A Subquadratic Approximation Scheme for Partition

Marcin Mucha*  Karol Węgrzycki*  Michał Włodarczyk*

Abstract

The subject of this paper is the time complexity of approximating Knapsack, Subset Sum, Partition, and some other related problems. The main result is an $\tilde{O}(n + 1/\varepsilon^{5/3})$ time randomized FPTAS for Partition, which is derived from a certain relaxed form of a randomized FPTAS for Subset Sum. To the best of our knowledge, this is the first NP-hard problem that has been shown to admit a subquadratic time approximation scheme, i.e., one with time complexity of $\tilde{O}((n + 1/\varepsilon)^{2-\delta})$ for some $\delta > 0$. To put these developments in context, note that a quadratic FPTAS for Partition has been known for 40 years.

Our main contribution lies in designing a mechanism that reduces an instance of Subset Sum to several simpler instances, each with some special structure, and keeps track of interactions between them. This allows us to combine techniques from approximation algorithms, pseudopolynomial algorithms, and additive combinatorics.

We also prove several related results. Notably, we improve approximation schemes for 3SUM, (min, +)-convolution, and TreeSparsity. Finally, we argue why breaking the quadratic barrier for approximate Knapsack is unlikely by giving an $\Omega((n + 1/\varepsilon)^{2-o(1)})$ conditional lower bound.

*Institute of Informatics, University of Warsaw, Poland, {mucha, k.wegrzycki, m.wlodarczyk}@mimuw.edu.pl
1 Introduction

The Knapsack-type problems are among the most fundamental optimization challenges. These problems have been studied for more than a century already, as their origins can be traced back to the 1897’s paper by Mathews [52].

The Knapsack problem is defined as follows:

**Definition 1.1 (Knapsack).** Given a set of \( n \) items \( E_n = \{1, \ldots, n\} \), with item \( j \) having a positive integer weight \( w_j \) and value \( v_j \), together with knapsack capacity \( t \). Select a subset of items \( E \subseteq E_n \), such that the corresponding total weight \( w(E) = \sum_{i \in E} w_i \) does not exceed the capacity \( t \) and the total value \( v(E) = \sum_{i \in E} v_i \) is maximized.

Knapsack is one of the 21 problems featured in Karp’s list of NP-complete problems [43]. We also study the case where we are allowed to take each element multiple times, called Unbounded Knapsack. Let \( \Sigma(S) \) denote the sum of elements \( S \). Subset Sum is defined as follows:

**Definition 1.2 (Subset Sum).** Given a set \( S \subseteq \mathbb{N} \) of \( n \) numbers (sometimes referred to as items) and an integer \( t \), find a subset \( S' \subseteq S \) with maximal \( \Sigma(S') \) that does not exceed \( t \).

Subset Sum is a special case of Knapsack, where item weights are equal to item values. This problem is NP-hard as well. In fact, it remains NP-hard even if we fix \( t \) to be \( \Sigma(S)/2 \). This problem is called the Number Partitioning Problem (or Partition, as we will refer to it):

**Definition 1.3 (Partition).** Given a set \( S \subseteq \mathbb{N} \) of \( n \) numbers, find a subset \( S' \subseteq S \) with maximal \( \Sigma(S') \) not exceeding \( \Sigma(S)/2 \).

The practical applications of Partition problem range from scheduling [42] to minimization of circuits sizes, cryptography [53], or even game theory [35, 54]. The decision version of this problem is sometimes humorously referred to as “the easiest NP-complete problem” [35]. In this paper we will demonstrate that there is a grain of truth in this claim.

All the aforementioned problems are weakly NP-hard and admit pseudo-polynomial time algorithms. The first such an algorithm for the Knapsack was proposed by Bellman [15] and runs in time \( O(nt) \). This bound was improved for the Subset Sum [48] and the current best (randomized) time complexity for this problem is \( O(n + t) \), due to Bringmann [17] (for more on these and related results see Section 2). The strong dependence on \( t \) in all of these algorithms makes them impractical for a large \( t \) (note that \( t \) can be exponentially larger than the size of the input). This dependence has been shown necessary as an \( O(\text{poly}(n)^{0.99}) \) algorithm for the Subset Sum would contradict both the SETH [4] and the SetCover conjecture [28].

One possible approach to avoid the dependence on \( t \) is to settle for approximate solutions. The notion of approximate solution we focus on in this paper is that of a Polynomial Time Approximation Scheme (PTAS). A PTAS for a maximization problem is an algorithm that, given an instance of size \( n \) and a parameter \( \varepsilon > 0 \), returns a solution with value \( S \), such that \( \text{OPT}(1 - \varepsilon) \leq S \leq \text{OPT} \). It also needs to run in time polynomial in \( n \), but not necessarily in \( 1/\varepsilon \) (so, e.g., we allow time complexities like \( O(n^{1/\varepsilon}) \)). A PTAS is a Fully Polynomial Time Approximation Scheme (FPTAS) if it runs in time polynomial in both \( n \) and \( 1/\varepsilon \). Equivalently, one can require the running time to be polynomial in \( (n + 1/\varepsilon) \). For example, \( O(n^2/\varepsilon^4) = O((n + 1/\varepsilon)^6) \). For definitions of problems in both exact and approximate sense see Appendix C.

The first approximation scheme for Knapsack (as well as Subset Sum and Partition as special cases) dates back to 1975 and is due to Ibarra and Kim [39]. Its running time is \( O(n/\varepsilon^2) \). After
a long line of improvements \cite{29, 30, 45, 44, 50, 47}, the current best algorithms for each problem are:

- the $O\left(\min\{n/\varepsilon, n+1/\varepsilon^2\}\right)$ algorithm for \textsc{Partition} due to \cite{31}, the \(O\left(\min\{n/\varepsilon, n+1/\varepsilon^2 \log (1/\varepsilon)\}\right)\) algorithm for \textsc{Subset Sum} due to \cite{46}, and a very recent \(\tilde{O}(n+1/\varepsilon^{12/5})\) for \textsc{Knapsack}, due to \cite{19}.

Observe that all of these algorithms work in $\Omega\left((n+1/\varepsilon)^2\right)$ time. In fact, we are not aware of the existence of any FPTAS for an NP-hard problem working in time $O\left((n+1/\varepsilon)^{2-\delta}\right)$.

**Open Question 1.** Can we get an $O\left((n+1/\varepsilon)^{2-\delta}\right)$ FPTAS for any \textsc{Knapsack}-type problem (or any other NP-hard problem) for some constant $\delta > 0$ or justify that it is unlikely?

In this paper we resolve this question positively, by presenting the first such algorithm for the \textsc{Partition} problem. This improves upon almost 40 years old algorithm by Gens and Levner \cite{31}. On the other hand, we also provide a conditional lower bound suggesting that similar improvement for the more general \textsc{Knapsack} problem is unlikely.

After this paper was announced, Bringmann \cite{18} showed that for any $\delta > 0$, an $O\left((n+1/\varepsilon)^{2-\delta}\right)$ algorithm for \textsc{Subset Sum} would contradict the $(\min, +)$-convolution-conjecture. This not only shows a somewhat surprising separation between the approximate versions of \textsc{Partition} and \textsc{Subset Sum}, but also explains why our techniques do not seem to transfer to approximating \textsc{Subset Sum}.

1.1 Related Work

In this paper we avoid the dependence on $t$ by settling on approximate instead of exact solutions. Another approach is to allow running times exponential in $n$. This line of research has been very active with many interesting results. The naive algorithm for \textsc{Knapsack} works in $O^*(2^n)$ time by simply enumerating all possible subsets. Horowitz and Sahni \cite{36} introduced the meet-in-the-middle approach and gave an exact $O^*(2^{n/2})$ time and space algorithm. Schroeppel and Shamir \cite{56} improved the space complexity of that algorithm to $O^*(2^{n/4})$. Very recently Bansal et al. \cite{11} showed an $O^*(2^{0.86n})$-algorithm working in polynomial space.

An interesting question (and very relevant for applications in cryptography) is how hard \textsc{Knapsack} type problems are for random instances. For results in this line of research see \cite{6, 7, 8, 37}.

1.2 History of Approximation Schemes for \textsc{Knapsack}-type problems

To the best of our knowledge, the fastest approximation for \textsc{Partition} dates back to 1980 \cite{31} with $\tilde{O}(\min\{n/\varepsilon, n+1/\varepsilon^2\})$ running time\textsuperscript{1}. The majority of later research focused on matching this running time for the \textsc{Knapsack} and \textsc{Subset Sum}. In this section we will present an overview of the history of the FPTAS for these problems.

The first published FPTAS for the \textsc{Knapsack} is due to Ibarra and Kim \cite{39}. This naturally gives approximations for the \textsc{Subset Sum} and \textsc{Partition} as special cases. In their approach, the items were partitioned into large and small classes. The profits are scaled down and then the problem is solved optimally with dynamic programming. Finally, the remaining empty space is filled up greedily with the small items. This algorithm has a complexity $O(n/\varepsilon^2)$ and requires $O(n+1/\varepsilon^3)$ space. Lawler \cite{50} proposed a different method of scaling and obtained $O(n+1/\varepsilon^4)$ running time.\textsuperscript{2}

\textsuperscript{1} As is common for \textsc{Knapsack}-type problems, the $\tilde{O}$ notation hides terms poly-logarithmic in $n$ and $1/\varepsilon$, but not in $t$.

\textsuperscript{2} In \cite{45, Section 4.6} there are claims, that \cite{1973Karp} Karp \cite{44} also gives $O(n/\varepsilon^2)$ approximation for \textsc{Subset Sum}.
Table 1: Brief history of FPTAS for Knapsack-type problems. Since Partition is a special case of Subset Sum, and Subset Sum is a special case of Knapsack, an algorithm for Knapsack also works for Subset Sum and Partition. We omit redundant running time factors for clarity, e.g., [46] actually runs in $\tilde{O}(\min\{n/\varepsilon, n+1/\varepsilon^2\})$ time but [32, 29] gave $O(n/\varepsilon)$ algorithm earlier. For a more robust history see [47, Section 4.6]. A star (*) marks the papers that match the previous best $\tilde{O}((n+1/\varepsilon)^2)$ complexity for Partition problem.

| Running Time | Problem     | Reference |
|--------------|-------------|-----------|
| $O(n^2/\varepsilon)$ | Knapsack    | [15, 47]  |
| $O(n/\varepsilon^2)$ | Knapsack    | [39, 44]  |
| $O(n/\varepsilon)$ | Subset Sum  | * [32, 29] |
| $O(n+1/\varepsilon^4)$ | Knapsack    | [50]      |
| $O(n+1/\varepsilon^2)$ | Partition   | * [31]    |
| $O(n+1/\varepsilon^3)$ | Subset Sum  | [30]      |
| $O(n+1/\varepsilon^2)$ | Subset Sum  | * [46]    |
| $O(n+1/\varepsilon^{12/5})$ | Knapsack    | * [19]    |
| $O(n+1/\varepsilon^{5/3})$ | Partition   | This Paper |

Later, Gens and Levner [32, 29] obtained an $O(n/\varepsilon)$ algorithm for the Subset Sum based on a different technique. Then, in 1980 they proposed an even faster $O(\min\{n/\varepsilon, n+1/\varepsilon^2\})$ algorithm [31] for the Partition. To the best of our knowledge this algorithm remained the best (until this paper). Subsequently, Gens and Levner [30] managed to generalize their result to Subset Sum with an increase of running time and obtained $O(\min\{n/\varepsilon, n+1/\varepsilon^3\})$ time and $O(\min\{n/\varepsilon, n+1/\varepsilon^2\})$ space algorithm [30]. Finally, Kellerer et al. [46] improved this algorithm for Subset Sum by giving $O(\min\{n/\varepsilon, n+1/\varepsilon^2\log(1/\varepsilon)\})$ time and $O(n+1/\varepsilon)$ space algorithm. This result matched (up to the polylogarithmic factors) the running time for Partition.

For the Knapsack problem Kellerer and Pferschy [45] gave an $O(n \min\{\log n, \log (1/\varepsilon)\} + 1/\varepsilon^2 \log(1/\varepsilon) \min\{n, 1/\varepsilon \log (1/\varepsilon)\})$ time algorithm (note that the exponent in the parameter $(n+1/\varepsilon)$ is 3 here) and for Unbounded Knapsack Jansen and Kraft [40] gave an $O(n+1/\varepsilon^2 \log^2 (1/\varepsilon))$ time algorithm (the exponent in $(n+1/\varepsilon)$ is 2, see Appendix C for the definition of Unbounded Knapsack). Very recently Chan [19] presented the currently best $\tilde{O}(n+1/\varepsilon^{12/5})$ algorithm for the Knapsack.

### 1.3 Our Contribution

Our main result is the design of the mechanism that allows us to merge the pseudo-polynomial time algorithms for Knapsack-type problems with algorithms on dense Subset Sum instances. The most noteworthy application of these reductions is the following.

**Theorem 1.4.** There is an $\tilde{O}(n+1/\varepsilon^2)$ randomized time FPTAS for Partition.

This improves upon the previous, 40 year old bound of $\tilde{O}(n+1/\varepsilon^2)$ for this problem, due to Gens and Levner [31]. Our algorithm also generalizes to a weak $(1-\varepsilon)$-approximation for Subset Sum.

**Theorem 1.5.** There is a randomized weak $(1-\varepsilon)$-approximation algorithm for Subset Sum running in $\tilde{O}(n+1/\varepsilon^2)$ time.

---

Footnote: Weak approximation can break the capacity constraint by a small factor. Definition 2.1 specifies formally what weak $(1-\varepsilon)$-approximation for Subset Sum is.
For a complete proof of these theorems see Section 5. We also present a conditional lower bound for Knapsack and Unbounded Knapsack.

**Theorem 1.6.** For any constant $\delta > 0$, an FPTAS for Knapsack or Unbounded Knapsack with $O((n + 1/\varepsilon)^{2-\delta})$ running time would refute the $(\min, +)$-convolution conjecture.

This means that a similar improvement is unlikely for Knapsack. Also, this shows that the algorithm of [40] for Unbounded Knapsack is optimal (up to polylogarithmic factors). This lower bound is relatively straightforward and follows from previous works [24, 49] and was also observed in [19]. The lower bound also applies to the relaxed, weak $(1 - \varepsilon)$-approximation for Knapsack and Unbounded Knapsack, which separates these problems from weak $(1 - \varepsilon)$-approximation approximation for Subset Sum. This result was recently extended by Bringmann [18] who showed a conditional hardness for obtaining a strong subquadratic approximation for Subset Sum, which explains why we need to settle for a weak approximation.

Lately it has been shown that the exact pseudo-polynomial algorithms for Knapsack and Unbounded Knapsack are subquadratically equivalent to the $(\min, +)$-convolution [24, 49]. Therefore, as a possible first step towards obtaining an improved FPTAS for Knapsack, we focus our attention on $(\min, +)$-convolution.

**Theorem 1.7.** $(1 + \varepsilon)$-approximate $(\min, +)$-convolution can be computed in $\tilde{O}((n/\varepsilon) \log W)$ time.

This also entails an improvement for the related Treesparcity problem (see Section 7). The best previously known algorithms for both problems worked in time $\tilde{O}((n/\varepsilon^2) \text{polylog}(W))$ (see Backurs et al. [10]).

The techniques used to improve the approximation algorithm for $(\min, +)$-convolution also apply to approximation algorithms for 3SUM. For this problem we are able to show an algorithm that matches its asymptotic lower bounds.

**Theorem 1.8.** There is a deterministic algorithm for $(1 + \varepsilon)$-approximate 3SUM running in time $\tilde{O}((n + 1/\varepsilon) \text{polylog}(W))$.

