SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES FOR
s-GEOMETRICALLY CONVEX FUNCTIONS AND THEIR
APPLICATIONS

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Abstract. In this paper, some new integral inequalities of Hermite-Hadamard type related to the s-geometrically convex functions are established and some applications to special means of positive real numbers are also given.

1. Introduction

In this section, we firstly list several definitions and some known results.

Definition 1. Let $I$ be an interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard integral inequality [4]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$ \hspace{1cm} (1.1)

In [5], Hudzik and Maligranda considered the following class of functions:

Definition 2. A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ where $\mathbb{R}_+ = [0, \infty)$, is said to be $s$-convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in I$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $s$ fixed in $(0, 1]$. They denoted this by $K_s^2$.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$. For some recent results and generalizations concerning s-convex functions see [11] [2] [5] [8] [9].

Recently, In [11], the concept of geometrically and s-geometrically convex functions was introduced as follows:

Definition 3. A function $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}_+$ is said to be a geometrically convex function if

$$f\left(x^ty^{1-t}\right) \leq f(x)^t f(y)^{1-t}$$

for $x, y \in I$ and $t \in [0, 1]$.

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Definition 4. A function \( f : I \subseteq \mathbb{R}_+ \times (0, \infty) \to \mathbb{R}_+ \) is said to be a \( s \)-geometrically convex function if
\[
 f \left( x^s y^{1-t} \right) \leq f(x)^t f(y)^{(1-t)s}
\]
for some \( s \in (0, 1] \), where \( x, y \in I \) and \( t \in [0, 1] \).

In [11], the authors have established some integral inequalities connected with the inequalities (1.1) for the \( m \)-functions. In [10], the authors have introduced concepts of the type for these classes of functions.

Theorem 1. Let \( f \subseteq I \subseteq \mathbb{R}_+ \to \mathbb{R} \) be a differentiable mapping on \( I \), and \( a, b \in I \), with \( a < b \). If \( f' \in L[a, b] \), then one has the inequalities:
\[
 f \left( \sqrt{ab} \right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x) f'(ab/x)}{x} dx \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx
\]

\[
 \leq \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} \leq \frac{f(a) + f(b)}{2}.
\]

Lemma 1. Let \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R} \) be a differentiable mapping on \( I \), and \( a, b \in I \), with \( a < b \). If \( f' \in L[a, b] \), then
\[
f \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx
\]

\[
= \frac{(\ln b - \ln a)}{4} \left[ a \int_0^1 t \left( \frac{b}{a} \right)^t f' \left( a^{1-t} (ab)^t \right) dt - b \int_0^1 t \left( \frac{a}{b} \right)^t f' \left( b^{1-t} (ab)^t \right) dt \right],
\]

\[
= \frac{(\ln b - \ln a)}{2} \left[ a \int_0^1 t \left( \frac{b}{a} \right)^t f' \left( a^{1-t} b^t \right) dt - b \int_0^1 t \left( \frac{a}{b} \right)^t f' \left( b^{1-t} a^t \right) dt \right].
\]

Theorem 2. Let \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+ \) be differentiable on \( I \), and \( a, b \in I \) with \( a < b \) and \( f' \in L[a, b] \). If \( |f'|^q \) is geometrically convex on \( [a, b] \) for \( q \geq 1 \), then
\[
f \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x) f'(ab/x)}{x} dx \leq \frac{\ln b - \ln a}{4} \left( \frac{1}{2} \right)^{1 - \frac{1}{q}}
\]

\[
\times \left\{ a |f'(a)| \left[ g_1 \left( \frac{q}{2} \right) \right]^\frac{1}{q} + b |f'(b)| \left[ g_1 \left( \frac{q}{2} \right) \right]^\frac{1}{q} \right\},
\]

\[
\left| f(a) + f(b) \right| - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq \frac{\ln b - \ln a}{2} \left( \frac{1}{2} \right)^{1 - \frac{1}{q}}
\]
Theorem 3. Let \( f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+ \) be differentiable on \( I^o \), and \( a, b \in I^o \) with \( a < b \) and \( f' \in L[a, b] \). If \( |f'|^q \) is geometrically convex on \([a, b]\) for \( q > 1 \), then

