Abstract—This paper initially discusses the definition of new rough statistical convergence with Pascal Fibonacci binomial matrix. Some general properties of rough statistical convergence are inspected. Further, approximation theory worked as a rate of the rough statistical convergence has been presented.

Keywords—rough statistical convergence, natural density, triple entire sequences, Korovkin type approximation theorems, Pascal Fibonacci matrix, positive linear operator.

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I. INTRODUCTION

The triple Pascal matrix is an infinite matrix comprising the binomial coefficients as the elements. This can be achieved as any of the below three types of matrices. The $4 \times 4$ truncation of these are demonstrated below.

Triple upper triangular

$$U_4 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 27 & 96 \\
0 & 0 & 1 & 500 \\
0 & 0 & 0 & 1
\end{pmatrix};$$

Triple lower triangular

$$L_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 27 & 1 & 0 \\
1 & 96 & 500 & 1
\end{pmatrix};$$

Symmetric

$$A_4 = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 27 & 500 & 8575 \\
1 & 96 & 3375 & 87808 \\
1 & 250 & 15435 & 592704
\end{pmatrix};$$

These matrices have the pleasing relationship $A_n = L_n^T U_n$. It can be perceived that all 3 matrices possess determinant 1. The symmetric triple Pascal matrix has its elements as the binomial coefficients. (i.e.)

$$A_{ijk} = \binom{r}{m} \binom{s}{n} \binom{t}{k} = \frac{r!}{m!(r-m)!n!(n-s)!k!(k-t)!},$$

where $r, s, t = i + j + k$ and $m = i, n = j, k = t$. In other words

$$A_{ijk} = C_{ijk} = \binom{i+j+k}{i!j!k!}.$$
It follows from (4) that if \( \lambda \) is an eigen value of \( P_n \), then so is \( \lambda^{-1} \).

II. THE TRIPLE PASCAL MATRIX OF INVERSE AND TRIPLE PASCAL SEQUENCE SPACES

Let \( P \) denote the Pascal means defined by the Pascal matrix as is defined by

\[
P = \left[ \binom{r}{m} \binom{r}{n} \binom{r}{k} \right], \text{ if } 0 \leq (m,n,k) \leq (r,s,t)
\]

and the inverse of Pascal’s matrix

\[
P^{-1} = \left\{ \begin{array}{ll} \binom{r}{m} \binom{s}{n} \binom{t}{k} & \text{if } (m,n,k) \leq (r,s,t) \\ 0 & \text{if } (m,n,k) > (r,s,t) \end{array} \right.
\]

There is some interesting properties of Pascal matrix. For example, we can form three types of matrix; symmetric, lower triangular and upper triangular; for any integer \( i,j,k > 0 \). The symmetric Pascal matrix of order \( n \times n \times n \) is defined by

\[
A_{ijk} = \binom{i + j + k}{i} \binom{i + j + k}{j} \binom{i + j + k}{k}
\]

for \( i,j,k = 0,1,2,\ldots,n \).

We can define the lower triangular Pascal matrix of order \( n \times n \times n \) by

\[
L_{ijk} = \binom{1}{i} \binom{1}{j} \binom{1}{k}, \text{ if } i,j,k = 1,2,\ldots,n
\]

and the upper triangular Pascal matrix of order \( n \times n \times n \) is defined by

\[
U_{ijk} = \binom{1}{i} \binom{1}{j} \binom{1}{k}, \text{ if } i,j,k = 1,2,\ldots,n
\]

We know that \( U_{ijk} = L_{ijk}^T \) for any positive integer \( i,j,k \).

(i). Let \( A_{ijk} \) be the symmetric Pascal matrix of order \( n \times n \times n \) defined by (5), \( L_{ijk} \) be the lower triangular Pascal matrix of order \( n \times n \times n \) defined by (6), then \( A_{ijk} = L_{ijk} U_{ijk} \) and \( \det(A_{ijk}) = 1 \).

(ii). Let \( A \) and \( B \) be \( n \times n \times n \) matrices.\( A \) is treated similar to \( B \) when invertible \( n \times n \times n \) matrix \( P \) occursso that \( P^{-1} A P = B \).

(iii). Let \( A_{ijk} \) be the symmetric Pascal matrix of order \( n \times n \times n \) defined by (6), then \( A_{ijk} \) is similar to its inverse \( A_{ijk}^{-1} \).

