AN EXTENSION OF HEAT HIERARCHY

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Abstract. We propose a formally completely integrable extension of heat hierarchy based on the space of symmetries isomorphic to the Weyl algebra \( A_1 \). The extended heat hierarchy will be the basic model for the analysis of the extension property of KP hierarchy, and other integrable equations.

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1. Overview

Given a differential equation and its subsequent infinite prolongation, two basic invariants are symmetries and conservation laws: roughly speaking, a symmetry is a canonical solution to the linearization of the original equation, and a conservation law, which is conceptually somewhat dual to symmetry, is a certain cohomological invariant related to the moment conditions for solving the associated boundary value problems. In fact, from the works of Tsujishita, Vinogradov, Bryant & Griffiths, there exists a canonically attached spectral sequence to a differential equation, and this gives rise to a variety of local cohomological invariants including the symmetries and conservation laws, [6][8][9][11].

The class of integrable equations by definition possess many inherent symmetric structures. They admit various forms of algebraic and geometrical representations that are relevant to diverse areas of mathematics and physics. The presence of infinite sequence of higher order symmetries and conservation laws is in fact one of the characteristic properties of integrable equation.

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The purpose of this paper is to make the initial step towards solving an integrable equation by the following scheme: first, extend the differential equation by symmetries to an associated hierarchy, second, find the additional (generalized) symmetries and further extend the hierarchy to a formally completely integrable system, and finally, solve for the solutions which are stationary for a subalgebra of enlarged symmetries. We refer to [7] for the example of solutions to KdV equations arising from string theory which fit in this framework.

In this paper we consider heat hierarchy (1) associated to the familiar heat equation. As mentioned above, symmetry is defined as a local cohomological invariant of a differential equation. On the other hand, there exist various types of generalized symmetries and they already appear in heat hierarchy. Furthermore, heat hierarchy is up to dressing the Lax pair for KP hierarchy. Our analysis reveals the geometric structures which underlie the additional symmetries of KP hierarchy. Considering the universal role played by KP hierarchy, we suspect that heat hierarchy example would serve as the model for the analysis of symmetry extension of the other integrable equations.

Heat hierarchy for a scalar function $u$ of an infinite sequence of independent variables $(t_1 = x, t_2, t_3, \ldots)$ is the sequence of evolution equations

$$\frac{\partial u}{\partial t_j} = p_j, \quad j \geq 1,$$

where $p_j = \frac{\partial^j u}{\partial x^j}$. Let $X_0 = \mathbb{R}^\infty$ with coordinates $(p_0 = u, p_1, p_2, \ldots; t_1 = x, t_2, \ldots)$. Then Eqs. (1) define a formally completely integrable Pfaffian system $I_0$ on $X_0$.

Let $\mathfrak{g}_1$ be the Weyl algebra of differential operators with polynomial coefficients in 1 variable. Let $G_{\mathfrak{g}_1}$ be the corresponding formal Lie group with Lie algebra $\mathfrak{g}_1$. Let $\pi : X = G_{\mathfrak{g}_1} \times \mathbb{R} \times X_0 \to X_0$ be the product bundle. Based on the extended symmetry algebra of $(X_0, I_0)$ isomorphic to $\mathfrak{g}_1$, we show that heat hierarchy admits an extension to a formally completely integrable system $I$ defined on $X$ such that

$$\pi^* I_0 \subset I.$$

Moreover, $X$ is foliated by submanifolds on which $I$ reduces to the original heat hierarchy, Thm. 4.1.

Consider now the following diagram.

\begin{center}
\begin{tikzcd}
\text{heat hierarchy} & \text{dressed heat hierarchy} = \text{Lax pair for KP hierarchy} \\
\uparrow \text{symmetry extension} & \\
\text{extended heat hierarchy} & \text{extended dressed heat hierarchy} = \text{Lax pair for extended KP hierarchy}
\end{tikzcd}
\end{center}

The corresponding extension of KP hierarchy will be the subject of the next study.

