Spin Decoherence from Hamiltonian dynamics in Quantum Dots

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The dynamics of a spin-1/2 particle coupled to a nuclear spin bath through an isotropic Heisenberg interaction is studied, as a model for the spin decoherence in quantum dots. The time-dependent polarization of the central spin is calculated as a function of the bath-spin distribution and the polarizations of the initial bath state. For short times, the polarization of the central spin shows a gaussian decay, and at later times it revives displaying nonmonotonic time dependence. The decoherence time scale depends on moments of the bath-spin distributions, and also on the polarization strengths in various bath-spin channels. The bath polarizations have a tendency to increase the decoherence time scale. The effective dynamics of the central spin polarization is shown to be described by a master equation with non-markovian features.

I. INTRODUCTION

Decoherence is ubiquitous in quantum systems, either because of the interaction with the environment \((I)\), or because of interventions from the measuring apparatus \((\mathcal{E})\). An understanding of the phenomenon of decoherence is, therefore, essential – from the view point of the foundational issues as well as the dynamics of open quantum systems. In this context, \(N\)-level systems are particularly interesting; the state is defined in a finite dimensional Hilbert space (the associated phase space is compact), which makes the theoretical analysis simpler. More importantly, there is an abundance of \(N\)-level systems as encountered in NMR systems \((\mathcal{E})\), NQR systems, and polarized photons. In addition, one also has the so called pseudo \(N\)-level systems – occurring in atomic, semiconductor and quantum dot environments. Although the states of the latter systems are defined in infinite dimensional Hilbert spaces, energetics (at sufficiently low temperatures) effectively restrict the states to lie in a finite dimensional subspace. It is no surprise that the dynamics of \(N\)-level systems (spin systems in short) has been extensively studied.

Decoherence in spin systems has acquired a new importance in view of the rapid developments which are taking place in quantum information theory. Controllable quantum gates are central to quantum computation, and are built of qubits which are physically realized through spin-half (or pseudo spin-half) systems. Proposals for qubits include the spin of electrons \((I)\), nuclear spins \((\mathcal{E})\) and squids \((\mathcal{E})\). Photons are also potential candidates since highly entangled states have been prepared experimentally by employing the polarization degree of freedom \((I)\). Recently, fullerenes-based single-electron transistors \((\mathcal{E})\) have also been proposed where the electron spin acts as the qubit. Finally, we mention yet another important example involving the spin of the electrons, viz, quantum dot quantum computers (QDQC) \((\mathcal{E})\). An additional feature of QDQC is that the number of qubits (electrons) can be controlled precisely: starting from zero, electrons can be added one by one. The preparation of quantum gates and their manipulation should be simpler in QDQC.

Minimising the decoherence involving the electron spin in various environments is, therefore, of great importance. We study in this paper the dynamics of decoherence with a special focus on QDQC. However, our method possesses wider applicability to instances such as decoherence in fullerenes, and to spin systems where a given reference spin - the central spin - decoheres due to its interaction with the bath constituted by the other spins. In QDQC systems, the environment is constituted by GaAs, (Ga,Al)As, and InAs. The nuclei \(^{69}Ga\), \(^{71}Ga\), \(^{27}Al\) and \(^{115}In\) have non-vanishing spins and magnetic moments, as shown in the parentheses. The magnetic moments are written in units of nucleon Bohr magneton. Since the qubit itself carries a large magnetic moment, the dominant contribution to the decoherence comes from the interaction between the spins. Indeed, decoherence times have been measured experimentally for qubits both in quantum dot ensembles and single quantum dots. These studies find a long decoherence time \((\sim 100\,\text{ns})\) in quantum dot ensembles \((\mathcal{E})\). Similarly, studies on a single quantum dot \((\mathcal{E})\) indicate that the spin flip rate is low, implying a long decoherence time again.

Theoretically, there have been a large number of investigations, summarized in a recent paper by Schliemann et al \((\mathcal{E})\). A common feature that underlies these investigations is the assumption of a mean magnetic field \(\mathcal{B}\) that is produced by the nuclei which constitute the environment. This assertion is based on the observation that there are \(10^3 - 10^9\) nuclei, all with non-vanishing magnetic moments, that interact with the qubit \((\mathcal{E})\). In this semiclassical approach, it is the time dependence in \(\mathcal{B}\) that causes the decoherence in the spin state. Reliable estimates for the decoherence rates have been obtained both analytically and numerically.

This paper revisits the problem of decoherence with a rather different perspective. We show that given the phenomenology of QDQC, it is possible to map the hyperfine interaction between the qubit and the nuclei to an isotropic Heisenberg model with a global interaction between the qubit spin and the total nuclear spin. More precisely, the qubit interacts with a spin...
bath, whose initial state is described by a density matrix \( \rho_B = \sum \lambda_I \rho_{I_B} \), where the coefficients \( \lambda_I \) are the weights for the bath to be in a state with a total spin \( I_B \).

Within each spin sector of the bath, the hyperfine interaction between the bath spin and the qubit spin gets recast into an isotropic Heisenberg interaction. With this simplification, the dynamics becomes exactly solvable: the state of the qubit+bath can be determined in a closed form explicitly. A partial trace yields the reduced density matrix of the qubit. The decoherence rate is easily inferred thereafter.

In the next section we obtain an effective Hamiltonian from the basic hyperfine interaction (for QDQC), and elaborate on the model in some detail. We then illustrate the dynamics in the simplest of the cases, viz., of two qubits in section III. This example is of practical importance, exhibiting the so called interaction induced entanglement. After the warm up, the general problem of a qubit interacting with a spin bath (as described above) will be addressed. The expression for the reduced density matrix and the dynamics of decoherence will be investigated at length. Section IV is devoted to examples and special cases. Section V shows how the results can be extended to more general local Heisenberg interactions perturbatively, at small times. In Section VI, we shall set up a master equation satisfied by the reduced density matrix. The non-markovian nature of the evolution, the time-dependent decoherence rates, and the unitary part of the evolution will be clearly identified.

II. THE MODEL

A dominant interaction in QDQC between the electron spin and the nuclei is given by the hyperfine coupling of the electron spin to the nuclear spins:

\[
H_{hf} = \sum_i K_i \vec{S} \cdot \vec{I}_i
\]  

where \( K_i \) is the interaction strength between the qubit spin \( \vec{S} \) and the \( i \)'th nuclear spin \( \vec{I}_i \). The coupling constant \( K_i \) depends on the basic coupling strength and the wave function of the electron, \( K_i = K|\Psi(r)|^2 |\vec{p} = \vec{p}_i| \), where the electron density is evaluated at the location of the nucleus. Merkulov et al. [20] have pointed out that the spin orbit interaction, which is otherwise dominant in spin decoherence, is suppressed here because the qubit is strongly localized. For the same reason the contribution of the phonon mediated interactions is also suppressed at low temperatures [20]. Thus the hyperfine interaction between the electron and the nuclei emerges as the dominant mechanism for spin relaxation.

