Approximate Sparsity Pattern Recovery: Information-Theoretic Lower Bounds

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Abstract—Recovery of the sparsity pattern (or support) of a sparse vector from a small number of noisy linear projections (or samples) is a common problem that arises in signal processing and statistics. In this paper, the high-dimensional setting is considered. It is shown that if the sampling rate and per-sample signal-to-noise ratio (SNR) are finite constants independent of the length of the vector, then the optimal sparsity pattern estimate will have a constant fraction of errors. Lower bounds on the sampling rate needed to attain a desired fraction of errors are given in terms of the SNR and various key parameters of the unknown vector. The tightness of the bounds in a scaling sense, as a function of the SNR and the fraction of errors, is established by comparison with existing achievable bounds. Near optimality is shown for a wide variety of practically motivated signal models.

Index Terms—compressed sensing, information-theoretic bounds, random matrices, random projections, regression, sparse approximation, sparsity, subset selection.

I. INTRODUCTION

Recovery of sparse or compressible signals from a limited number of noisy linear projections is a problem that has received considerable attention in signal processing and statistics. Suppose, for instance, that a vector $x$ of length $n$ is known to have exactly $k$ nonzero elements, but the values and locations of these elements are unknown and must be estimated from a set of $m$ noisy linear projections (or samples $\{y_i\}$) of the form

$$y_i = \langle \phi_i, x \rangle + W_i \quad \text{for} \quad i = 1, \ldots, m \quad (1)$$

where $\phi_i$ are known sampling vectors, $\langle \cdot, \cdot \rangle$ denotes the usual euclidean inner product, and $W_i$ is additive white Gaussian noise. Then, a key insight from sparse signal recovery is that the number of samples required for reliable estimation depends primarily on the number of nonzero elements, and is potentially much less than the length of the vector.

One estimation problem of particular interest is to determine which elements of the vector $x$ are nonzero. This problem, which is referred to as sparsity pattern recovery in this paper, is known variously throughout the literature as support recovery or model-selection and has applications in compressed sensing [2–4], sparse approximation [5], subset selection in regression [7], and structure estimation in graphical models [8].

A large body of work [8–23] has considered exact recovery of the sparsity pattern by deriving necessary and sufficient conditions on scalings of the tuple $(n, k, m)$ to ensure that the probability of exact recovery tends to one as the vector length $n$ becomes large. It is now well understood that there exist two fundamentally different scalings depending on whether or not the samples are corrupted by noise: the noiseless setting requires $m_n = k_n + 1$ and the noisy setting requires $m_n = k_n + 1 + C \cdot k_n \log n$ where $C$ is a constant (whose exact value is typically unknown but bounded) that depends on the signal-to-noise ratio (SNR) and various other assumptions about the values of the nonzero elements. Although these scaling results provide valuable insights for a variety problem settings and serve as a benchmark for the development of computationally efficient algorithms, they have two important limitations.

The first limitation is that in many practically relevant settings, the cost (in terms of number of samples and SNR) of exact sparsity pattern recovery far exceeds the cost of other estimation tasks. For example, suppose that $k_n/n \to \Omega$ for some positive sparsity rate $\Omega$ and $m_n/n \to \rho$ for some sampling rate $\rho$. Then, a central result from compressed sensing [24, 25] is that the vector $x$ can be estimated with bounded mean squared error (MSE) even if the sampling rate $\rho$ is finite (and possibly much less than one) and the per-sample SNR is a fixed value that does not depend on $n$. By contrast, the scaling results outlined above show that exact recovery of the sparsity pattern is not possible unless either the sampling rate $\rho$ is infinite or the per-sample SNR increases without bound with $n$. If noise is due to quantization, this means that accurate estimation with respect to MSE requires only fixed bit-rate whereas exact recovery of the sparsity pattern requires an unbounded bit-rate.

The second limitation is that scaling results in terms of the dimensions $(n, k, m)$ do not tell the whole story. Often, one needs to know the exact constants involved in the bounds and the dependence of these constants on parameters such as the SNR or various assumptions about the vector $x$. For many of the estimation tasks considered throughout the compressed sensing literature, these properties are not well understood. As a result, the majority of sufficient conditions are far more conservative than those suggested by empirical evidence, and the optimality (or gap from optimality) of existing algorithms is difficult to determine due to the potential looseness of the necessary conditions.

In the present work, we derive bounds for approximate recovery of the sparsity pattern. In particular, we derive lower
bounds on the number of samples required to ensure that the sparsity pattern can be estimated with no more than $\alpha \cdot k/n$ errors for some error rate $\alpha$. Corresponding upper bounds are derived in the companion paper [26]. An example of these bounds is shown in Figure 1.

The contributions of this paper directly address the limitations of the scaling results for exact recovery outlined above. With respect to the first limitation, we show that if the sampling rate and SNR are both finite, then the error rate $\alpha$ must be a positive constant independent of $n$. In other words, recovery up to an asymptotically vanishing fraction of errors is as hard as perfect recovery, a gap left open by previous work. This means that in order to have the same scaling behavior as in the MSE recovery problem, one must consider sparsity pattern recovery subject to a constant fraction of errors.

With respect to the second limitation, our lower bounds are derived with an explicit dependence on various key problem parameters such as the SNR, the sparsity rate, and the relative size of the smallest nonzero elements. These bounds allow us to consider a wide variety of problem settings where the unknown vectors may be deterministic or stochastic and the magnitude the smallest nonzero element may tend to zero as the vector length becomes large. Our framework allows us to address a number of important questions:

- What is the effect of prior information? The upper bounds in [26] correspond to estimators that know the exact number of nonzero elements, but have no prior information about their values. The lower bounds in this paper apply to settings where the estimator may know statistical information such as the average power, range of values, or distribution. Interestingly, the resulting bounds show that in many cases, this additional knowledge does not significantly improve the ability to estimate the sparsity pattern.
- What happens as the desired error rate tends to zero? Our bounds show that the sampling rate depends on the inverse of the error rate $1/\alpha$. If the magnitudes of the nonzero elements have a fixed lower bound that is independent of $n$ then this dependence is logarithmic. Otherwise, the dependence is polynomial.
- How does recovery depend on the SNR? We show that the sampling rate $\rho$ must scale like $1/\log(1 + \text{SNR})$. In particular, our bounds show that performance is dominated by the size of the smallest nonzero elements at low SNR and by the entropy of the nonzero elements at high SNR.
- What happens in the noiseless setting? It is straightforward to see that $\rho \geq \Omega$ is a lower bound whenever the nonzero values are unconstrained. By contrast, when the nonzero values are drawn from a discrete and finite set (known to the estimator), then $\rho > 0$ is the best universal lower bound. We show that if the nonzero values are drawn from a known distribution with sufficiently large (differential) entropy, then the condition $\rho \geq \Omega$ remains necessary.

This paper is organized as follows: Section II gives the precise problem formulation. Section III gives information-theoretic lower bounds on the sampling rate for the noiseless setting. Section IV gives corresponding bounds for the noisy setting. Section V compares the scaling properties of these lower bounds with those of the upper bounds in [26]. Section VI provides specific examples and illustrations, and proofs are given in the Appendices. The following section provides a brief, and necessarily incomplete, overview of work related to this paper.

A. Related Work

One line of related research has focused on the design and analysis of computationally efficient algorithms for sparse signal approximation [21–30], [4], [6]. A key theoretical result [24], [25] is that any $k$-sparse vector $x$ of length $n$ can be approximated with bounded mean squared error $\|x - \hat{x}\|^2/n < C_1/\text{SNR}$ using $m = \lceil C_2 \cdot k \log(n/k) \rceil$ samples and a quadratic program known as Basis Pursuit [6] where $C_1, C_2$ are finite constants. In the absence of any sampling noise, this result guarantees exact recovery. In the presence of noise, however, bounds on the mean squared error are insufficient to determine the accuracy of the estimated sparsity pattern. Work focusing directly on recovery of sparsity pattern [8], [12], [13] has derived various sufficient conditions for exact recovery using a particular convex relaxation known in the statistics literature as the Lasso [30].

In conjunction with the results outlined above, another line of research has focused on the fundamental limitations of sparse signal approximation that apply to any algorithm, regardless of computational complexity. For the special case of exact recovery in the noiseless setting, these limitations have been well understood: recovery of any $k$-sparse vector requires exactly $m = 2k$ samples for deterministic guarantees and only $m = k + 1$ samples for almost sure guarantees [9]–[11], regardless of the vector length $n$. In both cases, recovery corresponds to an NP-hard exhaustive search through all possible sparsity patterns. In Section III of this paper, we address the extent to which an even smaller number of samples
are needed when there exists prior knowledge about the vector $x$, or when only partial recovery is needed.

Although the noiseless setting provides insight into the limitations of sparse approximation that cannot be overcome simply by increasing the SNR, consideration of the noisy setting is crucial for cases where noise is intrinsic to the problem or where real-valued numbers are subject to rate constraints. From an information-theoretic perspective, a number of works have studied the rate-distortion behavior of sparse sources [38–33]. Most closely related to this paper, however, is work that has addressed sparsity pattern recovery directly. An initial necessary bound based on Fano’s inequality was provided by Gastpar [11] who considered Gaussian signals and deterministic sampling vectors. Necessary and sufficient conditions in the ways described in the introduction. Comparable improvements to the previous sufficient conditions are given in the companion paper [26].

II. Problem Formulation

In this paper, we assume that $x$ is an arbitrary (non-random) element from some subset $\mathcal{X}^{n} \subset \mathbb{R}^{n}$. The sparsity pattern $s \subset \{1, 2, \ldots, n\}$ is the set of integers indexing the nonzero elements of $x$,

$$s := \{i : x_i \neq 0\},$$

and the sparsity $k = |s|$ is the number of nonzero elements.

We assume that $x$ is sampled using the noisy linear observation model given in (1). In matrix form, the vector of samples $Y = [Y_1, \ldots, Y_m]^T$ can be expressed as

$$Y = Ax + W$$

where the sampling matrix $A \in \mathbb{R}^{m \times n}$ has rows $\phi_i^T$ and the noise vector $W = [W_1, \ldots, W_m]^T$ has i.i.d. standard Gaussian elements. We further assume that an estimator is given the set $(Y, A, k)$, and the goal is to recover the sparsity pattern $s$ of $x$. In some cases, additional information about the set $\mathcal{X}^{n}$ is also provided.

To quantify the distortion between a sparsity pattern $s$ and its estimate $\hat{s}$, it is important to observe that there are two different error events: one type of error occurs when an element in $s$ is omitted from the estimate $\hat{s}$ and the other occurs when an element not present in $s$ is included in $\hat{s}$. In this paper, we focus on recovery at the point where there is an equal number of each error type. We assume throughout that any estimate $\hat{s}$ has the same size $k$ as the true sparsity pattern $s$, and we define the distortion to be relative overlap

$$d(s, \hat{s}) := 1 - \frac{|s \cap \hat{s}|}{|s|}.$$ 

We say that recovery is successful with respect to distortion $\alpha \in [0, 1]$ if $d(s, \hat{s}) \leq \alpha$. Exact recovery corresponds to the case $\alpha = 0$.

We are interested in performance guarantees that hold uniformly for any $x \in \mathcal{X}^{n}$. It is important to note, however, that for any particular sampling matrix $A$, there may exist a degenerate subset of $\mathcal{X}^{n}$ for which recovery is particularly difficult. To overcome the effects of these sets, we allow $A$ to be a random matrix (denoted using boldface) distributed independently of $x$. Given any sparsity pattern estimator $\hat{s}(y, A, k)$, the probability of error corresponds to the worst case $x \in \mathcal{X}^{n}$ with respect to the distribution on $A$.

$$P_e(n) = \inf_{x \in \mathcal{X}^{n}} \Pr\{d(s, \hat{s}(Y, A, k)) > \alpha\}.$$ 

Estimation in the presence of noise depends critically on the size of the entries in the sampling matrix. In this paper, we assume that

$$\mathbb{E}[\text{tr}(AA^T)] = m.$$ (2)

This scaling is consistent with the related work [3, 4]. [15, 17], [19] and corresponds to the setting where each sampling vector (i.e. row of $A$) has unit magnitude. Thus, one useful property of this scaling, is that the SNR of the linear samples given in (1) can be compared directly that of classical samples of the form $y_i = x_i + w_i$. Another useful property is that the SNR does not depend of the number of samples $m$.

We caution the reader that various other scalings of the sampling matrix are also used in the literature, and thus extra care is needed when comparing results. For instance, in [12], [14], [21] each element of $A$ has unit power, and the squared magnitude of each sampling vector is thus proportional to the vector length $n$.

