THE VACUUM ENERGY DENSITY FOR SPHERICAL AND CYLINDRICAL UNIVERSES

E. ELIZALDE
Institute of Theoretical Physics, Chalmers University of Technology,
S-412 96 Göteborg, Sweden

Abstract

The vacuum energy density (Casimir energy) corresponding to a massless scalar quantum field living in different universes (mainly no-boundary ones), in several dimensions, is calculated. Hawking’s zeta function regularization procedure supplemented with a very simple binomial expansion is shown to be a rigorous and well suited method for performing the analysis. It is compared with other, much more involved techniques. The principal-part prescription is used to deal with the poles that eventually appear. Results of the analysis are the absence of poles at four dimensions (for a 4d Riemann sphere and for a 4d cylinder of 3d Riemann spherical section), the total coincidence of the results corresponding to a 3d and a 4d cylinder (the first after pole subtraction), and the fact that the vacuum energy density for cylinders is (in absolute value) over an order of magnitude smaller than for spheres of the same dimension.

1On leave of absence from and permanent address: Department E.C.M., Faculty of Physics, Barcelona University, Diagonal 647, E-08028 Barcelona, Spain; e-mail: eli @ ebubecm1.bitnet
1 Introduction

The investigation of the vacuum energy density (or Casimir energy) \([1]\) corresponding to a quantum field theory defined in a certain manifold (spacetime) with or without boundary, is one of the most basic issues of quantum field theory. The Casimir force is probably the simplest and most spectacular of the different manifestations of the vacuum energy — better, of the modification of the vacuum energy when boundaries, curvature or a background field are superimposed to the manifold \([1, 2]\). To understand the correspondence between the sign of the effect (associated with an attractive or repulsive force, depending on the sign being negative or positive, respectively) and the specific topology of the boundary is a most challenging point which requires explanation. A historical failure in relation with this issue occurred some years ago, when the Casimir force was thought to contain the explanation of quark confinement in the context of the bag model. The idea was that this vacuum effect would provide the attractive force giving tension to the bag that contained the quarks. But it turns out that the calculation of the force for curved boundaries is very tricky, plagued with infinities and needing appropriate regularization. Sometimes cut-offs remain, zeta-function regularization is said to be insufficient, and it is difficult to extract uncontroverted, physically meaningful results \([2]\). Actually, this subject has been the source of several sound errors in the scientific literature. In any case, when it was proven, without doubt, that the force for a closed sphere in three-dimensional space is repulsive (as it is for a closed cube, and not attractive as for a pair of plates) the brilliant idea of supplementing the bag model with the Casimir force had to be abandoned.

Notwithstanding that, the presence of the Casimir force in very different phenomena of condense matter, solid state and laser physics has been rigorously established, both theoretically and experimentally \([3]\). Its relevance for possible models of our universe is also without discussion. Maybe owing to the difficulties encountered when trying to give a plausible answer to the question: what are the boundary conditions of our universe?, some of the most popular models of spacetime nowadays are given by manifolds without boundaries.

Riemann spheres are to be counted among the simplest and most important of these manifolds. This is the reason why we have chosen them as the spaces of our models. We will study these models in different dimensions, with the hope to find some characteristic that may singularize some of the manifolds considered and some particular dimension, among all of them.

From a more technical point of view, the specifications of our study are as follows. We
shall use the most elegant, rigorous and simple of the regularizations known to date, namely zeta function regularization \([4]\). Moreover, we are going to supplement it with a most easy technique, which is binomial expansion \([4]\). Then we shall compare our findings with those coming from other approaches, which turn out to be much more artificial, lengthy and less rigorous. Hard to believe as this statement may seem, it is the plain truth. Little merit about it, again this is just a confirmation of the famous general principle, attributed to Einstein, which asserts that nature always follows the most simple path among the ones that are available to her. That such path turns out to be in this case, at the same time, the most rigorous mathematically is however a rewarding small surprise (since sometimes physical intuition and beauty has been associated with mathematical sloppiness). Even from the pure calculational point of view, the method here developed yields very rapidly convergent expressions which allow us to obtain 6 or 8 digit precision with just a few first terms of a series, in a home computer and with a standard computation package (Mathematica, for instance, but never use it for dealing with the Hurwitz zeta function \(\zeta(s,a)\) when \(a > 1\)).

As side products of our analysis, some asymptotic expansions for Hurwitz and Epstein zeta functions, that had been obtained by the author previously, are here checked and numerically contrasted with other results in the literature. As has been pointed out already, this is a necessary exercise in this field, because of the discrepancies and errors that so frequently appear. In particular, a small table of derivatives of the Riemann zeta function, which are repeatedly used in our numerical calculations, is given.

