Dynamic Analysis of Ecological Model on Typical Red Tide Algae in Bohai Bay

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Abstract. In this paper, based typical red tide algae in Bohai Bay—skeletonema costatum and considering the effects of nutrient substance, light and zooplankton predation, Bohai Bay ecodynamic model is established. Using the delay as parameter, the stability and bifurcation behaviors of system are analyzed by applying stability and bifurcation theories. The critical values of delay are obtained, such that the system changes from the stable equilibrium to stable periodic solution, which confirms the periodic outburst phenomenon of red tide. At last, it carries out some effective numerical simulations under Bohai Bay red tide data, and finds the existence of periodic solutions in large scale, whose results have some reference values to the red tide formation in Bohai.

Introduction

Red tide is a complicated ecological phenomena, whose causes are complex. Scientists consider, red tide is caused by the organic pollution of coastal waters. In general case, the nutrient content is low in the ocean which limits the growth of phytoplankton, however, after a lot of nutrients into the sea and other physical and chemical factors conducive to the biological growth and reproduction, red tide plankton will dramatically multiply and form red tide.

Bohai Bay belongs to the semi-closed inland sea, water exchange ability is poor in the Bay. With the rapid development of coastal economy, a large amount of waste water enters the Bohai Bay leading to serious water eutrophication and red tide takes place frequently. Control of Bohai sea pollution and protection of marine ecological environment become the focus of attention. In Bohai Bay phytoplankton population, diatoms accounts for the important position, especially, skeletonema costatum has high abundance of cells and great effects to the whole Bohai Bay ecosystem. In 2012, Bohai Bay had outbreak of skeletonema costatum red tides, hence the study can not be ignored [1-3]. In this paper, combining red tide data in bohai bay and considering the effects of the factors of nutrients, light and zooplankton predation, the following ecological dynamic model is established:

\[
\begin{align*}
\dot{N}(t) &= k(N_0 - N(t)) - \frac{N(t)}{e + N(t)} \frac{a}{b + cP(t)} P(t), \\
\dot{P}(t) &= \varepsilon_1 \frac{N(t)}{e + N(t)} \frac{a}{b + cP(t)} P(t) - \mu(1 - e^{-\alpha P(t)}) Z(t) - (s + k) P(t), \\
\dot{Z}(t) &= \varepsilon_2 \mu[1 - e^{-\alpha P(t-r)}] Z(t) - dZ(t),
\end{align*}
\]

where $N(t)$ denotes the sum of nitrogen, phosphorus and other nutrients in $t$, $P(t)$ denotes the density of skeletonema costatum in $t$, $Z(t)$ denotes the density of zooplankton in $t$, respectively. $a$ is the growth rate of skeletonema costatum, $b$ is the light decay rate in the water, $c$ is the effect of self-shield of algae, $d$ is the predation rate of higher animal, $e$ is the half saturated constant that the
algae absorbs nutrient, $k$ is the commutative rate through jump layer, $s$ is the sedimentation rate of algae, $\mu$ is the maximum ingestion rate of zooplankton, $\alpha$ is the half saturated constant, $N_0$ is the density of nutrient, $\varepsilon_1, \varepsilon_2$ is the nutrition conversion, $\tau$ is the time of nutrients transformation of zooplankton. Holling II functional response function [4] is used as considering the predator effect, Steele function [5] is used as considering the light effect and Michaelis-Menten function is used as considering the effect of algae grow.

Based on biological sense, the initial conditions are chosen as follows:

$$N(\theta) = \varphi_1(\theta) \geq 0, P(\theta) = \varphi_2(\theta) \geq 0, Z(\theta) = \varphi_3(\theta) \geq 0, \varphi_i(0) > 0,$$

$$\varphi_i(\theta) \in C = C([-\tau, 0], R^+) \quad \theta \in [-\tau, 0], i = 1, 2, 3.$$  \hspace{1cm} (2)

**Equilibrium**

The equilibrium $(N, P, Z)$ of system (1) satisfies:

$$\begin{cases}
k(N_0 - N) - \frac{N}{e + N} \frac{a}{b + cP} P = 0, \\
\varepsilon_1 \frac{N}{e + N} \frac{a}{b + cP} P - \mu(1 - e^{-\alpha P})Z - (s + k)P = 0, \\
\varepsilon_2 \mu(1 - e^{-\alpha P})Z - dZ = 0.
\end{cases}$$  \hspace{1cm} (3)

From (3), the existence of equilibrium is as follows.