In addition to the hyperfine interaction, the electron spin may also be subjected to an external magnetic field which is responsible for its polarization. Alternatively, a spin polarized electron can be directly injected into a quantum dot, as Cortez et al. [21] have done. Remarkably, they have also succeeded in ‘writing’ and then subsequently ‘reading’ a quantum dot. We shall be interested in this “zero field” scenario. In either case, the nuclear coupling to the field is relatively negligible because of the smaller values of the nuclear magnetic moments. Finally, mention should be made of the internuclear dipolar coupling. This coupling (strength is of order \( 10^{-12} \text{eV} \)) is a small perturbation to the other interaction terms. But it has an important role in determining the initial spin state of the nuclei, and its effect on the subsequent dynamics is negligible [13].

As mentioned earlier, each electron is in an environment of \( 10^4 \) – \( 10^6 \) nuclei. It has been assumed, therefore, that the cumulative effect of the nuclear spins is to produce an effective magnetic field \( B \), the Overhauser field, at the site of the qubit. Likewise, the action of the electron spin on the nuclei makes \( B \) time dependent, which in turn causes spin flip transitions in the qubit [17, 18, 20]. For an explicit determination of the transition amplitudes, computational schemes in this semiclassical approach involve either taking a gaussian distribution for \( B \) or considering special configurations for the initial state [22, 23]. The contribution of intra-nuclear interaction to the time-dependent magnetic field, and the resulting spectral diffusion of the qubit polarization were considered by de Sousa and Das Sarma [21], and Deng and Hu [22]. Let us summarize the semiclassical results in brief. Of the two decoherence times, it is is found that the phase decoherence time is larger; standard spin flip transition calculations (using the Fermi golden rule) yield decay profiles which are nonexponential [22]. More interestingly, the same authors claim that for a large Zeeman field, the polarization of the electron decays \( \sim t^{-\frac{3}{2}} \). This surprising result is obtained for a special choice for the initial state of the qubit-nuclear system. Numerical simulations by Schliemann et al. [19] for a variety of nuclear spin states have yielded several interesting results: (i) the polarization of the electron decreases in magnitude, with accompanying oscillations whose period is estimated to be \( 4r/K \), with only a weak dependence on the initial state; (ii) if the nuclei are initially only randomly correlated, the spin dynamics of the electron is significantly different in that the decay is much faster. A master equation for the qubit density matrix was set up, and the dynamics was investigated perturbatively by Coish and Loss [20], where the effect of the electronic wave function on the qubit decoherence was also considered.

Interesting that these results are, the semiclassical approximation needs a validation from a more rigorous analysis. Indeed, the hyperfine Hamiltonian is itself not easily amenable to a full quantum treatment. We nevertheless argue that the phenomenology of QDQC allows us to map the problem to the more tractable isotropic Heisenberg interaction. To that end, consider a circular geometry for the quantum dot containing the electron [19]. Taking the confinement to be either parabolic or coulombic, the electron wave function \( \psi(\vec{r}) \) is either a gaussian or an exponential with the distance. The ef-
fective coupling to the nuclear spins which are farther is accordingly suppressed, with the maximum contribution coming from the nearest nuclei, all of which are roughly equidistant from the electron. Indeed, in GaAs there are about 45 nuclei in a volume of 1 nm$^3$, and the electron wave function is roughly uniform over a distance of 2-3 nm (while the size of the quantum dot is about 20 nm). This translates into about a few hundred nuclei that interact with the qubit with the same coupling strength $K_i$ in Eq. 1 and thus the Hamiltonian assumes a simple form

$$H_{eff} = K \vec{S} \cdot \vec{I}_B,$$

(2)

where $\vec{I}_B = \sum i \vec{I}_i$ is the total spin of the (nearest) nuclei. If we were to consider the nuclei in the nearest circle (next to nearest neighbors in these two dimensional samples), the Hamiltonian acquires additional terms with another effective coupling constant $K'$ whose value is exponentially suppressed relative to $K$, and may be treated as a (small) perturbation. We shall ignore all such higher order contributions here. The dynamics has been effectively mapped, to an excellent approximation, to an isotropic Heisenberg interaction between the qubit spin and the total nuclear bath spin.

The above Hamiltonian can have a more general role: consider an assembly of $N$ identical spin-half particles interacting through a Heisenberg interaction. We can write the Hamiltonian of the spin system as

$$H_{spin} = \sum_{i,j} K_{ij} \vec{S}_i \cdot \vec{S}_j,$$

(3)

where $K_{ij}$ is the interaction strength of a pair of spins $\vec{S}_i$ and $\vec{S}_j$. For any chosen spin, denote it by $\vec{S}$, the other spins would constitute a bath. We can split the above into terms involving the interaction of the spin and the bath, and an intra-bath term, as

$$H_{spin} = K_1 \vec{S} \cdot \vec{I}_{B_1} + K_2 \vec{S} \cdot \vec{I}_{B_2} + \ldots H_{bath},$$

(4)

where we displayed the interaction strength $K_1$ ($K_2$) of the central spin with the total spin of the first (second) neighbours $\vec{I}_{B_1}$ ($\vec{I}_{B_2}$). And similarly there are interactions with further neighbours. The last term is the intra-bath interaction. In these spin systems typical interaction strengths are $K_1 \sim 0.1eV$ and $K_2 < K_1$. The number of nearest neighbours is about 6, depending on the lattice structure. Keeping the dominant interaction (the first term) of the qubit with the bath, the problem reduces to the effective Hamiltonian given in Eq.2. Consequently, our fore-going analysis for the qubit dynamics is applicable to these spin systems as well. The dynamics in these spin systems has been studied at length in the context of entanglement generation and propagation.

