CURVATURE OF FIELDS OF QUANTUM HILBERT SPACES

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ABSTRACT. We show that using the family of adapted Kähler polarizations of the phase space of a compact, simply connected, Riemannian symmetric space of rank-1, the obtained field $H^{corr}$ of quantum Hilbert spaces produced by geometric quantization including the half-form correction is flat if $M$ is the 3-dimensional sphere and not even projectively flat otherwise.

1. Introduction. Suppose an $m$–dimensional compact Riemannian manifold $M$ is the classical configuration space of a mechanical system, the metric corresponding to twice the kinetic energy. To quantize it according to the prescriptions of Kostant and Souriau [Ko,So,Wo], one first passes to phase space $N$, which for the moment is taken $TM \approx T^*M$, a symplectic manifold with an exact symplectic form $\omega$, equal to $\sum dq_j \wedge dp_j$ in the usual local coordinates. The prequantum line bundle is a Hermitian line bundle $E \to N$ with a connection whose curvature is $-i\omega$. If $M$ is simply connected, the bundle is unique up to a connection preserving Hermitian isomorphism. In any case, one such line bundle is obtained from a real 1–form $a$ on $N$ such that $da = -\omega$, by letting $E = N \times \mathbb{C} \to N$ to be the trivial line bundle with $h^E(x,\gamma) = |\gamma|^2$ the trivial metric on it. If sections are identified with functions $\psi : N \to \mathbb{C}$, the connection $\nabla^E$ is defined by

$$\nabla^E_\zeta \psi = \zeta \psi + ia(\zeta)\psi, \quad \zeta \in \text{Vect } N.$$ 

A choice of a Kähler structure on $N$ with Kähler form $\omega$ induces on $E$ the structure of a holomorphic line bundle. This gives rise to the quantum Hilbert space $H$, consisting of holomorphic sections of $E$ that are $L^2$ with respect to the volume form $\omega^m/m!$.

Often one is forced to include in this construction the so called half-form correction. Suppose $\kappa$ is a square root of the canonical bundle $K_N$. Then the corrected quantum Hilbert space $H^{corr}$ consists of the $L^2$ holomorphic sections of $E \otimes \kappa$.

When $M$ is a real-analytic Riemannian manifold, there is a natural Kähler polarization on (some subset of) $N$. In [Sz1,GS] the second author and Guillemin–Stenzel construct a canonical complex structure (“adapted complex structure” or “Grauert tube”) on a neighborhood $X \subset TM$ of the zero section, in which $\omega$ becomes a Kähler form (see also [H-K]). In good cases $X = N$. One gets examples of this sort when $M$ is a compact normal Riemannian homogeneous space, but there are nonhomogeneous examples as well, see [A,Sz1, Sz2].

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In fact the adapted complex structure is just one member in a natural family of Kähler structures on \( N \) [L-Sz2]. To see this it is advantageous to adhere to Souriau’s philosophy ([So]) and define the phase space \( N \) of a compact Riemannian manifold not as \( TM \) or \( T^*M \) but as the manifold of parametrized geodesics \( x : \mathbb{R} \to M \). Any \( t_0 \in \mathbb{R} \) induces a diffeomorphism \( N \ni x \mapsto \dot{x}(t_0) \in TM \), and the pull back of the canonical symplectic form of \( TM \approx T^*M \) is independent of \( t_0 \); we denote it by \( \omega \). We identify \( M \) with the submanifold of zero speed geodesics in \( N \). Affine reparametrizations \( t \mapsto a + bt, a, b \in \mathbb{R} \), act on \( N \) and define a right action of the Lie semigroup \( \Sigma \) of affine reparametrizations.

Given a complex manifold structure on \( \Sigma \), a complex structure on \( N \) is called adapted if for every \( x \in N \) the orbit map \( \Sigma \ni \sigma \mapsto x\sigma \in N \) is holomorphic ([L-Sz2]). An adapted complex structure on \( N \) can exist only if the initial complex structure on \( \Sigma \) is left invariant. Left invariant complex structures on \( \Sigma \) are parametrized by the points of \( \mathbb{C} \setminus \mathbb{R} \). For each \( s \in \mathbb{C} \setminus \mathbb{R} \) and corresponding left invariant complex structure \( I(s) \) on \( \Sigma \), if an \( I(s) \) adapted complex structure \( J(s) \) exists on \( N \), then this structure is unique and if \( J(i) \) exists, then \( J(s) \) also exists for all \( s \in \mathbb{C} \setminus \mathbb{R} \). The points of the upper half plane (denoted from now on by \( \mathbb{R} \)) correspond to \( J(s) \) in which \( \omega \) is a Kahler form. The original definition of adapted complex structures in [L-Sz1, Sz1] corresponds to the parameter \( s = i \).

Now suppose for the compact Riemannian manifold \( M \) the adapted complex structure \( J(i) \) exists on \( N \). With the help of the corresponding family of Kähler structures \( J(s) \) on \( N \), geometric quantization produces a family \( H_s \) of quantum Hilbert spaces. Our main concern is how (and when) can one define a natural (projective) isomorphism among these Hilbert spaces.

