On the power structure over the Grothendieck ring of varieties and its applications

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Abstract

We discuss the notion of a power structure over a ring and the geometric description of the power structure over the Grothendieck ring of complex quasi-projective varieties and show some examples of applications to generating series of classes of configuration spaces (for example, nested Hilbert schemes of J. Cheah) and wreath product orbifolds.

To a pre-λ ring there corresponds a so called power structure. This means, in particular, that one can give sense to an expression of the form

\[(1 + a_1 t + a_2 + ...)^m\]

for \(a_i\) and \(m\) from the ring \(R\). (Generally speaking, on a ring there are many pre-λ structures which correspond to one and the same power structure.) A natural pre-λ structure on the Grothendieck ring \(K_0(\mathcal{V}_C)\) of complex quasi-projective varieties is defined by the Kapranov zeta-function

\[\zeta_X(t) = 1 + [X]t + [S^2X]t^2 + [S^3X]t^3 + \ldots,\]

where \(S^kX = X^k/S_k\) is the \(k\)-th symmetric power of the variety \(X\). In [8], there was given a geometric description of the corresponding power structure.
over the Grothendieck ring $K_0(V_C)$. In some cases this permits to give new (short and somewhat more transparent) proofs and also certain refinements of formulae for generating series of classes of moduli spaces in the ring $K_0(V_C)$ and/or of their invariants: the Euler characteristic and the Hodge–Deligne polynomial. An application of this sort (for the generating series of classes of Hilbert schemes of 0-dimensional subschemes of a smooth quasi-projective variety) was described in [9].

The aim of this paper is to describe the concept of a power structure (in a somewhat more general context introduced in [10]) and to show its applications to proofs and also some improvements of results by J. Cheah in [11] about nested Hilbert schemes, by W.P. Lin and Zh. Qin in [12] about moduli spaces of 1-dimensional subschemes. Finally we rewrite some results of W. Wang, J. Zhou in [13] and [14] on generating series of orbifold generalized Euler characteristic of wreath product orbifolds in terms of the power structure.

1 Power structures

Definition: A pre-$\lambda$ structure on a ring $R$ is given by a series $\lambda_a(t) \in 1 + t \cdot R[[t]]$ defined for each $a \in R$ so that

1. $\lambda_a(t) = 1 + at \pmod{t^2}$.
2. $\lambda_{a+b}(t) = \lambda_a(t)\lambda_b(t)$ for $a, b \in R$.

Example. One has the following important examples of pre-$\lambda$ structures.

1. $R$ is the ring $\mathbb{Z}$ of integers, $\lambda_k(t) = (1 - t)^{-k}$.
2. $R = \mathbb{Z}$ and $\lambda_k(t) = (1 + t)^k$.
3. $R = \mathbb{Z}[u_1, \ldots, u_r]$ (the ring of polynomials in $r$ variables $u_1, \ldots, u_r$), for a polynomial $P = P(u) = \sum p_k u^k$, $k \in \mathbb{Z}_{\geq 0}^r$ and $p_k \in \mathbb{Z}$,

$$\lambda_P(t) = \prod_{k \in \mathbb{Z}_{\geq 0}^r} (1 - u^k t)^{-p_k},$$

where $u = (u_1, \ldots, u_r)$, $k = (k_1, \ldots, k_r)$, $u^k = u_1^{k_1} \cdots u_r^{k_r}$, (see [9]).
4. (A more geometric example.) Let $R$ be the $K$-functor $K(X)$ of the space $X$, i.e., the Grothendieck ring of (say, real or complex) vector bundles over $X$. For a vector bundle $E$ over $X$, let $\Lambda^k E$ be the $k$-th exterior power of the bundle $E$. The series

$$\lambda_E(t) = 1 + [E]t + [\Lambda^2 E]t^2 + [\Lambda^3 E]t^3 + \ldots$$

defines a pre-$\lambda$ structure on the ring $K(X)$.

To a pre-$\lambda$ structure on a ring $R$ one can associate a power structure over $R$: a notion introduced in [8].

**Definition:** A power structure over a (semi)ring $R$ with a unit is a map $(1 + t \cdot R[[t]]) \times R \to 1 + t \cdot R[[t]]$: $(A(t), m) \mapsto (A(t))^m$, which possesses the following properties:

1) $(A(t))^0 = 1$,
2) $(A(t))^1 = A(t)$,
3) $(A(t) \cdot B(t))^m = (A(t))^m \cdot (B(t))^m$,
4) $(A(t))^{m+n} = (A(t))^m \cdot (A(t))^n$,
5) $(A(t))^{mn} = ((A(t))^n)^m$,
6) $(1 + t)^m = 1 + mt +$ terms of higher degree,
7) $(A(t^k))^m = (A(t))^m |_{t \mapsto t^k}$.

**Remark.** For a ring property 1) follows from the other ones. It is necessary to keep it only for a semiring.

**Definition:** A power structure is finitely determined if for each $M > 0$ there exists a $N > 0$ such that for any series $A(t)$ the $M$-jet of the series $(A(t))^m$ (i.e., $(A(t))^m \mod t^{M+1}$) is determined by the $N$-jet of the series $A(t)$.

**Proposition 1** To define a finitely determined power structure over a ring $R$ it is sufficient to define the series $(A_0(t))^m$ for any fixed series $A_0(t)$ of the form $1 + t +$ terms of higher degree, and for each $m \in R$, so that:

1) $(A_0(t))^m = 1 + mt +$ terms of higher degree;
2) \((A_0(t))^{m+n} = (A_0(t))^m(A_0(t))^n\).

