Quantum Equivalence of $f(R)$ Gravity and Scalar-tensor Theories in the Jordan and Einstein Frames

Nobuyoshi Ohta*

Department of Physics, Kindai University, Higashi-Osaka, Osaka 577-8502, Japan
and
Maskawa Institute for Science and Culture, Kyoto Sangyo University, Kyoto 603-8555, Japan

Abstract

The $f(R)$ gravity and scalar-tensor theory are known to be equivalent at the classical level. We study if this equivalence is valid at the quantum level. There are two descriptions of the scalar-tensor theory in the Jordan and Einstein frames. It is shown that these three formulations of the theories give the same determinant or effective action on shell, and thus they are equivalent at the quantum one-loop level on shell in arbitrary dimensions. We also compute the one-loop divergence in $f(R)$ gravity on an Einstein space.

*e-mail address: ohtan@phys.kindai.ac.jp
1 Introduction

It is always of great interest to consider various modifications of Einstein gravity for phenomenological applications. Among others, what is called $f(R)$ gravity attracts much attention especially in the context of inflationary scenario in cosmology. For early attempts, see [1]-[3] and [4, 5] for reviews. This class of theories has a nice feature that even though the theory involves higher derivatives, there is no ghost introduced. In addition to the massless spin two graviton, the theory involves an additional scalar degree of freedom. The simplest way to see this is to rewrite the theory using scalar field coupled to the Einstein theory, leading to scalar-tensor theory. It has been known for long time that the equivalence is valid at the classical level on shell (see, for example [6, 7, 4]), but it has not been much discussed at the quantum level.

One possible way to understand quantum properties of gravity is the asymptotic safety scenario [8]-[12]. The idea is that the theory of quantum gravity is searched for within a large class of theories, and one single theory is chosen by the condition that it corresponds to a fixed point of the renormalization group flow. The use of functional renormalization group equation has given a considerable evidence in support of the existence of a nontrivial fixed point. The asymptotic safety scenario is discussed for $f(R)$ gravity in [13]-[23], and for scalar-tensor theory in [24]-[29].

In fact, Benedetti and Guarnieri considered the problem of the equivalence by using the functional renormalization group approach [30]. They rewrite the $f(R)$ gravity in the scalar-tensor theory in the form without kinetic term for the scalar, and then introduce kinetic term for the scalar with constant coefficient $\omega$. After deriving functional renormalization group equation in Feynmann and Landau gauges, they try to find fixed points in these gauges, in particular for $\omega = 0$ case. They found that the results disagree with each other, and argued that this is an evidence against the equivalence to $f(R)$ theory. However this might be just a gauge artifact.

While our work is in progress, a paper appeared in which one-loop divergence in $f(R)$ gravity on arbitrary background has been computed in a specific gauge [31]. Based on this result, it is argued that $f(R)$ gravity and classically equivalent scalar-tensor theory are also equivalent on shell at the quantum level [32]. It would be interesting to further check the equivalence at the level of effective potential and/or functional renormalization group satisfied by the effective average action. As the equivalence holds on shell classically, it is expected that the equivalence only holds on shell also at the quantum level. It is a general belief that the effective potential is gauge independent on shell [33], so it is expected that the equivalence holds on shell in any gauges. We also expect that the result is independent of the parametrization of the metric. In this paper, we discuss this problem by studying the effective actions or equivalent determinants obtained in the path integral formulation in arbitrary dimensions.

There are two equivalent (at the classical level) formulations of scalar-tensor theories related by conformal transformation. They are known as scalar-tensor theories in the Jordan and Einstein frames. There is still ongoing debate about the quantum equivalence of these theories in the different frames [34]-[41]. We also study the relation in this paper. We find that all these formulations are equivalent on shell at the quantum level.

This paper is organized as follows. In sect. 2, we first review how the $f(R)$ theory is rewritten into a theory of scalar field coupled to the Einstein theory in the Jordan frame. Then we make conformal transformation to map the theory in the Einstein frame. In sect. 3, we start studying
the effective actions in these theories in the background field formalism. In subsect. 3.1, we derive Hessian for the metric fluctuation $h_{\mu\nu}$ in $f(R)$ gravity on Einstein background, which is assumed throughout this paper. Using the exponential parametrization of the metric which has nice feature that it gives results rather close to on-shell [42, 43, 28, 21, 22, 44, 45], we calculate the determinant with general linear gauge having two gauge parameters $\alpha$ and $\beta$. We show that the resulting effective action or determinant after path integral does not depend on the gauge parameters (if we make partial gauge fixing $h_{\mu\mu} = 0$). For completeness, we also give the one-loop divergent part of the effective action and the resulting functional renormalization group equation. The first agrees with the recent calculation [31]. In subsect. 3.2, we repeat the calculation in the scalar-tensor theory in the Jordan frame. The Hessian has matrix structure but after taking the determinant, we find that the result precisely agrees with that in the $f(R)$ theory if we use field equations for the background. In subsect. 3.3, we go on to the scalar-tensor theory in the Einstein frame, and find that the resulting determinant is different off shell but becomes the same on shell. This is to be expected because classically the theory is equivalent only on shell. We take these facts as evidence of the quantum equivalence of these theories. In sect. 4, we give conclusions and discussions. In the appendix, we give a formula of conformal transformation.

2 Classical Equivalence

Let us consider the Euclidean theory

$$S_f = \int d^4x \sqrt{g} f(R),$$

(2.1)

where $g = \det(g_{\mu\nu})$. Classically it is known that this theory is equivalent to a scalar field $\phi$ coupled to the Einstein gravity.

