Infra-nilmanifolds modeled on the group of uni-triangular matrices

Younggi Choi¹ · Jong Bum Lee² · Kyung Bai Lee³

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Abstract Let \( N_m \) be the group of \( m \times m \) upper triangular real matrices with all the diagonal entries 1. Then it is an \((m - 1)\)-step nilpotent Lie group, diffeomorphic to \( \mathbb{R}^{1\frac{1}{2}m(m-1)} \). It contains all the integer matrices as a lattice \( \Gamma_m \). The automorphism group of \( N_m \) (\( m \geq 4 \)) turns out to be extremely small. In fact, \( \text{Aut}(N) = I \times \text{Out}(N) \), where \( I \) is a connected, simply connected nilpotent Lie group, and \( \text{Out}(N) = \tilde{K} = (\mathbb{R}^*)^{m-1} \rtimes \mathbb{Z}_2 \). With a nice left-invariant Riemannian metric on \( N \), the isometry group is \( \text{Isom}(N) = N \rtimes K \), where \( K = (\mathbb{Z}_2)^{m-1} \rtimes \mathbb{Z}_2 \subset \tilde{K} \) is a maximal compact subgroup of \( \text{Aut}(N) \). We prove that, for odd \( m \geq 4 \), there is no infra-nilmanifold which is essentially covered by the nilmanifold \( \Gamma_m \setminus N_m \). For \( m = 2n \geq 4 \) (even), there is a unique infra-nilmanifold which is essentially (and doubly) covered by the nilmanifold \( \Gamma_m \setminus N_m \).

Keywords Almost Bieberbach group · Almost crystallographic group · Almost flat manifold · Infra-nilmanifold · Nilpotent Lie group · Uni-triangular matrix

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1 Introduction

Let $\mathcal{N}_m$ be the group of uni-triangular (upper-triangular unipotent) matrices of size $m$, i.e., $\mathcal{N}_m$ consists of all $m \times m$ upper triangular real matrices with all the diagonal entries 1. Then it is an $(m - 1)$-step nilpotent Lie group, diffeomorphic to $\mathbb{R}^{\frac{1}{2}m(m-1)}$. We note that $\mathcal{N}_2$ is the abelian group $\mathbb{R}$, and $\mathcal{N}_3$ is the Heisenberg group. We will suppress $m$ whenever no confusion is likely. We shall show that $\text{Aut}(\mathcal{N}_m)$, $m \geq 4$, contains a maximal compact subgroup $K = (\mathbb{Z}_2)^{m-1} \rtimes \mathbb{Z}_2$.

By Gordon–Wilson [4, Corollary 5.3], with any left-invariant metric on $\mathcal{N}$, the group of isometries, $\text{Isom}(\mathcal{N})$, lies in $\mathcal{N} \ltimes \text{Aut}(\mathcal{N})$. Let $K$ be a maximal compact subgroup of $\text{Aut}(\mathcal{N})$. Since a stabilizer of the isometry group must be compact, $\text{Isom}(\mathcal{N})$ can be conjugated so that it lies in $\mathcal{N} \ltimes K$. Conversely, starting with any left-invariant metric on $\mathcal{N}$, by a standard averaging method over the compact group $K$, one can find a left-invariant metric on $\mathcal{N}$ whose group of isometries is exactly

$$\text{Isom}(\mathcal{N}) = \mathcal{N} \ltimes K.$$ 

Therefore, it is important to understand a maximal compact subgroup of $\text{Aut}(\mathcal{N})$.

As is well known, a discrete cocompact subgroup of $\mathcal{N} \ltimes K$ is called an almost crystallographic group of $\mathcal{N}$. A torsion free almost crystallographic group is an almost Bieberbach group.

Let $\Gamma \subset \mathcal{N}$ be the subgroup consisting of all matrices with integer entries. Then $\Gamma$ is a lattice of $\mathcal{N}$. The quotient $\Gamma \backslash \mathcal{N}$ is a nilmanifold, and a finite quotient of $\Gamma \backslash \mathcal{N}$ is an infra-nilmanifold. By the works of Gromov [5] and Ruh [11], infra-nilmanifold is synonymous to almost flat manifold.

It is the purpose of this work to classify all infra-nilmanifolds that are covered by the (standard) nilmanifold $\Gamma_m \backslash \mathcal{N}_m$ for every $m$. We prove that (Theorem 5.1), for odd $m \geq 4$, there is no infra-nilmanifold which is essentially covered by the nilmanifold $\Gamma_m \backslash \mathcal{N}_m$. For $m = 2n \geq 4$ (even), there is a unique infra-nilmanifold which is essentially (and doubly) covered by $\Gamma_m \backslash \mathcal{N}_m$.

2 Bieberbach theorems on nilpotent Lie groups

Let $G$ be a Lie group, and let $\text{Aut}(G)$ be the group of continuous automorphisms of $G$. The group $\text{Aff}(G)$ is the semi-direct product $\text{Aff}(G) = G \ltimes \text{Aut}(G)$ with multiplication

$$(a, A) \cdot (b, B) = (a \cdot A(b), AB).$$

It has a Lie group structure and acts on $G$ by

$$(a, A) \cdot x = a \cdot A(x)$$

for all $x \in G$. With the linear connection on $G$ defined by the left invariant vector fields, it is known that $\text{Aff}(G)$ is the group of connection-preserving diffeomorphisms of $G$.

Celebrated works of Bieberbach on $\mathbb{R}^n$ have been generalized to nilpotent groups.

**Theorem 2.1** [Generalization to nilpotent groups (see [10, Chapter 8])]

(A) [1] Let $G$ be a connected, simply connected nilpotent Lie group, and let $C$ be a compact subgroup of $\text{Aut}(G)$. If $\Pi \subset G \ltimes C$ is a lattice, then $\Gamma = \Pi \cap G$ is a lattice of $G$, and $\Gamma$ has finite index in $\Pi$.

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Let $G$ be a connected, simply connected nilpotent Lie group. Let $\Pi, \Pi' \subset \text{Aff}(G)$ be finite extensions of lattices in $G$. Then every isomorphism $\theta: \Pi \to \Pi'$ is a conjugation by an element of $\text{Aff}(G)$.

