A REMARK ON THE CONCENTRATION PHENOMENON FOR THE $L^2$-CRITICAL NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We observe a link between the window size of mass concentration and the rate of explosion of the Strichartz norm by revisiting Bourgain’s mass concentration for the $L^2$-critical nonlinear Schrödinger equations.

1. Introduction

Consider the initial value problem for the $L^2$-critical case of nonlinear Schrödinger equation, $NLS^\pm_p(\mathbb{R}^d)$, $p = \frac{4}{d} + 1$,

$$\begin{cases}
iu_t + \Delta u = \pm |u|^{p-1} u, \\
u(0, x) = u_0(x),
\end{cases}$$

where $u = u(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ and $u_0 \in L^2(\mathbb{R}^d)$. $NLS^+_p(\mathbb{R}^d)$ is called defocusing; $NLS^-_p(\mathbb{R}^d)$ is focusing.

Denote by $[0, T^*)$ the maximal (forward) existence time interval of solution $u(t, x)$. For our purposes this means that any Strichartz norm $\|u\|_{L^q_t L^r_x([0, T^*) \times \mathbb{R}^d)} = \infty$ and $\|u\|_{L^q_t L^r_x([0, T^*) \times \mathbb{R}^d)} < \infty$ for all $t < T^*$. Here, the pair $(q, r)$ is admissible, i.e. $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$.

In a breakthrough work [2], Bourgain established the mass concentration phenomenon for finite time blowup solutions of the cubic $NLS$ in $d = 2$ with an $L^2$ initial data (independent of focusing or defocusing case): consider $NLS^+_2(\mathbb{R}^2)$ in (1.1) with $u_0 \in L^2(\mathbb{R}^2)$; if the blow up time $T^* < \infty$, then $L^2$-norm concentration on a parabolic window occurs

$$\limsup_{t \nearrow T^*} \sup_{a \in \mathbb{R}^2} \int_{B(a, c(T^* - t)^{\frac{d}{4}})} |u(t, x)|^2 \, dx \geq \epsilon,$$

where $\epsilon = \|u_0\|_{L^2}^{-M}$ for some $M > 0$. The proof used a refinement of the Strichartz estimate obtained by Moyua, Vargas and Vega [2] and mass conservation. Compactness properties of blowup solutions modulo symmetries were obtained by Merle and Vega [6] for the two dimensional case. Recently, Carles and Keraani in [3] obtained the corresponding results for quintic $NLS$ in $d = 1$ and Bégout and Vargas in [1] extended the mass concentration and compactness modulo symmetries results for all dimensions $d$ with nonlinearities $p = \frac{4}{d} + 1$ ($L^2$-critical). The necessary refinement of the Strichartz inequality for all dimensions comes from the bilinear Fourier restriction theorem obtained by Tao [10, Theorem 1.1].

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\footnotetext{The pair $q = \infty, r = 2$ is obviously omitted from this claim since the $L^2$ norm is conserved under the \textsc{ft} evolution.}
In this note we investigate the dependence of the window of mass concentration upon the growth of the $L_{t,x}^{\frac{2(d+2)}{d+1}}([0,t] \times \mathbb{R}^d)$-norm. We show that if, close to the blow up time $T^*$, the $L_{t,x}^{\frac{2(d+2)}{d+1}}([0,t] \times \mathbb{R}^d)$-norm grows in time no slower than $(T^* - t)^{-\beta}$, then the window of concentration is of width $(T^* - t)^{\frac{1}{2}+\frac{\beta}{2}}$. We also obtain the opposite direction, namely, if the mass concentration has the concentration window of size $(T^* - t)^{\frac{1}{2}+\frac{\beta}{2}}$, then the growth of $L_{t,x}^{\frac{2(d+2)}{d+1}}([0,t], \times \mathbb{R}^d)$-norm is no slower than $(T^* - t)^{-\beta}$. For the first direction we revisit the argument of Bourgain and the extension to all dimensions by Bégout and Vargas. For the opposite direction we use a restriction on frequencies (which shows up implicitly in Bourgain’s argument) in order to connect the $L^2_{x}$-concentration with the space-time $L_{t,x}^{\frac{2(d+2)}{d+1}}$-norm explosion. We also generalize the above results to the setting of non-polynomial growth and concentration rates. The result we obtain shows that if the $L_{t,x}^{\frac{2(d+2)}{d+1}}$ norm explodes like $f(T^* - t)$ for certain $f \uparrow \infty$ as $t \nearrow T^*$, then the concentration window shrinks at the rate $\left[-(\partial_t f)(T^* - t)\right]^{-\frac{1}{2}}$ and vice versa. As a corollary to [2] and [1] which proved that parabolic mass concentration occurs, we obtain that the blow up of the diagonal Strichartz norm must be at least as fast as $|\ln(T^* - t)|$ (see Corollary [5,6]).

**Remark 1.1.** Explicit blowup solutions for (1.1) in the focusing case have been obtained as the pseudoconformal image of ground and excited state solitons. These solutions have mass concentration windows shrinking like $(T^* - t)^{-\frac{3}{2}}$ and their Strichartz norm $L_{t,x}^{\frac{2(d+2)}{d+1}}([0,t] \times \mathbb{R}^d)$ explodes like $(T^* - t)^{-1}$ with $\beta = 1$. Another family of blowup solutions is known (see [8] and [11, 5]) which concentrates mass slightly faster (by $\sqrt{\log \log(T^* - t)}$) than $\beta = 0$. It would be interesting to observe or rule out other blowup concentration/explosion rates.

**Remark 1.2.** It is conjectured that the defocusing problem (1.1) with the minus sign is globally well-posed and scatters for all $L^2$ data. We hope that the results obtained here may be useful in proving that no concentration occurs in the defocusing problem. For example, in light of Corollary [2.6] global well-posedness and scattering follows if finite time blowup solutions are shown to have sub-logarithmic Strichartz norm explosion. Also, a result which rules out very tight concentration windows would imply upper bounds on the blowup rate of the Strichartz norm. No general upper bounds on the rate of blowup are known.