We will investigate in Sections III and IV the qubit dynamics exactly for the model Hamiltonian given in Eq.2, as a function of the initial nuclear bath spin distribution and nuclear polarization strengths in various spin sectors. As we will see later, through the time evolution, the qubit polarization displays a decoherence regime (characterized by a gaussian decay with a decoherence time scale $\tau$) for short times. The decoherence time depends on the initial bath state. For longer times, the qubit polarization revives, as a consequence of the quantum coherent evolution of the total system of the qubit and the bath. The revival time (Poincare time scale) for the subsystem is large in comparison with $\tau$. However, for times larger than $\tau$ the sub-dominant interactions, which were ignored in the simplified model Hamiltonian, will come into play. The revival time becomes larger when the next to leading interaction is considered, as the revival time scale is set by the smallest interaction strength. Thus the revival time in a model Hamiltonian dynamics has no significance for the real system, as the weaker interactions are ignored in the model, which will push the revival time further. The decoherence regime boundary, determined by the dominant interactions contained in the model Hamiltonian, is not affected too much by the inclusion of the sub-dominant interactions. As we will see in Section IV, the decoherence time scale from our model Hamiltonian compares very well with the experimental numbers, implying that our model is an excellent approximation.

III. DECOHERENCE WITH HEISENBERG INTERACTIONS

A. The initial state

It is straightforward to specify the initial state of the central spin and the nuclear spin bath. As mentioned earlier, we consider the case where a prepolarised electron is injected into the quantum dot. Its state is initially uncorrelated with the nuclear state. We shall take the state of the total system to be a direct product of the qubit state and the nuclear bath state. At $T = 0$, the bath will be in its ground state which is sensitive to the inter-nuclear interactions. For example, a ferromagnetic interaction coupling would lead to a fully-polarised bath state. This would give rise to a large nuclear magnetic field acting on the qubit spin. At nonzero temperatures, the bath will be in a mixed state with contributions from many bath spin sectors with weights for each spin sector $I_B$ (to be fixed by the energetics and the temperature). We will consider the case where the intra-bath dynamics conserves the total bath spin. The density matrix of the bath spins may be written, on general considerations as $\rho_B = \sum \lambda_I \rho_I$ where the sum is over all the possible values of the total bath spin that the dominantly-interacting nuclei can take. In each spin sector the bath can have polarizations of various rank. The dynamics of the Heisenberg interaction between the qubit and the bath, does not mix different bath spin sectors. We can determine the dynamics of the qubit and bath in different spin sectors separately.
A further comment is in order here. Consider the nuclei in a state with a given value for the total spin $I_B$. It is customary to consider two extreme situations: when the nuclear magnetic field has a nonzero expectation value, and when only the fluctuating values survive. In the latter case the bath is usually dubbed as unpolarized. For a maximally-unpolarised bath, the density matrix is $\rho_B = \frac{1}{\mathcal{I}}$. Then, not only does $\langle I_B \rangle$ vanish, so do all higher order moments involving the multilinear operators in the spin operators. By projection theorem, the expectation values of all the moments of the magnetic field operator also vanish. There is no a priori nuclear magnetic field, and it has to be necessarily induced by the qubit. On the other hand, should the system be prepared in such a way that $\langle I_B \rangle = 0$, but $\langle I_B^n I_B^m \rangle$ is non vanishing, the system is not unpolarized; only the vector polarization is zero. It is of course possible that higher rank tensor polarizations may survive. We shall clarify the contributions coming from tensor polarizations of various ranks, i.e., of various higher order magnetic field fluctuations later.

Finally, we shall take all the nuclei to be spin-1/2 particles for simplicity. In reality, the nuclei do have higher spins, but the robustness of the analysis is unaffected by this approximation. As a warm-up example, let the bath be comprised of a single spin-half nucleus, as in the following.

**B. Interaction between two spin-half particles**

Though simple, this case is important because it shows how entanglement is generated by interactions. The Hamiltonian is given by $H = K S \cdot I_B$. Let the initial state be a direct product state,

$$\rho(0) = \frac{1}{2} (| I \rangle \langle I | + \vec{P}_A(0) \cdot \vec{\sigma}_A) \otimes \frac{1}{2} (| I \rangle \langle I | + \vec{P}_B(0) \cdot \vec{\sigma}_B),$$

where $\vec{P}_A$ and $\vec{P}_B$ denote the initial polarization vectors of the qubit and the bath respectively. The polarization of the qubit at subsequent times, $\vec{P}_A(t) = Tr \rho(t) \vec{\sigma}_A$, is easily evaluated to yield,

$$\vec{P}_A(t) = \cos^2(Kt/2)\vec{P}_A(0) + \sin^2(Kt/2)\vec{P}_B(0)$$

$$+ \frac{1}{2} \sin(Kt)\vec{P}_B(0) \times \vec{P}_A(0).$$

The polarization for the nuclear spin B is obtained by interchanging the labels $A$ and $B$ in the above expression. Now starting with an initial polarization $P_A(0) = 1$, if $P_A(t) < 1$ at a later time, the it implies a decoherence. However, after decoherence the polarization (of the particle A) will build up again, as there is an overall periodicity due to the Hamiltonian dynamics.

It is easy to see from the above equation that $\vec{P}_A(t = \pi/K) = \vec{P}_B(0)$, and vice versa, which shows that the polarizations get swapped between the qubit and the nucleus at that time. The decoherence is illustrated in Fig. (1a) where we have plotted the variation of $P_A(t)$ with time for four different choices of the initial state of the total system. Anticipating the analysis in section VI, one may note that $\vec{P}_B$ has a dual role: to cause a precession of $\vec{P}_A$ (represented by the cross product term in the above equation) and also change its magnitude. It may also be pointed out that since the evolution of the total system is unitary, there is a revival of the initial state with a period $2\pi/K$. The “decoherence”, however, takes place at a much smaller time scale, determined by
the first term in Eq[3]. If $\vec{P}_A = \vec{P}_B$, the state remains invariant in time.

Now, the initially-polarized qubit (i.e. in a pure state) loses its polarization (i.e. the state is mixed), due to the interaction. In other words the total state of the qubit and the nuclear spin picks up entanglement. Through the time evolution, the entanglement between the two keeps changing nonmonotonically. The concurrence measure of the entanglement is easily calculated. In Fig.1(b) we have plotted the entanglement measure, along with the polarization $P_A(t)$ as a function of time. From the figure it is clear that decoherence sets in (a decrease in the polarization close to zero time) along with the generation of entanglement, and at latter times the polarization picks up again at the cost of the entanglement.

The fidelity measure captures how close is $\rho_A(t)$, the reduced density matrix of the qubit, to its original state $\rho_A(0)$. The average fidelity at time $t$ is given by

$$F_A(t) = \langle Tr[\rho_A(t)\rho_A(0)]/Tr[\rho_A^2(0)] \rangle,$$

(7)

where the average is indicated over a distribution of initial states of the qubit. We have normalized the fidelity to unity at time $t = 0$. If the averaging is done with a uniform distribution over all possible initial pure states, we have

$$F_A^{\text{pure}}(t) = \frac{1}{2}(1 + \cos^2(Kt/2)).$$

The average fidelity over all possible initial pure and mixed states is obtained as,

$$F_A(t) = 1 - 3\left(\frac{\pi}{4} - \frac{2}{3}\right) \sin^2(Kt/2).$$

(8)

The decoherence is, of course, not easily apparent in the above expressions. The fidelity can be less than unity due to either the state is decohered or the state has changed coherently (a precession of the polarization vector in an external magnetic field). It may not be possible to separate the contributions from a coherent unitary evolution and decoherence in the above equation.

