THE BROWN-COMENETZ DUAL OF THE $K(2)$-LOCAL SPHERE AT THE PRIME 3

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Abstract. We calculate the homotopy type of the Brown-Comenetz dual $I_2$ of the $K(2)$-local sphere at the prime 3 and show that there is an equivalence in the $K(2)$-local category between $I_2$ and a smash product of the determinant twisted sphere and an exotic element $P$ in the Picard group. We give a characterization of $P$ as well. A secondary aim of the paper is to extend our library of calculations in the $K(2)$-local category.

1. Introduction

Because $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group, the functor

$$X \mapsto \text{Hom}(\pi_* X, \mathbb{Q}/\mathbb{Z})$$

defines a cohomology theory on spectra represented by a spectrum $I$; the Brown-Comenetz dual of $X$ is then the function spectrum $IX = F(X, I)$. In particular, $I$ itself is the Brown-Comenetz dual of the sphere spectrum. This duality on spectra was introduced in [3] and some of the basic properties are outlined there. Here we are interested in Brown-Comenetz duality for the $K(n)$-local category. There are a number of reasons to restrict to this category; for example, the work of Gross and Hopkins [10] indicates that in the $K(n)$-local category, Brown-Comenetz duality is a homotopical analog of Grothendieck-Serre duality.

In order to discuss our results we need a bit of notation. Fix a prime $p$. Let $K(n)$ be the $n$th Morava $K$-theory at that prime and let $E_n$ be the associated Lubin-Tate theory. Both theories are complex orientable; to be concrete, we specify that the formal group over $K(n)$ is the Honda formal group $\Gamma_n$ of height $n$, and the formal group over $(E_n)_*$ is a choice of a universal deformation of the Honda formal group. We write

$$(E_n)_* X \overset{\text{def}}{=} \pi_* L_{K(n)}(E_n \wedge X).$$

For all $X$, $(E_n)_* X$ is a twisted $\mathbb{G}_n$-module, where

$$\mathbb{G}_n = \text{Aut}(\Gamma_n) \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

is the big Morava stabilizer group. Indeed, by the Hopkins-Miller Theorem, $\mathbb{G}_n$ acts on the spectrum $E_n$ itself; this induces the action on homology. If $p$ and $n$ are understood we may also write $E$ for $E_n$ and $E_* X$ for $(E_n)_* X$. We write $L_{K(n)}$ for localization with respect to $K(n)$.

If $X$ is a $K(n)$-local spectrum, then $IX$ need not be $K(n)$-local. To address this, let $L_n$ be localization at $K(0) \vee \ldots \vee K(n)$ and define the $n$th monochromatic layer $M_n X$ of $X$ to be the fiber of $L_n X \to L_{n-1} X$. Then, for $K(n)$-local $X$, we define the Brown-Comenetz dual to be

$$I_n X = F(M_n X, I) = IM_n X.$$
This version of Brown-Comenetz duality was first laid out by Hopkins (see [19]). The basic properties are worked out in §10 of [14].

Define $I_n = I_n L_{K(n)} S^0$. One of the main results of [10] (see also [20]) asserts that

$$ (E_n)_* I_n \cong \Sigma^{n^2-n} (E_n)_* \langle \det \rangle $$

(1.1)

where $(E_n)_* \langle \det \rangle$ is the twisted $\mathbb{G}_n$-module obtained by twisting $(E_n)_* S^0$ by a determinant action. This action is explained in detail in Section 5.

In particular, $(E_n)_* I_n$ is a free module of rank 1 over $(E_n)_*$ and, hence, by the results of [19] and [13], $I_n$ is an element of the Picard group $\text{Pic}_n$ of invertible elements in the $K(n)$-local category. The formula (1.1) determines the homotopy type of $I_n$ for large primes or, indeed, for any prime for which $(E_n)_*(-)$ differentiates the elements of the Picard group. In this case, there is an equivalence in the $K(n)$-local category

$$ I_n \simeq \Sigma^{n^2-n} S^0 \langle \det \rangle $$

where $S^0 \langle \det \rangle$ is the determinant twisted sphere. The construction of $S^0 \langle \det \rangle$ is reviewed in Section 6; it has the property that $(E_n)_* S^0 \langle \det \rangle \cong (E_n)_* \langle \det \rangle$.

More information is needed when the $K(n)$-local Picard group has exotic elements. Let $\kappa_n$ be the subgroup of the $K(n)$-local Picard group with elements the weak equivalence classes of invertible spectra $X$ so that $(E_n)_* X \cong (E_n)_* S^0$ as twisted $\mathbb{G}_n$-modules. If $\kappa_n \neq 0$ the computation of $I_n$ is no longer a purely algebraic problem. For example, if $p = 2$ and $n = 1$ it is fairly simple to show that $I_1 \wedge V(0) \simeq \Sigma^{-2} L_{K(1)} V(0)$, where $V(0)$ is the mod 2 Moore spectrum. Since $S^0 \langle \det \rangle = S^2$ in this case, there must be some contribution from $\kappa_1$. Using the results of [12], one can show $I_1 = \Sigma^2 L_{K(1)} DQ$ where $L_{K(1)} DQ \in \kappa_1 \cong \mathbb{Z}/2$ is the generator. The spectrum $L_{K(1)} DQ$ is the $K(1)$-localization of the “dual question mark complex”; it is characterized by the fact that it is in $\kappa_1$ and $KO \wedge DQ \simeq \Sigma^4 KO$. This example is discussed in some detail in [13] §6.

A similar phenomenon occurs at $p = 3$ and $n = 2$, but it is harder to characterize the exotic element in $\kappa_2$. The group $\kappa_2$ has been calculated to be isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$ in [7]; we review that calculation in Section 5. Let $V(1)$ be the four cell complex obtained as the cofiber of the Adams map $\Sigma^4 V(0) \to V(0)$ of the mod 3 Moore spectrum. The spectrum $V(1)$ is a basic example of a type 2 complex. Our main results can be encapsulated in the following theorem. Here and below we will write $X \wedge Y$ for $L_{K(n)} (X \wedge Y)$ whenever we work in the $K(n)$-local category. This is the smash product internal to the $K(n)$-local category.

**Theorem 1.1.** Let $n = 2$ and $p = 3$. There is a unique spectrum $P \in \kappa_2$ so that $P \wedge V(1) \simeq \Sigma^{18} L_{K(2)} V(1)$. The Brown Comenetz dual $I_2$ of $L_{K(2)} S^0$ is given as

$$ I_2 \simeq S^2 \wedge S^0 \langle \det \rangle \wedge P. $$

Furthermore $I_2 \wedge V(1) \simeq \Sigma^{-22} L_{K(2)} V(1)$.

The calculation that $\pi_* I_2 \wedge V(1) \cong \pi_* \Sigma^{-22} V(1)$ was known to Mark Behrens (see [1], §1.8) and is the starting point of our argument. The spectrum $P$ is discussed in Theorem 5.5 below. The other parts of the proof of Theorem 1.1 are spread over Theorem 7.1 and Corollary 7.2.

Let $G_{24} \subseteq \mathbb{G}_2$ be a maximal finite subgroup containing an element of order 3. Then it is true that $E_2^{hG_{24}} \wedge P \simeq \Sigma^{18} E_2^{hG_{24}}$. Since $E_2^{hG_{24}}$ is periodic of period 72, this is analogous to the statement at $n = 1$ and $p = 2$ that $KO \wedge DQ \simeq \Sigma^4 KO$. However, this is not enough to characterize the spectrum $P$. In [7] we constructed non-trivial elements $Q \in \kappa_2$ so that $E_2^{G_{24}} \wedge Q \simeq E_2^{hG_{24}}$. If we fix one such $Q$, then it follows from the main theorem of [7] that
every element of $\kappa_2$ may be written $P^{a} \wedge Q^{b}$ with $(a, b) \in \mathbb{Z}/3 \times \mathbb{Z}/3$. It then follows from Equation (1.1) that we have an equivalence

\[(1.2) \quad I_2 = \Sigma^2 S^0(\det) \wedge P^a \wedge Q^b\]

for some $(a, b) \in \mathbb{Z}/3 \times \mathbb{Z}/3$. In Proposition 5.3 we show that $Q^{ab} \wedge V(1)$ has the homotopy groups of a suspension of $L_{K(2)}V(1)$ only if $b = 0$. In Theorem 6.4 we show that $S^0(\det) \wedge V(1) \simeq \Sigma^2 L_{K(2)}V(1)$; the calculation of $I_2$ then follows. As this summary indicates, a secondary aim of this paper is to add to our store of $K(2)$-local calculations.

The paper ends with an appendix which displays graphically the homotopy groups of $E_2^{hG_2} \wedge V(1)$ and $E_2^{hG_2^1} \wedge V(1)$, where $G_2^1 \leq G_2$ is the kernel of a reduced determinant map. (See (2.1).) By [18] and [5] we know

$$\pi_*L_{K(2)}V(1) \cong \Lambda(\zeta) \otimes \pi_*(E_2^{hG_2} \wedge V(1))$$

where $\Lambda(\zeta)$ is an exterior algebra on a class of degree $-1$.

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## 2. Background

We are working in the $K(n)$-local category and all our spectra are implicitly localized. In particular, we emphasize that $X \wedge Y = L_{K(n)}(X \wedge Y)$, as this is the smash product internal to the $K(n)$-local category. In addition, we will write $V(1)$ for $L_{K(2)}V(1)$; this will greatly economize notation in calculations and in the statements of our results. If we work with the unlocalized version of $V(1)$, we will explicitly say so. That said, we will write $L_{K(2)}S^0$ for the localized sphere.

We will write $E$ for $E_n$, where $E_n$ is the Lubin-Tate spectrum with

$$E_\ast = (E_n)_\ast \cong \mathcal{W}(u_1, \ldots, u_{n-1})[u^{\pm 1}]$$

where $\mathcal{W}$ is the Witt vectors on $\mathbb{F}_{p^n}$, the $u_i$ are in degree 0, and $u$ is in degree $-2$. Note that $E_0$ is a complete local ring with residue field $\mathbb{F}_{p^n}$; the formal group over $E_0$ is a choice of universal deformation of the Honda formal group $\Gamma_n$ over $\mathbb{F}_{p^n}[u^{\pm 1}]$. We make the $p$-typical choice of the coordinate of the deformation so that $v_i = u_i^{-p^{i+1}}$, $1 \leq i \leq n-1$, $v_n = u^{-p^{n+1}}$ and $v_i = 0$ for $i > n$. The endomorphism ring of $\Gamma_n$ is given by the non-commutative polynomial ring

$$\text{End}(\Gamma_n) \cong \mathcal{W}(S)/(S^n - p, Sa = \phi(a)S)$$

where $S$ is the endomorphism given by $S(x) = x^p$ and $\phi: \mathcal{W} \to \mathcal{W}$ is the lift of the Frobenius. Then the automorphism group $\mathcal{G}_n = \text{Aut}(\Gamma_n)$ is the group of units in this ring, and the extended group $\mathcal{G}_n = \text{Aut}(\Gamma_n) \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on $E$, by the Hopkins-Miller theorem. The groups $\mathcal{S}_n$ and $\mathcal{G}_n$ are the Morava stabilizer group and the big Morava stabilizer group, respectively.

