EULER AND THE Γ-FUNCTION

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We present and discuss Euler’s results on the Γ-function. We will explain, how Euler obtained them and how Euler’s ideas anticipate more modern approaches and theories.

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1 Introduction

1.1 Motivation

According to Leibniz, here are two arts in mathematics: The "Ars inveniendi" (Art of Finding) and the "Ars Demonstrandi". Nowadays, the latter is clearly dominating the mathematical education, whereas the first is almost completely neglected.

Leonhard Euler’s mathematical works are special not only for their quality and quantity, but also for his pedagogical style rendering them easily understandable. This is summarized by two famous quotes, the first due to Laplace: "Read Euler, read Euler, he is the Master of us all", the other due to Gauß: "The study of Euler’s works cannot be replaced by anything else."

Moreover, Euler does not only provide the proofs of certain theorems, but also tells us how he found the theorem in the first place. In other words, Euler’s work gives us both, the Ars Demonstrandi and the Ars Inveniendi.

In this article we want to review and discuss some of the properties of the Γ-function, defined as

\[ \Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt \quad \text{for Re}(x) > 0, \]

that were already discovered by Euler himself. More precisely, we will explain, how Euler, the discoverer of (1) \[E19\], derived several different expressions for the Γ-function which are usually attributed to others.

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1W. Dunham even wrote a book with a title resembling this [Du99]
2Being written in the 18th century, Euler’s work do not meet the modern standards of mathematical rigor, but in most cases it is not a lot of work to formulate Euler’s proofs rigorously.
3His famous book on calculus of variations [E65], the first ever written on the subject, has the title "Methodus inveniendi lineas curvas maximi minimae proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti” even contains Methodus inveniendi in the title
4Most books devoted to the Γ-function start from the integral representation. We mention [Ar15], [Fr06] as an example. [Ni05] is an exception.
5To be completely precise, in [E19] Euler introduced the integral \[ \int_0^1 \left( \log \frac{1}{t} \right)^{x-1} dt \] which goes over into the above representation by the substitution \( t = e^{-u} \)
1.2 Euler’s idea concerning the $\Gamma$-function

Euler found his expressions for the $\Gamma$-function basically from two different sources.

1. Interpolation theory
2. The function equation $\Gamma(x + 1) = x\Gamma(x)$

But eventually they all boil down to the solution of the functional equation by different methods. The first approach, outlined in [E212] (Chapter 16 and 17 of the second part) and [E613], is based on difference calculus and lead him essentially to the Weierstraß product expansion of $\frac{1}{\Gamma(x)}$ and also to the Taylor series expansion of $\log \Gamma(1 + x)$. Indeed, Euler implicitly uses the functional equation even in this approach. But we want to separate it from the other approaches, in which he tries to solve the functional equation explicitly and says so.

Concerning the direct solution of the functional equation, first, Euler solved functional equation satisfied by $\Gamma(x)$

$$\Gamma(x + 1) = x\Gamma(x)$$

or equivalently the difference equation

$$\log \Gamma(x + 1) - \log \Gamma(x) = \log x$$

by applying the Euler-Maclaurin formula, he discovered in [E25] and proved in [E47]. Later he attempted a solution by conversion of the difference equation into a differential equation of infinite order with constant coefficients, see [E189], by the methods he developed in [E62], [E188]. Both ideas led him to the Stirling-formula for the factorial, i.e.,

$$x! = \Gamma(x + 1) = \frac{x^x}{e^x} \sqrt{2\pi x} \quad \text{for} \quad x \to \infty.$$ 

In [E123] he explains a method how to solve the functional equation by an educated guess. Euler applied it to the factorial in §13 of [E594], basically adding some more examples to those of [E123], we will see that his ideas

---

6This is the more natural approach, since the functional equation is one of the defining properties of the factorial.

7We say "attempted", since the solution Euler found this way is incorrect. This will be discussed in more detail in the corresponding section.
lead to the integral representation. Finally, in [E652], he uses the functional equation to derive a product representation he first stated without proof in [E19].

1.3 ORGANISATION OF THE PAPER

1.3.1 General Overview

This paper is organized as follows:
It can roughly be subdivided into two parts. The first part contains the modern introduction of the \( \Gamma \)-function and the classification theorems. Furthermore, we dedicated a whole section to the rigorous solution of the difference equation \( F(x + 1) = xF(x) \).
The large second part is then devoted to Euler’s several approaches he used or could have used to arrive at the \( \Gamma \)-function (most of the time, he intended something entirely different and the \( \Gamma \)-function just was a special case). We will discuss Euler’s idea to solve general homogeneous difference equations with linear coefficient by an idea we will refer to as moment-ansatz and will discuss how he solved the general difference equation by converting it into a differential equation of infinite order. Another idea of Euler was to derive solutions of the general difference equation by difference calculus, which we will also discuss in detail. We also devoted a complete section to the relation among the \( \Gamma \)- and \( B \)-function and how also found those connections. After this we will conclude and try to summarize Euler’s vast output.

Given the time in which Euler wrote his papers, some of his arguments are not completely rigorous and some are even incorrect. Therefore, when necessary or appropriate, we will show, how his ideas can be formulated in modern setting and at some points also give the necessary rigorous proof. Sometimes, we will not give all the details, since this would simply take too long and would lead too far away from our actual intention.
We will always try to put Euler’s results and ideas in contrast to these modern ideas and it will turn out that Euler actually anticipated a lot that came after him (if understood in the modern context). Thus, we added some sections just containing some historical notes. Furthermore, since this is mainly a paper on Euler’s works, we also included some quotes from his papers, translated from his Latin original into English. They will help to understand Euler’s way of thinking and his persona a bit more.
1.3.2 Notation

Euler invented many of the modern notation, e.g., $\sum$ for a sum, $f : x$ for a function of $x$ etc. Nevertheless, most of the times he did not use the compact notation, but wrote things out explicitly, e.g., for $\sum_{k=1}^{\infty} \frac{1}{k}$ he wrote $1 + \frac{1}{2} + \cdots + \frac{1}{k}$. When referring to Euler’s papers, we stick to his notation as close as possible and only resort to the modern compact notation, if things become more clear that way. Euler also never used the symbol $\Gamma(x)^8$ to denote he factorial nor did he write $x!$, his notation varies from paper to paper. We will always stick to the modern notation concerning this issue.

Furthermore, the notion of limits as today did not exist at that time. Euler often speaks of infinitesimal or infinitely large numbers. In this case, we will use the modern symbol $\lim_{n \to 0}$ and $\lim_{n \to \infty}$, respectively.

\footnote{This letter was introduced by Legendre in [Le26a]}

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2 Short Modern Introduction to the Gamma-Function

We briefly mention the modern definition of $\Gamma(x)$ following \cite{Fr06} (pp. 194-197). In this book, as in \cite{Ar15}, we start from the integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt.$$

2.1 Definition and Simple Properties

2.1.1 Definition

\textbf{Definition 2.1} ($\Gamma$-integral). We define the $\Gamma$-function as

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt.$$

Here $t^{z-1} := e^{(z-1)\log(t)}$, $\log t \in \mathbb{R}$, $\text{Re}(z) > 0$.

We have the following simple theorem:

\textbf{Theorem 2.1.} The $\Gamma$-integral

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt$$

converges absolutely for $\text{Re}(z) > 0$ and represents an analytic function on the domain. The derivatives are given (for $k \in \mathbb{N}$) by

$$\Gamma^{(k)}(z) = \int_0^\infty t^{z-1}(\log t)^k e^{-t}dt.$$

\textbf{Proof.} We split the $\Gamma$ integral into the two integrals

$$\Gamma(z) = \int_0^1 t^{z-1}e^{-t}dt + \int_1^\infty t^{z-1}e^{-t}dt$$

and use the relation
\[ |t^{-1}e^{-t}| = t^{t-1}e^{-t} \]
where we wrote \( x \) for \( \text{Re}(z) \). Let us consider both integrals separately. In general, for each \( x_0 > 0 \) there is a number \( C > 0 \) with
\[
t^{x-1} \leq Ce^t \quad \forall \ x \text{ with } 0 < x \leq x_0 \text{ and } t \geq 1.
\]
Thus the integral
\[
\int_{1}^{\infty} t^{x-1}e^{-t} \, dt
\]
converges absolutely for all \( z \in \mathbb{C} \).

For the other integral, we use the estimate
\[
|t^{x-1}e^{-t}| < t^{x-1} \quad \text{for } t > 0
\]
and the existence of the integral
\[
\int_{0}^{1} \frac{1}{t^s} \, dt \quad \text{for } s < 1.
\]
From these estimates it follows that the sequence of functions
\[
f_n(z) := \int_{\frac{1}{n}}^{1} t^{x-1}e^{-t} \, dt
\]
converges uniformly to \( \Gamma \) for \( n \to \infty \). Therefore, \( \Gamma \) is an analytic function.
The formula for the \( k \)-th derivative follows from the application of the Leibniz rule (for the differentiation) and then taking the limit \( n \to \infty \). \( \square \)

### 2.1.2 Simple Properties

We have

**Theorem 2.2** (Elementary Properties of the \( \Gamma \)-integral). The \( \Gamma \)-function can be analytically continued to the whole complex plane except the points
\[
z \in S := \{0, -1, -2, -3, \ldots \}
\]
and there satisfies the functional equation:

\[ \Gamma(z + 1) = z\Gamma(z). \]

All singularities are poles of first order with the residues:

\[ \text{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}. \]

**Proof.** We show the functional equation first. Obviously, we have

\[ \Gamma(1) = \int_{0}^{\infty} e^{-t} dt = \left[ -e^{-t} \right]_{0}^{\infty} = 1. \]

By integration by parts one arrives at the functional equation

\[ \Gamma(z + 1) = z\Gamma(z) \quad \text{for } \Re(z) > 0. \]

Using the functional equation iteratively, we find

\[ \Gamma(z) = \frac{\Gamma(z + n + 1)}{z(z + 1) \cdots (z + n)}. \]

The right-hand side of the equation has a large domain where it can be defined, i.e

\[ \Re(z) > -(n + 1) \quad \text{and} \quad z \neq 0, -1, -2, -3, \ldots, -n. \]

Therefore, the above equation is an analytic continuation of \( \Gamma \) into a larger domain.

Finally, let us consider the residues. Using the functional equation, we have

\[ \text{Res}(\Gamma; -n) = \lim_{z \to -n} (z + n)\Gamma(z) = \frac{\Gamma(1)}{(-n)(-n + 1) \cdots (-1)} = \frac{(-1)^n}{n!}. \]
2.2 Classification Theorems

The Γ-function was invented by Euler to interpolate the factorial in [E19]. The integral representation obviously fulfills this task, since \( \Gamma(n + 1) = n! \). The factorial has the two properties \( 0! = 1 \) and \( n! = n(n - 1)! \). Therefore, this automatically raises the question, whether the \( \Gamma \)-function is the only holomorphic function with \( \Gamma(z + 1) = z\Gamma(z) \) and \( \Gamma(1) = 1 \). The answer to this question is no, since, e.g.,

\[
f(z) := (1 + \sin(2\pi z))\Gamma(z)
\]

also has the two properties. Below we will encounter several other expressions also satisfying the functional equation and \( f(1) = 1 \). Therefore, it will be useful to have theorems that tell us immediately that the new expression is indeed the \( \Gamma \)-function without showing the equality to the integral representation directly. This is provided by classification theorems. They state that the \( \Gamma \)-function can be uniquely defined by the two obvious properties \( \Gamma(1) = 1 \) and \( \Gamma(z + 1) = z\Gamma(z) \) and an additional third one. We will present two theorems, Wielandt’s theorem and the Bohr-Mollerup theorem. In Bourbaki [Bo51], the Bohr-Mollerup theorem is the starting point for the \( \Gamma \)-function. There, one does not start from a specific representation.

### 2.2.1 Wielandt’s Theorem

Wielandt’s Theorem is one possible characterisation of the \( \Gamma \)-function. Wielandt’s original proof can be found in his collected papers [Wi96]. Other proofs can be found, e.g., in the books [Kn41] (pp. 47-49) and [Fr06] (pp. 198-199) which we will present here, and the paper [Re96].

We have:

**Theorem 2.3 (Wielandt’s Theorem).** Let \( D \subseteq \mathbb{C} \) be a domain containing the vertical strip

\[
1 \leq x < 2.
\]

Let \( f : D \rightarrow \mathbb{C} \) be a function with the following properties:

1) \( f \) is bounded in the vertical strip

2) We have
\[ f(z + 1) = zf(z) \quad \text{for} \quad z, z + 1 \in D \]

Then we have:

\[ f(z) = f(1)\Gamma(z) \quad \text{for} \quad z \in D. \]

Proof. Applying the functional equation, it is easily seen that the function \( f \) can be analytically continued to the whole complex plane except for the points:

\[ z \in S = \{0, -1, -2, -3, \cdots \} \]

and satisfies

\[ f(z + 1) = zf(z). \]

All \( z \in S \) are either poles of first order or removable singularities, and we have:

\[ \text{Res}(f; -n) = \frac{(-1)^n}{n!} f(1). \]

Therefore, the function \( h(z) := f(z) - f(1)\Gamma(z) \) is an entire function. Furthermore, it is bounded in the vertical strip \( 0 \leq x \leq 1 \), what follows immediately from the boundedness in the strip \( 1 \leq x < 2 \) and the functional equation for \( |\text{Im}(z)| \geq 1 \). The domain \( |\text{Im}(z)| \leq 1, 0 \leq \text{Re}(z) \leq 1 \) is compact.

We want to use Liouville’s theorem and observe that from the functional equation for \( h \), i.e. \( h(z)z = h(z + 1) \), if

\[ H(z) = h(z)h(1 - z), \]

we find \( H(z + 1) = -H(z) \). But the strip \( 0 \leq x \leq 1 \) is not changed under the transformation \( z \rightarrow 1 - z \). Thus, \( H \) is bounded on this strip and, because of the periodicity, it is bounded on \( C \). Therefore, Liouville’s theorem implies that \( H \) is constant. But \( h(1) = 0 \), so \( H = 0 \) and hence also \( h = 0 \) for all \( z \in C \).

The \( \Gamma \)-integral obviously satisfies all three properties. We will see this below, when we find the integral representation from the moment-ansatz.

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2.2.2 Bohr-Mollerup Theorem

The Bohr-Mollerup Theorem also states that the Gamma-function can be uniquely classified by three properties. In other words, aside from the two obvious ones $\Gamma(x + 1) = x\Gamma(x)$, $\Gamma(1) = 1$, we, as in the case of Wielandt’s theorem, need one additional property. This is the so-called logarithmic convexity. For the sake of completeness, let us define convexity first and show that $\Gamma(x)$ has the property of logarithmic convexity, before we get to the theorem.

**Definition 2.2 (Logarithmic Convexity).** Let $X, Y$ be open subsets of the real numbers $\mathbb{R}$. Further, let $f : X \to Y$ be a function. Then, $f$ is called convex, if the following inequality holds:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad \forall x, y \in X, \quad \forall t \in [0, 1]$$

Furthermore, $f$ is called logarithmically convex, if $\log f(x)$ is convex.

Let us state a theorem which can be used if $f$ additionally is two times continuously differentiable.

**Theorem 2.4.** If the second derivative of a two times continuously differentiable function function is always $\geq 0$ in the interval $(a, b)$, then the function $f$ is convex in this interval. The converse of this theorem is also true.

The proof can be found in every book on analysis of one variable, one can also find a proof in \[Ar15\] (pp. 6-7). We will need the following corollary.

**Corollary 2.4.1.** If $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and the following inequalities are satisfied for all $x \in (a, b)$

$$f(x) > 0, \quad f(x)f''(x) - (f'(x))^2 \geq 0,$$

then $\log f$ is convex, i.e. $f$ is logarithmically convex, in this interval.

For a proof one just has to apply the previous theorem to $\log f$. Further, we have

**Corollary 2.4.2.** The sum of two logarithmically convex function is also logarithmically convex.
Now, we want to go over to the logarithmic convexity of $\Gamma$. For this, consider $f(t, x)$, continuous in both variables $x$ and $t$. Let $a \leq t \leq b$ and $x$ live in another interval. If $f(t, x)$ now is a logarithmically convex for all $t$ and twice continuously differentiable of $x$, define:

$$F_n(x) = h \left\{ f(a, x) + f(a + h, x) + f(a + 2h, x) + \cdots + f(a + (n-1)h, x), \quad h = \frac{b - a}{n} \right\}$$

Then, $F_n(x)$ is also logarithmically convex for all $n \in \mathbb{N}$. Therefore, also

$$\lim_{n \to \infty} F_n(x) = \int_a^b f(t, x) dt$$

is logarithmically convex. This also holds for improper integrals, if the integral exists. Therefore, we especially have:

**Theorem 2.5.** The $\Gamma$-function, given as

$$\int_0^\infty t^{x-1} e^{-t} dt$$

is logarithmically convex for $x > 0$.

But having mentioned all this in advance, we can finally state the Bohr-Mollerup theorem.

**Theorem 2.6 (Bohr-Mollerup Theorem).** If a function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfies the following three properties, it is identical to the $\Gamma$-function in the region where it is defined:

1) $f(x + 1) = xf(x)$
2) $f$ is logarithmically convex on the whole domain where it is defined
3) $f(1) = 1$.

**Proof.** We have shown that $\Gamma$ satisfies all conditions. Therefore, let $f$ be another function with the above properties. From the functional equation we find

$$f(x + n) = (x + n - 1)(x + n - 2) \cdots (x + 1)xf(x).$$
Since \( f(1) = 1 \) we have \( f(n) = \Gamma(n) \ \forall n \in \mathbb{N} \). We only need to show \( f = \Gamma \) for the interval \( 0 < x \leq 1 \), because of the functional equation. Thus, let \( x \) be a number in that interval and \( n \) a natural number \( \geq 2 \). Then, we have the following inequality

\[
\log(f(-1 + n)) - \log(f(n)) \leq \frac{\log(f(x + n)) - \log(f(n))}{(x + n) - n} \leq \frac{\log(f(1 + n)) - \log(f(n))}{(1 + n) - n}
\]

which follows from the logarithmic convexity. We can simplify the last equation:

\[
\log(n - 1) \leq \frac{\log(f(x + n)) - \log(f(n))}{(x + n) - n} \leq \log n
\]

or

\[
\log((n - 1)x(n - 1)! \leq f(x + n) \leq \log(n!x(n - 1)).
\]

Using the above equation for \( f(x + n) \):

\[
\frac{(n - 1)x(n - 1)!}{x(x + 1) \cdots (x + n - 1)} \leq \frac{n^x(n - 1)!}{x(x + 1) \cdots (x + n - 1)} = \frac{n^x n!}{x(x + 1) \cdots (x + n)} \cdot \frac{x + n}{n}.
\]

Since we assumed \( n \geq 2 \), we can replace \( n \) by \( n + 1 \) and find:

\[
\frac{n^x n!}{x(x + 1) \cdots (x + n)} \leq \frac{n^x n!}{x(x + 1) \cdots (x + n)} \cdot \frac{x + n}{n}.
\]

Therefore,

\[
f(x) \frac{n}{n + x} \leq \frac{n^x n!}{x(x + 1) \cdots (x + n)} \leq f(x).
\]

Taking the limit \( n \to \infty \):

\[
f(x) = \lim_{n \to \infty} \frac{n^x n!}{x(x + 1) \cdots (x + n)}.
\]

Since the function \( f \) only had to satisfy the three conditions in the theorem and was arbitrary otherwise, we conclude \( f(x) = \Gamma(x) \). \( \square \)

We have the following corollary:
Corollary 2.6.1.

\[ \Gamma(x) = \lim_{n \to \infty} \frac{n^x n!}{x(x + 1) \cdots (x + n)}. \]

We will find other ways to get to this product representation below. But it is interesting that it follows directly from the proof. Additionally, we want to point out that Gauß in \[Ga28\] used the last corollary as a definition for the \(\Gamma\)-function. We will see that this product formula has already been discovered by Euler.

In summary, we see that the Bohr-Mollerup theorem is more conveniently applied to product formula representations of \(\Gamma\), whereas Wielandt’s theorem is more convenient, when dealing with integral representations.
3 Solution of the Difference Equation

\[ F(x + 1) = xF(x) \]

In this section we will solve the functional equation in general and descend to the \( \Gamma \)-function. This will lead us to the Weierstrass product expansion. Our exposition follows [Ni05].

3.1 Weierstrass’s Definition of the \( \Gamma \)-function

It was Weierstrass’s [We56] idea to define the \( \Gamma \)-function as solution of the difference equation

\[ F(x + 1) = xF(x) \]

with the additional condition

\[ \lim_{n \to \infty} \frac{F(x + n)}{(n - 1)!n^x} = 1. \]

The above condition effectively states that the \( \Gamma \)-function does not involve a periodic part, as the condition of logarithmic convexity does in the Bohr-Mollerup theorem\(^{10}\).

Anyhow, in this section we want to solve the difference equation in general and want to show that it indeed defines the \( \Gamma \)-function as claimed.

3.2 A Remark Concerning the Solution of the Difference Equation

Let us begin with the following remark:

**Remark 1.** In order to solve the difference equation \( F(x + 1) = xF(x) \), we essentially only need one particular solution.

**Proof.** For, let \( F_1(x) \) and \( F_2(x) \) be two particular solutions of the difference equation. Then, one has

\[ F_2(x) = F_1(x) \left( \frac{F_2(x)}{F_1(x)} \right) \]

\(^9\)Weierstrass even added the condition \( F(1) = 1 \) which is not necessary to define the \( \Gamma \)-function uniquely.

\(^{10}\)This already indicates that this is another possible way to introduce the \( \Gamma \)-function. It is the only function with these two properties.
\[ \frac{F_2(x + 1)}{F_1(x + 1)} = \frac{x F_2(x)}{x F_1(x)} = \frac{F_2(x)}{F_1(x)}, \]
i.e. the quotient of the two solutions is a periodic function with period +1. In other words, if \( F_1(x) \) is a solution of the difference equation, then every other solution \( F_2(x) \) is connected to it by

\[ F_2(x) = \omega(x) F_1(x) \quad \text{with} \quad \omega(x + 1) = \omega(x). \]

\[ \square \]

### 3.3 General solution of the equation \( F(x + 1) = x F(x) \)

#### 3.3.1 Introduction of an auxiliary function

Having mentioned the remark in advance, we can now proceed to the actual solution of the difference equation. For this aim, let us introduce the following function:

**Definition 3.1.** We define a function \( \Sigma : \mathbb{C} \setminus \{0, -1, -2, -3, \ldots \} \rightarrow \mathbb{C} \) by the sum

\[ \Sigma(x) := \sum_{s=0}^{\infty} \left( \frac{1}{s + 1} - \frac{1}{x + s} \right) \]

This series is easily seen to converge uniformly. Thus, we are allowed to integrate it term by term with respect to \( x \), provided the path of integration is of finite length and does not pass through any of the poles.

Furthermore, we have

**Theorem 3.1.** \( \Sigma \) as defined above satisfies the functional equation

\[ \Sigma(x + 1) = \Sigma(x) + \frac{1}{x}. \]

**Proof.** Consider the difference \( \Sigma(x + 1) - \Sigma(x) \); it reads

\[ \Sigma(x + 1) - \Sigma(x) = \frac{1}{x}. \]

\[ \square \]

\[ ^{11} \text{We will meet this function again, when we talk about Euler’s ideas on interpolation of so-called inexplicable functions, a term he coined in chapter 16 of [E212].} \]
\[ \sum_{s=0}^{\infty} \left( \frac{1}{s+1} - \frac{1}{x+1+s} \right) - \sum_{s=0}^{\infty} \left( \frac{1}{s+1} - \frac{1}{x+s} \right) \]

\[ = \sum_{s=0}^{\infty} \left( \frac{1}{s+1} - \frac{1}{x+1+s} \right) - \left(1 - \frac{1}{x}\right) - \sum_{s=0}^{\infty} \left( \frac{1}{s+2} - \frac{1}{x+1+s} \right) \]

\[ = 1 - 1 + \frac{1}{x} = \frac{1}{x} \]

since the sums involving \( x \) cancel and the sums involving only \( s \) are telescoping sums.

\[ \square \]

### 3.3.2 Product representation of the function \( F \)

**Theorem 3.2.** Every function satisfying the equation \( F(x+1) = xF(x) \) has a product expansion of the form

\[ F(x+1) = \omega(x) \cdot \frac{e^{x}}{x} \cdot \prod_{s=1}^{\infty} \frac{e^{s}}{1 + \frac{s}{x}} \]

where \( K \) is a constant and \( \omega : \mathbb{C} \to \mathbb{C} \) a periodic integrable function with period \(+1\).

**Proof.** Let \( \omega(x) \) be as above, and let it be integrable from 0 to \( x \). Define

\[ \omega_1(x) := \int_{0}^{x} \omega(x) dx. \]

Then \( \omega_1(x) \) satisfies the functional equation:

\[ \omega_1(x + 1) = \omega_1(x) + K, \]

\( K \) being a constant. Now from our function \( \Sigma \) we have

\[ \log F(x + 1) = \int_{0}^{x} \Sigma(x + 1) dx + K \cdot (x + 1) \]

or, equivalently substituting the series for \( \Sigma \) and integrating it term by term, we have
\[
\log F(x + 1) = \sum_{s=0}^{\infty} \left( \frac{x}{s + 1} - \log \left( 1 + \frac{x}{s + 1} \right) \right) + K \cdot (x + 1).
\]

Thus, taking the exponentials, we find
\[
F(x) = \omega(x) \cdot \frac{e^{Kx}}{x} \cdot \prod_{s=1}^{\infty} \frac{e^{\frac{x}{s}}}{1 + \frac{x}{s}}.
\]

Thus, to summarize the proof: We basically solved the simpler equation \( f(x + 1) - f(x) = \frac{1}{2} \) first. A particular solution is given by our function \( \Sigma \). Thus, by integrating, we can then deduce the solution of \( g(x + 1) - g(x) = \log(x) \) and by taking the exponentials we arrive at the functional equation for \( F(x) \). It is helpful to keep this in mind, since this is also basically what Euler did in [E613] to find the product representation of the \( \Gamma \)-function. In other words, this proof can easily constructed from Euler’s ideas in that paper.

