COMMON ENVELOPE: ENTHALPY CONSIDERATION

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ABSTRACT

In this Letter, we discuss a modification to the criterion for the common envelope (CE) event to result in envelope dispersion. We emphasize that the current energy criterion for the CE phase is not sufficient for an instability of the CE, nor for an ejection. However, in some cases, stellar envelopes undergo stationary mass outflows, which are likely to occur during the slow spiral-in stage of the CE event. We propose the condition for such outflows, in a manner similar to the currently standard \( \alpha_{\text{CE}} \)-prescription but with an addition of \( P/\rho \) term in the energy balance equation, accounting therefore for the enthalpy of the envelope rather than merely the gas internal energy. This produces a significant correction, which might help to dispense with an unphysically high value of energy efficiency parameter during the CE phase, currently required in the binary population synthesis studies to make the production of low-mass X-ray binaries with a black hole companion to match the observations.

Key words: binaries: close – stars: evolution – X-rays: binaries

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1. INTRODUCTION

The most important event in the formation and evolution of most close interacting binaries is the so-called common envelope (CE) event during which the components of a binary system are engulfed by a common gaseous envelope, and the resulting interaction dramatically shrinks their orbit (Bisnovatyi-Kogan & Syunyaev 1971; Ostriker 1976; Paczynski 1976). Depending on the envelope structure and companion masses, the envelope is ejected to leave behind a close binary, or the two stars merge. Dividing the parameter space into binaries that survive CE and those that do not, and determining the final separation for the former, is critical for calculating formation rates of low-mass X-ray binaries (LMXBs), \( \gamma \)-ray bursts, as well as for LISA and LIGO events (Belczynski et al. 2008).

In the standard treatment of CE outcomes, the final separation of the binary is determined via “energy formalism” (Webbink 1984; Livio & Soker 1988) in which the binding energy of the (shunned) envelope is equated to the decrease in the orbital energy \( E_{\text{orb}} \):

\[
E_{\text{bind}} = E_{\text{orb},i} - E_{\text{orb},f} = -\frac{Gm_1m_2}{2a_i} + \frac{Gm_1m_{1,c}m_2}{2a_f}.
\]

Here, \( a_i \) and \( a_f \) are the initial and final binary separations, \( m_1 \) and \( m_2 \) are the initial star masses, and \( m_{1,c} \) is the final mass of the star that lost its envelope.

\( E_{\text{bind}} \) is assumed to be the energy expense needed to remove the envelope to infinity and is commonly adopted to be the sum of the potential energy of the envelope and its internal energy. To characterize the donor envelope central concentration and simplify calculations, specifically for population synthesis, a parameter \( \lambda \) was introduced:

\[
E_{\lambda,\text{bind}} = -\int_{\text{surface}}^{\text{core}} (\Psi(m) + \epsilon(m)) dm = \frac{Gm_1m_{1,c}}{\lambda R_1}.
\]

Here, \( m_{1,c} \) is the mass of the removed giant envelope and \( R_1 \) is the radius of the giant star at the onset of CE, \( \epsilon \) is the specific internal energy and \( \Psi(m) = -Gm/r \) is the (gravitational) force potential (or specific potential energy). \( E_{\lambda,\text{bind}} \) can be found directly from stellar structure for any accepted core mass.

Another parameter, \( \alpha_{\text{CE}} \), is introduced as a measure of the energy transfer efficiency from the orbital energy into envelope expansion, and the balance of energy is written as

\[
\alpha_{\text{CE}} \left( \frac{Gm_1m_2}{2a_i} - \frac{Gm_1m_{1,c}m_2}{2a_f} \right) = \frac{Gm_1m_{1,c}}{R_1}.
\]

Many authors choose to accept \( \alpha_{\text{CE}} \approx 1 \), since for many (at least low-mass) stars \( \lambda = 1 \), as can be found from detailed stellar structure, and \( \alpha_{\text{CE}} \) is bounded above by 1.

