]

In terms of these homotopic quantities, the finite gauge transformation of the \(L_{(d+1)}^{CS}[A, Y]\) can be expressed as

\[
\frac{4\kappa^2(d-2)!}{(-1)^{d+1}d!} \left( L_{(d+1)}^{CS}[\tilde{A}, \tilde{Y}] - L_{(d+1)}^{CS}[A, Y] \right)
= -\int_0^1 d\hat{s} \epsilon_{\hat{\alpha_0}\hat{\alpha_1}...\hat{\alpha_d}} \hat{A}^{\hat{\alpha_0}} \wedge \hat{F}^{\hat{\alpha_1}\hat{\alpha_2}}(\hat{s}) \wedge \hat{D}(\hat{s})Y^{\hat{\alpha_2}...\hat{\alpha_d}}
+ d\left[ \int_0^1 ds \int_0^{1-s} d\hat{s} \epsilon_{\hat{\alpha_0}\hat{\alpha_1}...\hat{\alpha_d}} \hat{A}^{\hat{\alpha_0}} \wedge \hat{A}^{\hat{\alpha_1}\hat{\alpha_2}} \wedge D(s, \hat{s})Y^{\hat{\alpha_2}...\hat{\alpha_d}} \right]. \tag{28}
\]

Obviously, the bulk term vanishes for infinitesimal gauge transformations. It means that the perturbative gauge invariance is unbroken up to the boundary term. However, for large gauge transformations, the bulk term is no longer zero and the full \(SO(2, d)\) invariance is broken.

### 3.1.2 Topological term and bulk consistent action

Due to the existence of the ruler field, one can always introduce an \(A\)-independent term as

\[
S_{(d+1)}^{\text{top}} = \int_M L_{(d+1)}^{\text{top}} = \frac{d}{\kappa^2(d+1)!\ell^d} \int_M \epsilon_{\hat{\alpha_0}...\hat{\alpha_d+1}} dY^{\hat{\alpha_0}} \wedge ... \wedge dY^{\hat{\alpha_d}} Y^{\hat{\alpha_d+1}}. \tag{29}
\]

It is a topological term since \(L_{(d+1)}^{\text{top}}\) is closed

\[
d(\epsilon_{\hat{\alpha_0}...\hat{\alpha_d+1}} dY^{\hat{\alpha_0}} \wedge ... \wedge dY^{\hat{\alpha_d}} Y^{\hat{\alpha_d+1}}) = (-1)^{d+1} \epsilon_{\hat{\alpha_0}...\hat{\alpha_d+1}} dY^{\hat{\alpha_0}} \wedge ... \wedge dY^{\hat{\alpha_d+1}} = 0. \tag{30}
\]

Due to the topological natural, the variation of \(\text{(29)}\) is just a boundary term

\[
\delta S_{(d+1)}^{\text{top}} = \frac{(-1)^d}{\kappa^2\ell^d(d-1)!} \int_{\Sigma} \epsilon_{\hat{\alpha_1}...\hat{\alpha_d+2}} \partial Y^{\hat{\alpha_1}} Y^{\hat{\alpha_2}} Y^{\hat{\alpha_3}} \wedge ... \wedge Y^{\hat{\alpha_d+2}}. \tag{31}
\]

Especially, \(S_{(d+1)}^{\text{top}}\) is invariant under the bulk perturbative \(SO(2, d)\) gauge transformations.

Under the finite \(SO(2, d)\) gauge transformations, \(L_{(d+1)}^{\text{top}}\) gives rise to

\[
\frac{4\kappa^2(d-2)!}{(-1)^{d+1}d!} \left( L_{(d+1)}^{\text{top}}[\tilde{A}, \tilde{Y}] - L_{(d+1)}^{\text{top}}[A, Y] \right)
= -\int_0^1 d\hat{s} \epsilon_{\hat{\alpha_0}...\hat{\alpha_d+1}} \hat{A}^{\hat{\alpha_0}} \wedge \hat{F}^{\hat{\alpha_1}\hat{\alpha_2}}(\hat{s}) \wedge \hat{D}(\hat{s})Y^{\hat{\alpha_2}...\hat{\alpha_d}}
+ \frac{2(-1)^d}{(d-1)!\ell^d} \left[ \int_0^1 d\hat{s} \epsilon_{\hat{\alpha_0}...\hat{\alpha_d+1}} \hat{A}^{\hat{\alpha_0}} \wedge F(\hat{s})Y^{\hat{\alpha_1}...\hat{\alpha_d}} \right]. \tag{32}
\]

