A General Framework of Multi-Armed Bandit Processes by Arm Switch Restrictions*

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Abstract

This paper proposes a general framework of multi-armed bandit (MAB) processes by introducing a type of restrictions on the switches among arms evolving in continuous time. The Gittins index process is constructed for any single arm subject to the restrictions on switches and then the optimality of the corresponding Gittins index rule is established. The Gittins indices defined in this paper are consistent with the ones for MAB processes in continuous time, integer time, semi-Markovian setting as well as general discrete time setting, so that the new theory covers the classical models as special cases and also applies to many other situations that have not yet been touched in the literature. While the proof of the optimality of Gittins index policies benefits from ideas in the existing theory of MAB processes in continuous time, new techniques are introduced which drastically simplify the proof.

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1 Introduction

Multi-armed bandit processes (MAB in short) model the resource allocation problem with uncertainties where a decision maker attempts to optimize his decisions based on the existing knowledge, so as to maximize his expected total reward over time (Gittins et al., 2011). It has extensive applications in clinical trials, design of experiments, manufacturing systems, economics, queuing and communication networks, control theory, search theory, scheduling, machine learning/reinforcement learning etc.

In this paper we are concerned with a general multi-armed bandit problem with restricted random stopping time sets, which can be roughly described as follows: There is a multi-armed bandit process consisting of a set of \( d \) statistically independent arms evolving in continuous time among which a resource (time, effort) has to be allocated. Every arm is associated with a restricted stopping time set, in the sense that the arm must be engaged exclusively if its operation time does not belong to the stopping time set. The allocation respects the restrictions and any engaged arm accrues rewards that are represented as a general stochastic process. The objective is to maximize the total expected discounted reward over an infinite time horizon.

The early versions of discrete-time MAB processes in Markovian and semi-Markovian fashions have been well understood due to the pioneer work of Gittins and Jones (1972) and subsequently the seminal contributions of Gittins (1979, 1989) and Whittle (1980, 1982). The significance of Gittins’ contribution is the drastic dimension reduction: Instead of solving the optimal problems of the Markov (or semi-Markov) decision models formed by all arms, one only needs to compute an index function of the states based merely on the information delivered by each arm itself and then picks an arm with the highest index to operate. That index function, known generally as Gittins Indices today, was defined by Gittins as the maximum reward rate over all arm-specified stopping times, Whittle (1980) provided a mathematically elegant proof by showing that Gittins index policies solve the optimality
equations of the corresponding dynamic programming modeling the multi-bandit processes. For general reward processes in integer time (without Markovian assumption), Varaiya et al. (1984) defined an optimal policy in abstract terms by reducing every $d$-armed problem to $d$ independent stopping problems of the type solved by Snell (1952). Mandelbaum (1986) proposed a technically convenient framework by formulating a control problem with time parameters in a multidimensional, partially ordered set. EL Karoui and Karatzas (1993) presented a mathematically rigorous proof of Gittins index policies for arbitrary stochastic processes evolving in integer times by combining the formulation of Mandelbaum (1986) with ideas from Whittle (1980). The most general treatments for discrete time setting can be found in Cai, et al. (2014, Section 6.1) and Cowan and Katehakis (2015) by dropping the Markovian property from the semi-Markovian model so that switches from one arm to another can only take place at certain time points and the intervals between any pair of consecutive points are random quantities. One key feature in discrete time setting is that the switches from any arm can only occur in countably many time instants, even though the arms can evolve continuously over the time horizon. This type of problems are referred to as general discrete time setting. Some aspects of the theory in the discrete time version and applications in searching, job scheduling, etc., can also be found in the comprehensive monograph by Gittins et al. (2011).

The parallel theory for MAB processes in continuous time was not developed until a later time, due to mainly the technical intricacy in mathematics, where the term “continuous time” emphasizes not only that rewards can be continuously collected but, most significantly in mathematics, that switches from one arm to another are allowed to be made at arbitrary time points in $(0, \infty)$ also, such that the time set for an arm from which switches can be made is the whole positive axis, i.e., essentially uncountable, sharply in contrast to the discrete time version in which the switches are essentially countable. It is consensus that continuous time stochastic processes are far more difficult to attack than their discrete time versions, due to the difficulties in dealing with the measurability of the quantities involved. As to the continuous time version of the problem in a Markovian case, relevant results were first obtained by Karatzas (1984) and Eplett (1986). By insightfully formulating the model as a stochastic
control problem for certain multi-parameter processes, Mandelbaum (1987) extended the problem to a general dynamic setting. Based on Mandelbaum’s formulation, EL Karoui and Karatzas (1994) derived general results by combining martingale-based methodologies with the retirement option designed by Whittle (1980) for his elegant proof of the optimality of Gittins index policies in discrete time. These results were further revisited by Kaspi and Mandelbaum (1998) with a relatively short and rigorous proof by means of excursion theory.

To sum up, studies on MAB processes have treated only the two regular ends: the discrete time version (including the general discrete time setting) in which switches from any arm to another are at most countably infinite, and the continuous time version in which the controller can switch from one arm to another in any time point in the positive time horizon, with technically different methods.

Clearly, in between the two regular ends, there exist many real-life situations that could not be put in the framework formed by solely either of the two versions, especially when there are technical restrictions on the switch times of the arms. As an example, consider a simple job scheduling scenario subject to machine breakdowns (see, e.g., Cai et al, 2014), in which a single unreliable machine is to process a set of jobs and, in serving the jobs, the machine may be subject to breakdowns from time to time, caused by, for instance, damage of components of the machine or power supply. When the machine is workable, a job can be processed and the processing can be preempted so as to switch the machine to any one of the unfinished jobs. Once the machine is broken down, it must be continuously repaired until it can resume its operation again. In this scenario, the stopping times for the machine to be switched from one job to another are restricted to the time interval in which the machine is in good condition. By associating the repairing duration of the machine to the job being processed, this problem can be modeled by a multi-armed bandit process. This bandit process, however, cannot be put in any of the frameworks of discrete time and continuous time bandit processes, owing to two significant features: First, for any job, the set of its potential switching times are essentially continuum in the interval in which the machine is workable so that the framework cannot be the discrete time version. Second, in the time intervals of machine reparation, a switch from the job is prohibited so that the
framework cannot be the continuous time version. As another example that the classical MAB models cannot accommodate, consider a second job scheduling problem in which some of the jobs can be preempted at any time points, whereas the other jobs consist of a number of nonpreemptable components so that, once a job is selected to process, it could not be preempted until the completion of a component. This problem can be translated to such an MAB formula that some arms evolve in continuous time setting and the others respect to a discrete time mechanism. Furthermore, one can even image such situations where jobs consist of possibly preemptable and nonpreemptable components, so that, being represented as MAB models, the arms can be in continuous time, discrete time version or in a mixture mode in which the switch times contain both continuum and discrete time parts. Clearly, the existing optimality theory of MAB processes is not applicable to these situations.

This paper is dedicated to propose a new MAB process model so as to accommodate these situations. This is accomplished by introducing a type of restrictions on switch times, or equivalently the arm-specified stopping times as what discussed recently in Bao et al (2017) for restricted optimal stopping problems. Firstly, it turns out that this new model also unifies the existing versions of MAB processes. Specifically, for the sole discrete time version, the switching times of every arm are only the integer times, for the general discrete time version, the switching times are clearly the end points of the intervals during which no switch is allowed and the purely continuous time setting corresponds simply to the case of no restriction (see Section 2 for details). Moreover, an obvious merit of this new framework is, by introducing different restrictions on different arms, it can give the optimal solution to irregular cases in which some of the arms follow continuous time, some others follow discrete time and still others even respect more complicated mixtures; see the examples above. Such important types of MAB processes have not yet been touched in the existing literature.

To successfully tackle this problem, we will combine the martingale techniques as employed by EL Karoui and Karatzas (1994) with the excursion method similar to that used by Kaspi and Mandelbaum (1998), but now under the new framework of general \(d\)-armed bandit processes with each arm attached with a restricted stopping time set.

The main contribution of this paper consists of the following:
(1) We develop a general and new framework of MAB processes, suggest correspondingly a
general definition of Gittins indices and demonstrate their optimality in arm allocation
under switch time restrictions. This framework generalizes and unifies the models,
methodologies and theory for all versions of MAB processes and can apply to more
other situations.

(2) While the proof follows the ideas partly from EL Karoui and Karatzas (1994) and
partly from Kaspi and Mandelbaum (1998), new techniques (e.g., the discounted gain
process \[3.2\] and Lemma \[4.1\]) are introduced such that the proof is drastically shorter
than the ones for the unrestricted MAB processes in continuous
time.

The reminder of the paper is organized as follows. Section \[2\] formulates the restricted
MAB processes with each arm associated with a restriction on stopping times. After a
concise review of the theory of optimal stopping times with restrictions in Section \[3.1\] so as to
prepare some necessary theoretical foundation, Section \[3.2\] associates each arm with a Gittins
index process defined under the restrictions on stopping times, which unifies and extends
the classical definitions for discrete time, continuous time and semi-Markovian setting. The
properties of the Gittins index process are also addressed there. Section \[4\] is dedicated to
demonstrate the optimality of Gittins index policies. The paper is concluded in Section \[5\]
with a few remarks.