**Theorem 1.9.** Assuming the Strong-3SUM conjecture, there is no $\tilde{O}((n + 1/\varepsilon^{1-\delta}) \text{polylog}(W))$ algorithm for $(1 + \varepsilon)$-approximate 3SUM, for any constant $\delta > 0$.

For proofs of these theorems and detailed running times see Sections 8.

### 1.4 Organization of the Paper

In the Section 2 we present the building blocks of our framework and a sketch of the approximation scheme for Partition. Section 3 contains the notation and preliminaries, and the main proof is divided into Sections 4 and 5. In Sections 6 and 7 we present the algorithms for $(\min, +)$-convolution and Treesparcity. In the Section 8 we present the algorithms for 3SUM. The proofs of technical lemmas can be found in Appendix A and Appendix B. In Appendix C we give formal definitions of all problems.

### 2 Connecting Dense, Pseudo-polynomial and Approximation Algorithms for KNAPSACK-type problems: An Overview

In this section we describe main building blocks of our framework. We also briefly discuss the recent advances in the pseudo-polynomial algorithms for SUBSET SUM and discuss how to use them. Then,
we explain the intuition behind the trade-off we exploit and give a sketch of the main algorithm. The formal arguments are located in Section 5.

Difficulties with Rounding for Subset Sum

There is a strong connection between approximation schemes and pseudo-polynomial algorithms [59]. For example, a common theme in approximating knapsack is to reduce the range of the values (while keeping the weights intact) and then apply a pseudo-polynomial algorithm. Rounding the weights would be tricky because of the hard knapsack constraint. In particular, if one rounds the weights down, some feasible solutions to the rounded instance might correspond to infeasible solutions in the original instance. On the other hand, when rounding up, some feasible solutions might become infeasible in the rounded instance.

Recently, new pseudo-polynomial algorithms have been proposed for Subset Sum (see Koiliaris and Xu [48] and Bringmann [17]). A natural idea is to use these to design an improved approximation scheme for Subset Sum. However, this seems to be difficult due to rounding issues discussed above. After this paper was announced, Bringmann [18] explained this difficulty by giving a conditional lower bound on a quadratic approximation of Subset Sum.

2.1 Weak Approximation for Subset Sum and Application to Partition

Because of these rounding issues, it seems hard to design a general rounding scheme that, given a pseudo-polynomial algorithm for Subset Sum, produces an FPTAS for Subset Sum. What we can do, however, is to settle for a weaker notion of approximation.

**Definition 2.1 (Weak apx for Subset Sum).** Let \( Z^* \) be the optimal value for an instance \((Z, t)\) of Subset Sum. Given \((Z, t)\), a weak \((1 - \varepsilon)\)-approximation algorithm for Subset Sum returns \(Z^H\) such that \((1 - \varepsilon)Z^* \leq Z^H < (1 + \varepsilon)t\).

Compared to the traditional notion of approximation, here we allow a small violation of the packing constraint. This notion of approximation is interesting in itself. Indeed, it has been already considered in the stochastic regime for Knapsack [16].

Before going into details of constructing the weak \((1 - \varepsilon)\)-approximation algorithms for the Subset Sum, let us establish a relationship with the approximation for the Partition.

**Corollary 2.2.** If we can weakly \((1 - \varepsilon)\)-approximate Subset Sum in time \(\widetilde{O}(T(n, \varepsilon))\), then we can \((1 - \varepsilon)\)-approximate Partition in the same \(\widetilde{O}(T(n, \varepsilon))\) time.

This is because of the symmetric structure of Partition problem: If a subset \(Z'\) violates the hard constraint \(t \leq \Sigma(Z') \leq (1 + \varepsilon)t\), then the set \(Z - Z'\) is a good approximation and does not violate it (recall that in Partition problem we always have \(t = \Sigma(Z)/2\)). For a formal proof see Section A.

2.2 Constructing Weak Approximation Algorithms for Subset Sum: A Sketch

**Fact 2.3.** Given an \(\widetilde{O}(T(n, t))\) exact algorithm for Subset Sum, we can construct a weak \((1 - \varepsilon)\)-approximation algorithm for Subset Sum working in time \(\widetilde{O}(T(n, \frac{t}{2\varepsilon}))\).

Proof. We assume that the exact algorithm for the Subset Sum works also for multisets. We will address this issue in more detail in Section 4.1.
Let $Z = \{v_1, \ldots, v_n\}$ and $t$ constitute a Subset Sum instance. Let $I$ be the set of indices of elements of some optimal solution, and let $OPT$ be their sum. Let us also introduce a scaled approximation parameter $\varepsilon' = \frac{\varepsilon}{4}$.

Let $k = \frac{2\varepsilon'}{t}$. Define a rounded instance as follows: the (multi)-set of $\tilde{Z}$ contains a copy of $\tilde{v}_i = \left\lfloor \frac{v_i}{k} \right\rfloor$ for each $i \in \{1, \ldots, n\}$, and $\tilde{t} = \left\lfloor \frac{t}{k} \right\rfloor$.

Apply the exact algorithm $A$ to the rounded instance $(\tilde{Z}, \tilde{t})$. Let $I'$ be the set of indices of elements of the solution found.

We claim that $\{v_i : i \in I'\}$ is a weak $(1 - \varepsilon)$ approximation for $Z$ and $t$. First let us show that this solution is not much worse than $OPT$:

$$\sum_{i \in I'} v_i \geq k \sum_{i \in I'} \tilde{v}_i \geq k \sum_{i \in I} \left\lfloor \frac{v_i}{k} \right\rfloor \geq \sum_{i \in I} (v_i - k) \geq OPT - nk = OPT - 2\varepsilon't \geq OPT(1 - \varepsilon).$$

The last inequality holds because we can assume $OPT \geq t/2$ (see Section 4.3 for details).

Similarly, we can show that this solution does not violate the hard constraint by too much:

$$\sum_{i \in I'} v_i \leq \sum_{i \in I'} (k\tilde{v}_i + k) \leq nk + k \sum_{i \in I'} \tilde{v}_i \leq nk + \tilde{t}k \leq nk + k + t \leq 3\varepsilon't + t \leq t(1 + \varepsilon).$$

Finally, since the exact algorithm is applied to a (multi)-set of $n$ items with $\tilde{t} = \left\lfloor \frac{t}{k} \right\rfloor = \left\lfloor \frac{t}{2\varepsilon'} \right\rfloor$, the resulting algorithm runs in the claimed time.

We state the above proof only to give the flavour of the basic form of reductions in this paper. Usually reductions that we will consider are more complex for technical reasons. One thing to note in particular is that the relation between $k$ and $\varepsilon$ is dictated by the fact, that there may be as many as $n$ items in the optimal solution. Given some control over the solution size, one can improve this reasoning (see Lemma 4.7).

### 2.3 Approximation via Pseudo-polynomial time Subset Sum algorithm

Currently, the fastest pseudo-polynomial algorithm for Subset Sum runs in time $\tilde{O}(n + t)$, randomized. $S(Z, t)$ denotes the set of all possible subsums of set $Z$ up to integer $t$ (see Section 3).

**Theorem 2.4** (Bringmann [17]). There is a randomized, one-sided error algorithm with running time $O(n + t \log t \log^3 \frac{n}{\varepsilon} \log n)$, that returns a set $Z' \subseteq S(Z, t)$, containing each element from $S(Z, t)$ with probability at least $1 - \delta$.

This suffices to solve Subset Sum exactly with high probability. Here $S(Z, t)$ is represented by a binary array which for a given index $i$ tells whether there is a subset that sums up to $i$ (see Section 3 for a formal definition). For our trade-off, we actually need a probabilistic guarantee on all elements of $S(Z, t)$ simultaneously. Fortunately, this kind of bound holds for this algorithm as well (see [24, Appendix B.3.2] for detailed analysis).

**Corollary 2.5.** There is a randomized $\tilde{O}(n + t)$ algorithm that computes $S(Z, t)$ with a constant probability of success.
The first case where this routine comes in useful occurs when all items are in the range \([\gamma t, t]\) (think of \(\gamma\) as a trade-off parameter set to \(\varepsilon^{-2/3}\)). Note, that any solution summing to at most \(t\) can consist of at most \(1/\gamma\) such elements. This observation allows us to round the elements with lower precision and still maintain a good approximation ratio, as follows:

\[
v'_i = \left\lfloor \frac{2v_i}{\gamma \varepsilon t} \right\rfloor, \quad t' = \left\lfloor \frac{2t}{\gamma \varepsilon t} \right\rfloor = \left\lfloor \frac{2}{\gamma \varepsilon} \right\rfloor.
\]

Bringmann’s [17] algorithm on the rounded instance runs in time \(\tilde{O}(n + t') = \tilde{O}(n + \frac{1}{\gamma \varepsilon})\) and returns an array of solutions with an additive error \(\pm \varepsilon t\) with high probability (see Lemma 5.1). Similar reasoning about sparseness also applies if the number of items is bounded (i.e., when \(n = \tilde{O}(\frac{1}{\varepsilon})\)). In that case Bringmann’s [17] algorithm runs in time \(\tilde{O}(\frac{1}{\varepsilon})\) and provides the same guarantees (see Lemma 5.2 and also the next section).

### 2.4 Approximation via Dense Subset Sum

Now we need a tool to efficiently solve the instances where all items are in range \([0, \gamma t]\), so-called dense instances. More formally, an instance consisting of \(m\) items is dense if all items are in the range \([1, m^{O(1)}]\). Intuitively, rounding does not work well for these instances since it introduces large rounding errors. On the other hand, if an instance contains many distinct numbers on a small interval, one can exploit its additive structure.

**Theorem 2.6** (Galil and Margalit [27]). Let \(Z\) be a set of \(m\) distinct numbers in the interval \((0, \ell]\) such that

\[
m > 1000 \cdot \sqrt{\ell} \log \ell,
\]

and let 

\[L := \frac{100 \cdot \Sigma(Z) \ell \log \ell}{m^2}.
\]

Then in \(O(m + ((\ell/m) \log \ell)^2)\) preprocessing time we can build a structure that can answer the following queries in constant time. In a query the structure receives a target number \(t \in (L, \Sigma(Z) - L)\) and decides whether there is a \(Z' \subseteq Z\) such that \(\Sigma(Z') = t\). The structure is deterministic.

In fact we will use a more involved theorem that can also construct a solution in \(O(\log(\ell))\) time but we omit it here to keep this section relatively free of technicalities (see Section 5.2 for a discussion regarding these issues).

Observe that \(L = \tilde{O}(\ell^{1.5})\) (because \(\Sigma(Z) < m \ell\)) and the running time is bounded by \(\tilde{O}(m + \ell)\) (because \(\ell/m = \tilde{O}(\sqrt{\ell})\)). We will apply this result for the case \(\ell = \gamma t\) (see Lemma 5.5). Recall, that Bringmann’s [17] algorithm runs in time \(\tilde{O}(m + t)\), which would be slower by the factor \(\gamma\) (the trade-off parameter). For simplicity, within this overview we will assume, that Theorem 2.6 provides a data structure that can answer queries with the target numbers in \([0, \Sigma(Z)]\). In the actual proof, we need to overcome this obstacle, by merging this data structure with other structures, responsible for targets near the boundary, which we call marginal targets (see Lemma 5.3).

Suppose our instance consists of \(m\) elements in the range \([0, \gamma t]\). We use the straightforward rounding scheme, as in the proof of Fact 2.3:

\[
v'_i = \left\lfloor \frac{2mv_i}{\varepsilon t} \right\rfloor, \quad t' = \left\lfloor \frac{2mt}{\varepsilon t} \right\rfloor = \left\lfloor \frac{2m}{\varepsilon} \right\rfloor.
\]
We chose $\gamma t$ as the upper bound on item size, so that $\ell' = m\gamma/\varepsilon$ is an upper bound on $v'_i$. Now, if the number of items satisfies the inequality $\ell' < m^2$, then we can use the Theorem 2.6 with running time $\tilde{O}(m + \ell') = \tilde{O}(m + m\gamma/\varepsilon)$. This provides a data structure that can answer queries from the range that is of our interest (for a careful proof see Section 5).

Still, it can happen that most of the items are in the sparse instance (i.e., $\ell' \geq m^2$) and we cannot use the approach from [27]. In that case we use Theorem 2.4 again, with running time $\tilde{O}(m + \frac{\gamma}{\varepsilon^2})$ (see Lemma 5.2).

In the end, we are able to compute an array of solutions, for items in range $[0, \gamma t]$ in time $\tilde{O}(m + \gamma \varepsilon^2 + \frac{m^2}{\varepsilon^2})$ with additive error $\pm \varepsilon t$ and high probability (see Lemma 5.6). The last term in time complexity comes from handling the marginal queries.

### 2.5 A Framework for Efficient Approximation

In this section we will sketch the components of our mechanism (see Algorithm 1). The mechanism combines pseudo-polynomial Bringmann’s [17] algorithm with Galil and Margalit [27] algorithm for dense instances of Subset Sum.

#### Algorithm 1 Roadmap for the weak $(1-\varepsilon)$-approximation for Subset Sum. Input: item set $Z, t, \varepsilon$

1: ensure $OPT \geq t/2$
2: reduce $|Z|$ to $\tilde{O}(1/\varepsilon)$
3: repeat
4: partition items into $Z_{\text{large}}$ and $Z_{\text{small}}$
5: divide $[0, \gamma t]$ into $\ell = O(\gamma \log(n)/\varepsilon) \cdot |Z_{\text{small}}|$ segments
6: round down small items
7: remove item repetitions in $Z_{\text{small}}$
8: until $\ell = O(\gamma \log(n)/\varepsilon) \cdot |Z_{\text{small}}|$
9: build a data structure for large items
10: if $|Z_{\text{small}}| = \tilde{O}(\sqrt{\ell})$ then
11: build a data structure for small items
12: else
13: build data structures for marginals
14: exploit the density of the instance to cover the remaining case
15: end if
16: merge the data structures for large and small items

We begin by reducing the number of items in the instance $Z$ to roughly $\tilde{O}(1/\varepsilon)$ items to get a near linear running time (see Lemma 4.2). After that our goal is to divide items into small and large and process each part separately, as described earlier.

However, Theorem 2.6 requires a lower bound on the number of distinct items. To control this parameter, we merge identical items into larger ones, until each item appears at most twice. However, this changes the number of items, and so the procedure might have to be restarted. Lemma 4.4 guarantees that we require at most $\log n$ such refinement steps.

In the next phase we decide which method to use to solve the instance depending on its density (line 10). We encapsulate these methods into data structures (lines 11-14). Finally we will need to merge the solutions. For this task we introduce the concept of membership oracles (see Defi-
nition [4,5] that are based on FFT and backtracking to retrieve solutions (see Lemma 4.6). The simplified trade-off schema is presented on the Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{schema.png}
\caption{Overall schema of trade-off and usage of building blocks. The parameter $m$ denotes number of items in the dense instance, $n$ is the number of all elements, $\gamma$ is the trade-off parameter, $\ell$ is the upper bound on the item size after rounding, $t$ is the target sum. The buckets in the sparse/dense instance depict the rounding scheme for small and large items.}
\end{figure}

The final running time of our framework is $\tilde{O}(n + 1/\gamma\varepsilon + \gamma\varepsilon^2 + \gamma^2/\varepsilon^3)$ with high probability for any $\gamma(n, \varepsilon) > 0$ (see Lemma 5.9). For $\gamma = \varepsilon^{-2/3}$, this gives us an $\tilde{O}(n + \varepsilon^{-5/3})$ time weak $(1 - \varepsilon)$-approximation approximation for SUBSET SUM.