To deal with this problem, a key idea, following [ADW] and [Hi], is that the collection \( \{ H_s : s \in S \} \) resembles a holomorphic Hermitian vector bundle, in which one can try to construct a Chern-like canonical connection, and use its parallel transport canonically to identify the different fibers \( H_s \). To what extent this can be done was explored in [L-Sz3]. The starting point is that the family of adapted complex structures \( J(s), s \in S \) on \( N \) can all be put together to form a holomorphic fibration \( \pi : Y \to S \); where the fibers \( Y_s = \pi^{-1}s \) are biholomorphic to \( (N, J(s)) \). In fact, as a differentiable manifold, \( Y = S \times N \), and the projection \( pr : Y \to N \) realizes the biholomorphisms \( Y_s \to (N, J(s)) \), ([L-Sz2, Theorem 5]).

Armed with this fibration one can perform geometric quantization simultaneously. As we shall see shortly, the object we get is what we call a field of Hilbert spaces. A field of Hilbert spaces is simply a map \( p : H \to S \) of sets with each fiber \( H_s = p^{-1}(s) \) endowed with the structure of a Hilbert space. When \( S \) is a smooth or real analytic manifold, one can introduce the notion of a smooth or analytic structure on \( p : H \to S \), by specifying a set \( \Gamma^\infty \) resp. \( \Gamma^\omega \) of sections of \( p \), together with operators \( \nabla_\xi : \Gamma^\infty \to \Gamma^\infty \) for all vector fields \( \xi \) on \( S \). The set \( \Gamma^\infty \) and the operators \( \nabla_\xi \) are supposed to satisfy certain axioms (see [L-Sz3, Sect.2]).

Smooth fields of Hilbert spaces are looser structures than Hilbert bundles, but the notion is strong enough to define curvature, (projective) flatness and (local) triviality of the field. In particular a smooth Hilbert field \( H \to S \) is called projectively flat if the curvature operator

\[
R(\xi, \eta) = \nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi, \eta]} : \Gamma^\infty \to \Gamma^\infty
\]

is multiplication by a function \( r(\xi, \eta) : S \to \mathbb{C} \). Just like with vector bundles, \( r \) is in fact a smooth closed 2-form on \( S \), and a simple twisting will reduce projectively flat
smooth Hilbert fields to flat ones. Flatness and projective flatness are important, because in a flat and analytic field a parallel transport can be introduced that identifies the fibers canonically. Similarly, in projectively flat analytic fields the corresponding parallel transport identifies the projectivized fibers ([L-Sz3, Theorem 2.3.2, Theorem 2.4.2]).

Now back to geometric quantization of a compact Riemannian manifold $M$, assuming the adapted complex structure $J(s)$ exists on the entire manifold $N$ of geodesics. To quantize $(N,J(s))$ simultaneously, construct a Hermitian holomorphic line bundle $E \to Y$ with curvature $-i\tilde{\omega} := -ipr^*\omega$. The restriction of $E$ to $Y_s$ yields the prequantum line bundle corresponding to $(N,J(s),\omega)$. The restriction of the form $\nu = \tilde{\omega}^m/m!$ to a fiber $Y_s$ is a volume form. The spaces of holomorphic $L^2$-sections of $E|Y_s$ form the Hilbert field $H \to S$. Assuming now that $M$ is simply connected, there is a unique Hermitian holomorphic line bundle $\kappa$ on $Y$, so that $\kappa \otimes \kappa \approx K_\pi$ (the relative canonical bundle of $Y$ with $K_\pi|Y_s$ being the canonical bundle of $Y_s$). The spaces of holomorphic $L^2$-sections of $E \otimes \kappa|Y_s$ form the corrected Hilbert field $H_{\text{corr}} \to S$.

More generally Hilbert fields naturally arise as direct images of holomorphic vector bundles. Suppose $\pi : Y \to S$ is a surjective holomorphic submersion of finite dimensional complex manifolds, not necessarily proper. Let $\nu$ be a smooth form on $Y$ that restricts to a volume form on each fiber $Y_s = \pi^{-1}s$ and let $(E,h^H) \to Y$ be a Hermitian holomorphic vector bundle of finite rank. Let $H_s$ be the Hilbert space of $L^2$ holomorphic sections of $E|Y_s$. The spaces $H_s$ form a Hilbert field $H_{\text{corr}} \to S$.

Under certain conditions on $Y$ and $E$, the field comes naturally endowed with a smooth structure ([L-Sz3, Sect. 6, 7]). In the problem of geometric quantization by adapted complex structures, these conditions are known to be satisfied in the special case when $M$ is a compact, simply connected, normal Riemannian homogeneous space. In fact, in this case $H_{\text{corr}} \to S$ turns out to be analytic ([L-Sz3, Theorem 11.1.1]).

Our main result is the following:

**Theorem 1.1.** Let $M$ be a compact, simply connected, Riemannian symmetric space of rank-1. Then the corresponding field $H_{\text{corr}}$ of quantum Hilbert spaces is flat if $M$ is the 3-dimensional sphere and not even projectively flat otherwise.

