A note on the existence of \(\{k,k\}\)-equivelar polyhedral maps

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Abstract. A polyhedral map is called \(\{p,q\}\)-equivelar if each face has \(p\) edges and each vertex belongs to \(q\) faces. In [12], it was shown that there exist infinitely many geometrically realizable \(\{p,q\}\)-equivelar polyhedral maps if \(q \geq p = 4\), \(p > q = 4\) or \(q - 3 \geq p = 3\).

It was shown in [6] that there exist infinitely many self dual \(\{k,k\}\)-equivelar polyhedral maps. In [1], it was shown that \(\{5,5\}\)- and \(\{6,6\}\)-equivelar polyhedral maps exist. In this note, examples are constructed, to show that infinitely many self dual \(\{k,k\}\)-equivelar polyhedral maps exist for each \(k \geq 5\). Also vertex-minimal non-singular \(\{p,p\}\)-pattern are constructed for all odd primes \(p\).

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1 Introduction and results

A polyhedral complex (of dimension 2) is collection of cycles (finite connected 2-regular graphs) together with the edges and the vertices in the cycles such that the intersection of any two cycles is empty, a vertex or an edge. The cycles are called the faces of the polyhedral complex. For a polyhedral complex \(K\), \(V(K)\) denotes its vertex-set and \(EG(K)\) denotes its edge-graph or 1-skeleton. We say \(K\) finite if \(V(K)\) is finite. If \(EG(K)\) is connected then \(K\) is said to be connected.

A polyhedral complex is called a polyhedral 2-manifold (or an abstract polyhedron) if for each vertex \(v\) the faces containing \(v\) are of the form \(F_1,\ldots,F_m\), where \(F_1 \cap F_2,\ldots,F_{m-1} \cap F_m, F_m \cap F_1\) are edges for some \(m \geq 3\). A connected finite polyhedral 2-manifold is called a polyhedral map. A combinatorial 2-manifold is a polyhedral 2-manifold whose faces are 3-cycles. A polyhedral map is called \(\{p,q\}\)-equivelar if each face is a \(p\)-cycle and each vertex is in \(q\) faces. A polyhedral map is called equivelar if it is \(\{p,q\}\)-equivelar for some \(p, q\) (cf. [10, 3, 4, 11]).

To each polyhedral complex \(K\), we associate a pure 2-dimensional simplicial complex \(B(K)\) (called the barycentric subdivision of \(K\)) whose 2-faces are of the form \(ueF\), where \((u,e,F)\) is a flag (i.e., \(e\) is an edge of the face \(F\) and \(u\) is a vertex of \(e\)) in \(K\). The geometric carrier of \(B(K)\) is called the geometric carrier of \(K\) and is denoted by \([K]\). Clearly, \(K\) is a polyhedral 2-manifold if and only if \(B(K)\) is a combinatorial 2-manifold (equivalently, \([K]\) is a 2-manifold). A polyhedral 2-manifold \(K\) is called orientable if \([K]\) is orientable.

An isomorphism between two polyhedral complexes \(K\) and \(L\) is a bijection \(\varphi: V(K) \to V(L)\) such that \((v_1,\ldots,v_m)\) is a face of \(K\) if and only if \((\varphi(v_1),\ldots,\varphi(v_m))\) is a face of
L. Two complexes are called isomorphic if there is an isomorphism between them. We identify two isomorphic polyhedral complexes. An isomorphism from $K$ to itself is called an automorphism of $K$. The set $\Gamma(K)$ of automorphisms of $K$ form a group. A polyhedral 2-manifold $K$ is called combinatorially regular if $\Gamma(K)$ is transitive on flags (cf. [10]).

For a polyhedral 2-manifold $K$, consider the polyhedral complex $\tilde{K}$ whose vertices are the faces of $K$ and $(F_1, \ldots, F_m)$ is a face of $\tilde{K}$ if $F_1, \ldots, F_m$ have a common vertex and $F_1 \cap F_2, \ldots, F_{m-1} \cap F_m, F_m \cap F_1$ are edges. Then $\tilde{K}$ is a polyhedral 2-manifold and called the dual of $K$. If $\tilde{K}$ is isomorphic to $K$ then $K$ is called self dual.

