Entropy bound and causality violation in higher curvature gravity

Ishwaree P Neupane$^1$ and Naresh Dadhich$^2$

$^1$ Department of Physics and Astronomy, University of Canterbury, Private Bag 4800, Christchurch 8020, New Zealand
$^2$ Inter-University Centre for Astronomy and Astrophysics, Pune 411 007, India

E-mail: ishwaree.neupane@canterbury.ac.nz and nkd@iucaa.ernet.in

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Abstract
In any quantum theory of gravity we do expect corrections to Einstein gravity to occur. Yet, at a fundamental level, it is not apparent what the most relevant corrections are. We argue that the generic curvature square corrections present in the lower dimensional actions of various compactified string theories provide a natural passage between the classical and quantum realms of gravity. The Gauss–Bonnet and $(\text{Riemann})^2$ gravities, in particular, provide concrete examples in which inconsistency of a theory, such as a violation of microcausality, and a classical limit on black hole entropy are correlated. In such theories the ratio of the shear viscosity to the entropy density, $\eta/s$, can be smaller than for a boundary conformal field theory with Einstein gravity dual. This result is interesting from the viewpoint that nuclear matter or quark-gluon plasma produced (such as at RHIC) under extreme densities and temperatures may violate the conjectured KSS bound $\eta/s \gtrsim 1/4\pi$, albeit marginally so.

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(Some figures in this article are in colour only in the electronic version)

1. The problem of universality

Black holes are thermal objects and hence they radiate as pure black bodies with characteristic thermodynamic properties such as temperature and entropy. In pure Einstein gravity, all black holes do satisfy the famous Bekenstein–Hawking entropy law [1]

$$ S = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}, $$

(1.1)
where $A$ is the area of the horizon corresponding to the surface at $r = r_*$. There are two important implications of this result. First, the entropy depends on both Planck’s constant $\hbar$ and Newton’s constant $G_N$, hinting that black hole thermodynamics unites quantum mechanics and gravity. Second, in accordance with Wheeler’s famous dictum that ‘black holes have no hair’, formula (1.1) is universal to all kinds of black holes irrespective of their charges, shapes and rotation [2, 3].

Black hole solutions in curved backgrounds, especially in an anti-de Sitter (AdS) background, exhibit several interesting new features which can be related to certain class of boundary field theories residing in one dimension lower. Notably, the AdS conformal field theory (CFT) (or gravity-gauge theory) correspondence [4, 5] has provided several interesting insights into the dynamics of strongly coupled gauge theories in the limit $N_c \to \infty$ and $\lambda = g_{YM}^2 N_c \to \infty$, where $N_c$ is the number of colours (or the rank of the gauge group) and $\lambda$ is the ‘t Hooft coupling. Through AdS holography one can also study the hydrodynamic properties of a certain class of boundary CFTs. In particular, it has been possible to compute, for a large class of four-dimensional CFTs with an Einstein gravity dual, the ratio of the shear viscosity $\eta$ to the entropy density $s$, which (in units where $\hbar = k_B = 1$) is given by $^3 [6–8],$

$$\frac{\eta}{s} = \frac{1}{4\pi} \approx 0.08. \quad (1.2)$$

This result has been conjectured to be a universal lower bound (the so-called Kovtun–Starinets–Son (KSS) bound [9]) in nature$^4$. While this bound is below the value found for ordinary substances (especially, cold atomic gases, including water and liquid helium), a great deal of interest has been generated by results from the Relativistic Heavy Ion Collider (RHIC) experiments suggesting that just above the deconfinement phase transition (or infrared limit), QCD is very close to saturating this bound (see, e.g., [11]).

For finite temperature quantum field theories, in particular, for thermal super Yang–Mills theories at the weak coupling limit, $g_{YM} \ll 1$, both the shear viscosity and the entropy density scale being proportional to $T^3$, namely $\eta \sim \frac{N_c}{\pi} T^3$ and $s \sim \frac{N_c}{\pi} T^3$. This scaling behaviour effectively looks similar to the CFT one, for which $\eta/s \sim 1/4\pi$ [6]. Notwithstanding this result, supergravity calculations are generally carried out in an opposite end, i.e., in the strong coupling regime where $\lambda \equiv g_{YM}^2 N_c \gg 1$ and $N_c \to \infty$. This limit is also known in the gauge theory as the ‘t Hooft limit, where one takes the rank of the gauge group $N_c$ to infinity while keeping $\lambda$ fixed. Presumably, without much surprise, all gauge theories operate at a finite coupling. For example, QCD is a gauge theory with $N_c = 3$. Thus, to establish better contact with QCD via AdS holography, it is essential to understand the effect of $1/N_c$ corrections which arise from curvature square corrections to Einstein–Hilbert action in the holographic framework (see, for example, [12–14]).

Given that we do expect corrections to Einstein gravity to occur in any consistent quantum theory of gravity, it is perhaps natural to expect modifications to both the entropy law (1.1) and the ratio $\eta/s = 1/4\pi$. As the simplest modification to classical Einstein gravity, the leading-order higher-derivative corrections may be written as

$$I_g = \frac{1}{16\pi G_N} \int d^4 x \sqrt{-g} \left[ R - 2\Lambda + \alpha' L^2 (\alpha R^2 + b R_{\mu\nu} R^{\mu\nu} + c R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho}) \right], \quad (1.3)$$

$^3$ The shear viscosity term $\eta$ appears as a coefficient of frictional force in the spatial traceless (dissipative) part of the stress–energy tensor $T_{\mu\nu}(x) = T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu}$, where the first term describes the perfect (non-dissipative) fluid dynamics, $T^{(0)}_{\mu\nu} = [\rho(x) + p(x)] u_{\mu}(x) u_{\nu}(x) - p(x) \delta_{\mu\nu}$ and $T^{(1)}_{\mu\nu} = -\eta \left( \frac{1}{2} \nabla_{\mu} u_{\nu}(x) + \frac{1}{2} \nabla_{\nu} u_{\mu}(x) - \frac{1}{2} \delta_{\mu\nu} \nabla \cdot u(x) \right)$, where the collective flow velocity $u_{\mu}(x) \equiv (1, -u(x))/\sqrt{1 - u^2}$.

$^4$ The KSS bound $\eta/s \gg 1/4\pi$ is perhaps related to a similar limit of low energy absorption cross sections for classical black holes [10]. Since $\eta \approx \frac{\Delta m^2}{a r}$, a large (black hole) scattering cross section $\sigma$ yields a small shear viscosity.
where $\alpha'$ is a dimensionless coupling. For simplicity, the bulk cosmological term $\Lambda$ is fixed in terms of the length scale $L$, namely $\Lambda = -(d-1)(d-2)/2 L^2$, which is a sort of fine tuning! One may supplement the above action with analogue Gibbons–Hawking surface terms [15], but we do not need these terms in our present discussion. In fact, the Gauss–Bonnet (GB) term obtained by setting $a = c = 1$ and $b = -4$ in (1.3) produces the most general action, retaining only second-order field equations and hence admits exact solutions in numerous cases [16–18]. Since the GB term is topological in $d = 4$, especially, with a constant coupling, $\alpha' L^2 = \text{const}$, one considers a spacetime for which $d \geq 5$. The Gauss–Bonnet term is relevant not only because of the solvability of the model but also due to its natural appearance in the lower dimensional actions of various compactified string theories [19]. Moreover, it is a unique combination of quadratic curvature invariants which is free of ghost not only in a flat Minkowski background but also when expanded about a warped AdS braneworld background [20].