**Proof.** By properties 6 and 7, each series \(A(t) \in 1 + t \cdot R[[t]]\) can be written in a unique way as a product of the form \(\prod_{i=1}^{\infty} (A_0(t^i))^{b_i}\), with \(b_i \in R\). Then by properties 3 and 7 (and the finite determinacy of the power structure)

\[
(A(t))^m = \prod_{i=1}^{\infty} (A_0(t^i))^{b_i}.
\] (1)

In the other direction, one can easily see that the power structure defined by the equation (1) possesses properties 1)–7). \(\square\)

Proposition 1 means that a pre-\(\lambda\) structure on the ring \(R\) defines a finitely determined power structure over \(R\). In the other direction, there are many pre-\(\lambda\) structures on the ring \(R\) which give one and the same power structure: those defined by the series \((A_0(t))^m\) for any fixed series \(A_0(t)\) of the form \(1 + t + \ldots\) terms of higher degree. In what follows we shall prefer to use the series \(A_0(t) = (1 - t)^{-1} = 1 + t + t^2 + \ldots \in R[[t]]\).

Let \(R[[t]] = R[[t_1, \ldots, t_r]]\) be the ring of series in \(r\) variables \(t_1, \ldots, t_r\) with coefficients from the ring \(R\) and let \(m\) be the ideal \((t_1, \ldots, t_r)\). A power structure over the ring \(R\) in a natural way permits to give sense to expressions of the form \((A(t))^m\), where \(A(t) \in 1 + mR[[t]]\). Namely, the series \(A(t)\) can be in a unique way represented in the form

\[
A(t) = \prod_{k \in \mathbb{Z}_{\geq 0} \setminus \{0\}} (1 - t^k)^{-b_k}
\]

\((t_1^k = t_1^{k_1} \ldots t_r^{k_r})\). Then

\[
(A(t))^m = \prod_{k \in \mathbb{Z}_{\geq 0} \setminus \{0\}} (1 - t^k)^{-b_k m}.
\]

Let \(R_1\) and \(R_2\) be rings with power structures over them. A ring homomorphism \(\varphi : R_1 \to R_2\) induces the natural homomorphism \(R_1[[t]] \to R_2[[t]]\) (also denoted \(\varphi\)) by \(\varphi(\sum a_i t^i) = \sum \varphi(a_i) t^i\). One has:

**Proposition 2** If a ring homomorphism \(\varphi : R_1 \to R_2\) is such that \((1 - t)^{-\varphi(m)} = \varphi((1 - t)^{-m})\) for any \(m \in R\), then \(\varphi((A(t))^m) = (\varphi(A(t)))^{\varphi(m)}\) for \(A(t) \in 1 + mR[[t]], m \in R\).
Definition: The Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$ of complex quasi-projective varieties is the abelian group generated by classes $[X]$ of all quasi-projective varieties $X$ modulo the relations:

1) if varieties $X$ and $Y$ are isomorphic, then $[X] = [Y]$;
2) if $Y$ is a Zariski closed subvariety of $X$, then $[X] = [Y] + [X \setminus Y]$.

The multiplication in $K_0(\mathcal{V}_\mathbb{C})$ is defined by the Cartesian product of varieties.

Remark. One can also consider the concept of the Grothendieck semiring $S_0(\mathcal{V}_\mathbb{C})$ of complex quasi-projective varieties substituting the word “group” above by the word “semigroup”. Elements of the semiring $S_0(\mathcal{V}_\mathbb{C})$ have somewhat more geometric sense: they are represented by “genuine” quasi-projective varieties (not by virtual ones).

The class $[\mathbb{A}_\mathbb{C}^1] \in K_0(\mathcal{V}_\mathbb{C})$ of the complex affine line is denoted by $\mathbb{L}$. In a number of cases it is reasonable (or rather necessary) to consider the localization $K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}]$ of the Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$ by the class $\mathbb{L}$.

For a complex quasi-projective variety $X$, let $S^k X = X^k/S_k$ be the $k$-th symmetric power of the space $X$ (here $S_k$ is the group of permutations on $n$ elements; $S^k X$ is a quasi-projective variety as well).

Definition: The Kapranov zeta function of a quasi-projective variety $X$ is the series

$$
\zeta_X(t) = 1 + [X] \cdot t + [S^2 X] \cdot t^2 + [S^3 X] \cdot t^3 + \ldots \in K_0(\mathcal{V}_\mathbb{C})[[t]]
$$

(1).

One can see that

$$
\zeta_{X+Y}(t) = \zeta_X(t) \cdot \zeta_Y(t).
$$

(2)

This follows from the relation $S^k(X \amalg Y) = \bigoplus_{i=0}^{k} S^i X \times S^{k-i} Y$. Also one has

$$
\zeta_{\mathbb{L}^n}(t) = \frac{1}{1 - \mathbb{L}^n t}.
$$

As an example this implies that

$$
\zeta_{\mathbb{CP}^n}(t) = \prod_{i=0}^{n} \frac{1}{1 - \mathbb{L}^i t}.
$$
Equation (2) means that the series \( \zeta_X(t) \) defines a pre-\( \lambda \) structure on the Grothendieck ring \( K_0(\mathcal{V}_C) \). The geometric description of the corresponding power structures over the ring \( K_0(\mathcal{V}_C) \) was given in [8]. We shall formulate it here in the form adapted for series in \( r \) variables ([10]).