Lemma 1. (i) The equilibrium $E_0(N_0, 0, 0)$ always exists.

(ii) The equilibrium $E_i(N_i, P_i, 0)$ exists iff $N_i < N_0$ and hold.

$$\begin{cases}
ck\varepsilon_i N_i^2 + \left[\varepsilon_i a - b(s + k) + ckek(e - N_0)\right]N_i - c[b(s + k) + ckekN_0] = 0,
\\
P_i = \frac{\varepsilon_k k}{s + k}(N_0 - N_i),
\end{cases}$$  \hspace{1cm} (4)

(iii) The positive equilibrium $E^*(N^*, P^*, Z^*)$ exists iff hold, where

$$\varepsilon_2 \mu > d, N_m > 0, f(N_m) > 0$$  \hspace{1cm} (5)

$$\begin{cases}
P^* = \frac{1}{\alpha} \ln \frac{\varepsilon_2 \mu}{\varepsilon_2 \mu - d}, \\
N^* = \frac{(N_0 - e - l) + \sqrt{(N_0 - e - l)^2 + 4N_0e}}{2}, \\
Z^* = \frac{\varepsilon_k k \left(N_0 - N^* - (s + k)P^*\right)}{\mu \left(1 - e^{-\alpha P^*}\right)},
\end{cases}$$

and

$$l = \frac{aP^*}{b + cP^*}, N_0 - \frac{(s + k)\ln \frac{\varepsilon_2 \mu}{\varepsilon_2 \mu - d}}{\alpha \varepsilon_k k}, f(N) = kN^2 + [l + k(e - N_0)]N - keN_0.$$  

In fact, $E^*(N^*, P^*, Z^*)$ satisfies
\[
\begin{align*}
\frac{N^*}{e+N^*} \frac{aP^*}{b+cP^*} &= k(N_0 - N^*), \\
\epsilon_i \frac{N^*}{e+N^*} \frac{aP^*}{b+cP^*} &= \mu(1-e^{-\alpha P})Z^* + (s+k)P^*, \\
\epsilon_i \mu[1-e^{-\alpha P^*}] &= d.
\end{align*}
\]  

(6)

From the third equation of (6), it has \( P^* = \frac{1}{\alpha} \ln \left( \frac{\epsilon_i \mu}{\epsilon_i \mu - d} \right) \) and \( N^* \) satisfies
\[
kN^* + [l+k(e-N_0)]N^*-keN_0 = 0,
\]
i.e., the unique positive root is
\[
N^* = \frac{[l+k(e-N_0)] + \sqrt{(l+k(e-N_0))^2 + 4k^2N_0e}}{2k}
\]
From the first two equations of (6), it has
\[
Z^* = \frac{\epsilon_i k(N_0 - N^*) - (s+k)P^*}{\mu(1-e^{-\alpha P^*})}.
\]

Hence
\[
P^* > 0 \Leftrightarrow \epsilon_i \mu > d, Z^* > 0 \Leftrightarrow \epsilon_i k(N_0 - N^*) - (s+k)P^* > 0,
\]
That is,
\[
0 < N^* < N_0 - \frac{(s+k) \ln \frac{\epsilon_i \mu}{\epsilon_i \mu - d}}{\alpha \epsilon_i k} = N_m.
\]
By \( f(N_m) > 0 \), then (1) has a unique positive equilibrium.

**Local Stability and Hopf Bifurcation Analysis**

In this part, using the stability and Hopf bifurcation methods in [6,7], it will discuss the stability of equilibrium of system (1).

