ZIPF’S LAW AND CITY SIZES: A SHORT TUTORIAL REVIEW
ON MULTIPLICATIVE PROCESSES IN URBAN GROWTH

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We address the role of multiplicative stochastic processes in modeling the occurrence of power-law city size distributions. As an explanation of the result of Zipf’s rank analysis, Simon’s model is presented in a mathematically elementary way, with a thorough discussion of the involved hypotheses. Emphasis is put on the flexibility of the model, as to its possible extensions and the relaxation of some strong assumptions. We point out some open problems regarding the prediction of the detailed shape of Zipf’s rank plots, which may be tackled by means of such extensions.

Keywords: Multiplicative stochastic processes; Zipf’s law; Simon’s model.

1. Introduction

Biological populations—and, among them, human communities—are subject, during their existence, to a multitude of actions of quite disparate origins. Such actions involve a complex interplay between factors endogenous to the population and external effects related to the interaction with the ecosystem and with physical environmental factors. The underlying mechanism governing the growth or decline of the population size (i.e., the number of individuals) is however very simple in essence, since it derives from the elementary events of reproduction: at a given time, the growth rate of the population is proportional to the population itself. This statement must be understood in the sense that two populations formed by the same organisms and under the same ecological conditions, one of them—say—twice as large as the other, will grow by amounts also related by a factor of two. Such proportionality between population and growth rate, which is empirically verified in practically all instances of biological systems, defines a multiplicative process [11].

Populations whose size is governed by multiplicative processes and which, at the same time, are subject to environmental random-like fluctuations, are known to display universal statistical regularities in the distribution of certain features. Specifically, those traits which are transmitted vertically, from parents to their off-
spring, exhibit broad, long-tailed distributions with stereotyped shapes—typically, log-normal or power laws. For instance, consider a human society where, except for some unfrequent exceptions, the surname of each individual is inherited from the father. Consider moreover the subpopulations formed by individuals with the same surname. It turns out that the frequency of subpopulations of size $n$ is approximately proportional to $n^{-2}$ [19, 4]. Or take, from the whole human population, the communities whose individuals speak the same language, which in the vast majority of the cases is learnt from the mother. The sizes of those communities are distributed following a log-normal function [12]. Such statistical regularities are generally referred to as Zipf’s law [18, 19]. The derivation of Zipf’s law from the underlying multiplicative processes was first worked out in detail by the sociologist H. A. Simon, within a set of assumptions which became known as Simon’s model [9].

A well-documented instance of occurrence of Zipf’s law involves the distribution of city sizes [3, 14, 11, 2], where “size” is here identified with the number of inhabitants. In practically any country or region over the globe, the frequency of cities of size $n$ decays as $n^{-z}$, where the exponent $z$ is approximately equal to 2—as in the case of surnames. The occurrence of Zipf’s law in the distribution of city sizes can be understood in terms of multiplicative processes using Simon’s model. Inspection of current literature on the subject of city size distributions, however, suggests that the potential of Simon’s model as an explanation of Zipf’s law, as well as its limitations, are not well understood. In a recently published handbook on urban economics [2], for instance, we read: “Simon’s model encounters some serious problems. In the limit where it can generate Zipf’s law, it ... requires that the number of cities grow indefinitely, in fact as fast as the urban population.” It turns out that this assertion is wrong; the truth, in fact, happens to be exactly the opposite! Leaving aside the derivation that may have led to this false conclusion [1], we note that such strong statements risk to become dogmatic for the part of the scientific community which does not have the tools for their critical analysis.

With this motivation, the present short review will be devoted to give a pedagogical presentation of Simon’s model in the frame of the evolution of city size distributions. The emphasis will be put on a qualitative description of the basic processes involved in the modeling. The explicit statement of the hypotheses that define the model should already expose its limitations but, at the same time, should clarify its flexibility regarding possible generalizations. In the next section, an elementary model for the evolution of a population based on stochastic processes is introduced, and the concurrent role of multiplicative and additive mechanisms in the appearance of power-law distributions is discussed. After an outline of the main features of Zipf’s rank plots in the distribution of city sizes, Simon’s model is presented in its original version, describing its implications as for the population distribution in urban systems. Then, we discuss a few extensions of the model, aimed at capturing some relevant processes not present in its original formulation. Finally, we close with a summary of the main results and some concluding remarks.
2. Multiplicative processes and population growth

