ON SOME GEOMETRIC PROPERTIES OF SEQUENCE SPACE DEFINED BY DE LA VALLÉE-POUSSIN MEAN

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Abstract. In this work, we investigate $k$-nearly uniform convex ($k$-NUC) and the uniform Opial properties of the sequence space defined by de la Vallée-Poussin mean. Also we give some corollaries concerning the geometrical properties of this space.

1. Introduction

In summability theory, de la Vallée-Poussin’s mean is first used to define the $(V,\lambda)$-summability by Leindler [13]. Malkowsky and Savaş [16] introduced and studied some sequence spaces which arise from the notion of generalized de la Vallée-Poussin mean. Also the $(V,\lambda)$-summable sequence spaces have been studied by many authors including [7] and [23].

In literature, there have been many papers on the geometrical properties of Banach spaces. Some of them are as follows: In [20], Opial defined the Opial property with his name mentioned and he proved that $\ell_p(1 < p < \infty)$ satisfies this property but the spaces $L^p[0,2\pi]$ ($p \neq 2, 1 < p < \infty$) do not. Franchetti [9] has shown that any infinite dimensional Banach space has an equivalent norm satisfying the Opial property. Later, Prus [21] has introduced and investigated uniform Opial property for Banach spaces. In [10], the notion of nearly uniform convexity for Banach spaces was introduced by Huff. It is an infinite dimensional counterpart of the classical uniform convexity. Also Huff [10] proved that every nearly uniformly convex Banach space is reflexive and it has the uniformly Kadec-Klee property. However, Kutzarova [12] defined and studied $k$-nearly uniformly convex Banach spaces.

Recently, there has been a lot of interest in investigating geometric properties of sequence spaces. Some of the recent work on sequence spaces and their geometrical properties is given in the sequel: Shue [24] first defined the Cesáro sequence spaces with a norm. In [5], it is shown that the Cesáro sequence spaces $ces_p (1 < p < \infty)$ have $k$-nearly uniform convex and uniform Opial properties. Şimşek and Karakaya...
studied the uniform Opial property and some other geometric properties of generalized modular spaces of Cesáro type defined by weighted means. In addition, some related papers on this topic can be found in [1], [3], [11], [14], [17], [18], [19] and [22].

Quite recently, Şimşek et al [26] introduced a new sequence space defined by de la Vallée-Poussin’s mean and investigated some geometric properties as Kadec-Klee and Banach-Saks of type $p$. Moreover, the sequence space involving de la Vallée-Poussin’s mean is more general than Cesáro sequence space defined by Shue [24] and investigated by Cui and Hudzik [4].

The main purpose of this paper is to investigate uniform Opial property and $k$-nearly uniformly convex property of the sequence space defined in [26]. In addition it will be given some corollaries concerning this space.

2. Preliminaries and Notation

Let $(X,\| \cdot \|)$ (for the brevity $X = (X,\| \cdot \|)$ ) be a normed linear space and let $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. unit sphere) of $X$. The space of all real sequences is denoted by $\ell_0$. For any sequence $\{x_n\}$ in $X$, we denote by $\mathrm{conv}(\{x_n\})$ the convex hull of the elements of $\{x_n\}$ (see [2]).

A Banach space $X$ is called uniformly convex (UC) if for each $\varepsilon > 0$, there is $\delta > 0$ such that for $x, y \in S(X)$, the inequality $\|x - y\| > \varepsilon$ implies that

$$\| \frac{1}{2}(x + y) \| < 1 - \delta.$$ 

Recall that for a number $\varepsilon > 0$ a sequence $\{x_n\}$ is said to be an $\varepsilon$–seperated sequence if

$$\mathrm{sep}(\{x_n\}) = \inf \{\|x_n - x_m\|, \ n \neq m \} > \varepsilon.$$ 

A Banach space $X$ is said to have the Kadec – Klee property (H property) if every weakly convergent sequence on the unit sphere is convergent in norm.

A Banach space $X$ is said to have the uniform Kadec – Klee property (UKK) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x$ is the weak limit of a normalized $\varepsilon$-separated sequence, then $\|x\| < 1 - \delta$ (see [10]). We have that every (UKK) Banach space have the Kadec-Klee property.

