Multi-state asymmetric simple exclusion processes

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Abstract

It is known that the Markov matrix of the asymmetric simple exclusion process (ASEP) is invariant under the $U_q(sl_2)$ algebra. This is the result of the fact that the Markov matrix of the ASEP coincides with the generator of the Temperley-Lieb (TL) algebra, the dual algebra of the $U_q(sl_2)$ algebra. Various types of algebraic extensions have been considered for the ASEP. In this paper, we considered the multi-state extension of the ASEP, by allowing more than two particles to occupy the same site. We constructed the Markov matrix by dimensionally extending the TL generators and derived explicit forms of particle densities and currents on steady states. Then we showed how decay lengths differ from the original two-state ASEP under closed boundary conditions.

1 Introduction

The asymmetric simple exclusion process (ASEP) is a one-dimensional exclusion process that describes discrete one-dimensional random walks. Among far-from-equilibrium systems, the ASEP is one of few examples which allows us the exact analysis. This model is first introduced in order to discuss the dynamics of ribosome translocation along mRNA [1]. Then the model was applied in the context of traffic and transport systems [2]. As the exactly solvable non-equilibrium systems, the ASEP has been attracted for decades; After being used in the context of a diffusion process [3][4], the connection to the solvable two-dimensional lattice system, the six-vertex model associated with the $U_q(sl_2)$ algebra, was pointed out in [5][6]. Based on the $U_q(sl_2)$-invariance of the Markov matrix, the dynamical exponent was discussed besides the particle-density profiles and the particle currents were computed [7].

The standard procedure to analyze the ASEP with general boundary conditions is the matrix product ansatz (MPA) [8]. By assigning a matrix to each of the empty and occupied state, the steady state is given by the combination of those matrices which satisfy quadratic relations. Employing this method, various physical quantities such as particle-density profiles and particle currents, as well as steady states, were exactly computed. The existence of the phase transition in the particle densities with respect to incoming and outgoing hopping rates was obtained from the exact calculation [9][10][11][12], as the evidence of strong dependence on boundary conditions of non-equilibrium systems. Matrices casted in a matrix product state satisfies quadratic relations which are understood in the context of the $q$-boson [13] and later whose connection with the Askey-Wilson polynomials was pointed out [14].
The Bethe ansatz method, first introduced to solve many body problems \cite{15}, was also applied to the computation of physical quantities on the steady state of the ASEP. The periodic boundary condition \cite{16}, the system with infinite length \cite{17, 18}, and the open boundary case \cite{19} were discussed in each literature. The relation to the Kardar-Parisi-Zhang (KPZ) universality classes was also discussed through the dynamical exponent derived from the analysis of the Bethe-ansatz equations for the periodic boundary case \cite{16} and through the asymptotic behavior of correlation functions for the closed boundary case \cite{7}. Even though the connection between the MPA and the Bethe vectors have not been clear for years, this question was unveiled in \cite{20}, by expressing creation and annihilation operators of the algebraic Bethe ansatz in terms of the operators in the matrix product states.

The model was algebraically extended to be associated with various algebras. The possible realizations of the Hecke algebra were pointed out which describe the time evolution of reaction-diffusion processes \cite{21}. Among various algebraic extensions, the ASEP with multi-species has been closely studied \cite{22, 23, 24}. The steady state was constructed by the MPA, whose matrix elements satisfy similar quadratic relations as in the one-species case \cite{25}. The spectrum of the Markov matrix was also studied on the periodic system, in which the dynamical exponent was found to be the same as the one-species case \cite{26}. It has been also pointed out that various models are mapped onto the ASEP. The interesting example is the zero-range process \cite{27}, which often appears as another traffic model with hopping rates depending on the number of particles in each box.

In this paper, another algebraic extension of the ASEP, the ASEP associated with the higher-dimensional representation of the $U_q(sl_2)$ algebra is proposed. The original two-state ASEP is characterized by the two-dimensional basis associated with the fundamental representations of the $U_q(sl_2)$ algebra defined by the empty state and the one-particle state. The idea is based on the dimensional extension of the Markov matrix, which results in the multi-state extension of the ASEP. Although it is not straightforward how to extend the Markov matrix itself to the higher-dimensional one by keeping integrability of the model, we found the similarity transformation which maps the Markov matrix onto generators of the Temperley-Lieb (TL) algebra \cite{28}. The dimensional extension of the TL generators was proposed in \cite{29}, by using the notion of fusion. Applying the projection operator $q$-symmetrizes $\ell$ spaces of the two-dimensional vector spaces by allowing to take out the $(\ell + 1)$-dimensional representation, the highest weight representation. The virtue of this dimensional extension employing the fusion procedure is that, while keeping the commutativity with the $U_q(sl_2)$ generators, the TL generator is dimensionally extended. Consequently, the Markov matrices associated with higher-dimensional TL generators have the $U_q(sl_2)$-invariance but describe multi-state stochastic processes. This construction of the multi-state ASEP's suggests the existence of a new family of integrable stochastic models associated with the higher-dimensional representations.

It is an interesting question how the number of states in the model affects on physical behaviors. For example, we would expect a different exponent in decay lengths for the ASEP's with different number of states. Due to the possible variety of incoming and outgoing hopping rates, the rich structure of phases in the particle-density profiles is expected under the general boundary conditions. This paper focuses on phenomena in the steady state under the closed boundary conditions, in which the whole system is invariant under the $U_q(sl_2)$ algebra. We first derive the steady states of the multi-state ASEP's, which are given by the basis of the $U_q(sl_2)$ algebra. Although computations are cumbersome if one works with high-dimensional representations, we show how one comes down onto two-dimensional representations by means
of the projection operators. We derive the explicit forms of particle-density profiles, which shows the transition from the zero-particle domain to the high-density domain. Then we prove that the transition domain depends on the number of states of the system. Consequently, we also show that the existence of right-moving currents and left-moving currents, although they compensate with each other, at the transition domain from the zero density to the high density.

This paper is organized as follows. In Section 2, we review the algebraic aspects of the ASEP. Section 3 is devoted to the model settings of the multi-state ASEPs, including the construction of the higher-dimensional TL generators and the similarity transformations in order for a Markov matrix to satisfy the principle of preservation of probability. In Section 4, we show the exact calculation of the particle-density profiles and the particle currents as the examples of physical quantities that can be computed on the multi-state ASEPs. The last section is devoted to concluding remarks and open problems.

2 A brief review of ASEP

The asymmetric simple exclusion process (ASEP) is a stochastic process defined on a one-dimensional lattice consisting of \( N \) sites with a variable \( \tau_i \in \{0, 1\} \) attached to each site \( i \). This variable \( \tau_i \) is, in a physical sense, considered as the number of particles admitted in the \( i \)th box. The transition rules is determined by the local transition rates defined for the configuration of variables on two neighboring sites \((\tau_i, \tau_{i+1})\); the transition \((1, 0) \rightarrow (0, 1)\) occurs with a rate \( p_R \), while the transition \((0, 1) \rightarrow (1, 0)\) with a rate \( p_L \).

The time evolution of the configuration of variables is given by the differential-difference equation. By writing the state of the whole system as

\[
|\tau_1, \ldots, \tau_N\rangle := |\tau_1\rangle \otimes \cdots \otimes |\tau_N\rangle,
\]

where \( |\tau_i\rangle \in \mathbb{C}^2 \), the vector of configurations at time \( t \) is expressed as

\[
|P(t)\rangle = \sum_{\tau_i \in \{0, 1\}} p(t; \tau_1, \ldots, \tau_N)|\tau_1, \ldots, \tau_N\rangle
\]

with the probabilities \( p(t; \tau_1, \ldots, \tau_N) \) to obtain each configuration. Then the time evolution of \( |P(t)\rangle \) is simply given by the differential-difference equation of \( p(t; \tau_1, \ldots, \tau_N) \):

\[
\frac{d}{dt} p(t; \tau_1, \ldots, \tau_N) = \sum_{i=1}^{N} \Theta(\tau_{i+1} - \tau_i) p(t; \tau_1, \ldots, \tau_{i+1}, \tau_i, \ldots, \tau_N)
\]

\[\]

\[
- \sum_{i=1}^{N} \Theta(\tau_i - \tau_{i+1}) p(t; \tau_1, \ldots, \tau_i, \tau_{i+1}, \ldots, \tau_N),
\]

where

\[
\Theta(x) = \begin{cases} 
-p_R & x < 0 \\
0 & x = 0 \\
 p_L & x > 0.
\end{cases}
\]

