Noise-tolerant Signature of $Z_N$ Topological Orders in Quantum Many-body States

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Topologically ordered states are fundamentally important for theoretical physics, and are regarded as promising candidates to build fault-tolerant quantum devices (quantum memories, quantum computers, etc.). However, it is still elusive how topological orders can be affected or detected under the action of noise. The existing methods (such as ones based on detecting topological Renyi entropy) were shown to work when the noise respects the corresponding symmetries. In this work, we find a quantity (termed as ring degeneracy $D$) that is robust under pure noise for detecting both trivial and intrinsic topological orders. The ring degeneracy is defined as the degeneracy of the solutions of the self-consistent equations that encode the contraction of the corresponding tensor network. For $Z_N$ orders, we find that the ring degeneracy satisfies a simple relation $D = (N + 1)/2 + d$, with $d = 0$ for odd $N$ and $d = 1/2$ for even $N$. By simulating different models (the 2D statistical Ising model, $Z_N$ topological states, and resonating valence bond states), we show that the ring degeneracy can tolerate noises of around $O(10^{-1})$ order of magnitude.

Introduction.— Topological states [1,4] are exotic states of matter that cannot be described by conventional order parameters, such as those within the Landau-Ginzburg paradigm. This kind of states have been considered as promising candidates to realize fault-tolerant quantum devices, such as quantum computers [5,7] and quantum memories [5,8]. Taking the Kitaev model as an example [9], its two-fold degenerated ground state provides a subspace that can store the information of one qubit. Since the degenerated states are connected by non-local operations that wind the whole system, that is, they are protected by a large gap, local perturbations will not be able to induce any errors to the stored information [3,10,11] as long as the perturbations are small compared with the energy gap [5].

Several methods and signatures have been proposed to detect the topological orders. The most widely applied ones are: i) topological entanglement entropy (TEE) [9,12], ii) topological Renyi entropy [13], and iii) topological ground-state degeneracy [14,15]. For the symmetry protected topological (SPT) states [16], for instance, the fixed-point tensors from tensor-entanglement-filtering renormalization [16] is used to characterize symmetry breaking and SPT phase transitions.

However, due to the high computational complexity, the investigations on realistic higher-dimensional systems are still rare, particularly for those systems, which do not admit known analytical solutions. The ground-state degeneracy of 1D system can be efficiently reached by the bipartite entanglement spectrum [17,18], which has been applied to detect 1D SPT states such as Haldane phase [16,17,19,20]. But, for two- and higher-dimensional systems, the applications are sparse [21,23], essentially due to the complexity of calculating the entanglement. Such difficulty also hinders the applications of TEE and topological Renyi entropy for detecting topological orders in higher dimensions.

Moreover, it remains elusive how the noise affects the topological properties, which is an important issue to the utilization of topological systems to develop novel quantum technologies [5,8]. Chen et al. showed that topological Renyi entropy fails to characterize $Z_2$ topological order, when the perturbation breaks the $Z_2$ symmetry [24]. It is not clear whether the topological properties are destroyed by the noise, or just the signature fails to correctly reveal the topological features. Therefore noise-tolerant signatures of topological orders need to be developed.

In this work, we propose a quantity named ring degeneracy (RD, denoted by $D$) that robustly detects the topological orders even under a noise that breaks the symmetries. RD is defined as the degeneracy of the fixed-point solutions of the self-consistent eigenvalue equations constructed from the TN representation of the quantum system [Fig. 1(a)] [25]. We show that the topological order (either trivial or intrinsic) will lead to a degeneracy of the RD. On the 2D statistical Ising model, our results show $D = 1$ in the high-temperature disordered phase and $D = 2$ in the low-temperature ordered phase, where there exist two degenerate ground states that are connected by $Z_2$ transformation. We apply our scheme to the states that have intrinsic topological orders, including resonating valence bond state on kagomé lattice, which possesses $Z_3$ order [26–29], and the $Z_N$ string-net states [12,22,23,30]. We find that the RD for $Z_N$ ordered states satisfies $D = (N + 1)/2 + d$, with $d = 0$ for odd $N$, and $d = 1/2$ for even $N$. We investigate the robustness of RD under pure noises, and demonstrate how the noise affects the stability of RD. We show that the RD can be reached robustly up to a noise of about $O(10^{-1})$ order of magnitude.

