The Cuntz semigroup of unital commutative AI-algebras

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Abstract. We provide an abstract characterization for the Cuntz semigroup of unital commutative AI-algebras, as well as a characterization for abstract Cuntz semigroups of the form \( \text{Lsc}(X, \overline{\mathbb{N}}) \) for some \( T_1 \)-space \( X \). In our investigations, we also uncover new properties that the Cuntz semigroup of all AI-algebras satisfies.

1 Introduction

The celebrated Effros-Handelman-Shen theorem [11, Theorem 2.2] characterizes when a countable ordered abelian group \( G \) is order isomorphic to the ordered \( K_0 \)-group of an approximately finite-dimensional (AF) \( C^* \)-algebra. More explicitly, it states that \( G \) is unperforated and has the Riesz interpolation property if and only if \( G \) is order isomorphic to the \( K_0 \)-group of such a \( C^* \)-algebra.

In analogy to the definition of an AF-algebra, a \( C^* \)-algebra \( A \) is said to be an AI-algebra if \( A \) is \( * \)-isomorphic to an inductive limit whose building blocks have the form \( C[0,1] \otimes F_n \) with \( F_n \) finite-dimensional for every \( n \). In the unital commutative setting, an AI-algebra is of the form \( C(X) \), with \( X \) homeomorphic to an inverse limit of (possibly increasing) finite disjoint unions of unit intervals.

In this paper, we use the Cuntz semigroup of a \( C^* \)-algebra, a refinement of \( K_0 \) introduced by Cuntz in [10] and used successfully in the classification of not necessarily simple \( C^* \)-algebras (see, for example, [7, 8, 19, 21]). We provide an abstract characterization for the Cuntz semigroup of unital commutative AI-algebras and introduce new properties that the Cuntz semigroup of every AI-algebra satisfies. This settles the range problem for the Cuntz semigroup of this class of commutative \( C^* \)-algebras; the corresponding problem in the setting of general AI-algebras was posed during the 2018 mini-workshop on the Cuntz semigroup in Houston, and is studied in [25].

Abstracting some of the properties that the Cuntz semigroup of a \( C^* \)-algebra always satisfies, the subcategory \( \text{Cu} \) of partially ordered monoids was introduced in [9]. This subcategory, whose objects are often called Cu-semigroups, contains the Cuntz
semigroup of all C*-algebras, and has been studied extensively in [1, 3, 4, 5], among others. A relevant family in Cu is that of countably based Cu-semigroups satisfying (O5) (see Paragraph 2.3), which contains the Cuntz semigroups of all separable C*-algebras (see [2, 22]).

The range problem for this class of algebras consists of finding a list of properties that a Cu-semigroup S satisfies if and only if S is Cu-isomorphic to the Cuntz semigroup of a unital commutative AI-algebra. In order to do this, we first investigate when a compact metric space X is such that C(X) is an AI-algebra, that is to say, we analyze when X is homeomorphic to an inverse limit of finite disjoint unions of unit intervals. To this end, and in analogy to the definition of a chainable continuum (see, for example, [17, Chapter 12]), we introduce almost chainable and generalized arc-like spaces (see Definitions 3.4 and 3.7, respectively). We prove the following theorem.

**Theorem 1.1** (3.15) Let X be a compact metric space. The following are equivalent:

(i) X is almost chainable.
(ii) X is a generalized arc-like space.
(iii) X is an inverse limit of finite disjoint copies of unit intervals.
(iv) C(X) is an AI-algebra.

The dimension of the spaces appearing in Theorem 1.1 is at most one, and in this case the Cuntz semigroup of C(X) is isomorphic to the semigroup Lsc(X, N) of lower semicontinuous functions from X to \{0, 1, \ldots, \infty\} (see, e.g., [20]). Thus, the next step in our approach is to characterize those Cu-semigroups of the form Lsc(X, N) for some T₁-space X. In Section 4, we define the notion of an Lsc-like Cu-semigroup. Given such a semigroup S, we prove in Section 5 that S has an associated T₁-topological space Xₛ (Paragraph 5.1) and that many properties defined for Cu-semigroups have a topological counterpart whenever the semigroup is Lsc-like (see Proposition 5.6). For example, an Lsc-like Cu-semigroup S has a compact order unit (in the sense of Paragraph 2.3) if and only if Xₛ is countably compact. In Theorem 6.4, we show that S is Lsc-like if and only if it is Cu-isomorphic to Lsc(Xₛ, N).

Using this characterization, together with the notion of covering dimension for Cu-semigroups introduced in [23], we obtain the following result.

**Theorem 1.2** (6.5) Let S be a Cu-semigroup satisfying (O5), and let n ∈ N ∪ {∞}. Then, S is Cu-isomorphic to Lsc(X, N) with X a compact metric space such that \text{dim}(X) = n if and only if S is Lsc-like, countably based, has a compact order unit, and \text{dim}(S) = n.

In particular, a Cu-semigroup S is Cu-isomorphic to the Cuntz semigroup of the C*-algebra C(X) with X compact metric and \text{dim}(X) ≤ 1 if and only if S is Lsc-like, countably based, satisfies (O5), has a compact order unit, and \text{dim}(S) ≤ 1.

With these results at hand, we introduce in Section 7 notions of chainability for Cu-semigroups. These conditions aim at modeling, at the level of abstract Cuntz semigroups, the concepts of cover and chain for a topological space. We prove the following theorem.

**Theorem 1.3** (7.11) Let S be a Cu-semigroup. Then, S is Cu-isomorphic to the Cuntz semigroup of a unital commutative AI-algebra if and only if S is countably based, Lsc-like, weakly chainable, has a compact order unit, and satisfies (O5).
Finally, in Section 8, we generalize some of the properties used in Theorem 7.11 and show that these generalizations are satisfied for the Cuntz semigroup of every AI-algebra (see Theorem 8.7).

In [25], we continue the study of Cuntz semigroups of AI-algebras and develop tools toward a similar characterization of Theorem 7.11 in the general case.

2 Preliminaries

Given a C$^*$-algebra $A$ and two positive elements $a, b \in A$, recall that $a$ is Cuntz subequivalent to $b$, and write $a \simeq b$, if there exists a sequence $(r_n)_n$ in $A$ such that $a = \lim_n r_n b r^*_n$. Moreover, if $a \simeq b$ and $b \simeq a$, we say that $a$ and $b$ are Cuntz equivalent, in symbols $a \sim b$.

The Cuntz semigroup of $A$, denoted by $\text{Cu}(A)$, is defined as the quotient $(A \otimes K)_{+}/\sim$, where we denote by $[a]$ the class of an element $a \in (A \otimes K)_{+}$. Equipped with the order induced by $\leq$ and the addition induced by $[a] + [b] = \left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right]$, the Cuntz semigroup becomes a positively ordered monoid (see [9, 10]).

2.1 Given two elements $x, y$ in a partially ordered set, we say that $x$ is way-below $y$, and write $x \ll y$, if for every increasing sequence $(z_n)_n$ whose supremum exists and is greater than or equal to $y$ there exists $n \in \mathbb{N}$ such that $x \leq z_n$.

In [9], it was shown that the Cuntz semigroup $S$ of every $C^*$-algebra always satisfies the following properties:

(O1) Every increasing sequence in $S$ has a supremum.
(O2) Every element in $S$ can be written as the supremum of a $\ll$-increasing sequence.
(O3) For every $x' \ll x$ and $y' \ll y$, we have $x' + y' \ll x + y$.
(O4) For every pair of increasing sequences $(x_n)_n$ and $(y_n)_n$, we have $\sup_n x_n + \sup_n y_n = \sup_n (x_n + y_n)_n$.

2.2 In a more abstract setting, we say that a positively ordered monoid is a Cu-semigroup if it satisfies (O1)–(O4).

We also say that a map between two Cu-semigroups is a Cu-morphism if it is a positively ordered monoid morphism that preserves suprema of increasing sequences and the way-below relation. As proved in [9], every $*$-homomorphism $\varphi$ between $C^*$-algebras induces a Cu-morphism $\text{Cu}(\varphi)$ between their Cuntz semigroups.

Thus, one can consider the subcategory Cu of the category of positively ordered monoid as the category with Cu-semigroups and Cu-morphisms as its objects and morphisms, respectively. By the results from [9], the assignment $\text{Cu}: C^* \rightarrow \text{Cu}$ is functorial.

Moreover, we know from [3, Corollary 3.2.9] that the category Cu has arbitrary inductive limits and that the functor Cu is arbitrarily continuous (see also [9]).

2.3 A Cu-semigroup is said to have weak cancellation if $x \ll y$ whenever $x + z \ll y + z$ for some element $z$. Given a compact element $z$, that is, an element such that $z \ll z$, weak cancellation implies that $x \leq y$ whenever $x + z \leq y + z$. Stable rank one $C^*$-algebras have weakly cancellative Cuntz semigroups by [22, Theorem 4.3].
It was also proved in [22] that the Cuntz semigroup of a $C^*$-algebra always satisfies the following property: for every $x' \ll x \leq z$, there exists $c$ such that

$$x' + c \leq z \leq x + c.$$ 

Moreover, it was shown in [3, Proposition 4.6] that a stronger property termed (O5) was also satisfied in the Cuntz semigroup of any $C^*$-algebra:

(O5) Given $x' \ll x$, $y' \ll y$ and $x + y \leq z$, there exists an element $c$ such that $y' \ll c$ and $x' + c \leq z \leq x + c$.

Under the assumption of weak cancelation, (O5) and the weaker property from [22] coincide.

We will say that a Cu-semigroup is countably based if there exists a countable subset such that every element in the semigroup can be written as the supremum of an increasing sequence of elements in the subset. It follows from [2, Lemma 1.3] that Cuntz semigroups of separable $C^*$-algebras are countably based.

2.4 Recall that a $C^*$-algebra $A$ is an AI-algebra if it is $\ast$-isomorphic to an inductive limit of the form $\lim_n C[0,1] \otimes F_n$ with $F_n$ finite-dimensional for every $n$.

By [24], every unital commutative AI-algebra is $\ast$-isomorphic to an inductive limit of the form

$$C([0,1])^{n_1} \xrightarrow{\varphi_{i+1,1}} C([0,1])^{n_2} \xrightarrow{\varphi_{i+2,2}} C([0,1])^{n_3} \xrightarrow{\varphi_{i+3,3}} \cdots$$

with $n_i \leq n_{i+1}$ for each $i$ and where all the homomorphisms are in standard form. That is to say, for every $i$, we have

$$\pi_{j,i+1} \varphi_{i+1,i}(f_1, \ldots, f_{n_i}) = f_r \sigma_{j,i},$$

where $\pi_{j,i+1}: C([0,1])^{n_{i+1}} \to C([0,1])$ is the $j$th projection map, $\sigma_{j,i}: [0,1] \to [0,1]$ is a continuous function, and $r \leq n_i$.

Thus, every unital commutative AI-algebra is isomorphic to $C(X)$, where $X$ is an inverse limit of (possibly increasing) finite disjoint unions of unit intervals. Conversely, it is easy to see that $C(X)$ is an AI-algebra for such an inverse limit $X$.

Since the space $X$ above will always have dimension at most one, we know by [20, Theorem 1.1] that $\text{Cu}(C(X)) \cong \text{Lsc}(X, \mathbb{N})$.

Moreover, note that, with the above notation, if there exists some $i$ such that $n_1 = n_j$ for every $j \geq i$, we can write $A \cong \oplus_{k=1}^{n_i} C(X_k)$ with $X_k$ an inverse limit of unit intervals.

3 Chainable and almost chainable spaces

In this section, we will prove that a compact metric space $X$ is homeomorphic to an inverse limit of finite disjoint copies of unit intervals (i.e., $C(X)$ is an AI-algebra) if and only if $X$ satisfies an abstract property, which we call almost chainability (see Theorem 3.15). This is done in analogy to [17, Chapter 12], where it is shown that a continuum is homeomorphic to the inverse limit of unit intervals if and only if it is chainable, as defined below.
3.1 Recall that a compactum is a compact metric space, and that a continuum is a connected compactum. As in [17, Chapter 12], a chain in a continuum $X$ will be a finite nonempty indexed collection $C = \{C_1, \ldots, C_k\}$ of open subsets of $X$ such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

The mesh of a chain $C = \{C_1, \ldots, C_k\}$, in symbols mesh$(C)$, is defined as $\text{mesh}(C) = \max\{\text{diam}(C_i)\}$.

A chain of mesh less than $\epsilon$ is called an $\epsilon$-chain.

Following [17, Chapter 12], one can now define chainable continua.

**Definition 3.1** A continuum $X$ is said to be chainable if for every positive $\epsilon$ there exists an $\epsilon$-chain covering $X$.

We will say that a compactum is piecewise chainable if it can be written as the finite disjoint union of closed chainable subspaces.

Chainability may also be defined in a general setting, and one can check that the previous and following definitions coincide whenever $X$ is a continuum (see Lemma 3.5).

**Definition 3.2** A topological space $X$ is said to be topologically chainable if any finite open cover of $X$ can be refined by a chain, that is, an open cover $C_1, \ldots, C_k$ such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

**Remark 3.2** Note that both of the abovementioned definitions imply that the dimension of the space is at most one, and that the space is connected whenever it is compact.

The following proposition shows why we consider chainable continua. More on chainable spaces can be found in [17, Chapter 12].

**Proposition 3.3** A compactum is an inverse limit of unit intervals if and only if it is chainable.

**Proof** Let $X$ be the inverse limit of unit intervals. Since the unit interval is compact and connected, $X$ is connected and, consequently, a continuum.

If $X$ is degenerate (i.e., a point), it is clearly chainable, so we may assume otherwise.

If $X$ is nondegenerate, then [17, Theorems 12.11 and 12.19] and the comments following Theorem 12.19 of [17] imply that $X$ is chainable.

Conversely, if $X$ is chainable, it can either be degenerate (in which case we are done) or nondegenerate. By [17, Theorem 12.11], a nondegenerate chainable continuum is an inverse limit of unit intervals, so the result follows. ■

**Definition 3.3** A unital AI-algebra $A$ will be said to be block stable if it is isomorphic to $C(X)$ with $X$ a compact metric piecewise chainable space, that is, if $A \cong \bigoplus_{k=1}^{n} C(X_k)$ with $X_k$ a chainable continuum for each $k$.

3.1 Generalized arc-like spaces

We now generalize the previous results and characterize the topological spaces arising from unital commutative AI-algebras in terms of a weaker property.
3.4 Let $X$ be a compactum. In analogy with the definition of chains, an \textit{almost chain} in $X$ will be a finite nonempty indexed collection $C = \{C_1, \ldots, C_k\}$ of open subsets of $X$ such that

$$C_i \cap C_j = \emptyset \text{ whenever } |i - j| \geq 2.$$

The \textit{mesh} of an almost chain will be $\text{mesh}(C) = \max\{\text{diam}(C_i)\}$, and an almost chain of mesh less than $\varepsilon$ will be called an $\varepsilon$-\textit{almost chain}.

\textbf{Definition 3.4} A compact metric space $X$ will be said to be \textit{almost chainable} if, for every $\varepsilon > 0$, there exists an $\varepsilon$-almost chain covering $X$.

\textbf{Definition 3.5} A topological space $X$ will be said to be \textit{topologically almost chainable} if any finite open cover of $X$ can be refined by an almost chain, that is, an open refinement $C_1, \ldots, C_k$ such that $C_i \cap C_j = \emptyset$ whenever $|i - j| \geq 2$.

\textbf{Lemma 3.5} Definitions 3.4 and 3.5 coincide whenever $X$ is a compact metric space.

The same proof can be applied to chainability and topological chainability.

\textbf{Proof} If $X$ is topologically almost chainable, let $\varepsilon > 0$ and take an open cover of $\varepsilon$-balls in $X$. Since $X$ is compact, there exist finitely many points $x_1, \ldots, x_n$ such that their $\varepsilon$-balls cover $X$.

By topological almost chainability, this finite open cover can be refined by open subsets $C_1, \ldots, C_k$ such that $C_i \cap C_j = \emptyset$ whenever $|i - j| \geq 2$. Since $C_i$ is contained in some $\varepsilon$-ball, it follows that $C_1, \ldots, C_k$ is an $\varepsilon$-almost chain.

Conversely, assume that $X$ is almost chainable and take a (finite) open cover $U_1, \ldots, U_n$.

Since $X$ is a compact metric space, the cover has a nonzero Lebesgue number $\delta$. That is, every subset of $X$ having diameter less than $\delta$ is contained in some $U_j$.

Set $\varepsilon < \delta$ and consider an $\varepsilon$-almost chain $C = \{C_1, \ldots, C_k\}$ covering $X$. Since $\text{diam}(C_i) \leq \varepsilon < \delta$ for every $i$, it follows that each $C_i$ is contained in some $U_j$. Consequently, $C_1, \ldots, C_k$ is an open refinement of $U_1, \ldots, U_n$ with the required property.