In the physics realm, the time evolution of \( |P(t)\rangle \) is often written in a matrix form, which is known as the master equation:

\[
\frac{d}{dt} |P(t)\rangle = M|P(t)\rangle,
\]
where $M$ is the Markov matrix obtained from the update rules (3):

$$M = \sum_{i=1}^{N-1} M_{i,i+1},$$

$$M_{i,i+1} = \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes \cdots \mathbf{1}_{i-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -p_L & p_R & 0 \\ 0 & p_L & -p_R & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathbf{1}_{i+2} \otimes \cdots \otimes \mathbf{1}_N,$$

where the matrix $M_{i,i+1}$ nontrivially acts only on the $i$th and $(i+1)$th spaces of the $N$-fold tensor product of the fundamental representations. The equation (7) indeed describes the process of particle-hopping to the right with a rate $p_R$ and to the left with a rate $p_L$ under the choice of the one-site state $|\tau_i\rangle$ as

$$|0\rangle_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i, \qquad |1\rangle_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i.$$  

Let us remark that under the closed boundary conditions, no incoming or outgoing particle is obtained on the system.

### 2.1 $U_q(sl_2)$-invariance of the Markov matrix

The ASEP is an integrable stochastic process since the update operator $M_{i,i+1}$ is identified with a Temperley-Lieb (TL) generator:

$$e_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1},$$

which satisfies the following algebraic relations:

$$e_i^2 = (q + q^{-1})e_i,$$

$$e_ie_{i+1}e_i = e_i,$$

$$e_ie_j = e_je_i, \quad |i - j| \geq 2.$$  

Applying the following similarity transformation:

$$U = \otimes_{i=1}^{N} U_i = \otimes_{i=1}^{N} \begin{pmatrix} 1 & 0 \\ 0 & q^{-1-1} \end{pmatrix}_i, \quad q = \sqrt{\frac{p_R}{p_L}} > 0$$

leads to the update operator (7) written by the TL generator; the update operator is related to the TL generator via

$$M_{i,i+1} = -\sqrt{p_Rp_L} U_{i,i+1}^{-1} e_i U_{i,i+1}^{-1},$$

for which one can easily check that the relations (10) hold up to overall factors.

Let us introduce the spin operators given by

$$S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad q^{S_z} = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$  


The spin operators are known to generate the $U_q(\mathfrak{sl}_2)$ algebra. The TL algebra is a dual algebra of the $U_q(\mathfrak{sl}_2)$ algebra in the sense that the TL generators commute with those of the $U_q(\mathfrak{sl}_2)$ algebra:

$$[e_i, \Delta(X)] = 0, \quad X \in \{S^\pm, q^{S_z}\},$$

(14)

where $\Delta$ represents the coproduct defined in Appendix A. The commutativity of these generators allows us to compute the exact steady state of the ASEP, and consequently, the exact physical quantities.

In our notations (8), the empty state $|0\rangle$, which is naturally invariant under the time development, is identified with the highest weight vector of the $U_q(\mathfrak{sl}_2)$ algebra. Namely, the steady state of the ASEP with no particle is given by the highest weight vector of the $U_q(\mathfrak{sl}_2)$ algebra:

$$\frac{d}{dt}|0\rangle = M|0\rangle = 0.$$  

(15)

Taking into account that the $U_q(\mathfrak{sl}_2)$ generators commute with the TL generators, and consequently, with the Markov matrix up to the similarity transformation, we have

$$MU(\Delta^{(N)}(S^-))^nU^{-1}|0\rangle = U(\Delta^{(N)}(S^-))^nU^{-1}M|0\rangle = 0,$$

(16)

where $\Delta^{(N)}$ is the $N$th coproduct (Appendix A). Thus, a series of steady states is obtained as the basis of the $U_q(\mathfrak{sl}_2)$ algebra constructed by applying the generator $S^-$ to the highest weight vector.

3 Multi-state ASEP

As we reviewed in the previous section, the update operators of the ASEP satisfy the TL relations. Based on this fact, the extension of the integrable stochastic process to multi-state cases is, taking account that a state of each box is given by (8), achieved by constructing the higher-dimensional update matrices which still commute with the $U_q(\mathfrak{sl}_2)$ generators. The dimensional extension of the TL generators have been discussed by P. Zinn-Justin in [29], in which they have constructed an $(\ell + 1)$-dimensional representation from an $\ell$-fold tensor product of the fundamental representations. Although the higher-dimensional TL generators themselves cannot be update operators, as they enjoy neither the probability conservation nor the positivity of probability in nature, we found similarity transformations, which make the TL generators of an arbitrary dimensional representation describe stochastic processes, and proper combinations of different types of the TL generators that satisfy the positivity of probability.

3.1 Higher-spin TL generators

Higher-dimensional representations are constructed from tensor products of fundamental representations. For instance, applying the projection operator to an $\ell$-fold tensor product of the fundamental representations allows one to take out the $(\ell + 1)$ dimensional representation, i.e. the irreducible highest weight representation.

In the case of the TL algebra, the projection operator is recursively defined from the TL
Figure 1: The TL generator of the \((\ell + 1)\)-dimensional representation. \(\ell\) spaces are fused by the projection operator \(Y\). \(j\) links are included in the operator \(e^{(\ell,j)}_i\).

generators:

\[
Y_k^{(\ell)} = Y_k^{(\ell-1)}\left(1 - \frac{U_{\ell-1}(\tau)}{U_{\ell}(\tau)} e_{k+\ell-1}\right) Y_k^{(\ell-1)},
\]

\[
Y_k = Y_k^{(1)} = 1.
\]

The functions \(U_k(\tau)\) are the Chebyshev polynomials of the second kind with a parameter \(\tau\) given by \(\tau = (q + q^{-1})/2\). The superscripts of the projection operator denote how many spaces the operator acts on. The projection operator \(Y_k^{(\ell)}\) \(q\)-symmetrizes from the \(k\)th to the \((k + \ell - 1)\)th spaces. The TL generators of an \((\ell + 1)\)-dimensional representation are defined through the projection operators as

\[
e^{(\ell,j)}_k = Y_{\ell k}^{(\ell)} Y_{\ell (k+1)}^{(\ell)} e_{\ell k} e_{\ell k-1} e_{\ell k+1} \cdots e_{\ell k-j+1} \cdots e_{\ell k j-1} e_{\ell k+1} e_{\ell k} Y_{\ell k}^{(\ell)} Y_{\ell (k+1)},
\]

where \(j = 1, 2, \ldots, \ell\) indicates the type of the TL generators (Figure 1). Let us remark that each of \(\ell\) kinds of the TL generators constructed in this way still commutes with the \(U_q(\mathfrak{sl}_2)\) generators of the \((\ell + 1)\)-dimensional representations:

\[
[e_i^{(\ell,j)}, \Delta(X)] = 0, \quad X \in \{S^\pm, q S^z\}.
\]

### 3.2 Update operators of multi-state ASERPs

Now we construct update operators of the multi-state ASERPs. In order to be an update operator, the following two conditions should be satisfied:

(i) The sum of each column should be zero (the principle of probability preservation).

(ii) The non-diagonal elements should be negative values, while the diagonal elements should be positive values (positivity of probability).

#### 3.2.1 Probability conservation

As the simplest example, we first show how the 3-dimensional TL generator of the type-1 \(e_i^{(2,1)}\) is modified to satisfy the probability conservation. The operator is given by the 9-by-9
matrix since it acts on a two-fold tensor product of 3-dimensional vector spaces:

\[
E_{i}^{(2;1)} = \left( \begin{array}{cccccccc}
0 & 0 & -\frac{q^{-2}}{q+q^{-1}} & 0 & \frac{q^2}{(q+q^{-1})^2} & 0 & 0 & 0 \\
0 & -\frac{1}{q} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{q^{-2}}{q+q^{-1}} & 0 & \frac{q^2}{(q+q^{-1})^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right). 
\] (21)

This matrix apparently does not satisfy the probability conservation and, in general, the same is true for the other TL generators. However, we found for any dimensional representation the existence of a similarity transformation which makes the TL generators satisfy the probability preservation.