We prove this result in Sect.5. It shows quantization is unique for the 3-sphere and in the rest of the cases quantization does depend on the choice of the Kähler polarization.

Flatness also implies $H_{\text{corr}} \to S$ is a genuine Hilbert bundle (trivial in this case), something that is not known to be true for the other rank-1 symmetric spaces.

The situation for the higher rank symmetric spaces is more complicated and will be treated in a separate publication [L-Sz4].

2. Curvature calculations.

Consider a simply connected, compact, Riemannian symmetric space $(M^m, g)$ and $H_{\text{corr}} \to S$ the corresponding field of quantum Hilbert spaces. Let $U$ denote the identity component of the isometry group of $M$ and $K \subset U$ the isotropy group of a fixed $o \in M$. Let $\mathfrak{u}$ and $\mathfrak{k}$ be the Lie algebras of $U$ and $K$ and let $\mathfrak{p}_* \subset \mathfrak{u}$ be the orthogonal complement of $\mathfrak{k}$.

$U$ acts on $(N,J(i))$ by biholomorphisms and this action induces a representation $\hat{\pi}$ on $\mathcal{O}(N,J(i))$, by the formula $av = (a^{-1})^*v$ (pull back by $a^{-1}$), where $a \in U$,
\[ v \in \mathcal{O}(N, J(i)). \] The same formula defines a unitary representation \( \pi \) on \( L^2(M) \). The restrictions \( V_\chi|_M \) of the isotypical subspaces of \( \hat{\pi} \) are precisely the isotypical subspaces of \( \pi \) and the latter are well known to be finite dimensional. Since \( M \) is a maximal dimensional, totally real submanifold in \( N \), we get that \( V_\chi \) are also finite dimensional. The isotypical subspaces of \( \pi \) are parametrized by the irreducible spherical (w.r.t. \( K \)) representations of \( U \) ([He2, Theorem 4.3]). In fact the restrictions of \( \hat{\pi} \) to the isotypical subspaces \( V_\chi \) (or equivalently the restrictions of \( \pi \) to \( V_\chi|_M \)) are precisely these spherical representations.

Flatness of the field \( \mathcal{H}^{corr} \to S \) can be understood in terms of certain operators \( P_\chi(s) \) on \( V_\chi \). Namely \( \mathcal{H}^{corr} \to S \) is flat (resp. projectively flat) if and only if \( P_\chi(s) \) are of the form \( P_\chi(s) = P_\chi(s)Id_\chi \) and \( \bar{\partial}\partial \log P_\chi(s) = 0 \) for all \( \chi \) (resp. \( \bar{\partial}\partial \log P_\chi(s) \) is independent of \( \chi \)), see [L-Sz3, Theorem 9.2.1]. According to [L-Sz3, Lemma 11.2.1] and [L-Sz3, Sect. 12.1],

\[ p_\chi(s) = \frac{c_\chi}{(\text{Im} s)^{m/2}} \int \int_{p^*} e^{-\frac{|\zeta|^2}{2}} \chi(k \exp(-2i\zeta)) \sqrt{\eta(\zeta)} dk d\zeta, \]

where \( c_\chi \) is independent of \( s \), \( dk \) is normalized Haar measure on \( K \), \( d\zeta \) translation invariant Lebesgue measure on \( p^* \), and

\[ \eta(\zeta) := \det \left( \frac{\sin 2ad\zeta}{a \zeta} \right) \bigg|_{\mathbb{C} \otimes p^*}. \]

The function \( f_\chi(g) = \int_K \chi(kg^{-1})dk \), occurring in (2-1), is known as spherical function, corresponding to the character \( \chi \), see [He2, IV., Theorem 4.2]. This function has a holomorphic extension to the complexified group \( U_\mathbb{C} \) that we also denote by \( f_\chi \).

**Proposition 2.1.** The function \( f_\chi \circ \exp \) is \( \text{Ad}_K \) invariant on the Lie algebra \( u_\mathbb{C} \) of \( U_\mathbb{C} \).

**Proof.** For any \( k, k_0 \in K, \zeta \in u_\mathbb{C} \)

\[ \chi(k \exp(-\text{Ad}(k_0)\zeta)) = \chi(kk_0 \exp(-\zeta)k_0^{-1}) = \chi(k_0^{-1}kk_0 \exp(-\zeta)). \]

Thus

\[ f_\chi(\exp(\text{Ad}(k_0)\zeta)) = \int_K \chi(k_0^{-1}kk_0 \exp(-\zeta))dk = f_\chi(\exp(\zeta)). \]

**Proposition 2.2.** Let \( F \in \mathcal{O}(\mathbb{C}) \) be an even function and \( v \in p_* \). Then \( F(\text{ad}(v)) \) (defined by its power series) maps \( \mathbb{C} \otimes p_* \) into itself and \( \text{det}(F(\text{ad}(v)))\big|_{\mathbb{C} \otimes p_*} \) is an \( \text{Ad}_K \) invariant function.