In Sect. 2 we summarize our method of zeta function regularization and describe the way in which the binomial expansion is used. We compare it with other approaches that have been employed, in other to prove the advantages of our procedure. In Sect. 3 we obtain the Casimir energy density for Riemann spheres in \(d = 1, 2, 3, 4\) dimensions. These are manifolds without boundary, but also the situation when one has a half such manifold with Dirichlet and Neumann boundary conditions, respectively, is considered for comparison. Cylindrical manifolds whose sections are Riemann spheres, with total dimension \(d = 2, 3, 4, 5\) are investigated in Sect. 4. The discussion of our results and the conclusions of our analysis are given in Sect. 5.
2 Analytical approach to the zeta functions for Riemann surfaces

As has been observed elsewhere [6], the formula [7] (see also [8])

\[ F(s; a, b) \equiv \infty \sum_{n=1}^{\infty} \left( (n+a)^2 + b \right)^{-s} = \frac{b^{-s}}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+s)}{n!} b^{-n} \zeta(-2n, a) \]

and its generalizations to higher dimensional manifolds [9], are extremely useful (owing to exponentially quick convergence) when \( b > 0 \). However, it turns out to be rather difficult to apply them when \( b < 0 \). And this is usually the relevant case which appears when one tries to calculate, e.g., the determinants of the Laplacian operators on Riemann spheres by using zeta functions. One has to regularize (i.e., analytically continue) expressions of the general form

\[ f(s; a, b, c) \equiv \sum_{l=1}^{\infty} l^{-s+b}(l+a)^{-s+c}, \]

where \( a \) turns out to be positive.

An alternative expression to (2) has been obtained in [9], by making use, essentially, of the usual integral representation for the Hurwitz zeta function together with the Mellin transformation (actually the same ingredients, aside from adequate series commutation, which were used in [7] for the derivation of (2)). Specifically, for

\[ \zeta^{(n)}(s) \equiv \infty \sum_{l=1}^{\infty} \left[ l^{-s}(l+2n+1)^{1-s} + l^{1-s}(l+2n+1)^{-s} \right], \]

the following interesting result was obtained

\[ \frac{d}{ds} \zeta^{(n)}(s) \bigg|_{s=0} = 4\zeta'(-1) - \frac{1}{2} (2n+1)^2 + \sum_{k=1}^{2n+1} (2k - 2n - 1) \log k, \]

from which, in particular, for the zeta function of the Laplacian on the hemisphere with Dirichlet and Neumann boundary conditions, respectively,

\[ \zeta_D(s) = \sum_{l=1}^{\infty} l[l(l+1)]^{-s}, \quad \zeta_N(s) = \sum_{l=1}^{\infty} (l+1)[l(l+1)]^{-s}, \]

one gets

\[ \zeta'_D(0) = 2\zeta'(-1) + \frac{1}{2} \log(2\pi) - \frac{1}{4}, \quad \zeta'_N(0) = 2\zeta'(-1) - \frac{1}{2} \log(2\pi) - \frac{1}{4}. \]
Those are nice results, indeed. However, the derivation of these expressions in [3] is not free from difficulties. In fact, to start with, the analysis is rather lengthy and, on the other hand, a highly arbitrary, additional regularization is needed at some point: a certain infrared convergence factor $t^s$ ($t$ is the integration variable) must be introduced and the exponent $s$ can be let to go to zero only after performing a convenient combination of different terms. That these manipulations are not so obviously accepted (even if in the end they turn out to be right, as is the case here) is proven by the continuous recourse to specific checking of the final numbers with well-known results [4, 10, 11]. In certain cases, in fact, use of these integral transforms can lead to discrepancies with already known results. So, the value of $\zeta'(-1)$ obtained in [12] as a byproduct of an original method there developed seems to be quite far from the best accepted value as given, for instance, in [13] (see expressions (50) below). Our value is coincident (at least to 8 digits) with the one obtained by Salomonson using a different formula [14]. On the other hand, however, the remarkable expression obtained in [12] for $\zeta'(-2)$ appears to be right (see (50)).

