Distribution of orbits of unipotent groups on $S$-arithmetic homogeneous spaces

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Abstract

We will prove an $S$-arithmetic version of a theorem of Dani-Margulis on the convergence of ergodic averages of a given bounded continuous function, when the initial point is outside certain compact subsets of the singular set associated to the unipotent flow.

1 Introduction

The goal of this note is to prove a version of a theorem of Dani and Margulis in an $S$-arithmetic context. In [2], Dani and Margulis proved the following uniform version of Ratner’s theorem:

**Theorem 1.1.** [2] Let $G$ be a connected Lie group, and $\Gamma$ be a lattice in $G$. Let $U = (u_t)$ denote a one-parameter Ad-unipotent subgroup of $G$. Consider the data consisting of a bounded continuous function $\phi : G/\Gamma \to \mathbb{R}$, a compact set $K \subseteq G/\Gamma$, and $\epsilon > 0$. Then there exists a finite number of proper closes subgroups $H_1, \ldots, H_k$ such that $H_i \cap \Gamma$ is a lattice in $H_i$ for all $1 \leq i \leq k$, and compact sets $C_i \subseteq X(H_i, U)$, $1 \leq i \leq k$, such that for every compact set $F \subseteq K - \bigcup_{i=1}^k C_i \Gamma/\Gamma$, for all $x \in F$ and $T \gg 0$, the following holds:

$$\left| \frac{1}{T} \int_{[0,T]} \phi(u_t x) \, dt - \int_{G/\Gamma} \phi \, d\mu \right| < \epsilon.$$  

Here, and in the rest of the article, for a closed subgroups $H$ and $W$ of a Lie group $G$, the set $X(H, W)$ is defined by $X(H, W) = \{ g \in G : g^{-1}Wg \subseteq H \}$. It is clear that if $H \cap \Gamma$ is a lattice in $H$, then for any $g \in X(H, W)$ the orbit $Wg\Gamma$ is included in $gH\Gamma$, which is a closed set carrying a finite $H$-invariant measure. Such points are called singular points. The set of singular points will be denoted by $\mathcal{S}(W)$. The complement in $G/\Gamma$ of the set of singular points is called the set of genetic points and is denoted by $\mathcal{G}(W)$. In [2], it is shown that if $W$ is connected and generated by Ad-unipotent elements, then $\mathcal{S}(W)$ is the union of $X(H, W)\Gamma/\Gamma$, where $H$ runs over all closed connected subgroups of $G$, such that $H \cap \Gamma$ is a lattice in $H$, and $\text{Ad}(H \cap \Gamma)$ is Zariski-dense in $\text{Ad}(H)$.

Perhaps one of the striking features of this theorem is that for the equidistribution up to an error of size $\epsilon$ to be achieved, only a compact subset of a union of finitely many singular sets need to be removed. In other words, all but finitely many singular orbits behave as dense ones, for a given test function $\phi$ and error tolerance $\epsilon$. In [2], Dani and Margulis use this theorem to give asymptotically exact lower bounds for the number of integer vectors in a given ball satisfying $Q(v) \in (a, b)$, where $Q$ varies over a compact family of indefinite quadratic forms. In accordance with what was said before, all but finitely many rational quadratic forms obey the asymptotic behavior for a given tolerance $\epsilon$.

Note that $G$ is an arbitrary (and not necessarily algebraic, or even linear) Lie group, and $\Gamma$ does not have to be arithmetic. The version of the theorem proven in this paper involves $S$-arithmetic groups, which are sufficient for many applications. It turns out that in this setting, a more restricted class
of algebraic subgroups can appear as the orbit closures. This class was introduced in [7] under the name class $\mathcal{F}$ (see definition 2.4). To state the theorem we also need a substitute for the domain of the unipotent flow (or $S$-adic time). The related definitions are given in Section 2.

**Theorem 1.2.** Let $G$ be a $k$-algebraic group, $G = G(k_S)$, and $\Gamma$ an $S$-arithmetic lattice in $G$, and $\mu$ denote the $G$-invariant probability measure on $G/\Gamma$. Let $U = \{(u_v(t_v))_{u_v \in S}| t_v \in k_v\}$ be a one-parameter unipotent $k_T$-subgroup of $G$, and let $\phi: G/\Gamma \rightarrow \mathbb{R}$ be a bounded continuous function. Let $\mathcal{K}$ be a compact subset of $G/\Gamma$, and let $\epsilon > 0$. Then there exist finitely many proper subgroups $P_1, \ldots, P_k$ of class $\mathcal{F}$, and compact subsets $C_i \subseteq X(P_i, U)$, where $P_i = P_i(k_S), 1 \leq i \leq k$, such that the following holds: for any compact subset $\mathcal{F}$ of $\mathcal{K} - \cup_i C_i \Gamma/\Gamma$ there exists $T_0$ such that for all $x \in \mathcal{F}$ and $T > T_0$, we have

$$\left| \frac{1}{\lambda_S(I(T))} \int_{I(T)} \phi(u(t)x) d\lambda_T(t) - \int_{G/\Gamma} \phi d\mu \right| \leq \epsilon.$$ 

The general line of argument is similar to the one in [2]. There are a number of places in which technicalities arise that need to be handled differently.

## 2 Preliminaries

In this section we will introduce some notation and recall a number of theorems from [7] that will be used in this paper.