2 Model Specification

The MAB processes for which the switches among arms are subject to restrictions are referred
to as “restricted multi-armed Bandit processes” (RMAB processes).

In this paper, a RMAB process refers to a stochastic control process governed by the
following mechanism. The primitives are \(d\) stochastic processes \((X^k, \mathcal{F}^k), k = 1, 2, \ldots, d,\)
evolving on \(\mathbb{R}_+ = [0, +\infty),\) all of which are defined on a common probability space \((\Omega, \mathcal{F}, P)\)
to represent \(d\) arms, meeting the following formulation:

(a) **Filtrations.** For every \(k \in \{1, 2, \ldots, d\}, \mathcal{F}^k = \{\mathcal{F}^k_t, t \in \mathbb{R}_+\}\) is a quasi-left-continuous
filtration satisfying the usual conditions and \( \mathcal{F}_0 = \{\emptyset, \Omega\} \), mod \( P \). The collection \( \{\mathcal{F}^1, \mathcal{F}^2, \ldots, \mathcal{F}^d\} \) of filtrations are assumed to be mutually independent.

(b) **Rewards.** For every \( k \in \{1, 2, \ldots, d\} \), \( X_t^k \geq 0 \), the instant reward rate obtained at the moment when arm \( k \) has just been pulled for \( t \) units of time, is assumed to be \( \mathcal{F}^k \)-progressive and, with no loss of generality, satisfies \( E \left[ \int_0^\infty e^{-\beta t} X_t^k dt \right] < \infty \).

(c) **Restrictions.** Let \( \mathcal{M}^k \) be an \( \mathcal{F}^k \)-adapted random time set, referred to as the feasible time set of arm \( k \), satisfying \( 0, \infty \in \mathcal{M}^k \) and \( \mathcal{M}^k(\omega) = \{t : (t, \omega) \in \mathcal{M}^k\} \) is closed for every \( \omega \in \Omega \). For an \( \mathcal{F}^k \)-stopping time \( \tau \), also write \( \tau \in \mathcal{M}^k \) almost surely; the symbol \( \mathcal{M}^k \) refers to both a random set and the set of stopping times \( \tau \) with \( (\tau, \omega) \in \mathcal{M}^k \) a.s.. Here \( \mathcal{M}^k \) may vary over \( k \), subject to different requirements.

(d) **Policies under restrictions.** An allocation policy \( T \) is characterized by a \( d \)-dimensional stochastic process \( T := \{T(t) : t \in \mathbb{R}_+\} = \{(T^1(t), T^2(t), \ldots, T^d(t)) : t \in \mathbb{R}_+\} \), where \( T^k(t) \) is the total amount of time that \( T \) allocates to arm \( k \) during the first \( t \) units of calendar time, satisfying the following technical requirements:

1. \( T(t) \) is component-wise nondecreasing in \( t \geq 0 \) with \( T(0) = 0 \).
2. \( T^1(t) + T^2(t) + \cdots + T^d(t) = t \) for every \( t \geq 0 \).
3. For any nonnegative vector \( s = (s_1, s_2, \ldots, s_d) \in \mathbb{R}^n_+ \), \( \{T(t) \leq s\} \in \mathcal{F}^{s_1}_1 \vee \cdots \vee \mathcal{F}^{s_d}_d \).
4. \( \frac{d^{+}T^k(t)}{dt} = 1 \) if \( (T^k(t), \omega) \in \mathcal{M}^k_c := \mathbb{R}_+ \times \Omega - \mathcal{M}^k \), where \( \frac{d^k}{dt} \) indicates the right derivative.

(e) **Objective.** With any policy \( T \), the total reward of the bandit in calendar time interval \([t, t+dt]\) is \( \sum_{k=1}^d X^k_{T^k(t)} dT^k(t) \), so that the total expected present value of this \( d \)-armed bandit system is

\[
v(T) = \sum_{k=1}^d E \left[ \int_0^\infty e^{-\beta t} X^k_{T^k(t)} dT^k(t) \right],
\]

(2.1)

where \( \beta > 0 \) indicates the interest rate. The objective is to find a policy \( \hat{T} \) such that \( v(\hat{T}) = \max_T v(T) \), where the maximization is taken over all the policies characterized above.
The following remarks give more details on the formulation of RMAB processes.

(a) For the reward processes, the requirement \( E \left[ \int_0^\infty e^{-\beta t} X_k^t dt \right] < \infty, k = 1, 2, \ldots, d \) makes the problem nontrivial, because, supposing it does not hold for some \( k \), then one can optimally obtain an infinite expected reward by operating arm \( k \) all the time.

(b) While, from a practical point of view, policies satisfying \( T_1^1(t) + T_2^2(t) + \cdots + T_d^d(t) \leq t \) for every \( t \geq 0 \) allow for machine idle and are also practically feasible and can contain more policies than those defined by condition (2) in the “Policies under restrictions” which does not allow for machine idle. Nevertheless, by introducing a dummy arm with constantly zero reward rate, constant filtration and the trivial feasible random time set \([0, \infty)\), the setting in condition (2) can model this more realistic situation.

(c) Conditions (1) – (3) in “Policies under restrictions” are similar to those in Kaspi and Mandelbaum (1998), whereas condition (4) that is new captures the feature of restricted policies that the machine can operate arm \( k \) at a rate strictly less than 1 only when its operation time is in \( \mathcal{M}^k \); in other words, if \( T^k(t) \in \mathcal{M}^k \), then at time \( t \), the machine can only be occupied by arm \( k \) exclusively.

(d) Clearly, the setting we have just formulated subsumes classical versions in discrete time, continuous time and general discrete time setting, as discussed below:

i) Because \( \mathcal{M}^k = (\mathbb{N} \cup \{\infty\}) \times \Omega \) indicates that arm \( k \) can be switched at only integer times, an integer time MAB process corresponds to a RMAB process in which \( \mathcal{M}^k = (\mathbb{N} \cup \{\infty\}) \times \Omega \) for every \( k = 1, 2, \ldots, d \).

ii) In the case of a semi-Markov process, let \( G^k_i \) be the state of the process and denote by \( \tau^k_n, n = 0, 1, \ldots \), the time instants at which \( G^k_i \) makes transitions, with \( \tau^k_0 = 0 \).

Arm \( k \) can only be switched only at the time instants \( \tau^k_n, n = 0, 1, \ldots \), so that

\[
\mathcal{M}^k = \{(\tau^k_n(\omega), \omega) : n = 0, 1, \ldots, \omega \in \Omega\} \cup \{ (\infty, \omega) : \omega \in \Omega \}. \tag{2.2}
\]

A semi-Markovian MAB corresponds to a RMAB process with every \( \mathcal{M}^k, k = 1, 2, \ldots, d \) of the form in (2.2).
iii) We in this item show how the RMAB processes can be reduced to general discrete
time MAB processes. Let \( \{s_n : n \geq 1\} \) be a sequence of increasing \( \mathcal{F}^k \)-stopping
times at which arm \( k \) can be can be stopped to switch to another arm, satisfying
\( \Pr(s_n \geq s_{n-1}) = 1 \) for all \( n = 1, 2, \ldots \) and \( \lim_{n \to \infty} s_n = \infty \) a.s.. Clearly, for this
example,
\[
\mathcal{M}^k = \{(s_n(\omega), \omega) : n = 0, 1, \ldots, \omega \in \Omega\} \cup \{ (\infty, \omega) : \omega \in \Omega\}.
\] (2.3)

Also, an general discrete time MAB corresponds to a RMAB process with every
\( \mathcal{M}^k \) having the form in Equation (2.3). This model extends the semi-Markov model
by dropping the Markovian property in the transition. Note that this model essentially covers MAB in discrete time, because the evolving of the process in between
s_n and s_{n-1} are irrelevant for the purpose of making decision on stopping at those
stopping times \( s_k, k = 1, 2, \ldots \). It was discussed in Cai et al (2014, Section 6.1)
and Cowan and Katehakis (2015) when they discussed their multi-armed bandit
processes. Clearly, RMAB process clearly covers general discrete time model as a
special case, but not vice versa because, as just stated, RMAB process covers the
continuous time version of MAB whereas that discrete time version of MAB does
not.

iv) If \( \mathcal{M}^k = [0, \infty] \times \Omega \), arm \( k \) is an arm in continuous time in which one can stop at
any time,and for optimal stopping problem in discrete time.