### 3.3.3 Finding the constant \( K \)

Finally, we need to find the constant \( K \). Hence let us introduce the function \( G_n : \mathbb{C} \setminus \{0, -1, -2, \ldots, -(n-1)\} \to \mathbb{C} \) defined by
\[
G_n(x) := \frac{e^{Kx} \cdot \prod_{s=1}^{n-1} \frac{e^{\frac{x}{s}}}{1 + \frac{x}{s}}}{x}.
\]

From this
\[
G_n(x + 1) = \frac{e^{Kx} \cdot e^K}{x + 1} \cdot \prod_{s=1}^{n-1} \frac{e^{\frac{x}{s}} \cdot e^{\frac{1}{s+1}}}{1 + \frac{x}{s+1}} \cdot \frac{s}{s + 1}
\]

Or, writing this in a more convenient form
\[
G_n(x + 1) = \frac{e^{Kx} \cdot e^K}{x + 1} \cdot \prod_{s=1}^{n-1} \frac{e^{\frac{x}{s}} \cdot e^{\frac{1}{s+1}}}{1 + \frac{x}{s+1}} \cdot e^{\frac{s}{s+1} - \log(1 + \frac{1}{s})}
\]

Now using the well-known result that
\[
\lim_{n \to \infty} \sum_{s=1}^{n-1} \left( \frac{1}{s} - \log \left( 1 + \frac{1}{s} \right) \right) = \gamma
\]
where $\gamma$ is the Euler-Mascheroni constant, we know this limit to exist[12]. Define

$$\gamma_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$$

so that also $\gamma = \lim_{n \to \infty} \gamma_n$. Then, we have

$$G_n(x) = \frac{e^{-\gamma x + \frac{x}{n}}}{x} \prod_{s=1}^{n-1} \frac{e^\frac{x}{s+1}}{1 + \frac{x}{s+1}}$$

and the functional equation

$$G_n(x + 1) = x \cdot G_n(x) \cdot \frac{n}{n + x}.$$

Therefore, using theorem 3.2 we have:

**Theorem 3.3.** Let $\gamma$ be the Euler-Mascheroni constant. Further, let $\omega : \mathbb{C} \to \mathbb{C}$ an arbitrary function satisfying $\omega(x + 1) = \omega(x)$; then, the most general solution of the difference equation $F(x + 1) = xF(x)$ is given by

$$F(x) = \omega(x) \cdot \frac{e^{-\gamma x}}{x} \cdot \prod_{s=1}^{\infty} \frac{e^\frac{x}{s}}{1 + \frac{x}{s}}.$$

### 3.4 Application to $\Gamma(x)$-Weierstrass Product Representation.

Now that we found the most general solution of the difference equation $F(x + 1) = xF(x)$, we want to descend to $\Gamma(x)$ from this. This is, e.g., possible by an application of the Bohr-Mollerup theorem.

Doing so, we will arrive at the following theorem

**Theorem 3.4** (Weierstrass Product Expansion of $\Gamma(x)$). The $\Gamma$-function has the following product expansion:

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \cdot \prod_{s=1}^{\infty} \frac{e^\frac{x}{s}}{1 + \frac{x}{s}}.$$

[12] This constant is discussed in the appendix. There it is also shown that the limit exists.
Proof. We need to check whether the three conditions in the Bohr-Mollerup theorem are fulfilled. Therefore, let us check $\Gamma(1) = 1$ first.

We have

$$\Gamma(1) = e^{-\gamma} \cdot \prod_{s=1}^{\infty} \frac{e^{\frac{1}{s}}}{1 + \frac{1}{s}}$$

or, by taking logarithms,

$$\log \Gamma(1) = -\gamma + \sum_{s=1}^{\infty} \left( \frac{1}{s} - \log \left(1 + \frac{1}{s}\right)\right)$$

But, as we have seen above, the sum evaluates to $\gamma$, whence $\log \gamma(1) = 1$ or $\Gamma(1) = 1$. Hence the first condition is satisfied.

The second condition of the Bohr-Mollerup Theorem, i.e. $\Gamma(x + 1) = x\Gamma(x)$ is satisfied, since it solves the general difference equation $F(x + 1) = xF(x)$.

Finally, let us check logarithmic convexity. Obviously, $\log \Gamma(x)$ is twice continuously differentiable, since the resulting sum converges uniformly. We find

$$\frac{d}{dx} \log \Gamma(x) = -\gamma - \frac{1}{x} + \sum_{s=1}^{\infty} \left( \frac{1}{s} - \frac{1}{1 + \frac{1}{s}} \cdot \frac{1}{s} \right)$$

and

$$\frac{d^2}{dx^2} \log \Gamma(x) = \frac{1}{x^2} + \sum_{s=1}^{\infty} \frac{1}{s^2} \cdot \frac{1}{(1 + \frac{1}{s})^2}.$$ 

Therefore, obviously $\frac{d^2}{dx^2} \log \Gamma(x) > 0 \forall x > 0$.

Hence the Bohr-Mollerup theorem applies and the function defined by the infinite product is indeed the familiar $\Gamma$-function.

3.5 Euler on Weierstrass’s condition

Here, we want to show how Euler already arrived at the condition for the $\Gamma$-function Weierstraß used to introduce it. Note that since we obtained that $F(x) = \Gamma(x)$, we proved that:

\[ \boxed{\quad} \]

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Theorem 3.5. The $\Gamma$-function can also be defined by Weierstraß’s conditions, i.e. the $\Gamma$ function is the unique function satisfying

$$\Gamma(x + 1) = x\Gamma(x)$$

and

$$\lim_{n \to \infty} \frac{\Gamma(x + n)}{(n - 1)!n^x} = 1.$$

We will present how Euler obtained this condition in [E652]. Weierstraß attributed it to Gauß who introduced this condition in [Ga28]. We will show it for $\Gamma(x + 1)$, since in [E652] Euler also did it for $x!$. Euler’s idea is to consider $n$ as a large natural number and $x$ as fixed finite natural number with $x \ll n$ and evaluate $\Gamma(x + n + 1)$ in two ways. Using the functional equation $x$ times, we have

$$\Gamma(x + 1 + n) = (x + n)(x + n - 1) \cdots (n + 1)\Gamma(n + 1).$$

But, since $x \ll n$, the finite parts added to $n$ in each factor can be ignored so that

$$\Gamma(x + 1 + n) = n^x\Gamma(n + 1).$$

On the other hand, we can use functional equation $n$ times, to find:

$$\Gamma(x + n + 1) = (n + x)(n + x - 1) \cdots (x + 1)\Gamma(x + 1).$$

Therefore, dividing both expressions expressions for $\Gamma(x + n + 1)$, using $\Gamma(n + 1) = n!$ and solving for $\Gamma(x + 1)$, we arrive at the formula:

$$\Gamma(x + 1) = \frac{n^x n!}{(x + 1) \cdots (n + x - 1)(n + x)} \text{ for } x \ll n.$$

In modern notation, this is precisely the above condition in the theorem. One just has to use the functional equation on $\Gamma(x + n)$ times to arrive at Euler’s and Gauß’s formula. Note that although the proof required $x$ and $n$ to be natural numbers, the right-hand side does not demand $x$ to be a natural number. Therefore, it can also be used to interpolate $x! = \Gamma(x + 1)$. Additionally, we point out again that Gauß [Ga28] used this formula to introduce the $\Gamma$-function.
Euler’s reasoning, using infinitely large numbers, is obviously not rigorous enough for modern times. But, in possession of the Bohr-Mollerup theorem, one could start from this expression and check, whether all conditions are satisfied or not\textsuperscript{13}. But since we already did it for the Weierstraß product and know this expression to be equivalent to it, we do not want to repeat this here. Furthermore, we also already arrived at this precise theorem, when we proved the Bohr-Mollerup theorem above.

\textsuperscript{13}In fact this is easily done, but we will not do it here
4 Euler’s direct Solution of the Equation
\( \Gamma(x + 1) = x\Gamma(x) - \text{The Moment-Ansatz} \)

We now go over to Euler’s different approaches leading him to an explicit formula of the \( \Gamma \)-function. We will start with the moment-ansatz, a name that will become clear later. First, we want to explain briefly, where the method actually originated from.

4.1 Origin of the Idea

Euler uses a technique we will refer here to as moment-ansatz to solve difference equations of the kind:

\[(a + \alpha x)f(x) = (b + \beta x)f(x + 1) + (c + \gamma x)f(x + 2),\]

where \( \alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\} \) and \( a, b, c \in \mathbb{R} \), in his papers [E123] and [E594]. His original intention was to derive continued fractions from this. For, dividing the above equation by \( (a + ax) \) and \( f(x + 1) \), one will find

\[
\frac{f(x)}{f(x + 1)} = \frac{b + \beta x}{a + ax} + \frac{c + \gamma x}{a + ax} \frac{f(x + 2)}{f(x + 1)}
\]

or

\[
\frac{f(x)}{f(x + 1)} = \frac{b + \beta x}{a + ax} + \frac{c + \gamma x}{a + ax} \frac{1}{\frac{f(x + 1)}{f(x + 2)}}.
\]

Replacing \( x \) by \( x + 1 \) one will get a similar equation for the quotient \( \frac{f(x + 1)}{f(x + 2)} \) which can be inserted in the above equation. Repeating this procedure infinitely often, one will get a continued fraction for \( \frac{f(x)}{f(x + 1)} \).

Euler was interested in the continued fraction arising from this and he hence tried to solve the difference equation. In the following we will explain how he did this.

4.2 Euler’s Idea

Let us discuss his idea on the concrete example of the above difference equation. The generalisation to the general difference equation with linear coefficients is immediate. Euler’s assumed that the solution is given as an integral of the form
\[
\int_{a}^{b} t^{x-1} P(t) dt,
\]
whence we have to determine the limits of the integration and the function \( P(t) \). In order to do so, Euler considers the auxiliary equation

\[
(a + ax) \int t^{x-1} P(t) dt = (b + bx) \int t^{x} P(t) dt + (c + cx) \int t^{x+1} P(t) dt + t^{x} Q(t);
\]

here \( \int \) is supposed to denote the indefinite integral over \( t \) and \( Q(t) \) is another function we have to determine, the use of which will become clear in a moment.

Euler then differentiates the auxiliary equation with respect to \( t \):

\[
(a + ax)t^{x-1} P(t) = (b + bx)t^{x} P(t) + (c + cx)t^{x+1} P(t) + xt^{x-1} Q(t) + t^{x} Q'(t).
\]

Now divide by \( t^{x-1} \):

\[
(a + ax) P(t) = (b + bx)t P(t) + (c + cx)t^{2} P(t) + xQ(t) + tQ'(t).
\]

Comparing the coefficients of the powers of \( x \), we will get the following systems of coupled equations:

1. \( aP(t) = btP(t) + ct^{2} P(t) + tQ'(t) \)
2. \( \alpha P(t) = \beta t P(t) + \gamma t^{2} P(t) + Q(t) \)

Solving both equations for \( P \), we find

1. \( P(t) = \frac{tQ'(t)}{a - bt - ct^{2}} \)
2. \( P(t) = \frac{Q(t)}{\kappa - \beta t - \gamma t^{2}} \)

Therefore, we obtain the following equation for \( Q(t) \)
\[ \frac{tQ'(t)}{Q(t)} = \frac{a - bt - ct^2}{\alpha - \beta t - \gamma t^2} \]

Although this equation can be solved in general, we will not do this here, because it will be more illustrative to consider examples. Anyhow, having found \( Q(t) \), we can also find \( P(t) \) substituting the value of \( Q(t) \) in one of the above equations.

Finally, we need the term \( t^x Q(t) \) to vanish in the auxiliary equation. Hence the limits of integration are found from the solutions of the equation \( t^x Q(t) = 0 \).

### 4.2.1 Application to the \( \Gamma \)-function - the integral representation

As mentioned, everything becomes a lot more clear in certain examples. Therefore, let us consider the \( \Gamma \)-function, i.e. the functional equation \( f(x + 1) = xf(x) \). Euler considers the factorial in §13 in [E594].

We make the ansatz

\[ f(x) = \int_a^b t^{x-1} P(t) dt. \]

Hence we need to determine \( P(t) \) and the limits of integration \( a \) and \( b \). Let us introduce the auxiliary equation:

\[ \int_t^1 t^x P(t) dt = x \int t^{x-1} P(t) dt + t^x Q(t) \]

Differentiating with respect to \( t \) gives

\[ t^x P(t) = xt^{x-1} P(t) + xt^{x-1} Q(t) + t^x Q'(t). \]

Dividing by \( t^{x-1} \)

\[ P(t) = xP(t) + xtQ(t) + tQ'(t). \]

Therefore, comparing the coefficients of \( x \):
1. \( P(t) = tQ'(t) \)
2. \( 0 = P(t) + tQ(t) \)

Solving both equations for \( P(t) \):

1. \( P(t) = tQ'(t) \)
2. \( P(t) = -tQ(t) \)

Hence we obtain the following differential equation for \( Q(t) \):

\[
\frac{Q'(t)}{Q(t)} = -1.
\]

This equation is easily integrated and gives

\[
\log(Q(t)) = -Ct \quad \text{or} \quad Q(t) = Ce^{-t},
\]

where \( C \neq 0 \) is an arbitrary constant of integration. From this \( P \) is found to be

\[
P(t) = -e^{-t}.
\]

Finally, we need to find the limits of integration. For this, we consider the equation \( t^x Q(t) = Ct^x e^{-t} = 0 \). For \( x > 0 \) we find the two solution \( t = 0 \) and \( t = \infty \). Therefore, the term \( t^x Q(t) \) in the auxiliary equation vanishes in these cases and we find:

\[
C \int_0^\infty t^x e^{-t} dt = xC \int_0^\infty t^{x-1} e^{-t} dt.
\]

In other words, the equation \( f(x + 1) = xf(x) \) is satisfied by:

\[
f(x) = C \int_0^\infty t^{x-1} e^{-t} dt.
\]

This is, of course, almost the famous integral representation of the \( \Gamma \)-function. (There the constant \( C \) is one.)

\(^{14}\text{Note that this is precisely the condition on } x \text{ we need for the integral to converge!} \)
Finding the integral representation of $\Gamma(x)$

We can force the function $f$ to be the $\Gamma$ by demanding it to satisfy all conditions of Wielandt’s theorem\(^{15}\). The condition $\Gamma(1) = 1$ forces $C = 1$. More precisely, we have the theorem:

**Theorem 4.1** (Integral Representation of $\Gamma(x)$). The $\Gamma$-function is given by the following integral:

$$
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad \text{for } \Re x > 0.
$$

*Proof.* We have to check all conditions of Wielandt’s theorem\(^{16}\). First, find $\Gamma(1)$.

$$
\Gamma(1) = \int_0^\infty e^{-t}dt = [-e^t]_0^\infty = 0 - (-1) = 1.
$$

Secondly, the functional equation is satisfied, as demonstrated in the last section.

Finally, we have to check holomorphy (which is obvious, see also in the introduction) and that $\Gamma(x)$ is bounded in the strip $S := \{x|1 \leq x < 2\}$. Hence consider

$$
|\Gamma(x)| = \left| \int_0^\infty t^{x-1}e^{-t}dt \right| \leq \int_0^\infty \left| t^{x-1} \right| e^{-t}dt
$$

In other words, we have

$$
|\Gamma(x)| \leq \Re(\Gamma(x)) \quad \text{for } \Re x > 0.
$$

For checking Wielandt’s theorem we hence have to consider

$$
\int_0^\infty t^{x-1}e^{-t}dt \quad \text{for } 1 \leq x < 2.
$$

But these integrals are obviously bounded, whence Wieldlandt’s theorem applies. \(\square\)

\(^{15}\)As we mentioned above, it is more convenient to use Wielandt’s theorem, if dealing with integrals. At least, if one is not in position of all the auxiliary theorems, how to prove logarithmic convexity of parameter integrals.

\(^{16}\)So at this point, we basically prove that the integral representation is indeed a correct definition for $\Gamma(x)$.
4.2.2 Some Remarks on the ansatz

We assume the solution to have the form $\int t^{x-1}P(t)\,dt$, which explains the name moment-ansatz\(^{17}\). But one can, of course, make other choices for the integrand. For the sake of an example, one can set $\int R(t)^{x-1}$. By the same produce one would then arrive at the equation

$$\Gamma(x) = \int_0^1 \left(\log \frac{1}{t}\right)^{x-1} \,dt.$$  

This was Euler’s preferred integral representation and actually the first he found in $[E19]$. It follows from the one above by setting $e^{-t} = u$.

Furthermore, one can even generalize the ansatz to

$$f(x) = \int R(t)^{x-1}P(t)\,dt,$$

where $R(t)$ is another function to be determined\(^{18}\). Carrying out the procedure as above one would arrive at certain conditions on the function $R(t)$ which are trivially satisfied by $R(t) = t$. Indeed, Euler tried this most general ansatz in $[E123]$, but realizing that $R(t) = t$ meets all requirements, he quickly focused on that special case.

4.3 Examples of Other Equations Which Can be Found by This Method

Having found the integral representation of the $\Gamma$-function from its difference equation, let us apply Euler’s method to more complicated but still familiar difference equations in order to find some interesting integral representations which seem to be completely new in the literature.

4.3.1 1. Example: Legendre Polynomials

The Legendre polynomials satisfy the following difference equation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

\(^{17}\)A moment is defined as $M_n := \int x^n d\mu$, $\mu$ being some integration measure.

\(^{18}\)It is indeed convenient to use this ansatz in the case of hypergeometric series, for example.
Together with the conditions \( P_0(x) = 1 \) and \( P_1(x) = x \), this difference equation determines them completely. The moment-ansatz can be used to find an explicit formula for \( P_n(x) \). More precisely, we have the theorem:

**Theorem 4.2** (Integral representation for the \( n \)-th Legendre Polynomial). We have

\[
P_n(x) = \frac{1}{\pi i} \int_{x - \sqrt{x^2 - 1}}^{x + \sqrt{x^2 - 1}} \frac{t^n}{\sqrt{1 - 2xt + t^2}} dt.
\]

**Proof.** This expression can be found by the moment-ansatz. Since this is our first concrete example of a second order difference equation and one has to be more careful than in the case of the \( \Gamma \)-function, let us present the calculation in detail. We start with the auxiliary equation again which reads

\[
(n + 1) \int x \, R(x, t) \, dt = (2n + 1) x \int t^{n-1} R(x, t) \, dt - n \int t^{n-2} R(x, t) \, dt + Q(x, t) t^n.
\]

We wish to find \( R(x, t) \) and \( Q(x, t) \) and the limits of integration. Let us differentiate that equation

\[
(n + 1) t^n R(x, t) = (2n + 1) x t^{n-1} R(x, t) - n t^{n-2} R(x, t) - n t^{n-2} R(x, t) + Q(x, t) t^n.
\]

Dividing by \( t^{n-2} \) and comparing the coefficients of the powers of \( n \), we obtain the following system of equations

\[
1. \quad t^2 R(x, t) = 2t R(x, t) x - R(x, t) + t Q(x, t) \\
2. \quad t^2 R(x, t) = t R(x, t) x + \dot{Q}(x, t) t^2
\]

Solving both for \( R(x, t) \)

\[
1. \quad R(x, t) = \frac{t Q(x, t)}{t^2 - 2tx + 1} \\
2. \quad R(x, t) = \frac{t^2 \dot{Q}(x, t)}{t^2 - xt}
\]

\(^{19}\)Although we wrote \( R(x, t) = \dot{R}(t) \) instead of \( R(t) \), this does not alter the procedure at all.

It will just turn out that \( R \) depends also on \( x \) which is to be considered as a parameter in the difference equation. The same goes for \( Q \).
Therefore,

\[ \frac{tQ(t)}{t^2 - 2xt + 1} = \frac{t^2Q(x,t)}{t^2 - xt} \]

whence we find

\[ Q(t) = C(x) \sqrt{t^2 - 2xt + 1}. \]

\( C(x) \) being an arbitrary function of \( x \) that entered via integration with respect to \( t \). Therefore,

\[ R(x,t) = C(x) \frac{t}{\sqrt{1 - 2xt + t^2}}. \]

Integrating the differentiated auxiliary equation again from \( a \) to \( b \), we would have

\[ P_n(x) = C(x) \int_a^b \frac{t^n}{\sqrt{1 - 2xt + t^2}} dt \]

if we determine \( a \) and \( b \) in such a way that \( Q(x,t)t^n \) vanishes for \( a \) and \( b \) for all \( n \). Since \( t^0 = 1 \) has no zeros, we have to put \( Q(x,t) = 0 \). This gives

\[ a = x - \sqrt{x^2 - 1} \quad \text{and} \quad b = x + \sqrt{x^2 - 1}. \]

Therefore, it remains to find \( C(x) \). For this we use the special case \( P_0(x) = 1 \). We calculate

\[ C(x) \int_{x - \sqrt{x^2 - 1}}^{x + \sqrt{x^2 - 1}} \frac{t^0}{\sqrt{1 - 2xt + t^2}} dt = C(x) \left[ \log \left( \frac{\sqrt{1 - 2xt + t^2} + t - x}{x + \sqrt{x^2 - 1}} \right) \right]_{x - \sqrt{x^2 - 1}}^{x + \sqrt{x^2 - 1}} \]

Therefore,

\[ C(x) = \frac{1}{\log(-1)} = \frac{1}{\pi i}, \]

where we used the principal branch of the logarithm, of course. The explicit formula for the \( n \)-th Legendre polynomial hence reads
\[ P_n(x) = \frac{1}{n!} \int_{x-\sqrt{x^2-1}}^{x+\sqrt{x^2-1}} \frac{t^n}{\sqrt{1-2xt+t^2}} \, dt. \]

It is easily checked that the explicit formula also gives \( P_1(x) = x \). Therefore, both initial conditions and the functional equations are satisfied and hence the above formula gives the \( n \)-th Legendre polynomial.

### 4.3.2 Historical Remark

The Legendre polynomials were named after Legendre because of his paper [Le85]; he discovered them in his investigations on the gravitational potential. Nowadays, they are important in electrodynamics, more precisely, the multipole expansion. But they were in fact already discovered by Euler in his paper [E551] in a completely different context. In [E606] Euler even gave the explicit formula we found. But since in that work he was mainly interested in continued fractions for the quotients of two consecutive integrals, he did not find the constant \( C(x) = \frac{1}{\pi} \). Further, it seems that he did not notice the connection between the findings of [E551] and [E606]. In other words, he was not aware that he already obtained an explicit formula for \( P_n(x) \). This is even more interesting, because in [E551] he said that it is not possible for him to find such an explicit formula, although he provided all necessary tools in his earlier papers [E123] and [E594]. In summary, it seems that Euler was not aware that in those papers he basically discovered an general method to find a particular solution of the general homogeneous difference equation with linear coefficients.

### 4.3.3 2. Example: Hermite Polynomials

The Hermite polynomials satisfy the recurrence relation

\[ H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \]

with the additional conditions \( H_0(x) = 1, H_1(x) = 2x \). We then have the formula

**Theorem 4.3** (Explicit Formula for the \( n \)-th Hermite Polynomial). The following formula holds:
\[ H_n(x) = \frac{i^n e^{x^2}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} t^n e^{\frac{x^2}{2} \left(-\frac{e^2}{2} - 2xit\right)} dt \]

**Proof.** We use the moment-ansatz. We will not carry out the calculation since it is similar to the case of the Legendre polynomials. We will only state the intermediate results.

Of course, we start from the auxiliary equation:

\[ \int t^n P(t) dt = 2x \int t^n P(t) dt - 2n \int t^{n-1} P(t) dt + t^n Q(t). \]

From this we derive the following equations for \( P(t) \) and \( Q(t) \):

1. \( P(t) = \frac{Q'(t)}{t^2 - 2xt} \)
2. \( P(t) = \frac{Q(t)}{2} \)

whence

\[ Q(t) = C(x)e^{\frac{1}{2} \left(\frac{x^2}{2} - 2xt\right)} \quad \text{and} \quad P(t) = \frac{1}{2} C(x)e^{\frac{1}{2} \left(-\frac{x^2}{2} - 2xt\right)} \]

In order to find the limits of integration we need to solve \( t^n e^{\frac{x^2}{2} \left(-\frac{x^2}{2} - 2xt\right)} = 0 \), which leads to \( t = \pm i\infty \), if we want \( n \) to be an arbitrary integer number. Therefore, up to this point we have:

\[ H_n(x) = \frac{C(x)}{2} \int_{-\infty}^{\infty} t^n e^{\frac{x^2}{2} \left(-\frac{x^2}{2} - 2xt\right)} dt. \]

It is convenient to get rid of the imaginary limits by the substitution \( t = iy \). This gives

\[ H_n(x) = \frac{C(x)}{2} i^{n+1} \int_{-\infty}^{\infty} y^n e^{\frac{1}{2} \left(-\frac{y^2}{2} - 2xy\right)} dy. \]

From the initial condition \( H_0(x) = 1 \) we find

\[ \frac{C(x)}{2} = \frac{e^{x^2}}{2i\sqrt{\pi}}. \]
Therefore, we arrive at

\[
H_n(x) = \frac{e^{x^2 - i\pi}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} t^n e^{\frac{t^2}{2} - 2itx} dt.
\]

The condition \(H_1(x) = 2x\) is easily checked to be satisfied by the explicit formula. The general formula we arrived at seems to be completely new. It is obvious that one can find similar explicit formulas for other orthogonal polynomials defined by second order homogeneous difference equations with linear coefficients, like, e.g., the Laguerre and Chebyshev polynomials.

4.3.4 3. Example: Beta-Function

Let us consider the \(B\)-function, which is defined as

**Definition 4.1 (B-function).** The \(B\)-function, also referred to as Eulerian integral of the first kind, is defined as:

\[
B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt \quad \text{for } \Re x, \Re y > 0
\]

From its definition it is immediate that \(B\) satisfies the functional equation

\[
B(x + 1, y) = \frac{x}{x+y} B(x, y).
\]

And one could start from this functional equation to obtain the integral representation via the moment-ansatz. Euler did this in §17 of [E594]20. But after the above examples we do not want to do this here.

Here we want to use the results obtained up to this point to show:

**Theorem 4.4.** We have

\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\]

**Proof.** We start from the functional equation, of course. Further, we assume that \(B\) can be written as product of two functions \(B_1, B_2\). We demand those to satisfy the equations:

---

20He even considered a slightly more general example.
\[ B_1(x + 1, y) = xB_1(x, y) \quad \text{and} \quad (x + y)B_2(x + 1, y) = B_2(x, y). \]

For the sake of brevity, we will drop the \( y \) in the argument in the following; since we consider the functional equation only in \( x, y \) can be seen as a parameter.

