Topologies on the future causal completion

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Abstract

On the Geroch-Kronheimer-Penrose future completion $IP(X)$ of a spacetime $X$, there are two frequently used topologies. We systematically examine $\tau_+$, the stronger (metrizable) of them, which is the coarsest causally continuous topology, obtaining a variety of novel results, among them a complete characterization of the difference in convergence between both topologies. In our framework, we can allow for $X$ being a chr. space and consequently for the interpretation of $IP$ as an idempotent functor on a category that includes spacetimes of very low regularity. Furthermore, we explicitly calculate $(IP(X), \tau_+)$ for multiply warped chronological spaces.

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1 Introduction

A central notion of mathematical relativity, frequently used to define concepts related to black holes, is the one of future null infinity. Its definition requires a conformal equivalence of (a part of) the spacetime $M$ to a manifold-with-boundary. Closely related is the recently developed notion of 'conformal future-compact extension' (called 'conformal extension' in some references), which turned out to be useful e.g. for the proof of global existence of conformally equivariant PDEs [25] like Yang-Mills-Higgs-Dirac equations (the type of the standard model of particle physics), for the proof of existence of black holes in a strong sense in Einstein-Maxwell theory [30], as well as in quantization [14]. Let us review the definition following [34] and [25]: A subset $A$ of a spacetime $M$ is called future compact iff $A$ is contained in $J^-(C)$ for some compact $C \subset M$, and it is called causally convex iff there is no causal curve in $M$ leaving and re-entering $A$. A $C^k$ conformal future-compact extension (CFE) $E$ of a globally hyperbolic spacetime $(M, g)$ is an open conformal embedding of $(I^+(S_0), g)$ (where $S_0$ is a Cauchy surface of $(M, g)$) into a g.h. spacetime $(N, h)$ with a Lorentzian metric of class $C^k$ such that the closure of the image is future compact and causally convex. This generalizes the usual notion of 'conformal compact extension' (CCE) by requiring only future compactness of the closure of the image instead of compactness. From the work of Friedrich, Anderson-Chrusciel, Lindblad-Rodnianski and others ([22], [1], [13], [15], [31], [40]) it follows that there is a weighted Sobolev neighborhood $U$ around zero in the set of

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vacuum Einstein initial values such that, for any \( u \in U \), the maximal vacuum Cauchy development of \( u \) admits a smooth CFE (as opposed to CCEs of class \( C^2 \), whose nonexistence for every nonflat Einstein-Maxwell solution follows from the positive mass theorem, see Thm. 6 in [36]). Given a CFE \( E : M \to N \), its future boundary is \( \partial^+(E(M), N) := \{ x \in \partial(E(M)) | I^-(x) \cap E(M) \neq \emptyset \} \subset \partial(E(M), N) \). Still, many important examples of spacetimes do not admit CFEs. Luckily, there is a classical intrinsic notion of future completion \( IP(M) \) and future boundary \( \partial^+M \) of a spacetime \( M \) as defined by Geroch, Kronheimer and Penrose in [24]. Budic and Sachs [9] then defined an appropriate chronological structure \( \leq_{BS} \) on \( IP(M) \). Below we give the precise definition \( IP(M) \) and \( \leq_{BS} \) and revise the well-known construction of a map \( i_M : M \to IP(M) \) assigning to \( p \in M \) the set \( I^-(p) \), and the classical fact that that \( IP(M) \setminus i_M(M) \) consists of the past \( \Gamma^-(c) \) of future-inextendible curves \( c \), and construct the end-point map \( \varepsilon_E : IP(M) \to \text{cl}(E(M), N) \) assigning to every IP \( I^-(c) \) the future endpoint (continuous extension) of \( E \circ c \).

The present article focuses firstly on future completions, in contrast to e.g. [19], and secondly, on the upper part of the causal ladder (assuming at least causal continuity), in contrast to e.g. [27]. All our constructions can almost effortlessly be made for spaces defined by Harris [26] called 'chronological spaces' that generalize strongly causal spacetimes and could play a role of limit spaces in Lorentzian geometry similar to metric spaces in Gromov-Hausdorff theory for Riemannian geometry, cf. [37] for related ideas. Following a definition by Harris, a chronological or chr. set is a tuple \( (X, \ll) \) where \( X \) is a set and \( \ll \) is a binary transitive anti-reflexive relation on \( X \) such that \( (X, \ll) \) is chronologically separable, meaning that there is a countable subset \( S \subset X \) that is chronologically separating, i.e., \( \forall (x, y) \in \ll \exists s \in S : x \ll s \ll y \).

A chr. set \( X \) is called \( I^\pm\)-distinguishing iff \( I^- : X \to 2^X, I^+(x) := \{ y \in X | y \leq x \} \forall x \in X \) is injective. For a chr. set \( X \), a subset \( A \) of \( X \) is called past if \( I^-(A) = A \). A nonempty past subset \( U \) of a chr. set \( X \) is called indecomposable iff for any two past sets \( A, B \) we have \( U = A \cup B \Rightarrow U = A \lor U = B \), and \( X \) is called prerregular iff \( \forall p \in X : I^+(p) \) are indecomposable. We call a chr. set \( (X, \ll) \) past-reflecting iff \( \forall x, y \in X : (I^-(x) \subset I^-(y) \Rightarrow I^+(y) \subset I^+(x)) \). If \( X \) is a spacetime, the purely chr. property of being distinguishing and past-reflecting is equivalent to the mixed chr.-topological property of outer continuity of \( I^- \), cf. [33], Lemmas 3.42 and 3.46. We define the category \( C \) of \( I^-\)-distinguishing prerregular past-reflecting chr. sets, whose morphisms are \( f : M \to N \) s.t. for \( x_1, x_2 \in M \), we have \( f(x_1) \ll f(x_2) \iff x_1 \ll x_2 \), and isomorphisms are invertible such.

On the set \( N(X) := 2^X \setminus \{ \emptyset \} \) of nonempty subsets of \( X \) the Budic-Sachs relation \( \leq_{BS} \) with

\[
\forall A, B \subset X : A \leq_{BS} B \iff \exists x \in B : I^-(x) \supset A.
\]

fails to be chronologically separable in general: eg. in the case \( X := \mathbb{R}^{1,1} \), we have \( I^-(\{0\}) \leq_{BS} \{0\} \), but there is no subset \( A \) of \( X \) with \( I^-(\{0\}) \leq_{BS} A \leq_{BS} \{0\} \). It is easy to see that the

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1. i.e., \( \ll \) is disjoint from the diagonal: \( \neg(x \ll x) \forall x \in X \).
2. Harris’ original definition also requires that \( \ll \) is connex, i.e. \( \forall x \in X \exists y \in X : x \ll y \lor y \ll x \). Here we will renounce this requirement as we want to include e.g. compact causal diamonds, which contain points with empty timelike future and past.
3. This is a stronger condition than the notion of isocausality issued by García-Parrado and Senovilla [23], insofar as it requires that the images of the chronological future cones are not subsets of but coincide exactly with the chronological future cones in the range. However, in [21] it has been shown that the future boundary is not invariants under isomorphisms in the weaker notion from [23]. To get uniqueness, one can either add the requirement of future continuity of the morphisms and appeal to Harris’ result [26] or proceed as we do here.
restriction of $\ll_{BS}$ to the set $P(X)$ of nonempty past subsets is a chr. relation, and as well the restriction to the set $IP(X)$ of indecomposable subsets of $X$, which we call $\ll_{IP(X)}$. However, it will turn out to be more practical to consider another chr. relation $\ll_{+} \supseteq \ll_{BS}$ on $P(X)$ equally restricting to $\ll_{IP(X)}$ on $IP(X)$, defined by

$$\forall A, B \in P(X) : A \ll_{+} B :\Rightarrow (\forall U \subset A : U \in IP(X) \Rightarrow U \ll_{BS} B),$$

more on this in the last section. Whereas in $X := (-1;1) \times \mathbb{R} \subset \mathbb{R}^{1+1}$, the past subset $A_{a} := I^{-}([-1,0] \times (-a;0))$ has empty chr. future w.r.t. the chr. relation $\ll_{BS}$ for all $a > 0$, the future w.r.t. $\ll_{+}$ contains e.g. the sets $I^{-}([-1,0] \times (-a;0))$ for all $r > 0$.

$IP_{BS} : X \mapsto (P(X), \ll_{BS}), P_{+} : X \mapsto (P(X), \ll_{+})$ and $IP : X \mapsto (IP(X), \ll_{IP(X)})$ are functors $C \mapsto C$. For every object $X$ of $C$ we have the morphism $i_{X} : X \mapsto IP(X)$, $x \mapsto I^{-}(x) \forall x \in X$.

There are functorial ways to define a causal relation from a chr. relation $\ll$, e.g. $\alpha$ defined by $x \ll_{\alpha}(y) \iff I^{+}(y) \subset I^{+}(x) \land I^{-}(x) \subset I^{-}(y)$, see [33, Def. 2.22 and Th. 3.69 showing that for a causally simple spacetime $(M,g)$, with the usual definition of $\ll$ resp. $\leq$ via temporal resp. causal curves, we get $x \leq y \iff x \ll_{\alpha}(y) \iff I^{+}(y) \subset I^{+}(x) \lor I^{-}(x) \subset I^{-}(y)$: The implication from left to right is true in any spacetime, for the other direction calculate $I^{+}(y) \subset I^{+}(x) \Rightarrow y \in \text{cl}(I^{+}(x)) = \text{cl}(I^{-}(x)) = I^{+}(x)$. It is well-known that $\alpha$ may introduce spurious causal relations in the non-causally simple case, see the causally continuous example $\mathbb{R}^{1+1} \setminus \{0\}$, where $(-1,-1) \geq (1,1)$ but $(-1,-1) \ll_{\alpha}(1,1)$. We will also see in Sec. 7 that the definition of the causal relation via $\alpha$ entails the *push-up property* $I^{+} \circ J^{+} = I^{+}$.

Thus, until Sec. 7 we just renounce speaking of causality and restrict ourselves to the chr. relation. Let $Y$ be a chr. set. A map $c : \mathbb{N} \to Y$ is called (future) chr. chain iff, for all $m, n \in \mathbb{N}$, we have $m < n \Rightarrow c(n) \ll_{\alpha} c(m)$. $Y$ is called future properly causal (cf. [19]) or future chr. complete iff for every future chr. chain $n \mapsto c_n$ in $Y$ there is $c_{\infty} \in Y$ with $I^{-}(c_{\infty}) = \bigcup_{n=0}^{\infty} I^{-}(c_n)$, and it is a classical fact that $IP(X)$ is future chr. complete. If $X$ is preregular, the map $i_{X} : p \mapsto I^{-}(p)$ is a morphism of $C$ (due to chr. separability) with $i_{X}(X) = I^{-}(\partial^{+}(IP(X)))$, where $\partial^{+}(Y) := \{y \in Y | I^{+}(y) = \emptyset\}$ for a chr. set $Y$. As we show in Sec. 2, for every preregular chr. set $X$, the set $(IP(X), \ll_{IP(X)})$ is past-reflecting.

A 3-tuple $(X, \tau, \ll)$ where $(X, \tau)$ is a topological space and $(X, \ll) \in \text{Obj}(C)$ is called a chronological or chr. space. It is called regular iff it is preregular and $\forall \tau \subset X : I^{\pm}(\tau)$ open. Causal spacetimes are regular chr. spaces. We define the category $CS$ of distinguishing, past-reflecting and regular chr. spaces whose morphisms are the continuous morphisms of $C$ and whose isomorphisms are the invertible such. We enrich the functor $IP : X \mapsto IP(X)$ by adding topologies, i.e., prolong $IP$ to a functor $F_{+} = \tau_{+} \circ IP$ with a functor $\tau_{+} : C \to CS$ which is a left inverse of the forgetful functor $CS \to C$. For such a functor $F$ we define three desiderata:

- $F$ is future chain complete iff for every future chr. chain $c$ in an object $X$ of $F(C)$, $c(n) \to_{n \to \infty} c_{\infty}$;

- $F$ is marginal iff for every causally continuous spacetime $M$, the map $i_{M}$ is a homeomorphism onto $i_{M}(M)$, which is open and dense in $F(M)$;

- $F$ respects CFEs iff the end-point map $\varepsilon_{E}$ of any CFE $E : M \to N$ of $M$ maps $\partial^{F}(M) := F(M) \setminus i_{M}(M)$ homeomorphically to $\partial^{+}(\text{cl}(E(M)), N)$, preserving causal relations.
The history of topologies on $IP(M)$ is strangely involved. The first proposal for a topology, here denoted by $\tau_+$, was by Beem [5], for the entire (future and past) causal completion of a causally continuous spacetime, which consists of the union $IP(X)$ and its time dual modulo some identifications — the proofs in Beem’s article however transfer verbatim to $IP(X)$. He stated without proof that $\tau_+$ is compatible with the metric defined in Th. 11, Item 2, and showed:

1. If $X$ is a causally continuous spacetime then $\tau_+$ is metrizable on $IP(X)$ and $i_X(X)$ is open and dense in $IP(X)$;
2. If $X$ is g.h., then $i_X^{-1}(\tau_+)$ is the Alexandrov (manifold) topology on $X$.

After Beem’s article, $\tau_+$ seems to have fallen into oblivion in the Lorentzian community, considering that in the 1990s a lively and extensive debate about the choice of topology on $X$ began without ever taking $\tau_+$ into account. Another very agreeable candidate, the ’chronological topology’, here denoted by $\tau_-$, was elaborated by Flores [17] inspired by ideas of Harris [27]. Flores, Herrera and Sánchez [19] extended its definition to the entire causal completion and related it to the conformally standard stationary situation [20]. $\tau_-$ is in general not Hausdorff — but still T1. The main advantage of $\tau_-$ is that it recovers the manifold topology on a spacetime $M$ not only if $M$ is causally continuous (like $\tau_+$ does), but in the larger class of strongly causal spacetimes.

In 2014, unaware of Beem’s article, the author defined a metric on $IP(X)$ as in Th. 11, Item 2, using it as a tool in the context of conformal future-compact extensions to show that the future boundary of any CFE is homeomorphic to the intrinsic future boundary $\partial^+ X$ defined in terms of $IP(X)$. Then in 2018 Costa e Silva, Flores and Herrera, also unaware of Beem’s result, explored $\tau_+$, re-proving many facts independently and more systematically, and contributed other interesting aspects, e.g. the fact that if $(IP(X), \tau_-)$ is Hausdorff then $\tau_- = \tau_+$, and applied the causal boundary topologized by $\tau_-$ to generalize the notion of black hole showing a version of Hawking’s theorem that each trapped surface is contained in the complement of the past of future null infinity under appropriate conditions on the future boundary. Shortly after publication of their article, the author and they discussed the topology again discovering Beem’s article. This led the author to work out a streamlined introduction into the topic, comparing the topologies systematically. While writing this review article, the author obtained several novel results, also laid down here.

We will see that $\tau_-$ and $\tau_+$ applied to $IP(X)$ satisfy all desiderata. Whereas many previous approaches defined $\tau_+$ only on $IP(X)$, we present a definition of $\tau_+$ for each chr. set., give a self-contained introduction to $IP(X)$, compare $\tau_+$ and $\tau_-$, showing, e.g.:

1. several steps of an analogue of the causal ladder for chr. spaces (cf. Sec. 4 and Sec. 7);
2. the fact that for all causally continuous well-behaved chr. space $X$, the chr. space $\text{cl}(i_X(X), (P(X), \tau_+)) \supset IP(X)$ is well-behaved and causally continuous again (Th. 12);
3. equivalence of several definitions of $\tau_+$, some of which appear in the previous literature, among them the equivalence stated without proof in Beem’s article (Th. 11);
4. future-compactness of $\tau_+$ on $P(X)$ and implications like existence of minimal and maximal TIPs in such futures (Th. 13).

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4 defined below, satisfied e.g. by each connected causal spacetime and each connected Lorentzian length space
Moreover, as an example we calculate \((IP(X), \tau_+)\) for \(X\) being a multiply-warped chr. space (see below, generalizing multiply-warped spacetimes), extending a result by Harris (see Sec. [4]).

The article is structured as follows: Sections 2 and 3 are largely expository; Section 2 reviews the construction of the functor \(IP\) assigning to a chronological set \(M\) its future completion \(IP(M)\) as a chr. set, Section 3 resumes facts about \(\tau_-\) on \(IP(M)\). Section 4 explores parts of the causal ladder for chronological spaces. Section 5 introduces the topology \(\tau_+\) on a chr. set, compares it to \(\tau_-\) and to the Alexandrov topology and applies it to \(IP(M)\). In Section 6, in a close analogy to a proof by Harris, we compute \(\tau_+\) in the future completions of multiply warped products. In Section 7, we develop the topic from the perspective of the causal relation instead of the chr. relation, develop a functor \(\tilde{\alpha} \neq \alpha\) from the chr. category to a causal one, show that \(\tilde{\alpha}\), when applied to arbitrary causally continuous spacetimes, avoids the spurious causal relations known from \(\alpha\), and give a simple one-line definition of the future causal completion as a causal space with topology \(\tau_+\), and finally, draw a conclusion passing a solomonic judgement between the two topologies.