---

1. By Lax pair we mean the linear differential equation for zero curvature representation.
Throughout the paper, the convergence issue of the infinite series of various functions, (pseudo) differential operators, and differential forms involved shall be ignored. We work in the framework of formal differential geometry.

For basic reference to the theory of KP hierarchy we refer to [2], [7].

2. Heat hierarchy

Heat hierarchy for a scalar function $u(x): \mathbb{R} \to \mathbb{C}$ is by definition the sequence of evolution equations:

$$\frac{\partial}{\partial t_j} u = p_j, \quad j = 2, 3, \ldots,$$

where $p_0 = u, p_j = \frac{\partial^j}{\partial x^j} u$. Since

$$\frac{\partial^2}{\partial t_i \partial t_j} u = p_{i+j}, \quad i, j \geq 1 \text{ (set } t_1 = x),$$

the set of differential equations are compatible. It can therefore be written as a formally integrable, determined, first order system of linear differential equations as follows. Let $p$ be the row vector $p = (p_0, p_1, p_2, \ldots)$. Then heat hierarchy is equivalent to

$$dp = p \omega,$$

where $\omega$ is the lower-triangular $\infty$-by-$\infty$ matrix valued 1-form (here $\cdot$ denotes 0)

$$\omega = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}.$$

It is formally integrable in the sense that $\omega$ satisfies the compatibility equation

$$d\omega + \omega \wedge \omega = 0$$

(in the present case $d\omega = 0, \omega \wedge \omega = 0$ separately).

Note the first component of (3) gives

$$du - u_t dx - u_{xx} dt_2 = \sum_{j=3}^{\infty} p_j dt_j.$$

Since the heat equation $u_t = u_{xx}$ is linear and homogeneous, each $p_j, j \geq 0$, is a symmetry and heat hierarchy can be considered as an extension of the heat equation by its (commuting) symmetries. This is summarized by the following commutation relations

$$\left[ \frac{\partial}{\partial t_j} - \frac{\partial^2}{\partial x^2}, \partial_x^j \right] = 0, \quad (\partial_x = \frac{\partial}{\partial x}),$$

$$\left[ \partial_x^j, \partial_x^k \right] = 0, \quad j, k \geq 0.$$

---

A symmetry of a differential equation is roughly a canonical solution to its linearization. From Eq. (6), $p_j = \partial_x^j(u)$ is a symmetry for each $j \geq 0$. 
Example 2.1. For an indeterminate $z$, consider the wave function for the original heat equation

$$w = e^{\xi}, \quad \xi = \sum_{k=1}^{2} t_{k}z^{k}.$$  

Substituting $u = w$ in (5), we get

$$0 = \sum_{j=3}^{\infty} dt_{j}z^{j},$$  

i.e., $dt_{j} = 0, j \geq 3$. Combining (7) with Eq. (3), heat hierarchy reduces to heat equation.

3. Extended heat hierarchy

Repeating this extension process, we now wish to further extend heat hierarchy by its symmetry.

Let $T$ be the differential operator

$$T = \sum_{j=1}^{\infty} j t_{j}\partial_{x}^{j-1}.$$  

One finds

$$[\frac{\partial}{\partial t_{i}} - \partial_{x}^{i}, T] = 0, \quad \forall i \geq 1.$$  

From the obvious relation $[\frac{\partial}{\partial t_{i}}, \partial_{x}^{i}, T] = 0, i, k \geq 1$, consider the algebra of operators generated by

$$\mathcal{S} := \{ V_{m,j} = T^{m} \circ \partial_{x}^{j} | m, j \geq 0 \},$$  

which commute with the entire heat hierarchy. Similarly as before, they generate the symmetries of heat hierarchy.

As an algebra, $\mathcal{S}$ is generated by two elements $\{\partial_{x}, T\}$. From the commutation relation

$$[\partial_{x}, T] = [\frac{\partial}{\partial t_{1}}, T] = 1,$$

it follows that $\mathcal{S}$ is isomorphic to the Weyl algebra $\mathfrak{A}_{1}$, the algebra of differential operators with polynomial coefficients in 1 variable.

