Normalized solutions for the fractional NLS with mass supercritical nonlinearity

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June 2, 2020 12:51am Z

Abstract

We investigate the existence of solutions to the fractional nonlinear Schrödinger equation

$$(-\Delta)^s u = f(u)$$

with prescribed $L^2$-norm $\int_{\mathbb{R}^N} |u|^2 \, dx = m$ in the Sobolev space $H^s(\mathbb{R}^N)$. Under fairly general assumptions on the nonlinearity $f$, we prove the existence of a ground state solution and a multiplicity result in the radially symmetric case.

1 Introduction

In this paper we investigate the existence of solutions to the fractional Nonlinear Schrödinger Equation (NLS in the sequel)

$$i \frac{\partial \psi}{\partial t} = (-\Delta)^s \psi - V(|\psi|)\psi,$$  \hspace{1cm} (1.1)

where $i$ denotes the imaginary unit and $\psi = \psi(x, t): \mathbb{R}^N \times (0, \infty) \to \mathbb{C}$. An important family of solutions, known under the name of travelling or standing waves, is characterized by the ansatz

$$\psi(x, t) = e^{i\mu t} u(x)$$  \hspace{1cm} (1.2)

for some (unknown) function $u: \mathbb{R}^N \to \mathbb{R}$. Clearly, these solutions have the remarkable property that they conserve their mass along time, i.e.

$$\|\psi(t)\|_{L^2(\mathbb{R}^N)} = \|\psi(0)\|_{L^2(\mathbb{R}^N)}$$

for any $t \in (0, \infty)$. It is therefore natural and meaningful to seek solutions having a prescribed $L^2$-norm. This type of Schrödinger equation was introduced by Laskin in [10], and the interest in its analysis has grown over the years.

Coupling (1.1) with (1.2), we arrive at the problem

$$\begin{cases}
(-\Delta)^s u = V(|u|)u - \mu u & \text{in } \mathbb{R}^N, \\
\|u\|_{L^2(\mathbb{R}^N)}^2 = m
\end{cases}$$

Both authors are supported by GNAMPA, project “Equazioni alle derivate parziali: problemi e modelli.”

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where $s \in (0,1)$, $N > 2s$, $\mu \in \mathbb{R}$, $m > 0$ is a prescribed parameter, and $(-\Delta)^s$ denotes the usual fractional laplacian. Namely,

$$(-\Delta)^s u(x) = C(N, s) \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,$$

where

$$C(N, s) = \left( \int_{\mathbb{R}^N} \frac{1 - \cos \zeta_1}{|\zeta|^{n+2s}} d\zeta \right)^{-1}.$$

For further details on the fractional laplacian we refer to [6]. For our purposes, and since the parameter $s$ is kept fixed, we will always work with a rescaled fractional operator, in such a way that $C(N, s) = 1$.

In order to ease notation, we will write $f(u) = V(|u|)u$, and study the problem

$$\begin{cases}
(-\Delta)^s u = f(u) - \mu u & \text{in } \mathbb{R}^N, \\
\|u\|_{L^2(\mathbb{R}^N)}^2 = m.
\end{cases} \quad (P_m)$$

The rôle of the real number $\mu$ is twofold: it can either be prescribed, or it can arise as a suitable parameter during the analysis of $(P_m)$. In the present work we will choose the second option, and $\mu$ will arise as a Lagrange multiplier.

Since we are looking for bound-state solutions whose $L^2$-norm must be finite, it is natural to build a variational setting for $(P_m)$. Since this is by now standard, we will be sketchy. We introduce the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \mid [u]_{H^s(\mathbb{R}^N)}^2 < +\infty \right\},$$

where

$$[u]_{H^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy$$

is the so-called Gagliardo semi-norm. The norm in $H^s(\mathbb{R}^N)$ is defined by

$$\|u\| = \sqrt{\|u\|_{L^2}^2 + [u]_{H^s(\mathbb{R}^N)}^2},$$

which arises from an inner product. We then (formally) introduce the energy functional

$$I(u) = [u]_{H^s(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} F(u) dx$$

where $F(t) = \int_0^t f(\sigma) d\sigma$. A standard approach for studying $(P_m)$ consists in looking for critical points of $I$ constrained on the sphere

$$S_m = \left\{ u \in H^s(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |u|^2 dx = m \right\}.$$

The convenience of this variational approach depends strongly on the behavior of the nonlinearity $f$. If $f(t)$ grows slower than $|t|^{1+\frac{4s}{N}}$ as $t \to +\infty$, then $I$ is coercive and bounded from below on $S_m$; this is the mass subcritical case, and the minimization problem

$$\min \{ I(u) \mid u \in S_m \}$$

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is the natural approach. On the other hand, if $f(t)$ grows faster than $|t|^{1+\frac{4}{N}}$ as $t \to +\infty$ then $I$ is unbounded from below on $S_m$, and we are in the mass supercritical case. Since constrained minimizers of $I$ on $S_m$ cannot exist, we have to find critical points at higher levels.

When $s = 1$, that is when the fractional Laplace operator $(-\Delta)^s$ reduces to the local differential operator $-\Delta$, the literature for $(P_m)$ is huge. The particular case of a combined nonlinearity of power type, namely $f(t) = t^{p-2} + \mu t^{q-2}$ with $2 < q < p < 2N/(N - 2)$ has been widely investigated. The interplay of the parameters $p$ and $q$ add some richness to the structure of the problem.

The situation is different when $0 < s < 1$, and few results are available. Feng et al. in [7] deal with particular nonlinearities. Stanislavova et al. in [17] add the further complication of a trapping potential. In the recent paper [18] the author proves some existence and asymptotic results for the fractional NLS when a lower order perturbation to a mass supercritical pure power in the nonlinearity is added. It is also worth mentioning [12], where Zhang et al. studied the problem when the nonlinear term consists in the sum of two pure powers of different order. They provide some existence and non-existence results analysing separately what happens in the mass subcritical and supercritical case for both the leading term and the lower order perturbation.

Very recently, Jeanjean et al. in [9] provided a thorough treatment of the local case $s = 1$ via a careful analysis based on the Pohozaev identity. In the present paper we extend part of their results to the non-local case $0 < s < 1$. Since we deal with a fractional operator, our conditions on $f$ must be adapted correspondingly.

We collect here our standing assumptions about the nonlinearity $f$; we recall that

$$F(t) = \int_0^t f(\sigma) \, d\sigma$$

and define the auxiliary function

$$\tilde{F}(t) = f(t)t - 2F(t).$$

(f0) $f: \mathbb{R} \to \mathbb{R}$ is an odd and continuous function;

(f1) $\lim_{t \to 0} \frac{f(t)}{|t|^{1+4s/N}} = 0$;

(f2) $\lim_{t \to +\infty} \frac{f(t)}{|t|^{(N+2s)/(N-2s)}} = 0$;

(f3) $\lim_{t \to +\infty} \frac{F(t)}{|t|^{2+4s/N}} = +\infty$;

(f4) The function $t \mapsto \frac{\tilde{F}(t)}{|t|^{2+4s/N}}$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, +\infty)$;

(f5) $f(t)t < \frac{2N}{N-2s} F(t)$ for all $t \in \mathbb{R} \setminus \{0\}$;

(f6) $\lim_{t \to 0} \frac{tf(t)}{|t|^{2N/(N-2s)}} = +\infty.$
Remark 1.1. The oddness of $f$ is necessary in order to use the classical genus theory and to get a desired property on the fiber map that we will introduce in detail in the next section (see for instance Lemma 2.6 below). Assumption $(f_2)$ guarantees a Sobolev subcritical growth, whereas $(f_3)$ characterises the problem as mass supercritical. At one point we will need $(f_5)$ to establish the strict positivity of the Lagrange multiplier $\mu$.

Example 1.2. As suggested in [9], an explicit example can be constructed as follows. Set $\alpha_N = \frac{8}{N(N-2)}$ for simplicity, and define
\[
f(t) = \left(2 + \frac{4}{N}\right) \log (1 + |t|^\alpha_N) + \frac{\alpha_N |t|^{\alpha_N}}{1 + |t|^{\alpha_N}} |t|^{\frac{N}{2}}.
\]

We briefly outline our results. Firstly, we show that the ground state level is attained with a strictly positive Lagrange multiplier.

Theorem 1.3. Assume that $f$ satisfies $(f_0)$-$(f_5)$. Then $(P_m)$ admits a positive ground state for any $m > 0$. Moreover, for any ground state the associated Lagrange multiplier $\mu$ is positive.

Furthermore, we can prove some remarkable properties of the ground state level energy with respect the variable $m$ and its asymptotic behaviour.

Theorem 1.4. Assume that $f$ satisfies $(f_0)$-$(f_6)$. Then the function $m \mapsto E_m$ is positive, continuous, strictly decreasing. Furthermore, $\lim_{m \to 0^+} E_m = +\infty$ and $\lim_{m \to \infty} E_m = 0$.

Finally, we have a multiplicity result for the radially symmetric case.

Theorem 1.5. If $(f_0)$-$(f_5)$ hold, then $(P_m)$ admits infinitely many radial solutions $(u_k)_k$ for any $m > 0$. In particular,
\[
I(u_{k+1}) \geq I(u_k)
\]
for all $k \in \mathbb{N}$ and $I(u_k) \to +\infty$ as $k \to +\infty$.

Our paper is organised as follows. Section 2 contains the proofs of some preliminary lemmas that will be useful during the whole remaining part of the paper. Moreover, we introduce a fiber map that will play a crucial role for our purposes. In Section 3 we define the ground state level energy for a fixed mass $m$ and we start analysing its asymptotic behaviour near zero and infinity. Section 4 is devoted to prove our main existence theorem. Using a min-max theorem of linking type and the fiber map cited previously, we construct a Palais-Smale sequence whose value on the Pohozaev functional is zero and we show that a sequence of this kind must be necessarily bounded. Finally, in Section 5, for the sake of completeness, we discuss the existence of radial solutions. Here, we use a variant of the min-max theorem already cited in Section 4, but this time we are helped by the fact that the space of the radially symmetric functions with finite fractional derivative is compactly embedded in $L^p(\mathbb{R}^N)$ for $p \in (2, 2^*_s)$.

2 Preliminary results

We define the Pohozaev manifold
\[
P_m = \{ u \in S_m \mid P(u) = 0 \},
\]
where
\[ P(u) = [u]_{H^s(\mathbb{R}^N)}^2 - \frac{N}{2s} \int_{\mathbb{R}^N} \tilde{F}(u) \, dx. \]

Let us collect some technical results that we will frequently used in the paper. The first two Lemmas will be proved in the Appendix. We use the shorthand
\[ B_m = \left\{ u \in H^2(\mathbb{R}^N) \mid \| u \|_{L^2(\mathbb{R}^N)}^2 \leq m \right\}. \]

**Lemma 2.1.** Assuming \((f_0), (f_1), (f_2)\), the following statements hold

(i) for every \(m > 0\) there exists \(\delta > 0\) such that
\[ \frac{1}{4} [u]_{H^s(\mathbb{R}^N)}^2 \leq I(u) \leq [u]_{H^s(\mathbb{R}^N)}^2 \]
where \(u \in B_m\) and \([u]_{H^s(\mathbb{R}^N)} \leq \delta\).

(ii) Let \((u_n)_n\) be a bounded sequence in \(H^s(\mathbb{R}^N)\). If \(\lim_{n \to +\infty} \| u_n \|_{L^{2+4s/N}(\mathbb{R}^N)} = 0\) we have that
\[ \lim_{n \to +\infty} \int_{\mathbb{R}^N} F(u_n) \, dx = 0 = \lim_{n \to +\infty} \int_{\mathbb{R}^N} \tilde{F}(u_n) \, dx. \]

(iii) Let \((u_n)_n, (v_n)_n\) two bounded sequences in \(H^s(\mathbb{R}^N)\). If \(\lim_{n \to +\infty} \| v_n \|_{L^{2+4s/N}} = 0\) then
\[ \lim_{n \to +\infty} \int_{\mathbb{R}^N} f(u_n)v_n \, dx = 0. \]

**Remark 2.2.** An inspection of the proof of this Lemma shows that the inequality
\[ \int_{\mathbb{R}^N} \tilde{F}(u) \, dx \leq \frac{8}{N} [u]_{H^s(\mathbb{R}^N)}^2 \]
holds true if \(u \in B_m\) and \([u]_{H^s(\mathbb{R}^N)} \leq \delta\). It follows that
\[ P(u) \geq \frac{1}{2} [u]_{H^s(\mathbb{R}^N)}^2 \]
for every \(u \in B_m\) with \([u]_{H^s(\mathbb{R}^N)} \leq \delta\).

In order to prove the next result we introduce for every \(u \in H^s(\mathbb{R}^N)\) and \(\rho \in \mathbb{R}\) the scaling map
\[
(\rho * u)(x) = e^{\frac{N\rho}{2s}} u(e^\rho x) \quad x \in \mathbb{R}^N.
\]

It easy to verify that \(\rho * u \in H^s(\mathbb{R}^N)\) and \(\| \rho * u \|_{L^2(\mathbb{R}^N)} = \| u \|_{L^2(\mathbb{R}^N)}\).