Let \( A_n, n = (n_1, \ldots, n_r) \in \mathbb{Z}^r_{\geq 0} \setminus \{0\} \), and \( M \) be quasi-projective varieties and \( A(t) = 1 + \sum_{n \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}} [A_n] t^n \in K_0(\mathcal{V}_C)[[t]] \). Let \( \mathfrak{A} \) be the disjoint union \( \bigsqcup_{k \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}} A_k \), and let \( k : \mathfrak{A} \to \mathbb{Z}^r_{\geq 0} \) be the tautological map on it: it sends points of \( A_k \) to \( k \in \mathbb{Z}^r_{\geq 0} \).

**Geometric description of the power structure over the ring \( K_0(\mathcal{V}_C) \).**

The coefficient at \( t^n \) in the series \( A(t)^{[M]} \) is represented by the configuration space of pairs \((K, \varphi)\), where \( K \) is a finite subset of the variety \( M \) and \( \varphi \) is a map from \( K \) to \( \mathfrak{A} \) such that \( \sum_{x \in K} k(\varphi(x)) = n \). To describe such a configuration space as a quasi-projective variety one can write it as

\[
\sum_{k : \sum i k_i = n} \left[ \left( \prod_i M^{k_i} \right) \setminus \Delta \right] \times \prod_i A_{k_i} / \prod_i S_{k_i},
\]

where \( k = \{k_i : i \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}, k_i \in \mathbb{Z}\} \) and \( \Delta \) is the “large diagonal” in \( M^{\Sigma k} \) which consists of \( (\sum k_i) \)-tuples of points of \( M \) with at least two coinciding ones; the permutation group \( S_{k_i} \) acts by permuting corresponding \( k_i \) factors in \( \prod_i M^{k_i} \supset (\prod_i M^{k_i}) \setminus \Delta \) and the spaces \( A_{k_i} \) simultaneously (the connection between this formula and the description above is clear).

One can shows (see [8]) that the described operation really gives a power structure over \( K_0(\mathcal{V}_C) \), i.e. it satisfies conditions 1) – 7) of the definition. The fact that this structure corresponds to the Kapranov zeta function follows from the equation

\[
(1 + t + t^2 + \ldots)^{[M]} = 1 + [M] \cdot t + [S^2 M] \cdot t^2 + [S^3 M] \cdot t^3 + \ldots
\]

Indeed, since there is only one map from \( M \) to a point (the coefficients in the series \( 1 + t + t^2 + \ldots \)), the coefficient at \( t^n \) in the LHS of equation (4) is represented by the space a point of which is a finite set of points of the variety \( M \) with positive multiplicities such that the sum of these multiplicities is equal to \( n \). This is just the definition of the \( n \)-th symmetric power of the variety \( M \).
It is also useful to describe the binomial \((1 + t)^M\). The coefficient at \(t^n\) in it is represented by the space a point of which is a finite subset of \(M\) with \(n\) elements, i.e. the configuration space \((M^n \setminus \Delta)/S_n\) of unordered \(n\)-tuples of distinct points of \(M\).

It seems that the power structure can be used to prove some combinatorial identities. For instance, applying formula (3) to a finite set \(M\) with \(m\) elements one gets a formula for the power of a series:

\[
\left(1 + \sum_{n \in \mathbb{Z}_{\geq 0}} a_n t^n\right)^m = 1 + \sum_{n \in \mathbb{Z}_{\geq 0}} \left(\sum_{\sum_{a_i} = m} \frac{m!}{k! \prod_i a_i^k} \prod_{i} a_i^k\right) t^n.
\]

There are two natural homomorphism from the Grothendieck ring \(K_0(\mathcal{V}_C)\) to the ring \(\mathbb{Z}\) of integers and to the ring \(\mathbb{Z}[u, v]\) of polynomials in two variables: the Euler characteristic (with compact support) \(\chi: K_0(\mathcal{V}_C) \to \mathbb{Z}\) and the Hodge–Deligne polynomial \(e: K_0(\mathcal{V}_C) \to \mathbb{Z}[u, v]; e(X)(u, v) = \sum e^{p,q}(X) u^p v^q\).

The formula of I.G. Macdonald \[13\]:

\[
\chi(1 + [X] t + [S^2 X] t^2 + [S^3 X] t^3 + \ldots) = (1 - t)^{\chi(X)}
\]

and the corresponding formula for the Hodge–Deligne polynomial (see \[2, 3, Proposition 1.2\]):

\[
e(1 + [X] t + [S^2 X] t^2 + \ldots)(u, v) = \prod_{p,q} \left(\frac{1}{1 - u^p v^q t}\right)^{e^{p,q}(X)}
\]

implies that these homomorphisms respect the power structures on these rings described above (see Example 3 and Proposition \[2\] or \[9\]). Therefore a relation between series from \(K_0(\mathcal{V}_C)[[t]]\) written in terms of the power structure yields the corresponding relations between the Euler characteristics and the Hodge–Deligne polynomials of these series.

**Remark.** It is also possible to define the power structure and to describe it in the *relative setting*, i.e. over the Grothendieck ring \(K_0(\mathcal{V}_S)\) of complex quasi-projective varieties over a variety \(S\). The ring \(K_0(\mathcal{V}_S)\) is generated by classes of varieties with maps (“projections”) to \(S\). In this case the coefficient of the series \((A(\mathcal{U}))^M\) is the configuration space a point of which is a pair \((K, \phi)\) consisting of a finite subset \(K \subset M\) which is contained in the preimage of one point of \(S\) and the map \(\phi\) commutes with the projections to \(S\).
2 Nested Hilbert schemes of J. Cheah

Let \( \text{Hilb}_X^n \), \( n \geq 1 \), be the Hilbert scheme of zero-dimensional subschemes of length \( n \) of a complex quasi-projective variety \( X \); for \( x \in X \), let \( \text{Hilb}_{X,x}^n \) be the Hilbert scheme of subschemes of \( X \) supported at the point \( x \).

