ON SOME CONJECTURES BY LU AND WENZEL

JIANQUAN GE, FAGUI LI, ZHIQIN LU, AND YI ZHOU

Abstract. In order to give a unified generalization of the BW inequality and the DDVV inequality, Lu and Wenzel proposed three Conjectures 1, 2, 3 and an open Question 1 in 2016. In this paper we discuss further these conjectures and put forward several new conjectures which will be shown equivalent to Conjecture 2. In particular, we prove Conjecture 2 and hence all conjectures in some special cases. For Conjecture 3 we obtain a bigger upper bound $2 + \sqrt{10}/2$, and we also give a weaker answer for the more general Question 1. In addition, we obtain some new simple proofs of the complex BW inequality and the condition for equality.

1. Introduction

In 2005, Böttcher and Wenzel [4] raised the so-called BW conjecture that if $X$, $Y$ are real square matrices, then

$$\|XY - YX\|^2 \leq 2\|X\|^2\|Y\|^2,$$

where $\|X\| = \sqrt{\text{Tr} XX^*}$ is the Frobenius norm (here $X^*$ is the conjugate transpose of $X$). For real $2 \times 2$ matrices, the proof was obtained by Böttcher and Wenzel in [4], and László [21] proved the $3 \times 3$ case. The first proof for the real $n \times n$ case was found by Vong and Jin [28] and independently by Lu [23]. After that Böttcher and Wenzel found another proof (cf. [5, 29]) that also extends to the case of complex matrices. Then immediately Audenaert [2] gave a simplified proof by probability method and Lu [24] also got a different simple proof by eigenvalue method. The complete characterization of the equality was given in [7] and another unitarily invariant norm attaining the minimum norm bound for commutators was given in [13]. Some generalizations of the BW-type inequalities were obtained by Wenzel and Audenaert [30], also by Fong, Lok, Cheng [6] and Cheng, Liang [8].

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In comparison with the BW inequality that estimates the Frobenius norm of the commutator between two arbitrary matrices, the DDVV inequality estimates the Frobenius norm of the commutators among arbitrary many real symmetric matrices. Recall that the DDVV inequality comes from the normal scalar curvature conjecture (DDVV conjecture) in submanifold geometry posed by De Smet, Dillen, Verstraelen and Vrancken [10] in 1999: Let $M^n \to N^{n+m}(\kappa)$ be an isometric immersed $n$-dimensional submanifold in the real space form with constant sectional curvature $\kappa$. Then there is a pointwise inequality

$$\rho + \rho^\perp \leq \|H\|^2 + \kappa,$$

where $\rho$ is the scalar curvature (intrinsic invariant), $H$ is the mean curvature vector field and $\rho^\perp$ is the normal scalar curvature (extrinsic invariants). Dillen, Fastenakels and Veken [11] then transformed this conjecture into an equivalent algebraic version (DDVV inequality):

$$\sum_{\alpha,\beta=1}^m \| [B_\alpha, B_\beta] \|^2 \leq c \left( \sum_{\alpha=1}^m \|B_\alpha\|^2 \right)^2,$$

here $c = 1$ when $B_1, \cdots, B_m$ are real $n \times n$ symmetric matrices. There were many researches on the DDVV conjecture (cf. [12, 9, 16, 22] etc.). Finally Lu [23] and Ge-Tang [15] proved the DDVV inequality (and hence the DDVV conjecture) independently and differently. After then various of DDVV-type inequalities were obtained such as: $c = \frac{1}{3}$ ($n = 3$) and $c = \frac{2}{3}$ ($n \geq 4$) for real skew-symmetric matrices (cf. [14]); $c = \frac{4}{9}$ for Hermitian matrices (cf. [17]) and also for arbitrary real or complex matrices (cf. [18]).

With the BW inequality and the DDVV inequality on both hands, Lu and Wenzel ([25, 26]) summarized the commutator estimates and considered a unified generalization of them. They proposed the following three conjectures and an open question. Let $M(n, \mathbb{K})$ be the space of $n \times n$ matrices in the field $\mathbb{K}$.

**Conjecture 1.** Let $B_1, \cdots, B_m \in M(n, \mathbb{R})$ be real $n \times n$ matrices subject to

$$\text{Tr} \left( B_\alpha [B_\gamma, B_\beta] \right) = 0$$

for any $1 \leq \alpha, \beta, \gamma \leq m$, then

$$\sum_{\alpha,\beta=1}^m \| [B_\alpha, B_\beta] \|^2 \leq \left( \sum_{\alpha=1}^m \|B_\alpha\|^2 \right)^2. \tag{1.1}$$

**Conjecture 2.** (LW Conjecture) Let $B, B_2, \cdots, B_m \in M(n, \mathbb{R})$ be matrices with

(i) $\text{Tr}(B_\alpha B^*_\beta) = 0$ (i.e., $B_\alpha \perp B_\beta$) for any $\alpha \neq \beta$;

(ii) $\text{Tr} \left( B_\alpha [B, B_\beta] \right) = 0$ for any $2 \leq \alpha, \beta \leq m$. 
Then
\[
\sum_{\alpha=2}^{m} \| [B, B_\alpha] \|_2^2 \leq \left( \max_{2 \leq \alpha \leq m} \| B_\alpha \|_2^2 + \sum_{\alpha=2}^{m} \| B_\alpha \|_2^2 \right) \| B \|_2^2.
\]

**Conjecture 3.** For \( X \in M(n, \mathbb{R}) \) with \( \| X \| = 1 \), let \( T_X \) be the linear map on \( M(n, \mathbb{R}) \) defined by \( T_X(Y) = [X^*, [X, Y]] \) and \( \lambda(T_X) := \{ \lambda_1(T_X) \geq \lambda_2(T_X) \geq \lambda_3(T_X) \cdots \} \) be the set of eigenvalues of \( T_X \). Then
\[
\lambda_1(T_X) + \lambda_3(T_X) \leq 3.
\]

**Question 1.** What is the upper bound of \( \sum_{i=1}^{k} \lambda_{2i-1}(T_X) \)?

If \( k = 1 \), the bound is 2 by the BW inequality, i.e., \( \lambda_1(T_X) \leq 2 \), since we have
\[
\lambda_1(T_X) = \max_{\| Y \| = 1} \langle T_X Y, Y \rangle = \max_{\| Y \| = 1} \| [X, Y] \|_2^2 \leq 2.
\]

If \( k = 2 \), the bound is supposed to be 3 by Conjecture 3. On the other hand, when restricted to real symmetric matrices, Conjecture 1 reduces to the DDVV inequality. It turns out that not only the BW inequality and the DDVV inequality but also both Conjectures 1 and 3 are implied by Conjecture 2 (cf. [25]). Moreover, we will show that Conjecture 2 is equivalent to assigning \( k + 1 \) as the upper bound of \( \sum_{i=1}^{k} \lambda_{2i-1}(T_X) \) for \( k \geq 1 \), which is nothing but the following Conjecture 4 because we can prove \( \lambda_{2i-1}(T_X) = \lambda_{2i}(T_X) \) for any \( i \) (See Proposition 2.6). Hence, Conjecture 2 as well as its equivalent Conjectures 4-6 in the following, takes exactly the role of a unified generalization of the BW inequality and the DDVV inequality for real matrices. We call Conjecture 2 the Fundamental Conjecture of Lu and Wenzel, or simply the (real) LW Conjecture.

**Conjecture 4.** For \( X \in M(n, \mathbb{R}) \) with \( \| X \| = 1 \), we have
\[
\sum_{i=1}^{2k} \lambda_i(T_X) \leq 2k + 2, \quad k = 1, \cdots, \left[ \frac{n^2}{2} \right].
\]

In fact, the summation \( \sum_{i=1}^{2k} \lambda_i(T_X) \) in Conjecture 4 cannot exceed \( 2n \). We explain this by introducing the following Conjecture 5 which looks stronger but in fact is equivalent to Conjecture 4. Before that, we introduce some notations.

Let \( x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \). We rearrange the components of \( x \) in decreasing order and obtain a vector \( x^\downarrow = (x_1^\downarrow, x_2^\downarrow, \cdots, x_n^\downarrow) \) where
\[
x_1^\downarrow \geq x_2^\downarrow \geq \cdots \geq x_n^\downarrow.
\]
Definition 1. For \( x = (x_1, x_2, \cdots, x_n) \) and \( y = (y_1, y_2, \cdots, y_n) \) in \( \mathbb{R}^n \), we say that \( x \) is weakly majorized by \( y \), written as \( x \preceq y \), if
\[
\sum_{i=1}^{k} x_i^\downarrow \leq \sum_{i=1}^{k} y_i^\downarrow, \quad k = 1, 2, \cdots, n.
\]

Definition 2. A multiset may be formally defined as a \( 2 \)-tuple \((A, m)\) where \( A \) is the underlying set of the multiset, formed from its distinct elements, and \( m: A \to \mathbb{N}_{\geq 1} \) is a function from \( A \) to the set of the positive integers, giving the multiplicity, that is, the number of occurrences, of the element \( a \) in the multiset as the number \( m(a) \).

If \( A = \{a_1, a_2, \ldots, a_n\} \) is a finite set, the multiset \((A, m)\) is often represented as \( \{a_1^{m(a_1)}, a_2^{m(a_2)}, \ldots, a_n^{m(a_n)}\} \). For example, the multiset \( \{a, a, b\} \) is written as \( \{a^2, b\} \).

Conjecture 5. For \( X \in M(n, \mathbb{R}) \) with \( \|X\| = 1 \), the set \( \lambda(T_X) \) of eigenvalues of \( T_X \) is weakly majorized by the multiset \( \{2^2, 1^{2n-4}, 0^{(n-1)^2+1}\} \).

It is just
\[
2k \sum_{i=1}^{2k} \lambda_i(T_X) \leq 2n, \quad \text{for } k \geq n,
\]
that looks stronger in the assertion here than in that of Conjecture 4. Another equivalent conjecture that also looks stronger is the following Conjecture 6 by omitting the second assumption of Conjecture 2.

Conjecture 6. Let \( B, B_2, \cdots, B_m \in M(n, \mathbb{R}) \) be matrices with \( \text{Tr}(B_\alpha B_\beta^*) = 0 \) for any \( 2 \leq \alpha \neq \beta \leq m \). Then
\[
\sum_{\alpha=2}^{m} \|B, B_\alpha\|^2 \leq \left( 2 \max_{2 \leq \alpha \leq m} \|B_\alpha\|^2 + \sum_{\alpha=2}^{m} \|B_\alpha\|^2 \right) \|B\|^2.
\]

We summarize the relations of these conjectures in the following theorem.

Theorem 1.1. (1) Conjectures 2, 4, 5 and 6 are equivalent to each other.
(2) If one of the above conjectures is true, then Conjectures 1 and 3 hold.