Let \( E(\bar{N}, \bar{P}, \bar{Z}) \) be any equilibrium and \( u_1 = N - \bar{N}, u_2 = P - \bar{P}, u_3 = Z - \bar{Z} \), the linearization system of system (1) at \( E \) is
\[
\begin{align*}
\dot{u}_1(t) &= - \left( k + \frac{ea\bar{P}}{(e+\bar{N})^2(b+c\bar{P})} \right) u_1(t) - \frac{ab\bar{N}}{(e+\bar{N})(b+c\bar{P})^2} u_2(t), \\
\dot{u}_2(t) &= \frac{\epsilon_i ea\bar{P}}{(e+\bar{N})^2(b+c\bar{P})} u_1(t) + \frac{\epsilon_i ab\bar{N}}{(e+\bar{N})(b+c\bar{P})^2} - (s+k) - \alpha \mu e^{-\alpha P} \bar{Z} u_2(t) - \mu(1-e^{-\alpha P}) u_3(t), \\
\dot{u}_3(t) &= \alpha \epsilon_i e^{-\alpha P} \bar{Z} u_2(t - \tau) + \left[ \epsilon_i \mu(1-e^{-\alpha P}) - d \right] u_3(t),
\end{align*}
\]
whose characteristic equation is
\[
\Delta(\lambda) = (\lambda + A) \left[ (\lambda + B)(\lambda - \epsilon_i \mu(1-e^{-\alpha P}) + d) + \alpha \epsilon_i \mu e^{-\alpha P} \bar{Z} e^{-\lambda \tau} \mu(1-e^{-\alpha P}) \right] \\
+ \frac{a^2bc\epsilon_i e^{\bar{N}P}}{(e+\bar{N})^2(b+c\bar{P})^3} \left[ \lambda - \epsilon_i \mu(1-e^{-\alpha P}) + d \right] = 0
\]
(7)
where \( A = k + \frac{ea\bar{P}}{(e+\bar{N})^2(b+c\bar{P})} \), \( B = s + k + \alpha \mu e^{-\alpha P} \bar{Z} - \frac{\epsilon_i ab\bar{N}}{(e+\bar{N})(b+c\bar{P})^2} \). At \( E_0 \), (7) is equivalent to
\[(\lambda + k)(\lambda + d)[\lambda + s + k - \frac{\varepsilon_i a N_0}{b(e + N_0)}] = 0.\]

Define \(\Phi = \frac{\varepsilon_i a N_0}{b(e + N_0)} - (s + k)\), it has the following theorem.

**Theorem 1.** When \(\Phi > 0\), \(E_0\) is unstable; when \(\Phi < 0\), \(E_0\) is stable; when \(\Phi = 0\), \(E_0\) is linear stable.

**Remark.**
\[
\Phi > 0 \iff N_0 > \frac{be(s + k)}{\varepsilon_i a - b(s + k)} \quad \text{and} \quad \varepsilon_i a - b(s + k) > 0;
\]
\[
\Phi < 0 \iff \varepsilon_i a - b(s + k) < 0 \quad \text{or} \quad \varepsilon_i a - b(s + k) > 0 \quad \text{and} \quad N_0 < \frac{be(s + k)}{\varepsilon_i a - b(s + k)}.
\]

At \(E_1\), (7) is equivalent to
\[
\lambda^2 + A_1 \lambda + B_1 = 0,
\]
where
\[
A_1(k(s + k) + \frac{seaP_1}{(e + N_1)(b + cP_1)} + \frac{ka}{(e + N_1)(b + cP_1)} \left[ \frac{eP_1}{e + N_1} - \frac{e_i bN_1}{b + cP_1} \right],
\]
\[
B_1 = s + 2k + \frac{a}{(e + N_1)(b + cP_1)} \left[ \frac{eP_1}{e + N_1} - \frac{e_i bN_1}{b + cP_1} \right].
\]

Hence all roots of (8) have negatively real parts if and only if \(A_1 > 0, B_1 > 0\) and \(\varepsilon_i \mu(1 - e^{-\alpha r}) < d\).