A Banach space $X$ is said to be the nearly uniformly convex (NUC) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence $\{x_n\} \subset B(X)$ with $\mathrm{sep}(\{x_n\}) > \varepsilon$, we have

$$\mathrm{conv}(\{x_n\}) \cap (1 - \delta)B(X) \neq \emptyset.$$ 

Let $k \geq 2$ be an integer. A Banach space $X$ is said to be $k$–nearly uniformly convex ($k$–NUC) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence
\{x_n\} \subset B(X) with \textit{sep}(\{x_n\}) > \varepsilon, there are \(n_1, n_2, \ldots, n_k \in \mathbb{N}\) such that
\[
\left\| \frac{x_{n_1} + x_{n_2} + \ldots + x_{n_k}}{k} \right\| < 1 - \delta.
\]

Of course a Banach space \(X\) is (\textit{NUC}) whenever it is (\(k - \text{NUC}\)) for some integer \(k \geq 2\). Clearly, (\(k - \text{NUC}\)) Banach spaces are (\textit{NUC}) but the opposite implication does not hold in general (see [12]).

A Banach space \(X\) is said to have the \textit{Opial property} if every sequence \(\{x_n\}\) weakly convergent to \(x_0\) satisfies
\[
\liminf_{n \to \infty} \|x_n\| < \liminf_{n \to \infty} \|x_n + x\|
\]
for every \(x \in X\) (see [20]).

A Banach space \(X\) is said to have the \textit{uniform Opial property} if every \(\varepsilon > 0\) there exists \(\delta > 0\) such that for each weakly null sequence \(\{x_n\} \subset S(X)\) and \(x \in X\) with \(\|x\| \geq \varepsilon\), we have (see [21])
\[
1 + \tau \leq \liminf_{n \to \infty} \|x_n + x\|.
\]

A point \(x \in S(X)\) is called an \textit{extreme point} if for any \(y, z \in B(X)\) the equality \(2x = y + z\) implies \(y = z\).

A Banach space \(X\) is said to be \textit{rotund} (abbreviated as (\(R\))) if every point of \(S(X)\) is an extreme point.

A Banach space \(X\) is said to be \textit{fully k-rotund} (write \(kR\)) (see [8]) if for every sequence \(\{x_n\} \subset B(X)\),
\[
\|x_{n_1} + x_{n_2} + \ldots + x_{n_k}\| \to k \quad \text{as} \quad n_1, n_2, \ldots, n_k \to \infty
\]
implies that \(\{x_n\}\) is convergent.

It is well known that (\textit{UC}) implies (\(kR\)) and (\(kR\)) implies ((\(k + 1)\)R), and (\(kR\)) spaces are reflexive and rotund, and it is easy to see that (\(k - \text{NUC}\)) implies (\(kR\)).

In this paper, we will need the following inequalities in the sequel;
\[
|a_k + b_k|^p \leq 2^{p-1} (|a_k|^p + |b_k|^p),
\]
for \(p \geq 1\).

Let \(\Lambda = (\lambda_k)\) be a nondecreasing sequence of positive real numbers tending to infinity and let \(\lambda_1 = 1\) and \(\lambda_{k+1} \leq \lambda_k + 1\).

The generalized de la Vallée-Poussin means of a sequence \(x = \{x_k\}\) are defined as follows:
\[
t_k(x) = \frac{1}{\lambda_k} \sum_{j \in I_k} x_k \quad \text{where} \quad I_k = [k - \lambda_k + 1, k] \quad \text{for} \quad k = 1, 2, \ldots.
\]

We write
\[
[V, \lambda]_0 = \left\{ x \in \ell^0 : \lim_{k \to \infty} \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| = 0 \right\}
\]
\[
[V, \lambda] = \{ x \in \ell^0 : x - le \in [V, \lambda]_0, \text{ for some } l \in \mathbb{C} \}.
\]
and

\[ [V, \lambda]_\infty = \left\{ x \in \ell^0 : \sup_k \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| < \infty \right\} \]

for the sequence spaces that are strongly summable to zero, strongly summable and strongly bounded by the de la Vallée-Poussin method, resp. (see [13]). In the special case where \( \lambda_k = k \) for \( k = 1, 2, \ldots \), the spaces \([V, \lambda]_0, [V, \lambda] \) and \([V, \lambda]_\infty \) reduce to the spaces \( w_0, w \) and \( w_\infty \) introduced by Maddox [15].