\[
\left(\frac{1}{q} - \frac{1}{2q - 1}\right)^{1 - \frac{1}{q}} \times \left\{ a |f'(a)| \left[ g_2 \left( \alpha \left( \frac{q}{2} \right) \right) \right] \right. \left. \frac{q - 1}{2q - 1} \times \frac{q}{2} \right\},
\]

where

\[
g_2(\alpha) = \begin{cases} 
1, & \alpha = 1 \\
\frac{q - 1}{2q - 1}, & \alpha \neq 1
\end{cases}
\]

and \( \alpha(u), \gamma(u) \) are the same as in (2.7).

In this paper, we will establish some new integral inequalities of Hermite-Hadamard-like type related to the \( s \)-geometrically convex functions and then apply these inequalities to special means.

2. Main Results

Theorem 4. Let \( f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+ \) be differentiable on \( I^o \), and \( a, b \in I^o \) with \( a < b \) and \( f' \in L[a, b] \). If \( |f'|^q \) is \( s \)-geometrically convex on \([a, b]\) for \( q \geq 1 \) and \( s \in (0, 1) \), then

\[
\left(\frac{1}{q} - \frac{1}{2q - 1}\right)^{1 - \frac{1}{q}} \times \left\{ a |f'(a)| \left[ g_2 \left( \alpha \left( \frac{q}{2} \right) \right) \right] \right. \left. \frac{q - 1}{2q - 1} \times \frac{q}{2} \right\},
\]

where

\[
g_2(\alpha) = \begin{cases} 
1, & \alpha = 1 \\
\frac{q - 1}{2q - 1}, & \alpha \neq 1
\end{cases}
\]
\begin{equation}
|f(\sqrt{ab}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{3-\frac{3}{q}} H_1(s, q; g_1(\theta_3), g_1(\theta_4)),
\end{equation}

where

\[ g_1(u) = \begin{cases} \frac{1}{u}, & u = 1 \\ \frac{2}{a \ln u - u + 1}, & u \neq 1 \end{cases}, \]

\begin{equation}
\begin{align*}
\theta_1 &= \left( \frac{b |f'(b)|^s}{a |f'(a)|^s} \right)^q, \\
\theta_2 &= \left( \frac{a |f'(a)|^s}{b |f'(b)|^s} \right)^q,
\end{align*}
\end{equation}

\begin{equation}
\begin{array}{l}
(2.3) \\
(2.4) \\
(2.5)
\end{array}
\end{equation}

\textbf{Proof.} (1) Since $|f'|^q$ is $s$-geometrically convex on $[a, b]$, from lemma 1 and power mean inequality, we have

\begin{equation}
\begin{align*}
&\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \\
&\leq \frac{\ln \left( \frac{b}{a} \right)}{2} \left[ a \int_0^1 t \left( \frac{b}{a} \right)^t |f'(a^{1-t}b^t)| \, dt + b \int_0^1 t \left( \frac{a}{b} \right)^t |f'(b^{1-t}a^t)| \, dt \right] \\
&\leq \frac{a \ln \left( \frac{b}{a} \right)}{2} \left( \int_0^1 t \left( \frac{b}{a} \right)^t dt \right)^{1-\frac{3}{q}} \left( \int_0^1 t \left( \frac{a}{b} \right)^t \left| f'(a^{1-t}b^t) \right|^q \, dt \right)^{\frac{1}{q}} \\
&\quad + \frac{b \ln \left( \frac{b}{a} \right)}{2} \left( \int_0^1 t \left( \frac{a}{b} \right)^t dt \right)^{1-\frac{3}{q}} \left( \int_0^1 t \left( \frac{a}{b} \right)^t \left| f'(b^{1-t}a^t) \right|^q \, dt \right)^{\frac{1}{q}} \\
&\leq \frac{a \ln \left( \frac{b}{a} \right)}{2} \left( \frac{1}{2} \right)^{1-\frac{3}{q}} \left( \int_0^1 t \left( \frac{b}{a} \right)^t \left| f'(a) \right|^q t^{(1-t)s} \left| f'(b) \right|^{qt} \, dt \right)^{\frac{1}{q}} \\
&\quad + \frac{b \ln \left( \frac{b}{a} \right)}{2} \left( \frac{1}{2} \right)^{1-\frac{3}{q}} \left( \int_0^1 t \left( \frac{a}{b} \right)^t \left| f'(b) \right|^q t^{(1-t)s} \left| f'(a) \right|^{qt} \, dt \right)^{\frac{1}{q}}.
\end{align*}
\end{equation}
If $0 < \mu \leq 1 \leq \eta$, $0 < \alpha, s \leq 1$, then