C. Interaction of a qubit with N spin-half nuclei

The central problem of the paper will be addressed here. As already mentioned, we shall take all the nuclei to be spin-half, for the sake of simplicity. As argued in section II, the Heisenberg interaction involves, not all the nuclei, but only those in the immediate neighborhood of the localized qubit.

The Hamiltonian is then simply given by

$$H_{q-B} = K \vec{S} \cdot \vec{I}_B$$

(9)

The interaction strength $K$ can be written in terms of the hyperfine parameters; in general it depends on the value of the total spin of the environment.

Let the initial qubit-bath state be a direct product, $\rho(0) = \rho_A(0) \otimes \rho_B(0)$, where the qubit is denoted by $A$, and the bath by $B$. The initial state of the qubit has the standard form $\rho_A(0) = \frac{1}{2}(I + \sigma \cdot \vec{P}_A(0))$ where $\vec{P}_A(0) \equiv T\rho_A(0)\vec{\sigma}_A$. The initial bath state $\rho_B$ can be written in terms of density matrices corresponding to the various bath-spin sectors, $\rho_B(0) = \sum \lambda_{ib} \rho_{ib}(0)$, which displays the initial bath state as an incoherent sum of states labelled by bath spin $I_B$, with weights $\lambda_{ib}$. The nuclear bath state is in general a mixed state in each sector of the total bath spin. The bath density matrix in a total bath-spin sector $I_B$ can be written in terms of tensor polarizations, and spin operators of various ranks as given by 30.

$$\rho_{ib} = \frac{1}{2I_B + 1}[I + \frac{\vec{P}_{ib} \cdot \vec{I}_B}{I_B} + 3 \sum_{m,n=1}^{3} \Pi_{ib}^{mn} \hat{Q}_{ib}^{mn} I_B(I_B + 1) / 3]$$

(10)

here, $\vec{P}_{ib}$ is the vector polarization of the bath in the bath-spin sector $I_B$, and $\Pi_{ib}^{mn}$ is a component of the rank-two tensor polarization with the cartesian indices $m$ and $n$. In the above expression we have not shown explicitly the terms involving higher-rank tensor polarizations. We shall see later that only these polarization terms will be relevant to the time evolution of the qubit polarization at all times. The components of the second rank tensor operator are defined by $Q_{ib}^{mn} = (I_B I_B + I_B I_B - 1/2)/3 I_B^2 \delta_{mn}/3$. The polarizations $\vec{P}_{ib}$ and $\Pi_{ib}$ are determined by the expectation values of the vector and tensor spin operators respectively in the $I_B$ sector, $(\vec{I}_B) = \vec{P}_{ib}(I_B + 1)/3$ and $(Q_{ib}^{mn}) = \Pi_{ib}^{mn} (2I_B - 1)(2I_B + 3)/3$. With the above definitions the components of both vector and tensor polarization are of order unity.

The dynamical evolution of the system is governed by the equation $\rho(t) = U \rho(0) U^\dagger$, where the time evolution operator is given by (in a sector with the bath spin $I_B$, and setting $\hbar = 1$)

$$U = e^{-i H t} = e^{-i \hat{H}_t (a_{ib}(t) + b_{ib}(t) \vec{S} \cdot \vec{I}_B)}.$$  

(11)

The time-dependent coefficients are $a_{ib}(t) = \cos(\Lambda t) - i \sin(\Lambda t)/(2I_B + 1)$ and $b_{ib}(t) = -4i \sin(\Lambda t)(2I_B + 1)$, where $2\Lambda = K(I_B + 1/2)$. The Heisenberg Hamiltonian makes the dynamics exactly solvable. The various components of the total density matrix, corresponding to different bath spin $I_B$, can be time evolved separately. After determining the state at any time $t$, a partial trace over the bath degrees of freedom yields the expression for the reduced density matrix $\rho_A(t)$ of the qubit. We can represent the reduced density matrix as

$$\rho_A(t) = \frac{1}{2}(1 + \vec{P}_A(t) \cdot \vec{\sigma}_A).$$

(12)

The polarization vector $\vec{P}_A(t)$ carries all the information about the qubit. $P_A(t) = 1$ for a pure state, $P_A(t) < 1$ for
a mixed state. It depends on the bath-spin distribution, and the polarization strengths of the bath in each total spin channel. The polarization vector of the qubit at any time $t$ is obtained as

$$\vec{P}_A(t) = f(t)\vec{P}_A(0) + g(t) + \vec{h}(t) \times \vec{P}_A(0) + \sum_{m,n} \tilde{\Pi}_{mn}(t)P^m_A(0)\hat{e}_n.$$  \hspace{1cm} (13)

In the above the sum is over cartesian components $(m, n = x, y, z)$ and $\hat{e}_n$ stands for a cartesian unit vector. The time-dependent coefficients depend on the bath-spin distribution, and are given by

$$f(t) = 1 - \sum_{I_B} \lambda_{I_B} \frac{4(I_B + \frac{1}{2})}{3(I_B + \frac{3}{2})} \sin^2[(I_B + \frac{1}{2})\frac{Kt}{2}],$$

$$g(t) = \sum_{I_B} \lambda_{I_B} \frac{2(I_B + \frac{1}{2})}{3(I_B + \frac{3}{2})} \sin^2[(I_B + \frac{1}{2})\frac{Kt}{2}]\vec{P}_B,$$

$$h(t) = \sum_{I_B} \lambda_{I_B} \frac{4(I_B + \frac{1}{2})}{3(I_B + \frac{3}{2})} \sin[(I_B + \frac{1}{2})\frac{Kt}{2}]\vec{P}_B,$$

$$\tilde{\Pi}_{mn}(t) = \sum_{I_B} \lambda_{I_B} \frac{4(I_B + \frac{1}{2})}{3(I_B + \frac{3}{2})} \sin^3[(I_B + \frac{1}{2})\frac{Kt}{2}]\Pi^m_{ln}.$$  \hspace{1cm} (14)

The coefficient $f$ depends only on the spin distribution, and is independent of bath polarizations. The other coefficients depend on various polarizations that may be present in the initial nuclear bath state. It is clear from Eq.13 that $\vec{P}_A$ couples utmost to the second-rank tensor polarization. It is a straightforward consequence of the Wigner Eckart theorem that the higher rank tensors do not couple. We conclude that all nuclear states which have the same values of vector and second rank tensor polarizations have indistinguishable dynamics. However, the dynamics of a central spin-one particle (its state will be described by a vector polarization and a rank-2 tensor polarization) will have contributions from rank-3 and tensor polarizations. Nonzero bath polarizations have a tendency to increase the time scale over which the gaussian decay takes place. However, the dominant contribution to decoherence comes from the first term (i.e., from the function $f(t)$ in Eq.14) which does not depend on bath polarizations.

