Orthounimodal Distributionally Robust Optimization: Representation, Computation and Multivariate Extreme Event Applications

Henry Lam* Zhenyuan Liu* Xinyu Zhang*

Abstract

This paper studies a basic notion of distributional shape known as orthounimodality (OU) and its use in shape-constrained distributionally robust optimization (DRO). As a key motivation, we argue how such type of DRO is well-suited to tackle multivariate extreme event estimation by giving statistically valid confidence bounds on target extremal probabilities. In particular, we explain how DRO can be used as a nonparametric alternative to conventional extreme value theory that extrapolates tails based on theoretical limiting distributions, which could face challenges in bias-variance control and other technical complications. We also explain how OU resolves the challenges in interpretability and robustness faced by existing distributional shape notions used in the DRO literature. Methodologically, we characterize the extreme points of the OU distribution class in terms of what we call OU sets and build a corresponding Choquet representation, which subsequently allows us to reduce OU-DRO into moment problems over infinite-dimensional random variables. We then develop, in the bivariate setting, a geometric approach to reduce such moment problems into finite dimension via a specially constructed variational problem designed to eliminate suboptimal solutions. Numerical results illustrate how our approach gives rise to valid and competitive confidence bounds for extremal probabilities.

Keywords — multivariate extreme event analysis, orthounimodality, distributionally robust optimization, nonparametric, shape constraint

1 Introduction

Distributionally robust optimization (DRO) is a methodology to tackle optimization under uncertainty that has gathered substantial attention in recent years. This methodology advocates a robust perspective that, when facing uncertain or unknown parameters in decision-making, the modeler looks for a decision that optimizes over the worst-case scenario (Delage and Ye (2010); Goh and Sim (2010); Kuhn et al. (2019)). More precisely, DRO can be considered as a special case of classical robust optimization (RO) (Bertsimas et al.)

*Department of Industrial Engineering and Operations Research, Columbia University.
In which the uncertain parameter is the underlying probability distribution in a stochastic problem, and the worst case is over constraints constituting a so-called uncertainty set or ambiguity set that, roughly speaking, contains the true distribution with high confidence. In this paper, we will use the term DRO broadly to refer to worst-case optimization over a class of distributions defined via the uncertainty set.

Our focus of study is a particular uncertainty set represented by a shape constraint on a multivariate probability distribution known as orthounimodality (OU). On a high level, this geometric property means that each marginal density of the underlying distribution is monotonically non-increasing away from its mode. OU is arguably the most basic multivariate shape constraint, and has appeared in works such as Devroye (1997); Biou and Devroye (2003); Sager (1982); Polonik (1998); Gao and Wellner (2007). Yet, a systematic investigation of its geometric properties, and the associated optimization strategies for the corresponding DRO, appears open. More specifically, we will build the mixture, or more precisely the so-called Choquet representation that allows us to represent an OU distribution as a mixture of more “elementary” distributions which act as the extreme points in convex analysis. This subsequently allows us to reformulate the associated DRO in terms of decision variables that correspond to the mixing distribution (without the OU constraint). However, as we will demonstrate, the Choquet representation of OU differs from the range of distributional shapes discussed in the literature (both in probability theory, e.g., Dharmadhikari and Joag-Dev (1988), and in DRO, e.g., Van Parys et al. (2016); Li et al. (2019)), which substantially increases the complexity of the reformulated DRO to contain function-valued random variables and in turn necessitates new variational arguments to reduce the problem to a tractable form.

Our interest in OU is motivated from multivariate extreme event analysis (Resnick (2013)). This discipline studies the estimation of tail probabilities or other risk quantities from data, and is evidently at the core of risk analytics and management. A beginning well-known challenge in this task is that, by its own definition, there are little data thatinform the tail of a distribution. To this end, multivariate extreme event analysis has proposed a range of statistical methods to extrapolate data to their tail, under some principled modeling assumptions. However, the challenges of these methods are also well-documented, and exacerbated especially in multivariate settings. A key motivation of our paper is to advocate DRO as a well-grounded alternative for multivariate extreme event analysis. We show in particular how OU, in contrast to other more well-studied distributional shapes, constitutes the most natural uncertainty set for this purpose. This is argued in terms of both interpretability on the multivariate tail and robustness against the misspecification of the distributional mode, two main challenges critically faced by other established distributional shapes as we will illustrate. From a broader perspective, the power of DRO in tackling uncertainty under partial distributional information has been actively investigated (e.g., Wiesemann et al. (2014); Hanasusanto et al. (2015); Doan et al. (2015); Ghaoui et al. (2003)) – Our current work follows this perspective, but also distinguishes from it by taking a specialized step to justify the choice of uncertainty sets, develop both the probability and optimization theories, in an important application that has traditionally been handled using parametric statistical models. While our Choquet theory applies to arbitrary-dimensional problems, we will focus our tractable reduction and optimization method on the bivariate setting which, as we will discuss momentarily, already forms a challenging case for traditional statistical approaches, and also requires intricate geometric arguments to handle its DRO. We hope that these developments would open the door to higher-dimensional extremal estimation problems in the future.
In the following, we first discuss the challenges in multivariate extreme event analysis that motivates the use of our OU-DRO (Section 2). We then introduce in detail the main geometric properties of OU that pertain to optimization (Section 3). After that, we present our main DRO formulation (Section 4), and our reformulation approaches and optimization methods (Section 5). Next we motivate and discuss the generalization of OU-DRO to situations where the considered monotonicity only holds for part of all dimensions (Section 6). We present some numerical examples (Section 7). Supplemental details and all proofs are delegated to the Appendix.

2 Motivation

As discussed in the Introduction, our study is motivated from multivariate extreme event analysis. In this section, we first discuss the challenges in conventional methods in this discipline, starting with the univariate case (Section 2.1) which sets the stage to transit to multivariate (Section 2.2). We then propose DRO as an alternative approach, discuss the literature, and also present the challenges in using existing DRO formulations (Section 2.3). Motivated from these challenges, we finally propose OU-DRO as our solution approach (Section 3).

2.1 Challenges in Conventional Extreme Event Analysis: Univariate Case

Extreme event analysis refers to the estimation of tail probabilities, quantiles or other measures, and is a core common task in analyzing and managing risks. For instance, in the maritime industry, estimating the extremes of the metocean climate is essential to design oil rigs and other marine structures (Zachary et al. (1998)). In finance, prediction of tail risk measures such as value-at-risk is used to manage portfolio losses (Longin (2000); McNeil (1999)). In insurance, product pricing is stress-tested by modeling large claims and estimating ruin probabilities (McNeil (1997); Beirlant and Teugels (1992)). In transportation, safety analysis is built on the estimation of crashes and other defined conflicts that are rare events by nature (Jonasson and Rootzén (2014); Songchitruksa and Tarko (2006)).

A recurrent challenge in extreme event analysis is that, by its very definition, there are few data available to fit the tail distribution. A dominant approach in the statistics literature is to use extreme value theory, which suggests the use of parametric models to extrapolate data based on principled asymptotics. Below we will discuss these models and their documented challenges, starting from the univariate case and then transiting to the multivariate case, which is more subtle and constitutes our focus.

Extreme value theory in the univariate setting hinges on two important asymptotic theorems that characterize the parametric distributions one should use in fitting tails, leading to the so-called annual-maxima method (Gumbel (1958)) and peak-over-threshold method (Smith (1984)) respectively. To set the stage, let us denote \( \{X_1, \ldots, X_n\} \) as i.i.d. data or random variables in \( \mathbb{R} \). The first theorem, known as the Fisher–Tippett–Gnedenko theorem (Fisher and Tippett (1928), Gnedenko (1943)), states that under some technical conditions (see, e.g., Embrechts et al. (2013) Section 3.3 and 3.4), the maximum of i.i.d. random variables, namely \( \max\{X_1, \ldots, X_n\} \), converges to the generalized extreme value (GEV) distribution

\[
G_{\xi, \mu, \sigma}^{\text{GEV}}(x) = \begin{cases} 
\exp \left[- \left(1 + \frac{x - \mu}{\sigma} \right)^{-1/\xi}\right] & \text{if } \xi \neq 0 \\
\exp \left[- \exp \left(- \frac{x - \mu}{\sigma}\right)\right] & \text{if } \xi = 0
\end{cases}
\]

(1)
under suitable normalization, where $\mu$ is a location parameter and $\sigma > 0$ is a scale parameter. Depending on the value of $\xi$, this distribution is categorized into three regimes known as Gumbel ($\xi = 0$), Fréchet ($\xi > 0$) and Weibull ($\xi < 0$), each of which classifies a random variable according to its so-called maximum domain of attraction. This theorem suggests the fitting of data into $G^{\text{GEV}}_{\xi, \mu, \sigma}$, provided that the data are first batched into blocks in which the maximum is taken from each block. The second theorem, known as the Pickands–Balkema–de Haan theorem (Pickands III (1975), Balkema and De Haan (1974)), states that, under the same technical conditions as Fisher–Tippett–Gnedenko, the excess of a considered random variable $X$ over a high threshold $u$, defined as $X - u$ given $X > u$, converges to the generalized Pareto (GP) distribution

$$G^{\text{GP}}_{\xi, \sigma}(x) = 1 - \left(1 + \frac{x}{\sigma}\right)^{-\frac{1}{\xi}}.$$

This theorem, while theoretically equivalent to Fisher–Tippett–Gnedenko as hinted by the same needed technical conditions, suggests an alternate approach to fit the tail. More concretely, we choose a high threshold $u$ and fit the excess of data above $u$ into $G^{\text{GP}}_{\xi, \sigma}$. In both methods, the asymptotically justified distributions $G^{\text{GEV}}_{\xi, \mu, \sigma}(x)$ and $G^{\text{GP}}_{\xi, \sigma}(x)$ are parametrized by a small number of parameters, which can be estimated by maximum likelihood estimation (MLE) and other parametric methods (Embrechts et al. (2013) Chapter 6). These methods have been used for decades by hydrologists, insurers, financial managers and modelers in various other industries (see, e.g., Smith (1986), Rootzén and Tajvidi (1997), Danielsson and De Vries (1997) and Solari and Losada (2012)).

While powerful, the two methods in extreme value theory are known to face a bias-variance tradeoff that is not always easy to handle. The bias comes from the use of a parametric distribution that is valid only asymptotically, but in finite sample (in the case of annual-maxima) or finite threshold (in the case of peak-over-threshold) it incurs a model misspecification error. The variance refers to the estimation variability of the parameters that arises from a limited data size. More precisely, in the case of annual-maxima, given a total sample size, there is a tradeoff between the number of blocks and the sample size per block and, if we choose a large sample size per block to lower the model bias, we must necessarily use fewer blocks that increases the variance of parameter fitting in the GEV. In the case of peak-over-threshold, if we choose a high excess threshold to lower the model bias, then the amount of data above the threshold must necessarily decrease, leading to again a higher variance for fitting the GP. This bias-variance leads to two issues. First is that the optimal, or even a good choice of block size or threshold value relies on intricate second-order distributional properties of the data (Smith (1987); Bladt et al. (2020)). Second is that, when data is limited, there simply may not exist a good choice of block size or threshold value to control the bias and variance simultaneously. In the literature, there exists a variety of visualization and diagnostic tools that, though ad hoc in nature, demonstrably provide good guidance in tuning these prior parameters for model fitting (Embrechts et al. (2013) Chapter 6; McNeil et al. (2015) Chapter 5).

### 2.2 Challenges in Conventional Extreme Event Analysis: Multivariate Case

We have seen the intricacy in univariate extreme event estimation. In the multivariate case, these challenges evidently continue to hold. More importantly, new difficulties arise.

To explain in more detail, the two approaches in univariate extreme value theory both have multivariate analogs. For annual-maxima, under technical conditions (Resnick (2013) Section 5.4), the component-wise
maximum of i.i.d. random vectors converges to the multivariate extreme value distribution given by

\[ G(x) = \exp(-l(- \log G_1(x_1), - \log G_2(x_2), \ldots, - \log G_d(x_d))), \quad (2) \]

under suitable normalization, where \( G_i \)'s are the marginal distributions and belong to the GEV family (1), and \( l(\cdot) \) is called the stable tail dependence function. This latter function \( l \), which is defined on \([0, \infty]^d\), summarizes the dependence structure among the \( d \) components of \( G(x) \) and needs to satisfy the following necessary conditions (Beirlant et al. (2006) Section 8.2.2):

(C1) \( l(s \cdot) = sl(\cdot) \) for \( 0 < s < \infty \);
(C2) \( l(e_j) = 1 \) for \( j = 1, \ldots, d \), where \( e_j \) is the \( j \)th unit vector in \( \mathbb{R}^d \);
(C3) \( \max\{v_1, \ldots, v_d\} \leq l(v) \leq v_1 + \cdots + v_d \) for \( v \in [0, \infty)^d \);
(C4) \( l \) is a convex function.

The above result suggests the multivariate annual-maxima method (Gumbel and Goldstein (1964)) where we divide data into blocks like in the univariate case, then fit the component-wise maxima of each block into \( G(x) \). On the other hand, the multivariate peak-over-threshold method (Beirlant et al. (2006)) relies on the concept of copula which captures the dependency among marginals (e.g., Nelsen (2007)), more precisely the convergence

\[ \lim_{t \to \infty} C_F^{1/t}(v_1^{1/t}, \ldots, v_d^{1/t}) = C_G(v), \quad v = (v_1, \ldots, v_d) \in [0,1]^d, \quad (3) \]

where \( C_F \) and \( C_G \) are the copulas of the sample distribution \( F \) and the multivariate extreme value distribution \( G \) respectively. This yields the approximation

\[ C_F(v) \approx C_G(v) = \exp(-l(- \log(v_1), \ldots, - \log(v_d))). \quad (4) \]

for high enough values of \( v \), where the equality follows a provable equivalence with the representation (2). This suggests that we choose a high threshold level \( u \in \mathbb{R}^d \) such that the approximation (4) is reliable and, like in the univariate case, employ the data which exceeds \( u \) to fit (4).

The multivariate methods inherit the bias-variance tradeoff as the univariate case regarding the block size or threshold value (Ledford and Tawn (1996)). Moreover, there are two new difficulties that make multivariate case even more challenging. One is the opaqueness on the technical conditions of the underlying theorems, which are all formulated in terms of the asymptotic behaviors of the distribution in the tail region which is exactly the problem target (Resnick (2013)). The second difficulty is the lack of information on the function \( l \). Since \( l \) does not admit a finite-dimensional parametrization, an approach is to restrict it to a parametric subfamily (e.g., logistic model and its variations in Gumbel (1960); Tawn (1988); Joe et al. (1992)) so that parametric inference tools like MLE can be used, but this encounters the risk of model misspecification. An alternate approach is to use nonparametric methods (e.g., Pickands (1981); Capéraà et al. (1997); de Oliveira (1989); Hall and Tajvidi (2000); Drees and Huang (1998); Capéraà and Fougères (2000)). However, the construction of nonparametric estimators of \( l \) that satisfy all the necessary properties (C1)-(C4) remains open (Beirlant et al. (2006)). Although in the bivariate case there are some ad hoc modifications to the nonparametric methods listed above, e.g., taking the convex minorant for the non-convex estimator in Hall and Tajvidi (2000), these modifications appear difficult to generalize to higher-dimensional cases, and moreover even the performances of the bivariate estimators are unclear (Beirlant et al. (2006)).
2.3 DRO for Multivariate Extreme Event Analysis

Motivated by the challenges in conventional statistical methods for tackling multivariate extremes, we consider DRO as a well-grounded alternative. As described before, all existing approaches extrapolate tail by using models justified from asymptotic theory. In finite sample, these methods can face difficult error tradeoff. Moreover, in the multivariate case, these asymptotic models are beyond parametric as they involve the stable tail dependence function that does not admit a finite-dimensional parametrization. Our key idea is to replace these models with geometric tail property that is placed as constraints in a worst-case optimization. More precisely, suppose we focus on the estimation of a tail probability $P((X_1, \ldots, X_d) \in S)$ where $S$ is an extreme set. We consider optimization roughly speaking in the form

$$\max_F P_F((X_1, \ldots, X_d) \in S)$$

subject to

- geometric property of the distribution $F$ for $X \geq u$
- auxiliary constraints on $F$

(5)

where $X = (X_1, \ldots, X_d)$, $u = (u_1, \ldots, u_d)$, and the inequality is defined component-wise. The decision variable in (5) is the unknown distribution $F$. At least some of the thresholds $u_i$ is large, so that the geometric property corresponds to the (positive) tail of the distribution. Problem (5) can be viewed as a DRO with an uncertainty set on the unknown distribution $F$ that is characterized by the geometric conditions and auxiliary constraints.

We hold off the discussion of a suitable geometric property for the tail and why we need the auxiliary constraints at the moment, and first explain conceptually how to use (5) and why this bypasses the challenges of the conventional methods. This geometric property acts as general, nonparametric replacement of the parametric (or semiparametric) models in extreme value theory, which as we have seen faces several challenges in usage, and in particular relies on a good choice of exceedance threshold $u$. When we use the geometric property, we will not hinge on the asymptotic theory that relies on a high enough $u$, thus bypassing the bias-variance tradeoff faced by the peak-over-threshold method. However, the catch is that there can be many distributions that satisfy such general geometric conditions, and consequently we take the worst-case value of the target performance measure to construct a bound.

For the last point above, we note the following trivial guarantee:

**Lemma 1.** Suppose that

$$P \left( \begin{array}{c}
\text{geometric property of the distribution } F \text{ holds for } X \geq u, \\
\text{auxiliary constraints on } F \text{ holds}
\end{array} \right) \geq 1 - \alpha$$

(6)

for some confidence level $1 - \alpha$. Then the optimal value of (5), called $Z^*$, satisfies

$$P(Z^* \geq Z) \geq 1 - \alpha$$

where $Z$ is the true value of $P((X_1, \ldots, X_d) \in S)$.

In Lemma 1, the conditions inside the probability in (6) are calibrated from data, and the probability is with respect to the randomness from data. Lemma 1 concludes that if the uncertainty set contains the true distribution with high confidence, then the optimal value of the associated DRO would be an upper bound for the true target value with at least the same confidence level. The guarantee in Lemma 1 is well-established in data-driven DRO (Delage and Ye (2010); Ben-Tal et al. (2013); Esfahani and Kuhn (2018); Bertsimas et al. 2019).
Moreover, a corresponding lower bound guarantee holds analogously and we have skipped to avoid repetition.

Thus, DRO provides confidence bounds on target tail performance measure as long as the uncertainty set is a valid confidence region on the unknown true distribution. The question then is what constitutes a good choice of uncertainty set, which we discuss next.

2.4 Choices of Uncertainty Set

In data-driven DRO, uncertainty sets can be generally categorized into two major types. The first type is a neighborhood ball surrounding a baseline distribution, where the ball size is measured via a statistical distance. Common choices of distance include the class of $\phi$-divergence (Glasserman and Xu (2014); Gupta (2019); Bayraksan and Love (2015); Iyengar (2005); Hu and Hong (2013); Duchi et al. (2021); Gotoh et al. (2018); Lam (2016, 2018); Ghosh and Lam (2019)) which also covers in particular the Renyi divergence (Atar et al. (2015); Dey and Juneja (2010)) and total variation distance (Jiang and Guan (2018)), and the Wasserstein metric (Esfahani and Kuhn (2018); Blanchet and Kang (2021); Gao and Kleywegt (2016); Xie (2019); Shafieezadeh-Abadeh et al. (2019); Chen and Paschalidis (2018)). The ball sizes using these distances are calibrated from either density and entropy estimation (Jiang and Guan (2016)), using goodness-of-fit statistics (Ben-Tal et al. (2013); Bertsimas et al. (2018)), or employing or developing nonparametric empirical likelihood theory (Lam and Zhou (2017); Duchi et al. (2021); Lam (2019); Blanchet et al. (2019, 2021)). However, these approaches do not apply naturally to tail estimation. The first approach requires substantial amount of data due to the need of using kernel estimation, while the second approach could be conservative, both imposing challenges in the tail region. The third approach, on the other hand, builds on a statistical theory that ties to the objective function in the DRO, and its validity in tail estimation is not established.

The second major type of uncertainty sets constitutes partial information on the distribution, including moments and support (Delage and Ye (2010); Bertsimas and Popescu (2005); Wiesemann et al. (2014); Goh and Sim (2010); Ghaoui et al. (2003)), marginal constraints (Doan et al. (2015); Dhara et al. (2021)), and shape constraints (Van Parys et al. (2016); Li et al. (2019); Lam and Mottet (2017); Mottet and Lam (2017); Chen et al. (2021)). The first two subtypes require calibration, i.e., setting bounds on the moments, supports or the marginal distributions. The last subtype does not require calibration, and thus can be used even when no data is available. At the same time, it reflects the geometric belief on the distribution and, when correctly imposed, it advantageously alleviates conservativeness.

Thanks to the power of reducing estimation conservativeness with few data, shape constraints appear suitable to be the primary choice in constructing uncertainty sets for extremal estimation. This observation is in line with some documented motivation in the literature (e.g., Li et al. (2019)). Before we argue the specific shape constraint to be used, we also mention several works in specializing DRO in extremal estimation. The most relevant is Lam and Mottet (2017) that considers convex tail extrapolation, focusing on the univariate setting. It investigates the light-versus-heavy tail properties in the worst-case distributions and the associated computation procedures. Blanchet et al. (2020) considers robustification of GEV using Renyi divergence ball and studies the domain-of-attraction properties of the worst-case distributions, which aims to alleviate the reliance on the validity of asymptotics in justifying the GEV model. Engelke and Ivanovs (2017) derives robust asymptotic bounds on exceedance probabilities subject to $\chi^2$-ball and first moment, and Birghila et al. (2021) studies bounds on both probabilities and tail indices based on the Wasserstein distance and $f$-divergence.
around a heavy-tailed distribution. Moreover, as we have seen, copula or dependence structure of random vectors plays an important role in multivariate extreme event analysis. Motivated by this, there are works on robust bounds for extremal performance measures when the marginal distributions are given but dependence structure is unknown. These measures include tail probabilities for functions of random vectors that can be interpreted as financial risks (Embrechts and Puccetti (2006a,b); Puccetti and Rüschendorf (2013)), expected values for convex functions of sums (Wang and Wang (2011)) and conditional value-at-risk (Dhara et al. (2021)).

2.5 Challenges of Existing Shape Constraints

A commonly used shape condition in the multivariate setting is unimodality (Dharmadhikari and Joag-Dev (1988)). In the univariate case, a unimodal probability density can be readily intuited as having a unique mode (or connected set of modes) with monotonically decreasing density when moving away from the mode. In the multivariate case, defining unimodality becomes more subtle as the notion of monotonicity is primarily one-dimensional and different definitions can be drawn depending on how one defines monotonicity. The three most widely used multivariate unimodality notions are star unimodality, block unimodality and $\alpha$-unimodality (Dharmadhikari and Joag-Dev (1988) Sections 2.2 and 3.2). In the following, we will introduce these notions which would then help understand their limitations and our motivations for proposing our OU notion.

First we discuss star unimodality. Given a mode \( x_0 := (x_{10}, x_{20}, \ldots, x_{d0}) \in \mathbb{R}^d \), a probability distribution with density on \( \mathbb{R}^d \) is called star unimodal if this density is non-increasing along any ray pointing away from \( x_0 \). That is,

**Definition 1** (Star unimodal density). A probability distribution with density (with respect to the Lebesgue measure) is **star unimodal** about mode \( x_0 \) if the density is non-increasing along any ray pointing away from \( x_0 \) (i.e., \( tx + x_0, t > 0 \) for any nonzero vector \( x \in \mathbb{R}^d \)).

Star unimodal distribution can also be defined using a mixture representation which does not require the existence of the density. This representation requires us to define star-shaped sets, detailed as follows.

**Definition 2** (Mixture representation of star unimodal distribution). We have:

1. A set \( K \) is said to be star-shaped about \( x_0 \) if for every \( x \in K \), the line segment joining \( x \) to \( x_0 \) is completely contained in \( K \).

2. A probability distribution on \( \mathbb{R}^d \) is called star unimodal about \( x_0 \) if it belongs to the closed convex hull of the set of all uniform distributions on sets that are star-shaped about \( x_0 \).

Definition 2 is equivalent to Definition 1 when the distribution has a density (Dharmadhikari and Joag-Dev (1988), the criterion in Section 2.2 or Theorem 3.6).

In parallel to star unimodality, a block unimodal distribution is defined as a mixture of uniform distributions on rectangles instead of star-shaped sets. That is,

**Definition 3** (Mixture representation of block unimodal distribution). A probability distribution on \( \mathbb{R}^d \) is called **block unimodal** about \( x_0 \) if it belongs to the closed convex hull of the set of all uniform distributions on rectangles that contain \( x_0 \) and have edges parallel to the coordinate axes.
It is easy to see that block unimodality satisfies that the density is non-increasing along any ray pointing away from the mode. By either this observation or combining the fact that rectangle is star-shaped and the mixture representation, we see that block unimodal distributions form a subclass of star unimodal distributions.

On the other hand, $\alpha$-unimodality can be viewed as a generalization of the star unimodality notion, by allowing the density to increase on a ray pointing away from the mode but at a controlled rate. More concretely,

**Definition 4 ($\alpha$-unimodal density).** A probability distribution with density on $\mathbb{R}^d$ is $\alpha$-unimodal about $x_0$ if the density $f$ is such that $t^{d-\alpha} f(tx + x_0)$ is non-increasing in $t \in (0, \infty)$ for any nonzero vector $x \in \mathbb{R}^d$.

We argue that all of the star unimodality, block unimodality and $\alpha$-unimodality notions encounter issues in extreme event estimation using DRO, when we place them as the “geometric property” in problem (5). To facilitate discussion, let us focus on $u > x_0$ in (5), i.e., in the positive tail region, or in other words it suffices to define unimodality about $x_0$ on $D_0 = \{x \in \mathbb{R}^d : x \geq x_0\}$ by requiring the rays and sets in the definitions above to be contained in $D_0$. Below, we describe the issues of the existing unimodality under this regime.