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2 The set of indecomposable past subsets

Let \(X\) be a chr. set. Then arbitrary unions of past subsets of \(X\) are past. For each subset \(A\) of \(X\), the subset \(I^-(A)\) is past, i.e., \(I^-\) is idempotent: Transitivity of \(\ll\) implies \(I^-(I^-(A)) \subset I^-(A)\) and chr. separability implies the reverse inclusion. A chr. chain \(c\) has a (chr.) limit \(x \in X\) iff \(I^-(x) = \bigcup\{I^-(c(n))|n \in \mathbb{N}\}\). Analogously to chr.chains, (future) chr.curves are defined by replacing \(\mathbb{N}\) with a real interval. Geroch, Kronheimer, Penrose \([24]\) [Th. 2.1] and Harris \([26]\) show:

A nonempty subset of a chr. set \(X\) is an IP if and only if it is the past of the image of a chr. chain.

For the sake of self-containedness let us give a proof: For the 'if' part, let \(B \cup C = A = I^-(c(\mathbb{N}))\) for a chronological chain \(c\) and \(B, C\) past. As \(c\) is a \(\{\text{chr.}\}\) limit \(x \in X\) iff \(I^-(x) = \bigcup\{I^-(c(n))|n \in \mathbb{N}\}\). Analogously to chr.chains, (future) chr.curves are defined by replacing \(\mathbb{N}\) with a real interval. Geroch, Kronheimer, Penrose \([24]\) [Th. 2.1] and Harris \([26]\) show:

A nonempty subset of a chr. set \(X\) is an IP if and only if it is the past of the image of a chr. chain.

If \(X\) is preregular, then also finite intersections \(\bigcap_{k=1}^n A_n\) of past subsets \(A_n\) are past: By induction it suffices to consider \(n = 2\), and then for \(p \in A_1 \cap A_2\) there are \(p^+ \in I^+(p) \cap A_1\), and future synopticity of \(I^+(p)\) implies that we can find \(p^+ \in I^+(p) \cap \bigcap_{k=1}^n I^-(p^+_k) \subset I^+(p) \cap A_1 \cap A_2\).
We define, for any $R \in N(X)$, by $I^+_X \cap (R)$ the joint chronological future $\cap \{I^+(r) | r \in R\}$ of $A$ in $X$, correspondingly for the pasts. Whereas $I^\pm$ are monotonically increasing maps from $(N(X), \subset)$ to itself, $I^\pm_\cap$ are monotonically decreasing, and e.g. $I^+ \cap (A \cap B) \supset I^+(A \cup I^+(B))$. We then have $A \ll_{BS} B \Rightarrow A \subset B$ and

$$A \ll_{BS} B \Leftrightarrow I^+_X \cap (A) \cap B \neq \emptyset \ \forall A, B \in 2^X.$$ 

For $A \in \{I, I^+, I^\pm\}$, we call a chr. space $X$ $A$-distinguishing iff $i_X : p \mapsto A(x)$ is injective. With essentially the same proof as in [17], Th.4.3, we see that $IP(X)$ is $I^\pm$-distinguishing if $X$ is $I^\pm$-distinguishing, because $i_X^{-1}(IP(X)) = x \ \forall x \in IP(X)$. Even more, $I_{IP(X)} : P(X) \rightarrow 2^{P(X)}$ is injective. Let $A, B \subset X$ past with $I^+_P(X)A \cap \emptyset = I^+_P(X)B \cap \emptyset \neq \emptyset$; now let $x \in A = I^\pm(A)$, then there is $y \gg x$ with $y \in A$, and if we apply (*) to $D := I^-(x)$ and $y \in I^+(D) \cap A$, there is $b \in B$ with $I^-(x) \subset I^-(b)$, which means by $X$ past-reflecting $I^+(b) \subset I^+(x)$. Now by $B$ being past, there is $c \in B$ with $c \gg b \gg x$, which means $x \in I^-(c) \subset B$. A point $x \in X$ is called past-full resp. future-full if $I^-(x) \neq \emptyset$ resp. $I^+(x) \neq \emptyset$ and full if $I^-(x) \neq \emptyset I^+(x)$. Let $C$ be the category of $I^\pm$-distinguishing preregular past-reflecting chr. sets.

Let $(X, \ll)$ be an $I^\pm$-distinguishing preregular chronological space. Then the above defined map $i_X : (X, \ll) \rightarrow (IP(X), \ll_{BS})$ is an injective morphism of $C$ (i.e., preserves the chr. relation).

The functor $IP$ assigning $IP(X)$ to any chr. set $X$ is idempotent and maps (preregular resp. past-full resp. $I^\pm$-distinguishing) chr. sets to future complete (preregular resp. past-full resp. $I^\pm$-distinguishing) chr. sets. Moreover, $i_X$ preserves limits of chr. chains.

A straightforward proof of this assertion uses $\forall A \in P(X) : A = \bigcup \{C \subset IP(X) | C \ll_{BS} A\}$ to show past-distinguishing, and to show past-reflecting one can use $I_{BS}(A) \subset I_{BS}(B) \Rightarrow A \subset B$ which one can show by contraposition: Assume $\emptyset \neq A \cap B \ni p$, then there is $p_+ \in I^+(A \cap B)$, then by $C := I^-(p)$ we get $C \in I_{BS}(A) \cap I_{BS}(B)$, contradiction.

The functor $IP$ is idempotent, as $IP(X)$ is future-complete for each chr. space $X$, and $IP(S) = S$ for all future-complete chr. sets $S$, as every chr. chain has a future chr. limit in $S$.

For a CFE $E : M \rightarrow N$, we define a map $\epsilon_E$ assigning to an IP $I^-(c)$ the endpoint of $E \circ c$ in $N$. Clearly, the choice of $c$ is not canonical, but for two chr. chains $c, k$ with $I_M(c) = I_M(k)$ we have $E(M) \cap I_M(E \circ k) = E(I_M(k)) = E(I_M(c)) = I_N(E \circ c)$, and thus $\epsilon_E(c) = \epsilon_E(k)$ as $(N, h)$ is distinguishing (the existence of an endpoint is obvious from future-compactness). Thus $\epsilon_E$ is well-defined. In any regular chr. space, past subsets are open, and past subsets of spacetimes have a Lipschitz boundary ([4], Th. 3.9).

Having defined $IP(X)$ as a chr. set, we want to equip it with two topologies $\tau_+, \tau_-$. Our aim is not to use the topology on $X$ in their definitions but only the chr. structure on $IP(X)$.
3 The topology \( \tau_- \)

We define a topology \( \tau_- \) by letting \( C \subset \text{IP}(M) \) be \( \tau_- \)-closed iff for every sequence \( \sigma \) in \( C \) we have

\[
C \supset L_-(\sigma) := \{ P \in \text{IP}(M) \setminus \{ \emptyset \} | P \subset \lim \inf(\sigma) \land P \text{ maximal IP in } \lim \sup(\sigma) \},
\]

where \( \lim \sup \) and \( \lim \inf \) are defined set-theoretically, see Sec. 4. There is a construction that unites future and past boundary, and then, one can single out the future part of the boundary and equip it with the relative topology; the latter coincides with the future chronological topology if the initial spacetime \((M, g)\) was globally hyperbolic [19]. The definition of \( \tau_- \) works for \( \text{IP}(X) \), where \( X \) is any chronological space, and is functorial in the category of \( \text{IP} \)s of chronological spaces and the \( \text{IP} \) prolongations \( I^- \circ f \) of chronological morphisms \( f \). Furthermore, \( \tau_- \) is locally compact and even future-compact, which follows from the fact shown in [18] (Theorem 5.11) that every sequence of IPs not converging to \( \emptyset \) has a subsequence convergent to some IP. Defining a functor \( F_- := (\text{IP}, \tau_-) \) between the categories of causally continuous spacetimes and chronological spaces we get:

**Theorem 1 (see [18], [19])** \( F_- \) is sequentially future-compact, marginal and respects CFEs.

**Proof.** Sequential future compactness is shown in [18], Th. 5.11, marginality in [19], Th. 3.27. Now, Theorem 4.16 in [19] assures that the end-point map is a chronological homeomorphism if

1. \( E \) is future chr. chain complete,
2. each point \( p \in \partial^+ E(M) \) is timelike transitive, i.e. there is an open neighborhood \( U \) of \( p \) in \( N \) such that for all \( x, y \in V := \text{cl}(E(M)) \cap U \) the following push-up properties hold: \( x \ll_E z \ll_E y \Rightarrow x \ll_E y, x \ll_E z \ll_E y \Rightarrow x \ll_E y \). Here, \( p \ll_E q \iff \) there is a continuous curve \( c \) in \( \text{cl}(E(M)) \) between \( p \) and \( q \) that is a smooth future timelike curve in \( M \) apart from the endpoints of the interval.
3. Each point \( p \in \partial^+ E(M) \) is timelike deformable, i.e. there is an open neighborhood \( U \) of \( p \) in \( N \) such that \( I^-(p, E(M)) = I^-(E(c)) \) for a \( C^0 \)-inextendible future timelike curve \( c : [a; b] \to M \) with \( E(c(a)) \ll_E q \) for all \( q \in V := \text{cl}(E(M)) \cap U \).

These properties are satisfied by CFEs: The first one follows from future compactness, the second one from the push-up properties in \( N \) and the fact that \( \leq_{E(M)} \subset \leq_N \) and for \( q \in \partial^+ E(M) \) we have \( p \ll_{E(M)} q \iff p \ll_N q \) as \( I^-(q) \subset E(M) \). The same argument works also for the last item. \( \square \)

The topology \( \tau_- \) does not in general inherit the \( \mathbb{R} \)-action of the flow of the timelike Killing vector field even for standard static spacetimes, without the additional hypothesis that \( \tau_- \) is Hausdorff. Remark 3.40 in [19] shows that \( \tau_- \) is in general not Hausdorff nor even first countable; in Sec. 5 we will see an example (the ultrastatic spacetime over the unwrapped-grapefruit-on-a-stick) well-known to have a non-Hausdorff future causal boundary if the latter is equipped with \( \tau_- \).
4 Topologies on power sets, causal ladder for chr. spaces

As $IP(X)$ consists of subsets of $X$, we now examine more closely topologies on power sets. For topological spaces $(W, \sigma), (X, \tau)$, a map $F : W \to \tau(X)$ from $W$ to the open sets of $X$ is called inner (resp., outer) continuous at a point $p \in Y$ iff for all compact sets $C \subset (F(p))$ (resp., for all compact sets $K \subset X \setminus cl(F(p))$), there is a $\sigma$-open set $U$ containing $p$ such that for all $q \in U$, we have $C \subset (F(q))$ (resp., $K \subset X \setminus cl(F(q))$). Translated to notions of convergence, this means that for every net $a$ valued in $\tau(W)$ convergent to $p$, the net $F \circ a$ inner (resp., outer) converges to $F(p)$, where a net $a : b \to \tau(X)$ inner converges to $A \in \tau(X)$ iff for all compact sets $C$ in $A$ there is $n \in b$ such that for all $m \geq n$ we have $C \subset a(m)$ (resp., for all compact sets $C$ in $int(X \setminus A)$ there is $n \in b$ such that for all $m \geq n$ we have $C \subset int(X \setminus a(m))$). We denote the topology determined by inner and outer convergence (i.o.-convergence) by $\tau_{io}$ (which is, somewhat confusingly at a first sight, a topology on $\tau(X)$). Sometimes we will consider the i.o. topology as a topology on the entire power set $2^X$; there, it is induced as the initial topology $int^{-1}(\tau_{io})$ via the map $int : 2^X \to \tau(X)$ taking the interior w.r.t. $\tau$. A net $a$ converges in this topology to a subset $U$ iff $int \circ a$ converges to $int(U)$. Of course, this topology is not Hausdorff any more: For each $A \in \tau(X)$, $A$ and $cl(A)$ cannot be separated by open neighborhoods.

Let $C(X)$ denote the set of closed resp. compact subsets of $X$; if $X$ is a metric space, until further notice we always equip $K(X)$ with the topology defined by the Hausdorff distance $d_H$ (recall that the definition of the Hausdorff distance implies that $d_H(\emptyset, A) = \infty$, which makes the Hausdorff distance a generalized metric).

**Theorem 2** Let $X$ be a Heine-Borel metric space, let $a : Y \to \tau(X)$ be a net. If $a$ i.o.-converges to $a_\infty \in \tau(X)$, then for all compact $A \subset X$ such that $\forall K \subset X$ compact : $a_K := cl \circ (\cdot \cap K) \circ a : Y \to (K(X), d_H)$ converges to $cl(a_\infty \cap K)$.

**Remark.** The reverse implication does not hold, an instructive counterexample with $a$ taking values in the interiors of closed connected subsets being $a(n) := B(0, 1) \setminus B(x(n), 1/n) \subset R^k$ for $k \geq 2$, where $x : N \to \mathbb{Q}^k \cap B(0, 1)$ is any bijective sequence; in this case, for $K := cl(B(0, 1))$, the sequence $a_K$ converges to $cl(B(0, 1))$ in $d_H$, but $a$ does not converge in $\tau_{io}$.

**Proof.** We have to show $cl(K \cap a(m)) \to_{m \to \infty} cl(K \cap a_\infty)$ in $d_H$. Inner convergence implies that for all $L \subset a_\infty$ compact, there is $m_L \in Y$ such that $L \subset a(n) \forall n \geq m_L$, and outer convergence implies that for all $S \subset int(X \setminus a_\infty)$ compact, there is $\mu_S \in Y$ such that $S \subset int(X \setminus a(n)) \forall n \geq \mu_S$. The proof is completed if we apply this to compact subsets $C_\varepsilon$ and $C_\varepsilon'$ with

\[
C_\varepsilon \subset a_\infty \quad \land \quad a_\infty \cap K \subset B(C_\varepsilon, \varepsilon), \quad C_\varepsilon' \subset \text{int}(X \setminus a_\infty) \quad \land \quad cl(X \setminus a_\infty) \cap K \subset B(C_\varepsilon', \varepsilon).
\]

To construct $C_\varepsilon$, consider a finite open cover \{ $B(p_i, \varepsilon) | i \in \mathbb{N}_n^\ast$ \} of $a_\infty \cap K$ with $p_i \in a_\infty \forall i \in \mathbb{N}_n^\ast$. Then for all $i \in \mathbb{N}_n^\ast$ there is $\delta_i > 0$ with $B(p_i, \delta_i) \subset a_\infty$. Now we define $C_\varepsilon := \bigcup_{i=1}^{n} cl(B(p_i, \delta_i))$. The subset $C_\varepsilon'$ is constructed analogously.

We write $M \nrightarrow N$ for a strictly monotonically increasing map between partially ordered spaces $M$ and $N$. For $A : \mathbb{N} \to 2^X$ we then define
Even for sequences of open subsets, these inclusions are not equalities: Consider

\[ \limsup_{n \to \infty} A(n) := \{ q \in X \mid \exists a : N \to X \exists j : N \接手 N : a \circ j(n) \in A(j(n))\forall n \in N \land a(j(n)) \to_n \to q \}, \]

\[ \liminf_{n \to \infty} A(n) := \{ q \in X \mid \exists a : N \to X : a(n) \in A(n)\forall n \in N \land a(n) \to_n \to q \}. \]

It is easy to see that \( \limsup \) and \( \liminf \) take values in the set of closed subsets, and \( \limsup(\text{cl} \circ a) = \limsup a \) as well as \( \liminf(\text{cl} \circ a) = \liminf a \), for details see \cite{8}. We want to connect those limits to the usual definitions of \( \limsup \) and \( \liminf \) in a directed set \( Y \): For any sequence \( a : N \to Y \),

\[ \liminf(a) := \sup_{n \in N} \inf_{m \geq n} a(m), \quad \limsup(a) := \inf_{n \in N} \sup_{m \geq n} a(m). \]

Specialized to the case of subsets \( Y = 2^X \) of \( X \), where \( \inf = \bigcap \) and \( \sup = \bigcup \), this reads

\[ \liminf(a) := \bigcup_{n \in N} \bigcap_{m \geq n} a(m), \]

\[ \limsup(a) := \bigcap_{n \in N} \bigcup_{m \geq n} a(m). \]

With those definitions, we see that

\[ \liminf(a) \supset \text{cl}(\lim inf a), \quad \limsup(a) \supset \text{cl}(\lim sup a). \]

Even for sequences of open subsets, these inclusions are not equalities: Consider \( a(n) := (\frac{1}{n+1}, \frac{1}{n+2}) \subset \mathbb{R} \), then the \( a(n) \) are pairwise disjoint, so \( \lim sup(a) = \emptyset \) but \( \lim \sup a = \{0\} \).

