A maximum principle for Markov-modulated SDEs of mean-field type and mean-field game

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Abstract

In this paper, we analyze mean-field game modulated by finite states markov chains. We first develop a sufficient stochastic maximum principle for the optimal control of a Markov-modulated stochastic differential equation (SDE) of mean-field type whose coefficients depend on the state of the process, some functional of its law as well as variation of time and sample. As coefficients are perturbed by a Markov chain and thus random, to study such SDEs, we analyze existence and uniqueness of solutions of a class of mean-field type SDEs whose coefficients are random Lipschitz as well as the property of propagation of chaos for associated interacting particles system with method parallel to existing results as a byproduct. We also solve approximate Nash equilibrium for the Markov-modulated mean-field game by mean-field theory.

Keywords: Mean-field type SDEs, interacting particle systems, Markov-modulated, stochastic control, stochastic maximum principle, mean-field game, approximate Nash equilibrium

1 Introduction

Mean-field type stochastic differential control (games) has become a very popular topic and developed rapidly during recent years, for instance, see Huang et al. (2003, 2006) and Lasry & Lions (2007). Tools to analyze such problems include mean-field games method (e.g. see Huang et al. 2006) which solves a standard control problem for a deterministic function firstly, and then utilize fixed point method to determine that there exists such function which is the distribution of the state process, and partial differential equation (e.g. see Borkar & Kumar 2010) and stochastic maximum principle of mean-field type (e.g. see Andersson & Diehlhe 2011) which solve mean-field type control problem in a direct manner.

This paper aims to generalize the stochastic maximum principle of mean-field type developed in Andersson & Diehlhe (2011) to the case modulated by a homogenous Markov chain defined finite state space, and analyze Markov-modulated mean-field game. We consider the stochastic problem of a Markov-modulated SDE of mean-field type whose coefficients depend on the state of the process,
some functional of its law as well as variation of time and sample (see Problem 2.1 for details). Under suitable assumptions, the Markov-modulated SDE of mean-field type can be obtained as a limit of an interacting particles system modulated by independent, identically distributed (i.i.d.) Markov chains defined on finite state space (see the system (1) for details). We provide sufficient conditions for maximum principle of mean-field type modulated by a Markov chain. Compared to Andersson & Djehiche (2011), the adjoint equation in this paper involves Markov regime-switching jumps which inhabit property of martingale. As in Andersson & Djehiche (2011), we establish the sufficient conditions for maximum principle in general set-up that only requires predictable processes and some integrability conditions without any other special conditions for admissible controls, which makes it suitable to handle time inconsistency of the mean-field type control problem. Stochastic maximum principles of mean-field type include Buckdahn et al. (2011), Hosking (2012), Li (2012), Shen & Siu (2013). For standard maximum principles modulated by Markov chains, please see Zhang et al. (e.g. 2012).

As coefficients of mean-field type SDEs are left continuous with right limit, and perturbed by a Markov chain and thus random, to study such SDEs, it is necessary to analyze existence and uniqueness of solutions of a class of mean-field type SDEs whose coefficients are random Lipschitz as well as the property of propagation of chaos for associated interacting particles system although methods used by us are simply parallel to the ones from existing results in Jourdain et al. (2008).

We also analyze asymptotic errors between stochastic dynamic game and its limiting optimal problem. By mean-field game theory, we find approximate Nash equilibrium for our model.

The remaining of this paper is organized as follows. In Section 2 we introduce Markov-modulated diffusion, weakly coupled stochastic dynamic game and mean-field type SDEs, and analyze existence and uniqueness of mean-field type SDEs as well as the property of propagation of chaos of the associated interacting particles system. In Section 3 we provide sufficient conditions for maximum principle. In Section 4 we make asymptotic analysis and verify approximate Nash equilibrium. In Section 5 we comment concluding remarks.

## 2 Problem formulation

In this section, we define the Markov-modulated diffusion model and formulate the Markov-modulated weakly coupled interacting particles control system governed by \( n \) dimensional nonlinear stochastic system and its limit system governed by Markov-modulated mean-field type SDEs. Meanwhile, we analyze existence and uniqueness of two systems as well as approximation of two systems. Assume that \( T > 0 \) is time horizon and \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a complete probability space. We shall make use of the following notation:

\[
\begin{align*}
\mathbb{R} & : \text{the real Euclidean space;} \\
\mathbb{R}^n & : \text{the } n\text{-dimensional real Euclidean space;} \\
|\cdot| & : \text{the Euclidean norm;} \\
M^* & : \text{the transpose of any matrix or vector;} \\
Diag(y) & : \text{the diagonal matrix with the elements of } y;
\end{align*}
\]
\[ L^2_M(0, T; \mathbb{V}) : \] the space of all measurable, \( \mathcal{M}_t \)-predictable processes

\[ f : [0, T] \times \Omega \rightarrow \mathbb{V} \text{ such that } \mathbb{E} \int_0^T |f(t)|^2 dt < \infty, \]

where \( \mathcal{M}_t \) is a \( \sigma \)-field and \( \mathbb{V} \) is a subset of \( \mathbb{R} \);

\( \mathbb{U} \) : the action space which is a nonempty, closed and convex subset of \( \mathbb{R} \);

\( C \) : a constant which may change from line to line.

### 2.1 The Markov-modulated diffusion model

Let the time-homogeneous Markov chain \( \alpha \) take values in finite state space \( \mathbb{S} \triangleq \{1, 2, \ldots, d\} \) associated with the generator \( \Lambda \triangleq [\lambda_{ij}]_{i,j=1}^{d} \). We assume that that chain starts in a fixed state \( i_0 \in \mathbb{S} \) such that \( \alpha_0 = i_0 \). Let \( N_t(i, j) \) be the number of jumps from state \( i \) to state \( j \) up to time \( t \). Denote by \( 1 \) the indicator function. Then

\[ N_t(i, j) = \sum_{0 < s \leq t} 1_{\{\alpha_s = i\}} 1_{\{\alpha_s = j\}}, \forall t \in [0, T]. \]

Define the intensity process \( m_t(i, j) \triangleq \lambda_{ij} 1_{\{\alpha_t = i\}} \). If we compensate \( N_t(i, j) \) by \( \int_0^t m_s(i, j) ds \), then the resulting process \( N_t(i, j) - \int_0^t m_s(i, j) ds \) is a purely discontinuous, square integrable martingale which is null at the origin. Let

\[ \bar{\Phi}_t(j) \triangleq \sum_{i=1, i \neq j}^d \left( N_t(i, j) - \int_0^t m_s(i, j) ds \right), \quad j = 1, \ldots, d, \]

\( \bar{\Phi}_t(j) \) is a martingale. Denote \( \bar{\Phi}_t = (\bar{\Phi}_t(1), \ldots, \bar{\Phi}_t(d)) \) and \( m_t = (m_t(1), \ldots, m_t(d)) \), where \( m_t(j) \triangleq \sum_{i=1, i \neq j}^d \int_0^t m_s(i, j) ds, \) for \( j = 1, \ldots, d \).

