Self-adjointness, Group Classification and Conservation Laws of an Extended Camassa-Holm Equation

Nadjafikhah M* and Pourrostami N2

1Department of Pure Mathematics, School of Mathematics, Iran University of Science and Technology, Narmak-16, Tehran, Iran
2Department of Complementary Education, Payam Noor University, Tehran, Iran

Abstract

In this paper, we prove that equation \( E = u_t - u_{xx} + uu_x - uu_{xx} - b u u_x = 0 \) is self-adjoint and quasi self-adjoint, then we construct conservation laws for this equation using its symmetries. We investigate a symmetry classification of this nonlinear third order partial differential equation, where \( f \) is smooth function on \( u \) and \( a, b \) are arbitrary constans.

We find three special cases of this equation, using the Lie group method.

Keywords: Lie symmetry analysis; Self-adjoint; Quasi self-adjoint; Conservation laws; Camassa-Holm equation; Degas peris-Procesi equation; Fornberg whitham equation; BBM equation

Introduction

A new procedure for constructing conservation laws was developed by Ibragimov [1]. For Camassa-Holm equation are calculated in studies of Ibragimov, Khamitova and Valenti [2]. In this paper, we study the following third-order nonlinear equation

\[
E = u_t - u_{xx} + uu_x - uu_{xx} - b u u_x = 0,
\]

and we show that this equation is self-adjoint and quasi self-adjoint. Therefore we find Lie symmetries and conservation laws. There are three cases to consider: 1) \( b = 0, a = \) arbitrary constant, 2) \( b = 0, a \neq 0 \), and 3) \( b = 0, a = 0 \). Clarkson, Mansfield and Priestly [3] are concerned with symmetry reductions of the non-linear third order partial differential equation given by \( u_t - u_{xx} + ku_{xx} - bu_x u_x = 0 \), where \( k, \alpha, \kappa, b \) are arbitrary constants. Symmetry classification and conservation laws for higher order Camassa-Holm equation are calculated in framework of Nadjafikhah and Shirvani-Sh [4].

The special cases of \((1)\) are:

Camassa-Holm (CH) equation \( u_t - u_{xx} + ku_{xx} - bu_x u_x = 0 \), \( k \)-arbitrary (real), describing the unidirectional propagation of shallow water waves over a flat bottom (let \( k = 3, a = 2, b = 1 \) in \((1)\)).

Degas peris-Procesi (DP) equation \( u_t - u_{xx} + ku_{xx} - bu_x u_x = 0 \), \( k \)-arbitrary (real), is another equation of this class (let \( k = 4, a = 3, b = 1 \) in \((1)\)).

Fornberg Whitham (FW) equation \( u_t - u_{xx} + ku_{xx} - bu_x u_x = 0 \), \( k \)-arbitrary (real), is another equation of this class (let \( k = 5, a = 4, b = 2 \) in \((1)\)).

BBM equation \( u_t - u_{xx} + ku_{xx} - bu_x u_x = 0 \), \( k \)-arbitrary (real), is another equation of this class (let \( k = 1, a = 0, b = 0 \) in \((1)\)).

Preliminaries

In this section, we recall the procedure in literature of Ibragimov [1]. Let us introduce the formal Lagrangian

\[
L = \nu E,
\]

where \( \nu = \nu(t, x) \) is a new dependent variable.

We define the adjoint equation by \( E^* = E ' + \delta L / \delta u = 0 \). Here

\[
\delta = \frac{\partial}{\partial u} + D_1 \frac{\partial}{\partial u_1} + D_2 \frac{\partial}{\partial u_2} - C_1 D_1 \frac{\partial}{\partial u_1} + C_2 D_2 \frac{\partial}{\partial u_2} + \cdots, \quad i, j = 1, 2,
\]

for \( \delta \) is the variational derivative and \( D_i \) is the operator of total differentiation.

An equation \( E = 0 \) is said to be self-adjoint [5] if the equation obtained from the adjoint equation by substitution \( \nu = u \) is identical with the original equation.

An equation \( E = 0 \) is said to be quasi- self-adjoint [5] if there exists a function \( \nu = \phi(u), \phi(u) \neq 0 \) such that \( E^* = E ' + \lambda E \) with an undetermined coefficient \( \lambda \). Eq.(1) is said to have a nonlocal conservation law if there exits a vector \( C = (C^i, C^j) \) satisfying the equation

\[
D_i(C^i) + D_j(C^j) = 0,
\]

on any solution of the system of differential equations comprising \( E \) and the adjoint equation \( E^* \). We say that original equation has a local conservation law if (3) is satisfied on any solution of Eq.(1). In studies of Ibragimov [1], the conserved vector associated with the Lie point symmetry \( \nu = \xi(x,t,u) \), \( \xi(x,t,u) \) is obtained by the following formula:

\[
C^i = \xi(x,t,u) \frac{\partial}{\partial u} - \frac{\partial \xi}{\partial t} \frac{\partial}{\partial u_t} + \frac{\partial \xi}{\partial u} \frac{\partial}{\partial u},
\]

where \( i, j = 1, 2 \) and \( \phi = \xi(u) \). (Here \( \partial_i \) means \( \frac{\partial}{\partial x_i} \)).