To move to such a formulation, let us first consider the theory

$$S = \int d^4x \sqrt{\tilde{g}} \left[ f'(\chi)(R - \chi) + f(\chi) \right],$$

(2.2)

where $\chi$ is a new scalar field. If we take the variation with respect to $\chi$ and $g^{\mu\nu}$, we get

$$f''(\chi)(R - \chi) = 0,$$

$$f'(\chi)R_{\mu\nu} - \frac{1}{2}f'(\chi)(R - \chi) + f'(\chi)g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(\chi) + g_{\mu\nu} \nabla^2 f'(\chi) = 0.$$

(2.3)

We assume that $f''(\chi) \neq 0$, and then we get $\chi = R$. Substituting this into (2.2) or the second equation in (2.3), we find that it reduces to

$$f'(R)R_{\mu\nu} - \frac{1}{2}f'(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \nabla^2 f'(R) = 0,$$

(2.4)

which is nothing but the field equation obtained from the action (2.1). Thus the theory (2.2) is classically equivalent to (2.1).

We can now rewrite the theory further using another scalar field $\phi$. We set

$$Z_N \phi = -f'(\chi),$$

(2.5)
where \( Z_N = \frac{1}{16\pi G} \) with \( G \) being the Newton constant. It should be understood that we solve (2.5) for the field \( \chi \) in terms of \( \phi \). Then Eq. (2.2) takes the form

\[
S = \int d^d x \sqrt{g} \left[ Z_N \phi \{ \chi(\phi) - R \} + f(\chi(\phi)) \right].
\]  

(2.6)

The field equations following from this action are

\[
\delta \phi : \quad Z_N (\chi(\phi) - R + \phi \chi'(\phi)) + f'(\chi(\phi)) \chi'(\phi) = 0,
\]

\[
\delta g^{\mu \nu} : \quad Z_N (-\phi R_{\mu \nu} + \nabla_{\mu} \nabla_{\nu} \phi - g_{\mu \nu} \nabla^2 \phi) - \frac{1}{2} \left[ Z_N \phi \{ \chi(\phi) - R \} + f(\chi(\phi)) \right] g_{\mu \nu} = 0.
\]  

(2.7)

Here and in what follows, the primes should be understood as differentiations with respect to the arguments, so \( f'(\chi) = \frac{df(\chi)}{d\chi} \) and \( \chi'(\phi) = \frac{d\chi}{d\phi} \), and they should not be confused. Using (2.5) in the first equation, we find \( \chi(\phi) = R \). Together with (2.5) again, the second equation in (2.7) then recovers (2.4). So this theory is also classically equivalent to (2.1).

We define a potential by

\[
V(\phi) = Z_N \phi \chi(\phi) + f(\chi(\phi)).
\]  

(2.8)

Using (2.5), the derivatives of the potential is found to be

\[
V'(\phi) = Z_N \chi(\phi),
\]

\[
V''(\phi) = Z_N \chi'(\phi).
\]  

(2.9)

We also have

\[
\chi'(\phi) = -\frac{Z_N}{f''(\chi)}.
\]  

(2.10)

The action (2.6) is what is known as a theory of scalar field coupled to gravity in the Jordan frame. We refer to this theory as scalar-tensor theory in the Jordan frame.

We can go to the Einstein frame by setting

\[
\ln \phi = \sqrt{\frac{d-2}{d-1}} \varphi,
\]  

(2.13)

and get

\[
S_{EF} = \int d^d x \sqrt{\tilde{g}} \left[ Z_N \left\{ -\tilde{R} + \frac{d-1}{d-2} (\partial_{\mu} \varphi)^2 + e^{-\frac{4}{\sqrt{(d-1)(d-2)}}} \tilde{\chi}(\varphi) \right\} + e^{-\frac{d}{\sqrt{(d-1)(d-2)}}} f(\chi(\varphi)) \right].
\]  

(2.14)

We also refer to this theory as scalar-tensor theory in the Einstein frame. This should be again equivalent to (2.1). Thus we have two equivalent formulations of the theory (2.1) at the
classical level. Note that this equivalence is valid on shell, i.e. when we use the field equations. The question that we would like to address is whether these descriptions are also equivalent at quantum level. We expect that the equivalence is also valid only on shell.

From (2.13), we have
\[
\varphi = \sqrt{\frac{d-1}{d-2}} \ln \left( \frac{-f'(\chi)}{Z_N} \right),
\]
and
\[
\chi'(\varphi) = \sqrt{\frac{d-2}{d-1}} f'(\chi), \quad \chi''(\varphi) = \frac{d-2}{d-1} \frac{f''(\chi)}{f'(\chi)}\frac{f''(\chi)^2 - f'(\chi)f'''(\chi)}{f''(\chi)^2} \text{ etc.}
\]
We can get all the equations for \( \chi^{(n)}(\varphi) \) in terms of \( f \) and its derivatives. Then if we define the potential by
\[
U(\varphi) = e^{-\sqrt{\frac{d}{(d-1)(d-2)}} \varphi} \left\{ Z_N e^{\frac{\varphi}{\sqrt{d-2}}} \chi(\varphi) + f(\chi(\varphi)) \right\},
\]
we can express the condition of a minimum of the potential in terms of \( f(\chi) \):
\[
U'(\varphi) = \left( -\frac{f'(\chi)}{Z_N} \right)^{-\frac{d-2}{2}} 2\chi f'(\chi) - d f(\chi) \frac{\sqrt{(d-2)(d-1)}}{\sqrt{(d-2)(d-1)}}.
\]
In addition we also have that
\[
U(\varphi) = \left( -\frac{f'(\chi)}{Z_N} \right)^{-\frac{d-2}{2}} (f(\chi) - \chi f'(\chi)).
\]
The Einstein equations for \( \tilde{g}_{\mu\nu} \) is
\[
-Z_N \tilde{R}_{\mu\nu} = \frac{1}{2} \tilde{g}_{\mu\nu} \left[ -Z_N \tilde{R} + Z_N (\partial_{\rho}\varphi)^2 + U(\varphi) \right] + Z_N \partial_{\mu}\varphi \partial_{\nu}\varphi = 0.
\]
For constant backgrounds, this gives
\[
Z_N \tilde{R} = \frac{d}{d-2} U(\varphi).
\]
On the other hand, by the transformation (2.11), we also have that
\[
\tilde{R} = \left( -\frac{f'(\chi)}{Z_N} \right)^{-\frac{d-2}{2}} R
\]
At the minimum of the potential, we have
\[
U(\varphi)_{\text{min}} = \left( -\frac{f'(\chi)}{Z_N} \right)^{-\frac{d-2}{2}} d \frac{d-2}{d} Z_N \chi.
\]
and we get the Einstein equation
\[
\left( -\frac{f'(\chi)}{Z_N} \right)^{-\frac{d-2}{2}} R = \left( -\frac{f'(\chi)}{Z_N} \right)^{-\frac{d-2}{2}} \chi,
\]
or
\[
R = \chi.
\]
3 Quantum equivalence