### 2.2 Weakly coupled stochastic dynamic game and mean-field type SDEs

We consider weakly coupled system of \( n \) interacting particles modulated by independent Markov chains. The dynamic of each particle is given by

\[
\begin{cases}
    dx_t^{i,n} = b(t, x_t^{i,n}, \frac{1}{n} \sum_{j=1}^n \psi(x_t^{j,n}), u_t^i) \, dt + \sigma(t, x_t^{i,n}, \frac{1}{n} \sum_{j=1}^n \phi(x_t^{j,n}), u_t^i) \, dw_t^i, \\
    x_0^{i,n} = x_0^i, \quad i = 1, \ldots, n, \quad t \in [0, T],
\end{cases}
\]

(1)

where \( w_1^i, \ldots, w_n^i, \) \( i \in [0, T] \), are \( n \) independent standard scalar Brownian motions and \( \alpha_1^i, \ldots, \alpha_n^i, \) \( i \in [0, T] \), are \( n \) i.i.d. time-homogeneous Markov chains taking values in \( \mathbb{S} \). The initial states \( x_0^1, \ldots, x_0^n \) are mutually independent and satisfy \( \mathbb{E}|x_0^i|^2 < \infty, \) \( i = 1, \ldots, n \). We also assume that \( \{x_0^1, \ldots, x_0^n\}, \{w_1^1, \ldots, w_1^n\} \) and \( \{\alpha_1^1, \ldots, \alpha_1^n\} \) are mutually independent. The control \( u^i \in L^2_M(0, T; \mathbb{U}), \) \( i = 1, \ldots, n, \) where

\[ G_t \triangleq \sigma(\{x_0^1, \ldots, x_0^n\}, \{w_1^1, \ldots, w_1^n\}, \{\alpha_1^1, \ldots, \alpha_1^n\}, s \leq t). \]
The functions $b$, $\sigma$, $\psi$ and $\phi$ are given as follows:

\[
\begin{align*}
    b & : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \rightarrow \mathbb{R}, \\
    \sigma & : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \rightarrow \mathbb{R}, \\
    \psi & : \mathbb{R} \rightarrow \mathbb{R}, \\
    \phi & : \mathbb{R} \rightarrow \mathbb{R}, \\
    r & : \mathbb{R} \rightarrow \mathbb{R}.
\end{align*}
\]

Each particle is often called an agent (or a player).

The cost functional for the $i$th agent is given by

\[
J^i(u^1, \ldots, u^n) \triangleq \mathbb{E} \left( \int_0^T h \left( t, x_t^{i,n}, \frac{1}{n} \sum_{j=1}^n \varphi(x_t^{j,n}, u_t^j), \alpha_t^i \right) dt + g \left( x_T^{i,n}, \frac{1}{n} \sum_{j=1}^n \chi(x_T^{j,n}) \right) r(\alpha_T^i) \right),
\]

where

\[
\begin{align*}
    g & : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \\
    h & : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \rightarrow \mathbb{R}, \\
    \varphi & : \mathbb{R} \rightarrow \mathbb{R}, \\
    \chi & : \mathbb{R} \rightarrow \mathbb{R}.
\end{align*}
\]

The objective of each agent is to minimize his own cost by properly controlling his own dynamics. Due to the interaction between agents, the computation of a Nash equilibrium is highly complicated, especially for a large population of agents. In practice, a convenient computable strategy is highly demanded. Instead of Nash equilibrium, an approximate $\varepsilon$-Nash equilibrium which was introduced successfully solve this problem (e.g. see Huang et al. [2006]).

**Definition 2.1.** For the $n$ agents, a sequence of controls $u^i \in L^2_G(0, T; \mathbb{U})$ (resp., $u^i \in L^2_{\mathcal{F}_T}(0, T; \mathbb{U})$) which is a Lipschitz feedback, where $\mathcal{F}^i_t \triangleq \sigma(x_t^i, u_t^i, \alpha_t^i, s \leq t)$, $i = 1, \ldots, d$, is called $\varepsilon$-Nash equilibrium with respect to the cost $J^i(u^1, \ldots, u^n)$ if there exists $\varepsilon > 0$ such that for any fixed $1 \leq i \leq n$, we have

\[
J^i(u^1, \ldots, u^n) \leq J^i(u^1, \ldots, u^{i-1}, v^i, u^{i+1}, \ldots, u^n) + \varepsilon,
\]

when any alternative control $v^i \in L^2_G(0, T; \mathbb{U})$ (resp., $v^i \in L^2_{\mathcal{F}_T}(0, T; \mathbb{U})$) which is another Lipschitz feedback is applied by the $i$th agent.

By the mean field game theory, a candidate for $\varepsilon$-Nash equilibrium can be solved via solving the following limiting problem.

**Problem 2.1.** Find an control strategy $\hat{u} \in L^2_{\mathcal{F}}(0, T; \mathbb{U})$, minimize

\[
J(\hat{u}) \triangleq \mathbb{E} \left( \int_0^T h \left( t, x_t, \mathbb{E}\varphi(x_t), \hat{u}_t \right) r(\alpha_t) dt + g \left( x_T, \mathbb{E}\chi(x_T) \right) r(\alpha_T) \right)
\]

subject to

\[
\begin{align*}
    dx_t &= b(t, x_t, \mathbb{E}\psi(x_t), \hat{u}_t)dt + \sigma(t, x_t, \mathbb{E}\phi(x_t), \hat{u}_t)\,dw_t, \\
    x_0 &= x(0),
\end{align*}
\]

(2)
for any control \( \bar{u} \in L^2_{\mathcal{F}}(0, T; U) \), where \( \mathcal{F}_t \triangleq \sigma(x(0), w_s, \alpha_t, s \leq t) \). We assume that \( w_t, t \in [0, T] \), is a standard scalar Brownian motion. \( \alpha_t, t \in [0, T] \), is a time-homogeneous Markov chain defined on \( S \) and independent of \( w_t \). The initial state satisfies \( \mathbb{E}|x(0)|^2 < \infty \) and is independent of \( w_t \) and \( \alpha_t \). In the above equation, the expectation means the conditional expectation conditioned on \( \{x_0 = x(0), \alpha_0 = i_0\} \).

Once Problem 2.1 were solved with \( \hat{u} \), we could obtain controls \( u^i \in L^2_{\mathcal{F}_i}(0, T; U) \), \( i = 1, \ldots, n \). \( u^1, \ldots, u^n \) are independent. It can be shown that \( (u^1, \ldots, u^n) \) is a \( \epsilon \)-Nash equilibrium. In this paper, we shall develop a stochastic maximum principle of mean-field type for Problem 2.1 which generalizes the result in Andersson & Djehiche (2011). The following assumptions will be imposed throughout this paper, where \( x \) denotes the state variable, \( y \) the "expected value", \( v \) the control variable and \( t \) the time.

\( \text{(A.1)} \) \( \psi, \phi, \chi \) and \( \varphi \) are continuously differentiable. \( g \) is continuously differentiable with respect to \( (x, y) \). \( b, \sigma \) and \( h \) are continuously differentiable with respect to \( (x, y, v) \). \( b \) and \( \sigma \) are left continuous with right limit with respect to \( t \). \( r \) is positive continuous function.

\( \text{(A.2)} \) All the derivatives in (A.1) are Lipschitz continuous and bounded.

\( \text{(A.3)} \) \( \int_0^T |(b(s, 0, \psi(0), 0))^2 + |\sigma(s, 0, \phi(0), 0)|^2| \) \( ds < \infty \).

Under the assumptions (A.1), (A.2) and (A.3), the solutions to the system (1) and equation (2) are unique.

\textit{Remark 2.1.} If \( b \) and \( \sigma \) are continuous with respect to \( t \), then the assumption (A.3) can be relaxed.