(iv). Let \( L_{ijk} \) be the lower triangular Pascal matrix of order \( n \times n \times n \) defined by (7), then \( L_{ijk}^{-1} = L_{ijk}^T = (-1)^{i+j+k} \).

We wish to introduce the Pascal sequence spaces \( P_\lambda^3 \) and \( P_\chi^3 \) as the set of all sequences such that \( P \) transforms of them are in the spaces \( \Lambda^3 \) and \( \chi^3 \), respectively, that is

\[
\lambda(x) = \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \binom{r}{m} \binom{s}{n} \binom{t}{k} |x_{mnk}|^{1/n} < \infty
\]

and

\[
\chi(x) = \sup_{r \rightarrow \infty} \left( \frac{1}{r} \right) \sum_{m=0}^{r} \sum_{n=0}^{s} \sum_{k=0}^{t} \binom{r}{m} \binom{s}{n} \binom{t}{k} |x_{mnk}|^{1/n} < \infty
\]

A. Densities and rough statistical convergence

Several definitions of density exists related to the theory of numbers and the most prevalent is asymptotic density. However, asymptotic density cannot be estimated for several sequences. So it is necessary to define densities to cover such areas.

It is known intuitively that the positive integers are much more compared to the perfect squares. The set of perfect squares as well as the set of positive integers are infinite, countable and may have one-to-one correspondence. The squares become increasingly rare when natural numbers are considered. Natural density aids us in this instance and makes this intuition accurate.

Let \( \alpha \) and \( \beta \) be the subsets of the set of positive integers \( \mathbb{Z}^+ \). Consider the interval \([1,n]\). Choose an integer in this interval. The ratio of the count of elements of \( \alpha \in [1,n] \) to over all count of elements in \([1,n]\) belongs to \( \alpha \), probably. For \( n \rightarrow \infty \), if this probability occurs, i.e., it inclines to certain limit, it is considered as the asymptotic density of the set \( \alpha \). It should be noted that \( \alpha \) is asymptotically equal to \( \beta \) (\( \alpha \approx \beta \)) if the symmetric difference \( \alpha \Delta \beta \) is finite. Freedman and Sember presented the notion of a lower asymptotic density.

Definition 2.1

Let \( f \) be a function which defined for all sets of natural numbers and take values in the interval \([0,1]\). Then, the function \( f \) is said to a lower asymptotic density, if the following conditions hold:

(i) \( f(\alpha) = f(\beta) \), if \( \alpha \approx \beta \),
(ii) \( f(\alpha) + f(\beta) \leq f(\alpha \cup \beta) \), if \( \alpha \cap \beta = \phi \),
(iii) \( f(\alpha) + f(\beta) \leq 1 + f(\alpha \cap \beta) \), for all \( \alpha \),
(iv) \( f(\mathbb{Z}) = 1 \).

The upper density can be stated based on the statement related to lower density as below:

Let \( \alpha \) be some density. The function \( \tilde{f} \) is upper density
related with $f$, if $\bar{f}(\alpha) = 1 - f(\mathbb{Z} \setminus \alpha)$ for any natural number set.

Consider the case of set $\alpha \subset \mathbb{Z}^+$. If $f(\alpha) = \bar{f}(\alpha)$, then the set $\alpha$ has natural density with respect to $\alpha$. The term asymptotic density if frequently used for the function
\[
d(\alpha) = \lim_{u,v,w \to \infty} \inf_{u,v,w} \frac{\alpha(u,v,w)}{u,v,w},
\]
where $\alpha \subset \mathbb{N}$ and $\alpha(u,v,w) = \sum_{(a,b,c) \in \alpha(u,v,w)} 1$.

And the natural density of $\alpha$ is
\[
d(\alpha) = \lim_{u,v,w \to \infty} \frac{\alpha(u,v,w)}{u,v,w},
\]
where $\alpha(u,v,w)$ represents the number of elements in $\alpha(u,v,w)$.

Steinhaus [26] and Fast [13] brought up the concept of statistical convergence for real sequences. A triple sequence can be expressed as $x: \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$, and $\mathbb{C}$ represent the sets comprising natural numbers, real numbers, and complex numbers respectively. Various kinds of concepts of triple sequences were discussed by Bihan Hazarika et al. [2], Sahiner et al. [17, 18], Esi et al. [3, 4, 5, 6, 7, 8, 9, 10], Dutta et al. [11], Subramanian et al. [19, 20, 21, 22, 23, 24], Velmurgan [25], Deb Nath et al. [12] and many others.

