REGULARITY OF $\bar{\partial}$ ON PSEUDOCONVEX DOMAINS IN $\mathbb{C}^2$

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ABSTRACT. We construct a linear solution operator to the $\bar{\partial}$-equation on smoothly bounded weakly pseudoconvex domains, $\Omega$, in $\mathbb{C}^2$. The solution operator is bounded as a map $W_{0,1}^s(\Omega) \cap \ker \bar{\partial} \to W^{s+1/2}(\Omega)$.

1. INTRODUCTION

We investigate the regularity of solutions, $u$, to the $\bar{\partial}$-equation $\bar{\partial}u = f$, for $\bar{\partial}$-closed $(0,1)$-forms $f$ on smoothly bounded weakly pseudoconvex domains $\Omega \subset \mathbb{C}^2$. Regularity of the forms and functions are measured in terms of Sobolev norms: we denote by $W^s(\Omega)$, respectively $W_{0,1}^s(\Omega)$, the space of functions, respectively $(0,1)$-forms, whose derivatives of order $\leq s$ are in $L^2(\Omega)$. In the case of smoothly bounded strictly pseudoconvex domains, the canonical solution (the solution of minimal $L^2$ norm) can be shown to provide a solution operator which preserves the Sobolev spaces, $W^s(\Omega)$ for all $s \geq 0$; estimates for the canonical solution are due to Kohn (see [4] and [5]). This is not the case in the situation of smoothly bounded weakly pseudoconvex domains as shown by Barrett in [1]. And it is not just a loss of derivatives which takes place; Christ has shown that the canonical may not even be in $C^\infty(\Omega)$ even if the data form $f$ is in $C^\infty(0,1)(\Omega)$ [3].

On the other hand, using weighted Sobolev spaces, Kohn showed that for any given $s \geq 0$, there exists a weight $\varphi$ and a solution operator $K_{s,\varphi}$ (which depends on the weight as well as level of the Sobolev norm) such that $K_{s,\varphi} : W^k(\Omega) \to W^k(\Omega)$ for all $k \leq s$ and such that $\bar{\partial} \circ K_{s,\varphi} = I$ when restricted to $\bar{\partial}$-closed forms [6]. These operators can then be used to construct a solution operator which maps $C^0_{(0,1)}(\Omega)$ to $C^\infty(\Omega)$, but with this method a continuous solution operator between Sobolev spaces can only be obtained with a resulting loss of regularity. This suggests the question whether a linear solution operator which maps $W^s_{(0,1)}(\Omega)$ to $W^s(\Omega)$ simultaneously for all $s \geq 0$ (see the discussion in Section 5.2 in [8]):

Question. Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Let $f \in W^s_{(p,q)}(\Omega)$ be a $\bar{\partial}$-closed $(p,q)$-form for $0 \leq p \leq n$ and $1 \leq q \leq n$. Does there exist a solution operator $K$ such that

$$K : W^s_{p,q}(\Omega) \to W^s_{p,q-1}(\Omega)$$

for all $s$, and such that $\bar{\partial}Kf = f$?

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It is this question which we study in this article in the case of weakly pseudoconvex domains in $\mathbb{C}^2$, we provide an affirmative answer to the question if we restrict our attention to the subspace $W_{(0,1)}^s(\Omega) \cap \ker \bar{\partial}$ by constructing a solution operator with a gain of a half-derivative. We define the space $A_{(0,1)}^{s}(\Omega) = W_{(0,1)}^{s}(\Omega) \cap \ker \bar{\partial}$ and assign to it the norm from $W_{(0,1)}^{s}(\Omega)$.

**Main Theorem.** Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. There exists a solution operator $K$ such that $\bar{\partial}Kf = f$ for all $f \in A_{(0,1)}^{s}(\Omega)$ and

$$K : A_{(0,1)}^{s}(\Omega) \to W_{(0,1)}^{s+1/2}(\Omega).$$

for all $s \geq 0$.

As the counterexample of [1] took place in dimension 2, overcoming the irregularity in this dimension 2 is of importance.

The idea behind the proof is to base the construction of the solution operator on the solution to a boundary value problem, much as the solution to the canonical solution is based on the $\bar{\partial}$-Neumann problem. The $\bar{\partial}$-Neumann problem is defined as follows. Let $\vartheta$ denote the formal adjoint of $\bar{\partial}$. Let $\Box = \vartheta \bar{\partial} + \bar{\partial} \vartheta$. The $\bar{\partial}$-Neumann problem is the boundary value problem:

$$\Box v = f \quad \text{in } \Omega$$

with the boundary conditions

$$\bar{\partial}v|\bar{\partial}\rho = 0,$$

$$v|\bar{\partial}\rho = 0,$$

where $\rho$ is a smooth defining function: $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$. Let $N$ denote the solution operator to the $\bar{\partial}$-Neumann problem, i.e. as written above, $v = Nf$. Then $\partial N$ provides a solution operator to the $\partial$-equation.

As mentioned above the solution operator $\partial N$ does not satisfy the conclusions of the Main Theorem (not even the weaker conclusion of being a map from $W_{(0,1)}^s(\Omega)$ to $W^s(\Omega)$). Our approach in this article is to add a correction term, $B_\epsilon$ to the interior operator $\Box$ as well as to relax the boundary conditions (we eliminate the first, Dirichlet-type, condition). The correction term $B_\epsilon$ has order $< 2$ and so the resulting equations upon reducing to the boundary remain non-elliptic (they include a vanishing coefficient on the highest order term of one of the tangential directions). As a result the theory of pseudodifferential operators is of little use in constructing parametrices (see however the possibility of parametrices under the condition of certain added restrictions in [2]).

The advantage of adding the interior operator $B_\epsilon$ is that the resulting boundary operators have similar properties to elliptic operators; in particular, when dividing by their symbols an increase in regularity is observed. These operators we name $\bar{\Psi}$ operators. They have the property that they are not elliptic, but contain an elliptic operator of order lower than that of the operator itself.
The properties of $\Psi$ operators are worked out in Section 5 bearing their name. In that section we develop their properties and analogues to parametrices.

As a note to the reader familiar with the technique of reducing to the boundary, we mention a few differences with the standard case (of the $\bar{\partial}$-Neumann problem). Section 6 regarding the Dirichlet to Neumann operator (DNO) contains the well-known result that the highest order term of the inward normal derivative to the solution to a Dirichlet problem is given by the square root of the highest order tangential terms in the elliptic interior operator. What is new here is that an intermediate term (of order $<1$) is included in the DNO and leads to the observation above that the boundary operators exhibit parametrix-like approximate inverses. The approach of [2] could be used here as well to calculate the DNO, but we take another approach based on pseudodifferential operators on half-spaces, an approach which also helps us in calculating the symbol of the normal to the Green’s operator. In fact, it is the property of the Green’s operator which suggests that a weakening of the $\bar{\partial}$-Neumann boundary conditions leads to an increase of regularity. A boundary condition can be appropriately chosen to replace the Dirichlet-type $\bar{\partial}$-Neumann condition which exploits the properties of the Green’s operator leading to the increase in regularity.

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2. Notation and Background

We take a moment to fix the notation used throughout the article. Our notation for derivatives is $\partial_t : \frac{\bar{\partial}}{i}$. We also use the index notation for derivatives: let $a = (a_1, \ldots, a_n)$ be a multi-index. Then

$$\partial^a = \partial_{x_1}^{a_1} \cdots \partial_{x_n}^{a_n}.$$  

We follow [9] in using the notation $D_{x_j} = -i\partial_{\xi_j}$, with the index notation $D^a_{\xi} = (-i)^{|a|}\partial_{\xi}^a$. This notation has the benefit that under Fourier transforms we can write $\hat{D^a_{\xi} f(\xi)} = \xi^a \hat{f}(\xi)$.

We let $\Omega \subset \mathbb{R}^n$ and define pseudodifferential operators on $\Omega$ as in [9].

**Definition 2.1.** We denote by $S^a(\Omega)$ the space of symbols $a(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ which have the property that for any given compact set, $K$, and for any $n$-tuples $k_1$ and $k_2$, there is a constant $c_{k_1, k_2}(K) > 0$ such that

$$\left| D_{\xi}^{k_1} D_x^{k_2} a(x, \xi) \right| \leq c_{k_1, k_2}(K) (1 + |\xi|)^{|a| - |k_1|} \quad \forall x \in K, \xi \in \mathbb{R}^n.$$
Associated to the symbols in class $S^\alpha(\Omega)$ are the pseudodifferential operators, denoted by $\Psi^\alpha(\Omega)$ defined in

**Definition 2.2.** We say an operator $A : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is in class $\Psi^\alpha(\Omega)$ if $A$ can be written as an integral operator with symbol $a(x, \xi) \in S^\alpha(\Omega)$:

$$A \phi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} a(x, \xi) \hat{\phi}(\xi) e^{ix \cdot \xi} d\xi.$$ 

In our use of Fourier transforms and equivalent symbols we find it useful to use cutoffs. Let $\chi(\xi) \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ be such that $\chi \equiv 1$ in a neighborhood of 0 and $\chi \equiv 0$ outside of a compact set which includes 0. Then we reserve the notation $\chi'$ to denote functions which are 0 near the origin: $\chi'(\xi) := 1 - \chi(\xi)$ where $\chi(\xi)$ is as defined above.

We also use $\tilde{\cdot}$ to indicate transforms in tangential directions. Let $p \in \partial \Omega$ and let $(x_1, \ldots, x_{n-1}, \rho)$ be local coordinates around $p$, ($\rho < 0$). Let $\chi_p(x, \rho)$ denote a cutoff which is $\equiv 1$ near $p$ and vanishes outside a small neighborhood of $p$ on which the local coordinates $(x, \rho)$ are valid. Then with $v \in L^2(\Omega)$ we write

$$\tilde{\chi}_p \tilde{v}(\xi, \eta) = \int \chi_p v(x, \rho) e^{-ix\xi} e^{-i\rho \eta} dx d\rho$$

$$\tilde{\chi}_p \tilde{v}(\xi, 0) = \int \chi_p v(x, \rho) e^{-ix\xi} dx.$$ 

We use the $\tilde{\cdot}$ notation when describing transforms of functions supported on the boundary. With notation and coordinates as above, we let $v_b(x) \in L^2(\partial \Omega)$ and write

$$\tilde{\chi}_p(0) v_b(\xi) = \int \chi_p(x, 0) v_b(x) e^{-ix\xi} dx.$$ 

### 3. SETUP

We follow [2] in setting up our boundary value problem (which is similar to the setup of the $\bar{\partial}$-Neumann problem in [2]). We let $\rho$ denote the geodesic distance (with respect to the standard Euclidean metric) to the boundary function for $\Omega \subset \mathbb{C}^2$, a smoothly bounded pseudoconvex domain. We let $U$ be an open neighborhood of $\partial \Omega$ such that

$$\Omega \cap U = \{ z \in U | \rho(z) < 0 \};$$

$$\nabla \rho(z) \neq 0 \quad \text{for} \; z \in U.$$ 

We define an orthonormal frame of $(1, 0)$-forms on a neighborhood $U$ with $\omega_1, \omega_2$ where $\omega_2 = \sqrt{2} d\rho$, and $L_1, L_2$ the dual frame. We thus can write

$$L_1 = \frac{1}{2} (X_1 - iX_2) + O(\rho)$$

$$L_2 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho} + iT + O(\rho) \quad (3.1)$$
where \( \partial/\partial \rho \) is the vector field dual to \( d\rho \), and \( X_1, X_2, \) and \( T \) are tangential fields. We can expand the vector fields \( L_1 \) and \( T \) as in [2] as
\[
L_1 = L_1^0 + \rho L_1^1 + \cdots \\
L_2 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho} + i \left( T_0^0 + \rho T_0^1 + \cdots \right).
\]

We then choose coordinates \((x_1, x_2, x_3)\) on \( \partial \Omega \) near a point \( p \in \partial \Omega \), in terms of which the vector fields \( L_0^1 \) and \( T_0^0 \) are given by
\[
T_0^0 = \frac{\partial}{\partial x_3}, \\
L_0^1 = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) + O(x - p).
\]

We also define the scalar function \( s \) by
\[
\bar{\partial} \bar{\omega}_1 = s \bar{\omega}_1 \wedge \bar{\omega}_2.
\]

With respect to the coordinates, \( z_1 \) and \( z_2 \)
\[
\bar{\omega}_1 = \sqrt{2} \left( \frac{\partial \rho}{\partial z_2} d\bar{z}_1 - \frac{\partial \rho}{\partial z_1} d\bar{z}_2 \right), \\
\partial \bar{\omega}_1 = -\sqrt{2} \left( \frac{\partial^2 \rho}{\partial z_1 \partial z_1} + \frac{\partial^2 \rho}{\partial z_2 \partial z_2} \right) d\bar{z}_1 \wedge d\bar{z}_2 \\
= -2\sqrt{2} \left( \frac{\partial^2 \rho}{\partial z_1 \partial z_1} + \frac{\partial^2 \rho}{\partial z_2 \partial z_2} \right) \bar{\omega}_1 \wedge \bar{\omega}_2,
\]
and so
\[
s(z_1, z_2) = -2\sqrt{2} \left( \frac{\partial^2 \rho}{\partial z_1 \partial z_1} + \frac{\partial^2 \rho}{\partial z_2 \partial z_2} \right).
\]

We will also need to calculate the formal adjoint of \( \bar{\partial} \), denoted by \( \vartheta \). For \( \varphi, \psi \in C^\infty_0(U) \) we use integration by parts to write, with some \( h_1 \) and \( h_2 \) to be determined,
\[
\int_U \varphi L_j \psi dV = \int_U (-L_j + h_j) \varphi \psi dV.
\]

In calculating the adjoint of
\[
T_1 = 2\sqrt{2} \left( \frac{\partial \rho}{\partial z_2} \frac{\partial}{\partial z_1} - \frac{\partial \rho}{\partial z_1} \frac{\partial}{\partial z_2} \right)
\]

we use
\[
\int_U \varphi \frac{\partial \rho}{\partial z_i} \frac{\partial \psi}{\partial z_j} dV = -\int_U \frac{\partial \varphi}{\partial z_j} \frac{\partial \psi}{\partial z_i} dV - \int_U \frac{\partial^2 \rho}{\partial z_i \partial z_j} \varphi \psi dV.
\]

We see immediately that
\[
h_1 = 0.
\]
Similarly from
\[ \int_U \frac{\partial \rho}{\partial z_j} \frac{\partial \psi}{\partial z_j} dV = - \int_U \frac{\partial \rho}{\partial z_j} \frac{\partial \psi}{\partial z_j} dV - \int_U \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_j} \frac{\partial \phi}{\partial \overline{z}_j} dV \]
and
\[ L_2 = 2\sqrt{2} \left( \frac{\partial \rho}{\partial z_1} \frac{\partial \overline{\partial}}{\partial \overline{z}_1} + \frac{\partial \rho}{\partial z_2} \frac{\partial \overline{\partial}}{\partial \overline{z}_2} \right) \]
we have
\[ h_2 = s. \]

We now create a boundary value problem based on the \( \overline{\partial} \)-Neumann boundary value problem. The \( \overline{\partial} \)-Neumann problem is the vector-valued boundary-value problem:
\[ \square u = f \quad \text{in } \Omega, \]
where \( \square = \partial \overline{\partial} + \overline{\partial} \theta \), with the boundary conditions
\[ L_2(u_1) - su_1 = 0 \]
\[ u_2 = 0 \]
on \( \partial \Omega \). As we will see later in the paper, we can achieve better regularity results if we relax the second boundary condition. Eliminating the condition \( u_2 = 0 \) on the boundary leads to the consideration of forms \( u \) which are no longer in the domain of \( \overline{\partial}^* \) and it is for this reason we describe the operator \( \square \) in terms of the formal adjoint, rather than with \( \overline{\partial}^* \) as is common in the theory of the \( \overline{\partial} \)-Neumann problem (note that on \( \text{dom}(\overline{\partial}^*) \), \( \overline{\partial}^* = \theta \)). The boundary conditions of the \( \overline{\partial} \)-Neumann problem are written more generally as
\[ \overline{\partial} u|\partial \rho = 0, \]
\[ u|\partial \rho = 0. \]
We only consider the first condition, written in terms of the coefficients \( u_1 \) and \( u_2 \) as
\[ (3.2) \quad L_2(u_1) - su_1 - L_1(u_2) = 0. \]

From (2.19) in [2] the operator \( \square \) has the form
\[ (3.3) \quad 2\square = - \frac{\partial^2}{\partial \rho^2} + \tilde{C} \frac{\partial}{\partial \rho} + \square \rho, \]
where
\[ \tilde{C} = \begin{bmatrix} \sqrt{2}(s - \overline{s} + h_2) & 0 \\ 0 & \sqrt{2}h_2 \end{bmatrix} \]
\[ = \sqrt{2}s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]
and from (2.22) in [2]

\[
\square_p \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -2(T^2 + \overline{T}L_1) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - 2 \left[ L_1, \overline{T} \right] \begin{bmatrix} 0 \\ u_2 \end{bmatrix} + \frac{2i}{\sqrt{2}} \begin{bmatrix} \partial \overline{T} \\ \partial \rho \end{bmatrix} \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} + 2isT \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} + 2 \left[ \left[ L_2, \overline{L}_1 \right] u_2 \\ \left[ L_1, \overline{L}_2 \right] u_1 \right] + 2s \left[ \overline{T}_1 u_2 \\ -L_1 u_1 \right] 
\]

(3.4)

We will see that if we add a term to \(\square\) the boundary operators resulting from a reduction to the boundary using the boundary condition (3.2) we produce operators which exhibit approximate inverses which allow for us to choose a particular solution to the boundary value problem with desired regularity. We define the operator \(|d t|^{3/2}\) to be the pseudodifferential operator whose symbol is given by

\[
\sigma(|d t|^{3/2}) = |\xi_1^2 + \xi_2^2 + \xi_3^2|^{3/4},
\]

where \(\xi_j\) is the transform variable corresponding to coordinate \(x_j\) given above for \(j = 1, 2, 3\). To the operator \(\square\) in (3.3) we add a first order operator and obtain a new operator, denoted \(\square_\varepsilon\). To \(\square\) we add the operator,

\[
B_\varepsilon = \varepsilon \begin{bmatrix} |d t|^{3/2} & 0 \\ 0 & 0 \end{bmatrix}
\]

so that

\[\square_\varepsilon = \square + B_\varepsilon.\]

The boundary value problem we study in subsequent sections is

\[\square_\varepsilon u = f \quad \text{on} \quad \Omega\]

for \((0,1)\)-forms, with the boundary condition (3.2).

4. PSEUDODIFFERENTIAL OPERATORS ON HALF-SPACES

In this section we develop some of the properties of pseudodifferential operators on half-spaces. Of particular importance for the reduction to the boundary techniques are the boundary values of pseudodifferential operators on half-spaces, as well as pseudodifferential operators acting on distributions supported on the boundary.

We define the half-space \(\mathbb{H}_{n+1}^{-} := \{(x, \rho) \in \mathbb{R}^{n+1} : \rho < 0\}\). The space of distributions, \(\mathcal{D}'(\mathbb{H}_{n+1}^{-})\) is defined as the distributions in \(\mathcal{D}'(\mathbb{R}^{n+1})\) with support in \(\mathbb{H}_{n+1}^{-}\). The topology of \(\mathcal{D}'(\mathbb{H}_{n+1}^{-})\) is inherited from that of \(\mathcal{D}'(\mathbb{R}^{n+1})\). We endow \(C^\infty(\mathbb{R}^{n+1})\) with the topology defined in terms of the semi-norms

\[p_{l,K}(\phi) = \max_{(x,\rho) \in K \subset \mathbb{R}^{n+1}} \sum_{|\alpha| \leq l} |\partial^\alpha \phi(x, \rho)|.\]
A regularizing operator, $\Psi^{-\infty}(\mathbb{R}^{n+1})$, is a continuous linear map

\[ (4.1) \quad \Psi^{-\infty} : \mathcal{E}'(\mathbb{R}^{n+1}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^{n+1}). \]

Continuity of (4.1) can be shown, for instance, by using the density of $L^p_c(\mathbb{R}^{n+1})$ in $\mathcal{E}'(\mathbb{R}^{n+1})$ for $1 \leq p \leq +\infty$ so that continuity is read from estimates of the form

\[ \max_{(x,\rho) \in K} \sum_{|\alpha| \leq l} |\partial^\alpha \phi(x,\rho)| \lesssim \|\phi\|_{L^p} \]

for each $l \geq 0$.

In working with pseudodifferential operators on half-spaces, with coordinates $(x_1, \ldots, x_n, \rho)$, $\rho < 0$, we will have the need to show that by multiplying symbols of inverses of elliptic operators with smooth cutoffs with compact support, which are functions of transform variables corresponding to tangential coordinates, we produce operators which are smoothing after restriction to the boundary ($\rho = 0$); these operators are then maps $\mathcal{E}'(\mathbb{H}^{n+1}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n)$.

**Lemma 4.1.** Let $A \in \Psi^k(\mathbb{R}^{n+1})$ for $k \leq -1$ be the inverse to an elliptic operator with non-vanishing symbol, such that the symbol, $\sigma(A)(x,\rho,\xi,\eta)$ has poles at $\eta = q_1(x,\rho,\xi), \ldots, q_k(x,\rho,\xi)$ with $q_i(x,\rho,\xi)$ themselves symbols of elliptic operators of order 1 on $\mathbb{R}^n$.

Let $A_{\chi}$ denote the operator with symbol

\[ \chi(\xi)\sigma(A), \]

where $\chi(\xi) \in \mathcal{C}^\infty_0(\mathbb{R}^n)$. Then $A_{\chi}$ is regularizing on distributions supported on the boundary:

\[ A_{\chi} : \mathcal{E}'(\mathbb{R}^n) \times \delta(\rho) \rightarrow \mathcal{C}^\infty(\mathbb{H}^{n+1}). \]

**Proof.** Without loss of generality we suppose $\phi_\rho(x) \in L^2_c(\mathbb{R}^n)$, and let $\phi = \phi_\rho \times \delta(\rho)$. We estimate derivatives of $A_\chi(\phi)$. We let $a(x,\rho,\xi,\eta)$ denote the symbol of $A$, and $a_\chi(x,\rho,\xi,\eta)$ that of $A_\chi$.

We first note that derivatives with respect to the $x$ variables pose no difficulty due to the $\chi$ term in the symbol of $A_\chi$: from

\[ A_\chi \phi = \frac{1}{(2\pi)^{n+1}} \int a_\chi(x,\rho,\xi,\eta) \tilde{\phi}(\xi,\eta) e^{ix\xi} e^{i\rho\eta} d\xi d\eta \]

\[ = \frac{1}{(2\pi)^{n+1}} \int a_\chi(x,\rho,\xi,\eta) \phi_{\rho}(\xi) e^{ix\xi} e^{i\rho\eta} d\xi d\eta \]
we calculate
\[
|\partial_\xi^\alpha A_\chi \phi| \lesssim \int |\partial_\xi^\alpha a_\chi(x, \rho, \bar{\zeta}, \eta)|||\xi|^{a_1}| \tilde{\phi}_b(\xi)| \, d\xi d\eta
\]
\[
\lesssim \|\phi_b\|_{L^2} \left( \int |\xi|^{2a_1} |\tilde{\chi}(\xi)||\partial_\xi^\alpha a(x, \rho, \bar{\zeta}, \eta)||^2 \, d\xi d\eta \right)^{1/2}
\]
\[
\lesssim \|\phi_b\|_{L^2} \left( \int |\xi|^{2a_1} |\tilde{\chi}(\xi)| \frac{1}{(1+\xi^2+\eta^2)^k} \, d\xi d\eta \right)^{1/2}
\]
\[
\lesssim \|\phi_b\|_{L^2} \left( \int |\xi|^{2a_1} |\tilde{\chi}(\xi)| \, d\xi \right)^{1/2}
\]
where \(a_1 + a_2 = \alpha\). For any mixed derivative \(\partial_\xi^\beta \phi_b\), the \(x\) derivatives can be handled in the manner above and so we turn to derivatives of the type \(\partial_\rho^{\beta_x} A_\chi \phi\).

We use the residue calculus to integrate over the \(\eta\) variable in
\[
A_\chi \phi = \frac{1}{(2\pi)^{n+1}} \int a_\chi(x, \rho, \bar{\zeta}, \eta) \tilde{\phi}_b(\xi) e^{ix \bar{\zeta}} e^{i\eta \bar{\eta}} d\xi d\eta.
\]
For \(\rho < 0\), we integrate over a contour in the lower-half plane and analyze a typical term resulting from a simple pole at \(\eta = q-(x, \rho, \bar{\zeta})\) of \(a_\chi(x, \rho, \bar{\zeta}, \eta)\). Let
\[
a_{\eta}(x, \rho, \bar{\zeta}) = \frac{1}{2\pi i} \text{Res}_{\eta = q^{-}} a_\chi(x, \rho, \bar{\zeta}, \eta)
\]
and
\[
A_{\chi, \eta} \phi = \frac{1}{(2\pi)^{n+1}} \int \chi(\xi) a_{\eta}(x, \rho, \bar{\zeta}) \tilde{\phi}_b(\xi) e^{ix \bar{\zeta}} e^{i\eta \bar{\eta}} d\xi,
\]
We can now estimate \(\partial_\rho^{\beta_x} A_{\chi, \eta} \phi\) by differentiating under the integral. For \(\rho < 0\) we have the estimates
\[
|\partial_\rho^{\beta_x} A_{\chi, \eta} \phi| \lesssim \int |\partial_\rho^{\beta_1} a_{\eta}(x, \rho, \bar{\zeta})|||{\eta}|^{\beta_2}| \tilde{\phi}_b(\xi)| \, d\xi
\]
\[
\lesssim \|\phi_b\|_{L^1},
\]
where \(\beta_1 + \beta_2 + \beta_3 = \beta\). The integral over \(\xi\) converges due to the factor of \(\chi(\xi)\) contained in \(a_{\eta}\).