Another peculiar feature is that the time scale also has the initial qubit state dependence through the appearance of $P_A(0)$ in the above expression, which is the hallmark of nonmarkovian dynamics. Starting with qubit pure states ($P_A(0) = 1$), initially there will be a decay, i.e., decoherence, over a time scale $\tau$. At later times, the polarization will again grow, showing a nonmonotonic behaviour, as we will see in the next section in specific examples of the bath-spin distribution. We will also consider more general local interactions, unlike the global interaction considered above, where the qubit can interacting with several nuclear spins with the different interaction strengths. We will calculate the gaussian decoherence time scale by a direct expansion of the time evolution operator in powers of time.

It should be noted that, for a generic markovian evolution, an exponential decay is expected. However, the case we have considered where the central spin-$1/2$ is evolved with the spin bath through Hamiltonian dynamics, a gaussian decay is typical as the process of elimination of the bath degrees of freedom is a nonmarkovian process. A comparison of markovian and nonmarkovian dynamics will be done in the last section, where we construct a master equation that describes the effective dynamics of the central spin-1/2 particle.

Before we proceed to discuss special cases and examples, we write the expression for the average fidelity for the qubit, averaging over all possible initial states of the qubit,

$$\langle F_A(t) \rangle = 1 - \left(\frac{1}{4} - \frac{2}{3}\right) \sum_{I_B} \lambda_{I_B} I_B(I_B + 1) \frac{Kt}{2} \sin^2(I_B + \frac{1}{2}) \frac{Kt}{2}$$  \hspace{1cm} (20)

If the averaging is done over all possible initial pure states only, we have is confined to the pure states only,

$$\langle F_A(t) \rangle = 1 - \left(\frac{1}{4} - \frac{2}{3}\right) \sum_{I_B} \lambda_{I_B} I_B(I_B + 1) \frac{Kt}{2} \sin^2(I_B + \frac{1}{2}) \frac{Kt}{2}$$  \hspace{1cm} (21)
IV. EXAMPLES AND SPECIAL CASES

A. Unpolarized bath

For an unpolarized bath, polarizations of all ranks are identically zero i.e., \( \langle I_B^A I_B^B \rangle = \langle I_B^A I_B^B \cdots \rangle \equiv 0 \). The initial bath state is \( \rho_B(0) = \frac{I_B}{2^{N}I_B+1} \). For this case, the polarization of the qubit at any time is given by

\[
\bar{P}_A(t) = f(t) \bar{P}_A(0)
\]

where \( f(t) \) is given in Eq.14 with \( \lambda_{I_B} = \frac{1}{2^N} C_N^N \frac{(2I_B+1)^2}{I_B^2} \). We note that the decay time scale is determined by small time behaviour of \( f(t) \). Expanding \( f(t) \) up to \( O(t^2) \) we get

\[
\bar{P}_A(t) = (1 - NK^2t^2/4)\bar{P}_A(0) \approx e^{-t^2/\tau} \bar{P}_A(0)
\]

where the decay time scale is \( \tau = \frac{2}{K\sqrt{N}} \). Using the known experimental decay time scales of order 10-100 nanoseconds, and estimating the number of nuclei interacting with the qubit to be, \( N \sim 100 \) (see Section II), the typical effective coupling strength relevant for QDQC systems works out to be, \( K \sim 10^{-3}eV \). These results are in broad agreement with the results obtained by [27]. This agreement justifies writing an effective Heisenberg interaction of the qubit with the total bath spin.

In Fig.2 we have plotted \( P_A(t) \) for an unpolarized bath composed of \( N=10, 30, 50 \) particles. One can clearly see the \( N \) dependence on the initial decay. As \( N \) grows large, the value of \( P_A(t) \) falls to \( 1/3 \) of its initial value rapidly, and stays there for a long time. In general when \( \rho_B(0) = \sum \lambda_{I_B} \bar{I}_B/(2I_B +1) \), the gaussian decay time scale is given by \( Kt = \sqrt{<\bar{P}_B>}/3 \). We can also see in Fig.2 (inset) the revival of the qubit polarization over a time scale of the order \( \sqrt{<\bar{P}_B>/h} \), which is a consequence of the coherent quantum evolution of the total system (the qubit and the bath). However, this revival time scale is not a physical time scale for the qubit, as we include the next to leading interaction strength \( K' \) (which is exponentially smaller than the dominant interaction strength \( K \), see Section II), the revival time will become larger \( \sim h/K' \). The decoherence time scale shown above, however, will change only slightly due to the inclusion of the sub-dominant interactions. Thus though our model Hamiltonian is an excellent approximation for the decoherence time scale, the revival times in the dynamics are not physical for the real systems.

In the above examples, the initial bath state is completely isotropic, and consequently there is no magnetic field produced by the bath. By projection theorem, all the moments of the magnetic field also vanish identically as argued earlier. We conclude that the decoherence is a higher order effect in the following sense: the magnetic field produced by the qubit polarizes the bath, which in turn produces a magnetic field at the site of the qubit. There is no \textit{a priori} fluctuating magnetic field, contrary to the statement found in literature. This conclusion holds, of course for a larger class of initial nuclear states whose vector polarization and rank-2 tensor polarization are absent; we have already remarked that the higher rank tensors are of no consequence. To illustrate, let \( \rho_B(0) = \sum \lambda_{I_B} \bar{I}_B/(2I_B +1) \). The bath is maximally unpolarized in each spin sector. The polarization of the bath in a sector \( I_B \) can be found easily (by a partial trace over the qubit degree of freedom), and the total polarization of the bath, \( \bar{P}_B(t) = \sum \lambda_{I_B} \bar{P}_B \), is given by

\[
\bar{P}_B(t) = \bar{P}_A(0) \sum \lambda_{I_B} \frac{2I_B}{(I_B +1/2)^2} \sin^2 \left( I_B + \frac{1}{2} \right) \frac{Kt}{2}
\]

From the above expression we can see that the induced polarization in each spin sector grows with time; this in turn effects the dynamics of the qubit spin. The subsequent evolution leads to the decoherence in the spin state. This is illustrated in Fig.3, where we have plotted the total induced bath polarization, \( P_B(t) \), for initially-unpolarized nuclear bath with a gaussian bath-spin distribution.