Consider $\mathcal{K}$, a subset of the set $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Let's represent the set $\{(m, n, k) \in \mathbb{N} : m \leq u, n \leq v, k \leq w\}$ by $K_{uvw}$. Now the natural density of $\mathcal{K}$ is $\delta(\mathcal{K}) = \lim_{u,v,w \to \infty} \mathcal{K}_{uvw}$ where $|K_{uvw}|$ symbolizes the number of elements in $K_{uvw}$. Noticeably, a finite subset has natural density zero, and therefore $\delta(\mathcal{K}') = 0$ for any $\mathcal{K} \notin \mathcal{K}$ is the complement of $\mathcal{K}$. If $K_1 \subset K_2$, then $\delta(K_2) \leq \delta(K_1)$.

Let $x = (x_{m,n,k})$ such that $x_{m,n,k} \in \mathbb{R}$. A triple sequence $x = (x_{m,n,k})$ is regarded statistically convergent to $0 \in \mathbb{R}$, written as $x \rightarrow \lim x = 0$, if the set $\{(m, n, k) \in \mathbb{N}^3 : |x_{m,n,k} - 0| \geq \varepsilon\}$ contains natural density zero for any $\varepsilon > 0$. Zero is known as the statistical limit of the triple sequence $x$ in this situation.

Consider a triple sequence that is statistically convergent. For every $\varepsilon > 0$, infinitely several terms of the sequence may stay outside the $\varepsilon -$ neighbourhood of the statistical limit, provided the natural density of the set involving the indices of these terms is zero. This identity differentiates statistical and ordinary convergence. It can be concluded that every ordinary convergent sequence is statistically convergent as the natural density is zero for finite set.

A Pascal triple sequence $x = (x_{m,n,k})$ is statistically analytic if there exists a positive number $M$ such that
\[
\delta(\{(m, n, k) \in \mathbb{N}^3 : |x_{m,n,k}|^{1/m+n+k} \geq M\}) = 0
\]
In present work, we define the Pascal Fibonacci binomial matrix $F = (f_{m,n,k})_{m,n,k=1}^{\infty}$ which is different from existing Pascal Fibonacci binomial matrix by employing Fibonacci numbers $f_{ij,k}$ and present certain new triple sequence space of $P_{r^3}$ and $P_{A^3}$. We define the Pascal Fibonacci binomial matrix $Ab_{r^3} = Ab_{uvw,mnk}$ where
\[
= \left( f_{m,n,k}^{(u)} \overline{m} \overline{(v)} \overline{(w)} \ast (u-m)^{\ast} \overline{(v-n)^{\ast}} \overline{(w-k)^{\ast}} \right), \quad \text{if } m \leq u, n \leq v, k \leq w
\]
\[
= \left( f_{m,n,k}^{(u)} \overline{m} \overline{(v)} \overline{(w)} \ast (u-m)^{\ast} \overline{(v-n)^{\ast}} \overline{(w-k)^{\ast}} \right), \quad \text{if } m > u, n > v, k > w
\]
Phu [16] came up with the concept of rough convergence. This idea has remarkable applications. This idea was extended by Ayta [1] into rough statistical convergence. Pal et al. [15] elaborated the view of rough convergence employing the notion of ideals. In this paper, we present the concept of rough statistical convergence of triple sequences. Pascal Fibonacci binomial matrix criteria associated with this set of rough statistical convergence has been obtained. All through this paper ris taken as nonnegative real number.

Definition 2.2
A Pascal triple sequence $\mu = (\mu_{m,n,k})$ is said to be rough convergent $(r \rightarrow \text{convergent})$ to $l$ (Pringsheim’s sense), denoted as $\mu_{m,n,k} \rightarrow r$, provided that
\[
\forall \varepsilon > 0, \exists \ i_\varepsilon \in \mathbb{N} : m,n,k \geq i_\varepsilon \Rightarrow |\mu_{m,n,k} - l| < r + \varepsilon,
\]
or equivalently, if
\[
\lim sup |\mu_{m,n,k} - l| \leq r.
\]
The symbol $l$ is known as the roughness degree. The ordinary convergence of a Pascal triple sequence will be attained if $r = 0$.