2.3 Solution to mean-field type SDEs and propagation of chaos

In this subsection, we analyze solvability of the system (1) and the equation (2). Instead of directly analyzing the system (1) and the equation (2), we study a more general case driven by square integrable Lévy processes \( \{z_t, t \in [0, T]\} \) parallel to Jourdain et al. (2008). To do this, we introduce nonlinear stochastic differential equation of mean-field type modulated by a Markov chain defined on \( S \) and corresponding system of \( n \) interacting particles as follows:

\[
\begin{aligned}
  d\tilde{x}_t &= \tilde{\sigma}(t, \tilde{x}_{t-}, \mathbb{P}_{t-}) r(\tilde{\alpha}_{t-})d\tilde{z}_t, \quad t \in [0, T] \\
  \tilde{x}_0 &= \tilde{x}(0),
\end{aligned}
\]

(3)

where \( \{z_t, t \in [0, T]\} \) is a Lévy process with value in \( \mathbb{R} \), \( \{\tilde{\alpha}_t, t \in [0, T]\} \) is a Markov chain on \( S \). For \( t \in [0, T] \), \( \mathbb{P}_t \) denotes the probability distribution of \( \tilde{x}_t \), and \( \mathbb{P}_s^{-1} = \mathbb{P} \circ \tilde{x}_s^{-1} \) is the weak limit of \( \mathbb{P}_t \) as \( t \to s \) increasingly. The initial state \( \tilde{x}(0) \) takes values in \( \mathbb{R} \), distributed according to \( \pi \), and satisfies \( \mathbb{E}|\tilde{x}(0)|^2 < \infty \). Furthermore, \( \tilde{x}(0), \{z_t, t \in [0, T]\} \) and \( \{\tilde{\alpha}_t, t \in [0, T]\} \) are mutually independent. The functions \( \tilde{\sigma} \) is given as follows:

\[
\tilde{\sigma} : [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R},
\]

5
where \( \mathcal{P}(\mathbb{R}) \) is the space of probability measures on \( \mathbb{R} \). By choosing \( \sigma \) linear in the third variable, the classical McKean-Vlasov model studied in [Sznitman (1991)] can be obtained as a special case of (3). Let \( \tilde{\sigma}(t, \omega, \cdot, \mathcal{P}_{t-}) \triangleq \sigma(t, \cdot, \mathcal{P}_{t-})r(\tilde{\alpha}_{t-}(\omega)) \). It is noted that the above equation have more general coefficient which depends not only on the state process and the probability distribution of the state process, but also on the time and the sample.

For \( i = 1, \ldots, n \), let \( (\tilde{x}_i, z^i) \) be a sequence of independent copies of \( (\tilde{x}(0), z) \). Define the weakly coupled system of \( n \) interacting particles

\[
\begin{cases}
    d\tilde{x}^i_t = \tilde{\sigma}(t, \tilde{x}^i_{t-}, \mu^n_{t-})dz^i_t, & t \in [0, T], \ i = 1, \ldots, n, \\
    \tilde{x}^i_0 = \tilde{x}_0^i, & \mu^n \triangleq \frac{1}{n} \sum^n_{j=1} \delta_{\tilde{x}^j}, \ \text{is the empirical distribution,} \ \delta_x \ \text{is Dirac measure.} 
\end{cases}
\tag{4}
\]

We shall show that as \( n \to \infty \), for \( i = 1, \ldots, n \), \( \tilde{x}^i, n \) converge to a limit \( \tilde{x}^i \) which is an independent copy of the solution to equation (3).

Let \( \mathcal{P}_2(\mathbb{R}) \) be the space of probability measures on \( \mathbb{R} \) with finite second order moments. For \( \mu, \nu \in \mathcal{P}_2(\mathbb{R}) \), we define the Vaserstein metric as follows:

\[
d(\mu, \nu) = \inf \left\{ \left( \int_{\mathbb{R} \times \mathbb{R}} |x - y|^2 Q(dx, dy) \right)^{1/2} : Q \in \mathcal{P}(\mathbb{R} \times \mathbb{R}) \text{ with marginals } \mu \text{ and } \nu \right\}.
\]

It induces the topology of weak convergence together with convergence of moments up to order 2. Due to \( r(\cdot) \) being continuous and thus bounded on any compact set, if for each \( t \), \( \tilde{\sigma}(t, \cdot) \) is Lipschitz continuous when \( \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \) is endowed with the product of the canonical metric on \( \mathbb{R} \) and the Vaserstein metric on \( \mathcal{P}_2(\mathbb{R}) \), then for each \( (t, \omega, \nu) \), \( \tilde{\sigma}(t, \omega, \cdot, \nu) \) is random Lipschitz continuous with respect to the canonical metric on \( \mathbb{R} \).

The solvability of the equation (3) and the system (4) is given as follows:

**Proposition 2.1.** Assume that \( \{z_t, t \in [0, T]\} \) is square integrable, and that for each \( t \), \( \tilde{\sigma}(t, \cdot, \cdot) \) is Lipschitz continuous when \( \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \) is endowed with the product of the canonical metric on \( \mathbb{R} \) and the Vaserstein metric on \( \mathcal{P}_2(\mathbb{R}) \). In addition, for fixed \( x \) and \( \nu \), \( \tilde{\sigma}(\cdot, x, \nu) \) is left continuous with right limit and \( \int^T_0 |\tilde{\sigma}(s, 0, \delta_0)|^2 ds < \infty \), where \( \nu \in \mathcal{P}_2(\mathbb{R}) \) and \( \delta_0 \) is Dirac measure. Then equation (3) admits a unique strong solution such that \( \mathbb{E}(\sup_{t \leq T} |\tilde{x}_t|^2) < \infty \).

Since for \( \xi = (x_1, \ldots, x_n) \) and \( \zeta = (y_1, \ldots, y_n) \) in \( \mathbb{R}^n \), we have

\[
d \left( \frac{1}{n} \sum_{j=1}^n \delta_{x_j}, \frac{1}{n} \sum_{j=1}^n \delta_{y_j} \right) \leq \left( \frac{1}{n} \sum_{j=1}^n |x_j - y_j|^2 \right)^{1/2} = \frac{1}{\sqrt{n}} ||\xi - \zeta||. \tag{5}
\]

Thus, let \( \vartheta(t, \omega, x_1, \ldots, x_n) \triangleq \tilde{\sigma}(t, \omega, x_1, \frac{1}{n} \sum_{j=1}^n \delta_{x_j}) \), we have \( \vartheta : [0, T] \times \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R} \) is random Lipschitz continuous. It induces a process Lipschitz operator which is therefore functional Lipschitz. Hence, existence of a unique solution to the system (4), with finite second order moments, follows from Theorem 7, p.253, in [Protter (2004)].

Next, we give the trajectorial propagation of chaos result for the system (4).
Proposition 2.2. Under the assumptions of Proposition 2.1

\[ \lim_{n \to \infty} \sup_{i \leq n} \mathbb{E} \left( \sup_{t \leq T} |\tilde{x}^{i,n}_t - \tilde{x}^i_t|^2 \right) = 0. \]

Moreover, if \( \bar{\sigma}(t, \tilde{x}, \nu) = \int_{\mathbb{R}} \eta(t, \tilde{x}, \tilde{y}) \nu(d\tilde{y}) \), where \( \eta : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a Lipschitz continuous function with respect to \((\tilde{x}, \tilde{y})\) and left continuous with right limit with respect to \(t\), then

\[ \sup_{i \leq n} \mathbb{E} \left( \sup_{t \leq T} |\tilde{x}^{i,n}_t - \tilde{x}^i_t|^2 \right) \leq \frac{C}{n} \]

where \( C \) does not depend on \( n \).

Remark 2.2. In Proposition 2.1 and 2.2 we only state 1-dimensional case. In fact, the results can be similarly generalized to multi-dimensional case.

In Jourdain et al. (2008), the coefficients are defined on \( \mathbb{R} \times \mathcal{P}(\mathbb{R}) (\mathbb{R}^k \times \mathcal{P}(\mathbb{R}^k)) \). We consider an extended case in which the coefficients are defined on \([0, T] \times \Omega \times \mathbb{R} \times \mathcal{P}(\mathbb{R})\), depending on variation of the time and sample. Proofs of Propositions 2.1 and 2.2 are similar to Jourdain et al. (2008). With the help of Protter (2004) on general stochastic differential equations, we mimic proofs of Jourdain et al. (2008) to prove our results. For detailed proofs, please see the Appendix.

