Abrikosov Vortex and Branes

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Abstract

In this contribution in honor of A.P. Balchandran we give a brief description of application of topological solutions in field theory, a subject which has been central to many of Balachandran's important contributions to physics.

1 Introduction

Beautiful mathematics and its application in physics is one of the main characteristics of Balachandran's work. The application of topological ideas in physics permeates throughout most of Bal's papers. It is a pleasure to read them and I personally have always found them enlightening. A collection of some of these can be found in the monograph. [1]

Topological configurations have played important role in our understanding of condensed matter as well as particle physics. In high energy physics they appear as finite energy or finite (Euclidean-) action solutions of non linear classical field equations while in string theory they are p-brane configurations with non zero charges defined by space integrals of differential forms of appropriate ranks. [2], [3] Magnetic systems are understood in terms of domain formation, phase transitions in two dimensional systems with continuous symmetries are explained in terms of formation of vortices. [4] In particle physics our only understanding of the famous $U(1)$ problem is in terms of the instanton solution. [5] Formation of vortices, domain walls and magnetic monopoles have relevance in cosmology. [6] Also one of the ideas which recently acquired considerable interest is the possibility that our 4-dimensional space time may be a topological defect of a 3-brane type extending in a higher dimensional space time. [7] Unlike the classical Kaluza Klein theory where one assumed that the extra dimensions should be small and cover a compact
manifold in this paradigm the extra dimensions can be large and non compact. [8], [9] This freedom allows a reformulation of one of the basic questions of fundamental physics, namely, why the planck mass is so much larger than the electroweak symmetry breaking scale. [10]. In this contribution we shall give a very brief review of some of these ideas. In section 2 we start from the Kosterliz-Thouless description of phase transitions in two dimensional systems with continuous symmetries in terms of vortex formations and go on to the vortex solutions in 3 and higher dimensions. In section 3 we discuss very briefly the formation of walls, and fermion localization on them. In section 4 we mention some examples in string theory.

2 Vortices in 1+1 and higher dimensions

2.1 1+1 dimensions

Consider a field $\theta(x)$ in 1 + 1 dimensions which takes its values in a circle. The Hamiltonian is

$$H = J/2 \int d^2x \nabla_i \theta \nabla_i \theta$$ (1)

The non triviality of this system is related to the fact that $\theta(x)$ ia an angle. Shifting $\theta(x)$ by an $x-$ independent angle does not change $H$. Thus there is a continuous $U(1)$ symmetry in this problem. It is not hard to show that this symmetry can not be broken spontaneously in 1+1 dimensions. The low temperature fluctuations are spin waves with power law correlations. Therefore there can not be a phase transitions which is triggered by spontaneous symmetry breaking. There is a continuous line of critical points with temperature dependent exponent. It was shown by Kosterlitz and Thouless that actually a sharply defined transition temperature does exist and the phase transition proceeds by the formation of vortices. [4] These configurations are defined by those $\theta(x)$ for which $\nabla \theta(x)$ are multiple valued, viz. $\int_C dl \nabla \theta = 2n\pi n$, where $C$ is a contour which encircles the origin and $n$ is any integer. Away from the origin one can write (we take n=1 case)$\nabla_i \theta(x) = (0, 1/r)$. The energy of this configuration will be $E = \pi J ln(L/a)$. Here $a$ is a short distance cut off, the lattice spacing in spin systems to which these considerations are normally applied, and $L$ is the size of the system. Since the center of
the vortex can be located at any one of the \((L/a)^2\) points on the lattice. The entropy for the creation of a vortex will be \(S = \ln(L/a)^2\). The free energy associated with the creation of a vortex can easily be obtained to be \(F = E - TS = (\pi J - 2T)\ln(L/a)\). There is thus a critical temperature \(T_c = \pi J/2\) above which the quasi order of the spin waves are unstable against formation of single vortices. Below \(T_c\) we have only vortex-antivortex pair. As \(T\) increases the size of vortex-antivortex pair increases and above \(T_c\) the pair becomes unstable. This destabilize the quasi order of the low \(T\) spin waves and makes the correlations to decay exponentially. This is the KT description of phase transitions in two dimensional spin systems with continuous symmetries. Note that the formation of vortices in this system is entirely due to topology, namely, the fact that \(\theta\) takes its values on a \(S^1\) and that \(\pi_1(S^1) = \mathbb{Z}\).