A mathematically clean, rigorous and, at the same time, much more simple procedure to deal with any expression of the general form (2) can be devised which makes use of the simplest of ideas: binomial expansion [5]. This goes as follows

$$f(s; a, b, c) \equiv \sum_{l=1}^{\infty} l^{-s+b+c}(l+a)^{-s+c} = \sum_{l=1}^{\infty} l^{-2s+b}(1+al^{-1})^{-s+c} = \sum_{l=1}^{[a]} \{ \} + \sum_{l=[a]+1}^{\infty} \{ \}$$

$$= g(s; a, b, c) + \sum_{l=[a]+1}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma(1-s+c)}{k!\Gamma(1-s-k+c)} a^k l^{-2s-k+b+c},$$

being $[a]$ the integer part of $a$, so that $g(s)$ is an integer function of $s$, while the second, truncated series is absolutely convergent (since $al^{-1} < 1$ there). The final result is

$$f(s; a, b, c) = \sum_{l=1}^{[a]} l^{-2s+b+c}(1+al^{-1})^{-s+c} + \sum_{k=0}^{\infty} \frac{\Gamma(1-s+c)}{k!\Gamma(1-s-k+c)} a^k$$

$$\times \left[ \zeta(2s+k-b-c) - \sum_{l=1}^{[a]} l^{-2s-k+b+c} \right].$$

(7)

In particular,

$$f(0; a, b, c) = \sum_{l=1}^{[a]} l^b(1+al^{-1})^c + \sum_{k=0}^{\infty} \frac{\Gamma(1+c)}{k!\Gamma(1-k+c)} a^k \left[ \zeta(k-b-c) - \sum_{l=1}^{[a]} l^{-k+b+c} \right],$$

(9)

and

$$f'(0; a, b, c) = -\sum_{l=1}^{[a]} \left[ 2\log l + \log(1+al^{-1}) \right] l^b(1+al^{-1})^c$$

(8)
It is easy to see that care, by performing first expansions around the poles and zeros of these functions at \( s = 0 \) (this will be shown in full detail below).

For the zeta function \( \zeta_D(s) \), eq. (9), we obtain
\[
\zeta_D(s) = \sum_{l=1}^{\infty} l^{1-2s}(1 + l)^{-s} = 2^{-s} + \sum_{k=0}^{\infty} \frac{\Gamma(1 - s)}{k!\Gamma(1 - s - k)} \left[ \zeta(2s + k - 1) - 1 \right],
\]
and
\[
\zeta'_D(0) = 2\zeta'(-1) + \frac{5}{4} + \frac{\gamma}{2} - \log 2 - \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k) - 1}{k + 1}.
\]

The power of our simple method will be now demonstrated by showing that this last expression coincides with the first of (11). In fact, we have
\[
\sum_{k=2}^{\infty} (-1)^k \zeta(k) = \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k+1} \int_0^\infty dt \frac{t^{k-1}}{e^t - 1} \equiv \varphi(1),
\]
with
\[
\varphi(u) \equiv \int_0^\infty \frac{dt}{e^t - 1} \sum_{k=2}^{\infty} \frac{(-u)^{k+1}t^{k-1}}{(k+1) (k-1)!}.
\]

It is easy to see that \( \varphi'(u) = u[\psi(1) - \psi(u + 1)] \) and integrating, \( \varphi(1) = -\gamma/2 - 1 + (1/2)\log(2\pi) \). Finally, adding the rest of the series:
\[
\sum_{k=2}^{\infty} \frac{(-1)^k}{k + 1} = -\frac{1}{2} + \log 2,
\]
we obtain the desired result, i.e. that (12) coincides with the first of eqs. (11). The numerical value is
\[
\zeta'_D(0) = 0.338096.
\]

A second particular example is the following. For the case of a rectangle (of sides \( a \) and \( b \)) with Dirichlet boundary conditions, the spectrum of the Laplacian is \( \lambda_{mn} = \pi^2(m^2/a^2 + n^2/b^2) \), and the zeta function
\[
\zeta_{rec}(s) = \pi^{-2s} \sum_{m,n=1}^{\infty} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{-s} = -\frac{1}{2} \left( \frac{b}{a} \right)^{2s} \left( \frac{b}{a} \right) \zeta(2s) + \frac{a}{2\sqrt{\pi}} \left( \frac{b}{a} \right)^{2s-1} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \zeta(2s - 1)
\]
\[
+ \frac{2}{\Gamma(s)} \left( \frac{m}{n} \right)^s \sqrt{\frac{a}{b}} \sum_{m,n=1}^{\infty} \left( \frac{m}{n} \right)^{s-1/2} K_{s-1/2}(2\pi mna/b),
\]

\[
(17)
\]
where, again, the corresponding asymptotic expansion for the Epstein zeta function in \[7\] has been used. Taking now the derivative, we get