Now, we apply Proposition 2.1 and related discussion to analyze solvability of the system (1) and equation (2). Since \( b \) and \( \sigma \) are Lipschitz continuous with respect to \( x \), it remains to verify they are also Lipschitz continuous with respect to the Vaserstein metric. Noticing that \( b, \sigma, \psi \) and \( \phi \) are all Lipschitz continuous, we have

\[ |b(t, \cdot, \int \psi(x) \mu(dx)) r(\alpha t(\cdot)) - b(t, \cdot, \int \psi(y) \nu(dy)) r(\alpha t(\cdot))| \]
\[ \leq C |\int \psi(x) d\mu(x) - \int \psi(y) d\nu(y)| \]
\[ \leq C d(\mu, \nu) \]

and similarly for \( \sigma r \). Hence, for given control, under assumptions (A.1), (A.2) and (A.3), Proposition 2.1 implies equation (2) admits a unique strong solution. Replacing \( \mu \) and \( \nu \) by empirical measures in the above inequality together with equation (5), we have that the coefficients in the system (1) satisfy the property of functional Lipschitz. Thus, for given controls, the system (1) admits a unique solution.

3 Sufficient conditions for maximum principle

In this section, we develop sufficient conditions for maximum principle for Problem 2.1. Define the Hamiltonian

\[ \mathcal{H}(t, x, \mu, u, i, p, q) \triangleq h(t, x, \int \varphi d\mu(u)) r(i) + b(t, x, \int \psi d\mu(u)) r(i) p + \sigma(t, x, \int \phi d\mu(u)) r(i) q \]

For notational convenience, whenever \( x \) is random variable associated with probability law \( \mu \), we rewrite the Hamiltonian as

\[ H(t, x, u, \alpha, p, q) \triangleq h(t, x, \mathbb{E} \varphi(x), u) r(\alpha) + b(t, x, \mathbb{E} \psi(x), u) r(\alpha) p + \sigma(t, x, \mathbb{E} \phi(x), u) r(\alpha) q. \]
We also denote by $b_x$, $b_y$ and $b_v$ the derivative of $b$ with respect to the state variable, the "expected value" and the control variable, respectively, and similarly for $\sigma$, $h$, $g$ and $\psi$, $\phi$, $\varphi$, $\chi$. We shall use short-hand notation $b(t) = b(t, x_t, \mathbb{E}(\psi(x_t)), u_t)$ and similarly for other functions. Let $\tilde{u}_t$ be an equilibrium strategy to Problem 2.1 and $\tilde{x}_t$ be the associated state variable. We define $\hat{\psi}(t) \triangleq \psi(\tilde{x}_t)$ and $\hat{b}(t) \triangleq b(t, \tilde{x}_t, \mathbb{E}(\hat{\psi}(t)), \tilde{u}_t)$ and similarly for the other functions and their derivatives. Then, the adjoint equation is given by

\[
\begin{align*}
\frac{dp(t)}{dt} &= - \left( \hat{b}_x(t) r(\alpha_t) \hat{p}_t + \hat{\sigma}_x(t) r(\alpha_t) \hat{q}_t + \tilde{h}_x(t) r(\alpha_t) \right) dt \\
&
\quad - \left( \mathbb{E} \left( \hat{b}_y(t) r(\alpha_t) \hat{p}_t \right) \hat{\psi}_x(t) + \mathbb{E} \left( \hat{\sigma}_y(t) r(\alpha_t) \hat{q}_t \right) \hat{\phi}_x(t) + \mathbb{E} \left( \tilde{h}_y(t) r(\alpha_t) \right) \hat{\varphi}_x(t) \right) dt \\
&
\quad + \hat{q}_t dw_t + s(t) d\Phi_t \\
\hat{p}_T &= \tilde{g}_x(T) r(\alpha_T) + \mathbb{E}(\tilde{g}_y(T) r(\alpha_T)) \hat{\chi}_x(T),
\end{align*}
\]

which is a backward stochastic differential equation (BSDE). For the existence and uniqueness of solutions to BSDEs, see Pardoux & Peng (1990). For existence and uniqueness of solutions to BSDEs driven by Markov chains, see Cohen et al. (2010).

We impose the following assumptions for sufficient conditions for maximum principle:

(A.4) the function $g$ is convex in $(x, y)$.

(A.5) the Hamiltonian is convex in $(x, y, v)$.

(A.6) $\psi, \phi, \varphi, \chi$ are convex.

(A.7) the functions $b_y, \sigma_y, h_y, g_y$ are nonnegative.

**Theorem 3.1.** Assume the assumptions (A.1)–(A.7) are satisfied and let $\tilde{u} \in L^2(0, T; \mathbb{U})$ with corresponding state process $\tilde{x}_t$ and suppose there exists solutions $(\hat{p}_t, \hat{q}_t, \hat{s}_t)$ to the adjoint equation satisfying for all $u \in L^2(0, T; \mathbb{U}),$

\[
\begin{align*}
\mathbb{E} \int_0^T |\sigma(t) r(\alpha_t) \hat{p}_t|^2 dt &< \infty \quad (6) \\
\mathbb{E} \int_0^T |(\tilde{x}_t - x_t) \hat{q}_t|^2 dt &< \infty \quad (7) \\
\mathbb{E} \int_0^T |(\tilde{x}_t - x_t) \hat{s}_t \text{Diag}(m_t) \hat{s}_t |(\tilde{x}_t - x_t)| dt &< \infty \quad (8)
\end{align*}
\]

Then, if

\[
H(t, \tilde{x}_t, \tilde{u}_t, \alpha_t, \hat{p}_t, \hat{q}_t) = \inf_v H(t, \tilde{x}_t, v, \alpha_t, \hat{p}_t, \hat{q}_t)
\]

for all $t \in [0, T]$, $\mathbb{P}$-a.s., $\tilde{u}$ is an optimal strategy to Problem 2.1.

**Proof:** Let $H(t) \triangleq H(t, x_t, \alpha_t, \hat{p}_t, \hat{q}_t)$ and $\hat{H}(t) \triangleq H(t, \tilde{x}_t, \tilde{u}_t, \alpha_t, \hat{p}_t, \hat{q}_t)$. For any $u \in L^2(0, T; \mathbb{U})$, we have

\[
J(\tilde{u}) - J(u) = \mathbb{E} \int_0^T \left( \tilde{h}(t) - h(t) \right) r(\alpha_t) dt + \mathbb{E}(\tilde{g}(T) - g(T)) r(\alpha_T))
\]