The bulk term in \(\text{(22)}\) is exactly same as the one appeared in \(\text{(28)}\). Therefore, the large gauge transformation of the Chern-Simons like action can be compensated by adding the topological term

\[
S_{(d+1)}^{\text{con}} = \int_M L_{(d+1)}^{\text{con}} = \int_M [L_{(d+1)}^{CS} - L_{(d+1)}^{\text{top}}]. \tag{33}
\]

Up to the boundary terms, this bulk consistent action \(S_{(d+1)}^{\text{con}}\) is invariant under both perturbative and large gauge transformations.
Since $\delta S_{(d+1)}^{\text{top}}$ is just a boundary term, the variation of the consistent action gives rise to the same bulk term as in $\delta S_{(d+1)}^{\text{CS}}$ and $\delta S_{(d+1)}^{\text{cov}}$. We have

$$\delta S_{(d+1)}^{\text{con}} = \int_M \left[ \delta A^{\hat{\alpha} \hat{\beta}} \wedge (J_{(d+1)})_{\hat{\alpha} \hat{\beta}} + \delta Y^{\hat{\alpha}} (J_{(d+1)})_{\hat{\alpha}} \right] + \int_{\Sigma} \left[ \delta A^{\hat{\alpha} \hat{\beta}} \wedge (K^{\text{con}}_{(d+1)})_{\hat{\alpha} \hat{\beta}} + \delta Y^{\hat{\alpha}} (K^{\text{con}}_{(d+1)})_{\hat{\alpha}} \right], \quad (34)$$

where $(K^{\text{con}}_{(d+1)})_{\hat{\alpha} \hat{\beta}} = (K^{\text{CS}}_{(d+1)})_{\hat{\alpha} \hat{\beta}}$ since $S_{(d+1)}^{\text{top}}$ is $A$-independent and only $(K_{(d+1)})_{\hat{\alpha}}$ is modified by the topological term

$$(K^{\text{con}}_{(d+1)})_{\hat{\alpha}} = (K^{\text{CS}}_{(d+1)})_{\hat{\alpha}} - \frac{(-1)^d}{\kappa^2 \ell^d (d-1)!} \epsilon^{\hat{\alpha} \hat{\beta}_1 \ldots \hat{\alpha}_d} Y^{\hat{\beta}_1} \wedge \cdots \wedge dY^{\hat{\alpha}_d}. \quad (35)$$

The pullback of the consistent Noether potentials gives rise to the Bardeen-Zumino polynomial $[14]

$$(P_{(d)})_{\hat{\alpha} \hat{\beta}} = -f_{S}^{\text{CS}}[(K^{\text{con}}_{(d+1)})_{\hat{\alpha} \hat{\beta}}], \quad (P_{(d)})_{\hat{\alpha}} = -f_{S}^{\text{CS}}[(K^{\text{con}}_{(d+1)})_{\hat{\alpha}}], \quad (36)$$

which are basically the differences between the boundary covariant and consistent currents.

### 3.2 Boundary effective action

Similar to [22], one can also construct the boundary relative effective action by the homotopic integration over the boundary current $[13]$. That is,

$$W_{(d)} = \int_{\Sigma} L_{(d)}^{\text{eff}} = \int_{\Sigma} \int_{0}^{1} ds A^{\hat{\alpha} \hat{\beta}} \wedge (J_{(d)})_{\hat{\alpha} \hat{\beta}}(A(s), Y)$$

$$= -\frac{1}{2 \kappa^2 \ell^d (d-1)!} \int_{\Sigma} \int_{0}^{1} ds \epsilon^{\hat{\alpha} \hat{\beta}_1 \ldots \hat{\alpha}_d} A^{\hat{\alpha} \hat{\beta}} \wedge D(s) Y^{\hat{\beta}_1} \wedge \cdots \wedge D(s) Y^{\hat{\alpha}_d}. \quad (37)$$

