A necessary and sufficient condition for conservation of angular momentum at foot strike during passive dynamic walking

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ABSTRACT
Legged robots can walk stably down slopes without any actuation or control. The locomotion is called passive dynamic walking. Its dynamics at foot strike are governed by impact forces. The impulsive dynamics have been modelled in several approaches, where all the approaches assume that the stance leg instantaneously loses contact with the ground at foot strike. The loss-of-contact assumption has been introduced so as to guarantee conservation of angular momentum at foot strike; however, it is shown in this paper that the loss-of-contact assumption is neither necessary nor sufficient for that purpose. This paper provides a necessary and sufficient condition for conservation of angular momentum at foot strike in a modelling framework for multibody dynamics with impact. A compass-like biped robot is considered in this paper. Its impulsive dynamics are modelled by the relationship between impact forces and change of momentum with constraints. A necessary and sufficient condition for conservation of angular momentum is derived by examining the validity of foot velocities and impact forces in the impulsive model.

1. Introduction
Legged robots can walk stably down slopes without any actuation or control. The locomotion is called passive dynamic walking [1]. Passive dynamic walking provides human-like motions, and it has the potential to produce useful models of human-like locomotion for legged robots with actuation [2–4].

Walking dynamics are affected by impact forces at foot strike. The impulsive dynamics at foot strike have been modelled in several approaches [5–10], where all the approaches assume that only one leg is the stance leg at a time. The assumption implies that a leg in contact with the ground before foot strike instantaneously loses the contact at foot strike. The assumption is called the loss-of-contact assumption in this paper. The loss-of-contact assumption has been introduced so as to guarantee the conservation of angular momentum at foot strike; however, it is shown in this paper that the loss-of-contact assumption is neither necessary nor sufficient for that purpose.

This paper derives a necessary and sufficient condition for conservation of angular momentum at foot strike in a modelling framework for multibody dynamics with impact formulated in [11,12]. The derived condition is different from loss-of-contact. In particular, this paper shows that the loss-of-contact assumption is neither necessary nor sufficient for conservation of angular momentum at foot strike. This implies that the loss-of-contact assumption used in existing works is inappropriate. Note that recent researches such as [13–21] remain based only on the walking models derived in [5–10] where the loss-of-contact assumption is assumed. Therefore, the issue addressed in this paper is left unresolved at present, while the walking models have been derived several decades ago.

Details are as follows. This paper considers a compass-like biped robot. Its impulsive dynamics at foot strike are modelled by using the modelling framework for multibody dynamics with impact. In particular, the model is given by the relationship between impact forces and change of momentum with constraints. It is shown that the impulsive model obtained in this paper is equivalent to one derived from conservation laws of angular momentum in existing works. This confirms the validity of the approaches used in this paper. A necessary and sufficient condition that the derived model holds is obtained by examining the validity of foot velocities and impact forces in the impulsive model. The obtained condition is also a necessary and sufficient condition for conservation of angular momentum.

This paper is an extended version of a conference paper [22], and two extensions are provided. First, a compass-like biped robot with a mass at the hip is considered in this paper, while a robot without the hip mass has been done in [22]. The first extension...
enables comparisons between the model in this paper and existing models, since compass-like biped robots considered in [5–10] have a mass at the hip. Second, a necessary and sufficient condition for conservation of angular momentum is derived from multibody dynamics with impact. Conservation of angular momentum has not been explored in [22], while the dynamics have been investigated there. The second extension is the main contribution in this paper, and it is described in Section 4.

2. Compass-like biped robot

This paper considers a compass-like biped robot that walks on a rigid ground illustrated in Figure 1, and uses notations in Table 1. The robot has a hip mass and two rigid legs connected with a passive revolute joint. It is assumed that (i) the centre of mass of the hip is located at the joint, and (ii) the two legs have the same physical parameters. The end of each leg is called a foot. It is supposed that one leg is in contact with the ground, the leg pivots on the foot of the leg, and then the other leg strikes the ground. The leg in contact with the ground before strike is called the stance leg. The other leg is called the non-stance leg, while it is often called the swing leg. The state transition at foot strike is represented as

\[
\dot{\theta}_{n-} \rightarrow \dot{\theta}_{n+} \quad \text{and} \quad \dot{\theta}_{s-} \rightarrow \dot{\theta}_{s+}
\]

in this paper, while it is written as

\[
\dot{\theta}_{n-} \rightarrow \dot{\theta}_{n+} \quad \text{and} \quad \dot{\theta}_{s-} \rightarrow \dot{\theta}_{s+}
\]

in [6,7], where the leg labels are replaced in (2).