The fluctuating nature of the many environmental actions which modulate the growth of a population calls for a description based on stochastic –i.e., random– processes. Within this kind of formulation, it is explicitly assumed that the parameters that govern the evolution can change with time in irregular ways. For instance, the change in the number \( n(t) \) of individuals during a certain time interval \( \Delta t \) can be modeled by means of the discrete stochastic equation

\[
 n(t + \Delta t) - n(t) = a(t)n(t) + f(t)
\]

where \( a(t) \) and \( f(t) \) are random variables. At each time step, their values are drawn from suitably chosen probability distributions. As a consequence of the random variation of \( a(t) \) and \( f(t) \), the number \( n(t) \) also displays stochastic evolution. Equation (1) is used to predict the statistical properties of \( n(t) \), for instance, finding the probability distribution \( P(n,t) \) that the population has a value \( n \) at time \( t \). This kind of equation has been studied in detail by several authors in various contexts [11, 10].

The first term in the right-hand side of Eq. (1), \( a(t)n(t) \), represents the contributions to the evolution of \( n \) that are proportional to the population itself, i.e. the multiplicative effects referred to in the Introduction. If the population is closed, multiplicative processes are restricted to birth and death, and \( a(t) \) stands for the difference between the birth and death rates per individual in the interval \( \Delta t \). In open populations, the number of individuals is also affected by migration processes. Emigration flows are generally proportional to \( n(t) \) because, on the average, each individual has a certain probability of leaving the population per time unit. On the other hand, immigration has both multiplicative and additive effects. In fact, immigration can be favored by a large preexisting population –as in big cities– but a portion of arrivals may also occur as a consequence of individual decisions that do not take into account how large the population is. Additive contributions are described by the second term in Eq. (1). This term can also stand for negative effects on the population growth, such as catastrophic events where a substantial part of the population dies irrespectively of the value of \( n(t) \) [5].

The probability distribution \( P(n,t) \) of the population \( n \), as derived from Eq. (1), can have a complicated analytical form depending on the specific distributions chosen for \( a(t) \) and \( f(t) \). It is nevertheless known that, for large times, it decreases as a power law,

\[
P(n,t) \sim n^{-1-\gamma},
\]

over a substantial interval of values of \( n \). The exponent \( \gamma > 0 \) is given by the distribution \( p(a) \) for the random variable \( a(t) \), as the solution of the equation [10]

\[
1 = \int p(a)(a + 1)^\gamma da.
\]

Equation (2) holds under very general conditions on the probability distributions of \( a(t) \) and \( f(t) \), provided that \( f(t) \) is not identically equal to zero. In other words, a
power-law distribution is obtained when both multiplicative and additive processes are in action. The case \( f(t) \equiv 0 \) is, mathematically speaking, a singular limit. In the absence of additive processes, \( P(n, t) \) becomes a log-normal distribution.

The empirical observation of a power-law distribution in a real system would require to have access to many realizations of the evolution of the same population—which, in practice, is rarely possible—or, alternatively, to follow the parallel evolution of several sub-populations of the same type. This second instance is often met in human populations, which are naturally divided into communities of different nature, determined by historical, geographical, sociocultural, and/or economic factors. One of such divisions is given, precisely, by urban settlements. In this case, \( P(n, t) \) can be interpreted as the probability of having a city of size \( n \) at time \( t \) within the region that encompasses the whole population under study. In view of the above discussion, it is expected that the populations of different cities follow, under suitably homogeneous conditions over the studied region, a power-law distribution. As is well known, in fact, they do. Power laws in the population distribution of human groups of various kinds have been reported by several authors and, notably, by the philologist G. K. Zipf [19]. As advanced in the Introduction, the power-law dependence of the frequency of groups as a function of their population is now known as Zipf’s law.

Zipf’s law is often presented in an alternative formulation which, in the frame of the distribution of city sizes, goes as follows. Take all the cities under consideration, and rank them in order of decreasing population, so that rank \( r = 1 \) corresponds to the largest city, \( r = 2 \) to the second largest, and so on. Then, the population \( n \) of a city decreases with its rank as a power law,

\[
n(r) \sim r^{-z},
\]

over a wide range of values of \( r \). The exponent \( z \) is usually referred to as the Zipf exponent. Equations (2) and (4) are closely related. In fact, the formulation of Zipf’s law in terms of the probability distribution \( P(n, t) \) and the rank formulation are equivalent, though the latter is much less significant than the former from the viewpoint of a statistical description. To understand the connection between the two formulations, it is first useful to recall that—in our interpretation of the stochastic growth equation (1) as describing the parallel evolution of several sub-populations—the sum

\[
\sum_{n=n_1}^{n_2} P(n, t)
\]

is the probability of having a city with population between \( n_1 \) and \( n_2 \). Accordingly, the product of this sum times the total number of cities under consideration, is the number of cities with populations within that interval. Since the rank \( r \) of a city of population \( n \) equals the number of cities with populations larger than or equal to
In Eq. (2), we have

\[ r(n) = M \sum_{n'=n}^{\infty} P(n',t)dn', \]  

where \( M \) is the total number of cities. This establishes the relation between the rank-frequency dependence and the probability distribution \( P \). In particular, replacing Eq. (2) into (5), we get \( r(n) \sim n^{-\gamma} \), implying that the Zipf exponent and \( \gamma \) are related as

\[ z = \frac{1}{\gamma}. \]  

