Embedded Defects

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Abstract:

We give a prescription for embedding classical solutions and, in particular, topological defects in field theories which are invariant under symmetry groups that are not necessarily simple. After providing examples of embedded defects in field theories based on simple groups, we consider the electroweak model and show that it contains the \(Z\) string and a one parameter family of strings called the \(W(\alpha)\) string. It is argued that, although the members of this family are gauge equivalent when considered in isolation, each member should be considered distinct when multi-string solutions are considered. We then turn to the issue of stability of embedded defects and demonstrate the instability of a large class of such solutions in the absence of bound states or condensates. The \(Z\) string is shown to be unstable when the Weinberg angle \((\theta_w)\) is \(\pi/4\) for all values of the Higgs mass. The \(W\) strings are also shown to be unstable for a large range of parameters. Embedded monopoles suffer from the Brandt-Neri-Coleman instability. A simple physical understanding of this instability is provided in terms of the phenomenon of W-condensation. Finally, we connect the electroweak string solutions to the sphaleron: “twisted” loops of \(W\) string and finite segments of \(W\) and \(Z\) strings collapse into the sphaleron configuration, at least, for small values of \(\theta_w\).
1. INTRODUCTION

Topological defects are classical solutions of certain field theories and have been known for nearly three decades. These include domain walls, strings and monopoles. However, few field theories admit the required topology and the standard model of the electroweak interactions [1] lacks any topological defects. Over the last few years, it has been realized [2] that even if the non-trivial topology required for the existence of a defect is absent in a field theory, it may be possible to have defect-like solutions. The idea is simply that topological defects may be “embedded” in such topologically trivial field theories. Embedded defect solutions are very common and even the electroweak model admits string solutions. It is the properties of these solutions that we wish to explore in this paper.

A crucial difference between topological and embedded defects is that the stability of the former is guaranteed by topology while the embedded defect is generally unstable under small perturbations. Therefore, if embedded defects are to be significant, some mechanism by which they can be stabilized must be found. At least one embedded defect - the semilocal string [3, 4] - is stable by itself and electroweak strings can be locally stable also [5, 6, 7]. A general mechanism for stabilizing embedded defects was proposed in Ref. 8, where it was shown that scalar bound states on electroweak strings vastly improve their stability. It was also argued that fermionic bound states would improve the string stability and that this mechanism of stabilizing solutions would apply to other saddle point solutions as well. Hence, the possibility that stable embedded defects exist in the real world must be taken seriously.

In this paper we shall investigate the existence and stability of embedded defects with particular emphasis on defects in the electroweak model. We shall first consider an arbitrary symmetry breaking $G \to H$ and derive the conditions under which embedded
defects are possible (Section 2). In doing this, we clarify the analysis in Ref. 2 where we had only considered the case of a simple group \( G \); here we also treat the case when \( G \) is not simple. This extension has direct relevance since the electroweak model is based on \( G = SU(2) \times U(1) \) which is not simple. We also find that a suitable choice of basis in the Lie algebra of \( G \) reduces the six conditions of Ref. 2 to two non-trivial conditions.

In Section 3, we apply the general analysis of Section 2 to a few specific examples. These include the simplest embedded defect we could think of - a domain wall embedded in a global \( U(1) \) model; then we construct the \( O(3) \rightarrow O(2) \) string of Ref. 2 and the known string solutions of the electroweak model. We provide further insight into electroweak strings and show that there is a one parameter family of string solutions - the \( W(\alpha) \) string, with \( \alpha \) being the parameter. All these string solutions are gauge equivalent in isolation but should be considered as being distinct when multi-string solutions are considered.

We turn to the issue of stability in Section 4. We first show that embedded global defects are unstable by constructing a sequence of field configurations that continuously lower the energy of the embedded global defect. This very construction is then applied to embedded gauge defects when the group \( G \) is simple and we find that they are unstable provided a certain condition on the group generators is satisfied. This analysis immediately shows that the \( O(4) \rightarrow O(3) \) monopole [2], the \( O(3) \rightarrow O(2) \) string and the electroweak \( Z^{-} \)-string at \( \sin^2 \theta_W = 0.5 \) are all unstable.

Embedded monopoles fall into the class of “non-topological” monopoles considered by Brandt and Neri [9] and by Coleman [10], who showed that such monopoles always suffer from a long range instability. We have found a simple physical understanding of the Brandt-Neri-Coleman instability in terms of the phenomenon of W-condensation [11]. This connection is described in Section 5.
It requires a little more cleverness to show that the electroweak \( W \) string is unstable. Fortunately, our realization that the sphaleron and electroweak strings are equivalent (discussed in Section 7) and also the elegant derivation of the sphaleron instability can be combined to show that the \( W \) string is unstable in the absence of bound states (Section 6).

In addition to strings, the electroweak model is known to contain a saddle-point solution called the “sphaleron”. The sphaleron is an important solution because it mediates baryon number violating processes. We discuss the connection [12] of the sphaleron to the electroweak string in Section 7. Our arguments show that the sphaleron can be interpreted as a collapsed segment or loop of electroweak string.

We summarize our findings in Section 8.

2. EMBEDDING SOLUTIONS

In this section we study the conditions for the existence of embedded defects. We are going to consider a field theory invariant under a symmetry group \( G \) that contains \( n \) simple factors, i.e. \( G = G_1 \times ...G_k \times ...G_n \). This is a generalization of Ref. 2 in which only a simple group \( G \) was considered.