Two points of \((x_h, y_h)\) and \((x_j, y_j)\) are defined by different variables, and they are constrained to the same point by \(f_{y_h}\) and \(f_{y_j}\). The two legs are connected at the joint with \(f_{x_j}\) and \(f_{y_j}\). Therefore, \(f_{x_h}\) and \(f_{y_j}\) are different from each other, while they are acting at the same point in the same direction.

In Figure 1, the \(y\) axis is directed in the normal direction to the slope. Note that the impulsive dynamics are independent of the slope angle, since it is not affected by gravity [11,12]. It is naturally assumed that the leg

![Figure 1. The biped robot that walks on a rigid ground. Left: The leg in contact with the ground before strike is called the stance leg. The other leg is called the non-stance leg in this paper, while it is often called the swing leg. Right: A coordinate system and some variables used in this paper. See also Table 1 and the text for their precise definitions. Two points of \((x_h, y_h)\) and \((x_j, y_j)\) are defined by different variables, and they are constrained to the same point. The \(y\) axis is directed in the normal direction to the slope. The angles \(\theta_n\) and \(\theta_s\) are defined positive in the anticlockwise direction.](image)

Table 1. Notations used in this paper.

| \(x_h, y_h\) | The position of the centre of mass of the non-stance leg. |
| \(x_s, y_s\) | The position of the centre of mass of the stance leg. |
| \(x_h, y_h\) | The position of the foot of the non-stance leg. |
| \(x_s, y_s\) | The position of the foot of the stance leg. |
| \(x_h, y_h\) | The position of the joint. |
| \(x_h, y_h\) | The position of the centre of mass of the hip. |
| \(\theta_n, \theta_s\) | The angles of the non-stance leg and the stance leg. They are defined positive in the anticlockwise direction. |
| \(s_-, s_+\) | Values of variable \(s\) before strike and after strike. In particular, \(s_-\) denotes the limit of \(s\) as the time approaches the time of foot strike from the left. Also, \(s_+\) denotes the limit from the right. |
| \(f_{x_h}, f_{y_h}\) | Impact forces at \((x_h, y_h)\) in the \(x\) and \(y\) directions. |
| \(m_h, m_l\) | Mass of the hip and mass of the legs. |
| \(J\) | Moment of inertia about the centre of mass for the legs. |
| \(\ell_f\) | The length between the foot and the centre of mass of the legs. |
| \(\ell_l\) | The length between the joint and the centre of mass of the legs. |
| \(Q_{0 \times m}\) | The zero matrix of size \(n \times m\). |
| \(x \geq 0\) | For a given vector \(x\), all the elements of \(x\) are non-negative. |
| \((X)_{ij}, (X)_i\) | For a given matrix \(X\), the \((i,j)\)-th element of \(X\), and the \(i\)-th row vector of \(X\). |
angles are within
\[ |\theta_n| < \frac{\pi}{2}, \quad \text{and} \quad |\theta_s| < \frac{\pi}{2}. \] (3)

The following assumptions are made.

**Assumption 2.1:** (a) One foot is in contact with the ground before strike.
(b) If a foot is in contact with the ground, then no slip occurs between the foot and the ground.
(c) Collisions are perfectly inelastic.