A pattern is an ordered pair $(M,G)$, where $M$ is a connected closed surface in some Euclidean space and $G$ is a finite graph on $M$ such that each component of $M \setminus G$ is simply connected. The closure of each component of $M \setminus G$ is called a face of $(M,G)$. For a face $F$, the closed path (in $G$) consisting of all the edges and the vertices in $F$ is called the boundary of $F$. A pattern $(M,G)$ is said to be non-singular if the boundary of each face is a cycle. A non-singular pattern is said to be a polyhedral pattern if the intersection of any two faces is empty, a vertex or an edge. A pattern $(M,G)$ is called a $\{p,q\}$-pattern if each face contains $p$ edges and the degree of each vertex in $G$ is $q$ (cf. [7]).

If $(M,G)$ is a polyhedral pattern then clearly the boundaries of the faces of $(M,G)$ form a polyhedral map. Conversely, for a polyhedral map $K$, let $M = |K|$ and $G = EG(K)$. Then $(M,G)$ is a polyhedral pattern and the faces of $K$ are the boundaries of the faces of $(M,G)$. This pattern $(M,G)$ is called a geometric realization of $K$. A geometric realization $(M,G)$ (in some $\mathbb{R}^3$) is called linear if each face of $M$ is a convex polygon and no two adjacent faces (i.e., faces which share a common edge) lie in the same plane. If a polyhedral map has a linear geometric realization in $\mathbb{R}^3$ then it is called geometrically realizable.

If $f_0(K)$, $f_1(K)$ and $f_2(K)$ are the number of vertices, edges and faces respectively of a polyhedral complex $K$ then the number $\chi(K) := f_0(K) - f_1(K) + f_2(K)$ is called the Euler characteristic of $K$. Observe that $\chi(B(K)) = \chi(K)$. If $u$ and $v$ are vertices of a face $F$ and $uv$ is not an edge of $F$ then $uv$ is called a diagonal. Clearly, if $d(K)$ is the number of diagonals of a polyhedral complex $K$ then $d(K) + f_1(K) \leq (f_0(K))/2$ and in the case of equality each pair of vertices belong to a face. A polyhedral map $K$ is called a weakly neighbourly polyhedral map (in short, wpn map) if each pair of vertices belong to a common face.

We know (cf. [6]) that there exists a unique $\{p,q\}$-equivelar polyhedral map if $(p,q) = (3,3)$, $(3,4)$ or $(4,3)$ and there are exactly two $\{p,q\}$-equivelar polyhedral maps if $(p,q) = (3,5)$ or $(5,3)$. In [12], McMullen et al. constructed infinitely many geometrically realizable $\{p,q\}$-equivelar polyhedral maps for each $(p,q) \in \{(r,4) : r \geq 5\} \cup \{(4,s) : s \geq 5\} \cup \{(3,k) : k \geq 7\}$. In [6], it was shown that there exist infinitely many $\{4,4\}$- and $\{3,6\}$-equivelar polyhedral maps. It was also shown that there are exactly two neighbourly $\{3,8\}$-equivelar polyhedral maps and there are exactly 14 neighbourly $\{3,9\}$-equivelar polyhedral maps.

In [5], Coxeter constructed a geometrically realizable combinatorially regular infinite polyhedral 2-manifold whose faces are hexagons and each vertex is in six faces (namely, $\{6,6|3\}$). In [9], Grünbaum constructed another combinatorially regular infinite polyhedral 2-manifold of type $\{6,6\}$ (namely, $\{6,6|4\}$) (cf. [10]). In [8], Gott constructed a geometrically realizable infinite polyhedral 2-manifold whose faces are pentagons and each vertex is in five faces. If $K$ is a $\{p,q\}$-equivelar polyhedral map on $n$ vertices then $d(K) = nq(p-3)/2$ and $f_1(K) = nq/2$. Therefore, if $K$ is an $n$-vertex $\{p,p\}$-equivelar polyhedral map then $np(p-3)/2 + np/2 \leq n(n-1)/2$ and hence $n \geq (p-1)^2$. Clearly, equality holds if and only if $K$ is a wpn map. Let $\alpha(p)$ denote the smallest $n$ such that there exist an $n$-vertex $\{p,p\}$-equivelar polyhedral map. Clearly, the 4-vertex 2-sphere (the boundary of a 3-simplex) is the unique $\{3,3\}$-equivelar wpn map. In [1], Brehm proved that there exist exactly
three $\{4,4\}$-equivelar wnp maps and constructed the 16-vertex $\{5,5\}$-equivelar polyhedral map $M_{5,16}$ (of Example 1). It was shown in [2] that $M_{5,16}$ is the unique $\{5,5\}$-equivelar polyhedral map on 16 vertices. So, $\alpha(k) = (k-1)^2$ for $k \leq 5$. In [1], Brehm also constructed the 26-vertex $\{6,6\}$-equivelar polyhedral map $M_{6,26}$ (of Example 1). Here we show:

**Theorem 1.** For each $m \geq 3$ and $n \geq 0$, there exist a $2(3^{m-1} + 2n - 1)$-vertex self dual $\{2m - 1, 2m - 1\}$-equivelar polyhedral map and a $(3^m + 2n - 1)$-vertex self dual $\{2m, 2m\}$-equivelar polyhedral map.

Thus $(2m - 2)^2 \leq \alpha(2m - 1) \leq 2(3^{m-1} - 1)$ and $(2m - 1)^2 \leq \alpha(2m) \leq 3^m - 1$ for all $m \geq 3$. In [13], using a computer, Nilakantan has shown that there does not exist any 25-vertex $\{6,6\}$-equivelar polyhedral map. So, $\alpha(6) = 26$ and hence there does not exist any $\{6,6\}$-equivelar wnp map. We believe the following is true:

**Conjecture 1.** There does not exist any $\{k,k\}$-equivelar wnp map for $k \geq 7$.

For the existence of an $n$-vertex $\{k,k\}$-pattern $n$ must be at least $k + 1$. Here we show:

**Theorem 2.** There exists a $(p+1)$-vertex non-singular $\{p,p\}$-pattern for each odd prime $p$.

## 2 Examples and proofs of the results

We first construct infinitely many $\{k,k\}$-equivelar polyhedral maps. We need these to prove our results. We identify a polyhedral complex with the set of faces in it.

**Example 1.** For $m \geq 3$ and $n \geq 0$, let

$$M_{2m-1,2(3^{m-1}+2n-1)} = \{F_{i,2m-1} : 1 \leq i \leq 2(3^{m-1} + 2n - 1)\},$$

$$M_{2m,3^{m}+2n-1} = \{F_{i,2m} : 1 \leq i \leq 3^m + 2n - 1\},$$

where $b_{2l-1} = 3^{l-1} - 1$, $b_{2l} = 2 \times 3^{l-1} - 1$, for $l \geq 1$ and

$$F_{i,2m-1} = (i + b_1, i + b_2, \ldots, i + b_{2m-3}, i + b_{2m-2} + n, i + b_{2m-1} + 2n),$$

$$F_{i,2m} = (i + b_1, i + b_2, \ldots, i + b_{2m-2}, i + b_{2m-1}, i + b_{2m} + n)$$

are cycles ($2m-1$)-cycles and $(2m)$-cycles respectively with vertices from $\mathbb{Z}_{2(3^{m-1}+2n-1)}$ and $\mathbb{Z}_{3^{m}+2n-1}$ respectively. Clearly, there are $2m - 1$ faces through each vertex in $M_{2m-1,2(3^{m-1}+2n-1)}$ and there are $2m$ faces through each vertex in $M_{2m,3^{m}+2n-1}$. So, $\chi(M_{2m-1,2(3^{m-1}+2n-1)}) = (3^{m-1} + 2n - 1)(2m - 1)$ and $\chi(M_{2m,3^{m}+2n-1}) = (3^m + 2n - 1)m$.

Thus, $\chi(M_{2m-1,2(3^{m-1}+2n-1)}) = (3^{m-1} + 2n - 1)(5 - 2m)$ and $\chi(M_{2m,3^{m}+2n-1}) = (3^m + 2n - 1)(2m - 1)$. By Lemma 2 below, $M_{2m+1,2(3^{m-1}+2n-1)}$ and $M_{2m,3^{m}+2n-1}$ are polyhedral maps. But, by Lemma 4, none of these polyhedral maps are combinatorially regular.