**Proposition 1.** The TL generators of an \((\ell+1)\)-dimensional representation satisfy the principle of probability conservation after a similarity transformation given by the following matrix:

\[
U^{(\ell)} = \bigotimes_{i=1}^{N} U_i = \bigotimes_{i=1}^{N} \left( \begin{array}{c}
a_0^{(i)} \\
a_1^{(i)} \\
\vdots \\
a_k^{(i)} \\
\vdots \\
a_{\ell-1}^{(i)} \\
a_{\ell}^{(i)} \\
\end{array} \right) 
\] (22)

whose matrix elements are given by \(a_k^{(i)} = q^{k\ell(i-1)}\).

The important property is that this diagonal matrix simultaneously transforms any kinds of the TL generators, as long as they belong to the same dimensional representation, in order to satisfy the probability conservation law. As an illustration, let us show the transformed three-dimensional TL generator of the type 1:

\[
U_{i,i+1}^{(2)} e_{i}^{(2;1)} (U_{i,i+1}^{(2)})^{-1} = \left( \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{q^{-2}}{q+q^{-1}} & 0 & \frac{q^2}{(q+q^{-1})^2} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{q} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{q^{-2}}{q+q^{-1}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right). 
\] (23)
and of the type 2:

\[
U_{i,i+1}^{(2)} e_1^{(2;2)} (U_{i,i+1}^{(2)})^{-1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{q-1}{\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -q^4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -q^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -q^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\
\end{pmatrix}_{i,i+1},
\]

both of which, indeed, satisfy the probability conservation.

### 3.2.2 Positivity of probability

Besides the principle of probability preservation, the update matrix should satisfy the positivity of probability which is realized by such conditions that the non-diagonal elements of the update matrix are positive values and the diagonal elements are negative. As was shown in \((23)\), the transformed operator \(M_{i,i+1}^{(2,1)} = U_{i,i+1}^{(2)} e_i^{(2,1)} (U_{i,i+1}^{(2)})^{-1}\) in nature satisfies the positivity of probability, since we chose \(q > 0\) \((11)\). However, the operator \(M_{i,i+1}^{(2,2)} = U_{i,i+1}^{(2)} e_i^{(2,2)} (U_{i,i+1}^{(2)})^{-1}\) has, as is obtained in \((24)\), negative values at the \((3,7)\) and \((7,3)\)-elements.

**Proposition 2.** Consider the following combination:

\[
M_{i,i+1}^{(2)} = b_1^{(2)} M_{i,i+1}^{(2,1)} + b_2^{(2)} M_{i,i+1}^{(2,2)}.
\]

The matrix \(M_{i,i+1}^{(2)}\) gives the update rules of the three-state ASEP as long as \(\beta = \frac{b_2^{(2)}}{b_1^{(2)}}\) satisfies the following conditions:

\[
\begin{align*}
-\frac{q^2}{q + q^{-1}} &< \beta < 0 & (0 < q < 1) \\
-\frac{q^{-2}}{q + q^{-1}} &< \beta < 0 & (1 < q).
\end{align*}
\]

**Proof.** The update matrix \(M_{i,i+1}^{(2)}\) is written as

\[
M_{i,i+1}^{(2)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{q^2(q+q^{-1})} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{q+\beta}{q^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{q^2(q+q^{-1})} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{q^3 + \beta q(q+q^{-1})}{q^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -q^4 & q^4 + \beta q^4 & q^4 + \beta q^4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{q^2(q+q^{-1})} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{q^2(q+q^{-1})}{q^3(q+q^{-1})^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{q^2(q+q^{-1})} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^2 & 0 \\
\end{pmatrix}_{i,i+1}.
\]

In order to make all non-diagonal elements positive at the same time with making all diagonal elements negative values, \(\beta\) needs to satisfy the following six conditions:

\[
\begin{align*}
q(q^3 + \beta + q^2 \beta) &> 0, & -q\beta &> 0, \\
1 + q\beta + q^3 \beta &> 0, & q(q + \beta) &> 0, \\
1 - q^2 + q^4 + q^3 \beta &> 0, & 1 + q\beta &> 0.
\end{align*}
\]
Figure 2: Behaviors of each function in (28) with respect to the parameter $q$. Since the green line crosses with the blue line at $q = 1$, we have $-q^2/(q + q^{-1}) < \beta < 0$ for $0 < q < 1$ and $q^{-2}/(q + q^{-1}) < \beta < 0$ for $1 < q$.

Taking into account that our $q$ takes only positive values, the inequalities (28) can be solved as

$$
\beta < 0, \\
\beta > \max \left\{ -\frac{q^2}{q+q^{-1}}, -\frac{q^{-2}}{q+q^{-1}}, -q, -q^{-1}, -\frac{q^2+q^{-2}}{q+q^{-1}} \right\}.
$$

(29)

From Figure 2, we finally obtain the condition (29) for $\beta$. \hfill \Box

3.3 Special limits of multi-state ASEPs

Now we consider special limits of multi-state ASEPs. Here we focus on the three-state case, which is the simplest but already show different features from the original ASEP. In this subsection, we consider two special limits; the $q \to 1$ limit and the $q \to 0$ limit. In the former limit, the original ASEP shows the symmetric hopping rates to the right and to the left. On the other hand, the latter limit becomes the totally asymmetric simple exclusion process (TASEP), in which each particle is allowed to move only to the left. The special care is required to take this limit for avoiding singularities coming from $q^{-1}$.

3.3.1 $q \to 1$ limit

First, we consider the $q \to 1$ limit of the multi-state ASEP. In this limit, the original ASEP becomes symmetric simple exclusion process (SSEP) in which each particles hops to the right and to the left with the same rate as long as neighboring sites are unoccupied. In the multi-
state case, for instance in the three-state case, we have the following update matrix:

\[
\lim_{q \to 1} M^{(2)}_{i,i+1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\beta - 1 & 0 & \frac{1+2\beta}{4} & 0 & -\beta & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 2\beta + 1 & 0 & -\beta + \frac{1}{2} & 0 & 2\beta + 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & -\beta & 0 & \frac{1+2\beta}{4} & 0 & -\beta - 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}_{i,i+1}, \quad (30)
\]

from which one easily obtains that, as in the original ASEP, the matrix (30) gives the symmetric transition.

The condition (26) is still valid for having the positive probabilities; in the \(q \to 1\) limit, the condition (26) is given by \(-\frac{1}{2} < \beta < 0\). Since we always have \(\frac{1+2\beta}{4} < 2\beta + 1\), the process of this limit forces to reduce the number of boxes occupied by two particles. However, for \(-\frac{1}{2} < \beta < -\frac{1}{3}\), we have \(2\beta + 1 < -\beta\) which implies that two particles in the same box tend to move together.

3.3.2 \(q \to 0\) limit

Next, we consider the \(q \to 0\) limit of the multi-state ASEP. In order for this limit to be well-defined, we set \(q = x/y\) and then take \(x \to 0\). By rescaling the parameter \(\beta\) as \(\beta = \frac{q^2}{q+q^{-1}} \tilde{\beta}\), we obtain the following update matrix:

\[
\lim_{q \to 0} q^3(q + q^{-1})^2 M^{(2)}_{i,i+1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{\beta} + 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -\tilde{\beta} & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}_{i,i+1}, \quad (31)
\]

which describes the TASEP-like transition, namely, particles are allowed to move only to the left. The process is well-defined for such \(\beta\) that satisfies \(-1 < \tilde{\beta} < 0\), which determines the strength of the coupling of two particles in the same box; for larger \(|\tilde{\beta}|\), more hoppings of two particles at the same time, while for smaller \(|\tilde{\beta}|\), we obtain more hoppings of each particle separately.

4 Steady states

In this section, we discuss the steady state of multi-state processes constructed in the previous section. Due to invariance of the TL algebra with respect to a transformation \(q \leftrightarrow q^{-1}\), we restrict our attention to the regime \(q > 1\). This condition is physically interpreted as more hopping to the right than to the left.
A steady state and physical quantities of a closed two-state ASEP have been intensively studied in [7] based on the $U_q(sl_2)$ algebraic relations. Starting from the zero-particle state, which is obviously a steady state of the ASEP, they subsequently obtained an $n$-particle state for arbitrary $n$ by applying the $S^-$-operator to the zero-particle state. A series of steady states of a multi-state ASEP is derived in a similar way from the zero-particle state, as the update matrix still commutes with the $U_q(sl_2)$ generators. However, multi-state extension requires much more cumbersome computation than the two-state case even for the norms. In order to resolve this difficulty, we use a property of a projection operator $Y(\ell)|0\rangle = |0\rangle$ and a map from $(\ell + 1)$-dimensional representations onto the fundamental representations of the $U_q(sl_2)$ algebra, which allows us to proceed all calculations on the $(\ell + 1)$-state ASEP in terms of the two-state system.