Since the BW inequality (resp. the DDVV inequality) holds also for complex (resp. complex symmetric) matrices (cf. [3], [18]), we can also expect for the same conjectures as above with all matrices being complex matrices. In fact we will prove the relations of Theorem 1.1 between these conjectures in complex version. Hence we call Conjecture 2 for complex matrices the complex LW Conjecture. Obviously, the

\[\text{Notice that for the complex version, the vanishing conditions in the conjectures should be in the form of taking trace other than Hermitian inner product, since trace is complex linear while Hermitian inner product is not.}\]
On some conjectures by Lu and Wenzel

It turns out that the methods we developed in the study of the conjectures above lead us to some new simple proofs of the complex BW inequality and the condition for equality, which we will discuss first in Section 3 as it is just the first eigenvalue estimate $\lambda_1(T_X) \leq 2$, the basic case $k = 1$ of the complex LW Conjecture [7]. In Section 2 we prepare several useful lemmas and properties of $T_X$. In Section 4 we prove the equivalence between Conjectures 4-6 and Conjecture 2, i.e., Theorem 1.1 in the complex version. In Section 5 we prove the conjectures for the special cases of Theorem 1.2 and for general cases, we show the partial results Theorems 1.3 and 1.4.

Although the inequalities we study in this paper are matrix inequalities, it is not hard to generalize them as inequalities of bounded operators on separable Hilbert
spaces. In quantum physics, these inequalities are related to the *Uncertainty Principle*, or more precisely, the Robertson-Schrödinger relations. The classical Uncertainty Principle, in our notations, can be formulated by

$$\| [A, B] \|_{OP}^2 \leq 2 \| A \|_{OP}^2 \cdot \| B \|_{OP}^2,$$

where $\| \cdot \|_{OP}$ is the operator norm. In this context, the BW-type inequality can be viewed as another version of the Uncertainty Principle. There are literature in physics provides various of generalization of the Uncertainty Principle; see [27] for example. In our paper, we study the optimal version of all these inequalities.

2. Preliminaries

In this section, we will introduce some necessary notations and lemmas which are interesting in themselves. To avoid needless duplication, we discuss the complex version directly so as to include the real version.

Let $T$ be a linear mapping on a complex $N$-dimensional vector space $V$ with Hermitian inner product $\langle \cdot, \cdot \rangle$. In this paper, we always denote by

$$\lambda(T) := \{ \lambda_1(T) \geq \cdots \geq \lambda_N(T) \}, \quad \sigma(T) := \{ \sigma_1(T) \geq \cdots \geq \sigma_N(T) \geq 0 \}$$

the ordered sets of real eigenvalues (if available) and singular values of $T$ respectively, where singular values are square roots of eigenvalues of $T^*T$.

Now suppose $T \geq 0$ be self-dual and positive semi-definite. Then by elementary linear algebra, we have

**Lemma 2.1.** The multiplicity of each positive eigenvalue of $T$ is even if and only if there exists a unitary skew-symmetric mapping $S$ (i.e., $U^*SU$ is real skew-symmetric for some unitary matrix $U$) such that $T = S^*S = -S^2$. In addition, $Tx = 0$ if and only if $Sx = 0$.

**Proof.** The sufficiency is clear. Now suppose that there are $g$ distinct positive eigenvalues $\lambda(T) = \{ t_1 = s_1^2 > \cdots > t_g = s_g^2 > 0 \}$ with multiplicities $2n_1, \cdots, 2n_g$, and denote by $r = 2 \sum_{j=1}^{g} n_j$ the rank of $T$. Then we can diagonalize $T$ by a unitary matrix $U$ as

$$T = U \text{ diag} \left( t_1 I_{2n_1}, \cdots, t_g I_{2n_g}, O_{N-r} \right) U^*,$$
where \( O_{N-r} \) denotes the zero matrix of order \( N - r \). Then the required unitary skew-symmetric matrix can be defined as

\[
S := U \begin{pmatrix}
O & -s_1I_{n_1} \\
-s_1I_{n_1} & O \\
& \\
& \\
& \\
& \\
& \\
O & -s_gI_{n_g} \\
s_gI_{n_g} & O \\
& \\
& \\
& \\
O_{N-r}
\end{pmatrix} U^*.
\]

The proof is complete. \( \square \)

Now let \( T \) be self-dual and positive semi-definite with even multiplicities of positive eigenvalues (i.e., \( \lambda_{2i-1}(T) = \lambda_{2i}(T) \) for any \( i \) with \( \lambda_{2i-1}(T) > 0 \)), and \( S \) be the unitary skew-symmetric mapping as in Lemma 2.1. Then we have the following lemmas.

**Lemma 2.2.** Let \( y \in V \) with \( |y| = 1 \). Then

\[
\langle T^2y, y \rangle \geq (Ty, y)^2.
\]

**Proof.** Since \( T = S^*S = -S^2 \), the inequality above is equivalent to

\[
(T^2y, y) \geq (Ty, y)^2.
\]

Let \( \{e_i\}_{i=1}^N \) be an orthonormal basis of \( V \) such that \( e_i \) is a unit eigenvector corresponding to \( \lambda_i(T) \). Setting \( y = \sum_{i=1}^N y_i e_i \), then \( \sum_{i=1}^N y_i^2 = 1 \) and we have

\[
(T^2y, y) = \sum_{i=1}^N y_i^2 \lambda_i^2(T) = \left( \sum_{i=1}^N y_i^2 \right) \left( \sum_{i=1}^N \lambda_i^2(T) \right) \geq \left( \sum_{i=1}^N y_i^2 \lambda_i(T) \right)^2 = (Ty, y)^2.
\]

The proof is complete. \( \square \)

**Lemma 2.3.** Let \( W \subseteq V \) be a complex \( m \)-dimensional isotropic subspace of \( S \), i.e., \( S(W) \subseteq W^\perp \) ( \( (Sw_1, w_2) = 0 \) for any \( w_1, w_2 \in W \) ). Then we have

\[
\text{Tr } T|_W \leq \text{Tr } T|_{S(W)}, \quad \text{Tr } T|_W \leq \sum_{i=1}^m \lambda_{2i-1}(T).
\]

**Proof.** We will find a suitable basis to compare the traces by using Lemma 2.2. Let \( \{E_i\}_{i=1}^N \) be an orthonormal basis of \( V \) such that \( \{E_i\}_{i=1}^m \) is a basis of \( W \), and under this basis we identify \( V \cong \mathbb{C}^N \). Denote

\[
\text{rank}(SE_1, \ldots, SE_m) = \dim S(W) =: k \leq m.
\]
Assume \( k \geq 1 \), otherwise we have \( S|_W = 0 \) and thus \( \text{Tr} T|_W = 0 \) by Lemma 2.1. By singular value decomposition, there exist \( P \in U(N) \) and \( Q \in U(m) \) such that

\[
P^*(SE_1, \ldots, SE_m)Q = \Lambda =: \begin{pmatrix} \Lambda_{k \times k} & 0 \\ 0 & O \end{pmatrix}_{N \times m},
\]

where \( \tilde{\Lambda} =: \text{diag}(\Lambda_1, \ldots, \Lambda_k) \), \( \Lambda_i > 0 \) for \( 1 \leq i \leq k \). Setting

\[
PA =: (F_1, \ldots, F_m),
\]

we have \( \langle F_i, F_j \rangle = \Lambda_i \Lambda_j \delta_{ij} \) for \( 1 \leq i, j \leq k \) and \( F_i = 0 \) for \( i > k \). Thus \( \{\tilde{F}_i\}_{i=1}^k \) is an orthonormal basis of \( S(W) \), where \( \tilde{F}_i := \Lambda_i^{-1} F_i \). Let

\[
(\tilde{E}_1, \ldots, \tilde{E}_m) := (E_1, \ldots, E_m)Q,
\]

then \( \{\tilde{E}_i\}_{i=1}^m \) is an orthonormal basis of \( W \) and satisfies

\[
(F_1, \ldots, F_m) = PA = (SE_1, \ldots, SE_m)Q = (SE_1, \ldots, SE_m).
\]

Therefore, Lemma 2.2 implies

\[
\text{Tr} T|_W = \sum_{i=1}^m \langle T\tilde{E}_i, \tilde{E}_i \rangle \leq \sum_{i=1}^k \langle T\tilde{F}_i, \tilde{F}_i \rangle = \text{Tr} T|_{S(W)}.
\]

Since \( S(W) \subset W^\perp \), \( \{\tilde{E}_i\}_{i=1}^m \bigcup \{\tilde{F}_i\}_{i=1}^k \) is an orthonormal basis of \( W \oplus S(W) \). Hence,

\[
\text{Tr} T|_W + \text{Tr} T|_{S(W)} = \text{Tr} T|_{W \oplus S(W)} \leq \sum_{i=1}^{m+k} \lambda_i(T) \leq \sum_{i=1}^{2m} \lambda_i(T),
\]

\[
\text{Tr} T|_W \leq \frac{1}{2} \sum_{i=1}^{2m} \lambda_i(T) = \sum_{i=1}^m \lambda_{2i-1}(T).
\]

The proof is complete. \( \square \)

Now we consider the linear operator \( T_X \) as in Conjecture 3. More specifically, for any \( n \times n \) complex matrix \( X \) with \( \|X\| = 1 \), we define

\[
T_X : M(n, \mathbb{C}) \to M(n, \mathbb{C}),
\]

\[
Y \mapsto [X^*, [X, Y]].
\]

It turns out that \( T_X \) is exactly an operator of the same type as \( T \) in the preceding lemmas with \( V = M(n, \mathbb{C}) \), \( \dim V = n^2 =: N \) (cf. \[23\]). For the sake of completeness, we repeat the properties as follows.

**Proposition 2.4.** \[23\] \( T_X \) is an self-dual and positive semi-definite linear map.
Proof. This is because of the following straightforward computations:

\[ \langle Y_1, [X^*, [X, Y_2]] \rangle = \langle [X, Y_1], [X, Y_2] \rangle = \langle [X^*, [X, Y_1]], Y_2 \rangle \]

and

\[ \langle T_X Y, Y \rangle = \|[X, Y]\|^2. \]

\[ \square \]

It follows immediately from the definition (2.1) that

(2.2) \[ T_{U^*XU} (U^*YU) = U^*(T_X Y)U, \quad \text{for } U \in U(n), \]

thus we have

Lemma 2.5. The set of eigenvalues \( \lambda(T_X) := \{ \lambda_1(T_X) \geq \cdots \geq \lambda_N(T_X) \} \) is invariant under unitary congruences of \( X \).

Proposition 2.6. The multiplicity of each positive eigenvalue of \( T_X \) is even, i.e., \( \lambda_{2i-1}(T_X) = \lambda_{2i}(T_X) \) for any \( i \) with \( \lambda_{2i-1}(T_X) > 0 \).

Proof. Let \( \lambda > 0 \) be a positive eigenvalue of \( T_X \) and \( E_\lambda \) be its eigenspace. We will show that the complex dimension of \( E_\lambda \) is even.