To characterize the number of samples that are needed, we consider the high dimensional setting where the vector length $n$ becomes large. We use $\mathcal{X}$ to denote a sequence of subsets $\{\mathcal{X}^{n}\}$ and refer to $\mathcal{X}$ as a vector source. The main question we address is whether or not recovery is possible when the number of samples is given by $m_n = \lfloor \rho \cdot n \rfloor$ for some finite sampling rate $\rho$ that is a fixed constant independent of $n$.

Definition 1. A sampling rate distortion pair $(\rho, \alpha)$ is said to be achievable for a source $\mathcal{X}$ if for each integer $n$ there exists an estimator $\hat{s}(y, A, k)$ and a $\lfloor \rho \cdot n \rfloor$ sampling matrix $A$ such that

$$P_e(n) \to 0 \quad \text{as} \quad n \to \infty.$$ 

The sampling rate distortion function $\rho(\alpha)$ is the infimum of rates $\rho \geq 0$ such that the pair $(\rho, \alpha)$ is achievable.

We focus exclusively on the scaling regime where the sparsity $k$ scales linearly with the vector length $n$.

Definition 2. Given any sparsity rate $0 \leq \Omega \leq 1/2$, the set $\mathcal{X}^{n}(\Omega)$ consists of all vectors $x \in \mathbb{R}^{n}$ with sparsity $\lfloor \Omega \cdot n \rfloor$. 
The vector source \( X(\Omega) \) denotes the sequence \( \{X^n(\Omega)\} \) for all \( n \).

From a sampling perspective, the sparsity rate \( \Omega \) measures the degrees of freedom per dimension of \( x \) and is analogous to the rate of innovation \([44]\) or “bandwidth” of an infinite length discrete time sequence.

One limitation of the general source \( X(\Omega) \) is that the nonzero values may be arbitrarily small, thus making recovery in the presence of noise impossible. In previous work \([12],[14]\), this issue is addressed by placing a lower bound on the magnitudes of the nonzero elements of \( x \). This paper uses the more general approach outlined below where the nonzero elements are characterized by a set of distribution functions.

We define \( \mathcal{F}_0 \) to be the set of all cumulative distribution functions (henceforth referred to simply as distributions) \( F(x) = \text{Pr}\{X \leq x\} \) for a random variable \( X \) such that \( \mathbb{E}X^2 < \infty \) and \( \text{Pr}\{X = 0\} = 0 \).

**Definition 3.** Given any distribution \( F \in \mathcal{F}_0 \), the vector source \( X(\Omega, F) \) consists of all sequences of vectors \( \{x^n \in X^n(\Omega)\} \) for which \( \sup_{x \in \mathbb{R}} |F_{x^n(a)}(x) - F(x)| \to 0 \) as \( n \to \infty \), where \( F_{x^n(a)} \) denotes the empirical distribution of the non-zero elements in \( x^n \),

\[
F_{x^n(a)}(x) = \frac{1}{k} \sum_{i \in a} 1(x_i \leq x).
\]

Given any subset \( \mathcal{F} \subseteq \mathcal{F}_0 \), the vector source \( X(\Omega, \mathcal{F}) \) denotes the union \( \cup_{F \in \mathcal{F}} X(\Omega, F) \).

To be consistent with previous work, we may for example consider the source \( X(\Omega, F) \) where \( \mathcal{F} \) denotes the set of all distributions whose support is bounded away from zero. However, one advantage of our approach is that we may also consider a source \( X(\Omega, F) \) where \( F \) has a density around zero, and thus a small number of nonzero elements may be arbitrarily small.

### III. The Noiseless Setting

In this section, we lower bound the achievable sampling rate distortion region in the absence of any measurement error. We use \( \rho_0(\alpha) \) to denote the noiseless sampling rate distortion function. The results in this section give insight into the fundamental limitations of the sampling process that cannot be overcome simply by increasing the signal-to-noise ratio. These results also serve as a useful starting point for the noisy setting considered in Section IV.

We first consider the general vector source \( X(\Omega) \). It is well known that exact recovery requires \( m = k + 1 \) samples and hence \( \rho_0(0) = \Omega \). However, if a distortion \( \alpha > 0 \) is allowed, then the following result shows that recovery using fewer samples is possible using a “rate sharing” strategy. The proof is given in Appendix A-A.

**Theorem 1.** The noiseless sampling rate distortion function \( \rho_0(\alpha) \) of the vector source \( X(\Omega) \) is given by

\[
\rho_0(\alpha) = \begin{cases} \Omega - \frac{\Omega - \alpha}{1 - \alpha} & \text{if } \alpha < 1 - \Omega, \\ 0, & \text{if } \alpha \geq 1 - \Omega. \end{cases}
\]  

The tradeoff between \( \rho \) and \( \alpha \) exhibited in Theorem 1 requires that the sampling matrix has some subset of columns equal to zero. Since the number of such columns depends on \( \alpha \), it is not possible for a single random sampling matrix to uniformly achieve all the points in the achievable region.

In contrast to the matrix constructions used in Theorem 1, a great deal of the work in compressed sensing has focused on matrices whose elements are independently and identically distributed with zero mean. We henceforth refer to these matrices as i.i.d. sampling matrices. The following result shows i.i.d. sampling matrices are optimal for exact recovery, but suboptimal for any nonzero distortion. The proof is given in Appendix B-E.

**Proposition 1.** If the sampling matrix is i.i.d., then the noiseless sampling rate distortion function \( \rho_0(\alpha) \) of the vector source \( X(\Omega) \) is given by

\[
\rho_0(\alpha) = \begin{cases} \Omega & \text{if } \alpha < 1 - \Omega, \\ 0, & \text{if } \alpha \geq 1 - \Omega. \end{cases}
\]  

Next, we consider recovery for a vector source \( X(\Omega, F) \) characterized by a distribution function \( F \). In some cases, the constraints imposed by \( F \) significantly alter the nature of the estimation problem.

**Proposition 2.** Suppose that the distribution \( F \) is supported on a discrete and finite set \( \Sigma \subset \mathbb{R} \setminus \{0\} \). Then, \( m = 1 \) sample is sufficient for exact recovery, and the noiseless sampling rate distortion function \( \rho_0(\alpha) \) of the vector source \( X(\Omega, F) \) is given by \( \rho_0(\alpha) = 0 \) for all \( \alpha \).

**Proof:** Suppose that \( A \) is an \( 1 \times n \) “matrix” whose elements are drawn i.i.d. from continuous distribution with finite power. Then, with probability one, the projection \( x \mapsto Ax \) maps each of the \( \binom{n}{k} \) possible realizations of \( x \) to a unique real number.

The fact that only one sample is needed for discrete distributions is not due to the sparsity in the problem (after all, the results do not depend on the sparsity rate \( \Omega \)) and Proposition 2 provides little insight into cases where the nonzero values are continuous. To address these cases, we use the following property.

**Definition 4.** Given any sparsity rate \( \Omega \) and distribution \( F \) with mean \( \mu_F \), variance \( \sigma_F^2 \), and differential entropy \( h(F) \), the function \( \theta(\Omega, F) \in [0, 1] \) is given by

\[
\theta(\Omega, F) = \frac{2(e\pi)\alpha^{-1}\exp(2h(F))}{\sigma_F^2 + (1 - \Omega) \mu_F},
\]  

The quantity \( \theta(\Omega, F) \) measures the normalized entropy power of the source \( X(\Omega, F) \) and is equal to one if and only if \( F \) is a zero mean Gaussian distribution. Roughly speaking, one may interpret \( \theta(\Omega, F) \) as a relative “distance” between \( X(\Omega, F) \) and a source characterized by a discrete distribution.

Another property we use is the information rate (given in nats per dimension) required to encode a sparsity pattern to within distortion \( \alpha \). Although the following result is likely to exist elsewhere in the literature, a simple proof is given in Appendix B-E.

The tradeoff between \( \rho \) and \( \alpha \) exhibited in Theorem 1 requires that the sampling matrix has some subset of columns equal to zero. Since the number of such columns depends on \( \alpha \), it is not possible for a single random sampling matrix to uniformly achieve all the points in the achievable region.
Lemma 1. Let $S^0_\alpha$ be set of all subsets of $\{1, 2, \ldots, n\}$ of size $|\Omega \cdot n|$, and let $N_n(\Omega, \alpha)$ denote the cardinality of the smallest subset $S \subseteq S^0_\alpha$ such that for any $s \in S^0_\alpha$ there exists $s' \in S$ satisfying $d(s, s') \leq \alpha$. Then,
\[
\frac{1}{n} \log N_n(\Omega, \alpha) = R(\Omega, \alpha)
\]
where
\[
R(\Omega, \alpha) = \begin{cases} \frac{H(\Omega)}{\Omega} - \Omega H(\alpha) - (1-\Omega)H\left(\frac{\Omega}{\alpha + \Omega}\right) & \alpha < 1 - \Omega \\ 0 & \alpha \geq 1 - \Omega \end{cases}
\]
and $H(p) = -p \log p - (1-p) \log (1-p)$ is binary entropy.

Using the above properties, it is possible to give a nontrivial lower bound for i.i.d. sampling matrices and any distribution $F$ with a density. The proof of the following result is given in Appendix B-D.

Theorem 2. If the sampling matrix is i.i.d., then a sampling rate distortion pair $(\rho, \alpha)$ is not achievable for the vector source $\mathcal{X}(\Omega, F)$ in the noiseless setting if $\rho < \Omega$ and
\[
\frac{\rho}{2} \log \left(\frac{1}{\theta(\Omega, F)} \cdot \frac{\Delta(\rho)}{\Delta(\rho/F)}\right) < R(\Omega, \alpha)
\]
where $\theta(\Omega, F)$ is given by (5), $R(\Omega, \alpha)$ is given by (4), and
\[
\Delta(\rho) = \begin{cases} (1-r)^{1-1/r} & \text{if } r < 1 \\ 1 & \text{if } r = 1 \end{cases}
\]
The main intuition suggested by Theorem 2 is that the difficulty of recovering the support is related to the normalized entropy of the elements of $x$. Additionally, one consequence of Theorem 2 is that there is a simple test to see whether the sampling rate needed for a source $\mathcal{X}(\Omega, F)$ is any less than for the general source $\mathcal{X}(\Omega)$.

Corollary 1 (Theorem 2). If the sampling matrix is i.i.d., then the noiseless sampling rate distortion function $\rho_0(\alpha)$ of the vector source $\mathcal{X}(\Omega, F)$ is given by $\rho_0(\alpha) = \Omega$ for all $\alpha < 1 - \Omega$ such that
\[
\theta(\Omega, F) > \Delta(\Omega) \exp \left(-\frac{2}{\Omega} R(\Omega, \alpha)\right).
\]

IV. THE NOISY SETTING

In this section, we lower bound the achievable sampling rate distortion region in the presence of additive white Gaussian noise. Unlike the noiseless setting considered in Section III, it is shown that recovery in the noisy setting depends significantly on the size the nonzero elements. We first derive a genie-aided lower bound that applies to any possible sampling matrix. We then derive stronger results for i.i.d. matrices.

A. Bounds for Arbitrary Sampling Matrices

To begin, we note that recovery for the general source $\mathcal{X}(\Omega)$ with distortion $\alpha < 1 - \Omega$ is not possible in the noisy setting since the nonzero elements of $x$ may be arbitrarily small with respect to the noise. Throughout this section, we focus exclusively on a source $\mathcal{X}(\Omega, F)$ characterized by a distribution function $F$ and use the following properties.

Definition 5. The power and variance of a vector source $\mathcal{X}(\Omega, F)$ are given by
\[
P(\Omega, F) = \Omega(\mu_F^2 + \sigma_F^2) \quad \text{and} \quad V(\Omega, F) = \Omega(1 - \Omega)\mu_F^2 + \Omega\sigma_F^2
\]
where $\mu_F$ and $\sigma_F$ are the mean and variance of $F$.

Due to the scaling of the sampling matrix given by (2), the power $P(\Omega, F)$ represents the SNR of the samples, that is
\[
P(\Omega, F) = \lim_{n \to \infty} \frac{\mathbb{E}[\|Ax\|^2]}{\mathbb{E}[\|W\|^2]}
\]
The variance is closely related and obeys
\[
(1 - \Omega) P(\Omega, F) \leq V(\Omega, F) \leq P(\Omega, F)
\]
with equality on the left when $\sigma_F^2 = 0$ and equality on the right when $\mu_F = 0$.

