Strongly regular graphs with parameters \((81, 30, 9, 12)\) and a new partial geometry

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Abstract

Twelve new strongly regular graphs with parameters \((81, 30, 9, 12)\) are found as graphs invariant under certain subgroups of the automorphism groups of the two previously known graphs that arise from 2-weight codes. One of these new graphs is geometric and yields a partial geometry with parameters \(pg(5, 5, 2)\) that is not isomorphic to the partial geometry discovered by J. H. van Lint and A. Schrijver [13] in 1981.

Keywords Strongly regular graph · Partial geometry · Automorphism group

Mathematics Subject Classification 05E30 · 51E14 · 05E18

1 Introduction

A strongly regular graph with parameters \((v, k, \lambda, \mu)\) (or \(srg(v, k, \lambda, \mu)\) for short) is an undirected graph with \(v\) vertices, having no multiple edges or loops, such that every vertex has exactly \(k\) neighbours, every two adjacent vertices have exactly \(\lambda\) common neighbours, and every two nonadjacent vertices have exactly \(\mu\) common neighbours.

A partial geometry with parameters \(s, t, \alpha\), or shortly, \(pg(s, t, \alpha)\), is a pair \((P, L)\) of a set \(P\) of points and a set \(L\) of lines, with an incidence relation between points and lines, satisfying the following axioms:
(1) A pair of distinct points is not incident with more than one line.
(2) Every line is incident with exactly \( s + 1 \) points \((s \geq 1)\).
(3) Every point is incident with exactly \( t + 1 \) lines \((t \geq 1)\).
(4) For every point \( p \) not incident with a line \( l \), there are exactly \( \alpha \) lines \((\alpha \geq 1)\) which are incident with \( p \), and also incident with some point incident with \( l \).

Partial geometries and strongly regular graphs were introduced by R. C. Bose [2]. In the original Bose’s notation, the number \( t + 1 \) of lines incident with a point is denoted by \( r \), and the number \( s + 1 \) of points incident with a line is denoted by \( k \). A survey on strongly regular graphs is given by Brouwer in [5], and for a survey on partial geometries, see Thas [15].

In terms of \( s, t, \alpha \), the number \( v = |P| \) of points and the number \( b = |L| \) of lines of a partial geometry \( pg(s, t, \alpha) \) are given by Eq. (1).

\[
v = \frac{(s + 1)(st + \alpha)}{\alpha}, \quad b = \frac{(t + 1)(st + \alpha)}{\alpha}.
\]  

If \( G = (P, L) \) is a partial geometry \( pg(s, t, \alpha) \), the incidence structure \( G' \) having as points the lines of \( G \), and having as lines the points of \( G \), where a point and a line are incident in \( G' \) if and only if the corresponding line and a point of \( G \) are incident, is a partial geometry \( pg(t, s, \alpha) \), called the dual of \( G \).

If \( G = (P, L) \) is a partial geometry \( pg(s, t, \alpha) \) with point set \( P \) and line set \( L \), the point graph \( \Gamma_p \) of \( G \) is the graph with vertex set \( P \), where two vertices are adjacent if the corresponding points of \( G \) are collinear. The line graph \( \Gamma_L \) of \( G \) is the graph having as vertices the lines of \( G \), where two lines are adjacent if they share a point. Both \( \Gamma_p \) and \( \Gamma_L \) are strongly regular graphs [2]. The parameters \( v, k, \lambda, \mu \) of \( \Gamma_p \) are given by (2).

\[
v = (s + 1)(st + \alpha)/\alpha, \quad k = s(t + 1), \quad \lambda = s - 1 + t(\alpha - 1), \quad \mu = \alpha(t + 1).
\]  

A strongly regular graph \( \Gamma \) whose parameters \( v, k, \lambda, \mu \) can be written as in Eq. (2) for some integers \( s, t, \alpha \) is called pseudo-geometric, and \( \Gamma \) is called geometric if there exists a partial geometry \( G \) with parameters \( s, t, \alpha \) such that \( \Gamma \) is the point graph of \( G \). A pseudo-geometric strongly regular graph \( \Gamma \) with parameters (2) is geometric if and only if there exists a set \( S \) of \( b = (t + 1)(st + \alpha)/\alpha \) cliques of size \( s + 1 \) such that every two cliques from \( S \) share at most one vertex [2].