Define a quasi-linear map by

\[ \tilde{S}_X : M(n, \mathbb{C}) \rightarrow M(n, \mathbb{C}), \]

\[ Y \mapsto [X, Y]^*. \]

Then it follows easily that \( \tilde{S}_X(zY) = \bar{z}\tilde{S}_X(Y) \) for \( z \in \mathbb{C} \), \( \tilde{S}_X \) is anti-self-dual and \( T_X = -\tilde{S}_X^2 \) because

\[ \langle \tilde{S}_X Y_1, Y_2 \rangle = \text{Re} \text{ Tr}[X, Y_1]Y_2 = \text{Re} \text{ Tr} X[Y_1, Y_2] = -\langle Y_1, \tilde{S}_X Y_2 \rangle, \]

\[ -\tilde{S}_X^2 Y = -[X, [X, Y]^*]^* = [X^*, [X, Y]] = T_X Y. \]

Now for any eigenvector \( Y \in E_\lambda \), i.e., \( T_X Y = \lambda Y \), we claim that \( \tilde{S}_X Y \) is also an eigenvector in \( E_\lambda \) which is \( \mathbb{C} \)-independent (even \( \mathbb{C} \)-orthogonal) to \( Y \). In fact, since \( T_X = -\tilde{S}_X^2 \) we have

\[ T_X \tilde{S}_X Y = \tilde{S}_X T_X Y = \lambda \tilde{S}_X Y, \quad \|\tilde{S}_X Y\|^2 = \langle T_X Y, Y \rangle = \lambda \|Y\|^2 > 0, \]

\[ \text{Tr} \left( Y(\tilde{S}_X Y)^* \right) = \text{Tr} \left( Y[X, Y] \right) = 0, \quad \text{and thus} \quad \langle Y, \tilde{S}_X Y \rangle = \langle iY, \tilde{S}_X Y \rangle = 0, \]

where \( i = \sqrt{-1} \) here and for the rest of this paper.

For \( k \geq 1 \), suppose that \( \text{Span}_\mathbb{C} \{ Y_i, \tilde{S}_X Y_i \}_{i=1}^k \subset E_\lambda \) and \( Y_{k+1} \in E_\lambda \) is orthogonal to \( \text{Span}_\mathbb{C} \{ Y_i, \tilde{S}_X Y_i \}_{i=1}^k \). Then it suffices to prove

\[ \tilde{S}_X Y_{k+1} \perp \text{Span}_\mathbb{C} \{ Y_i, \tilde{S}_X Y_i \}_{i=1}^k. \]
This is easily verified as follows:

\[
\begin{align*}
\text{Tr} \left( Y_i(\tilde{S}_X Y_{k+1})^* \right) &= - \text{Tr} \left( Y_{k+1}(\tilde{S}_X Y_i)^* \right) = 0, \\
\text{Tr} \left( \tilde{S}_X Y_i(\tilde{S}_X Y_{k+1})^* \right) &= \text{Tr} \left( Y_{k+1}(T_X Y_i)^* \right) = \lambda \text{Tr} \left( Y_{k+1}(Y_i)^* \right) = 0.
\end{align*}
\]

The proof is complete. \(\Box\)

As the pair \((T, S)\) in Lemmas 2.1-2.3, we can define a unitary skew-symmetric linear operator \(S_X\) on \(V = M(n, \mathbb{C})\) such that \(T_X = S_X^* S_X = -S_X^2\) as follows. Taking an orthonormal basis \(\{v_i\}_{i=1}^N\) of \(V\) such that \(v_i\) is an eigenvector of the eigenvalue \(\lambda_i(T_X)\), we define \(S_X\) on this basis by \(S_X(v_i) := \tilde{S}_X v_i = [X, v_i]^*\) and then extend it linearly to the whole space as

\[
S_X : M(n, \mathbb{C}) \longrightarrow M(n, \mathbb{C}),
\]

\[
Y = \sum_{i=1}^N y_i v_i \longmapsto \sum_{i=1}^N y_i [X, v_i]^*, \quad \text{for } y_1, \cdots, y_N \in \mathbb{C}.
\]

In particular, by the proof of Proposition 2.6 we can choose the second half of the eigenvectors \(v_i\)'s of those positive eigenvalues \(\lambda_i(T_X)\) to be the image of \(\tilde{S}_X\), namely,

\[
v_{i+n_i} := \tilde{S}_X v_i / \|\tilde{S}_X v_i\| = \tilde{S}_X v_i / \sqrt{\lambda_i(T_X)},
\]

where \(2n_i\) is the even multiplicity of the positive eigenvalues \(\lambda_i(T_X)\). As in Lemma 2.1, suppose that there are \(g\) distinct positive eigenvalues \(\lambda(T_X) = \{t_1 = s_1^2 > \cdots > t_g = s_g^2 > 0\}\) with multiplicities \(2n_1, \cdots, 2n_g\), and denote by \(r = 2 \sum_{j=1}^g n_j\) the rank of \(T_X\). Under the special basis above, the linear operator \(S_X\) can be represented by the real skew-symmetric matrix

\[
S_X = \begin{pmatrix}
O & -s_1 I_{n_1} \\
\varepsilon_1 I_{n_1} & O \\
\varepsilon_2 I_{n_2} & \varepsilon_3 I_{n_3} & O \\
\varepsilon_g I_{n_g} & O & O \\
o & O_{N-r}
\end{pmatrix},
\]
while $T_X$ is represented by

$$T_X = \text{diag}\left( t_{1}I_{2n_1}, \ldots, t_{g}I_{2n_g}, O_{N-r} \right).$$

One can also reorder the basis in the way $v_{2i} = \tilde{S}_X v_{2i-1}/\sqrt{\lambda_{2i-1}(T_X)}$ such that

$$S_X = \begin{pmatrix} I_{n_1} \otimes \begin{pmatrix} 0 & -s_1 \\ s_1 & 0 \end{pmatrix} \\ \vdots \\ I_{n_g} \otimes \begin{pmatrix} 0 & -s_g \\ s_g & 0 \end{pmatrix} \end{pmatrix} O_{N-r}^{-1}.$$

(2.4)

Hence, Lemma 2.3 is suitable for the pair $(T_X, S_X)$ and will be applied in the proof of the equivalence between Conjecture 2 and Conjecture 4.

We will also need the following notations and useful lemmas. Let Vec be the canonical isomorphism from $M(n, \mathbb{C})$ to $\mathbb{C}^N$, i.e.

$$\text{Vec} : M(n, \mathbb{C}) \rightarrow \mathbb{C}^N, \quad X = (x_{ij}) \mapsto (x_{11}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2}, \ldots, x_{1n}, \ldots, x_{nn})^t,$$

where $X^t$ is the transpose of $X$. Using Kronecker product of matrices, we have

**Lemma 2.7.** [19] $\text{Vec}(AYB) = (B^t \otimes A) \text{Vec}(Y)$.

Moreover, Vec is an isometry since $\langle X, Y \rangle = \langle \text{Vec}(X), \text{Vec}(Y) \rangle$, and thus we can calculate the eigenvalues of $T_X$ by

$$\lambda(T_X) = \lambda(\text{Vec} \circ T_X \circ (\text{Vec})^{-1}).$$

**Proposition 2.8.** $\lambda(T_X) = \lambda(K_X^* K_X) = \lambda(K_1 + K_2)$, where $K_X = I \otimes X - X^t \otimes I$ and $K_1 = I \otimes X^* X + \overline{X} X^t \otimes I$, $K_2 = -X^t \otimes X^* - \overline{X} \otimes X$.

**Proof.** By Lemma 2.7 we have $\text{Vec}([X,Y]) = K_X \text{Vec}(Y)$, where

$$K_X = I \otimes X - X^t \otimes I$$

is regarded as a linear operator on $\mathbb{C}^N$, or equivalently as a $N \times N$ matrix. It is easily seen that $K_X^* = K_X$.

Define $\Phi_X(Y) := [X,Y]$, then

$$\text{Vec} \circ \Phi_X \circ (\text{Vec})^{-1} = K_X, \quad T_X = \Phi_X^* \circ \Phi_X.$$

In particular, we have

$$\text{Vec} \circ T_X \circ (\text{Vec})^{-1} = K_X^* K_X = K_X^* K_X,$$

hence

$$\lambda(T_X) = \lambda(K_X^* K_X).$$
By direct calculation, we have \( K^*_X K_X = K_1 + K_2 \), where
\[
K_1 = I \otimes X^* X + \overline{X} X^t \otimes I, \quad K_2 = -(X^t \otimes X^* + \overline{X} \otimes X)
\]
are Hermitian matrices.

**Corollary 2.9.** For \( X \in M(n, \mathbb{C}) \) with \( \|X\| = 1 \), we have \( \text{Tr} T_X = 2n - 2|\text{Tr} X|^2 \). In particular, for \( n = 2 \), \( \lambda_1(T_X) = \lambda_2(T_X) = 2 - |\text{Tr} X|^2 \) and \( \lambda_3(T_X) = \lambda_4(T_X) = 0 \).

**Proof.** It follows immediately from Proposition 2.8 that
\[
\text{Tr} T_X = \text{Tr} K_1 + \text{Tr} K_2 = 2n \|X\|^2 - 2|\text{Tr} X|^2 = 2n - 2|\text{Tr} X|^2.
\]
For \( n = 2 \), the conclusion follows from Proposition 2.6 and the fact that \( T_X X = 0 \) and thus \( T_X \) must have a zero eigenvalue. \( \square \)

To end this section, we prepare two useful lemmas about eigenvalues of Kronecker product and sum of two matrices.

**Lemma 2.10.** [31] Let \( A \) and \( B \) be \( m \times m \) and \( n \times n \) complex matrices with eigenvalues \( \lambda_1, \ldots, \lambda_m \) and \( \mu_1, \ldots, \mu_n \), respectively. Then the eigenvalues of \( A \otimes B \) are
\[
\lambda_i \mu_j, 1 \leq i \leq m, 1 \leq j \leq n,
\]
and the eigenvalues of \( A \otimes I_n + I_m \otimes B \) are
\[
\lambda_i + \mu_j, 1 \leq i \leq m, 1 \leq j \leq n.
\]

**Lemma 2.11.** [31] Let \( A, B \) be \( n \times n \) Hermitian matrices and \( C = A + B \). If \( \alpha_1 \geq \cdots \geq \alpha_n, \beta_1 \geq \cdots \geq \beta_n \), and \( \gamma_1 \geq \cdots \geq \gamma_n \) are the eigenvalues of \( A, B, \) and \( C \), respectively. Then for any sequence \( 1 \leq i_1 < \cdots < i_k \leq n \),
\[
\sum_{t=1}^{k} \alpha_{i_t} + \sum_{t=1}^{k} \beta_{n-i_t+t} \leq \sum_{t=1}^{k} \gamma_{i_t} \leq \sum_{t=1}^{k} \alpha_{i_t} + \sum_{t=1}^{k} \beta_t.
\]

### 3. Some new proofs of the complex BW inequality

In this section, we will give some new simple proofs of the complex BW inequality by eigenvalue estimates of \( T_X \) in (2.1) for \( X \in M(n, \mathbb{C}) \) with \( \|X\| = 1 \). Each estimate implies \( \lambda_1(T_X) \leq 2 \) and thus the complex BW inequality since for \( \|Y\| = 1 \),
\[
\|X, Y\| \leq \max_{\|Y\| = 1} \|(X, Y)\| = \max_{\|Y\| = 1} \langle T_X Y, Y \rangle = \lambda_1(T_X) \leq 2 = 2\|X\|\|Y\|^2.
\]
As a matter of fact, the core of our approach lies in the fact that the multiplicity of positive eigenvalues of \( T_X \) is even by Proposition 2.6.
Theorem 3.1. Let $X = A + B \in M(n, \mathbb{C})$ be the canonical decomposition and $\|X\| = 1$, where $A$ is Hermitian, $B$ is skew-Hermitian. Then

$$\lambda_1(T_X) \leq 2 \left( \max_{i,j} \{-a_i a_j\} + \max_{i,j} \{-b_i b_j\} \right) + \left( \sigma_1^2(X) + \sigma_2^2(X) \right) \leq 2,$$

where $\sigma_1(X) \geq \cdots \geq \sigma_n(X)$ are singular values of $X$ and $\lambda(A) = \{a_1, \cdots, a_n\}$, $\lambda(B) = \{b_1, \cdots, b_n\}$ are eigenvalues of $A$, $B$ respectively.