Under the finite $SO(2, d)$ gauge transformations, $W_{(d)}$ transforms as

$$W_{(d)}[A, Y] - W_{(d)}[\hat{A}, \hat{Y}] = \frac{(-1)^{d+1} \ell}{4 \kappa^2 (d-2)!} \int_{\Sigma} \int_{0}^{1} d\hat{s} \epsilon^{\hat{\alpha} \hat{\beta}_1 \ldots \hat{\alpha}_d} \left[ \int_{0}^{1-\hat{s}} ds A^{\hat{\alpha} \hat{\beta}} \wedge A^{\hat{\alpha}_1 \hat{\alpha}_2} \wedge D(s, \hat{s}) Y^{\hat{\alpha}_3} \wedge \cdots \wedge D(s, \hat{s}) Y^{\hat{\alpha}_d} \right]$$

$$+ \frac{2}{(d-1) \ell^2} A^{\hat{\alpha} \hat{\beta}} Y^{\hat{\alpha}_1} \wedge \hat{D}(\hat{s}) Y^{\hat{\alpha}_2} \wedge \cdots \wedge \hat{D}(\hat{s}) Y^{\hat{\alpha}_d}. \quad (38)$$

Comparing with [28] and [32], we find [35] is explicitly the boundary term produced by the gauge transformation of the bulk consistent action $S_{(d+1)}^{\text{con}}$. Thus we have the expected finite descendent relation

$$\Delta_{U} L_{(d+1)}^{\text{con}}[A, Y] = d(\Delta_{U} L_{(d)}^{\text{eff}}[A, Y]). \quad (39)$$

At the infinitesimal limit, [35] gives rise to the covariant anomaly

$$\int_{\Sigma} u^{\hat{\alpha} \hat{\beta}} (A_{(d)})_{\hat{\alpha} \hat{\beta}} = \delta W_{(d)}[A, Y]$$

$$= \frac{(-1)^d \ell}{4 \kappa^2 (d-2)!} \int_{\Sigma} u^{\hat{\alpha} \hat{\beta}} \epsilon^{\hat{\alpha} \hat{\beta}_1 \ldots \hat{\alpha}_d} \left[ \int_{0}^{1} ds (1-s) d(A^{\hat{\alpha}_1 \hat{\alpha}_2} \wedge D(s) Y^{\hat{\alpha}_3} \wedge \cdots \wedge D(s) Y^{\hat{\alpha}_d}) \right]$$

$$+ \frac{2}{(d-1) \ell^2} dY^{\hat{\alpha}_1} \wedge \cdots \wedge dY^{\hat{\alpha}_d}, \quad (40)$$
where the total derivative terms have been subtracted since \( \partial \Sigma = \partial \partial M = 0 \). The boundary consistent anomaly satisfies that the infinitesimal descendent relation

\[
\delta_u I^{\text{con}}_{(d+1)} = dI^{\text{con}}_{(d)},
\]

where we denote that

\[
I^{\text{con}}_{(d)} = u^{\hat{\alpha} \hat{\beta}} (A^{\text{con}}_{(d)})_{\hat{\alpha} \hat{\beta}}, \quad I^{\text{con}}_{(d+1)} = L^{\text{con}}_{(d+1)}.
\]

The corresponding consistent currents are

\[
(\mathcal{J}^{\text{con}}_{(d)})_{\hat{\alpha} \hat{\beta}} = \frac{\delta W_{(d)}}{\delta A^{\hat{\alpha} \hat{\beta}}} = \frac{(-1)^d}{2\kappa^2 \ell (d-1)!} \epsilon_{\hat{\alpha} \hat{\beta} \hat{\alpha}_1 \cdots \hat{\alpha}_d} [Y^{\hat{\alpha}_1} \text{DY}^{\hat{\alpha}_2} \wedge \cdots \wedge \text{DY}^{\hat{\alpha}_d}
- \frac{\ell^2 (d-1)}{2} \int_0^1 ds A^{\hat{\alpha}_1 \hat{\alpha}_2} (s) \wedge D(s) Y^{\hat{\alpha}_3} \wedge \cdots \wedge D(s) Y^{\hat{\alpha}_d}],
\]