A partial geometry \( pg(s, t, \alpha) \) with \( \alpha = s + 1 \) is a Steiner 2-\((v, s + 1, 1)\) design, or dually, if \( \alpha = t + 1 \), then the dual geometry is a Steiner 2-\((b, t + 1, 1)\) design. If \( \alpha = s \), or dually, \( \alpha = t \), then \( G \) is a net of order \( s + 1 \) and degree \( t + 1 \) [2]. A partial geometry with \( \alpha = 1 \) is a generalized quadrangle [2,14].

A partial geometry \( pg(s, t, \alpha) \) is called proper if \( 1 < \alpha < \text{min}(s, t) \).

The known proper partial geometries are divided into eight types, four of which are infinite families, and there are four sporadic geometries that do not belong to any known infinite family [[15], Theorem 41.31]. One of the four sporadic examples is a partial geometry \( pg(5, 5, 2) \) discovered by van Lint and Schrijver [13]. The point graph of the van Lint–Schrijver geometry has parameters \( v = 81, \ k = 30, \lambda = 9, \mu = 12, \) and is invariant under the elementary abelian group of order 81 acting regularly on
the set of vertices. By a result of Delsarte [10], any graph with these parameters that is invariant under the elementary abelian group of order 81 acting regularly on the set of vertices can be obtained from a ternary linear \([30, 4, 9]\) two-weight code with weight distribution \(a_9 = 50, a_{12} = 30\) (see also [6], [4], [8]). Up to equivalence, there are exactly two such codes (Hamada and Helleseth [11]) that give rise to two nonisomorphic \(srg(81, 30, 12, 9)\): a graph \(\Gamma_1\), being isomorphic to the van Lint–Schrijver graph and having full automorphism group of order 116640, and a second graph \(\Gamma_2\) having full automorphism group of order 5832. According to [4,6,7], \(\Gamma_1\) and \(\Gamma_2\) appear to be the only previously known strongly regular graphs with parameters \(v = 81, k = 30, \lambda = 9, \mu = 12\).

In this paper, we use a method for finding strongly regular graphs based on orbit matrices that was developed in [1] and [9], to show that there are exactly three nonisomorphic graphs \(srg(81, 9, 12)\) which are invariant under a subgroup of order 360 of the automorphism group of \(\Gamma_1\), and exactly eleven nonisomorphic graphs \(srg(81, 30, 9, 12)\) which are invariant under a subgroup of order 972 of the automorphism group of \(\Gamma_2\). One of the newly found graphs invariant under a group of order 972 gives rise to a new partial geometry \(pg(5, 5, 2)\) that is not isomorphic to the van Lint–Schrijver partial geometry. An isomorphic partial geometry was simultaneously and independently constructed by V. Krčadinac [12] by using a different method. The adjacency matrices of the two previously known graphs and the twelve newly found graphs are available online at http://www.math.uniri.hr/~asvob/SRGs81_pg552.txt

The lines of the new partial geometry \(pg(5, 5, 2)\) are given in Appendix.

2 New strongly regular graphs \(srg(81, 30, 9, 12)\)

The graphs \(\Gamma_1\) and \(\Gamma_2\) described in the introduction have full automorphism groups \(G_1\) and \(G_2\) of order 116640 and 5832, respectively. In this section, we construct twelve new strongly regular graphs with parameters \((81, 30, 9, 12)\). These new graphs are constructed by expanding orbit matrices with respect to the action of certain subgroups of \(G_1\) or \(G_2\). We have checked subgroups of \(G_1\) or \(G_2\) that act on graphs \(\Gamma_1\) and \(\Gamma_2\) in a small number of orbits, and present the results obtained by subgroups that gave us new strongly regular graphs. For more information on orbit matrices of strongly regular graphs, we refer the reader to [1,9]. We used Magma [3] for all computations involving groups and codes in this paper.

