DUALITY BETWEEN MEASURE AND CATEGORY
OF ALMOST ALL SUBSEQUENCES OF A GIVEN SEQUENCE

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Abstract. Let $S$ be the set of subsequences $(x_{n_k})$ of a given real sequence $(x_n)$ which preserve the set of statistical cluster points. It has been recently shown that $S$ is a set of full (Lebesgue) measure. Here, on the other hand, we prove that $S$ is meager if and only if there exists an ordinary limit point of $(x_n)$ which is not also a statistical cluster point of $(x_n)$. This provides a non-analogue between measure and category.

1. Introduction

Oxtoby’s classical book [14] examines analogues and non-analogues of statements about measure and category. To make some examples, recall that every number $\omega \in (0, 1]$ has a unique binary representation

$$\omega = \sum_{n \geq 1} \frac{d_n(\omega)}{2^n}$$

(1)

such that $d_n(\omega) = 1$ for infinitely many positive integers $n$. Then, it is well known that the set of normal numbers

$$\mathcal{N} := \left\{ \omega \in (0, 1] : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} d_k(\omega) = \frac{1}{2} \right\}$$

has full Lebesgue measure, that is, $\lambda(\mathcal{N}) = 1$, where $\lambda$ stands for the (completion of the) Lebesgue measure on $\mathbb{R}$. On the other hand, $\mathcal{N}$ is a first category set. Hence $\mathcal{N}$ is “big” from a measure theoretic viewpoint, but “small” in the topological sense. This gives a non-analogue between measure and category.

In a different direction, let $A, B \subseteq \mathbb{R}$ be two sets with positive inner Lebesgue measure, i.e., they contain a closed set with positive Lebesgue measure. Then, by a famous result of Steinhaus, the sumset $A + B := \{a + b : a \in A, b \in B\}$ has non-empty interior (in particular, it is “big” in the measure sense), cf. e.g. [7, Theorem 3.7.1]. As a topological analogue, if $A, B \subseteq \mathbb{R}$ are two second category sets with the Baire property, then the sumset $A + B$ has non-empty interior as well (in particular, it is “big” also in the category sense), cf. [7, Theorem 2.9.1].

The aim of this note is to provide another example of non-analogue between measure and category related to statistical cluster points. Here, we recall that a real $\ell$ is said to

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be a statistical cluster point of a real sequence \( x = (x_n) \) provided that
\[
\forall \varepsilon > 0, \quad d^*(\{n \in \mathbb{N} : |x_n - \ell| \leq \varepsilon\}) > 0,
\]
where \( d^* \) stands for the upper asymptotic density, i.e., the function
\[
d^* : \mathcal{P}(\mathbb{N}) \to \mathbb{R} : X \mapsto \limsup_{n \to \infty} \frac{|X \cap [1, n]|}{n}.
\]

Hereafter, we denote the set of statistical cluster points of a real sequence \( x \) by \( \Gamma_x \) and the set of ordinary limit points by \( L_x \) (clearly, \( \Gamma_x \subseteq L_x \)).

Statistical cluster points were introduced by Fridy [4] and then studied by many authors, see e.g. [1, 5, 6, 9, 11, 12, 13]. However, this notion has been studied much before under a different name. Indeed, as it follows by [10, Theorem 4.2], statistical cluster points of a real sequence \( x \) correspond to classical “cluster points” of a filter \( \mathcal{F} \) on \( \mathbb{R} \) (depending on \( x \)), cf. [3, Definition 2, p.69].

2. Main results

For each \( \omega \in (0, 1] \) and real sequence \( x \), let \( x \upharpoonright \omega \) be the subsequence of \( (x_n) \) obtained by choosing all the indexes \( n \) such that \( d_k(\omega) = 1 \) in the representation (1), cf. [2, Appendix A31] and [11]. Accordingly, the following result has been recently shown by the authors, see [12] and [9, Theorem 3.1]:

**Theorem 2.1.** Let \( x \) be a real sequence. Then \( \lambda(\{\omega \in (0, 1] : \Gamma_x = \Gamma_x \upharpoonright \omega\}) = 1 \).

In other words, almost all subsequences of \( x \) preserve the set of statistical cluster points, from a measure theoretic viewpoint.