Star unimodality is highly sensitive to its mode. That is, when the mode is misspecified, the intended geometric property can become incorrect, even when star unimodality itself holds. To put it in another way, the two conditions “$F$ is star unimodal about mode $x_0$ for $X \geq x_0$” and “$F$ is star unimodal about mode $x_0'$ for $X \geq x_0'$”, for two different $x_0, x_0'$ (where each of them is component-wise smaller than $u$), can result in very different uncertainty sets. In fact, even when $x_0$ and $x_0'$ differ only slightly, the difference in the uncertainty sets could be huge. For instance, in Figure 1(a) the shaded area represents the region $\{X \geq u\}$ in which star unimodality is used to capture the tail geometry. With different modes $x_0$ and $x_0'$, the sets of ray directions on which the density is monotonically non-increasing are different. Moreover, when the shaded area is far away from the mode, a small misspecification of its location can cause a huge difference in the set of directions.

Since not all problems have clearly defined modes to begin with, and estimation of the mode, even though statistically possible, causes sensitive impacts on the tail geometry, star unimodality can be difficult to apply in practice.

Next we discuss the challenges of block unimodality. While having a clear mixture representation, block
unimodal distribution owns a “differencing” property that resembles the requirement of a multivariate cumulative distribution function. In the bivariate case for instance, a distribution $F(x, y)$ with a continuous density $f(x, y)$ that is block unimodal about the mode $(x_0, y_0)$ must satisfy that
\[
    f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) + f(x_2, y_2)
\]
is nonnegative for any $x_0 < x_1 < x_2$ and $y_0 < y_1 < y_2$. This requirement is unintuitive and difficult to check in general. In fact, it is difficult to reason why a distribution should behave this way. Because of this, block unimodality is also difficult to apply, and its differencing requirement flags that it could be too stringent as a geometric property to be used.

Lastly, $\alpha$-unimodality runs into several challenges similar to star unimodality and block unimodality. Similar to star unimodality, misspecification of the mode can cause sensitive impact to the implied geometric property in the tail region. Similar to block unimodality, it is unclear why a distribution should behave in the way that the notion is specified, namely that the density changes in the precisely controlled way in Definition 3. Moreover, even if such a property is true, the specification of $\alpha$ can be a challenge. As an example, we consider a distribution $F$ with density
\[
    f(x_1, x_2) = C \exp \left( - \max \left( \arctan \left( \frac{x_1}{x_2} \right), \arctan \left( \frac{x_2}{x_1} \right) \right) \right) (x_1 + x_2), x_1 \geq 0, x_2 \geq 0, \tag{7}
\]
where $C$ is a normalizing constant to make $f$ a probability density. For this density, it is not easy to tell if it could be $\alpha$-unimodal for some mode $x_0$ at a first glance. In fact, when $\alpha < 2$, $f$ cannot be $\alpha$-unimodal for any $x_0$ since $\lim_{t \to 0} t^{2-\alpha} f(tx + x_0) = 0$ would contradict the condition that $t^{2-\alpha} f(tx + x_0)$ is non-increasing in $t \in (0, \infty)$. When $\alpha \geq 2$, $f$ can be $\alpha$-unimodal ($\alpha$-unimodality reduces to star unimodality when $\alpha = 2$) but the choice of mode is very subtle. We can show by routine calculus that $f$ is $\alpha$-unimodal about $x_0 \in \mathbb{R}_+^2$ if $x_0$ belongs to the diagonal while $f$ is not $\alpha$-unimodal about $x_0 \in \mathbb{R}_+^2$ if $|x_0 - x_1| > 4(\alpha - 2)/(4 - \pi)$. This example illustrates the main drawbacks discussed above, that it is hard to judge whether, or reason why, a distribution is $\alpha$-unimodal, and the specification of $\alpha$ and the mode is not an easy task.

3 Resolution via Orthounimodality

Due the limitations of the existing multivariate unimodality notions discussed in Section 2.5, we propose another multivariate unimodality notion called orthounimodality (OU) which, as we will argue, is natural for extreme event analysis and resolves the statistical challenges faced by the existing unimodality notions.

First, we define OU for probability densities. Here for simplicity, we only define OU in the positive region of the mode since we only focus on the geometric property in the positive tail part of $u$ in Problem (5) (we leave the discussion on more general OU distributions in Appendix B). Given a mode $x_0 \in \mathbb{R}^d$ and its positive region $D_0 = \{x \in \mathbb{R}^d : x \geq x_0\}$, a probability $f$ on $D_0$ is called OU if $f(x)$ is non-increasing in any component of $x$ on $D_0$, i.e.,

**Definition 5** (Orthounimodal density). A probability distribution on $D_0$ with density $f$ (with respect to the Lebesgue measure) is OU about mode $x_0$ if $f(x') \geq f(x)$ for $x \geq x' \geq x_0$.

The definition of OU above is very intuitive in that any point that is more “extreme” than a point in the tail should appear even more rarely, where the extremeness is measured simply by the marginal positions.
of the points. This avoids the additional “differencing” property possessed by block unimodality, and the intricate density change property possessed by \( \alpha \)-unimodality, both of which are hard to interpret. Moreover, compared to star unimodality and \( \alpha \)-unimodality, OU is insensitive to the misspecification of the mode, in the sense that the OU property imposed on a tail region remains correct regardless of the exact position of the mode. To see this, in Figure 1(b), the shaded area represents the tail region \( \{ X \geq u \} \) where OU is imposed. Although the modes \( x_0 \) and \( x'_0 \) are different, their requirements on the tail region are the same: the density is non-increasing along any ray parallel to the axes. Therefore, the geometric requirement on the tail region is independent of the choice of the mode as long as the mode is less than or equal to \( u \) component-wise. Because of the interpretability and the robustness to mode misspecification presented above, OU appears suitable as the multivariate unimodality for use in extreme value analysis.

Next, like star unimodal and block unimodal distributions, an OU distribution can also be defined as a mixture of uniform distributions on what we call OU sets.

**Definition 6** (Mixture representation of orthounimodal distribution). We have:

1. A set \( K \subset D_0 \) is said to be OU about \( x_0 \) if for every \( x \in K \), we have \( x' \in K \) if \( x \geq x' \geq x_0 \).
2. A distribution on \( D_0 \) is called OU about \( x_0 \) if it belongs to the closed convex hull of the set of all uniform distributions on subsets of \( D_0 \) that are OU about \( x_0 \).

We establish the equivalence of Definitions 5 and 6 as well as a Choquet representation theorem for OU. In convex analysis, Choquet theory establishes that any point in a compact convex set \( C \) can be written as the mixture of the extreme points of \( C \) (Phelps (2001) Section 3). In the field of unimodal distributions, Choquet representation means any distribution in a certain class of unimodal distributions can be written as the mixture of the extreme points of this class of distributions. For example, Choquet representation has been established for the three unimodal distributions presented in Section 2.5 (Dharmadhikari and Joag-Dev (1988) Theorem 2.2 and Theorem 3.5). For \( \alpha \)-unimodal distribution \( P \) about the origin (including star unimodality when \( \alpha = d \) ), it can be written as

\[
P = \int_{\mathbb{R}^d} W_{\alpha-uni}(z)dQ(z),
\]

where \( W_{\alpha-uni}(z) \) is the distribution of \( U^{1/\alpha}z \), \( U \) is the uniform distribution on \((0,1)\), and \( Q \) is a probability measure on \( \mathbb{R}^d \) uniquely determined by \( P \). For block unimodal distribution \( P \) about the origin, it can be be written as

\[
P = \int_{\mathbb{R}^d} W_{rect}(z)dQ(z),
\]

where \( W_{rect}(z) \) is the uniform distribution on the rectangle with edges parallel to the axes and opposite vertices \( 0 \) and \( z \), and \( Q \) is again a probability measure on \( \mathbb{R}^d \) uniquely determined by \( P \). Here, \( W_{\alpha-uni}(z) \), \( z \in \mathbb{R}^d \) and \( W_{rect}(z) \), \( z \in \mathbb{R}^d \) are exactly the extreme points in the class of \( \alpha \)-unimodal distributions and block unimodal distributions respectively. The Choquet representation that we will prove in the following Theorem 1 is a natural analog of (8) and (9) for OU distributions. We note that Choquet representation is not the same as the mixture representations in Definitions 2, 3 and 6 since these definitions do not tell us if the unimodal distributions can be generated by the mixture of only the extreme points. To derive our results for OU distributions, we begin with some notations. For a set \( K \), we define \( W_K \) as the uniform distribution on \( K \) and define \( \lambda(K) \) as its Lebesgue measure. Besides, we denote the interior, closure and boundary of \( K \) by \( K^\circ \), \( \bar{K} \) and \( \partial K \) respectively. We show that the OU set has the following properties.
Lemma 2. Suppose that $K \subset \mathcal{D}_0$ is an OU set about $x_0$. Then $K$ is Lebesgue measurable. Besides, $\bar{K}$ and $K^\circ$ (closure of $K^\circ$) are also OU sets about $x_0$ and $K^\circ, K, \bar{K}$ have the same Lebesgue measure, i.e., $\lambda(K^\circ) = \lambda(K) = \lambda(\bar{K})$.

The following theorem justifies the equivalence of Definitions 6 and 5 and also establishes the Choquet representation for OU distributions in the presence of density.

Theorem 1. Suppose a distribution $P$ on $\mathcal{D}_0$ is absolutely continuous with respect to Lebesgue measure. Then $P$ is OU about $x_0$ if and only if there is a density $f(x)$ of $P$ such that for every $s > 0$, the set

$$C_s = \{x \in \mathcal{D}_0 : f(x) \geq s\}$$

is OU about $x_0$, or equivalently, if and only if

$$f(x') \geq f(x) \quad \text{for} \quad x \geq x' \geq x_0.$$  

Besides, $P$ has the following Choquet representation:

$$P(B) = \int_0^\infty W_{C_s}(B)g(s)ds$$ (10)

for any Lebesgue measurable set $B$, where $C_s$ is the closure of $C_s$ and $g(s) = \lambda(\bar{C}_s)$ is a probability density on $(0, \infty)$.

In the proof of Theorem 1, we need the following lemma about the extreme point in the class of OU distributions on $\mathcal{D}_0$ to ensure (10) is indeed a Choquet representation.

Lemma 3. If $K \subset \mathcal{D}_0$ is an OU set about $x_0$ with $\lambda(K) > 0$, then $W_K$ is an extreme point in the class of OU distributions on $\mathcal{D}_0$.

Note that the Choquet representation for OU obviously implies its mixture representation. Comparing the mixture representation of OU with those of star and block unimodality, we also see that the class of OU distributions in fact lies in between star unimodality and block unimodality as stated in the following proposition. In other words, the additional “differencing” property of block unimodality that is hard to interpret can be viewed as overly stringent and unnecessary if we use the OU notion.

Proposition 1. For any $x_0 \in \mathbb{R}^d$, the following is true:

$$\{ \text{distributions on } \mathcal{D}_0 \text{ that is block unimodal about } x_0 \} \subset \subset \{ \text{distributions on } \mathcal{D}_0 \text{ that is OU about } x_0 \}$$

As a notion of multivariate unimodality, Definition 5 of OU has been studied in a range of fields with different names such as “orthounimodal” and “block decreasing”. For example, Devroye (1997) proposes several general algorithms for random vector generation based on the accept-reject algorithm when the sample density is OU. Sager (1982) proves the existence and consistency of the nonparametric MLE for the probability density under the OU constraint. Polonik (1998) derives a new graphical representation of the nonparametric MLE for the probability density under the OU constraint and proves the equivalence of MLE and a density
estimator called “silhouette”. Biau and Devroye (2003) studies the minimax lower bound with the $L_1$ distance for estimating an OU density and proposes Birgé’s multivariate histogram estimate which is minimax optimal. Gao and Wellner (2007) derives upper and lower bounds for the metric entropy and bracketing entropy of the set of bounded OU functions on $[0,1]^d$ under $L_p$ norms. However, none of these works propose the more general Definition 6 that does not require the existence of density. To our best knowledge, we are the first to propose the mixture representation in Definition 6, prove the equivalence between Definitions 5 and 6 in the presence of probability densities (Theorem 1), and study the relation among star unimodality, block unimodality and OU in general, i.e., without requiring the existence of probability densities (Proposition 1), as well as the extensions in Appendix B. Moreover, by the same token, we are also the first to study DRO with the OU constraint.

Finally, before we go to the optimization details in the next section, let us discuss the difficulties and our main contributions on solving DRO with OU constraint on a high level. First of all, there is no off-the-shelf method to solve (an infinite-dimensional) DRO problem with geometric shape constraints. In the literature, e.g., Van Parys et al. (2016), DRO problems with $\alpha$-unimodality are successfully reduced to finite-dimensional semidefinite programs by means of the Choquet representation in (8). However, there is a significant difference between the Choquet representation of $\alpha$-unimodal distributions in (8) and OU distributions in (10), which imposes additional difficulties on the reduction of the OU-DRO problem via the Choquet representation. In (8), the uniform distributions $W_{\alpha-un}(z)$ are parametrized by $z$ in an obvious way. On the contrary, although the distributions $W_{C_s}$ in (10) is formally parametrized by $s$ (or $\bar{C}_s$), we only know $C_s$ is OU but do not known what $\bar{C}_s$ looks like. Moreover, $\bar{C}_s$ even depends on the underlying density $f$. So the Choquet representation of OU distributions gives us less information and is not as ready to use as (8). This difficulty essentially results from the complication of the extreme points of OU distributions. As Lemma 3 shows, all the uniform distributions on the OU sets with positive Lebesgue measure are extreme points in the class of OU distributions. However, there is no obvious way to parametrize OU sets, and hence parametrize the extreme points of OU distributions.

In view of the above challenge, our main contribution in terms of optimization methodology is that, in the bivariate case, we succeed in transforming the challenging OU-DRO problem into a finite-dimensional moment problem which can then be solved by methods such as generalized linear programming (GLP). Roughly speaking, we first use Choquet representation (10) to rewrite the DRO problem as an optimization problem whose feasible solutions are distributions on OU sets. This optimization problem is still difficult since the space of OU sets is too large (infinite-dimensional). Fortunately, each (closed) OU set in $\mathcal{D}_0$ is uniquely characterized by a left-continuous non-increasing function, which allows us to eliminate many suboptimal OU sets by analyzing a variational problem in the space of these functions. The remaining feasible OU sets then have a finite-dimensional parametrization, which gives us an equivalent finite-dimensional moment problem. Although such reduction can only be done in the bivariate case, note that multivariate extreme event analysis already faces all the drawbacks discussed in Section 2.2 in the bivariate case.

4 Orthounimodal Distributionally Robust Optimization

In this and the next section, we will study the shape-constrained DRO problem (5) where the geometric property in the tail part $\{X \geq u\}$ is characterized by OU. We call this problem orthounimodal distributionally
robust optimization (OU-DRO). In the following, we first present the detailed formulation of OU-DRO and the associated rationale (Section 4.1). Then we analyze its reformulation to a program with decision variables that are distributions on OU sets (Section 4.2). Section 5 will continue to present the methodology in further reducing the reformulation to a finite-dimensional moment problem in the bivariate case.

### 4.1 Formulation

First, as explained in Section 3, OU retains the same requirement on \( \{X \geq u\} \) regardless of the mode \( x_0 \) as long as it is less than or equal to \( u \). For simplicity, we just choose \( x_0 = u \) as the mode and the tail part is exactly \( D_0 = \{x \geq x_0\} \) (hereafter we will use \( x_0 \) instead of \( u \) when discussing the tail part). Our OU-DRO problem is formulated as

\[
\max \ P((X_1, \ldots, X_d) \in S)
\]

subject to
\[
\begin{align*}
l_{X_i} & \leq f_{X_i}(x_{i0}) \leq u_{X_i}, i = 1, \ldots, d \\
\end{align*}
\]

\[
\begin{align*}
a_i \bar{F}(x_0) & \leq P(\bar{x}_j \leq X_j \leq \bar{x}_j, j = 1, \ldots, d) \leq b_i \bar{F}(x_0), i = 1, \ldots, n \\
f(x') & \geq f(x) \text{ for } x \geq x' \geq x_0
\end{align*}
\]

The decision variable of the problem is the unknown distribution \( F \) of the random vector \( X \). In the following, we will explain the notations and logic of the formulation (11) and justify its statistical validity.

For the objective probability, as we focus on the extreme event analysis, we assume \( S \) is a subset of the tail part \( D_0 \). Further, we assume \( S \) has the following representation:

\[
S = \{(x_1, \ldots, x_d) \in D_0 : x_d \geq g(x_1, \ldots, x_{d-1})\},
\]

for some known function \( g : [x_{i0}, \infty) \times \cdots \times [x_{(d-1)0}, \infty) \rightarrow (-\infty, \infty] \). Without loss of generality, we can assume the range of \( g \) is \([x_{d0}, \infty] \); otherwise we can replace \( g \) with \( \max(g, x_{d0}) \), which will not change the set \( S \).

Next we discuss the constraints. The last one is the OU property. The others are auxiliary constraints on \( F \) used to reduce conservativeness. Overall, we have two types of auxiliary constraints: density constraints and moment constraints, both of which are used to control the magnitude of the distribution in the tail. Due to the requirement that OU density \( f(x) \) is non-increasing in each component of \( x \) on \( D_0 \), we can control the behavior of each component of \( X \) in the tail by restricting the marginal densities at \( x_{i0}, i = 1, \ldots, d \). This leads to the following density constraints in (11):

\[
l_{X_i} \leq f_{X_i}(x_{i0}) \leq u_{X_i}, i = 1, \ldots, d,
\]

where \( f_{X_i}(x_i) \)'s are the truncated marginal densities defined by

\[
f_{X_i}(x_i) = \int_{x_{i0}}^{\infty} \cdots \int_{x_{(i-1)0}}^{\infty} \int_{x_{(i+1)0}}^{\infty} \cdots \int_{x_{d0}}^{\infty} f(x_1, \ldots, x_d) \, dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_d,
\]

and \( 0 \leq l_{X_i} \leq u_{X_i} \) are constants. On the other hand, the moment constraints are used to control the magnitudes of some tail probabilities which play an important role in making the DRO problem nontrivial and reducing conservativeness (see Proposition 2 below). In (11), we use the following moment constraints:

\[
l_F \leq \bar{F}(x_0) \leq u_F,
\]
where $\bar{F}$ is the tail distribution function defined by $\bar{F}(x) = P(X \geq x)$ and $0 \leq l \leq u, 0 \leq a_i \leq b_i \leq 1, x_0 \leq \bar{x}_j \leq \bar{\bar{x}}_j$ are constants ($\bar{x}_j$ can be infinity). Constraint (13) controls the probability of the entire tail part and constraint (14) provides additional information about how the probability mass is distributed in this part. We note that constraint (14) in fact regards conditional probabilities since it can be written as

$$a_i \leq P(\bar{x}_j \leq X \leq \bar{x}_j, j = 1, \ldots, d) \leq b_i, i = 1, \ldots, n,$$

We can also consider constraints for unconditional probabilities as follows:

$$a_i \leq P(\bar{x}_j \leq X \leq \bar{x}_j, j = 1, \ldots, d) \leq b_i, i = 1, \ldots, n. \quad (15)$$

However, for ease of illustration, we only consider the form (14) in most of our subsequent discussion, and will explain how to deal with the constraint (15) at the end of Section 5.

Note that formulation (11) only depends on the values in $D_0$, so we can restrict our attention to truncated distributions on $D_0$. By Lemma 1, the OU-DRO problem (11) provides a statistically valid upper bound on the true probability as long as the constraints in (11) are statistically valid (joint) confidence intervals. We summarize this as:

**Corollary 1.** Suppose that

$$P($$ constraints in (11) holds for the true distribution $$) \geq 1 - \alpha$$

for some confidence level $1 - \alpha$. Then the optimal value of (11), called $Z^*$, satisfies

$$P(Z^* \geq Z) \geq 1 - \alpha,$$

where $Z$ is the true rare event probability $P((X_1, \ldots, X_d) \in S)$. If the condition holds in the asymptotic sense, i.e.,

$$\lim \inf P($$ constraints in (11) holds for the true distribution $$) \geq 1 - \alpha, $$

then we have

$$\lim \inf P(Z^* \geq Z) \geq 1 - \alpha \quad (16)$$

where $\lim \inf$ refers to the limit as the data size grows to $\infty$.

All the parameters in formulation (11) can be readily calibrated using data so that (16) holds and hence also the guarantee (17). This requires constructing confidence regions for expectation-type quantities and densities that is quite standard in statistics, and hence we delegate this discussion to Appendix A.

### 4.2 Reduction of the Problem

The OU-DRO problem (11) is an infinite-dimensional optimization program, and to proceed we need to reduce it to a tractable form. To begin with, we show that the lower bound density constraints $l_{X_i} \leq f_{X_i}(x_i0)$ are redundant. In fact, for a density $f$ which only violates these lower bound density constraints, we can increase its values on $\partial D_0$, i.e., $\cup_{i=1}^d \{ (x_1, \ldots, x_d) : x_i = x_i0, x_j \geq x_j0, j \neq i \}$. Note that such modification will only
Thus, the OU-DRO problem becomes

\begin{align*}
\text{max } & \quad P((X_1, \ldots, X_d) \in S) \\
\text{subject to } & \quad l_F \leq \bar{F}(x_0) \leq u_F \\
& \quad f_{X_i}(x_{i0}) \leq u_{X_i}, i = 1, \ldots, d \\
& \quad a_i \bar{F}(x_0) \leq P(x_{i0} \leq X_j \leq \bar{x}_{j}, j = 1, \ldots, d) \leq b_i \bar{F}(x_0), i = 1, \ldots, n \\
& \quad f(x') \geq f(x) \text{ for } x \geq x' \geq x_0
\end{align*}

(18)

As a byproduct, formulation (18) reveals that, for many choices of the target rare-event set \( S \), the moment constraint (14) is essential to make the OU-DRO problem nontrivial. This is stated by the following proposition.

**Proposition 2.** Consider the problem (18) without the moment constraint (14). Suppose \( 0 < u_F \leq 1 \) and \( u_{X_i} > 0 \). Suppose the rare-event set \( S \) satisfies

\[ S = \{ (x_1, \ldots, x_d) \in D_0 : x_d \geq g(x_1, \ldots, x_{d-1}) \}, \]

where \( g : [x_{i0}, \infty) \times \cdots \times [x_{(d-1)0}, \infty) \mapsto [x_{d0}, \infty) \) is bounded on compact sets. Then the optimal value of this problem is \( u_F \).

Since the choice of the density of a distribution is not unique, we can choose a good one to ease our analysis for further reduction of the problem (18). For our convenience, we add the following constraint to the problem (18):

\[ f(x) = \limsup_{y \uparrow x, y \in D_0} \sup \ f(y) \text{ for } x \in \partial D_0. \]  

(19)

For a feasible density \( f \) of problem (18) that only violates (19), we can reset its values on \( \partial D_0 \) according to (19). Such changes will not affect the distribution and the feasibility of the OU constraint so it will not affect the optimal value of the problem (18). Moreover, let us focus on the subproblem with the equality constraint \( \bar{F}(x_0) = c \) instead of \( l_F \leq \bar{F}(x_0) \leq u_F \), where \( c \in [l_F, u_F] \) is a fixed positive number (which can be chosen at the end by solving the DRO repeatedly at different \( c \) and applying a simple one-dimensional line search). Thus, the OU-DRO problem becomes

\begin{align*}
\text{max } & \quad P((X_1, \ldots, X_d) \in S) \\
\text{subject to } & \quad \bar{F}(x_0) = c \\
& \quad f_{X_i}(x_{i0}) \leq u_{X_i}, i = 1, \ldots, d \\
& \quad a_i \bar{F}(x_0) \leq P(x_{i0} \leq X_j \leq \bar{x}_{j}, j = 1, \ldots, d) \leq b_i \bar{F}(x_0), i = 1, \ldots, n \\
& \quad f(x') \geq f(x) \text{ for } x \geq x' \geq x_0 \\
& \quad f(x) = \limsup_{y \uparrow x, y \in D_0} \sup \ f(y) \text{ for } x \in \partial D_0.
\end{align*}

(20)

Next, Choquet representation (10) helps us rewrite problem (20) by means of the sets \( \bar{C}_s \) and the probability density \( g(s) \). We introduce some needed notations. For any OU set \( K \) about \( x_0 \), we define \( K_i \) as the slice of \( K \) on the plane \( x_i = x_{i0} \), i.e., \( K_i = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) : (x_1, \ldots, x_{i-1}, x_{i0}, x_{i+1}, \ldots, x_d) \in K \} \).
For clarity, we write $\lambda_d(\cdot)$ and $\lambda_{d-1}(\cdot)$ as the Lebesgue measure on $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$ respectively. We define $\lambda_{d-1}(K_i)/\lambda_d(K) = 0, i = 1, \ldots, d$ if $K = \emptyset$ and define $\lambda_d(K' \cap K)/\lambda_d(K) = 0$ for any measurable set $K' \subset \mathbb{R}^d$ if $\lambda_d(K) = 0$. By Choquet representation (10), we reformulate the problem (20) as follows:

**Lemma 4.** Problem (20) can be rewritten as

$$\max \ c \int_0^\infty \frac{\lambda_d(S \cap C_s)}{\lambda_d(C_s)} g(s) ds$$

subject to

$$\int_0^\infty \frac{\lambda_{d-1}(C_s, i)}{\lambda_d(C_s)} g(s) ds \leq \frac{u_i x_i}{c}, i = 1, \ldots, d$$  \hspace{1cm} (21)

$$a_i \leq \int_0^\infty \frac{\lambda_d(\{x \in D_0 : x_{ji} \leq x_j \leq \bar{x}_{ji}, j = 1, \ldots, d\} \cap C_s)}{\lambda_d(C_s)} g(s) ds \leq b_i, i = 1, \ldots, n,$$

where $C_s = \{x \in D_0 : f(x) \geq s\}$, $\bar{C}_s$ is the closure of $C_s$, $\bar{C}_s, i$ is the slice of $\bar{C}_s$ on the plane $x_i = x_{i0}$, $g(s) = \lambda_d(\bar{C}_s)/c$ is a probability density on $(0, \infty)$ and $f(x)$ is a density on $D_0$ with total mass $c$ and satisfies the last two constraints in the problem (20).