**Proposition 3.** Let us consider the $(\ell + 1)$-state ASEP on $N$ sites. The vacuum (zero-particle) state is a steady state of the $(\ell + 1)$-state ASEP and mapped onto the vacuum state of the two-state ASEP with length $\ell N$:

$$|\ell; 0\rangle_{1,\ldots,N} \mapsto \otimes^\ell N \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle_{1,\ldots,\ell N},$$

(32)

$$1,\ldots,N|\ell; 0\rangle \mapsto \otimes^\ell N \begin{pmatrix} 1 & 0 \end{pmatrix} Y^{(\ell)} = |1,\ldots,\ell N\rangle_0.$$  

Proof. It is obvious that the zero-particle state is time invariant. Since we defined the empty state and the occupied state of the two-state ASEP as in (11), the zero-particle state of the two-state ASEP with length $N$ is given by an $N$-fold tensor product of the two-dimensional highest weight vectors:

$$|0\rangle_{1,\ldots,N} = \otimes^N \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$  

(33)

Similarly, the vacuum of the $(\ell + 1)$-state ASEP is given by an $N$-fold tensor product of the $(\ell + 1)$-dimensional highest weight vectors. Taking into account that the $(\ell + 1)$-dimensional highest weight vector is expressed by an $\ell$-fold tensor product of the two-dimensional highest weight vectors with the projection operator, the following relation holds:

$$|\ell; 0\rangle_{1,\ldots,N} \mapsto \otimes^\ell N \left( Y^{(\ell)} \otimes^\ell \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$

(34)

Since the projection operator $Y^{(\ell)}$ trivially acts on the highest-weight vector, we obtain the relation (32).

The dual vector of the zero-particle state is obtained in the same way. \Box

From now on, we abbreviate the subscripts $1,\ldots,N$ attached to each vector unless confusion occurs. An $n$-particle steady state of the two-state ASEP is then subsequently created from the vacuum state by applying the operator $U^{(\ell)} \Delta^{(N)}(S^+)(U^{(\ell)})^{-1}$ [7]. Since the similarity transformation operator trivially acts on the vacuum $(U^{(\ell)})^{-1}|\ell; 0\rangle = |\ell; 0\rangle$, we define the following “untwisted” $n$-particle state:

$$|n\rangle = \frac{1}{[n]!}(\Delta^{(N)}(S^-))^n|0\rangle, \quad \langle n| = \frac{1}{[n]!}\langle 0|(\Delta^{(N)}(S^+))^n.$$  

(35)

Here we introduced the $q$-factorials $[n]! = [n][n-1]\ldots[1]$ consisting of the $q$-integers defined by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$  

(36)
Proposition 4. An $n$-particle steady state of the $(\ell + 1)$-state ASEP is mapped onto a tensor product of two-dimensional vectors:

$$|\ell; n\rangle \mapsto \frac{1}{[n]!}(\Delta^{(\ell)}(S^-))^n|0\rangle, \quad \langle \ell; n| \mapsto \frac{1}{[n]!}(\Delta^{(\ell)}(S^+))^n.$$  \hspace{1cm} (37)

Proof. An untwisted $n$-particle state of the $(\ell + 1)$-state ASEP is generated by applying the operator $S^{\pm}$ to the vacuum:

$$|\ell; n\rangle = \frac{1}{[n]!}(\Delta^{(N)}(S^-)^n|\ell; 0\rangle), \quad \langle \ell; n| = \frac{1}{[n]!}(\Delta^{(N)}(S^+)^n|\ell; 0\rangle.$$  \hspace{1cm} (38)

Here $S^{\pm, (\ell)}$ are the $U_q(sl_2)$ generator of $(\ell + 1)$-dimensional representations.

Let us remind that $U_q(sl_2)$ generators have spatial extension called co-multiplication:

$$\Delta^{(\ell)}(S^{\pm}) = \sum_{j=1}^{\ell} q^{S^j_+} \otimes \cdots \otimes q^{S_{j-1}^+} \otimes S^j_- \otimes q^{-S^j_+} \otimes \cdots \otimes q^{-S^{\ell}_-}.$$  \hspace{1cm} (39)

From the definition, $\Delta^{(\ell)}(S^{\pm})$ $q$-symmetrically acts on an $\ell$-fold tensor product spaces. On the other hand, an $(\ell + 1)$-dimensional vector space associated with the $U_q(sl_2)$ algebra is mapped onto a $q$-symmetric $\ell$-fold tensor product of two-dimensional vector spaces, the operators $S^{\pm, (\ell)}$ admit the following map:

$$S^{\pm, (\ell)} \mapsto \Delta^{(\ell)}(S^{\pm}).$$  \hspace{1cm} (40)

Subsequently, coproduct of $U_q(sl_2)$ generators are invariant under the $q$-symmetrizer, i.e. the projection operator:

$$Y^{(\ell)}\Delta^{(\ell)}(S^{\pm})Y^{(\ell)} = \Delta^{(\ell)}(S^{\pm}).$$  \hspace{1cm} (41)

Combining (38) and (40), we obtain

$$|\ell; n\rangle \mapsto \frac{1}{[n]!}\left(\Delta^{(N)} \circ \cdots \circ \Delta^{(N)} \Delta^{(\ell)}(S^-)\right)^n|0\rangle,$$

$$= \frac{1}{[n]!}(\Delta^{(\ell)}(S^-))^n|0\rangle,$$

reads (37). Here we have chosen the normalization constant as $1/[n]!$. \hfill \Box

4.1 Norms of the steady states

In derivation of steady states, we showed that an $n$-particle state of the $(\ell + 1)$-state ASEP is expressed in terms of two-dimensional $U_q(sl_2)$ basis. Although it is often difficult to proceed analytical computation on higher-dimensional representations, now we can use various formulae having already achieved for the two-state case.

For instance, the norm of an $n$-particle steady state of the two-state ASEP was computed in [7]:

$$\langle n|UU^{-1}|n\rangle = \langle n|n\rangle = \begin{bmatrix} N \\ n \end{bmatrix} = \frac{[N]!}{[N-n][n]!}.$$  \hspace{1cm} (43)

Using relations (37), we straightforwardly obtain multi-state extension of the norm.
Proposition 5. The norm of an $n$-particle steady state of the $(\ell + 1)$-state ASEP is given by
\[
\langle \ell; n | U^{(\ell)}(U^{(\ell)})^{-1} | n; \ell \rangle = \langle \ell; n | \ell; n \rangle = \left[ \frac{\ell N}{n} \right]_! \left[ \frac{n}{n} \right]_! \langle 0 | (\Delta^{(\ell N)}(S^+))^n(\Delta^{(\ell N)}(S^-))^n | 0 \rangle \tag{44} \]
\[
= \left[ \frac{\ell N}{n} \right] \cdot \tag{45}
\]
Proof. The first equation is proved from Proposition 4. The second equation is derived from the commutation relations among $U_q(sl_2)$ generators:
\[
[\Delta(S^+), \Delta(S^-)] = \frac{\Delta(q^{2S_z}) - \Delta(q^{-2S_z})}{q - q^{-1}}. \tag{45}
\]
Detail proof is given in Appendix B.

Here we define the normalized vector by
\[
|\ell; n\rangle_{\text{norm}} \equiv \left[ \frac{\ell N}{n} \right]^{-1} |\ell; n\rangle. \tag{46}
\]
in order to have a normalization condition: \( \langle \ell; n | \ell; n \rangle_{\text{norm}} = 1 \).

4.2 Particle-density profiles

In this subsection, we derive particle-density profiles of an $n$-particle steady state of the multi-state ASEP. Particle-density profiles of the two-state ASEP have been closely studied and known to show transitions in a system with general boundary conditions [9, 10, 11, 12]. We show, although our boundary conditions are closed ones, on which the number of particles is conserved, how particle-density profiles depend on the number of states of the process.