### 3. Dynamics with impact

The following equations are obtained from Figure 1:

\[ x_s = x_{sf} - \ell_s \sin \theta_s, \quad y_s = y_{sf} + \ell_s \cos \theta_s, \] (4)
\[ x_j = x_{sj} - \ell_j \sin \theta_j, \quad y_j = y_{sj} + \ell_j \cos \theta_j, \] (5)
\[ x_{nj} = x_{nj} + \ell_n \sin \theta_n, \quad y_{nj} = y_{nj} - \ell_n \cos \theta_n, \] (6)
\[ x_j = x_{j} - \ell_j \sin \theta_j, \quad y_j = y_{j}, \] (8)

This paper considers only motion at foot strike from here until the end of the paper, and it is assumed that each variable has a value at foot strike. The joint constraints (5), (6), and (8) can be written by

\[ \dot{x}_n - \ell_u \dot{\theta}_n \cos \theta_n = x_{n-} - \ell_u \dot{\theta}_n \cos \theta_n, \] (9)
\[ \dot{y}_n - \ell_u \dot{\theta}_n \sin \theta_n = y_{n-} - \ell_u \dot{\theta}_n \sin \theta_n, \] (10)
\[ \dot{x}_n = x_{n-} - \ell_u \dot{\theta}_n \cos \theta_n, \] (11)
\[ \dot{y}_n = y_{n-} - \ell_u \dot{\theta}_n \sin \theta_n, \] (12)
\[ \dot{x}_n - \ell_u \dot{\theta}_n \cos \theta_n = x_{n+} - \ell_u \dot{\theta}_n \cos \theta_n, \] (13)
\[ \dot{y}_n - \ell_u \dot{\theta}_n \sin \theta_n = y_{n+} - \ell_u \dot{\theta}_n \sin \theta_n, \] (14)
\[ \dot{x}_n = x_{n-} - \ell_u \dot{\theta}_n \cos \theta_n, \] (15)
\[ \dot{y}_n = y_{n-} - \ell_u \dot{\theta}_n \sin \theta_n, \] (16)

Differentiating (4) and (7) yields

\[ \dot{x}_{nj} = \dot{x}_n + \ell \dot{\theta}_n \cos \theta_n = 0, \] (17)
\[ \dot{y}_{nj} = \dot{y}_n + \ell \dot{\theta}_n \sin \theta_n = 0, \] (18)
\[ \dot{x}_j = \dot{x}_{nj} + \ell \dot{\theta}_n \cos \theta_n = 0, \] (19)
\[ \dot{y}_j = \dot{y}_{nj} + \ell \dot{\theta}_n \sin \theta_n = 0, \] (20)

where each last equation in (17) and (19) follows from Assumption (b), (18) from Assumption (a), and (20) from Assumption (c).

The following lemma provides a necessary and sufficient condition for impact occurrence, and it can be proven in the same manner as Lemma 1 in [22].

**Lemma 3.1:** Consider the compass-like biped robot, and suppose that Assumption (a) is satisfied. Then, the foot of the non-stance leg strikes the ground, if and only if it holds that

\[ \theta_j = -\theta_n, \] (21)

and

\[ (\dot{\theta}_n - \dot{\theta}_n) \sin \theta_n < 0. \] (22)

Impulsive dynamics at foot strike are modelled by the relationship between impact forces and change of momentum with constraints, where the constraints are provided in (9)–(20). The impulsive model is physically valid, if and only if all of \( y_{nj} + x_{nj} f_{nj}, \) and \( f_{nj} \) are non-negative, as can be seen in the right of Figure 1. The impulsive model and its validity condition are respectively given as (23) and (24) in the following theorem, when no impact force arises at the foot of the stance leg.

**Theorem 3.1:** Consider the compass-like biped robot at foot strike, and suppose that Assumptions (a)–(c) are satisfied. If an impact force does not arise at the foot of the stance leg, then it holds that

\[ \dot{\theta}_+ = C \dot{\theta}_-, \] (23)
\[ D \dot{\theta}_- \geq 0, \] (24)

where

\[ \dot{\theta}_- = \begin{bmatrix} \dot{\theta}_{n-} \\ \dot{\theta}_{s-} \end{bmatrix}, \] (25)
\[ \dot{\theta}_+ = \begin{bmatrix} \dot{\theta}_{n+} \\ \dot{\theta}_{s+} \end{bmatrix}, \] (26)

and \( C \in \mathbb{R}^{2 \times 2} \) and \( D \in \mathbb{R}^{2 \times 2} \) are defined, respectively, by (42) and (46) in the proof.