In order to discuss quantum theory, we use the background field method and expand the metric and the scalar fields as

\[ g_{\mu\nu} = g_{\mu\rho}(e^h)^\rho_{\nu}, \quad \phi = \tilde{\phi} + \tilde{\phi}, \quad \varphi = \varphi + \tilde{\varphi}. \]  

(3.1)

We will consider constant background \( \bar{\Delta} \) Laplacians on the symmetric tensor, vector and scalar respectively, defined by

\[ \Delta L = \bar{\Delta} L, \quad \text{respectively}, \]

where we have suppressed the overall factor \( \sqrt{\gamma} \) for the metric. This is because this parametrization has various virtues like least gauge-dependence.

For the one-loop calculation, we have to derive the Hessian. Henceforth we assume that the background space is an Einstein space with

\[ \bar{R}_{\mu\nu} = \frac{\bar{R}}{d} g_{\mu\nu}, \quad \bar{R} = \text{const}. \]  

(3.2)

3.1 \( f(R) \) gravity

For the \( f(R) \) gravity, we have the quadratic term [22]

\[ I^{(2)}_{f(R)} = -\frac{1}{4} f'(\bar{R}) h^{TT}_{\mu\nu} \left( \Delta L_2 - \frac{2}{d} \bar{R} \right) h^{TT}_{\mu\nu} \]

\[ + \frac{d - 1}{4d} \left[ \frac{2(d-1)}{d} f''(\bar{R}) \left( \Delta L_0 - \frac{1}{d-1} \bar{R} \right) \right] \Delta L_0^2 \left( \Delta L_0 - \frac{1}{d-1} \bar{R} \right) \]

\[ + \frac{1}{d} \left[ \frac{2(d-1)}{d} f''(\bar{R}) \left( \Delta L_0 - \frac{1}{d-1} \bar{R} \right)^2 \right] + \frac{(d-1)(d-2)}{d^2} f'(\bar{R}) \left( \Delta L_0 - \frac{2}{d-2} \bar{R} \right) + \frac{1}{d} f'(\bar{R}) \]  

\[ + \frac{d - 1}{2d} \left[ \frac{2(d-1)}{d} f''(\bar{R}) \left( \Delta L_0 - \frac{1}{d-1} \bar{R} \right) + \frac{d - 2}{d} f'(\bar{R}) \right] \Delta L_0 \left( \Delta L_0 - \frac{1}{d-1} \bar{R} \right), \]  

(3.3)

where we have suppressed the overall factor \( \sqrt{g} \) and \( \Delta L_2, \Delta L_1 \) and \( \Delta L_0 \) are the Lichnerowicz Laplacians on the symmetric tensor, vector and scalar respectively, defined by

\[ \Delta L_2 T_{\mu\nu} = -\nabla^2 T_{\mu\nu} + R_{\mu}^\rho T_{\rho\nu} + R_{\nu}^\rho T_{\mu\rho} - R_{\mu\rho\sigma} T^{\rho\sigma} - R_{\mu\rho\sigma\rho}, \]

\[ \Delta L_1 V_{\mu} = -\nabla^2 V_{\mu} + R_{\mu}^\rho V_{\rho}, \]

\[ \Delta L_0 S = -\nabla^2 S. \]  

(3.4)

We have also used the York decomposition defined by

\[ h_{\mu\nu} = h_{\mu\nu}^{TT} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + \nabla_\mu \nabla_\nu \sigma - \frac{1}{d} \bar{g}_{\mu\nu} \nabla^2 \sigma + \frac{1}{d} \bar{g}_{\mu\nu} h, \]  

(3.5)

where

\[ \nabla_\mu h_{\mu\nu}^{TT} = \bar{g}^{\mu\nu} h_{\mu\nu}^{TT} = \nabla_\mu \xi_\mu = 0. \]  

(3.6)

The above formula agrees with [46]. In terms of \( s = \Delta L_0 \sigma + h \), Eq. (3.3) is put into

\[ I^{(2)}_{f(R)} = -\frac{1}{4} f'(\bar{R}) h^{TT}_{\mu\nu} \left( \Delta L_2 - \frac{2}{d} \bar{R} \right) h^{TT}_{\mu\nu} \]

\[ + \frac{(d-1)^2}{2d^2} s \left[ f''(\bar{R}) \Delta L_0 + \frac{(d-2)f'(\bar{R}) - 2\bar{R}f''(\bar{R})}{2(d-1)} \right] \left( \Delta L_0 - \frac{1}{d-1} \bar{R} \right) s \]

\[ + \frac{df'(\bar{R}) - 2\bar{R}f''(\bar{R})}{8d} h^2 \]  

(3.7)
We then consider the gauge fixing term

$$S_{GF} = \frac{1}{2\alpha} \int d^d x \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu F_\nu,$$

with

$$F_\mu = \nabla_\mu h^\rho \mu - \frac{\beta + 1}{d} \nabla_\mu h. \tag{3.9}$$

Following [33], it is convenient to reparametrize the scalar sector in terms of the gauge-invariant variable $s$ and a new degree of freedom $u$ defined as

$$s = h + \Delta_{L0} \sigma, \quad u = \frac{(d - 1) \Delta_{L0} - \bar{R} \sigma + \beta h}{(d - 1 - \beta) \Delta_{L0} - \bar{R}}. \tag{3.10}$$

The gauge fixing action then becomes

$$S_{GF} = \frac{1}{2\alpha} \int dx \sqrt{\bar{g}} \left[ \xi_\mu \left( \Delta L_1 - \frac{2 \bar{R}}{d} \right)^2 \xi_\nu \left( \Delta L_0 - \frac{\bar{R}}{d - 1 - \beta} \right)^2 u \right]. \tag{3.11}$$

On shell, the last term in (3.7) is zero, so the quadratic part of the action is written entirely in terms of the physical degrees of freedom $h^{TT}$ and $s$, and the gauge fixing entirely in terms of the gauge degrees of freedom $\xi$ and $u$.