As poles of other orders are handled similarly, we conclude the proof of the lemma. \(\square\)

Without the assumption of the cutoff function \(\chi(\xi)\) in the symbol of \(A_\chi\) we can still prove

**Lemma 4.2.** Let \(g \in \mathcal{D}(\mathbb{R}^{n+1})\) of the form \(g(x, \rho) = g_b(x) \delta(\rho)\) for \(g_b \in W^s(\mathbb{R}^n)\). Let \(A \in \Psi^k(\mathbb{R}^{n+1})\), \(k \leq -1\) be as in Lemma [4.1] Then
\[
\|A g\|_{W^s_{loc}(\mathbb{H}^{n+1})} \lesssim \|g_b\|_{W^{s+k+1/2}(\mathbb{R}^n)}.
\]
Proof. We follow and use the notation of the proof of Lemma 4.1 and analyze a typical term resulting from a simple pole at \( \eta = q_-(x, \rho, \xi) \) of \( a(x, \rho, \xi, \eta) \). With

\[
a_{q_-(x, \rho, \xi)} = \frac{1}{2\pi i} \text{Res}_{q=q_-} a(x, \rho, \xi, \eta)
\]

and

\[
A_{q_-} g = \frac{1}{(2\pi)^{n+1}} \int a_{q_-(x, \rho, \xi)} \tilde{g}_b(\xi) e^{ix \xi} e^{i \rho q_-} d\xi,
\]

we estimate \( \partial^\alpha_x \partial^\beta_{\rho} A_{q_-} g \) by differentiating under the integral. For \( \rho < 0 \) we have the estimates

\[
\left\| \partial^\alpha_x \partial^\beta_{\rho} A_{q_-} \phi \right\|_{L^2_{rc}}^2 \lesssim \int \left( \partial^\alpha_x \partial^\beta_{\rho} a_{q_-(x, \rho, \xi)} \right)^2 \partial^\alpha_x q_- |2| \beta_2 | \partial^\beta_{\rho} q_- |2| |2| d\xi \lesssim \int \left( \partial^\alpha_x \partial^\beta_{\rho} \phi \right)^2 d\xi
\]

where \( \beta_1 + \beta_2 + \beta_3 = \beta \) and \( \alpha_1 + \cdots + \alpha_4 = \alpha \) and \( |\alpha| + |\beta| = s \). We assume \( a \) includes a factor \( \chi'(\xi) \) (it can be shown including \( \chi'(\xi) \) into the symbol produces smoothing errors), so that when we integrate over \( \rho \) we get a factor of \( \frac{1}{|\text{Im } \rho|} \sim \frac{1}{|\rho|} \) which lowers the order of the norm in the tangential directions by 1/2:

\[
\left\| \partial^\alpha_x \partial^\beta_{\rho} A_{q_-} \phi \right\|_{L^2_{rc}}^2 \lesssim \int |1 + \xi^2|^{k+1} |\xi^2| \chi'(\xi) |\tilde{g}_b(\xi)|^2 \left( \frac{\chi'(\xi)}{|\xi|} \right)^2 d\xi
\]

\[
\lesssim \int |1 + \xi^2|^{k+1/2} |\xi^2| |\tilde{g}_b(\xi)|^2 d\xi
\]

\[
\lesssim \| \tilde{g}_b \|_{W^{s+1/2}(\mathbb{R}^n)}.
\]

We use the notation \( \Psi^k_b(\mathbb{R}^n) \) (resp. \( \Psi^k(\partial \Omega) \)) to denote the space of pseudodifferential operators of order \( k \) on \( \mathbb{R}^n = \partial \Omega \) (resp. on \( \partial \Omega \)). Further following our use of the notation \( \Psi^k \) to denote any operator belonging to the family \( \Psi^k(\mathbb{H}^{n+1}) \) (resp. \( \Psi^k(\Omega) \)) when acting on distributions \( \mathcal{E}'(\mathbb{H}^{n+1}) \) (resp. \( \mathcal{E}'(\Omega) \)) we write for \( \mathcal{E}'(\mathbb{R}^n) \) (resp. \( \mathcal{E}'(\partial \Omega) \)) \( \Psi^k_b \phi_b \) \( \Psi^k_b \phi_b \) \( \Psi^k_b \phi_b \) \( \Psi^k_b \phi_b \) denoting any pseudodifferential operator of order \( k \) on the appropriate (boundary of a) domain.

With coordinates \( (x_1, \ldots, x_n, \rho) \) in \( \mathbb{R}^{n+1} \), let \( R \) denote the restriction operator, \( R : \mathcal{D}(\mathbb{R}^{n+1}) \to \mathcal{D}(\mathbb{R}^n) \), given by \( R \phi = \phi|_{\rho=0} \).

**Lemma 4.3.** Let \( g \in \mathcal{D}(\mathbb{R}^{n+1}) \) of the form \( g(x, \rho) = g_b(x) \delta(\rho) \) for \( g_b \in \mathcal{D}(\mathbb{R}^n) \). Let \( A \in \Psi^k(\mathbb{R}^{n+1}) \), be an operator of order \( k \), for \( k \leq -2 \), with symbol \( a(x, \rho, \xi, \eta) \). Then \( R \circ A \) induces a pseudodifferential operator of order \( k + 1 \) on \( g_b \) via

\[
R A g_b = \Psi^{k+1}_b g_b.
\]

**Proof.** The symbol

\[
a(x, \rho, \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(x, \rho, \xi, \eta) d\eta
\]
for any fixed $\rho$ belongs to the class $S^{k+1}(\mathbb{R}^n)$, which follows from the properties of $a(x, \rho, \xi, \eta)$ as a member of $S^k(\mathbb{R}^{n+1})$ and differentiating under the integral. The composition $R \circ Ag$ is given by

$$\frac{1}{(2\pi)^{n+1}} \int a(x,0,\xi,\eta)\overline{g_b(\xi)}e^{ix\xi}d\xi d\eta = \frac{1}{(2\pi)^n} \int \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} a(x,0,\xi,\eta)d\eta \right] \overline{g_b(\xi)}e^{ix\xi}d\xi$$

$$= \frac{1}{(2\pi)^n} \int a(x,0,\xi)\overline{g_b(\xi)}e^{ix\xi}d\xi = \Psi^{k+1}_b g_b.$$

Remark 4.4. We can generalize Lemma 4.3 to operators of order $k = -1$ in the case of inverses to elliptic operators satisfying the hypotheses of Lemma 4.1 by using the residue calculus to integrate out the $\eta$ variable.

If we choose coordinates $(x_1, \ldots, x_{2n-1}, \rho)$ on a domain $\Omega \subset \mathbb{R}^{n+1}$ in a neighborhood, $U$, around a boundary point, $p \in \partial \Omega$, such that $\Omega \cap U = \{z : \rho(z) < 0\}$ and $\rho|_{\partial \Omega \cap U} = 0$ we obtain the obvious analogues of Lemmas 4.1 and 4.3 with $\Omega$ replacing $\mathbb{H}^{n+1} \subset \mathbb{R}^{n+1}$ and $\partial \Omega$ replacing $\mathbb{R}^n$.

If we do not restrict in Lemma 4.3 we can obtain tangential estimates for operators of order $\leq -1$. For these estimates we use the following notations: The operator $D_\tau$ is the tangential operator whose symbol, $\sigma(D_\tau)$ is given by

$$\Lambda_\tau(\xi) = (1 + \xi_1^2 + \cdots + \xi_n^2)^{1/2}.$$ 

The norm $\| \cdot \|_{W^s(\mathbb{H}^{n+1})}$ is equivalent to

$$\| \phi \|_{W^s(\mathbb{H}^{n+1})} \sim \| \Lambda_\tau^s \phi \|_{L^2(\mathbb{R}^{n+1})}.$$ 

Lemma 4.5. Let $g \in \mathcal{D}(\mathbb{R}^{n+1})$ of the form $g(x, \rho) = g_b(x)\delta(\rho)$ for $g_b \in W^s(\mathbb{R}^n)$. Let $A \in \Psi^k(\mathbb{R}^{n+1})$ for $k \leq -1$, with symbol $a(x, \rho, \xi, \eta)$. Then

$$\| |A|g\|_{W^{s+k+1/2}(\mathbb{R}^n)} \lesssim \|g_b\|_{W^{s+k+1/2}(\mathbb{R}^n)}.$$ 

Proof. We have

$$\| |Ag\|_{W^{s+k+1/2}(\mathbb{R}^n)} \lesssim \|D_\tau^s Ag\|_{L^2(\mathbb{H}^{n+1})}$$

$$\lesssim \int |a(x, \rho, \xi, \eta)|^2 \Lambda_\tau^{2s}|\overline{g_b(\xi)}|^2 d\xi d\eta d\rho$$

$$\lesssim \int \left(1 + \frac{|\xi|^{2s}}{(1 + \xi_1^2 + \cdots + \xi_n^2)^k} \right) |\overline{g_b(\xi)}|^2 d\xi d\eta$$

$$\lesssim \left(1 + \frac{|\xi|^{2s-2k+1}}{(1 + \xi_1^2 + \cdots + \xi_n^2)^k} \right) |\overline{g_b(\xi)}|^2 d\xi d\eta$$

$$\lesssim \|g_b\|_{W^{s+k+1/2}(\mathbb{R}^n)}.$$
where the $x$ and $\rho$ integrals are done over compact sets.

**Lemma 4.6.** Let $g \in \mathcal{D}(\mathbb{R}^{n+1})$ of the form $g(x, \rho) = g_b(x)\delta(\rho)$ for $g_b \in \mathcal{D}(\mathbb{R}^n)$. Let $A \in \Psi^k(\mathbb{R}^{n+1})$, be a pseudodifferential operator of order $k$ with symbol $a(x, \rho, \xi, \eta)$. Let $\rho$ denote the operator of multiplication with $\rho$. Then $\rho \circ A$ induces a pseudodifferential operator of order $k - 1$ on $g$:

$$\rho A g = \Psi^{k-1} g.$$

**Proof.** We let $\alpha_\rho = \rho \cdot a(x, \rho, \xi, \eta)$. Then $\alpha_\rho$ is also of order $k$, and

$$\rho \circ A(g) = \int \rho a(x, \rho, \xi, \eta)\bar{g}_b(\xi) e^{i x \cdot \xi} e^{i \rho \eta} d\xi d\eta
= -i \int a(x, \rho, \xi, \eta)\bar{g}_b(\xi) e^{i x \cdot \xi} \frac{\partial}{\partial \eta} e^{i \rho \eta} d\xi d\eta
= i \int \frac{\partial}{\partial \eta} \left(a(x, \rho, \xi, \eta)\right)\bar{g}_b(\xi) e^{i x \cdot \xi} e^{i \rho \eta} d\xi d\eta
= \Psi^{k-1} g,$$

since $\frac{\partial}{\partial \eta} \left(a(x, \rho, \xi, \eta)\right)$ is a symbol of class $S^{k-1}(\mathbb{R}^{n+1})$.

Lemma 4.3 concerned itself with the restrictions of pseudodifferential operators (applied to distributions supported on the boundary) to the boundary, while Lemma 4.2 allows us to consider pseudodifferential operators applied to restrictions of distributions. A special case of Lemma 4.2 is

**Lemma 4.7.** Let $A \in \Psi^k(\mathbb{R}^{n+1})$, for $k \leq -1$, be as in Lemma 4.1. Then

$$A \circ R\Psi^{-\infty} : \mathcal{E}'(\mathbb{H}^{n+1}) \to C^\infty(\mathbb{H}_n^{n+1})$$

i.e., $A \circ R\Psi^{-\infty} = \Psi^{-\infty}$.

**Proof.** Let $f \in \mathcal{E}'(\mathbb{H}_n^{n+1})$ and apply Lemma 4.2 with $g_b = R \circ \Psi^{-\infty} f$. Then for all $s$

$$\|A \circ R\Psi^{-\infty} f\|_{W^s(\mathbb{H}_n^{n+1})} \lesssim \|R \circ \Psi^{-\infty} f\|_{W^{s+1/2}(\mathbb{R}^n)}
\lesssim \|\Psi^{-\infty} f\|_{W^{s+k+1}(\mathbb{R}^n)}
\lesssim \|f\|_{W^{-\infty}(\mathbb{H}_n^{n+1})}.$$

The lemma thus follows from the Sobolev Embedding Theorem.

Similarly proven is the

**Lemma 4.8.** Let $A \in \Psi^k(\mathbb{R}^{n+1})$, for $k \leq -1$, be as in Lemma 4.7. Then

$$A \circ \Psi^{-\infty} : \mathcal{E}'(\mathbb{R}^n) \to C^\infty(\mathbb{H}_n^{n+1})$$

i.e., $A \circ \Psi^{-\infty} \circ R = \Psi^{-\infty}$. 
We end this section with an illustration of how our analysis lends itself to the proof of useful theorems on harmonic functions. A result of Ligocka states that multiplication with the defining function of the solution to a Dirichlet problem leads to an increase in smoothness:

**Theorem 4.9** (Ligocka, see Theorem 1 [7]). Let $\Omega$ be a smooth bounded domain with defining function $\rho$, and $H^s(\Omega)$ be the space of harmonic functions belonging to $W^s(\Omega)$. Let $T_ku = \rho^ku$, where $k$ is a positive integer. Then for each integer $s$, the operator $T_k$ maps $H^s(\Omega)$ continuously into $H^{s+k}(\Omega)$.

**Proof.** It is not any more difficult to work in the more general situation of functions satisfying

$$\Gamma u = 0$$

for some elliptic operator $\Gamma$ of order $m$ whose parametrix $\Gamma^{-1}$ satisfies the hypothesis of Lemma 4.1 and we provide here a sketch of the proof of Theorem 4.9 using the techniques from this work. We make a change of coordinates so that locally $\Omega = \{(x, \rho)|\rho < 0\}$, and $\partial\Omega = \{(x, 0)\}$. We define

$$g^j_b = \frac{\partial^j u}{\partial \rho^j}|_{\partial\Omega}$$

$$g^j = g^j_b \times \delta(\rho) \quad j = 1, 2, \ldots m - 1.$$

Let $\Gamma^{-1}$ denote a parametrix for $\Gamma$.

The solution $u$ can be written locally as

$$\Gamma^{-1}\left(\sum_{j=0}^{m-1} \Psi^j_0 g^{m-1-j}\right),$$

modulo smoothing terms, where $\Psi^j_0$ is used to denote a pseudodifferential operator whose symbol is independent of $\rho$ (see for instance, (6.14)). Furthermore, locally,

$$\rho^k u = \rho^k \Gamma^{-1}\left(\sum_{j=0}^{m-1} \Psi^j_0 g^{m-1-j}\right)$$

$$= \rho^k \left(\sum_{j=0}^{m-1} \Psi^{-m+j} g^{m-1-j}\right)$$

$$= \sum_{j=0}^{m-1} \Psi^{-m+j-k} g^{m-1-j}$$

by Lemma 4.6. The operators $\Psi^{-m+j-k}$ have the form of elliptic operators of order $-m(k+1)$ composed with operators of order $(m-1)k + j$. The proof of Lemma 4.2 applies to such operators
and we estimate
\[
\left\| \Psi^{m+j-k} \delta^{m-1-j} \right\|_{W^{k+s}(\Omega)} \lesssim \left\| \delta^{m-1-j} \right\|_{W^{m+j+s+1/2}(\partial \Omega)}
\]
\[
\lesssim \left\| \frac{\partial^{m-1-j} u}{\partial \rho^{m-1-j}} \right\|_{W^{m+j+s+1}(\Omega)}
\]
\[
\lesssim \| u \|_{W^{s}(\Omega)},
\]
where we use the Trace Theorem in the second to last step. \(\square\)

5. Operators of type \(\Psi\)

As in the \(\bar{\delta}\)-Neumann problem, the boundary conditions of the problem
\[
\Box_e u = f \quad \text{on } \Omega,
\]
where \(\Box_e = \bar{\partial} \bar{\partial} + \bar{\partial} \partial + B_e\), with the boundary conditions
\[
\bar{\partial} u |_{\partial \Omega} = 0 \quad \text{on } \partial \Omega,
\]
lead to non-elliptic equations on the boundary. For this purpose, we introduce the \(\Psi\) operators, which will help in constructing solutions to such boundary value problems, and derive some of their useful properties. We let \(q_{\gamma,\delta}(x, \xi)\) be a polynomial of rational powers of the \(\xi\) variables and \(|\xi|\), with the properties (uniform in \(x\))
\[
|q_{\gamma,\delta}(x, \xi)| \lesssim |q_{\gamma,\delta}(x, \xi - \xi) + |q_{\gamma,\delta}(x, \xi)|
\]
(5.1)
\[
(1 + |\xi|)^\gamma \lesssim |q_{\gamma,\delta}(x, \xi)| \lesssim (1 + |\xi|)^{\gamma - \delta}
\]
(5.2)
for \(1 \leq \delta < 2\) and such that \(\gamma \geq 1\). By definition we can (and will) assume \(\delta\) and \(\gamma\) are rational.

We define a new class of symbols with respect to the function \(q_{\gamma,\delta}\). We use the notation \(D^{k}_{\xi} = (-i)^{|k|} \delta^{k}_{\xi}\).

**Definition 5.1.** Let \(q_{\gamma,\delta}(x, \xi)\) be as above. For \(\alpha \geq 0\), and \(\beta = (\beta_1, \beta_2)\) with \(0 < \beta_1\) and \(\beta_2 \geq 0\) such that \(\frac{\delta - 1}{\delta} \beta_2 \leq 1\), we denote by \(\mathcal{S}^{a,\beta}_{q_{\gamma,\delta}}(\mathbb{R}^n)\) the space of symbols \(\tilde{a}(x, \xi) \in C_0(\mathbb{R}^n \times \mathbb{R}^n)\) which have the property that for any given compact set, \(K\), and for any \(n\)-tuples \(k_1\) and \(k_2\), there is a constant \(c_{k_1,k_2}(K) > 0\) such that
\[
\left| D^{k_1}_{\xi} D^{k_2}_{\xi} \tilde{a}(x, \xi) \right| \leq c_{k_1,k_2}(K) \left( \frac{1}{1 + |q_{\gamma,\delta}(x, \xi)|} \right)^{a + \frac{1}{2} \beta_1 |k_1| - \frac{1}{2} d \frac{1}{d+2} \beta_2 |k_2|}
\]
(\(\forall x \in K, \xi \in \mathbb{R}^n\).

Associated to the symbols in class \(\mathcal{S}^{a,\beta}_{q_{\gamma,\delta}}(\mathbb{R}^n)\) are the operators, denoted by \(\Psi^{a,\beta}_{q_{\gamma,\delta}}(\mathbb{R}^n)\) defined in
Theorem 5.2. We say an operator $A : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ is in class $\Psi^{-\alpha,\beta}_{q,\gamma,d}(\mathbb{R}^n)$ if $A$ can be written as an integral operator with symbol $a(x,\zeta) \in \tilde{\Psi}^{-\alpha,\beta}_{q,\gamma,d}(\mathbb{R}^n)$:

$$A\phi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} a(x,\zeta) \hat{\phi}(\zeta) e^{ix\cdot\zeta} d\zeta.$$ 

For example, the case $q,\gamma,d = 1 + |\zeta|^2$ with $\gamma = 2$ and $d = 1$ shows that symbols in the classes $\mathcal{S}^{-2\alpha}_{\gamma,d}(\mathbb{R}^n)$ for $\alpha \geq 0$ belong as well to the class $\Psi^{-\alpha,1}_{\gamma,d}(\mathbb{R}^n)$ for any $0 \leq \beta$ as defined above.

We note for a given symbol, $\tilde{a}(x,\zeta) \in \tilde{\Psi}^{-\alpha,\beta}_{q,\gamma,d}(\mathbb{R}^n)$, we have directly from the definition

$$D^j_{\zeta} \tilde{a}(x,\zeta) \in \tilde{\Psi}^{-\alpha-\beta j}_{q,\gamma,d}(\mathbb{R}^n).$$

Another important consequence of the definition is the following type of commuting of derivatives

**Theorem 5.3.** Let $q,\gamma,d(x,\zeta)$ be as above and $A \in \tilde{\Psi}^{-\alpha,\beta}_{q,\gamma,d}(\mathbb{R}^n)$ for $\alpha \geq 0$. For $u \in W^{k_1}(\mathbb{R}^n)$ the estimates

$$\|D^j_{x} A u\|_{L^2(\mathbb{R}^n)} \lesssim \|u\|_{W^{k_1}\times^{\gamma,d}(\mathbb{R}^n)}$$

hold.

**Proof.** Let $a(x,\zeta) = \sigma(A)$. We have

$$A u = \frac{1}{(2\pi)^n} \int a(x,\zeta) \hat{u}(\zeta) e^{ix\cdot\zeta} d\zeta.$$ 

Then, by differentiating under the integral, we can write $D^j_{x} A u$ as a sum of terms, in which $k_1$ and $k_2$ are multi-indices, such that $k_1 + k_2 = k$, of the form

$$\int D^j_{x} a(x,\zeta) \zeta^{k_2} \hat{u}(\zeta) e^{ix\cdot\zeta} d\zeta.$$ 

We let $\varphi \in C^\infty_0(\mathbb{R}^n)$ and estimate

$$\|\varphi D^j_{x} A u\|_{L^2(\mathbb{R}^n)} \lesssim \int |\varphi(x)|^2 \left( \frac{1}{1 + |q,\gamma,d(x,\zeta)|} \right)^{2\alpha - \frac{\gamma - 1}{\gamma} |\beta_1| |k_1| |\zeta|^{2|k_2|} |\hat{u}(\zeta)|^2 dxd\zeta$$

$$\lesssim \int |\varphi(x)|^2 \left( \frac{1}{1 + |\zeta|} \right)^{2\gamma \alpha} \left( 1 + |\zeta|^{2|k|} \right) |\hat{u}(\zeta)|^2 dxd\zeta$$

$$\lesssim \int \left( 1 + |\zeta|^{2|k| - 2\gamma \alpha} \right) |\hat{u}(\zeta)|^2 d\zeta,$$

where we use (5.2) in the second inequality. Thus

$$\|\varphi D^j_{x} A u\|_{L^2(\mathbb{R}^n)} \lesssim \left( \int \left( 1 + |\zeta|^{2|k| - \gamma \alpha} \right) |\hat{u}(\zeta)|^2 d\zeta \right)^{1/2}$$

$$\lesssim \|u\|_{W^{k_1}\times^{\gamma,d}(\mathbb{R}^n)}.$$
We now define symbols and operators in respective classes $\mathcal{S}^{a,\beta}_{q,\gamma}(\mathbb{R}^n)$ and $\mathcal{P}^{a,\beta}_{q,\gamma}(\mathbb{R}^n)$ for $\alpha > 0$ in a way which deviates from the definitions belonging to pseudodifferential operator theory to take advantage of the fact that we will be working with symbols whose $x$ derivatives can be bounded by fixed growth.

**Definition 5.4.** Let $q_{\gamma,\delta}(x, \zeta)$ be as above. For $\alpha > 0$ and $\beta = (\beta_1, \beta_2)$ with $0 < \beta_1$ and $\beta_2 \geq 0$ such that $\frac{\delta-1}{\delta} \beta_1 \leq 1$, we denote by $\mathcal{S}^{a,\beta}_{q,\gamma}(\mathbb{R}^n)$ the space of symbols $\tilde{a}(x, \zeta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which have the property that for any given compact set, $K$, and for any $n$-tuples $k_1$ and $k_2$, there is a constant $c_{k_1,k_2}(K) > 0$ such that

\[
D^k_x D^\zeta_\xi \tilde{a}(x, \zeta) \leq c_{k_1,k_2}(K) \left( 1 + |q_{\gamma,\delta}(x, \zeta)| \right)^{a - \frac{\delta}{\delta} \beta_1 |k_1| + \frac{\delta}{\delta} \beta_2 \min(|k_2|, \delta \gamma),
\]

for all $x \in K$ and $\xi \in \mathbb{R}^n$.

Again, to illustrate how this definition fits with that of the theory of standard pseudodifferential operators, we take the example in which $q_{\gamma,\delta}(x, \zeta) = 1 + |\zeta|^2$ and $\gamma = 2$, $\delta = 1$. In this special case we see the symbols in the classes $\mathcal{S}^{2a,\beta}_{q,\gamma}(\mathbb{R}^n)$ for $\alpha > 0$ belong as well to the class $\mathcal{S}^{a,\beta}_{q,\gamma}(\mathbb{R}^n)$ for any $\beta_2 \geq 0$ as defined above.