For a lattice $\Gamma$ of a connected, simply connected nilpotent Lie group $G$, there are only finitely many extensions of $\Gamma$ by finite groups containing $\Gamma$ as a discrete nil-radical (i.e., maximal normal, nilpotent subgroup).

These can be interpreted as topological statements.

**Corollary 2.2**

(A) Every almost flat Riemannian manifold is finitely covered by a nilmanifold.

(B) Homotopy equivalent almost flat manifolds are affinely diffeomorphic.

(C) Under each nilmanifold $M$, there are only finitely many almost flat manifolds which are essentially covered by $M$.

A covering is an essential covering if no deck transformation is homotopically trivial.

In this paper we shall consider $N$ as our connected and simply connected nilpotent Lie group and take $\Gamma$ as the lattice consisting of matrices with integer entries. Then we will study almost Bieberbach groups $\Pi$ of $N$ having $\Gamma$ as its nil-radical. Hence $\Pi$ fits in the following commutative diagram

$$
\begin{array}{c}
1 \longrightarrow N \longrightarrow N \rtimes C \longrightarrow C \longrightarrow 1 \\
\uparrow U \quad \uparrow U \quad \uparrow U \\
1 \longrightarrow \Gamma \longrightarrow \Pi \longrightarrow \Phi \longrightarrow 1
\end{array}
$$

where $C$ is a maximal compact subgroup of $\text{Aut}(N)$ and $\Phi$ is a finite group, called the holonomy group of $\Pi$.

### 3 Automorphism group of $N$

The group of automorphisms $\text{GL}(m, \mathbb{R})$ of $\mathbb{R}^m$ is obtained after a lattice $\mathbb{Z}^m$ is fixed. Likewise, in order to calculate the group of automorphisms of $N_m$, we fix a lattice of $N_m$ first. Let $\Gamma_m$ be the subgroup of all integer matrices. Then clearly $\Gamma_m$ is a lattice of $N_m$.

Let $\mathfrak{N}$ be the Lie algebra of $N$ (We suppress $m$). Then $\mathfrak{N}$ is the algebra of $m \times m$ strictly upper triangular real matrices. We use the notation $\mathfrak{N}^{(k+1)} = [\mathfrak{N}, \mathfrak{N}^{(k)}]$, and the same for $N$. Let $e_{i,j} \in \mathfrak{N}$ ($i < j$) be the matrix whose entries are all zero, except for the $(i, j)$-entry which is 1. When we use the notation $e_{i,j}$, we assume $i < j$.

We also define $E_{i,j}$ as $\exp e_{i,j}$,

$$E_{i,j} = \exp e_{i,j} = I_m + e_{i,j} \in N,$$

(because $e_{i,j}^2 = 0$ for $i < j$). We use the notation for commutator

$$[x, y] = xy - yx, \quad \text{in the Lie algebra} \ \mathfrak{N}$$

$$[X, Y] = X Y X^{-1} Y^{-1}, \quad \text{in the Lie group} \ N$$

Then

$$e_{i,j} e_{p,q} = \begin{cases} e_{i,q} & \text{if } j = p \\ 0 \text{ (zero matrix)} & \text{otherwise.} \end{cases}$$
Therefore,
\[
\begin{bmatrix}
  e_{i,j}, e_{p,q}
\end{bmatrix} = \begin{cases}
  e_{i,q} & \text{if } j = p \\
  -e_{p,j} & \text{if } i = q \\
  0 & \text{otherwise}
\end{cases}
\] (3.1)

It is easy to observe that
\[
\begin{bmatrix}
  E_{i,j}, E_{p,q}
\end{bmatrix} = \begin{cases}
  E_{i,q} & \text{if } j = p \\
  E_{p,j}^{-1} & \text{if } i = q \\
  e (= \text{identity matrix in } N) & \text{otherwise}
\end{cases}
\] (3.2)

Let
\[
L_1 = \{ e_{1,2}, e_{2,3}, \ldots, e_{m-1,m} \} \quad \text{linear basis of } \mathcal{N}/\mathcal{N}^{(2)}
\]
\[
L_2 = \{ e_{1,3}, e_{2,4}, \ldots, e_{m-2,m} \} \quad \text{linear basis of } \mathcal{N}^{(2)}/\mathcal{N}^{(3)}
\]
\[
\vdots
\]
\[
L_{m-1} = \{ e_{1,m} \} \quad \text{linear basis of } \mathcal{N}^{(m-1)}/\mathcal{N}^{(m)}
\]

**Lemma 3.1** The set
\[
L_1 = \{ e_{1,2}, e_{2,3}, \ldots, e_{m-1,m} \}
\]
generates the Lie algebra \( \mathcal{N} \).

**Proof** Observe that \( e_{p,p+q} \) can be expressed using repeated commutators of the elements of \( L_1 \) only. Suppose \( e_{p,p+q-1} \in \mathcal{N}^{(q-1)} \). Then
\[
e_{p,p+q} = [e_{p,p+q-1}, e_{p+q-1,p+q}] \in \mathcal{N}^{(q)},
\]
the \( q \)-fold commutator of \( \mathcal{N} \). \( \square \)

Consider the natural homomorphism
\[
\vartheta : \text{Aut}(\mathcal{N}) \longrightarrow \text{Aut}(\mathcal{N}/\mathcal{N}^2) = \text{Aut}(\mathbb{R}^{m-1}).
\]

First we study the image of \( \vartheta \). Suppose that we are given an automorphism \( A \in \text{GL}(m - 1, \mathbb{R}) \) of \( \mathbb{R}^{m-1} \). In general, there does not exist \( \tilde{A} : \mathcal{N} \rightarrow \mathcal{N} \) which induces \( A \) on the quotient \( \mathbb{R}^{m-1} \), unless \( A \) satisfies some very specific requirements.

**Lemma 3.2** ([2, Lemma 3.9]) Assume \( m \geq 4 \). For \( A \in \text{GL}(m - 1, \mathbb{R}) = \text{Aut}(\mathcal{N}/[\mathcal{N}, \mathcal{N}]) \), \( A \) is lifted to \( \tilde{A} \in \text{Aut}(\mathcal{N}) \) if and only if \( A \) is either diagonal or anti-diagonal; that is, the only possible \( A \in \tilde{K} \) are
\[
\begin{pmatrix}
  a_{1,1} & 0 & \cdots & 0 \\
  0 & a_{2,2} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & a_{m-1,m-1}
\end{pmatrix}
\]
or
\[
\begin{pmatrix}
  0 & \cdots & 0 & a_{1,m-1} \\
  0 & \cdots & a_{2,m-2} & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m-1,1} & \cdots & 0 & 0
\end{pmatrix}
\]

with all \( a_{i,j} \neq 0 \).