The following new paranormed sequence space defined in [26].

\[ V(\lambda; p) = \left\{ x = (x_j) \in \ell^0 : \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^p < \infty \right\}. \]

If we take \( p_k = p \) for all \( k \); the space \( V(\lambda; p) \) reduced to normed space \( V_p(\lambda) \) defined by

\[ V_p(\lambda) = \left\{ x = (x_j) \in \ell^0 : \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_k} \sum_{j \in I_k} |x_j| \right)^p < \infty \right\}. \]

The details of the sequence spaces mentioned above can be found in [26].

3. Main Results

In this section we show that the space \( V_p(\lambda) \) is \((k - NUC)\) and have uniform Opial property. Firstly we need an important lemma.

**Lemma 3.1.** Let \( X \subset V_p(\lambda) \). For any \( \varepsilon > 0 \) and \( L > 0 \), there exists \( \delta > 0 \) such that for all \( x, y \in X \),

\[ \left| \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x(i) + y(i)| \right)^p - \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x(i)| \right)^p \right| < \varepsilon, \]

whenever

\[ \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x(i)| \right)^p < L \]

and

\[ \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |y(i)| \right)^p \leq \delta. \]

**Proof.** (See [5]) Let \( ||.|| \) denote the norm in \( V_p(\lambda) \). Then \( ||y||^p < \delta \) implies

\[ ||x + y|| - ||x|| \leq ||(x + y) - x|| = ||y|| \leq \delta^{\frac{1}{p}}. \]

Since the function \( g(t) = t^p \) is uniformly continuous on the interval \( [0, L^{\frac{1}{p}} + 1] \), we get the assertion of the lemma.

**Theorem 3.2.** The space \( V_p(\lambda) \) is \((k - NUC)\) for any integer \( k \geq 2 \) where \((1 < p < \infty)\).
Proof. Let \( \varepsilon > 0 \) and \( (x_n) \subset B(V_p(\lambda)) \) with \( \text{sep}\{\{x_n\}\} > \varepsilon \). Let \( \lambda^m = (0, 0, \ldots, x_n(m), x_n(m+1), \ldots) \) for each \( m \in \mathbb{N} \). Since for each \( i \in \mathbb{N} \), \( \{x_n(i)\}_{i=1}^{\infty} \) is bounded therefore using the diagonal method one can find a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that the sequence \( \{x_{n_k}(i)\} \) converges for each \( i \in \mathbb{N} \).

Therefore, there exists an increasing sequence of positive integer \( (k_m) \) such that

\[
\text{sep}\{\{x_{n_k}^{m}\}_{k=k_m}\} \geq \varepsilon.
\]

Hence there is a sequence of positive integers \( (n_m)_{m=1}^{\infty} \) with \( n_1 < n_2 < n_3 < \ldots \) such that

\[
\|x_{n_m}^{m}\| \geq \frac{\varepsilon}{2}
\]

for all \( m \in \mathbb{N} \).

Write \( I_p(x) = \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x(i)| \right)^p \) and put \( \varepsilon_1 = \frac{k^{p-1}}{2k^{p-1} (\frac{\varepsilon}{2})} \). Then by Lemma \textbf{3.1}, there exists \( \delta > 0 \) such that

\[
|I_p(x + y) - I_p(x)| < \varepsilon_1
\]

whenever \( I_p(x) \leq 1 \) and \( I_p(y) \leq \delta \) (see\textbf{[6]}).

There exists \( m_1 \in \mathbb{N} \) such that \( I_p(x_1^{m_1}) \leq \delta \). Next there exists \( m_2 > m_1 \) such that \( I_p(x_2^{m_2}) \leq \delta \). In such a way, there exists \( m_2 < m_3 < \ldots < m_{k-1} \) such that \( I_p(x_j^{m_j}) \leq \delta \) for all \( j = 1, 2, \ldots, k-1 \). Define \( m_k = m_{k-1} + 1 \). By condition \textbf{3.1}, there exists \( n_k \in \mathbb{N} \) such that \( I_p(x_n^{m_k}) \geq \left( \frac{\varepsilon}{2} \right)^p \). Put \( n_i = i \) for \( 1 \leq i \leq k-1 \). Then in virtue of \textbf{3.1}, \textbf{3.2} and convexity of the function \( f(u) = |u|^p \), we get