This defines the connection between the power laws in Eqs. (2) and (4).

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3. Zipf’s law in the distribution of city sizes

The application of Zipf’s rank analysis to urban settlements implicitly assumes that individual cities are well-defined entities. Actually, however, the modern city is such a complex of intermingled systems that it defies a definition in terms of traditional classification schemes, and requires a wider concept of class [8]. Figure 1 illustrates the fact that, while individual urban settlements can be distinctly identified in some regions, in other places the situation is by far less obvious. Anyway, it is currently accepted that the entities to be considered in Zipf’s analysis are the clusters resulting from the growth and aggregation of initially separated settlements. A plot of the population \( n \) versus the rank \( r \) for the cities of a given country or region usually reveals three regimes. For the lowest ranks, corresponding to the largest cities, the variation of \( n \) with \( r \) is generally irregular, with a marked descending step between the first one to three cities and the following. The biggest urban settlements in any large country or region often lie outside any significant statistical regularity, both within the region in question and between different regions. As the rank becomes
higher, these irregularities smooth out, and the plot enters the power-law regime. In the usual representation of \( n \) versus \( r \) in log-log scales, this regime is revealed by a linear profile, typically extending from \( r \approx 10 \) to ranks of the order of a few to several hundreds. The Zipf exponent \( z \), given by the slope of the linear profile in the log-log plot, is considerably uniform between different regions. It is customary to quote the value \( z \approx 1 \), though it may vary between 0.7 and 1.2. Finally, for the highest ranks the power-law regime is cut off, and \( n \) declines faster as \( r \) grows. Figure 2 illustrates these typical features for 276 metropolitan areas in the USA, according to data from the year 2000 census. Note carefully that the class of “metropolitan areas” does not necessarily include all urban settlements above a certain size. Below, we comment on related methodological problems in the construction of rank plots.

![Zipf rank plot](image)

Fig. 2. Zipf rank plot for 276 metropolitan areas in the United States, after results of the census in 2000. Source: factfinder.census.gov. The straight line has slope 1.11.

It is clear from the discussion on multiplicative stochastic processes in Section 2 that, among the three regimes identified in rank plots, the natural candidate to be explained in terms of such mechanism is the central power-law range. It also results from our discussion that, to derive a power-law city size distribution, it is necessary to take into account both multiplicative and additive contributions to the evolution of the population. These ingredients are captured by Simon’s model, which is presented in next section. Let us here point out that, to explain Zipf’s law in the distribution of city sizes, a model solely based on multiplicative processes—namely, Gibrat’s model— is often invoked. As already commented, however, purely multiplicative mechanisms can only produce a log-normal distribution. While over restricted ranges a log-normal distribution may seem to exhibit a power-law decay, \( P(n,t) \sim n^{-\lambda} \) with \( \lambda \approx 1 \), it certainly cannot fit the variety of Zipf exponents found in real city size distributions.

Before passing to the formulation of Simon’s model for the power-law regime of
_rank plots, it is pertinent to discuss a few aspects regarding the description of the two remaining regimes—those corresponding to the lowest and the highest ranks. As for the former, the biggest cities in a large country or region are, almost invariably, special cases that elude inclusion in any statistical description. It would be hopeless to pretend that, for instance, Paris, Berlin, or Rome enter the same statistical class as the European cities whose present population is below, say, one million. The political and economic role of those cities has been—and still is—markedly peculiar. In consequence, their individual evolution is exceptional among urban settlements and must be dealt with as such. While it would make little sense to discuss the case of the biggest cities in the frame of a statistical model for the distribution of city sizes, it is nevertheless interesting to advance that Simon’s model assigns a special role to those cities: their sizes bear information on the initial state of the urban system, before the smaller settlements played any significant role in the population statistics. In this sense, Simon’s model also recognizes that the biggest cities are special cases._