(e) Moreover, the restrictions allow one to tackle many more situations. Here is a selection
of some examples, for all of which but the first the existing theory for MAB processes
cannot apply.

i) If the case \( \mathcal{M}^k = \{0, +\infty\} \), then \( \mathcal{M}^k_c = \mathbb{R}_+ \times \Omega \), so that the arm \( k \) will be operated
exclusively forever once it is picked. Obviously, it corresponds a nonpreemptable
arm.

ii) If \( \mathcal{M}^k = [0, \tau] \cup \{n : n \text{ is positive integer in between } [\tau, \{+\infty\}] \cup [+\infty]\}, \) where \( \tau \) is
an \( \mathcal{F}^k \)-stopping time, so that \( \mathcal{M}^k_c = (\tau, \infty) \), then switches from arm \( k \) are all time
points no larger than \( \tau \) and the integer time points larger than \( \tau \).
iii) Let \( s^k_n, n = 1, 2, \ldots, \infty \) be a sequence of \( \mathcal{F}^k \)-stopping times increasing in \( n \) and
\[
\mathcal{M}^k = \bigcup_{n=1}^{\infty} [s_{2n-1}, s_{2n}] \bigcup \{0, \infty\} \times \Omega.
\]
Then arm \( k \) can only be switched from at its private random time intervals \([s_{2n-1}, s_{2n}], n = 1, 2, \ldots\) whereas in its private time intervals \((s_{2n-2}, s_{2n-1}), n = 1, 2, \ldots\), the occupation of machine by this arm is exclusive.

iv) One can treat MAB processes of multiple types of arms, where operation on some of the arms can be switched to other arms at any time (corresponding to a continuous time setting) but operation on some other arms can only be switched when the machine has been served for integer amount of time (discrete time setting) or when the state of the arm is just transferred in the case of the semi-Markovian setting. Some arms can even be nonpreemptable.

3 Gittins Indices for A Single-Arm Process

After the RMAB processes were formulated in the last section, we now associate each arm with an appropriately defined Gittins index process, which unifies and extends the classical definitions for discrete time, continuous time and general discrete time setting. Because we consider only a single arm so as to define the associated Gittins index process and demonstrate its desired properties, for the time being, the arm identifier \( k \) is suppressed for the time being for notation convenience. Hence we work only with a single stochastic process \( G = (G_t)_{t \in \mathbb{R}_+} \) that is \( \mathcal{F} \)-adapted on a filtered probability space \((\Omega, \mathcal{F}, P)\), equipped with a quasi-left-continuous filtration \( \mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+} \) satisfying the usual conditions of right continuity and augmentation by the null sets of \( \mathcal{F}_\infty \), where \( \mathbb{R}_+ = [0, +\infty] \). To \((\Omega, \mathcal{F}, P)\) is associated with a random set \( \mathcal{M} \) to represent the restricted feasibility on the stopping times, as defined in Section 2.

This section consists of two parts: Section 3.1 gives a concise review of restricted optimal stopping times with some material taken from Bao et al. (2017), which is put here for easy reference and in Section 3.2 we define the Gittins index process induced over a single arm and gives its details.
3.1 Optimal stopping times under the restrictions

The optimal stopping time problem with restrictions, denoted by \((\Omega, \mathcal{F}, P, \mathcal{M})\), is defined as the following: For an arbitrary stopping time \(\nu \in [0, \infty]\) (unnecessarily in \(\mathcal{M}\)), find an optimal stopping times \(\tau^* \in \mathcal{M}\) such that

\[
Z_{\nu} := E[G_{\tau^*} | \mathcal{F}_{\nu}] = \text{esssup}_{\tau \in \mathcal{M}_{\nu}} E[G_{\tau} | \mathcal{F}_{\nu}],
\]

(3.1)

where esssup stands for the operation of essential supremum, \(\mathcal{M}_{\nu} = \{\tau \geq \nu : \tau \in \mathcal{M}\}\) and \(G\) is assumed to satisfy the following assumptions:

**Assumption 3.1**

1. \(G\) has almost surely right continuous paths.
2. \(E\left[\sup_{t \in \mathbb{R}^+} |G_t| \right] < \infty\).
3. \(E[G_{\infty}] \geq \limsup_{t \to \infty} E[G_t]\).

By Bao et al. (2017), problem (3.1) is solved by the following two theorems that are cited here for later reference. The first theorem characterizes the optimal stopping times should they exist.

**Theorem 3.1** The following three statements are equivalent for any \(\tau_\ast \in \mathcal{M}_{\nu}:

(a) \(\tau_\ast\) is optimal for problem (3.1), i.e., \(Z_{\nu} = E[G_{\tau_\ast} | \mathcal{F}_{\nu}]\);
(b) The stochastic process \(\{Z_{\tau_\ast \wedge (\nu \vee t)} : t \in \mathbb{R}^+\}\) is an \(\mathcal{F}_{\nu \vee t}\)-martingale and \(Z_{\tau_\ast} = G_{\tau_\ast}\) a.s.;
(c) \(Z_{\tau_\ast} = G_{\tau_\ast}\) a.s. and \(Z_{\nu} = E[Z_{\tau_\ast} | \mathcal{F}_{\nu}]\).

For any \(\lambda \in (0, 1)\) and stopping time \(\nu\), define \(D^\lambda_{\nu} = \text{essinf}\{\tau \in \mathcal{M}_{\nu} : \lambda Z_{\tau} \leq G_{\tau}\}\) and \(D^1_{\nu} = \lim_{\lambda \uparrow 1} D^\lambda_{\nu}\). The following theorem indicates the existence of the required stopping time \(\tau^*\).

**Theorem 3.2** If \(G_t\) is quasi-left continuous, then

(1) \(D^1_{\nu}\) is optimal for the stopping problem (3.1), that is, \(Z_{\nu} = E[G_{D^1_{\nu}} | \mathcal{F}_{\nu}]\) a.s.,
(2) \(D^1_{\nu} = \text{essinf}\{\tau \in \mathcal{M}_{\nu} : Z_{\tau} = G_{\tau}\} = \min\{t \geq \nu(\omega) : (\omega, t) \in \mathcal{M}, Z(\omega, t) = G(\omega, t)\}\) a.s., and
(3) \(Z_t\) is also quasi-left-continuous.
3.2 Gittins index process

For the instant reward rate process \( X_t \) and an arbitrary stochastic processes \( q = \{q_t\} \) that is \( \mathcal{F} \)-adapted, pathwise right continuous, nonincreasing, bounded and nonnegative, introduce a discounted gain process

\[
G_q(t; m) = \int_0^t e^{-\beta u} q_u X_u du + \beta m \int_t^\infty e^{-\beta u} q_u du, \quad t \in [0, \infty].
\]  

(3.2)

Note that \( q_t \equiv 1 \) gives the well-known gain process with retirement option \( G(t; m) = \int_0^t e^{-\beta u} X_u du + me^{-\beta t}, \quad t \in [0, \infty] \), which was introduced by Whittle (1980). To any finite \( \mathcal{F} \)-stopping time \( \eta \), associate a class of optimal stopping problems

\[
V_q(\eta, m) = \text{esssup}_{\tau \in \mathcal{M}_{\eta}} \mathbb{E} \left[ \int_{\eta}^{\tau} e^{-\beta(u-\eta)} q_u X_u du + \beta m \int_{\tau}^{\infty} e^{-\beta(u-\eta)} q_u du \middle| \mathcal{F}_\eta \right],
\]  

(3.3)

indexed by \( m \in [0, \infty) \), indicating the optimal expected rewards from \( \eta \) onwards. Then, for every fixed \( m \in [0, \infty) \), the optimal stopping time theory reviewed in Section 3.1 can be translated for \( V_q(\eta, m) \) to:

(1). The process \( Z_q(t, m) = \int_0^t e^{-\beta u} q_u X_u du + e^{-\beta t} V_q(t, m), \quad t \in [0, \infty] \) is a quasi-left-continuous supermartingale.

(2). The feasible stopping time

\[
\sigma_\eta(m) = \text{essinf}\{\tau \in \mathcal{M}_{\eta} : Z_q(\tau; m) = G_q(\tau; m)\}
\]

\[
= \text{essinf} \left\{ \tau \in \mathcal{M}_{\eta} : V_q(\tau; m) = \beta m \mathbb{E} \left[ \int_\tau^\infty e^{-\beta(u-\tau)} q_u du \middle| \mathcal{F}_\eta \right] \right\} \in \mathcal{M}_{\eta}
\]  

(3.4)

is an optimal solution for \( V_q(\eta; m) \).

(3). \( \{Z_q(\tau; m) : \tau \text{ is } \mathcal{F}\text{-stopping time satisfying } \eta \leq \tau \leq \sigma_\eta(m)\} \) is a martingale family.