$$8$$
By the convexity of $g$ and $\chi$ as well as $g_y \geq 0$ and $r > 0$, we obtain
\[
\mathbb{E}((\hat{g} - g)r) \leq \mathbb{E}(\hat{g}_2(T)r(\alpha_T)(\hat{x}_T - x_T) + \hat{g}_y(T)r(\alpha_T)\mathbb{E}(\hat{\chi}(T) - \chi(T))) \\
\leq \mathbb{E}(\hat{g}_2(T)r(\alpha_T)(\hat{x}_T - x_T) + \hat{g}_y(T)r(\alpha_T)\mathbb{E}(\hat{\chi}_x(T)(\hat{x}_T - x_T))) \\
= \mathbb{E}(\hat{p}_T(\hat{x}_T - x_T)).
\]
Apply Irô’s formula to expand $\hat{p}(\hat{x}_T - x_T)$ to get
\[
\hat{p}_T(\hat{x}_T - x_T) = \int_0^T (\hat{x}_t - x_t) d\hat{p}_t + \int_0^T \hat{p}_t d\hat{x}_t - x_t) + [\hat{p}, \hat{x}_T - x_T](T) \\
= \int_0^T (\hat{x}_t - x_t) \left\{ - \left( \mathbb{E} \left( \hat{b}_y(t)r(\alpha_t)\hat{p}_t + \hat{\sigma}_x(t)(r(\alpha_t)\hat{q}_t + \hat{h}_x(t)r(\alpha_t) \right) dt \\
- \left( \mathbb{E} \left( \hat{b}_y(t)r(\alpha_t)\hat{p}_t \right) \hat{\psi}_x(t) + \mathbb{E} \left( \hat{\sigma}_y(t)(r(\alpha_t)\hat{q}_t + \hat{h}_x(t)r(\alpha_t) \right) \phi_x(t) \right) dt \\
+ \hat{p}_t dW_t + s(t)d\Phi_t \right\} \\
+ \int_0^T \hat{p}_t (\hat{b}(t) - b(t)) r(\alpha_t) dt + \int_0^T \hat{q}_t (\hat{\sigma}(t) - \sigma(t)) r(\alpha_t) dt \\
+ \int_0^T \hat{q}_t (\hat{\sigma}(t) - \sigma(t)) r(\alpha_t) dt.
\]
Due to the integrability condition \[6], \[7] and \[8], the Brownian motion and Markov chain martingale integrals in the above equation are square integrable martingales which are null at the origin, we have
\[
\mathbb{E}(\hat{p}_T(\hat{x}_T - x_T)) \\
= -\mathbb{E} \int_0^T (\hat{x}_t - x_t)(\hat{b}_x(t)r(\alpha_t)\hat{p}_t + \mathbb{E} \left( \hat{b}_y(t)r(\alpha_t)\hat{p}_t \right) \hat{\psi}_x(t) \\
+ \hat{\sigma}_x(t)r(\alpha_t)\hat{q}_t + \mathbb{E} \left( \hat{\sigma}_y(t)(r(\alpha_t)\hat{q}_t + \hat{h}_x(t)r(\alpha_t) \right) \phi_x(t) \right) dt \\
+ \mathbb{E} \int_0^T \hat{q}_t (\hat{b}(t) - b(t)) r(\alpha_t) dt + \int_0^T \hat{q}_t (\hat{\sigma}(t) - \sigma(t)) r(\alpha_t) dt \\
+ \int_0^T \hat{q}_t (\hat{\sigma}(t) - \sigma(t)) r(\alpha_t) dt.
\]
Hence,
\[
J(\dot{u}) - J(u) \\
\leq \mathbb{E} \int_0^T (\hat{h}(t) - h(t)) r(\alpha_t) dt + \mathbb{E}(\hat{p}_T(\hat{x}_T - x_T)) \\
= \mathbb{E} \int_0^T (\hat{H}(t) - H(t)) dt - \mathbb{E} \int_0^T \hat{p}_t (\hat{b}(t) - b(t)) r(\alpha_t) dt - \mathbb{E} \int_0^T \hat{q}_t (\hat{\sigma}(t) - \sigma(t)) r(\alpha_t) dt \\
+ \mathbb{E}(\hat{p}_T(\hat{x}_T - x_T)) \\
= \mathbb{E} \int_0^T (\hat{H}(t) - H(t)) dt - \mathbb{E} \int_0^T (\hat{x}_t - x_t)(\hat{b}_x(t)r(\alpha_t)\hat{p}_t + \mathbb{E} \left( \hat{b}_y(t)r(\alpha_t)\hat{p}_t \right) \hat{\psi}_x(t) \\
+ \hat{\sigma}_x(t)r(\alpha_t)\hat{q}_t + \mathbb{E} \left( \hat{\sigma}_y(t)(r(\alpha_t)\hat{q}_t + \hat{h}_x(t)r(\alpha_t) \right) \phi_x(t) \right) dt \\
+ \mathbb{E} \int_0^T \hat{q}_t (\hat{b}(t) - b(t)) r(\alpha_t) dt + \int_0^T \hat{q}_t (\hat{\sigma}(t) - \sigma(t)) r(\alpha_t) dt.
\]
On the other hand, we differentiate the Hamiltonian and use the convexity of the functions to get for all $t \in [0, T]$, $\mathbb{P}$-a.s.
\[
\left( \hat{H}(t) - H(t) \right) \\
\leq \hat{H}_x(t)(\hat{x}_t - x_t) + \hat{h}_x(t)r(\alpha_t)\mathbb{E}(\hat{\psi}(t) - \psi(t)) \\
+ \hat{b}_y(t)r(\alpha_t)\mathbb{E}(\hat{\psi}(t) - \psi(t))\hat{p}_t + \hat{\sigma}_y(t)r(\alpha_t)\mathbb{E}(\hat{\psi}(t) - \psi(t))\hat{q}_t + \hat{H}_a(t)(\hat{u}_t - u_t) \\
\leq \hat{H}_x(t)(\hat{x}_t - x_t) + \hat{h}_x(t)r(\alpha_t)\mathbb{E}(\hat{\psi}_x(t)(\hat{x}_t - x_t)) \\
+ \hat{b}_y(t)r(\alpha_t)\mathbb{E}(\hat{\psi}_x(t)(\hat{x}_t - x_t))\hat{p}_t + \hat{\sigma}_y(t)r(\alpha_t)\mathbb{E}(\hat{\psi}_x(t)(\hat{x}_t - x_t))\hat{q}_t + \hat{H}_a(t)(\hat{u}_t - u_t) \\
\leq \hat{H}_x(t)(\hat{x}_t - x_t) + \hat{h}_x(t)r(\alpha_t)\mathbb{E}(\hat{\psi}(t) - \psi(t))\hat{p}_t \\
+ \hat{b}_y(t)r(\alpha_t)\mathbb{E}(\hat{\psi}(t) - \psi(t))\hat{q}_t + \hat{\sigma}_y(t)r(\alpha_t)\mathbb{E}(\hat{\psi}(t) - \psi(t))\hat{q}_t + \hat{H}_a(t)(\hat{u}_t - u_t)
\]
where in the last inequality, we have used that $\hat{H}_a(t)(\hat{u}_t - u_t) \leq 0$ due to the minimization condition \[9\]. Therefore, $J(\dot{u}) - J(u) \leq 0$. Thus, $\hat{u}$ is an optimal strategy.
4 Approximate Nash equilibrium

We assume that the assumptions of Theorem 3.1 are satisfied, therefore, Problem 2.1 has an optimal strategy. In this section, we shall show that the optimal feedback strategy $u^i_t \triangleq u(t, x^i_t, \alpha^i_t)$ solved from Problem 2.1 is an $\varepsilon$-Nash equilibrium. In order to show that $(u^1, \ldots, u^n)$ is an $\varepsilon$-Nash equilibrium, we prove that for any $\varepsilon > 0$, there exists $N > 0$ such that whenever $n > N$, the definition 2.1 is satisfied. We impose an additional assumption for feedback control strategy: for $i = 1, \ldots, n$, $u^i$ satisfies Lipschitz condition. We write the closed-loop equation and the associated McKean-Vlasov equation as follows: for $i = 1, \ldots, n$,

$$dx_t^{i,n} = b(t, x_{i,n}^t, \frac{1}{n} \sum_{j=1}^{n} \psi(x_{j,n}^t), u(t, x_{i,n}^t, \alpha_{i,n}^t)) dt + \sigma(t, x_{i,n}^t, \frac{1}{n} \sum_{j=1}^{n} \phi(x_{j,n}^t), u(t, x_{i,n}^t, \alpha_{i,n}^t)) r(\alpha_{i,n}^t) dw^i_t. \quad (10)$$

$$dx_t^i = b(t, x_t^{i}, \mathbb{E}\psi(x_t^{i}), u(t, x_t^{i}, \alpha_t^{i})) r(\alpha_t^{i}) dt + \sigma(t, x_t^{i}, \mathbb{E}\phi(x_t^{i}), u(t, x_t^{i}, \alpha_t^{i})) r(\alpha_t^{i}) dw^i_t. \quad (11)$$

Then, we have that $x^{i,n}$ can be approximated by $x^i$ as $n \to \infty$.

