On Arithmetic Progressions in Model Sets

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Abstract
We establish the existence of arbitrary-length arithmetic progressions in model sets and Meyer sets in Euclidean $d$-space. We prove a van der Waerden-type theorem for Meyer sets. We show that subsets of Meyer sets with positive density and pure point diffraction contain arithmetic progressions of arbitrary length.

Keywords Arithmetic progressions · Model sets · Meyer sets · Cut-and-project schemes

Mathematics Subject Classification 11B25 · 52C23

1 Introduction

The discovery of quasicrystals in the early 1980s [30] has triggered increased interest in structures with long range aperiodic order, usually shown via pure point diffraction. The best mathematical idealisations of quasicrystals are model sets. Introduced by Meyer in the 1970s [20], model sets and their subsets, now called Meyer sets, have been popularised in the area of aperiodic order by Lagarias [14,15] and Moody [21,22]. They are constructed via so-called cut-and-project schemes (CPS), which involve cutting a
piece of a higher dimensional lattice, bounded around the real space, and projecting the points into the real space (see Definition 2.3 below for a precise definition).

Model sets with regular windows show long-range order via a clear pure point diffraction spectrum [2,7,12,16,27–29], which can even be traced to the underlying lattice [28]. Meyer sets show long range order, in the form of non-trivial pure point diffraction spectrum with relatively dense support [32–36], which itself is highly ordered [33,35].

The goal of this paper is to show the existence of arbitrarily long arithmetic progressions in model sets and Meyer sets, results which are of the same nature as classical theorems for subsets of $\mathbb{Z}$. This is evidence of high coherence inside Meyer sets, which is a leftover of the lattice from the underlying CPS.

The existence of arithmetic progressions of arbitrary length in subsets of $\mathbb{Z}$ is well studied. In 1927, van der Waerden proved [38] the following result.

**Theorem 1.1** (van der Waerden’s theorem) Given any natural numbers $k, r$, there exists a number $W(r, k)$, such that, for any colouring of $\mathbb{Z}$ with $r$ colours, and for any $N \geq W(r, k)$, the set $\{1, \ldots, N\}$ contains a monochromatic arithmetic progression of length $k$.

Intuitively, this theorem says that, if we split the integers into $r$ disjoint sets, at least one of the sets will have arithmetic progressions of arbitrary length. Moreover, there exists a bound $W(r, k)$ on how far one needs to go to find such an arithmetic sequence, which depends on $k$ and $r$ but is independent of the splitting. In 1975, Szemerédi extended the result, proving the following well-known conjecture of Erdős and Turán [9].

**Theorem 1.2** (Szemerédi’s theorem [37]) Let $\Lambda \subset \mathbb{N}$ be a subset with the property that

$$\text{dens}(\Lambda) := \limsup_n \frac{\text{card}(\{1, 2, 3, \ldots, n\} \cap \Lambda)}{n} > 0.$$ 

Then, $\Lambda$ contains arithmetic progressions of arbitrary length.

It is easy to see that Szemerédi’s theorem implies van der Waerden’s theorem; indeed, any finite partition of $\mathbb{N}$ contains a set with positive density. It is also easy to construct subsets of $\mathbb{N}$ of zero density which do not contain arithmetic progressions of large length. For example, the set $\Lambda = \{2^n : n \in \mathbb{N}\}$ cannot contain arithmetic sequences of length 3. Nevertheless, under suitable extra conditions, one can still hope to find arbitrarily long arithmetic progressions in sets of zero upper density. In 2004, Green and Tao proved the following landmark result, solving the long-standing conjecture that the set of primes contains arbitrarily long arithmetic progressions.

**Theorem 1.3** (Green–Tao theorem [11]) Let $\Lambda \subset \mathbb{P}$ be a subset of the primes with the property that

$$\limsup_n \frac{\text{card}(\{1, 2, 3, \ldots, n\} \cap \Lambda)}{\pi(n)} > 0,$$

where $\pi(n)$ is the number of primes up to $n$. 

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where \( \pi(N) \) is the prime counting function. Then, \( \Lambda \) contains arithmetic progressions of arbitrary length.

A stronger version of Szemerédi’s theorem, which would imply the Green–Tao theorem, is the following famous conjecture of Erdős.

**Conjecture 1.4** Let \( \Lambda \subset \mathbb{N} \) be a set such that \( \sum_{n \in \Lambda} 1/n = \infty \). Then, \( \Lambda \) contains arithmetic progressions of arbitrary length.

In all the results above, the existence of the arithmetic progressions can be traced to the high translational order present in \( \mathbb{Z} \) and \( \mathbb{N} \). The goal of our paper is to extend these results to highly ordered aperiodic structures, by studying the existence of arithmetic progressions of arbitrary length in model sets and Meyer sets. This is a natural generalisation, as model sets are usually considered the natural candidate for “aperiodic” lattices, and Meyer sets are the natural candidate for “aperiodic” lattice subsets.

Given a set \( \Lambda \subset \mathbb{R}^d \), by an arithmetic progression of length \( k \) we understand a sequence \( a_1, a_2, \ldots, a_k \in \Lambda \) with the property that there exists some \( r \neq 0 \) such that, for all \( 1 \leq j \leq k \), we have \( a_j = a_1 + (j - 1)r \). This is equivalent to

\[
a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} \neq 0.
\]

Given a model set \( \mathcal{X}(W) \) (see Definition 2.4), we first prove the following result of van der Waerden type.

**Theorem 1.5** (van der Waerden-type theorem for model sets) Let \( \Lambda \subset \mathbb{R}^d \) be a model set. Let \( r \) and \( k \) be natural numbers. Then, there is some \( R > 0 \) such that, for any colouring of \( \Lambda \) with \( r \) colours and for any \( x \in \mathbb{R}^d \), the set \( \Lambda \cap B_R(x) \) contains a monochromatic arithmetic progression of length \( k \).