We can express many results about noncompact subsets in a more elegant way via Busemann's following result (see \cite{8}) on a metric \( d_1 \) on the space \( C(X) \) of the closed nonempty subsets of a Heine-Borel metric space \( (X, d_0) \), which differs from Hausdorff metrics like the continuous extension of \( \arctan d_H \) from \( d_H \), by blurring the behavior at infinity:

**Theorem 3** (Compare with \cite{8} Sec. I.3) Let \( (X, d_0) \) be a Heine-Borel metric space, let \( x_0 \in X \). Then for \( C(X) := \{ A \subset X \mid \emptyset \neq A \text{ closed} \} \), the map

\[ d_1 = d_1^{o,d_0} : C(X) \times C(X) \to \mathbb{R} \cup \{\infty\}, \]

\[ d_1(A, B) := \sup_{\psi} \left\{ d_0(\{x\}, A) - d_0(\{x\}, B) \cdot \exp(-d_0(x_0, x)) ; x \in X \mid \forall A, B \in C(X) \right\}. \]

takes only finite values and is a metric on \( C(X) \) that makes \( C(X) \) a Heine-Borel metric space.

For every point \( y \) we have \( d_1(\{x_0\}, \{y\}) = d_0(x, y) \). For two points \( x_0, x_1 \), the metrics \( d_1^{o} \) and \( d_1^{\circ} \) are uniformly equivalent (so by abuse of notation we henceforth occasionally suppress this dependence). A sequence \( a : N \to 2^X \) converges in the topology induced by the extended metric \( d_1 \) to \( a_\infty \) iff \( \lim \inf_{n \to \infty} a(n) = \lim \sup_{n \to \infty} a(n) = \text{cl}(a_\infty) \). Limits w.r.t. \( d_1 \) are monotone in the sense that if two \( d_1 \)-convergent sequences \( a \) and \( b \) in \( C(X) \) with limits \( a_\infty \), \( b_\infty \), respectively, satisfy \( a(n) \subseteq b(n) \forall n \in N \) then \( a_\infty \subseteq b_\infty \) (which entails an obvious sandwich principle for convergence).

For each \( y \in X \) there is \( K > 0 \) such that for all \( A, B \in C(X) \) we have \( y \in A \cap B \Rightarrow d_1^{o}(A, B) < K \); even more, if, for \( r > 0 \), we define \( H(r) := \sup \{ d_1^{o}(A, B) ; A \cap B \supset B(y, r) \} \in [0; \infty) \), we get \( H(r) \to_{r \to \infty} 0 \).
Theorem 6

almost future set $B_{\text{cl}}$:
P are almost past, and $\partial$ future-compact extensions, where we have $X$ a chr. space $I$.

Proof. Let $X,d$ be a Heine-Borel metric space. Then $d_1$-convergence of a sequence $a : \mathbb{N} \to C(X)$ is equivalent to $d_H$-convergence of $a_L := L \cap a$ for all $L \subset X$ compact. Thus $f : Y \to (C(X),d_1)$ is continuous iff $f_L : Y \to (K(L),d_H)$ (as in Th. 4) is continuous for all $L \in K(X)$.

Proof. The first direction, to show that $d_1$-convergence of $a$ implies $d_H$-convergence of $a_L$, follows from the topological equivalence of $d_1$ and $d_H$ on compacta. For the reverse implication, let $\varepsilon > 0$ be given, then choose $K := \text{cl}(B(x_0,r))$ for $2re^{-r} < \varepsilon$ and $n \in \mathbb{N}$ such that for all $m \geq n$ we have $d_H(a_k(m),b_k) < \varepsilon$. Then $d_1(a(m),b) < \varepsilon$. \hfill $\square$

Now we want to connect the previous notions to causality. For a sequence $a$ of past sets, in general neither $\liminf(a)$ nor $\limsup(a)$ is a past set: Consider a Lorentzian product $X := (0;1) \times M$, let us take $M := (0;1)$ for simplicity. As $(0;1) \times (0;1)$ is isometrically embedded in $\mathbb{R}^{1,1}$, the future causal completion is isomorphic in the category $\mathcal{C}$ to the subset $V := (0;1) \times [0;1]$ of $\mathbb{R}^{1,1}$. Now consider the sequence $a$ in $V$ defined by $a(n) := (1,\frac{1}{n} + \frac{1}{n})$ and a corresponding sequence $A$ of indecomposable past subsets of $X$ defined by $A(n) := I^-(a(n))$. Then we get $\liminf(A) = I^-((1,\frac{1}{2}) \cup \{(1,\frac{1}{2}) + t(-1,1)|0 < t < \sqrt{2}\})$, and the second subset in the union is a part of the boundary of the first subset, preventing the liminf to be open and so to be a past set. Analogous examples exist in Kruskal spacetime. However, we can define a subset $B$ of a chr. set $X$ to be almost past resp. almost future if $I^-((1,\frac{1}{2}) \cup \{(1,\frac{1}{2}) + t(-1,1)|0 < t < \sqrt{2}\})$, and the second subset in the union is a part of the boundary of the first subset, preventing the liminf to be open and so to be a past set. Then we get easily:

Theorem 5

Let $a$ be a sequence of past sets in a chr. set $X$, i.e. $I^- \circ a = a$. Then $\limsup(a)$ and $\liminf(a)$ are almost past. Let $X$ be a regular chr. space. Then $\limsup(a)$ and $\liminf(a)$ are almost past, and $\limsup(a) = \text{cl}(\limsup a)$ and $\liminf(a) = \text{cl}(\liminf a)$. If $X$ is regular and chr. dense, then the complement of an almost past subset is almost future, each almost past or almost future set $B$ satisfies $B \subset \text{int}(B)$, thus $\text{cl}(\text{int}(B)) = \text{cl}(B)$, $\text{int}(B) = \text{int}(\text{cl}(B))$. The maps $\text{cl} : P(X) \to \text{CAP}(X)$ and $I^+ : \text{CAP}(X) \to \text{CAP}(X)$ are inverse to each other. \hfill $\square$

Theorem 6

1. If, in a chr. dense chr. space, $I^\pm(p)$ are open $\forall p \in Y$, then $Y$ is regular.

2. For a regular chr. space $(X,\ll,\tau)$ the maps $I^\pm_X$ are inner-continuous.

Proof. For the first assertion, we show that every $I^-(x)$ is synoptic and thus indecomposable: Let $y,z \in I^-(x)$, then $I^+(y) \cap I^+(z)$ is an open neighborhood of $x$, so chr. denseness shows that $I^-(x) \cap I^+(y) \cap I^+(z) \neq \emptyset$. For the second assertion, let $p \in X$ be fixed and $p \ll q$, then
chronological separability ensures some \( r_q \in I^+(p) \cap I^-(q) \), and regularity implies that \( I^+(r_q) \) is an open neighborhood of \( q \) whereas \( I^-(r_q) \) is an open neighborhood of \( p \). Now let \( C \in I^+(p) \) be compact, then the open covering \( \{ I^+(r_q) \mid q \in C \} \) of \( C \) has a finite subcovering \( \{ I^+(r_q(i)) \mid i \in \mathbb{N} \} \), and \( U := \bigcap_{i=0}^n I^-(r_q(i)) \) is an open neighborhood of \( p \) such that for every \( q \in U \) we have \( C \in I^+(q) \).

\[ \blacksquare \]

So, in a regular chr. space, \( p \mapsto I^\pm(p) \) are inner continuous. A future- or past-distinguishing chr. space \( X \) is called future resp. past causally continuous iff \( I^\pm \) is inner and outer continuous. A past subset of \( X \) is determined by its closure: Let \( A, B \subseteq X \) be past. If \( \text{cl}(A) = \text{cl}(B) \) then \( A = B \): Let \( p \in A \), then \( U := I^+(p) \cap A \neq \emptyset \) is open; now \( U \subseteq \text{cl}(A) = \text{cl}(B) \), so \( U \cap B \neq \emptyset \), thus \( p \in B \) (and vice versa).

For a chr. set \( X \) we define the future resp. past boundary \( \partial^+ X := \{ x \in X \mid I^+(x) = \emptyset \} \) resp. \( \partial^- X := \{ x \in X \mid I^-(x) = \emptyset \} \), which is related in an obvious way to the definitions \( \partial^+ (\text{cl}(E(M)), N) \) in the Introduction, which in turn were restricted to the cases that there is a conformal future-compact extension \( E : M \to N \). In general, for a chr. space \( X \) and \( A \subseteq X \) we put \( \partial^+(A, X) := \{ x \in \partial(A, X) \mid I^+_X(x) \cap A \neq \emptyset \} \), and the fact that \( E(M) \cup \partial^+(E(M), N) \) is a chronological space chronologically embedded into \( N \) (due to chr. convexity) implies the consistency of the notions: \( \partial^+(M \cup \partial^+(E(M), N)) = \partial^+(E(M), N) \). In contrast to \( E(M) \cup \partial^+(E(M), N) \), the subset \( \text{cl}(E(M), N) \) is not a chr. set, as it is not semi-full due to the presence of \( i_0 \) as in the Penrose conformal compact extension of \( \mathbb{R}^{1,n} \).

We say that a chronology \( \ll \) on a topological space \( X \) is path-generated iff \( p \ll q \) implies that there is a path (i.e., continuous map) \( c \) from a real interval \( I \) to \( X \) with \( c(0) = p \) and \( c(1) = q \) with \( \forall s, t \in I : s < t \Rightarrow c(s) \ll c(t) \). Spacetime chronologies are path-generated. Of course, every path-generated chr. space is chronologically dense. By a little argument, each semi-full regular path-generated chr. space is locally path-connected (and recall that causal diamonds e.g. in Minkowski spacetime are not semi-full because of their spatial infinities). If the chronology of a distinguishing chr. space \( X \) is path-generated, \( A \subseteq X \) is indecomposable if and only if \( A \) is the past of the image of a chronological future curve \( c \) in \( A \).

A regular path-generated chr. space \((X, \ll, \tau)\) is called well-behaved iff \((X, \tau)\) is locally compact, connected and locally arcwise connected (and recall that, by classical arguments using the pointwise compactness radius, each connected locally compact topological space is also sigma-compact).

Each chr. spacetime and each connected Lorentzian length space is well-behaved: From connectedness, local compactness and paracompactness we can conclude sigma-compactness, and chr. separability follows from localizability. Via Hausdorffness and chr. denseness we see that every well-behaved chr. space is \( I \)-distinguishing, i.e. for \( I(A) := I^+(A) \cup I^-(A) \) we get that \( I : X \to 2^X \) and \( I^+ \) and \( I^- : X \to 2^X \times 2^X \) are injective (whereas \( I^\pm \) are not in general, due to possible nonempty causal boundaries). For the next theorem, let a chr. set \( X \) be full iff \( I^+(x) \neq \emptyset \neq I^-(x) \) \( \forall x \in X \). For a chr. set \( X \), we denote its largest full subset by \( X := X \setminus (\partial^+ X \cup \partial^- X) \).

A classical result by Vaughan ([42]) says that each sigma-compact and locally compact topological space admits a compatible Heine-Borel metric. Even more, we can combine this with the result obtained in [41] that for each locally compact, connected, locally connected and separable metrizable space \((X, T)\) there is a \( T \)-compatible complete length metric on \( X \) and incorporate it in Vaughan’s proof to obtain a stronger statement:
Theorem 7  Let \( X \) be a sigma-compact, locally compact, connected and locally connected metrizable topological space. Then there is a compatible intrinsic metric on \( X \) that is Heine-Borel.

Proof. There is a sequence \( U : \mathbb{N} \to 2^X \) of open relatively compact subsets with \( \text{cl}(U(n)) \subset U(n+1) \) for all \( n \in \mathbb{N} \). Let \( d \) be a compatible complete length metric on \( X \) and \( a_n := d(\text{cl}(U(n)), X \setminus U(n+1)) = d(\text{cl}(U(n)), \text{cl}(U(n+1) \setminus U(n))) > 0 \).

Here the equality holds as \( d \) is a length metric and the inequality because of compactness of \( \text{cl}(U(n+2)) \setminus U(n+1) \subset \text{cl}(U(n+2)) \).

For each \( n \in \mathbb{N} \) we define \( f_n := \frac{1}{a_n} \max\{d(\text{cl}(U(n)), A), a_n\} \).

Then \( f_n \) is \( a_n^{-1} \)-Lipschitz continuous with \( f_n|_{\text{cl}(U_n)} = 0 \), \( f_n|_{X \setminus U(n+1)} = 1 \), and \( g := \sum_{k=1}^{\infty} f_k \) is well-defined and locally Lipschitz continuous, and the intrinsification \( d'' \) of \( d'\) \( : X \times X \to \mathbb{R} \) defined by \( d'(x, y) := d(x, y) + |f(x) - f(y)| \) is finite and compatible by equivalence to \( d \leq d'' \). If \( u : \mathbb{N} \to X \) converges to \( v \in X \) w.r.t. \( d \) then also w.r.t. \( d'' \) by local Lipschitzness of \( g \). And \( d'' \) is Heine-Borel: Each \( d'' \)-bounded subset is contained in some \( U_n \) and thus has compact closure.

For any well-behaved causally continuous chr. space we can pick such an intrinsic Heine-Borel metric \( d_0 \), which determines corresponding metrics \( d_H \) on compact subsets and \( d_1 \) on closed subsets. Recall that for a length metric we always have \( \text{cl}(B(x,r)) = B(x,r) := \{ y \in X \mid d(x,y) \leq r \} \) \( \forall r > 0 \), and that the curve \( c : [0; \infty) \to C(X) \) with \( c(t) := B(x,t) \) for each \( t > 0 \) and \( c(0) := \{ x \} \) is \( d_1 \)-continuous. Theorem 3 tells us that the topology induced by \( d_1 \) (henceforth denoted \( \tau_1 \)) does not depend on the metric chosen. The following theorem shows an interesting link between \( d_1 \)-convergence and i.o.-convergence for almost past sets. We define \( T^\pm(p) := I^\pm(p) \cup \{ p \} \) for all \( p \in X \). Let \( \text{CAP}(X) \) be the set of closed almost past subsets of \( X \) and \( \text{CAF}(X) \) be the set of closed almost future subsets of \( X \). The map \( \cap : (\tau(X), \tau_{io}) \times (\tau(X), \tau_{io}) \to (\tau(X), \tau_{io}) \) is continuous.

The corresponding statement for \( d_1 \)-convergence is still true for almost past sets, cf. Item 2 of the following theorem:

Theorem 8  Let \( X \) be a well-behaved chr. space. Let \( d_0 \) be an intrinsic Heine-Borel metric on \( X \), and let \( d_1 := d_{1,0,0} \) be the corresponding metric on the closed subsets as defined in Theorem 3.

1. \( \text{cl} \cdot d_1 \) generates \( \tau_{io} \) on \( P(X) \) for \( \text{cl} : P(X) \to \text{CAP}(X) \).
2. \( \text{CAP}(X) \) is closed in \( (C(X), d_1) \).
3. \( \cap : (\text{CAP}(X) \times \text{CAF}(X), d_1 \times d_1) \to (C(X), d_1) \) is continuous.
4. \( X \) is causally continuous \( \iff I^\pm : (K(X), d_1) \to (\tau_X, \tau_{io}) \) is continuous \( \iff \text{cl} \cdot T^\pm : (K(X), d_1) \to (C(X), d_1) \) is continuous \( \iff \text{cl} \cdot T^\pm : X \to (C(X), d_1) \) is continuous.

Furthermore, if \( X \) is causally continuous, then it is locally chr. convex, i.e., every neighborhood of a point \( x \in X \) contains a chronologically convex subneighborhood of \( x \).