The simplicity of the standard prescription resulted in its popular use in the binary population synthesis calculations. However, once accurate values of \( \lambda \) from stellar structure calculations were determined, this approach has shown inconsistencies with the observations, especially large for the formation of black hole LMXBs: in massive giants \( \lambda \ll 0.1 \) (Podsiadlowski et al. 2003), and it has been shown that with \( \alpha_{\text{CE}} \leq 1 \) only an intermediate-mass companion could avoid a merger; this challenges the formation of an LMXB with a low-mass companion in general (Justham et al. 2006).

Here, we revise the energy requirements necessary to force a CE to disperse and discuss its possible application for LMXBs formation.

2. ENTHALPY CONSIDERATIONS

2.1. Total Energy and Instability

While the definition of the gravitational binding energy is unequivocal (e.g., Chandrasekhar 1939), there exists no authoritative source defining what expression should be used as the binding energy of a gaseous sphere in the sense of Section 1. To clarify, in treating the CE problem we are interested in the additional energy to be deposited in the envelope in order to disperse material to infinity. The “industry standard” at the moment is to use the total of the gravitational \( U \) and the gas...
thermal $E$ energies, which, as we shall argue in this section, is not correct.

As an illustrative limiting case, consider a star with the positive total energy\(^3\) \(W_{\text{t}} > 0\) to begin with, which is “kinetically stable”; i.e., an energy barrier has to be overcome between the bound and unbound states (Bisnovatyi-Kogan & Zel’Dovich 1967). It is clear that in such cases, the additional energy $\Delta$ required to unbind the star is the magnitude of the energy barrier, not the (unphysical) $-W_{\text{t}} < 0$.

The secular stability of a star against small adiabatic perturbations is not defined by the sign of the total energy $W$, but rather by the variational conditions: an extremum of an energy is an extremum of the total energy (\(\delta W) = 0\), and secularly stable configurations are then found at local minima: \(\delta^2 W > 0\) (Chiu 1968).

If the first adiabatic exponent $\Gamma_1$ is approximately a constant throughout the star, the secular stability criterion reduces to the condition (Chiu 1968)

\begin{equation}
\Gamma_1 = \left( \frac{\partial \ln T}{\partial \ln \rho} \right)_\text{ad} > 4/3.
\end{equation}

Here, $P$ is the pressure and $\rho$ is the density.

For a one-zone model of a stellar envelope, a linearized version of the above condition, the Baker’s model (Baker 1966) gives a similar criterion for the volume-averaged adiabatic exponent $\Gamma_1 > 4/3$ in the envelope.

On the other hand, the virial theorem, applied to the entire star, gives us the condition for the total energy $W < 0$ in the case of constant third adiabatic exponent $\Gamma_3$, as (e.g., Hansen et al. 2004)

\begin{equation}
\Gamma_3 := \left( \frac{\partial \ln T}{\partial \ln \rho} \right)_\text{ad} + 1 > 4/3.
\end{equation}

Here, $T$ is the temperature. We note that this simple form depends crucially on Newton’s third law applied to every pair of particles inside the star, and thus needs to be substantially modified if we only consider the envelope.

We reiterate the importance of distinguishing these conditions: instability does not have to occur in a state with positive total energy, and $\Gamma_1$ does not have to coincide with $\Gamma_3$; they are related as

\begin{equation}
\Gamma_3 = 1 + \Gamma_1 \left( \frac{\partial \ln T}{\partial \ln P} \right)_\text{ad}.
\end{equation}

In particular, the ionization zones, where $n_{\text{ad}} = 0.4$ and can become as low as 0.1, have local value of $\Gamma_3 < 4/3$ while $\Gamma_1 > 4/3$.

It is unfortunately non-trivial to find the volume-averaged $\Gamma_1$ in the giant envelope after depositing some heat in it. However, it is clear that the criterion of the total energy in the envelope to be $W = 0$ is neither applicable for the envelope stability nor a sufficient condition for the envelope to be dispersed to infinity. We will consider instead the stability of a stellar envelope toward creating outflows.

