Low-dimensional irreducible rational representations of classical algebraic groups

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20th November 2018

Abstract

Let $G$ be an algebraic group of classical type of rank $l$ over an algebraically closed field $K$ of characteristic $p$. We list and determine the dimensions of all irreducible $KG$-modules $L$ with $\dim L < \binom{l+1}{4}$ if $G$ is of type $A_l$, and with $\dim L < 16\binom{l}{4}$, if $G$ is of type $B_l$, $C_l$ or $D_l$.

1 Introduction

Let $K$ be an algebraically closed field of characteristic $p > 0$. For every simply connected simple linear algebraic group $G$ over $K$ of rank $l$, the irreducible $KG$-modules with dimension below a bound proportional to $l^2$ were determined by Liebeck in [8]. Lübeck [9] extended these results taking a bound proportional to $l^3$. For groups of type $A_l$, this bound was $l^3/8$; for types $B_l$, $C_l$ and $D_l$, the bound was $l^3$. Extending this classification further is desirable for some applications (see for example [4, 7]). The bound we take here is $\binom{l+1}{4}$ if $G$ is of type $A_l$, and $16\binom{l}{4}$ if $G$ is of type $B_l$, $C_l$ or $D_l$.

The irreducible $KG$-modules are parameterised by dominant weights $\lambda$, we denote them by $L(\lambda)$. Due to Steinberg’s tensor product theorem, we need only consider the case where $\lambda$ is $p$-restricted. For small ranks ($l \leq 20$ if $G$ has type $A_l$ and $l \leq 11$ if $G$ has type $B_l$, $C_l$ or $D_l$), lists of weights $\lambda$ with $\dim L(\lambda)$ under the bound we consider can be found in [10]. There, similar lists for groups of exceptional type are also provided. We only consider groups of classical type.

Our results are summarised in the following two theorems. Throughout, $\epsilon_p(k)$ will denote 1 if $p$ divides $k$ and 0 otherwise.

**Theorem 1.1.** Let $G$ be a simply connected simple algebraic group of type $A_l$ and let $l \geq 9$. Table 1 contains all nonzero $p$-restricted dominant weights $\lambda$ up to duals such that $\dim L(\lambda) < \binom{l+1}{4}$, as well as the dimensions of the corresponding modules $L(\lambda)$.

**Theorem 1.2.** Let $G$ be a simply connected simple algebraic group of type $B_l$, $C_l$ or $D_l$ and let $l \geq 9$. Tables 2, 3 and 4 contain all nonzero $p$-restricted dominant weights $\lambda$ such that $\dim L(\lambda) < 16\binom{l}{4}$, as well as the dimensions of the corresponding modules $L(\lambda)$. Note that for $p = 2$ the modules for type $B_l$ and for type $C_l$ have the same dimensions; we only list them in Table 3.
| $\lambda$ | $\dim L(\lambda)$ |
|----------|------------------|
| $\lambda_1$ | $l + 1$ |
| $\lambda_2$ | $\binom{l+1}{2}$ |
| $2\lambda_1$ | $\binom{l+2}{2}$ |
| $\lambda_1 + \lambda_l$ | $(l + 1)^2 - 1 - \epsilon_p(l + 1)$ |
| $\lambda_3$ | $\binom{l+1}{3}$ |
| $3\lambda_1$ | $\binom{l+3}{3}$ |
| $\lambda_1 + \lambda_2$ | $2\binom{l+2}{3} - \epsilon_p(3)\binom{l+1}{3}$ |
| $\lambda_1 + \lambda_{l-1}$ | $3\binom{l+2}{3} - \binom{l+2}{2} - \epsilon_p(l)(l+1)$ |
| $2\lambda_1 + \lambda_l$ | $3\binom{l+2}{3} + \binom{l+1}{2} - \epsilon_p(l+2)(l+1)$ |

Table 1: Type $A_l$
Nonzero $p$-restricted dominant weights $\lambda$ such that $\dim L(\lambda) < \binom{l+1}{4}$ for $l \geq 9$.

| $\lambda$ | $\dim L(\lambda)$ |
|----------|------------------|
| $\lambda_1$ | $2l + 1$ |
| $\lambda_2$ | $\binom{2l+1}{2}$ |
| $2\lambda_1$ | $\binom{2l+2}{2} - \epsilon_p(2l + 1)$ |
| $\lambda_3$ | $\binom{2l+1}{3}$ |
| $3\lambda_1$ | $\binom{2l+3}{3} - (2l + 1) - \epsilon_p(2l + 3)(2l + 1)$ |
| $\lambda_1 + \lambda_2$ | $2^4\binom{l+1}{3} - \epsilon_p(l)(2l + 1) - \epsilon_p(3)(\binom{2l+1}{3})$ |
| $\lambda_l$ ($l \leq 13, p = 2$) | $2^l$ |

Table 2: Type $B_l$, $p \neq 2$
Nonzero $p$-restricted dominant weights $\lambda$ such that $\dim L(\lambda) < 16\binom{l}{4}$ for $l \geq 9$.

| $\lambda$ | $\dim L(\lambda)$ |
|----------|------------------|
| $\lambda_1$ | $2l$ |
| $\lambda_2$ | $\binom{2l}{3} - 2l - \epsilon_p(l)$ |
| $2\lambda_1$ | $\binom{2l+1}{2}$ |
| $\lambda_3$ | $\binom{2l}{3} - 2l - \epsilon_p(l - 1)(2l)$ |
| $3\lambda_1$ | $\binom{2l+2}{3}$ |
| $\lambda_1 + \lambda_2$ | $2^4\binom{l+1}{3} - \epsilon_p(2l + 1)(1 - \epsilon_p(3))(2l) - \epsilon_p(3)(\binom{2l}{3} - 2l)$ |
| $\lambda_l$ ($l \leq 13, p = 2$) | $2^l$ |

Table 3: Type $C_l$
Nonzero $p$-restricted dominant weights $\lambda$ such that $\dim L(\lambda) < 16\binom{l}{4}$ for $l \geq 9$.  