It is easily seen that \( B_1(x) \cdot B_2(x) \) satisfies the functional equation for \( B(x, y) \). Therefore, we need to solve the equations for \( B_1 \) and \( B_2 \) to find an expression for \( B(x, y) \).

Let us consider \( B_1 \) first. It satisfies the functional equation of the \( \Gamma \)-function in \( x \). Therefore, we immediately have

\[ B_1(x) = C_1(y) \Gamma(x), \]

\( C(y) \) being an arbitrary function of \( y \).

To solve the functional equation for \( B_2(x) \), let us introduce \( D(x) = \frac{1}{B_2(x)} \).

Then, \( D(x) \) satisfies the functional equation:

\[ D(x + 1) = (x + y)D(x). \]

This equation is easily seen to be solved by

\[ D(x) = C_2(y) \Gamma(x + y). \]

\( C_2(y) \) is an arbitrary function of \( y \). And hence

\[ B_2(x) = \frac{1}{C_2(y) \Gamma(x + y)}. \]

In total, we have found

\[ B(x, y) = C(y) \frac{\Gamma(x)}{\Gamma(x + y)}. \]

\( C(y) = \frac{C_1(y)}{C_2(y)}. \)

It remains to define the function \( C(y) \). From the definition of \( B(x, y) \) we find:

\[ B(1, y) = \int_0^1 (1 - t)^{y-1}dt = \frac{1}{y}, \]

37
which can be used as an initial condition for the functional equation. First, from our solution we find

\[ B(1, y) = C(y) \frac{\Gamma(1)}{\Gamma(y + 1)} = C(y) \frac{1}{y \cdot \Gamma(y)}, \]

where we used \( \Gamma(1) = 1 \) and \( \Gamma(y + 1) = y \Gamma(y) \). Therefore, \( C(y) \) must satisfy:

\[ C(y) \frac{1}{y \Gamma(y)} = \frac{1}{y} \quad \text{or} \quad C(y) = \frac{1}{\Gamma(y)}. \]

Therefore, we finally arrived at the formula:

\[ B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}. \]

\[ \square \]

4.3.5 Some Remarks

The relation among the \( B^- \) and \( \Gamma^- \)-function was already discovered by Euler. He states it, e.g., in [E421]. But his argument given there is not by any means a rigorous proof, but rather he only proves the formula for integers \( x \) and \( y \) and then, without any further explanation, replaces the factorials by the integral representation of \( \Gamma(x) \).

Rigorous proofs were first given by Jacobi and Dirichlet, but they both use the theory of double integrals, which Euler did not know. Indeed, Euler only wrote one single paper on multiple integrals [E391] and ran into some trouble, basically since he did not know about the antisymmetry of the wedge product. See Katz’s article in [Du07]. We will discuss Dirichlet’s and Jacobi’s proof below.

Our reasoning to obtain this fundamental relation on the other hand does not require double integrals and our proof certainly was in Euler grasp. In [E591] he even considered similar questions, but never made the connection to the \( B^- \) and \( \Gamma^- \)-function.

Furthermore, concerning the solution of the difference equations for \( B_1(x) \) and \( B_2(x) \). The most general solution of for \( B_1(x) \) is given by:

\[ B_1(x) = \omega_1(x) \Gamma(x) C_1(y) \quad \text{with} \quad \omega_1(x + 1) = \omega_1(x). \]
And similarly for $B_2(x)$, as we saw discussing Weierstraß’s ideas concerning the $\Gamma$-function. But we omitted the periodic function in this solution, since otherwise we would have:

$$B(x, y) = \frac{1}{0} \int dt t^{x-1}(1-t)^{y-1} = \Omega_1(x) \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$$

$\Omega_1(x)$ is a function with period 1. But $\Omega_1(x)$ can be found from the special case $B(x, 1)$. For, in this case we have on the one hand

$$B(x, 1) = \frac{1}{0} \int dt t^{x-1} = \frac{1}{x}.$$ 

But on the other hand

$$B(x, 1) = \frac{\Gamma(x)}{\Gamma(x+1)} \Omega_1(x) = \frac{\Omega_1(x)}{x}.$$ 

Here we used the functional equation of the $\Gamma$-function and $\Gamma(1) = 1$. This already implies $\Omega_1(x) = 1$ for all $x$. Hence the periodic function is simply $= 1$ and the above relation among $\Gamma$- and $B$-function is correct.

4.3.6 4. Example: Hypergeometric Series

Finally, let us mention the hypergeometric series. It was first defined by Euler in [E710]

**Definition 4.2 (Hypergeometric Series).** For $a, b, c \in \mathbb{C} \setminus \{0, -1, -2, -3, \cdots \}$ the hypergeometric series is defined by

$$\, _2F_1(a, b; c; z) = 1 + \frac{ab \, z}{c \, 1!} + \frac{a(a+1)b(b+1) \, z^2}{c(c+1) \, 2!} + \cdots \quad \text{for} \quad |z| < 1.$$ 

We will drop the subscripts 2 and 1, and write simply $F$, if there is no chance for confusion. The first systematic study was done by Gauß [Ga28], whence the above series is often referred to as Gaussian hypergeometric series. Many people contributed to the nowadays highly developed theory of this function. We mention Kummer [Ku36] and Riemann [Ri57] as some of the contributors. A
modern treatise is [Ao11]. What is of interested for us is that one can derive the an integral representation of the hypergeometric series attributed to Euler from a certain difference equation the hypergeometric satisfies. Gauß in his paper called them contiguous relations and gives a complete list of 15 of such equations. We will need one following from those gave in the mentioned paper. We have

**Theorem 4.5.** We have the following equation:

\[
B(b + 2, c - b)F(a, b + 2, c + 2; x) = \left(\frac{b}{x(a - c - 1)}\right) B(b, c - b)F(a, b, c; x) + \left(\frac{(b - a + 1)x + c}{x(c - a + 1)}\right) B(b + 1, c - b)F(a, b + 1, c + 1; x)
\]

The proof is simply by expanding each hypergeometric function into a power series and compare coefficients. We will not do this here. We will consider the above equation as an equation in \(b\). Note that by dividing both sides by \(B(b, c - b)\) and applying the relation to the \(\Gamma\)-function and its functional equation, the coefficients become linear functions in \(b\). Then, it is a homogeneous difference equation with linear coefficients. \(a, c, x\) are parameters. Thus, we can solve this equation by the moment-ansatz. Indeed, proceeding as in the previous cases (with the condition \(F(a, 0, c; x) = 1\)), after a long and tedious calculation we arrive at:

\[
F(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - xt)^{-a} dt.
\]

This is the famous Eulerian integral representation of the hypergeometric series.

Finally, let us mention one drawback of the moment ansatz. In Gauß’s paper one also finds contiguous relations, relating \(F(a, b, c; x), F(a + 1, b, c; x)\) and \(F(a - 1, b, c; x)\) (see, e.g. equation [1] in §7 of [Ga28]). The coefficients are also linear in \(a\). But in this case the moment-ansatz does not produce a nice solution, if one just uses the ansatz.

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21We have to say some things about that later.
22Below in the section on intersection numbers we will arrive precisely at this relation starting from the integral representation.
23It produces one, but actually that thing is quite a mess.
\[ \int t^{a-1} P(t) dt, \]
since \( P(t) \), as we have seen, also does depend on \( a \). Therefore, despite its success in all the examples, the moment ansatz does only produces a nice solution, if it is actually possible to express it as an integral of the given form conveniently.

### 4.3.7 Historical Note on the Integral Representation

We want to mention a little on the origin of the the integral representation of the hypergeometric function. It is often ascribed to Euler, but it is actually not that simple. What can be said for certain is that that Euler never wrote down the above equality explicitly. Therefore, let us briefly discuss, what Euler actually did.

First, as we already mentioned, Euler was indeed the first who studied the hypergeometric series and defined it as the power series above in [E710]. He proceeds to find the differential equation satisfied by it and finds a transformation, now bearing his name, of the hypergeometric series in that paper. But he did NOT state the above integral representation anywhere. Nevertheless, on several instances, he did more general investigations, from which the formula would easily follow. Those investigations were mainly concerned with differential equations. We mention [E274] and especially chapter 12 of his second book on integral calculus [E366]. His paper one the hypergeometric series [E710] was written later, but nevertheless he did not make the connection to his earlier investigations. So, in conclusion, Euler could have written down the above equation, but he did not do so.

Therefore, let us turn to the people, who actually did. The first to write down the integral representation was Legendre [Le17], although Abel [Ab27] is often credited for it\(^2\). Kummer also found it in [Ku37a] and [Ku37b]. In those papers he gives a general method to convert certain integrals into series and vice versa.

Having mentioned all this, it is up to personal preference, whether one calls the integral formula Eulerian representation or not.

---

\(^2\)Unfortunately, the second half of his paper is lost. But reading the first few paragraphs, no one will doubt that Euler’s investigations would have led him to a formula containing the integral representation of the hypergeometric series as a special case.

\(^2\)In some sense this is true, since Abel was the first to prove that uniformly convergent series can be integrated term-by-term.
We want to sketch an inverse approach to the solution of difference equations with linear coefficients. This approach is referred to as intersection theory and was originally developed in the context of hypergeometric functions. For this, we have to explain the main ingredient of Euler’s idea a bit more. Then, we will, skipping a lot of technicalities, explain the intersection theory and finally apply it to some familiar examples.

### 4.4.1 Main Ingredient of Euler’s approach

As mentioned, Euler explains his idea in §§49 – 53, mainly focusing on continued fractions. But his main idea, aside from the ansatz

\[
\int_a^b t^{x-1} P(t) dt,
\]

is the assumption of an auxiliary equation, i.e. adding the extra term \( t^x Q(t) \). This was necessary to find the limits of integration. Then, we differentiated this auxiliary equation just to integrate it again later. Therefore, what we have essentially done, is to express the integral

\[
\int_a^b t^{x+1} P(t) dt
\]

as a linear combination of the one or two others of the same kind. More precisely, we found an equation

\[
\int_a^b t^{x+1} P(t) dt = C_1(x) \int_a^b t^x P(t) dt + C_2(x) \int_a^b t^{x-1} P(t) dt.
\]

\( C_1(x) \) and \( C_2(x) \) are functions of \( x \), linear in \( x \) in our case. In physics such equations are referred to as integration-by-parts-identities or IBPs for short.

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26 We say inverse since the method we are about to explain was actually (also) invented to find the recursion relations of integrals of the type we considered above.

27 The intersection theory we mean has nothing to do with the classical theory of Poincare and the intersection of two or more curves.

28 It will turn out that it is very difficult to find the limits using intersection theory, since it actually starts from the integrals, i.e. the integration domain is given.
and are frequently employed in the calculation of Feynman diagrams. The name stems from the fact that the exponent of $t$ is lowered by 1 by partial integration one time. Therefore, the extra term $t^{x}Q(t)$ can be understood as a term that has to be added to ensure that we arrive at the IBP given by the propounded difference equation. Intersection theory now turns this on its head and starts from an integral like the one above. It tells how many integrals of the same type one needs to express the given integral (e.g., it would immediately tell that one requires two integrals to express $\int_{a}^{b} t^{x+1}P(t)dt$) and furthermore it provides one with a tool to calculate the coefficients ($C_1(x)$ and $C_2(x)$ in our example). But we will use it to solve difference equations again, which will again lead us to differential equations for the function $P(t)$. But let us explain the origin of intersection theory first.

4.4.2 Definition of the relevant cohomology class

Since we are mainly interested in the application of the theory, we will skip the mathematical rigorous construction. For this, the interested reader is referred to the first two chapters of [Ao11]. Our explanation will follow the more application-oriented paper [Fr19].

In intersection theory one mainly considers integrals of the form

$$I = \int_{C} u(\bar{z}) \varphi(\bar{z})$$

where $\bar{z} = (z_1, z_2, z_3, \cdots, z_m)$ and $C$ is the integration contour, $\varphi(\bar{z}) = \hat{\varphi}(z) d^m \bar{z}$ is a $m$-form. We assume that $u(\bar{z})$ vanishes on the boundaries of $C$, i.e. $u(\partial C) = 0$. This guarantees that we have no surface terms after partial integration, in other words in the above language the auxiliary term like $t^{x}Q(t)$ is not necessary. Integrals of the above form are referred to as Gel’fand-Aomoto hypergeometric functions, see [Ao11], [Ao77]. Obviously, there are many choices of $\varphi$ that lead to the same result as $I$. To see this, consider a total derivative of $u$ times an arbitrary $(m-1)$-form $\chi$

$$\int_{C} d(u \chi) = 0.$$
This equality is true by Stokes’s theorem and our definition of the domain \( C \). Let us now massage the integral into a more convenient form:

\[
0 = \int_C d(u \chi) = \int_C (du \wedge \chi + u d\chi) = \int_C u \left( \frac{du}{u} \wedge + d \right) \chi
\]

\( \wedge \) means the wedge product of differential forms. This leads us to the following definition and theorem and definition

**Definition 4.3** (Connection \( \nabla_\omega \)). We define the connection \( \nabla_\omega \) as

\[
\nabla_\omega = d + \omega \wedge \quad \text{with} \quad \omega = d \log u.
\]

Furthermore, we have

**Theorem 4.6.**

\[
\int_C u \varphi = \int_C u (\varphi + \nabla_\omega \chi)
\]

In other words, \( \varphi \) and \( \varphi + \nabla_\omega \) contain the same information and it is natural to define equivalence classes, more precisely, cohomology classes.

**Definition 4.4** (Cohomology classes of forms). Let \( \varphi \) and \( \nabla_\omega \) be as above, then we define a class of forms as all forms that integrate to the same result. We indicate them by \( \omega \langle \chi \rangle \) and we have

\[
\omega \langle \chi \rangle : \varphi \sim \varphi + \nabla_\omega \chi.
\]

Therefore, two forms are equivalent, if they are equal to each other up to an integration by parts. In the literature, this class is referred to as twisted cocycle.

**4.4.3 Intersection numbers**

We can now pair \( \langle \varphi \rangle \) and \( \langle C \rangle \) to obtain an integral of the above form. We define

\[
\langle \varphi \rangle \langle C \rangle := \int_C u \varphi.
\]

This defines a bilinear and hence can be used to find linear relations among hypergeometric functions. To see this, let \( \nu \) be the number of linearly independent cocycles, and denote an arbitrary basis of forms by
\[ \langle e_1 \rangle, \langle e_2 \rangle, \cdots, \langle e_\nu \rangle. \]

Then, as known from linear algebra, a decomposition is achieved by expressing the arbitrary cocycle \( \langle \varphi \rangle \) as a linear combination of the base elements. And this can be done as follows. We need a dual space of twisted cocycles whose basis we want to denote \( |h_i\rangle \) with \( i \in \{1, 2, \cdots, \nu\} \). This will lead us to the intersection number:

**Definition 4.5** (Intersection Number). Let \( \langle e_i \rangle \) with \( i \in \{1, 2, \cdots, \nu\} \) be a basis element of the space of cocycles and \( |h_i\rangle \) with \( i \in \{1, 2, \cdots, \nu\} \) a basis element of the dual space of the space of cocycles, then we define the intersection number of the base elements as

\[ C_{ij} := \langle e_i | h_j \rangle. \]

These are the entries of a \( \nu \times \nu \) matrix.

**4.4.4 Derivation of the decomposition formula**

Having defined the intersection numbers, our next task is to find the decomposition formula, i.e. how to find the coefficients in the linear combination of \( \langle \varphi \rangle \) in terms of the basis elements. For this, let us define a \( (\nu + 1) \times (\nu + 1) \) matrix \( \mathcal{M} \) as follows:

\[
\mathcal{M} = \begin{pmatrix}
    \langle \varphi | \psi \rangle & \langle \varphi | h_1 \rangle & \langle \varphi | h_2 \rangle & \cdots & \langle \varphi | h_\nu \rangle \\
    \langle e_1 | \psi \rangle & \langle e_1 | h_1 \rangle & \langle e_1 | h_2 \rangle & \cdots & \langle e_1 | h_\nu \rangle \\
    \langle e_2 | \psi \rangle & \langle e_2 | h_1 \rangle & \langle e_2 | h_2 \rangle & \cdots & \langle e_2 | h_\nu \rangle \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \langle e_\nu | \psi \rangle & \langle e_\nu | h_1 \rangle & \langle e_\nu | h_2 \rangle & \cdots & \langle e_\nu | h_\nu \rangle
\end{pmatrix} \equiv \begin{pmatrix}
    \langle \varphi | \psi \rangle & A^\top \\
    B & C
\end{pmatrix}
\]

Each entry is given by a pairing or bilinear. The matrix \( C \) is clearly a submatrix of \( \mathcal{M} \). \( B \) is columnvector, \( A^\top \) is a row vector.

Since we have \( \nu + 1 \) cocycles labelling the rows and columns and the corresponding vector spaces have dimension \( \nu \), the determinant of \( \mathcal{M} \) must vanish. Therefore, from the common formula for the determinant of a matrix with block matrices:

\[
\det(\mathcal{M}) = \det(C) \cdot \left( \langle \varphi | \psi \rangle - A^\top C^{-1} B \right) = 0.
\]
But by definition the matrix $C$ cannot be zero, whence the other factor must be zero. From this we conclude

$$\langle \varphi | \psi \rangle = \mathbf{A}^\top C^{-1} \mathbf{B}$$

or writing out the products

$$\langle \varphi | \psi \rangle = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle \left( C^{-1} \right)_{ji} \langle e_i | \psi \rangle$$

Therefore, since $|\psi\rangle$ is arbitrary, we arrive at the decomposition formula:

**Theorem 4.7 (Decomposition Formula).** Let $\langle e_i |$ with $i \in \{1, 2, \cdots, \nu\}$ be a basis element of the space of cocycles and $\langle e_i |$ with $i \in \{1, 2, \cdots, \nu\}$ a basis element of the dual space of the space of cocycles, then we have the decomposition formula

$$\langle \varphi | = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle \left( C^{-1} \right)_{ji} \langle e_i |$$

The decomposition formula provides us with a projection of $\langle \varphi |$ onto the basis of $\langle e_i |$. Contracting both sides with the twisted cocycle $|C\rangle$ (this means multiplying by $u$ and integrating over $C$), we find the formula

**Corollary 4.7.1.**

$$\int_C u \varphi = \sum_{i,j=1}^{\nu} \langle \varphi | h_j \rangle \left( C^{-1} \right)_{ji} \int_C u e_i.$$

Therefore, we find the coefficients of the decomposition

**Corollary 4.7.2.** Given

$$I = K \langle \varphi | C \rangle = \sum_{i=1}^{\nu} c_i J_i,$$

where

$$J_i \equiv KE_i \quad \text{with} \quad E_i \equiv \langle e_i | C \rangle,$$

we have

$$c_i = \sum_{j=1}^{\nu} \langle \varphi | h_j \rangle \left( C^{-1} \right)_{ji}.$$
4.4.5 Some Remarks

This is basically the formula, we will need in the following. But before going over to calculations, we add some more things. First, it is helpful to think of $|e_i\rangle$ and $|h_j\rangle$ as basis elements of a vector space of inequivalent integrands of hypergeometric functions. Then, $C$ defines in metric on this space. Furthermore, we note that the dual space of twisted cocycles has a straight-forward interpretation as the following equivalence classes:

$$|\varphi\rangle_\omega: \varphi \sim \varphi + \nabla_{-\omega}\chi.$$ This also defines a bilinear

$$\langle \varphi_L | \varphi_R \rangle_\omega$$
is, in analogy to above, called the intersection number of $\langle \varphi_L \rangle$ and $|\varphi_R\rangle$. It was convenient to note this, since this term appears frequently in the literature on hypergeometric functions. We already mentioned that there is no direct relation to the intersection number of two curves and there is no reason why our intersection number should be an integer. Indeed, it will be a function of the parameters of the integrals under consideration. For a more detailed review of twisted cohomologies the reader is referred to [Ao11].

4.4.6 The case of 1-forms

In this section we descend to 1-forms and tell (but not prove) how to calculate intersection numbers and to apply the decomposition formula. First, we need to find the dimension $\nu$ of the vector space under consideration. This is addressed by the following theorem.

**Theorem 4.8.** The dimension $\nu$ of the vector space of cocycles is

$$\nu = \{\text{number of solutions of } \omega = 0\}.$$ Following [Ch95] and [Ma98], we define the intersection number $\langle \varphi_L | \varphi_R \rangle_\omega$ as follows:

**Definition 4.6 (Intersection Number $\langle \varphi_L | \varphi_R \rangle_\omega$).** Further, let $P$ be the set of poles of $\omega$, i.e
\[ P = \{ z | z \text{ is a pole of } \omega \} \]

Then, we have:

\[ \langle \varphi_L | \varphi_R \rangle_\omega = \sum_{p \in P} \text{Res}_{z=p} (\psi_p \varphi_R) , \]

where \( \psi_p \) is a function and solves the differential equation \( \nabla_\omega \psi = \psi_L \) around \( p \), i.e.

\[ \nabla_\omega \psi_p = \psi_L, \]

In general \( f_p \) denotes the Laurent expansion about \( p \).

### 4.4.7 1. Example - B-function

Having now prepared everything, let us see it in action. We will start with the most simple hypergeometric integral - the B-function, defined as the integral

\[ B(x, y) = \int_0^1 dz z^{x-1} (1 - z)^{y-1}. \]

Moreover, we found the relation:

\[ B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}. \]

We will discuss the B-function in much detail, to see, how the method works.

### 1. Approach: Direct Integration

We need to formulate it in the language of hypergeometric integrals. Let us put

\[ I_n = \int_C u z^n dz, \quad u = z^\gamma (1 - z)^\gamma, \quad C = [0, 1]. \]

We can use the relation among \( \Gamma \) and \( B \), to express \( I_n \) directly:

\[ I_n = \frac{\Gamma(1 + \gamma) \Gamma(1 + \gamma + n)}{\Gamma(2 + 2\gamma + n)}. \]

Therefore,
\[ I_n = \frac{\Gamma(1 + \gamma + n)\Gamma(2 + 2\gamma)}{\Gamma(1 + \gamma)\Gamma(2 + 2\gamma + n)} I_0. \]

For \( n = 1 \), this reduces to

\[ I_1 = \frac{1}{2} I_0. \]

2. **Approach: Integration-by-Parts**

Let us see, whether we can find the last relation among \( I_1 \) and \( I_0 \) by integration by parts. For this aim, note that

\[ \int_C d \left( (z(1-z))^{\gamma+1}z^{n-1} \right) = 0. \]

Expanding the integrand gives:

\[ (\gamma + n)I_{n-1} - (1 + 2\gamma + n)I_n = 0. \]

Hence

\[ I_n = \frac{(\gamma + n)}{(1 + 2\gamma + n)} I_{n-1}, \]

which for \( n = 1 \) gives

\[ I_1 = \frac{1}{2} I_0. \]

3. **Approach: Intersection Numbers**

We want to find the relation among the \( B \)-integrals again. We define

\[ I_n = \int u \phi_{n+1} \equiv \omega \langle \phi_{n+1} | C \rangle, \quad \phi_{n+1} \equiv z^n dz. \]

Additionally,

\[ u = z^\gamma (1-z)^\gamma, \quad \omega = d \log u = \gamma \left( \frac{1}{z} + \frac{1}{z-1} \right) dz. \]

The equation \( \omega = 0 \) obviously has only 1 solution. Hence the dimension of the space under consideration is \( \nu = 1 \). Further, we need to find the poles of
The poles $z = 0$ and $z = 1$ are immediate. To consider $z = \infty$, perform the substitution $t = \frac{1}{z}$, whence $dt = -\frac{1}{z^2}$. Therefore,

$$\omega = -\gamma \left( t + \frac{t}{1-t} \right) \frac{dt}{T^2}.$$  

Since this tends to $\infty$ for $t \to 0$, the set of poles is:

$$\mathcal{P} = \{0, 1, \infty\}.$$  

we want to express $I_1$ in terms of some integrals of the same class. Since $\nu = 1$, we only need one integral and choose it to be $I_0$. Therefore, we know

$$I_1 = c_1 I_0 \iff \omega \langle \phi_2 | C \rangle = c_1 \cdot \omega \langle \phi_1 | C \rangle$$

and we need to find $c_1$.

Since $\nu = 1$, the matrix $C_{ij}$ is simply a number, i.e. $\langle \phi_1 | \phi_1 \rangle$. Hence, in total we need to evaluate the numbers $\langle \phi_1 | \phi_1 \rangle$, $\langle \phi_2 | \phi_1 \rangle$.

For this we need the residues. For each pole $p \in \mathcal{P}$ we need $\phi_{i,p}$ - the series expansion of $\phi_i$ about $z = p$ - and $\psi_{i,p}$ - the series expansion of $\psi_i$ about $z = p$, which is found from the differential equation

$$\nabla \omega_p \psi_{i,p} = \phi_{i,p}.$$  

Note, that the following Laurent expansions are known

$$\phi_{i,p} = \sum_{k=\min}^{\max} \phi_{i,p}^{(k)} z^k \quad \text{and} \quad \omega_p = \sum_{k=-1}^\infty \omega_p^{k} z^k$$

and need to find

$$\psi_p = \sum_{k=\min}^{\max} \alpha_k z^k.$$

In other words, the coefficients $\alpha_k$ are to be found. $\max(\phi_i) = \text{ord}_p(\phi_i) + 1$ and $\min(\phi_i) = -\text{ord}_p(\phi_i) - 1$. We introduced all min and max, since those determine the range of coefficients necessary to determine the residue we need. We do not need the rest of the Laurent series.