**Proof:** The impact forces \( f_{sj}, f_{sj}, f_{sh}, \) and \( f_{sh} \) are defined without loss of generality as follows: (i) \( f_{sj} \) and \( f_{sj} \) come from the non-stance leg to the stance leg, and their reactions are applied from the stance leg to the non-stance leg. (ii) \( f_{sh} \) and \( f_{sh} \) come from the non-stance leg to the hip, and their reactions are applied from the hip to the non-stance leg. Then, linear and angular momenta satisfy

\[ m_h \dot{x}_h = m_h \dot{x}_n + f_{hn}, \] (27)
\[ m_h \dot{y}_h = m_h \dot{y}_n + f_{hy}, \] (28)
\[ m_e \dot{x}_e = m_e \dot{x}_n + f_{ej} - f_{ej} - f_{eh}, \] (29)
\[ m_e \dot{y}_e = m_e \dot{y}_n + f_{ey} - f_{ey} - f_{eh}, \] (30)
\[ J \dot{\theta}_h = (y_{nj} - y_n) f_{nj} + (x_{nj} - x_n) f_{nj} + (y_j - y_n) f_{nj} - (x_j - x_n) f_{nj}, \] (31)
\[ m_e \dot{x}_e = m_e \dot{x}_j + f_{sj}, \] (32)
\[ m_e \dot{y}_e = m_e \dot{y}_j + f_{sj}, \] (33)
\[ J \dot{\theta}_j = (y_j - y_n) f_{nj} + (x_j - x_n) f_{nj}. \] (34)
The Equations (13)–(16), (19), (20) and (27)–(34) can be written in a matrix form of

\[ A\lambda = b = B\dot{\theta}_n, \]

with

\[ A = \begin{bmatrix} M & N \\ N^T & 0_{6\times 6} \end{bmatrix}, \]

\[ M = \text{diag} (m_h, m_h, m_\ell, m_\ell, J, m_\ell, m_\ell), \]

\[ N = \begin{bmatrix} 0 & 0 & 0 \\ -\ell_\ell \cos \theta_n & -\ell_\ell \sin \theta_n & -\ell_u \cos \theta_n \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & \ell_u \cos \theta_n & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \]

\[ b = \begin{bmatrix} m_h \dot{x}_h \\ m_h \dot{y}_h \\ m_\ell \dot{x}_n \\ m_\ell \dot{y}_n \\ J \dot{\theta}_n \\ m_\ell \dot{x}_s \\ m_\ell \dot{y}_s \\ J \dot{\theta}_s \end{bmatrix}, \]

\[ \lambda = \begin{bmatrix} \dot{x}_h \\ \dot{y}_h \\ \dot{x}_n \\ \dot{y}_n \\ \dot{x}_s \\ \dot{y}_s \end{bmatrix}, \]

where the first equality in (35) follows from (5)–(8) and (21), and the second from (9)–(12), (17), (18) and (21). It is straightforward to see that

\[ \det A = (J + m_\ell \ell_\ell \ell_\ell) \alpha + 2m_\ell \ell_\ell \ell_\ell \beta, \]

\[ + m_h \ell_\ell \ell_\ell \beta + m_\ell^2 \ell_\ell \ell_\ell^2 \sin^2 \theta_n > 0. \]

Let \( C \) be defined by

\[ C = \begin{bmatrix} (A^{-1}B)_{5,1} & (A^{-1}B)_{5,2} \\ (A^{-1}B)_{8,1} & (A^{-1}B)_{8,2} \end{bmatrix}. \]

Then, (23) is derived from (35) and (42).

