Spectra and Laplacian spectra of arbitrary powers of lexicographic products of graphs

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\textbf{Abstract}

Consider two graphs $G$ and $H$. Let $H^k[G]$ be the lexicographic product of $H^k$ and $G$, where $H^k$ is the lexicographic product of the graph $H$ by itself $k$ times. In this paper, we determine the spectrum of $H^k[G]$ and $H^k$ when $G$ and $H$ are regular and the Laplacian spectrum of $H^k[G]$ and $H^k$ for $G$ and $H$ arbitrary. Particular emphasis is given to the least eigenvalue of the adjacency matrix in the case of lexicographic powers of regular graphs, and to the algebraic connectivity and the largest Laplacian eigenvalues in the case of lexicographic powers of arbitrary graphs. This approach allows the determination of the spectrum (in case of regular graphs) and Laplacian spectrum (for arbitrary graphs) of huge graphs. As an example, the spectrum of the lexicographic power of the Petersen graph with the googol number (that is, $10^{100}$) of vertices is determined. The paper finish with the extension of some well known spectral and combinatorial invariant properties of graphs to its lexicographic powers.

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1 Introduction

The lexicographic product of a graph $H$ by himself several times is a very special graph product, it is a kind of fractal graph which reproduces its copy in each of the positions of its vertices and connects all the vertices of each copy with another copy when they are placed in positions corresponding to adjacent vertices of $H$. This procedure can be repeated, reproducing a copy of the previous iterated graph in each of the positions of the vertices of $H$ and so on. Despite the spectrum and Laplacian spectrum of the lexicographic product of two graphs (with some restrictions regarding the spectrum) expressed in terms of the two factors be well known (see [4], where a unified approach is given), it is not the case of the spectra and Laplacian spectra of graphs obtained by iterated lexicographic products, herein called lexicographic powers, of regular and arbitrary graphs, respectively. A lexicographic power $H^k$ of a graph $H$ can produce a graph with a huge number of vertices whose spectra and Laplacian spectra may not be determined using their adjacency and Laplacian matrices, respectively. The expressions herein deduced for the spectra and Laplacian spectra of lexicographic powers can be easily programmed, for example, in Mathematica, and the results can be obtained immediately. For instance, the spectrum of the 100-th lexicographic power of the Petersen graph, presented in Section 3, was obtained by Mathematica and the computations lasted only a few seconds. Notice that such lexicographic power has the googol number (that is, $10^{100}$) of vertices.

The paper is organized as follows. In the next section, the notation is introduced and some preliminary results are given. The main results are introduced in Section 3, where the spectra (Laplacian spectra) of $H^k[G]$ and $H^k$, when $G$ and $H$ are regular (arbitrary) graphs, are deduced. Particular attention is given to the Laplacian index and algebraic connectivity of the lexicographic powers of arbitrary graphs. In Section 4, the obtained results are applied to extend some well known properties and spectral relations of combinatorial invariants of graphs $H$ to its lexicographic powers $H^k$.

2 Preliminaries

In this work we deal with simple and undirected graphs. If $G$ is such a graph of order $n$, its vertex set is denoted by $V(G)$ and its edge set by $E(G)$. The elements of $E(G)$ are denoted by $ij$, where $i$ and $j$ are the extreme vertices of the edge $ij$. The degree of $j \in V(G)$ is denoted by $d_G(j)$, the maximum and minimum degree of the vertices in $G$ are $\delta(G)$ and $\Delta(G)$ and the set of the neighbors of a vertex $j$ is $N_G(j)$. The adjacency matrix of $G$ is the $n \times n$ matrix $A_G$ whose $(i, j)$-entry is equal to 1 whether $ij \in E(G)$ and is equal to 0 otherwise. The Laplacian matrix of $G$ is the matrix $L_G = D - A_G$, where $D$ is the diagonal matrix whose diagonal
Theorem 2.1. Let $H$ be a graph such that $V(H) = \{1, \ldots, n\}$ and, for
Let $j = 1, \ldots, n$, let $G_j$ be a $p_j$-regular graph with order $m_j$. Then
\[
\sigma_A(H[G_1, \ldots, G_n]) = \left( \bigcup_{j=1}^{n} (\sigma_A(G_j) \setminus \{p_j\}) \right) \cup \sigma(C),
\]
where
\[
C = \begin{pmatrix}
p_1 & c_{12} & \cdots & c_{1(n-1)} & c_{1n} \\
c_{21} & p_2 & \cdots & c_{2(n-1)} & c_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_{(n-1)1} & c_{(n-1)2} & \cdots & p_{n-1} & c_{(n-1)n} \\
c_{n1} & c_{n2} & \cdots & c_{n(n-1)} & p_n
\end{pmatrix}
\]
and
\[
c_{ij} = \begin{cases} 
\sqrt{m_im_j} & \text{if } ij \in E(H), \\
0 & \text{otherwise}.
\end{cases}
\]

Let $H$ be a graph of order $n$ and $G$ be an arbitrary graph. If, for $1 \leq i \leq n$, $G_i$ is isomorphic to $G$, it follows immediately that $H[G_1, \ldots, G_n] = H[G]$, a fact also noted in [2]. In particular case of a regular graph $G$, Theorem 2.3 implies the corollary below.