**Notation.** Denote by $l(J)$ the side length of a cube $J \subset \mathbb{R}^d$ and $|J|$ its Lebesque measure; $\mathcal{D}$ is the set of dyadic cubes in $\mathbb{R}^d$ with $\tau_k^d = \prod_{i=1}^{d} \left[ \frac{k_i}{2^j}, \frac{k_i+1}{2^j} \right]$ a dyadic cube, and when there is no confusion the indices will be dropped $\tau = \tau_k^d$; for $a \in \mathbb{R}^d$ and $r > 0$ the set $B(a,r) = \{x \in \mathbb{R}^d : |x-a| < r\}$ is an open ball of radius $r$.

For a measurable set $E \subset \mathbb{R}^d$, denote by $P_E$ the Fourier restriction with respect to the $x$-variable: $P_E \psi = \hat{\psi} \chi_E$. The linear evolution of the Schrödinger equation in (1.1) is denoted by $e^{it\Delta}$, i.e.

$$e^{it\Delta} f(x) = \int_{\mathbb{R}^d} e^{2\pi i(x \cdot \xi - 2\pi t |\xi|^2)} \hat{f}(\xi) \ d\xi.$$
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2. Strichartz norm explosion $\Rightarrow$ tight concentration window

First, we show the dependence of the size of mass concentration window upon the divergence rate of $\|u\|_{L^{2(d+2)}([0,t] \times \mathbb{R}^d)}$-norm.

Proposition 2.1. Suppose that $T^* < \infty$ and

$$\|u\|_{L^{2(d+2)}([0,t] \times \mathbb{R}^d)} \gtrsim \frac{1}{(T^* - t)^\beta} \text{ for some } \beta > 0.$$  \hfill (2.1)

Then there exists $\epsilon > 0$ such that $\epsilon = \|u_0\|_{L^{2}([0,T])}$, when $c(d) = O(d^4)$,

$$\limsup_{t \nearrow T^*} \sup_{\text{cubes } J \in \mathbb{R}^d : l(J) < (T^* - t)^{\frac{3}{d} + \frac{3}{2}}} \int_J |u(t,x)|^2 \, dx \geq \epsilon.$$ \hfill (2.2)

Furthermore, for any $0 < t < T^*$ there exists a cube $\tau(t) \subseteq \mathbb{R}^d$ of size $l(\tau(t)) \gtrsim (T^* - t)^{-\left(\frac{1}{d} + \frac{3}{2}\right)}$ such that

$$\limsup_{t \nearrow T^*} \sup_{\text{cubes } J \in \mathbb{R}^d : l(J) < (T^* - t)^{\frac{3}{d} + \frac{3}{2}}} \int_J |P_{\tau(t)}u(t,x)|^2 \, dx \geq \epsilon.$$ \hfill (2.3)

Thus, a lower bound on the Strichartz explosion implies tight mass concentration along a sequence of times. Moreover, the tight concentration may be frequency localized to the natural scale.

Proof. We follow [2] where the mass concentration in the space dimension $d = 2$ is established keeping in mind the generalization to all space dimensions from [1].

First, recall that a time sequence $\{t_n\} \nearrow T^*$ is chosen such that for any $n$

$$\|u\|_{L^{2(d+2)}((t_n,t_{n+1}) \times \mathbb{R}^d)} = \eta$$ \hfill (2.4)

for some small $\eta$. The decomposition $[0,T^*) = \bigcup_{n=0}^{\infty} (t_n,t_{n+1})$ will play an important role throughout this paper. If $\eta > 0$ is small enough, then on the interval $(t_n,t_{n+1})$ the nonlinear part of the evolution $u(t_n) \mapsto u(t)$ is insignificant compared to the linear flow $e^{i(t-t_n)\Delta}u(t_n)$:

$$\|u - e^{i(t-t_n)\Delta}u(t_n)\|_{L^{2(d+2)}((t_n,t_{n+1}) \times \mathbb{R}^d)} \leq \|u\|_{L^{2(d+2)}((t_n,t_{n+1}) \times \mathbb{R}^d)}^{\frac{d+1}{2}} = \eta^\frac{d+1}{2},$$ \hfill (2.5)

and thus,

$$\|e^{i(t-t_n)\Delta}u(t_n)\|_{L^{2(d+2)}((t_n,t_{n+1}) \times \mathbb{R}^d)} \sim \eta.$$ \hfill (2.6)

\[b\] For example, for $\mathbb{R}^2$ the argument in (2.2) gives $c(d) \sim 292$; see the proof for general $d$.  

\[\]
We impose \( \eta < \min(1, \frac{1}{2\pi}(T^*)^{-\beta}) \), where \( c \) is the implicit constant in (2.1). Then the bound (2.1) implies

\[
(2.7) \quad \eta = \|u\|_{L^{\frac{2(d+2)}{d}}((t_n, t_{n+1}) \times \mathbb{R}^d)} \gtrsim \frac{(t_{n+1} - t_n)}{(T^* - t_n)^{\beta+1}},
\]

deleting the sequence \( \{t_n\} \) has the following property

\[
(2.8) \quad t_{n+1} - t_n \lesssim \eta (T^* - t_n)^{\beta+1}.
\]

Fix \( n \in \mathbb{N} \) and the time interval \((t_n, t_{n+1})\). Denote \( f(x) = u(t_n, x) \), note that \( \|f\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)} \) by mass conservation. Using the Squares Lemma ([32 in [2] and [1] Lemma 3.1]), we obtain the following localizations in frequency.