2.2. The Condition for the Envelope to Outflow

Since we are interested in the energy requirements, it is natural to consider a lower bound on $\Delta$: all orbital energy is converted into heat, and the velocity of ejecta at infinity is zero. How precise these bounds are is not of great concern here and is a separate problem on its own, although we expect them to be reasonably close: the timescale for viscous friction between differentially rotating regions can exceed the spiral-in time, but most angular momentum is lost in the outer envelope while most of the energy is released at closer orbits, so the overall effect of angular momentum conservation can at least be constructed to be small.

As any other thermodynamical system undergoing a steady process, the material in a star obeys the first law of thermodynamics, which states that the change of the internal energy comes not only from the heat transferred into system but also from the work done by the system:

\begin{equation}
\delta E = \delta Q - P \delta \left( \frac{1}{\rho} \right).
\end{equation}

Applying this equation for each mass shell in the envelope (see, e.g., Colgate & White 1966; Kutter & Sparks 1972) gives the energy conservation equation in Lagrangian coordinates as

\begin{equation}
\frac{\partial}{\partial t} \left( \frac{u^2}{2} + \Psi + \epsilon \right) + P \frac{\partial (1/\rho)}{\partial t} = 0.
\end{equation}

Here, $u$ is the velocity.

The initial condition is that at the start the mass shells in the envelope are not moving, but every shell in the envelope received some heat, in this case deposited from the companion’s orbital contraction. The heat has some arbitrary distribution $\delta q(m) > 0$ in the envelope $(\delta q(m))$ is per mass unit), such that

\begin{equation}
\int_0^{\Sigma} \delta q(m) dm = Q.
\end{equation}

A given Lagrangian shell, once it has been heated and started expansion, will have reached the point of no return in its expansion when its $\Sigma > 0$ (e.g., Bisnovatyi-Kogan & Zel’Dovich 1967; Sparks & Kutter 1972):

\begin{equation}
\left( \delta q(m) + \Psi + \epsilon + \frac{P}{\rho} \right)_{\text{start}} = \left( \frac{1}{2} u^2 + \Psi + \epsilon + \frac{P}{\rho} \right)_{\text{exp}} = \Sigma = \text{const}.
\end{equation}

In this form, the equation can be also recognized as a version of the Bernoulli equation.

The quantity $\Sigma$, for stability analysis purposes, is more important than the total energy: bipolitropic stellar model that has $\Sigma < 0$ in its envelope but positive total energy is metastable as a whole, even though the runaway to infinity is energetically allowable; but a star with $\Sigma > 0$ in its envelope will be always quasi-steadily outflowing (Bisnovatyi-Kogan & Zel’Dovich 1967; Bisnovatyi-Kogan 2002). It has also been noted that it is a general feature of low-mass giants during double shell burning to establish $\Sigma > 0$ in larger and larger parts of their envelopes as they evolve (with He shell providing large energy inflow to the envelope with each He-shell flash), and could possibly being responsible for envelope outflows and/or ejections with planetary nebula formation (Sparks & Kutter 1972).

In line with results of past stability analyses of when a star starts an outflow (Bisnovatyi-Kogan 2002, and the references therein), we suggest that once a part of the CE has obtained positive $\Sigma$, it will start outflowing, notwithstanding the sign of the envelope’s total energy. A stronger criterion would be to

\(^3\) With zero of energy defined for all material evacuated to infinity, and pressure set to zero.
require the *entire* envelope to have positive $\Sigma$. Under assumption of the minimum energy requirement (velocity of gas at infinity is zero), we can write the energy conservation for a whole envelope as

$$Q + \int_{\text{core}}^\text{surface} \left( \Psi(m) + \epsilon(m) + \frac{P(m)}{\rho(m)} \right) dm = 0. \quad (11)$$