**Lemma 2.3.** Assuming \((f_0), (f_1), (f_2)\) and \((f_3)\), we have:

(i) \(I(\rho * u) \to 0^+\) as \(\rho \to -\infty\),

(ii) \(I(\rho * u) \to -\infty\) as \(\rho \to \infty\).

**Remark 2.4.** Assume \(f \in C(\mathbb{R}, \mathbb{R})\), \((f_1)\) and \((f_2)\). Then the function \(g: \mathbb{R} \to \mathbb{R}\) defined as
\[ g(t) = \begin{cases} 
\frac{f(0)t - 2F(t)}{|t|^{s+\frac{\rho}{2}}} & t \neq 0 \\
0 & t = 0
\end{cases} \]
is continuous, strictly increasing in \((0, \infty)\) and strictly decreasing in \((-\infty, 0)\).
Lemma 2.5. Assuming \( f \in C(\mathbb{R}, \mathbb{R}) \), \((f_1), (f_3)\) and \((f_4)\), we have

(i) \( F(t) > 0 \) if \( t \neq 0 \),

(ii) there exists \((\tau_n^+)_{n} \subset \mathbb{R^+}\) and \((\tau_n^-)_{n} \subset \mathbb{R^-}, |\tau_n^\pm| \to 0\) as \( n \to +\infty \) such that

\[
f(\tau_n^\pm) > \left( 2 + \frac{4s}{N} \right) F(\tau_n^\pm) \quad n \geq 1,
\]

(iii) there exists \((\sigma_n^+)_{n} \subset \mathbb{R^+}\) and \((\sigma_n^-)_{n} \subset \mathbb{R^-}, |\tau_n^\pm| \to +\infty\) as \( n \to +\infty \) such that

\[
f(\sigma_n^+) > \left( 2 + \frac{4s}{N} \right) F(\sigma_n^+) \quad n \geq 1,
\]

(iv) \( f(t)t > \left( 2 + \frac{4s}{N} \right) F(t) \quad t \neq 0 \).

Proof. (i) By contradiction suppose \( F(t_0) \leq 0 \) for some \( t_0 \neq 0 \). Because of \((f_1)\) and \((f_3)\) the function \( F(t)/|t|^{2+4s/N} \) must attain its global minimum in a point \( \tau \neq 0 \) such that \( F(\tau) \leq 0 \). It follows that

\[
\frac{d}{dt} \frac{F(t)}{|t|^{2+4s/N}} \bigg|_{t=\tau} = \frac{f(\tau)\tau - \left( 2 + \frac{4s}{N} \right) F(\tau)}{|\tau|^{3+\frac{4s}{N}} \text{sgn}(\tau)} = 0.
\]

From Remark 2.4 it follows that \( f(t)t > 2F(t) \) if \( t \neq 0 \). Indeed, were the claim false, there would exists \( \tau \) such that \( f(\tau)\tau \leq 2F(\tau) \). Choosing without loss of generality \( \tau < 0 \), we have that \( g(\tau) \leq 0 \). This and the fact that \( g(0) = 0 \) show that \( g \) must be strictly increasing on an interval between \( \tau \) and 0. Finally, we can have a contradiction observing that

\[
0 < f(\tau)\tau - 2f(\tau) = \frac{4s}{N} F(\tau) \leq 0.
\]

(ii) We start with the positive case. By contradiction we suppose there is \( T_\alpha > 0 \) small enough such that

\[
f(t)t \leq \left( 2 + \frac{4s}{N} \right) F(t)
\]

for every \( t \in (0, T_\alpha] \). Remembering the expression of (2.1) computed in the step (i) we have that the derivative of \( F(t)/|t|^{2+4s/N} \) is negative on \((0, T_\alpha]\), then

\[
\frac{F(t)}{t^{2+\frac{4s}{N}}} \geq \frac{F(T_\alpha)}{T_\alpha^{2+\frac{4s}{N}}} > 0 \quad \text{for every} \quad t \in (0, T_\alpha],
\]

that is in contradiction with \((f_1)\). The negative case is similar.

(iii) Being the two cases similar, we will prove only the negative one. Again, by contradiction we suppose there is \( T_\gamma > 0 \) such that

\[
f(t)t \leq \left( 2 + \frac{4s}{N} \right) F(t) \quad \text{for every} \quad t \leq -T_\gamma.
\]
Since the derivative of $F(t)/|t|^{2+4s/N}$ is negative on $(-\infty, -T_\gamma]$, we can deduce
\[
\frac{F(t)}{t^{2+\frac{4s}{N}}} \leq \frac{F(-T_\gamma)}{T_\gamma^{2+\frac{4s}{N}}} \quad \text{for every } t \in (-\infty, -T_\gamma],
\]
which contradicts $(f_3)$.

$(iv)$ We start proving that the inequality holds weakly. By contradiction we assume
\[
f(t_0)t_0 < 2 + \frac{4s}{N} \quad \text{for some } t_0 \neq 0 \text{ and without loss of generality we can suppose } t_0 < 0.
\]
By step $(ii)$ and $(iii)$ there are $\tau_{\min}, \tau_{\max} \in \mathbb{R}$, where $\tau_{\min} < t_0 < \tau_{\max} < 0$ such that
\[
f(t)t < \left( 2 + \frac{4s}{N} \right) F(t) \quad \text{for every } t \in (\tau_{\min}, \tau_{\max}) \quad (2.2)
\]
and
\[
f(t)t = \left( 2 + \frac{4s}{N} \right) F(t) \quad \text{for every } t \in \{\tau_{\min}, \tau_{\max}\}. \quad (2.3)
\]
By $(2.2)$ we have
\[
\frac{F(\tau_{\min})}{|\tau_{\min}|^{2+\frac{4s}{N}}} < \frac{F(\tau_{\max})}{|\tau_{\max}|^{2+\frac{4s}{N}}}. \quad (2.4)
\]
Besides, by $(2.3)$ and $(f_4)$ must be
\[
\frac{F(\tau_{\min})}{|\tau_{\min}|^{2+\frac{4s}{N}}} = \frac{\tilde{F}(\tau_{\min})}{\frac{4s}{|\tau_{\min}|^{2+\frac{4s}{N}}} > \frac{\tilde{F}(\tau_{\max})}{\frac{4s}{|\tau_{\max}|^{2+\frac{4s}{N}}} = \frac{F(\tau_{\max})}{|\tau_{\max}|^{2+\frac{4s}{N}}}. \quad (2.5)
\]
and clearly $(2.4)$ and $(2.5)$ are in contradiction. From what we have just proved, we have that $F(t)/|t|^{2+4s/N}$ is non-increasing in $(-\infty, 0)$ and non decreasing in $(0, \infty)$. Hence, by virtue of $(f_4)$ the function $F(t)/|t|^{1+4s/N}$ must necessarily be strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, \infty)$. Then
\[
\left( 2 + \frac{4s}{N} \right) F(t) = \left( 2 + \frac{4s}{N} \right) \int_0^t \frac{f(\kappa)}{|\kappa|^{1+\frac{4s}{N}}} |\kappa|^{1+\frac{4s}{N}} d\kappa < \left( 2 + \frac{4s}{N} \right) \frac{f(t)}{|t|^{1+\frac{4s}{N}}} \int_0^t |\kappa|^{1+\frac{4s}{N}} d\kappa = f(t)t
\]
completes the proof. \qedhere

**Lemma 2.6.** Assume $(f_0) - (f_4)$, $u \in H^s(\mathbb{R}^N) \setminus \{0\}$. Then the following hold:

(i) There is a unique $\rho(u) \in \mathbb{R}$ such that $P(\rho(u) * u) = 0$.

(ii) $I(\rho(u) * u) > I(u)$ for any $\rho \neq \rho(u)$. Moreover $I(\rho(u) * u) > 0$.

(iii) The map $u \to \rho(u)$ is continuous for every $u \in H^s(\mathbb{R}^N)$.

(iv) $\rho(u) = \rho(-u)$ and $\rho(\cdot + y) = \rho(u)$ for ever $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ and $y \in \mathbb{R}^N$. 

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Proof. (i) Since
\[
I(\rho \ast u) = \frac{1}{2} e^{2\rho s} [u]_{H^s(\mathbb{R}^N)}^2 - e^{-N\rho} \int_{\mathbb{R}^N} F(e^{N\rho} u) \, dx
\]
it is easy to check that $I(\rho \ast u)$ is $C^1$ with respect to $\rho$. Now, computing
\[
\frac{d}{d\rho} I(\rho \ast u) = \rho e^{2\rho s} [u]_{H^s(\mathbb{R}^N)}^2 - \frac{N}{2} e^{-N\rho} \int_{\mathbb{R}^N} \tilde{F} \left( e^{N\rho} u \right) \, dx.
\]
and observing that
\[
P(\rho \ast u) = e^{2\rho s} [u]_{H^s(\mathbb{R}^N)}^2 - \frac{N}{2s} e^{-N\rho} \int_{\mathbb{R}^N} \tilde{F} \left( e^{N\rho} u \right) \, dx
\]
we deduce
\[
\frac{d}{d\rho} I(\rho \ast u) = sP(\rho \ast u).
\]
Remembering that by lemma 2.3
\[
\lim_{\rho \to -\infty} I(\rho \ast u) = 0^+ \quad \text{and} \quad \lim_{\rho \to \infty} I(\rho \ast u) = -\infty
\]
we can conclude that $I(\rho \ast u)$ must reach a global maximum in a point $\rho(u) \in \mathbb{R}$, and that, together with the fact that
\[
0 = \frac{d}{d\rho} I(\rho(u) \ast u) = sP(\rho(u) \ast u),
\]
imply that $P(\rho(u) \ast u) = 0$. To see the uniqueness, remembering the function $g$ defined in Remark 2.4, we observe that $\tilde{F}(t) = g(t)|t|^{2+\frac{4s}{N}}$ for every $t \in \mathbb{R}$, thus we obtain
\[
P(\rho \ast u) = e^{2\rho s} [u]_{H^s(\mathbb{R}^N)}^2 - \frac{N}{2s} e^{2\rho s} \int_{\mathbb{R}^N} g(e^{N\rho} u)|u|^{2+\frac{4s}{N}} \, dx
\]
\[
= e^{2\rho s} \left[ [u]_{H^s(\mathbb{R}^N)}^2 - \frac{N}{2s} \int_{\mathbb{R}^N} g(e^{N\rho} u)|u|^{2+\frac{4s}{N}} \, dx \right] = \frac{1}{s} \frac{d}{d\rho} I(\rho \ast u).
\]
Fixing $t \in \mathbb{R} \setminus \{0\}$, thanks to Remark 2.4 and $(f_4)$, we notice that the function $\rho \mapsto g \left(e^{\frac{N\rho}{2}} t\right)$ is strictly increasing. Thus, by virtue of the computations we have done above, it follows that $\rho(u)$ must be unique.

(ii) It is immediate for what we have already seen.

(iii) By step (i) the function $u \mapsto \rho(u)$ is well defined. Let $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ and $(u_n)_n \subset H^s(\mathbb{R}^N) \setminus \{0\}$ a sequence such that $u_n \to u$ in $H^s(\mathbb{R}^N)$ as $n \to +\infty$. We set $\rho_n = \rho(u_n)$ for any $n \geq 1$. Let us show that up to a subsequence we have $\rho_n \to \rho(u)$ as $n \to +\infty$.

Claim. The sequence $(\rho_n)_n$ is bounded.

We recall that the function $h_\lambda$ defined in (6.4) noticing that by lemma 2.5 (i) $h_0(t) \geq 0$ for every $t \in \mathbb{R}$. We assume by contradiction that up to a subsequence $\rho_n \to +\infty$. By Fatou’s lemma and the fact that $u_n \to u$ a.e. in $\mathbb{R}^N$, we have that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} h_0 \left( e^{\frac{N\rho_n}{2}} u_n \right) |u_n|^{2+\frac{4s}{N}} \, dx = \infty.
\]
As a consequence of that, by (6.5) with \( \lambda = 0 \) and step (ii), we obtain
\[
0 \leq e^{-2\rho_n s} I(\rho_n * u_n) = \frac{1}{2} |u_n|_{H^s(\mathbb{R}^N)}^2 - \int_{\mathbb{R}^N} h_0 \left( \frac{\rho_n}{2} u_n \right) |u_n|^{2+ \frac{4}{N}} dx \to -\infty
\]
as \( n \to +\infty \) that is evidently not possible. Then \( (\rho_n)_n \) must be upper bounded. Instead now, assume again by contradiction that \( \rho_n \to -\infty \). By step (ii) we observe that
\[
I(\rho_n * u_n) \geq I(\rho(u) * u_n)
\]
and since \( \rho(u) * u_n \to \rho(u) * u \) in \( H^s(\mathbb{R}^N) \), it follows that
\[
I(\rho(u) * u_n) = I(\rho(u) * u) + o_n(1)
\]
from which we can deduce
\[
\liminf_{n \to +\infty} I(\rho_n * u_n) \geq I(\rho(u) * u) > 0.
\]
(2.7)
Since for \( m \) large enough we have that \( \rho_n * u_n \subset B_m \), Lemma 2.1 (i) imply that there exists \( \delta > 0 \) such that if \( [\rho_n * u_n]_{H^s(\mathbb{R}^N)} \leq \delta \), we have
\[
\frac{1}{4} |\rho_n * u_n|_{H^s(\mathbb{R}^N)}^2 \leq I(\rho_n * u_n) \leq [\rho_n * u_n]_{H^s(\mathbb{R}^N)}^2.
\]
(2.8)
Since
\[
[\rho_n * u_n]_{H^s} = e^{\rho n s} [u_n]_{H^s(\mathbb{R}^N)}
\]
there exists \( n \) so large that (2.8) holds. Passing to the limit we obtain
\[
\liminf_{n \to +\infty} I(\rho_n * u_n) = 0,
\]
in contradiction to (2.7). The claim is proved.

