Metric Coordinate Systems

Craig Calcaterra, Axel Boldt, Michael Green
Department of Mathematics; Metropolitan State University;
St. Paul, MN 55106; craig.calcaterra@metrostate.edu

David Bleecker
Department of Mathematics; University of Hawaii;
Honolulu, HI 96822

Abstract

Coordinate systems are defined on general metric spaces with the purpose of generalizing vector fields on a manifold. Conversion formulae are available between metric and Cartesian coordinates on a Hilbert space. Nagumo’s Invariance Theorem is invoked to prove the analogue of the classical Cauchy-Lipschitz Theorem for vector fields on a locally compact coordinatized space. A metric space version of Nagumo’s Theorem is one consequence. Examples are given throughout.

Key Words: metric coordinates; distance coordinates; metric space; vector field; Cauchy-Lipschitz Theorem; Nagumo invariance

AMS Subject Classification: Primary 37C10, 34G99; Secondary 54E45

1 Introduction

The notion of a metric coordinate system is offered here to extend the methods of calculus and differential equations to metric spaces. The inspiration behind metric coordinates is quite simple. On the plane $\mathbb{E}^2$, for example, choose three non-colicinear points $a$, $b$, and $c$. Then every point $x \in \mathbb{E}^2$ is distinguished by three numbers, $d(x,a)$, $d(x,b)$, and $d(x,c)$, which we call the metric coordinates of $x$. In a similar manner any metric space may be coordinatized.

The idea of metric coordinates has been put forth in the past to study static problems in Euclidean spaces: [1], [2], [3]. There have been several recent and notable efforts to develop generalizations of differential equations in the context of metric spaces: quasi-differential equations [4], mutational analysis [5], and arc fields [6]. These largely commensurable approaches have each succeeded in producing a generalization of the
Cauchy-Lipschitz Theorem. In each of these schemes the idea of velocity at a point \( x \) in a metric space \((X, d)\) is represented by a curve issuing from \( x \). The method of this paper is different.

With metric coordinate systems, \( X \) is embedded into a Banach subspace \( E \) of \( \mathbb{R}^C \) where \( C \) is the set of coordinatizing points. \( E \) is used to define vector fields on \( X \). Under suitable assumptions, the vector field can be extended to a Lipschitz continuous vector field on \( E \). Then the traditional Cauchy-Lipschitz Theorem on Banach spaces yields unique solutions. The Nagumo Invariance Theorem then promises that solutions with initial conditions in the embedded subset remain there. The proof allows arbitrary coordinatizing sets \( C \), but uses local compactness. We expect a version without this restriction is possible.

One of the strengths of metric coordinate systems is that, due to the embedding, solving metric-coordinate vector fields on \( X \) reduces to solving an ODE on \( \mathbb{R}^C \). Our vector fields and solutions will depend on the choice of \( C \). This coordinate dependence may be an advantage because it allows us to capture dynamics that cannot easily be described otherwise. Also metric coordinates, like other types of coordinate systems, are often more convenient than Cartesian coordinates for solving certain problems. Spheres, ellipses and hyperbolae are the loci of linear equations in metric coordinates.

Throughout the paper, examples are explored on Euclidean and non-Euclidean spaces. Several open lines of research are detailed in the concluding section.

## 2 Metric coordinatizing sets

**Definition 1** Let \((M, d)\) be a metric space with \( X \subset M \). A **metric coordinatizing set** for \( X \) is a set of points \( C \subset M \) with the property that for all \( x, y \in X \) with \( x \neq y \), there is some \( c \in C \) such that \( d(x, c) \neq d(y, c) \).

We then call \((M, d, X, C)\) a **metric coordinate system**.

As any point \( x \in X \) in a metric coordinate system \((M, d, X, C)\) is represented by a \( C \)-tuple of real numbers \( x_C = (x_c)_{c \in C} \), this will be called the **\( C \) embedding** of \( X \) into \( \mathbb{R}^C \). We are using the term “embedding” loosely here; it is not necessarily a homeomorphism onto its image as the inverse is not necessarily continuous.

Throughout the paper we will be using arbitrary sets \( C \subset M \) which may be infinite or even unbounded. Most examples, however, suffice with finite sets \( C \) as in the following:

**Example 2** We begin with Euclidean spaces. Consider the open half-plane \( H^2 \) in the Euclidean plane \( \mathbb{E}^2 \) with the Euclidean metric \( d \). Pick
any two distinct points $a$ and $b$ on the boundary. We can locate any
point $x$ in $H^2$ if we know its distances to $a$ and $b$, say $x_a = d(x,a)$ and $x_b = d(x,b)$. Then $(\mathbb{E}^2, d, H^2, \{a,b\})$ is a bona-fide metric coordinate system.

Equations in $(\mathbb{E}^2, d, H^2, \{a,b\})$ are naturally different from those in Cartesian or polar coordinate systems. E.g., for any $r > d(a,b)$, the locus of the equation

$$x_a + x_b = r$$

in metric coordinates is the set

$$\{x \in H^2 : d(x,a) + d(x,b) = r\}.$$  

The graph of (1) is half of an ellipse with foci at $a$ and $b$.

$\mathbb{E}^2$, the plane, requires 3 non-colinear points for a metric coordinatizing set. $H^3$ (the half-space) is metrically coordinatized with 3 non-colinear points on its boundary, and $\mathbb{E}^3$ needs 4 non-coplanar points. Many geometrical objects are readily described in metric coordinates on $\mathbb{E}^3$:

Sphere (center $a$, radius $r$) 

$$x_a = r \quad r \geq 0$$

Ellipsoid (foci $a$, $b$) 

$$x_a + x_b = r \quad r \geq d(a,b)$$

Hyperboloid (foci $a$, $b$) 

$$|x_a - x_b| = r \quad 0 < r < d(a,b)$$

Infinite Cylinder 

(with axis $\vec{ab}$, radius $\frac{2r}{d(a,b)}$) 

$$\sqrt{s(s-x_a)(s-x_b)(s-d(a,b))} = r$$

where $s = \frac{x_a + x_b + d(a,b)}{2}$

Infinite Cone 

$$x_b^2 = d(a,b)^2 + x_a^2 - 2x_a d(a,b) \cos \theta$$

(with axis $\vec{ab}$, vertex $a$, angle $\theta$)

Plane (perpendicular to $\vec{ab}$) 

$$x_a = x_b$$

Segment $\vec{ab}$ 

$$x_a + x_b = d(a,b)$$

Ray $\vec{ab}$ 

$$x_a \pm x_b = d(a,b)$$

Line $\vec{ab}$ 

$$|x_a \pm x_b| = d(a,b)$$

The equation for the cylinder comes from Heron’s formula for area of a triangle. The equation for the cone is simply the cosine angle formula for a triangle and represents only one half of a two sided cone; the other half is given when $\theta$ is replaced with $\pi - \theta$. More general equations for lines and planes are available but are not so concise. Choosing the coordinates according to the problem simplifies the formulae.
As each of the above formulae use only metric coordinates, they may serve as definitions for the various geometrical objects in general metric spaces.

**Proposition 3** Let \((M, d, X, C)\) be a metric coordinate system. The metric coordinates \(x_c := d(x, c)\) of any point \(x \in X\) satisfy

\[
x_c \geq 0 \\
|x_a - x_b| \leq d(a, b) \\
x_a + x_b \geq d(a, b)
\]

for all \(a, b, c \in C\).

**Proof.** Triangle inequality. ■

Proposition 3, though mathematically trivial is used in every example. It shows that care must be taken when defining curves in terms of metric coordinates since not all \(C\)-tuples describe points in \(X\).

**Example 4** On any metric space \((M, d)\) there is at least one metric coordinatizing set for any subset \(X\). The worst-case scenario is the discrete metric, defined on any set \(M\) as

\[
d(x, y) := \begin{cases} 
1 & \text{if } x \neq y \\ 
0 & \text{if } x = y.
\end{cases}
\]

This metric requires all of the points in \(X\) save one for its metric coordinatizing set.

**Example 5** On a separable metric space \(M\), any subset \(X\) may be coordinatized with countably many points.

**Example 6** Take \(M = X = \mathbb{R}^2\) with the supremum metric

\[
d_\infty(x, y) := \max_{i=1,2} \{|x_i - y_i|\}.
\]

No bounded set is a coordinatizing set for \(X\). If \(C\) is contained in some square, then two vertically aligned points placed far enough to the left of the square cannot be distinguished by \(C\). \(X\) may, however, be coordinatized by

\[
C := (\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N})
\]

where \(\mathbb{N} := \{1, 2, 3, \ldots\}\)
3 Conversion formulae for Hilbert spaces

On the Euclidean plane \( \mathbb{E}^2 \) choose metric coordinates \( a, b, c \) so the rays \( \overrightarrow{ca} \) and \( \overrightarrow{cb} \) are perpendicular with \( d(a, c) = 1 = d(b, c) \). Define a Cartesian coordinate system on the plane with the origin \((0, 0)\) at \( c \), the positive \( x \)-axis along the ray \( \overrightarrow{ca} \) and the positive \( y \)-axis along the ray \( \overrightarrow{cb} \). The conversion formulae\(^1\) are easy to find:

\[
\begin{align*}
\text{Metric} & \quad (w_a, w_b, w_c) = w = (x, y) \quad \text{Cartesian} \\
\quad w_c & = \sqrt{x^2 + y^2} \\
\quad w_b & = \sqrt{x^2 + (y - 1)^2} \\
\quad w_a & = \sqrt{(x - 1)^2 + y^2}.
\end{align*}
\]

Solving these same equations for \( x \) and \( y \) yields the inverse formulae

\[
\begin{align*}
x &= \frac{w_c^2 - w_a^2 + 1}{2} \\
y &= \frac{w_c^2 - w_b^2 + 1}{2}.
\end{align*}
\]

More generally, on a Hilbert space we have:

**Theorem 7** Let \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) be a real Hilbert space with orthonormal basis \( B \). The set \( C := B \cup \{0\} \subset \mathcal{H} \) is a metric coordinatizing set.