We now define the notion of $(\varepsilon, \delta)$-maps and generalized arc-like spaces. This is done in analogy with [17, Theorem 12.11].

Recall that a continuous map $f: X \to Y$ is an $\varepsilon$-\textit{map} if $\text{diam}(f^{-1}(y)) \leq \varepsilon$ for every $y \in Y$, where $\text{diam}(\emptyset) = 0$ by definition.

\textbf{Definition 3.6} Let $X, Y$ be metric spaces. Given $\varepsilon, \delta > 0$, we will say that a continuous map $f: X \to Y$ is an $(\varepsilon, \delta)$-\textit{map} if $\text{diam}(f^{-1}(Z)) < \varepsilon$ for every $Z \subseteq Y$ with $\text{diam}(Z) < \delta$.

Recall also that a continuum $X$ is \textit{arc-like} if, for every $\varepsilon > 0$, there exists an $\varepsilon$-map from $X$ onto $[0, 1]$ (see [17, Definition 2.12]).

\textbf{Definition 3.7} A compactum $X$ is said to be a \textit{generalized arc-like space} if, for every $\varepsilon > 0$, there exists $\delta > 0$ and an $(\varepsilon, \delta)$-map $f: X \to I_1 \sqcup \cdots \sqcup I_n$ with $I_j = [0, 1]$ for each $j$.

\textbf{Remark 3.6} Given a finite disjoint union of unit intervals $I_1 \sqcup \cdots \sqcup I_n$ and some $\delta > 0$, one can clearly construct a $(\delta, \delta')$-map $r: I_1 \sqcup \cdots \sqcup I_n \to [0, 1]$ (by simply rescaling $I_1 \sqcup \cdots \sqcup I_n$ until it fits in $[0, 1]$).
Thus, a compactum \( X \) is a generalized arc-like space if and only if, for every \( \varepsilon > 0 \), there exists an \((\varepsilon, \delta)\)-map \( f: X \to [0,1] \) for some \( \delta > 0 \).

**Lemma 3.7** Inverse limits of finite disjoint unions of unit intervals are generalized arc-like spaces.

**Proof** Let \( X \) be an inverse limit of finite disjoint unions of unit intervals, and let \( ([0,1] \sqcup \cdots \sqcup [0,1], f_{i,j}) \) be its associated inverse system.

Recall that the metric on \( X \) is defined to be

\[
d((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty) = \sum_{i=1}^\infty \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)},
\]

where \( d_i \) is the distance in the \( i \)th component of the inverse system.

Moreover, recall that the distance between two points \( x, y \) in \([0,1] \sqcup \cdots \sqcup [0,1]\) is either the usual distance if \( x, y \) belong to the same connected component or 2 if they do not.

Given \( \varepsilon > 0 \), let \( n \) be such that \( \sum_{i>n} 2/2^i < \varepsilon/2 \).

Then, since for every fixed \( n \) the maps \( f_{i,n} \) are uniformly continuous for every \( i \leq n \), there exists \( \delta > 0 \) such that

\[
d_i(f_{i,n}(x), f_{i,n}(y)) \leq \varepsilon/2n
\]

whenever \( d_n(x, y) \leq \delta \).

Let \( \pi_n: X \to [0,1] \sqcup \cdots \sqcup [0,1] \) be the \( n \)th canonical projection map, and take \( Z \subseteq \pi_n(X) \) with \( \text{diam}_n(Z) \leq \delta \).

Take \( x, y \in \pi_n^{-1}(Z) \). Then, we have

\[
d(x, y) = \sum_{i=1}^{n} \frac{1}{2^i} \frac{d_i(f_{i,n}(x), f_{i,n}(y))}{1 + d_i(f_{i,n}(x), f_{i,n}(y))} + \sum_{i>n} \frac{1}{2^i} \frac{d_i(x, y)}{1 + d_i(x, y)} \leq n \frac{\varepsilon}{2n} + \sum_{i>n} \frac{1}{2^i} \leq \varepsilon,
\]

since any pair of elements is at distance at most 2.

This shows that \( \text{diam}(\pi_n^{-1}(Z)) \leq \varepsilon \), and so \( \pi_n \) is an \((\varepsilon, \delta)\)-map, as required. \( \square \)

**Proposition 3.8** A compactum \( X \) is a generalized arc-like space if and only if it is almost chainable.

**Proof** We follow the proof of [17, Theorem 12.11] while making some minor adjustments.

Assume first that \( X \) is almost chainable and take \( \varepsilon > 0 \). Let \( C \) be an \( \varepsilon/2 \)-almost chain covering \( X \), and decompose it as \( C = C_1 \sqcup \cdots \sqcup C_r \) with

\[
C_1 = \{C_{i,1}\}_{i=1}^\infty, \ldots, C_r = \{C_{i,r}\}_i
\]

\( \varepsilon/2 \)-chains in \( X \). Note that, for any \( i, I \) we have \( C_{i,j} \cap C_{i,k} = \emptyset \) whenever \( j \neq k \) and \( \text{diam}(f_{i,j}^{-1}(Z)) < \varepsilon \) whenever \( \text{diam}(Z) < 1 \).

Take \( s \leq r \). If \( |C_s| \leq 2 \), consider the map \( f_s: \bigcup_j C_{i,s} \to [0,1] \) sending every element to 0. Note that \( \text{diam}(f_s^{-1}(Z)) < \varepsilon \) whenever \( \text{diam}(Z) < 1 \).
If \(|C_i| \geq 3\), use the techniques in [17, Theorem 12.11] to obtain an \((\epsilon, \delta_i)\)-map \(f_i: \bigcup_j C_{i,j} \to [0,1]\).

Now, define \(f = f_1 \sqcup \cdots \sqcup f_I: X \to I_1 \sqcup \cdots \sqcup I_r\), where \(I_l = [0,1]\) for every \(l\). Note that this is clearly continuous.

Setting \(\delta < \min\{1, \delta_i\}\), one gets that, if the diameter of \(Z \subseteq I_1 \sqcup \cdots \sqcup I_r\) is less than \(\delta\), \(Z\) is a subset of some \(I_l\) (since the distance between disjoint components is 2).

This implies that \(f^{-1}(Z) = f_l^{-1}(Z)\) has diameter at most \(\epsilon\), as required.

Conversely, if \(X\) is a generalized arc-like, then given any \(\epsilon\) there exists \(\delta\) with \(f: X \to I_1 \sqcup \cdots \sqcup I_r\) an \((\epsilon, \delta)\)-map.

For each \(I_l\), consider a \(\delta\)-chain \(C_l\). The inverse image of these chains through \(f\) gives the desired \(\epsilon\)-almost chain for \(X\).

**Corollary 3.9** Inverse limits of disjoint unions of unit intervals are almost chainable. In particular, each connected component is arc-like.

**Proof** This follows from our previous two results. Note, however, that only one implication of Proposition 3.8 has been used.

We will now show that the converse of Corollary 3.9 also holds, thus obtaining a characterization of inverse limits of finite disjoint copies of unit intervals (see Theorem 3.15).

We first recall the following result from [13], even though this particular formulation is from [24, Lemma 2.14, p. 184].

**Lemma 3.10** Let \(C\) be a closed nonempty subset of \([0,1]\), and let \(\epsilon > 0\). Then, there exist a subset \(Y \subseteq C\) which is a finite disjoint union of closed intervals (possibly degenerate) and a map \(\alpha: C \to Y\) such that \(\alpha(y) = y\) for every \(y \in Y\) and \(|\alpha(x) - x| < \epsilon\) for every \(x \in C\).

**Remark 3.11** With the previous notation, note that \(\alpha\) is an onto \(2\epsilon\)-map from \(C\) to \(Y\).

Indeed, for any \(y \in Y\) and for any \(x \in C\) with \(\alpha(x) = y\), we must have \(|x - y| < \epsilon\).

Since \(y \in \alpha^{-1}(y)\), it follows that \(\text{diam}(\alpha^{-1}(y)) \leq \text{diam}(B_\epsilon(y)) < 2\epsilon\).

From now on, by a **closed interval**, we will mean a possibly degenerate closed interval (that is to say, either a point or a nondegenerate closed interval).

**3.12** Given any \((\epsilon, \delta)\)-map \(f: X \to [0,1]\), consider the induced onto \(\epsilon\)-map \(\text{Im}(f)\).

Since \(X\) is compact, so is \(\text{Im}(f)\). Using Lemma 3.10, we can find an onto \(\delta\)-map \(\alpha: \text{Im}(f) \to Y\) with \(Y\) the finite disjoint union of closed intervals and, consequently, we get an onto \(\epsilon\)-map \(\alpha f: X \to Y\).

This shows that given any generalized arc-like compactum \(X\) (equivalently, any almost chainable compactum, by Proposition 3.8) and any \(\epsilon > 0\), there exists an onto \(\epsilon\)-map \(f: X \to Y\) with \(Y\) a finite disjoint union of closed intervals. Indeed, given any \(\epsilon > 0\), we know by Definition 3.7 and Remark 3.6 that there exists an \((\epsilon, \delta)\)-map \(f: X \to Y\). The conclusion now follows from the previous argument.

Using Paragraph 3.12, we will see that \(C(X)\) is a commutative AI-algebra for any generalized arc-like compactum \(X\). Since we already know that \(X\) is almost chainable whenever \(C(X)\) is an AI-algebra (see Corollary 3.9), this will show that \(X\) is almost chainable if and only if \(C(X)\) is a commutative AI-algebra.
The following lemma is a natural generalization of [17, Lemma 12.17]. We follow both the structure and the notation of the proof of said lemma.

**Lemma 3.13** Let X be a compactum, let \( g_1: X \to Y_1 \) be an onto continuous map with \( Y_1 \) a finite disjoint union of closed intervals, and let \( \eta > 0 \).

Then, there exists \( \varepsilon > 0 \) such that, for any onto \( \varepsilon \)-map \( g_2: X \to Y_2 \) with \( Y_2 \) a finite disjoint union of closed intervals, there exists a continuous map \( \varphi: Y_2 \to Y_1 \) such that \( |g_1(x) - \varphi g_2(x)| < \eta \) for every \( x \in X \).

**Proof** First, write

\[
Y_1 = I_1 \sqcup \cdots \sqcup I_n \sqcup \{ q_1 \} \sqcup \cdots \sqcup \{ q_m \}
\]

with \( J_k \) closed nondegenerate intervals for every \( k \).

Fix \( m \in \mathbb{N} \) such that \( 1/m < \eta/2 \) and define \( s_i = i/m \) for every \( 0 \leq i \leq m \). Since \( g_1 \) is uniformly continuous, there exists some \( \gamma > 0 \) such that \( \text{diam}(g_1(A)) < 1/m \) whenever \( \text{diam}(A) < \gamma \).

Let \( \delta = \gamma/2 \) and fix an onto \( \delta \)-map \( g_2: X \to Y_2 \) as in the statement of the lemma. Recall that there exists \( \delta > 0 \) such that \( \text{diam}(g_2^{-1}(Z)) < 2\varepsilon = \gamma \) whenever \( \text{diam}(Z) < \delta \).

Now, fix \( n \in \mathbb{N} \) such that \( 1/n < \delta/2 \) and define \( t_j = j/n \) for every \( 0 \leq j \leq n \).

As before, write \( Y_2 \) as

\[
Y_2 = I_1 \sqcup \cdots \sqcup I_n \sqcup \{ p_1 \} \sqcup \cdots \sqcup \{ p_m \}
\]

with \( I_l \) closed nondegenerate intervals for every \( l \).

For each \( l, k \), consider the subsets

\[
A^k_l = [s_0, s_1), \quad A^k_l = (s_{l-1}, s_{l+1}), \quad A^m_l = (s_{m-1}, s_m) \subseteq J_k,
\]

\[
B^l_j = [t_0, t_1), \quad B^l_j = (t_{j-1}, t_{j+1}), \quad B^n_l = (t_{n-1}, t_n) \subseteq I_l.
\]

By construction, we know that \( \text{diam}(B^l_j) < \delta \) for every fixed \( j, l \). This implies that \( \text{diam}(g_2^{-1}(B^l_j)) < \gamma \) and, consequently,

\[
g_1(g_2^{-1}(B^l_j)) \subseteq A^k_l \quad \text{or} \quad g_1(g_2^{-1}(B^l_j)) \subseteq \{ q_{r'} \}
\]

for some \( i, k, r' \). Here, we have taken \( m \) large enough so that the distance between the connected components of \( Y_1 \) is greater than \( 1/m \).

Note that, by the same argument, we also have that for every \( r \leq m_2 \) there exist \( i, k \) with \( g_1(g_2^{-1}(p_r)) \subseteq A^k_l \) or \( g_1(g_2^{-1}(p_r)) \subseteq \{ q_{r'} \} \) for some \( r' \).

Moreover, since for every fixed \( l \) and every \( j \) we have \( B^l_j \cap B^{l+1}_j \neq \emptyset \), we get

\[
\emptyset \neq g_1(g_2^{-1}(B^l_j)) \cap g_1(g_2^{-1}(B^{l+1}_j))
\]

because \( g_2 \) is onto.

It follows that, for every fixed \( l \), the sets \( g_1(g_2^{-1}(B^l_j)) \) belong to the same connected component of \( Y_1 \) for every \( j \). Thus, for every \( l \), there either exists \( k \) such that \( g_1(g_2^{-1}(I_l)) \subseteq J_k \) or there exists \( r' \) such that \( g_1(g_2^{-1}(I_l)) = \{ q_{r'} \} \).

Given a connected component \( Y \) of \( Y_2 \), we define the map \( \varphi_Y: Y \to Y_1 \) as follows:

If \( g_1(g_2^{-1}(Y)) = \{ q_{r'} \} \) for some \( r' \), define \( \varphi_Y: Y \to \{ q_{r'} \} \) as the constant map.
Else, there exists some \( k \) such that \( g_1(g_2^{-1}(Y)) \subseteq J_k \). If \( Y \) is degenerate, we can find \( A_i^k \subseteq I_k \) such that \( g_1(g_2^{-1}(Y)) \subseteq A_i^k \) for some \( i, k \). Define \( \varphi: Y \rightarrow A_i^k \subseteq I_k \) as the constant map \( \varphi_Y \equiv s_i \).

Finally, if \( Y \) is nondegenerate, it is of the form \( Y = J_l \). Then, for every \( j \), fix \( i(j) \) such that \( g_1(g_2^{-1}(B_j)) \subseteq A_i^{k(j)} \), and recall that

\[
\varnothing \neq g_1(g_2^{-1}(B_j)) \cap g_1(g_2^{-1}(B_{j+1}))
\]

for every \( j \). This shows that \( |i(j) - i(j + 1)| \leq 1 \).

Define \( \varphi_{i_l}: I_l \rightarrow J_k \) as \( \varphi_{i_l}(t_j) = s_{i(l)} \) and extend it linearly.

Let \( \varphi := \varphi_{i_1} \sqcup \cdots \sqcup \varphi_{i_{n_2}} \sqcup \varphi_{p_1} \sqcup \cdots \sqcup \varphi_{p_{n_2}}: Y_2 \rightarrow Y_1 \), which is clearly continuous. We will now see that \( |g_1(x) - \varphi g_2(x)| < \eta \).

Thus, let \( x \in X \) and let \( B \subseteq Y_2 \) such that \( g_2(x) \in B \) with \( B \) being either \( B_l \) for some \( l, j \) or \( \{p_r\} \) for some \( r \). Note that \( g_1(x) \in g_1(g_2^{-1}(B)) \).

Thus, if \( g_1(g_2^{-1}(B)) = \{q_{r'}\} \) for some \( r' \), we have \( g_1(x) = q_{r'} \) and, consequently,

\[
|g_1(x) - \varphi g_2(x)| = |q_{r'} - q_{r'}| = 0
\]

by the definition of \( \varphi \).

Finally, if \( g_1(g_2^{-1}(B)) \subseteq A_i^k \), we have that \( g_1(x) \in A_i^k \). Therefore, one gets \( |g_1(x) - s_i| < 1/2 \). There are now two different situations:

If \( B = \{p_r\} \) for some \( r \), we have defined \( \varphi_p \), as the constant map \( s_i \). Thus, one gets

\[
|g_1(x) - \varphi g_2(x)| = |g_1(x) - s_i| < 1/m \leq \eta/2
\]

Else, if \( B = B_j \) for some \( l \), \( j \) such that \( i(j) \) be the previously fixed integer such that

\[
g_1(g_2^{-1}(B)) \subseteq A_i^{k(j)}
\]

Then, since \( g_2(x) \in B \), we either have \( t_{j-1} \leq g_2(x) \leq t_j \) or \( t_j \leq g_2(x) \leq t_{j+1} \). This implies that \( \varphi(g_2(x)) \) is either between \( s_{i(j-1)} \) and \( s_{i(j)} \) or between \( s_{i(j)} \) and \( s_{i(j+1)} \).