**Proposition 4.1.** As $n \to \infty$, we have that $\sup_{t \leq T} \mathbb{E} \left( \sup_{t \leq T} |x^i_t - x^i_t| \right) \to 0$. Moreover, if $b$ and $\sigma$ is linear in the third variable as in Proposition 2.2, then, $\sup_{t \leq T} \mathbb{E} \left( \sup_{t \leq T} |x^{i,n}_t - x^i_t|^2 \right) = O \left( \frac{1}{n} \right)$

For the proof, see Appendix 2. Now, let $\varepsilon^1_n \triangleq \sup_{t \leq T} \mathbb{E} \left( \sup_{t \leq T} |x_t^{i,n} - x_t^i|^2 \right)$. Proposition 4.1 implies $\lim_{n \to \infty} \varepsilon^i_n \to 0$ and $\mathcal{J}^i(u^1, \ldots, u^n) = J^i(u^i) + O(\sqrt{\varepsilon_n})$, where $J^i(u^i)$ formulated as Problem 2.1 is the limiting optimization problem corresponding to $\mathcal{J}^i(u^1, \ldots, u^n)$. $\varepsilon_n$ will be determined.

**Theorem 4.1.** $(u^1, \ldots, u^n)$ is an $\varepsilon$-Nash equilibrium of the cost $\mathcal{J}^i$ subject to the system (11), for $i = 1, \ldots, n$. That is, for any fixed $1 \leq i \leq n$, we have

$$\mathcal{J}^i(u^1, \ldots, u^n) \leq \mathcal{J}^i(u^1, \ldots, u^{i-1}, v^i, u^{i+1}, \ldots, u^n) + \sqrt{\varepsilon_n},$$

when any alternative control $v^i \in L^2_T, (0, T; \mathcal{U})$ which is another Lipschitz feedback is applied by the $i$th agent.

**Remark 4.1.** If $b$, $\sigma$, $h$ and $g$ are linear in the third variable as in Proposition 2.2, then $\varepsilon_n$ will be specified as $\frac{1}{n}$.

For the proof of Theorem 4.1, see Appendix 2. We simply interpret the above theorem as follows. If a given agent changes its control, it results in state process variations for other agents. These variations and the initial control will affect the dynamics of that agent.

5 Concluding remark

We have proved sufficient stochastic maximum principle for Markov-modulated diffusion model. We modulate the dynamics in a special way by multiplying coefficients by a positive function of a
Markov chain. It is possible that a analogy to more general coefficients involving a Markov chain which satisfy suitable Lipchitz condition. On the other hand, a generalization to Markov-modulated jump diffusion for maximum principle is also possible. On the other hand, developing necessary conditions for maximum principle is also possible.
Appendix 1: proofs for solutions of mean-field type SDEs

Proof of Proposition 2.1 Let $D$ denote the space of càdlàg functions from $[0,T]$ to $\mathbb{R}$, $\mathcal{P}_2(D)$ the space of probability measures $Q$ on $D$ such that $\int_{D} \sup_{t \leq T} |\tilde{y}_t|^2 Q(dy) < \infty$. For $P, Q \in \mathcal{P}_2(D)$, define the Vaserstein metric, for $t \in [0,T]$,

$$D_t(P, Q) = \inf \left\{ \left( \int_{D \times D} \sup_{s \leq t} |\tilde{y}_s - \tilde{w}_s|^2 R(d\tilde{y}, d\tilde{w}) \right)^{1/2} : R \in \mathcal{P}(D \times D) \text{ with marginals } P \text{ and } Q \right\}.$$  

Under the above metric, $\mathcal{P}_2(D)$ is a complete space.

For any fixed $Q \in \mathcal{P}_2(D)$ with time-marginals $\{Q_t : t \in [0,T]\}$, we show

$$\tilde{x}_t^Q = \tilde{x}(0) + \int_0^t \tilde{\sigma}(s, \tilde{x}_s^Q, Q_{s-}) r(\tilde{\alpha}_s) dz_s, \ t \in [0,T]$$  \hspace{1cm} (12)

admits a unique solution, where $Q_{s-} = Q \circ \tilde{y}_s^{-1}$ is the weak limit of $Q_t$ as $t \to s$ increasingly. Noticing that by Lebesgue’s theorem, as $s \to t$ increasingly, the distance

$$d(Q_{t-}, Q_{s-}) \leq \int_{D} |\tilde{y}_t - \tilde{y}_s|^2 Q(d\tilde{y})$$

converges to 0. Similarly,

$$d(Q_t, Q_{s-}) \leq \int_{D} |\tilde{y}_t - \tilde{y}_s|^2 Q(d\tilde{y})$$

converges to 0 as $s \to t$ decreasingly. Hence, we obtain that the mapping $t \in [0,T] \mapsto Q_t$ is càdlàg under the metric $d$ defined on $\mathcal{P}_2(\mathbb{R})$. Thus, for fixed $x \in \mathbb{R}$, the mapping $t \in [0,T] \mapsto \tilde{\sigma}(t, x, Q_t)$ is càdlàg. On the other hand, $r(\cdot)$ is continuous function and $\tilde{\alpha}_t$ is càdlàg. Hence, $\tilde{\sigma}(t, x, Q_t) r(\tilde{\alpha}_t(\omega))$ is càdlàg. Then, according to Theorem 6, p. 249, in Protter (2004), equation (12) admits a unique strong solution such that $\tilde{\sigma}(t, x, Q_t) r(\tilde{\alpha}_t(\omega))$ is random continuous.

Let $\Phi$ denote the mapping on $\mathcal{P}_2(D)$ which associates the law of $\tilde{x}_t^Q$ with $Q$. To use fixed point method, we shall verify that $\Phi$ is a mapping from $\mathcal{P}_2(D)$ to $\mathcal{P}_2(D)$. Indeed, for $K > 0$, we set $\tau_K = \inf \{ s \leq T : |\tilde{x}_s^Q| \geq K \}$. By Theorem 66, p.339 in Protter (2004) and Lipschitz property of $\tilde{\sigma}(t, \cdot, \cdot) r(\tilde{\alpha}(\cdot))$, we have

\[
\mathbb{E} \left( \sup_{s \leq t} \left| \tilde{x}_s^Q \right|^2 \right) 
\leq C \left( \mathbb{E} |\tilde{x}(0)|^2 + \int_0^T \mathbb{E} \left( 1_{\{s \leq \tau_K\}} \left| \tilde{\sigma}(s, \tilde{x}_s^Q, Q_s) - \tilde{\sigma}(s, 0, \delta_0) \right|^2 + |r(\tilde{\alpha}_s)|^2 \right) ds \right) 
\leq C \left( \mathbb{E} |\tilde{x}(0)|^2 + \int_0^T \mathbb{E} \left( 1_{\{s \leq \tau_K\}} \left| \tilde{\sigma}(s, \tilde{x}_s^Q, Q_s) - \tilde{\sigma}(s, 0, \delta_0) \right|^2 + |r(\tilde{\alpha}_s)|^2 \right) ds \right) 
\leq C \left( \mathbb{E} |\tilde{x}(0)|^2 + \int_0^T \mathbb{E} \left( \sup_{u \leq s} \left| \tilde{x}_u^Q \right|^2 \right) ds + t \int_{D} \sup_{t \leq T} |\tilde{y}_t|^2 Q(d\tilde{y}) + \int_0^T \mathbb{E} \left( \sup_{u \leq s} \left| \tilde{x}_u^Q \right|^2 \right) ds \right) .
\]

By Gronwall’s Lemma, it follows that

\[
\mathbb{E} \left( \sup_{s \leq t} \left| \tilde{x}_s^Q \right|^2 \right) \leq C \left( \mathbb{E} |\tilde{x}(0)|^2 + \int_{D} \sup_{t \leq T} |\tilde{y}_t|^2 Q(d\tilde{y}) + \int_0^T |\tilde{\sigma}(s, 0, \delta_0)|^2 ds \right) ,
\]
where $C$ does not depend on $K$. Let $K \to \infty$, by Fatou’s Lemma,

$$
\int_D \sup_{t \leq T} |\tilde{y}_t|^2 d\Phi(Q)(\tilde{y}) = \mathbb{E} \left( \sup_{s \leq t} \left| x_s^Q \right|^2 \right) \leq C \left( \mathbb{E} |\tilde{x}(0)|^2 + \int_D \sup_{t \leq T} |\tilde{y}_t|^2 Q(d\tilde{y}) + \int_0^t |\tilde{\sigma}(s, 0, \delta_0)|^2 ds \right). \tag{13}
$$

Hence, $\Phi$ is a mapping from $\mathcal{P}_2(D)$ to $\mathcal{P}_2(D)$.