\[
(\mathcal{J}^{\text{con}}_{(d)})_{\hat{\alpha}} = \frac{\delta W_{(d)}}{\delta Y^{\hat{\alpha}}} = \frac{(-1)^d}{2\kappa^2 \ell (d-1)!} \epsilon_{\hat{\alpha} \hat{\alpha}_1 \cdots \hat{\alpha}_d} Y^{\hat{\alpha}_3} \left[ \frac{2}{\ell^2} (\text{DY}^{\hat{\alpha}_2} \wedge \cdots \wedge \text{DY}^{\hat{\alpha}_d} - d\text{Y}^{\hat{\alpha}_2} \wedge \cdots \wedge d\text{Y}^{\hat{\alpha}_d})
- (d-1) \int_0^1 ds D(s) [A^{\hat{\alpha}_1 \hat{\alpha}_2} \wedge D(s) Y^{\hat{\alpha}_3} \wedge \cdots \wedge D(s) Y^{\hat{\alpha}_d}] \right].
\]

One can further verify that the consistent currents satisfy the off-shell consistent conservation law

\[
D(\mathcal{J}^{\text{con}}_{(d)})_{\hat{\alpha} \hat{\beta}} + Y_{\hat{\beta}} (\mathcal{J}^{\text{con}}_{(d)})_{\hat{\alpha}} = (A^{\text{con}}_{(d)})_{\hat{\alpha} \hat{\beta}},
\]

as well as the relations \[14\]

\[
(\mathcal{J}^{\text{con}}_{(d)})_{\hat{\alpha} \hat{\beta}} = (\mathcal{J}^{\text{cov}}_{(d)})_{\hat{\alpha} \hat{\beta}} - (P_{(d)})_{\hat{\alpha} \hat{\beta}}, \quad (\mathcal{J}^{\text{con}}_{(d)})_{\hat{\alpha}} = (\mathcal{J}^{\text{cov}}_{(d)})_{\hat{\alpha}} - (P_{(d)})_{\hat{\alpha}}.
\]

Since the Bardeen-Zumino polynomial does not contribute to the homotopic integration

\[
A^{\hat{\alpha} \hat{\beta}} \wedge (P_{(d)})(s, A, Y) \propto \int_0^1 ds \epsilon_{\hat{\alpha}_1 \hat{\alpha}_2 \cdots \hat{\alpha}_d} A^{\hat{\alpha}_1 \hat{\alpha}_2} \wedge A^{\hat{\alpha}_3} \wedge \cdots \wedge D(s) Y^{\hat{\alpha}_d} = 0,
\]

the effective action can also be constructed in terms of the homotopic integration over the consistent current \[15\]

\[
W_{(d)} = \int \int_0^1 ds A^{\hat{\alpha} \hat{\beta}} \wedge (\mathcal{J}^{\text{cov}}_{(d)})_{\hat{\alpha} \hat{\beta}} (A(s), Y) = \int \int_0^1 ds A^{\hat{\alpha} \hat{\beta}} \wedge (\mathcal{J}^{\text{con}}_{(d)})_{\hat{\alpha} \hat{\beta}} (A(s), Y).
\]

### 3.3 Characteristic class

The characteristic class of the \( SO(2, d) \) anomaly is given by the exterior derivative of the bulk consistent Lagrangian

\[
I_{(d+2)} = dI^{\text{con}}_{(d+1)} = \frac{(-1)^{d+1} \ell}{8\kappa^2 (d-2)} \epsilon_{\hat{\alpha}_1 \hat{\alpha}_2 \cdots \hat{\alpha}_d} F^{\hat{\alpha}_1 \hat{\alpha}_2} \wedge \text{DY}^{\hat{\alpha}_3} \wedge \cdots \wedge \text{DY}^{\hat{\alpha}_d}.
\]
As shown in [10], the exterior derivative of the bulk covariant Lagrangian gives rise to the same results

\[ I_{(d+2)} = d\mathcal{L}^{\text{cov}}_{(d+1)} . \]  

(50)

In fact, the bulk consistent action is just the summation of the bulk covariant action and the boundary effective action

\[ \mathcal{L}^{\text{cov}}_{(d+1)} = \mathcal{L}^{\text{con}}_{(d+1)} - d\mathcal{L}_{(d)} = \mathcal{L}^{\text{CS}}_{(d+1)} - \mathcal{L}^{\text{top}}_{(d+1)} - d\mathcal{L}_{(d)} . \]  

(51)

Under the finite gauge transformations, the bulk and boundary terms of \( \Delta U S^{\text{CS}}_{(d+1)} \) are compensated respectively by the transformations of topological term and the boundary effective action. This is consistent with the fact that \( \mathcal{L}^{\text{cov}}_{(d+1)} \) is manifestly \( SO(2, d) \) gauge invariant.