Proof. Let $\sigma_1(X) \geq \cdots \geq \sigma_n(X)$ be singular values of $X$, then

$$\lambda(X^*X) = \lambda(XX^t) = \{\sigma_1^2(X), \cdots, \sigma_n^2(X)\}.$$ 

Hence for $K_1 = I \otimes X^*X + XX^t \otimes I$ in Proposition 2.8, we have by Lemma 2.10 (3.1)

$$\lambda(K_1) = \{\sigma_1^2(X) + \sigma_2^2(X) : 1 \leq i, j \leq n\}.$$ 

In particular, $\lambda_2(K_1) = \sigma_1^2(X) + \sigma_2^2(X)$. Let $X = A + B$, where $A$ is Hermitian, $B$ is skew-Hermitian. Thus for $K_2$ in Proposition 2.8, we have

$$K_2 = -X^t \otimes X^* - X \otimes X = 2 (B^t \otimes B - A^t \otimes A).$$

Then by Lemma 2.10

$$\lambda(-A^t \otimes A) = \{-a_i a_j : 1 \leq i, j \leq n\},$$

$$\lambda(B^t \otimes B) = \{-b_i b_j : 1 \leq i, j \leq n\},$$

where $\lambda(A) = \{a_1, \cdots, a_n\}$, $a_1 \geq \cdots \geq a_n$; $\lambda(B) = \{b_1, \cdots, b_n\}$, $b_1 \geq \cdots \geq b_n$. Therefore

$$\lambda_1(-A^t \otimes A) = \max_{i,j} \{-a_i a_j\} = \max\{\max_{i,j} \{-a_i a_j\}, \max_{i \neq j} \{-a_i^2\}\}$$

$$\leq \max_{i \neq j} \{-a_i a_j\}, 0 \leq \max_{i \neq j} \{|a_i a_j|\}$$

$$\leq \frac{1}{2} \max_{i \neq j} \{a_i^2 + a_j^2\} \leq \frac{1}{2} \|A\|^2,$$

Similarly

$$\lambda_1(B^t \otimes B) = \max_{i,j} \{-b_i b_j\} \leq \frac{1}{2} \max_{i \neq j} \{b_i^2 + b_j^2\} \leq \frac{1}{2} \|B\|^2.$$

Since $B^t \otimes B$ and $-A^t \otimes A$ are Hermitian, by Lemma 2.11 we have

$$\lambda_1(K_2) \leq 2 \left( \lambda_1(B^t \otimes B) + \lambda_1(-A^t \otimes A) \right) = 2 \left( \max_{i,j} \{-a_i a_j\} + \max_{i,j} \{-b_i b_j\} \right)$$

$$\leq \|A\|^2 + \|B\|^2 = \|X\|^2 = 1.$$ 

Moreover, for $K_X^* K_X = K_1 + K_2$ in Proposition 2.8, again by Lemma 2.11, we have

$$\lambda_2(K_X^* K_X) \leq \lambda_2(K_1) + \lambda_1(K_2) \leq \sigma_1^2(X) + \sigma_2^2(X) + \|X\|^2 \leq 2 \|X\|^2.$$
Finally by Proposition 2.6 and 2.8 we have the required estimation
\[ \lambda_1(T_X) = \lambda_2(T_X) = \lambda_2(K_X^* K_X) \leq 2\|X\|^2 = 2. \]

The proof is complete. \qed

For \( X \in M(n, \mathbb{C}) \) with \( \|X\| = 1 \), we have the following characterization of when \( \lambda_1(T_X) \) attains the upper bound 2.

**Theorem 3.2.** \( \lambda_1(T_X) = 2 \) if and only if \( X = U \) diag\( (X_0, O_{n-2})U^* \) for some \( U \in U(n) \), where \( X_0 \in M(2, \mathbb{C}) \) and Tr\( (X_0) = 0 \).

**Proof.** We first prove the necessity. All the inequalities in the proof of Theorem 3.1 achieve equality when \( \lambda_1(T_X) = 2 \). Thus by the equality conditions of (3.2) and (3.3), we have \( \lambda_1 = -\lambda_n =: a \geq 0 \), \( \lambda_2 = -\lambda_{n-1} =: b \geq 0 \), and \( \lambda_i = \lambda_i \) for \( 1 < i < n \). Therefore,

\[ \lambda(A^t \otimes A) = \{a^2, a^2, 0, \ldots, 0, -a^2, -a^2\}, \]
\[ \lambda(B^t \otimes B) = \{b^2, b^2, 0, \ldots, 0, -b^2, -b^2\}, \]

and there exist \( U, V \in U(n) \) such that

\[ U^* AU = \text{diag}(a, -a, 0, \ldots, 0), \]
\[ V^* BV = \text{diag}(b, -b, 0, \ldots, 0). \]

Hence

\[ \text{Tr}(X) = \text{Tr}(A) + \text{Tr}(B) = 0. \]

Because (3.4) achieves equality, the eigenspaces of \( \lambda_1(B^t \otimes B) \) and \( \lambda_1(-A^t \otimes A) \) have a nontrivial intersection. Let \( U = (u_1, u_2, \ldots, u_n), V = (v_1, v_2, \ldots, v_n) \), we have

\[ Au_1 = au_1, \quad Au_2 = -au_2, \quad Au_j = 0, \quad 3 \leq j \leq n; \]
\[ Bv_1 = bv_1, \quad Bv_2 = -bv_2, \quad Bv_j = 0, \quad 3 \leq j \leq n. \]

Since \( A \) is Hermitian and \( B \) is skew-Hermitian, we have

\[ A^t \overline{u_1} = a \overline{u_1}, \quad A^t \overline{u_2} = -a \overline{u_2}, \quad A^t \overline{u_j} = 0, \quad 3 \leq j \leq n; \]
\[ B^t \overline{v_1} = b \overline{v_1}, \quad B^t \overline{v_2} = -b \overline{v_2}, \quad B^t \overline{v_j} = 0, \quad 3 \leq j \leq n. \]

By the property of Kronecker product, the eigenspace of \( \lambda_1(-A^t \otimes A) \) is \( \text{Span}_{\mathbb{C}} \{ \overline{u_1} \otimes u_2, \overline{u_2} \otimes u_1 \} \); the eigenspace of \( \lambda_1(B^t \otimes B) \) is \( \text{Span}_{\mathbb{C}} \{ \overline{v_1} \otimes v_2, \overline{v_2} \otimes v_1 \} \). Therefore, there exist \( k_1, k_2, l_1, l_2 \in \mathbb{C} \) and \( |k_1|^2 + |k_2|^2 = |l_1|^2 + |l_2|^2 \neq 0 \) such that

\[ k_1 \overline{u_1} \otimes u_2 + k_2 \overline{u_2} \otimes u_1 = l_1 \overline{v_1} \otimes v_2 + l_2 \overline{v_2} \otimes v_1. \]

Recall that \( U, V \in U(n) \), so we have

\[ k_2 u_2 = l_1 (u_1^t v_2) v_1 + l_2 (u_1^t v_1) v_2, \]
by left multiply $I \otimes u_1^*$ and conjugate $[3.5]$. Similarly,

\[
\begin{align*}
\overline{k_1 u_1} &= \overline{l_1(u_2^* v_2)v_1} + \overline{l_2(u_2^* v_1)v_2}, \\
\overline{l_1 v_1} &= k_1(v_2^* u_2)u_1 + k_2(v_2^* u_1)u_2, \\
\overline{l_2 v_2} &= k_1(v_2^* u_2)u_1 + k_2(v_2^* u_1)u_2.
\end{align*}
\]

There are two cases to discuss:

- If $k_1k_2 \neq 0$, it is easy to see that $\text{Span}_\mathbb{C}\{u_1, u_2\} = \text{Span}_\mathbb{C}\{v_1, v_2\}$.
- If one of $k_1, k_2$ is zero, we can assume without loss of generality that $k_1 \neq 0$ and $k_2 = 0$. Then we claim that one of $l_1, l_2$ is zero, otherwise

\[
\overline{l_1 v_1} = k_1(v_2^* u_2)u_1, \quad \overline{l_2 v_2} = k_1(v_2^* u_2)u_1
\]

will lead to a contradiction. So we can also assume without loss of generality that $l_1 \neq 0, l_2 = 0$, thus

\[
k_1 \overline{v_1} \otimes u_2 = \overline{l_1} \overline{v_2} \otimes v_2.
\]

Since $U, V \in U(n)$, we have $|k_1/l_1| = 1$ and

\[
1 = (v_1^t \otimes v_2^t)(\overline{l_1} \otimes v_2) = (k_1/l_1)(v_1^t \otimes v_2^t)(\overline{l_1} \otimes u_2) = (k_1/l_1)(v_1^t \overline{v_1} \otimes v_2 u_2).
\]

The equality condition of Cauchy-Schwartz inequality implies that $u_1, v_1$ are linear dependent and $u_2, v_2$ are linear dependent.

In both cases, we have $\text{Span}_\mathbb{C}\{u_1, u_2\} = \text{Span}_\mathbb{C}\{v_1, v_2\}$. Therefore

\[
U^* X U = U^* A U + U^* B U = \text{diag}(a, -a, 0, \cdots, 0) + \text{diag}(B_0, O_{n-2}),
\]

where $B_0 \in M(2, \mathbb{C})$. Setting $X_0 := \text{diag}(a, -a) + B_0$, we have the necessity.

To prove the sufficiency, since $X_0 \in M(2, \mathbb{C})$ and $\text{Tr} X_0 = 0$, it follows from Lemma 2.3 and Corollary 2.4 that

\[
\lambda_1(T_X) = \lambda_1(T_{\text{diag}(X_0, O_{n-2})}) = \lambda_1(T_{X_0}) = 2 - |\text{Tr}(X_0)|^2 = 2.
\]

This completes the proof. \qed

Now we give a new proof of the equality condition for the complex BW inequality.