**Proof.** For \( u, v \in \mathcal{H} \) assume \( d(u, c) = d(v, c) \) for all \( c \in C \). Then since \( 0 \) is in \( C \) we have \( \langle u, u \rangle = \langle v, v \rangle \). Further

\[
\begin{align*}
\langle u - c, u - c \rangle &= \langle v - c, v - c \rangle \\
\langle u, u \rangle - 2 \langle c, u \rangle + \langle c, c \rangle &= \langle v, v \rangle - 2 \langle c, v \rangle + \langle c, c \rangle \\
\langle c, u \rangle &= \langle c, v \rangle
\end{align*}
\]

for all \( c \in B \) so that \( u = v \). \( \blacksquare \)

Using the basis \( B \) write an element \( w \in \mathcal{H} \) in orthonormal coordinates as \( w = (\tilde{w}_c) \) where \( \tilde{w}_c = \langle w, c \rangle \) for each \( c \in B \). Any point \( w \in \mathcal{H} \) is given in metric coordinates by \( w = (w_c)_{c \in B \cup \{0\}} \) where \( w_c := \|w - c\| = d(w, c) \).

With this, the conversion formulae are

\[
\begin{align*}
\tilde{w}_c &= \frac{w_0^2 - w_c^2 + 1}{2} \quad c \in B \\
w_c &= (\|w\|^2 - 2\tilde{w}_c + 1)^{1/2} \quad c \in B \\
w_0 &= \|w\|.
\end{align*}
\]

\(^1\)To write \((w_a, w_b, w_c) = w = (x, y)\) is technically abuse of notation. \((w_a, w_b, w_c)\) and \((x, y)\) are actually representations of \( w \), and in the sequel we write \( w_C = (w_a, w_b, w_c) \) to make this distinction explicit.
a straightforward generalization of the finite dimensional formulae, (4) and (5).

(5) results from the easy calculation
\[ w_c = \|w - c\| = (w - c, w - c)^{1/2} \]
\[ = (\langle w, w \rangle - \langle w, c \rangle - \langle c, w \rangle + \langle c, c \rangle)^{1/2} \]
\[ = (\|w\|^2 - 2\tilde{w}_c + 1)^{1/2}. \]

Solving this equation for \( \tilde{w}_c \) yields (5).

Example 8 One must be careful in applying these formulae. They do not necessarily work on non-Hilbert vector spaces. The finite dimensional Banach space \( \mathbb{R}^2 \) with the infinity norm has basis \( \{(1, 0), (0, 1)\} \) which does not produce a coordinatizing set in the above manner. Refer to Example 6.

4 Derivatives

A curve in a metric space is a map \( \phi : (t_1, t_2) \to X \) continuous with respect to the metric on \( X \) where \((t_1, t_2)\) is a subinterval of \( \mathbb{R} \).

Definition 9 Let \((M, d, X, C)\) be a metric coordinate system and let \( \phi \) be a curve in \( X \). Write \( \phi \) in metric coordinates as \( \phi_C(t) = (\phi_c(t))_{c \in C} \). Assuming the limits exist, we define the \textbf{metric-coordinate derivative} of \( \phi \) with respect to \( C \) to be \( \phi'_c(t) := (\phi'_c(t))_{c \in C} \in \mathbb{R}^C \) where

\[ \phi'_c(t) = \lim_{h \to 0} \frac{\phi_c(t + h) - \phi_c(t)}{h}. \]

Similarly, the \textbf{forward metric-coordinate derivative} of \( \phi \) with respect to \( C \) is \( \phi^+_c(t) := (\phi^+_c(t))_{c \in C} \) where

\[ \phi^+_c(t) = \lim_{h \to 0^+} \frac{\phi_c(t + h) - \phi_c(t)}{h}. \]

Two curves \( \phi \) and \( \psi \) are said to be (forward) \textbf{metric-coordinate-wise tangent} at \( t_0 \) with respect to \( C \) if they meet at \( t_0 \) and have the same (forward) metric-coordinate derivative, i.e.,

\[ \phi(t_0) = \psi(t_0) \]
\[ \phi'_C(t_0) = \psi'_C(t_0) \quad \text{(or } \phi^+_C(t_0) = \psi^+_C(t_0) \text{)}. \]

If there exists \( r < \infty \) such that \( |\phi'_c(t_0)| \leq r \) for all \( c \) then \( \phi \) is said to have \textbf{bounded metric-coordinate speed} at \( t_0 \).

\[ \text{There are other inequivalent notions of speed such as metric speed } s(t) := \lim_{h \to 0} \frac{d(\phi(t+h), \phi(t))}{|h|} \text{ or length speed } s^*(t) := \lim_{h \to 0} \frac{L(\phi(t)+h)}{|h|} \text{ where } L \text{ refers to the length of the curve.} \]
Remark 10  For finite coordinatizing sets \( C \), every metric-coordinate-wise differentiable curve has bounded metric-coordinate speed at any particular \( t \) in its domain.

Remark 11  A curve \( \phi \) in a metric coordinate system which runs through a coordinatizing point with positive metric speed \( s(t) \) can never be differentiable in all of its coordinates; when the curve hits \( c \in C \). The \( c \)-th coordinate derivative of \( \phi \) discontinuously changes from negative to positive. \( \phi \) may still be metric-coordinate-wise differentiable with respect to another nonintersecting metric coordinatizing set. Choosing \( C \) outside of the region of interest is the reason for the artifice \((M,d,X,C)\) instead of simply \((X,d,C)\).

Such representational problems are nothing new. E.g., polar coordinates make do with a continuum-sized discontinuity in representing position.

The next theorem shows that Definition 3 faithfully generalizes the traditional derivative on \( \mathbb{R}^n \).

Theorem 12  Let \( U \) be an open subset of \( \mathbb{R}^n \). Let \( C \) be a coordinatizing set for \( U \) with respect to the Euclidean metric. Let \( \phi : (t_1,t_2) \to U \) be a curve and \( t \in (t_1,t_2) \) such that \( \phi(t) \notin C \). Then \( \phi \) is differentiable at \( t \) (in the traditional sense) iff it is metric-coordinate-wise differentiable at \( t \).

Proof.  First assume \( \phi \) is differentiable in the traditional sense at \( t \). Then for any \( c \in C \)

\[
\phi'_c(t) = \lim_{h \to 0} \frac{\phi_c(t + h) - \phi_c(t)}{h}
= \lim_{h \to 0} \frac{d(\phi(t + h),c) - d(\phi(t),c)}{h}
= \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}
\]

where \( f(t) = d(\phi(t),c) \). The Euclidean distance is differentiable except at 0. Since \( \phi(t) \neq c \) the function \( f \) is the composition of two differentiable functions and hence differentiable. Thus \( \phi \) is metric-coordinate-wise differentiable at \( t \).

The converse is slightly more difficult. Assume \( \phi \) is metric-coordinate-wise differentiable at \( t \) with respect to \( C \). We prove that \( \phi \) is differentiable in the traditional sense in the context of \( \mathbb{R}^2 \); the generalization to \( \mathbb{R}^n \) is immediate. There exist two points from \( C \), say \( a \) and \( b \in \mathbb{R}^2 \),
which together with $\phi(t) \in \mathbb{R}^2$ are non-collinear—else $C$ would not effectively discriminate between all points of $U$. Define $(f_g) : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) := \|x - a\|$, and $g(x) := \|x - b\|$ so that $(f_g)$ is differentiable at any point not equal to $a$ or $b$. We will show that $(f_g)^{-1}$ and $(f_g) \circ \phi$ are differentiable so that $\phi$ is the composition of two differentiable functions.

Our assumption that $\phi$ is coordinate-wise differentiable at $t$ immediately gives the differentiability of $(f_g) \circ \phi : (t_1, t_2) \to \mathbb{U}$ at $t$. To prove the differentiability of $(f_g)^{-1}$, we show $D(f_g)$ is nonsingular at $x = \phi(t)$:

$$D(f_g)(x) = \begin{bmatrix} f_u(x) g_u(x) \\ f_v(x) g_v(x) \end{bmatrix} = \begin{bmatrix} x_1 - a_1 & x_1 - b_1 \\ \|x - a\| & \|x - b\| \\ \|x - a\| & \|x - b\| \end{bmatrix} \begin{bmatrix} x_2 - a_2 \\ x_2 - b_2 \end{bmatrix}$$

which is singular if and only if one column vector is a multiple of the other, i.e.,

$$\left( \frac{x_1 - a_1}{\|x - a\|}, \frac{x_2 - a_2}{\|x - a\|} \right) = \lambda \left( \frac{x_1 - b_1}{\|x - b\|}, \frac{x_2 - b_2}{\|x - b\|} \right)$$

or equivalently that

$$0 = (x_1 - a_1, x_2 - a_2) - \lambda_1 (x_1 - b_1, x_2 - b_2) = (1 - \lambda_1) x - a - \lambda_1 b.$$

In this case $x, a$ and $b$ are collinear, which cannot happen. Thus $D(f_g)$ is nonsingular, $(f_g)$ is locally invertible, and $(f_g)^{-1}$ is differentiable. Hence $(f_g)^{-1} \circ (f_g) \circ \phi = \phi$ is differentiable. $\blacksquare$

In view of this theorem we could use “differentiable” in lieu of the awkward phrasing “metric-coordinate-wise differentiable”. But in order to be perfectly clear in this nascent setting we usually employ the full term.