Since we have obtained any steady states of the $(\ell + 1)$-state ASEP in terms of the fundamental representations, correlation functions of the multi-state ASEP are also computed on the fundamental representations if physical quantities can be expressed by two-dimensional representations.

For this reason, the first aim of this subsection is to write an $(\ell + 1)$-by-$(\ell + 1)$ matrix by a tensor product of two-by-two matrices. By means of the method used in [30], the following proposition holds for an $(\ell + 1)$-by-$(\ell + 1)$ single-entry matrix with the $(r, s)$-entry 1.

Proposition 6. An $(\ell + 1)$-by-$(\ell + 1)$ single-entry matrix with the $(r, s)$-entry 1 is given by $|\ell; r - 1 \rangle \otimes |\ell; s - 1 \rangle$. Using Proposition 4 a single-entry matrix of an $(\ell + 1)$-dimensional representation is expressed by an $\ell$-fold tensor product of the fundamental representations.

Using this proposition, we compute particle-density profiles of the multi-state ASEP. For simplicity, we first discuss the three-state case and then later show a general case.

4.2.1 Density profiles on the three-state ASEP

Under the presence of $n$ particles, particle density at the $x$th site is given by the following quantity:
\[
\rho_n^{(2)}(x) = \langle 2; n | U^{(2)} \text{diag.}(0, 1, 2)_x (U^{(2)})^{-1} | 2; n \rangle_{\text{norm}} \tag{47}
\]
\[
= \langle 2; n | \text{diag.}(0, 1, 2)_x | 2; n \rangle_{\text{norm}},
\]
where $\text{diag.}(0,1,2)_x$ is a three-by-three diagonal matrix which acts on the $x$th space of an $N$-fold tensor product of three-dimensional vector spaces.

Using Proposition 6, the matrix $\text{diag.}(0,1,2)$, which counts the number of particles at the $x$th site in a steady state, is written by
\[
\text{diag.}(0,1,2) = 1 \cdot |2;1\rangle_{\text{norm}} \otimes |2;1\rangle + 2 \cdot |2;2\rangle_{\text{norm}} \otimes |2;2\rangle.
\] (48)

Using the relations (37), the right-hand side of (48) is expressed in terms of the two-dimensional $U_q(sl_2)$-basis. For instance, $|2;2\rangle_{\text{norm}} \otimes |2;2\rangle$ is expressed as
\[
|2;2\rangle_{x;\text{norm}} \otimes x|2;2\rangle 
\Rightarrow |1\rangle_{2x-1;\text{norm}}|2\rangle_{2x;\text{norm}} \otimes 2x-1|1\rangle_{2x}|1\rangle
= \text{diag.}(0,1)_{2x-1} \otimes \text{diag.}(0,1)_{2x}
= n_{2x-1}n_{2x}.
\] (49)

Hence we have
\[
\rho_n^{(2)}(x) = \langle n | \frac{1}{2} \left( q(1-n_{2x-1})n_{2x} + S_{2x-1}^+ S_{2x}^- + S_{2x-1}^- S_{2x}^+ + q^{-1}n_{2x-1}(1-n_{2x}) \right) |n\rangle_{\text{norm}}
+ 2 \cdot \langle n | n_{2x-1}n_{2x} | n \rangle_{\text{norm}}.
\] (50)

The first line of (50) is written only by particle-counting operators using the relations (89):
\[
\frac{q}{2} \cdot \langle n | n_{2x} | n \rangle_{\text{norm}} + \frac{q^{-1}}{2} \cdot \langle n | n_{2x-1} | n \rangle_{\text{norm}} - \langle n | n_{2x-1}n_{2x} | n \rangle_{\text{norm}}
+ 2 \cdot \left[ \frac{|n|q^{-N+2(2x-1)}}{N-n+1} \right] \cdot (n-1)(1-n_{2x-1})(1-n_{2x}) | n-1 \rangle_{\text{norm}}.
\] (51)

The second line of (50) is decomposed into a summation of one-point functions using the formula (91). Then, we obtain that particle-density profiles of the three-state ASEP are expressed in terms of one-point functions:
\[
\rho_n^{(2)}(x) = \langle n | n_{2x-1} | n \rangle_{\text{norm}} + \langle n | n_{2x} | n \rangle_{\text{norm}},
\] (52)

whose explicit expression is evaluated from the formula (88):
\[
\rho_n^{(2)}(x) = \left[ \frac{2N}{n} \right]^{-1} \sum_{k=0}^{n-1} (-1)^{n-k+1} \left[ \frac{2N}{k} \right] q^{-(n-k)(2N-4x+2)} \left( q^{n-k} + q^{-(n-k)} \right).
\] (53)

The resulting relation (52) is understood as follows: We have constructed three-dimensional basis of the $U_q(sl_2)$ algebra by $q$-symmetrizing a two-fold tensor product of two-dimensional $U_q(sl_2)$ vector spaces. It is equivalent to work on the double length of two-state ASEP but with $q$-symmetrizing $(2x-1)$th and $2x$th sites, instead of working on three-state ASEP. As a result, particle density at the $x$th site of the three-state model is given by a summation of particle densities at the $(2x-1)$th site and the $2x$th site of the two-state ASEP.

### 4.2.2 Density profiles on the $(\ell + 1)$-state ASEP

As the analogue of the three-state ASEP, particle-density profiles of the $(\ell + 1)$-state ASEP is also expressed by a summation of one-point functions of the two-dimensional representation. Now we prove the following proposition:
Proposition 7. Particle-density profiles of the $(\ell + 1)$-state ASEP is evaluated through the following expression:

$$\rho_n^{(\ell)}(x) = \sum_{j=1}^{\ell} \langle n | (U^{(\ell)})^{-1} n_{j(x-1)+j} U^{(\ell)} | n \rangle_{\text{norm}} = \sum_{j=1}^{\ell} \langle n | n_{j(x-1)+j} | n \rangle_{\text{norm}}. \quad (54)$$

In the proof of this proposition, we need the following proposition besides Proposition 10:

Proposition 8. An expectation value of a pair of spin operators $S^\pm \sigma$ ($\sigma = \pm$) is written by particle-counting operators in two different ways:

$$\langle n | S_{x_1}^\sigma S_{x_2}^{-\sigma} | n \rangle_{\text{norm}} = q^{x_2-x_1} \cdot \langle n | n_{x_1} (1 - n_{x_2}) | n \rangle_{\text{norm}} \quad (55)$$

$$= q^{-(x_2-x_1)} \cdot \langle n | (1 - n_{x_1}) n_{x_2} | n \rangle_{\text{norm}}. \quad (56)$$

Proof. Since the spin operators act on an $n$-particle state as (89), the following relations are obtained between an expectation value of the spin operators and that of the particle-counting operators:

$$\langle n | S_{x_1}^\sigma S_{x_2}^{-\sigma} | n \rangle_{\text{norm}} = \left[ \frac{N}{n} \right]^{-1} \cdot \langle n | S_{x_1}^\sigma S_{x_2}^{-\sigma} | n \rangle$$

$$= \left[ \frac{N}{n} \right]^{-1} q^{(-N-1+x_1+x_2)} \cdot \langle n-1 | (1-n_{x_1})(1-n_{x_2}) | n \rangle_{\text{norm}}$$

$$= \left[ \frac{n}{N-n+1} \right] q^{-N-1+x_1+x_2} \cdot \langle n-1 | (1-n_{x_1})(1-n_{x_2}) | n \rangle_{\text{norm}}. \quad (57)$$

Applying the recursion relation (90) to the operator $(1 - n_{x_1})$, one obtains the relations (55), while by applying (90) to $(1 - n_{x_2})$, the relation (56) is obtained. \qed

Let us remind that Proposition 6 allows us to map the $(j, j)$-element of a $(2\ell + 1)\times(2\ell + 1)$ matrix of the $(\ell + 1)$-dimensional representation of the $U_q(sl_2)$ algebra onto the $\ell$th fold of the two-dimensional tensor spaces:

$$| \ell; j \rangle_{\text{norm}} \otimes | \ell; j \rangle = | j \rangle_{1, \ldots, \ell; \text{norm}} \otimes \ldots \otimes | j \rangle. \quad (58)$$

Then we obtain the following lemmas.