We also note that the coefficient multiplying the GB term or terms quadratic in curvature has a mass dimension of 2 or (length)$^2$; in (1.3) we have replaced this coefficient by $\alpha' L^2$, so that $\alpha'$ is dimensionless. We focus most of our discussions below on the gravity sector in AdS$_5$, for which we have (from AdS/CFT dictionary [4, 13])

$$\frac{1}{16\pi G_N} \equiv \frac{N_c^2}{4\pi^2 L^2}.$$ 

Moreover, the curvature radius of AdS$_5$ can be defined by

$$L = (4\pi g_s N_c)^{1/4} \ell_s,$$

with $\ell_s$ being the string scale and $g_s$ the string coupling. In terms of the ’t Hooft coupling $\lambda = g_s^2 N_c$, the dimensionless scale $\alpha'$ of string theory, for instance, on AdS$_5 \times S^5$, is related to the SYM parameters by $\alpha' \sim 1/\sqrt{\lambda}$. The latter implies that a small $\alpha'$ corresponds to the strong coupling limit, i.e. $g_s^2 N_c \gg 1$ in dual supergravity description. Our focus here is to compute the limits on black hole entropy and shear viscosity nonperturbatively in the Gauss–Bonnet coupling $\lambda_{\text{GB}}$ (or $\alpha' \equiv \lambda_{\text{GB}}/2$), so we hardly use the above relations, but they are useful for expressing our results in terms of $N_c$ and the AdS curvature $L$.

In fact, far from the classical limit ($N_c \to \infty$), or with $\alpha' L^2 > 0$, the ratio of shear viscosity to entropy in equation (1.2) is expected to be modified. Accordingly, the KSS bound $\eta/s \geq 1/(4\pi)$ could be violated at high energies or short distances, or in the presence of generic higher-derivative corrections, $\alpha' > 0$. The reason is simple to understand. In contrast to the entropy density $s$ which is the zeroth-order parameter in ideal fluid equations, the first-order transport coefficients in macroscopic hydrodynamic equations for non-equilibrium systems, such as shear viscosity $\eta$ and heat conductivity $\kappa$, can take a nearly vanishing value at high densities and temperatures. The ratio $\eta/s$ is perhaps not limited by any quantum bound.

In [21], it has been argued that in the very centre of the collision zone at RHIC, $\eta/s$ may take a value as small as $\eta/s \approx 0.4/(4\pi)$, which is less than the conjectured KSS bound. Further, in [22] it has been shown that, for a class of CFTs in flat space ($\epsilon = 0$) with Gauss–Bonnet gravity dual, the ratio $\eta/s$ is given by

$$\frac{\eta}{s} = \frac{1}{4\pi} \left( 1 - \frac{2(d-1)\lambda_{\text{GB}}}{(d-3)} \right),$$

where $\lambda_{\text{GB}} \equiv (d-3)(d-4)\alpha'$, which is smaller than $1/4\pi$ for $\lambda_{\text{GB}} > 0$. In the $d = 5$ case, $\lambda_{\text{GB}} < 1/4$ is required to keep $\eta/s$ positive definite. In [22], based on bulk causal structures

\footnote{In gravitational theories, the high-energy effects arise through higher-order curvature effects.}
of an AdS₅ black brane solution, a more stronger bound for \( \lambda_{\text{GB}} \) was proposed, namely

\[
\lambda_{\text{GB}} < \frac{9}{100}
\]

(1.5)

or equivalently

\[
\frac{\eta}{s} \geq \frac{16}{25} \left( \frac{1}{4\pi} \right),
\]

(1.6)

which otherwise violates a microcausality in the dual CFT defined on a flat hypersurface. In [22], in the context of Gauss–Bonnet gravity, it was found hard to understand the sudden change in behaviour at \( \lambda = \frac{9}{100} \). In this paper we provide a plausible explanation for this change in behaviour. We actually establish that the critical value of \( \lambda_{\text{GB}} \) beyond which the theory becomes inconsistent is related to the entropy bound for a class of AdS GB black holes.

We also find that, in the holographic context, the AdS-GB black hole solutions with spherical and hyperbolic event horizons \( (\epsilon = \pm 1) \) allow much wider possibilities for \( \eta/s \).

2. Gauss–Bonnet gravity and causality violation

Let us start with a pure AdS-GB black hole solution (i.e. without any electric or magnetic charge), for which the entropy and Hawking temperature are given by [14]

\[
S = \frac{A}{4G_N} \left( 1 + \frac{2(d-2)\epsilon\lambda_{\text{GB}}}{(d-4)x^2} \right), \quad T = \frac{(d-1)x^4 + \epsilon(d-3)x^2 + (d-5)x^2 \lambda_{\text{GB}}}{4\pi L x(x^2 + 2\epsilon \lambda_{\text{GB}})}
\]

(2.1)

(in units \( \hbar = k_B = 1 \)) where \( x \equiv r_s/L, A \equiv V_{d-2}r_+^{d-2}, \) with \( V_{d-2} \) being the unit volume of the base manifold or the hypersurface \( M \). The above entropy relation holds with an arbitrary \( \Lambda \), i.e. even if \( \Lambda = 0 \).

Note that, unlike in Einstein’s general relativity, the entropy of a GB black hole depends on the curvature constant \( \epsilon \), whose value determines the geometry of event horizon

\[
M = \begin{cases} 
S^{d-2}: & \text{Euclidean de Sitter space} \ (\epsilon = +1) \\
\mathbb{R}^{d-2}: & \text{Ricci flat space} \ (\epsilon = 0) \\
H^{d-2}: & \text{Euclidean Anti de Sitter space} \ (\epsilon = -1).
\end{cases}
\]

(2.2)

The hypersurface \( M \) is related to the \( M' \) on which the dual field theory is defined only by a rescaling of the metric. As a consequence, the Hawking temperature of a boundary conformal field theory can be different by some constant, say \( N_* \), which specifies the speed of light of the boundary theory. In the particular case where \( \epsilon = 0 \), the Hawking temperature is found to be proportional to black hole horizon size, namely

\[
T_{\text{CFT}} = N_* \frac{(d-1)r_+}{4\pi L^2}.
\]

(2.3)

Particularly, on a flat hypersurface at \( r = \infty \), we have (see equation (2.8) below)

\[
f(r) \to \frac{r^2}{a^2L^2}, \quad \text{with} \quad a^2 = \frac{1}{2}(1 + \sqrt{1 - 4\lambda_{\text{GB}}}).
\]

(2.4)

If we choose \( N_* = a \), then the boundary speed of light is unity. In the \( \epsilon = 0 \) case, we then find \( s \equiv S/V \sim \frac{1}{4\lambda_{\text{GB}}}r_+^{d-2} \sim \frac{x^2}{2}N_*^2T^{d-2} \). However, such a scaling relation does not hold, in general, with \( \alpha' \) corrections, especially, when \( \epsilon = \pm 1 \). This behaviour can be seen also

6 Classical properties of Gauss–Bonnet black holes with Maxwell-type electric or magnetic charges will be briefly discussed in the appendix.
Figure 1. The Hawking temperature as a function of $x$ (left plot) with $d = 5, \lambda_{GB} = 0.1$ and (right plot) with $d = 6, \lambda_{GB} = 0.25$. The solid (green), short-dash (purple) and long-dash (red) lines correspond, respectively, to $\epsilon = +1$, $\epsilon = 0$ and $\epsilon = -1$.

Figure 2. The Hawking temperature versus entropy with $\epsilon = +1$ (spherical black hole). (Left plot) $d = 5, \lambda_{GB} = 1/12, 0.15, 0.25$ (top to bottom) and (right plot) $d = 6, \lambda_{GB} = 0.135, 0.25, 0.5$ (top to bottom). In each plot the dotted line corresponds to $\lambda_{GB} = 0$.

The Hawking temperature roughly scales in between $r_+ / \pi L^2$ and $r_+ / \pi L^2 \lambda_{GB}$. When $\epsilon = +1$, the entropy density is $s = S / V = \frac{1}{4G_5} \left( r_+^3 + 6\lambda_{GB} r_+ L^2 \right)$, which increases with $T$ more rapidly (as compared to that in Einstein gravity). It is quite plausible that the ratio $\eta / s$ becomes less than $1/4\pi$ in the regime where the Gauss–Bonnet contribution to entropy exceeds (or becomes comparable) to that of the Einstein–Hilbert term.

We also note that, for $\epsilon = +1$, the Hawking temperature of an $AdS_5$ GB black hole is always less than that of a Schwarzschild black hole, for a given (fixed) entropy. Also, the entropy vanishes as $T \to 0$. In the $d = 6$ case, the Hawking temperature not only exceeds that of a Schwarzschild black hole, at a small $r_+$, but it also diverges as $S \to 0$ (see figure 2). From this observation, we can argue that the $d = 5$ case is the most relevant one.