The matrices in Lemma 3.2 form a subgroup
\[
\tilde{K} = (\mathbb{R}^*)^{m-1} \rtimes \mathbb{Z}_2 \subset \text{GL}(m - 1, \mathbb{R}),
\] (3.3)
where

\[(\mathbb{R}^*)^{m-1} = \left\{ \begin{pmatrix} a_{1,1} & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m-1,m-1} \end{pmatrix} \right\} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix} \]

generates \( \mathbb{Z}_2 \). The image of \( \vartheta \) is exactly \( \tilde{K} \). Let \( I \) be the kernel of \( \vartheta \). Since \( \tilde{K} \) is sitting in \( \text{Aut}(\mathfrak{N}) \) already, the homomorphism \( \vartheta : \text{Aut}(\mathfrak{N}) \to \tilde{K} \) splits so that

\[ \text{Aut}(\mathfrak{N}) = I \rtimes \tilde{K}. \]

Inside \( \tilde{K} \) of equality (3.3), we have

\[ K = \tilde{K} \cap \text{GL}(m - 1, \mathbb{Z}). \]

Then

\[ K = (\mathbb{Z}_2)^{m-1} \rtimes \mathbb{Z}_2, \]

where

\[ \mathbb{Z}_2^{m-1} = \left\{ \begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & \pm 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pm 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathbb{Z}_2 \text{ is generated by } \tau. \]

Clearly, \( K \subset \tilde{K} \) is the totality of torsion elements, which forms a subgroup. Thus, \( K \) is a fully normal subgroup of \( \tilde{K} \). Furthermore, it is a unique maximal compact subgroup of \( \tilde{K} \).

We turn our attention to the group \( \mathcal{I} \) of automorphisms of \( \mathfrak{N} \) which induce the identity map on \( \mathfrak{N}/\mathfrak{N}^2 \).

**Lemma 3.3** \( \mathcal{I} \) is a connected, simply connected nilpotent Lie group.

**Proof** We use the (ordered) linear basis of \( \mathfrak{N} \)

\[ \mathcal{L} = \{ L_1, L_2, \ldots, L_{m-1} \}. \]

With respect to the basis \( \mathcal{L} \) (of \( \frac{1}{2}m(m-1) \) elements), any automorphism of \( \mathcal{I} \) becomes a lower triangular matrix with diagonal entries 1. One can show that \( \mathcal{I} \) is connected by induction on \( m \) in \( \mathcal{N}_m \). Then, \( \mathcal{I} \) is a connected subgroup of the group of all lower triangular matrices, which is a connected, simply connected nilpotent Lie group. Any such subgroup is simply connected as well.

Alternatively, one can use the classification of automorphisms of \( \mathfrak{N} \) in [2]. By [2, Theorem 3.14],

\[ \text{Aut}(\mathcal{N}) \cong ((\text{Inn}(\mathcal{N}) \rtimes (\mathcal{C} \times \mathcal{E})) \rtimes \mathcal{D}) \rtimes \mathcal{G} \]

where

\[ \mathcal{C} \cong \mathbb{R}^{m-1}, \mathcal{E} \cong \mathbb{R}^2, \mathcal{D} \cong (\mathbb{R}^*)^{m-1}, \mathcal{G} \cong \mathbb{Z}_2. \]

Hence \( \text{Out}(\mathcal{N}) \cong ((\mathcal{C} \times \mathcal{E})) \rtimes \mathcal{D} \rtimes \mathcal{G} \). Remark also that \( \text{Aut}(\mathcal{N}) \to \text{Aut}(\mathcal{N}/[\mathcal{N}, \mathcal{N}]) \) has image \( \mathcal{D} \rtimes \mathcal{G} \) and kernel \( \mathcal{I} = \text{Inn}(\mathcal{N}) \rtimes (\mathcal{C} \times \mathcal{E}). \)
Proposition 3.4 Let $F \subset \text{Aut}(\mathfrak{N})$ be a finite group. Then $F$ can be conjugated into $K \subset \tilde{K}$.

Proof Let $\pi : \text{Aut}(\mathfrak{N}) = \mathcal{I} \rtimes \tilde{K} \rightarrow \tilde{K}$ be the projection, and $F_1 = \pi(F) \subset \tilde{K}$. Since $\mathcal{I}$ is torsion free and $F$ is finite, $\pi$ is injective on $F$. Define

$$\lambda : F_1 \rightarrow \mathcal{I}$$

by $\lambda(A) = a$, if $(a, A) \in F$. Then $\lambda \in Z^1(F_1, \mathcal{I})$, a crossed homomorphism.

Now we apply [8, p. 436, Theorem]. Set the space $W$ to be a singleton space, and $Q$ to be a finite group. Then $Q$ acts on $W$ properly discontinuously. The theorem states exactly

Lemma 3.5 Let $Q$ be a finite group, $L$ a connected, simply connected nilpotent Lie group. Then $H^i(Q; L) = 0$, for $i = 1, 2$.

The proof of the above lemma uses induction on the nilpotency of $L$ together with the fact that $H^i(Q; L) = 0$ for a finite group $Q$ and a real vector group $L$.

Thus we have $H^1(F_1, \mathcal{I}) = 0$. Consequently, $\lambda$ is principal, and there exists $b \in \mathcal{I}$ such that $\lambda(A) = \delta(b)$ so that $\lambda(A) = b^{-1} \cdot A(b)$ for all $A \in F_1$. Therefore,

$$(b, I)(a, A)(b^{-1}, I) = (b \cdot A^{-1} \cdot A(b^{-1}), A)$$

for all $(a, A) \in F$. Thus we have shown that $b \in \mathcal{I}$ conjugates $F$ into $\tilde{K}$.

Since $K \subset \tilde{K}$ is the totality of torsion elements, the conjugation image lies inside $K$. This finishes the proof of Preposition 3.4 as well as the following.