2
\[
\begin{array}{|c|c|}
\hline
\lambda & \dim L(\lambda) \\
\hline
\lambda_1 & 2l \\
\lambda_2 & (\frac{2l}{2}) - \epsilon_p(2)(1 + \epsilon_p(l)) \\
2\lambda_1 & (\frac{2l+1}{2}) - 1 - \epsilon_p(l) \\
\lambda_3 & (\frac{2l}{3}) - \epsilon_p(2)(1 + \epsilon_p(l))(2l) \\
3\lambda_1 & (\frac{2l+2}{3}) - 2l - \epsilon_p(l+1)(2l) \\
\lambda_1 + \lambda_2 & 2^4(l^3 + 1) - \epsilon_p(2l-1)(2l) - \epsilon_p(3)(\frac{2l}{3}) \\
\lambda_{l-1} & 2^{l-1} \\
\lambda_l & 2^{l-1} \\
\hline
\end{array}
\]

Table 4: Type $D_l$

Nonzero $p$-restricted dominant weights $\lambda$ such that $\dim L(\lambda) < 16\binom{l}{4}$ for $l \geq 9$.

Remark 1.1. The bounds have been taken so that (using the notation in Section 3) the tables include exactly the $\lambda$ with $\kappa(\lambda) \leq 3$. In particular, they should exclude the weight $\lambda_4$, and in fact, $|\mathcal{W}\lambda_4| = \binom{l+1}{4}, 16\binom{l}{4}$ respectively if $G$ has type $A_l$ or one of the other types.

In Section 3 it is shown that Tables 1, 2, 3 and 4 contain all the weights that need to be considered. The dimensions of the modules in the tables are established in Section 4.

Acknowledgements

Above all I would like to thank my supervisor Professor Martin Liebeck for his encouragement, guidance and precious comments. I am also thankful for the financial support of the Undergraduate Research Opportunities Program at Imperial College London.

2 Preliminaries

Let $p$ and $K$ be as in the introduction and let $G$ be a simply connected cover of a simple classical algebraic group of rank $l$ over $K$. Let $B = UT$ be a Borel subgroup containing the maximal torus $T$ and with unipotent radical $U$, and let $B^-$ be the opposite Borel subgroup. Denote by $X(T), Y(T)$ respectively the character and cocharacter groups of $T$. Let $\Phi \subset Y(T)$ be the root system of $G$ and denote by $S = \{\alpha_1, ..., \alpha_l\} \subset \Phi$ the base of simple roots for $B$, where we label Dynkin diagrams as in [1]. We denote by $\lambda_1, ..., \lambda_l$ the corresponding fundamental weights with respect to the usual pairing on $X(T) \times Y(T)$. The Weyl group $\mathcal{W} = N_G(T)/T$ is generated by the simple reflections $s_\alpha$ associated to the simple roots $\alpha \in S$. We denote by $w_0 \in \mathcal{W}$ the longest element of the Weyl group.

We recall some standard facts. The irreducible $KG$-modules are parameterised by dominant weights $\lambda \in X(T)$, that is, weights of the form $\lambda = \sum a_i \lambda_i$ with all $a_i \geq 0$. We denote the irreducible module with highest weight $\lambda$ by $L(\lambda)$. Given a $KG$-module $M$, we say that the element $\mu \in X(T)$ is a weight of $M$ if and only if the weight space $M_\mu = \{m \in M : tm = \mu(t)m \text{ for all } t \in T\}$ is
nonzero, and we say that the multiplicity of \( \mu \) in \( M \) is \( \dim M_\mu \). If \( \lambda \) is dominant, we denote the multiplicity of \( \mu \) in \( L(\lambda) \) by \( m_\lambda(\mu) \). A dominant weight as above is \( p \)-restricted if each \( a_i \) satisfies \( 0 \leq a_i < p \). Steinberg’s tensor product theorem [15] allows one to express any \( L(\lambda) \) as a tensor product of twists of \( KG \)-modules with \( p \)-restricted highest weights, therefore we only consider \( p \)-restricted dominant weights.

In order to understand the module \( L(\lambda) \), it can be useful to understand the related induced module \( H^0(\lambda) \) and Weyl module \( V(\lambda) \). In [8], the group \( G \) is regarded as a group scheme. Given a dominant \( \lambda \in X(T) \) and the corresponding \( B^- \)-module \( K_\lambda \), one constructs a left exact functor \( \text{ind}_B^-(-) \) whose derived functors are denoted by \( H^i(\lambda) \). The induced module is simply \( H^0(\lambda) \). In this framework, the Weyl module is defined thus: \( V(\lambda) = H^0(-w_0\lambda)^* \). The induced module \( H^0(\lambda) \) has a unique irreducible submodule isomorphic to \( L(\lambda) \), that is, \( L(\lambda) = \text{soc} H^0(\lambda) \). Dually, \( V(\lambda) \) has \( L(\lambda) \) as its unique irreducible quotient, so we can also realise the latter as \( L(\lambda) = V(\lambda)/\text{rad} V(\lambda) \).

Write \( e(\lambda) \in \mathbb{Z}[X(T)] \) for the element of the group ring of \( X(T) \) corresponding to \( \lambda \in X(T) \). Denote the formal character of a (finite-dimensional) \( KG \)-module \( M \) by \( \chi(M) = \sum_{\mu \in X(T)} \dim M_\mu e(\mu) \). Observe that \( \chi(M) \) encodes all information about weight multiplicities of \( M \) and in particular, it yields \( \dim M \). For any dominant \( \lambda \in X(T) \), one can compute \( \chi V(\lambda) = \chi H^0(\lambda) \) using Weyl’s character formula. There is an Euler characteristic defined for each \( \mu \in X(T) \) as \( \chi(\mu) = \sum_{i \geq 0} (-1)^i \chi H^i(\lambda) \).