![Diagram](attachment:image.png)
Lemma 1. For a collection $\mathcal{C}$ of cycles, let $\mathcal{C}$ be the 2-dimensional pure simplicial complex whose 2-faces are of the form $xyF$, where $F \in \mathcal{C}$ and $xy$ is an edge in $F$. If $B(\mathcal{C})$ is as defined earlier then the following three are equivalent.

(i) $B(\mathcal{C})$ is a combinatorial 2-manifold.

(ii) $\mathcal{C}$ is a combinatorial 2-manifold.

(iii) For any vertex $v$, the cycles containing $v$ are of the form $F_1 = (v, v_1, \ldots, v_{1,m})$, $\ldots$, $F_m = (v, v_{m,1}, \ldots, v_{m,m})$ such that $v_{1,n_1} = v_{2,1}, \ldots, v_{m-1,n_{m-1}} = v_{m,1}$, $v_{m,n_m} = v_{1,1}$ for some $m \geq 2$.

Proof. Clearly, $B(\mathcal{C})$ is a subdivision of $\mathcal{C}$. Therefore, (i) and (ii) are equivalent.

For a 2-dimensional pure simplicial complex $X$, the link of a vertex $v$ is the graph $lk_X(v)$ whose vertex-set is $\{u \in V(X) : uv \in X\}$ and edge-set is $\{xy : xyv \in X\}$. Clearly, $X$ is a combinatorial 2-manifold if and only if $lk_X(v)$ is a cycle for each $v \in V(X)$.

Let $v$ be a vertex of $\mathcal{C}$. If $v = F \in \mathcal{C}$ then $lk_\mathcal{C}(v)$ is $F$ itself. Let $v$ be a vertex of $\mathcal{C}$ which is not a cycle in $\mathcal{C}$. If the cycles containing $v$ are of the form $F_1 = (v, v_1, \ldots, v_{1,m}), \ldots, F_m = (v, v_{m,1}, \ldots, v_{m,m})$ such that $v_{1,n_1} = v_{2,1}, \ldots, v_{m-1,n_{m-1}} = v_{m,1}$, $v_{m,n_m} = v_{1,1}$ for some $m \geq 2$ then $lk_\mathcal{C}(v)$ is the cycle $v_1F_1v_2F_2 \cdots v_mF_m$. Conversely, if $lk_\mathcal{C}(v)$ is a cycle then, from the definition of $\mathcal{C}$, $lk_\mathcal{C}(v)$ must be of the form $v_1F_1v_2F_2 \cdots v_mF_m$, where $F_1 = (v, v_1, \ldots, v_{1,n_1}), \ldots, F_m = (v, v_{m,1}, \ldots, v_{m,m})$ such that $v_{1,n_1} = v_{2,1}, \ldots, v_{m-1,n_{m-1}} = v_{m,1}$, $v_{m,n_m} = v_{1,1}$. This proves that (ii) and (iii) are equivalent.

Lemma 2. $M_{2m-1,2(3^m-1+2n-1)}$ and $M_{2m,3^m+2n-1}$ are polyhedral maps for $m \geq 3$, $n \geq 0$.

Proof: Since $\{i, i+1\}$ is an edge in $M_{2m-1,2(3^{m-1}+2n-1)}$ for each $i$, $EG(M_{2m-1,2(3^{m-1}+2n-1)})$ is connected. Similarly, $EG(M_{2m,3^m+2n-1})$ is connected.

Observe that the faces in $M_{2m-1,2(3^{m-1}+2n-1)}$ containing $i$ are $F_1, F_{i-2}, F_{i-3}, F_{i-4}, \ldots, F_{i-b_{2m-3}}, F_{i-b_{2m-2}}$, where $F_i = F_{i-1} = (i + b_1, i + b_2, \ldots, i + b_{2m-3}, i + b_{2m-2} + n, i + b_{2m-1} + 2n)$. Clearly, $F_i \cap F_{i-k} = \cdots = F_i \cap F_{i-b_{2m-2}-n} = \cdots = F_i \cap F_{i-b_{2m-1}-2n}$. Since $b_{2j-1} = 2b_{2j-1}$ for all $j$, $F_i \cap F_{i-b_{2j-1}}$ is the edge $\{i, i + b_{2j} - b_{2j-1}\}$.