As for the cut-off region of highest ranks, let us mention that it is found not only in rank plots for city sizes, but also in many other instances where Zipf’s law holds for intermediate ranks. A classical example occurs in the frequency of words in human languages [7, 16]. In the case of city sizes, the appearance of the cut-off is well known but, to our knowledge, there is no systematic study regarding the population-rank functional dependence in that regime. This lack of quantitative empirical results discourages modelling of city sizes for high ranks, as there is no reference data to validate potential models. Moreover, as pointed out in connection with Fig. 2, the regime of high ranks is susceptible of methodological errors related to possible data incompleteness. While, arguably, the lists of large cities provided by most sources of demographic information are exhaustive, the same sources may result to be less reliable when it comes to smaller urban settlements. Inspection of many public-domain databases immediately reveals lack of completeness in the lists of cities for high ranks. The direct effect of these “gaps” is that the assigned ranks are lower than in reality, with the consequent reinforcement of the cut-off (cf. Fig. 2). Avoiding this effect without restricting too much the range of ranks under consideration, requires relying on presumably complete data sets—typically, from official census reports. This, in turn, limits the corpus of data, because such databases are not always available. In any case, as stated above, the cut-off in rank plots can be observed in other systems where this kind of methodological error is not present. In Section 5, we adapt to the case of city sizes an extension of Simon’s model put forward to give a semi-quantitative explanation of the cut-off regime for the case of word frequencies in language.

4. Simon’s model: Hypotheses and main results

Elaborating on an idea previously advanced by Willis and Yule [13], H. A. Simon proposed in 1955 the model that now bear his name [9], as an explanation for
the origin of power-law distributions and Zipf’s law. Simon presented his model by referring to the case of word frequencies, which Zipf himself had discussed in detail in his publications [18]. Here, we introduce the original Simon’s model adapted to the framework of city growth. In practice, this just implies a change in the vocabulary employed to express the dynamical rules that define the model.

Simon’s model describes the evolution of a population divided into well-defined groups—the cities. We characterize this division by means of the quantity \(m(n,t)\), which gives the number of cities with exactly \(n\) inhabitants at time \(t\). This quantity is closely related to the probability distribution \(P(n,t)\) introduced in Section 2. In fact, we have \(m(n,t) = M(t)P(n,t)\), where \(M(t)\) is the total number of cities in the system at the same time. Instead of using the real time \(t\), Simon’s model proceeds by discrete steps, which are identified by means of a discrete variable \(s = 0, 1, 2\ldots\) Each step corresponds to the time interval needed for the total population to increase by exactly one inhabitant. The actual duration of an evolution step—which is determined by a balance between birth and immigration on one side, and death and emigration on the other—is irrelevant to the model. The growth of the population in real time is a separate problem which can be specified and solved independently. As for the model, thus, the elementary evolution event is the addition of a single inhabitant to the total population. Accordingly, the quantity \(m(n,t)\) is replaced by \(m(n,s)\), the number of cities with exactly \(n\) inhabitants at step \(s\). The starting point of the evolution is given by the initial distribution \(m(n,0)\), at \(s = 0\), which describes a preexistent population distributed over a certain number of cities.

The evolution is governed by the following stochastic rules, which imply making a decision at each step \(s\), i.e. each time a new inhabitant is added to the total population. (i) With probability \(\alpha\), the new inhabitant founds a new city. In this case, the number of cities \(M\) grows by one, and the new city has initially a single inhabitant. (ii) With the complementary probability, \(1 - \alpha\), the new inhabitant is added to an already populated city. In this case, the destination city is chosen with a probability proportional to its current population. It increases its population by one, and the number of cities \(M\) does not vary. Clearly, rule (ii) stands for the multiplicative contribution to the evolution of the individual population of cities. Larger cities have higher probability of incorporating new inhabitants than smaller ones. As it grows, the population is preferentially assigned to those groups which are already relatively large. Rule (i), on the other hand, represents a contribution independent of the preexisting distribution and, thus, stands for additive effects. In particular, it implies that the number of cities grows, on the average, at a constant rate \(\alpha\). Hence, the average number of cities at step \(s\) is \(M(s) = M(0) + \alpha s\). Meanwhile, since exactly one inhabitant is added per time step, the total population in the system is \(N(s) = N(0) + s\).