Now, the sequence \( (\rho_n)_n \) being bounded, we can assume up to a subsequence that \( \rho_n \to \rho^* \) for some \( \rho^* \) in \( \mathbb{R} \). Hence, \( \rho_n * u_n \to \rho^* * u \) in \( H^s(\mathbb{R}^N) \) and since \( P(\rho_n * u_n) = 0 \) we have
\[
P(\rho^* * u) = 0.
\]
By the uniqueness proved at step (ii) we obtain \( \rho^* = \rho(u) \).

(iv) Since \( f \) is odd by \( (f_0) \), the fact that
\[
P(\rho(u) * (-u)) = P(\rho(-u) * \rho(u)) = P(\rho(u) * u) = 0
\]
imply \( \rho(u) = \rho(-u) \). Similarly, changing the variables in the integral, we can verify that it is invariant under translation, and it easy to check that
\[
P(\rho(u) * u(\cdot + y)) = P(\rho(u) * u) = 0,
\]
thus \( \rho(u(\cdot + y)) = \rho(u) \).

As we are going to see, the functional \( I \) constrained on \( \mathcal{P}_m \) has some crucial properties.

Lemma 2.7. Assuming \( (f_0) - (f_4) \), the following statements are true:
(i) $\mathcal{P}_m \neq \emptyset$,

(ii) $\inf_{u \in \mathcal{P}_m} \|u\|_{H^s(\mathbb{R}^N)} > 0$,

(iii) $\inf_{u \in \mathcal{P}_m} I(u) > 0$,

(iv) $I$ is coercive on $\mathcal{P}_m$, i.e. $I(u_n) \to \infty$ if $(u_n)_n \subset \mathcal{P}_m$ and $\|u_n\|_{H^s(\mathbb{R}^N)} \to \infty$ as $n \to +\infty$.

**Proof.** Statement (i) follows directly from Lemma 2.6 (i).

(ii) Were the assertion not true, we would be able to take a sequence $(u_n)_n \subset \mathcal{P}_m$ such that $\|u_n\|_{H^s(\mathbb{R}^N)} \to 0$, and so, by Lemma 2.1 (i) we could also find $\delta > 0$ and $\overline{\mathbb{N}}$ big enough such that $\|u_n\|_{H^s(\mathbb{R}^N)} \leq \delta$ for every $n \geq \overline{\mathbb{N}}$. By remark 2.2 we would have

$$0 = P(u_n) \geq \frac{1}{2} \|u_n\|_{H^s(\mathbb{R}^N)}^2$$

that is possible only for $(u_n)_n$ constant, but this is not admissible since $u \in S_m$. Hence the statement must hold.

(iii) For every $u \in \mathcal{P}_m$ Lemma 2.6 (ii) and (iii) implies that

$$I(u) = I(0 \ast u) \geq I(\rho \ast u) \quad \text{for every } \rho \in \mathbb{R}.$$

Let $\delta > 0$ be the number given by Lemma 2.2 (i) and set $\rho := 1/s \log \left( \delta / \|u\|_{H^s(\mathbb{R}^N)} \right)$. Since $\delta = \|\rho \ast u\|_{H^s(\mathbb{R}^N)}$, using again Lemma 2.1 (i) we obtain

$$I(u) \geq I(\rho \ast u) \geq \frac{1}{4} \|\rho \ast u\|_{H^s(\mathbb{R}^N)}^2 = \frac{1}{4} \delta^2$$

proving the statement.

(iv) By contradiction we suppose the existence of $(u_n)_n \subset \mathcal{P}_m$ such that $\|u_n\|_{H^s(\mathbb{R}^N)} \to \infty$ with $\sup_{n \geq 1} I(u_n) \leq c$ for some $c \in (0, \infty)$. For any $n \geq 1$ we set

$$\rho_n = \frac{1}{s} \log \left( \|u_n\|_{H^s(\mathbb{R}^N)} \right) \quad \text{and} \quad v_n = (-\rho_n) \ast u_n.$$

Evidently $\rho_n \to +\infty$, $(v_n)_n \subset S_m$ and $\|v_n\|_{H^s(\mathbb{R}^N)} = 1$. We denote with

$$\alpha = \limsup_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |v_n|^2 \, dx$$

and we distinguish two cases.

**Non vanishing:** $\alpha > 0$. Up to a subsequence we can assume the existence of a sequence $(y_n)_n \subset \mathbb{R}^N$ and $\omega \in H^s(\mathbb{R}^N) \setminus \{0\}$ such that

$$\omega_n = v_n(\cdot + y_n) \to \omega \in H^s(\mathbb{R}^N) \quad \text{and} \quad \omega_n \to \omega \text{ a.e. in } \mathbb{R}^N.$$

Recalling the continuous function $h_\lambda$ with $\lambda = 0$, remembering that $\rho_n \to +\infty$ as $n \to +\infty$ and using the Fatou’s lemma we have

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} h_0 \left( e^{\frac{\omega_n^2}{\rho_n}} \omega_n \right) |\omega_n|^2 + \frac{\delta}{\rho_n} \, dx = \infty.$$
By step (iii) and (2.1), after changing the variables in the integral, we obtain

\[
0 \leq e^{-2\rho s} I(u_n) = e^{-2\rho s} I(\rho_n * v_n) = \frac{1}{2} - \int_{\mathbb{R}^N} h_0 \left( e^{\frac{N\rho}{2} v_n} \right) |v_n|^{2^*} \, dx
\]

\[
= \frac{1}{2} - \int_{\mathbb{R}^N} h_0 \left( e^{\frac{N\rho}{2} \omega} \right) |\omega_n|^{2^*} \, dx \to -\infty
\]
as \( n \to +\infty \).

**Vanishing:** \( \alpha = 0 \). By [16, Lemma II.4], we have that \( v_n \to 0 \) in \( L^{2^*} (\mathbb{R}^N) \) and by Lemma 2.1 (ii) we see that

\[
\lim_{n \to +\infty} e^{N\rho} \int_{\mathbb{R}^N} F \left( e^{\frac{N\rho}{2} v_n} \right) = 0 \quad \text{for every} \quad \rho \in \mathbb{R}.
\]

Since \( P(\rho_n * v_n) = P(u_n) = 0 \), by Lemma 2.6 (ii) and (iii), we obtain

\[
c \geq I(u_n) = I(\rho_n * v_n)
\]

\[
\geq P(\rho * v_n) = \frac{1}{2} e^{2\rho s} - e^{-N\rho} \int_{\mathbb{R}^N} F \left( e^{\frac{N\rho}{2} v_n} \right) \, dx = \frac{1}{2} e^{2\rho s} - o_n(1).
\]

We can conclude choosing \( \rho > \log(2c)/2s \) and letting \( n \to +\infty \).

We conclude with a splitting result à la Brezis-Lieb. A proof is included for the reader’s convenience.

**Lemma 2.8.** Let \( f: \mathbb{R} \to \mathbb{R} \) continuous, odd and let \( (u_n)_n \subset H^s (\mathbb{R}^N) \) a bounded sequence such that \( u_n \to u \) pointwise almost everywhere in \( \mathbb{R}^N \). If there exists \( C > 0 \) such that

\[
|f(t)| \leq C \left( |t| + |t|^{2^*-1} \right),
\]

then

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |F(u_n) - F(u_n - u) - F(u)| \, dx = 0
\]

**Proof.** Let \( a, b \in \mathbb{R} \) and \( \varepsilon > 0 \). We compute

\[
|F(a + b) - F(a)| = \left| \int_0^1 \frac{d}{d\tau} F(a + \tau b) \, d\tau \right|
\]

\[
= \left| \int_0^1 F'(a + \tau b) b \, d\tau \right|
\]

\[
\leq C \int_0^1 \left( |a + \tau b| + |a + \tau b|^{2^*-1} \right) |b| \, d\tau
\]

\[
\leq C \left( |a| + |b| + 2^{2^*-1} \left( |a|^{2^*-1} + |b|^{2^*-1} \right) \right) |b|
\]

\[
\leq C \left( |a| + |b| + 2^{2^*} \left( |a|^{2^*-1} + |b|^{2^*-1} \right) \right) |b|
\]

\[
\leq C \left( |ab| + b^2 + 2^{2^*} \left( |a|^{2^*-1} + |b|^{2^*-1} \right) \right).
\]

We have used that \( \tau \leq 1 \) and the convexity inequality

\[
|a + b|^{2^*-1} \leq 2^{2^*-1} \left( |a|^{2^*-1} + |b|^{2^*-1} \right).
\]
Now we use Young’s inequality twice:

\[ |ab| \leq \varepsilon \frac{a^2}{2} + \frac{1}{2\varepsilon} |b|^2 \]

\[ |a|^{2s-1}|b| \leq \eta \frac{2s}{2s-1} \left( \frac{|a|^{2s}}{2s} + \frac{1}{2s} |b|^{2s} \right) \]

Hence, choosing

\[ \eta = \varepsilon \frac{2s-1}{2s} \]

we get

\[ |ab| + b^2 + 2^{2s} \left( |a|^{2s-1}|b| + |b|^{2s} \right) \leq \varepsilon \frac{a^2}{2} + \frac{1}{2\varepsilon} b^2 + b^2 + 2^{2s} \left( |a|^{2s-1}|b| + |b|^{2s} \right) \]

\[ \leq \varepsilon C \left( a^2 + 2a^{2s} \right) + C \left[ \left( 1 + \varepsilon^{-1} \right) b^2 + \left( 1 + \varepsilon^{-2} \right) |b|^{2s} \right] \]

\[ = \varepsilon \varphi(a) + \psi(b). \]

Applying \([4, \text{Theorem 2}]\) with \( g_n = u_n - u \) and \( f = u \) we have the assertion. \( \square \)

3 Behavior of the map \( m \mapsto E_m \)

Under our standing assumptions \((f_0)-(f_4)\), for every \( m > 0 \) we can define the least level of energy

\[ E_m = \inf_{u \in P_m} P(u). \]

This section is devoted to the analysis of the quantity \( E_m \) as a function of \( m > 0 \).

Lemma 3.1. If \((f_0)-(f_4)\) hold true, then \( m \mapsto E_m \) is continuous.