Proof. For the first and second item, let \( x \in A_\infty \). We want to show that \( I^- \subset \text{int}(A_\infty) \). As \( x \in A_\infty \), there is \( a : \mathbb{N} \to X \) with \( a(n) \in A(n) \forall n \in \mathbb{N} \) and \( a(n) \to_{n \to \infty} x \). Let \( z \ll x \), then

\( \text{The necessity to use } T \text{ instead of } I \text{ stems from the fact that we do not know fullness at the points and we have to avoid the empty set. In the causal setting explored in the last chapter this difficulty does not appear, as we can assume fullness for } J. \)
by chr. separability there is \( w \in I^+(z) \cap I^-(x) \), and \( I^+(w) \) is an open neighborhood of \( x \), so there is \( N \in \mathbb{N} \) s.t. \( \forall n \geq N : a(n) \in I^+(w) \). As \( A_n \) is past, \( w \in A_n \) \( \forall n \geq N \), so \( w \in A_{\infty} \), and \( z \in I^-(w) \subset \text{int}(A_{\infty}). \)

For the second assertion of the first item, the statement on i.o.-convergence, let a compact \( C \subset I^-(A_{\infty}) \) be given, then there is a finite covering of \( C \) by subsets of the form \( I^-(z_i) \) with \( z_i \in I^-(A_{\infty}) \), there are \( y_i \in I^-(A_{\infty}) \) with \( y_i \succ z_i \) for all \( i \in \mathbb{N}_m \). Then for all \( i \in \mathbb{N}_m \) there is a sequence \( a^{(i)} \) with \( a^{(i)}(n) \in A(n) \) for all \( n \in \mathbb{N} \) and \( a^{(i)}(n) \to_{n \to \infty} y_i \). By proceeding as before we show that there is \( N \in \mathbb{N} \) such that \( C \subset A(n) \) \( \forall n \geq N \). Analogously for outer convergence, where we use that \((X \setminus (X \setminus A)) \) d1-converges to \( (X \setminus (X \setminus A)) \) (as the Hausdorff distance and thus also \( d_1 \) is compatible by construction with complements), and by Theorem \( 3 \) \( X \setminus A_n \) and \( X \setminus A_{\infty} \) are almost future for all \( n \in \mathbb{N} \) and \( X \setminus I^-(A_{\infty}) = I^+(X \setminus A_{\infty}). \)

To prove the second item, for all \( n \in \mathbb{N} \), let \( A(n) \) be almost past, i.e., \( I^- (A(n)) \subset A(n) \), and \( B(n) \) be almost future, i.e., \( I^+ (B(n)) \subset B(n) \). We want to show that \( d_1 (A(n) \cap B(n), A_{\infty} \cap B_{\infty}) \to_{n \to \infty} 0 \), or, equivalently by Th. \( 4 \) for all \( K \subset X \) compact and all \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \) s.t. for all \( n \geq N : \)

\[
A_{\infty} \cap B_{\infty} \cap K \subset B(A(n) \cap B(n) \cap K, \varepsilon) \quad \text{and} \quad A(n) \cap B(n) \cap K \subset B(A_{\infty} \cap B_{\infty} \cap K, \varepsilon).
\]

For the first inclusion, let \( X = \{x_1, ..., x_M\} \) be a finite \( \varepsilon/2 \)-grid of \( A_{\infty} \cap B_{\infty} \cap K \), i.e., \( A_{\infty} \cap B_{\infty} \cap K \subset B(X, \varepsilon/2) \). For all \( i \in \mathbb{N}_M \) we find points \( y_i \ll x_i \ll z_i \) with \( d(x_i, y_i), d(x_i, z_i) < \varepsilon/2 \), then \( Y := \{y_1, ..., y_M\} \) and \( Z := \{z_1, ..., z_m\} \) are \( \varepsilon \)-grids of \( A_{\infty} \cap B_{\infty} \cap K \). As \( I^+(x_i) \) are open neighborhoods of \( z_i \in A_{\infty} \), we find \( P_i \in \mathbb{N} \) s.t. for all \( n \geq P_i \) we have \( A(n) \cap I^+(x_i) \neq \emptyset \), and as \( A(n) \) is almost past, we have \( x_i \in A(n) \) \( \forall n \geq P_i \). Analogously we find \( Q_i \in \mathbb{N} \) s.t. for all \( n \geq Q_i \) we have \( x_i \in B(n) \). Defining \( N := \max\{\max\{P_i | i \in \mathbb{N}_M\}, \max\{Q_i | i \in \mathbb{N}_M\}\} \), the first inclusion of Eq. \( 3 \) is true if \( n \geq N \).

The second inclusion in Eq. \( 4 \) can be shown by quite general (non-causal) arguments: For all \( \varepsilon > 0 \) we know, for large enough \( n \), \( A(n) \subset B(A_{\infty}, \varepsilon) \) and \( B(n) \subset B(B_{\infty}, \varepsilon) \) and therefore \( X \setminus A(n) \subset X \setminus B(A_{\infty}, \varepsilon) \) and \( X \setminus B(n) \subset X \setminus B(B_{\infty}, \varepsilon) \), thus the second inclusion follows from

\[
X \setminus (A_n \cap B_n \cap K) = (X \setminus A_n) \cup (X \setminus B_n) \cup (X \setminus K) \subset B(X \setminus A_{\infty}, \varepsilon) \cup B(X \setminus B_{\infty}, \varepsilon) \cup B(X \setminus K, \varepsilon) = B((X \setminus A_{\infty}) \cup (X \setminus B_{\infty}) \cup (X \setminus K), \varepsilon) = B(X \setminus (A_{\infty} \cap B_{\infty} \cap K), \varepsilon).
\]

For Item 3, we first number the statements whose equivalence is to be shown from (i) to (iv). We will show (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv), and we first observe that causal continuity is defined as continuity of \( I^\pm : X \to (\tau_X, \tau_\infty) \) and that (as the domain of every involved map is metrizable) it suffices to show sequential continuity in every case. (i) \( \Rightarrow \) (ii): Inner continuity: Let a sequence of compacta \( K_n \to K_{\infty} \) in \( d_1 \) and let \( L \in I^+(K_{\infty}). \) Then compactness of \( L \) implies that there is a finite set \( \{p_1, ..., p_n\} \subset K_{\infty} \) with \( L \subset \bigcup_{k=1}^n I^+(p_k) \). By chr. denseness, for every \( i \in \mathbb{N}_n \), there is \( q^{(i)} : \mathbb{N} \to I^+(p_i) \) with \( q^{(i)}(n) \to_{n \to \infty} p_i \), and causal continuity of \( X \) implies that for each \( i \) there is \( n_i \in \mathbb{N} \) with \( L \subset \bigcup_{i=1}^n I^+(q(n_i)) \). As \( I^- (q(n_i)) \) is open for each \( n_i \), there is \( r_i \) such that \( q(n_i) \in I^+(K_{\infty}) \) for all \( M \geq r_i \), then with \( r := \max\{r_1, ..., r_n\} \) we get \( q(n_i) \in I^+(K_{\infty}) \forall M \geq r_i \forall i \in \mathbb{N}_n \), thus \( L \subset I^+(K_{\infty}) \). For outer continuity proceed analogously: Let \( S \in X \setminus I^+(K_{\infty}) \) and let \( z_1, ..., z_n \in X \) with \( U := \bigcup_{i=1}^n I^+(z_i) \subset K_{\infty} \) s.t. \( S \cap I^+(z_i) = \emptyset \) for all \( i \). Then there is \( r \in \mathbb{N} \) s.t. for all \( N \geq r \) we have \( K_N \subset U \), and we conclude as above \( L \cap I^+(K_N) = \emptyset \).
(ii) ⇒ (iii) follow directly from Theorems 2 and Theorem 4.

(iii) ⇒ (iv) holds because for a sequence \(a \in X\) convergent to some \(a_\infty\), the sequence of compacta \(n \to \{a(n)\}\) converges in \(d_1\) to \(\{a_\infty\}\).

(iv) ⇒ (i): Let \(a \in X^N\) with \(a(n) \to_{n \to \infty} p\). Item 1 then implies that \(I^-(a(n)) \to_{n \to \infty} I^-(\text{cl}(I^-(p)))\), whose last equality \((*)\) can be shown as follows: "\(\supset\)" holds as \(I^-(p) = I^-(I^-(\text{cl}(I^-(p))))\), and "\(\subset\)" holds as well: Let \(x \in I^-(\text{cl}(I^-(p)))\), i.e., there are \(y \gg x\) and \(z : N \to I^-(p)\) with \(z(n) \to_{n \to \infty} y\). As \(I^+(x)\) is open, there is \(n \in N\) such that for all \(n \geq n\) we have \(z_N \in I^+(x)\), thus \(x \ll p\).

For the first statement of the last item, let \(U_1\) be a neighborhood of \(x \in X\). Local compactness of \(X\) ensures that there is a compact subneighborhood \(U_2 \subset U_1\) of \(x\). Metrizability and local compactness of \(X\) imply that \(X\) satisfies the \(T_3\) separation axiom. Therefore there is an open neighborhood \(U_3\) of \(x\) with \(V := \text{cl}(U_3) \subset \text{int}(U_2)\). The sets \(V\) and \(L := U_2 \setminus U_3\) are compact.

Now, defining \(CB(x, r) := \{y \in X | d(y, x) \leq r\}\) for \(x \in X, r \geq 0\), let \(\Phi(r) := \text{cl}(I^+(CB(x, r))) \cap \text{cl}(I^-(CB(x, r)))\). For all \(r > 0\), \(\Phi(r)\) is a neighborhood of \(x\). As \(\Phi(0) = \{x\}\), we have \(L \subset \text{int}(X \setminus \Phi(0))\), so there is \(r > 0\) with \(\Phi(r) \cap L = \emptyset\) due to outer continuity applied to \(U_2\) (here we need that \(r \mapsto \Phi(r)\) is continuous at \(r = 0\) and the second and the third item). Local path-connectedness of \(X\) implies that \(\Phi(r) \subset U_2\), thus \(\Phi(r)\) is the desired neighborhood of \(x\).

For the last assertion, the previous facts imply that there is \(\varepsilon > 0\) s.t. \(U_\varepsilon := \text{cl}(I^+(B(x, \varepsilon))) \cap \text{cl}(I^-(B(x, \varepsilon)))\) is contained in \(U\) (as \(U_0 = \{x\} \subset U\)), it is chr. convex and contains \(x\). □

5 The topology \(\tau_+\)

This section presents the topology \(\tau_+\) on a chr. set \(Y\) and applies it to the case \(Y = IP(X)\) or \(Y = P(X)\) for a chr. set \(X\). Equivalence of the various definitions is shown in Th. 11.

Let \((Y, \ll)\) be any distinguishing chr. set. We define \(\liminf_\pm, \limsup_\pm : Y^N \to P(Y)\) by

\[
\liminf_\pm(a) := \bigcup_{n \in N} I_\pm \cap \{\{a(m) | m \geq n\}\},
\]

\[
\limsup_\pm(a) := \bigcup\{I_\pm \cap \{a \circ j(N) | j : N \to N\}\}
\]

for every \(a : N \to Y\). Clearly \(\limsup_\pm(a) \geq \liminf_\pm(a)\), as the union in the definition of the latter is not over all subsequences but only over those of the form \(n \mapsto c + n\) for some constant \(c \in \mathbb{N}\). Furthermore it is clear that for a monotonically increasing sequence \(\limsup_\pm\) and \(\liminf_\pm\) coincide.

We define a limit operator \(L_+ : Y^N \to P(Y)\) by

\[
L_+(a) := \{v \in X | I^-(\liminf_\pm(a)) = I^-(\limsup_\pm(a)) = I^-(v)\}
\]

for every \(a \in Y^N\), and \(S \subset Y\) is \(\tau_+\)-closed if for every sequence \(a\) in \(S\) we have \(L_+(a) \subset S\). \(L_+\) is called the limit operator of \(\tau_+\). As \(Y\) is distinguishing, we have \(\sharp(L_+(a)) \in \{0, 1\}\) for all sequences \(a\) in \(Y\).

This indeed defines a topology, due to a classical construction by Fréchet and Urysohn (see e.g. [16], p. 63): We only have to show that for all \(a : N \to Y\):
1. If $a(N) = \{p\}$ we have $L_+(a) = \{p\}$;

2. If $L_+(a) = \{p\}$ then for every subsequence $b$ of $a$ we have $L_+(b) = \{p\}$;

3. If $p \notin L_+(a)$ then there is a subsequence $b$ of $a$ s.t. every subsequence $c$ of $b$ has $p \notin L_+(c)$.

These conditions are easy to verify: The second one is a consequence of $\liminf^- (a) \leq \liminf^- (b) \leq \limsup^- (b) \leq \limsup^- (a)$ for any subsequence $b$ of a sequence $a$ (recall that $\leq=\subset$ on $P(Y)$). To prove the third one, either we have $I^- (\limsup^- (a)) \neq I^- (\liminf^- (a))$ in which case we choose $b$ as a subsequence making up this difference, i.e. with $\liminf^- a \neq \bigcup_{n \in N} I^- \cap (b(N))$; or $I^- (\limsup^- (a)) = I^- (\liminf^- (a))$ but then this holds for all subsequences, which finishes the argument. So indeed $\tau_+: C \to CS$ is a well-defined functor, a left inverse of the forgetful functor.

A warning example: There are strongly causal spacetimes whose convergence structure is not described correctly by $L_+$. Let $\mathbb{R}^{1,1} := (\mathbb{R}^2, -dx^2 + dt^2)$ and consider $M := \mathbb{R}^{1,1} \setminus (\{0\} \times [0; \infty))$. This is a classical example of a strongly causal spacetime that is not past-reflecting. And indeed, for $z(n) := (1 + 1/n, 1)$ and $z_\infty := (1, 1)$, the sequence $z$ converges to $z_\infty$ in the manifold topology but not in $L_+$, as $A := I^\pm_{\mathbb{R}^{1,1}}(0) \subset I^\pm_{\mathbb{R}^{1,1}}(z_n)\forall n \in \mathbb{N}$ but $A \cap I^\pm_{\mathbb{R}^{1,1}}(z_\infty) = \emptyset$. Theorem[12] will show that this only happens in the non-causally continuous case.

We want to connect the previous notions of limsup and liminf to the set-theoretical ones in the case of a chr. set $Y = IP(X)$, where $X$ is a regular and $I^-$-distinguishing chr. space. To this aim, we consider the relation $\ll_{BS}$ on $2^X$ and the sets $I^\pm_{2^X}$ defined by it[9]. This is necessary, as $I^\pm_X(\liminf(a))$ is not indecomposable in general for a sequence $a$ of indecomposable subsets: Consider, e.g., the sequence $a : \mathbb{N} \to IP(\mathbb{R} \times S^1)$ into the two-dimensional Einstein cylinder given by $a(n) := (0, (-1)^n)$ for all $n \in \mathbb{N}$, then $\liminf(a)$ is the union of two PIPs.

**Lemma 1** Let $X$ be a past-full chr. set. For all $a : \mathbb{N} \to P(X)$ we have

$$I^\pm_{P(X)}(I^-_X(\liminf a)) = I^\pm_{P(X)}(\liminf_{c \in P(X)} \underline{a}), \quad I^\pm_{P(X)}(I^-_X(\limsup a)) = I^\pm_{P(X)}(\limsup_{c \in P(X)} \underline{a}) .$$

**Proof.** Let us focus first on the first equality. To show the inclusion $\subset$ from left to right, let $p \in I^\pm_{P(X)}(I^-_X(\liminf(a)))$, this is equivalent to $I^\pm_X(p) \cap I^-_X(\liminf a) \neq \emptyset$. Equivalently again,

$$\exists q \in X : (p \subset I^\pm_X(q) \land \exists r \gg q \exists M \in \mathbb{N} \forall n \geq M : r \in a(n)). \quad (5.1)$$

On the other hand, we want to show $p \in I^\pm_{P(X)}(\liminf^- a)$, or, equivalently

$$\exists y \in \liminf^- a \subset P(X) : I^\pm_X(p) \cap y \neq \emptyset .$$

This previous statement is equivalent to the assertion that there is a $z \in y \subset X$ such that $p \subset I^- (z)$ and $y \in \liminf^- (a)$, i.e., $\exists N \in \mathbb{N} \forall n \geq N : y \subset a(N)$ (*). And this last statement can indeed be verified by setting $y := I^- (r) , z := q$ and $N := M$.

---

*Here we just infer the definition $I^- (A) := \{B \in 2^X \mid B \ll_{BS} A\}$ but we recall that $\ll_{BS}$ is not a chr. relation on $2^X$ in general, as chr. separability might fail.*
For the other inclusion $\supset$ we assume the existence of $N, y, z$ as in (*) and conclude Eq. 5.1 for $M := N$ and $q \ll r \in y = I^-(y)$ arbitrary (recall that $y \in P(X)$).

The same proof works for $\limsup$ instead of $\liminf$, replacing the existence of a threshold value $N$ with the existence of a subsequence.

**Theorem 9** Let $X$ be a past-full chr. set. Then for all $a \in (P(X))^\mathbb{N}$ we get:

$L_+(a) = \{a_\infty\} \Leftrightarrow I_X^-(\liminf a) = a_\infty = I_X^-(\limsup a)$.

**Proof. From left to right.** If $I^- (\liminf^+ a) = I^- (\limsup^+ a) = I^- (a_\infty)$, Lemma 1 implies

$$I_X^- (\liminf a) = I_X^- (\liminf^+ a) = I_X^- (\liminf^- a) = I_X^- (\limsup^+ a) = I_X^- (\limsup a),$$

As moreover $I_{P(X)}^-$ is injective on $P(X)$ (see Sec. 2), we get the assertion.

**From right to left:** We calculate

$$I_X^+ (\liminf^\pm a) = I_X^+ (\liminf a) = I_X^+ (a_\infty) = I_X^+ (\limsup a) = I_X^+ (\limsup^\pm a).$$

Recall that a liminf or limsup of a sequence of past sets is almost past but not a past set in general. This is the reason why we have to take $I^-$ three times in the definition of the limit operator. As a convergent example in $IP(\mathbb{R}^{1,1})$ in which $\liminf a \neq \limsup a$, consider $a(n) := I^- ((0, (-1)^n n^{-1})$.