Let $k$ be a number field, i.e., a finite extension of $\mathbb{Q}$, and let $v$ be a valuation of $k$, and $| \cdot |_v$ denoted the associated norm. A standing assumption is this paper is that $v$ is normalized, i.e. $v(k^*) = \mathbb{Z}$. The completion of $k$ with respect to $v$ is denoted by $k_v$. The set of elements of $x \in k_v$ satisfying $|x|_v \leq 1$ is called the ring of integers of $k_v$. Note that $(k_v, +)$ is an abelian locally compact group. The Haar measure on $(k_v, +)$ normalized such that it assigns 1 to the ring of integers of $k_v$ is denoted by $\lambda_v$.

Let us mention in passing that $k_v$ contains a competition of the $p$-adic field $\mathbb{Q}_p$, where $p = p(v)$ is a prime when $v$ is non-archimedean and $p = \infty$ when $v$ is archimedean. Let $S$ be a finite set of normalized valuations of $k$ containing the set $S_\infty$ of Archimedean ones. We write $S_f = S - S_\infty$, and $k_T = \oplus_{v \in T} k_v$ for any $T \subseteq S$. We will also denote by $\mathcal{O}_S$ (or simply $\mathcal{O}$) the ring of $S$-integers in $k$. Likewise, given a subset $T \subseteq S$, we will fix the Haar measure $\lambda_T = \prod_{v \in T} \lambda_v$ on $K_T$. We also equip $K_T$ with the supremum norm

$$\| (x_v) \| = \sup_{v \in T} |x_v|_v.$$ 

Throughout this paper, we will use bold upper scale letters (such as $G$, $P$, etc.) for algebraic groups defined over $k$. The $k_S$ points of these groups are denoted by the corresponding letter case (such as $G$, $P$, etc.). Having fixed a $k$-algebraic group $G$, and a set $S$ of places as above, we denote by $\Gamma$ an $S$-arithmetic subgroup of $G$. This means that $\Gamma$ and $G(\mathcal{O}_S)$ are commensurable subgroups of $G(k)$. When $G/\Gamma$ is a lattice, we denote by $\mu$ the unique $G$-invariant probability measure on $G/\Gamma$.

The $S$-arithmetic analogue of Theorem 1.1 will naturally involve averaging $\phi(u_x)$, where $t$ ranges over an $S$-interval. Let us give some fixes definitions. For a subset $T \subseteq S$, let $T = (T_v)_{v \in T} \in (\mathbb{R}^+)^T$, and $a = (a_v)_{v \in T} \in k_T$. The $T$-interval centered at $a$ of radius $T$ is the subset of $k_T$ defined by

$$I(a, T) = \{(x_v)_{v \in T} \in k_T : |x_v - a_v|_v \leq T(v), \forall v \in T\}.$$ 

For a fixed $v \in S$ and $r > 0$, we define the $k_v$-interval

$$I_v(r) = \{x \in k_v : |x|_v \leq r\}.$$ 

Set $T = (T_v)_{v \in S}$, where $T_v$ is an integer power of $v$ for $v \in S_f$ and a real number for $v \in S_\infty$. Call $T$ an $S$-time. The magnitude of an $S$-time is defined by

$$|T| = \prod_{v \in S} T_v.$$
Note that $\lambda_T(I(a,T)) = |T|$ for all $a \in k_T$. We will also write $m(T) = \min_{v \in S} T_v$, where $T_v$ is considered as a real number. The set of all $S$-time vectors is denoted by $T_S$. For $T = (T_v)$, $T' = (T'_v) \in T_S$, we write $T \succ T'$ if $T_v \geq T'_v$ for all $v \in S$. We write $T_i = (T_{v,i})_{v \in S} \to \infty$ if $T_{v,i} \to \infty$ for each $v \in S$. For $v \in S_f$, assume that $\varpi \in k_v$ is such that $v(\varpi) = -1$. For an interval $L = I_v(r)$, we will write $\hat{L} = I_v(\varpi r)$, so that $\lambda_v(\hat{L}) \leq |\varpi|_v \lambda_v(L)$.

**Remark 2.1.** Let $v$ be a non-archimedean place. The ultrametric property of the norm implies that if $b \in I_v(a,r)$, we have $I_v(b,r) = I_v(a,r)$. This in particular implies that if $J_1$ and $J_2$ are two intervals with a non-empty intersection, then $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$.

We will need the following easy lemma.

**Lemma 2.2.** Let $L_1, \ldots, L_r \subseteq k_v$ be disjoint intervals. Then

$$\sum_{i=1}^r \lambda_v(L_i) \leq \lambda_v\left(\bigcup_{i=1}^r \hat{L}_i\right).$$

Note that $G(k_v)$ is naturally embedded in $G = \prod_{v \in S} G(k_v)$. By a one-parameter $k_v$-subgroup $U_v = \{u_v(t)\}$ of $G$ we mean a non-trivial $k_v$-rational homomorphism $u_v : k_v \to G(k_v)$. Let $T \subseteq S$ and for each $v \in T$, let $U_v = \{u_v(t_v) : t_v \in k_v\}$ be a one-parameter unipotent $k_v$-subgroup. Then the direct sum $u_T : k_T \to G$ defined by $u_T((t_v)_{v \in T}) = (u_v(t_v))_{v \in T}$ is called a one-parameter unipotent $k_T$-subgroup of $G$. One of the key properties of the unipotent subgroups is the non-divergence properties of the unipotent flow that plays an essential role in the measure classification results for the actions of $U$ with a non-empty intersection, then $J_1 \subseteq J_2$ or $J_2 \subseteq J_1$.