Proof. Let \( m > 0 \) and \((m_k)_k \subset \mathbb{R}\) such that \( m_k \to m \) in \( \mathbb{R} \). We want to show that \( E_{m_k} \to E_m \) as \( k \to +\infty \). Firstly, we will prove that

\[ \limsup_{k \to +\infty} E_{m_k} \leq E_m. \tag{3.1} \]

For any \( u \in P_m \) we define

\[ u_k := \sqrt{\frac{m_k}{m}} \in S_{m_k}, \quad k \in \mathbb{N}. \]

It is easy to see that \( u_k \to u \) in \( H^s(\mathbb{R}^N) \), thus, by Lemma 2.6 \((iii)\) we get

\[ \lim_{k \to +\infty} \rho(u_k) = \rho(u) = 0. \]

Therefore

\[ \rho(u_k) \ast u_k \to \rho(u) \ast u = u \quad \text{in} \ H^s(\mathbb{R}^N) \]

as \( k \to +\infty \) and as a consequence

\[ \limsup_{k \to +\infty} E_{m_k} \leq \limsup_{k \to +\infty} I(\rho(u_k) \ast u_k) = I(u). \]

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Since this holds for any $u$, we obtain (3.1). The next step consists in proving
\[
\liminf_{k \to +\infty} E_{m_k} \geq E_m. \tag{3.2}
\]
By definition of infimum, for every $k \in \mathbb{N}$ there exists $v_k \in \mathcal{P}_{m_k}$ such that
\[
I(v_k) \leq E_{m_k} + \frac{1}{k}. \tag{3.3}
\]
We set
\[
t_k := \left( \frac{m}{m_k} \right)^{\frac{1}{N}} \quad \text{and} \quad \tilde{v}_k := v_k \left( \frac{1}{t_k} \right) \in S_m.
\]
By Lemma 2.6 and (3.3) we get
\[
E_m \leq I(\rho(\tilde{v}_k) * \tilde{v}_k) \leq I(\rho(v_k) * \tilde{v}_k) + |I(\rho(\tilde{v}_k) * \tilde{v}_k) - I(\rho(\tilde{v}_k) * v_k)|
\leq I(v_k) + |I(\rho(\tilde{v}_k) * \tilde{v}_k) - I(\rho(\tilde{v}_k) * v_k)|
\leq E_{m_k} + \frac{1}{k} + |I(\rho(\tilde{v}_k) * \tilde{v}_k) - I(\rho(\tilde{v}_k) * v_k)|
=: E_{m_k} + \frac{1}{k} + C(k).
\]
In order to prove (3.2) we show that
\[
\lim_{k \to +\infty} C(k) = 0. \tag{3.4}
\]
Indeed, as a first step we notice that $\rho * (v(\overline{1})) = (\rho * v)(\overline{1})$, and after a change of variable we get
\[
C(k) = \left| \frac{1}{2} \left( t_k^{N-2s} - 1 \right) [\rho(\tilde{v}_k) * v_k]^2_{H^s(\mathbb{R}^N)} - (t_k^N - 1) \int_{\mathbb{R}^N} F(\rho(\tilde{v}_k) * v_k) \, dx \right|
\leq \frac{1}{2} \left( t_k^{N-2s} - 1 \right) |[\rho(\tilde{v}_k) * v_k]^2_{H^s(\mathbb{R}^N)}| + (t_k^N - 1) \int_{\mathbb{R}^N} |F(\rho(\tilde{v}_k) * v_k)| \, dx
=: \frac{1}{2} \left( t_k^{N-2s} - 1 \right) A(k) + (t_k^N - 1) B(k).
\]
Since $t_k \to 1$ as $k \to +\infty$, it suffices to prove that
\[
\limsup_{k \to +\infty} A(k) < \infty, \quad \limsup_{k \to +\infty} B(k) < \infty. \tag{3.5}
\]
We divide the proof of (3.5) in three claims.

Claim 1: $(v_k)_k$ is bounded in $H^s(\mathbb{R}^N)$.
Recalling (3.1) and (3.3) we have that
\[
\limsup_{k \to +\infty} I(v_k) \leq E_m.
\]
Thus, observing that $v_k \in \mathcal{P}_{m_k}$ and $m_k \to m$ if the claim does not hold, we obtain a contradiction with Lemma 2.6 (iv).

Claim 2: $(\tilde{v}_k)_k$ is bounded in $H^s(\mathbb{R}^N)$, and there are a sequence $(y_k)_k \subset \mathbb{R}$ and $v \in H^s(\mathbb{R}^N) \setminus \{0\}$ such that $\tilde{v}(\cdot + y_k) \to v$ a.e. in $\mathbb{R}^N$ up to a subsequence.
To see the boundedness of $(\tilde{v}_k)_k$ it suffices to notice that $t_k \to 1$ and the statement follows by claim 1. Now, we set
\[
\alpha = \limsup_{k \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |\tilde{v}_k|^2 \, dx.
\]
If $\alpha = 0$, by [16, Lemma II.4] we get $\tilde{v}_k \to 0$ in $L^{2+\frac{4}{N}}(\mathbb{R}^N)$. As a consequence we have that
\[
\int_{\mathbb{R}^N} |v_k|^{2+\frac{4}{N}} \, dx = \int_{\mathbb{R}^N} |\tilde{v}_k(t_k \cdot)|^{2+\frac{4}{N}} \, dx = t_k^{-N} \int_{\mathbb{R}^N} |\tilde{v}_k|^{2+\frac{4}{N}} \, dx \to 0
\]
as $k \to +\infty$, and since $P(v_k) = 0$, by Lemma 2.3 (i), we deduce that
\[
[v_k]^2_{H^s(\mathbb{R}^N)} = \frac{N}{2s} \int_{\mathbb{R}^N} F(v_k) \, dx \to 0.
\]
In this case, by virtue of Remark 2.2, we see that
\[
0 = P(v_k) \geq [v_k]^2_{H^s(\mathbb{R}^N)},
\]
which is admissible only if $v_k$ is constant. But this is in contradiction with the fact that $v_k \in \mathcal{P}_{m_k}$. Hence $\alpha$ must be strictly positive.

**Claim 3**: $\limsup_{k \to +\infty} \rho(\tilde{v}_k) < \infty$.

By contradiction we assume that up to a subsequence $\rho(\tilde{v}_k) \to \infty$ as $k \to +\infty$. By Claim 2 we can suppose the existence of a sequence $(y_k)_k \subset \mathbb{R}^N$ and $v \in H^s(\mathbb{R}^N) \setminus \{0\}$ such that
\[
\tilde{v}_k(\cdot + y_k) \to v \quad \text{a.e. in } \mathbb{R}^N. \tag{3.6}
\]
Instead, by Lemma 2.6 we get
\[
\rho(\tilde{v}_k(\cdot + y_k)) = \rho(\tilde{v}_k) \to \infty \tag{3.7}
\]
and
\[
I(\rho(\tilde{v}_k(\cdot + y_k)) \ast \tilde{v}_k(\cdot + y_k)) \geq 0. \tag{3.8}
\]
Now, taking into account (3.6), (3.7), (3.8) and arguing similarly as we have already done to prove (2.6) we have a contradiction. The proof concludes observing that by Claims 1 and 3
\[
\limsup_{k \to +\infty} \|\rho(\tilde{v}_k) \ast v_k\|_{H^s(\mathbb{R}^N)} < \infty. \tag{3.9}
\]
Hence, by virtue of $(f_0) - (f_2)$ and (3.9), (3.5) holds true. \hfill \square

The next result provides a weak monotonicity property for $E_m$.

**Lemma 3.2.** If $(f_0) - (f_4)$ hold, then $m \mapsto E_m$ is non-increasing in $(0, \infty)$.

**Proof.** It suffices to show that for every $\varepsilon > 0$ and $m$, $m' > 0$ with $m > m'$ we have
\[
E_m \leq E_{m'} + \frac{\varepsilon}{2}. \tag{3.10}
\]
Now, we take $\chi \in C^\infty_c(\mathbb{R}^N)$ radial such that

$$
\chi(x) = \begin{cases} 
1 & |x| \leq 1 \\
[0, 1) & 1 < |x| \leq 2 \\
0 & |x| > 2 
\end{cases}
$$

and for every $\delta > 0$ we set $u_\delta(x) = u(x)\chi(\delta x)$. By a result of Palatucci et al., see [14, Lemma 5 of Section 6.1], we know that $u_\delta \to u$ as $\delta \to 0^+$, and using Lemma 2.6 (iii) we obtain

$$
\lim_{\delta \to 0^+} \rho(u_\delta) = \rho(u) = 0.
$$

As a consequence of that, we obtain

$$
\rho(u_\delta) * u_\delta \to \rho(u) * u \quad \text{in } H^s(\mathbb{R}^N) \tag{3.11}
$$

as $\delta \to 0^+$. Now, fixing $\delta > 0$ small enough, by virtue of (3.11) we have

$$
I(\rho(u_\delta) * u_\delta) \leq I(u) + \frac{\varepsilon}{4}. \tag{3.12}
$$

After that, we choose $v \in C^\infty_c(\mathbb{R}^N)$ with $\text{supp}(v) \subset B(0, 1 + \frac{4}{\delta}) \setminus B(0, \frac{4}{\delta})$ and we set

$$
\tilde{v} = \frac{m - \|u_\delta\|_{L^2(\mathbb{R}^N)}}{\|v\|_{L^2(\mathbb{R}^N)}}
$$

For every $\lambda \leq 0$ we also define $\omega_\lambda = u_\delta + \lambda * \tilde{v}$. We observe that choosing $\lambda$ appropriately we have

$$
\text{supp}(u_\delta) \cap \text{supp}(\lambda * \tilde{v}) = \emptyset
$$

thus $\omega_\lambda \in S_m$.

**Claim:** $\rho(\omega_\lambda)$ is upper bounded as $\lambda \to -\infty$.

If the claim does not hold we observe that by lemma 2.6 (ii) $I(\rho(\omega_\lambda) * \omega_\lambda) \geq 0$ and that $\omega_\lambda \to u_\delta$ a.e. in $\mathbb{R}^N$ as $\lambda \to -\infty$. Hence, arguing as we have already done to obtain (2.6) we reach a contradiction. Then the claim must hold.

By virtue of the claim

$$
\rho(\omega_\lambda) + \lambda \to -\infty \quad \text{as } \lambda \to -\infty,
$$

thus

$$
[(\rho(\omega_\lambda) + \lambda) * \tilde{v}]_{H^s(\mathbb{R}^N)}^2 = e^{2s(\rho(\omega_\lambda) + \lambda)} [\tilde{v}]_{H^s(\mathbb{R}^N)}^2 \to 0
$$

implying

$$
\|\rho(\omega_\lambda) + \lambda) * \tilde{v}\|_{L^{2s}([0, 4\delta)](\mathbb{R}^N)} \leq C\|\rho(\omega_\lambda) + \lambda) * \tilde{v}\|_{L^2(\mathbb{R}^N)} \|[(\rho(\omega_\lambda) + \lambda) * \tilde{v}]_{H^s(\mathbb{R}^N)} \to 0.
$$

As a consequence, by Lemma 2.1 (ii), for a suitable $\lambda$

$$
I([(\rho(\omega_\lambda) + \lambda) * \tilde{v}) \leq \frac{\varepsilon}{4}, \tag{3.13}
$$
Finally, by Lemma 2.6 and using (3.10), (3.12) and (3.13) it easy to see that
\[
E_m \leq I(\rho(\omega_\lambda) * \omega_\lambda) = I(\rho(\omega_\lambda) * u_\delta) + I(\rho(\omega_\lambda) * (\lambda * \tilde{v}))
\]
\[
\leq I(\rho(u_\delta) * u_\delta) + I((\rho(\omega_\lambda) + \lambda) * \tilde{v})
\]
\[
\leq I(u) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq E_{m'} + \varepsilon
\]
completing the proof.

The strict monotonicity of $E_m$ holds true only locally, as we now show.

**Lemma 3.3.** Assume $(f_0) - (f_4)$ hold true. Moreover, let $u \in S_m$ and $\mu \in \mathbb{R}$ such that
\[
(-\Delta)^s + \mu u = f(u)
\]
and $I(u) = E_m$. Then $E_m > E_{m'}$ for every $m' > m$ close enough if $\mu > 0$ and for any $m' < m$ close enough if $\mu < 0$.

**Proof.** Let $t > 0$ and $\rho \in \mathbb{R}$. Defining $u_{t,\rho} := u(\rho * (tu)) \in S_{m't}$ and
\[
\alpha(t, \rho) := I(u_{t,\rho}) = \frac{1}{2} \int_{\mathbb{R}^N} e^{2ps} |u|_{H^s(\mathbb{R}^N)}^2 - e^{-N\rho} \int_{\mathbb{R}^N} \left( t e^\frac{N\rho}{2} u \right) \cdot e^\frac{N\rho}{2} u \, dx
\]
it is straightforward to verify that
\[
\frac{\partial}{\partial t} \alpha(t, \rho) = te^{2ps} |u|_{H^s(\mathbb{R}^N)}^2 - e^{-N\rho} \int_{\mathbb{R}^N} f \left( t e^\frac{N\rho}{2} u \right) e^\frac{N\rho}{2} u \, dx
\]
\[
= t^{-1} I'(u_{t,\rho}) [u_{t,\rho}].
\]
In the case $\mu > 0$, we observe that $u_{t,\rho} \to u$ in $H^s(\mathbb{R}^N)$ as $(t, \rho) \to (1, 0)$. Moreover, we notice that
\[
I'(u) [u] = -\mu \|u\|_{L^2(\mathbb{R}^N)}^2 = -\mu m < 0
\]
and so, choosing $\delta > 0$ small enough we have
\[
\frac{\partial \alpha}{\partial t}(t, \rho) < 0 \quad \text{for any } (t, \rho) \in (1, 1 + \delta) \times [-\delta, \delta].
\]
Using the Mean Value Theorem, there exists $\xi \in (1, t)$ such that
\[
\frac{\partial \alpha}{\partial t}(\xi, \rho) = \frac{\alpha(t, \rho) - \alpha(1, \rho)}{t - 1}
\]
whenever $(t, \rho) \in (1, 1 + \delta) \times [-\delta, \delta]$, hence
\[
\alpha(t, \rho) = \alpha(1, \rho) + (t - 1) \frac{\partial \alpha}{\partial t}(\xi, \rho) < \alpha(1, \rho).
\]
(3.14)
Since by Lemma 2.6 (iii) $\rho(tu) \to \rho(u) = 0$ as $t \to 1^+$, setting for any $m' > m$ close enough to $m$
\[
t := \sqrt{\frac{m'}{m}} \in (1, 1 + \delta) \quad \text{and} \quad \rho := \rho(tu) \in [-\delta, \delta],
\]
and using (3.14) together with Lemma 2.6 (ii) we obtain that
\[
E_m \leq \alpha(t, \rho(tu)) < \alpha(1, \rho(tu)) = I(\rho(tu) * u) \leq I(u) = E_m.
\]
The proof for $\mu < 0$ is similar, and we omit it. □
As a direct consequence of the previous two lemmas we have the following result.