**Corollary 2.2.** Let $H$ be a graph of order $n$ with $\sigma_A(H) = \{\lambda_1^{[h_1]}(H), \ldots, \lambda_i^{[h_i]}(H)\}$ and let $G$ be a $p$-regular graph of order $m$ such that $\sigma_A(G) = \{\lambda_1^{[p]}(G), \ldots, \lambda_i^{[p]}(G)\}$. Then
\[
\sigma_A(H[G]) = \{p^{[n(g_1-1)]}, \ldots, \lambda_i^{[n\phi_3]}(G)\} \cup \{(m\lambda_1(H) + p)^{[h_1]}, \ldots, (m\lambda_i(H) + p)^{[h_i]}\}.
\]

Now it is worth to recall the following result.

**Theorem 2.3.** [3] Let $H$ be a graph such that $V(H) = \{1, \ldots, n\}$ and, for each $j \in \{1, \ldots, n\}$, let $G_j$ be a graph of order $m_j$ with Laplacian spectrum $\sigma_L(G_j)$. Then the Laplacian spectrum of $H[G_1, \ldots, G_n]$ is given by
\[
\sigma_L(H[G_1, \ldots, G_n]) = \left( \bigcup_{j=1}^{n} (s_j + (\sigma_L(G_j) \setminus \{0\})) \right) \cup \sigma(C),
\]
where
\[
s_j = \begin{cases} 
\sum_{i \in N_H(j)} m_i, & \text{if } N_H(j) \neq \emptyset, \\
0, & \text{otherwise}
\end{cases}
\]
and $s_j + (\sigma_L(G_j) \setminus \{0\})$ means that the number $s_j$ is added to each element of $\sigma_L(G_j) \setminus \{0\}$, and
\[
C = \begin{pmatrix}
s_1 & -c_{12} & \cdots & -c_{1(n-1)} & -c_{1n} \\
-c_{21} & s_2 & \cdots & -c_{2(n-1)} & -c_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-c_{(n-1)1} & -c_{(n-1)2} & \cdots & s_{n-1} & -c_{(n-1)n} \\
-c_{n1} & -c_{n2} & \cdots & -c_{n(n-1)} & s_n
\end{pmatrix}
\]
with
\[ c_{ij} = \begin{cases} \sqrt{m_i m_j} & \text{if } ij \in E(H), \\ 0 & \text{otherwise} \end{cases} \] (5)

Assuming that \( G_1, \ldots, G_n \) are all isomorphic to a particular graph \( G \) we have the next corollary, which was also proved in [2].

**Corollary 2.4.** Let \( H \) be a graph of order \( n \) with \( \sigma_L(H) = \{\mu_1(H), \ldots, \mu_n(H)\} \) and let \( G \) be a graph of order \( m \) such that \( \sigma_L(G) = \{\mu_1(G), \ldots, \mu_m(G)\} \). Then

\[ \sigma_L(H[G]) = \left( \bigcup_{j=1}^{n} \{md_H(j) + \mu_i(G) : 1 \leq i \leq m - 1\} \right) \cup \{m\mu_1(H), \ldots, m\mu_n(H)\}. \]

### 3 The spectra and Laplacian spectra of iterated lexicographic products of graphs

Let us consider the graphs obtained by an arbitrary number of iterations of the lexicographic product of a graph by another as follows:

\( H^0[G] = G \), \( H^1[G] = H[G] \) and \( H^k[G] = H[H^{k-1}[G]] \), for all integer \( k \geq 2 \).

**Example 3.1.** Let us consider the graph \( H = C_4 \) (the cycle with four vertices) and \( G = K_2 \) (the complete graph with two vertices). Then \( H^0[G] = K_2 \) and \( H[G] = C_4[K_2] \) are depicted in Figure [3]. Furthermore, the Figure [2] depicts the graph \( H^2[G] = C_4^2[K_2] = C_4[C_4[K_2]] \).

In what follows, we adopt the traditional notation of the union of sets for denoting the union of multisets, where the repeated elements of the multisets \( A \) and \( B \) appear in \( A \cup B \) as many times as we count them in \( A \) and \( B \).

![Figure 1: The graphs \( H^0[G] = K_2 \) and \( H^1[G] = C_4[K_2] \).](image-url)
3.1 The spectrum in the case of a $p$-regular graph $G$ and a $q$-regular graph $H$

The next theorem states the regularity degree, order and spectrum of $H^k[G]$, for $k \geq 0$, when $G$ and $H$ are both regular connected graphs.

