A BPS Skyrme model

C Adam¹, J Sanchez-Guillen¹ and A Wereszczynski²
¹ Departamento de Física de Partículas, Universidad de Santiago, and Instituto Galego de Física de Altas
Enerxías (IGFAE) E-15782 Santiago de Compostela, Spain
² Institute of Physics, Jagiellonian University, Reymonta 4, Kraków, Poland
E-mail: adam@fpaxp1.usc.es, joaquin@fpaxp1.usc.es, wereszczynski@th.if.uj.edu.pl

Abstract. Within the set of generalized Skyrme models, we identify a submodel which has both infinitely
many symmetries and a Bogomolny bound which is saturated by infinitely many exact soliton solutions.
Concretely, the submodel consists of the square of the baryon current and a potential term only. Further,
already on the classical level, this BPS Skyrme model reproduces some features of the liquid drop model
of nuclei. Here, we review the properties of the model and we discuss the semiclassical quantization of the
simplest Skyrmion (the nucleon).

1. Introduction
The derivation of the correct low energy hadron dynamics from the underlying fundamental theory
(QCD) is, without doubt, one of the most prominent challenges of strong interaction physics, due to the
non-perturbative nature of the quark and gluon interactions in the low energy limit. In the large \( N_c \) limit it
is known that QCD becomes equivalent to an effective theory of mesons [1], [2], where baryons (hadrons
as well as atomic nuclei) appear as solitonic excitations, where the topological charge is identified with
the baryon number [3]. One of the most popular realizations of this idea is the Skyrme model [4]-[6], i.e.,
a version of a phenomenological chiral Lagrangian where the primary fields are mesons. The original
Lagrangian has the following form (here \( U \) is an SU(2) valued field which is related to the meson fields)

\[
L = L_2 + L_4 + L_0
\]  

(1)

(the subindices refer to the number of derivatives in each term), where the sigma model part is

\[
L_2 = -\frac{f^2}{4} \text{Tr} \left( U^\dagger \partial_\mu U U^\dagger \partial^\mu U \right)
\]  

(2)

and a quartic term, referred to as the Skyrme term, has to be added, to circumvent the standard Derrick
argument against the existence of static solutions

\[
L_4 = -\frac{1}{32e^2} \text{Tr} \left( [U^\dagger \partial_\mu U, U^\dagger \partial_\nu U] \right)^2.
\]  

(3)

The last term is a potential

\[
L_0 = -\mu^2 V(U, U^\dagger),
\]  

(4)

which explicitly breaks the the chiral symmetry. Its particular form is usually adjusted to a concrete
physical situation. Finally, static properties of baryons as well as nuclei are derived with the help of the
semiclassical quantization of the solitonic zero modes [7], [8], [9]. Within this framework, it has been established that topological solitons (Skyrmions) possess qualitative properties of baryons [7], [8]. Quantitatively, for the original Skyrme model nucleon properties are reproduced within a typical precision of about 30%. For higher nuclei, one of the principal problems of the Skyrme model is that the binding energies of the nuclei result too large already at the classical level. This problem may be traced back to the fact that the solitons in the Skyrme model are not BPS, i.e., they do not saturate the Bogomolny bound of the model. Other related problems are too strong internuclear forces at intermediate distances and the crystal-like behaviour of large nuclei in the standard Skyrme model. The situation seems to be better for the excitation spectra of nuclei. Indeed, by identifying the symmetries of higher Skyrmions and by the use of the semiclassical quantization method the (iso-)rotational excitations of several light nuclei have been described rather successfully in [9].