**Theorem 10 (see also [11])** Let $X$ be a past-full chr. set. On $IP(X)$, we have $\tau_- \subseteq \tau_+$. 

**Proof.** Let $a : \mathbb{N} \to IP(X)$. The synopsis of the Definitions 2.3, 2.4 and 6.1 from [17] shows that $L_-(a)$ is precisely the set of those $v \in X$ such that for every $A \subset X$ that is a maximal IP (resp. maximal IF) in $I^-(v)$ (resp. $I^+(x)$) satisfy: $A \subset \liminf a$ and $A$ is a maximal IP (resp. IF) in $\limsup a$. The characterization of $L_+$ on $IP(X)$ in Theorem 9 shows that $L_+ \subset L_-$ as partial maps from $(IP(X))^\mathbb{N}$ to $P(IP(X))$ (because $I^- (\liminf^- (a)) = I^- (v)$ implies $A \subset I^- (x) \Rightarrow A \subset \liminf a$ and $I^- (\limsup^- (a)) = I^- (v)$ implies that if $A$ is a maximal IP in $I^- (x)$, then $A$ is a maximal IP in $\limsup a$), which implies the claim. The fact that the two topologies do not coincide in general, even on globally hyperbolic spacetimes, is shown by Harris’ example of the unwrapped-grapefruit-on-a-stick [28] accounted for at the end of this section. 

The next task is to examine $\tau_+$ on $IP(X)$. First, the definition of the metrizable topology in [34] was made for $IP(M)$ where $M$ is a g.h. spacetime. In this case, when defining an appropriate measure, we can use Lemma 3.3 of [12] stating that, for a globally hyperbolic manifold $(N, h)$ and for any compactly supported $\psi \in C^0(N, [0, \infty])$ with $\int_M \psi(x) d\text{vol}(x) = 1$, the function $\tau_\psi$ with $\tau_\psi(p) := \int_{I^-(p)} \psi \text{dvol}_h$ is continuously differentiable. We choose a locally finite countable covering of $M$ by open precompact sets $U_i$ and define, for $\phi_i \in C^\infty(M, (0, \infty))$ with $\phi_i^{-1}(0) = M \setminus U_i$,

$$\phi = \sum_{i \in \mathbb{N}} 2^{-i} (\|\phi_i\|_{C^1} + \|\phi_i\|_{L^1})^{-1} \phi_i,$$

therefore $\phi \in C^1(M) \cap L^1(M)$ with $\|\phi\|_{C^1} \leq 1$; we rescale $\phi$ such that $\|\phi\|_{L^1} = 1$. When treating general chronological spaces $X$, we have to ensure the existence of appropriate measures on them: A synopsis of [30] (Corollary after Th. 3) and [29] (Cor. 2.8) ensures that on...
every Polish (i.e., complete-metrizable separable topological) space without isolated points there is an admissible measure, i.e., a finite non-atomic strictly positive Borel measure (the finiteness it not mentioned in the corollary but clear from the construction via the Stone-Cech compactification). A classical result \[3\] (Lemma 26.2) ensures that every finite Borel measure \(\mu\) on a Polish space is Radon, that is, locally finite, outer-regular and inner-regular: for every Borel set \(B\) we have \(\mu(B) = \sup\{\mu(K)|K \subset B\text{ compact}\} = \inf\{\mu(U)|B \subset U\text{ open}\}\). In summary, on every Polish space \(X\) we have a non-atomic strictly positive finite Radon Borel measure \(\mu\), which we can apply instead of \(\Phi_{\text{vol}}\) above. Now, we can induce a metric \(\delta_\mu : IP(X) \times IP(X) \to [0;\infty)\) by defining

\[
\delta_\mu(A, B) := \mu(\Delta(A, B)),
\]

where \(\Delta(A, B) := (A \setminus B) \cup (B \setminus A)\) is the symmetric difference of \(A\) and \(B\).



For the last statement of the following theorem, assuming that \(X = M\) is a g.h. spacetime, we use a Cauchy temporal function \(t\) (whose existence is ensured by the nowadays classical result \[9\]), for a short self-contained account and extended results see \[35\]) to define a diffeomorphism \(D : \mathbb{R} \times S \to M\) with \(D^*t = pr_1\) (where \(S\) is a Cauchy surface of \(M\)) and \(D^*g = -L^2dt^2 + pr_3(h \circ t)\) where \(L\) is a function on \(\mathbb{R} \times S\) and \(h : \mathbb{R} \to \text{Riem}(M)\) is a one-parameter family of Riemannian metrics on \(S\). As for every IP \(A\) in \((\mathbb{R} \times S, D^*g)\), \(\partial A\) is an achronal boundary, which, as a subset of \(\mathbb{R} \times S\), is well-known (see e.g. \[32\], Th. 2.87(iii)) to be (the graph of) a locally Lipschitz partial function \(f(t, A) : S \to \mathbb{R}\). The Hausdorff metric w.r.t. a metric \(d\) is denoted by \(d_{H(g)}\).

**Theorem 11** Let \(X\) be a well-behaved past-full chr. space, let \(a \in (P(X))^\mathbb{N}\), let \(a_\infty \in P(X)\). Let \(d\) be a compatible Heine-Borel metric on \(X\). Then the pullback \(\text{cl}^\ast d_1\) of the corresponding metric \(d_1\) along cl is a metric on \(P(X)\). The following are equivalent:

1. \(\text{cl} \circ a\) converges in \((C(X), d_1)\) to \(\text{cl}(a_\infty)\),
2. For every finite non-atomic strictly positive Borel measure \(\mu\) on \(X\) and for every compact subset \(K\) of \(X\), \(\lim_{n \to \infty}(\mu(\Delta(a_\infty \cap K, a(n) \cap K))) = 0\) and \(\delta_\mu : IP(X) \times IP(X) \to \mathbb{R}\) defined by \(\delta_\mu(A, B) := \mu(\Delta(A, B))\) is a metric topologically equivalent to \(d_1\),
3. \(\forall C \in K(X) : \lim_{n \to \infty}(d_{H(g)}(\text{cl}(a_\infty \cap C), \text{cl}(a(n) \cap C))) = 0\ \forall g \in \text{Metr}(C)\ \text{compatible},\)
4. \(I^-(\lim \sup a) = I^- (\lim \inf a) = a_\infty\),
5. \(\text{int}(\lim \sup(a)) = \text{int}(\lim \inf(a)) = a_\infty\),
6. \(\text{cl}(\lim \sup(a)) = \text{cl}(\lim \inf(a)) = \text{cl}(a_\infty)\),
7. \(\lim \sup(a) = \lim \inf(a) = \text{cl}(a_\infty)\),
8. \(L_+(a) = \{a_\infty\}\),
9. \(a\) converges in \((P(X), \tau_+)\) to \(a_\infty\).

In particular, 8. \(\iff\) 9. shows that \((P(X), \tau_+, L_+)\) is of first order. If \(X\) is a g.h. spacetime, all nine conditions are equivalent to: For \(t\) a continuous Cauchy time function and \(f(t, A)\) defined as above, directly before this theorem, \(f(t, a(n))\) converges to \(f(t, a_\infty)\) pointwise, or equivalently, in the compact-open topology (*).
Remark. Beem’s original definition of the topology in [5] corresponds to Item 7, and he shows in his Proposition 3 the equivalence of Items 1 and 7. Moreover, he states without proof equivalence of Items 1 and 2 for globally hyperbolic manifolds with compact Cauchy surface.

Remark. Different from the topology $\tau_+$, neither the metrics inducing it nor even their uniform structures are functorial, as the choice of $\mu$ is highly arbitrary. However, it is possible to construct natural metrics e.g. in the category of tuples consisting of g.h. spacetimes and their Cauchy surfaces or in the category of temporally compact g.h. spacetimes, cf [37].

Proof. First we want to show that Items 1,3,4,6,7 and 8 are equivalent without further conditions.

(1) $\iff$ (3): see Th. [4]

(1) $\implies$ (7): This is the second last assertion of Th. [3]

(1) $\implies$ (4): Let $p \in a_\infty$, then the open subset $A := I^+(p) \cap a_\infty$ of $X$ contains a nonempty open subset $U$ with $K = \text{cl}(U,X) \subset A$ compact. By the assumption (1), there is $n \in \mathbb{N}$ such that for all $N \geq n$ we have $a_N \cap U \neq \emptyset$, implying $p \in a_N$. Consequently, $p \in \text{lim inf}(a)$. Conversely, if there is $n \in \mathbb{N}$ such that for all $N \geq n$ we have $p \in a_N$, then $p \in \text{cl}(a_\infty)$, then for an open and precompact neighborhood $B$ of $p$ whose closure is contained in $X \setminus \text{cl}(a_\infty)$, we have $B := I^-(p) \cap K \subset X \setminus \text{cl}(a_\infty)$; on the other hand, $B \subset a_n \forall n \in \mathbb{N}$ as the $a_n$ are past; thus if a ball of radius $r$ is contained in $U \subset B(x_0)$ then $d_1(a_\infty,a_n) \geq re^{-R} > 0 \forall n \in \mathbb{N}$, in contradiction to (2). Thus $\text{lim inf}(a) \subset \text{cl}(a_\infty)$, so $\text{int}(\text{lim inf}(a)) \subset a_\infty$, and with the above $a_\infty = \text{int}(\text{lim inf}(a)) \subset \text{int}(\text{lim sup}(a))$.

For the remaining inclusion $\text{int}(\text{lim sup}(a)) \subset a_\infty$, we use that on one hand, $a_\infty$ is a past set and therefore open, on the other hand we show $\text{int}(\text{lim sup}(a)) \subset \text{cl}(a_\infty)$:

Let $p \in \text{int}(\text{lim sup}(a))$, then there is a strictly monotonic $j : \mathbb{N} \to \mathbb{N}$ with $p \in \bigcap_{k=0}^\infty a(j(k))$. We want to show that $p \in \text{cl}(a_\infty)$ by constructing a sequence $y : \mathbb{N} \to a_\infty$ with $\lim(y) = p$. As $X$ is past-full, there is a sequence $b$ in $I^-(p) \cap \text{int}(\text{lim sup}(a))$ converging w.r.t. $d_1$ to $p$. For all $n \in \mathbb{N}$ we define $U_n := I^+(b(n)) \cap I^-(p)$. We have a ball of radius $r_n > 0$ contained in $U_n \subset B(x_0,R)$ for all $n$, so

$$\forall n \in \mathbb{N} \exists l \in \mathbb{N} \forall L \geq l : d_1(a(L),a_\infty) < r_ne^{-R}.$$  

As, on the other hand, $U_n \subset a(j(k))$ for $j(k) > L$, we have $\emptyset \neq a_\infty \cap U_n \ni y_n$, which defines indeed a sequence $y$ that converges to $p$ by application of the "causal sandwich principle" in $(P(X),d_1)$ implied by Th. [3]. If $p \in (I^-(p_\infty))^{\mathbb{N}}$ converges to $p_\infty$ and if $q \in X^{\mathbb{N}}$ with $p(n) \ll q(n) \ll p_\infty \forall n \in \mathbb{N}$ then also $q$ converges to $p_\infty$.

(4) $\iff$ (8): This is Theorem [9].

(6) $\iff$ (7): For sequences $a$ with $J^- \circ a = a$ we have $\text{cl}(\text{lim sup}(a)) = \text{lim sup}(a)$ and $\text{cl}(\text{lim inf}(a)) = \text{lim inf}(a)$. Chronological denseness implies that there is a sequence $k : \mathbb{N} \to I^+(k(n))$ (which we can w.l.o.g. choose as a chr. chain) converging to $p$. Openness of the $I^+(k(n))$ and the condition that the $a(n)$ are past imply that there is a subsequence $l$ of $k$ with $l(n) \in a(n)$. The reverse inclusion is trivial.

(7) $\iff$ (1) follows immediately from the fact that $d_1$-convergence is equivalent to the topology determined by convergence of $\text{lim sup}$ and $\text{lim inf}$ (second last assertion in Th. [3]) and the fact that $\text{lim sup}(\text{cl} \circ a) = \text{lim sup}(a)$ (and correspondingly for $\text{lim inf}$) mentioned directly after the definition of $\text{lim sup}$ above.
Thus we get (1) ⇒ (4) ⇒ (6) ⇐ (7) ⇐ (1) ⇐ (3), showing pairwise equivalence of (1), (3), (4), (6), (7), (8). As the topology in (1) is metric, the limit operator is first order and we get equivalence to (9).

(2) ⇐ (3): First of all, any σ-compact topological space is separable. Also any σ-compact metrizable topological space is complete-metrizable. Consequently, every well-behaved space is Polish. Now, for the first assertion, δμ does not vanish between different subsets. By regularity, past subsets are always open. But in general, for A, B two different open past subsets, we have (A \ B) ∪ (B \ A) is nonempty and open: Let, w.r.o.g., be A \ B. Openness of A allows to choose a nonempty open precompact U ⊂ X with cl(U) ⊂ A ∩ I+(x). By the fact that B is past and does not contain x we know that U ∩ ∂B = ∅. Consequently, U ⊂ A \ B, and μ(Δ(A, B)) ≥ μ(U) > 0. Thus δμ is not only a pseudometric but a true one. Now, the claim follows from inner and outer regularity of μ which implies d1-continuity of μ on the set of closures of past subsets according to Theorem 8.

(5) ⇐ (6): We apply int resp. cl to each side and use Theorem 5

(9) is equivalent to (8) due to the equivalence of (8) and (1): The limit operator L+ is equal to the limit operator w.r.t. the metric d1, therefore the closed sets are the same, thus the topology.

(4) ⇐ (∗) for g.h. spacetimes (see also [11]. Th. 3.9) follows from the fact that pointwise convergence is equivalent to compact-open convergence on a compact set for locally Lipschitz maps.

Here it is more practical to show (∗) ⇐ (3): Assume that (3) holds and let C ⊂ X be compact, let K := C ∩ t−1{(0)}. Then we choose a product metric as a compatible metric, and then, with fn := f(t, An) and f∞ := f(t, A∞), we have that f∞|K and fn|K are Lipschitz continuous for all n ∈ N, with the same Lipschitz constant. Due to compactness of K and Lipschitz continuity, for every ε > 0 there is δ > 0 s.t. for each p ∈ K with |fn(p) − f∞(p)| > ε there is a ball around (p, fn(p)) ∈ ∂an of radius δ not intersecting a∞ (if fn(p) > f∞(p)) or there is a ball around (p, f∞(p)) ∈ ∂a∞ of radius δ not intersecting a0 (if fn(p) < f∞(p)). In either case, dh(C ∩ cl(∀a0), C ∩ cl(∀a0)) ≥ δ. So (3) ⇒ (∗). The other implication is the well-known fact that for a compact set K, if fn → f∞ uniformly, dh(fn|K, f∞|K) →n→0 0 as subsets of K × R.

The previous theorem shows that the topologies from [34] and [11] coincide. The concrete definition in [34] is appropriate e.g. for calculations of future boundaries of spacetimes as those done in [34], whereas the original definition of τ+ due to Beem shows that the topology is functorial in the purely chr. category.

**Theorem 12** Let (X, ≪, τ) be a causally continuous well-behaved past-full chr. space. Then τ = i−1(τ+) in particular iX : X → (iX(X), τ+) is a homeomorphism. Moreover, (P(X), τ+) and IP+(X) := cl(iX(X), (P(X), δ)) (with induced chr. structure and topology) are again past-full causally continuous well-behaved chr. spaces both containing IP(X). The relative topology of the subset IP(X) of full points of (IP(X), τ+) is the Alexandrov topology. Finally, iX(c(n)) →n→0 I−(c[M]) for any chr. chain c in X.

**Remark:** So τ+ applied to a causally continuous spacetime recovers the manifold topology. Moreover, Theorem 12 allows to define a functor F+ = (IP+, τ+) from the category CW of causally continuous well-behaved past-full chr. spaces to itself, and F+ is marginal by definition.
Furthermore, the theorem implies that $\tau_+$ is second-countable on $P(X), IP^+(X)$ and $IP(X)$ for $X$ well-behaved (for $X$ spacetime this is in Th. 5.2. (ii) of [19]), as metrizable and sigma-compact topological spaces are second countable (compact metric spaces are second countable, and a countable union of countable sets is countable).