**Example 13** Any curve in $(\mathbb{E}^2, d, H^2, \{a, b\})$ from Example 2 which satisfies the conditions of Proposition 3

$$\phi_a, \phi_b \geq 0$$

$$|\phi_a(t) - \phi_b(t)| \leq 1$$

$$\phi_a(t) + \phi_b(t) \geq 1,$$

and is differentiable in each of its coordinates will be a metric-coordinate-wise differentiable curve.
Example 14 In the Hilbert space \( L^2(\mathbb{R}) \) define the curve \( \phi : (-\infty, \infty) \rightarrow L^2 \), by \( \phi(t)(x) := \chi_{[0,1]}(x-t) \) where \( \chi_{[0,1]} \) is the characteristic function of the unit interval. \( \phi \) is not Frechet differentiable, but it is metric-coordinate-wise differentiable with respect to certain metric coordinatizing sets as we show.

The difference quotient

\[
\frac{\phi(t+h) - \phi(t)}{h}
\]

does not converge in \( L^2 \) to any \( g \in L^2 \); it does however converge in the distribution sense to the difference of Dirac deltas \( \delta(t+1) - \delta(t) \).

Choose an orthonormal basis \( B \) of \( L^2 \) consisting of continuous functions. A metric coordinate system is then automatically given by \( C := B \cup \{0\} \) by Theorem 7. Then

\[
\phi'_c(t) = \lim_{h \to 0} \frac{\phi_c(t+h) - \phi_c(t)}{h} = \lim_{h \to 0} \frac{\|\phi(t+h) - c\|_2 - \|\phi(t) - c\|_2}{h}
\]

(7)

\[
= \lim_{h \to 0} \frac{\int ([\phi(t+h)^2 - \phi(t)^2] - 2c[\phi(t+h) - \phi(t)])}{h(\|\phi(t+h) - c\|_2 + \|\phi(t) - c\|_2)}
\]

(8)

and since \( \phi(t)^2 = \phi(t) \) by the nature of the characteristic function, we have

\[
= \lim_{h \to 0} \frac{\int ([\phi(t+h) - \phi(t)] - 2c[\phi(t+h) - \phi(t)])}{h(\|\phi(t+h) - c\|_2 + \|\phi(t) - c\|_2)}
\]

(9)

\[
= \lim_{h \to 0} \frac{\int (1 - 2c) [\phi(t+h) - \phi(t)]/h}{\|\phi(t+h) - c\|_2 + \|\phi(t) - c\|_2}
\]

(10)

\[
= \frac{2[c(t+1) - c(t)]}{2\|\phi(t) - c\|_2} = \frac{c(t) - c(t+1)}{\phi_c(t)}.
\]

(11)

The second to last equality in line (11) is the reason we require the continuous basis.

Example 15 (Observer dependence of smoothness) On general metric spaces, metric-coordinate-wise differentiability is crucially dependent on the particular coordinate system. E.g., let \( M := \mathbb{R}^2 \) with Euclidean metric \( d \).

\[
X := \{(x, |x|) : |x| \leq 1\}.
\]
Two different metric coordinate systems for $X$ are given by the singletons $C_1 = \{(-2,0)\}$ and $C_2 = \{(1,1)\}$. The curve $\psi : (-1,1) \to X$ given by $\psi(t) := (t,|t|)$ is metric-coordinate-wise differentiable at $t = 0$ with respect to $\{(-2,0)\}$, but not with respect to $\{(1,1)\}$. I.e., an observer at $(1,1)$ measures the jarring difference in distance at time $t = 0$, whereas an observer at $(-2,0)$ measures a smoothly changing distance.

One could give a more involved definition of metric-coordinate-wise differentiability that eliminates coordinate dependence, but we will not pursue it here.

5 Vector fields

A map $f : (X,d_X) \to (Y,d_Y)$ from one metric space to another is called $K$-Lipschitz (or just Lipschitz) if

$$d_Y(f(x),f(y)) \leq K d_X(x,y)$$

for all $x,y \in X$. The map $f$ is called locally Lipschitz if for each point there is a $K \geq 0$ and a neighborhood on which $f$ is $K$-Lipschitz.

Consider the autonomous ordinary differential equation

$$\dot{x} = f(x) \quad (12)$$

on a Banach space $B$. $f$ is called the vector field associated with the differential equation (12) and is a map $f : B \to B$. The Cauchy-Lipschitz Theorem on Banach spaces guarantees that if $f$ is locally Lipschitz then unique solutions exist for short time from any initial condition $x_0 \in B$. I.e., there exists $x : (-\delta,\delta) \to B$ for some $\delta > 0$ with $x(0) = x_0$ satisfying (12). The goal of this section is to achieve a similar result for metric coordinate systems using the fact that $X$ may be associated with a subset of $\mathbb{R}^C$ via the $C$ embedding.

In order to achieve this goal we use a new metric $d_C$ on $X$. We will see that in many important cases $(X,d_C)$ is homeomorphic to $(X,d)$. For a metric coordinate system $(M,d,X,C)$ define $d_C : X \times X \to \mathbb{R}$ by

$$d_C(x,y) := \sup_{c \in C} |x_c - y_c| \quad (13)$$

for $x,y \in X$. To see that this gives a finite number for arbitrary coordinatizing sets $C$ notice that

$$|x_c - y_c| = |d(x,c) - d(y,c)| \leq d(x,y) \quad (14)$$

\[3\]Our focus in this paper is on autonomous dynamics–i.e., vector fields which do not change in time–but the mechanics of generating time-dependent flows may be extended with little extra effort with the standard trick. Simply work on the metric space $X \times \mathbb{R}$ with $\mathbb{R}$ representing the time coordinate, then carefully project solutions on $X$. 

10
by the triangle inequality. This shows that $d_C \leq d$. Further, a subset of $X$ is bounded with respect to $d$ if and only if it is bounded with respect to $d_C$.

**Definition 16** Let $(M, d, X, C)$ be a metric coordinate system. Let $X^+$ represent the space of curves $\phi : [0, \delta) \to (X, d_C)$ which are forward metric-coordinate-wise differentiable with bounded metric-coordinate-wise speed at $t = 0$ and define an equivalence relation $\sim$ on $X^+$ by $\phi \sim \psi$ if $\phi$ is forward tangent to $\psi$ at $t = 0$. The space of equivalence classes for $\sim$ is the **tangent bundle** of $X$ and is written with the symbol $TX$.

The set of equivalence classes of curves under $\sim$ for which $\phi(0) = x \in X$ is the **tangent space** of $X$ at $x$ and is referred to with the symbol $T_xX$.

We also define a metric $d_T^C$ on the tangent bundle $TX$ by

$$d_T^C([\phi], [\psi]) := \max \left\{ d_C(\phi(0), \psi(0)), \sup_{c \in C} |\phi_c^+(0) - \psi_c^+(0)| \right\}$$

where $\phi \in [\phi] \in TX$ and $\psi \in [\psi] \in TX$.

Forward derivatives are used because of the abundance of metric spaces with boundaries. Henceforth we only consider forward derivative, but everything could be formulated in terms of two-sided derivatives as well.

Clearly $TX$ is the disjoint union $\coprod_{x \in X} T_xX$. Notice that $TX$ depends on $C$, not just $X$ and that we use the metric $d_C$ instead of $d$.

**Remark 17** Any member $[v] \in T_xX$ is by definition an equivalence class of curves, but may be represented with a single element of $\mathbb{R}^C$. This is true since any two members $v, w \in [v]$ have the same forward metric-coordinate derivatives at 0, i.e.,

$$v_c^+(0) = (v_c^+(0))_{c \in C} = (w_c^+(0))_{c \in C} = w_c^+(0) \in \mathbb{R}^C.$$ 

It would not be too egregious an abuse of notation to write $T_xX \subset \mathbb{R}^C$.

**Remark 18** Though the symbol $T_xX$ represents a vector space in the context of differentiable manifolds, this is often not true in metric coordinate systems. E.g., from Example 3 we consider the closed half space $\mathbb{H}^2$ and the metric coordinate system $(\mathbb{E}^2, d, \mathbb{H}^2, \{a, b\})$. Then $T_x\mathbb{H}^2$ is naturally identified with $\mathbb{R}^2$, a vector space, for any interior point $x$ of $\mathbb{H}^2$, but this is not true for $x$ on the boundary. Notice however that for any system $(M, d, X, C)$ the tangent space $T_xX$ consists of rays emanating from the origin, since curves may be reparametrized to have greater or smaller metric coordinate derivative.
Definition 19  On a metric coordinate system \((M,d,X,C)\) a **metric-coordinate vector field** is a map \(V : X \to TX\) such that \(V(x) \in T_xX\) for each \(x \in X\) with \(V(x)\) uniformly bounded in \(x\) for each \(x\), i.e., \(V(x)\) has bounded metric coordinate speed for each \(x\).

Such a vector field is called (locally) **Lipschitz** if \(V : (X,d_C) \to (TX,d_T)\) is (locally) Lipschitz.

A solution to \(V\) with initial condition \(x \in X\) is a curve \(\sigma : [0,\delta) \to (X,d)\) for some \(\delta > 0\) with \(\sigma(0) = x\) such that \(\sigma^+(t) = V(\sigma(t))\) for all \(t \in [0,\delta)\).

A metric-coordinate vector field on \((M,d,X,C)\) is said to have **unique solutions** if for any point \(x \in X\), there exists a solution \(\sigma : [0,\delta) \to (X,d)\) with \(\sigma(0) = x\), and if \(\tau : [0,\epsilon) \to (X,d)\) is another solution with \(\tau(0) = x\), then for \(t \in [0,\min\{\delta,\epsilon\}]\) we have \(\tau(t) = \sigma(t)\).

Remark 20  Any solution to a metric-coordinate vector field with unique solutions may be continued to produce a solution with maximal domain using a straightforward analytic argument.

