SCHWARZIAN QUANTUM MECHANICS AS A DRINFELD-SOKOLOV REDUCTION OF BF THEORY

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ABSTRACT. We give an interpretation of the holographic correspondence between two-dimensional BF theory on the punctured disk with gauge group PSL(2, R) and Schwarzian quantum mechanics in terms of a Drinfeld-Sokolov reduction. The latter, in turn, is equivalent to the presence of certain edge states imposing a first class constraint on the model. The constrained path integral localizes over exceptional Virasoro coadjoint orbits. The reduced theory is governed by the Schwarzian action functional generating a Hamiltonian $S^1$-action on the orbits. The partition function is given by a sum over topological sectors, each of which is computed by the Duistermaat-Heckman integration formula. We recover the same results in the operator formalism.

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1. INTRODUCTION

In their seminal paper [20], Maldacena, Stanford and Yang showed that Jackiw-Teitelboim (JT) gravity on a disk is dual to Schwarzian quantum
mechanics on the boundary. This is an instance of an $\text{AdS}_2/\text{CFT}_1$ correspondence and serves as a toy model for quantum gravity. Notably, the only dynamical variables of the theory are (orientation preserving) diffeomorphisms of the boundary. More recently, the partition function of JT gravity on two-dimensional compact surfaces of arbitrary genus and with arbitrary many boundaries has been computed exactly by describing it as a matrix model and employing topological recursion methods [24].

JT gravity is a two-dimensional dilatonic gravity theory [14, 27]. It is well-known that the model admits a first order formulation in terms of a two-dimensional $BF$ theory with gauge group $\text{SL}(2,\mathbb{R})$ [15]. In fact, this is a special example of a more general construction of two-dimensional dilatonic gravity theories, which were classified in [12, 25] by means of so-called Poisson sigma models.

Schwarzian quantum mechanics is a very rich and interesting theory in itself due to its strong ties with the Sachdev-Ye-Kitaev model [19] (and references therein) where it arises as a low energy limit. Notably, its partition function is one-loop exact and was calculated in [26]. In particular, the path integral is taken over the space of orientation preserving diffeomorphisms $\text{Diff}^+(S^1)$ modulo an $\text{SL}(2,\mathbb{R})$-action. Now, $\text{Diff}^+(S^1)/\text{SL}(2,\mathbb{R})$ can be identified with an exceptional Virasoro orbit [2, 28] on which the Schwarzian action functional generates a Hamiltonian $S^1$-action by rotating the (source) circle. The one-loop exactness of the partition function is therefore a consequence of an analog of the Duistermaat-Heckman integration formula.

Motivated by the above observations, this article is devoted to the study of the holographic dual of two-dimensional $BF$ theory on a punctured disk (or cylinder) with gauge group $\text{PSL}(2,\mathbb{R})$. The holographic duality between the Schwarzian theory and $BF$ theory on a disk for gauge group $\text{SL}(2,\mathbb{R})$ was already explored in [3, 6, 7, 21]. The duality was derived from the holographic correspondence of three-dimensional gravity and two-dimensional Liouville theory by dimensional reduction. It was shown that the configuration space of the $BF$ theory on a disk reduces to the space of contractible based loops of the gauge group. Furthermore, constraining the gauge group to $\text{SL}^+(2,\mathbb{R})$ recovers the correct density of states [6] which was previously calculated in [26].

Our approach is different: In passing from the disk to the cylinder we are no longer restricted to contractible loops. Rather, we consider loops of arbitrary winding number $n$. In addition, the constraint we impose is different in nature as it keeps loops with $n > 0$ in the theory. Indeed, we find that the partition function picks up contributions from all corresponding topological sectors. For $n = 1$ we recover the results of [26].

In more detail: following [6], we implement the boundary conditions by adding a Hamiltonian on the boundary. In addition we assume that half of the fields vanish at the puncture and we restrict the holonomy of the gauge field to be trivial. After shortly recalling some background material in Section 2, we recall in Section 3 how integration over the scalar fields localizes the $BF$ theory over the space of flat connections modulo gauge transformations which are trivial on the boundary. The latter can be naturally
identified with the space of based maps from the boundary to PSL(2, R), while the action functional reduces to quantum mechanics of a free particle moving in the group manifold. Furthermore, we explain how the presence of edge states, in the sense of [11], constrains the path integral.

A detailed analysis of the constrained model is given in Section 4. In particular, we show that the constraint can be understood in terms of a Drinfeld-Sokolov reduction of the space of based loops in PSL(2, R). After the reduction, the path integral localizes to a sum of integrals over exceptional Virasoro coadjoint orbits. Each orbit is associated to a topologically distinct sector corresponding to different windings of the (based) loop. At the same time, the theory reduces to Schwarzian quantum mechanics whose action functional generates an $S^1$-action on the orbits. Inspired by [22, 26], we define the partition function in terms of an infinite-dimensional version of the Duistermaat-Heckman integration formula. It turns out that there is a unique fixed point of the $S^1$-action and we find

$$Z(c) \propto \sum_{n \geq 1} n \exp \left( \frac{i c n^2}{24 \hbar} \right).$$

Furthermore, we show that in the presence of a fixed edge state the constrained partition function reduces to a sum of integrals over non-exceptional Virasoro coadjoint orbits. Again the reduced action generates a Hamiltonian $S^1$-action. Once more, we compute the partition function by means of a Duistermaat-Heckman integration.

Since the holographic dual of the two-dimensional BF theory in the presence of edge states is (constrained) quantum mechanics, the partition function can be computed also in the operator formalism. This computation is carried out in Section 4.5. The Hamiltonian of the model coincides (up to a constant) with the quadratic Casimir operator of $\mathfrak{sl}(2, \mathbb{R})$ and the Hilbert space is given by certain unitary irreducible representations of PSL(2, R). The constraint singles out a special eigenvalue of the generator of the non-compact subgroup of upper triangular matrices. The corresponding states were described by Lindblad and Nagel [18] and constitute the spectrum of a particle moving in a quantized magnetic field on the upper half plane [8]. Identifying the Hilbert space with the positive discrete series representation of PSL(2, R), we compute the constrained partition function in the operator formalism and find the same result as in the path integral approach.

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2. Preliminaries

2.1. Finite-dimensional Duistermaat-Heckman integration. Since the aim of this note is to calculate the partition function of a quantum mechanical system on the group manifold PSL(2, R) by means of an infinite-dimensional version of Duistermaat-Heckman integration, we first want to give a non-exhaustive recollection of finite-dimensional Duistermaat-Heckman integration. For more details we refer the interested reader to [1, 4, 22].

Let $(M, \omega)$ be a compact symplectic $2n$-dimensional manifold endowed with an action of $S^1$. Suppose that this circle action is Hamiltonian, that is
we assume that the action is generated by a vector field $\xi$ and the existence of a smooth function $H$ on $M$ which satisfy the relation
\[ t_\xi \omega + dH = 0. \]
Moreover, suppose that $H$ has only isolated critical points. Then, Duistermaat and Heckman showed in [10] that the integral
\[ I(\varepsilon) = \int_M e^{i\varepsilon H} \frac{\omega^n}{n!} \]
localizes over the fixed points of $H$:
\[ I(\varepsilon) = \sum_m e^{i\varepsilon H(m)} \prod_{j=1}^n \frac{i\varepsilon}{2\pi} w_j(m), \]
where the sum runs over all (isolated) critical points $m$ of $H$ and the $w_j(m)$ are the weights of the $S^1$-action on the tangent space $T_mM$ of $M$ at $m$. Finally, the integration measure is taken to be the Liouville measure $\omega^n/n!$ defined by the symplectic form $\omega$.

2.2. Kac-Moody orbits. Let $G$ be a semi-simple Lie group with Lie algebra $\mathfrak{g}$. Denote by $LG = C^\infty(S^1, G)$ the loop space of $G$. The space $LG$ is itself a Lie group, whose Lie algebra $L\mathfrak{g}$ coincides with the algebra of smooth $\mathfrak{g}$-valued function on the circle.