- For any \( \epsilon_0 > 0 \) there exist \( N_0 = N_0(\|f\|_{L^2}, d, \epsilon_0) \) and a finite collection \( \{f_j\}_{j=1}^{N_0} \in L^2(\mathbb{R}^d) \) with supp \( \hat{f}_j \subseteq \tau_j \) - a cube in \( \mathbb{R}^d \), \( l(\tau_j) \leq c(\|f\|_{L^2}, \eta, \epsilon_0) \).

\[
(2.9) \quad \|e^{\iota \Delta} f - \sum_{j=1}^{N_0} e^{\iota \Delta} f_j\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R} \times \mathbb{R}^d)} < \epsilon_0.
\]

Expand (2.4) and apply (2.5) and (2.6)

\[
\eta^{\frac{2(d+2)}{d}} = \int_{(t_n, t_{n+1}) \times \mathbb{R}^d} \left[ u(t,x) \left( e^{\iota (t-t_n) \Delta} f + (u(t,x) - e^{\iota (t-t_n) \Delta} f) \right) \right.
\]

\[
\times \left| e^{\iota (t-t_n) \Delta} f + (u(t,x) - e^{\iota (t-t_n) \Delta} f) \right|^2 \left. \right] dx \, dt
\]

\[
= \int_{(t_n, t_{n+1}) \times \mathbb{R}^d} u(t,x) \left| e^{\iota (t-t_n) \Delta} f \right|^2 \left( e^{\iota (t-t_n) \Delta} f \right) \left. \right] dx \, dt
\]

\[
+ O(\eta^{2(\beta+1)}) \quad \text{if} \quad d \leq 4,
\]

\[
+ O(\eta^{2+\beta(\beta+1)}) \quad \text{if} \quad d > 4.
\]

Choose \( \epsilon_0 = \eta^{\frac{d+4}{d+4}}. \) Then the above estimate together with (2.8) may be rewritten as

\[
(2.10) \quad \eta^{\frac{2(d+2)}{d}} = \int_{(t_n, t_{n+1}) \times \mathbb{R}^d} u(t,x) \left( \sum_{j=1}^{N_0} e^{\iota (t-t_n) \Delta} f_j(x) \right) \left[ \sum_{j=1}^{N_0} e^{\iota (t-t_n) \Delta} f_j(x) \right]^2 \left. \right] dx \, dt
\]

\[
+ O(\eta^{2(\beta+1)}) \quad \text{if} \quad d \leq 4,
\]

\[
+ O(\eta^{2+\beta(\beta+1)}) \quad \text{if} \quad d > 4.
\]

Note that \( 2(\beta+1) > \frac{2(d+2)}{d} \) or \( 2 + \frac{d}{4} + 1 > \frac{2(d+2)}{d} \) for any \( d > 0 \). Since \( N_0 \) is finite, it follows from (2.10) that there exists \( j_0 \) \((1 \leq j_0 \leq N_0)\) such that

\[
(2.11) \quad \frac{\eta^{\frac{2(d+2)}{d}}}{2 N_0^{\frac{d+4}{d} \cdot \beta}} \leq \int_{(t_n, t_{n+1}) \times \mathbb{R}^d} u(t,x) \left( e^{\iota (t-t_n) \Delta} f_{j_0}(x) \right) \left( e^{\iota (t-t_n) \Delta} f_{j_0}(x) \right)^2 \left. \right] dx \, dt.
\]
Denote $\tau = \tau_0$, $A = A_{\tilde{\eta}} \left( I(\tau) \leq c_{0} A \right)$ and the center of $\tau$ by $\xi_0$. Since $|\hat{f}_{\eta}| < A^{-d/2}$ and $\hat{f}_{\eta}$ is supported in $\tau$ with $l(\tau) = c_0(\|f\|_{L^2}, \eta) \cdot A$, we obtain

$$\sup_t \|e^{i(\tau-t_n)\Delta} f_{\eta}\|_{L^2} \leq A^{-d/2} |\tau| = c A^{d/2} \quad \text{with} \quad c = c(\|f\|_{L^2(R^d)}, \eta).$$

Hence, (2.11) becomes

$$\frac{1}{2} \frac{2(d+2)}{N_0^{1+4/d}} \leq A^2 \left| \int_{(t_n,t_{n+1}) \times \mathbb{R}^d} u(t,x) (e^{i(t-t_n)\Delta} \hat{f}_{\eta}(x)) \, dx \, dt \right|. $$

Using Plancherel and supp $\hat{f}_{\eta} \subseteq \tau$, we obtain a refined version of the previous inequality

$$\frac{1}{2} \frac{2(d+2)}{N_0^{1+4/d}} \leq A^2 \int_{(t_n,t_{n+1}) \times \mathbb{R}^d} |P_\tau u(t,x)| \| e^{i(t-t_n)\Delta} f_{\eta}(x) \| dx \, dt. $$

Using the Tubes Lemma ($\S 3$ in $[2]$ and $[1]$, Lemma 3.3$)$, we obtain a further space-time localization:

- Let $\epsilon_1 = \left( \frac{1}{4N_0} \right)^{d/4} \eta$. Then there exist $N_1 = N_1(\|f\|_{L^2}, d, \epsilon_1)$ and a sequence of tubes $\{ Q_k \} = \{ I_k \times K_k(t) \} \subseteq \mathbb{R} \times \mathbb{R}^d$, where $|I_k| = \frac{1}{N_0}$ and $K_k(t) = 4\pi t \xi_0 + C, C \in D$, with $l(C) = \frac{1}{N_0^4}$ such that

$$\| e^{i(t-t_n)\Delta} f_{\eta} \|_{L^2(\mathbb{R} \times \mathbb{R}^d \setminus \bigcup_{k=1}^{N_1} Q_k)} < \epsilon_1. $$

By (2.12) and (2.13), we obtain

$$\frac{1}{4} \frac{2(d+2)}{N_0^{1+4/d}} \leq A^2 \int_{(t_n,t_{n+1}) \times \mathbb{R}^d \cap Q_{k_0}} \left| P_\tau u(t,x) \right| \| e^{i(t-t_n)\Delta} f_{\eta}(x) \| dx \, dt. $$

Since the number of tubes $N_1$ is finite, there exists a tube $Q_{k_0} = I \times K(t)$ such that (2.14) produces

$$\int_{(t_n,t_{n+1}) \times \mathbb{R}^d \cap Q_{k_0}} \left| P_\tau u(t,x) \right| \| e^{i(t-t_n)\Delta} f_{\eta}(x) \| dx \, dt. $$