6  **Cauchy-Lipschitz Theorem for metric coordinate systems**

Theorem 21  Let \((M,d,X,C)\) be a metric coordinate system and assume \((X,d)\) is locally compact. Let \(V : (X,d_C) \to (TX,d_T)\) be a locally Lipschitz metric-coordinate vector field. Then \(V\) has unique solutions.

This section is devoted to the proof. Much of the following could be conceptually simplified by considering only finite metric coordinatizing sets. But the setting of a metric space is so abstract that it is a great advantage to consider arbitrary \(C\).

The outline of the proof begins by viewing \(X\) and \(TX\) as subsets of \(\mathbb{R}^C\). We then extend the vector field \(V\) to a map \(V^1 : \mathbb{R}^C \to \mathbb{R}^C\), use the traditional Cauchy–Lipschitz Theorem to guarantee solutions, and verify that restrictions of these solutions to \(X\) remain in \(X\) for short time with the Nagumo Invariance Theorem.

The problem with this plan is that \(\mathbb{R}^C\) with the supremum norm is not a Banach space when \(C\) is infinite, and so the standard Cauchy–Lipschitz theorem (Theorem 22 below) does not apply. However, the space of bounded \(C\)-tuples

\[
\mathbb{R}_B^C := \left\{ x_C \in \mathbb{R}^C : \sup_{c \in C} |x_c| < \infty \right\}
\]
is a Banach space for any set $C$ with norm\footnote{We reserve the notation $\|\cdot\|$ for this supremum norm henceforth.}
\[ \|x\| := \sup_{c \in C} |x_c|. \]

To carry out our plan, we embed $X$ into $\mathbb{R}^C_b$ instead.

Let $w \in X$ be a distinguished element (arbitrarily chosen), and for each $x \in X$ define the embedding $i : (X, d_C) \to (\mathbb{R}^C_b, \|\cdot\|)$ by
\[ i(x)_c := x_c - d(c, w). \]

Then $i(x) \in \mathbb{R}^C_b$ since
\[ i(x)_c = d(x, c) - d(c, w) \leq d(x, w) \]
which is uniformly bounded in $C$. Subtracting $d(c, w)$ in the definition of $i$ is only necessary in the case that $C$ is unbounded in the metric sense. Finally $i$ is an isometry (in particular it is injective) since
\[
\|i(x) - i(y)\| = \sup_{c \in C} |i(x)_c - i(y)_c| \\
= \sup_{c \in C} |d(x, c) - d(c, w) - [d(y, c) - d(c, w)]| \\
= \sup_{c \in C} |d(x, c) - d(y, c)| = d_C(x, y).
\]

We will need the following results.

**Theorem 22 (Cauchy-Lipschitz)** A locally Lipschitz vector field on a Banach space has unique solutions.

Here we are referring to the traditional notion of vector field, not metric-coordinate vector fields. Proofs are legion.

**Remark 23** The uniqueness of one-sided solutions, required for this section, is also true. See \cite{3}, e.g.

**Theorem 24 (Lipschitz Extension)** If $S$ is a subset of a metric space $(X, d)$, and if $f : S \to \mathbb{R}$ is $K$-Lipschitz, then $\overline{f} : X \to \mathbb{R}$ defined by
\[ \overline{f}(x) := \sup \{ f(y) - K \cdot d(x, y) \mid y \in S \} \]
equals $f$ on $S$ and is $K$-Lipschitz.

**Proof.** Given in \cite{4}. \qed
Lemma 25 If $S$ is a subset of a metric space $(X,d)$, and if $f : S \to \mathbb{R}_b^C$ is $K$-Lipschitz, then there exists a $K$-Lipschitz extension $\overline{f} : (X,d) \to \mathbb{R}_b^C$.

Proof. Use the Lipschitz Extension Theorem on each coordinate to get $\overline{f} : X \to \mathbb{R}_b^C$ which is $K$-Lipschitz in each coordinate. We need to check that $\overline{f}(X) \subset \mathbb{R}_b^C$. For any $x \in X$ and $y \in S$

$$\sup_{c \in C} |\overline{f}_c(x)| \leq \sup_{c \in C} \{ |\overline{f}_c(x) - \overline{f}_c(y)| + |\overline{f}_c(y)| \} \leq Kd(x,y) + \| f(y) \| < \infty.$$ 

Therefore $\overline{f}_c(x) \in \mathbb{R}_b^C$. ■

The upper forward derivative of a function $f : [a,b] \to \mathbb{R}$ is defined by

$$D^+ f(t) := \lim_{h \to 0^+} \frac{f(t+h) - f(t)}{h}.$$

Lemma 26 Let $f : [a,b] \to \mathbb{R}$ be continuous with $D^+ f(t) \leq K f(t)$. Then $f(t) \leq f(a)$ for all $t \in [a,b]$.

Proof. See [11, p. 354] for the following result: Let $F : [a,b] \to \mathbb{R}$ be continuous with $D^+ F \leq 0$ for all $t \in [a,b]$. Then $F(a) \geq F(b)$.

Now apply this to $F(t) := e^{-Kt} f(t)$. ■

Definition 27 A subset $S$ of a normed vector space $E$ is said to be **positively invariant** with respect to the vector field $V : E \to E$ if any forward solution $\sigma : [0, \delta) \to E$ to $V$ with initial condition $\sigma(0) \in S$ has $\sigma(t) \in S$ for all $t \in [0, \delta)$.

For a point $x$ in a metric space $(X,d)$ and a subset $S$, the distance from $x$ to $S$ is defined as

$$d(x,S) := \inf_{y \in S} \{ d(x,y) \} =: d(S,x).$$

It is easy to check that $d(x,S) \leq d(x,y) + d(y,S)$ for any $y \in X$. As a consequence the distance is continuous in $x$.

Theorem 28 (Nagumo Invariance) Let $E$ be a normed vector space space, let $V : E \to E$ be a map, and let $S$ be a closed subset of $E$. Suppose that at each $a \in S$ the vector field $V$ is tangent to $S$ in the
following sense: there exists an open neighborhood \( \Omega_a \) and \( K_a > 0 \) such that

\[
\lim_{h \to 0^+} \frac{d(x + hV(x), S) - d(x, S)}{h} \leq K_a d(x, S)
\]

for all \( x \in \Omega_a \), where \( d \) is the metric induced by the norm on \( E \).

Then \( S \) is positively invariant with respect to the vector field \( V \).

**Proof.** This generalization of Nagumo’s result on \( \mathbb{R}^n \) is due to Volkmann and is given in [12] under more general conditions. We adapt his proof to this context. Similar results are surveyed in [10, pp. 70-71,98].

Assume \( S \) is not positively invariant. Then there is a solution \( \sigma : [0, \delta) \to E \) with \( \sigma(0) \in S \) and \( \sigma(t_0) \notin S \) for some \( t_0 \in [0, \delta) \). Let

\[
t_1 := \sup \{ t : \sigma([0, t)) \subset S \}.
\]

Since \( S \) is closed, \( a = \sigma(t_1) \in S \) and \( 0 \leq t_1 < t_0 < \delta \).

For the point \( \sigma(t_1) \) choose \( \Omega \) according to the assumptions of the theorem and let \( t_2 \) be chosen greater than \( t_1 \) such that \( \sigma(t) \in \Omega \) for \( t \in [t_1, t_2] \). By the definition of \( t_1 \) there exists some \( t_3 \in (t_1, t_2) \) such that \( \sigma(t_3) \notin S \).

Define \( \eta : [t_1, t_2) \to [0, \infty) \) by

\[
\eta(s) := d(\sigma(s), S) .
\]

Certainly \( \eta \) is continuous, positive, and \( \eta(t_1) = 0 \). We prove that the upper forward derivative of \( \eta \) is less than \( K_a \eta \) on its domain, so that \( \eta(s) \equiv 0 \) by the previous lemma. To this end, fix \( s \in [t_1, t_2) \). Then for \( h > 0 \)

\[
\eta(s + h) = d(\sigma(s + h), S)
\]

\[
\leq d(\sigma(s + h), \sigma(s) + hV(\sigma(s))) + d(\sigma(s) + hV(\sigma(s)), S)
\]

\[
= \| \sigma(s + h) - \sigma(s) - hV(\sigma(s)) \| + d(\sigma(s) + hV(\sigma(s)), S)
\]

\[
= o(h) + d(\sigma(s) + hV(\sigma(s)), S)
\]

The last equality results from the fact that \( \sigma \) is a solution to \( V \). Thus the upper forward derivative of \( \eta \) is

\[
\overline{D}_{\eta}^+(s) := \lim_{h \to 0^+} \frac{\eta(s + h) - \eta(s)}{h}
\]

\[
\leq \lim_{h \to 0^+} \frac{d(\sigma(s) + hV(\sigma(s)), S) - d(\sigma(s), S)}{h}
\]

\[
\leq K_a d(\sigma(s), S) = K_a \eta(s) .
\]
The last inequality is from [15]. Thus \( \eta(s) \equiv 0 \) so that \( \sigma(t) \in S \) for all \( t \in [t_1, t_2] \), contradicting \( \sigma(t_3) \notin S \). □

Finally we are ready to prove the major result.

**Proof of Theorem 21.** Let \( x_0 \in X \). Since \((X, d_C)\) is locally compact, there exists a compact ball \( B := B_{d_C}(x_0, r) \) for some \( r > 0 \). We may assume \( r \) is chosen small enough so that \( V \) is \( K \)-Lipschitz on \( B \). Notice that the imbedding map \( i \) gives \( i(B) \subset B_{∥·∥}(i(x_0), r) \) where \( B_{d_C} \) refers to a ball in \((X, d_C)\) and \( B_{∥·∥} \) refers to a ball in \( \mathbb{R}^C_b \). Further \( i(B) \) is compact, being the continuous (isometric) image of the compact space \( B \).