Writing out the above differential equation for differential forms, we arrive at the following simpler one (just for functions):

$$\frac{d}{dz} \psi_p + \omega_p \psi_p = \phi_{i,p}.$$  

50
Now we can start with the calculation of the intersection numbers. We start with $\langle \phi_1 \vert \phi_1 \rangle$. For $p = 0$ and $p = 1$ we find that $\min = 1 > \max = -1$. Therefore, the resulting residues are zero. For $p = \infty$ on the other hand, we find $\min = -1$ and $\min = 1$. Therefore, our series for $\psi_\infty$ has the following three terms:

$$\psi_\infty = \alpha_1 \frac{1}{z} + \alpha_0 + \alpha_1 z^1.$$  

Substituting this series in the above differential equations together with the series expansion of $\phi_{1,\infty}$ and $\omega_\infty$, we find by comparing coefficients

$$\alpha_{-1} = \frac{1}{2\gamma + 1}, \quad \alpha_0 = -\frac{1}{2(2\gamma + 1)}, \quad \alpha_1 = \frac{\gamma}{2(2\gamma - 1)(2\gamma + 1)}.$$  

Therefore, the intersection number is just

$$\langle \phi_1 \vert \phi_1 \rangle = \text{Res}_{z=\infty}(\psi_\infty \phi_1) = \frac{\gamma}{2(2\gamma - 1)(2\gamma + 1)}.$$  

Let us go over to $\langle \phi_2 \vert \phi_1 \rangle$. As in the first case, the differential equation has no solution for $p = 0$ and $p = 1$. For $p = \infty$ we find that the series must look as follows:

$$\psi_\infty = \alpha_{-2} \frac{1}{z^p} + \alpha_{-1} \frac{1}{z} + \alpha_0 + \alpha_1 z.$$  

The coefficients are found by the same procedure as above and are calculated to be

$$\alpha_{-2} = \frac{1}{2(\gamma + 1)}, \quad \alpha_{-1} = \frac{\gamma}{2(\gamma + 1)(\gamma + 2)}, \quad \alpha_0 = \frac{1}{4(2\gamma + 1)}, \quad \alpha_1 = -\frac{\gamma}{4(2\gamma - 1)(2\gamma + 1)}.$$  

Thus,

$$\langle \phi_2 \vert \phi_1 \rangle = \text{Res}_{z=\infty}(\psi_\infty \phi_1) = \frac{\gamma}{4(2\gamma - 1)(2\gamma + 1)}.$$  

Now, we can apply the decomposition formula and find

$$c_1 = \langle \phi_2 \vert \phi_1 \rangle \left(\langle \phi_1 \vert \phi_1 \rangle\right)^{-1} = \frac{1}{2},$$  

i.e.
\[ I_1 = \frac{1}{2} I_0 \]

in agreement with the approaches above.

### 4.4.8 2. Example: Hypergeometric function

Above we saw that we can express the Gaußian hypergeometric series as

\[
B(b, c - b) \binom{a, b, c; x}{} = \frac{1}{\binom{b}{b-1}} \int_0^1 z^{b-1} (1-z)^{c-b-1} (1-xz)^{-a} dz.
\]

Hence the integration contour is \( C = [0, 1] \). \( B(a, b) \) is the B-function. We want to use intersection theory to find a difference equation for hypergeometric series. We write:

\[
B(b, c - b) \binom{a, b, c; x}{} = \int_C u \varphi = \omega \langle \varphi | C \rangle,
\]

with

\[
u = 2 \quad \text{and} \quad \mathcal{P} = \left\{ 0, 1, \frac{1}{x}, \infty \right\}.
\]

Therefore, this indicates that we can one hypergeometric integrals by two others. This is not surprising, since we know Gauß’s contiguous relations. But let us find a specific one by using intersection theory.

We chose the basis \( \{ \langle \phi_1 | , \langle \phi_2 | \} \)\), where, as above \( \phi_{i+1} = z'dz \).

In this example the matrix \( C \) is a \( 2 \times 2 \) matrix and looks as follows:
The intersection numbers are calculated as above, we just list the results:

\[
\begin{align*}
\langle \phi_1 | \phi_1 \rangle &= \left( x^2(-(a-b+1))(b-c+1) - 2ax(-b+c-1) + a(c-2) \right) / \left( x^2(a - c + 1)(a-c+2)(a-c+3) \right), \\
\langle \phi_1 | \phi_2 \rangle &= \left( x^3(-(a-b+1)(a-b+2)(b-c+1)) - ax^2(-b+c+1)(2a - 3b + c + 2) + ax(a+2c-5)(-b+c+1) - a(c-3)(c-2) \right) / \left( x^3(a-c+1)(a-c+2)(a-c+3)(a-c+4) \right), \\
\langle \phi_2 | \phi_1 \rangle &= \left( x^3(-(a-b))(a-b+1)(b-c+1) - ax^2(-b+c-1)(2a-3b+c) + ax(a+2c-3)(-b+c+1) - a(c-2)(c-1) \right) / \left( x^3(a-c)(a-c+1)(a-c+2)(a-c+3) \right), \\
\langle \phi_2 | \phi_2 \rangle &= \left( -ax^2(a^2b-a^2c+a^2-3ab^2+7abc-8ab-4ac^2+9ac-5a-3b^2c + 6b^2+4bc^2-10bc+6b-c^2+2c^2-c) + x^4(-(a^3-3a^2b+3a^2+3ab^2 - 6ab+2a-b^3+3b^2-2b))(b-c+1) + 2ax^3(a-b+1)(ab-ac+a - 2b^2+3bc-2b-c^2+c) + 2a(c-2)x(a+c-2)(b-c+1) + a(c^3-6c^2 + 11c-6) \right) / \left( x^4(a-c)(a-c+1)(a-c+2)(a-c+3)(a-c+4) \right). 
\end{align*}
\]

Now we found everything necessary for the application of the decomposition formula, which in this case reads

\[
\langle \phi_n \rangle = \sum_{i,j=1}^{2} \langle \phi_n | \phi_j \rangle \left( C^{-1} \right)_{ji} \langle \phi_i \rangle .
\]

As an example let us take \( \langle \phi_3 | C \rangle = B(b+2,c-b)2F_1(a,b+2,c+2;x) \) in terms of \( B(b,c-b)2F_1(a,b,c;x) \) and \( B(b+1,c-b)2F_1(a,b+1,c+1;x) \). Then, we find
\( B(b + 2, c - b) \binom{a}{b} F_1(a, b + 2, c + 2; x) = \left( \frac{b}{x(a - c - 1)} \right) B(b, c - b) \binom{a}{b} F_1(a, b, c; x) \)

\( + \left( \frac{(b - a + 1)x + c}{x(c - a + 1)} \right) B(b + 1, c - b) \binom{a}{b} F_1(a, b + 1, c + 1; x) \)

which relation can also be derived from the contiguous relations for hypergeometric functions.

### 4.4.9 Example: Γ-function

Having explained the idea, let us apply the idea to the Γ-function and solve its functional equation. We still assume that it is possible to write \( \Gamma(x) \) as an integral of the form

\[
\Gamma(x) = \int_0^\infty t^{x-1} P(t) dt
\]

and we have to determine \( P(t) \). We want to solve \( \Gamma(x + 1) = x \Gamma(x) \). In the language of intersection theory, we have

\[
u = t^x P(t) \quad \text{and hence} \quad \omega = d \log u = \left( \frac{x}{t} + \frac{P'(t)}{P(t)} \right) dt.
\]

Since we seek to express \( \Gamma(x + 1) \) by one other integral, this forces us to arrange that

\[
\frac{x}{t} + \frac{P'(t)}{P(t)} = 0
\]

has precisely one solution for \( t \). This in turn implies

\[
\frac{P'(t)}{P(t)} = C \quad \text{or} \quad P(t) = B e^{Ct}.
\]

for some constants \( B \) and \( C \neq 0 \) we have to determine from the remaining conditions. We need to find the poles of \( \omega \).\[31\] Inserting the result we found for \( P(t) \), we have

\[31\]Of course, at this point we could use Wielandt’s theorem directly to force the integral to become \( \Gamma(x) \) - it would imply \( B = 1 \) and \( C = -1 \). But we will use intersection theory to the end and see, how we have to find \( B \) and \( C \) this way.
\[ \omega = \left( \frac{x}{t} + C \right) dt. \]

One pole is obviously given by \( t = 0 \). But we also have to consider \( t = \infty \).

Therefore, put \( u = \frac{1}{t} \). Hence locally about \( t = \infty \)

\[ \omega = -(xu + C) \frac{du}{u^2} \]

which is infinite for \( u = 0 \) and hence indicates a pole at \( t = \infty \). Therefore,

\[ \mathcal{P} = \{0, \infty\}. \]

As in the first example, we need the intersection numbers \( \langle \phi_1 | \phi_1 \rangle \) and \( \langle \phi_2 | \phi_1 \rangle \).

Since we forced \( \nu = 1 \), the intersection matrix is just \( C_{11} = \langle \phi_1 | \phi_1 \rangle \), i.e. a \( 1 \times 1 \)-matrix. The two necessary intersection numbers are calculated as in the preceding examples. As in the first example only the point \( p = \infty \) actually contributes and we find:

\[ \langle \phi_1 | \phi_1 \rangle = \frac{x}{Cx} \]

\[ \langle \phi_2 | \phi_1 \rangle = -\frac{x^2}{C^3} \]

Therefore, decomposition formula yields

\[ c_1 = \langle \phi_2 | \phi_1 \rangle \langle \phi_1 | \phi_1 \rangle^{-1} = -\frac{x}{C}. \]

Recall that we want to solve the equation

\[ \Gamma(x + 1) = x \Gamma(x). \]

Therefore, this gives the condition

\[ -\frac{x}{C} = x \quad \Rightarrow \quad C = -1. \]

Hence, we finally arrive at the solution:

\[ \Gamma(x) = B \int_0^\infty t^{x-1} e^{-t} dt. \]

Using the initial condition \( \Gamma(1) = 1 \), one finds the integral representation of \( \Gamma(x) \) again.
4.4.10 Remarks

This last example shows that intersection theory can also be used backwards to solve certain difference equations, although it is quite a lot of work compared to Euler’s method. Furthermore, one has to insert the integration limits from the beginning in contrast to Euler’s method, where you calculate them explicitly. Nevertheless, intersection theory stands on a mathematically solid foundation\textsuperscript{32}, whereas the moment-ansatz is rather a technique for the solution of homogeneous difference equations with linear coefficients. But this method does not seem to be well-established in the literature, although it provides an complete solution to the above class of equations.

\textsuperscript{32}This is mainly due to the fact that it intended to find the difference equations for the integrals in questions and we just applied it the other way around
5 Solution of the Equation
\[ \log \Gamma(x + 1) - \log \Gamma(x) = \log x \] by Conversion into a Differential Equation of Infinite Order

5.1 Overview

We solve the general difference equation

\[ f(x + 1) - f(x) = g(x) \]

by converting it into a differential equation of infinite order via Taylor’s theorem. This was done by Euler in [E189]. Unfortunately, there is an error in Euler’s approach that we will explain and correct, before we solve the equation by the more modern approach of Fourier analysis.

5.2 Euler’s Idea

Euler’s reasoning involves some purely formal operations. Therefore, we will not add the conditions under which the operations are valid here. It is nevertheless a beautiful example of the "ars inveniendi".

5.2.1 Presentation of his idea

As mentioned, the main source for this section is [E189]. Here, Euler actually intended to solve various interpolation problems. One of them is to interpolate the function \( f \) defined for positive integers:

\[ f(n) := \sum_{k=0}^{n-1} g(k), \]

\( g \) being an arbitrary function. Obviously, \( f \) satisfies the functional equation:

\[ f(n + 1) - f(n) = g(n) \quad \forall n \in \mathbb{N}. \]

Now, one possible way to interpolate \( f \) is to solve the above difference equation for general \( x \in \mathbb{C} \) which was Euler’s intention in [E189]. For this he used Taylor’s theorem to write:

33 [E62] He considered other examples of differential equations of infinite order in [E62] and his second book on integral calculus [E366].
34 Euler actually assumed \( x \) to be real, but this restriction is not necessary at all.
Therefore, the above difference equation becomes:

$$\sum_{n=0}^{\infty} \frac{f(n)(x)}{n!} = g(x).$$

This is a inhomogeneous ordinary differential equation of infinite order with constant coefficients.

In [E62] and [E188] Euler explained the method how to solve such differential equations, if the order is finite. The method is still the same we use today and can be found in any modern textbook on differential equations.

5.2.2 Actual Solution

Now, let us present Euler’s solution. This first step to solve an equation of such a kind (at least in the case of finite order) is to consider the characteristic polynomial, i.e., the polynomial resulting by substituting $\frac{d^n}{dx^n}$ for $z^n$ in the differential operator acting on $f$. This “polynomial” in our case reads:

$$P(z) = e^z - 1.$$

Next, Euler wants to find the zeros of $P$. Using the theory of complex logarithms he developed in [E168] and [E807], he finds the zeros to be

$$z_k = 2k\pi i \quad k \in \mathbb{Z}.$$

All zeros are easily seen to be simple. From this Euler (assuming that the case of infinite order can be treated as the case of finite order) concluded that each zero will lead to a term

$$e^{z_k x} \int e^{-z_k t} g(t) dt.$$

$\int$ means that we to put $x = t$ after the integration. Therefore, Euler stated the solution of the general difference equation to be

\[35\] [E62] considers the homogeneous case, whereas [E188] treats the inhomogeneous case.
\[ f(x) = \sum_{k=-\infty}^{\infty} e^{2k\pi ix} \int_{x}^{\infty} e^{-2k\pi it} g(t) \, dt. \]

Note that each integral leads to an integration constant \( c_k \), whence this solution is not particular but the complete solution.

### 5.2.3 Mistake in Euler’s Approach

Unfortunately, Euler’s solution is incorrect. It gives the correct solution in the homogeneous case. One finds

\[ f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2k\pi ix} \]

as solution of \( f(x + 1) = f(x) \). \( c_k \) are arbitrary constants of integration. Interestingly, Euler found that each periodic function has what we nowadays call a Fourier series. But Euler was at that time not realizing what a broad field of mathematics he entered and did not pursue this any further.

But his solution formula already fails to give the correct result in the case \( g(t) = 1 \). The correct formula, as we will prove in the following, reads:

\[ f(x) = -\frac{1}{2} g(x) + \sum_{k=-\infty}^{\infty} c_k e^{2k\pi ix} \]

or, presented in the more convenient form,

\[ f(x) = \int_{x}^{\infty} g(t) \, dt - \frac{1}{2} g(x) + \sum_{k=-\infty}^{\infty} c_k e^{2k\pi ix} \int_{x}^{\infty} e^{-2k\pi it} g(t) \, dt \]

where ‘ indicates that \( k = 0 \) is skipped in the sum.

Furthermore, Euler’s mistake is not a computational but a conceptual one. His approach to construct the solution from the zeros of the characteristic “polynomial” (in analogy to the finite case where this is possible) simply does not work in the case of infinite order. Instead of the zeros one has to use the partial fraction decomposition\(^{36}\).

\(^{36}\)The following partial fraction decomposition is proved in the appendix.
\[
\frac{1}{e^z - 1} = -\frac{1}{2} + \frac{1}{z} + \sum_{k=-\infty}^{\infty} ' \frac{1}{z - 2k\pi i}.
\]
Comparing this to the solution it is easily seen that each term \(\frac{1}{z - 2k\pi i}\) leads to a term \(e^{2k\pi ix} \int e^{-2k\pi it} g(t) dt\) in the solution, whereas \(-\frac{1}{2}\) explains the term \(-\frac{1}{2}g(x)\). We will prove this in the following section, but need to give some definitions and state some auxiliary theorems in advance.

5.3 Correction of Euler’s Approach

5.3.1 Preparations - Lemmata and Definitions

**Theorem 5.1** (Fundamental Theorem of Algebra). Let \(P(z) = a_0 + a_1 z + \cdots + a_n z^n\) an non-constant polynomial of degree \(n\) with complex coefficients, then \(P(z)\) has exactly \(n\) zeros in \(\mathbb{C}\), where the zeros have to be counted with multiplicity.

This theorem is usually proved by applying Liouville’s theorem. We refer the reader to any modern book on complex analysis for a proof.

As a historical note we add that Euler also tried to prove the fundamental theorem of Algebra in \([E170]\). But his proof is incomplete as pointed out by Gauß\([Ga99]\). Confer also \([Du91]\) for a review from the modern perspective.

**Definition 5.1** (Schwartz Space). We denote by \(S\) the set of all functions \(f \in C^\infty(\mathbb{R}^n)\), such that
\[
\sup_{x \in \mathbb{R}^n} \max_{|\alpha|, |\beta| < N \in \mathbb{N}} |x^\alpha D^\beta f(x)| < \infty
\]
holds. The linear space \(S\) endowed with the convergence
\[
f_j \to 0 \Leftrightarrow \sup_{x \in \mathbb{R}^n} \max_{|\alpha|, |\beta| < N \in \mathbb{N}} |x^\alpha D^\beta f_j(x)| \to 0
\]
is called the Schwartz space.

Next we define the notion of a functional.
Definition 5.2 (Functional). Let $V$ be a $\mathbb{R}$-vector space, then a functional $T$ is a function

$$T : V \to \mathbb{R}.$$ 

We will mainly need continuous linear functionals which we will define next.

Definition 5.3 ((Continuous linear functional in $S$)). A linear functional $u$ is called continuous on $S$, if the following implication holds:

$$f_j \to 0 \text{ in } S \Rightarrow u(f_j) \to 0$$

Having introduced the notion of a functional, we can now introduce the dual space.

Definition 5.4 (Dual Space). Let $V$ be a $\mathbb{R}$-vector space, then the vector space, containing all linear mappings from $V$ to $\mathbb{R}$ is called to dual space to $V$ and is denoted by $V'$.

This now allows to define a "scalar product", often referred to a dual pairing.

Definition 5.5 (Dual Pairing). Let $V$ be a $\mathbb{R}$-vector space and $V'$ be the corresponding dual space, then the mapping:

$$\langle \cdot, \cdot \rangle : V \times V' \to \mathbb{R}, \quad \langle x, l \rangle := l(x)$$

is called a dual pairing.

Next we introduce tempered distributions we will need to solve the general difference equation.

Definition 5.6 (Tempered Distribution). A continuous linear functional $u$ on $S$ is called a tempered distribution. We denote the set of all tempered distributions by $S'$.

Finally, we introduce the main tool of computation, the Fourier transform and the Fourier inversion formula.

Definition 5.7 (Fourier Transform and Fourier Inversion Formula). We define the Fourier transformation of a function $f \in L_1(\mathbb{R}^n)$ as:
\[ \hat{f}(p) := \int_{\mathbb{R}^n} e^{-ix \cdot p} f(x) dx, \]

where \( x \cdot p = \sum_{i=1}^{n} x_k p_k. \)

If also \( \hat{f} \) is integrable, the following formula, the Fourier inversion formula holds:

\[ f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot p} \hat{f}(p) dp \]

Let us briefly consider a well-known example. The \( \delta \)-distribution is a tempered distribution, that for all \( f \in \mathcal{S} \) we have

\[ \delta : \mathcal{S} \mapsto \mathbb{R} \quad \text{and} \quad \delta(f) = f(0) \]

But we will mainly need the Fourier transform of tempered distributions.

**Definition 5.8 (Fourier Transform of a tempered Distribution).** Let \( h \in \mathcal{S}' \) and a tempered distribution, then the Fourier Transform of \( \hat{u} \) is defined as:

\[ \langle \hat{u}, f \rangle := \langle u, \hat{f} \rangle \]

**Theorem 5.2 (Fourier Transform of the \( \delta \)-distribution).** We have

\[ \hat{\delta} = 1 \quad \text{and} \quad \frac{1}{(2\pi)^n} \hat{\delta} = \delta. \]

**Proof.** We have by definition for a function \( f \in \mathcal{S} \):

\[ \langle \hat{\delta}, f \rangle := \langle \delta, \hat{f} \rangle \]

And by comparison we find

\[ \hat{\delta} = 1 \]

Hence we also have

\[ \frac{1}{(2\pi)^n} \hat{\delta} = \delta. \]
Corollary 5.2.1. In one dimension and written out, this reduces to the familiar representation

\[ \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{i\pi y} dy \]

Theorem 5.3 (Convolution Theorem). Let \( \mathcal{F} \) be the operator of the Fourier transform, that \( \mathcal{F}\{f\} = \hat{f} \) and \( \mathcal{F}\{g\} = \hat{g} \) are the Fourier transforms of the functions \( f \) and \( g \in \mathcal{S} \), then we have

\[ \mathcal{F}\{f \ast g\} = (2\pi)^{\frac{n}{2}} \mathcal{F}\{f\} \cdot \mathcal{F}\{g\} \quad \text{and} \quad (2\pi)^{\frac{n}{2}} \mathcal{F}\{f \cdot g\} = \mathcal{F}\{f\} \ast \mathcal{F}\{g\} \]

where \((f \ast g)(x)\) means the convolution product defined via

\[ (f \ast g)(x) := \int_{-\infty}^{\infty} f(\tau)g(x - \tau)d\tau = \int_{-\infty}^{\infty} f(x - \tau)g(\tau)d\tau \]

The theorem is straight-forward by a simple application of Fubini’s theorem, so that we omit it here. One finds a proof in almost every standard math textbook, that covers the Fourier transform.

Theorem 5.4 (Convolution Theorem for the inverse Fourier Transform). Let \( \mathcal{F}^{-1} \) the operator of the inverse Fourier transform, so that \( \mathcal{F}^{-1}\{f\} \) and \( \mathcal{F}^{-1}\{g\} \) the inverse Fourier transforms \( f \) and \( g \in \mathcal{S} \), then we have

\[ f \ast g = (2\pi)^{\frac{n}{2}} \mathcal{F}^{-1}\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\} \quad \text{and} \quad (2\pi)^{\frac{n}{2}} f \cdot g = \mathcal{F}^{-1}\{\mathcal{F}\{f\} \ast \mathcal{F}\{g\}\} \]

The proof is just by applying the inverse Fourier transform to the equations in the convolution theorem.

5.3.2 Solution Algorithm for the general difference equation - Finding a particular solution

Having stated all the definitions and theorems in advance, we can now finally give the solution for the general difference equation with constant coefficients.
Theorem 5.5. Let the following equation be propounded:

\[ \sum_{k=0}^{N} a_k f(x + k) = g(x), \]

with \( f \) and \( g \in S \) and let \( z_k \) the zeros, which we, for the sake of brevity (the generalisation to non-simple zeros is not difficult) assume to be simple, of the polynomial

\[ P(z) = \sum_{k=0}^{N} a_k z^k. \]

And let \( p_l \) be all solutions of the equation

\[ e^{-ip_l} = z_k. \]

Further, find the partial fraction decomposition:

\[ \sum_{k=0}^{N} \left[ a_k e^{-ip_k} \right]^{-1} = C + \sum_{l \in \mathbb{Z}} \frac{b_l}{i(p - p_l)}. \]

Then, a particular solution of the difference equation is given by

\[ f(x) = Cg(x) + \sum_{l \in \mathbb{Z}} b_l \int_{\tau}^{x} e^{-ip_l(x-\tau)} g(\tau)d\tau, \]

where \( \int_{\tau}^{x} f(\tau)d\tau \) means, that we integrate with respect to \( \tau \) and then put \( x \) for \( \tau \) in the integrated function. This solution is unique up to a function, satisfying the equation:

\[ \sum_{k=0}^{N} a_k f(x + k) = 0. \]

Proof. To give a proof, it is easier to start from the solution and derive the difference equation. We do not present the calculation in detail. All steps are justified, by using the theorems in the preparations. With \( p_k \) defined as above, consider

\[ \text{The fundamental theorem of algebra guarantees, that we always have exactly } N \text{ solutions} \]
\[
f(x) := Cg(x) + \sum_{l \in \mathbb{Z}} b_{l}e^{-ip_{l}x} \int_{0}^{x} e^{ip_{l}\tau} g(\tau) d\tau.
\]

Taking the Fourier transform of this expression, simplifying it by using the convolution theorem and the inverse convolution theorem, we arrive at the following expression:

\[
\hat{f}(p) = \hat{g}(p) \left( C + \sum_{l \in \mathbb{Z}} b_{l} \frac{1}{i(p - p_{l})} \right).
\]

The expression in brackets is just the partial fraction decomposition of the polynomial in \(e^{ipk}\), hence we have

\[
\hat{f}(p) = \hat{g}(p) \left[ \sum_{k=0}^{N} a_{k} e^{-ipk} \right]^{-1}.
\]

Solving for \(\hat{g}(p)\) gives

\[
\hat{g}(p) = \frac{\hat{f}(p)}{\left[ \sum_{k=0}^{N} a_{k} e^{-ipk} \right]}. 
\]

Finally, taking the inverse Fourier transform, we get

\[
g(x) = \sum_{k=0}^{N} a_{k} f(x + k),
\]

which is the difference equation propounded and completes our proof. That we can add a function satisfying

\[
\sum_{k=0}^{N} a_{k} f(x + k) = 0,
\]

is obvious.

5.3.3 Note on the case of multiple Roots

Although we only proved the theorem for simple zeros of the polynomial, we can directly generalize it to multiple zeros. Suppose, that \(z_{k} = e^{-ip_{l}}\) is a multiple zero of order \(m\). Then in the formula
\[
\sum_{k=0}^{N} \left[ a_k e^{-ipk} \right]^{-1} = C + \sum_{l \in \mathbb{Z}} \frac{b_l}{i(p - p_l)}
\]

we just have to replace

\[
\frac{b_l}{p - p_l} \text{ by } \sum_{j=1}^{m} b_j \frac{1}{(p - p_j)}.
\]

In general we have, that

\[
\frac{j!}{(ip - ip_l)^{j+1}} g(p)
\]

leads to the term

\[
\int_{x}^{\infty} (\tau - x) e^{ip_l(\tau - x)} g(\tau) d\tau
\]

in the final solution.

### 5.3.4 Application to the difference equation \( f(x + 1) - f(x) = g(x) \).

The difference equation we are interested in is a special case of the theorem we just proved. Here,

\[ P(z) = z - 1. \]

Considering the solutions of

\[ e^{-ip_l} = z_k = 1, \]

\( p_l \) is found to satisfy:

\[ -ip_l = 2k\pi i \quad k \in \mathbb{Z}. \]

Therefore, we need to find the partial fraction composition of

\[ \frac{1}{e^{-ip} - 1} \]

which (comparing to the result we mentioned above already) is found to be
or in terms of $p_l$:

$$
\frac{1}{e^{-ip} - 1} = \frac{1}{2} + \sum_{l=\infty}^{\infty} \frac{b_l}{i(p - p_l)}.
$$

Therefore, the solution of the difference equation is concluded to be

$$
f(x) = -\frac{1}{2} + \sum_{l=\infty}^{\infty} e^{2\pi i x} \int e^{-2\pi i t} g(t) \, dt.
$$

Precisely, as we stated the solution above.

5.4 A Solution Euler could have given

Fourier analysis obviously came after Euler, i.e. he had no access to it. But here we argue that he could have derived the correct solution from his results only, if we overlook some details of mathematical rigor.