Applying Cauchy-Schwarz we obtain

$$\frac{1}{4} \frac{2(d+2)}{N_0^{1+4/d}} \leq A \left( \int_{(t_n,t_{n+1}) \times \mathbb{R}^d \cap Q_{k_0}} \left| P_\tau u(t,x) \right|^2 \, dx \, dt \right)^{1/2}. $$

Since $N_0^{\frac{d+1}{2}} \sim \| f \|_{L^2}^{(d)}$, using the conservation of mass, we get

$$c \leq A \| u_0 \|_{L^2} (t_{n+1} - t_n)^{1/2}, \quad \text{where} \quad c = c(\|u_0\|_{L^2}, d, \eta).$$

By (2.8), we obtain

$$\frac{1}{A} \leq c(T^{*} - t_n)^{\frac{d+1}{2}} \quad \text{again with} \quad c = c(\|u_0\|_{L^2}, d, \eta).$$

c) We estimate $c(d) \approx \frac{(d+4)(d+3)(d^2 + 3d + 4)^2}{(d+1)(d+2)}$. 
Considering (2.16) again, we have

\begin{equation}
\tag{2.17}
c \eta^{2(d+2)} \leq A^2 \int_{I \cap (t_n, t_{n+1})} \int_{K(t)} |\mathcal{P}_r u(t, x)|^2 dx \, dt
\end{equation}

\begin{equation}
\tag{2.18}
\leq \sup_{t \in I \cap (t_n, t_{n+1})} \int_{K(t)} |\mathcal{P}_r u(t, x)|^2 dx.
\end{equation}

Therefore, there exists a mass concentration time $t_n^* \in I \cap (t_n, t_{n+1})$ such that

\begin{equation}
\tag{2.19}
\int_{K(t_n^*)} |\mathcal{P}_r u(t_n^*, x)|^2 dx \geq c' \eta^{4(d+2)}.
\end{equation}

The limsup claim in (2.3) will be realized along the sequence $t_n^* \nearrow T^*$. Recall that $l(K(t_n^*)) = \frac{1}{A} < c(T^* - t_n)^{\frac{d+1}{2}}$, and therefore, $K(t_n^*) \subseteq B(a, \sqrt{d}c(T^* - t_n)^{\frac{d+1}{2}})$ for some $a \in \mathbb{R}^d$. Observe that

\begin{equation}
\tag{2.20}
T^* - t_n^* > T^* - t_{n+1} = T^* - t_n - (t_{n+1} - t_n)
\end{equation}

\begin{equation}
\geq (T^* - t_n)(1 - c\eta(T^* - t_n)^{\beta}) > \frac{1}{2}(T^* - t_n),
\end{equation}

where the last estimate follows from $c \eta(T^* - t_n)^{\beta} < c \eta(T^*)^{-\beta} < \frac{1}{2}$. Hence, $B(a, \sqrt{d}c(T^* - t_n)^{\frac{d+1}{2}})$ (and thus, $K(t_n^*)$) can be covered by a finite number of balls (or cubes) of radius (side length) $(T^* - t_n)^{\frac{d+1}{2}}$ (and this number is independent of $n$). Choosing one of them, and noting that $l(\tau) \geq \frac{c}{(T^* - t_n)^{\frac{d+1}{2}}} \approx \frac{c}{(T^* - t_n)^{\frac{d+1}{2}}}$, we get

\begin{equation}
\int_{B(a, \sqrt{d}(T^* - t_n)^{\frac{d+1}{2}})} |\mathcal{P}_r u(t_n^*, x)|^2 dx \geq \epsilon,
\end{equation}

and since $n$ is arbitrary, the proposition follows.

\begin{remark}
We did not use the splitting of the interval $(t_n, t_{n+1})$ as on page 261 in [2], since we had the estimate (2.3) of $(t_{n+1} - t_n)$ in terms of $(T^* - t_n)^{\beta+1}$, $\beta + 1 > 1$, which gives a nonzero bound in (2.20). In Bourgain's argument $\beta = 0$, i.e. $(t_{n+1} - t_n) < (T^* - t_n)$, which is not enough to conclude mass concentration with the above argument, and thus, a more careful splitting of the time interval is needed.
\end{remark}

Note that the construction of $t_n^*$ given in the proof above provides more information about the mass concentration than is claimed in (2.3). For example, we know that there is a concentration time $t_n^*$ in each of the time intervals $[t_n, t_{n+1})$. The next statement contains a strengthened conclusion which shows that the concentration actually holds on a thickened interval of times containing $t_n^*$ of size proportional to $t_{n+1} - t_n$.

\begin{corollary}
Assume the hypotheses of Proposition (2.4). The conclusion (2.3) may be strengthened as follows: There exist $0 < \sigma < \frac{1}{4}$ and a sequence of time

\begin{itemize}
\item[(d)] If $\sigma = 0$, then this statement coincides with the theorem by choosing $I_n = \{t_n^*\}$.
\end{itemize}

\end{corollary}
intervals \(\{I_n\}\) with \(I_n \subset (t_n, t_{n+1})\) and \(|I_n| = \sigma (t_{n+1} - t_n)\), uniform for all \(n\), such that for some \(\tilde{\sigma} = \tilde{\sigma}(\sigma) > 0\) we have

\[
\lim_{n \to \infty} \inf_{t \in I_n} \sup_{\text{cubes } J \in \mathbb{R}^d : l(J) < (T^* - t)^{\frac{d}{d+2}}} \int_J |P_{\tau(t)} u(x, t)|^2 \, dx \geq (1 - \tilde{\sigma}) \varepsilon. \tag{2.21}
\]

**Proof.** Recall the inequalities (2.17) - (2.18) from the proof of Proposition 2.1. From (2.18) only one concentration time \(t^*_n \in I \cap (t_n, t_{n+1})\) was selected such that (2.19) holds. However, (2.17) contains a stronger statement, namely, on each set \(I \cap (t_n, t_{n+1})\) there exists a subset \(E_n\) such that for any \(t \in E_n\) we have

\[
c' \eta^{\frac{4(d+2)}{d}} \leq \int_{K(t)} |P_\tau u(x, t)|^2 \, dx. \tag{2.22}
\]