**Lemma 3.4.** Assume \((f_0) - (f_4)\) hold true. In addition let \(u \in S_m\) and \(\mu \in \mathbb{R}\) such that

\[
(-\Delta)^s u + \mu u = f(u)
\]

with \(I(u) = E_m\). Then \(\mu \geq 0\) and if \(\mu > 0\) it is \(E_m > E_{m'}\) for any \(m' > m > 0\).

To make a step ahead, we describe the asymptotic behaviour of \(E_m\) as \(m \to 0^+\) and and \(m \to +\infty\).

**Lemma 3.5.** Assume \((f_0) - (f_4)\) hold true, then \(E_m \to +\infty\) as \(m \to 0^+\).

**Proof.** In order to prove the Lemma, we will show that for every sequence \((u_n)\) such that

\[
\text{Assume Lemma 3.5.}
\]

it must be \(I(u_n) \to +\infty\). We set

\[
\rho_n := \frac{1}{s} \log \left( \left[ u_n \right]_{H^s(\mathbb{R}^N)} \right) \quad \text{and} \quad v_n := (-\rho_n) \ast u_n
\]

Trivially \([v_n]_{H^s(\mathbb{R}^N)} = 1\) and \(||v_n||_{L^2(\mathbb{R}^N)} \to 0\). Moreover, thanks to these two facts we also have by interpolation that \(v_n \to 0\) in \(L^{2^*_s}(\mathbb{R}^N)\), thus, by Lemma 2.1 (ii) we have

\[
\lim_{n \to +\infty} e^{-N\rho} \int_{\mathbb{R}^N} F \left( e^{\frac{N}{2}s} v_n \right) \, dx = 0.
\]

Since \(P(\rho_n \ast v_n) = P(u_n) = 0\), using Lemma 2.6 (i) and (ii) we obtain that

\[
I(u_n) = I(\rho_n \ast v_n) \geq I(\rho \ast v_n) = \frac{1}{2} e^{2\rho} - e^{N\rho} \int_{\mathbb{R}^N} F \left( e^{\frac{N}{2}s} v_n \right) \, dx
\]

\[
= \frac{1}{2} e^{2\rho} + o_n(1).
\]

Since \(\rho\) is arbitrary, we get the statement as \(\rho \to +\infty\). \(\square\)

**Lemma 3.6.** Assume \((f_0) - (f_4)\) and \((f_6)\). Then \(E_m \to 0\) as \(m \to +\infty\).

**Proof.** We fix \(u \in L^\infty(\mathbb{R}^N) \cap S_1\) and we set \(u_m = \sqrt{m} u \in S_m\). By Lemma 2.6 (ii) we can find a unique \(\rho(m) \in \mathbb{R}\) such that \(\rho(m) \ast u_m \in P_m\). Since by Lemma 2.5 (i) \(F\) is non negative, we get

\[
0 < E_m \leq I(\rho(m) \ast u_m) \leq \frac{1}{2} e^{2\rho(m)s} \left[ u \right]_{H^s(\mathbb{R}^N)}^2.
\]

Thus, by (3.15) it suffices to show that

\[
\lim_{m \to +\infty} \sqrt{m} e^{\rho(m)s} = 0.
\]

Recalling the function \(g\) defined in Remark 2.4, and recalling that \(P(\rho(m) \ast u_m) = 0\) we get

\[
\left[ u \right]_{H^s(\mathbb{R}^N)}^2 = \frac{N}{2s} \int_{\mathbb{R}^N} g \left( \sqrt{m} e^{\frac{N}{2}s} u \right) \left| u \right|^{2^*_s} \, dx,
\]

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that implies
\[
\lim_{m \to \infty} \sqrt{me^{N\rho(u)}} = 0. \tag{3.17}
\]
Now, using \((f_6)\) for any \(\varepsilon > 0\) we can find \(\delta > 0\) such that
\[
\tilde{F}(t) \geq \frac{4s}{N} F(t) \geq \frac{1}{\varepsilon} |t|^N_{s-2s}
\]
if \(|t| \leq \delta\). Hence, taking into account the fact that \(P(\rho(m) * u_m) = 0\) and \((3.17)\), we get
\[
[u]^2_{H^s(\mathbb{R}^N)} = \frac{N}{2s} e^{-(N+2s)\rho(m)} \int_{\mathbb{R}^N} \tilde{F} \left( \sqrt{me^{N\rho(m)}} u \right) dx \geq \frac{N}{2s} \varepsilon \left( \sqrt{me^{\rho(m)}} \right) \int_{\mathbb{R}^N} \tilde{F} \left( \sqrt{me^{\rho(m)}} u \right) dx
\]
for \(m\) large enough. Then \((3.16)\) holds.

\[\Box\]

4 Ground states

We introduce the restricted functional
\[
\Psi(u) = I(\rho(u) * u) = \frac{1}{2} e^{2\rho(u)s} [u]^2_{H^s(\mathbb{R}^N)} - e^{-N\rho(u)} \int_{\mathbb{R}^N} F \left( e^{-N\rho(u)} u \right) dx.
\]

**Lemma 4.1.** \(\Psi : H^s(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}\) is of class \(C^1\), and
\[
d\Psi(u) [\varphi] = dI(\rho(u) * u) [\rho(u) * \varphi]
\]
for every \(u \in H^s(\mathbb{R}^N) \setminus \{0\}\) and \(\varphi \in H^s(\mathbb{R}^N)\).

**Proof.** A proof appears in [9] for the case \(s = 1\). Since only minor adjustments are needed in the fractional case, we omit the details.

Let \(m > 0\), we consider the constrained functional \(J : S_m \to \mathbb{R}\), where \(J = \Psi|_{S_m}\). Lemma 4.1 implies immediately the following statement.

**Lemma 4.2.** The functional \(J : S_m \to \mathbb{R}\) is \(C^1\) and
\[
dJ(u) [\varphi] = d\Psi(u) [\varphi] = dI(\rho(u) * u) [\rho(u) * \varphi]
\]
for any \(u \in S_m\) and \(\varphi \in T_u S_m\), where \(T_u S_m\) is the tangent space at \(u\) to the manifold \(S_m\).

We recall from [8, Definition 3.1] a definition that will be useful to construct a min-max principle.

**Definition 4.3.** Let \(B\) be a closed subset of a metric space \(X\). We say that a class \(\mathcal{G}\) of compact subsets of \(X\) is a homotopy stable family with closed boundary \(B\) provided

(i) every set in \(\mathcal{G}\) contains \(B\),

(ii) for any set \(A\) in \(\mathcal{G}\) and any homotopy \(\eta \in C([0, 1] \times X, X)\) that satisfies \(\eta(t, u) = u\) for all \((t, u) \in ([0] \times X) \cup ([0, 1] \times B)\), one has \(\eta([1] \times A) \in \mathcal{G}\).

We remark that \(B = \emptyset\) is admissible.
Lemma 4.4. Let $G$ be a homotopy stable family of compact subset with (with $B = \emptyset$). We set
\[
E_{m,G} = \inf_{A \in G} \max_{u \in A} J(u).
\]
If $E_{m,G} > 0$, then there exists a Palais-Smale sequence $(u_n)_n \in \mathcal{P}_m$ for the constrained functional $I|_{S_m}$ at level $E_{m,G}$. In particular, if $G$ is the class of all singletons in $S_m$, one has that $\|u_n\|_{L^2(\mathbb{R}^N)} \to 0$ as $n \to +\infty$.

**Proof.** Let $(A_n)_n \subset G$ be a minimizing sequence of $E_{m,G}$. We define the map
\[
\eta: [0,1] \times S_m \to S_m
\]
where $\eta(t,u) = (t \rho(u)) * u$ is continuous and well defined by lemma 2.6 (ii) and (iii). Noticing $\eta(t,u) = u$ for every $(t,u) \in \{0\} \times S_m$ we obtain that
\[
D_n := \eta(1, A_n) = \{\rho(u) * u \mid u \in A_n\} \in G.
\]
In particular we can see that $D_n \subset \mathcal{P}_m$ for any $m > 0$, with $m > 0$. Since $J(\rho(u) * u) = J(u)$ for every $\rho \in \mathbb{R}$ and $u \in S_m$, we can observe that
\[
\max_{u \in D_n} J(u) = \max_{u \in A_n} J(u) \to E_{m,G}
\]
thus, $(D_n)_n$ is another minimizing sequence for $E_{m,G}$. Now, using [8, Theorem 3.2] we get a Palais-Smale sequence $(v_n)_n \subset S_m$ for $J$ at level $E_{m,G}$ such that $\text{dist}_{H^s(\mathbb{R}^N)}(v_n, D_n) \to 0$ as $n \to +\infty$. We will denote
\[
\rho_n := \rho(v_n) \quad \text{and} \quad u_n := \rho_n * v_n.
\]
**Claim:** There exists $C > 0$ such that $e^{-2\rho_n s} \leq C$ for any $n \in \mathbb{N}$.

We start pointing out that
\[
e^{-2\rho_n s} = \frac{[v_n]_{H^s(\mathbb{R}^N)}^2}{[u_n]_{H^s(\mathbb{R}^N)}^2}.
\]
By virtue of the fact that $(u_n)_n \subset \mathcal{P}_m$, using lemma 2.7 (ii) we obtain that $\left\{[u_n]_{H^s(\mathbb{R}^N)}\right\}_n$ is bounded from below. Moreover, since $D_n \subset \mathcal{P}_m$ and the fact that
\[
\max_{u \in D_n} I = \max_{u \in D_n} J \to E_{m,G},
\]
Lemma 2.7 (iv) implies that $D_n$ is uniformly bounded in $H^s(\mathbb{R}^N)$. Finally, from $\text{dist}(v_n, D_n) \to 0$ we can deduce that $\sup_{n \in \mathbb{N}} [v_n]_{H^s(\mathbb{R}^N)} < \infty$. Thus the claim holds.

Now, from $(u_n) \subset \mathcal{P}_m$ we get
\[
I(u_n) = J(u_n) = J(v_n) \to E_{m,G}.
\]
Instead, for any $\psi \in T_{u_n} S_m$ we have
\[
\int_{\mathbb{R}^N} v_n \left[(-\rho_n) * \psi\right] dx = \int_{\mathbb{R}^N} v_n e^{-\frac{\rho_n u_n}{2}} \psi(e^{-\rho_n x}) dx = \int_{\mathbb{R}^N} e^{-\frac{\rho_n u_n}{2}} v_n (e^{\rho_n x}) \psi dx
\]
\[
= \int_{\mathbb{R}^N} (\rho_n * v_n) \psi dx = \int_{\mathbb{R}^N} u_n \psi dx = 0
\]
implying \((-\rho_n \ast \psi) \in T_{v_n}S_m\). Besides, by the claim
\[
\|(-\rho_n) \ast v_n\|_{H^s(\mathbb{R}^N)} \leq \max\{C, 1\}\|\psi\|_{H^s(\mathbb{R}^N)}.
\]

Denoting with \(\| \cdot \|_{u,s}\) the dual norm of the space \((T_uS_m)^*\) and using Lemma 2.8 we get
\[
\|dI(u_n)\|_{u_n,*} = \sup_{\psi \in T_{u_n}S_m} |dI(u_n) [\psi]| = \sup_{\psi \in T_{u_n}S_m} |dI(\rho_n \ast v_n) [\rho_n \ast ((-\rho_n) \ast \psi)]| = \sup_{\psi \in T_{u_n}S_m} |dJ(v_n) [(-\rho_n) \ast \psi]| \\
\leq \|dJ(v_n)\|_{v_n,*} \sup_{\psi \in T_{u_n}S_m} \|((-\rho_n) \ast \psi)\|_{H^s(\mathbb{R}^N)} \leq \|dJ(v_n)\|_{v_n,*} \rightarrow 0
\]
as \(n \rightarrow +\infty\) remembering that \((v_n)_n\) is a Palais-Smale sequence for the functional \(J\). We have just proved \((u_n)_n\) is a Palais-Smale sequence for the functional \(I|_{S_m}\) at level \(E_m,G\) with the additional property that \((u_n)_n \subset P_m\). Finally, noticing that the family of singleton of \(S_m\) is a particular homotopy stable family of compact subsets of \(S_m\), and doing this particular choice as \(G\), arguing similarly as we have just done, we can obtain a minimizing sequence \((D_n)_n\) with the additional property that its elements are non negative (up to replacing the functions with their absolute value). Moreover, \((A_n)_n\) will inherit this property, and as a consequence of that, recalling that \(\text{dist}(v_n,D_n) \rightarrow 0\) as \(n \rightarrow +\infty\) we have
\[
\|u_n^-\|_{L^2(\mathbb{R}^N)} = \|\rho_n \ast v_n^-\|_{L^2(\mathbb{R}^N)} = \|v_n^-\|_{L^2(\mathbb{R}^N)} \rightarrow 0.
\]
This concludes the proof of the lemma. \(\square\)

** Lemma 4.5.** We assume \((f_0) - (f_5)\) hold. Then there exists a Palais-Smale sequence \((u_n)_n \subset P_m\) for the constrained functional \(I|_{S_m}\) at level \(E_m\) such that \(\|u_n^-\|_{L^1(\mathbb{R}^N)} \rightarrow 0\) as \(n \rightarrow +\infty\).

