Transformation of the nonrelativistic quantum system under transition from one inertial reference frame to another

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We consider the problem of internal particle state transformation, which is a bound state of several particles, from the rest frame of a composite particle to the system in which it is relativistic. It is assumed that in the rest frame of the composite particle, its internal state could be considered in the nonrelativistic approximation. It has been shown, that this internal state is unchanged during the transition from one reference frame to another. Namely, the spherically symmetric particle in the rest frame stays spherically symmetric in any other reference frame. We discuss the possible application of these results for description of the hadrons scattering processes like bound states of quarks.

I. INTRODUCTION

Recently it has been shown that the hadron-hadron inelastic scattering processes can be successfully described with the Laplace’s method [1]. However, calculations in [1] have been performed using a simplified scalar theory, rather than the quantum chromodynamics (QCD). This allowed to reproduce the experimental data for hadron-hadron scattering cross section on a qualitative level only [2]. Obtained results force us to try to apply this method within the framework of more realistic theory, in QCD. Herewith we run into the well-known problem: there are quark and gluon lines in Feynman diagrams in QCD, whereas there are no lines representing hadrons. In “ordinary” scattering process the initial and finite state of the scattering particles are taken to be non-interacting, but for quarks inside interacting hadrons it is not true. We surely can decompose the bound state of the particles over the momentum eigenstates, notably over the states of free particles, however two problems appears.

The first one lays in the possibility to find coefficients of this series in the arbitrary reference frame. The second one – the Hamiltonian of that system in the initial and finite states does not approximate asymptotically to the free quarks Hamiltonian, hence the interaction process does not “turn on” in the initial state and does not “turn off” in the finite state,
like in the ordinary scattering theory. This leads to the fact that even we decompose the bound state over free particles states, the asymptotic time dependence of these states will not match with the dependence of free particles states. As the result, we cannot use the usual interaction representation, and we need to find another one, where the role of Hamiltonian of the free fields plays the Hamiltonian of the non-interaction hadrons, with accounting their quarks structure.

We can simplify solution of the mentioned problems with the following arguments. If we assume, that the free hadron in the initial and the finite state of the scattering process consists of certain numbers of specific quarks, then a new particles cannot be produced as the result of these quarks interactions. Therefore, the internal state of the free hadron in the rest frame system can be described in the nonrelativistic approximation. Accordingly the Hamiltonian of the quarks inside this free hadron at the hadron rest frame system can be described in the nonrelativistic approximation. However, the initial and the finite state of the scattering process consists of a few hadrons, hence generally speaking we cannot find a reference frame that it would be the rest frame for all these hadrons.

In this way, it appears a problem of the transformation of the nonrelativistic internal state and the Hamiltonian in the transition from the rest frame of the particle into the reference frame where this particle is relativistic. The essence of this problem can be illustrated with the following simple example. Consider the most trivial quantum system – the hydrogen atom in the simplest spherically symmetric ground state. Imagine an inertial observer moving with relativistic velocity with respect to it. The observer measures the coordinates (momenta) of particles that make up the system. What would be the result of this measurement? That is what probability amplitude can describe the results of these measurements and how this probability amplitude is connected with the probability amplitude in the rest frame of the hydrogen atom, namely with the center of mass frame of the particles which compose this atom. Further in the paper, for instance, we will consider a meson as a two-particle system of a quark and an antiquark and afterwards we will apply obtained results to more complex three-quark systems, i.e. baryons.

Outlined problem is rather non-typical. Usually one deals with the measurements of a specific quantities associated with the same event, which are performed in the different reference frames. This is not the case if we deal with the probability amplitude for many-particle system. Consider two inertial observers, which we call unprimed and primed respectively. Then, the probability amplitude for two-particle system in the reference frame of unprimed observer (denoted as \( \Psi (t, r_1, r_2) \)) describes the result of simultaneous measurement of particles’ coordinates, performed at a certain moment of time \( t \) in this frame. Likewise, the probability amplitude \( \Psi' (t', r'_1, r'_2) \) in the reference frame of the primed observer describes the results of measurements, which are simultaneous with respect to this observer at his clock time \( t' \). However, measurements that are simultaneous in the reference frame of the primed observer will not be simultaneous in the reference frame of the unprimed one, and vice versa. This is the essential distinction between the considered problem in comparison with the classical problem of the Lorentz contraction, where the coordinate measurement of the rod ends should be synchronized with the reference frame relative to which the rod moves, but it cannot be simultaneous in the rod rest frame. Due to this fact the rod length can be calculated by the coordinates of the same events but in different reference frames. In our case, the probability amplitudes \( \Psi (t, r_1, r_2) \) and \( \Psi' (t', r'_1, r'_2) \) are associated with the different realizations of the measurement process. As a result, one cannot conclude any kind of relation between the magnitudes \( t \) and \( t' \), since such a relations could be established only
between the time coordinates of the same event, measured with respect to different reference frames. Accordingly, it is impossible to establish a tie between the values of $r_1$ and $r_2$, as well as between $r'_1$ and $r'_2$. Hence, there is no relations similar to Lorentz transformations between the arguments of the probability amplitudes in both reference frames. Therefore, the notions of length contraction and time dilation, which are the consequences of Lorentz transformations, are not applicable in our case.

Thus, in the problem of state transformation at the transition from one inertial reference frame to another, it is improper to consider connections between the values of probability amplitudes, corresponding to the same events in different reference frames. Rather, one should examine the relation between the values of probability amplitude in the different reference frames, corresponding to the same values of arguments, similar to that in dealing with the internal symmetries. Taking this into account, we will denote the probability amplitude in the primed reference frame through $\Psi'(t, r_1, r_2)$.

The principle method to solve the state transformation problem at the transition from one inertial reference frame to another is provided by the field quantization postulate, established in [3]. However that postulate is applied to the case of the relativistic quantum system state, which cause problem. This means, for instance, that an internal state of the meson should be described not by the two-particle probability amplitude $\Psi(t, r_1, r_2)$, but by the Fock space element, i.e. by the set of probability amplitudes of the type:

$$|\Psi\rangle = \{\Psi_{n_q, n_g}(t, r_1, r_2; q_1, \ldots, q_{n_q}; g_1, \ldots, g_{n_g})\},$$  \hspace{1cm} (1)

which take into account that system besides the constituent quark and antiquark (the values $r_1, r_2$ can be obtained during measurements of the coordinates) can contain an arbitrary number $n_q$ of current quarks and antiquarks (obtained by measuring their coordinates and denoted $q_1, \ldots, q_{n_q}$). The notations $n_g$ and $g_1, \ldots, g_{n_g}$ have a similar meaning but for gluons.

To avoid difficulties connected with the necessity to define a large number of unknown functions we want to use advantage of the above circumstance, which consists in the fact that hadron internal state in the rest frame can be described in the non-relativistic approximation. This in turn mean that there is a possibility to transform operators and the state vector Eq.1 thus that all observed values remained unchanged. While all functions included in Eq.1, except $\Psi_{n_q=0, n_g=0}(t, r_1, r_2) \equiv \Psi(t, r_1, r_2)$ were taking small values and may be set equal to zero with a high accuracy.

But then we also have to build a corresponding nonrelativistic approximation for the generators, which are included in the unitary operator with help of which the state can be transformed in the transition to the new reference frame according to the quantization postulate [3]. Our own formulation of such approximations is the content of the next part of the work. It is well known that the arbitrary Lorentz transformation can be made as the product of rotations and boost. Obviously, the above mentioned problems concern exactly to the boost. Therefore further we will consider only state transformations at the boost. At the same time we will consider that the boost is realized along $z$ axis.