The metric-coordinate vector field \( V : X \to TX \) transfers to a map \( V^1 \) on \( i(B) \) via the following diagram:

\[
\begin{array}{ccc}
V : (\mathbb{B}, d_C) & \xrightarrow{K\text{-Lip}} & (TX, d_C^T) \\
(isometry) \ i \downarrow & \downarrow \pi(\text{weak contraction}) \\
V^1 : (i(B), ∥·∥) & \to & (\mathbb{R}^C_b, ∥·∥)
\end{array}
\]

where \( \pi([φ]) := φ^+_C(0) \). The map \( π \) is a weak contraction (i.e., \( K \)-Lipschitz with \( K \leq 1 \)) since

\[
∥\pi([φ]) − π([ψ])∥ = ∥φ^+_C(0) − ψ^+_C(0)∥ \leq d_C^T([φ],[ψ])
\]

Notice \( V^1 \circ i = π \circ V \) so we see that \( V^1 \) is \( K \)-Lipschitz since

\[
\begin{align*}
∥V^1(i(x)) − V^1(i(y))∥ &= ∥π(V(x)) − π(V(y))∥ \\
&\leq d_C^T(V(x), V(y)) \leq K d_C(x, y) = K ∥i(x) − i(y)∥.
\end{align*}
\]

Extend \( V^1 \) to a Lipschitz vector field \( V^2 \) on all of \( \mathbb{R}^C_b \) via Lemma 23. We will prove that a solution to \( V^2 \) starting at \( i(x_0) \) remains in \( i(B) \) for short time. Modify \( V^2 \) to be an invariant vector field on \( B_{∥·∥}(i(x_0), r) \) by shrinking the speed to 0 near its boundary. To do this define the new vector field \( V^3 : \mathbb{R}^C_b \to \mathbb{R}^C_b \) to be

\[
V^3(w) := \begin{cases} 
V^2(w) & w \in B_{∥·∥}(i(x_0), r/2) \\
0 & w \notin B_{∥·∥}(i(x_0), r) \\
(2 - \tfrac{3}{4}∥w − i(x_0)∥) V^2(w) & \text{otherwise}
\end{cases}
\]

which is again Lipschitz (which is verified in Lemma 24 below), say with constant \( K_1 \). The Cauchy-Lipschitz Theorem on Banach spaces then provides unique solutions to \( V^3 \).

For the penultimate step of the proof we invoke the Nagumo Invariance Theorem to demonstrate that the solutions to \( V^3 \) which begin in
\( \Omega = \mathbb{R}_b^C \). We use the metric \( d_\infty \) derived from the norm \( \| \cdot \| \) on \( \mathbb{R}_b^C \). First consider \( w \in \mathcal{B} \); we get

\[
\frac{d_\infty \left( w + hV^3(w), \mathcal{B} \right) - d_\infty \left( w, \mathcal{B} \right)}{h} = \frac{d_\infty \left( w + hV^3(w), \mathcal{B} \right)}{h} \leq \frac{d_\infty \left( w + hV^3(w), \phi(h) \right) + d_\infty \left( \phi(h), \mathcal{B} \right)}{h}
\]

where \( \phi : [0, \delta) \to \mathcal{B} \) is a curve with \( \phi(0) = w \) and \( \phi^+(0) = V^3(w) \). It is not immediately clear that there is such a curve which remains in \( \mathcal{B} \). To see that such a \( \phi \) exists consider the three cases:

1. If \( w \in \mathcal{B} \) or \( \mathbb{R}_b^C \) then \( V^3(w) = \pi(V(i^{-1}(w))) \) so that there exists a member of the equivalence class \( V(i^{-1}(w)) \), call it \( \psi : [0, \delta) \to X \) with \( \psi^+(0) = \pi(V(i^{-1}(w))) \). We assume \( \delta > 0 \) is chosen small enough that \( \psi \) remains in \( \mathcal{B} \) which may be done since \( \psi \) is continuous with respect to \( d_\infty \). Then \( \phi := \iota \circ \psi \) is the desired curve.

2. If \( w \in \mathcal{B} \) or \( \mathbb{R}_b^C \) then \( \phi \) works again; just reparametrize with multiplicative factor

\[
\left( 2 - \frac{2}{r} \| w - i(x_0) \| \right).
\]

3. If \( w \notin \mathcal{B} \) use the constant curve \( \phi(t) \equiv w \). This seems simple, but it is the reason we modified \( V^2 \) to \( V^3 \); when \( w \) is on the boundary of \( \mathcal{B} \) we do not necessarily have such representatives of \( V^2 \) which remain in \( \mathcal{B} \).

With this curve \( \phi \) we have \( d_\infty \left( \phi(h), \mathcal{B} \right) = 0 \) and \( \| w + hV^3(w) - \phi(h) \| / h \to 0 \) as \( h \to 0^+ \). Thus \( (13) \) is satisfied for \( w \in \mathcal{B} \).

For \( w \notin \mathcal{B} \) let \( v \in \mathcal{B} \) be such that \( d_\infty \left( w, \mathcal{B} \right) = d_\infty \left( w, \mathcal{B} \right) \). Such a \( v \) exists by the compactness of \( \mathcal{B} \). We may now apply the...
previous case to \( v \). Thus we have
\[
\frac{d_{\infty} \left( w + hV^3 (w), i (\overline{B}) \right) - d_{\infty} (w, i (\overline{B}))}{h}
\]
\[
\leq \frac{d_{\infty} (w + hV^3 (w), v + hV^3 (v)) + d_{\infty} (v + hV^3 (v), i (\overline{B}))}{h}
\]
\[
- \frac{d_{\infty} (w, i (\overline{B}))}{h}
\]
\[
= \frac{\| w - v + h [V^3 (w) - V^3 (v)] \| + d_{\infty} (v + hV^3 (v), i (\overline{B}))}{h}
\]
\[
- \frac{d_{\infty} (w, i (\overline{B}))}{h}
\]
\[
\leq \frac{\| w - v \| + hK_{1} \| w - v \| - d_{\infty} (w, i (\overline{B}))}{h} + \eta (h)
\]
\[
(\text{where } \eta (h) \to 0 \text{ as } h \to 0)
\]
\[
= K_{1}d_{\infty} (w, i (\overline{B})) + \eta (h)
\]
and (13) is satisfied for \( w \notin i (\overline{B}) \). Thus the unique solutions to \( V^3 \) with initial conditions in \( i (\overline{B}) \) remain in \( i (\overline{B}) \).

Thus the solution \( \sigma \) of \( V^3 \) with initial condition \( i (x_0) \) exists and remains in \( i (\overline{B}) \). By the continuity of \( \sigma \), there exists \( \delta > 0 \) such that \( \sigma ([0, \delta)) \subseteq \overline{B} \left( \| i (x_0), r/2 \| \right) \) on which \( V^3 (w) = V^2 (w) = V^1 (w) \) when \( w \in i (\overline{B}) \) so that \( i^{-1} \circ \sigma : [0, \delta) \to X \) is a solution to \( V \) with initial condition \( x_0 \). Note that this solution is continuous with respect to \( d_C \) (as required), but not necessarily with respect to \( d \). \( \blacksquare \)

**Lemma 29** Let \( E \) be a normed vector space and let \( f : E \to E \) be a \( K \)-Lipschitz map. For some fixed \( x_0 \in E \) and \( r > 0 \) let \( f^* : E \to E \) to be defined as

\[
f^* (x) = \begin{cases} f (x) & x \in B (x_0, r/2) \\ \left( 2 - \frac{2}{r} \| x - x_0 \| \right) f (x) & x \in B (x_0, r) \setminus B (x_0, r/2) \\ 0 & x \notin B (x_0, r) \end{cases}
\]

Then \( f^* \) is Lipschitz.

**Proof.** Clearly \( f^* \) is \( K \)-Lipschitz inside \( B (x_0, r/2) \) and 0-Lipschitz outside \( B (x_0, r) \). Hence the analysis breaks down into the following four cases:
Case 1. \(x, y \in B(x_0, r) \setminus B(x_0, r/2)\).

\[
\|f^*(x) - f^*(y)\| = \left\| f(x) \left(2 - \frac{2}{r} \|x-x_0\|\right) - f(y) \left(2 - \frac{2}{r} \|y-x_0\|\right)\right\|
\]

\[
\leq 2 \|f(x) - f(y)\| + \frac{2}{r} \|f(x)\| \|x-x_0\| - \|f(y)\| \|y-x_0\|
\]

\[
= 2 \|f(x) - f(y)\| + \frac{2}{r} \left(\|f(x)\| \|x-x_0\| + f(y) \|\|x-x_0\| - \|y-x_0\|\right)\right\|
\]

\[
\leq 2 \|f(x) - f(y)\| + \frac{2}{r} \|f(x) - f(y)\| \|x-x_0\|
\]

\[
+ \frac{2}{r} \|f(y)\| \|x-x_0\| - \|y-x_0\|
\]

\[
\leq 2 \|f(x) - f(y)\| + \frac{2}{r} \|f(x) - f(y)\| \|x-x_0\| + \|f(y)\| \|x - y\|
\]

\[
\leq \left(2K + \frac{2}{r}Kr + \frac{2}{r}M\right) \|x - y\|
\]

where \(M = \sup \{\|f(y)\| : y \in B(x_0, r)\} < \infty\) since \(f\) is Lipschitz. Therefore \(f^*\) is \(K_1\)-Lipschitz with \(K_1 := 4K + \frac{2}{r}M\).

Case 2. \(x \in B(x_0, r/2)\) and \(y \in B(x_0, r) \setminus B(x_0, r/2)\).