The faces in $M_{2m,3^m+2n-1}$ containing $i$ are $C_i, C_i-b_1, C_i-b_2, \ldots, C_i-b_{2m-1}, C_i-b_{2m}$, where $C_i = F_i = (i + b_1, i + b_2, \ldots, i + b_{2m-1}, i + b_{2m} + n)$ and $C_i \cap C_{i-b_{2j}} = C_i \cap C_{i-b_{2m-1}} = \cdots = C_i \cap C_{i-b_{2m-2}-n} = \cdots = C_i \cap C_{i-b_{2m-2}} = \{i\}$. Also, since $2b_{2m} - b_{2m-1} + 2n \equiv 0$ (mod $3^n + 2n - 1$), $C_i \cap C_{i-b_{2j-1}}$ is the edge $\{i, i + b_{2j} - b_{2j-1}\}$ for $1 \leq l \leq m-1$. Thus, any pair of faces containing $i$ intersect in either at $i$ or on an edge through $i$ and the faces containing $i$ form a single cycle of adjacent faces (sharing a common edge). Therefore, $M_{2m-1,2(3^{m-1}+2n-1)}$ is a polyhedral map.

From the uniqueness of 16-vertex $\{5, 5\}$-equivelar polyhedral map it follows that $M_{5,16}$ is self dual. Here we prove.
Lemma 3. $M_{2m-1,2(3m-1+2n-1)}$ and $M_{2m,3^m+2n-1}$ are self dual for $m \geq 3$ and $n \geq 0$.

Proof. Let $\varphi: M_{2m-1,2(3m-1+2n-1)} \to \tilde{M}_{2m-1,2(3m-1+2n-1)}$ be the mapping given by $\varphi(i) = F_i := F_{-i,2m-1}$. Clearly $\varphi$ is a bijection. Consider the face $F_i = (i + b_1, \ldots, i + b_{2m-3}, i + b_{2m-2} + n, i + b_{2m-1} + 2n)$. Now, $(\varphi(i + b_1), \ldots, \varphi(i + b_{2m-3}), \varphi(i + b_{2m-2} + n), \varphi(i + b_{2m-1} + 2n)) = (F_{-i-b_1}, \ldots, F_{-i-b_{2m-3}}, F_{-i-b_{2m-2}+n}, F_{-i-b_{2m-1}+2n}) = F_{-i}$ (say). From the proof of Lemma 2, $F_{-i}$ is a cycle of adjacent faces (sharing a common edge) containing the common vertex $-i$. Therefore, by the definition, $F_{-i}$ is a face of $\tilde{M}_{2m-1,2(3m-1+2n-1)}$. This implies that $\tilde{M}_{2m-1,2(3m-1+2n-1)}$ is isomorphic to $M_{2m-1,2(3m-1+2n-1)}$. Similarly, $\tilde{\psi}: M_{2m,3^m+2n-1} \to \tilde{M}_{2m,3^m+2n-1}$, given by $\tilde{\psi}(i) = F_{-i,2m}$ defines an isomorphism. □

Clearly, $\Gamma(M_{2m-1,2(3m-1+2n-1)})$ and $\Gamma(M_{2m,3^m+2n-1})$ are transitive on the vertices and the faces. Here we prove.

Lemma 4. $M_{2m-1,2(3m-1+2n-1)}$ and $M_{2m,3^m+2n-1}$ are not combinatorially regular for all $m \geq 3$ and $n \geq 0$.

