A bound on the mixing rate of 2D perfect fluid flows

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Abstract
Using the $H^{-1}$ norm as a measure of mixing, we prove that 2D Euler flows on the torus mix passive scalars at most exponentially. The mixing rate is bounded linearly by the BMO norm of the vorticity (and thus by its $L^\infty$ norm). We also give an analogous bound on the growth rate of scalar gradients.

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1. Introduction

Let $v(x, t)$ be the velocity field of an incompressible inviscid fluid flow, governed by the Euler equation

$$\partial_t v + v \cdot \nabla v + \nabla p = 0 \quad \text{with} \quad \nabla \cdot v = 0. \quad (1.1)$$

For concreteness, we take $x \in \mathcal{M} := [0, 1]^d$ with periodic boundary conditions; for now, we take $d = 2$ or $3$. With no loss of generality, we assume that $v$ has zero integral over the domain $\mathcal{M}$. For our purpose here we assume that the initial data $v(\cdot, 0)$ is smooth, say, $C^3$; see, e.g., [3] for a discussion of cases with more general velocity. The evolution of a passive scalar $\theta(x, t)$ in this flow is then governed by

$$\partial_t \theta + v \cdot \nabla \theta = 0. \quad (1.2)$$

As with $v$, we assume that the integral of $\theta$ over $\mathcal{M}$ vanishes.

Assuming suitable boundedness of $v$ for all $t \geq 0$, one can show from (1.2) that $|\theta|_{L^p}$ is constant in time for all $p \in [1, \infty]$. It is clear that there are many solutions of (1.1), e.g., shear flows and (nearly) stable ones, for which the scalar $\theta$, with certain non-uniform initial data, will undergo little time evolution. If the solution of (1.1) is chaotic (in the sense of $v$ generating chaotic trajectories, as is often the case in many interesting situations), however, one expects that $\theta$ will become increasingly ‘mixed’ in time; mathematically, one expects that $\theta(\cdot, t)$ will converge weakly to $0$ as $t \to \infty$ in some suitable space. Of significant mathematical and physical interest is the rate of this mixing or convergence.
In [10], the authors introduced a coarse-grained ‘mix norm’ which they showed to be equivalent to the $H^{-1/2}$ norm, defined formally as

$$|w|_{H^{-1/2}}^2 := \sum_{k \neq 0} |k|^2 |\hat{w}_k|^2,$$

(1.3)

where $\hat{w}_k$ are the Fourier coefficients of $w$. (This is a seminorm in general, but is a proper norm when $w$ has zero integral over $M$, which will always be the case in this work.) They argued that this $H^{-1/2}$ norm is a good measure of mixing as it is weighted towards larger-scale features at the expense of fine details.

Subsequently, [9] proposed the use of the more convenient $H^{-1}$ norm to measure the mixing rate, having proved that for any $s < 0$ and for any $\theta(\cdot, t)$ with zero integral bounded uniformly in $L^2$, the weak convergence $\theta(\cdot, t) \rightharpoonup 0$ in $L^2$ as $t \to \infty$ is equivalent to $|\theta(\cdot, t)|_{H^s} \to 0$.

In this work, we follow their use of the $H^{-1}$ norm as our measure of mixing. Putting $\varphi := \Delta^{-1} \theta$, defined uniquely by imposing zero integral over $M$, and multiplying (1.2) by $-\varphi$ in $L^2$, we have

$$\frac{1}{2} \frac{d}{dt} |\nabla \varphi|_{L^2}^2 - (v \cdot \nabla \Delta \varphi, \varphi)_{L^2} = 0. \tag{1.4}$$

Assuming sufficient smoothness, we integrate the nonlinear term by parts

$$- (v \cdot \nabla \Delta \varphi, \varphi)_{L^2} = \sum_j (v \cdot \nabla \partial_j \varphi, \partial_j \varphi)_{L^2} + \sum_j ((\partial_j v) \cdot \nabla \varphi, \partial_j \varphi)_{L^2}$$

$$= \sum_j ((\partial_j v) \cdot \nabla \varphi, \partial_j \varphi)_{L^2}. \tag{1.5}$$