Equation (23) is valid, if and only if all of \( \dot{y}_{j+}, \dot{f}_{yj}, f_{yj} \) are non-negative. It is guaranteed that \( \dot{y}_{j+} = 0 \) and \( f_{yj} = 0 \), where the former follows from Assumption (c), and the latter from the assumption in the theorem. Therefore, a necessary and sufficient condition that (23) holds is given by

\[ \dot{y}_{j+} \geq 0, \quad \text{and} \quad f_{yj} \geq 0. \]

Differentiating (4) yields

\[ \dot{y}_{j+} = \dot{x}_{j+} + \ell_\ell \dot{\theta}_j \sin \theta_n. \]

Substituting (14), (20), (21) and (44) into the first inequality of (43) shows that the first inequality in (43) is equivalent to

\[ (\dot{\theta}_n + \dot{\theta}_s) \sin \theta_n \leq 0. \]

Let \( D \) be defined by

\[ D = \begin{bmatrix} \alpha & -\beta \\ (A^{-1}B)_{10,1} & (A^{-1}B)_{10,2} \end{bmatrix}, \]

\[ \alpha = -((C)_{1,1} + (C)_{2,1}) \sin \theta_n, \]

\[ \beta = -((C)_{1,2} + (C)_{2,2}) \sin \theta_n. \]

It is seen from (35), (45) and (46) that (43) is equivalent to (24). This concludes the proof.

4. Conservation of angular momentum

This section describes conservation of angular momentum at foot strike. Section 4.1 shows that (23) in Theorem 3.1 is equivalent to conservation of angular momentum. Section 4.2 investigates the impact force \( f_{yj} \) which is closely related to conservation of angular momentum. Section 4.3 finally discusses a necessary and sufficient condition for conservation of angular momentum at foot strike.
4.1. State transition at foot strike

The following corollary provides another form of (23).

**Corollary 4.1:** The Equation (23) is equivalent to

\[ Q_p \dot{\theta}_+ = Q_m \dot{\theta}_- , \]

where

\[ Q_p = \begin{bmatrix} f + m_\ell (\ell^2_\ell + \ell^2_t) + m_h \ell^2_\ell & -m_\ell \ell_u \ell_t \cos 2\theta_n \\ -m_\ell \ell_u \ell_t \cos 2\theta_n & f + m_\ell \ell^2_t \end{bmatrix} , \]

\[ Q_m = \begin{bmatrix} f - m_\ell \ell \ell_u \ell_t \ell(2m_\ell \ell_t + m_h \ell^2_\ell) \cos 2\theta_n \\ 0 & f - m_\ell \ell \ell_u \ell_t \end{bmatrix} . \]

**Proof:** It is straightforward to see that

\[ \det Q_p = \det A > 0 , \]

and

\[ C = Q_p^{-1} Q_m , \]

where \( \det A \) can be found in (41).

The matrices \( Q_p \) and \( Q_m \) can be obtained by eliminating the intermediate variables from (35) one by one, where the intermediate variables denote the elements of \( \lambda \) except for \( \dot{\theta}_{n+} \) and \( \dot{\theta}_{s+} \). This method is similar to the elimination of variables for solving a system of linear equations.

The Equation (49) provides a simple form of the state transition at foot strike. Its size is reduced from (35). In addition, the elements of \( Q_p \) and \( Q_m \) are simpler than those of the closed-form solution of \( C \). Therefore, (49) is suitable for computation.

The following corollary confirms conservation of angular momentum at foot strike.

**Corollary 4.2:** Equation (23) holds, if and only if the following two statements are true.

- The total angular momentum around the foot of the non-stance leg is conserved at foot strike.
- The angular momentum of the stance leg around the joint is conserved at foot strike.

**Proof:** Let \( L_{t-} \) and \( L_{t+} \) denote the total angular momenta around the foot of the non-stance leg before strike and after strike, respectively. They are given by

\[ L_{t-} = -m_h (y_h - y_n) \dot{x}_{h-} + m_h (x_h - x_n) \dot{y}_{h-} \]
\[ + m_\ell (y_{s-} - y_n) \dot{x}_{s-} + m_\ell (x_{s-} - x_n) \dot{y}_{s-} \]
\[ + J \dot{\theta}_{n-} - m_\ell (y_{s-} - y_n) \dot{x}_{n-} + m_\ell (x_{n-} - x_n) \dot{y}_{n-} \]
\[ + J \dot{\theta}_{s-} - m_\ell (y_{s-} - y_n) \dot{x}_{s-} + m_\ell (x_{s-} - x_n) \dot{y}_{s-} , \]