Before we proceed to the resolving of the mentioned problem, let’s turn ours attention to the further simplification of this problem. Let us consider the hadron in its rest frame. In this system the state $|\Psi\rangle$ must be eigenstate for total momentum $\hat{P}$ of the all particles that make up this system. And at the same time this state must correspond to zero eigenvalue. Before the transition to non-relativistic approximation the temporal progress of state $|\Psi\rangle$ of
the particles system, that make up the hadron, can be represented in next form
\[
|\Psi (t)\rangle = \exp \left( -i\hat{H}t \right) |\Psi (t = 0)\rangle ,
\]  
(2)

where \( \hat{H} \) is the relativistic Hamiltonian of a system of fields, whose quants are making up the hadron. In the reference frame that obtained from outcome system by applying the boost, according to [3] we will obtain:
\[
|\Psi' (t)\rangle = \hat{U}(Y) \left( \exp \left( -i\left( \hat{H}t - \left( \hat{P} \cdot \hat{R} \right) \right) \right) |\Psi (t = 0)\rangle \right),
\]  
(3)

Here \( \hat{U}(Y) \) is the unitary state transformation operator of Eq.1 from [3] as a result of boost with rapidity \( Y \). Taking into account that we are dealing with the eigenstate of total momentum of the system and this state corresponds the zero eigenvalue we can write relation Eq.3 in the following form:
\[
|\Psi' (t)\rangle = \hat{U}(Y) \hat{u}(x) \hat{U}^{-1}(Y) \hat{U}(Y) |\Psi (t = 0)\rangle ,
\]  
(4)

where \( \hat{u}(x) \equiv \exp \left( -i\left( \hat{H}t - \left( \hat{P} \cdot \hat{R} \right) \right) \right) \).  
(6)

Expression \( \hat{U}(Y) \hat{u}(x) \hat{U}^{-1}(Y) \) in Eq.5 formally coincides with the expression which appears at transformation of operator field functions [3]. Therefore, denoting the matrix of boost along \( z \) direction by \( \Lambda^{(0)}(Y) \) we get
\[
\hat{U}(Y) \hat{u}(x) \hat{U}^{-1}(Y) = \hat{u} \left( \Lambda^{(0)}(Y) x \right).
\]  
(7)

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Then, instead of Eq.5, we can write:
\[
|\Psi' (t)\rangle = \exp \left( -it \left( \text{ch} \left( Y \right) \hat{H} + \text{sh} \left( Y \right) \hat{P}_z \right) \right) \exp \left( iR_x \left( \text{sh} \left( Y \right) \hat{H} + \text{ch} \left( Y \right) \hat{P}_z \right) \right) \times 
\]
\[
\times \exp \left( i \left( R_x \hat{P}_x + R_y \hat{P}_y \right) \right) \hat{U}(Y) |\Psi (t = 0)\rangle .
\]  
(8)

Pay attention, that operators \( \hat{H} \) and \( \hat{P} \) are included in relation Eq.8 that belongs to the original reference frame, in which we can apply the nonrelativistic approximation. Using this approximation these operators may be replaced by nonrelativistic internal Hamiltonian of the quarks system which make up hadron, and nonrelativistic momentum operator of this system accordingly. The quantity \( |\Psi (t = 0)\rangle \) in such nonrelativistic approximation may be replaced by coordinate part of the probability amplitude of energy eigenstate for two-particle system (quark and antiquark). In addition, if we consider a extreme case of small rapidities \( Y \), one may note that we must choose a coordinates of the center of mass as an arbitrary coordinates of the vector \( \hat{R} \):
\[
\mathbf{R} = \mathbf{R} \left( \mathbf{r}_1, \mathbf{r}_2 \right) = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}.
\]  
(9)
Thus, the simplification discussed above and achieved throughout the Eq. 4-8 means that we don’t need to describe the transformation of the whole probability amplitude in the energy eigenstate during the transition from the center-of-mass (quark and antiquark) system to another reference system, but we can limit that calculation only to the transformation of the coordinate part of this probability amplitude.

II. APPROXIMATION OF LORENTZ TRANSFORMATION GENERATORS USING DIFFERENTIAL OPERATORS

As it is known from [3] the component $\hat{M}_{03}$ of the angular momentum tensor is the generator of transformation $\hat{U}(Y)$. That is

$$\hat{U}(Y) = \exp \left(i\hat{M}_{03}Y\right).$$

(10)

According to [3] the representation of $\hat{M}_{03}$ by the differential operators looks as follows:

$$\hat{M}_{03} = i \left(t \frac{\partial}{\partial z} + z \frac{\partial}{\partial t}\right).$$

(11)

Therefore, we note that the representation of generators through differential operators can be obtained by considering some function of coordinates and time with a corresponding substitution of the independent variables in the function. But, as was mentioned in the previous section, in our case the replacement of the independent variables is impossible. Therefore, relation Eq. 11 can be understood only as a limit to which the “proper” relativistic operator $\hat{M}_{03}$ is approached in the transition to a nonrelativistic approximation. The question then arises: ”To what limit should this operator approach in case of many-particle system?”. Given that the spatial components of the momentum are presented as a sum of the corresponding single-particle operators, we can make an assumption that the components in which one of indices is equal to zero are also additive. Then, for the two-particle system we have:

$$\hat{M}_{03} = i \left(t \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial t} + (z_1 + z_2) \frac{\partial}{\partial t}\right).$$

(12)

As already noted, a quantity $|\Psi(t = 0)\rangle$, which is included in Eq. 8, can be replaced in our case by coordinates part of energy eigenstate, which we denote as $\psi(r_1, r_2)$, at the transition to a nonrelativistic approximation. This function does not depend on time and is the operator eigenfunction of the total momentum of system, that corresponds to zero eigenvalue. If we consider that operator Eq. 12 can be written as:

$$\hat{M}_{03} = -t\hat{P}_z + (z_1 + z_2) i \frac{\partial}{\partial t},$$

(13)

Hence, we will come to the conclusion that function $\psi(r_1, r_2)$ is also the eigenfunction of operator $\hat{M}_{03}$, which corresponds to zero eigenvalue.

This can be explained by the following reflections. Since the original reference frame is the center of mass frame of quark and antiquark, we have:

$$\psi(r_1, r_2) = \psi(r_2 - r_1).$$

(14)
If in the expression

\[ i \left( t \left( \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right) + (z_1 + z_2) \frac{\partial}{\partial t} \right) \psi(r_2 - r_1), \tag{15} \]

from \( r_1 \) and \( r_2 \) we move to the new variables

\[ r_+ = r_1 + r_2, \quad r_- = r_1 - r_2, \tag{16} \]

in this way, operator in Eq.15 will depend only on \( z \) component of vector \( r_+ \) as \( z_+ \), and function on which this operator acts will depends only on \( z \) component of vector \( r_- \) as \( z_- \).

Thereby we can make the next conclusion

\[ \exp \left( i \hat{M}_{03} \right) \psi(r_2 - r_1) = \psi(r_2 - r_1). \tag{17} \]

Namely, the internal state of meson does not change in the transition to a new reference frame.

In all previous discussion we have considered generator \( \hat{M}_{03} \) as a component of the orbital momentum tensor. We note that for free bispinor field with explicit appearance of component spin in tensor of angular momentum \[ 3 \] it is seen that this components equal to zero for tensor's components which have at least one zero index. The operator of interaction of bispinor field with gauge field does not contain derivatives from components of bispinor field, thus does not give contribution to the spin angular momentum tensor. Therefore spin component is equal to zero for “proper” relativistic operator \( \hat{M}_{03} \). It means that in the transition to the nonrelativistic bound, we can consider only the orbital contribution to the \( \hat{M}_{03} \).