In the above problem, both $\{C_s, s > 0\}$ and $g(s)$ are generated by $f$ and they possess some special structures, e.g., $\{C_s, s > 0\}$ is a sequence of non-increasing closed OU sets. These structures are not easy to fully characterize and thus impose difficulties on solving problem (20). So the next key step is to disentangle the dependence of $\{C_s, s > 0\}$ and $g(s)$ on the density $f$ and generalize the choices of $\{C_s, s > 0\}$ and $g(s)$.

We have the following lemma:

**Lemma 5.** Consider the optimization problem

$$\max \ c \int_0^\infty \frac{\lambda_d(S \cap R_s)}{\lambda_d(R_s)} dG(s)$$

subject to

$$\int_0^\infty \frac{\lambda_{d-1}(R_s, i)}{\lambda_d(R_s)} dG(s) \leq \frac{u_i x_i}{c}, i = 1, \ldots, d$$  \hspace{1cm} (22)

$$a_i \leq \int_0^\infty \frac{\lambda_d(\{x \in D_0 : x_{ji} \leq x_j \leq \bar{x}_{ji}, j = 1, \ldots, d\} \cap R_s)}{\lambda_d(R_s)} dG(s) \leq b_i, i = 1, \ldots, n,$$

all the integrands are measurable

with the decision variables $\{R_s \subset D_0, s > 0\}$ and $G(s)$, where $\{R_s, s > 0\}$ is a sequence of closed OU sets about $x_0$ satisfying $\lambda_d(R_s) \in (0, \infty), \lambda_{d-1}(R_s, i) \in (0, \infty)$ for any $i = 1, \ldots, d$ and $s > 0$, and $G(s)$ is a probability distribution on the index set $(0, \infty)$. Then the optimal value of problem (22) is not less than the optimal value of problem (21).

If we can solve problem (22) and show its optimal solution is also feasible to problem (21), then we know that the optimal values of both problems are the same and thus problem (21) is also solved. Section 5 shows that it is possible to solve problem (22) in the bivariate case, which is our next focus.

### 5 Reduction to Tractable Form for the Bivariate Case

In this section, we will show how to reduce problem (22) to a finite-dimensional moment problem for the bivariate case. To simplify notations, we will use $(X, Y)$, $(x, y)$, $(x_0, y_0)$ instead of $(X_1, X_2)$, $(x_1, x_2)$, $(x_{10}, x_{20})$. 


For reference, we explicitly write the DRO problem when $d = 2$:

$$
\begin{align*}
\max P((X, Y) \in S) \\
\text{subject to } & F(x_0, y_0) = c \\
& f_X(x_0) \leq u_X \\
& f_Y(x_0) \leq u_Y \\
& a_i F(x_0, y_0) \leq P(x_{i1} \leq X \leq x_{2i}, y_{i1} \leq Y \leq y_{2i}) \leq b_i F(x_0, y_0), i = 1, \ldots, n \\
& f(x', y') \geq f(x, y) \text{ if } x_0 \leq x' \leq x \text{ and } y_0 \leq y' \leq y,
\end{align*}
$$

(23)

where $c \in [\bar{f}, u_F]$ is a fixed positive number. For any OU set $K \subset \mathcal{D}_0$ about $(x_0, y_0)$, we define $K^X = \{x : (x, y) \in K \} - x_0$ and $K^Y = \{y : (x, y) \in K \} - y_0$ as the $x$-intercept and $y$-intercept of $K$ within the domain $\mathcal{D}_0$ respectively. In the bivariate case, Lemma 5 reduces to the following corollary.

**Corollary 2.** The optimal value of problem (23) is not greater than the optimal value of the following problem:

$$
\begin{align*}
\max c \int_0^\infty \frac{\lambda(S \cap R_s)}{\lambda(R_s)} dG(s) \\
\text{subject to } & \int_0^\infty \frac{R^Y}{\lambda(R_s)} dG(s) \leq \frac{u_X}{c} \\
& \int_0^\infty \frac{R^X}{\lambda(R_s)} dG(s) \leq \frac{u_Y}{c} \\
& a_i \leq \int_0^\infty \frac{\lambda(\{(x, y) : x_{i1} \leq x \leq x_{2i}, y_{i1} \leq y \leq y_{2i}\} \cap R_s)}{\lambda(R_s)} dG(s) \leq b_i, i = 1, \ldots, n
\end{align*}
$$

(24)

all the integrands are measurable

where the decision variables are the sequence of closed OU sets $\{R_s \subset \mathcal{D}_0, s > 0\}$ and the distribution $G(s)$ on the index set $(0, \infty)$, and moreover $R_s$ satisfies $\lambda(R_s) \in (0, \infty), R^X_s \in (0, \infty), R^Y_s \in (0, \infty)$ for any $s > 0$.

We aim to solve problem (24) and show that problems (23) and (24) have the same optimal value. As explained at the end of Section 3, our main idea to solve the problem (24) is to eliminate suboptimal OU sets and show the remaining OU sets have a finite-dimensional parametrization. To be more specific, for any closed OU set $R_0$ satisfying $\lambda(R_0) \in (0, \infty), R^X_0 \in (0, \infty), R^Y_0 \in (0, \infty)$, we will find an alternative closed OU set $\tilde{R}$ (called the dominating OU set of $R_0$) such that by replacing $R_0$ with $\tilde{R}$, the constraints in (24) are still satisfied and moreover the objective value is at least as good. In other words, we want $\tilde{R}$ to satisfy $\lambda(\tilde{R}) = \lambda(R_0), \tilde{R}^X \leq R^X_0, \tilde{R}^Y \leq R^Y_0, \lambda(\{(x, y) : x_{i1} \leq x \leq x_{2i}, y_{i1} \leq y \leq y_{2i}\} \cap \tilde{R}) = \lambda(\{(x, y) : x_{i1} \leq x \leq x_{2i}, y_{i1} \leq y \leq y_{2i}\} \cap R_0)$ for $i = 1, \ldots, n$ and $\lambda(S \cap \tilde{R}) \geq \lambda(S \cap R_0)$. Thus, instead of considering all the closed OU sets in problem (24), it suffices to consider all the dominating OU sets $\tilde{R}$ obtained in this way. Besides, we will show $\tilde{R}$ has a finite-dimensional characterization, which reduces problem (24) to a finite-dimensional moment problem.

Now, let us explain how to obtain $\tilde{R}$. Since $R_0$ is a closed OU set, it can be represented by

$$
R_0 = \{(x, y) : y_0 \leq y \leq h_0(x), x_0 \leq x \leq x_0 + R^X_0\}
$$

for some non-increasing left-continuous function $h_0 : [x_0, x_0 + R^X_0] \mapsto [y_0, y_0 + R^Y_0]$ with $h_0(x_0) = y_0 + R^Y_0$. We define $h_0^{-1}(y) = \sup\{x : h_0(x) \geq y\}$ for $y$ such that $\{x : h_0(x) \geq y\} \neq \emptyset$. We sort $x_0, x_0 + R^X_0, x_{i1}, x_{2i}, h_0^{-1}(y_{i1})$
and \( h_0^{-1}(y_{2i}) \) (only consider \( x_{2i} < \infty \) and \( h_0^{-1}(y_{i1}) \), \( h_0^{-1}(y_{2i}) \) that are well-defined) and only keep one if some numbers are repeated. Suppose the sorted sequence is \( x_0 < x_1 < \cdots < x_{nR_0} < x_{nR_0+1} = x_0 + R_0^X \), where \( nR_0 \) is the number of distinct values (except \( x_0 \) and \( x_0 + R_0^X \)) satisfying \( nR_0 \leq 4n \) in general. Since we fix \( R_0 \), these \( x_i \)'s and \( h_0(x) \) are also fixed. Now we consider the following constraints of a function \( h : [x_0, x_0 + R_0^X] \mapsto [y_0, y_0 + R_0^Y] \):
\[
\begin{align*}
&\int_{x_i}^{x_{i+1}} (h(x) - y_0) dx = \lambda(\{(x, y) : x_i \leq x \leq x_{i+1}, y \geq y_0\} \cap R_0), i = 0, 1, \ldots, n_R \\
&h_0(x_{i+1}) \leq h(x) \leq h_0(x_i+), x \in [x_i, x_{i+1}], i = 0, 1, \ldots, n_R \\
&h \text{ is non-increasing and left-continuous with } h(x_0) = h(x_0+) ,
\end{align*}
\]
where \( h(x+) \) is the right limit of the function \( h \) at \( x \). Corresponding to this function \( h \), we can define a closed OU set
\[
R := \{(x, y) : y_0 \leq y \leq h(x), x_0 \leq x \leq x_0 + R_0^X\} .
\]

Then we have the following claim.

**Lemma 6.** If an OU set \( R \) is defined by (26) with \( h(x) \) satisfying the constraints (25), then \( R \) satisfies \( \lambda(R) = \lambda(R_0), R^X \leq R_0^X, R^Y \leq R_0^Y \) and \( \lambda(\{(x, y) : x_{i1} \leq x \leq x_{2i}, y_{i1} \leq y \leq y_{2i}\} \cap R) = \lambda(\{(x, y) : x_{i1} \leq x \leq x_{2i}, y_{i1} \leq y \leq y_{2i}\} \cap R_0) \) for \( i = 1, \ldots, n \).

Lemma 6 tells us an OU set \( R \) defined by (26) with \( h(x) \) satisfying the constraints (25) already satisfies all the requirements of the dominating OU set of \( R_0 \) except \( \lambda(S \cap R) \geq \lambda(S \cap R_0) \). Notice that \( R_0 \) and its corresponding function \( h_0 \) also satisfy the representation (26) and the constraints (25). Therefore, if we optimize over all the OU sets satisfying the conditions in Lemma 6 with the objective function \( \lambda(S \cap R) \), the optimal solution must be a dominating OU set of \( R_0 \). By the representation (26) and the form of the rare-event set \( S = \{(x, y) \in \mathcal{D}_0 : y \geq g(x)\} \) for some known function \( g : [x_0, \infty) \mapsto [y_0, \infty] \), the objective function \( \lambda(S \cap R) \) can be equivalently written as
\[
\lambda(S \cap R) = \int_{x_0}^{x_0 + R_0^X} (h(x) - g(x))_+ dx.
\]

Therefore, the construction of the dominating OU set can be formulated as a variational problem as stated in the following lemma.

**Lemma 7.** Suppose \( h^* \) is the optimal solution of the following variational problem:
\[
\begin{align*}
\max & \int_{x_0}^{x_0 + R_0^X} (h(x) - g(x))_+ dx \\
\text{subject to} & \int_{x_i}^{x_{i+1}} (h(x) - y_0) dx = \lambda(\{(x, y) : x_i \leq x \leq x_{i+1}, y \geq y_0\} \cap R_0), i = 0, 1, \ldots, n_R \\
&h_0(x_{i+1}) \leq h(x) \leq h_0(x_i+), x \in [x_i, x_{i+1}], i = 0, 1, \ldots, n_R \\
&h \text{ is non-increasing and left-continuous with } h(x_0) = h(x_0+)
\end{align*}
\]
Then the closed OU set \( \hat{R} \) defined by the representation (26) with \( h = h^* \) is a dominating OU set of \( R_0 \).

Next we explain how to characterize the optimal solution \( h^* \) to problem (28). Notice that (28) is separable, i.e., it can be divided into \( nR_0 + 1 \) subproblems on different intervals \( (x_i, x_{i+1}] \) for \( i = 0, 1, \ldots, n_R \):
\[
\max \int_{x_i}^{x_{i+1}} (h(x) - g(x))_+ dx
\]
subject to $\int_{x_i}^{x_{i+1}} h(x) dx = y_0(x_{i+1} - x_i) + \lambda(\{(x,y) : x_i \leq x \leq x_{i+1}, y \geq y_0\} \cap R_0)$ \hspace{1cm} (29) 

$h_0(x_{i+1}) \leq h(x) \leq h_0(x_i+), x \in (x_i, x_{i+1}]$

$h$ is non-increasing and left-continuous

Suppose $h^*_i(x)$ is the optimal solution to the $i$th subproblem. If we define $h^* : [x_0, x_0 + r_X] \mapsto [y_0, y_0 + r_Y]$ by

$$h^*(x) = h^*_i(x), x \in (x_i, x_{i+1}]$$

and $h^*(x_0) = h^*(x_0+)$, we can see $h^*$ satisfies all the constraints in (28) and it is optimal in each interval $(x_i, x_{i+1}]$, which implies it is the optimal solution to (28). Therefore, in order to characterize $h^*$, it suffices to characterize each $h^*_i$. Notice that all the subproblems (29) have the following form

$$\max \int_{\bar{x}}^{x} (h(x) - g(x))_+ dx$$

subject to $\int_{\bar{x}}^{x} h(x) dx = C$

$$\bar{b} \leq h(x) \leq \bar{b}, x \in [\bar{x}, \bar{x}]$$

$h$ is non-increasing and left-continuous

for $b(\bar{x} - x) \leq C \leq \bar{b}(\bar{x} - x)$. The following lemma characterizes the optimal solution to problem (31).

**Lemma 8.** Given a function $g : [\bar{x}, \bar{x}] \mapsto (-\infty, \infty]$, the optimal solution $h^*(x)$ to problem (31) exists and has the following form

$$h^*(x) = \begin{cases} 
y'_1, & \bar{x} < x \leq x'_1 
y'_2, & x'_1 < x \leq x'_2 
y'_3, & x'_2 < x \leq \bar{x} 
\end{cases}$$

for some $\bar{b} \leq y'_3 \leq y'_2 \leq y'_1 \leq \bar{b}$ and $\bar{x} \leq x'_1 \leq x'_2 \leq \bar{x}$.

By (30), $h^*$ can be constructed by “combining” $n_{R_0} + 1$ optimal solution $h^*_i$’s, each of which is a step function with at most three steps by Lemma 8. In general, we have an upper bound on $n_{R_0} : n_{R_0} \leq 4n$. This gives us the structure of $h^*$ and also a finite-dimensional parametrization of all the dominating OU sets.

**Corollary 3.** Given a function $g : [x_0, \infty) \mapsto [y_0, \infty]$, the optimal solution $h^*(x)$ to the problem (28) exists and can be represented by the following step function:

$$h^*(x) \equiv h^*(x; z, w) = \begin{cases} 
y_0 + \sum_{i=1}^{12n+3} w_i, & x_0 \leq x \leq x_0 + z_1 
y_0 + \sum_{i=1}^{12n+2} w_i, & x_0 + z_1 < x \leq x_0 + z_1 + z_2 
y_0 + \sum_{i=1}^{12n+1} w_i, & x_0 + z_1 + z_2 < x \leq x_0 + z_1 + z_2 + z_3 
\ldots 
y_0 + w_i, & \sum_{i=1}^{12n+2} z_i < x \leq x_0 + \sum_{i=1}^{12n+3} z_i 
\end{cases}$$

for some $(z, w) \equiv (z_1, \ldots, z_{12n+3}, w_1, \ldots, w_{12n+3}) \in (0, \infty)^{12n+4} \times [0, \infty)^{12n+2}$. Moreover, all the dominating OU sets are contained in the class $R^* = \{R_{z, w} : R_{z, w} = \{(x, y) \in D_0 : y_0 \leq y \leq h^*(x; z, w)\} \text{ with } h^* \text{ defined in (32)}\}$ and thus are fully parametrized by $(z, w)$.
From the view of constructing dominating OU sets, it suffices to consider $R_x \in \mathcal{R}^*$ instead of all the closed OU sets in problem (24). Corollary 3 gives us a finite-dimensional parametrization of the OU sets in $\mathcal{R}^*$. This helps us reduce problem (24) to an equivalent finite-dimensional moment problem.

**Proposition 3.** Problem (24) is equivalent to the following moment problem:

$$
\max c E_Q \left[ \sum_{i=1}^{2n+3} \int \left( y_0 + \sum_{j=1}^{12n+4-i} W_j - g(x) \right) I \left( x_0 + \sum_{j=1}^{i-1} Z_j < x \leq x_0 + \sum_{j=1}^{i} Z_j \right) dx \right]
$$

$$
s.t. \ E_Q \left[ \frac{\sum_{i=1}^{12n+3} W_i}{\sum_{i=1}^{12n+3} \sum_{j=1}^{12n+4-i} Z_j W_j} \right] \leq \frac{u_X}{c}
$$

$$
E_Q \left[ \frac{\sum_{i=1}^{12n+3} Z_i}{\sum_{i=1}^{12n+3} \sum_{j=1}^{12n+4-i} Z_j W_j} \right] \leq \frac{u_Y}{c}
$$

$$
a_k \leq E_Q \left[ \frac{\sum_{i=1}^{12n+3} Z_i W_j}{\sum_{i=1}^{12n+3} \sum_{j=1}^{12n+4-i} Z_j W_j} \right] \left\{ \sum_{i=1}^{12n+3} \int I \left( \sum_{j=1}^{i-1} Z_j < x - x_0 \leq \sum_{j=1}^{i} Z_j, x_{1k} \leq x \leq x_{2k} \right) \right\} \leq b_k, k = 1, \ldots, n
$$

where the decision variable $Q$ is the probability distribution of $(Z, W) \in (0, \infty)^{12n+4} \times [0, \infty)^{12n+2}$.

Note that the upper bound on $n_{R_0}$ determines the number of steps of $h^*$ in (32) and further determines the dimension of $(Z, W)$ in the moment problem (33). If in some cases we can get a sharper upper bound on $n_{R_0}$ instead of $n_{R_0} \leq 4n$, then we are able to reduce the dimension of the moment problem (33). We will discuss this point in more detail at the end of this section.

By Corollary 2 and Proposition 3, we know the optimal value of the OU-DRO problem (23) is not greater than the optimal value of the moment problem (33). Now let us show the other direction, i.e., the optimal value of the OU-DRO problem (23) is not less than the optimal value of the moment problem (33). Consider any feasible solution $Q$ to the problem (33). We define an absolutely continuous OU distribution $P$ with total mass $c$ by

$$
P = c \int W_{\text{step}}(z, w)dQ(z, w), \tag{34}
$$

where $W_{\text{step}}(z, w)$ is the uniform distribution on the closed OU set $R_{z, w} = \{(x, y) \in D_0 : y_0 \leq y \leq h^*(x; z, w)\}$ with $h^*$ defined in (32). Then we can see the objective value and the constraints for the density $f_P$ in problem (23) are just the translation of those for $Q$ in problem (33). Since $Q$ is feasible to (33), $f_P$ must be feasible to (23) with the same objective value, which means the optimal value of (23) is not less than the optimal value of (33). Combining the above two directions, we see that the optimal value of problem (23) is equal to that of problem (33). Finally, according to Theorem 3.2 in Winkler (1988), to find an optimal solution to (33), it suffices to consider discrete probability measures with at most $n + 3$ points in the support. So (33) is equivalent to the following non-linear optimization:

$$
\max c \sum_{i=1}^{n+3} \sum_{j=1}^{12n+4-i} \int \left( y_0 + \sum_{j=1}^{12n+4-i} w_{ij} - g(x) \right) I \left( x_0 + \sum_{j=1}^{i-1} z_{ij} < x \leq x_0 + \sum_{j=1}^{i} z_{ij} \right) dx \frac{\sum_{i=1}^{12n+3} \sum_{j=1}^{12n+4-i} z_{ij} w_{ij}}{\sum_{i=1}^{12n+3} \sum_{j=1}^{12n+4-i} z_{ij} w_{ij}}
$$

21
Theorem 2. We have the following:

\[ \begin{align*}
&\text{s.t. } \sum_{l=1}^{n+3} p_l \frac{\sum_{i=1}^{12n+3} w_{li} \sum_{j=1}^{12n+4-i} z_{lj} u_{lj}}{z_{li}} \leq \frac{u_X}{c} \\
&\sum_{l=1}^{n+3} p_l \frac{\sum_{i=1}^{12n+3} z_{li}}{z_{li}} \leq \frac{u_Y}{c} \\
&\sum_{l=1}^{n+3} \frac{p_l}{l} \sum_{i=1}^{12n+3} \frac{z_{li}}{z_{li}} \sum_{j=1}^{12n+4-i} w_{lj} u_{lj} \leq \sum_{i=1}^{12n+3} \left\{ \sum_{j=1}^{i-1} \int \left( \frac{\sum_{j=1}^{i-1} z_{lj} x - x_0 \leq \sum_{j=1}^{i} z_{lj}, x_{1k} \leq x \leq x_{2k} \right) \\
&\times \left( \min \left( y_0 + \sum_{j=1}^{12n+4-i} w_{lj}, y_{2k} \right) \right) - \min \left( y_0 + \sum_{j=1}^{12n+4-i} w_{lj}, y_{1k} \right) \right) \right\} \leq b_k, k = 1, \ldots, n \\
&\sum_{l=1}^{n+3} p_l = 1, p_l \geq 0, l = 1, \ldots, n + 3 \\
w_{li} > 0, w_{li} \geq 0, l = 1, \ldots, n + 3, i = 2, \ldots, 12n + 3, \\
z_{li} > 0, l = 1, \ldots, n + 3, i = 1, \ldots, 12n + 3,
\end{align*} \]

where \( p_l, w_{li}, z_{li} \) are the decision variables.

The results in this section are summarized in the following main theorem.

**Theorem 2.** We have the following:

1. The OU-DRO problem (23), the moment problem (33) and the non-linear optimization problem (35) have the same optimal value. If \( Q^* \) is the optimal solution to the moment problem (33), then the density of \( P^* \) defined in (34) with \( Q \) replaced by \( Q^* \) is the optimal solution to the OU-DRO problem (23).

2. Consider the OU-DRO problem (23) with \( \bar{F}(x_0, y_0) = c \) replaced by \( l_F \leq \bar{F}(x_0, y_0) \leq u_F \). Its optimal value is equal to the optimal value of the non-linear optimization problem (35) with an additional decision variable \( c \) and an additional constraint \( l_F \leq c \leq u_F \). If \( c^*, p_l^*, w_{li}^*, z_{li}^* \) is the optimal solution to this non-linear optimization, then the optimal solution to the DRO problem is the density of \( P^* \) defined in (34) where \( c = c^* \) and \( Q \) is the discrete distribution on \( (z_{11}^*, \ldots, z_{12n+3}^*, w_{11}^*, \ldots, w_{12n+3}^*), l = 1, \ldots, n + 3 \) with probability mass \( p_l^*, l = 1, \ldots, n + 3 \).

Theorem 2 suggests two approaches to solve the OU-DRO problem (23). One is to solve the moment problem (33) by methods such as GLP described in, e.g., Section 3 of Birge and Dula (1991). Another is to solve the non-linear program (35). Similarly, in order to solve the DRO problem (23) with \( \bar{F}(x_0, y_0) = c \) replaced by \( l_F \leq \bar{F}(x_0, y_0) \leq u_F \), one way is to solve the non-linear program described in Theorem 2. An alternative way is to discretize \( c \in [l_F, u_F] \), solve a collection of moment problems (33) each with a discretized value of \( c \) via GLP, and search for the best discretized \( c \), which would lead to an approximation of the optimal value of the DRO problem. In practice, we observe that solving the moment problem (33) by GLP gives a solution with better quality so we will use this method in numerical experiments.

Now we discuss some variants of Theorem 2. First, we explain how to handle the unconditional moment constraint (15). Suppose in the DRO problem (23), some of the moment constraints
\[ a_i \bar{F}(x_{0}, y_0) \leq P(x_{1i} \leq X \leq x_{2i}, y_{1i} \leq Y \leq y_{2i}) \leq b_i \bar{F}(x_{0}, y_0). \]
are replaced by the unconditional version:
\[ a_i \leq P(x_{1i} \leq X \leq x_{2i}, y_{1i} \leq Y \leq y_{2i}) \leq b_i. \]
Since the constraint $\tilde{F}(x_0, y_0) = c$ is included in the DRO problem (23), (37) is equivalent to
\[
\frac{a_k}{c} \tilde{F}(x_0, y_0) \leq P(x_{1i} \leq X \leq x_{2i}, y_{1i} \leq Y \leq y_{2i}) \leq \frac{b_k}{c} \tilde{F}(x_0, y_0), i = 1, \ldots, n.
\]
Therefore, to handle the constraint (37), we simply replace $a_k$ and $b_k$ with $a_k/c$ and $b_k/c$ in the third constraint of the problem (33) or (35) and Theorem 2 still holds.

Second, the dimension of the moment problem (33) can be reduced in some cases. Suppose we have $n$ moment constraints in total which can be in the form of (36) or (37). We define $n'$ as the total number of $x_{1i}, x_{2i}, y_{1i}, y_{2i}$ that are equal to $x_0, \infty, y_0, \infty$ respectively, i.e.,
\[
n' = \sum_{i=1}^{n} (I(x_{1i} = x_0) + I(x_{2i} = \infty) + I(y_{1i} = y_0) + I(y_{2i} = \infty)).
\]
Then in (25) we have a sharper bound on $n_{R_0}$ given by $n_{R_0} \leq 4n - n'$. It follows that the function $h^{*}$ in (32) can be changed into a step function with at most $3(4n - n' + 1)$ steps. So the dimension of $Z$ and $W$ can be reduced to $3(4n - n' + 1)$ instead of $12n + 3$.