In any case, as an effective field theory the Skyrme model should be viewed as a derivative expansion where higher derivative terms have been neglected. Further, baryons, i.e., extended (solitonic) solutions, have regions where derivatives are not small, so it may not be justified to omit these higher terms. Indeed, many generalized Skyrme models, i.e., extensions of the original Skyrme Lagrangian with higher derivative terms have been investigated [10]-[16]. Among them, the simplest and most natural generalization of the Skyrme model is provided by the addition of the following sextic derivative term

\[ L_6 = \frac{\lambda^2}{24\pi^2} \left( \text{Tr} \left( e^{\mu\nu\rho\sigma} U^\dagger \partial_\mu U U^\dagger \partial_\nu U U^\dagger \partial_\rho U \right) \right)^2. \]  

(5)

Phenomenologically, this term effectively appears if one considers a massive vector field coupled to the chiral field via the baryon density [10], [16]. It turns out that such a modified Skyrme model gives slightly better predictions [10], [11], [13], [15]. Besides, this sextic term is at most quadratic in time derivatives, so it allows for a standard Hamiltonian interpretation. It is, in fact, nothing but the topological (or baryon) current density squared,

\[ L_6 = \lambda^2 \pi^4 \mathbb{B}_\mu \mathbb{B}^\mu \]  

(6)

where

\[ \mathbb{B}_\mu = \frac{1}{24\pi^2} \text{Tr} \left( e^{\mu\nu\rho\sigma} U^\dagger \partial_\nu U U^\dagger \partial_\rho U \right). \]  

(7)

Within the set of generalized Skyrme models, the standard Skyrme model may be viewed as an approximation which is mode tractable and sufficient for the description of certain aspect of nuclei. The BPS Skyrme model is another approximation proposed in [17], [18], which consists of the potential and the sextic term only. In this contribution, we want to discuss its main properties as well as its viability for the description of nuclei. The lagrangian is

\[ L_{06} = \frac{\lambda^2}{24\pi^2} \left( \text{Tr} \left( e^{\mu\nu\rho\sigma} U^\dagger \partial_\mu U U^\dagger \partial_\nu U U^\dagger \partial_\rho U \right) \right)^2 - \mu^2 V(U, U^\dagger), \]  

(8)

where the subindex 06 refers to the fact that only a potential term without derivatives and a term sextic in derivatives appear in this lagrangian. The model is, by construction, more topological in nature than the standard Skyrme model. As in the standard Skyrme model, soliton solutions are stabilized by a higher derivative term. The model is an example of a BPS theory where the topological soliton solutions saturate its Bogomolny bound and, therefore, binding energies are zero at the classical level. Further, this BPS model has an infinite number of symmetries, among which there are the volume preserving diffeomorphisms on base space, which endows the BPS Skyrme model with the symmetries of an incompressible liquid. Moreover, the solitons of this theory are of the compact type which results (classically) in a finite range of their interactions (contact-like interactions). As a consequence, the BPS Skyrme model apparently is a good guess for the description of certain properties of nuclei, as we
discuss below. For a more detailed discussion we refer to [17], [18]. We want to remark that there exists another BPS generalization of the Skyrme model [19], which is based on a dimensional reduction from a (4+1) dimensional Yang-Mills theory, where the $SU(2)$ Skyrme field is accompanied by an infinite tower of vector and tensor mesons.

2. The BPS Skyrme model

2.1. Field equations

We use the standard parametrization of $U$ by means of a real scalar $\xi$ and a three component unit vector $\vec{n}$ field ($\vec{\tau}$ are the Pauli matrices), $U = \exp[i\xi \vec{n} \cdot \vec{\tau}]$. Further, the vector field may be related to a complex scalar $u$ by the stereographic projection

$$\vec{n} = \frac{1}{1 + |u|^2} \left( u + \bar{u}, -i(u - \bar{u}), |u|^2 - 1 \right).$$

For the topological current the resulting expression is

$$B^\mu = \frac{-i}{\pi^2} \frac{\sin^4 \xi}{(1 + |u|^2)^2} \epsilon^{\mu\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma,$$

and, assuming for the potential $V = V(\text{tr}(U + U^\dagger)) = V(\xi)$ (which we assume for the rest of the paper) we get for the lagrangian

$$L_{06} = -\lambda^2 \sin^2 \xi (1 + |u|^2)^2 \partial_\mu (\sin^2 \xi H^\mu) + \mu^2 V_\xi = 0,$$