The first idea is to consider $\frac{d}{dx}$ and the higher derivatives as operators acting on the function $f(x)$. Then, solving a differential equation is equivalent to finding the inverse operator to the operator acting on $f$ and applying it to both sides of the equation. Let us again write $\frac{d}{dx} = z$. The fundamental theorem of calculus implies

$$
\frac{1}{z} f = \int f \, dx
$$

And the difference equation we want to solve then becomes

$$
f(x) = \frac{1}{e^z - 1} X.
$$

We can now simplify this equation inserting the partial fraction decomposition of $(e^z - 1)^{-1}$. We obtain

$$
f(x) = \left( \frac{1}{z} - \frac{1}{2} + \sum_{k \in \mathbb{Z}} \frac{1}{z - 2k\pi i} \right) X.
$$

The $'$ indicates that $k = 0$ is left out in the summation.
We only need to find out what $\frac{1}{z - \alpha} X$ is. For this, let us write

$$\frac{1}{z - \alpha} X = \frac{1}{z} \left(1 - \frac{\alpha}{z}X\right) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{\alpha^n}{z^n} X$$

where we used the geometric series in the second step. $\frac{1}{z+1} X$ is the $n + 1$ times iterated integral of $X$. Euler considered such kind of integrals in [E679]. His formulas yield

$$\int^n X \, dx = \int^x \frac{(x - t)^{n-1}}{(n-1)!} X(t) \, dt.$$  

Here $\int^n$ denotes the $n$ times iterated integral, $\int^x$ indicates that one has to put $x = t$ after the integration. Substituting this formula and using the Taylor series for $e^x$ we find

$$\frac{1}{z - \alpha} X = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int^x (x - t)^n X(t) \, dt = \int^x e^{\alpha(x-t)} X(t) \, dt$$

Therefore, we can write our solution as

$$f(x) = \int X \, dx - \frac{1}{2} X(x) + \sum_{k \in \mathbb{Z}} \int e^{2k\pi i (x-t)} X(t) \, dt$$

$$= \int X \, dx - \frac{1}{2} X(x) + 2 \sum_{k=1}^{\infty} \int \cos (2k\pi (x - t)) X(t) \, dt$$

where we used the formula $\cos x = \frac{e^{ix} - e^{-ix}}{2}$ in the second step. Note that this is almost the formula Euler gave, in Euler’s formula only the term $-\frac{1}{2}X$ is missing making his result incorrect.

Although we have operated completely non-rigorously, the formula, as we found it, is correct, see [We14].

This formal calculus is a beautiful example of the "ars inveniendi" and is made rigorous by the techniques from Fourier analysis.
5.4.1 An Application - Derivation of the Stirling Formula for the Factorial

In the last paragraphs of \[E_{189}\] Euler tried to derive the Stirling formula for \(\Gamma(x)\) from his general solution of the general difference equation. But since he missed the term \(-\frac{1}{2} g(x)\) in the solution, his formula is incorrect. He gives a correct derivation in \[E_{212}\]. We will present his idea from \[E_{189}\] and derive the Stirling formula from the solution of the difference equation:

\[
\log \Gamma(x + 1) - \log \Gamma(x) = \log(x).
\]

More precisely, we prove the formula

**Theorem 5.6 (Stirling Formula for \(\Gamma(x + 1)\)).**

\[
\Gamma(x + 1) = \sqrt{2\pi x} \frac{x^x}{e^x} \quad \text{for} \quad x \to \infty
\]

**Proof.** The idea is to simplify the solution we derived, i.e.

\[
\log \Gamma(x + 1) = -\frac{1}{2} \log x + \int x \log(t) dt + \sum_{k = -\infty}^{\infty} \int e^{2k\pi i} \log(t) dt
\]

and check the conditions of the Bohr-Mollerup theorem.

Let us simplify the general solution following Euler in \[E_{189}\]. First, we have

\[
\int x \log(t) dt = x \log x - x + C.
\]

\(C\) being the constant of integration. Secondly, by iterated partial integration we obtain

\[
\int x \log(t) dt = C_k - \frac{\log x e^{-2k\pi i}}{2k\pi i} + \sum_{n = 1}^{\infty} \frac{(-1)^n (n - 1)!}{(2k\pi i)^n + 1} \frac{x^n}{x^n} e^{-2k\pi i}
\]

\(C_k\) being the constant of integration. Therefore,

\[
y(x) = x \log x - x + \Pi(x) - \frac{1}{2} \log x + \sum_{k \in \mathbb{Z}} \left[ \frac{\log x}{2k\pi i} + \sum_{n = 1}^{\infty} \frac{(-1)^n (n - 1)!}{(2k\pi i)^n + 1} \frac{x^n}{x^n} \right]
\]

\(38\)He was even aware of this but tries to argue it away by another incorrect argument
where \( \Pi(x) \) is an arbitrary periodic function with period 1, i.e., a solution of the homogeneous equation \( y(x + 1) - y(x) = 0 \). Now note that for a natural number \( m \)

\[
\sum'_{k \in \mathbb{Z}} \frac{1}{k^{2m-1}} = 0 \quad \text{and} \quad \sum'_{k \in \mathbb{Z}} \frac{1}{k^{2m}} = 2 \sum_{k=1}^{\infty} \frac{1}{k^{2m}}.
\]

Therefore,

\[
y(x) = x \log x - x + \Pi(x) - \frac{1}{2} \log x + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} 2^{m-1} (2m-2)!}{x^{2m-1} (2m\pi)^{2m}}.
\]

Now recall Euler’s famous formula for the even \( \zeta \)-values

\[
\sum_{k=1}^{\infty} \frac{1}{k^{2m}} = (-1)^{m-1} B_{2m} (2\pi)^{2m} \frac{2}{(2m)!},
\]

where \( B_m \) are the Bernoulli numbers. Hence, having substituted those explicit values for the sums, we finally arrive at

\[
y(x) = x \log x - x - \frac{1}{2} \log x + \Pi(x) + \sum_{k=1}^{\infty} \frac{B_{2m}}{2m(2m-1)x^{2m-1}}.
\]

Having simplified the solution this far, let us now check the conditions of the Bohr-Mollerup theorem. We start with logarithmic convexity.

Here we want to show that the periodic function is actually a constant. It will turn out to be Stirling’s constant, i.e., \( \frac{1}{2} \log(2\pi) \) later.

One way of arguing that \( \Pi(x) = A \), i.e., a constant, is as follows. Recall the following condition for convexity.

**Definition 5.9** (Convexity for a differentiable function). A differentiable function \( y(x) \) is convex on the interval \( I \) if and only if

\[
\frac{y(a) - y(b)}{a - b} \geq y'(b) \quad \forall a, b \in I
\]

He never wrote it down like this explicitly but gave a list of the first 13 values. He certainly was aware of this general formula.
Let us denote the convex part of the right-hand side in the final equation, i.e., the sum of every function except the periodic function $\Pi(x)$ since all of them are easily seen to be convex on the positive real axis, by $C(x)$. Then, since $y(x) = \Pi(x) + C(x)$ and $y(x)$ is convex by assumption, we have

$$\frac{\Pi(a) - \Pi(b) + C(a) - C(b)}{a - b} \geq \frac{\Pi(a) - \Pi(b)}{a - b} + C'(b)$$

and

$$\frac{\Pi(a) - \Pi(b) + C(a) - C(b)}{a - b} \geq C'(b) + \Pi'(b)$$

Therefore, we must have either

$$\frac{\Pi(a) - \Pi(b)}{a - b} + C'(b) \geq C'(b) + \Pi'(b)$$

in which case $\Pi(x)$ would already be convex and every works out nicely, since the only differentiable periodic function that is convex on the whole positive real axis is the constant function, or we have

$$C'(b) + \Pi'(b) \geq C'(b) + \frac{\Pi(a) - \Pi(b)}{a - b}$$

or equivalently

$$\Pi'(b) \geq \frac{\Pi(a) - \Pi(b)}{a - b}.$$ 

Since this inequality must hold for all positive $a$ and $b$, let us put $b = a + 1$. Then, since $\Pi(x)$ and hence also $\Pi'(x)$ is periodic,

$$\Pi'(a) \geq 0.$$

This condition must hold for all positive $a$. Hence $\Pi(x)$ is a monotonically increasing function on the positive real axis. But the only differentiable periodic function that is monotonically increasing on the whole positive real axis is, again, the constant function. This completes the proof that $\Pi(x)$ is indeed a constant.

As mentioned several times, the condition of logarithmic convexity in the Bohr-Mollerup theorem was actually added by the authors precisely for that reason that $\Gamma$ does not somehow involve a periodic function.
In the next step, we have to determine the constant correctly. This is done by the condition \( y(1) = 0 \). Therefore,

\[
0 = y(1) = -1 + A + \sum_{k=1}^{\infty} \frac{B_{2m}}{2m(2m - 1)}.
\]

This equation, at least in principle, allows to find the constant \( A \). We say "in principle", since the sum diverges because of the rapid growth of the Bernoulli numbers. Nevertheless, applying techniques to sum divergent series, see, e.g., \([Ha48]\), the sum can be evaluated and we find \( A = \log \sqrt{2\pi} \).

The evaluation of this constant was Stirling’s great contribution to the Stirling-formula - he gave a more general formula in \([St30]\) and essentially used the Wallis product formula for \( \pi \) to find the constant \( A \). But the formula we want to prove and named after him, was actually discovered by de Moivre \([dM18]\) who could at that time only evaluate the constant numerically. See, e.g., \([Du91]\), \([Le86]\), \([Pr24]\) for a discussion on this matter.

In order to avoid the use of divergent series here, let us present Euler’s strategy to find \( A \), taken from \([E212]\) §158 chapter 6 of the second part.

Euler argues that, since \( A \) is constant, we can determine it from any case. First, let us assume that \( x \) is an integer and \( x \gg 1 \) so that the term \( \sum_{k=1}^{\infty} \frac{B_{2m}}{2m(2m - 1)x^{2m-1}} \) is negligible. Then, we have the summation

\[
\sum_{k=1}^{x} \log k = A + \left( x + \frac{1}{2} \right) \log x - x
\]

and hence

\[
\sum_{k=1}^{2x} \log k = A + \left( 2x + \frac{1}{2} \right) \log(2x) - 2x.
\]

Moreover,

\[
\sum_{k=1}^{x} \log(2k) = \sum_{k=1}^{x} \log k + \sum_{k=1}^{x} \log 2 = x \log 2 + A + \left( x + \frac{1}{2} \right) \log x - x.
\]

Therefore, finally

\[
\sum_{k=1}^{x} \log(2k - 1) = \sum_{k=1}^{2x} \log k - \sum_{k=1}^{x} \log(2k) = x \log x + \left( x + \frac{1}{2} \right) \log 2 - x.
\]
Now, Euler’s idea also was to use the Wallis product formula for \( \pi \), i.e.

\[
\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \frac{8 \cdot 10}{9 \cdot 9} \cdot \text{etc.}
\]

Therefore,

\[
\log \frac{\pi}{2} = \lim_{x \to \infty} \left( 2 \sum_{k=1}^{x} \log(2k) - \log(2x) - 2 \sum_{k=1}^{x} \log(2k - 1) \right)
\]

Thus, combining the corresponding equations and taking the limit, we find

\[
\log \frac{\pi}{2} = 2A - 2 \log 2
\]

and hence

\[
A = \log \sqrt{2\pi}.
\]

Finally, we can write down our expression for \( y(x) = \log \Gamma(x) \) in its final form:

\[
\log \Gamma(x) = x \log x - x - \frac{1}{2} \log x + \log \sqrt{2\pi} + \sum_{k=1}^{\infty} \frac{B_{2m}}{2m(2m - 1)x^{2m-1}}
\]

Finally, taking exponentials we arrive at the limit formula.

\[\square\]

### 5.4.2 Generalized Factorials

In \([E661]\), as the title suggests, Euler studied more general factorials. The generalisation is that the difference equation \( f(n) = nf(n - 1) \) is now replaced by the slightly more general one \( f(n) = (a + bn)f(n - 1) \) with positive real numbers \( a, b \). Having discussed the moment-ansatz, we can easily solve such equations now by integrals, but Euler had other ideas. He wanted to find Stirling like formulas and used the Euler-Maclaurin summation formula\(^{40}\). Otherwise, the ideas in this paper are not new. We just want to state his results for the sake of completeness. He introduces the following functions:

\(^{40}\)This will be discussed in the following section, but it turns out that the summation formula is just a special case of the formula we found here.
\[ \Gamma(i) = a(a + b)(a + 2b)(a + 3b) \cdots (a + (i - 1)b) \]
\[ \Delta(i) = a(a + 2b)(a + 4b) \cdots (a + (2i - 2)b) \]
\[ \theta(i) = (a + b)(a + 3b) \cdots (a + (2i - 1)b). \]

He establishes several relations among them and finds the Stirling-like formulas

\[ \Gamma(i) = Ae^{-i(a - b + bi)} + \frac{1}{2} \]
\[ \Delta(i) = Be^{-i(a - 2b + 2bi)} + \frac{1}{2} \]
\[ \theta(i) = Ce^{-i(a - b + 2bi)} + i \]

All equations hold only for \( i \to \infty \). He could not determine the constants as in the case of Stirling’s formula, since he did not have access to a Wallis-like product and could not argue analogously. We will not attempt this here either, but keep this task for another occasion.
6 Solution of $\log \Gamma(x + 1) - \log \Gamma(x) = \log x$ via the Euler-Maclaurin formula

6.1 Overview

We will present Euler’s and the modern derivation of the Euler-Maclaurin summation formula. Euler also saw it as the solution of the difference equation, i.e., as a tool to calculate finite sums approximately.

6.2 Euler’s Derivation of the Formula

Euler derived the Euler-Maclaurin summation formula on various occasions. The first occasion was $E_{25}$, but he also gives derivations in $E_{47}$ and his book $E_{212}$.

He saw it as a tool to calculate finite sums and this helps to understand his derivation. For, we already mentioned that

$$f(n) := \sum_{k=1}^{n} g(k)$$

satisfies the functional equation

$$f(n) - f(n - 1) = g(n).$$

And the idea to find is still the same as above (or in $E_{189}$), i.e. to solve the above difference equation. Using Taylor’s theorem, he writes

$$f(n - 1) = \sum_{k=0}^{\infty} \frac{(-1)^k f^{(k)}(n)}{k!}.$$

This, as above, leads him to an difference equation of infinite order, i.e.

$$\sum_{k=1}^{\infty} \frac{(-1)^k f^{(k)}(n)}{k!} = g(n).$$

Now, it is important to note that at the time of his first proof in $E_{25}$ and later in $E_{47}$, Euler had not developed the theory how to solve differential equations with constant coefficients. But Euler had another idea: He made an educated guess. More precisely, he assumed the solution to be of the following form:
\[ g(n) = \alpha \int g(k) dk + \beta \frac{d^2 g(n)}{dn^2} + \gamma \frac{d^3 g(n)}{dn^3} + \delta \frac{d^4 g(n)}{dn^4} + \text{etc.} \]

Inserting this ansatz into the differential equation of infinite order gives recursive relations to define the coefficients \( \alpha, \beta, \gamma \) etc.

In his book \([E212]\), Euler then also proves that the coefficients are generated by:

\[ \frac{z}{e^z - 1} \]

if it is expanded into a Taylor series about \( z = 0 \). This is almost the modern definition of the Bernoulli numbers \([41]\).

**Definition 6.1 (Bernoulli Numbers).** The Bernoulli numbers are defined via a generating function. More precisely, we define the \( n \)th Bernoulli number \( B_n \) via

\[ \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}. \]

Euler calculated many of the Bernoulli numbers and proved some elementary properties about them, e.g., that all odd Bernoulli numbers except \( B_1 = -\frac{1}{2} \) vanishes. He was mainly interested in them since they appear in his formula for \( \zeta(2n) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \), a connection he realized the first time in \([E130]\). See also Sandifer’s article in \([Br07]\) (pp. 279 - 303).

Using the Bernoulli numbers, one can write the Euler-Maclaurin formula as

\[ f(n+1) = \sum_{k=0}^{n} g(k) = \int g(k) dk + \sum_{k=0}^{\infty} \frac{B_{k+1}}{(k+1)!} \frac{d^k g(n)}{dn^k}. \]

The constant of integration must be determined in such a way that the condition \( f(1) = g(0) \) is satisfied. This is at least, how Euler used the formula. The modern formula avoid this problem, essentially by subtracting two sums from each other.

41In his earlier papers he does not realize this.
42Indeed, Euler introduced that name for those numbers in \([E212]\). Furthermore, in \([E746]\) he arrived the generating function \( \frac{z}{e^z - 1} \), which is used nowadays to introduce the Bernoulli numbers.
6.2.1 Some Remarks

The derivation of the above series is purely formal and the convergence of the sum is not guaranteed by any means. Indeed, since the Bernoulli numbers increase rapidly (roughly as \((2n)!\) which follows from Euler’s formula for \(\zeta(2n)\)), the series does actually converge very rarely. Euler was aware of this and only used it for numerical calculations truncating the sum after a certain number of terms. The Euler-Maclaurin summation formula leads to the notion of a semi-convergent series, a term coined by Gauß in [Ga28].

\(B_n\) is small for the first few \(n\), whence the series seems to converge taking only a few terms\(^{13}\), although the sum if continued to infinity must ultimately diverge, if the derivatives of \(g\) to not vanish, of course. An explicit formula for the remainder term, if the sum is truncated at some point, was given by Jacobi \([Ja34d]\) and Poisson.

The issue of semi-convergence troubled Euler and many others, including Gauß \([Ga28]\). Nowadays, the right-hand side of the above series is understood as an asymptotic expansion of the sum on the right.

Leaving the issues of convergence aside, let us discuss the nature of the formula. Recalling its origin, as an solution of a differential equation of infinite order, it has to be particular solution of it. In other words, it has to be a special case of the solution given above in the previous section. Staying purely formal, it is easier to understand this connection, what we will do in the following section.

6.2.2 Purely formal Derivation

We will use the idea that we can replace \(\frac{d}{dx}\) by \(z\) and an integral by \(\frac{1}{z}\) and vice versa\(^{14}\). Above we saw that a solution of

\[f(x + 1) - f(x) = g(x)\]

is given by

\[f(x) = \frac{1}{e^z - 1} g(x).\]

\(^{13}\)Indeed, the value found that way is often very accurate

\(^{14}\)Note that using the language of Fourier analysis this can be made completely rigorous as we have seen above.
And next, we expanded \( \frac{1}{e^z - 1} \) into partial fractions and derived the complete solution of the simple difference equation. But nobody can stop us to expand \( \frac{1}{e^z - 1} \) differently, i.e. into a Laurent series around \( z = 0 \). Using the definition of the Bernoulli numbers above, we find

\[
\frac{1}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{B_0}{z} + \sum_{n=0}^{\infty} \frac{B_{n+1}}{(n+1)!} z^{n+1}.
\]

Therefore, replacing \( \frac{1}{e^z - 1} \) by the right-hand side of this equation and then replacing \( z^n \) by \( \frac{d^n}{dx^n} \) and \( \frac{1}{z} \) by \( \int \) in the above equation, we have

\[
f(x) = B_0 \int g(t) dt + \sum_{n=0}^{\infty} \frac{B_{n+1}}{(n+1)!} \frac{d^n g(x)}{dx^n}.
\]

It is easy to see that this is the Euler-Maclaurin summation formula again. Therefore, the naive and formal derivation easily reproduces both, the general solution and the particular solution provided by the Euler-Maclaurin series, from the general difference equation.

### 6.3 Historical Overview

For the better understanding it will be convenient, to give at least a short overview about the results in the theory of differential equations of infinite order. There we quickly see, that the overview can be subdivided in formal results and rigorously proved results. Naturally the formal results extend much further, but will we see, that the general formulas, derived in purely formal manner, do not hold in every case. And the study of the first rigorous results will reveal, why the the formal approaches are not correct in the most general case.

#### 6.3.1 Formal Approaches - From Euler to Bourlet

Although many people contributed to the formal theory of differential equations of infinite order, including Lagrange \([La72]\), Laplace \([La20]\) and many others, the most general result was derived by Bourlet \([Bo97]\) and \([Bo99]\), who basically treated the problem of solving a differential equation as a problem to find the left-inverse operator of the corresponding differential operator as we did in the case of the general difference equation. He wrote \( z = \frac{d}{dx} \), and considered the equation
\[ F(x, z)f(x) = \sum_{n=0}^{\infty} a_n(x) \frac{d^n}{dx^n} f(x) = g(x), \]

where Bourlet, same as Euler, did not specify the functions \( a_n(x), f(x), g(x) \).

Now Bourlet’s simple idea was, that, since \( F(x, z) \) is an operator, it has (in modern language), a left inverse \( X(x, z) \). And treating \( z \) as a variable quantity, he derived a partial differential equation, that determines \( X(x, z) \). It reads as follows:

\[ \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n X}{\partial z^n} \cdot \frac{\partial^n F}{\partial x^n} = 1. \]

As appealing as this formula might look, it is not true in general. For this consider the equation:

\[ \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \frac{d^n}{dx^n} f(x) = g(x). \]

On the one hand we see, that the left hand side is by Taylor’s theorem equal to:

\[ f(0). \]

Thus, \( g(x) \) cannot be chosen arbitrarily. On the other hand Bourlet’s formula would let to an inverse function, because the corresponding differential equation can be solved. So it gives a solution to an ill-defined question.

**6.3.2 Rigorous Results - From Carmichel to today**

So formally the theory is established by Bourlet’s formula, a beautiful account of this is given in [Da36], where the main focus is put on solving the equations by mostly formal means. Because of the problems with the formal procedure, mathematicians considered more special problems and considered more special classes of functions, for which rigorous results can be proved. One of the first overview papers on the rigorous results was [Ca36]. Most of the results, described there only consider the existence of a solution, but do not give explicit solution formulas. One exception is the theorem we want to quote here.
Theorem 6.1. In the linear differential equation of infinite order

\[ a_0 y + a_1 y' + \cdots = \phi(x) \]

let the constants \( a_\nu \) be such constants that the function

\[ F(z) = a_0 + a_1 z + a_2 z^2 + \cdots \]

is analytic in the region \(|z| \leq q\)\(^{45}\) where \( q \) is a given positive constant or zero, and let \( \phi(x) \) be a function of exponential type not exceeding \( q \). If \( F(z) \) vanishes at least once in the region \(|z| \leq q\), let \( n \) be the number of its zeros in this region (each counted according to its multiplicity) and let \( P(z) \) be the polynomial of degree \( n \) with leading coefficient unity, that \( \frac{F(z)}{P(z)} \) does not vanish in the region. If \( F(z) \) does not vanish in the region, let \( P(z) \) be identically equal to 1. When \( P(z) \equiv 1 \), let \( P_{n-1}(z) \) be identically equal to zero; otherwise, let it be an arbitrary polynomial of degree \( n - 1 \) (including the case of an arbitrary constant when \( n = 1 \)). Then the general solution \( y(x) \), subject to the condition that it shall be a function of exponential type not exceeding \( q \), may be written in the form

\[ y(x) = \frac{1}{2\pi i} \int_{C_\rho} \frac{\Psi(s)}{F(s)} ds + \frac{1}{2\pi i} \int_{C_\rho} \frac{P_{n-1}(s)}{P_n(s)} ds \]

where

\[ \Psi(s) = \sum_{\nu=0}^{\infty} \frac{\phi^{(\nu)}(0)}{s^{\nu+1}}, \]

and where \( C_\rho \) is a circle of radius \( \rho \) about 0 as center, \( \rho \) being greater than \( q \) and such that \( F(z) \) is analytic in the region \( q > |z| \leq \rho \) and does not vanish there.

If \( \phi(x) \) is exactly of exponential type \( q \), then the named solution \( y(x) \) is also of exponential type \( q \).

Having stated this theorem, we see, how careful it is formulated and that the class of functions is quite restricted. But we have to keep in mind, that in the time of this paper the notion and theory of distributions did not exist, making the theorems quite difficult to state. The more modern treatments like \([Du94],[Du10]\) start with constructing the appropriate function spaces,\(^{45}\)

\(^{45}\)To be precise we should rather write \(|z| < q\), because for differentiation we need open domains, but Carmichel’s paper states the theorem with \( \leq \), that we simply adopted here.
before solving any differential equations. Hence before stating the results in full
generality, we will also have to define the appropriate function spaces as we did above.

6.3.3 The necessity for ultra distributions - at least for the calculation

As we already saw in the first example, considering Euler’s and Bourlet’s
results, we need functions that can differentiated infinitely often. And to be
able to state the results in complete generality as the authors of [Du94] clearly
had in mind, we need the appropriate function spaces. Ultra distributions
were considered at first in [Ho83] and [Ho85] and then formally introduced
and investigated in the famous papers [Sa59] and [Sa60]. A monograph is
[Ka88].
The theory of ultra distributions requires a lot of preparation, which is, why
we consider the simplest cases, requiring only tempered distributions, which
are on a more elementary level.

6.4 Modern Derivation

It will be illustrative to compare a modern proof of the Euler-Maclaurin sum-
mation formula to Euler’s idea. Most modern proofs in modern introductory
textbooks are similar. We will present it as it is found in [Koe00] (pp. 223-226).
The proof in [Va06] is the same. The proof of the general formula proceeds
in several steps, slowly ascending from special cases to the general formula.

6.4.1 Euler Maclaurin Formula for $C^1$-functions

We first have to define an auxiliary function

**Definition 6.2.** We define a function $H : \mathbb{R} \to \mathbb{R}$ as follows

$$H(x) := \begin{cases} 
  x - \lfloor x \rfloor - \frac{1}{2} & \text{for } x \in \mathbb{R} \setminus \mathbb{Z} \\
  0 & \text{for } x \in \mathbb{Z}
\end{cases}$$

$\lfloor x \rfloor$ is the Gauss bracket of $x$ and expresses the integer part of $x$.