In [4], J. Cheah considered nested Hilbert schemes on a smooth \( d \)-dimensional complex quasi-projective variety \( X \). For \( n = (n_1, \ldots, n_r) \in \mathbb{Z}_{\geq 0}^r \), the nested Hilbert scheme \( Z_X^n \) of depth \( r \) is the scheme which parametrizes collections of the form \( (Z_1, \ldots, Z_r) \), where \( Z_i \in \text{Hilb}_X^{n_i} \) and \( Z_i \) is a subscheme of \( Z_j \) for \( i < j \). The scheme \( Z_X^n \) is non-empty only if \( n_1 \leq n_2 \leq \ldots \leq n_r \); notice that \( Z_X^{(n)} = \text{Hilb}_X^n \cong Z_X^{(n, \ldots, n)} \).

For \( Y \subset X \), let \( Z_X^n_{X,Y} \) be the scheme which parametrizes collections \( (Z_1, \ldots, Z_r) \) from \( Z_X^n \) with \( \text{supp} Z_i \subset Y \). For \( Y = \{x\}, x \in X \), we shall use the notation \( Z_X^n_{X,x} \).

For \( r \geq 1 \), let \( t = (t_1, \ldots, t_r) \) and

\[
Z_X^{(r)}(t) := \sum_{n \in \mathbb{Z}_{\geq 0}^r} [Z_X^n] t^n, \quad Z_X^{(r)}_{X,X}(t) := \sum_{n \in \mathbb{Z}_{\geq 0}^r} [Z_X^n_{X,x}] t^n,
\]

be the generating series of classes of nested Hilbert schemes \( Z_X^n \) (resp. supported at the point \( x \)) of depth \( r \).

**Theorem 1** For a smooth quasi-projective variety \( X \) of dimension \( d \), the following identity holds in \( S_0(\mathcal{V}_C)[[t]] \) (and therefore also in \( K_0(\mathcal{V}_C)[[t]] \)):

\[
Z_X^{(r)}(t) = \left( Z_{A^d,0}^{(r)}(t) \right)^{[X]}.
\]

**Proof.** For a Zariski closed subset \( Y \subset X \), one has \( Z_X^{(r)}(t) = Z_{X,Y}^{(r)}(t) \cdot Z_{X,X,Y}^{(r)}(t) \). Therefore it is sufficient to prove equation (5) for Zariski open subsets \( U \) of \( X \) which form a covering of \( X \) and for their intersections.

One can take \( U \) which lies in an affine chart \( A^d_C \) and such that its projection to a \( d \)-dimensional coordinate space (say, generated by the first \( d \) coordinates) is everywhere non-degenerate (i.e. is an étale morphism). For any point \( x \in U \), this projection identifies \( n \)-nested Hilbert schemes of \( Z_{U,x}^n \) with \( Z_{k^d,x}^n \).

A nested (zero-dimensional) subscheme of \( U \) of type \( n \) is defined by a finite subset \( K \subset U \) with a nested subscheme from \( Z_{X,x}^{k(x)} \) at each point \( x \in K \).
such that \( \sum_{x \in K} k(x) = n \). This coincides with the description of the coefficient at \( t^n \) in the RHS of the equation (5). □

Similar considerations permit to give a short proof of a somewhat refined version of the main result of [4]. Following J. Cheah, let

\[
\mathcal{F}_X^n = \{(x, Z) \in X \times \text{Hilb}_X^n : x \in \text{supp } Z\},
\]

\[
\mathcal{F}_X^{n-1,n} = \{(x_1, x_2, Z_1, Z_2) \in X \times X \times Z_X^{(n-1,n)} : x_i \in \text{supp } Z_i, i = 1, 2\},
\]

\[
\mathcal{T}_X^n = \{(x_1, x_2, Z) \in X \times X \times \text{Hilb}_X^n : x_i \in \text{supp } Z, i = 1, 2\},
\]

\[
\mathcal{G}_X^n = \{(x, Z_1, Z_2) \in X \times Z_X^{(n-1,n)} : x \in \text{supp } Z_2\}.
\]

Let the series \( \mathfrak{P}_X(t_0, t_1, t_2, t_3) \) and \( f_d(t_0, t_1, t_2, t_3) \) from \( K_0(\mathcal{V}_c)[[t_0, t_1, t_2, t_3]] \) be defined by

\[
\mathfrak{P}_X(t_0, t_1, t_2, t_3) := \left[ \sum_{n \geq 1} [\text{Hilb}_X^n] t_0^n \right] t_1 + \left[ \sum_{n \geq 1} [\mathcal{F}_X^n] t_0^n \right] t_2 + \left[ \sum_{n \geq 1} [\mathcal{T}_X^n] t_0^n \right] t_3 + \left[ \sum_{n \geq 2} [Z_X^{(1,n-1,n)}] t_0^n \right] t_1 t_3 + \left[ \sum_{n \geq 2} [\mathcal{G}_X^n] t_0^n \right] t_2 t_3 \cdot
\]

\[
f_d(t_0, t_1, t_2, t_3) := \sum_{k \geq 0} [\text{Hilb}_{\mathcal{A}_d,0}^k] t_0^k + \sum_{k \geq 1} [Z_{\mathcal{A}_d,0}^{(k-1,k)}] t_0^k t_3 + \sum_{k \geq 1} [\text{Hilb}_{\mathcal{A}_d,0}^k] t_0^k t_2 + \sum_{k \geq 1} [Z_{\mathcal{A}_d,0}^{(k-1,k)}] t_0^k t_2 t_3 + \sum_{k \geq 1} [\text{Hilb}_{\mathcal{A}_d,0}^k] t_0^k t_1 + \sum_{k \geq 1} [Z_{\mathcal{A}_d,0}^{(k-1,k)}] t_0^k t_1 t_3 + \sum_{k \geq 2} [Z_{\mathcal{A}_d,0}^{(k-1,k)}] t_0^k t_1 t_2 + \sum_{k \geq 2} [Z_{\mathcal{A}_d,0}^{(k-1,k)}] t_0^k t_1 t_2 t_3 \cdot
\]