If \( \lambda \) is dominant, then Kempf’s vanishing theorem states that \( H^i(\lambda) = 0 \) for all \( i > 0 \). Lastly, recall that both the \( \chi(\lambda) = \chi H^0(\lambda) \) and the \( \chi L(\lambda) \) with \( \lambda \) dominant form \( \mathbb{Z} \)-bases of \( \mathbb{Z}[X(T)]^\mathcal{W} \).

Finally, we remark that there is an isogeny \( \varphi \) between the groups of type \( B_l \) and \( C_l \) when \( p = 2 \), which is an isomorphism of abstract groups but not of algebraic groups. If \( L(\lambda) \) is the irreducible module with highest weight \( \lambda \) for one of the groups, then composing the action of the group with \( \varphi \) yields the corresponding irreducible module for the other group. In particular, the weight multiplicities and dimensions are the same for both types. For this reason, we exclude the case \( p = 2 \) in Table [2].

### 3 Dimensional bounds and reduction

We recall some basic weight theory. There is a partial order \( \leq \) on \( X(T) \) defined by: \( \mu \leq \lambda \) if and only if \( \lambda - \mu \) is a nonnegative linear combination of simple roots. Denote the (usual) action of the Weyl group on \( X(T) \) by \( (w, \mu) \mapsto w\mu \). Then \( m_\lambda(\mu) = m_\lambda(w\mu) \). Assume \( \lambda \) and \( \mu \) are both dominant. If \( \mu \leq \lambda \) and \( \mu \neq \lambda \), we say that \( \mu \) is subdominant to \( \lambda \). The weights of \( L(\lambda) \) form a subset of the weights of \( V(\lambda) \), and every dominant weight of \( V(\lambda) \) is subdominant to \( \lambda \). It follows that \( \dim L(\lambda) = \sum_{\mu \leq \lambda} m_\lambda(\mu) \dim W(\lambda) \), where \( \mu \) runs over all dominant weights. The stabiliser of \( \mu \) in \( \mathcal{W} \) is the subgroup \( \mathcal{W}_\mu \leq \mathcal{W} \) generated by the reflections \( s_\alpha \) such that \( \langle \mu, \alpha \rangle = 0 \). Hence if \( \mu \) is a weight of \( L(\lambda) \) we have the bound

\[
\dim L(\lambda) \geq |\mathcal{W}_\mu| = |\mathcal{W} : \mathcal{W}_\mu|.
\]

Finally, Premet’s theorem [13] asserts that if \( (G, p) \) is not special and \( \lambda \) is \( p \)-restricted, then any \( \mu \leq \lambda \) is a weight of \( L(\lambda) \). For the classical types, the pair \( (G, p) \) is special only if \( p = 2 \) and \( G \) has type \( B_l \) or \( C_l \).

We now proceed to show that any dominant weight \( \lambda \) not in Tables [1][2][3] and [4] must satisfy \( \dim L(\lambda) \geq (l+1)^2 \) if \( G \) has type \( A_l \) or \( \dim L(\lambda) \geq 16l^4 \) if \( G \) has type \( B_l, C_l \) or \( D_l \). Following [11],
for a \( p \)-restricted dominant weight \( \mu = \sum_{i=1}^l a_i \lambda_i \in X(T) \), we define the integers

\[
\kappa(\mu) = \begin{cases} 
\sum_{i=1}^l i a_i & \text{if } G \text{ is of type } B_l, C_l \text{ or } D_l \\
\sum_{i=1}^l \min\{i, l+1-i\} a_i & \text{if } G \text{ is of type } A_l 
\end{cases}
\]

\[
r_{\mu} = \begin{cases} 
0 & \text{if } a_c = 0 \text{ for all } c > \frac{l+1}{2} \\
\max\{c : 1 \leq c < \frac{l+1}{2} \text{ and } a_{l+1-c} \neq 0\} & \text{otherwise.}
\end{cases}
\]

**Proposition 3.1.** Let \( G \) be of type \( A_l \) and let \( \lambda \) be a \( p \)-restricted dominant weight. Set \( \kappa := \kappa(\lambda) \). Then,

\[
\dim L(\lambda) \geq \begin{cases} 
\frac{(l+1)}{\kappa} & \text{if } \kappa \leq \frac{l+1}{2}, \\
\frac{(l+1)}{\left\lfloor \frac{\kappa}{2} \right\rfloor} & \text{otherwise.}
\end{cases}
\]

**Proof.** Both bounds are respectively part of Proposition 4.7 and Lemma 4.8 in [1].

**Corollary 3.2.** Let \( G \) be of type \( A_l \), and assume \( l \geq 9 \). Any nonzero \( p \)-restricted dominant weight \( \lambda \in X(T) \) not listed in Table 1 satisfies \( \dim L(\lambda) \geq \frac{(l+1)}{4} \).

**Proof.** The result follows from Proposition 3.1 and the observation that Table 1 contains precisely the nonzero \( p \)-restricted dominant weights \( \lambda \) with \( \kappa(\lambda) \leq 3 \).

The following can be seen as an analogue of Proposition 3.1 for types \( B_l, C_l \) and \( D_l \).  

**Proposition 3.3.** Let \( G \) be of type \( B_l, C_l \) or \( D_l \), assume \( l \geq 7 \) and let \( \lambda \) be a \( p \)-restricted dominant weight \( \sum_{i=1}^l a_i \lambda_i \). Set \( \kappa := \kappa(\lambda) \). Then, the following hold.