We then extend this result to Meyer sets (see Definition 2.6).

**Theorem 1.6** (van der Waerden-type theorem for Meyer sets) Let \( \Lambda \subset \mathbb{R}^d \) be a Meyer set. Let \( r \) and \( k \) be natural numbers. Then, there is some \( R > 0 \) such that, for any colouring of \( \Lambda \) with \( r \) colours and for any \( x \in \mathbb{R}^d \), the set \( \Lambda \cap B_R(x) \) contains a monochromatic arithmetic progression of length \( k \).

We should emphasise here that, while Theorem 1.6 implies Theorem 1.5, the proof of Theorem 1.6 uses Theorem 1.5, so proving Theorem 1.5 is necessary for our approach. This generalises some partial results in this direction obtained in [19, 26].

We complete the paper by showing that pure point diffractive subsets of Meyer sets of positive density contain arbitrarily long arithmetic progressions as well. In particular, weak model sets of maximal density have this property.

## 2 Preliminaries

Here we briefly review some basic definitions and properties about point sets in \( \mathbb{R}^d \). For more details we refer the reader to the monograph [2].
Recall that a set \( \Lambda \subseteq \mathbb{R}^d \) is called \( R \)-relatively dense (or simply relatively dense) if there exists some \( R > 0 \) such that \( \Lambda + B_R(0) = \mathbb{R}^d \). Here, for two sets \( A, B \subset \mathbb{R}^d \), we denote by \( A \pm B \) the Minkowski sum and difference:

\[
A \pm B := \{a \pm b : a \in A, b \in B\}.
\]

A set \( \Lambda \subseteq \mathbb{R}^d \) is called uniformly discrete if there exists some \( r > 0 \) such that, for all \( x, y \in \Lambda \) with \( x \neq y \), we have \( d(x, y) \geq r \). Finally, \( \Lambda \subset \mathbb{R}^d \) is called locally finite if, for each \( R > 0 \), the set \( \Lambda \cap B_R(0) \) is finite. This is equivalent to \( \Lambda \) being discrete and closed in \( \mathbb{R}^d \).

**Definition 2.1** A finite sequence \( a_1, a_2, \ldots, a_n \) in \( \mathbb{R}^d \) is called an arithmetic progression if

\[
a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1} \neq 0.
\]

**Remark 2.2** \( a_1, a_2, \ldots, a_n \) is an arithmetic progression in \( \mathbb{R}^d \) if and only if there exist some \( s, t \in \mathbb{R}^d \) such that \( t \neq 0 \) and for all \( 1 \leq k \leq n \), we have \( a_k = s + (k - 1)t \).

Next, we review the notions of a cut-and-project scheme, model sets, and Meyer sets. For a general overview of these topics we recommend the monographs [2,3], as well as [14,16,20–22,27,29,32–36].

**Definition 2.3** By a cut-and-project scheme, or simply CPS, we understand a triple \((\mathbb{R}^d, H, \mathcal{L})\) consisting of \( \mathbb{R}^d \), a locally compact Abelian group (LCAG) \( H \), together with a lattice (i.e., a discrete co-compact subgroup) \( \mathcal{L} \subset \mathbb{R}^d \times H \), with the following two properties:

- The restriction \( \pi^\mathbb{R}^d |_\mathcal{L} \) of the canonical projection \( \pi^\mathbb{R}^d : \mathbb{R}^d \times H \to \mathbb{R}^d \) to \( \mathcal{L} \) is a one-to-one function.
- The image \( \pi^H(\mathcal{L}) \) of \( \mathcal{L} \) under the canonical projection \( \pi^H : \mathbb{R}^d \times H \to H \) is dense in \( H \).

Given a CPS \((\mathbb{R}^d, H, \mathcal{L})\), we define \( L := \pi^\mathbb{R}^d(\mathcal{L}) \). This is a subgroup of \( \mathbb{R}^d \), which is typically dense in \( \mathbb{R}^d \). The first condition in the definition of a CPS implies that we can define a mapping \( \star : L \to H \), called the \( \star \)-mapping, as \( \star = \pi^H \circ (\pi^\mathbb{R}^d |_\mathcal{L})^{-1} \). Then, we can reparametrise \( \mathcal{L} \) as \( \mathcal{L} = \{(x, x^\star) : x \in L\} \), and the range of the \( \star \)-mapping is \( L^\star := \pi^H(\mathcal{L}) \). We can summarise a CPS in the following diagram.

\[
\begin{array}{ccccccc}
\mathbb{R}^d & \xrightarrow{\pi^\mathbb{R}^d} & \mathbb{R}^d \times H & \xrightarrow{\pi^H} & H \\
\cup & & \cup & & \parallel \\
\pi^\mathbb{R}^d(\mathcal{L}) & \xleftarrow{1-1} & \mathcal{L} & \xrightarrow{\text{dense}} & H \\
\parallel & & \parallel & & \cup \\
L & \xrightarrow{\star} & L^\star
\end{array}
\]
Definition 2.4  Given a CPS \((\mathbb{R}^d, H, \mathcal{L})\) and some subset \(W \subset H\), we denote by \(\bigstar(W)\) its pre-image under the \(\ast\)-mapping, that is
\[
\bigstar(W) := \{x \in L : x^* \in W\} = \{x \in \mathbb{R}^d : \exists y \in W \text{ such that } (x, y) \in \mathcal{L}\}.
\]
When \(W\) is compact, the set \(\bigstar(W)\) is called a weak model set. If \(W\) has non-empty interior and compact closure, the set \(\bigstar(W)\) is called a model set. An example of a CPS and a model set is illustrated in Fig. 1.