Moreover, for any finite \( \eta \in \mathcal{M} \) and \( m \in [0, \infty) \), write

\[
\varphi_q(\eta, m) = \text{esssup}_{\tau \in \mathcal{M}_{\eta}} \mathbb{E} \left[ \int_{\eta}^{\tau} e^{-\beta u} q_u (X_u - \beta m) du \middle| \mathcal{F}_\eta \right].
\]  

(3.5)

It is then immediate that

\[
\varphi_q(\eta, m) = e^{-\beta \eta} \left[ V_q(\eta; m) - \beta m \mathbb{E} \left( \int_\eta^\infty e^{-\beta(u-\eta)} q_u du \middle| \mathcal{F}_\eta \right) \right].
\]  

(3.6)
Remark 3.1 Given a stopping time \( \eta \), owing to the the esssup operation, even though for any couple of nonnegative numbers \( m_1 < m_2 \) and \( \lambda \in [0, 1] \), it is clear that

\[
\Pr \{ \omega : \varphi_q(\eta, m_1) \geq \varphi_q(\eta, m_2) \text{ and } \varphi_q(\eta, \lambda m_1 + (1 - \lambda) m_2) \geq \lambda \varphi_q(\eta, m_1) + (1 - \lambda) \varphi_q(\eta, m_2) \} = 1,
\]

definition (3.5) does not necessarily ensure pathwise monotonicity and convexity of \( \varphi_q(\eta, m) \) in \( m \). This difficulty can be overcome by a procedure as follows. First, order the rationals arbitrarily as \( Q = \{ r_1, r_2, \ldots, r_n, \ldots \} \) and write \( Q_n = \{ r_1, r_2, \ldots, r_n \} \). Let \( \Omega_1 = \Omega \). For \( n \geq 2 \), denote \( \Omega_n = \{ \omega : \varphi_q(\eta, \cdot) \text{ is nonincreasing and convex on } Q_n \} \). Then \( \Omega_n \) is decreasing in \( n \) and \( \Pr(\Omega_n) = 1 \) for all \( n \geq 1 \). Let \( \bar{\Omega} := \bigcap_{n=1}^{\infty} \Omega_n \) such that \( \Pr(\bar{\Omega}) = 1 \), and for every \( \omega \in \bar{\Omega} \), \( \varphi_q(\eta, m) \) is decreasing along set \( Q \). For the other (real) numbers \( m \), take \( \varphi_q(\eta, m; \omega) \) as the limit of \( \varphi_q(\eta, r; \omega) \) along \( Q \), so that \( \varphi_q(\eta, m; \omega) \) defined as such is a decreasing and convex function of \( m \) for every \( \omega \in \bar{\Omega} \). That is, we get a version of \( \varphi_q(\eta, r; \omega) \) that is pathwise decreasing and convex in \( m \) almost surely. We will thoroughly work with this version of \( \varphi_q(\eta, m) \).

The following is a fundamental property of \( \sigma(m) \).

Lemma 3.1 Given a stopping time \( \eta \in \mathcal{M} \), \( \sigma_q(m) \) is nonincreasing and right-continuous in \( m \).

Proof. The monotonicity of \( \sigma_q(m) \) follows from the fact that

\[
\varphi_q(\sigma_q(m_2); m_1) \leq \varphi_q(\sigma_q(m_2); m_2) \leq 0 \text{ for } m_1 > m_2,
\]

so that \( \sigma_q(m_1) = \text{essinf} \{ \tau \in \mathcal{M}_\eta : \varphi_q(\tau; m_1) \leq 0 \} \leq \sigma_q(m_2) \). For the right-continuity of \( \sigma_q(m) \) in \( m \), consider a decreasing sequence \( \delta_n \downarrow 0 \) of real numbers. By the monotonicity above, the sequence \( \sigma_q(m + \delta_n) \) is a nondecreasing sequence dominated by \( \sigma_q(m) \). Then there exists \( \sigma_* \in \mathcal{M}_\eta \) such that \( \sigma_* = \lim_{n \to \infty} \sigma_q(m + \delta_n) \leq \sigma_q(m) \). On the other hand, thanks to the quasi-left-continuity of \( \varphi_q \) (implied by that of \( Z \), cf. Theorem 3.2 (3)) and the fact that \( \varphi_q(\sigma_q(m + \delta_l); m + \delta_k) \leq 0 \) for any \( l > k \), we see that \( \varphi_q(\sigma_*; m + \delta_k) = \lim_{l \to \infty} \varphi_q(\sigma_q(m + \delta_l); m + \delta_k) \leq 0 \). Hence, the continuity of \( \varphi_q(\sigma_*, m) \) in \( m \) implies that
\[
\varphi_q(\sigma; m) = \lim_{k \to \infty} \varphi_q(\sigma; m + \delta_k) \leq 0, \text{ which in turn implies } \sigma_\ast \geq \sigma_\eta(m). \quad \text{Consequently,} \\
\sigma_\ast = \sigma_\eta(m), \text{ that is, } \lim_{n \to \infty} \sigma_\eta(m + \delta_n) = \sigma_\eta(m).
\]

This completes the proof. 

Thanks to this lemma, with a procedure similar to Remark 3.1 we can work with the version of \(\sigma_\eta(m)\) that is nonincreasing and right continuous in \(m\) for every \(\omega \in \Omega\), so that we can speak of its pathwise inverse

\[
M^q_t = \sup \{ m \geq 0 : \sigma_\eta(m) > t \}, \quad t \geq \eta,
\]

and write particularly

\[
M^q_\eta = M^q_0(\eta) \quad \text{and} \quad M^q(t) = M^q_\Delta(t).
\]

The following lemma explains what these quantities indicate and states that \(M_\eta := M^1_\eta\) is a direct extension of the classical Gittins index to the setting with restricted stopping times.

**Lemma 3.2** Given \(\eta \in \mathcal{M}\), the following properties hold for the stochastic process \(\{M^q_t(t)\}\):

(a). \(M^q_\eta(t)\) is \(\mathcal{F}\)-adapted.

(b). \(M^q_\eta = \inf\{m > 0 : \varphi_q(\eta, m) \leq 0\}\).

(c). \(M^q_\eta(\rho) = \text{essinf}_{\tau \in \mathcal{M}_\eta, \tau \leq \rho} M^q_\tau\) for \(\rho \in \mathcal{M}_\eta\).

(d). \(\beta M^q_\eta(\rho) = \text{esssup}_{\tau > \eta, \tau \in \mathcal{M}} E\left[\int_\eta^\tau e^{-\beta(s-u)q_s X_u} du \mid \mathcal{F}_\eta\right] / E\left[\int_\eta^\tau e^{-\beta(s-u)q_s X_u} du \mid \mathcal{F}_\eta\right]\).

**Proof.** (a). For any finite \(m > 0\) and \(t \in [0, \infty)\), if follows that

\[
\{\omega : M^q_t(t) > m\} = \{\omega : \eta > t\} \cup \{\omega : \sigma_\eta(m) > t \text{ and } \eta \leq t\} = \{\omega : \eta > t\} \cup \left\{\omega : V_q(u, m) > \beta m E\left[\int_u^\infty e^{-\beta(s-u)q_s X_u} ds \mid \mathcal{F}_u\right] \text{ for all } (u, \omega) \in ([\eta, t] \times \omega) \cap \mathcal{M}(\omega)\right\},
\]

where the first equality is a straightforward result of definition (3.7) and the second from equality (3.14). Note that the first equality implies the adaptedness of \(\{M^q_t(t)\}\), i.e., \(M^q_\eta(t) \in \mathcal{F}_t\) for all \(t \in \mathbb{R}_+\). This proves (a).
(b). For \( \eta \in \mathcal{M} \), it is clear that
\[
M_{\eta}^q = \inf \left\{ m \geq 0 : V_q(\eta, m) = \beta m E \left[ \int_{\eta}^{\infty} e^{-\beta(s-\eta)} q_s ds \right] F_\eta \right\} = \inf \{ m > 0 : \varphi_q(\eta, m) \leq 0 \}.
\] (3.10)

(c). Note that, by \([3.9]\), for \( t \geq \eta(\omega) \),
\[
M_{\eta}^q(t) > m \iff V_q(u, m) > \beta m E \left[ \int_{\eta}^{\infty} e^{-\beta(s-\eta)} q_s ds \right] F_\eta 	ext{ for all } u \in [\eta, t] \cap \mathcal{M}(\omega)
\]
\[
\iff M_{\eta}^q > m \text{ for all } u \in [\eta, t] \cap \mathcal{M}(\omega).
\]
That is, \( M_{\eta}^q(t) = \inf_{\eta \leq u \leq t, (u, \omega) \in \mathcal{M} M_\eta^q} \). Re-expressing this in terms of stopping times leads to the desired equality \( M_{\eta}(\rho) = \text{essinf}_{\tau \in \mathcal{M}, \tau \leq \rho} M_{\eta}^q \) for \( \eta \in \mathcal{M} \) and \( \rho \in \mathcal{M}_\eta \).