The following $S$-arithmetic version of a quantitative non-divergence theorem will later be needed in this paper:

**Theorem 2.3** ([7], Theorem 3.3). Let $G$ be a $k$-algebraic group, $G = G(k_S)$, and $\Gamma$ an $S$-arithmetic lattice in $G$, and $\mu$ denote the $G$-invariant probability measure on $G/\Gamma$. Let $U = \{u_v(t_v) | t_v \in k_v\}$ be a one-parameter unipotent $k_T$-subgroup of $G$. Let $\epsilon > 0$ and $\mathcal{X} \subseteq G/\Gamma$ be a compact set. Then there exists a compact subset $\mathcal{X}$ such that for any $x \in \mathcal{X}$ and any $T$-interval $I(T)$ in $k_T$, we have

$$\frac{1}{\lambda(I(T))} \lambda_T \{t \in I(T) | u(t)x \not\in \mathcal{X}\} < \epsilon.$$

**Definition 2.4.** A connected $k$-algebraic subgroup $P$ of $G$ is a subgroup of class $F$ relative to $S$ if for each proper normal $k$-algebraic subgroup $Q$ of $P$ there exists $v \in S$ such that $(P/Q)(k_v)$ contains a non-trivial unipotent element.

Let $\Gamma$ be an $S$-arithmetic lattice in $G$. If $P$ is a subgroup of class $F$ in $G$ then for any subgroup $P'$ of finite index in $P(k_S)$, we have $P' \cap \Gamma$ is an $S$-arithmetic lattice in $P'$. The following theorems have been proven in [7].

**Proposition 2.5** ([7], Theorem 4.2). Let $M \subseteq k_v^m$ be Zariski closed. Given a compact set $A \subseteq M$ and $\epsilon > 0$, there exists a compact set $B \subseteq M$ containing $A$ such that the following holds: for a compact neighborhood $\Phi$ of $B$ in $k_v^m$, there exists a neighborhood $\Psi$ of $A$ in $k_v^m$ such that for any one-parameter unipotent subgroup $\{u(t)\}$ in $\text{GL}_m(k_v)$, and any $w \in k_v^m - W_0$, and any interval $I \subseteq k_v$ containing 0, we have

$$\lambda_v\{t \in I : u(t)w \in \Psi\} \leq \epsilon \cdot \lambda_v\{t \in I : u(t)w \in \Phi\}.$$
\[ u^{(i)}(t) \rightarrow u(t) \text{ for any } t \text{ as } i \rightarrow \infty. \] Let \( x_i \rightarrow G/\Gamma \) converge to the point \( x \in G/\Gamma \), and let \( T_i \rightarrow \infty \).

For any bounded continuous function \( \phi : G/\Gamma \rightarrow \mathbb{R} \), we have
\[
\frac{1}{|T_i|} \int_{I(T_i)} \phi(u^{(i)}_t x_i) \, d\lambda_S(t) \rightarrow \int_{G/\Gamma} \phi \, d\mu.
\]

Let us briefly sketch the proof of this theorem. The main ingredient of the proof is the quantitative non-divergence theorem, whose \( S \)-adic analogue, Theorem 2.3, is proven in [7]. Arguing by contradiction, one assumes that there exists a sequence \( x_i \) of points for which the statement is not true. Using the density of the set of generic points, one can easily show that \( x_i \) could be assumed to be generic for \( u_t \). Also using the quantitative non-divergence, one can prove that there is no escape of mass to infinite, and then one easily shows that the limiting measure is invariant under the action of \( u_t \). The measure classification of Ratner will then finish the proof. For details, we refer the reader to [2].

### 3 \( S \)-adic linearization

Let \( P \) be a subgroup of class \( F \) in \( G \). Using Chevalley’s theorem, there exists a \( k \)-rational representation \( \rho : G \rightarrow GL(V_P) \) such that \( N_G(P) \) equals the stabilizer of a line in \( V \) spanned by a vector \( m \in V(k) \). This representation and the vector \( m \) is fixed throughout this paper. Let \( \chi \) be the \( k \)-rational character of \( N_G(P) \) defined by \( \chi(g)m = g.m \), for \( g \in N_G(P) \). We denote \( N = \{g \in G : gm = m\} \) and \( N = N(k_S) \). We also set \( \Gamma_N = \Gamma \cap N \) and \( \Gamma_P = \Gamma \cap N_G(P) \). The orbit map \( \eta : G \rightarrow Gm \subseteq V_G \) is defined by \( \eta(g) = gm \). \( Gm \) is isomorphic to the quasi-affine variety \( G/N \) and \( \eta \) is a quotient map.

Set \( X = \{g \in G : U g \subseteq gP\} \) and let \( A_P \) denote the Zariski closure of \( \eta(X(P,U)) \). Clearly \( X \) is an algebraic variety of \( G \) defined over \( k_S \) and \( X(k_S) = X(P,U) \). It is not hard to show (see [7]) that
\[
\eta^{-1}(A_P) = X(P,U).
\]

It will be useful to consider the map \( \mathcal{R} : G/\Gamma \rightarrow V_P \) defined as follows. For each \( x \in G/\Gamma \), we define
\[
\mathcal{R}(x) = \{\eta_P(g) : g \in G, x = g\Gamma\}.
\]
For \( D \subseteq A_P \) and for \( \gamma \in \Gamma \), we define the \( \gamma \)-overlaps of \( D \) by
\[
\mathcal{O}^\gamma(D) = \{g\Gamma : \eta_P(g) \in D, \eta_P(g\gamma) \in D\} \subseteq G/\Gamma.
\]
Finally, we set
\[
\mathcal{O}(D) = \bigcup_{\gamma \in \Gamma - \Gamma_P} \mathcal{O}^\gamma(D) \subseteq G/\Gamma.
\]

Throughout this paper, we will use a number of properties of the overlaps. These are formulated in the following lemma, whose proof is straightforward:

**Lemma 3.1.** For \( \gamma \in \Gamma \) and \( \gamma_1 \in \Gamma_P \), and \( D \subseteq A_P \) we have

1. \( \mathcal{O}^\rho(D) = \{x \in G/\Gamma : \mathcal{R}(x) \cap D \neq \emptyset\} \).
2. \( \mathcal{O}^\gamma(D) = \mathcal{O}^{\gamma \gamma_1}(D) \).