\[
\zeta_{\text{rec}}'(0) = \frac{1}{2} \log(2b) + \frac{\pi a}{12b} + 2 \sqrt{\frac{a}{b}} \sum_{m,n=1}^{\infty} \sqrt{\frac{n}{m}} K_{-1/2}(2\pi mna/b),
\]

which is best for numerical computations when \( a \geq b \). In the particular case \( a = b \) (square), this reduces to

\[
\zeta_{\text{sq}}'(0) = \frac{1}{2} \log(2a) + \frac{\pi}{12} + 2 \sqrt{\frac{1}{2}} \sum_{m,n=1}^{\infty} \sqrt{\frac{n}{m}} K_{-1/2}(2\pi mn),
\]

which is just another expression for the same result obtained in \[10\] (cf. eq. (A11.3))

\[
\zeta_{\text{sq}}'(0) = \frac{1}{2} \log(2a) + \frac{1}{4} \log(8\pi) + \frac{1}{2} \log \Gamma(3/4) \Gamma(1/4).
\]

The numerical value is, in both cases,

\[
\zeta_{\text{sq}}'(0) = \frac{1}{2} \log(2a) + 0.263672.
\]

Notice, however, that for the general rectangle, expression (18) is of much more practical use than the well-known one in terms of Dedekind’s modular form \( \eta \), i.e. \[10, 11\]

\[
\zeta_{\text{rec}}'(0) = \frac{1}{4} \log(ab) - \log \left( \frac{1}{\sqrt{2}} \left( \frac{b}{a} \right)^{1/4} \eta(q) \right), \quad \eta(q) = q^{1/24} \prod_{m=1}^{\infty} (1-q^m), \quad q = \exp \left( -2\pi \sqrt{\frac{b}{a}} \right).
\]

In fact, a few first terms of the series in (18) suffice to obtain extremely accurate numerical results (just as for the case of the square).

3 The vacuum energy density for Riemann surfaces in different dimensions

We shall here calculate the vacuum energy density corresponding to a massless scalar field living in a Riemann sphere (manifold without boundary) or in a part thereof (namely a half-sphere, with Dirichlet or Neumann boundary conditions). The calculations become rather simple and precise numerical results are easy to obtain, by making exhaustive use of the formulas and considerations of the preceding section. But rather than attributing it to the specific manipulations we have carried out there (nothing special, in fact), this simplicity is to be interpreted as a success of the zeta-function regularization procedure itself \[4\].
### 3.1 Dimension d=1

This case is rather trivial and deserves no comment. For the one-dimensional Riemann sphere (no boundary, hence no boundary conditions), the vacuum energy density (or Casimir energy) is

\[
E_1 = \frac{\hbar}{2r^2} \sum_{n=1}^{\infty} 2\sqrt{n^2} = \frac{\hbar}{r^2} \sum_{n=1}^{\infty} n = \frac{\hbar}{r^2} \zeta(-1),
\]

(23)

being \(r\) (here and in what follows) the radius of the Riemann sphere, and

\[
\zeta(s) = \sum_{n=1}^{\infty} n^{-s},
\]

(24)

just the ordinary Riemann zeta function; therefore

\[
E_1 = -\frac{\hbar}{12r^2}.
\]

(25)

For the semicircumference, the eigenmodes are shared by the cases of Dirichlet and Neumann boundary conditions, respectively. We obtain

\[
E_1^D = E_1^N = -\frac{\hbar}{24r^2}.
\]

(26)

### 3.2 Dimension d=2

For the ordinary Riemann sphere (i.e., the one which appears in string theory, see for instance \[10, 15\]), we have, in the case of the half-sphere with Dirichlet and Neumann boundary conditions,

\[
E_2^D = -\frac{\hbar}{2r^2} \sum_i m_i^D \lambda_i, \quad E_2^N = -\frac{\hbar}{2r^2} \sum_i m_i^N \lambda_i,
\]

(27)

respectively, where the eigenvalues \(\lambda_i\) and eigenmode multiplicities \(m_i^D, m_i^N\) are

\[
\lambda_i = \frac{\sqrt{l(l+1)}}{r}, \quad m_i^D = l, \quad m_i^N = l + 1.
\]

(28)

We get

\[
E_{2,D,N}^D = \frac{\hbar}{2r^3} \zeta^{D,N}(-1/2), \quad \zeta_2^D(s) = \sum_{l=1}^{\infty} l[l(l+1)]^{-s}, \quad \zeta_2^N(s) = \sum_{l=1}^{\infty} (l+1)[l(l+1)]^{-s},
\]