**Proof.** For every \( k \in K, \text{Ad}(k) \) is in \( \text{Aut}(u) \). Thus for every \( v \in u, l = 0, 1, \ldots \)

\[ (\text{ad}(\text{Ad}(k)v))^l = \text{Ad}(k) \circ (\text{ad}(v))^l \circ \text{Ad}(k)^{-1}. \]

Hence

\[ F(\text{ad}(\text{Ad}(k)v)) = \text{Ad}(k) \circ F(\text{ad}(v)) \circ \text{Ad}(k)^{-1}. \]
Since \( p_\ast \) is both \( \text{Ad}(k) \) and \( (\text{ad}(v))^{2l} \) invariant \((l = 0,1\ldots)\), the statement follows. \( \square \)

From now on we shall assume that \( M \) is a rank–1 symmetric space.

Let \( H_0 \in p_\ast \) with \( \| H_0 \| = 1 \). Then \( a_\ast = \mathbb{R}H_0 \) is maximal Abelian in \( p_\ast \) (resp. \( a := i\ast a_\ast \) in \( p := i\ast p_\ast \)). Let \( \Sigma \) be the set of restricted roots corresponding to \( (g_0 := \mathfrak{t} + p, a) \). Let \( a^+ := \{ i\ast H_0 \mid r > 0 \} \) be the Weyl chamber and \( \Sigma^+ \) the set of positive restricted roots. Then \( \Sigma^+ = \{ \beta, \beta/2 \} \) with an appropriate \( \beta \) in the dual of \( p \) with \( B := \beta(iH_0) > 0 \). The corresponding multiplicities are \( m_\beta \) and \( m_{\beta/2} \), where our convention is that the latter is zero when \( \Sigma \) is reduced (i.e. when \( M \) is a sphere with the round metric).

Let \( Z_+ = \{ 0, 1, 2, \ldots \} \). According to Helgason’s theorem ([He2, Theorem 4.1,(ii), p.535 and Sect.3, p.542]), the set of linear functionals \( \{ \mu = n\beta : n \in Z_+ \} \) is precisely the set of the highest weights of all irreducible spherical representations of \( U \) (w.r.t. \( K \)) restricted to \( a \). Now in light of what was said at the beginning of Sect.2 about the relationship of \( V_\chi \) and the irreducible spherical representations of \( U \), we can conclude that the isotypical subspaces \( V_\chi \) of \( \hat{\pi} \) are parametrized by the elements \( n_\chi \) of \( Z_+ \). Let

\[
(2-2) \quad a_\chi = \frac{1}{2} m_{\beta/2} + m_\beta + n_\chi, \quad b_\chi = -n_\chi, \quad c_\chi = \frac{m_{\beta/2} + m_\beta + 1}{2} = \frac{m}{2},
\]

and denote by \( F_\chi \) the Gauss hypergeometric function, corresponding to these parameters

\[
(2-3) \quad F_\chi(x) = F(a_\chi, b_\chi, c_\chi, x).
\]

(see Sect.5 for more on hypergeometric functions).

Let \( S^{m-1}_{p_\ast} \) be the unit sphere in the euclidean space \( p_\ast \).

**Theorem 2.3.** Let \( M^m \) be a compact, simply connected, rank-1 symmetric space. Then

\[
(2-4) \quad p_\chi(s) = \frac{c_\chi \text{Vol}(S^{m-1}_{p_\ast})2^{m+1}}{B^m(\text{Im } s)^{m/2}} \int_0^\infty e^{-\frac{t}{\text{Im } s}} F_\chi(-\text{sh}^2(t)) t^{\frac{m-1}{2}} (\text{sh}(t))^{\frac{m-1}{2}} (\text{ch}(t))^{\frac{m}{2}} dt.
\]

**Proof.** Suppose \( f \in L^1(p_\ast) \) depends only on \( \| \zeta \| \). Using polar coordinates we obtain

\[
(2-5) \quad \int_{p_\ast} f(\zeta) d\zeta = \text{Vol}(S^{m-1}_{p_\ast}) \int_0^\infty f(rH_0) r^{m-1} dr.
\]

According to [He2, formula (25), p.543], the spherical function \( f_\chi \) can be expressed as

\[
(2-6) \quad f_\chi(\exp(2i\ast H_0)) = F_\chi(-\text{sh}^2(\beta(i\ast H_0))) = F_\chi(-\text{sh}^2(rB)).
\]

Prop.2.2 applied to \( F(z) = \frac{\sin 2z}{z} \) shows that \( \eta \) is \( \text{Ad}_K \) invariant. Since the rank is 1, \( \text{Ad}_K \) acts transitively on each sphere with center the origin in \( p_\ast \). In light of Prop.2.1, (2-5) applies to the integrand in (2-1) and we get

\[
(2-7) \quad p_\chi(s) = \frac{c_\chi \text{Vol}(S^{m-1}_{p_\ast})}{(\text{Im } s)^{m/2}} \int_0^\infty e^{-\frac{t}{\text{Im } s}} F_\chi(-\text{sh}^2(rB)) r^{m-1} \sqrt{\eta(rH_0)} dr.
\]
For $H \in \mathfrak{a}$, $\text{ad}H^2 : \mathfrak{p} \to \mathfrak{p}$ has eigenvalues 0 with multiplicity 1, $\beta(H)^2$ with multiplicity $m_\beta$ and $(\beta(H)/2)^2$ with multiplicity $m_{\beta/2}$ ([He1, Lemma 2.9, p288]). Thus

\begin{equation}
\eta(H) = 2 \left( \frac{\sin(2\beta(H))}{\beta(H)} \right)^{m_\beta} \left( \frac{\sin(\beta(H))}{\beta(H)/2} \right)^{m_{\beta/2}}.
\end{equation}