**Theorem 3.2.** Let $H$ be a $q$-regular connected graph of order $n$ with $\sigma_A(H) = \{q, \gamma_2^{[h_2]}(H), \ldots, \gamma_2^{[h_t]}(H)\}$ and $G$ be a $p$-regular connected graph of order $m$ with $\sigma_A(G) = \{p, \gamma_2^{[g_2]}(G), \ldots, \gamma_2^{[g_s]}(G)\}$. Then for each integer $k \geq 0$, $H^k[G]$ is a $r_k$-regular graph of order $\nu_k$ with

$$r_k = \frac{mq \cdot n^k - 1}{n - 1} + p,$$

$$\nu_k = mn^k,$$

$$\sigma_A(H^k[G]) = \{\gamma_2^{[n^k \cdot g_2]}(G), \ldots, \gamma_2^{[n^k \cdot g_s]}(G)\} \cup \{r_k\} \cup \Lambda_k$$

where

$$\Lambda_k = \bigcup_{i=0}^{k-1} \{(mn^i \gamma_2(H) + r_i)^{[n^k - 1 - h_2]}, \ldots, (mn^i \gamma_4(H) + r_i)^{[n^k - 1 - h_t]}\},$$

assuming that $\Lambda_0 = \emptyset$.

**Proof.** Since $H^0[G] = G$, the case $k = 0$ follows. Furthermore, the case $k = 1$ follows from Corollary 2.2 since $H^1[G] = H[G]$ (notice that $r_1 = mq + p = m \gamma_2(H) + p$). Let us assume that the result holds for $k - 1$ iterations, with $k \geq 2$. 


By definition of lexicographic product, we obtain

\[ r_k = \nu_{k-1}q + r_{k-1} \]
\[ = mq(n^{k-1} + n^{k-2} + \cdots + n + 1) + p = mq\frac{n^k - 1}{n - 1} + p \]

and \( \nu_k = \nu(H^k[G]) = \nu_{k-1}n = mn^k \). Additionally, replacing in the Corollary 2.4 the graph \( G \) by \( H^{k-1}[G] \) it follows that

\[ \sigma_A(H^k[G]) = \{\gamma_2^{[k^g2]}(G), \ldots, \gamma_a^{[k^g_a]}(G)\} \cup \{r_k\} \cup \Lambda_k, \]

where \( \Lambda_k = \bigcup_{i=0}^{k-1} \{mn^i\gamma_2(H) + r_i)^{[n^k-1-h_2]}, \ldots, (mn^i\gamma_l(H) + r_i)^{[n^k-1-h_l]} \}. \]

\[ \square \]

**Example 3.3.** For the graphs of Figure 4 we have \( m = 2, n = 4, p = 1 \) and \( q = 2 \). From Theorem 3.2, we obtain the following degree, order and spectra for \( C_4[H] \) for a given \( k \geq 1 \) integer:

\[ r_k = 2 \times 2 \frac{4^k - 1}{4 - 1} + 1 = \frac{4^{k+1} - 1}{3}, \]
\[ \nu_k = \nu(H^k[G]) = 2 \times 4^k, \]
\[ \sigma_A(H^k[G]) = \{(-1)^{[t]}\} \cup \left\{\frac{4^{k+1} - 1}{3} \right\} \cup \Lambda_k, \]

where \( \Lambda_k = \bigcup_{i=0}^{k-1} \left\{\left(2 \times 4^i \times 0 + \frac{4^{i+1} - 1}{3}\right)^{[4^k-1-i-2]} \right, \left.\left(2 \times 4^i(-2) + \frac{4^{i+1} - 1}{3}\right)^{[4^k-1]} \right\} \)

\[ = \bigcup_{i=0}^{k-1} \left\{\left(\frac{4^{i+1} - 1}{3}\right)^{[4^k-1-i-2]}, \left(-\frac{4^{i+1} + 4^{i+1} - 1}{3}\right)^{[4^k-1]} \right\}. \]

In particular, for \( k = 2 \) (the graph of Figure 3), it follows that

\[ r_2 = 21, \]
\[ \nu_2 = \nu(H^2[G]) = 32 \]
\[ \sigma_A(H^2[G]) = \{21, (5)^{[2]}, (1)^{[8]}, (-1)^{[16]}, (-3)^{[4]}, -11\}. \]

We may consider the graph obtained by an arbitrary number of iterations of the lexicographic product of a graph for itself. In fact, for a given \( q \)-regular graph \( H \) of order \( n \), we assume that \( H^0 = K_1, H^1 = H \) and that \( H^k = H^{k-1}[H] \) for \( k \geq 2 \). Then, as immediate consequence of Theorem 3.2 we have the following corollary.