Lemma 1. Probability of the presence of $j$ particles at the $x$th site in an $n$-particle steady state is calculated as

$$\langle n | \left( | \ell; j \rangle_x \otimes \ldots \otimes | \ell; j \rangle \right) | n \rangle_{\text{norm}}$$

$$= \sum_{\{x_1, \ldots, x_j\} \in S_1 \setminus S_j} \langle n | \prod_{k=1}^{\ell} (1 - n_{\ell(x-1)+k}) \prod_{i=1}^{j} n_{\ell(x-1)+x_1} | n \rangle_{\text{norm}}. \quad (59)$$
Figure 3: Particle-density profiles of the (A) two-state ASE P, (B) three-state ASEP, and (C) four-state ASEP. Plots are given for a system with size $N = 100$ under the presence of $n = 60$ particles with $q = 2$. The profiles show step-function like behaviors. If one chooses $q^{-1} = 2$, one obtains plots reflected at $N = 50$.

**Lemma 2.** Particle density at the $x$th site of the $(\ell + 1)$-state ASEP is written in terms of the fundamental representation:

$$
\rho^{(\ell)}_n(x) = \sum_{j=0}^{\ell} j \cdot \langle n|\ell; j\rangle_{x;\text{norm}} \otimes x\langle \ell; j|n\rangle
$$

$$
= \sum_{j=0}^{\ell} \sum_{\{x_1, \ldots, x_j\} \in \mathcal{S}_{\ell} \setminus \mathcal{S}_j} \langle n| \prod_{k=x_1, \ldots, x_j}^{\ell} (1 - n_{\ell(x-1)+k}) \prod_{i=1}^{j} n_{\ell(x-1)+x_i}|n\rangle_{\text{norm}}
$$

$$
= \sum_{p=1}^{\ell} \left( \sum_{r=0}^{p-1} (-1)^r (p-r) p C_{p-r} \sum_{\{x_1, \ldots, x_p\} \in \mathcal{S}_{\ell} \setminus \mathcal{S}_p} \epsilon_N \langle n| \prod_{j=1}^{p} n_{\ell(x-1)+x_j}|n\rangle_{\ell N;\text{norm}} \right).
$$

(60)

Using the following relation:

$$
\sum_{r=0}^{p-1} (-1)^r (p-r) p C_{p-r} = 0 \quad p \in \mathbb{Z}_{\geq 2},
$$

(61)

we obtain (61). From the formula for a one-point function (88), we finally obtain an expression for particle-density profile of the $(\ell + 1)$-state ASEP:

$$
\rho^{(\ell)}_n(x) = \sum_{j=1}^{\ell} \left[ \frac{\ell N}{n} \right]^{-1} \sum_{k=0}^{n-1} (-1)^{n-k+1} \left[ \frac{\ell N}{k} \right] q^{-(n-k)(\ell N+1-2(\ell(x-1)+j))}.
$$

(62)

The Figure 3 is particle-density profiles of steady states of the two, three, and four-state ASEP's. The profiles show step-function like behaviors with decay lengths inversely proportional to the number of states of the processes. The step-function like behaviors are phenomenologically understood as a result that we chose a bigger hopping rate to the right than to the left. Detailed analysis of decay lengths is given later with asymptotic analysis.
**4.3 Currents**

In this subsection, we calculate particle currents on the multi-state ASEP. At the $x$th site of the two-state ASEP, current is defined through the following quantity:

$$J(x) := J_R(x) - J_L(x),$$

$$J_R(x) := q \langle n | U^{-1} n_x (1 - n_{x+1}) U | n \rangle_{\text{norm}} = q \langle n | n_x (1 - n_{x+1}) | n \rangle_{\text{norm}},$$

$$J_L(x) := q^{-1} \langle n | U^{-1} (1 - n_x) n_{x+1} U | n \rangle_{\text{norm}} = q^{-1} \langle n | (1 - n_x) n_{x+1} | n \rangle_{\text{norm}}.$$  \hspace{1cm} (63)

By definition, $J_R(x)$ gives an expectation value for the $x$th site being occupied at the same time with the $(x+1)$th site being empty, while $J_L(x)$ gives an expectation value for the $(x+1)$th site being occupied at the same time with the $x$th site being empty up to overall factors (Figure 4). However, in the multi-state case, there are several possible ways for a particle to move to the right or the left; For instance, in the three-state ASEP, there are five types of different hoppings which contribute to right-moving currents and another five to left-moving currents (Figure 4). Since our Markov matrix of the three-state ASEP is given

![Figure 4: Allowed configurations of particles and hopping types. Five different types of right-hopping exist, while another five types for left-hopping in the three-state model. The number of particles which move together is also denoted. Explicit forms of hopping rates $w(ab|a'b')$ are given in (65).](image-url)
by (27), we define currents on a steady state of the three-state ASEP by

\[
J_R^{(2)}(x) = w(10|01)\langle 2; n| \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_x \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_{x+1} |2; n\rangle_{\text{norm}} + \frac{q}{q+q^{-1}}, \quad w(20|11)\langle 2; n| \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_x \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_{x+1} |2; n\rangle_{\text{norm}} + w(10|02)\langle 2; n| \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_x \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_{x+1} |2; n\rangle_{\text{norm}} + w(11|02)\langle 2; n| \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_x \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_{x+1} |2; n\rangle_{\text{norm}} + w(20|02)\langle 2; n| \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_x \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_{x+1} |2; n\rangle_{\text{norm}},
\]

\[
J_L^{(2)}(x) = w(01|10)\langle 2; n| \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_x \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{x+1} |2; n\rangle_{\text{norm}} + w(01|10)\langle 2; n| \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_x \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{x+1} |2; n\rangle_{\text{norm}} + w(11|20)\langle 2; n| \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_x \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)_{x+1} |2; n\rangle_{\text{norm}} + w(20|20)\langle 2; n| \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_x \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)_{x+1} |2; n\rangle_{\text{norm}}
\]

Here \( \beta \) is chosen to satisfy the conditions given by (26). The coefficients \( w(ab|a'b') \) are transition rates obtained in the update matrix (27):\n
\[
w(10|01) = \frac{q^2}{q+q^{-1}}, \quad w(20|11) = q + \beta q^3(q + q^{-1}),
\]
\[
w(11|02) = \frac{q^3 + \beta q(q + q^{-1})}{(q+q^{-1})^2}, \quad w(21|12) = \frac{q^2}{q+q^{-1}},
\]
\[
w(20|02) = -\beta q^2, \quad w(01|10) = \frac{1}{q^2(q+q^{-1})},
\]
\[
w(02|11) = q^{-1} + \beta q^{-3}(q + q^{-1}), \quad w(11|20) = \frac{1+\beta q^2(q + q^{-1})}{q^2(q+q^{-1})^2},
\]
\[
w(12|21) = \frac{1}{q^2(q+q^{-1})}, \quad w(02|20) = -\beta q^{-4}.
\]

Computation is cumbersome but straightforward. Using formulae (91), (55), and (56) and a map onto the fundamental representations, one obtains that currents of the three-state ASEP are expressed by particle-counting operators (Appendix D). Substituting (97) into (64), one obtains

\[
J(x) = J_R(x) - J_L(x) = 0.
\]

Instead of giving explicit forms, we show plots of right-moving currents and left-moving currents (Figure 5). One obtains non-zero right-moving currents, which are compensated by non-zero left-moving ones, at the surface of the high-density domain, as can be easily expected from particle-density profiles (Figure 3). In the high-density domain, particles are “frozen” since each site does not admit more than \( \ell \) particles.

5 Large-volume limit

Representative behaviors of particle density and currents obtained in Figure 3 and 5 are understood from asymptotic analysis of the exact expressions of density profiles and currents. In this section, we analyze these physical quantities in a steady state in the large-volume limit with a large enough number of particles.

5.1 Asymptotics of density profiles

Since there are no particles going in and out, we expect as typical behaviors of particle-density profiles that a larger hopping rate to the right than to the left \( q > 1 \) brings a high-density
domain for $x > x_H$. At the same time, it is expected that a zero-density domain exists for $x < x_L$. Thresholds $x_{H,L}$ are determined from a decay length.