Definition 2.3
It is obvious that the $r \rightarrow$ limit set of a Pascal triple sequence is not unique. The $r \rightarrow$ limit set of the Pascal triple sequence $\mu = (\mu_{m,n,k})$ is defined as $LIM^r \mu_{m,n,k} = \{ l \in \mathbb{R} : \mu_{m,n,k} \rightarrow^r l \}$.

Definition 2.4
A Pascal triple sequence $\mu = (\mu_{m,n,k})$ is said to be $r \rightarrow$ convergent if $LIM^r \mu \neq \emptyset$. In this case, $r$ is known as the convergence degree of the Pascal triple sequence $\mu = (\mu_{m,n,k})$. For $r = 0$, we obtain the ordinary convergence.

Definition 2.5
A Pascal triple sequence $(\mu_{m,n,k})$ is said to be $r \rightarrow$ statistically convergent to $l$, denoted by $\mu_{m,n,k} \rightarrow^{rst} l$, provided that the set
\[
\{(m, n, k) \in \mathbb{N}^3 : |\mu_{m,n,k} - l| \geq r + \varepsilon\}
\]
has natural density zero for every \( \varepsilon > 0 \), or equivalently, if the condition

\[
\text{st - lim sup } |\mu_{mnk} - l| \leq r
\]

is satisfied. Additionally, \( \mu_{mnk} \to \gamma \), if and only if the inequality

\[
|\mu_{mnk} - l| < r + \varepsilon
\]

holds for all \( \varepsilon > 0 \) and almost all \((m, n, k)\).

**Definition 2.6**

A Pascal triple sequence \( \mu = (\mu_{mnk}) \) is said to be rough statistically Cauchy sequence if for every \( \varepsilon > 0 \) and \( r \) be a positive number there is positive integer \( N = N(r + \varepsilon) \) such that

\[
d\left(\{(m, n, k) \in \mathbb{N}^3 : |\mu_{mnk} - \gamma_{mnk}| > r + \varepsilon\}\right) = 0.
\]

Assuming that a Pascal triple sequence \( \gamma = (\gamma_{mnk}) \) is statistically convergent and cannot be estimated accurately. An approximated triple sequence \( \mu = (\mu_{mnk}) \) has to be used fulfilling \( |\mu_{mnk} - \gamma_{mnk}| \leq r \) for all \((m, n, k)\) (or for almost every \((m, n, k)\), i.e.,

\[
d\left(\{(m, n, k) \in \mathbb{N}^3 : |\mu_{mnk} - \gamma_{mnk}| > r\}\right) = 0.
\]

Then the Pascal triple sequence \( \mu = (\mu_{mnk}) \) is not statistically convergent no longer, but because the inclusion

\[
\left(\{(m, n, k) \in \mathbb{N}^3 : |\mu_{mnk} - \gamma_{mnk}| \geq r + \varepsilon\}\right) \supset \left(\{(m, n, k) \in \mathbb{N}^3 : |\mu_{mnk} - \gamma_{mnk}| \geq r + \varepsilon\}\right)
\]

holds and we have

\[
d\left(\{(m, n, k) \in \mathbb{N}^3 : |\mu_{mnk} - \gamma_{mnk}| \geq r + \varepsilon\}\right) = 0,
\]

i.e., we get

\[
d\left(\{(m, n, k) \in \mathbb{N}^3 : |\mu_{mnk} - \gamma_{mnk}| \geq r + \varepsilon\}\right) = 0,
\]

i.e., the Pascal triple sequence spaces \( \mu \) is \( r \) - statistically convergent.

**B. Approximation theory**

Korovkin type approximation theorems can be used to verify a specified Pascal triple sequence \( (\alpha_{mnk})_{mnk \leq 1} \) of positive linear operators on \([a, b]\) of all continuous functions on the real interval \([a, b]\) is an approximation process. They introduced a variety of test functions. These functions will provide the approximation property that is true on the whole space. Such an identity was presented by Korovkin for the functions \( u, v, w \) in the space \( \mathbb{R} \). Also, he discussed on the functions \( 1, \cos x, \sin x \) in the space of all continuous \( 2\pi \) periodic functions on the real line.