Lastly, we explain why we cannot directly apply our above reduction procedure to $d \geq 3$. As we have seen, the reduction procedure for $d = 2$ mainly relies on finding and parametrizing the dominating OU sets, but both aspects face difficulties when $d \geq 3$. For the first aspect, when $d = 2$, the quantity $\lambda_{d-1}(R_{s,i})$ in (22) has a simple geometric meaning, i.e., x-intercept or y-intercept within $D_0$, which allows us to carefully design a solvable variational problem (28) to find the dominating OU set. However, for $d \geq 3$, the geometric meaning of $\lambda_{d-1}(R_{s,i})$ is more opaque and not easy to handle, making it challenging to solve the analog of problem (28) in higher dimensions. For the second aspect, the dominating OU sets for $d = 2$ are characterized by univariate non-increasing left-continuous step functions with bounded numbers of steps. Such step functions are finite-dimensionally parametrizable as it suffices to parametrize the length and height of each step. However, when $d \geq 3$, even if we can show the dominating OU sets can be characterized by multivariate step functions, the multivariate steps might bear “shapes” that are not encoded in finite dimension. The generalization to $d \geq 3$ thus appears to require new techniques and is worth a separate future work.

6 Generalization to Partially Orthounimodal Distributionally Robust Optimization

In Sections 4 and 5, we have discussed the formulation and reduction of the OU-DRO problem motivated by extremal estimation where all the dimensions of the random vector are in their tails. Nevertheless, for an event to be in the extreme, it is possible that only some but not all of the dimensions are in their tails. For example, the sets $\{(x, y) : x \geq 4, y \geq -1\}$ and $\{(x, y) : x \geq 4, x + y \geq 5\}$ are seen as rare events for the standard bivariate normal distribution $N(0, I_{2 \times 2})$. However, the density is only non-increasing in $x$ but not in $y$ in these regions since only $x$ is in its tail. OU cannot capture the feature of the probability density in such regions as it requires monotonicity in all dimensions. To remedy this, we design another new notion of multivariate unimodality called \textit{partial orthounimodality} (POU) which can be seen as the generalization of OU that allows for monotonicity in only part of all dimensions. From this, we formulate a POU-DRO problem in parallel to Section 4 and, like in Section 5, we demonstrate how to solve it in a special case where only one dimension is in its tail. Due to paper length, the details of this investigation are delegated to Appendix C.
7 Numerical Experiments

In this section, we present some numerical experiments for OU-DRO problems in the bivariate case. In Section 7.1, we consider obtaining the upper bounds of rare-event probabilities when the constraints in the DRO problem (23) is calibrated by the true distribution. In Section 7.2, we consider estimating rare-event probabilities in data-driven scenarios. In the experiments, we solve the moment problem (33) by GLP.

7.1 OU-DRO Problems Calibrated by True Distributions

We consider the DRO problem (23) calibrated by the true distribution. We choose the true distribution of \((X,Y)\) to be a bivariate normal distribution \(N(0, 16I_{2\times 2})\), where \(I_{2\times 2}\) is the identity matrix in \(\mathbb{R}^{2\times 2}\). The tail part is chosen as \([8, \infty)^2\). We aim to obtain upper bounds of the rare-event probabilities \(P(X \geq 8, Y \geq 1.5X - 2)\) and \(P(X \geq 8, Y \geq X + 5)\). The OU-DRO problem is formulated as follows:

\[
\begin{align*}
\max \quad & P((X,Y) \in S) \\
\text{s.t.} \quad & \bar{F}(8, 8) = 5.176 \times 10^{-4} \\
& f_X(8) \leq 3.071 \times 10^{-4} \\
& f_Y(8) \leq 3.071 \times 10^{-4} \\
& P(8 \leq X \leq 8 + i, Y \geq 8) = a_i, i = 1, 2, 3, 4, 5 \\
& P\left(X \geq 8, 8 \leq Y \leq 8 + \frac{3}{5}i\right) = b_i, i = 1, 2, 3, 4, 5, \\
& f(x', y') \geq f(x, y) \text{ if } 8 \leq x' \leq x \text{ and } 8 \leq y' \leq y,
\end{align*}
\]

where \(a = 10^{-4} \times [2.395, 3.763, 4.498, 4.869, 5.044]\) and \(b = 10^{-4} \times [1.586, 2.736, 3.551, 4.115, 4.498]\). We choose the set \(S\) as \(S_1 = \{(x, y) : x \geq 8, y \geq 1.5x - 2\}\) and \(S_2 = \{(x, y) : x \geq 8, y \geq x + 5\}\) corresponding to two target rare-event probabilities.

We solve each DRO problem 50 times (To solve the DRO problem easily, we perturb the moment constraint \(P(8 \leq X \leq 8 + i, Y \geq 8) = a_i\) a little, i.e., we replace them with \(0.995a_i \leq P(8 \leq X \leq 8 + i, Y \geq 8) \leq 1.005a_i\); the constraint \(P(X \geq 8, 8 \leq Y \leq 8 + 3i/5) = b_i\) is similarly perturbed). As a benchmark, the ground truths are \(P(X \geq 8, Y \geq 1.5X - 2) = 5.028 \times 10^{-5}\) and \(P(X \geq 8, Y \geq X + 5) = 5.341 \times 10^{-6}\). Figure 2 shows the optimal values of the DRO problems. For \(P(X \geq 8, Y \geq 1.5X - 2)\), the upper bounds are valid and within twice the ground truth. For \(P(X \geq 8, Y \geq X + 5)\), most upper bounds are valid and they are within one order (10 times) of the ground truth.

7.2 Data-Driven OU-DRO Problems

We consider data-driven OU-DRO problems. In other words, we can only get access to the data \((X_1, Y_1), \ldots, (X_m, Y_m)\) generated from the unknown true distribution. Throughout this subsection, we consider the DRO problem (18) where the moment constraints are specified as follows

\[
\begin{align*}
a_{X,i} & \leq P(x_0 \leq X \leq x_i, Y \geq y_0 | X \geq x_0, Y \geq y_0) \leq b_{X,i}, i = 1, \ldots, n_X \\
a_{Y,j} & \leq P(X \geq x_0, y_0 \leq Y \leq y_j | X \geq x_0, Y \geq y_0) \leq b_{Y,j}, j = 1, \ldots, n_Y
\end{align*}
\]
All the constraints in the DRO can be calibrated by the methods described in Appendix A. We still choose the true distribution to be the bivariate normal distribution $N(0, 16I_{2 \times 2})$. We consider three data sizes: $m = 10^4, 10^5, 10^6$. We choose the number of moment constraints as $n_X = n_Y = 5$. For $x_0, x_1, x_2, x_3, x_4, x_5$, we consider three different choices: the top 20, 16, 13, 9.5, 6, 2.5 percentiles of $X_i$, the top 10, 8, 6, 4, 2, 1 percentiles of $X_i$, and the top 5, 4, 3, 2, 1, 0.5 percentiles of $X_i$. For $y_0, y_1, y_2, y_3, y_4, y_5$, we choose them to be the same percentiles of $Y_i$ as in the choices of $x_i$’s. We refer to these three settings as “DRO truncated at 80% percentile”, “DRO truncated at 90% percentile” and “DRO truncated at 95% percentile” respectively. We aim at obtaining the upper bounds of three rare-event probabilities $P(X \geq 7, Y \geq X + 1)$, $P(X \geq 8, Y \geq 1.5X - 2)$ and $P(X \geq 8, Y \geq X + 5)$ with ground truths $4.35 \times 10^{-4}$, $5.0275 \times 10^{-5}$ and $5.3408 \times 10^{-6}$ respectively. To solve the DRO problem via the moment problem $(33)$, we discretize the constraint $l_F \leq \bar{F}(x_0, y_0) \leq u_F$ as $\bar{F}(x_0, y_0) = c$ with $c = l_F, (l_F + u_F)/2, u_F$.

Figures 3-5 show the results. We observe that the results are better when we use higher levels to define $x_i$’s and $y_i$’s, which is reasonable since larger $x_i$’s and $y_i$’s provide more information about the distribution in the rare-event sets. Regarding the quality of the upper bounds, it becomes better with larger data sizes as we expect. Besides, we see that the upper bounds for $P(X \geq 7, Y \geq X + 1)$ appear accurate, especially for DRO truncated at 90% percentile and 95% percentile, most of which are within 2 or 3 times of the ground truth. As for the results of $P(X \geq 8, Y \geq 1.5X - 2)$, DRO truncated at 95% percentile performs reasonably well, whose optimal values are within one order (10 times) of the ground truth. But the results for $P(X \geq 8, Y \geq X + 5)$ seem more conservative, which can be attributed to the very small magnitude of this probability which makes the estimation more challenging. Nonetheless, the upper bounds provided by DRO across all settings correctly bound the ground truths, thus validating the statistical correctness of our approach.
Figure 3: Data-driven DRO problems with $10^4$ samples from a bivariate normal distribution. Each problem is solved 50 times.
Figure 4: Data-driven DRO problems with $10^5$ samples from a bivariate normal distribution. Each problem is solved 50 times.
Figure 5: Data-driven DRO problems with $10^6$ samples from a bivariate normal distribution. Each problem is solved 50 times.
8 Conclusion

This paper studied OU-DRO as a nonparametric alternative to existing methods for multivariate extreme event analysis. Our approach bypassed the bias-variance tradeoff and other technical complications faced by conventional multivariate extreme value theory, via the computation of worst-case upper bounds subject to shape constraints. We explained why OU is a suitable and natural shape constraint choice compared to other well-known multivariate unimodality notions such as star unimodality, block unimodality and α-unimodality.

We formulated the OU-DRO problem and presented how it can be reduced to a moment problem in the bivariate case by solving a specially designed variational problem to rule out suboptimal solutions. To extend our analysis to rare-event sets where only some of the dimensions are in their tails, we also proposed the use of POU-DRO and investigated its reduction to a tractable moment problem when only one dimension is in the tail. We demonstrated numerical results to show the statistical correctness and performance of our approach.

We suggest two directions for future work. One is the reduction of OU-DRO for general dimensions. This may require a deeper understanding on the geometry of OU sets in order to design a suitable variational problem to eliminate suboptimal solutions. Related to this is the reduction of POU-DRO for any number of dimensions being in the tail. Another direction is the alleviation of conservativeness in using DRO for extremal estimation, which involves further study on finding appropriate auxiliary constraints that can be calibrated from data while informative on tail behaviors.

Acknowledgments

We gratefully acknowledge support from the National Science Foundation under grants CAREER CMMI-1834710 and IIS-1849280.

References

Atar, R., Chowdhary, K., Dupuis, P., 2015. Robust bounds on risk-sensitive functionals via rényi divergence. SIAM/ASA Journal on Uncertainty Quantification 3 (1), 18–33.

Balkema, A. A., De Haan, L., 1974. Residual life time at great age. The Annals of probability, 792–804.

Bayraksan, G., Love, D. K., 2015. Data-driven stochastic programming using phi-divergences. In: The Operations Research Revolution. INFORMS, pp. 1–19.

Beirlant, J., Goegebeur, Y., Segers, J., Teugels, J. L., 2006. Statistics of extremes: theory and applications. John Wiley & Sons.

Beirlant, J., Teugels, J. L., 1992. Modeling large claims in non-life insurance. Insurance: Mathematics and Economics 11 (1), 17–29.

Ben-Tal, A., Den Hertog, D., De Waegenaere, A., Melenberg, B., Rennen, G., 2013. Robust solutions of optimization problems affected by uncertain probabilities. Management Science 59 (2), 341–357.

Ben-Tal, A., El Ghaoui, L., Nemirovski, A., 2009. Robust Optimization. Princeton University Press.
Bertsimas, D., Brown, D. B., Caramanis, C., 2011. Theory and applications of robust optimization. SIAM Review 53 (3), 464–501.

Bertsimas, D., Gupta, V., Kallus, N., 2018. Robust sample average approximation. Mathematical Programming 171 (1), 217–282.

Bertsimas, D., Popescu, I., 2005. Optimal inequalities in probability theory: A convex optimization approach. SIAM Journal on Optimization 15 (3), 780–804.

Biau, G., Devroye, L., 2003. On the risk of estimates for block decreasing densities. Journal of multivariate analysis 86 (1), 143–165.

Birge, J. R., Dulá, J. H., 1991. Bounding separable recourse functions with limited distribution information. Annals of Operations Research 30 (1), 277–298.

Birgila, C., Aigner, M., Engelke, S., 2021. Distributionally robust tail bounds based on wasserstein distance and $f$-divergence. arXiv preprint arXiv:2106.06266.

Bladt, M., Albrecher, H., Beirlant, J., 2020. Threshold selection and trimming in extremes. Extremes 23 (4), 629–665.

Blanchet, J., He, F., Murthy, K., 2020. On distributionally robust extreme value analysis. Extremes , 1–31.

Blanchet, J., Kang, Y., 2021. Sample-out-of-sample inference based on wasserstein distance. To appear in Operations Research.

Blanchet, J., Kang, Y., Murthy, K., Sep 2019. Robust wasserstein profile inference and applications to machine learning. Journal of Applied Probability 56 (3), 830–857.

Blanchet, J., Murthy, K., Si, N., 2021. Confidence regions in wasserstein distributionally robust estimation. To appear in Biometrika.

Capéraà, P., Fougères, A.-L., 2000. Estimation of a bivariate extreme value distribution. Extremes 3 (4), 311–329.

Capéraà, P., Fougères, A.-L., Genest, C., 1997. A nonparametric estimation procedure for bivariate extreme value copulas. Biometrika 84 (3), 567–577.

Chen, R., Paschalidis, I. C., 2018. A robust learning approach for regression models based on distributionally robust optimization. Journal of Machine Learning Research 19 (13), 1–48.

Chen, X., He, S., Jiang, B., Ryan, C. T., Zhang, T., 2021. The discrete moment problem with nonconvex shape constraints. Operations Research 69 (1), 279–296.

Chen, Y.-C., 2017. A tutorial on kernel density estimation and recent advances. Biostatistics & Epidemiology 1 (1), 161–187.

Danielsson, J., De Vries, C. G., 1997. Tail index and quantile estimation with very high frequency data. Journal of empirical Finance 4 (2-3), 241–257.
de Oliveira, J. T., 1989. Intrinsic estimation of the dependence structure for bivariate extremes. Statistics & probability letters 8 (3), 213–218.

Delage, E., Ye, Y., 2010. Distributionally robust optimization under moment uncertainty with application to data-driven problems. Operations research 58 (3), 595–612.

Devroye, L., 1997. Random variate generation for multivariate unimodal densities. ACM Transactions on Modeling and Computer Simulation (TOMACS) 7 (4), 447–477.

Dey, S., Juneja, S., 2010. Entropy approach to incorporate fat tailed constraints in financial models. Available at SSRN 1647048.

Dhara, A., Das, B., Natarajan, K., 2021. Worst-case expected shortfall with univariate and bivariate marginals. INFORMS Journal on Computing 33 (1), 370–389.

Dharmadhikari, S., Joag-Dev, K., 1988. Unimodality, convexity, and applications. Elsevier.

Doan, X. V., Li, X., Natarajan, K., 2015. Robustness to dependency in portfolio optimization using overlapping marginals. Operations Research 63 (6), 1468–1488.

Drees, H., Huang, X., 1998. Best attainable rates of convergence for estimators of the stable tail dependence function. Journal of Multivariate Analysis 64 (1), 25–46.

Duchi, J. C., Glynn, P. W., Namkoong, H., 2021. Statistics of robust optimization: A generalized empirical likelihood approach. To appear in Mathematics of Operations Research.

Embrechts, P., Klüppelberg, C., Mikosch, T., 2013. Modelling extremal events: for insurance and finance. Vol. 33. Springer Science & Business Media.

Embrechts, P., Puccetti, G., 2006a. Bounds for functions of dependent risks. Finance and Stochastics 10 (3), 341–352.

Embrechts, P., Puccetti, G., 2006b. Bounds for functions of multivariate risks. Journal of multivariate analysis 97 (2), 526–547.

Engelke, S., Ivanovs, J., 2017. Robust bounds in multivariate extremes. The Annals of Applied Probability 27 (6), 3706–3734.

Esfahani, P. M., Kuhn, D., 2018. Data-driven distributionally robust optimization using the wasserstein metric: Performance guarantees and tractable reformulations. Mathematical Programming 171 (1), 115–166.

Fisher, R. A., Tippett, L. H. C., 1928. Limiting forms of the frequency distribution of the largest or smallest member of a sample. In: Mathematical proceedings of the Cambridge philosophical society. Vol. 24. Cambridge University Press, pp. 180–190.

Gao, F., Wellner, J. A., 2007. Entropy estimate for high-dimensional monotonic functions. Journal of Multivariate Analysis 98 (9), 1751–1764.
Gao, R., Kleywegt, A. J., 2016. Distributionally robust stochastic optimization with wasserstein distance. arXiv preprint arXiv: 1604.02199.

Ghaoui, L. E., Oks, M., Oustry, F., 2003. Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. Operations research 51 (4), 543–556.

Ghosh, S., Lam, H., 2019. Robust analysis in stochastic simulation: Computation and performance guarantees. Operations Research 67 (1), 232–249.

Glasserman, P., Xu, X., 2014. Robust risk measurement and model risk. Quantitative Finance 14 (1), 29–58.

Gnedenko, B., 1943. Sur la distribution limite du terme maximum d’une serie aleatoire. Annals of mathematics, 423–453.

Goh, J., Sim, M., 2010. Distributionally robust optimization and its tractable approximations. Operations research 58 (4-part-1), 902–917.

Gotoh, J., Kim, M. J., Lim, A. E. B., 2018. Robust empirical optimization is almost the same as mean–variance optimization. Operations Research Letters 46 (4), 448 – 452.

Gumbel, E. J., 1958. Statistics of extremes, columbia univ. Press, New York 201.

Gumbel, E. J., 1960. Bivariate exponential distributions. Journal of the American Statistical Association 55 (292), 698–707.

Gumbel, E. J., Goldstein, N., 1964. Analysis of empirical bivariate extremal distributions. Journal of the American Statistical Association 59 (307), 794–816.

Gupta, V., 2019. Near-optimal bayesian ambiguity sets for distributionally robust optimization. Management Science 65 (9), 4242–4260.

Hall, P., Tajvidi, N., 2000. Distribution and dependence-function estimation for bivariate extreme-value distributions. Bernoulli , 835–844.

Hanasusanto, G. A., Roitch, V., Kuhn, D., Wiesemann, W., 2015. A distributionally robust perspective on uncertainty quantification and chance constrained programming. Mathematical Programming 151 (1), 35–62.

Hu, Z., Hong, L. J., 2013. Kullback-leibler divergence constrained distributionally robust optimization. Available at Optimization Online.

Iyengar, G. N., 2005. Robust dynamic programming. Mathematics of Operations Research 30 (2), 257–280.

Jiang, R., Guan, Y., 2016. Data-driven chance constrained stochastic program. Mathematical Programming 158 (1), 291–327.

Jiang, R., Guan, Y., 2018. Risk-averse two-stage stochastic program with distributional ambiguity. Operations Research 66 (5), 1390–1405.
Joe, H., Smith, R. L., Weissman, I., 1992. Bivariate threshold methods for extremes. Journal of the Royal Statistical Society: Series B (Methodological) 54 (1), 171–183.

Jonasson, J. K., Rootzén, H., 2014. Internal validation of near-crashes in naturalistic driving studies: A continuous and multivariate approach. Accident Analysis & Prevention 62, 102–109.

Kuhn, D., Esfahani, P. M., Nguyen, V. A., Shafieezadeh-Abadeh, S., 2019. Wasserstein distributionally robust optimization: Theory and applications in machine learning. In: Operations Research & Management Science in the Age of Analytics. INFORMS, pp. 130–166.

Lam, H., 2016. Robust sensitivity analysis for stochastic systems. Mathematics of Operations Research 41 (4), 1248–1275.

Lam, H., 2018. Sensitivity to serial dependency of input processes: A robust approach. Management Science 64 (3), 1311–1327.

Lam, H., 2019. Recovering best statistical guarantees via the empirical divergence-based distributionally robust optimization. Operations Research 67 (4), 1090–1105.

Lam, H., Mottet, C., 2017. Tail analysis without parametric models: A worst-case perspective. Operations Research 65 (6), 1696–1711.

Lam, H., Zhou, E., 2017. The empirical likelihood approach to quantifying uncertainty in sample average approximation. Operations Research Letters 45 (4), 301 – 307.

Lavrić, B., 1993. Continuity of monotone functions. Archivum Mathematicum 29 (1), 1–4.

Ledford, A. W., Tawn, J. A., 1996. Statistics for near independence in multivariate extreme values. Biometrika 83 (1), 169–187.

Li, B., Jiang, R., Mathieu, J. L., 2019. Ambiguous risk constraints with moment and unimodality information. Mathematical Programming 173 (1), 151–192.

Longin, F. M., 2000. From value at risk to stress testing: The extreme value approach. Journal of Banking & Finance 24 (7), 1097–1130.

McNeil, A. J., 1997. Estimating the tails of loss severity distributions using extreme value theory. ASTIN Bulletin: The Journal of the IAA 27 (1), 117–137.

McNeil, A. J., 1999. Extreme value theory for risk managers. Departement Mathematik ETH Zentrum 12 (5), 217–37.

McNeil, A. J., Frey, R., Embrechts, P., 2015. Quantitative risk management: concepts, techniques and tools-revised edition. Princeton university press.

Mottet, C., Lam, H., 2017. On optimization over tail distributions. arXiv preprint arXiv:1711.00573.

Nelsen, R. B., 2007. An introduction to copulas. Springer Science & Business Media.
Phelps, R. R., 2001. Lectures on Choquet’s theorem. Springer Science & Business Media.

Pickands, J., 1981. Multivariate extreme value distribution. Proceedings 43th, Session of International Statistical Institution, 1981.

Pickands III, J., 1975. Statistical inference using extreme order statistics. the Annals of Statistics, 119–131.

Polonik, W., 1998. The silhouette, concentration functions and ml-density estimation under order restrictions. Annals of statistics, 1857–1877.

Puccetti, G., Rüschendorf, L., 2013. Sharp bounds for sums of dependent risks. Journal of Applied Probability 50 (1), 42–53.

Resnick, S. I., 2013. Extreme values, regular variation and point processes. Springer.

Rootzén, H., Tajvidi, N., 1997. Extreme value statistics and wind storm losses: a case study. Scandinavian Actuarial Journal 1997 (1), 70–94.

Sager, T. W., 1982. Nonparametric maximum likelihood estimation of spatial patterns. The Annals of Statistics, 1125–1136.

Shafieezadeh-Abadeh, S., Kuhn, D., Esfahani, P. M., 2019. Regularization via mass transportation. Journal of Machine Learning Research 20 (103), 1–68.

Smith, R. L., 1984. Threshold methods for sample extremes. In: Statistical extremes and applications. Springer, pp. 621–638.

Smith, R. L., 1986. Extreme value theory based on the r largest annual events. Journal of Hydrology 86 (1-2), 27–43.

Smith, R. L., 1987. Estimating tails of probability distributions. The annals of Statistics, 1174–1207.

Solari, S., Losada, M., 2012. A unified statistical model for hydrological variables including the selection of threshold for the peak over threshold method. Water Resources Research 48 (10).

Songchitruksa, P., Tarko, A. P., 2006. The extreme value theory approach to safety estimation. Accident Analysis & Prevention 38 (4), 811–822.

Tawn, J. A., 1988. Bivariate extreme value theory: models and estimation. Biometrika 75 (3), 397–415.

Van Parys, B. P., Goulart, P. J., Kuhn, D., 2016. Generalized gauss inequalities via semidefinite programming. Mathematical Programming 156 (1-2), 271–302.

Wang, B., Wang, R., 2011. The complete mixability and convex minimization problems with monotone marginal densities. Journal of Multivariate Analysis 102 (10), 1344–1360.

Wiesemann, W., Kuhn, D., Sim, M., 2014. Distributionally robust convex optimization. Operations Research 62 (6), 1358–1376.
Winkler, G., 1988. Extreme points of moment sets. Mathematics of Operations Research 13 (4), 581–587.

Xie, W., 2019. Tractable reformulations of distributionally robust two-stage stochastic programs with \( \infty \)-Wasserstein distance. arXiv preprint arXiv: 1908.08454.

Zachary, S., Feld, G., Ward, G., Wolfram, J., 1998. Multivariate extrapolation in the offshore environment. Applied Ocean Research 20 (5), 273–295.
Appendix

The appendix is organized as follows. Appendix A details the calibration methods for all parameters using data in OU-DRO. Appendix B discusses OU distributions defined on the whole space $\mathbb{R}^d$ as the generalization of OU distributions defined on $D_0$. Appendix C provides the details of POU distributions and POU-DRO discussed in Section 6. Appendix D contains the proofs of all our results.