A particularly interesting case is $\epsilon = -1$, for which the Hawking temperature of an $AdS$-GB black hole diverges to $-\infty$ as $x \to 0$. But in this limit the black hole entropy also diverges to $-\infty$. Thus, in order to properly understand the behaviour of temperature at a small value of $r_+ / L$, which corresponds to a large $z \equiv r / r_+$ or infrared limit of a dual
Figure 3. The Hawking temperature versus entropy with $\epsilon = -1$. (Left plot) $d = 5, \lambda_{GB} = 0.0834, 0.25$ and (right plot) $d = 6, \lambda_{GB} = 0.138, 0.25$ (top to bottom).

CFT, one should actually study the behaviour of temperature as a function of entropy. The plots in figure 3 show that, for $\epsilon = -1$, the entropy $S$ can have an extremum as a function of Hawking temperature. In this case, beyond a critical coupling $\lambda_{GB} > \lambda_{crit}$ the entropy $S$ becomes negative at zero Hawking temperature, indicating a violation of cosmic censorship or the second law of the thermodynamics. In the AdS$_5$ case, we find that $\lambda_{crit} = 1/12$. It is not coincident that this critical value of $\lambda_{GB}$ above which the theory is inconsistent nearly coincides with the bound $\lambda_{GB} < 0.09$ required for a consistent formulation of a class of CFTs in flat space with Gauss–Bonnet gravity dual.

2.1. Shear viscosity for Gauss–Bonnet gravity

The effect of Gauss–Bonnet coupling $\lambda_{GB}$ on shear viscosity $\eta$ can be studied by considering small metric fluctuations $\phi = h_{1/2}$ around an AdS black hole solution of the form

$$ ds^2 = -f(r)N^2 dt^2 + \frac{1}{f(r)} dr^2 + \frac{r^2}{L^2} \left( \frac{dx_3^2}{1 - \epsilon x_3^2} + \sum_{i=1}^{2} dx_i^2 + 2\phi(t, x_3, r) dx_1 dx_2 \right) $$

(2.6)

where $\epsilon = 0, \pm 1$ and

$$ f(r) = \epsilon + \frac{r^2}{L^2 2\lambda_{GB}} \left[ 1 \pm \sqrt{1 - 4\lambda_{GB} + \frac{4\lambda_{GB} r^2}{r^4}} \left( 1 + \frac{\epsilon L^2}{r_+^2} + \frac{\lambda_{GB}\epsilon L^4}{r_+^2} \right) \right]. $$

(2.7)

Although there are two distinct vacuum solutions, we shall consider only the negative root in equation (2.7) which has a smooth limit to Einstein gravity, $\lambda_{GB} \to 0$. Note that as $r \to \infty$, $f(r) \to \frac{r^2}{\tilde{a}^2 L^2}$, $\frac{1}{\tilde{a}^2} = \left[ \frac{\epsilon}{z^2} + \frac{2}{1 + \sqrt{1 - 4\lambda_{GB}}} \right]$, (2.8)

where $\tilde{\epsilon} \equiv \epsilon/x^2, x \equiv r_+/L$ and $z \equiv r/r_+$. To allow a black hole interpretation, the GB coupling must satisfy $\lambda_{GB} \leq 1/4$ (see, e.g., [14, 23]); beyond this value, the above solution does not admit a consistent vacuum AdS solution (see the appendix).

In the $\epsilon = \pm 1$ case, the curvature of the boundary or the hypersurface $S^3$ (or $H^3$) introduces a new scale into the problem. However, the result in flat space is reproduced as a limit, which

7 For an electrically charged AdS Gauss–Bonnet black hole, these two branches of solutions may actually join together without developing a physical singularity (see the appendix).
can be characterized as the high temperature limit \((z \to \infty)\). To be specific, we make the following ansatz:
\[
\phi(t, x, r) = \int \frac{dw \, dq}{(2\pi)^3} \phi(r; k) e^{-iwr + iqx}, \quad \phi(r; -k) = \phi^*(r, k)
\]  
(2.9)

\((where \(k = (w, 0, 0, q))\). The quadratic action for \(\phi\) takes the form (up to the surface terms)
\[
\delta^2 I \propto \int dz \frac{dw \, dq}{(2\pi)^3}(K(\partial z \phi)^2 - K_2 \phi^2),
\]
(2.10)

where
\[
K = z^2 \tilde{f}(z - \lambda_{GB} \partial_z \tilde{f}), \quad K_2 = \frac{z^2 \bar{\omega}^2}{N_z^2 \tilde{f}} (z - \lambda_{GB} \partial_z \tilde{f}) - z(1 - \lambda_{GB} \partial_z \tilde{f})(\tilde{q}^2 + 2\tilde{\epsilon})
\]

and
\[
\tilde{f}(z) = \frac{L^2}{r^2} \tilde{f}(r) = \tilde{\epsilon} + \frac{z^2}{2\lambda_{GB}} \left( 1 - \sqrt{1 - 4\lambda_{GB}^2} + \frac{4\lambda_{GB}^2}{z^4} \left( 1 + \tilde{\epsilon} + \lambda_{GB} \tilde{\epsilon}^2 \right) \right),
\]
(2.12)

where \(\tilde{\epsilon} = \epsilon/x^2, x \equiv r/L, z \equiv r/r_*, \tilde{\omega} \equiv wL/x, \tilde{q} \equiv qL/x\) and \(\tilde{f} = f/x^2\). The equation of motion following from (2.10) is
\[
K \partial_z^2 \phi + \partial_z K \partial_z \phi + K_2 \phi = 0.
\]
(2.13)

Especially, for \(\epsilon = 0\), in the limit \(z \to 1\), we find
\[
K_2 \left( \frac{\bar{\omega}^2}{N_z^2 \tilde{f}} - \frac{\lambda_{GB} \tilde{q}^2 + 2\tilde{\epsilon}}{a^2 z^4} \right) \left( \frac{1}{z - 1} + O(1) \right),
\]
(2.14)

\[
\frac{\partial_z K}{K} = \frac{1}{z - 1} + \frac{5 + 24\lambda_{GB}}{2} + O(z - 1).
\]

For \(\epsilon \neq 0\), the limiting values of these functions are regular at \(z = 1\), but their explicit expressions will not be important in the present discussion.

From the viewpoint of gravity–gauge theory duality, it is instructive to study the large \(z\) behaviour of AdS solutions. In the limit \(z \to \infty\), we find
\[
K_2 = \frac{\bar{\omega}^2 \lambda_{GB}^2}{N_z^2 a^4 z^4} - \frac{\lambda_{GB} \tilde{q}^2 + 2\tilde{\epsilon}}{a^2 z^4} + O(z^{-6}),
\]
(2.15)

\[
\frac{\partial_z K}{K} = \frac{5}{z - a^2 z^3} + O(z^{-5}),
\]

where \(a^2 \equiv \frac{1}{2}(1 + \sqrt{1 - 4\lambda_{GB}^2})\). The effect of the curvature \(\tilde{\epsilon}\) on a dual field theory can appear only as a small correction to the results in flat space \([22]\). To be precise, one solves equation (2.13) with the boundary condition
\[
\phi(z; k) = a_{in}(k)a_{in}(z; k) + a_{out}(k)\phi_{out}(z; k),
\]
(2.16)

\(a_{out} \equiv 0\) and \(a_{in} \equiv J(k)\), where \(J(k)\) is an infinitesimal boundary source for the fluctuating field. Following \([22]\), we make the following low-frequency expansion:
\[
\phi_{in}(z; w, q) = \exp \left[ -i\tilde{\omega} \frac{\ln \left( \frac{\bar{\omega}^2 \tilde{f}}{z^2} \right)}{4N_z} \right] \times \left( 1 - i\tilde{\omega} g_1(z) + O(\tilde{\omega}^2, \tilde{\omega}^3) \right).
\]
(2.17)

\(^8\) Especially, in \(d = 5\), the linearized equation of motion can be expressed in this simple form. In dimensions \(d \geq 6\), various functions \((K, K_2)\) receive extra contributions, being proportional to \((\tilde{\epsilon} - \tilde{f})/(\partial_z \tilde{f})\); see also \([24]\) for a discussion of the gravitational instability of six-dimensional asymptotically flat EGB black holes.