### III PASCAL FIBONACCI BINOMIAL OF ROUGH STATISTICAL CONVERGENCE

A Pascal sequence \( \eta = (\eta_{mnk}) \) is said to be triple analytic if

\[
\sup_{m, n, k} |\eta_{mnk}|^{m + n + k} < \infty.
\]

The Pascal triple sequence space \( P_{\alpha}^{3} \) is a metric space with the metric

\[
d(x, y) = \sup_{m, n, k} \left\{ |\mu_{mnk} - \gamma_{mnk}|^{m + n + k} : m, n, k: 1, 2, 3, \ldots \right\}
\]

for all \( \mu = (\mu_{mnk}) \) and \( \gamma = (\gamma_{mnk}) \) in \( P_{\alpha}^{3} \). Then,

\[
P_{\alpha}^{3}(Ab_{uvw}^{\alpha}m_{mnk}) = \{ \mu = (\mu_{mnk}) \in w : (Ab_{uvw}^{\alpha}m_{mnk}) \in P_{\alpha}^{3} \}.
\]

If \( P_{\alpha}^{3} \) is a linear space then \( P_{\alpha}^{3}(Ab_{uvw}^{\alpha}m_{mnk}) \) is also a linear space.

If \( P_{\alpha}^{3} \) is a complete metric space, then \( P_{\alpha}^{3}(Ab_{uvw}^{\alpha}m_{mnk}) \) is also a complete metric space with the metric

\[
d(x, y) = \sup_{m, n, k} \left\{ |\mu_{mnk} - \gamma_{mnk}| : m, n, k = 1, 2, 3, \ldots \right\}
\]

**Lemma 3.1**

If \( P_{\alpha}^{3} \subset P_{\alpha}^{3} \gamma \); then \( P_{\alpha}^{3}(Ab_{uvw}^{\alpha}m_{mnk}) \subset P_{\alpha}^{3}(Ab_{uvw}^{\alpha}m_{mnk}) \).

**Proof.** It is trivial.

**Theorem 3.1**

Consider that \( P_{\alpha}^{3} \) is a complete metric space and \( \alpha \) is closed subset of \( P_{\alpha}^{3} \). Then \( \alpha(\alpha_{mnk}) \) is also closed in \( P_{\alpha}^{3}(Ab_{uvw}^{\alpha}m_{mnk}) \).

**Proof.** Because \( \alpha \) is a closed subset of \( P_{\alpha}^{3} \) from Lemma 3.1,

\[
\alpha(\alpha_{mnk}) = \alpha(\alpha_{mnk}), \quad \alpha(\alpha_{mnk}) \text{ denote the closure of } \alpha(\alpha_{mnk}) \text{ and } \alpha \text{ respectively. It is enough to prove that } \alpha(\alpha_{mnk}) = \alpha(\alpha_{mnk}).
\]

Firstly, we take \( \mu \in \alpha(\alpha_{mnk}) \), there exists a sequence \( (\mu_{uvw}) \in \alpha(\alpha_{mnk}) \) such that \( |\mu_{uvw} - x|_{\alpha_{mnk}} \to 0 \) in \( \alpha(\alpha_{mnk}) \) for \( u, v, w \to \infty \). Hence, \( |\mu_{uvw} - x|_{\alpha_{mnk}} \to 0 \) as \( u, v, w \to \infty \) in \( \alpha \). Therefore, \( Ab_{uvw}^{\alpha}m_{mnk} \in \alpha \) and so \( \mu \in \alpha(\alpha_{mnk}) \).

Conversely, if we take \( \mu \in \alpha(\alpha_{mnk}) \), then \( \mu \in \alpha(\alpha_{mnk}) \).

Since \( \alpha \) is closed. Then \( \alpha(\alpha_{mnk}) = \alpha(\alpha_{mnk}) \). Hence \( \alpha(\alpha_{mnk}) \) is a closed subset of \( P_{\alpha}^{3}(Ab_{uvw}^{\alpha}) \).

**Corollary 3.1**

If \( P_{\alpha}^{3} \) is a separable space, then \( P_{\alpha}^{3}(Ab_{uvw}^{\alpha}) \) is also a separable space.