**Proof.** We apply lemma 4.4 with \(G\) the class of all singletons in \(S_m\). Lemma 2.7 imply that \(E_m > 0\), thus the only thing it remains to prove is \(E_m = E_m,G\). In order to do that, as a first step we notice that
\[
E_m,G = \inf_{A \in G} \max_{u \in A} J(u) = \inf_{u \in S_m} I(\rho(u) \ast u).
\]

Since for every \(u \in S_m\) we have that \(\rho(u) \ast u \in P_m\) it must be \(I(\rho(u) \ast u) \geq E_m\), thus \(E_m,G \geq E_m\). On the other hand, if \(u \in P_m\) we have \(\rho(u) = 0\) and \(I(u) \geq E_m,G\), that implies \(E_m \geq E_m,G\). \(\square\)

** Lemma 4.6.** Let \((u_n)_n \subset S_m\) be a bounded Palais-Smale sequence for the constrained functional \(I|_{S_m}\) at level \(E_m > 0\) such that \(P(u_n) \rightarrow 0\) as \(n \rightarrow +\infty\). Then we have the existence of \(u \in S_m\) and \(\mu > 0\) such that, up to a subsequence and translations in \(\mathbb{R}^N\), \(u_n \rightarrow u\) strongly in \(H^s(\mathbb{R}^N)\) and
\[
(-\Delta)^s u + \mu u = f(u).
\]
Proof. It is clear that \((u_n)_n \subset S_m\) is bounded in \(H^s(\mathbb{R}^N)\) and is a Palais-Smale sequence. Together, these two facts enable us to assume without loss of generality that \(\lim_{n \to +\infty} [u_n]_{H^s(\mathbb{R}^N)}\), \(\lim_{n \to +\infty} \int_{\mathbb{R}^N} F(u_n) \, dx\), and \(\lim_{n \to +\infty} \int_{\mathbb{R}^N} f(u_n) u_n \, dx\) exist. Besides, \([3, \text{Lemma 3}]\) implies
\[
(-\Delta)^s u_n + \mu_n u_n - f(u_n) \to 0 \quad \text{in } H^s(\mathbb{R}^N)^*,
\]
where we denoted
\[
\mu_n = \frac{1}{m} \left( \int_{\mathbb{R}^N} f(u_n) u_n \, dx - [u_n]_{H^s(\mathbb{R}^N)}^2 \right).
\]
By the assumptions done above we can see that \(\mu_n \to \mu\) for some \(\mu \in \mathbb{R}\) and we also have that for any \((y_n)_n \subset \mathbb{R}^N\)
\[
(-\Delta)^s u_n (\cdot + y_n) + \mu u_n (\cdot + y_n) - f(u_n (\cdot + y_n)) \to 0 \quad \text{in } H^s(\mathbb{R}^N)^*.
\]
Claim: \((u_n)_n\) is non vanishing.

Otherwise by \([16, \text{Lemma II.4}]\) we would get \(u_n \to 0\) in \(L^{2+\frac{4s}{N}}(\mathbb{R}^N)\). Taking into account that \(P(u_n) \to 0\) and using lemma \(2.1\) \((ii)\) we get
\[
[u_n]_{H^s(\mathbb{R}^N)}^2 = P(u_n) + \frac{N}{2s} \int_{\mathbb{R}^N} \tilde{F}(u_n) \, dx \to 0
\]
and as a consequence of that,
\[
E_m = \lim_{n \to +\infty} I(u_n) = \frac{1}{2} \lim_{n \to +\infty} [u_n]_{H^s(\mathbb{R}^N)}^2 - \lim_{n \to +\infty} \int_{\mathbb{R}^N} F(u_n) \, dx
\]
contradicting \(E_m > 0\). Then the claim must hold.

Since \((u_n)_n\) in non vanishing we can find \((y^1_n)_n \subset \mathbb{R}^N\) and \(\omega_1 \in B_m \setminus \{0\}\) such that \(u_n (\cdot + y^1_n) \rightharpoonup \omega_1\)
in \(H^s(\mathbb{R}^N)\), \(u_n (\cdot + y^1_n) \rightharpoonup \omega_1\) in \(L^p(\mathbb{R}^N)\) for \(p \in [1, 2^*_s]\) and \(u_n (\cdot + y^1_n) \to \omega\) a.e. in \(\mathbb{R}^N\). Now, we want to apply \([2, \text{Lemma A.1}]\) with \(P(t) = f(t)\) and \(Q(t) = |t|^{(N+2s)/(N-2s)}\) and we notice that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} |[f(u_n (\cdot + y^1_n) - f(\omega_1)) \varphi| \, dx
\]
\[
\leq \|\varphi\|_{L^\infty(\mathbb{R}^N)} \lim_{n \to +\infty} \int_{\text{supp}(\varphi)} |f(u_n (\cdot + y^1_n) - f(\omega_1))| \, dx \quad \text{(4.2)}
\]
for any \(\varphi \in C^\infty_c(\mathbb{R}^N)\). Hence, by (4.1) and (4.2) we get
\[
(-\Delta)^s \omega_1 + \mu_1 \omega_1 = f(\omega_1)
\]
and through the Pohozaev Identity (see for instance \([5, \text{Proposition 4.1}]\) associated to (4.3) we also have \(P(\omega_1) = 0\). Now, we set \(v^1_n := u_n - \omega_1 (\cdot - y^1_n)\) for every \(n \in \mathbb{N}\). Clearly \(v^1_n (\cdot + y^1_n) = u_n (\cdot + y^1_n) - \omega_1 \to 0\) in \(H^s(\mathbb{R}^N)\), thus
\[
m = \lim_{n \to +\infty} \|u_n (\cdot + y^1_n)\|_{L^2(\mathbb{R}^N)} = \lim_{n \to +\infty} \|v^1_n\|_{L^2(\mathbb{R}^N)}^2 + \|\omega_1\|_{L^2(\mathbb{R}^N)}^2.
\]
By lemma \(2.8\) we also have
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} F(u_n (\cdot + y^1_n)) \, dx = \int_{\mathbb{R}^N} F(\omega_1) \, dx + \lim_{n \to +\infty} \int_{\mathbb{R}^N} F(v^1_n (\cdot + y^1_n)) \, dx
\]
hence
\[ E_m = \lim_{n \to +\infty} I(u_n) = \lim_{n \to +\infty} I(u_n(\cdot + y_n^1)) = \lim_{n \to +\infty} I(v_n(\cdot + y_n^1)) + I(\omega_1) \] (4.5)
\[ = \lim_{n \to +\infty} I(v_n^1) + I(\omega_1). \]

Claim: \( \lim_{n \to +\infty} I(v_n^1) \geq 0. \)
If the claim does not hold, i.e \( \lim_{n \to +\infty} I(v_n^1) < 0, \) \((v_n^1)\) is non vanishing, then there exists \((y_n^2)\) such that
\[ \lim_{n \to +\infty} \int_{B(y_n^2,1)} |v_n^1|^2 > 0. \]
Since \( v_n^1(\cdot + y_n^1) \to 0 \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \), it must be \( |y_n^2 - y_n^1| \to \infty \), and up to a subsequence \( v_n^1(\cdot + y_n^2) \to \omega_2 \) in \( H^s(\mathbb{R}^N) \) for some \( \omega_2 \in B_m \setminus \{0\} \). We notice
\[ u_n(\cdot + y_n^1) = v_n^1(\cdot + y_n^2) + \omega_1(\cdot - y_n^1 + y_n^2) \to \omega_2 \]
thus, arguing as before, we get \( P(\omega_2) = 0 \) and \( I(\omega_2) > 0 \). We set
\[ v_n^2 = v_n^1 - \omega^2(\cdot - y_n^2) = u_n - \sum_{\ell=1}^2 \omega_\ell(\cdot - y_\ell_n) \]
and we observe that
\[ \lim_{n \to +\infty} [v_n^1]^2_{H^s(\mathbb{R}^N)} = \lim_{n \to +\infty} [v_n^1]^2_{H^s(\mathbb{R}^N)} + [\omega_2]^2_{H^s(\mathbb{R}^N)} - 2 \lim_{n \to +\infty} \langle v_n^1, \omega_2(\cdot - y_n^2) \rangle_{H^s(\mathbb{R}^N)} \]
\[ = \lim_{n \to +\infty} [v_n^1]^2_{H^s(\mathbb{R}^N)} + [\omega_2]^2_{H^s(\mathbb{R}^N)} - 2 \lim_{n \to +\infty} \langle v_n^1(\cdot + y_n^1), \omega_2 \rangle_{H^s(\mathbb{R}^N)} \]
\[ = \lim_{n \to +\infty} [u_n]^2_{H^s(\mathbb{R}^N)} + [\omega_1]^2_{H^s(\mathbb{R}^N)} - 2 \lim_{n \to +\infty} \langle u_n(\cdot + y_n^1), \omega_1 \rangle_{H^s(\mathbb{R}^N)} \]
\[ - \sum_{\ell=1}^2 [\omega_\ell]^2_{H^s(\mathbb{R}^N)} \]
and
\[ 0 > \lim_{n \to +\infty} I(v_n^1) = I(\omega_2) + \lim_{n \to +\infty} I(v_n^2) > \lim_{n \to +\infty} I(v_n^2). \]
Iterating, we can build an infinite sequence \((\omega_k) \subset B_m \setminus \{0\} \) such that \( P(\omega_k) = 0 \) and
\[ \sum_{\ell=1}^k [\omega_k]^2_{H^s(\mathbb{R}^N)} \leq [u_n]^2_{H^s(\mathbb{R}^N)} < \infty \]
for every \( k \in \mathbb{N} \). Though, this is a contradiction. Indeed, recalling remark 2.2, for any \( \omega \in B_m \setminus \{0\} \) such that \( P(\omega) = 0 \), we can find \( \delta > 0 \) such that \( [\omega]^2_{H^s(\mathbb{R}^N)} \geq \delta \). Hence, the claim must hold and \( \lim_{n \to +\infty} I(v_n^1) \geq 0 \).
Now, we denote with \( h := [\omega_1]^2_{L^2(\mathbb{R}^N)} \in (0, m] \). By virtue of the claim, (4.5) and the fact that \( \omega_1 \in \mathcal{P}_h \), we get
\[ E_m = I(\omega_1) + \lim_{n \to +\infty} I(v_n^1) \geq I(\omega_1)^1 \geq E_h \]

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but, recalling that $E_m$ in non-increasing by lemma 3.2, we obtain

$$I(\omega_1) = E_m = E_h$$

and

$$\lim_{n \to +\infty} I(v^1_n) = 0.$$  \hspace{1cm} (4.7)

To prove that $\mu \geq 0$ it suffices to put together (4.3), (4.6) and Lemma 3.4. Instead, to see that $\mu$ is strictly positive, using (f5), lemma 2.3 and the Pohozaev Identity corresponding to (4.3), we get

$$\mu = \frac{1}{m} \int_{\mathbb{R}^N} \left( \frac{N}{s} F(\omega_1) - \frac{N - 2s}{2} f(\omega_1) \omega_1 \right) dx > 0.$$  \hspace{1cm} (4.8)

At this point, we suppose by contradiction that $h < m$, but taking into account (4.3), (4.8) and Lemma (6.5) we would have

$$I(\omega_1) = E_h > E_m$$

which is not compatible with (4.7). Thus $h = m$. Moreover, by (4.4) $v^1_n \to 0$ in $L^2(\mathbb{R}^N)$. It remains only to prove the strong convergence of $(v^1_n)_n$ in $H^s(\mathbb{R}^N)$. To do that, it is sufficient to notice that by lemma 2.1 (ii) we have $\lim_{n \to +\infty} \int_{\mathbb{R}^N} F(v^1_n) dx$, and so we obtain the assertion thanks to (4.7).

**Proof of theorem 1.3.** Applying lemma 4.5 we obtain a Palais-Smale sequence $(u_n)_n \subset \mathcal{P}_m$ at level $E_m > 0$ for the constrained functional $I|_{S_m}$. This sequence is bounded in $H^s(\mathbb{R}^N)$ by Lemma 2.7 and through Lemma 4.6 we get a critical point $u \in S_m$ at the level $E_m > 0$ that results to be a ground state energy. Finally, since $\|u_n\|_{L^2(\mathbb{R}^N)} \to 0$ we deduce that $u \geq 0$ and after applying the strong maximum principle we obtain $u > 0$.