Hence all considerations are applicable not only for mesons but for baryons, because the presence of this non-zero spins does not change anything. In the case of baryons, taking assumption that all components of momentum are additive, instead of Eq.12 we obtain:

\[ \hat{M}_{03} = i \left( t \left( \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} \right) + (z_1 + z_2 + z_3) \frac{\partial}{\partial t} \right). \tag{18} \]

This operator is expressed through the operator of \( z \)-component of total momentum of the system. Therefore, when we act by this operator on the eigenfunction of the total momentum operator that corresponds to zero eigenvalue we get zero.

The considerations in this section suffer two essential shortcomings. First, the “proper” relativistic operator \( \hat{M}_{03} \) is not realized through differential operators, but it is given in the second quantization representation. Therefore, it is more convenient to look for the nonrelativistic limit of this operator in this representation. In addition, we have essentially used the assumptions Eq.12 and Eq.18. These assumptions are not required in the second quantization representation, owing to the fact that the expression for the operators does not depend on whether these operators are set on the single-particle or on the many-particle space.

Therefore, the arguments in this section may be considered only as auxiliary. In the next section we will demonstrate that considering this problem in the secondary quantization representation one can get the same result.
III. APPROXIMATION OF LORENTZ TRANSFORMATION GENERATORS IN THE SECOND QUANTIZATION REPRESENTATION

We denote $\hat{q}^{+}(f, \nu, c, r)$ as nonrelativistic quark creation operator in the coordinate representation of the second quantization. Indices $f, \nu, c$ set flavor, spin and color state of quarks, respectively, where quarks are created in the radius eigenvector state corresponding to the eigenvalue $r$. Creation antiquark operator in the same state denoted as $\hat{q}^{-}(f, \nu, c, r)$ and annihilation operators as $\hat{q}^{-}(f, \nu, c, r)$ and $\hat{q}^{-}(f, \nu, c, r)$ respectively.

In these notations, the coordinate of the internal state of meson as quark-antiquark system can be represented in the form:

$$|\mu\rangle = \int d\mathbf{r}_2 d\mathbf{r}_1 \psi(|\mathbf{r}_2 - \mathbf{r}_1|) s(\nu_1, \nu_2) c(c_1, c_2) a(f_1, f_2) \times$$

$$\times \hat{q}^{+}(f_1, \nu_1, c_1, r_1) \hat{q}^{+}(f_2, \nu_2, c_2, r_2) |0\rangle \tag{19}$$

In this relation have denoted the spin, color and flavor of probability amplitudes through $s(\nu_1, \nu_2) c(c_1, c_2) a(f_1, f_2)$, respectively, whereas the function $\psi(|\mathbf{r}_2 - \mathbf{r}_1|)$ describes the coordinate dependence of probability amplitude in the center of mass frame of quark and antiquark. Since we consider the coordinate part of the energy eigenstate as $\psi(|\mathbf{r}_2 - \mathbf{r}_1|)$, we should consider the eigenfunctions of the nonrelativistic Hamiltonian of quark and antiquark system. Besides, as is usually assumed the summation goes over repeated indices. Also was used the usual notation for vacuum state $|0\rangle$.

Since the dependence of all quantities on internal indices is insufficient, we hereinafter denote the set of indices $\{\nu, c, f\}$ by $\xi$, and the dependence of probability amplitude on internal indices as:

$$s(\nu_1, \nu_2) c(c_1, c_2) a(f_1, f_2) \equiv F(\xi_1, \xi_2) \tag{20}$$

Notably, instead of Eq.19 we can write

$$|\mu\rangle = F(\xi_1, \xi_2) \int d\mathbf{r}_2 d\mathbf{r}_1 \psi(|\mathbf{r}_2 - \mathbf{r}_1|) \hat{q}^{+}(\xi_1, r_1) \hat{q}^{+}(\xi_2, r_2) |0\rangle \tag{21}$$

As it is known, in the field theory the operator $\hat{M}_{03}$ is presented in the form:

$$\hat{M}_{03} = \int d\mathbf{r} \left(x_3 \hat{T}_{00} (\mathbf{r}) - x_0 \hat{T}_{30} (\mathbf{r})\right) \tag{22}$$

where $\hat{T}_{00}(\mathbf{r})$ and $\hat{T}_{30}(\mathbf{r})$ are the operators of the corresponding components of the energy-momentum tensor, $x_0 \equiv t$ is the time component of the coordinate 4-vector, and $x_3 \equiv (-z)$ -- is covariant component of the coordinate 4-vector along the $z$-axis. Hence, the relation Eq.22 clearly can be obviously rewritten in the form:

$$\hat{M}_{03} = -t \hat{P}_z + \int d\mathbf{r} \left(x_3 \hat{T}_{00} (\mathbf{r})\right) \tag{23}$$

where $\hat{P}_z$ is operator of the $z$ component of the total momentum of the system.

Note, that relations Eq.22 and Eq.23 are accurate and do not require any assumptions and approximations. Herewith, the dependence on $t$ in Eq.23 coincides with Eq.13, while Eq.13 is a consequence of the assumptions Eq.12 and Eq.18. Thus, we can conclude that Eq.23 proves the validity of these assumptions.
The state Eq. 21 is an eigenstate for the total momentum of the system, which corresponds to the zero eigenvalue, so the action of the first summand of Eq. 23 on this state trivially gives zero. Therefore, we will represent the second summand of Eq. 23 as follows

\[
\tilde{M}_{03} (\hat{T}_{00}) = \int x_3 \hat{T}_{00} (\mathbf{r}) \, d\mathbf{r}. \tag{24}
\]

In order to act by this operator on the state of two-particle system Eq. 21, we shall construct a nonrelativistic approximation for the energy density \( T_{00} (\mathbf{r}) \). For the solution of this problem it is most convenient to use the representation of second quantization, because in this representation the Hamiltonian is written as an integral from some operator-valued function, which can be taken as nonrelativistic limit of the energy density.

Nonrelativistic Hamiltonian of the quark-antiquark system in the second quantization representation can be written in the form:

\[
\hat{H} = \hat{H}^{(0)} + \hat{H}^{(V)},
\]

\[
\hat{H}^{(0)} = \int d\mathbf{r} \left( \hat{\bar{q}}^+ (\xi, \mathbf{r}) \left( -\frac{1}{2m}\Delta \right) \hat{q}^- (\xi, \mathbf{r}) + \int d\mathbf{r}' \left( \hat{q}^+ (\xi, \mathbf{r}') \left( -\frac{1}{2m}\Delta \right) \hat{\bar{q}}^- (\xi, \mathbf{r}) \right) \right), \tag{25}
\]

\[
\hat{H}^{(V)} = \frac{1}{2} \int d\mathbf{r}_1 d\mathbf{r}_2 V (\mathbf{r}_2 - \mathbf{r}_1) \hat{q}^+ (\xi_1, \mathbf{r}_1) \hat{q}^+ (\xi_2, \mathbf{r}_2) \hat{\bar{q}}^- (\xi_2, \mathbf{r}_2) \hat{\bar{q}}^- (\xi_1, \mathbf{r}_1),
\]

where \( V (\mathbf{r}_2 - \mathbf{r}_1) \) is the potential energy of the quark-antiquark interaction, \( m \) is the mass of quark or antiquark, which is approximately independent of the flavor, because the bound state exists due to the strong interaction and other types of interactions are neglected.