\[
I_p(x_{n_1} + x_{n_2} + \ldots + x_{n_k}) = \sum_{n=1}^{m_1} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} \left| \frac{x_{n_1}(i) + x_{n_2}(i) + \ldots + x_{n_k}(i)}{k} \right| \right)^p + \varepsilon_1
\]

\[
\leq \sum_{n=m_1+1}^{m_2} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} \left| \frac{x_{n_1}(i) + x_{n_2}(i) + \ldots + x_{n_k}(i)}{k} \right| \right)^p + \varepsilon_1
\]

\[
= \sum_{n=m_1+1}^{m_2} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} \left| \frac{x_{n_1}(i) + x_{n_2}(i) + \ldots + x_{n_k}(i)}{k} \right| \right)^p + \varepsilon_1
\]
\[
\leq \sum_{n=1}^{m_1} \frac{1}{k} \sum_{j=1}^{k} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x_{n_j}(i)| \right)^p + \sum_{n=m_1+1}^{m_2} \frac{1}{k} \sum_{j=2}^{k} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x_{n_j}(i)| \right)^p + \\
+ \sum_{n=m_2+1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} \frac{|x_{n-1}(i) + x_{n}(i) + \ldots + x_{n_k}(i) + x_{n_k-1}(i) + x_{n_k}(i)|}{k} \right)^p + 2\varepsilon_1
\]

\[
\leq \frac{I_p(x_{n_1}) + \ldots + I_p(x_{n_k-1})}{k} + \frac{1}{k} \sum_{n=1}^{m_k-1} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x_{n_k}(i)| \right)^p + \\
+ \sum_{n=m_k-1+1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x_{n_k}(i)| \right)^p + (k-1)\varepsilon_1
\]

\[
\leq \frac{k-1}{k} + \frac{1}{k} \sum_{n=1}^{m_k-1} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x_{n_k}(i)| \right)^p + \frac{1}{k^p} \sum_{n=m_k-1+1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x_{n_k}(i)| \right)^p + (k-1)\varepsilon_1
\]

\[
= 1 + \frac{1}{k} \left( 1 - \sum_{n=m_k-1+1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x_{n_k}(i)| \right)^p \right) + \frac{1}{k^p} \sum_{n=m_k-1+1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x_{n_k}(i)| \right)^p + (k-1)\varepsilon_1
\]

\[
\leq 1 + (k-1)\varepsilon_1 - \left( \frac{k_p-1}{k^p} \right) \sum_{n=m_k-1+1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x_{n_k}(i)| \right)^p
\]

\[
\leq 1 + (k-1)\varepsilon_1 - \left( \frac{k_p-1}{k^p} \right) \left( \frac{\varepsilon}{2} \right)^p
\]

\[
= 1 - \frac{1}{2} \left( \frac{k_p-1}{k^p} \right) \left( \frac{\varepsilon}{2} \right)^p
\]

Under the condition $\varepsilon > 0$, $V_p(\lambda)$ is $(k - NUC)$ for any integer $k \geq 2$. \[\square\]

**Theorem 3.3.** For any $(1 < p < \infty)$, the space $V_p(\lambda)$ has the uniform Opial property.

**Proof.** Let $\varepsilon > 0$ and $\varepsilon_0 \in (0, \varepsilon)$. Also let $x \in X$ and $||x|| \geq \varepsilon$. There exists $n_1 \in \mathbb{N}$ such that

\[
\sum_{n=n_1+1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x(i)| \right)^p < \left( \frac{\varepsilon_0}{4} \right)^p.
\]

Hence we have

\[
\left\| \sum_{i=n_1+1}^{\infty} x(i)e_i \right\| < \frac{\varepsilon_0}{4},
\]
where \( e_i = (0, 0, ..., i^{th}, 1, 0, 0, ...) \). Furthermore, we have

\[
\varepsilon^p \leq \sum_{n=1}^{n_1} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x(i)| \right)^p + \sum_{n=n_1+1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x(i)| \right)^p
\]

\[
< \sum_{n=1}^{n_1} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x(i)| \right)^p + \left( \frac{\varepsilon_0}{4} \right)^p
\]

\[
\varepsilon^p - \left( \frac{\varepsilon_0}{4} \right)^p \leq \sum_{n=1}^{n_1} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x(i)| \right)^p.
\]

whence

\[
\varepsilon^p - \left( \frac{\varepsilon_0}{4} \right)^p \leq \sum_{n=1}^{n_1} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x(i)| \right)^p.
\]