(a) Assume \( r_{\lambda} \neq 0 \) and, if \( G \) has type \( D_l \), assume \( r_{\lambda} \geq 3 \). Then

\[
|\mathcal{W}_\lambda| \geq 2^{l-r_{\lambda}+1}\left(\frac{l}{r_{\lambda}-1}\right).
\]

(b) If \( r_{\lambda} = 0 \) and \( \kappa \leq (l+1)/2 \), then

\[
\dim L(\lambda) \geq 2^{\kappa\left(\frac{l}{\kappa}\right)}.
\]

(c) If \( r_{\lambda} = 0 \) and \( \kappa > (l+1)/2 \), then

\[
\dim L(\lambda) \geq 2^{\left\lfloor \frac{l+2}{2} \right\rfloor\left(\frac{l}{\kappa}\right)}.
\]

**Proof.** For part (a), note that the stabiliser \( \mathcal{W}_\lambda \) is contained in \( \mathcal{W}_{\lambda,-r_{\lambda}+1} \) and use Bound (4). Part (b) follows from Proposition 4.7 in [4] and the observation that \( (l+1)/2 \leq l-3 \). For (c), we argue as in the proof of 4.9(c) in [4]. Define \( d = \max\{i : a_i \neq 0\} \). We consider two cases.

If \( a_d = 1 \), then Lemma 4.5 from [4] ensures that \( L(\lambda) \) has a subdominant weight (with nonzero multiplicity) of the form \( \sum_{i=1}^{\left\lfloor \frac{l+1}{2} \right\rfloor} b_i \lambda_i + \lambda_{\left\lfloor \frac{l+1}{2} \right\rfloor+1} \) and so by (a), we have \( \dim L(\lambda) \geq |\mathcal{W}_\mu| \geq 2^{\left\lfloor \frac{l+2}{2} \right\rfloor\left(\frac{l}{\kappa}\right)} \). Otherwise if \( a_d > 1 \), then \( \mu' = \mu - \alpha_d = \sum_{i=1}^{\left\lfloor \frac{l+1}{2} \right\rfloor} c_i \lambda_i + \lambda_{d+1} \) is subdominant to \( \lambda \).

By the previous case, there exists in turn some \( \nu \leq \mu' \) of the form \( \sum_{i=1}^{\left\lfloor \frac{l+1}{2} \right\rfloor} b'_i \lambda_i + \lambda_{\left\lfloor \frac{l+1}{2} \right\rfloor+1} \). To see that \( \nu \) has nonzero multiplicity in \( L(\lambda) \), observe that \( \lambda \) is \( p \)-restricted and \( a_d > 1 \), hence \( p > 2 \) and Premet’s theorem applies. Again, applying (a) to \( \nu \) yields the desired inequality.

**Corollary 3.4.** Let \( G \) be of type \( B_l, C_l \) or \( D_l \) and assume \( l \geq 9 \). Any nonzero \( p \)-restricted dominant weight \( \lambda \in X(T) \) not listed in Tables 2, 3 and 4 satisfies \( \dim L(\lambda) \geq 16\left(\frac{l}{4}\right) \).
Proof. Write \( \lambda = \sum_{i=1}^l a_i \lambda_i \). First note that all the weights with \( \kappa(\lambda) < 4 \) appear in the three tables, so assume \( \kappa(\lambda) \geq 4 \). Now, if \( r_\lambda = 0 \), the result directly follows from (b) and (c) of Proposition 3.3, so we further assume \( r_\lambda \neq 0 \). If \( r_\lambda \geq 2 \) or, if \( G \) has type \( D_l \), if \( r_\lambda \geq 3 \), then Proposition 3.3(a) shows that \( \dim L(\lambda) \geq 2^{l-r_\lambda + 1}(r_\lambda - 1) \). An elementary check shows that this is greater or equal to \( 2^l \binom{l}{q} \) for all \( l \geq 9 \). We discuss the remaining possibilities separately.

Types \( B_l \) and \( C_l \), \( r_\lambda = 1 \)

In view of the tables, the weight \( \lambda = \lambda_l \) needs only be considered when \( G \) has type \( C_l \) and \( p \neq 2 \).

In this case by Premet’s theorem \( \lambda_l - (\alpha_l - 1 + \alpha_l) = \lambda_{l-2} \) has nonzero multiplicity in \( L(\lambda) \) so Proposition 3.3(a) yields the result. Next, if \( a_j > 0 \) for some \( j < l \), the stabiliser \( W_\lambda \leq W \) is a subgroup of \( W_{\lambda_j + \lambda_l} \). But then in both types \( |W_\lambda| \geq |W(\lambda_j + \lambda_l)| = 2^l \binom{l}{j} \geq 2^l \geq 2^4 \binom{l}{4} \). Finally, if \( a_l > 1 \), then \( p \neq 2 \) and by Premet’s theorem \( \mu = \lambda - \alpha_l - 1 \) is a subdominant weight for both types; but then \( r_\mu = 2 \) and Proposition 3.3(a) yields the inequality.

Remark 3.5. The weights considered in Theorem 5.1 in [9] are precisely the \( p \)-restricted weights \( \lambda \) with \( \kappa(\lambda) \leq 2 \).

4 Dimensions of irreducible \( KG \)-modules

An effective method to compute the multiplicities of the weight spaces in characteristic \( p \) was already observed in [2]. One considers a certain bilinear form on \( V(\lambda)_Z \), a minimal admissible \( Z \)-lattice of the Weyl module. Restricting this form to the weight space of \( \mu \leq \lambda \), reducing it modulo \( p \) and computing its rank yields the multiplicity \( m_\lambda(\mu) \). However, these computations can be lengthy and may not a priori provide much structural information about \( V(\lambda) \) or \( L(\lambda) \). For some cases we will instead find the constituents of modules having \( L(\lambda) \) as a composition factor. Given a \( KG \)-module \( M \), we write \( M = N_1 \mid N_2 \mid \cdots \mid N_k \) to indicate that \( M \) has a filtration \( M = M_1 \supset M_2 \supset \cdots \supset M_{k+1} = 0 \) such that \( M_i/M_{i+1} \cong N_i \) for each \( i = 1, \ldots, k \). Note that the \( N_i \) are not required to be irreducible.