\(^9\) At this point, one also notes the relation \(\partial_z \tilde{f} = [4\epsilon^3 + 2z(\tilde{\epsilon} - \tilde{f})]/[\epsilon^2 + 2\lambda_{GB}(\tilde{\epsilon} - \tilde{f})]\).
Note that as \( z \to \infty \),
\[
\frac{\tilde{a}^2 j}{z^2} \to 1 - \frac{C}{\sqrt{1 - 4\lambda_{GB} z^6}} + O(1/z^8),
\]
where \( \tilde{a}^2 \equiv \left( \frac{\tilde{\epsilon}}{z^2} + \frac{1}{a^2} \right)^{-1} \).

\[ \ln \left( \frac{\tilde{a}^2 j}{z^2} \right) \to -\frac{C a^2}{\sqrt{1 - 4\lambda_{GB} z^6}} + \frac{C \tilde{\epsilon} a^2}{\sqrt{1 - 4\lambda_{GB} z^6}} + O(1/z^8), \]

where \( C \equiv 1 + \tilde{\epsilon} + \lambda_{GB} \tilde{\epsilon}^2 \). To find the large \( z \) behaviour of \( g_1(z) \), one effectively solves the equation \( K \phi'' + \partial_z K \phi' = 0 \), which yields
\[ g_1(z) = \frac{(C - 1 + 4\lambda_{GB}) a^2}{\sqrt{1 - 4\lambda_{GB} z^6}} - \frac{C \tilde{\epsilon} a^2}{\sqrt{1 - 4\lambda_{GB} z^6}} + \frac{2\lambda_{GB} \tilde{\epsilon} \sqrt{1 - 4\lambda_{GB}}}{3z^6} + O(1/z^8). \]

Substituting this value of \( g_1(z) \), along with equation (2.18), into equation (2.17), we can see that the effect of curvature on \( \phi(z; k) \) is only sub-leading in the limit \( z \to \infty \). More precisely,
\[
\phi(z; k) = J(k) \left[ 1 + \frac{i\tilde{\epsilon}}{4N_c} a^2 \sqrt{1 - 4\lambda_{GB}} \left( \frac{1}{z^2} - \frac{4\lambda_{GB} \tilde{\epsilon}}{3(1 + \sqrt{1 - 4\lambda_{GB}})} \frac{1}{z^6} + O(1/z^8) \right) + O(\tilde{\epsilon}^2) \right].
\]

From this we can see that the curvature on a boundary does not affect the shear viscosity
\[ \eta = \frac{1}{16\pi G_N} \left( \frac{r_s^2}{L^3} \right) (1 - 4\lambda_{GB}) \]

obtained by the Kubo formula
\[ \eta = \lim_{w \to 0} \frac{1}{2i w} \left[ G^A_{12,12}(w,0) - G^R_{12,12}(w,0) \right] \equiv \lim_{w \to 0} \frac{1}{i w} \Im G^R_{12,12}(w,0), \]

which relates \( \eta \) to zero spatial momentum \( (q = 0) \), low frequency limit of the retarded two-point Green’s function\(^{10}\)
\[ G^R_{12,12}(w,0) = -i \int d^4x e^{iwz} \theta(t) \left[ \{ T_{12}(t,\bar{z}), T_{12}(0,\bar{0}) \} \right] = i\eta w + O(w^2) \]

(modulo contact terms) and \( G^A(w,\vec{q}) \equiv G^R(w,\vec{q})^* \). One also notes that, in the limit \( \lambda_{GB} \to 1/4 \), the retarded Green’s function in AdS spacetimes do not receive any corrections from the massive Kaluza–Klein modes. This result is consistent with the discussion in [20], in reference to Randall–Sundrum-type warped braneworld models.

Finally, the ratio \( \eta/s \) is given by\(^{11}\)
\[ \frac{\eta}{s} = \frac{1}{4\pi} \left( 1 - 4\lambda_{GB} \right). \]

\(^{10}\)AdS/CFT relates every field in supergravity (or gravity solutions in AdS spaces) to a corresponding gauge invariant operator of dual gauge theory, such as SYM theory [5]. In particular, the two-point correlation function of \( T_{12} \) corresponds to scalar fluctuations \( \phi' \) in gravity, whose boundary values at infinity \( (z \to \infty) \) are \( \phi'_c \). The CFT partition function corresponds to the AdS action, \( \exp \left[ \oint \phi_c(O) \right] = \exp \left[ -\int_{s=0}^{s=1} \sigma(d^4x) \right] \).

\(^{11}\)For EGB gravity, the on-shell action (which utilizes the equations of motion) reduces to surface contribution:
\[ I[\phi^\infty(z)] = -\frac{\sqrt{G_N}}{\sqrt{2\pi}} \int_{S^3} d^4x \left( K \partial_z \phi(z) + \cdots \right)_{\text{surface}} \] (see [22] for other details).

A decade ago, studies by Gubser–Klebanov–Polyakov/Witten [5] showed that AdS/CFT works well for spherically symmetric AdS black holes. Nevertheless, the Kubo formula could appear somewhat formal on \( S^3 \), the reason being that \( S^3 \) is a finite-size manifold. In turn, one could ask whether there exists a proper hydrodynamic limit for a class of CFTs defined on \( S^3 \). In that respect, hyperbolic black holes may introduce new and fruitful feature. Here we only assume that the Kubo formula is applicable to a slightly negatively curved spacetime, namely on \( \mathcal{H} \), but it would be really nice to check this point. Our expectation seems reasonable from the viewpoints that hyperbolic black holes are usually described as thermal Rindler states of the dual conformal field theory in flat space and the Kubo formula is a good approximation in the AdS asymptotic \( (z \to \infty) \) where the effect of curvature becomes negligibly small.
Given that $0 < \lambda_{GB} < 1/4$, for a hyperbolic AdS-GB black hole, the ratio $\eta/s$ can be larger than $1/4\pi$, while for a spherical AdS-GB black hole, $\eta/s$ may further be decreased (as compared to the $\epsilon = 0$ case [22, 28]). It seems possible to saturate the KSS bound $\eta/s \geq 1/4\pi$ in the $\epsilon = -1$ case; specifically, at some fixed value of $\tilde{\epsilon}$, namely $\epsilon = -1$ and $x = \sqrt{3/2}$, one finds $\eta/s = \frac{1}{4\pi}$. Of course, with $\epsilon = 0$, it also is possible to obtain $\eta/s > 1/4\pi$ by taking $\lambda_{GB} < 0$, but in this case the background solution may contain ghosts.

Although the result (2.21) is obtained in a high temperature limit, let us assume that in a thought experiment one can minimize the entropy of a given state. The minimum of entropy density occurs for the $\epsilon = -1$ extremal solution at $x = \sqrt{17/2}$, which is given by

$$s = \frac{1}{G_N} \frac{1}{2^{5/2}} (1 - 12\lambda_{GB}).$$

At this extremal state the shear viscosity is given by

$$\eta = \frac{1}{4\pi G_N} \frac{1}{2^{5/2}} (1 - 4\lambda_{GB}).$$

Hence, with $\lambda_{GB} \leq 1/12$ (as implied by the positivity of extremal entropy), we find

$$\eta/s > \frac{2}{3} \left( \frac{1}{4\pi} \right) \ (\epsilon = 0), \quad \eta/s \leq \frac{5}{3} \left( \frac{1}{4\pi} \right) \ (\epsilon = -1).$$

Remarkably, the lower bound $\eta/s \approx 0.66/4\pi$ is similar to a lower value of $\eta/s$ anticipated in the very centre of collision at RHIC (see, e.g., [21]).