Since \( |i(j) - i(j + 1)| \leq 1 \), the triangle inequality implies that

\[
|g_1(x) - \varphi g_2(x)| \leq |g_1(x) - s_{i(j)}| + |s_{i(j)} - \varphi g_2(x)| \leq 1/2 \eta < \eta,
\]

as required.

In order to prove our next result, we will need the following proposition, a proof of which can be found in \([17, Proposition 12.18] \).

**Proposition 3.14** Let \((X, d)\) be a compactum, and let \( Y = \lim_{i \rightarrow} (Y_i, f_i) \) be an inverse limit of compacta \((Y_i, d_i)\) with \( f_i: Y_{i+1} \rightarrow Y_i \).

Assume that there exist two sequences of strictly positive real numbers \((\delta_i), (\varepsilon_i)\) with \( \lim \varepsilon_i = 0 \) and a family of onto \( \varepsilon_i \)-maps \( g_i: X \rightarrow Y_i \) such that the following conditions hold:

(i) For every pair \( i < j \), we have \( \text{diam}(f_i, j(A)) \leq \delta_i/2^{-i} \) for any \( A \subseteq Y_j \) with \( \text{diam}(A) \leq \delta_j \).

(ii) \( d_i(g_i(x), g_i(y)) > 2\delta_i \) whenever \( d(x, y) \geq 2\varepsilon_i \).

(iii) \( d_i(g_i, f_i g_{i+1}) \leq \delta_i/2 \).

Then, \( X \cong Y \).
We summarize our results in the following.

**Theorem 3.15** Let $X$ be a compactum. The following are equivalent:

(i) $X$ is almost chainable.
(ii) $X$ is a generalized arc-like space.
(iii) $X$ is homeomorphic to an inverse limit of finite disjoint copies of unit intervals.
(iv) $C(X)$ is an AI-algebra.

**Proof** Conditions (i) and (ii) are equivalent by Proposition 3.8, whereas (iii) is equivalent to (iv) by the arguments in Paragraph 2.4. Furthermore, that (iii) implies (i) follows from Corollary 3.9. Thus, we are left to prove that (i) implies (iii).

Let $X$ be an almost chainable compactum. Then, for every $\varepsilon > 0$, there exists an onto $\varepsilon$-map $f: X \to Y$ with $Y$ a finite disjoint union of closed intervals.

As in [17, Theorem 12.19], we will inductively construct sequences of maps and real numbers satisfying the conditions of Proposition 3.14. We give the proof here for the sake of completeness, although the only difference with the original proof is that we replace $[0,1]$ with $Y$.

Let $0 < \varepsilon_1 \leq 1$, and consider an onto $\varepsilon_1$-map $g_1$ from $X$ to a finite disjoint union of closed intervals $Y_1$.

Note that, in particular, we must have $g_1(x) \neq g_1(y)$ whenever $d(x, y) \geq 2\varepsilon_1$. By compactness of $X$, this implies that there exists some $\delta_1 > 0$ such that $|g_1(x) - g_1(y)| > 2\delta_1$ whenever $d(x, y) \geq 2\varepsilon_1$.

Setting $\eta = \delta_1/2$, let $\varepsilon > 0$ be as in our previous lemma and set $\varepsilon_2 = \min\{1/2, \varepsilon\}$. Then, let $g_2: X \to Y_2$ be an onto $\varepsilon_2$-map given by the almost chainability of $X$.

By our choice of $\varepsilon_2$, there exists some $f_1: Y_2 \to Y_1$ such that

$$|g_1(x) - f_1g_2(x)| < \eta = \delta_1/2$$

for every $x \in X$.

As before, there exists $\delta_2 > 0$ such that $|g_2(x) - g_2(y)| > 2\delta_2$ whenever $d(x, y) \geq 2\varepsilon_2$. Furthermore, by the uniform continuity of $f_1$, we can choose $\delta_2 > 0$ so that $\text{diam}(f_1(A)) \leq \eta$ whenever $\text{diam}(A) \leq \delta_2$.

This shows that $g_1, g_2$ satisfy conditions (i)–(iii) of Proposition 3.14.

Proceeding as in [17, Theorem 12.19], one can now inductively find $\varepsilon_i, \delta_i > 0$ and maps $g_i: X \to Y_i, f_i: Y_{i+1} \to Y_i$ satisfying conditions (i)–(iii) of Proposition 3.14.

Applying Proposition 3.14, we have $X \cong \lim(Y_k, f_k)$ with $Y_k$ finite disjoint unions of closed intervals. Clearly, this implies that $C(X)$ is an AI-algebra, as required. ■

### 4 Lsc-like Cu-semigroups

In this section, we introduce Lsc-like Cu-semigroups (Definition 4.3) and prove some of their main properties. As we shall prove in Theorem 6.4, such Cu-semigroups are exactly those that are Cu-isomorphic to the Cu-semigroup of lower-semicontinuous functions $\text{Lsc}(X, \mathbb{N})$ for some $T_1$ topological space.

Using Lemma 4.18, we also prove that the semigroup $\text{Lsc}(X, \mathbb{N})$ is a Cu-semigroup whenever $X$ is compact and metric (see Corollary 4.19).
Definition 4.1 A Cu-semigroup $S$ will be called distributively lattice ordered if $S$ is a distributive lattice such that, given any pair of elements $x, y \in S$, we have $x + y = (x \lor y) + (x \land y)$.

We will say that a distributively lattice ordered Cu-semigroup is complete if suprema of arbitrary sets exist.

Remark 4.1 Given two increasing sequences $(x_n)_n$ and $(y_n)_n$ in a complete distributively lattice ordered Cu-semigroup, we have that $(\sup_n x_n) \lor (\sup_n y_n) = \sup_n (x_n \lor y_n)$.

Indeed, by definition, we know that $\sup_n x_n = \lor_{n=1}^\infty x_n$ and the equality $(\lor_{n=1}^\infty x_n) \lor (\lor_{n=1}^\infty y_n) = \lor_{n=1}^\infty (x_n \lor y_n)$ holds in any complete lattice.

Throughout the paper, we will say that a sum of finitely many indexed elements $x_1 + \cdots + x_n$ is ordered if the sequence $(x_i)_{i=1}^n$ is increasing or decreasing.

Lemma 4.2 Let $S$ be a distributively lattice ordered Cu-semigroup. Given two finite decreasing sequences $(x_i)_{i=1}^m$, $(y_i)_{i=1}^m$, the following equality holds:

$$\sum_{i=1}^m (x_i + y_i) = \sum_{i=1}^{2m} \lor_{j=0}^m (x_j \land y_{1-j}),$$

where, on the right-hand side, $x_i \land y_k = x_i$, and $x_k \land y_i = y_i$ whenever $k \leq m$ and $y_k = x_k = 0$ whenever $k > m$.

Note that the sum on the right-hand side is decreasingly ordered.

Proof This is a generalization of the equality $x + y = (x \lor y) + (x \land y)$ and is proved by induction. 

Remark 4.3 Given any finite sum $x_1 + \cdots + x_n$, we can apply Lemma 4.2 iteratively (first to $x_1 + x_2$, then to $(x_1 \lor x_2) + (x_1 \land x_2) + x_3$, etc.) to obtain an ordered sum.

Therefore, every sum in a distributively lattice ordered semigroup can be written as an ordered sum.

Definition 4.2 Let $S$ be a Cu-semigroup, and let $H$ be a subset of $S$. We say that $H$ is topological if, given two finite increasing sequences $(x_i)_{i=1}^m$, $(y_i)_{i=1}^m$ in $H$, we have

$$\sum_{i=1}^m x_i \leq \sum_{i=1}^m y_i$$

if and only if $x_i \leq y_i$ for every $i$.

Similar notions to that of topological order appear in other contexts, such as in [6, Definition 4.1].

Given any element $r$ in partially ordered set $P$, we denote by $\downarrow r$ the set $\{ s \in P \mid s \leq r \}$ (see, for example, [14, Definition O-1.3]).

Notation 4.4 Given a Cu-semigroup $S$ and an element $y \in S$, we write $\infty y := \sup_n ny$. Furthermore, if $S$ has a greatest element, we denote it by $\infty$. 
**Definition 4.3** A Cu-semigroup $S$ will be said to be *Lsc-like* if it is a complete distributively lattice ordered Cu-semigroup such that the following conditions hold:

(C1) For every pair of idempotent elements $y, z$ in $S$, $y \geq z$ if and only if
\[
\{ x < \infty \mid x \text{ maximal idempotent, } x \geq y \} \subseteq \{ x < \infty \mid x \text{ maximal idempotent, } x \geq z \}.
\]

(C2) There exists a topological subset of the form $\downarrow e$ such that the finite sums of elements in $\downarrow e$ are sup-dense in $S$.

The following example justifies our terminology.

**Example 4.5** Any Cu-semigroup of the form $\text{Lsc}(X, \mathbb{N})$ with $X$ a $T_1$-space is Lsc-like. Indeed, $\text{Lsc}(X, \mathbb{N})$ is clearly a complete distributively lattice ordered semigroup.

Furthermore, the maximal idempotent elements $s < \infty$ are of the form $s = \infty\chi_{X\setminus\{x\}}$, and given any pair of elements $\infty f = \infty\chi_{\text{supp}(f)}$ and $\infty g = \infty\chi_{\text{supp}(g)}$, we know that $\infty\chi_{\text{supp}(f)} \leq \infty\chi_{\text{supp}(g)}$ if and only if $\text{supp}(f) \subseteq \text{supp}(g)$. That is, if and only if for every $x \in X$ such that $\text{supp}(g) \subseteq X\setminus\{x\}$ we have $\text{supp}(f) \subseteq X\setminus\{x\}$. This shows that $\text{Lsc}(X, \mathbb{N})$ satisfies (C1).

To see (C2), consider the subset $\downarrow 1$. One can see that the order is topological, and it clearly generates a semigroup that is dense in $\text{Lsc}(X, \mathbb{N})$.

Recall that by an *order unit* we mean an element $e$ such that $x \leq \infty e$ for every $x \in S$.

**Remark 4.6** Let $S$ be an Lsc-like Cu-semigroup. Since the semigroup generated by $\downarrow e$ is dense in $S$, $e$ is an order unit.

Furthermore, let $f$ be another order unit, and let $(f_m)_m$ be an increasing sequence of ordered finite sums of elements below $e$ such that $f = \sup_m f_m$ (the sums can be assumed to be ordered by Remark 4.3). Write $f_m = \sum_{i=1}^{r_m} g_{i,m}$ with $e \geq g_{i,m} \geq g_{i+1,m}$ for each $i < r_m$.

Moreover, consider a $\ll$-increasing $(e_n)_n$ whose supremum is $e$.

Then, since $e \leq \infty f$, one has that for every $n \in \mathbb{N}$ there exist some $k, m \in \mathbb{N}$ with $e_n \leq k f_m = \sum_{i=1}^{r_m} k g_{i,m}$. Thus, since the order in $\downarrow e$ is topological and $\sum_{i=1}^{r_m} k g_{i,m}$ is an ordered sum of elements below $e$ (the first $k$ greatest elements are $g_{1,m}$, the next $k$ are $g_{2,m}$, etc.), it follows that $e_n \leq g_{1,m} \leq f$.

This shows $e \leq f$ and, consequently, $e$ is the least order unit of $S$.

**4.7** Note that an Lsc-like Cu-semigroup $S$ has no maximal idempotent elements $x < \infty$ if and only if $S = \{0\}$. Indeed, if $S$ has no maximal idempotents, then every idempotent $z$ satisfies $z \leq 0$ by condition (C1). This implies that every idempotent element is zero and, since for every $s \in S$ the element $\infty s$ is idempotent and $\infty s \geq s$, we get $s = 0$ for every $s$, as desired. The converse is trivial.

Similarly, an element $s \in S$ satisfies $\infty s = \infty$ if and only if there are no maximal idempotents $x < \infty$ with $\infty s \leq x$.

**Lemma 4.8** Let $S$ be an Lsc-like Cu-semigroup, and let $y \leq ne$, where $e$ is the least order unit of $S$ and $n \in \mathbb{N}$. Then, $y$ can be written as an ordered sum of at most $n$ nonzero terms in $\downarrow e$.

**Proof** Write $y$ as $y = \sup_r y_r$, where each $y_r$ is a finite ordered sum of elements in $\downarrow e$ (see Lemma 4.2 and Remark 4.3).
As \( y_r \leq n e \) for each \( r \) and \( \downarrow e \) is topological, each \( y_r \) has at most \( n \) nonzero summands.

Furthermore, since \( y_r \leq y_{r+1} \), the \( k \)th summands of \( (y_r)_r \) form an increasing sequence for every \( k \).

Taking their suprema, this shows that \( y \) is an ordered sum of at most \( n \) elements in \( \downarrow e \).

**Corollary 4.9** Let \( S \) be an Lsc-like Cu-semigroup. Given \( y_1, \ldots, y_n \in S \) such that \( y_1 + \cdots + y_n \geq s \) for some \( s \leq e \), we have \( y_1 \lor \cdots \lor y_n \geq s \).

**Proof** Take any \( s' < s \), and let \( y'_1 \ll y_1 \) be such that \( y'_1 + \cdots + y'_n \geq s' \). Since \( 0 \leq y_1 + \cdots + y_n \gg y'_1 + \cdots + y'_n \), there exists some \( k \in \mathbb{N} \) such that \( k e \geq y'_1 + \cdots + y'_n \).

By Lemma 4.8, each \( y'_i \) can be written as a finite ordered sum of elements below \( e \). That is, we can write \( y'_i = \sum_{j=1}^{k} y_{j,i} \) with \( e \geq y_{j,i} \geq y_{j+1,i} \) for each \( j < k \).

Applying Lemma 4.2 iteratively, we have that \( y'_1 + \cdots + y'_n \) can be written as a finite ordered sum of elements below \( e \), and that the greatest summand of the sum is \( y_1, y \lor \cdots \lor y_1 \).

Since the order in \( \downarrow e \) is topological, we get \( y_1, y \lor \cdots \lor y_1 \geq s' \). It follows that
\[
s' \leq y_1, y \lor \cdots \lor y_1 \leq y_1, y \lor \cdots \lor y_n \leq y_1 \lor \cdots \lor y_n.
\]
Since this holds for every \( s' < s \), we have \( s \leq y_1 \lor \cdots \lor y_n \) as required.

The next two lemmas will be of particular importance when working with the induced topology of an Lsc-like Cu-semigroup (see Paragraph 5.1). In particular, Lemma 4.11 gives an alternative version of (C1) in Definition 4.3.

**Lemma 4.10** Let \( S \) be an Lsc-like Cu-semigroup with least order unit \( e \), and let \( x \in S \). Then, \( 0 \infty x = \infty (x \wedge e) \) and \( (0 \infty x) \wedge e = x \wedge e \).

In particular, if \( x < \infty \) is a maximal idempotent, we must have \( x \wedge e \neq e \).

**Proof** By Lemma 4.8 and condition (C2) in Definition 4.3, \( x \) can be written as the supremum of finite ordered sums of elements in \( \downarrow e \). Moreover, using that the order in \( \downarrow e \) is topological, it follows that the sequence formed by the greatest element of each ordered sum is increasing. Let \( x' \) be the supremum of such a sequence.

Then, it is clear that \( 0 \infty x = 0 \infty x' \) and \( x \wedge e = x' \). Therefore, we have
\[
0 \infty x = 0 \infty x' = 0 \infty (x \wedge e), \quad \text{and} \quad (0 \infty x) \wedge e = (0 \infty x') \wedge e = x' = x \wedge e,
\]
as desired.

**Lemma 4.11** Let \( S \) be an Lsc-like Cu-semigroup with least order unit \( e \). Let \( y, z \leq e \). If \( z \leq x \) for every maximal element \( x < e \) such that \( y \leq x \), then \( z \leq y \).

**Proof** If there are no maximal idempotent elements \( x < \infty \), we know that \( S = \{0\} \), so we may assume otherwise.

We claim that the maximal idempotent elements \( x < \infty \) are precisely the elements \( 0 \infty s \) with \( s < e \) maximal.

To see this, let \( x < \infty \) be a maximal idempotent and take \( s = x \wedge e \). By Lemma 4.10, we have \( 0 \infty s = 0 \infty (x \wedge e) = 0 \infty x = x \). Let us check that \( s \) is maximal, and thus take \( t \in S \) with \( s \leq t \leq e \).
By maximality of \( x \), we either have \( \infty t = \infty \) or \( \infty t = x \). Using Lemma 4.10 once again, it follows that we have \( t = (\infty t) \land e = \infty \land e = e \) or \( t = x \land e = s \), as required.

Conversely, let \( s < e \) be maximal and consider the element \( \infty s = \sup_n ns \). Let \((s_k)_k\) be a \( \ll \)-increasing sequence with supremum \( s \).