The Euler–Lagrange equations read

$$\lambda^2 \sin^2 \xi \frac{\partial (\sin^2 \xi H^\mu)}{(1 + |u|^2)^2} + \mu^2 V_\xi = 0,$$

where

$$H^\mu = \frac{\partial (\epsilon^{\alpha\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma)}{\partial \xi^\mu}, \quad K^\mu = \frac{\partial (\epsilon^{\alpha\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma)}{\partial \bar{u}^\mu}.$$

These objects obey the useful formulae

$$H^\mu \bar{u}^\mu = H_\mu \bar{u}^\mu = 0, \quad K^\mu \xi^\mu = K^\mu u^\mu = 0, \quad H^\mu \xi^\mu = K^\mu \bar{u}^\mu = 2(\epsilon^{\alpha\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma)^2.$$

2.2. Symmetries

Apart from the standard Poincare symmetries, the model has an infinite number of target space symmetries. The sextic term alone is the square of the pullback of the volume form on the target space $S^3$, where this target space volume form reads explicitly

$$dV = -i \frac{\sin^2 \xi}{(1 + |u|^2)^2} d\xi dud\bar{u},$$

and the exterior (wedge) product of the differentials is understood. Therefore, the sextic term alone is invariant under all target space diffeomorphisms which leave this volume form invariant (the volume-preserving diffeomorphisms on the target $S^3$). The potential term in general does not respect all these symmetries, but depending on the specific choice, it may respect a certain subgroup. Concretely, for $V = V(\xi)$, the potential is invariant under those volume-preserving target space diffeomorphisms which do not change $\xi$,

$$\xi \rightarrow \xi, \quad u \rightarrow \bar{u}(u, \bar{u}, \xi), \quad (1 + |\bar{u}|^2)^{-2} d\xi d\bar{u} = (1 + |u|^2)^{-2} d\xi d\bar{u}.$$
Since \( u \) spans a two-sphere in target space, these transformations form a one-parameter family of the groups of the area-preserving diffeomorphisms on the corresponding target space \( S^2 \) (one-parameter family because the transformations may still depend on \( \xi \), although they act nontrivially only on \( u, \bar{u} \)). Both the Poincare transformations and this family of area-preserving target space diffeomorphisms are symmetries of the full action, so they are Noether symmetries with the corresponding conserved currents. The energy functional for static fields has the additional infinite-dimensional group of volume-preserving diffeomorphisms on base space as symmetries, as we want to discuss now. These symmetries are not symmetries of the full action, so they are not of the Noether type, but nevertheless they are very interesting from a physical point of view, as we will see below. The energy functional for static fields reads

\[
E = \int d^3x \left( \frac{\lambda^2 \sin^4 \xi}{(1 + |u|^2)^4} (\epsilon^{mnl} i \xi_m u_n \bar{u}_l)^2 + \mu^2 V \right)
\]

and we observe that both \( d^3x \) and \( \epsilon^{mnl} i \xi_m u_n \bar{u}_l \) are invariant under coordinate transformations of the base space coordinates \( x_j \) which leave the volume form \( d^3x \) invariant.

### 2.3. Bogomolny bound

In the BPS Skyrmie model, there exists a Bogomolny bound for static finite energy solutions [20], [17], [18], [21]. The most concise version of the proof has been given by M. Speight [21], so here we use this version of the proof. In a first step, the energy functional has to be rewritten as a complete square plus a remainder,

\[
E = \int d^3x \left( \frac{\lambda^2 \sin^4 \xi}{(1 + |u|^2)^4} (\epsilon^{mnl} i \xi_m u_n \bar{u}_l)^2 + \mu^2 V \right)
\]

\[
= \int d^3x \left( \frac{\lambda \sin^2 \xi}{(1 + |u|^2)^2} \epsilon^{mnl} i \xi_m u_n \bar{u}_l \pm \mu \sqrt{V} \right)^2
\]