The following bound is general in the sense that it depends only on the average variance (or power) of the vector source. The proof is given in Appendix B-A.

Proposition 3. The sampling rate distortion function $\rho(\alpha)$ of the vector source $\mathcal{X}(\Omega, F)$ is lower bounded by
\[
\rho(\alpha) \geq \frac{2R(\Omega, \alpha)}{\log (1 + V(\Omega, F))}
\]
where the functions $R(\cdot, \cdot)$ and $V(\cdot, \cdot)$ are defined by (6) and (11) respectively.

Versions of Proposition 3 have been derived previously for various signal models in the special case of exact recovery [11], [21], [40], as well as for approximate recovery in the special case of binary signals [17]. Also, the techniques used to derive the result (standard information inequalities) extend readily to other distortion measures. For example, under Hamming distortion, Proposition 3 can be restated with the function $R(\Omega, \alpha)$ replaced by $H(\Omega) - H(\alpha)$.

It is important to note, however, that Proposition 3 does not reflect the true difficulty of sparsity recovery with small distortions $\alpha$. For example, for $\alpha = 0$, the bound in Proposition 3 is finite even though it has been shown that an infinite sampling rate is needed [13]. Among other things, this discrepancy leaves open the possibility that the total number of recovery errors could grow sublinearly with the length $n$ such that the fraction of errors is asymptotically zero.

To overcome the shortcomings of Proposition 3 outlined above, it is useful to consider the smallest nonzero elements. We use the following definition.

Definition 6. For any $0 \leq \beta \leq 1$, the $\beta$-truncated distribution $F_{\beta}$ of a distribution $F$ is defined as
\[
F_{\beta}(x) := \Pr\{X \leq x | Z = 0\}
\]
where $X$ has distribution $F$ and $Z \in \{0, 1\}$ obeys
\[
\Pr\{Z = 1\} = \begin{cases} 1, & \text{if } |X| > t_{\beta} \\ p_{\beta}, & \text{if } |X| = t_{\beta} \\ 0, & \text{if } |X| < t_{\beta} \end{cases}
\]
with $t_{\beta}$ and $p_{\beta}$ chosen such that $\Pr\{Z = 0\} = \beta$. 

The $\beta$-truncated distribution $F_\beta$ characterizes the empirical distribution of the smallest (in magnitude) $\beta k$ nonzero elements of $x$. For instance, if $F(x)$ has a nonzero density that is flat in a neighborhood around $x = 0$ then $F_\beta$ converges to a uniform distribution as $\beta \to 0$.

The $\beta$-truncated distribution allows us to use the following argument: Suppose that $F$ is continuous, and that for some fraction $\beta \in (0,1)$ a “genie” tells the estimator the values and locations of all the nonzero elements of $x$ whose magnitude exceeds $t_\beta$. Then, there remain approximately $\beta \cdot k$ unknown nonzero elements whose values are characterized by $F_\beta$. It can be shown that estimation of these elements is equivalent to the original problem with altered parameters, and maximizing over all possible $\beta$ gives the following result. The proof is given in Appendix B-B.

**Theorem 3.** The sampling rate distortion function $\rho(\alpha)$ of the vector source $\mathcal{X}(\Omega, F)$ is lower bounded by

$$\rho(\alpha) \geq \max_{\alpha \leq \beta \leq 1} \frac{2(1 - (1 - \beta)\Omega)}{\log(1 + V(\beta \Omega, F_\beta))} (15)$$

where the functions $R(\cdot, \cdot)$ and $V(\cdot, \cdot)$ are defined by (6) and (11) respectively.

In some cases, the right hand side of (15) is maximized by $\beta = 1$ and Theorem 3 is equivalent to Proposition 3. However, as $\alpha$ becomes small, the maximizing value of $\beta$ is eventually less than one and Theorem 3 is much stronger. In particular, it is shown in Proposition 4 in Section V that the scaling of the bound (15) as $\alpha \to 0$ is tight in the sense that it has the same scaling as the upper bounds given in (26). For any distribution $F$, this scaling is at least log$(1/\alpha)$, and thus one consequence of Theorem 3 is that any estimator must have at least a (nonzero) fraction of errors if both the sampling rate and SNR are finite.

As the SNR becomes large, the lower bound in Theorem 3 tends to zero at a rate proportional to $1/\log(1 + P)$ regardless of the distribution $F$. For discrete distributions, this limiting behavior makes sense since the noiseless sampling rate distortion function is equal to zero (see Proposition 2). However, the best known upper bound for sources with continuous valued elements scales like $\Omega + C/\log(1 + P)$ for some constant $C \in (0, \infty)$. Thus, in general there is a disconnect between the upper and lower bounds in the high SNR setting. In the following section, we address this issue for the special case of i.i.d. sampling matrices.

**B. Bounds for I.I.D. Sampling Matrices**

This section derives improved lower bounds for matrices whose elements are independently and identically distributed with zero mean. These results are consistent with the noiseless bound (Theorem 2) given in Section III and provide a tight characterization of the high SNR scaling.

One useful fact about about i.i.d. matrices is that their spectrum (i.e. the empirical distribution of their singular values) converges as $n$ becomes large to a deterministic density known as the Marčenko Pastur law [15]. This convergence allows us to more accurately describe certain aspects of our bounds.

To begin, we present the following analog of Proposition 3 which serves as a building block for our further results. The proof is given in Appendix B-B.

**Proposition 4.** If the sampling matrix is i.i.d., then the sampling rate distortion function $\rho(\alpha)$ of the vector source $\mathcal{X}(\Omega, F)$ must satisfy

$$G(\rho(\alpha), \Gamma(\Omega, F)) \geq R(\Omega, \alpha) (16)$$

where the functions $R(\cdot, \cdot)$ and $\Gamma(\cdot, \cdot)$ are defined by (6) and (11) respectively, and

$$G(r, \gamma) := \frac{1}{2} \left[ r \log \left( 1 + \gamma - \xi(r, \gamma) \right) + \log \left( 1 + \gamma - \xi(r, \gamma) \right) - \frac{1}{2} \xi(r, \gamma) \right] (17)$$

with $\xi(r, \gamma) = \frac{1}{2} \left( \sqrt{\gamma(\sqrt{r} + 1)^2 + 1} - \sqrt{\gamma(\sqrt{r} - 1)^2 + 1} \right)^2$.

The difference between Propositions 4 and 3 is the function $G(r, \gamma)$ which obeys $G(r, \gamma) \leq r \log(1 + \gamma)$. At high SNR, this difference is relatively small and thus Proposition 4 suffers both the low distortion and high SNR shortcomings of Proposition 3.

The next result significantly strengthens Proposition 4 in the special case where $F$ is Gaussian. The proof is given in Appendix B-C.

**Proposition 5.** Suppose that $F$ is a Gaussian distribution with mean $\mu_F$ and variance $\sigma^2_F$. If the sampling matrix is i.i.d., then the sampling rate distortion function $\rho(\alpha)$ of the vector source $\mathcal{X}(\Omega, F)$ must satisfy

$$G(\rho(\alpha), V(\Omega, F)) \geq R(\Omega, \alpha) + \Omega G(\rho(\alpha)/\Omega, \Omega \sigma^2_F) (18)$$

where the functions $R(\cdot, \cdot)$, $V(\cdot, \cdot)$, and $G(\cdot, \cdot)$ are defined by (6), (11), and (17) respectively.

The term $G(\rho(\alpha)/\Omega, \Omega \sigma^2_F)$ on the right hand side of (18) increases with the variance $\sigma^2$. As a result, Proposition 5 is significantly stronger than the previously stated bounds in the high SNR setting. To extend these improvements to other distributions, we use the following property.

**Definition 7.** The entropy power of a source $\mathcal{X}(\Omega, F)$ is defined by

$$V_h(\Omega, F) = \frac{1}{2\pi} \exp \left( 2h(F) \right) (19)$$

if $F$ has a density where $h(F)$ denotes differential entropy of the distribution $F$. If $F$ does not have a density, then $V_h(\Omega, F) \equiv 0$.

The entropy power obeys $0 \leq V_h(\Omega, F) \leq V(\Omega, F)$ with equality on the left if $F$ does not have a density and equality on the right if and only if $F$ is a zero mean Gaussian. Moreover, the ratio $V_h(\Omega, F)/V(\Omega, F)$ is equal to the function $\theta(\Omega, F)$ given in Definition 4.

Using the entropy power, it is possible to give an improved lower bound for any distribution $F$ with a density. The proof of the following result is given in Appendix B-C.
**Proposition 6.** If the sampling matrix is i.i.d., then the sampling rate distortion function $\rho(\alpha)$ of the vector source $\mathcal{X}(\Omega, F)$ must satisfy

$$G(\rho(\alpha), V(\Omega, F)) \geq R(\Omega, \alpha) + \Omega V(\rho(\alpha), V(\Omega, F))$$

(20)

where the functions $R(\cdot, \cdot)$, $V(\cdot, \cdot)$, $G(\cdot, \cdot)$ and $V_h(\cdot, \cdot)$ are defined by (6), (11), (17), and (19) respectively, and

$$V(r, \gamma) := \begin{cases} \frac{e}{\gamma} \log (1 + \frac{\Delta(r)}{\gamma}) & r \leq \gamma \\ \frac{1}{\gamma} \log (1 + \frac{\Delta(r)}{\gamma}) & r > \gamma \end{cases}$$

(21)

with $\Delta(\cdot)$ given by (6).

The function $V(r, \gamma)$ obeys $V(r, \gamma) \leq G(\rho(\alpha), V(\Omega, F))$ with $V(r, \gamma)/G(r, \gamma) \rightarrow 1$ as $\gamma \rightarrow \infty$, and thus the high SNR gains of Proposition 6 are comparable to those of Proposition 8. Taking the limit as $V_h(\Omega, F) \rightarrow \infty$ recovers the bound for the noiseless setting given in Theorem 2.

Additionally, the concavity of the logarithm gives $V(r, \gamma) \geq \min\{r, 1\} \frac{e}{\gamma} \log (1 + \frac{\Delta(r)}{\gamma})$. Applying this inequality to Proposition 6 gives the following simplified, although necessarily weaker, lower bound,

$$\rho(\alpha) \geq \frac{2R(\Omega, \alpha) + \min\{\rho(\alpha), \Omega\} \log (1 + \frac{2V_h(\Omega, F)}{\log (1 + V(\Omega, F))})}{\log (1 + V(\Omega, F))}$$

(22)

Taking the limit as $V_h(\Omega, F) \rightarrow \infty$ leads to the following simplified bound for the noiseless setting,

$$\rho_0(\alpha) \geq \min\left\{\Omega, \frac{2R(\Omega, \alpha)}{1 + \log (1/\theta(\Omega, F))}\right\}$$

(23)

Lastly, combining the low distortion improvement used in Theorem 1 with the high SNR improvement of Proposition 6 gives to the following bound. The proof is given in Appendix B-E.

**Theorem 4.** If the sampling matrix is i.i.d., then the sampling rate distortion function $\rho(\alpha)$ of the vector source $\mathcal{X}(\Omega, F)$ must satisfy

$$G(\rho(\alpha), V(\beta\Omega, F_\beta)) \geq R\left(\frac{\beta\Omega}{1 - (1 - \beta)^\Omega}, \beta \right) + \frac{\beta\Omega}{1 - (1 - \beta)^\Omega} V\left(\rho(\alpha), V_h(\beta\Omega, F_\beta)\right)$$

(24)

for all $\alpha \leq \beta \leq 1$ where the functions $R(\cdot, \cdot)$, $V(\cdot, \cdot)$, $G(\cdot, \cdot)$, $V(\cdot, \cdot)$, $V_h(\cdot, \cdot)$, and $F_\beta$ are defined by (6), (11), (17), (21), (19), and (14) respectively.

For the most part, Theorem 4 represents our strongest lower bound. Strictly speaking though, Proposition 5 may be slightly stronger than Theorem 4 in the special case where $F$ is Gaussian and the maximization of (23) occurs at $\beta = 1$.

**V. SCALING BEHAVIOR**

Although the bounds given in the previous two sections are computable, their complexity makes it difficult to understand how in a scaling sense the sampling rate distortion function $\rho(\alpha)$ depends on the distortion $\alpha$ or various properties of the source $\mathcal{X}(\Omega, F)$ such as the sparsity rate $\Omega$ or the power $\rho(\alpha)$ becomes small and how $\rho(\alpha)$ converges to the noiseless rate $\rho_0(\alpha)$ as the SNR becomes large.