The ghost action for this gauge fixing contains a non-minimal operator

$$S_{gh} = \int dx \sqrt{\bar{g}} \bar{C}_\mu \left( \partial_\nu \nabla^2 + \left( 1 - 2 \frac{\beta + 1}{d} \right) \nabla_\mu \nabla^\nu + \bar{R}_\mu \bar{C}_\nu \right) C_\nu. \tag{3.12}$$

Upon decomposing the ghost into transverse and longitudinal parts

$$C_\nu = C_\nu^T + \nabla_\nu C^L = \frac{1}{\sqrt{\Delta_{L0}}} C^T, \quad \bar{C}_\nu = \bar{C}_\nu^T + \nabla_\nu \frac{1}{\sqrt{\Delta_{L0}}} C^L, \tag{3.13}$$

and the same for $\bar{C}$, the ghost action splits in two terms

$$S_{gh} = \int dx \sqrt{\bar{g}} \left[ -\bar{C}_\mu \left( \Delta L_1 - \frac{2 \bar{R}}{d} \right) C_\mu^T - 2 \frac{d - 1 - \beta}{d} \bar{C}^\mu \left( \Delta L_0 - \frac{\bar{R}}{d - 1 - \beta} \right) C^L \right]. \tag{3.14}$$

Now if we make a partial gauge fixing to set $h = 0$, which can be done by sending $\beta \to \infty$, we get the following one-loop determinants:

$$\text{Det} \left[ \Delta_{L2} - \frac{2 \bar{R}}{d} \right]^{-1/2} \text{Det} \left[ \left( f''(\bar{R}) \Delta_{L0} + \frac{(d - 2) f'(\bar{R}) - 2 \bar{R} f''(\bar{R})}{2(d - 1)} \right) \left( \Delta_{L0} - \frac{\bar{R}}{d - 1} \right) \right]^{-1/2}, \tag{3.15}$$

from (3.7),

$$\text{Det} \left[ \Delta_{L1} - \frac{2 \bar{R}}{d} \right]^{-1} \text{Det} \left[ \Delta_{L0} \right]^{-1/2} \text{Det} \left[ \Delta_{L0} - \frac{\bar{R}}{d - 1 - \beta} \right]^{-1}, \tag{3.16}$$

from the gauge fixing term (3.11), and

$$\text{Det} \left[ \Delta_{L1} - \frac{2 \bar{R}}{d} \right] \text{Det} \left[ \Delta_{L0} - \frac{\bar{R}}{d - 1 - \beta} \right]. \tag{3.17}$$
from the ghost terms (3.14). The York decomposition has Jacobian
\[
\text{Det}\left(\Delta_{L1} - \frac{2}{d} \bar{R}\right)^{1/2} \text{Det}|\Delta_{L0}|^{1/2} \text{Det}\left(\Delta_{L0} - \frac{\bar{R}}{d - 1}\right)^{1/2}
\]

(3.18)

whereas the subsequent transformation \((\sigma, h) \rightarrow (s, u)\) has unit Jacobian. We see many of these cancel and we are left with
\[
\frac{\text{Det}\left[\Delta_{L1} - \frac{2}{d} \bar{R}\right]^{1/2}}{\text{Det}\left[\Delta_{L2} - \frac{2\bar{R}}{d}\right]^{1/2} \text{Det}\left[\Delta_{L0} + \frac{(d-2)f'(R)-2Rf''(R)}{2(d-1)f''(R)}\right]^{1/2}}.
\]

(3.19)

As observed in [22] and confirmed in [45], this result does not depend on the gauge parameters \(\alpha\) and \(\beta\), and the result is close to on-shell once the partial gauge choice \(h = 0\) is made. This is the advantage of the exponential parametrization [42, 43, 28, 21, 22, 44, 45]. The above determinant is what governs the quantum theory at the one-loop level, in particular effective action.

Given the above result, we can evaluate the effective action which is related to the partition function by \(Z(\bar{g}) = e^{-\Gamma(\bar{g})}\). Neglecting field-independent terms, we find
\[
\Gamma(\bar{g}) = \frac{1}{2} \log \text{Det} \left(\Delta_{L2} - \frac{2\bar{R}}{d}\right) - \frac{1}{2} \log \text{Det} \left(\Delta_{L1} - \frac{2\bar{R}}{d}\right)
\]
\[
+ \frac{1}{2} \log \text{Det} \left(\Delta_{L0} - \frac{\bar{R}}{d - 1} + \frac{(d-2)f'(R)}{2(d-1)f''(R)}\right).
\]

(3.20)

The divergent part of the effective action can be computed by standard heat kernel methods [47]. On an Einstein background in four dimensions, the logarithmically divergent part is
\[
\Gamma_{\log}(\bar{g}) = \frac{1}{12(4\pi)^2} \int d^4x \sqrt{\bar{g}} \log \left(\frac{\Lambda^2}{\mu^2}\right) \left(-\frac{71}{10}\bar{R}^2_{\mu\nu\rho\sigma} + \frac{433}{120}\bar{R}^2 - \frac{f'(\bar{R})^2}{3f''(\bar{R})^2} + \frac{\bar{R}f'(\bar{R})}{f''(\bar{R})}\right),
\]