Given an idempotent element \( x \) such that \( \infty s \leq x \leq \infty \), we know by Lemma 4.8 that there exists a \( \ll \)-increasing sequence \((x_m)_m\) with supremum \( x \) such that each \( x_m \) can be written as a finite increasing sum of elements in \( \downarrow e \).

Let \((m_k)_k\) be an increasing sequence of integers such that \( ks_k \leq x_{m_k} \) for every \( k \).

Using that \( \downarrow e \) is topological, each \( s_k \) is less than or equal to each of the first \( k \) summands of \( x_{m_k} \). As in the proof of Lemma 4.10, note that the largest summands of each sum form an increasing sequence of elements below \( e \). Letting \( x' \) be the supremum of this sequence, we get \( s \leq x' \leq e \).

By maximality of \( s \), we either have \( s = x' \) or \( x' = e \). Thus, it follows from Lemma 4.10 that \( \infty s = x \) or \( x = \infty x' = \infty \). This proves that \( \infty s \) is a maximal idempotent, as desired.

A similar argument shows that for any \( y, z \leq e \) we have \( y \leq z \leq e \) if and only if \( \infty y \leq \infty z \). Consequently, if \( z \leq s \) for every maximal element \( s < e \) such that \( y \leq s \), we know that \( \infty z \leq \infty y \). Taking the infimum with \( e \), one gets \( z = (\infty z) \land e \leq (\infty y) \land e = y \), as required.

**Remark 4.12** Note that the previous lemma implies that \( y < e \) if and only if there exists a maximal element \( x < e \) with \( y \leq x \).

Recall that a complete lattice \((P, \leq)\) is said to be a *complete Heyting algebra* if, for every element \( s \in P \) and every subset \( T \subseteq P \), the following holds:

\[
 s \land (\lor_{t \in T} t) = \lor_{t \in T}(s \land t).
\]

**Lemma 4.13** Let \( S \) be an Lsc-like Cu-semigroup with least order unit \( e \). The subset \( \downarrow e \subseteq S \) is a complete Heyting algebra.

**Proof** By definition, we have to see that for every subset \( T \subseteq \downarrow e \), one has \( s \land (\lor_{t \in T} t) = \lor_{t \in T}(s \land t) \) for every \( s \in \downarrow e \). Thus, let \( x < e \) be maximal with \( x \geq \lor_{t \in T}(s \land t) \), which happens if and only if \( x \geq s \land t \) for every \( t \in T \). Since \( x \neq e \) and \( x \lor x \geq x \lor (s \land t) = (x \lor s) \land (x \lor t) \), we either have \( x \lor s = x \) or \( x \lor t = x \) (since otherwise both of these unions would be equal to \( e \) and then \( x \geq e \), a contradiction).

If \( x = x \lor s \geq s \), we have \( x \geq s \land (\lor_{t \in T} t) \). Else, \( x \geq t \) for each \( t \in T \), so we also get \( x \geq s \land (\lor_{t \in T} t) \).

By Lemma 4.11, this shows that \( s \land (\lor_{t \in T} t) \leq \lor_{t \in T}(s \land t) \).

The other inequality holds in any lattice.

Let \( S \) be a Cu-semigroup, and let \( y, z \in S \) be a pair of elements such that \( y \leq z \). We say that an element \( u \) is the *almost complement of \( z \) by \( y \)* if, for every \( x \in S \), we have \( x + y \leq z \) if and only if \( x \leq u \). If it exists, the almost complement is unique, and we denote by \( y \backslash z \).

Given a Cu-semigroup \( S \) where every pair of elements has a supremum, we say that \( S \) is *sup-semilattice ordered* if for every \( x, y, z \in S \) we have \( x + (y \lor z) = (x + y) \lor (x + z) \).
A natural question to ask about Lsc-like Cu-semigroups is if they satisfy such a property. This is indeed the case.

**Lemma 4.14** Let $S$ be an Lsc-like Cu-semigroup, and let $e$ be its least order unit. Given $y \leq z$ with $y \leq ne$ for some $n$, the almost complement $y \setminus z$ exists.

In particular, given $x, y, z \in S$, we have $x + (y \lor z) = (x + y) \lor (x + z)$.

**Proof** We will construct our almost complement $y \setminus z$ in three steps.

**Step 1.** Let $e$ be the least order unit of $S$ and assume that $y \leq z \leq e$. Then, consider the subset $T := \{x \in S \mid y + x \leq z\}$ and, since arbitrary suprema exist in $S$, we define

$$y \setminus z := \lor\{x \in S \mid y + x \leq z\} = \lor_{x \in T} x.$$  

Furthermore, note that, for $x \in T$, $y + x = (y \lor x) + (y \land x)$ and, as $z \leq e$ and $\lor e$ is topological, we have $y \lor x \leq z$ and $y \land x = 0$.

Using that $\lor e$ is a complete Heyting algebra, we get $(\lor_{x \in T} x) \lor y = \lor_{x \in T} (x \lor y) \leq z$ and $(\lor_{x \in T} x) \land y = \lor_{x \in T} (x \land y) = 0$. Consequently, we get

$$y + (y \setminus z) = y \lor (y \setminus z) \lor (y \setminus z) = y \lor (y \setminus z) \leq z.$$  

This shows that $x \leq y \setminus z$ if and only if $y + x \leq z$.

**Step 2.** Now, given any pair of elements $y \leq z \leq ne$ for some $n \in \mathbb{N}$, write them as finite ordered sums of elements in $\lor e$, say $y = \sum_{i=1}^n y_i$ and $z = \sum_{j=1}^n z_j$.

Since $\lor e$ is topological, $y_i \leq z_i \leq e$ for every $i$, so we can define $y_i \setminus z_i := \sum_{i=1}^n y_i \setminus z_i$.

Note that, given any element $x$ such that $y + x \leq z \leq ne$, we have $x \leq ne$ and so, by Lemma 4.8, $x$ can be written as a finite ordered sum $\sum_{i=1}^n x_i$ with $x_i \leq e$ for every $i$.

Moreover, we also have

$$\sum_{i=1}^n y_i \lor x_i \leq y + x \leq \sum_{i=1}^n z_i$$

and so $y_i \lor x_i \leq z_i$, which happens if and only if $x_i \leq y_i \setminus z_i$.

This shows that $x + y \leq z \leq ne$ for some $n$ if and only if $x \leq y \setminus z$.

**Step 3.** Given $y \leq z \land ne$ for some $n$, we have that $y \leq z \land me$ for every $m \geq n$.

Thus, we can consider the element $y \setminus (z \land me)$. Furthermore, it is easy to check that $y \setminus (z \land me) \leq y \setminus (z \land (m + 1)e)$ for every $m$. We define $y \setminus z := \sup_m y \setminus (z \land me)$, which has the required property.

Finally, to see that $x + (y \lor z) = (x + y) \lor (x + z)$ for any given $x, y, z$, note that “$\geq$” is clear. To prove “$\leq$,” let $x' \ll x$ and let $s$ be such that $x' + (y \lor z) \leq s$.

Since $x' \ll x \leq \infty e$, there exists some $n$ with $x' \leq ne$, so we can consider the element $x' \setminus s$.

Thus, we know that $x' + (y \lor z) \leq s$ holds if and only if $y \lor z \leq x' \setminus s$, which in turn holds if and only if $x' + y, x' + z \leq s$. Consequently, we get

$$x' + (y \lor z) = (x' + y) \lor (x' + z).$$

Since the equality holds for every $x' \ll x$, it also holds for $x$. \hfill \Box

**Lemma 4.15** Let $S$ be an Lsc-like Cu-semigroup with least order unit $e$, and let $x, y, z \leq e$ with $x + y \leq x + z$. Then, $y \leq z$. 

Proof First, note that \( y \leq x \lor y \leq x \lor z \). Indeed, since \( x, y, z \leq e \) and we know that \( x + y = (x \lor y) + (x \land y) \) and \( x + z = (x \lor z) + (x \land z) \), we have that \( (x \lor y) + (x \land y) \leq (x \lor z) + (x \land z) \).

Since the right- and left-hand sides of the previous inequality are ordered sums of elements below \( e \), we can use that the order in \( \downarrow e \) is topological to get \( x \lor y \leq x \lor z \), as desired.

Therefore, the sum \((z \lor x) + y\) is ordered, as we have \( y \leq z \lor x \).

Using Lemma 4.14 at the first and third steps, the inequality \( x + y \leq x + z \) at the second step, and that \( S \) is distributively lattice ordered at the last step, one obtains
\[
(z \lor x) + y = (y + z) \lor (y + x) \leq (y + z) \lor (z + x) = z + (y \lor x) = (z \lor y) \lor x.
\]

Using once again that the order in \( \downarrow e \) is topological, it follows that \( y \leq z \land (y \lor x) \leq z \), as required.

\[ \qed \]

By extending the previous proof, one can check that whenever \( y, z \in S \) and \( x \leq ne \) for some \( n \), the same cancelation property holds. In particular, it follows that every Lsc-like Cu-semigroup has weak cancelation.

Remark 4.16 Using the previous form of cancelation and the equality
\[
((x + y) \lor (x + z)) + (x + (y \land z)) = x + y + x + z
\]
\[
= ((x + y) \lor (x + z)) + ((x + y) \land (x + z)),
\]
one can check that \( S \) is inf-semilattice ordered, that is to say, \( x + y \land z = (x + y) \land (x + z) \) for every \( x, y, z \) (see [1]). For elements \( x, y, z \leq ne \) for some \( n \), note that this simply follows by canceling the term \((x + y) \lor (x + z))\).

Using that every element in \( S \) is the supremum of a \( \ll \)-increasing sequence of finite sums of elements below \( e \), one can then prove that the equality \( x + (y \land z) = (x + y) \land (x + z) \) is always satisfied.

As one might expect, having a topological order also affects the way below relation.

Lemma 4.17 Let \( S \) be an Lsc-like Cu-semigroup with least order unit \( e \). If \( y, z, y', z' \leq e \) are such that
\[
y + z \ll y' + z'
\]
with \( y \geq z \) and \( y' \geq z' \), we have that \( y \ll y' \) and \( z \ll z' \).

The same holds for any pair of finite sums (i.e., with more than two summands).

Proof Write \( y' = \sup_n y'_n \) and \( z' = \sup_n z'_n \) with \( (y'_n) \) and \( (z'_n) \) \( \ll \)-increasing. Since \( y + z \ll y' + z' \), we have \( y + z \leq y'_n + z'_n \) for some \( n \).

Moreover, as we have that \( z'_n \ll z' \leq y' \), there exists a \( k \) such that \( z'_n \leq y'_k \). This implies that
\[
y + z \leq y'_n + z'_n \leq y'_{\max\{n,k\}} + z'_n,
\]
and, since the order is topological, we obtain \( y \leq y'_{\max\{n,k\}} \ll y' \) and \( z \leq z' \ll z' \).

Even though the following lemma is probably well known, we prove it here for the sake of completeness. For second countable finite-dimensional compact Hausdorff
spaces, it follows from a much more general result that $\text{Lsc}(X, \mathbb{N}) \in \text{Cu}$ (see [2, Theorem 5.15]). As a consequence of Lemma 4.18, we will have that $\text{Lsc}(X, \mathbb{N})$ is a Cu-semigroup whenever $X$ is a compact metric space (see Corollary 4.19). Recently, this result has been further generalized by Elliott and Im (see [12, Proposition 1.16]).

Given $f \in \text{Lsc}(X, \mathbb{N})$ and $n \in \mathbb{N}$, we write $\{f \geq n\}$ to denote the open set $f^{-1}([n, \infty])$. For an open set $U \subseteq X$, we denote by $\chi_U$ the indicator function of $U$.

**Lemma 4.18**  For any topological space $X$ and any pair $f, g \in \text{Lsc}(X, \mathbb{N})$, one has $f \ll g$ if and only if $\chi_{\{f \geq n\}} \ll \chi_{\{g \geq n\}}$ for every $n$ and $\sup(f) < \infty$.

**Proof**  First, let us assume that $f \ll g$. Fix $n \in \mathbb{N}$ and consider an increasing sequence $(h_k)_k$ such that

$$\chi_{\{g \geq n\}} \leq \sup_k h_k,$$

which happens if and only if $\chi_{\{g \geq n\}} \leq \chi_{\cup_k \text{supp}(h_k)}$.

Define the increasing sequence of functions

$$G_k := (n - 1) + \chi_{\text{supp}(h_k)} \sum_{r=0}^{\infty} \chi_{\{g \geq n+r\}},$$

and note that $g \leq \sup_k G_k$.

Since $f \ll g$, we get that $f \leq G_k$ for some $k$ and, consequently,

$$\{f \geq n\} \subseteq \{G_k \geq n\} = \text{supp}(h_k) \cap \{g \geq n\} \subseteq \text{supp}(h_k).$$

This in turn implies $\chi_{\{f \geq n\}} \leq \chi_{\text{supp}(h_k)} \leq h_k$, so it follows that $\chi_{\{f \geq n\}} \ll \chi_{\{g \geq n\}}$. That $\sup(f) < \infty$ is clear.

Conversely, if $\sup(f) < \infty$, we know that $f = \sum_{i=1}^{m} \chi_{\{f \geq i\}}$, for some $m < \infty$. Furthermore, given an increasing sequence $(h_k)_k$ with $g \leq \sup_k h_k$, it follows that

$$\{g \geq n\} \subseteq \bigcup_k \{h_k \geq n\},$$

so $\chi_{\{g \geq n\}} \leq \sup_k \chi_{\{h_k \geq n\}}$. Since $\chi_{\{f \geq n\}} \ll \chi_{\{g \geq n\}}$ for every $n$, we get that for each $i$ there exists an integer $k_i$ with

$$\chi_{\{f \geq i\}} \leq \chi_{\{h_{k_i} \geq i\}}.$$

Taking $k = \max_{i=1,\ldots,m} \{k_i\}$, we have

$$f \leq \sum_{i=1}^{m} \chi_{\{h_{k_i} \geq i\}} \leq h_k,$$

as desired.  

**Corollary 4.19**  Let $X$ be a compact metric space. Then $\text{Lsc}(X, \mathbb{N})$ is a Cu-semigroup with pointwise order and addition.

**Proof**  Axioms (O1) and (O4) are always satisfied in $\text{Lsc}(X, \mathbb{N})$, so we only need to prove (O2) and (O3).
Let $U$, $V$ be open subsets of $X$, and note that $\chi_U \ll \chi_V$ if and only if $U$ is compactly contained in $V$. Indeed, if $\chi_U \ll \chi_V$, we can write $V$ as a countable increasing union of open sets $V_n$ such that $V_n$ is compactly contained in $V_{n+1}$ for every $n$. Thus, one gets $\chi_U \ll \sup_n \chi_{V_n}$, which implies that $U$ is contained in $V_n$ for some $n$. Conversely, if $\chi_U \ll \chi_V$, we can write $V$ as a countable increasing union of open sets $V_n$ such that $V_n$ is compactly contained in $V_{n+1}$ for every $n$. Thus, one gets $\chi_U \ll \sup_n \chi_{V_n}$, which implies that $U$ is contained in $V_n$ for some $n$. This shows $\chi_U \ll \chi_V$, as required.

In particular, it follows that every indicator function can be written as the supremum of a $\ll$-increasing sequence. Since every element in $S = \text{Lsc}(X, \mathbb{N})$ is the supremum of finite sums of indicator functions, one can check that $S$ satisfies (O2).

Now, let $f' \ll f$ and $g' \ll g$ in $S$, which by Lemma 4.18 implies that $\sup(f), \sup(g) < m \leq \infty$ and $\{f' \geq n\}, \{g' \geq n\}$ are compactly contained in $\{f \geq n\}, \{g \geq n\}$, respectively. Thus, we have

$$\bigcup_{k=0}^{m} (\{f' \geq k\} \cap \{g' \geq n - k\}) \subseteq \bigcup_{k=0}^{m} (\{f \geq k\} \cap \{g \geq n - k\})$$

for every $n \leq \sup(f) + \sup(g)$, where note that the left-hand side is equal to $\{f' + g' \geq n\}$ and the right-hand side is contained in $\{f + g \geq n\}$. By Lemma 4.18, we have $f' + g' \ll f + g$, which shows that $S$ satisfies (O3).

$\square$

5 The topological space of an Lsc-like Cu-semigroup

In this section, we associate with each Lsc-like Cu-semigroup $S$ a topological space $X_S$. In Proposition 5.3, we prove some of the properties that such a topological space must satisfy and, using these, we show that $\text{Lsc}(X, \mathbb{N})$ is always a Cu-semigroup (see Theorem 5.9). In Theorem 6.4, we will see that $S$ and $\text{Lsc}(X_S, \mathbb{N})$ are in fact Cu-isomorphic.

We also introduce notions for Cu-semigroups that have a topological equivalent when the semigroup is Lsc-like. More explicitly, given an Lsc-like Cu-semigroup $S$, we characterize when $X_S$ is second countable, normal, and metric in terms of algebraic properties of $S$ (see Proposition 5.6).