For most of the operators belonging to a $\mathcal{P}^{a,\beta}_{q,\gamma}(\mathbb{R}^n)$ class which we will use the index $\beta = (1, 1)$, and for such operators, we will simply write

\[
\mathcal{P}^{a}_{q,\gamma}(\mathbb{R}^n) := \mathcal{P}^{a,(1,1)}_{q,\gamma}(\mathbb{R}^n).
\]

Naturally occurring the in study of inverses to operators (or in our case, approximate inverses) are theorems on compositions of operators. In this regard we prove the

**Theorem 5.5.** Let $q_{\gamma,\delta}(x, \zeta)$ be a polynomial of rational powers of the $\xi$ variables and of $|\zeta|$, with coefficients smooth functions of $x$ and which satisfies (5.1) and (5.2). Let $0 < \beta \leq \alpha$, $A \in \mathcal{P}^{\alpha}_{q,\gamma}(\mathbb{R}^n)$ and let $B \in \mathcal{P}^{\beta}_{q,\gamma}(\mathbb{R}^n)$ be properly supported. Then the composition $A \circ B$ can be written as a sum

\[
A \circ B = \mathcal{P}^{a,\beta}_{q,\gamma}(\mathbb{R}^n) + \sum_{k \in N} \mathcal{P}^{a,\beta-k\left(\frac{2}{\delta}-\frac{1}{\gamma}\right)}_{q,\gamma}(\mathbb{R}^n) + \mathcal{P}^{a,\alpha-\delta\beta+1}\left(1,1\right)(\mathbb{R}^n).
\]

Specifically, with $N$ chosen to satisfy

\[
N > \delta \gamma (\delta \beta + \delta - 1) - (\beta - 1) - 1,
\]

(5.5) \[\sigma(A \circ B)(x, \zeta) = \sum_{|k| \leq N} \frac{1}{k!} D^k_\xi \sigma(A) D^k_\zeta \sigma(B) + \mathcal{S}^{a,\alpha-\beta-1}\left(1,1\right)(\mathbb{R}^n),\]

where

\[
D^k_\xi \sigma(A) D^k_\zeta \sigma(B) \in \mathcal{S}^{a,\alpha-\beta-k\left(\frac{2}{\delta}-\frac{1}{\gamma}\right)}_{q,\gamma}(\mathbb{R}^n).
\]
**Proof.** Let \( a(x, \xi) \) and \( b(x, \xi) \) be the symbols of the operators \( A \) and \( B \), respectively. We calculate for a test function \( \varphi \in C_0^\infty(\mathbb{R}^n) \), using a change in order of integration since \( B \) is assumed to be properly supported:

\[
A \circ B \varphi(x) = \frac{1}{(2\pi)^n} \int a(x, \xi) \left[ \int b(x', \zeta) \varphi(\zeta) e^{ix' \cdot \xi} d\zeta e^{-ix' \cdot \xi} dx' \right] e^{ix \cdot \xi} d\xi
\]

\[
= \frac{1}{(2\pi)^n} \int \int a(x, \xi) b(x', \zeta) e^{i(x-x') \cdot (\xi-\zeta)} dx' d\zeta \varphi(\zeta) e^{ix \cdot \xi} d\xi.
\]

Thus the symbol of \( A \circ B \) is given by

\[
(5.6) \quad (A \circ B)(x, \xi) = \frac{1}{(2\pi)^n} \int \int a(x, \xi) b(x', \zeta) e^{i(x-x') \cdot (\xi-\zeta)} dx' d\zeta.
\]

We write a Taylor expansion with remainder of \( b(x', \zeta) \) in the \( x' \) variable, around a point \( x \):

\[
b(x', \zeta) = \sum_{|l| \leq N} \frac{1}{l!} \partial_x^l b(x, \xi) (x' - x)^l
\]

\[
+ (N + 1) \sum_{|l| = N+1} \frac{1}{l!} (x' - x)^l \int_0^1 (1 - t)^N \partial_x^l b(x' + t(x - x'), \zeta) dt,
\]

where we use the index notation:

\[
l! = (l_1! \cdots l_n!)
\]

for the index \( l = (l_1, \ldots, l_n) \). When we insert a term, \( \partial_x^l b(x, \xi) (x' - x)^l \) into the integral in (5.6) we get

\[
\frac{1}{(2\pi)^n} \int \int a(x, \xi) \partial_x^l b(x, \xi) (x' - x)^l e^{i(x-x') \cdot (\xi-\zeta)} dx' d\zeta
\]

\[
= \frac{1}{(2\pi)^n} \int \int a(x, \xi) \partial_x^l b(x, \xi) D_a^l \varphi(\zeta) e^{i(x-x') \cdot (\xi-\zeta)} dx' d\zeta
\]

\[
= \frac{1}{(2\pi)^n} \int \int \partial_x^l a(x, \xi) \partial_x^l b(x, \xi) e^{i(x-x') \cdot (\xi-\zeta)} dx' d\zeta
\]

\[
= D_x^l b(x, \xi).
\]

We now want to show that each such term, \( D_x^l b(x, \xi) \partial_x^l b(x, \xi) \) belongs to class \( \mathcal{S}_{\alpha, \beta}^{\alpha + \beta - 1}(\mathbb{R}^n) \).

We thus estimate \( D_x^l D_x^{k_2} \left( D_x^k a(x, \xi) D_x^l b(x, \xi) \right) \) which is made up of a sum of terms

\[
D_x^{k_1} \left( D_x^{l+k_1} a(x, \xi) \right) D_x^{k_2} D_x^{k_2+1} b(x, \xi), \quad k_1 + k_2 = k_j, \quad \text{for } j = 1, 2.
\]

We estimate on compact sets, and for ease of notation suppress the constants of inequality, preferring instead to write simply \( \lesssim \), but keeping in mind the constants of inequalities depend on the derivative indices.
By definition, since $D_x^{l+k_{11}} a(x, \zeta) \in S_{q,\delta}^{-\alpha - \frac{1}{2q} |l + k_{11}|} (\mathbb{R}^n)$ by (5.3), we have

$$D_x^{k_{21}} D_x^{l+k_{11}} a(x, \zeta) \lesssim \left( \frac{1}{1 + |q\gamma,\delta(x, \zeta)|} \right)^{a + \frac{1}{2q} (|l| + |k_{11}| - (\delta - 1)|k_{21}|)} \delta^\gamma,$$

and

$$D_x^{k_{12}} D_x^{l+k_{11}} b(x, \xi) \lesssim \left( 1 + |q\gamma,\delta(x, \zeta)| \right)^{\beta - \frac{1}{2q} |k_{12}| + \frac{1}{2q} \min(|k_{22}| + |l|, |\delta \gamma|)} \lesssim \left( 1 + |q\gamma,\delta(x, \zeta)| \right)^{\beta + \frac{1}{2q} ((\delta - 1)|k_{22}| + |l|) - |k_{12}|},$$

by (5.4).

We can now estimate $D_x^{k_1} D_x^{k_2} \left( D_x^l a(x, \zeta) D_x b(x, \xi) \right)$ by

$$\left| D_x^{k_1} D_x^{k_2} \left( D_x^l a(x, \zeta) D_x b(x, \xi) \right) \right| \lesssim \sum_{k_{12} < k_{21} - k_2} \sum_{k_{11} < k_{12} - k_1} \left( \frac{1}{1 + |q\gamma,\delta(x, \zeta)|} \right)^{a - \beta + \frac{1}{2q} (|l| + |k_{11}| + |k_{12}|) - \frac{1}{2q} |(k_{21} + |k_{22}| + |l|)} \lesssim \left( \frac{1}{1 + |q\gamma,\delta(x, \zeta)|} \right)^{a - \beta + \frac{1}{2q} (|l| + \frac{1}{2q} |k_1| - \frac{1}{2q} |k_2|)}.$$

To handle the terms in the symbol (5.6) stemming from the remainder term of (5.7), we denote

$$I_1(x, x', \zeta) = \int_0^1 (1 - t)^N \partial_x^\alpha b(x + t(x' - x), \zeta) dt$$

for $|l| = N + 1$ and insert the terms

$$(x' - x)^l I_1(x, x', \zeta)$$

into (5.6). We have

$$\int \int a(x, \zeta) (x' - x)^l I_1(x, x', \zeta) e^{i(x - x')(\zeta - \xi)} dx' d\zeta = \int \int D_x^l a(x, \zeta) I_1(x, x', \zeta) e^{i(x - x')(\zeta - \xi)} dx' d\zeta,$$

and hence

$$D_x^{k_1} D_x^{k_2} \int \int a(x, \zeta) (x' - x)^l I_1(x, x', \zeta) e^{i(x - x')(\zeta - \xi)} dx' d\zeta$$

is a sum of terms (ignoring constants of multiplication) of the form

$$\int \int \left( D_x^{l+k_{12}} a(x, \zeta) \right) \left( D_x^{k_{11}} D_x^{k_{22}} I_1(x, x', \zeta) \right) \left( D_x^{k_{12}} D_x^{k_{23}} e^{i(x - x')(\zeta - \xi)} \right) dx' d\zeta = \int \int \left( D_x^{l+k_{12}} D_x^{k_{21}} a(x, \zeta) \right) \left( D_x^{k_{11}} D_x^{k_{22}} D_x^{k_{23}} I_1(x, x', \zeta) \right) e^{i(x - x')(\zeta - \xi)} dx' d\zeta,$$

where $k_{11} + k_{12} = k_1$ and $k_{21} + k_{22} + k_{23} = k_2$. 
We now let $\Lambda^{M}_{\gamma \delta}$ denote the differential operator
\[
\left( 1 + \sum_{j} D_{x_{j}}^{2} \right)^{\delta \gamma M/2},
\]
where $M$ is a positive integer such that $M' := \delta \gamma M/2$ is an integer, and write
\[
\int \int \left( D_{\xi}^{1+k_{12}} D_{x}^{k_{21}} a(x, \xi) \right) \left( D_{\xi}^{k_{11}} D_{x}^{k_{23}} D_{x}^{k_{22}} I_{l}(x, x', \xi) \right) e^{i(x-x')(\xi-\xi')} \, dx' d\xi
\]
\[
= \int \int \left( D_{\xi}^{1+k_{12}} D_{x}^{k_{21}} a(x, \xi) \right) \left( D_{\xi}^{k_{11}} D_{x}^{k_{23}} D_{x}^{k_{22}} I_{l}(x, x', \xi) \right) \frac{\Lambda^{M}_{\gamma \delta} e^{i(x-x')(\xi-\xi')}}{(1 + |\xi - \xi'|^{2})^{M'}} \, dx' d\xi.
\]
We denote by $\left( \Lambda^{M}_{\gamma \delta} \right)^{*}$ the adjoint of $\Lambda^{M}_{\gamma \delta}$. It follows that
\[
\int \int \left( D_{\xi}^{1+k_{12}} D_{x}^{k_{21}} a(x, \xi) \right) \left( D_{\xi}^{k_{11}} D_{x}^{k_{23}} D_{x}^{k_{22}} I_{l}(x, x', \xi) \right) e^{i(x-x')(\xi-\xi')} \, dx' d\xi
\]
\[
= \int \int \left( D_{\xi}^{1+k_{12}} D_{x}^{k_{21}} a(x, \xi) \right) \left( \Lambda^{M}_{\gamma \delta} \right)^{*} \left( D_{\xi}^{k_{11}} D_{x}^{k_{23}} D_{x}^{k_{22}} I_{l}(x, x', \xi) \right) \frac{e^{i(x-x')(\xi-\xi')}}{(1 + |\xi - \xi'|^{2})^{M'}} \, dx' d\xi.
\]
For large enough $|l| = N + 1$ (large enough so that $|l| > \delta \gamma$) we have
\[
\left| D_{x_{j}}^{M_{j}} D_{\xi}^{k_{11}} D_{\xi}^{k_{23}} D_{x}^{k_{22}} D_{x}^{k_{21}} b(x + t(x' - x), \xi) \right| \lesssim \frac{(1 + |q_{\gamma, \delta}(x + t(x' - x), \xi)|)^{\beta + \delta - 1}}{(1 + |q_{\gamma, \delta}(x + t(x' - x), \xi)|)^{\frac{1}{2}|k_{11}|}}
\]
\[
\lesssim \frac{(1 + |\xi|)^{\delta \gamma (\beta + \delta - 1)}}{(1 + |\xi|)^{\frac{1}{2}|k_{11}|}}
\]
\[
= (1 + |\xi|)^{\delta \gamma (\beta + \delta - 1) - \frac{1}{2}|k_{11}|},
\]
(5.8)
where $M_{j} \leq M'$ and with the constant of inequality depending only on the indices of the derivatives. It follows that
\[
\left( \Lambda^{M}_{\gamma \delta} \right)^{*} \left( D_{\xi}^{k_{11}} D_{x}^{k_{23}} D_{x}^{k_{22}} I_{l}(x, x', \xi) \right) \lesssim (1 + |\xi|)^{\delta \gamma (\beta + \delta - 1) - \frac{1}{2}|k_{11}|}
\]
as well.
Thus using the estimates on compact sets for the symbol $D_{\xi}^{1+k_{12}} D_{x}^{k_{21}} a(x, \xi)$,
\[
\left| D_{\xi}^{1+k_{12}} D_{x}^{k_{21}} a(x, \xi) \right| \lesssim \left( \frac{1}{1 + |q_{\gamma, \delta}(x, \xi)|} \right)^{a + \frac{1}{2}(N+1+|k_{12}|)(\delta - 1)k_{21})}
\]
(5.9)
for $|l| = N + 1$ as well as the estimates above, and the estimate
\[
\frac{1}{(1 + |\xi - \xi'|^{2})^{M'}} \lesssim \frac{1}{(1 + |q_{\gamma, \delta}(x, \xi - \xi)|)^{M}}
\]
following from the definitions, we have

\[
\left| D_{\xi}^{k_1} D_{\zeta}^{k_2} \int \int a(x, \xi)(x' - x) I_1(x, x', \xi) e^{i(x-x')(\xi - \zeta)} dx' d\zeta \right|
\]

\[
\lesssim \sum_{k_{21} + k_{12} = k_1} \int \int \left( \frac{1}{1 + |q_{\gamma, \delta}(x, \xi)|} \right)^{a + \frac{1}{\delta_2}(N+1+|k_{12}|-(\delta-1)|k_{21}|)} \frac{(1 + |\xi|)^{\delta_2(\beta + \delta - 1) - \frac{\beta}{2} |k_{12}|}}{(1 + |q_{\gamma, \delta}(x, \xi - \xi)|)^M} dx' d\zeta.
\]

Since \( B \) is properly supported we can assume that the integration over \( x' \) is over a compact set. From the property in (5.1), we have for any \( m \) the estimate

(5.10) \[
\frac{1}{(1 + |q_{\gamma, \delta}(x, \xi|)^m} \lesssim \frac{(1 + |q_{\gamma, \delta}(x, \xi)|)^m}{(1 + |q_{\gamma, \delta}(x, \xi|)^m},
\]

and, likewise

(5.11) \[
\frac{1}{(1 + |q_{\gamma, \delta}(x, \xi - \xi)|)^m} \lesssim \frac{(1 + |q_{\gamma, \delta}(x, \xi)|)^m}{(1 + |q_{\gamma, \delta}(x, \xi|)^m}.
\]

Then with two applications of the above estimates (the first with \( m = \alpha + \frac{1}{\delta_2}(N + 1 + |k_{12}|) \), and the second with \( m = \frac{\beta}{2} |k_{21}| \)) we can estimate the above integral, with \( M \) chosen large enough so that

\[
M \geq (n + 1) + \alpha + \frac{1}{\delta_2}(N + 1 + |k_{12}|) + \frac{\beta}{\delta_2} |k_{21}|
\]

and

\[
\frac{1}{(1 + |q_{\gamma, \delta}(x, \xi - \xi)|)^M} \lesssim \frac{1}{(1 + |q_{\gamma, \delta}(x, \xi)|)^{a + \frac{1}{\delta_2}(N+1+|k_{12}|)} (1 + |q_{\gamma, \delta}(x, \xi)|)^{\frac{\beta}{\delta_2} |k_{21}|}}{(1 + |q_{\gamma, \delta}(x, \xi - \xi)|)^{a + \frac{1}{\delta_2}(N+1+|k_{12}|)} (1 + |q_{\gamma, \delta}(x, \xi|)^{\frac{\beta}{\delta_2} |k_{21}|}.}
\]
by
\[
\int \frac{1}{1 + |\xi|^{n+1}} \left( \frac{1}{1 + |q_{\gamma,\delta}(x, \xi)|} \right) \frac{1}{\frac{1}{\alpha} + \frac{1}{N+1} + |k_2|} (1 + |q_{\gamma,\delta}(x, \xi)|)^{\frac{1}{\alpha}} |\xi|^{\frac{1}{\alpha}} |k_2| d\xi
\]
\[
\lesssim \left( \frac{1}{1 + |q_{\gamma,\delta}(x, \xi)|} \right) \frac{1}{\frac{1}{\alpha} + \frac{1}{N+1} + |k_2|} (1 + |q_{\gamma,\delta}(x, \xi)|)^{\frac{1}{\alpha}} |k_2| d\xi,
\]
Therefore, the remainder term in (5.5) is in the class of symbols $\mathcal{S}_{\Psi,\delta}^{-\alpha - \beta} (\mathbb{R}^n)$. □

**Remark 5.6.** We thus conclude that the terms $\sum_{k_1 \geq 1} 1 \int D^{k_1} \alpha(A) \partial^{k_2} \beta(B)$ in the summation of (5.5) all belong to $\mathcal{S}_{\Psi,\delta}^{-\alpha - \beta} (\mathbb{R}^n)$. Since, by assumption,

\[
\frac{2}{\delta \gamma} - \frac{1}{\gamma} > 0
\]

it also holds that these terms, as well as the last term in (5.5), belong to $\mathcal{S}_{\Psi,\delta}^{-\alpha + \beta} (\mathbb{R}^n)$ for $\delta < -\alpha + \beta$, and as such can be considered remainder terms.

We will also have occasion to compose $\Psi (\mathbb{R}^n)$ operators with pseudodifferential operators. For this purpose, we note that we may write a standard pseudodifferential operator of positive order $\alpha$, $\Psi^\alpha (\mathbb{R}^n)$ as

(5.12) \[ \Psi^\alpha (\mathbb{R}^n) = \Psi_{\Psi,\delta}^{\alpha / \gamma} (\mathbb{R}^n); \]

let $a(x, \xi) \in \mathcal{S}^a (\mathbb{R}^n)$, then the estimates

\[
\left| D^{k_1} \partial^{k_2} a(x, \xi) \right| \lesssim \frac{(1 + |\xi|)^a}{(1 + |\xi|)^{k_1}} (1 + |q_{\gamma,\delta}(x, \xi)|)^{a / \gamma}
\lesssim \frac{(1 + |q_{\gamma,\delta}(x, \xi)|)^{a / \gamma}}{(1 + |q_{\gamma,\delta}(x, \xi)|)^{\frac{1}{\alpha} k_1}}
\lesssim (1 + |q_{\gamma,\delta}(x, \xi)|)^{a / \gamma - \frac{1}{\alpha} k_1} + \frac{a}{\alpha} \min(|k_2|, \delta \gamma)
\]

hold for $x$ belonging to a given compact set $K$ with a constant of inequality depending on the indices $k_1$ and $k_2$ and on $K$. 
Corollary 5.7. For \( 0 < \beta \leq \alpha \) let \( A \in \Psi^{-\alpha}_{q_{\gamma,\delta}}(\mathbb{R}^n) \) and let \( B \) be a properly supported operator in the class \( \Psi^{\beta}(\mathbb{R}^n) \). Then
\[
A \circ B \in \Psi^{-\alpha+\beta/\gamma}_{q_{\gamma,\delta}}(\mathbb{R}^n) + \Psi^{-\alpha+\beta/\gamma-1/(1/\delta,1)}_{q_{\gamma,\delta}}(\mathbb{R}^n).
\]

For specific values of \( \delta \) and \( \gamma \), we can compose operators with the positive order operator coming first:

Theorem 5.8. Let \( A \in \Psi^{-\alpha}_{q_{\gamma,\delta}}(\mathbb{R}^n) \) be properly supported and \( B \in \Psi^{\beta}(\mathbb{R}^n) \) for \( 0 < \beta \leq \alpha \) and \( \delta(\delta-1) < 2/3 \). Then for sufficiently large \( N \),
\[
B \circ A \in \Psi^{-\alpha+\beta}_{q_{\gamma,\delta}}(\mathbb{R}^n) + \sum_{1 \leq k \leq N} \Psi^{-\alpha+\beta-k\left(\frac{2}{\gamma}-\frac{1}{\gamma}\right)}_{q_{\gamma,\delta}}(\mathbb{R}^n) + \Psi^{-\alpha+\beta-1}_{q_{\gamma,\delta}}(\mathbb{R}^n)
\]
with symbol representation
\[
(5.13) \quad \sigma(B \circ A)(x, \xi) = \sum_{0 \leq k \leq N} \frac{1}{k!} D^k_x \sigma(B) \partial^k_x \sigma(A) + \mathcal{S}_{q_{\gamma,\delta}}^{-\alpha+\beta-1,1}(\mathbb{R}^n).
\]

Remark 5.9. The inequality \((\delta-1) < 1 \leq \gamma\) following from the definitions ensures that the parameter \((1/\delta, \delta)\) controlling the growth of derivatives conforms to the requirements in Definition 5.1 and thus so that Theorem 5.3 applies. The need for the condition \(\delta(\delta-1) < 2/3\) will be apparent from the proof of the theorem.

Proof. The proof of Theorem 5.5 may be followed to write \( \sigma(B \circ A)(x, \xi) \) as a finite sum plus a remainder integral term. The proof follows through with obvious changes and shows the finite sum is of the given form in (5.13) and that
\[
D^k_x \sigma(B) \partial_x \sigma(A) \in \mathcal{S}_{q_{\gamma,\delta}}^{-\alpha+\beta-k\left(\frac{2}{\gamma}-\frac{1}{\gamma}\right)}(\mathbb{R}^n).
\]

We leave these details to the reader.

For the remainder integral term, the estimate (5.8) needs to be replaced with
\[
\left| D^{M_1}_{x'} D^{k_{11}}_{x} D^{k_{22}}_{x} D^{k_{23}}_{x} D^l_x a(x + t(x' - x), \xi) \right| \lesssim \left( \frac{1}{1 + |q_{\gamma,\delta}(x + t(x' - x), \xi)|} \right)^{\alpha + \frac{1}{\gamma} |k_{11}| - \frac{1}{\gamma} |(k_{22} + k_{23} + l)| + M'}
\]
by (5.4) so that for
\[
I_l(x, x', \xi) = \int_0^1 (1-t)^N D^l_x a(x + t(x' - x), \xi) dt
\]
Similarly, \((5.9)\) needs to be replaced with
\[
\frac{1}{(1 + |q_{\gamma,\delta}(x, \xi)|)^{a + \frac{1}{\delta\gamma}}|k_{11}| - \frac{\delta - 1}{\delta\gamma} (k_{22} + k_{23} + |l| + M')}
\]
assuming that, as
\[
(1 + |q_{\gamma,\delta}(x, \xi)|)^{a + \frac{1}{\delta\gamma}}|k_{11}| - (\delta - 1)(k_{22} + k_{23} + |l| + M')
\]
set. We choose \(M(1)\) is an integer. Then using the relations \((5.10)\) and \((5.11)\), the above term is bounded by
\[
\frac{1}{(1 + |q_{\gamma,\delta}(x, \xi)|)^{\frac{\delta}{2} + \frac{1}{\delta\gamma} |k_{11}| - \frac{\delta - 1}{\delta\gamma} (k_{22} + k_{23} + |l| + M')}
\]
\[
\frac{1}{1 + |q_{\gamma,\delta}(x, \xi)|})^{\frac{\delta}{2} + \frac{1}{\delta\gamma} |k_{11}| - \frac{\delta - 1}{\delta\gamma} (k_{22} + k_{23} + |l| + M')}
\]
\[
|\frac{1}{1 + |q_{\gamma,\delta}(x, \xi)|})^{\frac{\delta}{2} + \frac{1}{\delta\gamma} |k_{11}| - \frac{\delta - 1}{\delta\gamma} (k_{22} + k_{23} + |l| + M')}
\]
\[
|\frac{1}{1 + |q_{\gamma,\delta}(x, \xi)|})^{\frac{\delta}{2} + \frac{1}{\delta\gamma} |k_{11}| - \frac{\delta - 1}{\delta\gamma} (k_{22} + k_{23} + |l| + M')}
\]
assuming that, as \(A\) is properly supported, the integration over the \(x'\) variable is over a compact set. We choose \(M \geq n + 1 + (\delta - 1) + \beta + \frac{1}{\delta\gamma} (N + 1 + |k_{12}|)\), so that \(M' = \delta\gamma M/2\) is an integer; we write
\[
M = n' + 1 + (\delta - 1) + \beta + \frac{1}{\delta\gamma} (N + 1 + |k_{12}|),
\]
where \(n \leq n'\) \((n')\) is a function of \(\delta\gamma\) is chosen so that \(M\) is a large enough integer guaranteeing that \(M'\) is an integer. Then using the relations \((5.10)\) and \((5.11)\), the above term is bounded by
\[
\frac{1}{(1 + |q_{\gamma,\delta}(x, \xi)|)^{\frac{\delta}{2} + \frac{1}{\delta\gamma} (|k_{11}| + |k_{12}|) + \frac{\delta(\delta - 1)}{2\delta\gamma}} (N + 1) - (\delta - 1) - \frac{1}{\delta\gamma} |k_{22} + k_{23} + M'|
\]
\[
\frac{1}{(1 + |q_{\gamma,\delta}(x, \xi)|)^{\frac{\delta}{2} - \frac{1}{\delta\gamma} (2 + \delta(\delta - 1) + \frac{1}{\delta\gamma} |k_{11}| + \frac{1}{\delta\gamma} (2 - \delta(\delta - 1)) |k_{12}| + \frac{1}{\delta\gamma} (2 - \delta(\delta - 1)) (N + 1) - (\delta - 1) - \frac{1}{\delta\gamma} |k_{22} + k_{23} + (\delta - 1) (n' + 1)}
\]
\[
\frac{1}{(1 + |q_{\gamma,\delta}(x, \xi)|)^{\frac{\delta}{2} - \frac{1}{\delta\gamma} (2 + \delta(\delta - 1)) - \frac{1}{\delta\gamma} (2 - \delta(\delta - 1) + \frac{1}{\delta\gamma} |k_{11}| - \frac{1}{\delta\gamma} |k_{12}| - \frac{\delta(\delta - 1)}{2} (n' + 1))}
\]
\[
\frac{1}{(1 + |q_{\gamma,\delta}(x, \xi)|)^{\frac{\delta}{2} - \frac{1}{\delta\gamma} (2 + \delta(\delta - 1)) - \frac{1}{\delta\gamma} (2 - \delta(\delta - 1) + \frac{1}{\delta\gamma} |k_{11}| - \frac{1}{\delta\gamma} |k_{12}| - \frac{\delta(\delta - 1)}{2} (n' + 1))}
\]

As $\frac{2-3\delta(\delta-1)}{2\delta} > 0$ we can choose $N = N(\alpha, \beta, \gamma, \delta)$ large enough so that the remainder integral term is of type $\Psi_{q_\gamma,\delta}^{-\alpha+\beta-1}(1, \delta)$ $(\mathbb{R}^n)$.