In order to translate into mathematical terms the evolution rules (i) and (ii), we must take into account the following remarks. First, rule (i) only affects the number of cities with exactly one inhabitant, \(m(1,s)\). When it applies, which happens with frequency \(\alpha\) per evolution step, \(m(1,s)\) grows by one. Second, when rule (ii) applies,
which happens with frequency $1 - \alpha$ per evolution step, the probability that any city of size $n$ is chosen as destination is proportional to $n/N(s)$ and to the number of cities of that size. Since the chosen city changes its population from $n$ to $n + 1$, this event represents a positive contribution to the number of cities of size $n + 1$ at the next step, $m(n + 1, s + 1)$, and a negative contribution to $m(n, s + 1)$. The contribution to $m(n, s + 1)$ will be positive when the chosen destination is a city of size $n - 1$. Summing up these considerations, the average change per step in the number of cities of size $n$ is

$$m(1, s + 1) - m(1, s) = \alpha - \frac{1 - \alpha}{N(s)} m(1, s)$$

(7)

for $n = 1$, and

$$m(n, s + 1) - m(n, s) = (1 - \alpha) \left[ \frac{n - 1}{N(s)} m(n - 1, s) - \frac{n}{N(s)} m(n, s) \right]$$

(8)

for $n = 2, 3, 4 \ldots$. These are the equations that govern the evolution of Simon’s model in its original formulation [9].

From a mathematical viewpoint, Eqs. (7) and (8) are not complicated. First of all, they form a linear system, which can therefore be tackled with a host of well-tested analytical and numerical methods. Moreover, they can be solved recursively. In fact, the solution to Eq. (7) gives the number of cities with one inhabitant. Once $m(1, s)$ has been found, $m(2, s)$ and, successively, $m(n, s)$ for larger $n$, are obtained from Eq. (8). The only difficulty is that the equations involve the function $N(s) = N(0) + s$, which depends explicitly on the variable $s$. Consequently, the system is non-autonomous.

In his original paper, Simon was able to prove that—as we show below—Eqs. (7) and (8) imply a power-law decay for $m(n, s)$ as a function of $n$. The presentation of the solution will differ from Simon’s in that we first introduce a continuous approximation to the model equations, replacing the discrete variables $n$ and $s$ by continuous variables $\eta$ and $\xi$, respectively. This approximation has the advantage of transforming the infinitely many equations (8) into a single evolution law. The disadvantage is that the new problem is differential, instead of algebraic. Replacing discrete by continuous variables is justified by the fact that, in the distribution of city sizes, we are mainly interested in the range of large values for both $n$ and $s$, where $m(n, s)$ is expected to vary smoothly. To the first order, we approximate the differences in Eq. (8) by derivatives; for instance,

$$m(n, s + 1) - m(n, s) \approx \frac{\partial m}{\partial \xi} (\eta, \xi).$$

This approximation can be systematically improved by considering higher order terms, as discussed elsewhere [6]. The resulting partial-differential equation is

$$\frac{\partial \mu}{\partial \xi} + (1 - \alpha) \frac{\eta}{N(0) + \xi} \frac{\partial \mu}{\partial \eta} = 0,$$

(9)
with \( \mu(\eta, \xi) = \eta m(\eta, \xi) \). This equation has to be solved for \( \eta > 1 \), with the initial condition \( \mu(\eta, 0) = \eta m(\eta, 0) \) and a boundary condition at \( \eta = 1 \) derived from Eq. (4), namely,

\[
\mu(1, \xi) = \frac{\alpha}{1 - \alpha} (N(0) + \xi). \tag{10}
\]

We do not discuss the details of the solution method for this linear equation. It is enough to say that, by means of a change of variables \([6]\), the equation reduces to a standard one-dimensional wave equation and is then solved by the so-called “method of characteristics.” In the following, we describe the result in terms of the original variables \( n \) and \( s \).

As a function of the population \( n \), the number \( m(n, s) \) of cities of size \( n \) at step \( s \), solution of the Simon’s equations, shows two distinct regimes. The boundary between both regimes is situated at

\[
n_B(s) = \left(1 + \frac{s}{N(0)}\right)^{1-\alpha}. \tag{11}
\]

We see that this boundary depends on \( s \), and shifts to higher populations as the evolution proceeds. For populations above the boundary, \( n > n_B \), the solution to the continuous approximation of Simon’s model is

\[
m(n, s) = \frac{1}{n_B(s)} m \left( \frac{n}{n_B(s)}, 0 \right). \tag{12}
\]

Thus, \( m(n, s) \) is directly given by the initial condition \( m(n, 0) \). As a matter of fact, this regime can be seen to encompass those cities that were already present at \( s = 0 \). In Simon’s model, preexisting cities—not unlike the oldest cities of real urban systems—are those that reach the largest sizes, i.e., those that are assigned the lowest rank values in Zipf’s analysis. We realize that, as advanced in Section 3, information about the initial state of the urban system is stored in the size distribution at the lowest ranks. A detailed study of the effects of the initial condition in the large-size regime, referring to the discrete equations (7) and (8), has been presented elsewhere \([15]\).