The group is characterized by the Lie algebra

\[
[\tau^{a_k}, \tau^{b_k}] = f^{a_k b_k}_{c_k} \tau^{c_k},
\]

\[
[\tau^{a_k}, \tau^{b_{k'}}] = 0 \quad k \neq k',
\]

(2.1)

where \( \tau^{a_k} \) and \( f^{a_k b_k}_{c_k} \) are the generators and structure constants of the symmetry group \( G_k \). The energy functional for a static field configuration with only one Higgs field is

\[
E = \int d^3x \left[ \frac{1}{4} Tr(G_{ij}G_{ij}) + [D_i \phi]^\dagger [D_i \phi] + V[\phi] \right]
\]

(2.2)
with
\[ G_{ij} \equiv G_{ij}^{a_k} \tau^{a_k} \] (2.3)
and where a sum over repeated indices is implied, \( a_k \) runs over all the generators of \( G_k \) with \( k = 1 \ldots n \) and \( i, j = 1, 2, 3 \) are spatial indices. The Higgs potential, \( V[\phi] \), is required to lead to spontaneous symmetry breaking but is otherwise arbitrary.

In what follows, we shall choose our group generators to satisfy
\[ \text{Tr}(\tau^{a_k} \tau^{b_l}) = \delta^{a_k b_l} \delta^{kl}. \] (2.4)
In this basis the structure constants are antisymmetric in all three indices and the indices can be freely raised or lowered.

The covariant derivatives and field strength are defined by
\[ D_i = \partial_i + ig_k A_i^{a_k} \tau^{a_k} \] (2.5)
\[ G_{ij}^{a_k} = \partial_i A_j^{a_k} - \partial_j A_i^{a_k} + g_k f^{a_k b_k c_k} A_i^{b_k} A_j^{c_k} \]
where all indices are summed over in the covariant derivative and only \( b_k \) and \( c_k \) are summed over in the field strength (no sum over \( k \)). \( A_i^{a_k} \) is the gauge field associated with the generator \( \tau^{a_k} \) and \( g_k \) is the gauge coupling constant for \( G_k \).

To simplify our notation we are going to define
\[ T^{a_k} = g_k \tau^{a_k} \] (2.6)
\[ F^{a_k b_k c_k} = g_k f^{a_k b_k c_k}. \]
for every \( k \). From (2.1) the commutation relations for the generators are,
\[ [T^a, T^b] = F^{abc} T^c, \] (2.7)
where unsubscripted Roman indices run over all group generators. With these rescalings,
\[ D_i = \partial_i + iA_i^{a_k} \tau^{a_k} \] (2.8)
\[ G_{ij}^{a_k} = \partial_i A_j^{a_k} - \partial_j A_i^{a_k} + F^{abc} A_i^{b_k} A_j^{c_k}. \]
We are going to consider the possibility that the gauge fields that describe the embedded defect may be a linear combination of the $A_j^a$. For this purpose, let us define

$$A_j^a = \Lambda_0^a B_j^b$$

(2.9)

where $\Lambda$ is an orthogonal matrix ($\Lambda^T \Lambda = 1$). In terms of the new (rotated) gauge fields, the field strength is,

$$G_{ij}^a = \Lambda_0^a(\partial_i B_j^b - \partial_j B_i^b) + \mathcal{F}^{abc} \Lambda_d^b \Lambda^c_e B_i^d B_j^e.$$  

(2.10)

We define

$$\Lambda^a_0 \tilde{\mathcal{F}}^{bde} = \mathcal{F}^{abc} \Lambda_d^b \Lambda^c_e$$

(2.11)

so that

$$G_{ij}^a = \Lambda_0^a \left[ \partial_i B_j^b - \partial_j B_i^b + \tilde{\mathcal{F}}^{bde} B_i^d B_j^e \right].$$

(2.12)

Since $\Lambda$ is orthogonal, this gives,

$$G_{ij}^a G_{ij}^a = \left[ \partial_i B_j^a - \partial_j B_i^a + \tilde{\mathcal{F}}^{abc} B_i^b B_j^c \right]^2.$$  

(2.13)

We next look at the kinetic term of (2.2). If we define

$$\mathcal{T}^a = \Lambda_0^a T^b,$$

(2.14)

the covariant derivative will be

$$D_i \phi = \partial_i + i \mathcal{T}^a B_i^a.$$  

(2.15)

It is easy to check that $\tilde{\mathcal{F}}^{abc}$ are the structure constants for the $\mathcal{T}$ generators. Hence, in terms of the rescaled generators and fields, the general energy functional in (2.2), is identical to the energy functional for a field theory which is invariant under transformations belonging to a simple group. But the conditions for the existence of embedded defects in
the case of a simple symmetry group were already obtained in Ref. 2 and so we can
directly use those results. Instead of stating those conditions, however, we will follow a
more pedagogical approach and describe how embedded solutions can be constructed and
finally state the conditions that need to be satisfied.

The first step is to look for a subgroup $G_{emb}$ of $G$. Let the generators of $G_{emb}$ be $T^\alpha$
where the index on the generators of $G_{emb}$ are unbarred Greek indices. Then, since the
subalgebra \{T^\alpha\} closes, we have $\tilde{F}^{\alpha\beta\gamma} = 0$, where barred Greek indices denote generators
other than those generating $G_{emb}$. (Recall that we are working in the basis given by (2.4)
and so the structure constants are antisymmetric in all their indices. This means that
$\tilde{F}^{\alpha\beta\gamma} = 0 = \tilde{F}^{\alpha\beta\gamma}$, simply because $G_{emb}$ is a subgroup [2]). Using (2.11) we can write
these conditions in terms of the structure constants that correspond to the “un-mixed”
generators,

$$\sum_k g_k \Lambda^\alpha_{ak} \Lambda^\beta_{bk} \Lambda^\gamma_{ck} f^{akbkck} = 0.$$  \tag{2.16}

Note that, in this form, the condition depends on the coupling constants $g_k$.