5.1 Let $S$ be an Lsc-like Cu-semigroup with least order unit $e$. The topological space $X_S$ of $S$ is defined as

$$X_S := \{ x \in S \mid x < e \text{ maximal} \},$$

with closed subsets

$$C_y := \{ x \in X_S \mid x \geq y \}, \quad y \leq e.$$

We check that this is indeed a topology for $X_S$.

Lemma 5.2 Let $S$ be an Lsc-like Cu-semigroup. Then, $\{ X_S \setminus C_y \mid y \leq e \}$ is a $T_1$-topology for $X_S$.

Proof First, note that $C_0 = X_S$ and that $C_e = \emptyset$. Moreover, $C_x = \{ x \}$ for every $x \in X_S$. Thus, our topology is $T_1$. 

To see that arbitrary intersections of $C_y$'s are of the form $C_z$ for some $z \leq e$, simply note that
\[
\bigcap_i C_{y_i} = C_{\vee_i y_i}.
\]

Furthermore, one also has that
\[
\bigcup_{i=1}^n C_{y_i} = C_{\wedge^n_{i=1} y_i}.
\]

Indeed, given $x \in X_S$ with $x \geq \wedge^n_{i=1} y_i$, we have
\[
x = x \vee x \geq x \vee (\wedge^n_{i=1} y_i) = (x \vee y_1) \wedge \cdots \wedge (x \vee y_n).
\]

Since $x$ is maximal, for each $i$, we either have $x \vee y_i = x$ or $x \vee y_i = e$.

However, note that the previous inequality implies that we cannot have $x \vee y_i = e$ for every $i$, so $x = x \vee y_j \geq y_j$ for some $j$.

The other inclusion is clear. ■

Retaining the above notation, for every $y \leq e$, we will denote by $U_y$ its associated open subset. That is, $U_y = X \setminus C_y$.

We list some properties of these sets. Recall that, for every pair of elements $y \leq z \leq e$, the element $y \setminus z$ denotes the almost complement of $z$ by $y$, as constructed in Lemma 4.14.

**Proposition 5.3** Let $S$ be an Lsc-like $Cu$-semigroup with least order unit $e$. Then:

(i) For every $y, z \leq e$, $C_y \subseteq C_z$ if and only if $y \geq z$.

(ii) For every $y \leq e$, $U_y = \{ x \in X_S \mid y \vee x = e \}$.

(iii) Given $y, z \leq e$ such that $U_y \subseteq C_z$, we have $y \wedge z = 0$.

(iv) The closure of $U_y$, denoted by $\overline{U_y}$, is $C_{y \setminus e}$ for every $y \leq e$.

(v) Given $y \leq e$, we have $\text{Int}(C_y) = X_S \setminus (X_S \setminus C_y) = U_{y \setminus e}$, where $\text{Int}(C_y)$ stands for the interior of $C_y$.

(vi) For every $y, z \leq e$, $C_y \subseteq U_z$ if and only if $y \vee z = e$.

**Proof** To see (i), recall that, by definition, $C_y \subseteq C_z$ if and only if $x \geq z$ for every $x \geq y$ with $x < e$ maximal. Using Lemma 4.11, we see that this is equivalent to $y \geq z$.

For (ii), let $y \leq e$ and take $x < e$ be maximal. Thus, we have $x \leq y \vee x \leq e$. Since $x$ is maximal, we either have $x = y \vee x \geq y$ (i.e., $x \in C_y$) or $y \vee x = e$. Thus, $U_y = \{ x \in X_S : y \vee x = e \}$.

To prove (iii), let us assume, for the sake of contradiction, that $y \wedge z \neq 0$. Then, $U_{y \wedge z}$ is nonempty and we can consider a maximal element $x \in U_{y \wedge z}$.

By (ii), we have $x \vee (y \wedge z) = e$. Thus, one gets $x \vee y = e$ and, consequently, $x \geq z$ from our assumption that $U_y \subseteq C_z$. However, we also have
\[
e = x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) = e \wedge x = x,
\]
which is a contradiction, as required.

Let us now prove (iv) and, consequently, (v). First, note that, if $y \vee x = e$, we have
\[
y + x = (y \vee x) + (y \wedge x) \geq e \geq y \setminus e + y.
\]
Canceling $y$ (see Lemma 4.15), we have that $x \geq y \setminus e$. This shows $U_y \subseteq C_{y \setminus e}$.

Conversely, let $z$ be such that $U_y \subseteq C_z$. By (iii), we know that this implies $y \wedge z = 0$ and, consequently, $y + z \leq e$. Thus, $z \leq y \setminus e$ or, equivalently, $C_{y \setminus e} \subseteq C_z$.

Finally, for (vi), assume first that $C_y \subseteq U_z$. Furthermore, assume for the sake of contradiction that $y \vee z \neq e$. Then, there exists $x \in X_S$ with $x \geq y \vee z$. This implies $x \geq y$ and, consequently, $x \vee z = e$ from $C_y \subseteq U_z$. However, we have

$$x = x \vee y \vee z \geq x \vee z = e,$$

which is a contradiction.

Conversely, if $y \vee z = e$, take $x \geq y$, which implies that $x \vee z \geq y \vee z = e$. In particular, $x \in U_z$.

**Example 5.4** Let $X$ be a $T_1$ topological space. Recall from Example 4.5 that $S = \text{Lsc}(X, \overline{\mathbb{N}})$ is an Lsc-like Cu-semigroup with least order unit 1. Then, the topological space of $S$ is homeomorphic to $X$.

Indeed, note that the maximal elements below 1 are the characteristic functions $\chi_{X \setminus \{x\}}$. Thus, we have

$$X_S = \{ \chi_{X \setminus \{x\}} \mid x \in X \}$$

and

$$U_{\chi_x} = X_S \setminus C_{\chi_x} = X_S \setminus \{ \chi_{X \setminus \{x\}} \mid \chi_{X \setminus \{x\}} \geq \chi_u \}$$

$$= X_S \setminus \{ \chi_{X \setminus \{x\}} \mid x \in X \setminus \{x\} \} = \{ \chi_{X \setminus \{x\}} \mid x \in \{x\} \}$$

for every open subset $\{x\} \subseteq X$.

It should now be clear that $\varphi: X_S \to X$ defined as $\chi_{X \setminus \{x\}} \mapsto x$ is a homeomorphism between $X$ and $X_S$.

The following characterizes compact containment under certain conditions.

**Lemma 5.5** Let $S$ be an Lsc-like Cu-semigroup with least order unit $e$. Assume that $e$ is compact and that $S$ satisfies (O5). Then $X_S$ is normal.

Furthermore, given $y, z \leq e$, we have $U_y \subseteq U_z$ if and only if $y \ll z$.

**Proof** Let $x, y < e$ be two elements such that $C_x \cap C_y = C_e = \emptyset$. In terms of the elements in $S$, this is equivalent to $x + y \geq e \gg e$. Then, we can take $x' \ll x$ and $y' \ll y$ such that $x' + y' \gg e$.

Using (O5), there exist $c, d \leq e$ such that

$$x' + c \leq e \leq x + c, \quad y' + d \leq e \leq y + d.$$

Consequently, we also have $x' + c + y' + d \leq e + e$ with $x' + y' \gg e$. Since every Lsc-like Cu-semigroup has weak cancelation, it follows that $c + d \leq e$.

Since our order is topological, we get $c \wedge d = 0$, $x \vee c = e$, and $y \vee d = e$.

Using the properties listed in Proposition 5.3, the previous inequalities imply that $C_x \subseteq U_c$, $C_y \subseteq U_d$, and $U_c \cap U_d = U_{c \wedge d} = \emptyset$. Thus, $X_S$ is normal.

Now, let $y, z \leq e$ and assume that $U_y \subseteq U_z$, which by (iv) in Proposition 5.3 happens if and only if $(y \setminus e) \vee z = e$. Furthermore, since $y \setminus e + z \geq e$, we have $y \setminus e + z \geq e \gg e \geq y \setminus e + y$. As elements below $e$ have cancelation (see Lemma 4.15), one gets $z \gg y$. 

Conversely, assume that \( e \geq z \gg y \). Then, by (O5), there exists an element \( x \) such that \( y + x \leq e \) and \( e \leq z + x \).

From the first inequality, it follows that \( x \leq y \land e \), so \( e \leq z + x \leq y \land e \) as required.

In Proposition 5.6, we study notions for Cu-semigroups that have a topological equivalent when the semigroup is Lsc-like. Recall from Paragraph 2.3 that a Cu-semigroup is said to be \( \text{countably based} \) if it has a countable sup-dense subset.

**Definition 5.1** We say that an inf-semilattice ordered Cu-semigroup \( S \) is normal if there exists an order unit \( z \in S \) such that, whenever \( x + y \geq z \) for some \( x, y \in S \), there exist \( s, t \in S \) with

\[
x + s \geq z, \quad y + t \geq z, \quad s \land t = 0.
\]

**Proposition 5.6** Let \( S \) be an Lsc-like Cu-semigroup, and let \( X_S \) be its associated topological space. Then:

(i) \( X_S \) is second countable if and only if \( S \) is countably based.

(ii) \( X_S \) is countably compact if and only if \( S \) has a compact order unit.

(iii) \( X_S \) is normal if and only if \( S \) is normal.

(iv) \( X_S \) is a metric space whenever \( S \) is countably based and normal.

(v) \( X_S \) is a compact metric space whenever \( S \) is countably based, has a compact order unit, and satisfies (O5).

**Proof** Let \( e \in S \) be the least order unit of \( S \). To show (i), assume first that \( S \) is countably based with a countable basis \( B \), and let \( \sum'(\downarrow e) \) denote the set of finite sums of elements in \( \downarrow e \). Naturally, the set \( B' := B \cap \sum'(\downarrow e) \) is also a countable basis for \( S \).

Given an open set \( U_y \) with \( y \leq e \), write \( y = \sup_n y_n \) with \( y_n \in B' \). We have \( \cup_n U_{y_n} = U_y \), and so \( X_S \) is second countable.

Conversely, assume that \( X_S \) is second countable with basis \( C = \{ U_{z_n} \}_n \). Note that the family \( C' \) consisting of all the finite unions of sets in \( C \) is also countable.

Then, any open subset \( U_y \) can be written as the countable union of increasing open subsets \( U_{z_n}' \) of \( C' \). We know that this is equivalent to \( y = \sup_n z_n \). This implies that the set \( \sum'(z_n')_n \) of finite sums from \( \{ z_n' \}_n \) is a countable basis for \( \sum'(\downarrow e) \) and, since \( \sum'(\downarrow e) \) is dense in \( S \), \( \sum'(z_n')_n \) is a countable basis for \( S \).

To prove (ii), note that it is easy to check that \( e \in S \) is compact if and only if \( X_S \) is countably compact. Therefore, we are left to prove that, if there exists a compact order unit in \( S \), then \( e \) must also be compact.

To see this, let \( p \) be a compact order unit in \( S \), which implies that \( p \leq ne \) for some \( n \in \mathbb{N} \). Since we know that \( p \) can be written as a finite ordered sum of elements below \( e \), there exist \( m \in \mathbb{N} \) and elements \( q_m \leq \cdots \leq q_1 \leq e \) such that \( p = q_1 + \cdots + q_m \).

By weak cancelation applied to \( q_1 + \cdots + q_m \leq q_1 + \cdots + q_m \), the element \( q_1 \) is compact and satisfies \( q_1 = \infty \). Thus, \( q_1 \) is a compact order unit with \( e \geq q_1 \). By minimality of \( e \), one gets \( e = q_1 \) compact as required.

Let us now show (iii). First, assume that \( S \) is normal, and let \( z \) be the associated order unit. Let \( C_x, C_y \) be closed subsets of \( X_S \) with \( x, y \leq e \), and recall that \( C_x, C_y \) are disjoint if and only if \( x \lor y = e \).
Since $e$ is an order unit, we have $\infty x + \infty y = \infty z$. Thus, we get $s, t$ such that $\infty x + s \geq z$, $\infty y + t \geq z$, and $s \land t = 0$. As $z \geq e$, we know by Corollary 4.9 that $\infty x \lor s \geq e$ and $\infty y \lor t \geq e$.

Since $x, y \leq e$, taking the infimum with $e$ and using Lemma 4.10, we have $e = (\infty x \land e) \lor (s \land e) = x \lor (s \land e), y \lor (t \land e) = e$, and $(s \land e) \lor (t \land e) = 0$. By (vi) in Proposition 5.3, it follows that $C_x \subseteq U_{\infty e}, C_y \subseteq U_{t \land e}$, and $U_{s \land e} \cap U_{t \land e} = \emptyset$. This implies that $X_S$ is normal.

Conversely, if $X_S$ is normal, it is easy to see that $S$ is normal by setting $z = e$ in the definition of normality.

To prove (iv), we have that $X_S$ is second countable and normal by (i) and (iii), and that $C_x = \{x\}$ for any $x \in X_S$. Thus, points are closed in our topology, so $X_S$ is Hausdorff. We can now use Urysohn’s metrization theorem to conclude that $X_S$ is metric (see, e.g., [16, Theorem 34.1]).

For (v), note that $e \ll e$. Thus, Lemma 5.5 implies that $X_S$ is normal. Following the arguments above, one gets that $X_S$ is metric. Moreover, we also know that $X_S$ is second countable and countably compact by (i) and (ii) above. Thus, $X_S$ is compact. \hfill \blacksquare

We will now show that $\operatorname{Lsc}(X_S, \mathbb{N})$ is a Cu-semigroup with the usual way-below relation for every Lsc-like Cu-semigroup $S$. Note that (O1) and (O4) are always satisfied, so we are left to prove (O2) and (O3).

**Lemma 5.7** Let $S$ be an Lsc-like Cu-semigroup with least order unit $e$. Given $y, z \leq e$, we have $\chi_{U_y} \ll \chi_{U_z}$ in $\operatorname{Lsc}(X_S, \mathbb{N})$ if and only if $y \ll z$ in $S$.

**Proof** Assume $y \ll z$, and let $(f_n)_n$ be an increasing sequence in $\operatorname{Lsc}(X_S, \mathbb{N})$ such that $\chi_{U_y} \leq \sup_n f_n$. In particular, note that this holds if and only if $\chi_{U_y} \leq \chi_{\cup_n \supp(f_n)}$ or, equivalently, if

$$\bigcap_n (X_S \setminus \supp(f_n)) \subseteq C_z.$$  

Denote by $z_n$ the elements in $\downarrow e$ with $C_z_n = X_S \setminus \supp(f_n)$. Given that $\supp(f_n) \subseteq \supp(f_{n+1})$, (i) in Proposition 5.3 implies that $z_n$ is increasing.

Using the proof of Lemma 5.2 in the first step, we can rewrite the previous inclusion as

$$C_{\sup_n(z_n)} = \bigcap_n C_{z_n} \subseteq C_z.$$  

Applying (i) in Proposition 5.3 once again, one gets $z \leq \sup_n(z_n)$ and, consequently, $y \leq z_n$ for some $n$. This implies that $U_y \subseteq U_{z_n}$ or, equivalently, that $\chi_{U_y} \leq \chi_{\sup_n(f_n)} \leq f_n$. This shows $\chi_{U_y} \ll \chi_{U_z}$.

Now, let $y, z \leq e$ be such that $\chi_{U_y} \ll \chi_{U_z}$, and consider an increasing sequence $(h_n)_n$ in $S$ such that $z \leq \sup_n(h_n)$. Note that, by taking $z \land h_n$ instead of $h_n$, we can assume $h_n \leq e$ for every $n$.

Applying again (the proof of) Lemma 5.2, one gets

$$\bigcap_n C_{h_n} = C_{\sup_n(h_n)} \subseteq C_z.$$
and, consequently, we have \( \sup_n \chi_{U_n} \geq \chi_{U_i} \) since \( \chi_{U_i} \ll \chi_{U_n} \); i.e., \( C_{U_n} \subseteq C_{U_i} \).

Using (i) in Proposition 5.3 one last time, one sees that \( y \leq h_n \) as required. 

**Proposition 5.8** Let \( S \) be a \( \text{Cu-semigroup} \). If \( S \) is \( \text{Lsc-like} \), then \( \text{Lsc}(X_S, \overline{N}) \) satisfies (O2).

**Proof** Take \( f \in \text{Lsc}(X_S, \overline{N}) \), and let \( (y_i)_i \) be the sequence on \( \downarrow e \) such that
\[
\{ f \geq i \} = U_{y_i},
\]
where recall that the sequence is decreasing as a consequence of (i) in Proposition 5.3. Since \( S \) satisfies (O2), we have \( y_i = \sup_n y_{i,n} \) with \( y_{i,n} \ll y_{i,n+1} \) for every \( n \).