**Corollary 5.10.** Let $A$ and $B$ be properly supported with $A \in \Psi_{q_\gamma,\delta}^{-\alpha}(\mathbb{R}^n)$ and $B \in \Psi_{q_\gamma,\delta}^\beta(\mathbb{R}^n)$ for $0 < \beta \leq \alpha$, and $\delta(\delta-1) < 2/3$. Then

$$[A, B] = \Psi_{q_\gamma,\delta}^{-\alpha+\beta-1}(1, \delta) + \Psi_{q_\gamma,\delta}^{-\alpha+\beta-1}(1, \delta) \Psi_{q_\gamma,\delta}^{-\alpha+\beta}(\mathbb{R}^n).$$

**Proof.** Subtract (5.13) from (5.5). □

In this section $\Psi$ operators were defined on $\mathbb{R}^n$. Localization arguments lead to the obvious definitions for operators $\Psi(\Omega)$ with the properties of the $\Psi$ operators but on a domain $\Omega \subset \mathbb{R}^n$.

### 6. THE DIRICHLET TO NEUMANN OPERATOR

The Dirichlet to Neumann operator (DNO), which is the boundary value operator giving the inward normal derivative of the solution to a Dirichlet problem, arises from the normal derivative in our boundary condition (3.2). It is the symbol of the DNO which contains a term (of order $< 1$) which allows for the construction of parametrix-like approximate inverses later.

We start with the DNO corresponding to the operator $\Box_\varepsilon$. We look for an expression for $v = v_1 \omega_1 + v_2 \omega_2$ which solves

$$2 \Box_\varepsilon v = 0 \quad \text{on } \Omega$$

$$v = g_b \quad \text{on } \partial \Omega,$$

(modulo smooth terms) and we obtain an expression for $\frac{\partial g_b}{\partial \rho}$ near a given point $p \in \partial \Omega$ in terms of $g_b$.

If $\chi_p$ is a smooth cutoff function with support in a small neighborhood of $p$ and $\chi_p'$ a smooth cutoff such that $\chi_p' \equiv 1$ on supp $\chi_p$, we have

$$2 \Box_\varepsilon (\chi_p v) = \Psi_1 (\chi_p' v) \quad \text{on } \Omega$$

$$\chi_p v = \chi_p g_b \quad \text{on } \partial \Omega.$$

We choose a coordinate system, with $\rho$ as one of the coordinates, and write out each operator composing $2 \Box_\varepsilon$ in with respect to our chosen coordinates. We recall from Section 3 that we can expand the vector fields $L_1$ and $T$ as

$$L_1 = L_1^0 + \rho L_1^1 + \cdots$$

$$L_2 = \frac{1}{\sqrt{2} \partial \rho} + i \left(T^0 + \rho T^1 + \cdots \right)$$
and choose coordinates \((x_1, x_2, x_3)\) on \(\partial \Omega\), in terms of which the vector fields \(L^1_i\) and \(T^0\) are given by

\[
T^0 = \frac{\partial}{\partial x_3}
\]

\[
L^1_i = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) + O(x-p).
\]

We write

\[
(6.2) \quad L_1 = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) + \frac{3}{2} \ell_j(x) \frac{\partial}{\partial x_j} + \rho L^1_j + \cdots,
\]

where \(\ell_j(x) = O(x-p)\) for \(j = 1, 2, 3\). Then we have

\[
\mathcal{T}_1 L_1 = \frac{1}{4} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 4 \frac{\partial^2}{\partial x_3^2} \right) + \sum_{j=1}^3 \mathcal{T}_1 (\ell_j(x)) \frac{\partial}{\partial x_j} + \frac{3}{2} \sum_{j=1}^3 \overline{\ell}_j(x) \frac{\partial^2}{\partial x_1 \partial x_j} - \frac{i}{2} \sum_{j=1}^3 \overline{\ell}_j(x) \frac{\partial^2}{\partial x_2 \partial x_j}
\]

\[
+ \frac{3}{2} \sum_{j=1}^3 \ell_j(x) \frac{\partial^2}{\partial x_1 \partial x_j} + \frac{i}{2} \sum_{j=1}^3 \ell_j(x) \frac{\partial^2}{\partial x_2 \partial x_j} + \sum_{j,k=1}^3 \overline{\ell}_j(x) \ell_k(x) \frac{\partial^2}{\partial x_j \partial x_k} + O(\rho).
\]

Similarly,

\[
T^2 = \frac{\partial^2}{\partial x_3^2} + O(\rho).
\]

Hence, the operator \(-2(\mathcal{T}_1 L_1 + T^2)\) occurring in \(\Box_\rho\) in (6.3) can be written as

\[
(6.3) \quad - \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 4 \frac{\partial^2}{\partial x_3^2} \right) - 2 \sum_{j=1}^3 \mathcal{T}_1 (\ell_j(x)) \frac{\partial}{\partial x_j} - 2 \sum_{j=1}^3 \text{Re} \, \ell_j(x) \frac{\partial^2}{\partial x_1 \partial x_j} + 2 \sum_{j=1}^3 \text{Im} \, \ell_j(x) \frac{\partial^2}{\partial x_2 \partial x_j}
\]

\[
- 2 \sum_{j,k=1}^3 \overline{\ell}_j(x) \ell_k(x) \frac{\partial^2}{\partial x_j \partial x_k} + \rho \sum_{j,k=1}^3 \tau_{jk}(x, \rho) \frac{\partial^2}{\partial x_j \partial x_k}.
\]

For simplicity we collect first order tangential derivatives in (6.3) and (6.4) into a matrix \(A\), and use (6.3) to write \(2\Box_\rho(\chi_{p}v)\) as

\[
2\Box_\rho(\chi_{p}v) = - \left( \frac{\partial^2}{\partial x_1^2} + \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 4 \frac{\partial^2}{\partial x_3^2} \right) \right) (\chi_{p}v) + 2B_\ell(\chi_{p}v)
\]

\[
- 2 \left( \sum_{j=1}^3 \text{Re} \, \ell_j(x) \frac{\partial^2}{\partial x_1 \partial x_j} - \sum_{j=1}^3 \text{Im} \, \ell_j(x) \frac{\partial^2}{\partial x_2 \partial x_j} + \sum_{j,k=1}^3 \overline{\ell}_j(x) \ell_k(x) \frac{\partial^2}{\partial x_j \partial x_k} \right) (\chi_{p}v)
\]

\[
+ \sqrt{2} \text{Re} \, \ell_j(x) \frac{\partial}{\partial x_j} (\chi_{p}v) + A(\chi_{p}v) + \rho \sum_{j,k=1}^3 \tau_{jk}(x, \rho) \frac{\partial^2}{\partial x_j \partial x_k} (\chi_{p}v).
\]
We analyze each term on the right-hand side of (6.4) as a pseudodifferential operator acting on $\chi_{\rho}v$ which is supported on the half-space $\rho \leq 0$. We use the calculations
\[
\frac{\partial^2 (\chi_{\rho}v)}{\partial \rho^2} (\xi, \eta) = \int_{-\infty}^{0} \frac{\partial^2 (\chi_{\rho}v)}{\partial \rho^2} e^{-ix\xi} e^{-i\eta \eta} d\rho dx
\]
\[
= \frac{\partial (\chi_{\rho}v)}{\partial \rho} (\xi, 0) + i\eta (\chi_{\rho}v) (\xi, 0) - \eta^2 (\chi_{\rho}v) (\xi, \eta)
\]
\[
(6.5)
\]
\[
= \left(\frac{\partial (\chi_{\rho}v)}{\partial \rho}\right|_{\rho=0} \cdot \delta (\rho)\right) + i\eta \left(\chi_{\rho} \left( g_b \times \delta (\rho) \right) \right) - \eta^2 (\chi_{\rho}v) (\xi, \eta),
\]
in the following expressions. We start with the first component of (6.4) (in the end, we are only interested in the normal derivative of the first component). We use the notation
\[
v = v_1 \tilde{\omega}_1 + v_2 \tilde{\omega}_2
\]
\[
g_b = g^1_b \tilde{\omega}_1 + g^2_b \tilde{\omega}_2,
\]
and consider first the terms of order greater than 1,
\[
- \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 4 \frac{\partial^2}{\partial x_3^2} \right) \right] - 2\varepsilon |d_1|^{3/2}
\]
\[
+ 2 \left[ \sum_{j=1}^{3} \text{Re} \ell_j (x) \frac{\partial^2}{\partial x_1 \partial x_j} - \sum_{j=1}^{3} \text{Im} \ell_j (x) \frac{\partial^2}{\partial x_2 \partial x_j} + \sum_{j,k=1}^{3} \ell_j (x) \ell_k (x) \frac{\partial^2}{\partial x_j \partial x_k} \right] (\chi_{\rho}v_1),
\]
and write them as a sum of operators on $\chi_{\rho}v_1$, $\frac{\partial (\chi_{\rho}v_1)}{\partial \rho}\bigg|_{\rho=0} \times \delta (\rho)$, and $g^1_b \times \delta (\rho)$. For the second derivative with respect to $\rho$ we use (6.5) to obtain
\[
\frac{1}{(2\pi)^4} \int (\eta^2 + \varepsilon^2 \xi^2) e^{i\xi \xi} e^{i\eta \eta} d\xi d\eta
\]
\[
- \frac{1}{(2\pi)^4} \int \left[ \left( \frac{\partial (\chi_{\rho}v_1)}{\partial \rho}\bigg|_{\rho=0} \times \delta (\rho) \right) + i\eta \left(\chi_{\rho} \left( g^1_b \times \delta (\rho) \right) \right) \right] e^{i\xi \xi} e^{i\eta \eta} d\xi d\eta,
\]
where
\[
E = \frac{1}{2} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{array} \right] + \left[ \begin{array}{ccc} \text{Re} \ell_1 & \text{Re} \ell_2 & \text{Re} \ell_3 \\ -\text{Im} \ell_1 & -\text{Im} \ell_2 & -\text{Im} \ell_3 \end{array} \right] + \left[ \begin{array}{ccc} \ell_1 \ell_1 & \ell_2 \ell_1 & \ell_3 \ell_1 \\ \ell_1 \ell_2 & \ell_2 \ell_2 & \ell_3 \ell_2 \\ \ell_1 \ell_3 & \ell_2 \ell_3 & \ell_3 \ell_3 \end{array} \right].
\]
Similarly the terms which are \( O(\rho) \) in the first component of (6.4) will be written as the operator, \( \rho \circ \tau \) with symbol:

\[
- \rho \sum_{j,k=1}^{3} \tau^{jk}(x,\rho) \frac{\partial^{2}}{\partial x_{j} \partial x_{k}}(\chi_{p}v)
\]

\[= - \frac{1}{(2\pi)^{4}} \int \rho \sum_{j,k=1}^{3} \tau^{jk}(x,\rho) \xi_{j} \xi_{k} \]

\[= \frac{1}{(2\pi)^{4}} \int \left( \frac{\partial (\chi_{p}v)}{\partial \rho} \right)_{\rho=0} \times \delta(\rho) + i\eta \left( \chi_{p} \left( \mathbf{g}^{1}_{1} \times \delta(\rho) \right) \right) e^{ix} e^{i\rho \eta} d\xi d\eta.
\]

Putting all this together we rewrite the first component of (6.4) in operator form:

\[
\frac{1}{(2\pi)^{4}} \int \left( \eta^{2} + \xi^{T} \mathbf{E} \xi + 2e|\xi_{3}|^{3/2} \right) \left( \chi_{p}v \right)(\xi,\eta) e^{ix} e^{i\rho \eta} d\xi d\eta
\]

\[= \frac{1}{(2\pi)^{4}} \int \left[ \left( \frac{\partial (\chi_{p}v)}{\partial \rho} \right)_{\rho=0} \times \delta(\rho) + i\eta \left( \chi_{p} \left( \mathbf{g}^{1}_{1} \times \delta(\rho) \right) \right) \right] e^{ix} e^{i\rho \eta} d\xi d\eta
\]

\[+ \sqrt{2} \int \frac{\partial (\chi_{p}v)}{\partial \rho} + (A(\chi_{p}v))_{1} + \rho \tau(\chi_{p}v) = \Psi^{1}(\chi_{p}v).
\]

In order to apply a parametrix (over which we can integrate), we add to both sides a 0 order term:

\[
\frac{1}{(2\pi)^{4}} \int \left( 1 + \eta^{2} + \xi^{T} \mathbf{E} \xi + e|\xi_{3}|^{3/2} \right) \left( \chi_{p}v \right)(\xi,\eta) e^{ix} e^{i\rho \eta} d\xi d\eta
\]

\[= \frac{1}{(2\pi)^{4}} \int \left[ \left( \frac{\partial (\chi_{p}v)}{\partial \rho} \right)_{\rho=0} \times \delta(\rho) + i\eta \left( \chi_{p} \left( \mathbf{g}^{1}_{1} \times \delta(\rho) \right) \right) \right] e^{ix} e^{i\rho \eta} d\xi d\eta
\]

\[+ \sqrt{2} \int \frac{\partial (\chi_{p}v)}{\partial \rho} + (A(\chi_{p}v))_{1} + \rho \tau(\chi_{p}v) = \Psi^{1}(\chi_{p}v).
\]
We define
\[
\Xi^2(x, \xi) = \xi \partial \xi + 2\varepsilon |\xi|^{3/2}
\]
and apply a parametrix for the operator, \( \Gamma = Op(1 + \eta^2 + \Xi^2(x, \xi)) \), whose principal term, \( \Gamma^\sharp \), has as symbol
\[
(6.7) \quad \sigma(\Gamma^\sharp) = \frac{1}{1 + \eta^2 + \Xi^2(x, \xi)}
\]
to both sides of (6.6). From (6.6) we then have
\[
\chi(p) v_1 = \frac{1}{(2\pi)^d} \int \frac{1}{1 + \eta^2 + \Xi^2(x, \xi)} \left[ \frac{\partial (\chi(p) v_1)}{\partial \rho} \right]_{\rho = 0} \times (\rho)
\]
\[
+ \frac{1}{(2\pi)^d} \int \frac{1}{1 + \eta^2 + \Xi^2(x, \xi)} \left[ \frac{\partial (\chi(p) v_1)}{\partial \rho} \right]_{\rho = 0} \times (\rho)
\]
\[
- \sqrt{2\Gamma^\sharp} \circ S \left( \frac{\partial}{\partial \rho} (\chi(p) v_1) \right) - \Gamma^\sharp \circ (A(\chi(p) v)) - (\rho \tau(\chi(p) v)) + \Psi^{-1}(\chi(p) v).
\]

According to Lemma 4.1 we can now introduce the cutoff \( \chi'(\xi) \) into the symbols:
\[
\chi(p) v_1 = \frac{1}{(2\pi)^d} \int \frac{\chi'(\xi)}{1 + \eta^2 + \Xi^2(x, \xi)} \left[ \frac{\partial (\chi(p) v_1)}{\partial \rho} \right]_{\rho = 0} \times (\rho)
\]
\[
+ \frac{1}{(2\pi)^d} \int \frac{\chi'(\xi)}{1 + \eta^2 + \Xi^2(x, \xi)} \left[ \frac{\partial (\chi(p) v_1)}{\partial \rho} \right]_{\rho = 0} \times (\rho)
\]
\[
- \sqrt{2\Gamma^\sharp} \circ S \left( \frac{\partial}{\partial \rho} (\chi(p) v_1) \right) - \Gamma^\sharp \circ (A(\chi(p) v)) - (\rho \tau(\chi(p) v)) + \Psi^{-1}(\chi(p) v),
\]
and then by expanding the symbol
\[
\frac{\chi'(\xi)}{1 + \eta^2 + \Xi^2(x, \xi)}
\]
we see that it has as principal symbol
\[
\frac{\chi'(\xi)}{\eta^2 + \Xi^2(x, \xi)}
\]
so that we may write
\[
\chi(p) v_1 = \frac{1}{(2\pi)^d} \int \frac{\chi'(\xi)}{\eta^2 + \Xi^2(x, \xi)} \left[ \frac{\partial (\chi(p) v_1)}{\partial \rho} \right]_{\rho = 0} \times (\rho)
\]
\[
+ \frac{1}{(2\pi)^d} \int \frac{\chi'(\xi)}{\eta^2 + \Xi^2(x, \xi)} \left[ \frac{\partial (\chi(p) v_1)}{\partial \rho} \right]_{\rho = 0} \times (\rho)
\]
\[
- \sqrt{2\Gamma^\sharp} \circ S \left( \frac{\partial}{\partial \rho} (\chi(p) v_1) \right) - \Gamma^\sharp \circ (A(\chi(p) v)) - (\rho \tau(\chi(p) v)) + \Psi^{-1}(\chi(p) v).
\]
We are now in position to prove a theorem about the Poisson operator. In order to consolidate the various smoothing terms which arise, we write $R_b^{-\infty}$ to include the restriction to $\rho = 0$ of any sum of smoothing operators in $\Psi^{-\infty}(\Omega)$ applied to $v$, or smoothing operators in $\Psi_b^{-\infty}(\partial \Omega)$ applied to $g_b$ or $\frac{\partial g_b}{\partial \rho} |_{\rho=0}$. We also write $R^{-\infty}$ to include any sum of smoothing operators in $\Psi^{-\infty}(\Omega)$ applied to $v$, or smoothing operators in $\Psi^{-\infty}(\Omega)$ applied to $g_b \times \delta(\rho)$ or $\frac{\partial g_b}{\partial \rho} |_{\rho=0} \times \delta(\rho)$. From the definitions we have $R(\Omega) = R_b^{-\infty}$.

We define the Poisson operator corresponding to $\square_e$ as a solution operator, $P_e$ mapping $(0,1)$-forms on $\partial \Omega$ to $(0,1)$-forms on $\Omega$, to

$$\square_e \circ P_e = R^{-\infty},$$

$$R \circ P_e = R_b^{-\infty}.$$

**Theorem 6.1.** Let $g_b$ be a $(0,1)$-form on $\partial \Omega$; each component, $g_b^j$, for $j = 1,2$, is a distribution supported on $\partial \Omega$. Then

$$P_e g = \Psi^{-1} g + R^{-\infty}.$$  

**Proof.** We first derive the theorem which applies to the first coordinate, $v_1$ of the Poisson solution. From (6.9), we have

$$\chi_p v_1 = \frac{1}{(2\pi)^4} \int \chi'(\xi) \frac{\partial (\chi p v_1)}{\partial \rho}(\xi,0) + i \eta \chi_p g_b^1(\xi) \eta^2 + \frac{\xi^2}{\xi} e^{i \xi \eta} d\xi d\eta$$

$$- \Gamma^2 \circ \left( \sqrt{2} (\frac{\partial}{\partial \rho}(\chi_p v_1)) + (A(\chi_p v))_1 + (\rho \tau(\chi_p v_1)) \right)$$

$$+ \Psi^{-1} \left( \frac{\partial (\chi_p v_1)}{\partial \rho} \bigg|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-1} \left( \chi_p (g_b^1 \times \delta(\rho)) \right) + \Psi^{-1}(\chi_p v).$$

The terms $\Gamma^2 \circ S \left( \frac{\partial}{\partial \rho}(\chi_p v_1) \right)$ and $\Gamma^2 \circ (A(\chi_p v))_1$ obviously contribute a term $\Psi^{-1}(\chi_p v)$.

For the term

$$\Gamma^2 \circ (\rho \tau(\chi_p v_1))$$

we rearrange (6.9) as

$$\chi_p v_1 = \frac{1}{(2\pi)^4} \int \chi'(\xi) \frac{\partial (\chi p v_1)}{\partial \rho}(\xi,0) + i \eta \chi_p g_b^1(\xi) \eta^2 + \frac{\xi^2}{\xi} e^{i \xi \eta} d\xi d\eta$$

$$+ \frac{1}{(2\pi)^4} \int \mathcal{L}_{k=1}^{\sum j=1} \frac{\tau^{jk}(\xi,\eta)}{1+\eta^2+\xi^2} \chi_p v_1(\xi,\eta) e^{i \xi \eta} d\xi d\eta$$

$$+ \Psi^{-1} \left( \frac{\partial (\chi_p v_1)}{\partial \rho} \bigg|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-1} \left( \chi_p (g_b^1 \times \delta(\rho)) \right) + \Psi^{-1}(\chi_p v).$$
as the terms involving the operators $S$ and $A$ are included in the $\Psi^{-1}$ operator. We then bring the second term on the right to the left-hand side:

\[
\frac{1}{(2\pi)^4} \int \left( 1 - \rho \sum_{j,k=1}^{3} \tau_{jk}(x,\rho) \xi_j \xi_k \right) \chi_{\rho \psi_1}(\xi, \eta) e^{ix\xi} e^{ip\eta} d\xi d\eta =
\]
\[
\frac{1}{(2\pi)^4} \int \chi'(\xi) \frac{\partial(\chi_{\rho \psi_1})}{\partial \rho}(\xi, 0) + i\eta \chi_{\rho \psi_1}(\xi) \eta^2 + e^{ix\xi} e^{ip\eta} d\xi d\eta
\]
\[
+ \Psi^{-3} \left( \left. \frac{\partial(\chi_{\rho \psi_1})}{\partial \rho} \right|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2} \left( \chi_{\rho} \left( g_{\rho}^1 \times \delta(\rho) \right) \right) + \Psi^{-1}(\chi'_{\rho} \psi).
\]

For small enough $\rho$ the symbol

\[
1 - \rho \sum_{j,k=1}^{3} \tau_{jk}(x,\rho) \xi_j \xi_k \frac{1}{1 + \eta^2 + \mathcal{E}_x^2(x, \xi)}
\]

is non-zero, and so we can apply a parametrix of the operator with symbol \((6.12)\) to both sides of \((6.11)\). We note the symbol of such an operator is of the form \(1 + \mathcal{O}(\rho)\), where \(\mathcal{O}(\rho)\) refers to a symbol of class \(S^0(\Omega)\), whose symbol is \(\mathcal{O}(\rho)\). We obtain

\[
\chi_{\rho \psi_1} = \frac{1}{(2\pi)^4} \int (1 + \mathcal{O}(\rho)) \chi'(\xi) \frac{\partial(\chi_{\rho \psi_1})}{\partial \rho}(\xi, 0) + i\eta \chi_{\rho \psi_1}(\xi) \eta^2 + e^{ix\xi} e^{ip\eta} d\xi d\eta
\]
\[
+ \Psi^{-3} \left( \left. \frac{\partial(\chi_{\rho \psi_1})}{\partial \rho} \right|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2} \left( \chi_{\rho} \left( g_{\rho}^1 \times \delta(\rho) \right) \right) + \Psi^{-1}(\chi'_{\rho} \psi).
\]

From Lemma 4.6, we have that

\[
\frac{1}{(2\pi)^4} \int \mathcal{O}(\rho) \chi'(\xi) \frac{\partial(\chi_{\rho \psi_1})}{\partial \rho}(\xi, 0) + i\eta \chi_{\rho \psi_1}(\xi) \eta^2 + e^{ix\xi} e^{ip\eta} d\xi d\eta
\]
\[
= \Psi^{-3} \left( \left. \frac{\partial(\chi_{\rho \psi_1})}{\partial \rho} \right|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2} \left( \chi_{\rho} \left( g_{\rho}^1 \times \delta(\rho) \right) \right),
\]

Returning to \((6.13)\) we write

\[
\chi_{\rho \psi_1} = \frac{1}{(2\pi)^4} \int \chi'(\xi) \frac{\partial(\chi_{\rho \psi_1})}{\partial \rho}(\xi, 0) + i\eta \chi_{\rho \psi_1}(\xi) \eta^2 + e^{ix\xi} e^{ip\eta} d\xi d\eta
\]
\[
+ \Psi^{-3} \left( \left. \frac{\partial(\chi_{\rho \psi_1})}{\partial \rho} \right|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2} \left( \chi_{\rho} \left( g_{\rho}^1 \times \delta(\rho) \right) \right) + \Psi^{-1}(\chi'_{\rho} \psi).
\]

An immediate consequence of \((6.14)\) is obtained by summing over a partition of unity; for a partition of unity, \(\{\chi_j\}_{j}\), applied to the domain \(\Omega\), a relation as in \((6.14)\) is obtained for each \(\chi_j\) in place
of $\chi_p$. Summing over these relations gives an expression for $v$ on $\Omega$: from (6.14) we write

$$\chi_j v_1 = \Psi^{-2} \left( \left. \frac{\partial v_1}{\partial \rho} \right|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-1}(g^1_p \times \delta(\rho)) + \Psi^{-1}(\chi'_p v),$$

whose sum gives

$$v_1 = \Psi^{-2} \left( \left. \frac{\partial v_1}{\partial \rho} \right|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-1}(g^1_p \times \delta(\rho)) + \Psi^{-1}v.$$

The same arguments can be applied to the second component, $v_2$, (by setting $\varepsilon = 0$, and changing the appropriate indices) to obtain as well

$$v_2 = \Psi^{-2} \left( \left. \frac{\partial v_2}{\partial \rho} \right|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-1}(g^2_p \times \delta(\rho)) + \Psi^{-1}v.$$

Hence,

$$(6.15) \quad v = \Psi^{-2} \left( \left. \frac{\partial v}{\partial \rho} \right|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-1}g + \Psi^{-\infty}v,$$

as a vector-valued relation, with matrix-valued pseudodifferential operators.