Now $G_{emb}$ acts on a subspace of the vacuum manifold and not necessarily on the whole
manifold. Then the Higgs field may be decomposed: $\phi = \psi + \phi_\perp$ where $\psi$ forms a non-
trivial irreducible representation of $G_{emb}$ and $\phi_\perp$ is orthogonal to all $\psi$: $\text{Re}(\psi^\dagger \phi_\perp) = 0$.
The generators $T^\alpha$ mix the components of $\psi$ but do nothing to $\phi_\perp$ while the generators $T^{\bar{\alpha}}$
will mix the components of $\phi_\perp$ and also $\psi$ and $\phi_\perp$ among themselves. We will be interested
in the case when $\phi$ acquires a vacuum expectation value such that $G_{emb}$ is spontaneously
broken down to a subgroup $H_{emb}$ and when the symmetry breaking $G_{emb} \rightarrow H_{emb}$ contains
topological defects*. Let this topological defect solution be denoted by $\phi_{emb}$, $[B_\mu^\alpha]_{emb}$.
Since the topological defect is due to the non-trivial topology associated with the breaking

* Our considerations apply equally well to solutions that are not topological but are still present in the embedded theory $G_{emb} \rightarrow H_{emb}$. 
of $G_{emb}$, $\phi_{emb}$ consists of a non-trivial $\psi$ and vanishing $\phi_\perp$. That is, $\phi_{emb} = \psi_{emb} + (\phi_\perp)_{emb}$ where $(\phi_\perp)_{emb} = 0$.

The second step in the construction of the embedded defect is to set up a candidate embedded defect configuration:

$$\phi_{emb} = \psi_{emb}, \quad B_\alpha^\nu = [B_\alpha^\nu]_{emb}, \quad B_\mu = 0 \quad (2.17)$$

The third and final step is to check if the candidate configuration satisfies the constraints derived in Ref. 2:

$$\psi^\dagger \mathcal{T}_{\alpha} \phi_{emb} = 0 \quad (2.18)$$

for all $\psi$, and

$$V_\perp (\phi_{emb}) = 0 . \quad (2.19)$$

where the derivative is with respect to directions along $\phi_\perp$. If the candidate solution does satisfy these conditions, then it is a legitimate embedded defect.

The condition (2.18) has the interpretation that infinitesimal group elements not belonging to $G_{emb}$ should translate points in the subspace of the vacuum manifold covered by $\psi$ in directions that are orthogonal to the subspace. And the condition (2.19) ensures that the potential term in the energy functional is extremized by the candidate solution.

For completeness, we give the condition (2.18) in terms of the original unscaled variables occurring in the energy functional (2.2):

$$\sum_k g_k \Lambda_k \tau^a_k \phi_{emb} = 0 . \quad (2.20)$$

Once again the condition depends on the gauge coupling constants.
3. EXAMPLES

We are going to describe several examples in this section that will clarify the idea of embedded defects. We will first consider solutions of field theories that are invariant under a simple group $G$ and then study the more complicated case of the Weinberg-Salam model in which the group $G = SU(2)_L \times U(1)_Y$ is not simple.

**Walls** - The most trivial embedded solution is a domain wall embedded in a global $G = U(1)$ model. We express the Higgs field in terms of two real scalar fields $\phi^a$, $a = 1, 2$. A Lagrangian that is invariant under the global $U(1)$ rotation and describes static field configurations is,

$$L = \partial_i \phi^a \partial^i \phi_a - \lambda \left( \phi^a \phi^a - \eta^2 \right)^2,$$

with $i$ labeling the spatial coordinates.

As outlined in the previous section, the first step in constructing the embedded domain wall solution is to identify a $Z_2$ subgroup. Let us consider the $Z_2$ subgroup defined by the transformation: $(\phi_1, \phi_2) \to (-\phi_1, \phi_2)$. Any non-zero vacuum expectation value of $\phi_1$ will break this $Z_2$ subgroup completely and so the embedded symmetry breaking is $Z_2 \to 1$. This symmetry breaking has topological domain walls: $\phi_1 = \eta \tanh(x)$ and so this configuration for $\phi_1$ together with $\phi_2 = 0$ is our candidate solution.

The final step is to check if the conditions (2.18) and (2.19) are satisfied. Here, since we are considering global symmetries, (2.18) is trivially satisfied since the gauge coupling constants in (2.20) are zero. The condition on the potential is also easily checked to be satisfied.

**Strings** - We will now construct a $U(1) \to 1$ string solution [13, 14] embedded in a model with $G = SO(3)$ symmetry and where the Higgs is in the adjoint representation. The generators of $SO(3)$ are $(T^a)_{bc} = i \epsilon^{abc}$, with $a, b, c = 1, 2, 3$. $T^a$ is also the generator of $O(2)$
rotations around the $a$-th axis of group space. Let us consider the $O(2)$ subgroup generated by $T^3$. This subgroup will be broken completely if either of the first two components of $\Phi$ acquires a vacuum expectation value and hence there will be topological string solutions in the embedded symmetry breaking. So our candidate string configuration is:

$$\Phi = f_{\text{vor}}(r)e^{iT^3\theta} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A_i^3 = (A_i)_{\text{vor}}, \quad A_i^{\bar{\alpha}} = 0, \quad \bar{\alpha} = 1, 2$$

(3.2)

where $r, \theta$ are cylindrical coordinates and the subscript $\text{vor}$ indicates the Nielsen-Olesen vortex solution. It is easily checked that the conditions (2.18) and (2.19) are satisfied by this candidate configuration.

**Monopoles** - Next we consider the embedding of a ‘t Hooft-Polyakov monopole in a model with $O(4)$ symmetry. In this model the Higgs field $\Phi$ is in the adjoint representation and has four real components. The generators of $O(4)$ are: $(T^\alpha)_{JK} = i\epsilon^{\alpha JK}$, $(T^{\bar{\alpha}})_{JK} = \frac{1}{2}(\delta^{\bar{\alpha}I}\delta^{J4} - \delta^{\bar{\alpha}J}\delta^{I4})$ where $\alpha, \bar{\alpha} = 1, 2, 3$ and $I, J = 1, 2, 3, 4$. The $T^{\alpha}$ are also the generators of an $O(3)$ subgroup and if $\Phi$ gets a vacuum expectation value that is non-zero in the first three components, this subgroup breaks down to $O(2)$. Since the symmetry breaking $O(3) \rightarrow O(2)$ is known to lead to topological magnetic monopoles, we can at once write down the candidate embedded monopole solution:

$$\Phi = \left( \begin{array}{c} \vec{\phi}_{\text{tP}} \\ 0 \end{array} \right), \quad A_i^\alpha = [A_i^\alpha]_{\text{tP}}, \quad A_i^{\bar{\alpha}} = 0,$$

(3.3)

where the subscript $\text{tP}$ indicates the ‘t Hooft-Polyakov solution.