**Corollary 3.4.** Let \( H \) be a connected \( q \)-regular graph of order \( n \) such that \( \sigma_A(H) = \{q, \gamma_2^{[b_2]}(H), \ldots, \gamma_l^{[b_l]}(H)\} \). Then, for each integer \( k \geq 1 \), \( H^k \) is a \( r_k \)-regular graph.
of order \( \nu_k \), such that

\[
  r_k = q \frac{n^k - 1}{n - 1},
  \nu_k = n^k \text{ and}
\]

\[
  \sigma_A(H^k) = \left( \bigcup_{i=0}^{k-1} \{ (n^i \gamma_2(H) + r_i)[n^{k-1-i}h_2], \ldots, (n^i \gamma_t(H) + r_i)[n^{k-1-i}h_t] \} \right) \cup \{ r_k \}.
\]

**Remark 3.5.** The least eigenvalue of \( H^k \) is \( \lambda_{n^k}(H^k) = n^{k-1} \lambda_n(H) + q \frac{n^{k-1} - 1}{n - 1} \).

**Proof.** In fact, based on the Corollary 3.4, we obtain

\[
  \lambda_{n^k}(H^k) = \min_{0 \leq i \leq k-1} \{ n^i \gamma_t(H) + r_i \} = \min_{0 \leq i \leq k-1} \{ n^i \gamma_t(H) + q \frac{n^i - 1}{n - 1} \}
  = n^{k-1} \gamma_t(H) + q \frac{n^{k-1} - 1}{n - 1} = n^{k-1} \lambda_n(H) + q \frac{n^{k-1} - 1}{n - 1}.
\]

The third equality above is obtained taking into account that for every \( i \in \{ 0, \ldots, k - 1 \} \), \( n^{k-1} \gamma_t(H) + q \frac{n^{k-1} - 1}{n - 1} \leq n^i \gamma_t(H) + q \frac{n^i - 1}{n - 1} \iff (n^{k-1} - n^i) \gamma_t(H) \leq -q \frac{n^{k-1} - 1}{n - 1} \iff \gamma_t(H) \leq -q \frac{1}{n - 1} \) and last inequality holds since the graph \( H \) has at least one edge and then (see [3]) \( \gamma_t(H) \leq -1 \leq -q \frac{1}{n - 1} \). \( \Box \)

**Remark 3.6.** Let \( H \) be a \( p \)-regular graph of order \( n \). Then for all \( k \in \mathbb{N} \) and for all nonnegative integer \( q \), \( \sigma_A(H^k) \setminus \{ r_k \} \subset \sigma_A(H^{k+q}) \), where \( r_k \) is the regularity of \( H^k \) and this inclusion means that all eigenvalues with the respective multiplicities of the multiset \( \sigma_A(H^k) \setminus \{ r_k \} \) belong to the multiset \( \sigma_A(H^{k+q}) \).

**Proof.** This is a direct consequence of Corollary 3.4. \( \Box \)

**Example 3.7.** Let us apply the Corollary 3.4 to the powers of the Petersen graph \( P^k \), with \( k \in \{ 2, 3, 100 \} \).

| \( k \) | Spectrum of \( P^k \) |
|-------|------------------|
| \( k = 1 \) | 3, 1[^{[5]}], -2[^{[4]}] |
| \( k = 2 \) | 33, 13[^{[5]}], 1[^{[50]}], -2[^{[4]}], -17[^{[4]}] |
| \( k = 3 \) | 333, 13[^{[5]}], 1[^{[50]}], -2[^{[4]}], -17[^{[4]}], -167[^{[4]}] |
| \( k = 100 \) | \( 3 \times \sum_{i=0}^{99} 10^i, 1[^{[5\times10^{99}]}], -2[^{[4\times10^{99}]}], 10^m + 3 \sum_{i=0}^{m-1} 10^i[^{[5\times10^{99-m}]}], m = 1, \ldots, 99, 7 + 10^m + 6 \sum_{i=1}^{m-1} 10^i[^{[4\times10^{99-m}]}], m = 1, \ldots, 99. \) |

Notice that the graph \( P^k \) has \( 10^k \) vertices, in particular \( P^{100} \) has the googol number of vertices \( 10^{100} \). All the computations were done by Mathematica and lasted just a few seconds.
3.2 The Laplacian spectra

In this section we characterize the Laplacian spectrum of the iterated lexicographic product $H^k[G]$, where $G$ and $H$ are arbitrary graphs. The particular cases of the Laplacian spectra of these iterated lexicographic products, when $H$ is regular and when $H$ is arbitrary but equal to $G$ are also presented.