\[
\geq \pm (2\lambda \mu \pi^2) \left[ -\frac{i}{\pi^2} \int d^3x \frac{\sin^2 \xi \sqrt{V}}{(1 + |u|^2)^2} \epsilon^{mnl} i \xi_m u_n \bar{u}_l \right] \equiv E_{\text{BPS}},
\]

and in a second step it remains to prove that the remainder \( E_{\text{BPS}} \) is topological and does not depend on the field configuration. For this we note that the integrand in the above expression is not only a three-form on the base space but, in addition, it is the pullback under the map \( U \) of a three-form on target space. Therefore, the base space integral of the base space three-form is equal to the target space integral of the target space three-form times the number of times the target space is covered while the base space is covered once (i.e., times the winding number \(|B|\)),

\[
E_{\text{BPS}} = 2\lambda \mu \pi^2 |B| \frac{i}{\pi^2} \int_{S^3} d\xi d\bar{u} \frac{\sin^2 \xi}{(1 + |u|^2)^2} \sqrt{V}(\xi)
\]

\[
\equiv 2\lambda \mu \pi^2 < \sqrt{V} >_{S^3} |B|
\]

where \(< \sqrt{V} >_{S^3}\) is the average value of \( \sqrt{V} \) on the target space \( S^3 \),

\[
< \sqrt{V} >_{S^3} = \frac{\int_{S^3} d\Omega \sqrt{V}}{\int_{S^3} d\Omega}
\]

and \( d\Omega \) is the volume form on \( S^3 \) (and \( \int_{S^3} d\Omega = \pi^2 \)). The corresponding Bogomolny (first order) equation is

\[
\frac{\lambda \sin^2 \xi}{(1 + |u|^2)^2} \epsilon^{mnl} i \xi_m u_n \bar{u}_l = \mp \mu \sqrt{V}
\]

and is satisfied by all soliton solutions which we will encounter below.
2.4. Exact solutions
We are interested in static topologically nontrivial solutions. Thus \( u \) must cover the whole complex plane (\( \vec{n} \) covers at least once \( S^2 \)) and \( \xi \in [0, \pi] \). The natural axially symmetric ansatz is

\[
\xi = \xi(r), \quad u(\theta, \phi) = g(\theta)e^{in\phi}.
\]

(15)

Then, the field equation for \( u \) reads

\[
\frac{1}{\sin \theta} \partial_\theta \left( \frac{g^2 g_{\theta}}{(1 + g^2)^2 \sin \theta} \right) - \frac{g g_{\theta}^2}{(1 + g^2)^2 \sin^2 \theta} = 0,
\]

and the solution with the right boundary condition is

\[
g(\theta) = \tan \frac{\theta}{2}.
\]

(16)

Observe that this solution holds for all values of \( n \). The equation for the real scalar field reads

\[
\frac{n^2 \lambda^2 \sin^2 \xi}{2r^2} \partial_r \left( \frac{\sin^2 \xi \xi_r}{r^2} \right) - \mu^2 V_\xi = 0
\]

or, after introducing the new variable \( z = (\sqrt{2}\mu r^3)/(3|n|\lambda) \),

\[
\sin^2 \xi \partial_z \left( \sin^2 \xi \xi_z \right) - V_\xi = 0.
\]

(17)

It may be integrated to

\[
\frac{1}{2} \sin^4 \xi \xi_z^2 = V,
\]

(18)

where we chose a vanishing integration constant to get finite energy solutions. This first integration of the field equation is just the dimensionally reduced Bogomolny equation.

For a further evaluation we have to specify a potential. Here we shall restrict ourselves to the case of the standard Skyrme potential

\[
V = \frac{1}{2} \text{Tr}(1 - U) \quad \rightarrow \quad V(\xi) = 1 - \cos \xi.
\]

(19)

Imposing the boundary conditions for topologically non-trivial solutions we get

\[
\xi = \begin{cases} 
2 \arccos \sqrt{\frac{3}{4}} & z \in [0, \frac{4}{3}] \\
0 & z \geq \frac{4}{3} 
\end{cases}
\]

(20)

The solution is of the compacton type, i.e., it has a finite support (see [17], [18] for details; similar compact solutions in the baby Skyrme model have been found in [22], [23], [24]).