**Definition 3.** [5] A pair $(X, Y)$ of $M(n, \mathbb{C})$ is said to be maximal if $X \neq O$, $Y \neq O$ and $\|XY - YX\|^2 = 2\|X\|^2\|Y\|^2$ is satisfied.

**Corollary 3.3.** Let $X, Y \in M(n, \mathbb{C})$ be nonzero matrices. Then $(X, Y)$ is maximal if and only if there exists a unitary matrix $U \in U(n)$ such that

\[
X = U \text{diag}(X_0, 0)U^* \quad \text{and} \quad Y = U \text{diag}(Y_0, 0)U^*
\]

with a maximal pair $(X_0, Y_0)$ in $M(2, \mathbb{C})$, i.e., $X_0 \perp Y_0$ and $\text{Tr} X_0 = \text{Tr} Y_0 = 0$. 
Proof. Without loss of generality, we assume \( \|X\| = \|Y\| = 1 \). If \((X, Y)\) is maximal, by definition, we have
\[
\langle TXY, Y \rangle = \langle TYX, X \rangle = \| [X, Y] \|_2^2 = 2.
\]
Thus \( \lambda_1(T_X) = \lambda_1(T_Y) = 2 \) and hence by Theorem 3.2 there exist unitary matrices \( U_1, U_2 \in U(n) \) such that
\[
X = U_1 \operatorname{diag}(X_0, 0) U_1^* \quad \text{and} \quad Y = U_2 \operatorname{diag}(\tilde{Y}_0, 0) U_2^*
\]
with \( \operatorname{Tr} X = \operatorname{Tr} Y = 0 \). Since \( Y \) is an eigenvector of the maximal eigenvalue \( \lambda_1(T_X) = 2 \) and \( X \) is an eigenvector of the zero eigenvalue of \( T_X \), we know immediately \( X \perp_C Y \).

Moreover, by (2.2) and Lemma 2.5 we know \( U_1^* Y U_1 \) is an eigenvector of the maximal eigenvalue \( \lambda_1(T_{U_1^* X U_1}) = \lambda_1(T_{X_0}) = 2 \), which implies \( U_1^* Y U_1 = \operatorname{diag}(Y_0, 0) \) for some \( Y_0 \in M(2, \mathbb{C}) \). This completes the proof of the necessity.

The sufficiency can be verified by direct computation (cf. [5]).

Let \( \|X\|_{(2), 2} \) be the \((2, 2)\)-norm defined by
\[
\|X\|_{(2), 2} = \sqrt{\sigma_1^2(X) + \sigma_2^2(X)}.
\]
For \( X \in M(n, \mathbb{R}) \), Lu [24] has already proved
\[
\lambda_1(T_X) \leq 2 \|X\|_{(2), 2}^2.
\]
In fact, we can show this inequality holds also for \( X \in M(n, \mathbb{C}) \).

**Theorem 3.4**. For \( X \in M(n, \mathbb{C}) \) with \( \|X\| = 1 \), we have \( \lambda_1(T_X) \leq 2 \|X\|_{(2), 2}^2 \leq 2 \).

Proof. For \( Y \in M(n, \mathbb{C}) \), by Proposition 2.8 we have
\[
\langle WW^* \tilde{v}, \tilde{v} \rangle = \langle T_X Y, Y \rangle,
\]
where
\[
W = \begin{pmatrix} I \otimes X^* & O \\ -X \otimes I & O \end{pmatrix}_{2N \times 2N}, \quad \tilde{v} = \begin{pmatrix} \operatorname{Vec} Y \\ \operatorname{Vec} Y \end{pmatrix}.
\]
Noticing that
\[
W^* W = I \otimes XX^* + X^*X \otimes I,
\]
we have by Proposition 2.6 and Lemma 2.10 that
\[
\lambda_1(T_X) = \lambda_2(T_X) \leq 2 \lambda_2(WW^*) = 2 \lambda_2(W^*W) = 2(\sigma_1^2(X) + \sigma_2^2(X)) = 2 \|X\|_{(2), 2}^2.
\]
This completes the proof.
Denote the upper bound in Theorem 3.1 by
\[ C_X := 2\max_{i,j} \{-a_i a_j + b_i b_j\} + \|X\|_2^2. \]

It worths remarking that \( C_X \leq 2\|X\|_2^2 \) if \( \text{rank}(X) \leq 2 \). In general, \( C_X \) is not necessarily less than \( 2\|X\|_2^2 \), since \( \{|a_j - ib_{n-j+1}|^2\}_{j=1}^n \) is majorized by \( \{\sigma_j^2(X)\}_{j=1}^n \) due to Ando-Bhatia [1]. Therefore these two upper bounds are strictly different. Combining Theorems 3.1 and 3.4, we have the following estimate.

**Corollary 3.5.** For \( X \in M(n, \mathbb{C}) \) with \( \|X\| = 1 \), we have
\[ \lambda_1(TX) \leq \min\{C_X, 2\|X\|_2^2\} \leq 2. \]

Furthermore, our approach can be used to estimate all eigenvalues of \( TX \) by that of \( K_1 \) in Proposition 2.8. Recall that the set of eigenvalues of \( K_1 \) is given in (3.1):
\[ \lambda(K_1) = \{\sigma_i^2(X) + \sigma_j^2(X) : 1 \leq i, j \leq n\}. \]

**Theorem 3.6.** For \( X \in M(n, \mathbb{C}) \) with \( \|X\| = 1 \), we have \( \lambda_i(TX) \leq 2\lambda_i(K_1) \) for all \( i \).

**Proof.** Recall that \( K_1 = I \otimes X^* X + \overline{X} X^t \otimes I \), \( K_2 = -(X^t \otimes X^* + \overline{X} \otimes X) \), and
\[ \text{Vec} \circ TX \circ (\text{Vec})^{-1} = K_1 + K_2. \]

Let \( \hat{K}_X := I \otimes X + X^t \otimes I \). Then we observe
\[ 2K_1 - \text{Vec} \circ TX \circ (\text{Vec})^{-1} = K_1 - K_2 = \hat{K}_X^* \hat{K}_X \geq 0, \]
which implies
\[ \lambda_i(TX) \leq 2\lambda_i(K_1), \quad \text{for all } i. \]
The proof is complete. \( \square \)

In particular, Theorem 3.6 implies Theorem 3.4 since
\[ \lambda_1(TX) = \lambda_2(TX) \leq 2\lambda_2(K_1) = 2(\sigma_1^2 + \sigma_2^2) = 2\|X\|_2^2. \]

### 4. Equivalence of the conjectures with the LW conjecture

In this section, we prove the equivalence between Conjectures 4.6 and Conjecture 2, i.e., Theorem 1.1 in the complex version. This theorem will be divided into the following propositions.

**Proposition 4.1.** Conjecture 2 is equivalent to Conjecture 4.
Proof. Assume Conjecture 2 is true at first. Setting $B = X$ and $B_\alpha$ be a unit eigenvector of $\lambda_{2\alpha-3}(T_X)$ for $\alpha = 2, \cdots, m$, by the last expression of $S_X$ in (2.3, 2.4) we know $S_X B_\alpha = [B, B_\alpha]^*$ is exactly an eigenvector of $\lambda_{2\alpha-2}(T_X)$. Therefore the conditions (i,ii) of Conjecture 2 are satisfied and thus we have the inequality (1.2). Then the inequality (1.3) of Conjecture 4 for $k = m - 1$ follows by Proposition 2.6 and the following

$$\sum_{i=1}^{2k} \lambda_i(T_X) = 2 \sum_{\alpha=2}^{m} \lambda_{2\alpha-3}(T_X) = 2 \sum_{\alpha=2}^{m} (T_B B_\alpha, B_\alpha) = 2 \sum_{\alpha=2}^{m} \|[B, B_\alpha]\|^2$$

$$\leq 2 \left( \max_{2 \leq \alpha \leq m} \|B_\alpha\|^2 + \sum_{\alpha=2}^{m} \|B_\alpha\|^2 \right) \|B\|^2 = 2m = 2(k + 1).$$

Now we assume Conjecture 4 is true. Without loss of generality, we assume $1 = \|B\| \geq \|B_2\| \geq \cdots \geq \|B_m\| > 0$. Using summation by parts, we can write

$$\sum_{\alpha=2}^{m} \|[B, B_\alpha]\|^2 = \sum_{\alpha=2}^{m} (T_B B_\alpha, B_\alpha) = \sum_{\alpha=2}^{m} (T_B \frac{B_\alpha}{\|B_\alpha\|}, \frac{B_\alpha}{\|B_\alpha\|}) \|B_\alpha\|^2$$

$$= \sum_{\beta=2}^{m} (\|B_\beta\|^2 - \|B_{\beta+1}\|^2) \sum_{\alpha=2}^{\beta} (T_B \frac{B_\alpha}{\|B_\alpha\|}, \frac{B_\alpha}{\|B_\alpha\|}),$$

where $B_{m+1} = 0$. Setting $X = B$, the conditions (i,ii) of Conjecture 2 show that the subspace $W := \text{Span}_C \{B_\alpha\}_{\alpha=2}^{m}$ is isotropic about $S_X$, i.e., $S_X(W) \perp C W$. Then by the formula above, Lemma 2.3 and the inequality (1.3) of Conjecture 4 we have

$$\sum_{\alpha=2}^{m} \|[B, B_\alpha]\|^2 \leq \sum_{\beta=2}^{m} (\|B_\beta\|^2 - \|B_{\beta+1}\|^2) \sum_{\alpha=2}^{\beta} \lambda_{2\alpha-3}(T_X)$$

$$\leq \sum_{\beta=2}^{m} (\|B_\beta\|^2 - \|B_{\beta+1}\|^2) \sum_{\alpha=2}^{\beta} \lambda_{2\alpha-3}(T_X)$$

$$= \|B_2\|^2 + \sum_{\alpha=2}^{m} \|B_\alpha\|^2,$$

which is the inequality (1.2) of Conjecture 2.

The proof is complete. \qed

**Proposition 4.2.** Conjecture 4 is equivalent to Conjecture 5.

Proof. Obviously Conjecture 5 implies Conjecture 4 by definition. Suppose Conjecture 4 be true. To prove Conjecture 5 we only need to prove the following four parts:

(i) $\lambda_1(T_X) \leq 2$;
(ii) $\sum_{i=1}^{2k} \lambda_i(T_X) \leq 2k + 2$;
\begin{align*}
\sum_{i=1}^{2k-1} \lambda_i(T_X) &\leq 2k + 1; \\
\sum_{i=1}^N \lambda_i(T_X) & = 2n - 2|\text{Tr } X|^2 \leq 2n,
\end{align*}

where (i) and (iv) are ensured by the complex BW inequality (e.g., Theorem 3.1) and Corollary 2.9 and (ii) is assumed by Conjecture 4, respectively. We are left to show the inequality (iii). We prove it by contradiction in the following.