Ring degeneracy of infinite 2D tensor network— An infinite TN state (TNS) (also called projected entangled pair state) [31,32] in 2D system with translation invariance can be writ-
FIG. 1. (Color online) (a) Self-consistent eigenvalue equations of TRD. (b) and (c) Two orthogonal degeneracy of TRD ring tensors.

The Latin letters represent the physical indexes that correspond to the physical Hilbert space of the quantum state, and the Greek letters represent the geometrical indexes that will be contracted (dummy indexes). The TN of the inner product between the state and its conjugate \( \langle \psi | \psi \rangle \) gives a 2D TN, where all physical indexes are contracted. Such a TN is formed by infinite copies of \( T_{\eta_1 \eta_2} = \sum_{\alpha_1, \alpha_2, \alpha_3} P_{\alpha_1, \alpha_2, \alpha_3} \alpha_1 \alpha_2 \alpha_3 \) with \( \eta_1 = (\alpha_1, \alpha_2, \alpha_3) \); it is in fact the zero-temperature partition function of the system, and conveys many physical properties of the TNS, such as the effective action and critical point \([33]\).

Tensor ring decomposition (TRD) \([25, 31]\) is an efficient way of computing the TN contraction. Unlike the methods based on tensor renormalization group (see, e.g., Refs. \([34–39]\]), TRD “encodes” the TN contraction problem to a set of self-consistent eigenvalue equations. With the inversion symmetries, the solution of TRD contains two tensors dubbed by \( A \) and \( B \). The eigenvalue equations \([\text{see the first two sub-figures of Fig. } 1(a)]\) that \( A \) and \( B \) satisfy are

\[
\sum_{\eta_1, \eta_2} T_{\eta_1, \eta_2, \eta_3} \tilde{B}_{\eta_1, \eta_2, \eta_3} \alpha_{\eta_1, \eta_2, \eta_3} = \lambda_A \alpha_{\eta_1, \eta_2, \eta_3}, \tag{2}
\]

\[
\sum_{\eta_1, \eta_2} T_{\eta_1, \eta_2, \eta_3} \alpha_{\eta_1, \eta_2, \eta_3} \tilde{B}_{\eta_1, \eta_2, \eta_3} = \lambda_B \alpha_{\eta_1, \eta_2, \eta_3}, \tag{3}
\]

with \( \lambda_A \) and \( \lambda_B \) the eigenvalues. The third sub-figure is the QR decomposition \( \tilde{B}_{\eta_1, \eta_2, \eta_3} = \sum_{\eta_4} \tilde{B}_{\eta_1, \eta_2, \eta_4} R_{\eta_3, \eta_4} \), which ensures that \( A \) and \( B \) converges to the non-trivial fixed points \([25]\). After randomly initializing \( A \) and \( B \), the fixed point can be reached by recursively solving the above equation.

To determine the degeneracy of the fixed-point solutions of TRD, we define the ring tensor \( R \) from \( A \) and \( B \) \([\text{Fig. } 1(b)]\) as

\[
R_{\eta_1, \eta_2, \eta_3} = \sum_{\eta_4, \eta_5} A_{\eta_1, \eta_2, \eta_4}^* A_{\eta_1, \eta_3, \eta_5} B_{\eta_2, \eta_4, \eta_5}^* R_{\eta_3, \eta_4, \eta_5}^*. \tag{4}
\]

From the TRD, we know that the ring tensor \( R \) actually is the optimal approximation of the environment of \( T \), i.e., the tensor after contracting all the TN except \( T \). Thus, the contraction \( Z = \sum_{\eta_1, \eta_2, \eta_3} T_{\eta_1, \eta_2, \eta_3} R_{\eta_1, \eta_2, \eta_3} \) gives approximately the whole

\[
F(R, R') = | \sum_{\eta_1, \eta_2, \eta_3} R_{\eta_1, \eta_2, \eta_3} R_{\eta_1, \eta_2, \eta_3}^* / \sqrt{|R||R'|}. \tag{5}
\]