As is well known, in the representation of the two-particle Hamiltonian through the differential operators is considered by introducing Jacobi coordinates

\[
\mathbf{R} = \frac{1}{2} (\mathbf{r}_1 + \mathbf{r}_2), \quad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \tag{26}
\]

which helps to represent the Hamiltonian as a sum of two commuting operators – center of mass Hamiltonian, which depends only on \( \mathbf{R} \) and internal Hamiltonian which depends only on \( \mathbf{r} \). We want to achieve a similar representation in case when the Hamiltonian is written through the creation and annihilation operators. For this purpose, it is convenient to rewrite one-particle part of Hamiltonian \( \hat{H}^{(0)} \) in the two-particle operator form. As we examine the nonrelativistic approximation, all operators can be considered on the subspace of the Fock space with a fixed number and composition of particles. In our case, we consider the subspace of states which contain one quark and one antiquark. The basis states of this subspace can be written in the form:

\[
|\xi_1, \xi_2, \mathbf{r}_1, \mathbf{r}_2 \rangle = \hat{q}^+ (\xi_1, \mathbf{r}_1) \hat{q}^+ (\xi_2, \mathbf{r}_2) |0 \rangle. \tag{27}
\]

If one will act on arbitrary linear combination of states Eq. 27 by the operator

\[
\hat{\bar{E}} = \int d\mathbf{r} \hat{\bar{q}}^+ (\xi, \mathbf{r}) \hat{q}^- (\xi, \mathbf{r}), \tag{28}
\]

one may note that on the subspace of states which contain one quark and one antiquark the operator in Eq. 28 acts as unity operator. The same holds for the operator

\[
\hat{\bar{E}}' = \int d\mathbf{r} \hat{q}^+ (\xi, \mathbf{r}) \hat{\bar{q}}^- (\xi, \mathbf{r}). \tag{29}
\]
which act on the same subspace. If the first summand in the one-particle part $\hat{H}^{(0)}$ of Hamiltonian Eq.25 multiply by the unit operator Eq.29, and the second summand multiply by Eq.28 we will get the expression of the one-particle part in two-particle effective form

$$\hat{H}^{(0)} = \left(-\frac{1}{2m}\right) \int dr_1 dr_2 \left( \hat{q}^+ (\xi_1, r_1) \hat{q}^+ (\xi_2, r_2) \Delta_1 \hat{q}^- (\xi_2, r_2) \hat{q}^- (\xi_1, r_1) + \right.$$

$$+ \hat{q}^+ (\xi_1, r_1) \hat{q}^+ (\xi_2, r_2) \Delta_2 \hat{q}^- (\xi_2, r_2) \hat{q}^- (\xi_1, r_1) \left.). \tag{30}$$

Next, let us replace the one-particle part of the Hamiltonian Eq.25 by Eq.30 and, after this replacement, pass to the variables Eq.26. We also introduce the following notations

$$r_1 (R, r) = R - \frac{1}{2} r, \quad r_2 (R, r) = R + \frac{1}{2} r,$$

$$\hat{q}^+ (\xi_1, r_1 (R, r)) = \hat{q}^+_1, \quad \hat{q}^+ (\xi_2, r_2 (R, r)) = \hat{q}^+_2,$$

$$\hat{q}^- (\xi_2, r_2 (R, r)) = \hat{q}^-_2, \quad \hat{q}^- (\xi_1, r_1 (R, r)) = \hat{q}^-_1. \tag{31}$$

Then, instead of Hamiltonian Eq.25 we get

$$\hat{H} = \hat{H}^{(R)} + \hat{H}^{(r, V)},$$

$$\hat{H}^{(R)} = \left(-\frac{1}{4m}\right) \int dR d\hat{q}^+_1 \hat{q}^+_2 \Delta_R \hat{q}^-_2 \hat{q}^-_1, \tag{32}$$

$$\hat{H}^{(r, V)} = \int dR d\hat{q}^+_1 \hat{q}^+_2 \left(-\frac{1}{m} \Delta_r + V (r) \right) \hat{q}^-_2 \hat{q}^-_1.$$ 

The operator $\hat{H}^{(R)}$ we define as the center of mass Hamiltonian, and the operator $\hat{H}^{(r, V)}$ we define as the internal Hamiltonian of the system. Thus the Hamiltonian Eq.32 can be written as

$$\hat{H} = \int \hat{T}_{00} (R) dR, \tag{33}$$

where the energy density operator $\hat{T}_{00} (R)$ can be written in the form:

$$\hat{T}_{00} (R) = T_{00}^{(R)} (R) + T_{00}^{(r)} (R) + T_{00}^{(V)} (R), \tag{34}$$

with the help of the following denotations:

$$T_{00}^{(R)} (R) = \left(-\frac{1}{4m}\right) \int d\hat{q}^+_1 \hat{q}^+_2 \Delta_R \hat{q}^-_2 \hat{q}^-_1,$$

$$T_{00}^{(r)} (R) = \left(-\frac{1}{m}\right) \int d\hat{q}^+_1 \hat{q}^+_2 \Delta_r \hat{q}^-_2 \hat{q}^-_1, \tag{35}$$

$$T_{00}^{(V)} (R) = \int d\hat{q}^+_1 \hat{q}^+_2 V (r) \hat{q}^-_2 \hat{q}^-_1.$$ 

Relations in Eqs.33-35 define the nonrelativistic approximation for the operator $\hat{T}_{00} (r)$ in the representation of second quantization on Fock subspace. That approximation can be
used to construct nonrelativistic approximation for generator Eq. 24. Using these expressions, we have:

\[ \hat{M}_{03} \left( \hat{T}_{00} \right) = \hat{M}_{03}^{(R)} + \hat{M}_{03}^{(r)} + \hat{M}_{03}^{(V)}, \]  

(36)

where

\[ \hat{M}_{03}^{(a)} = \int dR \left( R_3 T_{00}^{(a)} \left( R \right) \right), \]  

(37)

and index \( a \) takes three possible values \( a = R, r, V \).

Having a nonrelativistic approximation for the generator Eq. 22 we can act by the associated operator Eq. 21 on the nonrelativistic approximation for state Eq. 10 and obtain the probability amplitude of this state in the new reference frame. In order to simplify the action of operator’s exponent Eq. 10 on Eq. 21 we consider that state Eq. 21 is the eigenstate of internal Hamiltonian \( \hat{H}^{(r,V)} \). We are interested in the ground state of the bound system of quarks, that in center of mass system of quarks corresponds to the eigenvalue which equals the mass of hadron. However, for the aforementioned ground state this eigenvalue is not degenerate. Therefore, since that the generator Eq. 22 commutes with the internal Hamiltonian, we get that the state Eq. 21 is the eigenstate for both the boost generator Eq. 22 and the operator Eq. 10.

So, our next goal is to prove that the operators \( \hat{M}_{03} \left( \hat{T}_{00} \right) \) and \( \hat{H}^{(r,V)} \) commute.