### 2.2 Higher than 2-dimensions

For the application in particle physics and cosmology the interesting case arises when we couple gauge fields to scalars in some representation of the gauge group. The scalars are assumed to have a potential which allows spontaneous breaking of the gauge symmetry \(G\) to one of its subgroups \(H\). It is the topology of the vacuum manifold \(G/H\) which plays the key role in ensuring the existence of topologically non-trivial configurations. The simplest case is to have a U(1) gauge theory coupled to a complex scalar field \(\Phi\). In 3 space dimensions, provided that the charge of \(\Phi\) is twice that of the electron, this will give the Ginzburg-Landau description of the BCS theory of superconductivity. The spontaneous breaking of the \(U(1)\) symmetry gives rise to a finite penetration depth. This so called Meissner effect can also be seen in 2 + 1 dimensional gauge theories of Chern Simons type without a need for spontaneous symmetry breaking. However, unlike the Higgs induced order which disappears at some finite temperature, the CS induced Meissner effect does not seem to vanish at any \(T_c\). The penetration depth remains finite at all temperature, at least in perturbation theory. [12] Abrikozov demonstrated the formation of vortices in the LG model and discussed their physical relevance for type II superconductivity. [13] The same system was then studied by Nielsen and Olesen in the context of 3 + 1 dimensional scalar electrodynamics. [14] Here the vortices will appear as static solutions. Their formation was thought to be relevant for quark confinement. Another very
interesting consequence of this type of solution is the localization of fermionic zero modes to the core of the vortex. The fact that this does indeed happen was shown in Ref. [20]. We shall come back to this issue in the following section and discuss it in some detail in 6 dimensions. If the localized fermions are electrically charged they can lead to the formation of superconduction cosmic strings. [15]

One can proceed to higher than 4 dimensions and apply these ideas to brane world scenarios. The simplest model for our world as a brane extending in a higher dimensional space time is perhaps the one constructed by Rubakov and Shaposhnikov. [7] In the absence of gravity this model consists of a scalar field with an appropriate self coupling to allow for the spontaneous breaking of a $Z_2$ symmetry. Denote the fifth coordinate by $r$ and the first four coordinates by $x^\mu$. The $r$-dependent (and $x^\mu$ independent) kink solution for the scalar field will be seen as a 3-brane localized along the $r$ axis. Its thickness will be given by the thickness of the kink, which in turn is determined by the Compton wavelength of the scalar particle. The interesting result of Rubakov and Shaposhnikov was to show that if we study the solutions of the 5-dimensional Dirac equation in the kink background we shall find two, 4-dimensional chiral configurations. Only one of them has a normalizable kinetic energy and is localized to the wall. This result should be contrasted with the Kaluza Klein ideas according to which the extra dimensions are assumed to be compact contrary to the non compact $r$ coordinate. Also to obtain chiral fermions one needed to start from an even dimensional theory with fundamental gauge fields coupled to chiral fermions in the higher dimensional space time. [16] [17] If we couple a charged scalar to the bosonic part of the same 6-dimensional system we can search for vortex solution. Such a solution has indeed been found and has been interpreted as a 4-dimensional universe sitting at the core of the vortex. [18] In order for the 4-dimensional physics to reside at the core one needs to find physical mechanisms which localize the higher-dimensional fields to the 3-dimensional brane residing at the core and whose world volume is our 4-dimensional space-time. In general the localization of fermions is relatively straightforward in the presence of warped geometry and non trivial Yang-Mills backgrounds. [19] In the specific vortex background the relevant equations can be reduced to a form which agrees with the familiar case of $3 + 1$ dimensional problem near the core of the vortex. [20] Far from the core they are different. [21] It turns out that the gauge field fluctuations also do localize. However, there is no mass gap
in the spectrum of the localized gauge fields to separate them from the bulk excitations. The fermion spectrum on the other hand does have a mass gap. In the absence of such a gap for the gauge fields it is unclear how one can neglect the bulk modes and consider only the localized modes. Many aspects of the localization problem recently has been a subject of an intense study. [22] Allowing for non compact internal dimensions raised some early hopes for a solution of the cosmological constant problem. [8] [9] As mentioned before, recently the idea of a non compact extra dimension raised the hope of reformulating the well known hierarchy problem. [10] Another reformulation of the same problem is in the context of a large but compact extra dimension [11], although the idea of a large compact extra dimensions is not new.

2.3 Vortices in 6-dimensions

Now we go a little more into the detail of the 6-dimensional vortex solution. We are interested in a warped geometry for the space-time and vortex configuration for the gauge-scalar system. The background geometry is defined by the metric,

$$ds^2 = e^{A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + e^{B(r)}a^2d\theta^2$$

where $\mu, \nu = 0, 1, 2, 3$ and $a$ is the radius of $S^1$ covered by the $\theta$ coordinate. The ansatz for the gauge field $A_M$ and the scalar $\Phi$ will be a Nielsen-Olesen vortex solution. $\Phi = f(r)e^{im\theta}$ and $aeA = (\varepsilon(r) - n)d\theta$. The functions $f(r)$ and $\varepsilon(r)$ satisfy the following boundary conditions

$$f(0) = 0, \quad f(\infty) = f_0$$
$$\varepsilon(0) = n, \quad \varepsilon(\infty) = 0$$ \hspace{1cm} (2.3)

In Ref. [18] it was shown that the above ansatz solves the bosonic background equations. There are solutions with different boundary conditions for the metrical fields $A$ and $B$. The boundary conditions of interest for us which localize fields of spin $0, 1/2$ and $1$ are, as $r \to 0$, given by $A(r) \to 0$ and $B(r) \to 2 \ln(r/a)$, while as $r \to \infty$ we should demand $A(r) = B(r) \to -2\varepsilon r$, where $\varepsilon$ is a positive constant.

