Another Complex Bateman Equation

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Abstract

A further class of complex covariant field equations is investigated. These equations possess several common features: they may be solved, or partially solved in terms of implicit functional relations, they possess an infinite number of inequivalent Lagrangians which vanish on the space of solutions of the equations of motion, they are invariant under linear transformations of the independent variables, and thus are signature-blind and are consequences of first order equations of hydrodynamic type.

1 Introduction

This paper is another in a series devoted to an investigation of simple equations exhibiting covariance of solutions. These equations have arisen in the study of generalisations of the Bateman equation \([1]\), in the equations arising from continuations of the String and Brane Lagrangians to the situation where the target space has fewer dimensions than the base space \([2]\) and a complex form of these equations \([3]\). The simplest example of these is a complexification of the Bateman equation. What we have called the Complex Bateman equation is the following equation for a real function \(\phi\) defined over the space of variables \((x_1, x_2; \bar{x}_1, \bar{x}_2)\):

\[
\phi_{x_1}\phi_{x_2}\phi_{x_2\bar{x}_1} + \phi_{x_2}\phi_{x_2}\phi_{x_1\bar{x}_1} - \phi_{x_1}\phi_{x_2}\phi_{x_1x_2} - \phi_{x_2}\phi_{x_2}\phi_{x_1x_2} = 0.
\]

(Here subscripts denote differentiation). This equation was shown to be completely integrable \([4][5]\), with solution given by solving for \(\phi\) the following constraint upon two arbitrary functions of three variables, \(F, G\):

\[
F(\phi; x_1, x_2) = G(\phi; \bar{x}_1, \bar{x}_2).
\]

From the form of the solution, or from the equation itself, it is manifest that if \(\phi\) is a solution, any function of \(\phi\) will also be a solution and thus

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that the equation exhibits covariance. It is also invariant under separate
diffeomorphisms of the pairs of variables \((x_1, x_2)\) and \((\bar{x}_1, \bar{x}_2)\). In fact a sub-
class of solutions is given by the sum of ‘holomorphic’ and ‘antiholomorphic’
functions
\[
\phi = f(x_1, x_2) + g(\bar{x}_1, \bar{x}_2).
\]
A general characteristic of such equations is that they possess an infinite
number of inequivalent Lagrangians. The equations of motion are partially,
or sometimes fully solvable in implicit form, as in the examples cited. The
fully integrable equations arise from kinematical first order equations of hy-
drodynamic type.

2 Another Complex Bateman Equation

Now there is another possibility for complexification; we could take instead
\[
\bar{\phi}_x \phi_x \phi_{tt} - \bar{\phi}_x \phi_t \phi_{tx} - \bar{\phi}_t \phi_x \phi_{tx} + \bar{\phi}_t \phi_t \phi_{xx} = 0,
\]
(2)
together with its complex conjugate. These equations also exhibit covariance;
\(\phi\) may be replaced by any function of itself, and the same for \(\bar{\psi}\) and the
equations remain invariant. Where do these equations come from? Take the
hydrodynamic equations
\[
\frac{\partial u}{\partial t} = v \frac{\partial u}{\partial x},
\]
(3)
\[
\frac{\partial v}{\partial t} = u \frac{\partial v}{\partial x},
\]
(4)
and set \(u = \frac{\bar{\phi}_t}{\phi_x}, \ v = \frac{\phi_t}{\phi_x}\), and the equation (2), together with its complex
conjugate are reproduced. Indeed, all that is necessary is to set in an alterna-
tive reduction, \(u = \bar{\phi}\) and \(v = \phi\) and the same equations arise in consequence.
These equations admit an infinite number of conserved quantities \([6]\): If \(S_n\)
denotes the symmetric polynomial of degree \(n\) in \(u, v\), then
\[
\frac{\partial}{\partial t} S_n = \frac{\partial}{\partial x} (uv S_{n-1})
\]
(5)
is a conservation law. This is easily proved by induction and from the iterative
definition: \(S_n = u^n + v S_{n-1}\). These equations can be integrated by the usual
hodographic method of interchanging dependent and independent variables, where they become
\[
\frac{\partial x}{\partial v} + v \frac{\partial t}{\partial v} = 0, \quad (6)
\]
\[
\frac{\partial x}{\partial u} + u \frac{\partial t}{\partial u} = 0, \quad (7)
\]
which can be solved in terms of two arbitrary functions \(f, g\) to give
\[
t = f'(u) + g'(v); \quad x = f(u) - uf'(u) + g(v) - vg'(v)
\]
with primes denoting differentiation with respect to the argument. If \(u = \bar{\phi}\) and \(v = \phi\) this parametrisation is a solution to the alternative complexification. Note that the requirements that \((t, x)\) be real imposes a further constraint upon the functions \((f, g)\). Of course, if \((\phi, \bar{\phi})\) are treated as independent real functions, no such restriction exists. Now the second order equations (2) are Poincaré invariant; indeed are covariant under general inhomogeneous linear transformations of the independent variables. This must be true also for the first order equations (6), (7). They are clearly translation invariant; if \((t, x)\) transform as
\[
t' = at + bx, \quad x' = ct + dx,
\]
then invariance will be maintained if
\[
u' = \frac{du - c}{a - bu}, \quad v' = \frac{dv - c}{a - bv}.
\]