For this purpose it is better to use the momentum representation for differentiation operator functions of the coordinates, which are included in the Laplace operator in the expression for the Hamiltonian Eq. 32. For transition to the momentum representation we will write the operators Eq. 31 in form:

\[ \hat{q}_1^+ = \frac{1}{(2\pi)^{3/2}} \int dp_1 \hat{q}_1^+ (\xi_1, p_1) \exp \left( ip_1 \left( R - \frac{1}{2} r \right) \right), \]

\[ \hat{q}_2^+ = \frac{1}{(2\pi)^{3/2}} \int dp_2 \hat{q}_2^+ (\xi_2, p_2) \exp \left( ip_2 \left( R + \frac{1}{2} r \right) \right), \]

\[ \hat{q}_3^- = \frac{1}{(2\pi)^{3/2}} \int dp_3 \hat{q}_3^- (\xi_2, p_3) \exp \left( -ip_3 \left( R + \frac{1}{2} r \right) \right), \]

\[ \hat{q}_4^- = \frac{1}{(2\pi)^{3/2}} \int dp_4 \hat{q}_4^- (\xi_1, p_4) \exp \left( -ip_4 \left( R - \frac{1}{2} r \right) \right), \]  

(38)

where \( \hat{q}_1^+ (\xi_1, p_1), \hat{q}_2^+ (\xi_2, p_2), \hat{q}_3^- (\xi_2, p_3), \hat{q}_4^- (\xi_1, p_4) \) are creation and annihilation operators of quarks in the states, which are eigenstates of momentum. Analogously, we present potential energy

\[ V \left( r \right) = \frac{1}{(2\pi)^{3/2}} \int dk V \left( k \right) \exp \left( ik r \right). \]  

(39)

If we substitute these expressions into Eq. 35 we will get the integrals over the variables \( p_1, p_2, p_3, p_4 \), in which is convenient to make the following replacements:

\[ p_1 + p_2 = P_{12}, \quad \frac{p_2 - p_1}{2} = p_{12}, \quad p_3 + p_4 = P_{34}, \quad \frac{p_4 - p_3}{2} = p_{34}, \]  

(40)
Jacobian of which is equal to 1. Considering this substitution, we will obtain:

\[ T^{(R)}_{00}(R) = \frac{1}{(2\pi)^3} \int dP_{12}dP_{34}dP_{12}dP_{34} \left( \frac{(P_{34})^2}{4m} \right) \]
\[ \times \hat{q}^+ \left( \xi_1, p_1 = \frac{1}{2}P_{12} - p_1 \right) \hat{q}^+ \left( \xi_2, p_2 = \frac{1}{2}P_{12} + p_1 \right) \]
\[ \times \hat{q}^- \left( \xi_2, p_3 = \frac{1}{2}P_{34} - p_3 \right) \hat{q}^- \left( \xi_2, p_4 = \frac{1}{2}P_{34} + p_3 \right) \]
\[ \times \exp(i(P_{12} - P_{34})R) \delta(p_{12} + p_{34}). \]

Here \( \delta(p_{12} + p_{34}) \) is denotation of Dirac \( \delta \)-function. Due to this function we may do \( P_{34} \) integration. Suppose that
\[ p_{12} = p, \quad p_{34} = -p, \]
we will get
\[ T^{(R)}_{00}(R) = \frac{1}{(2\pi)^3} \int dP_{12}dP_{34}dP \left( \frac{(P_{34})^2}{4m} \right) \]
\[ \times \hat{q}^+ \left( \xi_1, p_1 = \frac{1}{2}P_{12} - p \right) \hat{q}^+ \left( \xi_2, p_2 = \frac{1}{2}P_{12} + p \right) \]
\[ \times \hat{q}^- \left( \xi_2, p_3 = \frac{1}{2}P_{34} + p \right) \hat{q}^- \left( \xi_2, p_4 = \frac{1}{2}P_{34} - p \right) \exp(i(P_{12} - P_{34})R). \]

The expression for \( T^{(r)}_{00}(R) \) is analogous to \( T^{(R)}_{00}(R) \), but considering that operator \( \Delta_r \) provides the other variable differentiation of exponent:
\[ T^{(r)}_{00}(R) = \frac{1}{(2\pi)^3} \int dP_{12}dP_{34}dP \left( \frac{P^2}{m} \right) \]
\[ \times \hat{q}^+ \left( \xi_1, p_1 = \frac{1}{2}P_{12} - p \right) \hat{q}^+ \left( \xi_2, p_2 = \frac{1}{2}P_{12} + p \right) \]
\[ \times \hat{q}^- \left( \xi_2, p_3 = \frac{1}{2}P_{34} + p \right) \hat{q}^- \left( \xi_2, p_4 = \frac{1}{2}P_{34} - p \right) \exp(i(P_{12} - P_{34})R). \]

Considering Eq.39 for \( T^{(V)}_{00}(R) \) we have
\[ T^{(V)}_{00}(R) = \frac{1}{(2\pi)^{9/2}} \int dP_{12}dP_{34}dP_{12}dP_{34}dP \cdot V(k) \]
\[ \times \hat{q}^+ \left( \xi_1, p_1 = \frac{1}{2}P_{12} - p_1 \right) \hat{q}^+ \left( \xi_2, p_2 = \frac{1}{2}P_{12} + p_1 \right) \]
\[ \times \hat{q}^- \left( \xi_2, p_3 = \frac{1}{2}P_{34} - p_3 \right) \hat{q}^- \left( \xi_2, p_4 = \frac{1}{2}P_{34} + p_3 \right) \]
\[ \times \exp(i(P_{12} - P_{34})R) \delta(p_{12} + k + p_{34}). \]

Now to \( p_{34} \) integrate taking into account \( \delta \)-function we will suppose
\[ p_{12} = p, \quad p_{34} = -p - k. \]
Then
\[
T_{00}^{(V)}(R) = \frac{1}{(2\pi)^{9/2}} \int dP_{12} dP_{34} dp dk \cdot V(k)
\]
\[
\times \tilde{q}^+ \left( \xi_1, p_1 = \frac{1}{2} P_{12} - p \right) \tilde{q}^+ \left( \xi_2, p_2 = \frac{1}{2} P_{12} + p \right) \]
\[
\times \tilde{q}^- \left( \xi_2, p_3 = \frac{1}{2} P_{34} + p + k \right) \tilde{q}^- \left( \xi_2, p_4 = \frac{1}{2} P_{34} - p - k \right) \exp (i (P_{12} - P_{34}) R). \tag{47}
\]

Substituting the expressions for \( T_{00}^{(r)}(R) \) and \( T_{00}^{(V)}(R) \) into Eq.34 considering Eq.33 and Eq.32 we will obtain the expression for internal Hamiltonian \( H(R) \). Along with this, \( R \) integration appears, which leads again to appearance of Dirac \( \delta \)-function. This enable us to integrate over \( P_{34} \). Besides we set
\[
P_{12} = P, \quad P_{34} = \bar{P}. \tag{48}
\]

As a result we will obtain:
\[
\hat{H}^{(r)} = \int T_{00}^{(r)}(R) dR = \int dP dp \left( \frac{P^2}{m} \right)
\]
\[
\times \tilde{q}^+ \left( \xi_1, p_1 = \frac{1}{2} P - p \right) \tilde{q}^+ \left( \xi_2, p_2 = \frac{1}{2} P + p \right) \]
\[
\times \tilde{q}^- \left( \xi_2, p_3 = \frac{1}{2} P + p + k \right) \tilde{q}^- \left( \xi_2, p_4 = \frac{1}{2} P - p \right),
\tag{49}
\]
\[
\hat{H}^{(V)} = \int T_{00}^{(V)}(R) dR = \frac{1}{(2\pi)^{3/2}} \int dP dp dk \cdot V(k)
\]
\[
\times \tilde{q}^+ \left( \xi_1, p_1 = \frac{1}{2} P - p \right) \tilde{q}^+ \left( \xi_2, p_2 = \frac{1}{2} P + p \right) \]
\[
\times \tilde{q}^- \left( \xi_2, p_3 = \frac{1}{2} P + p + k \right) \tilde{q}^- \left( \xi_2, p_4 = \frac{1}{2} P - p \right).
\]

The internal Hamiltonian of the system is represented by this operators in next way
\[
\hat{H}^{(r,v)} = \hat{H}^{(r)} + \hat{H}^{(V)}. \tag{50}
\]