The candidate configuration (3.3) can be checked to satisfy all the necessary conditions to be an embedded magnetic monopole solution. An important point that should be noted here is that the long range gauge field of the embedded monopole is one of the three massless gauge fields that remain in the symmetry breaking $O(4) \rightarrow O(3)$ and that these three massless gauge fields transform in the adjoint representation of $O(3)$. 

**Electroweak strings** - As an example of a group $G$ that is not simple we consider the Weinberg-Salam [1, 15] model of the electroweak interactions. The symmetry breaking is: \( SU(2)_L \times U(1)_Y \rightarrow U(1) \). This symmetry breaking pattern is achieved in a Lagrangian of the general form corresponding to the energy functional in (2.2) if we take the Higgs field to be in the fundamental representation of \( SU(2) \). From the definition (2.14), the generators are \( T^a = \cos \theta w \tau^a \), for \( a = 1, 2, 3 \) and \( \tau^a \) are the \( 2 \times 2 \) Pauli matrices, \( T^4 = \sin \theta w I \) where \( I \) is a two dimensional unit matrix. The gauge field associated with these generators are \( W^a_\mu \) and \( B_\mu \) respectively.

As described in the previous section, the first step is to choose a subgroup. We choose this to be the \( U(1) \) subgroup generated by

\[
\mathcal{T}^3 = -\cos \theta w T^3 + \sin \theta w T^4 = \text{diag}(-\cos 2\theta w, 1). \tag{3.4}
\]

In order for the matrix \( \Lambda \) to be orthogonal we have to choose,

\[
\mathcal{T}^4 = \sin \theta w T^3 + \cos \theta w T^4. \tag{3.5}
\]

Then the non-trivial elements of the \( \Lambda \) matrix are

\[
\Lambda^1_1 = \Lambda^2_2 = 1, \quad \Lambda^3_3 = -\Lambda^4_4 = -\cos \theta w, \quad \Lambda^3_4 = \Lambda^4_3 = \sin \theta w. \tag{3.6}
\]

Now the candidate embedded string solution may be written down:

\[
\phi_{emb} = f_{vor}(r)e^{i T^3 \theta} \varphi_0. \tag{3.7}
\]

where, we take,

\[
\varphi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.8}
\]

Here, \((r, \theta)\) are polar coordinates.
It is easy to check that the condition (2.18) for the existence of the embedded defect is satisfied because
\[ \varphi_0^\dagger T^1 \varphi_0 = \varphi_0^\dagger T^2 \varphi_0 = \varphi_0^\dagger T^4 \varphi_0 = 0. \] (3.9)
The potential condition can also be easily checked to be satisfied.

With the choice of (3.8) for \( \varphi_0 \), it is easy to see that the solution corresponds to a string with \( Z \) magnetic flux because \( Z_\mu \) is the gauge boson associated with the generator \( T^3 \). Therefore the full embedded \( Z \) string solution is:
\[ \phi_{emb} = f_{vor}(r) \begin{pmatrix} 0 \\ i \end{pmatrix} , \quad Z_\mu = (A_i)_{vor} , \quad W^1_i = W^2_i = 0 = (A_i)_{em} . \] (3.10)

Different choices of the (“embedded”) subgroup leads to other string solutions. The choice that we now consider is the subgroup sitting entirely in the \( SU(2) \) factor of the electroweak model and generated by: \( T_\alpha \equiv \sin \alpha \tau^1 + \cos \alpha \tau^2 \), where \( \alpha \) is some constant. Then the corresponding embedded string solution is:
\[ \phi_{emb} = f_{vor}(r)e^{i T_\alpha \theta} \varphi_0 , \quad \sin \alpha W^1_i + \cos \alpha W^2_i = (A_i)_{vor} , \] (3.11)
and all other orthogonal combinations of gauge fields vanish.

The one parameter family of string solutions in (3.11) is called the \( W \) string since the flux in the string is purely in the \( SU(2) \) sector. Furthermore, by a global gauge transformation, any single string solution in the family - that is, a string with any value of \( \alpha \) - may be transformed into the string configuration with \( \alpha = 0 \). Explicitly, this gauge transformation is:
\[ \phi' = \exp \left[ -i \frac{(1 + \tau^3)}{2} \alpha \right] \phi \] (3.12)

Together with a corresponding transformation of the gauge fields. This does not, however, mean that multi-string solutions of different \( \alpha \) can be gauge transformed to another multi-string solution with all strings having the same value of \( \alpha \). The simplest way to see that
\( \alpha \) is a non-trivial parameter is to consider a loop of \( W \) string such that \( \alpha \) runs from 0 to \( 2\pi \) as we go around the loop. The winding of \( \alpha \) around the loop is a discrete number and cannot be altered by any non-singular gauge transformation. Hence, a loop with varying \( \alpha \) is not gauge equivalent to one with a constant value of \( \alpha \).

There are two other arguments that led us to the conclusion that \( \alpha \) is not a gauge artifact but a genuine label for different string solutions. The first of these is that if we consider two \( W \) strings with \( \alpha = 0 \), they would combine to form a winding number 2 string with \( \alpha = 0 \) and, in particular, would not annihilate each other when brought together. On the other hand, an \( \alpha = \pi \) string is the anti- of the \( \alpha = 0 \) string and these would annihilate each other if brought together. Therefore the system with two \( \alpha = 0 \) strings must be gauge inequivalent to the system with one \( \alpha = 0 \) string and another \( \alpha = \pi \) string. The second argument is that if we try and construct a straight \( W \) string with varying \( \alpha \), we necessarily find that the gradients in \( \alpha \) cause electromagnetic fields to emanate from the string and hence the variations in \( \alpha \) cannot be gauge artifacts.