Using the standard estimate

$$|(\partial_j v) \cdot \nabla \varphi, \partial_j \varphi)_{L^2} | \leq c |\nabla v|_{L^\infty} |\nabla \varphi|_{L^2}^2,$$

(1.6)

we have

$$\frac{d}{dt} |\nabla \varphi|_{L^2}^2 \geq -c_1 |\nabla v|_{L^\infty} |\nabla \varphi|_{L^2}^2. \tag{1.7}$$

Without assuming that $v$ is a solution of (1.1), the possibility of perfect mixing in a finite time $T$ was not ruled out in [9], where lower bounds for perfect mixing time were given for physically relevant cases; here perfect mixing is defined by $|\theta(\cdot, T)|_{H^s} = |\nabla \varphi(\cdot, T)|_{L^2} = 0$. They did, however, point out that if $|\nabla v(\cdot, t)|_{L^\infty}$ is uniformly bounded, this will give a bound on the mixing rate.

When $v$ is the solution of (1.1), however, one can use the argument of [1, 6] to rule out perfect mixing as long as the solution remains regular. Unlike the results in the following sections, this argument works both in 2D and 3D. We fix $s \geq 3$ and assume that $v(\cdot, 0) \in H^s$. By Sobolev embedding, this implies $\nabla v(\cdot, 0) \in L^\infty$. Using the differential inequality

$$\frac{d}{dt} |v|_{H^s}^2 \leq c_2 |\nabla v|_{L^\infty} |v|_{H^s}^2,$$

(1.8)

we have

$$|v(\cdot, t)|_{H^s}^2 \leq |v(\cdot, 0)|_{H^s}^2 \exp \left( c_2 \int_0^t |\nabla v(\cdot, t')|_{L^\infty} \, dt' \right). \tag{1.9}$$

Now (1.7) can also be integrated to give

$$|\nabla \varphi(\cdot, t)|_{L^2}^2 \geq |\nabla \varphi(\cdot, 0)|_{L^2}^2 \exp \left( -c_1 \int_0^t |\nabla v(\cdot, t')|_{L^\infty} \, dt' \right). \tag{1.10}$$
In the 3D case [1], one can bound $|\nabla v|_{L^\infty}$ essentially by $|\text{curl } v|_{L^\infty} \log |v|_{H^1}$, to show that the integral in (1.9) is finite for all $t \geq 0$ such that $\text{curl } v \in L^1([0, t]; L^\infty)$. In other words, no perfect mixing is possible before the blow-up time if the latter is finite. In the 2D case, it is shown in [6] that $v(\cdot, t) \in H^s$ for all $t \geq 0$, implying that the integral in (1.10) is finite and $|\nabla \psi(\cdot, t)|_{L^2} > 0$ for all $t \geq 0$. Therefore, no perfect mixing is possible in finite time when $v$ is a solution of the 2D Euler equation (1.1).

2. A uniform bound on the mixing rate

The inequality (1.10) does not tell us much about the mixing rate since we do not have a bound uniform in $t$ for $|\nabla v|_{L^\infty}$. As noted above, [1] bounded $|\nabla v|_{L^\infty}$ essentially by $|\text{curl } v|_{L^\infty} \log |v|_{H^1}$, which is not bounded uniformly in $t$—in fact, one expects $|v(\cdot, t)|_{H^1}$ to grow without bound for 'generic' flows.