For every fixed \( k \), we have
\[
y_1 \geq \cdots \geq y_k \gg y_{k,n} \text{ for all } n.
\]

Thus, for every \( i \), one can choose inductively \( n_{i,k} \) with \( k \geq i \) such that
\[
y_{i,n_{i,k}} \ll y_{i,n_{i,k+1}}, \quad \text{and} \quad y_{i,n_{i,k}} \geq \cdots \geq y_{k,n_{k,k}}.
\]

Indeed, we begin by setting \( n_{1,1} = 1 \) (i.e., \( y_{1,n_{1,1}} = y_{1,1} \)). Then, assuming that we have defined \( n_{i,k} \) for every \( i, k \leq m - 1 \) (and \( k \geq i \)) for some fixed \( m \), we set \( n_{m,m} = 1 \), that is, \( y_{m,n_{m,m}} = y_{m,1} \). We then set \( n_{m-1,m} \) large enough so that \( y_{m-1,n_{m-1,m}} \geq y_{m,n_{m,m}} \) and \( n_{m-1,m} \geq n_{m-1,m-1} \). Similarly, we set \( n_{m-2,m} \geq n_{m-2,m-1} \) such that \( y_{m-2,n_{m-2,m}} \geq y_{m-1,n_{m-1,m}} \) and define \( n_{i,m} \) for every \( i \leq m - 2 \) in the same fashion.

Now, consider the sums \( f_k = \sum_{i=1}^{k} \chi_{U_{y_{i,n_{i,k}}}} \), which are ordered by construction. Thus, one has \( U_{y_{i,n_{i,k}}} = \{ f_k \geq i \} \) for every \( i \). Since \( y_{i,n_{i,k}} \ll y_{i,n_{i,k+1}} \) for every \( i \), it follows from Lemmas 4.18 and 5.7 that
\[
f_k = \sum_{i=1}^{k} \chi_{U_{y_{i,n_{i,k}}}} \ll \sum_{i=1}^{k} \chi_{U_{y_{i,n_{i,k+1}}}} \leq f_{k+1}.
\]

It is now easy to check that \( \sup_k f_k = f \).

**Theorem 5.9** Let \( S \) be a \( \text{Cu-semigroup} \). If \( S \) is \( \text{Lsc-like} \), the monoid \( \text{Lsc}(X_S, \overline{N}) \) is a \( \text{Cu-semigroup} \).

**Proof** Note that the semigroup \( \text{Lsc}(X_S, \overline{N}) \) always satisfies (O1) and (O4). Moreover, we already know that (O2) is also satisfied by Proposition 5.8. Thus, we are left to prove (O3).

Let \( f \ll f' \) and \( g \ll g' \). By Lemma 4.18, this implies that
\[
\chi_{\{ f \geq i \}} \ll \chi_{\{ f' \geq i \}} \quad \text{and} \quad \chi_{\{ g \geq i \}} \ll \chi_{\{ g' \geq i \}}
\]
for every \( i \). We also know that there exists \( m < \infty \) such that \( \sup(f), \sup(g) \leq m \).

Let \( y_i, y'_i, z_i, \) and \( z'_i \) be elements in \( \downarrow e \) such that
\[
U_{y_i} = \{ f \geq i \}, \quad U_{y'_i} = \{ f' \geq i \}, \quad U_{z_i} = \{ g \geq i \}, \quad \text{and} \quad U_{z'_i} = \{ g' \geq i \}.
\]
By Lemma 5.7, we have $y_i \ll y'_i$ and $z_i \ll z'_i$ for every $i$ and, since $S$ satisfies (O3), one gets

$$\sum_{i=1}^{m} (y_i + z_i) \ll \sum_{i=1}^{m} (y'_i + z'_i).$$

By Lemma 4.2, these sums can be rewritten as

$$\sum_{j=0}^{2^m} \vee_{j=0}^{m} (y_j \wedge z_{i-j}) \ll \sum_{j=0}^{2^m} \vee_{j=0}^{m} (y'_j \wedge z'_{i-j})$$

and, since both the right- and left-hand sides of the previous inequality are ordered, we can use Lemma 4.17 to obtain

$$\vee_{j=0}^{m} (y_j \wedge z_{i-j}) \ll \vee_{j=0}^{m} (y'_j \wedge z'_{i-j})$$

for every $i$.

Note that

$$\{f + g \geq i\} = \bigcup_{j=0}^{m} (\{f \geq j\} \cap \{g \geq i - j\}),$$

so, by the equalities in the proof of Lemma 5.2, one gets

$$X_S \setminus \{f + g \geq i\} = \bigcap_{j=0}^{m} (X_S \setminus \{f \geq j\} \cup (X_S \setminus \{g \geq i - j\})) = \bigcap_{j=0}^{m} (C_{y_j \wedge z_{i-j}})$$

and, consequently, $\chi_{\{f + g \geq i\}} = \chi_{\vee_{j=0}^{m} (y_j \wedge z_{i-j})}$.

The same argument also shows that $\chi_{\{f + g' \geq i\}} \geq \chi_{\vee_{j=0}^{m} (y'_j \wedge z'_{i-j})}$.

By using Lemma 5.7, we get $\chi_{\{f + g \geq i\}} \ll \chi_{\{f' + g' \geq i\}}$ for every $i$. Lemma 4.18 then implies that $f + g \ll f' + g'$, as desired. 

6 An abstract characterization of Lsc(X,N)

In this section, we prove that every Lsc-like Cu-semigroup $S$ is Cu-isomorphic to the semigroup of lower-semicontinuous functions $\text{Lsc}(X_S, \overline{N})$ (see Theorem 6.4). To do so, we first define a map $\varphi' : \text{Lsc}(X_S, \overline{N})_{fs} \to S$, where $\text{Lsc}(X_S, \overline{N})_{fs}$ is the subsemigroup of functions with finite supremum.

We then extend this map to a Cu-morphism $\varphi : \text{Lsc}(X_S, \overline{N}) \to S$. Finally, the following (probably well-known) lemma will be used to complete the proof.

**Lemma 6.1** Let $S$, $H$ be Cu-semigroups, and let $\varphi : S \to H$ be a Cu-morphism such that:

(i) $\varphi$ is an order embedding on a basis of $S$.
(ii) $\varphi(S)$ is a basis for $H$.

Then, $\varphi$ is a Cu-isomorphism

**Proof** It is easy to see that $\varphi$ is a global order embedding.
To prove surjectivity, let \( h \in H \). Since \( \varphi(S) \) is a basis for \( H \), we can write \( h = \sup_n \varphi(s_n) \) for some \( s_n \in S \). Furthermore, as we know that \( \varphi \) is an order embedding, the sequence \((s_n)_n\) is increasing in \( S \), so \( \sup_n \varphi(s_n) = \varphi(s) \) for \( s = \sup_n s_n \).

**Definition 6.1** Let \( S \) be an Lsc-like Cu-semigroup. Given \( f \in \text{Lsc}(X_S, \mathbb{N})_{f_S} \), there exists \( m < \infty \) such that we can write

\[
f = \sum_{i=1}^{m} \chi_{\{f \geq i\}}.
\]

We define the map \( \varphi' : \text{Lsc}(X_S, \mathbb{N})_{f_S} \to S \) as \( \varphi'(f) = \sum_{i=1}^{m} z_i \), where \( \{f \geq i\} = U_{z_i} \).

**Lemma 6.2** \( \varphi' \) is a positively ordered monoid morphism and an order embedding.

**Proof** Let \( f = \sum_{i=1}^{m} \chi_{U_i} \) as above and take \( g = \sum_{j=1}^{n} \chi_{U_j} \). We will first prove by induction on \( m \) that \( \varphi'(f + g) = \varphi'(f) + \varphi'(g) \).

For \( m = 1 \), \( f \) is simply \( \chi_{U_i} \) for some open subset \( U_i \). Since \( \text{Lsc}(X_S, \mathbb{N}) \) is distributively lattice ordered and \( 0 \leq 0 \leq \cdots \leq 0 \leq f \) is an increasing sequence, we can apply Lemma 4.2 to get

\[
f + \sum_{j=1}^{n} \chi_{U_j} = \chi_{(\cup_{i=1}^{m} U_i) \cup U_{12}} + \cdots + \chi_{(\cap_{j=1}^{n} U_{j})}.
\]

Applying \( \varphi' \) and the equalities in the proof of Lemma 5.2 at the first step, and Lemma 4.2 at the second step, one gets

\[
\varphi'(f + g) = (z \lor y_1) + (z \land y_1 \lor y_2) + \cdots + (z \land y_1 \land \cdots \land y_n) = \varphi'(f) + \varphi'(g).
\]

Now, fix any finite \( m \) and assume that the result has been proved for any \( k \leq m-1 \). Then, using the induction hypothesis at the second step, and the case \( m = 1 \) at the third and fourth steps, we have

\[
\varphi'(f + g) = \varphi'((f - \chi_{U_m}) + (g + \chi_{U_m})) = \varphi'(f - \chi_{U_m}) + \varphi'(g + \chi_{U_m}) = \varphi'(f - \chi_{U_m}) + \varphi'(g) + \varphi'((\chi_{U_m}) = \varphi'((f - \chi_{U_m} + \chi_{U_m}) + \varphi'(g) = \varphi'(f) + \varphi'(g),
\]

as desired.

Note that this could have also been proved using Lemma 4.2.

To see that \( \varphi' \) is an order embedding, let \( f \leq g \) in \( \text{Lsc}(X_S, \mathbb{N})_{f_S} \) and note that \( f \leq g \) if and only if \( \{f \geq i\} \subseteq \{g \geq i\} \) for every \( i \). Let \( z_i, y_i \in S \) be such that \( U_{z_i} = \{f \geq i\} \) and \( U_{y_i} = \{g \geq i\} \).

By (i) in Proposition 5.3, \( U_{z_i} \subseteq U_{y_j} \) if and only if \( z_i \leq y_j \) for every \( i \). Furthermore, note that the sequences \((z_i)_{i=1}^{m}, (y_i)_{i=1}^{m}\) are both decreasing. Since we have a topological order, \( z_i \leq y_i \) for every \( i \) if and only if

\[
\sum_{i=1}^{m} z_i \leq \sum_{i=1}^{m} y_i,
\]
where note that the right- and left-hand sides correspond to $\varphi'(f)$ and $\varphi'(g)$, respectively.

Using $\varphi'$, one can now construct a Cu-isomorphism.

**Theorem 6.3** Let $S$ be an Lsc-like Cu-semigroup. Then, the Cu-morphism $\varphi'$ extends to a Cu-isomorphism.

**Proof** We will need the following claims.

**Claim 1** Let $(f_n)_n$ and $(g_n)_n$ be two $\ll$-increasing sequences with the same supremum in $\Lsc(X_S, \mathbb{N})$. Then, $\sup_{n} \varphi'(f_n) = \sup_{n} \varphi'(g_n)$.

Since $(f_n)_n$ and $(g_n)_n$ have the same supremum, we know that for every $n$ there exist $k, m$ such that $f_n \leq g_m$ and $g_n \leq f_k$.

Applying $\varphi'$, we get $\varphi'(f_n) \leq \varphi'(g_m) \leq \sup_{m} \varphi'(g_m)$, and $\varphi'(g_n) \leq \varphi'(f_k) \leq \sup_{k} \varphi'(f_k)$, and so $\sup_{k} \varphi'(f_k) = \sup_{m} \varphi'(g_m)$ as desired.

By Claim 1, we can define the map $\varphi : \Lsc(X_S, \mathbb{N}) \to S$ as $\varphi(f) = \sup_{n} \varphi'(f_n)$, where $(f_n)_n$ is a $\ll$-increasing sequence with supremum $f$. We will see that $\varphi$ is a Cu-morphism that extends $\varphi'$ and that it satisfies the conditions in Lemma 6.1 (i.e., $\varphi$ is a Cu-isomorphism).

**Claim 2** Let $(f_n)_n$ be an increasing sequence in $\Lsc(X_S, \mathbb{N})$, with supremum $f = \sup f_n \in \Lsc(X_S, \mathbb{N})$. Then, we have $\varphi'(f) = \sup_{n} \varphi'(f_n)$.

To prove the claim, let $f_n = \chi_{U_i_n}$ for every $n$, and recall that $\sup_{n} \chi_{U_i_n} = \chi_{\cup_{n} U_i_n \sup_{n} \varepsilon_n}$. This is equivalent to

$$\varphi'(\sup_{n} f_n) = \sup_{n} \chi_{U_i_n} = \sup_{n} \varphi'(f_n).$$

Now, given any increasing sequence as in the statement of the lemma with supremum $f$, we know that $\sup(f) < \infty$, say $\sup(f) = m \in \mathbb{N}$.

Thus, given $U_{i,n} = \{f_n \geq i\}$ for $1 \leq i \leq m$, we can write

$$f_n = \sum_{i=1}^{m} \chi_{U_{i,n}}$$

with some possibly empty $U_{i,n}$’s.

We have

$$f = \sup_{n} (f_n) = \sum_{i=1}^{m} \chi_{U_{i,n}} \cup_{n} U_{i,n},$$

where $\cup_{n} U_{i,n} = \{f \geq i\}$.

Using that $\varphi'$ preserves suprema of indicator functions, we have

$$\varphi'(f) = \sum_{i=1}^{m} \varphi'(\chi_{U_{i,n}}) = \sup_{n} \varphi'(\sum_{i=1}^{m} \chi_{U_{i,n}}) = \sup_{n} \varphi'(f_n),$$

as required.
Since \( \varphi' \) preserves addition and \( S \) is a Cu-semigroup, it is clear that \( \varphi \) also preserves addition. Note that the proof of Claim 2 also shows that \( \varphi \) extends \( \varphi' \) and that \( \varphi \) is order-preserving.

To see that \( \varphi \) preserves suprema, let \( \varphi(f) = \sup_n \varphi'(f_n) \) with \( (f_n)_n \ll \langle \) increasing with supremum \( f \) and consider an increasing sequence \( (g_n)_n \) whose supremum is also \( f \).

Then, for every \( n \), there exists an \( m \) with \( f_n \leq g_m \) and, consequently, \( \varphi'(f_n) = \varphi(f_n) \leq \varphi(g_m) \). It follows that \( \varphi(f) = \sup_n \varphi'(f_n) \leq \sup_m \varphi(g_m) \).

On the other hand, \( f \geq g_m \) for every \( m \), so \( \varphi(f) = \sup_m \varphi(g_m) \).

Now, let \( f, g \in \text{Lsc}(X_\mathbb{S}, \mathbb{N}) \) be such that \( f \ll g \). Then, we know by Lemma 4.18 that this happens if and only if

\[
\chi_{\{f \geq i\}} \ll \chi_{\{g \geq i\}} \quad \text{for every } i \quad \text{and} \quad \sup(f) = m < \infty.
\]

Letting \( y_i, z_i \in S \) such that \( U_{y_i} = \{ f \geq i \} \) and \( U_{z_i} = \{ g \geq i \} \), we know that

\[
\varphi(f) = \varphi'(f) = y_1 + \cdots + y_m \ll z_1 + \cdots + z_m \leq \sup_n \sum_{i=1}^n z_i = \varphi(g).
\]

Finally, note that the image of \( \varphi \) is clearly dense in \( S \), since \( \varphi' \) is surjective on \( \downarrow \ e \). Furthermore, \( \varphi \) is an order embedding in \( \text{Lsc}(X_\mathbb{S}, \mathbb{N}) \), since \( \varphi \) coincides with \( \varphi' \) in this basis of \( \text{Lsc}(X_\mathbb{S}, \mathbb{N}) \).

Thus, since the conditions in Lemma 6.1 are satisfied, it follows that \( \varphi \) is a Cu-isomorphism. \( \blacksquare \)

**Theorem 6.4** Let \( S \) be a Cu-semigroup. Then, \( S \) is Lsc-like if and only if \( S \) is Cu-isomorphic to \( \text{Lsc}(X, \mathbb{N}) \) for a \( T_1 \) topological space \( X \).

**Proof** We already know that \( \text{Lsc}(X, \mathbb{N}) \) is Lsc-like whenever it is a Cu-semigroup (see Example 4.5), and the converse follows from Theorem 6.3. \( \blacksquare \)

In [23], a notion of covering dimension for Cu-semigroups is introduced. This dimension satisfies many of the expected permanence properties [23, Proposition 3.10], and is related to other dimensions, such as the nuclear dimension of \( C^* \)-algebras [23, Theorem 4.1] and the Lebesgue covering dimension (see [23, Proposition 4.3 and Corollary 4.4]).

Using such a notion, one can prove the following.

**Theorem 6.5** Let \( S \) be a Cu-semigroup satisfying (O5), and let \( n \in \mathbb{N} \cup \{ \infty \} \). Then, \( S \) is Cu-isomorphic to \( \text{Lsc}(X, \mathbb{N}) \) with \( X \) a compact metric space such that \( \dim(X) = n \) if and only if \( S \) is Lsc-like, countably based, has a compact order unit, and \( \dim(S) = n \).