One key property of the source is the power $P(\Omega, F)$. To describe scalings of the power we use $\mathcal{X}(\Omega, F; P)$ to denote a source characterized by a distribution $F$ that is scaled to have power $P$. Another key property of the source is the following.

**Definition 8.** The decay rate $L \in [0, \infty]$ of a distribution function $F$ is defined as

$$L := \lim_{\epsilon \rightarrow 0} \frac{\log \epsilon}{\log (F(\epsilon) - F(-\epsilon))}$$

(25)

if the limit exists.

The decay rate $L$ is independent of the power of the distribution $F$ and characterizes the relative size the smallest nonzero elements drawn from a source $\mathcal{X}(\Omega, F)$. For instance, if $X$ is a random variable with decay rate $L < \infty$, and we define

$$x_\epsilon = \inf \{x \geq 0 : \Pr(|X| \leq x) \geq \epsilon\},$$

then $e^{-L} \cdot x_\epsilon \rightarrow c$ as $\epsilon \rightarrow 0$ for some $c \in (0, \infty)$. The decay rate is $L = 0$ if $X$ is bounded away from zero and $L = \infty$ if and only if $\Pr(X = 0) > 0$. Thus, $L$ is finite for any $F \in \mathcal{F}_0$.

One useful property of the decay rate is that it can be used to bound the relative power of the $\beta$-truncated distribution given in Definition 7. The proof of the following result is given in Appendix C-A.

**Lemma 2.** Given any distribution function $F \in \mathcal{F}_0$ with decay rate $L$, there exist constants $0 < C_F^- < C_F^+ < \infty$ such that

$$C_F^- \cdot \beta^{2L} \leq \frac{P(\Omega, F_\beta)}{P(\Omega, F)} \leq C_F^+ \cdot \beta^{2L}$$

(26)

for any $0 \leq \beta \leq 1$.

Using the above properties, we are able to provide the following simplified version of Theorem 7. The proof is given in Appendix C-B.

**Proposition 7.** Given any distribution $F \in \mathcal{F}_0$, there exists a constant $C_F > 0$ such that the sampling rate distortion function $\rho(\alpha)$ of the vector source $\mathcal{X}(\Omega, F; P)$ is lower bounded by

$$\rho(\alpha) \geq C_F \cdot \frac{\alpha \Omega \log(\frac{1}{\alpha})}{\log (1 + \alpha^{2L+1} P)}$$

(27)

for all distortions $\alpha \in (0, 1/4)$ where $F$ has decay rate $L$.

Combining Proposition 7 with bounds from (26) provides a tight characterization of the scaling behavior of $\rho(\alpha)$ as $\alpha$ becomes small.

**Proposition 8.** Given any distribution function $F \in \mathcal{F}_0$ and sparsity rate $\Omega$, there exist constants $0 < C_{F, \Omega}^- \leq C_{F, \Omega}^+ < \infty$ such that the sampling rate distortion function $\rho(\alpha)$ of the vector source $\mathcal{X}(\Omega, F)$ obeys

$$C_{F, \Omega}^- \cdot \Omega^{-2L} \log (\frac{1}{\alpha}) \leq \rho(\alpha) \leq C_{F, \Omega}^+ \cdot \Omega^{-2L} \log (\frac{1}{\alpha})$$

(28)

for all distortions $\alpha \in (0, 1/4)$ where $F$ has decay rate $L$. 
Proposition 9 resolves a gap in the existing literature: It has been known that perfect sparsity pattern recovery in the presence of noise requires an infinite sampling rate, and it has also been known that for any fraction $\alpha > 0$ of errors, a finite sampling rate is sufficient. This leaves open the case of a vanishing fraction of errors. Proposition 9 shows that the latter is as hard as perfect recovery: again, an infinite sampling rate is needed.

Next, we consider the setting of i.i.d. sampling matrices and provide a simplified version of Theorem 4. The proof is given in Appendix C-C.

**Proposition 9.** If the sampling matrix is i.i.d., then there exists a constant $C > 0$ such that the sampling rate distortion function $\rho(\alpha)$ of the vector source $X(\Omega, F; P)$ is lower bounded by

$$\rho(\alpha) \geq \Omega + C \cdot \frac{\Omega \log(\frac{\alpha}{\Omega})}{\log(1 + \rho)}$$

for all distortions $\alpha \in (0, 1/4)$ that satisfy

$$\theta(\Omega, F) > \exp\left(1 - \frac{1}{2}R(\Omega, \alpha)\right)$$

where $\theta(\Omega, F)$ and $R(\Omega, \alpha)$ are defined by (5) and (6) respectively.

Combining Proposition 9 with bounds from [26] provides tight characterization of the scaling behavior of $\rho(\alpha)$ as a function of the SNR.

**Proposition 10.** If the sampling matrix is i.i.d., then given any distribution $F \in \mathcal{F}_0$, sparsity rate $\Omega$, and distortion $\alpha \in (0, 1/4)$ that satisfy Inequality (30), there exist constants $0 < C_{P, \Omega, \alpha}^- \leq C_{P, \Omega, \alpha}^+ < \infty$ such the sampling rate distortion function $\rho(\alpha)$ of the vector source $X(\Omega, F; P)$ obeys

$$\frac{C_{P, \Omega, \alpha}^-}{\log(1 + \rho)} \leq \rho(\alpha) - \Omega \leq \frac{C_{P, \Omega, \alpha}^+}{\log(1 + \rho)}$$

One may think of the difference $\rho(\alpha) - \Omega$ as the excess sampling rate due to noise. Proposition 10 shows that if $\theta(\Omega, F)$ is large relative to the distortion $\alpha$ then this excess rate scales like $1/\log(1 + \rho)$ for all SNR.

**VI. EXAMPLES AND ILLUSTRATIONS**

This section provides specific examples and illustrations of the bounds developed in Section IV-B for i.i.d. sampling matrices. We first highlight various aspects of the bounds using sources characterized by a single distribution function $F$. We then show how the bounds derived for these sources can be applied to a more general source characterized by a set of distribution functions $\mathcal{F}$. The properties of the distributions used in this section that are needed to compute the bounds are described explicitly in Appendix D.

**A. Comparison of Lower Bounds**

To begin, we consider a vector source $X(\Omega, F)$ characterized by a zero mean Gaussian distribution. The properties of this distribution needed to compute the bounds in this paper are given explicitly in the following example.

**Example 1.** Suppose that $F$ is a Gaussian distribution with zero mean and variance $\sigma_F^2$. Then, the mean, variance, and entropy of the $\beta$-truncated distribution $F_\beta$ are given by

$$\mu_{F_\beta} = 0$$

$$\sigma_{F_\beta}^2 = r_\beta \sigma_F^2$$

$$h(F_\beta) = \frac{1}{2} \left[\log(2\pi\beta^2 \sigma_F^2) + r_\beta\right]$$

respectively, where $r_\beta = 1 - (t_\beta/\beta)(2/\pi)^{1/2} \exp(-t_\beta^2/2)$ with $t_\beta = Q^{-1}(1 - \beta/\beta)$, and $Q(x) = \int_x^\infty (2\pi)^{-1/2} \exp(-x^2/2) dx$.

Since the density of the Gaussian distribution is flat and continuous around $x = 0$, the decay rate is $L = 1$ and the power $P(\Omega, F_\beta)$ scales like $\beta^2$ for small $\beta$. Applying a Taylor expansion to the expression for $\sigma_{F_\beta}^2$ given in the above example gives the more precise characterization

$$\lim_{\beta \to 0} \frac{1}{\beta^2} P(\Omega, F_\beta) = \frac{\pi}{6} \Omega.$$

The bounds in in Propositions 4, 5 and 6 are shown in Figure 2 which plots the distortion sampling rate function $\alpha(\rho)$ as a function of the sampling rate $\rho$ at high SNR (50 dB). Also shown is a corresponding upper bound from the companion paper [26].

Figure 2 shows that Propositions 4 and 5 are significantly stronger than Proposition 6, and are relatively close to the upper bound, especially for small (but nonzero) distortions. Although the Proposition 4 is strictly stronger than Proposition 6, the difference between the bounds too small to be discerned. Together, the upper and lower bounds show that the slope of $\rho(\alpha)$ is very steep over a range of $\rho$. Hence in certain settings, a small increase in the sampling rate provides a relatively large increase in accuracy. This behavior is qualitatively consistent with the noiseless setting shown in Section III.

One aspect that is difficult to see in Figure 2 is what happens when $\alpha$ is small. This setting is illustrated more clearly in Figure 3 which compares Proposition 6, Theorem 4 and the upper bound from [26] at a lower SNR (0 dB). As $\alpha$ becomes small, Theorem 4 is significantly stronger than Proposition 6 and is comparable to the upper bound. Together, the upper and lower bounds show that if $\alpha$ is small (relative to the SNR) then a small decrease in $\alpha$ requires a relatively large increase in $\rho$.
B. Low Distortion Behavior

Next, we consider a source \( \mathcal{X}(\Omega, F) \) characterized by a continuous uniform distribution.

**Example 2.** Suppose that \( F \) is a continuous uniform distribution with mean \( \mu_F > 0 \) and variance \( \sigma_F^2 \). Then, the mean, variance, and entropy of the \( \beta \)-truncated distribution \( F_\beta \) are given by

\[
\begin{align*}
\mu_{F_\beta} &= \max \{ 0, \mu - (1 - \beta) \sqrt{3} \sigma_F^2 \} \\
\sigma_{F_\beta}^2 &= \beta^2 \sigma_F^2 \\
h(F_\beta) &= \frac{1}{2} \log(12 \beta^2 \sigma_F^2)
\end{align*}
\]

respectively.

The support of the uniform distribution is bounded away from zero if and only if \( \mu_F^2 > 3 \sigma_F^2 \), and the decay rate is thus

\[
L = \begin{cases} 
1, & \text{if } \mu_F^2 \leq 3 \sigma_F^2 \\
0, & \text{if } \mu_F^2 > 3 \sigma_F^2
\end{cases}
\]

A comparison between the cases \( \mu_F^2/\sigma_F^2 = 2.3 \) and \( \mu_F^2/\sigma_F^2 = 4 \) is shown in Figure 4 which plots the bounds given Theorem 4 and [26] using a log-log scale. The low distortion behavior shown in Figure 4 is consistent with Proposition 8 and demonstrates the impact of the decay rate on the scaling of \( \rho(\alpha) \) as \( \alpha \) becomes small.

C. Bounds for General Sources

For our last example, we consider a more general source that corresponds to a set of distributions obeying certain constraints. In particular, we consider the source \( \mathcal{X}(\Omega, F(\eta, \gamma)) \) where, for any parameters \( \eta \in [0, 1] \) and \( \gamma \in (0, \infty) \), we define \( F(\eta, \gamma) \subset F_0 \) to be the set of all distributions with power \( \gamma \) and a lower bound \( \sqrt{\eta} \) on the magnitude of any realization.

The constraints imposed by this source correspond directly to the assumptions typically used in literature for exact recovery [3, 12–15, 18, 20, 21]. However, one difference between this paper and previous work, is that our framework allows us to bound this source explicitly in terms of sources characterized by a single distribution. More precisely, we take advantage of the simple fact that the sampling rate distortion function of \( \mathcal{X}(\Omega, F(\eta, \gamma)) \) is lower bounded by the sampling rate distortion function of \( \mathcal{X}(\Omega, F) \) for any candidate distribution \( F \in F(\eta, \gamma) \).

In the following examples, two different candidate distributions are provided. Each of these distributions has power \( \gamma \) and a lower bound \( b \) on the magnitude of any realization and is therefore an element of the set \( F(\eta, \gamma) \) provided that \( b \geq \sqrt{\eta} \).

**Example 3 (Point-Mass).** Suppose that \( F \) is the distribution function of a discrete random variable with probability mass function

\[
p(x) = \begin{cases} 
\epsilon/2, & \text{if } x^2 = [\gamma - (1 - \epsilon)b^2]/\epsilon \\
(1 - \epsilon)/2, & \text{if } x^2 = b^2
\end{cases}
\]

where \( b^2 \in [0, \gamma] \) and \( \epsilon \in (0, 1) \). Then, the mean and variance of the \( \beta \)-truncated distribution \( F_\beta \) are given by

\[
\begin{align*}
\mu_{F_\beta} &= 0 \\
\sigma_{F_\beta}^2 &= \begin{cases} 
\gamma - \frac{1 - \beta}{\beta \epsilon} (1 - \epsilon) b^2 & \beta > 1 - \epsilon \\
b^2 & \beta \leq 1 - \epsilon
\end{cases}
\]

respectively.