A Statistical Calibration of Constraints

To illustrate how to calibrate the constraints of formulation (11) with data, we consider the following example used in Section 7.2:

$$\max P((X, Y) \in S)$$
$$\text{subject to } l_{\hat{F}} \leq \hat{F}(x_0, y_0) \leq u_{\hat{F}}$$
$$f_X(x_0) \leq u_X$$
$$f_Y(y_0) \leq u_Y$$
$$a_{X,i} \leq P(x_0 \leq X \leq x_i, Y \geq y_0 | X \geq x_0, Y \geq y_0) \leq b_{X,i}, i = 1, \ldots, n_X,$$
$$a_{Y,j} \leq P(X \geq x_0, y_0 \leq Y \leq y_j | X \geq x_0, Y \geq y_0) \leq b_{Y,j}, j = 1, \ldots, n_Y,$$

$$f(x', y') \geq f(x, y) \text{ if } x_0 \leq x' \leq x \text{ and } y_0 \leq y' \leq y$$

Here we do not include the lower bound density constraints since we see in Section 4.2 that they are redundant. Suppose we have $m$ samples $(X_i, Y_i), i = 1, \ldots, m$ and the confidence level is set to be $\alpha$. Suppose also that $x_0$ and $y_0$ are set as some coordinate-wise top percentiles of the data (e.g., top 10 percentiles) in order to locate the tail region. Then the constants in problem (38) can be calibrated as

$$l_{\hat{F}} = \hat{F}(x_0, y_0) - \frac{\Phi^{-1}(1 - \alpha/14)}{\sqrt{m}} \hat{\sigma}_{F(x_0, y_0)}, \quad u_{\hat{F}} = \hat{F}(x_0, y_0) + \frac{\Phi^{-1}(1 - \alpha/14)}{\sqrt{m}} \hat{\sigma}_{F(x_0, y_0)},$$
$$u_X = \hat{f}_{1-\alpha/7, X | Y \geq y_0}(x_0) \left( \hat{P}(Y \geq y_0) + \frac{\Phi^{-1}(1 - \alpha/7)}{\sqrt{m}} \hat{\sigma}_{P(Y \geq y_0)} \right),$$
$$u_Y = \hat{f}_{1-\alpha/7, Y | X \geq x_0}(y_0) \left( \hat{P}(X \geq x_0) + \frac{\Phi^{-1}(1 - \alpha/7)}{\sqrt{m}} \hat{\sigma}_{P(X \geq x_0)} \right),$$
$$a_{X,i} = \hat{F}_X(x_i) - \frac{q}{\sqrt{m_1}}, \quad b_{X,i} = \hat{F}_X(x_i) + \frac{q}{\sqrt{m_1}},$$
$$a_{Y,j} = \hat{F}_Y(y_j) - \frac{q}{\sqrt{m_1}}, \quad b_{Y,j} = \hat{F}_Y(y_j) + \frac{q}{\sqrt{m_1}}.$$

We will explain each notation and provide justifications. First, we have

$$\hat{F}(x_0, y_0) = \frac{1}{m} \sum_{i=1}^{m} I(X_i \geq x_0, Y_i \geq y_0), \quad \hat{\sigma}_{F(x_0, y_0)} = \sqrt{\frac{1}{m-1} \sum_{i=1}^{m} (I(X_i \geq x_0, Y_i \geq y_0) - \hat{F}(x_0, y_0))^2}$$

as the point estimate of $\hat{F}(x_0, y_0)$ given by the sample mean of $I(X_i \geq x_0, Y_i \geq y_0)$’s, and the sample standard deviation of $I(X_i \geq x_0, Y_i \geq y_0)$’s respectively. We also denote $\Phi$ as the cumulative distribution function of the standard normal distribution. Thus, the constants $l_{\hat{F}}$ and $u_{\hat{F}}$ are set as the asymptotically exact lower and
upper confidence bounds of $\tilde{F}(x_0, y_0)$ induced by the standard central limit theorem (CLT) at the confidence level $1 - \alpha/7$. Here, the confidence level $1 - \alpha/7$ is due to a Bonferroni correction which we will explain later.

Next, the constant $u_X$ is calibrated as the $(1 - \alpha/7)$-level upper confidence bound on $f_X(x_0)$ by the bootstrap as follows. Note that the truncated marginal density $f_X(x_0)$ can be written as $f_X(x_0) = P(Y \geq y_0) f_X|Y \geq y_0(x_0)$, where $f_X|Y \geq y_0$ is the density of $X$ conditional on $\{Y \geq y_0\}$. We need to find the asymptotically exact $(1 - \alpha/7)$-level upper bound for both factors. For $f_X|Y \geq y_0(x_0)$, we select the data $(X_i, Y_i)$ with $Y_i \geq y_0$. Suppose the number of such data is $m_0$. We resample the $x$-coordinates of the selected data with replacement to the size $m_0$, and use the resample to construct a kernel density estimate for $X$ at $x_0$. Repeat the procedure many times and the $(1 - \alpha/7)$-th quantile $\hat{f}_{1-\alpha/7}(X|Y \geq y_0)(x_0)$ of these estimates is an asymptotically exact $(1 - \alpha/7)$-level upper bound of $f_X|Y \geq y_0(x_0)$ (Chen (2017)). On the other hand, an asymptotically exact $(1 - \alpha/7)$-level upper confidence bound of $P(Y \geq y_0)$ is obtained by the CLT like for $\tilde{F}(x_0, y_0)$ discussed earlier, where

$$\hat{P}(Y \geq y_0) = \frac{1}{m} \sum_{i=1}^{m} I(Y_i \geq y_0), \quad \hat{\sigma}_{P(Y \geq y_0)} = \sqrt{\frac{1}{m-1} \sum_{i=1}^{m} (I(Y_i \geq y_0) - \hat{P}(Y \geq y_0))^2}.$$ 

The calibration of $u_Y$ is justified similarly.

Furthermore, to calibrate the constants $a_{X,i}, b_{X,i}$, we use the Kolmogorov–Smirnov statistic. We collect all data such that $X \geq x_0$ and $Y \geq y_0$. Suppose these data have size $m_1$. The Kolmogorov–Smirnov test gives us

$$\sqrt{m_1} \sup_{x \geq x_0} |F_X(x) - \hat{F}_X(x)| \leq q$$

(39) an asymptotically exact $(1 - \alpha/7)$-level confidence region on $F_X(x)$, where $F_X(x) = P(X \leq x | X \geq x_0, Y \geq y_0)$ and $\hat{F}_X$ is the empirical distribution of $X$ restricted to $X \geq x_0, Y \geq y_0$, and $q$ is the $(1 - \alpha/7)$-th quantile of $\sup_{t \in [0,1]} |BB(t)|$ where $BB$ is the standard Brownian bridge. By considering the left and right limits at $x = x_i$ in (39), we can rewrite (39) to obtain the lower and upper bounds of $F_X(x_i)$ as $a_{x,i}$ and $b_{X,i}$ defined earlier, where $\hat{F}_X(x_i-)$ is the left limit of $\hat{F}_X$ at $x_i$. The constants $a_{Y,j}, b_{Y,j}$ can be calibrated similarly.

Finally, the Bonferroni correction is applied since there are in total 7 sets of parameters that we need to calibrate, and we want to control the familywise Type I error to be at most $\alpha$, thus giving each individual confidence level discussed above to be $1 - \alpha/7$. If the constraints are calibrated via the methods above and the true density is OU about $(x_0, y_0)$ on $[x_0, \infty) \times [y_0, \infty)$, then the constraints in (38) will hold simultaneously with confidence level $1 - \alpha$ asymptotically. By Corollary 1, the optimal value will then be an asymptotically valid upper confidence bound on the true probability with level $1 - \alpha$.

**B OU Distribution on $\mathbb{R}^d$**

In this section, we consider a generalization of the OU distribution defined in Definition 6. We will enlarge the support from $\mathcal{D}_0$ to $\mathbb{R}^d$ and keep the monotonicity property for each variable. Like the OU distribution on $\mathcal{D}_0$, we will provide two definitions for the OU distribution on $\mathbb{R}^d$. One is defined in terms of the density and the other is defined via mixture representation.

**Definition 7** (Orthounimodal density on $\mathbb{R}^d$). A probability distribution on $\mathbb{R}^d$ with density $f$ (with respect to the Lebesgue measure) is OU about mode $x_0 = (x_{10}, \ldots, x_{d0})$ if for each $i = 1, \ldots, d$ and any fixed $x_j \in \mathbb{R}, j \neq i$, the function $x_i \mapsto f(x)$ is non-decreasing on $(-\infty, x_{i0})$ and non-increasing on $[x_{i0}, \infty)$. 

Definition 8 (Mixture representation of OU distribution on $\mathbb{R}^d$). We have:

1. A set $K \subseteq \mathbb{R}^d$ is said to be orthounimodal about $x_0$ if for every $x \in K$, the closed rectangle with edges parallel to the axes and with opposite vertices $x_0$ and $x$ is in $K$, i.e., $(\eta_1 x_1 + (1 - \eta_1)x_{10}, \ldots, \eta_d x_d + (1 - \eta_d)x_{d0}) \in K$ for any $\eta_i \in [0,1], i = 1, \ldots, d$.

2. A distribution on $\mathbb{R}^d$ is called OU about $x_0$ if it belongs to the closed convex hull of the set of all uniform distributions on subsets of $\mathbb{R}^d$ which are OU about $x_0$.

From Definition 7 and Definition 8, we can see OU distribution has a linear transformation property: if $X$ is OU about $x_0$, then $X + x'_0$ is OU about $x_0 + x'_0$. Besides, analogies of Lemma 2, Theorem 1 and Lemma 3 hold for OU distributions on $\mathbb{R}^d$.

Lemma 9. Suppose that $K \subseteq \mathbb{R}^d$ is an OU set about $x_0$. Then $K$ is Lebesgue measurable. Besides, $\bar{K}$ and $K^\circ$ (closure of $K^\circ$) are also OU sets about $x_0$ satisfying $\lambda(K^\circ) = \lambda(K) = \lambda(\bar{K})$.

The following theorem justifies the equivalence of Definition 8 and Definition 7 and establishes the Choquet representation for OU distributions on $\mathbb{R}^d$ in the presence of density. To simplify the notations, we assume without loss of generality that the mode is the origin. If $X$ is OU about $x_0$, we can see $X - x_0$ is OU about the origin and Theorem 3 applies. Recall that $W_K$ denotes the uniform distribution on $K$.

Theorem 3. Suppose a distribution $P$ on $\mathbb{R}^d$ is absolutely continuous with respect to Lebesgue measure. Then $P$ is OU about the origin if and only if there is a density $f(x)$ of $P$ such that for every $s > 0$, the set

$$C_s = \{x \in D_0 : f(x) \geq s\}$$

is OU about the origin, or equivalently, if and only if $f$ is an OU density about the origin on $\mathbb{R}^d$. Besides, let $O_1, \ldots, O_{2^d}$ be the $2^d$ (closed) orthants of $\mathbb{R}^d$. Then $P$ has the following Choquet representation:

$$P(B) = \sum_{1 \leq i \leq 2^d, P(O_i) > 0} P(O_i) \int_0^\infty W_{C^*_i}(B) g_i(s) ds$$

(40)

for any Lebesgue measurable set $B$, where $C_i = \{x \in O_i : f_i(x) \geq s\}$, $f_i(x) = f(x)I(x \in O_i)/P(O_i)$ is the density of the conditional distribution $P(\cdot | O_i)$, $C^*_i$ is the closure of $C_i^*$ and $g_i(s) = \lambda(C^*_i)$ is a probability density on $(0, \infty)$.

The following Lemma tells us about the extreme point in the class of OU distributions about the origin on $\mathbb{R}^d$.

Lemma 10. If $K \subseteq O_i$ is an OU set about the origin with $\lambda(K) > 0$ for some orthant $O_i$, then $W_K$ is an extreme point in the class of OU distributions about the origin on $\mathbb{R}^d$.

The condition $K \subseteq O_i$ is essential for Lemma 10. Consider an example in the bivariate case. Define $K = \{(x,y): -1 \leq x \leq 1, -1 \leq y \leq 1\}$, $K_1 = \{(x,y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$, $K_2 = \{(x,y): -1 \leq x \leq 0, 0 \leq y \leq 1\}$, $K_3 = \{(x,y): -1 \leq x \leq 0, -1 \leq y \leq 0\}$ and $K_4 = \{(x,y): 0 \leq x \leq 1, -1 \leq y \leq 0\}$. We can see all the sets are OU about the origin. However, $W_{K_i}$ is not an extreme point in the class of OU distributions about the origin on $\mathbb{R}^2$ since it can be written as

$$W_K = \frac{1}{4} W_{K_1} + \frac{1}{4} W_{K_2} + \frac{1}{4} W_{K_3} + \frac{1}{4} W_{K_4}.$$
This explains why the Choquet representation in Theorem 3 is represented as (40) instead of

\[ P(B) = \int_0^\infty W_{C_s}(B)g(s)ds \]

although (41) still holds in the setting of Theorem 3.

C Theory of Partial Orthounimodality and Partially Orthounimodal Distributionally Robust Optimization

In this section, we provide the details on the theory of POU distributions and the POU-DRO problem discussed in Section 6.

C.1 POU Distribution

In this section, we introduce POU and establish its Choquet representation by following the same flow as in the OU case. As we have discussed, POU models a density that is non-increasing in only some of the dimensions. For simplicity, we suppose the density is non-increasing in only the first \( d' \) components. In other words, only the first \( d' \) components of the random vector are in their tails. For other cases, we can simply permute the components to transform them into this case. Let \( D_0 = \{ x \geq x_0 \} \) where \( x_0 = (x_{10}, \ldots, x_{d0}) \in \mathbb{R}^{d'} \times [-\infty, \infty)^{d-d'} \) for some \( d' < d \). When \( x_{j0} = -\infty \), \( x_j \geq x_{j0} \) is interpreted as \( x_j \in \mathbb{R} \). Similar to OU distributions, POU distributions can be defined in terms of monotonicity of the probability density or mixture representation.

Definition 9 (\( d' \)-partially orthounimodal density). A probability distribution on \( D_0 \) with density \( f \) (with respect to the Lebesgue measure) is \( d' \)-partially orthounimodal (\( d' \)-POU) about mode \( x_0 \) if \( f \) is non-increasing with respect to \( x_1, \ldots, x_{d'} \) on \( D_0 \) for any fixed \( (x_{d'+1}, \ldots, x_d) \) with \( (x_{d'+1}, \ldots, x_d) \geq (x_{d'+1,0}, \ldots, x_{d0}) \).

Definition 10 (Mixture representation of \( d' \)-POU distribution). We have:

1. A measurable set \( K \subset D_0 \) is said to be \( d' \)-POU about \( x_0 \) if for every \( x \in K \), we have \( x' \in K \) if \( x_i \geq x'_{i} \geq x_{i0} \) for \( i = 1, \ldots, d' \) and \( x_i = x'_{i} \geq x_{i0} \) for \( i = d' + 1, \ldots, d \).
2. A distribution on \( D_0 \) is called \( d' \)-POU about \( x_0 \) if it belongs to the closed convex hull of the set of all uniform distributions on subsets of \( D_0 \) which are \( d' \)-POU about \( x_0 \).

The following theorem justifies the equivalence of Definitions 9 and 10 in the presence of a density.

Theorem 4. Suppose a distribution \( P \) on \( D_0 \) is absolutely continuous with respect to Lebesgue measure. Then \( P \) is \( d' \)-POU about \( x_0 \) if and only if there is a density \( f(x) \) of \( P \) such that for every \( s > 0 \), the set

\[ C_s = \{ x \in D_0 : f(x) \geq s \} \]

is \( d' \)-POU about \( x_0 \), or equivalently, if and only if \( f \) is a \( d' \)-POU density about \( x_0 \) on \( D_0 \).
Theorem 6. Without the presence of a density, the Choquet representation of \( d' \)-POU distributions is slightly more complicated than that of the OU distributions. We first introduce the notations. For a \( d' \)-POU set \( K \) about \( x_0 \) on \( \mathcal{D}_0 \), we define its slice with respect to the subspace \( \{ y \in \mathbb{R}^d : y_{d+1} = x_{d+1}, \ldots, y_d = x_d \} \) as \( K_{x_{d+1},\ldots,x_d} = \{(y_1, \ldots, y_d) : (y_1, \ldots, y_d, x_{d+1}, \ldots, x_d) \in K \} \subset \mathbb{R}^d \). In other words, \( K \cap \{ y \in \mathbb{R}^d : y_{d+1} = x_{d+1}, \ldots, y_d = x_d \} = K_{x_{d+1},\ldots,x_d} \times \{(x_{d+1}, \ldots, x_d) \} \). When \( K \) is \( d' \)-POU, \( K_{x_{d+1},\ldots,x_d} \) is either empty or a nonempty OU set about \((x_{10}, \ldots, x_{d0})\) on \( \{ y \in \mathbb{R}^d : y_i \geq x_{i0}, i = 1, \ldots, d' \} \). Recall that \( W_K \) denotes the uniform distribution on \( K \) and \( K \) could be in a lower-dimensional subspace of \( \mathbb{R}^d \). In the presence of a density, we have the following Choquet representation theorem:

**Theorem 5.** Suppose a distribution \( P \) on \( \mathcal{D}_0 \) is absolutely continuous with respect to Lebesgue measure and \( P \) is \( d' \)-POU about \( x_0 \). Let \( f \) be a \( d' \)-POU density of \( P \) and \( C_s = \{ x \in \mathcal{D}_0 : f(x) \geq s \} \). Then the Choquet representation of \( P \) is given by:

\[
P(B) = \int_{0}^{\infty} \cdots \int_{x_{d+1}0}^{\infty} W_{C_s \cap \{ y \in \mathbb{R}^d : y_{d+1} = x_{d+1}, \ldots, y_d = x_d \}}(B)g(s, x_{d+1}, \ldots, x_d)dx_{d+1}\cdots dx_d ds
\]

for any Lebesgue measurable set \( B \), where \( g(s, x_{d+1}, \ldots, x_d) = \lambda_d(C_{s,x_{d+1},\ldots,x_d}) \) is a probability density on \( \{ (s, x_{d+1}, \ldots, x_d) : s > 0, x_i \geq x_{i0}, i = d' + 1, \ldots, d \} \), \( C_{s,x_{d+1},\ldots,x_d} \) is the slice of \( C_s \) with respect to the subspace \( \{ y \in \mathbb{R}^d : y_{d+1} = x_{d+1}, \ldots, y_d = x_d \} \) and \( \lambda_d(\cdot) \) is the Lebesgue measure on \( \mathbb{R}^d \).

In the proof of Theorem 5, we need the following lemma to ensure the representation of \( P \) is indeed a Choquet representation.

**Lemma 11.** Suppose \( K \subset \mathcal{D}_0 \) is a \( d' \)-POU set about \( x_0 \). If \( K \) is fully contained in the subspace \( \{ y \in \mathbb{R}^d : y_{d+1} = x_{d+1}, \ldots, y_d = x_d \} \) for \( x_i \geq x_{i0}, i = d' + 1, \ldots, d \) with \( \lambda_d(K_{x_{d+1},\ldots,x_d}) > 0 \), then \( W_K \) is an extreme point in the class of \( d' \)-POU distributions about \( x_0 \) on \( \mathcal{D}_0 \).

When we consider \( d' = 1 \), i.e., 1-POU distributions, we can see the slice \( K_{x_2,\ldots,x_d} \) is either \([x_{10}, x_1]\) or \([x_{10}, x_1]\) for some \( x_1 \geq x_{10} \). Such a simple form of \( K_{x_2,\ldots,x_d} \) gives us a cleaner Choquet representation even without the presence of a density.

**Theorem 6.** Suppose a random vector \( X = (X_1, \ldots, X_d) \) takes values on \( \mathcal{D}_0 \). Then the probability measure \( P \) of \( X \) is 1-POU about \( x_0 \) if and only if \( X \) is distributed as \((x_{10} + U(Z_1 - x_{10}), Z_2, \ldots, Z_d)\) where \( U \) is the uniform distribution on \((0, 1)\) and independent of \((Z_1, \ldots, Z_d) \in \mathcal{D}_0 \). Consequently, \( P \) is 1-POU about \( x_0 \) if and only if it has the following Choquet representation

\[
P = \int_{\mathcal{D}_0} W_{1-POU}(z) dQ(z), \tag{42}
\]

where \( W_{1-POU}(z) \) is the distribution of \((x_{10} + U(Z_1 - x_{10}), z_2, \ldots, z_d)\), \( U \) is the uniform distribution on \((0, 1)\) and \( Q \) is the distribution of \((Z_1, \ldots, Z_d)\) uniquely determined by \( P \). Moreover, if \( P(X_1 = x_{10}) = 0 \), then \( X_1 \) is an absolutely continuous random variable with the probability density \( E_Q[I(Z_1 \geq x)/(Z_1 - x_{10})], x \geq x_{10} \) which is continuous at \( x = x_{10} \).

**C.2 POU-DRO Problem: Formulation and Reduction**

In this section, we discuss the formulation of the \( d' \)-POU-DRO problem and the reduction of the 1-POU-DRO problem. With these, for the bivariate case, theories in this paper would cover all the extreme cases: if two
dimensions are both in the tail, we can handle it by the OU-DRO problem; if only one dimension is in the tail, we can handle it by the 1-POU-DRO problem. Since the general \( d' \)-POU-DRO problem involves the OU-DRO problem (when \( d' = d \)) and the latter hasn’t been solved completely for any dimension, we will leave the reduction of the general \( d' \)-POU-DRO problem as the future work.

Recall that for \( d' \)-POU, only the first \( d' \) dimensions are in the tail and the tail region is given by \( D_0 = \{ x \geq x_0 \} \) with \( x_0 = (x_{10}, \ldots, x_{d0}) \in \mathbb{R}^{d'} \times [-\infty, \infty)^{d-d'} \). Imitating the OU-DRO problem, we formulate the \( d' \)-POU as follows:

\[
\begin{align*}
\max & \quad P((X_1, \ldots, X_d) \in S) \\
\text{subject to} & \quad l_{\bar{F}} \leq \bar{F}(x_0) \leq u_{\bar{F}} \quad (43) \\
& \quad f_{X_i}(x_{i0}) \leq u_{X_i}, i = 1, \ldots, d' \\
& \quad a_i \bar{F}(x_0) \leq P(\bar{x}_{ji} \leq X_j \leq \bar{x}_{ji}, j = 1, \ldots, d) \leq b_i \bar{F}(x_0), i = 1, \ldots, n \\
& \quad f(x') \geq f(x) \text{ if } x_i \geq x'_i \geq x_{i0}, i \leq d' \text{ and } x_i = x'_i \geq x_{i0}, i > d' \\
\end{align*}
\]

The interpretation and calibration of the constraints in (43) are the same as the OU-DRO problem (11). However, we do not restrict the truncated marginal densities for the last \( d - d' \) components because these components are not in the tail in the motivating problems.

Now we focus on the reduction of the 1-POU-DRO problem. Here we make a small modification that we will use Definition 10 instead of Definition 9 as the 1-POU shape constraint. In other words, we will not assume the existence of the density of \( X \). As we will see later in Theorem 7 and Corollary 4, this is because we may not be able to recover an absolutely continuous \( X \) for the 1-POU-DRO problem from the optimal solution to the reduced moment problem. As in the OU-DRO problem, we will replace \( l_{\bar{F}} \leq \bar{F}(x_0) \leq u_{\bar{F}} \) by \( \bar{F}(x_0) = c \) in the reduction procedure. For reference, the 1-POU-DRO problem is formulated as

\[
\begin{align*}
\max & \quad P((X_1, \ldots, X_d) \in S) \\
\text{subject to} & \quad \bar{F}(x_0) = c \\
& \quad f_{X_i}(x_{i0}) \leq u_{X_i} \quad (44) \\
& \quad a_i \bar{F}(x_0) \leq P(\bar{x}_{ji} \leq X_j \leq \bar{x}_{ji}, j = 1, \ldots, d) \leq b_i \bar{F}(x_0), i = 1, \ldots, n \\
& \quad (X_1, \ldots, X_d) \text{ is 1-POU about } x_0 \text{ when restricted to } D_0 \text{ and } P(X_1 = x_{10}) = 0 \\
\end{align*}
\]

Here \( f_{X_1}(x_1) \) denotes the marginal density of \( X_1 \) within \( D_0 \), i.e., for any measurable \( A \subset [x_{10}, \infty) \), \( \int_A f_{X_1}(x_1)dx_1 = P(X_1 \in A, X_i \geq x_{i0}, i \geq 2) \). The existence of \( f_{X_1}(x_1) \) is ensured by Theorem 6. We require \( f_{X_1}(x_1) \) to be continuous at \( x_1 = x_{10} \) in order to avoid the arbitrariness of the density at this point. When \( X \) has a continuous density \( f \) satisfying Definition 9 on \( D_0 \), \( f_{X_1} \) is just given by

\[
f_{X_1}(x_1) = \int_{x_{20}}^{\infty} \cdots \int_{x_{d0}}^{\infty} f(x_1, \ldots, x_d)dx_2 \cdots dx_d
\]

as in the formulation of OU-DRO problem. For the rare-event set \( S \), we assume it is in the form

\[
S = \{ x \in D_0 : g_1(x_2, \ldots, x_d) \leq x_1 \leq g_2(x_2, \ldots, x_d) \} \quad (45)
\]

where \( g_i : [x_{20}, \infty) \times \cdots \times [x_{d0}, \infty) \mapsto [x_{10}, \infty) \) is a measurable function for \( i = 1, 2 \) and \( g_1(x_2, \ldots, x_d) \leq g_2(x_2, \ldots, x_d) \) for any \( (x_2, \ldots, x_d) \). We can see the formulation (44) only depends on the behavior in \( D_0 \) so we
can restrict our attention to the truncated distribution on $\mathcal{D}_0$. The idea for solving (44) is to use the Choquet representation (42) to rewrite it as a moment problem with respect to the probability measure $Q$.

**Theorem 7.** The 1-POU-DRO problem (44) is equivalent to the following moment problem:

$$\max \ cE_Q \left[ \frac{\min(g_2(Z_2, \ldots, Z_d), Z_1) - \min(g_1(Z_2, \ldots, Z_d), Z_1)}{Z_1 - x_{10}} \right]$$

subject to $E_Q \left[ \frac{1}{Z_1 - x_{10}} \right] \leq \frac{u_{X_1}}{c}$

$$a_i \leq E_Q \left[ \frac{\min(Z_1, x_{1i}) - \min(Z_1, x_{1i})}{Z_1 - x_{10}} \right] \leq b_i, i = 1, \ldots, n$$

where the decision variable $Q$ is the probability distribution of $(Z_1, \ldots, Z_d) \in \mathcal{D}_0$. If $(Z_1^*, \ldots, Z_d^*) \sim Q^*$ is the optimal solution to the moment problem (46), then $cP^*$ is the optimal solution to the 1-POU-DRO problem (44) where $P^*$ is the distribution of $(x_{10} + U(Z_1^* - x_{10}), Z_2^*, \ldots, Z_d^*)$ with $U$ being the uniform distribution on $(0, 1)$ and independent of $(Z_1^*, \ldots, Z_d^*)$.