Now for $H = rH_0$ we have $\beta(rH_0) = -irB$. Hence (2-8) yields

\begin{equation}
\eta(rH_0) = \frac{2^m}{(rB)^{m-1}}(\text{sh}(rB))^{m-1}(\text{ch}(rB))^{m}.
\end{equation}

Substituting this into (2-7) and changing the variable $r$ in the integral to $t = rB$ we finally get formula (2-4). □

From (2-4) we see that $p_\chi(s)$ depends only on $\tau = B^2 \text{Im} s$. In light of our earlier characterization of the (projective) flatness of $H^{\text{corr}} \to S$ in terms of $\partial\bar{\partial} \log p_\chi$, we obtain:

**Corollary 2.4.** Let $M$ be a compact, simply connected rank-1 symmetric space. For $\tau > 0$ let

\begin{equation}
q_\chi(\tau) := \int_0^\infty e^{-t^2/\tau} F_\chi(-\text{sh}^2(t)) t^{m-1} (\text{sh}(t))^{m-1} (\text{ch}(t))^{m} \, dt.
\end{equation}

Then the field of quantum Hilbert spaces $H^{\text{corr}} \to S$ is

\begin{equation}
\text{flat iff } (\log q_\chi(\tau))'' \equiv 0 \text{ for every } \chi,
\end{equation}

projectively flat iff $(\log q_\chi(\tau))''$ does not depend on $\chi$.

The integral in (2-9) can be explicitly calculated only in very special cases. To be able to decide whether (2-10) holds, we shall use asymptotic methods to investigate the behavior of $q_\chi(\tau)$ as $\tau \to 0$ and $\tau \to \infty$. As we will see in Sect.5, the function $F_\chi(x)$ is a polynomial of degree $n_\chi$. This motivates our investigations in the next sections.

### 3. $Q_P$ functions and central polynomial sequences.

Let $P(x) = c_n x^n + \cdots + c_1 x + c_0$ be a polynomial ($c_j \in \mathbb{C}$), $\tau > 0$ and $\mu, \kappa, \nu \in \mathbb{C}$, with $\text{Re}(\mu + \kappa) > -1$. Define the corresponding $Q_P$ function by the formula

\begin{equation}
Q_P(\tau) := \int_0^\infty e^{-t^2/\tau} P(-\text{sh}^2(t)) t^\mu (\text{sh}(t))^{\kappa} (\text{ch}(t))^{\nu} \, dt.
\end{equation}

The integral converges absolutely and $Q_P$ depends holomorphically on the parameters $\mu, \kappa, \nu$. The function

\begin{equation}
f_P(t) := P(-\text{sh}^2(t)) \left( \frac{\text{sh}(t)}{t} \right)^\kappa (\text{ch}(t))^{\nu}
\end{equation}
is even and extends holomorphically to a neighborhood of the real line. Let
\( r := \kappa + \mu + 1. \) Then
\[
Q_P(\tau) = \int_0^\infty e^{-\frac{t^2}{r^2}} t^{r-1} f_P(t) dt.
\]

Applying Watson’s lemma [W] to this integral we get
\[
Q_P(\tau) = \frac{\tau^{\frac{r}{2}}}{2} \left( \Gamma \left( \frac{r}{2} \right) f_P(0) + \Gamma \left( \frac{r}{2} + 1 \right) \frac{f''_P(0)}{2} \tau + o(\tau) \right), \quad \tau \to 0,
\]
where \( \Gamma \) denotes the usual gamma function. Now \( f_P(0) = c_0 \) and a straightforward calculation shows
\[
\frac{f''_P(0)}{2} = -c_1 + \frac{\kappa}{6} + \frac{\nu}{2},
\]
and we get:

**Proposition 3.1.**

\[
Q_P(\tau) = \frac{\tau^{\frac{r}{2}}}{2} \left( \Gamma \left( \frac{r}{2} \right) c_0 + \Gamma \left( \frac{r}{2} + 1 \right) \left( -c_1 + \frac{\kappa}{6} + \frac{\nu}{2} \right) \tau + o(\tau) \right), \quad \tau \to 0,
\]

The next definition is motivated by Corollary 2.4.

**Definition 3.2.** Let \( \{ P_n(x) = c_{n,n}x^n + \cdots + c_{n,1}x + 1 \}_{n=0}^\infty, c_{n,n} \neq 0 (c_{n,j} \in \mathbb{C}) \) be a sequence of polynomials. The sequence is called **central** (w.r.t. the parameters \( \mu, \kappa, \nu \)) if the function \( (\log Q_{P_n})'' \) does not depend on \( n \).