**Example 4 (Sliced-Gaussian).** Suppose that \( F \) is the distribution function of a random variable \( X = Z + \text{sgn}(Z) b \) where \( b^2 \in (0, \gamma) \) and \( Z \) has a zero mean Gaussian distribution with variance \( \sigma_Z^2 \) satisfying

\[
b^2 + (2/\pi)^{-1/2} e b \sigma_z + \sigma_Z^2 = \gamma.
\]

Then, the mean, variance, and entropy of the \( \beta \)-truncated distribution \( F_\beta \) are given by

\[
\begin{align*}
\mu_{F_\beta} &= 0 \\
\sigma_{F_\beta}^2 &= b^2 + r_\beta \sigma_Z^2 + \bar{r}_\beta b \sigma_z \\
h(F_\beta) &= \frac{1}{2} \left[ \log(2 \pi \beta^2 \sigma_Z^2) + r_\beta \right]
\end{align*}
\]

respectively where \( \bar{r}_\beta = (2/\pi)^{-1/2} [1 - \exp(-t_\beta^2/2)]/\beta \) and \( r_\beta \) and \( t_\beta \) are defined as in Example 1.
To lower bound the sampling rate distortion function of the source \( X(\Omega, F(\eta, \gamma)) \) we apply Theorem 4 to the two sources characterized by the above distributions. The corresponding bounds are shown as a function of the distortion \( \alpha \) in Figures 5 and 6; the maximum of the bounds is shown as a function of the SNR in Figure 1. In all cases, the bounds for the point-mass distribution correspond to the limit \( \epsilon \to 0 \). Also shown are upper bounds from [26].

One difference between the two candidate distributions is that the entropy power of the sliced-Gaussian source increases with the SNR whereas the entropy power of the point-mass source is zero. Not surprisingly, the sliced-Gaussian provides a significantly stronger bound for the high SNR (50 dB) setting shown in Figure 5. In this case, our bounds support the intuition that the difficulty of estimation in the high SNR setting is due to entropy of the nonzero values.

Another difference between the two candidate distributions is that the power of the \( \beta \)-truncated point-mass source is significantly larger than the power of the \( \beta \)-truncated sliced-Gaussian source for small \( \beta \). As a result, the point-mass distribution provides a significantly stronger bound in the low SNR (-20 dB) setting shown in Figure 6. In this case, our bounds support the intuition that the difficulty of estimation in the low distortion setting is due to the size of the nonzero elements.

**APPENDIX A**

**Proofs for Arbitrary Sources**

This appendix gives the proofs of Proposition 1 and Theorem 1. For the lower bounds, we assume that the sparsity pattern is a random set \( S \) distributed uniformly over all subsets of size \( k \). Since the probability of error for the vector source \( X(\Omega) \) corresponds to the worst case prior distribution on \( x \), this assumption provides a valid lower bound.

We provide the following comments on notation: For sets \( s, u \), we use \( s \setminus u \) to denote the difference set \( \{ s \in s : s \notin u \} \). We assume throughout that any sparsity pattern \( s \) belongs to the set of all subsets of \( \{1, 2, \ldots, n\} \) of size \( k \) where \( k = \lfloor \Omega \cdot n \rfloor \). Also, for any matrix \( M \), we use the notation \( R(M) \) to denote the range space \( M \).

**A. Proof of Proposition 1**

**Upper Bound:** Let \( x \) be a fixed vector with sparsity pattern \( s \) of size \( k \), and let \( A \) be a random \( k + 1 \times n \) sampling matrix, drawn independently of \( x \), with elements i.i.d. Gaussian. Given the set \( (Ax, A, k) \), the estimator knows that the true sparsity pattern must be an element of the set

\[
S(A) = \{ s' : Ax \in R(Aw) \}. \tag{32}
\]

We observe that for any sparsity pattern \( s' \neq s \),

\[
\Pr\{Ax \in R(Aw)\} = \Pr\{Axu \in R(Aw)\} = 0
\]

where \( u = s \setminus s' \), since the \((k+1)\)-dimensional Gaussian vector \( Axu \) is independent of the \( k \)-dimensional linear subspace defined by \( R(Aw) \). Thus, with probability one, \( S(A) = \{ s \} \) and exact recovery is attained.

**Lower Bound:** Let \( x \) be a vector whose sparsity pattern \( S \) is distributed uniformly over all subsets of size \( k \) and whose nonzero values \( \{x_i\}_{i \in S} \) are chosen arbitrarily, and let \( A \) be a \( k \times n \) random matrix, drawn independently of \( x \), whose entries are i.i.d. with zero mean and unit variance. Given the set \( (Ax, A, k) \), the set of admissible sparsity patterns is given by the set \( S(A) \) defined in (32).

If the elements of \( A \) are continuous random variables, then it is straightforward to see that

\[
\Pr\{\text{rank}(A_s) = m\} = 1 \quad \text{for all} \quad s,
\]

and thus \( |S(A)| = \binom{n}{k} \) almost surely. In this case, estimation of \( S \) is equivalent to selecting an estimate \( \hat{S} \) uniformly at random, and using a version of Hoeffding’s inequality for sampling without replacement [46] shows that \( d(\hat{S}, S) \to 1 - \Omega \) in probability as \( n \to \infty \).

However, if the elements of \( A \) are discrete random variables then there may exist sparsity patterns \( s \) for which \( \text{rank}(A_s) < k_n \). The existence of such sets means that it may be possible for the estimator to discard certain sparsity patterns from consideration, and thus outperform the random guessing estimator outlined above.

To show that the effects of discrete distributions discussed above are insignificant in the high dimensional setting, we...
define
\[ \gamma(A) := \left| \{ s : \text{rank}(A_s) < k \} \right| \]
to be the fraction of degenerate submatrices of a realization \( A = A \), and observe that the admissible set \( S(A) \) is lower bounded by
\[ |S(A)| \geq (1 - \gamma(A)) \binom{n}{k}. \]

Furthermore, the distortion of any estimate \( \hat{S} \) can be lower bounded by considering the case where \( \hat{S} \) is uniform on \( S(A) \) and the sparsity patterns \( s \notin S(A) \) happen to be the sparsity patterns with the the greatest distortion \( d(S, s) \) from the true sparsity pattern \( S \). This argument gives the lower bound
\[ \mathbb{E}[d(S, \hat{S}) | A = A] \geq 1 - \Omega - \gamma(A). \]
for any estimator \( \hat{S} \).

Next, using the linearity of expectation gives
\[ \mathbb{E}[\gamma(A)] = \mathbb{E}\left[ \frac{1}{\binom{n}{k}} \sum_s \mathbb{1}(\text{rank}(A_s) < k) \right] = \Pr\{ \text{rank}(A_{[k]}) < k \} \]
where \( A_{[k]} \) denotes the first \( k \) columns of \( A \). Using only the fact that elements of \( A \) are independent and have finite and nonzero second moments, it is possible to show (see for example Theorem 2.1 in [47]) that \( \Pr\{ \text{rank}(A_{[k]}) < k \} \to 0 \) as \( n \to \infty \). Hence, we conclude that
\[ \mathbb{E}[d(S, \hat{S})] \to 1 - \Omega \quad \text{as} \quad n \to \infty. \]
for any i.i.d. sampling matrix. By Markov’s inequality, this convergence in expectation is sufficient to prove that the probability of error with respect to distortion \( \alpha < 1 - \Omega \) is bounded away from zero for any sampling rate \( \rho < \Omega \).

**B. Proof of Theorem [7]**

**Upper Bound:** Let \( x \) be a fixed vector with sparsity pattern \( s \) of size \( k \), and let \( B \) be a random \( m \times n \) matrix, drawn independently of \( x \), with elements i.i.d. Gaussian and \( m = \lceil \rho \cdot n \rceil \) where \( \rho < \Omega \). Additionally, for any \( \epsilon > 0 \), let \( U \) be distributed uniformly over all subsets of \( \{1, 2, \ldots, n\} \) of size \( \lceil 1 - (1 - \epsilon)\rho/\Omega \rceil \cdot n \), independently of \( x \) and \( B \), and let \( A \) be the \( m \times n \) random matrix whose columns obey
\[ A_i = \begin{cases} 0, & \text{if } i \in U \\ B_i, & \text{if } i \notin U \end{cases} \]
where \( 0 \) denotes a column vector of zeros.

Given the set \( \{ Ax, A, k \} \), suppose that the estimator performs the following two-stage procedure. First, the estimator identifies the smallest integer \( k_0 \) such that \( Ax \in \mathcal{R}(A_s) \) for some subset \( s_0 \subset \{1, 2, \ldots, n\} \setminus U \) of size \( k_0 \). If there exist multiple subsets of size \( k_0 \) satisfying this condition, the estimator declares and error. Otherwise, the estimator begins with the unique set \( u_0 \) and constructs an estimate \( \hat{S} \) by selecting \( k - k_0 \) additional indices uniformly at random and without replacement from \( U \).

To show that the procedure outlined above achieves the upper bound, we define the event
\[ \mathcal{E} = \left\{ \left| \frac{1}{n} |s| \right| - (1 - \epsilon)\rho < \epsilon^2 \cdot \rho \right\} \]
and note the \( \Pr \{ \mathcal{E}^c \} \to 0 \) as \( n \to \infty \) by Hoeffding’s inequality for sampling without replacement [46]. Conditioned on the event \( \mathcal{E} \), the number nonzero elements in \( Ax \) is less then \( m \), and the same arguments we used in the proof of Proposition [1] show that with probability one, \( s_0 = s \setminus U \) is the unique smallest subset for which \( Ax \in \mathcal{R}(A_s) \).

Thus, conditioned on the event \( \mathcal{E} \), the expected distortion of the estimate \( \hat{S} \) is due entirely to the number of errors in the \( |s \cap U| \) remaining elements drawn from \( U \), and can be lower bounded as
\[ \mathbb{E}[d(s, \hat{S}) | \mathcal{E}] = \frac{1}{k} \cdot \mathbb{E}\left[ (k - |s \setminus U|) \left( 1 - \frac{k - \|s \setminus U\|}{k} \right) \right] \leq (1 - \frac{\rho}{\Omega}) (1 - \Omega) + \epsilon' \]
where \( \epsilon' \to 0 \) as \( \epsilon \to 0 \). Using Hoeffding’s inequality for sampling without replacement [46] we conclude that for any \( \rho < \Omega \) and \( \epsilon > 0 \), there exists a sequence of random \( \lceil \rho \cdot n \rceil \times n \) sampling matrices \( A \) such that
\[ \Pr \{ d(s, \hat{S}) > (1 - \frac{\rho}{\Omega}) (1 - \Omega) + \epsilon \} \to 0 \]
which completes the proof.

**Lower Bound:** Let \( x \) be a vector whose sparsity pattern is a random set \( S \) is distributed uniformly over all subsets of size \( k \) and whose nonzero values \( \{ x_i \}_{i \in S} \) are chosen arbitrarily, and let \( A \) be a random \( m \times n \) sampling matrix, drawn independently of \( x \), with an arbitrary distribution with \( \mathbb{E}[\text{tr}(A^T A)] < \infty \) and \( m \leq k \). Furthermore, suppose that in addition to the set \( \{ Ax, A, k \} \), a “genie” provides the estimator with a subset \( U \subseteq S \) that indexes \( |U| = \text{rank}(A_s) \) linearly independent columns of the matrix \( A \). Since \( \mathcal{R}(A_U) = \mathcal{R}(A_S) \), the set \( \{ U, A, k \} \) is a sufficient statistic for estimation of \( S \).