**Proposition 9.** Particle-density profiles of an $n$-particle steady state asymptotically behave as

$$\rho_n^{(\ell)}(N - \frac{n - 1}{\ell} + 1 - r) = q^{-2r} + O(q^{-2r}) \sim \exp[-r/\xi]$$

(67)

$$\rho_n^{(\ell)}(N - \frac{n + 1}{\ell} + r) = \ell - q^{-2r} + O(q^{-2r}) \sim \ell - \exp[-r/\xi],$$

(68)

for large enough $N$ and $n$ with a decay length given by

$$\xi = \frac{1}{2\ell \ln q}.$$  

(69)

**Proof.** Remind that particle-density profile in an $n$-particle steady state of the $(\ell + 1)$-state ASEP is expressed by the formula (62). Since $q$-binomials asymptotically behave as

$$\left[\frac{\ell N}{n}\right]^{-1} \left[\frac{\ell N}{k}\right] \sim q^{-\frac{1}{2}(\ell N - n)(\ell N - n + 1)} q^{-\frac{1}{2}n(n+1)} q^{\frac{1}{2}((\ell N - k)(\ell N - k + 1))} q^{\frac{1}{2}(k+k+1)}$$

(70)

density profiles show asymptotic behaviors for $q > 1$ as

$$\rho_n^{(\ell)}(x) \sim \sum_{j=0}^{\ell} \sum_{k=0}^{n-1} (-1)^{n-k+1} q^{-(n-k)(\ell N + 1 - 2(\ell(x-1) + j) - n - k + \ell N)}$$

$$= \sum_{j=1}^{\ell} \sum_{k=0}^{n-1} (-1)^{n-k+1} q^{f(k)},$$

(71)

where

$$f(k) = -(n-k)(\ell N + 1 - 2(\ell(x-1) + j) - n - k + \ell N).$$

(72)

Thus, asymptotic behavior of particle density is governed by the maximum value of $f(k)$.

The function $f(k)$ is a quadratic function whose zeros are located at $k_1^* = n$ and $k_2^* = 2\ell N - 2(\ell(x-1) + j) - n + 1$ (Figure 6). Then the following two cases are to be considered:
Figure 6: Behaviors of a function $f(k)$ in the case (i) and (ii). Taking the values depicted by filled circles, $k$ is not allowed to take values at dotted circles. The red circles and the blue ones are compensated each other accompanied by opposite signs as is obtained in particle density.

(i) $k_1^* < k_2^*$ iff $x < N - \frac{j+n}{\ell} + \frac{1}{2\ell} + 1$

(ii) $k_1^* > k_2^*$ iff $x > N - \frac{j+n}{\ell} + \frac{1}{2\ell} + 1$

Taking account that the transition from the zero-particle domain to the high-density domain is assumed to occur around $x \sim \lfloor N - \frac{n}{\ell} \rfloor$, the exponent of the rising edge is analyzed from the case (i), while the falling edge from (ii).

In the case (i), the maximum value of $f(k)$ is given by $k = k_1^* - 1$, since the summation is taken over $k \in \{0, \ldots, n-1 = k_1^* - 1\}$. The maximum exponent is then derived as

$$f(k_1^* - 1) = -(k_2^* - k_1^* + 1),$$

which takes an negative value as $k_1^* < k_2^*$ in the case (i). Thus, we have the rising exponent as

$$\rho_{n,1}^{(\ell)}(x) \sim \sum_{j=0}^{\ell} q^{-(k_2^* - k_1^* + 1)} \sim q^{-(2\ell N - n + 2 - 2\ell(x-1))}. \tag{74}$$

On the other hand, in the case (ii), the maximum value of $f(k)$ is given by $k = \lceil k_1^* + k_2^*/2 \rceil$ and $k = \lceil k_1^* + k_2^*/2 \rceil$. Nevertheless, $q^{f(k_1^* + k_2^*/2)}$ and $q^{f(k_1^* + k_2^*/2)}$ appear in (71) accompanied by different signs, and thus these two terms compensate each other. The same happens for $k \in \{k_2^* + 1, \ldots, k_1^* - 1\}$ due to

$$f(k_2^* + k') = f(k_1^* - k'), \quad k' \in \{1, \ldots, \lfloor k_1^* + k_2^*/2 \rfloor \}. \tag{75}$$

Thus, the maximum exponent is given by $k = k_2^*$, i.e. $f(k_2^*) = 0$. The second leading term is then given by $k = k_2^* - 1$, and we obtain the falling exponent as

$$\rho_{n,1}^{(\ell)}(x) \sim \sum_{j=0}^{\ell-1} \left[ 1 - q^{-(n-k_2^* + 1)} \right] \sim \ell - q^{-(2\ell N + n + 2n - 2\ell(x-1))}. \tag{76}$$

From (74) and (76), asymptotic behaviors are obtained for particle-density profiles, respectively as in (67) and (68).
Asymptotic behaviors (67) and (68) imply that particle density exponentially decays for \( x < x_L \) with \( x_L = N - \frac{n-1}{2} + 1 \), while one obtains the high-density domain for \( x > x_H \), i.e. \( x_H = N - \frac{n+1}{2} \). Thus we conclude that the transition domain antiproportionally shrinks with respect to the number of states. At the same time, the decay length (69) also suggests that the multi-state system shows faster decay in density profiles than the two-state ASEP.

6 Conclusions

In this paper, we have constructed the multi-state asymmetric simple exclusion processes based on the fusion procedure of the Temperley-Lieb algebra, satisfied by the Markov matrices. Motivated by higher-spin extension of integrable quantum spin chains, we have discussed a new family of integrable stochastic processes. As is in the case of ASEP, the multi-state processes proposed in this paper have corresponding quantum spin chains, some of whose wave functions are exactly discussed by the Bethe ansatz method. This fact implies that the multi-state ASEP also admits exact calculation of physical quantities through the Bethe ansatz method.

Existence of the similarity transformation which makes the Markov matrix to satisfy probability conservation is significant, since otherwise the higher-dimensional Temperley-Lieb generators do not describe any stochastic processes. It is also important to find a proper combination of the twisted generators such that positivity of probability is satisfied. All these are possible due to beautiful algebraic structure of the Temperley-Lieb algebra, which still holds for the extended model. Based on this algebraic structure, we computed particle-density profiles and particle currents on a steady state. Then a specific feature of the multi-state processes has been obtained in decay lengths which strongly depend on the number of states of the system. The decay lengths were defined from the rising and falling exponent of particle-density profiles obtained from asymptotic analysis.

Although we focused on the closed model on which the Temperley-Lieb algebraic structure holds for the whole system, it is more interesting to consider general boundary conditions. The open system would be solved via the matrix product ansatz, although we did not find the matrix product steady state [31]. Another interesting open problem is how the multi-state TASEPs relate with combinatorial problems. It has been shown that current fluctuations of TASEP with the step initial condition, which are also considered as fluctuations of the surface growth model, obey the Tracy–Widom (TW) distribution [32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. This TW distribution appears in various context of probability theory such as distribution of the longest increasing subsequences [32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. We hope that these interesting problems will be soon resolved.

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Appendix A  The $U_q(sl_2)$ algebra

The $U_q(sl_2)$ algebra is generated by $S^\pm$ and $q^{S_z}$ which satisfy the following commutation relations:

$$q^{S_z} S^\pm q^{-S_z} = q^\pm S^\pm, \quad [S^+, S^-] = \frac{q^{2S_z} - q^{-2S_z}}{q - q^{-1}}. \quad (77)$$

As the Hopf algebra, the following comultiplication holds for the $U_q(sl_2)$ generators $X \in \{S^\pm, q^{S_z}\}$:

$$(1 \otimes \Delta) \circ \Delta(X) = (\Delta \otimes 1) \circ \Delta(X). \quad (78)$$

By choosing $\Delta(S^\pm) = S^\pm \otimes q^{S_z} + q^{S_z} \otimes S^\pm$ and defining $\Delta^{(N)}$ as

$$\Delta^{(N)} = (1 \otimes \Delta) \cdots (1 \otimes \Delta)(1 \otimes 1) \cdots (1 \otimes \Delta),$$

we have the spatially extended generators:

$$\Delta^{(N)}(S^\pm) = \sum_{j=1}^N q^{S_z^1} \otimes \cdots \otimes q^{S_z^{j-1}} \otimes S^\pm_j \otimes q^{-S_z^{j+1}} \otimes \cdots \otimes q^{-S_z^N}, \quad (80)$$

$$\Delta^{(N)}(q^{S_z}) = q^{S_z^1} \otimes \cdots \otimes q^{S_z^N}. \quad (81)$$