In the range of small populations, \( n < n_B \), the solution to the continuous approximation of Simon’s model is

\[
m(n, s) = \frac{\alpha}{1 - \alpha} (N(0) + s) n^{-1-1/(1-\alpha)}. \tag{13}
\]

This regime encompasses the cities founded during the evolution of the urban system, corresponding to higher rank values in Zipf’s analysis. We see that their size distribution follows a power law with exponent \( \gamma = 1/(1-\alpha) \) [cf. Eq. (2)]. According to Eq. (6), the Zipf exponent is

\[
z = 1 - \alpha. \tag{14}
\]

Since, being a probability, \( \alpha \) is positive and lower than one, this formulation of Simon’s model predicts a Zipf exponent \( 0 < z < 1 \). The characteristic value \( z \approx 1 \) is
obtained for very small $\alpha$, i.e. when the frequency of city foundation is very small as compared with the growth rate of the population. As advanced in the Introduction, this conclusion is in full disagreement with the bibliographic quotation given there.

In summary, the main results obtained in this section for the original version of Simon’s model, within the continuous first-order approximation, are the following. At any evolution stage, the distribution of city sizes shows two well-differentiated regimes. For large cities, which correspond to low rank values, the distribution depends sensibly on the initial condition. This range keeps information on the early state of the urban system and, thus, results to be specific for each realization of the model. On the other hand, the size distribution of small cities, within the range of high rank values, exhibits a universal power-law decay whose exponent is completely determined by the rate $\alpha$ at which new cities are founded. The respective Zipf exponent is always less than one, and the limit $z = 1$ is approached when $\alpha$ is vanishingly small. The two regimes, whose mutual boundary recedes towards high populations as the evolution proceeds, can be immediately identified with two of the three regions of rank plots, described in Section 3. The cut-off region, on the other hand, remains unexplained by this version of Simon’s model. Moreover, as presented in this section, the model is not able to produce Zipf exponents larger than one (cf. Fig. 2). Some of the generalizations discussed in the next section are aimed at alleviating these limitations.

5. Generalization of Simon’s model

It is clear that, in the original formulation of Simon’s model, both rules (i) and (ii) involve strong assumptions on the parameters that govern the evolution of the urban system. Specifically, rule (i) establishes that the rate at which new cities are founded is constant, i.e. does not vary with time. Rule (ii), in turn, makes a concrete hypothesis on the size dependence of the probability for a city to be chosen as destination for a new inhabitant. Not without reason, it may be argued that these assumptions are unrealistically simple. Reinforcing this impression, we have just shown that Simon’s model is not able to predict some basic features in the rank plots of real urban systems, such as the cut-off at high ranks and the possibility that the Zipf exponent is larger than one.

It has to be understood, however, that the assumptions implicit in the evolution rules have been introduced by Simon, mainly, to facilitate the analytical treatment of the equations and to show, as straightforwardly as possible, that a couple of elementary mechanisms are enough to explain the occurrence of power-law distributions and Zipf’s law. If one intends to be more realistic, those strong assumptions can be immediately relaxed, without inherently modifying the basic dynamical processes that define the evolution. In this section, we present a small collection –by no means exhaustive– of generalizations of Simon’s model, based on relaxing the evolution rules. Some of these extensions have already been introduced in the literature to solve the above discussed limitations of the model, regarding the detailed
prediction of Zipf’s rank distributions. Our main aim is, nevertheless, to emphasize
the flexibility of Simon’s model as for possible extensions towards a more realistic
description of city growth.

5.1. **Time-dependent rate of city foundation**

A straightforward generalization of Simon’s model consists in assuming that the
probability \( \alpha \) of foundation of a city when a new inhabitant is added to the system
depends on time. Indeed, it is expected that the rate at which new cities appear
in a urban system decreases as the total number of cities grow. In the model, a
time-dependent rate of city foundation amounts at admitting that \( \alpha \) depends on
the variable \( s \). In this way, \( \alpha(s) \) gives the probability per evolution step that a city
is founded by the inhabitant added at step \( s \).

To mathematically implement this generalization, we do not need to to rewrite
the evolution equations. It is just enough to take into account that, in Eqs. (7) and
(8), the parameter \( \alpha \) may depend on \( s \). In principle, there are no limitations on
the functional form of this dependence. Of course, however, whether the resulting
evolution equations are analytically tractable and whether they produce a power-
law distribution is a matter to be ascertained in each particular case. In any case,
the problem can be dealt with by numerical means.