2.1 Graphs invariant under subgroups \(A_6\) of \(\text{Aut}(\Gamma_1)\)

There are exactly four conjugacy classes of subgroups of order 360 in the group \(G_1 = \text{Aut}(\Gamma_1)\) of order 116640, the representatives of which will be denoted by \(H_1^1, \ldots, H_4^1\). Each of these four representatives is isomorphic to the simple group \(A_6\).
The subgroup $H_1^1$ is acting in two orbits on the set of vertices of $\Gamma_1$, one of size 36 and the other of size 45, giving an orbit matrix $OM_1^1$ (3).

$$OM_1^1 = \begin{pmatrix} 15 & 15 \\ 12 & 18 \end{pmatrix}$$ (3)

The orbit matrix $OM_1^1$ expands to the (0,1)-adjacency matrices of two nonisomorphic strongly regular graphs: the graph $\Gamma_1$ and a new graph denoted by $\Gamma_{14}$ with full automorphism group of order 360.

The subgroup $H_2^1$ is acting in three orbits of sizes 6, 15 and 60, respectively, giving an orbit matrix $OM_2^1$.

$$OM_2^1 = \begin{pmatrix} 0 & 0 & 30 \\ 0 & 6 & 24 \\ 3 & 6 & 21 \end{pmatrix}$$

The orbit matrix $OM_2^1$ gives rise to two nonisomorphic strongly regular graphs, $\Gamma_1$, and a second graph $\Gamma_{13}$ having full automorphism group of order 720.

The subgroup $H_3^1$ is acting in three orbits with sizes 6, 15 and 60, and orbit matrix $OM_3^1$.

$$OM_3^1 = \begin{pmatrix} 5 & 5 & 20 \\ 2 & 8 & 20 \\ 2 & 5 & 23 \end{pmatrix}$$

The orbit matrix $OM_3^1$ can be expanded to only one (up to isomorphism) strongly regular graph, namely the original graph $\Gamma_1$.

The subgroup $H_4^1$ acts in four orbits, with sizes 1, 20, 30 and 30, and orbit matrix $OM_4^1$.

$$OM_4^1 = \begin{pmatrix} 0 & 0 & 0 & 30 \\ 0 & 9 & 9 & 12 \\ 0 & 6 & 12 & 12 \\ 1 & 8 & 12 & 9 \end{pmatrix}$$

Up to isomorphism, the matrix $OM_4^1$ is the orbit matrix of only one strongly regular graph, that is, $\Gamma_1$.

### 2.2 Graphs invariant under subgroups of order 972

There are exactly five conjugacy classes of subgroups of order 972 in the group $G_2 = \text{Aut}(\Gamma_2)$ of order 5832, with representatives $H_1^2, \ldots, H_5^2$.

The subgroups $H_1^2$ and $H_2^2$ act transitively on the 81 vertices and produce two nonisomorphic strongly regular graphs, isomorphic to $\Gamma_1$ and $\Gamma_2$. 

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The subgroup $H_2$ partitions the set of vertices of $\Gamma_2$ in three orbits of length 27 and gives an orbit matrix $OM_2^2$.

$$OM_2^2 = \begin{pmatrix} 12 & 9 & 9 \\ 9 & 12 & 9 \\ 9 & 9 & 12 \end{pmatrix}$$

The orbit matrix $OM_2^2$ gives rise to two nonisomorphic strongly regular graphs, the graph $\Gamma_2$ and a new graph $\Gamma_4$, having full automorphism group of order 1944.

The subgroups $H_2^4$ and $H_2^5$ are acting in the same way, with two orbits of size 27 and 54, respectively, and gives an orbit matrix $OM_4^2$.

$$OM_4^2 = \begin{pmatrix} 12 & 18 \\ 9 & 1 \\ 2 & 9 \end{pmatrix}$$

The subgroup $H_2^4$ leads to four nonisomorphic strongly regular graphs, including $\Gamma_2$. Two of these graphs, $\Gamma_5$ and $\Gamma_6$, have full automorphism groups of order 972, and $\Gamma_4$ has full automorphism group of order 1944. The group $H_2^5$ gives rise to 12 nonisomorphic strongly regular graphs, among them $\Gamma_1$, $\Gamma_2$ and $\Gamma_4$. Eight of these graphs, denoted by $\Gamma_5, \ldots, \Gamma_{12}$, have full automorphism groups of order 972, and $\Gamma_3$ has full automorphism group of order 3888.