Working now in 2D, with $x = (x, y)$ and $v = (u, v)$, we take advantage of the special structure of the Jacobian to bound (1.6) by $|\nabla v|_{\text{BMO}}$ instead of $|\nabla v|_{L^\infty}$ and use a time-uniform bound on $|\nabla v|_{\text{BMO}}$ to obtain a bound on the mixing rate. To this end, we introduce the vorticity $\omega := \partial_x v - \partial_y u$, whose integral over $\mathcal{M}$ vanishes by Stokes' theorem, and the streamfunction $\psi := \Delta^{-1} \omega$ (defined uniquely by requiring that its integral over $\mathcal{M}$ vanishes), in terms of which $v = \nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi)$. Taking the curl of (1.1), we have

$$\partial_t \omega + \partial(\psi, \omega) = 0$$

(2.1)

where the Jacobian $\partial(\psi, \omega) := \nabla^\perp \psi \cdot \nabla \omega = \partial_x \psi \partial_y \omega - \partial_y \omega \partial_x \psi$. Using (1.5), we can write (1.4) as

$$\frac{1}{2} \frac{d}{dt} |\nabla \psi|_{L^2}^2 + \sum_j (\partial(\partial_j \psi, \partial_j \omega), \partial_j \psi)_{L^2} = 0.$$ 

(2.2)

For our purpose here, the following definition suffices; we refer the reader to [13] for more details. Let $\mathcal{N} \subset \mathbb{R}^2$. For $w \in L^1_{\text{loc}}(\mathcal{N})$ and any ball $B \subset \mathcal{N}$, let $w_B$ denote the average of $w$ in $B$,

$$w_B := \frac{1}{|B|} \int_B w(x) \, dx.$$ 

(2.3)

We say that $w \in \text{BMO}(\mathcal{N})$, the space of functions of bounded mean oscillations in $\mathcal{N}$, if

$$|w|_{\text{BMO}} := \sup_{B \subset \mathcal{N}} \frac{1}{|B|} \int_B |w(x) - w_B| \, dx < \infty.$$ 

(2.4)

We note that $| \cdot |_{\text{BMO}}$ as defined here is a seminorm (constants have BMO norm zero), but since we only deal with functions of zero average, (2.4) defines a proper norm for all relevant quantities. We also note that (2.4) implies

$$|w|_{\text{BMO}} \leq |w|_{L^\infty}.$$ 

(2.5)

Our main result is the following:

**Theorem.** Let the passive scalar $\theta$ be the solution of (1.2) with $\theta(\cdot, 0) \in L^2(\mathcal{M})$ and $\nu$ that of the Euler equation (1.1) with initial vorticity $\omega(\cdot, 0) \in L^\infty(\mathcal{M})$. Then $\theta$ satisfies

$$|\theta(\cdot, t)|_{H^s}^2 \geq |\theta(\cdot, 0)|_{H^s}^2 \exp\left(-\lambda \int_0^t |\omega(\cdot, \tau)|_{\text{BMO}} \, d\tau\right)$$

(2.6)

for some constant $\lambda$ depending only on the domain $\mathcal{M}$. Moreover, the mixing rate is bounded by the sup-norm of the initial vorticity as

$$|\theta(\cdot, t)|_{H^1}^2 \geq |\theta(\cdot, 0)|_{H^1}^2 \exp\left(-\lambda |\omega(\cdot, 0)|_{L^\infty}\right).$$

(2.7)
We note that our result does not address the more difficult issue of the existence of exponentially mixing solutions of (1.1), or whether the bound (2.7) is attained even qualitatively. See [5] and references therein for further discussion.

**Proof.** Parts of this proof were inspired by [8]. As usual, \( c \) denotes a generic positive constant whose value may differ each time the symbol appears. Assuming for now the following estimate for the Jacobian,

\[
|\partial(\xi, \varphi)|_{L^2} \leq c |\nabla \xi|_{BMO} |\nabla \varphi|_{L^2},
\]

we take \( \zeta = \partial_1 \psi \) and \( \xi = \partial_2 \psi \) in turn in (2.2) to obtain (see (1.7))

\[
\frac{d}{d\tau}|\nabla \varphi|_{L^2}^2 \geq -c |\nabla \psi|_{BMO} |\nabla \varphi|_{L^2}^2.
\]

This implies the analogue of (1.10) with \( BMO \) in place of \( L^\infty \). Unlike the \( L^\infty \) case where \( |\omega|_{L^\infty} \) does not bound \( |\nabla \psi|_{L^\infty} \), here we have

\[
|\nabla \psi|_{BMO} \leq c |\omega|_{BMO},
\]

(2.10)

whose proof will follow shortly. Using (2.10) in (2.9) and integrating gives us (2.6). Using (2.5) and the fact that \( |\omega(\cdot, t)|_{L^\infty} = |\omega(\cdot, 0)|_{L^\infty} \) gives us (2.7).