**Proof.** First we want to show regularity of $i_X^{-1}(\tau_+)$, i.e. that $i_X(I^+(x))$ is $\tau_+$-open for all $x$. Given metrizability and thus sequentiality of $\tau_+$, it is enough to show that $I^+(x)$ is sequentially open, or, equivalently, that $X \setminus I^+(x)$ is sequentially closed. Therefore, let $a \in (X \setminus I^+(p))^\mathbb{N}$ converge to some $v \in X$, and we have to show that $v \in X \setminus I^+(p)$. As $a(n) \in X \setminus I^+(p) \forall n \in \mathbb{N}$, we know $p \not< a(n) \forall n \in \mathbb{N}$. Convergence of $a$ to $v$ means $I^\pm(\lim \inf^+(a)) = I^\pm(\lim \sup^+(a)) = I^\pm(v)$, and we want to show $p \not< I^-(v)$, which with the convergence condition is equivalent to $p \not< I^-(\lim \inf^+ a) = I^-(\bigcup_{n \in \mathbb{N}} I^\ominus \{(a(m) | m \geq n\})$. Assume $p \in I^-(\bigcup_{n \in \mathbb{N}} I^\ominus \{(a(m) | m \geq n\})$, then there is $n \in \mathbb{N}$ such that for all $m \geq n$ we have $p \in I^-(I^-(a(m))) = I^-(a(m))$, which is a contradiction.

$IP(X)$ is path-generated: Let $P \subseteq Q \subseteq IP(X)$, then either $P, Q \subseteq i_X(X)$, in which case $P := i_X(p), Q := i_X(q)$ for some $p, q \in X$ with $p \not< q$. Then there is a chronological path $c : p \leadsto q$, and $i_X \circ c$ is a chronological path from $P$ to $Q$. If, on the other hand, $P = i_X(p), Q \subseteq \partial^+ X$, then $Q := I^-(c)$ for a chronological path $c : (a; b) \rightarrow X$, i.e. there is $s \in \mathbb{R}$ with $p \in I^-(c(s))$, so there is a chronological path $k_1 : p \leadsto c(s)$ and for $k_2 := c|_{[s,b]}$, $k := k_2 * k_1, C : t \rightarrow I^-(k(t))$ is a chronological path from $P$ to $Q$.

Moreover, the last statement of Theorem[3] implies that $(CAP(X), d_1)$ is locally compact (if $(X, d)$ was locally compact). Theorem[11] equivalence of Items 1 and 9, implies that the topology $\tau_+ = cl^*(\tau_{d_1})$ on $P(X)$, generated by $\delta = cl^*d_1$, also is locally compact, as $cl$ is a homeomorphism. Metrizability has been shown in Th. [11] If $X$ is locally arcwise connected, w.l.o.g. consider a precompact neighborhood $U$ at $\partial^+ X$ contained in some $I^+(q)$.

The assertion on $IP(X)$ follows directly from the last assertion (local chr. convexity) in Th. [8] and chr. denseness.

**Lemma 2 (Non-imprisonment)** In a well-behaved locally chr. convex chr. space $X$, every chr. chain $c$ in a compact subset has an endpoint $p_e$, i.e. with $I^-(p_e) = I^-(c)$.

**Proof of the lemma.** The last statement in Th. [8] implies that there is a chr. convex neighborhood basis, by which one shows easily that an accumulation point is actually a limit (as each chr. chain cannot leave and re-enter a chronologically convex set).

**Lemma 3 (Continuity of $i_X$)** Any sequence $x : \mathbb{N} \rightarrow X$ that $\tau$-converges to $x_\infty \in X$ also converges to $x_\infty$ in $i_X^{-1}(\tau_+)$. **Proof of the lemma.** Directly from causal continuity and Th. [8] Item 4.

**Lemma 4 (Openness of $i_X$)** Let $p : \mathbb{N} \rightarrow (IP(X), \tau_+)$ converge to a point $I^-(x) \in i_X(X)$. Let $U$ be a $\tau$-open precompact neighborhood of $x$. Then there is $m \in \mathbb{N}$ with $\forall n \geq m : p(n) \in i_X(U)$.

**Proof of the lemma.** $X$ is Alexandrov, due to the previous assertion based on Theorem[3] Item 5, thus we find $p_{\pm} \in X$ s.t. $A := cl(I^+(p_-) \cap I^-(p_+)) \subset U$ is an compact neighborhood of $x$, and two other points $p_{\pm}$ with
\[ p_- \ll p'_- \ll x \ll p'_+ \ll p_+. \] (5.2)

Eq. 5.2 implies the existence of \( N_1, N_2 \in \mathbb{N} \) with

\[ p'_- \in \lim_{n \to \infty} p_n \subset \bigcap_{n \geq N_1} p_n, \] (5.3)

\[ p'_+ \in \bigcap_{n \geq N_2} \text{cl}(I^+(p_n)). \] (5.4)

(last assertion due to openness of \( I^+_P(X)(x) \)). Thus there is \( N := \max\{N_1, N_2\} \in \mathbb{N} \) with \( p_- \in \text{cl}(I^-(p_n)) \land p_+ \in \text{cl}(I^+(p_n)) \forall n \geq N \), so for all \( n \geq N \) we have \( p_n = I^-(c_n) \) for a chronological chain \( c_n : \mathbb{N} \to \text{cl}(I^+(p_-) \cap I^-(p_+)) = A \). As \( A \) is a compact subneighborhood of \( U \), and \( X \) is nonimprisoning as shown in the Lemma above, the \( c_n \) converge to some point in \( U \). Thus, as we desired to show, \( p_n = I^-(x_n) \) with \( x_n \in U \).

This concludes the proof of the first assertion of the theorem. \( Local \ compactness \) of \( (IP(X), \tau_+) \) follows as \( (C(X), d_1) \) is locally compact and \( \text{cl} : (P(X), \delta := \text{cl}^* d_1) \to (\text{CAP}(X), d_1) \) is a homeomorphism.

Finally, every point of \( \partial^+X \) can be connected through a chr. path to \( i_X(X) \), then precompactness of \( U \) implies \( locally \ arcwise \ connectedness \ of \ IP(X) \).

\[ \square \]

If a topology \( \tau \) on \( X \) is causally continuous, then obviously every \( \tau \)-converging net \( \tau_+ \)-converges, thus \( \tau_+ \) is the coarsest causally continuous topology.

The next theorem (in parts adapted from [34], compare also [19], Th. 4.16 for the corresponding assertion on \( \tau_- \)) shows that the intrinsic future completion defined above is homeomorphic to the union of the image of a conformal extension \( E \) and its future boundary. The map \( \varepsilon_E \) is defined as follows: Let \( A \in IP(M) \), then choose a chronological chain \( c : \mathbb{N} \to M \) with \( A = I^-(c(\mathbb{N})) \).

We define \( \varepsilon_E(A) \) to be the limit of \( E \circ c \) in \( \text{cl}(E(M), N) \) (which exists due to future compactness of \( E(M) \) in \( N \)). As argued before, this does not depend on the choice of \( c \). Moreover, for the statement below it makes no difference which Cauchy surface \( S \) one chooses in the definition of a CFE or if we even replace \( I^+(S) \) with \( M \). For the definition of \( X \) see the second last paragraph before Th. 8.

The relation between \( \tau_- \)-convergence and \( \tau_+ \)-convergence is completely described in the following theorem:

**Theorem 13** \( (P(X), \text{cl}^* d_1) \) is future compact, i.e. for all \( p \in P(X) \) (and all compact \( C \subset P(X) \)), we have that \( J^+(p) \) (resp., \( J^+(C) \)) is compact: Every sequence \( a : \mathbb{N} \to J^+(p) \) has a subsequence \( b = a \circ f \) convergent in \( (P(X), \text{cl}^* d_1) \) to some \( A \in P(X) \) which is, in general, decomposable. Let \( U \) be a maximal indecomposable past subset in \( A \). Then \( b(n) \to_{n \to \infty} U \).

**Proof.** Future compactness follows from the fact that for every \( p \in X \), the set \( C_p := \{ A \in C(X) | p \in A \} \) is compact in the metric \( d_1 \). Then by Theorem 11 \( \text{cl}(b(n)) \to_{n \to \infty} A \) implies
The previous theorem shows that \((P(X), \tau_+)\) contains at least as much information about the chr. structure of \(X\) as \((IP(X), \tau_-)\). Actually, the former topology contains strictly more information than the latter, which can be seen in the example of \(X := I^-((c(0; \infty)))\) for \(c : (0; \infty) \rightarrow \mathbb{R}^{1,2}\) given by \(c(t) := (t, \sin(1/t), 0)\): Consider the sequence \(a : \mathbb{N} \rightarrow \partial^+ X\) given by \(c(n) := I^-(c(1/n))\), then \(a\) does not converge in \(\tau_+\), as \(\limsup(a) = I^-((0; 1) \times [-1; 1] \times \{0\})\) but \(\liminf(a) = x_3^1((-\infty; -1))\). In \(\tau_-\), however, the sequence converges to each \(I^-((0, t, 0))\) with \(t \in [0; 1]\), whereas, in contrast, we will see an example (due to Harris) of a sequence displaying the same behaviour (convergence to more than one element of \(IP(X)\)) w.r.t. \(\tau_-\) while not converging at all in \(P(X)\) with \(\tau_+\).

The previous theorem also entails an interesting aspect of the causal boundary: it admits to determine maximal and minimal elements (w.r.t. the order given by inclusion) in the future of a compact subset.

**Theorem 14** Let \(C \subset IP(X)\) be compact, then there are minimal and maximal elements in \(J^+(C) \cap \partial^+ X\).

**Remark.** This problem has brought to the author’s attention by Leonardo García-Heveling.

**Proof.** Consider a \(\mu\)-infimizing sequence \(a\) in \(J^+(C)\), i.e., \(\mu(a_n) \rightarrow_{n \rightarrow \infty} \inf\{\mu(b) : b \in J^+(C, P(X))\}\). Then \(a\) has a subsequence \(u\) that converges to some \(v \in P(X)\). Now as \(\text{cl}(v) \cap C \neq \emptyset\), there is some \(w \in IP(X)\) with \(w \subset v\) and \(\text{cl}(w) \cap C \neq \emptyset\). Let \(x \in \text{cl}(w), c \in C \cap \text{cl}(I^+(x))\) and \(y : \mathbb{N} \rightarrow v\) with \(y(n) \rightarrow_{n \rightarrow \infty} x\), let \(c : x \xrightarrow{\tau_-} \text{past chr. curve}\). Then, by openness of the \(I^-((0; t))\) it is easy to see that \(c((0; t))) \in v\), thus with \(w := I^-(c((0; t)))\) we get the desired element of \(IP(X)\). Now as \(a\) was \(\mu\)-infimizing, a posteriori we see \(v = w\), \(w \in \text{cl}(I^+(C, P(X)))\) and \(w\) minimal: If \(z < w\) then \(d_1(\text{cl}(z), \text{cl}(w)) > 0\), and \(\text{cl}(z) \setminus \text{cl}(w)\) contains a ball, which is of positive measure, in contradiction to the assumption of an infimizing sequence. For the maximal case we correspondingly consider a \(\mu\)-supremizing sequence, a subsequence converging to some \(V \in P(X)\), then each maximal indecomposable past subset \(W\) of \(V\) is a maximal element (here we cannot conclude \(V = W\)).

**Theorem 15 (see [34])** The functor \(F_+\) respects conformal future-compact extensions. More precisely, let \((M, g)\) be a globally hyperbolic spacetime and \(E : (M, g) \rightarrow (N, h)\) be a conformal future-compact extension. Then the end-point map \(\varepsilon_E\) is a homeomorphism from \((IP(M), \tau_+)\) to \(E(M) \cup \partial^+(E(M), N) \subset N\) taking \(i_M(M)\) to \(E(M)\). Its continuous inverse is the map \(\overline{\varepsilon} : p \mapsto E^{-1}(I_N^+(p))\).

**Proof.** We want to show that \(\overline{\varepsilon}\) is a right and left inverse of \(\varepsilon_E\). First we have to show that \(\overline{\varepsilon}|_{\partial^+(E(M), N)}\) takes values in the IPs. Given a point \(p\) in \(\partial^+(E(M), N)\), then there is a timelike future curve \(c : [0; 1] \rightarrow N\) from a point \(q\) in \(E(M)\) to \(p\). Causal convexity of \(\text{cl}(E(M))\) implies that the curve is contained in \(\text{cl}(E(M))\) and that \(k := c|[0; 1]\) even takes values in \(E(M)\) and thus recall that an element \(x\) of an ordered set \((X, \leq)\) is called **maximal** iff \(\forall y \in X : \neg(x < y)\), correspondingly for minimal elements. On \(IP(X)\) inclusion corresponds precisely to causal relatedness. More on this in Sec. [34].
$k = E \circ \kappa$ for some $C^0$-inextendible timelike future curve in $M$. Then $E^{-1}(I_N(p)) = E^{-1}(I_N(c)) = E^{-1}(I_N(E \circ \kappa)) = I_N(k)$, where, in the last step, we use causal convexity. Therefore, indeed, for $q \in \partial^+ E(M)$, the set $E^{-1}(I^{-}(q) \cap E(M))$ is an IP in $M$. And $\overline{\varepsilon}$ is a right inverse of $\varepsilon_E$ as $N$ is distinguishing, and a left inverse as $c$ generating the IP is a chr. chain.

The map $\overline{\varepsilon}$ is continuous: First, the assignment $p \mapsto I^{-}(p)$ is inner continuous in any regular chr. space (Theorem 11) and is outer continuous in causally continuous chr. spaces, and g.h. manifolds are causally continuous (see 4, e.g.). Now let $a$ be a sequence in $N$ convergent w.r.t. the manifold topology. Then by i.o.-continuity of $I_N$ the sequence $k \mapsto I_N(a(k))$ and also the sequence $k \mapsto I_N(a(k)) \cap E(M)$ converges in $\tau_\infty$ and thus also in $d_1$ (following Theorems 2 and Theorem 4). As $E$ is a homeomorphism onto its image, mapping compacta to compacta, the sequence $b : k \mapsto E^{-1}(I_N(a(n)))$ converges in $\tau_\infty$ as well. Then Th. 2 and Th. 4 imply convergence of $b$ in $d_1$ and Theorem 11 implies convergence of $b$ in $\tau_\infty$. For the converse direction, let $A_n \in IP(M)$, then each $A_n$ is a PIP $I^{-}(q_n)$ in $N$, so we want to show that convergence of $n \mapsto A_n$ implies convergence of $n \mapsto q_n$, but this is just the openness statement of Th. 12.

A further positive feature of $\tau_\infty$ is that $\tau_\infty$ inherits conformal standard-stationarity, and mimicking the proofs in 27 and 19 one gets (see also the corresponding statement in 11) the following theorem. For its statement, recall that the Busemann function $b_c : M \to \mathbb{R} \cup \{\infty\}$ of an arclength-parametrized curve $c : I \to M$ is defined as $b_c(x) := \lim_{t \to \infty}(t - d(c(t), x))$. The finite-valued Busemann functions are 1-Lipschitz, if we denote their set by $B(M)$ then the Busemann boundary is defined as $\partial_B M := B(M) \setminus j(M)$ equipped with the compact-open topology, where for $p \in M$ we have $j(p) = -d(p, \cdot)$; for details see 20 and 28. The following theorem is from 28, we include a proof for the sake of self-containedness.

**Theorem 16** (28) Let $(M, g)$ be a conformally standard static spacetime with standard slice $N$. Then $(\partial^+ M, \tau_\infty) \setminus \{M\}$ is homeomorphic to $\partial_B N$.

**Proof.** As a Cauchy temporal function $t$ for the definition of the $f(t, A)$ (as in Theorem 11) for the IPs $A$ we take the static temporal function whose level sets are the standard slices isometric to $N$. The graph metric (i.e., Hausdorff distance in $\mathbb{R} \times N$) of any set $\{(t, x) | t < b_c(x)\}$ (which is $I^{-}(\hat{c})(\mathbb{R})$) for $\hat{c}$ with $\hat{c}(t) := (t, c(t))$ being the unique null lift of $c$ starting at $t = 0$) is after restriction to a compact subset equivalent to the $C^0$ (function space) topology on the Busemann functions, as the latter are 1-Lipschitz. The rest is as in the cited articles (in particular Theorem 6 in 28).

Thus $IP(M)$ and $\partial^+ IP(M)$ are cones with tip $M$, and the $\mathbb{R}$-action on $M$ mapping $(r, (t, x))$ to $(r + t, x)$ has a canonical continuous extension to $IP(M)$, which fixes the point $M$.

**Remark:** A corresponding fact holds for the conformally standard stationary case, as in 20.

As an example, we consider a manifold $G$, which is a slight variant of Harris’ unwrapped grapefruit-on-a-stick: $G = (\mathbb{R}^2, g_G := (f^2 \circ pr_2) \cdot g_0)$ where $g_0$ is the Euclidean metric and $f \in C^\infty(\mathbb{R}, (0; \infty))$ with $f(\mathbb{R} \setminus [-2; 2]) = 1$, $f(-x) = f(x) \forall x \in \mathbb{R}$, $f(x) \in (1; 2) \forall x \in (-2; 2) \setminus [-1; 1]$ and $f(x) = 2 \forall x \in [-1; 1]$, and $M := (\mathbb{R} \times G, -dt^2 + g_G)$.