As an illustration, we consider here a phenomenological model for the time
variation of \( \alpha \) put forward in the framework of word frequencies in language [7, 16].
In this model, \( \alpha \) decreases with \( s \) as a power law of the form
\[
\alpha(s) = \alpha_0 s^{-\nu},
\]
where \( \alpha_0 \) is a constant, and \( 0 < \nu < 1 \). In the problem of city growth, this form of
\( \alpha(s) \) implies that the total number of cities increases slower than linearly, \( M(s) \sim
s^\nu \), instead of displaying linear growth as in the original version of Simon’s model.
In the relevant limit where \( \alpha(s) \ll 1 \) for all \( s \) – a condition which is insured if
the constant \( \alpha_0 \) is very small – it is possible to find the solution of the first-order
continuous approximation, Eq. (9). As a function of the population \( n \), the resulting
distribution \( m(n, s) \) shows again two regimes. As in the case of constant \( \alpha \), the large-
population regime is determined by the initial condition and, thus, bears information
on the initial state of the urban system. The small-population regime, in turn,
corresponds again a power-law distribution, but its exponent has changed:
\[
m(n, s) = \alpha_0 N(0) \left( 1 + \frac{s}{N(0)} \right)^\nu n^{-1-\nu}.
\]
The associated Zipf exponent is [cf. Eqs. (2) and (6)]
\[
z = \frac{1}{\nu}.
\]
Since \( \nu < 1 \), we have \( z > 1 \). We conclude that allowing the rate of city foundation
to depend on time, the restriction in the resulting Zipf exponent can be removed.
The effect of more general forms of \( \alpha(s) \) may be assessed numerically.
5.2. The cut-off regime

Another extension of Simon’s model makes it possible to predict the presence of a faster population decay for high ranks, thus providing a plausible explanation for the cut-off observed in the rank plot. Here, we limit ourselves to a semi-quantitative description of this generalization, as technical details have already been given elsewhere [7, 16].

The generalization is based on a realistic consideration regarding the foundation of cities as new inhabitants are added to the population. It can be argued that a single inhabitant is not enough to define the existence of a new city. Rather, there should be a minimal population for a city to enter the regime where the multiplicative process of Simon’s rule (ii) acts. This effect can be implemented by modifying the probability that a newly founded city is chosen as destination by new inhabitants. Namely, the probability that a city of size \( n \) is chosen by a new inhabitant can be taken to be proportional to \( \max\{n, n_{\text{min}}\} \), where \( n_{\text{min}} \) is the threshold population. In this way, a given city must attract \( n_{\text{min}} \) new inhabitants before multiplicative growth begins to act. Until then, the probability that the city is chosen as destination is a constant. Note that the threshold \( n_{\text{min}} \) may be different for each city. Within this extension, the cut-off of Zipf’s plot is interpreted as corresponding to those cities whose size has not yet attained the threshold.

This generalization of Simon’s model has originally been introduced in the framework of word frequencies [7]. Numerical simulations of the model with an exponential distribution for the value of \( n_{\text{min}} \) assigned to each city, combined with an \( s \)-dependent probability \( \alpha \) of the type discussed in Section 5.1 have provided excellent fittings of Zipf rank plots for several texts in different languages. In view of these encouraging results, it would be interesting to try these combined extensions of Simon’s model for city size distributions.

5.3. Size-dependent choice of the destination city

As mentioned above, rule (ii) in the original formulation of Simon’s model involves the very special assumption that the probability for a city to be chosen as destination by a new inhabitant is proportional to its size. In other words, the specific growth rate of cities per time step –relative to their current individual populations– is constant all over the system.

This assumption can be relaxed supposing that the probability that a city receives a new inhabitant is not proportional to its population \( n \), but to a function \( \phi(n) \). If \( \phi(n) \) grows with \( n \) faster than linear, the specific growth rate of large cities will be higher than for small cities, and vice versa. In the original formulation of the model, one has \( \phi(n) = n \). The function \( \phi(n) \) stand thus for a nonlinear effect in the multiplicative process. The probability that a city of size \( n \) is chosen as destination
is given by the ratio $\phi(n)/\Phi(s)$, where the normalization factor is given by

$$\Phi(s) = \sum_{n=1}^{\infty} \phi(n)m(n,s).$$

(18)

This normalization insures that the sum of the probabilities over the whole ensemble of cities equals one. In the original model, the normalization factor equals the total population, $\Phi(s) = N(s)$.