The effect of curvature at $z \to \infty$ can be known also by studying the propagation speed of tensor modes on a constant $r$-hypersurface. In particular,

$$c_2^T(z) = \frac{N_s^2 f \lambda_{GB}}{z^2} \frac{(1 - \lambda_{GB} \partial^2_x f)(1 + 2\tilde{\epsilon}/\tilde{q}^2)}{1 - \lambda_{GB} \partial^2_x / z}$$

can be interpreted as the square of local speed of graviton on a constant $r$-hypersurface by identifying $N_s \equiv a$. The speed of graviton increases with the strength of the coupling $\lambda_{GB}$; this increase is maximum for a dual field theory defined on $\epsilon = +1$ hypersurfaces (see figure 4). In the $\epsilon = 0$ case, the square of local speed of light, $c_2^T \equiv \frac{N_s^2 f}{z}$, is always sub-luminal, but it can be super-luminal for $\epsilon \neq 0$ (see figure 5). This is due to a well-known fact that in the presence of higher curvature terms the graviton wave packets in general do not propagate on the light cone of a given background geometry.

2.2. Black hole thermodynamics and entropy bound

We now briefly discuss the thermodynamics properties such as free energy and entropy of AdS Gauss–Bonnet black holes. Indeed, spherical AdS black holes present a different qualitative feature, namely a phase transition at finite temperatures. This can be seen, for instance, by considering the free energy of a spherically symmetric AdS$_5$ GB black hole in $d$-spacetime dimensions, the free energy is given by (see, e.g., [26])

$$F = \frac{V_{d-2}(xL)^{d-3}}{16\pi G_N} \left[ (\epsilon - x^2 + \frac{(d - 2) \lambda_{GB}}{(d - 4)} x^2 \right] - \frac{\epsilon \lambda_{GB}(2x^2 + \epsilon)}{(d - 4)} x^2 + \epsilon \lambda_{GB} \right] - M_{extr} \delta_{\epsilon, -1}.$$ 

\[\text{Eq. (2.29)}\]

\[\text{In } d = 5, \text{ the location of the extremal horizon of an AdS GB black hole does not depend on the strength of the coupling } \lambda_{GB}.\]

\[\text{In the } \epsilon = +1 \text{ case, the lower bound on } \eta/s \text{ could be very close to that in flat space which, however, arises as a consequence of boundary causality.}\]
where the extremal ADM (Arnowitt–Deser–Misner) mass $M_{\text{extr}}$, which is nonzero for $\epsilon = -1$, is given by

$$M_{\text{extr}} \equiv \frac{(d - 2)V_d L^2}{16\pi G_N} \mu_{\text{extr}} = \frac{(d - 2)V_d L^2}{16\pi G_N} L_{d-3} \left[ (\epsilon + x^2) x^{d-3} + \epsilon^2 \lambda_{\text{GB}} x^{d-5} \right]_{x = x_{\text{extr}}},$$

(2.30)

where, as above, $x \equiv r_s/L$. For a definiteness, we shall take $d = 5$. This choice is again motivated from AdS/CFT correspondence [4]. In the limit $x \to 0$, we find

$$F_{\epsilon = 1} = \frac{3V_3 L^2}{16\pi G_N} \lambda_{\text{GB}}.$$

(2.31)

This corresponds to the free energy of a boundary field theory defined on $S^3$. However, for $\epsilon = 0$ and $\epsilon = -1$, the free energy always vanishes in the limit $x \to 0$. In $d = 5$, and with $\lambda_{\text{GB}} < 1/4$, the total thermodynamic energy [26]

$$E = TS + F = \frac{3V_3 L^2}{16\pi G_N} \left[ x^4 + \epsilon x^2 + \lambda_{\text{GB}} \epsilon^2 + \frac{1}{4} (1 - 4\lambda_{\text{GB}}) \delta_{\epsilon, -1} \right]$$

(2.32)

is a positive concave function of the black hole’s temperature for all values of $\epsilon$ ($= 0, \pm 1$).
Next, we consider the change in entropy, which is given by
\[
\frac{dS}{dr_+} = \frac{(d-2)V_{d-2}r_+^{d-1}}{4\pi G_N} \left( 1 + \frac{2\lambda_{GB}\epsilon}{\lambda^2} \right).
\] (2.33)

With \(\lambda_{GB} > 0\), \(dS/dr_+\) is always positive for \(\epsilon \geq 0\). In fact, \(dS/dr_+\) is positive also for \(\epsilon = -1\), since the minimal horizon of a GB black hole is \(\lambda_{\text{min}}^2 = -2\epsilon\lambda_{GB}\). Nevertheless, at small distances, like \(x < \sqrt{T/2}\), the temperature of a spherically symmetric Gauss–Bonnet black hole becomes larger than that of a Schwarzschild black hole, at a given (fixed) entropy, indicating a classical instability of the system. An earlier study by Odintsov and Nojiri in [27] showed that the negative entropy state corresponds to the maximum rather than to the minimum of Euclidean action or free energy. Although the discussion in [27] was restricted to \(c = 0\) in equation (1.3), similar arguments hold for \(c \neq 0\), or a nonzero \(\lambda_{GB}\).

When \(\epsilon = -1\), there exists an extremal solution with zero Hawking temperature at \(r_+ = r_{\text{extr}} = \frac{\lambda}{\sqrt{2}}\) and \(\mu_{\text{extr}} = \frac{\lambda^2}{4}(4\lambda_{GB} - 1)\). It is easy to check that
\[
S|_{T \to 0} = \frac{V_3}{G_N^{2/3}} \frac{L^3}{27/2} (1 - 12\lambda_{GB}), \quad E|_{T \to 0} = 0.
\] (2.34)

The positivity of entropy requires \(\lambda_{GB} < 1/12\); when \(\lambda_{GB} > 1/12\), the solutions may violate the second law (of black hole thermodynamics), rendering the theory inconsistent. This observation is consistent with a recent discussion in [22], where, for a class of \(3+1\)-dimensional CFTs in flat space, the GB gravity violates a micro-causality for \(\lambda_{GB} > 0.9\); particularly, the local speed of graviton can exceed the speed of light in this limit.

We also note that, for a \((3 + 1)\)-dimensional CFT duals of \((4 + 1)\)-dimensional Gauss–Bonnet gravity with \(\epsilon = -1\), the consistency of the theory may require
\[
\lambda_{GB} < 1/12, \quad \frac{\eta}{s} \geq 2/3 \left( \frac{1}{4\pi} \right),
\] (2.35)
which puts a 2.6% weaker limit on \(\eta/s\) than in [22] but a 33.4% stronger limit than the KSS bound. When \(d = 6\) and \(d = 7\), the consistency of Gauss–Bonnet gravity requires, respectively,
\[
\lambda_{GB}^{d=6} \leq 0.1380, \quad \lambda_{GB}^{d=7} \leq 0.1905.
\] (2.36)

Similar results hold with (Riemann)\(^2\) corrections to classical Einstein gravity.

### 3. Entropy bound for (Riemann)\(^2\) gravity

Another interesting theory of extended gravity is obtained by setting \(a = b = 0\) in equation (1.3). When \(d = 5\), the action (1.3) corresponds to an effective AdS\(_5\) action deduced from a 10d heterotic string theory via heterotic type I duality [13]
\[
I = \frac{N^2}{4\pi^2 L^3} \int d^5x \left( R + \frac{12}{L^2} + \frac{L^2}{16N_c} R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} \right),
\] (3.1)
where, using the AdS/CFT dictionary, the coefficient of (Riemann)\(^2\) term has been fixed as \(\alpha' L^5 = L^2/(16N_c) = \lambda_{\text{Riem}} L^2/2\) and \(N^2_c = \pi L^3/4G_N\) for AdS\(_5\). Although, at a linearized level, the effective theory defined by (3.1) behaves differently from that defined by Gauss–Bonnet action, for instance, in terms of effective degrees of freedom or graviton propagators, they both modify the corresponding dual field theory variables including the ratio \(\eta/s\) in a qualitatively similar way.