**Definition 3.1**

A Pascal triple sequence \( \mu = (\mu_{mnk}) \) is said to be Pascal Fibonacci binomial matrix on rough statistically Cauchy if there is a number \( \theta \) such that for every \( \varepsilon > 0 \) and \( r \) be a positive number the set

\[
\text{K}_{r+\varepsilon}(Ab_{uvw}^{\alpha}) = \{ (m, n, k) \leq (u, v, w) : |Ab_{uvw}^{\alpha}m_{mnk} - l| \geq r + \varepsilon \}
\]

has natural density zero, i.e.;

\[
d(\text{K}_{r+\varepsilon}(Ab_{uvw}^{\alpha})) = 0.
\]

That is \( \lim_{u,v,w \to \infty} \frac{1}{u,v,w} |\text{K}_{r+\varepsilon}(Ab_{uvw}^{\alpha})| = 0 \). Then we have \( d(\text{K}_{r+\varepsilon}(Ab_{uvw}^{\alpha})) = 0 \) or \( \text{lim}_{u,v,w \to \infty} l(\text{rs}(Ab_{uvw}^{\alpha})) \). The set of \( Ab_{uvw}^{\alpha} \) - rough statistically convergent Pascal triple sequence space will be denoted by \( rs(\alpha(\alpha_{mnk})) \). Here \( \alpha = 0 \), we will write \( rs(\alpha) \).

**Definition 3.2**

A Pascal triple sequence \( \mu = (\mu_{mnk}) \) is said to be Pascal Fibonacci binomial matrix on rough statistically Cauchy if there exists a number \( N = N(r + \varepsilon) \) such that for every \( \varepsilon > 0 \) and \( r \) be a positive number the set

\[
\text{K}_{r+\varepsilon}(Ab_{uvw}^{\alpha}) = \{ (m, n, k) \leq (u, v, w) : |Ab_{uvw}^{\alpha}m_{mnk} - l| \geq r + \varepsilon \}
\]

has natural density zero, i.e.;

\[
d(\text{K}_{r+\varepsilon}(Ab_{uvw}^{\alpha})) = 0.
\]

That is \( \lim_{u,v,w \to \infty} \frac{1}{u,v,w} |\text{K}_{r+\varepsilon}(Ab_{uvw}^{\alpha})| = 0 \). Then we have \( d(\text{K}_{r+\varepsilon}(Ab_{uvw}^{\alpha})) = 0 \) or \( \text{lim}_{u,v,w \to \infty} l(\text{rs}(Ab_{uvw}^{\alpha})) \). The set of \( Ab_{uvw}^{\alpha} \) - rough statistically convergent Pascal triple sequence space will be denoted by \( rs(\alpha(\alpha_{mnk})) \). Here \( \alpha = 0 \), we will write \( rs(\alpha) \).
Theorem 3.2
If a Pascal triple sequence space \( \mu \) is a Pascal Fibonacci binomial matrix on rough statistically convergent sequence then \( \mu \) is a Pascal Fibonacci binomial matrix on rough statistically Cauchy sequence.

Proof. Let \( \varepsilon > 0 \) and \( r \) be a positive real number. Assume that \( (\mu_{mnk}) \rightarrow (rs(\mathcal{A}^{rs})) \). Then
\[
|\mathcal{A}^{rs}\mu_{mnk} - l| < \frac{r + \varepsilon}{2} \quad \text{for almost all } m, n, k.
\]
If \( N \) is chosen so that
\[
|\mathcal{A}^{rs}\mu_{N} - l| < \frac{r + \varepsilon}{2}
\]
then we have
\[
|\mathcal{A}^{rs}\mu_{mnk} - \mathcal{A}^{rs}\mu_{N}| < |\mathcal{A}^{rs}\mu_{mnk} - l| + |\mathcal{A}^{rs}\mu_{N} - l| < \left( \frac{r + \varepsilon}{2} + \frac{r + \varepsilon}{2} \right) = r + \varepsilon
\]
for almost all \( m, n, k \).

\[ \Rightarrow \mu \text{ is Pascal Fibonacci binomial matrix on rough statistically Cauchy sequence.} \]

Theorem 3.3
If \( \mu \) is Pascal triple sequence for which there is a Pascal Fibonacci binomial matrix on rough statistically convergent sequence \( \gamma = (\gamma_{mnk}) \) such that \( \mathcal{A}^{rs}\mu_{mnk} = \mathcal{A}^{rs}\gamma_{mnk} \) for almost all \( m, n, k \), then \( \mu \) is Pascal Fibonacci binomial matrix on rough statistically convergent sequence.