For our purpose, it is more convenient to use an anti-isomorphism $\mathcal{S} \rightarrow \mathfrak{A}_{1}$ given as follows. Let $z$ be the indeterminate for the Weyl algebra $\mathfrak{A}_{1}$. Define the anti-isomorphism $s \equiv \mathfrak{A}_{1}$ in terms of the basis elements by

$$V_{m,j} \rightarrow z^{j} \left( \frac{d}{dz} \right)^{m}, \quad m, j \geq 0.$$  

In this way $\mathfrak{A}_{1}$ has an anti-representation (as a Lie algebra) on the infinite jet space of solutions to heat hierarchy.
Let us determine the structure equation of $s$ as a Lie algebra. Note first the identity

$$
\left( \frac{d}{dz} \right)^m \circ z^k = \sum_{r=0}^{\infty} c_r^{m,k} z^{k-r} \left( \frac{d}{dz} \right)^{m-r},
$$

$$
c_r^{m,k} = \frac{m!}{r!} \binom{k}{r}
$$

**Lemma 3.1.** The generators satisfy the commutation relations

$$
[V_{m,j}, V_{n,k}] = - \sum_{r=0}^{\infty} (c_r^{m,k} - c_r^{n,j}) V_{m+n-r,j+k-r}.
$$

Note the negative sign because (11) is an anti-isomorphism of Lie algebras.

**Corollary 3.2.** Let $\{ \eta^{\theta,i} \}$ be formally the left invariant 1-forms of the Lie algebra $s$ dual to the basis $\{ V_{q,i} \}$. They satisfy the structure equation

$$
z^i \left( \frac{d}{dz} \right)^q \eta^{\theta,i} + \left( z^k \left( \frac{d}{dz} \right)^n \right) \circ \left( z^l \left( \frac{d}{dz} \right)^m \right) \eta^{m,j} \wedge \eta^{n,k} = 0.
$$

**Corollary 3.3.** For any $q, i \geq 0$,

$$
d\eta^{\theta,i} \equiv 0 \mod \{ \eta^{m,j} \}_{m \geq 1} \cup \{ \eta^{0,0} \}.
$$

**Proof.** It follow from $[V_{0,i}, V_{0,j}] = [\partial_z^i, \partial_z^j] = 0$ for all $i, j \geq 0$. □

This suggests the following extension of heat hierarchy. Consider the first order system of equations given by

$$
du = \sum_{m,j \geq 0} V_{m,j}(u) \eta^{m,j},
$$

$$
dp_i = \sum_{m,j \geq 0} L_{V_{0,j}} \left( V_{m,j}(u) \eta^{m,j} \right)
$$

$$
= \sum_{m,j \geq 0} V_{0,j} \circ V_{m,j}(u) \eta^{m,j} + V_{m,j}(u) L_{V_{0,j}} (\eta^{m,j})
$$

$$
= \sum_{m,j \geq 0} V_{m,j} \circ V_{0,i}(u) \eta^{m,j} = \sum_{m,j \geq 0} V_{m,j}(p_i) \eta^{m,j},
$$

where $p_i = V_{0,i}(u)$. Here in the fourth equality, we used the formula

$$
L_{V_{0,j}} (\eta^{m,j}) = \sum_{q,l} \eta^{m,j} ([V_{q,l}, V_{0,j}]) \eta^{q,l}.
$$

It follows that one may rewrite (16) as

$$
dp_i = \sum_{m,j \geq 0} V_{m,j}(p_i) \eta^{m,j}, \quad i \geq 0.
$$

In terms of (3), this is equivalent to an extension

$$
dp = p \hat{\omega}$$
for an $\infty$-by-$\infty$ matrix valued 1-form $\hat{\omega}$. The equation (17) shows that $\hat{\omega}$ also has the lower triangular form as in (11).