**Theorem 3.8.** Let $G$ be a graph of order $m$ such that $\sigma_L(G) = \{\mu_1(G), \ldots, \mu_m(G)\}$ and let $H$ be a graph such that $V(H) = [n]$ and $\sigma_L(H) = \{\mu_1(H), \ldots, \mu_n(H)\}$. Then, for each integer $k \geq 1$, $H^k[G]$ is a graph of order $\nu_k = mn^k$ such that

$$\sigma_L(H^k[G]) = \Omega^k_G \cup \Gamma^k_H$$

where

$$\Omega^k_G = \bigcup_{(j_1, j_2, \ldots, j_k) \in [n]^k} \left\{ \mu_l(G) + m \sum_{i=1}^k n^{i-1}d_H(j_i) : 1 \leq l \leq m - 1 \right\}$$

and

$$\Gamma^k_H = \bigcup_{i=2}^k \left( \bigcup_{(j_1, \ldots, j_k) \in [n]^{k-1}} \left\{ mn^{i-2}\mu_l(H) + m \sum_{r=1}^{k-1} n^{r-1}d_H(j_r) : 1 \leq l \leq n - 1 \right\} \right) \cup \left\{ mn^{k-1}\mu_j(H) : 1 \leq j \leq n \right\}$$

**Proof.** Corollary 2.4 give us the assertion in case $k = 1$. Given an integer $k \geq 2$, let suppose that the

$$\sigma_L(H^{k-1}[G]) = \Omega^{k-1}_G \cup \Gamma^{k-1}_H,$$

where

$$\Omega^{k-1}_G = \bigcup_{(j_1, \ldots, j_{k-1}) \in [n]^{k-1}} \left\{ \mu_l(G) + m \sum_{i=1}^{k-1} n^{i-1}d_H(j_i) : 1 \leq l \leq m - 1 \right\}$$

and

$$\Gamma^{k-1}_H = \bigcup_{i=2}^{k-1} \left( \bigcup_{(j_1, \ldots, j_{k-1}) \in [n]^{k-1}} \left\{ mn^{i-2}\mu_l(H) + m \sum_{r=1}^{k-1} n^{r-1}d_H(j_r) : 1 \leq l \leq n - 1 \right\} \right) \cup \left\{ mn^{k-2}\mu_j(H) : 1 \leq j \leq n \right\}$$

Then, by Corollary 2.4

$$\sigma_L(H^k[G]) = \sigma_L(H[H^{k-1}[G]]) =$$

$$= \left( \bigcup_{j_k=1}^n \{ mn^{k-1}d_H(j_k) + x : x \in \Omega^{k-1}_G \} \right) \cup \left( \bigcup_{j_k=1}^n \{ mn^{k-1}d_H(j_k) + y : y \in \Gamma^{k-1}_H \} \right),$$

where

$$\bigcup_{j_k=1}^n \{ mn^{k-1}d_H(j_k) + x : x \in \Omega^{k-1}_G \} =$$
\[
\begin{align*}
&= \bigcup_{j_k=1}^{n} \left( \bigcup_{(j_1, \ldots, j_{k-1}) \in [n]^{k-1}} \left\{ \sigma_{H}^{k-1}(j_k) + \mu_i(G) + m \sum_{i=1}^{k-1} n^{i-1} d_H(j_i) : 1 \leq l \leq n \right\} \right) \\
&= \bigcup_{(j_1, \ldots, j_{k-1}, j_k) \in [n]^k} \left\{ \mu_i(G) + m \sum_{i=1}^{k} n^{i-1} d_H(j_i) : 1 \leq l \leq n \right\} = \Omega_G^k \quad \text{and} \\
&\bigcup_{j_k=1}^{n} \left\{ \sigma_{H}^{k-1}(j_k) + y : y \in \Gamma_{H}^{k-1} \right\} = \\
&= \bigcup_{j_k=1}^{n} \left( \bigcup_{i=2}^{k-1} \bigcup_{(j_1, \ldots, j_{k-1}) \in [n]^{k-1}} \left\{ \sigma_{H}^{k-2}(j_k) + \sigma_{H}^{k-1}(j_i) + \mu_i(G) + m \sum_{r=i}^{k-1} n^{r-1} d_H(j_r) : 1 \leq l \leq n - 1 \right\} \\
&\bigcup \left\{ \sigma_{H}^{k-1}(j_k) + \mu_i(H) : 1 \leq l \leq n - 1 \right\} \right) \bigcup \left\{ \sigma_{H}^{k-1}(j_k) + \mu_i(H) : 1 \leq j \leq n \right\} = \\
&= \bigcup_{i=2}^{k} \bigcup_{(j_1, \ldots, j_k) \in [n]^{k+1}} \left\{ \sigma_{H}^{k-2}(j_k) + \mu_i(H) + m \sum_{r=i}^{k} n^{r-1} d_H(j_r) : 1 \leq l \leq n - 1 \right\} \bigcup \\
&\left\{ \sigma_{H}^{k-2}(j_k) + \mu_i(H) : 1 \leq j \leq n \right\} = \Gamma_{H}^k.
\end{align*}
\]