The timescale to start such quasi-steady surface outflows is one on which the envelope redistributes the dumped heat, i.e., the thermal timescale of the envelope—recall our earlier note about most energy being deposited in lower orbits. This time is about few hundred years (e.g., it is about 1000 years for a 20 $\odot$; see Figure 1 in Ivanova 2011). This makes the appearance of mass outflows *natural* during the self-regulating spiral-in stage; such a stage could last for up to a thousand years (Podsiadlowski 2001); but they likely will not take place if the CE event occurs on the dynamical time, as is the case, e.g., in the case of a physical collision, or if the swallowed companion is too small to expand the giant envelope and establish a self-regulated slow spiral-in phase.

Note that

1. due to the presence of a non-negative term $P/\rho$, this condition occurs before the envelope’s total energy become positive;
2. the master Equation (3) of the “$\lambda$-formalism” is a version of Equation (10) where all the velocities, as well as the work $P/\rho$ are simply neglected.

The term $P/\rho$ is of the order of magnitude of $\epsilon$; for example, for an ideal gas, with no radiation pressure and ionization taken into account, $P/\rho = 2/3 \epsilon$. The quantity $h = \epsilon + P/\rho$ is generally known as enthalpy.

In line with the classical “$\lambda$-formalism” and for ease of comparison, we introduce

$$E_{h,\text{bind}} = - \int_{\text{core}}^\text{surface} (\Psi(m) + h(m)) dm = \frac{Gm_1 m_2}{\lambda h R_1} \quad (12)$$

Following Sparks & Kutter (1972) in considering the detailed stellar models of giants with respect to their $\Sigma$, we also note that normal single giants have a “boiling pot” zone (BPZ), where $\Sigma(m) = \Psi(m) + h(m) > 0$ due to high value of $P/\rho$ (Figure 1). Once sufficient amount of the “lid”—the star matter above this zone—is removed, all material with positive $\Sigma$ could freely stream away without the need to convert any additional mechanical energy. The outer boundary of the BPZ almost does not change during the giant stage, and the inner boundary does not change once the convective envelope is established (e.g., in case of low-mass red giants, RGs). The bottom of the BPZ roughly (but not always exactly) coincides with the bottom of the outer convective zone. Illustratively, the mass contained in this zone is $\sim 0.5 M_\odot$ for a 2 $M_\odot$ RG, up to 7.7 $M_\odot$ for a 30 $M_\odot$ giant evolved without wind mass loss, and about three times less for a 30 $M_\odot$ giant evolved with mass loss. We anticipate that the presence of the BPZ could be a key to understanding why RGs are expanding when losing their mass.

### Table 1: CE Outcomes Comparison

| $m_1$/($m_1$.ams) | $R_1$ | $m_1$.X | $m_1$.cp | $\lambda$ | $\lambda_h$ | $m_2$.X | $m_2$.h |
|-------------------|-------|---------|---------|-----------|-------------|---------|---------|
| 25.59(30)         | 900   | 9.381   | 11.44   | 0.026     | 0.085       | 6.33    | 1.53    |
| 25.53(30)         | 1500  | 10.223  | 11.39   | 0.026     | 0.064       | 2.59    | 0.92    |
| 18.5(20)          | 600   | 5.59    | 6.48    | 0.065     | 0.299       | 2.84    | 0.46    |
| 18.5(20)          | 750   | 5.70    | 6.48    | 0.133     | 0.309       | 0.82    | 0.32    |
| 16.8(20)          | 850   | 6.75    | 6.92    | 0.067     | 0.142       | 0.72    | 0.31    |
| 9.75(10)          | 200   | 1.69    | 1.95    | 0.148     | 0.274       | 1.87    | 0.86    |
| 9.75(10)          | 300   | 1.73    | 2.04    | 0.136     | 0.244       | 1.28    | 0.62    |
| 9.74(10)          | 360   | 1.95    | 2.10    | 0.143     | 0.253       | 0.74    | 0.37    |
| 5.09(10)          | 380   | 2.87    | 2.94    | 0.061     | 0.109       | 0.16    | 0.09    |
| 4.99(5)           | 40    | 0.575   | 0.725   | 0.402     | 0.815       | 1.9     | 0.75    |
| 4.99(5)           | 80    | 0.702   | 0.784   | 0.425     | 0.822       | 0.56    | 0.25    |
| 2                 | 10    | 0.253   | 0.271   | 0.288     | 2.804       | 0.39    | 0.13    |
| 2                 | 40    | 0.526   | 0.529   | 0.730     | 1.652       | 0.04    | 0.02    |
| 1                 | 10    | 0.253   | 0.254   | 0.941     | 2.29        | 0.04    | 0.02    |