To lower bound the performance of the setting outlined above, we define the following quantities for any realization \( A = A \) and set \( s \):
\[ r_s := \text{rank}(A_s) \]
\[ t_s := |i \in s^c : A_i \in \mathcal{R}(A_s)|. \]
If \( r_s = k \), then \( U = S \) and exact recovery is achieved. However, if \( r_s < k \) then the estimator must choose \( k - r_s \) additional indices corresponding to the set \( S \setminus U \). Since the estimator knows the range space \( \mathcal{R}(A_s) \), it may exclude from consideration any index \( i \) for which \( A_i \notin \mathcal{R}(A_s) \). Thus, the number of admissible indices is given by \( k - r_s + t_s \). Since it is impossible to distinguish which elements in the admissible set correspond to \( S \), the expected distortion of any estimate \( \hat{S} \) is proportional to the fraction \( t_s/(k - r_s + t_s) \) and is given by
\[ \mathbb{E}[d(S, \hat{S}) | S] = \frac{1}{k} \cdot \frac{t_s \cdot (k - r_s)}{t_s + (k - r_s)}. \]
(33)
To evaluate the distortion given in (33) in terms of the random sparsity pattern \( S \), we need the following result.
Lemma 3. For any sampling matrix $A$, 
\[ E[r_s] \geq (n/k - 1)(k - E[r_s]). \] (34)
Proof: Let $i$ be a random index distributed uniformly over the set $S^c$ and define 
\[ p_{up} = \Pr \{ A_i \notin R(A_S) \} = \Pr \{ r_{S \cup i} = r_s + 1 \} = E[r_{S \cup i}] - E[r_s]. \]
Additionally, let $V$ be distributed uniformly over all subset of \{1, 2, \ldots, $n \} of size $k+1$, let $j$ be distributed uniformly over $V$, and define 
\[ p_{down} = \Pr \{ A_j \notin R(A_V) \} = \Pr \{ r_{V \cup j} = r_V - 1 \} = E[r_{V \cup j}] - E[r_v]. \]
Since $S \cup i$ and $V \cup j$ are equal in distribution, 
\[ p_{up} = E[r_v] - E[r_s] = p_{down}. \] (35)
Moreover, for any set $V$ there exists at most $r_v$ indices $j \in V$ for which $r_{V \cup j} = r_V - 1$, and hence 
\[ p_{down} \leq \frac{E[r_v]}{k+1}. \] (36)
Combining (35) and (36) gives 
\[ E[r_v] \leq \left( \frac{k+1}{k} \right) E[r_s], \]
and 
\[ p_{up} \leq \frac{E[r_s]}{k}. \]
Noting that $(n-k)(1-p_{up})$ is equal to the left hand side of (34) completes the proof.

Next, for any $\epsilon > 0$, we define the events 
\[ E_1 = \{ |d(S, \hat{S}) - E[d(S, \hat{S})]| \leq \epsilon \}, \]
\[ E_2 = \{ |r_s - E[r_s]| \leq \epsilon \cdot (k - E[r_s]) \}, \]
\[ E_3 = \{ t_s \geq (1-\epsilon) \cdot E[t_s] \}. \]
Conditioned on $E_1 \cap E_2 \cap E_3$ the distortion $d(S, \hat{S})$ can be lower bounded as follows:
\[ d(S, \hat{S}) \geq \frac{1}{k} \left[ t_s^{-1} + (k - r_s)^{-1} \right]^{-1} - \epsilon \] (37)
\[ \geq \frac{1}{k} \left[ (E[t_s])^{-1} + (k - E[r_s])^{-1} \right]^{-1} (1 - \epsilon - \epsilon \] (38)
\[ \geq (1 - E[r_s]/k)(1 - k/n) - 2\epsilon \] (39)
\[ \geq (1 - \min\{m, k\}/k)(1 - k/n) - 2\epsilon \] (40)
where (37) follows from $E_1$ and (33); (38) follows from $E_2$ and $E_3$; (39) follows from Lemma 3 and (40) follows from the fact that $r_s \leq \min\{m, k\}$. Hence, for any $\epsilon > 0$, and sampling rate $\rho < \Omega$, the distortion obeys 
\[ \Pr \{ d(S, \hat{S}) \leq (1 - \frac{\rho}{\Omega})(1 - \Omega) - 2\epsilon \} \leq \frac{3}{i=1} \Pr \{ E_i \}. \] (41)
To conclude, we show that the right hand side of (41) is bounded away from one for all $n$. For the event $E_1$, the convergence $\Pr \{ E_1 \} \rightarrow 0$ as $n \rightarrow \infty$ follows directly from Hoeffding’s inequality for sampling without replacement [40]. To handle the event $E_2$, we use the following generalization of McDiarmid’s inequality to sampling without replacement given in [48].

Lemma 4. Let $Z_1, Z_2, \ldots, Z_k$ be a sequence of random variables, sampled from an underlying set $Z$ of $n$ elements without replacement, and let $\phi : Z^k \rightarrow \mathbb{R}$ be a symmetric function such that for all $i \in \{1, 2, \ldots, k\}$ and for all $z_1, z_2, \ldots, z_k \in Z$ and $z_1', z_2', \ldots, z_k' \in Z$, 
\[ |\phi(z_1, \ldots, z_k) - \phi(z_1', \ldots, z_{i-1}, z_i', z_{i+1}, \ldots, z_k) | \leq c. \]
Then, for all $\epsilon > 0$, 
\[ \Pr \{ |\phi(Z) - E[\phi(Z)]| \geq \epsilon \} \leq 2 \exp \left( \frac{-2\epsilon^2 \gamma(k, n-k) c^2}{\gamma(k, n-k) \cdot \frac{1}{k}} \right) \]
where $\gamma(k, n-k) = \frac{k(n-k)}{2n(n-1)} \cdot \frac{1}{k(\max(k, n-k))}$.
To apply Lemma 4 to our setting we let $Z = \{1, 2, \ldots, n\}$ and $\phi(z) = \text{rank}(A_z)$. Since replacing any index in $z$ can change the rank of $A_z$ by at most one, $c = 1$, and thus, $\Pr \{ E_2 \} \rightarrow 0$ as $n \rightarrow \infty$.

Lastly, consider the event $E_3$. Ideally, it would be preferable to show that $\Pr \{ E_3 \} \rightarrow 0$ and thus conclude that the right hand side of (41) tends to zero and $n \rightarrow \infty$. However, such convergence appears to be nontrivial, and for the current version of the proof we instead show that $\Pr \{ E_3 \}$ is bounded away from one. To do so, we apply Markov’s inequality to the positive random variable $n - k - t_s$ to attain 
\[ \Pr \{ E_3 \} \leq \frac{\epsilon E[t_s]}{n-k} \leq 1 - \epsilon \cdot (1 - E[r_s])/k \leq 1 - \epsilon \cdot (1 - m/k) \]
where the last two steps follow from Lemma 3 and the fact that $r_s \leq m$. Thus, we have shown that 
\[ \limsup_{n \rightarrow \infty} \sum_i \Pr \{ E_i \} < 1 \]
which completes the proof.

APPENDIX B
PROOFS FOR SOURCES WITH A DISTRIBUTION

This appendix gives the proofs of Propositions 3-6 and Theorems 2-4. Each of these results provides a lower bound on the sampling rate distortion function $\rho(\alpha)$ of a vector source $\mathcal{X}(\Omega, F)$ characterized by a single distribution $F$. However, instead of analyzing the source $\mathcal{X}(\Omega, F)$ directly, we find it convenient to consider the following stochastic analog.

Definition 9. For each integer $n$ the stochastic vector source $\mathcal{X}_n(\Omega, F)$ outputs a random vector $X \in \mathbb{R}^n$ whose sparsity pattern is distributed uniformly over all sets of \{1, 2, \ldots, $n\} of size \lfloor \Omega \cdot n \rfloor$, and whose nonzero elements $(X_i)_{i \in S}$ are i.i.d. with distribution function $F$. 

\[
\begin{align*}
E[r_s] &\geq (n/k - 1)(k - E[r_s]). \\
\Pr \{ d(S, \hat{S}) \leq (1 - \frac{\rho}{\Omega})(1 - \Omega) - 2\epsilon \} &\leq \frac{3}{i=1} \Pr \{ E_i \}. \\
\end{align*}
\]
We use the same definition of achievability for the stochastic source $X_n(\Omega, F)$ as for the deterministic source, except that the probability of error is also taken with respect to the random vector $X$. Based on the fact that any sequence of random vectors $\{X^{(n)}\}$ drawn from $X_n(\Omega, F)$ is almost surely an element of $X(\Omega, F)$, we immediately obtain the useful property that the sampling rate distortion function of $X(\Omega, F)$ is lower bounded by that of the stochastic version $X_n(\Omega, F)$.

We provide the following comments on notation: For a random vector $X$, we use $\mathbb{E}_X$ to denote the expectation. For sets $S, U$, we use $S \setminus U$ to denote the difference set $\{s \in S : s \not\in U\}$. We use $S^n$ to denote the set of all subsets of $\{1, 2, \ldots, n\}$ of size $\lfloor n/2 \rfloor$. For any random vector $X$, we use $\mathbb{P}(X)$ to denote the probability distribution of $X$. For a matrix $A$, we use $\text{tr}(A)$ to denote the determinant. To do so, we consider a given problem size uniformly for any sequence of sampling matrices obeying the normalization (2). To do so, we consider a given problem size uniformly for any sequence of sampling matrices obeying the normalization (2). To do so, we consider a given problem size uniformly for any sequence of sampling matrices obeying the normalization (2). To do so, we consider a given problem size uniformly for any sequence of sampling matrices obeying the normalization (2). To do so, we consider a given problem size uniformly for any sequence of sampling matrices obeying the normalization (2).

**A. Proof of Proposition 3**

We begin with the following lemma which shows that asymptotically reliable recovery is impossible if the mutual information between the sparsity pattern $S$ and the samples $Y$ is too small. The proof is given in Appendix B-G.

**Lemma 5.** For any stochastic vector source $X_n(\Omega, F)$ and sequence of sampling matrices $\{A^{(n)}\}$, the sampling rate distortion pair $(\rho, \alpha)$ is not achievable if

\[
\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}_A I(S; Y) < R(\Omega, \alpha)
\]

where $R(\Omega, \alpha)$ is given by (6).

The next step is to upper bound the left hand side of (42) uniformly for any sequence of sampling matrices obeying the normalization (2). To do so, we consider a given problem size $n$ and a condition on a random matrix $A = A$. Based on the definition of $X_n(\Omega, F)$, $S \to AX \to Y$ forms a Markov chain. Thus, by the data processing inequality,

\[
I(S; Y) \leq I(AX; Y).
\]

One way to upper bound the right hand side of (43) is to observe that

\[
I(AX; Y) \leq \max_{\|Z\|^2 \leq V(\Omega, F)} \mathbb{E}_Z I(Z; Z + W)
\]

where the maximization is over all random vectors $Z$ satisfying the same average power constraint as $AX - EAX$. This information term is maximized when $Z_i$ are i.i.d. Gaussian (see e.g. (49)), and thus

\[
I(AX; Y) \leq \frac{1}{m} \log \left(1 + \limsup_{n \to \infty} \mathbb{E}_A \text{tr}(AA^T)\right).
\]

For any random matrix $A$, Jensen’s inequality gives

\[
\mathbb{E}_A \log \left(1 + \frac{1}{m} \text{tr}(AA^T)\right) \leq \log \left(1 + \frac{1}{m} \text{tr}(\Gamma(\Omega, F)E_A \text{tr}(AA^T))\right).
\]

Hence, for any sequence of random matrices satisfying the power constraint $\mathbb{E}_m \text{tr}(AA^T) = 1$, we conclude that

\[
\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}_A I(S; Y) \leq \rho \log \left(1 + V(\Omega, F)\right).
\]

Combining this result with Lemma 5 concludes the proof of Proposition 3.

**B. Proof of Proposition 4**

For this proof, we begin with Inequality (43) from the proof of Proposition 3 and derive an improved upper bound on the information $I(S; Y)$. In particular, observe that

\[
I(AX; Y) \leq \max_{\|Z\|^2 = V(\Omega, F)} \mathbb{E}_A I(Z; Z + W)
\]

where, this time, the maximization is over all random vectors $Z$ with covariance exactly equal to that of $AX$. The right hand side of (46) is maximized when $Z_i$ are jointly Gaussian with covariance $AA^T$ (see e.g. (49)), and thus

\[
I(AX; Y) \leq \frac{1}{2} \log \left(1 + V(\Omega, F)AA^T\right).
\]

For any random sampling matrix $A$, the expected value of the right hand side of (47) may be difficult to compute in general. However, a central result from random matrix theory is that if the elements of $A$ are i.i.d. with zero mean and variance $1/n$, then the empirical distribution of the singular values of the random sequence $\{A^{(n)}\}$ converge almost surely to a non-random limit given by the Marchenko Pastur law. This convergence leads to the following result.

**Lemma 6.** Let $M$ denote an $m \times n$ random matrix whose entries are i.i.d. with zero mean and unit variance. If $m/n \to \infty$, then

\[
\lim_{n \to \infty} \frac{1}{m} \mathbb{E}_A \log \left|I_m + \frac{1}{n} \Gamma(\Omega, F)AA^T\right| = \mathcal{G}(r, \gamma)
\]

where $\mathcal{G}(r, \gamma)$ is given by (17).