To be specific, we consider the black hole entropy and Hawking temperature of a (Riemann)\(^2\) corrected AdS\(_5\) black hole, which are given by [14] (see also [29, 30])...
Figure 6. Hawking temperature versus entropy. Left plot: $\epsilon = 0$ and $\lambda_{\text{Riem}} = 0, 0.15, 0.3, 0.4$ (top to bottom). Right plot: $\epsilon = +1$ and $\lambda_{\text{Riem}} = 0, 0.05, 0.1, 0.15$ (top to bottom).

\[ S = \frac{A}{4G_N} \left[ 1 + 2\lambda_{\text{Riem}} (2 + 3\tilde{\epsilon}) + O(\lambda_{\text{Riem}}^2) \right], \]

\[ T = \frac{x}{2\pi L} \left[ 2 + \tilde{\epsilon} - 2\lambda_{\text{Riem}}\tilde{\epsilon}^2 (1 + \tilde{\epsilon})^2 + O(\lambda_{\text{Riem}}^2) \right], \]

where $A \equiv V_3r^3$ and, as above, $\tilde{\epsilon} \equiv \epsilon/x^2$. Note that, with $|\lambda_{\text{Riem}}| > 0$, the entropy is not simply given by one-quarter the area even for black brane solutions ($\epsilon = 0$), indicating a breaking of conformal symmetry of the boundary field theory. This also implies that a bulk gravity other than with Einstein–Hilbert and Gauss–Bonnet gravity actions may not have a well-defined boundary CFT dual; with $(\text{Riemann})^2$ terms, the boundary conditions alter the symmetry algebras of a pure AdS space or of the general relativity.

Following the perturbative metric solution found in [14] and the methods discussed in [22, 31] for calculating shear viscosity, we find

\[ \eta = \frac{r_3N^2}{4\pi^2L^3} \left( 1 - 4\lambda_{\text{Riem}} + O(\lambda_{\text{Riem}}^2) \right). \]

The entropy density of a $(\text{Riemann})^2$ corrected black hole is given by [14, 25]

\[ s = \frac{r_3N^2}{\pi L^3} \left( 1 + 4\lambda_{\text{Riem}} \left( 1 + \frac{3}{2}\tilde{\epsilon} \right) + O(\lambda_{\text{Riem}}^2) \right). \]

From these results we can infer that for a class of boundary field theories in flat space ($\epsilon = 0$) with $(\text{Riemann})^2$-gravity dual, the ratio $\eta/s$ (up to the linear term in $\lambda_{\text{Riem}}$) is

\[ \frac{\eta}{s} = \frac{1}{4\pi} \left( 1 - 4\lambda_{\text{Riem}} \right) \approx \frac{1}{4\pi} (1 - 8\lambda_{\text{Riem}}). \]

Our estimation of $\eta/s$ is different from that in [22] by a factor of $2^{14}$ but it agrees with the final result (equation (5.6)) in [31].

In the $\epsilon = 0$ and $\epsilon = +1$ cases, the Hawking temperature of a $(\text{Riemann})^2$-corrected black hole is always found to be less than that of a Schwarzschild black hole, at a given entropy. Also, the black hole entropy is non-negative in the limit $T \to 0$ (see figure 6). In the $\epsilon = -1$ case, however, the entropy becomes negative at zero Hawking temperature, especially, for $\lambda_{\text{Riem}} > 0.15$ (see also figure 7), indicating a possible violation of the second law of thermodynamics.

\[ 14 \text{ The source of this difference is that in our discussion the entropy of a (Riemann)$^2$-corrected black hole is not given by one-quarter the area even in the } \tilde{\epsilon} = 0 \text{ case; the coupling } \lambda_{\text{Riem}} \text{ here is related to the coefficient of (Riemann)$^2$-term } (\alpha_3) \text{ introduced in [22] via } \lambda_{\text{Riem}} = 2\alpha_3. \]
With $\epsilon = -1$, the Hawking temperature vanishes at $x \simeq \sqrt{\frac{1}{2}(1 + \lambda_{\text{Riem}})}$. The positivity of extremal black hole entropy density

$$s_{(\text{AdS})}\big|_{T \to 0} = \frac{1}{G_N} \frac{L^3}{2^{7/2}} (1 - 8\lambda_{\text{Riem}}),$$

which now requires $\lambda_{\text{Riem}} < 1/8$, ensures that the total thermodynamic energy [25]

$$E = \frac{3V_5L^2}{16\pi G_N} \left[ x^2(\epsilon + x^2) + \lambda_{\text{Riem}}(\epsilon + x^2)(\epsilon + 3x^2) + \frac{1}{4}(1 + \lambda_{\text{Riem}}) \right]$$

is non-negative. For a class of CFTs in flat space with (Riemann)$^2$ gravity dual, we find

$$\frac{\eta}{s} = \frac{1}{4\pi} \left( \frac{N_c - 1/2}{N_c + 1/2} \right) \simeq \frac{1}{4\pi} \left( 1 - \frac{1}{N_c} \right).$$

In the limit $N_c \to \infty$, one has $\eta/s \approx 1/4\pi$, but $\eta/s < 1/4\pi$ for a finite $N_c$. However, it does not mean that for a class of CFTs with (Riemann)$^2$ gravity dual, the ratio $\eta/s$ is always smaller than $1/4\pi$. In fact, the minimum of entropy density (and hence the maximum of $\eta/s$) occurs for the $\epsilon = -1$ extremal solution at $x \simeq \sqrt{(1 + \lambda_{\text{Riem}})/2}$, which is given by

$$\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 + 4\lambda_{\text{Riem}} - O(\lambda_{\text{Riem}}^2) \right].$$

Taking into account all three possibilities for the boundary topology that $\epsilon = 0$ or $\epsilon = \pm 1$, we find that consistency of (Riemann)$^2$ gravity requires

$$0 < \frac{\eta}{s} \leq \frac{3}{2} \left( \frac{1}{4\pi} \right).$$

It is an open question whether either of these limits applies to nuclear matter at extreme densities and temperatures, or heavy ion collision experiments exhibiting perfect fluid behaviour. Nevertheless, it is intriguing that a general consideration based on AdS black hole solutions with curvature square corrections provides such bounds on $\eta/s$.

We end this section with a couple of remarks. In particular, in the AdS$_5 \times S^5$ string background dual to $\mathcal{N} = 4SU(N_c)$ SYM gauge theory at strong coupling ($g_{\text{YM}}^2 N_c \to \infty$),

$$15 \text{ Unlike for Gauss–Bonnet gravity, the location of the extremal horizon of a (Riemann)$^2$ corrected AdS}_5 \text{ black hole depends on the coupling } \lambda_{\text{GB}}.$$
the ratio $\eta/s$ has been found to increase once the $\alpha'^3 R^4$ terms are added in the effective action of type IIB string theory, as implied by [32, 33]

$$\frac{\eta}{s} = \frac{1}{4\pi} \left( 1 + \frac{15\zeta(3)}{\lambda^{3/2}} + \cdots \right),$$

where $\zeta(3) = 1.202$. Even if this result is correct (including the precise coefficient and the sign of the correction), the conjectured KSS bound $\eta/s \geq 1/4\pi$ may be violated at a finite coupling, since the $\alpha'^3 R^4$-type corrections are suppressed relative to the $1/N_c$ corrections arising from (Riemann)$^2$ terms. One may argue that for a particular version of string theory, for example, type IIB string theory, the $R^2$-type corrections could be absent due to some supersymmetric conditions and hence the KSS bound $\eta/s \geq 1/4\pi$ may hold. Nevertheless, one cannot deny the role of $R^2$-type corrections in a full quantum theory of gravity, with all possible higher-derivative and higher-order curvature contributions. It would be interesting to check a universality of the result like $\eta/s \approx 1/4\pi$ through numerical hydrodynamic simulations of data from RHIC and LHC.

4. Conclusion

The Gauss–Bonnet gravity with a coupling $\lambda_{GB} < 1/4$ could be viewed as a classical limit of a consistent theory of quantum gravity. Such a theory is consistent with the prediction of some low-energy effective superstring models or the supergravity approximation of string theory. In a full quantum theory of gravity, with all possible higher-derivative and higher-order curvature contributions, though, we cannot find such explicit bounds for the couplings—the solutions can be known only at perturbative level. We must instead find new conditions strong enough to ensure the presence of a black hole, but weak enough to be allowed by the positivity of (extremal) entropy and the total thermodynamic energy.