Assume that there is a positive number \( m \geq 2 \) such that
\[
\sum_{i=1}^{2m-1} \lambda_i(T_X) > 2m + 1.
\]
Then
\[
2m + 1 < \sum_{i=1}^{2m-1} \lambda_i(T_X) = \lambda_{2m-1}(T_X) + \sum_{i=1}^{2m-2} \lambda_i(T_X) \leq \lambda_{2m-1}(T_X) + 2m.
\]
Thus
\[
\lambda_{2m}(T_X) = \lambda_{2m-1}(T_X) > 1,
\]
and
\[
\sum_{i=1}^{2m} \lambda_i(T_X) = \lambda_{2m}(T_X) + \sum_{i=1}^{2m-1} \lambda_i(T_X) > 1 + 2m + 1 = 2m + 2.
\]
This leads to the contradiction to (ii) and completes the proof. \( \square \)

**Proposition 4.3.** Conjecture 4 is equivalent to Conjecture 6.

**Proof.** The proof is similar to that of Proposition 4.1, without using Lemma 2.3 now since we have no condition (ii) of Conjecture 2. For the sake of clearness, we repeat it as follows.

Assume Conjecture 6 is true. Setting \( B = X \) and \( B_\alpha \) be a unit eigenvector of \( \lambda_{\alpha-1}(T_X) \) for \( \alpha = 2, \ldots, m \), we know \( B_\alpha \)'s are C-orthogonal and therefore we have the inequality of Conjecture 6. Then the inequality (1.3) of Conjecture 4 for \( m = 2k + 1 \) follows by
\[
\sum_{i=1}^{m-1} \lambda_i(T_X) = \sum_{\alpha=2}^{m} \lambda_{\alpha-1}(T_X) = \sum_{\alpha=2}^{m} \langle TB_\alpha, B_\alpha \rangle = \sum_{\alpha=2}^{m} \| [B, B_\alpha] \|^2 \\
\leq \left( 2 \max_{2 \leq \alpha \leq m} \| B_\alpha \|^2 + \sum_{\alpha=2}^{m} \| B_\alpha \|^2 \right) \| B \|^2 = m + 1.
\]

Now we assume Conjecture 4 is true and hence Conjecture 5 is true by Proposition 4.2. In particular, we have
\[
\sum_{i=1}^{m} \lambda_i(T_X) \leq m + 2, \quad \text{for any } m.
\]
Without loss of generality, we assume $1 = \|B\| \geq \|B_2\| \geq \cdots \geq \|B_m\| > 0$ and set $B_{m+1} = 0$. Then using summation by parts, we have
\[
\sum_{\alpha=2}^{m} \| [B, B_{\alpha}] \|^2 = \sum_{\alpha=2}^{m} \langle T_B B_{\alpha}, B_{\alpha} \rangle = \sum_{\alpha=2}^{m} \langle T_B \frac{B_{\alpha}}{\|B_{\alpha}\|}, \frac{B_{\alpha}}{\|B_{\alpha}\|} \rangle \|B_{\alpha}\|^2
\]
\[
= \sum_{\beta=2}^{m} (\|B_{\beta}\|^2 - \|B_{\beta+1}\|^2) \sum_{\alpha=2}^{\beta} \langle T_B \frac{B_{\alpha}}{\|B_{\alpha}\|}, \frac{B_{\alpha}}{\|B_{\alpha}\|} \rangle,
\]
\[
\leq \sum_{\beta=2}^{m} (\|B_{\beta}\|^2 - \|B_{\beta+1}\|^2) \sum_{\alpha=1}^{\beta-1} \lambda_\alpha(T_X)
\]
\[
\leq \sum_{\beta=2}^{m} (\|B_{\beta}\|^2 - \|B_{\beta+1}\|^2) (\beta + 1)
\]
\[
= 2\|B_2\|^2 + \sum_{\alpha=2}^{m} \|B_{\alpha}\|^2,
\]
which is the inequality of Conjecture 6.

The proof is complete. \(\square\)

**Proposition 4.4.** [25] The LW Conjecture 2 implies Conjectures 1 and 3.

**Proof.** Conjecture 3 is trivially implied by Conjecture 4 and thus by Conjecture 2.

As for Conjecture 1 for the sake of completeness, we copy the proof of the real version from [25] for our complex version now.

We first observe that the inequality (1.1) is invariant under the transformations
\[
M(n, \mathbb{C}) \rightarrow M(n, \mathbb{C}),
\]
\[
A_{\alpha} \mapsto QA_{\alpha}Q^*,
\]
\[
A_{\alpha} \mapsto \sum_{\beta=1}^{m} p_{\alpha\beta}A_{\beta},
\]
for all unitary $n \times n$ matrices $Q$ and $m \times m$ matrices $P = (p_{\alpha\beta})$.

Now, let $a > 0$ be the largest positive real number such that
\[
(\sum_{\alpha=1}^{m} \|A_{\alpha}\|^2)^2 \geq 2a(\sum_{\alpha<\beta} \|[A_{\alpha}, A_{\beta}]\|^2)
\]
for all matrices $A_{\alpha}$’s satisfying the condition of Conjecture 1.

Since $a$ is maximal, by the invariance we can find matrices $A_1, \cdots, A_m$ such that
\[
(\sum_{\alpha=1}^{m} \|A_{\alpha}\|^2)^2 = 2a(\sum_{\alpha<\beta} \|[A_{\alpha}, A_{\beta}]\|^2)
\]
with the following additional properties:
(1) $\text{Tr } A_\alpha A_\beta^* = 0$ for any $\alpha \neq \beta$;
(2) $\text{Tr } A_\alpha [A_\gamma, A_\beta] = 0$ for any $1 \leq \alpha, \beta, \gamma \leq m$;
(3) $0 \neq ||A_1|| \geq ||A_2|| \geq \cdots \geq ||A_m||$.

We let $t^2 = ||A_1||^2$ and let $A' = A_1/t$. Then (4.1) becomes a quadratic expression in terms of $t^2$:

$$t^4 - 2t^2 \left( a \sum_{1<\alpha} ||[A', A_\alpha]]||^2 - \sum_{1<\alpha} ||A_\alpha||^2 \right) + \left( \sum_{\alpha=2}^m ||A_\alpha||^2 \right)^2 - 2a \left( \sum_{1<\alpha<\beta} ||[A_\alpha, A_\beta]\|^2 \right) = 0.$$

Since the left-hand side of the above is nonnegative for all $t^2 \geq 0$ and is zero for $t^2 = ||A_1||^2$, we have

$$a \sum_{1<\alpha} ||[A', A_\alpha]]||^2 - \sum_{1<\alpha} ||A_\alpha||^2 > 0,$$

and

$$||A_1||^2 = a \sum_{1<\alpha} ||[A', A_\alpha]]||^2 - \sum_{1<\alpha} ||A_\alpha||^2.$$  

By Conjecture 2, we have

$$\sum_{1<\alpha} ||[A', A_\alpha]]||^2 \leq \sum_{\alpha=2}^m ||A_\alpha||^2 + ||A_2||^2 \leq \sum_{\alpha=1}^m ||A_\alpha||^2,$$

which proves that $a \geq 1$ and this completes the proof.

5. Partial results on the complex LW Conjecture

In this section, we prove the complex LW Conjecture separately for those special cases (Theorem 1.2), and for general cases, we give some non-sharp upper bounds for the inequalities of Conjectures 3 and 7 (Theorems 1.3 and 1.4).

Firstly we prove the complex version of Conjecture 3 for the first special case of Theorem 1.2. We remind that Conjecture 3 is also the first step of the complex LW Conjecture after the solution of the BW inequality (i.e., $\lambda_1(T_X) \leq 2$).

**Theorem 5.1.** *Conjecture 3 is true when $X$ is a normal matrix.*

**Proof.** Since $X$ is a normal matrix, there exists a unitary matrix $U$ such that

$$U^*XU = \text{diag}(x_1, \cdots, x_n), \text{ for some } x_1, \cdots, x_n \in \mathbb{C} \text{ with } \sum_i |x_i|^2 = 1.$$

Direct calculations show that for any $1 \leq i, j \leq n$,

$$T_{U^*XU}(E_{ij}) = |x_i - x_j|^2 E_{ij},$$
where $E_{ij} \in M(n, \mathbb{C})$ is the standard basis matrix with the $(i, j)$-element being 1 and the others being 0. Then by the identity (2.2):

$$T_{U^*XU}(U^*UY) = U^*TX(Y)U,$$

we have

$$T_X(UE_{ij}U^*) = UT_{U^*XU}(E_{ij})U^* = |x_i - x_j|^2UE_{ij}U^*.$$

It follows that

$$\lambda(T_X) = \{ |x_i - x_j|^2 : 1 \leq i, j \leq n \} = \{ \lambda_1 \geq \cdots \geq \lambda_n \}.$$

Suppose $\lambda_1 = \lambda_2 = |x_a - x_b|^2$, $\lambda_3 = \lambda_4 = |x_c - x_d|^2$, where $1 \leq a, b, c, d \leq n$. There are two cases need to be discussed:

- If $a, b, c, d$ are four different integers, then

$$\lambda_1 + \lambda_3 = |x_a - x_b|^2 + |x_c - x_d|^2 \leq 2(|x_a|^2 + |x_b|^2 + |x_c|^2 + |x_d|^2) \leq 2.$$

- If one of $a, b$ is equal to one of $c, d$, we can assume $a = c, b \neq d$. Then

$$\lambda_1 + \lambda_3 = |x_a - x_b|^2 + |x_a - x_d|^2
= |x_a|^2 - \overline{x_a}x_b + |x_b|^2 + |x_c|^2 - \overline{x_c}x_d + |x_d|^2
\leq 2|x_a|^2 + 2|x_a|(|x_b| + |x_d|) + |x_b|^2 + |x_d|^2
\leq 3|x_a|^2 + (|x_b| + |x_d|)^2 + |x_b|^2 + |x_d|^2
\leq 3(|x_a|^2 + |x_b|^2 + |x_d|^2) \leq 3.$$

The equality holds if and only if $|x_a| = \sqrt{\frac{\lambda_1}{3}}$, $|x_b| = |x_d| = \sqrt{\frac{\lambda_3}{6}}$, other $x_c = 0$ and $x_a, x_b, x_d$ are co-linear in the complex plane.