We remark that the compact nature of the solutions is a consequence of the specific potential we chose (more concretely, its quadratic behaviour near the vacuum). Other potentials (with a different behaviour near the vacuum) lead to non-compact solutions.

The corresponding energy is

\[
E = 8\sqrt{2}\pi \mu \lambda |n| \int_0^{4/3} \left( 1 - \left( \frac{3z}{4} \right)^{\frac{2}{3}} \right) dz = \frac{64\sqrt{2}\pi}{15} \mu \lambda |n|
\]

(21)
and is linear in the topological charge $|B| = |n|$. The energy density and topological charge density per unit volume are

\[ E = 8\sqrt{2}\mu\lambda(1 - |n|^{-\frac{2}{3}}\bar{r}^2) \quad \text{for} \quad 0 \leq \bar{r} \leq |n|^{\frac{1}{3}} \]
\[ = 0 \quad \text{for} \quad \bar{r} > |n|^{\frac{1}{3}}. \tag{22} \]

\[ B = \text{sign}(n) \frac{4}{\pi^2}(1 - |n|^{-\frac{2}{3}}\bar{r}^2)^\frac{1}{2} \quad \text{for} \quad 0 \leq \bar{r} \leq |n|^{\frac{1}{3}} \]
\[ = 0 \quad \text{for} \quad \bar{r} > |n|^{\frac{1}{3}} \tag{23} \]

where

\[ \bar{r} = \left( \frac{\sqrt{2}\mu}{4\lambda} \right)^{\frac{1}{3}} \frac{r}{R_0} \tag{24} \]

(here $R_0$ is the compacton radius for the soliton with charge one), see Figure 1.

**Figure 1.** Normalized energy density (left figure) and topological charge density (right figure) as a function of the rescaled radius $\bar{r}$, for topological charge $n=1$. For $|n| > 1$, the height of the densities remains the same, whereas their radius grows like $|n|^{\frac{1}{3}}$.

### 3. Some phenomenology of nuclei
For simplicity, we choose the standard Skyrme potential $V = 1 - \cos \xi$ throughout this section.

#### 3.1. Classical aspects
Already the classical solutions of the BPS Skyrme model describe quite well some static properties of nuclei [17], [18]. Concretely, we find

- A linear mass - baryon number relation: $E = 64\sqrt{2}\mu|B|/15 \equiv E_0|B|$, which is a well-known relation for nuclei

- No binding energies due to the BPS property of soliton solutions. Binding energies for physical nuclei are small ($\leq 1\%$), while in the standard Skyrme model they are significantly bigger.

- No forces between solitons: due to the BPS and compact nature of solutions there are no long range forces. Physical nuclei have a very short range interaction.
• Radii of nuclei: the compact solutions have a definite radius \( R = (2\sqrt{2}\lambda|B|/\mu)^{(1/3)} \equiv R_0 \sqrt{|B|} \), which, again, is a well-known relation for physical nuclei.

• Incompressible fluid: the static energy functional has the volume-preserving diffeos (VPDs) as symmetries, which are the symmetries of an incompressible ideal fluid. Physical nuclei do not have this symmetry; they have definite shape, and deformations cost energy. But volume-preserving deformations cost much less energy, as a consequence of the liquid drop model of nuclei.

To summarize, the classical model already reproduces some features of the nuclear liquid drop model (mass and volume proportional to baryon charge, strictly finite size, VPDs). Clearly, this picture is still too naive (there are no pions and, therefore, no long range interactions and no pion cloud; there are no quantum corrections, etc.), and both the quantization and the inclusion of further terms in the lagrangian will be necessary for a more precise description of nuclear properties.