Proof. Suppose that \( \mathcal{A}^{rs}\mu_{mnk} = \mathcal{A}^{rs}\gamma_{mnk} \) for almost all \( m, n, k \), and \( (\gamma_{mnk}) \rightarrow (l(\mathcal{A}^{rs})) \). Then, \( \varepsilon > 0 \) and \( r \) be a positive real number and for each \( u, v, w \),
\[
((m, n, k) \leq (u, v, w)) : |\mathcal{A}^{rs}\mu_{mnk} - l| \leq r + \varepsilon
\]
\[ \mathcal{A}^{rs} \text{symn} \geq r + \varepsilon \]
\[ ((m, n, k) \leq (u, v, w)) : |\mathcal{A}^{rs}\mu_{mnk} - l| \leq r + \varepsilon \]

Since \( (\gamma_{mnk}) \rightarrow l(\mathcal{A}^{rs}) \), the latter set contains a fixed number of integers, say \( g = g(r + \varepsilon) \). Therefore, for \( \mathcal{A}^{rs}\mu_{mnk} = \mathcal{A}^{rs}\gamma_{mnk} \) for almost all \( m, n, k, \)
\[
\lim_{u \rightarrow v \rightarrow w} \frac{1}{u v w} \left| \left( (m, n, k) \leq (u, v, w) : |\mathcal{A}^{rs}\mu_{mnk} - l| \geq r + \varepsilon \right) \right| = 0
\]
\[ \mathcal{A}^{rs} \text{symn} \geq \lim_{u \rightarrow v \rightarrow w} \text{mod} u v w = 0. \]

Hence \( (\mu_{mnk}) \rightarrow l(\mathcal{A}^{rs}) \).

Definition 3.3
A Pascal triple sequence \( \mu = (\mu_{mnk}) \) is said to be rough statistically analytic if there exists some \( l \geq 0 \) such that
\[
d \left( \left\{ (m, n, k) : |\mu_{mnk}|^{1/m+n+k} > \frac{1}{l} \right\} \right) = 0,
\]
i.e., \( |\mu_{mnk}|^{1/m+n+k} \leq l a. a. k. \)

Analytic sequences are clearly rough statistically analytic because of the zero natural density of empty set. But the converse is not correct.

For example, consider the Pascal triple sequence \( \mu = (\mu_{uwv}) = (u w v)^{u v w} + w, \) if \((m, n, k)\) is a square
\[
\begin{cases} 0 & \text{if } (m, n, k) \text{ is not a square} \\
\end{cases}
\]
clearly the Pascal triple sequence \( (\mu_{mnk}) \) is not a analytic sequence. However,
\[
d \left( \left\{ (m, n, k) : |\mu_{mnk}|^{1/m+n+k} \geq \frac{1}{6} \right\} \right) = 0,
\]

because the of squares has zero natural density and therefore the Pascal triple sequence \( (\mu_{mnk}) \) is rough statistically analytic.

Corollary 3.2
Every convergent sequence is rough statistically triple Pascal analytic.

Corollary 3.3
Every rough statistical convergent sequence is rough statistically triple Pascal analytic.

Corollary 3.4
Every Pascal Fibonacci binomial matrix of rough statistical convergent sequence is Pascal Fibonacci binomial matrix of rough statistically triple Pascal analytic.

IV. RATE OF PASCAL FIBONACCI BINOMIAL MATRIX ON ROUGH STATISTICAL CONVERGENCE & RESULTS

Let \( F(\mathbb{R}) \) represent the linear space of real value function on \( \mathbb{R} \). Let \( C(\mathbb{R}) \) be space of all real-valued continuous functions \( f \) on \( \mathbb{R} \). \( C(\mathbb{R}) \) with the metric given as follows:
\[
d(f, g) = \sup_{t \in \mathbb{R}} |f(t) - g(t)|^{1/m+n+k}, f \in C(\mathbb{R})
\]
and we denote \( C_{\mathcal{A}}(\mathbb{R}) \) the space of all \( 2\pi \)- periodic functions \( f \in C(\mathbb{R}) \) with the metric is given by
\[
d(f, g)_{C_{\mathcal{A}}} = \sup_{t \in [0, 2\pi]} |f(t) - g(t)|^{1/m+n+k}, f \in C_{\mathcal{A}}(\mathbb{R})
\]
We calculate rate of Pascal Fibonacci binomial matrix on rough statistical convergence of a triple Pascal sequence of positive linear operators defined \( C_{\mathcal{A}}(\mathbb{R}) \) into \( C_{\mathcal{A}}(\mathbb{R}) \).