(d). It is obvious that \( V_q(\eta; m) = V_q^+(\eta; m) \bigvee \beta m E \left[ \int_{\eta}^{\infty} e^{-\beta(u-\eta)} q_u du \right] F_\eta \) for \( \eta \in \mathcal{M} \), where
\[
V_q^+(\eta, m) = \text{esssup}_{\tau \in \mathcal{M}, \tau > \eta} E \left[ \int_{\eta}^{\tau} e^{-\beta(u-\eta)} q_u X_u du + \beta m \int_{\tau}^{\infty} e^{-\beta(u-\eta)} q_u du \right] F_\eta. \tag{3.11}
\]
The assertion in (d) thus follows from the equivalence
\[
V_q(\eta; m) \leq \beta m E \left[ \int_{\eta}^{\infty} e^{-\beta(u-\eta)} q_u du \right] F_\eta \iff V_q^+(\eta; m) \leq \beta m E \left[ \int_{\eta}^{\infty} e^{-\beta(u-\eta)} q_u du \right] F_\eta
\]
\[
\iff \beta m \geq \text{esssup}_{\tau > \eta, \tau \in \mathcal{M}} \frac{E[\int_{\eta}^{\tau} e^{-\beta u q_u X_u du} | F_\eta]}{E[\int_{\eta}^{\infty} e^{-\beta u q_u du} | F_\eta]}.
\]
The proof is thus completed. \hfill \blacksquare

The following lemma establishes a crucial expression for \( E \left[ \int_{\eta}^{\infty} e^{-\beta t q_t X_t dt} \right] \) by means of the right derivative of \( V_q(\eta, m) \) with respect to \( m \).

**Lemma 3.3** For any stopping time \( \eta \in \mathcal{M} \), \( V_q(\eta; m) \) is increasing in \( m \) with right-hand derivative
\[
\frac{\partial^+ V_q(\eta; m)}{\partial m} = \beta E \left[ \int_{\sigma_{\eta}(m)}^{\infty} e^{-\beta(u-\eta)} q_u du \right] F_\eta. \tag{3.12}
\]
As a result,
\[
E \left[ \int_{\eta}^{\infty} e^{-\beta u q_u X_u du} \right] = \beta E \left[ \int_{\eta}^{\infty} e^{-\beta u q_u M_{\eta}^q(u) du} F_\eta \right]. \tag{3.13}
\]
Proof. The monotonicity of \( V_q(\eta, m) \) in \( m \) is straightforward and we first examine equality (3.12). For \( \delta > 0 \), Theorem 3.1 (b) and Lemma 3.1 simply state that \( Z_q(\eta; m) = E[Z_q(\sigma_\eta(m + \delta); m)|\mathcal{F}_\eta] \), so that

\[
V_q(\eta; m) = E \left[ \int_{\eta}^{\sigma_\eta(m + \delta)} e^{-\beta(u-\eta)} q_u X_u du + e^{-\beta(\sigma_\eta(m + \delta) - \eta)} V_q(\sigma_\eta(m + \delta); m) \right]_{\mathcal{F}_\eta}
\]

\[
\geq E \left[ \int_{\eta}^{\sigma_\eta(m + \delta)} e^{-\beta(u-\eta)} q_u X_u du + \beta m \int_{\sigma_\eta(m + \delta)}^{\infty} e^{-\beta(u-\eta)} q_u du \right]_{\mathcal{F}_\eta}
\]

\[
= V_q(\eta; m + \delta) - \beta \delta E \left[ \int_{\sigma_\eta(m + \delta)}^{\infty} e^{-\beta(u-\eta)} q_u du \right]_{\mathcal{F}_\eta}.
\]

Consequently,

\[
V_q(\eta; m + \delta) - V_q(\eta; m) < \beta \delta E \left[ \int_{\sigma_\eta(m + \delta)}^{\infty} e^{-\beta(u-\eta)} q_u du \right]_{\mathcal{F}_\eta}. \tag{3.14}
\]

On the other hand, the relationship

\[
Z_q(\eta; m + \delta) = E[Z_q(\sigma_\eta(m + \delta); m + \delta)|\mathcal{F}_\eta] \geq E[Z_q(\sigma_\eta(m); m + \delta)|\mathcal{F}_\eta],
\]

which is obtained from the supermartingale property of \( Z_q(t; m + \delta) \), implies that

\[
V_q(\eta; m + \delta) \geq E \left[ \int_{\eta}^{\sigma_\eta(m)} e^{-\beta(u-\eta)} X_u du + e^{-\beta(\sigma_\eta(m) - \eta)} V_q(\sigma_\eta(m); m + \delta) \right]_{\mathcal{F}_\eta}
\]

\[
\geq V_q(\eta; m) + \beta \delta E \left[ \int_{\sigma_\eta(m)}^{\infty} e^{-\beta(u-\eta)} q_u du \right]_{\mathcal{F}_\eta}.
\]

Hence,

\[
V_q(\eta; m + \delta) - V_q(\eta; m) \geq \beta \delta E \left[ \int_{\sigma_\eta(m)}^{\infty} e^{-\beta(u-\eta)} q_u du \right]_{\mathcal{F}_\eta}. \tag{3.15}
\]

Combining (3.14) with (3.15) and letting \( \delta \to 0+ \) lead to the desired equality (3.12).

By (3.12) and the equality \( V_q(\eta; M_{\eta}^q) - V_q(\eta; 0) = \int_0^{M_{\eta}^q} \frac{\partial^+ V_q(\eta; m)}{\partial m} dm \), it follows that

\[
V_q(\eta; M_{\eta}^q) - V_q(\eta; 0) = \beta \int_0^{M_{\eta}^q} E \left[ \int_{\sigma_\eta(m)}^{\infty} e^{-\beta(u-\eta)} q_u du \right]_{\mathcal{F}_\eta} dm
\]

\[
= M_{\eta}^q \beta E \left[ \int_{\eta}^{\infty} e^{-\beta(u-\eta)} q_u du \right]_{\mathcal{F}_\eta} - \beta E \left[ \int_0^{M_{\eta}^q} \int_{\eta}^{\sigma_\eta(m)} e^{-\beta(u-\eta)} q_u du dm \right]_{\mathcal{F}_\eta}.
\]

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Noting that $V_q(\eta; M^q_\eta) = \beta M^q_\eta \mathbb{E}\left[ \int_\eta^\infty e^{-\beta(u-\eta)} q_u du \bigg| \mathcal{F}_\eta \right]$, it is immediate that

$$V_q(\eta; 0) = \beta \mathbb{E}\left[ \int_0^{M^q_\eta} \int_\eta^{\sigma_{\eta}(m)} e^{-\beta(u-\eta)} q_u dm \bigg| \mathcal{F}_\eta \right].$$

(3.16)

Due to the relationship

$$\{ (m, u) : \eta \leq u \leq \sigma_{\eta}(m), 0 \leq m \leq M^q_\eta \} = \{ (m, u) : 0 \leq m \leq \overline{M}^q_\eta(u), \eta \leq u < \infty \},$$

it follows by interchanging the integrations in (3.16) that

$$V_q(\eta; 0) = \beta \mathbb{E}\left[ \int_\eta^\infty e^{-\beta(u-\eta)} q_u \overline{M}^q_\eta(u) du \bigg| \mathcal{F}_\eta \right].$$

Thus the desired equality in (3.13) follows.

We will need to treat the case where one has an extra $\sigma$-algebra $\mathcal{G}'$ that is independent of the filtration $\mathcal{F}$. This introduces a new filtration $\mathcal{G} = \{ \mathcal{G}_t \}$ by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{G}'$, generally called an initial enlargement (or augmentation) of $\mathcal{F}$ by $\mathcal{G}'$. Denote the set of all $\mathcal{G}$-stopping times taking values a.s. in $\mathcal{M}$ by $\mathcal{M}_\mathcal{G}$ and those taking values in $\mathcal{M}$ and larger than or equal to $\tau$ by $\mathcal{M}_\mathcal{G}^\tau$. Consider the setting in which

(a) $X_t$ is $\mathcal{F}$-adapted and

(b) $q_t$ is $\mathcal{G}$-adapted, almost surely right continuous, and right decreasing at such time $t$ with $(t, w) \in \mathcal{M}$.

Under the augmented filtration $\mathcal{G}$, taking the right continuous version of $Z(t, m)$, we can extend the notation $Z(\tau, m)$ to any $\mathcal{G}$-stopping times $\tau$ by $Z(\tau, m) = Z(\tau(\omega), m)$. Define a new optimization problem $\bar{Z}(\tau, m) = \text{esssup}_{\nu \in \mathcal{M}_\mathcal{G}^\tau} \mathbb{E}[G(\nu, m)|\mathcal{G}_\tau]$. Then it is straightforward that $Z(\tau, m) = \bar{Z}(\tau, m)$ for any $\mathcal{G}$-stopping time $\tau$, which states that, regardless of the enlargement of the domain of stopping times by initially introducing extra information, the optimal stopping problem basically remains if the additionally obtained information is independent of the original information filtration $\mathcal{F}$ and $X_t$ is $\mathcal{F}$-adapted. The following lemma holds for any $\mathcal{G}$-adapted, right continuous $q_u$ that is right decreasing only when $u \in \mathcal{M}$.
Lemma 3.4 \textit{Let} $\tilde{X}_t$ \textit{be an arbitrary $\mathcal{F}$-adapted process and $q_t$ be $\mathcal{G}$-adapted, right continuous, and right decreasing at time $t \in \mathcal{M}$. Then, for any $\mathcal{F}$-stopping times $\eta \in \mathcal{M}$, the inequality $\operatorname{esssup}_{\nu \in \mathcal{M}, \eta} E[\int_\eta^\nu e^{-\beta t} \tilde{X}_t dt | \mathcal{F}_\eta] \leq 0$ implies $E[\int_\eta^\infty e^{-\beta t} q_t \tilde{X}_t dt | \mathcal{G}_\eta] \leq 0$.}

\textit{Proof.} Introduce the right continuous inverse $q^{-1}(s) = \min\{u : q_u \leq s\} = \max\{u : q_u > s\}$. Then, for any $s$, $q^{-1}(s)$ is a $\mathcal{G}$-stopping time because $\{\omega : q^{-1}(s) \leq u\} = \{\omega : q_u \leq s\} \in \mathcal{G}_u$. In addition,

(a) for any $t \geq \eta$, the relationship $s < q_t(\leq q_\eta)$ implies $q^{-1}(s) > \eta$ and

(b) for any $\omega$, $q^{-1}(s) \in \mathcal{M}(\omega)$ because $q_u$ is right decreasing only when $u \in \mathcal{M}$.