\[
\hat{\omega} = \begin{pmatrix}
\hat{\omega}_0 & \hat{\omega}_1 & \hat{\omega}_2 & \hat{\omega}_3 & \hat{\omega}_4 & \cdots \\
\hat{\omega}_0 & \hat{\omega}_0 & \hat{\omega}_1 & \hat{\omega}_2 & \hat{\omega}_3 & \cdots \\
\hat{\omega}_0 & \hat{\omega}_0 & \hat{\omega}_0 & \hat{\omega}_1 & \hat{\omega}_2 & \cdots \\
\hat{\omega}_0 & \hat{\omega}_0 & \hat{\omega}_0 & \hat{\omega}_0 & \hat{\omega}_1 & \cdots \\
\hat{\omega}_0 & \hat{\omega}_0 & \hat{\omega}_0 & \hat{\omega}_0 & \hat{\omega}_0 & \cdots \\
\end{pmatrix}.
\]

Note in particular $\hat{\omega} \wedge \hat{\omega} = 0$, and the compatibility equations for (17) reduces to

\[
d \hat{\omega} = 0.
\]

It should be remarked that a component of $\hat{\omega}$ is a formal sum of infinite number of 1-forms in $\eta^{m,j}$'s with coefficients in the (finite) polynomial ring $\mathbb{R}[t_1, t_2, ...]$. It is thus necessary to introduce the variables $t_i$'s to the extended system such that:

a) it is compatible with (17),

b) the system of equations (17) reduces to the original heat hierarchy under the constraint

\[
\text{(21)} \quad \{ \eta^{m,j} = 0 \}_{m \geq 1} \cup \{ \eta^{0,0} = 0 \}.
\]

In view of Ex. 2.1, we give an example of such an extension via a wave function for heat hierarchy.

### 4. Extension by a wave function for heat hierarchy

Consider the wave function for heat hierarchy

\[
w = e^{\xi}, \quad \text{where } \xi = \sum_{k=0}^{\infty} t_k z^k.
\]

Here $z$ is an auxiliary indeterminate (one may identify this with 'z' from (11)). By expanding in a power series in $z$, one finds that $w$ is a formal solution to the original heat hierarchy in the sense that each coefficient of the series, which is a well defined element of the polynomial ring $\mathbb{R}[t_1, t_2, ...]$ up to scaling by $e^{0_0}$, is a solution to heat hierarchy. Note that the sum starts with $j = 0$ and $t_0$ is a new variable.

Observe first

\[
\partial_j^i w = z^i w, \quad T^m w = \left( \frac{d}{dz} \right)^m w.
\]

From the first equation for $du$ in (17), substitute $u = w$. Then

\[
dw = w \left( \sum_{k=0}^{\infty} \frac{d}{dz} \right)^k = \sum_{m,j \geq 0} T^m \circ \partial_j^i (w) \eta^{m,j}
\]

\[
= \sum_{m,j \geq 0} T^m (z^j w) \eta^{m,j} = \sum_{m,j \geq 0} z^j \left( \frac{d}{dz} \right)^m (w) \eta^{m,j}.
\]

The last expression is of the form

\[
w \left( \sum_{k=0}^{\infty} z^k \sum_{m,j} T^k \eta^{m,j} \right), \quad T^k_{m,j} \in \mathbb{R}[t_1, t_2, t_3, ...].
\]
It follows that each $dt_k, k \geq 0$, is uniquely determined by (23).

**Definition 4.1.** Extended heat hierarchy by the wave function (22) is the linear differential system (17) for the infinite sequence of variables $p = (p_0, p_1, p_2, ...)$ combined with the system of equations (23) for the sequence of time variables $(t_0, t_1, t_2, ...)$.

Our main result in this section is that the integrability conditions for the combined linear differential system of equations (17), (23) are satisfied and the extended heat hierarchy by wave function is formally completely integrable.

Let $X_0 = \mathbb{R}^\infty$ denote the manifold with coordinates $(p_0, p_1, ...)$. Let $I_0$ be the (completely integrable) Pfaffian system on $X_0$ for the heat hierarchy (2).