**Proof of theorem 1.4.** The proof is a direct consequence of Theorem 1.3 and Lemmas 2.7, 3.1, 3.2, 3.5, 3.6.

### 5 Existence of radial solutions

This section is devoted to prove the existence of infinitely many radial solutions to problem $(P_m)$. Before doing this, we recall some basic definitions and we provide some notation.

Denote by $\sigma: H^s(\mathbb{R}^N) \to H^s(\mathbb{R}^N)$ the transformation $\sigma(u) = -u$ and let $X \subset H^s(\mathbb{R}^N)$. A set $A \subset X$ is called $\sigma$-invariant if $\sigma(A) = A$. A homotopy $\eta$: $[0, 1] \times X \to X$ is $\sigma$-equivariant if $\eta(t, \sigma(u)) = \sigma(\eta(t, u))$ for all $(t, u) \in [0, 1] \times X$. Next definition is in [8, Definition 7.1].

**Definition 5.1.** Let $B$ be a closed $\sigma$-invariant subset $X \subset H^s(\mathbb{R}^N)$. We say that a class $\mathcal{G}$ of compact subsets of $X$ is a $\sigma$-homotopy stable family with closed boundary $B$ provided

(i) every set in $\mathcal{G}$ is $\sigma$-invariant.

(ii) every set in $\mathcal{G}$ contains $B$,  

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(iii) for any set $A$ in $\mathcal{G}$ and any $\sigma$-equivariant homotopy $\eta \in C([0,1] \times X, X)$ that satisfies $\eta(t,u) = u$ for all $(t,u) \in \{(0) \times X\} \cup \{(0,1] \times B\}$, one has $\eta(\{1\} \times A) \in \mathcal{G}$.

We denote with $H^s_*(\mathbb{R}^N)$ the space of radially symmetric functions in $H^s(\mathbb{R}^N)$ and recall that $H^s_*(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R})$ compactly for all $p \in (2, 2^*_s)$ (see [11, Proposition I.1]).

In order to prove the main result of this section, we need to build a sequence of $\sigma$-homotopy stable families of compact subsets of $S_m \cap H^s_*(\mathbb{R}^N)$. We point out that in the definition above, the case in which $B = \emptyset$ is not excluded. The idea is borrowed from [9]. Let $(V_k)$ be a sequence of finite dimensional linear subspaces of $H^s_*(\mathbb{R}^N)$ such that $V_k \subset V_{k+1}$, $\dim V_k = k$ and $\bigcap_{k \geq 1} V_k$ is dense in $H^s_*(\mathbb{R}^N)$. Denote by $\pi_k$ the orthogonal projection from $H^s_*(\mathbb{R}^N)$ onto $V_k$. We recall to the reader the definition of the genus of $\sigma$-invariant sets introduced by M. A. Krasnoselskii and we refer to [15, Section 7] or [1, chapter 10] for its basic properties.

**Definition 5.2.** Let $A$ be a nonempty closed $\sigma$-invariant subset of $H^s_*(\mathbb{R}^N)$. The genus $\gamma(A)$ of $A$ is the least integer $k$ such that there exists $\phi \in C(H^s_*(\mathbb{R}^N), \mathbb{R}^k)$ such that $\phi$ is odd and $\phi(x) \neq 0$ for all $x \in A$. We set $\gamma(A) = \infty$ if there are no integers with the above property and $\gamma(\emptyset) = 0$.

Let $\mathcal{A}$ be the family of closed $\sigma$-invariant subset of $S_m \cap H^s_*(\mathbb{R}^N)$. For each $k \in \mathbb{N}$, set

$$\mathcal{G}_k := \{A \in \mathcal{A} \mid \gamma(A) \geq k\}$$

and

$$E_{m,k} = \inf_{A \in \mathcal{A}} \max_{u \in A} J(u).$$

Next, we give a result about the weak convergence of the nonlinearity $f$.

**Lemma 5.3.** Assume $(f_0)-(f_2)$ hold true. Let $(u_n)_n \subset H^s_*(\mathbb{R}^N)$. If $u_n \to u$ in $H^s_*(\mathbb{R}^N)$ for some $u \in H^s_*(\mathbb{R}^N)$, then $f(u_n) \to f(u)$ in $L^{\frac{2N}{N+4s}}(\mathbb{R}^N)$.

**Proof.** We borrow some ideas from [13, Theorem 2.6]. We start exploiting that $H^s_*(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ compactly for $p \in (2, 2^*_s)$. Hence, up to a subsequence, $u_n \to u$ in $L^p(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. From equation (6.3), we get

$$|f(u_n)|^{\frac{2N}{N+4s}} \leq C_\varepsilon |u_n|^{\frac{2N}{N+4s}} + C |u_n|^{\frac{2N}{N+4s}}$$

for some $C_\varepsilon, C > 0$. As a consequence of that, recalling the fractional Sobolev inequality and observing that $2\frac{N+4s}{N+4s} \in (2, 2^*_s)$, we obtain that $(f(u_n))_n$ is bounded in $L^{\frac{2N}{N+4s}}(\mathbb{R}^N)$. Thus, there exists $y \in L^{\frac{2N}{N+4s}}(\mathbb{R}^N)$ such that $f(u_n) \rightharpoonup y$. At this point, we fix a cover $(\Omega_j)_j$ of $\mathbb{R}^N$ made of subsets with finite measure. For any $\varepsilon > 0$, Severini-Egorov’s Theorem yields the existence of $B_\varepsilon^j \subset \Omega_j$, with measure $|B_\varepsilon^j| < \varepsilon$, such that $u_n \to u$ uniformly in $\Omega_j \setminus B_\varepsilon^j$. Clearly $y = f(u)$ in $\Omega_j \setminus B_\varepsilon^j$. Now, we set

$$Q := \{x \in \mathbb{R}^N \mid y \neq f(u)\} \quad \text{and} \quad Q_j := \{x \in \Omega_j \mid y \neq f(u)\}.$$  

Since $\varepsilon$ is arbitrary and $Q_j \subset B_\varepsilon^j$, we have that $Q_j$ is a set of measure zero. Furthermore, it is easy to see that $Q = \bigcup_{j=1}^{\infty} Q_j$, thus $Q$ has measure zero and the proof is complete. □
From now on, we will always assume \((f_0) - (f_5)\) hold until the end of the section.

**Lemma 5.4.** Let \(\mathcal{G}\) be a \(\sigma\)-homotopy stable family of compact subset of \(S_m \cap H^s_r(\mathbb{R}^N)\) (with \(B = \emptyset\)) and set

\[
E_{m,\mathcal{G}} := \inf_{A \in \mathcal{G}} \max_{u \in A} J(u).
\]

If \(E_{m,\mathcal{G}}\) then there exists a Palais-Smale sequence \((u_n)_n\) in \(P_m \cap H^s_r(\mathbb{R}^N)\) for \(I|_{S_m \cap H^s_r(\mathbb{R}^N)}\) at level \(E_{m,\mathcal{G}}\).

**Proof.** It suffices to replace Theorem 3.2 with 7.2 of [8] in the proof of Lemma 4.4. 

**Lemma 5.5.** For any \(k \in \mathbb{N}\) we have,

(i) \(G_k \neq \emptyset\) and \(G_k\) is a \(\sigma\)-homotopy stable family of compact subsets of \(S_m \cap H^s_r(\mathbb{R}^N)\) (with \(B = \emptyset\)),

(ii) \(E_{m,k+1} \geq E_{m,k}\).

**Proof.** (i) It suffices to notice that for any \(k \in \mathbb{N}\) one has \(S_m \cap V_k \in \mathcal{A}\) and that by [1, Theorem 10.5]

\[
\gamma(S_m \cap V_k) = k.
\]

Thus \(G_k \neq \emptyset\). The conclusion is a direct consequence of the definition of \(\mathcal{A}\).

(ii) By the previous step \(E_{m,k}\) is well defined. Furthermore, recalling that \(\rho(u) * u \in P_m\) for all \(u \in A\), where \(A\) is chosen arbitrarily in \(\mathcal{G}\), we have

\[
\max_{u \in A} J(u) = \max I(\rho(u) * u) = \inf_{v \in P_m} I(v),
\]

hence \(E_{m,k} > 0\). The other part of the statement follows easily from \(G_{k+1} \subset G_k\). 

**Lemma 5.6.** Let \((u_n)_n \subset S_m \cap H^s_r(\mathbb{R}^N)\) be a bounded Palais-smale sequence for \(I|_{S_m}\) at an arbitrary level \(c > 0\) satisfying \(P(u_n) \to 0\). Then there exists \(u \in S_m \cap H^s_r(\mathbb{R}^N)\) and \(\mu > 0\) such that, up to a subsequence, \(u_n \rightharpoonup u\) strongly in \(H^s_r(\mathbb{R}^N)\) and

\[
(-\Delta)^s + \mu u = f(u).
\]

**Proof.** By the boundedness of the Palais-Smale sequence we may assume \(u_n \to u\) in \(H^s_r(\mathbb{R}^N)\), \(u_n \to u\) in \(L^p(\mathbb{R}^N)\) for any \(p \in (2, 2^*_s)\) and a.e. in \(\mathbb{R}^N\). Besides, as already seen in the previous section, using [3, Lemma 3] we get

\[
(-\Delta)^s u_n + \mu u_n - f(u_n) \to 0 \quad \text{in} \quad H^s_r(\mathbb{R}^N)
\]

(5.1)

where

\[
\mu_n := \frac{1}{m} \left( \int_{\mathbb{R}^N} f(u_n) u_n \, dx - [u_n]^2_{H^s_r(\mathbb{R}^N)} \right).
\]

Again, similarly to the proof of Lemma 4.6, we can assume the existence of \(\mu \in \mathbb{R}\) such that \(\mu_n \to \mu\), from which we derive

\[
(-\Delta)^s + \mu u = f(u).
\]

(5.2)
Claim: $u \not= 0$.
If $u = 0$, then by the compact embedding $u_n \to 0$ in $L^{2 + \frac{4s}{N}}(\mathbb{R}^N)$. Hence, using Lemma 2.1 (ii) and the fact that $P(u_n) \to 0$, we have $\int_{\mathbb{R}^N} F(u_n) \, dx \to 0$ and

$$[u_n^2]_{H^s(\mathbb{R}^N)} = P(u_n) + \frac{N}{2s} \int_{\mathbb{R}^N} \tilde{F}(u_n) \, dx \to 0,$$

from which

$$c = \lim_{n \to +\infty} I(u_n) = \frac{1}{2} \lim_{n \to +\infty} [u_n^2]_{H^s(\mathbb{R}^N)} - \lim_{n \to +\infty} F(u_n) = 0,$$

that contradicts the hypothesis of $c > 0$. Now, since $u \not= 0$, as we obtained (4.8), we get

$$\mu := \frac{1}{m} \int_{\mathbb{R}^N} \left( \frac{N}{s} F(u) - \frac{N - 2s}{2} f(u)u \right) \, dx > 0.$$

Since $u_n \to u$ in $H^s_0(\mathbb{R}^N)$, by Lemma 5.3

$$\int_{\mathbb{R}^N} [f(u_n) - f(u)] u \, dx \to 0.$$

Indeed, the fractional Sobolev inequality implies that $u \in L^{\frac{2sN}{N - 2s}}(\mathbb{R}^N)$, and the multiplication by $u$ turns out to be a continuous linear operator from $L^{\frac{2sN}{N - 2s}}(\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$. Now, observing that $\int_{\mathbb{R}^N} (u_n - u) \, dx \to 0$ by Lemma 2.1 (iii) we get

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} f(u_n) u_n \, dx = \int_{\mathbb{R}^N} f(u) u \, dx.$$

Finally, from (5.1) and (5.2) one has

$$[u]_{H^s(\mathbb{R}^N)}^2 + \mu \int_{\mathbb{R}^N} u^2 \, dx = \int_{\mathbb{R}^N} f(u) u \, dx$$

$$= \lim_{n \to +\infty} \int_{\mathbb{R}^N} f(u_n) u_n \, dx = \lim_{n \to +\infty} [u_n]_{H^s(\mathbb{R}^N)}^2 + \mu m,$$

and since $\mu > 0$,

$$\lim_{n \to +\infty} [u_n]_{H^s(\mathbb{R}^N)}^2 = [u]_{H^s(\mathbb{R}^N)}^2, \quad \lim_{n \to +\infty} \int_{\mathbb{R}^N} u_n^2 \, dx = m = \int_{\mathbb{R}^N} u^2 \, dx.$$ 

Thus $u_n \to u$ in $H^s_0(\mathbb{R}^N)$.