**Proof:** Almost sure convergence is shown in [50]. The extension to convergence in expectation is straightforward using, for example, Hadamard’s inequality and the dominated convergence theorem.

Using Lemma 6 gives

\[
\lim_{n \to \infty} \frac{1}{m} \mathbb{E}_A \log |I_m + \frac{1}{n} \Gamma(\Omega, F)AA^T| = \mathcal{G}(\rho, \Gamma(\Omega, F)).
\]

Combining this result with (47) and Lemma 5 completes the proof of Proposition 4.

**C. Proof of Propositions 5 and 6**

The shortcomings of Propositions 3 and 4 are due, in part, to the fact that the data-processing inequality (43) is not tight in general. In this proof, we begin with the information $I(S; Y)$ and derive a stronger upper bound that takes into account the fact that values of the nonzero elements are unknown. Using the chain rule for mutual information, $I(AX, S; Y)$ can be written two ways as

\[
I(S, AX, Y) = I(S, Y) + I(AX, Y|S) = I(S, Y|AX) + I(AX, Y).
\]

Since $S \to AX \to Y$ forms a Markov chain, the term $I(S; Y|AX)$ is equal to zero and thus

\[
I(S, Y) = I(AX; Y) - I(AX; Y|S).
\]

Intuitively, one may think of the the information $I(AX; Y|S)$ as quantifying the amount of $I(AX; Y)$ that is “used up” describing the values of the nonzero elements, and hence cannot contribute to estimation of the sparsity pattern.
Since the asymptotic limit of $I(AX; Y)$ is upper bounded in the proof of Proposition 4, the remaining challenge is to find a non-trivial lower bound on the asymptotic limit of $I(AX; Y|S)$. To do so, define the $k$-dimensional vector $Z = X_S$, and observe that

$$I(AX; Y|S) = \frac{1}{|S|} \sum_{s \in S} I(A_s Z; A_s Z + W).$$

If $A$ is a random matrix whose elements are i.i.d., then each submatrix $A_s$ is identically distributed. Hence, by the linearity of expectation,

$$E_A I(AX; Y|S) = E_B I(BZ; BZ + W)$$

(50)

where $B$ is an $m \times k$ matrix whose elements have the same distribution as the elements of $A$.

In the remainder of the proof, we lower bound the asymptotic limit of the right hand side of (50). We first consider the special case where the distribution of the nonzero elements is Gaussian, and then we consider the general case.

**Gaussian Distributions:** Suppose, that the nonzero elements are i.i.d. Gaussian with mean $\mu$ and variance $\sigma^2$. Conditioned on any realization $B = b$ the samples $Y = bV + W$ are Gaussian with covariance $I_m + \sigma^2 Bb^T$ and hence

$$I(BV; bV + W) = \frac{1}{2} \log |I_m + \sigma^2 Bb^T|.$$

Using Lemma 6 gives

$$\lim_{n \to \infty} \frac{1}{n} E_B \log |I_m + \sigma^2 Bb^T| = \Omega(G(\rho/\Omega, \Omega \sigma^2)).$$

which completes the proof of Propositions 4.

**General Distributions:** Unlike the Gaussian setting, it does not appear possible in general to give an exact expression for the information $I(BV; bV + W)$. To lower bound this term, we first consider the case $m \leq k$. Conditioned on any realization $B = b$,

$$I(BV; bV + W) = h(bV + W) - \frac{m}{2} \log(2\pi e)$$

where $h(\cdot)$ denotes differential entropy [49].

If we define the entropy power of an $n$-dimensional random vector $X$ to be

$$N(X) := \frac{1}{2\pi e} \exp(\frac{1}{2} h(X)),$$

then two applications of the entropy power inequality (see e.g. [49]) give

$$I(BV; bV + W) \geq \frac{m}{2} \log(2\pi e N(bV + W)) \geq \frac{m}{2} \log(2\pi e [N(bV) + N(W)]) \geq \frac{m}{2} \log (1 + N(bV)) \geq \frac{m}{2} \log (1 + N(U_1)|Bb^T|^{\frac{1}{2}}).$$

(51)

Next we consider the case $m > k$. For any realization $B = b$,

$$I(BV; bV + W) = I(V; V + B^TW)$$

where $B^\dagger$ denotes the Moore-Penrose pseudoinverse of $B$. Hence, following the same steps as for the case $m \leq k$, we may conclude that

$$I(BV; bV + W) \geq \frac{1}{2} \log (1 + N(U_1)|Bb^T|^{\frac{1}{2}}).$$

(52)

Although the expectations, with respect to a random matrix $B$, of the right hand sides of (51) and (52) are difficult to compute for finite dimensions, their asymptotic behavior can be characterized using the Marčenko Pastur law. A proof of the following result is given in the paper [51].

**Lemma 7.** Let $M$ denote an $m \times n$ random matrix whose entries are i.i.d. with zero mean and unit variance. If $m/n \to r$ as $n \to \infty$ with $r \geq 1$, then

$$\lim_{n \to \infty} \frac{1}{n} M^T M^{1/2} = \begin{cases} \left( \frac{r}{r-1} \right)^{r-1} e^{-1} & r > 1 \\ e^{-1} & r = 1 \end{cases}$$

(53)

almost surely.

Combining Lemma 7 and the bounds (51) and (52), it can be shown that

$$\lim_{n \to \infty} E_B I(BV; bV + W) \geq \Omega(\rho/\Omega, \Omega N(U_1))$$

which completes the proof of Proposition 6 since $V_h(\Omega, F) = \Omega N(U_1)$.

**D. Proof of Theorem 2**

In this proof, we begin with the lower bound for the noisy setting given in Proposition 6 and take the limit as the SNR tends to infinity. For notational convenience, we use $V$ to denote the variance $V(\Omega, F)$ given in Definition 5. Hence, the entropy power, given in Definition 7, can be expressed $V_h(\Omega, F) = \Theta(\Omega, F)V$ and a lower bound for the noiseless setting corresponds to the limit of (20) as $V \to \infty$.

To begin, it is straightforward to verify that

$$\lim_{r \to \infty} V(r, \gamma) - r \log(\gamma/e) = \frac{1}{2} \log(\Delta(r))$$

for $r \leq 1$. Likewise, it can be shown (see Example 2.15 on pg. 44 of [50]) that the above statement is also true if the function $V(r, \gamma)$ is replaced with $G(r, \gamma)$. Thus, for any $\rho \leq \Omega$, the difference between the left and right hand sides of (20) obeys

$$\lim_{V \to \infty} G(\rho, V) - R(\Omega, \alpha) - \Omega V(\rho/\Omega, \theta(\Omega, F)V)$$

$$= \frac{1}{2} \log(\Delta(\rho)) - R(\Omega, \alpha) - \frac{1}{2} \log \left( \frac{1}{\theta(\Omega, F)} \cdot \Delta(\rho) \right)$$

$$= \frac{1}{2} \log \left( \frac{1}{\theta(\Omega, F)} \cdot \Delta(\rho/\Omega) \right) - R(\Omega, \alpha)$$

which completes the proof.

**E. Proofs of Theorems 3 and 4**

These proofs are based on the following “genie” argument: we first suppose that a genie provides the estimator with the locations and values of some fraction of the nonzero elements, and we then characterize the number of samples required to identify the remaining elements (up to allowable distortion $\alpha$).

To be more precise, suppose that for each index $i \in S$, a genie reports the pair $(i, X_i)$ to the estimator automatically if
$|X_t| > T$ and with probably $\gamma$ if $X_t = T$ where $T$ and $\gamma$ are chosen such that probability of reporting is equal to $1 - \beta$. We define $U \subseteq S$ to be the set of reported indices and use $X_U$ to denote their corresponding values. In this setting, the number of elements that are reported is a random variable $B = |U|$ with $E_B = (1 - \beta)k$ and the elements of $X_U$ are i.i.d. with the $\beta$-truncated distribution $F_\beta$ given in Definition 6.

Conditioned on any realization $B = b$, the values $X_U$ of the reported elements are, by construction, independent of the reported indices $U$, the unreported indices $S \setminus U$, the unknown values $X_{S \setminus U}$ and the noise $W$. Hence, $(Y, U, X_U) \rightarrow (Y - A_U X_U, U) \rightarrow S \setminus U$ forms a Markov chain.

Moreover, conditioned on any realization $U = u$ with $|u| = b$, the set of unreported indices $S \setminus U$ is distributed uniformly over all $\binom{n - b}{k - b}$ possibilities, and the unreported values $X_{S \setminus U}$ are independent of $S \setminus U$ and i.i.d. with the $\beta$-truncated distribution given in Definition 6.

Using the above observations, we may conclude that estimation problem more difficult, any lower bound on the sampling rate of the new problem must satisfy

$$G(\hat{\rho}, \Gamma(\beta \Omega, F_\beta)) \geq R(\beta, \alpha) + \Omega \mathbb{V}(\hat{\rho}/\Omega, \Gamma_\beta(\beta \Omega, F_\beta)).$$

Taking the maximum over all choices of $\beta$ concludes the proof of Proposition 3.

### F. Proof of Lemma 7

Let $k = \lfloor \Omega \cdot n \rfloor$ and note that $S_{\Omega}^n$ has cardinality $\binom{n}{k}$. For any $s \in S_{\Omega}^n$, a simple counting argument shows that the number of subsets $s' \in S_{\Omega}^n$ with $d(s, s') \leq \alpha$ is given by

$$|S_{\Omega}^n| = \sum_{a=0}^{\lfloor \alpha k \rfloor} \binom{k}{a} \binom{n - k}{a}.$$  

Hence, $N_n(\alpha, \alpha) \geq \binom{n}{k}/N_n$. To characterize the limit of this term we use the following fact which can be found in [49].

**Lemma 8.** If $k/n \to p$ as $n \to \infty$ for some $0 \leq p \leq 1$, then

$$\lim_{n \to \infty} \frac{1}{n} \log \binom{n}{k} = H(p)$$

where $H(p)$ is binary entropy.

Applying Lemma 8 shows that

$$\liminf_{n \to \infty} \frac{1}{n} \log N_n(\Omega, \alpha) \geq R(\Omega, \alpha).$$

To show the upper bound, we use a random covering argument. Let $S$ be a random subset of $S_{\Omega}^n$ with $\binom{n}{k}/N_n + 2n$ elements chosen uniformly at random. For any $s \in S_{\Omega}^n$, the probability that there does not exist $s' \in S$ with $d(s, s') \leq \alpha$ is given by

$$p_n = \left(1 - \frac{n}{\binom{n}{k}}\right)\binom{n}{k}/N_n + 2n \leq \exp(-2n).$$

Applying a union bound shows that the probability that $S$ does not cover $S_{\Omega}^n$ is upper bounded by $\binom{n}{k}p_n$ which is strictly less than one. Hence, this proves that $N_n(\Omega, \alpha) \leq \binom{n}{k}/N_n + 2n$. Taking the limit and applying Lemma 8 shows that

$$\limsup_{n \to \infty} \frac{1}{n} \log N_n(\Omega, \alpha) \leq R(\Omega, \alpha).$$

### G. Proof of Lemma 5

This proof follows the proof of Fano’s inequality given in [49] with some modifications to handle our error criterion.

For a given problem size $n$, realization $A = A$, and estimator $\hat{x}(k, y, A)$, define the error event $E = \{d(X, \hat{X}) > \alpha\}$. Using the chain rule for entropy [49], $H(E | S | Y)$ can be written two ways as

$$H(E, S | Y) = H(S | Y) + H(E | S, Y) = H(S | Y) + H(E | \mathcal{S}, Y).$$

The entropy $H(E | S, Y)$ is equal to zero since $E$ is defined by $S$ and $Y$. Also, since conditioning reduces entropy, $H(E | Y) \leq H(E) \leq \log 2$ and

$$H(S | E, Y) \leq H(S | E) \leq \Pr\{E\} H(S | E) + \Pr\{E^c\} H(S | E^c).$$
On the event $\mathcal{E}^c$ the entropy of $\mathbf{S}$ is trivially upper bounded by a uniform distribution on all possible sparsity patterns. Hence $H(\mathbf{S}|\mathcal{E}^c) \leq H(\mathbf{S}) = \log(N)$. On the event $\mathcal{E}^c$, the entropy of $\mathbf{S}$ is upper bounded by a uniform distribution on the set of all sparsity patterns with distortion less than or equal to $\alpha$. Hence $H(\mathbf{S}|\mathcal{E}) \leq \log N$, where $N$ is given by (54).