**Theorem 4.1.** Let $\mathfrak{U}_1$ be the Weyl algebra of differential operators with polynomial coefficients in 1 variable. Let $G_{\mathfrak{U}_1}$ be the corresponding formal Lie group with Lie algebra $\mathfrak{U}_1$. Let $X = G_{\mathfrak{U}_1} \times \mathbb{R} \times X_0 \to X_0$ be the product bundle (here the middle $\mathbb{R}$ factor is for the variable $t_0$).

a) The extended heat hierarchy by the wave function (22) defines a completely integrable Pfaffian system $I$ on $X$.

b) Under the projection $\pi : X \to X_0$, we have

$$\pi^* I_0 \subset I.$$ 

In this sense, $(X, I)$ is an integrable extension of $(X_0, I_0)$.

c) When restricted to a leaf of the foliation defined by the constraint (21), the Pfaffian system $I$ reduces to the original heat hierarchy.

We remark that, since the extended heat hierarchy contains infinite number of both dependent and independent variables, the meaning of complete integrability here is different from the usual Frobenius condition in finite dimensional situation. As mentioned in the introduction, it should be understood in the sense of formal differential geometry.

**Proof of Thm. 4.1.** The claim is that, by taking the exterior derivative of Eqs. (17), (23), we get (symbolically)

$$d(17), d(23) \equiv 0 \mod (17), (23).$$

We check each equation $d(23), d(17)$ in turn.

4.1. Eq. (23). We show that

$$(24) \quad d \left( \sum_{m,j \geq 0} z^j \left( \frac{d}{dz} \right)^m (w) \eta^{m,j} \right) \equiv 0 \mod (23), (14).$$

Note here the exterior derivative $d$ treats $z$ as constant and commutes with $z^j \left( \frac{d}{dz} \right)^m$. Hence the left hand side of (24) is

$$= \sum_{m,j \geq 0} z^j \left( \frac{d}{dz} \right)^m (dw) \wedge \eta^{m,j} + \sum_{m,j \geq 0} z^j \left( \frac{d}{dz} \right)^m (w) d\eta^{m,j}$$

$$= \sum_{n,k \geq 0} \left( z^k \left( \frac{d}{dz} \right)^n (w) \eta^{n,k} \wedge \eta^{m,j} + \sum_{q \geq 0} z^q \left( \frac{d}{dz} \right)^q (w) d\eta^{q,i} \right)$$

$$= 0 \text{ from (14).}$$
4.2. Eq. (17). As remarked earlier, the integrability equation for \( d(p_i) = 0, \ i \geq 0, \) is
\[
(25) \quad d \hat{\omega}_0^k = 0, \quad k \geq 0,
\]
where \( d \hat{\omega}_0^k \) is a series of 2-forms of the form
\[
dt \wedge \eta^{m/l}, \eta^{m/l} \wedge \eta^{p/f}
\]
with coefficients in \( \mathbb{R}[t_1, t_2, ...] \). Substituting \( u = w \), we have \( p_i = z^i w \). The claim follows from Sec. 4.1 above by expanding (25) as a series in \( z \).

To complete the proof of theorem, it is left to show that extended heat hierarchy reduces to heat hierarchy under the constraint (21). Note when \( u = w \), we have \( p_i = z^i w \). Expand the equation
\[
(26) \quad dw = w \sum_{k=0}^{\infty} dt_k z^k = \sum_{k=0}^{\infty} p_k \hat{\omega}_0^k
\]
as a series in \( z \), and one gets
\[
dt_k = \hat{\omega}_0^k, \quad k \geq 0.
\]
Moreover, set \( m = 0 \) from (23) and one finds that under the constraint (21),
\[
dt_0 \equiv 0, \\
dt_k \equiv \eta^{0,k}, \quad \text{mod } \{ \eta^{m/l} \}_{m \geq 1} \cup \{ \eta^{0,0} \}, \quad k \geq 1.
\]
This completes the proof.

Remark 4.2. Compare this with Ex. 2.1.

One may construct other formally completely integrable extended heat hierarchies by assigning the 1-forms \( dt_k \)'s differently from (23). The integrability condition (20) implies that, at least locally, they always have the structure of integrable extension of heat hierarchy.

5. KP hierarchy and its extension

After a review of the theory of KP hierarchy, we give an indication on how to derive extended KP hierarchy from the integrability equations of dressed extended heat hierarchy. The study of extension property of KP hierarchy will be reported elsewhere.