\section{Preliminaries}

For a finite multiset $Z \subset \mathbb{N}$ we denote its size as $|Z|$, the number of distinct elements as $||Z||$, and the sum of its elements as $\Sigma(Z)$.

For a number $x$ we define $\text{pow}(x)$ as the largest power of 2 not exceeding $x$. If $x < 2$ we set $\text{pow}(x) = 1$.

For sets $A, B \subset \mathbb{N}$ their bounded algebraic sum $A \oplus_t B$ is a set $\{a + b : a \in \{0\} \cup A, b \in \{0\} \cup B\} \cap [0, t]$.

\begin{definition}[Subsums] For a finite multiset $Z \subset \mathbb{N}$ we define $S(Z)_k$ as a set of all possible subset sums of $Z$ of size at most $k$, i.e., $x \in S(Z)_k$ iff there exists $S' \subseteq Z$, such that $\Sigma(S') = x$ and $|S'| \leq k$. $S(Z)$ is the set without the constraint on the size of the subsets, i.e., $S(Z) := S(Z)_\infty$.

The capped version is defined as $S(Z, t)_k := S(Z)_k \cap [0, t]$ and $S(Z, t) := S(Z) \cap [0, t]$.

We call two multisets $Z_1, Z_2 \subset \mathbb{N}$ equivalent if $S(Z_1) = S(Z_2)$.

Note that $0 \in S(Z, t)_k$ for all sets $Z$ and $t, k > 0$.
\end{definition}

\begin{definition}[$(\varepsilon, t)$-closeness] We say that set $B$ is $(\varepsilon, t)$-close to $A$ if there is a surjection $\phi : A \to B$ such that $x - \varepsilon t \leq \phi(x) \leq x + \varepsilon t$. A SUBSET SUM instance $(Z_2, t)$ is $\varepsilon$-close to $(Z_1, t)$ if $S(Z_2, t)$ is $(\varepsilon, t)$-close to $S(Z_1, t)$.

Sometimes, when there is no other notation on $t$, we will use the notion of $\varepsilon$-closeness as a $(\varepsilon, t)$-close.

Usually the surjection from the definitions will come by rounding down the item sizes and each item set will get a moderately smaller total size. We will also apply the notion of $(\varepsilon, t)$-closeness to binary arrays having in mind the sets they represent.


Fact 3.3. If $A$ is $(\varepsilon, t)$-close to $S(Z_1, t)$ and $B$ is $(\varepsilon, t)$-close to $S(Z_2, t)$ then $A \oplus_t B$ is $(2\varepsilon, t)$-close to $S(Z_1 \cup Z_2, t)$

We will also need to say, that there are no close elements in a set. It will come in useful to show, that after rounding down all the elements are distinct.

Definition 3.4 ((x)-distinctness). The set $S$ is said to be $(x)$-distinct if every interval of length $x$ contains at most one item from $S$. The set $S$ is said to be $(x, 2)$-distinct if every interval of length $x$ contains at most two items from $S$.

4 Preprocessing

This section is devoted to simplify the instance of Subset Sum in order to produce a more readable proof of the main algorithm. In here we will deal with:

- multiplicities of the items,
- division of the instance into large and small items,
- proving that rounding preserves $\varepsilon$-closeness,
- reducing a number of items from $n$ to $\tilde{O}(1/\varepsilon)$ items.

The solutions to these problems are rather technical and well known in the community \cite{46, 47, 48, 17}. We include it in here because these properties are used in approximation algorithms \cite{47, 46} and exact pseudo-polynomial algorithms \cite{48, 17} communities separately. We expect that reader may not be familiar with both of these technical toolboxes simultaneously and accompany this section with short historical references and pointers to the original versions of proofs.

4.1 From Multisets to Sets

The general instance of Subset Sum may consists of plenty of items with equal size. Intuitively, these instances seem to be much simpler than instances where almost all items are different. The next lemma will allow us to formally capture this intuition with the appropriate reduction. This lemma was proposed in \cite[Lemma 2.2]{48} but was also used in \cite{17}.

Lemma 4.1 (cf. Lemma 2.2 from \cite{48}). Given a multiset $S$ of integers from $\{1, \ldots, t\}$, such that $|S| = n$ and the number of distinct items $||S||$ is $n'$, one can compute, in $O(n \log n)$ time, a multiset $T$, such that:

- $S(S, t) = S(T, t)$
- $|T| \leq |S|$
- $|T| = O(n' \log n)$
- no element in $T$ has multiplicity exceeding two.
Proof. We follow the proof from [48, Lemma 2.2], however the claimed bound on $|T|$ is only $O(n' \log t)$ therein. Consider an element $x$ with the multiplicity $2k + 1$. We can replace it with a single copy of $x$ and $k$ copies of $2x$ while keeping the multiset equivalent. If the multiplicity is $2k + 2$ we need 2 copies of $x$ and $k$ copies of $2x$. We iterate over items from the smallest one and for each with at least 3 copies we perform the replacement as described above. Observe that this procedure generates only elements of form $2^i x$ where $i \leq \log n$ and $x$ is an element from $S$. This yields the bound on $|T|$. The routine can be implemented to take $O(\log n)$ time for creating each new item using tree data structures. \hfill \Box

4.2 From n Items to $\tilde{O}(1/\varepsilon)$ Items

To reduce number of items $n$ to $\tilde{O}(1/\varepsilon)$ Kellerer et al. [46] gave a very intuitive construction that later found applications in Knapsack-type problems [47].

Intuitively, rounding scheme described in Section 2 could divide the items into $O(n/\varepsilon)$ intervals and this would result with an $\varepsilon$-close instance to the original one. In here we start similarly but we want to get rid of factor $O(n)$. We divide an instance to $k = \lceil \frac{n}{\varepsilon} \rceil$ intervals of length $\varepsilon t$, i.e., $I_j := (jt, (j + 1)t]$. Next notice that for interval $I_j$ we do not need to store more than $O(\lceil \frac{n}{\varepsilon t} \rceil)$ items, because their sum would exceed $t$ (this is the step where $\varepsilon$ factor will come in). Finally, the number of items is upper bounded (up to the constant factors):

$$\sum_{j=1}^{k} \left\lfloor \frac{k}{j} \right\rfloor \leq k \sum_{j=1}^{k} \frac{1}{j} < k \log k = O(1/\varepsilon \log (1/\varepsilon))$$

The last inequality is just an upper bound on harmonic numbers. This was a very informal sketch of the proof of [46] construction to give some intuition. The next technical lemma is based on their trick.

Lemma 4.2. Given a Subset Sum instance $(Z, t)$, $|Z| = n$, one can find an $\varepsilon$-close instance $(Z_2, t)$ such that $|Z_2| = O\left(\frac{1}{\varepsilon} \log(\frac{n}{t}) \log(n)\right)$. The running time of this procedure is $O(|Z| + |Z_2|)$.

Proof. We begin with constructing $Z_1$ as follows. For $i = 1, \ldots, \log(\frac{2n}{t})$ we round down each element in $Z \cap (\frac{t}{2^i}, \frac{t}{2^{i-1}})$ to the closest multiplicity of $\lceil \frac{t}{2^i} \rceil$. We neglect elements smaller than $\frac{t}{2n}$. Observe that $||Z_1|| = O(\frac{1}{\varepsilon} \log(\frac{n}{t}))$.

We argue that $(Z_1, t)$ is $\varepsilon$-close to $(Z, t)$. To see this, consider any subset $I \subseteq Z$ summing to at most $t$ and its counterpart $Y_1 \subseteq Z_1$. We lose at most $n \cdot \frac{\varepsilon t}{2n} = \frac{\varepsilon t}{2}$ by omitting items smaller than $\frac{t}{2n}$. Let $k_i = |I \cap (\frac{t}{2^i}, \frac{t}{2^{i-1}})|$ and $t_i$ denote the sum of elements in $I \cap (\frac{t}{2^i}, \frac{t}{2^{i-1}})$. Since each element in $[\frac{t}{2^i}, \frac{t}{2^{i-1}})$ has been decreased by at most $\frac{\varepsilon t}{2n}$ and $k_i \cdot \frac{t}{2^i} \leq t_i$, we have

$$\Sigma(I) - \Sigma(Y_1) \leq \frac{\varepsilon t}{2} + \sum_{i=1}^{\log(2n)/t} k_i \cdot \frac{\varepsilon t}{2^{i+1}} \leq \frac{\varepsilon t}{2} + \sum_{i=1}^{\log(2n)/t} \frac{\varepsilon t_i}{2} \leq \varepsilon t.$$

In the end we take advantage of Lemma 4.1 to transform $Z_1$ into an equivalent multiset $Z_2$ such that $|Z_2| \leq ||Z_1|| \log(||Z_1||) = O\left(\frac{1}{\varepsilon} \log(\frac{n}{t}) \log(n)\right)$. \hfill \Box

Note, that we discarded items smaller than $\frac{t}{2n}$. We do this because sum of these elements is just too small to influence the worst case approximation factor. We do not consider them just for the simplicity of analysis. To make this algorithm competitive in practice, one should probably just greedily add these small items to get a little better solution.
4.3 From One Instance to Small and Large Instances

First we need a standard technical assumption, that says that we can cheaply transform an instance to one with a lower bounded solution. We will need it just to simplify the proofs (e.g., it will allow us to use Lemma 4.2 multiple times).

**Lemma 4.3.** One may assume w.l.o.g. that for any **Subset Sum** instance OPT $\geq \frac{1}{2}$.

**Proof.** Let us remove from the item set $Z$ all elements exceeding $t$ since they cannot belong to any solution. If $\Sigma(Z) \leq t$ then the optimal solution consists of all items. Otherwise consider a process in which $Y_1 = Z$ and in each turn we obtain $Y_{k+1}$ by dividing $Y_k$ into two arbitrary non-empty parts and taking the one with a larger sum. We terminate the process when $Y_{last}$ contains only one item. Since $\Sigma(Y_1) > t$, $\Sigma(Y_{last}) \leq t$, and in each step the sum decreases by at most factor two, for some $k$ it must be $\Sigma(Y_k) \in [\frac{1}{2}, t]$. Because there is a feasible solution of value at least $\frac{1}{2}$, OPT cannot be lower.

One of the standard ways of solving **Subset Sum** is to separate the large and small items. Usually these approximations consider items greater and smaller than some trade-off parameter. Our techniques require a bound on the multiplicities of small items, which is provided by the next lemma.

**Lemma 4.4** (Partition into Small / Large Items). Given an instance $(Z, t)$ of **Subset Sum**, an approximation factor $\varepsilon$, and a trade-off parameter $\gamma$, one can deterministically transform the instance $(Z, t)$, in time $O(n \log^2 n)$, to an $\varepsilon$-close instance $(Z_{small} \cup Z_{large}, t)$ such that:

- $\forall z_s \in Z_{small}$, it holds that $z_s < \gamma t$,
- $\forall z_l \in Z_{large}$, it holds that $z_l \geq \gamma t$,
- The set $Z_{small}$ is $(\frac{\varepsilon t}{m \log n}, 2)$-distinct where $m = O(|Z_{small}|)$, i.e., after rounding there can be at most 2 occurrences of each item.

**Proof.** We call an item $x$ large if $x \geq \gamma t$ and small otherwise. Let $Y_0$ be the initial set of small items and $m_0 = |Y_0|$, $q_0 = \text{pow}(\frac{\varepsilon t}{m_0 \log n})$. We round down the size of each small item to the closest multiplicity of $q_0$. Then we apply Lemma 4.1 to the set of small items to get rid of items with 3 or more copies. Note that this operation might introduce new items that are large. We obtain a new set of small items $Y_1$ and repeat this procedure with notation $m_i = |Y_i|$, $q_i = \text{pow}(\frac{\varepsilon t}{m_i \log n})$. It holds that $m_{i+1} \leq m_i$ and $q_i \mid q_{i+1}$. We stop the process when $m_{i+1} \geq \frac{m_i}{2}$, which means there can be at most $\log n$ iterations. Let $m$ denote the final number of small items and $q \geq \frac{\varepsilon t}{4m \log n}$ – the last power of 2 used for rounding. All small items now occur with multiplicities at most 2.

Let us fix $Z_{small}$ as the set of small items after the modification above and likewise $Z_{large}$. In the $i$-th rounding step values of $m_i$ items are being decreased by at most $\frac{\varepsilon t}{m_i \log n}$, so each new instance is $\frac{\varepsilon}{\log n}$-close to the previous one. There are at most $\log n$ steps and the removal of copies keeps the instance equivalent, therefore $(Z_{small} \cup Z_{large}, t)$ is $\varepsilon$-close to $(Z, t)$.

Our algorithm works independently on these two instances and produces two arrays $\varepsilon$-close to them. The construction below allows us to join these solutions efficiently. We want to use them even if we have only access to them by queries. We formalize this as an $(\varepsilon, t)$-membership-oracle. The asymmetry of the definition below will become clear in Lemma 4.7.
Lemma 4.6 (Merging solutions). Given $S(Z_1, t)$ and $S(Z_2, t)$ as $(\epsilon, t)$-membership-oracles

1. if $X$ contains an element in $[q - \epsilon t, q + \epsilon t]$, then the answer is yes,

2. if the answer was yes, then $X$ contains an element in $[q - 2\epsilon t, q + 2\epsilon t]$.

A query to the oracle takes $\tilde{O}(1)$ time. Moreover, if the oracle answers yes, then it can return a witness $x$ in $O(1)$ time.

Below we present an algorithm that can efficiently join the solutions. We assume, that we have only query-access to them and want to produce an $(\epsilon, t)$-membership-oracle of the merged solution.

Lemma 4.6 (Merging solutions). Given $S(Z_1, t)$ and $S(Z_2, t)$ as $(\epsilon, t)$-membership-oracles

- $S_1$ that is $(\epsilon, t)$-close instance to $S(Z_1, t)$,
- $S_2$ that is $(\epsilon, t)$-close instance to $S(Z_2, t)$,

we can, deterministically in $\tilde{O}(\frac{1}{\epsilon})$ time, construct a $(2\epsilon, t)$-membership-oracle for $S(Z_1 \cup Z_2, t)$.

Proof. For an ease of presentation, only in this proof we will use interval notation of inclusion, i.e.,

we will say that $[a, b] \cap A$ iff $\exists x \in (a, b) \wedge x \in A$. Let $p = O(\epsilon t)$. For each interval $(ip, (i + 1)p]$ where $i \in \{0, \ldots, \lfloor \frac{1}{\epsilon t} \rfloor \}$ we query oracles whether $S(Z_1, t)$ and $S(Z_2, t)$ contain some element in the interval, having in mind that the answer is approximate. The number of intervals is $O(\frac{1}{\epsilon})$.

We store the answers in arrays $S_1$ and $S_2$, namely $S_j[i] = 1$ if the oracle for $S(Z_j, t)$ answers yes for interval $(ip, (i + 1)p]$.

\[
S'_j[i] = \begin{cases} 1 & \text{if the oracle for } (ip, (i + 1)p] \cap S_j \text{ or } i = 0 \\ 0 & \text{otherwise} \end{cases}
\]

Then we perform a fast convolution on $S_1, S_2$ with FFT.

If $x \in S(Z_1 \cup Z_2, t) \cap (kp, (k + 1)p]$, then there is some $x_1 \in S(Z_1, t)$ and $x_2 \in S(Z_2, t)$ such that $x = x_1 + x_2$. We have $(S_1 \oplus_{\text{FFT}} S_2)[k] = \sum_{i=0}^{k} S_1[i] \cdot S_2[k - i]$ and thus $(S_1 \oplus_{\text{FFT}} S_2)[k']$ is nonzero for $k' = k$ or $k' = k + 1$. This defines the rule for the new oracle. The additive error of the oracle gets doubled with the summation. On the other hand, if one of these fields is nonzero, then there are corresponding indices $i_1, i_2$ summing to $k$ or $k + 1$. The second condition from Definition 4.5 allows the corresponding value $x_1$ to lie within one of the intervals with indices $i_1 - 1, i_1$, or $i_1 + 1$ and likewise for $x_2$. Therefore, the additive error is $O(p) = O(\epsilon t)$.

iff there is $i$ such that $(ip, (i + 1)p] \cap S_1 \cup \{0\}$ and $((k - i)p, (k - i + 1)p] \cap S_2 \cup \{0\}$.