Since $\mathcal{G}_n$ acts on $E$, $\mathcal{G}_n$ acts on $E_\ast X$. The $E_\ast$-module $E_\ast X$ is equipped with the maximal topology where $m$ is the maximal ideal in $E_0$. This topology is always topologically complete, but need not be separated. With respect to this topology, the group $\mathcal{G}_n$ acts through continuous maps and the action is twisted because it is compatible with the action of $\mathcal{G}_n$ on the coefficient ring $E_\ast$. See [6] §2 for some precise assumptions which guarantee that $E_\ast X$ is complete and separated. All modules which will be used in this paper will in fact satisfy these assumptions, and we will call these modules twisted $\mathcal{G}_n$-modules.
We will work at $p = 3$ and $n = 2$ exclusively. The right action of $\text{Aut}(\Gamma_2)$ on $\text{End}(\Gamma_2)$ defines a determinant map $\text{det} : \text{Aut}(\Gamma_2) \to \mathbb{Z}_3^\times$ which extends to a determinant map

$$G_2 \cong S_2 \times \text{Gal}(\mathbb{F}_9/\mathbb{F}_3) \xrightarrow{\text{det} \times \text{id}} \mathbb{Z}_3^\times \times \text{Gal}(\mathbb{F}_9/\mathbb{F}_3) \xrightarrow{p_1} \mathbb{Z}_3^\times.$$  

Let $G_2^1 \subseteq G_2$ be the kernel of the composition

$$G_2 \longrightarrow \mathbb{Z}_3^\times \longrightarrow \mathbb{Z}_3^\times / \{ \pm 1 \}.$$  

The center of $G_2$ is isomorphic to $\mathbb{Z}_3^\times \subseteq W^\times \subseteq \text{Aut}(\Gamma_n)$; there is a further decomposition $\mathbb{Z}_3^\times = U_1 \times \{ \pm 1 \}$ where $U_1 = 1 + 3\mathbb{Z}_3$ and there is a non-canonical isomorphism $U_1 \cong \mathbb{Z}_3$. We fix such an isomorphism (it will not matter in the sequel which one we choose) and thus we get a fixed isomorphism

$$\mathbb{Z}_3 \times G_2^1 \cong U_1 \times G_2^1 \cong G_2.$$  

In addition, $G_2$ contains two important finite subgroups $G_{24}$ and $SD_{16}$ of orders 24 and 16 respectively. The group $SD_{16}$ is generated by the Frobenius $\phi$ in $\text{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ and $\omega \in W^\times \subseteq \text{Aut}(\Gamma_2)$, a primitive 8th root of unity. The group $G_{24}$ is a choice for a maximal finite subgroup containing an element of order 3. Specifically, the element

$$a = -\frac{1}{2} (1 + \omega S)$$

is our usual choice of an element of order 3. Then $G_{24}$ is generated by $a$, $\omega^2$, and $\omega \phi$. All of the subgroups and the homotopy fixed points $E_{hF}$ where $F$ is finite are discussed extensively in [5] and [6].

**Remark 2.1.** A very basic fact we will need about the Smith-Toda complex $V(1)$ is that the identity $V(1) \to V(1)$ has order 3 even before $K(2)$-localization. To see this, note that $0 = 3 : V(0) \to V(0)$, hence $3 : V(1) \to V(1)$ factors through $V(1)/V(0) \cong \Sigma^5 V(0)$. Since $\pi_5 V(1) = \pi_6 V(1) = 0$, we have $[\Sigma^5 V(0), V(1)] = 0$, which gives the assertion. This implies

$$\pi_*(X \land V(1))$$

is an $\mathbb{F}_3$-vector space for all $X$, localized or not.

### 3. The calculation of $\pi_* L_{K(2)} V(1)$

We recapitulate and expand the results from [5] using a slightly different order of ideas.

Let $G_2^0 \subseteq G_2$ be the index 2 subgroup given as the kernel of the composition

$$G_2 \xrightarrow{\text{det}} \mathbb{Z}_3^\times \longrightarrow C_2 = \{ \pm 1 \}$$

where $\text{det} : G_2 \to \mathbb{Z}_3^\times$ is the extended determinant map of (2.1) and the second map is reduction modulo 3. For our calculation of $\pi_* (S^0(\text{det}) \land V(1))$ in §6, it will be helpful to make calculations with $G_2^0$.

In summary, we are calculating the Adams-Novikov Spectral Sequence

$$H^*(G_2^0, \mathbb{F}_9[u^{\pm 1}]) \Rightarrow \pi_* E_{hG_2^0} V(1)$$

but there are subtleties in both the cohomology calculation and in the calculation of the differentials. See Theorem 3.5. From this we can recover $\pi_* L_{K(2)} V(1)$.
3.1. Coherent detection by centralizers. Let $S_2 \subseteq S_2$ be the 3-Sylow subgroup; then
\[ G_2 = S_2 \rtimes SD_{16} \]
where $SD_{16}$ is generated by $\omega \in \mathbb{F}_9^\times$ and the Frobenius $\phi$ in the Galois group. We also have $G_2^0 \cong S_2 \times D_8$ where $D_8 \subseteq SD_{16}$ is generated by $\omega^2$ and $\phi$.

Let $C_{G_2}(a)$ be the centralizer in $G_2$ of the subgroup $C_3$ generated by our chosen element $a \in G_2$ of order 3 and let $N_{G_2}(a)$ be its normalizer in $G_2$. The inclusion $C_{G_2}(a) \subseteq N_{G_2}(a)$ is of index 2 and the quotient is generated by the image of $\omega^2$. The structure of $C_{G_2}(a)$ and of $N_{G_2}(a)$ has been discussed in Proposition 20 of [9] (see also [8] and [5]). If we write $C_n(x)$ for the cyclic group of order $n$ generated by $x$ then we have an isomorphism
\[ C_{G_2}(a) \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times C_3(a) \times C_4(\omega \phi) \]
which is compatible with the conjugation action by $\omega^2$ if we let $\omega^2$ act on the right hand side by multiplication with $-1$ on $C_3$, on $C_4$ and on one of the factors $\mathbb{Z}_3$, and trivially on the other factor $\mathbb{Z}_3$.

If $C := C_{S_2}(a)$ denotes the centralizer of $a$ in $S_2$ then this isomorphism restricts to an isomorphism
\[ C_3(a) \times \mathbb{Z}_3 \times \mathbb{Z}_3 \to C. \]
If $N := N_{S_2}(a)$ is the normalizer of $a$ in $S_2$ then we get isomorphisms $N \cong C \rtimes C_4(\omega^2)$ and $N_{S_2}(a) \cong C \times Q_8$ with $Q_8$ the quaternion group of order 8 generated by $\omega^2$ and $\omega \phi$.

The action of $S_2$ on $\mathbb{F}_9[u^{\pm 1}]$ is trivial and therefore we get
\[ H^*(C, \mathbb{F}_9[u^{\pm 1}]) \cong H^*(C_3, \mathbb{F}_9[u^{\pm 1}]) \otimes \Lambda(a_1, \zeta_1) \cong \mathbb{F}_9[y_1, u^{\pm 1}] \otimes \Lambda(x_1, a_1, \zeta_1) \]
where $x_1 \in H^1(C_3, \mathbb{F}_3)$ and $y_1 \in H^2(C_3, \mathbb{F}_3)$ are multiplicative generators of the cohomology of a cyclic group of order 3, and $a_1$ and $\zeta_1$ are one dimensional classes accounting for the two factors $\mathbb{Z}_3$ of $C$. Conjugation by $\omega$ defines an isomorphism
\[ \omega_* : H^*(C, \mathbb{F}_3) \to H^*(\omega C \omega^{-1}, \mathbb{F}_3) \]
and we will write $x_2 = \omega_* x_1$ and so on. Note: In what follows we use the subscripts to distinguish between cohomology classes of the two centralizers, except possibly when referring to the class $u$.

The bedrock calculation is the following result from Theorem 4.2 of [8].

**Theorem 3.1.** The restriction maps determine a monomorphism of $\mathbb{F}_9$-algebras
\[ \rho : H^*(S_2, \mathbb{F}_9[u^{\pm 1}]) \longrightarrow H^*(C, \mathbb{F}_9[u^{\pm 1}]) \times H^*(\omega C \omega^{-1}, \mathbb{F}_9[u^{\pm 1}]) \]
\[ \cong \left( \prod_{i=1,2} \mathbb{F}_3[y_i] \otimes \Lambda(a_i, x_i, \zeta_i) \right) \otimes \mathbb{F}_9[u^{\pm 1}] \]
whose image is the free $\mathbb{F}_9[y_1 + y_2, u^{\pm 1}] \otimes \Lambda(\zeta_1 + \zeta_2)$-module generated by the elements
\[ 1, \quad x_1, \quad x_2, \quad y_1, \quad x_1 a_1 - x_2 a_2, \quad y_1 a_1, \quad y_2 a_2, \quad y_1 x_1 a_1. \]

Note that in this result $y_1$ could be replaced with $y_2$ and $y_1 x_1 a_1$ with $y_2 x_2 a_2$.

The conjugation action of $G_2$ on the normal subgroup $S_2$ induces an action of $SD_{16} \cong G_2/S_2$ on $H^*(S_2, \mathbb{F}_9[u^{\pm 1}])$. Likewise the conjugation action of $N_{G_2}(a)$ on $C$ induces an action of the quotient group $Q_8$ on $H^*(C, \mathbb{F}_9[u^{\pm 1}])$. This action of $Q_8$ extends to an action of $SD_{16}$ on $H^*(C, \mathbb{F}_9[u^{\pm 1}]) \times H^*(\omega C \omega^{-1}, \mathbb{F}_9[u^{\pm 1}])$. 

\[
\begin{align*}
\end{align*}
\]
The monomorphism of (3.1) is $SD_{16}$-linear. The element $\omega$ acts through an $F_9$-algebra map, while $\phi$ acts through an $F_3$-linear algebra map which is $F_9$-antilinear, i.e. $\phi_\lambda(\lambda z) = \lambda^3 \phi_\lambda(z)$ if $\lambda \in F_9$. The action is then determined by the formulas (see (4.6) and (4.7) of [6])

$$
\begin{align*}
\omega_\lambda(x_1) &= x_2, & \omega_\lambda(y_1) &= y_2, & \omega_\lambda(a_1) &= a_2 \\
\omega_\lambda(x_2) &= -x_1, & \omega_\lambda(y_2) &= -y_1, & \omega_\lambda(a_2) &= -a_1 \\
\phi_\lambda(x_1) &= -x_2, & \phi_\lambda(y_1) &= -y_2, & \phi_\lambda(a_1) &= -a_2 \\
\phi_\lambda(x_2) &= -x_1, & \phi_\lambda(y_2) &= -y_1, & \phi_\lambda(a_2) &= -a_1 .
\end{align*}
$$

Furthermore,

$$
\begin{align*}
\omega_\lambda(u) &= \omega u, & \omega_\lambda(\zeta_1) &= \zeta_2, & \omega_\lambda(\zeta_2) &= \zeta_1 \\
\phi_\lambda(u) &= u, & \phi_\lambda(\zeta_1) &= \zeta_2, & \phi_\lambda(\zeta_2) &= \zeta_1 .
\end{align*}
$$

3.2. The cohomology of $G_2$ and $G_2^0$ with coefficients in $F_9[u^{\pm 1}]$. Theorem 3.1 and the formulas of (3.1)-(3.4) suffice to deduce Theorem 3.2 below.