B. Fully-polarized bath

We now consider the other extreme, a fully polarized nuclear bath. All the nuclear spins are parallel. This example is of relevance when the inter nuclear coupling is ferromagnetic. The resultant magnetic field at the site of the qubit can be large, and may be identified with the Overhauser field. The semiclassical approach may be expected to work the best in this case. We shall present
below an exact analysis. The initial bath state here is a pure state with bath spin \( I_B = N/2 \). \( \rho_B(0) = | \uparrow \uparrow \ldots \uparrow \rangle \langle \uparrow \uparrow \ldots \uparrow | \). The vector and tensor polarizations for this state are

\[
P^A_\perp(0) = \frac{3N}{N+2} \; \hat{\mathbf{z}}; \; \Pi^A_\perp(0) = \frac{5N}{6(N+3)} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\]

Substituting these values in Eq.\( \text{19} \) we get (with \( \Lambda = (N+1)K/4 \)),

\[
P^A(t) = P^A_\parallel(0) + \frac{2N}{(N+1)^2} \sin^2(\Lambda t)(1 - 2P^A_\parallel(0))
\]

\[
P^A_\perp(t) = \frac{N}{N+1} \left( \frac{1}{N} \cos 2\Lambda t - \sin 2\Lambda t \right) \hat{P}_\perp(0)
\]

FIG. 3: The nuclear bath vector polarization \( P_B(t) \) is plotted against time for one complete period, for the case where a spin-1/2 particle is interacting with an initially-unpolarized bath composed of \( N=100 \) spin-1/2 particles, and a gaussian bath-spin distribution \( \lambda_{IB} \sim \exp(-0.1I_B^2) \).

In the above we have denoted the transverse component of \( \hat{P}_A \) by a two-component column vector \( \hat{P}_\perp \). The expressions for the transverse components of the electron spin are the same as those obtained by placing the electron in a constant magnetic field \( \vec{B} = (N+1)K/2\hat{z} \) apart from terms of order \( 1/N \). For large \( N \), the polarization hardly changes from its initial value. For an initial qubit pure state with \( \hat{P}_A(0) = -\hat{z} \), we get from Eq.\( \text{19} \), the gaussian decoherence time scale, \( \tau = \sqrt{2}/K\sqrt{N} \). These results are consistent with the work of Taylor et al.\( \text{31} \) (however, there is an apparent discrepancy, which we trace to a trivial algebraic error in that paper).

The reason for such a long-lived polarization of the qubit, in the above example, can be traced to the energetics of the dynamics. Consider the total state of the system to be a direct product of the qubit in down spin state and the bath in a fully polarized state. As the total z-component of the spin \( S^z + I^z \) is conserved in the evolution, the state at any later time can be written as \( |\psi(t)\rangle = c_1(t)|\downarrow\rangle|\frac{N}{2}\rangle + c_2(t)|\uparrow\rangle|\frac{N}{2} - 1\rangle \), where \( |\frac{N}{2}\rangle \) and \( |\frac{N}{2} - 1\rangle \) denote the nuclear bath states with \( I^z = \frac{N}{2}, \frac{N}{2} - 1 \) respectively. The energy difference of this state with the initial state (with \( c_2(0)=0 \)) is given by \( \delta E/K = N(1 - |c_1(t)|^2) + \sqrt{N}Re(c_2(t)c_1(t)) \). Thus, for large \( N \), if the coefficient \( c_1(t) \) changes even slightly, there is a large change in the energy, hence the corresponding transition is not favorable. This implies that the qubit polarization cannot change much through the evolution. In contrast, if we apply a magnetic field of order \( KN/\mu_B \) for the qubit along direction opposite to the nuclear polarization, then the qubit polarization is given by \( \rho^A(t) = \cos(\sqrt{N}Kt)\rho^A_\parallel(0) \), which shows that the initial polarization is not preserved. In fact, there will be coherent oscillations between the two states shown above. This situation can be used in nuclear memory where the information coded into the qubit initial state at \( t = 0 \) can be transferred to the nuclear state through the time evolution. These considerations were treated by Taylor et al.\( \text{31} \).

In the case of a fully polarized bath, the role played by the tensor polarization is often glossed over. In fact due to the presence of the tensor polarizations (see Eq.\( \text{25} \)), the bath is in a pure state. To highlight the role played by the tensor polarization in the above case, and show its importance, let us consider the following mixed state \( \rho_B(0) = \frac{1}{N+1}(I + \frac{1}{N}P^{I_B}_\parallel I^B_\parallel) \). For \( P^{I_B}_\parallel = \sum_{1_B,N/2}^N \delta_{1_B,N/2} I^B_\parallel \) this state has a vector polarization same as that of the

FIG. 4: The polarization of a spin-1/2 particle interacting with a nuclear spin bath with a given total spin, is plotted against time. The initial polarization is \( P_\parallel(0) = -\hat{z} \). The two different cases shown are a fully-polarized bath in a pure state with the vector and tensor polarizations as shown in Eq.\( \text{25} \) (solid line), and a bath in a mixed state with a vector polarization as shown in Eq.\( \text{25} \) and zero tensor polarization (dashed line). A nonzero tensor polarization increases the initial gaussian decay time scale.
fully-polarized bath that we considered above, but zero tensor polarization. Again considering an initial qubit pure state with \( \hat{P}_A(0) = -\hat{z} \), from Eq.19 we get the decoherence time scale as \( \tau = 2\sqrt{3}/k\sqrt{N(N+5)} \sim 1/N \). In Fig.4 we have plotted \( P_A(t) \) when the qubit is interacting with baths of same vector polarization but different tensor polarizations. In the absence of the tensor polarization one can see that \( P_A \) decreases rapidly to zero, and in contrast \( P_A \) changes very little in the presence of tensor polarization of the spin bath. The decoherence time scale, or the gaussian decay time scale, is much larger when tensor polarization is nonzero.

In Fig.5, we have shown the short-time behaviour of \( P_A(t) \), with the initial qubit polarization \( \hat{P}_A(0) = (\hat{x} + \hat{z})/\sqrt{2} \) for a few different bath states, in each case with a gaussian bath-spin distribution, \( \lambda_{ist} \sim \exp(-0.1\hat{P}_B^2) \).