**Proposition 3.3.** Suppose \( \{ P_n \}_{n=0}^\infty \) is a central sequence of polynomials. Then
\[
Q_{P_n}(\tau) = e^{-\frac{r}{2} c_{n,1} \tau} Q_1(\tau), \quad n \geq 1, \quad \text{where} \quad r = \mu + \kappa + 1.
\]

**Proof.** From our assumption
\[
(\log Q_{P_n} - \log Q_P)'' \equiv 0,
\]
for every \( n \). Hence there exist constants \( \alpha_n \) and \( \beta_n \) such that
\[
Q_{P_n}(\tau) = \beta_n e^{\alpha_n \tau} Q_P(\tau)
\]
Substituting this into the asymptotic formula in Prop.3.1 we get
\[
\beta_n (1 + \alpha_n \tau + o(\tau)) \frac{\tau^{\frac{r}{2}}}{2} \left( \Gamma \left( \frac{r}{2} \right) + \Gamma \left( \frac{r}{2} + 1 \right) \left( \frac{\kappa}{6} + \frac{\nu}{2} \right) \tau + o(\tau) \right) =
\]
\[
= \frac{\tau^{\frac{r}{2}}}{2} \left( \Gamma \left( \frac{r}{2} \right) + \Gamma \left( \frac{r}{2} + 1 \right) \left( -c_{n,1} + \frac{\kappa}{6} + \frac{\nu}{2} \right) \tau + o(\tau) \right).
\]
Now dividing by \( \frac{\tau^{\frac{r}{2}}}{2} \) and comparing the constant term and the coefficient of \( \tau \) on both sides of (3-6) we get
\[
\beta_n = 1,
\]
and
\[
\beta_n \alpha_n \Gamma \left( \frac{r}{2} \right) + \Gamma \left( \frac{r}{2} + 1 \right) \beta_n \left( \frac{\kappa}{6} + \frac{\nu}{2} \right) = \Gamma \left( \frac{r}{2} + 1 \right) \left( -c_{n,1} + \frac{\kappa}{6} + \frac{\nu}{2} \right).
\]
Hence \( \alpha_n = -rc_{n,1}/2. \) \( \square \)
4. Asymptotics at infinity.

**Proposition 4.1.** Let $a \geq 0, \lambda > 0, \mu > -1, \tau > 0$. Then

$$I(\tau) := \int_a^\infty e^{-t^2/\tau} t^\mu e^{\lambda t} dt = \frac{\lambda \sqrt{\pi}}{2^\mu} \tau^{\mu+1/2} e^{\lambda^2/4} (1 + o(1)), \quad \tau \to \infty.$$  

**Proof.** We can rewrite $I(\tau)$ as follows:

$$I(\tau) = e^{\lambda^2/4} \int_a^\infty e^{-\left(\frac{t}{\sqrt{\tau}} - \frac{\lambda \sqrt{\tau}}{2}\right)^2} t^\mu dt.$$  

After the substitution $x = \frac{t}{\sqrt{\tau}} - \frac{\lambda \sqrt{\tau}}{2}$, we get

$$I(\tau) = \int_a^\infty e^{-x^2} \left(\frac{x}{\sqrt{\tau}} + \frac{\lambda}{2}\right)^\mu dx =: \tau^{\mu+1/2} e^{\lambda^2/4} I_2(\tau).$$  

Lebesgue’s dominated convergence theorem applied to $I_2(\tau)$ yields:

$$I_2(\tau) \to \left(\frac{\lambda}{2}\right)^\mu \int_{-\infty}^{+\infty} e^{-x^2} dx = \left(\frac{\lambda}{2}\right)^\mu \sqrt{\pi}, \quad \tau \to \infty,$$

finishing the proof.  

**Proposition 4.2.** Let $\nu > 0, \kappa > 0, \mu + \kappa > -1 \tau > 0$. Then

$$\int_0^\infty e^{-t^2/\tau} t^{\mu} (sht)^\kappa (cht)\nu dt = O(1) = \tau^{\mu+1/2} e^{\frac{(\kappa+\nu)^2}{4}} (1 + o(1)), \quad \tau \to \infty.$$  

**Proof.** Let $a > 0$ be an arbitrary fixed constant. Then

$$\int_0^a e^{-t^2/\tau} t^{\mu} (sht)^\kappa (cht)\nu dt = O(1) = \tau^{\mu+1/2} e^{\frac{(\kappa+\nu)^2}{4}} (1 + o(1)), \quad \tau \to \infty.$$  

Let $h(x) = (1-x)^\kappa (1+x)^\nu$. Then $h \in C^\infty(-1, 1)$, $h(0) = 1$. Hence $h(x) = 1 + x g(x)$, where $g \in C^\infty(-1, 1)$. Therefore

$$(sht)^\kappa (cht)\nu = e^{(\kappa+\nu)t} (1 - e^{-2t})^\kappa (1 + e^{-2t})\nu = e^{(\kappa+\nu)t} (1 + e^{-2t} g(e^{-2t})).$$