Theorem 3.6 $K = (\mathbb{Z}_2)^{m-1} \rtimes \mathbb{Z}_2$ is a maximal compact subgroup of $\text{Aut}(\mathfrak{N})$.

4 Action of $K$ on $\mathcal{N}$

It is well known that $\text{Aut}(\mathcal{N}_3) \cong \mathbb{R}^2 \rtimes \text{GL}(2, \mathbb{R})$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R})$ is an automorphism of $\mathcal{N}_3$ given by

$$A : \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & ax_1 + bx_2 & x_3' \\ 0 & 1 & cx_1 + dx_2 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.1)$$

where

$$x_3' = \frac{1}{2} (ax_1(cx_1 + 2dx_2) + x_2(bdx_2 - 2x_1)) + (ad - bc)x_3.$$

For $m \geq 4$, we use the commutative diagram

$$\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\tilde{A}} & \mathcal{N} \\
\log \uparrow & & \downarrow \exp \\
\mathcal{N} & \longrightarrow & \mathcal{N}
\end{array}$$

to obtain an automorphism of $\mathcal{N}$, $\exp \circ \tilde{A} \circ \log$, determined by $A$. 

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Here is an explicit description of the automorphisms $A \in K$ when $m = 4$ and 5.

For $x = \begin{pmatrix} 1 & x_1 & x_4 & x_6 \\ 0 & 1 & x_2 & x_5 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in N_4$,

\[
\left( \begin{array}{llll}
\epsilon_1 & 0 & 0 & 0 \\
0 & \epsilon_2 & 0 & 0 \\
0 & 0 & \epsilon_3 & 0 \\
0 & 0 & 0 & \epsilon_4 \\
\end{array} \right) \cdot x = \begin{pmatrix} 1 & \epsilon_1x_1 & \epsilon_1\epsilon_2x_4 & \epsilon_1\epsilon_2\epsilon_3x_6 \\ 0 & 1 & \epsilon_2x_2 & \epsilon_2\epsilon_3x_5 \\ 0 & 0 & 1 & \epsilon_3x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

where

\[
x_6' = \epsilon_1\epsilon_2\epsilon_3(x_1x_2x_3 - x_1x_5 - x_3x_4 + x_6).
\]

For $x = \begin{pmatrix} 1 & x_1 & x_5 & x_8 & x_{10} \\ 0 & 1 & x_2 & x_6 & x_9 \\ 0 & 0 & 1 & x_3 & x_7 \\ 0 & 0 & 0 & 1 & x_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in N_5$,

\[
\left( \begin{array}{lllll}
\epsilon_1 & 0 & 0 & 0 & 0 \\
0 & \epsilon_2 & 0 & 0 & 0 \\
0 & 0 & \epsilon_3 & 0 & 0 \\
0 & 0 & 0 & \epsilon_4 & 0 \\
0 & 0 & 0 & \epsilon_5 & 0 \\
0 & \epsilon_6 & 0 & 0 & 0 \\
\end{array} \right) \cdot x = \begin{pmatrix} 1 & \epsilon_1x_1 & \epsilon_1\epsilon_2x_5 & \epsilon_1\epsilon_2\epsilon_3x_8 & \epsilon_1\epsilon_2\epsilon_3\epsilon_4x_{10} \\ 0 & 1 & \epsilon_2x_2 & \epsilon_2\epsilon_3x_6 & \epsilon_2\epsilon_3\epsilon_4x_9 \\ 0 & 0 & 1 & \epsilon_3x_3 & \epsilon_3\epsilon_4x_7 \\ 0 & 0 & 0 & 1 & \epsilon_4x_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},
\]

where

\[
\begin{align*}
x_8' &= \epsilon_1\epsilon_2\epsilon_3(x_2x_3x_4 - x_2x_7 - x_4x_6 + x_9) \\
x_9' &= \epsilon_2\epsilon_3\epsilon_4(x_1x_2x_3 - x_1x_6 - x_3x_5 + x_8) \\
x_{10}' &= \epsilon_1\epsilon_2\epsilon_3\epsilon_4(x_1x_2x_3x_4 - x_1x_2x_7 - x_1x_4x_6 - x_3x_4x_5 \\
&\quad + x_1x_9 + x_4x_8 + x_5x_7 - x_{10}).
\end{align*}
\]
We introduce notation \( \text{diag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1}) \), and \( \text{adiag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1}) \):

\[
\begin{pmatrix}
\epsilon_1 & 0 & \cdots & 0 \\
0 & \epsilon_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \epsilon_{m-1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & \cdots & 0 & \epsilon_1 \\
0 & \cdots & \epsilon_2 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\epsilon_{m-1} & \cdots & 0 & 0
\end{pmatrix}
\]

Also, we introduce \( Z[z] \):

\[
Z[z] = \begin{pmatrix}
1 & 0 & \cdots & z \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix} \in N_m \subset GL(m, \mathbb{R}),
\]

the central element of \( N_m \) with the entry \( z \).

**Lemma 4.1** The matrices in \( K \subset GL(m - 1, \mathbb{Z}) \) act on \( Z[z] \) by

\[
\text{diag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1}) \cdot Z[z] = Z[(\epsilon_1 \epsilon_2 \cdots \epsilon_{m-1}) z],
\]

\[
\text{adiag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1}) \cdot Z[z] = Z[(-1)^m (\epsilon_1 \epsilon_2 \cdots \epsilon_{m-1}) z].
\]

For \( m = 2n \), \( \det(\text{adiag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1})) = (-1)^{n+1}(\epsilon_1 \epsilon_2 \cdots \epsilon_{m-1}) \).