Using the residue calculus, we can take the inverse transform in (6.14) with respect to $\eta$. For $\rho \to 0^+$, we have

$$0 = \frac{1}{(2\pi)^2} 2\pi i \int \frac{\chi'(\xi)}{2|\Xi_e(x, \xi)|} \frac{\partial \langle \chi_p v \rangle_1}{\partial \rho} (\xi, 0) e^{i \xi \cdot \eta} d\xi - \int \chi'(\xi) \left[ \langle \Xi_e(x, \xi) \rangle \langle \chi_p v \rangle_1 (\xi, 0) \right] e^{i \xi \cdot \eta} d\xi$$

$$+ R\Psi^{-3} \left( \left. \frac{\partial \langle \chi_p v \rangle_1}{\partial \rho} \right|_{\rho=0} \times \delta(\rho) \right) + R\Psi^{-2}(g^1_p \times \delta(\rho)) + R\Psi^{-1}(\chi'_p v)$$

$$= \frac{1}{(2\pi)^3} \int \left( \frac{\chi'(\xi)}{2|\Xi_e(x, \xi)|} \left. \frac{\partial \langle \chi_p v \rangle_1}{\partial \rho} \right|_{\rho=0} (\xi, 0) - \chi'(\xi) \frac{1}{2} \chi_p g^1_b (\xi) \right) e^{i \xi \cdot \eta} d\xi$$

$$+ \Psi^{-2} \left( \left. \frac{\partial \langle \chi_p v \rangle_1}{\partial \rho} \right|_{\rho=0} \right) + \Psi^{-1}g^1_b + R\Psi^{-1}(\chi'_p v),$$

where we apply Lemma 4.3 to the terms with operators $R\Psi^{-3}$ and $R\Psi^{-2}$ in the second step. This gives (after multiplying by an inverse to the boundary operator with symbol $\chi'(\xi)/|\Xi_e(x, \xi)|$)

$$(6.16) \quad \frac{\partial \langle \chi_p v \rangle_1}{\partial \rho} (x, 0) = \int \chi'(\xi) \Xi_e(x, \xi) \langle \chi_p g^1_b (\xi) \rangle_1 e^{i \xi \cdot \eta} d\xi + \Psi^{-1}g^1_b + \Psi^{-1}(\chi'_p v).$$

Alternatively, we could in a similar manner use the residue calculus to take an inverse transform with respect to $\eta$ in (6.14) and calculate for $\rho \to 0^-$ with the same result.
For the term $\Psi_b^1 R \Psi^{-1}(\chi^\prime_p,v)$ we insert (6.15) in the argument:

\[
\Psi^1_b R \Psi^{-1}(\chi^\prime_p,v) = \Psi^1_b R \Psi^{-1} \left( \Psi^{-2} \left( \frac{\partial v}{\partial \rho} \Big|_{\rho=0} \right) \times \delta(\rho) \right) + \Psi^{-1} g + \Psi^{-\infty} v
\]

\[
= \Psi^1_b R \Psi^{-3} \left( \frac{\partial v}{\partial \rho} \Big|_{\rho=0} \right) \times \delta(\rho) + \Psi^1_b R \Psi^{-2} g + \Psi^1_b R \Psi^{-\infty} v
\]

\[
= \Psi^1_b \Psi^{-2} \left( \frac{\partial v}{\partial \rho} \Big|_{\rho=0} \right) + \Psi^1_b \Psi^{-1} g + \Psi^1_b R \Psi^{-\infty}
\]

\[
= \Psi^{-1} \left( \frac{\partial v}{\partial \rho} \Big|_{\rho=0} \right) + \Psi^0_b g + R^{-\infty}_b.
\]

Thus (6.16) above leads to the well-known result that the DNO is a first order operator on the boundary data, with principal term $|\Xi_\varepsilon(x,\xi)|$:

\[
\chi_p \left. \frac{\partial v_1}{\partial \rho} \right|_{\rho=0} = \int \chi^\prime(\xi) |\Xi_\varepsilon(x,\xi)| \widehat{\chi_p g_b}(\xi) e^{i \xi \cdot \xi} d\xi + \Psi^0_b g + \Psi^{-1} \left( \frac{\partial v}{\partial \rho} \Big|_{\rho=0} \right) + R^{-\infty}_b.
\]

The analogous expression for the second component reads

\[
\chi_p \left. \frac{\partial v_2}{\partial \rho} \right|_{\rho=0} = \int \chi^\prime(\xi) |\Xi_0(x,\xi)| \widehat{\chi_p g_b}(\xi) e^{i \xi \cdot \xi} d\xi + \Psi^0_b g + \Psi^{-1} \left( \frac{\partial v}{\partial \rho} \Big|_{\rho=0} \right) + R^{-\infty}_b.
\]

Once again summing both expressions over the partition of unity and solving for (the vector) \( \left. \frac{\partial v}{\partial \rho} \right|_{\rho=0} \), absorbing extra terms into the remainder term $R^{-\infty}_b$ leads to the expression for the first component:

(6.17)

\[
\left. \frac{\partial v_1}{\partial \rho} \right|_{\partial \Omega} = |D_b| g_b + \Psi^0_b g + R^{-\infty}_b
\]

where $|D_b|$ is defined as the first order operator with symbol locally given by $\sigma(|D_b|) = |\Xi_\varepsilon(x,\xi)|$. 
We can now insert (6.17) in (6.14):

\[
\chi_p v_1 = \frac{1}{(2\pi)^4} \int \chi'(\xi) \left( \frac{|\Xi(x,\xi)|}{\eta^2 + |\Xi|^2(x,\xi)} \right) e^{ix\xi} e^{i\eta \eta} d\xi d\eta
\]

\[
+ \Psi^{-3} \left( \frac{\partial (\chi_p v_1)}{\partial \rho} \bigg|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2} \left( \chi_p (g_b \times \delta(\rho)) \right) + \Psi^{-1} (\chi_p' v) + \Psi^{-2} R_b^{-\infty}
\]

\[
= \frac{i}{(2\pi)^4} \int \chi'(\xi) \chi_p g_1^b(\xi) \left( \frac{1}{\eta + i|\Xi|^2(x,\xi)} \right) e^{ix\xi} e^{i\eta \eta} d\xi d\eta
\]

\[
+ \Psi^{-3} \left( (\Psi_1^1 g_b^1 + R_b^{-\infty}) \times \delta(\rho) \right) + \Psi^{-2} \left( \chi_p (g_b \times \delta(\rho)) \right) + \Psi^{-1} (\chi_p' v) + R^{-\infty},
\]

which, together with the analog relation for \( v_2 \), we write as

\[
(6.18)
\]

From the proof of the Theorem we also have the principal symbol of the operator \( \Psi^{-1} \) acting on \( g_b \times \delta(\rho) \); it is given by

\[
(6.19)
\]

\[
\begin{pmatrix}
 i \frac{\chi'(\xi)}{\eta + i|\Xi|^2(x,\xi)} & 0 \\
 0 & i \frac{\chi'(\xi)}{\eta + i|\Xi|^2(x,\xi)}
\end{pmatrix},
\]

which we note for future reference. Using the representation as in (6.17),

\[
(6.20)
\]

\[
\frac{\partial v}{\partial \rho} \bigg|_{a\Omega} = \Psi_1^1 g_b + R_b^{-\infty},
\]

we can obtain with Lemma 4.2 the following estimates for the Poisson operator:

**Theorem 6.2.**

\[
\|P_\epsilon(g)\|_{W^{s+1/2}(\Omega)} \lesssim \|g_b\|_{W^s(a\Omega)} + \|P_\epsilon(g)\|_{-\infty}.
\]

**Proof.** Let \( v = P_\epsilon(g) \), the solution to \( \Box_\epsilon v = 0 \), with the boundary condition \( v|_{a\Omega} = g_b \). As in Theorem 4.9, \( v \) can be written as

\[
(6.21)
\]

\[
 v = \Gamma^{-1} \left( \Psi_1^1 g_b + \Psi_0^0 a' \right),
\]

where \( \Gamma^{-1} \) represents a parametrix of the interior operator \( \Box_\epsilon \), where \( \Psi_j^l \) for \( j = 0, 1 \) are pseudodifferential operators of order 0 and 1, respectively, whose symbols are independent of \( \rho \), and
where
\[ v' = \frac{\partial v}{\partial \rho} \bigg|_{\partial \Omega} \times \delta(\rho). \]

Now using (6.20) in (6.21) we have
\[ v = \Gamma^{-1} \left( \Psi_0^1 g \right) + R^{-\infty}. \]

As \( \Gamma^{-1} \) is the inverse of an elliptic operator of order 2, and can be realized as the sum of properly supported operators as in the hypothesis of Lemma 4.2, the estimates of Lemma 4.2 apply:
\[
\|v\|_{W^{s+1/2}(\Omega)} \lesssim \left\| \Gamma^{-1} \left( \Psi_0^1 g \right) \right\|_{W^{s+1/2}(\Omega)} + \|v\|_{-\infty} \\
\lesssim \|g_b\|_{W^{s}(\partial\Omega)} + \|v\|_{-\infty}.
\]

\[ \square \]

Included in the proof of Theorem 6.1 is the calculation of the highest order term of the DNO; from (6.17) we have in particular the first component of the DNO, which we write as, \( N^-_\varepsilon \):

**Theorem 6.3.**

(6.22)
\[ N^-_\varepsilon g = |D_\varepsilon|g_1^1 + \Psi_0^0(g_b) + R^{-\infty}. \]

In the case \( \varepsilon = 0 \), our interior operator, \( \square_\varepsilon \) is the same as \( \square \) considered in [2], and the result in Theorem 6.3 reduces to the (first order term in the result of) Theorem 1.2 in [2] in this case.

7. **The Green’s operator**

In this section we obtain an expression for the normal derivative restricted to the boundary of the first component of the solution to the inhomogeneous Dirichlet problem, that is, we look at the normal derivative of the Green’s function restricted to the boundary. This expression is used when the normal derivative of the Green’s operator arises as a result of our boundary condition (3.2). We obtain an expression for \( v = v_1\tilde{\omega}_1 + v_2\tilde{\omega}_2 \) which solves
\[
2\square_\varepsilon v = g \quad \text{on } \Omega \\
v = 0 \quad \text{on } \partial\Omega,
\]
for a \((0,1)\)-form, \( g = g_1\tilde{\omega}_1 + g_2\tilde{\omega}_2 \), modulo smoothing terms. The smoothing terms which arise are somewhat different than those in the case of the Poisson operator, and we change the notation for these remainder terms as follows. We write \( I^{-\infty}_b \) to include the restriction to \( \rho = 0 \) of any sum of smoothing operators in \( \Psi^{-\infty}(\Omega) \) applied to \( v \) or \( g \), or smoothing operators in \( \Psi^{-\infty}_b(\partial\Omega) \) applied to \( \frac{\partial v}{\partial \rho} \bigg|_{\rho=0} \). We also write \( I^{-\infty} \) to include any sum of smoothing operators in \( \Psi^{-\infty}(\Omega) \) applied to \( v \) or \( g \), or smoothing operators in \( \Psi^{-\infty}(\Omega) \) applied to \( \frac{\partial v}{\partial \rho} \bigg|_{\rho=0} \times \delta(\rho) \). From the definitions we have \( R(I^{-\infty}) = I^{-\infty}_b \).
We define the Green’s operator corresponding to $\Box_\varepsilon$ as a solution operator, $G_\varepsilon$ mapping $(0,1)$-forms on $\Omega$ to $(0,1)$-forms on $\Omega$, to

$$\Box_\varepsilon \circ G_\varepsilon = I + I^{-\infty}$$

$$R \circ G_\varepsilon = I_b^{-\infty}.$$  

If $\chi_p$ is a smooth cutoff function with support in a small neighborhood of $p$ and $\chi'_p$ a smooth cutoff such that $\chi'_p \equiv 1$ on supp $\chi_p$, we have

\begin{equation}
2\Box_\varepsilon (\chi_p v) = \chi_p g + \Psi^1(\chi'_p v) \quad \text{on } \Omega \tag{7.1}
\end{equation}

\(\chi_p v = 0\) on $\partial \Omega$ (modulo smoothing terms) for $v = G_\varepsilon(g)$.

We use the same notation and representation for the operators contained in $\Box_\varepsilon$ as in Section 6. We thus rewrite the first component of (7.1) in the form:

\begin{equation}
\frac{1}{(2\pi)^4} \int \frac{1}{1 + \eta^2 + \Xi^2(\xi, \eta)} e^{ix\xi} e^{i\rho \eta} d\xi d\eta - \frac{1}{(2\pi)^4} \int \left[ \left( \frac{\partial (\chi_p v_1)}{\partial \rho} \right) \bigg|_{\rho=0} \times \delta(\rho) \right] e^{ix\xi} e^{i\rho \eta} d\xi d\eta 
\end{equation}

\begin{equation}
+ \sqrt{2}S \left( \frac{\partial}{\partial \rho} (\chi_p v) \right) + (A(\chi_p v))_1 + \rho \tau(\chi_p v) = \chi_p g_1 + \Psi^1(\chi'_p v). \tag{7.2}
\end{equation}

Again, we apply a parametrix for the operator, $\Gamma = Op(1 + \eta^2 + \Xi^2(\xi, \eta))$, whose principal term, $\Gamma^\sharp$, has as symbol

\begin{equation}
\sigma(\Gamma^\sharp) = \frac{1}{1 + \eta^2 + \Xi^2(\xi, \eta)} \tag{7.3}
\end{equation}

to both sides of (7.2). From (7.2) we then have

\begin{equation}
\chi_p v_1 = \frac{1}{(2\pi)^4} \int \frac{1}{1 + \eta^2 + \Xi^2(\xi, \eta)} (\chi_p g_1)(\xi, \eta) e^{ix\xi} e^{i\rho \eta} d\xi d\eta + \Psi^{-3}(\chi_p g_1)
\end{equation}

\begin{equation}
+ \frac{1}{(2\pi)^4} \int \frac{\chi'_p(\xi)}{\eta^2 + \Xi^2(\xi, \eta)} \frac{\partial (\chi_p v_1)}{\partial \rho}(\xi, 0) e^{ix\xi} e^{i\rho \eta} d\xi d\eta + \Psi^{-3} \left( \frac{\partial (\chi_p v_1)}{\partial \rho} \bigg|_{\rho=0} \times \delta(\rho) \right) \tag{7.4}
\end{equation}

\begin{equation}
- \sqrt{2} \Gamma^\sharp \circ S \left( \frac{\partial}{\partial \rho} (\chi_p v_1) \right) - \Gamma^\sharp \circ (A(\chi_p v))_1 - \Gamma^\sharp \circ (\rho \tau(\chi_p v))_1 + \Psi^{-1}(\chi'_p v),
\end{equation}

where we use Lemma 4.1 in introducing the cutoff $\chi'_p(\xi)$ into the third term on the right as we did in Section 6.
From (7.4), we have

\[ \chi_{p^1} = \frac{1}{(2\pi)^4} \int (1 + O(\rho)) \frac{\left(\chi_{p^1}\right)(\xi, \eta)}{1 + \eta^2 + \Sigma^2(x, \xi)} e^{ix\xi e^{ip\eta} d\xi} + \frac{1}{(2\pi)^4} \int \frac{\chi'_{p^1}(\xi)}{\eta^2 + \Sigma^2(x, \xi)} \frac{\partial (\chi_{p^1})}{\partial \rho} (\xi, 0) e^{ix\xi e^{ip\eta} d\xi} \]

\[ \Gamma^\circ S \left( \frac{\partial}{\partial \rho} (\chi_{p^1}) \right) \]

(7.5)

\[ \Psi^{-3} \left( \frac{\partial (\chi_{p^1})}{\partial \rho} \right) \bigg|_{\rho = 0} + \Psi^{-3} (\chi_{p^1}) + \Psi^{-1} (\chi_{p^1}). \]

As in Section 6, the terms \( \Gamma^\circ S \left( \frac{\partial}{\partial \rho} (\chi_{p^1}) \right) \) and \( \Gamma^\circ (A(\chi_{p^1})) \) are of the form \( \Psi^{-1} (\chi_{p^1}) \). We follow Section 6 in bringing the term

(7.6)

\[ \Gamma^\circ (\rho \chi_{p^1}) \]

to the left-hand side, and restrict ourselves to a neighborhood with \( \rho \) sufficiently small so that the symbol, given by (6.12), of the resulting operator on the left-hand side is non-zero, and so we can apply a parametrix (see (6.13) to obtain

\[ \chi_{p^1} = \frac{1}{(2\pi)^4} \int (1 + O(\rho)) \frac{\left(\chi_{p^1}\right)(\xi, \eta)}{1 + \eta^2 + \Sigma^2(x, \xi)} e^{ix\xi e^{ip\eta} d\xi} + \frac{1}{(2\pi)^4} \int (1 + O(\rho)) \chi'_{p^1}(\xi) \frac{\partial (\chi_{p^1})}{\partial \rho} (\xi, 0) e^{ix\xi e^{ip\eta} d\xi} \]

(7.7)

\[ \Psi^{-3} \left( \frac{\partial (\chi_{p^1})}{\partial \rho} \right) \bigg|_{\rho = 0} \times \delta (\rho) \bigg) + \Psi^{-3} (\chi_{p^1}) + \Psi^{-1} (\chi_{p^1}), \]

where \( O(\rho) \) refers to a symbol of class \( S^0(\Omega) \), whose symbol is \( O(\rho) \).

From Lemma 4.6 we have that

\[ \frac{1}{(2\pi)^4} \int O(\rho) \frac{\chi'_{p^1}(\xi)}{\eta^2 + \Sigma^2(x, \xi)} \frac{\partial (\chi_{p^1})}{\partial \rho} (\xi, 0) e^{ix\xi e^{ip\eta} d\xi} = \Psi^{-3} \left( \frac{\partial (\chi_{p^1})}{\partial \rho} \right) \bigg|_{\rho = 0} \times \delta (\rho), \]

and so we write (7.7) as

(7.8)

\[ \chi_{p^1} = \frac{1}{(2\pi)^4} \int (1 + O(\rho)) \frac{\left(\chi_{p^1}\right)(\xi, \eta)}{1 + \eta^2 + \Sigma^2(x, \xi)} e^{ix\xi e^{ip\eta} d\xi} + \frac{1}{(2\pi)^4} \int \chi'_{p^1}(\xi) \frac{\partial (\chi_{p^1})}{\partial \rho} (\xi, 0) e^{ix\xi e^{ip\eta} d\xi} \]

(7.8)

\[ \Psi^{-3} \left( \frac{\partial (\chi_{p^1})}{\partial \rho} \right) \bigg|_{\rho = 0} \times \delta (\rho) \bigg) + \Psi^{-3} (\chi_{p^1}) + \Psi^{-1} (\chi_{p^1}). \]
Finding the analogous expression for the second component, \( v_2 \), (by setting \( \varepsilon = 0 \)) and summing over equations analogous to (7.8) for cutoffs \( \chi_j \) forming a partition of unity leads to the expression

\[
v = \Psi^{-2} g + \Psi^{-2} \left( \frac{\partial v}{\partial \rho} \bigg|_{\rho=0} \times \delta(\rho) \right) + I^{-\infty},
\]

where the principal symbol of the \( \Psi^{-2} \) operators is given by

\[
(1 + O(\rho)) \begin{bmatrix} \frac{1}{1+\eta^2+\Xi^2(x,\xi)} & 0 \\ 0 & \frac{1}{1+\eta^2+\Xi^2(x,\xi)} \end{bmatrix}.
\]

This gives an expression, modulo smoothing terms, for the Green’s solution in terms of the data function as well as the restriction to the boundary of the normal derivative of the solution. We are interested in the normal derivative at the boundary of the Green’s solution in the form of the

**Theorem 7.1.** Let \( \Theta^-_{\varepsilon} \in \Psi^{-1}(\Omega) \) be the operator with symbol

\[
\sigma(\Theta^-_{\varepsilon}) = \frac{i}{\eta - i|\Xi(x,\xi)|}.
\]

Then

\[
R \frac{\partial}{\partial \rho} \circ G_1(g) = R \circ \Theta^-_{\varepsilon}(g_1) + R \circ \Psi^{-2} g + I^{-\infty}.
\]

**Proof.** We apply a normal derivative to (7.8):

\[
\frac{\partial (\chi RW_{\rho} v_1)}{\partial \rho} = \frac{1}{(2\pi)^4} \int (1 + O(\rho)) \frac{i\eta}{1 + \eta^2 + \Xi^2(x,\xi)} \chi_{\rho 1}(\xi,\eta) e^{ix\xi} e^{ip\eta} d\xi d\eta
\]

\[
+ \frac{1}{(2\pi)^4} \int \chi'_{\rho 1}(\xi) \frac{i\eta}{\eta^2 + \Xi^2(x,\xi)} \frac{\partial (\chi RW_{\rho} v_1)}{\partial \rho} \bigg|_{\rho=0} (\xi,0) e^{ix\xi} e^{ip\eta} d\xi d\eta
\]

\[
+ \Psi^{-2} \left( \frac{\partial (\chi RW_{\rho} v_1)}{\partial \rho} \bigg|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2} (\chi RW_{\rho} v_1) + \Psi^0 (\chi RW_{\rho} v_0).
\]