The proof is complete. \[\square\]

For more general cases, we need Lu’s lemma in the complex version:

**Lemma 5.2.** Suppose $\eta_1, \ldots, \eta_n$ are complex numbers and

$$\eta_1 + \cdots + \eta_n = 0, \quad |\eta_1|^2 + \cdots + |\eta_n|^2 = 1.$$  

Let $r_{ij} \geq 0$ be nonnegative numbers for $i < j$. Then we have

$$(5.1) \quad \sum_{i<j} |\eta_i - \eta_j|^2 r_{ij} \leq \sum_{i<j} r_{ij} + \max_{i<j}(r_{ij}).$$

**Corollary 5.3.** The complex LW Conjecture \[7\] is true when $X$ is a normal matrix.

**Proof.** Let $X$ be a normal matrix and $r_{ij} \in \{0, 1\}$, then it follows from the proof of Theorem 5.1 that

$$\lambda(T_X) = \{ |\eta_i - \eta_j|^2 : 1 \leq i, j \leq n \},$$

where $E_{ij} \in M(n, \mathbb{C})$ is the standard basis matrix with the $(i, j)$-element being 1 and the others being 0. Then by the identity (2.2):

$$T_{U^*XU}(U^*UY) = U^*TX(Y)U,$$

we have

$$T_X(UE_{ij}U^*) = UT_{U^*XU}(E_{ij})U^* = |x_i - x_j|^2UE_{ij}U^*.$$
where $\lambda(X) = \{\eta_1, \eta_2, \cdots, \eta_n\}$. Thus Corollary 5.2 applies to tell us
\[ \sum_{\alpha=1}^{k} \lambda_{2\alpha-1} \leq k + 1, \]
where $\lambda_{2\alpha-1}$ equals some $|\eta_i - \eta_j|^2$ and $r_{ij} = 1$ for $k$ pairs of $(i < j)$.

This completes the proof. \quad \square

**Corollary 5.4.** Let $B_1, \cdots, B_m \in M(n, \mathbb{C})$ be Hermitian matrices. Assume that
\[ \text{Tr} \left( B_\alpha [B_\gamma, B_\beta] \right) = 0 \]
for any $1 \leq \alpha, \beta, \gamma \leq m$, we have
\[ \sum_{\alpha,\beta=1}^{m} \| [B_\alpha, B_\beta] \|^2 \leq \left( \sum_{\alpha=1}^{m} \| B_\alpha \|^2 \right)^2. \]

**Proof.** As Hermitian matrices are normal matrices, by Corollary 5.3 above, the complex LW Conjecture 7 holds for this case. This in turn by Theorem 1.1 implies the complex version of Conjecture 1. \quad \square

**Remark 5.5.** When $B_1, \cdots, B_m$ are real symmetric matrices, (5.2) is valid for all $\alpha, \beta, \gamma$. Thus the corollary generalizes the DDVV inequality and is sharp under the trace condition (5.2). We remind that for general Hermitian matrices the optimal constant $c = \frac{4}{3}$ is bigger than 1 here (cf. Section 7, [17, 18]).

Next we prove Conjecture 3 for the second special case rank$(X) = 1$. We will need the following lemma.

**Lemma 5.6.** [3] Let $M \in M(n, \mathbb{C})$ be a complex matrix. Then
\[ \lambda_i \left( \frac{M^* + M}{2} \right) \leq \sigma_i(M), \quad i = 1, \cdots, n, \]
where $\lambda_i$ and $\sigma_i$ are eigenvalues and singular values, respectively.

**Theorem 5.7.** The complex LW Conjecture 7 is true when rank$(X) = 1$.

**Proof.** Recall Proposition 2.8 that we have $K_X^* K_X = K_1 + K_2$, where $K_1 = I \otimes X^* X + \overline{X} X^t \otimes I$, $K_2 = -(X^t \otimes X^* + X \otimes X)$. Denote $K_3 = -X^t \otimes X^*$, then $K_2 = K_3^* + K_3$ and by Lemma 5.6
\[ \lambda_i(K_2) = 2\lambda_i \left( \frac{K_3^* + K_3}{2} \right) \leq 2\sigma_i(K_3), \quad i = 1, \cdots, n. \]
Let $\sigma_1(X) \geq \cdots \geq \sigma_n(X)$ be singular values of $X$, then by Lemma 2.10
\[ \sigma(K_3) = \{ \sigma_i(X)\sigma_j(X) : 1 \leq i, j \leq n \}. \]
In particular, now \( \text{rank}(X) = 1 \) implies \( \sigma_1(X) = 1 \) and \( \sigma_i(X) = 0 \) for \( 2 \leq i \leq n \). Thus we have \( \sigma(K_4) = \{1, 0^{N-1}\} \) and by (3.1)

\[
\lambda(K_1) = \{\sigma_i(X)^2 + \sigma_j(X)^2 : 1 \leq i, j \leq n\} = \{2^1, 1^{2(n-1)}, 0^{(n-1)^2}\}.
\]

Finally by Propositions 2.6, 2.8 and Lemma 2.11, we have

\[
\sum_{i=1}^{k} \lambda_{2i-1}(T_X) = \sum_{i=1}^{k} \lambda_{2i}(T_X) \leq \sum_{i=1}^{k} \lambda_i(K_1) + \sum_{i=1}^{k} \lambda_{2i}(K_3)
\]

\[
\leq \sum_{i=1}^{k} \lambda_i(K_1) + \sum_{i=1}^{k} 2\sigma_{2i}(K_3)
\]

\[
= \sum_{i=1}^{k} \lambda_i(K_1) \leq k + 1,
\]

which completes the proof.

Furthermore, we can get the characteristic polynomial of \( T_X \) if \( \text{rank}(X) = 1 \).

**Proposition 5.8.** Let \( K_X = I \otimes X - X^t \otimes I \). Then the sets of singular values

\[
\sigma(K_X) = \sigma\left(I \otimes \Lambda - (\Lambda \otimes I) \left(Q^t \otimes Q^a\right)\right),
\]

where \( X = Q_1 \Lambda Q_2 \) is the singular value decomposition of \( X \) and \( Q = Q_2 Q_1 \).

**Proof.** Direct calculations show

\[
K_X = I \otimes X - X^t \otimes I
\]

\[
= I \otimes (Q_1 \Lambda Q_2) - (Q_2^t \Lambda Q_1^t) \otimes I
\]

\[
= (Q_2^t \otimes Q_1) \left[I \otimes \Lambda - (\Lambda \otimes I) \left(Q^t \otimes Q^a\right)\right] (Q_2 \otimes Q_2^t).
\]

This completes the proof by the invariance of singular values under congruences.

**Theorem 5.9.** Let \( X \) be a complex square matrix of order \( n \) (\( \geq 2 \)) with \( \|X\| = 1 \) and \( \text{rank}(X) = 1 \), then the characteristic polynomial of \( T_X \) is

\[
\det(\lambda I - T_X) = \left(\lambda - 2 + |\text{Tr} X|^2\right)^2 (\lambda - 1)^{2n-4} \lambda^{(n-1)^2+1}.
\]

**Proof.** Let \( X = Q_1 \Lambda Q_2 \) be the singular value decomposition and \( Q = Q_1^t Q_2^t =: (q_{ij}) \). Proposition 5.8 implies \( \sigma(K_X) = \sigma(\widetilde{K_X}) \), where \( \widetilde{K_X} = I \otimes \Lambda - (\Lambda \otimes I) \left(Q \otimes \overline{Q}\right) \). By Proposition 2.8 we have

\[
\lambda(T_X) = \lambda(K_X K_X^*) = \lambda(\widetilde{K_X} \widetilde{K_X}^*),
\]

where direct calculations show

\[
\widetilde{K_X} \widetilde{K_X}^* = I \otimes \Lambda^2 + \Lambda^2 \otimes I - (Q^* \Lambda) \otimes (\Lambda Q^t) - (\Lambda Q) \otimes (\overline{Q} \Lambda).
\]
Since $\|X\| = 1$ and $\text{rank}(X) = 1$, it implies $\Lambda = \text{diag}(1, 0, \cdots, 0)$. By direct calculations, we have $I \otimes \Lambda^2 + \Lambda^2 \otimes I = \text{diag}(I + \Lambda, \Lambda, \cdots, \Lambda)$ and thus

$$\lambda(I \otimes I) - \widetilde{K}_X \widetilde{K}_X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A := (\lambda - 1) I - \Lambda + q_{11} \overline{Q} \Lambda + q_{11} \Lambda Q^t, \quad B := (q_{12} \overline{Q} \Lambda, q_{13} \overline{Q} \Lambda, \cdots, q_{1n} \overline{Q} \Lambda),$$

$$C := B^*, \quad D := \text{diag}(\lambda I - \Lambda, \lambda I - \Lambda, \cdots, \lambda I - \Lambda).$$

Without loss of generality, suppose that the determinant of matrix $D$ is not zero, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det (A - BD^{-1}C) \cdot \det D$$

$$= \det \left( A - \left( 1 - |q_{11}|^2 \right) \overline{Q} \Lambda \hat{D} \Lambda Q^t \right) \cdot \det D,$$

where $\hat{D} = \text{diag}(\frac{1}{\lambda - 1}, \frac{1}{\lambda} \cdots, \frac{1}{\lambda})$.

Thus

$$A - BD^{-1}C = A - \left( 1 - |q_{11}|^2 \right) \overline{Q} \Lambda \hat{D} \Lambda Q^t = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix},$$

where $\hat{A} := \lambda - 2 + 2 |q_{11}|^2 - \frac{1 - |q_{11}|^2}{\lambda - 1} |q_{11}|^2$, $\hat{B} := (q_{12} \overline{q}_{21} \frac{\lambda - 2 + |q_{11}|^2}{\lambda - 1} \cdots, q_{11} q_{21} \frac{\lambda - 2 + |q_{11}|^2}{\lambda - 1})$, $\hat{C} := \hat{B}^*$, $\hat{D} := (\lambda - 1) I - \frac{1 - |q_{11}|^2}{\lambda - 1} u^* u$, $u := (q_{21}, q_{31}, \cdots, q_{n1})$.

Similarly,

$$\det \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} = \det (\hat{A} - \hat{C} \hat{A}^{-1} \hat{B}) \cdot \det \hat{A}$$

$$= \det \left( (\lambda - 1) I - \frac{1}{\lambda - 1 + |q_{11}|^2} u^* u \right) \cdot \det \hat{A}$$

$$= \left[ \frac{\lambda - 2 + |q_{11}|^2}{\lambda - 1 + |q_{11}|^2} (\lambda - 1)^{n-2} \right] \left[ \frac{1}{\lambda - 1} (\lambda - 1 + |q_{11}|^2) (\lambda - 2 + |q_{11}|^2) \right]$$

$$= (\lambda - 2 + |q_{11}|^2)^2 (\lambda - 1)^{n-3} \lambda.$$

So we have

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det (A - BD^{-1}C) \cdot \det D$$

$$= \det (\hat{A} - \hat{C} \hat{A}^{-1} \hat{B}) \cdot \det \hat{A} \cdot \det D$$

$$= (\lambda - 2 + |q_{11}|^2)^2 (\lambda - 1)^{n-3} \lambda (\lambda - 1)^{n-1} \lambda^{n-1}$$

$$= (\lambda - 2 + |q_{11}|^2)^2 (\lambda - 1)^{2n-4} \lambda^{(n-1)^2+1}.$$
Finally we observe that $q_{11} = \text{Tr} X$. The proof is complete. \hfill \square

Immediately we obtain

**Corollary 5.10.** Let $X$ be a complex square matrix of order $n \geq 2$ with $\|X\| = 1$ and $\text{rank}(X) = 1$. Then $\lambda_1(T_X) = 2$ if and only if $\text{Tr}(X) = 0$.