**Proof** Let \( A \in K \). By Lemma 3.2, \( A \) induces an automorphism \( \tilde{A} \in \text{Aut}(\Gamma) \subset \text{Aut}(\mathcal{N}) = \text{Aut}(\mathfrak{M}) \). Since the center of \( \mathfrak{M} \) is one-dimensional and is generated by \( e_{1,m} \), it suffices to compute \( \tilde{A}(e_{1,m}) = \lambda e_{1,m} \). For simplicity, we consider only the case where \( A = \text{adiag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_{m-1}) \). Recall that

\[
e_{1,m} = [\cdots[[e_{1,2}, e_{2,3}], e_{3,4}], \cdots, e_{m-1,m}] .
\]

Applying \( \tilde{A} \) on both sides, we obtain

\[
\tilde{A}(e_{1,m}) = [\cdots[[A(e_{1,2}), A(e_{2,3})], A(e_{3,4})], \cdots, A(e_{m-1,m})]
\]

\[
= [\cdots[[\epsilon_{m-1} e_{m-1,m}, \epsilon_{m-2} e_{m-2,m-1}], \epsilon_{m-3} e_{m-3,m-2}], \cdots, \epsilon_1 e_{1,2}]
\]

\[
= \epsilon_1 \epsilon_2 \cdots \epsilon_{m-1} \cdot [\cdots[[e_{m-1,m}, e_{m-2,m-1}], e_{m-3,m-2}], \cdots, e_{1,2}]
\]

\[
= (-1)^m \epsilon_1 \epsilon_2 \cdots \epsilon_{m-1} [\cdots[[e_{1,2}, e_{2,3}], e_{3,4}], \cdots, e_{m-1,m}] \quad \text{(by (3.1))}
\]

\[
= (-1)^m \epsilon_1 \epsilon_2 \cdots \epsilon_{m-1} e_{1,m} .
\]

\( \square \)

**Lemma 4.2** For \( m \geq 4 \), every automorphism in \( K \) maps the lattice \( \Gamma_m \) onto itself.

By Theorem 2.1 (A), an almost crystallographic group \( \Pi \subset \mathcal{N} \times K \) for which \( \Gamma \) is the discrete nil-radical is generated by \( \Gamma \) together with a finitely many elements of \( \mathcal{N} \times K \): \( \Pi = \langle \Gamma, (a_1, A_1), \cdots, (a_k, A_k) \rangle \). Since \( \Gamma \) is a normal subgroup of \( \Pi \), we must have that

\[
(a_i, A_i)\Gamma(a_i, A_i)^{-1} = \Gamma, \text{ for all } i.
\]
For any \( g \in \Gamma \) and \( (a, A) \in \mathcal{N} \times C \),

\[
(a, A)(g, I)(a, A)^{-1} = (a \cdot A(g), A^{-1}(a^{-1}), A^{-1}) = (a \cdot A(g) \cdot a^{-1}, I).
\]

This shows that \((a, A)\) normalizes \( \Gamma \) if and only if \( \mu(a) \circ A \) maps \( \Gamma \) onto itself, where \( \mu(a) \) is conjugation by \( a \). In particular, if \( A \in C \subset \text{Aut}(\mathcal{N}) \) preserves \( \Gamma \) (this is the case of \( m \geq 4 \)) then \( a \in \mathcal{N} \) should normalize \( \Gamma \). Consequently the following result is crucial in our discussion.

**Lemma 4.3** The normalizer of \( \Gamma \) in \( \mathcal{N} \) is the group \((\text{Center of } \mathcal{N}) \cdot \Gamma\). That is,

\[
\mathcal{N}_\mathcal{N}(\Gamma) = \mathcal{Z}(\mathcal{N}) \cdot \Gamma \cong \mathbb{R} \cdot \Gamma.
\]

**Proof** Clearly, \( \mathcal{Z}(\mathcal{N}) \) is 1-dimensional, consisting of all matrices, the identity matrix \( I_m \) with the \((1, m)\)-entry replaced by any real.

It is clear that the subgroup \( \mathcal{Z}(\mathcal{N}) \cdot \Gamma \) normalizes \( \Gamma \). For the reverse inclusion, we will use induction on \( m \). If \( m = 2 \), then \( \mathcal{N} \cong \mathbb{R} \) is abelian and the result is trivial. Assume \( m \geq 3 \).

For \( x \in \mathcal{N} \), we write \( x \) as

\[
x = \begin{pmatrix} \bar{x}_{m-1} & \bar{x} \\ \bar{x} & 1 \end{pmatrix}.
\]

Assume \( x \in \mathcal{N} \) normalizes \( \Gamma \). Then for any \( g \in \Gamma \), we have

\[
\begin{pmatrix} \bar{x} & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{g} & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \bar{x}g & x + \bar{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x}^{-1} & -\bar{x}^{-1} \\ 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} \bar{x}g & x + \bar{x} \\ 0 & 1 \end{pmatrix} \in \Gamma.
\]

Hence \( \bar{x} \in \mathcal{N}_{m-1} \) normalizes \( \Gamma_{m-1} \), and the column vector \(-\bar{x}g\bar{x}^{-1}x + \bar{x}g + x\) is integral for all \( g \in \Gamma \). By induction hypothesis, we can write \( \bar{x} = \bar{z}h \) where \( h \in \Gamma_{m-1} \), and \( \bar{z} \in \mathcal{Z}(\mathcal{N}_{m-1}) \). Let’s say \( \bar{z} = \mathcal{Z}(\mathcal{N}_{m-1}) \). We claim that \( c = 0 \) so that \( \bar{z} = I_{m-1} \).

Let \( e_{i,j} \in \mathbb{N} \) \((i < j)\) be the matrix whose whose entries are all zero, except for the \((i, j)\)-entry which is 1. Choose \( g = I_m + e_{m-1,m} \in \Gamma \). Then \( \bar{g} = I_{m-1} \) and \( g = (0, 0, \cdots, 0, 1)^t \in \mathbb{Z}^{m-1} \). Then

\[
\begin{pmatrix} \bar{x} & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{g} & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \bar{x} & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x} & x \\ 0 & 1 \end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix} \bar{x} & x + x' \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} \bar{x} & x \\ 0 & 1 \end{pmatrix} \right)^{-1} \quad (x' = (m-1)-\text{st column of } \bar{x})
\]

\[
= \left( \begin{pmatrix} \bar{x} & x' \\ 0 & 1 \end{pmatrix} \right) + \left( \begin{pmatrix} \bar{x} & x \\ 0 & 1 \end{pmatrix} \right)^{-1}
\]

\[
= \begin{pmatrix} \bar{x} & x' \\ 0 & 1 \end{pmatrix} + \left( \begin{pmatrix} \bar{x} & x \\ 0 & 1 \end{pmatrix} \right)^{-1}
\]

\[
= \begin{pmatrix} \bar{x} & x' \\ 0 & 1 \end{pmatrix} + I
\]
Now we see that the \((1, m)\)-entry of the right hand side is the \((1, m - 1)\)-entry of \(\bar{x}\) which is \(x_{1,m-1}\). Since \(xgx^{-1} \in \Gamma\), this entry must be an integer, say \(k\). Let \(J = I_{m-1} + ke_{1,m-1}\). We can rewrite \(\bar{x}\) as \(\tilde{z} = (\tilde{z} J^{-1}) (Jh) \in \mathcal{Z}(N_{m-1}) \cdot \Gamma_{m-1}\). Notice that our new \(\tilde{z}\) (which is now \(\tilde{z} J^{-1}\)) has \((1, m - 1)\)-entry 0, and hence is equal to \(I_{m-1}\). Thus we may assume \(\tilde{z} = I_{m-1}\).