(3.21)

where \(\Lambda\) stands for a cutoff and we introduced a reference mass scale \(\mu\). On shell, we have \(f'(\bar{R}) = 2f(\bar{R})\), and \(\bar{R}^2_{\mu\nu\rho\sigma} = \frac{\bar{R}^2}{6}\) for maximally symmetric space, this reduces to
\[
\Gamma_{\log}(\bar{g}) = \frac{1}{24(4\pi)^2} \int d^4x \sqrt{\bar{g}} \log \left(\frac{\Lambda^2}{\mu^2}\right) \left(-\frac{97}{20}\bar{R}^2 - \frac{8f(\bar{R})^2}{3f''(\bar{R})^2} + \frac{4f(\bar{R})}{f''(\bar{R})}\right),
\]

(3.22)

in agreement with [31].

The flow equation for the \(f(R)\) theory on the 4-sphere was derived in [21, 22]. The result using spectral sum is
\[
32\pi^2(\Phi - 2r\Phi' + 4\Phi) = \frac{d_1}{6 + (6\alpha + 1)r} - \frac{d_2}{\Phi'} \left(\frac{2r\Phi'' - 2\Phi' - \Phi'}{\Phi'}\right) + \frac{d_3}{(3 + (3\beta - 1)r)\Phi'' + \Phi'} + \frac{d_5}{4 + (4\gamma - 1)r},
\]

(3.23)

where
\[
d_1 = \frac{5[6 + (6\alpha - 1)r][12 + (12\alpha - 1)r]}{12},
\]
\[
d_2 = \frac{5[6 + (6\alpha - 1)r][3 + (3\alpha - 2)r]}{108},
\]
\[
d_3 = \frac{5[6 + (6\alpha - 1)r][12 + (12\alpha - 1)r]}{12},
\]
\[
d_5 = \frac{5[6 + (6\alpha - 1)r][3 + (3\alpha - 2)r]}{108},
\]
\[
d_6 = \frac{5[6 + (6\alpha - 1)r][12 + (12\alpha - 1)r]}{12},
\]
\[
d_7 = \frac{5[6 + (6\alpha - 1)r][3 + (3\alpha - 2)r]}{108}.
\]
Next we discuss the one-loop determinant in the scalar-tensor theory in the Jordan frame.

\[ d_3 = \frac{[2 + (2\beta + 3)r] [3 + (3\beta - 1)r] [6 + (6\beta - 5)r]}{72}, \]
\[ d_4 = \frac{[2 + (2\beta - 1)r] [12 + (12\beta + 11)r]}{8}, \]
\[ d_5 = \frac{-72 - 18r(1 + 8\gamma) + r^2 (19 - 18\gamma - 72\gamma^2)}{6}. \] (3.24)

and we have used the dimensionless quantities \( r = \bar{R}k^{-2} \) and \( \Phi(r) = k^{-4}f(\bar{R}) \), and \( \alpha, \beta \) and \( \gamma \) are the parameters of endomorphism, not to be confused with the gauge parameters.

### 3.2 Scalar-tensor theory in the Jordan frame

Next we discuss the one-loop determinant in the scalar-tensor theory in the Jordan frame.

We find from (2.9) that the quadratic terms in the fluctuations in the scalar field are

\[ \sqrt{g}V(\phi) \simeq \sqrt{\frac{N}{2}} \chi'(\bar{\phi})(\bar{\phi}^2 + h\bar{\phi}) + \frac{1}{8}V(\bar{\phi})h^2. \] (3.25)

Together with the contribution from the rest of the terms, we find, using the York decomposition (3.5),

\[
I^{(2)} = -Z_N\bar{\phi} \left[ \frac{1}{4}h^{TT}_\mu \nu \left( \Delta_{L2} - \frac{2\bar{R}}{d} \right) h^{TT}_\mu \nu + \frac{(d-2)(d-2)}{4d^2} s \left( \Delta_{L0} - \frac{\bar{R}}{d-1} \right) s + \frac{d-2}{8d} \bar{R}h^2 \right] \\
- Z_N \left[ \frac{\bar{R}}{2} h\bar{\phi} + \frac{d-1}{d} \phi \left( \Delta_{L0} - \frac{\bar{R}}{d-1} \right) s \right] + \frac{Z_N}{2} \chi'\bar{\phi}(\bar{\phi}^2 + h\bar{\phi}) + \frac{1}{8}V(\bar{\phi})h^2. \] (3.26)

We employ the same gauge fixing as in the preceding subsection. Here we also make the partial gauge fixing \( h = 0 \). Then we again find that the quadratic part of the action is written entirely in terms of the physical degrees of freedom \( h^{TT}, s \) and \( \bar{\phi} \) and the gauge fixing entirely in terms of the gauge degrees of freedom \( \xi \) and \( u \). The one-loop determinant from the scalar sector \( (s, \bar{\phi}) \) is

\[
\text{Det} \left[ \left( \frac{(d-1)(d-2)}{2d^2} Z_N \bar{\phi} \left( \Delta_{L0} - \frac{\bar{R}}{d-1} \right) \right)^{-1/2} \right] \\
- \frac{d-1}{d} Z_N \bar{\phi} \left( \Delta_{L0} - \frac{\bar{R}}{d-1} \right) \frac{Z_N \chi'(\bar{\phi})}{d-1} \right]^{-1/2} \] (3.27)

It follows from (2.5) and (2.10) that

\[ \phi\chi'(\phi) = \frac{f'(\chi)}{f''(\chi)}. \] (3.28)

So, writing out the resulting whole one-loop determinant, we get

\[
\text{Det} \left[ \frac{\Delta_{L1} - \frac{2\bar{R}}{d}}{\Delta_{L2} - \frac{2\bar{R}}{d}} \right]^{1/2} \text{Det} \left[ \frac{\Delta_{L0} + \frac{(d-2)f'(\chi)}{d(d-1)f''(\chi)} - \frac{\bar{R}}{d-1}}{\Delta_{L0} + \frac{(d-2)f'(\chi)}{d(d-1)f''(\chi)} - \frac{\bar{R}}{d-1}} \right]^{1/2}. \] (3.29)

If we use \( \bar{\chi} = \bar{R} \), this precisely agrees with (3.19), the result for the \( f(\bar{R}) \) theory. Thus we conclude that this formulation by scalar-tensor theory is equivalent to the original \( f(\bar{R}) \) theory at the quantum (at least) one-loop level on shell.
3.3 Scalar-tensor theory in the Einstein frame

Next we discuss the Hessian and one-loop determinant in the scalar-tensor theory in the Einstein frame.