For every arc-length parametrized curve $c$ in $G$, we define a null curve $\hat{c}$ in $M$ by $\hat{c}(t) := (t, c(t))$. For every curve $c : I \to G$ whose Busemann function is finite, $c(t)/|c(t)|$ converges to some $s \in S^1$. 23
Conversely, if for two curves $c_1, c_2$, this limit is the same, the corresponding points $I^-(c_1), I^-(c_2) \in \partial^+M$ are in the same $\mathbb{R}$-orbit. By distinction of the cases that $(pr_2 \circ c)^{-1}(0)$ is finite (in which case the curve stays in $pr_2^{1,1}([-1; 1])$ or infinite, we easily see that only curves $k$ in $M$ with $I^-(k) = M$ can $G$-project to ‘oscillatory’ curves in $G$, i.e., curves $\gamma$ intersecting $G \setminus B(0, r)$ for all $r > 0$ in both semiplanes: \(\forall r > 0: \gamma(I) \cap (G \setminus B(0, r)) \cap \mathbb{R} \times (\pm 0; \infty) \neq \emptyset\). Consequently, if for each ‘asymptotic angle’ \(\lim_{t \to \infty} c(t)/|c(t)| \in S^1\) we define \([c^\alpha]\) as the Busemann function of the standard Euclidean ray with angle $\alpha$ to the positive $x$-axis, \(d([c^\alpha], [c^{\beta}]) > 2\) whenever $\alpha \in (0; \pi)$ and $\beta \in (\pi; 2\pi)$, and for $\alpha = 0$ or $\alpha = \pi$ there are two different classes of curves \([c^0_\pm]\) and \([c^\pi_\pm]\) with finite Busemann function, depending on whether the curve stays above or below $[-2; 2]$. And also \(d([c^0_+], [c^0_-]) \geq 2\), so the two classes are distant. Thus the Busemann boundary consists of two cones over intervals $[0; \pi]$ and $[\pi; 2\pi]$ identified at their tips, parametrized by the asymptotic angle $\alpha$ (so at the $x$-axis at the values $0$ and $\pi$ each appear twice) and the $\mathbb{R}$-action. Thus Theorem 16 implies that $(\partial^+M, \tau_\pm)$ consists of two cones over an interval identified at their tips (i.e., at $M$) as well. In contrast, $\tau_-$ is in this case non-Hausdorff (see 18, whose Section 2 is emphatically recommended to the reader interested in learning more on the stationary case). Flores and Harris (in Sec. 2 of 18, p. 1219 of the journal article, p.8 in the arXiv version) consider (mutatis mutandis) the sequence $\sigma : \mathbb{N} \to M$ defined by $\sigma(n) := (\frac{n}{2} + w, n, 0)$ for $w := \int_{-2}^{2} f(t)dt$, which is not a chr. chain, but still $\sigma(n) \in I^+(\sigma(0)) \forall n \in \mathbb{N}$. In $\tau_+$, the sequence $\sigma$ does not converge at all, which shows that $\tau_+$ is in general not future-compact. Even more, it shows that also $J^+(p) \cap \partial^+X$ is in general noncompact, even if $X$ is g.h., as the $\sigma(n)$ can be made part of $\partial^+X$ by considering $X := M \setminus \bigcup_{k=1}^{\infty} J^+(\sigma(k))$, which is an open chr. convex subset of $M$.

Here we can see an interesting similarity to the case of a Riemannian manifold $(N, h)$. Its Gromov boundary $\partial_G N$ is defined as

\[\text{cl}\{[d_h(\cdot, x)]|x \in N\}, L_1(N, d_h)/\mathbb{R}\} \setminus \{d_h(\cdot, x)|x \in N\}\]

where $L_1(N, d_h)$ is the space of 1-Lipschitz functions on $N$ (cf. 20 for details). Now, if $(N, h)$ is asymptotically flat, i.e. \(|g - g_0| \in O(r)|\) in a polar coordinate chart, then the Gromov and the Busemann boundary coincide, as in a Euclidean chart, for every proper Busemann function $b_c$ to a curve $c$, the map $u_c : [0; \infty) \to S^{w-1}, t \mapsto c(t)/|c(t)|$ must converge, and $\lim_{t \to \infty} u_c(t) = \lim_{t \to \infty} u_k(t)$ implies $b_c = b_k$ for any two such curves, unlike in $\mathbb{R} \times G$.

The example is very instructive as here $IP^+(X) = \text{cl}(i_X(X), (P(X), \tau_+))$ is strictly larger than $IP(X)$. As $\tau_-$ is coarser than $\tau_+$ we have $\text{cl}(i_X(X), (P(X), \tau_+))$ strictly larger than $IP(X)$ as well. But each limit set of a sequence in $X$ w.r.t. the non-Hausdorff topology $\tau_-$ contains at least one element in $IP(X)$, thus $\partial^+X = \text{cl}(i_X(X), (P(X), \tau_-)) \cap IP(X)$ is still future complete and sequentially future compact. In contrast, $(IP(X), \tau_+)$ is not future compact. This indicates that, using $\tau_+$, one should consider $IP^+(X)$ instead of $IP(X)$. Of course, often the two spaces coincide, e.g. in the asymptotically flat case or in certain warped products over bounded intervals, cf. next section.

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6 Application to multiply warped chronological spaces

We want to transfer statement and proof of \cite{27}, Prop. 5.2, from \( \tau_- \) to \( \tau_+ \), generalizing it at the same time from multiply warped spacetimes to multiply warped chronological spaces.

Let \( I \) be a real interval. A multiply warped chronological space over \( I \) is a space \( X = I \times K, K := \times_{i=1}^n K_i \) where \((K_i, d_i)\) are complete length metric spaces, and \( f_i \in C^1(\text{int}(I), (0; \infty)) \) for \( i \in N^*_m \), and let \( \text{pr}_i : I \times K \to K_i \) be the standard projections, \( \text{pr}_0 = \text{pr}_I : I \times K \to I \) and \( \text{pr}_K : I \times K \to K \). For \( p \in X \), we write \( p_i := \text{pr}_i(p) \). The topology of \( X \) is the product topology \( \tau_p \). There is a chronology \( \ll_0 \) defined by \((t, x) \ll_0 (s, y)\) if and only if there is a good path from \((t, x)\) to \((s, y)\), which is a continuous path \( c : [t; s] \to X \) from \((t, x)\) to \((s, y)\) with \( c_0 := \text{pr}_0 \circ c \in C^1 \) with \( c_0(u) = u \forall u \) and s.t. for all \( r \in (t; s) \) we have a neighborhood \( U(r) \) of \( r \) s.t. each \( \text{pr}_i \circ c|_{U(r)} \) is \( k_i,U \)-Lipschitz with respect to \( f_i(r)d_i \), with \( \sum_{i=1}^n k_i^2 < 1 \). We denote the multiply warped chron. space above by \( I \times^f \times_{j=1}^m K_j \). The next theorem shows that the definition of the chron. relation generalizes the usual chronology of multiply warped spacetimes, i.e., in a multi-warped spacetime, the future timelike curves are exactly the good paths. For \( a, b \in \{ -\infty \} \cup R \cup \{ \infty \} \), \((K_i, h_i)\) complete Riemannian manifolds and \( f_i : (a; b) \to (0; \infty) \), the multiply-warped spacetime \((a; b) \times^f \times_{j=1}^m K_j \) is defined as \((a; b) \times^f \times_{j=1}^m K_j \) with the metric \( g := -dt^2 + \sum_{j=1}^n (f_j \circ \text{pr}_0)(\tau_j^* h_j) \).

One example of this situation is (interior) Schwarzschild spacetime with \((after an obvious conformal relation generalizes the usual chronology of multiply warped spaces, i.e., in a multi-warped spacetime, the future timelike curves are exactly the good paths). For \( a, b \in \{ -\infty \} \cup R \cup \{ \infty \} \), \((K_i, h_i)\) complete Riemannian manifolds and \( f_i : (a; b) \to (0; \infty) \), the multiply-warped spacetime \((a; b) \times^f \times_{j=1}^m K_j \) is defined as \((a; b) \times^f \times_{j=1}^m K_j \) with the metric \( g := -dt^2 + \sum_{j=1}^n (f_j \circ \text{pr}_0)(\tau_j^* h_j) \).

Theorem 17 \( \forall x, y \in I \times^f \times_{j=1}^m K_j : x \ll y \iff \text{there is a good path } x \leadsto y \text{ in } I \times^f \times_{j=1}^m K_j. \)

Proof: A timelike curve \( c : (u; v) \to M \) can be reparametrized s.t. \( c_0(t) = t \) and thus \( c_0' = 1 \), then \( 0 > g(c'(s), c'(s)) = -1 + \sum_{j=1}^n f_j(s)h_j(c_j'(s), c_j'(s)) \).

Any such \( c \) is a good path: At each \( r \in I \) we have \( 0 > g(c'(r), c'(r)) = -1 + \sum_{j=1}^n f_j(r)h_j(c_j'(r), c_j'(r)) \).

By continuity, there is \( \varepsilon > 0 \) s.t.

\[
\sum_{j=1}^n \max \{ f_j(r)g(c_j'(u), c_j'(u)) : u \in [r - \varepsilon; r + \varepsilon] \} < 1,
\]

so \( c \) is good. Conversely, let a good path \( c \) from \( x \) to \( y \) be given, then for all \( r \in (a; b) \) we want to show that there is \( \varepsilon > 0 \) with \( c(r - \varepsilon) \ll c(r) \ll c(r + \varepsilon) \) for all \( u \in (0; \varepsilon) \). For \( U(r) \) as in the definition of goodness of a curve, let \( \kappa := c(U(r)) \). Then the \( \kappa_i \) are \( k_i \)-Lipschitz w.r.t. \( f_i(r)d_i \) for all \( i \in N^*_m \) and \( \rho = \sum_{j=1}^n k^2_i < 1 \). We find \( \delta > 0 \) with \((1 + \delta)\rho < 1 \) and we define \( \varepsilon > 0 \) s.t. \( f_i(c(x)) < (1 + \delta)f_i(c(r)) \forall u \in [r - \varepsilon; r + \varepsilon] \). Then we can construct the required timelike curve \( \gamma \) between \( c(r) \) and \( c(r + u) \) as a \( d_i \)-geodesic curve in each factor \( M_i \) From \( g_i(\gamma_i'(x), \gamma_i'(x)) < k_i^2 \) we conclude

\[
g'(\gamma_K(x), \gamma_K(x)) = \sum_{i=1}^m f_i(x)g_i(\gamma_i'(x), \gamma_i'(x)) < (1 + \varepsilon)\sum_{i=1}^m k_i < 1,
\]

which shows that \( \gamma \) is timelike. \( \square \)
Theorem 18 Let $X = I × \prod_{j=1}^{m} K_j$, $I = (a;b)$ with $b < \infty$. Assume that for all $i \in \mathbb{N}_m$, $(K_i,d_i)$ complete, and (*) $\int_c^b f_i^{-1/2}(x)dx < \infty$ for some (hence any) $c \in (a;b)$. Then $(IP(X), \ll_{BS}, \tau_+)$ is chronologically homeomorphic to $(X := (a;b) × K_1 × ... × K_n, \ll_0)$.

Remark. Schwarzschild spacetime satisfies the additional finiteness condition (*) on the $f_j$, the Grapefruit Spacetime does not because of the infinite range of the time parameter.

Proof. The map $L : x \mapsto I^-(x)$ is a chr. isomorphism from $((a;b) × K, \ll_0)$ to $(IP(X), \ll_{BS})$:
The map $i_X(p) \mapsto p$, $I^-(c(N)) \mapsto \lim_{n \to \infty} X_n$ is inverse to $L$. Eq. (*) implies that for every chr. chain $c$ in $X$, $pr_K \circ c$ converges to some $k_c \in K$, thus to each $c$ without endpoint in $X$ we can assign the point $(b,k_c)$, and as $i_X$ is chronological, we only have to check whether $y \gg_0 i_X(z)$ for all $y \in \partial^+ IP(X)$ and all $z \notin y$, but this is true as every chr. chain in $X$ can be interpolated and equipped with an endpoint as above to become a good path in $\overline{X}$.

Both sides are metrizable. Now it is an easy exercise to see that convergence of the closures of past sets in $X$ is coordinate-wise, i.e. for a sequence $A : \mathbb{N} \rightarrow P(X)$ we distinguish the cases $B := \lim sup_{n \in \mathbb{N}} (\sup \{pr_0(x) | x \in A_n\}) < b$ and $B = b$ and show that in each case the limsup is a true limit, and that, for $i_X(X) \ni A(n) = I^-(p_n)$, the projection to the other factors have to converge as well, showing also that $IP^+(X) = IP(X)$.

7 Causal perspective, solomonic conclusion, open questions

Finally, one can look at the causal completion from the perspective of the causal relation $\ll$ instead of the chr. relation $\ll_0$. One obvious advantage of this procedure is that we can require semi-fullness (of $\ll$) right from the beginning and still include causal diamonds $J^+(p) \cap J^-(q)$. With $\alpha$, the functor $IP$ would map every causally continuous well-behaved chr. space $X$ to a causally simple chr. space $IP(X)$: Let $p \in X$ and $a : \mathbb{N} \rightarrow J^+_{IP(X)}(p)$, i.e., $a(n) \supset p \forall n \in \mathbb{N}$ and $a(n) \rightarrow_{n \to \infty}^+ v$, i.e., $I_\mathbb{N}(\lim inf a) = v = I_\mathbb{N}(\lim sup a)$. Then we conclude $p = I_\mathbb{N}(p) \subseteq I_\mathbb{N}(\bigcap_{j \in \mathbb{N}} a(j)) \subseteq I_\mathbb{N}(\lim inf a) = v$. The case $a : \mathbb{N} \rightarrow J^-_{IP(X)}(p)$ is proven analogously, here the last line is $p = I^-_\mathbb{N}(p) \supseteq I^-_\mathbb{N}(\bigcup_{j \in \mathbb{N}} a(j)) \supseteq I^-_\mathbb{N}(\lim sup a) = v$.

This corresponds to the fact mentioned above that $\alpha$ introduces spurious causal relations in $i_X(X)$.

Fortunately, we can refine $\alpha$ to a functor $\tilde{\alpha}$ from chr. to ordered spaces $(X, \ll) \mapsto (X, \tilde{\alpha}(\ll))$ with $\alpha(\ll) \supseteq \tilde{\alpha}(\ll)$ that avoids this undesired phenomenon: We define, for any real interval $I = [a;b]$ and any chr. space $(X, \ll)$ a curve $c : I \rightarrow X$ to be chr. coherent iff $\forall t \in (a;b) : I^-(c(t)) = \bigcup_{s \in [a;t]} I^-(c(s))$ and $I^+(c(t)) = I^+(\bigcap_{s \in [t;b]} I^-(c(s)))$ for all $t \in [a;b]$.

The definition of $\ll = \tilde{\alpha}(\ll)$ implies $\ll \subseteq \tilde{\alpha}(\ll) \subseteq \alpha(\ll)$ and the push-up property $x \leq y \Leftarrow z \forall x \ll y \ll z \Rightarrow x \ll z \forall x,z,y \in X$ (valid even for $\alpha(\ll)$ and in every spacetime, see e.g. [33]).

The pus-up property entails $\int (J^+(A)) = I^+(A)$ for each $A \subset X$.

An advantage of $\tilde{\alpha}$ over $\alpha$ is the absence of spurious causal relations for causally continuous spacetimes even if they are not causally simple:

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*This has been chosen for the sake of simplicity of the presentation and can be arranged by a reparametrization without disturbing the other properties.*

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Theorem 7.1 Let \((X, g)\) be a causally continuous spacetime with \(\ll \subset M \times M\) resp. \(\leq \subset M \times M\) being the usual chr. resp. causal relation defined by connectability along timelike resp. causal curves. Then \(\hat{\alpha}(\ll) = \leq\).

**Proof.** The inclusion of the right-hand side in the left-hand side follows from the fact that each \(\leq\)-causal curve is an \(\alpha(\ll)\)-chr. coherent curve. For the other inclusion, let \(p, q \in M\) with \(p \hat{\alpha}(\ll) q\). Then there is an \(\alpha(\ll)\)-chr. coherent curve \(c : I \to M, p \sim q\). Now \(c\) is continuous. Let a sequence \(t \mapsto I\) with \(t(n) \to_{n \to \infty} t_\infty \in I\). Then any subsequence \(t \circ j\) has a monotonous subsequence \(s := t \circ j \circ k\), such that \(I^-(c \circ s)\) converges to the union or intersection of the respective subsets equalling \(I^-(c(t_\infty))\), and in the manifold topology (which equals \(\tau_+\)) any sequence of points whose pasts converge converges. Then \(c\) is also a (continuous) causal curve, as every spacetime is locally causally simple, which allows to use [33] Th. 3.69.

A **causal set** is a (partially) ordered set \((X, \leq)\) that is semi-full and causally separable (defined like chr. separable, only replacing \(\ll\) with \(\prec\)) and nowhere total, i.e., \(\forall x \in X : X \neq J(x)\).