**Theorem 2** (cf. Main Theorem in [4]) Let \( X \) be a smooth quasi-projective variety of dimension \( d \). Then

\[
\mathfrak{P}_X(t_0, t_1, t_2, t_3) = (f_d(t_0, t_1, t_2, t_3))^{[X]} \mod (t_1^2, t_2^2, t_3^2). \quad (6)
\]
**Proof.** Using the arguments of the proof of Theorem 1, we may suppose that $X$ lies in an affine chart $A_N^C$ and its projection to a $d$-dimensional coordinate space is nondegenerate. This identifies $\text{Hilb}_{X,x}^s$ and $Z_{X,x}^{(s-1,s)}$ with $\text{Hilb}_{k^d,0}^s$ and $Z_{k^d,0}^{(s-1,s)}$ respectively for each point $x \in X$. To prove equation (6) one has to give an interpretation of the coefficients at the monomials $t_0^nt_1^nt_3$, . . . , $t_0^nt_1t_2t_3$ in the RHS of (6). Let us make this for the coefficients at $t_0^nt_3$ and at $t_0^nt_1t_2t_3$ (other cases are treated in the same way).

The coefficient at $t_0^nt_3$ is represented by the space a point of which is defined by a point $x_0$ of $X$ with a zero-dimensional nested scheme from $Z_{X,x_0}^{(k(x_0)-1,1)}$ at it plus several other points of $X$ with zero-dimensional schemes from $\text{Hilb}_{X,x}^{k(x)} \cong Z_{X,x}^{(k(x),k(x))}$ at each of them, such that $k(x_0) + \sum k(x) = n$. This is just the definition of a point of the space $Z_{X,n-1}^{1,n}$.

The monomial $t_0^nt_1t_2t_3$ can be obtained either as a product of two monomials of the form $t_0^nt_2$, $t_0^nt_3$ and of several monomials of the form $t_0^n$ or as a product of a monomial of the form $t_0^nt_2t_3$ and of several monomials of the form $t_0^n$. Therefore the coefficient at the monomial $t_0^nt_1t_2t_3$ is represented by the space consisting of two parts.

A point of the first part is defined by a point $x_1$ of $X$ with a scheme from $\text{Hilb}_{X,x}^{k(x_1)} \cong Z_{X,x_1}^{(k(x_1),k(x_1))}$ at it, with $k \geq 1$ (i.e. it is not empty: $x_1$ belongs to the support of it), a point $x_2 \in X$ with a scheme from $Z_{X,x_2}^{(k(x_2)-1,1)}$ at it plus several points of $X$ with 0-dimensional schemes from $\text{Hilb}_{X,x}^{k(x)}$ at each of them such that $k(x_1) + k(x_2) + \sum k(x) = n$.

A point of the second part is defined by a point $x_1$ of $X$ with the scheme $(z_1, z_2)$ from $Z_{X,x_1}^{(k(x_1)-1,1)}$ at it (in this case $z_2$ is not empty: $x_1$ belongs to the support of it) plus several points of $X$ with 0-dimensional schemes from $\text{Hilb}_{X,x}^{k(x)}$ at each of them such that $k(x_1) + \sum k(x) = n$. Therefore a point of the union of these two subspaces can be described by a nested scheme $(Z_1, Z_2)$ from $Z_{X,n-1}^{1,n}$ plus a point which belongs to $Z_2$. This is just the description of the space $G^n X$. □

Applying the Hodge–Deligne homomorphism to (6) one gets the Main Theorem of [1].

**Example.** Let $S$ be a smooth quasi-projective surface. Consider the incidence variety $Z_S^{(n-1,n)} = \{(Z_1, Z_2) \in \text{Hilb}_S^{n-1} \times \text{Hilb}_S^n : Z_1 \subset Z_2\}$. Using the results of J. Cheah on the cellular decomposition of $Z_{k^d,0}^{(n-1,n)}$ ([3]), one gets
the result of L. Göttsche ([7, Theorem 5.1]):

\[\sum_{n \geq 1} [Z_{S}^{n-1,n}] t^n = \frac{[S] \cdot t}{1 - \prod_{k \geq 1} \frac{1}{1 - \frac{t}{1 - L^k}}}.\]

3 On moduli spaces of curves and points - (after W.-P. Li and Zh. Qin)

In [12], there were considered certain moduli spaces of 1-dimensional subschemes in a smooth \(d\)-dimensional projective complex variety. Let \(X\) be a smooth \(d\)-dimensional projective complex variety with a Zariski locally trivial fibration \(\mu : X \to S\) where \(S\) is smooth of dimension \(d - 1\) and fibres are smooth irreducible curves of genus \(g\). Let \(\beta \in H_2(X, \mathbb{Z})\) be the class of the fibre.