According to Theorem 3.2 in Winkler (1988), to solve the moment problem (46), it suffices to consider discrete probability measures with at most $n + 2$ points in the support. So (46) is equivalent to the following non-linear optimization:

$$\max \ c \sum_{l=1}^{n+2} p_l \frac{\min(g_2(z_{1l}, \ldots, z_{dl}), z_{1l}) - \min(g_1(z_{1l}, \ldots, z_{dl}), z_{1l})}{z_{1l} - x_{10}}$$

subject to $\sum_{l=1}^{n+2} p_l \frac{z_{1l} - x_{10}}{z_{1l} - x_{10}} \leq \frac{u_{X_1}}{c}$

$$a_i \leq \sum_{l=1}^{n+2} p_l \frac{\min(z_{1l}, x_{1i}) - \min(z_{1l}, x_{1i})}{z_{1l} - x_{10}} \leq b_i, i = 1, \ldots, n$$

$$\sum_{l=1}^{n+2} p_l = 1, p_l \geq 0, l = 1, \ldots, n + 2$$

$$z_{li} \geq x_{i0}, l = 1, \ldots, n + 2, i = 1, \ldots, d$$

where $p_l, z_{li}$ are the decision variables. The non-linear optimization (47) is capable to handle the 1-POU-DRO problem (44) when the constraint $\bar{F}(x_0) = c$ is replaced by $l_{\bar{F}} \leq \bar{F}(x_0) \leq u_{\bar{F}}$. In this case, we simply make $c$ an additional decision variable and an additional constraint $l_{\bar{F}} \leq c \leq u_{\bar{F}}$. If $c^*, p_l^*, z_{li}^*$ is the optimal solution to this non-linear optimization problem and we write $(Z_1^*, \ldots, Z_d^*) \sim Q^*$ as the discrete distribution on $(z_{1l}^*, \ldots, z_{dl}^*), l = 1, \ldots, n + 2$ with corresponding probability $p_l^*$, then $c^*P^*$ is the optimal solution to this 1-POU-DRO problem where $P^*$ is the distribution of $(x_{10} + U(Z_1^* - x_{10}), Z_2^*, \ldots, Z_d^*)$ with $U$ being the uniform distribution on $(0, 1)$ and independent of $(Z_1^*, \ldots, Z_d^*)$.

**Corollary 4.** We have the following:

1. The 1-POU-DRO problem (44), the moment problem (46) and the non-linear optimization problem (47) are equivalent.

2. Consider the 1-POU-DRO problem (44) with $\bar{F}(x_0) = c$ replaced by $l_{\bar{F}} \leq \bar{F}(x_0) \leq u_{\bar{F}}$. Then this 1-POU-DRO problem is equivalent to the non-linear optimization problem (47) with an additional decision variable $c$ and an additional constraint $l_{\bar{F}} \leq c \leq u_{\bar{F}}$. If $c^*, p_l^*, z_{li}^*$ is the optimal solution to this non-linear optimization problem and we write $(Z_1^*, \ldots, Z_d^*) \sim Q^*$ as the discrete distribution on $(z_{1l}^*, \ldots, z_{dl}^*), l = 1, \ldots, n + 2$ with corresponding probability $p_l^*$, then $c^*P^*$ is the optimal solution to this 1-POU-DRO problem where $P^*$ is the distribution of $(x_{10} + U(Z_1^* - x_{10}), Z_2^*, \ldots, Z_d^*)$ with $U$ being the uniform distribution on $(0, 1)$ and independent of $(Z_1^*, \ldots, Z_d^*)$. 42
Corollary 4 reveals why we do not assume the existence of the density in the formulation of the 1-POU-DRO problem (44). Suppose $p_i^*, z_{i1}$ is the optimal solution to the non-linear optimization problem (47), i.e., the optimal solution to the moment problem (46) can be taken as the discrete distribution $(Z_1^*, \ldots, Z_d^*) \sim Q^*$ on $(z_{i1}, \ldots, z_{iD})$, $l = 1, \ldots, n + 2$ with corresponding probability $p_i^*$. Then if we recover the optimal solution $cP^*$ in the way described in Theorem 7, we will find $cP^*$ is not absolutely continuous on $D_0$ as its support has Lebesgue measure 0. Due to this reason, we do not assume the existence of the density in (44).

We close this section by discussing a variant of formulation (44). As in the discussion of OU-DRO problem, we can replace the moment constraint in (44) by the unconditional version:

$$a_i \leq P(x_{ji} \leq X_j \leq \bar{x}_{ji}, j = 1, \ldots, d) \leq b_i. \quad (48)$$

In this case, we just need to replace $a_i$ and $b_i$ in the problem (46) or (47) by $a_i/c$ or $b_i/c$ and Theorem 7 and Corollary 4 still hold since (48) is equivalent to the conditional version constraint:

$$\frac{a_i}{c} F(x_0) \leq P(x_{ji} \leq X_j \leq \bar{x}_{ji}, j = 1, \ldots, d) \leq \frac{b_i}{c} F(x_0)$$

given $F(x_0) = c$.

### D Proofs

**Proof of Lemma 1.** The conclusion follows since

\[
\begin{align*}
\{ & \text{geometric property of the distribution } F \text{ holds for } X \geq u, \\
& \text{auxiliary constraints on } F \text{ holds} \}
\end{align*}
\]

implies $\{Z^* \geq Z\}$. □

**Proof of Lemma 2.** Let us first show $K$ and $K^\circ$ are OU sets about $x_0$. In order to show $K$ is OU, it suffices to show that if $x$ is a limit point of $K$, then $x' \in K$ if $x_0 \leq x' \leq x$. Suppose that the sequence $\{x_j, j \geq 1\} \subset K$ converges to $x$. Consider the sequence $\{\min(x, x_j), j \geq 1\}$ where the minimum is taken component-wise. By the definition of OU sets, this sequence must be in $K$. Besides, we notice that the limit point of this sequence is $\min(x', x) = x'$. Thus $x' \in K$, i.e., $K$ is an OU set about $x_0$. Now let’s show $K^\circ$ is also OU. Suppose $x \in K^\circ$, i.e., there exists a sequence $\{x_j\} \subset K^\circ$ s.t. $x_j \to x$. We need to show $x' \in K^\circ$ if $x_0 \leq x' \leq x$. By the definition of OU, $x_j \in K^\circ$ implies $x_0 < x_j$ and $x'' \in K^\circ$ for $x_0 < x'' \leq x$. Therefore, if $x_0 < x' \leq x$, we have $\min(x_j, x') \in K^\circ$. Since $\min(x_j, x') \to x'$, we know $x' \in K^\circ$ for all $x'$ satisfying $x_0 < x' \leq x$. Moreover, since $K$ is a closed set, we can get $x' \in K^\circ$ for $x_0 \leq x' \leq x$, which means $K^\circ$ is OU about $x_0$.

Then let us consider other claims in the lemma. Since Lebesgue measure is complete, in order to show $K$ is Lebesgue measurable and $\lambda(K^\circ) = \lambda(K) = \lambda(K)$, it suffices to show that $\lambda(\partial K) = 0$. This is already proven in Lavrić (1993), which concludes our proof. □

**Proof of Theorem 1.** Suppose there is a density $f(x)$ of $P$ such that for every $s > 0$, the set

$$C_s = \{x \in D_0 : f(x) \geq s\}$$

is OU about $x_0$. Write $W_{C_s}$ as the uniform distribution on $C_s$ and let $g(s) = \lambda(C_s) \geq 0$. Notice that

$$f(x) = \int_0^{f(x)} 1 ds = \int_0^\infty I(x \in C_s) ds.$$
Thus for any measurable set $B$, $$P(B) = \int_B f(x)dx = \int_B \int_0^\infty I(x \in C_s)dsdx = \int_0^\infty \lambda(B \cap C_s)ds.$$ By Lemma 2, we know that $\lambda(\partial C_s) = 0$. Thus, $\lambda(B \cap C_s) = \lambda(B \cap \bar{C}_s)$, which means $$P(B) = \int_0^\infty \lambda(B \cap \bar{C}_s)ds = \int_0^\infty W_{\bar{C}_s}(B)g(s)ds.$$ Setting $B = D_0$, we can see $P(B) = 1 = \int_0^\infty g(s)ds$ and thus $g(s)$ is a probability density on $(0, \infty)$. This proves the correctness of (10). By Lemma 2, we know that $\bar{C}_s$ is an OU set. So $W_{\bar{C}_s}$ is the uniform distribution on the OU set $\bar{C}_s$. Since Lebesgue measure is translation invariant, we can get $\lambda(K \cap \bar{C}_s(x')) = \lambda(K \cap \bar{C}_s(x))$. Clearly, this relation also holds under the convex combinations of the uniform distributions on OU sets. Since $P$ is OU about $x_0$, by definition, there is a sequence $\{Q_m, m \geq 1\}$ such that $Q_m$ converges weakly to $P$, where $Q_m$’s are the convex combinations of the uniform distributions on OU sets. Therefore, these $Q_m$’s satisfy (49). Since $P$ has a density, say $f_0$, we have $$P(\partial N_\delta(x')) = P(\partial N_\delta(x)) = 0.$$ Weak convergence and (49) imply that $$P(N_\delta(x')) \geq P(N_\delta(x)).$$ For $x \in D_0^\circ$, we define $$f(x) = \limsup_{\delta \downarrow 0} \frac{P(N_\delta(x))}{\lambda(N_\delta(x))} = \limsup_{\delta \downarrow 0} \frac{P(N_\delta(x))}{(2\delta)^d}.$$ By the Lebesgue differentiation theorem, $f(x) = f_0(x)$ a.e., which means $f(x)$ is also a density for $P$. Besides, (50) implies $f(x') \geq f(x)$ for $x, x' \in D_0^\circ$ with $x \geq x'$. For $x \in \partial D_0$, we simply define $$f(x) = \sup_{y \in D_0^\circ} f(y).$$ Then the density $f(x)$ is what we want.

Finally, note that only $W_{\bar{C}_s}$ with $\lambda(\bar{C}_s) > 0$ contributes to (10). Thus, to show (10) is indeed a Choquet representation, it suffices to show $W_K$ is an extreme point in the class of OU distributions on $D_0$ if $K$ is an OU set about $x_0$ with $\lambda(K) > 0$. This is proved in Lemma 3. 

□
Proof of Lemma 3. Suppose $W_K$ is not an extreme point, i.e., there exist two different OU distributions $P_1, P_2$ and a number $\eta \in (0, 1)$ s.t.

$$W_K = \eta P_1 + (1 - \eta) P_2. \quad (51)$$

By Lemma 2, we know $W_K$ and $W_{K^c}$ are the same distribution. Therefore, the support of $W_K$ is the OU set $\bar{K}^\circ$ (closure of $K^\circ$). Let $K_1$ and $K_2$ be the support of $P_1$ and $P_2$ respectively. We have $\bar{K}^\circ = K_1 \cup K_2$. Since $W_K$ is absolutely continuous with respect to Lebesgue measure, $P_1$ and $P_2$ are also absolutely continuous with respect to Lebesgue measure. Let $f_{W_K}, f_1$ and $f_2$ be densities of $W_K$, $P_1$ and $P_2$ respectively. By Theorem 1, we can without loss of generality assume that $f_1(x)$ and $f_2(x)$ are non-increasing in each component of $x$ on $\mathcal{D}_0$, which also implies $K_1$ and $K_2$ are OU sets about $x_0$. Then (51) can be written as

$$f_{W_K} = \eta f_1 + (1 - \eta) f_2, \text{ a.e.}$$

However, since $f_{W_K}$ is constant on the OU set $\bar{K}^\circ$ and $f_1(x)$, $f_2(x)$ are non-increasing in each component of $x$, $f_1(x)$ and $f_2(x)$ must also be constant a.e. on $\bar{K}^\circ$. Thus, $P_1$ and $P_2$ are uniform distributions on $K_1$ and $K_2$ respectively. Since $\bar{K}^\circ$, $K_1$ and $K_2$ are all OU sets with positive Lebesgue measure, there exists an open set (close to $x_0$) $K' \subset \bar{K}^\circ \cap K_1 \cap K_2$. By (51), we have

$$\frac{\lambda(K')}{{\lambda}(K^\circ)} = \eta \frac{\lambda(K')}{{\lambda}(K_1)} + (1 - \eta) \frac{\lambda(K')}{{\lambda}(K_2)},$$

which implies $\lambda(K^\circ) = \lambda(K_1) = \lambda(K_2)$. So we know $W_K$, $P_1$ and $P_2$ are the same distribution, which contradicts our assumption that $P_1$ and $P_2$ are different distributions. Therefore, $W_K$ is an extreme point in the class of OU distributions about $x_0$.

Proof of Proposition 1. Note that a closed rectangle in $\mathcal{D}_0$ which contains $x_0$ and have edges parallel to the coordinate axes is OU about $x_0$. Moreover, an OU set about $x_0$ is also star-shaped about $x_0$. Therefore, by the definitions of these three unimodalities, we have

$$\{\text{distributions on } \mathcal{D}_0 \text{ which is block unimodal about } x_0\}$$
$$\subset \{\text{distributions on } \mathcal{D}_0 \text{ which is OU about } x_0\}$$
$$\subset \{\text{distributions on } \mathcal{D}_0 \text{ which is star unimodal about } x_0\}.$$

It remains to prove that the inclusions above are proper. For simplicity, we only show it in the bivariate case. Higher dimensional cases give us no additional insight but only the complication of notations. Without loss of generality, we assume $x_0$ is the origin.

To show the first inclusion is proper, consider a probability distribution $P$ with the following density:

$$f(x, y) = \begin{cases} 
1/7 & \text{if } (x, y) \in (1, 2] \times (1, 2] \\
2/7 & \text{if } (x, y) \in \{(0, 2] \times [0, 2]\} \setminus \{(1, 2] \times (1, 2]) \\
0 & \text{otherwise.}
\end{cases}$$

Let $P_1$ be the uniform distribution on $[0, 2] \times [0, 2]$ and $P_2$ be the uniform distribution on $\{(0, 2] \times [0, 2]\} \setminus \{(1, 2] \times (1, 2])$. Then $P$ is a convex combination of $P_1$ and $P_2$ ($P = 4P_1/7 + 3P_2/7$). Note that both sets are OU about the origin. Therefore, $P$ is OU about the origin by definition. However, $P$ is not block unimodal about
the origin. To see this, consider four points \((1/2, 1/2), (3/2, 3/2), (3/2, 1/2)\) and \((1/2, 3/2)\) at which \(f(x, y)\) is continuous. If \(P\) is block unimodal, we must have

\[ f \left( \frac{1}{2}, \frac{1}{2} \right) + f \left( \frac{3}{2}, \frac{3}{2} \right) \geq f \left( \frac{3}{2}, \frac{1}{2} \right) + f \left( \frac{1}{2}, \frac{3}{2} \right) \]

according to Section 2.2 in Dharmadhikari and Joag-Dev (1988) (or we can show this by the similar proof of the “only if” part of Theorem 1). However,

\[ f \left( \frac{1}{2}, \frac{1}{2} \right) + f \left( \frac{3}{2}, \frac{3}{2} \right) = \frac{3}{7} < \frac{4}{7} = f \left( \frac{3}{2}, \frac{1}{2} \right) + f \left( \frac{1}{2}, \frac{3}{2} \right), \]

which means \(P\) is not block unimodal about the origin. This proves the first inclusion is proper.

To show the second inclusion is also proper, we consider the density mentioned in Section 2.5:

\[ f(x, y) = C \exp \left( -\max \left( \arctan \left( \frac{y}{x} \right), \arctan \left( \frac{x}{y} \right) \right) (x + y) \right), x \geq 0, y \geq 0, \]

where \(C\) is the normalizing constant. For any ray \(\{(r \cos \theta, r \sin \theta) : r \geq 0\}, \theta \in [0, \pi/2]\), we can see \(f(x, y)\) is non-increasing along it. So according to Section 2.2 in Dharmadhikari and Joag-Dev (1988), \(f(x, y)\) is star unimodal about the origin. However, it’s not OU about the origin because by direct calculation we have

\[ f(1, 2) \approx 0.036C < 0.043C \approx f(2, 2), \]

which violates Theorem 1. Thus the second inclusion is also proper.

\(\square\)

**Proof of Corollary 1.** This follows directly from Lemma 1.

\(\square\)

**Proof of Proposition 2.** The proof is constructive. Suppose we remove the moment constraint (14) from (18) and get the following problem

\[
\begin{align*}
\max & \quad P((X_1, \ldots, X_d) \in S) \\
\text{subject to} & \quad l_F \leq \bar{F}(x_0) \leq u_F \\
& \quad f_{X_i}(x_0) \leq u_{X_i}, i = 1, \ldots, d \\
& \quad f(x') \geq f(x) \text{ for } x \geq x' \geq x_0
\end{align*}
\]

(52)

We can see there is a trivial bound for the optimal value of the problem (52), i.e.,

\[ P((X_1, \ldots, X_d) \in S) \leq \bar{F}(x_0) \leq u_F. \]

(53)

We will construct a sequence of feasible distributions of \((X_1, \ldots, X_d)\) s.t.

\[ P((X_1, \ldots, X_d) \in S) \to u_F, \]

(54)

which implies the optimal value of (52) is exactly \(u_F\).

Consider a sequence of rectangles \(\{R_i, i \geq 1\}\) in \(\mathcal{D}_0\) defined as

\[ R_i = \{(x_1, \ldots, x_d) : x_{j0} \leq x_j \leq x_{j0} + m, j = 1, \ldots, d - 1 \text{ and } x_{d0} \leq x_d \leq x_{d0} + i\}, \]

46
where $1/m < \min(u_{X_1}, \ldots, u_{X_d})$ is a fixed number. Correspondingly, we define $F_i$ as the uniform distribution on $R_i$ with total mass $u_{F_i}$. The density $f_i$ of the distribution $F_i$ is given by

$$f_i(x_1, \ldots, x_d) = \frac{u_{F_i}}{m^{d-1}} I(x_{j_0} \leq x_j \leq x_{j_0} + m, j = 1, \ldots, d - 1)I(x_{d_0} \leq x_d \leq x_{d_0} + i).$$

Thus, the constraint that $f(x') \geq f(x)$ for $x \geq x' \geq x_0$ is satisfied. Besides, the truncated marginal density $f_{X_j}$ at point $x_{j_0}$ satisfies

$$f_{X_j}(x_{j_0}) = \frac{u_{F_i}}{m} \leq \frac{1}{m} < u_{X_j} \text{ for } j = 1, \ldots, d - 1,$$

and

$$f_{X_d}(x_{d_0}) = \frac{u_{F_i}}{i} \leq \frac{1}{i} < u_{X_d} \text{ when } i \text{ is large enough.}$$

So when $i$ is large enough, the distribution $F_i$ is feasible to the problem (52).

Now let us consider the objective value of $F_i$. Recall that $S = \{(x_1, \ldots, x_d) \in D_0 : x_d \geq g(x_1, \ldots, x_{d-1})\}$ and $g(x_1, \ldots, x_{d-1}) \geq x_{d_0}$.

$$P_{F_i}((X_1, \ldots, X_d) \in S) = \int \frac{u_{F_i}}{m^{d-1}} I(x_j \in [x_{j_0}, x_{j_0} + m], j = 1, \ldots, d - 1, x_d \in [g(x_1, \ldots, x_{d-1}), x_{d_0} + i])dx_1 \cdots dx_d.$$

According to our assumption, $g$ is bounded on compact sets. So $\exists M \geq x_{d_0}$ s.t. $g(x_1, \ldots, x_{d-1}) \leq M$ when $x_{j_0} \leq x_j \leq x_{j_0} + m, j = 1, \ldots, d - 1$. Then when $i > M - x_{d_0}$, we have

$$P_{F_i}((X_1, \ldots, X_d) \in S) = \int \frac{u_{F_i}}{m^{d-1}} I(x_{j_0} \leq x_j \leq x_{j_0} + m, j = 1, \ldots, d - 1)I(M \leq x_d \leq x_{d_0} + i)dx_1 \cdots dx_d$$

$$= \frac{u_{F_i} i + x_{d_0} - M}{i} \to u_{F_i},$$

i.e., a sequence of feasible distributions satisfying (54) exists.

Finally, it follows by the trivial bound (53) that the optimal value of the problem (52) is exactly $u_{F_i}$. $\square$  

**Proof of Lemma 4.** Suppose $f$ is a feasible solution to the problem (20) and $C_s = \{x \in D_0 : f(x) \geq s\}$. We will show the objective function and the constraints in the problem (21) are equivalent to those in the problem (20).

Notice that $\{C_s, s > 0\}$ is a non-increasing sequence of OU sets by Theorem 1. First, we show two properties of $\{C_s, s > 0\}$: $\lambda_d(C_s) < \infty$ for any $s > 0$, and $\lambda_d(C_s) = 0$ implies $C_t = \emptyset$ for $t > s$. The first one is obviously true otherwise $F(x_0) \geq s\lambda_d(C_s) = \infty$. Now let us show the second property. Suppose that $\lambda_d(C_s) = 0$ and $t > s$. If $C_t \neq \emptyset$, then $x_0 \in C_t$ since $x_0$ is in any nonempty OU set about $x_0$. By the definition $C_t = \{x \in D_0 : f(x) \geq t\}$, we know that $f(x_0) \geq t > s$. Since $f(x_0) = \lim_{y \downarrow x_0, y \in D_0} \sup f(y)$, there exists $y_0 \in D_0$ s.t. $f(y_0) > s$. It follows from the OU property that $f(x) > s$ for $x_0 \leq x \leq y_0$. Therefore, $\lambda_d(C_s) > 0$, which contradicts our assumption. So we must have $C_t = \emptyset$.

Define $s_0 = \sup\{s > 0 : \lambda(C_s) > 0\}$. From the last paragraph, we know that $0 < \lambda_d(C_s) < \infty$ for $0 < s < s_0$ and $C_s = \emptyset$ for $s > s_0$. For the probability measure $P$ induced by the density $f$, from the proof of
Theorem 1, we know that for any measurable set $B$, the following representation holds

$$ P(B) = \int_0^\infty \lambda_d(B \cap \bar{C}) ds = \int_0^s \lambda_d(B \cap \bar{C}) ds = \int_0^s \frac{\lambda_d(B \cap \bar{C})}{\lambda_d(C)} \lambda_d(\bar{C}) ds, $$

where the last equality holds because $0 < \lambda_d(C) < \infty$ for $0 < s < s_0$ and $\lambda_d(C) = \lambda_d(\bar{C})$ by Lemma 2. When $s > s_0$, $C = \bar{C} = \emptyset$ and thus $\lambda_d(B \cap \bar{C}) = \lambda_d(\bar{C}) = 0$ by our definition. Then we can extend the integral interval from $(0, s_0)$ to $(0, \infty)$ and get

$$ P(B) = c \int_0^\infty \frac{\lambda_d(B \cap \bar{C})}{\lambda_d(C)} g(s) ds, \tag{55} $$

where $g(s) = \lambda_d(\bar{C})/c$. In fact, $g(s)$ is a probability density on $(0, \infty)$, which can be verified by setting $B = D_0$ in (55) and $P(D_0) = c$. The representation (55) ensures the correctness of the representations of the objective function and the last constraint in the problem (21).