Appendix B  Derivation of the norms

In derivation of the norms of an $n$-particle steady state of the $(\ell + 1)$-state ASEP, we use the commutation relation of the $U_q(sl_2)$ generators:

$$[\Delta(S^+), \Delta(S^-)] = \frac{\Delta(q^{2S_z}) - \Delta(q^{-2S_z})}{q - q^{-1}}. \quad (82)$$

This relation leads to

$$(\Delta(S^+))^n(\Delta(S^-))^n = (\Delta(S^+))^{n-1} \left\{ \Delta(S^-)\Delta(S^+) + \frac{\Delta(q^{2S_z}) - \Delta(q^{-2S_z})}{q - q^{-1}} \right\} (\Delta(S^-))^{n-1}$$

$$= (\Delta(S^+))^{n-1}(\Delta(S^-))^n \Delta(S^+)$$

$$+ (\Delta(S^+))^{n-1}(\Delta(S^-))^{n-1} \sum_{j=1}^n q^{-2(n-j)}\Delta(q^{2S_z}) - q^{2(n-j)}\Delta(q^{-2S_z}).$$

Using the fact that $\Delta(S^+)|0\rangle = 0$ and $\Delta(q^{\pm2S_z})|0\rangle = q^{\pm N}$, we obtain the following recursion relation:

$$\langle 0| (\Delta^{(\ell N)}(S^+))^n(\Delta^{(\ell N)}(S^-))^n |0\rangle$$

$$= \sum_{j=1}^n q^{\ell N - 2(n-j)} - q^{-\ell N + 2(n-j)} q^{-2(n-j)}\Delta(q^{2S_z}) - q^{2(n-j)}\Delta(q^{-2S_z}). \quad (84)$$
Taking account that the initial condition:

$$\langle 0|\Delta^{(\ell N)}(S^+)(S^-)|0 \rangle = \frac{q^{\ell N} - q^{-\ell N}}{q - q^{-1}},$$

we obtain the expression (44) for the norm.

### Appendix C Correlation functions

We introduce a particle-counting operator defined on a two-dimensional vector space:

$$n_j = S^+_j S^-_j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_j, \quad 1 - n_j = S^-_j S^+_j = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_j. \quad (86)$$

In the case of the two-state ASEP, important physical quantities such as particle densities and particle currents are expressed by $n_j$.

For instance, an $l$-point correlation function of the two-state ASEP is written by means of particle-counting operators in the following way:

$$\langle n|[U_{n_{x_1} n_{x_2} \ldots n_{x_l}} U^{-1}|n \rangle_{\text{norm}} = \langle n|n_{x_1} n_{x_2} \ldots n_{x_l}|n \rangle_{\text{norm}}. \quad (87)$$

Useful formulae have been obtained in [7]: The one-point correlation functions is given by

$$\langle n|[U^{-1} n_x U|n \rangle = \langle n|n_x|n \rangle = \left[ \frac{N}{n} \right]^{-1} \sum_{k=0}^{N-1} (-1)^{n-k+1} q^{-(n-k)(N+1-2x)} \left[ \frac{N}{k} \right], \quad (88)$$

which was derived using the following relations:

$$S^+_x|n \rangle = q^{(-N-1+2x)/2}(1 - n_x)|n - 1 \rangle, \quad \langle n|S^-_x = q^{(-N-1+2x)/2}(n - 1)(1 - n_x). \quad (89)$$

The relations (89) lead to a recursion relation for an $l$-point correlation function with respect to $n$:

$$\langle n|n_{x_1} \ldots n_{x_l}|n \rangle_{\text{norm}} = \left[ \frac{n q^{N-1+2x}}{N-n+1} \right] \langle n-1|n_{x_1} \ldots n_{x_l-1}(1 - n_x)|n - 1 \rangle_{\text{norm}}. \quad (90)$$

In contrast to the recursion relation (90), by which one needs to compute correlation functions in basis of different particle-sectors, we found another recursion relation which does not change the number of particles:

**Proposition 10.** An $l$-point function is decomposed into one-point functions:

$$\langle n|n_{x_1} n_{x_2} \ldots n_{x_l}|n \rangle_{\text{norm}} = \sum_{j=1}^{l} \prod_{k=1}^{\ell} \frac{q^{(x_k-x_j)}}{q^{(x_k-x_j)} - q^{-(x_k-x_j)}} \cdot \langle n|n_{x_j}|n \rangle_{\text{norm}}. \quad (91)$$

**Proof.** The proof is given by an induction on $l$. Before starting the proof, let us remark the following lemma:

**Lemma 3.** A two-point function is decomposed into one-point functions:

$$\langle q^{x_2-x_1} - q^{-(x_2-x_1)}\rangle \cdot \langle n|n_{x_1} n_{x_2}|n \rangle_{\text{norm}} = q^{x_2-x_1} \cdot \langle n|n_{x_1}|n \rangle_{\text{norm}} - q^{-(x_2-x_1)} \cdot \langle n|n_{x_2}|n \rangle_{\text{norm}}. \quad (92)$$
Proof. This lemma can be proved by considering two expressions of the following function;

\[
\langle n - 1 | (1 - n_{x_1})(1 - n_{x_2}) | n - 1 \rangle_{\text{norm}}. \tag{93}
\]

First applying the formula (90) to the operator \((1 - n_{x_2})\), one obtains

\[
\langle n - 1 | (1 - n_{x_1})(1 - n_{x_2}) | n - 1 \rangle_{\text{norm}} = \langle n - 1 | (1 - n_{x_2}) | n - 1 \rangle_{\text{norm}} - \langle n - 1 | n_{x_1}(1 - n_{x_2}) | n - 1 \rangle_{\text{norm}} \tag{94}
\]

and then applying to the operator \((1 - n_{x_1})\), one has

\[
\langle n - 1 | (1 - n_{x_1})(1 - n_{x_2}) | n - 1 \rangle_{\text{norm}} = \langle n - 1 | (1 - n_{x_1}) | n - 1 \rangle_{\text{norm}} - \langle n - 1 | n_{x_2}(1 - n_{x_1}) | n - 1 \rangle_{\text{norm}} \tag{95}
\]

From (94) and (95), decomposition of two-point functions (92) is obtained.

Assume the relation (91) holds for an \(l\)-point function. Then an \((l + 1)\)-point function is evaluated as

\[
\langle n | n_{x_1}n_{x_2} \cdots n_{x_l}n_{x_{l+1}} | n \rangle_{\text{norm}} = \sum_{j=1}^{l} \prod_{\substack{k=1 \atop k \neq j}}^{l} \frac{q^{x_k-x_j}-q^{-(x_k-x_j)}}{q^{x_k-x_j}} \cdot \langle n | n_{x_j}n_{x_{l+1}} | n \rangle_{\text{norm}}. \tag{96}
\]

Using the decomposition formula of two-point functions into one-point functions (92) to the right-hand side, we obtain the relation (91) for an \((l + 1)\)-point function.

Substituting the expression for the one-point function (88) into (91), one obtains the expression for the \(l\)-point function.
Appendix D  Currents in terms of the particle-counting operators

Here we give useful expressions for the seven types of expectation values in (53) in terms of particle-counting operators.

\[
\langle 2; n| \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_x \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{x+1} |2; n\rangle_{\text{norm}}
\]

\[
= -\frac{q^{-5}(q+q^{-1})}{(q^{-1} - q)(q^{-2} - q^{-3})(q^{-2} - q^{-3})} \langle n| n_{2x-1} |n\rangle_{\text{norm}} - \frac{q^{-3}(q+q^{-1})}{(q^{-1} - q)(q^{-2} - q^{-3})} \langle n| n_{2x} |n\rangle_{\text{norm}}
\]

\[
- \frac{q^{-1}(q+q^{-1})}{(q^{-1} - q)(q^{-2} - q^{-3})(q^{-1} - q)} \langle n| n_{2x+1} |n\rangle_{\text{norm}} - \frac{q(q+1)}{(q^{-3} - q^3)(q^{-2} - q^{-3})(q^{-1} - q)} \langle n| n_{2x+2} |n\rangle_{\text{norm}},
\]
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