Then \(\tilde{z}h = I_{m-1}h = h\), and

\[
\begin{pmatrix}
\bar{x} \cdot x \\
0 \cdot 1
\end{pmatrix}
= \begin{pmatrix}
I \cdot x \\
0 \cdot 1
\end{pmatrix} = \begin{pmatrix}
h \cdot x \\
0 \cdot 1
\end{pmatrix}.
\]

Since the product \(\begin{pmatrix}
I \cdot x \\
0 \cdot 1
\end{pmatrix} \begin{pmatrix}
h \cdot 0 \\
0 \cdot 1
\end{pmatrix}\) normalizes \(\Gamma\), and \(\begin{pmatrix}
h \cdot 0 \\
0 \cdot 1
\end{pmatrix} \in \Gamma\), \(\begin{pmatrix}
I \cdot x \\
0 \cdot 1
\end{pmatrix}\) must normalize \(\Gamma\). Therefore,

\[
\begin{pmatrix}
I \cdot x \\
0 \cdot 1
\end{pmatrix} \begin{pmatrix}
g \cdot x \\
0 \cdot 1
\end{pmatrix}^{-1} \begin{pmatrix}
I \cdot x \\
0 \cdot 1
\end{pmatrix} = \begin{pmatrix}
g \cdot -gx + g + x \\
0 \cdot 1
\end{pmatrix} \in \Gamma.
\]

Thus

\[
-\tilde{g}x + g + x = (I - \tilde{g})x + g \in \mathbb{Z}^{m-1}
\]

for all \(\tilde{g} \in \Gamma_{m-1}\) and \(g \in \mathbb{Z}^{m-1}\). This is equivalent to

\[
(I - \tilde{g})x \in \mathbb{Z}^{m-1}
\]

for all \(\tilde{g} \in \Gamma_{m-1}\). Since \((I - \tilde{g})\) takes all possible upper-triangular integral matrices with diagonal entries all 0, the above condition readily implies

\[
x = (r, n_2, n_3, \ldots, n_{m-1})', \; r \in \mathbb{R}, \; n_i \in \mathbb{Z}.
\]

Let

\[
x = r + n = (r, 0, 0, \ldots, 0)' + (0, n_2, n_3, \ldots, n_{m-1})'.
\]

Then

\[
x = \begin{pmatrix}
\bar{x} \\
0 \cdot 1
\end{pmatrix} = \begin{pmatrix}
h \cdot x \\
0 \cdot 1
\end{pmatrix} = \begin{pmatrix}
h \cdot r + n \\
0 \cdot 1
\end{pmatrix} = \begin{pmatrix}
I \cdot r \\
0 \cdot 1
\end{pmatrix} \begin{pmatrix}
h \cdot n \\
0 \cdot 1
\end{pmatrix} \in \mathcal{Z}(N_m) \cdot \Gamma.
\]

**Proposition 4.4** Let \(m \geq 3\) and let \(A(\neq I) \in K \subset \text{GL}(m - 1, \mathbb{Z})\) be a diagonal matrix. There is no torsion free almost crystallographic group of \(N\), with discrete nil-radical \(\Gamma\), and with holonomy group containing \(A\).
Proof Let $A = \text{diag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_d)$. It acts on $\mathcal{Z}[a] / \Gamma_1$ by

$$A : \mathcal{Z}[a] \to \mathcal{Z}[\prod \epsilon_j a].$$

If $\prod \epsilon_j = -1$, then

$$A(\mathcal{Z}[a]) = \mathcal{Z}[-a].$$

This immediately implies that $\alpha^2 = (a, A)^2 = (e, I)$, for any choice of $a \in \mathcal{Z}(\mathcal{N})$, so that the group generated by $\Gamma$ and $\alpha = (a, A)$ has a torsion. Therefore, we may assume now that $\prod \epsilon_j = +1$.

Suppose the $p$-th entry is the first diagonal entry of $A$ which is $-1$. Then $\prod_{j=1}^p \epsilon_j = -1$, and then $\prod_{j=p+1}^{m-1} \epsilon_j = -1$. Now $\{E_{i, p+1}, E_{p+1, i}, E_{1, i}\}$ generates a subgroup $\Gamma' \mathcal{N}$ of $\Gamma$ isomorphic to the standard lattice $\Gamma_3$ of the Heisenberg group $\mathcal{N}$; that is,

$$[E_{i, p+1}, E_{p+1, i}] = E_{1, i},$$

$$[E_{i, p+1}, E_{1, i}] = 1,$$

$$[E_{p+1, i}, E_{1, i}] = 1.$$

Moreover,

$$A(E_{1, p+1}) = A([\cdots [E_{1, 2}, E_{2, 3}, E_{3, 4}], \cdots, E_{p, p+1}])$$

$$= [\cdots [E_{1, 2}^{\epsilon_{1, 2}}, E_{2, 3}^{\epsilon_{2, 3}}, \cdots, E_{p, p+1}^{\epsilon_{p, p+1}}]$$

$$= \cdots = E_{1, p+1}^{\epsilon_{1, p+1}} = E_{1, p+1}^{-1},$$

$$A(E_{p+1, i}) = E_{p+1, i}^{\epsilon_{p+1, i}} = E_{p+1, i}^{-1},$$

$$A(E_{1, i}) = E_{1, i}^{\epsilon_{1, i}} = E_{1, i}^{-1}.$$
Proof If there is a crystallographic group $\Pi$ of $\mathcal{N}$ whose holonomy group contains $A$, then its holonomy will contain $A^2$. But $A^2$ is diagonal and $A^2 \neq I$ if $A$ is not symmetric. Thus by Proposition 4.4, the group $\Pi$ must contain a torsion.