If we make one more step in the discussion in sect. 2, we find that the second derivative of the potential is given by

\[ U''(\varphi) = \left( -f'(\chi) \right) Z_N \frac{(d-2)f'' - 2\chi f'''}{(d-1)f''} - \frac{d}{\sqrt{(d-2)(d-1)}} U'(\varphi), \]  

(3.30)

so the Hessian for the fluctuation \( \tilde{\varphi} \) is

\[ I^{(2)}_{\tilde{\varphi}\tilde{\varphi}} = Z_N \left( \Delta L_0 + \frac{(d-2)f' - 2\chi f''}{2(d-1)f''} \right) - \frac{d}{2\sqrt{(d-2)(d-1)}} U'(\tilde{\varphi}). \]  

(3.31)

Note that if we exploit the equations of motion for the background \( U'(\bar{\varphi}) = 0 \) and \( \bar{\chi} = \bar{R} \), we find that this is proportional to that of the field \( s \) in (3.7):

\[ S^{(2)}_{ss} \propto \frac{f''}{Z_N} S^{(2)}_{\tilde{\varphi}\tilde{\varphi}} \bigg|_{\text{on-shell}}. \]  

(3.32)

There are also some mixing terms between graviton \( h \) and the scalar \( \varphi \), but these drop out in the gauge \( h = 0 \) in the exponential parametrization.

The rest of the theory is the usual Einstein theory. The Hessian for this theory can be obtained from the result in the preceding subsection by setting \( f(R) = -Z_N R \). We thus find

\[ I^{(2)}_E = -Z_N \left[ -\frac{1}{4} h^{TT}_{\mu\nu} \left( \Delta L_2 - \frac{2}{d \bar{R}} \right) h^{TT \mu\nu} + \frac{(d-1)(d-2)}{4d^2} s \left( \Delta L_0 - \frac{\bar{R}}{d-1} \right) s + \frac{d-2}{8d} \bar{R}h^2 \right]. \]  

(3.33)

The gauge fixings can be taken as in the preceding subsections. With the partial gauge fixing \( h = 0 \), we again find the separation of the physical degrees of freedom and the gauge fixing terms. It is now straightforward to derive the one-loop determinant

\[ \frac{\text{Det} \left[ \Delta L_1 - \frac{2}{d} \bar{R} \right]^{1/2}}{\text{Det} \left[ \Delta L_2 - \frac{2\bar{R}}{d} \right]^{1/2}} \frac{\text{Det} \left[ Z_N \left( \Delta L_0 + \frac{(d-2)f' - 2\chi f''}{2(d-1)f''} \right) - \frac{d}{2\sqrt{(d-2)(d-1)}} U'(\varphi) \right]^{1/2}}{\Delta L_0}. \]  

(3.34)

As noted above, on shell, this is equivalent to the result of \( f(R) \) gravity.

However before concluding that the theory is equivalent at the quantum level, we have to take the Jacobian from the transformation (2.13) into account. The path integral would produce divergences in the form \( \delta(0) \) times the volume from this change of variable. However such terms would affect the power-law divergence coefficients, which are subject to regularization scheme. The coefficients of logarithmic divergence are not affected and are universal. It is true that we have to take into account the difference in the definition of the scales in different frames, but that can be easily incorporated since the form of the effective action are the same. One may worry that the conformal transformation introduces the change of path integral measure and hence leads to a trace anomaly. However it will be taken into account in a form of the
logarithmic ultraviolet (UV) cutoff dependence when the determinant is regularized with UV cutoff, which is dependent on the conformal transformation in terms of the scalar field \[40\]. We will discuss this problem of the difference in the scale in the next section. All the results for one-loop divergence, effective action and flow equation can be derived from these determinants. The scale dependence in different frames would introduce a formal difference in the resulting functional renormalization group equations, but should not affect the physical results. We thus conclude that the theory is also equivalent to the \(f(R)\) theory at quantum level on shell.

4 Conclusions and discussions

In this paper, we first summarized the relation of the \(f(R)\) theory and the reformulations of the theory in the form of a scalar field coupled to the Einstein theory in the Jordan and Einstein frames, and then calculated determinants, which correspond to the effective actions, after path integral over the fluctuation fields in the background field formalism. It turns out that all three formulations give the same determinant on shell. If we evaluate the determinant with a suitable cut off, this gives the divergences in the theory. After renormalization, this produces an effective action. We have also given the one-loop divergent term in the effective action. One could also try to derive the functional renormalization group equation by introducing suitable cutoff function. The fact that the determinant, from which these are all derived, are the same is a strong evidence that these theories are equivalent at the quantum level, at least at one loop.