Since a process $\{\tilde{x}_t : t \in [0, T]\}$ such that $\mathbb{E} \left( \sup_{t \leq T} |\tilde{x}_t|^2 \right) < \infty$ solves equation (13) if and only if its law is a fixed point of $\Phi$. In the following, we shall verify that $\Phi$ admits a unique fixed point. For $P, Q \in \mathcal{P}_2(D)$, by a localization procedure similar to the one used above, we have that

$$
\mathbb{E} \left( \sup_{s \leq t} \left| x_s^P - x_s^Q \right|^2 \right) \leq C \int_0^t d^2(P, Q) ds.
$$

It is noted that $D_t^2(\Phi(P), \Phi(Q)) \leq \mathbb{E} \left( \sup_{s \leq t} \left| x_s^P - x_s^Q \right|^2 \right)$ and $d(P, Q) \leq D_s(P, Q)$, we have that

$$
D_t^2(\Phi(P), \Phi(Q)) \leq \int_0^t D_s^2(P, Q) ds.
$$

Iterating this inequality and denoting by $\Phi^n$ the $n$-fold composition of $\Phi$, we obtain that $n = 1, 2, \ldots,$

$$
D_T^n(\Phi^n(P), \Phi^n(Q)) \leq C^n \int_0^T \left( \frac{T - s}{n - 1} \right) D_s^2(P, Q) ds \leq \frac{C^n T^n}{n!} D_T^2(P, Q).
$$

Hence, for sufficiently large $n$, $\Phi^n$ is a contraction, therefore, $\Phi$ admits a unique fixed point. □

**Proof of Proposition 2.2** Let $P^n = \frac{1}{n} \sum_{j=1}^n \delta_{x_i}$ be the empirical measure of the independent nonlinear process

$$
\begin{align*}
   \frac{d\tilde{x}^i_t}{ds} &= \tilde{\sigma}(t, \tilde{x}^i_t, \tilde{\mu}_t) \tilde{\alpha}^i_t ds_t, \quad t \in [0, T] \\
   \tilde{x}^i_0 &= \tilde{x}^i_0, \quad \forall t \in [0, T],
\end{align*}
$$

$P_t$ denotes the probability distribution of $\tilde{x}^i_t$.

By a localization procedure similar to the one used in the proof of Proposition 2.1, we have

$$
\mathbb{E} \left( \sup_{s \leq t} \left| \tilde{x}_s^{i, n} - \tilde{x}_s^i \right|^2 \right) \leq C \int_0^t \mathbb{E} \left( \left| \tilde{\sigma}(s, \tilde{x}_s^{i, n}, \mu_s^n) - \tilde{\sigma}(s, \tilde{x}_s^{i}, \mu_s^n) \right|^2 |\tilde{\alpha}_s^i|^2 \right) ds \\
   + C \int_0^t \mathbb{E} \left( \left| \tilde{\sigma}(s, \tilde{x}_s^{i, n}, \mu_s^n) - \tilde{\sigma}(s, \tilde{x}_s^{i}, \mu_s^n) \right|^2 |\tilde{\alpha}_s^i|^2 \right) ds \\
   \leq C \int_0^t \mathbb{E} \left( \left| \tilde{\sigma}(s, \tilde{x}_s^{i, n}, \mu_s^n) - \tilde{\sigma}(s, \tilde{x}_s^{i}, \mu_s^n) \right|^2 \right) ds \\
   + C \int_0^t \mathbb{E} \left( \left| \tilde{\sigma}(s, \tilde{x}_s^{i, n}, \mu_s^n) - \tilde{\sigma}(s, \tilde{x}_s^{i}, \mu_s^n) \right|^2 \right) ds.
$$

(14)
Due to the Lipschitz property of $\bar{\sigma}$, equation (5) and exchangeability of the couples $(\bar{x}^i, \bar{x}^{i,n})$, $i = 1, \ldots, n$, the first term of the right in the above inequality is less than $C \int_0^t \mathbb{E} \left( \sup_{s \leq t} |\bar{x}^{i,n}_s - \bar{x}^i_s|^2 \right) ds$.

By Gronwall’s Lemma and Lipschitz assumption on $\bar{\sigma}$, we have

$$
\mathbb{E} \left( \sup_{s \leq t} |\bar{x}^{i,n}_s - \bar{x}^i_s|^2 \right) \leq C \int_0^t \mathbb{E} \left( |\bar{\sigma}(s, \bar{x}^i_s, \mathbb{P}^n_s) - \bar{\sigma}(s, \bar{x}^{i,n}_s, \mathbb{P}_s)|^2 \right) ds \leq C \int_0^t \mathbb{E}(\mathbb{d}^2(\mathbb{P}^n_s, \mathbb{P}_s))ds,
$$

From Lemma 4 in [Jourdain et al., 2008], the upper bounds of the second order moments in Proposition 2.2, we yield the first assertion.

Moreover, if $\bar{\sigma}(t, \bar{x}, \nu) = \int_{\mathbb{R}} \eta(t, \bar{x}, \bar{y})\nu(d\bar{y})$, we have $\mathbb{E} \left( |\bar{\sigma}(s, \bar{x}^i_s, \mathbb{P}^n_s) - \bar{\sigma}(s, \bar{x}^{i,n}_s, \mathbb{P}_s)|^2 \right)$ is equal to

$$
\frac{1}{n} \sum_{j,l=1}^n \mathbb{E} \left( \left[ \eta(s, \bar{x}_s^i, \bar{x}_s^{i,j}) - \int_{\mathbb{R}} \eta(s, \bar{x}_s^i, \bar{y})\mathbb{P}_s(d\bar{y}) \right] \left[ \eta(s, \bar{x}_s^{i,j}, \bar{x}_s^{i,l}) - \int_{\mathbb{R}} \eta(s, \bar{x}_s^{i,l}, \bar{y})\mathbb{P}_s(d\bar{y}) \right] \right).
$$

By the independence of the random variables $\bar{x}^i_s, \ldots, \bar{x}^n_s$ with common law $\mathbb{P}_s$, the expectation in the above summation vanishes as long as $j \neq l$. As a consequence, the result follows.

