SOME INTEGRAL REPRESENTATIONS AND PROPERTIES OF LAH NUMBERS

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ABSTRACT. In the paper, the authors find some integral representations and properties of Lah numbers.

1. INTRODUCTION

In combinatorics, Lah numbers, discovered by Ivo Lah in 1955 and usually denoted by \( L(n, k) \), count the number of ways a set of \( n \) elements can be partitioned into \( k \) nonempty linearly ordered subsets and have an explicit formula

\[
L(n, k) = \frac{(n-1)!}{k!} \cdot \frac{n}{k!}.
\]

(1.1)

Lah numbers \( L(n, k) \) may also be interpreted as coefficients expressing rising factorials \( (x)_n \) in terms of falling factorials \( \langle x \rangle_n \), where

\[
(x)_n = \begin{cases} 
  x(x+1)(x+2) \cdots (x+n-1), & n \geq 1, \\
  1, & n = 0
\end{cases}
\]

(1.2)

and

\[
\langle x \rangle_n = \begin{cases} 
  x(x-1)(x-2) \cdots (x-n+1), & n \geq 1, \\
  1, & n = 0
\end{cases}
\]

(1.3)

Lah numbers \( L(n, k) \) may be generated by

\[
\frac{1}{k!} \left( \frac{x}{1-x} \right)^k = \sum_{n=0}^{\infty} L(n, k) \frac{x^n}{n!}.
\]

(1.4)

For more information on Lah numbers \( L(n, k) \), please refer to [2, p. 156].

In the theory of special functions, it is well known that the modified Bessel function of the first kind \( I_\nu(z) \) may be defined [1, p. 375, 9.6.10] by

\[
I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)} \left( \frac{z}{2} \right)^{2k+\nu}
\]

(1.5)

for \( \nu \in \mathbb{R} \) and \( z \in \mathbb{C} \), where \( \Gamma \) represents the classical Euler gamma function which may be defined [1, p. 255] by

\[
\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} \, dt
\]

(1.6)

for \( \Re z > 0 \).

In this paper, we will find some integral representations and properties of Lah numbers \( L(n, k) \).

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2. INTEGRAL REPRESENTATIONS OF LAH NUMBERS

We first establish integral representations of Lah numbers $L(n, k)$, in which the exponential function $e^{-1/x}$ and the modified Bessel function of the first kind $I_1$ is involved.

**Theorem 2.1.** For $1 \leq m \leq n$, we have

$$
\sum_{k=1}^{n} L(n, k)x^k = \frac{e^{-x}}{x^n} \int_{0}^{\infty} I_1(2\sqrt{t}) t^{n-1/2} e^{-xt} \, dt
$$

(2.1)

and

$$
L(n, m) = \frac{1}{m!} \lim_{x \to 0} \int_{0}^{\infty} I_1(2\sqrt{t}) t^{n-1/2} \frac{d^m}{dx^m} \left( \frac{e^{-x-t/x}}{x^n} \right) \, dt.
$$

(2.2)

**Proof.** In [25, Theorem 1.2], among other things, it was obtained that the function

$$
H_k(z) = e^{1/z} - \sum_{m=0}^{k} \frac{1}{m!} \frac{1}{z^m}
$$

for $k \in \{0\} \cup \mathbb{N}$ and $z \neq 0$ has the integral representation

$$
H_k(z) = \frac{1}{k!(k+1)!} \int_{0}^{\infty} F_2(1; k+1, k+2; t) t^k e^{-zt} \, dt
$$

(2.4)

for $\Re(z) > 0$, where $pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x)$ stands for the generalized hypergeometric series which may be defined by

$$
pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!}
$$

(2.5)

for complex numbers $a_i$ and $b_j \notin \{0, -1, -2, \ldots\}$ and for positive integers $p, q \in \mathbb{N}$. See also [17, Section 1.2] and [19, Lemma 2.1]. When $k = 0$, the integral representation (2.4) becomes

$$
e^{1/z} = 1 + \int_{0}^{\infty} \frac{I_1(2\sqrt{t})}{\sqrt{t}} e^{-zt} \, dt
$$

(2.6)

for $\Re(z) > 0$. Hence, for $n \in \mathbb{N}$, we have

$$
(e^{1/x})^{(n)} = (-1)^n \int_{0}^{\infty} I_1(2\sqrt{t}) t^{n-1/2} e^{-xt} \, dt.
$$

(2.7)

In [18, Theorem 2] and its formally published paper [28, Theorem 2.2], the following explicit formula for computing the $n$-th derivative of the exponential function $e^{\pm 1/x}$ was inductively obtained:

$$
(e^{\pm 1/x})^{(n)} = (-1)^n e^{\pm 1/x} \sum_{k=1}^{n} \frac{(\pm 1)^k}{x^{n+k}} L(n, k) \frac{1}{x^{n+k}}.
$$

(2.8)

By the way, the formula (2.8) have been applied in [8, 9, 15, 17, 19, 10, 25]. Combing (2.7) and (2.8) and rearranging yield

$$
e^{1/x} \sum_{k=1}^{n} L(n, k) \frac{1}{x^{n+k}} = \int_{0}^{\infty} I_1(2\sqrt{t}) t^{n-1/2} e^{-xt} \, dt,
$$

$$
\sum_{k=1}^{n} L(n, k) \frac{1}{x^k} = \int_{0}^{\infty} I_1(2\sqrt{t}) t^{n-1/2} x^n e^{-xt-1/2} \, dt,
$$

which may be rewritten as (2.1).