(29)

and by using our method as described in the preceding section,

\[
\zeta_2^D(s) = 2^{-s} + \sum_{k=0}^{\infty} \frac{\Gamma(1-s)}{k!\Gamma(1-s-k)} [\zeta(2s+k-1) - 1],
\]

\[
\zeta_2^N(s) = 2^{1-s} + \sum_{k=0}^{\infty} \frac{\Gamma(2-s)}{k!\Gamma(2-s-k)} [\zeta(2s+k-1) - 1].
\]

(30)
Now, when evaluating $\zeta_{D,N}^2(s = -1/2)$, a pole for $k = 3$ appears, in each case. The usual prescription in zeta function regularization is to take the principal part of the pole [2, 16, 17].

With this in mind, it is easy to obtain

$$\zeta_2^D(-1/2) = \frac{1}{32(s + 1/2)} + 0.033532, \quad \zeta_2^N(-1/2) = -\frac{1}{32(s + 1/2)} - 0.298630. \quad (31)$$

This yields for the energy

$$E_2^D = 0.016766 \cdot \frac{\hbar}{r^3}, \quad E_2^N = -0.149314 \cdot \frac{\hbar}{r^3}. \quad (32)$$

For the whole, ordinary Riemann sphere (no boundary), we have

$$\zeta_2(s) = \sum_{l=1}^{\infty} (2l + 1)[l(l + 1)]^{-s} = \zeta_2^D(s) + \zeta_2^N(s) = -0.265096, \quad (33)$$

and thus we obtain

$$E_2 = -0.132548 \cdot \frac{\hbar}{r^3}. \quad (34)$$

Summing up, we see that in $2 + 1$ spacetime dimensions, where space is the ordinary Riemann sphere (no boundaries), the Casimir energy density is negative. It is the sum of the energies corresponding to two half-spheres, one with Dirichlet and the other with Neumann boundary conditions on the one-dimensional boundary. The poles have opposite sign, and they annihilate when performing the sum. This is certainly an interesting result. It should be compared with the Casimir energy in $3 + 1$ spacetime and boundary conditions imposed on a two-dimensional spherical surface. It is clear that the sign of the Casimir energy density for Riemannian manifolds can have important global cosmological consequences in plausible models of our universe. In fact, it could account for a lessening of the expansion ratio of the universe.

### 3.3 Dimension d=3

The three-dimensional Riemann sphere is a manifold without boundary that could perfectly well correspond to the spatial part of our universe, as a whole. The eigenvalues of the Laplacian operator are $\lambda_i^2 = l(l + 2)/r^4$, with degeneracies $m_i = (l + 1)^2$. Thus, the vacuum energy density for a massless scalar field is given by

$$E_3 = -\frac{\hbar}{2r^4} \zeta_3(s = -1/2), \quad \zeta_3(s) = \sum_{l=1}^{\infty} (l + 1)^2[l(l + 2)]^{-s}. \quad (35)$$
We can write,
\[ \zeta_3(s) = \sum_{l=2}^{\infty} l^{2(1-s)} (1 - l^{-2})^{-s} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(1 - s)}{k! \Gamma(1 - s - k)} [\zeta(2s + 2k - 2) - 1], \] (36)
and
\[ \zeta_3(-1/2) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(3/2)}{k! \Gamma(3/2 - k)} [\zeta(2k - 3) - 1] = -\frac{1}{16(s + 1/2)} - 0.411502, \] (37)
which has a pole, for \( k = 2 \). Doing as above, we obtain the following numerical result:
\[ E_3 = -0.205751 \cdot \frac{\hbar}{r^4}. \] (38)

### 3.4 Dimension d=4

For the four-dimensional Riemann sphere, the corresponding eigenvalues and multiplicities are \( \lambda_i^2 = l(l + 3)/r^6 \) and \( m_i = (l + 1)(l + 2)(2l + 3)/6 \). The vacuum energy density is now
\[ E_4 = -\frac{\hbar}{2r^5} \zeta_4(s = -1/2), \quad \zeta_4(s) = \frac{1}{6} \sum_{l=1}^{\infty} (l + 1)(l + 2)(2l + 3)[l(l + 3)]^{-s}. \] (39)