Now consider the expression for the momentum representation of generator Eq.36. For \( \hat{M}_{03}^{(R)} \) we have
\[
\hat{M}_{03}^{(R)} = \frac{1}{(2\pi)^{3}} \int dP_{12} dP_{34} dp \left( \frac{(P_{34})^2}{4m} \right)
\]
\[
\times \tilde{q}^+ \left( \xi_1, p_1 = \frac{1}{2} P_{12} - p \right) \tilde{q}^+ \left( \xi_2, p_2 = \frac{1}{2} P_{12} + p \right) \]
\[
\times \tilde{q}^- \left( \xi_2, p_3 = \frac{1}{2} P_{34} + p \right) \tilde{q}^+ \left( \xi_2, p_4 = \frac{1}{2} P_{34} - p \right) \int R_{3} \exp (i (P_{12} - P_{34}) R) dR. \tag{51}
\]

Integration variable \( P_{12} \) will be represented as \( P \), and instead of \( P_{34} \) we will enter a new variable of integration \( \varepsilon \) with the help of this expression:
\[
P_{34} = P - \varepsilon. \tag{52}
\]
After this transformations:

\[
\hat{M}^{(R)}_{03} = -i \int dP d\epsilon dp \left( \frac{\partial \delta (\epsilon)}{\partial \epsilon_3} \right) \left( \frac{(P - \epsilon)^2}{4m} \right) \\
\times q^+ \left( \xi_1, p_1 = \frac{1}{2} P - p \right) \hat{q}^+ \left( \xi_2, p_2 = \frac{1}{2} P + p \right) \\
\times \hat{q}^- \left( \xi_2, p_3 = \frac{1}{2} (P - \epsilon) + p \right) \hat{q}^- \left( \xi_2, p_4 = \frac{1}{2} (P - \epsilon) - p \right). 
\]

(53)

Analogously,

\[
\hat{M}^{(x)}_{03} = -i \int dP d\epsilon dp \left( \frac{\partial \delta (\epsilon)}{\partial \epsilon_3} \right) \left( \frac{p^2}{m} \right) \\
\times q^+ \left( \xi_1, p_1 = \frac{1}{2} P - p \right) \hat{q}^+ \left( \xi_2, p_2 = \frac{1}{2} P + p \right) \\
\times \hat{q}^- \left( \xi_2, p_3 = \frac{1}{2} (P - \epsilon) + p \right) \hat{q}^- \left( \xi_2, p_4 = \frac{1}{2} (P - \epsilon) - p \right). 
\]

\[
\hat{M}^{(V)}_{03} = \frac{-i}{(2\pi)^{3/2}} \int dP d\epsilon dp dk \frac{\partial \delta (\epsilon)}{\partial \epsilon_3} V(k) \\
\times q^+ \left( \xi_1, p_1 = \frac{1}{2} P - p \right) \hat{q}^+ \left( \xi_2, p_2 = \frac{1}{2} P + p \right) \\
\times \hat{q}^- \left( \xi_2, p_3 = \frac{1}{2} (P - \epsilon) + p + k \right) \hat{q}^- \left( \xi_2, p_4 = \frac{1}{2} (P - \epsilon) - p - k \right). 
\]

(54)

Now we have the expressions for generator \( \hat{M}_{03} \) and for internal Hamiltonian we are able to calculate their commutator. Due to the fact that each of the operators is composed of a few components, we will consider the commutators between this components. To simplify the calculating of commutators by Wick’s theorem we will lead the products of operators in different order to the normal form.

Previously, we have used the fact that all operators are considered on Fock subspace state which contains one quark and one antiquark. Therefore operators, which in normal form containing two or more quark/antiquark operators (creation or annihilation) will have zero matrix elements for all basic elements on the Fock subspace, which we actually consider. Therefore, we will discard such operators. Given that each of the operators, which we consider contain one quark and one antiquark operators, their product will contain two quark operators and two antiquarks operators. So this product will contain “extra” operators. By applying Wick’s theorem to this product, we find that nonzero matrix elements on the considered space will have only those summands for which the “extra” operators are paired together.

Let’s consider the product \( \hat{H}^{(x)} \hat{M}^{(R)}_{03} \). After reduction to normal form and discarding of
terms with zero matrix elements, this product can be written in the form

\[
\hat{H}^{(r)} \hat{M}^{(R)}_{0,3} = -i \int d\mathbf{p} d\mathbf{p}' d\mathbf{p}'' d\varepsilon \left( \frac{p^2}{m} \right) \left( \frac{(P' - \varepsilon)^2}{4m} \right) \left( \frac{\partial \delta (\varepsilon)}{\partial \varepsilon_3} \right) \\
\times \delta \left( \left( \frac{1}{2} P + p \right) - \left( \frac{1}{2} P' + p' \right) \right) \delta \left( \left( \frac{1}{2} P - p \right) - \left( \frac{1}{2} P' - p' \right) \right) \\
\times \hat{q}^+ (\xi_1, p_1) \hat{q}^+ (\xi_2, p_2) \hat{q}^- (\xi_1, p_4 = \frac{1}{2} (P' - \varepsilon) - p') \\
\times \hat{q}^+ (\xi_2, p_3 = \frac{1}{2} (P - \varepsilon) + p') \hat{q}^- (\xi_1, p_2 = \frac{1}{2} P + p). 
\] (55)

Here we have already considered the subsum on the internal indexes \( \xi \) with the Kronecker \( \delta \)-symbols that appear in paired operators. Please note that during the integration we returned back to the old variables

\[
p_1 = \frac{1}{2} P - p, \quad p_2 = \frac{1}{2} P + p, \\
p_3 = \frac{1}{2} P' + p', \quad p_4 = \frac{1}{2} P' - p'. 
\] (56)

Afterwards, due to the presence of \( \delta \)-functions one may easily perform the integration of components momenta \( p_3 \) and \( p_4 \). And as a result, obtain

\[
\hat{H}^{(r)} \hat{M}^{(R)}_{03} = -i \int d\mathbf{p}_1 d\mathbf{p}_2 d\varepsilon \left( \frac{\partial \delta (\varepsilon)}{\partial \varepsilon_3} \right) \left( \frac{(p_2 - p_1)^2}{4m} \right) \left( \frac{(p_1 + p_2 - \varepsilon)^2}{4m} \right) \\
\times \hat{q}^+ (\xi_1, p_1) \hat{q}^+ (\xi_2, p_2) \hat{q}^- (\xi_2, p_2 - \frac{1}{2} \varepsilon) \hat{q}^- (\xi_1, p_1 - \frac{1}{2} \varepsilon). 
\] (57)

In the case of multiplication of the same operators in reverse order after the analogous transformations we have

\[
\hat{M}^{(R)}_{03} \hat{H}^{(r)} = -i \int d\mathbf{p}_1 d\mathbf{p}_2 d\varepsilon \delta \left( \frac{(p_1 - \frac{1}{2} \varepsilon)}{4m} \right) \left( \frac{(p_1 + p_2 - \varepsilon)^2}{4m} \right) \left( \frac{\partial \delta (\varepsilon)}{\partial \varepsilon_3} \right) \\
\times \hat{q}^+ (\xi_2, p_3 - \frac{1}{2} \varepsilon) \hat{q}^+ (\xi_1, p_1) \hat{q}^- (\xi_2, p_2) \hat{q}^- (\xi_1, p_1 - \frac{1}{2} \varepsilon) \hat{q}^- (\xi_2, p_3). 
\] (58)

If we execute this expression and integrating the components \( p_3 \) and \( p_4 \), we will get the result that equal to Eq.57. Hence we conclude that the operators \( \hat{M}^{(R)}_{03} \) and \( \hat{H}^{(r)} \) commute with each other.