Note that since \(u_0 \in L^2_\varepsilon\), by the local well-posedness and mass conservation, \(u \in C^0_\varepsilon(L^2_\varepsilon)\), and so \(P_\tau u(t)\) is also continuous in time, which means that the set \(E_n\) above can be chosen to be an interval, denote it by \(I_n\). Next we estimate how large \(I_n\) can be in comparison with \(I \cap (t_n, t_{n+1})\). First, recall that \(|I| = \frac{1}{A}\) and \(\frac{1}{A} \lesssim (t_{n+1} - t_n)\). Since the cube \(\tau \in \mathbb{R}^d\) has the center \(\xi_0\) and side length \(l(\tau) \leq c_0 A\), the function \(P_\tau u\) (on the time interval \((t_n, t_{n+1})\)) contains frequencies \(\xi \in \tau\), and thus, \(|\xi| \leq c(\xi_0, c_0) \cdot A\). By the uncertainty principle (for example, p. 332 of [11]) \(P_\tau u\) is approximately constant on spatial balls of radius \(\frac{1}{A}\) for some small \(c\), in particular, since \(l(K(t)) = \frac{1}{A}\), it will be approximately constant on some fixed part of \(K(t)\). By the propagation of Schrödinger waves, this set will persist for an interval of times of measure \(\sim \frac{1}{A}\) (after which it may disperse). This length scale is exactly comparable with the size of \(I\) (note independently of the step \(n\)), so we can find \(0 < \sigma < 1\) such that \(|I_n| = \sigma |I \cap (t_n, t_{n+1})|\) for all \(n\) and (2.22) holds for all \(t \in I_n\) and some \(\tilde{\sigma} > 0\):

\[
(1 - \tilde{\sigma}) c \eta^{\frac{4(d+2)}{d}} \leq \int_{K(t)} |P_\tau u(x, t)|^2 \, dx. \tag{2.23}
\]

For the above heuristics we need the following lemma

**Lemma 2.4.** Let \(f \in L^2_\varepsilon(\mathbb{R}^d)\) and \(\text{supp } f \subset [0, 1]^d\). Suppose that for some constant \(c_1 > 0\)

\[
\int_{[0, 1]^d} |f(x)|^2 \, dx \geq c_1.
\]

Then for \(|t| < c_1, \|f\|_{L^2}\) the same concentration holds for the linear Schrödinger evolution of \(f\), i.e.,

\[
\int_{[0, 1]^d} |e^{it\Delta} f(x)|^2 \, dx \geq \frac{c_1}{2}.
\]

**Proof.** A basic calculation yields

\[
c_1 \leq \int_{[0, 1]^d} |f(x)|^2 \, dx \leq \int_{[0, 1]^d} |f(x) - e^{it\Delta} f(x)|^2 + e^{it\Delta} f(x)|^2 \, dx
\]

\[
\leq 2 \left( \int_{[0, 1]^d} |f(x) - e^{it\Delta} f(x)|^2 \, dx + \int_{[0, 1]^d} |e^{it\Delta} f(x)|^2 \, dx \right) := A + B.
\]

\(\) with \(\tilde{\sigma} = 0\) if \(\sigma = 0\) and \(\tilde{\sigma} = 1\) when \(\sigma = 1\).
We reexpress the integrand in A using the Fourier transform
\[ |f(x) - e^{it\Delta} f(x)|^2 \leq \int_{[0,1]^d} |(e^{-4\pi^2 t|\xi|^2} - 1) e^{2\pi ix\xi} \hat{f}(\xi)|^2 \, d\xi \]
\[ \leq \sup_{\xi \in [0,1]^d} \left| e^{-4\pi^2 t|\xi|^2} - 1 \right|^2 \int_{[0,1]^d} |\hat{f}(\xi)|^2 \, d\xi \leq 2 (4\pi^2 t)^2 \| f \|_{L^2}^2. \]
Here we used the estimate
\[ |e^{4\pi^2 t} - 1|^2 \leq (\cos(4\pi^2 t) - 1)^2 + \sin^2(4\pi^2 t) \leq 2 (1 - \cos^2(4\pi^2 t)) \leq 2 (4\pi^2 t)^2. \]
If we restrict t such that |t| \leq \frac{1}{4\pi^2 \| f \|_{L^2}^2} \sqrt{c_1}, then \( A \leq \frac{c_1}{2} \), and we obtain the conclusion of the lemma. □

We return to the proof of Corollary 2.3. The preceding lemma shows that \( L^2 \) functions which are band limited to unit scale and lower frequencies and which are mass concentrated at unit scale remain mass concentrated at unit scale for unit time under the linear Schrödinger flow. Applying the dilation invariance shows that \( L^2 \) functions which are band limited to frequencies \(|\xi| \leq A\) and which are mass concentrated on \(|x| \leq \frac{1}{A}\) will remain mass concentrated for time \(|t| \leq \frac{1}{\pi A^2}\) under the linear Schrödinger flow. Finally, using the translation and Galilean invariances, we observe that this parabolic mass concentration persistence property holds without special reference to the frequency or spatial origin.

Now the rest of the argument in the proof repeats for any \( t \in I_n \) (for example, \( (2.20) \) holds for any \( t \in I_n \) because of the fixed proportion \( \sigma \) to \((T^* - t_n)^{3+1}\) and we obtain \( (2.21) \). This completes the proof of Corollary 2.3. □

**Corollary 2.5.** The above results can be extended to a more general form of the lower bound on the Strichartz norm in (2.7). Suppose
\[ \| u \|_{L^{\frac{4\pi^2 (\sigma + 2)}{2\pi^2} + \frac{d}{2} ([0,t] \times \mathbb{R}^d)}} \gtrsim G(T^* - t), \]
where \( G(s) \to +\infty \) as \( s \to 0 \) and \( G \in C^1(0, 1) \). Then the window in the mass concentration (2.2) changes as follows:
1. if \( G(T^* - t) \gtrsim |\ln(T^* - t)|^\gamma \) with \( \gamma \geq 1 \), then \( l(J) < [-(\partial_t G)(T^* - t)]^{-1/2} \),
2. otherwise, \( l(J) < (T^* - t)^{1/2} \).