In particular, a Cu-semigroup \( S \) is Cu-isomorphic to the Cuntz semigroup of \( C(X) \) with \( X \) compact metric and \( \dim(X) \leq 1 \) if and only if \( S \) is Lsc-like, countably based, satisfies (O5), has a compact order unit, and \( \dim(S) \leq 1 \).

**Proof** The forward implication follows from Examples 4.5 and 5.4, Proposition 5.6, and [23, Corollary 4.4].

To prove the converse, use Theorem 6.4 and (v) in Proposition 5.6 to deduce that \( S \cong \text{Lsc}(X_\mathbb{S}, \mathbb{N}) \) with \( X_\mathbb{S} \) compact metric. Then, it follows from [23, Corollary 4.4] that \( \dim(X_\mathbb{S}) = \dim(S) = n \), as required.
Now, assume that $S$ is Cu-isomorphic to the Cuntz semigroup of $C(X)$ with $X$ compact metric and $\dim(X) \leq 1$. By [20, Theorem 1.1] we know that $Cu(C(X)) \cong Lsc(X, \mathbb{N})$. In particular, $S$ satisfies (O5) (e.g., [22]).

Thus, it now follows from our previous argument that $S$ is Lsc-like, countably based, satisfies (O5), has a compact order unit, and $\dim(S) \leq 1$.

Conversely, if $S$ satisfies the list of properties in the second part of the statement, note that $S \cong Lsc(X_S, \mathbb{N})$ with $\dim(X_S) \leq 1$ again by Theorem 6.4. Using [20, Theorem 1.1] a second time, it follows that $Lsc(X_S, \mathbb{N}) \cong Cu(C(X_S))$, as desired. □

7 Chain conditions and the Cuntz semigroup of commutative AI-algebras

In this section, we introduce the notions of piecewise chainable and weakly chainable Cu-semigroups and prove that, together with some additional properties, these notions give a characterization of when $S$ is Cu-isomorphic to the Cuntz semigroup of a unital commutative block stable AI-algebra and a unital commutative AI-algebra, respectively (see Theorems 7.4 and 7.11).

We also show that the Cuntz semigroup of any AI-algebra is weakly chainable, thus uncovering a new property that the Cuntz semigroup of any AI-algebra satisfies (see Corollary 7.9).

We first prove the following categorical proposition, which summarizes the results of the above sections. We denote by Top the category of topological spaces, and by $T^1_{Cu}$ the subcategory of Top whose objects $X$ are the $T^1$ spaces such that $Lsc(X, \mathbb{N}) \in Cu$. Note that, by Corollary 4.19, this includes all compact, metric spaces.

**Proposition 7.1** Let Lsc be the subcategory of Cu consisting of Lsc-like Cu-semigroups. There exists a faithful and essentially surjective contravariant functor $T: T^1_{Cu} \to Lsc$ that is full on isomorphisms.

**Proof** For every topological space $X \in T^1_{Cu}$, define $T(X) = Lsc(X, \mathbb{N})$.

Furthermore, given any continuous map $f: X \to Y$, set $T(f): Lsc(Y, \mathbb{N}) \to Lsc(X, \mathbb{N})$ as the unique Cu-morphism such that $T(f)(\chi_U) = \chi_{f^{-1}(U)}$ for every open subset $U$ of $Y$.

Note that, given $f: X \to Y$ and $g: Y \to Z$ in $T^1_{Cu}$, we have

$$T(g \circ f)(\chi_U) = \chi_{(g \circ f)^{-1}(U)} = \chi_{f^{-1}g^{-1}(U)} = (T(f) \circ T(g))(\chi_U).$$

Thus, $T$ is a contravariant functor, which is clearly faithful by construction.

Moreover, we know by Theorem 6.4 that for every Lsc-like Cu-semigroup $S$ there exists a $T^1$-space $X_S$ with $S \cong Lsc(X_S, \mathbb{N})$. Therefore, $T$ is essentially surjective.

Now, let $\phi: S \to T$ be a Cu-isomorphism of Lsc-like Cu-semigroups. Using Theorem 6.4, we get a Cu-isomorphism of the form $\phi: Lsc(X_S, \mathbb{N}) \to Lsc(X_T, \mathbb{N})$.

Since $\phi(1) = 1$, indicator functions must map to indicator functions. Since $\phi$ is a Cu-isomorphism, maximal elements below 1 must map to maximal elements below 1. More explicitly, for every $x \in X_S$, there exists $y \in X_T$ such that $\phi(\chi_{X_S \setminus \{x\}}) = \chi_{X_T \setminus \{y\}}$.

We define the map $f: X_T \to X_S$ as $y \mapsto x$, which is bijective because $\phi$ is a Cu-isomorphism.
To see that it is continuous, let \( U \) be an open subset of \( X_S \) and let \( V \subseteq X_T \) be such that \( \phi(\chi_U) = \chi_V \). Then, given \( y \in X_T \), we have that \( y \in V \) if and only if
\[
1 \leq \chi_{X_T \setminus \{y\}} + \chi_V = \phi(\chi_{X_S \setminus \{f(y)\}}) + \chi_U.
\]

Since \( \phi \) is a Cu-isomorphism, this in turn holds if and only if \( (X_S \setminus \{f(y)\}) \cup U = X_S \) or, equivalently, if \( f(y) \in U \).

This shows that \( f^{-1}(U) = V \) and, consequently, that \( f \) is continuous.

Finally, let \( V \subseteq X_T \) be open. Since \( \phi \) is an isomorphism, there exists some open subset \( U \subseteq X_S \) such that \( \chi_V = \phi(\chi_U) \).

By the argument above, one has \( V = f^{-1}(U) \) and, since \( f \) is bijective, it follows that \( f(V) = f(f^{-1}(U)) = U \). This shows that \( f \) is open.

Thus, \( f \) is a homeomorphism between \( X_S \) and \( X_T \), as required. 

We now introduce chainable and piecewise chainable inf-semilattice ordered Cu-semigroups.

Definition 7.1 Let \( S \) be an inf-semilattice ordered Cu-semigroup. An element \( x \in S \) is said to be chainable if for every sum \( y_1 + \cdots + y_n \geq x \), there exist elements \( z_1, \ldots, z_m \) such that:

(i) For every \( i \), there exists some \( k \) with \( z_i \leq y_k \).

(ii) \( z_i \land z_j \neq 0 \) if and only if \( |i - j| \leq 1 \).

(iii) \( z_1 + \cdots + z_m \geq x \).

\( S \) will be called chainable if it has a chainable order unit.

Moreover, we will say that \( S \) is piecewise chainable if there exist chainable elements \( s_1, \ldots, s_n \) such that \( s_1 + \cdots + s_n \) is an order unit and \( s_i \land s_j = 0 \) whenever \( i \neq j \).

Lemma 7.2 Given an Lsc-like Cu-semigroup \( S \) with least order unit \( e \) and an element \( y \leq e \), \( U_y \) is topologically chainable if and only if \( y \) is chainable.

In particular, \( S \) is chainable if and only if \( X_S \) is topologically chainable.

Proof If \( y \) is chainable, take a finite cover \( U_{y_1} \cup \cdots \cup U_{y_n} = U_y \). We have that \( y = y_1 \lor \cdots \lor y_n \leq y_1 + \cdots + y_n \). Thus, applying the chainability of \( y \), one gets elements \( z_1, \ldots, z_m \) such that for every \( i \) there exists \( k \) with \( z_i \leq y_k \leq y \). This shows that \( z_1 \lor \cdots \lor z_m \leq y \).

By Corollary 4.9 and (iii) in Definition 7.1, we have \( z_1 \lor \cdots \lor z_m \geq y \) and, consequently, \( z_1 \lor \cdots \lor z_m = y \). This shows that \( U_{z_1}, \ldots, U_{z_m} \) is a cover for \( U_y \).

Using the equalities in the proof of Lemma 5.2 and conditions (i)–(iii) in Definition 7.1, one sees that \( U_{z_1}, \ldots, U_{z_m} \) is a chain that refines our original cover in the sense of Definition 3.1.

Conversely, if \( U_y \) is topologically chainable and we have a sum \( y_1 + \cdots + y_n \geq y \), we can apply Corollary 4.9 once again to obtain
\[
(y_1 \land y) \lor \cdots \lor (y_n \land y) = y.
\]

This shows that \( U_{y_1 \land y} \cup \cdots \cup U_{y_n \land y} = U_y \), and we can use the chainability of \( U_y \) to obtain a chain refining this cover. Using Lemma 5.2, it is easy to check that the elements below \( e \) corresponding to the open subsets of the chain satisfy conditions (i)–(iii) in Definition 7.1.
The Cuntz semigroup of unital commutative AI-algebras

In particular, the previous argument shows that \( e \) is chainable whenever \( X_S \) is topologically chainable. By definition, this implies that \( S \) is chainable.

Conversely, if \( S \) is chainable, we have a chainable order unit \( s \). Let us now show that \( e \) is also chainable, which by the above arguments will imply that \( U_e = X_S \) is topologically chainable.

Thus, let \( y_1 + \cdots + y_n \geq e \), which by Corollary 4.9 implies that \( y_1 \lor \cdots \lor y_n \geq e \). Since \( s \) is an order unit, one has \( \infty y_1 \lor \cdots \lor \infty y_n \geq \infty = \infty s \geq s \).

Using that \( s \) is chainable, we obtain elements \( z_1, \ldots, z_m \) satisfying (i)–(iii) in Definition 7.1.

Lemma 7.3 A countably based Lsc-like Cu-semigroup \( S \) with a compact order unit is piecewise chainable if and only if \( X_S \) is.

Proof If \( X_S \) is piecewise chainable, there exist chainable components \( Y_1, \ldots, Y_n \) such that \( X_S = Y_1 \cup \cdots \cup Y_n \). Since chainability implies connectedness (whenever the space is compact), there is a finite number of connected components, and so these are clopen.

By Lemma 7.2, the disjoint chainable components correspond to disjoint chainable elements, so \( S \) is piecewise chainable by definition.

Conversely, if \( S \) is piecewise chainable, each element \( s_i \) in the definition of chainable corresponds to a chainable open subset of \( X_S \), which is disjoint from the other chainable open subsets by construction.

Theorem 7.4 Let \( S \) be a Cu-semigroup. Then, \( S \) is Cu-isomorphic to the Cuntz semigroup of a unital block stable commutative AI-algebra if and only if \( S \) is countably based, Lsc-like, piecewise chainable, has a compact order unit, and satisfies (O5).

Proof Let \( S \) be Cu-isomorphic to the Cuntz semigroup of a unital commutative block stable AI-algebra. Then, we know from [20, Theorem 1.1] and Definition 3.3 that \( S \cong \text{Lsc}(X, \mathbb{N}) \) with \( X \) a compact, metric, piecewise chainable space. In particular, \( S \) satisfies (O5), has a compact order unit, is countably based, and is Lsc-like. Using Lemma 7.3, it also follows that \( S \) is piecewise chainable.

Conversely, assume that \( S \) satisfies all the conditions in the list. By Theorem 6.4 and (v) in Proposition 5.6, we have \( S \cong \text{Lsc}(X, \mathbb{N}) \) with \( X \) a compact metric space.

Then, it follows from Example 5.4 and Lemma 7.3 that \( X \) is piecewise chainable. In particular, it has dimension less than or equal to one by Remark 3.2. Thus, \( \text{Cu}(C(X)) \cong \text{Lsc}(X, \mathbb{N}) \) by [20, Theorem 1.1], so \( S \) is isomorphic to the Cuntz semigroup of a unital commutative block stable AI-algebra.
We now define weak chainability for any Cu-semigroup and prove that every Cu-semigroup of an AI-algebra satisfies such a condition. Moreover, we also show that an Lsc-like Cu-semigroup is weakly chainable if and only if its associated space is almost chainable.

**Definition 7.2** We will say that a Cu-semigroup $S$ is weakly chainable, or that it satisfies the weak chainability condition if, for any $x, y, y_1, \ldots, y_n$ such that

$$x \ll y \ll y_1 + \cdots + y_n,$$

there exist $x', z_1, \ldots, z_m \in S$ such that $x' \leq y, x \prec x'$ and:

(i) For any $i$, there exists $j$ such that $z_i \leq y_j$.

(ii) $z_i + z_j \leq x'$ whenever $|i - j| \geq 2$.

(iii) $z_1 + \cdots + z_m \geq x'$.

**Lemma 7.5** Let $X$ be a compact metric space. Then, $\text{Lsc}(X, \mathbb{N})$ is weakly chainable if and only if $X$ is almost chainable.

**Proof** First, recall that $S = \text{Lsc}(X, \mathbb{N})$ is a Cu-semigroup whenever $X$ is compact and metric by Corollary 4.19. Furthermore, also recall from Example 4.5 that $S$ is an Lsc-like Cu-semigroup with least order unit 1, and that $X \simeq X_S$ by Example 5.4.

Now, assume that $S$ is weakly chainable, and let $U_{y_1}, \ldots, U_{y_n}$ be a cover of $X_S$. Then, the elements $y_1, \ldots, y_n \leq 1$ satisfy $y_1 \vee \cdots \vee y_n \geq 1$ and, consequently, $y_1 + \cdots + y_n \geq 1$.

Set $x = y = 1$, and apply Definition 7.2 to obtain elements $x', z_1, \ldots, z_m$ satisfying the conditions in the definition. Note that, since $x'$ satisfies $x' \leq 1 \leq kx'$ for some $k \in \mathbb{N}$, it follows from the second inequality that $x' \geq 1$ and, therefore, $x' = 1$. Let $U_{z_1}, \ldots, U_{z_m}$ be the open subsets of $X_S$ corresponding to $z_1, \ldots, z_m$, respectively. Using (i)–(iii) in Definition 7.2, it is easy to see that such sets form an almost chain refining the original cover. This implies that $X_S$ is almost chainable and, since $X \simeq X_S$, so is $X$.

Conversely, assume that $X$ is almost chainable and let $x, y, y_1, \ldots, y_n \in S$ be as in Definition 7.2. Set $x' = x \wedge 1$. Then, we know by Corollary 4.9 that $y_1 \vee \cdots \vee y_n \geq y \gg x'$. Taking the infimum with 1, one has $(y_1 \wedge 1) \vee \cdots \vee (y_n \wedge 1) \gg x'$. Moreover, note that $x'$ satisfies $x' \leq y$ and $x \prec x'$. Since $X$ is compact, metric, and almost chainable, we know by Remark 3.2 that its dimension is less than 2. This implies by [20, Theorem 1.1] that $\text{Cu}(C(X)) \cong \text{Lsc}(X, \mathbb{N})$ and, in particular, that $S$ satisfies (O5).

Thus, by the proof of Lemmas 5.2 and 5.5, $(y_1 \wedge 1) \vee \cdots \vee (y_n \wedge 1) \gg x'$ corresponds to a cover $\overline{U_{x'}} \subseteq U_{y_1} \cup \cdots \cup U_{y_n}$. In particular, the open sets $X \setminus U_{x'}, U_{y_1}, \ldots, U_{y_n}$ form a cover of $X$ and, since $X$ is almost chainable, there exists an almost chain $C_1, \ldots, C_m$ covering $X$ and refining $X \setminus U_{x'}, U_{y_1}, \ldots, U_{y_n}$.

Now, take the almost chain $C_1 \cap U_{x'}, \ldots, C_m \cap U_{x'}$, which clearly covers $U_{x'}$. For each $i$, let $z_i \in S$ be the associated element to $C_i \cap U_{x'}$. These elements satisfy the desired conditions in Definition 7.2.

Indeed, to see condition (ii), note that $z_i \leq x'$ for every $i$, so it follows that $z_i + z_j \leq x'$ if and only if $z_i \wedge z_j = 0$. By the proof of Lemma 5.2, this is equivalent to
(C_i \cap U_{x'}) \cap (C_i \cap U_{x'}) = \emptyset. Since \{C_i \cap U_{x'}\}_i is an almost chain, this condition is satisfied.

Conditions (i) and (iii) follow similarly using that \{C_i \cap U_{x'}\}_i is a cover of U_{x'} refining \{U_{y_j}\}_j.

Using Remark 3.2, one gets the following result.

**Corollary 7.6** Given X compact, metric, and connected, the Cu-semigroup Lsc(X, \mathbb{N}) is weakly chainable if and only if X is chainable.

It would be interesting to know whether a general Lsc-like Cu-semigroup S is weakly chainable if X_S is almost chainable.

**Lemma 7.7** Given two weakly chainable Cu-semigroups S and T, their direct sum S \oplus T is also weakly chainable.

**Proof** Take x, y, y_1, \ldots, y_n \in S \oplus T as in Definition 7.2. Write x = (x_1, x_2), y = (y_1, y_2), and y_j = (y_{j,1}, y_{j,2}) with x_1, y_1, y_{j,1} \in S and x_2, y_2, y_{j,2} \in T.