**Proposition 4.6** For $m \geq 5$ odd, there is no torsion free almost crystallographic group of $\mathcal{N}$ with discrete nil-radical $\Gamma$ and with non-trivial holonomy group.

Proof Let $A = \text{diag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_d)$, $d = m - 1$. If $m$ is odd, then $d + 1 = (m - 1) + 1 = m$ is odd, and
\[
A \cdot \mathbb{Z}[z] = \mathbb{Z}[-1]^{d+1}(\epsilon_1 \epsilon_2 \cdots \epsilon_d) z] = \mathbb{Z}[-(\epsilon_1 \epsilon_2 \cdots \epsilon_d) z].
\]
By Proposition 4.5, we may assume that $A$ is symmetric. Then $\prod \epsilon_i = +1$ and,
\[
A \cdot \mathbb{Z}[z] = \mathbb{Z}[-z].
\]
Now notice that $A$ has order 2. Therefore,
\[
(\mathbb{Z}[z], A)^2 = (\mathbb{Z}[z] \cdot A(\mathbb{Z}[z]), I) = (\mathbb{Z}[z - z], I) = (e, I).
\]
Consequently, $(\mathbb{Z}[z], A)$ is a torsion for any choice of $z$. □

Remark that by the mapping (4.1), Lemma 4.1 is true even when $m = 3$. Hence Propositions 4.4, 4.5, and 4.6 are true when $m = 3$. Thus we have:

**Corollary 4.7** Let $A(\neq I_2) \in K \subset \text{GL}(2, \mathbb{Z})$ (diagonal or anti-diagonal). Then there is no torsion free almost crystallographic group of $\mathcal{N}_3$ with discrete nil-radical $\Gamma_3$ and with holonomy group containing $A$.

5 Bieberbach groups of $\mathcal{N}_m$ ($m \geq 4$)

For our $\mathcal{N}$, $\Gamma \backslash \mathcal{N}$ will be our “standard” nilmanifold. A covering of an aspherical manifold $\tilde{M} \to M$ is called an essential covering if no element of the deck transformation group is homotopic to the identity.

We intend to classify all infra-nilmanifolds which are essentially covered by $\Gamma \backslash \mathcal{N}$. This is the same as classifying torsion free cocompact subgroups
\[
\Pi \subset \mathcal{N} \rtimes C,
\]
(where $C \subset \text{Aut}(\mathcal{N})$ is a compact subgroup), which contain $\Gamma$ as a discrete nil-radical. Recall that a maximal compact subgroup
\[
K = \mathbb{Z}_2^{m-1} \rtimes \mathbb{Z}_2 \subset \text{Aut}(\mathcal{N}).
\]

had been found. We have a complete classification of all Bieberbach groups of $\mathcal{N}_m$ ($m \geq 4$) containing the standard lattice consisting of matrices with integer entries as the discrete nil-radical.

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Theorem 5.1 (Classification) For odd $m \geq 4$, there is no infra-nilmanifold which is essentially covered by $\Gamma_m \backslash N_m$.

For $m = 2n \geq 4$, there is a unique infra-nilmanifold which is essentially covered by the nilmanifold $\Gamma_m \backslash N_m$. This manifold has the covering group $\mathbb{Z}_2$ generated by $\alpha = (a, J) \in N \times K$, where $a = \mathbb{Z}\left[\frac{1}{2}\right]$ and

$$J = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix} \in \text{GL}(m - 1, \mathbb{Z}).$$

When $m = 2n$, note $\alpha^2 = (\mathbb{Z}[1], I) \in \Gamma$, where $\mathbb{Z}[1] \in \mathcal{Z}(\Gamma_m)$, and this yields a 2-fold covering:

$$\Gamma_m \backslash N_m \xrightarrow{\mathbb{Z}_2} \text{(infra-nilmanifold)}.$$ 

The resulting infra-nilmanifold is orientable/non-orientable depending on whether $(n - 1)$ is even/odd.

Proof By Proposition 4.6, for $m \geq 5$ odd, there is no torsion free almost crystallographic group of $N$ with discrete nil-radical $\Gamma$ and with non-trivial holonomy group. Assume $m = 2n$.

Let $A \in K$ be a symmetric and anti-diagonal matrix of size $m - 1$. Note that $A$ maps $\Gamma$ onto itself. Therefore, for any $\alpha = (a, A)$ to conjugate $\Gamma$ onto itself, by Lemma 4.3, it is necessary and sufficient that $a \in \mathbb{Z}(N)$. Let $a = \mathbb{Z}[z]$.

Since $A$ is symmetric, $A^2 = I$. By Lemma 4.1, $\alpha^2 = (\mathbb{Z}[z + \epsilon z], I)$, where $\epsilon = \prod \epsilon_i$. If $\epsilon = -1$ then $\alpha$ is a torsion. Hence $\epsilon = 1$. Then $\alpha^2 = (\mathbb{Z}[2z], I)$. For this to be in $\Gamma$, $2z \equiv 0 \mod \mathbb{Z}$. Thus the only non-trivial $z$ is $\frac{1}{2}$, and $(a, A)^2 = (e_d, I) = t_d$, where $d = \frac{1}{2}m(m - 1)$.

With the most general element of $\Gamma$

$$s = \begin{pmatrix}
1 & n_1 & n_m & * & * & n_d \\
0 & 1 & n_2 & n_{m+1} & * & * \\
0 & 0 & 1 & n_3 & \ddots & * \\
0 & 0 & 0 & 1 & * & n_\ell \\
0 & 0 & 0 & 0 & 1 & n_{m-1} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \in \text{GL}(m - 1, \mathbb{Z}),$$

denote the matrix $s \cdot \mathbb{Z}[z]$ by $t$. Then

$$(s, I) \cdot \alpha = (s, I)(a, A) = (t, A).$$

We try to solve the equation

$$(t, A)^2 = (e, I)$$

for $n_i$’s.