Using (7.9) in the second to last term in (7.11) gives

\[
\frac{\partial (\chi RW_{\rho} v_1)}{\partial \rho} = \frac{1}{(2\pi)^4} \int (1 + O(\rho)) \frac{i\eta}{1 + \eta^2 + \Xi^2(x,\xi)} \chi_{\rho 1}(\xi,\eta) e^{ix\xi} e^{ip\eta} d\xi d\eta
\]

\[
+ \frac{1}{(2\pi)^4} \int \chi'_{\rho 1}(\xi) \frac{i\eta}{\eta^2 + \Xi^2(x,\xi)} \frac{\partial (\chi RW_{\rho} v_1)}{\partial \rho} \bigg|_{\rho=0} (\xi,0) e^{ix\xi} e^{ip\eta} d\xi d\eta
\]

\[
+ \Psi^{-2} \left( \frac{\partial v}{\partial \rho} \bigg|_{\rho=0} \times \delta(\rho) \right) + \Psi^{-2} g + I^{-\infty}.
\]
We now calculate
\[
\frac{1}{(2\pi)^4} \int (1 + O(\rho)) \frac{in}{1 + \eta^2 + \Xi^2(x, \xi)} \left(\overline{\chi_p g_1}(\xi, \eta) e^{ix\xi} e^{i\rho \eta} d\xi d\eta \right)
\]
\[
= \frac{1}{(2\pi)^4} \int (1 + O(\rho)) \frac{in}{1 + \eta^2 + \Xi^2(x, \xi)} \int_{-\infty}^{0} \left(\overline{\chi_p g_1}(\xi, t) e^{-i\eta t} d\xi d\eta \right)
\]
\[
= \frac{1}{(2\pi)^4} \int (1 + O(\rho)) \int_{-\infty}^{0} \left(\overline{\chi_p g_1}(\xi, t) \frac{in\eta e^{i(\rho - t)} \eta}{1 + \eta^2 + \Xi^2(x, \xi)} d\eta d\xi \right)
\]
\[
= \frac{1}{(2\pi)^3} \frac{1}{2} \int (1 + O(\rho)) \chi'(\xi) \times
\]
\[
\int_{-\infty}^{0} (\text{sgn}(t - \rho)) \left(\overline{\chi_p g_1}(\xi, t) e^{-i\rho - t|1 + \Xi^2(x, \xi)} d\xi d\eta \right)
\]
which, in the limit \(\rho \to 0\), tends to
\[
- \frac{1}{(2\pi)^3} \frac{1}{2} \int \int_{-\infty}^{0} \left(\overline{\chi_p g_1}(\xi, t) e^{i(\rho - t)\sqrt{1 + \Xi^2(x, \xi)}} d\xi d\eta \right)
\]
Similarly, using the residue calculus, we have, for \(\rho < 0\)
\[
\frac{1}{(2\pi)^4} \int \chi'(\xi) \frac{in}{\eta^2 + \Xi^2(x, \xi)} \frac{\partial (\overline{\chi_p g_1})}{\partial \rho}(\xi, 0) e^{ix\xi} e^{i\rho \eta} d\xi d\eta
\]
\[
= \frac{1}{(2\pi)^3} \frac{1}{2} \int \chi'(\xi) \frac{\partial (\overline{\chi_p g_1})}{\partial \rho}(\xi, 0) e^{ix\xi} e^{i\rho |\Xi(x, \xi)|} d\xi
\]
\[
= \frac{1}{2} \frac{\partial (\overline{\chi_p g_1})}{\partial \rho}(x, 0) + l_b^{-\infty},
\]
and thus, letting \(\rho \to 0^-\) in (7.12), we have
\[
\frac{\partial (\overline{\chi_p g_1})}{\partial \rho}(x, 0) = - \frac{1}{(2\pi)^3} \int \left(\overline{\chi_p g_1}(\xi, i\sqrt{1 + \Xi^2(x, \xi)}) e^{ix\xi} d\xi \right)
\]
\[
+ R \circ \Psi^{-2} \frac{\partial v}{\partial \rho}_{\rho=0} \times \delta(\rho) + R \circ \Psi^{-2} g + l_b^{-\infty}.
\]
We note that
\[
- \frac{1}{(2\pi)^3} \int \left(\overline{\chi_p g_1}(\xi, i\sqrt{1 + \Xi^2(x, \xi)}) e^{ix\xi} d\xi \right) =
\]
\[
i \frac{1}{(2\pi)^4} \int \frac{1}{\eta - i\sqrt{1 + \Xi^2(x, \xi)}} \left(\overline{\chi_p g_1}(\xi, \eta) e^{ix\xi} d\xi d\eta \right).
\]
Introducing a function \(\chi'(\xi)\) can be shown to produce only smoothing error terms (this can be seen by differentiating the difference under the integral as in the proof of Lemma 4.1 and furthermore
an expansion of
\[ \frac{i \chi'(\xi)}{\eta - i \sqrt{1 + \alpha^2(x, \xi)}} \]
shows that its principal symbol is that of \( \sigma(\Theta^-) (x, \xi, \eta) \). Thus, the first term on the right-hand side of (7.14) can be written as
\[ R \circ \Theta^- (\chi_p g_1) + R \circ \Psi^{-2} g + I_b^{-\infty}. \]
Furthermore, using
\[ R \circ \Psi^{-2} \left( \left( \frac{\partial v}{\partial \rho} \bigg|_{\rho=0} \times \delta(\rho) \right) = \Psi_b^{-1} \left( \frac{\partial v}{\partial \rho} \bigg|_{\rho=0} \right) \]
from Lemma 4.3, we have
\[ \frac{\partial (\chi_p v_1)}{\partial \rho}(x, 0) = R \circ \Theta^- (\chi_p g_1) + \Psi_b^{-1} \left( \frac{\partial v}{\partial \rho} \bigg|_{\rho=0} \right) + R \circ \Psi^{-2} g + I_b^{-\infty} \]
\[ = R \circ \chi_p \circ \Theta^- (g_1) + \Psi_b^{-1} \left( \frac{\partial v}{\partial \rho} \bigg|_{\rho=0} \right) + R \circ \Psi^{-2} g + I_b^{-\infty}, \]
since \( \Theta^- \in \Psi^{-1} \). Going through the calculations for the second component, then summing over similar relations using a partition of unity \( \{ \chi_j \} \) and isolating \( \frac{\partial v_1}{\partial \rho} \) leads to
\[ \frac{\partial v_1}{\partial \rho}(x, 0) = R \circ \Theta^- (g_1) + R \circ \Psi^{-2} g + I_b^{-\infty}. \]

We note that by Lemma 4.2, we have the following estimates for the Green’s operator:

**Theorem 7.2.**
\[ \| G_\varepsilon (g) \|_{W^{s+2} (\Omega)} \lesssim \| g \|_{W^{s+2} (\Omega)} + \| G_\varepsilon (g) \|_{-\infty}. \]

**Proof.** Let \( v = G_\varepsilon (2g) \), the solution to \( \Box_\varepsilon v = g \), with Dirichlet boundary conditions. As in Theorem 4.9, \( v \) can be written as
\[ v = \Gamma^{-1} \left( 2g + \Psi_0^0 v' \right), \]
where \( \Gamma^{-1} \) represents a parametrix of the interior operator \( \Box_\varepsilon \), \( \Psi_0^0 \) is a pseudodifferential operator of order 0 with symbol independent of \( \rho \), and where
\[ v' = \frac{\partial v}{\partial \rho} \bigg|_{\partial \Omega} \times \delta(\rho). \]

With analogous results for \( v_2 \), we obtain from Theorem 7.1
\[ R \frac{\partial v}{\partial \rho} = R \circ \Psi^{-1} g + I_b^{-\infty}. \]
Inserting this expression into (7.15) we derive the following expression of the Green’s solution operator:

\[ v = \Psi^{-2}g + \Psi^{-2} \left( R \circ \Psi^{-1}g \right) + I^{-\infty}. \]

Both \( \Psi^{-2} \) operators are inverses to elliptic operators of order 2 composed with 0 order operators, for which the estimates of Lemma 4.2 apply. Taking Sobolev norms gives

\[
\|v\|_{W^{s,2}(\Omega)} \lesssim \|\Psi^{-2}g\|_{W^{s,2}(\Omega)} + \|\Psi^{-2} \left( R \circ \Psi^{-1}g \right)\|_{W^{s,2}(\Omega)} + \|v\|_{-\infty}
\]

\[ \lesssim\|g\|_{W^{s,2}(\Omega)} + \|R \circ \Psi^{-1}g\|_{W^{s+1/2}(\partial \Omega)} + \|v\|_{-\infty} \]

\[ \lesssim\|g\|_{W^s(\Omega)} + \|\Psi^{-1}g\|_{W^{s+1}(\Omega)} + \|v\|_{-\infty} \]

\[ \lesssim\|g\|_{W^s(\Omega)} + \|v\|_{-\infty}. \]

\[ \square \]

8. The Boundary Operators and Approximate Inverses

In this section we construct approximate inverses to operators which will appear in subsequent sections of the paper. In particular, we study the operators \( \frac{1}{\sqrt{2}}N_{\ell}^- + iT^0 \) and \( \frac{1}{2}(N_{\ell}^-)^2 + (T^0)^2 \), which belong to \( \Psi^1(\partial \Omega) \) and \( \Psi^2(\partial \Omega) \), respectively.

We start with \( \frac{1}{\sqrt{2}}N_{\ell}^- + iT^0 \) and consider the following boundary equation:

\[ (\frac{1}{\sqrt{2}}N_{\ell}^- + iT^0) v = g. \]

We apply the operator \( \frac{1}{\sqrt{2}}N_{\ell}^- - iT^0 \) to both sides of (8.1) and obtain

\[ \left( \frac{1}{2}(N_{\ell}^-)^2 + (T^0)^2 \right) v + i\frac{1}{\sqrt{2}} \left[ N_{\ell}^-, T^0 \right] v = \left( \frac{1}{\sqrt{2}}N_{\ell}^- - iT^0 \right) g. \]

From Theorem 6.22 we have

\[
\sigma \left( \frac{1}{2}(N_{\ell}^-)^2 + (T^0)^2 \right) = \frac{1}{2} \xi^2 + \xi_3^2 + \mathcal{S}^1(\partial \Omega)
\]

\[
= \frac{1}{4}(\xi_1^2 + \xi_2^2) + \frac{\xi_3}{2} |\xi|^3/2 + \xi_3 \sum_{j=1}^3 \Re \xi_j \xi_j - \xi_2 \sum_{j=1}^3 \Im \xi_j \xi_j + \sum_{j,k} \ell_j \ell_k \xi_j \xi_k + \mathcal{S}^1(\partial \Omega).
\]

Using the calculations for the symbol of \( \left( \frac{1}{2}(N_{\ell}^-)^2 + (T^0)^2 \right) \) above, we write (8.2) as

\[ Q_2 v + \Psi^1 v = \left( \frac{1}{\sqrt{2}}N_{\ell}^- - iT^0 \right) g,
\]

with

\[
\sigma(Q_2) = 1 + \sigma \left( \frac{1}{2}(N_{\ell}^-)^2 + (T^0)^2 \right).
\]
As in Lemma 4.1 up to smoothing operators we can multiply symbols composing $Q_2$ with a cutoff in the $\xi$ variable vanishing near 0, and so we can take $Q_2$ to be given by

$$\sigma(Q_2) = 1 + \chi'(\xi) \left( \frac{1}{4} (\xi_1^2 + \xi_2^2) + \frac{\epsilon}{2} |\xi|^{3/2} + \xi_1 \sum_{j=1}^{3} \text{Re} \ell_j \tilde{\xi}_j - \xi_2 \sum_{j=1}^{3} \text{Im} \ell_j \tilde{\xi}_j + \sum_{j,k} \ell_{jk} \tilde{\xi}_j \tilde{\xi}_k \right),$$

with $\chi'(\xi) \in C^\infty(\mathbb{R}^3)$ such that $\chi' \equiv 0$ in a neighborhood of the origin and $\chi' \equiv 1$ outside a compact set containing the origin. As localizations are used to express the operators in terms of coordinates $(x, \rho)$ as in Section 6, we work with functions $\ell_j$ which are smooth with compact support. We note that $Q_2$ and $\Psi_1$ in (8.3) are properly supported (as will all the operators we consider on $\partial \Omega$) and thus the composition theorems of Section 5 apply.

We denote the operator corresponding to the symbol in $1/\sigma(Q_2)$ by $Q_2^{-1}$. We seek an identity of the type

$$Q_2^{-1} \circ Q_2 = 1 + \hat{R}$$

where the operator $\hat{R}$ is more smoothing (as a $\hat{\Psi}$ operator of negative order) than an order 0 pseudodifferential operator. That $Q_2^{-1}$ is such an approximate inverse can be shown by an application of Theorem 5.5 to the composition $Q_2^{-1} \circ Q_2$.

The first step is to show estimates for $\sigma(Q_2^{-1})$ which, as

$$\sigma(Q_2^{-1}) = \frac{1}{\sigma(Q_2)},$$

follow from estimates on the symbol, $\sigma(Q_2)$.

**Proposition 8.1.** There exist constants $c_1, c_2 > 0$ such that

$$c_1 \left( 1 + \xi_1^2 + \xi_2^2 + \xi_3^2 \right)^{3/4} \leq \sigma(Q_2)(x, \xi) \leq c_2 \left( 1 + \xi_1^2 + \xi_2^2 + \xi_3^2 \right).$$

**Proof.** The estimates follow from the definition of $Q_2$. From the discussion preceding the Proposition above, we have

$$\sigma(Q_2) = 1 + \chi'(\xi) \left( \frac{1}{2} (\xi_1 - i \xi_2) + \sum_{j=1}^{3} \ell_j \tilde{\xi}_j \right)^2 + \chi'(\xi) \frac{\epsilon}{2} |\xi|^{3/2}. $$

The estimates then follow from this expression. \qed

As a result of Proposition 8.1 we see that the symbol $\sigma(Q_2)(x, \xi)$ satisfies (5.1) and (5.2) above with $\gamma = 3/2, \delta = 4/3$, and we set

$$q_{3/2,4/3}(x, \xi) = \sigma(Q_2)(x, \xi) = 1 + \chi'(\xi) \left( \frac{1}{2} (\xi_1 - i \xi_2) + \sum_{j=1}^{3} \ell_j \tilde{\xi}_j \right)^2 + \chi'(\xi) \frac{\epsilon}{2} |\xi|^{3/2}. $$
In this section as shorthand for \( q_{3/2,4/3}(x, \xi) \) we define \( q_2(x, \xi) = \sigma(Q_2)(x, \xi) \). Thus \( q_2(x, \xi) \) is of the type \( q_{3/2,4/3} \) for \( \gamma = 3/2 \) and \( \delta = 4/3 \). We reserve the use of \( q_2(x, \xi) \) to denote the symbol of \( Q_2 \) and \( q_{3/2,4/3} \) in the use of the description of the \( \Psi \) operators. As a corollary of Proposition 8.1, the symbol

\[
q_2^{-1}(x, \xi) := \frac{1}{\sigma(Q_2)(x, \xi)}
\]
satisfies the estimate

\[
\left| q_2^{-1}(x, \xi) \right| = \frac{1}{\sigma(Q_2)(x, \xi)} \lesssim \frac{1}{(1 + |\xi|)^{3/2}}.
\]

We now want to show the operators with symbols \( q_2(x, \xi) \) and \( q_2^{-1}(x, \xi) \) belong to the class of \( \Psi \) operators. As a corollary of Proposition 8.1, the symbol

\[
q_{1/2,1}(x, \xi) := \frac{1}{\sigma(Q_2)(x, \xi)}
\]
satisfies the estimate

\[
\left| q_{1/2,1}(x, \xi) \right| = \frac{1}{\sigma(Q_2)(x, \xi)} \lesssim \frac{1}{(1 + |\xi|)^{3/2}}.
\]

We now want to show the operators with symbols \( q_2(x, \xi) \) and \( q_2^{-1}(x, \xi) \) belong to the class of \( \Psi \) operators, respectively, defined in Section 5.

**Theorem 8.2.** The operators \( Q_2 \) and \( Q_2^{-1} \) as defined above have the following properties:

\[
Q_2^{-1} \in \Psi_{-1/2,1}^{1/2}(\mathbb{R}^3), \quad Q_2 \in \Psi_{1/2,1}^{1/2}(\mathbb{R}^3).
\]

For the proof of Theorem 8.2 we need to consider and keep track of the occurrence of \( x \) derivatives of \( \ell_3 \). With this in mind, we define a class of polynomials in the transform variables; we let \( p_{n, m}(y_1, \ldots, y_5) \) be a homogeneous polynomial with coefficients functions of \( x \), of order \( n \) in the \( y \) variables (which we will take to be composed of the \( \xi \) variables), and with \( m_3, m_4, \) and \( m_5 \) the respective orders of \( y_3, y_4, \) and \( y_5, \) of order \( m = m_3 + m_4 + m_5 \) in the last three variables.

With \( \ell := (\ell_1, \ell_2, \ell_3) \), we use the notation

\[
\ell_\ell := \frac{1}{2} (\xi_1 - i \xi_2) + \sum_{j=1}^{3} \ell_j \xi_j
\]

and define \( P(n, m) \), \( n \geq m \) to be the collection of polynomials which are sums of polynomials of the form

\[
\sum_{k \leq n, \ell \leq \min(m,k)} p_{k, \ell}(\xi_\ell, \xi_\ell, \xi_\ell, \xi_\ell, \xi_\ell).
\]

We write \( p(n, m) \) for any polynomial belonging to class \( P(n, m) \). Thus, for instance,

\[
\sigma(Q_2)(x, \xi) = 1 + \chi'(\xi) \left( p(2, 0) + \frac{\xi}{2} |\xi|^{3/2} \right)
\]

\[
\xi_3 = p(1, 1).
\]

Mainly to ease notation, we will extend the definition to let \( p(n, m) \) for \( m > n \) denote \( p(n, n) \).

We collect here some properties of our \( p(n, m) \) polynomials which follow directly from the definition:
Lemma 8.3.

\begin{align*}
(8.5) & \quad p(n_1, m_1) \cdot p(n_2, m_2) \in P(n_1 + n_2, m_1 + m_2), \\
(8.6) & \quad \partial_{\xi_j} p(n, m) \in P(n - 1, m), \quad \text{for } j = 1, 2, 3 \\
(8.7) & \quad \partial_{x_j} p(n, m) \in P(n, m + 1), \quad \text{for } j = 1, 2, 3 \\
& \quad P(n_2, m_2) \subset P(n_1, m_1) \quad n_2 \leq n_1, m_2 \leq m_1.
\end{align*}

From the proof of Proposition 8.1 we note that
\begin{align*}
|\xi_\ell| & \lesssim |\sigma(Q_2)|^{1/2}, \\
|\xi| & \lesssim |\sigma(Q_2)|^{2/3}.
\end{align*}

We will use repeatedly the estimates which follow from those above
\begin{align*}
|p(n, m)| & \lesssim |\xi_\ell|^{|n - m|} |\xi|^m \\
& \lesssim |\sigma(Q_2)|^{\frac{2}{3} + \frac{m}{6}}.
\end{align*}

We are now in a position to prove Theorem 8.2. Theorem 8.2 follows from the more general

Proposition 8.4. For
\[ \frac{n}{2} + \frac{m}{6} - l \leq -\alpha, \]

in which \( \alpha \geq 0 \), a symbol of the form
\[ \frac{p(n, m)}{\sigma^l(Q_2)} \]

belongs to \( S_{q/3, 2, 4/3}(\mathbb{R}^3) \).

Proof. We need to show estimates of the form (see Definition 5.1)
\[
\left| D_{\xi}^k x_{\ell}^{k_1} \left( \frac{p(n, m)}{\sigma^l(Q_2)} \right) \right| \lesssim \left( \frac{1}{1 + \|q_2(x, \xi)\|} \right)^{\alpha + \frac{1}{2\gamma} |k_1| - \frac{1}{2 \delta} |k_2|}
\]

hold with \( \gamma = 3/2 \) and \( \delta = 4/3 \) (we add the 1 and absolute values in the denominator only to make the estimates similar in appearance to those in Section 5).

We start with estimates on purely \( x \) derivatives, which we prove by induction. We assume
\[ D_x^k \left( \frac{p(n, m)}{\sigma^l(Q_2)} \right), \]

is a sum of terms of the form
\[ \frac{p(n', m')}{\sigma^l(Q_2)}, \]

(8.9)
for which
\[ \frac{n'}{2} + \frac{m'}{6} - l' \leq \frac{|k|}{6} - \alpha. \]

Each such term can be estimated using (8.8) by \((1 + |q_2|)^{-\alpha + |k|/6}\). Note that for \(\gamma = 3/2\) and \(\delta = 4/3\), we have
\[ \frac{1}{\gamma} \frac{\delta - 1}{\delta} = \frac{2}{3} \frac{1}{4/3} = \frac{1}{6}. \]

We apply an \(x\) derivative to a term of the form (8.9), using (8.5) and (8.7), and obtain that \(D_{k'}^x\left(\frac{p(n,m)}{\sigma^l(Q_2)}\right)\) for \(|k'| = |k| + 1\) is made up of terms of the form
\[ \frac{p(n',m' + 1)}{\sigma^{l+1}(Q_2)} + \frac{p(n',m')p(2,1)}{\sigma^{l+1}(Q_2)} = \frac{p(n',m' + 1)}{\sigma^l(Q_2)} + \frac{p(n' + 2,m' + 1)}{\sigma^{l+1}(Q_2)} \]
each term of which is of the form
\[ \frac{p(n'',m'')}{\sigma^{l''}(Q_2)} \]
for which
\[ \frac{n''}{2} + \frac{m''}{6} - l'' \leq \frac{|k| + 1}{6} - \alpha \]
holds, and thus, by (8.8), are estimated by
\[(1 + |q_2|)^{-\alpha + (|k| + 1)/6}.\]

Now we handle pure \(\xi\) derivatives, again proving the estimates by induction. The derivative
\[(8.10)\]
\[D_{k'}^\xi\left(\frac{p(n,m)}{\sigma^l(Q_2)}\right)\]
is made up of terms of the form
\[(8.11)\]
\[\left(D_{k}^\xi p(n,m)\right) \frac{1}{q_2^{l+p}} \left(D_{k_2}^{k_2+1} q_2\right) \cdots \left(D_{k_p}^{k_p} q_2\right) \sum |k_2| = |k_2|, \quad |k_1| + |k_2| = |k|.\]

We prove by induction that derivatives in (8.10) are of the form (8.11) in which the terms exhibit the estimates
\[(8.12)\]
\[\frac{1}{q_2^{l+p}} \left(D_{k}^{k_2+1} q_2\right) \cdots \left(D_{k_p}^{k_p} q_2\right) \lesssim (1 + |q_2|)^{-\beta} \]
such that
\[ -\beta + \frac{n}{2} + \frac{m}{6} - \frac{|k_1|}{2} + \frac{|k|}{2} \leq -\alpha - \frac{|k|}{2} \]
can be estimated by $\gamma \delta = 1/2$. Applying a derivative to (8.11) yields
\[
\left( D_{\xi}^{k_1}p(n, m) \right) \frac{1}{q_2^{l+p}} \left( D_{\xi}^{k_21}q_2 \right) \cdots \left( D_{\xi}^{k_{2p}}q_2 \right)
\]
\[
- \left( D_{\xi}^{k_1}p(n, m) \right) \frac{1}{q_2^{l+p+1}} \left( D_{\xi}^{k_20}q_2 \right) \left( D_{\xi}^{k_21}q_2 \right) \cdots \left( D_{\xi}^{k_{2p}}q_2 \right)
\]
\[
+ \left( D_{\xi}^{k_1}p(n, m) \right) \frac{1}{q_2^{l+p}} \left( D_{\xi}^{k_21}q_2 \right) \cdots \left( D_{\xi}^{k_{2p}}q_2 \right),
\]
where $|k_1| = |k_1| + 1$, $|k_{20}| = 1$, and $\sum |k_{2j}'| = |k_2| + 1$. The claim that estimates of the form (8.12) hold now follows from the induction hypothesis:
\[
\frac{1}{q_2^{l+p+1}} \left( D_{\xi}^{k_20}q_2 \right) \left( D_{\xi}^{k_21}q_2 \right) \cdots \left( D_{\xi}^{k_{2p}}q_2 \right) \lesssim (1 + |q_2|)^{-\beta} \frac{D_{\xi}^{k_20}q_2}{q_2}
\]
\[
\lesssim (1 + |q_2|)^{-\beta-1} \left( |p(1, 0)| + |\xi|^{1/2} \right)
\]
\[
\lesssim (1 + |q_2|)^{-\beta-1}(1 + |q_2|)^{1/2}
\]
\[
\lesssim (1 + |q_2|)^{-\beta-1/2}
\]
and
\[
\frac{1}{q_2^{l+p}} \left( D_{\xi}^{k_21}q_2 \right) \cdots \left( D_{\xi}^{k_{2p}}q_2 \right) \lesssim \frac{(1 + |q_2|)^{-\beta}}{1 + |\xi|}
\]
\[
\lesssim (1 + |q_2|)^{-\beta-1/2}.
\]
The first inequality immediately above can be shown by the inequality
\[
\frac{D_{\xi}^{k_2}q_2}{D_{\xi}^{k_{2j}}q_2} \lesssim \frac{1}{1 + |\xi|'}
\]
for $k_{2j}' = k_{2j} + s$ where $|s| = 1$.

Thus a derivative of (8.11) is a sum of terms of the form
\[
\left( D_{\xi}^{k_1}p(n, m) \right) \frac{1}{q_2^{l+p'p}} \left( D_{\xi}^{k_21}q_2 \right) \cdots \left( D_{\xi}^{k_{2p}}q_2 \right)
\]
\[
\sum |k_{2j}'| = |k_2'|, \quad |k_1'| + |k_{2j}'| = |k| + 1
\]
in which the terms
\[
\frac{1}{q_2^{l+p'p}} \left( D_{\xi}^{k_21}q_2 \right) \cdots \left( D_{\xi}^{k_{2p}}q_2 \right)
\]
can be estimated by
\[
\frac{1}{q_2^{l+p'p}} \left( D_{\xi}^{k_21}q_2 \right) \cdots \left( D_{\xi}^{k_{2p}}q_2 \right) \lesssim (1 + |q_2|)^{-\beta'}
\]
for
\[ -\beta' + \frac{n}{2} + \frac{m}{6} - \frac{|k_1'|}{2} \leq -\alpha - \frac{|k| + 1}{2} \]
completing the proof of (8.12) by induction.