Now we consider the commutator \( [\hat{M}^{(R)}_{03}, \hat{H}^{(r)}] \). After reduction to normal form and return to the “old” variables we have

\[
\hat{M}^{(R)}_{03} \hat{H}^{(r)} = -i \frac{1}{(2\pi)^{3/2}} \int d\mathbf{p} d\mathbf{p}' d\mathbf{k} d\varepsilon \left( \frac{\partial \delta (\varepsilon)}{\partial \varepsilon_3} \right) \left( \frac{(p_2 + p_1 - \varepsilon)^2}{4m} \right) V (k) \\
\times \hat{q}^+ (\xi_1, p_1) \hat{q}^+ (\xi_2, p_2) \hat{q}^- (\xi_2, p_3 = (p_2 - \frac{1}{2} \varepsilon) + k) \hat{q}^- (\xi_1, p_4 = (p_1 - \frac{1}{2} \varepsilon) - k). 
\] (59)
Calculation of product $\hat{H}^{(v)\hat{M}_{03}^{(R)}}$ after similar transformations leads to the expression which coincides with the right-hand side of equality Eq. 59. Hence, operator $\hat{M}_{03}^{(R)}$ commutes with each one of the two summands of internal Hamiltonian and thus commutes with the Hamiltonian of the bound quarks system.

Let us now calculate the commutators of the operator $\hat{M}_{03}^{(r)}$ with the summands of internal Hamiltonian. With the help of transformations considered above a product $\hat{H}^{(r)\hat{M}_{03}^{(r)}}$ is reduced to the form:

$$\hat{H}^{(r)\hat{M}_{03}^{(r)}} = -i \int dp_1 dp_2 dp_3 dp_4 d\epsilon \left( \frac{\partial \delta(\epsilon)}{\partial \epsilon} \right) \left( \frac{(p_2 - p_1)^2}{4m} \right) \left( \frac{(p_2 - p_1)^2}{4m} \right)$$

$$\times \hat{q}^+ (\xi_1, p_1) \hat{q}^+ (\xi_2, p_2) \hat{q}^- (\xi_2, p_3 = p_2 + k - \frac{1}{2} \epsilon) \hat{q}^- (\xi_1, p_4 = p_1 - k - \frac{1}{2} \epsilon).$$

The same result is obtained if one construct the normal form from these operators product but in inverse order. Thus $\hat{M}_{03}^{(r)}$ and $\hat{H}^{(r)}$ commutes with each other.

Calculation of the product $\hat{H}^{(v)\hat{M}_{03}^{(r)}}$ leads to result

$$\hat{H}^{(v)\hat{M}_{03}^{(r)}} = \frac{-i}{(2\pi)^{3/2}} \int dp_1 dp_2 dp_3 dp_4 d\epsilon d\kappa V(\kappa) \left( \frac{\partial \delta(\epsilon)}{\partial \epsilon} \right) \left( \frac{(p_2 - p_1 + 2\kappa)^2}{4m} \right)$$

$$\times \hat{q}^+ (\xi_1, p_1) \hat{q}^+ (\xi_2, p_2) \hat{q}^- (\xi_2, p_3 = p_2 + k - \frac{1}{2} \epsilon) \hat{q}^- (\xi_1, p_4 = p_1 - k - \frac{1}{2} \epsilon).$$

Calculating the product of these operators in inverse order $\hat{M}_{03}^{(r)} \hat{H}^{(v)}$ gives the result that does not coincide with Eq. 61, therefore operators $\hat{M}_{03}^{(r)}$ and $\hat{H}^{(v)}$ do not commute with each other:

$$\hat{M}_{03}^{(r)} \hat{H}^{(v)} = \frac{-i}{(2\pi)^{3/2}} \int dp_1 dp_2 dp_3 dp_4 d\epsilon d\kappa d\kappa V(\kappa) \left( \frac{\partial \delta(\epsilon)}{\partial \epsilon} \right) \left( \frac{(p_2 - p_1)^2}{4m} \right)$$

$$\hat{q}^+ (\xi_1, p_1) \hat{q}^+ (\xi_2, p_2) \hat{q}^- (\xi_2, p_3 = p_2 + k - \frac{1}{2} \epsilon) \hat{q}^- (\xi_1, p_4 = p_1 - k - \frac{1}{2} \epsilon).$$

But

$$\hat{M}_{03}^{(v)} \hat{H}^{(r)} = \frac{-i}{(2\pi)^{3/2}} \int dp_1 dp_2 dp_3 dp_4 d\epsilon d\kappa d\kappa \left( \frac{\partial \delta(\epsilon)}{\partial \epsilon} \right) \left( \frac{(p_2 - p_1 + 2\kappa)^2}{4m} \right)$$

$$\times \hat{q}^+ (\xi_1, p_1) \hat{q}^+ (\xi_2, p_2) \hat{q}^- (\xi_2, p_3 = p_2 + k - \frac{1}{2} \epsilon) \hat{q}^- (\xi_1, p_4 = p_1 - k - \frac{1}{2} \epsilon).$$

and

$$\hat{H}^{(r)\hat{M}_{03}^{(v)}} = \frac{-i}{(2\pi)^{3/2}} \int dp_1 dp_2 dp_3 dp_4 d\epsilon d\kappa d\kappa \left( \frac{\partial \delta(\epsilon)}{\partial \epsilon} \right) \left( \frac{(p_2 - p_1)^2}{m} \right)$$

$$\times \hat{q}^+ (\xi_1, p_1) \hat{q}^+ (\xi_2, p_2) \hat{q}^- (\xi_2, p_3 = p_2 + k - \frac{1}{2} \epsilon) \hat{q}^- (\xi_1, p_4 = p_1 - k - \frac{1}{2} \epsilon).$$
Given that the expressions $\hat{H}^{(V)} \hat{M}_{03}^{(r)}$ in Eq.61 and $\hat{M}_{03}^{(V)} \hat{H}^{(r)}$ (see Eq.63) are included in the general expression for commutator $[\hat{H}^{(r,V)}, \hat{M}_{03}]$ with opposite signs. And, since, same thing can be said about Eq.62 and Eq.64, we conclude:

$$[\hat{H}^{(r)}, \hat{M}_{03}^{(V)}] + [\hat{H}^{(V)}, \hat{M}_{03}^{(r)}] = 0. \quad (65)$$

Finally, computation of products $\hat{M}_{03}^{(V)} \hat{H}^{(V)}$ and $\hat{H}^{(V)} \hat{M}_{03}^{(V)}$ give us the same result:

$$\hat{M}_{03}^{(V)} \hat{H}^{(V)} = \hat{H}^{(V)} \hat{M}_{03}^{(V)} = -\frac{i}{(2\pi)^3} \int d\mathbf{p}_1 d\mathbf{p}_2 d\varepsilon d\mathbf{p}' d\varepsilon' \frac{\partial \delta (\varepsilon)}{\partial \varepsilon} V(\mathbf{k}) V(\mathbf{k}') \times \hat{q}^+ (\xi_1, \mathbf{p}_1) \hat{q}^+ (\xi_2, \mathbf{p}_2) \hat{q}^- \left( \xi_2, \mathbf{p}_3 = \mathbf{p}_2 + \mathbf{k}' + \mathbf{k} - \frac{1}{2} \varepsilon \right) \hat{q}^- \left( \xi_1, \mathbf{p}_4 = \mathbf{p}_1 - \mathbf{k}' - \mathbf{k} - \frac{1}{2} \varepsilon \right). \quad (66)$$

Therefore, if we decompose the commutator $[\hat{H}^{(r,V)}, \hat{M}_{03}]$ on the terms that corresponds to the summands of internal Hamiltonian and generator, then the sum of all these summands is equal to zero, hence

$$[\hat{M}_{03}, \hat{H}^{(r,V)}] = 0. \quad (67)$$

So, as noted early in this paper, the state Eq.21 is the eigenstate for the internal Hamiltonian, which corresponds to non-degenerate eigenstate. As consequence of Eq.67 the state Eq.21 must be the eigenstate for the generator of boost:

$$\hat{M}_{03} |\mu\rangle = m_{03} |\mu\rangle, \quad (68)$$

where $m_{03}$ is the eigenvalue of generator $\hat{M}_{03}$ that correspond to the state $|\mu\rangle$.