Of importance to us is the following result.

Lemma 2.5  Let \((\mathbb{R}^d, H, \mathcal{L})\) be a CPS and \(W \subset H\). If \(W\) has non-empty interior, then \(\bigstar(W)\) is relatively dense.

Next, we briefly review the concept of Meyer sets. For a more general review, we recommend the paper [21] (or [35] for arbitrary LCAG).

Definition 2.6  A subset \(\Lambda \subset \mathbb{R}^d\) is called a Meyer set if \(\Lambda\) is relatively dense and \(\Lambda - \Lambda\) is uniformly discrete.

Later, we will need the following characterisation of Meyer sets.

Proposition 2.7  Let \(\Lambda \subset \mathbb{R}^d\) be relatively dense. Then the following are equivalent:

(i) \(\Lambda\) is a Meyer set.

(ii) There exists a model set \(\bigstar(W)\) such that \(\Lambda \subseteq \bigstar(W)\).

(iii) \(\Lambda\) is locally finite and there exists a finite set \(F\) such that \(\Lambda - \Lambda \subset \Lambda + F\).

2.1 Diffraction

Here we briefly review the notion of pure point diffraction, which will be used in Sect. 6. Note that the results in Sect. 6 are deduced from condition (2), and this condition is equivalent, by Proposition 2.11, to pure point diffraction. We restrict our presentation to point sets in \(\mathbb{R}^d\), and refer the reader to [2] for the more general case.

Recall that a subset \(\Lambda \subset \mathbb{R}^d\) has finite local complexity (or FLC) if the set \(\Lambda - \Lambda\) is locally finite.

Definition 2.8  Let \(\Lambda \subset \mathbb{R}^d\) be a set with FLC, and \(A_n = [-n, n]^d\). We say that the autocorrelation \(\gamma\) of \(\Lambda\) exists with respect to \(A = \{A_n\}\), if, for all \(z \in \Lambda - \Lambda\), the following limit exists:
\[
\eta(z) := \lim_n \frac{\text{card}(\{(x, y) : x, y \in \Lambda \cap A_n, x - y = z\})}{(2n)^d}.
\]
In this case, we define \(\gamma := \sum_{z \in \Lambda - \Lambda} \eta(z) \delta_z\).

Given a set \(\Lambda \subset \mathbb{R}^d\) with FLC, the autocorrelation \(\gamma\) always exists with respect to some subsequence \(B\) of \(\Lambda\) [2,5,7,29]. Furthermore, there exists a positive measure \(\widehat{\gamma}\)
on $\mathbb{R}^d \simeq \mathbb{R}^d$ such that [1,8,25]
\[ \int_{\mathbb{R}^d} |\tilde{f}(t)|^2 \, d\hat{\gamma}(t) = \int_{\mathcal{G}} f * \tilde{f}(s) \, d\gamma(s), \tag{1} \]
holds for all $f \in C_c(\mathbb{R}^d)$, that is, for all continuous compactly supported functions. Here, for $f, g \in C_c(\mathbb{R}^d)$, we use the standard notations $\tilde{f}(x) := f(-x)$ for the complex conjugate of reflection operators, $\hat{f}(x) := \int_{\mathbb{R}^d} e^{2\pi i x \cdot y} f(y) \, dy$ for the inverse Fourier Transform of $f$ and $f * g(x) = \int_{\mathbb{R}^d} f(x-t) g(t) \, dt$ for the convolution product of $f$ and $g$.

**Definition 2.9** We call the measure $\hat{\gamma}$ from (1) the *diffraction of $\Lambda$ (with respect to $B$).*

We say that $\Lambda$ is *pure point diffractive with respect to $B$* if the measure $\hat{\gamma}$ is a pure point measure.

Next, we review an important metric for point sets, which has many names. For two uniformly discrete point sets $\Lambda$ and $\Gamma$, and $B$ being a fixed subsequence of $A_n = [-n,n]^d$, we define
\[ d_B(\Lambda, \Gamma) := \limsup_{n \to \infty} \frac{\text{card}((\Lambda \Delta \Gamma) \cap B_n)}{\text{vol}(B_n)} =: \text{dens}_B(\Lambda \Delta \Gamma), \]
where $\Delta$ denotes the symmetric difference of two sets. The topology induced by this metric on the hull of a point set is called the autocorrelation topology in [5,6]. In [10], the author refers to this as the Patterson topology, in [24] it is called the statistical coincidence topology, while in [17] this is called the mean topology. Whenever the sequence $B$ is clear from context, we will simply write $d_B(\Lambda, \Gamma)$. We will use the following properties of this metric.

**Lemma 2.10** [17,24] *Let $B$ be a fixed subsequence of $A$. For each $r > 0$, $d_B$ defines a translation-invariant semi-metric on $UD_r(\mathbb{R}^d)$, the set of $r$-uniformly discrete subsets of $\mathbb{R}^d$.***

Of importance to us is the following result.

**Proposition 2.11** [7, Thm.5] *Let $\Lambda \subset \mathbb{R}^d$ be such that $\Lambda - \Lambda$ is uniformly discrete, and let $B$ be any subsequence of $A_n = [-n,n]^d$ with respect to which the autocorrelation of $\Lambda$ exists. Then, $\Lambda$ is pure point diffractive with respect to $B$ if and only if, for each $\epsilon > 0$, the set
\[ P_\epsilon := \{ t \in \mathbb{R}^d : d_B(\Lambda, t + \Lambda) < \epsilon \} \]
is relatively dense.*

For more general versions of this proposition see [10,17].