3 2-dimensional Born-Infeld equation

We may remark parenthetically that the same equations (3), (4) also yield the general solution to one form of the so-called Born-Infeld equation in two dimensions in light-cone co-ordinates [6] [7];
\[
\phi_x^2 \phi_{tt} + \phi_x^2 \phi_{tt} - (\lambda + 2\phi_x\phi_t)\phi_{xt} = 0.
\]

This is achieved by setting
\[
\frac{\partial \phi}{\partial x} = \frac{\sqrt{\lambda}}{\sqrt{u} - \sqrt{v}}, \quad \frac{\partial \phi}{\partial t} = \frac{\sqrt{\lambda uv}}{\sqrt{u} - \sqrt{v}}.
\]
The integrability constraints upon these equations is just the Born-Infeld equation itself. Thus the primacy of the first order hydrodynamic equations is again manifest. This is a phenomenon which has been noticed before; that the same first order equations yield different second order ones depending upon the assumptions made about the dependency of the unknown functions in the first order equations upon the functions which enter into the second order equations [8][3].

4 Lagrangian

The construction of a Lagrangian for (2) follows along the lines of [3]. Introduce an auxiliary field \( \psi \) and consider the singular Lagrangian

\[
\mathcal{L} = \left( \frac{\partial \bar{\phi} \partial \psi}{\partial t \partial x} - \frac{\partial \psi \partial \bar{\phi}}{\partial t \partial x} \right) \frac{\partial \phi}{\partial x}.
\]

(8)

The equation of motion corresponding to variations in the field \( \psi \) is simply equation (2). Similarly for the variations with respect to \( \bar{\phi} \) we obtain

\[
\psi_x \phi_x \phi_t - \psi_x \phi_t \phi_{tx} - \psi_t \phi_x \phi_{tx} + \psi_t \phi_t \phi_{xx} = 0,
\]

(9)

i.e. a similar equation with \( \bar{\phi} \) replaced with \( \psi \). But the third equation, corresponding to variations with respect to \( \phi \) is just

\[
\frac{\partial}{\partial t} \left[ (\bar{\phi}_t \psi_x - \bar{\phi}_x \psi_t) \left( \frac{1}{\phi_x} \right) \right] - \frac{\partial}{\partial x} \left[ (\bar{\phi}_t \psi_x - \bar{\phi}_x \psi_t) \left( \frac{\phi_t}{\phi_x^2} \right) \right] = 0.
\]

(10)

This is satisfied if \( \psi \) is a function of \( \bar{\phi} \); then equation (9) is the same as equation (2). Incidentally, we see here a situation which has been remarked upon before in the context of free field equations [4], and equations arising from Born-Infeld Lagrangians, namely that the Lagrangian itself is a constant, or else a divergence on the space of solutions of the equations of motion. It is also evident that the factor \( \frac{\partial \phi}{\partial x} \) may be replaced by any homogeneous function of \( \left( \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x} \right) \) of weight zero, without affecting the equations of motion.
5 Multi-field Lagrangian

The Lagrangian can be constructed along similar lines to that for the single field; one choice is

\[ L = \frac{\partial(\bar{\phi}^1, \bar{\phi}^2, \theta)}{\partial(x_1, x_2, x_3)} \left( \begin{array}{cc} \frac{\partial(\phi^1, \phi^2)}{\partial(x_1, x_2)} \\ \frac{\partial(\phi^3)}{\partial(x_1, x_3)} \end{array} \right), + \text{cc.} \]  

(11)