We recall the general procedure for determining symmetries for an arbitrary system of partial differential equations [6]. Let us consider the general system of a nonlinear system of partial differential equations of order \( n \), containing \( p \) independent and \( q \) dependent variables is given as follows

\[
\Delta_p(x, u^{(n)}) = 0, \quad \nu = 1, \ldots, l,
\]

*Corresponding authors: Nadjafikhah M, Department of Pure Mathematics, School of Mathematics, Iran University of Science and Technology, Narmak-16, Tehran, Iran, Tel: +98 2173225426; Fax: +982173228426; E-mail: m_nadjafikhah@iust.ac.ir

Received April 21, 2015; Accepted December 22, 2015; Published December 24, 2015

Citation: Nadjafikhah M, Pourrostami N (2015) Self-adjointness, Group Classification and Conservation Laws of an Extended Camassa-Holm Equation. J Generalized Lie Theory Appl S2: 004. doi:10.4172/1736-4337.S2-004

Copyright: © 2015 Nadjafikhah M, et al. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.
where \( \phi' \) are the infinitesimals of the transformations for the independent and dependent variables respectively. The associated vector fields is of the form 

\[
\mathbf{v} = (\zeta(t,x,u)\partial_t + \tau(t,x,u)\partial_x, \phi(t,x,u)\partial_u),
\]

and the third prolongation of \( \mathbf{v} \) the vector field

\[
v^{(3)} = v + \phi' \partial_u + \phi'' \partial_{uu} + \phi''' \partial_{u^2u} + \ldots + \phi^{(n)} \partial_{u^n},
\]

with coefficient

\[
\phi^{(n)} = D_1(\phi - \sum_{i=1}^{n} \zeta_{ui} u^i + \sum_{i=1}^{n} \phi u^i),
\]

where \( D_1 \) is the total derivative with respect to independent variables. The invariance condition (6) for Eq. (1) is given by,

\[
v^{(n)}[u - u_x + u_f - au_{u_j} - au_{u_{u_j}}] = 0,
\]

whenever \( E = 0 \). The condition (12) is equivalent to

\[
\phi'' - bu_{u_x} - \phi' + 2(\phi' + \phi'' - bu) = 0,
\]

whenever \( E = 0 \). Substituting (11) into (13), yields the determining equations. There are three cases to consider:

- \( a \) and \( b \neq 0 \) are arbitrary constants

In this case, complete set of determining equation is:

\[
\zeta_{u} = 0,
\]

\[
\tau_{u} = 0,
\]

\[
\phi_{u} = 0,
\]

\[
3b\phi_{uu} - 3a\phi_{u} - 2a\phi = 0,
\]

\[
2\phi_{u} = 0,
\]

\[
\psi_{u} = -b(\psi_{u} - bu) + 3bu_{uu} = 0,
\]

\[
2\phi_{uu} = 0,
\]

\[
\phi_{u} = \phi_{uu} = -b(\phi_{u} - bu) = 0.
\]

With substituting (14) – (18) into (18) – (23) we have

\[
\phi = c_1 + \frac{1}{b}\phi'(t), \quad \tau = -\epsilon t + c_2, \quad \zeta = \alpha(t),
\]

With substituting (26) into (24) – (25) we have

\[
f = -1 + K(bu + 1),
\]

where \( c_1, c_2 \) and \( K \) are arbitrary constants. With substituting (27) into determining system, we have

\[
\phi = -c_1 \frac{bu + 1}{b}, \quad \tau = \epsilon t + c_1, \quad \zeta = -\epsilon t + c_1,
\]

where \( c_1, i = 1, 2, 3 \) are arbitrary constants.