Combining all the inequalities and using the fact that $I(\mathbf{S};\mathbf{Y}) = H(\mathbf{S}) - H(\mathbf{S}|\mathbf{Y})$ shows that the probability of error with respect to any random matrix $A$ is lower bounded by

$$
\mathbb{E}_A \Pr\{\mathcal{E}\} \geq 1 - \frac{\mathbb{E}_A I(\mathbf{S};\mathbf{Y}) + \log 2}{\log N_n(\Omega, \alpha)}.
$$

Taking the limit as $n \to \infty$ and applying Lemma 1 completes the proof.

## Appendix C

### Proofs of Scaling Bounds

#### A. Proof of Lemma 2

Without loss of generality we assume that $F$ has unit power. Also, we define

$$
\tau(\beta) = \beta^{-2L} P(\Omega, F_\beta)/P(\Omega, F).
$$

Since $0 \leq P(\Omega, F_\beta) \leq P(\Omega, F)$ with equality on the left if and only if $\beta = 0$, we know that $0 < \tau(\beta) < \infty$ for any $0 < \beta \leq 1$. Hence, to prove our desired result we must consider the case $\beta \to 0$.

Let $X$ be a random variable with distribution $F$, and define

$$
F_{X^2}^{-1}(p) := \inf \{ x \geq 0 : \Pr\{X^2 \leq x\} \geq p \}
$$

to be the quantile function of $X^2$. Using the definition of $F_\beta$, the function $\tau(\beta)$ can be expressed as

$$
\tau(\beta) = \frac{1}{\beta^{2L+1}} \int_0^\beta F_{X^2}^{-1}(p) dp.
$$

Since $F_{X^2}^{-1}(p)$ is a non-negative and non-decreasing function,

$$
\frac{\beta}{2} F_{X^2}^{-1}(\frac{\beta}{2}) \leq \int_0^\beta F_{X^2}^{-1}(p) dp \leq \beta F_{X^2}^{-1}(\beta),
$$

and thus

$$
\limsup_{\beta \to 0} \tau(\beta) \leq \limsup_{\beta \to 0} \beta^{-2L} F_{X^2}^{-1}(\beta),
\liminf_{\beta \to 0} \tau(\beta) \geq \frac{1}{2L+1} \liminf_{\beta \to 0} \beta^{-2L} F_{X^2}^{-1}(\beta).
$$

To conclude, we note that since the decay rate $L$ of any distribution $F \in \mathcal{F}_0$ must be finite, $\lim_{\beta \to 0} \beta^{-2L} F_{X^2}^{-1}(\beta) \in (0, \infty)$.

#### B. Proof of Proposition 4

This proof begins with the lower bound in Theorem 3. Since $\alpha < 1/4$ we may attain a simplified bound by considering the case $\beta = 2\alpha$:

$$
\max_{\alpha \leq \beta \leq 1} \frac{2(1 - (1 - \beta)\Omega R(\frac{\beta\Omega}{1-(1-2\alpha)\Omega} \cdot \frac{2}{2}))}{\log (1 + V(\beta\Omega, F_\beta))} \geq \frac{2(1 - (1 - 2\alpha)\Omega R(\frac{2\alpha\Omega}{1-(1-2\alpha)\Omega} \cdot \frac{1}{2}))}{\log (1 + V(2\alpha\Omega, F_{2\alpha}))} \geq \frac{R(\frac{2\alpha\Omega}{1-(1-2\alpha)\Omega} \cdot \frac{1}{2})}{\log (1 + V(2\alpha\Omega, F_{2\alpha}))}.
$$

where the last step uses the fact that $\Omega \in (0, 1/2)$.

Next, we consider the numerator of (55). With a bit of work, it can be verified that $2\alpha\Omega/[1 - (1 - 2\alpha)\Omega] < 1/3$. Hence it suffices to lower bound the function $R(1, 1/2)$ for all $x \in [0, 1/3]$. Since $0 \leq R(x, 1/2) \leq H(x)$ with equality on the left if and only if $x = 0$, and since $\lim_{x \to 0} R(x, 1/2) = 1/2$, there exists some constant $C > 0$ such that

$$
R(\frac{2\alpha\Omega}{1-(1-2\alpha)\Omega} \cdot \frac{1}{2}) \geq C \cdot H(\frac{2\alpha\Omega}{1-(1-2\alpha)\Omega} \cdot \frac{1}{2}) \geq C \cdot H(\alpha\Omega) \geq C \cdot \alpha \Omega \log(\alpha\Omega).
$$

Lastly, we consider the denominator of (55). Using the fact that $V(\Omega, F) \leq P(\Omega, F)$, Lemma 2, and the concavity of the logarithm gives

$$
\log (1 + V(2\alpha\Omega, F_{2\alpha})) \leq \log(1 + 2\alpha P(\Omega, F_{2\alpha})) \leq \log(1 + C_F P(2\alpha P(2\alpha))) \leq C_F P(2\alpha P(2\alpha)) \leq C_F P(2\alpha P(2\alpha)).
$$

Combining (55), (56) and (57) completes the proof.

#### C. Proof of Proposition 6

This proof follows directly from Proposition 6. Combining the noiseless bound (23) with the assumption (30) shows that the noiseless sampling rate distortion function is $\rho_0(\alpha) = \Omega$, and hence $\rho(\alpha) \geq \Omega$ for all $P$. Combining this fact with the simplified bound (22) gives

$$
\rho(\alpha) \geq \frac{2R(\Omega, \alpha) + \Omega \log (1 + e^{-1}V_\alpha(\Omega, F))}{\log (1 + V(\Omega, F))}.
$$

Next, we note that since $0 \leq e^{-1}V_\alpha(\Omega, F) \leq V(\Omega, F)$,

$$
\frac{1 + V(\Omega, F)}{e^{-1}V_\alpha(\Omega, F)} \leq \frac{V(\Omega, F)}{e^{-1}V_\alpha(\Omega, F)} - \frac{e}{\theta(\Omega, F)}.
$$

Starting with (58) we can write

$$
\log(1 + V(\Omega, F)[\rho(\alpha) - \Omega] \geq 2R(\Omega, \alpha) - \Omega \log \left( \frac{1 + V(\Omega, F)}{e^{-1}V_\alpha(\Omega, F)} \right) \geq 2R(\Omega, \alpha) - \Omega \log \left( \frac{e}{\theta(\Omega, F)} \right) \geq R(\Omega, \alpha)
$$

where (60) follows from (59) and (61) follows from (30).

Since $\alpha \leq 1/4$, we have $R(\Omega, \alpha) \geq R(\Omega, 1/4) + 1/4$. Additionally, since $0 \leq R(\Omega, 1/4) \leq H(\Omega)$ with equality on the left if...
and only if $\Omega = 0$, and since $\lim_{\Omega \to 0} R(\Omega, 1/4)/H(\Omega) = 3/4$, we conclude that there exists some constant $C > 0$ such that

\[ R(\Omega, \alpha) \geq C \cdot H(\Omega) \geq \Omega \log(1/\Omega). \]  

(62)

Using (62) and the fact that $V(\Omega, F) \leq P(\Omega, F)$ completes the proof.

\section*{Appendix D

Example Distributions

\subsection*{A. Zero-Mean Gaussian (Example 1)}

Let $X$ be a Gaussian random variable with zero mean and variance $\sigma^2$. By definition, the $\beta$-truncated distribution of $F$ is given by

\[ F_\beta(x) = \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(\frac{-x^2}{2\sigma^2}\right) 1(|x| < \sigma t_\beta) dx \]

where $t_\beta = Q^{-1}(\frac{1}{\beta})$ satisfies $\Pr\{|X| < \sigma t_\beta\} = \beta$. Using integration by parts shows that the power of $F_\beta$ is

\[ \sigma_\beta^2 = \int_{-\infty}^{\infty} x^2 dF_\beta(x) = \int_{-t_\beta}^{t_\beta} x^2 \frac{1}{\sqrt{2\pi}\sigma^2} \exp(-x^2/2) dx \]

\[ = \frac{\sigma^2}{\beta} \left[ \int_{-t_\beta}^{t_\beta} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) dx - \sqrt{\frac{2}{\pi}} t_\beta \exp(-t_\beta^2/2) \right] \]

\[ = r_\beta^2 \sigma^2 \]

where $r_\beta$ is given in Example 1. The differential entropy is given by

\[ h(F_\beta) = \int_{-\sigma t_\beta}^{\sigma t_\beta} -\log\left(\frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(\frac{-x^2}{2\sigma^2}\right)\right) dF_\beta(x) \]

\[ = \int_{-\sigma t_\beta}^{\sigma t_\beta} \beta \log(\sqrt{2\pi}) dF_\beta(x) + \frac{1}{2} \int_{-\sigma t_\beta}^{\sigma t_\beta} x^2 dF_\beta(x) \]

\[ = \frac{1}{2} \log(2\pi\beta) + r_\beta. \]

\subsection*{B. Continuous Uniform (Example 2)}

Let $X$ be a continuously uniform random variable with mean $\mu$ and variance $\sigma^2$. The support of $X$ is the interval $[a, b]$ with $a = \mu - \sqrt{3}\sigma$ and $b = \mu + \sqrt{3}\sigma$. If we let $t_\beta$ be the solutions to $\min(b, t_\beta) = -\max(a, -t_\beta) = \beta (b - a)$, then $\Pr\{|X| < t_\beta\} = \beta$. Thus, the $\beta$-truncated distribution $F_\beta$ is uniform over the interval $[a_\beta, b_\beta]$ where $a_\beta = \max(a, -t_\beta)$ and $b_\beta = \min(b, t_\beta)$, and

\[ \mu_\beta = (b_\beta + a_\beta)/2 = \max\{0, \mu - (1 - \beta)\sqrt{3}\sigma\} \]

\[ \sigma_\beta^2 = \frac{(b_\beta - a_\beta)^2}{12} = \beta^2 \sigma^2 \]

\[ h(F_\beta) = \log(b_\beta - a_\beta) = \frac{1}{2} \log(12\beta^2\sigma^2). \]

\subsection*{C. Point-Mass (Example 3)}

Let $X$ be a random variable with the point-mass distribution given in Example 3 with parameters $b$, $\gamma$, and $\epsilon$. By definition, the probability mass function of the $\beta$-truncated random variable $X_\beta$ is given by

\[ p(x) = \begin{cases} \min(\beta, 1 - \epsilon) x - (1 - \epsilon) b \beta^2 / \epsilon, & \text{if } x^2 = [\gamma - (1 - \epsilon)\beta^2] / \epsilon, \\ 2 \min(\beta, 1 - \epsilon), & \text{if } x^2 = b^2 \end{cases} \]

\[ \text{D. Sliced-Gaussian (Example 4)} \]

Let $X$ be a continuous random variable with the sliced-Gaussian distribution given in Example 4 with parameters $\sigma_\beta^2$ and $b$. Let $t_\gamma = Q^{-1}(\frac{1}{\gamma})$ and observe that

\[ \Pr\{|X| \leq B + \sigma_\beta t_\gamma\} = \Pr\{|Z| \leq \sigma_\gamma t_\gamma\} = \beta. \]

Thus, $F_\beta$ is the distribution of $X_\beta = Z_\beta + \text{sgn}(Z_\beta)\sqrt{B}$ where $Z_\beta$ has the $\beta$-truncated zero mean Gaussian distribution. By symmetry, we have $\mu_\beta = 0$ and by linearity of expectation we write

\[ \sigma_\gamma^2 = \mathbb{E}X_\beta^2 \]

\[ = \mathbb{E}\left(Z_\beta + \text{sgn}(Z_\beta)\sqrt{B}\right)^2 \]

\[ = B + \mathbb{E}Z_\beta^2 + 2\sqrt{B}\mathbb{E}|Z_\beta|. \]

From Example 1 we know that $\mathbb{E}Z_\beta^2 = r_\beta\sigma^2$. To calculate the remaining term we write

\[ \mathbb{E}|Z_\beta| = \frac{2\sigma_\gamma}{\beta} \int_0^{t_\gamma} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = \tilde{r}_\beta \sigma_\gamma. \]

where $\tilde{r}_\beta$ is given in Example 3. Finally, we observe that the differential entropy is

\[ h(F_\beta) = h(Z_\beta) = \frac{1}{2} \log(2\pi\beta\sigma_\beta^2) + r_\beta. \]

\section*{Acknowledgment}

This work was supported in part by ARO MURI No. W911NF-06-1-0076.

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