\begin{equation}
\mu^{\alpha s} \leq \mu^{\alpha s}, \quad \eta^{\alpha s} \leq \eta^{\alpha s+1-s}.
\end{equation}

(i) If $1 \geq |f'(a)|, |f'(b)|$, by (2.6) we obtain that

\[\int_{0}^{1} t \left(\frac{b}{a}\right)^{qt} |f'(a)|^{q(1-t)^{s}} |f'(b)|^{qt^{s}} dt \leq \int_{0}^{1} t \left(\frac{b}{a}\right)^{qt} |f'(a)|^{qs(1-t)} |f'(b)|^{qt^s} dt = |f'(a)|^{qs} g_1(\theta_1),\]

(ii) If $1 \leq |f'(a)|, |f'(b)|$, by (2.6) we obtain that

\[\int_{0}^{1} t \left(\frac{b}{a}\right)^{qt} |f'(a)|^{q(1-t)^{s}} |f'(b)|^{qt^{s}} dt \leq \int_{0}^{1} t \left(\frac{b}{a}\right)^{qt} |f'(a)|^{qs(1-t)} |f'(b)|^{qt^s} dt = |f'(b)|^{qs} g_1(\theta_2).\]

(iii) If $|f'(a)| \leq 1 \leq |f'(b)|$, by (2.6) we obtain that

\[\int_{0}^{1} t \left(\frac{b}{a}\right)^{qt} |f'(a)|^{q(1-t)^{s}} |f'(b)|^{qt^{s}} dt \leq \int_{0}^{1} t \left(\frac{b}{a}\right)^{qt} |f'(a)|^{qs(1-t)} |f'(b)|^{qt^s} dt = \left(|f'(a)|^s |f'(b)|^{1-s}\right)^q g_1(\theta_1),\]
From (2.5) to (2.10), (2.1) holds.

(iv) If \( |f'(b)| \leq 1 \leq |f'(a)| \), by (2.6) we obtain that
\[
\int_0^1 t \left( \frac{a}{b} \right) q^t |f'(b)|^{q(1-t)r} |f'(a)|^{q\tau r} \, dt
\leq \int_0^1 t \left( \frac{a}{b} \right) q^t |f'(a)|^{q(1-st)} |f'(b)|^{q\tau st} \, dt = |f'(b)|^q g_1(\theta_2).
\]

(2.10) \[
\int_0^1 t \left( \frac{a}{b} \right) q^t |f'(b)|^{q(1-t)r} |f'(a)|^{q\tau r} \, dt
\leq \int_0^1 t \left( \frac{a}{b} \right) q^t |f'(b)|^{q(1-t)} |f'(a)|^{q(st+1-s)} \, dt = \left( |f'(b)|^s |f'(a)|^{1-s} \right)^q g_1(\theta_2).
\]

From (2.5) to (2.10), (2.1) holds.