**Ring degeneracy and global symmetry.**—We use the global symmetry of the TN to explain the ring degeneracy. Suppose the TNS \( |\psi \rangle \) satisfies a global symmetry \( G \), which requires the tensor \( P \) satisfies the following condition \([40]\) \([\text{Fig. } 2]\)

\[
\sum_{\eta} g_{\eta, \eta} P_{\eta, \eta', \eta''} = \sum_{\eta_1, \eta_2, \eta_3} U_{\eta_1, \eta_2} U_{\eta_2, \eta_3} W_{\eta_3, \eta_4} W_{\eta_4, \eta_5} P_{\eta_5, \eta_6, \eta_7}. \tag{6}
\]

Here \( g \) is a group element of \( G \); \( U \) and \( W \) are the projective representation of the group respected to \( g \). Then, the tensor \( T \) in the inner product TN of TNS \( |\psi \rangle \) possesses the corresponding symmetry \([\text{Fig. } 2]\) that reads

\[
T_{\eta_1, \eta_2, \eta_3} = G(T) = \sum_{\eta'_1, \eta'_2, \eta'_3} U_{\eta_1, \eta'_1} U_{\eta_2, \eta'_2} U_{\eta_3, \eta'_3} T_{\eta'_1, \eta'_2, \eta'_3}. \tag{7}
\]

While \( T \) has a ring tensor \( R \), one can immediately define another tensor \( R' = G(R) \), which is also the ring tensor of \( T \) (i.e., the fixed-point solution of the eigenvalue equations).

When the TN satisfies the global symmetry, it is not necessary to have the ring degeneracy. While the ring tensor satisfies the same symmetry, i.e., \( G(R) = R \), there will be no ring degeneracy. One example is the TN representing the 2D statistical Ising model in the high-temperature disordered phase (the results are given below). The ring degeneracy appears when \( F(R, G(R)) = 0 \). The TN in the ordered phase of 2D Ising model belongs to such a case (see below). We believe that the symmetry of \( R \) (i.e. the ring degeneracy) is reflected by the boundary theory of the TN \([33, 34, 43]\). It is known that from TRD (or other methods such as DMRG or CTMRG), one can calculate the boundary matrix product states (MPS’s) of a 2D TN. The boundary MPS’s, which are given by \( A \) and \( B \) in TRD, reveals the non-local bulk properties of the system. In the disordered phase, the system does not spontaneously break the \( Z_2 \) symmetry, leading to the ring tensor satisfying \( G(R) = R \). In the ordered phase, the \( Z_2 \) symmetry is spontaneously broken, and so is the \( Z_2 \) symmetry of the ring tensor. Still, it is to be further investigated theoretically whether one would have \( G(R) = R \) or not when the TN has the global symmetry.
We shall stress that the ring degeneracy is essentially different from the degeneracy of the eigenvalue equations of matrices. In our case, each of the eigenvalue equations in Eq. 3 has a unique dominant eigenvalue, while the equation set have degenerated fixed points.

Statistical Ising model.— We first apply our method on 2D statistical Ising model, where the TN satisfies the $Z_2$ symmetry. The 2D statistical Ising model on square lattice was investigated by Gu et al. [16] as the very first example that inspired the symmetry-protected topological orders. The interaction of this model is described by $H = \sum_{i,j} \eta_j \eta_i$, where $\eta_i$ represents the Ising spin on the $i$-th site, and the summation runs over all nearest-neighbor pairs of spins. The partition function $Z = Tr(e^{-\beta H})$ can be written as a TN, where we have $T_{\eta_1,\eta_2,\ldots,\eta_N} = e^{-\beta(\eta_1+\ldots+\eta_N)}$, which is the partition in a square. This Hamiltonian is invariant under a global $Z_2$ transformation, hence the partition function $Z$ and the tensor $T$ are also invariant under $Z_2$ transformation. When the temperature $T$ is higher than the critical temperature $T_c \approx 2.26919$, the system is in a disordered state; when $T < T_c$, it has two degenerated ordered states.

Applying TRD on the TN at different temperatures, the results (with bond dimension cut-off $\chi = 40$) are shown in Fig. 3. We found for $T > T_c$, we have $D = 1$ and there is only one fixed point representing the high-temperature disordered phase. For $T < T_c$, we obtain $D = 2$, where the two ring tensors give the same $Z$ and are orthogonal to each other with the fidelity $F \approx 0$.