Since \( x_m(i) \to 0 \) for \( i = 1, 2, ..., m > m_0 \), we choose any weakly null sequences \( \{ x_m \} \) such that \( \lim_{m \to \infty} \| x_m \| \geq 1 \). Then there exists \( m_0 \in \mathbb{N} \) such that

\[
\left\| \sum_{i=1}^{n_1} x_m(i)e_i \right\| < \frac{\varepsilon_0}{4}
\]

when \( m > m_0 \). Therefore,

\[
\| x_m + x \| = \left\| \sum_{i=1}^{n_1} (x_m(i) + x(i)) e_i + \sum_{i=n_1+1}^{\infty} (x_m(i) + x(i)) e_i \right\|
\]

\[
\geq \left\| \sum_{i=1}^{n_1} x(i)e_i + \sum_{i=n_1+1}^{\infty} x_m(i)e_i \right\| - \left\| \sum_{i=1}^{n_1} x_m(i)e_i \right\| - \left\| \sum_{i=n_1+1}^{\infty} x(i)e_i \right\|
\]

\[
\geq \left\| \sum_{i=1}^{n_1} x(i)e_i + \sum_{i=n_1+1}^{\infty} x_m(i)e_i \right\| - \frac{\varepsilon_0}{2}
\]

Moreover, we have

\[
\left\| \sum_{i=1}^{n_1} x(i)e_i + \sum_{i=n_1+1}^{\infty} x_m(i)e_i \right\|^p
\]

\[
= \sum_{n=1}^{n_1} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x(i)e_i| \right)^p + \sum_{n=n_1+1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{i \in I_n} |x_m(i)| \right)^p
\]

\[
\geq 1 + \varepsilon^p - 2 \left( \frac{\varepsilon_0}{4} \right)^p.
\]

Since \( (2 \left( \frac{\varepsilon_0}{4} \right)^p - 1 + (1 + \varepsilon_0)^p)^\frac{1}{p} \geq \varepsilon_0 \) for \( 1 < p < \infty \), we can choose \( \varepsilon \geq (2 \left( \frac{\varepsilon_0}{4} \right)^p - 1 + (1 + \varepsilon_0)^p)^\frac{1}{p} \)

and we have

\[
\left\| \sum_{i=1}^{n_1} x(i)e_i + \sum_{i=n_1+1}^{\infty} x_m(i)e_i \right\| \geq \left( 1 + \varepsilon^p - 2 \left( \frac{\varepsilon_0}{4} \right)^p \right)^\frac{1}{p}
\]

\[
\geq 1 + \varepsilon_0
\]
Therefore, combining this result with the previous inequality, we get
\[
\|x_m + x\| \geq \left\| \sum_{i=1}^{n_1} x(i) e_i + \sum_{i=n_1+1}^{\infty} x_m(i) e_i \right\| - \frac{\varepsilon_0}{2} \\
\geq 1 + \varepsilon_0 - \frac{\varepsilon_0}{2} = 1 + \frac{\varepsilon_0}{2}
\]
This means that \(V_p(\lambda)\) has the uniform Opial property.

From the Theorem 3.2, we get that \(V_p(\lambda)\) is \((k-NUC)\). Clearly \((k-NUC)\) Banach spaces are \((NUC)\), and \((NUC)\) implies property \((H)\) and reflexivity holds, \(^{10}\). Also, Huff proved that \(X\) is \((NUC)\) if and only if \(X\) is reflexive and \((UKK)\) (see in \(^{10}\)).

On the other hand, it is well known that

\((UC) \Rightarrow (kR) \Rightarrow (k + 1)R,\)

and \((kR)\) spaces are reflexive and rotund, and it is easy to see that

\((k - NUC) \Rightarrow (kR).\)

By the facts presented in the introduction and the just above; we get the following corollaries:

**Corollary 3.4.** The space \(V_p(\lambda)\) \((1 < p < \infty)\) is \((NUC)\) and then is reflexive.

**Corollary 3.5.** The space \(V_p(\lambda)\) \((1 < p < \infty)\) is \((UKK)\).

**Corollary 3.6.** The space \(V_p(\lambda)\) \((1 < p < \infty)\) is \((kR)\).

**Corollary 3.7.** The space \(V_p(\lambda)\) \((1 < p < \infty)\) is rotund.

**References**

[1] F. Basar, B. Altay and M. Mursaleen, *Some generalizations of the space \(bv_p\) of \(p\)-bounded variation sequences*, Nonlinear Analysis:Theory, Methods & Applications, Volume 68, (2), (2008) 273-287.