After taking invariants with respect to the subgroup $D_8$ generated by $\omega^2$ and $\phi$ the map $\rho$ of (3.1) induces a monomorphism

$$\tilde{\rho} : H^*(G_2^0, F_9[u^{\pm 1}]) \cong H^*(S_2, F_9[u^{\pm 1}])^{D_8} \to (H^*(N, F_9[u^{\pm 1}]) \times H^*(\omega N\omega^{-1}, F_9[u^{\pm 1}]))^{C_2(\phi)} .$$

In this subsection we will write elements in the target as couples. Then the following elements

$$
\begin{align*}
&v_{1/2}^2 := (u^{-4}, u^{-4}) & w := (\omega^2 u^{-4}, \omega^{-2} u^{-4}) \\
&\zeta := (\zeta_1, \zeta_2) & a_{35} := (\omega_1 u^{-18}, \omega^{-1} a_{2} u^{-18}) \\
&\alpha := (\omega_1 u^{-18}, \omega^{-1} x_2 u^{-2}) & \beta := (\omega^3 y_1 u^{-6}, \omega^{-3} y_2 u^{-6})
\end{align*}
$$

are easily checked to be $D_8$-invariant, and with the exception of $v_{1/2}^2$ they are even $SD_{16}$-invariant. We have $\omega_\lambda(v_{1/2}^2) = -v_{1/2}^2$, so $v_{1/2}^2$ is still an eigenvector for the action of $SD_{16}$. Note also that $w^2 = -v_2$. The reason for the power $u^{-18}$ as well as the name of the class $a_{35}$ will become clear in $\S$3.4. The elements $\alpha$ and $\beta$ detect (up to sign) the images of the elements $\alpha_1$ and $\beta_1$ of $\pi_1 S^0$.

By using (3.3) and (3.4) we can easily check that the target of $\tilde{\rho}$ is the free module over $F_9[\beta, v_{1/2}^2] \otimes \Lambda(\zeta, \alpha, a_{35})$ on the class $1 := (1, 1)$. (Note that we need to twist the $F_9$-module structure on the second factor $H^*(\omega N\omega^{-1}, F_9[u^{\pm 1}])$ by Frobenius for this to be true.) It is also the free module over $F_9[\beta, v_{1/2}^2] \otimes \Lambda(\zeta, \alpha, a_{35})$ on the classes $1$ and $w$. The second point of view is better adapted for understanding the residual action of $\omega$ on the $D_8$-invariants and for describing the image of $\tilde{\rho}$ after passing to those invariants.

Furthermore, by (3.2), (3.3) and (3.4) the image of $\tilde{\rho}$ is the free module over $F_9[\beta, v_{1/2}^2] \otimes \Lambda(\zeta)$ on the 8 classes

$$
\begin{align*}
1 := (1, 1) & \quad \alpha & \quad w\alpha & \quad w\beta \\
\alpha a_{35} & \quad \beta a_{35} & \quad w\beta a_{35} & \quad w\beta a_{35} .
\end{align*}
$$

Thus, missing from the image is the free $F_9[v_{1/2}^2] \otimes \Lambda(\zeta)$-submodule of the target of $\tilde{\rho}$ generated by the 4 classes

$$w, a_{35}, w a_{35}, w \alpha a_{35} .$$

Note that all of the 8 classes in (3.6) are eigenvectors for the residual action of $\omega$ on the $D_8$-invariants. Together with the fact that $v_{1/2}^2$ is an eigenvector with eigenvalue $-1$ for this residual action this yields the following result which is an extension of Corollary 19 of [5].
Theorem 3.2. 1) The restriction map
\[ H^*(G_2, \mathbb{F}_9[u^{±1}]) \to \left( H^*(N, \mathbb{F}_9[u^{±1}]) \times H^*(\omega N \omega^{-1}, \mathbb{F}_9[u^{±1}]) \right)^{C_2(\phi)} \]
is a monomorphism. Its target is isomorphic to \( \mathbb{F}_9[\beta, v_2^{±1/2}] \otimes \Lambda(\zeta, \alpha, a_{35}) \) and its image is the free module over \( \mathbb{F}_9[\beta, v_2^{±1/2}] \otimes \Lambda(\zeta) \) on the 8 classes \( 1, \alpha, w\alpha, w\beta, \alpha a_{35}, \beta a_{35}, w\beta a_{35} \) and \( w\alpha a_{35} \).

2) The restriction map
\[ H^*(G_2, \mathbb{F}_9[u^{±1}]) \to \left( H^*(N, \mathbb{F}_9[u^{±1}]) \times H^*(\omega N \omega^{-1}, \mathbb{F}_9[u^{±1}]) \right)^{C_2(\phi) \times C_2(\omega)} \]
is a monomorphism. Its target is isomorphic to the free module over \( \mathbb{F}_9[\beta, v_2^{±1}] \otimes \Lambda(\zeta, \alpha, a_{35}) \) generated by 1 and \( w \), and its image is the free module over \( \mathbb{F}_9[\beta, v_2] \otimes \Lambda(\zeta) \) on the 8 classes \( 1, \alpha, w\alpha, w\beta, \alpha a_{35}, \beta a_{35}, w\beta a_{35} \) and \( w\alpha a_{35} \).

3) The residual action of \( \omega \) on \( H^*(G_2, \mathbb{F}_9[u^{±1}]) \) induces an eigenspace decomposition
\[ H^*(G_2, \mathbb{F}_9[u^{±1}]) \cong H^*(G_2, \mathbb{F}_9[u^{±1}]) \oplus v_2^{1/2} H^*(G_2, \mathbb{F}_9[u^{±1}]) \]
where the first summand is is the \((+1)\)-eigenspace and the second summand is the \((-1)\)-eigenspace. \(\square\)

3.3. The homotopy groups of \( E^hG_24 \wedge V(1) \). In this subsection we recall the calculation in [5] and add some information on Toda brackets and \( \alpha \)-multiplications.

We have that \( G_24 = C_3 \rtimes Q_8 \) where \( C_3 \) is generated by our element \( a \) of order 3 and \( Q_8 \) is generated by \( \omega^2 \) and \( \omega\phi \). The element of order 3 acts trivially on \( \mathbb{F}_9[u^{±1}] \) and
\[ H^*(G_24, \mathbb{F}_9[u^{±1}]) \cong H^*(C_3, \mathbb{F}_9[u^{±1}])^{Q_8}. \]
We have
\[ H^*(C_3, \mathbb{F}_9[u^{±1}]) \cong \mathbb{F}_9[y_1, u^{±1}] \otimes \Lambda(x_1). \]
The action of \( \omega^2 \) is \( \mathbb{F}_9 \)-linear; however, the action of \( \omega\phi \) is \( \mathbb{F}_9 \)-antilinear.

By abuse of notation we define
\[ (3.7) \quad \alpha = \omega x_1 u^{-2}, \quad \beta = \omega^3 y_1 u^{-6}, \quad w = \omega^2 u^{-4} \]
of bidegrees \((1, 4)\), \((2, 12)\) and \((0, 8)\) respectively. As before these elements \( \alpha \) and \( \beta \) will detect (up to sign) the images of the elements \( \alpha_1 \) and \( \beta_1 \) of \( \pi_* S^0 \).

Using the formulas (3.3) we have
\[ (3.8) \quad H^*(G_24, \mathbb{F}_9[u^{±1}]) = \mathbb{F}_9[\beta, w^{±1}] \otimes \Lambda(\alpha). \]

By the calculations of many people (see [6] §3, for example), the spectrum \( E^hG_24 \) has an invertible class \( \Delta^3 \in \pi_7 E^hG_24 \); this class reduces to \(-w^9\). In the homotopy fixed point spectral sequence
\[ H^*(G_24, \mathbb{F}_9[u^{±1}]) \Rightarrow \pi_*(E^hG_24 \wedge V(1)) \]
all differentials commute with multiplication by \( \alpha \), \( \beta \), and \( w^9 \) and are determined by
\[ (3.9) \quad d_5(w^{i+3}) = \pm \alpha \beta^2 w^i, \quad 0 \leq i \leq 5; \]
\[ d_9(w^{i+6} \alpha) = \pm \beta^5 w^i, \quad 0 \leq i \leq 2. \]

In particular \( w^i, 0 \leq i \leq 2 \), and \( w^i \alpha, 0 \leq i \leq 5 \), are permanent cycles. The first of these differentials implies that the Toda bracket
\[ (3.10) \quad z_i := \langle \alpha, \alpha, \beta w^i \rangle, \quad 0 \leq i \leq 2 \]
Theorem 3.3. There is an isomorphism of modules over $\mathbb{F}_3[w^\pm 9, \beta]
abla^\mathbb{F}_3[w^\pm 9] \otimes (\mathbb{F}_3[\beta]/(\beta^5)\{1, w, w^2\} \oplus \mathbb{F}_3[\beta]/(\beta^2)\{\alpha, w\alpha, w^2\alpha, z_0, z_1, z_2\})$.

The homotopy groups $\pi_*(E^{hG_{24}} \wedge V(1))$ are 72-periodic on $w^9 = \omega^2v_2^{9/2}$.

We summarize this discussion in the following theorem, which is a slightly refined version of Theorem 9 of [5]. The calculation is also displayed graphically in Figure 1 of the Appendix. The notation $A\{a, b, \ldots\}$ means the free $A$-module on the indicated generating set.

**Theorem 3.3.** There is an isomorphism of modules over $\mathbb{F}_3[w^\pm 9, \beta]$ $\pi_*(E^{hG_{24}} \wedge V(1)) \cong \mathbb{F}_3[w^\pm 9] \otimes (\mathbb{F}_3[\beta]/(\beta^5)\{1, w, w^2\} \oplus \mathbb{F}_3[\beta]/(\beta^2)\{\alpha, w\alpha, w^2\alpha, z_0, z_1, z_2\})$.

The class $w^9$ has order 3. Since $\pi_{72+i} E^{hG_{24}} \wedge V(1) = 0$ for $i = 4$ and $i = 5$, we can then extend the map $w^9 : S^{72} \to E^{hG_{24}} \wedge V(1)$ to a map

$$\Sigma^{72}V(1) \to E^{hG_{24}} \wedge V(1).$$

Since $E^{hG_{24}}$ is a ring spectrum, this can be extended to an equivalence

$$\Sigma^{72}E^{hG_{24}} \wedge V(1) \xrightarrow{\cong} E^{hG_{24}} \wedge V(1).$$

Thus we say $E^{hG_{24}} \wedge V(1)$ is 72-periodic. Of course, it is a standard fact that $E^{hG_{24}}$ is itself 72-periodic. For one reference among many, see [6] §3.

We will also need the homotopy groups of $E^{hG_{12}} \wedge V(1)$, where $G_{12} = C_3 \times C_4(\omega^2)$. The $E_2$-term of the homotopy fixed point spectral sequence is

$$H^*(G_{12}, F_9[w^{\pm 1}]) = H^*(C_3, F_9[w^{\pm 1}])\mathbb{C}_4 = F_9[\beta, v_2^{1/2}] \otimes \Lambda(\alpha)$$

and all the differentials are determined by (3.9). In fact,

$$\pi_*(E^{hG_{12}} \wedge V(1)) \cong \pi_*(E^{hG_{24}} \wedge V(1)) \otimes F_9$$

and

$$\pi_*(E^{hG_{24}} \wedge V(1)) \cong [\pi_* E^{hG_{12}} \wedge V(1)]^{C_2(\phi_0)}.$$  

3.4. **The homotopy groups of $\pi_* E^{hN} \wedge V(1)$.** Recall that $N$ denotes the normalizer of $a$ in $S_2$ and that $N = C \times C_4(\omega^2)$. By (3.3) we have

$$H^*(C, F_9[u^{\pm 1}]) \cong H^*(C_3, F_9[u^{\pm 1}]) \otimes \Lambda(a_1, \zeta_1)$$

with $(\omega^2)_a a_1 = -a_1$ and $(\omega^2)_a \zeta_1 = \zeta_1$. By continued abuse of notation we name the class $u^{-18} a_1$ of bidegree (1, 36) simply by $a_{35}$, $u^{-4}$ of bidegree (0, 8) by $v_2^{1/2}$ and $\zeta_1$ of bidegree (1, 0) by $\zeta$. With this and the notation used in §3.3 we have

$$H^*(N, F_9[u^{\pm 1}]) \cong [H^*(C, F_9[u^{\pm 1}])\mathbb{C}_4(\omega^2) \cong F_9[\beta, v_2^{1/2}] \otimes \Lambda(\alpha, a_{35}, \zeta)].$$