Let \(z_0\) be a point with \(\|z_0 - x_0\| = r/2\) and \(z_0 = t_0x + (1-t_0)y\) for some \(0 \leq t_0 < 1\) (such a \(z_0\) exists by continuity). Then

\[
\|f^*(x) - f^*(y)\| \leq \|f^*(x) - f^*(z_0)\| + \|f^*(z_0) - f^*(y)\|
\]

\[
\leq K \|x - z_0\| + K_1 \|z_0 - y\|
\]

\[
\leq K_1 (\|x - z_0\| + \|z_0 - y\|)
\]

\[
= K_1 (\|x - (t_0x + (1-t_0)y)\| + \|t_0x + (1-t_0)y - y\|)
\]

\[
= K_1 (\|1 - t_0\| \|x - y\| + t_0 \|x - y\|) = K_1 \|x - y\|.
\]

Case 3. \(x \in B(x_0, r) \setminus B(x_0, r/2)\) and \(y \notin B(x_0, r)\).

Similar to Case 2.

Case 4. \(x \in B(x_0, r/2)\) and \(y \notin B(x_0, r)\).

Like before, there exists a point \(z_1\) with \(\|z_1 - x_0\| = r/2\) and \(z_1 = t_1x + (1-t_1)y\) for some \(0 < t_1 < 1\). Then let \(z_2\) be a point with \(\|z_2 - x_0\| = r\) and \(z_2 = t_2x + (1-t_2)y\) for some \(0 \leq t_2 < 1\) and \(t_2 < t_1\).
Then
\[ \| f^*(x) - f^*(y) \| \]
\[ \leq \| f^*(x) - f^*(z_1) \| + \| f^*(z_1) - f^*(z_2) \| + \| f^*(z_2) - f^*(y) \| \]
\[ \leq K \| x - z_1 \| + K_1 \| z_1 - z_2 \| + 0 \]
\[ \leq K_1 (\| x - z_1 \| + \| z_1 - z_2 \|) \]
\[ = K_1 (\| x - (t_1 x + (1 - t_1) y) \| + \| t_1 x + (1 - t_1) y - [t_2 x + (1 - t_2) y] \|) \]
\[ = K_1 ([1 - t_1] \| x - y \| + (t_1 - t_2) \| x - y \|) = K_1 (1 - t_2) \| x - y \| \]
so that \( f^* \) is again \( K_1 \)-Lipschitz.

**Remark 30** We only use the local compactness of \((X, d_C)\) at one line in the proof of Theorem 21. Perhaps a better analyst can complete the proof assuming \((X, d_C)\) is only locally complete, in the sense that every element \( x \in X \) is contained in a complete neighborhood. Open subsets of a complete metric space are locally complete, and it is straightforward to show that every locally complete metric space is an open subset of its metric completion.

**Remark 31** There should be skepticism about using \( d_C \) instead of \( d \). For example, we define the Lipschitz continuity of \( V \). But in using metric coordinates, the formulae for the vector field \( V_C(x) \) will automatically be in terms of \( C \). The most natural metric to use when checking that the formulae are Lipschitz is \( d_C \) and in each of the examples the calculation is straightforward or automatic.

Still we need \( X \) to be locally compact in \( d_C \) not \( d \) and our solutions are only guaranteed to be continuous with respect to \( d_C \). Therefore it is important to study the connection between \((X, d)\) and \((X, d_C)\) which is the purpose of the following section.

### 7 \( d_C \) versus \( d \)

For a metric coordinate system \((M, d, X, C)\) the metric \( d_C \) on \( X \) defined by \((K)\) can behave rather unintuitively. E.g., there exist sequences \( x_n, y_n \subset \mathbb{R}^2 \) with Euclidean metric \( d \) for which \( d_C (x_n, y_n) \to 0 \) but \( d (x_n, y_n) \to \infty \). As case in point, choose \( C := \{(0,0), (1,0)\} \) and \( x_n := (n, 2^n) \) and \( y_n := (-n, 2^n) \). Therefore we are very interested in the answer to the following:

**Problem 32 (Open question)** Characterize the metric coordinate systems \((M, d, X, C)\) for which \((X, d_C)\) is locally compact.
Example 33 \((X,d)\) being locally compact does not guarantee that \((X,d_C)\) is locally compact. Take \(M = \mathbb{R}^2\) with Euclidean metric \(d\),

\[
X := \{(x,y) \in \mathbb{R}^2 : y > 1 \text{ and } x \neq 0\} \cup \{(0,-1)\}
\]

and coordinatizing set \(C := \{(0,0),(1,0)\}\). Then \((X,d)\) is locally compact, but \((X,d_C)\) is not locally complete (and therefore not locally compact) since \((0,-1)\) has no complete neighborhood. Notice \((0,-1)\) is identified with the point \((0,1)\) by \(d_C\).

In particular this shows there exist metric coordinate systems \((M,d,X,C)\) such that \((X,d)\) is not homeomorphic to \((X,d_C)\).

It can be shown that \(T_{(0,-1)}X\) is naturally identified with a half plane (via \(\pi\) from the proof of Theorem 21). Thus we can give a locally Lipschitz vector field on \((X,d_C)\) that gives vertical translation for its flow, but no solution exists for the initial condition \((0,-1)\). Thus the assumption that \(X\) be locally compact in \(d_C\) instead of in \(d\) is the correct condition for Theorem 21.

Yet another vector field \(V : X \to TX\) may be given that has a diagonal flow. Then a solution does exist with initial condition \((0,-1)\); but this solution is discontinuous in \((X,d)\), jumping from \((0,-1)\) to \((0,1)\). Such discontinuous solutions do not exist in the case that the closed balls of \((X,d)\) are compact which follows from Theorem 34, below.

A similar setup shows that \((X,d_C)\) being locally compact does not guarantee \((X,d)\) is locally compact. Take \(M = \mathbb{R}^2\),

\[
X := \{(x,y) \in \mathbb{R}^2 : y \geq 1 \text{ and } x \neq 0\} \cup \{(0,1)\} \cup \{(0,y) : y < -1\}
\]

and The coordinatizing set is \(C := \{(0,0),(1,0)\}\). \(d_C\) “sees” the points \(\{(0,y) : y < -1\}\) as if they were reflected across the \(x\)-axis. Now \((X,d_C)\) is locally compact, but \((X,d)\) is not locally complete (and therefore not locally compact) since \((0,1)\) has no complete neighborhood.

Despite the pessimism of this example, we have a good beginning on answering the open question with:

**Theorem 34** If all closed balls are compact in \((X,d)\), then \((X,d)\) is homeomorphic to \((X,d_C)\), and in particular \((X,d_C)\) is locally compact.

**Proof.** Since we know \(d_C \leq d\) by (14), we need to show that if a sequence converges in \((X,d_C)\), then it also converges in \((X,d)\). So pick a sequence \(x_n \to x\) in \((X,d_C)\). Pick a \(c \in C\). Then the sequence \(d(x_n,c)\) converges to \(d(x,c)\) and is therefore bounded, implying that the sequence \(x_n\) is bounded in \((X,d)\) and thus contained in a closed, therefore compact, ball \(Q\). Now assume \(x_n\) does not converge in \((X,d)\).
Then it has at least two subsequences which converge towards different points of \( Q \), say \( x_{n_i} \to u \) and \( x_{n_j} \to v \) (both with respect to \( d \)), \( u \neq v \). Since \( d_C \leq d \), we know that \( x_{n_i} \to u \) and \( x_{n_j} \to v \) with respect to \( d_C \) as well. But \( x_n \) converges to \( x \) in \((X,d_C)\), so \( u = x = v \) contradicting \( u \neq v \). 

Thus any closed subset \( X \) of \( \mathbb{R}^n \) with the Euclidean metric \( d \) gives a locally compact metric space \((X,d_C)\). This is also true for any open subset of \( \mathbb{R}^n \) as is proven below in Proposition 36.

**Corollary 35** Let \((M,d,X,C)\) be a metric coordinate system and assume the closed balls of \((X,d)\) are compact. Then there exist unique solutions for any locally Lipschitz metric-coordinate vector field \( V : (X,d_C) \to (TX,d_C^1) \). In addition all solutions are continuous with respect to \( d \).

**Proposition 36** Let \((M,d,X,C)\) be a metric coordinate system with \( M \subset \mathbb{R}^n \). If \( X \) is an open subset of \( \mathbb{R}^n \) and \( d \) is the Euclidean metric, then \((X,d)\) is homeomorphic to \((X,d_C)\).

**Proof (Sketch).** Again pick a sequence \( x_n \to x \) in \((X,d_C)\) and assume it does not converge in \((X,d)\). This sequence is bounded in \((X,d)\) and therefore there exists a subsequence \( x_{n_j} \to y \) in \((X,d)\) for \( y \in \mathbb{R}^n \) with \( y \neq x \). Also \( d_C(x,y) = 0 \) so \( y \notin X \). Since \( C \) doesn’t distinguish metrically between \( x \) and \( y \), \( C \) must be contained in the hyperplane of \( \mathbb{R}^n \) perpendicular to \( xy \) through its midpoint. Further there exists \( \epsilon > 0 \) such that \( B_d(x,\epsilon) \subset X \) since \( X \) is open. Then \( B_d(y,\epsilon) \) is symmetric with respect to the hyperplane to \( B_d(x,\epsilon) \). Therefore every point in \( B_d(y,\epsilon) \) has a counterpart in \( B_d(x,\epsilon) \) which are not distinguished by \( C \). (This uses the geometry of the Euclidean balls.) Thus \( B_d(y,\epsilon) \subset \mathbb{R}^n \setminus X \), so \( x_{n_j} \not\to y \) in \((X,d)\) which is the desired contradiction.

The proposition relies heavily on the geometry of \( \mathbb{R}^n \) with the Euclidean metric as the following example shows.