In the following, we will be interested in the central extension $\hat{L}\mathfrak{g}$. Elements of $\hat{L}\mathfrak{g}$ are of the form $(u(x), k)$, where $u(x)$ is a $\mathfrak{g}$-valued function of $S^1$ and $k \in \mathbb{R}$ is central. The Lie bracket on $\hat{L}\mathfrak{g}$ is defined by
\[ [u, v](x) = \frac{1}{2\pi} \text{tr} \int_0^x v(x) u'(x) dx. \]
where $[u, v]_\mathfrak{g}$ denotes the bracket in the Lie algebra $\mathfrak{g}$ and tr denotes the normalized Killing form on $\mathfrak{g}$. Elements of the dual space $\hat{L}\mathfrak{g}^*$ can be described equally by pairs $(v(x), -k)$, where $v(x)$ is again $\mathfrak{g}$-valued function on $S^1$ and $k$ a real number. The number $k$ is called the level.

The pairing between $\hat{L}\mathfrak{g}^*$ and $\hat{L}\mathfrak{g}$ is
\[ \langle (v, -k), (u, \ell) \rangle = \text{tr} \int v(x) u(x) dx - k \ell. \]
The coadjoint action of $LG$ on $\hat{L}\mathfrak{g}^*$ is then
\[ (v, -k)_g \equiv \text{Ad}_g^*(v, -k) = \left( gvg^{-1} + \frac{k}{2\pi} g'g^{-1}, -k \right). \]
The coadjoint orbit is naturally symplectic, so is $\mathcal{O}_v$. The symplectic form on $\mathcal{O}_v$ descends from the left-invariant pre-symplectic form on $LG$, which can be described as follows: For $X, Y \in T_v(LG) = L\mathfrak{g}$, one defines
\[ \omega_v(X, Y) = \langle (v, -k), [(X, 0), (Y, 0)] \rangle = \text{tr} \int \left( v[X, Y]_\mathfrak{g} + \frac{k}{2\pi} XY' \right) dx. \]

\footnote{For the sake of readability, we will often refrain from writing the explicit dependence on the level, e.g. we write $\mathcal{O}_v$ instead of $\mathcal{O}_{(v, -k)}$ and $\text{Stab}(v)$ instead of $\text{Stab}(v, -k)$.}
Using left translation in $LG$, the pre-symplectic form can then be defined at any $g \in LG$:

$$\omega_g = \text{tr} \oint \left( v(g^{-1} \delta g)^2 + \frac{k}{4\pi} g^{-1} \delta g \wedge (g^{-1} \delta g)' \right) dx,$$

where $g^{-1} \delta g$ denotes the left-invariant Maurer-Cartan element of $LG$.

It is instructive to look at two examples in detail.

**Example 2.1.** Suppose that $v = 0 \in Lg^*$. Then the stabilizer $\text{Stab}(v)$ is the group $G$ itself and the coadjoint orbit passing through 0 is isomorphic to the space of based loops $O_0 \cong LG/G \cong \Omega G = \{g \in LG \mid g(0) = e \in G\}$, and the symplectic form, at level $k$, reduces to

$$\omega_g = \frac{k}{4\pi} \text{tr} \oint (g^{-1} \delta g \wedge (g^{-1} \delta g)') dx.$$

Rotating the loop defines an $S^1$-action on $\Omega G$ by $^2$: $g(x) \mapsto g(x + t)$, $t \in S^1$. This action is Hamiltonian, see e.g. [4, 23]. Its generating Hamiltonian, with respect to the symplectic form (7), is the energy function $H : \Omega G \rightarrow \mathbb{R}$ of the loop defined by

$$H(g) = -\frac{k}{4\pi} \text{tr} \oint (g' g^{-1})^2 dx.$$

Moreover, any closed subgroup $H \subset G$ acts on $\Omega G$ by conjugation. This action is as well Hamiltonian with respect to (7). If $H$ has Lie algebra $\mathfrak{h}$, then the moment map is defined by the projection of $g' g^{-1} \in Lg \cong Lg^*$ to $L\mathfrak{h}^*$ (see for instance [23])

$$\mu : \Omega G \rightarrow L\mathfrak{h}^*$$

$$g \mapsto \text{pr}_{\mathfrak{h}^*} (g' g^{-1})$$

**Example 2.2.** Let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t}$, and consider a constant regular element $v_0 \in \mathfrak{t}^* \subset Lg^*$. Here, **regular** means that the stabilizer subgroup of $v_0$ is a maximal torus. Then the orbit through $v_0$ is isomorphic to the homogeneous space $LG/T$. In this case, the symplectic form can be conveniently written in terms of the quasi-periodic element $h(x) = g(x) \exp(-\frac{4\pi}{k} v_0 x)$:

$$\omega_g = \frac{k}{4\pi} \text{tr} \oint h^{-1} \delta h \wedge (h^{-1} \delta h)' dx.$$

The $S^1$-action which rotates the loop is again Hamiltonian, with

$$H(g) = -\frac{k}{4\pi} \text{tr} \oint (h' h^{-1})^2 dx = -\frac{k}{4\pi} \text{tr} \oint (v_0 + g^{-1} g')^2 dx.$$

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$^2$Here, and henceforth, we take $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. In particular, we will use the additive notation for the group structure on $S^1$. 
2.3. Virasoro orbits. This section follows closely [2, 28]. Let \( \text{Vect}(S^1) \) be the Lie algebra of the orientation preserving diffeomorphism group of the circle, denoted by \( \text{Diff}^+(S^1) \). Elements \( \phi \in \text{Diff}^+(S^1) \) can be seen as increasing quasi-periodic maps, i.e. \( \phi : \mathbb{R} \to \mathbb{R} \) satisfying
\[
\phi(x + 2\pi) = \phi(x) + 2\pi.
\]
Let \( \text{Vir} \) be the central extension of \( \text{Vect}(S^1) \) by \( \mathbb{R} \). Elements of \( \text{Vir} \) are pairs \((v, r)\) where \( v \in \text{Vect}(S^1) \) stands for the Lie bracket in \( \text{Vect}(S^1) \).

The Lie bracket on \( \text{Vir} \) is defined by
\[
[(v_1, r_1), (v_2, r_2)] = \left( [v_1, v_2]_{\text{Vect}(S^1)}, \frac{1}{48\pi} \oint v''_1 v_2 - v_1 v''_2 \right),
\]
where \([v_1, v_2]_{\text{Vect}(S^1)} = (v_1 v'_2 - v'_1 v_2) \partial_x\) stands for the Lie bracket in \( \text{Vect}(S^1) \).

In particular, if we define
\[
L_m = i e^{imx} \partial_x
\]
then the commutators
\[
[(L_m, a), (L_n, b)] = \left( (m - n)L_{m+n}, \frac{m^3}{12}\delta_{m+n,0} \right)
\]
define the Virasoro algebra relation, which, in the literature, is more commonly written as
\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m^3\delta_{m+n,0}.
\]

**Remark 1.** Shifting \( L_0 \) by \( \frac{c}{24} \) amounts to replacing \( m^3 \) by \( m^3 - m \), which is found more often in the literature.

Elements of the dual space \( \text{Vir}^* \) are pairs \((b, c)\) where \( b = b(x)dx^2 \) is a quadratic differential on the circle and \( c \in \mathbb{R} \). The pairing between \( \text{Vir} \) and \( \text{Vir}^* \) is given by
\[
\langle (b, c), (v, d) \rangle = \oint b(x)v(x)dx + cd.
\]
The coadjoint action of \( \text{Diff}^+(S^1) \) on \( \text{Vir}^* \) is described as follows: infinitesimally, i.e. for \( v \in \text{Vect}(S^1) \), one has
\[
\text{ad}^*_v(b, c) = \left( (2v'(x)b(x) + v(x)b'(x) - \frac{c}{24}v'''(x))dx^2, 0 \right).
\]
This integrates to
\[
(b, c)_\phi \equiv \text{Ad}^*_{\phi^{-1}}(b, c) = \left( \phi^*b - \frac{c}{24}\{\phi, x\}dx^2, c \right)
\]
where \( \phi \in \text{Diff}^+(S^1) \) and
\[
\{\phi, x\} = \frac{\phi'''}{\phi'^2} - \frac{3}{2} \left( \frac{\phi''}{\phi'} \right)^2 = \left( \frac{\phi''}{\phi'} \right)' - \frac{1}{2} \left( \frac{\phi''}{\phi'} \right)^2
\]
denotes the Schwarzian derivative of \( \phi \).