**Theorem 3.1.1.** Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:

\[
v_i = -t \partial_t + \epsilon \partial_x - \frac{(bu + 1)}{b} \partial_u, \quad v_i = \partial_i, \quad v_i = \partial_i.
\]

We want to construct the conservation law associated with the symmetry

\[
v_i = -t \partial_t + \epsilon \partial_x - \frac{(bu + 1)}{b} \partial_u.
\]
We have
\[ W = -u - \frac{1}{b} t_u + tw_u. \]  
(28)

The right-hand side of (4) is written
\[ C^2 = W(v - D_1(v)) + (D_1W)(D_1v) - D_2^2(W)v, \]
(29)

\[ C^2 = W(\partial_v - au_{,2} - D_1(W)) - 2D_2D_1(v) \]

We eliminate the term \( \xi \) since the Lagrangian \( L \) is equal to zero on solution of Eq.(1). Substituting in (29), the expression (7) for \( W \) and (28) for \( W \), we obtain
\[ C^2 = -w - \frac{1}{b} t_u + tw_u + tw_{,u} + w_{,u} - \frac{1}{b} t_{uu} + t_{uu} - tw_{,u} - tw_u + \]

\[ -u_{,v}v + tw_{,v} + tw_{,u} + u_{,v} + tw_{,v} - tw_{,u}. \]
(30)

Now, by considering Eq. (33) – (42) it is not to hard to find that the components \( \xi, \tau \) and \( \varphi \) of infinitesimal generators become
\[ \phi = \frac{df(t)}{dt} - \frac{d^2f(t)}{dt^2} + f(t), \quad \tau = \frac{d^2f(t)}{dt^2} + c_2, \quad \xi = \xi. \]
(45)

To find complete solution of the above system, we consider Eq. (43) and
\[ l = \dim \text{Span} \{\varphi, f, 1\}. \]

Three general cases are possible:

3.2.1) \( l = 1 \), then \( f = \text{constant}; \)
3.2.ii) \( l = 2 \), then \( f = af + \beta \);
3.2.iii) \( l = 3 \), then \( af + \beta f + \gamma \neq 0, \alpha \neq 0 \).

Case 3.2.1). With substituting \( f = \text{constant} \) in determining system (33)-(44), we have \( \varphi = c_1, \tau = c_2, \xi = c_2 \), where \( c_1, c_2 \) are arbitrary constants.

**Theorem 3.2.1.** Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:

\[ v_1 = \partial, \quad v_2 = \partial, \quad v_3 = \partial. \]

Case 3.2.ii). With integrating from \( f = af + \beta \) with respect to \( u \), we obtain
\[ f = \frac{-\beta}{a} + Ce^{\alpha u}, \]
(46)

where \( C \) is an integrating constant. With substituting (46) into Eq. (43)-(44) and Eq. (45), we have
\[ \xi = \xi, \quad \tau = -c_1, \quad \phi = \frac{c(Ca - e^{\alpha u} \beta)}{Ca^2}. \]
(47)

**Theorem 3.2.2.** Infinitesimal generator of every one parameter Lie group of point symmetries in this case is:

\[ v = \partial, \quad v = -c_1 t, \quad \phi = \frac{c(Ca - e^{\alpha u} \beta)}{Ca^2}. \]
(48)

Case 3.2.iii). The Eq. (43) leads to \( \varphi = 0, \tau = c_2, \xi = c_2 \).

**Theorem 3.2.3.** Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:

\[ v_1 = \partial, \quad v_2 = \partial, \quad v_3 = 0. \]

Complete set of determining equation is
\[ \xi = 0. \]
(49)
\[ \xi = 0, \]
\[ \tau = 0, \]
\[ \phi = 0, \]
\[ \phi = 0, \]
\[ \phi = 2 \xi, \]
\[ 2\phi = \xi, \]
\[ \phi + \phi = \phi, \]
\[ f(\tau + f) = 0. \]

To find a complete solution of the above system we consider Eq. (58) and with assumption \( f \neq f \) we rewrite:
\[ \phi = -\frac{f}{f}(\tau + \xi). \]

Two general cases are possible:
\[ 3.3(i) \quad \frac{f}{f} = c, \quad 3.3(ii) \quad \frac{f}{f} = h(u), \]
where \( c \) is constant.

**Case 3.3.ii).**

With integrating from \( f / f \neq c \) with respect to \( u \), we have
\[ f = Le^{cu}, \]
where \( L \) is an integrating constant. Then the Eq. (58) reduce to
\[ \phi = -c(\tau + \xi). \]

With substituting Eq. (61) into determining equation, we have
\[ \xi = c, \quad \tau = c\xi + c, \quad \phi = -c\xi, \]
where \( c, i = 1,2,3 \) are arbitrary constants.