[2] W. L. Bynum, *Normal structure coefficient for Banach spaces*, J. Math., 86(1980), 427-436.

[3] S. T. Chen, *Geometry of Orlicz spaces*, Dissertationes Math., 356 (1996).

[4] Y. Cui, H. Hudzik, *Some geometric properties related to fixed point theory in Cesàro spaces*, Collect. Math. 50, 3(1999), 277-288.

[5] Y. Cui, H. Hudzik, *Packing constant for Cesàro sequence spaces*, Nonlinear Analysis, 47 (2001) 2695-2702.

[6] Y. Cui, R. Pluciennik, *Local uniform nonsquareness in Cesàro sequence spaces*, Comment. Math., 37(1997), 47-58.

[7] M. Et, *Spaces of Cesàro difference sequences of order \(r\) defined by a modulus function in a locally convex space*, Taiwanese J. Math., 10 (2006) no. 4, 865-879.

[8] K. Fan and I. Glicksberg, *Fully convex normed linear spaces*, Proc. Nat. Acad. Sci. USA. 41(1955), 947-953.

[9] C. Franchetti, *Duality mapping and homeomorphisms in Banach theory*, in "Proceedings of Research Workshop on Banach Spaces Theory", University of Iowa, 1981.
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[10] R. Huff, *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math., 10(1980), 473-479.

[11] V. Karakaya, *Some geometric properties of sequence spaces involving Lacunary sequence*, Journal of Inequalities and Applications, Volume 2007, Article ID 81028, 8 pages doi:10.1155/2007/81028.

[12] D. N. Kutzarova, *k-β and k-nearly uniformly convex Banach spaces*, J. Math. Anal. Appl. 162(1991), 322-338.

[13] L. Leindler, *Über die verallgemeinerte de la Vallée-Poussinsche summierbarkeit allgemeiner Orthogonalreihen*, Acta Math. Acad. Sci. Hungar., 16, no.3-4 (1965) 375–387.

[14] Y. Q. Liu, B. E. Wu and Y. P. Lee, *Method of sequence spaces*, Guangdong of Science and Technology Press, (1996) (in Chinese).

[15] I. J. Maddox, *On Kuttners theorem*, J. London Math. Soc., 43 (1968), 285-290.

[16] E. Malkowsky, E. Savaş, *Some λ-sequence space defined by a modulus*, Arch.Math. (BRNO), 36 (2000), 219-228.

[17] M. Mursaleen, F. Başar, B. Altay, *On the Euler sequence spaces which include the spaces ℓp and ℓ∞ II*, Nonlinear Analysis, 65 (2006) 707-717.

[18] M. Mursaleen, Rifat Çolak and Mikail Et, *Some geometric inequalities in a new Banach sequence space*, Journal of Inequalities and Applications, Volume 2007 (2007), Article ID 86757, 6 pages, doi:10.1155/2007/86757.

[19] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, Vol.1034, Springer-Berlin, 1983.

[20] A. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73(1967), 591-597.

[21] S. Prus, *Banach spaces with uniform Opial property*, Nonlinear Anal., 8(1992), 697-704.

[22] E. Savaş, V. Karakaya, N. Şimşek, *Some ℓ(μ)-type new sequence spaces and their geometric properties*, Abstract and Applied Analysis, Volume 2009, Article ID 696971, 12 pages doi:10.1155/2009/696971.

[23] E. Savaş, R. Savaş, *Some λ-sequence spaces defined by Orlicz functions*, Indian J. Pure Appl. Math., 34 (2003), no. 12, 1673–1680.

[24] J. S. Shue, *Cesáro sequence spaces*, Tamkang J. Math., 1 (1970) 143-150.

[25] N. Şimşek and V. Karakaya, *On some geometrical properties of generalized modular spaces of Cesáro type defined by weighted means*, Journal of Inequalities and Applications, Volume 2009, Article ID 932734, 13 pages doi:10.1155/2009/932734.

[26] N. Şimşek, E. Savaş and V. Karakaya, *Some geometric and topological properties of a new sequence space defined by de la Vallée-Poussin mean*, Journal of Computational Analysis and Applications, to appear.

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