Now let us show the correctness of the representation of the first constraint in the problem (21). Take $i = 1$ as an example. Notice that

$$ f(x_{10}, x_2, \ldots, x_d) = \int_0^{f(x_{10}, x_2, \ldots, x_d)} 1 ds = \int_0^\infty I((x_{10}, x_2, \ldots, x_d) \in C_s) ds. $$

Since $C_s = \emptyset$ for $s > s_0$, it can also be written as

$$ f(x_{10}, x_2, \ldots, x_d) = \int_0^{s_0} I((x_{10}, x_2, \ldots, x_d) \in C_s) ds. $$

Thus, the marginal density $f_{X_1}(x_{10})$ is given by

$$ f_{X_1}(x_{10}) = \int_{x_{20}}^\infty \cdots \int_{x_{d0}}^\infty f(x_{10}, x_2, \ldots, x_d) dx_2 \cdots dx_d $$

$$ = \int_{x_{20}}^\infty \cdots \int_{x_{d0}}^\infty \int_0^{s_0} I((x_{10}, x_2, \ldots, x_d) \in C_s) ds dx_2 \cdots dx_d $$

$$ = \int_0^{s_0} \int_{x_{20}}^\infty \cdots \int_{x_{d0}}^\infty I((x_{10}, x_2, \ldots, x_d) \in C_s) dx_2 \cdots dx_d ds $$

$$ = \int_0^{s_0} \lambda_{d-1}(C_{s,1}) ds, $$

where $C_{s,1} = \{(x_2, \ldots, x_d) : (x_{10}, x_2, \ldots, x_d) \in C_s\}$ is slice of $C_s$ on the plane $x = x_{10}$. Next, let’s show $\lambda_{d-1}(C_{s,1}) = \lambda_{d-1}(\bar{C}_{s,1})$, where $\bar{C}_{s,1}$ is the slice of $\bar{C}_s$ on the plane $x = x_{10}$. We will show $\bar{C}_{s,1} \subset C_{s,1}$, where $C_{s,1}$ is the closure of $C_{s,1}$. Suppose $(x_2, \ldots, x_d) \in \bar{C}_{s,1}$, i.e., $(x_{10}, x_2, \ldots, x_d) \in \bar{C}_s$. Then there is a sequence $(x_{1k}, x_{2k}, \ldots, x_{dk}, k \geq 1) \subset C_s$ such that

$$(x_{1k}, x_{2k}, \ldots, x_{dk}) \to (x_{10}, x_2, \ldots, x_d).$$

By the OU property, $(x_{10}, x_{2k}, \ldots, x_{dk}, k \geq 1)$ are also in $C_s$, which implies $(x_{2k}, \ldots, x_{dk}, k \geq 1) \subset C_{s,1}$. Thus, its limiting point $(x_2, \ldots, x_d)$ is in $C_{s,1}$. This means $C_{s,1} \subset C_{s,1}$. Clearly, $C_{s,1} \subset \bar{C}_{s,1}$ and $C_{s,1}$ is OU about $(x_{20}, \ldots, x_{d0})$ in $\mathbb{R}^{d-1}$. By Lemma 2, we can get

$$ \lambda_{d-1}(C_{s,1}) \leq \lambda_{d-1}(\bar{C}_{s,1}) \leq \lambda_{d-1}(C_{s,1}) = \lambda_{d-1}(\bar{C}_{s,1}) \Rightarrow \lambda_{d-1}(C_{s,1}) = \lambda_{d-1}(\bar{C}_{s,1}). $$

Thus, we can rewrite $f_{X_1}(x_{10})$ as

$$ f_{X_1}(x_{10}) = \int_0^{s_0} \lambda_{d-1}(\bar{C}_{s,1}) ds = \int_0^{s_0} \frac{\lambda_{d-1}(\bar{C}_{s,1})}{\lambda_d(C_s)} \lambda_d(\bar{C}_s) ds = c \int_0^{s_0} \frac{\lambda_{d-1}(\bar{C}_{s,1})}{\lambda_d(C_s)} g(s) ds, $$

48
where \( g(s) = \lambda_d(C_s)/c \) is defined above. When \( s > s_0 \), \( C_s = \bar{C}_s = \emptyset \) and thus \( \lambda_{d-1}(\bar{C}_{s,1})/\lambda_d(\bar{C}_s) = 0 \) by our definition. So we can extend the integral interval from \((0, s_0)\) to \((0, \infty)\) and get

\[
f_{x_1}(x_{10}) = c \int_0^\infty \frac{\lambda_{d-1}(\bar{C}_{s,1})}{\lambda_d(\bar{C}_s)} g(s) ds.
\]

Thus, the representation of the first constraint of the problem (21) is correct. This concludes our proof of this lemma. \( \square \)

**Proof of Lemma 5.** It suffices to show for any feasible solution to the problem (21), there exists a feasible solution to the problem (22) with the same objective value. Consider a density \( f(x) \) on \( D_0 \) with total mass \( c \) and satisfying the last two constraints in (20). Let \( \{C_s, s > 0\} \) and \( g(s) \) be the sequence of OU sets and the probability density defined in Lemma 4. In the proof of Lemma 4, we know there exists \( s_0 \in (0, \infty) \) s.t. \( 0 < \lambda_d(C_s) = \lambda_d(\bar{C}_s) < \infty \) for \( 0 < s < s_0 \) and \( C_s = \emptyset \) for \( s > s_0 \). Therefore, \( g(s) = 0 \) for \( s > s_0 \) by its definition. Consider the slice of \( \bar{C}_s \), e.g., \( \bar{C}_{s,1} \). By the property of OU sets, we know \( \bar{C}_s \subset \mathbb{R} \times \bar{C}_{s,1} \). Therefore, \( \lambda_d(\bar{C}_s) > 0 \) implies \( \lambda_{d-1}(\bar{C}_{s,1}) > 0 \) for \( 0 < s < s_0 \). Similarly, we can also get \( \lambda_{d-1}(\bar{C}_{s,i}) > 0 \) for \( 0 < s < s_0 \) and \( i = 1, \ldots, d \). Moreover, we must have \( \lambda_{d-1}(\bar{C}_{s,i}) < \infty \) for any \( s > 0 \) and \( i = 1, \ldots, d \). If not, suppose without loss of generality that \( \lambda_{d-1}(\bar{C}_{s,1}) = \infty \) for some \( s_1 > 0 \). Since \( \bar{C}_{s,1} \) is non-increasing in \( s \), we have \( \lambda_{d-1}(\bar{C}_{s,1}) = \infty \) for \( 0 < s \leq s_1 \). Now we consider the first constraint in (21) with \( i = 1 \):

\[
\frac{u_{x_1}}{c} \geq \int_0^\infty \frac{\lambda_{d-1}(\bar{C}_{s,1})}{\lambda_d(\bar{C}_s)} g(s) ds \geq \int_0^{\min(s_0, s_1)} \frac{\lambda_{d-1}(\bar{C}_{s,1})}{\lambda_d(\bar{C}_s)} g(s) ds = \frac{1}{c} \int_0^{\min(s_0, s_1)} \lambda_{d-1}(\bar{C}_{s,1}) ds = \infty,
\]

which contradicts to the assumption that \( f \) is a feasible solution to the problem (20). In summary, we know that \( \lambda_d(\bar{C}_s) \in (0, \infty) \), \( \lambda_{d-1}(\bar{C}_{s,i}) \in (0, \infty) \) for any \( s \in (0, s_0) \) and \( i = 1, \ldots, d \), and \( g(s) = 0 \) for \( s > s_0 \).

Now we construct a feasible solution to the problem (22) with the same objective value as \( f(x) \). Notice that \( s \mapsto \tan(\pi s/2s_0) \) is a bijection from \((0, s_0)\) into \((0, \infty)\) and

\[
\int_0^{s_0} \varphi(s) g(s) ds = \int_0^\infty \varphi \left( \frac{2s_0 \arctan(u)}{\pi} \right) g \left( \frac{2s_0 \arctan(u)}{\pi} \right) \frac{2s_0}{\pi(1 + u^2)} du
\]

for any non-negative measurable function \( \varphi \) by the change of variable \( u = \tan(\pi s/2s_0) \). Therefore, if we define \( R_s \) by

\[
R_s = \frac{\bar{C}_{2s_0 \arctan(s)} - \pi}{\pi}, \quad s \in (0, \infty)
\]

and define \( G(s) \) as the probability distribution with density

\[
G'(s) = g \left( \frac{2s_0 \arctan(s)}{\pi} \right) \frac{2s_0}{\pi(1 + s^2)}, \quad s \in (0, \infty),
\]

it follows by (56) that the objective value and the constraints in (21) and (22) are exactly the same. Further, \( R_s \) satisfies \( \lambda_d(R_s) \in (0, \infty) \), \( \lambda_{d-1}(R_{s,i}) \in (0, \infty) \) for any \( i = 1, \ldots, d \) and \( s > 0 \), which means \( \{R_s, s > 0\} \) and \( G(s) \) are feasible to the problem (22). This concludes our proof. \( \square \)

**Proof of Corollary 2.** This follows directly from Lemma 5 when \( d = 2 \). \( \square \)

**Proof of Lemma 6.** By the definition of \( R \) in (26) and the constraints of \( h \) in (25), we can easily see that \( R \) is a closed OU set with \( R^X \leq R_0^X \) and \( R^Y = h(x_0) \leq h_0(x_0+) \leq R_0^Y \). By the first constraint in (25), we have

\[
\lambda(R) = \int_{x_0}^{x_0 + \tau_X} (h(x) - y_0) dx = \sum_{i=0}^{\frac{h_0}{x_{i+1}}} \int_{x_i}^{x_{i+1}} (h(x) - y_0) dx
\]
If

Note that

Suppose

Lemma 12.

Proof of Lemma 7. 

So it remains to verify

sequence

are similar and thus are omitted. This concludes our proof.

To prove Lemma 8, we first prove the following simple lemma.

Lemma 12. Suppose \( a_3 > a_2 > a_1 \) and \( c \) are some constants in \( \mathbb{R} \), and \( g : (a_1, a_3] \rightarrow (-\infty, \infty] \). Consider the following function

\[
H(y) = \int_{a_1}^{a_2} (y - g(x))_+ \, dx + \int_{a_2}^{a_3} \left( \frac{c - (a_2 - a_1)}{a_3 - a_2} - g(x) \right)_+ \, dx
\]

50
Therefore, \( H(\eta y_1 + (1 - \eta) y_2) \) is a convex function. For \( y_1, y_2 \in [\underline{y}, \bar{y}] \) and \( \eta \in [0, 1] \), we have

\[
H(\eta y_1 + (1 - \eta) y_2)
= \int_{a_1}^{a_2} (\eta y_1 + (1 - \eta) y_2 - g(x))_+ dx + \int_{a_2}^{a_3} \left( c - \frac{(a_2 - a_1)(\eta y_1 + (1 - \eta) y_2)}{a_3 - a_2} - g(x) \right)_+ dx
= \int_{a_1}^{a_2} (\eta (y_1 - g(x)) + (1 - \eta) (y_2 - g(x)))_+ dx
+ \int_{a_2}^{a_3} \left( \eta \left( \frac{c - (a_2 - a_1) y_1}{a_3 - a_2} - g(x) \right) + (1 - \eta) \left( \frac{c - (a_2 - a_1) y_2}{a_3 - a_2} - g(x) \right) \right)_+ dx
\leq \int_{a_1}^{a_2} [\eta (y_1 - g(x))_+ + (1 - \eta) (y_2 - g(x))_+] dx
+ \int_{a_2}^{a_3} \left[ \eta \left( \frac{c - (a_2 - a_1) y_1}{a_3 - a_2} - g(x) \right) + (1 - \eta) \left( \frac{c - (a_2 - a_1) y_2}{a_3 - a_2} - g(x) \right) \right] dx
= \eta H(y_1) + (1 - \eta) H(y_2).
\]

Therefore, \( H(y) \) is a convex function. \( \square \)

Now, let us show Lemma 8.

**Proof of Lemma 8.** We define a subproblem of the problem (31) by adding an additional constraint:

\[
\max \int_{\underline{x}}^\bar{x} (h(x) - g(x))_+ dx
\text{ subject to } \int_{\underline{x}}^\bar{x} h(x) dx = C \tag{57}
\]

\( b \leq h(x) \leq b, x \in [\underline{x}, \bar{x}] \)

\( h \) is a non-increasing left-continuous step function

where the step function is in the form of

\[
h(x) = \begin{cases} 
y_1, & a_0 < x \leq a_1 
y_2, & a_1 < x \leq a_2 
\vdots & 
y_n, & a_{n-1} < x \leq a_n 
\end{cases}
\]

where \( n \in \mathbb{N} \), \( \bar{x} = a_0 < a_1 < \cdots < a_n = \bar{x} \) and \( y_i \)'s are some constants. To clarify, here we abuse the notations \( a_i \) and \( n \). They are not the constants in the formulation of our DRO problem (11). We aim at showing the optimal value of the problem (57) is the same as that of the problem (31). It suffices to show that for any feasible solution \( h \) to the problem (31), there exists a sequence of feasible solutions \( \{h_n, n \geq 1\} \) to the problem (57), s.t.

\[
\int_{\underline{x}}^\bar{x} (h_n(x) - g(x))_+ dx \to \int_{\underline{x}}^\bar{x} (h(x) - g(x))_+ dx. \tag{58}
\]
Now fix any feasible solution $h$ to the problem (31). For $n \geq 1$, let’s construct $h_n$. For simplicity, we extend the definition of $h$ to $[\underline{x}, \bar{x}]$ by defining $h(x) = h(x^+)$. For $i \in \{0, 1, \ldots, n-1\}$, since $h$ is non-increasing, we have

$$
\int_{\underline{x}}^{\bar{x}} \left( \frac{h \left( x + \frac{(i+1)(\bar{x} - x)}{n} \right) - h \left( x + \frac{i(\bar{x} - x)}{n} \right) }{n} \right) dx \leq \int_{\underline{x}}^{\bar{x}} \left( h(x) \right)_+ dx \leq \int_{\underline{x}}^{\bar{x}} \left( h \left( x + \frac{(i+1)(\bar{x} - x)}{n} \right) - h \left( x + \frac{i(\bar{x} - x)}{n} \right) \right) dx.
$$

Therefore, there is a real number $h_{i,n}$ satisfying

$$
h_{i,n} \in \left[ h \left( x + \frac{(i+1)(\bar{x} - x)}{n} \right), h \left( x + \frac{i(\bar{x} - x)}{n} \right) \right]
$$

s.t.

$$
\int_{\underline{x}}^{\bar{x}} h(x)_+ dx = h_{i,n} \left( \frac{\bar{x} - x}{n} \right).
$$

Then our $h_n$ is defined as

$$
h_n(x) = h_{i,n}, \forall x \in \left[ \bar{x} + \frac{i(\bar{x} - x)}{n}, \bar{x} + \frac{(i+1)(\bar{x} - x)}{n} \right].
$$

We can see $h_n$ is a feasible solution to the problem (57) by our construction. Let us verify (58).

$$
\left| \int_{\underline{x}}^{\bar{x}} \left( h_n(x) - g(x) \right)_+ dx - \int_{\underline{x}}^{\bar{x}} \left( h(x) - g(x) \right)_+ dx \right|
$$

$$
\leq \int_{\underline{x}}^{\bar{x}} \left| \left( h_n(x) - g(x) \right)_+ - \left( h(x) - g(x) \right)_+ \right| dx
$$

$$
\leq \int_{\underline{x}}^{\bar{x}} \left| \left( h_n(x) - h(x) \right) - \left( h(x) - g(x) \right) \right| dx
$$

$$
= \sum_{i=0}^{n-1} \int_{\underline{x}}^{\bar{x}} \left( h_n(x) - h(x) \right)_+ dx
$$

$$
\leq \sum_{i=0}^{n-1} \int_{\underline{x}}^{\bar{x}} \left( h \left( \bar{x} + \frac{i(\bar{x} - x)}{n} \right) - h \left( \bar{x} + \frac{(i+1)(\bar{x} - x)}{n} \right) \right) dx
$$

$$
= \frac{\bar{x} - x}{n} \left( h(\bar{x}) - h(x) \right) \to 0
$$

as $n \to \infty$. Thus, the optimal value of the problem (57) is the same as that of the problem (31).

Next, we define a subproblem of the problem (57) by adding an additional constraint again:

$$
\max \int_{\underline{x}}^{\bar{x}} \left( h(x) - g(x) \right)_+ dx
$$

subject to

$$
\int_{\underline{x}}^{\bar{x}} h(x) dx = C
$$

(59)

$$
h \leq h(x) \leq \bar{b}, x \in [\underline{x}, \bar{x}]
$$

$h$ is a non-increasing left-continuous step function with at most three steps

Again, we aim at showing the optimal value of the problem (59) is the same as that of the problem (57). It suffices to show that for any feasible solution $h_1$ to the problem (57), there is a feasible solution $h_2$ to the problem (59) s.t.

$$
\int_{\underline{x}}^{\bar{x}} (h_1(x) - g(x))_+ dx \leq \int_{\underline{x}}^{\bar{x}} (h_2(x) - g(x))_+ dx.
$$

(60)
Notice that for any feasible solution $h_1$ of the problem (57), it can be uniquely written as

$$h_1(x) = \begin{cases} 
y_1, & a_0 < x \leq a_1 
y_2, & a_1 < x \leq a_2 
\vdots 
y_n, & a_{n-1} < x \leq a_n 
\end{cases},$$

where $b \leq y_n < y_{n-1} < \cdots < y_1 \leq b$ and $x = a_0 < a_1 < \cdots < a_n = \bar{x}$. Now let’s prove (60) by induction with respect to $n$ (hereafter we will call $n$ the number of steps of $h_1$). When $n \leq 3$, $h_1(x)$ is also a feasible solution to the problem (59). So we can choose $h_2 = h_1$ and then (60) holds. Assume that when $n \leq k$ ($k \geq 3$), the conclusion is true. Now suppose $n = k + 1 \geq 4$.

Let’s consider the steps $y_{k-1}$ and $y_k$, i.e., consider the interval $(a_{k-2}, a_k]$. So the objective value in this interval is given by

$$\int_{a_{k-2}}^{a_{k-1}} (y_{k-1} - g(x))_+ dx + \int_{a_{k-1}}^{a_k} (y_k - g(x))_+ dx.$$ 

We want to change the function values in $(a_{k-2}, a_{k-1}]$ and $(a_{k-1}, a_k]$ to get a larger objective value while keeping the function feasible for the problem (59). Suppose $y_{k-1}$ is changed into $y$. To make $\int_{\bar{x}}^{\bar{x}} h(x) dx$ fixed, we must change $y_k$ into

$$\frac{y_{k-1}(a_{k-1} - a_{k-2}) + y_k(a_k - a_{k-1}) - (a_k - a_{k-2})y}{(a_k - a_{k-1})}.$$ 

To make the function still non-increasing, $y$ should satisfy

$$y_{k+1} \leq \frac{y_{k-1}(a_{k-1} - a_{k-2}) + y_k(a_k - a_{k-1}) - (a_k - a_{k-2})y}{(a_k - a_{k-1})} \leq y \leq y_{k-2},$$

i.e.,

$$y \in \left[ y_{k-1} \frac{a_{k-1} - a_{k-2}}{a_k - a_{k-1}} + y_k \frac{a_k - a_{k-1}}{a_k - a_{k-2}}, \min \left( y_{k-2}, y_{k-1} + (y_k - y_{k-1}) \frac{a_k - a_{k-1}}{a_k - a_{k-2}} \right) \right].$$

Now the objective value in $(a_{k-2}, a_k]$ becomes

$$H(y) = \int_{a_{k-2}}^{a_{k-1}} (y - g(x))_+ dx + \int_{a_{k-1}}^{a_k} \left( \frac{y_{k-1}(a_{k-1} - a_{k-2}) + y_k(a_k - a_{k-1}) - (a_k - a_{k-2})y}{(a_k - a_{k-1})} - g(x) \right)_+ dx.$$ 

Therefore, by Lemma 12, we have that

$$\int_{a_{k-2}}^{a_{k-1}} (y_{k-1} - g(x))_+ dx + \int_{a_{k-1}}^{a_k} (y_k - g(x))_+ dx = H(y_{k-1})$$

$$\leq \max \left( H \left( y_{k-1} \frac{a_{k-1} - a_{k-2}}{a_k - a_{k-2}} + y_k \frac{a_k - a_{k-1}}{a_k - a_{k-2}}, \min \left( y_{k-2}, y_{k-1} + (y_k - y_{k-1}) \frac{a_k - a_{k-1}}{a_k - a_{k-2}} \right) \right), H \left( \min \left( y_{k-2}, y_{k-1} + (y_k - y_{k-1}) \frac{a_k - a_{k-1}}{a_k - a_{k-2}} \right) \right) \right).$$

By considering which one attains the maximal value, there are three cases.

Case 1:

$$H \left( y_{k-1} \frac{a_{k-1} - a_{k-2}}{a_k - a_{k-2}} + y_k \frac{a_k - a_{k-1}}{a_k - a_{k-2}} \right) > H \left( \min \left( y_{k-2}, y_{k-1} + (y_k - y_{k-1}) \frac{a_k - a_{k-1}}{a_k - a_{k-2}} \right) \right).$$

In this case,

$$H(y_{k-1}) \leq H \left( y_{k-1} \frac{a_{k-1} - a_{k-2}}{a_k - a_{k-2}} + y_k \frac{a_k - a_{k-1}}{a_k - a_{k-2}} \right).$$
which means we can get a better objective value by changing both \( y_{k-1} \) and \( y_k \) into

\[
y_{k-1} \frac{a_{k-1} - a_{k-2}}{a_k - a_{k-2}} + y_k \frac{a_k - a_{k-1}}{a_k - a_{k-2}}.
\]

In other words, the new function

\[
h(x) = \begin{cases} y_1, & a_0 < x \leq a_1 \\ \vdots & \\ y_{k-2}, & a_{k-3} < x \leq a_{k-2} \\ y_{k-1} \frac{a_{k-1} - a_{k-2}}{a_k - a_{k-2}} + y_k \frac{a_k - a_{k-1}}{a_k - a_{k-2}}, & a_{k-2} < x \leq a_k \\ y_{k+1}, & a_k < x \leq a_{k+1} \end{cases}
\]

has a better objective value than the original function \( h_1(x) \). Since the number of steps of \( h(x) \) is \( k \), by the induction hypothesis, there is a feasible solution \( h_2 \) to the problem (59) s.t.

\[
\int_{\bar{x}}^x (h(x) - g(x))_+ \, dx \leq \int_{\bar{x}}^x (h_2(x) - g(x))_+ \, dx,
\]

which implies

\[
\int_{\bar{x}}^x (h_1(x) - g(x))_+ \, dx \leq \int_{\bar{x}}^x (h_2(x) - g(x))_+ \, dx.
\]

So (60) holds in this case.

Case 2:

\[
H \left( y_{k-1} \frac{a_{k-1} - a_{k-2}}{a_k - a_{k-2}} + y_k \frac{a_k - a_{k-1}}{a_k - a_{k-2}} \right) \leq H \left( \min \left( y_{k-2}, y_{k-1} + (y_k - y_{k+1}) \frac{a_k - a_{k-1}}{a_{k-1} - a_{k-2}} \right) \right),
\]

and

\[
\min \left( y_{k-2}, y_{k-1} + (y_k - y_{k+1}) \frac{a_k - a_{k-1}}{a_{k-1} - a_{k-2}} \right) = y_{k-2}.
\]

This is to say

\[
H \left( y_{k-1} \right) \leq H \left( y_{k-2} \right),
\]

and

\[
y_k + (y_{k-1} - y_{k-2}) \frac{a_{k-1} - a_{k-2}}{a_k - a_{k-1}} \geq y_{k+1},
\]

which means we can get a better objective value by changing \( y_{k-1} \) into \( y_{k-2} \). In other words, the new function

\[
h(x) = \begin{cases} y_1, & a_0 < x \leq a_1 \\ \vdots & \\ y_{k-2}, & a_{k-3} < x \leq a_{k-1} \\ y_k + (y_{k-1} - y_{k-2}) \frac{a_{k-1} - a_{k-2}}{a_k - a_{k-1}}, & a_{k-1} < x \leq a_k \\ y_{k+1}, & a_k < x \leq a_{k+1} \end{cases}
\]

has a better objective value than the original function \( h_1(x) \). We can see the number of steps of \( h(x) \) is \( k \).

By the same argument in case 1, (60) also holds in this case.

Case 3:

\[
H \left( y_{k-1} \frac{a_{k-1} - a_{k-2}}{a_k - a_{k-2}} + y_k \frac{a_k - a_{k-1}}{a_k - a_{k-2}} \right) \leq H \left( \min \left( y_{k-2}, y_{k-1} + (y_k - y_{k+1}) \frac{a_k - a_{k-1}}{a_{k-1} - a_{k-2}} \right) \right),
\]

54
By the construction of dominating OU sets, in problem (24), we can restrict the choice of
some non-increasing left-continuous step function with at most 3 steps. In total we have
By Lemma 8, we know that each
Proof of Corollary 3. By Lemma 8, we know that each \( h^* \) is a step function with at most three steps. In total we have \( nR_0 + 1 \leq 4n + 1 \) subproblems and \( h^* \) is just a “combination” of these \( h^*_1 \)'s. Therefore, \( h^* \) is a non-increasing left-continuous step function with at most 3\((4n + 1)\) steps and can be represented by (32) for some \((z, w)\). Consequently, \( \tilde{R} \) defined by
\[
\tilde{R} = \{(x, y) \in D_0 : y_0 \leq y \leq h^*(x; z, w)\}
\] is a dominating OU set of \( R_0 \) by Lemma 7. Here the requirement \((z, w) \in (0, \infty)^{12n+4} \times [0, \infty)^{12n+2} \) is due to \( \lambda(\tilde{R}) = \lambda(R_0) > 0 \). Note that the number of steps \( 12n + 3 \) doesn’t rely on the choice of \( R_0 \) so the dominating OU set of any closed OU set \( R_0 \) with \( \lambda(R_0) \in (0, \infty), R_0^X \in (0, \infty), R_0^Y \in (0, \infty) \) can be represented by (61) with possibly different \((z, w)\). In other words, all dominating OU sets are contained in the class \( \mathcal{R}^* = \{ R_{x, w} : R_{z, w} = \{(x, y) \in D_0 : y_0 \leq y \leq h^*(x; z, w)\} \text{ with } h^* \text{ defined in (32)} \}. \)

Proof of Proposition 3. By the construction of dominating OU sets, in problem (24), we can restrict the choice of \( R_* \) to \( \mathcal{R}^* \) which is fully parameterized by \((z, w)\). Further, the cardinality of \((z, w) \in (0, \infty)^{12n+4} \times [0, \infty)^{12n+2} \)
and the cardinality of \( s \in (0, \infty) \) are both continuum. So the distribution \( G(s) \) on the index \( s \) in problem (24) is equivalent to a distribution \( Q \) on \((z, w)\). Thus, problem (24) can be rewritten as the following problem:

\[
\begin{align*}
\max \; & c \int \frac{\lambda(S \cap R_{z,w})}{\lambda(R_{z,w})} dQ(z, w) \\
\text{subject to} \; & \int \frac{R_{z,w}^y}{\lambda(R_{z,w})} dQ(z, w) \leq \frac{u_X}{c} \\
& \int \frac{R_{z,w}^x}{\lambda(R_{z,w})} dQ(z, w) \leq \frac{u_Y}{c} \\
& a_i \leq \int \frac{\lambda\{(x, y) : x_{1i} \leq x \leq x_{2i}, y_{1i} \leq y \leq y_{2i}\} \cap R_{z,w}}{\lambda(R_{z,w})} dQ(z, w) \leq b_i, i = 1, \ldots, n
\end{align*}
\]

where \( R_{z,w} \) is the closed OU set represented by \( R_{z,w} = \{(x, y) : y_0 \leq y \leq h^*(x; z, w)\} \) with \( h^* \) defined in (32). In view of the form of \( h^* \) and the representation of \( S = \{(x, y) \in D_0 : y \geq g(x)\} \) for some known function \( g : [x_0, \infty) \to [y_0, \infty] \), we can obtain

\[
\lambda(R_{z,w}) = \sum_{i=1}^{12n+3} \sum_{j=1}^{12n+4-i} z_j w_j, \quad R_{z,w}^y = \sum_{i=1}^{12n+3} w_i, \quad R_{z,w}^x = \sum_{i=1}^{12n+3} z_i,
\]

\[
\lambda(S \cap R_{z,w}) = \sum_{i=1}^{12n+3} \int \left( y_0 + \sum_{j=1}^{12n+4-i} w_j - g(x) \right) I\left( x_0 + \sum_{j=1}^{i-1} z_j < x \leq x_0 + \sum_{j=1}^{i} z_j \right) dx,
\]

\[
\lambda(\{(x, y) : x_{1i} \leq x \leq x_{2i}, y_{1i} \leq y \leq y_{2i}\} \cap R_{z,w})
\]

\[
= \sum_{i=1}^{12n+3} \int I\left( \sum_{j=1}^{i-1} z_j < x - x_0 \leq \sum_{j=1}^{i} z_j, x_{1k} \leq x \leq x_{2k} \right) \times \left( \min \left( y_0 + \sum_{j=1}^{12n+4-i} w_j, y_{2k} \right) - \min \left( y_0 + \sum_{j=1}^{12n+4-i} w_j, y_{1k} \right) \right) dx.
\]

Plugging them into problem (62) and noticing that “all the integrands are measurable” automatically holds, we get the moment problem (33).