We can identify the coadjoint orbit \( \mathcal{O}_b \) through the point \((b, c) \in \text{Vir}^* \) with
\[
\mathcal{O}_b \cong \text{Diff}^+(S^1)/\text{Stab}(b)
\]
where \( \text{Stab}(b) \subset \text{Diff}^+(S^1) \) is the stabilizer subgroup of \( b \) under the coadjoint action (18).

The stabilizers of a general quadratic differential \( b \in \text{Vir}^* \) are hard to calculate. However, if \( b = b_0 dx^2 \), \( b_0 \in \mathbb{R} \) is constant, the calculation becomes feasible: A point in the orbit \( \mathcal{O}_{b_0} \) can be written as

\[
(b_0, c)_{\phi} = \left( \left( \phi^2 b_0 - \frac{c}{24\pi}, \phi, x \right) dx^2, c \right).
\]

For generic \( b_0 \in \mathbb{R} \), \( (b_0, c) \) is only invariant under a (constant) translation, namely \( \phi(x) = x + a \). This shows that \( \text{Stab}(b_0) = S^1 \) such that the orbit is isomorphic to the homogeneous space \( \text{Diff}(S^1)/S^1 \).

However, for exceptional values of \( b_0 \) one finds that the stabilizer subgroup of \( b_0 \) is larger than \( S^1 \): choosing the vector field \( v \) to be \( L_n = i e^{int} \partial_x \) in Equation (17), one has for constant \( b_0 \)

\[
ad^*_L(b_0, c) = \left( \pm \left( \frac{1}{2} \cdot i e^{int} b_0 - \frac{c}{24\pi} \cdot 3 i e^{int} \right) dx^2, 0 \right)
\]

\[
= \left( \mp 2n e^{int} \left( b_0 + \frac{c n^2}{48\pi} \right) dx^2, 0 \right).
\]

Therefore, for the special values

\[
b_0 = -\frac{c n^2}{48\pi},
\]

the stabilizer is generated by \( \{ L_0, L_{\pm n} \} \). This integrates to the subgroup \( \text{SL}(n)(2, \mathbb{R}) \subset \text{Diff}^+(S^1) \) which is the \( n \)-fold cover of \( \text{SL}(2, \mathbb{R}) \) [28]. Therefore, for the exceptional values (23), the orbits are identified with the homogeneous space \( \text{Diff}^+(S^1)/\text{SL}(n)(2, \mathbb{R}) \):

\[
\mathcal{O}_n \cong \text{Diff}^+(S^1)/\text{SL}(n)(2, \mathbb{R}).
\]

Being a coadjoint orbit, \( \mathcal{O}_{b_0} \) is naturally symplectic. In terms of the left-invariant Maurer-Cartan element \( Y \) of \( \text{Diff}^+(S^1) \) [2], the symplectic form at any point \( \phi \) is given by

\[
\omega_{\phi} = \langle (b_0, c)_{\phi}, [Y(\phi), Y(\phi)] \rangle.
\]

Explicitly, we have \( Y(\phi) = \frac{\delta \phi}{\partial x} \), where \( \delta \) denotes the de Rham differential on \( \text{Diff}^+(S^1) \). Then, for \( b_0 = -\frac{c b^2}{48\pi} \) with \( b \in \mathbb{R} \), one finds (c.f. [2, 26])

\[
\omega_{\phi} = -\frac{c}{48\pi} \oint \left( 2 b^2 \delta \phi \wedge \delta \phi' - \frac{\delta \phi' \wedge \delta \phi''}{\phi'^2} \right) dx.
\]

In the following, we will be interested in an infinite-dimensional version of Duistermaat-Heckman integration over the orbits \( \mathcal{O}_{b_0} \). To this end, let us remark that there exists again an \( S^1 \)-action on \( \mathcal{O}_{b_0} \) which rotates the source circle: \( \phi(x) \mapsto \phi(x + a) \). As it turns out, this \( S^1 \)-action is Hamiltonian with respect to the symplectic form (26). The Hamiltonian in this case is

\[
H(\phi) = \frac{c}{24\pi} \oint \left( \frac{b^2}{2} \phi'^2 + \{ \phi, x \} \right) dx.
\]

\[\text{By abuse of notation, we will often write } b_0 \text{ instead of } b_0 dx^2.\]
3. BF THEORY ON A PUNCTURED DISK

3.1. Quantum mechanics in PSL(2, R) as a holographic dual. Let $D^* = D - \{0\}$ be the punctured unit disk with the origin removed and let $G = \text{PSL}(2, \mathbb{R})$. We denote by $\mathfrak{g}$ its Lie algebra and by tr the non-degenerate Killing form. The BF theory we are interested in is defined by a $\mathfrak{g}$-valued scalar field $X$ and a $\mathfrak{g}$-valued one form $A$. The action functional of the model is

$$S(X, A) = \int_{D^*} \text{tr} X F_A + \oint_{\partial D^*} \text{tr} X A,$$

where $F_A = dA + \frac{1}{2} [A, A]$ denotes the curvature of $A$. In order to have a well-defined Dirichlet problem for the variational principle, we need to specify boundary conditions. Following [6], we choose to implement these boundary conditions by adding a Hamiltonian on the boundary:

$$S(X, A) = \int_{D^*} \text{tr} X F_A + \oint_{\partial D^*} \text{tr} X A - \frac{1}{2} \text{tr} X^2 dx,$$

where $dx$ denotes a volume form on $S^1 = \partial D^*$ and the Hamiltonian $\text{tr} X^2$ is, up to a constant, the quadratic Casimir of PSL(2, R). Moreover, we assume that the scalar field $X$ vanishes at the puncture.

Remark 2. We can think of the punctured disk as a semi-infinite cylinder $(-\infty, t_0) \times S^1$. Let $t \in (-\infty, t_0)$ be the coordinate along the cylinder and $x$ the angle coordinate. Then the origin, i.e., the puncture of the disk corresponds to $t \to -\infty$. The assumption that $X$ vanishes at the puncture of the disk is therefore equivalent to the assumption that $X$ vanishes at infinity.

The equations of motion for $X$ are

$$F_A = 0 \quad \text{(bulk)},$$
$$A|_{\partial D^*} = X dx|_{\partial D^*} \quad \text{(boundary)}.$$

Hence, integrating out the scalar field $X$, the path integral localizes to the moduli space of flat connection on $D^*$. To determine the moduli space in question, we first point out that the action is gauge invariant only under those gauge transformations that are trivial on the boundary. Indeed, let $G = C^\infty(D^*, G)$ be the full gauge group, acting on the fields $X$ and $A$ as

$$X^g = gXg^{-1}, \quad A^g = gAg^{-1} - dg g^{-1}.$$

Then the action is invariant only up to a boundary term:

$$\delta_g S = S(X^g, A^g) - S(X, A) = -\oint_{\partial D^*} \text{tr} X g^{-1} dg.$$

However, the normal subgroup $G_0 = \{ g \in G \mid g|_{\partial D^*} = Id \}$ of gauge transformations which are trivial on the boundary leaves the action invariant. The path integral therefore localizes over the space of flat connections $A_{\text{flat}}$ modulo the aforementioned gauge transformations: $M_0 = A_{\text{flat}}/G_0$. Flat connections on $D^*$ are characterized by their holonomy around the boundary.