**X** is called **future** resp **past causally simple** iff \(J^+(p)\) is closed for every \(p \in X\), or equivalently, that \(J^+ \subset X \times X\) is closed: The nontrivial implication \(\Rightarrow\) follows from \(J^+ = \text{cl}(I^+)\), whose nontrivial inclusion \(\Rightarrow\) follows from \((x, y) \in \text{cl}(I^+) \Rightarrow y \in \text{cl}(I^+(x)) = J^+(x)\).

Analogously to the above, a **causal space** is a causal set equipped with a topology. A preregular causally continuous, locally causally simple path-generated causal space \((X, \leq, \tau)\) is called **well-behaved** iff \((X, \tau)\) is locally compact, \(\sigma\)-compact and locally arcwise connected.

Conversely to the functors \(\alpha\) and \(\hat{\alpha}\), there are at least two possible ways \(\beta, \gamma\) to define a chr. relation from a causal one. Firstly, following a suggestion of Miguzzi and Sánchez [33], \(\beta(\leq)\) defined by

\[(x, y) \in \beta(\leq) : \Leftrightarrow (x \leq y \land (\exists u, v \in X : x < u < v < y \land J^+(u) \cap J^-(v) \text{ not totally ordered}))\]

is a chr. relation which encodes the fact from Lorentzian geometry that a null geodesic ceases to be maximal after the first cut point and manifestly has the push-up property. Secondly, we define

\[p(\gamma(\ll))q : \Leftrightarrow \forall a > p \forall c < q \exists b \in J(a, p) \exists d \in J(c, q) : b \leq d.\]

Both \(\beta\) and \(\gamma\) recover the right chr. relation from the causal relation on causally continuous spacetimes. Under which conditions on a causal set they are equal and/or inverses of \(\hat{\alpha}\) is an interesting question that cannot be pursued further here. The warning example of \(\mathbb{R}^{1,2} \setminus I(0)\) shows that semi-fullness of \(\beta(\leq)\) or \(\gamma(\leq)\) is not automatically satisfied. The brute-force way here would be to just assume a partial order relation \(\leq\) on \(X\) such that \(\beta(\leq) = \gamma(\leq)\) and \(\tau_+\) make \(X\) a well-behaved chr. space. But already with much weaker hypotheses we can show that for a well-behaved causally continuous causal space \(X\), the causal space \((\text{ICP}(X), \subset, \tau_+)\) of indecomposable causal past subsets is well-behaved and causally continuous by the proof of Theorem [12] mutatis mutandis. Again, all central statements work also in the subset \(CP(X)\) of causal past subsets instead of the indecomposable ones \(\text{ICP}(X)\).

The causal perspective to induce \(\ll\) from the primordial object \(\leq\) allows for a reinterpretation of \(\text{IP}(X)\) as the set of filters on the poset \((X, \leq)\), and for a simple one-line description for a future completion as a causal-chr. space with a metric \(\delta\), where \(d_1\) is the above pointed Hausdorff metric:
\((IP(X), \leq := \subset, \ll := \gamma(\leq), \delta := \text{cl}^*(d_1))\).

\(IP(X)\) is \(J^-\)-distinguishing, as \(x = \bigcup \{J^-(x) | x \in IP(X)\}\). The push-up property entails \(\bigcap \{J^+(r) | r \in R\} = \bigcap \{J^+(r) | r \in R\}\) for every past subset \(R\) of \(X\), and consequently we can and will define \(\tau_+\) in our new context using \(J^+\cap\) instead of \(I^+\cap\). The following theorem shows that the diagram

\[
\begin{array}{ccc}
(X, \ll) & \longrightarrow & (IP(X), \ll_{BS}) \\
\delta & | & \delta \\
(X, \leq) & \longrightarrow & (IP(X), \subset)
\end{array}
\]  

(7.1)

is commutative, but one should keep in mind that the construction of \(IP(X)\) also in the upper line involves the chr. relation, which we will see can defined as \(\gamma(\leq)\) there. Alternatively, we can define causal past sets \(CP(X)\) and define indecomposability analogously to the chr. case.

**Theorem 19** For every chr. set \(X\), we have \(A\alpha(\ll_{IP(X)}) B \Leftrightarrow A \subset B\).

**Proof.** For the implication from the right to the left, assume \(U \ll_{BS} A\), then \(\emptyset \neq I^+_X(U) \cap A \subset I^+_X(U) \cap B\), and if \(U \ll_{BS} B\), we have \(I^+_X(U) \cap A \subset I^+_X(U) \cap B\). But also \(I^+_X(B) \subset I^+_X(A)\): For \(U \in I^+_X(B)\) we have \(\emptyset \neq I^+_X(B) \subset I^+_X(A)\) due to the monotonicity of \(I^+_X\). For the reverse implication, let \(A\alpha(\ll_{BS})B\), i.e. \(I^-_X(A) \subset I^-_X(B)\), for every \(U \in IP(X)\) we have \(I^+_X(U) \cap A \neq \emptyset \Rightarrow I^+_X(U) \cap B \neq \emptyset\). Now let \(a \in A\). Then \(U := I^-(a) \in IP(X)\) and \(a \in I^+_X(U) \cap A \neq \emptyset\), so \(I^+_X(U) \cap B \neq \emptyset\) and \(a \in B\). \(\square\)

We define a chr. space \(X\) to be **globally hyperbolic** or g.h. iff \(J^+(K) \cap J^-(C)\) is compact for any two compact sets \(C, K\) in \(X\). In the globally hyperbolic case, the future boundary is achronal:

**Theorem 20**

1. Let \((X, \ll, \tau_+)\) be a well-behaved past-full g.h. chr. space. Let \(p \in IP(X) \setminus i_X(X)\). Then \(J^+(p) \cap i_X(X) = \emptyset = I^+(p)\) for \((\ll_{IP(X)}, \leq_{IP(X)}) = (\ll_{BS}, \alpha(\ll_{BS}) = \subset)\).

2. Let \((X, \leq)\) be a causal set s.t. \((X, \leq, \gamma(\leq), \tau_+)\) is a well-behaved g.h. chr. space. Let \(p \in IP(X) \setminus i_X(X)\). Then \(J^+(p) \cap i_X(X) = \emptyset = I^+(p)\), for \((\ll_{BS}, \leq_P(X)) = (\gamma(\subset), \subset)\).

**Proof.** The first item is immediate from the theorems in Sec. 5. For the second item, we first show the first equality: Assume the opposite, then \(p \subset J^+(x) \cap J^-(y)\) for \(x, y \in i_X(X)\). As \(i_X : (X, \leq) \to (IP(X), \subset)\) is a monotonic map, mapping the \(J^\pm\) to each other, we get for \(c\) being a chr. chain generating \(p\) that \(c(N) \subset J^+(x) \cap J^-(y)\), which is compact. But \(c\) has no endpoint, contradiction. For the second equality we use that \(I^+(p)\) is open, and then, if \(q \in I^+(p)\), we have a chronological chain in \(X\) generating \(q\), which therefore enters \(I^+(p) \cap J^-(q)\), and then we can use the first equality to show a contradiction. \(\square\)

**Theorem 21** Let \((Y, \ll, \tau)\) be a chr. dense chr. space, then \(J^\pm(p) \subset \text{cl}(I^\pm(p)) \forall p \in X\).

**Proof:** Directly from the push-up property. \(\square\)
Theorem 22 Let $(X,\ll,\tau)$ be a chr. dense regular chr. space. Then
\[ x\alpha(\ll)y \Rightarrow i_X(x)\alpha(\ll_{BS})i_X(y). \]
Moreover, if $(X,\ll,\tau)$ is $\alpha$-causally simple then
\begin{enumerate}
  
  1. $J^\pm(K) = \text{cl}(I^\pm(K))$ for all compact subsets $K$ of $X$, in particular $J^\pm(p) = \text{cl}(I^\pm(p))\forall p \in X$.
  
  2. $(X,\ll,\tau)$ is causally continuous.
  
  3. $x\alpha(\ll_X)y \Leftrightarrow i_X(x)\alpha(\ll_{BS})i_X(y)$
\end{enumerate}

Proof. The first assertion follows from $x \leq y \Rightarrow I^-(x) \subset I^-(y)$ \[ Eq.\ 6 \] $i_X(x) \leq i_X(y)$.

The first item applied to points follows from the first assertion of Th. \[ Eq.\ 6 \] For the first item applied to general compact sets, let $y \in \text{cl}(J^+(C))$, then there is a sequence $a : \mathbb{N} \rightarrow X$ and a sequence $b : \mathbb{N} \rightarrow C$ with $a(n) \rightarrow_{n \rightarrow \infty} y$ and $b(n) \leq a(n) \forall n \in \mathbb{N}$. Then a subsequence $b \circ j$ of $b$ converges to some $x \in C$. As causal simplicity implies that $J^+ \subset X \times X$ is closed and the (in $X \times X$) convergent sequence $(a, b \circ j)$ takes values in $J^+$ we can conclude $y \in J^+(x) \subset J^+(C)$. So $J^+(C)$ is closed, and obviously $J^+(C) \subset \text{cl}(I^+(C))$, which implies the assertion.

For the second item, $\text{cl}(I^+(x)) = J^+(x) \forall x \in X$ implies $\forall p,q \in \text{cl}(I^+(p)) \Rightarrow p \in \text{cl}(I^+(q)) \ (*)$. Now assume that $J^-$ is not outer continuous, then there is $p \in X$ and a compact $C \subset X$ such that $C \cap \text{cl}(J^-(p)) = \emptyset \ (**)$ yet there is a sequence $a$ with $a(n) \rightarrow_{n \rightarrow \infty} p$ such that $C \cap \text{cl}(I^-(a(n))) \neq \emptyset \forall n \in \mathbb{N}$. Choose $c(n) \in C \cap \text{cl}(I^-(a(n)))$. Then a subsequence $c \circ j$ converges to some $c_\infty \in C$. For all $y \leq c_\infty$, openness of $J^+(y)$ implies that $y \ll a(n)$ for $n$ sufficiently large, thus $p \in \text{cl}(I^+(y))$. This implies with (*) that $y \in \text{cl}(I^-(p)) = J^-(p)$. Now let $z(n) \rightarrow_{n \rightarrow \infty} c_\infty$ with $z(n) \ll c_\infty \forall n \in \mathbb{N}$, then $z(n) \in J^-(p) \forall n \in \mathbb{N}$, which together with closedness of $J^-(p)$ implies that $c_\infty \in J^-(p)$. As $c_\infty \in C$, we have $J^-(p) \cap C \neq \emptyset$, contradicting (**).

For the missing implication in the third item, consider $i_X(x) \leq i_X(y) \Rightarrow \text{Eq.\ 22} \ I^{-}_X(x) \subset I^{-}_X(y)$, which implies by chr. denseness $x \in \text{cl}(I^-(y)) = J^-(y)$. \[ \square \]

To transfer the results to $\hat{\alpha}$, note that a causally coherent path maps to a causally coherent path under $i_X$, and that Th. \[ Eq.\ 21 \] implies that every $\hat{\alpha}$-causally simple chr. space is $\alpha$-causally simple. Now we want to define strong causality for chr. spaces. This point is a bit subtle, as the embedding of strong continuity on the causal ladder relies on the non-imprisonment property for causal curves, which in turn is proven via local coordinates. The same holds for the proof that any strongly causal spacetime carries the Alexandrov topology (for both see \[ Eq.\ 23 \]). A chr. space $X$ is almost strongly causal iff every neighborhood of any $x \in X$ contains a causally convex subneighborhood of $x$. A causal chain $c$ in an almost strongly causal chr. space $X$ converges to $p \in X$ if and only if $I^-(c) = I^-(p)$. $X$ called strongly causal iff every neighborhood $U$ of any $x \in X$ has a subneighborhood of the form $J^+(K) \cap J^-(L)$ for compact subsets $K,L \subset X$ (we can even choose $K,L \subset X$ if $X$ is locally compact, locally path-connected and path-generated). If $X$ is causally simple, by intersection with $J^+(x)$ we can ensure that $K \subset J^-(x), L \subset J^+(x)$ if $J^+(x), J^-(x) \neq \emptyset$.

Finally, $X$ is called Alexandrov iff every neighborhood of any point $x \in X$ has a subneighborhood of $x$ of the form $J^+(y) \cap J^-(z)$ for some $y,z \in X$, i.e., iff its topology is the Alexandrov topology. The three notions are different: The weakest one, almost strong causality, is equivalent to requiring that the subneighborhood can be chosen to be the intersection of a future and a past subset (write a convex subneighborhood $V$ as $I^+(V) \cap I^-(V)$). Secondly, $X_0 := \mathbb{R}^{1,2} \setminus I^+(0)$ with the induced chronology and topology is strongly causal (even causally simple) but not Alexandrov. For an example of an almost strongly causal yet non-strongly causal chr. space, replace spatial $\mathbb{R}^2$ in $X_0$ with the infinite-dimensional separable Hilbert space $l_2$, losing local compactness.
It is easy to see that for $X$ well-behaved, we have $\text{int}(J^\pm(A)) = I^\pm(A)$ for all $A \subset X$ (where either $X$ is a chr. space and $\leq := \Delta(\ll)$ or $X$ is a causal space and $\ll := \beta(\leq)$ or $\ll := \gamma(\leq)$).

If $X$ is causally continuous, then $X$ is strongly causal, and $X$ is Alexandrov: Let $x \in U_2 \subset X$ be open. We define $U_3$ and $L$ as in the proof of Th. 8 while replacing $CB(x,r)$ with $A(r) := J^+(c(-t)) \cap J^-(c(t))$ for a $C^0$ future timelike curve $c$ with $c(0) = x$, then we argue as in the proof of the last item of that theorem, as $A(0) = \{ x \}$.

So if $(X, \leq)$ is a causal set with $\alpha \circ \gamma(\leq) = \leq$ and such that $(X, \leq, \gamma(\leq), \tau)$ is a well-behaved past-full g.h. chr. set, then for $ICP(X)$ we have $(ICP(X), \subset, \gamma(\subset)) = (IP(X), \subset, \ll_{BS})$ where $\ll_{BS}$ is defined via the chr. relation $\gamma(\leq)$ on $X$: We have the commuting diagram

$$(X, \leq) \xrightarrow{\gamma} (ICP(X), \subset) \xrightarrow{\gamma} (X, \ll = \gamma(\leq)) \xrightarrow{\gamma} (IP(X), \ll_{BS})$$

where we identify $A \in ICP(X)$ with $I^-(A) \in IP(X)$.

The assignment of the topologies $\tau_\beta := \tau_+ \circ \beta$ and $\tau_\gamma := \tau_+ \circ \gamma$ constitutes functors from the category POS of partially ordered sets to the category TOP of topological spaces, and, as opposed to the Alexandrov topology, yielding the desired topology on causal spaces with future boundary.

After so many facts about $\tau_-$ and $\tau_+$, our question remains which topology is more advantageous. Unsurprisingly, the author is convinced that the choice between $\tau_-$ and $\tau_+$ has revealed itself more and more as a pure matter of taste. Those who enjoy applicability to the wide class of strongly causal spacetimes, future compactness and compactifications to both time directions (keep in mind that $M \cup \partial^+ M$ alone is noncompact for both topologies!) might prefer $\tau_-$, whereas those attracted by metrizability and interested in the future completion as a quick practical tool are likely to tend to $\tau_+$, in particular if one widens the scope to include all of $P(X)$. Theorem 13 shows that the two topologies on $IP(X)$ carry basically the same information (whereas on $P(X)$ or even $\partial^+ P(X)$ the topology $\tau_+$ carries slightly but strictly more information, so whereas $(IP(X), \tau_-)$ is a good minimal future completion, maybe $(P(X), \tau_+)$ is a good larger future completion with some additional advantages).

An interesting question is whether there is a holographic principle at work allowing to reconstruct the chr. structure of $X$ from $(\partial^+ X, \ll_{BS}, \tau_-)$ or $(\partial^+ X, \ll_{BS}, \tau_+)$. In a recent work 37, the author proved a Lorentzian finiteness theorem, defining a functor from a Lorentzian to a Riemannian category and using Cheeger finiteness, which in turn uses Gromov compactness of the space of compact metric spaces. It is quite obvious that such a functor is restricted to timelike compact Lorentzian manifolds, e.g. Cauchy slabs. But there is an refined approach to Gromov compactness: metric measure spaces, of which Berger writes in his book 19 that they are, in his opinion the geometry of the future. The causal perspective shows that they could have an appropriate Lorentzian analogue: (partially) ordered measure spaces, forming a category $PM$, whose objects comprise Lorentzian spacetimes but also possible degeneracies. There is an obvious injective functor from the category of causal Lorentzian manifolds into $PM$: The order allows to reconstruct the conformal structure and together with the volume form one obtains the entire Lorentzian metric. To obtain finiteness theorems, it is fundamental to know compactness of appropriate subsets of $\text{Obj}(PM)$, subject of a row of forthcoming articles starting with 37, 38.
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