These two points actually further state that, for any $s \in [0, 1)$, $q^{-1}(s)$ is a $\mathcal{G}$-stopping time in $\mathcal{M}_\eta^G$. Note that $q_\eta \in \mathcal{G}_\eta$. Therefore,

$$E\left[\int_\eta^\infty e^{-\beta t} \tilde{X}_t dt | \mathcal{G}_\eta\right] = E\left[\int_\eta^\infty e^{-\beta t} \tilde{X}_t \int_0^{q_t} ds dt | \mathcal{G}_\eta\right] = \int_\eta^{q_\eta} E\left[\int_\eta^{q^{-1}(s)} e^{-\beta t} \tilde{X}_t dt | \mathcal{G}_\eta\right] ds \leq 0,$$

This completes the proof. \hfill \blacksquare

With Lemma 3.4 setting $\tilde{X}_t = X_t - \beta m$ and replacing $q_u$ in Lemma 3.4 by $\tilde{q}_u = I_{(\eta, \tau)} q_u$, it is immediate that $\varphi(\eta, m) \leq 0$ implies $\varphi_\eta(\eta, m) \leq 0$. Consequently,

$$M_\eta^q = \inf\{m > 0 : \varphi_\eta(\eta, m) \leq 0\} \leq M_\eta = \inf\{m > 0 : \varphi(\eta, m) \leq 0\} \text{ for all } \eta \in \mathcal{M}. \quad (3.17)$$

Now let $T(t)$ be a generic component of a policy (i.e., $T^k(t)$ in the policy formulation with some $k \in \{1, 2, \ldots, d\}$). We address the particular choice

$$q_u = \exp[-\beta(\zeta(u) - u)], \quad (3.18)$$

where

$$\zeta(u) = \inf\{t : T(t) > u\} \quad \text{(hence, } T(t) < u \iff \zeta(u) < t) \quad (3.19)$$

is the right continuous inverse of $u = T(t)$, indicating the calendar time of the system at which the current arm, which has been operated for $u$ units of time, is to be selected for further operation, so that $\zeta(u) - u$ is the time spent on other arms and thus is also
nondecreasing in $u$. Clearly, the particular $q_u$ in (3.18) is $G$-adapted, right continuous, and right decreasing at time $u \in \mathcal{M}$. The following lemma gives an bound for the expected discounted reward from a single arm under any policy.

**Lemma 3.5** Let $T(t)$ be a generic component of a policy. Then

$$E \left[ \int_{0}^{\infty} e^{-\beta t} X_{T(t)} dt \right] \leq E \left[ \int_{0}^{\infty} \beta e^{-\beta t} M(T(t)) dt \right].$$

(3.20)

**Proof.** First note that, by the definition $q$ in (3.18),

$$E \left[ \int_{0}^{\infty} e^{-\beta t} X_{T(t)} dt \right] = E \left[ \int_{0}^{\infty} e^{-\beta \zeta(t)} X_t dt \right] = E \left[ \int_{0}^{\infty} e^{-\beta t} q_t X_t dt \right].$$

Because $M(t)$ and $M^q(t)$ are both nonincreasing, $M^q(t) \leq M(t)$ follows from (3.17). Therefore, an application of equality (3.13) indicates that

$$E \left[ \int_{0}^{\infty} e^{-\beta t} X_{T(t)} dt \right] = E \left[ \int_{0}^{\infty} \beta e^{-\beta t} q_t M^q(t) dt \right] \leq E \left[ \int_{0}^{\infty} \beta e^{-\beta t} q_t M(t) dt \right].$$

Using again the definition of $q$ leads to

$$E \left[ \int_{0}^{\infty} e^{-\beta t} X_{T(t)} dt \right] \leq E \left[ \int_{0}^{\infty} \beta \exp(-\beta \zeta(t)) M(t) dt \right] = E \left[ \int_{0}^{\infty} \beta e^{-\beta t} M(T(t)) dt \right].$$

This proves the lemma.

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**4 Optimal Allocation of RMAB Processes**

We are now ready to state and prove the results on the optimal policies for the RMAB processes. The identifier $k$ of arms has to be added back.

We will see that the solution to this problem is still the celebrated Gittins index policy, with Gittins indices generalized as follows. Note that for $\eta \in \mathcal{M}^k$, the Gittins index is the same as $M^q_\eta$ introduced in the previous section (by (3.7) and (3.8)) with $q \equiv 1$, see especially Lemma 3.2 (d), whereas for $\eta$ not in $\mathcal{M}^k$, the definition is new.
Definition 4.1 The Gittins indices \( \{M^k_\eta : \eta \text{ is a finite } F^k\text{-stopping time} \} \) of arm \( k \) are defined by two steps:

Step 1. For \( F^k\)-stopping times \( \eta \in M^k \), compute

\[
M^k_\eta = \text{esssup}_{\tau > \eta, \tau \in M^k} \frac{\mathbb{E}[\int_{\eta}^{\tau} e^{-\beta u} X^k_u du | F^k_\eta]}{\beta \mathbb{E}[\int_{\eta}^{\tau} e^{-\beta u} du | F^k_\eta]},
\]

where the essential supremum is taken over the set \( \{\tau : \tau > \eta \text{ and } \tau \in M^k\} \) of \( F^k\)-stopping times.

Step 2. For other \( F^k\)-stopping times \( \eta \), find \( \eta = \text{esssup} \{\tau \leq \eta : \tau \in M^k\} \) \((\in M^k)\) and define \( M^k_\eta = M^k_\eta \).

Since \( M^k_\eta \) is defined for all stopping times \( \eta \), we can construct an associated process \( M^k = \{M^k_t(\omega) : t < \infty, \}\) to \( M^k_\eta \) (called Gittins indices process) and all the processes \( \{M^k, k = 1, 2, \ldots, d\} \) serve to select an arm to operate, as will be illustrated later on.

Remark 4.1 Here note that Definition 4.1 unifies and extends all the classical definitions of Gittins indices in discrete time, continuous time and general discrete time setting to the current RMAB process situation. For example, in the case \( M^k = \mathbb{N} \cup \{\infty\} \), \( (M^k_\eta) \) is a stochastic sequence of Gittins indices in integer times, coinciding with what were defined by Gittins and Jones (1974) and Gittins (1979).

With the lower envelope \( M^k(t) = \min_{0 \leq u \leq t} M^k(u) \) (see (3.8) and Lemma 3.2 (c)), let \( \mathcal{M}^k_t(\omega) = \text{closure} \{t : (t, \omega) \in \mathcal{M}^k(\omega), M^k_t(\omega) = M^k(t, \omega)\} \) and call the times in the complement of \( \mathcal{M}^k_t(\omega) \) excursion times of \( M^k \) from its lower envelope \( \underline{M}^k \). It is obvious that, for fixed \( \omega \), the set of excursion times is a union of countably many open intervals.

The next presents the definition of (Gittins) index policy.

Definition 4.2 A restricted policy \( \tilde{T} = (\tilde{T}^1, \ldots, \tilde{T}^d) \) is a (Gittins) index policy if, for each \( k \), \( \tilde{T}^k = \{\tilde{T}^k(t) \geq 0\} \) right increases at time \( t \geq 0 \) only when

\[
M^k_{\tilde{T}^k(\omega)} = \sqrt[\text{d}]{\prod_{j=1}^{d} M^j_{\tilde{T}^j(\omega)}},
\]

and time must be allocated exclusively to a single arm over its excursion interval without switching.
Because $\mathcal{M}_1^k \subset \mathcal{M}^k$, it is clear that an index policy satisfies the restrictions on policies. As observed by Mandelbaum (1987), index policies need not be unique. The solution to the RMAB process is stated in the theorem below.