**Theorem 2.** When \(A_1 > 0, B_1 > 0\) and \(\varepsilon_i \mu(1 - e^{-\alpha r}) < d\), \(E_1\) is locally asymptotically stable; when \(A_1 < 0\) or \(B_1 < 0\) or \(\varepsilon_i \mu(1 - e^{-\alpha r}) > d\), \(E_1\) is unstable.

Furthermore, if \(ebP_1 + ecP_1^2 \geq \varepsilon_i beN_1 + e_i bN_1^2\) and \(\varepsilon_i \mu(1 - e^{-\alpha r}) < d\), \(E_1\) is locally asymptotically stable. At \(E^*\), (7) is equivalent to
\[
\lambda^3 + Q_1 \lambda^2 + Q_0 (\lambda + A^*) e^{-\lambda \tau} = 0,
\]
where
\[
Q_1 = A^* + B^*, Q_i = A^* B^* + \frac{\varepsilon_i e a bN_1^*}{(e + N^*)^2(b + cP^*)}, Q_0 = \alpha \varepsilon_i \mu^* e^{-\alpha r^*} (1 - e^{-\alpha r^*}) Z^*,
\]
\[
A^* = k + \frac{e a P^*}{(e + N^*)^2(b + cP^*)}, B^* = s + k + \alpha \mu e^{-\alpha r^*} Z^* - \frac{\varepsilon_i abN^*}{(e + N^*)(b + cP^*)}.
\]

When \(\tau = 0\), (9) becomes
\[
\lambda^3 + Q_2 \lambda^2 + (Q_1 + Q_0) \lambda + Q_0 A^* = 0,
\]
Then all roots of (10) have negatively real parts if and only if \(Q_2 > 0\) and \(Q_2(Q_1 + Q_0) > Q_0 A^*\), \(E^*\) is locally asymptotically stable. When \(\tau > 0\), let \(i \omega (\omega > 0)\) be the root of (10), then \(\omega\) satisfies
\[
Q_0[A^* \sin \omega \tau - \omega \cos \omega \tau] = -\omega^3 + \omega Q_1, Q_0[A^* \cos \omega \tau + \omega \sin \omega \tau] = Q_2 \omega^2,
\]
Squaring and adding both sides of the equation (11), it has
\[
\omega^6 + [Q_2^2 - 2Q_1] \omega^4 + [Q_1^2 - Q_0^2] \omega^2 - Q_0^2 A^2 = 0.
\]
Since \( Q_0^2 A^2 > 0 \), (12) has at least one positive root. Without loss of generality, it assumes that (12) has three positive roots defined as \( \omega_n (n = 1, 2, 3) \). Substituting \( \omega_n \) into (12), it has

\[
\cos \omega_n \tau = \frac{\omega_n^4 + (A Q_0 - Q_0^2) \omega_n^2}{Q_0 (A^2 + \omega_n^2)} := R(\omega_n), \quad \sin \omega_n \tau = \frac{(Q_0 - A^*) \omega_n^3 + \omega A^* Q_0}{Q_0 (A^2 + \omega_n^2)} := R(\omega_n),
\]

Furthermore,

\[
\tau_n = \begin{cases} \frac{1}{\omega_n} [\arccos(R(\omega_n)) + 2j\pi], & R(\omega_n) \geq 0, \\ \frac{1}{\omega_n} [2\pi - \arccos(R(\omega_n)) + 2j\pi], & R(\omega_n) < 0, j = 0, 1, 2, \ldots. \end{cases}
\]

Define \( \tau_0 = \min(\tau_n) \), \( \omega(\tau_0) = \omega_0 \), \( n = 1, 2, 3 \); \( j = 0, 1, 2, \ldots \). Let \( \lambda(\tau) = \alpha(\tau) + i\omega(\tau) \) be the root of (10) satisfying \( \alpha(\tau_n) = 0, \omega(\tau_n) = \omega_n (n = 1, 2, 3; j = 0, 1, 2, \ldots) \). Deriving both sides of (9) to \( \tau \), it has the following theorem.