Similar changes hold for (2.3) and also in Corollary 2.3.

Observe that by the argument of Bourgain [2] and Bézout-Vargas [1] we always have the case (2). The case (1) is an improvement of (2) when \( G \) grows faster than \(|\ln(T^* - t)|\), otherwise, the argument of Proposition 2.1 gives a weaker statement, i.e. the width of the window of concentration given by \( G_t \) is narrower than the parabolic window.

**Proof.** The proof of this corollary follows the proof of Proposition 2.1 (and Corollary 2.3) with the following changes. Given \( G \) as above, the estimate (2.7) changes to
\[ \eta = \| u \|_{L^{\frac{2d + 2}{4\pi^2} + \frac{d}{2} \times \mathbb{R}^d}} \gtrsim (t_{n+1} - t_n) \left\{ -(\partial_t G)(T^* - t_n) \right\}, \]
with the constant independent of \( n \), and thus,
\[ t_{n+1} - t_n \lesssim \eta [-(\partial_t G)(T^* - t_n)]^{-1}. \]
Hence, the size of \( \tau = \tau_{j_0} \) is estimated as
\[
\frac{1}{A} \leq c\left( \|u_0\|_{L^2}^2, \eta, d \right) \left[ -(\partial_t G)(T^*-t_n) \right]^{-1/2},
\]
which implies the result in (1). Note that if \( G \) has a faster grows than \( |\ln(T^*-t)| \), then to get the parabolic window of concentration, we need the extra splitting of the interval \((t_n, t_{n+1})\) as on page 261 in [2] or in Step 3 of Prop. 4.1 in [1]. This finishes the proof. \( \square \)

As an example, consider \( G(T^*-t) = |\ln(T^*-t)|^{1+\epsilon} \), \( \epsilon > 0 \), then \( l(J) < (T^*-t)^{1/2} |\ln(T^*-t)|^{-\epsilon/2} \) which is wider than the parabolic window. If \( G(T^*-t) = \ln |\ln(T^*-t)| \), then the case (2) holds and the window of concentration is parabolic.

3. Tight concentration window \( \implies \) Strichartz norm explosion

The following statement shows how the radius of mass concentration affects the divergence rate of the \( L^{2(d+2)/d} \)-norm. We will use the shorthand notation \( P_{L(t)} \) to denote the Fourier restriction operator \( P_{|\xi| \leq L(t)} \) and \( F(t) = \|u\|_{L^{2(d+2)/d}(\mathbb{R}^d)} \).

**Proposition 3.1** (Local Estimate). Let \( u \in C([0, T^*]; L^2(\mathbb{R}^d)) \cap L^{2(d+2)/d}([0, T^*]; L^{2(d+2)/d}(\mathbb{R}^d) \) be the maximal solution of NLS \( \text{NLS}_\lambda^d(\mathbb{R}^d), p = \frac{4}{d} + 1 \), with \( u_0 \in L^2(\mathbb{R}^d) \).

For \( \epsilon > 0 \) let \( \kappa(\epsilon) = 2^{-(d+2)} \left( \epsilon \|u_0\|_{L^2}^{-2}/8 \right)^{1/d} \) and for \( \alpha > 0 \) define \( L(t) = \frac{1}{2} \kappa(\epsilon) \left( T^*-t \right)^{\alpha} \).

Suppose there exists \( \alpha \geq \frac{1}{2} \) and \( \epsilon > 0 \) such that
\[
\limsup_{t \nearrow T^*} \sup_{\text{cubes } J \subset \mathbb{R}^d: l(J) < (T^*-t)^{\alpha}} \int_J |P_{L(t)}u(t, x)|^2 \, dx \geq \epsilon.
\]

Then there exists \( t_n \searrow T^* \) such that
\[
F'(t_n) \geq \frac{1}{(T^*-t_n)^{2\alpha}}.
\]

**Remark 3.2.** If in (3.1) one has \( \liminf \) instead of \( \limsup \), then (3.3) holds for any sequence \( t_n \nearrow T^* \).

**Proof.** The condition (3.1) implies that there exists a sequence of times \( \{t_n\}_{n=1}^\infty \) with \( t_n \nearrow T^* \) and a sequence of cubes \( \{J_n\}_{n=1}^\infty \subset \mathbb{R}^d \) with \( l(J_n) < (T^*-t_n)^{\alpha} \) such that
\[
\frac{\epsilon}{2} \leq \int_{J_n} |P_{L(t_n)}u(x, t_n)|^2 \, dx.
\]

Since \( F(t) \) is an increasing function, by the monotonicity theorem (e.g. see [9]), it follows that \( F' \) exists for a.e. \( t \). We may assume that \( F'(t_n) \) exists and is finite: for any \( \epsilon > 0 \) there exists \( t_n^* \in (t_n, t_n + \epsilon) \) such that \( F'(t_n^*) \) exists and is finite, so we choose \( \bar{\epsilon}_n \) such that \( \bar{\epsilon}_n < 2^{d(d+3)-2} \epsilon \frac{(T^*-t_n)^{d\alpha}}{(T^*-t_n+\epsilon)^{2\alpha}} \) (note that \( \bar{\epsilon}_n \to 0 \) as \( n \to \infty \)). Then using the triangle inequality in (3.3), we obtain
\[
\frac{\epsilon}{2} \leq \frac{1}{2} \left( \int_{J_n} |P_{L(t_n)}u(x, t_n) - P_{L(t_n)}u(x, t_n^*)|^2 \, dx + \int_{J_n} |P_{L(t_n^*)}u(x, t_n^*)|^2 \, dx \right)
\]
\[ \leq \frac{\epsilon}{4} + 2 \int_{J_n} |P_{L(t_n^*)}u(x, t_n^*)|^2 \, dx, \]

where the bound on the first term in the right hand side is discussed below, and thus, we obtain (3.3) with \( \frac{\epsilon}{4} \) on the left-hand side. Observe that