Resonating valence bond and $Z_N$ string-net states.— It is known that the nearest-neighbor resonating valence bond (RVB) state on Kagomé lattice is a quantum spin liquid state with intrinsic $Z_2$ topological order [26,29]. Its TN representation is formed by the infinite copies of tensors $P$ and $B$, whose non-zero elements are $[32]:$

$$P_{0,0000} = P_{1,1211} = P_{1,1211} = P_{1,1112} = 1$$

$$B_{00} = B_{12} = 1, B_{21} = -1$$

The TN of $\langle \psi | \psi \rangle$ can be similarly obtained by contracting the physical indexes. We calculate the TRD of such a TN with $\chi = 40$ and find two-fold ring degeneracy. The fidelity between the two degenerated ring tensors is $F \sim 10^{-9}$.

We then apply our theory to $Z_N$ string-net states [12,22,23,30], which possess intrinsic $Z_N$ topological orders [11,24]. On a square lattice, the TNS of a $Z_N$ string-net state can be defined by the tensor as

$$T_{\alpha\beta\gamma\delta} = \begin{cases} 1, & (\alpha + \beta + \gamma + \delta) \mod N = 0 \\ 0, & \text{otherwise} \end{cases}$$

We find that the ring degeneracy $D$ satisfies

$$D = \begin{cases} (N + 2)/2, & N \text{ is even} \\ (N + 1)/2, & N \text{ is odd} \end{cases}$$

The relation in Eq. 9 can be understood by the representation theory of the $Z_N$ group. The group elements can be written as $[1, g_1, g_2, \ldots, g_{N-1}]$, where all the irreducible representations are denoted by $g_0 = \exp(ik\theta)$ with $N\theta = 0(mod 2\pi)$. Thus for $N > 2$, the representations will be complex. However, when applying TRD in the complex space, we meet with a convergence problem: the fixed points may “drift” due to the gauge degrees of freedom of the TN. Our results show that there exist several fixed points, where the fidelity between each two can be any values between 0 and 1. Thus, we restrain ourselves in the real space, and the fidelity takes only 0 or 1. In this case, the gauge degrees of freedom are fixed due to the uniqueness of the dominant eigenvectors of the two eigenvalue problems.

We give an intuitive picture in Fig. 4 to show how the degeneracy is reduced by taking $Z_2, Z_3, Z_4$ and $Z_5$ as examples. In the even cases, we always have two real transformation operators: identity $g_0 = 1$ and inversion $g_{N/2} = -1$, which will give us two real ring tensors noted as $R$ and $G(R)$. Besides, a real solution can be defined by adding each complex ring tensor $G(R)$ with its conjugate $G^*(R)$. In this way, $(N - 2)$ complex tensors will give us $(N - 2)/2$ real ring tensors. In

FIG. 3. (Color online) For the temperature $T > T_c$ (with $T_c$ the critical temperature), there is only one fixed point. When $T \leq T_c$, the overlap rapidly vanishes $O(10^{-5})$, indicating the existence of two degenerated fixed points that are orthogonal to each other. The relative errors of the free energy (compared with the analytical solution) are also shown, which is about $O(10^{-9})$ at the critical temperature and soon decays to $O(10^{-15})$ away from $T_c$.

FIG. 4. (Color online) An intuitive picture explaining the relation between the ring degeneracy and $Z_N$ orders, by taking $Z_2, Z_3, Z_4$ and $Z_5$ as examples. The black line represents the real space and the blue dash circle represents the complex space. The blue dots represent the fixed-point solutions (ring tensors) in the complex space, and the red dots show the projections in the real space by combining two conjugate solutions.
total, there will be $(N - 2)/2 + 2 = (N + 2)/2$ real fixed points. When $N$ is odd, there is only one real operator as the identity $\mathbb{1}$, and $N - 1$ complex transformations will give $(N - 1)/2$ real ring tensors. Thus the degeneracy of the ring tensors will be $(N - 1)/2 + 1 = (N + 1)/2$ in total.