We restrict ourselves to flat connections $A \in A_{\text{flat}}^0$ whose holonomy around

\footnote{It is customary to think of $A$ as a connection of a trivial principle $G$-bundle over $D^*$.}
the boundary is the identity. The moduli space $\mathcal{M}_0 = A^0_{\text{flat}}/\mathcal{G}_0$ can be identified with the space of based loops $\Omega G = \{g \in C^\infty(S^1, G) \mid g(1) = Id\}$ as follows: let

$$\text{Maps}_0(D^*, G) = \{f : D^* \to G \mid f(1) = Id\}$$

be the space of based maps from $D^*$ to $G$. We define a map

$$\mathcal{M}_0 \to \text{Maps}_0(D^*, G)/\mathcal{G}_0 \cong \Omega G$$

$$A \mapsto (f_A : x \mapsto P_A(1 \leadsto x))$$

where $P_A(1 \leadsto x)$ denotes the parallel transport defined by $A$ from $1 \in D^*$ to $x \in D^*$. This map is well-defined and in particular independent of the chosen path $1 \leadsto x$. Indeed, since the connection is flat, the parallel transport map only depends on the homotopy type of the path. If now $\gamma, \gamma' : 1 \leadsto x$ are any two paths connecting $1$ and $x$, then their difference is a loop (which possibly winds several times around the puncture at the origin), c.f. Figure 1. This loop is always homotopic to a multiple of a loop winding around the boundary. Since the holonomy of the connection around the boundary is trivial, the parallel transport maps along $\gamma$ and $\gamma'$ are the same. The

![Figure 1](image.png)

**Figure 1.** Construction of the map $A^0_{\text{flat}} \to \text{Map}_0(D^*, G)$ via parallel transport along a path. Two paths differ by a parallel transport along a loop winding around the boundary.

inverse map is constructed as follows: given a based loop $g \in \Omega G$, we can extend it to the punctured disk. Indeed, if $r : D^* \to S^1$ is a retraction, then $\tilde{g} = g \circ r : D^* \to G$ is an extension of the based loop $g$ to $D^*$. We then get a flat connection by setting $A = -d\tilde{g}^{-1}\tilde{g}^{-1}$. The path integral thus localizes over $\Omega G$ and any connection is parametrized by an element $g \in \Omega G$ as $A = -dgg^{-1}$. Using the boundary equations of motion $A|_{\partial D^*} = Xdx|_{\partial D^*}$ the boundary action becomes

$$S_0(g) = \frac{1}{2} \int_{S^1} \text{tr}(g'g^{-1})^2 dx$$

which describes a quantum mechanical free particle moving in the group manifold $\text{PSL}(2, \mathbb{R})$. Notice that the boundary action coincides with the Hamiltonian (8).

### 3.2. Edge states and larger gauge groups

Recall that due to the presence of the boundary the action was invariant only under the smaller gauge group $\mathcal{G}_0 \subset \mathcal{G}$ of gauge transformations which are trivial on the boundary. Following [11] and reference therein, one can reinstate the missing gauge degrees of freedom by allowing so-called edge states: let $\Lambda \in C^\infty(\partial D^*, G)$
which transforms under a gauge transformation as $\Lambda^g = g\Lambda$. Then, one can add the following extra term to the action:

\begin{equation}
S(X, A, \Lambda) = S(X, A) + \oint_{\partial D^*} \text{tr} \, X d\Lambda \Lambda^{-1}.
\end{equation}

The new action is indeed invariant under any gauge transformation:

\begin{align*}
\delta_g S(X, A, \Lambda) &= -\oint_{\partial D^*} \text{tr} \, X g^{-1} dg + \oint_{\partial D^*} \text{tr} \, X g^{-1} dg = 0.
\end{align*}

The action (33) admits a global $G$-symmetry which acts on the loops $g$ by conjugation. Allowing the edge states $\Lambda$ to take values only in a certain subgroup $H \subset G$ gauges the global symmetry corresponding to the subgroup $H$, i.e. the gauge group is enhanced to $G_H = \{ g : D^* \to G \mid g|_{\partial D^*} \in H \}$.

Considering the action (34), integration over the scalar field $X$ imposes the equations of motion

\begin{align*}
F_A &= 0 \quad \text{(bulk),}
A|_{\partial D^*} - X dx|_{\partial D^*} + d\Lambda \Lambda^{-1} &= 0 \quad \text{(boundary)}.
\end{align*}

Parametrizing $A = -dgg^{-1}$, the action on the boundary becomes

\begin{align*}
S_\partial(g, \Lambda) &= \frac{1}{2} \oint_{\partial D^*} \text{tr} \, (g'g^{-1} - \Lambda'\Lambda^{-1})^2 \, dx \\
&= S_\partial(g) + S_\partial(\Lambda) - \oint_{\partial D^*} \text{tr} \left( \Lambda'\Lambda^{-1} g'g^{-1} \right),
\end{align*}

where $S_\partial(g)$ is given as in Equation (33).

In particular, if we choose an Iwasawa decomposition $G = NAK$, where $N$ is the group of upper triangular matrices with ones on the diagonal, $A$ diagonal with unit determinant and $K = SO(2)$ a compact subgroup, one can restrict the edge state $\Lambda$ to take values only in $N$:

\begin{equation}
\Lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad \lambda \in C^\infty(S^1, \mathbb{R}).
\end{equation}

The path integral now localizes over the space of flat connections modulo the augmented gauge group $G_N = \{ g : D^* \to G \mid g|_{\partial D^*} \in N \}$. This moduli space can be identified with the space of based loops, modulo the action of based loops in $N$, i.e. the space of based loops in $AK \subset G$. Importantly, one finds $S_\partial(\Lambda) = 0$. The boundary action in the presence of edge states then becomes

\begin{equation}
S_\partial = \int_{\partial D^*} \left( \frac{1}{2} \text{tr} (g'g^{-1})^2 - \lambda' J_-(g) \right) \, dx, \quad g \in \Omega(AK)
\end{equation}

where

\begin{equation}
J_-(g) = \text{tr} \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g'g^{-1} \right).
\end{equation}

After an integration by parts, $\lambda$ takes over the role of a Lagrangian multiplier which imposes the condition $J_-(g) = \text{cst}$. We will see in the next section that the above constraint arises as a first class constraint in the theory without edge states and can be equivalently seen as a Drinfeld-Sokolov reduction of the moduli space $M_0 \cong \Omega G$. Finally, we will show how the constraint leads to Schwarzian quantum mechanics.
4. Quantum Mechanics on PSL(2, \mathbb{R})

4.1. The constrained model. As before, let \( G = \text{PSL}(2, \mathbb{R}) \). The considerations of the preceding section lead us to the study of quantum mechanics of a free particle moving in \( G \) (for compactified time). The action\(^5\) of the model is given by (33)

\[
S(g) = -\frac{k}{4\pi} \text{tr} \oint (g'g^{-1})^2 dx, \quad g \in \Omega G.
\]

Under a small variation of the loop, \( g \to g + \delta g \) the action changes by

\[
\delta S \propto \text{tr} \oint ((g'g^{-1})'(\delta gg^{-1}) dx.
\]

The equations of motion, therefore, imply that the current \( J(g) = g'g^{-1} \) is conserved. To fix notation, let

\[
J_-(g) = \text{tr} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g'g^{-1}.
\]

The action admits a global \( G \) symmetry acting by conjugation on the loops \( g \in \Omega G \). We will be interested in gauging part of this symmetry. Notice that \( LG \) acts on \( \Omega G \) by

\[
(hg)(x) = h(x)g(x)h^{-1}(0), \quad h \in LG, \; g \in \Omega G.
\]

The action satisfies

\[
\delta_h S(g) = S(hg) - S(g) - \frac{k}{4\pi} \oint (h^{-1}h'g'g^{-1}) dx.
\]

In the case that \( h \) takes values in the subgroup \( N \subset G \) of upper triangular matrices, one finds

\[
\delta_h S(g) = \frac{k}{4\pi} \oint u \partial_x J_-(g) dx, \quad h(x) = \begin{pmatrix} 1 & u(x) \\ 0 & 1 \end{pmatrix},
\]

Therefore, imposing the first class constraint \( J_-(g) = 1 \), the action acquires a local \( LN \) symmetry.

\textbf{Remark 3.} Equivalently, according to the previous section, we could study the model in the presence of edge states taking values in \( N \subset G \). Upon integration, the edge states impose the above constraint.