As in the usual treatment of chiral anomalies, one can also establish the descendent structure

\[ \delta u I^{\text{CS}}_{(d+1)} = d I^{\text{CS}}_{(d)} \]  

(52)

directly by the bulk Chern-Simons term

\[ I^{\text{CS}}_{(d+1)} = \mathcal{L}^{\text{con}}_{(d+1)} , \quad I^{\text{CS}}_{(d)} = u^{\alpha \beta} (A^{\text{CS}}_{(d)})_{\alpha \beta} . \]  

(53)

The corresponding anomaly

\[ (A^{\text{CS}}_{(d)})_{\alpha \beta} \tilde{\gamma} = \frac{(-1)^d}{4\kappa^2 (d-2)!} \epsilon_{\alpha \beta \tilde{\gamma} \alpha_1 \cdots \alpha_d} \int_0^1 ds \left( 1 - s \right) d \left( A^{\alpha_1 \alpha_2} \wedge D(s) Y^{\alpha_3} \wedge \cdots \wedge D(s) Y^{\alpha_d} \right) \]  

(54)

appears from the infinitesimal gauge transformation of the topologically improved effective action

\[ \tilde{W} = W_{(d)} + S^{\text{top}}_{(d+1)} . \]  

(55)

Due to the existence of the bulk term in \( \Delta U S^{\text{CS}}_{(d+1)} \) and \( \Delta U S^{\text{top}}_{(d+1)} \), the Chern-Simons descendent relation is correct only for the perturbative gauge transformations. The finite descendent relation like (39) is absent in this approach.

4 Summary

In this paper, we establish the descendent structure of the holographic \( SO(2, d) \) anomaly. Due to the existence of the ruler field, one can write down the explicit form of the boundary covariant and consistent currents. The bulk Chern-Simons like action and the boundary effective action are constructed in terms of the homotopic integration method. To compensate the large gauge transformation of the Chern-Simons like action, a gauge field independent topological term is introduced as a function of the ruler field. We conjecture that the topological term plays a similar role as the \( \eta \)-invariant [16] in chiral anomalies. It is very curious to discuss the effect of this topological term in the story of cobordism invariance [17].
In the Einstein gravity, the \( SO(2, d) \) anomaly is governed by the characteristic class
\[
I_{(d+2)} = \frac{(-1)^{d+1}f}{8\kappa^2(d-2)!}\epsilon_{\hat{a}_0\ldots\hat{a}_{d+1}} F^{\hat{a}_0\hat{a}_1} \wedge F^{\hat{a}_2\hat{a}_3} \wedge DY^\hat{a}_4 \wedge \cdots \wedge DY^\hat{a}_{d+1}.
\] (56)

One can naturally generalize the form of the characteristic class to
\[
I_{(d+2)} = \sum_{i=0}^{\lfloor d/2 \rfloor+1} \lambda_i \epsilon_{\hat{a}_0\ldots\hat{a}_{d+1}} F^{\hat{a}_0\hat{a}_1} \wedge \cdots \wedge F^{\hat{a}_{2i-2}\hat{a}_{2i-1}} \wedge DY^\hat{a}_{2i} \wedge \cdots \wedge DY^\hat{a}_{d+1}.
\] (57)

Now the corresponding bulk theory should be the \( SO(2, d) \) gauge theory uplifting of the Lovelock gravity [18] for which the bulk covariant action is in the form
\[
S_{\text{cov}}^{(d+1)} = \int_M \sum_{i=0}^{\lfloor d/2 \rfloor} \lambda_i \epsilon_{\hat{a}_0\ldots\hat{a}_{d+1}} F^{\hat{a}_0\hat{a}_1} \wedge \cdots \wedge F^{\hat{a}_{2i-2}\hat{a}_{2i-1}} \wedge DY^\hat{a}_{2i} \wedge \cdots \wedge DY^\hat{a}_{d+1}.
\] (58)

The recent work [19] suggests that the string low energy effective action could always be recast as Lovelock type of theory to all orders in \( \alpha' \). Therefore, by considering the gauge field and ruler field associated with the stringy gauge symmetries, it is possible to recast the string low energy effective theory as a gauge theory in the form of (58).

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