To summarize, we have constructed the \( Z \) string and a one parameter family of distinct \( W \) strings present in the Weinberg-Salam model of the electroweak interactions.

4. STABILITY

In this section we consider the stability of embedded defects. Although the question of which embedded solutions are stable against small perturbations is not answered in its full generality, a large class of embedded solutions are shown to be unstable by explicitly indicating a particular instability. Qualitatively, this mode can be described as a combination of a dilatation and a rotation of the embedded solution into the trivial vacuum. The instability that we have found only applies when the embedded gauge group \( G_{emb} \)
acts trivially on the subspace spanned by $\phi_\perp$ and when the potential is of the Mexican hat variety. One outcome is that all global embedded defects in models with a Mexican hat potential are unstable to small perturbations.

Let us consider an embedded solution in a model with the energy functional given in (2.2) and with the specific form of the potential:

$$V[\phi] = \lambda \left( \phi^\dagger \phi - \eta^2 \right)^2.$$  \hfill (4.1)

The embedded defect solution is given in (2.17) and the Higgs field for the solution will be denoted by $\phi_0$. Now consider the sequence of configurations labeled by the parameter $\xi$:

$$\begin{align*}
\phi(x; \xi) &= \cos \xi \phi_0(\cos \xi x) + \sin \xi \phi_\perp \\
A_j(x; \xi) &= \cos \xi A_j(\cos \xi x)
\end{align*}$$  \hfill (4.2)

where $\phi_\perp$ is constant with $\phi_\perp^\dagger \phi_\perp = \eta^2$ and $\phi_\perp^\dagger \phi_0(x) = 0$. For $\xi = 0$, the configuration is the embedded defect solution and for $\xi = \pi/2$, the configuration describes the vacuum.

We then have,

$$\begin{align*}
\phi^\dagger(x; \xi)\phi(x; \xi) &= \cos^2 \xi \phi_0^\dagger \phi_0 + \sin^2 \xi \eta^2 \\
G_{ij}^a(x; \xi)G_{ij}^a(x; \xi) &= \cos^4 \xi G_{ij}^a(x)G_{ij}^a(x), \\
D_i \phi(x; \xi) &= \cos^2 \xi \left[ \partial_i + i A_i^a(x) T^a \right] \phi_0(x) + i \cos \xi \sin \xi A_i^a(x) T^a \phi_\perp, \\
V[\phi(x, \xi)] &= \cos^4 \xi V[\phi_0(x)]
\end{align*}$$  \hfill (4.3)

If the orthogonality condition

$$T^\alpha \phi_\perp = 0$$  \hfill (4.4)

is satisfied, we get

$$\begin{align*}
[D_i \phi(x; \xi)]^\dagger [D_i \phi(x; \xi)] &= \cos^4 \xi (D_i \phi_0(x))^\dagger (D_i \phi_0(x)) \,.
\end{align*}$$  \hfill (4.5)

and hence, the total energy of the configuration is

$$E(\xi) = \cos^d \xi E(\xi = 0),$$  \hfill (4.6)
where $d$ is the dimension of the world-surface of the defect. ($d = 1, 2, 3$ for monopoles, strings, and walls respectively.) For $d > 0$ this shows that the solution is quadratically unstable because $\xi$ parameterizes a smooth sequence of configurations with monotonically decreasing energy starting at the embedded defect solution ($\xi = 0$) and ending at the vacuum ($\xi = \pi/2$).

We are now prepared to consider some specific examples. For a global embedded defect the condition (4.4) is trivially satisfied because there are no gauge fields. (Recall that the generators $\mathcal{T}^\alpha$ have been rescaled with the gauge coupling constants as in (2.6) and so, in the global case, $\mathcal{T}^\alpha = 0$.) Therefore, all embedded global defects are unstable.

At this juncture we should point out that the existence of a sequence of lower energy configurations such as that given in (4.6) does not say anything about the time scales associated with the dynamical instability [16]. This is because the inertia associated with the instability could be large and this would cause the instability to be slower. The issue is even more tricky when we consider global defects since the energy of global strings and global monopoles diverges and there is a possibility that the inertia associated with the above instability would also diverge. In this case, although (4.6) is valid, the time scale associated with the instability is infinite and the defect would not decay. The best way to see that this is not the case is to consider perturbations that are truncated beyond a certain distance. For example, we could consider $\xi$ to have the following form:

$$\xi(r < R, t) = \xi_0(t)$$
$$\xi(r > R, t) = \xi_0(t) \exp \left[ \frac{-(r - R)}{l} \right]$$

(4.7)

where, $R$ and $l$ are some length scales that are large compared to the core of the defect. We have reconstructed the energy of such configurations for small values of $\xi_0$ (without the rescaling of the coordinate in (4.2)) and kept the kinetic term due to the time derivative
of $\xi_0(t)$. This calculation confirms that the inertia of the truncated perturbation (4.7) is not infinite and that the instability is on a finite time scale.

We next consider the embedded gauge monopole solution constructed in Sec. 3 and given by (3.3). The fourth component of the vector $\Phi$ is annihilated by the action of $T^\alpha$ the generators of $G_{\text{emb}} = O(3)$. Hence, (4.4) shows that this embedded monopole is unstable.

We now consider two important cases for which the procedure for demonstrating instability described above fails. Let us first consider the $W$ string in the electroweak model with $G = SU(2)_L \times U(1)_Y$. Here the condition (4.4) cannot be satisfied because $U(1)_Y$ does not annihilate any nonvanishing Higgs field and so the instability does not apply.

For the embedded $Z$-string solutions in the electroweak model the situation is slightly more complicated. It turns out that the above argument for instability fails except for the special case $\theta_W = \pi/4$. The string is generated by $T^3 = \text{diag}(-\cos 2\theta_w, 1)$ From the orthogonality condition $\text{Re}(\phi_\perp \phi_0(x)) = 0$, we can choose $\phi_\perp = (1, 0)$. Therefore, the condition for instability, eqn. (4.4), becomes $\cos 2\theta_W = 0$, that is, $\theta_W = \pi/4$.