(54)

\[ L_{t+} = -m_h (y_h - y_n) \dot{x}_{h+} + m_h (x_h - x_n) \dot{y}_{h+} \]
\[ + m_\ell (y_{s+} - y_n) \dot{x}_{s+} + m_\ell (x_{s+} - x_n) \dot{y}_{s+} \]
\[ + J \dot{\theta}_{n+} - m_\ell (y_{s+} - y_n) \dot{x}_{n+} + m_\ell (x_{n+} - x_n) \dot{y}_{n+} \]
\[ + J \dot{\theta}_{s+} - m_\ell (y_{s+} - y_n) \dot{x}_{s+} + m_\ell (x_{s+} - x_n) \dot{y}_{s+} . \]

(55)

Substituting (4)–(8) into the above equations yields

\[ L_{t-} = \{(Q_m)_{1+} + (Q_m)_{2+}\} \dot{\theta}_{-} , \]

(56)

\[ L_{t+} = \{(Q_p)_{1+} + (Q_p)_{2+}\} \dot{\theta}_{+} . \]

(57)

Let \( L_{t-} \) and \( L_{t+} \) denote the angular momenta of the stance leg around the joint before strike and after strike, respectively. As similar to the above, they are obtained as

\[ L_{s-} = -m_\ell \ell (y_s - y_n) \dot{x}_{s-} + m_\ell \ell (x_s - x_n) \dot{y}_{s-} \]
\[ = (Q_m)_{2s} \dot{\theta}_{-} , \]

(58)

\[ L_{s+} = -m_\ell \ell (y_s - y_n) \dot{x}_{s+} + m_\ell \ell (x_s - x_n) \dot{y}_{s+} \]
\[ = (Q_p)_{2s} \dot{\theta}_{+} . \]

(59)

Corollary 4.1 with (56)–(59) concludes the proof.

The state-transition equation in [6,7] has been given directly from the conservation law. It is represented as

\[ \tilde{Q}_p \dot{\theta}_+ = \tilde{Q}_m \dot{\theta}_- , \]

(60)

where

\[ \tilde{Q}_p = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} Q_p \]
\[ = \begin{bmatrix} f + m_\ell (\ell^2_\ell + \ell^2_t) + m_h \ell^2_\ell - m_\ell \ell_u \ell_t \cos 2\theta_n \\ -m_\ell \ell_u \ell_t \cos 2\theta_n \end{bmatrix} , \]

(61)

\[ \tilde{Q}_m = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} Q_m \]
\[ = \begin{bmatrix} f - m_\ell \ell \ell_u \ell_t \ell(2m_\ell \ell_t + m_h \ell^2_\ell) \cos 2\theta_n \\ 0 \end{bmatrix} , \]

(62)

in their framework. Each row in (60) corresponds to a conservation law in Corollary 4.2. It is straightforward
to see that
\[ C = Q_p^{-1} Q_m = \tilde{Q}_p^{-1} \tilde{Q}_m \]  
(63)
This shows that some terms can be deleted from the first row of (60). The state-transition Equation (49) is simpler than (60) in the existing works, while they are equivalent.

Although the approach in this paper is different from the existing works, the derived state-transition equations are equivalent. This confirms the validity of (23).

4.2. Impact force

Conservation of angular momentum is closely related to (24), or equivalently, (43), as will be shown in the next subsection. The impact force \( f_{yn_f} \) that appears in (43) is an important factor, and it is investigated below.

The following corollary gives another reduced form of (35).