We can write,
\[ \zeta_4(s) = \frac{1}{3} \sum_{l=1}^{\infty} u(u^2 - 1/4)(u^2 - 9/4)^{-s} = \frac{1}{3} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(1 - s)}{k! \Gamma(1 - s - k)} \times \left(\frac{9}{4}\right)^k [\zeta(2s + 2k - 3, 5/2) - \zeta(2s + 2k - 1, 5/2)/4], \] (40)
being \( u = l + 3/2 \) and \( \zeta(s,a) \) Hurwitz’s zeta function
\[ \zeta(s,a) = \sum_{n=0}^{\infty} (n + a)^{-s}. \] (41)

Again, nothing else has been done here but to apply the procedure as described in the preceding section. We thus obtain
\[ \zeta_4(-1/2) = \frac{1}{3} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(3/2)}{k! \Gamma(3/2 - k)} \left(\frac{9}{4}\right)^k [\zeta(2k - 4, 5/2) - \zeta(2k - 2, 5/2)/4] = -0.424550. \] (42)

It might seem that the term \( (9/4)^k \) could spoil convergence. This is not true: as always, we get a quickly convergent series. This is guaranteed in advance by the procedure itself, but it is rewarding to check this property numerically and see explicitly the rapid convergence (warning to the reader: one cannot use Mathematica to compute this expression since the Hurwitz zeta functions \( \zeta(s,a) \) are very ill defined in this program for values of \( a > 1 \); this applies at least to version 2.0). Another important surprise is the fact that expression \( \frac{\hbar}{2r^5} \zeta_4(s = -1/2) \) is the solution to the problem of finding \( E_4 \) for the four-dimensional Riemann sphere.
is finite, no pole appears in this case (contrary to previous situations, in (42) there is no
principal part reduction).

Finally,

$$E_4 = -0.212275 \cdot \frac{\hbar}{r^5}.$$  \hfill (43)

Before closing this section, the following observation is in order. As explained before, an
alternative treatment of the zeta functions above would be simply to split the polynomial
in powers of the summation indices and then use the method of [9]. It is easy to check that
this procedure is much more lengthy than the one developed here. On the other hand, the
cancellation of poles in this method must be done explicitly (resorting to expansions around
all poles and zeros), while it is immediate in our procedure (actually, no pole is ever formed).
We conclude that the most direct way (the one we use) turns out to be here, at the same
time, the shortest, most rigorous and best suited for numerical evaluation.

The numerical results corresponding to the different cases here considered are depicted
in Fig. 1. The vacuum energy density for a massless scalar field living in Riemann spheres
(no-boundary manifolds) is represented as a function of the space dimension, $d$, in units of
$\hbar r^{-(d+1)}$, for $d = 1, 2, 3, 4$. We just observe that the function is monotonically decreasing,
showing no other remarkable feature than the small plateau for $d = 3, 4$. In fact, the values
for the three- and four-dimensional Riemann spheres are very similar. There is a sort of
stabilization of the Casimir force for this number of dimensions.

4 The vacuum energy density for cylinders of Riemann
surface section

As a sort of simplified study of stability of the vacuum energy density against deformations
of the space manifold considered, we shall now investigate how the preceding values change
when we consider the cylinders, with and without boundary conditions, which are obtained
by adding a flat suplementary dimension to the Riemann spheres and half-spheres considered
above.

4.1 Dimension $d=2$

We start with the case of a two-dimensional cylinder whose sections are one-dimensional Riemann
spheres (no boundaries) or semicircumferences with Dirichlet and Neumann boundary
We obtain immediately (no pole appears):  
\[
E_{1,1}^{D,N} = \frac{\hbar}{4\pi r^2} \int_{-\infty}^{+\infty} dk \sum_{n=1,0}^{\infty} \sqrt{k^2 + n^2} = \frac{\hbar}{4\sqrt{\pi r^3}} \zeta_{1,1}^{D,N}(s = -1/2),
\]
being  
\[
\zeta_{1,1}^{D,N}(s) = \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{n=1,0}^{\infty} (n^2)^{1/2-s} = \frac{\Gamma(s - 1/2)}{\Gamma(s)} \zeta(2s - 1).
\]
We obtain immediately (no pole appears):  
\[
E_{1,1}^{D,N} = \frac{\hbar \zeta'(-2)}{4\pi r^3} = -0.0024 \cdot \frac{\hbar}{r^3}, \quad E_{1,1} = -0.0048 \cdot \frac{\hbar}{r^3}.
\]