This function obviously has period 1. Having introduced $H$ this way, we
can state the elementary form of the summation formula

**Theorem 6.2** (Euler summation formula (simple version)). Let $f : [1,n] \to \mathbb{C}$,
n $\in \mathbb{N}$ be a one time continuously differentiable function, then

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\[
\sum_{k=1}^{n} f(k) = \int_{1}^{n} f(x)dx + \frac{1}{2}(f(1) + f(n)) + \int_{1}^{n} H(x)f'(x)dx.
\]

**Proof.** By integration by parts over the interval \([k, k+1]\), one has

\[
\int_{k}^{k+1} f(x)dx = \left[ \left( x - k - \frac{1}{2} \right) f(x) \right]^{k+1}_{k} - \int_{k}^{k+1} \left( x - k - \frac{1}{2} \right) f'(x)dx.
\]

Since the function \((x - k - \frac{1}{2})f'\) is identical to \(Hf'\) in the interval \([k; k+1]\), their integrals over the same intervals are identical, thus,

\[
\int_{k}^{k+1} f(x)dx = \frac{1}{2}(f(k+1) - f(k)) - \int_{k}^{k+1} H(x)f'(x)dx.
\]

Summation over \(k\) from 1 to \(n-1\) and addition of \(\frac{1}{2}(f(1) + f(n))\) then gives the formula. \(\square\)

### 6.4.2 General Euler-Maclaurin Summation formula

As in the simple case we need to introduce some auxiliary functions

**Definition 6.3.** We define functions \(H_k : \mathbb{R} \to \mathbb{R}\) recursively as follows:

1) \(H_k\) is a primitive of \(H_{k-1}\), \(k \geq 2 \in \mathbb{N}\) and \(H_1 := H\)

2) \(\int_{0}^{1} H_k(x)dx = 0\).

Now can state the general Euler-Maclaurin summation formula

**Theorem 6.3** (Euler-Maclaurin Summation Formula). Let \(f : [1, n] \to \mathbb{C}\) be a \(C^{2k+1}\)-function and \(k \geq 1\). Then we have

\[
\sum_{\nu=1}^{n} f(\nu) = \int_{1}^{n} f(x)dx + \frac{1}{2}(f(1) + f(n)) + \sum_{k=1}^{n} H_{2k}(0)f^{(2k-1)}(1) + R(f);
\]

with

\[
R(f) = \int_{1}^{n} H_{2k+1}f^{(2k+1)}dx.
\]
Proof. One just has to note that $H_k$ is a periodic function for all $k$, which
is seen as follows by induction. We know that $H_1 = H$ is periodic. Thus,
consider

$$H_{k+1}(x + 1) - H_{k+1}(x) = \int_x^{x+1} H_k(t) \, dt = \int_0^1 H_k(t) \, dt = 0,$$

where we used the defining properties of $H_k$ and the induction assumption
that $H_k$ is periodic. Having noticed this, one proceeds just as in the case of
the simple Euler-Maclaurin formula, but integrates by parts $2k + 1$-times to
arrive at the formula.

\[\square\]

6.4.3 Comparison to Euler’s Idea

First, we want to mention that this idea of the modern proof is basically
Jacobi’s. See [Ja34a]. Furthermore, we want to point out that we have in
general

$$H_k(0) = \frac{1}{k!} B_k.$$

Thus, Euler’s result and the modern result agree. Anyhow, the modern proof
does not obtain the formula from the solution of a difference equation but
rather starts from the results and proves it to be correct. Therefore, it only
works, if one knows the result in advance. In Euler’s case, the formula re-
sulted as an answer to a more general question.

Finally, let us mention that we used the procedure of iterated integration
by parts above in our derivation of the Stirling formula for $n!$ and without
mentioning it explicitly at that point assumed that the term $R(f)$ vanishes.
7 INTERPOLATION THEORY AND DIFFERENCE CALCULUS

This section mainly discusses chapter 16 and 17 of [E212] and the paper [E613], which Euler stated to be an elaboration of the mentioned chapters.

7.1 OVERVIEW

Euler again tries to solve the difference equation

$$f(x + 1) - f(x) = g(x)$$

in order to find an explicit formula for

$$\sum_{k=1}^{x} g(k - 1).$$

The sum only defined for integer $x$ is interpolated by the solution of the difference equation. But this time, Euler tries to use the rules of difference calculus he developed in [E212] to solve the difference equation. Interestingly, addressing issues of convergence, this led him to the concept of Weierstraß factors.

7.2 EULER'S IDEA

Euler’s idea, outlined in [E613], is best explained by an example. Euler tries to find the sum

$$\sum_{k=1}^{x} g(k).$$

This time he simply adds 0 in a clever way. For, we formally have

$$\sum_{k=1}^{x} g(k) = g(1) + g(2) + g(3) + g(4) + \text{etc.}$$

$$- g(x + 1) - g(x + 2) - g(x + 3) - g(x + 4) - \text{etc.}$$

Or, in short notation

$$\sum_{k=1}^{x} g(k) = \sum_{k=1}^{\infty} (g(k) - g(x + k)).$$
As Euler observes, this can only work, if \( g(k) - g(x + k) \) converges to zero for \( k \to \infty \).
If the series does not converge, one can again add 0 in a clever way. If the resulting series does still not converge, one can do it again. Indeed, one can repeat the process arbitrarily often to increase the convergence as much as you want.

In [E613] Euler divides sums into classes according to the behaviour of their infinitesimal terms. More precisely, the first class contains those series, in which we have \( \lim_{k \to \infty} g(k) = 0 \). The second class contains those series whose differences of infinitesimal terms vanish, i.e. \( \lim_{k \to \infty} g(x + k + 1) - g(x + k) = 0 \). The third class contains the series, whose second differences vanish, etc. From this definition it immediately follows that the series of the \( i + 1 \)-th class are a subset of the series of the \( i \)-th class.

Euler then explicitly gives formulas for the first, second and third class and explains how the general formula for the \( i \)-th class can be constructed. We will only need the the first and second class for our discussion. The formula for the first class was stated above. Therefore, we will give the formula for the second class. It reads

\[
\sum_{k=1}^{x} g(k) = (1 - x)g(1) + (1 - x)g(2) + (1 - x)g(3) + \text{etc.} \\
+ xg(1) + xg(2) + xg(3) + xg(4) + \text{etc.} \\
- g(x + 1) - g(x + 2) - g(x + 3) - \text{etc.}
\]

Or in short notation:

\[
\sum_{k=1}^{x} g(k) = xg(1) + \sum_{k=1}^{\infty} ((1 - x)g(k) + xg(k + 2) - g(x + k))
\]

Whereas the left-hand side only makes sense for integer \( x \), this restriction is not necessary on the right-hand side. Therefore, we can interpolate the sum this way.

### 7.3 Examples

#### 7.3.1 Harmonic series

Euler chooses the harmonic series as his first example, i.e.
\[ f(x) = \sum_{k=1}^{x} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{x} \]

This series belongs to the first class and hence can write

\[ f(x) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.} \]

\[ - \frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{x+4} - \text{etc.} \]

And adding each two terms written above each other, we find:

\[ f(x) = \sum_{k=1}^{\infty} \frac{x}{k(x+k)} = \frac{x}{1(x+1)} + \frac{x}{2(x+2)} + \frac{x}{3(x+3)} + \frac{x}{4(x+4)} + \text{etc.} , \]

Trying to find the derivative of the function, in his book [E212], Euler arrived at the Taylor series expansion of this series. For this, he just expanded each term \( \frac{x}{k(x+k)} \) into a power series via the geometric series and summed the resulting series columnwise.

### 7.3.2 \( \Gamma(x) \)-function

But we are mainly interested in the expression for the \( \Gamma \)-function arising from this. Since we have \( \Gamma(x+1) = x\Gamma(x) \), we also have \( \log \Gamma(x+1) - \log \Gamma(x) = \log x \). Therefore, we can, as we did above, consider \( \log \Gamma(x) \) as the solution of the simple difference equation and for integer \( x \) we have

\[ \log \Gamma(x+1) = \sum_{k=1}^{x} \log(k) . \]

Since \( \log(k+1) - \log(k) = \log(1 + \frac{1}{k}) \) tends to zero for infinite \( k \), the series belongs to the second class. Applying the corresponding formula, we obtain

\[ \sum_{k=1}^{x} \log(k) = (1 - x) \log(1) + (1 - x) \log(2) + (1 - x) \log(3) + \text{etc.} \]

\[ + x \log(1) + x \log(2) + x \log(3) + x \log(4) + \text{etc.} \]

\[ - \log(x+1) - \log(x+2) - \log(x+3) - \text{etc.} \]
Collecting the columns, we arrive at:

\[
\log(\Gamma(x + 1)) = \log(1^{1-x}) + \log \left( \frac{1^{1-x}2x}{x+1} \right) + \log \left( \frac{2^{1-x}3x}{x+2} \right) + \log \left( \frac{3^{1-x}4x}{x+3} \right) + \text{etc.}
\]

Taking exponentials, we arrive at Euler’s first formula for \( x! \) in \[E19\]:

\[
x! = \Gamma(x + 1) = 1^{1-x} \cdot \frac{1^{1-x}2x}{x+1} \cdot \frac{2^{1-x}3x}{x+2} \cdot \frac{3^{1-x}4x}{x+3} \cdot \text{etc.} = \prod_{k=1}^{\infty} \frac{k^{1-x}(k+1)^x}{x+k}
\]

### 7.4 Difference Calculus according to Euler

Here, we will explain in a bit more detailed, how Euler arrived at his formulas. For this, we need to explain his results on difference calculus. He outlines his ideas in his book \[E212\] and also in \[E613\]. We want to state his main formula from \[E613\] for the finite sum of \( x \) terms. For this, we need to introduce some notation.

**Definition 7.1** (n-th Difference). We denote the \( n \)-th difference by \( \Delta^n g(k) \) and define it recursively by

\[
\Delta^n g(k) := \Delta^{n-1} g(k + 1) - \Delta^{n-1} g(k)
\]

\( k \) being an arbitrary number \[47\] and

\[
\Delta^0 g(k) := g(k).
\]

Additionally, we write sometimes write simply \( \Delta \) for \( \Delta^1 \).

Now, it is easily seen that we can express each term \( g(k) \) using only the differences of \( g(1) \). More precisely, we have:

**Theorem 7.1.** We have for a natural number \( k \):

\[
g(k) = g(1) + \frac{k-1}{1} \Delta g(1) + \frac{k-1}{2} \cdot \frac{k-2}{1} \Delta^2 g(1) + \frac{k-1}{3} \cdot \frac{k-2}{2} \cdot \frac{k-3}{3} \Delta^3 g(1) + \text{etc.}
\]

or in compact notation

\[47\text{There he does not prove the formula, but just gives an heuristic argument, why it is true.}

\[48\text{In most cases } k \text{ will be a natural number}

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\[ g(k) = \sum_{n=0}^{\infty} \binom{k-1}{n} \Delta^ng(1). \]

where \( \binom{n}{k} \) is the binomial coefficient.

Proof. Since the binomial coefficients vanish if \( n > k - 1 \), the sum is finite and hence converges. The proof is given by induction. The formula is obviously true for \( k = 1 \). So let us assume that it is true for a fixed \( k \in \mathbb{N} \). Consider

\[ g(k+1) = g(k) + \Delta g(k) = g(k) + \sum_{n=0}^{\infty} \binom{k-1}{n} \Delta^ng(k), \]

where we used the definition of \( \Delta \) and the induction assumption. Using the formula

\[ \Delta g(k) = \sum_{n=1}^{\infty} \binom{k-1}{n-1} \Delta^ng(1) \]

we arrive at

\[ g(k+1) = \sum_{n=0}^{\infty} \left( \binom{k-1}{n} \Delta^ng(1) + \sum_{n=1}^{\infty} \binom{k-1}{n-1} \Delta^ng(1) \right). \]

or

\[ g(k+1) = \sum_{n=0}^{k-1} \binom{k-1}{n} \Delta^ng(1) + \sum_{n=1}^{k} \binom{k-1}{n-1} \Delta^ng(1). \]

Let us extract the first term of the first sum and the last term of the second sum so that

\[ g(k+1) = g(1) + \sum_{n=1}^{k-1} \binom{k-1}{n} \Delta^ng(1) + \sum_{n=1}^{k-1} \binom{k-1}{n} \Delta^ng(1) + \Delta^kg(k). \]

Contracting the sums and using the well-known identity \( \binom{k-1}{n} + \binom{k-1}{n-1} = \binom{k}{n} \), we have

\[ g(k+1) = g(1) + \sum_{n=1}^{k-1} \binom{k}{n} \Delta^ng(1) + \Delta^kg(k). \]

\footnote{The formula follows by taking \( \Delta \) on both sides of the equation for \( g(k) \).}
Finally, absorbing the two isolated terms into the sum and using the vanishing of the binomial coefficients for $n > k$ we arrive at.

$$g(k + 1) = \sum_{n=0}^{\infty} \binom{k}{n} \Delta^n g(1).$$

Next, we want to determine the sum $\sum_{k=1}^{x} g(k)$. This is done in the following theorem.

**Theorem 7.2.** We have

$$\sum_{k=1}^{x} g(k) = \sum_{k=1}^{\infty} \binom{x}{k} \Delta^k g(k)$$

The proof is by induction again along the same lines as the last. So, we omit it here. Instead, we want to make some remarks.

Although the proof explicitly assumes $x$ to be a natural number, the right-hand side of the above equation does not require $x$ to be an integer and hence interpolates the sum on the left-hand side. In other words, the right-hand side solves the functional equation of the finite sum, i.e. $\sum_{k=1}^{x} g(k) - \sum_{k=1}^{x-1} g(k) = g(x)$ with the initial condition $\sum_{k=1}^{1} g(k) = g(1)$.

And Euler precisely did this replacement, even addressing the issue of convergence of the then infinite series.

But let us now go over to Euler’s idea of adding zero in a clever way to enforce convergence. First note, that from the above theorem we also have:

$$g(x + 1) = \sum_{k=0}^{\infty} \binom{x}{k} \Delta^k g(1).$$

But we can find similar expressions for $g(x + 2)$ using $g(2)$. We have:

$$g(x + 2) = \sum_{k=0}^{\infty} \binom{x}{k} \Delta^k g(2).$$

And in general for $n$:  

50The binomial coefficients can be defined for non-integer numbers replacing the factorials by $Γ$-functions. Indeed, this is a consequence of Newton’s generalized binomial theorem proved by Euler in [E465].
\[ g(x + n) = \sum_{k=0}^{\infty} \left( \begin{array}{c} x \\ k \end{array} \right) \Delta^k g(n). \]

Therefore, using the sum representation we found above, we arrive at the following equation:

**Theorem 7.3** (Equation for sum in terms of differences).

\[ \sum_{k=1}^{x} g(k) = \]

\[ \left( \begin{array}{c} 1 \\ 1 \end{array} \right) g(1) + \left( \begin{array}{c} x \\ 2 \end{array} \right) \Delta^1 g(1) + \left( \begin{array}{c} x \\ 3 \end{array} \right) \Delta^2 g(1) + \cdots \]

\[ + g(1) + \left( \begin{array}{c} x \\ 2 \end{array} \right) \Delta^1 g(1) + \left( \begin{array}{c} x \\ 3 \end{array} \right) \Delta^2 g(1) + \cdots - g(x + 1) \]

\[ + g(2) + \left( \begin{array}{c} x \\ 3 \end{array} \right) \Delta^1 g(2) + \left( \begin{array}{c} x \\ 4 \end{array} \right) \Delta^2 g(2) + \cdots - g(x + 2) \]

\[ + g(3) + \left( \begin{array}{c} x \\ 4 \end{array} \right) \Delta^1 g(3) + \left( \begin{array}{c} x \\ 5 \end{array} \right) \Delta^2 g(3) + \cdots - g(x + 3) \]

\[ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \]

\[ + g(n) + \left( \begin{array}{c} x \\ n \end{array} \right) \Delta^1 g(n) + \left( \begin{array}{c} x \\ n+1 \end{array} \right) \Delta^2 g(n) + \cdots - g(x + n) \]

**Proof.** The proof is immediate from the preceding. The first row is just the alternate representation of the sum. Each following row is simply \( 0 \) by the results we stated. \( \square \)

And this is Euler’s fundamental formula. For, he proceeds to sum the series column by column from \( n = 1 \) to \( n = \infty \). And hence it is easily seen that the definition of the classes we mentioned above makes sense. For, if any iterated difference vanishes, one only has a finite number of columns to sum and the series converges.\(^{51}\) It is easily seen, how the examples we gave (from the first and second class) result from this formula. In the next section we will explain, how this idea actually anticipates the idea of Weierstraß factors.

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\(^{51}\) Under some mild additional assumptions
7.5 MODERN IDEA - WEIERSTRASS PRODUCT

We introduce, but not prove, Weierstraß’s product theorem and show the connection to Euler’s ideas outlined in the last sections. The exposition of Weierstraß’s theory follows [Fr06] (pp. 213-217)

7.5.1 Introduction to the Problem

Weierstraß considered the following problem: Given a domain $D \subset \mathbb{C}$ and a discrete subset $S$ in $D$, can one construct an analytic function $f : D \to \mathbb{C}$ with zeros of a given order $m_s$ precisely in $S$? The answer to this question is yes and the task can be solved by Weierstraß products. Let us see, how to arrive at the concept.

For the sake of simplicity, let us take $D = \mathbb{C}$. First, we note that closed disks are compact sets and hence there are only finitely many $s \in S$ with $|s| \leq N \in \mathbb{N}$. Thus, $S$ is a countable set and the elements can be ordered with respect to their magnitude

$$S = \{s_1, s_2, \cdots \}, \quad |s_1| \leq |s_2| \leq |s_3| \leq \cdots .$$

If $S$ is a finite set, we know how to solve the problem, the solution is given by the polynomial

$$\prod_{s \in S} (z - s)^{m_s} .$$

For infinite sets on the other hand, the product obtained this way can not converge in general. But we can assume $S$ to not contain zero, since we can multiply by $z^{m_0}$ at the end. We want to do this, since we can focus on products of the form

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{s_n}\right)^{m_n}, \quad m_n := m_{s_n} .$$

Indeed, Euler already had this idea and it led him to the discovery of the sine product in [E41] and its proof in [E61] \(^{52}\).

This product still does not always converge (it converges, e.g., for $s_n = n^2$, $m_n = 1$ but diverges for $s_n, m_n = 1$).

\(^{52}\)The discussion of his proof is contained in the appendix.
Now Weierstraß had the idea to multiply with factors not changing the zeros but force the product to converge. He made the ansatz:

\[ f(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z}{s_n}\right)^{m_n} \cdot e^{P_n(z)}. \]

\( P_n(z) \) is a polynomial still to be determined. We have to ensure that

\[ \lim_{n \to \infty} \left(1 - \frac{z}{s_n}\right)^{m_n} e^{P_n(z)} = 1 \quad \forall z \in \mathbb{C}. \]

This is possible, since it is easily seen that we can find an analytic function \( A_n(z) \)

\[ \left(1 - \frac{z}{s_n}\right)^{m_n} e^{A_n(z)} = 1 \quad \forall z \in U(s_n)(0) \]

with \( A_n(0) = 0 \). The power series \( A_n \) converges uniformly in each compact subset of the disk \( U(s_n)(0) \). Therefore, truncating this power series for \( A_n \), we can easily find a polynomial \( P_n(z) \) with

\[ \left|1 - \left(1 - \frac{z}{s_n}\right)^{m_n} e^{P_n(z)}\right| \leq \frac{1}{n^2} \quad \text{for all } z \text{ with } |z| \leq \frac{1}{2} |s_n|. \]

Since the series \( 1 + \frac{1}{4} + \frac{1}{9} + \cdots \) converges we arrive at the theorem:

**Theorem 7.4.** The series

\[ \sum_{n=1}^{\infty} \left|1 - \left(1 - \frac{z}{s_n}\right)^{m_n} e^{P_n(z)}\right| \leq \frac{1}{n^2} \quad \text{for all } z \text{ with } |z| \leq \frac{1}{2} |s_n| \]

converges normally.

Concerning the problem propounded initially, we can now formulate Weierstraß’s factorisation theorem.

**Theorem 7.5 (Weierstraß’s factorisation theorem).** Let \( S \subseteq \mathbb{C} \) be a discrete subset. Further, let the following map be given

\[ m : S \rightarrow \mathbb{N}, \quad s \mapsto m_s. \]

Then, there exists an analytic function

\[ f : \mathbb{C} \rightarrow \mathbb{C} \]
with the properties:

1) \( S := \{ z \in \mathbb{C} | f(z) = 0 \} \)

2) \( m_s = \text{ord}(f; s) \). (Order of the zero.)

### 7.5.2 Comparison to Euler’s idea

Let us compare both, Weierstraß’s and Euler’s, ideas. Euler in \( E613 \) added zero and expressed the series under consideration in an alternative way to obtain a more convergent series, whereas Weierstraß tells us to multiply by additional exponential of polynomials to ensure the converge. But it is easily seen that both ideas are actually the same. The following quote from the Introduction written by G. Faber in Volume 16,2 of the first series of Euler’s Opera Omnia confirms this:

*Tatsächlich hat Euler nicht nur die Produktdarstellung (12) [this means the product expansion of the \( \Gamma \)-function], sondern sogar den Gedanken der Konvergenz erzeugenden Faktoren von Weierstraß vorweggenommen. Denn es bedeutet keinen Unterschied, ob man den Gliedern des divergenten Produktes \( \prod_{\nu=1}^{\infty} \left( 1 + \frac{x}{\nu} \right) \) die Konvergenz erzeugenden Faktoren \( e^{-\frac{x}{\nu}} \) oder den Gliedern der divergenten unendlichen Reihe \( \sum_{\nu=1}^{\infty} \log \left( 1 + \frac{1}{\nu} \right) \) die Konvergenz erzeugenden Summanden \( -\frac{x}{\nu} \) oder auch \( -x \log \left( 1 + \frac{1}{\nu} \right) \) beifügt. Das tat aber Euler mit voller Absicht in der Abhandlung 613.*

For, Euler’s idea translates into the one of Weierstraß by considering sums of logarithms. Euler even tells us, how to find those factors you need to enforce convergence\(^{53}\). Therefore, Euler actually anticipated the idea of Weierstraß factors without actually intending it. His intention, as we saw above, was the interpolation of a sum from the integers to all numbers. Nevertheless, Weierstraß’s name is attached to the idea, since he constructed a rigorous theory of infinite products and by solving the problem propounded above also provided the mathematical community with a large class of analytic functions. Euler’s contribution was maybe overlooked, since he was interested in something completely different and did not point out the generality of his method clearly enough.

\(^{53}\)Indeed, there is even a prescription, how to find those factors for the Weierstraß product. Confer, e.g. \( Fr06 \)
8 Relation of $\Gamma$ and $B$

This section is entirely devoted to the connection between the $\Gamma$- and $B$-function.

8.1 From $B$ to $\Gamma$ - Euler’s first Way to the Integral Representation

8.1.1 Euler’s thought Process

Euler describes his thought process, how he got the idea that the $\Gamma$-function is can be given as an integral. This provide us with a beautiful example of the Methodus inveniendi. He explains his thoughts in §§3 – 7. He writes:

I had believed before that the general term of the series $1, 2, 6, 24$ etc., if not algebraic, is nevertheless given as an exponential. But after I had understood that certain terms depend on the quadrature of he circle, I realized that neither algebraic nor exponential quantities suffice to express it. [...].

But after I had considered that among differential quantities there are formulas of such a kind, which admit an integration in certain cases and then yield algebraic quantities, but in other do not admit an integration and then exhibited quantities depending on quadratures, it came to mind hat maybe formulas of this kind are apt to express the general terms of the mentione and other progressions. [...].

But the differential formula must contain a certain variable quantity. [...] For the sake of clarity, I say that $\int pdx$ is the general terms of the progression to be found as follows from it; but let $p$ denote a function of $x$ and constants, amongst which here $n$ must be contained. Imagine $pdx$ integrated and such a constant to be add that for $x = 0$ the whole integral vanishes; then put $x$ equal to a certain known quantity. Having done this, if in the found integral only quantities extending to to progression remain, it will express the term, whose index is $n$. In other words, the integral determined like this will be the general term. [...]

Therefore, I considered many differential formulas only admitting an integration, if one takes $n$ to be a positive integer number, so that the principal terms become algebraic, and hence formed progressions.

Euler wants this the parameter integral to express the $n$-th term of the progression. Thus, the parameter integral must contain $n$, which is not to be integrated over.
To summarize Euler’s idea in modern formulation: He propounded that there is a function \( p(x, n) \) so that

\[
  n! = \int_a^b p(x, n) \, dx \quad \text{for } n \in \mathbb{N}.
\]

And in the following paragraphs he really tries out several different functions and integrals (which essentially all boil down to the \( B \)-function) and eventually arrives at the integral representation of the \( \Gamma \)-function. We will discuss his proof in the following sections.

But here we want to stress that the theory of definite integrals did not really exist at the time. Euler also started to develop this theory. Furthermore, this is one, if not the first paper, in which parameter integrals or functions defined through have be discussed.

**8.1.2 Euler’s Mathematical Argument**

It is interesting, how Euler arrived at the integral representation of \( \Gamma(x) \) for the first time in \([E19]\). He repeats his argument in \([E42]\). For a review of the following argument, using Euler’s notation, confer, e.g., the article on the \( \Gamma \)-function in \([Sa15]\), \([Va06]\) or \([Du99]\). He shows first that

\[
\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g) \cdots (f+ng)} = \frac{f + (n+1)g}{g^{n+1}} \int_0^1 x^n dx (1 - x)^n.
\]

This is easily proved by induction. Now it is easy to see that one arrives at an expression for \( 1 \cdot 2 \cdots n = n! \), if one puts \( g = 0 \), i.e.

\[
\frac{n!}{f^n} = \lim_{g \to 0} \frac{1}{g^{n+1}} \int_0^1 \frac{x^n (1 - x)^n}{g^{n+1}} \, dx
\]

To get rid of the \( g \) in the denominator, Euler puts \( x = y^{\frac{1}{n+1}} \) (and uses then \( y = x \) again) and arrives at:

\[
\frac{n!}{f^n} = \lim_{g \to 0} \int_0^1 \frac{g}{f + g} \left(1 - x^{\frac{1}{n+1}}\right)^n \, dx
\]

We are mainly interested in the case \( f = 1 \), i.e.
\[ n! = \lim_{g \to 0} \int_0^1 \frac{g}{1 + g} \frac{(1 - x^{1/g})^n}{g^{n+1}} \, dx \]

But this is the same as:

\[ n! = \lim_{g \to 0} \int_0^1 \left( \frac{1 - x^{1/g}}{g^n} \right)^n \, dx. \]

We can pull the limit into the integral, i.e.

\[ n! = \int_0^1 \lim_{g \to 0} \left( \frac{1 - x^{1/g}}{g^n} \right)^n \, dx. \]

Note, that we already took the limit in the denominator of the power of \( x^{55} \).

\[ n! = \int_0^1 \lim_{g \to 0} \left( \frac{1 - x^{1/g}}{g^n} \right)^n \, dx. \]

Let us rewrite the a bit more as

\[ n! = \int_0^1 \left( \lim_{g \to 0} \frac{1 - x^{1/g}}{g^n} \right)^n \, dx. \]

This is allowed, since \( x^n \) is a continuous function. But for natural \( n \) this limit can be found by l’Hospital’s rule. And one finds:

\[ n! = \int_0^1 \left( \log \frac{1}{x} \right)^n \, dx. \]

And this is the integral representation of \( \Gamma(n + 1) \). Hence it is easily understood why Euler preferred to work with this representation. He was led naturally to it.

\[ ^{55}\text{This is allowed since all functions are continuous.} \]
8.1.3 Using the $B$-function

Euler did not consider the $B$-function as an independent function at the time he wrote $[E19]$ and $[E421]$ is devoted to the integral representation of the $\Gamma$-function. Therefore, let us see, how Euler’s argument can be formulated using the known properties of $B$.