The graph $\Gamma_{12}$ is geometric and produces a new partial geometry $pg(5, 5, 2)$.

### 3 A new partial geometry $pg(5, 5, 2)$

The results presented in Sect. 2 can be summarized as follows.

**Theorem 3.1**

1. Up to isomorphism, there are exactly 3 strongly regular graphs with parameters $(81, 30, 9, 12)$ invariant under a subgroup of order 360 of the automorphism group of the graph $\Gamma_1$.
2. Up to isomorphism, there are exactly twelve strongly regular graphs with parameters $(81, 30, 9, 12)$ invariant under a subgroup of order 972 of the automorphism group of the graph $\Gamma_2$.
3. One of the twelve graphs from part (2) yields a new partial geometry $pg(5, 5, 2)$.

Details about these strongly regular graphs are given in Table 2: the order of the full automorphism group of the graph, the 3-rank of the $(0, 1)$-adjacency matrix, the maximum clique size and the number of 6-cliques. For every graph $\Gamma_i$ that contains at least 81 6-cliques, we define a graph $\Gamma_i^*$ having as vertices the 6-cliques of $\Gamma_i$, where two 6-cliques are adjacent if they share at most one vertex. The last but one column of Table 2 contains the maximum clique size of $\Gamma_i^*$, and if this maximum clique size is 81, the last column contains the total number of 81-cliques in $\Gamma_i^*$.

Only two of the fourteen graphs, $\Gamma_1$ and $\Gamma_{12}$, are geometric. The graph $\Gamma_1^*$ contains two 81-cliques, each consisting of 81 6-cliques of $\Gamma_1$ that are the lines of a partial geometry $pg(5, 5, 2)$ with full automorphism group of order 58320 acting transitively.
Table 1  New partial geometry \( pg^* (5, 5, 2) \)

A generator \( f \) of order 6

\[
(1, 4, 3)(2, 6, 9, 18, 7, 13)(5, 20, 10, 21, 11, 12)(8, 24, 19, 16, 15, 27)
(14, 17, 23, 26, 22, 25)(28, 29, 32)(30, 47, 42, 39, 36, 34)(31, 33, 35, 46, 37, 38)
(40, 41, 54, 49, 50)(43, 48, 51, 53, 44, 52)(55, 71, 72, 65, 59, 57)
(56, 67, 77, 68, 63, 73)(58, 60, 75, 74, 62, 61)(64, 78, 66)(69, 81, 76, 79, 70, 80)
\]

Line orbit representatives

\[
\{ 4, 15, 24, 28, 31, 46 \}, \{ 32, 51, 52, 64, 70, 81 \}, \{ 4, 17, 22, 57, 72, 78 \},
\{ 31, 39, 41, 58, 67, 71 \}, \{ 2, 11, 24, 66, 71, 72 \}, \{ 7, 14, 20, 36, 43, 54 \},
\{ 5, 13, 26, 36, 44, 50 \}, \{ 29, 43, 52, 68, 71 \}, \{ 9, 11, 27, 68, 74, 80 \},
\{ 33, 42, 49, 71, 73, 75 \}, \{ 35, 41, 47, 66, 76, 81 \}, \{ 1, 19, 24, 30, 44, 54 \},
\{ 9, 10, 24, 56, 62, 70 \}, \{ 4, 14, 25, 56, 69, 75 \}, \{ 11, 13, 25, 32, 37, 46 \}
\]