The following tool provides information about \( \text{rad} V(\lambda) \), and it can be interpreted as providing the determinants of the bilinear forms above (this is explained in detail in II.8.17 of [6]). Denote by \( H^0(Z) \) and \( V_Z(Z) \) the induced and Weyl modules over \( Z \). Then in II.8.16 of [6] one defines a homomorphism \( V_Z(\lambda) \to H^0_Z(\lambda) \) which we denote by \( T_\lambda \) such that \( \text{Im}(T_\lambda \otimes Z 1_K) = L(\lambda) \). Now, let \( D \) be the group of divisors of \( Z \), that is, the abelian group generated by the formal elements \( [q] \) for each prime \( q \). Given a finitely generated torsion abelian group \( N \) and a prime \( q \), denote by \( \nu_q(N) \) the composition length of \( N \otimes Z \lambda(q) \) as a \( \lambda(q) \)-module. For a \( G_Z \)-module \( M \), define

\[
\nu^c(M) = \sum_{\mu \in X(T)} \nu(M_\mu)e(\mu) \in D \otimes Z \lambda[X]\]

6
where $\nu(M_\mu) = \sum_{q \text{ prime}} \nu_q(M_\mu)[q] \in D$.

Then, writing $\nu^c(T_\lambda)$ for $\nu^c(\text{coker}(T_\lambda))$, for each subdominant $\mu \leq \lambda$ one has that $L(\mu)$ is a composition factor of $\text{rad} V(\lambda)$ if and only if the coefficient of $[p] \cdot \chi L(\mu)$ in $\nu^c(T_\lambda)$ is nonzero (recall that the $[p] \cdot \chi L(\mu)$ form a basis of $X(T)^p$). In fact, the coefficient of $[p] \cdot c(\mu)$ in $\nu^c(T_\lambda)$ is the $p$-adic valuation of the determinant of the bilinear form mentioned above restricted to the weight space of $\mu$. If this coefficient is $1$ then the composition factor $L(\mu)$ has multiplicity one in $V(\lambda)$, but the converse does not hold in general. We remark that for certain dominant weights $\lambda$, the character $\nu^c(T_\lambda)$ has been evaluated for arbitrary rank (see e.g. [5], [12]).

We now establish the dimension of $L(\lambda)$ for each $\lambda$ in Tables 1, 2, 3 and 4. Note that by Remark 3.5, we only need to consider the weights with $\kappa(\lambda) > 2$.

### 4.1 Type $A_l$

Note first that if $G$ is of type $A_l$, the weights of the form $\lambda_k$, $k \lambda_1$ correspond respectively to the exterior power $(\dim L(k\lambda_1) = \binom{l+1}{k})$ and symmetric power $(\dim L(k\lambda_1) = \binom{l+k}{k})$ of the natural module, which are irreducible (as $\lambda$ is $p$-restricted). The remaining weights $\lambda$ in Table 1 with $\kappa(\lambda) > 2$ are $\lambda_1 + \lambda_2$, $\lambda_1 + \lambda_{l-1}$ and $2\lambda_1 + \lambda_l$. We will make use of the following result, which is part of 8.6 of [14].

**Lemma 4.1.** Let $\lambda = a_i \lambda_1 + a_j \lambda_j$, $i < j$, be $p$-restricted with $a_i a_j \neq 0$. Suppose $\mu = \lambda - (\alpha_i + \ldots + \alpha_j)$. Then $m(\mu) = j - i + 1 - \epsilon_p(a_i + a_j + j - i)$.

The dimensions stated in Table 1 easily follow from this, as the only subdominant weights to those listed either satisfy the conditions of the lemma or they have multiplicity one in the Weyl module (hence in $L(\lambda)$ by Premet’s theorem). We give as an example the weight $\lambda = 2\lambda_1 + \lambda_l$; the other cases can be dealt with in a similar fashion. The subdominant weights are $\lambda - \alpha_1 = \lambda_2 + \lambda_l$ and $\lambda - (\alpha_1 + \ldots + \alpha_l) = \lambda_1$. The multiplicity of $\lambda - \alpha_1$ in the Weyl module is $1$, hence so it is in $L(\lambda)$. By Lemma 4.1 the subdominant weight $\lambda - (\alpha_1 + \ldots + \alpha_l)$ has multiplicity $l - \epsilon_p(l + 2)$. This implies $\text{ch} L(\lambda) = \chi(\lambda) - \epsilon_p(l + 2) \chi(\mu)$ and therefore $\dim L(\lambda) = 3 \binom{l+2}{2} + \binom{l+1}{2} - \epsilon_p(l + 2)(l + 1)$.

### 4.2 Types $B_l$, $C_l$ and $D_l$

We now consider $G$ of type $B_l$, $C_l$ and $D_l$. By Remark 3.5, the weights that need to be considered are the ones with $\kappa(\lambda) > 2$ in Tables 2, 3 and 4 that is, the weights $3\lambda_1$, $\lambda_3$, $\lambda_1 + \lambda_2$, $\lambda_{l-1}$ and $\lambda_l$. The stated dimensions for the module $L(3\lambda_1)$ immediately follow from Proposition 4.7.4 in [12] if $G$ is of type $B_l$ or $D_l$. For $G$ of type $C_l$, consider the natural embedding of $G$ into a group $G$ of type $A_{2l-1}$. The Weyl module with highest weight $3\lambda_1$ is an irreducible $K\tilde{G}$-module and it remains irreducible for $G$ by 8.1(c) of [14] (note that $\lambda$ is $p$-restricted). Similarly, 8.1(a) and 8.1(b) of [14] show that if $p \neq 2$, the module $V(\lambda_3)$ is irreducible for $G$ of type $B_l$ or $D_l$. For $p = 2$ and $G$ of type $D_l$, the dimension of $L(\lambda_3)$ is found in 7.2.5 of [3]. For $G$ of type $C_l$ and any $p$ (so in particular, for type $B_l$ and $p = 2$), the dimension of $L(\lambda_3)$ follows from 4.8.2 in [12].