### 3 Arithmetic Progressions in the Fibonacci Model Set

Before looking at the general case, let us first consider the Fibonacci model set. For a detailed overview, see [2, Chap. 7]. Consider the following CPS:
Here, $\tau = (1 + \sqrt{5})/2$ and $\tau' = (1 - \sqrt{5})/2$, and the $\ast$-mapping is the Galois conjugation

$$(m + n\tau)^\ast = m + n\tau' \quad \forall m, n \in \mathbb{Z}.$$ 

With this CPS, the Fibonacci model set is

$$\land([-1, \tau - 1]) := \{x \in L : x^\ast \in [-1, \tau - 1]\},$$

see Fig. 1. With this particular CPS and window, the Fibonacci model set coincides with the left end points of the geometric realisation of the substitution rule

$$a \mapsto ab \quad \text{and} \quad b \mapsto a$$

with seed $b|a$, see [2, Chap. 7]. Now, we can prove the following result.
Lemma 3.1  Let $n \in \mathbb{N}$, and let $\Lambda = \bigvee([-1, \tau - 1))$ be the Fibonacci model set. Then there exist $s, t \in \mathbb{R}$ such that

$$s, s + t, s + 2t, \ldots, s + nt \in \Lambda.$$  

Proof  For arbitrary but fixed $n$, pick some $t \in \bigvee([0, (\tau - 1)/n))$. Then, by construction, $0, t^*, 2t^*, \ldots, nt^* \in [-1, \tau - 1)$ and hence

$$0, 0 + t, 0 + 2t, \ldots, 0 + nt \in \Lambda. \quad \Box$$

An example of arithmetic progression of length 5 can be seen in Fig. 2.

Remark 3.2  The basic idea in the proof of Lemma 3.3 is the following: Since the window of our model set has non-empty interior, we can find a small open set such that small shifts of this open set stay inside the window. This type of argument is used in the study of repetitivity of model sets with open windows (see for example the proof of [2, Thm. 7.1]). We will employ the same argument later in the article to prove Theorem 1.5.

Next, we show that any element of the Fibonacci model set can be the first term of such a sequence, and, for a fixed element $s \in \Lambda$, we list all possible values of the common difference $t$.

Lemma 3.3  Let $n \in \mathbb{N}$, and let $\Lambda = \bigvee([-1, \tau - 1))$ be the Fibonacci model set. Let $s \in \Lambda$ be arbitrary. Then:

(a) There exists some $t \in \mathbb{R}$ such that

$$s, s + t, s + 2t, \ldots, s + nt \in \Lambda.$$

(b) For each $s \in \Lambda$ we have

$$\{t : s, s + t, s + 2t, \ldots, s + nt \in \Lambda\} = \bigvee\left(\left[\frac{-1 - s^*}{n}, \frac{\tau - 1 - s^*}{n}\right]\right).$$

In particular, this set is relatively dense.

Proof  First note that, to get $s, s + t \in \Lambda$, we must have $t \in \Lambda - \Lambda \subset L$. Then, for all $s \in \Lambda$, $t \in L$, the definition of $\Lambda$ implies

$$s, s + t, \ldots, s + nt \in \Lambda \iff s^*, s^* + t^*, \ldots, s^* + nt^* \in [-1, \tau - 1).$$

This condition is equivalent to $(-1 - s^*)/n \leq t^* < (\tau - 1 - s^*)/n$. Since the interval $\left((1 - s^*)/n, (\tau - 1 - s^*)/n\right)$ has non-empty interior, the claims follow now from Lemma 2.5.  \(\Box\)

Fig. 2  An arithmetic progression of length $n = 5$ in the Fibonacci model set. The common ratio is $r = 3\tau + 2$, with conjugate $r' \approx 0.1459$. Note that $-1 \leq 0, r', 2r', 3r', 4r' < \tau - 1$. 

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Next, we provide a stronger version of Proposition 3.3. This will allow us to prove a van der Waerden-type theorem for the Fibonacci model set.

**Lemma 3.4** For each \( n \in \mathbb{N} \), there exists some \( R > 0 \) such that, for all \( x \in \mathbb{R} \), the set \( \Lambda \cap [x, x + R] \) contains a non-trivial arithmetic progression of length \( n \).

**Proof** Since \([-1, \tau/2-1)\) has non-empty interior, the set \( \Lambda \cap ([-1, \tau/2-1]) \) is relatively dense by Lemma 2.5. Therefore, there exists some \( A > 0 \) such that \( \Lambda \cap ([-1, \tau/2-1]) + [-A, 0] = \mathbb{R} \). Next, as \((0, \tau/(2n))\) has non-empty interior, the set \( \Lambda \cap ((0, \tau/(2n))) \) is relatively dense by Lemma 2.5. Therefore, there exists some \( t \in \Lambda \cap ((0, \tau/(2n))) \cap (0, \infty) \).

Set \( R := A + nt \). We claim that this \( R \) has the desired property. Indeed, let \( x \in \mathbb{R} \). Then, since \( \Lambda \cap ([-1, \tau/2-1]) \) has non-empty interior, there exists some non-zero \( s \in \Lambda \cap ([-1, \tau/2-1]) \cap [x, x + A] \). Then, for all \( 0 \leq k \leq n \), we have \( x \leq s + kt \leq x + A + nt = x + R \), as well as \( s^* + kt^* \in [-1, \tau/2-1] \cup (0, k\tau/(2n)) \subset [-1, \tau-1) \). In particular, we have \( s, s + t, \ldots, s + nt \in \Lambda \cap [x, x + R] \).

Next, let us prove the following simple fact.

**Fact 3.5** In the Fibonacci CPS, if \( a < b \), then \( \Lambda \cap ((a, b)) \) is \((\tau^3/(b - a))-relatively dense.