Thus

$$\int_a^\infty e^{-t^2/\tau} t^{\mu} (sht)^\kappa (cht)\nu dt =$$

$$\int_a^\infty e^{-t^2/\tau} t^{\mu} e^{(\kappa+\nu)t} dt + \frac{1}{2^{\kappa+\nu}} \int_a^\infty e^{-t^2/\tau} t^{\mu} e^{(\kappa+\nu-2)t} g(e^{-2t}) dt = I_1 + I_2.$$
Now by Proposition 4.1

\begin{equation}
I_1 = \frac{\sqrt{\pi}(\nu + \kappa)^\mu}{2^{\mu+\nu+\kappa}} \tau^{\mu+\frac{1}{2}} e^{\frac{(\kappa+\nu)^2}{4}} \tau (1 + o(1)).
\end{equation}

The function $g(e^{-2t})$ is bounded on $[a, \infty)$, thus for an appropriate constant $A$ we have

$$|I_2| \leq A \int_0^\infty e^{-\frac{t^2}{2}} t^{\mu+\nu-\kappa} dt = AI_3.$$ 

If $\kappa + \nu - 2 > 0$, by Proposition 4.1

$$I_3 = \tau^{\mu+\frac{1}{2}} e^{\frac{(\kappa+\nu-2)^2}{4}} \tau O(1) = \tau^{\mu+\frac{1}{2}} e^{\frac{(\kappa+\nu)^2}{4}} \tau o(1).$$

If $\kappa + \nu - 2 \leq 0$, after the substitution $t = \sqrt{\tau}x$ we obtain

$$I_3 \leq \int_0^\infty e^{-\frac{t^2}{2}} t^{\mu+\nu-\kappa} dt = \int_0^\infty e^{-x^2} x^{\mu+\nu-2} dx = \frac{\tau^{\mu+1}}{2} \Gamma \left( \frac{\mu+1}{2} \right).$$

Thus in both cases

\begin{equation}
I_2 = \tau^{\mu+\frac{1}{2}} e^{\frac{(\kappa+\nu)^2}{4}} \tau o(1).
\end{equation}

Then (4-1), (4-2), (4-3) and (4-4) together prove our claim. □

**Proposition 4.3.** Let $\nu > 0$, $\kappa > 0$, $\mu + \kappa > -1$ and $P(t) = c_n t^n + \cdots + c_0$ be a polynomial. Let $Q_P$ be the corresponding $Q$ function (see (3-1)). Then

$$Q_P(\tau) = (-1)^n c_n \frac{\sqrt{\pi}(\nu + \kappa + 2n)^\mu}{2^{\mu+\nu+\kappa+2n}} \tau^{\mu+\frac{1}{2}} e^{\frac{(\kappa+\nu+2n)^2}{4}} \tau (1 + o(1)), \ \tau \to \infty.$$ 

**Proof.** It is enough to show the statement for the special case $P(t) = t^k$. Then

$$Q_P(\tau) = \int_0^\infty e^{-\frac{t^2}{2}} (-\sin^2 t)^k t^{\mu}(\sinh t)^\kappa (\cosh t)^\nu dt = (-1)^k \int_0^\infty e^{-\frac{t^2}{2}} t^{\mu}(\sinh t)^{\kappa+2k} (\cosh t)^\nu dt$$

and Proposition 4.2 proves our claim. □

**Theorem 4.4.** Let $\nu > 0$, $\kappa > 0$, $\mu + \kappa > -1$. Suppose $\{P_n\}_{n=0}^\infty$ is a central sequence of polynomials. Then for all $n$

\begin{equation}
Q_{P_n}(\tau) = e^{n(\nu+\kappa+n)}\tau Q_1(\tau),
\end{equation}

\begin{equation}
c_{n,1} = -\frac{2n(\nu + \kappa + n)}{\mu + \kappa + 1}, \quad c_{n,n} = (-1)^n 4^n \left( \frac{\nu + \kappa}{\nu + \kappa + 2n} \right)^\mu.
\end{equation}
Proof. Centrality implies (see (3-4)) that \( Q_P(\tau) = A_n e^{B_n \tau} Q_1(\tau) \) for appropriate constants \( A_n, B_n \). Substituting into this the asymptotics of Proposition 4.3 and comparing the leading terms on both sides we get

\[
c_{n,n}(-1)^n (\nu + \kappa + 2n) \mu \frac{e^{(\nu + \kappa + 2n)^2 \tau}}{4^n} = A_n e^{B_n \tau} (\nu + \kappa) \mu \frac{e^{(\nu + \kappa)^2 \tau}}{4^n}.
\]

Therefore

\[
B_n = \frac{(\nu + \kappa + 2n)^2 - (\nu + \kappa)^2}{4} = n(\nu + \kappa + n)
\]

and

\[
A_n = c_{n,n}(-1)^n \left( \frac{\nu + \kappa + 2n}{\nu + \kappa} \right)^\mu \frac{1}{4^n}.
\]

Now comparing these expressions of \( A_n, B_n \) with (3-4) we get (4-5) and (4-6). □

Taking \( n = 1 \) in (4-6) we get two different expressions for the same coefficient and thus we obtain the following.

