A simple approximation for the modified Bessel function of zero order $I_0(x)$

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Abstract. New efficient analytic approximations have been found for the modified zero-order Bessel functions $I_0(x)$. The method, used herein, improves prior techniques, such as multipoint quasi-rational approximations, MPQA. The present work is also an improvement of previous works, in the sense that the form of the approach is more efficient, that is, a smaller relative error is obtained using a lower number of parameters. As in the Pade method rational functions are used with coefficients determined from the powers series, but now also asymptotic expansions are also used simultaneously with that series, and consequently the rational functions have to be combined with elementary functions, in such a way, that the approach is a bridge between both expansions. The approximation now found is a rational function combined with a hyperbolic function. Only three parameters are needed to obtain a relative error smaller than 0.6 percent.

1. Introduction

The Bessel functions are important in Physics and in particular the modified Besel function $I_0(x)$ [1,3]. Its power series and asymptotic expansion are known, the first is good for small values of the variable $x$, and the second for large values. Although the power series is entire, this is, with an infinite radius of convergence, however, many terms need to be used for large and intermediate values of $x$. It is the object of the present work to find an analytic approximation of this function for every value of $x$, which can be derived and integrated, as it is required in Thermodynamics, Optics and other areas of Physics. This will be simple and with a small relative error, so that, it can be used in almost all cases where this function is of interest in Physics. For this purpose, the Multipoint Quasirational Approximation, MPQA, has been used and improved. This technique has been successfully used in several physics problems such as Plasma dispersion function, Zeemman Quadratic effect in 2D, and others [4]. The method uses rational functions, as in Pade, but now combined with elementary functions. The reason for introducing these types of functions is that in addition to use potential series as in Pade, asymptotic series are also used simultaneously with the power series. Asymptotic series are usually expressed in terms of sometimes fractional negative powers combined with non-potential elementary functions. What is desired with the quasirational approximations is to have a function like a bridge, between the power series and the asymptotic expansion.

The usual fact that asymptotic series are not valid in the whole circle of infinity, but only in one sector, does not present problems, because although the approximation contains all the singularities of the exact function, this is only true in the sector of the complex plane wherein the
asymptotic expansion is valid. It is not desired that the approximation and the exact function have the same singularities in the whole complex plane, because in that case, as demonstrated in complex variable theory, both functions would differ only in one constant.

2. Theoretical development
The potential series of $I_0(x)$ is

$$I_0(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}x^2\right)^k}{k \Gamma(k+1)}$$

(1)

and the asymptotic series is

$$I_0(x) = \frac{e^x}{\sqrt{2\pi x}} \left( 1 + \frac{1}{8x} + \frac{9}{128x^2} + \cdots \right)$$

(2)

The approximant used here will be

$$\tilde{I}_0(x) = (1 + \lambda^2 x^2)^{-1/4} \cosh x \frac{p_0 + p_1 x^2}{1 + q_1 x^2}$$

(3)

The form of this approximant is selected by the following reasons:

(i) It is necessary that when $x$ tends to infinity the factor that accompanies the rational function has the factor $x^{-1/2}$. Choosing this factor as an auxiliary function does not work, because since fractional powers have two branching points, the second point would be a branch point at $x = 0$, which the function $I_0(x)$ does not have. Thus, the second branching point must be placed outside the region of interest, which in this case has been chosen to be such that the approximant is good for every positive value of $x$. One way to do this would be to select the auxiliary function as $(1 + x)^{1/2}$, so that the second branch point would be in $x = -1$, which clearly is outside the region of interest. While this selection is good and has been done well in other problems, however is not the best, as in this paper is shown. The reason for this is that we also have to observe the potential series that has only paired powers, so that, if we want to have more efficiency we should select the auxiliary function in order to obtain only even powers, such as the function $(1 + x^2)^{1/4}$ which in infinity behaves as $x^{-1/2}$, and at the same time by expanding it around $x$ we would have only even functions. In this way we make more efficient the approximate in its totality. Using an additional $\lambda$ parameter is also a way of introducing a degree of freedom that allows us to obtain more accurate approximations as shown below.

(ii) The asymptotic expansion has the factor $e^x$, which must also be included as an auxiliary function; but this function is not even, and since we want the function to be even for the reasons given in the previous paragraph, we choose $\cosh x$, instead of $e^x$, as an auxiliary function, so that the behavior at infinite is correct and at the same time, its power series will have are even powers as desired.

(iii) Since the auxiliary functions are even and $I_0(x)$ is also even, the rational function of $x$, will have only even powers.