The estimates on (8.10) now follow easily from (8.11) with the estimates (8.12):
\[
\left| D_{\xi}^k \left( \frac{p(n,m)}{\sigma (Q_2)} \right) \right| \lesssim \sum_{k_1 + k_2 = k} \left| \left( D_{\xi}^{k_1} \frac{p(n,m)}{q_2} \right) \frac{1}{q_2^{1+\beta}} \left( D_{\xi}^{k_2} q_2 \right) \right| \\
\lesssim (1 + |q_2|)^{\frac{n'}{2} + \frac{m'}{6} - l'} + \beta (1 + |q_2|)^{-\beta} \\
\lesssim (1 + |q_2|)^{-\alpha - \frac{|k|}{2}}.
\]
\[ (8.13) \]
Estimates on both types of derivatives (with respect to \(x\) and \(\xi\)) simultaneously,
\[
D_{\xi}^{k_1} D_{\xi}^{k_2} \left( \frac{p(n,m)}{q_2} \right),
\]
follow by applying the \(x\) derivatives first and using (8.9) to reduce the estimates to
\[
\left| D_{\xi}^{k_1} \left( \frac{p(n',m')}{q_2} \right) \right|,
\]
where
\[
\frac{n'}{2} + \frac{m'}{6} - l' \leq \frac{|k_2|}{6} - \alpha := -\alpha',
\]
and then applying the estimates (8.13) which show
\[
\left| D_{\xi}^{k_1} \left( \frac{p(n',m')}{q_2} \right) \right| \lesssim (1 + |q_2|)^{-\alpha' - \frac{|k_1|}{2}} \\
= (1 + |q_2|)^{-\alpha - \frac{|k_1|}{2} + \frac{|k_1|}{2}}.
\]
\[ \square \]

**Proof of Theorem 8.2** We consider \(x\) and \(\xi\) derivatives of the symbol, \(\sigma (Q_{2,-1})\); we want to show
\[
\left| D_{\xi}^{k_1} D_{\xi}^{k_2} \sigma (Q_{2,-1}) \right| \lesssim \left( \frac{1}{1 + |q_2(x,\xi)|} \right)^{1 + \frac{1}{2} |k_1| - \frac{1}{2} |k_2|}.
\]
The case \(k_2 = 0\) is proved in Proposition 8.4 (see (8.13) with \(n = m = 0\) and \(l = 1\)).

In the case \(k_2 \neq 0\), we first apply an \(x\) derivative first to obtain
\[
D_{\xi}^{k_1} D_{\xi}^{k_2} \sigma (Q_{2,-1}) = D_{\xi}^{k_1} D_{x}^{k_2} \chi'(\xi) \frac{p(2,1)}{q_2} \\
= \chi'(\xi) \left( D_{\xi}^{k_1} D_{x}^{k_2} \frac{p(2,1)}{q_2} \right) + S^{-\alpha} (\partial \Omega),
\]
where $|k_2'| = |k_2| - 1$. From Proposition 8.4 it follows that

$$p(2, 1) \in \mathcal{S}^{-1+1/6}(\mathbb{R}^3),$$

and thus,

$$\left| D_{\xi}^{k_1} D_x^{k_2} \sigma(Q_{2,-1}) \right| \lesssim \left| D_{\xi}^{k_1} D_x^{k_2} \frac{p(2, 1)}{q_2^2} \right| \lesssim \left( \frac{1}{1 + |q_2(x, \xi)|} \right)^{1 - 1/6 + \frac{1}{2} |k_1| - \frac{1}{6} (|k_2| - 1)} \lesssim \left( \frac{1}{1 + |q_2(x, \xi)|} \right)^{1 + \frac{1}{2} |k_1| - \frac{1}{6} |k_2|},$$

as desired.

If we use the convention (without loss of any generality) that $p(n, m) = 0$ for $n < 0$, that $Q_2 \in \Psi^{-1,3/2,4/3}(\mathbb{R}^3)$ can be seen from by writing $q_2 = \chi'(\xi) p(2, 0) + b_{3/2}$, where we write $b_k$ for a term which can be estimated by $|b_k| \lesssim (1 + |\xi|)^k$, and calculating

$$D_{\xi}^{k_1} D_x^{k_2} q_2 = \chi'(\xi) p(2 - \min(2, |k_1|), \min(k_2, 2 - \min(2, |k_1|))) + b_{3/2 - |k_1|} + \mathcal{S}^{-\infty}(\partial \Omega) \lesssim |\xi|^{2 - |k_1| - \min(k_2, 2 - \min(2, |k_1|))} + (1 + |\xi|)^{3/2 - |k_1|} \lesssim (1 + |q_2|) \min(k_2, 2 - \min(2, |k_1|)) + (1 + |q_2|)^{-1/2} |k_1| \lesssim (1 + |q_2|)^{1 - \frac{1}{2} |k_1|} + \frac{1}{6} \min(k_2, 2).$$

Theorem 8.5.

$$Q_{2,-1} \circ Q_2 = 1 + \Psi^{-1,3/2,4/3}_{\Psi^{-1,3/2,4/3}}(\mathbb{R}^3) + \Psi^{-1,3/2,4/3}_{\Psi^{-1,3/2,4/3}}(\mathbb{R}^3)$$

Proof. We apply Theorem 5.9 with $A = Q_{2,-1}$, $B = Q_2$. By Theorem 8.2 we have $Q_{2,-1} \in \Psi^{-1}_{\Psi^{-1,3/2,4/3}}(\mathbb{R}^3)$ and $Q_2$ is properly supported with the property $Q_2 \in \Psi^{1}_{\Psi^{1,3/2,4/3}}(\mathbb{R}^3)$, and thus from Theorem 8.5 we have the expansion

$$\sigma(Q_{2,-1} \circ Q_2) = \sum_{|k_1| \leq 2} \frac{1}{|k_1|!} D_{\xi}^{k_1} \sigma(Q_{2,-1}) \partial_x^{k_2} \sigma(Q_2^2) + \mathcal{S}_{\Psi^{-1,3/2,4/3}}^{-1/3}(\mathbb{R}^3) + \mathcal{S}_{\Psi^{-1,3/2,4/3}}^{-1,3/4,1}(\mathbb{R}^3) = 1 + \mathcal{S}_{\Psi^{-1,3/2,4/3}}^{-1/3}(\mathbb{R}^3) + \mathcal{S}_{\Psi^{-1,3/2,4/3}}^{-1,3/4,1}(\mathbb{R}^3).$$
9. Applications of $\Psi$ Operators

Given the equation

\[ Q_2 v + A_b v = h, \]

for $h \in W^s(\partial \Omega)$ where $A_b \in \Psi^1_b(\partial \Omega)$, for any given $k$, we give a method to find a specific $\phi_k$ which satisfies

\[ Q_2 \phi_k + A_b \phi_k = h + E_k(h), \]

where we use the notation $E_k$ to denote any linear operator mapping $W^s(\partial \Omega)$ to $W^{s+k}(\partial \Omega)$, i.e. $E_k(h)$ is a term depending linearly on $h$ with the estimates

\[ \| E_k(h) \|_{W^{s+k}(\partial \Omega)} \lesssim \| h \|_{W^s(\partial \Omega)}. \]

In this way $\phi_k$ may be considered to be an approximate solution to (9.1).

It holds that $E_{k_1} \circ E_{k_2} = E_{k_1+k_2}$ and from Theorem 5.3 we have that $\Psi^{-a_0}_{\delta_{\alpha,\beta}} = E_{a_0}$ for $\alpha \geq 0$.

**Proposition 9.1.** Let $h \in W^s(\partial \Omega)$. For all $k \geq 0$ there exist approximate linear solution operators, $F^k_2$ to (9.1) such that

\[ \left\| (Q_2 + A_b) F^k_2 h - h \right\|_{W^{s+k}(\partial \Omega)} \lesssim \| h \|_{W^s(\partial \Omega)}. \]

**Proof.** Set

\[ v_0 = Q_{2,-1} h. \]

Then if $h \in W^s(\partial \Omega)$ we have $v_0 = \Psi^{-1}_{\delta_{3/2,4/3}} h \in W^{s+3/2}(\partial \Omega)$ by Theorems 5.3 and 8.2. To see how $v_0$ approximates a solution to (9.1) we calculate $Q_2 v_0 = Q_2 \circ Q_{2,-1} h$, using Corollary 5.10 (note that with $\delta = 4/3, \delta(\delta - 1) = 4/9 < 2/3$) to write

\begin{align*}
Q_2 \circ Q_{2,-1} &= Q_{2,-1} \circ Q_2 + [Q_2, Q_{2,-1}] \\
&= 1 + \Psi^{-1/3}_{\delta_{3/2,4/3}} (\partial \Omega) + \Psi^{-1/3,3/4,1}_{\delta_{3/2,4/3}} (\partial \Omega) + [Q_2, Q_{2,-1}] \\
&= 1 + \Psi^{-1/3}_{\delta_{3/2,4/3}} (\partial \Omega) + \Psi^{-1/3,3/4,1}_{\delta_{3/2,4/3}} (\partial \Omega) + \Psi^{-1,3/4,4/3}_{\delta_{3/2,4/3}} (\partial \Omega) \\
&= 1 + \Psi^{-1/3}_{\delta_{3/2,4/3}} (\partial \Omega) + \Psi^{-1,3/4,4/3}_{\delta_{3/2,4/3}} (\partial \Omega).
\end{align*}

The expression for $Q_2 v_0$ thus becomes

\[ Q_2 v_0 = h + \Psi^{-1/3}_{\delta_{3/2,4/3}} h + \Psi^{-1,3/4,4/3}_{\delta_{3/2,4/3}} h. \]
We also have that $A_b v_0$ is given by
\[
A_b v_0 = \Psi^1 v_0 \nonumber \\
= \Psi^1 \circ \Psi^{-1}_{q_{3/2,4/3} h} \nonumber \\
= \Psi^{2/3} \circ \Psi^{-1}_{q_{3/2,4/3} h} \nonumber \\
= \Psi^{-1/3}_{q_{3/2,4/3} h} h + \Psi^{-4/3,(3/4,4/3)}_{q_{3/2,4/3} h},
\]
by (5.12) and Theorem 5.8. Putting $Q_2 v_0$ and $A_b v_0$ together yields
\[
Q_2 v_0 + A_b v_0 = h + \Psi^{-1/3}_{q_{3/2,4/3} h} h + \Psi^{-4/3,(3/4,4/3)}_{q_{3/2,4/3} h},
\]
We set now
\[
h_1 = h - (Q_2 v_0 + A_b v_0)
\]
(note that $h_1 = E_{1/2}(h)$) and
\[
v_1 = Q_{2,-1} h_1.
\]
As such $v_1$ has the property
\[
v_1 = \Psi^{-1}_{q_{3/2,4/3} h_1} h_1 \nonumber \\
= E_{3/2}(h_1) \nonumber \\
= E_2(h).
\]
As above,
\[
Q_2 v_1 + A_b v_1 = h_1 + \Psi^{-1/3}_{q_{3/2,4/3} h_1} h_1 + \Psi^{-1/3,(3/4,4/3)}_{q_{3/2,4/3} h_1} h_1
\]
Inductively, we define
\[
h_{j+1} := h_j - (Q_2 v_j + A_b v_j)
\]
and
\[
v_{j+1} := Q_{2,-1} h_{j+1}
\]
with the properties
\[
h_{j+1} = E_{1/2} h_j = E_{j+1} h \nonumber \\
v_{j+1} = E_{j+1} h,
\]
and finally
\[ Q_2^\# \left( \sum_{j=0}^{N} v_j \right) + A_b \left( \sum_{j=0}^{N} v_j \right) = h - h_{N+1} \]
\[ = h + E_{N+1} h. \]

For the operator \( F^k \), choose \( N \) above large enough so that \( N \geq 2k - 1 \). Then define \( F^k h \) by
\[ F^k h = \sum_{j=0}^{N} v_j. \]

Similarly, given the equation
\[ \left( \frac{1}{\sqrt{2}} N_{-}^\varepsilon + iT^0 \right) v = h \]
with \( h \in W^s(\partial\Omega) \), for any given \( k \) we give a method to find a specific \( \phi_k \) which satisfies
\[ \left( \frac{1}{\sqrt{2}} N_{-}^\varepsilon + iT^0 \right) \phi_k = h + E_k(h) \]
We first set
\[ v_0 = \left( \frac{1}{\sqrt{2}} N_{-}^\varepsilon - iT^0 \right) \circ Q_{-1}h. \]
We have the estimate
\[ v_0 = \Psi^1_b \circ \Psi^{-1}_{q_{3/2,4/3}} h \]
\[ = E_{1/2} h. \]

Furthermore, using
\[ \left( \frac{1}{\sqrt{2}} N_{-}^\varepsilon + iT^0 \right) \left( \frac{1}{\sqrt{2}} N_{-}^\varepsilon - iT^0 \right) = Q_2 + \Psi^1_b \]
we have
\[ \left( \frac{1}{\sqrt{2}} N_{-}^\varepsilon + iT^0 \right) v_0 = Q_2 \circ Q_{-1}h + \Psi^1_b \circ \Psi^{-1}_{q_{3/2,4/3}} h \]
\[ = h + E_{1/2} h. \]

We then define inductively, with \( h_0 = h \),
\[ h_{k+1} = h_k - \left( \frac{1}{\sqrt{2}} N_{-}^\varepsilon + iT^0 \right) v_k \]
(9.3)
\[ v_{k+1} = \left( \frac{1}{\sqrt{2}} N_{-}^\varepsilon - iT^0 \right) \circ Q_{-1}h_{k+1}. \]
Thus the properties

\[ h_{k+1} = E_{1/2} h_k = E_{k+1} h \]
\[ v_{k+1} = E_{1/2} h_{k+1} = E_{k+1} h \]

and

\[ \left( \frac{1}{\sqrt{2}} N_{\epsilon}^\frac{1}{2} + iT^0 \right) v_{k+1} = h_{k+1} + E_{k+1} h \]

hold. From (9.3), it follows that

\[ \left( \frac{1}{\sqrt{2}} N_{\epsilon}^\frac{1}{2} + iT^0 \right) \left( \sum_{j=0}^{N} v_j \right) = h - h_{N+1} = h + E_{N+1} h. \]

Defining an operator \( F^k_1 h \) by

\[ F^k_1 h = \sum_{j=0}^{N} v_j \]

for \( N \) above large enough so that \( N \geq 2k - 1 \) gives us in the manner of Proposition 9.1 the

**Proposition 9.2.** Let \( h \in W^s(\partial \Omega) \). There exists approximate linear solution operators, \( F^k_1 \) to (9.1) such that

\[ \left\| \left( \frac{1}{\sqrt{2}} N_{\epsilon}^\frac{1}{2} + iT^0 \right) F^k_1 h - h \right\|_{W^{s+k}(\partial \Omega)} \lesssim \| h \|_{W^s(\partial \Omega)}. \]

Propositions 9.1 and 9.2 dealt with constructing solutions to (9.1). For known solutions to (9.1) we can deduce apriori estimates in the form

**Proposition 9.3.** Let \( v \in W^k(\partial \Omega) \) be a solution to

(9.4) \[ Q_2 v + A_b v = h, \]

for \( h \in W^k(\partial \Omega) \). Then \( v \in W^{k+3/2}(\partial \Omega) \) and

(9.5) \[ \| v \|_{k+3/2} \lesssim \| h \|_k + \| v \|_0. \]

**Proof.** We apply the operator \( Q_{2,-1} \) to both sides of (9.4) using Theorem 8.5 to obtain

(9.6) \[ v + \Psi_{\frac{1}{3}/2/4/3}^{-1/3} v + \Psi_{\frac{1}{3}/2/4/3}^{-1,(3/4,1)} v + Q_{20,-1} \circ \Psi v = Q_{20,-1} h. \]

From Corollary 5.7 we can write terms of the form \( Q_{20,-1} \circ \Psi \) as

\[ Q_{20,-1} \circ \Psi = \Psi_{\frac{1}{3}/2/4/3}^{-1/3} + \Psi_{\frac{1}{3}/2/4/3}^{-4/3,(3/4,1)}, \]
and thus (9.6) becomes
\[ v + \psi^{-1/3}_{\Omega^{3/2,4/3}} v + \psi^{-1,(3/4,1)}_{\Omega^{3/2,4/3}} v = \psi^{-1}_{\Omega^{3/2,4/3}} h, \]
which we write as
\[ (9.7) \quad v = \psi^{-1}_{\Omega^{3/2,4/3}} h + \psi^{-1/3}_{\Omega^{3/2,4/3}} v + \psi^{-1,(3/4,1)}_{\Omega^{3/2,4/3}} v. \]

Theorem 5.3 leads to the apriori estimates. We apply \( k \) derivatives to (9.7) and take \( L^2 \)-norms:
\[ \|v\|_k \lesssim \|h\|_{k-3/2} + \|v\|_{k-1/2}. \]
Using the inequality
\[ \|v\|_{k-1/2} \lesssim \epsilon \|v\|_k + \|v\|_0 \]
for any (small) \( \epsilon > 0 \) and bringing the \( \epsilon \|v\|_k \) term to the left hand side results in (9.5).

Of course if we start with a solution known to be in \( L^2(\partial\Omega) \) with data in \( W^k(\partial\Omega) \), then we can apply Proposition 9.3 in steps of 3/2 to arrive at

**Corollary 9.4.** Assume \( v \in L^2(\partial\Omega) \) and \( h \in W^k(\partial\Omega) \) for which (9.4) holds. Then \( v \in W^{k+3/2}(\partial\Omega) \) and
\[ \|v\|_{k+3/2} \lesssim \|h\|_k + \|v\|_0. \]

### 10. The Boundary Value Problem

Following [2], we use a Green’s operator and Poission operator to reduce the following boundary value problem to the boundary:
\[ (10.1) \quad \Box_k u = f, \]
where \( \Box_k = \bar{\partial} \bar{\partial} + \bar{\partial} \bar{\partial} + B_k \), with the boundary conditions
\[ (10.2) \quad L_2 u_1 - s_0 u_1 - L_1 u_2 = 0. \]

While it is not necessary to add further restrictions on the data, \( f \), our solution here will help us with regard to the \( \bar{\partial} \) problem if we further assume that \( f \) is \( \bar{\partial} \)-closed. There are possibly many solutions to the boundary value problem (note that as stated we leave the Dirichlet type condition open in contrast to the \( \bar{\partial} \)-Neumann problem), and we will isolate one particular approximate solution, with a particular boundary value for the component \( u_2 \).

With the solution \( u \) written \( u = u_1 \bar{\omega}_1 + u_2 \bar{\omega}_2 \), we write its restriction to \( \partial\Omega \) as
\[ u_b = u_1^b \bar{\omega}_1 + u_2^b \bar{\omega}_2. \]

The Green’s operator
\[ G_\epsilon = \begin{bmatrix} G_{11}^\epsilon & G_{12}^\epsilon \\ G_{21}^\epsilon & G_{22}^\epsilon \end{bmatrix} \]
solves
\[
2\square_\varepsilon \circ G_\varepsilon = I \quad \text{on } \Omega
\]
\[R \circ G_\varepsilon = 0 \quad \text{on } \partial \Omega,\]
modulo smooth terms, where \(R\) is the restriction to the boundary operator as above. If \(f = f_1\bar{\omega}_1 + f_2\bar{\omega}_2\), we write
\[
G_\varepsilon(f) = G_1^\varepsilon(f)i + G_2^\varepsilon(f)\bar{i},
\]
where
\[
G_1^\varepsilon(f) = G_1^\varepsilon(f_1) + G_1^\varepsilon(f_2),
\]
\[
G_2^\varepsilon(f) = G_2^\varepsilon(f_1) + G_2^\varepsilon(f_2).
\]

A solution to (10.1) under condition (10.2) is then given by
\[
(10.3) \quad u = G_\varepsilon(2f) + P_\varepsilon(u_b),
\]
where \(P_\varepsilon\) is a Poisson’s operator for the boundary value problem
\[
\square \circ P_\varepsilon = 0 \quad \text{on } \Omega
\]
\[R \circ P_\varepsilon = I \quad \text{on } \partial \Omega,
\]
modulo smooth terms. As in Sections 6 and 7, we can localize the problem and operators and use coordinates \((x, \rho)\) in a neighborhood of a boundary point. Then locally we can write condition (10.2) as
\[
0 = \left( \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho} - iT^0 \right) u_1 - L_1 u_b^2
\]
\[= \left( \frac{1}{\sqrt{2}} \frac{\partial}{\partial \rho} - iT^0 \right) \left( G_1^\varepsilon(2f) + P_1^\varepsilon(u_b) \right) - L_1 u_b^2
\]
\[= \frac{1}{\sqrt{2}} R \circ \Theta_\varepsilon^{-1}(2f_1) + R \circ \Psi^{-2}f + \left( \frac{1}{\sqrt{2}} N^\varepsilon - iT^0 \right) u_b - L_1 u_b^2 + I_b^{-\infty} + R_b^{-\infty},
\]
using Theorem 7.1 in the last line. We rewrite this as
\[
\left( \frac{1}{\sqrt{2}} N^\varepsilon - iT^0 \right) u_b - L_1 u_b^2 = - \frac{2}{\sqrt{2}} R \circ \Theta_\varepsilon^{-1}(f_1) + R \circ \Psi^{-2}f,
\]
modulo the smooth terms \(R_b^{-\infty}\) and \(I_b^{-\infty}\).

Our approximate solution \(u\), will be determined via (10.3) by its boundary values. We first use approximations for the Green’s and Poisson operators. The approximations, respectively, \(\tilde{G}_\varepsilon\) and \(\tilde{P}_\varepsilon\), are pseudodifferential operators of order \(-2\) and \(-1\), respectively, and agree up to high order with \(G_\varepsilon\) and \(P_\varepsilon\), respectively. In particular the estimates of Theorems 6.2 and 7.2 hold as well for \(\tilde{G}_\varepsilon\) and \(\tilde{P}_\varepsilon\), respectively, without the smoothing error terms. With these approximations, the
approximate solution we find will depend linearly on the data. The error terms $R_b^{-\infty}$ and $I_b^{-\infty}$ will also no longer play a role. Equation (10.3) is thus to be replaced with the approximation

\[ u = \tilde{G}_\varepsilon(2f) + \tilde{P}_\varepsilon(u_b). \]

We work with $\tilde{G}_\varepsilon$ and $\tilde{P}_\varepsilon$ approximating $G_\varepsilon$ and $P_\varepsilon$ to high enough order that we have

\[ \Box_\varepsilon \left( \tilde{G}_\varepsilon(2f) + \tilde{P}_\varepsilon(u_b) \right) = f + \Psi^{-1} f + \Psi^{-1} u_b \]

and on the boundary

\[ \mathcal{T}_1 \left( \tilde{G}_\varepsilon^l(2f) + \tilde{P}_\varepsilon^l(u_b) \right) = \frac{1}{\sqrt{2}} R \circ \Theta_\varepsilon^l(2f) + R \circ \Psi^{-2} f + \left( \frac{1}{\sqrt{2}} N_\varepsilon^l - iT^0 \right) u_b^1 + \Psi_b^0 u_b \]

As above, the boundary condition is a condition on

\[ \mathcal{T}_2 u_1 - s_0 u_1 - \mathcal{T}_1 u_2 = \frac{1}{\sqrt{2}} R \circ \Theta_\varepsilon^l(2f) + R \circ \Psi^{-2} f + \left( \frac{1}{\sqrt{2}} N_\varepsilon^l - iT^0 \right) u_b^1 + \Psi_b^0 u_b - \mathcal{T}_1 u_b^2. \]

The idea is to choose $u_b^1$ and $u_b^2$ so that the right hand side of this equation on the boundary is sufficiently smooth, i.e.

\[ \left( \frac{1}{\sqrt{2}} N_\varepsilon - iT^0 \right) u_b^1 + \Psi_b^0 u_b - \mathcal{T}_1 u_b^2 = -\frac{2}{\sqrt{2}} R \circ \Theta_\varepsilon^l(f) + R \circ \Psi^{-2} f, \]

modulo (sufficiently smooth) error terms.

We begin by motivating our particular choice for $u_b^2$. The choice for $u_b^1$ will then follow naturally.