Furthermore, if $N^1 = S_2^1 \cap N$, then $N^1 = (C_3 \times \mathbb{Z}_3) \times C_4(\omega^2)$ and $\omega^2$ acts by multiplication with $-1$ on $C_3$ and on $\mathbb{Z}_3$ and thus we have

$$H^*(N^1, F_9[u^{\pm 1}]) \cong F_9[\beta, v_2^{1/2}] \otimes \Lambda(\alpha, a_{35}).$$

The inclusion $G_{12} \subseteq N^1$ gives the map on cohomology defined by sending $a_{35}$ to zero. The selection of degree 36 and the name $a_{35}$ is explained by the existence of a fiber sequence

$$\Sigma^{35}E^{hG_{12}} \to E^{hN^1} \to E^{hG_{12}}$$

which was established in Corollary 13 of [5]. By (3.13) and (3.14) there is a short exact sequence

$$0 \to a_{35} H^*(G_{12}, F_9[u^{\pm 1}]) \to H^*(N^1, F_9[u^{\pm 1}]) \to H^*(G_{12}, F_9[u^{\pm 1}]) \to 0.$$
of $E_2$-terms of spectral sequences abutting to a short exact sequence in homotopy groups

$$0 \to \pi_*(\Sigma^{35} E^{hG_{12}} \land V(1)) \to \pi_*(E^{hN_1} \land V(1)) \to \pi_*(E^{hG_{12}} \land V(1)) \to 0.$$  

This last statement is proved in Lemma 14 and Proposition 15 of [5] by a computation; it is not a conceptual result. We conclude that there is a splitting

$$(3.15) \quad \pi_*(E^{hN_1} \land V(1)) \cong \pi_*(E^{hG_{12}} \land V(1))\{1, a_{35}\}$$

where we have confused $a_{35}$ with some choice of homotopy class in $\pi_{35}E^{hN_1}$ detected by $a_{35}$. This should not be interpreted as a ring isomorphism, but only as an isomorphism of modules over $\mathbb{F}_{9}[\beta] \otimes \Lambda(\alpha)$.

To get the homotopy groups of $E^{hN} \land V(1)$ we use the fiber sequence

$$E^{hN} \to E^{hN_1} \xrightarrow{id-k} E^{hN_1}$$

where $k$ is a topological generator of $\mathbb{Z}_3 \cong N/N_1$. By Lemma 17 of [5] (again, a case-by-case calculation) we have that

$$0 = \pi_*(id - k) : \pi_*(E^{hN_1} \land V(1)) \to \pi_*(E^{hN_1} \land V(1))$$

and we can conclude that there is an isomorphism of $\mathbb{F}_3[\beta] \otimes \Lambda(\alpha, \zeta)$-modules

$$(3.16) \quad \pi_*(E^{hN} \land V(1)) \cong (\pi_*(E^{hN_1} \land V(1)) \otimes \Lambda(\zeta) \cong \pi_*(E^{hG_{12}} \land V(1)) \otimes \Lambda(\zeta)\{1, a_{35}\}.$$  

The spectrum $E^{hN} \land V(1)$ is periodic, although the argument is not quite as simple as in (3.12). We have the following result.

**Proposition 3.4.** The spectra $E^{hN} \land V(1)$ and $E^{hN_1} \land V(1)$ are 72-periodic on the class $v_{2}^{9/2}$.

**Proof.** We do the case of $N$: the case $N_1$ follows the same line of reasoning. To begin, we note $\pi_{72+4}(E^{hN} \land V(1)) = 0$. This is equivalent to noting that $\pi_{j}(E^{hG_{24}} \land V(1)) = 0$ for $j = 4, 5, 41, \text{ and } 42$. That these groups are zero follows from Theorem 3.3. See also Figure 1. But note $\pi_{72+5}E^{hN} \neq 0$ as $\pi_{43}(E^{hG_{24}} \land V(1)) \neq 0$. However multiplication by $\beta$ is a monomorphism on $\pi_{72+5}E^{hN}$ and we can appeal to Proposition 3.8 below to extend $v_{2}^{9/2}$ to a map

$$\Sigma^{72}V(1) \to E^{hN} \land V(1).$$

This map can be then be extended to an equivalence of $E^{hN}_2$-module spectra. □

3.5. **The homotopy groups of $E^{hG_2^0} \land V(1)$ and of $L_{K(2)}V(1)$**. The differentials and multiplicative extensions in the spectral sequence

$$H^*(G_2^0, \mathbb{F}_9[u^\pm 1]) \Rightarrow \pi_*(E^{hG_2^0} \land V(1))$$

are now forced by the homotopy fixed point spectral sequence of $\pi_*(E^{hN} \land V(1))$ and naturality.

In fact, there is a diagram of spectral sequences

$$\begin{array}{ccc}
H^*(G_2^0, \mathbb{F}_9[u^\pm 1]) & \to & [H^*(N, \mathbb{F}_9[u^\pm 1]) \times H^*(\omega^{-1}N\omega, \mathbb{F}_9[u^\pm 1])]^{C_2(\phi)} \\
\downarrow & & \downarrow \\
\pi_*(E^{hG_2^0}) \land V(1) & \to & \pi_*((E^{hN} \land E^{h\omega^{-1}N\omega})^{hC_2(\phi)} \land V(1))
\end{array}$$

and the differentials on the right hand side are determined by (3.9) and (3.16) (cf. [5] §3).
More precisely, by Theorem 3.2 the $E_2$-term of the right hand side spectral sequence is a free module over $\mathbb{F}_9[\beta, v_2^{+1/2}] \otimes \Lambda(\omega, \alpha, \alpha_{35})$ on the class $1$. This $E_2$-term can also be considered as a free module over $\mathbb{F}_3[\beta, v_2^{+1/2}] \otimes \Lambda(\omega, \alpha, \alpha_{35})$ on the classes $1$ and $w$. The $E_2$-term of the left hand side spectral sequence injects into this $E_2$-term as the free module over $\mathbb{F}_3[\beta, v_2^{+1/2}] \otimes \Lambda(\omega, \alpha, \alpha_{35})$ on the 8 classes $1, \alpha, w_\beta, \alpha_{35}, \beta_{35}, w_\beta \alpha_{35} \text{ and } w_\beta \alpha_{35}$ of Theorem 3.2. Note that this is only a graded $\mathbb{F}_3$-subvector space and not an $\mathbb{F}_9$-subvector space.

The differentials on the right hand side are $\mathbb{F}_9[v_2^{+9/2}]$-linear and are determined by (3.9) and Proposition 3.4 and the fact that $\alpha, \beta$ and $\zeta$ are obviously permanent cycles while $\alpha_{35}$ is a permanent cycle by (3.16). By using that $w = \omega^2 v_2^{1/2}$ we get, up to sign, the following non-trivial differentials on the left hand side

\begin{align}
&d_5(v_2^{(i+3)/2}) = v_2^{(i-1)/2} \beta^2 w_\alpha, \quad i \equiv 0, 1, 2, 3, 4, 5 \mod (9) \\
d_5(v_2^{(i+3)/2} \beta \alpha_{35}) = v_2^{(i-1)/2} \beta^2 w_\beta \alpha_{35}, \quad i \equiv 0, 1, 2, 3, 4, 5 \mod (9) \\
&d_5(v_2^{(i+3)/2} w_\beta) = v_2^{(i+1)/2} \beta \alpha_{35}, \quad i + 1 \equiv 0, 1, 2, 3, 4, 5 \mod (9) \\
d_5(v_2^{(i+3)/2} w_\beta \alpha_{35}) = v_2^{(i+1)/2} \beta^3 \alpha_{35}, \quad i + 1 \equiv 0, 1, 2, 3, 4, 5 \mod (9)
\end{align}

and

\begin{align}
&d_9(v_2^{(i+6)/2}) = v_2^{i/2} \beta^5, \quad i \equiv 0, 1, 2 \mod (9) \\
d_9(v_2^{(i+6)/2} \alpha \alpha_{35}) = v_2^{i/2} \beta^5 \alpha_{35}, \quad i \equiv 0, 1, 2 \mod (9) \\
&d_9(v_2^{(i+5)/2} w_\alpha) = v_2^{(i-1)/2} w_\beta \alpha_{35}, \quad i \equiv 0, 1, 2 \mod (9) \\
d_9(v_2^{(i+5)/2} w_\beta \alpha_{35}) = v_2^{(i-1)/2} w_\beta \alpha_{35}, \quad i \equiv 0, 1, 2 \mod (9).
\end{align}

The first part of the following result gives the outcome of this calculation. The generators have been chosen so that they correspond to those given in the calculation of $\pi_*(L_{K(2)} V(1))$ in the main theorem of [5]. The second part follows from Theorem 3.2.3.

**Theorem 3.5.** 1) As a module over $S := \mathbb{F}_3[v_2^{+9/2}, \beta] \otimes \Lambda(\omega)$ there is an isomorphism

\[\pi_*(E_{hG_2} \wedge V(1)) \cong \oplus S/(\beta^6) v_2^{l=0, 1, 1, 1} \oplus S/(\beta^6) v_2^{l=0, 1, 2, 2} \oplus S/(\beta^6) v_2^{l=0, 1, 3, 3} \oplus S/(\beta^6) v_2^{l=0, 1, 4, 4} \oplus S/(\beta^6) v_2^{l=0, 1, 5, 5} \oplus S/(\beta^6) v_2^{l=0, 1, 6, 6} \]

2) There is an isomorphism of $\pi_*(L_{K(2)} V(1))$-modules

\[\pi_*(E_{hG_2} \wedge V(1)) \cong \pi_*(L_{K(2)} V(1)) \oplus v_2^{9/2} \pi_*(L_{K(2)} V(1)) \]

which is compatible with the splitting

\[E_{hG_2} \overset{\sim}{=}(E_{hG_2})^+ \oplus (E_{hG_2})^- = E_{hG_2} \oplus (E_{hG_2})^- \]

of $E_{hG_2}$ into the $(+1)$- and $(-1)$-eigenspectra for the residual action of $C_2(\omega)$ on $E_{hG_2}$. \(\square\)

Figure 2 of the Appendix describes the homotopy groups of $E_{hG_2} \wedge V(1)$ including the $\alpha$-multiplications determined by (3.11). Those of $L_{K(2)} V(1)$ are obtained by taking the tensor product with the exterior algebra on $\zeta$. 


Remark 3.6. Notice that \( \pi_*(L_{K(2)}V(1)) \) is 144-periodic with periodicity generator \( v_2^9 \). Using Proposition 3.8 below, we can extend the map \( v_2^9 : S^{144} \to L_{K(2)}V(1) \) to a self-equivalence, giving a topological realization of the algebraic periodicity. Even before \( K(2) \)-localization, the spectrum \( V(1) \) has a \( v_2^3 \)-self map; see [2]. The arguments there are more difficult.

Similarly, note that \( \pi_*(E_2^{hG_2^0} \land V(1)) \) is periodic with periodicity generator \( v_2^{9/2} \). Again using Proposition 3.8 below, the map \( v_2^{9/2} : S^{72} \to E_2^{hG_2^0} \land V(1) \) extends to a map \( \Sigma^{72}V(1) \to E_2^{hG_2^0} \land V(1) \) and hence to an equivalence \( \Sigma^{72}E_2^{hG_2^0} \land V(1) \simeq E_2^{hG_2^0} \land V(1) \).