(2) Since \( |f'|^s \) is \( s \)-geometrically convex on \([a, b]\), from lemma 4 and power mean inequality, we have
\[
\left| f(\sqrt[2k]{ab}) - \frac{1}{\ln b - \ln a} \int_a^b f(x) dx \right|
\leq \ln \frac{b}{a} \left[ a \int_0^1 t \left( \frac{b}{a} \right)^{q} |f'(a^{1-t} (ab)^{\frac{q}{2}})| \, dt + b \int_0^1 t \left( \frac{a}{b} \right)^{q} |f'(b^{1-t} (ab)^{\frac{q}{2}})| \, dt \right]
\leq \frac{a \ln \frac{b}{a}}{4} \left( \int_0^1 t \, dt \right)^{\frac{1}{q}} \left( \int_0^1 t \left( \frac{b}{a} \right)^{q} |f'(a^{1-t} (ab)^{\frac{q}{2}})|^q \, dt \right)^{\frac{1}{q}}
+ \frac{b \ln \frac{b}{a}}{4} \left( \int_0^1 t \, dt \right)^{\frac{1}{q}} \left( \int_0^1 t \left( \frac{b}{a} \right)^{q} |f'(b^{1-t} (ab)^{\frac{q}{2}})|^q \, dt \right)^{\frac{1}{q}}
\leq \frac{a \ln \frac{b}{a}}{4} \left( \frac{1}{2} \right)^{\frac{1}{q}} \left( \int_0^1 t \left( \frac{b}{a} \right)^{q} |f'(b)|^{q(1/2)r} |f'(a)|^{q((2-t)/2)r} \, dt \right)^{\frac{1}{q}}
+ \frac{b \ln \frac{b}{a}}{4} \left( \frac{1}{2} \right)^{\frac{1}{q}} \left( \int_0^1 t \left( \frac{a}{b} \right)^{q} |f'(a)|^{q((2-t)/2)r} |f'(b)|^{q((2-t)/2)r} \, dt \right)^{\frac{1}{q}}.
\]
(i) If $1 \geq |f'(a)|, |f'(b)|$, by (2.8) we obtain that
\[
\int_0^1 t \left( \frac{b}{a} \right)^{\frac{a}{q}} |f'(b)|^{q(t-2)} |f'(a)|^{q(2-t)/2} \, dt \leq |f'(a)|^q g_1 (\theta_3),
\]
(2.12) \[
\int_0^1 t \left( \frac{a}{b} \right)^{\frac{b}{q}} |f'(a)|^{q(t-2)} |f'(b)|^{q(2-t)/2} \, dt \leq |f'(b)|^q g_1 (\theta_4).
\]
(ii) If $1 \leq |f'(a)|, |f'(b)|$, by (2.6) we obtain that
\[
\int_0^1 t \left( \frac{b}{a} \right)^{\frac{a}{q}} |f'(b)|^{q(t-2)} |f'(a)|^{q(2-t)/2} \, dt \leq \left( |f'(a)| |f'(b)|^{1-s} \right)^q g_1 (\theta_3),
\]
(2.13) \[
\int_0^1 t \left( \frac{a}{b} \right)^{\frac{b}{q}} |f'(a)|^{q(t-2)} |f'(b)|^{q(2-t)/2} \, dt \leq \left( |f'(b)| |f'(a)|^{1-s} \right)^q g_1 (\theta_4).
\]
(iii) If $|f'(a)| \leq 1 \leq |f'(b)|$, by (2.6) we obtain that
\[
\int_0^1 t \left( \frac{b}{a} \right)^{\frac{a}{q}} |f'(b)|^{q(t-2)} |f'(a)|^{q(2-t)/2} \, dt \leq \left( |f'(a)| |f'(b)|^{1-s} \right)^q g_1 (\theta_3),
\]
(2.14) \[
\int_0^1 t \left( \frac{a}{b} \right)^{\frac{b}{q}} |f'(a)|^{q(t-2)} |f'(b)|^{q(2-t)/2} \, dt \leq |f'(b)|^q g_1 (\theta_4).
\]
(iv) If $|f'(b)| \leq 1 \leq |f'(a)|$, by (2.6) we obtain that
\[
\int_0^1 t \left( \frac{b}{a} \right)^{\frac{a}{q}} |f'(b)|^{q(t-2)} |f'(a)|^{q(2-t)/2} \, dt \leq |f'(a)|^q g_1 (\theta_3),
\]
(2.15) \[
\int_0^1 t \left( \frac{a}{b} \right)^{\frac{b}{q}} |f'(a)|^{q(t-2)} |f'(b)|^{q(2-t)/2} \, dt \leq \left( |f'(b)|^s |f'(a)|^{1-s} \right)^q g_1 (\theta_4).
\]
From (2.11) to (2.15), (2.2) holds. This completes the required proof. \(\square\)

If taking $s = 1$ in Theorem 3, we can derive the following inequalities which are the same of the inequalities (1.2) and (1.3).