3.2. Quantization

Here we perform the first steps towards a semiclassical (collective coordinates) quantization of solitons, as is done for the standard Skyrme model. Concretely, the standard procedure consists in quantizing the (iso-)rotational degrees of freedom of each soliton. In principle, one can quantize all solitons in this way and obtain information about the binding energies and rotational spectra of nuclei, but here we shall restrict to the simplest \( B = 1 \) case (the nucleon). Following the standard treatment, we introduce the collective coordinates of the isospin by including a time-dependent iso-rotation of the classical soliton configuration

\[
U(x) = A(t)U_0(x)A^\dagger(t),
\]

where \( A(t) = a_0 + ia_1 \tau_i \in SU(2) \) and \( a_0^2 + \vec{a}^2 = 1 \). The \( B = 1 \) hedgehog is invariant under a combined rotation in base and isotopic space, therefore it is enough to introduce the collective coordinates of one of the two. The standard procedure consists in inserting these iso-rotations into the lagrangian, introducing the conjugate momenta to the SU(2) variables \( a_n \) and applying the standard canonical quantization to the latter. This results in an energy expression which is the sum of the static soliton energy \( E_0 \) and an isospin (or equivalently spin) contribution

\[
H = E_0 + \hbar^2 I^2 \frac{2I}{2I},
\]

where \( I^2 \) is the isospin operator (the spherical laplacian on \( S^3 \) [18], [15]. Further, \( I \) is the moment of inertia

\[
I = \frac{4\pi}{3} \int dr \sin^4 \xi \xi_r^2 = \frac{2^8 \sqrt{2\pi}}{15 \cdot 7} \lambda \mu \left( \frac{\Lambda}{\mu} \right)^2.
\]

The soliton with baryon number one is quantized as a fermion. Concretely, the nucleon has spin and isospin \( 1/2 \), whereas the \( \Delta \) resonance has spin and isospin \( 3/2 \), so we find for their masses

\[
M_N = E_0 + \frac{3\hbar^2}{8I}, \quad M_\Delta = E_0 + \frac{15\hbar^2}{8I} \quad \Rightarrow \quad M_\Delta - M_N = \frac{3\hbar^2}{2I},
\]

like in the standard Skyrme model. These expressions may now be fitted to the physical masses of the nucleon \( (M_N = 938.9 \text{ MeV}) \) and the \( \Delta \) resonance \( (M_\Delta = 1232 \text{ MeV}) \), which determines the coupling constants

\[
\lambda \mu = 45.70 \text{ MeV}, \quad \frac{\lambda}{\mu} = 0.2556 \text{fm}^3.
\]

These may now be used to “predict” further physical quantities like, e.g. the charge radii of the nucleons. For the isoscalar (baryon) charge density per unit \( r \) we find

\[
\rho_0 = 4\pi r^2 \mathcal{B}^0 = -\frac{2}{\pi} \sin^2 \xi \xi_r^t
\]

(27)
Table 1. Compacton radius and some charge radii and their ratios for the nucleon. The numbers for the massive Skyrme model are from Ref. [8]. All radii are in fm.

| radius                  | BPS Skyrme | massive Skyrme | exp. |
|-------------------------|------------|----------------|------|
| compacton               | 0.90       | -              | -    |
| electric isoscalar \(r_{e,0}\) | 0.64       | 0.68           | 0.72 |
| electric isovector \(r_{e,1}\) | 0.67       | 1.04           | 0.88 |
| magnetic isoscalar \(r_{m,0}\) | 0.71       | 0.95           | 0.81 |
| \(r_{e,1}/r_{e,0}\)    | 1.05       | 1.53           | 1.22 |
| \(r_{m,0}/r_{e,0}\)    | 1.12       | 1.40           | 1.13 |
| \(r_{e,1}/r_{m,0}\)    | 0.94       | 1.10           | 1.09 |

and for the isovector charge density per unit \(r\)

\[
\rho_1 = \frac{4\pi}{3} A^2 \sin^4 \xi x^2. \tag{28}
\]

The corresponding isoscalar and isovector mean square electric radii are

\[
\langle r^2 \rangle_{e,0} = \int dr r^2 \rho_0 = \left(\frac{\lambda}{\mu}\right)^{2/3}, \tag{29}
\]

\[
\langle r^2 \rangle_{e,1} = \int dr r^2 \rho_1 = \frac{10}{9} \left(\frac{\lambda}{\mu}\right)^{2/3}. \tag{30}
\]

Further, the isoscalar magnetic radius is defined as the ratio

\[
\langle r^2 \rangle_{m,0} = \frac{\int dr r^4 \rho_0}{\int dr r^2 \rho_0} = \frac{5}{4} \left(\frac{\lambda}{\mu}\right)^{2/3}. \tag{31}
\]

The numerical values are displayed in Table 1.