Within this generalization, the model evolution equations read

$$m(1,s+1) - m(1,s) = \alpha - (1 - \alpha) \frac{\phi(1)}{\Phi(s)}m(1,s)$$

(19)

for $n = 1$, and

$$m(n,s+1) - m(n,s) = (1 - \alpha) \left[ \frac{\phi(n-1)}{\Phi(s)}m(n-1,s) - \frac{\phi(n)}{\Phi(s)}m(n,s) \right]$$

(20)

for $n = 2, 3, 4 \ldots$. In the first-order continuous approximation introduced in Section 4, they transform into

$$\frac{\partial \psi}{\partial \xi} + (1 - \alpha) \frac{\phi(\eta)}{\Phi(\xi)} \frac{\partial \psi}{\partial \eta} = 0,$$

(21)

with $\psi(\eta, \xi) = \phi(\eta)m(\eta, \xi)$. As in the original model, this equation has to be solved for $\eta > 1$, with the initial condition $\psi(\eta, 0) = \phi(\eta)m(\eta, 0)$. The boundary condition at $\eta = 1$ is now

$$\psi(1, \xi) = \frac{\alpha}{1 - \alpha} \phi(1)\Phi(\xi).$$

(22)

Now, finally, we have managed to end up with a really complicated mathematical problem. Equations (19) to (21) are very similar to the evolution equations of the original model but, alas, the similitude is only formal. The key difficulty of our new equations for $m(n,s)$ resides in the fact that the function $\Phi(s)$ is generally not known beforehand. In the original model, on the other hand, it coincides with the total population $N(s) = N(0) + s$. Within the present generalization, $\Phi(s)$ can only be given in terms of $m(n,s)$ itself [cf. Eq. (18)]. Unfortunately, it is not possible to find an independent equation for the evolution of $\Phi(s)$ alone. The distribution of city sizes $m(n,s)$ and $\Phi(s)$ must therefore be found simultaneously and self-consistently.

We have been unable to find a form of $\phi(n)$ allowing us to give an analytical solution either to Eqs. (19) and (20) or to Eq. (21). It seems, not unexpectedly, that the problem must be treated numerically. We leave it open for the reader interested at studying the effects of nonlinear multiplicative processes. To our knowledge, this kind of processes have until now received relatively little attention.

6. Conclusion

This short review has been devoted to a presentation of the mathematics of Simon’s model in terms that, we hope, are accessible to a broad academic readership.
We have shown that, in its original version, Simon’s model is able to explain the occurrence of a power-law regime in the distribution of city sizes, though it fails at predicting some of the Zipf exponents observed in real urban systems, as well as other systematic features resulting from Zipf’s rank analysis. The extensions discussed later should have demonstrated that such limitation can be removed –at least, partially– by relaxing some of the assumptions of the model’s dynamical rules without modifying the key underlying mechanisms. These extensions were mainly aimed at illustrating the potential of Simon’s model with respect to possible generalizations in the direction of a better description of empirical data.

Several processes relevant to the evolution of city sizes have not been addressed at all in the presentation of Simon’s model. Let us point out three of them. In the first place, we have avoided a detailed description of death and emigration events. We have in fact assumed that the growth of the total population in the urban system is monotonous, the only effect of mortality and emigration being a lengthening of the duration of the evolution step (cf. Section 3). This excludes the possibility that the population might temporarily decrease—a necessary event if one aims at describing, for instance, the eventual disappearance of cities. As discussed elsewhere [6], a separate consideration of mortality and/or emigration implies a change in the Zipf exponent predicted by the model. Secondly, we have not taken into account the possibility of migration flows inside the urban system, between its cities. One can see that a purely multiplicative migration mechanism would exchange population between cities without modifying the city size distribution. On the other hand, additive and nonlinear mechanisms would imply a change in the distribution. This belongs to the class of open problems mentioned at the end of Section 5.3. Third, we have ignored possible events of coalescence of cities which, as indicated in Section 3, shape many modern urban systems. A particularly interesting open problem related to such events regards the persistence of Zipf’s law beyond the formation of urban agglomerations. A model for this persistence may shed light on the statistics of the coalescence process itself.

Finally, it is obvious that we have made no attempt to produce a quantitative fitting of real data from city size distributions with Simon’s model or any of its extensions. On the other hand, very good fittings have been reported for distributions of word frequencies in language [7, 16], musical notes in Western compositions [17], and surname abundance [6, 4], all of which share the dynamical basis of multiplicative processes. It would be nice if this work elicits similar initiatives in the statistical study of urban systems.

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