Since S and T are weakly chainable, one gets elements x'_1, z_1, \ldots, z_m, \in S and x', y_1, \ldots, z_{m',2} \in T satisfying the conditions in Definition 7.2. Define x' = (x'_1, x'_2) and note that x' \leq y and that there exists some k \in \mathbb{N} with x \leq kx'.

Now, set z_i = (z_{i,1}, 0) for i \leq m and z_i = (0, z_{i-m+1,2}) for i > m. We have that
\[ z_1 + \cdots + z_{m+m'-1} = (z_{1,1}, 0) + \cdots + (z_{m,1}, 0) + 0, (z_{1,2}) + \cdots + (0, z_{m',2}) \geq (x'_1, 0) + (0, x'_2) = x'. \]

As expected, we also get that for every i there exists a j such that z_i \leq (y_{j,1}, 0) \leq y_j or z_i \leq (0, y_{j,2}) \leq y_j.

Now, take z_i, z_j with |i - j| \geq 2. If i, j \leq m, we have z_i + z_j \leq (x'_1, 0) \leq x'. Similarly, z_i + z_j \leq (0, x'_2) \leq x' whenever i, j > m.

Moreover, if i \leq m and j > m, we know that z_i = (z_{i,1}, 0) \leq (x'_1, 0) and z_j = (0, z_{j-m+1,2}) \leq (0, x'_2). This implies z_i + z_j \leq (x'_1, 0) + (0, x'_2) = x'.

Since z_1, \ldots, z_{m+m'-1} satisfy all the required properties, S satisfies the weak chain-ability condition.

**Proposition 7.8** Let S = \lim_n S_n be a sequential inductive limit of Cu-semigroups. Assume that S_n is weakly chainable for each n. Then, S is also weakly chainable.

**Proof** Let S = \lim_n S_n with S_n weakly chainable for every n. Given an element x \in S_n, let us denote its image through the canonical map S_n \to S by [x].

Let x, y, y_1, \ldots, y_n \in S be as in Definition 7.2. Then, let m \in \mathbb{N} be such that there exist elements u, v, and v_j in S_m with [v_j] \ll y_j,
\[ x \ll [u] \ll [v] \ll [v_1] + \cdots + [v_n] \ll y_1 + \cdots + y_n, \]
and u \ll v \ll v_1 + \cdots + v_n.

Since S_m is weakly chainable, we obtain elements u', z_1, \ldots, z_m \in S_m satisfying the conditions in Definition 7.2. We have:

(i) u' \leq v, u \leq k u' for some k \in \mathbb{N}. This implies [u'] \leq [v] \leq y and x \leq [u] \leq k[u'].

(ii) For any i, there exists j such that z_i \leq v_j, which shows that [z_i] \leq [v_j] \leq y_j.
(iii) \( z_i + z_j \leq u' \) whenever \( |i - j| \geq 2 \). Consequently, \([z_i] + [z_j] \leq [u']\) whenever \( |i - j| \geq 2 \).

Since \( z_1 + \cdots + z_m \geq u' \), one also gets \([u'] \leq [z_1] + \cdots + [z_m] \). Thus, \( S \) is weakly chainable, as desired.

**Corollary 7.9** The Cuntz semigroup of any AI-algebra is weakly chainable.

**Example 7.10** The Cu-semigroups Lsc(\( T, \mathbb{N} \)) and Lsc([0,1]\(^2\),\( \mathbb{N} \)) do not satisfy the weak chainability condition.

Indeed, this follows clearly from Corollary 7.6, as \( T \) and [0,1]\(^2\) are not chainable continua.

Using the results developed thus far, one can now use an analogous proof to that of Theorem 7.4 to prove the following theorem.

**Theorem 7.11** Let \( S \) be a Cu-semigroup. Then, \( S \) is Cu-isomorphic to the Cuntz semigroup of a unital commutative AI-algebra if and only if \( S \) is countably based, Lsc-like, weakly chainable, has a compact order unit, and satisfies (O5).

### 8 New properties of the Cuntz semigroup of an AI-algebra

Inspired by the abstract characterization obtained above, in this section, we introduce properties that are satisfied by the Cuntz semigroups of all AI-algebras and that are not satisfied by other well-known Cu-semigroups. In Definition 7.2, we have already introduced one such property, which is not satisfied by Lsc(\( T, \mathbb{N} \)) (see Example 7.10). We now introduce the conditions of Cu-semigroups with refinable sums and almost ordered sums, and show that the Cuntz semigroups of all AI-algebras satisfy these properties. We also prove that \( Z \), the Cuntz semigroup of the Jiang–Su algebra \( Z \), does not have refinable sums (see Example 8.2).

**Definition 8.1** We say that a Cu-semigroup \( S \) has refinable sums if, given a finite \( \ll \)-increasing sequence

\[
x_1 \ll \cdots \ll x_n
\]

and elements \( x'_1, \ldots, x'_n \) such that \( x_i \ll x'_i \) for every \( i \), there exist finite decreasing sequences \( (y^j_i)_{j=1}^l \) such that:

(i) \( x'_{i+1} \geq y^1_i \) for every \( i \).
(ii) \( y^j_i \ll y^{j+1}_i \) for every \( i \) and \( j \).
(iii) \( x_i \ll y^1_i + \cdots + y^l_i \ll x_{i+1} \).

**Example 8.1** Any Cu-semigroup \( S \) of the form Lsc(\( X, \mathbb{N} \)) has refinable sums.

To see this, let \( x_i, x'_i \) as in Definition 8.1, and let \( \tilde{x}_i \) be such that

\[
x_1 \ll \tilde{x}_1 \ll x_2 \ll \tilde{x}_2 \ll x_3 \ll \cdots \ll x_n.
\]

Since \( \tilde{x}_i \in S_\infty \) for each \( i \), they can all be written as ordered finite sum of elements below one. Furthermore, by possibly adding some zeros, we may assume that all \( \tilde{x}_i \)'s have the same amount of summands. That is to say, we have

\[
x_1 \ll \tilde{x}_1 = y^1_i + \cdots + y^l_i \ll x_2 \ll \tilde{x}_2 = y^1_i + \cdots + y^l_i \ll x_3 \ll \cdots \ll x_n.
\]
Thus, we know by Lemma 4.17 that \( y_j^i \ll y_j^{i+1} \) for every \( i, j \). Moreover, since we have \( x_i \ll x_{i+1} \ll x_{i+1}' \), one can find \( x_i'' \ll x_i' \) such that \( x_i \approx x_i'' \).

Since \( x_i'' \in S_\infty \), we have that \( x_i \approx x_i'' \approx x_i''_{i+1} \). Applying the topological order in \( S \), we obtain \( y_i^1 \leq x_i''_{i+1} \wedge 1 \). As required.

**Example 8.2** Let \( Z = (0, \infty) \cup \mathbb{N} \), and denote by \( n^\infty \in (0, \infty] \) the associated element to \( n \in \mathbb{N} \). Order and addition in \( Z \) are defined normally in each component, and given \( x \in (0, \infty] \) and \( n \in \mathbb{N} \), we set \( x \leq n \) if and only if \( x \leq n^\infty \) in \((0, \infty]\); \( n \leq x \) if and only if \( n^\infty < x \); and \( x + n = x + n^\infty \). It was proved in \([18, \text{Theorem 3.1}]\) that \( Z \) is \( \text{Cu} \)-isomorphic to the \( \text{Cu} \)-isomorphism of the \( \text{Jiang–Su algebra} \ Z \), as defined in \([15]\).

We claim that \( Z \) does not have refinable sums. Indeed, assume for the sake of contradiction that it does, and consider the elements \( x_1 = x_1' = x_2 = 1 \) and \( x_2' = 0.5 \) in \( Z \). These satisfy

\[
x_1 \ll x_2, \quad x_1 \approx x_1', \quad \text{and} \quad x_2 \approx x_2'.
\]

Thus, applying Definition 8.1, we obtain a finite decreasing sequence \((y_j^i)_{j=1}^l\) such that

\[
1 = x_1 \ll y_1^1 + \cdots + y_1^l \ll x_2 = 1.\]

Since \((y_j^i)_{j=1}^l\) is decreasing, one must have \( y_1^i = 1 \) and \( y_1^1 = 0 \) whenever \( i \geq 2 \). However, it follows from (i) in Definition 8.1 that

\[
1 = y_1^1 \leq x_2' = 0.5,\]

which is a clear contradiction.

**Proposition 8.3** Let \( S \) be a \( \text{Cu} \)-semigroup that can be written as a sequential inductive limit \( \lim S_k \) of \( \text{Cu} \)-semigroups \( S_k \) that have refinable sums. Then, \( S \) also has refinable sums.

**Proof** Let \( S = \lim S_k \) where each \( S_k \) has refinable sums. As in the proof of Proposition 7.8, let us denote the image through the canonical map \( S_k \rightarrow S \) of an element \( x \in S_k \) by \([x] \).

Let \( x_1, \ldots, x_n \) and \( x_1', \ldots, x_n' \) be elements in \( S \) as in Definition 8.1. Let \( k \in \mathbb{N} \) such that, for every \( i \leq n-1 \), there exist elements \( u_{2i-1}, u_{2i}, v_{2i-1}, v_{2i} \in S_k \) satisfying

\[
x_i \ll [u_{2i-1}] \ll [u_{2i}] \ll x_{i+1}, \quad [v_{2i}] \leq x_{i+1},\]

and

\[
u_1 \ll \cdots \ll u_{2n-2}, \quad \text{and} \quad u_i \ll v_i.
\]

Since \( S_k \) has refinable sums, we obtain decreasing sequences \((y_j^i)_{j=1}^l\) for \( i \leq 2n-2 \) satisfying the properties of Definition 8.1. In particular, we get

\[
x_i \ll [u_{2i-1}] \ll [y_1^{2i-1}] + \cdots + [y_1^{2i-1}] \ll [u_{2i}] \ll x_{i+1},\]

and \([y_1^{2i-1}] \leq [v_{i+1}] \leq x_{i+1}'\).

It follows that \( S \) has refinable sums. \( \blacksquare \)
Definition 8.2  A Cu-semigroup $S$ is said to have almost ordered sums if for any finite set of elements $x_1, \ldots, x_n$ in $S$ there exist elements $y_{j,i}$ in $S$ such that
\[
x_1 + \cdots + x_n = \sup_i (y_{1,i} + \cdots + y_{n,i})
\]
and such that:
(i) $y_{1,i} \geq \cdots \geq y_{n,i}$,
(ii) $(y_{n,i})_i$ is increasing and bounded by $x_1, \ldots, x_n$.
(iii) If $x' \ll x_{j_1}, \ldots, x_{j_r} \leq z$ for $j_1, \ldots, j_r$ pairwise different, we have $x' \leq y_{r,i}$ and $y_{n+1-r,i} \leq z$ for every sufficiently large $i$.

Example 8.4  If $S$ is a distributively lattice ordered Cu-semigroup, $S$ has almost ordered sums. This applies, in particular, to Cu-semigroups such that $S \cong \text{Lsc}(X, \mathbb{N})$ for some $X$.

Indeed, given $x_1, \ldots, x_n$, set
\[
y_{1,i} = x_1 \lor \cdots \lor x_n,
\]
\[
y_{2,i} = (x_1 \land x_2) \lor \cdots \lor (x_{n-1} \land x_n),
\]
for every $i$, and note that by Lemma 4.2 we have
\[
x_1 + \cdots + x_n = y_{1,i} + \cdots + y_{n,i}.
\]
This implies that $S$ has almost ordered sums.

Example 8.5  Let $Z' = Z \cup \{1''\}$ with $1''$ a compact element not comparable with $1$ such that $1 + x = 1'' + x$ for every $x \in Z' \setminus \{0\}$ and $k1'' = k$ for every $k \in \mathbb{N}$. Then, $Z'$ does not have almost ordered sums.

To see this, consider the sum $1 + 1''$ and assume, for the sake of contradiction, that $Z'$ has refinable sums. Then, there exist elements $y_{1,i}, y_{2,i}$ such that $1 + 1'' = \sup_i y_{1,i} + y_{2,i}.
\]
Since $1 + 1'' = 2$ is compact, for every sufficiently large $i$, we have $1 + 1'' = y_{1,i} + y_{2,i}$.

This implies that $y_{1,i} = 2$ and $y_{2,i} = 0$, since we know that $1, 1'' \leq y_{1,i}$ and that $1, 1''$ are not comparable.

However, we also have $1, 1'' \leq 1.5$, so we get $2 = y_{1,i} \leq 1.5$, a contradiction.

Proposition 8.6 Sequential inductive limits of distributively lattice ordered Cu-semigroups have almost ordered sums.

Proof  Let $S = \lim_k (S_k, \varphi_{k+1,k})$ be the inductive limit of distributively lattice ordered Cu-semigroups $S_k$. As before, given an element $x \in S_k$, let us denote its image through the canonical map $S_k \to S$ by $[x]$.

Let $x_1, \ldots, x_n$ be elements in $S$. One can check that there exists an increasing sequence of integers $(k_i)$ and elements take $x^{1}_{j}, \ldots, x^{n}_{j} \in S_{k_i}$ such that $([x^{j}_{k_i}])_j$ are $\ll$-increasing sequences in $S$ with suprema $x_j$ for every $j \leq n$, in such a way that $\varphi_{k_i,k_{i+1}}(x^{j}_{k_i}) \ll x^{j+1}_{k_{i+1}}$ for every $j$ and $l$.

Since each $S_k$ is distributively lattice ordered, for every $l$, there exist elements $y_{1}^{j}, \ldots, y_{n}^{j}$ in $S_{k_{i}}$ with
\[
x^{j}_{1} + \cdots + x^{j}_{n} = y_{1}^{j} + \cdots + y_{n}^{j},
\]
satisfying the properties of Definition 8.2 (see Example 8.4). This implies, in particular, 
\[ \sup_l \left( [y^1_l] + \cdots + [y^n_l] \right) = x_1 + \cdots + x_n. \]

We will now check that the elements \([y^1_l], \ldots, [y^n_l]\) satisfy conditions (i)–(iii) in Definition 8.2.

By construction, one has \(y^1_l \geq \cdots \geq y^n_l\) for every \(l\), so condition (i) is satisfied. For condition (ii), let \(l \in \mathbb{N}\). Then, applying condition (ii) in \(S_{k_l}\), we have \(\varphi_{k_{l+1}, k_l}(y^l_n) \leq \varphi_{k_{l+1}, k_l}(x^l_1) + \cdots + \varphi_{k_{l+1}, k_l}(x^l_n) \ll x^l_{l+1} + \cdots + x^l_n\), and, by condition (iii) in \(S_{k_{l+1}}\), we get \(\varphi_{k_{l+1}, k_l}(y^l_n) \leq y^l_{l+1}\). It follows that condition (ii) is satisfied.

To prove (iii), take \(x', z \in S\) such that \(x' \ll x_{j_1}, \ldots, x_{j_r} \leq z\) for some pairwise different \(j_1, \ldots, j_r \leq n\). For a large enough \(l\), there exist \(u\) such that 
\[ u \ll x^l_{j_1}, \ldots, x^l_{j_r}, \]
and \(x' \ll [u]\) in \(S\).

This implies that, for every \(l' \geq l\), one has 
\[ \varphi_{k_{l'}, k_l}(u) \ll \varphi_{k_{l'}, k_l}(x^l_{j_1}) + \cdots + \varphi_{k_{l'}, k_l}(x^l_{j_r}) \]
in \(S_{k_{l'}}\).

Consequently, we have \(\varphi_{k_{l'}, k_l}(u) \leq d^l_{l'}\) and so \(x \ll [u] \ll [d^l_{l'}]\) in \(S\) for every \(l' \geq l\). Furthermore, since \(x_{j_1}, \ldots, x_{j_r} \leq z\), for every \(l\), there exists some \(z_l \in S_{k_l}\) with \([z_l] \ll z\) and 
\[ x^l_{j_1}, \ldots, x^l_{j_r} \leq z_l. \]

Therefore, one gets \(y^l_{n+1-r} \leq z_l\). This implies \([y^l_{n+1-r}] \leq [z_l] \ll z\) for every \(l\), as required.

\(\square\)

**Theorem 8.7** Let \(A\) be an AI-algebra. Then, its Cuntz semigroup \(\text{Cu}(A)\) is weakly chainable and has refinable sums and almost ordered sums.

**Proof** The Cuntz semigroup \(\text{Cu}(A)\) is weakly chainable by Corollary 7.9. Furthermore, using the same arguments as in Lemma 7.7, it is easy to see that finite direct sums of Cu-semigroups having refinable sums or almost ordered sums have refinable sums or almost ordered sums, respectively. Thus, it follows from Example 8.1 and Proposition 8.3 that \(S\) has refinable sums.

By Example 8.4 and Proposition 8.6, \(\text{Cu}(A)\) also has almost ordered sums.

\(\square\)

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