Definition 4.1
Let \( (a_{uwv}) \) be a positive non-increasing sequence. The triple Pascal sequence \( \mu = (\mu_{mnk}) \) is rate of Pascal Fibonacci binomial matrix on rough statistical convergence to \( l \) with the rate \( o(a_{uwv}) \) if for every \( \varepsilon > 0 \) and \( r \) be a real number such that
\[
\lim_{u \rightarrow v \rightarrow w} \frac{1}{u v w} \left| \left( (m, n, k) \leq (u, v, w) : |\mathcal{A}^{rs}\mu_{mnk} - l| \geq r + \varepsilon \right) \right| = 0.
\]

We can write \( (\mu_{mnk}) \rightarrow l = d(\mathcal{A}^{rs}) - o(a_{uwv}). \)

Lemma 4.1
Let \( (a_{uwv}) \) and \( (b_{uwv}) \) be two positive non-increasing sequences. Let \( \mu = (\mu_{mnk}) \) and \( \gamma = (\gamma_{mnk}) \) be two triple Pascal sequences such that \( (\mu_{mnk}) \rightarrow l_1 = d(\mathcal{A}^{rs}) - o(a_{uwv}) \) and \( (\gamma_{mnk}) \rightarrow l_2 = d(\mathcal{A}^{rs}) - o(b_{uwv}). \) Then we have

(i) \( (\mu_{mnk}) \rightarrow l_1 = d(\mathcal{A}^{rs}) - o(a_{uwv}), \)
(ii) \( (\mu_{mnk}) \rightarrow l_1 \pm \gamma_{mnk} = d(\mathcal{A}^{rs}) - o(c_{uwv}), \)
(iii) \( (\mu_{mnk} - l_1) \cdot (\gamma_{mnk} - l_2) = d(\mathcal{A}^{rs}) - o(c_{uwv} b_{uwv}), \)

where \( c_{uwv} = \max(a_{uwv}, b_{uwv}). \)

For any \( \delta > 0 \), the modulus of continuity of \( f, w(f, \delta) \) is defined by
w(f, \delta) = \sup_{\mu, \nu, \gamma} \left| \frac{w(f, \mu) - w(f, \nu)}{w(f, \delta)} \right|.

\hspace{1cm} (14)

A function \( f \in C[a, b] \), \( \lim_{w_{\text{urw}}} w(f, \delta) = 0 \). For any \( \delta > 0 \)
\[
|f(\mu) - f(\nu)| \leq w(f, \delta) \left( |\mu - \nu| \delta + 1 \right).
\]

Theorem 4.1

Let \( (l_{m_nk}) \) be triple Pascal sequence of positive linear operator from \( C_2(\mathbb{R}) \) into \( C_2(\mathbb{R}) \). Assume that

(i) \( d(l_{m_nk}((1, \mu) - f(\mu)), 0) \geq d(AB^T) - o(\epsilon_{\text{urw}}) \)

(ii) \( w(f, \theta_{\text{urw}}) = d(AB^T) - o(\epsilon_{\text{urw}}) \) where \( \theta_{\text{urw}} = \frac{\sum_{n=1}^{\infty} \sin^2 \left( \frac{\pi n}{2} \right)}{2} \).

Then for all \( f \in C_2(\mathbb{R}) \), we get
\[
d(l_{m_nk}((f(\mu) - f(\nu)))_2, 0) = d(AB^T) - o(\epsilon_{\text{urw}})
\]

where \( \epsilon_{\text{urw}} = \max\{\epsilon_{\text{urw}}, g_{\text{urw}}\} \).

Proof. Let \( f \in C_2(\mathbb{R}) \) and \( \mu \in [-\pi, \pi] \), we can write
\[
l_{m_nk}((f(\mu) - f(\nu))) \leq l_{m_nk}((f(\mu) - f(\nu))) + |f(\mu)|l_{m_nk}((1, \mu) - f(1)) + |f(\mu)|l_{m_nk}((1, \mu) - f(1))
\]

\[
l_{m_nk}(\mu + 1, \mu) w(f, \delta) + l_{m_nk}(\mu + 1, \mu) w(f, \delta) + l_{m_nk}(\mu + 1, \mu) w(f, \delta)
\]

By choosing \( \sqrt{\theta_{\text{urw}}} = \delta \), we get
\[
d(l_{m_nk}((f(\mu) - f(\nu)))_2, 0) \geq d((f(\mu), (f, \nu)))_2.
\]

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