In this section, we will use the same notation as above. For each subgroup \( P \) of class \( F \) relative to \( S \), we will denote \( I_P = \{g \in G : \rho_P m_P = m_P\} \). The proof of the following proposition is exactly the same as the proof of Proposition 7.1 in [2].

**Proposition 3.2.** Suppose \( P \) is a subgroup of class \( F \) relative to \( S \), and \( C \subseteq V_P \) be compact. Assume also that \( \mathcal{K} \subseteq G/\Gamma \) is compact. Then there exists a compact set \( \tilde{C} \subseteq G \) such that
\[
\pi(\tilde{C}) = \{x \in \mathcal{K} : \mathcal{R}(x) \cap C \neq \emptyset\}.
\]
Proposition 3.3. Let $P$ be a subgroup of class $F$ relative to $S$ and $D \subseteq A_P$ be compact. Let $\mathcal{K} \subseteq G/\Gamma$ be compact. Then the family $$\{\mathcal{K} \cap \mathcal{O}^\gamma(D)\}_{\gamma \in \Gamma}$$ contains only finitely many distinct elements. Moreover, for each $\gamma \in \Gamma$, there exists a compact set $\bar{C}_\gamma \subseteq \eta^{-1}_P(D) \cap \eta^{-1}_P(D)\gamma$ such that $$\mathcal{K} \cap \mathcal{O}^\gamma(D) = \pi(\bar{C}_k).$$

Proof. The argument for finiteness from Proposition 7.2. in [2] can be carried over verbatim to this case.

Let us denote by $\mathscr{E}$ the class of subsets of $G$ of the form $$E = \bigcap_{i=1}^r \eta^{-1}_P(D_i)$$ where $P_i$ are subgroup of class $F$ and $D_i \subseteq A_P$ are compact. For such a set $E$ (together with the given decomposition), we denote $\mathcal{N}(E)$ to be the family of all neighborhoods of the form $$\Phi = \bigcap_{i=1}^r \eta^{-1}_P(\Theta_i)$$ where $\Theta_i \supset D_i$ are neighborhoods in $V_{P_i}$. We will refer to these neighborhoods as components of $\Phi$.

We will now prove a theorem which is a stronger version of the theorem in Tomanov.

Theorem 3.4. Let $\mathcal{K} \subseteq G/\Gamma$ be compact and $\epsilon > 0$. Given $E \in \mathscr{E}$, there exists $E' \in \mathscr{E}$ such that the following holds: given $\Phi \in \mathcal{N}(E')$, there exists a neighborhood $\Omega \supset \pi(E)$ such that for any one-parameter unipotent subgroup $\{u_t\}$ of $G$, and any $g \in G$, and $r_0 > 0$, one of the following holds:

1. A component of $\Phi$ contains $\{u(t)g\gamma : t \in I_\epsilon(r)\}$ for some $\gamma \in \Gamma$.
2. For all $r > r_0$, we have $$\frac{1}{\lambda(T)} \lambda(\{t \in I_\epsilon(r) \setminus I_\epsilon(r_0) : u(t)g\gamma \in \Omega \cap \mathcal{K}\} \leq \epsilon.$$

Proof. It is clear that we can assume that $E = \eta^{-1}_P(C)$ and that $E$ is $S(v)$-small. We will now proceed by the induction on $\dim P$. The result is clearly valid for $\dim P = 0$. Let us assume that it is known for all $P$ with dimension at most $n - 1$ and that $C \subseteq A_P$, with $\dim P = n$. Applying Proposition 2.5 to the set $C$ (as a compact subset of the Zariski closed set of $A_P$), we obtain a compact subset $D$ of $A_P$ such that for a compact neighborhood $\Phi$ of $D$ in $A_P$, there exists a neighborhood $\Psi$ of $C$ in $A_P$ such that for any one-parameter subgroup $\{u_t\}$ of $GL(V_P)$ and any $w \in V_P - \Phi$, and any interval $I \subseteq k_v$ containing $0$, we have $$\lambda_v\{t \in I : u_tw \in \Psi\} \leq \epsilon \cdot \lambda_v\{t \in T : u_tw \in \Phi\}.$$

Note that since the set of the roots of unity in $K$ is finite, we can choose $D$ such that $\omega D = D$ for every root of unity $\omega \in K$. Note that $D$ can be chosen to be $S(v)$-small. Now, let $B = \eta^{-1}_P(D)$.

By Proposition 3.3 the family of sets $\{\mathcal{K} \cap \mathcal{O}^\gamma(D)\}_{\gamma \in \Gamma}$ is finite, hence consists of the sets $\mathcal{K} \cap \mathcal{O}^\gamma(D)$ for $1 \leq j \leq k$. We assume that $\gamma_1 = e$. Moreover, we can write $\mathcal{K} \cap \mathcal{O}^\gamma(D) = \pi(C_j)$ for some compact subset $C_j \subseteq B \cap B_j^{-1} \subseteq X(P \cap \gamma_j P \gamma_j^{-1}, W)$. We claim that $\gamma_j \notin \Gamma_P$ for $j \geq 2$. Assuming the contrary, we obtain $\rho(\gamma_j)P = \chi(\gamma_j)P$. Since $\chi(\gamma_j) \in \mathcal{O}^*$, we have $\eta(b\gamma_j) = \chi(\gamma_j)\eta(b) \in D$. Since $D$ is $S(v)$-small, we obtain that $\chi(\gamma_j)$ is a root of unity in $k_v^\times$. This shows that $B\gamma_j \subseteq \eta^{-1}(D) = B$. 