As immediate consequence of the above theorem, for a regular graph \( H \) it follows

**Corollary 3.9.** Let \( G \) and \( H \) as in Theorem 3.8, with \( H \) is q-regular. Then, for each integer \( k \geq 1 \),

\[
\sigma_L(H^k[G]) = \left\{ \left( \mu_i(G) + mq \frac{n^k-1}{n-1} \right)^{[n^k]} \right\} \cup \left\{ \left( \mu_i(G) + m \sum_{i=1}^{k} n^{i-1} d_H(j_i) : 1 \leq l \leq m - 1 \right) \right\} \cup \{ 0 \} \cup \\
\bigcup_{i=2}^{k+1} \left\{ \left( mn^{i-1} \mu_i(H) + m\sum_{r=i}^{k} n^{r-1} d_H(j_r) : 1 \leq l \leq n - 1 \right) \right\}^{[n^{k+1}]}. 
\]

**Proof.** From Theorem 3.8, for all integer \( k \geq 1 \), it follows that

\[
\sigma_L(H^k[G]) = \Omega_{H}^k \cup \Gamma_{H}^k,
\]

where

\[
\Omega_{H}^k = \bigcup_{(j_1, \ldots, j_k) \in [n]^k} \left\{ \mu_i(G) + mq \sum_{i=1}^{k} n^{i-1} : 1 \leq l \leq m - 1 \right\}^{[n^k]} = \left\{ \left( \mu_i(G) + mq \frac{n^k-1}{n-1} \right)^{[n^k]} : 1 \leq l \leq m - 1 \right\}.
\]

and
Now, let us consider the case $G = H$.

**Corollary 3.10.** Let $H$ be a graph such that $V(H) = [n]$, with $\sigma_L(H) = \{\mu_1(H), \ldots, \mu_n(H)\}$. Then $H^k$ is a graph of order $\nu_k = n^k$ such that

$$
\sigma_L(H^k) = \bigcup_{i=1}^{k-1} \left( \bigcup_{(j_1, \ldots, j_{k-i}) \in [n]^{k-i}} \left\{ \mu_l(H) + n \sum_{r=1}^{k-1} n^{r-1} d_H(j_r) : 1 \leq l \leq n-1 \right\} \right) \cup \left\{ mm^{k-1} \mu_j(H) : 1 \leq j \leq n \right\},
$$

for all $k \geq 2$.

**Proof.** The first statement is obvious. Regarding the second statement, applying again Theorem 3.8 for $k \geq 2$ we obtain

$$
\sigma_L(H^k) = \sigma_L(H^{k-1}[H]) =
$$

$$
= \bigcup_{(j_1, j_2, \ldots, j_{k-1}) \in [n]^{k-1}} \left\{ \mu_l(H) + n \sum_{i=1}^{k-1} n^{i-1} d_H(j_i) : 1 \leq l \leq n-1 \right\} \cup
$$

$$
= \bigcup_{i=1}^{k-1} \left( \bigcup_{(j_1, \ldots, j_{k-i}) \in [n]^{k-i}} \left\{ n^{i-1} \mu_l(H) + n \sum_{r=1}^{k-1} n^{r-1} d_H(j_r) : 1 \leq l \leq n-1 \right\} \right) \cup \left\{ mn^{k-2} \mu_j(H) : 1 \leq j \leq n \right\} =
$$

$$
= \bigcup_{i=1}^{k-1} \left( \bigcup_{(j_1, \ldots, j_{k-i}) \in [n]^{k-i}} \left\{ n^{i-1} \mu_l(H) + n \sum_{r=1}^{k-1} n^{r-1} d_H(j_r) : 1 \leq l \leq n-1 \right\} \right) \cup \left\{ n^{k-1} \mu_j(H) : 1 \leq j \leq n \right\}.
$$

Finally, the next proposition determines the algebraic connectivity and the largest Laplacian eigenvalue of $H^k$, for $k \geq 1$. 

11
Proposition 3.11. If $H$ is a connected graph of order $n$ with $\sigma_L(H) = \{\mu_1(H), \ldots, \mu_n(H)\}$ and $k \geq 1$, then

$$\mu_{n-1}(H^k) = n^{k-1}\mu_{n-1}(H) \quad \text{and} \quad \mu_1(H^k) = n^{k-1}\mu_1(H) \quad (6) \quad (7)$$