**Proof** Since \( \tau \) is an unit in \( \mathbb{Z}[\tau] \), for all intervals \( I \) we have

\[
\tau \Lambda (I) = \{ \tau x : x' \in I \} = \{ \tau x : (\tau x)' \in \tau' I \} = \{ y : y' \in I \} = \Lambda (\tau' I).
\]

Therefore, for all \( n \in \mathbb{N} \) we have

\[
\tau^{2n} \Lambda \cap (-1, \tau - 1)) = \Lambda \cap ((-\tau)^{2n}, (\tau)^{2n}(\tau - 1))).
\]

Since \( \Lambda \cap (-1, \tau - 1)) \) is \( \tau \)-relatively dense, the set \( \Lambda \cap ((-\tau)^{2n}, (\tau)^{2n}(\tau - 1))) \) is \( \tau^{2n} \)-relatively dense, where we observe that \((-\tau)^{2n}, (\tau)^{2n}(\tau - 1)) \) is an interval of length \( |\tau'|^{2n-1} \).

Next, fix any \( a < b \) in \( \mathbb{R} \) and pick \( n \in \mathbb{Z} \) such that

\[
|\tau'|^{2n-1} < b - a \leq |\tau'|^{2n-3}.
\]

This is possible because \( |\tau'|^{2n-1} \to 0 \) as \( n \to \infty \) and \( |\tau'|^{2n-1} \to \infty \) as \( n \to -\infty \). Since \( |\tau'|^{2n-1} < b - a \) and \( L^* \) is dense in \( \mathbb{R} \), there exists some \( (t, t^*) \in L \) such that \( t^* + [-\tau)^{2n}, (\tau)^{2n}(\tau - 1)) \) \( \subset (a, b) \). Then, we get

\[
\Lambda \cap ((-\tau)^{2n}, (\tau)^{2n}(\tau - 1))) \subset \Lambda (t^* + (a, b)) = -t + \Lambda (a, b)).
\]

Since \( \Lambda \cap ((-\tau)^{2n}, (\tau)^{2n}(\tau - 1))) \) is \( \tau^{2n} \)-relatively dense, it follows that \( \Lambda \cap ((a, b)) \) is \( \tau^{2n} \)-relatively dense as well. Finally, \( b - a \leq |\tau'|^{2n-3} \) gives

\[
\tau^{2n} = \frac{\tau^3}{|\tau'|^{2n-3}} \leq \frac{\tau^3}{b-a}.
\]
which proves the fact. □

**Remark 3.6** Using Fact 3.5, we can give an upper bound on $R$ from Lemma 3.4. Indeed, in the proof of Lemma 3.4, Fact 3.5 implies that we can choose $A = 2\tau^2$ and $0 < t \leq 2n\tau^2$, thus

$$R = 2\tau^2 + 2n^2\tau^2 = 2(n^2 + 1)\tau^2,$$

is a possible choice which works.

Next, let us prove the following van der Waerden-type result for point sets.

**Lemma 3.7** Let $\Lambda \subset \mathbb{R}^d$ be a point set such that, for each $n \in \mathbb{N}$, there exists some $R' > 0$ such that for all $x \in \mathbb{R}^d$, the set $\Lambda \cap B_R(x)$ contains a non-trivial arithmetic progression of length $n$. Let $r, k$ be two given natural numbers. Then, there exists some $R > 0$ such that, for each colouring of $\Lambda$ with $r$ colours, and for each $x \in \mathbb{R}^d$, the set $\Lambda \cap B_R(x)$ contains $k$ elements of the same colour in arithmetic progression.

**Proof** Let $N$ be chosen so that van der Waerden’s theorem (Theorem 1.1) holds for $r, k$ applied to $\{1, 2, \ldots, N\}$. By the condition in the statement, there exists some $R > 0$ such that, for all $x \in \mathbb{R}^d$, the set $\Lambda \cap B_R(x)$ contains a non-trivial arithmetic progression of length $N$.

We claim that this $R$ works. To see this, consider any colouring of $\Lambda$ with $r$ colours. Let $x \in \mathbb{R}^d$ be arbitrary. Then, we can find $s, t$ such that $t \neq 0$ and, for all $1 \leq k \leq N$,

$$a_j := s + jt \in \Lambda \cap B_R(x).$$

Now, colour $j \in \{1, 2, \ldots, N\}$ with the colour of $a_j$. Then, by Theorem 1.1, there exists an arithmetic progression $l_1 < l_2 < \ldots < l_k$ of length $k$ of the same colour. Consequently, $s + l_1t, s + l_2t, \ldots, s + l_kt \in \Lambda \cap B_R(x)$ are $k$ elements of the same colour in arithmetic progression. □

By combining Lemmas 3.4 and 3.7 we get the following result.

**Theorem 3.8** (van der Waerden-type theorem for Fibonacci) For any given natural numbers $r$ and $k$, there is some $R > 0$ such that, for each colouring of $\Lambda$ with $r$ colours, and for each $x \in \mathbb{R}$, the set $\Lambda \cap [x, x + R]$ contains $k$ elements of the same colour in arithmetic progression.

## 4 Arithmetic Progressions in Model Sets

In this section, we show that model sets in $\mathbb{R}^d$ contain arbitrarily long arithmetic progressions, and prove a van der Waerden-type result for model sets.

For this entire section, $(\mathbb{R}^d, H, L)$ is a fixed CPS and $\Lambda = \Lambda(W)$ is a fixed model set in this CPS.

We begin with a preliminary result.
Lemma 4.1  Let \((\mathbb{R}^d, H, \mathcal{L})\) be a CPS and let \(0 \in U \subset H\) be open. Then \(\mathcal{L}(U) \setminus \{0\}\) is relatively dense.