**Corollary 1.** Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ be differentiable on $I^\circ$, and $a, b \in I^\circ$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is geometrically convex on $[a, b]$ for $q \geq 1$, then
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^2 H_1 \left( 1, q; g_1 (\theta_1), g_1 (\theta_2) \right),
\]
\[
\left| f \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{3-\frac{1}{q}} H_1 \left( 1, q; g_1 (\theta_3), g_1 (\theta_4) \right),
\]
If taking \( q = 1 \) in Theorem 1 we can derive the following corollary.

**Corollary 2.** Let \( f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be differentiable on \( I^c \), and \( a, b \in I^c \) with \( a < b \) and \( f' \in L[a, b] \). If \( |f'| \) is s-geometrically convex on \( [a, b] \) for \( s \in (0, 1] \), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{2-\frac{s}{q}} H_2 (s, q; g_1(\theta_1), g_1(\theta_2)),
\]

where

\[
g_2(u) = \begin{cases} 
\frac{u}{u-1}, & u = 1 \\
\frac{u-1}{u}, & u \neq 1
\end{cases}, \quad u > 0
\]

\[
H_2 (s, q; g_1(\theta_1), g_1(\theta_2)) = \left\{ \begin{array}{l} 
\frac{a |f'(a)|^s g_2^{1/q} (\theta_1) + b |f'(b)|^s g_2^{1/q} (\theta_2)}{|f'(a)|, |f'(b)| \leq 1}, \\
\frac{a |f'(a)| |f'(b)|^{1-s} g_2^{1/q} (\theta_1) + b |f'(b)| |f'(a)|^{1-s} g_2^{1/q} (\theta_2)}{|f'(a)|, |f'(b)| \geq 1}, \\
\frac{a |f'(a)|^s |f'(b)|^{1-s} g_2^{1/q} (\theta_1) + b |f'(b)|^s g_2^{1/q} (\theta_2)}{|f'(a)| \leq 1 \leq |f'(b)|}, \\
\frac{a |f'(a)| g_2^{1/q} (\theta_1) + b |f'(b)|^s |f'(a)|^{1-s} g_2^{1/q} (\theta_2)}{|f'(b)| \leq 1 \leq |f'(a)|}.
\end{array} \right.
\]

and \( \theta_1, \theta_2, \theta_3, \theta_4 \) are the same as in \([2.3]\).

**Proof.** (1) Since \( |f'|^q \) is s-geometrically convex on \([a, b]\), from lemma 1 and Hölder inequality, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{2-\frac{s}{q}} H_2 (s, q; g_1(\theta_1), g_1(\theta_2)).
\]
\[
\begin{align*}
&\leq \frac{a \ln \left( \frac{b}{a} \right)}{2} \left( \int_{0}^{1} t^{\frac{q}{q-1}} dt \right)^{1-\frac{1}{q}} \left( \int_{0}^{1} \left( \frac{b}{a} \right)^{qt} |f'(a^{1-t}b')|^{q} dt \right)^{\frac{1}{q}} \\
&\quad + \frac{b}{2} \ln \left( \frac{b}{a} \right) \left( \int_{0}^{1} t^{\frac{q}{q-1}} dt \right)^{1-\frac{1}{q}} \left( \int_{0}^{1} \left( \frac{a}{b} \right)^{qt} |f'(b^{1-t}a')|^{q} dt \right)^{\frac{1}{q}} \\
&\leq \frac{a \ln \left( \frac{b}{a} \right)}{2} \left( \frac{q - 1}{2q - 1} \right)^{1-\frac{1}{q}} \left( \int_{0}^{1} \left( \frac{b}{a} \right)^{qt} |f'(b)|^{qt} |f'(a)|^{q(1-t)^{r}} dt \right)^{\frac{1}{q}} \\
&\quad + \frac{b}{2} \ln \left( \frac{b}{a} \right) \left( \frac{q - 1}{2q - 1} \right)^{1-\frac{1}{q}} \left( \int_{0}^{1} \left( \frac{a}{b} \right)^{qt} |f'(a)|^{qt} |f'(b)|^{q(1-t)^{r}} dt \right)^{\frac{1}{q}}. \\
(2.18)
\end{align*}
\]