**Proof of Theorem 2.** (1): By Corollary 2 and Proposition 3, we know the optimal value of the OU-DRO problem (23) is not greater than the optimal value of the moment problem (33). On the other hand, any feasible solution to the moment problem (33) can be transformed into a feasible solution to the OU-DRO problem (23) with the same objective value by means of the OU distribution \( Q \) defined in (34). Combining two directions together, we know that optimal values of the OU-DRO problem (23) and the moment problem (33) are the same. The equivalence of the moment problem (33) and the non-linear optimization (35) is ensured by Theorem 3.2 in Winkler (1988) and the conditions in Theorem 3.2 can be verified by Theorem 2.1 and Proposition 3.1 in the same paper. The optimality of \( P^* \) is clear since it has the same objective value as the optimal solution \( Q^* \).

(2): By part (1), the OU-DRO problem (23) has the same optimal value as the non-linear optimization (35). Thus, when the constraint \( F(x_0, y_0) = c \) is replaced by \( I_F \leq F(x_0, y_0) \leq u_F \), it suffices to make \( c \) vary
in the interval \([l_F, u_F]\) when solving the non-linear optimization (35). Thus, these two problems have the same optimal value. Finally, \(P^*\) is optimal since its objective value is exactly the optimal value of both the non-linear optimization and the DRO problem.

Proof of Lemma 9. The proof of \(\bar{K}\) and \(\bar{K}^\circ\) being OU is similar to the proof of Lemma 2 and thus is omitted. To show \(K\) is Lebesgue measurable and satisfies \(\lambda(K^\circ) = \lambda(K) = \lambda(\bar{K})\), it suffices to show \(\lambda(\partial K) = 0\). Without loss of generality, assume \(x_0\) is the origin. Let \(O_1, \ldots, O_{2^d}\) be the \(2^d\) (closed) orthants of \(\mathbb{R}^d\). We have

\[
K = \bigcup_{i=1}^{2^d}(K \cap O_i)
\]

and thus

\[
\partial K \subset \bigcup_{i=1}^{2^d} \partial(K \cap O_i).
\]

Note that by suitable reflection, each \(K \cap O_i\) can be transformed into an OU set about the origin on the first orthant \([0, \infty)^d\). Then by Lavrič (1993), we have \(\lambda(\partial(K \cap O_i)) = 0\) for each \(i = 1, \ldots, 2^d\), which implies \(\lambda(\partial K) = 0\).

Proof of Theorem 3. The first “if and only if” claim can be proved by the similar arguments in the proof of Theorem 1. The second “if and only if” claim is obvious. To prove the Choquet representation, we first note that

\[
P(B) = \sum_{1 \leq i \leq 2^d, P(O_i) > 0} P(O_i)P(B|O_i)
\]

by the law of total probability. By Theorem 1, \(P(B|O_i)\) can be written as

\[
P(B|O_i) = \int_0^\infty W_{\bar{C}_i}(B)g_i(s)ds,
\]

which justifies (40). Note that \(\bar{C}_i\) is an OU set which is fully contained in \(O_i\). To show (40) is indeed a Choquet representation, it suffices to show \(W_{\bar{C}_i}\) is an extreme point in the class of OU distributions about the origin if \(\lambda(\bar{C}_i) > 0\). This is proved in Lemma 10.

Proof of Lemma 10. The proof is similar to the proof of Lemma 3 and thus is omitted.

Proof of Theorem 4. Suppose there is a density \(f(x)\) of \(P\) such that for every \(s > 0\), the set

\[
C_s = \{x \in D_0 : f(x) \geq s\}
\]

is \(d'\)-POU about \(x_0\). Write \(W_{C_s}\) as the uniform distribution on \(C_s\) and let \(g(s) = \lambda(C_s) \geq 0\). Notice that

\[
f(x) = \int_0^{f(x)} 1ds = \int_0^\infty I(x \in C_s)ds.
\]

Thus for any measurable set \(B\),

\[
P(B) = \int_B f(x)dx = \int_B \int_0^\infty I(x \in C_s)dsdx = \int_0^\infty \lambda(B \cap C_s)ds = \int_0^\infty W_{C_s}(B)g(s)ds.
\]

Setting \(B = D_0\), we can see \(P(B) = 1 = \int_0^\infty g(s)ds\) and thus \(g(s)\) is a probability density on \((0, \infty)\). Since the probability measure with the density \(g(s)\) must be the limit of a sequence of discrete probability measures
with finite support, we can see $P$ is in the closed convex hull of the set of all uniform distributions on $d'$-POU subsets of $D_0$, i.e., $P$ is a $d'$-POU distribution.

To prove the “only if” part, we assume $P$ is $d'$-POU about $x_0$. Suppose that $Q$ is the uniform distribution on a $d'$-POU set $K \subset D_0$ and $0 < \lambda(K) < \infty$. For a point $x \in \mathbb{R}^d$ and $\delta > 0$, we define the neighborhood $N_\delta(x) = \{y \in \mathbb{R}^d : \max_{1 \leq i \leq d} |x_i - y_i| < \delta\}$ as in the proof of Theorem 1. For $x, x' \in D_0^c$ with $x_i \geq x'_i$ for $i = 1, \ldots, d'$, $x_i = x'_i$ for $i = d' + 1, \ldots, d$ and $0 < \delta < \min_{1 \leq i \leq d'} (x'_i - x_{10})$, we have

$$y \in K \cap N_\delta(x) \Rightarrow y + x' - x \in K \cap N_\delta(x').$$

Since Lebesgue measure is translation invariant, we can get

$$\lambda(K \cap N_\delta(x')) \geq \lambda(K \cap N_\delta(x)).$$

Dividing by $\lambda(K)$, we have

$$Q(N_\delta(x')) \geq Q(N_\delta(x)). \quad (63)$$

Clearly, this relation also holds under the convex combinations of the uniform distributions on $d'$-POU sets. Since $P$ is $d'$-POU about $x_0$, by definition, there is a sequence $\{Q_m, m \geq 1\}$ such that $Q_m$ converges weakly to $P$, where $Q_m$'s are the convex combinations of the uniform distributions on $d'$-POU sets. Therefore, these $Q_m$'s satisfy (63). Since $P$ has a density, say $f_0$, we have

$$P(\partial N_\delta(x')) = P(\partial N_\delta(x)) = 0.$$

Weak convergence and (63) imply that

$$P(N_\delta(x')) \geq P(N_\delta(x)). \quad (64)$$

For $x \in D_0^c$, we define

$$f(x) = \lim_{\delta \to 0} \sup_{x' \in D_0^c} \frac{P(N_\delta(x))}{\lambda(N_\delta(x))} = \lim_{\delta \to 0} \frac{P(N_\delta(x))}{(2\delta)^d}.$$

By the Lebesgue differentiation theorem, $f(x) = f_0(x)$ a.e., which means $f(x)$ is also a density for $P$. Besides, (64) implies $f(x') \geq f(x)$ for $x, x' \in D_0^c$ with $x_i \geq x'_i$ for $i = 1, \ldots, d'$, $x_i = x'_i$ for $i = d' + 1, \ldots, d$. For $x \in \partial D_0$, we simply define

$$f(x) = \sup_{y \in D_0^c} f(y).$$

Then the density $f(x)$ is $d'$-POU about $x_0$ on $D_0$.

The equivalence of $C_s$ being $d'$-POU about $x_0$ and $f$ being a $d'$-POU density about $x_0$ on $D_0$ is easy to see by definition. \qed

**Proof of Theorem 5.** From the proof of Theorem 4, we have

$$P(B) = \int_0^\infty \lambda(C_s \cap B)ds. \quad (65)$$

By Fubini’s theorem, we have

$$\lambda(C_s \cap B) = \int_{D_0} I(x \in C_s \cap B)dx_1 \cdots dx_d$$

58
to prove (67) is indeed a Choquet representation, it suffices to show \( W \) density. Since only \( W \) where \( g \)

Plugging (66) into (65), we get

\[
\int_{x_{d'+1}, \ldots, x_d} \lambda_d'(C_s \cap B)x_{d'+1}, \ldots, x_d) dx_{d'+1} \cdots dx_d.
\]

Notice that for any set \( K \) we have

\[
K_{x_{d'+1}, \ldots, x_d} \times \{ (x_{d'+1}, \ldots, x_d) \} = K \cap \{ y \in \mathbb{R}^d : y_{d'+1} = x_{d'+1}, \ldots, y_d = x_d \}.
\]

Therefore

\[
W_{C_s, x_{d'+1}, \ldots, x_d}(B_{x_{d'+1}, \ldots, x_d}) = W_{C_s, y_{d'+1} = x_{d'+1}, \ldots, y_d = x_d}(B \cap \{ y \in \mathbb{R}^d : y_{d'+1} = x_{d'+1}, \ldots, y_d = x_d \}) = W_{C_s, y_{d'+1} = x_{d'+1}, \ldots, y_d = x_d}(B).
\]

It follows that \( \lambda(C_s \cap B) \) can be represented as

\[
\lambda(C_s \cap B) = \int_{x_{d'+1},1}^{\infty} \cdots \int_{x_d,0}^{\infty} W_{C_s, y_{d'+1} = x_{d'+1}, \ldots, y_d = x_d}(B) \lambda_d'(C_s, x_{d'+1}, \ldots, x_d) dx_{d'+1} \cdots dx_d. \tag{66}
\]

Plugging (66) into (65), we get

\[
P(B) = \int_0^{\infty} \int_{x_{d'+1},1}^{\infty} \cdots \int_{x_d,0}^{\infty} W_{C_s, y_{d'+1} = x_{d'+1}, \ldots, y_d = x_d}(B) g(s, x_{d'+1}, \ldots, x_d) dx_{d'+1} \cdots dx_d ds. \tag{67}
\]

where \( g(s, x_{d'+1}, \ldots, x_d) = \lambda_d'(C_s, x_{d'+1}, \ldots, x_d) \). Letting \( B = D_0 \), we can see \( g(s, x_{d'+1}, \ldots, x_d) \) is a probability density. Since only \( W_{C_s, y_{d'+1} = x_{d'+1}, \ldots, y_d = x_d} \) with \( \lambda_d'(C_s, x_{d'+1}, \ldots, x_d) > 0 \) contributes to (67), in order to prove (67) is indeed a Choquet representation, it suffices to show \( W_{C_s, y_{d'+1} = x_{d'+1}, \ldots, y_d = x_d} \) with \( \lambda_d'(C_s, x_{d'+1}, \ldots, x_d) > 0 \) is an extremal point in the class of \( d' \)-POU distributions. This is proved in Lemma 11.

Proof of Lemma 11. Suppose there exist two \( d'- \)POU distributions \( P_1, P_2 \) and \( \eta \in (0,1) \) s.t.

\[
W_K = \eta P_1 + (1-\eta)P_2.
\]

Since the support of \( W_K \) is contained in \( \{ y \in \mathbb{R}^d : y_{d'+1} = x_{d'+1}, \ldots, y_d = x_d \} \), the supports of \( P_1 \) and \( P_2 \) are also contained in \( \{ y \in \mathbb{R}^d : y_{d'+1} = x_{d'+1}, \ldots, y_d = x_d \} \). Then we can reduce \( W_K, P_1 \) and \( P_2 \) to OU distributions on the subspace \( \{ y \in \mathbb{R}^d : y_{d'+1} = x_{d'+1}, \ldots, y_d = x_d \} \). Note that \( W_K \) is reducible to \( W_{K_{x_{d'+1}, \ldots, x_d}} \) on \( x \in \mathbb{R}^d : x_i \geq x_{i,0}, i = 1, \ldots, d' \) and \( K_{x_{d'+1}, \ldots, x_d} \) is an OU set about \( (x_{10}, \ldots, x_{d,0}) \) on \( x \in \mathbb{R}^d : x_i \geq x_{i,0}, i = 1, \ldots, d' \) with \( \lambda_d'(K_{x_{d'+1}, \ldots, x_d}) > 0 \). By Lemma 3, \( W_{K_{x_{d'+1}, \ldots, x_d}} \) is an extremal point in the class of OU distributions about \( (x_{10}, \ldots, x_{d,0}) \) on \( x \in \mathbb{R}^d : x_i \geq x_{i,0}, i = 1, \ldots, d' \). Therefore, the distributions of \( P_1 \) and \( P_2 \) on \( \{ y \in \mathbb{R}^d : y_{d'+1} = x_{d'+1}, \ldots, y_d = x_d \} \) are the same as \( W_{K_{x_{d'+1}, \ldots, x_d}} \), which means \( P_1 \) and \( P_2 \) are equal to \( W_K \). Therefore, \( W_K \) is an extremal point in the class of \( d'- \)POU distributions about \( x_0 \) on \( D_0 \).
Proof of Theorem 6. We use the similar arguments for star unimodality in Dharmadhikari and Joag-Dev (1988). We first prove the “if” part. Suppose $X \overset{d}{=} (x_{10} + U(Z_1 - x_{10}), Z_2, \ldots, Z_d)$. We need to prove $X$ is 1-POU. In fact, we only need to prove it when $(Z_1, \ldots, Z_d)$ is degenerate, i.e., $(Z_1, \ldots, Z_d)$ put mass 1 at a point $(z_1, \ldots, z_d) \in D_0$. Then by taking the convex mixture and weak limit and noticing that 1-POU distribution is closed under these operations, we can see it holds for any $(Z_1, \ldots, Z_d)$. When $(Z_1, \ldots, Z_d)$ is degenerate at $(z_1, \ldots, z_d) \in D_0$, $X \overset{d}{=} (x_{10} + U(z_1 - x_{10}), z_2, \ldots, z_d)$ whose distribution is uniform on $[x_{10}, z_1] \times \{(z_2, \ldots, z_d)\}$. Since $[x_{10}, z_1] \times \{(z_2, \ldots, z_d)\}$ is a 1-POU set on $D_0$, $X$ is clearly 1-POU. This proves the “if” part.

Now we consider the “only if” part. Suppose $X \equiv (X_1, \ldots, X_d)$ is the uniform distribution on a 1-POU set $K$ which can be written as

$$K = \{x \equiv (x_1, \ldots, x_d) \in D_0 : x_{10} \leq x_1 \leq g(x_2, \ldots, x_d)\},$$

where $g(x_2, \ldots, x_d) > x_{10}$ is continuous with domain $Dom(g) = [x_{20}, \infty) \times \cdots \times [x_{d0}, \infty)$ and $\lambda(K) \in (0, \infty)$. Then the density of $X$ is

$$f_X(x) = \frac{1}{\int_{Dom(g)}(g(x_2, \ldots, x_d) - x_{10})dx_2 \cdots dx_d}I(x_{10} \leq x_1 \leq g(x_2, \ldots, x_d)).$$

Therefore, the conditional density of $X_1$ given $(X_2, \ldots, X_d) = (x_2, \ldots, x_d)$ is

$$f_{X_1|(X_2, \ldots, X_d)}(x_1|x_2, \ldots, x_d) = \frac{1}{g(x_2, \ldots, x_d) - x_{10}}I(x_{10} \leq x_1 \leq g(x_2, \ldots, x_d)).$$

Let $U = (X_1 - x_{10})/(g(X_2, \ldots, X_d) - x_{10})$. Then we can see the conditional distribution of $U$ given $(X_2, \ldots, X_d) = (x_2, \ldots, x_d)$ is the uniform distribution on $(0, 1)$ and thus is independent of $(X_2, \ldots, X_d)$. So $X$ can be represented by

$$X \equiv (X_1, \ldots, X_d) = (x_{10} + U(g(X_2, \ldots, X_d) - x_{10}), X_2, \ldots, X_d)$$

$$= (x_{10} + U(Z_1 - x_{10}), Z_2, \ldots, Z_d),$$

where $(Z_1, Z_2, \ldots, Z_d) = (g(X_2, \ldots, X_d), X_2, \ldots, X_d)$ and $U$ is independent of $(Z_1, Z_2, \ldots, Z_d)$. By taking the convex mixture and weak limit, we can see “only if” part holds for any $X$.

The representation (42) is just a reformulation of $X = (x_{10} + U(Z_1 - x_{10}), Z_2, \ldots, Z_d)$ by conditioning on the values of $(Z_1, \ldots, Z_d)$. By Lemma 11, we know $W_{1-POU}(z)$ is an extreme point in the class of 1-POU distributions about $x_0$ on $D_0$. So (42) is indeed a Choquet representation. Next we will show $Q$ is uniquely determined by $P$, i.e., the Choquet representation (42) is unique. We write $\varphi_X$ and $\varphi_Z$ as the characteristic function of $X$ and $Z$ respectively. For $t \in \mathbb{R}^n$, we have

$$\varphi_X(t) = E[e^{it^T X}] = E[e^{it^T (x_{10} + U(Z_1 - x_{10}), Z_2, \ldots, Z_d)}] = \int_0^1 E[e^{it^T (x_{10} + U(Z_1 - x_{10}), Z_2, \ldots, Z_d)}]du$$

$$= \int_0^1 e^{it^T x_{10}(1-u)}\varphi_Z(ut_1, t_2, \ldots, t_d)du = e^{it_1 x_{10}} \int_0^1 e^{-it_1 u x_{10}} \varphi_Z(ut_1, t_2, \ldots, t_d)du.$$

So for $y > 0$, we have

$$\varphi_X(yt_1, t_2, \ldots, t_d) = e^{yt_1 x_{10}} \int_0^1 e^{-yt_1 u x_{10}} \varphi_Z(yut_1, t_2, \ldots, t_d)du$$

By change of variable $v = uy$, we have

$$\varphi_X(yt_1, t_2, \ldots, t_d) = \frac{e^{yt_1 x_{10}}}{y} \int_0^y e^{-yt_1 x_{10}} \varphi_Z(vt_1, t_2, \ldots, t_d)dv.$$
\[\phi_X(yt_1, t_2, \ldots, t_d) = \int_0^y e^{-it_1yt_0} \phi_Z(vt_1, t_2, \ldots, t_d) dv. \quad (68)\]

Since the right hand side of (68) is differentiable with respect to \(y\), so is the left hand side. Then we have

\[\frac{d}{dy} \left[ \frac{y}{e^{it_1yt_0}} \phi_X(yt_1, t_2, \ldots, t_d) \right] = e^{-it_1yt_0} \phi_Z(yt_1, t_2, \ldots, t_d)\]

\[\Rightarrow e^{it_1yt_0} \frac{d}{dy} \left[ \frac{y}{e^{it_1yt_0}} \phi_X(yt_1, t_2, \ldots, t_d) \right] = \phi_Z(yt_1, t_2, \ldots, t_d).\]

Letting \(y = 1\), we can see \(\phi_X\) determines \(\phi_Z\) and hence \(P\) determines \(Q\).

Finally, suppose \(P(X_1 = x_{10}) = 0\), we need to prove \(X_1\) is an absolutely continuous random variable. Note that \(X_1 \equiv x_{10} + U(Z_1 - x_{10})\) so we have \(P(Z_1 = x_{10}) = 0\). Consider any measurable set \(A\) with \(\lambda(\cdot) = 0\) and write \(Q_{Z_1}\) as the probability distribution of \(Z_1\). We have

\[P(X_1 \in A) = P(x_{10} + U(Z_1 - x_{10}) \in A) = \int_{\{x_{10}, \infty\}} P(x_{10} + U(z_1 - x_{10}) \in A) dQ_{Z_1}(z_1)\]

\[= \int_{\{x_{10}, \infty\}} 0 dQ_{Z_1}(z_1) = 0,\]

which means \(X_1\) is absolutely continuous. Now we prove \(E_Q[I(Z_1 \geq x)/(Z_1 - x_{10})], x > x_{10}\) is a probability density of \(X_1\). For any \(x_1 > x_{10}\), we have

\[P(X_1 \geq x_1) = P(x_{10} + U(Z_1 - x_{10}) \geq x_1) = \int_{[x_1, \infty)} P(x_{10} + U(z_1 - x_{10}) \geq x_1) dQ_{Z_1}(z_1)\]

\[= \int_{[x_1, \infty)} \frac{z_1 - x_{10}}{z_1 - x_{10}} dQ_{Z_1}(z_1) = \int_{[x_1, \infty)} \frac{1}{z_1 - x_{10}} dQ_{Z_1}(z_1)\]

\[= \int_{[x_1, \infty)} \int_{[x_{10}, \infty]} \frac{1}{z_1 - x_{10}} dQ_{Z_1}(z_1) dx = \int_{[x_1, \infty)} E_Q[I(Z_1 \geq x) / (Z_1 - x_{10})] dx.\]

Since this holds for any \(x_1 > x_{10}\), we know \(E_Q[I(Z_1 \geq x) / (Z_1 - x_{10})], x > x_{10}\) is indeed a probability density of \(X_1\). Moreover, by Monotone Convergence Theorem and \(P(Z_1 = x_{10}) = 0\),

\[\lim_{x \rightarrow x_{10}} E_Q[I(Z_1 \geq x) / (Z_1 - x_{10})] = E_Q[I(Z_1 \geq x_{10}) / (Z_1 - x_{10})] = E_Q[I(Z_1 \geq x_{10}) / (Z_1 - x_{10})],\]

which means \(E_Q[I(Z_1 \geq x) / (Z_1 - x_{10})]\) is continuous at \(x = x_{10}\).

**Proof of Theorem 7.** Without loss of generality suppose \(P\) is the probability measure induced by \(X\). Notice that we have \(\bar{F}(x_0) = c\) in (44). Therefore, \(P/c\) is a 1-POU probability measure on \(D_0\). By Theorem 6, we have the representation

\[P = c \int_{D_0} W_{1-POU}(z) dQ(z) \quad (69)\]

for some probability measure \(Q\) on \(D_0\). We write \(Z = (Z_1, \ldots, Z_d)\) as a random vector with the distribution \(Q\) and write \(U\) as the uniform distribution on \((0, 1)\) independent of \(Z\). Then \((x_{10} + U(Z_1 - x_{10}), Z_2, \ldots, Z_d)\) has the probability distribution \(P/c\) on \(D_0\).

We first consider the reduction of the objective function. Since \(S\) is in the form of (45), by (69), we have

\[P((X_1, \ldots, X_d) \in S) \equiv P(S) = c \int_{D_0} \min(g_2(z_2, \ldots, z_d), z_1) - \min(g_1(z_2, \ldots, z_d), z_1) / z_1 - x_{10} dQ(z)\]

\[= c E_Q \left[ \frac{\min(g_2(Z_2, \ldots, Z_d), Z_1) - \min(g_1(Z_2, \ldots, Z_d), Z_1)}{Z_1 - x_{10}} \right].\]
So the objective function is expressed by the expectation under $Q$.

Next we consider the density constraint $f_{X_1}(x_{10}) \leq u_{X_1}$. Since $f_{X_1}$ is the marginal density of $X_1$ within $D_0$, we can see $f_{X_1}/c$ is the probability density of $x_{10} + U(Z_1 - x_{10})$. Recall that we require $f_{X_1}$ to be continuous at $x = x_{10}$. So the value $f_{X_1}(x_{10})$ is uniquely determined and is actually given by Theorem 6, i.e.,

$$f_{X_1}(x_{10}) = \frac{c f_{X_1}(x_{10})}{c} = c E_Q \left[ \frac{I(Z_1 \geq x_{10})}{Z_1 - x_{10}} \right],$$

where the last inequality follows from the fact that $Z_1$ takes values in $[x_{10}, \infty)$. Therefore, the density constraint can be rewritten as

$$E_Q \left[ \frac{1}{Z_1 - x_{10}} \right] \leq \frac{u_{X_1}}{c}.$$

Moreover, this constraint also implies that $Q(Z_1 = x_{10}) = 0$ for any feasible probability measure $Q$. Therefore, $P(X_1 = x_{10}) = P(x_{10} + U(Z_1 - x_{10}) = x_{10}) = 0$. In other words, the constraint that $P(X_1 = x_{10}) = 0$ is already involved in the density constraint.

Then we consider the moment constraint

$$a_i \bar{F}(x_0) \leq P(\bar{x}_{ji} \leq X_j \leq \bar{x}_{ji}, j = 1, \ldots, d) \leq b_i \bar{F}(x_0).$$

By (69) and $\bar{F}(x_0) = c$, it can be represented by

$$a_i \leq \int_{D_0} \min(z_i, \bar{x}_{ii}) - \min(z_i, \bar{x}_{ii}) I(\bar{x}_{ji} \leq z_j \leq \bar{x}_{ji}, j \geq 2) dQ(z) \leq b_i,$$

i.e.,

$$a_i \leq E_Q \left[ \frac{\min(Z_1, \bar{x}_{ii}) - \min(Z_1, \bar{x}_{ii}) I(\bar{x}_{ji} \leq Z_j \leq \bar{x}_{ji}, j \geq 2)}{Z_1 - x_{10}} \right] \leq b_i.$$

Finally, by Theorem 6, the representation (69) is an “if and only if” condition for $P$ being 1-POU about $x_0$ on $D_0$. Hence, the moment problem (46) is equivalent to the 1-POU-DRO problem (44). Moreover, the optimal solution of the two problems are also related by the representation given in Theorem 6. This completes our proof.

**Proof of Corollary 4.** This follows directly from Theorem 3.2 in Winkler (1988).