Differentiating $1 \leq m \leq n$ times on both sides of (2.1) results in

$$
\sum_{k=m}^{n} L(n, k) \frac{k!}{(k-m)!} \frac{x^{k-m}}{x^k} = \int_{0}^{\infty} I_1(2\sqrt{t}) t^{n-1/2} \frac{d^m}{dx^m} \left( \frac{e^{-x-t/x}}{x^n} \right) \, dt.
$$

(2.9)

Letting $x \to 0$ in the above equation leads to (2.2). The proof of Theorem 2.1 is complete. □
With the help of the integral representation (2.1), we find some properties of Lah numbers \( L(n, k) \).

**Theorem 3.1.** For \( n \in \mathbb{N} \), the integer polynomial

\[
\mathcal{L}_n(x) = \sum_{k=0}^{n} L(n + 1, k + 1)x^k
\]

(3.1)
of degree \( n \) has no real zero. Concretely speaking,

1. If \( x \geq 0 \), then \( \mathcal{L}_n(x) > 0 \);
2. If \( x < 0 \), then \( \mathcal{L}_{2n-1}(x) < 0 \) and \( \mathcal{L}_{2n}(x) > 0 \).

**Proof.** From the integral representation (2.1), it follows that

\[
\int_{0}^{\infty} I_1 \left( \frac{2\sqrt{t}}{x+1} \right) t^{n+1/2} e^{-x-t/x} \, dt \neq 0
\]

for all \( x \in \mathbb{R} \setminus \{0\} \). By this, it is easy to verify Theorem 3.1. \( \square \)

An infinitely differentiable function \( f \) on an interval \( I \) is called absolutely convex on \( I \) if \( f^{(2k)}(x) \geq 0 \) on \( I \). See either [7, p. 375, Definition 3], or [16, p. 2731, Definition 4.5], or [26, p. 617, Definition 3], or [27, p. 3356, Definition 3]. A sequence \( \{\mu_n\}^\infty_{n=0} \) is said to be absolutely convex if its elements are non-negative and its successive differences satisfy

\[
\Delta^2_k \mu_n \geq 0
\]

for \( n, k \geq 0 \), where

\[
\Delta^k \mu_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \mu_{n+k-m}.
\]

**Theorem 3.2.** For \( n \in \mathbb{N} \), the total sum of Lah numbers

\[
\mathcal{L}_n = \sum_{k=1}^{n} L(n, k)
\]

(3.4)
is an absolutely convex sequence. Specially, the sequence \( \mathcal{L}_n \) is convex.

**Proof.** Letting \( x = 1 \) in (2.1) gives

\[
\mathcal{L}_n = \int_{0}^{\infty} I_1 (2\sqrt{t}) t^{n-1/2} e^{-(1+t)} \, dt.
\]

It is clear that the function \( t^x \) for \( t > 0 \) satisfies \( \frac{d^k t^x}{dx^k} = t^x (\ln t)^k \). As a result, when \( t > 0 \), the function \( t^x \) is absolutely convex with respect to \( x \). Consequently, the sequence \( t^n \) is absolutely convex. Hence, the sequence \( \mathcal{L}_n \) is absolutely convex. The proof of Theorem 3.2 is complete. \( \square \)

4. A RECOVERY OF THE FORMULA (1.1)

Finally, as by-product, a recovery of the formula (1.1) for Lah numbers \( L(n, k) \) may be carried out as follows.

The generating function (1.4) may be rewritten as

\[
(-1)^k \frac{1}{k!} \left( \frac{x}{1+x} \right)^k = \sum_{n=k}^{\infty} (-1)^n L(n, k) \frac{x^n}{n!}.
\]

(4.1)
The equation (4.1) may be reformulated as

\[
(-1)^k \frac{1}{k!} \frac{1}{(1+x)^k} = \sum_{n=k}^{\infty} (-1)^n L(n, k) \frac{x^{n-k}}{n!} = \sum_{n=0}^{\infty} (-1)^{n+k} L(n+k, k) \frac{x^n}{(n+k)!}.
\]
Because
\[
\frac{1}{(1+x)^k} = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-(1+x)t} \, dt,
\]  
we have
\[
\frac{1}{k! (k-1)!} \int_0^\infty t^{m+k-1} e^{-(1+x)t} \, dt = \sum_{n=0}^\infty (-1)^n L(n+k,k) \frac{x^n}{(n+k)!}.
\]
Differentiating \(m\) times with respect to \(x\) on both sides of the above equation gives
\[
(-1)^m \frac{1}{k! (k-1)!} \int_0^\infty t^{m+k-1} e^{-(1+x)t} \, dt = \sum_{n=m}^\infty (-1)^n L(n+k,k) \frac{n!}{(n-m)! (n+k)!} x^{n-m}.
\]
Taking \(x \to 0\) in the above equation yields
\[
(-1)^m \frac{1}{k! (k-1)!} \int_0^\infty t^{m+k-1} e^{-t} \, dt = (-1)^m L(m+k,k) \frac{m!}{(m+k)!}
\]
which may be rearranged as
\[
L(m+k,k) = \frac{(m+k)!}{m!} \frac{1}{k! (k-1)!} \int_0^\infty t^{m+k-1} e^{-t} \, dt
\]
\[
= \frac{(m+k)!}{m!} \frac{1}{k! (k-1)!} \left( \frac{m+k-1}{k-1} \right).
\]
The formula (1.1) is thus recovered.

Remark 4.1. In the early morning of 30 December 2013, the second author searched out the paper [3] in which the formula (2.8) was also found independently by five approaches. The motivation of the paper [3] is different from the ones of [28] and its preprint [18]. The motivations of the formula (2.8) in [18, 28] essentially originated from the articles [4, 5, 6] and their preprints [20, 21, 22]. For more information, please refer to the expository and survey articles [11, 23, 24] and their preprints [12, 13, 14].

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