One possible caveat is that the transformation into the Einstein frame involves conformal transformation. This transformation would produce \(\delta(0)\) type divergence, which could be removed by a local counterterm. However this also produce difference in the scales in different frames. In a given frame, short distance cutoff \(\ell\) may be defined by

\[
\ell^2 = g_{\mu\nu} \Delta x^\mu \Delta x^\nu.
\]  

(4.1)

When the metric is transformed as (2.11), the cutoff lengths in the Jordan and Einstein frames are related by

\[
\ell_J^2 = g^J_{\mu\nu} \Delta x^\mu \Delta x^\nu = (\bar{\phi})^{-2/(d-2)} g^E_{\mu\nu} \Delta x^\mu \Delta x^\nu = (\bar{\phi})^{-2/(d-2)} \ell^2_E.
\]  

(4.2)

The UV cutoff is then related by

\[
\Lambda^2_J = (\bar{\phi})^{2/(d-2)} \Lambda^2_E.
\]  

(4.3)

This would result in slightly different looking functional renormalization group equations in the two frames due to the different cutoffs. If this difference is dealt with suitably, the physical result should not depend on the difference because the effective action is the same. For related discussions, see \[40\].

To summarize, we have found strong evidence that the \(f(R)\) theory and the scalar-tensor theories are equivalent on shell in arbitrary dimensions. As a byproduct, we also find evidence that the theories in the Jordan and Einstein frames are equivalent. Note that our discussions are based on the Einstein space with the curvature (3.2). It would be interesting to try to extend our result to more general spacetime.
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A Conformal transformation

We give a formula relevant in the text here. Under the transformation

\[ g_{\mu\nu} = e^{-2\rho} \tilde{g}_{\mu\nu} , \]  

(A.1)

the Einstein term changes as

\[ \sqrt{g} R = \sqrt{\tilde{g}} e^{(2-d)\rho} \left( \tilde{R} + 2(d-1)\nabla_{\mu}^2 \rho - (d-1)(d-2)(\partial_{\mu}\rho)^2 \right) . \]  

(A.2)

Note that the contraction is made on the right hand side by \( \tilde{g} \).

References

[1] J. Hwang and H. Noh, “f(R) gravity theory and CMBR constraints,” Phys. Lett. B 506 (2001) 13 [astro-ph/0102423].

[2] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, “One-loop f(R) gravity in de Sitter universe,” JCAP 0502 (2005) 010 [hep-th/0501096].

[3] S. Capozziello, V. F. Cardone and A. Troisi, “Reconciling dark energy models with f(R) theories,” Phys. Rev. D 71 (2005) 043503 [astro-ph/0501426].

[4] T. P. Sotiriou and V. Faraoni, “f(R) Theories Of Gravity,” Rev. Mod. Phys. 82 (2010) 451 [arXiv:0805.1726 [gr-qc]].

[5] A. De Felice and S. Tsujikawa, “f(R) theories,” Living Rev. Rel. 13 (2010) 3 [arXiv:1002.4928 [gr-qc]].

[6] H. J. Schmidt, “Comparing selfinteracting scalar fields and R + R**3 cosmological models,” Astron. Nachr. 308 (1987) 183 [gr-qc/0106035].

[7] K. Maeda, “Towards the Einstein-Hilbert Action via Conformal Transformation,” Phys. Rev. D 39 (1989) 3159.

[8] S. Weinberg, “Ultraviolet Divergences In Quantum Theories Of Gravitation,” in Hawking, S.W., Israel, W.: General Relativity (Cambridge University Press), (1980) 790-831.

[9] M. Niedermaier and M. Reuter, “The Asymptotic Safety Scenario in Quantum Gravity,” Living Rev. Rel. 9 (2006) 5.
[10] R. Percacci, “Asymptotic Safety,” In *Oriti, D. (ed.): Approaches to quantum gravity* 111-128 [arXiv:0709.3851 [hep-th]].

[11] D. F. Litim, “Renormalisation group and the Planck scale,” Phil. Trans. Roy. Soc. Lond. A 369 (2011) 2759 [arXiv:1102.4624 [hep-th]].

[12] M. Reuter and F. Saueressig, “Quantum Einstein Gravity,” New J. Phys. 14 (2012) 055022 [arXiv:1202.2274 [hep-th]].

[13] A. Codello, R. Percacci and C. Rahmede, “Ultraviolet properties of \( f(R) \)-gravity,” Int. J. Mod. Phys. A 23 (2008) 143 [arXiv:0705.1769 [hep-th]].

[14] A. Codello, R. Percacci, C. Rahmede, “Investigating the Ultraviolet Properties of Gravity with a Wilsonian Renormalization Group Equation,” Ann. Phys. 324 (2009) 414 arXiv:0805.2275 [hep-th].

[15] M. Hindmarsh and I. D. Saltas, “\( f(R) \) Gravity from the renormalisation group,” Phys. Rev. D 86 (2012) 064029 [arXiv:1203.3957 [gr-qc]].

[16] K. Falls, D. Litim, K. Nikolakopoulos and C. Rahmede, “A bootstrap towards asymptotic safety,” arXiv:1301.4191 [hep-th].

“Further evidence for asymptotic safety of quantum gravity,” Phys. Rev. D 93 (2016) 104022 [arXiv:1410.4815 [hep-th]].

“On de Sitter solutions in asymptotically safe \( f(R) \) theories,” arXiv:1607.04962 [gr-qc].

[17] J. A. Dietz and T. R. Morris, “Asymptotic safety in the \( f(R) \) approximation,” JHEP 1301 (2013) 108 [arXiv:1211.0955 [hep-th]].

[18] J. A. Dietz and T. R. Morris, “Redundant operators in the exact renormalisation group and in the \( f(R) \) approximation to asymptotic safety,” JHEP 1307 (2013) 064 [arXiv:1306.1223 [hep-th]].

[19] A. Eichhorn, “The Renormalization Group flow of unimodular \( f(R) \) gravity,” JHEP 1504 (2015) 096 [arXiv:1501.05848 [gr-qc]].

[20] M. Demmel, F. Saueressig and O. Zanusso, “A proper fixed functional for four-dimensional Quantum Einstein Gravity,” JHEP 1508 (2015) 113 [arXiv:1504.07656 [hep-th]].