Let us parametrize elements \( g \in G \) by an Iwasawa decomposition of the form

\[
g = NAK = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix},
\]

where \( F \in \mathbb{R}, \; a \in (0, \infty), \; \theta \in [0, 2\pi] \). Elements of the based loop space \( \Omega G \) can now be parametrized in terms of real-valued functions

\[F: \mathbb{R} \to \mathbb{R}, \quad a: \mathbb{R} \to (0, \infty), \quad \theta: \mathbb{R} \to \mathbb{R}\]

satisfying the periodicity conditions

\[F(x + 2\pi) = F(x), \quad a(x + 2\pi) = a(x), \quad \theta(x + 2\pi) = \theta(x) + 2\pi n,\]

\(^5\)We scale the action in order to make contact with the formulas from Section 2.
in addition to
\[ F(0) = 0, \quad a(0) = 0, \quad \theta(0) = 0. \]

Here, \( n \in \mathbb{Z} \) is called the degree (or winding number) of \( \theta \).

In terms of these functions, the action (35) is
\[
S(F,a,\theta) = -\frac{k}{2\pi} \oint \left( \frac{1}{4} \theta'^2 + \left( \frac{a'}{a} \right)^2 + \frac{1}{2} a^2 \theta' F' \right) dx.
\]

It is now straightforward to read off the conjugate momentum for \( F \):
\[
\pi_F = \frac{1}{2} a^2 \theta'.
\]

Fixing \( \pi_F \) to be a constant imposes a first class constraint:
\[
2\pi F = 1 \iff a^2 \theta' = 1.
\]

Notice in particular that \( \pi_F = J_-(g) \).

As we have discussed before, the first class constraint generates a gauge symmetry in \( F \). Indeed, by setting \( a^2 \theta' = 1 \) in (38), \( F \) decouples from the action entirely, since it occurs as a total derivative. We are therefore interested in the following partition function:
\[
Z_0^{\text{red}} = \int_{\Omega G} \frac{d\lambda(g)}{\text{vol}(LN)} \delta(J_-(g) - 1/2) e^{iS(g)/\hbar},
\]
where the measure \( d\lambda(g) \) is taken to be the symplectic measure on \( \Omega G \). The above is a one-dimensional analog of the constrained WZW model studied by Bershadsky and Ooguri in [5].

Substituting \( F = 0 \), which fixes the gauge, and the constraint (40) into Equation (38), the action becomes
\[
S(\theta) = -\frac{k}{2\pi} \oint \left( -\frac{1}{4} \theta'^2 + \frac{1}{4} \left( \frac{\theta''}{\theta'} \right)^2 \right) dx = \frac{k}{4\pi} \oint \left( \frac{1}{2} \theta'^2 + \{\theta, x\} \right) dx,
\]
where we used Equation (19) to write the action explicitly in terms of the Schwarzian derivative.

As we have remarked before, \( \theta \) is a based map from \( \mathbb{R} \) to \( \mathbb{R} \) satisfying \( \theta(x + 2\pi) = \theta(x) + 2\pi n \) for some \( n \in \mathbb{Z} \). The integer \( n \) is called the degree (or winding number) and classifies \( \theta \) topologically. It counts how often the map winds around the target circle. The constraint (40) implies that \( \theta' > 0 \), i.e. \( \theta \) is an increasing map and hence has strictly positive degree. Any such map can be parametrized by some diffeomorphism \( \phi \in \text{Diff}^+(S^1) \) by setting \( \theta = n \phi \). The constrained path integral is therefore taken to be over \( \text{Diff}^+(S^1) \).

The partition function then splits into a sum over topologically distinct sectors which are labeled by the winding number. Each sector is governed by an action
\[
S_n(\phi) = \frac{k}{4\pi} \oint \left( \frac{n^2}{2} \phi'^2 + \{\phi, x\} \right) dx.
\]
This action, however, admits a residual $SL(n)(2,\mathbb{R})$ symmetry which, as was shown in [2], acts on the diffeomorphisms $\phi$ by
\begin{equation}
(a \ b) \ c d \ : \ e^{in\phi} \mapsto \frac{ae^{in\phi} + b}{ce^{in\phi} + d}.
\end{equation}
In particular, one needs to reduce the configuration space $\text{Diff}^+(S^1)$ by this global symmetry. The totally reduced configuration space, in each sector, will therefore be $\text{Diff}^+(S^1)/SL(n)(2,\mathbb{R})$ which is isomorphic to the exceptional Virasoro orbit $O_n$ passing through $b_0 = -\frac{ac^2}{48\pi}$, c.f. (24).

Now, the key observation is that the orbits $O_n$ are symplectic (see the discussion in Section 2.3) with symplectic structure
\begin{equation}
\omega_n = -\frac{c}{48\pi} \int n^2 \delta \phi \land \delta \phi' - \frac{\delta \phi' \land \delta \phi''}{\phi'^2}.
\end{equation}
Moreover, the actions $S_n$ are the Hamiltonians (27) for the $S^1$-action which rotates the loops of $O_n$. A comparison relates the level $k$ with the central charge $c$:
\[ c = 6k. \]

The constrained partition function is therefore the sum over all of the aforementioned topologically distinct sectors:
\begin{equation}
Z^\text{red}_0 = \sum_{n=1}^{\infty} Z_n, \quad Z_n = \int_{O_n} e^{iS_n(\phi)/\hbar} d\lambda_n,
\end{equation}
where $d\lambda_n$ is the symplectic measure of the reduced configuration space $O_n$.

Indeed, notice that the symplectic form on $\Omega G$ reduces to the symplectic form on $O_n$: Substituting the Iwasawa decomposition (37) into the symplectic form (7) yields:
\begin{equation}
\omega_0 = \frac{k}{4\pi} \int 2 \frac{\delta a \land \delta a'}{a^2} - \frac{\delta \theta \land \delta \theta'}{2} + \delta F \land (\ldots),
\end{equation}
which, after imposing the constraint (40) and gauging $F$ to zero, becomes
\begin{equation}
\omega^\text{red}_0 = \frac{k}{8\pi} \int \frac{\delta \theta' \land \delta \theta''}{\theta'^2} - \delta \theta \land \delta \theta'.
\end{equation}
With $\theta = n\phi$ and $c = 6k$ we therefore find that the symplectic form restricted to the (totally) reduced configuration space is given by
\begin{equation}
\omega_n = -\frac{c}{48\pi} \int n^2 \delta \phi \land \delta \phi' - \frac{\delta \phi' \land \delta \phi''}{\phi'^2}
\end{equation}
which coincides with the symplectic form on $O_n$, c.f. (26). In this way, the symplectic measure $d\lambda(g)$ on $\Omega G$ gives rise to a symplectic measure $d\lambda_n$ on the reduced spaces $O_n$.

**Remark 4.** The above reduction of degrees of freedom is well-known in the finite dimensional case. Let $(M, \omega)$ be a symplectic manifold endowed with a Hamiltonian $G$-action for some Lie group $G$ and let $\mu: M \to \mathfrak{g}^*$ be its moment map. Suppose that $0$ is a regular value of the moment map and consider the symplectic reduction $M^\text{red} = M//G = \mu^{-1}(0)/G$. Then
$$
\iota^*\omega = \pi^*\omega^\text{red},
$$
where $\iota: \mu^{-1}(0) \hookrightarrow M$ and $\pi: \mu^{-1}(0) \twoheadrightarrow M^\text{red}$. 

Finally, returning to (46), each of the $Z_n$ are integrals over an infinite-dimensional symplectic manifold endowed with a circle action. In each case, the integrand is the exponential of the Hamiltonian which generates this circle action. In the case of a finite-dimensional symplectic manifold, this would be precisely the setup suited for Duistermaat-Heckman integration. Hence, by analogy, we define the $Z_n$ by the right hand side of the Duistermaat-Heckman integration formula, namely

$$Z_n = \int_{\Omega_n} e^{iS_n/h} d\lambda_n := \sum_p e^{iS_n(p)/h} \prod_j \frac{2\pi}{w_j(p)} w_j(p),$$

with $p$ being the fixed points and $w_j(p)$ the corresponding weights of the $S^1$-action.