**Robustness under noise.**—To investigate the effects of noises, we add a random term $\epsilon T_p$ (with $T_p$ a random tensor and $\epsilon$ a constant to control the strength of the noise) to the TN, i.e., $\hat{T} = T_0 + \epsilon T_p$. Each element of $T_p$ is obtained randomly by the Gaussian distribution $N(0, 1)$. Our results show that even though the random term breaks the global symmetry of TNS (and the TN), the fixed points of the TRD (if exist) remain robust. In other words, the fixed points under a noise are still stable contractors of the recursive process in the TRD. As long as the initial guess is within the attraction domain, the corresponding fixed point will be reached eventually. In the following, we randomly choose about 50 pairs of $A$ and $B$ as the initial guesses to compute the fixed points. Different initial guesses may be within the attraction domain of different fixed points. We then check the resulting ring tensors by calculating the fidelity between each two of the fixed points, and obtain the ring degeneracy as the number of orthogonal ring tensors. To characterize the stability, we defined the probability as $p = N_{D=2}/N_{\text{ tot}}$, where $N_{\text{ tot}}$ is the total number of different random terms $T_p$ we added to $T_0$, and $N_{D=2}$ is the number of those terms with which the expected fixed points are successfully found.

We firstly calculate the 2D Ising model. The results [Fig. 5 (a)] show that the two fixed points remain stable (with $p = 1$) for $\epsilon < 0.1$. For $\epsilon > 0.1$, the probability $P$ drops rapidly, and finally decays to zero where the fixed points are totally destroyed by the noise.

Note that TRD applies to the TN that is translational invariant. The TN is formed by the copies of $\hat{T} = T_0 + \epsilon T_p$, meaning the random terms for different tensors are the same. To weaken such a translational invariance, we increase the unit cell, so that the random terms are translationally invariant for $L \times L$ tensor clusters. Inside the cluster, the random terms added to different tensors are independent to each other. Our results show that the stability persists for $L = 1, 2$ and 3.

We further test the robustness of ring degeneracy on the nearest-neighbor RVB state on kagomé lattice (which is a gapped $Z_2$ spin liquid) and the $Z_2$ string-net state on square lattice [11, 24]. Previous work show that when adding a perturbation that breaks the $Z_2$ symmetry on the $Z_2$ string-net state, the topological entanglement Renyi entropy failed to detect the $Z_2$ order [24].

Here we try more than 2000 different $T_p$ as noise and calculate the probability $P(D = 2)$ with $\chi = 20, 30, 40$ [Fig. 5 (b)]. By turning on the randomness strength $\epsilon$ from $10^{-3}$ to 10, for Kagomé RVB, $P(D = 2)$ is still almost 1 when $\epsilon = 1$. It decays to about 0.35 after $\epsilon \geq 10$. The probability of finding two fixed points on $Z_2$ string-net state starts to decay at $\epsilon = 0.1$ from $P(D = 2) \approx 1$ to $P(D = 2) \leq 0.05$ at $\epsilon = 1.2$.

We also investigate the $Z_N$ string-net states for $N = 3, 4, 5, 6$, and shows the robustness of the ring degeneracy in