In order to determine the eigenvalue $m_{03}$, we make use of symmetry features of the eigenstate $|\mu\rangle$. That is, in particular, this state should transform into itself upon an arbitrary inversion of coordinate exes. Besides, if we suppose that the potential of the quarks and antiquarks interaction is spherically symmetric, then the ground state of this system should also be spherically symmetric, i.e. the state that turn into itself upon an arbitrary rotation. If we denote the unitary operator that represents an inversion or a rotation on the Fock subspace by as $\hat{U}^{(I,R)}$, then we have

$$\hat{U}^{(I,R)} |\mu\rangle = |\mu\rangle. \quad (69)$$

Hence, Eq.68 can be written as

$$\hat{M}_{03} \hat{U}^{(I,R)} |\mu\rangle = m_{03} \hat{U}^{(I,R)} |\mu\rangle, \quad (70)$$

or

$$\left( \hat{U}^{(I,R)} \right)^{-1} \hat{M}_{03} \hat{U}^{(I,R)} |\mu\rangle = m_{03} |\mu\rangle. \quad (71)$$

Operator $\left( \hat{U}^{(I,R)} \right)^{-1} \hat{M}_{03} \hat{U}^{(I,R)}$ associated with $\hat{M}_{03}$ by the tensor transformation rules. It means that we choose an inverse or a rotation, that changes the orientation of $z$ axis to the opposite one, then, we have

$$\left( \hat{U}^{(I,R)} \right)^{-1} \hat{M}_{03} \hat{U}^{(I,R)} = -\hat{M}_{03}. \quad (72)$$
But then, inserting Eq. 72 into Eq. 71 with account of Eq. 68, we get

$$m_{03} = 0.$$ (73)

So, if we note that $|\mu'\rangle$ is the bound state of two-particle system in the reference frame, which is obtained from the c.m.s. of these particles by the boost transformation along $z$ axis with rapidity $Y$, then we have

$$|\mu'\rangle = \exp\left(i\hat{M}_{03}Y\right)|\mu\rangle.$$ (74)

Given Eq. 68, 73, we see that from the whole series, that represents the result of acting by the operator's exponent $\exp\left(i\hat{M}_{03}Y\right)$ on the state $|\mu\rangle$, there is the unity-operator summand only, which yields a non-zero result. So, we have

$$|\mu'\rangle = |\mu\rangle.$$ (75)

This result matches with the result shown in Eq. 17, which was obtained in the representation of the differential operators. Hence, we draw the conclusion, that the internal state of nonrelativistic system of bound particles does not change upon the boost transformation with boost to the reference frame, in which this bound system holds a relativistic energy-momentum.

IV. THE STATE TRANSFORMATION PROBLEM TREATMENT WITHIN THE COMMON GROUP-THEORETICAL CONSIDERATION

In the previous sections we have considered two different representations of the boost generator $\hat{M}_{03}$ which have led us to the identical results. The question therefore arises whether these results can be generalized. This generalization can be achieved if we review the problem using the general group-theoretical concerns.

Let us consider the generators of Poincare group. We have four generators of space-time translations $\hat{P}_a$, where $a = 0, 1, 2, 3$ and six Lorentz generators $\hat{M}_{ab} = -\hat{M}_{ba}$. Commutation relations between these generators depend only on the group multiplication law, whereas the above mentioned state-transformation features of quantum systems of interacting particles do not affect these commutation relations. In addition, these commutation relations do not depend on the exact choice of the representation of these generators, so we can examine them without using the explicit form of these generators. As it is known from [4] the operator $g^{ab}\hat{P}_a\hat{P}_b$ commutes with all generators of the Poincare group and particularly with the generator $\hat{M}_{03}$ that we are interested in. Take into account also that in accordance with the field quantization postulate [3] the generator $\hat{P}_0$ should match with the total Hamiltonian of the system, and the operators $\hat{P}_b$, where $b = 1, 2, 3$, should match with the operators of the momentum components. Then, one may note that the operator $g^{ab}\hat{P}_a\hat{P}_b$ matches with the square of the internal Hamiltonian of the system, since all eigenvalues of this operator are equal to the corresponding squared eigenvalues of the internal energy of the considered particle system.

As a result, the boost generator commutes with the square of the internal Hamiltonian irrespective of the possibility to apply the nonrelativistic approximation in the c.m.s of the examined particle system. If such approximation is possible as in the case of interest,
then the eigenstate that corresponds to the smallest eigenvalue of the square of the internal Hamiltonian is such that corresponds to the non-degenerate eigenvalue. Then, because of the commutativity of the operators $g^{ab} \hat{P}_a \hat{P}_b$ and $\hat{M}_{03}$, we get that this state is eigenstate also for $\hat{M}_{03}$. It implies as we have noticed that this state does not change with the boost. Thus as seen the most significant factor is the non-degeneracy of the ground state of the bound system.

Therefore we can say that from the commutation relations between the Poincare group generators it implies that if the internal state of the interacting particles system (i.e. the eigenstate of the squared internal Hamiltonian) is non-degenerate, then it will not change with the boost.

V. DISCUSSION OF RESULTS AND CONCLUSIONS

It is known that the ground state of the system with the central interaction is spherically symmetric. From the obtained result we conclude that this state will not change during the transition to the new reference frame. In other words, the state remains spherically symmetric and does not undergo to the Lorentz contraction.

Thus the Lorentz contraction in the bound quantum systems does not exist. This conclusion can be very important, since the description of the elastic scattering of hadron-hadron processes rely on the geometric models of discs [5], which are based on the assumption that the contraction takes place.

The physical reasoning for this follows from the previous discussion. Indeed, when we consider the problem of the rod contraction in the rod’s rest frame it is not sufficient whether the coordinates of its ends are measured simultaneously or not. Rather, it is sufficient only to ensure the simultaneity of measurements of the rod’s ends coordinates in the reference frame relative to which it is moving. Therefore, the problem of rod contraction can be considered in terms of the relations between the coordinates and time of the same events, measured in the two different reference frames. This is not the case for the problem considered in the paper. In measuring the coordinates of interacting particles in an arbitrary reference frame we have to ensure that the measurements are simultaneous with respect to the reference frame, relative to which the measurements are taken. This makes an essential difference to the problem of rod contraction. And unlike the rod case, there is no such reference frame, where the measurements of particle coordinates could be taken not at the same time. This leads to the fact that we should talk about different measurement events held by different observers. Then the coordinates and time of those events are not connected via Lorentz transformation because these transformations connect the coordinates and time of the same event measured in the different reference frames. Taking into account that via the Lorentz contraction is the exact result of the Lorentz transformation formulas, which do not hold in our case, there is no wonder that we have obtained the result that such a contraction does not take place in the considered problem.

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