Next, the weight $\lambda = \lambda_l$ is minuscule (i.e. it has no subdominant weights) for $G$ of types $B_l$ and $D_l$; in addition, if $G$ has type $D_l$, then $\lambda_{l-1}$ is minuscule too. Hence $V(\lambda)$ is irreducible in these cases. Clearly, if $\lambda$ is minuscule then $\dim L(\lambda)$ is just $|\Psi\lambda|$.

The only remaining weight is $\lambda_1 + \lambda_2$. The following is a direct consequence of 4.9.2 in [12].
Proposition 4.2. Let $\lambda = \lambda_1 + \lambda_2$ and assume $l \geq 5$ and $p > 3$. Set $t = l, 2l + 1, 2l - 1$ respectively if $G$ has type $B_l$, $C_l$ or $D_l$. Then $\text{ch} L(\lambda) = \chi(\lambda) - e_p(t)\chi(\lambda_1)$.

McNinch also provides the following computation for $l \geq 5$ (4.5.7 in [12]). Here, for any positive integer $k = q_1^{b_1} \ldots q_r^{b_r}$ (where $q_i$ are primes and each $b_i$ is a positive integer), the divisor of $k$ is defined as $\text{div}(k) = \sum_{i=1}^{r} b_i | q_i|$. Let $\lambda = \lambda_1 + \lambda_2$.

$$\nu^c(T_\lambda) = \begin{cases} \text{div}(3)\chi(\lambda_3) + \text{div}(2l)\chi(\lambda_1) + \text{div}(2)(\chi(2\lambda_1) + \chi(\lambda_2) - \chi(0)) & \text{if } G \text{ is of type } B_l, \\ \text{div}(3)\chi(\lambda_3) + \text{div}(2l+1)\chi(\lambda_1) & \text{if } G \text{ is of type } C_l, \\ \text{div}(3)\chi(\lambda_3) + \text{div}(2l-1)\chi(\lambda_1) & \text{if } G \text{ is of type } D_l. \end{cases}$$ (2)

Since for types $C_l$ and $D_l$, the coefficient of $[2]$ in $\nu^c(T_\lambda)$ is zero, it follows that for these types the Weyl module $V(\lambda_1 + \lambda_2)$ is irreducible if $p = 2$. Clearly $\text{dim} L(\lambda_1 + \lambda_2)$ is also determined for type $B_l$ and $p = 2$. Now set $t = 2l, 2l + 1, 2l - 1$ respectively if $G$ has type $B_l$, $C_l$ or $D_l$. In view of the computation of $\nu^c(T_\lambda)$, we see that if $3 \nmid t$, then $\text{rad} V(\lambda_1 + \lambda_2) = L(\lambda_3)$ and the dimensions in the tables for this case follow too.

Observe that Equation (2) is however not enough to determine $\text{dim} L(\lambda_1 + \lambda_2)$ when $p = 3$ and $3 \nmid t$. Instead, we will realise the module $L(\lambda_1 + \lambda_2)$ as a composition factor of $S^3 V$. In what follows, let $V$ be the natural module for $G$ and, for an integer $k \geq 1$, denote by $S^k V$ the $k$-th symmetric power of $V$. We write the elements of $S^k V$ as linear combinations of monomials $v_1 v_2 \cdots v_k$, where each $v_i \in V$.

Lemma 4.3. Let $p > 2$.

(a) If $G = \text{SL}_{t+1}(K)$ or $G = \text{Sp}_{2l}(K)$, then $H^0(p\lambda_1) = S^p V = L((p - 2)\lambda_1 + \lambda_2) \mid L(p\lambda_1)$.

(b) If $G = \text{SL}_{t+1}(K)$, then $H^0(p\lambda_1) = S^p(V^*) = L(\lambda_{l-1} + (p - 2)\lambda_1) \mid L(p\lambda_1)$.

Remark 4.4. Note that $L(p\lambda_1) \cong L(\lambda_1)^{(p)}$.

Proof. For (a), notice that when $\text{char} K = p$, the unique irreducible submodule of $H^0(p\lambda_1) = S^p V$ is $N = \{v^p : v \in V\}$, of highest weight $p\lambda_1$. This quotient is the $p$-th reduced symmetric power of $V$, which is irreducible for $G = \text{SL}_t(K), \text{Sp}_{2l}(K)$ respectively by 1.2 and 2.2 of [16]. Now observe that the highest weight of the quotient $S^p V / N$ is $(p - 2)\lambda_1 + \lambda_2$. Part (b) is analogous.

Corollary 4.5. If $G$ has type $C_l$ then $S^3 V = L(\lambda_1 + \lambda_2) \mid L(3\lambda_1)$.

The dimension of $L(\lambda_1 + \lambda_2)$ in Table 3 is now justified for all $p$. Note also that Lemma 4.3 yields again the dimension of $L(\lambda_1 + \lambda_2)$ in Table 1 for $p = 3$.

For types $B_l$ and $D_l$, we will use a result about type $A_l$. Let $G = \text{SL}_{t+1}(K)$. We denote by $e_1, \ldots, e_{l+1}$ the elements of the standard basis of $V$. We also write $e^*_1, \ldots, e^*_{l+1}$ for the corresponding basis elements of the dual module $V^*$.

Lemma 4.6. Let $G = \text{SL}_{t+1}(K)$ and assume $p = 3$ and $3 \nmid t + 2$.

(a) The module $S^2 V \otimes V^* = L(\lambda_1) \mid L(2\lambda_1 + \lambda_l) \mid L(\lambda_1)$ is indecomposable and $\text{soc}(S^2 V \otimes V^*) = \{\sum_{i=1}^{l+1} ve_i \otimes e_i^* : v \in V\}$.

(b) The module $S^2(V^*) \otimes V = L(\lambda_l) \mid L(\lambda_l + 2\lambda_l) \mid L(\lambda_l)$ is indecomposable and $\text{soc}(S^2(V^*) \otimes V) = \{\sum_{i=1}^{l+1} v^* e^*_i \otimes e_i : v^* \in V^*\}$. 