Let \(I_n(X, \beta)\) be the moduli space of 1-dimensional closed subschemes \(Z\) of \(X\) such that \(\chi(O_Z) = n\), \([Z] = \beta\), where \([Z]\) is the fundamental class of the scheme \(Z\) and let \(\mathcal{M}_n := I_{(1-g)+n}(X, \beta)\). Let \(\mathcal{M}^n_{X,C_x}\) be the moduli space of 1-dimensional closed subschemes \(\Theta\) in \(X\) such that \(I_\Theta \subset I_{C_x}\), the support \(\text{supp}(I_{C_x}/I_\Theta) = \{O\}\) and \(\dim_O(I_{(O)\times C}/I_\Theta) = n\). The number \(n\) will be called the length of the subscheme \(\Theta\). Let \(\mathcal{M}^n_{C^{d-1} \times C, \{O\} \times C, O}\) have the same meaning: it is the moduli space of 1-dimensional closed subschemes \(\Theta\) in \(\mathbb{C}^{d-1} \times \mathbb{C}\) such that \(I_\Theta \subset I_{(O)\times C}\), \(\text{supp}(I_{(O)\times C}/I_\Theta) = \{O\}\) and \(\dim_O(I_{(O)\times C}/I_\Theta) = n\).

\textbf{Theorem 3} (cf. Proposition 5.3, Lemma 6.1 and Proposition 6.2 in [12])

Let \(X\) be a smooth \(d\)-dimensional projective complex variety with a Zariski locally trivial fibration \(\mu : X \to S\) where \(S\) is smooth of dimension \(d - 1\) and fibres are smooth irreducible curves of genus \(g\). Then

\[\sum_{n \geq 0} [\mathcal{M}^n] t^n = [S] \left( \sum_{n \geq 0} [\text{Hilb}^n_{C^{d-1}}] t^n \right)^{|X|-|C|} \left( \sum_{n \geq 0} [\mathcal{M}^n_{C^{d-1} \times C, \{O\} \times C, O}] t^n \right)^{|C|}.\] (7)

\textbf{Proof}. A point of \(\mathcal{M}^n\) can be considered as consisting of a fibre \(C_s = \mu^{-1}(s)\) of the bundle \(\mu : X \to S\) and of several fixed points, both outside of \(C_s\) and on it, with a 0-dimensional subscheme (i.e. an element of \(\text{Hilb}^*_{X,s}\)) at each of
those points which are outside of $C_s$ and a subscheme of $\mathcal{M}_{X,C_s,x}$ at each of those points which lie on $C_s$ such that the sum of their lengths is equal to $n$. Thus, there is a natural map (projection) from $\mathcal{M}^n$ to $S$. Over a point $s \in S$, there are somewhat different objects (subschemes) at points outside of the curve $C_s$ and on this curve.

It is sufficient to prove equation (7) for preimages of elements of a covering of $S$ by Zariski open subsets and of their intersections. Therefore without any loss of generality we can suppose that $X = S \times C$. Moreover, let us choose a fixed point $s_0 \in S$. A constructible map which sends $\mathcal{M}^n_{X,C_s}$ to $\mathcal{M}^n_{X,C_s,0}$ and is an isomorphism of strata can be defined as follows. One takes a 0-dimensional subscheme which lies on $C_s$ and puts them to the corresponding points of $C_{s_0}$ and vice versa, one takes the elements of $\mathcal{M}^n_{X,C_s,x}$ and puts them to the corresponding points of $C_{s_0}$. Thus in the Grothendieck ring of algebraic varieties one has $[\mathcal{M}^n] = [S][\mathcal{M}^n_{X,C_{s_0}}]$.

Therefore to prove (7) one should show that

$$\sum_{n \geq 0} [\mathcal{M}^n_{X,C_{s_0}}] t^n = \left( \sum_{n \geq 0} [\text{Hilb}^n_{C_d,0}] t^n \right)^{|X|-[C_{s_0}]} \left( \sum_{n \geq 0} [\mathcal{M}^n_{C_{d-1}\times\mathbb{C},\{0\}\times\mathbb{C},\{0\}}] t^n \right)^{[C_{s_0}]}.$$  

(8)

Just as in the proofs in Section 2 we may suppose that at each point of $X$ the space $\text{Hilb}^k_{X,x}$ is identified with the space $\text{Hilb}^k_{C_d,0}$ and at each point of $C_{s_0} \subset X$ the space $\mathcal{M}^k_{X,C_{s_0},x}$ is identified with the space $\mathcal{M}^k_{C_{d-1}\times\mathbb{C},\{0\}\times\mathbb{C},\{0\}}$. The coefficient at $t^n$ in the RHS of equation (8) is represented by the space a point of which is defined by several points of the curve $C_{s_0} \subset X$ with a scheme from $\mathcal{M}^k_{X,C_{s_0},x}$ at each of them and several points from $X \setminus C_{s_0}$ with a scheme from $\text{Hilb}^k_{X,x}$ at each of them such that the sum of the lengths $k(x)$ over all the mentioned points is equal to $n$. This is just the description of a point of $\mathcal{M}^n_{X,C_{s_0}}$. □

4 Orbifold generalized Euler characteristic and the power structure

Here we rewrite some results of [15] and [16] in terms of the power structure. For that we need it over a somewhat modified version of the Grothendieck ring $K_0(V_C)$. For a fixed positive integer $m$, consider the ring $K_0(V_C)[L^{1/m}]$. 