Proof. Let $k \geq 1$ be fixed. From Corollary 3.10 it follows that the second least eigenvalue of $H^k$ is among the values $n^{k-1}\mu_{n-1}(H)$ and $n^{i-1}\mu_{n-1}(H) + \sum_{r=i}^{k-1} n^r d_H(j_r)$ for $1 \leq i \leq k - 1$. We may recall that $\delta(H) \geq \mu_{n-1}(H)$; then, for all $1 \leq i \leq k - 1$, it holds that

$$n^{i-1}\mu_{n-1}(H) + \sum_{r=i}^{k-1} n^r d_H(j_r) \geq n^{i-1}\mu_{n-1}(H) + \sum_{r=i}^{k-1} n^r \delta(H)$$

$$\geq n^{i-1}\mu_{n-1}(H) + \mu_{n-1}(H) \sum_{r=i}^{k-1} n^r = \mu_{n-1}(H) \left(n^{i-1} + \sum_{r=i}^{k-1} n^r\right)$$

$$= \mu_{n-1}(H) \sum_{r=i-1}^{k-1} n^r = \mu_{n-1}(H) \sum_{r=i-1}^{k-1} n^{r-1} \geq \mu_{n-1}(H)n^{k-1}.$$ 

Thus the equality (6) is proved. Now, let us prove the equality (7). Applying again Corollary 3.10 it follows that the largest Laplacian eigenvalue of $H^k$ is among the values $n^{k-1}\mu_1(H)$ and $n^{i-1}\mu_1(H) + \sum_{r=i}^{k-1} n^r d_H(j_r)$, $1 \leq i \leq k - 1$. Since $\mu_1(H) \geq \Delta(H) + 1$, for $1 \leq i \leq k - 1$, it follows that

$$n^{i-1}\mu_1(H) + \sum_{r=i}^{k-1} n^r d_H(j_r) \leq n^{i-1}\mu_1(H) + \sum_{r=i}^{k-1} n^r(\mu_1(H) - 1)$$

$$= \mu_1(H) \sum_{r=i-1}^{k-1} n^r - \sum_{r=i}^{k-1} n^r$$

$$= \mu_1(H)n^{i-1}\frac{n^{k-i+1} - 1}{n-1} - n^{i-1}\frac{n^{k-i} - 1}{n-1}$$

$$= \mu_1(H)n^{i-1}\left(\frac{n^{k-i} - 1}{n-1} + n^{k-i} - n^{k-i} - 1\right)$$

$$= \mu_1(H)n^{k-1} + \frac{n^{k-i} - 1}{n-1} n^{i-1}(\mu_1(H) - n)$$

$$\leq n^{k-1}\mu_1(H)$$

The last inequality is obtained taking into account that $\mu_1(H) - n \leq 0$. \(\square\)
4 Spectral and combinatorial invariant properties of lexicographic powers of graphs

In this section, a few well known spectral and combinatorial invariant properties of a graph $H$ are extended to the lexicographic powers of $H$. For instance, considering that $H$ has order $n \geq 2$, for all $k \geq 1$, we may deduce that

$$\delta(H^k) = \delta(H) \frac{n^k - 1}{n - 1} \quad \left(\Delta(H^k) = \Delta(H) \frac{n^k - 1}{n - 1}\right). \tag{8}$$

Notice that since $H$ has order $n$, then $H^k$ has order $n^k$. The equalities (8) can be proved by induction on $k$, taking into account that they are obviously true for $k = 1$. Assuming that the equalities (8) are true for $k - 1$, with $k \geq 2$, it is immediate that a vertex of $H^k$ with minimum (maximum) degree is a minimum (maximum) degree vertex of the copy of $H^{k-1}$ located in the position of a minimum (maximum) degree vertex of $H$, and then its degree in $H^k$ is equal to

$$\delta(H) \left(\frac{n^{k-1} - 1}{n - 1} + n^{k-1}\right) \quad \left(\Delta(H) \left(\frac{n^{k-1} - 1}{n - 1} + n^{k-1}\right)\right).$$

For an arbitrary graph $G$, let $q_1(G)$ and $q_n(G)$ be the largest and the least eigenvalue of the signless Laplacian matrix of $G$ (that is, the matrix $A_G + D$), respectively. Taking into account the relations $2\delta(G) \leq q_1(G) \leq 2\Delta(G)$, which were proved in [6], and also the inequality $q_n(G) < \delta(G)$ [8], for the lexicographic power $k$ of a graph $H$ we obtain the inequalities

$$2\delta(H) \frac{n^k - 1}{n - 1} \leq q_1(H^k) \leq 2\Delta(H) \frac{n^k - 1}{n - 1} \quad \text{and} \quad q_n(H^k) < \delta(H) \frac{n^k - 1}{n - 1}.$$

Denoting the distance between two vertices $x$ and $y$ in $G$ by $d_G(x, y)$ and the diameter of $G$ by $\text{diam}(G)$, we may conclude the following interesting result concerning the diameter of the iterated lexicographic products of graphs.