4.2. Drinfeld-Sokolov reduction. Before proceeding with the calculation of the partition function by means of an infinite-dimensional analog of Duistermaat-Heckman integration, it is instructive to take a step back and to analyze the geometric meaning of the constraint (40). According to the discussion in Section 2, the space of based loops $\Omega_G$ can be identified with the Kac-Moody coadjoint orbit $LG/G$ passing through $0 \in \hat{Lg}^*$ and is therefore naturally symplectic. We recall that its symplectic form is given by

$$\omega_0 = \frac{k}{4\pi} \text{tr} \oint (g^{-1}\delta g \wedge (g^{-1}\delta g)') \, dx.$$  

The action (35) is then the Hamiltonian for the $S^1$-action which rotates the loop:

$$S^1 \times LG/G \rightarrow LG/G, \quad (t, [g(x)]) \mapsto [g(x + t)].$$

Let now $N \subset G$ be the subgroup of upper triangular matrices. Recall from (9), that the coadjoint action of $LN$ on $LG/G$

$$LN \times LG/G \rightarrow LG/G, \quad (h, [g]) \mapsto [hg]$$

is Hamiltonian. Let $n$ be the Lie algebra of $N$. Then the moment map $\mu: LG/G \rightarrow Ln^*$ is given by projecting $g'g^{-1} \in Lg \cong Ln^*$ onto $Ln^* \cong C^\infty(S^1)$. Consider the preimage of any real positive number $q \in \mathbb{R}_+ \subset Ln^*$: in terms of the Iwasawa decomposition (37), elements of $\mu^{-1}(q)$ satisfy

$$g'g^{-1} = \begin{pmatrix} -\frac{a'}{a} + \frac{1}{2}a^2\theta'F & F' + 2a'F - \frac{1}{2}(a^{-2} + a^2F^2)\theta' \\ \frac{a'}{a} - \frac{1}{2}a^2\theta'F \\ \end{pmatrix} \begin{pmatrix} * & * \\ q & * \end{pmatrix},$$

which gives the condition

$$\frac{1}{2}a^2\theta' = q, \quad q > 0$$

which, in turn, is equivalent to the constraint (40). Notice that the constraint is fixed under the action of $LN$. Thus, the constrained theory, whose space of fields is $\mu^{-1}(q)$, exhibits a gauge symmetry, corresponding to a shift of $F$. The reduced configuration space is $\mu^{-1}(q)/LN$ which can be seen as the symplectic reduction of $LG/G$ with respect to the aforementioned moment map. In this special case, the reduction is known as Drinfeld-Sokolov.
reduction [9]. Notably, any class \([g] ∈ μ^{-1}(q)/LN\) has a unique representative of the form
$$
\begin{pmatrix}
0 & -\frac{1}{q} \left( \theta^2 + \{\theta, x\} \right) \\
q & 0
\end{pmatrix}
$$
which is obtained by using the gauge freedom to set \(F = \frac{1}{q} \frac{q}{a} \) in (52) and using the constraint \(q = \frac{1}{4} a^2 \theta'\) to express \(a\) in terms of \(\theta\). As before, the constraint imposes \(\theta' > 0\) and we can parametrize \(\theta = nφ\) by diffeomorphisms \(φ \in \text{Diff}^+(S^1)\). Therefore, if \(g\) has winding number \(n\), the orbit \([g] ∈ μ^{-1}(q)/LN\) can be mapped to a point \(\frac{n^2}{2} φ'^2 + \{φ, x\}\) in the exceptional Virasoro coadjoint orbit \(O_n\). To see that this establishes a one-to-one correspondence, let us show that \(φ\) transforms correctly under right translations of \(G\): By the Iwasawa decomposition, any \(g \in G\) can be written as
$$
g = \begin{pmatrix}
α & β \\
γ & δ
\end{pmatrix} = \begin{pmatrix} 1 & F \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 \\ \sin \theta/2 & \cos \theta/2 \end{pmatrix}
$$
where one can identify \(\tan(\theta/2) = γ/δ\). If we parametrize \(θ = nφ\) one has that \(\tan(nφ/2)\) transforms by fractional linear transformations
$$
G \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tan(nφ/2) \to \frac{a \tan(nφ/2) + c}{b \tan(nφ/2) + d},
$$
which indeed coincides with the \(\text{PSL}(2, \mathbb{R})\)-action on \(\text{Diff}^+(S^1)\).

In conclusion, one has that the reduced configuration space of the constrained theory is a disjoint union of Virasoro orbits: \(μ^{-1}(q)/LN \cong \bigsqcup_{n > 1} O_n\).

4.3. Calculation of the partition function. We recall that, in order to calculate the partition function of the model, we define the \(Z_n\) in Equation (46) by an analog of the Duistermaat-Heckman integration formula (1):

$$
Z_n = \int_{C_n} e^{iS_n/h} dλ_n := \sum_p e^{iS_n(p)/h} \prod_j \frac{1}{2πh} w_j(p),
$$
where the sum runs over all fixed points \(p\) of the \(S^1\)-action (rotating the loop) on \(O_n\) and \(w_j(p)\) denote the weights (at \(p\)) of the aforementioned circle action. We therefore have to compute two things: the fixed points \(p\) and the weights \(w_j(p)\).

Recall that \(O_n\) is the left quotient of the group \(\text{Diff}^+(S^1)\) by its subgroup \(\text{SL}^{(n)}(2, \mathbb{R})\). The \(S^1\)-action, which rotates the source circle, corresponds to the right multiplication by \(S^1 \subset \text{SL}^{(n)}(2, \mathbb{R})\): \(t \cdot φ(x) = φ(x + t)\). The fixed points on \(O_n\) correspond therefore to (the cosets of) elements \(φ \in \text{Diff}^+(S^1)\), s.t. for every \(t \in S^1\) there exists \(χ ∈ \text{SL}^{(n)}(2, \mathbb{R})\) satisfying \(φ \circ t = χ \circ φ\). Equivalently,
$$
I_φ(t) := φ \circ t \circ φ^{-1} ∈ \text{SL}^{(n)}(2, \mathbb{R}), \quad ∀ t ∈ S^1.
$$
Since \(I_φ\) is a homomorphism, its image \(I_φ(S^1)\) is a torus in \(\text{SL}^{(n)}(2, \mathbb{R})\), and therefore can be brought to the torus \(S^1 \subset \text{SL}^{(n)}(2, \mathbb{R})\) by conjugation, i.e.
$$
g \circ I_φ(S^1) \circ g^{-1} = I_{gφ}(S^1) \subset S^1, \quad \text{for some } g ∈ \text{SL}^{(n)}(2, \mathbb{R}).
$$
Replacing \(φ\) by \(g^{-1} \circ φ\) (they belong to the same equivalence class) we obtain a homomorphism \(I_φ : S^1 → S^1\), which implies that \(I_φ(t) = nt\) for some \(n ∈ \mathbb{Z}\).
Equivalently, \[ \phi(s + t) = \phi(s) + nt, \quad \forall s, t \in S^1. \]
Differentiating with respect to \( s \) we see that \( \phi' \) is a constant function and thus \( \phi \) is necessarily a (constant) rotation. Since rotations belong to the coset of the identity, the only fixed point is the identity \( \phi(x) = x \).

**Remark 5.** The same argument holds for the non-exceptional Virasoro orbits \( O_b \) passing through \((b, c) \in \text{Vir}^\ast\) with \( b \) constant. Hence, the \( S^1 \)-action on \( O_b \) which rotates the source circle has as well only one fixed point corresponding to the identity.