**Theorem 4.1** Any (restricted) index policy $\hat{T} = (\hat{T}_1, \ldots, \hat{T}_d)$ is optimal with the optimal value expressed in terms of the lower envelopes of the indices as

$$V = \mathbb{E} \left[ \int_0^\infty \beta e^{-\beta t} \left( \bigvee_{k=1}^d M^k(\hat{T}_k(t)) \right) dt \right]. \quad (4.3)$$

**Proof.** Define $\tilde{F}_i^k = \mathcal{F}_i^k \vee \bigcup_{j \neq k} \mathcal{F}_j^\infty$. Fix an arbitrary policy $T$ and let $\zeta^k(t) = \inf\{u : T^k(u) > t\}$ be the generalized inverse of $T^k(t)$. Define

$$v(T) = \mathbb{E} \left[ \sum_{k=1}^d \int_0^\infty e^{-\beta t} X^k_{T^k(t)} dT^k(t) \right]$$

and

$$\underline{v}(T) = \mathbb{E} \left[ \sum_{k=1}^d \int_0^\infty \beta e^{-\beta t} M^k(T^k(t)) dT^k(t) \right] \quad (4.4)$$

to represent respectively the expected values of the original bandit process and a deteriorating bandit process with reward rates $M^k(t), k = 1, 2, \ldots, d$ under the same policy $T$. Note that Lemma 3.5 simply states that

$$\mathbb{E} \left[ \int_0^\infty e^{-\beta t} X^k_{T^k(t)} dT^k(t) \right] \leq \mathbb{E} \left[ \int_0^\infty \beta e^{-\beta t} M^k(T^k(t)) dT^k(t) \right].$$

Summing it over all arms $k = 1, 2, \ldots, d$, it follows that under any policy $T$,

$$v(T) \leq \underline{v}(T).$$

Thus, to prove the optimality of an index policy $\hat{T}$, it suffices to prove $\underline{v}(T) \leq \underline{v}(\hat{T}) = v(\hat{T})$. This is done by the following Lemmas 4.1 and 4.2. \hfill \blacksquare

**Lemma 4.1** For any policy $T$ and index policy $\hat{T}$, $\underline{v}(T) \leq \underline{v}(\hat{T})$.

**Proof.** Under $T$, the total discounted reward $\underline{v}(T)$ for the reward rates $\{\beta M^k(t), k = 1, 2, \ldots, d\}$ is

$$\underline{v}(T) = \beta \sum_{k=1}^d \mathbb{E} \left[ \int_0^\infty e^{-\beta t} M^k(T^k(t)) dT^k(t) \right]. \quad (4.5)$$
Write \( h^k_u = \sup\{t : M^k(t) > u\} = \inf\{t : M^k(t) \leq u\} \) for the right-continuous inverse of \( M^k(t) \), which models the time needed to operate the arm such that its reward rate falls down to a level no more than \( u \). Because \( M^k(t) = M^k(\mathbf{t}) \), where \( \mathbf{t} \) is defined as in Definition 4.1, it is clear that \( h^k_u \in M^k \). Thus, in order that all \( M^k \) can fall down to level \( u \), one needs to spend in total \( \tilde{h}_u = \sum_{k=1}^d h^k_u \) units of time on the \( d \) arms.

We first examine the equality

\[
\sum_{k=1}^d \hat{T}^k(t) \wedge h^k_u = t \wedge \tilde{h}_u \tag{4.6}
\]

over the set of \( t \) at which all \( M^k(\hat{T}^k(t)) \), \( k = 1, 2, \ldots, d \), are continuous.

Since \( \sum_{k=1}^d \hat{T}^k(t) = t \) and \( \tilde{h}_u = \sum_{k=1}^d h^k_u \), it suffices to show that there exists no pair \((k, p)\) of identifiers such that

\[
\hat{T}^p(t) > h^p_u \quad \text{and} \quad \hat{T}^k(t) < h^k_u \tag{4.7}
\]

if both \( M^k(\hat{T}^k(t)) \) and \( M^p(\hat{T}^p(t)) \) are continuous at point \( t \). We prove it by contradiction. If (4.7) were true, then \( M^p(\hat{T}^p(t)) \leq u < M^k(\hat{T}^k(t)) \). Define

\[
\tau = \sup\{s : s < t, M^p(\hat{T}^p(t)) \geq M^k(\hat{T}^k(s))\} = \inf\{s : s \leq t, M^p(\hat{T}^p(t)) < M^k(\hat{T}^k(s))\}
\]

with the convention \( \sup\emptyset = 0 \). Because \( M^p(\hat{T}^p(s)) < M^k(\hat{T}^k(s)) \) for all \( s \in (\tau, t] \), the feature of \( \hat{T} \) following the leader indicates that

\[
\hat{T}^p(t) = \hat{T}^p(\tau). \tag{4.8}
\]

If \( \tau = 0 \) then \( \hat{T}(t) = 0 \leq h^p_u \), contradicting (4.7). Otherwise, i.e., \( \tau > 0 \), find a sequence of nonnegative sequence \( \alpha_n \to 0 \) such that \( M^p(\hat{T}^p(\tau - \alpha_n)) \geq M^k(\hat{T}^k(\tau - \alpha_n)) \geq M^k(\hat{T}^k(t)) > u \); if \( M^p(\hat{T}^p(\tau)) \geq M^k(\hat{T}^k(\tau)) \), then \( \alpha_n \) all take value 0. Therefore, \( \hat{T}^p(\tau - \alpha_n) < h^p_u \). Setting \( n \to \infty \) and using (4.8) lead to \( \hat{T}^p(t) = \hat{T}^p(\tau) \leq h^p_u \), contradicting (4.7) again. Thus (4.6) is proved.

We now turn to check the desired inequality of the lemma. For any policy \( T \),

\[
\sum_{k=1}^d (T^k(t) \wedge h^k_u) \leq \sum_{k=1}^d T^k(t) \wedge \tilde{h} = \sum_{k=1}^d (\hat{T}^k(t) \wedge h^k_u). \tag{4.9}
\]
Thus, by (4.5),
\[ v(T) = \beta \sum_{k=1}^{d} E \left[ \int_0^\infty e^{-\beta t} M_k(T^k(t)) dT^k(t) \right] \]
\[ = \beta \sum_{k=1}^{d} E \left[ \int_0^\infty e^{-\beta t} \int_0^\infty I_{\{0<u<\infty, M_k(T^k(t))\}} dudT^k(t) \right]. \]

By Fubini’s theorem,
\[ v(T) = \beta \sum_{k=1}^{d} E \left[ \int_0^\infty \int_0^\infty e^{-\beta t} I \{ T^k(t) < h^k_u \} dT^k(t) du \right] \]
\[ = \beta \sum_{k=1}^{d} E \left[ \int_0^\infty \int_0^\infty e^{-\beta t} d(T^k(t) \wedge h^k_u) du \right]. \]

Further using the partial integration and the inequality in (4.9) yields
\[ v(T) = \beta^2 E \left[ \int_0^\infty \int_0^\infty e^{-\beta t} \sum_{k=1}^{d} (T^k(t) \wedge h^k_u) dtdu \right] \]
\[ \leq \beta^2 E \left[ \int_0^\infty \int_0^\infty e^{-\beta t} \sum_{k=1}^{d} (\hat{T}^k(t) \wedge h^k_u) dtdu \right] \]
\[ = v(\hat{T}). \]

This proves the desired result.

Remark 4.2 An arm is said deteriorating if its reward rate is pathwise nonincreasing in time and a bandit is deteriorating if all its arms are deteriorating. In this case, the optimal policy is myopic in the sense that it plays the arms with the highest immediate reward rate. In fact, this lemma can be generalized as: \( v(T) \leq v(\hat{T}) \) for any restricted deteriorating bandits \( \{(X^k_t, M_k), k = 1, 2, \ldots, n\} \), where \( M_k \) is the feasible set of stopping times associated with arm \( k \).

Lemma 4.2 For any index policy \( \hat{T} \), \( v(\hat{T}) = v(\hat{T}) \).