This last result is of some importance because the stability analysis of Ref. 6 did not consider the case of very low Higgs masses and it was not clear from the given stability diagram if it would be possible to find some value of the Higgs mass for which the $Z$ string in the standard model with $\sin^2 \theta_w = 0.23$ could be stable. Our result here shows that the $Z$ string is unstable for all Higgs masses at $\sin^2 \theta_w = 0.5$, making it extremely unlikely for there to be stable solutions for smaller values of $\sin^2 \theta_w$.

So far we have ignored the possibility that there may be bound states on the embedded defects. It has been shown [8] that such bound states can considerably enhance the stability of the defect. The physical reason behind this enhancement is the same as the reason behind the existence of non-topological solitons [17] and is discussed in some detail in
Ref. 8. Mathematically, the introduction of a bound state would result in the presence of additional terms in the varied energy functional (4.6) that would be proportional to $\sin^2 \xi$. With these additional terms, it is possible that $\xi = 0$ describes a local minimum of the energy and so there is no instability towards increasing $\xi$.

### 5. INSTABILITY OF EMBEDDED MONOPOLES

The stability of monopoles has been studied by Brandt and Neri [9] and by Coleman [10]. They find that the asymptotic magnetic field of the monopole has an instability unless the monopole is topological. Here we will give a simple explanation of the Brandt-Neri-Coleman instability in the context of embedded monopoles [18].

The key to understanding the instability is already present in the work on W-condensation [11]. The idea is that a spin $s$ particle in a uniform magnetic field $\vec{B} = B\hat{z}$ has energy levels given by:

$$E_n^2 = (2n + 1)eB - 2eBs + k_3^2 + m^2$$  \hspace{1cm} (5.1)

where the mass of the particle is $m$, the electric charge is $e$ and the momentum in the $z$ direction is $k_3$. The first term in (5.1) is the Landau level term and is due to the orbital motion of the particle, the second term is the spin-magnetic field coupling, the third term is the kinetic energy due to the motion along the magnetic field and the last term is due to the rest mass of the particle. Note that the g-factor, that is, the numerical factor in the spin-magnetic field coupling, has been taken to be 2. For our purpose, since the magnetic field and the spin 1 field all belong to the adjoint representation of a non-Abelian group, this will be true [11]. The crucial observation is that for $s = 1$ and for large enough magnetic fields, $E_{n=0}^2$ can be negative and the system can decay to a state of lower energy by the creation of spin 1 particles. The critical magnetic field strength needed for the
instability is:

\[ B_c = \frac{m^2}{e} \]  

(5.2)

Now consider the embedded monopole discussed in Sec. 3. The magnetic field of the monopole is one of the three gauge fields of the \( O(3) \) residual symmetry group. Hence, far away from the monopole, the effective Lagrangian for the gauge fields is:

\[ L = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \]  

(5.3)

where \( a = 1, 2, 3 \). In the asymptotic region with, say, \( z > 0 \), the monopole magnetic field is simply like a uniform magnetic field. Therefore, in this region, the field configuration is:

\[ F_{12}^{a=3} = B(z) \]  

(5.4)

and the \( a = 1, 2 \) components are zero. But now we can apply the W-condensation arguments to this field since there are other spin 1 gauge fields present in the system - namely, the two other gauge fields of the \( O(3) \) theory. These gauge fields are all massless and hence the critical field needed for an instability is zero (eqn. (5.2)). This means that the magnetic field of the embedded monopole suffers from the same instability that leads to W-condensation.

The literature on W-condensation makes a further point that the phenomenon of W-condensation actually \textit{anti-screens} the applied magnetic field. Naively, this would imply that the magnetic field of the monopole would increase due to the instability. However, this is not true since, as pointed out by Ambjorn and Olesen [11], only U(1) fields are anti-screened. The non-Abelian gauge fields become larger due to the instability but in such a way that the field strengths become smaller. This can also be checked to be the case for the embedded monopole and hence, the magnetic field of the embedded monopole is diminished due to the instability.
The instability of the core structure of global defects described in Sec. 4 also applies to the embedded monopole. Therefore the embedded monopole suffers from two instabilities: the first is that the core spreads out and the second is that the long range magnetic field gets screened due to W-condensation. By considering the stability of embedded monopoles in the presence of bound states, it seems to us that the first of these instabilities can be avoided but that the second instability is incurable.

6. W STRING INSTABILITY

Here we will show that the bare W string is unstable for a large range of parameters. The idea of the proof follows from the observation that the sphaleron and the W string are closely related (see the following section) and so the instability of the sphaleron found by Manton \[19\] might well apply to the W string also.

The energy functional we want to consider is that given in (2.2) with \(SU(2) \times U(1)\) symmetry and the Higgs in the fundamental representation of \(SU(2)\). The potential is taken to be the Mexican hat potential given in (4.4) together with \(\eta^2 = 1\). (This amounts to suitably rescaling the fields and the coordinates.) The W string solution is given in (3.11) and, to be specific, we consider the \(\alpha = \pi/2\) solution.

Now consider the family of Higgs field configurations labeled by the index \(\mu\):

\[
\Phi(\mu, r, \theta, z) = (1 - f_{\text{vor}}(r)) \left( e^{-i\mu \cos \mu} \right) + f_{\text{vor}}(r) \left( e^{-i\mu (\cos \mu + i\sin \mu \cos \theta)} \sin \mu \sin \theta \right)
\]

(6.1)

and, the family of gauge field configurations:

\[
A_j(\mu, r, \theta, z) = -iv_{\text{vor}}(r)(\partial_j U)^{-1}
\]

(6.2)

where

\[
U = \begin{pmatrix}
  e^{i\mu (\cos \mu - i\sin \mu \cos \theta)} & \sin \mu \sin \theta \\
  -\sin \mu \sin \theta & e^{-i\mu (\cos \mu + i\sin \mu \cos \theta)}
\end{pmatrix}
\]

(6.3)
Note that this family of configurations is almost identical to the family considered by Manton for the case of the sphaleron. The differences are that we are working in cylindrical coordinates and that we have discarded the \( \phi \) (the spherical azimuthal angle) dependence that is present in the sphaleron.