It is conceivable that the bound on the shear viscosity of any fluid in terms of its entropy density is saturated, $\eta/s = 1/4\pi$, for gauge theories at large 't Hooft coupling, which correspond to the cases where all higher-order curvature contributions are absent. But this bound is naturally in immediate threat of being violated in the presence of generic higher-derivative or higher-order curvature corrections to the Einstein–Hilbert action. It is remarkable that by tuning of the Gauss–Bonnet coupling, the ratio $\eta/s$ can be adjusted to a small positive value or even to zero.

In our work, limits on $\lambda_{GB}$ are imposed by demanding the positivity of extremal black hole entropy and non-violation of boundary causality. Results such as (2.25) may receive nontrivial corrections in the presence of further higher-derivative terms, such as quartic terms in Weyl tensors. Similar remarks could be made in regard to (3.7). However, this kind of reasoning cannot be used to minimize or dim the significance of extended gravity theories with Gauss–Bonnet or (Riemann)$^2$ terms alone. We have not found any obvious explicit bound on $\lambda_{GB}$ from the thermodynamics of spherically symmetric AdS Gauss–Bonnet black holes, which may, however, arise as a consequence of boundary causality. What we find really interesting is that a critical value of $\lambda_{GB}$ beyond which the theory becomes inconsistent is related to the entropy bound for a class of AdS GB black holes with a hyperbolic or Euclidean anti-de Sitter event horizon. We have come to similar outcomes for Riemann squared gravity. In our discussions, inconsistency of the (Gauss–Bonnet and Riemann squared gravity) theory, such as a violation of micro-causality, has been shown to be related to a classical limit on black hole entropy.

Our results also indicate an interesting connection between the thermodynamics of black hole horizons and the quadratic Euler invariants present in extended theories of gravity, where
the relations $S = |\beta|(E - F)$ and $dS = \beta dE$ may have a greater domain of validity than that of classical Einstein gravity; see also [34] for related discussions. An interesting open question is whether Hawking radiation and black hole evaporation can also fit into extended gravity theories. It would be interesting to know how generic higher-derivative corrections modify various (dual) gauge theory observables both at finite and strong coupling limits. If similar bounds on coupling constants can be found for spinning AdS-GB black holes and/or generalized Gauss–Bonnet black holes in the presence of Maxwell charges or electromagnetic field strengths in the matter action, it would represent important progress.

In [35], it has just been reported that, in the presence of a GB term and a Maxwell-type charge $q$, the ratio $\eta/s$ is given by $4\pi(\eta/s) = 1 - 4\lambda_{\text{GB}}(1 - a/2)$, where $a \equiv q^2 L^6/r_2^2$. In the extremal limit ($a \to 2$), one can restrict $\lambda_{\text{GB}}$ such that $\lambda_{\text{GB}} \leq 1/24$, the latter ensures that the gravitational potential of a black brane is positive and bounded. The bound $4\pi(\eta/s) \geq 5/6$ reported in [35] for a nonzero charge is somewhat stronger than for pure EGB gravity in flat space, namely $4\pi(\eta/s) \geq 2/3$.

One of the main motivations for studying higher-curvature or higher-derivative corrected AdS black holes is gauge theory–gravity duality or more generally the implications of AdS black holes to dual CFTs as well as the thermal transport properties of low-energy QCD; the latter provide a real-time dynamics at strong coupling limits. It is perhaps true that limits on higher curvature couplings can be placed also by studying de Sitter (black hole) spacetimes, but it is not understood yet whether any de Sitter GB black holes would find direct application to the real-time dynamics of dual QFTs. Studying the GB black holes in de Sitter (dS) spaces has some significance in its own right. Indeed, for de Sitter GB black holes, there could arise a cosmological event horizon in addition to a black hole horizon. In turn, one would expect a separate set of thermodynamic variables for the cosmological horizon. In general, the set of thermodynamics quantities, for instance, the entropy densities, associated with the black hole horizon and cosmological horizon are not equal. As a consequence, the spacetime for a GB black hole in dS space may not be stable semi-classically. Also, an ambiguity could arise as to whether the thermal transport properties of low-energy QCD correspond to that defined on cosmological or black hole horizons. Nevertheless, we find some earlier discussions in [36] particularly encouraging, which indicate that the entropy of a de Sitter GB black hole become negative for $\lambda_{\text{GB}} < 1/12 < 1/4$, leading to a possible violation of unitarity in this range. This suggests some deeper connections between dS and AdS Gauss–Bonnet black hole solutions.

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Appendix A. Charged AdS Gauss Bonnet black holes

Consider the $d$-dimensional Einstein–Gauss–Bonnet gravitational action

$$I_g = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \left[ R + \frac{(d - 1)(d - 2)}{L^2} + \alpha' L^2 \mathcal{R}^2 \right],$$

(A.1)

where $\mathcal{R}^2 = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$, along with the standard Maxwell-type action

$$I_{\text{EM}} = -\frac{1}{4} \int d^d x \sqrt{-g} F_{ab} F^{ab},$$

(A.2)
Figure 8. The metric potential $\tilde{f}$ as a function of $z$ with $Q^2 = 0, \lambda_{\text{GB}} = 0.1, d = 5$ (left plot), $d = 6$ (right plot). The solid (green), short-dash (purple) and long-dash (red) lines correspond, respectively, to $\epsilon = +1, \epsilon = 0$ and $\epsilon = -1$. In $d = 5$, $\tilde{f}(z)$ is regular for all values of $z$. In $d = 6$, $\tilde{f}(z)$ diverges to $+\infty (-\infty)$ for $\kappa = +1 (-1)$ branch, as $z \to 0$.

Figure 9. The function $\tilde{f}(z)$ with $\lambda_{\text{GB}} = 0.245, Q^2 = 0.01$ (or $P^2 = 0.01$), $d = 5$ (left plot) and $d = 6$ (right plot). The solid (green), short-dash (purple) and long-dash (red) lines correspond, respectively, to $\tilde{\epsilon} = +1, \tilde{\epsilon} = 0$ and $\tilde{\epsilon} = -1$. For an electrically charged black hole, the $\kappa = \pm 1$ branches join at $z \ll 1$, while, for a magnetically charged black hole, they are disjoint; in the latter case, $\tilde{f}(z)$ diverges to $+\infty (-\infty)$ for $\kappa = +1 (\kappa = -1)$ branch, as $z \to 0$.

In any $d$-dimensional AdS spacetimes admitting $(d - 2)$-dimensional Euclidean subspace

$$h^E = h^E_{\mu\nu}dx^\mu dx^\nu = \frac{d\chi^2}{1 - \epsilon \chi^2} + \chi^2d\Omega^2_{d-3}$$  \hspace{1cm} (A.3)

with the constant curvature $\epsilon = 0, +1, -1$, the solution of Einstein field equations following from the total action, $I_g + I_{\text{EM}}$, is isometric to

$$ds^2 = -f(r)dr^2 + \frac{dr^2}{f(r)} + r^2h^E_{\mu\nu}dx^\mu dx^\nu,$$  \hspace{1cm} (A.4)

with electric field strength

$$F \equiv \frac{q^2}{4\pi}r^{4-2d}dt \wedge dr$$  \hspace{1cm} (A.5)
Figure 10. As in figure (9) but with $\lambda_{GB} = 0.255$. For $\lambda_{GB} > 1/4$, and with an electrically charged black hole, the two branches may join and form a closed loop.

and the metric potential

$$f(r) = \epsilon + \frac{r^2}{L^2} \frac{1}{2\lambda_{GB}} \left[ 1 + \kappa \sqrt{1 - 4\lambda_{GB} + \frac{4\lambda_{GB} L^2}{r^{d-1}} (\mu - Q^2 r^{3-d})} \right], \quad (A.6)$$

where $\lambda_{GB} \equiv (d-3)(d-4)\alpha'$ and $\kappa = \pm 1$. We refer to [23, 36–38] for further discussions on the black hole thermodynamic of charged AdS-GB black holes; [38] presents a comprehensive analysis for all three possible horizon topologies ($\epsilon = 0$ or $\pm 1$).