**Theorem 3.3.1.** Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:
\[ v_1 = t_0, \quad v_2 = \hat{0}, \quad v_3 = \hat{0}. \]

We want to construct the conservation law associated with the symmetry
\[ v_1 = t_0, \quad v_2 = \hat{0}, \quad v_3 = \hat{0}. \]

We have
\[ W = -tu_1, \]

The right-hand side of (4) is written
\[ C = W(v - v_1) - D(W)[v_1]D(W)[v], \]
\[ C = W[v] - D(W)[v_1]D(W)[v] - 2D, D(W)[v]. \]

Substituting in (64) and (65), the expression (7) for \( L \) and (63) for \( W \), we obtain
\[ C = -uv - tu_1, \]
\[ C = 2uv - tu_1, \]
\[ -vu_1 - tu_1, \]
\[ -fu_1 - tu_1, \]
\[ -fu_1 - tu_1, \]
\[ -fu_1 - tu_1. \]

We can eliminate \( u_1 \) by using Eq. (1) and obtain
\[ C = -uv + fu_1 + tu_1u_1 - fu_1u_1 - tu_1u_1. \]

Now, we substitute in (68) and (??) the expression \( v = u \), therefore arrive at the conserved vector with the following components:
\[ C = -uv + tu_1, \]
\[ C = -uv + tu_1, \]
\[ C = -uv + tu_1 + 2u + 2u_1u_1 - 2u_1u_1, \]
\[ C = -uv + tu_1 + 2u + 2u_1u_1 - 2u_1u_1, \]
where \( f = L e^{cu} \).

**Case 3.3.ii).** By considering Eq. (49) – (54), we find that the components \( \xi, \tau \) and \( \phi \) are \( \xi = \xi(x), \tau = \tau(t) \) and \( \phi = \phi(x)u + B(x,t) \). By considering Eq. (55) and (56) we have
\[ \xi = c_1 \exp 2x + c_2 \exp -2x + c_3, \]
\[ \tau = c_1 \exp 2x - c_2 \exp -2x + c_4, \]
\[ \phi = -c_1 \exp 2x + c_2 \exp -2x + c_5. \]

By considering Eq. (57) we have
\[ \tau = \beta (c_1 \exp 2x + c_2 \exp -2x + c_5) + c_6, \]
where \( c_6, i = 1,6 \) are arbitrary constants.

From the following identity:
\[ A(x)u + B(x,t) = \frac{-f}{f}(\tau + f) \],
we find that \( c_1 = c_2 = 0 \) and \( \phi = -\left( f / f \right)_{\xi} \). Hence we have two particular cases:
\[ \frac{f}{f} = K, \quad \frac{f}{f} = Ku = g(u), \]
where \( K \) is an arbitrary nonzero constant. For the first case, we have
\[ \xi = c_1, \quad \tau = c_2 + c_3, \quad \phi = -Ku_1, \]
and for the second case, we have
\[ \xi = c_1, \quad \tau = c_2, \quad \phi = 0. \]

**Theorem 3.2.** Infinitesimal generators of every one parameter Lie group of point symmetries in this case, when \( f / f \neq Ku \) are
\[ v_1 = \hat{0}, \quad v_2 = \hat{0}, \quad v_3 = t_0 - u_0, \]
and when \( f / f \neq Ku = g(u) \) are
\[ v_1 = \hat{0}, \quad v_2 = \hat{0}, \quad v_3 = \hat{0}, \]
where \( K \) is an arbitrary nonzero constant.

To construct the conservation law associated with the symmetry \( v = t_0, -u_0 \), we find that \( W = -u - t_1 \). Therefore, we have the conserved vector with the following components:
\[ C = -u - u_0, \]
\[ -fu_0 - u_0 - tu_0 + tu_1, \]
\[ C = -u - u_0, \]
\[ -fu_0 - u_0 + 4u - 2u + 2u_1, \]
where \( f / f \neq Ku = g(u) \).

**Acknowledgements**

The authors wish to express their sincere gratitude to Prof. N.H. Ibragimov for his useful advise and suggestions and helpful comments.
References

1. Ibragimov NH (2007) A new conservation theorem. J Math Anal Appl 333: 311-328.

2. Ibragimov NH, Khamitova RS, Valenti A (2011) Self-adjointness of generalized Camassa-Holm equation. J Applied Mathematics and Computation 218: 2579-2583.

3. Clarkson PA, Mansfield EL, Priestley TJ (1997) Symmetries of a Class of Nonlinear Third Order Partial Differential Equations. Math Comput Modelling 25: 195-212.

4. Nadjafikhah M, Shirvani-Sh V (2011) Symmetry classification and conservation laws for higher order Camassa-Holm equation.

5. Ibragimov NH (2007) Quasi-self-adjoint differential equations. Arch ALGA 4: 55-60.

6. Olver PJ (1986) Applications of Lie Group for Differential Equations. Springer-Verlag, New York.

7. Olver PJ (1995) Equivalence, invariant and symmetry. Cambridge University Press, Cambridge.