Variation with respect to \( \theta \) gives a combination of the equations of motion for \( \phi^1 \) and \( \phi^2 \) and their complex conjugates; variations with respect to the fields \( \bar{\phi}^1 \) and \( \bar{\phi}^2 \) yields other linear combinations which together imply the following equations, where \( j \) takes the values (1, 2);

\[
\det \begin{vmatrix}
0 & 0 & \tilde{\phi}_{x_1}^1 & \tilde{\phi}_{x_2}^1 & \tilde{\phi}_{x_3}^1 \\
0 & 0 & \tilde{\phi}_{x_1}^2 & \tilde{\phi}_{x_2}^2 & \tilde{\phi}_{x_3}^2 \\
\phi_{x_1}^1 & \phi_{x_2}^1 & \phi_{x_1x_1}^j & \phi_{x_1x_2}^j & \phi_{x_1x_3}^j \\
\phi_{x_2}^1 & \phi_{x_2}^2 & \phi_{x_2x_1}^j & \phi_{x_2x_2}^j & \phi_{x_2x_3}^j \\
\phi_{x_3}^1 & \phi_{x_3}^2 & \phi_{x_3x_1}^j & \phi_{x_3x_2}^j & \phi_{x_3x_3}^j \\
\end{vmatrix} = 0. \]  

(12)

and that \( \theta \) is a function of \( \tilde{\phi}^1 \) and \( \tilde{\phi}^2 \), in much the same manner as \( \psi \) is a function of \( \phi \) in the single field case. As in the case of a single pair of complex fields, these equations follow from a set of hydrodynamic equations.

\[
\frac{\partial u^i}{\partial x_1} + v^1 \frac{\partial u^i}{\partial x_3} + v^2 \frac{\partial u^i}{\partial x_2} = 0 \quad i = 1, 2 \]  

(13)

\[
\frac{\partial v^i}{\partial x_1} + u^1 \frac{\partial v^i}{\partial x_3} + u^2 \frac{\partial v^i}{\partial x_2} = 0 \quad i = 1, 2. \]  

(14)

Once again, these equations remain the same up to a constant factor under a general linear transformation of co-ordinates; this may be seen most easily if they are written in a homogeneous notation by introducing vectors \( \xi^\mu, \eta^\mu; \mu = 0, 1, 2 \) such that \( u^i = \frac{\xi^i}{\xi^0}, \quad v^i = \frac{\eta^i}{\eta^0} \) so that the equations may be written as

\[
\sum_0^2 \xi^\mu \frac{\partial v^i}{\partial x_\mu} = 0; \quad \sum_0^2 \eta^\mu \frac{\partial u^i}{\partial x_\mu} = 0,
\]

making the invariance up to a factor of the hydrodynamic equations under linear transformations of the co-ordinates and the vectors \( \vec{\xi}, \vec{\eta} \) manifest.
Set $u^i = \phi_i$ and choose the following set of these equations and their derivatives:

\[
\begin{align*}
\phi^1_{x_1} + v^1 \phi^1_{x_2} + v^2 \phi^1_{x_3} &= 0 \\
\phi^2_{x_1} + v^1 \phi^2_{x_2} + v^2 \phi^2_{x_3} &= 0 \\
\phi^1_{x_1 x_1} + v^1 \phi^1_{x_1 x_2} + v^2 \phi^1_{x_1 x_3} + v^1 \phi^1_{x_2 x_1} + v^2 \phi^1_{x_2 x_3} &= 0 \\
\phi^1_{x_1 x_2} + v^1 \phi^1_{x_1 x_2} + v^2 \phi^1_{x_2 x_3} + v^1 \phi^1_{x_2 x_2} + v^2 \phi^1_{x_2 x_3} &= 0 \\
\phi^1_{x_1 x_3} + v^1 \phi^1_{x_1 x_3} + v^2 \phi^1_{x_3 x_3} + v^1 \phi^1_{x_3 x_2} + v^2 \phi^1_{x_3 x_3} &= 0.
\end{align*}
\]

Eliminate the first derivatives $(\phi^1_{x_1}, \phi^1_{x_2})$ from the final three equations, solve the first pair of equations for $(v^1, v^2)$ and substitute in the undifferentiated terms of the result, setting $v^1 = \bar{\phi}^1$, $v^2 = \bar{\phi}^2$. The ensuing equation is just one member of (12).

6 The fundamental hydrodynamic equations

All second order integrable equations of the type discussed here and in earlier work [3, 13] are consequences of the general first order equations, for which an implicit solution may be constructed following Leznov [10]. Consider a $2n$ dimensional Euclidean space with independent co-ordinates $(x_i, \bar{x}_i, i = 1 \ldots n)$ and construct the differential operators

\[
D = \frac{\partial}{\partial x_n} + \sum_{j=1}^{n-1} u^j \frac{\partial}{\partial x_j}, \quad \bar{D} = \frac{\partial}{\partial \bar{x}_n} + \sum_{j=1}^{n-1} v^j \frac{\partial}{\partial \bar{x}_j} \quad (15)
\]

Since $D \bar{x}_i = 0$, $\bar{D} x_i = 0$, $D$ may be considered a holomorphic differential operator and $\bar{D}$ an antiholomorphic operator. Now imposing the zero curvature condition, $[D, \bar{D}] = 0$ requires that

\[
D v^j \equiv v^j_{x_n} + \sum u^j v^j_{x_j} = 0, \quad \bar{D} u^j \equiv u^j_{x_n} + \sum v^j u^j_{\bar{x}_j} = 0. \quad (16)
\]