Proof. Let $\mu = 2(3^m - 1 + 2n - 1)$. If $m > 3$ then consider the flags $F_1 = (0, \{0, b_m - b_{m+1}\}, F_{-b_{m+1}})$ and $F_2 = (0, \{b_{m+2} - b_{m+1}\}, F_{-b_{m+1}})$ in $M_{2m-1,1}$. If possible let there exist $\varphi \in \Gamma(M_{2m-1,1})$ such that $\varphi(F_1) = F_2$. Then $\varphi(0) = 0, \varphi(F_{-b_{m+1}}) = F_{-b_{m+1}}$ and hence $\varphi(1 - b_{m+1}) = -b_{m+1}$ and $\varphi(1) = 1$. If $m > 5$ then, by considering the faces containing 1, $\varphi(F_{1-b_{m+2}}) = F_{1-b_{m+2}}, \varphi(F_{1-b_{m+1}}) = F_{1-b_{m+3}}$. These imply $1 + b_1 - b_{m+3} = \varphi(1 - b_{m+1}) = -b_{m+1} + 2n \in Z_\mu$, a contradiction. If $m = 5$ then $\varphi(F_{1-b_5}) = F_{1-b_5} - 2n$ and hence $1 + b_1 - b_5 - n = \varphi(1 - b_5) = -b_5$ in $Z_\mu$. This is not possible. If $m = 4$ then $\varphi(F_{1-b_4}) = F_{1-b_4} - 2n$ and hence $1 + b_1 - b_7 - 2n = \varphi(1 - b_5) = -b_5$ in $Z_\mu$, a contradiction.

For $m = 3$, if $\psi \in \Gamma(M_{5,\mu})$ such that $\psi((0, \{0, 3+n\}, F_{-b_{4-n}})) = (0, \{13+3n\}, F_{-b_{4-n}})$ then $\psi(12 + 3n) = 11 + 3n$ and $\psi(F_{1-b_4-n}) = F_1$ and hence $3 = \psi(12 + 3n) = 11 + 3n$ in $Z_\mu$. This is also not possible.

Thus, $M_{2m-1,2(3m-1+2n-1)}$ always has a pair of flags $F_1$ and $F_2$ such that $\varphi(F_1) \neq F_2$ for all $\varphi \in \Gamma(M_{2m-1,2(3m-1+2n-1)})$. So, $M_{2m-1,2(3m-1+2n-1)}$ is not combinatorially regular.

Let $\eta = 3^m + 2n - 1$ and $C_i = F_{i,2m}$. Consider the flags $C_1 = (0, \{0, (1)^m(b_{m+2} - b_{m+1})\}, C_{-b_{m+1}})$ and $C_2 = (0, \{0, (1)^m(b_{m+2} - b_{m+1})\}, C_{-b_{m+1}})$ in $M_{2m,\eta}$. If $\varphi \in \Gamma(M_{2m,\eta})$ such that $\varphi(C_1) = C_2$. Then $\varphi(C_2) = C_1, \varphi(1 - b_{m+2}) = -b_{m+1}$ and $\varphi(1) = 1$. If $m > 3$ then $\varphi(C_{1-b_{m+2}}) = C_{1-b_{m+2}}$ and hence $1 + b_1 - b_{m+3} = \varphi(1 - b_{m+2}) = -b_{m+1}$ in $Z_\eta$, a contradiction. If $m = 3$ then $\varphi(C_{1-b_5}) = C_{1-b_5} - n$ and hence $1 + b_1 - b_5 - n = \varphi(1 - b_5) = -b_4$ in $Z_\eta$. This is not possible. Therefore, by similar argument as before, $M_{2m,3^m+2n-1}$ is not combinatorially regular. □

Example 2. Let $C_4$ be the collection of 4-cycles of the complete graph $K_5$ on the vertex set $Z_4 \cup \{u\}$ given by $C_4 = \{(0, 1, i, i+3), (u, i, i+1, i+3) : i \in Z_4\}$. Then $|C_4|$ is the torus and hence $(|C_4|, K_5)$ is a non-singular $(4, 4)$-pattern.