We also start from

$$n! = \lim_{g \to 0} \int_0^1 x^n (1-x)^n g^{n+1} \, dx.$$ 

Let us rewrite the integral as a $B$-function:

$$n! = \lim_{g \to 0} B \left( \frac{1}{g} + 1, n + 1 \right) g^{n+1} \, dx.$$ 

Using the functional equation $B(x+1,y) = \frac{x}{x+y} B(x,y)$, we have

$$n! = \lim_{g \to 0} \frac{1}{g} B \left( \frac{1}{g}, n + 1 \right).$$ 

It is more convenient to put $\frac{1}{g} = h$ and consider the following limit

$$n! = \lim_{h \to \infty} \frac{h}{n + h + 1} h^{n+1} B (h, n + 1).$$ 

This is the same limit as

$$n! = \lim_{h \to \infty} h^{n+1} B (h, n + 1).$$ 

Now, let use the relation $B(x,y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$,

$$n! = \lim_{h \to \infty} h^{n+1} \frac{\Gamma(h) \Gamma(n + 1)}{\Gamma(n + h + 1)}.$$ 

Using the functional equation of the $\Gamma$-function $h$ times in the denominator, we find

$$n! = \lim_{h \to \infty} h^{n+1} \frac{\Gamma(h) \Gamma(n + 1)}{\Gamma(n + 1)(n + h)(n + h - 1) \cdots (n + 1)}.$$ 

Or equivalently

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\[ n! = \lim_{h \to \infty} \frac{h^{n+1} \Gamma(h)}{(n + h)(n + h - 1) \cdots (n + 1)}. \]

Interestingly, we arrived at the condition Weierstraß used to define \( \Gamma \).

### 8.1.4 Gauß’s Idea

Now that we have seen that it is possible to get to the \( \Gamma \)-function from the \( B \)-function, for the sake of completeness, let us also briefly mention Gauß’s idea. His idea is basically the same as Euler’s. We already mentioned that Gauß defined the \( \Gamma \)-function as Weierstraß as the above limit. I.e. he defined

\[ \Gamma(x) = \lim_{n \to \infty} \frac{n^x n!}{(x + 1) \cdot (x + 2) \cdots (x + n)}. \]

And Gauß observed, essentially as Euler in [E19], that

\[ \frac{n^x n!}{(x + 1) \cdot (x + 2) \cdots (x + n)} = \int_0^n t^{x-1} \left( 1 - \frac{t}{n} \right)^{n-1} dt \]

and hence

\[ \Gamma(x) = \lim_{n \to \infty} \frac{n^x n!}{(x + 1) \cdot (x + 2) \cdots (x + n)} = \int_0^n t^{x-1} \left( 1 - \frac{t}{n} \right)^{n-1} dt \quad \text{for } \Re(x) > 0. \]

Hence he concluded:

\[ \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad \text{for } \Re(x) > 0. \]

It is interesting that, although starting from the same idea, the same identity even, Gauß and Euler got to the integral representation so differently. Euler’s proof is even rigorous by today’s standards, whereas Gauß’s approach is harder to make it rigorous. A rigorous proof was given by Schlömilch [Sc79] and is also presented in Nielsen’s book [Ni05].

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56To be completely accurate, Weierstraß followed Gauß’s example and explicitly said so.
57Obviously, Euler did not know how to reason rigorously that the limit can be pulled inside the integral, he simply did it without thinking about it. But it can be justified by the machinery of the Lebesgue integral.
8.2 Expressing $B$ via $\Gamma$-functions - Fundamental Relation

We want to start with the formula, already proved above, relating the $\Gamma$- and $B$-function, i.e. the formula

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.$$  

8.2.1 Euler’s Proof

As already indicated above, Euler’s proof is actually not rigorous, and more or less based on the extension of the validity of the formula for natural numbers $x$ and $y$ to all numbers. Nevertheless, we present Euler’s arguments here, since it is interesting to see, how he discovered the result. In [E421] (§26) he states the following equation

$$\int_{0}^{1} dx \left( \log \frac{1}{x} \right)^{n-1} \cdot \int_{0}^{1} dx \left( \log \frac{1}{x} \right)^{m-1} = k \int_{0}^{1} \int_{0}^{\infty} e^{-\left(t + u\right)} t^{x-1} u^{y-1} dt du,$$

which reduces to the desired relation for $m = 1$. But in his paper only established this equation for natural numbers $n$ and $m$ and not in general. He certainly was sure this was enough to prove it also for all positive real numbers $m$ and $n$ and he basically used the left-hand side to interpolate the right-hand side. In the following paragraphs he then establishes the formula for some more fractional numbers, but not for the general case. In the following we will consider Dirichlet’s and Jacobi’s proof and see, why it was difficult for Euler to prove the identity in general.

8.2.2 Jacobi’s Proof

We present Jacobi’s proof of the fundamental relation [Ja34].

Proof. Let $x, y$ be $> 0$ and consider:

$$\Gamma(x)\Gamma(y) = \int_{0}^{\infty} t^{x-1} e^{-t} dt \cdot \int_{0}^{\infty} u^{y-1} e^{-u} du = \int_{0}^{\infty} \int_{0}^{\infty} e^{-(t+u)} t^{x-1} u^{y-1} dt du.$$
Make the substitution:

\[ t + u = \alpha(u,t), \quad u = \alpha(u,t) \beta(u,t). \]

This gives:

\[ t = \alpha(1 - \beta), \quad u(1 - \beta) = t \beta, \]
\[ dt = (1 - \beta)da, \quad (1 - \beta)du = \alpha d\beta, \]

which immediately leads to the formula:

\[
\Gamma(x)\Gamma(y) = \int_0^\infty e^{-x \alpha} \cdot \frac{\beta x^{y-1}}{(1 - \beta)^{y-1}} d\beta.
\]

This implies the desired formula. \( \square \)

Obviously, the hard part is to find the substitution and actually it is only possible, if you know in advance how the final result looks like.

8.2.3 Dirichlet’s Proof

Dirichlet’s proof \([D39]\) is a bit more straight-forward.

**Proof.** We start from the following expression of \( B \)

\[
B(x,y) = \int_0^\infty \frac{t^{x-1} dt}{(1 + t)^{x+y}}.
\]

It is obtained from \( B(x,y) = \int_0^1 dt t^{x-1}(1 - t)^{y-1} \) by setting \( t = \frac{1}{2} \) and then \( z = u + 1 \). Furthermore, setting \( t = ky \) in the integral representation of the \( \Gamma \)-function, we have

\[
\int_0^\infty e^{-tx} t^{x-1} dt = \frac{\Gamma(x)}{k^x}.
\]

Therefore, from the above representation of the \( B \)-function

\[
B(x,y) = \frac{1}{\Gamma(x+y)} \cdot \int_0^\infty t^{x-1} dt \int_0^\infty e^{-(1+t)u} u^{y-1} du.
\]

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It is easy to see that we can exchange the order of integration here by Fubini’s theorem so that

\[
B(x, y) = \frac{1}{\Gamma(x+y)} \cdot \int_0^\infty e^{-u}u^{x+y-1}du \cdot \int_0^\infty e^{-t}t^{x-1}dt
\]

Performing the integral over \( t \):

\[
B(x, y) = \frac{1}{\Gamma(x+y)} \int_0^\infty e^{-u}u^{x+y-1} \cdot \frac{\Gamma(x)}{u^x}.
\]

Finally, we arrive at:

\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\]

\[\square\]

8.2.4 Discussion

Considering these proofs, Jacobi’s proof was out of Euler’s reach. For, he had theory, how to handle double integrals. More precisely, he did not know how to perform a substitution, or in other words, he did not know the concept of the Jacobi determinant. Additionally, the anticommutativity of the wedge product puzzled him in his only paper devoted explicitly to double integrals. See also Katz’s article in [Du07] on the subject.

Dirichlet’s proof was definitely within Euler’s reach. Especially, since he also knew the formula

\[
\frac{\Gamma(x)}{k^x} = \int_0^\infty e^{-kt}t^{x-1}dt
\]

and derived many extraordinary integrals from it in [E675], including the Fresnel-integrals. Although Fubini’s theorem was proven a lot later, this would not have troubled Euler, since he basically considered integrals as

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\[58\] Obviously, the concept of the wedge product was introduced much later. Indeed, Euler, could not explain why his results seem to indicate that \( dx dy = -dy dx \) and tries to argue it away. He did not realize that he encountered the wedge product of the first time.
ordinary sums, just over infinitesimally small numbers. In general, the exchange of limits did not bother him at all. The necessity to prove the validity of such a procedure just came after Abel’s proof of what we now call Abel’s limit theorem for series.

Interestingly, although Euler obviously knew the formula

\[ B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \]

and devoted own papers to examine it, e.g., [E321], [E640] and even a chapter in his second book on integral calculus, he never explicitly stated the formula:

\[ B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \int_0^\infty \frac{t^{x-1}}{(1+t)^x+y}, \]

which would have simplified his investigations and results on definite integrals a lot. Indeed, almost all his papers on definite integrals (contained in the Opera Omnia, Series I, Volumes 17-19) can be understood very easily using the B- and Γ-functions and their derivatives and the fundamental relation connecting them. But Euler seems to have not realized this, at least he does not mention it in any of his papers or books.

8.3 Relation to the Beta Function - Expressing Γ via B

In [E19] and [E122] Euler stated, but not proved, a formula which expressed \( \Gamma\left(\frac{p}{q}\right) \) in terms of a product of several B-functions. This is interesting, since the Γ-function involves a transcendental integrand, whereas B is an integral over an algebraic function (if \( p \) and \( q \) are integers) and hence is a period in the sense of Zagier and Kontsevich [Ko01]. Euler states the formula as follows in [E19]

\[ \int_0^1 (-\log x)^p dx = \sqrt{1 \cdot 2 \cdot 3 \cdots p} \left( \frac{2p}{q} + 1 \right) \left( \frac{3p}{q} + 1 \right) \left( \frac{4p}{q} + 1 \right) \cdots \left( \frac{qp}{q} + 1 \right) \]

\(^{59}\)Euler wants \( p \) and \( q \) to be natural numbers, but will see that this restriction is not necessary.
\[
\times \sqrt{\frac{1}{\pi} \int_0^1 dx (x-xx)^{\frac{p}{q}} \cdot \frac{1}{\pi} \int_0^1 dx (x^2-x^3)^{\frac{p}{q}} \cdot \frac{1}{\pi} \int_0^1 dx (x^3-x^4)^{\frac{p}{q}} \cdot \frac{1}{\pi} \int_0^1 dx (x^4-x^5)^{\frac{p}{q}} \ldots \frac{1}{\pi} \int_0^1 dx (x^{q-1}-x^q)^{\frac{p}{q}}.}
\]

Replacing the integral \( \frac{1}{\pi} \int_0^1 (-\log x)^{\frac{p}{q}} dx \) by \( \Gamma \left( \frac{p}{q} + 1 \right) \) and \( 1 \cdot 2 \cdot 3 \cdots p \) by \( \Gamma (p + 1) \), we can formulate the theorem as follows:

**Theorem 8.1.** Let \( p > 0 \) and \( q \in \mathbb{N} \). Then, the following formula holds:

\[
\Gamma \left( \frac{p}{q} + 1 \right) = \sqrt{\Gamma (p + 1) \times \prod_{k=2}^{q} \left( \frac{kp}{q} + 1 \right) \times \prod_{i=1}^{q-1} \int_0^1 dx (x^{i-1} - x^i)^{\frac{p}{q}}}
\]

**Proof.** The idea is to express the integrals as \( B \)-function, and then rewrite them in terms of \( \Gamma \)-functions. Let us consider the \( q \)-th power of the right-hand side and call it \( G(p, q) \). Then

\[
G(p, q) = \Gamma (p + 1) \times \prod_{k=2}^{q} \left( \frac{kp}{q} + 1 \right) \times \prod_{i=2}^{q} \int_0^1 x^{i-1} dx (1 - x)^{\frac{p}{q}}.
\]

The integrals can be rewritten in terms of \( B \):

\[
G(p, q) = \Gamma (p + 1) \times \prod_{k=2}^{q} \left( \frac{kp}{q} + 1 \right) \times \prod_{i=2}^{q} B \left( \frac{(i-1)p}{q} + 1, \frac{p}{q} + 1 \right)
\]

Next express the \( B \)-functions via \( \Gamma \)-functions:

\[
G(p, q) = \Gamma (p + 1) \times \prod_{k=2}^{q} \left( \frac{kp}{q} + 1 \right) \times \prod_{i=2}^{q} \frac{\Gamma \left( \frac{(i-1)p}{q} + 1 \right) \Gamma \left( \frac{p}{q} + 1 \right)}{\Gamma \left( \frac{kp}{q} + 1 \right)}
\]

\( \Gamma \left( \frac{p}{q} \right) \) does not depend on the multiplication index. Therefore, we can pull it out of the product. Additionally, let us use the functional equation to rewrite the denominator. Then, we arrive at:

\[
G(p, q) = \Gamma (p + 1) \cdot \left( \Gamma \left( \frac{p}{q} + 1 \right) \right)^{q-1} \times \prod_{k=2}^{q} \left( \frac{kp}{q} + 1 \right) \times \prod_{i=2}^{q} \frac{\Gamma \left( \frac{(i-1)p}{q} + 1 \right)}{\Gamma \left( \frac{kp}{q} + 1 \right) \Gamma \left( \frac{p}{q} + 1 \right)}
\]
Finally, writing everything as one product

\[
G(p, q) = \Gamma(p + 1) \cdot \left( \Gamma\left(\frac{p}{q} + 1\right) \right)^{q^{-1}} \times \prod_{i=2}^{q} \frac{\left(\frac{ip}{q} + 1\right) \cdot \Gamma\left(\frac{(i-1)p}{q} + 1\right)}{\left(\frac{ip}{q} + 1\right) \cdot \Gamma\left(\frac{ip}{q} + 1\right)}
\]

Almost every factor in the product cancels. Indeed, it remains

\[
\frac{\Gamma\left(\frac{p}{q} + 1\right)}{(p + 1) \cdot \Gamma\left(\frac{pq}{q} + 1\right)} = \frac{\Gamma\left(\frac{p}{q} + 1\right)}{\Gamma(p + 1)}.
\]

Hence we arrive at

\[
G(p, q) = \left( \Gamma\left(\frac{p}{q} + 1\right) \right)^{q}.
\]

Therefore, we arrived at the desired result.

**8.4 Reflection Formula**

**Theorem 8.2** (Reflection formula for the Γ-function). We have:

\[
\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin(\pi x)}
\]

We want to discuss the proof that Euler gave himself in [E421] and one he could have given. For Euler’s proof, we need to prove the product formula for the sine first.

**8.4.1 Euler’s proof of the sine product formula**

The product formula for the sine, i.e.

\[
\sin(\pi x) = x\pi \prod_{k=1}^{\infty} \left( 1 - \frac{x^2}{k^2} \right),
\]

is without a doubt one of Euler’s most beautiful discoveries. He discovered it in [E41] and used it in the solution of the Basel problem, i.e. the summation over the reciprocals of the squares. Due to this criticism of Bernoulli and Cramer, he gave a proof in [E41] which is repeated in [E101]. Concerning the criticism, Euler in [E61] §6 wrote:
This almost completely oppressed worry has recently been renewed by letters from Daniel Bernoulli, in which he shared the same worries concerning my method and also mentioned that Cramer has the same doubts that my method is right.

The main point of criticism was that in Euler did not prove that the sine only has the integer numbers times $\pi$ as roots. In other words, Euler could not rule out complex roots at that time.

But in Euler then offered a proof for the sine product formula. For this, he considered the expression

$$a^n - b^n,$$

$n$ being a natural number, and its factorization into real factors. The complex factors are easily found by de Moivre’s theorem. They appear in complex conjugate pairs and hence combing each two a real factor is

$$a^2 - 2ab \cos \frac{2k}{n} \pi + b^2$$

$k$ being a natural number with $2k < n$. If $n$ is odd, one has the additional factor $a - b$.

In the next step Euler uses the famous identity named after him to write

$$\sin s = \frac{e^{is} - e^{-is}}{2i}$$

and the definition of $e^s$

$$e^s := \lim_{n \to \infty} \left(1 + \frac{s}{n}\right)^n.$$

Therefore, Euler then considered the expression

$$\frac{(1 + \frac{is}{\pi})^n - (1 - \frac{is}{\pi})^n}{2i}$$

and noted that for infinite $n$ this expression goes over into $\sin s$.

This expression has the form of the general product and hence each factor is given by the form we ascribed to it above. Here, $a = 1 + \frac{is}{\pi}$, $b = 1 - \frac{is}{\pi}$ and hence each factor has the form

\[60\text{The worry concerns the correctness of his product formula for the sine and the solution of the Basel problem derived from it}

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\[ 2 - \frac{2s^2}{n^2} - 2 \left( 1 + \frac{s^2}{n^2} \right) \cos \frac{2k\pi}{n} \]

Therefore, all the factors of the sine are obtained, if all natural numbers are substituted for \( k \).

But since now \( n \) is infinite we have

\[ \cos \frac{2k\pi}{n} = 1 - \frac{2k^2\pi^2}{n^2}, \]

whence our factor becomes

\[ -\frac{4s^2}{n^2} + \frac{4k^2\pi^2}{n^2}, \]

or, if we want each factor to have the form \( 1 - a_k \), the general factor will be

\[ 1 - \frac{s^2}{k^2\pi^2}. \]

Therefore, we already have

\[ \sin s = A s \prod_{k=1}^{\infty} \left( 1 - \frac{s^2}{k^2\pi^2} \right). \]

The constant \( A \) is easily seen to be \( = 1 \) from the well-known limit \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \). This completes Euler’s proof of the sine product formula.

Although some arguments are not completely rigorous, the general idea is correct and one can work out a proof meeting today’s standard of rigor. Confer, e.g., [Val66] for a modern proof based on Euler’s ideas. Let us mention that in [E66] Euler started from the right-hand side of the last equation and and proved that it is equal to \( \sin x \). \[^{61}\]

### 8.4.2 Euler’s Proof of the Reflection Formula

The proof, we are about to give, is the one from [E421].

**Proof.** Above we had the formula

\[^{61}\text{More precisely, he used the corresponding expression of } \cos x, \text{ since the proof is easier working with the product of } \cos x.\]
Therefore, replacing \( x \) by \( -x \):

\[
\Gamma(1 - x) = 1^{1+x} \cdot \frac{1^{1+x}2^{-x}}{1-x} \cdot \frac{2^{1+x}3^{-x}}{2-x} \cdot \frac{3^{1+x}4^{-x}}{3-x} \cdot \text{etc.} = \prod_{k=1}^{\infty} \frac{k^{1-x}(k+1)^x}{k-x}.
\]

Therefore,

\[
\Gamma(1 + x)\Gamma(1 - x) = \frac{1^2}{x^2 - 1^2} \cdot \frac{2^2}{x^2 - 2^2} \cdot \frac{3^2}{x^2 - 3^2} \cdot \text{etc.}
\]

or in compact notation:

\[
\Gamma(1 + x)\Gamma(1 - x) = \prod_{k=1}^{\infty} \frac{k^2}{k^2 - x^2} = \prod_{k=1}^{\infty} \frac{1}{1 - \frac{x^2}{k^2}}
\]

The product is the well-known product formula for the sine. So that

\[
\Gamma(x + 1)\Gamma(1 - x) = \frac{x\pi}{\sin(\pi x)}.
\]

Using the functional equation of the \( \Gamma \)-function, one arrives at the desired formula.

\[\square\]

8.4.3 Proof Euler could have given

In \([E59]\) Euler stated and in \([E60]\), \([E61]\) and \([E462]\) Euler proved the following identity:

\[
\int_0^1 \frac{t^{x-1} - t^{-x}}{1 + t} dt = \frac{\pi}{\sin(\pi x)}.
\]

He does this by considering integrals over rational fractions and solving them by integration by parts. But using the partial fraction decomposition of \( \frac{\pi}{\sin(\pi x)} \), you will easily see the identity to be true by expanding the denominator into a geometric series and integrating term by term. Therefore, we will not give
the prove here.
Instead, we want to prove the reflection formula in the most simple way possible using only formulas at Euler’s disposal. The theorem to be proved is still the same, i.e that

\[ \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}. \]

**Proof.** We have

\[ \Gamma(x)\Gamma(1-x) = B(x, 1-x) = \int_0^\infty \frac{t^{x-1}}{1+t} \, dt \]

by using the relation between the \( \Gamma \)- and \( B \)-function and using the integral representation with limits 0 and \( \infty \) for \( B \). Next, split the integral into two integrals \( \int_0^1 + \int_1^\infty \) and put \( t = \frac{1}{u} \) in the second. Then, calling \( u \) \( t \) again, we arrive at:

\[ \Gamma(x)\Gamma(1-x) = \int_0^1 \frac{t^{x-1} - t^{-x}}{1+t} \, dt = \pi \frac{\sin \pi x}{\sin(\pi x)}. \]

\[ \square \]

### 8.5 Multiplication Formula

One of the fundamental properties of the \( \Gamma \)-function is so-called *multiplication formula* that reads, in the modern notation

\[ \Gamma \left( \frac{x}{n} \right) \Gamma \left( \frac{x+1}{n} \right) \cdots \Gamma \left( \frac{x+n-1}{n} \right) = \frac{(2\pi)^{n-1}}{n^{x-\frac{1}{2}}} \cdot \Gamma(x). \]

For \( n = 2 \) one obtains the *duplication formula* that is usually ascribed to Legendre [Le26].

The multiplication formula was first proven rigorously by Gauß in his influential paper [Ga28] on the hypergeometric series, in which he also gives a complete account of the factorial function \( \Pi(x) := \Gamma(x+1) = x! \). Gauß cited Euler’s results very often, but apparently he was not aware of the lesser-known paper [E421] of Euler. In that paper Euler presents a formula that is
essentially equivalent, as we will explain now. We need to introduce some notation.

### 8.5.1 The function \( \left( \frac{p}{q} \right) \)

In §3 of [E321] and §44 of [E421], Euler studies properties of the function

\[
\left( \frac{p}{q} \right) := \int_0^1 \frac{x^{p-1}dx}{(1-x^q)^{\frac{1}{n}}}.
\]

In his notation the variable \( n \) is left implicit, and Euler shows the nice symmetry property

\[
\left( \frac{p}{q} \right) = \left( \frac{q}{p} \right).
\]

Of course, by the substitution \( x^n = y \) this function is just the Beta-function in disguise:

\[
\left( \frac{p}{q} \right) = \frac{1}{n} \int_0^1 y^{\frac{p}{n}-1} (1-y)^{\frac{q}{n}-1} = \frac{1}{n} B \left( \frac{p}{n}, \frac{q}{n} \right),
\]

where the Beta-function is defined as

\[
B(x,y) = \int_0^1 t^{x-1} dt (1-t)^{y-1} \quad \text{for Re}(x), \text{Re}(y) > 0.
\]

Euler implicitly assumes \( p \) and \( q \) to be natural numbers, but this restriction is of course not necessary.

As mentioned several times, Euler already knew the relation between Beta-integral and the \( \Gamma \)-function:

\[
B(x,y) = \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)}.
\]

This result is also given in the supplement to [E421].

### 8.5.2 Applying the reflection formula

Euler’s version of the reflection formula for the \( \Gamma \)-function,

\[
\frac{\pi}{\sin \pi x} = \Gamma(x) \Gamma(1-x),
\]
can be found in §43 of [E421] and reads

\[ [\lambda] \cdot [-\lambda] = \frac{\pi \lambda}{\sin \pi \lambda}, \]

where \([\lambda]\) stands for \(\lambda!,\) that is \(\Gamma(1 + \lambda).\)

If one applies the reflection formula for \(x = \frac{i}{n}, i = 1, 2, \ldots, n - 1,\) we obtain

\[
\begin{align*}
\Gamma \left( \frac{1}{n} \right) \Gamma \left( \frac{n - 1}{n} \right) &= \frac{\pi}{\sin \frac{\pi}{n}}, \\
\Gamma \left( \frac{2}{n} \right) \Gamma \left( \frac{n - 2}{n} \right) &= \frac{\pi}{\sin \frac{2\pi}{n}}, \\
\Gamma \left( \frac{3}{n} \right) \Gamma \left( \frac{n - 3}{n} \right) &= \frac{\pi}{\sin \frac{3\pi}{n}}, \\
\vdots &= \ldots \\
\Gamma \left( \frac{n - 1}{n} \right) \Gamma \left( \frac{1}{n} \right) &= \frac{\pi}{\sin \frac{(n-1)\pi}{n}}.
\end{align*}
\]

Multiplying these equations together gives our first auxiliary formula

\[
\prod_{i=1}^{n-1} \Gamma \left( \frac{i}{n} \right)^2 = \frac{\pi^{n-1}}{\prod_{i=1}^{n-1} \sin \left( \frac{i\pi}{n} \right)}.\]

Our second auxiliary formula is

\[
\prod_{i=1}^{n-1} \sin \left( \frac{i\pi}{n} \right) = \frac{n}{2^{n-1}},
\]

which is a nice exercise and which was certainly known to Euler. For example, in §7 of [E562] and in §240 of [E101], he states the more general formula

\[
\sin n\varphi = 2^{n-1} \sin \varphi \sin \left( \frac{\pi}{n} - \varphi \right) \sin \left( \frac{\pi}{n} + \varphi \right) \\
\sin \left( \frac{2\pi}{n} - \varphi \right) \sin \left( \frac{2\pi}{n} + \varphi \right) \cdot \text{etc.}
\]

The product has \(n\) factors in total. If we divide by \(2^{n-1} \sin \varphi,\) use \(\sin \left( \frac{\pi(n-i)}{n} \right) = \sin \left( \frac{i\pi}{n} \right)\) and take the limit \(\varphi \to 0,\) we obtain the second auxiliary formula.