We can apply this discussion to identify the homotopy groups of the Brown-Comenetz dual of \( L_{K(2)}V(1) \). Since \( L_{K(2)}V(1) \) is the localization of a type 2 complex, \( M_2V(1) = L_{K(2)}V(1) \); hence

\[
\pi_*I_2V(1) \cong \pi_*IL_{K(2)}V(1) \cong \text{Hom}(\pi_*L_{K(2)}V(1), \mathbb{Q}/\mathbb{Z}).
\]

In the sequel we will return to our convention of writing \( V(1) \) for \( L_{K(2)}V(1) \), leaving the localization implicit.

The following result is also in [1], §1.8.

**Theorem 3.7.** 1) The \( K(2) \)-local spectrum \( V(1) \) is Brown-Comenetz self-dual on homotopy groups; more precisely, there is an isomorphism

\[
\pi_*\Sigma^{-28}V(1) \cong \pi_*I_2V(1)
\]

of modules over \( \mathbb{F}_3[\beta, v_2^{\pm 9}] \otimes \Lambda(\alpha, \zeta) \).

2) There is also an isomorphism

\[
\pi_*\Sigma^{-22}V(1) \cong \pi_*I_2 \land V(1)
\]

of modules over \( \mathbb{F}_3[\beta, v_2^{\pm 9}] \otimes \Lambda(\alpha, \zeta) \).

**Proof.** The two statements are equivalent, since

\[
I_2V(1) \simeq F(V(1), I_2) \simeq F(V(1), S^0) \land I_2 \simeq \Sigma^{-6}V(1) \land I_2.
\]

To prove statement (1), note there is an isomorphism

\[
\pi_*V(1) \cong \Lambda(\zeta) \otimes \pi_*(E^{hG_2} \land V(1)).
\]

The homotopy groups of \( E^{hG_2} \land V(1) \) are displayed in Figure 2 of the Appendix. Inspection of this chart shows there is an isomorphism

\[
\pi_*\Sigma^{-172}V(1) \cong \text{Hom}(\pi_*V(1), \mathbb{Q}/\mathbb{Z})
\]

with respect to which \( 1 \in \pi_0V(1) \) corresponds to the class named

\[
\beta^4\zeta[w_2^5w^5a_{35}] \in \pi_{172}V(1)
\]

in the conventions of Theorem 3.5. This is \( \zeta \) times the class labeled \( v_2^5w^5a_{35} \) in Figure 2 of the Appendix. Now use that \( \pi_*V(1) \) is 144-periodic to get the result. \( \square \)

3.6. Creating \( v_2 \)-self maps. We would now like to give a simple criterion for creating \( K(2) \)-local equivalences out of \( V(1) \). The first result, distilled from far more subtle arguments given in [2], is about extending homotopy classes over \( V(1) \); it works in the unlocalized stable category. Let \( W \) be the cofiber of \( v_1 : S^4 \to V(0) \). Then \( W \) is a 5-skeleton of the unlocalized \( V(1) \) and we define \( g : S^5 \to W \) to be attaching map of the 6-cell.
Proposition 3.8. Let $X$ be a spectrum, not necessarily $K(2)$-local, let $y \in \pi_n X$ be of order 3 and suppose $\pi_{n+4} X = 0$. Then there is an extension of $y$ to a map

$$\bar{y}: \Sigma^n W \to X$$

and the composite

$$S^{n+15} \xrightarrow{\beta} S^{n+5} \xrightarrow{g} \Sigma^n W \xrightarrow{\bar{y}} X$$

is null-homotopic. In particular, if the multiplication

$$\beta: \pi_{n+5} X \to \pi_{n+15} X$$

is injective, then $y$ extends to a map $\Sigma^n V(1) \to X$.

Proof. The assumptions guarantee the existence of $\bar{y}$, so we need only prove that $\bar{y}g\beta = 0$. This will follow from the fact that $\pi_{15} W = 0$. The relevant part of the long exact sequence for computing this group is

$$\pi_{11} S^0 \xrightarrow{v_1} \pi_{15} V(0) \to \pi_{15} W \xrightarrow{v_1} \pi_{10} S^0 \to \pi_{14} V(0).$$

The 3-primary homotopy groups of $S^0$ and $V(0)$ are well-known in this range; see, for example [17], Table A.3.4. The group $\pi_{15} V(0)$ is $\mathbb{Z}/3$ generated by $\alpha_4 = v_1^3 \alpha$ modulo 3. But $\pi_{11} S^0 \cong \mathbb{Z}/9$ generated by $\alpha_3/2$ and $\alpha_3/2 = v_1^2 \alpha$ modulo 3. The group $\pi_{10} S^0$ is generated by $\beta = \beta_1$ and modulo 3, $v_1 \beta_1 \neq 0$; in fact $v_1 \beta = \pm \alpha_3 \beta$ where $\beta$ maps to $\beta$ under the pinch map $V(0) \to S^1$.

Proposition 3.8 now extends to a result about equivalences out of $V(1)$.

Theorem 3.9. Let $X$ be a $K(2)$-local spectrum with a chosen isomorphism of twisted $\mathbb{G}_2$-modules

$$g : E_* V(1) \cong E_* X.$$

Let $\iota_X \in H^0(\mathbb{G}_2, E_0 X)$ be the image of the generator of $H^0(\mathbb{G}_2, E_0 V(1))$ under the map induced by $g$. Suppose that

1. $\iota_X$ is a permanent cycle in the Adams-Novikov Spectral Sequence and detects an element of order 3;
2. $\pi_4 X = 0$;
3. multiplication by $\beta$ is monomorphic on $\pi_5 X$.

Then $g$ can be realized by a $K(2)$-local equivalence

$$f : V(1) \to X.$$

Proof. Let $S^0 \to X$ be the map of order 3 detected by $\iota_X$. Then Proposition 3.8 applies and yields a map $f : V(1) \to X$ which, by construction, realizes $g$. Since $g$ is an isomorphism, $f$ is a weak equivalence, after localization.

Remark 3.10. Theorem 3.7 does not quite yield a weak equivalence $\Sigma^{-28} V(1) \simeq I_2 V(1)$, as Theorem 3.9 does not immediately apply. We do have an isomorphism of twisted $\mathbb{G}_2$-modules. In fact, using the equivalence $I_2 V(1) = F(V(1), I_2) \simeq \Sigma^{-6} V(1) \land I_2$, the Gross-Hopkins formula (1.1), and the fact that $E_* (\det) / (3, v_1) \cong \Sigma^8 E_* V(1)$ as twisted $\mathbb{G}_2$-modules, we have

$$E_* I_2 V(1) \cong E_* \Sigma^{-6} V(1) \otimes_{E_*} E_* I_2 \cong \Sigma^{-4} E_* V(1) \otimes_{E_*} E_* (\det) \cong \Sigma^4 E_* V(1) \cong \Sigma^{-28} E_* V(1).$$
calculations in Proposition 5.3 of the next section. The aim is to be able to compare and contrast algebraic and topological information for our \( \Sigma \) algebraic resolution of the trivial \( \Sigma \) and detects the element specified in the previous proof. Thus we have a candidate for a permanent cycle in \( H^0(G_2, E_{-28}I_2V(1)) \), but we would need to check that this is a permanent cycle and hence, \( \Sigma^{-28}V(1) \simeq I_2V(1) \).

4. The centralizer resolution

The results of this section are a revisiting of work of the second author from [9]. The aim is to be able to compare and contrast algebraic and topological information for our calculations in Proposition 5.3 of the next section.

Let \( Z_3[[G_2]] \) be the completed group ring of the group \( G_2 \). By Theorem 10 of [9] there is an algebraic resolution of the trivial \( Z_3[[G_2]]-\)module

\[
0 \to C_3 \to C_2 \to C_1 \to C_0 \to Z_3 \to 0
\]

with

\[
\begin{align*}
(1) & \quad C_0 = Z_3[[G_2]/G_{24}], \\
(2) & \quad C_1 = Z_3[[G_2]] \otimes_{Z_3[SD_{16}]} \chi \oplus Z_3[[G_2]] \otimes_{Z_3[SD_{16}]} \tilde{\chi}, \\
 & \quad \text{where } \chi \text{ is the } Z_3[SD_{16}]-\text{module whose underlying } Z_3-\text{module is } Z_3 \text{ and on which } \omega \text{ and } \phi \text{ act by } -1, \text{ and where } \tilde{\chi} \text{ is the } Z_3[G_24] \text{ module whose underlying } Z_3-\text{module is } Z_3 \text{ and on which } \omega \text{ and } \phi \text{ act trivially}, \\
(3) & \quad C_2 = Z_3[[G_2]] \otimes_{Z_3[SD_{16}]} \lambda, \\
 & \quad \text{where } \lambda \text{ is the } Z_3[SD_{16}]-\text{module whose underlying } Z_3-\text{module is } W \text{ and on which } \omega \text{ acts by multiplication with } \omega^2 \text{ and } \phi \text{ acts by Frobenius}, \\
(4) & \quad C_3 = Z_3[[G_2]/SD_{16}]].
\]

We get a slightly larger resolution of \( Z_3 \) as \( G_2 \)-module by using the decomposition \( Z_3 \times G_2 \cong G_2 \) of (2.3) and then tensoring the resolution (4.1) with the small resolution

\[
0 \to Z_3[[Z_3]] \to Z_4[[Z_3]] \to Z_4 \to 0.
\]

The resolution of (4.1) leads to a trigraded spectral sequence

\[
E_1^{p,q,t}(G_2^1) = E_1^{p,q,t} = \text{Ext}^q_{Z_3[[G_2]]}(C_p, F_9[u^{\pm 1}]) \implies H^{p+q}(G_2, F_9[u^{\pm 1}]).
\]

By Shapiro’s lemma and by using that the modules \( \chi, \tilde{\chi} \) and \( \lambda \) are self-dual this \( E_1 \)-term can be rewritten as

\[
\begin{align*}
E_1^{0,q,*} & \simeq H^q(G_24, F_9[u^{\pm 1}])b_0, \\
E_1^{1,q,*} & \simeq \text{Ext}^q_{Z_3[G_2]}(\tilde{\chi}, F_9[u^{\pm 1}]) \oplus \text{Ext}^q_{Z_3[SD_{16}]}(\chi, F_9[u^{\pm 1}]) \\
& \quad \cong H^q(G_24, F_9[u^{\pm 1}] \otimes \chi) \oplus H^q(SD_{16}, F_9[u^{\pm 1}] \otimes \chi) \\
& \quad \cong H^q(G_24, F_9[u^{\pm 1}])c_36 + H^q(SD_{16}, F_9[u^{\pm 1}])c_8, \\
E_1^{2,q,*} & \simeq \text{Ext}^q_{Z_3[SD_{16}]}(\lambda, F_9[u^{\pm 1}]) \cong H^q(SD_{16}, F_9[u^{\pm 1}] \otimes \lambda) \\
& \quad \cong H^q(SD_{16}, F_9[u^{\pm 1}])c_36 + H^q(SD_{16}, F_9[u^{\pm 1}])c_44, \\
E_1^{3,q,*} & \simeq H^q(SD_{16}, F_9[u^{\pm 1}])c_48
\end{align*}
\]

where \( R_{cn} \subseteq E_1^{p,q,*} \) means a free module over the bigraded ring \( R \) on a generator of tridegree \( (p, 0, n) \). The shift

\[
H^*(G_24, F_p[u^{\pm 1}] \otimes \tilde{\chi}) \cong H^*(G_24, F_p[u^{\pm 1}])b_{36}
\]
follows from the isomorphism $E_0 \otimes \tilde{\chi} \cong E_{-36}$ as $G_{24}$-modules. Similar explanations apply to the other shifts.