**Corollary 4.3:** The variables \( \dot{\theta}_+ \) and \( f_{yn_f} \) obtained from (35) are equivalent to those from
\[
S_p \begin{bmatrix} \dot{\theta}_+ \\ f_{yn_f} \end{bmatrix} = S_m \dot{\theta}_-, \tag{64}
\]
where
\[
S_p = \begin{bmatrix} Q_p \\ -q \end{bmatrix}, \quad S_m = \begin{bmatrix} Q_m \\ -p \end{bmatrix},
\]
\[
q = \sin \theta_n [m_t \ell_t + (m_t + m_b) \ell_t] m_t \ell_u, \tag{65}
\]
\[
p = [(q)_{1,2} (q)_{1,1}]. \tag{66}
\]

**Proof:** It is straightforward to see that
\[
\det S_p = -\det A < 0, \tag{69}
\]
and
\[
S_p^{-1} S_m = \begin{bmatrix} (A^{-1} B)_{5,1} & (A^{-1} B)_{5,2} \\ (A^{-1} B)_{8,1} & (A^{-1} B)_{8,2} \\ (A^{-1} B)_{10,1} & (A^{-1} B)_{10,2} \end{bmatrix}. \tag{70}
\]
This concludes the proof.

4.3. Necessary and sufficient condition

The inequality (43) is a necessary and sufficient condition that (23) holds, and it is equivalent to (24) as described in the proof of Theorem 3.1. This with Lemma 3.1 and Corollary 4.2 implies that the intersection of (22) and (24) is a necessary and sufficient condition for conservation of the two angular momenta that appear in Corollary 4.2.

Note that the loss-of-contact assumption assumes that the stance leg instantaneously loses contact with the ground at foot strike, and it can be represented as
\[
\dot{y}_{y_+} > 0. \tag{73}
\]
Comparing (73) with (43) shows that the loss-of-contact assumption is neither necessary nor sufficient for conservation of angular momentum. In particular, the two angular momenta that appear in Corollary 4.2 may be conserved even for \( \dot{y}_{y_+} = 0 \).

The preceding discussion implies that (43) can be checked by any one of the following four methods, where note that \( \dot{y}_{y_+} > 0 \) is equivalent to (45).

- Use (24) directly.
- Obtain \( \dot{\theta}_+ \) and \( f_{yn_f} \) from (35), and use (45).
- Obtain \( \dot{\theta}_+ \) and \( f_{yn_f} \) from (64), and use (45).
- Obtain \( \dot{\theta}_+ \) from (49), and use (45) and (71).

One of the four can be chosen arbitrarily, while \( D \) in (24) is not simple and the size of \( A \) in (35) is large.

5. Concluding remarks

This paper investigated impulsive dynamics of a compass-like biped robot at foot strike based on the modelling framework for multibody dynamics with impact. A necessary and sufficient condition for conservation of angular momentum was formulated as the intersection of (22) and (24). The condition is different from loss-of-contact, and the loss-of-contact assumption adopted in existing works is neither necessary nor sufficient for conservation of angular momentum. In addition, three alternatives to (24) were given, since the matrix \( D \) in (24) is not simple.

This paper focused only on a passive compass-like biped robot in the sagittal plane. The results in this paper follow from a general-purpose modelling framework for multibody dynamics with impact formulated in [11,12]. Therefore, the process in this paper may be extended to general legged robots under control in the three dimensional space, and general mechanical systems with multiple contact points, which is left as future work. This paper provides a basis for further studies.

For modelling the dynamics at foot strike, a similar modelling approach was used in [9]. The reference [9] is based on the modelling framework established in [23].
The modelling approach in [11,12] is simpler than [23], since the model in [11,12] is simply given by the relationship between impact forces and change of momentum with constraints, as described in (35). In contrast, the approach in [23] requires a complicated procedure such as deriving an extended inertia matrix, as shown in [9].

If the condition derived in this paper is not satisfied, then the dynamics at foot strike are governed by different state-transition equations. This means that, if the condition is not satisfied, then state-transition equations used in existing researches are invalid, and the all results in existing researches are also invalid. In other words, the condition derived in this paper shows whether the results in existing researches are valid or not. In actual studies, there may have been a gap between theory and practice in the past. The gap may be caused by the violation of the condition for conservation of angular momentum. The results in this paper improve the efficiency of actual studies, since they close the gap.

**Disclosure statement**

The authors report there are no competing interests to declare.

**Notes on Contributors**

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