### 4.2 Dimension d=3

For a cylinder made up of ordinary Riemann half-spheres with Dirichlet or Neumann boundary conditions, what we have is  
\[
E_{2,1}^{D,N} = \frac{\hbar}{4\pi r^3} \int_{-\infty}^{+\infty} dk \sum_{l}^{D,N} \sqrt{k^2 + l(l + 1)} = \frac{\hbar}{4\sqrt{\pi r^4}} \zeta_{2,1}^{D,N}(s = -1/2),
\]
being  
\[
\zeta_{2,1}^{D}(s) = \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{l=1}^{\infty} l[l(l + 1)]^{1/2-s}, \quad \zeta_{2,1}^{N}(s) = \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{l=1}^{\infty} (l + 1)[l(l + 1)]^{1/2-s}.
\]
We obtain  
\[
\begin{align*}
\zeta_{2,1}^{D}(s) &= \frac{\Gamma(s - 1/2)}{\Gamma(s)} \left\{ 2^{1/2-s} + \sum_{k=0}^{\infty} \frac{\Gamma(3/2-s)}{k!\Gamma(3/2-s-k)} [\zeta(2s + k - 2) - 1] \right\}, \\
\zeta_{2,1}^{N}(s) &= \frac{\Gamma(s - 1/2)}{\Gamma(s)} \left\{ 2^{3/2-s} + \sum_{k=0}^{\infty} \frac{\Gamma(5/2-s)}{k!\Gamma(5/2-s-k)} [\zeta(2s + k - 2) - 1] \right\}.
\end{align*}
\]
Here a pole appears in both cases, with the same residue $-1/(60\sqrt{\pi})$. Again, one must be here very careful when putting $s = -1/2$ in these expressions, since one has to expand the zeroes and the poles of the gamma and zeta functions in terms of $s + 1/2$. After doing so, and using the following table of values for the derivative of the zeta function  
\[
\begin{align*}
\zeta'(-1) &= -0.16542115, \quad \zeta'(-2) = -0.0304, \quad \zeta'(-3) = 0.0054, \\
\zeta'(-4) &= 0.0080, \quad \zeta'(-5) = 0.00066, \ldots
\end{align*}
\]
we obtain  
\[
E_{2,1}^{D} = -0.0042 \cdot \frac{\hbar}{r^4}, \quad E_{2,1}^{N} = -0.0073 \cdot \frac{\hbar}{r^4}, \quad E_{2,1} = -0.0115 \cdot \frac{\hbar}{r^4}.
\]
4.3 Dimension d=4

For the case of the four-dimensional cylinder whose sections are three-dimensional Riemann spheres, we will only consider the no-boundary situation. Now

\[ E_{3,1} = \frac{\hbar}{4\pi r^5} \int_{-\infty}^{+\infty} dk \sum_{l=1}^{\infty} (l+1)^2 \sqrt{k^2 + l(l+2)} = \frac{\hbar}{4\sqrt{\pi r^5}} \zeta_{3,1}(s = -1/2), \]  

being

\[ \zeta_{3,1}(s) = \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{l=1}^{\infty} (l+1)^2[l(l+2)]^{1/2-s} = \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(3/2 - s)}{k! \Gamma(3/2 - s - k)} [\zeta(2s+2k-3) - 1]. \]  

This case is also finite, no pole appears when computing \( \zeta_{3,1}(-1/2) \). On the other hand, it turns out that the final value for the energy density is remarkably small:

\[ E_{3,1} = -0.0115 \cdot \frac{\hbar}{r^5}, \]  

actually, it exactly coincides with the value obtained in the former case. There seem to be an intriguing stability of the vacuum energy density for these manifolds at this number of dimensions, more remarkable because the formulas for the zeta functions leading to these values look very different indeed (actually, the three-dimensional case needed a principal part evaluation of the pole while the four-dimensional case is finite). Such a plateau almost appeared for Riemann spheres, but here the values obtained are identical and the plateau is completely horizontal (see Fig. 2).

4.4 Dimension d=5

Finally, we shall consider the five dimensional cylinder whose sections are four dimensional Riemann spheres. Here

\[ E_{4,1} = \frac{\hbar}{4\pi r^6} \int_{-\infty}^{+\infty} dk \sum_{l=1}^{\infty} \frac{(l+1)(l+2)(2l+3)}{6} \sqrt{k^2 + l(l+3)} = \frac{\hbar}{4\sqrt{\pi r^6}} \zeta_{4,1}(s = -1/2), \]  

where

\[ \zeta_{4,1}(s) = \frac{\Gamma(s - 1/2)}{3\Gamma(s)} \sum_{l=1}^{\infty} u^{2(1-s)}(u^2 - 1/4)[1 - 9/(4u^2)]^{1/2-s}, \quad u = l + 3/2. \]  