5
which is a contradiction to the choice of $\gamma_j$. This shows that for $j \geq 2$, $P \cap \gamma_j P \gamma_j^{-1}$ is a proper subgroup of $P$. Hence there exists a subgroup $P_j$ of class $F$ which is contained in the connected component of $P \cap \gamma_j P \gamma_j^{-1}$. Note that $P_j$ is of dimension less than $n$, and $C_j \subseteq X(P_j, W)$. We now set $E_j = \eta P_j^{-1}(\eta P_j(C_j))$ and apply the induction hypothesis to obtain $E'_j \in \mathcal{E}$ such that for any choice of $\Phi_j \in \mathcal{N}(E'_j)$, we can find neighborhoods $\Omega_j$ of $E_j$ such that for any one-parameter subgroup $(u(t))_{t \in k_v}$ of $G$, $g \in G$ and $r > 0$, we have

$$
\lambda_v \{ t \in I_v(r) : u(t) g \Gamma \subseteq \Omega_j \} \leq (\epsilon/2) r
$$

unless there exists $\gamma \in \Gamma$ such that $\{ u(t) g \gamma : t \in I_v(r) \}$ is contained in a component of $\Phi_j$. Set $E''_j = \bigcup_{j=2}^n E'_j \in \mathcal{E}$, and $E'' = E''_j \cap B$. Consider $\Phi \in \mathcal{N}(E'')$. This shows that there exists a neighborhood $\Omega'$ of $\pi(E'')$ such that for any one-parameter unipotent subgroup $\{ u(t) \}_{t \in k_v}$, and every $g \in G$ and $r_0 > 0$, we have

$$
\lambda_v \{ t \in I_v(r_0) : t(t) g \Gamma \subseteq \Omega' \} \leq \frac{\epsilon r}{2}
$$

unless $\{ u(t) g \gamma : t \in I_v(r_0) \}$ is in a component of $\Phi$ for some $\gamma \in \Gamma$.

Set $\mathcal{H}_1 = \mathcal{H} - \Omega'$, and choose a compact subset $K' \subseteq G$ such that $\pi(K') = \mathcal{H}_1$. Let $\Phi_1$ be a neighborhood of $D$ in $V$ such that $\eta P_j^{-1}(\Phi_1) \subseteq \Phi$ and $O(\Phi_1) \cap \mathcal{H}_1 = \emptyset$. Since $D$ is $S(v)$-small, we can clearly choose $\Phi_1$ to be $S(v)$-small. Note that since $\rho(u(t))$ is a one-parameter unipotent subgroup of $GL(V)$, and $\eta P(C)$ is of relative size less than $\epsilon/4$ in $D$, we can find a neighborhood $\Psi$ of $C$ in $V$ satisfying

$$
\lambda_v \{ t \in I_v(r) : \rho(u(t)) v \subseteq \Psi \} \leq \frac{\epsilon}{4} \lambda_v \{ t \in I_v(r) : \rho(u(t)) v \subseteq \Phi_1 \},
$$

for all $v \in V - \Phi_1$, $r > 0$ and unipotent subgroups $\{ u(t) \}_{t \in k_v}$. Let $\Omega = \pi(\eta P(\Psi)) \subseteq G/\Gamma$. Assuming that (1) does not hold for $g \in G$, a one-parameter subgroup $\{ u(t) \}_{t \in k_v}$, and $r_0 > 0$. This implies that for every $\gamma \in \Gamma$, there exists $t \in I_v(r_0)$ such that $u(t) g \gamma \in G - \Phi$. For $q \in M$, we consider the following sets:

$$
J_1(q) = \{ t \in I_v(r) - I_v(r_0) : \rho(u(t)) g q \in \Phi_1 \},
$$

$$
J_2(q) = \{ t \in I_v(r) - I_v(r_0) : \rho(u(t)) g q \in \Psi, \pi(u(t)) g \in \mathcal{H}_1 \}.
$$

Note that $J_1(q)$ is an open subset of $k_v$ and is hence a disjoint union of intervals. We will also define $J_3(q) = \emptyset$ if $q_1, q_2 \in M$, unless $q_2 = \omega q_1$ for some root of unity $\omega \in K^*$. In the archimedean case, if $t \in J_3(q_1) \cap J_3(q_2)$, then there exists $a \geq 0$ such that $[t,t+a] \subseteq J_1(q_1) \cap J_1(q_2)$ and $\pi(u(t+a) g) \in \mathcal{H}_1$. If $q_j = \eta(\gamma_j)$ for $j = 1,2$, then $\eta(u(t+a) g \gamma_1) = \eta(u(t+a) g \gamma_2) \in \Phi_1$, we have $q_1,q_2 \in O(\Phi) \cap \mathcal{H}_1 = \emptyset$, unless $\gamma_1^{-1} \gamma_2 \in \Gamma_P$, which implies that $q_2 = \omega q_1$.

In the non-archimedean case, if $t \in J_3(q_1) \cap J_3(q_2)$, then $t \in J_1(q_1) \cap J_1(q_2)$ and there exist intervals $J(q_1), J(q_2) \subseteq k_v$ containing $t$ and $t' \in J(q_1)$ such that $\pi(u(t_1) g), \pi(u(t_2) g) \in \mathcal{H}_1$. Note that since $J(q_1)$ and $J(q_2)$ intersect one contains the other, hence, without loss of generality, we can assume that $t'_1 \in J(q_1) \cap J(q_2)$, and $\pi(u(t'_1) g) \in \mathcal{H}_1$. The rest of the argument is as in the archimedean case.