Proof. For every arm \( k \), introduce two policy-free quantities
\[ D^k_t := D^k_t(\omega) = \inf \{ u > t : u \in M^k_1(\omega) \} \]
\[ g^k_t := g^k_t(\omega) = \sup \{ u < t : u \in M^k_1(\omega) \} \]
for \( t > 0 \),
so that, for every \( t > 0 \) and \( u > t \),
\[
g_u^k \leq t \iff u \leq D_t^k
\]
and
\[
M^k_{D_t^k}(u) = M^k(u).
\]

Note that, under any index policy \( \hat{T} = (\hat{T}^1, \ldots, \hat{T}^d) \), \( \zeta^k(t) = \zeta^k(g_t^k) + t - g_t^k \). See (3.19) for the definition of \( \zeta^k(t) \), where the identifier \( k \) was suppressed. Define \( H_k(u) = \zeta^k(u) - u \) and its inverse \( H_k^{-1}(s) = \inf\{u : H_k(u) > s\} \), which is the time spent on arm \( k \) when \( s \) units of time has been spent on the other \( d - 1 \) arms and an arm other than \( k \) is to be selected to operation, so that \( H_k(u) \leq s \iff u \leq H_k^{-1}(s) \). Consequently, \( H_k(g_u^k) \leq s \iff H_k^{-1}(s) \geq g_u^k \iff u \leq D_{H_k^{-1}(s)} \). Therefore, the expected reward under the index policy \( \hat{T} \) can be computed by
\[
v(\hat{T}) = \sum_{k=1}^{d} \mathbb{E} \left[ \int_0^\infty e^{-\beta \zeta^k(u)} X_u^k du \right] = \sum_{k=1}^{d} \mathbb{E} \left[ \int_0^\infty e^{-\beta (\zeta^k(g_u^k) - g_u^k)} e^{-\beta u} X_u^k du \right].
\]

A bit algebraic computation gives rise to
\[
v(\hat{T}) = \beta \sum_{k=1}^{d} \mathbb{E} \left[ \int_0^\infty e^{-\beta u} X_u^k \int_0^\infty e^{-\beta s} ds du \right] = \beta \sum_{k=1}^{d} \mathbb{E} \left[ \int_0^\infty e^{-\beta s} \left( \int_0^{D_{H_k^{-1}(s)}} e^{-\beta u} X_u^k du \right) ds \right].
\]

By the first equality in (3.13) under \( \tilde{F} \) (see Lemma 3.3), it then follows that
\[
\mathbb{E} \left( \int_0^{D_{H_k^{-1}(s)}} e^{-\beta u} X_u^k du \right) = \mathbb{E} \left[ \int_0^\infty e^{-\beta u} X_u^k du - \mathbb{E} \left( \int_0^{D_{H_k^{-1}(s)}} e^{-\beta u} X_u^k du \right| \tilde{F}^k_{D_{H_k^{-1}(s)}} \right] = \beta \mathbb{E} \left[ \int_0^\infty e^{-\beta u} M^k(u) du - \mathbb{E} \left( \int_0^{D_{H_k^{-1}(s)}} e^{-\beta u} M^k_{D_{H_k^{-1}(s)}}(u) du \right| \tilde{F}^k_{D_{H_k^{-1}(s)}} \right] = \beta \mathbb{E} \left[ \int_0^\infty e^{-\beta u} M^k(u) du - \mathbb{E} \left( \int_0^{D_{H_k^{-1}(s)}} e^{-\beta u} M^k(u) du \right| \tilde{F}^k_{D_{H_k^{-1}(s)}} \right] = \beta \mathbb{E} \left[ \int_0^{D_{H_k^{-1}(s)}} e^{-\beta u} M^k(u) du \right].
\]
where the last equality follows from the equality in (4.11). Consequently,

\[ v(\hat{T}) = \beta^2 \sum_{k=1}^{\infty} E \left[ \int_0^\infty e^{-\beta s} \int_0^{D H_{k-1}(s)} e^{-\beta u M_k(u)} du ds \right] = \beta(\hat{T}). \]

This proves the desired equality.

5 Conclusions

By the extended optimal stopping theory to the problem with restricted stopping times, and combining the martingale techniques and Whittle (1980)'s retirement option, we have developed a general and new framework that generalizes and unifies the theories of multi-armed bandit processes in discrete time, continuous time, which apply also to other mixed settings that can occur practically but are not solved by the existing theory of MAB processes.

While the mathematical form of Gittins indices has obtained in terms of an essential supremum of the reward rate over a class of stopping times, it is generally difficult to precisely or numerically compute the Gittins indices, even with no restriction on switches. The only exceptions are the cases of MAB in discrete time and semi-Markovian fashion with finite states by means of achievable region method, and MAB in continuous time where every arm is a standard job, namely, it has a processing time and a constant reward is collect on the completion of the job (Gittins et al., 2011). Under the general framework of RMAB processes, it is challenging how the Gittins indices should be computed, even if an arm evolves in a Markovian chain or semi-Markovian fashion, both with finite number of states, or an arm acts as a standard job. This difficulty reflects the impact of the restrictions on switches, which imposes a challenging task for the Gittins index rule developed in this paper to be applied in real world problems.

In this study, the MAB processes considered are standard in the sense that the number \( d \) of arms is fixed, the arms that are inactive are frozen and contribute no reward and the switches from arm to arm cause neither operation delay nor switch cost and the arms are statistically independent. When these standard conditions are violated, the behaviors of the bandit processes and thus their optimal policies are quite different. The models extend
the standard MAB processes include, for example, restless MAB processes (Whittle, 1988 and Bertsimas and Nio-Mora, 2000) in which the inactive arms evolve also, open bandit processes (Whittle 1981, Weiss, 1988 and Wu and Zhou, 2013) in which new arms can arrive from time to time, MAB processes with switching cost/delay (Banks and Sundaram, 1994 and Jun, 2004). It has turned out that the optimality of the Gittins index policies for the standard RMAB processes are largely dependent on those conditions. It is unclear how the restrictions on switches among arms could impact the policies of those nonstandard MAB processes.

References

[1] Banks, J. S. and Sundaram, R. K. (1994). Switching costs and the Gittins index. Econometrica: Journal of the Econometric Society, 62 (3), 687-694.

[2] Bao, W., Wu, X. and Zhou, X. (2017). Optimal stopping problems with restricted stopping times, Journal of Industrial and Management Optimization, 13 (1), 399-411.

[3] Bertsimas, D. and Nio-Mora, J. (2000). Restless bandits, linear programming relaxations, and a primal-dual index heuristic. Operations Research, 48 (1), 80-90.

[4] Cai, X., Wu, X. and Zhou, X. (2009). Stochastic scheduling subject to preemptive-repeat breakdowns with incomplete information. Operations Research, 57(5): 1236-1249.

[5] Cai, X., Wu, X. and Zhou, X. (2014). Optimal Stochastic Scheduling, Springer Science+Business Media New York.

[6] Cowan, W. and Katehakis, M.N. (2015). Multi-armed bandits under general depreciation and commitment, Probability in the Engineering and Informational Sciences, 29 (1), 51-76.

[7] Jun, T. (2004). A survey on the bandit problem with switching costs. De Economist, 152 (4), 513-541.
[8] EL Karoui, N. and Karatzas, I. (1993). General Gittins index processes in discrete time. *Proceedings of the National Academy of Sciences of the United States of America*, 90, 1232-1236.

[9] EL Karoui, N. and Karatzas, I. (1994). Dynamic allocation problem in continuous time. *Annals of Applied Probability*, 4, 255-286.

[10] Eplett, W.J.R. (1986). Continuous-time allocation and their discrete-time approximations. *Advances in Applied Probability*, 18, 724-746.

[11] Gittins, J. and Jones, D. (1974) A dynamic allocation index for the sequential allocation of experiments, in (J. Gani, et al., Eds.) *Progress in Statistics*, North Holland, Amsterdam, pages 241-266, read at the 1972 European Meeting of Statisticians, Budapest.

[12] Gittins, J. C. (1979). Bandit processes and dynamic allocation indices (with Discussion). *Journal of Royal Statistical Society. B*, 41, 148-164.

[13] Gittins, J. C. (1989). *Multi-Armed Bandit Allocation Indices*. John Wiley & Sons, Inc.

[14] Gittins, J. C., Glazebrook, K. D. and Webber, R. R. (2011). *Multi-Armed Bandit Allocation Indices*. John Wiley & Sons, Inc.

[15] Karatzas, I. (1984). Gittins indices in the dynamic allocation problem for diffusion processes. *Annals of Probability*, 12, 173-192.

[16] Kaspi, H. and Mandelbaum, A. (1998). Multi-armed bandits in discrete and continuous time. *Annals of Applied Probability*, 8 (4), 1270-1290.

[17] Mandelbaum A. (1986). Discrete multiarmed bandits and multiparameter processes. *Probability Theory and Related Fields*, 71, 129-147.

[18] Mandelbaum, A. (1987). Continuous multi-armed bandits and multiparameter processes. *Annals of Probability*, 15 (4), 1527-1556.

[19] Snell, L. (1952). Applications of martingale systems theorems. *Transactions of American Mathematics Society.*, 73, 293-312.
[20] Varaiya, P., Walrand, J. and Buyukkoc, C. (1985). Extensions of the multiarmed bandit problem: The discounted case, IEEE Transactions on Automatic Control, 230, 426-439.

[21] Weiss, G. (1988). Branching bandit processes. Probability in the Engineering and Informational Sciences, 2(3), 269-278.

[22] Whittle, P. (1980). Multi-armed bandits and Gittins index. Journal of Royal Statistics Society, B, 42, 143-149.

[23] Whittle, P. (1981). Arm-acquiring bandits. The Annals of Probability, 9(2), 284-292.

[24] Whittle, P. (1982). Optimization over Time: Dynamic Programming and Stochastic Control. John Wiley, New York.

[25] Whittle, P. (1988). Restless bandits: Activity allocation in a changing world. Journal of applied probability, 25(A), 287-298.

[26] Wu, X. and Zhou, X. (2013). Open bandit processes with uncountable states and time-backward effects. Journal of Applied Probability, 50 (2), 388-402.