**Remark 5.11.** Actually, the conditions $\|X\| = 1$, $\text{rank}(X) = 1$ and $\text{Tr}(X) = 0$ in Corollary 5.10 implies that $X$ is unitary similar to $\text{diag}(X_0, O)$, where

$$X_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ 

Here, we give a simple calculation. Suppose $X = Q_1\Lambda Q_2$ is the singular value decomposition of $X$ and $Q = Q_2Q_1$, then $Q_1^*XQ_1 = \Lambda Q$. Due to $\|X\| = 1$, $\text{rank}(X) = 1$ and $\text{Tr}(X) = 0$, we can assume

$$\Lambda Q = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix},$$

where $q = (q_{12}, q_{13}, \cdots, q_{1n})$ and $\|q\| = 1$. Extend $q$ to be a unit orthogonal basis $\{q, p_1, p_2, \cdots, p_{n-2}\}$ of $\mathbb{C}^{n-1}$ and let

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ q^* & p_1^* & p_2^* & \cdots & p_{n-2}^* \end{pmatrix},$$

then $U^*U = I$ and $U^*Q_1^*XQ_1U = \text{diag}(X_0, O)$.

The last special case of Theorem 1.2 is a simple consequence of Corollary 2.9

**Theorem 5.12.** The complex LW Conjecture 1 is true when $n = 2, 3$.

**Proof.** The case $n = 2$ is an immediate consequence of Corollary 2.9 since it implies the set of eigenvalues $\lambda(T_X)$ is weakly majorized by $\{2^2, 0^2\}$.

The case $n = 3$ is similar, since Corollary 2.9 shows that

$$\sum_{i=1}^{2k} \lambda_i(T_X) \leq \text{Tr} T_X = 6 - 2|\text{Tr} X|^2 \leq 6 \leq 2k + 2 \quad \text{for any } k \geq 2,$$

and for $k = 1$ it follows from the BW inequality (e.g., Theorem 3.1) that

$$\sum_{i=1}^{2k} \lambda_i(T_X) = 2\lambda_1(T_X) \leq 4 = 2k + 2.$$

The proof is complete. \hfill \square
Now we come to prove the partial results Theorems 1.3 and 1.4.

**Proof of Theorem 1.3** By Lemma 2.11 and the proof of Theorem 3.1 for the fixed sequence $i_1 = 2, i_2 = 3, i_3 = 4$, we have

$$
\sum_{i=2}^{4} \lambda_i(T_X) \leq \sum_{i=2}^{4} \lambda_i(K_1) + \sum_{i=1}^{3} \lambda_i(K_2) \\
\leq 3 \left( \sigma_1^2(X) + \sigma_2^2(X) \right) + 2 \sum_{i=1}^{3} \left( \lambda_i(-A^t \otimes A) + \lambda_i(B^t \otimes B) \right),
$$

as $K_2 = -X^t \otimes X^* - X \otimes X = 2(B^t \otimes B - A^t \otimes A)$ for the decomposition $X = A + B$ with $A$ Hermitian and $B$ skew-Hermitian. Similarly we have

$$
\lambda_1(T_X) = \lambda_2(T_X) \leq \sigma_1^2(X) + \sigma_2^2(X) + 2 \left( \lambda_1(-A^t \otimes A) + \lambda_1(B^t \otimes B) \right).
$$

This implies

$$
\sum_{i=1}^{4} \lambda_i(T_X) \leq 4 \left( \sigma_1^2(X) + \sigma_2^2(X) \right) + \phi(X),
$$

where $\phi(X) := \varphi(A) + \tilde{\varphi}(B)$ and

$$
\varphi(A) := 4\lambda_1(-A^t \otimes A) + 2 \sum_{i=2}^{3} \lambda_i(-A^t \otimes A),
$$

$$
\tilde{\varphi}(B) := 4\lambda_1(B^t \otimes B) + 2 \sum_{i=2}^{3} \lambda_i(B^t \otimes B).
$$

Let $\lambda(A) = \{a_1, \ldots, a_n\}$, $a_1 \geq \cdots \geq a_n$, $\lambda(B) = \{b_1, \ldots, b_n\}$, $b_1 \geq \cdots \geq b_n$. Then by Lemma 2.10

$$
\lambda(-A^t \otimes A) = \{ -a_i a_j : 1 \leq i, j \leq n \},
$$

$$
\lambda(B^t \otimes B) = \{ -b_i b_j : 1 \leq i, j \leq n \}.
$$

We claim that

$$
\phi(X) = \varphi(A) + \tilde{\varphi}(B) \leq \sqrt{10} \left( \|A\|^2 + \|B\|^2 \right).
$$

We only need to show $\varphi(A) \leq \sqrt{10} \|A\|^2$ since the case for $\tilde{\varphi}(B)$ is similar. Obviously $\varphi(A)$ would be non-positive unless $a_1 > 0 > a_n$, in which case we have

$$
\lambda_1(-A^t \otimes A) = \lambda_2(-A^t \otimes A) = a_1 |a_n| = \max_{i,j} \{-a_i a_j\}
$$

and we can also assume without loss of generality that

$$
\lambda_3(-A^t \otimes A) = \lambda_4(-A^t \otimes A) = a_2 |a_n| \geq 0.
$$
Then
\[
\varphi(A) = 6a_1|a_n| + 2a_2|a_n| = 2\|a_n\|(3a_1 + a_2) \\
\leq 2\sqrt{10}|a_n|\sqrt{a_1^2 + a_2^2} \leq \sqrt{10}(a_n^2 + a_1^2 + a_2^2) \\
\leq \sqrt{10}\|A\|^2.
\]

In conclusion,
\[
\sum_{i=1}^{4} \lambda_i(T_X) \leq 4\left(\sigma_1^2(X) + \sigma_2^2(X)\right) + \sqrt{10}\left(\|A\|^2 + \|B\|^2\right) \leq (4 + \sqrt{10})\|X\|^2.
\]

By Proposition 2.6 this completes the proof. \hfill \Box

Remark 5.13. From the proof, one can see that the (non-sharp) upper bounds for the complex version and real version of Conjecture 3 are no different, both \(2 + \sqrt{10}/2\).

Remark 5.14. The reason of why we did not get the optimal upper bound 3 of Conjecture 3 mainly comes from that we divided the Hermitian matrix \(K_X^*XK_X\) into three parts and estimated them separately. The following example explains that the upper bound \(2 + \sqrt{10}/2\) we got in this way cannot be optimal. Set
\[
X = \begin{pmatrix}
-0.1236 & 0.0334 & 0.0647 \\
-0.4343 & 0.1029 & -0.8833 \\
0 & 0 & 0
\end{pmatrix}.
\]

By numerical calculation we see
\[
\sum_{i=1}^{4} \lambda_i(T_X) \approx 5.9814 < 6 < 4\left(\sigma_1^2(X) + \sigma_2^2(X)\right) + \varphi(X) \approx 7.0554 < 4 + \sqrt{10}.
\]

To estimate higher order eigenvalues, we need the following lemma.

Lemma 5.15. Suppose \(\eta_1, \eta_2, \cdots, \eta_{n_1}\) and \(\omega_1, \omega_2, \cdots, \omega_{n_2}\) are nonnegative real numbers and \(r_{ij} \in \{0, 1\}\) such that
\[
\sum_{i=1}^{n_1} \eta_i^2 + \sum_{i=1}^{n_2} \omega_i^2 = 1, \quad \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} r_{ij} = m.
\]

Then we have
\[
\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \eta_i \omega_j r_{ij} \leq \frac{\sqrt{m}}{2}.
\]

Proof. Suppose \(\eta_1 \geq \cdots \geq \eta_{n_1} \geq 0\) and \(\omega_1 \geq \cdots \geq \omega_{n_2} \geq 0\), without loss of generality we can select the following \(m\) elements with non-vanishing \(r_{ij}\)’s:
\[
\bullet \ \eta_1 \omega_1 \geq \eta_1 \omega_2 \geq \cdots \geq \eta_1 \omega_{p_1} \\
\bullet \ \eta_2 \omega_1 \geq \eta_2 \omega_2 \geq \cdots \geq \eta_2 \omega_{p_2}
\]
where \( p_1 + p_2 + \cdots + p_t = m \). Thus we complete the proof by

\[
\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \eta_i \omega_j r_{ij} = \sum_{i=1}^{t} \sum_{j=1}^{p_i} \omega_j \leq \sqrt{\sum_{i=1}^{t} \eta_i^2} \leq \sqrt{\sum_{i=1}^{t} (\sum_{j=1}^{p_i} \omega_j)^2} \leq \sqrt{\sum_{i=1}^{t} \sum_{j=1}^{p_i} \omega_j^2} \leq \sqrt{m} \frac{m}{2} = \frac{m}{2}.
\]

**Proof of Theorem 1.4.** The proof is similar to that of Theorem 1.3. Briefly, by Lemma 2.11 and Lemma 5.15, we have

\[
\sum_{i=1}^{2k} \lambda_i(T_X) \leq \sum_{i=1}^{2k} \lambda_i(K_1) + \sum_{i=1}^{2k} \lambda_i(K_2) = 2k + 1 + 2 \left( \sqrt{k} \|A\|^2 + \sqrt{k} \|B\|^2 \right) = 2k + 1 + 2\sqrt{k},
\]

where \( \sum_{i=1}^{2k} \lambda_i(K_1) \leq 2k + 1 \) follows from

\[
\lambda_1(K_1) = 2\sigma_1^2(X) \leq 2, \quad \lambda_i(K_1) \leq \lambda_2(K_1) = \sigma_1^2(X) + \sigma_2^2(X) \leq 1 \text{ for } i \geq 2;
\]

and

\[
\sum_{i=1}^{2k} \lambda_i(-A^t \otimes A) \leq 2 \sum_{r=1}^{k} \lambda_{2r-1}(-A^t \otimes A) \leq \sqrt{k} \|A\|^2
\]

(similar for \( \sum_{i=1}^{2k} \lambda_i(B^t \otimes B) \leq \sqrt{k} \|B\|^2 \)) follows by setting in Lemma 5.15

\[
\begin{aligned}
\eta_i &:= a_i/\|A\|, & 1 \leq i \leq n_1, \\
\omega_j &:= -a_{n_1+j}/\|A\|, & 1 \leq j \leq n - n_1,
\end{aligned}
\]

for \( a_1 \geq \cdots \geq a_{n_1} \geq 0 \geq a_{n_1+1} \geq \cdots \geq a_n \) and noticing that now the nonnegative eigenvalues \( \lambda_{2r-1}(-A^t \otimes A) = \lambda_{2r}(-A^t \otimes A) = -a_i a_{n_1+j} \) appear in pairs.

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School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R. CHINA.

E-mail address: jqge@bnu.edu.cn

School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R. CHINA.

E-mail address: faguili@mail.bnu.edu.cn

Department of Mathematics, University of California, Irvine, Irvine, CA 92697, USA.

E-mail address: zlu@math.uci.edu

School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, P.R. CHINA.

E-mail address: zhou.yi@mail.bnu.edu.cn