**Theorem 3.** \( \text{Sign}(\alpha(\tau_n)) = \text{Sign}(W_1 W_3 + W_2 W_4) \), where

\[
W_1 = \omega_n^2 [Q_0^2 \omega_n^2 + A^*(Q_0 - \omega_n^2)], \quad W_3 = Q_0 A^* - (Q_0^2 A^* + 3A^* + Q_0) \omega_n^2 - (Q_0 - \omega_n^2) \omega_n^2 \tau_n, \\
W_2 = \omega_n^2 [A^* Q_0 + (Q_0 - \omega_n^2)], \quad W_4 = \omega_n (Q_0 - 3\omega_n^2) + 2A^* \omega_n Q_0 - \omega_n (Q_0 - \omega_n^2)(1 - \tau_n A^*) - \tau_n Q_0 \omega_n^3.
\]

**Theorem 4.** It assumes that \( Q_2 > 0, Q_2 (Q_0 + Q_0) > Q_0 A^* \) and (5) hold. If \( \lambda(\tau_0) > 0 \), then \( E^- \) is asymptotically stable when \( \tau \in [0, \tau_0) \) and unstable when \( \tau > \tau_0, \tau_n \) is Hopf bifurcation value.

**Numerical Simulation**

According to red tides data in Bohai Bay, the parameters are chosen as follows [2]:

\[
a = 0.6, b = 0.04, c = 0.3, d = 0.25, e = 0.03, k = 0.13, \\
s = 0.04, \mu = 0.6, \alpha = 0.1, \epsilon_1 = 0.9, \epsilon_2 = 0.9, N_0 = 30.
\]

The system (1) exists equilibria:

\[
E_0 (30, 0, 0), E_1 (14.8402, 10.4335, 0), \quad E^- = (14.9685, 6.2169, 2.5265).
\]

From the equilibria, it can see the predator of zooplankton can reduce greatly the concentration of skeletonema costatum, which shows the importance that zooplankton to red tide generating and vanishing process. By computing, it obtains \( \Phi = 13.3165 > 0 \), \( E_0 \) is unstable. \( \epsilon_1 \mu (1 - e^{-\alpha t}) - d = 0.0998 > 0, \) \( E_1 \) is unstable. When \( \tau = 0 \), it has \( Q_1 = 0.3757 > 0, \) \( Q_2 (Q_1 + Q_0) - Q_0 A^* = 0.017 > 0 \), hence \( E^- \) is asymptotically stable. When \( \tau > 0 \), (12) has a unique positive root \( \omega_0 = 0.0787 \), furthermore, \( \tau_0 = 16.0316 + 6.2832 j \), \( W_1 W_3 + W_2 W_4 > 0 \), which shows the system occurs Hopf bifurcation at \( \tau_0 \). Choosing \( \tau = 10 \in (0, \tau_0) \), \( E^- \) is stable (see Fig. 1). Choosing \( \tau = 16.03 \), \( E^- \) is unstable and the system exists a stable periodic solution (see Fig. 2). Furthermore, choosing \( \tau = 20, 25, 30, 36 \), it can find that the periodic solutions continue to exist and the amplitudes increase with the increasing of delay, which shows that the concentrations of three variables reach higher maximum and lower minimum (see Fig. 3). From the above simulation results, it can find there exists at least one periodic solution between two Hopf bifurcations, which may mean the existence of periodic solutions in large scale. For the find, the detailed discussion will complete in the future. The initial conditions are chosen as \( \varphi_1 = 0.4, \varphi_2 = 0.1, \varphi_3 = 0.05 \).
Figure 1. $\tau = 10$, $E^*$ is stable.

Figure 2. $\tau = 16.3$, $E^*$ is unstable and there exists a stable periodic solution.

Figure 3. When $\tau = 20 \in (\tau_0, \tau_1)$ (upleft), $\tau = 25 \in (\tau_1, \tau_2)$ (upright), $\tau = 30 \in (\tau_2, \tau_3)$ (downleft) and $\tau = 36 \in (\tau_3, \tau_4)$ (downright), $E^*$ is unstable and exists stable periodic solutions, and the amplitudes of periodic solutions increase with the increasing of delay.
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