As has been explained in Ref. [21] in order to find the correct anzatz for the $U(1)$ gauge field we need to identify the unbroken symmetry group which
leaves the background geometry as well as the vortex configuration invariant. This leads us to the identification

\[ V_i^{(1)} = \frac{1}{ae} P(r) W_i(x), \quad \text{and} \quad h_{i\theta} = e^B W_i(x) \quad (2.6) \]

where \( W_i \) is a function of \( x^\mu \) only, and \( i=1,2 \) are the transverse directions in \( D = 4 \). It is the gauge field of the unbroken \( U(1) \). This anstaz leads to the following kinetic energy for \( W_i \) fields [21]

\[ S(W_\mu) = \frac{1}{2ae^2} \int d^6 x \ e^{\frac{1}{2} B} \left( P^2(r) + \frac{a^2 e^2}{\kappa^2} e^B \right) \partial_\nu W_i \partial^\nu W_i \quad (2.7) \]

The \( r \)-integral is converging quite rapidly and we obtain a normalizable effective action for \( W_i \).

Note that in the above formula we have the bilinear part of the Maxwell action in the light cone gauge. For that reason only the transverse components of the gauge field appear in the action. In the non-covariant gauge used in this calculation the light cone components \( V_- \) of the gauge field fluctuations and \( h_{-M} \) of the gravitational fluctuations are set to zero. The equations of motions for the remaining longitudinal components \( V_+ \) and \( h_{+M} \) then are determined in terms of the transverse components \( V_i \) and \( h_{ij} \), etc. In this way the redundant components disappear completely from the formalism. This simplifies the spectrum analysis of the massless modes. [21]

### 3 Fermions

Fermion localization on defects has many interesting physical implications. Here we consider three examples. The first is basically a two dimensional phenomena which results from the fact that the 1 + 3 dimensional Dirac equation in the background of a Nielsen-Olesen string has localized zero modes. [20] These modes are massless and propagate with the speed of light along the string. It was shown by Witten that, if charged, they can lead to superconducting cosmic strings. [15] Their presence in grand unifying theories have significant cosmological implications.

The second example concerns the existence of fermion zero modes in \( D=6 \) in a vortex background and warped geometry. Their presence leads to \( D=4 \) QED near the core of the vortex. [21]. In the absence of gravity the only
way to find normalizable solutions of Dirac equation in the background of a vortex in $1+3$ dimensions is to introduce a Yukawa coupling as it was originally done in Re.[20]. In the presence of gravity and warped geometry, however, there exist normalizable zero mode solutions even in the absence of Yukawa couplings. For the consistency of the starting theory one needs to make the six dimensional theory free from chiral anomalies. This can be done. In fact there exist anomaly free supergravity models in $D = 6$ with ungauged [26] or gauged [27] R-symmetries. Furthermore it has been shown that the spectrum of localized fermions is separated by a mass gap from the bulk fermions. Thus unlike the vector bosons they have a very clean physical interpretation in the context of brane world scenarios.

To study the problem of fermion mass gap we need only to consider the large $r$ limit, for which, the Yukawa and Majorana mass terms in the fermionic part of the action can be neglected, and the equations for bulk modes read (we consider only $4_+$ six-dimensional fermion, the case of $4_-$ is treated in full analogy):

$$
e^{+i\theta} \left( \partial_r + \frac{i}{a} e^{-B/2} \partial_\theta \right) \psi^L = m_f e^{-\frac{1}{2}A - \frac{1}{2}f^r} e^{-\frac{2}{e}A} \bar{e}_L A_\theta d\rho \psi^R , \quad (2.7)$$

$$e^{-i\theta} \left( \partial_r - \frac{i}{a} e^{-B/2} \partial_\theta \right) \psi^R = -m_f e^{-\frac{1}{2}A + \frac{1}{2}f^r} e^{-\frac{2}{e}A} \bar{e}_L A_\theta d\rho \psi^L .$$