3. Results and Discussion
The determination of the parameters $p_0, p_1$ and $q_1$, are now made using the coefficients of the power series and the asymptotic expansion. From the coefficient in the leading term of the asymptotic expansion we obtain that
\[
\frac{1}{\sqrt{2\pi}} = \frac{p_1}{2\lambda^{1/2} q_1}
\]  \hspace{1cm} (4)

Since we should to have equal numbers of equations than of parameters, then we need only two equations to determine all the parameters, that is, we only need two coefficients of the power series, so the series for \( I_0(x) \) will be written as

\[
I_0(x) = 1 + \frac{x^2}{4} + O(x^4).
\]  \hspace{1cm} (5)

Rationalizing equation (3), it is obtained

\[
(1 + \lambda^2 x^2)^{1/4}(1 + q_1 x^2)(1 + \frac{1}{4} x^2) = \cosh x(p_0 + \sqrt{\frac{2}{\pi}} \lambda^{1/2} q_1 x^2) + O(x^4)
\]  \hspace{1cm} (6)

Replacing \( \cosh x \) by its series expansion \( (1 + \frac{x^2}{2}) \), using only two terms, and matching the coefficients in \( x^0 \) and \( x^2 \) in both sides of the equation two equations are obtained to find \( p_0 \) and \( q_1 \) as a function of \( \lambda \). In this way, the parameters \( p_0 \) and \( q_1 \) are

\[
p_0 = 1; \quad q_1 = \frac{\frac{1}{4} - \frac{1}{4} \lambda^2}{1 - \sqrt{\frac{2}{\pi}} \lambda^{1/2}}
\]  \hspace{1cm} (7)

So the approximant is

\[
\tilde{I}_0(x) = \frac{\cosh x}{(1 + \lambda^2 x^2)^{1/2}} \frac{1 + \sqrt{\frac{2}{\pi}} \left[ \frac{1 - \lambda^2}{1 - \sqrt{\frac{2}{\pi}} \lambda^{1/2}} \right] \lambda^{1/4} x^2}{1 + \left[ \frac{1 - \lambda^2}{1 - \sqrt{\frac{2}{\pi}} \lambda^{1/2}} \right]}
\]  \hspace{1cm} (8)

The value of \( \lambda \) must be defined using the condition that there are no poles of \( \tilde{I}_0(x) \) in the region of interest, so that \( q_1 \) is required to be positive. A graph of \( q_1 \) as a function of \( \lambda \) is shown in Figure 1 and 2.
Figure 1. The $q_1$ value is shown as function of $\lambda$

Figure 2. The $q_1$ values as function of $\lambda$ is shown in the interval $[0, 1]$

From the first graph we obtain that $\lambda$ must be $0 \leq \lambda \leq 1$ and $\frac{1}{2}\pi \leq \lambda \leq \infty$.

Of all these values the minimum relative error is obtained for $\lambda = \frac{1}{2}$. In this way the approximant is

$$\tilde{I}_0(x) = \frac{\cosh x}{(1 + \frac{1}{2}x^2)^\frac{3}{2}} \frac{1 + \sqrt{\frac{1}{2} \left[ \frac{1 - \frac{1}{2}^2}{1 - \sqrt{\frac{1}{2} \frac{3}{2}}} \right] \frac{1}{2} x^2}}{1 + \frac{1 - \frac{1}{2}^2}{1 - \sqrt{\frac{1}{2} \frac{3}{2}}}}$$

(9)
A graph of the relative error is shown in Figure 3. The maximum relative error is 0.00564, but if the region near this value is discarded, the relative error of the approximant is always less than 0.002. When calculating each term of the approach it is convenient to use only the digits necessary, so that, the relative error does not increase. With this criterion the final approximation is obtained as

$$\tilde{I}_0(x) = \frac{\cosh x}{(1 + \frac{1}{2} x^2)^{\frac{1}{4}}} \frac{1 + 0.24273 x^2}{1 + 0.43023 x^2}$$

(10)

**Figure 3.** Relative error as function of the variable $x$

### 4. Conclusions

A quasi-rational approximation has been found for the modified Bessel $\tilde{I}_0(x)$ function, very simple with only four parameters $\lambda, p_0, p_1, q_1$, but with a great precision lower than 0.005 for almost all $x$ value, except for those close to the value 11, where the relative error can be 0.002, also quite small value. To find the approximation, rational functions combined with the hyperbolic cosine function and a fractional power have been used. In the present technique, the power series and the asymptotic expansion of $I_0(x)$ are used simultaneously, so that the is a bridge between the two expansions. The validity of the approximant is for positive values of the variable $x$ and in a broader sense for the right half-plane of the complex plane. In the present work the MPQA approximation technique has been improved, in the sense of obtaining approximations with a parity required in order to make more efficient the obtaining of the parameters of the approver, as well as to use less number of equations and of parameters for get good approximations with a small relative error. The approximate here obtained analytical and valid for all positive value of $x$, which can be derived in integrated with confidence, and we think that the function $I_0(x)$ in the different applications in different areas of Physics.

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