**Notes.** A post-CE mass is adopted to be the divergence point $m_1$.cp; for a comparison the mass of the hydrogen-exhausted core $m_1$.X (where $X = 10^{-10}$) is shown. $\lambda$ and $\lambda_h$ connect the energies required to eject the envelope with their parameterizations in two formulations (Equations (2) and (12)). $m_2$.X and $m_2$.h are the minimum companions’ masses that could survive a CE event, where $m_2$.X is using standard prescription and $m_2$.h is with enthalpy consideration. $m_1$ are the current donor masses and $m_{1,\text{ams}}$ are donor masses at the zero-age main sequence (if different from $m_1$). $R_1$’s are the current donor radii. All masses are in $M_\odot$, $R_1$ is in $R_\odot$.

### 3. Comparison of $E_{\lambda,\text{bind}}$ and $E_{h,\text{bind}}$ in Stellar Models

Just like $\lambda$, $\lambda_h$ can be found from detailed stellar calculations. For the purpose of this comparison, we adopt the definition of the core mass for both cases as in Ivanova (2011): the post-CE core is the point with maximum compression $P/\rho$ inside the Hydrogen burning shell, $m_1$.cp. Such a core will not re-expand after the envelope loss, and it was shown that in most of the cases it is energetically beneficial to remove the envelope down to exactly $m_1$.cp.

In Table 1, we show the results for giants of different masses. We used the set of stellar models described in Ivanova (2011). We find that in most giants the ratios of $\lambda_h/\lambda$ are from ~2 to ~5, where the largest ratios, for a giant of the same mass, are for an earlier giant. The final separations allowed by the enthalpy consideration are then larger by the same factor.
The minimal companion mass—in the sense that no possible stripped core value exists such that a binary would not merge—changes by several times as well (Table 1). Specifically, we want to point out the difference between the enthalpy consideration and standard prescription for massive giants: while $\alpha_{\text{CE}} \lambda$-formalism would predict the minimum surviving companion mass of about several $M_\odot$, in line with the problem raised by Justham et al. (2006), the enthalpy formalism allows for a low-mass companion to survive and in the future form a black hole LMXB.

4. CONCLUSIONS

In this Letter, we considered the termination of a CE event at the moment it establishes a quasi-stationary mass outflow that would reach a point of no return. Such outflow develops during the slow self-regulating spiral-in phase of the CE event. We showed that if the CE is escaping the binary in this way, neglecting the $P/\rho$ term in the standard $\alpha_{\text{CE}} \lambda$ energy conservation prescription is too gross. If the enthalpy rather than internal energy is calculated in the energy balance, it makes a crucial difference in giants.

While we anticipate that the estimate of a proper energy requirement is paramount for the formation rates of all kinds of post-CE binaries, we especially emphasize the importance of this effect in the case of CE with a massive giant. When the standard energy formalism predicts that only a several solar mass donor could survive the CE, and so the formation of an LMXB with a black hole accretor is forbidden unless $\alpha_{\text{CE}}$ exceeds 1, the enthalpy consideration naturally allows for a low-mass companion survival.

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