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The pre-\(\lambda\) structure on (and therefore the corresponding power structure over) the ring \(K_0(\mathcal{V}_C)\) can be extended to one on \(K_0(\mathcal{V}_C)[\mathbb{L}^{1/m}]\) by the formula

\[
\zeta_{[X]L^{s/m}}(t) = \zeta_{[X]}(\mathbb{L}^{s/m}t).
\]

In a similar way the corresponding pre-\(\lambda\) structure on the ring \(\mathbb{Z}[u_1^{1/m}, \ldots, u_r^{1/m}]\) can be defined by the formula

\[
\lambda_P(t) = \prod_{k \in \mathbb{Z}_{\geq 0}} (1 - u^k t)^{-p_k}
\]

for a polynomial 

\[P = P(u) = \sum_{k \in (1/m)\mathbb{Z}_{\geq 0}} p_k u^k.
\]

There are natural homomorphisms \((\chi, e)\) from the ring \(K_0(\mathcal{V}_C)[\mathbb{L}^{1/m}]\) to the rings \(\mathbb{Z}\) and \(\mathbb{Z}[u^{1/m}, v^{1/m}]\) which send the element \(\mathbb{L}^{1/m}\) to 1 and \((uv)^{(1/m)}\) respectively. One can easily see that these are homomorphisms of the \(\lambda\)-rings and therefore they respect the power structures.

Let \(X\) be a smooth quasi-projective complex algebraic variety of dimension \(d\) with an action of a finite group \(G\) of order \(m\). For an element \(g \in G\), let \(X^g\) be the set \(\{x \in X : gx = x\}\) of \(g\)-invariant points of the action. If \(h = vgv^{-1}\) in \(G\), the element \(v\) defines an isomorphism \(v : X^g \to X^h\). Let \(G_*\) be the set of conjugacy classes of elements of the group \(G\). For a conjugacy class \(c \in G_*\) choose its representative \(g \in G\). Let \(C_G(g)\) be the centralizer of the element \(g\) in \(G\). The centralizer \(C_G(g)\) acts on the set \(X^g\) of fixed points of \(g\). Suppose that its action on the set of connected components of \(X^g\) has \(N_c\) orbits and let \(X^g_1, \ldots, X^g_{N_c}\) be unions of components of each of these orbits. At each point \(x \in X^g_{\alpha_c}\), the map \(dg\) is an automorphism of the tangent space \(T_xX\) which acts as a diagonal matrix \(\text{diag}(\exp(2\pi i\theta_1), \ldots, \exp(2\pi i\theta_d))\), where \(0 \leq \theta_i < 1, \theta_i \in (1/m)\mathbb{Z}\). The shift number \(F^g_{\alpha_c}\) associated to \(X^g_{\alpha_c}\) is

\[
F^g_{\alpha_c} := \sum_{j=1}^d \theta_j \in \mathbb{Z}/m\text{ (it was introduced by E. Zaslow in [17]).}
\]

**Definition:** The orbifold generalized Euler characteristic \([X, G]\) of the pair \((X, G)\) is

\[
[X, G] := \sum_{c \in G_*} \sum_{\alpha_c=1}^{N_c} [X^g_{\alpha_c}/C_G(g)] \cdot \mathbb{L}^{F^g_{\alpha_c}} \in K_0(\mathcal{V}_C)[\mathbb{L}^{1/m}].
\]
Applying the Euler characteristic morphism one gets the notion of orbifold Euler characteristic invented in the study of string theory of orbifolds by L. Dixon et al. [6]:

\[
\chi(X, G) := \sum_{c \in G^*} \sum_{\alpha = 1}^{N_c} \chi(X^g_{\alpha c}/C_G(g)) = \sum_{c \in G^*} \chi(X^g/C_G(g)).
\]

Applying the Hodge–Deligne polynomial one gets the orbifold E-function introduced by V. Batyrev in [1]:

\[
E_{orb}(X, G; u, v) := \sum_{c \in G^*} \sum_{\alpha = 1}^{N_c} e(X^g_{\alpha c}/C_G(g))(u, v)(uv)^{F_{\alpha c}} \in \mathbb{Z}[u^{1/m}, v^{1/m}].
\]

Let \( G^n = G \times \ldots \times G \) be the Cartesian power of the group \( G \). The symmetric group \( S_n \) acts on \( G^n \) by permutation of the factors: \( s(g_1, \ldots, g_n) = (g_{s^{-1}(1)}, \ldots, g_{s^{-1}(n)}) \). The wreath product \( G_n = G \sim S_n \) is the semidirect product of \( G^n \) and \( S_n \) defined by the described action. Namely the multiplication in the group \( G_n \) is given by the formula \( (g, s)(h, t) = (g \cdot s(h), st) \), where \( g, h \in G^n, s, t \in S_n \). The group \( G^n \) is a normal subgroup of \( G_n \) via the identification of \( g \in G^n \) with \( (g, 1) \in G_n \). For a variety \( X \) with a \( G \)-action, there is the corresponding action of the group \( G_n \) on the Cartesian power \( X^n \) given by the formula

\[
((g_1, \ldots, g_n), s)(x_1, \ldots, x_n) = (g_1 x_{s^{-1}(1)}, \ldots, g_n x_{s^{-1}(n)}),
\]

where \( x_1, \ldots, x_n \in X, g_1, \ldots, g_n \in G, s \in S_n \). One can see that the factor variety \( X^n/G_n \) is naturally isomorphic to \( (X/G)^n/S_n \). In particular, \([X^n/G_n] = [(X/G)^n/S_n]\) in \( K_0(\mathcal{V}_c)\). Therefore

\[
\sum_{n \geq 0} [X^n/G_n]t^n = (1 - t)^{[X/G]} \in K_0(\mathcal{V}_c)[[t]].
\]

**Theorem 4** (cf. [15],[16]) Let \( X \) be a smooth quasi-projective complex algebraic variety of dimension \( d \) with an action of a finite group \( G \) of order \( m \). Then

\[
\sum_{n \geq 0} [X^n, G_n]t^n = \left( \prod_{r=1}^{\infty} (1 - \frac{r-1}{2}t^r)^{-[X,G]} \right)^{1/\mu}.
\]
Proof. One can say that essentially the proof is already contained in [16] where invariants of the $G_n$-action on the space $X^n$ are related to those of the $G$-action on the space $X$ (see also [15] and [14]).

