On hereditary models of polymers

M. De Angelis

Abstract. An equivalence between an integro-differential operator $M$ and an evolution operator $L_n$ is determined. From this equivalence the fundamental solution of $L_n$ is estimated in terms of the fundamental solution related to the third-order operator $L_1$ whose behavior is now available. Moreover, properties typical of wave hierarchies can be applied to polymeric materials. As an example the case $n = 2$ is considered and results are applied to the Rouse model and the reptation model which describe different aspects of polymer chains.

1 Statement of the problem

The creep and relaxation processes related to the viscoelastic behavior of many polymeric materials are specified by means of memory functions of the form:

$$g_n(t) = \sum_{h=1}^{n} B_h e^{-\beta_h t}, \quad (1)$$

where $n$, $B_h$ and $\beta_h$ depend on the polymer physics and are determined so as to fit the experimental curves for $g_n(t)$ to a given approximation [1–4].

Let $\mathcal{B}$ be a linear, isotropic, homogeneous system and let $u(x, t)$ be the displacement field from an underformed homogeneous reference configuration $\mathcal{B}_0$. If $\rho_0$ denotes the mass density in $\mathcal{B}_0$, and $F = f I$ is the known body force, the one-dimensional linear motions of $\mathcal{B}$ are described by the higher order equation [5]

$$L_n u = \sum_{k=0}^{n} a_k \partial_t^k (u_{tt} - c_k^2 u_{xx}) = F, \quad (2)$$

where

$$c_k = \alpha_k / \rho_0 a_k, \quad F = (1/\rho_0) \sum_{k=0}^{n} a_k \partial_t^k f. \quad (3)$$

In (2) the constants $c_k$ are the characterized speeds depending on the material properties of the medium and in many physical problems $c_0^2 < c_1^2 < ... < c_{n-1}^2 < c_n^2$ and so the equation is typical of wave hierarchies [6].

When $n = 1$, (2) turns into a strictly hyperbolic third-order equation which models the evolution of the standard linear solid [7] and its behavior was discussed in [8]: the fundamental solution $\mathcal{E}_1$ was explicitly determined, together with maximum theorems and boundary layer estimates.
Moreover, the behavior of most viscoelastic media is also fairly well modelled by linear hereditary equations of the form

\[ \varepsilon(t) = J(0)\sigma(t) + \int_{-\infty}^{t} \dot{J}(t-\tau)\sigma(\tau)d\tau \]  

(4)

where \( J(t) \) denotes the creep-compliance and \( \sigma, \varepsilon \) are the only non-vanishing components of the stress and the strain tensors such that \( \rho_0 u_{tt} = \sigma_x + f, \quad \varepsilon = u_x \).

According to fading memory hypotheses [9,10], \( J(t) \) is a positive fast decreasing function and, for many real materials such as polymers, rubbers and bitumens, which can be represented by means of chains of S.L.S. elements in series or parallel [1,2], one has

\[ \dot{J}_n(t) = J_n(0)g_n(t), \]  

(5)

where \( n \) is the number of elements in the chain, \( J_n(0) \) denotes the elastic compliances and constants \( B_k \) and frequencies \( \tilde{\omega}_k \) satisfy

\[ 0 < \beta_1 < \beta_2 < \ldots < \beta_n \quad \text{and} \quad B_k > 0 \quad \forall k = 1, 2 \ldots n. \]  

(6)

The well-known creep representation of one-dimensional linear motions of \( \mathcal{B} \) is given by [11] as

\[ \mathcal{M}u = c^2u_{xx} - u_{tt} - \int_{0}^{t} g(t-\tau)u_{\tau\tau}d\tau = -F_s(x,t), \]  

(7)

where

\[ c^2 = [\rho_0 J_n(0)]^{-1}, \quad F_s = c^2[J_n(0)]f + \int_{-\infty}^{0} \dot{J}_n(t-\tau)\sigma_x(\tau)d\tau + \int_{0}^{t} \dot{J}(t-\tau)f(\tau)d\tau. \]  

(8)

For all \( n \), the fundamental solution \( E_n \) of the operator \( \mathcal{M} \) has been explicitly determined [11,12]. Moreover, let \( E_1 \) be the fundamental solution related to an appropriate S.L.S. \( \mathcal{B}_1 \) defined by

\[ g_1 = b e^{-\beta_1 t} \quad \text{with} \quad b = \beta_1 \sum_{k=1}^{n} \frac{B_k}{\beta_k}; \]  

(9)

the following theorem shows that the fundamental solution \( E_n \) can be rigorously estimated by means of \( E_1 \).

In fact, if \( \Gamma \) is the open forward characteristic cone \( \{(t,x) : t > 0 \ | \ x < ct\} \), and \( \chi_n = \prod_{k=2}^{n} \left( \frac{B_k}{\beta_k} \right)^2 \), then the following theorem holds.