In fact, we have

$$H^*(SD_{16}, \mathbb{F}_9[u^{\pm 1}]) \cong \mathbb{F}_3[u^{\pm 1}] ,$$

in particular this cohomology is concentrated in cohomological dimension 0 and it is 16-periodic. Hence the $t$-degree of the generator $e_{48}$ could be changed by any multiple of 16. Likewise the $t$-degree of the generators $e_8$, $e_{36}$ and $e_{44}$ could be changed by any multiple of 16. By (3.8) we have

$$H^*(G_{24}, \mathbb{F}_9[u^{\pm 1}]) \cong \mathbb{F}_3[u^{\pm 1}, \beta] \otimes \Lambda(\alpha)$$

and the $t$-degrees of the generators $b_0$ and $b_{36}$ could be changed by any multiple of 8. We have made our choices in order to simplify some of the formulas below. Note that the element $b_{36}$ corresponds to the element $a_{35}$ in section 3.

The spectral sequence (4.2) must converge to the result described in Theorem 3.2.2 with the exterior algebra on $\zeta$ removed. The differentials are then determined. To give a formula, we note that the differentials are $v_2$-linear and we have, up to sign,

$$d_1(wb_0) = e_8$$
$$d_1(b_{36}) = e_{36}$$
$$d_1(wb_{36}) = e_{44}$$
$$d_2(\alpha wb_{36}) = e_{48} .$$

Note that the $E_\infty$-page is concentrated only in columns $p = 0$ and $p = 1$.

There is a similar spectral sequence for $H^*(G_2, \mathbb{F}_9[u^{\pm 1}])$ with

$$E_1(G_2) \cong E_1(G_2) \otimes \Lambda(\zeta)$$

and for which all differentials are determined by the fact that this is part of a splitting of spectral sequences and $b_0 \otimes \zeta \in E^{0,1,0}(G_2)$ is a permanent cycle.

By Theorem 11 of [9] these algebraic resolutions have topological resolutions as sequences of spectra. Note that $E^{hSD_{16}}$ is 16-periodic and as above we have altered the suspensions of the original reference [9] by a multiple of 16 in order to simplify some formulas below:

$$E^{hG_2} \quad \rightarrow \quad \Sigma^8 E^{hSD_{16}} \quad \rightarrow \quad \Sigma^{36} E^{hSD_{16}} \quad \rightarrow \quad \Sigma^{44} E^{hSD_{16}} \quad \rightarrow \quad \Sigma^{48} E^{hSD_{16}}$$

and

$$L_{K(2)}S^0 \quad \rightarrow \quad E^{hG_{24}} \quad \rightarrow \quad \Sigma^{36} E^{hG_{24}} \quad \rightarrow \quad \Sigma^8 E^{hSD_{16}} \quad \rightarrow \quad \Sigma^{36} E^{hSD_{16}} \quad \rightarrow \quad \Sigma^{44} E^{hSD_{16}}$$

All compositions and all Toda brackets are zero modulo indeterminacy and, thus, these resolutions can be refined to towers of fibrations with $E^{hG_2}$ or $L_{K(2)}S^0$ at the top, as
needed. Thus if $Y$ is a $K(2)$-local spectrum, there is a tower of fibrations

\[
\begin{array}{cccccc}
Y & \rightarrow & Y_3 & \rightarrow & Y_2 & \rightarrow & Y_1 & \rightarrow & E^{hG_{24}} \wedge Y \\
\Sigma^{-4}F_4 \wedge Y & \rightarrow & \Sigma^{-3}F_3 \wedge Y & \rightarrow & \Sigma^{-2}F_2 \wedge Y & \rightarrow & \Sigma^{-1}F_1 \wedge Y
\end{array}
\]

with

\[F_1 = \Sigma^8 E^{hSD_{16}} \vee E^{hG_{24}} \vee \Sigma^{36} E^{hG_{24}}\]

and so on for $F_2, F_3, \text{ and } F_4$.

This leads to a spectral sequence $E_{\ast, \ast}^{i,j} \Rightarrow \pi_{i-j}Y$ with

\[E_0^{i,j} = \pi_i E^{hG_{24}} \wedge Y\]

\[E_1^{i,j} = \pi_i \Sigma^8 E^{hSD_{16}} \wedge Y \oplus \pi_i E^{hG_{24}} \wedge Y \oplus \pi_i \Sigma^{36} E^{hG_{24}} \wedge Y\]

\[E_2^{i,j} = \pi_i \Sigma^{36} E^{hSD_{16}} \wedge Y \oplus \pi_i \Sigma^8 E^{hSD_{16}} \wedge Y \oplus \pi_i \Sigma^{44} E^{hSD_{16}} \wedge Y \oplus \pi_i \Sigma^{36} E^{hG_{24}} \wedge Y\]

\[E_3^{i,j} = \pi_i \Sigma^{48} E^{hSD_{16}} \wedge Y \oplus \pi_i \Sigma^{36} E^{hSD_{16}} \wedge Y \oplus \pi_i \Sigma^{44} E^{hSD_{16}} \wedge Y\]

\[E_4^{i,j} = \pi_i \Sigma^{48} E^{hSD_{16}} \wedge Y\]

**Remark 4.1.** While we will not use this information below, it is worth noting that it is a simple matter to calculate the differentials in the spectral sequence of this tower, at least for $Y = V(1)$. If, as before, $N$ denotes the normalizer of our element $a$ in $S_2$ then there is a resolution of the trivial $N$-module $Z_3$ as

\[
0 \rightarrow Z_3[[N]] \otimes Z_3[G_{12}] \chi \rightarrow Z_3[[N]] \otimes Z_3[G_{12}] \chi \oplus Z_3[[N/G_{12}]] \rightarrow Z_3[[N/G_{12}]] \rightarrow Z_3 .
\]

As we did in passing from the resolution of (4.1) to the tower (4.4), this has a refinement to a tower of fibrations

\[
\begin{array}{ccccccc}
E^{hN} \wedge Y & \rightarrow & Z_1 & \rightarrow & E^{hG_{12}} \wedge Y \\
\Sigma^{34} E^{hG_{12}} \wedge Y & \rightarrow & \Sigma^{35} E^{hG_{12}} \wedge Y \vee \Sigma^{-1} E^{hG_{12}} \wedge Y
\end{array}
\]

There is an obvious projection from the tower (4.4) to the tower (4.6) calculating the map $Y \rightarrow E^{hN} \wedge Y$; algebraically this is given by the evident projection on resolutions. In the case $Y = V(1)$, the topological spectral sequence for $\pi_\ast(E^{hN} \wedge V(1))$ collapses by Propositions 14 and 17 of [5]. This and the differentials of (4.3) determine all differentials in the topological spectral sequence for $V(1)$ itself. Notice that because there are differentials in the homotopy fixed point spectral sequence

\[H^\ast(G_{24}, F_9[u^{\pm 1}]) \Rightarrow \pi_\ast E^{hG_{24}} \wedge V(1)\]

the homotopy gets assembled slightly differently than might be predicted by the spectral sequence for (4.2) for $H^\ast(G_{22}^1, F_9[u^{\pm 1}])$. This discussion is implicit in [5] and Theorem 25 of [9].

Our main use of these topological resolutions is the following result.

**Theorem 4.2.** Let $Y$ be a spectrum so that there are equivalences of module spectra over $E^{hG_{24}}$ and $E^{hSD_{16}}$ respectively

\[E^{hG_{24}} \wedge V(1) \simeq E^{hG_{24}} \wedge Y \quad \text{and} \quad E^{hSD_{16}} \wedge V(1) \simeq E^{hSD_{16}} \wedge Y.\]

Let $\epsilon_Y \in \pi_0(E^{hG_{24}} \wedge Y)$ be the image of $1 \in \pi_0(E^{hG_{24}} \wedge V(1))$ under the isomorphism

\[\pi_0(E^{hG_{24}} \wedge V(1)) \cong \pi_0(E^{hG_{24}} \wedge Y).\]

If $\epsilon_Y$ survives to a class $\overline{\epsilon_Y} : S^0 \rightarrow Y$ of order 3, then $\epsilon_Y$ extends to a $K(2)$-local equivalence $\overline{\epsilon_Y} : V(1) \rightarrow Y$. 
Proof. We will use Proposition 3.8 to produce a commutative square

\[
\begin{array}{ccc}
V(1) & \xrightarrow{\bar{e}Y} & Y \\
\downarrow & & \downarrow \\
E^{hG_{24}} \wedge V(1) & \xrightarrow{\cong} & E^{hG_{24}} \wedge Y
\end{array}
\]

where the bottom map is the given weak equivalence and both vertical maps are the natural maps given by the unit \(S^0 \to E^{hG_{24}}\). Since \(E\) is an \(E^{hG_{24}}\)-module, it will follow that \(E_* \bar{e}Y\) is an isomorphism and that \(\bar{e}Y\) is a \(K(2)\)-equivalence.

By hypothesis we have a commutative square

\[
\begin{array}{ccc}
S^0 & \xrightarrow{eY} & Y \\
\downarrow & & \downarrow \\
E^{hG_{24}} \wedge V(1) & \xrightarrow{\cong} & E^{hG_{24}} \wedge Y
\end{array}
\]

Using the calculation of \(\pi_* E^{hG_{24}}\) displayed in Figure 1, the fact that \(\pi_*(E^{hSD_{16}} \wedge V(1)) \cong \mathbb{F}_3[v_2^{-1}]\) and the spectral sequence of the tower (4.4), we see that \(\pi_3 Y = 0\) and \(\pi_5 Y\) is either zero or isomorphic to \(\mathbb{Z}/3\) on a class which supports a non-zero \(\beta\) multiplication. In the nomenclature above (with \(b_{36}\) changed to \(a_{35}\)) this class is detected by

\[
\zeta \alpha v_2^{-2} a_{35} \in \pi_7 \Sigma^{36} E^{hG_{24}} \cong E^{2,7}_1
\]

and both this class and \(\zeta \alpha \beta v_2^{-2} a_{35}\) must be permanent cycles, because potential differentials go to zero groups. There is a possibility, just based on degree considerations, that either of the two could be hit by a differential; however we see that the only possible such differentials are

\[
d_1(\alpha \beta v_2^{-2} a_{35}) = c_i \zeta \alpha \beta v_2^{-2} a_{35}, \quad i = 0, 1
\]

for some \(c_i \in \mathbb{Z}/3\). Then \(\beta\)-multiplication implies that \(c_0 = c_1\). Thus either both classes are zero at \(E_\infty\) or both are non-zero at \(E_\infty\). \(\square\)

5. Exotic elements of the Picard group and \(V(1)\)

In the Picard group of weak equivalence classes of \(K(2)\)-local spectra there is a subgroup \(\kappa_2\) of elements \(X\) so that \(E_* X \cong E_* = E_* S^0\) as twisted \(G_2\)-modules; these are the exotic elements of the Picard group. In [7], we computed this group to be \((\mathbb{Z}/3)^2\). This extended work of Kamiya and Shimomura ([16]). Here we briefly review those results and then discuss \(X \wedge V(1)\) for various \(X \in \kappa_2\).

If \(X \in \kappa_2\), then we have an Adams-Novikov Spectral Sequence

\[
H^*(G_2, E_*) \xrightarrow{f_*} H^*(G_2, E_* X) \Longrightarrow \pi_* X
\]

where \(f_*\) is determined by the chosen isomorphism \(E_* \cong E_* X\). If we let \(\iota_X\) be the image of \(1 \in H^0(G_2, E_0)\), then \(X \cong L_{K(2)} S^0\) if and only if \(\iota_X\) is a permanent cycle. Shimomura had noticed that the only possible non-zero differential on \(\iota_X\) is \(d_5\) and, in effect, calculated \(H^7(G_2, E_4) \cong (\mathbb{Z}/3)^2\). Kamiya and Shimomura further showed that some of the elements could be realized as targets of differentials from an element in \(\kappa_2\); we completed this task. The following result is found in Proposition 3.1 of [7] and its proof.