Table 2  Strongly regular graphs with parameters \((81, 30, 9, 12)\)

| Graph \( \Gamma \) | \(| Aut(\Gamma) | \) | 3-rank | Max. clique size of \( \Gamma \) | # 6-cliques | Max. clique size of \( \Gamma^* \) | # 81-cliques |
|----------------|----------------|-------|-------------------|------------|-------------------|-------------|
| \( \Gamma_1 \) | 116640         | 19    | 6                 | 162        | 81               | 2           |
| \( \Gamma_2 \) | 5832           | 19    | 4                 | 0          |                   |             |
| \( \Gamma_3 \) | 3888           | 21    | 6                 | 54         |                   |             |
| \( \Gamma_4 \) | 1944           | 21    | 6                 | 108        | 54               |             |
| \( \Gamma_5 \) | 972            | 21    | 6                 | 54         |                   |             |
| \( \Gamma_6 \) | 972            | 21    | 6                 | 108        | 54               |             |
| \( \Gamma_7 \) | 972            | 21    | 6                 | 54         |                   |             |
| \( \Gamma_8 \) | 972            | 21    | 6                 | 54         |                   |             |
| \( \Gamma_9 \) | 972            | 20    | 6                 | 81         | 54               |             |
| \( \Gamma_{10} \) | 972           | 20    | 6                 | 81         | 54               |             |
| \( \Gamma_{11} \) | 972           | 21    | 6                 | 108        | 54               |             |
| \( \Gamma_{12} \) | 972           | 21    | 6                 | 108        | 81               | 1           |
| \( \Gamma_{13} \) | 720           | 21    | 6                 | 90         | 45               |             |
| \( \Gamma_{14} \) | 360           | 25    | 6                 | 21         |                   |             |

on the sets of points and lines, and is isomorphic to the van Lint–Schrijver partial geometry [13].

The graph \( \Gamma_{12}^* \) contains only one 81-clique and yields a new partial geometry \( pg^* (5, 5, 2) \) that is not isomorphic to the van Lint–Schrijver partial geometry. The full automorphism group of the new partial geometry is of order 972 and partitions the set of points, as well as the set of lines, in two orbits of length 54 and 27. An automorphism \( f \in Aut(\Gamma_{12}) \) of order 6 and a set of orbit representatives of the lines of \( pg^* (5, 5, 2) \) under the action of \( < f > \) are listed in Table 1, where the first three orbits are of length 3, and the remaining 12 orbits are of length 6. The set of all 81 lines of \( pg^* (5, 5, 2) \) is given in Appendix.

The data in Table 2 distinguish as nonisomorphic all but the three graphs \( \Gamma_5, \Gamma_7, \Gamma_8 \), the graphs \( \Gamma_6 \) and \( \Gamma_{11} \), and the two graphs \( \Gamma_9 \) and \( \Gamma_{10} \). Let \( D_1 \) be the design on 81 points.
having as blocks the 6-cliques in $\Gamma_i$. We checked with Magma [3] that $|Aut(D_9)| = 972$, while $|Aut(D_{10})| = 1944$; hence, $\Gamma_9$ and $\Gamma_{10}$ are nonisomorphic. Moreover, $|Aut(D_6)| = 972$, while $|Aut(D_{11})| = 1944$; hence, $\Gamma_6$ and $\Gamma_{11}$ are nonisomorphic. Similarly, $|Aut(D_5)| = 972$, $|Aut(D_7)| = |Aut(D_8)| = 1944$ shows that $\Gamma_5$ is not isomorphic to $\Gamma_7$ or $\Gamma_8$. Finally, $\Gamma_7$ and $\Gamma_8$ can be shown to be nonisomorphic by comparing the weight distributions of the ternary linear codes spanned by their adjacency matrices. The ternary code of $\Gamma_7$ contains 32400 code words of weight 33, while the code of $\Gamma_8$ contains 44550 code words of weight 33.

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**Availability of data and material**  Data generated and analysed during this study are included in this article, and additional data are available online at [http://www.math.uniri.hr/ asvob/SRGs81_pg552.txt](http://www.math.uniri.hr/ asvob/SRGs81_pg552.txt).

**Compliance with ethical standards**

**Conflicts of interest**  The authors declare no conflict of interest.

**Code availability**  Not applicable.

**Appendix**

**Lines of the new partial geometry $pg^*(5, 5, 2)$**
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