**Proof** The set \(\mathcal{L}(U)\) is relatively dense by Lemma 2.5. It is obvious that removing a single point yields another relatively dense set. \(\square\)

Now, we can prove that model sets have arbitrarily long arithmetic progressions.

**Proposition 4.2** Let \(n \in \mathbb{N}\). Then, for each \(s \in \mathcal{L}(\text{Int}(W))\), there exists a neighbourhood \(U \in H\) of 0, such that \(\mathcal{L}(U) \setminus \{0\}\) is relatively dense in \(\mathbb{R}^d\) and that for each \(t \in \mathcal{L}(U) \setminus \{0\}\) we have

\[s, s + t, s + 2t, \ldots, s + nt \in \Lambda = \mathcal{L}(W).\]

**Proof** Since \(s^* \in \text{Int}(W)\), which is open, we can find a neighbourhood \(U\) of 0 such that

\[s^* + U + U + \cdots + U \subset \text{Int}(W)\]

From here, it follows immediately that

\[s, s + t, s + 2t, \ldots, s + nt \in \Lambda, \quad \text{for all } t \in \mathcal{L}(U). \tag{3}\]

The claim follows now from (3) and Lemma 4.1. \(\square\)

Next, we show that, for each \(n\), we can find arithmetic progressions of length \(n\) within bounded gaps.

**Lemma 4.3** For each \(n \in \mathbb{N}\), there exists some \(R > 0\) such that, for all \(x \in \mathbb{R}^d\), the set \(\Lambda \cap B_R(x)\) contains a non-trivial arithmetic progression of length \(n\).

**Proof** Since \(W\) has non-empty interior, we can find an open set \(V \neq \emptyset\) and some \(U\), a neighbourhood of 0, such that

\[V + U + U + \cdots + U \subset W.\]

Since \(V\) is open, there exists some \(R' > 0\) such that \(\mathcal{L}(V) + B_{R'}(0) = \mathbb{R}^d\). Since \(U\) is an open neighbourhood of 0, by Lemma 4.1, there exists some \(R'' > 0\) such that \((\mathcal{L}(U) \setminus \{0\}) + B_{R''}(0) = \mathbb{R}^d\). Define \(R := R' + nR''\). We show that this \(R\) works.

Let \(x \in \mathbb{R}^d\) be arbitrary. Then, there exists some \(s \in \mathcal{L}(V) \cap B_R(x)\). Let \(t \in (\mathcal{L}(U) \setminus \{0\}) \cap B_{R''}(0)\). Then, since \(0 \in U\), we have

\[
\underbrace{t^* + t^* + \cdots + t^*}_{k \text{ times}} + \underbrace{0 + 0 + \cdots + 0}_{n - k \text{ times}} + \underbrace{U + U + \cdots + U}_{n \text{ times}} \in \mathcal{L}(U) \setminus \{0\}.
\]
for all $0 \leq k \leq n$. Therefore, for all $0 \leq k \leq n$, we have

$$s^* + kt^* \in V + U + U + \cdots + U \subset W \quad \Rightarrow \quad s + kt \in \Lambda(W) = \Lambda.$$  

Denote the Euclidean norm on $\mathbb{R}^d$ by $\|\cdot\|$. Now, for all $0 \leq k \leq n$, we have

$$\| (s + kt) - x \| \leq \| s - x \| + k \| t \| = \| s - x \| + k \| t \| < kR' + R' \leq nR' + R' = R.$$  

This gives $s, s + t, s + 2t, \ldots, s + nt \in \Lambda \cap B_R(x)$.  

We now can prove our first main result.

**Proof of Theorem 1.5** This follows immediately from Lemmas 4.3 and 3.7.  

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### 5 Arithmetic Progressions in Meyer Sets

In this section, we show that any Meyer set $\Lambda \subset \mathbb{R}^d$ has arbitrarily long arithmetic progressions.

For this entire section, $\Lambda \subset \mathbb{R}^d$ is a fixed Meyer set.

**Proposition 5.1** Let $\Lambda \subset \mathbb{R}^d$ be a Meyer set. Then, for each positive integer $k$, there exists some $R > 0$, such that, for all $x \in \mathbb{R}^d$, the set $\Lambda \cap B_R(x)$ contains a non-trivial arithmetic progression of length $k$.

**Proof** Since $\Lambda$ is a Meyer set, Proposition 2.7 implies the existence of a model set $\Lambda(W)$ such that $\Lambda \subset \Lambda(W)$. By [35, Lem. 5.5.1], there exists a finite set $F = \{t_1, \ldots, t_r\}$ such that $\Lambda(W) \subset \Lambda + F$. Let $R'$ be such that Theorem 1.5 holds for the model set $\Lambda(W)$, and the positive integers $r, k$. Let

$$R := \max \{ R' + \| t_j \| : 1 \leq j \leq r \}.$$  

We show that this $R$ works.

We colour $\Lambda(W)$ with $r$ colours in the following way: Since $\Lambda(W) \subset \Lambda + F$, for each $x \in \Lambda(W)$, there exists some $1 \leq j \leq r$ such that $x \in t_j + \Lambda$. Then, one can colour each $x \in \Lambda(W)$ by the colour $\min \{ j : x \in t_j + \Lambda \}$. We use the minimum since some $x \in \Lambda(W)$ may belong to multiple sets $t_j + \Lambda$, in which case we need to make a choice (any choice here makes the rest of the proof work).