Let
\[
D^{(-1)} = \{ \sum_{n=1}^{\infty} f_n \partial_x^{-n} \}
\]
be the space of integral operators, and let
\[
1 + D^{(-1)}
\]
be the corresponding group of invertible pseudo differential operators. Under the dressing by the group \( 1 + D^{(-1)} \), heat hierarchy transforms to a system of linear differential equations (Lax pair) associated with KP hierarchy. From this point of view, the introduction of additional symmetries of KP hierarchy becomes natural and they correspond to the symmetries generated by the operator \( \mathcal{T} \) of heat hierarchy.
Let $g \in 1 + D^{(-1)}$. Set

\begin{equation}
L := g \circ \partial_x \circ g^{-1} = \partial_x + \sum_{a=1}^{\infty} v^a \partial_x^{-a}.
\end{equation}

Let $\tilde{u} = g(u)$ be the dressing, a formal change of variable. In order to determine first the transformation of heat hierarchy, note

\begin{equation}
\frac{\partial g}{\partial t_j} \circ g^{-1} = \frac{\partial}{\partial t_j} - \left( \frac{\partial g}{\partial t_j} g^{-1} + g \partial_x g^{-1} \right).
\end{equation}

We require that this is a differential operator on $\tilde{u}$. Since $g \in 1 + D^{(-1)}$, this forces

\begin{equation}
\frac{\partial^j}{\partial t_j} g^{-1} = -(L)_-,
\end{equation}

where $L^j = (L)_+ + (L)_- \in D^{(\infty)} \oplus D^{(-1)}$ is the decomposition of $L^j$ into differential, and integral parts. From this it follows that

\begin{equation}
\frac{\partial g}{\partial t_j} \circ g^{-1} = \frac{\partial}{\partial t_j} - (L)_+,
\end{equation}

and the transformed linear differential equation for $\tilde{u}$ becomes

\begin{equation}
d\tilde{u} = \sum_{j=1}^{\infty} (L)_+^j(\tilde{u}) dt_j.
\end{equation}

Note

\begin{equation}
(L)_+^j = \partial_x^j + \text{(lower order terms)},
\end{equation}

and the symbol of this system of linear differential equations is isomorphic to heat hierarchy.

Differentiating (27), we obtain from Eq. 28 the defining equations of KP hierarchy

\begin{equation}
\frac{\partial}{\partial t_j} L = [(L)_+^j, L], \quad j = 1, 2, \ldots.
\end{equation}

It is known that this equation is equivalent to (28).

The compatibility of the hierarchy of linear differential equations for $\tilde{u}$,

\begin{equation}
\frac{\partial}{\partial t_j} (L)_+^k(\tilde{u}) = \frac{\partial}{\partial t_k} (L)_+^j(\tilde{u}),
\end{equation}

follows from the identity

\begin{equation}
\frac{\partial}{\partial t_j} (L)_+^k - \frac{\partial}{\partial t_k} (L)_+^j = [(L)_+^j, (L)_+^k].
\end{equation}

The fact that this is implied by the first order equations 32 is by no means trivial and exhibits a special property of KP hierarchy.
Let \( \hat{p} \) be the row vector \( \hat{p} = (\hat{p}_0 = \hat{u}, \hat{p}_1, \hat{p}_2, \ldots) \). Since \( L_+ = \partial_x \) and each \((L)_+\) is a linear differential operator, it follows that the transformed heat hierarchy is equivalent to

\[
(34) \quad d\hat{p} = \hat{p}\phi,
\]

for an \( \infty \)-by-\( \infty \) matrix valued 1-form \( \phi \). Each component of \( \phi \) is now a linear combination of 1-forms in \( dt' \)s with coefficients in the ring of differential polynomials generated by \( v^a \)'s (\( v^a \)'s are the coefficients of \( L \)). The identities (33), which are again equivalent to the defining equations of KP hierarchy, [7], imply as in the heat hierarchy case that the following compatibility equation holds,

\[
d\phi + \phi \wedge \phi = 0.
\]