First, we assume that all of $A$’s entries are $+1$:

$$A = \begin{pmatrix}
0 & 0 & * & 0 & 1 \\
0 & 0 & * & 1 & 0 \\
* & * & \ddots & * & * \\
0 & 1 & * & 0 & 0 \\
1 & 0 & * & 0 & 0
\end{pmatrix} \in K.$$
Since $A$ maps
\[
\{n_1, n_2, \cdots, n_{m-1}\} \longrightarrow \{n_{m-1}, \cdots, n_2, n_1\},
\]
\[
t \cdot A(t) = \begin{pmatrix}
1 & n_1 + n_{m-1} & * & * & * \\
0 & 1 & n_2 + n_{m-2} & * & * \\
0 & 0 & 1 & 0 & * \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

The matrix $t \cdot A(t)$ has its entries one above the diagonal
\[
\{n_1 + n_{m-1}, n_2 + n_{m-2}, \cdots, n_{m-1} + n_1\}.
\]

For $(t, A)^2 = (e, I)$, we must have all these entries 0. Then the result is of the form
\[
t \cdot A(t) = \begin{pmatrix}
1 & 0 & n_p - n_q & * & * & * \\
0 & 1 & 0 & n_{p+1} - n_{q-1} & * & * \\
0 & 0 & 1 & 0 & * \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

The entries two above the diagonal are
\[
\{n_p - n_q, n_{p+1} - n_{q-1}, \cdots, n_q - n_p\}, \ p = m, \ q = (m-1) + (m-2).
\]
For $(t, A)^2 = (e, I)$, we must have all these entries 0.

Similarly, the entries three above the diagonal are
\[
\{n_p + n_q, n_{p+1} + n_{q-1}, \cdots, n_q + n_p\}, \ p = 2m - 2, \ q = 3m - 6.
\]
For $(t, A)^2 = (e, I)$, we must have all these entries 0.

In general, the entries $k$ above the diagonal are
\[
\{n_p \pm n_q, n_{p+1} \pm n_{q-1}, \cdots, n_q \pm n_p\}, \ p = \sum_{i=1}^{k-1} (m-i) + 1, \ q = \sum_{i=1}^{k} (m-i),
\]
where the signature is determined to be $(-1)^{k-1}$.

For $(t, A)^2 = (e, I)$, we must have all these entries 0.

After $(m-2)$ steps, we finally obtain
\[
t \cdot A(t) = Z[2z + 2n_d].
\]
Recall that $z \equiv \frac{1}{2} \mod \mathbb{Z}$, and $n_d \in \mathbb{Z}$. These imply that
\[
2z + 2n_d \neq 0.
\]
Thus, the equation $((s, I) \cdot \alpha)^2 = (e, I)$ does not have an integral solution for
\[
\{n_1, n_2, \cdots, n_d\}. \text{ We conclude that the group } \langle \Gamma, \alpha \rangle \text{ is torsion free.}
\]

Let $B \in K$ be an arbitrary element. Then $A(t) - B(t) \in \text{gl}(m, 2\mathbb{Z})$. This implies, after imposing the conditions layer by layer, the $(1, m)$-entry is still the same as before modulo 2, and we get the same result:
\[
2z + 2n_d \neq 0.
\]
We conclude that the group \( \langle \Gamma, (a, A) \rangle \) is torsion free for any choice of \( A \in K \), where \( A \) is one of the following:

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & \epsilon_1 \\
0 & 0 & \cdots & \epsilon_2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \epsilon_{m-2} & \cdots & 0 & 0 \\
\epsilon_{m-1} & 0 & \cdots & 0 & 0
\end{pmatrix}
\in \text{GL}(m - 1, \mathbb{Z}),
\]

\( \epsilon_j = \pm 1 \), where the signs are taken in such a way that the number of \(-1\)'s is even, and the matrix is symmetric.

Finally it remains to show that such a group \( \langle \Gamma, (a, A) \rangle \) is isomorphic to \( \langle \Gamma, (a, J) \rangle \).

Let

\[
X = \begin{pmatrix}
0 & 0 & \cdots & 0 & \delta_1 \\
0 & 0 & \cdots & \delta_2 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \delta_{m-2} & \cdots & 0 & 0 \\
\delta_{m-1} & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

where \( \delta_i = 1 \) for \( 1 \leq i \leq n - 1 \) and

\[
\delta_{m-i} = \begin{cases} 
1 & \text{if } \epsilon_i(= \epsilon_{m-i}) = 1; \\
-1 & \text{if } \epsilon_i(= \epsilon_{m-i}) = -1.
\end{cases}
\]

For example, if \( A = \text{diag}(1, -1, 1, -1, 1) \) then \( X = \text{diag}(1, 1, 1, -1, 1) \).

Then \( X \in K = (\mathbb{Z}_2)^{m-1} \times \mathbb{Z}_2 \subset \text{Aut}(\Gamma_m) \subset \text{Aut}(\mathcal{N}_m) \), and

\[
XAX^{-1} = J.
\]

Hence \( (e, X) \in \text{Aff}(\mathcal{N}_m) \) and it can be seen that

\[
(e, X)\langle \Gamma, (a, A) \rangle (e, X)^{-1} = \langle \Gamma, (a^{\pm 1}, J) \rangle.
\]

In fact,

\[
(e, X)\Gamma(e, X)^{-1} = e \cdot X(\Gamma) \cdot e = X(\Gamma) = \Gamma,
\]

\[
(e, X)(a, A)(e, X)^{-1} = (X(a), XAX^{-1}) = (a^{\pm 1}, J).
\]

Here, the last identity follows from Lemma 4.1:

\[
X(a) = \text{diag}({\delta_1, \cdots, \delta_{m-1}}) \cdot \mathbb{Z}[\frac{1}{2}] = \mathbb{Z}[(-1)^m({\delta_1, \cdots, \delta_{m-1}})] = a^{\pm 1}.
\]

Since \( (a^{-1}, J) t_d = (a, J) \),

\[
\langle \Gamma, (a^{-1}, J) \rangle = \langle \Gamma, (a, J) \rangle.
\]

Therefore, every \( \langle \Gamma, (a, A) \rangle \) is a conjugate of \( \langle \Gamma, (a, J) \rangle \). \( \square \)

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