**Corollary 4.5.** Let \( \nu, \kappa > 0, \mu + \kappa > -1 \). Suppose there exists a central sequence of polynomials corresponding to these parameters. Then

\[
\frac{\nu + \kappa + 1}{\mu + \kappa + 1} = 2 \left( \frac{\nu + \kappa}{\nu + \kappa + 2} \right)^\mu.
\]

5. Hypergeometric polynomials and the proof of Theorem 1.1.

Recall that Gauss’ hypergeometric functions are given by

\[
F(a, b, c, z) := 1 + \frac{ab}{c} z + \cdots + \frac{a(a+1) \cdots (a+k-1) b(b+1) \cdots (b+k-1)}{k! c(c+1) \cdots (c+k-1)} z^k + \ldots
\]

where \( a, b, c \in \mathbb{C}, c \notin \mathbb{Z}_- = \{0, -1, -2, \ldots \} \). The series converges at least in the unit disk. If \( n \in \mathbb{Z}_+ = \{0, 1, 2, \ldots \}, b = -n, A \in \mathbb{C} \setminus \mathbb{Z}_- \), and \( a = A + n \), then \( F \) is a polynomial (in \( z \)) of degree \( n \). Now assume \( A, c \in \mathbb{R} \setminus \mathbb{Z}_- \), \( n \in \mathbb{Z}_+ \) and consider this sequence of polynomials

\[
P_n(x) := F(A + n, -n, c, x) = \sum_{j=0}^n c_{n,j} x^j.
\]

**Proposition 5.1.** Suppose the polynomial sequence \( P_n(x) \) is central w.r.t. some choice of the parameters \( \nu, \kappa, \mu \) with \( \nu > 0, \kappa > 0, \mu + \kappa > -1 \). Then \( A > 0, c > 0 \),

\[
\frac{\Gamma(A + 2n)}{\Gamma(A + n)} \frac{\Gamma(c)}{\Gamma(c + n)} = 4^n \left( \frac{A}{A + 2n} \right)^\mu,
\]

and

\[
\kappa = 2c - \mu - 1, \quad \nu = A - \kappa.
\]
Proof. Assume \( \{P_n\}_0^\infty \) is central. Then (5-1) and (4-6) yield
\[
c_{n,1} = \frac{(A+n)(-n)}{c} = -\frac{2n(\nu + \kappa + n)}{\mu + \kappa + 1},
\]
for every \( n \), which implies
\[
(5-4) \quad c = (\mu + \kappa + 1)/2, \quad A = \nu + \kappa,
\]
proving (5-3) and \( A, c > 0 \). From (5-1) we also get
\[
c_{n,n} = (-1)^n \frac{\Gamma(A+2n)}{\Gamma(A+n)} \frac{\Gamma(c)}{\Gamma(c+n)},
\]
which together with formula (4-9) and (5-4) proves (5-2). □

Proof of Theorem 1.1. Suppose the field \( H^{corr} \) of quantum Hilbert spaces corresponding to \( M \) is projectively flat. As in Section 2, with \( M \) we associate its system of restricted roots \( \Sigma \). We denote the longer positive restricted root \( \beta \), and the multiplicities of \( \beta, \beta/2 \) by \( m_\beta, m_{\beta/2} \) (with the understanding that this latter is 0 if \( \Sigma \) is reduced, so that \( \beta/2 \) is not a root, i.e. when \( M \) is a sphere). Then \( m = m_\beta + m_{\beta/2} + 1 \). Set
\[
A = m_\beta + \frac{m_{\beta/2}}{2}, \quad c = \frac{m}{2},
\]
and \( P_n(x) = F(A+n, -n, c, x), n = 1, 2, \ldots, P_0(x) = 1 \) the corresponding sequence of hypergeometric polynomials. Let also \( \mu = \kappa = (m-1)/2, \nu = m_{\beta}/2 \). According to Corollary 2.4 and Definition 3.2, projective flatness implies that \( P_n \) is central with respect to \( \kappa, \mu, \nu \).

We now apply Proposition 5.1. Choose \( n = 2A \), so that \( A/(A+2n) = 1/5 \). The left hand side of (5-2) is rational, hence \( \mu = (m-1)/2 \) on the right must be an integer and so \( m \) must be odd. Then it follows from the classification of compact rank-1 symmetric spaces (see [He2, Ch.I, Sect.4.2]), that \( M \) must be an odd dimensional sphere and so \( \Sigma \) is reduced. Thus \( m_{\beta/2} = 0 \), and \( A = m_\beta = m - 1 \). Substitute \( n = 1 \) into (5-2):
\[
2 = \left( \frac{m + 1}{m - 1} \right)^{\frac{m-1}{2}} = \left( 1 + \frac{2}{m - 1} \right)^{\frac{m-1}{2}}.
\]
Clearly \( m = 3 \) solves this equation, and there is no other solution, because \( (1+1/x)^x \) is strictly increasing for \( x > 0 \). Thus \( M \) must be the 3 dimensional sphere for no other compact simply connected symmetric space of rank 1 can the Hilbert field \( H^{corr} \) be projectively flat.

On the other hand, when \( M \) is \( S^1 \), or more generally a compact Lie group with biinvariant metric, the associated Hilbert field is outright flat, see [L-Sz, Theorem 11.3.1]. □
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