The motivation starts with the application of the operator $\left( \frac{1}{\sqrt{2}} N_\varepsilon^l + iT^0 \right)$ to both sides of (10.5). On the right-hand side we have

\[ -\frac{2}{\sqrt{2}} R \circ \left( \frac{1}{\sqrt{2}} N_\varepsilon^l + iT^0 \right) \Theta_\varepsilon^l(f) + R \Psi^{-1}(f). \]
In writing out the pseudodifferential operators, we use a partition of unity \( \{ \chi_j \} \) with respect to the domain \( \Omega \). The term \( N^- \Theta^- (f_1) \) is given by

\[
N^- \Theta^- (f_1) = \frac{i}{(2\pi)^4} \sum_j \int \chi'(\xi) |\Xi(x, \xi)| \left( \frac{\hat{\chi}_j f_1}(\xi, \eta) \right) e^{ix \xi} e^{i\rho \eta} d\xi d\eta + \Psi^{-1}(f)
\]

\[
= -\frac{i}{(2\pi)^4} \sum_j \int \chi'(\xi) \frac{\hat{\chi}_j f_1}(\xi, \eta) e^{ix \xi} e^{i\rho \eta} d\xi d\eta + \Psi^{-1}(f)
\]

\[
= -\frac{i}{(2\pi)^4} \sum_j \int \chi'(\xi) \frac{\partial}{\partial \rho} \left( \frac{\hat{\chi}_j f_1}(\xi, \eta) \right) e^{ix \xi} e^{i\rho \eta} d\xi d\eta + \Psi^{-1}(f)
\]

since the second integral in the third step vanishes. For \( \bar{\partial} \)-closed \( f \), we have the relation

\[
(I_2 - s) f_1 - L_1 f_2 = 0,
\]

or, locally,

\[
\frac{1}{\sqrt{2}} \frac{\partial f_1}{\partial \rho} - iT f_1 = L_1 f_2 + s f_1.
\]
and so, for $\delta f = 0$,
\[
\left( \frac{1}{\sqrt{2}} N_{\varepsilon} - iT \right) \Theta^{-}_\varepsilon (f_1) = \frac{1}{\sqrt{2}} N_{\varepsilon} \Theta^{-}_\varepsilon (f_1) + i \Theta^{-}_\varepsilon (Tf_1) + \Psi^{-1}(f) \]
\[
= - \frac{i}{\sqrt{2}} \left( \frac{1}{(2\pi)^4} \sum_j \int \chi'_{\varepsilon}(\xi) \frac{\partial (\chi f_1)(\xi, \eta)}{\eta - i |\Sigma_{\varepsilon}(x, \xi)|} e^{i\xi_\varepsilon^j} e^{i\eta_\varepsilon^j} d\xi d\eta \right) \]
\[
= - \frac{1}{(2\pi)^4} \left( \frac{1}{2} \sum_j \int \chi'_{\varepsilon}(\xi) \frac{T(\chi f_1)(\xi, \eta)}{\eta - i |\Sigma_{\varepsilon}(x, \xi)|} e^{i\xi_\varepsilon^j} e^{i\eta_\varepsilon^j} d\xi d\eta \right) + \Psi^{-1}(f) \]
\[
= - \frac{i}{(2\pi)^4} \left( \frac{1}{2} \sum_j \int \chi'_{\varepsilon}(\xi) \frac{T_2(\chi f_1)\xi, \eta)}{\eta - i |\Sigma_{\varepsilon}(x, \xi)|} e^{i\xi_\varepsilon^j} e^{i\eta_\varepsilon^j} d\xi d\eta \right) + \Psi^{-1}(f) \]
\[
= - \frac{i}{(2\pi)^4} \left( \frac{1}{2} \sum_j \int \chi'_{\varepsilon}(\xi) \frac{T_1(\chi f_2)(\xi, \eta)}{\eta - i |\Sigma_{\varepsilon}(x, \xi)|} e^{i\xi_\varepsilon^j} e^{i\eta_\varepsilon^j} d\xi d\eta \right) + \Psi^{-1}(f) \]
\[
= - \Theta^{-}_\varepsilon (T_1 f_2) + \Psi^{-1}(f) \]
\[
= - \Theta^{-}_\varepsilon (T_1 f_2) + \Psi^{-1}(f). \]
Thus, when we apply $\frac{1}{\sqrt{2}} N_{\varepsilon} - iT$ to Equation (10.5), we obtain
\[
\left( \frac{1}{2} (N_{\varepsilon}^{-2} + (T^0)^2) \right) u^-_b = \left( \frac{1}{\sqrt{2}} N_{\varepsilon}^{-} + iT^0 \right) T_1 u^-_b + \Psi^{-1}_1 u_b = \frac{2}{\sqrt{2}} R \circ \left( \frac{1}{\sqrt{2}} N_{\varepsilon}^{-} + iT \right) \Theta^{-}_\varepsilon (f_1) = \frac{2}{\sqrt{2}} T_1 R \circ \Theta^{-}_\varepsilon (f_2) + R \Psi^{-1} f, \]
which we write as
\[
(10.7) \quad \left( \frac{1}{2} (N_{\varepsilon}^{-2} + (T^0)^2) \right) u^-_b = T_1 \left( \frac{1}{\sqrt{2}} N_{\varepsilon}^{-} + iT^0 \right) u^-_b + \Psi^{-1}_1 u_b = \frac{2}{\sqrt{2}} T_1 R \circ \Theta^{-}_\varepsilon (f_2) + R \Psi^{-1} f. \]

We now see how a particular choice of boundary value for $u_2$ leads to regularity for our solution. Namely, we choose $u^-_b$ so that the highest order terms on the right-hand side cancel out. We use Proposition 9.2 to set a boundary condition such that $u^-'_b$ is the solution according to Proposition 9.2 to an equation
\[
\left( \frac{1}{\sqrt{2}} N_{\varepsilon}^{-} + iT^0 \right) u^-'_b = \frac{2}{\sqrt{2}} R \circ \Theta^{-}_\varepsilon (f_2) + E_k \left( R \circ \Theta^{-}_\varepsilon (f_2) \right) \]
for sufficiently large $k$ as defined in Proposition 9.2
\[
(10.8) \quad u^-'_b := \frac{2}{\sqrt{2}} F^k_1 \left( R \circ \Theta^{-}_\varepsilon (f_2) \right). \]
From (9.2) $u_b^2$ has the property

$$u_b^2 = E_{1/2} \left( R \circ \Theta^- (f_2) \right).$$

With $u_b^2$ as in (10.8), Equation (10.7) thus reads

$$\left( \frac{1}{2} (N^-)^2 + (T^0)^2 \right) u_b^1 + \Psi_b^1 u_b^1 = R \circ \Psi^{-1} f + E_{-1/2} (R \circ \Theta^- (f_2)) \quad \text{(10.9)}$$

$$= R \circ \Psi^{-1} f + E_{-1/2} (R \circ \Theta^- (f_2)),$$

for $k \geq 1$.

With $u_b^1$ now defined according to Proposition 9.1 (which we apply in the case the operator $A_b$ is given by the $\Psi^1_b$ operator on the left-hand side of (10.9)) we obtain an approximate solution to (10.9) with a particular choice given by (10.10)

$$u_b^1 := F_k^2 g,$$

where $g$ is given by the right-hand side of (10.9). With this choice (10.9) gives

$$\left( \frac{1}{2} (N^-)^2 + (T^0)^2 \right) u_b^1 + \Psi_b^1 u_b^1 = R \circ \Psi^{-1} f + E_{-1/2} (R \circ \Theta^- (f_2)) + E_k \left( R \circ \Psi^{-1} f \right).$$

Returning to (10.5), we see with $u_b^1$ and $u_b^2$ so defined

$$\left( \frac{1}{\sqrt{2}} N^- + iT^0 \right) \left[ \left( \frac{1}{\sqrt{2}} N^- - iT^0 \right) u_b^1 - T_1 u_b^2 \right] = \frac{2}{\sqrt{2}} T_1 R \circ \Theta^- (f_2) + R \Psi^{-1} f + E_k \left( R \circ \Psi^{-1} f \right),$$

where the $\Psi^{-1}$ operator in the second term on the right is the same as that in (10.6). Thus with $u_b^1$ and $u_b^2$ defined by (10.10) and (10.8), respectively, and $u$ defined in terms of $u_b$ and $f$ as in (10.4), the relations (10.5), (10.7), and (10.9) show that on the boundary

$$\left( \frac{1}{\sqrt{2}} N^- + iT^0 \right) (T_2 u_1 - s_0 u_1 - T_1 u_2) = E_k \left( R \circ \Psi^{-1} f \right).$$

Applying the operator $\left( \frac{1}{\sqrt{2}} N^- - iT^0 \right)$ (a first order operator), and using Corollary 9.4 shows

$$(T_2 u_1 - s_0 u_1 - T_1 u_2) = E_{3/2} \left( \Psi^1 E_k \left( R \circ \Psi^{-1} f \right) \right) = E_{k+1/2} \left( R \circ \Psi^{-1} f \right).$$

We also can obtain estimates on our solution.

**Theorem 10.1.** Let $u$ be defined by (10.4), (10.8), and (10.10). Then $u$ satisfies

$$\Box_e u = f + \Psi^{-1} f + \Psi^{-1} u_b, \quad \text{on } \Omega,$$

with the boundary relation

$$T_2 u_1 - s_0 u_1 - T_1 u_2 = E_{k+1/2} \left( R \circ \Psi^{-1} f \right).$$
Furthermore, we have the estimates
\begin{align}
\|u_1^b\|_{W^{s+3/2}(\partial \Omega)} & \lesssim \|f\|_{W^s(\Omega)} \\
\|u_2^b\|_{W^{s+1}(\partial \Omega)} & \lesssim \|f\|_{W^s(\Omega)} \\
\|u\|_{W^{s+3/2}(\Omega)} & \lesssim \|f\|_{W^s(\Omega)}.
\end{align}

**Proof.** For $u_1^b$ defined as in (10.10) we have estimates as in Proposition 9.1 or Corollary 9.4:
\[ u_1^b = E_{3/2} \left( R \circ \Psi^{-1} f + E_{-1/2} \left( R \circ \Theta_{\epsilon}^{-1} (f_2) \right) \right), \]
and
\[
\|u_1^b\|_{W^{s+3/2}(\partial \Omega)} \lesssim \|R \circ \Psi^{-1} f\|_{W^s(\partial \Omega)} + \|R \circ \Psi^{-1} f_2\|_{W^{s+1/2}(\partial \Omega)} \\
\lesssim \|\Psi^{-1} f\|_{W^{s+1}(\Omega)} \\
\lesssim \|f\|_{W^s(\Omega)}.
\]
This proves (10.12). Similarly, with $u_2^b$ defined according to (10.8) we estimate according to Proposition 9.2 or Corollary 9.4:
\[ u_2^b = E_{1/2} \left( R \circ \Theta_{\epsilon}^{-1} (f_2) \right), \]
and
\[
\|u_2^b\|_{W^{s+1}(\partial \Omega)} \lesssim \|R \circ \Theta_{\epsilon}^{-1} (f_2)\|_{W^{s+1/2}(\partial \Omega)} \\
\lesssim \|\Theta_{\epsilon}^{-1} (f_2)\|_{W^{s+1}(\Omega)} \\
\lesssim \|f\|_{W^s(\Omega)},
\]
proving (10.13). Lastly, with $u$ defined by
\[ u = \tilde{G}_\epsilon(2f) + \tilde{P}_\epsilon(u_b) \]
we can use the estimates (10.12) and (10.13) in Theorems 6.2 and 7.2:
\[
\|u\|_{W^s(\Omega)} \lesssim \|\tilde{G}_\epsilon(2f) + \tilde{P}_\epsilon(u_b)\|_{W^s(\Omega)} \\
\lesssim \|f\|_{W^{s-2}(\Omega)} + \|u_b\|_{W^{s-1/2}(\partial \Omega)} \\
\lesssim \|f\|_{W^{s-2}(\Omega)} + \|f\|_{W^{s-3/2}(\Omega)}.
\]

\[ \square \]

11. CONSTRUCTION OF SOLUTION OPERATORS TO $\bar{\partial}$

In this section we construct a solution to the equation $\bar{\partial} \phi = f$ with $f$ a $(0,1)$-form. The form $f$ in this section will therefore satisfy the compatibility condition $\bar{\partial} f = 0$. This condition allows us to bring in the results of Section 10. In this section we prove our Main Theorem. We recall
our definition of the subspace $A^s_{(0,1)} := \{ f : f \in W^s_{(0,1)}(\Omega); \bar{\partial} f = 0 \}$ with a norm given by $\|f\|_{A^s_{(0,1)}(\Omega)} = \|f\|_{W^s_{(0,1)}(\Omega)}$. We prove the

**Theorem 11.1.** Let $\Omega \subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. Let $f \in W^s_{(0,1)}(\Omega)$ such that $\bar{\partial} f = 0$. There exists a solution operator, $K_\epsilon$, such that

$$\bar{\partial} K_\epsilon f = f$$

with the continuity property $K_\epsilon : A^s_{(0,1)}(\Omega) \to W^{s+1/2}$ for every $s \geq 0$.

We base our construction of the solution operator on our solution to the boundary value problem

(11.1) \[ \square u = f + \Psi^{-1} f + \Psi^{-1} u \] on $\Omega$,

with the boundary condition

(11.2) \[ \mathbf{T}_2 u_1 - s_0 u_1 - \mathbf{T}_1 u_2 = E_{k+1/2} \left( R \circ \Psi^{-1} f \right). \]

Theorem 10.1 gave estimates of our chosen solution; in addition we will use estimates on terms given by various operators composing $\square$ applied to the solution, $u$. We start with the operator $B_\epsilon$:

**Lemma 11.2.**

$$\|B_\epsilon u\|_{W^{s+1/2}(\Omega)} \lesssim \|f\|_{W^s(\Omega)}$$

**Proof.** From Theorem 6.1 we have

$$\tilde{P}_\epsilon(u_1) = \left( \Psi^{-1} u_1^1 \right) \omega_1 + \left( \Psi^{-1} u_2^1 \right) \omega_2 + \Psi^{-2} u_b.$$ 

Since $B_\epsilon \in \Psi^{3/2}_b(\partial \Omega)$ acting only on the first component, $B_\epsilon \tilde{P}_\epsilon(u_b)$ can be written as

(11.3) \[ B_\epsilon \tilde{P}_\epsilon(u_b) = \epsilon |d|^{3/2}_1 \circ \Psi^{-1} u_1^1 + \Psi^{3/2}_b \circ \Psi^{-2} u_b. \]

From Lemma 4.5 the term $\epsilon |d|^{3/2}_1 \circ \Psi^{-1} u_1^1$ is estimated

$$\| |d|^{3/2}_1 \circ \Psi^{-1} u_1^1 \|_{W^s(\Omega)} \lesssim \| \Psi^{-1} u_1^1 \|_{W^{s+3/2}(\Omega)} \lesssim \| u_1^1 \|_{W^{s+1}(\partial \Omega)}.$$ 

Similarly, the term $\Psi^{3/2}_b \circ \Psi^{-2} u_b$ in (11.3) can be estimated from Lemma 4.5 by

$$\| \Psi^{3/2}_b \circ \Psi^{-2} u_b \|_{W^s(\Omega)} \lesssim \| \Psi^{-2} u_b \|_{W^{s+3/2}(\Omega)} \lesssim \| u_b \|_{W^{s}(\partial \Omega)}.$$ 

Together with Theorem 10.1 these estimates show

$$\|B_\epsilon \tilde{P}_\epsilon(u_b)\|_{W^{s+1/2}(\Omega)} \lesssim \|u_1^1\|_{W^{s+3/2}(\partial \Omega)} + \|u_b\|_{W^{s+1/2}(\partial \Omega)} \lesssim \|f\|_{W^{s-1/2}(\Omega)}.$$
The estimate
\[
\| B_{\epsilon} \tilde{G}_{\epsilon}(f) \|_{W^{s+1/2}(\Omega)} = \| \Psi^{3/2} \tilde{G}_{\epsilon}(f) \|_{W^{s+1/2}(\Omega)} \\
\lesssim \| G_{\epsilon}(f) \|_{W^{s+1/2}(\Omega)} \\
\lesssim \| f \|_{W^{s}(\Omega)}
\]
follows from Theorem 7.2.

We now prove estimates for \( \bar{\partial} u \):

**Lemma 11.3.**
\[
\| \bar{\partial} u \|_{W^{s+2}(\Omega)} \lesssim \| f \|_{W^{s}(\Omega)}.
\]

**Proof.** On the boundary \( \bar{\partial} u \) has the property
\[
\bar{\partial} u \big|_{\partial \Omega} = E_{k+1/2} \left( R \circ \Psi^{-1} f \right)
\]
by Theorem 10.1 and (10.11). As \( u \) is a solution to
\[
\Box_{\epsilon} u = f + \Psi^{-1} f + \Psi^{-1} u_b
\]
and as \( f \) is \( \bar{\partial} \)-closed, we apply \( \bar{\partial} \) to both sides of (11.4) to obtain
\[
\Psi^0 f + \Psi^0 u_b = \bar{\partial} \Box_{\epsilon} u \\
= \bar{\partial} \Box u + \partial B_{\epsilon} u \\
= \Box \bar{\partial} u + B_{\epsilon} \bar{\partial} u + [\bar{\partial}, B_{\epsilon}] u
\]
i.e.
\[
\Box_{\epsilon} \bar{\partial} u = \Psi^{3/2} u + \Psi^0 f + \Psi^0 u_b
\]
with the Dirichlet condition (with respect to \( \bar{\partial} u \)):
\[
\bar{\partial} u \big|_{\partial \Omega} = E_{k+1/2} \left( R \circ \Psi^{-1} f \right).
\]
In terms of (approximations of) a Green’s operator and Poisson’s operator (on the level of \((0, 2)\)-forms; it is easy to show the (approximate) Green’s and Poisson operators acting on \((0, 2)\)-forms satisfy the estimates of Theorems 7.2 and 6.2, respectively) without the smooth error terms. We denote these operators \( \tilde{G}_{\epsilon}^2 \) and \( \tilde{P}_{\epsilon}^2 \), respectively. We have
\[
\bar{\partial} u = \tilde{G}_{\epsilon}^2 \left( \Psi^{3/2} u + \Psi^0 f + \Psi^0 u_b \right) + \tilde{P}_{\epsilon}^2 \left( \bar{\partial} u \big|_{\partial \Omega} \right)
\]
and so can be estimated by
\[
\|\tilde{\partial}u\|_{W^{s+\frac{1}{2}}(\Omega)} \lesssim \|\Psi^{3/2}u\|_{W^s(\Omega)} + \|f\|_{W^s(\Omega)} + \|\Psi^{-2}u_b\|_{W^{s+\frac{1}{2}}(\Omega)} + \left\|E_{k+1/2} \left( R \circ \Psi^{-1}f \right) \right\|_{W^{s+\frac{1}{2}}(\partial\Omega)} \\
\lesssim \|u\|_{W^{s+3/2}(\Omega)} + \|f\|_{W^s(\Omega)} + \|u_b\|_{W^{s+1/2}(\partial\Omega)} + \left\|R \circ \Psi^{-1}f \right\|_{W^{s-k+1}(\partial\Omega)} \\
\lesssim \|f\|_{W^s(\Omega)} + \left\|\Psi^{-1}f\right\|_{W^{s-k+3/2}(\Omega)} \\
\lesssim \|f\|_{W^s(\Omega)}
\]
for large enough \( k \geq 2 \) from Theorem 10.1 and in particular estimate 10.14. Note that to apply the results of Lemma 4.2 it can be checked that the proof of the lemma can be applied to the operator \( \Psi^{-2} \) (it can since the operator is constructed from the approximation to Green’s operator whose symbol has simple poles with respect to the transform of the normal coordinate).

**Proof of Theorem 11.1** We first consider
\[
\tilde{\partial} (\partial u) = \Box u - B_\epsilon(u) - \partial \tilde{\partial} u = f + \Psi^{-1}f + \Psi^{-1}u_b - B_\epsilon(u) - \partial \tilde{\partial} u.
\]
(11.5)

The terms \( B_\epsilon u \) and \( \partial \tilde{\partial} u \) can be estimated by \( B_\epsilon u = E_{1/2}f \) and \( \partial \tilde{\partial} u = E_1f \) by Lemmas 11.2 and 11.3 respectively. The term \( \Psi^{-1}u_b \) comes from the approximation of the Possion operator, and so is constructed from the inverse of an elliptic operator for which Lemma 4.2 applies, and from Theorem 10.1 we have \( \Psi^{-1}u_b = E_{3/2}f \). For \( \delta > 0 \), we let the operator \( S_{-\delta} : W^k(\Omega) \to W^{k-\delta}(\Omega) \) be the linear solution operator to
\[
\tilde{\partial}v = -\Psi^{-1}f - \Psi^{-1}u_b + B_\epsilon(u) + \partial \tilde{\partial} u
\]
(with the operators coming from (11.5)) given by Straube in [8] (see Theorem 5.3) (note that from (11.5) it follows that \( \Psi^{-1}f + \Psi^{-1}u_b + B_\epsilon(u) + \partial \tilde{\partial} u \) is \( \tilde{\partial} \)-closed), i.e. with \( v \) defined by
\[
v = S_{-\delta} \left( \Psi^{-1}f + \Psi^{-1}u_b + B_\epsilon(u) + \partial \tilde{\partial} u \right)
\]
we have
\[
\tilde{\partial}v = \Psi^{-1}f + \Psi^{-1}u_b + B_\epsilon(u) + \partial \tilde{\partial} u
\]
and
\[
v = E_{-\delta} \left( \Psi^{-1}f + \Psi^{-1}u_b + B_\epsilon(u) + \partial \tilde{\partial} u \right) = E_{-\delta} \left( E_1f + E_{3/2}f + E_{1/2}f + E_1f \right) = E_{1/2-\delta}f.
\]

Then, from (11.5), we have the solution \( \partial u + v \):
\[
(11.7) \quad \tilde{\partial} (\partial u + v) = f
\]
with estimates
\[ \| \vartheta u + v \|_{W^s(\Omega)} \lesssim \| f \|_{W^{s+1/2}(\Omega)}. \]

To write our solution operator, we recall the operators which went into the construction of our solution \( u \). The solution \( u \) was written
\[ u = \tilde{P}_\varepsilon(u_b) + \tilde{G}_\varepsilon(2f) \]
where \( u_b \) was chosen via (10.8) and (10.10) (where a sufficiently large \( k \) was to be determined).

From above, it suffices to locally choose
\[ u_1^b = F_2 (R \circ \Psi^{-1} f + E_{-1/2} (R \circ \Theta^{-} (f_2))) \]
\[ u_2^b = -\frac{2}{\sqrt{2}} F_1 (R \circ \Theta^{-} (f_2)), \]
where \( \Psi^{-1} \) and \( E_{-1/2} \) are the specific operators given in (10.9), corresponding to \( k = 1 \) (respectively, \( k = 2 \)) in the definitions for the operators \( F_2 \) and \( F_1 \) given in Propositions 9.1 and 9.2, respectively. Let us write \( u_1^b \bar{\omega}_1 + u_2^b \bar{\omega}_2 \) together as \( \bar{U}(f) \), where \( U \) represents the operators on the right hand side of the expressions above for \( u_1^b \) and \( u_2^b \). Thus, the solution operator, which we define as \( \tilde{N}_\varepsilon \), to the boundary value problem (11.1) and (11.2) is given by
\[ \tilde{N}_\varepsilon f = \tilde{P}_\varepsilon \bar{U}(f) + \tilde{G}_\varepsilon(f), \]
again, for the specific operators \( \Psi^{-1} \) and \( E_{-1/2} \) as given in (10.9). And finally, \( K_{\varepsilon,\delta} \) can be written according to (11.7) as
\[ K_{\varepsilon,\delta}(f) = \vartheta \tilde{N}_\varepsilon f + S_{-\delta} \circ \Psi^{-1} f + S_{-\delta} \circ \Psi^{-1} U(f) + S_{-\delta} \circ B_{\varepsilon}(\tilde{N}_\varepsilon f) + S_{-\delta} \circ \vartheta \tilde{\tilde{\vartheta}}(\tilde{N}_\varepsilon f), \]
where the \( \Psi^{-1} \) operators are those on the right-hand side of (11.5). As \( K_{\varepsilon,\delta} \) consists of compositions of linear operators, so is \( K_{\varepsilon,\delta} \) itself.

Once we have the linear operator \( K_{\varepsilon,\delta} \) at our disposal, we retrace the steps of the proof, this time replacing the operator \( S_{-\delta} \) used for obtaining \( v \) in (11.6) with our newly constructed \( K_{\varepsilon,\delta} \), thereby replacing \( v \) with a \( v' \) which satisfies \( v' = E_{1-\delta} f \) and
\[ \vartheta (\vartheta u + v') = f \]
with estimates
\[ \| \vartheta u + v' \|_{W^s(\Omega)} \lesssim \| f \|_{W^{s+1/2}(\Omega)}. \]

We write the desired solution operator in the theorem as
\[ K_{\varepsilon}(f) = \vartheta \tilde{N}_\varepsilon f + K_{\varepsilon,\delta} \circ \Psi^{-1} f + K_{\varepsilon,\delta} \circ \Psi^{-1} U(f) + K_{\varepsilon,\delta} \circ B_{\varepsilon}(\tilde{N}_\varepsilon f) + K_{\varepsilon,\delta} \circ \vartheta \tilde{\vartheta}(\tilde{N}_\varepsilon f). \]
\[ \square \]
We note that there are infinitely many solution operators. For instance, different values for $\varepsilon$ lead to different solutions.

We also note that with our methods of using $\tilde{\Psi}$ operators as approximate inverses, the gain of $1/2$ a derivative is optimal. That is not to say that the optimal gain of the solution to the $\bar{\partial}$-problem must be $1/2$, only that our above analysis with the correction term $B_\varepsilon = \varepsilon |d_t|^{3/2}$ replaced with a correction term of another power $B'_\varepsilon = \varepsilon |d_t|^\gamma$ leads to a lower gain of regularity.

**References**

[1] D. Barrett. Behavior of the Bergmann projection on the Diederich-Fornaess worm. *Acta. Math.*, 168:1–10, 1992.

[2] Nagel A. Chang, D. C. and E. Stein. Estimates for the $\bar{\partial}$-Neumann problem in pseudoconvex domains of finite type in $\mathbb{C}^2$. *Acta Math.*, 169:153–228, 1992.

[3] M. Christ. Global $C^\infty$ irregularity of the $\bar{\partial}$-Neumann problem for worm domains. *J. Amer. Math. Soc.*, 9(4):1171–1185, 1996.

[4] J.J. Kohn. Harmonic integrals on strongly pseudoconvex domains I. *Ann. Math.*, 78:112–148, 1963.

[5] J.J. Kohn. Harmonic integrals on strongly pseudoconvex domains II. *Ann. Math.*, 79:450–472, 1964.

[6] J.J. Kohn. Global regularity for $\bar{\partial}$ on weakly pseudo-convex manifolds. *Trans. Amer. Math. Soc.*, 181:273–292, 1973.

[7] Ewa Ligocka. The Sobolev spaces of harmonic functions. *Studia Math.*, 84:79–87, 1986.

[8] E. Straube. *Lectures on the $L^2$-Sobolev theory of the $\bar{\partial}$-Neumann problem*. ESI Lectures in Mathematics and Physics. European Mathematical Society, 2010.

[9] F. Treves. *Introduction to Pseudodifferential and Fourier Integral Operators*. The University Series in Mathematics. Plenum Press, 1980.

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