To prove (2.10), we start with the identity (see [12, p 59])

\[
\nabla \psi = \nabla \nabla^{1/2} |\omega|^{-1/2} \nabla \varphi = (R_x, R_y) \nabla \varphi,
\]

(2.11)

where \( R_x := \partial_x (-\Delta)^{-1/2} \) and analogously for \( R_y \). It is shown in [13, p 138] that the Riesz transforms \( (R_x, R_y) \) are bounded operators in the Hardy space \( h^1(\mathbb{R}^2) \), which is dual to \( BMO(\mathbb{R}^2) \). Boundedness of \( R := (R_x, R_y) \) in \( BMO(\mathbb{R}^2) \) then follows by duality. From the definition (2.4), it is clear that the norm in \( BMO(M) \) is identical to that for periodic functions in \( BMO(\mathbb{R}^2) \).

We now prove the Jacobian estimate (2.8). For bounded domains in \( \mathbb{R}^2 \) with Dirichlet boundary conditions, although not stated explicitly, the proof is essentially contained in [7]. Its extension to periodic boundary conditions is essentially done in [4]. We reproduce these proofs here (with minor modifications) for convenience.

Let \( M_1 := \bigcup_{\xi \in M_1} B_1(\xi) \) where \( B_1(\xi) \subset \mathbb{R}^2 \) is the ball of unit radius centred at \( \xi \). We first prove (2.8) for \( \xi \in H^1_0(M_1) \cap \{ \nabla \xi \in BMO(M_1) \} \) and \( \varphi \in H^1_0(M_1) \). Denoting the Fourier transform by a \( \hat{\cdot} \), we use the following result from [2, p 154].

Let \( \hat{\sigma} \) be in \( C^\infty(\mathbb{R}^2 \times \mathbb{R}^2 - \{0\}) \) satisfy

\[
|\hat{\omega}\xi_1|_0^\beta \hat{\sigma}(\xi, \eta) \leq C_{\sigma}(\xi|\xi| + |\eta|)^{-|\alpha|}|\eta| |
\]

∀ \( \xi, \eta \in \mathbb{R}^2 - \{0\} \)

(2.12)

for every multi-indices \( \alpha \) and \( \beta \), and \( \hat{\sigma}(0, \eta) = 0 \).

(2.13)

Then the bilinear operator

\[
\sigma(f, g)(x) := \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\xi \cdot \eta} \hat{\sigma}(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) \, d\eta \, d\xi
\]

(2.14)

is bounded in \( L^2(M_1) \) as

\[
|\sigma(f, g)|_{L^2} \leq c |f|_{BMO} |g|_{L^2}.
\]

(2.15)

For the Jacobian \( \partial(\xi, \varphi) \), we have \( \hat{\sigma}(\xi, \eta) = -\xi^\perp \cdot \eta \), so in order to satisfy (2.12)–(2.13) we split \( \hat{\sigma}(\xi, \varphi) \) as follows. Let \( \rho \in C^\infty \) be monotone increasing with \( \rho(t) = 0 \) for \( t \leq 1 \) and \( \rho(t) = 1 \) for \( t \geq 2 \), and let \( \hat{\sigma}(\xi, \varphi) = \hat{J}^\rho + \hat{J}^\ominus \) where

\[
J^\rho(x) = -\frac{1}{\pi^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\xi \cdot \eta} \rho(||\xi||) \xi^\perp \cdot \eta \hat{\varphi}(\eta) \, d\eta \, d\xi
\]

\[
J^\ominus(x) = -\frac{1}{\pi^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\xi \cdot \eta} [1 - \rho(||\xi||)] \xi^\perp \cdot \eta \hat{\varphi}(\eta) \, d\eta \, d\xi.
\]