Let $x \in \mathbb{R}^d$ be arbitrary. By Theorem 1.5, there exists a non-trivial monochromatic arithmetic progression $a_1, \ldots, a_k$ of length $k$ in $\Lambda(W) \cap B_{R'}(x)$. By our construction of the colouring, there exists some $j$ such that

$$a_1, \ldots, a_k \in t_j + \Lambda.$$
It follows that $a_1 - t_j, a_2 - t_j, \ldots, a_k - t_j \in \Lambda$ is a non-trivial arithmetic progression of length $k$. Moreover, for each $1 \leq i \leq k$, we have

$$\| (a_k - t_j) - x \| \leq \| a_k - x \| + \| t_j \| < R' + \| t_j \| \leq R,$$

which proves the claim. \hfill \Box

We now can prove our second main result.

**Proof of Theorem 1.6** This follows immediately from Proposition 5.1 and Lemma 3.7. \hfill \Box

### 6 Pure Point Diffractive Sets

In this section we study the existence of arbitrarily long arithmetic progressions in sets $\Lambda \subset \mathbb{R}^d$ such that $\Lambda - \Lambda$ is uniformly discrete. We should emphasise here that $\Lambda$ is not assumed to be relatively dense (as in that case $\Lambda$ would be a Meyer set and we would be in the situation of Proposition 5.1). In particular, the technique from Sect. 5 cannot be used, as no Delone set can be covered by finitely many translates of $\Lambda$.

To compensate for the lack of relative denseness, we will assume instead that $\Lambda$ is pure point diffractive. Pure point diffraction is equivalent to the existence of a relatively dense set $T$ of translates $t$ of $\Lambda$ which ’almost’ agree with $\Lambda$ (see Proposition 2.11 for the exact meaning, as well as [7,10,17] for more general statements). From here, we can conclude that for each $n$ we can find some $t$ such that the sets $\Lambda$, $t + \Lambda$, $2t + \Lambda$, $\ldots$, $nt + \Lambda$ ”almost” agree, and the existence of an arithmetic progression of length $n$ follows. This way we can prove the following weak Szemerédi-type theorem.

**Theorem 6.1** Let $\Lambda \subset \mathbb{R}^d$ be a point set, $A_n = [-n, n]^d$, and $B$ be a subsequence of $A := \{ A_n \}_n$. Assume that

(i) $\Lambda - \Lambda$ is uniformly discrete,
(ii) $\text{dens}_B(\Lambda) > 0$,
(iii) $\Lambda$ is pure point diffractive with respect to $B$.

Then, for each $\epsilon > 0$ and each $n \in \mathbb{N}$, there exists a relatively dense set $T \subset \mathbb{R}^d$ such that, for each $t \in T$,

$$\text{dens}_B(\Lambda \cap (t + \Lambda) \cap \ldots \cap (nt + \Lambda)) \geq (1 - \epsilon)\text{dens}_B(\Lambda).$$

In particular, $\Lambda$ has arithmetic progressions of arbitrary length.

**Proof** Let $\epsilon > 0$ and $n \in \mathbb{N}$ be fixed. Then, by Proposition 2.11, the set

$$T = \left\{ t \in \mathbb{R}^d : d_B(\Lambda, t + \Lambda) < \frac{\epsilon \text{dens}_B(\Lambda)}{n} \right\}.$$
is relatively dense. Let $t \in T$ be arbitrary and $\Gamma_t := \Lambda \cap (t + \Lambda) \cap \ldots \cap (nt + \Lambda)$. A simple computation shows that

$$\Lambda - \Gamma_t \subseteq (\Lambda \Delta (t + \Lambda)) \cup \ldots \cup (((n - 1)t + \Lambda) \Delta (nt + \Lambda)).$$

(4)

Indeed, if $s \in \Lambda \setminus \Gamma_t$, then $s \notin \Lambda \cap (t + \Lambda) \cap \ldots \cap (nt + \Lambda)$. Let $k$ be the smallest $0 \leq k \leq n$ such that $s \notin kt + \Lambda$. As $s \in \Lambda$, we have $k \geq 1$ and $s \in (k - 1)t + \Lambda$. This shows that

$$s \in ((k - 1)t + \Lambda) \Delta (kt + \Lambda) \subseteq (\Lambda \Delta (t + \Lambda)) \cup \ldots \cup (((n - 1)t + \Lambda) \Delta (nt + \Lambda)).$$

Now, since $\Gamma_t \subset \Lambda$, we have

$$\overline{\text{dens}}_B(\Lambda \setminus \Gamma_t) = \overline{\text{dens}}_B(\Lambda \Delta \Gamma_t) = d_B(\Lambda, \Gamma_t) \leq \sum_{k=0}^{n-1} d_B(kt + \Lambda, (k + 1)t + \Lambda)
= \sum_{k=0}^{n-1} d_B(\Lambda, t + \Lambda) < n \frac{\epsilon}{n},$$

with the last equality following from Lemma 2.10. Therefore, as $\Lambda \subset \Gamma_t \cup (\Lambda \setminus \Gamma_t)$ we have

$$\overline{\text{dens}}_B(\Lambda) \leq \overline{\text{dens}}_B(\Gamma_t) + \overline{\text{dens}}_B(\Lambda \setminus \Gamma_t) \leq \overline{\text{dens}}_B(\Gamma_t) + \epsilon \overline{\text{dens}}_B(\Lambda),$$

which gives

$$\overline{\text{dens}}_B(\Lambda \cap (t + \Lambda) \cap \ldots \cap (nt + \Lambda)) \geq (1 - \epsilon)\overline{\text{dens}}_B(\Lambda).$$