The configurations in (6.1)-(6.3) yield the \( W \) string when \( \mu = \pi/2 \) and at \( \mu = 0 \) or \( \mu = \pi \), the configuration is that of the trivial vacuum. The parameter \( \mu \) parametrizes a path from the \( W \) string configuration to the trivial vacuum.

The energy of the configurations can be found by inserting (6.1)-(6.3) into the energy functional (2.2). Then, after some algebra we get,

\[
E(\mu) = 2\pi \int dzdr \left( \sin^2 \mu \left( \frac{v'}{gr} \right)^2 + \sin^2 \mu f'^2 + \frac{\sin^2 \mu}{r^2} \left[ \cos^2 \mu \left\{ v^2(1-f)^2 - 2v(1-f)f(1-v) \right\} + f^2(1-v)^2 \right] \right) \tag{6.4}
\]

where the subscript \( vor \) on the functions \( f \) and \( v \) has been dropped for convenience. It is clear that \( E \) is an increasing function of \( \sin^2 \mu \) at \( \mu = \pi/2 \) provided

\[
E_1 \equiv \sin^2 \mu \cos^2 \mu \int \frac{dr}{r} [v^2(1-f)^2 - 2v(1-f)f(1-v)] + \sin^2 \mu \int \frac{dr}{r} f^2(1-v)^2 \tag{6.5}
\]

is a monotonically increasing function of \( \sin^2 \mu \) at \( \mu = \pi/2 \). Let us denote the two integrals in (6.5) by \( I_1 \) and \( I_2 \) respectively and write \( \xi \equiv \sin^2 \mu \in [0,1] \). By differentiating (6.5) with respect to \( \xi \), we find that \( E_1 \) is a monotonically increasing function of \( \xi \) if \( I_2 > (2\xi - 1)I_1 \). We have checked this condition numerically at \( \xi = 1 \) for certain values of the parameters \( (\lambda \text{ and } g) \) and always found it to be satisfied. This shows that the \( W \) string is unstable for the parameters we considered.

We wish to remark that if we could show that

\[
s(\xi) \equiv \xi(1-\xi)[v^2(1-f)^2 - 2vf(1-f)(1-v)] + \xi f^2(1-v)^2 \tag{6.6}
\]
is an increasing function of $\xi$, then the condition regarding $E_1$ would also be satisfied. Now it is straightforward to show that $s(\xi)$ is maximum at $\xi = 1$ if
\[(1 + \sqrt{2})f(1 - v) \geq v(1 - f)\] (6.7)
for all $r$. Numerical evaluations of the Nielsen-Olesen vortex profile have shown that (6.7) is satisfied for almost all $r$ for a large range of values of $\lambda$. The inequality (6.7) is violated only in the large $r$ region, where the integrands in (6.5) are exponentially small. So the contributions that could change the monotonic increase of $E_1$ with $\xi$ are exponentially suppressed and $E_1(\xi)$ is an increasing function of $\xi$ for a wide range of parameters.*

7. ELECTROWEAK STRINGS AND THE SPHALERON

In this section, we connect [12] the electroweak string solutions with the sphaleron solution discovered by Manton [19]. In the limit $\sin^2 \theta_W \to 0$, the sphaleron solution is [20]:
\[\Phi = f_s(r) \left( e^{i\phi} \sin \theta \right) \cos \theta \equiv f_s(r) U \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad A_\mu \equiv W^a_\mu \tau^a = -iv_s(r)(\partial_\mu U)U^{-1}\] (7.1)
where we are now using spherical coordinates, $\tau^a$ are the Pauli spin matrices and the subscript $s$ on the radial functions $f$ and $v$ denote that these functions are particular to the sphaleron.

The sphaleron solution in (7.1) necessarily has an accompanying electromagnetic field which can be calculated [20] to first order in $\theta_W$. Alternately, it can be derived from the gauge invariant definition of the electromagnetic field strength in the electroweak model [21]:
\[F^em_{\mu\nu} = \partial_\mu A^em_\nu - \partial_\nu A^em_\mu - i4g^{-1}\eta^{-2}\sin \theta_W [\partial_\mu \phi \dot{\phi} + \partial_\nu \phi \dot{\phi} - \partial_\nu \phi \dot{\mu} \partial_\mu \phi]\] (7.2)

* Note that a condition similar to (6.7) is assumed to hold in the case of the sphaleron [19].
where, the electromagnetic gauge potential is

$$A_{\mu}^{em} = \sin\theta_W n^a W^a_{\mu} + \cos\theta_W B_{\mu}$$

(7.3)

with \(n^a \equiv -2\hat{\tau}^a \phi/\eta^2\).

The back-reaction of the electromagnetic field on the sphaleron configuration is second order in \(\theta_W\) and can be ignored in the limit that we are considering.

Now the matrix \(U\) in (7.1) is a unitary matrix and may be written as:

$$U = \exp[i\hat{m} \cdot \vec{\tau} \theta]$$

(7.4)

where, \(\hat{m} = (\sin\phi, \cos\phi, 0)\) and \((\theta, \phi)\) are spherical angles. A comparison of (7.4) with (3.11) immediately suggests that the sphaleron configuration (7.1) is precisely that of a ("twisted") loop of W string in which \(\alpha = \phi\). One difference is that \(\theta\) is a spherical angle and ranges from 0 to \(\pi\) in (7.4) whereas in the W string it is a cylindrical angle and ranges from 0 to \(2\pi\). Another difference is that the Higgs field vanishes only at one point in the sphaleron while in the case of the W string loop, it vanishes along a one-dimensional closed curve. Both these differences can be reconciled by imagining a twisted loop of W string that has collapsed to a single point. In this case, one should indeed restrict \(\theta\) to go from 0 to \(\pi\) and have a vanishing Higgs field at only one point.