(2.16)
Taking $\hat{\sigma}(\xi, \eta) = \rho(|\xi|)$, we have from (2.15)
$$
|J^>_{L^2}| \leq c \|\nabla \chi\|_{\text{BMO}} \|\nabla \psi\|_{L^2}.
$$
(2.17)

And taking
$$
\hat{\sigma}(\xi, \eta) = \frac{1 - \rho(|\xi|)}{1 + |\xi|},
$$
(2.18)

(2.15) gives us
$$
|J^<_{L^2}| \leq c \|\nabla \chi\|_{\text{BMO}} \|\nabla \psi\|_{L^2}.
$$
(2.19)

Having proved (2.8) in $M_1$ with Dirichlet boundary conditions, we now extend it to the periodic case, taking $\zeta \in H^1_{\text{per}}(\mathbb{R}^2) \cap \{\nabla \zeta \in \text{BMO}(\mathbb{R}^2)\}$ and $\varphi \in H^1_{\text{per}}(\mathbb{R}^2)$. Let $\varrho \in C^\infty(\mathbb{R}^2; [0, 1])$ with $\varrho = 1$ in $\mathcal{M}$ and $\varrho = 0$ in $\mathbb{R}^2 - M_1$. By the above, we have
$$
|\partial(\varrho \zeta, \varrho \varphi)|_{L^2(\mathcal{M}_1)} \leq C(M, M_1, \varrho) \|\nabla \zeta\|_{\text{BMO}(\mathcal{M}_1)} \|\nabla \varphi\|_{L^2(\mathcal{M}_1)}.
$$
(2.20)

By the periodicity of $\zeta$ and smoothness of $\varrho$, we have
$$
|\nabla (\varrho \zeta)|_{\text{BMO}(\mathcal{M}_1)} \leq C(M, M_1, \varrho) \|\nabla \zeta\|_{\text{BMO}(\mathcal{M})},
$$
(2.21)

and similarly
$$
|\nabla (\varrho \varphi)|_{L^2(M_1)} \leq C(M, M_1, \varrho) |\nabla \varphi|_{L^2(\mathcal{M})}.
$$
(2.22)

One then takes the infimum over $\varrho$ in (2.21) and (2.22) to remove the dependence of the constants on $\varrho$. The conclusion follows from these and
$$
|\partial(\zeta, \varphi)|_{L^2(\mathcal{M}_1)} = |\partial(\varrho \zeta, \varrho \varphi)|_{L^2(\mathcal{M}_1)} \leq |\partial(\varrho \zeta, \varrho \varphi)|_{L^2(\mathcal{M}_1)}.
$$
(2.23)

□

3. Further remarks

One can obtain an upper bound on the scalar gradient using the same method: multiplying (1.2) by $-\Delta \theta$ and estimating, we find (see (1.5) and (2.9))
$$
\frac{d}{dt} \|\nabla \theta\|_{L^2}^2 = -2 \sum_j ((\partial_j v) \cdot \nabla \theta, \partial_j \theta)_{L^2} \\
\leq C(M) \|\nabla v\|_{\text{BMO}} \|\nabla \theta\|_{L^2}^2.
$$
(3.1)

As before, since $\|\nabla v\|_{\text{BMO}} \leq c |\omega|_{L^\infty}$ and using the fact that the latter quantity is time-invariant, we have (see (2.7))
$$
\|\nabla \theta(\cdot, t)\|_{L^2}^2 \leq \|\nabla \theta(\cdot, 0)\|_{L^2}^2 \exp(t \lambda(M) |\omega(\cdot, 0)|_{L^\infty}).
$$
(3.2)

Since the vorticity equation (2.1), or equivalently,
$$
\partial_t \omega + v \cdot \nabla \omega = 0,
$$
(3.3)

is of the same form as (1.2), these bounds evidently apply equally well to $|\nabla \omega(\cdot, t)|_{L^2}$. For this active scalar, one expects $\nabla \omega$ to grow without bound; see, e.g., [11] for estimates in Hölder spaces.

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