Now, we promised only oracle output to our array. When a query comes, we scale down the query interval, then we check whether any of adjacent interval in our structure is nonzero (we lose a constant factor of $O(\epsilon)$ accuracy here) and output yes if we found it and no otherwise.

Moreover, with additional polylogarithmic factors we can also retrieve the solution. The idea is similar to backtracking from [46]. Namely, the fast convolution algorithm can compute the table of witnessing indexes (of only one). We store a witnessing index if there is solution and $-1$ otherwise. Then we ask the oracles of $S(Z_1, t)$ and $S(Z_2, t)$ for a solution with a proper indexes and return the combination of those.
4.4 From Exact Solution to $\varepsilon$-close Instance

In Section 2 we presented an overall approach of rounding elements and explained why it gives us the weak approximation of Subset Sum. Here we will focus on formally proving these claims.

In our subroutines, we round down the items, execute the exact algorithm on the rounded instance, and retrieve the solution. We want to argue, that in the end we lose only an additive factor of $\pm \varepsilon t$. We presented a sketch of this reasoning in Fact 2.3. For our purposes we will describe the procedure in the case, when the number of items in any solution is bounded by $k$ (i.e., we are interested only in $S(Z,t)_{k}$). We can always assume $k \leq n$.

Lemma 4.7. Given an exact algorithm that outputs the set $S(Z,t)_{k}$ and works in time $T(n,t)$, where $n = |Z|$, we can construct an $(\varepsilon, t)$-membership-oracle of set $S(Z,t)_{k}$ in time $\tilde{O}(n + T(n,k/\varepsilon))$.

If the exact algorithm retrieves solution in $\tilde{O}(1)$ time, then so does the oracle.

Proof. For sake of legibility, we assume that we are interested in $S(Z,t)_{k-1}$ - this only allows us to write simpler formulas. Let $(z_i)$ denote the items. We perform rounding in the following way:

$$z'_i = \left\lfloor \frac{kz_i}{\varepsilon t} \right\rfloor, \quad t' = \left\lfloor \frac{kt}{\varepsilon t} \right\rfloor = \left\lfloor \frac{k}{\varepsilon} \right\rfloor.$$

We run the exact algorithm on the rounded instance $(Z',t')$. It takes time $T(n,t') = \tilde{O}(T(n,k/\varepsilon))$. This algorithm returns $S(Z',t')_{k-1}$, which we store in array $Q[1,t']$. We construct $(\varepsilon, t)$-membership-oracle in array $Q'[1,t']$ as follows: we set $Q'[i] = 1$ iff $Q$ contains 1 in range $(i-2k, i+k]$. If we want to be able retrieve a solution, we need to also remember a particular index $j(i) \in (i-2k, i+k]$ such that $Q[j(i)] = 1$. Such a data structure can be constructed in a linear time with a help of a queue. Given a query $q$, the oracle returns $Q'[q']$, where $q' = \left\lfloor \frac{kt}{\varepsilon t} \right\rfloor$. It remains to prove that Definition 4.5 is satisfied.

Let $I \subseteq Z$ be a set of at most $k-1$ items and $I'$ be the set of their counterparts after rounding. Since for all $z_i \in Z$ it holds

$$\frac{kz_i}{\varepsilon t} - 1 \leq z'_i \leq \frac{kz_i}{\varepsilon t},$$

we obtain

$$\frac{k \cdot \Sigma(I)}{\varepsilon t} - k + 1 \leq \Sigma(I') \leq \frac{k \cdot \Sigma(I')}{\varepsilon t}.$$ (1)

Therefore, if $\Sigma(I) \in [q - \varepsilon t, q + \varepsilon t]$, then

$$\frac{kq}{\varepsilon t} - 2k + 1 = \frac{k \cdot (q - \varepsilon t)}{\varepsilon t} - k + 1 \leq \Sigma(I'),$$

$$\Sigma(I') \leq \frac{k \cdot (q + \varepsilon t)}{\varepsilon t} = \frac{kq}{\varepsilon t} + k,$$

and $\Sigma(I') \in (q' - 2k, q' + k]$, because $\Sigma(I')$ is integer. On the other hand, we can invert relation (1) to obtain

$$\frac{\varepsilon t}{k} \cdot \Sigma(I') \leq \Sigma(I) \leq \frac{\varepsilon t}{k} \cdot (\Sigma(I') + k - 1).$$
To satisfy the second condition we assume \( \Sigma(I') \in (q' - 2k, q' + k] \) and check that
\[
q - 2\varepsilon t = \frac{\varepsilon t}{k} \cdot \left( \frac{kq}{\varepsilon t} - 2k \right) \leq \frac{\varepsilon t}{k} \cdot (q'-2k+1) \leq \Sigma(I),
\]
\[
\Sigma(I) \leq \frac{\varepsilon t}{k} \cdot (q' + 2k - 1) \leq q + 2\varepsilon t,
\]
what finishes the proof.

We apply Lemma B.2 with \( \{z_1, \ldots, z_k\} = Y, q = t/2 \) and \( k \) and \( \varepsilon \) as in the statement. It guarantees that:
\[
(1 - \varepsilon)\Sigma(Y) \leq \frac{\varepsilon t}{2k} \Sigma(Y').
\]

And finally, \( (1 - \varepsilon)\Sigma(Y) \geq \Sigma(Y) - \varepsilon t \) (because we are only interested in solutions smaller than \( t \)). So if \( Y \) is a optimal solution, then an exact algorithm after rounding would return something greater or equal \( \Sigma(Y) - \varepsilon t \).

Conversely, it can turn out that an exact algorithm would find something with a sum greater than \( q \) (this is where we can violate the hard constraint). We need to bound it as well (because the definition of \((\varepsilon, t)\)-membership-oracle requires that). Note, that analogous argument proves it. Namely, the solution can consist of at most \( k \) items and each of them lose only \( O(\varepsilon t/k) \). Moreover, exact oracle gave us only the solution that its rounded version sums up to exactly \( t' \). Formally, we prove it again with Lemma B.2 with the same parameters as before. By dividing both sides by \((1 - \varepsilon)\) we know that:
\[
\sum_{i=1}^{k} \left\lfloor \frac{2kz_i}{t\varepsilon} \right\rfloor = \left\lfloor k \right\rfloor \cdot \varepsilon.
\]

Once again, we can use Lemma B.2 with the same parameters (we divided both sides by \((1 - \varepsilon) > 0\)):
\[
\sum_{i=1}^{k} x_i \leq \left( \frac{1}{1 - \varepsilon} \right) \frac{\varepsilon t}{2k} \sum_{i=1}^{k} \left\lfloor \frac{2kz_i}{t\varepsilon} \right\rfloor.
\]

The right side satisfies:
\[
\left( \frac{1}{1 - \varepsilon} \right) \frac{\varepsilon t}{2k} \sum_{i=1}^{k} \left\lfloor \frac{2kz_i}{t\varepsilon} \right\rfloor = \left( \frac{1}{1 - \varepsilon} \right) \frac{\varepsilon t}{2k} \left\lfloor \frac{2k}{\varepsilon} \right\rfloor \leq \frac{1}{1 - \varepsilon} t < (1 + 2\varepsilon)t.
\]

The constant before \( \varepsilon \) does not change much since we only need \((O(\varepsilon), t)\)-membership-oracle (we can always rescale the approximation factor by setting \( \varepsilon' = \varepsilon/2 \) at the beginning).

The main obstacle with returning a solution that obeys the capacity constraint comes from the above lemma. If we could provide a reduction from an exact algorithm without widening the interval \([q - \varepsilon t, q + \varepsilon t]\), this would automatically entail a strong approximation for \textsc{Subset Sum}. This seems unlikely due to conditional hardness result for a strong subquadratic approximation for \textsc{Subset Sum} [18].

At the end, we need to prove, that an \((\varepsilon, t)\)-membership-oracle gives us the correct solution for weak \((1 - \varepsilon)\)-approximation \textsc{Subset Sum}.
Lemma 4.8. Given an \((\frac{\gamma}{\epsilon}, t)\)-membership-oracle of \(S(Z, t)\), we can read the answer to the weak \((1 - \epsilon)\)-approximation Subset Sum in time \(\tilde{O}(\frac{1}{\epsilon})\).

Proof. We query the oracle for \(q = i \cdot \frac{\epsilon t}{6}\) for \(i = 0, \ldots, \frac{6}{\epsilon}\). Each query takes time \(\tilde{O}(1)\) and if the interval \([q - \frac{\epsilon t}{6}, q + \frac{\epsilon t}{6}]\) contains an \(x \in S(Z, t)\), then the oracle returns an element within \([x - \frac{\epsilon t}{6}, x + \frac{\epsilon t}{6}]\). If \(OPT < (1 - \frac{\epsilon}{2})t\), then the oracle will return a witness within \((OPT - \frac{\epsilon t}{2}, OPT)\). Otherwise the witness might belong to \((t, (1 + \frac{\epsilon}{2})t)\).

By taking advantage of Lemma 4.3, we can assume that \(OPT \geq t/2\), therefore the relative error gets bounded with respect to \(OPT\). \(\square\)

5 The weak \((1-\epsilon)\)-approximation algorithm for Subset Sum

5.1 Large Items

We will use Theorem 2.4 to compute \(S(Z_{\text{large}}, t)\) on a large instance. On that instance, this algorithm is more efficient than Kellerer et al. \([16]\) algorithm because one can round items less aggressively.

Lemma 5.1 (Algorithm for Large Items). Given a large instance \((Z_{\text{large}}, t)\) of Subset Sum (i.e., all items are greater than \(\gamma t\)), we can construct an \((\epsilon, t)\)-membership-oracle of \(S(Z_{\text{large}}, t)\) in randomized \(\tilde{O}(n + \frac{1}{\epsilon t})\) time with a constant probability of success.

Proof. We use Bringmann’s \([17]\) algorithm, namely Corollary 2.5, that solves the Subset Sum problem exactly. Since all elements are greater than \(\gamma t\), any subset that sums up to at most \(t\) must contain at most \(\frac{1}{\gamma}\) items. The parameter \(k\) in Lemma 4.7 is an upper bound on number of elements in the solution, hence we set \(k = \frac{1}{\gamma}\). The Bringmann’s \([17]\) algorithm runs in time \(\tilde{O}(n + t)\) and Lemma 4.7 guarantees that we can build an \((\epsilon, t)\)-membership-oracle in time \(\tilde{O}(n + k/\epsilon) = \tilde{O}(n + 1/(\gamma \epsilon))\), which is what we needed. \(\square\)

5.2 Small Items

Now we need an algorithm that solves the problem for small items. As mentioned in Section 2, we will consider two cases depending on the density of instance. The initial Subset Sum instance consists of \(n\) elements. The \(m\) is the number of elements in the small instance and let \(m' = O(m \log n)\) be as in Lemma 4.4. For now we will assume, that the set of elements is \((\epsilon t/m')\)-distinct (we will deal with multiplicities 2 in Lemma 5.7).

Let \(q = \epsilon t/m'\) be the rounding parameter (the value by which we divide) and \(\ell = \gamma m'/\epsilon = O(\frac{\gamma m \log n}{\epsilon})\) be the upper bound on items’ sizes in the small instance after rounding. Parameter \(L = O(S(S) \cdot \frac{1}{m^2})\) describes the boundaries of Theorem 2.6. We deliberately use \(O\) notation to hide constant factors (note that Galil and Margalit \([27]\) algorithm requires that \(m > 1000 \cdot \sqrt{l \log l}\).

Lemma 5.2 (Small items and \(m^2 < \ell \log^2 \ell\)). Suppose we are given an instance \((Z_{\text{small}}, t)\) of Subset Sum (i.e., all items are smaller than \(\gamma t\)) with size satisfying \(m^2 < \ell \log^2 \ell\). Then we can compute \((\epsilon, t)\)-membership-oracle of \(S(Z_{\text{small}}, t)\) in randomized \(\tilde{O}(m + \frac{1}{\epsilon^2})\) time.

Proof. In here we need to deal with the case, where small instance is sparse. So just as in the proof of Lemma 5.1 we can use Bringmann’s \([17]\) algorithm.
We will use the reduction from exact to weak \((1 - \varepsilon)\)-approximation algorithm for \textsc{Subset Sum} from Lemma 4.7. We set \(m\) as the maximal number of items in the solution, as there are at most \(m\) small items. Recall that \(\ell\) is \(\tilde{O}(m\gamma/\varepsilon)\). This gives us \(m^2 = \tilde{O}(\ell) = \tilde{O}(\frac{m\gamma}{\varepsilon})\). After dividing both sides by \(m\) we obtain \(m = \tilde{O}(\frac{\gamma}{\varepsilon})\).

Combining Corollary 2.5 and Lemma 4.7 allows us to construct an \((\varepsilon, t)\)-membership-oracle in \(\tilde{O}(m + T(m, m/\varepsilon)) = \tilde{O}(m + \frac{m^2}{\varepsilon^2})\) randomized time.

Now we have to handle the harder \(m^2 \geq \ell \log^2 \ell\) case. In this situation we again consider two cases. The Galil and Margalit \cite{27} algorithm allows only to ask queries in the range \((L, \Sigma(S) - L)\) where \(L = \mathcal{O}(\Sigma(S) \cdot \frac{1}{m})\). In the next lemma we take care of ranges \([0, L]\) and \([\Sigma(S) - L, \Sigma(S)]\). We focus on the range \([0, L]\), because the sums within \([\Sigma(S) - L, \Sigma(S)]\) are symmetric to \([0, L]\).

**Lemma 5.3** (Small items, range \((0, L)\)). Given an instance \((Z_{\text{small}}, t)\) of \textsc{Subset Sum}, such that \(|Z_{\text{small}}| = m\) and the items' sizes are at most \(\gamma t\), we can compute an \((\varepsilon, t)\)-membership-oracle for \(S(Z_{\text{small}}, L)\) in time \(\tilde{O}(m + \frac{m^2}{\varepsilon^2})\).

**Proof.** We round down items with rounding parameter \(q = \varepsilon t/m = \Omega(\frac{\varepsilon t}{m \log n})\) and denote the set of rounded items as \(Z'_{\text{small}}\). After scaling down we have \(L' = \Sigma(Z'_{\text{small}}) \cdot \frac{\gamma \varepsilon}{m} \cdot t\) (note that we only replace \(\Sigma(Z_{\text{small}})\) with \(\Sigma(Z'_{\text{small}})\) and \(\ell\) remains the same). Recall that \(\ell = \mathcal{O}(\frac{m\gamma \varepsilon \log n}{\varepsilon})\).

The total sum of items in \(Z'_{\text{small}}\) is smaller or equal to \(\ell m\) (because there are \(m\) elements of size at most \(\ell\)). Hence \(L' = \mathcal{O}(\ell^2/m) = \mathcal{O}(\frac{\gamma^2 m \log^2 n}{\varepsilon^2})\). Therefore Bringmann’s \cite{17} algorithm runs in time \(\tilde{O}(m + L') = \tilde{O}(m + \frac{m^2}{\varepsilon^2})\). Combining it with the analysis of the Lemma 4.7 gives us an \((\varepsilon, t)\)-membership-oracle for \(S(Z_{\text{small}}, L)\). \square

### 5.3 Applying Additive Combinatorics

Before we proceed forward, we need to present the full theorem of Galil and Margalit \cite{26, Theorem 6.1} (in Section 2.4 we presented only a short version to keep it free from technicalities). We need a full running time complexity (with dependence on \(\ell, m, \Sigma(S)\)). We copied it in here with a slight change of notation (e.g., \cite{26} use \(S_A\) but we use notation from \cite{18} paper of \(\Sigma(A)\)).