The four cases considered here are (i) no bath polarizations (ii) no vector polarization but a nonzero tensor polarization (iii) a nonzero vector polarization but no tensor polarization (iv) nonzero vector and tensor polarizations. The polarizations when nonzero in each bath-spin sector are given by

\[
\hat{P}_{I_B}(0) = \hat{z}; \Pi_{I_B}(0) = \frac{1}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\]  

It can be seen that the decay of the qubit polarization is slowest when both vector and tensor polarizations of the bath are non-zero, and it is fastest when there are no bath polarizations.

**V. SHORT-TIME BEHAVIOUR WITH LOCAL INTERACTIONS**

One feature that has emerged from the above is that the decoherence (dephasing in the case of ensembles of quantum dots) is governed entirely by the short-time structure of the evolution operator. This suggests that we can use a perturbative solution, to capture the short-time behaviour of the polarization. Then we may as well enlarge the class of Hamiltonians to include local interactions and also include an external magnetic field \( \vec{B} \).

Consider the Hamiltonian

\[
H = \vec{S} \cdot \sum_{i=1}^{N} J_i \vec{I}_i + \vec{B} \cdot \vec{S},
\]

which is the most general Hamiltonian describing the hyperfine interactions in quantum dots in the presence of an external magnetic field. The exact dynamics of the qubit governed by the above Hamiltonian is difficult to solve as the time evolution operator does not acquire a simple form (given in Eq.11) as in the case of global isotropic interaction considered earlier. Taking recourse to a perturbative approach, we expand the time-evolution operator. The qubit polarization can be calculated up to \( O(t^2) \) as,

\[
\hat{P}_A(t) = \hat{P}_A(0) - \frac{t^2}{2} \left\{ \sum_{i} J_i (\hat{P}_A(0) \times \hat{P}_{B_i}(0)) + 2\hat{P}_A(0) \times \vec{B} \right\}
\]

\[
+ \frac{t^2}{4} \left\{ \sum_{i} J_i^2 (\hat{P}_{B_i}(0) - \hat{P}_A(0)) + \frac{1}{4} \sum_{i \neq j} J_i J_j (\hat{P}_A(0) \times \hat{P}_{B_i}(0)) \times \hat{P}_{B_j}(0) + 2(\hat{P}_A(0) \times \vec{B}) \times \vec{B} \right\}.
\]

In writing the above solution we have taken the initial state of the system to be a direct product, \( \rho(0) = \rho_A(0) \otimes \rho_B(0) \), where \( \rho_B(0) = \rho_{B_1} \otimes \rho_{B_2} \cdots \otimes \rho_{B_N} \). The state of the i’th nuclear spin is given by \( \rho_{B_i} = \frac{1}{2}(I + \vec{\sigma}_{B_i} \cdot \hat{P}_{B_i}(0)) \), where its polarization is denoted by \( P_{B_i} \).

For an unpolarized bath where all \( \hat{P}_{B_i}(0) = 0 \), the
above equation acquires a simpler form

\[
\hat{P}_A(t) = \hat{P}_A(0) - t\hat{P}_A(0) \times \vec{B} - \frac{t^2}{4} \sum_i J_i^2 \hat{P}_A(0) + \frac{t^2}{2} (\hat{P}_A(0) \times \vec{B}) \times \vec{B}
\]  

(29)

\[
\rho_A(t) = e^{-iH_c t/\hbar}\rho_A(0)e^{iH_c t/\hbar}
\]

(30)

Here the polarization is \(P_A(t) = [1 - \frac{t^2}{4} \sum_i J_i^2] \hat{P}_A(0)\), which is again independent of the external magnetic field. From this we can conclude that for short times, the gaussian decay of the qubit polarization is completely determined by the sum of squares of the interaction coupling strengths \(J_i^2\) of the qubit with different nuclear bath spins.

VI. MASTER EQUATION FOR THE DECOHERENCE OF THE QUBIT

In this section we will derive the master equation obeyed by the qubit, from the explicit solution for the time-dependent density matrix (or the polarization) of the qubit given in Eq.13. The advantage in setting up a master equation is that it displays the unitary and nonunitary parts of the qubit evolution, and their dependence on the initial bath state, viz. the nuclear spin distribution and the polarization strengths. Though the master equation carries the same information as the time-dependent polarization calculated above, it displays the characteristic effective magnetic field seen by the qubit, and the time scale for decoherence and polarization processes explicitly.

We now proceed to recast our results as the solution of a master equation satisfied by the qubit state. The most general master equation for a two-level system was given by Gorini et.al., \cite{32} and Lindblad \cite{33}. Using the notation of Gorini et. al, the master equation for the qubit density matrix can be written as

\[
\frac{\partial}{\partial t} \rho_A(t) = -i[H_c, \rho_A(t)] + \frac{1}{2} \sum_{i,j=1}^{3} \Gamma_{ij} \{[\sigma_i, \rho_A(t)\sigma_j] + [\sigma_i, \rho_A(t), \sigma_j]\}.
\]

(31)

where, the first term represents the unitary component of the evolution. The second term is responsible for non-unitary processes such as decoherence, polarization and equilibration. The coefficients \((\Gamma_{ij})\) determine the decay or growth of polarization.

Since we have an explicit solution for the time dependence of the polarization, the various terms in the above equation can be identified. As we will see below, the matrix elements \(\Gamma_{nm}\) are time dependent, implying a non-Markovian evolution, causing a gaussian decay of the polarization for small times. Also, the polarization will show temporal decay and growth periodically, displaying the underlying Hamiltonian evolution of the spin-1/2 and the bath taken together. Let us start by rewriting our general solution for the polarization of the spin-1/2 particle as a function of time given in Eq.13 as

\[
\hat{P}_A(t) - \hat{g}(t) = M\hat{P}_A(0),
\]

(32)

where \(\hat{P}_A, \hat{g}\) are the column vectors of \(\hat{P}_A, \hat{g}\) respectively, and \(M\) is a 3 \(\times\) 3 matrix given as

\[
M = \begin{pmatrix}
\tilde{f} + \tilde{\Pi}_{xx} & h_z + \tilde{\Pi}_{xy} & -h_y + \tilde{\Pi}_{x}\n-h_z + \tilde{\Pi}_{xy} & \tilde{f} + \tilde{\Pi}_{yy} & h_x + \tilde{\Pi}_{y}\nh_y + \tilde{\Pi}_{xz} & -h_x + \tilde{\Pi}_{yz} & \tilde{f} + \tilde{\Pi}_{zz}
\end{pmatrix}.
\]

(33)

Then the equation of motion for the polarization vector can be written as

\[
\frac{d\hat{P}_A(t)}{dt} = D\hat{P}_A(t) + \hat{R}(t),
\]