(i) If \(1 \geq |f'(a)|, |f'(b)|,\) by (2.6) we have
\[
\left( \int_{0}^{1} \left( \frac{b}{a} \right)^{qt} |f'(b)|^{qt} |f'(a)|^{q(1-t)^{r}} dt \right)^{\frac{1}{q}} = |f'(a)|^{q_{2}} g_{2}(\theta_{1}),
\]
\[
(2.19)
\]

(ii) If \(1 \leq |f'(a)|, |f'(b)|,\) by (2.6) we have
\[
\left( \int_{0}^{1} \left( \frac{b}{a} \right)^{qt} |f'(b)|^{qt} |f'(a)|^{q(1-t)^{r}} dt \right)^{\frac{1}{q}} \leq \left( |f'(a)| |f'(b)|^{1-s} \right)^{q} g_{2}(\theta_{1}),
\]
\[
(2.20)
\]

(iii) If \(|f'(a)| \leq 1 \leq |f'(b)|,\) by (2.6) we obtain that
\[
\left( \int_{0}^{1} \left( \frac{b}{a} \right)^{qt} |f'(b)|^{qt} |f'(a)|^{q(1-t)^{r}} dt \right)^{\frac{1}{q}} \leq \left( |f'(a)|^{s} |f'(b)|^{1-s} \right)^{q} g_{2}(\theta_{1}),
\]
\[
(2.21)
\]

(iv) If \(|f'(b)| \leq 1 \leq |f'(a)|,\) by (2.6) we obtain that
\[
\left( \int_{0}^{1} \left( \frac{b}{a} \right)^{qt} |f'(b)|^{qt} |f'(a)|^{q(1-t)^{r}} dt \right)^{\frac{1}{q}} \leq |f'(a)|^{q} g_{2}(\theta_{1}),
\]
From (2.18) to (2.22), (2.16) holds.

(2) Since $|f'|^q$ is s-geometrically convex on $[a, b]$, from lemma [4] and H"older inequality, we have

$$
\ln \left( \frac{b}{a} \right) \left| f' \left( \left(\frac{b}{a}\right)^{1/2} \right) - \int_a^b \frac{f(x)}{x} \, dx \right| 
\leq \frac{\ln b - \ln a}{4} \left( \frac{q - 1}{2q - 1} \right)^{1-rac{1}{q}} \left( \int_0^1 \left( \frac{b}{a} \right)^{1/2} \left| f' \left( \left(\frac{b}{a}\right)^{1/2} \right) \right| \, dt \right)^{\frac{1}{q}} + b \ln \left( \frac{b}{a} \right) \left( \frac{q - 1}{2q - 1} \right)^{1-rac{1}{q}} \left( \int_0^1 \left( \frac{a}{b} \right)^{1/2} \left| f' \left( \left(\frac{a}{b}\right)^{1/2} \right) \right| \, dt \right)^{\frac{1}{q}}.
$$

From (2.23) and similarly to (2.19) - (2.22), (2.17) holds. □

If taking $s = 1$ in Theorem [3], we can derive the following inequalities which are the same of the inequalities (1.3) and (1.5).

**Corollary 3.** Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be differentiable on $I^0$, and $a, b \in I^0$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is geometrically convex on $[a, b]$ for $q > 1$, then

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \leq \frac{1}{H_2(1, q; g_2(\theta_1), g_2(\theta_2))},
$$

$$
\left| \frac{f\left(\sqrt{ab}\right)}{2} - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} \, dx \right| \leq \frac{1}{4} \ln \left( \frac{b}{a} \right) \left( \frac{q - 1}{2q - 1} \right)^{1-rac{1}{q}} H_2(1, q; g_2(\theta_3), g_2(\theta_4)).
$$

3. Application to Special Means

Let us recall the following special means of two nonnegative number $a, b$ with $b > a$:

1. The arithmetic mean

$$
A = A(a, b) := \frac{a + b}{2}.
$$

2. The geometric mean

$$
G = G(a, b) := \sqrt{ab}.
$$

3. The logarithmic mean

$$
L = L(a, b) := \frac{b - a}{\ln b - \ln a}.
$$
(4) The p-logarithmic mean

\[
L_p = L_p(a, b) := \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}, \quad p \in \mathbb{R} \setminus \{-1, 0\}.
\]

Let \( f(x) = (x^s/s) \), \( x \in (0, 1] \), \( 0 < s < 1 \), \( q \geq 1 \) then the function \( |f'(x)|^q = x^{(s-1)q} \) is s-geometrically convex on \((0, 1]\) for \( 0 < s < 1 \) and \( |f'(a)| = a^{s-1} > b^{s-1} = |f'(b)| \geq 1 \) (see \([11]\)).

**Proposition 1.** Let \( 0 < a < b \leq 1 \), \( 0 < s < 1 \), and \( q \geq 1 \). Then

\[
|A(a^s, b^s) - L(a^s, b^s)| \leq \frac{s}{2G^{(s-1)^2}(a, b)} \left( \frac{b - a}{2L(a, b)} \right)^{1 - \frac{s}{q}} \left( \frac{1}{(s^2 - s + 1)q} \right)^{\frac{1}{q}}
\]

\[
\times \left[ \left\{ b^{(s^2 - s + 1)q} - L(a^{(s^2 - s + 1)q}, b^{(s^2 - s + 1)q}) \right\}^{\frac{1}{q}}
\right.
\]

\[
+ \left\{ L(a^{(s^2 - s + 1)q}, b^{(s^2 - s + 1)q}) - a^{(s^2 - s + 1)q} \right\}^{\frac{1}{q}}
\],

\[
|G^s(a, b) - L(a^s, b^s)| \leq \frac{s}{2G^{(s-1)^2}(a, b)} \left( \frac{b - a}{4L(a, b)} \right)^{1 - \frac{s}{q}} \left( \frac{1}{(s^2 - s + 1)q} \right)^{\frac{1}{q}}
\]

\[
\times \left[ G(a^s, b^{-(s-1)^2}) \left\{ b^{(s^2 - s + 1)q/2} - L(a^{(s^2 - s + 1)q/2}, b^{(s^2 - s + 1)q/2}) \right\}^{\frac{1}{q}}
\right.
\]

\[
+ G(b^s, a^{-(s-1)^2}) \left\{ L(a^{(s^2 - s + 1)q/2}, b^{(s^2 - s + 1)q/2}) - a^{(s^2 - s + 1)q/2} \right\}^{\frac{1}{q}}
\].

**Proof.** The assertion follows from the inequalities \([2.1]\) and \([2.2]\) in Theorem \([3]\) for \( f(x) = (x^s/s), \ x \in (0, 1], \ 0 < s < 1 \). \( \square \)

**Proposition 2.** Let \( 0 < a < b \leq 1 \), \( 0 < s < 1 \), and \( q > 1 \). Then

\[
|A(a^s, b^s) - L(a^s, b^s)|
\leq \frac{s(b - a)}{L(a, b)G^{(s-1)^2}(a, b)} \left( \frac{q - 1}{2q - 1} \right)^{1 - \frac{s}{q}} L^\frac{q}{s}(a^{(s^2 - s + 1)q}, b^{(s^2 - s + 1)q}),
\]

\[
|G^s(a, b) - L(a^s, b^s)|
\leq \frac{s(b - a)}{2L(a, b)G^{(s-1)^2}(a, b)} \left( \frac{q - 1}{2q - 1} \right)^{1 - \frac{s}{q}} L^\frac{q}{s}(a^{(s^2 - s + 1)q/2}, b^{(s^2 - s + 1)q/2})
\]

\[
\times A \left( G \left( a^{-(s-1)^2}, b^s \right), G \left( a^s, b^{-(s-1)^2} \right) \right),
\]

**Proof.** The assertion follows from the inequalities \([2.16]\) and \([2.17]\) in Theorem \([5]\) for \( f(x) = (x^s/s), \ x \in (0, 1], \ 0 < s < 1 \). \( \square \)
Acknowledgments

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