Lemma 5. Suppose $C(\pi_p) = \{(0, 1, \ldots, p-1), (u, i + \pi_p(1), \ldots, i + \pi_p(p-1)) : i \in Z_p\}$ is a collection of cycles of the complete graph $K_{p+1}$ on the vertex set $Z_p \cup \{u\}$, where $p$ is an odd prime and $\pi_p$ is a permutation of the complete graph $K_{p+1}$ on the vertex set $Z_p \cup \{u\}$, where $p$ is an odd prime and $\pi_p$ is a permutation of the complete graph $K_{p+1}$ on the vertex set $Z_p \cup \{u\}$. If

(PP1) $\pi_p(i) + \pi_p(p-i) = p$ for $1 \leq i \leq p-1$,
(PP2) $\pi_p(p+1) = p$ for $1 \leq i \leq p-1$,
(pp3) exactly one of \( j, -j \) is in \( \{ \pi_p(2) - \pi_p(1), \pi_p(3) - \pi_p(2), \ldots, \pi_p(\frac{p+1}{2}) - \pi_p(\frac{p-1}{2}) \} \)

then \( \bar{C}(\pi_p) \) is a connected combinatorial 2-manifold.

**Proof.** Since edges of cycles of \( C(\pi_p) \) form a connected graph, \( EG(\bar{C}(\pi_p)) \) is connected.

Let \( a_i = \pi_p(i+1) - \pi_p(i) \) for \( 1 \leq i \leq p-2 \). Then, by (pp1), \( a_i = a_{p-1-i} \). Let \( r = \frac{p-3}{2} \).

Then, by (pp1), (pp2), \( a_{r+1} = 1 \) and, by (pp3), \( \{ a_1, \ldots, a_{r+1} = a_1, \ldots, -a_{r+1} \} = \mathbb{Z}_p \setminus \{0\} \).

If \( r \) is even then the cycles containing \( i \) are \( (i, u, \ldots, i + a_1), (i, i + a_1, \ldots, i - a_2), (i, i - a_2, \ldots, i + a_3), \ldots, (i, i + a_{r-1}, \ldots, i - a_r), (i, i - a_r, \ldots, i + 1), (i, i + 1, i + 2, \ldots, i + p - 1), (i, i + p - 1, \ldots, i + a_{r+2}), \ldots, (i, i + a_{2r}, \ldots, i - a_{2r+1}), (i, i - a_{2r+1}, \ldots, u) \).

If \( r \) is odd then the cycles containing \( i \) are \( (i, u, \ldots, i + a_1), (i, i + a_1, \ldots, i - a_2), (i, i - a_2, \ldots, i + a_3), \ldots, (i, i - a_{r-1}, \ldots, i + a_r), (i, i + a_r, \ldots, i + p - 1), (i, i + p - 1, \ldots, i + 2, i + 1), (i, i + 1, \ldots, i - a_{r+2}), \ldots, (i, i + a_{2r}, \ldots, i - a_{2r+1}), (i, i - a_{2r+1}, \ldots, u) \).

The cycles containing \( u \) are \( (u, \pi_p(i_1), \ldots, \pi_p(i_p)), (u, 1 + \pi_p(i_1), \ldots, 1 + \pi_p(p-1)), \ldots, (u, p-1 + \pi_p(i_1), \ldots, p-1 + \pi_p(p-1)) \). Since \( \{ \pi_p(p-1), 1 + \pi_p(p-1), \ldots, p-1 + \pi_p(p-1) \} = \mathbb{Z}_p \), the cycles containing \( u \) can be arranged as \( (u, \pi_p(i_1), \ldots, \pi_p(j_1)), \ldots, (u, \pi_p(i_p), \ldots, \pi_p(j_p)) \), where \( j_1 = i_2, \ldots, j_{p-1} = i_p, j_p = i_1 \). The lemma now follows by Lemma 1. \( \square \)

Clearly, \( \pi_3 \) is the identity permutation and \( C(\pi_3) \) is the 4-vertex 2-sphere \( S^2_4 \). Also, \( \chi(\bar{C}(\pi_3)) = 2(p+1) - (\binom{p+1}{2} + p(p+1)) + p = (p+1)(4-p)/2 \). So, if \( p = 4k+1 \) for some \( k \geq 1 \) then \( \chi(\bar{C}(\pi_p)) \) is odd and hence \( \bar{C}(\pi_p) \) is non-orientable. Here we prove.

**Lemma 6.** \( \bar{C}(\pi_p) \) is non-orientable for \( p > 3 \).

**Proof.** Let \( F = (0, 1, \ldots, p-1) \) and \( F_i = (u, i + \pi_p(1), \ldots, i + \pi_p(p-1)) \) for \( 1 \leq i \leq p-1 \).