\[ |P_{L(t_n)}u(x, t_n) - P_{L(t_n^*)}u(x, t_n^*)| \leq \int_{|\xi| \leq L(t_n^*)} |\hat{u}(\xi, t_n) - \hat{u}(\xi, t_n^*)| \, d\xi \]

\[ \leq \left( \int_{\mathbb{R}^d} |\hat{u}(\xi, t_n) - \hat{u}(\xi, t_n^*)|^2 \, d\xi \right)^{1/2} \left( \int_{\{\xi \in \mathbb{R}^d; |\xi| \leq L(t_n^*)\}} 1 \, d\xi \right)^{1/2} \]

\[ \leq 2 \|u_0\|_{L^2(\mathbb{R}^d)} [2L(t_n^*)]^{d/2} \leq \left( \frac{\epsilon_n}{2} \frac{1}{2^{d+3}} \frac{1}{(T^*-t_n^*)^{d/2}} \right)^{1/2}, \]

and integrating over \( J_n \), we obtain

\[ 2 \int_{J_n} |P_{L(t_n)}u(x, t_n) - P_{L(t_n^*)}u(x, t_n^*)|^2 \, dx \leq \frac{\epsilon_n}{2} \frac{1}{2^{d+3}} \frac{(T^*-t_n)^{d/2}}{(T^*-t_n^*)^{d/2}} < \frac{\epsilon}{4}, \]

by the choice of \( \tilde{\epsilon}_n \). Now we may re-denote the sequence \( t_n^* \) by \( t_n \).

Returning to (3.3), we obtain

\[ \frac{\epsilon}{2} \leq \left( \int_{J_n} |P_{L(t_n)}u(x, t_n)| \frac{2^{(d+2)}}{l(J_n)^{2d/2}} \, dx \right)^{1/2} l(J_n)^{d/2}, \]

by Hölder’s inequality.

Fix \( n \in \mathbb{N} \) and let \( 0 < \delta < \delta_n = (T^*-t_n)^{2\alpha} \leq (T^*-t_n) \) (recall \( 2\alpha \geq 1 \)). Raising to the power \( \frac{d+2}{d} \), dividing (3.3) by \( l(J_n)^2 \) and integrating both sides with respect to \( t \) on \( (t_n, t_n + \delta) \), we obtain \[ (\epsilon) \frac{d+2}{d} \frac{\delta}{(T^*-t_n)^{2\alpha}} \leq \int_{(t_n, t_n + \delta) \times J_n} |P_{L(t_n)}u(x, t_n)| \frac{2^{(d+2)}}{l(J_n)^{d/2}} \, dx \, dt \]

\[ \leq \int_{(t_n, t_n + \delta) \times J_n} |P_{L(t_n)}u(x, t_n) - P_{L(t)}u(x, t)| \frac{2^{(d+2)}}{l(J_n)^{d/2}} \, dx \, dt \]

\[ + \int_{(t_n, t_n + \delta) \times J_n} |P_{L(t)}u(x, t)| \frac{2^{(d+2)}}{l(J_n)^{d/2}} \, dx \, dt = I + II. \]

Using the same estimate as in (3.4), we get

\[ |P_{L(t_n)}u(x, t_n) - P_{L(t)}u(x, t)| \leq 2 \|u_0\|_{L^2(\mathbb{R}^d)} [2L(t_n + \delta)]^{d/2} \]

by Hölder’s inequality and the conservation of mass. Then the bound on term \( I \) in (3.8) is obtained by using the definition of \( L(t) \) and the bound on \( l(J_n) \)

\[ I \leq \int_{(t_n, t_n + \delta) \times J_n} (2 \|u_0\|_{L^2(\mathbb{R}^d)}) \frac{2^{(d+2)}}{l(J_n)^{d/2}} [2L(t_n + \delta)]^{d+2} \, dx \, dt \]

\[ \text{1) To be strictly correct, we should raise both sides in (3.6-3.8) to the power } \frac{d}{d+2} \text{ to obtain norms so that we can apply the triangle inequality.} \]
Substituting all above estimates into (3.8), we obtain Corollary 2.3, i.e., for some \( \delta \):

\[
\lim_{t \to \infty} \frac{1}{(T^* - t_n)^{2\alpha}} |J_n| \delta
\]

Taking the first term on the right into the left-hand side, we obtain

\[
\frac{\delta}{(T^* - t_n)^{2\alpha}}
\]

which implies

\[
\frac{\delta}{(T^* - t_n)^{2\alpha}} + [F(t_n + \delta) - F(t_n)]
\]

The second term in (3.8) is estimated by the space-time norm on all \( \mathbb{R}^d \) (\( J_n \subset \mathbb{R}^d \))

\[
II \leq \int_{(t_n, t_n + \delta) \times \mathbb{R}^d} |u(t, x)|^{2(d+2)} dt dx = \|u\|_{L^{2(d+2)}((t_n, t_n + \delta) \times \mathbb{R}^d)}^{2(d+2)}
\]

Substituting all above estimates into (3.8), we obtain

\[
\left(\frac{\epsilon}{2}\right)^{\frac{d+2}{2}} \frac{\delta}{(T^* - t_n)^{2\alpha}} \leq \frac{1}{2} \left(\frac{\epsilon}{2}\right)^{\frac{d+2}{2}} \frac{\delta}{(T^* - t_n)^{d\alpha}} + [F(t_n + \delta) - F(t_n)]
\]

which implies

\[
\left(\frac{\epsilon}{2}\right)^{\frac{d+2}{2}} \leq \frac{1}{2} \left(\frac{\epsilon}{2}\right)^{\frac{d+2}{2}} \frac{(T^* - t_n)^{(d+2)\alpha}}{(T^* - (t_n + \delta))^{(d+2)\alpha}} + (T^* - t_n)^{2\alpha} \left(F(t_n + \delta) - F(t_n)\right)
\]

Taking \( \delta \searrow 0 \) (\( n \) is fixed) and recalling that \( F' \) exists at \( t_n \), then absorbing the first term on the right into the left-hand side, we obtain

\[
\frac{1}{2} \left(\frac{\epsilon}{2}\right)^{\frac{d+2}{2}} \frac{1}{(T^* - t_n)^{2\alpha}} \leq F'(t_n),
\]

which gives (3.9).