Our main result, which is proved in Section 3, provides the topological counterpart of Theorem 2.1:

**Theorem 2.2.** Let \( x \) be a real sequence. Then \( \{\omega \in (0, 1] : \Gamma_x = \Gamma_x \upharpoonright \omega\} \) is not meager if and only if every ordinary limit point is also a statistical cluster point, i.e., \( \Gamma_x = L_x \). Moreover, in this case, it is comeager.

Lastly, given a real sequence \( x \), we recall that a real \( \ell \) is said to be a statistical limit point of \( x \) provided that there exists a subsequence \( (x_{n_k}) \) converging (in the ordinary sense) to \( \ell \) and \( d^*(\{n_k : k \in \mathbb{N}\}) > 0 \). We denote by \( \Lambda_x \) the set of statistical limit points of \( x \).

It is known that the analogue of Theorem 2.1 holds also for statistical limit points, see [12, Theorem 3.3] and [8, Theorem 4.2]. In the same direction, we have the analogue of Theorem 2.2:

**Theorem 2.3.** Let \( x \) be a real sequence. Then \( \{\omega \in (0, 1] : \Lambda_x = \Lambda_x \upharpoonright \omega\} \) is not meager if and only if \( \Lambda_x = L_x \). Moreover, in this case, it is comeager.

Proofs follow in Section 3. We leave as an open question for the interested reader to check whether Theorem 2.2 can be extended to every ideal on \( \mathbb{N} \). To be precise, let \( \mathcal{I} \subseteq \mathcal{P}(\mathbb{N}) \) be a collection of subsets closed under taking subsets and finite unions, containing all the finite sets, and different from \( \mathcal{P}(\mathbb{N}) \) itself. Moreover, given a real sequence \( x \), we let \( \Gamma_x(\mathcal{I}) \) be the set of \( \mathcal{I} \)-cluster points of \( x \), that is, the set of all \( \ell \) such
that \( \{ n : |x_n - \ell| \leq \varepsilon \} \notin \mathcal{I} \) for all \( \varepsilon > 0 \), cf. [10]. (Note that, if \( \mathcal{I} \) is the ideal of zero asymptotic density sets, i.e.,
\[
\mathcal{I}_0 := \{ X \subseteq \mathbb{N} : d^*(X) = 0 \}
\]
then \( \Gamma_x = \Gamma_x(\mathcal{I}_0) \).) Accordingly, is it true that \( \{ \omega \in (0,1] : \Gamma_x(\mathcal{I}) = \Gamma_x \upharpoonright \omega(\mathcal{I}) \} \) is not meager if and only if \( \Gamma_x(\mathcal{I}) = L_x \)?

3. Proof of Theorems 2.2 and 2.3

We start with a preliminary lemma.

**Lemma 3.1.** Let \( x \) be a real sequence with an ordinary limit point \( \ell \). Then
\[
S(\ell) := \{ \omega \in (0,1] : \ell \in \Gamma_x \upharpoonright \omega \}
\]
is comeager.

**Proof.** Fix \( \omega \in (0,1] \) and note that \( \ell \) is a statistical cluster point of the subsequence \( x \upharpoonright \omega \) if and only if
\[
\forall m \in \mathbb{N}, \quad d^* \left( \left\{ n \in \mathbb{N} : |(x \upharpoonright \omega)_n - \ell| \leq \frac{1}{m} \right\} \right) > 0,
\]
where \( d^* \) stands for the upper asymptotic density defined in (2). Hence, \( \ell \in \Gamma_x \upharpoonright \omega \) if and only if
\[
\forall m \in \mathbb{N}, \exists k \in \mathbb{N}, \quad \frac{\left| \{ n \leq N : |(x \upharpoonright \omega)_n - \ell| \leq \frac{1}{m} \} \right|}{N} \geq \frac{1}{2k}
\]
for infinitely many \( N \in \mathbb{N} \). Setting
\[
q_{\omega,m}(N) := \frac{\left| \{ n \leq N : |(x \upharpoonright \omega)_n - \ell| \leq \frac{1}{m} \} \right|}{N}
\]
for each \( m, N \in \mathbb{N} \) and \( \omega \in (0,1] \), it follows that the set \( \{ \omega : \ell \in \Gamma_x \upharpoonright \omega \} \) can be rewritten as
\[
\bigcap_{m \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \left\{ \omega : \exists \infty N \in \mathbb{N}, q_{\omega,m}(N) \geq \frac{1}{2k} \right\}.
\]

Accordingly, we have equivalently to prove that the set
\[
\{ \omega : \ell \notin \Gamma_x \upharpoonright \omega \} = \bigcup_{m \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \left\{ \omega : \forall \infty N \in \mathbb{N}, q_{\omega,m}(N) < \frac{1}{2k} \right\}
\]

is a meager set. To this aim, it will be enough to show that the sets
\[
Q(m,k,n) := \bigcap_{N \geq n} \left\{ \omega : q_{\omega,m}(N) < \frac{1}{2k} \right\}
\]
are nowhere dense for each \( m, k, n \in \mathbb{N} \).