We can choose a \( p \)-gonal disc (not necessarily convex) in the plane for each cycle in \( \bar{C}(\pi_p) \) so that the disc corresponding to \( F_i \) is attached with that for \( F \) along the common edge \( \{ i + \pi_p(\frac{p-1}{2}), i + \pi_p(\frac{p+1}{2}) \} \) for each \( i \) and there are no other intersections. This gives us a \( p(p-1) \)-gonal disc \( D(\pi_p) \). Then there are two edges in \( D(\pi_p) \) corresponding to an edge \( jk \) \((j, k \in \mathbb{Z}_p, -1 \neq j - k \neq 1) \) in some cycle \( F_i \), and they appear in the same direction (clockwise or anti-clockwise). Since \( \bar{C}(\pi_p) \) is homeomorphic to the space obtained by identifying such pairs of edges (and some more) of \( D(\pi_p) \), \( |\bar{C}(\pi_p)| \) is non-orientable. \( \square \)

**Lemma 7:** Let \( p > 3 \) be a prime.

(a) If \( p = 4k+3 \) for some \( k \geq 1 \) then the permutation \( \sigma_p = (2, 4k+1)(4, 4k+1) \cdots (2k, 2k+3) \) of \( \mathbb{Z}_p \setminus \{0\} \) satisfies (pp1), (pp2) and (pp3) of Lemma 5.

(b) If \( p = 4l+1 \) for some \( l \geq 1 \) then the permutation \( \rho_p = (1, 4l)(3, 4l-2) \cdots (2l-1, 2l+2) \) of \( \mathbb{Z}_p \setminus \{0\} \) satisfies (pp1), (pp2) and (pp3) of Lemma 5.
Proof. Clearly, $\sigma_p$ and $\rho_p$ satisfy hypothesis (pp1) and (pp2).

Now, $\{\sigma_p(2) - \sigma_p(1), \ldots, \sigma_p(2/m) - \sigma_p(1/m)\} = \{2k - 4, 4k - 4, \ldots, 4, -2, 1\} = \{-2, 4, 6, \ldots, -(4k - 2), 4k - 4, -(4k + 2)\}$. Thus $\sigma_p$ satisfies (pp1).

Again, $\{\rho_p(2) - \rho_p(1), \ldots, \rho_p(2/m) - \rho_p(1/m)\} = \{-(4l - 2), 4l - 4, -(4l - 6), \ldots, 4, -2, 1\} = \{-2, 4, 6, \ldots, (4l - 4), -(4l - 2), -4l\}$. Thus $\rho_p$ satisfies (pp3).

Proof of Theorem 1: Let $m \geq 3$ and $n \geq 0$. By Lemma 2, $M_{2m-1,2(3m-1+2m-1)}$ is a $(2(3m^{-1} + 2n - 1))$-vertex polyhedral map and hence a $(2m - 1, 2m - 1)$-equivelar polyhedral map. Again, by Lemma 2, $M_{2m,3n+2n-1}$ is a $(3m + 2n - 1)$-vertex polyhedral map and hence a $(2m, 2m)$-equivelar polyhedral map. The theorem now follows from Lemma 3.

Proof of Theorem 2: Let $p > 3$ be a prime and $K_{p+1}$ be the complete graph on the vertex set $\mathbb{Z}_p \cup \{u\}$. By Lemma 7, there exists a permutation $\pi_p$ of $\mathbb{Z}_p \setminus \{0\}$ which satisfies (pp1), (pp2) and (pp3) of Lemma 5. Let $C(\pi_p)$ be as in Lemma 5. Then, by Lemma 5, $\overline{C}(\pi_p)$ is a connected combinatorial 2-manifold. So, if $N_p := |\overline{C}(\pi_p)|$ then $(N_p, K_{p+1})$ is a non-singular $\{p, p\}$-pattern and the cycles in $C(\pi_p)$ are the boundaries of the faces of $(N_p, K_{p+1})$. Finally, the 4-vertex 2-sphere $S^2_4$ gives a $\{3, 3\}$-pattern. This completes the proof.

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