![Fig. 5](image-url)
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[1] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45, 494 (1980)
[2] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. 48, 1559 (1982)
[3] X. G. Wen, Phys. Rev. B 40, 7387 (1989)
[4] X. G. Wen, Int. J. Mod. Phys. B, 04 239 (1990)
[5] A. Y. Kitaev, Ann. Phys. (N.Y.) 303, 2 (2003)
[6] M. H. Freedman, A. Kitaev, M. J. Larsen, and ZH. Wang, Bull. Amer. Math. Soc. 40, 31-38 (2003)
[7] R. W. Ogburn, and J. Preskill, Lect. Notes Comput. Sci. 1509, 341 (1999)
[8] E. Dennis, A. Y. Kitaev, A. Landahl, and J. Preskill, J. Math. Phys. 43, 4452 (2002)
[9] A. Kitaev, and J. Preskill, Phys. Rev. Lett. 96, 110404 (2006)
[10] S. Dusuel, M. Kamfor, R. Orus, K. P. Schmidt and J. Vidal, Phys. Rev. Lett. 106, 107203 (2011)
[11] X. Chen, Z. X. Liu, X. G. Wen, Phys. Rev. B 84, 235141 (2011)
[12] M. Levin, X. G. Wen, Phys. Rev. Lett. (96), 110405 (2006)
[13] S. T. Flammia, A. Hamma, T. L. Hughes, and X. G. Wen, Phys. Rev. Lett. 103, 261601 (2009)
[14] X. G. Wen, and Q. Niu, Phys. Rev. B 41, 9337 (1990)
[15] S. Depenbrock, I. P. McCulloch, and U. Schollwock, Phys. Rev. Lett. 109, 067201 (2012)
[16] Z. C. Gu, X. G. Wen, Phys. Rev. B 80 155131 (2009)
[17] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B 81, 064439 (2010)
[18] F. Pollmann, and A. M. Turner, Phys. Rev. B 86, 125441 (2012)
[19] F. D. M. Haldane, Phys. Lett. 93A, 464 (1983)
[20] F. D. M. Haldane, Phys. Rev. Lett. 50, 1153 (1983)
[21] J. M. Kosterlitz, and D. J. Thouless, J. Phys. C 6, 1181 (1973)
[22] Z. C. Gu, M. Levin, B. Swingle, and X. G. Wen, Phys. Rev. B, 79, 085518 (2009)
[23] M. A. Levin, and X. G. Wen, Phys. Rev. B 71, 045110 (2005)
[24] X. Chen, B. Zeng, Z. C. Gu, I. L. Chuang and X. G. Wen, Phys. Rev. B 82, 165119 (2010)
[25] S. J. Ran, Phys. Rev. E 93, 053310 (2016)
[26] P. W. Anderson, Mater. Res. Bull. 8, 153 (1973)
[27] X. G. Wen, Phys. Rev. B 44, 2664 (1991)
[28] R. Moessner, and S. L. Sondhi, Phys. Rev. Lett. 86, 1881 (2001)
[29] G. Misguich, D. Serban, and V. Osuquler, Phys. Rev. B, 79, 137202 (2002)
[30] O. Buerschaper, M. Aguado, and G. Vidal, Phys. Rev. B, 79, 085119 (2009)
[31] S. J. Ran, E. Tirrito, C. Peng, X. Chen, and G. Su, [arXiv:1708.09213]
[32] F. Verstraete, M. M. Wolf, D. Perez-Garcia, and J. I. Cirac, Phys. Rev. Lett. 96, 220601 (2006)
[33] S.-J. Ran, C. Peng, W. Li, M. Lewenstein, and G. Su, Phys. Rev. B 95, 155114 (2017).
[34] M. Levin and C. P. Nave, Phys. Rev. Lett. 99, 120601 (2007).
[35] G. Vidal, Phys. Rev. Lett. 98, 070201 (2007).
[36] R. Orus, and G. Vidal, Phys. Rev. B 80, 094403 (2009).
[37] G. Evenbly and G. Vidal, Phys. Rev. Lett. 115, 180405 (2015).
[38] S. Yang, Z.-C. Gu, and X.-G. Wen, Phys. Rev. Lett. 118, 110504 (2017).
[39] M. Bal, M. Mariën, J. Haegeman, and F. Verstraete, Phys. Rev. Lett. 118, 250602 (2017).
[40] D. Perez-Garcia, M. M. Wolf, M. Sanz, F. Verstraete, and J. I. Cirac, Phys. Rev. Lett. 100, 167202 (2008)
[41] X. G. Wen, Phys. Rev. B 65, 165113 (2002)
[42] X. Chen, Z. C. Gu, and X. G. Wen, Phys. Rev. B 84, 235128 (2011)
[43] J. I. Cirac, D. Poilblanc, N. Schuch, and F. Verstraete, Phys. Rev. B 83, 245134 (2011).
[44] N. Schuch, D. Poilblanc, J. I. Cirac, and D. Pérez-García, Phys. Rev. Lett. 111, 090501 (2013).
[45] S. Yang, L. Lehman, D. Poilblanc, K. Van Acoleyen, F. Verstraete, J. I. Cirac, and N. Schuch, Phys. Rev. Lett. 112, 036402 (2014).