Fix \( m, k, n \in \mathbb{N} \). It is well known that \( Q(m,k,n) \) is nowhere dense if and only if, for each non-empty interval \( I \subseteq (0,1] \), there exists a non-empty interval \( J \subseteq I \) such that
\[
Q(m,k,n) \cap J = \emptyset.
\]

(3)
Since $\ell$ is an ordinary limit point of $x$, there exists $\omega_0 \in I$ such that
\[
\forall N \in \mathbb{N}, \quad q_{\omega_0,m}(N) \geq \frac{1}{2}.
\]
In particular, for each $n \in \mathbb{N}$, there exists $N_n \in \mathbb{N}$ greater than $n$ such that $q_{\omega_0,m}(N_n) \geq 1/2$. Lastly, fix $n \in \mathbb{N}$ and define the non-empty interval
\[
J := \{ \omega \in (0,1] : \forall i \leq N_n, d_i(\omega) = d_i(\omega_0) \}.
\]
It follows by construction that (3) holds, completing the proof. □

At this point, we can prove our main result.

**Proof of Theorem 2.2.** Let $x$ be a sequence such that $\Gamma_x = L_x$. First, it is claimed that
\[
S := \{ \omega \in (0,1] : L_x = \Gamma_x | \omega \}
\]
is a comeager set. This is trivial if $L_x$ is empty. Hence, let us suppose hereafter that $L_x \neq \emptyset$.

Since $L_x$ is a non-empty closed set, there exists a non-empty countable dense subset \( \{ \ell_n : n \in \mathbb{N} \} \). Also, since $\Gamma_x | \omega$ is closed, we have that
\[
S = \bigcap_{n \in \mathbb{N}} S(\ell_n).
\]
By Lemma 3.1 each $S(\ell_n)$ is comeager, hence $S$ is comeager.

Conversely, assume that $\Gamma_x \subsetneq L_x$. Considering that $S$ is comeager, then also \( \{ \omega : \Gamma_x \neq \Gamma_x | \omega \} \) is comeager, thus \( \{ \omega : \Gamma_x = \Gamma_x | \omega \} \) is meager. □

**Proof of Theorem 2.3.** Analogous to the proof of Lemma 3.1, it can be verified that for each $k \in \mathbb{N}$ and $\ell \in L_x$ the set
\[
T_k(\ell) := \left\{ \omega : d^* (\{ n : |(x | \omega)_n - \ell | \leq \varepsilon \}) \geq \frac{1}{k} \text{ for all } \varepsilon > 0 \right\}
\]
is comeager. Moreover, for each $\omega \in (0,1]$ and $k \in \mathbb{N}$ the set
\[
\Lambda_k^{\ell,\omega} := \left\{ \ell : d^* (\{ n : |(x | \omega)_n - \ell | \leq \varepsilon \}) \geq \frac{1}{k} \text{ for all } \varepsilon > 0 \right\}
\]
is closed (note that $\Lambda^{\ell,\omega} = \bigcup_{k \in \mathbb{N}} \Lambda_k^{\ell,\omega}$ is not necessarily closed).

Letting \( \{ \ell_n : n \in \mathbb{N} \} \) be a countable dense subset of $L_x$, we conclude that
\[
\{ \omega : \Lambda_x | \omega = L_x \} = \bigcup_{k \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} T_k(\ell_n) \supseteq \bigcap_{n \in \mathbb{N}} T_1(\ell_n)
\]
is comeager.

The rest is identical to the proof of Theorem 2.2. □

Note that it follows by the proofs of Theorem 2.2 and 2.3 that, given a real sequence $x$, the set \( \{ \omega \in (0,1] : \Lambda^{\ell,\omega} = \Gamma^{\ell,\omega} = L_x \} \) is comeager.
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