Let $a = (g, s) \in G_n$, $g = (g_1, \ldots, g_n)$. Let $z = (i_1, \ldots, i_r)$ be one of the cycles in the permutation $s$. The cycle-product of the element $a$ corresponding to the cycle $z$ is the product $g_{i_r} g_{i_{r-1}} \ldots g_{i_1} \in G$. The conjugacy class of the cycle-product is well-defined by $g$ and $s$. For $c \in G_*$ and $r \geq 0$, let $m_r(c)$ be the number of $r$-cycles in the permutation $s$ whose cycle-products lie in $c$. Let $\rho(c)$ be the partition which has $m_r(c)$ summands equal to $r$, and let $\rho = (\rho(c))_{c \in G_*}$ be the corresponding partition-valued function on $G_*$. One has

$$\|\rho\| := \sum_{c \in G_*} |\rho(c)| = \sum_{c \in G_*, r \geq 1} r m_r(c) = n.$$ 

The function $\rho$ or, equivalently, the data $\{m_r(c)\}_{r,c}$ is called the type of the element $a = (g, s) \in G_n$. Two elements of the group $G_n$ are conjugate to each other iff they are of the same type.

In [16] it is shown that:

1. For a conjugacy class of elements of the group $G_n$ containing an element $a$ of type $\rho = \{m_r(c)\}_{r,c} \in G_*$ ($\sum_{r,c} r m_r(c) = n$), the subspace $(X^n)^a$ can be naturally identified with $\prod_{c,r} (X^c)^{m_r(c)}$. The factor space $(X^n)^a / Z_{G_n}(a)$ is naturally isomorphic to $\prod_{c \in G_*, r \geq 1} S^{m_r(c)} (X^c / Z_G(c))$. The connected components of the space $(X^n)^a / Z_{G_n}(a)$ are numbered by sets of integers $(m_{r,c}(1), \ldots, m_{r,c}(N_c))$ satisfying the relation $\sum_{c=1}^{N_c} m_{r,c}(\alpha_{c}) = m_r(c)$. They are

$$(X^n)^a_{\{m_{r,c}(\alpha_{c})\}} = \prod_{c \in G_*, r \geq 1} \prod_{\alpha_{c} = 1}^{N_c} S^{m_{r,c}(\alpha_{c})} (X^c_{\alpha_{c}} / Z_G(c)).$$

2. The shift for the component $(X^n)^a_{\{m_{r,c}(\alpha_{c})\}}$ is equal to

$$F_{\{m_{r,c}(\alpha_{c})\}} = \sum_{c \in G_*, r \geq 1} \sum_{\alpha_{c} = 1}^{N_c} m_{r,c}(\alpha_{c}) \left(F^c_{\alpha_{c}} (r - 1) d/2 \right).$$
These two facts imply that

\[
\sum_{n \geq 0}[X^n, G_n] t^n = \sum_{n \geq 0} \left( \sum_{m_r(c)} \prod_{c,r} \left[ S^{m_r,c} \left( \frac{X^g_{\alpha_c}}{Z_G(g)} \right) \right] \prod_{\alpha_c=1}^{N_c} \left( m_r(c) \left( F^g_{\alpha_c} + \frac{(r-1)d}{2} \right) \right) \right) t^n
\]

\[
= \sum_{m_r(c)} \prod_{c,r} \left( \sum_{\alpha_c=1}^{N_c} \left[ S^{m_r,c}(\alpha_c) \left( \frac{X^g_{\alpha_c}}{Z_G(g)} \right) \right] \prod_{\alpha_c=1}^{N_c} \left( m_r(c) \left( F^g_{\alpha_c} + \frac{(r-1)d}{2} \right) \right) \right) t^{m_r(c)}
\]

\[
= \prod_{c,r} \prod_{\alpha_c=1}^{N_c} \left( \sum_{m_r,c(\alpha_c)} \left[ S^{m_r,c}(\alpha_c) \left( \frac{X^g_{\alpha_c}}{Z_G(g)} \right) \right] \prod_{\alpha_c=1}^{N_c} \left( m_r(c) \left( F^g_{\alpha_c} + \frac{(r-1)d}{2} \right) \right) \right) - [X^g_{\alpha_c}/Z_G(g)]
\]

\[
= \prod_{c,r} \prod_{\alpha_c=1}^{N_c} \left( 1 - \left( F^g_{\alpha_c} + \frac{(r-1)d}{2} \right) t^r \right)^{- \left[ X^g_{\alpha_c}/Z_G(g) \right]}
\]

\[
= \prod_{r \geq 1} \left( 1 - \left( \frac{(r-1)d}{2} \right) t^r \right)^{- \left[ X,G \right]} = \prod_{r \geq 1} \left( 1 - \left( \frac{d}{2} \right) t^r \right)^{- \left[ X,G \right]}
\]

\[
\square
\]

Taking the Euler characteristic of both sides of the equation (9), one gets Theorem 5 of [15]:

\[
\sum_{n \geq 0} \chi(X^n, G_n) t^n = \prod_{r=1}^{\infty} (1 - t^r)^{- \chi(X,G)}.
\]

Applying the Hodge–Deligne polynomial homomorphism, one gets the main result of [16]:

\[
\sum_{n=1}^{\infty} e(X^n, G_n; u, v) t^n = \prod_{r=1}^{\infty} \prod_{p,q} \left( \frac{1}{(1 - u^p v^q t^r)^{(r-1)d/2}} \right)^{e^{p,q}(X,G)}
\]

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