In the BPS Skyrme model, all radii are bound by the compacton radius \(R_0 = \sqrt{2(\lambda/\mu)^{1/3}}\). Further, all radii in the BPS Skyrme model are significantly smaller than their physical values, as well as significantly smaller than the values predicted in the standard massive Skyrme model. This has to be expected, because we know already that the pion cloud is absent in the BPS Skyrme model. Besides, we see that the error caused by the absence of the pion cloud in the model partly cancels in the ratios, as one would expect.

4. Conclusions

In this contribution we reviewed the properties of a certain submodel within the set of generalized Skyrme models, the BPS Skyrme model. The model is quite interesting on its own, because of its infinitely many symmetries and conserved charges, its solvability for any form of the potential, and because all solutions are of the BPS type. They obey a first order differential equation and saturate a Bogomolny bound. Further, we found some evidence that the model may be a good starting point for an effective field theory description of nuclei. For a further development of this application of the model, however, the following problems have to be resolved or further investigated.

- Symmetry breaking: the infinitely many symmetries of the model are not shared by physical nuclei. In addition, it is not clear how to select the correct soliton from the infinitely many ones related by symmetries or how to quantize these infinitely many symmetries. Therefore, a realistic
phenomenological application will require the breaking of these symmetries. The challenge will be to identify a breaking mechanism which effectively breaks the unwanted symmetries without perturbing too much the good properties of the model (like weak binding energies, weak internuclear forces, etc.).

- Quantization of higher nuclei: the semiclassical quantization of higher solitons should be performed and applied to higher nuclei, along the lines of what was done for the standard Skyrme model, e.g., in [9]. Recently in [25], the authors used our BPS Skyrme model with a different potential for this purpose. Concretely, they calculated the exact static soliton solutions plus the (iso-) rotational energies in the rigid rotor quantization for general baryon charge $B = n$ for the axially symmetric ansatz. Then they allowed for small perturbations of the total energies by the quadratic and quartic Skyrme terms and fitted the resulting binding energies to the experimental binding energies of the most abundant isotopes of higher nuclei, assuming, as is usually done, that these correspond to the states with the lowest possible value of the isospin. The resulting agreement between calculated and experimentally determined masses and binding energies is impressive, lending further support to the viability of the BPS Skyrme model as the leading contribution to an effective theory for the properties of nuclear matter. Their specific choice of the potential was motivated by the two requirements to avoid the compact nature of the solitons in order to simplify the analytical calculations, on the one hand, and to have the standard pion tail in the full model with the quadratic Skyrme term included, on the other hand. One can see, however, that these two requirements are mutually incompatible (but this issue is completely irrelevant for the remaining results of that paper). Their potential has, in fact, two vacua and behaves quadratically about one vacuum, but sextic around the other. Further, their solitons approach the sextic vacuum at large distances. This implies that the pure BPS model solutions are not exactly compact but approach the (sextic) vacuum by a steep exponential tail. On the other hand, in the full model there is no pion tail (the pion tail would be induced by the quadratic vacuum which is, however, attained at the center of the soliton, and not in the asymptotic large distance region). In any case, a further investigation of the semiclassical quantization with the inclusion of additional physical effects like higher excitations of nuclei or electrostatic interactions, among others, is the next necessary step.