**Lemma 5.1.** The reduction map

\[
r_* : H^5(G_2, E_4) \longrightarrow H^5(G_2, E_9[u^\pm 1]_4)
\]

is an injection and the image is generated by the elements \(\beta^2 v_2^{-2} u a\) and \(\zeta \beta v_2^{-3} a a_{35}\).
Remark 5.2. In Proposition 3.1 of [7] we had two generators named $\alpha \beta^2 \Delta^{-1}$ and $\alpha$, where $\alpha$ was shorthand for a class which might have been better labeled $\alpha \zeta e_3$. In the current notation $\alpha \beta^2 \Delta^{-1}$ corresponds to $\beta^2 \nu_2^{-2} w \alpha$ and $\alpha \zeta e_3$ corresponds to $\zeta \nu_2^{-3} \alpha a_{35}$. The different notation resulted from the fact that [7] is built around the topological resolutions of [6], while here we are using the centralizer resolutions of [9].

If $X \in \kappa_2$, we may use Lemma 5.1 to write
\begin{equation}
\label{eq:5.1}
d_5(\iota_X) = [c_1 \beta^2 \nu_2^{-2} w \alpha + c_2 \zeta \beta \nu_2^{-3} \alpha a_{35}] \iota_X
\end{equation}
defining
\[
a_X \iota_X
\]
for some $c_1$ and $c_2$ in $\mathbb{Z}/3$. This expression is multiplicative in the following sense. Suppose $X$ and $Y$ are in $\kappa_2$ and we have chosen isomorphisms $E_*X \cong E_* \cong E_*Y$ as twisted $\mathbb{G}_2$-modules. These choices determine an isomorphism $E_*(X \wedge Y) \cong E_*$ and a multiplicative pairing of spectral sequences
\begin{equation}
\label{eq:5.2}
H^*(\mathbb{G}_2, E_*) \otimes H^*(\mathbb{G}_2, E_*) \longrightarrow \pi_*X \otimes \pi_* Y
\end{equation}
where $m$ is induced by the product on $E_*$. We have
\begin{equation}
\label{eq:5.3}
d_5(\iota_{X \wedge Y}) = (a_X + a_Y) \iota_{X \wedge Y}
\end{equation}
and the main result of [7] is that $d_5$ defines an isomorphism
\[
d_5 : \kappa_2 \cong H^5(\mathbb{G}_2, E_4).
\]
The restriction maps define a commutative square
\[
H^5(\mathbb{G}_2, E_4) \longrightarrow H^5(\mathbb{G}_4, E_4)
\]
with the map $r_* : H^5(\mathbb{G}_{24}, E_4) \longrightarrow H^5(\mathbb{G}_{24}, F_9[u^{+1}]_4)$ an isomorphism. The source of $r_*$ is isomorphic to $\mathbb{Z}/3$ generated by $\alpha \beta^2 \Delta^{-1}$, which reduces to $\pm \alpha \beta^2 w^{-3} = \beta^2 \nu_2^{-2} w \alpha$. Since $d_5 \Delta = \pm \alpha \beta^2$, it follows that if $X \in \kappa_2$, then $X \wedge E^h \mathbb{G}_{24} \cong \Sigma^{24k} E^h \mathbb{G}_{24}$, with $k \in \mathbb{Z}$. Furthermore, because $E^h \mathbb{G}_{24}$ is 72-periodic on $\Delta^3$ we may assume $k \in \{0, 1, 2\}$. The assignment $X \mapsto k \mod (3)$ defines a short exact sequence
\begin{equation}
\label{eq:5.4}
0 \rightarrow \mathbb{Z}/3 \rightarrow \kappa_2 \rightarrow \mathbb{Z}/3 \rightarrow 0.
\end{equation}
We call $X \in \kappa_2$ truly exotic if $k = 0$; this is equivalent to having $c_1 = 0$ in the formula (5.1). This sequence is split; part of what this section accomplishes is to provide a distinguished splitting.

Proposition 5.3. Let $X \in \kappa_2$ be a non-trivial truly exotic element in $\kappa_2$. Then $\pi_*(X \wedge V(1))$ is not a free module over $\Lambda(\zeta)$. In particular $\pi_*X$ cannot be isomorphic to a shift of $\pi_*V(1)$.

Proof. We will use both the algebraic and topological spectral sequences determined by the centralizer resolution. Since $E_*X \cong E_*$ as twisted $\mathbb{G}_2$-modules, the algebraic spectral sequence for computing $H^*(\mathbb{G}_2, E_*(X \wedge V(1))$ is isomorphic to the spectral sequence for $H^*(\mathbb{G}_2, E_*V(1))$. Thus we have exactly the differentials of (4.3). In particular
\[
d_2(\alpha w^{2+1} b_{36}) = \pm \nu_2^3 e_{48}.
\]
In the topological spectral sequence given by the tower (4.4) we must have
\[ d_2(t_X) = \pm \zeta v_2^{-3} a a_{35} t_X. \]
This means that at \( E_3 \) the class
\[ y = \beta v_2^{-3} a a_{35} t_X \in E_3^{1,*} \]
is annihilated by \( \zeta \). Since \( E_3^{s,t} = 0 \) for \( s > 2 \) and \( t - s = -1 \), we also have that \( y \) is a permanent cycle and detects a homotopy class annihilated by \( \zeta \). Since \( y \in E_1^{1,*} \) and is not divisible by \( \zeta \) at \( E_1 \), the homotopy class detected by \( y \) cannot be divisible by \( \zeta \). \( \square \)

**Lemma 5.4.** There is a unique element \( P \in \kappa_2 \) so that
\[ v_2^3 \in H^*(G_{24}, E_*(P \wedge V(1))) \]
is a permanent cycle in the Adams-Novikov Spectral Sequence converging to \( \pi_*(P \wedge V(1)) \).

**Proof.** There exist three elements \( Y \in \kappa_2 \) so that \( v_2^3 \in H^0(G_{24}, \mathbb{F}_9[u^\pm 1]) \) is a permanent cycle. Fix one. We conclude that the only possible non-zero differential is then
\[ d_5(v_2^3 t_Y) = c \zeta \beta a a_{35} t_Y \]
where \( c \in \mathbb{Z}/3 \). Using the diagram of spectral sequences (5.2) we then see that there is a unique truly exotic element \( Q \in \kappa_2 \) so that
\[ d_5(v_2^3 t_Q) = -c \zeta \beta a a_{35} t_Q \]
Then \( P = Y \wedge Q \). That \( P \) is unique follows from the exact sequence (5.4). \( \square \)

**Theorem 5.5.** There is a unique element \( P \in \kappa_2 \) so that there is a weak equivalence
\[ \Sigma^{48} V(1) \simeq P \wedge V(1). \]

**Proof.** We first prove existence. Let \( P \) the element of \( \kappa_2 \) identified in Lemma 5.4. Since \( v_2^3 \) is a permanent cycle in \( H^*(G_{24}, E_*(P)) \) by restriction, we have that
\[ \Sigma^{48} E^{hG_{24}} \simeq E^{hG_{24}} \wedge P. \]
It is also automatic that \( E^{hSD_{16}} \wedge P \simeq E^{hSD_{16}} \) as the spectral sequence
\[ H^s(SD_{16}, E_1 P) \Rightarrow \pi_{t-s}(E^{hSD_{16}}) \wedge P \]
collapses to the \( s = 0 \) line. Since \( E^{hSD_{16}} \) is 16-periodic, we can now apply Theorem 4.2 to the spectrum \( \Sigma^{-48} P \) to produce the equivalence \( V(1) \simeq \Sigma^{-48} P \wedge V(1) \).

To get uniqueness, notice that an equivalence \( \Sigma^{48} V(1) \rightarrow Y \wedge V(1) \) produces an isomorphism \( E_* \Sigma^{48} V(1) \cong E_*(Y \wedge V(1)) \). Since \( H^0(G_{24}, \mathbb{F}_9[u^\pm 1]_{|_{48}}) \cong \mathbb{Z}/3 \) generated by \( v_2^3 \), we have that \( v_2^3 \) is a permanent cycle in the Adams-Novikov spectral sequence for \( Y \wedge V(1) \) and Lemma 5.4 applies. \( \square \)

**Remark 5.6.** In [15], Ichigi and Shimomura calculated \( \pi_*(X \wedge V(1)) \) for some \( X \) in the Picard group. These results are closely related to the calculations done in this section.
6. The calculation of $S^0(\det) \wedge V(1)$

In this section we construct a weak equivalence $\Sigma^2 V(1) \simeq S^0(\det) \wedge V(1)$. We begin by reviewing the construction of $S^0(\det)$. This material is discussed in more detail in §2 of [7].

The determinant map $G_2 \to \mathbb{Z}_3^\times$ of (2.1) defines a $G_2$-module $\mathbb{Z}_3(\det)$. If $M$ is any other $G_2$-module, we write

$$M(\det) = M \otimes_{\mathbb{Z}_3} \mathbb{Z}_3(\det) \ .$$

Define $SG_2 \subseteq G_2$ to be the kernel of $\det: G_2 \to \mathbb{Z}_3^\times$. The image of the central element $\psi^4 := 4 + 0.S \in G_2$ with respect to this homomorphism is a topological generator of the subgroup $1 + 3\mathbb{Z}_3$.

Let $(E^{hSG_2})^-$ be the wedge summand of $E^{hSG_2}$ realizing the $(-1)$-eigenspace of the action of $C_2 \subseteq \mathbb{Z}_3^\times$. Then $S^0(\det)$ is defined as the fibre of the self map $\psi^4 - \det(\psi^4)id$, i.e. there is a fibration

$$S^0(\det) \xrightarrow{\psi^4 - \det(\psi^4)id} (E^{hSG_2})^-$$

from which one deduces the isomorphism $E_*(S^0(\det)) \cong E_*(\det)$.

We note that $SG_2$ is of index 2 in the subgroup $G_2^1$ and therefore the residual action of the quotient group gives a splitting

$$E^{hSG_2}_2 \simeq (E^{hSG_2}_2)^+ \vee (E^{hSG_2}_2)^- \simeq E^{hG_2}_2 \vee (E^{hSG_2}_2)^-$$

into its $(+1)$-eigenspectrum and its $(-1)$-eigenspectrum.

Likewise the inclusion of the index 2 subgroup $G_2^0 \subseteq G_2$ given as the kernel of the homomorphism

$$G_2 \xrightarrow{\det} \mathbb{Z}_3^\times \xrightarrow{\psi^4} \mathbb{Z}_3^\times / 1 + 3\mathbb{Z}_3 \cong C_2 \ .$$

gives a residual action of $C_2$ on $E^{hG_2}_2$ and an associated splitting

$$E^{hG_2}_2 \simeq (E^{hG_2}_2)^+ \vee (E^{hG_2}_2)^- \simeq E^{hG_2}_2 \vee (E^{hG_2}_2)^-$$

into its $(+1)$-eigenspectrum and its $(-1)$-eigenspectrum.