Let $L$ be the family of those components $L = I_v(a, r_1)$ of $J_1(q)$ such that $L \cap I_v(r_0) = \emptyset$, and $L_2$ the rest of components. Note that $L \not\subseteq I_v(r_0)$. This implies that

$$
\lambda_v (L \cap J_2(q)) \leq \lambda_v (L \cap J_3(q)).
$$
From here and using the above claim we have
\[
\sum_{L \in \mathcal{L}_1} \lambda_v(L \cap J_2(q)) \leq \sum_{L \in \mathcal{L}_1} \lambda_v(L \cap J_3(q)) \leq \lambda_v(I_v(r) - I_v(r_0)).
\]

We now claim that
\[
\sum_{L \in \mathcal{L}_2} \lambda_v(L) \leq \lambda_v(I_v(r)).
\]

In fact, if \( L \in \mathcal{L}_2 \), then either \( L \subseteq I_v(r_0) \) or \( I_v(r_0) \subseteq L \). If \( I_v(r_0) \subseteq L \) for some \( L \in \mathcal{L}_2 \), then since components are disjoint, \( \mathcal{L}_2 \) has precisely one element and the result follows. So, assume that for each \( L \in \mathcal{L}_2 \), we have \( L \subseteq I_v(r_0) \). Then the disjointness of components imply that
\[
\sum_{L \in \mathcal{L}_2} \lambda_v(L) \leq \lambda_v(I_v(r_0)).
\]

Proposition 3.5. Let \( G \) be a \( k \)-algebraic group, \( G = G(k_S) \), and \( \Gamma \) an \( S \)-arithmetic lattice in \( G \). Let \( U \) be a one-parameter unipotent subgroup of \( G \). Assume that \( P_1, \ldots, P_k \) are subgroups of class \( F \) for \( 1 \leq i \leq k \), and let \( D_i \) be a compact subset of \( A_{P_i} \), and \( \Theta_i \) be a compact neighborhood of \( D_i \) in \( V_{P_i} \). For a given compact set \( \mathcal{X} \subseteq G/\Gamma \), there exists \( P'_1, \ldots, P'_k \) of class \( F \) and compact subsets \( D'_i \) be a compact subset of \( A_{P'_i} \), \( 1 \leq i \leq k \), such that for any compact set \( \mathcal{F} \subseteq \mathcal{X} = \bigcup_{i=1}^k (\eta^{-1}_{P_i}(D_i) \cup \eta^{-1}_{P'_i}(D'_i)) \Gamma / \Gamma \), there exists \( T_0 \) such that for any \( g \in G \) with \( g \Gamma \in \mathcal{F} \), and \( 1 \leq i \leq k \), there exists \( t \in B(T_0) \) such that \( u_g \eta_{P'_i}(\Theta_i) \).

Proof. The proof of this proposition is very similar to the proof of Proposition 8.1. in [2]. Let us denote by \( I_P \) consists of \( g \in G \) with \( g \cdot mp = mp \). We first claim that there exists a subgroup \( P' \) of class \( F \) such that \( X(I_P, U) \subseteq X(P', U) \). In fact, let \( P' \) be the smallest connected algebraic subgroup of \( G \) which contains all the unipotent elements of \( I_P \). Note that since \( P' \) is generated by unipotent subgroups, we have \( X_k(G) = \{1\} \), where \( X_k(G) \) denotes the group of characters of \( G \) defined over \( k \). It follows from Theorem 12.3. of [4] that \( G' \cap \Gamma \) is a lattice in \( G' \). We can now show that \( G' \) is of class \( F \). Let \( P'_1, \ldots, P'_k \) be chosen as above such that \( X(I_P, U) \subseteq X(P', U) \) for all \( 1 \leq i \leq k \). We will also define
\[
Q_i = \{ w \in \Theta_i : \rho_{P_i}(u_i)w = w, \forall t \}.
\]
Using Proposition 3.2, we can find compact subsets \( C_i \subseteq \mathcal{F}^{-1}(Q_i) \), \( 1 \leq i \leq k \) such that
\[
\pi(C_i) = \{ x \in Q_i : \Re(x) \cap C_i \neq \emptyset \}.
\]
This implies that \( C_i \subseteq X(P'_i, U) \). Consider the compact subsets \( D'_i = \eta_{P'_i}(C_i) \subseteq A_{P'_i} \) and assume that \( \mathcal{F} \) is a compact subset of \( \mathcal{X} = \pi(\bigcup_{i=1}^k (\eta^{-1}_{P_i}(D_i) \cup \eta^{-1}_{P'_i}(D'_i)) \Gamma / \Gamma \) is given. Find a compact subset \( F' \subseteq G \) such that \( \mathcal{F} = \pi(F') \). From the fact that \( \rho_{P_i}(\Gamma)mp = mp \) is a discrete subset of \( V_{P_i} \), it follows that there are only finitely many \( \gamma \in \Gamma \) such that \( \eta_{P_i}(\gamma) \in \rho_{P_i}(F')^{-1} \Theta_i \) for some \( 1 \leq i \leq k \). It thus suffices to show that for each \( 1 \leq i \leq k \) and \( \gamma \in \Gamma \), for all large enough \( T \) the set \( \Theta_i \cap \rho_{P_i}(u_i)(\Theta_i \cap \rho_{P_i}(F' \gamma)mp) = \emptyset \).