To compute the weights of the \( S^1 \)-action on \( T_{id}O_n \), note that the latter is naturally identified with \( \text{Vect}(S^1)/\langle L_0, L_{\pm n} \rangle \). The \( S^1 \)-action is generated by \( L_0 \), which acts on \( L_{-m} \in \text{Vect}(S^1) \) by \([L_0, L_{-m}] = mL_{-m}\), i.e. with integer weights. Therefore, the denominator in the Duistermaat-Heckman formula (1) is given by the infinite product
\[
\prod_{k=1, k \neq n}^{\infty} \frac{ik}{2\pi \hbar} = \frac{2\pi}{n} \frac{\sqrt{-i\hbar}}{2n}
\]
where the product is understood in the zeta-regularized sense. With
\[ S_n(id) = \frac{cn^2}{24} \]
the regularized partition function (53) in the \( n \)-th topological sector is therefore given by
\[
Z_n = \frac{n}{2\pi} \cdot \left( \frac{i}{\hbar} \right)^{1/2} \exp \left( \frac{icn^2}{24\hbar} \right)
\]
and hence the constrained partition function is
\[
Z_{0\text{red}} = \frac{1}{2\pi} \left( \frac{i}{\hbar} \right)^{1/2} \sum_{n=1}^{\infty} n \exp \left( \frac{icn^2}{24\hbar} \right).
\]

**4.4. Sources of conserved charges and non-exceptional Virasoro orbits.** It turns out that one can treat more general actions than the one for a free particle moving in \( G \) by the same methods. Suppose that we fix a particular edge state \( \Lambda \) taking values in \( K \subset G \): let \( \Lambda = \exp(-xv_0) \) where
\[
v_0 = \frac{v}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad v \in \mathbb{R}^\ast.
\]
Then, the \( BF \) action (28) in the presence of \( \Lambda \) becomes
\[
S(X, A, v) = \int_{D^\ast} \text{tr} XF_A + \oint_{\partial D^\ast} \text{tr} X(A + v_0 dx) - \frac{1}{2} \text{tr} X^2 dx.
\]
The boundary equations of motion for \( X \) are
\[
X dx = A + v_0 dx.
\]
Integrating out $X$, the path integral localizes over the moduli space of flat connections and, parametrizing $A$ by $A = g^{-1}dg$, the boundary action becomes

$$S(A, v) = \frac{1}{2} \oint \text{tr}(g^{-1}g'v_0)^2 dx = -\frac{\pi v^2}{2} + \oint \left( \frac{1}{2} \text{tr}(g^{-1}g')^2 - \frac{v}{2} J_K(g^{-1}) \right) dx$$

where

$$J_K(g) = \text{tr} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) g'g^{-1}$$

is the current corresponding to the compact subgroup $K$. In particular, $v$ plays the role of a source for the charge $Q_K(g) = \oint J_K(g^{-1})$.

Indeed, taking derivatives of the partition function with respect to $v$, one obtains correlation functions of powers of the charge. For example, one has

$$\left. \frac{\partial Z(v)}{\partial v} \right|_{v=0} = \int d\mu(g) e^{iS(g)/\hbar} Q_K(g) = \langle Q_K(g) \rangle.$$

**Remark 6.** By choosing more general $v_0$, one obtains an action in presence of sources for more general conserved charges.

Motivated by the above considerations, let us now fix $v_0$ with $v > 0$ and consider the action

$$S(g) = -\frac{k}{2\pi} \oint (g^{-1}g' + v_0)^2 dx = -\frac{k}{4\pi} \oint (h'h^{-1})^2 dx,$$

where we introduced the quasi-periodic element

$$h = g \exp(\frac{4\pi}{k} v_0 x).$$

For any element $g \in LG$, we have an Iwasawa decomposition for $h$ (c.f. (37)):

$$h = \left( \begin{array}{cc} 1 & F \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} a^{-1} & 0 \\ 0 & a \end{array} \right) \left( \begin{array}{cc} \cos(\tilde{\theta}/2) & -\sin(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) & \cos(\tilde{\theta}/2) \end{array} \right), \quad \tilde{\theta} = \theta + \frac{4\pi v x}{k},$$

where $F$, $a$ and $\theta$ are, as before, real-valued periodic (respectively quasi-periodic) functions on $\mathbb{R}$. Note that if $\theta$ has winding number $n \in \mathbb{Z}$, then $\tilde{\theta}$ satisfies the quasi-periodicity condition

$$\tilde{\theta}(x + 2\pi) = \tilde{\theta}(x) + 2\pi \left( n + \frac{4\pi v}{k} \right).$$

Proceeding as in the previous section, we impose the first class constraint $a^2\tilde{\theta}' = 1$. In particular, $\tilde{\theta}$ is an increasing map, $\tilde{\theta}' > 0$. Equation (59) therefore implies that $n + 4\pi v / k > 0$. Any such map can be parametrized by an orientation preserving diffeomorphism $\phi \in \text{Diff}^+(S^1)$:

$$\tilde{\theta}(x) = \left( n + \frac{4\pi v}{k} \right) \phi(x) = (n_v + \kappa_v) \phi(x)$$

where $n_v$ is defined as the shift of $n$ by the integer part of $\frac{4\pi v}{k}$ and $\kappa_v$ denotes the corresponding fractional part. Since $\phi(x + 2\pi) = \phi(x) + 2\pi$, $\tilde{\theta}$ satisfies indeed the correct periodicity condition (59). In fact, since $\tilde{\theta}' > 0$, only those $\tilde{\theta}$ with $n_v \geq 0$ will contribute to the constrained path integral.
Remark 7. Let us point out that the path integral reduces to an integral over the Kac-Moody orbit $LG/K$, c.f. Example 2.2. As before, the action coincides with the Hamiltonian generating the $S^1$-action on the orbit. Again, the constraint $a^2 \tilde{\theta} = 1$ corresponds to a Drinfeld-Sokolov reduction for the $LN$-action on $LG/K$.

In terms of the fields $(F, a, \tilde{\theta})$, the action reads:

\[ S(g, v_0) = -\frac{k}{2\pi} \oint \left( -\frac{1}{4} \tilde{\theta}^2 + \left( \frac{a'}{a} \right)^2 + \frac{1}{2} a^2 \tilde{\theta} F' \right) dx. \]

After imposing the constraint, and using the resulting gauge symmetry to set $F$ to zero, the partition function splits again into distinct sectors, each governed by an action of the form

\[ S_{n_v}(\phi) = \frac{k}{4\pi} \oint \left( \frac{(n_v + \kappa_v)^2}{2} \phi'^2 + \{\phi, x\} \right) dx. \]

As before, there exists residual symmetry, this time, however, it is only a $S^1$ symmetry which acts by constant shifts of $\phi$: $\phi(x) \to \phi(x) + t$, for $t \in S^1$. Hence, the (totally) reduced configuration space is isomorphic to the more general Virasoro orbit passing through the point $b_0 = -c(n_v + \kappa_v)^2 / 48\pi$, which in turn is isomorphic to $\text{Diff}^+(S^1)/S^1$. The reduced configuration space is therefore again a symplectic space. Setting $c = 6k$, the actions $S_{n_v}$ again coincide with the Hamiltonians generating the $S^1$-action (which rotates the source circle) on the aforementioned Virasoro coadjoint orbits.

By the same arguments as in the previous section, the partition function splits into a sum of integrals over these more general Virasoro orbits which again are defined by the right hand side of the Duistermaat-Heckman integration formula (1):

\[ Z_{v}^{\text{red}} = \sum_{n_v=1}^{\infty} \int_{\text{Diff}^+(S^1)/S^1} e^{i S_{n_v}(\phi)/\hbar} = \sum_{n_v=1}^{\infty} \sum_p e^{i S_{n_v}(p)/\hbar} \prod_j \frac{1}{2\pi} w_j(p), \]

where the sum runs over all fixed points $p$ and the $w_j(p)$ are the weights of the $S^1$-action on the tangent space at $p$. As discussed previously, there is only one fixed point, namely $\phi = id$ at which the action takes the value $S(id) = \frac{c}{24} (n_v + \kappa_v)^2$. By the same argument as before, the denominator is given by the zeta-regularized product

\[ \prod_{k=1}^{\infty} \frac{ik}{2\pi \hbar} = \frac{2\pi}{\sqrt{-i\hbar}}. \]

Therefore, the partition function is

\[ Z_v^{\text{red}} = \sqrt{-i\hbar} \sum_{n+4\pi/k \geq 0} \exp \left( \frac{ic(n + 4\pi v/k)^2}{24\hbar} \right). \]

Remark 8. It is intriguing that the limit of taking $v$ to zero does not give back the constrained partition function $Z_0^{\text{red}}$. On the other hand, in this limit we recognize that $(\text{Diff}^+(S^1)/S^1, \omega_0)$ is only a pre-symplectic space. The kernel of $\omega_0$, in the $n$-th topological sector, is generated precisely by the Virasoro vector fields $L_{\pm n}$. It is the emergence of these zero modes,
which spoils the naive limit of the partition functions. In fact, we can fix the zero modes by considering an additional reduction, namely by taking the quotient with respect to \( \ker \omega_0 \). The resulting spaces are exactly the exceptional Virasoro orbits \( O_n \cong \text{Diff}^+(S^1)/\text{SL}(n)(2,\mathbb{R}) \), whose partition function has been calculated in Equation (46).