These are the general first order equations mentioned above. Since $D$, $\bar{D}$ commute, these equations imply that $\bar{D} v^j$ is a solution to the same equation as $v^i$ satisfies. The integration of the equation $D \bar{D} v^i = 0$ requires that $\bar{D} v^i$ is a general anti-holomorphic function, hence,

\[
\bar{D} v^i = v^i_{x_n} + \sum v^j v^i_{x_j} = V^i(v^i; \bar{x}_i), \quad D u^i = u^i_{x_n} + \sum w^j u^i_{\bar{x}_j} = U^i(u^i; x_i). \quad (17)
\]
Indeed \( f(\bar{D})v^i \), for arbitrary differentiable \( f \) is also a solution to the equation for \( v^i \). Suppose now we take \((n-1)\) functions \( \phi^i \) constrained by the \((n-1)\) relations
\[
Q^i(\phi^j; x_k) = P^i(\phi^j; \bar{x}_k), \quad i = 1 \ldots n-1.
\]
The arbitrary functions \( Q^i, P^i \) depend upon \((2n-1)\) co-ordinates. They imply straightforwardly
\[
\phi^j_{x_k} = (P^i_{\phi^j} - Q^i_{\phi^j})^{-1}Q^i_{x_k}, \quad \bar{\phi}^j_{\bar{x}_k} = -(P^i_{\phi^j} - Q^i_{\phi^j})^{-1}P^i_{\bar{x}_k}.
\]
Suppressing the vector indices, suppose \( u \) is a function \( u(\phi,x) \) and \( v \) is a function \( u(\phi,\bar{x}) \). Then the equations (19) imply that
\[
D\phi^j = \phi^j_{x_n} + \sum_{1}^{n-1} v^k \phi_{x_k} = 0, \quad \bar{D}\phi^j = \bar{\phi}^j_{\bar{x}_n} + \sum_{1}^{n-1} u^k \bar{\phi}_{\bar{x}_k} = 0,
\]
In other words this requires that \( \phi^j \) be both holomorphic and antiholomorphic in this definition of holomorphicity. Substituting the derivatives from (19) and multiplying on the left by the matrix \((P^i_{\phi^j} - Q^i_{\phi^j})\) the equations become
\[
Q^j_{x_n} + \sum_{1}^{n-1} v^k Q^j_{x_k} = 0, \quad P^j_{\bar{x}_n} + \sum_{1}^{n-1} u^k P^j_{\bar{x}_k} = 0,
\]
which leads to the identifications
\[
\nu = -(Q_x)^{-1}Q_{x_n}, \quad \mu = -(P_{\bar{x}})^{-1}P_{\bar{x}_n}.
\]
In consequence any function of \( \phi, \bar{x} \) \((\phi, x)\) will be annihilated by \( D (\bar{D}) \). In particular
\[
D\phi = \bar{D}\phi = DQ = \bar{D}P = DQ = \bar{D}P = 0.
\]
The last two results follow a forteriori from the equality \( P = Q \). The functions \( \phi^j \) satisfy the multi-field complex Bateman equation [3].

7 Partially integrable covariant equations.

It appears likely that in the case where the difference between the number of dimensions of the base space exceeds that of the target space by more than one, the equations of motion are only partially integrable, though this is by
no means a definitive conclusion. In the case of one field dependent on three co-ordinates, the equation which results from the Euclidean Lagrangian

$$\mathcal{L} = \sqrt{\phi_t^2 + \phi_x^2 + \phi_y^2}$$

is

$$\phi_t(\phi_x^2 + \phi_y^2) + \phi_{xx}(\phi_y^2 + \phi_t^2) + \phi_{yy}(\phi_t^2 + \phi_x^2) = 2\phi_{tx}\phi_t\phi_x + 2\phi_{yt}\phi_y\phi_t + 2\phi_{xy}\phi_x\phi_y.$$  

This equation possesses a large class of solutions given implicitly by solving

$$tF(\phi) + xG(\phi) + yK(\phi) = \text{constant}$$

for $\phi$. It comes from the following first order system;

$$uu_x + vv_x = u_t + v_y + v^2u_t - uv(u_y + v_t) + u^2v_y$$

$$uv_x - vu_x = v_t - u_y,$$

where $u = \frac{\phi_t}{\phi_x}, \ v = \frac{\phi_y}{\phi_x}$. This construction suggests further analysis to try to determine whether the system is fully integrable or not. It is also not known whether there exist other Lagrangian formulations of these equations.

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