(34)

where the matrix \(D = \frac{dM}{dt}M^{-1}\) and the inhomogeneous term is given by \(\hat{R}(t) = \frac{d\hat{g}}{dt} - \frac{d}{dt}\hat{g}(t)\). Now, a straightforward comparison of this with Eq.31 yields expressions for the matrix elements \(\Gamma_{nm}\) and the effective magnetic field \(\vec{B}_{eff}\) (in units of \(\hbar/\mu_B\)) as

\[
\Gamma_{nm} = D_{nm} + D_{ml} - \delta_{lm} Tr D - i \sum_n \epsilon_{lmn} \hat{R}_n,
\]

(35)

and

\[
\vec{B}_{eff} \hat{e}_i = \frac{1}{2} \sum_{mn} \epsilon_{lmn} D_{mn}.
\]

(36)

It is easy to check at small times, that \(\Gamma_{nm} \sim \alpha t\) and \(|\vec{B}_{eff}| \sim \beta(1 - \delta t^2)\).

In the above equation for the polarization vector, the time development of the different components are coupled. Let us rewrite the matrix \(D\) in terms of the symmetric part \(D_s\) and the antisymmetric part \(D_a\) as \(D = \)
$D_x + D_y$. The antisymmetric part gives the effective magnetic field, which causes a standard precession of the polarization, with no decay or decoherence. The symmetric part causes a decay or growth of the polarization components. We can change to the diagonal basis of the matrix $D_y$, and the transformed vectors are $\hat{P}, \hat{R}$. The time development of the polarization components can be found directly by integration, and $\gamma_i$ (eigenvalue of $D_y$) determines the decay or growth of the polarization component $\hat{P}_i$. For short times, we have $-\gamma_i \sim t$, and the polarization is given by

$$P^2(t) \approx \sum_i \hat{P}_i^2(0)e^{2J_0\gamma_i dt},$$

which implies a gaussian decay. If all the elements of the matrix $\Gamma$ (or $D$) are independent of time, implying time-independent decay cosuntants $\gamma_i$, then we would have an exponential decay which is the hallmark of a markovian evolution.

For an illustration, let us consider a purely vector-polarized bath, where the density matrix in each sector of the bath spin $I_B$ is given by $\rho_{I_B} = [I + \frac{1}{I_B}P^2_{I_B}I_B^2]/(2I_B + 1)$. Here, there is no tensor polarization, and $\hat{h}(t) = h(t)\hat{z}$ and $\hat{g}(t) = g(t)\hat{z}$. The expressions for the effective magnetic field and the $\Gamma$ matrix can be directly calculated for this case,

$$\hat{B}_{\text{eff}} = \frac{f\hat{h} - \hat{f}\hat{h}}{f^2 + \hbar^2};$$

$$\Gamma = \begin{pmatrix} -\frac{d}{dt}\log f & -if\frac{d}{dt}(\frac{f}{\hbar}) & 0 \\ if\frac{d}{dt}(\frac{f}{\hbar}) & -\frac{d}{dt}\log f & 0 \\ 0 & 0 & \frac{d}{dt}\log f/2\pi\hbar^2 \end{pmatrix}. \quad (39)$$

The diagonal elements of the above matrix are related to the decay functions through, $\Gamma_{ii} = 2\gamma_i - TrD$. We have,

$$\gamma_1 = \gamma_2 = \gamma_\perp = \frac{d}{dt}\log f; \quad (40)$$

$$\gamma_3 = \gamma_\parallel = \frac{1}{2}\frac{d}{dt}\log(f^2 + \hbar^2). \quad (41)$$

We see that the polarization of the bath does produce an effective magnetic field $B_{\text{eff}}$ which is time dependent, and causes a precession of $\hat{P}_A(t)$. For short times, the effective magnetic field is $B_{\text{eff}} \approx K/\mu_B$ which is about a few gauss using a typical interaction strength $K \sim 10^{-8}$eV in QDQC systems. In Fig.6 we have plotted the effective magnetic field, and the decay/growth functions with time, for a representative gaussian bath-spin distribution. As can be seen from the figure, the effective magnetic field and $\gamma_i$ show a nonmonotonic behavior, and they take both positive and negative values. A negative $\gamma_i$ implies a decay of the corresponding polarization components, and a positive $\gamma_i$ implies a growth of polarization. Though $\gamma_i$ are time dependent, they satisfy the condition $2\gamma_\perp > \gamma_\parallel$, consistent with the general constraint found by Gorini et al [2].

Finally, we determine the effective magnetic field when the initial bath state is given by $\rho_B(0) = |\uparrow\uparrow\rangle\langle\downarrow\downarrow|$ and $\rho_B(0) = \sum_i \lambda_i/(2I + 1)(I + \hat{P}_i\hat{z})$, respectively. The bath is composed of $N = 100$ particles with the bath spin distribution given by $\lambda_i = \exp(-0.1I^2)$. Note that, with $\hbar = 1$, both the decay functions $\gamma_\parallel$ and $\gamma_\perp$ have a dimension of energy.

$$\hat{B}_{\text{eff}} = \{1 - \frac{2}{N + 1}\sin^2(N + 1)Kt/4\} KN/2; \quad (42)$$

$$\gamma_\parallel = 2\gamma_\perp = -\frac{NK}{(N + 1)}\sin(N + 1)Kt/2. \quad (43)$$

Here the strength of the effective magnetic field is about a few hundred gauss for $N = 100$, $K \sim 10^{-8}$eV.

**VII. CONCLUSIONS**

The interaction of a central spin-1/2 particle with the nuclear spins in quantum dots is modeled by a Heisenberg exchange interaction. The dynamics of initial direct product states of the qubit and the nuclear spin bath are investigated. The decoherence of the spin-1/2 particle can be seen from the the decay of its polarization through the Hamiltonian time evolution. The qubit polarization as function of time is explicitly calculated for any nuclear bath with a spin-conserving internal dynamics, for any bath-spin distribution and any polarizations.
The time development of the qubit polarization has non-markovian features. The qubit polarization shows a gaussian decay/decoherence for short times. For a typical interaction strength of the qubit and the nuclear spins, $K \sim 10^{-9}$eV, the decoherence time scale is about 100 nanoseconds. For longer times, the polarization shows nonmonotonic behaviour, eventually displaying a periodicity in time. The gaussian decay time scale depends on the bath-spin distribution, a larger width of the distribution leading to smaller time scales. The vector and tensor spin polarizations that may be present in the nuclear spin bath have a tendency to increase the gaussain decay time scale.

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