**Proposition 4.1.** Let $H$ be a connected not complete graph and let $G$ be an arbitrary graph of order $m$. For very $k \in \mathbb{N}$

$$\text{diam}(H^{k+1}) = \text{diam}(H^k[G]) = \text{diam}(H).$$

**Proof.** Consider $V(H) = \{1, \ldots, n\}$ and $x, y \in V(H^k[G])$ ($x, y \in V(H^{k+1})$). Then we have two cases (a) they are both in the same copy of $H^{k-1}[G]$ ($H^k$) located in the position of the vertex $i \in V(H)$ or (b) they are in different copies of $H^{k-1}[G]$ ($H^k$) located in the positions of the vertices $r, s \in V(H)$.

(a) If $x$ and $y$ are adjacent, then $d_{H^k[G]}(x, y) = 1$ ($d_{H^{k+1}}(x, y) = 1$), otherwise since there exists a vertex $j \in V(H)$ such that $ij \in E(H)$ and then there is
a path \(x, z, y\), where \(z\) is a vertex of the copy of \(H^{k-1}[G] (H^k)\) located in the position of the vertex \(j \in V(H)\). Therefore, \(d_{H^{k}[G]}(x, y) = 2 (d_{H^{k+1}}(x, y) = 2)\).

(b) In this case, assuming that \(r, j_1, \ldots, j_t, s\) is a shortest path in \(H\) connecting the vertices \(r\) and \(s\), there are vertices \(z_1, \ldots, z_t\) in the copies of \(H^{k-1}[G] (H^k)\) located in the positions of the vertices \(j_1, \ldots, j_t\), respectively, such that \(x, z_1, \ldots, z_t, y\) is a path of length \(d_H(r, s)\).

\[\square\]

4.1 The stability number

Regarding the stability number \(\alpha(G)\) (the maximum cardinality of a vertex subset of an arbitrary graph \(G\) with pairwise nonadjacent vertices), according to [11], \(\alpha(H[G]) = \alpha(H)\alpha(G)\) for an arbitrary graph \(H\). Thus we may conclude that \(\alpha(H^k) = \alpha(H)^k\) (and, denoting the complement of graph \(F\) by \(\overline{F}\) and the clique number by \(\omega(F)\), since \(\overline{H[G]} = \overline{H[G]}, \omega(H^k) = \omega(H)^k\)). Furthermore, from the spectral upper bound \(\alpha(G) \leq n\frac{\mu_1(G) - \delta(G)}{\Delta(G)}\), independently deduced in [18] and [12] for an arbitrary graph \(G\), and taking into account (8) and (7), considering the \(k\)-th lexicographic power of a graph \(H\) of order \(n\) we obtain

\[\alpha(H^k) \leq n^k\frac{\mu_1(H^k) - \delta(H^k)}{\Delta(H^k)} \leq n^k \frac{n-1}{n} n^{k-1}\frac{\mu_1(H) - \delta(H)}{\Delta(H)}\]

4.2 The vertex connectivity

Considering a graph \(G\) of order \(m\) and a graph \(H\) of order \(n\), it is well known that the lexicographic product \(H[G]\) is connected if and only if \(H\) is a connected graph [14]. On the other hand, according to [11], if both \(G\) and \(H\) are not complete, then \(\nu(H[G]) = n\nu(H)\), where \(\nu(H)\) denotes the vertex connectivity of \(H\) (that is, the minimum number of vertices whose removal yields a disconnected graph). Therefore, \(\nu(H^k) = n^{k-1}\nu(H)\). Furthermore, we may conclude that when \(H\) is connected not complete (and then \(H^k\) is also connected not complete),

\[n^{k-1}\mu_{n-1}(H) \leq \nu(H^k) \leq \delta(H)\frac{n^k - 1}{n - 1}\]

In fact, it should be noted that \(\nu(G) \leq \delta(G)\) and, when \(G\) is not complete, \(\mu_{n-1}(G) \leq \nu(G)\), see [9]. Therefore, taking into account (3) and (8) we obtain

\[n^{k-1}\mu_{n-1}(H) = \mu_{n-1}(H^k) \leq \nu(H^k) \leq \delta(H^k) = \delta(H)\frac{n^k - 1}{n - 1}\]
4.3 The chromatic number

Concerning the relations of the chromatic number of a graph $G$ of order $n$ with its spectrum, the following lower bound due to Hoffman in [16] is well known.

$$\chi(G) \geq 1 - \frac{\lambda_1(G)}{\lambda_n(G)}.$$ 

As direct consequence, if a graph $H$ is $q$-regular of order $n$, taking into account the Remark (3.5), we may conclude the following lower bound on the chromatic number of $H^k$:

$$\chi(H^k) \geq 1 - \frac{r_k}{\lambda_{nk}(H^k)} = 1 - q \frac{n^k - 1}{(n - 1) \left( n^{k-1} \lambda_n(H) + q^{\frac{n^k - 1}{n-1}} \right)} = 1 - \frac{n^k - 1}{n^{k-1} \left( (n - 1) \frac{\lambda_n(H)}{q} + 1 \right) - 1}.$$ 

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