**Theorem 5.4** (Theorem 6.1 from \cite{26}). Let \(A\) be a set of \(m\) different numbers in interval \((0, \ell]\) such that

\[
m > 1000 \cdot \ell^{0.5} \log_2 \ell;
\]

then we can build in \(\mathcal{O}\left(m + ((\ell/m) \log \ell)^2 + \frac{\Sigma(A)}{m} \ell^{0.5} \log^2 \ell\right)\) preprocessing time a structure which allows us to solve the \textsc{Subset Sum} problem for any given integer \(N\) in the interval \((L, \Sigma(A) - L)\). Solving means finding a subset \(B \subseteq A\), such that \(\Sigma(B) \leq N\) and there is no subset \(C \subseteq A\) such that \(\Sigma(B) < \Sigma(C) \leq N\). An optimal subset \(B\) is build in \(\mathcal{O}(\log \ell)\) time per target number and is listed in time \(\mathcal{O}(\|B\|)\). For finding the optimal sum \(\Sigma(B)\) only, the preprocessing time is \(\mathcal{O}\left(m + ((\ell/m) \log \ell)^2\right)\) and only constant time is needed per target number.

In \cite{26} authors defined \(L := \frac{100 \Sigma(A) \ell^{0.5} \log_2 \ell}{m}\), however in the next \cite{27} the authors improved it to \(L := \mathcal{O}(\frac{\Sigma(\Lambda) \ell}{m})\) without any damage on running time \cite{28}. For both of these possible choices of \(L\) we obtain a subquadratic algorithm. We will use the improved version \cite{27} because it provides a better running time.
Lemma 5.5 (Small items, range \((L, \Sigma(S) - L)\)). Given a small instance \((Z_{\text{small}}, t)\) of Subset Sum (i.e., all items are smaller than \(\gamma t\)) such that \(Z_{\text{small}}\) is \((\varepsilon t/m')\)-distinct (where \(m' = \mathcal{O}(m \log n)\)), we can compute an \((\varepsilon, t)\)-membership-oracle of \(S(Z_{\text{small}}, t) \cap (L, \Sigma(Z_{\text{small}}) - L)\) in time \(\tilde{\mathcal{O}}(n + \left(\frac{t}{\varepsilon}\right)^2 + \frac{\gamma m}{\varepsilon} \cdot \left(\frac{m}{\varepsilon}\right)^{0.5})\).

Proof. We round items to multiplicities of \(q = \varepsilon t/m'\). Precisely:

\[z'_i = \left\lfloor \frac{z_i}{q} \right\rfloor, \quad t' = \left\lfloor \frac{t}{q} \right\rfloor = \left\lfloor \frac{m'}{\varepsilon} \right\rfloor.

We know that \(z_i < \gamma t\). Therefore

\[z'_i \leq \frac{z_i}{q} < \frac{\gamma t}{q} = \frac{\gamma m'}{\varepsilon} = \ell.

By the same inequalities as in the proof of Lemma 4.7 we know that if we compute \(S(Z'_{\text{small}}, t')\) and multiply all results by \(q\), we obtain an \((\varepsilon, t)\)-membership-oracle for \(S(Z_{\text{small}}, t)\).

Checking conditions of the algorithm Now we will check that we satisfy all assumptions of Galil and Margalit [27] algorithm on the rounded instance \(Z'_{\text{small}}\). First note that \(m^2 < \ell \log^2 \ell, \ell\) is the upper bound on the items’ sizes in \(Z'_{\text{small}}\) and we know that all items in \(Z'_{\text{small}}\) are distinct because we assumed that \(Z_{\text{small}}\) is \((\varepsilon t/m')\)-distinct.

Preprocessing Next Galil and Margalit [27] algorithm constructs a data structure on the set of rounded items \(Z'_{\text{small}}\). The preprocessing of Galil and Margalit [27] algorithm requires

\[\mathcal{O}(m + (\ell/m \log \ell)^2 + \frac{\Sigma(Z'_{\text{small}})}{m^2} \cdot \varepsilon^{0.5} \log^2 \ell)

time. If we put it in terms of \(m, \varepsilon, t\) and hide polylogarithmic factors we see that preprocessing runs in:

\[\tilde{\mathcal{O}} \left( m + \left(\frac{\gamma}{\varepsilon}\right)^2 + \frac{\gamma}{\varepsilon} \left(\frac{m^{0.5}}{\varepsilon}\right) \right)

because \(\Sigma(Z_{\text{small}}) \leq \ell m\).

Queries With this data structure we need to compute a set \(\varepsilon\)-close to \(S(Z_{\text{small}}, t) \cap (L, \Sigma(Z'_{\text{small}}) - L)\). After scaling down we have \(L' = \tilde{\mathcal{O}}\left(\Sigma(Z'_{\text{small}}) \cdot \frac{t}{m^2}\right) = \tilde{\mathcal{O}}(\ell^2) = \tilde{\mathcal{O}}(\frac{\gamma^2 m^2}{m^2}) = \tilde{\mathcal{O}}(m^2).

Naively, one could run queries for all elements in range \((L', \Sigma(Z'_{\text{small}}) - L')\) and check if there is a subset of \(Z'_{\text{small}}\) that sums up to the query value. However this is too expensive. In order to deal with this issue, we take advantage of the fact that each query returns the closest set whose sum is smaller or equal to the query value.

Since we have rounded down items with \(q = \frac{\varepsilon t}{m'}\), we only need to ask \(\frac{z_i}{q} = \tilde{\mathcal{O}}(m)\) queries in order to learn sufficient information. The queries will reveal if \(Z_{\text{small}}\) contains at least one element in each range \([i\varepsilon t, (i + 1)\varepsilon t)\), what matches the definition of the \((\varepsilon, t)\)-membership-oracle.
Retrieving the solution  Galil and Margalit \[27\] algorithm can retrieve the solution in time \(O(\log \ell)\).

This finalizes the construction of the \((\varepsilon, t)\)-membership-oracle. The running time is dominated by the preprocessing time.

\(\square\)

### 5.4 Combining the Algorithms

Now we will combine the algorithms for small items.

**Lemma 5.6** (Small Items). Given a \((Z_{\text{small}}, t)\) instance of \textsc{Subset Sum} (i.e., all elements in \(Z_{\text{small}}\) are smaller than \(\gamma t\)), such that the set \(Z_{\text{small}}\) is \((\varepsilon t/m)\)-distinct, we can compute an \((\varepsilon, t)\)-membership-oracle of \(S(Z_{\text{small}}, t)\) in time \(\tilde{O}(m + \frac{\gamma}{\varepsilon^2} + \frac{m\gamma^2}{\varepsilon^2})\) with high probability.

**Proof.** We will combine two cases:

**Case When** \(m^2 < \ell \log^2 \ell\): In such case we use Lemma \[5.2\] that works in \(\tilde{O}(m + \frac{\gamma}{\varepsilon^2})\) time.

**Case When** \(m^2 \geq \ell \log^2 \ell\): First we take advantage of Lemma \[5.5\] This gives us an \((\varepsilon, t)\)-membership-oracle that answers queries within set \(S(Z_{\text{small}}, t) \cap (L, \Sigma(Z_{\text{small}}) - L)\). It requires \(\tilde{O}(nm + \left(\frac{\gamma}{\varepsilon^2}\right)^2 + \frac{\gamma^2}{\varepsilon^2} \cdot \left(\frac{2m}{\varepsilon}\right)^{0.5})\) time.

We combine it (using Lemma \[4.6\]) with the \((\varepsilon, t)\)-membership-oracle that gives us answers to a set \(S(Z_{\text{small}}, t) \cap [0, L]\) from Lemma \[5.3\] This oracle can be constructed in time \(\tilde{O}(m + \frac{m\gamma^2}{\varepsilon^2})\). The oracle for interval set \([\Sigma(Z_{\text{small}}) - L, \Sigma(Z_{\text{small}})]\) is obtained by symmetry.

**Running Time:** The running time of merging the solutions from Lemma \[4.6\] is \(\tilde{O}(1/\varepsilon)\) which is suppressed by the running time of Lemma \[5.3\] and Lemma \[5.5\]. Factor \(\tilde{O}(\left(\frac{\gamma}{\varepsilon^2}\right)^2)\) is suppressed by \(\tilde{O}\left(\frac{m\gamma^2}{\varepsilon^2}\right)\). The algorithm is randomized because Lemma \[5.3\] is randomized.

\(\square\)

The Lemma \[4.4\] allowed us to partition our instance into small and large items. We additionally know that each interval of length \(\varepsilon t/m'\) contains at most 2 items. However in the previous proofs we assumed there can be only one such item, i.e., the set should be \((\varepsilon t/m')\)-distinct.

**Lemma 5.7** (From multiple to distinct items). Given an instance \((Z_{\text{small}}, t)\) of \textsc{Subset Sum}, where \(|Z_{\text{small}}| = m\) and \(zz_{\text{small}}\) is \((\varepsilon t/m', 2)\)-distinct for \(m' = O(m \log n)\), we can compute an \((\varepsilon, t)\)-membership-oracle for instance \((Z_{\text{small}}, t)\) in \(\tilde{O}(n + \frac{1}{\varepsilon^2} + \frac{m\gamma^2}{\varepsilon^2})\) time with high probability.

**Proof.** We divide the set \(Z_{\text{small}}\) into two sets \(Z_{\text{small}}^1\) and \(Z_{\text{small}}^2\) such that \(Z_{\text{small}} = Z_{\text{small}}^1 \cup Z_{\text{small}}^2\); the sets \(Z_{\text{small}}^1, Z_{\text{small}}^2\) are disjoint, \((\varepsilon t/m')\)-distinct, and have size \(\Omega(m)\). This can be done by sorting \(Z_{\text{small}}\) and dividing items into odd-indexed and even-indexed. It takes \(\tilde{O}(m)\) time.

Next we use Lemma \[5.6\] to compute an \((\varepsilon, t)\)-membership-oracle for \((Z_{\text{small}}^1, t)\) and \((Z_{\text{small}}^2, t)\), and merge them using Lemma \[4.6\]

\(\square\)
Now we will combine the solutions for small and large items.

**Theorem 5.8.** Let $0 < \gamma$ be a trade-off parameter (that depends on $n, \varepsilon$). Given an $(Z, t)$ instance of Subset Sum, we can construct the $(\varepsilon, t)$-membership-oracle of instance $S(Z, t)$ in $\tilde{O}(n + \frac{1}{\gamma \varepsilon} + \frac{\gamma}{\varepsilon^2} + \frac{n^2}{\varepsilon^2})$ time with high probability.

**Proof.** We start with Lemma 4.4 that in $O(n \log^2 n)$ time partitions the set into $Z_{\text{large}}$ and $Z_{\text{small}}$, such that $Z_{\text{small}}$ is $(\frac{\gamma}{\varepsilon^2 n \log n}, 2)$-distinct, where $m = |Z_{\text{small}}|$. To deal with small items, we use Lemma 5.7. The algorithm for small items returns an $(\varepsilon, t)$-membership-oracle of $S(Z_{\text{small}}, t)$. For large items we can use Lemma 5.1. It also returns an $(\varepsilon, t)$-membership-oracle of $S(Z_{\text{large}}, t)$.

Finally, we use Lemma 1.6 to merge these oracles in time $\tilde{O}(1/\varepsilon)$. All the subroutines run with a constant probability of success.

Finally, we have combined all the pieces and we can get a faster algorithm for weak $(1 - \varepsilon)$-approximation for Subset Sum.

**Corollary 5.9** (Subset Sum with tradeoff). There is a randomized weak $(1 - \varepsilon)$-approximation algorithm for Subset Sum running in $\tilde{O}(n + \frac{1}{\gamma \varepsilon} + \frac{\gamma}{\varepsilon^2} + \frac{n^2}{\varepsilon^2})$ time with high probability for any $\gamma(n, \varepsilon) > 0$.

**Proof.** It follows from Lemma 4.8 and Theorem 5.8.

The weak $(1 - \varepsilon)$-approximation Subset Sum gives us the approximation for Partition via Corollary 2.2.

**Corollary 5.10** (Partition with trade-off). There is a randomized $(1 - \varepsilon)$-approximation algorithm for Partition running in $\tilde{O}(n + \frac{1}{\gamma \varepsilon} + \frac{\gamma}{\varepsilon^2} + \frac{n^2}{\varepsilon^2})$ time with high probability for any $\gamma(n, \varepsilon) > 0$.

To get running time of form $\tilde{O}(n+1/\varepsilon^c)$ and prove our main result we need to reduce the number of items from $n$ to $\tilde{O}(1/\varepsilon)$ and choose the optimal $\gamma$.

**Theorem 5.11** (Weak apx for Subset Sum). There is a randomized weak $(1 - \varepsilon)$-approximation algorithm for Subset Sum running in $\tilde{O}(n + \varepsilon^{-2})$ time.

**Proof.** We apply Lemma 1.2 to ensure that the number of items is $\tilde{O}(\frac{1}{\varepsilon})$ and work with an $O(\varepsilon)$-close instance. Then we take advantage of Corollary 5.9 with $\gamma = \varepsilon^{-2}$.

Analogously for Partition we get that:

**Theorem 5.12** (Apx for Partition). There is a randomized $(1 - \varepsilon)$-approximation algorithm for Partition running in $\tilde{O}(n + \varepsilon^{-2})$ time.

### 6 Approximate $(\min, +)$-convolution

| Approximate $(\min, +)$-convolution |
|-------------------------------------|
| **Input:** Sequences $A[0, \ldots, n-1]$, $B[0, \ldots, n-1]$ of positive integers and approximation parameter $0 < \varepsilon < 1$ |
| **Task:** Let $\text{OPT}[k] = \min_{0 \leq i \leq k}(A[i] + B[k-i])$ be the $(\min, +)$-convolution of $A$ and $B$. Find a sequence $C[0, \ldots, n-1]$ such that $\forall_i \text{OPT}[i] \leq C[i] \leq (1 + \varepsilon)\text{OPT}[i]$ |
Backurs et al. [10] described a $(1 + \varepsilon)$-approximation algorithm for $(\min, +)$-convolution, that runs deterministically in time $O(\frac{n}{\varepsilon} \log n \log^2 W)$. In their paper [10] it is used as a building block to show a near-linear time approximation algorithm for TREESPARSITY. With the approximation algorithm for $(\min, +)$-convolution, they managed to solve TREESPARSITY approximately in $\tilde{O}(\frac{n}{\varepsilon})$ time, which in practical applications may be faster than solving this problem exactly in time $\tilde{O}(n^2)$.

We begin with explaining its connection with the SUBSET SUM problem. A natural generalization of SUBSET SUM is KNAPSACK. In this scheme each item has value and weight and our task is to pack items into the knapsack of capacity $C$, so that their cumulative weight does not exceed capacity and in the same time we want to maximize their total value. In the special case when all weights and values are equal we obtain the SUBSET SUM problem.

To certify the existence of a subset with a given sum, we have used the fast convolution using FFT as subroutine. If we want to generalize it and capture maximal value subset of items of a given weight, we require $(\max, +)$-convolution, which is computationally equivalent to $(\min, +)$-convolution.