Lemma 5.7. For any $c > 0$, there exists $\beta = \beta(c) > 0$ and $k(c) \in \mathbb{N}$ such that for any $k \geq k(c)$ and any $u \in P_m \cap H^s_0(\mathbb{R}^N)$

$$\|\pi u\|_{H^s(\mathbb{R}^N)} \leq \beta \quad \text{implies} \quad I(u) \geq c.$$

Proof. By contradiction, we assume that there exists $c_0$ such that for any $\beta > 0$ and any $k \in \mathbb{N}$ it is possible to find $\ell \geq k$ and $u \in S_m \cap H^s(\mathbb{R}^N)$ such that

$$I(u) < c_0 \quad \text{with} \quad \|\pi u\|_{H^s(\mathbb{R}^N)} \leq \beta.$$
In view of that, one can find a sequence $(k_j)_j \subset \mathbb{N}$, with $k_j \to \infty$ as $j \to \infty$, and a sequence $(u_j)_j \subset P_m \cap H^s_\ast(\mathbb{R}^N)$ such that
\[
\|\pi_{k_j} u_j\|_{H^s(\mathbb{R}^N)} \leq \frac{1}{j} \quad \text{and} \quad I(u_j) < c_0
\] (5.3)
for any $j \in \mathbb{N}$. Noticing that by Lemma 2.7 (iv) $(u_j)_j$ is bounded, up to a subsequence we have
\[u_j \rightharpoonup u \text{ in } H^s_\ast(\mathbb{R}^N) \text{ and } L^2(\mathbb{R}^N).\]

Claim: $u = 0$.

Since $k_j \to \infty$, it follows that $\pi_{k_j} u_j \to u$ in $L^2(\mathbb{R}^N)$, hence
\[\left(\pi_{k_j} u_j, u\right)_{L^2(\mathbb{R}^N)} = (u_j, \pi_{k_j} u)_{L^2(\mathbb{R}^N)} \to (u, u)_{L^2(\mathbb{R}^N)}\]
as $j \to \infty$. \hfill \Box

On the other hand, using (5.3) we get $\pi_{k_j} u_j \to 0$ in $L^2(\mathbb{R}^N)$, thus the claim must hold. Now, since $\|u_j\|_{L^2+\frac{4s}{N}(\mathbb{R}^N)} \to 0$ by the compact embedding, $(u_j)_j \subset P_m \cap H^s_\ast(\mathbb{R}^N)$, and Lemma 2.1 (ii), we obtain
\[\|u_j\|^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \tilde{F}(u_j) \, dx \to 0\]
as $j \to \infty$, which contradicts Lemma 2.7 (ii).

**Lemma 5.8.** $E_{m,k} \to \infty$ as $k \to +\infty$.

**Proof.** We assume by contradiction that there exists $c > 0$ such that
\[\liminf_{k \to +\infty} E_{m,k} < c.\]
Denote with $\beta(c)$ and $k(c)$ the numbers given in Lemma 5.7. Up to choose a bigger $c$, we can find $k > k(c)$ such that $E_{m,k} < c$. Moreover, by definition of $E_{m,k}$ there must be $A \in G_k$ such that
\[\max_{u \in A} I(\rho(u) * u) = \max_{u \in A} J(u) < c.\]
Now, recalling Lemma 2.6 (iii) and (iv) we get that the map $\varphi : A \to P_m \cap H^s_\ast(\mathbb{R}^N)$ defined by $\varphi(u) = \rho(u) * u$ is odd and continuous. Thus, setting $\bar{A} := \varphi(A) \subset P_m \cap H^s_\ast(\mathbb{R}^N)$ we have
\[\max_{v \in \bar{A}} I(v) < c\]
and
\[\gamma(\bar{A}) \geq \gamma(A) \geq k > k(c)\] (5.4)
by the properties of the genus. On the other hand, Lemma 5.7 implies that
\[\inf_{v \in \bar{A}} \|\pi_{k(c)} v\|_{H^s(\mathbb{R}^N)} \geq \beta(c) > 0,\]
and after setting \[\phi(v) := \frac{\pi_{k(c)} v}{\|\pi_{k(c)} v\|_{H^s(\mathbb{R}^N)}} \quad \text{for any } v \in \bar{A}\]
we get
\[ \gamma(\mathcal{A}) \leq \gamma(\phi(\mathcal{A})) \leq k(c) \]
noticing that \( \phi \) is odd, continuous and that \( \phi(\mathcal{A}) \subset V_{k(c)} \). That is against (5.4). Therefore \( E_{m,k} \to \infty \) as \( k \to +\infty \).

**Proof of Theorem 1.5.** For each \( k \in \mathbb{N} \), by Lemmas 5.4 and 5.5 one can find a Palais-Smale sequence \((u_n)_n \subset \mathcal{P}_m \cap H^s_\ast(\mathbb{R}^N)\) of the constrained functional \( I_{S_m \cap H^s_\ast(\mathbb{R}^N)} \) at level \( E_{m,k} > 0 \). By Lemma 2.7 \((u_n)_n\) is bounded and by virtue of Lemma 5.6 we deduce that \((P_m)\) has a radial solution \( u_k \) such that \( I(u_k) = E_{m,k} \). Moreover, using Lemma 5.5 (ii) and Lemma 5.8, we get
\[ I(u_{k+1}) \geq I(u_k) > 0 \quad \text{for any } k \geq 1 \]
and \( I(u_k) \to \infty \).

**6 Appendix**

**Proof of Lemma 2.1.** (i) It suffices to show that there exists \( \delta > 0 \) such that
\[ \int_{\mathbb{R}^N} |F(u)| \, dx \leq \frac{1}{4} [u]_{H^s(\mathbb{R}^N)}^2 \]
whenever \( u \in B_m \) and \([u]_{H^s(\mathbb{R}^N)} \leq \delta \). In order to show that, we start noticing that \((f_0), (f_1), \) and \((f_2)\) imply that for every \( \varepsilon > 0 \) we can find \( C_1 = C_1(\varepsilon) > 0 \) such that
\[ |F(u)| \leq \varepsilon |t|^{2+\frac{4}{N}} + C_1 |t|^{\frac{2N}{N-2}}. \quad (6.1) \]
Hence, by (6.1), using the interpolation inequality and the fractional Sobolev inequality (see for instance [6, Theorem 6.5]), we get
\[ \int_{\mathbb{R}^N} |F(u)| \, dx \leq \varepsilon \int_{\mathbb{R}^N} |u|^{2+\frac{4}{N}} \, dx + C_1 \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} \, dx \leq \varepsilon m^{\frac{2N}{N-2}} \|u\|_{L^2(\mathbb{R}^N)}^2 + C_1 \|u\|_{L^s(\mathbb{R}^N)}^{2s} \]
\[ \leq \varepsilon m^{\frac{2N}{N-2}} C_1 [u]_{H^s(\mathbb{R}^N)}^2 + C_2 [u]_{H^s(\mathbb{R}^N)}^{2s} = \left[ \varepsilon m^{\frac{2N}{N-2}} C_1 + C_2 [u]_{H^s(\mathbb{R}^N)}^{2s-2} \right] [u]_{H^s(\mathbb{R}^N)}^2. \]
Choosing
\[ \varepsilon = \frac{1}{8m^{\frac{2N}{N-2}} C_1} \quad \text{and} \quad \delta = \left( \frac{1}{C_2} \right)^{\frac{2}{s-2}} \]
the assertion is verified.

(ii) Since \((f_0), (f_1)\) and \((f_2)\) hold, for every \( \varepsilon > 0 \) there exists \( C_3, C_4 > 0 \) such that
\[ |f(t)| \leq \frac{\varepsilon}{2} |t|^{\frac{2N}{N-2}} + C_3 |t|^{2+\frac{4}{N}} \]
and
\[ |F(t)| \leq \frac{\varepsilon}{2} |t|^{\frac{2N}{N-2}} + C_4 |t|^{2+\frac{4}{N}}, \]
which implies
\[ |F(t)| \leq \varepsilon |t|^{\frac{2N}{N-2}} + (C_3 + C_4) |t|^{2+\frac{4}{N}}. \quad (6.2) \]
By (6.2) we have
\[
\int_{\mathbb{R}^N} |\tilde{F}(u_n)| \, dx \leq \varepsilon \int_{\mathbb{R}^N} |u_n|^{\frac{2N}{N-2s}} \, dx + \int_{\mathbb{R}^N} |u_n|^{2+\frac{4s}{N}} \, dx
\]
\[
\leq \varepsilon C_5 |u_n|_{H^s(\mathbb{R}^N)}^{\frac{2N}{N-2s}} + (C_3 + C_4) \|u_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)}^{2+\frac{4s}{N}}
\]
\[
\leq \varepsilon C_6 + (C_3 + C_4) \|u_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)}^{2+\frac{4s}{N}} \to 0
\]
as \( n \to +\infty \) and \( \varepsilon \to 0 \).

(iii). (f_0), (f_1) and (f_2) imply that for every \( \varepsilon > 0 \) we can find \( C_7 > 0 \) such that
\[
|f(t)| \leq \varepsilon |t|^\frac{N+2s}{N-2s} + C_7 |t|^{1+\delta}.
\] (6.3)
Hence, by (6.3), we obtain that
\[
\int_{\mathbb{R}^N} |f(u_n)||v_n| \, dx \leq \varepsilon \int_{\mathbb{R}^N} |u_n|^\frac{N+2s}{N-2s} |v_n| \, dx + C_7 \int_{\mathbb{R}^N} |u_n|^{1+\frac{4s}{N}} |v_n| \, dx
\]
\[
\leq \varepsilon \|u_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)} \|v_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)} + C_7 \|u_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)} \|v_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)}
\]
\[
\leq \varepsilon C_8 \|u_n\|_{H^s(\mathbb{R}^N)} \|v_n\|_{H^s(\mathbb{R}^N)} + C_9 \|u_n\|_{H^s(\mathbb{R}^N)} \|v_n\|_{H^s(\mathbb{R}^N)}
\]
\[
\leq \varepsilon C_{10} + C_{11} \|u_n\|_{L^{2+\frac{4s}{N}}(\mathbb{R}^N)} \to 0
\]
as \( n \to +\infty \) and \( \varepsilon \to 0 \). This completes the proof of the Lemma.

Proof of Lemma 2.3. (i) Let us fix \( m := \|u\|^2_{L^2(\mathbb{R}^N)} \). We observe that \( \rho * u \in S_m \) and after a change of variables we obtain
\[
[b * u]^2_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} e^{N\rho}(u(x) - u(y))^2 \, dx \, dy = e^{2\rho s}[u]^2_{H^s(\mathbb{R}^N)}.
\]
By virtue of the previous computation, choosing \( \rho \ll -1 \), Lemma 2.1 (i) guarantees the existence of a \( \delta > 0 \) such that if \( [u]^2_{H^s(\mathbb{R}^N)} \leq \delta \) then
\[
\frac{1}{4} e^{2\rho s}[u]^2_{H^s(\mathbb{R}^N)} \leq I(\rho * u) \leq e^{2\rho s}[u]^2_{H^s(\mathbb{R}^N)},
\]
thus
\[
\lim_{\rho \to -\infty} I(\rho * u) = 0^+.
\]
(ii) For every \( \lambda \geq 0 \) we define the function \( h_\lambda : \mathbb{R} \to \mathbb{R} \) as follows
\[
h_\lambda(t) = \begin{cases} \frac{F(t)}{|t|^{2+\frac{4s}{N}}} + \lambda & t \neq 0 \\ \lambda & t = 0. \end{cases}
\] (6.4)
It is straightforward to verify that \( F(t) = h_\lambda(t)|t|^{2+\frac{4s}{N}} - \lambda |t|^{2+\frac{4s}{N}} \). Moreover, from (f_0) and (f_1) it follows that \( h_\lambda \) is continuous, whereas thanks to (f_3) we have
\[
h_\lambda(t) \to \infty \quad \text{as} \quad t \to \infty.
\]
Putting together the divergence of the limit above at infinity and \((f_1)\), we can find \(\lambda > 0\) large enough such that \(h_\lambda(t) \geq 0\) for every \(t \in \mathbb{R}\). Now, applying the well known Fatou’s Lemma, we obtain

\[
\liminf_{\rho \to \infty} \int_{\mathbb{R}^N} h_\lambda(e^{N\rho} u)|u|^{2+\frac{4}{N}} \, dx \geq \int_{\mathbb{R}^N} \lim_{\rho \to \infty} h_\lambda(e^{N\rho} u)|u|^{2+\frac{4}{N}} \, dx = \infty.
\]

Then, we observe that

\[
I(\rho * u) = \frac{1}{2} \left[ |\rho * u|_{H^s(\mathbb{R}^N)}^2 + \lambda \int_{\mathbb{R}^N} |\rho * u|^{2+\frac{4}{N}} \, dx - \int_{\mathbb{R}^N} h_\lambda(\rho * u)|\rho * u|^{2+\frac{4}{N}} \, dx \right] \quad (6.5)
\]

from which it follows immediately that

\[
\lim_{\rho \to \infty} I(\rho * u) = -\infty.
\]

\[
\blacksquare
\]

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