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Proof. By Proposition 4.6.10 in [2], the modules are indecomposable and have the stated constituents. The fact that \( \text{soc}(S^2V \otimes V^*) = \{ \sum v e_i \otimes e_i^* : v \in V \} \) follows from the observation that it is a submodule isomorphic to \( V \) and the fact that \( S^2V \otimes V^* \) is indecomposable. The argument for \( \text{soc}(S^2(V^*) \otimes V) \) is analogous.

In the following, instead of taking the simply connected group of type \( B_l \) or \( D_l \), we take \( G \) to be the special orthogonal group \( \text{SO}_n(K) \) (\( n = 2l + 1, 2l \) respectively), realised as follows. Let \( e_1, \ldots, e_l, e_o, e_{-l}, \ldots, e_{-1} \) be the elements of the ordered standard basis of \( \text{SL}_n(K) \) (dropping \( e_0 \) for type \( D_l \)). We define \( \text{SO}_n(K) \) as the subgroup of \( \text{SL}_n(K) \) preserving the quadratic form \( q(\sum x_i e_i) = \sum_{i=1}^n x_i x_{-i} + \frac{1}{2} x_0^2 \) (dropping the term \( \frac{1}{2} x_0^2 \) for type \( D_l \)).

Given a \( KG \)-module, we denote by \( M \downarrow H \) the restriction of \( M \) to a subgroup \( H \) of \( G \).

**Proposition 4.7.** Let \( G = \text{SO}_n(K) \), where \( n = 2l \) or \( 2l + 1 \), and set \( p = 3 \).

(a) Suppose \( 3 \nmid n - 1 \). Then \( S^3V = L(\lambda_1 + \lambda_2) \mid (L(3\lambda_1) \oplus L(\lambda_1)) \).

(b) Suppose \( 3 \mid n - 1 \). Then \( S^3V = L(\lambda_1) \mid (L(\lambda_1 + \lambda_2) \mid (L(3\lambda_1) \oplus L(\lambda_1)) \).

**Proof.** Fix \( B \) to be the Borel subgroup of upper triangular matrices in \( G \). Define \( Q \in S^2V \) as \( Q = \frac{1}{2} e_0^2 + \sum_{i=1}^l e_i e_{-i}, \) dropping the term \( \frac{1}{2} e_0^2 \) if \( n = 2l \). Define also \( J = \{ v Q : v \in V \} \subset S^3V \). By II.2.18 of [3], we have that \( H^0(3\lambda_1) \cong S^3V/J \). Clearly \( J \cong L(\lambda_1) \). Also, as in the proof of Lemma 4.3, since \( \text{char} K = 3 \), we have the irreducible submodule \( N = \{ v^3 : v \in V \} \subset S^3V \), and again \( N \cong L(3\lambda_1) \). Since \( H^0(3\lambda_1) \) has a unique irreducible submodule and \( N \cap J = 0 \), it follows that \( \text{soc} S^3V = N \oplus J \).

Now, to see (a), note that \( 3 \nmid n - 1 \) implies \( 3 \nmid t \), thus \( \dim L(\lambda_1 + \lambda_2) \) is known by the discussion after Equation 4.3. Since \( S^3V / \text{soc} S^3V = S^3V / (N \oplus J) \) has a maximal vector with weight \( \lambda_1 + \lambda_2 \) (namely, \( e_1^2 e_2 + (N \oplus J) \)) and it has the same dimension as \( L(\lambda_1 + \lambda_2) \), the result follows. For (b), we separate into cases \( D_l \) and \( B_l \).

**Case \( n = 2l \)**

Assume \( 3 \mid 2l - 1 \). Let \( H \) be the subgroup of type \( A_{l-1} \) inside \( G \) stabilising the subspace in \( V \) spanned \( e_1, \ldots, e_l \in V \) as well as the subspace spanned by \( e_{-l}, \ldots, e_{-1} \). Denote by \( \tilde{V} \) the natural module for this subgroup, as well as \( \tilde{L}(\mu) \) for the irreducible module of \( H \) with highest weight \( \mu \). The restriction \( V \downarrow H = \tilde{V} \oplus \tilde{V}^* \) yields a decomposition

\[
S^3V \downarrow H = S^3\tilde{V} \oplus S^3(\tilde{V}^*) \oplus (S^2\tilde{V} \otimes \tilde{V}^*) \oplus (S^2(\tilde{V}^*) \otimes \tilde{V}).
\]

Visibly, \( N \) corresponds to \( \tilde{L}(3\lambda_1) \oplus \tilde{L}(3\lambda_l) \subset S^3\tilde{V} \oplus S^3(\tilde{V}^*) \). By Lemma 4.3, we have \( (S^3\tilde{V} \oplus S^3(\tilde{V}^*/N) \cong \tilde{L}(\lambda_1 + \lambda_2) \oplus \tilde{L}(\lambda_{l-1} + \lambda_l) \). It follows that

\[
(S^3V/N) \downarrow H = \tilde{L}(\lambda_1 + \lambda_2) \oplus \tilde{L}(\lambda_{l-1} + \lambda_l) \oplus (S^2\tilde{V} \otimes \tilde{V}^*) \oplus (S^2(\tilde{V}^*) \otimes \tilde{V}).
\]