Cygan et al. [24] exploited this idea to show subquadratic equivalence between exact $(\min, +)$-convolution, KNAPSACK, and other problems. Here we focus on the approximate setting. From [24] it follows that the $\tilde{O}(n + \frac{1}{\varepsilon^2} \log n)$ approximation algorithm for UNBOUNDED KNAPSACK is unlikely. This lower bounds proves the optimality of Jansen and Kraft [40] $\tilde{O}(n + \frac{1}{\varepsilon})$ FPTAS for UNBOUNDED KNAPSACK. The current best FPTAS for KNAPSACK is burdened with time complexity of $\tilde{O}(n + 1/\varepsilon^{12/5})$ Chan [19]. We hope, that our approximation schemes are a step towards a faster FPTAS for KNAPSACK.

In this section we improve upon the $\tilde{O}(n/\varepsilon^2)$ approximation algorithm for $(\min, +)$-convolution. Similar techniques have been exploited to obtain the $\tilde{O}(n^\omega/\varepsilon)$-time approximation for APSP [60] and they have found use in the approximate pattern matching over $l_\infty$ [51]. The basic idea is to propose a fast exact algorithm depending on $W$ (upper bound on the weights) and apply it after rounding weights into smaller space. Our result also applies to $(\max, +)$-convolution.

### 6.1 Exact $\tilde{O}(nW)$ algorithm

The $(\min, +)$-convolution admits a brute force $O(n^2)$-algorithm. From the other hand, when all values in sequences are binary, then applying FFT and performing convolution yields an $O(n \log n)$-algorithm. Our exact $\tilde{O}(nW)$ algorithm is an attempt to capture this trade-off. Note, that this algorithm is worse than a brute force whenever $W > n$ which is often the case. However, this algorithm turns out useful for approximation.

**Lemma 6.1.** The $(\min, +)$-convolution $(\max, +)$-convolution problem can be solved deterministically in $O(nW \log (nW))$ time and $O(nW)$ space.

**Proof.** Given sequences $A[0, \ldots, n - 1]$ and $B[0, \ldots, n - 1]$ with values at most $W$, we transform them into binary sequences of length $2nW$. We encode every number in the natural unary manner. For $0 \leq i < n$, $1 \leq k \leq W$ we define:

$$
\tilde{a}[2Wi + k] = \begin{cases} 
0 & \text{if } A[i] \neq k \\
1 & \text{if } A[i] = k 
\end{cases}
$$

and similarly we define sequence $\tilde{B}$. For example, sequence $(2, 3, 1)$ with $W = 3$ gets encoded as $010'000'001'000'100'000$ (the separators ' are used to visually separate sections of length $W$).

21
We compute convolution $\tilde{C} = \tilde{A} \oplus \tilde{B}$ using FFT in time $O(nW \log n \log W)$. Since $\tilde{C}[2Wi + k] = \sum_{k_1 + k_2 = k} \tilde{A}[2Wi_1 + k_1] \cdot \tilde{B}[2Wi_2 + k_2]$, the first nonzero occurrence in the $i$-th block of length $2W$ encodes the value of the $i$-th element of the requested $(\min, +)$-convolution. If we are interested in computing $(\max, +)$-convolution, we should similarly seek for last nonzero value in each block.

The time complexity is dominated by performing convolution with FFT. As the additional space we need $O(nW)$ bits for the transformed sequences.

### 6.2 Approximating Algorithm

We start with a lemma inspired by [60, Lemma 5.1] and [51, Lemma 1].

**Lemma 6.2.** For natural numbers $x, y$ and positive $q, \varepsilon$ satisfying $q \leq x + y$ and $0 < \varepsilon < 1$ it holds:

\[
x + y \leq \left(\left\lfloor \frac{2x}{q\varepsilon} \right\rfloor + \left\lfloor \frac{2y}{q\varepsilon} \right\rfloor\right) \frac{4\varepsilon}{2} < (x + y)(1 + \varepsilon),
\]

\[
(x + y)(1 - \varepsilon) < \left(\left\lfloor \frac{2x}{q\varepsilon} \right\rfloor + \left\lfloor \frac{2y}{q\varepsilon} \right\rfloor\right) \frac{4\varepsilon}{2} \leq x + y.
\]

**Proof.** The proof is a special case of Lemmas [B.1] and [B.2] for $k = 2$. \hfill \Box

**Lemma 6.3.** Assume the $(\min, +)$-convolution ($(\max, +)$-convolution) can be solved exactly in time $T(n, W)$. Then we can approximate $(\min, +)$-convolution ($(\max, +)$-convolution) in time $O((T(n, \frac{1}{\varepsilon}) + n) \log W)$.

**Algorithm 2** APPROXIMATEMINCONV($A, B$). We use a simplified notation to transform all elements in the sequences $A[i]$ and $B[i]$.

```plaintext
1: Output[i] = \infty
2: for $l = 2\lceil \log W \rceil, \ldots, 0$ do
3:     $q := 2^l$
4:     $A'[i] = \left\lfloor 2^l A[i] \right\rfloor$
5:     if $A'[i] > \left\lfloor 4/\varepsilon \right\rfloor$ then
6:         $A'[i] = \infty$
7:     end if
8:     $B'[i] = \left\lfloor 2^l B[i] \right\rfloor$
9:     if $B'[i] > \left\lfloor 4/\varepsilon \right\rfloor$ then
10:        $B'[i] = \infty$
11:    end if
12:    $V = \text{runExact}(A', B')$
13:    if $V[i] < \infty$ then
14:        Output[i] = $V[i] \cdot \frac{q}{2}$
15:    end if
16: end for
17: return Output[0, \ldots, n - 1]
```

**Proof.** The idea is based on [51, Section 6.2]. We focus on the variant with $(\min, +)$-convolution, however the proofs works alike for $(\max, +)$-convolution.

22
We iterate the precision parameter $q$ through $2W, W, \ldots, 4, 2, 1$. In each iteration we apply the transform from Lemma 6.2 ($x \rightarrow \lceil \frac{2x}{q} \rceil$) to all elements in $A, B$, we set $\infty$ for each value exceeding $\left\lceil \frac{4}{\varepsilon} \right\rceil$, and launch the exact algorithm on such input. We multiply all finite elements in the returned array by $\frac{4}{\varepsilon}$ and store them in the output array $C$, possibly overwriting some elements.

Assume the correct value of $C[k]$ equals $A[i] + B[k - i]$. For some iteration we get the precision parameter $q$ such that $q \leq C[k] < 2q$. The rounded numbers $\lceil \frac{2A[i]}{q} \rceil$, $\lceil \frac{2B[i-k]}{q} \rceil$ are at most $\left\lceil \frac{4}{\varepsilon} \right\rceil$, so we will update the $k$-th index in the output array. On the other hand, the assumption of Lemma 6.2 is satisfied, therefore the generated value lies between $C[k]$ and $C[k](1 + \varepsilon)$. In the following iterations, we will still have $q \leq C[k]$, therefore any further updates to the $k$-th index will remain valid.

The algorithm performs $O(\log W)$ iterations and in each step we run the exact algorithm in time $T(n, \frac{4}{\varepsilon})$, thanks to the pruning procedure. Transforming the sequences takes $O(n)$ time in each step.

**Theorem 6.4** (Apx for $(\min / \max, +)$-conv). There is a deterministic algorithm for $(1 + \varepsilon)$-approximate $(\min, +)$-convolution $[(\max, +)$-convolution] running in $O\left(\frac{n}{\varepsilon} \log \left(\frac{n}{\varepsilon}\right) \log W\right)$ time.

**Proof.** From Lemma 6.1 the running time of exact algorithm is $T(n, W) = O(nW \log n \log W)$. This quantity dominates the additive term $O(n \log W)$. Hence by replacing each $W$ with $1/\varepsilon$ we get the claimed running time.

7 Tree Sparsity

The TreeSparsity problem has been stated as follows: given a node-weighted binary tree and an integer $k$, find a rooted subtree of size $k$ with the maximal weight. Its approximation version comes with two flavors: as a head approximation where we are supposed to maximize the weight of the solution, and as a tail approximation where we minimize the total weight of nodes that do not belong to the solution. Note that a constant approximation for one of the variants does not necessarily yield a constant approximation for the other one. Backurs et al. [10] proposed an $O\left(\frac{n}{\varepsilon} \cdot \log^{12} n \cdot \log^{2} W\right)$ running time for $(1 - \varepsilon)$-head approximation, and an $O\left(\frac{n}{\varepsilon} \cdot \log^{9} n \cdot \log^{3} W\right)$ running time for $(1 + \varepsilon)$-tail approximation.

In this section we improve the running times for both variants relying on the $\tilde{O}(\frac{n}{\varepsilon})$ algorithm for approximating $(\min, +)$ and $(\max, +)$ convolutions. Our construction is based on the approach by Cygan et al. [24] which also results in a simpler analysis than for the previously known approximation schema [10]. In particular, a single proof suffices to cover both head and tail variants.

The following theorem, combined with our approximation for $(\min, +)$-convolution yields an $O\left(\frac{n}{\varepsilon} \cdot \log(n/\varepsilon) \cdot \log^{3} n \cdot \log W\right)$-time algorithm that computes the maximal weights of rooted subtrees for each size $k = 1, \ldots, n$ with a relative error at most $\varepsilon$ in both head and tail variant.

**Theorem 7.1.** If $(1 + \varepsilon)$-approximate $(\min, +)$-convolution can be solved in time $T(n, W, \varepsilon)$, then $(1 + \varepsilon)$-approximate TreeSparsity can be solved in time $O\left((n + T(n, W, \varepsilon/ \log^{2} n)) \log n\right)$.

**Proof.** We exploit the heavy-light decomposition introduced by Sleator and Tarjan [57]. This technique has been utilized by Backurs et al. [10] in their work on TreeSparsity approximation and later by Cygan et al. [24] in order to show a subquadratic equivalence between TreeSparsity and $(\min, +)$-convolution.
We construct a spine with a head $s_1$ at the root of the tree. We define $s_{i+1}$ to be the child of $s_i$ with the larger subtree (in case of draw we choose any child) and the last node in the spine is a leaf. The remaining children of nodes $s_i$ become heads for analogous spines so the whole tree gets covered. Observe that every path from a leaf to the root intersects at most $\log n$ spines because each spine transition doubles the subtree size.

At first we express the head variant in the convolutional paradigm. For a node $v$ with a subtree of size $m$ we define the sparsity vector $(x_v[0],x_v[1],\ldots,x_v[m])$ of weights of the heaviest subtrees rooted at $v$ with fixed sizes. This vector equals the $(\max,+)$-convolution of the sparsity vectors for the children of $v$. We are going to compute sparsity vectors for all heads of spines in the tree recursively. Having this performed we can read the solution from a sparsity vector of the root. Let $(s_i)_{i=1}^\ell$ be a spine with a head $v$ and let $u^i$ indicate the sparsity vector for the child of $s_i$ being a head (i.e., the child with the smaller subtree). If $s_i$ has less than two children we treat $u^i$ as a vector $(0)$.

For an interval $[a,b] \subseteq [1,\ell]$ let $u^{a,b} = u^a \oplus^{\max} u^{a+1} \oplus^{\max} \ldots \oplus^{\max} u^b$ and $y^{a,b}[k]$ be the maximum weight of a subtree of size $k$ rooted at $s_a$ and not containing $s_{b+1}$. Let $c = \left\lfloor \frac{a+b}{2} \right\rfloor$. The $\oplus^{\max}$ operator is associative so $u^{a,b} = u^{a,c} \oplus^{\max} u^{c+1,b}$. To compute the second vector we consider two cases: whether the optimal subtree contains $s_{c+1}$ or not.

\begin{equation}
\begin{aligned}
y^{a,b}[k] &= \max \left[ y^{a,c}[k], \sum_{i=a}^{c} x(s_i) + \max_{k_1+k_2=k-(c-a+1)} \left( u^{a,c}[k_1] + y^{c+1,b}[k_2] \right) \right] \\
&= \max \left[ y^{a,c}[k], \sum_{i=a}^{c} x(s_i) + \left( u^{a,c} \oplus^{\max} y^{c+1,b} \right)[k-(c-a+1)] \right]
\end{aligned}
\end{equation}

Using the presented formulas we reduce the problem of computing $x^v = y^{1,\ell}$ to subproblems for intervals $[1,\ell/2]$ and $[\ell/2+1,\ell]$ and results are merged with two $(\max,+)$-convolutions. Proceeding further we obtain $O(\log \ell)$ levels of recursion. Since there are $O(\log n)$ spines on a path from a leaf to the root, the whole computation tree has $O(\log^2 n)$ layers, each node being expressed as a pair of convolutions on vectors from its children. Each vertex of the graph occurs in at most $\log n$ convolutions so the sum of convolution sizes is $O(n \log n)$.

In order to deal with the tail variant we consider a dual sparsity vector $(\overline{x}^v[0],\overline{x}^v[1],\ldots,\overline{x}^v[m])$, where $\overline{x}^v[i]$ stands for the total weight of the subtree rooted at $v$ minus $x^v[i]$. The dual sparsity vector of $v$ equals the $(\min,+)$-convolution of the vectors for the children of $v$. We can use an analog of equation (2) and also express the problem as a computation tree based on convolutions.

We take advantage of Theorem 6.4 to perform each convolution with a relative error $\delta$. The formula (2) contains an additive term $\sum_{i=a}^{c} x(s_i)$ but this can only decrease the relative error. The cumulative relative error is bounded by $(1-\delta)^{\log^2 n}$ for head approximation and $(1+\delta)^{\log^2 n}$ for tail approximation, therefore setting $\delta = \Theta(\varepsilon/\log^2 n)$ guarantees that the sparsity vector for the root is burdened with relative error at most $\varepsilon$.

The sum of running times for all convolutions is $O(T(n,W,\delta) \log n)$, what gives the postulated running time for the whole algorithm. In order to retrieve the solution for a given $k$, we need to find the pair of indices that produced the value of the $k$-th index of the last convolution. Then we proceed recursively and traverse back the computation tree. Since finding $\arg\max$ and $\arg\min$ can be performed in linear time, the total time of analyzing all convolutions is $O(n \log n)$.

\[ \square \]
8 $\tilde{O}(n + 1/\varepsilon)$ approximation algorithm for 3SUM

In the abstract we have claimed that our result for PARTITION constitutes the first approximation algorithm for NP-hard problem that breaks the quadratic barrier. However this is not necessary the case for the problems in P. In this section we will show an $\tilde{O}(n + 1/\varepsilon)$ approximation algorithm for 3SUM and prove accompanying lower bound under a reasonable assumption. To the best of our knowledge, this is also the first nontrivial linear approximation algorithm for a natural problem.

$k$-SUM

**Input:** Sets $A_1, A_2, \ldots, A_{k-1}, S$, each with cardinality at most $n$.

**Task:** Decide if there is a tuple $(a_1, \ldots, a_{k-1}, s) \in A_1 \times \cdots \times A_{k-1} \times S$ such that $a_1 + \cdots + a_{k-1} = s$.

The 3SUM problem is a special case of $k$-SUM for $k = 3$. The 3SUM is one of the most notorious problems with a quadratic running time and has been widely accepted as a hardness assumption (see [58] for overview). The fastest known algorithm for 3SUM is slightly subquadratic: Jørgensen and Pettie [41] gave an $O(n^2 \log \log n/\log n)^{2/3}$-time deterministic algorithm and then independently Freund [25] and Gold and Sharir [34] improved this result by presenting an $O(n^2 \log \log n/\log n)$-time algorithm.

The approximation variant for 3SUM was considered by Gfeller [33] who showed a deterministic $\tilde{O}(2^{\frac{1}{3}})$ algorithm as a byproduct of finding longest approximate periodic patterns. If we are not interested in exact solution, the Gfeller [33] algorithm is polynomially faster than the best exact algorithm for 3SUM. In this section we show how to solve 3SUM approximately in time $\tilde{O}(n + 1/\varepsilon)$ time and prove this tight up to the polylogarithmic factors.