[21] N. Ohta, R. Percacci and G. P. Vacca, “Flow equation for \( f(R) \) gravity and some of its exact solutions,” Phys. Rev. D 92 (2015) 061501 [arXiv:1507.00968 [hep-th]].

[22] N. Ohta, R. Percacci and G. P. Vacca, “Renormalization Group Equation and scaling solutions for \( f(R) \) gravity in exponential parametrization,” Eur. Phys. J. C 76 (2016) 46 [arXiv:1511.09393 [hep-th]].

[23] K. Falls and N. Ohta, “Renormalization Group Equation for \( f(R) \) gravity on hyperbolic spaces,” Phys. Rev. D 94 (2016) 084005 [arXiv:1607.08460 [hep-th]].

[24] S. Nojiri and S. D. Odintsov, “Quantum dilatonic gravity in (\( D = 2 \))-dimensions, (\( D = 4 \))-dimensions and (\( D = 5 \))-dimensions,” Int. J. Mod. Phys. A 16 (2001) 1015 [hep-th/0009202].
[25] G. Narain and R. Percacci, “Renormalization Group Flow in Scalar-Tensor Theories. I,” Class. Quant. Grav. 27 (2010) 075001 [arXiv:0911.0386 [hep-th]].

[26] G. Narain and C. Rahmede, “Renormalization Group Flow in Scalar-Tensor Theories. II,” Class. Quant. Grav. 27 (2010) 075002 [arXiv:0911.0394 [hep-th]].

[27] T. Henz, J. M. Pawlowski, A. Rodigast and C. Wetterich, “Dilaton Quantum Gravity,” Phys. Lett. B 727 (2013) 298 [arXiv:1304.7743 [hep-th]].

[28] R. Percacci and G. P. Vacca, “Search of scaling solutions in scalar-tensor gravity,” Eur. Phys. J. C 75 (2015) 188 [arXiv:1501.00888 [hep-th]].

[29] T. Henz, J. M. Pawlowski and C. Wetterich, “Scaling solutions for Dilaton Quantum Gravity,” Phys. Lett. B 769 (2017) 105 [arXiv:1605.01858 [hep-th]].

[30] D. Benedetti and F. Guarnieri, “Brans-Dicke theory in the local potential approximation,” New J. Phys. 16 (2014) 053051 [arXiv:1311.1081 [hep-th]].

[31] M. S. Ruf and C. F. Steinwachs, “One-loop divergences for f(R)-gravity,” arXiv:1711.04785 [gr-qc].

[32] M. S. Ruf and C. F. Steinwachs, “Quantum equivalence of f(R)-gravity and scalar-tensor-theories,” arXiv:1711.07486 [gr-qc].

[33] D. Benedetti, “Asymptotic safety goes on shell,” New J. Phys. 14 (2012) 015005 [arXiv:1107.3110 [hep-th]].

[34] G. Magnano and L. M. Sokolowski, “On physical equivalence between nonlinear gravity theories and a general relativistic selfgravitating scalar field,” Phys. Rev. D 50 (1994) 5039 [gr-qc/9312008].

[35] S. Capozziello, P. Martin-Moruno and C. Rubano, “Physical non-equivalence of the Jordan and Einstein frames,” Phys. Lett. B 689 (2010) 117 [arXiv:1003.5394 [gr-qc]].

[36] J. He and B. Wang, “Modeling f(R) gravity in terms of mass dilation rate,” arXiv:1203.2766 [astro-ph.CO].

[37] X. Calmet and T. C. Yang, “Frame Transformations of Gravitational Theories,” Int. J. Mod. Phys. A 28 (2013) 1350042 [arXiv:1211.4217 [gr-qc]].

[38] T. Chiba and M. Yamaguchi, “Conformal-Frame (In)dependence of Cosmological Observations in Scalar-Tensor Theory,” JCAP 1310 (2013) 040 [arXiv:1308.1142 [gr-qc]].

[39] A. Y. Kamenshchik and C. F. Steinwachs, “Question of quantum equivalence between Jordan frame and Einstein frame,” Phys. Rev. D 91 (2015) 084033 [arXiv:1408.5769 [gr-qc]].

[40] Y. Hamada, H. Kawai, Y. Nakanishi and K. y. Oda, “Meaning of the field dependence of the renormalization scale in Higgs inflation,” Phys. Rev. D 95 (2017) 103524 [arXiv:1610.05885 [hep-th]].
[41] S. Karamitsos and A. Pilaftsis, “Frame Covariant Nonminimal Multifield Inflation,” Nucl. Phys. B 927 (2018) 219 [arXiv:1706.07011 [hep-ph]].

[42] A. Nink, “Field Parametrization Dependence in Asymptotically Safe Quantum Gravity,” Phys. Rev. D 91 (2015) 044030 [arXiv:1410.7816 [hep-th]].
M. Demmel and A. Nink, “Connections and geodesics in the space of metrics,” Phys. Rev. D 92 (2015) 104013 [arXiv:1506.03809 [gr-qc]].

[43] K. Falls, “Renormalization of Newton’s constant,” Phys. Rev. D 92 (2015) 124057 [arXiv:1501.05331 [hep-th]].

[44] N. Ohta, R. Percacci and A. D. Pereira, “Gauges and functional measures in quantum gravity I: Einstein theory,” JHEP 1606 (2016) 115 [arXiv:1605.00454 [hep-th]].

[45] N. Ohta, R. Percacci and A. D. Pereira, “Gauges and functional measures in quantum gravity II: Higher derivative gravity,” Eur. Phys. J. C 77 (2017) 611 [arXiv:1610.07991 [hep-th]].

[46] P. F. Machado and F. Saueressig, “On the renormalization group flow of $f(R)$-gravity,” Phys. Rev. D 77 (2008) 124045 [arXiv:0712.0445 [hep-th]].

[47] R. Percacci “An introduction to covariant quantum gravity and asymptotic safety”, World Scientific, Singapore (2017).