The integration constants $\mu$ and $Q^2$ are related, respectively, to the Arnowitt–Deser–Meisener (ADM) mass $M$ and the Maxwell charge $q$ via

$$M \equiv \frac{(d-2)V_{d-2}L^2}{16\pi G} \mu, \quad Q^2 \equiv \frac{q^2}{2\pi(d-2)(d-3)}. \quad (A.7)$$

Needless to say, the Maxwell charge $q$ is topological in $d = 3$. The negative root solution ($\kappa = -1$) has a smooth limit as $\lambda_{GB} \to 0$, which is often referred to as the ‘Einstein branch’. While the positive root solution ($\kappa = +1$), which has no smooth limit as $\lambda_{GB} \to 0$, represents a distinct new feature of Gauss–Bonnet gravity which is completely absent in pure Einstein gravity in any dimensions. As anticipated, for the negative root solution ($\kappa = -1$), the metric potential $f(z) \equiv f(r)L^2/r_+^2$ is negative inside the black hole horizon ($z \equiv r/r_+ < 1$) where the role of $t$ and $r$ coordinates will interchange. Under a modulo double wick rotation, we get

$$f(r) = \epsilon + \frac{r^2}{L^2} \frac{1}{2\lambda_{GB}} \left[ 1 + \kappa \sqrt{1 - 4\lambda_{GB} + \frac{4\lambda_{GB} L^2}{r^{d-1}} (\mu + P^2 r^{3-d})} \right], \quad (A.8)$$

with the magnetic field strength

$$F = -\frac{q^2}{4\pi} r^{4-2d} \, d\theta \wedge dr \equiv \frac{g^2}{4\pi} r^{4-2d} \, d\theta \wedge dr \quad (A.9)$$

where $g \equiv -iq$ is the magnetic charge and $P^2 = -Q^2$.

In general, one evaluates the mass parameter $\mu$ at the black hole event horizon ($r = r_+$) where $f(r) = 0$. However, one should note that for a given $\mu$, $f(r)$ vanishes at $r = r_+$ only for the $\kappa = -1$ branch. We also note that, with a suitable choice of $\lambda_{GB}$, and especially, for an electrically charged AdS-GB black hole, the $\kappa = \pm 1$ branches of solutions may join together without developing a metric singularity (see figures 9 and 10). In that sense, the $\kappa = +1$ branch does not represent a black hole solution. More importantly, the ADM formula,
as given in (A.7), is valid only for the Einstein branch \((\kappa = -1)\). For this branch, the metric potential \(f(r)\) can be written as

\[
\tilde{f}(z) \equiv f(r) \frac{L^2}{r^2} = \tilde{\epsilon} + \frac{z^2}{2\lambda_{GB}} \left[ 1 + \sqrt{1 - 4\lambda_{GB} + \frac{4\lambda_{GB}}{z^{d-1}} (1 + \tilde{\epsilon} + \lambda_{GB}\tilde{\epsilon}^2 + \tilde{Q}^2 (1 - z^{3-d}))} \right]
\]

(A.10)

where \(z = r/r_\ast\), \(x = r_\ast/L\), \(\tilde{\epsilon} = \epsilon/x^2\) and \(\tilde{Q}^2 = Q^2 L^2 r_\ast^{4-2d}\) (electric charge) or \(\tilde{Q}^2 = -P^2 L^2 r_\ast^{4-2d}\) (magnetic charge). Although the value \(\lambda_{GB} > 1/4\) may be allowed, especially, for the positive root solution, such a large coupling is generally not allowed for magnetically charged black holes, so we rule out this possibility.

In a more general setup, for instance, in Einstein–Gauss–Bonnet (EGB) braneworld models [40], the Einstein branch \((\kappa = -1)\) refers to a pure EGB gravity in higher dimensions \((d \geq 5)\), while the GB branch \((\kappa = +1)\) refers to matter being confined to a 3-brane and gravity leaking into extra dimensions. For the \(\kappa = +1\) branch, a gravitational source (such as the mass term \(\mu\)) could reside on the brane. That is why this branch has no Einstein limit. In that sense, only the matter-free case could be relevant for the \(\kappa = +1\) branch, for which the background is always AdS. With a nonzero \(\mu\), the \(\kappa = +1\) branch can produce a repulsive gravity at small distances (see also [41]). As far as the black hole solution is concerned, the Einstein branch \((\kappa = -1)\) is perhaps the only relevant one. On the other hand, if we have a brane-bulk system, then the GB branch \((\kappa = +1)\) also becomes relevant, especially, for a matter-free bulk.

We end this appendix by exploring the possibility that GB gravity connects classical Einstein gravity to something pertinent (or expected) in a consistent quantum gravity theory. To this end, let us first note that in \(d\) dimensions,

\[
R_{GB}^2 = x^{d-2} \frac{d}{dz} \left( (x^{d-5} \tilde{\psi}^2) ' + \text{surface terms} \right),
\]

(A.11)

where \(\psi \equiv (\tilde{\epsilon} - \tilde{f}(z))\). In \(d \geq 5\) dimensions, the GB contribution becomes dominant as singularity is approached \((z \to 0)\), which may, therefore, help in regularizing black hole solution by weakening the singularity at \(z = 0\) (see also [39]). Comparing the behaviour of solutions in \(d = 5\) and \(d = 6\), we will find that the \(d = 5\) case gives several desirable features including that the metric function \(f(r)\) and its derivative both remain finite and regular at \(r = 0\). The GB term has the important feature of weakening of singularity which is in consonance with what is expected in quantum gravity.

While the GB term appears as the leading correction to the effective low-energy action of the heterotic string theory, quantum gravity effects could easily induce an infinite number of corrections, including non-local terms. The latter endow some unphysical (ghost) modes. To avoid any ghost degrees of freedom, one could allow the quadratic curvature corrections in a Gauss–Bonnet combination. In the holographic framework, any ghost field, if exists, is expected to decouple in the large \(\tilde{\epsilon}\) 't Hooft coupling limit, \(g_s^2 \Lambda_{YM} N_c \to \infty \gg 1\). This is reminiscent of the fact that in the full string theory there is (or should be) no such problem.

In terming the GB gravity as intermediate between classical and quantum gravity, our main point is that what is expected of QG is to remove or smoothen the singularity at \(r = 0\). One would expect a consistent quantum gravity theory also to remove divergence at \(r = 0\) even of curvature too. The GB contribution has a right effect in removal of singularity—at least it weakens the singularity. This is accomplished through a change in \(r\)-dependence of the metric and its derivative as \(r \to 0\). For brevity, let us consider the \(d = 5\) case. Note that as \(r \to 0\),

\[
f(r) \to \epsilon = \frac{\sqrt{\mu\alpha}}{\alpha} + \frac{r^2}{2\alpha} \equiv \mathcal{O}(r^4),
\]

(A.12)
where $\alpha \equiv \lambda_{GB} L^2$. This shows that for either branch there is no $\lambda_{GB} \to 0$ limit for small $r$. This is true also in the $Q^2 \neq 0$ case, for which

$$f(r) \to \epsilon \mp \sqrt{-\alpha Q^2 r} + \frac{\mu}{2\alpha} r^2 + O(r^3).$$

(A.13)

The existence of a black hole horizon generally requires $\mu > \lambda_{GB} L^2$, indicating that $\lambda_{GB}$ is always non-ignorable in a small $r$ limit. Especially, in the $Q^2 = 0$ case,

$$f' \sim \frac{r}{\lambda_{GB} L^2}.$$

(A.14)

With $|\lambda_{GB}| > 0$, there is a change in radial dependence from $1/r^3$ to proportional to $r$, which is analogous to a situation in loop quantum gravity. A similar behaviour happens in quantum cosmology. We believe that a final QG theory will have a unique GB or rather a more general Lovelock realization because only then it would have a valid classical limit for which a proper initial value problem could be defined. The requirement of a unique evolution from initial data might also single out some significance of extended Lovelock gravity, where the GB term is a natural next step of iteration of self-interaction.

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