**Approximate 3SUM**

**Input:** Three sets $A, B, C$ of positive integers, each with cardinality at most $n$.

**Task:** The algorithm:

- concludes that no triple $(a, b, c) \in A \times B \times C$ with $a + b = c$ exists, or
- it outputs a triple $(a, b, c) \in A \times B \times C$ with $a + b \in [c/(1 + \varepsilon), c(1 + \varepsilon)]$.

This definition generalizes to $k$-SUM, however we are unaware about any previous works on approximate $k$-SUM.

8.1 Faster approximation algorithm for 3SUM

In this section we present an $\tilde{O}(n + 1/\varepsilon)$-time approximation scheme for 3SUM problem. We use a technique from Section 6 where we gave the fast approximation algorithm for $(\text{min}, +)$-convolution. As previously, we start with a fast $\tilde{O}(n + W)$ exact algorithm and then utilize rounding to get an approximation algorithm. In the Section 9 we will show a conditional optimality of this result.

8.1.1 Exact $\tilde{O}(n + W)$ algorithm for 3SUM

Let $W$ denote the upper bound on the integers in the sets $A, B$ and $C$. The exact $\tilde{O}(n + W)$-time algorithm for 3SUM is already well known [22, 20]. In here we will place the proof for completeness. For formal reasons we need to take care of the special symbol $\infty$. What is more, we will generalize this result to $k$-SUM.
**Theorem 8.1** (Based on [22, 20]). The $k$-SUM can be solved deterministically in $\tilde{O}(kn + kW \log W)$ time and $\tilde{O}(kn + W)$ space.

**Proof.** We will encode the numbers in the sets as binary arrays of size $O(W)$ and iteratively perform fast convolution using FFT. Because we will use only $O(1)$ tables at once, the space complexity will not depend on $k$. At the end we will need to check if any entry in the final array is in $S$.

**Encoding:** We iterate for every set $A_1, \ldots, A_{k-1}$ and for $l$-th iteration encode it as a binary vector $V_l$ of length $W + 1$, such that:

$$V_l[i] = \begin{cases} 1 & \text{ iff } t \in A_l \\ 0 & \text{ otherwise} \end{cases}$$

To save space we will use only one $V_l$ vector at the time. The encoding can be done in $O(n + W)$ time. If the special symbol $\infty \in A_l$ appears then we simply discard it.

**FFT:** We want to perform a convolution with FFT on all vectors $V_l$. We do it one at a time and discard all elements larger than $W$. Let $U_l$ be the result of up to $l$-th iteration. We know that the proper polynomial is $U_l(x) = \sum_{(a_1, \ldots, a_l) \in (A_1 \times \ldots \times A_l)} x^{a_1 + \ldots + a_l}$. And if we multiply it by the polynomial $V_{l+1} = \sum_{a_{l+1} \in A_{l+1}} x^{a_{l+1}}$, we get $U_{l+1}(x) = \sum_{(a_1, \ldots, a_{l+1}) \in (A_1 \times \ldots \times A_{l+1})} x^{a_1 + \ldots + a_{l+1}}$.

Hence at the end we obtain the vector $V_{k-1}$ that encodes all the sums of elements in subsets truncated up to $W$ place.

**Comparing** At the end we need to get the binary vector for $S$ and compare it with the resulting vector $V_{k-1}$.

**Time and Space** We did $k$ iterations. In each of them we transformed a set into a vector in time $O(n)$. The fast convolution works in $O(T \log T)$ by using FFT. Hence, the running time is $O(kn + kW \log W)$. Algorithm needs $O(nk)$ space to encode input and $O(W)$ space to store binary vectors. □

### 8.1.2 Approximation algorithm

Next we will use an exact algorithm to propose the fast approximation. We will use the same reasoning as in Section 6.4.

**Lemma 8.2.** Assume the $k$-SUM can be solved exactly in $T(n, k, W)$ time. Then approximate $k$-SUM can be solved in $O((T(n, k, k/\varepsilon) + nk) \log W)$ time.

Because the proof is just a small modification of the Lemma 6.3 we have included it in Section 8.1.3. At the end we need to connect the exact algorithm from Lemma 8.1 and the reduction from Lemma 8.2.

**Theorem 8.3.** There is a deterministic algorithm for $(1 + \varepsilon)$-approximate $k$-SUM running in $O(nk \log W + \frac{k^2}{\varepsilon} \log \frac{k}{\varepsilon} \log W)$ time.

**Proof.** From Lemma 8.1 the running time of $k$-SUM is $T(n, k, W) = O(nk + kW \log W)$. Applying this running time to the reduction in Lemma 8.2 results in the claimed running time, because the $O(nk)$ term is dominated by $O(nk \log W)$ term in the reduction. □
To get an approximate algorithm for 3SUM we set $k = 3$.

**Corollary 8.4.** The approximate 3SUM can be solved deterministically in $O((n + \frac{1}{\varepsilon} \log \frac{1}{\varepsilon}) \log W)$ time.

### 8.1.3 Proof of Lemma 8.2

**Algorithm 3** APPROXIMATEKSUM$(a_1, a_2, \ldots, a_{k-1}, s, \varepsilon)$. We use a shorten notation to transform all elements in the sequences $a_i[i]$ and $s[i]$.

1: Output $[i] = \infty$
2: for $l = 2\lceil\log W\rceil, \ldots, 0$ do
3: $q := 2^l$
4: for $l = 1, \ldots, k - 1$ do
5: $a'_l[i] = \left\lceil \frac{ka_l[i]}{q\varepsilon} \right\rceil$
6: if $a'_l[i] > \left\lceil \frac{4k}{\varepsilon} \right\rceil$ then
7: $a'_l[i] = \infty$
8: end if
9: end for
10: $s'[i] = \left\lceil \frac{ks[i]}{q\varepsilon} \right\rceil$
11: if $s'[i] > \left\lceil \frac{4k}{\varepsilon} \right\rceil$ then
12: $s'[i] = \infty$
13: end if
14: if runExactKsum$(a'_1, \ldots, a'_{k-1}, s')$ then
15: return True
16: end if
17: end for
18: return False

**Proof.** The proof basically follows the approach approximating $(\min, +)$-convolution in Lemma 6.3. Assume, that there is some number $s$, for each there exists a tuple $(a_1, a_2, \ldots, a_{k-1}) \in A_1 \times \ldots A_{k-1}$, that $s < \sum_{i=1}^k a_i < s(1 + \varepsilon)$. Then look at Algorithm 3 in which we iterate precision parameter $q$. Hence there is some $q$, such that $q \leq s < 2q$. From Lemma B.1 we know, that then, we can round the numbers $a'_l = \left\lceil \frac{ka_l}{q\varepsilon} \right\rceil$ and then their sum should be approximately:

$$\sum_{i=1}^k a_i \leq \sum_{i=1}^k \left\lceil \frac{ka_i}{q\varepsilon} \right\rceil < (1 + \varepsilon) \sum_{i=1}^k a_i$$

So if there is some number $s \in S$, then APPROXIMATEKSUM algorithm would find a tuple, that sum up to $s' \in [s, (1 + \varepsilon)s]$.

From the other hand, if for all $s \in S$ no tuple sums up to $[s, (1 + \varepsilon)s]$ then our APPROXIMATEKSUM can also return YES. It is because before rounding items could sum up to something in $[(1-\varepsilon)s, s]$ (see Section 2). However, if there for such a parameter there always exists a precision parameter $q$, that $q \leq s < 2q$. Then rounding the numbers according to Lemma B.1 gives only $(1 \pm \varepsilon)$
error and they cannot sum to \[ \frac{\epsilon n}{7/9} \]. Hence if for all \( s \in S \) no tuple sums up to \[ (1 - \epsilon)s, (1 + \epsilon)s \] then our \textsc{ApproximateKSum} will return NO.

A technicality is hidden in the Definition 8 we need to return approximation of the form \( \text{OPT}/(1 + \epsilon) \leq \text{OUR} \leq \text{OPT}(1 + \epsilon) \) but note that \( \frac{1}{1+\epsilon} \approx 1 - \epsilon \) so we can take care of if by adjusting \( \epsilon \).

## 9 Conditional Lower Bounds

Proving conditional lower bounds in P under a plausible assumption is a very active line of research \[ 58, 28, 1, 1, 1, 1 \]. One of the first problems with a truly subquadratic running time ruled out was \textsc{EditDistance} \[ 9 \]. It admits a linear time approximation algorithm for \( \epsilon = 1/\sqrt{n} \), that follows from the exact \( \mathcal{O}(n + d^2) \) algorithm. Subsequently, this (linear-time) approximation factor was improved by \cite{Bar-Yossef} to \( n^{3/7+\omega(1)} \), then by \cite{Batu} to \( n^{1/3+o(1)} \), and most recently \cite{Andoni} proposed an \( \mathcal{O}(n^{1+\epsilon}) \)-time algorithm with factor \( (\log n)^{O(1/\epsilon)} \) for every fixed \( \epsilon > 0 \). From the other hand \cite{Abboud} ruled out a truly subquadratic PTAS for \textsc{Longest Common Subsequence} using circuit lower bounds. Our results are somehow of similar flavor to this line of research.

### 9.1 Conditional Lower Bound for Approximate 3SUM

We have shown an approximate algorithm for 3SUM running in \( \tilde{\mathcal{O}}(n + 1/\epsilon) \) time. Is this the best we can hope for? Perhaps one could imagine an \( \tilde{\mathcal{O}}(n + 1/\sqrt{\epsilon}) \) time algorithm. In this subsection we rule out such a possibility and prove the optimality of Theorem 8.1.

To show the conditional lower bound we will assume the hardness of the exact 3SUM. The 3SUM conjecture says, that the \( \tilde{\mathcal{O}}(n^2) \) algorithm is essentially the best we can hope for up to subpolynomial factors.

**Conjecture 9.1 (3SUM conjecture \[ 58 \]).** In the Word RAM model with \( \mathcal{O}(\log n) \) bit words, any algorithm requires \( \Omega(n^{2-o(1)}) \) time in expectation to determine whether given set \( S \subset \{-n^{3+o(1)}, \ldots, n^{3+o(1)}\} \) of size \( n \) contains three distinct elements \( a, b, c \) such that \( a + b = c \).

This definition of 3SUM in \[ 58 \] is equivalent to the one in Section C (see discussion in \[ 13 \]). What is more, solving 3SUM with only polynomially bounded numbers can be reduced to solving it with the upper bound \( W = \mathcal{O}(n^3) \) \[ 13 \]. 3SUM can be solved in subquadratic time when \( W = \mathcal{O}(n^{2-\delta}) \) via FFT, but doing so assuming only \( W = \mathcal{O}(n^2) \) constitutes a major open problem. \cite{Hsu} and Umans \[ 38 \] have considered it as a yet another hardness assumption.

**Conjecture 9.2 (Strong-3SUM conjecture \[ 38 \]).** 3SUM on a set of \( n \) integers in the domain of \( \{-n^2, \ldots, n^2\} \) requires time \( \Omega(n^{2-o(1)}) \).

**Theorem 9.3.** Assuming the Strong-3SUM conjecture, there is no \( \tilde{\mathcal{O}}(n + 1/\epsilon^{1-\delta}) \) algorithm for \( (1 + \epsilon) \)-approximate 3SUM, for any constant \( \delta > 0 \).

**Proof.** Consider the exact variant of 3SUM within the domain \( \{-n^2, \ldots, n^2\} \). We can assume that the numbers are divided into sets \( A, B, C \) and we can restrict ourselves to triples \( a \in A, b \in B, c \in C \) \[ 13 \]. We add \( n^2 + 1 \) to all numbers in \( A \cup B \) and likewise \( 2n^2 + 2 \) to numbers in \( C \) to obtain an equivalent instance with all input numbers greater than 0 and \( W = \mathcal{O}(n^2) \).
Suppose, that for some small $\delta > 0$ the approximate 3SUM admits an $\tilde{O}(n + 1/\varepsilon^{1-\delta})$-algorithm. We can use it to solve the problem above exactly by setting $\varepsilon = \frac{1}{2^\frac{1}{\delta}} = \Omega(n^{\frac{2}{\delta}})$. The running time of the exact algorithm is strongly subquadratic, namely $\tilde{O}(n + 1/\varepsilon^{1-\delta}) = \tilde{O}(n^{2-2\delta})$. This contradicts the Strong-3SUM conjecture.

### 9.2 Conditional Lower Bounds for Knapsack-type Problems

The conditional lower bounds for Knapsack and Unbounded Knapsack are corollaries from [24]. We commence by introducing the main theorem from that work, truncated to problems that are of interest to us.

**Theorem 9.4 (Theorem 2 from [24]).** The following statements are equivalent:

1. There exists an $O(n^2-\varepsilon)$ algorithm for $(\min, +)$-convolution for some $\varepsilon > 0$.
2. There exists an $O((n + t)^2-\varepsilon)$ algorithm for Unbounded Knapsack for some $\varepsilon > 0$.
3. There exists an $O((n + t)^2-\varepsilon)$ algorithm for Knapsack for some $\varepsilon > 0$.

We allow randomized algorithms.

**Conjecture 9.5 ((min, +)-convolution conjecture [24]).** Any algorithm computing $(\min, +)$-convolution requires $\Omega(n^{2-o(1)})$ running time.

Basically [24, Theorem 2] says that assuming the $(\min, +)$-convolution conjecture both Unbounded Knapsack and Knapsack require $\Omega((n + t)^2-o(1))$ time. The pseudo-polynomial algorithm for Knapsack running in time $O(nt)$ can be modified to work in time $O(nv)$, where $v$ is an upper bound on value of the solution. In similar spirit, the reductions from [24] can use a hypothetical $O((n + v)^2-\delta)$ algorithm for Knapsack or Unbounded Knapsack to get a subquadratic algorithm for $(\min, +)$-convolution (modify Theorem 4 from [24]).

**Corollary 9.6 ([24]).** For any constant $\delta > 0$, an exact algorithm for Knapsack or Unbounded Knapsack with $O((n + v)^2-\delta)$ running time would refute the $(\min, +)$-convolution conjecture.

We need this modification because in the definition of FPTAS for Knapsack we consider relative error with respect to the optimal value (not weight). We can use a hypothetical faster approximation algorithm to get a faster pseudo-polynomial exact algorithm, what would contradict the $(\min, +)$-convolution conjecture. More formally:

**Theorem 9.7 (restated Theorem [1.6]).** For any constant $\delta > 0$, obtaining a weak $(1-\varepsilon)$-approximation for Knapsack or Unbounded Knapsack with $O((n + 1/\varepsilon)^2-\delta)$ running time would refute the $(\min, +)$-convolution conjecture.

**Proof.** Suppose, that for some $\delta > 0$ we have a weak $(1-\varepsilon)$-approximation for Knapsack (or Unbounded Knapsack) with running time $O((n + 1/\varepsilon)^2-\delta)$. If we set $\varepsilon = 2/v$, then the approximation algorithm would solve the exact problem because the absolute error gets bounded by $1/2$. By [24, Theorem 2] we know that such an algorithm contradicts the $(\min, +)$-convolution conjecture. The claim follows. □
A similar argument works for the Subset Sum problem. Abboud et al. [4] showed that assuming SETH there can be no $O(t^{1-\delta}\text{poly}(n))$ algorithm for Subset Sum (Cygan et al. [23] obtained the same lower bound before but assuming the SetCover conjecture).