The ansatz $\psi^L = e^{im\theta} \psi^m_L$, $\psi^R = e^{i(m+1)\theta} \psi^m_L$ removes the angular dependence and leads, for large $r$, to the following asymptotic solutions

$$\psi_L \rightarrow \frac{1}{\sqrt{z}} \exp \left( -\frac{1 - 2ne_L/e}{2am_f} \right) (C_1 \sin(Sz) + C_2 \cos(Sz)) , \quad (2.7)$$

where $z = \frac{me}{e} e^{cr}$ and $S^2 = 1 - \frac{(m - ne_L/e + \frac{1}{2})^2}{(am_f)^2}$. From this we immediately see that the charged bulk fermions are massive, $m_f > \frac{1}{a}|m - \frac{ne_L}{e} + \frac{1}{2}|$, since the combination of $m$ and $n$ in $S^2$ is exactly the electric charge. There are of course also charged localized fermions which are massless.

For our third example we mention only in passing how fermion localization on a domain wall has been employed to give a non perturbative definition of chiral gauge theories, i.e a way of defining chiral fermions on the lattice without running into the doubling problem.
Let us consider a 5-dimensional space-time. Assume that there is a 4-dimensional wall sitting at some point on the time axis, as opposed to the spatial r-axis discussed before. This system has a time dependent mass term in the Hamiltonian. The time dependence is simply given by \( \text{sign}(t) \), which is +1 for positive \( t \) and −1 for negative \( t \). There are a finite number of chiral fermion modes localized to the wall. Kaplan made the interesting suggestion that this fact may be used to solve the long standing problem of defining Euclidean lattice chiral gauge theories by identifying the wall with the 4-dimensional Euclidean space. [23] Narayanan and Neuberger removed the shortcomings of this idea and turned it into a formalism for a non perturbative definition of chiral gauge theories. They gave a formula for the chiral Dirac operator as the overlap between two vectors in the Hilbert space of this set up. [24] These vectors are denoted by \( |A-\rangle \) and \( |A+\rangle \) in the notation of Re [25], where \( A \) refers to the gauge field. The chiral Dirac determinant is then identified with their inner product and chiral anomalies have their root in the phase of this complex functional. Several aspects of this prescription has been examined in detail. In particular it has been shown that the formalism correctly reproduces all one loop amplitudes for slowly varying external gauge fields, including chiral gauge and gravitational anomalies [25].

4 String Theory

Superstring theory allows for phenomena which may not exist or not easy to observe in field theory. For example it is possible that the topology of a Calabi-Yau manifold undergoes a change without creating a singularity in space time physics. This happens by passing through a conifold point in the moduli space of CY manifolds. [28]Starting with a smooth CY with Euler number \( \chi \) one can shrink \( N \) \( S^3 \)'s to single points, thereby obtaining a singular manifold with the Euler number of \( \chi + N \) sitting at a conifold singularity in the moduli space. In the next step one replaces each singular point with a \( S^2 \) to obtain a smooth CY with an Euler number of \( \chi + 2N \).

A beautiful interpretation of this conifold transition is as follows: Consider a \( D_3 \) brane wrapped around a \( S^3 \) in compactification space of the type IIB string on some \( CY_3 \) three folds. The low energy dynamics of the moduli fields is described by a \( \sigma \) model targeted on \( CY_3 \). These fields in their time evolution encounter a conifold singularity which corresponds to vanishing
$S^3$’s. The vanishing volume of the wrapped $D_3$ brane in its turn gives rise to the emergence of the massless states in the low energy spectrum. Thus physically, the appearance of singularity is equivalent to the emergence of new massless multiplets in the 4-dimensional spectrum. [29]

Conifolds enter many other discussions in string theory. One very interesting case is the demonstration by Witten that the free energy of a $U(N)$ Chern-Simons theory on $S^3$ equals the free energy of topological open strings targeted on the cotangent bundle of $S^3$, which is a deformed conifold. [30] The free energy of the CS theory can be formally expanded as $E = \sum_{g=0, h=1} C_{g,h} N^{2g-2} \lambda^{2g-2+h}$, where $\lambda = \frac{k}{N+k}$. According to Witten each term in the above sum can be interpreted as the free energy of an open string diagram on a world sheet of genus $g$ and $h$ boundaries. Assume we perform the sum over $h$. The result should be an expression of the form $E = \sum_{g=0}^{\infty} N^{2g-2} F(\lambda)$ which looks like the topological sum over all closed string world sheet, each term representing the contribution of a genus $g$ surface. The string coupling constant $g_s$ is clearly proportional to $1/N$ and vanishes as $N$ goes to infinity. In fact one can calculate $F_g$ for all values of $g$ and show that each $F_g$ is identical to the free energy of a closed string targeted on a small resolution of a CY$_3$. Thus it seems that summing over the boundaries of the world sheet generates a transition from a $S^3$ deformed conifold through a singular conifold to a small resolution. [31]

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