**Example 37** Consider \( M := \mathbb{R}^2 \) with the supremum metric \( d_\infty \). Define \( u := (1,1) \), \( v := (0,-1) \) and

\[
X := \{ x \in \mathbb{R}^2 : d_\infty(x,u) < \frac{1}{3} \text{ or } d_\infty(x,v) < \frac{1}{3} \} \setminus \{ (s,t) \in \mathbb{R}^2 : s-t = 1 \}.
\]

\( X \) is open and may be shown to be coordinatized by

\[
C := \{ (0,0), (1,0), (-1,1), (2,-1) \}.
\]

The sequence \( x_n := \left( \frac{1}{n}, -1 - \frac{1}{n} \right) \) for \( n > 4 \) converges towards \( u \) in \((X,d_C)\), but does not converge in \((X,d)\). Therefore \((X,d_C)\) is not homeomorphic to \((X,d)\).
8 Invariance on metric coordinate systems

Definition 38 On a metric coordinate system \((M, d, X, C)\) a subset \(S \subset X\) is said to be positively invariant with respect to the metric-coordinate vector field \(V : X \to TX\) if any solution \(\sigma : [0, \delta) \to X\) to \(V\) with initial condition in \(S\) has \(\sigma(t) \in S\) for all \(t \in [0, \delta)\).

We present a new version of the Nagumo Invariance Theorem on metric coordinate systems which follows easily from work completed above.

Theorem 39 Let \((M, d, X, C)\) be a metric coordinate system for which \((X, d_C)\) is locally compact. Let \(S\) be a closed subset of \((X, d_C)\). Let \(V : (X, d_C) \to (TX, d^T_C)\) be a locally Lipschitz metric-coordinate vector field tangent to \(S\) in the following sense:

For each \(x \in S\), there exists a curve \(\phi^x : [0, \delta) \to S\) which is a member of the equivalence class \(V(x)\).

Then \(S\) is positively invariant with respect to \(V\).

Proof. \((M, d, S, C)\) is a metric coordinate system and \(V\) restricts to a locally Lipschitz vector field \(V|_S : (S, d_C) \to (TS, d^T_C)\) since \(TS\) is naturally embedded in \(TX\). Also \(S\) is locally compact, being a closed subset of a locally compact space. Hence unique maximal solutions to \(V|_S\) exist in \(S\) by Theorem 21 and coincide with solutions to \(V\). If a solution \(\sigma\) to \(V\) with initial condition in \(S\) ever leaves \(S\), then define

\[
    t_1 := \sup \{ t : \sigma([0, t)) \subset S \}.
\]

Since \(S\) is closed in \((X, d_C)\) and \(\sigma\) is continuous with respect to \(d_C\) we know \(\sigma(t_1) \in S\). Further, we know for the initial condition \(\sigma(t_1) \in S\), the vector field \(V|_S\) has a solution which remains in \(S\) for short time and coincides with the solution \(\sigma\) to \(V\), which is a contradiction. Thus \(S\) is positively invariant. ■

9 Further examples and counterexamples

With reference to Remark [1], if the coordinatizing set \(C\) is a subset of \(X\), vector fields on \(X\) usually cannot be nonzero and continuous on \(C\).

Example 40 We work on the open half space \(X = H^3 \subset \mathbb{E}^3 = M\). For a metric coordinatizing set choose 3 points \(C = \{a, b, c\}\) on the boundary forming a right triangle with legs of length \(d(a, c) = d(b, c) = 1\) and \(\overrightarrow{cb} \perp \overrightarrow{ca}\). E.g., if we were to use Cartesian coordinates we might designate...
$H^3$ as the half-space with $z$-coordinate positive, and specify $a = (1, 0, 0)$, $b = (0, 1, 0)$, and $c = (0, 0, 0)$.

For any point $x \in H^3$ we have $T_x H^3$ is naturally identified with $\mathbb{R}^3$. Therefore it is very easy to generate vector fields in metric coordinates. In fact a vector field $V : H^3 \to TH^3$ given by

$$V_C(x) := (f(x_a, x_b, x_c), g(x_a, x_b, x_c), h(x_a, x_b, x_c))$$

for any locally Lipschitz $f, g, h : \mathbb{R}^3 \to \mathbb{R}$ will always be well-defined; and by Theorem 21 and Proposition 36, $V$ is guaranteed to have unique solutions for any initial condition in $H^3$, and furthermore the solutions are continuous with respect to both metrics, $d_C$ and $d$. Finding the actual solutions amounts to solving the problem as if it were a traditional vector field $V : \mathbb{R}^3 \to \mathbb{R}^3$ and then restricting the solutions to their domains of definition in $H^3$.

If we stipulate for all $w$ that $V_b = -V_a$ then all solutions will have $\frac{d}{dt} [\sigma_a(t) + \sigma_b(t)] = 0$. With reference to Example 2, this means the solutions are restricted to ellipsoids with foci $a$ and $b$ since $\sigma_a + \sigma_b$ remains constant. Alternatively, if $V_a = V_b$ then the flows are restricted to hyperboloids with foci $a$ and $b$. When $V_a = 0$ then the flows are restricted to spheres with center $a$.

Define, for example, the vector field $V : H^3 \to TH^3$ by

$$V_a(x) := 1$$
$$V_b(x) := -V_a(x) = -1$$
$$V_c(x) := 0.$$  \hspace{1cm} (17)

For an initial condition $x$ in metric coordinates $x_C = (x_a, x_b, x_c) \in H^3$, the solution $\sigma$ follows the intersection of the ellipsoid with foci $a$ and $b$ which touches $x$ and the sphere centered at $c$ which touches $x$. The formula is found by regular integration of (17) to be

$$\sigma_C(t) = (x_a + t, x_b - t, x_c)$$

in metric coordinates. One particular solution $\sigma$ is graphed in Figure 40. That the graph of $\sigma$ is given by the intersection of an ellipsoid and sphere is illustrated in Figure 44.
Alternatively examine the metric-coordinate vector field given by

\[ V_a(x) := 1 \]
\[ V_b(x) := -V_a(x) = -1 \]
\[ V_a(x) := V_b(x) = 1. \]

Then for an initial condition \( x \in H^3 \), the solution \( \sigma \) follows the intersection of the ellipsoid with foci \( a \) and \( b \) which touches \( x \) and the hyperboloid with foci \( a \) and \( b \) touching \( x \). The formula is simply

\[ \sigma_C(t) = (x_a + t, x_b - t, x_c - t). \]

On the boundary metric-coordinate vector fields are not so easily generated since the tangent spaces are not all of \( \mathbb{R}^3 \).
Example 41 Let \( M = X := S^2 \) be the Euclidean sphere with radius 1 and intrinsic metric \( d(x, y) \) given by the length of a shortest geodesic connecting \( x \) and \( y \). Metrically coordinatize \( S^2 \) with \( C := \{a, b, c\} \) where the three points are chosen so that

\[
d(a, b) = d(b, c) = d(c, a) = \frac{\pi}{2}.
\]

We wish to have solutions follow hyperbolic paths on \( S^2 \) with foci \( a \) and \( b \). We thus need to define \( V : S^2 \to TS^2 \subset \mathbb{R}^3 \) (by suppressing the notation of \( \pi \)) with

\[
V_a(x) := f(d(x, a), d(x, b), d(x, c)) = f(C) \\
V_b(x) := g(d(x, a), d(x, b), d(x, c)) = g(C) \\
V_c(x) := h(d(x, a), d(x, b), d(x, c)) = h(C)
\]

where the functions \( f, g, h : \mathbb{R}^3 \to \mathbb{R} \) are Lipschitz with \( f = g \). Some further conditions on \( f, g, \) and \( h \) are necessary to get a bona-fide map into \( TS^2 \). The hyperbolic paths of the solutions to \( V \) will be perpendicular to the great circle \( S^1 \) through \( a \) and \( b \) which is given by

\[
S^1 := \left\{ x : d(x, c) = \frac{\pi}{2} \right\}
\]

Thus the rate of change of the distance from \( a \) to a solution curve \( \sigma \) will be 0 if \( \sigma \) passes through \( S^1 \); i.e., for \( \sigma(t) = x \in S^1 \) we need \( \sigma_a^+(t) = V_a(x) = 0 \). Thus make

\[
f(u, v, w) = 0 \quad \text{for} \quad w = d(x, c) = \frac{\pi}{2}.
\]  

(18)

E.g.,

\[
f(u, v, w) := \left( \frac{\pi}{2} - w \right)
\]

with \( g = f \), then \( h \) is determined by \( f \) and \( g \) and by continuity. This gives a hyperbolic flow with foci \( a \) and \( b \). The direction of the flow is determined by

\[
f(u, v, w) > 0 \quad \text{for} \quad w > \frac{\pi}{2} \quad \text{and} \quad f(u, v, w) < 0 \quad \text{for} \quad w < \frac{\pi}{2}.
\]  

(19) (20)

Thus on the hemisphere bounded by \( S^1 \) containing \( c \) the flow is toward \( a \) and \( b \). On the complimentary hemisphere solutions move away from \( a \) and \( b \).
Example 42 The fact that the metrics $d_C$ and $d$ are not always equivalent may be exploited to give discontinuous flows. Let $M := \mathbb{R}^2$ with Euclidean metric $d$ and let $X$ consist of the infinite strips

$$X := \mathbb{R} \times \left( (0, 1) \cup \bigcup_{n \in \mathbb{N}} ([2n, 2n + 1) \cup (-2n - 2, -2n - 1]) \right).$$

Let $C := \{a = (0, 0), b = (1, 0)\}$, so that $(X, d_C)$ may be identified with an open half plane and is locally compact. A locally Lipschitz vector field $V : (X, d_C) \to (TX, d^T_C)$ is given by $V_a(x) := 1$ and $V_b(x) := 1$ so that as before we get a hyperbolic flow on the half plane. Such solutions, however, are discontinuous with respect to $d$, jumping from strip to strip.