One could also get different interpretations of the sphaleron in terms of electroweak strings by considering various slices of (7.1). For example, the \(xz\) and \(yz\) slices of the sphaleron yield the W string for \(\alpha = \pi/2\) and \(\alpha = 0\) respectively. Therefore, "stretching" deformations of the sphaleron along any axis in the \(xy\) plane would yield finite segments of W strings. Reversing this argument tells us that a segment of W string (for any \(\alpha\)) collapses into a sphaleron.
The final interpretation of the sphaleron that we point out seems like the most interesting to us. It is possible to arrive at this interpretation in two ways which we now describe.

If we look at the \( xy \)-slice of the sphaleron, that is, if we set \( \theta = \pi/2 \) in (7.1), we find that the resulting configuration is precisely that of the \( Z \) string in (3.10) upto the profile functions and a global gauge transformation. Hence, if we were to stretch the sphaleron along the \( z \)-axis, we would get a segment of \( Z \) string. Or in other words, a finite segment of \( Z \) string collapses into a sphaleron. But we know from the work of Nambu [22] that the \( Z \) string ends on magnetic monopoles. Hence, the sphaleron must be equivalent to a monopole sitting adjacent to an antimonopole along the \( z \) axis.

This interpretation can be arrived at directly by looking at the Higgs field configuration of the electroweak monopole found by Nambu [22]:

\[
\Phi = f_m(r) \left( e^{i\phi} \sin\theta/2 \cos\theta/2 \right)
\]

and comparing to the sphaleron Higgs field configuration given in (7.1). (The gauge fields for the monopole are given by the same formula as in (7.1).) The two configurations are identical upto the profile functions and, more importantly, upto a factor of 2 wherever \( \theta \) appears. Hence, as \( \theta \) is varied from 0 to \( \pi/2 \) in the sphaleron configuration, the full monopole configuration is mapped out. Then as \( \theta \) is varied from \( \pi/2 \) to \( \pi \) in the sphaleron, an antimonopole configuration is mapped out. This directly confirms that the sphaleron can be viewed as a monopole and an antimonopole sitting adjacent to each other.

The above interpretations lead to a picture for the space of configurations in the vicinity of the sphaleron in the electroweak model. The sphaleron is a saddle point solution to the electroweak equations of motion and has one unstable mode [23]. Therefore it is useful to think of an ordinary 2-dimensional saddle embedded in 3-dimensional space with the
saddle point being the sphaleron. Now, when we deform the sphaleron to get segments of string, we are going to higher energy configurations and so we are climbing up the saddle. Of course, there are many ways of going to higher energy configurations but there is a special one - the one which goes along the ridge of the saddle. Furthermore, once we have found this configuration, it will have two unstable modes: the first one causes the configuration to roll down into the sphaleron while the second instability is towards rolling off in a direction orthogonal to the ridge. These instabilities are exactly what we see for a finite segment of $Z$ string when $\sin^2 \theta W$ is not too close to 1. The segment of string is dynamically unstable to collapsing to shorter lengths and this corresponds to rolling down the ridge into the sphaleron configuration. The infinite $Z$ string also has an unstable mode and this can be viewed as the mode that causes the string to slide off the ridge.

It is interesting to consider what might happen when $\sin^2 \theta W$ is close to 1. Then the infinite $Z$ string is metastable and the mode that corresponds to sliding off the ridge is absent. This means that the saddle is not of the usual kind - it contains two ridges that go up from the saddle point and the $Z$ string lies in the valley between the two ridges. Assuming that the connection of the sphaleron and $Z$ string remains valid at large $\theta W$ also, a finite segment of string will collapse into a sphaleron configuration by rolling in the valley between the two ridges.

The sphaleron is known to be the intermediate point in processes that violate baryon number. In the language of the saddle, if we consider a sequence of configurations that pass from one side of the saddle to the other side, the baryon number of the configuration changes [19]. However, in going from one side to the other, it is necessary to pass through either the sphaleron configuration - which would be the least energetically expensive way - or to pass through a string configuration. Therefore, electroweak strings will play the same role as the sphaleron in baryon number violating processes.
At low energies, it is unfavourable to have long strings and the shortest string - the sphaleron - will dominate all baryon number violating processes. However, the presence of electroweak strings can become interesting if there is a mechanism that can stabilize string segments for a sufficiently long duration [8]. In the past, mechanisms have been found that prevent topological superconducting loops [24, 25, 26, 27, 28] from collapsing and it is an open question as to whether those mechanisms will apply in the electroweak case too.

An issue that we have not investigated but feel could be very interesting is the possible connection of electroweak strings with the (deformed) sphaleron solutions found by Yaffe [23] for large values of the Higgs mass.

8. SUMMARY

We itemize our main results:

(i) We have described a procedure by which embedded defect solutions may be constructed in Section 2. The procedure applies whether the symmetry group of the theory is simple or not. Examples were provided in Section 3. In particular, it was shown that the electroweak model contains the \(Z\) string and a one parameter family of \(W\) strings.

(ii) In Section 4 we have considered the stability of embedded defects in the absence of bound states. By considering a specific perturbation of the embedded defect solution, it was shown that embedded global defects are unstable and that embedded gauge defects are unstable if condition (4.4) is satisfied. By an application of this condition, we showed that the electroweak \(Z\) string is unstable when \(\sin^2 \theta_W = 0.5\). By considering a separate argument (Section 6), we showed that the \(W\) string is unstable for a wide range of parameters.

(iii) In Section 5 we explained the Brandt-Neri-Coleman instability for embedded gauge
monopoles in terms of the phenomenon of W-condensation.

(iv) In Section 7, we showed that the sphaleron may be reinterpreted as segments or loops of electroweak strings.

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