**Proposition 6.1.** Let $X$ be any $K(2)$-local spectrum so that the identity of $X$ has order 3. Then

$$E^{hG_2}_2 \wedge X \simeq X \vee (S^0(\det) \wedge X)$$

and

$$(E^{hG_2}_2)^- \wedge X \simeq S^0(\det) \wedge X \ .$$

**Proof.** We observe that the assumption on $X$ shows that there is a fibration

$$S^0(\det) \wedge X \xrightarrow{\psi^4 - \det(\psi^4)id} (E^{hSG_2}_2)^- \wedge X \ .$$

By using the splitting (6.2) and by taking the wedge sum of this fibration with the fibration

$$X \simeq E^{hG_2}_2 \wedge X \xrightarrow{(\psi^4 - id)\wedge id} E^{hG_2}_2 \wedge X$$

we get a fibration

$$X \vee (S^0(\det) \wedge X) \xrightarrow{(\psi^4 - id)\wedge id} E^{hSG_2}_2 \wedge X \ .$$
Because $\psi^4$ lies in the subgroup $\mathbb{G}_2^0$ and its image in the quotient $\mathbb{G}_2^0/\mathbb{S}_2 \cong \mathbb{G}_2/\mathbb{S}_2 \cong \mathbb{Z}_3$ is a topological generator we see that the fibre of the self map of $E^{h\mathbb{G}_2} \wedge X$ given by $(\psi^4 - \text{id}) \wedge \text{id}_X$ is also equivalent to $E^{h\mathbb{G}_2} \wedge X$. This establishes the first part of the proposition.

The second part is a consequence of (6.4) and the fact that $\mathbb{Z}_3^h$ is an abelian group so that the actions of the two factors $C_2$ and $\mathbb{Z}_3$ on $E^{h\mathbb{G}_2}$ commute. \hfill \Box

We noted in Remark 2.1 that the identity of $V(1)$ has order 3 and so the proposition can be applied to $V(1)$. Together with Theorem 3.5 this immediately gives the following result.

**Corollary 6.2.** There is an isomorphism of modules over $\mathbb{F}_2[\beta, v^{\pm 9}_2] \otimes \Lambda(\alpha, \zeta)$

$$\pi_* \Sigma^{72} V(1) \cong \pi_* (S^0(\det) \wedge V(1)).$$

We need one more ingredient before we can identify $S^0(\det) \wedge V(1)$. Compare Remark 3.10.

**Lemma 6.3.** Multiplication by $v^{3/2}_2$ defines an isomorphism of twisted $\mathbb{G}_2$-modules

$$\Sigma^{72} E_* V(1) \cong E_* (S^0(\det) \wedge V(1)).$$

**Proof.** There is an isomorphism

$$E_* S^0(\det) \cong E_* (\det) = E_* \otimes \mathbb{Z}_3(\det).$$

If we write $g = a + bS \in \mathbb{S}_2$, then $\det(g) = a^4$ modulo 3. Furthermore, in $E_* V(1) \cong \mathbb{F}_2[\alpha^{\pm 1}]$, $g$ acts trivially on degree 0 and $gu = au$. Thus, in $E_* (S^0(\det) \wedge V(1))$,

$$g(u^{8i+4}) = a^{8i+8} u^{8i+4} = u^{8i+4}$$

as $a^8 = 1$ modulo 3. In particular $v^{9/2}_2 = u^{-36}$ is invariant, as needed. \hfill \Box

**Theorem 6.4.** There is a weak equivalence of $K(2)$-local spectra

$$\Sigma^{72} V(1) \simeq S^0(\det) \wedge V(1).$$

**Proof.** This follows from Theorem 3.9, Corollary 6.2, and Lemma 6.3. \hfill \Box

7. **The Brown-Comenetz dual of the sphere**

The results of this section complete the proof of Theorem 1.1. Let $P \in \kappa_2$ be the exotic element of the Picard group singled out in Theorem 5.5.

**Theorem 7.1.** Let $n = 2$ and $p = 3$. The Brown-Comenetz dual $I_2$ of $L_{K(2)} S^0$ is given as

$$I_2 \simeq S^2 \wedge S^0(\det) \wedge P.$$

**Proof.** We know from the Gross-Hopkins formula (1.1) and [7], that

$$I_2 \simeq S^2 \wedge S^0(\det) \wedge P^{\wedge n} \wedge Q$$

where $0 \leq n \leq 2$ and $Q \in \kappa_2$ has the property that $E^{hG_{24}} \wedge Q \simeq E^{hG_{24}}$. Combining Theorems 5.5 and 6.4 we have

$$S^2 \wedge S^0(\det) \wedge P^{\wedge n} \wedge V(1) \simeq \Sigma^{48n+74} V(1).$$

From this and Proposition 5.3, we can conclude that if $Q$ is non-trivial, then $\pi_*(I_2 \wedge V(1))$ is not free over $\Lambda(\zeta)$, which contradicts Theorem 3.7. Finally, again using Theorem 3.7, we need

$$48n + 74 \equiv -22 \pmod{144}$$

as $V(1)$ is 144-periodic. Thus $n = 1$. \hfill \Box
Corollary 7.2. There is a weak equivalence of $K(2)$-local spectra

$$\Sigma^{-22}V(1) \sim I_2 \wedge V(1).$$

Proof. Using Theorems 5.5, 6.4, and 7.1, we have

$$I_2 \wedge V(1) \simeq S^2 \wedge S^0 \langle \det \rangle \wedge P \wedge V(1) \simeq \Sigma^{2 + 72 + 48} V(1) \simeq \Sigma^{-22} V(1)$$

as $V(1)$ is 144-periodic. \qed

8. Appendix: The homotopy groups of $E^{hG_2} \wedge V(1)$

We display two charts, distilled from [5]. In these charts we adhere to the following conventions:

1. Each dot represents a group isomorphic to $\mathbb{Z}/3$.
2. The horizontal scale is by degree in homotopy groups; the vertical scale is cosmetic only.
3. Both spectra are periodic; the bold part of the chart is one copy of the basic pattern in homotopy; the lighter parts are the periodic copies.
4. Horizontal line segments represent multiplication by $\beta = \beta_1$; the diagonal line segments represent multiplication by $\alpha = \alpha_1$.
5. The class $w = \omega^2 v_2^{1/2}$ has the property that $w^2 = -v_2$.
6. In Figure 1, $z_i = (\alpha, \alpha, \beta^2 w^i)$, $0 \leq i \leq 2$.
7. In Figure 2, the leading term of each pattern is labeled with a name which arises from the calculations in the cohomology of various subgroups of $G_2$; see 3.5 for details.
8. The element 1 has degree 0, $\alpha$ has degree 3, $\beta$ had degree 10, $w^n$ has degree $8n$, $v_2^n$ has degree $16n$, and $a_{35}$ has degree 35. But note that $a_{35}$ itself does not appear as a homotopy class. As an example, in figure 2 the element $v_2^5 w_2 a_{35}$ has degree $5 \cdot 16 + 8 + 10 + 35 = 133$ and the pattern on that element extends to degree 173.

The first chart displays the homotopy groups of $E^{hG_2} \wedge V(1)$. This chart can also be used to read off $\pi_*(E^{hG_2} \wedge V(1))$ as

$$(8.1) \quad \pi_*(E^{hG_2} \wedge V(1)) \cong F_9 \otimes \pi_*(E^{hG_2} \wedge V(1)).$$

There is one basic pattern, beginning on 1, repeated three times. This spectrum is 72-periodic on the class $w^9 = \omega^2 v_2^{9/2}$.

![Figure 1: The homotopy groups of $E^{hG_2} \wedge V(1)$](image-url)

The second chart displays the homotopy of $E^{hG_2} \wedge V(1)$. There are two basic patterns, one beginning on 1, the other on $w \alpha$. To each of these two patterns there is also a dual...
pattern visible on $a_{35} \beta$ and $a_{35} \alpha$ respectively. This spectrum is 144-periodic on the class $v_2^9$.

The homotopy of $V(1)$ itself can be recovered from

$$\pi_* V(1) = \Lambda(\zeta) \otimes \pi_*(E^{hG_2} \wedge V(1))$$

where $\Lambda(\zeta)$ is the exterior algebra on a class of degree $-1$. 
Figure 2: The homotopy groups of $E_{hG_1 \vee V(1)}$. 

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REFERENCES

1. Behrens, Mark, “A modular description of the $K(2)$-local sphere at the prime 3”, Topology, 45 (2006), no. 2, 343–402.
2. Behrens, Mark and Pemmaraju, Satya, “On the existence of the self map $v_2$ on the Smith-Toda complex $V(1)$ at the prime 3”, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, Contemp. Math. 246, 9–49. Amer. Math. Soc., Providence RI 2004.
3. Brown, Jr., Edgar H. and Comenetz, Michael, “Pontrjagin duality for generalized homology and cohomology theories”, Amer. J. Math., 98 (1976), no. 1, 1–27.
4. Devinatz, Ethan S. and Hopkins, Michael J., “Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups”, Topology 43 (2004), no.1, 1–47.
5. Goerss, Paul and Henn, Hans-Werner and Mahowald, Mark, “The homotopy of $L_2 V(1)$ for the prime 3”, Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), Progr. Math., 213, 125–151, Birkhäuser, Basel, 2004.
6. Goerss, P. and Henn, H.-W. and Mahowald, M. and Rezk, C., “A resolution of the $K(2)$-local sphere at the prime 3”, Ann. of Math. (2) 162 (2005) no. 2, 777-822.
7. Goerss, P. and Henn, H.-W. and Mahowald, M. and Rezk, C., “On Hopkins' Picard groups for the prime 3 and chromatic level 2”, J. of Top. 8 (2015), 267-294.
8. Henn, Hans-Werner, “Centralizers of elementary abelian $p$-subgroups and mod-$p$ cohomology of profinite groups”, Duke Math. J., 91 (1998), no. 3, 561–585.
9. Henn, Hans-Werner, “On finite resolutions of $K(n)$-local spheres”, Elliptic cohomology, London Math. Soc. Lecture Note Ser., 342, 122–169, Cambridge Univ. Press, Cambridge, 2007.
10. Hopkins, M. J. and Gross, B. H., “The rigid analytic period mapping, Lubin-Tate space, and stable homotopy theory”, Bull. Amer. Math. Soc. (N.S.), 30 (1994), no. 1, 76-86.
11. Hopkins, M. J. and Gross, B. H., “Equivariant vector bundles on the Lubin-Tate moduli space”, Topology and representation theory (Evanson, IL, 1992), Contemp. Math., 158, 23–88, Amer. Math. Soc., Providence, RI, 1994.
12. Hopkins, Michael J. and Mahowald, Mark and Sadofsky, Hal, “Constructions of elements in Picard groups”, Topology and representation theory (Evanson, IL, 1992), Contemp. Math. 158, 89–126, Amer. Math. Soc., Providence, RI, 1994.
13. Hovey, Mark and Strickland, Neil P., Morava K-theories and localisation, Mem. Amer. Math. Soc., 139 (1999), no. 666.
14. Ichigi, Ippei and Shimomura, Katsumi, “$E(2)$-invertible spectra smashing with the Smith-Toda spectrum $V(1)$ at the prime 3”, Proc. Amer. Math. Soc., 132 (2004), no. 10, 3111–3119.
15. Kamiya, Yousuke and Shimomura, Katsumi, “A relation between the Picard group of the $E(n)$-local homotopy category and $E(n)$-based Adams spectral sequence”, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, Contemp. Math. 346, 321–333, Amer. Math. Soc., Providence, RI, 2004.
16. Ravenel, Douglas C., Complex cobordism and stable homotopy groups of spheres, Pure and Applied Mathematics, 121, Academic Press, Inc., Orlando, FL, 1986.
17. Shimomura, Katsumi, “The homotopy groups of the $L_2$-localized Toda-Smith spectrum $V(1)$ at the prime 3”, Trans. Amer. Math. Soc., 349 (1997), no. 5, 1821–1850.
18. Strickland, N. P., “On the $p$-adic interpolation of stable homotopy groups”, Adams Memorial Symposium on Algebraic Topology, 2 (Manchester, 1990), London Math. Soc. Lecture Note Ser., 176, 45–54, Cambridge Univ. Press, Cambridge, 1992.
19. Strickland, N. P., “Gross-Hopkins duality”, Topology, 39 (2000), no. 5, 1021–1033.

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