4.5. Operator formalism. The partition function of the quantum mechanical model corresponding to the exceptional Virasoro orbits can also be calculated within the operator formalism. The description of the model in first order formalism is given in Equation (28) which, after integrating the scalar field in the bulk gives the boundary action

\[
S = \oint_{S^1} \left( \text{tr} \ X g' g^{-1} - \frac{1}{2} \text{tr} \ X^2 \right) dx, \quad X \in L(\mathfrak{sl}(2,\mathbb{R})), \quad g \in \Omega G.
\]

The Hamiltonian is therefore (up to a constant) the quadratic Casimir operator

\[
\frac{1}{2} \text{tr} \ X^2 \text{ of } \text{PSL}(2,\mathbb{R}).
\]

If one parametrizes \( X \) as

\[
X = \begin{pmatrix} X_0 & X_+ \\ X_- & -X_0 \end{pmatrix},
\]

one finds

\[
H = \frac{1}{2} \text{tr} \ X^2 = X_0^2 + \frac{1}{2}(X_+ X_- + X_- X_+).
\]

In order to canonical quantize the theory, we need to determine the conjugate momenta. Using the Iwasawa decomposition (37) and setting \( a = e^{2\varphi} \) one finds

\[
\text{tr} \ X g' g^{-1} = X_- \left( F' + 2\varphi' F - \frac{1}{2} \varphi' e^{-2\varphi} - \frac{1}{2} \varphi' F^2 e^{2\varphi} \right) + X_0 \left( -2\varphi' + \theta' F e^{2\varphi} \right) + X_+ \left( \frac{1}{2} \varphi' e^{2\varphi} \right).
\]

From the above one can derive an expression of \( X_0, X_\pm \) in terms of the conjugate momenta \( \pi_\varphi, \pi_\theta, \pi_F \):

\[
X_- = \pi_F,
\]

\[
X_0 = -\frac{1}{2} \pi_\varphi + F \pi_F,
\]

\[
X_+ = 2e^{-2\varphi} \pi_\theta + (e^{-4\varphi} - F^2) \pi_F + F \pi_\varphi.
\]

Canonical quantization is done by enforcing the canonical commutation relations: \([x^i, \pi_j] = i\hbar \) where \( x^i \in \{ \varphi, \theta, F \} \) and \( \pi_i \in \{ \pi_\varphi, \pi_\theta, \pi_F \} \). With these commutation relations one finds

\[
[X_0, X_-] = i\hbar X_-, \quad [X_0, X_+] = -i\hbar X_+, \quad [X_-, X_+] = 2i\hbar X_0
\]

which are the commutation relations of \( \mathfrak{sl}(2,\mathbb{R}) \).

**Remark 9.** The canonical commutation relations should not be surprising: the phase space of the free particle moving in \( G \) can be identified with \( T^*G \cong \mathfrak{g}^* \times G \). Now, \( X_0, X_\pm \) are linear functions on \( \mathfrak{g}^* \) and hence their Poisson bracket is given by the Lie bracket on \( \mathfrak{g} \), which up on quantization give the commutation relations (65).
Now, the partition function can be defined as the trace over the Hilbert space $\mathcal{H}$ of the exponential of the Hamiltonian:

$$Z = \text{tr}_\mathcal{H} e^{-\beta H}.$$ 

Of course, one first needs to identify the Hilbert space of the model which in the case at hand coincides with unitary irreducible representations of $\text{PSL}(2, \mathbb{R})$. Unitary irreducible representations of $\text{PSL}(2, \mathbb{R})$ are labeled by the eigenvalue $j(j-1)$ of the quadratic Casimir. The corresponding states are, in addition, labeled by the eigenvalue of one of the generators [18]. The standard choice is to take the generator corresponding to the compact subgroup $K \subset G$. It is convenient define linear combinations of the generators $X_0, X_\pm$:

$$J_0 = \frac{1}{2}(X_--X_+), \quad J_1 = \frac{1}{2}(X_++X_+), \quad J_2 = X_0. \tag{66}$$

Then, $J_0$ corresponds to the generator of the compact subgroup $K \subset G$. Moreover, these generators satisfy the following commutation relations:

$$[J_2, J_0] = i\hbar J_1, \quad [J_2, J_1] = i\hbar J_0, \quad [J_0, J_1] = i\hbar J_2. \tag{67}$$

Let now $|j, m\rangle$ denote a standard basis$^6$ such that

$$\langle j, m | j, n \rangle = \delta_{m,n}, \quad H |j, m\rangle = j(j-1) |j, m\rangle, \quad J_0 |j, m\rangle = m |j, m\rangle, \quad J_+ |j, m\rangle = ((m-j+1)(m+j))^{1/2} |j, m+1\rangle, \quad J_- |j, m\rangle = ((m-j)(m+j-1))^{1/2} |j, m-1\rangle$$

with $J_\pm = J_1 \pm iJ_2$. The unitary irreducible representations of $\text{PSL}(2, \mathbb{R})$ then fall into two classes, c.f. [18]:

- The continuous principal series: $j = \frac{1}{2} + is$ for $0 < s < \infty$. Then $m \in \mathbb{Z}$ or $m \in \mathbb{Z} + 1/2$

- The discrete principal series: $j = \frac{1}{2}, 1, \frac{3}{2} \ldots$ with $m = j, j+1, j+2, \ldots$ positive discrete series $D^+$, $m = -j, -j-1, -j-2, \ldots$ negative discrete series $D^-$. 

For a detailed account of the representation theory of $\text{PSL}(2, \mathbb{R})$ we refer the interested reader to [17, 18]. The first class constraints $\pi_F = \text{cst}$, which we were imposing before, is now equivalent to restricting the representations to those, for which $\pi_F$ acts constantly. Since $\pi_F = X_- = J_0 + J_1$, it is therefore necessary to diagonalize the generator of the subgroup $N \subset \text{PSL}(2, \mathbb{R})$ rather than $J_0$. These representations were studied in [8, 18]. Let $|j, q\rangle$ be the basis that diagonalizes $J_0 + J_1$, i.e.,

$$H|j, q\rangle = j(j-1)|j, q\rangle, \quad (J_0 + J_1)|j, q\rangle = q|j, q\rangle.$$ 

$^6$Our convention for $j$ differs from [18] by a sign.
The states $|j, q\rangle$ can be expanded in the standard basis $|j, m\rangle$, in which they are expressed in terms of Whittaker functions $W_{a,b}(x)$ \[18]:

$$
(j, m | j, q) = \left(\frac{\Gamma(m - j)}{\Gamma(m + j + 1)}\right)^{1/2} \frac{|q|^{-1/2}}{\Gamma(\varepsilon m - j)} W_{\varepsilon m - j + 1/2, \varepsilon m - j + 1/2}(2|q|)
$$

where $\varepsilon = \text{sgn}(q)$.

Now, let us identify the Hilbert space with the positive discrete series $D^+$. Notice that the eigenvalues of the Hamiltonian are $j(j - 1) = \left(j - \frac{1}{2}\right)^2 - \frac{1}{4}$.

If we neglect the vacuum energy shift, i.e. we consider $H = (j - 1/2)^2$, the partition function is

$$
Z = \sum_{j-1/2 \geq 0} \rho(j; D^+) \text{tr}_{q=1}^{(j)} e^{-\beta H}
$$

$$
= \frac{1}{(2\pi)^2} \sum_{j-1/2 \geq 0} \left(j - \frac{1}{2}\right) \text{tr}_{q=1}^{(j)} e^{-\beta(j - 1/2)^2}
$$

$$
= \frac{1}{(2\pi)^2} \sum_{n \geq 1} n e^{-\beta n^2}
$$

(68)

where $\rho(n; D^+) = (2\pi)^{-2} (n - 1/2)$ is the Plancherel measure of the discrete principal series of $\text{PSL}(2, \mathbb{R})$ \[17]. Here, $\text{tr}_{q=1}^{(j)}$ denotes the trace in the subspace of Hilbert space of the representation $j$ which is singled out by a fixed eigenvalue $q = 1$ of $J_0 + J_1$. The partition function (68) matches with the previous answer (55) which was obtained by the path integral method.

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