Example 43 (Observer dependence of tangent spaces) On general metric spaces the tangent space at a point fundamentally depends on the choice of metric coordinatizing set. The metric space from Example 15 is an obvious candidate to consider and does give the result we seek: with respect to $\{a = (-2, 0)\}$ the tangent space at $(0, 0)$ may be identified with $\mathbb{R}$ (via the weak contraction $\pi$ where $\pi([\phi]) := \phi_C^*(0)$) and with respect to $\{b = (1, 1)\}$ the tangent space $T_{(0,0)}X$ is $\mathbb{R}^- := (-\infty, 0]$. The discrepancy arises because a curve issuing from $(0, 0)$ in the direction of the point $(-1, 1)$ with finite Euclidean speed will be tangent to the circle with center $b = (1, 1)$ and radius $\sqrt{2}$. Hence, the rate of change of distance will be zero with respect to the metric coordinate $b = (1, 1)$; therefore there is no positive representative in $T_{(0,0)}X$. With respect to $a = (-2, 0)$ however, a curve can issue from $(0, 0)$ with positive or negative $y$-metric-coordinate derivative.

If we were to allow curves with infinite speed at $t = 0$ to represent members of $T_{(0,0)}X$ we could recover all of $\mathbb{R}$ for the tangent space with respect to $\{(1, 1)\}$. For example, the curve $\phi(t) := (-\sqrt{t}, \sqrt{t})$ has

$$\phi_b'(0) = \frac{1}{\sqrt{2}}.$$ 

so that $\phi_b'(0) = \frac{1}{\sqrt{2}}$. Then the tangent spaces in this example would be topologically equivalent with respect to different coordinate systems. Still, there exist metric coordinate systems where this fails to patch up the disparity as in the following:

Example 44 In $M := \mathbb{R}^2$ with the Euclidean metric $d$, choose

$$X := \left\{ \left( t, t \sin \left( \frac{1}{t} \right) \right) \in \mathbb{R}^2 : 0 < |t| \leq 1 \right\} \cup \{(0, 0)\}.$$
Let $c_0 = (0, 1), c_1 = (1, 1), c_2 = (-1, 0), c_3 = (1, 0)$, and $x := (0, 0)$. Notice that $C_1 := \{c_0, c_1\}$ and $C_2 := \{c_2, c_3\}$ each metrically coordinatize $X$.

Notationally use $T^C_x X$ to denote the tangent space of $X$ at $x$ relative to $C_i$. Without providing the voluminous details, we claim that $T^{C_1}_x X$ is the singleton 0 while $T^{C_2}_x X$ is $\mathbb{R}$. Infinite or even 0 speed reparametrizations will not recover any other elements of $T^C_x X$.

**Example 45** To amplify the last example, we show that there are dynamics describable with a vector field with respect to one coordinatizing set which are not achievable by any vector field with respect to another coordinatizing set.

Consider $M := \mathbb{R}^2$ with the Euclidean metric $d$ and $X := \{(x, |x|) : -1 \leq x \leq 1\}$. Then $C_1 := \{(-1, 1)\}$ and $C_2 := \{(1, 1)\}$ each coordinatize $X$. The curve $\phi(t) := \begin{cases} (2\sqrt{t}, 2\sqrt{t}) & t \geq 0 \\ (t, |t|) & t < 0 \end{cases}$ has bounded derivative with respect to $C_1$ but the metric coordinate derivative of $\phi$ with respect to $C_2$ does not exist at $t = 0$. The derivative of $\phi$ then yields a vector field with respect to $C_1$ giving dynamics which cannot be described with respect to $C_2$.

10 Open questions and future directions

Is there a canonical method for coordinatizing a metric space with a minimum number of metric coordinates? A simpler question is: does every metric space $(X, d)$ have a discrete metric coordinatizing subset $C$? Minor headway on this latter question is given by:

**Remark 46** If $(M, d, X, C)$ is a metric coordinate system and $c$ is an accumulation point of $C$, then $C \setminus \{c\}$ is still a metric coordinatizing set for $X$.

**Proof.** If $c_i \to c$ and $d(x, c_i) = d(y, c_i)$ for all $i$ then $d(x, c) = d(y, c)$ by continuity of the metric. □

**Example 47** By the above remark, we can remove accumulation points, one at a time, from any metric coordinatizing set. However, it may not be possible to remove all of them. More succinctly, not every coordinatizing set has a discrete coordinatizing subset. E.g., the closed upper half plane in $\mathbb{R}^2$ given by $\mathbb{R} \times \mathbb{R}^+ = \{(x, y) : y \geq 0\}$ with the metric $d_\infty$ from
Example 6 is coordinatized by the line $C := \{(x, 0) : x \in \mathbb{R}\}$. But any discrete subset of $C$ fails to coordinatize $\mathbb{R} \times \mathbb{R}^+$. In fact any coordinatizing subset must be dense in $C$. The open question remains, however, since there does exist a discrete coordinatizing set on $(\mathbb{R} \times \mathbb{R}^+, d_\infty)$, namely the one from Example 2.

We also have the open question from the end of Section 6: characterize the metric coordinate systems $(M, d, X, C)$ for which $(X, d_C)$ is locally compact. A version of Theorem 21 which does not require local compactness is highly desirable. The imagined condition is that $(X, d_C)$ is locally complete. Thus we would also be pleased with a characterization of the metric coordinate systems $(M, d, X, C)$ for which $(X, d_C)$ is locally complete.

Next, in a metric coordinate system a new measure of the dimension of a metric space presents itself.

**Definition 48** Let $I$ be a cardinal number. The metric space $(M, d)$ is locally $I$-coordinatizable if for each $x \in M$ there exists a neighborhood $X$ and a set $C \subset M$ of cardinality $I$ which metrically coordinatizes $X$. The smallest such cardinal number $I$ is called the metric-coordinate dimension.

Metric-coordinate dimension is not a homeomorphic invariant. For example the Koch curve is homeomorphic to $\mathbb{R}$ but has metric-coordinate dimension 2.

**Conjecture 49** Metric-coordinate dimension is a lipeomorphic invariant.

Next, what is the most appropriate definition for metric-coordinate-wise differentiability of maps between metric spaces? Which brings us to question what conditions give an Inverse Function Theorem on coordinatized metric spaces (this has been done before on metric spaces using the structure of “mutations”, [3]).

Higher order derivatives are obviously defined with $\phi''_C(t) := \left(\frac{d}{dt}\phi_c(t)\right)$.

How do we analyze higher-order differential equations?

The directional derivative $D_Vf$ of a function $f : X \to \mathbb{R}$ on a metric coordinate system in the direction of a metric-coordinate vector field $V$ can be defined as

$$D_Vf(x) := \lim_{h \to 0^+} \frac{f(\phi(h)) - f(x)}{h} \quad (21)$$

---

5 A lipeomorphism between metric spaces $M$ and $N$ is a bijective Lipschitz map between $M$ and $N$ with Lipschitz inverse.
assuming the limit exists and does not depend on the representative \( \phi : [0, \delta) \to X \) of the equivalence class \( V(x) \). This notion is useful in analyzing the qualitative dynamics of metric-coordinate vector fields using Lyapunov functions \( f \) as will be demonstrated in a forthcoming paper. Perhaps we can also use the directional derivative to analyze extrema and extract the Lagrange multiplier method for constraints. Certainly the fundamental theorem of line integrals should have an expression on metric spaces with \( D_V f \). What can be made of Stokes’ Theorem?

PDE’s on metric spaces should be possible to formulate with these directional derivatives.

When we consider non-autonomous (i.e., time-dependent) metric-coordinate vector fields, we might allow the location of the metric coordinatizing points \( c \) to change in time as well; giving us bonus descriptive power not available with Cartesian coordinates.

Finally one might abandon the goal of finding coordinatizing sets. Begin with any set \( C \subset M \) and define the quotient space \( X/\sim \) with the equivalence relation \( x \sim y \) if \( d(x,c) = d(y,c) \) for all \( c \in C \). Then work in the metric space \( (X/\sim, d_C) \) which is identified with a subset of \( \mathbb{R}_0^C \).

References

[1] Aubin, J.-P., “Mutational and Morphological Analysis”, *Birkhäuser* (1999).
[2] Aubin, J.-P., “Viability Theory”, *Birkhäuser* (1991).
[3] Calcaterra, Craig and David Bleecker, Generating Flows on a Metric Space, *Journal of Mathematical Analysis and Applications* **248** (2000), 645-667.
[4] Ellis, David, On Separable Metric Spaces, *Revista. Series A. Matemática y física teorica* **8** (1951), 15-18.
[5] Goldberg, Karl, Distance Coordinates with Respect to a Triangle of Reference, *Journal of Research of the National Bureau of Standards, Section B Mathematical Sciences* **76B** (1972), 145-152.
[6] Havel, Timothy F., Some Examples of the Use of Distances as Coordinates for Euclidean Geometry, *Journal of Symbolic Computation* **11** n5-6 (1991), 579-593.
[7] McShane, E. J., Extensions of Range of Functions, *Bull. Am. Math. Soc.* **40** (1934), 837-842.
[8] Panasyuk, A. I., Quasidifferential Equations in Metric Spaces (Russian), *Differentsial’nye Uravneniya* **21** (1985), 1344-1353; English translation: *Differential Equations* **21** (1985), 914-921.
[9] Panasyuk, A. I., Quasidifferential Equations in a Complete Metric Space under Conditions of the Caratheodory Type. II, *Differential Equations* **31**, No. 8 (1995), 1308-1317.
[10] Pavel, N. H., “Differential equations, flow invariance and applications”, (Research notes in mathematics; 113), Pitman Publishing Inc., 1984.

[11] Titchmarsh, E. C., “The Theory of Functions”, 2nd Ed., Oxford University Press (1939).

[12] Volkmann, Peter, The Positive Invariant Set of the Differential Equation on Banach Space, *Ann. of Diff. Eqs.* 14 (1998), 267-270.