**Appendix 2: proofs for Proposition 4.1**

**Proof of Proposition 4.1.** Let $\bar{\sigma}(t, x^{i,n}_t, \mu^i_t, u^{i,n}_t) \triangleq \left( b \left( t, x^{i,n}_t, \frac{1}{n} \sum_{j=1}^n \psi(x^{j,n}_t), u(t, x^{i,n}_t, \alpha^i_t) \right), \sigma \left( t, x^{i,n}_t, \frac{1}{n} \sum_{j=1}^n \phi(x^{j,n}_t), u(t, x^{i,n}_t, \alpha^i_t) \right) \right)$, $z^i_t = (t, u^i_t)^\ast$, $\bar{\sigma}(t, x^{i}_t, \mu_t, u^i_t) \triangleq \left( b \left( t, x^{i}_t, \mathbb{E}\psi(x^{i}_t), u(t, x^{i}_t, \alpha^i_t) \right), \sigma \left( t, x^{i}_t, \mathbb{E}\phi(x^{i}_t), u(t, x^{i}_t, \alpha^i_t) \right) \right)$, where $\mu^i_t = \frac{1}{n} \sum_{j=1}^n \delta_{x^{i,j}_t}$, $\mu_t$ is the marginal distribution of $x^{i}_t$, $u^{i,n}_t = u(t, x^{i,n}_t, \alpha^i_t)$ and $u^i_t = u(t, x^{i}_t, \alpha^i_t)$. Let $\nu^i_t \triangleq \frac{1}{n} \sum_{j=1}^n \delta_{x^{j}_t}$. Then, we can rewrite equations (10) and (11) as follows:

$$
dx^{i,n}_t = \bar{\sigma}(t, x^{i,n}_t, \mu^i_t, u^{i,n}_t)r(\alpha^i_t)dz^i_t. $$

$$
dx^{i}_t = \bar{\sigma}(t, x^{i}_t, \mu_t, u^i_t)r(\alpha^i_t)dz^i_t. $$

Hence, we have

$$
x^{i,n}_t - x^i_t = \int_0^t \left[ \bar{\sigma}(s, x^{i,n}_s, \mu^i_s, u^{i,n}_s) - \bar{\sigma}(s, x^i_s, \mu_t, u^i_t) \right] r(\alpha^i_t)dz^i_t
$$

$$
= \int_0^t \left[ \bar{\sigma}(s, x^{i,n}_s, \mu^i_s, u^{i,n}_s) - \bar{\sigma}(s, x^i_s, \nu^i_s, u^i_t) + \bar{\sigma}(s, x^i_s, \nu^i_s, u^i_t) - \bar{\sigma}(s, x^i_s, \mu_t, u^i_t) \right] r(\alpha^i_t)dz^i_t.
$$

It is noted that $b$ and $\sigma$ are differentiable with respect to $(x, y, v)$, and thus satisfy Lipschitz condition. Since $u^{i,n}$ and $u^i$ satisfy Lipschitz condition, by a similar argument to the proof of Proposition 2.2, we yield Proposition 4.1.

**Proof of Theorem 4.1.** Due to symmetry of index $i$, we only need to consider a control strategy for the first agent. We first analyze the running cost, and then the terminal cost by a similar procedure. We write the system with changed control variable for the first agent as follows:

$$
dx^{1,n}_t = \bar{\sigma}(t, x^{1,n}_t, \mu^{1,n}_t, \nu^{1,n}_t)r(\alpha^1_t)dz^1_t,$$
\[ d\hat{x}^1_{t,n} = \tilde{\sigma}(t, \hat{x}^1_{t,n}, \mu^1_{t}, \hat{u}^1_{t,n})r(\alpha^1_{t})dz^1_{t}, \quad i = 2, \ldots, n, \]

where \( \hat{\mu}^n_{t} \) and \( \hat{u}^1_{t,n}, \hat{u}^2_{t,n}, \ldots, \hat{u}^{n,n}_{t,n} \) are defined along the same line with \( \mu^1_{t} \) and \( u^i_{t,n} \), for \( i = 1, \ldots, n \), in the proof of Proposition 4.1.

For \( i \neq 1 \), we have the following estimate

\[
\sup_{2 \leq j \leq n} \mathbb{E} \left( \sup_{s \leq T} |x^i_{s,n} - \hat{x}^i_{s,n}|^2 \right) \leq \varepsilon^2_n, \quad \text{for sufficiently large } n.
\]

Indeed, the above estimate can be verified by Gronwall’s lemma together with equation (5). The expectation in the above equation is less than \( C \int_0^T \mathbb{E} \left( \sup_{u \leq s} \left( |x^i_{u,n} - x^i_{u,n}|^2 + \frac{1}{n} |x^1_{u,n} - \hat{x}^1_{u,n}|^2 \right) \right) ds \)

by treating \( x^1_{u,n} \) and \( \hat{x}^1_{u,n} \) as additional quantities and applying equation (5). Finite ness of \( |x^1_{u,n} - \hat{x}^1_{u,n}|^2 \)

can be analyzed as same as equation (13). Then, the above inequality follows from Gronewall’s lemma. By the same procedure as in the proof of Proposition 4.1 and treating \( x^1_{u,n} \) and \( \hat{x}^1_{u,n} \) as additional quantities, we also have the following estimate

\[
\sup_{2 \leq j \leq n} \mathbb{E} \left( \sup_{s \leq T} |x^i_{s,n} - \hat{x}^i_{s,n}|^2 \right) \leq \varepsilon^3_n, \quad \text{for sufficiently large } n.
\]

We construct a new equation

\[ d\hat{x}^1_{t,n} = \tilde{\sigma}(t, \hat{x}^1_{t,n}, \nu^1_{t}, \hat{v}^1_{t,n})r(\alpha^1_{t})dz^1_{t}. \]

Then, we have the following estimate

\[
\mathbb{E} \left( \sup_{s \leq T} |\hat{x}^1_{s,n} - \hat{x}^1_{s,n}|^2 \right) \leq \varepsilon^4_n, \quad \text{for sufficiently large } n.
\]

Now, we define the equation corresponding

\[ d\hat{x}^1_{t} = \tilde{\sigma}(t, \hat{x}^1_{t}, \nu^1_{t}, \hat{v}^1_{t})r(\alpha^1_{t})dz^1_{t}. \]

Then, by a similar argument to the proof of Proposition 4.1, we obtain

\[
\mathbb{E} \left( \sup_{s \leq T} |\hat{x}^1_{s} - \hat{x}^1_{s}|^2 \right) \leq \varepsilon^5_n, \quad \text{for sufficiently large } n.
\]

Let \( \varepsilon_n = \max\{\varepsilon^1_n, \ldots, \varepsilon^5_n\} \) and \( \bar{h}(t, x^i_{t,n}, \mu^i_{t}, u^i_{t,n}) \triangleq h \left( t, x^i_{t,n}, \frac{1}{n} \sum_{j=1}^n \psi(x^j_{t,n}), u(t, x^i_{t,n}, \alpha^i_{t}) \right) \). Based on above estimates, we obtain

\[
\mathbb{E} \int_0^T \bar{h}(t, \hat{x}^1_{t,n}, \mu^1_{t}, \hat{u}^1_{t,n})dt \\
\geq \mathbb{E} \int_0^T \bar{h}(t, \hat{x}^1_{t,n}, \nu^1_{t}, \hat{v}^1_{t,n})dt - O(\sqrt{\varepsilon_n}) \\
\geq \mathbb{E} \int_0^T \tilde{h}(t, \hat{x}^1_{t}, \nu^1_{t}, \hat{v}^1_{t})dt - O(\sqrt{\varepsilon_n}) \\
\geq \mathbb{E} \int_0^T \tilde{h}(t, \hat{x}^1_{t}, \mu^1_{t}, \hat{v}^1_{t})dt - O(\sqrt{\varepsilon_n}) \\
\geq \mathbb{E} \int_0^T \tilde{h}(t, x^1_{t}, \mu^1_{t}, u^1_{t})dt - O(\sqrt{\varepsilon_n}),
\]
where last inequality results from the optimality assumption. Similarly, we can analyze the terminal cost. Hence, we get

\[
J(v^1, u^2, \ldots, u^n) = J^1(v^1) - O(\sqrt{\varepsilon_n}) \\
\geq J^1(u^1) - O(\sqrt{\varepsilon_n}) \\
= J^1(u^1, \ldots, u^n) - O(\sqrt{\varepsilon_n}),
\]

where last equality follows from Proposition 4.1.

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