- Motivation from QCD: it would be very interesting to see whether the rather good phenomenological properties of the model can be justified in a more rigorous manner from the fundamental theory of strong interactions, i.e., QDC. In this context we want to emphasize that the two term of the BPS Skyrme model are rather specific. The sextic term is essentially topological in nature and is, therefore, naturally related to collective excitations of the underlying microscopic degrees of freedom. The potential, on the other hand, provides the chiral symmetry breaking and might therefore be related to collective degrees of freedom of the quarks, like the quark condensate. A related issue is the behaviour for a large number of colors $N_c$. Indeed, the good phenomenological properties of the BPS Skyrme model seem to further improve in the limit of large $N_c$, but a deeper understanding of this fact is still missing. For a more detailed discussion we refer to [18]. In any case, these issues currently are under investigation.

To summarize, we have demonstrated that BPS solutions can be obtained in a Skyrme type theory. We think that we have found and solved an interesting submodel in the space of generalized Skyrme models, which is singled out both by its capacity to reproduce qualitative properties of the liquid drop model of nuclei and by its unique mathematical structure. The model directly relates the nuclear mass to the topological charge, and it naturally provides both a finite size for the nuclei and the liquid drop behaviour, which probably is not easy to get from an effective field theory. Finally, our exact solutions might provide a calibration for the demanding numerical computations in physical applications of more general Skyrme models.
Acknowledgments
C.A. and J.S.-G. thank the Ministry of Science and Investigation, Spain (grant FPA2008-01177), and the Xunta de Galicia (grant INCITE09.296.035PR and Conselleria de Educacion) for financial support. A.W. acknowledges support from the Ministry of Science and Higher Education of Poland grant N N202 126735 (2008–2010).

References
[1] t’Hooft G 1974 Nucl. Phys. B 72 461
[2] Witten E. 1979 Nucl. Phys. B 160 57
[3] Witten E 1983 Nucl. Phys. B 223 433
[4] Skyrme THR 1961 Proc. Roy. Soc. Lon. 260 127
[5] Skyrme THR 1962 Nucl. Phys. 31 556
[6] Skyrme THR 1971 J. Math. Phys. 12 1735
[7] Adkins GS, C. R. Nappi CR and Witten E 1983 Nucl. Phys. B 228 552
[8] Adkins GS and Nappi CR 1984 Nucl. Phys. B233 109
[9] Battye RA, Manton NS, Sutcliffe PM and Wood SW 2009 Phys. Rev. C 80 034323
[10] Adkins GS and Nappi CR 1984 Phys. Lett. B 137 251
[11] Jackson A, Jackson AD, Goldhaber AS, Brown GE and Castillejo LC 1985 Phys. Lett. B 154 101
[12] Ishii N, Aoki S and Hatsuda T 2007 Phys. Rev. Lett. 99 022001
[13] Floratos I and Piette B 2001 Phys. Rev. D 64 045009
[14] Floratos I and Piette B 2001 J. Math. Phys. 42 5580
[15] Ding GL and Yan ML 2007 Phys. Rev. C 75 034004
[16] Sutcliffe P 2009 Phys. Rev. D 79 085014
[17] Adam C, Sanchez-Guillen J and Wereszczynski A 2010 Phys. Lett. B 691 105 (Preprint arXiv:1001.4544)
[18] Adam C, Sanchez-Guillen J and Wereszczynski A 2010 Phys. Rev. D 82 085015 (Preprint arXiv:1007.1567)
[19] Sutcliffe P 2010 J. High Energy Phys. JHEP08(2010)019 (Preprint arXive:1003.0023)
[20] de Innocentis M and Ward RS 2001 Nonlinearity 14 663
[21] Speight JM 2010 J. Phys. A 43 405201 (Preprint arXiv:1006.3754)
[22] Gisiger T and Paranjape MB 1997 Phys. Rev. D 55 7731
[23] Adam C, Klimas P, Sanchez-Guillen J and Wereszczynski A 2009 Phys. Rev. D 80 105013
[24] Adam C, Romanczukiewicz T, Sanchez-Guillen J and Wereszczynski A 2010 Phys. Rev. D 81 085007
[25] Bonenfant E and Marleau L 2010 Phys. Rev. D 82 054023 (Preprint arXiv:1007.1396)