Next, note that as a \( KH \)-module, \( J = J_1 \oplus J_l \), where \( J_1 = \{ \sum_{i=1}^l v e_i \otimes e_{-i} : v \in \tilde{V} \} \subset \text{soc} S^2V \otimes V^* \) and \( J_l = \{ \sum_{i=1}^l v^* e_{-i} \otimes e_i : v^* \in \tilde{V}^* \} \subset \text{soc} S^2(\tilde{V}^*) \otimes \tilde{V} \). Lemma 4.6 shows that \( J = \text{soc}(S^2\tilde{V} \otimes \tilde{V}^*) \oplus \text{soc}(S^2(\tilde{V}^*) \otimes \tilde{V}) \cong \tilde{L}(\lambda_1) \oplus \tilde{L}(\lambda_l) \). Write \( M_1 = (S^2\tilde{V} \otimes \tilde{V}^*)/J_1 \) and \( M_l = (S^2(\tilde{V}^*) \otimes \tilde{V})/J_l \). Denoting \( M = S^3V/(N \oplus J) \), we have

\[
M \downarrow H = \tilde{L}(\lambda_1 + \lambda_2) \oplus \tilde{L}(\lambda_{l-1} + \lambda_l) \oplus M_1 \oplus M_l
\]
where again by Lemma 4.6, \( M_i \) is indecomposable and has composition factors \( \tilde{L}(\lambda_i) \mid \tilde{L}(2\lambda_i + \lambda_{-i+1}) \) for \( i = 1, l \).

Now, let \( I \) be a nonzero irreducible \( KG \)-submodule of \( M \). Observe that the action of the antidiagonal element \( s \in G \) gives (nonzero) linear maps \( \tilde{L}(\lambda_1 + \lambda_2) \leftrightarrow \tilde{L}(\lambda_{-1} + \lambda_1) \) and \( M_1 \leftrightarrow M_l \). This shows that \( I \) must contain at least one of \( R_1 = \tilde{L}(\lambda_1 + \lambda_2) \oplus \tilde{L}(\lambda_{-1} + \lambda_1) \) or \( R_2 = \tilde{L}(2\lambda_1 + \lambda_l) \oplus \tilde{L}(2\lambda_l + \lambda_1) \) (as \( M_1, M_l \) are indecomposable for \( H \)). We show that in fact \( I \) must contain \( R_1 \oplus R_2 \). Let \( r \in G \) be the element sending \( e_2 \mapsto e_2 + e_1, e_{-l} \mapsto e_{-l} - e_{-2} \) and fixing \( e_j \) for every \( j \neq 2, -l \). Define also \( x_1 = e_1^2 e_2 + (N \oplus J) \in R_1 \) and \( x_2 = e_l^2 e_l + (N \oplus J) \in R_2 \). Then \( rx_1 = x_1 + x_2 = r^t x_2 \), where \( r^t \) is the transpose of \( r \). It follows that \( R_1 \oplus R_2 \subset I \). Now, note that \( x_1 \in I \) is a maximal vector with weight \( \lambda_1 + \lambda_2 \), so that \( I = \tilde{L}(\lambda_1 + \lambda_2) \). In view of (3), it is clear that either \( M = \tilde{L}(\lambda_1) \mid \tilde{L}(\lambda_1 + \lambda_2) \) or \( M = \tilde{L}(\lambda_1 + \lambda_2) \). Now since \( 3 \mid t \), Equation (2) implies \( \dim \tilde{L}(\lambda_1 + \lambda_2) \leq \dim H^0(\lambda_1 + \lambda_2) - \dim L(\lambda_3) - \dim L(\lambda_1) = \dim M - 2l \), so in fact \( M = \tilde{L}(\lambda_1) \) or \( L(\lambda_1 + \lambda_2) \).

Case \( n = 2l + 1 \)
Assume \( 3 \mid l \). We now consider \( H \) to be the subgroup of type \( D_l \) inside \( G \) that fixes \( e_0 \), and as before denote by \( \tilde{V} \) its natural module, as well as \( \tilde{L}(\mu) \) for the irreducible \( H \)-module with highest weight \( \mu \). The restriction \( V \downarrow H = \tilde{V} \oplus (K e_0) \), yields \( S^2 V \downarrow H = S^2(\tilde{V}) \oplus S^2(\tilde{V}) \oplus \tilde{V} \oplus T \), where \( T \) is trivial for \( H \). Now \( S^3(\tilde{V}) \) has composition factors as described in (a). Visibly, the submodule \( N \subset S^3V \) corresponds to the direct sum of \( T \) and the copy of \( \tilde{L}(3\lambda_1) \) inside \( S^3(\tilde{V}) \). Now by Proposition 4.7.4 in [12] and since \( p \mid l \), we have that \( S^2(\tilde{V}) = \tilde{T}_1 \mid \tilde{L}(2\lambda_1) \) or \( \tilde{T}_2 \) is indecomposable, where \( \tilde{T}_1, \tilde{T}_2 \) are trivial for \( H \). Note also that \( J \downarrow H \) decomposes as the direct sum of \( \tilde{T}_2 \) and the copy of \( \tilde{L}(\lambda_1) \) inside \( S^3(\tilde{V}) \). Combining these observations, we have \( M = S^3 V / (N \oplus J) \downarrow H = \tilde{L}(\lambda_1 + \lambda_2) \oplus M_0 \oplus \tilde{V} \), where \( M_0 = \tilde{T}_1 \mid \tilde{L}(2\lambda_1) \) is indecomposable. As before, let \( I \subset M \) be an irreducible \( KG \)-module. Note that \( M \) has no trivial submodules. Comparing dimensions with Table 2, we see that \( I \) must contain \( \tilde{L}(\lambda_1 + \lambda_2) \). Comparing now the dimension of \( M/I \), we see that the only possibilities are \( I = \tilde{L}(\lambda_1 + \lambda_2) \oplus \tilde{L}(2\lambda_1) \) and \( I = M \). Again \( I \) contains a maximal vector with weight \( \lambda_1 + \lambda_2 \), so that \( I = \tilde{L}(\lambda_1 + \lambda_2) \). By the same dimensional argument as for type \( D_l \), \( M = \tilde{L}(\lambda_1) \) or \( L(\lambda_1 + \lambda_2) \).

\[ \square \]

Proposition 4.7 yields the remaining dimensions for the weight \( \lambda_1 + \lambda_2 \) stated in Tables 2 and 4. This concludes the proof of Theorem 1.2

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