The first and the second auxiliary formula were also given by Gauss in [Ga28].
and are used in his proof of the multiplication formula. Combining them and taking the square root, we obtain the beautiful formula

$$\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right) \cdots \Gamma\left(\frac{n-1}{n}\right) = \sqrt{\frac{(2\pi)^{n-1}}{n}}.$$  

This formula was also found by Euler in §46 of [E816], where he states it in the form

$$\int_0^1 dx \left(\log \frac{1}{x}\right)^{\frac{1}{n}} \int_0^1 dx \left(\log \frac{1}{x}\right)^{\frac{2}{n}} \cdots \int_0^1 dx \left(\log \frac{1}{x}\right)^{\frac{n-1}{n}} = 1 \cdot 2 \cdot 3 \cdots (n-1) \sqrt{\frac{2^{n-1}(n-1)^{n-1}}{n^{2n-1}}}.$$  

### 8.5.3 Euler’s version of the Multiplication Formula

In §53 of [E421] Euler gives the formula

$$\left[\frac{m}{n}\right] = \frac{m}{n} \sqrt{n^{n-m} \cdot 1 \cdot 2 \cdot 3 \cdots (m-1) \left(\frac{1}{m}\right) \left(\frac{2}{m}\right) \left(\frac{3}{m}\right) \cdots \left(\frac{n-1}{m}\right)}.$$  

As before, $[\lambda]$ is Euler’s notation for the factorial of $\lambda$, so that $\left[\frac{m}{n}\right] = \Gamma\left(\frac{m}{n} + 1\right)$. Euler assumes $m$ and $n$ to be natural numbers, but it is easily seen that we can interpolate $1 \cdot 2 \cdot 3 \cdots (m-1)$ by $\Gamma(m)$. Therefore, if we assume $x$ to be real and positive and write $x$ instead of $m$ in the above formula and express it in terms of the Beta-function, Euler’s formula becomes

$$\Gamma\left(\frac{x}{n}\right) = \sqrt{n^{n-x} \Gamma(x) \frac{1}{n^{n-1}} B\left(\frac{1}{n}, \frac{x}{n}\right) B\left(\frac{2}{n}, \frac{x}{n}\right) \cdots B\left(\frac{n-1}{n}, \frac{x}{n}\right)}.$$  

Expressing the Beta-function in terms of the $\Gamma$-function, then after some rearrangement under the $\sqrt{}$-sign we obtain

$$\Gamma\left(\frac{x}{n}\right) = \sqrt{n^{1-x} \Gamma(x) \frac{\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{x+1}{n}\right)} \cdot \frac{\Gamma\left(\frac{2}{n}\right) \Gamma\left(\frac{2}{n}\right)}{\Gamma\left(\frac{x+2}{n}\right)} \cdots \frac{\Gamma\left(\frac{n-1}{n}\right) \Gamma\left(\frac{n-1}{n}\right)}{\Gamma\left(\frac{x+n-1}{n}\right)}.$$  

By bringing all $\Gamma$-functions of fractional argument to the left-hand side, the expression simplifies to

---

62Gauß’s proof follows the same lines as ours concerning the use of the auxiliary formulas
The product on the right-hand side, \( \Gamma\left(\frac{1}{n}\right) \cdots \Gamma\left(\frac{n-1}{n}\right) \), was evaluated in (3) and thus we obtain
\[
\Gamma\left(\frac{x}{n}\right) \Gamma\left(\frac{x+1}{n}\right) \cdots \Gamma\left(\frac{x+n-1}{n}\right) = n^{1-x} \Gamma(x) \sqrt{\left(\frac{2\pi}{n}\right)^{n-1}}.
\]
Thus, we arrived at the multiplication formula.

### 8.5.4 Discussion of Euler’s result

From the above sketch it is apparent that in [E421] Euler had a result that is essentially equivalent to the multiplication formula for the \( \Gamma \)-function. He expressed it in terms of the symbol \( \left(\frac{p}{q}\right) \), which is in modern notation is the Beta-function. One may wonder why Euler did not express his result in terms of the \( \Gamma \)-function itself. Reading his paper it becomes clear that his main motivation was to express the factorial of rational numbers in terms of integrals of algebraic functions, and the formula given by Euler fulfills this purpose. For the same reason he probably did not replace \( 1 \cdot 2 \cdot 3 \cdots (m-1) \) by \( \Gamma(m) \).

Euler also expressed \( \Gamma\left(\frac{p}{q}\right) \) in terms of integrals of algebraic functions in §23 [E19] and §5 of [E122]. That formula reads
\[
\int_{0}^{1} (-\log x)^{\frac{p}{q}} dx = \sqrt{1 \cdot 2 \cdot 3 \cdots p} \left(\frac{2p}{q} + 1\right) \left(\frac{3p}{q} + 1\right) \left(\frac{4p}{q} + 1\right) \cdots \left(\frac{qp}{q} + 1\right)
\]
\[
\times \sqrt{\int_{0}^{1} dx(x - xx)^{\frac{p}{q}} \cdot \int_{0}^{1} dx(x^2 - x^3)^{\frac{p}{q}} \cdot \int_{0}^{1} dx(x^3 - x^4)^{\frac{p}{q}} \cdot \int_{0}^{1} dx(x^4 - x^5)^{\frac{p}{q}} \cdots \int_{0}^{1} dx(x^{q-1} - x^q)^{\frac{p}{q}}.}
\]

Despite the similarity to the first formula expressing \( \Gamma \) via \( B \), that formula is not as general as the multiplication formula.\(^{[63]}\) It appears that Euler was

\(^{[63]}\)We want to mention here that in the foreword of the Opera Omnia, series 1, volume 19, p. LXI A. Krazer and G. Faber claim that these two formulas are equivalent and both are a special case of the multiplication formula. This is incorrect, as it was shown in the preceding sections. The formula given in section 3 does not lead to the multiplication formula, it only interpolates \( \Gamma\left(\frac{p}{q}\right) \) in terms of algebraic integrals.
aware that the proofs he indicated in \textbf{[E421]} were not completely convincing. He expressed that with characteristic honesty in a concluding SCHOLIUM:

\begin{quote}
\textit{Hence infinitely many relations among the integral formulas of the form}

\[
\int \frac{x^{p-1} \, dx}{(1 - x^n) \, x^q} = \left( \frac{p}{q} \right)
\]

\textit{follow, which are even more remarkable, because we were led to them by a completely singular method. And if anyone does not believe them to be true, he or she should consult my observations on these integral formulas\textsuperscript{64} and will then hence easily be convinced of their truth for any case. But even if this consideration provides some confirmation, the relations found here are nevertheless of even greater importance, because a certain structure is noticed in them and they are easily generalized to all classes, whatever number was assumed for the exponent } n, \textit{whereas in the first treatment the calculation for the higher classes becomes continuously more cumbersome and intricate.}
\end{quote}

\textsuperscript{64}Here Euler refers to his paper \textbf{[E321]}.
9 SUMMARY

9.1 OVERVIEW OVER EULER’S RESULTS ON THE GAMMA-FUNCTION

As we have seen, Euler basically already found all common representations of the \( \Gamma \)-function ranging from the integral representation to the product representation, although he never got credit for the latter. The common denominator of all his ways to these representations is an attempt to solve the functional equation \( \Gamma(x + 1) = x\Gamma(x) \). He basically had four different approaches: The moment ansatz, which gave the integral representation, solution by conversion into a differential equation, which led to the Euler-Maclaurin formula and from it to the Stirling formula, difference calculus, which led him to Weierstraß product formula and direct iterative application of the functional equation, which led him to the Gauß product formula.

Furthermore, he discovered several special properties like the reflection formula, the multiplication formula and the relation to the \( B \)-function. Thus, it is safe to say that throughout his career he discovered all basic properties of the factorial.

He could have found some more results that were then later discovered by others. As the foremost example we want to mention the Fourier series expansion of \( \log(\Gamma(x)) \). This is usually attributed to Kummer [Ku47], but an equivalent result was obtained one year earlier by Malmsten [Ma46] (p. 25).\(^6\)

Fourier series were obviously introduced later in Fourier’s monumental treatise on heat. But certain examples appear at seemingly random places in Euler’s work and in [E189] he actually showed that every periodic function has a Fourier expansion. But at that time he did not pay any further attention to it. He gave the formula for the Fourier coefficients then later in [E704] but did not make he connection to his earlier findings and did not realize the importance of those findings as Fourier did later.

But there certainly are also results that were out of his reach. Those concern all results which need the theory of complex functions, like the classification theorems. Those are necessary to see, whether two different expressions for

\(^6\)Malmsten’s paper was published later than Kummer’s, but written earlier in 1846. Both authors clearly obtained their results independently, since the applied techniques are very different.
Γ are actually identical. Euler never felt the need to do so and never even addressed that issue. Indeed, it is quite difficult to show directly that all the different expressions are identical. Another property that Euler rarely talked about is that the Γ-function is a transcendental function and even such values as Γ (\( \frac{1}{2} \)) are transcendental. Although Euler had a notion of what a transcendental number is, he never actually gave an actual definition.

Unfortunately, Euler never pointed out the connections between all his findings concerning the Γ-function and never organized all his results with the exception of [E421]. But that paper mainly focused on the integral representation and the other representations are not discussed. Probably, the closest to an overview article might be the paper [E368], in which he lists almost all of the formulas, we discussed throughout the text. But in that paper he seems to be more interested in the evaluation of other values of the Γ-function than Γ (\( \frac{1}{2} \))\(^66\). It would have been really interesting to see, what Euler had to say about all the connections we pointed out. But despite all this, it was interesting to see, what Euler actually already knew about the Γ-function and how many ideas he actually anticipated and that this is not generally known. I think, we have added some interesting details to the history of the Γ-function, wonderfully told in [Da59]. Therefore, I hope, this article provides a motivation to go through other historical texts and see what treasures they might contain.

\(^66\)His attempts were fruitless at the end.
10 APPENDIX

10.1 THE SOLUTION OF THE BASEL PROBLEM THAT EULER MISSED (BUT COULD HAVE FOUND)

The solution of the Basel problem, i.e. the summation of the series

\[ \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots \]

was Euler’s first great claim to fame. He gave the correct result the first time in \[E41\]. But his result used the sine-product that he could not prove at that time. Therefore, the first complete solution of the problem he gave was in \[E61\]. In his career he gave at least different proofs of this formula. A paper on this is planned for the future. Here, we want to show, how Euler could have given another proof, i.e. we give a proof using only Euler’s results.

We showed that Euler had an explicit solution of the difference equation

\[ f(x+1) - f(x) = X(x). \]

But the same equation, for integer \( x \), is also solved by

\[ f(x) = \sum_{n=1}^{x} X(n-1). \]

To solve the Basel problem let us consider the special case \( X = x^2 \) or the equation

\[ f(x+1) - f(x) = x^2. \]

For integer \( x \) the solution is

\[ f(x) = \sum_{n=1}^{x} (n-1)^2 = \sum_{n=1}^{x-1} n^2 \]

But applying the Euler-MacLaurin summation formula this sum is easily found to be

\[ f(x) = \frac{1}{6}x(x-1)(2x-1) = \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{6}. \]

But this solution is seen to satisfy the propounded difference equation for all \( x \)! (The general solution is obtained by adding an arbitrary periodic function
to that solution).

But now let us also use the explicit solution of the differential equation. We find

\[ f(x) = \int t \, dt - \frac{x^2}{2} + \sum_{k \in \mathbb{Z}} e^{2\pi i x} \int e^{-2\pi i t} \, dt \]

\[ = \frac{x^3}{3} + C - \frac{x^2}{2} + \frac{x}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k \in \mathbb{Z}} C_k e^{2\pi i x} \]

\[ = \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} + A(x) \]

The ′ again indicates that the term for \( k = 0 \) is left out in the summation. The \( C_k \) are integration constants. \( A \) is just being an arbitrary periodic function with period one, i.e., a solution of the homogeneous difference equation \( f(x + 1) - f(x) = 0 \).

Above we already found the general solution of the difference equation. Since the solution of a difference equation must be unique (if the solution of the homogeneous equation is subtracted), comparing the coefficients of \( x \) and the last equations we find

\[ \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{6} \]

And this is the Basel sum again.

Although the way in which we obtained this result was rather non-straightforward, it nevertheless only used only results that Euler also obtained, whence we justly claim that he also could have given that solution.

Anyhow, this solution seems to be new, since it does not appear in [CH03].

### 10.2 Euler and the Partial Fraction Decomposition of Transcendental Functions

Euler devoted the whole paper [E592] to the expansion of transcendental functions into partial fractions. Although, his method is not correct in general, all formulas he states in his paper, are indeed right. The flaw in his method can easily be fixed. We will explain his method in the example of
where it works, but then will also give an example in which his method fails and will explain why.

10.2.1 Example: $\frac{\pi}{\sin \pi x}$

In [E592] Euler wrote down the formula

$$\pi \frac{2\lambda}{2\lambda \sin \lambda \pi} - \frac{1}{2\lambda^2} = \frac{1}{1 - \lambda^2} - \frac{1}{4 - \lambda^2} + \frac{1}{9 - \lambda^2} - \frac{1}{16 - \lambda^2} + \text{etc.}$$

To find the partial fraction decomposition of $\frac{1}{\sin \phi}$ and other similar transcendental functions, Euler argued as if they were rational functions. For those he outlined the procedure in [E101], [E162], [E163] and an improved version in the last chapter of [E212]. Sandifer wrote a nice paper on this method in [Sa07].

The first step is, as usual, to find the zeros of the denominator, i.e., $\sin \phi$. They are, as it has now been demonstrated, $\phi_i = i\pi$ with $i \in \mathbb{Z}$. Furthermore, all of those zeros are simple.

Hence Euler made the ansatz

$$\frac{1}{\sin \phi} = \frac{A_i}{\phi - i\pi} + R_i(\phi)$$

$A_i$ being a constant to be determined, $R_i(\phi)$ being a function that does not contain the factor $\phi - i\pi$ or any positive or negative powers of it. To determine the constant $A_i$, he wrote

$$A_i = \frac{\phi - i\pi}{\sin \phi} - R_i(\phi)(\phi - i\pi).$$

Since $R_i(\phi)$ does not contain $\phi - i\pi$ or any power of it, we have

$$A_i = \lim_{\phi \to i\pi} \frac{\phi - i\pi}{\sin \phi} = \lim_{\phi \to i\pi} \frac{1}{\cos \phi} = (-1)^i,$$

where l’Hospital’s rule was used in the second step.

Since this method works for all zeros Euler then concluded

$$\frac{1}{\sin \phi} = \frac{1}{\phi} + \frac{1}{\phi - \pi} - \frac{1}{\phi + \pi} + \frac{1}{\phi - 2\pi} + \frac{1}{\phi + 2\pi} - \frac{1}{\phi - 3\pi} - \frac{1}{\phi + 3\pi} + \cdots$$

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This follows from the integral representation we used above to prove the sine-product and can vice versa be used to show that the integral formula is correct. Although this formula turns out to be right, Euler’s reasoning is not quite correct. We will elaborate on this in the next section. For now let us simplify the result a bit.

Adding each two terms we find

\[
\frac{1}{\sin \varphi} - \frac{1}{\varphi} = \frac{2\varphi}{\pi^2 - \varphi^2} - \frac{2\varphi}{4\pi^2 - \varphi^2} + \frac{2\varphi}{9\pi^2 - \varphi^2} - \frac{2\varphi}{16\pi^2 - \varphi^2} + \cdots
\]

Diving by \(2\varphi\) and then setting \(\varphi = \lambda \pi\) we arrive at

\[
\frac{\pi}{\sin \lambda \pi} - \frac{1}{2\lambda^2} = \frac{1}{1 - \lambda^2} - \frac{1}{4 - \lambda^2} + \frac{1}{9 - \lambda^2} - \frac{1}{16 - \lambda^2} + \cdots
\]

This is the claimed formula and hence completes the proof.

### 10.2.2 Example in which Euler's method fails - \(\frac{1}{e^z - 1}\)

We already mentioned that Euler’s method to find the partial fraction decomposition of a transcendental function does not work in general. We want to illustrate this with an example that we actually already needed above.

Let us try to find the partial fraction decomposition of \((e^z - 1)^{-1}\).

The first step is again to find all the zeros of the denominator. They are \(z_k = 2k\pi i\) with an integer number \(k\).

Using Euler’s ansatz let us set

\[
\frac{1}{e^z - 1} = \frac{A_k}{z - 2k\pi i} + R_k(z)
\]

where \(A_k\) is a constant we want to determined and \(R_k\) is a function not containing \(z - 2k\pi i\) or any powers of it. Proceeding as above we find \(A_k = 1\).

Therefore, we can write

\[
\frac{1}{e^z - 1} = \sum_{k \in \mathbb{Z}} \frac{1}{z - 2k\pi i} + R(z)
\]

and we have to determine \(R(z)\). Euler simply would assumed \(R(z)\) to be zero in \([E592]\); this is not true in general as we will see soon. Consider
\[ R(z) = \frac{1}{e^z - 1} - \sum_{k \in \mathbb{Z}} \frac{1}{z - 2k\pi i}. \]

From this we find that \( R(z) \) is a bounded holomorphic function and hence constant by Liouville’s theorem. To find the value of \( R \) consider the Laurent series of \( (e^z - 1)^{-1} \) about \( z = 0 \). This series turns out to be

\[
\frac{1}{e^z - 1} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,
\]

where \( B_n \) are the Bernoulli numbers. Euler essentially found this generating function for the Bernoulli number already in \[E25\], but especially pointed it out in his studies concerning the Euler-Maclaurin summation formula, see, e.g., \[E47\], \[E55\] and in his book \[E212\].

Using this series we have

\[
R = \frac{B_0}{z} + B_1 + \frac{1}{z} \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n - \frac{1}{z} - \sum_{k \in \mathbb{Z}} \frac{1}{z - 2k\pi i}.
\]

The prime indicates that \( k = 0 \) is omitted in the summation. But from the definition of the Bernoulli numbers one easily finds \( B_0 = 1 \) and \( B_1 = -\frac{1}{2} \). Hence putting \( z = 0 \) in the equation for \( R \) we obtain

\[
R(z) = R(0) = R = B_1 = -\frac{1}{2}.
\]

Therefore, we obtain the partial fraction decomposition

\[
\frac{1}{e^z - 1} = -\frac{1}{2} + \sum_{k \in \mathbb{Z}} \frac{1}{z - 2k\pi i}.
\]

As we saw above, the extra term of \(-\frac{1}{2}\) turns out to be of major importance in the derivation of the solution of the simple difference equation.
10.3 $\gamma$ meets $\Gamma$ - Euler on the Euler-Mascheroni constant

We needed the Euler-Mascheroni constant $\gamma$ in the derivation of the Weierstrass product formula for $\Gamma(x)$, whence we want to take the opportunity, to briefly look at Euler’s contribution to the constant. The $\gamma$ constant appears frequently at various places in mathematics and appears when you least expect it. Therefore, it is natural that it also appears at various places in Euler’s works. But we want to focus mainly on three papers [E43], the first occurrence of $\gamma$, [E583], a paper devoted to $\gamma$, and [E629] considering one singular expression for $\gamma$ in more detail.

10.3.1 Euler’s discovery

Euler discovered the constant nowadays called $\gamma$ in [E43]. That paper is also interesting for another reason. Euler states the modern convergence criterion for the convergence of an infinite series in §2; he writes:

A series, which continued to infinity has a finite sum, even though it is continued twice as far, will never gain any increment, but that what is added after infinity, will actually be infinitely small. For, if it would not be like this, the sum, even though it is continued to infinity, would not be defined and hence not finite. Hence it follows, if that, what results beyond the infinitesimal term, is of finite magnitude, that the sum of the series is necessarily finite.

This is essentially Cauchy’s definition for the convergence of a series. But let us discuss, how Euler discovered $\gamma$. He encounters it the first time in §6, where he notes that if

$$s := \sum_{n=1}^{i} \frac{c}{a + (n-1)b}$$

and $i$ is a very large number, that

$$\frac{ds}{di} = \frac{c}{a + bi} \quad \text{and hence} \quad s = C + \frac{c}{b} \log(a + ib).$$

Mascheroni’s name was added to the constant, since he in his elaborations to Euler’s book on integral calculus found a new formula for $\gamma$ and used it to calculate several digits of it. Additionally, he found several other functions, in which Taylor series expansion it appears. Mascheroni’s ideas are reprinted in the Opera Omnia version of Euler’s book [E366].

For an interesting historical account, confer also Sandifer’s article in [Sa15].
The constant $C$ will turn out to be $\gamma$ in the case $a = b = c = 1$. By this reasoning, Euler found that

$$\lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \right)$$

is a finite number, although both the harmonic series and the logarithm become infinite for the same limit. Euler was certainly intrigued by this and tried to find $\gamma$. For this he first noted that:

$$\log \left( \frac{2}{1} \right) + \log \left( \frac{3}{2} \right) + \cdots + \log \left( \frac{n+1}{n} \right) = \log(n+1).$$

Next, he notes the series expansion:

$$\log \left( \frac{n+1}{n} \right) = \log \left( 1 + \frac{1}{n} \right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \text{etc.}$$

where this is just the Taylor series expansion of $\log(1 + x)$ about $x = 0$ applied for $x = \frac{1}{n}$. Since $n \in \mathbb{N}$, the convergence condition $|x| \leq 1$ is met. Therefore, we have

$$1 = \log 2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \text{etc.}$$

$$\frac{1}{2} = \log \left( \frac{3}{2} \right) + \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} - \text{etc.}$$

$$\frac{1}{3} = \log \left( \frac{4}{3} \right) + \frac{1}{2 \cdot 3^2} - \frac{1}{3 \cdot 3^3} + \frac{1}{4 \cdot 3^4} - \text{etc.}$$

$$\frac{1}{4} = \log \left( \frac{5}{4} \right) + \frac{1}{2 \cdot 4^2} - \frac{1}{3 \cdot 4^3} + \frac{1}{4 \cdot 4^4} - \text{etc.}$$

$$\vdots$$

$$\frac{1}{n} = \log \left( \frac{n+1}{n} \right) + \frac{1}{2 \cdot n^2} - \frac{1}{3 \cdot n^3} + \frac{1}{4 \cdot n^4} - \text{etc.}$$

Therefore, adding the columns, we arrive at:
\[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \log(n+1) + \frac{1}{2} \left( \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{etc.} \right) \]
\[ - \frac{1}{3} \left( \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \text{etc.} \right) \]
\[ - \frac{1}{4} \left( \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.} \right) \]

etc.

Therefore, if we call \( \zeta(m) = \sum_{n=1}^{\infty} \frac{1}{n^m} \), we conclude:

\[ \gamma = \sum_{m=2}^{\infty} \frac{(-1)^m}{m} \zeta(m). \]

By the Leibniz criterion this series is seen to converge and hence Euler proved that \( \gamma \) indeed exists and is finite.

### 10.3.2 Euler’s formulas for \( \gamma \)

As mentioned above \[E583\] is solely devoted to the determination of alternate expressions for \( \gamma \). We do not want to prove them here, but want to mention some of the results and ideas. Probably the most interesting formula is:

\[ 1 - \gamma = \sum_{m=2}^{\infty} \frac{1}{m} (\zeta(m) - 1) \]

Euler lists all other formulas, involving series, at the end.

He also gives an integral formula\[E69\], namely

\[ \gamma = \int_{0}^{1} \left( \frac{1}{1-z} + \frac{1}{\log z} \right) \, dz. \]

He found it in true Eulerian fashion, so that we will explain how he found it. First, he notes that \[E69\] is completely devoted to this one formula

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\[
\lim_{x \to 1} \log \frac{1 - x^n}{1 - x} = n
\]

and

\[
\log (1 - x^n) = -n \int \frac{x^{n-1} dx}{1 - x^n}
\]

in particular

\[
\log (1 - x) = - \int \frac{dx}{1 - x}.
\]

Moreover,

\[
\int_0^1 \frac{(1 - x^n)}{1 - x} dx = 1 + \frac{1}{2} + \cdots + \frac{1}{n}
\]

which is follows by termwise integration of the geometric sum

\[
\frac{1 - x^n}{1 - x} = 1 + x + x^2 + \cdots + x^{n-1}.
\]

Therefore, we have:

\[
\gamma = \lim_{n \to \infty} \left( - \int_0^1 \frac{(1 - x^n) dx}{1 - x} + n \int_0^1 \frac{x^{n-1} dx}{1 - x^n} - \int_0^1 \frac{dx}{1 - x} \right),
\]

or

\[
\gamma = \lim_{n \to \infty} \left( - \int_0^1 \frac{x^n dx}{1 - x} + n \int_0^1 \frac{x^{n-1} dx}{1 - x^n} \right).
\]

Next, Euler puts \( x^n = z \) to arrive at

\[
\gamma = \lim_{n \to \infty} \left( - \frac{1}{n} \int_0^1 \frac{z^{\frac{1}{n}} dz}{1 - z^n} + \int_0^1 \frac{dz}{1 - z} \right).
\]

Note the known limit

\[
\log(z) = \lim_{n \to} n \left( z^{\frac{1}{n}} - 1 \right).
\]
Pull the limit inside the integral and use the above limit and $z^{\frac{1}{n}} \to 0$ for $n \to \infty$ and we finally arrive at

$$\gamma = \int_0^1 \left( \frac{1}{1-z} + \frac{1}{\log z} \right) \, dz.$$ 

Therefore, we arrived at the formula, which is subject of [E629]. But the formulas Euler presents in that paper are not as useful as those in [E583]. So we will not discuss this here.

Finally, we want to add Euler’s conjecture that $\gamma$ is a logarithm of an important number (confer §2 of [E583]). Unfortunately, Euler did not further explain what important means in this context. In general, not much is known about the Euler-Mascheroni constant. Not even whether it is irrational or not. See, e.g., [Ha17] for an entertaining account on the history and current state of the art of Euler’s constant $\gamma$. 
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