Using the same methods as before, we obtain,

\[ \zeta_{4,1}(s) = \frac{\Gamma(s - 1/2)}{3\Gamma(s)} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(3/2 - s)}{k! \Gamma(3/2 - s - k)} \left( \frac{9}{4} \right)^k [\zeta(2s+2k-4, 5/2) - \zeta(2s+2k-2, 5/2)/4]. \]
Here we get again the usual pole. The final value is

\[
E_{4,1} = -0.0218 \cdot \frac{\hbar}{r^6}. \tag{58}
\]

The results of this section are summarized in Fig. 2, where the vacuum energy density corresponding to a massless scalar field living in cylinders whose sections are Riemann spheres (again manifolds without boundary) is represented as a function of the space dimension \(d\) and in units of \(\hbar r^{-(d+1)}\), for \(d = 2, 3, 4, 5\). A similar monotonic behavior with dimension as in the case of spheres is observed. Moreover, we notice an absolute stabilization of the numerical value of the energy density, which is exactly the same for \(d = 4\) as for \(d = 3\) (sections are two- and three-dimensional Riemann spheres). Also, the case \(d = 4\) is somehow special in the sense that it gives a finite result, no pole arises there. To be remarked as well is the fact that the Casimir energies for these cylinders are almost exactly by one order of magnitude smaller (in absolute value) than the ones obtained for the corresponding Riemann spheres.

5 Discussion and conclusions

We have calculated in this paper the vacuum energy density (also called Casimir energy density [2]) corresponding to a massless scalar quantum field living in different no-boundary universes, as a function of the number of dimensions. For models of the universe we have chosen mainly manifolds without boundary, since they seem to be the most accepted nowadays and, among them, Riemann spheres, since they are to be counted among the simplest and most important of these manifolds. However, several manifolds with a boundary, with Dirichlet and Neumann conditions, respectively, have also been considered. We have studied the variation of the vacuum energy density with the dimension of the space, hoping to find some characteristic that may singularize some of the manifolds considered or a particular dimension, among all of them. For this purpose, we have also compared the results obtained for spheres with the ones corresponding to cylinders of spherical section. It is in this respect that an alternative title of our work could have been: search for the most stable quantum universe without boundaries (stability in the sense of the vacuum energy density).

As for the methods of analysis employed, they have consisted in zeta function regularization supplemented with a simple and natural binomial expansion, which has been shown to be both rigorous and very well suited for performing the numerical calculations. The
principal part prescription has been used to deal with the poles that appeared in several cases.

The most remarkable results of our investigation are the following. (i) The absence of poles (completely finite result) for the case of the four-dimensional Riemann sphere and also for the four-dimensional cylinder of three-dimensional Riemann spherical section. Also in the low dimensional cases ($d = 1, 2$) poles are absent, but this is not true for the intermediate case $d = 3$. Of course also in this case the principal part prescription gives a well-defined, finite result, but this does not seem to be quite as satisfactory (for the very purists). (ii) The exact coincidence of the results corresponding to a 3d and a 4d cylinder even if the initial expressions are completely different, and the first result is obtained only after subtracting the pole. (iii) The fact that the vacuum energy density for cylinders is by an order of magnitude smaller than the one corresponding to spheres of the same dimension. Since all the results are negative, this means that when the no-boundary manifold is not spherical, the attractive Casimir force diminishes considerably. (iv) Finally, the elegance, simplicity and mathematical rigour of our method, as compared with other approaches, is very remarkable. As discussed before, the merit for this is to be given in full to the procedure of zeta function regularization itself, which is being confirmed as the most beautiful and useful existing regularization procedure [21].

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Figure captions

Figure 1. The vacuum energy density for Riemann spheres (no-boundary manifolds) as a function of the space dimension, \( d \), in units of \( \hbar r^{-(d+1)} \), for \( d = 1, 2, 3, 4 \). The function is monotonically decreasing, showing no other remarkable feature than the small plateau for \( d = 3, 4 \). In fact, the values for the three- and four-dimensional Riemann spheres are very similar. There is a sort of stabilization of the Casimir force for this number of dimensions.

Figure 2. The vacuum energy density corresponding to cylinders whose sections are Riemann spheres (again manifolds without boundary), as a function of the space dimension \( d \) and in units of \( \hbar r^{-(d+1)} \), for \( d = 1, 2, 3, 4 \). We see the same monotonical behavior as in the case of the spheres but it is remarkable the clear stabilization of the numerical value of the energy density which is exactly the same for \( d = 4 \) as for \( d = 3 \) (cylinders whose sections are two- and three-dimensional Riemann spheres, respectively).