We proceed to describe the additional symmetries, and the corresponding extension of KP hierarchy. Let

\[
S = g \circ \mathcal{T} \circ g^{-1},
\]

where \( \mathcal{T} \) is the symmetry operator (8). Conjugating the relations (9), (10) one finds

\[
(35)
\]

\[
\left[ \frac{\partial}{\partial t_i} - L_i', S \right] = 0, \quad \forall i \geq 1, \\
\left[ L, S \right] = 1.
\]

It follows that the two elements \{ \( L, S \) \} again generate a symmetry algebra for the dressed heat hierarchy which is isomorphic to the Weyl algebra \( \mathfrak{sl}_1 \). Based on this, the extended dressed heat hierarchy can be constructed similarly as in the heat hierarchy case. Let us postpone the details of this construction, and instead content ourselves with the following schematic diagram.

\[
\omega \rightarrow \hat{\omega} \quad (\text{extended heat}) \quad \text{dressing} \quad \phi \rightarrow \hat{\phi} \quad (\text{extended dressed heat}).
\]

Here the short arrow “\( \rightarrow \)” means the extension by additional symmetries. The extended KP hierarchy will be equivalent to the compatibility equation

\[
d\hat{\phi} + \hat{\phi} \wedge \hat{\phi} = 0.
\]

6. Concluding remark

a) We have examined the extension property of heat hierarchy, a basic model of hierarchy of scalar evolution equations which underlies KP hierarchy up to dressing transformation. The subsequent generalization, application, and refinement of the theory to KP hierarchy may provide an alternative viewpoint on the subject, especially regarding the solutions to string equations, [7]. For example, it is not clear at the moment how one could relate a class of special solutions to extended heat hierarchy, which are stationary for a subalgebra of \( \mathfrak{sl}_1 \) symmetry, to a class of solutions to KP hierarchy satisfying Virasoro type constraints.

b) Among the integrable equations, one may argue that the simplest and most important is KP hierarchy. It is a prototype of the integrable equations which possesses almost all of the rich algebraic, and geometrical properties shared by integrable equations. In fact, it is believed that many integrable equations can be derived and/or associated with KP hierarchy by reduction, extension, and generalized change of variables, [3]. It is expected from this that the construction
of extended hierarchy similar to extended heat hierarchy will exist for the other integrable equations.

The initial motivation for the present work was to apply the idea of extended symmetry to the problem of finding a new class of generalized finite type constant mean curvature (CMC) surfaces in a three dimensional space form, [4]. From the remarks above, the intrinsic structural properties of the extended heat (KP) hierarchy may persist in the differential equation for CMC surfaces, and the extended heat (KP) hierarchy will serve as a model for the analysis of the hierarchy of equations associated with the symmetries of CMC surfaces. It should be remarked that the PDE analogue of CMC hierarchy is the combined elliptic sinh-Gordon+mKdV hierarchy, [5].

The proposed analogy between heat hierarchy and CMC hierarchy is summarized in the following diagram.

(extended) heat hierarchy \[\xrightarrow{\text{dressing}}\] dressed (extended) heat hierarchy = Lax pair for (extended) KP hierarchy

(extended) CMC hierarchy with \(\gamma^2 = 0\) \[\xrightarrow{\text{dressing}}\] (extended) CMC hierarchy with \(\gamma^2 \neq 0\)

(Here \(\gamma^2\) is a structural constant for CMC surfaces).

The infinite sequence of local, higher order symmetries for the CMC equation can be derived from the consideration of (generalized) conformal CMC deformations preserving Hopf differential, [4]. The additional symmetries for the extension of CMC hierarchy, which would correspond in the diagram above to the symmetries for heat hierarchy generated by the operator \(T\), should be transversal to these symmetries. They may be derived by inserting the spectral parameter \((\lambda)\) as a new variable into the CMC equation. Perhaps in this case the fact that the extended Maurer-Cartan form for CMC surfaces is linear in \(\lambda, \lambda^{-1}\) accounts for the appearance of Weyl algebra type of extended symmetry algebra.

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