The preceding result shows that the time derivative of the Strichartz norm is lower bounded along the sequence of times where we have tight mass concentration. If we assume that the tight mass concentration persists as in the conclusion of Corollary 2.3, we can integrate to obtain lower bounds on the Strichartz norm itself.

**Lemma 3.3.** Suppose that instead of the concentration (3.1) with \( \limsup \), we have the concentration (2.21) with the thickened time intervals \( \{I_n\} \subset [t_n, t_{n+1}] \) as in Corollary 2.3, i.e., for some \( 0 < \sigma < 1 \) there exist \( \tilde{\sigma} > 0 \) and cubes \( \{J_n\} \subset \mathbb{R}^d \) with \( l(J_n) < (T^* - t)^\alpha \) and \( |I_n| = \sigma(t_{n+1} - t_n) \) such that

\[
\lim_{n \to \infty} \inf_{t \in I_n} \int_{J_n} |P_{L(t)}u(x, t)|^2 dx \geq (1 - \tilde{\sigma}) \epsilon.
\]

Then

\[
F'(t) \geq \frac{c(\sigma, \epsilon)}{(T^* - t)^{2\alpha}} \quad \text{for a.e. } t \in I_n.
\]

Furthermore, for all \( t \in \bigcup_n I_n \) we have

\[
F(t) \geq \begin{cases} 
\frac{1}{(T^* - t)^{2\alpha}} + \text{const}, & \alpha > 1/2, \\
\ln|T^* - t| + \text{const}, & \alpha = 1/2.
\end{cases}
\]
Proof. Denote \((t_n, t_n + \delta_n) = \text{int } I_n\), the interior of \(I_n\). Take \(t \in (t_n, t_n + \delta_n)\) and repeat previous proof for this \(t\) to obtain (3.11) (note that the first step of shifting \(t_n\) to \(t_n^*\) in order to have differentiability of \(F\) available is not needed here, we may initially consider \(t \in I_n\) such that \(F'(t)\) exists).

For the second statement fix \(t \in I_n\) and observe that since \(F\) is increasing, we have \(F(t) - F(t_n) \geq \int_{t_n}^t F'\). Integrating the expression from (3.11) (for \(\alpha > \frac{1}{2}\)) we obtain

\[
F(t) \geq \frac{c}{(T^* - t)^{2\alpha - 1}} - \frac{c}{(T^* - t_n)^{2\alpha - 1}} + F(t_n),
\]

where \(c = c(\alpha, \epsilon, \sigma)\). Next observe that \(F(t_n) \geq F(t_{n - 1} + \delta_{n - 1}) \geq \int_{t_{n - 1}^*}^{t_{n - 1}^* + \delta_{n - 1}} F' + F(t_{n - 1}).\) Iterating this process, say, till \(t_{n - k} = T^* - 1\), and using the property that \(\delta_k = \sigma(t_{k + 1} - t_k)\), we obtain that

\[
F(t) \geq \frac{c}{(T^* - t)^{2\alpha - 1}} + \text{const}.
\]

Making appropriate changes in the integration for \(\alpha = \frac{1}{2}\), we obtain the second part of (3.12).

Observe that for \(t \in (t_n, t_{n + 1}) \setminus I_n\), we have the following estimate on \(F\):

\[
F(t) \gtrsim \begin{cases} 
\frac{1}{(T^* - (t_n + \delta_n))^{2\alpha - 1}} + \text{const}, & \alpha > 1/2, \\
|\ln(T^* - (t_n + \delta_n))| + \text{const}, & \alpha = 1/2,
\end{cases}
\]

by using that \(F\) is increasing on the compliment of \(I_n\) in \((t_n, t_{n + 1})\).

Corollary 3.4. In analogy with Corollary 2.5, the statement of Proposition 3.1 can be generalized to include not only the polynomial powers in the window of concentration in (3.1) but a more general dependence on \((T^* - t)\). Suppose that both the concentration window in (3.1) is \(l(J) < g(T^* - t)\) and \(L(t) = \frac{1}{2} \frac{\kappa(\epsilon)}{g(T^* - t)}\), where the function \(g\) can be written as \(g(T^* - t) = \frac{1}{-\partial_t G(T^* - t)}\) for some \(C^1\) function \(G\) with the properties that as \(t \to T^*\) both \(G(T^* - t) \to \infty\) and \(-\partial_t G(T^* - t) \to \infty\).

Then the conclusion in (3.12) modifies to

(3.13) \[ F(t_n) \gtrsim G(T^* - t_n). \]

For example, \(g(T^* - t) = (T^* - t)^{\alpha} |\ln(T^* - t)|^{-\gamma} \) with \(\alpha > 1/2, \gamma \in \mathbb{R}\), or \(\alpha = 1/2\) and \(\gamma \geq 0\) would satisfy the above conditions. The last case produces the logarithmic divergence \(|\ln(T^* - t)|^{2\gamma + 1}\) of the Strichartz norm \(\|u\|_{L^2([0, t] \times \mathbb{R}^d)}^{2(\alpha + \gamma)}\).

Proof. To prove this general statement we repeat the proof of the above Proposition with appropriate modifications and instead of (3.9) we arrive to

\[
F'(t_n) \gtrsim \frac{1}{g(T^* - t_n)^{2\gamma}}.
\]

Proceeding as in Lemmas 3.3 and using the definition of \(g\), we obtain (3.13). \(\square\)
Corollary 3.5. If the blow up time $T^* < \infty$ for $\text{NLS}^p \pm (\mathbb{R}^d)$ with $p = \frac{4}{d} + 1$, then the diagonal Strichartz norm $\|u\|_{L^{2(\frac{d+2}{d})}([0,t]\times\mathbb{R}^d)}$ explodes at least as fast as $|\ln(T^* - t)|$.

The proof follows from [2], [1], where it is shown that finite time blowup solutions parabolically concentrate in $L^2$, and the previous corollary with $g(T^* - t) = (T^* - t)^{1/2}$.

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