Note that \( \rho_{P_i}(F' \gamma)mp \cap \Theta_i \) is a compact subset of \( V_{P_i} \) which does not contain any fixed point of the flow \( \rho_{P_i}(u_i) \). Since \( \rho_{P_i}(u_i) \) is a unipotent one-parameter subgroup of \( GL(V_{P_i}) \) the result follows.

Proof of Theorem 1.2. For a bounded continuous function \( \phi \) defined on \( G/\Gamma \), the one-parameter unipotent group \( (u_t) \) and time box \( T \), and \( x \in G/\Gamma \), we define
\[
\Delta(\phi, u_t, x, T) = \frac{1}{\lambda_T(I(T))} \int_{I(T)} \phi(u_t x) d\lambda_T(t) - \int_{G/\Gamma} \phi d\mu(T).
\]
Let us consider the above statement with $S$ replaced by $T \subseteq S$ everywhere. We will prove the statement first for the case $|T| = 1$. Then we will show that if the statement holds for $T_1$ and $T_2$, then it must also hold for $T_1 \cup T_2$. Let us start with the case $T = \{v\}$ for some $v \in S$. We argue by contradiction. Assume that the statement of the theorem is not true. This implies the existence of a bounded continuous function $\phi : G/\Gamma \to \mathbb{R}$, a compact subset $\mathcal{K}_1 \subseteq G/\Gamma$, and $\epsilon > 0$ such that for any proper subgroups $P_1, \ldots, P_k$ of class $\mathcal{F}$, and any compact subsets $C_i \subseteq X(P_i, U)$, where $P_i = P_i(k)$, $1 \leq i \leq k$, there exists a compact set $\mathcal{F}$ of $\mathcal{K}_1 \cup C_i \Gamma/\Gamma$ such that for all $T_0 > 0$, there exists $T$ with $m(T) > T_0$, and $x \in F$ such that

$$\Delta(\phi, u_t, x, T) > \epsilon.$$  

Without loss of generality, we can assume that $\phi$ has a compact support, and $\|\phi\| \leq 1$. There exists a compact subset $\mathcal{K} \subseteq G/\Gamma$ such that for all $x \in \mathcal{K}_1$ and $T$, we have

$$\lambda_T\{t \in I(T) : u_t x \not\in \mathcal{K}\} < \frac{1}{3}\lambda_T(I(T)).$$

We can now apply to construct an increasing sequence $E_i \subseteq E_{i+1}$ in $\mathcal{F}$ such that

1. The family $\{E_i\}_{i \geq 1}$ exhausts the singular set of $U$, i.e., $\bigcup_{i=1}^{\infty} E_i = \mathcal{K}(U)$.

2. For each $i \geq 1$, there exists an open neighborhood $\Omega_i \supseteq E_i \Gamma/E_i$ such that for any compact set $F \subseteq \mathcal{K} - E_{i+1} \Gamma/\Gamma$, there exists $T_{i+1}$ such that for all $x \in F$ and $T \supset T_{i+1}$ we have

$$\lambda_T\{t \in I(T) : u_t x \in \Omega_i \cap \mathcal{K}\} \leq \frac{1}{4^i}\lambda_T(I(T)).$$

For $i \geq 1$, write $\mathcal{K} \cap \pi(E_i) = \bigcup \pi(C_j)$ for some compact sets $C_j \in X(P_j, U)$, $1 \leq j \leq k$. As we are arguing by contradiction, we can find a compact subset $\mathcal{F}_i \subseteq \mathcal{K}_1 - \pi(E_{i+1})$ such that for each $T_0$ there exists $x \in \mathcal{F}_i$ and $T \supset T_0$ such that $\Delta(\phi, u_t, x, T) > \epsilon$. Without loss of generality, assume that $|T_1| \leq |T_2| \leq \cdots$. This implies that there exists $x_i \in F_i$ and $\sigma_i$ such that $\Delta(\phi, u_t, x_i, \sigma_i) > \epsilon$. From (1) and (2), we obtain for each $j \geq 1$, a time $t_j \in I(T_j)$ such that $u_t x_j \in \mathcal{K} - \bigcup_{i=1}^{j} \Omega_i$. This implies that

$$\Delta(\phi, u_t, \sigma_j, y_j) \geq \epsilon - 2\frac{|T_j|}{|\sigma_j|} \geq \frac{\epsilon}{3}.$$

As $y_j \in \mathcal{K}$ and $\mathcal{K}$ is compact, there exists a limit point $y \in \mathcal{K}$. By construction, $y \not\in \Omega_j$ for all $j \geq 1$. This shows that $y$ is not a singular point for $U$. Now, we can apply Theorem 2.6 to the convergent subsequence of $\{y_j\}$ and the corresponding subsequence of $\sigma_j$, to obtain a contradiction. Let us now turn to the general case. Assume that the statement is known for $T_1, T_2 \subseteq S$, and $T_1 \cap T_2 = \emptyset$. We write $U_1 = (u_v(t_v))_{v \in T_1}$ and $U_2 = (u_v(t_v))_{v \in T_2}$. Note that there exists a compact subset $\mathcal{K}_1 \subseteq G/\Gamma$ such that for all $x \in \mathcal{K}$ and any interval $I(T) \subseteq K_{T_j}$, $j = 1, 2$, we have

$$\frac{1}{\lambda_j(I(T))} \lambda_j\{t \in I(T) : u(t) x(t) \in \mathcal{K}_1\} \geq 1 - \epsilon/16.$$  

Here, we have used the shorthands $u_1(t) = (u_v(t_v))_{v \in T_1}$ and $d\lambda_1$ for the Haar measure on $\prod_{v \in T_1} k_v$.