Finally, for a fixed $0 < \epsilon < 1$, $T$ is relatively dense and hence infinite. Then, we can pick some $0 \neq t \in T$ and we get that

$$\overline{\text{dens}}_B(\Lambda \cap (t + \Lambda) \cap \ldots \cap (nt + \Lambda)) > 0,$$

which gives that $\Lambda \cap (t + \Lambda) \cap \ldots \cap (nt + \Lambda)$ is non-empty. Now, picking some $s \in \Lambda \cap (t + \Lambda) \cap \ldots \cap (nt + \Lambda)$ we then get

$$s, s - t, s - 2t, \ldots, s - nt \in \Lambda,$$

which proves the claim.

\[ \square \]

**Remark 6.2** Under the assumptions of Theorem 6.1, the autocorrelation $\gamma$ of $\Lambda$ exists with respect to $B$, so in particular the density of $\Lambda$ exists with respect to $B$. Moreover, the density is non-zero exactly when the diffraction is non-trivial.
Given a CPS \((\mathbb{R}^d, H, \mathcal{L})\) and a compact window \(W\), recall that \(\ldots(W)\) is called a weak model set of maximal density with respect to a subsequence \(B\) of \(A = \{[-n, n]^d\}_n\), if
\[
\lim_n \frac{\text{card}(\ldots(W) \cap B_n)}{\text{vol}(B_n)} = \text{dens}(\mathcal{L}) \cdot \theta_H(W).
\]

For an overview of maximal density model sets and their properties, see \([4,13]\). As an immediate consequence of Theorem 6.1, we get the following result.

**Corollary 6.3** Let \((\mathbb{R}^d, H, \mathcal{L})\) be a CPS, let \(W \subseteq H\) be compact, and let \(B\) be a subsequence of \(A_n = [-n, n]^d\). If \(\ldots(W)\) has maximal density with respect to \(B\), and if \(\theta_H(W) \neq 0\), then \(\ldots(W)\) contains arithmetic progressions of arbitrary length.

**Proof** Since \(\ldots(W)\) has maximal density with respect to \(B\) and \(\theta_H(W) \neq 0\), \(\ldots(W)\) has positive density. Moreover \(\ldots(W) - \ldots(W) \subseteq \ldots(W - W)\) is uniformly discrete as \(W - W\) is compact. Further, \(\ldots(W)\) is pure point diffractive with respect to \(B\) by \([4,\text{Thm. 7}]\) or \([13, \text{Cor. 6}]\). The claim follows now from Theorem 6.1. \(\square\)

By combining this with \([23, \text{Thm. 1}]\) we get:

**Corollary 6.4** Let \((\mathbb{R}^d, H, \mathcal{L})\) be a CPS, let \(W \subseteq H\) be a compact set, and let \(A_n = [-n, n]^d\). If \(\theta_H(W) \neq 0\), then the model set \(\ldots(t + W)\) contains arithmetic progressions of arbitrary length for almost all \((s, t) \in \mathbb{T} := (\mathbb{R}^d \times H)/\mathcal{L}\).

We complete the section by introducing a class of pure point diffractive Delone sets that do not contain arithmetic progressions of length 3.

**Proposition 6.5** Let \(\{a_n\}_{n \in \mathbb{Z}} \in \mathbb{R}\) be such that

(a) \(a_n \to 0\) as \(n \to \pm \infty\),
(b) \(\ldots,a_n, \ldots,a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots, a_n, \ldots\) and \(1\) are linearly independent over \(\mathbb{Q}\).

Define \(\Lambda := \{n + a_n : n \in \mathbb{Z}\}\). Then, \(\Lambda\) is pure point diffractive and does not contain three elements in arithmetic progression.

**Proof** First, since \(a_n\) are linearly independent, we have \(n + a_n \neq m + a_m\) for all \(n \neq m\). Indeed, if we assume by contradiction that there exists some \(n \neq m\) such that \(n + a_n = m + a_m\), then
\[
(m - n) \cdot 1 + 1 \cdot a_m + (-1) \cdot a_n = 0,
\]
contradicting the linear independence over \(\mathbb{Q}\).

The claim of not containing three points in arithmetic progression is also an immediate consequence of the linear independence. Assuming the contrary, if \(n + a_n, m + a_m, k + a_k\) are three distinct elements in arithmetic progression, we get
\[
2(m + a_m) = (n + a_n) + (k + a_k),
\]
which gives
\[
0 = (n + k - 2m) \cdot 1 + 1 \cdot a_n + 1 \cdot a_k - 2 \cdot a_m.
\]
Since \( m, n, k \) are distinct, this contradicts the linear independence of \( 1, a_m, a_n, a_k \).

Finally, a simple computation shows that \( \delta_A - \delta_Z \) is a measure vanishing at infinity, and hence null-weakly almost periodic [31, Lem. 15]. Therefore, \( A \) is pure point diffractive by [18, Thm. 7.6]. More generally, \( A \) has the same diffraction as \( Z \) [18, Lem. 7.11], hence \( \hat{\gamma} = \delta_Z \).

**Example 6.6** An explicit such example is

\[
A = \left\{ n + \frac{e^{2n+2}}{3^{2n+2}} : n \in \mathbb{N} \right\} \cup \left\{ -n + \frac{e^{2n+1}}{3^{2n+1}} : n \in \mathbb{N} \right\} \cup \{0\}.
\]

Let us mention here that we suspect that the Szemerédi theorem is true in model sets, but the proof is probably very long and tedious, and beyond the scope of this paper. We state this here as a conjecture.

**Conjecture 6.7** (Szemerédi-type conjecture for model sets) Let \( \mathcal{X}(W) \) be a model set in a CPS \((\mathbb{R}^d, H, \mathcal{L})\) and let \( A \) be a van Hove sequence in \( \mathbb{R}^d \). If \( \Lambda \subset \mathcal{X}(W) \) is a subset such that \( \text{dens}_A(\Lambda) > 0 \), then the set \( \Lambda \) contains arithmetic progressions of arbitrary length.

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