By the induction hypothesis, there exist finitely many proper subgroups $P_1, \ldots, P_k$ of class $\mathcal{F}$, and compact subsets $C_i \subseteq X(P_i, U_1)$, where $P_i = P_i(k)$, $1 \leq i \leq k$, such that the following holds: for any compact subset $\mathcal{F}$ of $\mathcal{K}_1 \cup C_i \Gamma/\Gamma$ there exists $T_0$ such that for all $x \in \mathcal{F}$ and $T$ with $m(T) > T_0$, we have

$$\left| \frac{1}{\lambda_1(I(T))} \int_{I(T)} \phi(u(t)x) d\lambda_1(t) - \int_{G/\Gamma} \phi d\mu \right| \leq \frac{\epsilon}{16}.$$  

Since $C_i \Gamma/\Gamma \subseteq G/\Gamma$ has measure zero, we can choose neighborhoods $N_i$ of $C_i \Gamma/\Gamma$ of measure at most $\epsilon/16k$. Now, let $\phi_i$, $1 \leq i \leq k$ be a continuous function such whose restriction to $U_i$ is 1, and

8
\[ \int_{G/\Gamma} \phi_i < \epsilon/8k. \]

By applying the induction hypothesis to \( \phi_1, \ldots, \phi_k \), we can find finitely many proper subgroups \( Q_1, \ldots, Q_l \) of class \( \mathcal{F} \), and compact subsets \( D_i \subseteq X(Q_i, U_2) \), where \( Q_i = Q_i(kS), 1 \leq i \leq l \), such that the following holds: for any compact subset \( \mathcal{F} \) of \( \mathbb{K} - \bigcup_{i=1}^l C_i \Gamma/\Gamma \) there exists \( T_1 \) such that for all \( x \in \mathcal{F} \) and \( T_2 \) with \( m(T_2) > T_1 \), we have

\[
\left| \frac{1}{\lambda_2(I(T))} \int_{I(T)} \phi_i(u_2(t)x) d\lambda_2(t) - \int_{G/\Gamma} \phi_i d\mu \right| \leq \frac{\epsilon}{16k}, \quad 1 \leq i \leq k.
\]

Since \( \phi_i(x) = 1 \) for all \( x \in U_i \), we obtain

\[
\lambda_2 \left\{ t_2 \in I(T_2) : u_2(t_2) x \in \bigcup_{i=1}^k N_i \right\} \leq \frac{\epsilon}{16}.
\]

Let \( A = \{ t_2 \in I(T_2) : u_2(t_2) x \in \mathcal{K}_1 \} \). Note that by the choice of \( \mathcal{K}_1 \), we have

\[
\lambda_2(A) \geq (1 - \epsilon/16)\lambda_2(I(T_2)).
\]

Combining the last two equations, we obtain

\[
\lambda_2 \left\{ t_2 \in I(T_2) : u_2(t_2) x \in \mathcal{K}_1 - \bigcup_{i=1}^k N_i \right\} \geq 1 - \frac{\epsilon}{8}.
\]

Since \( \mathcal{K}_1 - \bigcup_{i=1}^k N_i \) is a compact subset of \( \mathcal{K}_1 \), disjoint from \( \bigcup_{i=1}^k C_i \Gamma/\Gamma \), we have there exists \( T_2 \) such that for all \( x \in \mathcal{K}_1 - \bigcup_{i=1}^k N_i \) and \( T \) with \( m(T) > T_2 \), we have

\[
\left| \frac{1}{\lambda_1(I(T_1))} \int_{I(T_1)} \phi_1(u_1(t)x) d\lambda_1(t) - \int_{G/\Gamma} \phi_1 d\mu \right| \leq \frac{\epsilon}{16}.
\]

Let us now consider

\[
\frac{1}{\lambda(I(T))} \int_{I(T)} \phi(u(t)x) d\lambda(t) = \frac{1}{\lambda_1(I(T_1))} \lambda_2(I(T_2)) \int_{I(T_1) \times I(T_2)} \phi(u_1(t_1)u_2(t_2)x) d\lambda_1(t_1) d\lambda_2(t_2).
\]

Combining the last two inequalities show that

\[
\left| \frac{1}{\lambda(I(T))} \int_{I(T)} \phi(u(t)x) d\lambda(t) - \int_{G/\Gamma} \phi d\mu \right| \leq \frac{\epsilon}{4}.
\]

It follows that the union \( X(P_1, U), X(Q_j, U) \) will satisfy the conditions of the theorem. \( \square \)

References

[1] Borel, Armand and Prasad, Gopal, *Values of isotropic quadratic forms at S-integral points* Compositio Math. 83 (1992), no. 3, 347-372.

[2] Dani, S. G. and Margulis, G. A., *Limit distributions of orbits of unipotent flows and values of quadratic forms* I. M. Gelfand Seminar, Adv. Soviet Math., 16, 91–137, Amer. Math. Soc., Providence, RI, 1993.

[3] A. Eskin, G. Margulis and S. Mozes, *Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture*, Ann. of Math. (2) 147 (1998), no. 1, 93-141.
[4] A. Borel, Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. Math. 75 (1962) 485-535.

[5] V. Platonov, A. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, INC., 1994.

[6] M. Ratner, *Raghunathan’s conjectures for Catesian products of real and p-adic Lie groups*, Duke Math. J. 77 (1995), no. 2, 275-382.

[7] G. Tomanov, *Orbits on Homogeneous Spaces of Arithmetic Origin and Approximations*, Adv. Stud. Pure Math., 26, Math. Soc. Japan, Tokyo, 2000.