**Theorem 9.8 (Conditional Lower Bound for approximate Subset Sum).** For any constant $\delta > 0$, a weak $(1-\varepsilon)$-approximation for Subset Sum with running time $O\left(\text{poly}(n)\left(\frac{1}{\varepsilon}\right)^{1-\delta}\right)$ would refute SETH and SetCover conjecture.

**Proof.** We set $\varepsilon = 2/t$ and obtain an algorithm solving the exact Subset Sum, because all numbers are integers and the absolute error is at most $1/2$. The running time is $O(t^{1-\delta}\text{poly}(n))$, what refutes SETH due to [4] and the SetCover conjecture due to [23]. $\square$

10 Conclusion and Open Problems

In this paper we study the complexity of the Knapsack, Subset Sum and Partition. In the exact setting, if we are only concerned about the dependence on $n$, Knapsack and Subset Sum were already known to be equivalent up to the polynomial factors. Nederlof et al. [55, Theorem 2] showed, that if there exists an exact algorithm for Subset Sum working in $O^*(T(n))$ time and $O^*(S(n))$ space, then we can construct an algorithm for Knapsack working in the same $O^*(T(n))$ time and $O^*(S(n))$ space. In contrast, in the realm of pseudo-polynomial time complexity, Subset Sum seems to be simpler than Knapsack (see Bringmann [17], Cygan et al. [24]). In this paper, we show similar separation for Knapsack and Partition in the approximation setting.

After this paper was announced, Bringmann [18] showed that the current $O(n + 1/\varepsilon^2)$ algorithm for Subset Sum is optimal assuming (min, +)-convolution conjecture. Can we improve the approximation algorithm for Knapsack to an $O(n + 1/\varepsilon^2)$ and match the quadratic lower bound?

It also remains open whether 3SUM and (min, +)-convolution admit FPTAS algorithms with no dependence on $W$. To add weight to this open problem, note that it is this issue that makes the FPTAS algorithms for TreeSparsity inefficient in practice.

Closing the time complexity gap for Partition is another open problem, either by improving the $O((n + 1/\varepsilon)^{5/3})$ FPTAS or the $\Omega((n + 1/\varepsilon)^{1-o(1)})$ conditional lower bound. It is worth noting, that if the Freiman’s Conjecture [26] is true, then our techniques would automatically lead to even faster FPTAS for Partition.

Finally, one can also ask whether randomization is necessary to obtain subquadratic FPTAS for Partition. We believe that the randomized building blocks can be replaced with deterministic algorithms by Kellerer et al. [46] and Koiliaris and Xu [48].

11 Acknowledgements

This work is part of the project TOTAL that has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 677651). Karol Węgrzycki is supported by the grants 2016/21/N/ST6/01468 and 2018/28/T/ST6/00084 of the Polish National Science Center. We would like to thank Marek Cygan, Artur Czumaj, Zvi Galil, Oded Margalit and Piotr Sankowski for helpful discussions. Also we are grateful to the organizers and participants of the Bridging Continuous and Discrete Optimization program at the Simons Institute for the Theory of Computing, especially Aleksander Mądry.
References

[1] Amir Abboud and Arturs Backurs. Towards hardness of approximation for polynomial time problems. In Christos H. Papadimitriou, editor, *8th Innovations in Theoretical Computer Science Conference, ITCS 2017, January 9-11, 2017, Berkeley, CA, USA*, volume 67 of LIPIcs, pages 11:1–11:26. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017. ISBN 978-3-95977-029-3. URL http://www.dagstuhl.de/dagpub/978-3-95977-029-3.

[2] Amir Abboud, Ryan Williams, and Huacheng Yu. More applications of the polynomial method to algorithm design. In *Proceedings of the Twenty-sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA ’15, pages 218–230, Philadelphia, PA, USA, 2015. Society for Industrial and Applied Mathematics.

[3] Amir Abboud, Virginia Vassilevska Williams, and Joshua R. Wang. Approximation and fixed parameter subquadratic algorithms for radius and diameter in sparse graphs. In Robert Krauthgamer, editor, *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, January 10-12, 2016*, pages 377–391. SIAM, 2016.

[4] Amir Abboud, Karl Bringmann, Danny Hermelin, and Dvir Shabtay. Seth-based lower bounds for subset sum and bicriteria path. *arXiv preprint arXiv:1704.04546, to appear at SODA 2019*, 2019.

[5] Alexandr Andoni, Robert Krauthgamer, and Krzysztof Onak. Polylogarithmic approximation for edit distance and the asymmetric query complexity. In *51th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2010, October 23-26, 2010, Las Vegas, Nevada, USA*, pages 377–386. IEEE Computer Society, 2010.

[6] Per Austrin, Petteri Kaski, Mikko Koivisto, and Jussi Määttä. Space-time tradeoffs for subset sum: An improved worst case algorithm. In Fedor V. Fomin, Rusins Freivalds, Marta Z. Kwiatkowska, and David Peleg, editors, *Automata, Languages, and Programming - 40th International Colloquium, ICALP 2013, Riga, Latvia, July 8-12, 2013, Proceedings, Part I*, volume 7965 of Lecture Notes in Computer Science, pages 45–56. Springer, 2013.

[7] Per Austrin, Petteri Kaski, Mikko Koivisto, and Jesper Nederlof. Subset sum in the absence of concentration. In Ernst W. Mayr and Nicolas Ollinger, editors, *32nd International Symposium on Theoretical Aspects of Computer Science, STACS 2015, March 4-7, 2015, Garching, Germany*, volume 30 of LIPIcs, pages 48–61. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015.

[8] Per Austrin, Petteri Kaski, Mikko Koivisto, and Jesper Nederlof. Dense subset sum may be the hardest. In Nicolas Ollinger and Heribert Vollmer, editors, *33rd Symposium on Theoretical Aspects of Computer Science, STACS 2016, February 17-20, 2016, Orléans, France*, volume 47 of LIPIcs, pages 13:1–13:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.

[9] Arturs Backurs and Piotr Indyk. Edit distance cannot be computed in strongly subquadratic time (unless SETH is false). In Rocco A. Servedio and Ronitt Rubinfeld, editors, *Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing, STOC 2015, Portland, OR, USA, June 14-17, 2015*, pages 51–58. ACM, 2015.
[10] Arturs Backurs, Piotr Indyk, and Ludwig Schmidt. Better approximations for tree sparsity in nearly-linear time. In Philip N. Klein, editor, Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19, pages 2215–2229. SIAM, 2017.

[11] Nikhil Bansal, Sham Garg, Jesper Nederlof, and Nikhil Vyas. Faster space-efficient algorithms for subset sum and k-sum. In Hamed Hatami, Pierre McKenzie, and Valerie King, editors, Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, Montreal, QC, Canada, June 19-23, 2017, pages 198–209. ACM, 2017.

[12] Ziv Bar-Yossef, T. S. Jayram, Robert Krauthgamer, and Ravi Kumar. Approximating edit distance efficiently. In 45th Symposium on Foundations of Computer Science (FOCS 2004), 17-19 October 2004, Rome, Italy, Proceedings, pages 550–559. IEEE Computer Society, 2004.

[13] Ilya Baran, Erik D. Demaine, and Mihai Patrascu. Subquadratic algorithms for 3sum. Algorithmica, 50(4):584–596, 2008.

[14] Tugkan Batu, Funda Ergün, and Süleyman Cenk Sahinalp. Oblivious string embeddings and edit distance approximations. In Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2006, Miami, Florida, USA, January 22-26, 2006, pages 792–801. ACM Press, 2006.

[15] Richard Bellman. Dynamic Programming. Princeton University Press, Princeton, NJ, USA, 1957.

[16] Anand Bhalgat, Ashish Goel, and Sanjeev Khanna. Improved approximation results for stochastic knapsack problems. In Dana Randall, editor, Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2011, San Francisco, California, USA, January 23-25, 2011, pages 1647–1665. SIAM, 2011.

[17] Karl Bringmann. A near-linear pseudopolynomial time algorithm for subset sum. In Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’17, pages 1073–1084, Philadelphia, PA, USA, 2017. Society for Industrial and Applied Mathematics.

[18] Karl Bringmann. personal communication, April 2018.

[19] Timothy M. Chan. Approximation schemes for 0-1 knapsack. In Raimund Seidel, editor, 1st Symposium on Simplicity in Algorithms, SOSA 2018, January 7-10, 2018, New Orleans, LA, USA, volume 61 of OASICS, pages 5:1–5:12. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2018.

[20] Timothy M. Chan and Moshe Lewenstein. Clustered integer 3sum via additive combinatorics. In Proceedings of the Forty-seventh Annual ACM Symposium on Theory of Computing, STOC ’15, pages 31–40, New York, NY, USA, 2015. ACM.

[21] Ioannis Chatzigiannakis, Piotr Indyk, Fabian Kuhn, and Anca Muscholl, editors. 44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, July 10-14, 2017, Warsaw, Poland, volume 80 of LIPIcs, 2017. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik.
[22] Thomas H Cormen. *Introduction to algorithms*. MIT press, 2009.

[23] Marek Cygan, Holger Dell, Daniel Lokshtanov, Dániel Marx, Jesper Nederlof, Yoshio Okamoto, Ramamohan Paturi, Saket Saurabh, and Magnus Wahlstrom. On problems as hard as cnf-sat. In *Proceedings of the 2012 IEEE Conference on Computational Complexity (CCC)*, CCC ’12, pages 74–84, Washington, DC, USA, 2012. IEEE Computer Society.

[24] Marek Cygan, Marcin Mucha, Karol Wegrzycki, and Michal Wlodarczyk. On problems equivalent to (min, +)-convolution. In Chatzigiannakis et al. [21], pages 22:1–22:15.

[25] Ari Freund. Improved subquadratic 3sum. *Algorithmica*, 77(2):440–458, 2017.

[26] Zvi Galil and Oded Margalit. An almost linear-time algorithm for the dense subset-sum problem. *SIAM J. Comput.*, 20(6):1157–1189, 1991.

[27] Zvi Galil and Oded Margalit. An almost linear-time algorithm for the dense subset-sum problem. In Javier Leach Albert, Burkhard Monien, and Mario Rodríguez-Artalejo, editors, *Automata, Languages and Programming, 18th International Colloquium, ICALP’91, Madrid, Spain, July 8-12, 1991, Proceedings*, volume 510 of *Lecture Notes in Computer Science*, pages 719–727. Springer, 1991. ISBN 3-540-54233-7.

[28] Zvi Galil and Oded Margalit. personal communication, 2017.

[29] George Gens and Eugene Levner. Computational complexity of approximation algorithms for combinatorial problems. In Jirí Becvár, editor, *Mathematical Foundations of Computer Science 1979, Proceedings, 8th Symposium, Olomouc, Czechoslovakia, September 3-7, 1979*, volume 74 of *Lecture Notes in Computer Science*, pages 292–300. Springer, 1979.

[30] George Gens and Eugene Levner. A fast approximation algorithm for the subset-sum problem. *INFOR: Information Systems and Operational Research*, 32(3):143–148, 1994.

[31] Georgii V Gens and Eugenii V Levner. Fast approximation algorithms for knapsack type problems. In *Optimization Techniques*, pages 185–194. Springer, 1980.

[32] GV Gens and EV Levner. Approximation algorithm for some scheduling problems. *Eng. Cybernetics*, 6:38–46, 1978.

[33] Beat Gfeller. Finding longest approximate periodic patterns. In Frank Dehne, John Iacono, and Jörg-Rüdiger Sack, editors, *Algorithms and Data Structures - 12th International Symposium, WADS 2011, New York, NY, USA, August 15-17, 2011. Proceedings*, volume 6844 of *Lecture Notes in Computer Science*, pages 463–474. Springer, 2011.

[34] Omer Gold and Micha Sharir. Improved bounds for 3sum, k-sum, and linear degeneracy. In Kirk Pruhs and Christian Sohler, editors, *25th Annual European Symposium on Algorithms, ESA 2017, September 4-6, 2017, Vienna, Austria*, volume 87 of *LIPIcs*, pages 42:1–42:13. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017.

[35] Brian Hayes. Computing science: The easiest hard problem. *American Scientist*, 90(2):113–117, 2002.
[36] Ellis Horowitz and Sartaj Sahni. Computing partitions with applications to the knapsack problem. *J. ACM*, 21(2):277–292, 1974.

[37] Nick Howgrave-Graham and Antoine Joux. New generic algorithms for hard knapsacks. In Henri Gilbert, editor, *Advances in Cryptology - EUROCRYPT 2010, 29th Annual International Conference on the Theory and Applications of Cryptographic Techniques, French Riviera, May 30 - June 3, 2010. Proceedings*, volume 6110 of *Lecture Notes in Computer Science*, pages 235–256. Springer, 2010.

[38] Chloe Ching-Yun Hsu and Chris Umans. On multidimensional and monotone k-sum. *To appear at MFCS 2017*, 2017.

[39] Oscar H. Ibarra and Chul E. Kim. Fast approximation algorithms for the knapsack and sum of subset problems. *J. ACM*, 22(4):463–468, 1975.

[40] Klaus Jansen and Stefan Erich Julius Kraft. A faster FPTAS for the unbounded knapsack problem. In Zsuzsanna Lipták and William F. Smyth, editors, *Combinatorial Algorithms - 26th International Workshop, IWOCA 2015, Verona, Italy, October 5-7, 2015, Revised Selected Papers*, volume 9538 of *Lecture Notes in Computer Science*, pages 274–286. Springer, 2015.

[41] Allan Grønlund Jørgensen and Seth Pettie. Threesomes, degenerates, and love triangles. In *55th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2014, Philadelphia, PA, USA, October 18-21, 2014*, pages 621–630. IEEE Computer Society, 2014.

[42] Edward G. Coffman Jr. and George S. Lueker. *Probabilistic analysis of packing and partitioning algorithms*. Wiley-Interscience series in discrete mathematics and optimization. Wiley, 1991.

[43] Richard M. Karp. Reducibility among combinatorial problems. In Raymond E. Miller and James W. Thatcher, editors, *Proceedings of a symposium on the Complexity of Computer Computations, held March 20-22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York.*, The IBM Research Symposia Series, pages 85–103. Plenum Press, New York, 1972.

[44] R.M. Karp. The fast approximate solution to hard combinatorial problems. *Proceedings of the 6th Southeastern Conference on Combinatorics, Graph Theory and Computing*, pages 15–31, 1975.

[45] Hans Kellerer and Ulrich Pferschy. Improved dynamic programming in connection with an FPTAS for the knapsack problem. *J. Comb. Optim.*, 8(1):5–11, 2004.

[46] Hans Kellerer, Ulrich Pferschy, and Maria Grazia Speranza. An efficient approximation scheme for the subset-sum problem. In Hon Wai Leong, Hiroshi Imai, and Sanjay Jain, editors, *Algorithms and Computation, 8th International Symposium, ISAAC ’97, Singapore, December 17-19, 1997, Proceedings*, volume 1350 of *Lecture Notes in Computer Science*, pages 394–403. Springer, 1997.

[47] Hans Kellerer, Ulrich Pferschy, and David Pisinger. *Knapsack problems*. Springer, 2004.
[48] Konstantinos Koiliaris and Chao Xu. A faster pseudopolynomial time algorithm for subset sum. In Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’17, pages 1062–1072, Philadelphia, PA, USA, 2017. Society for Industrial and Applied Mathematics.

[49] Marvin Künnemann, Ramamohan Paturi, and Stefan Schneider. On the fine-grained complexity of one-dimensional dynamic programming. In Chatzigiannakis et al. [21], pages 21:1–21:15.

[50] Eugene L Lawler. Fast approximation algorithms for knapsack problems. Mathematics of Operations Research, 4(4):339–356, 1979.

[51] Ohad Lipsky and Ely Porat. Approximate pattern matching with the $L_1$, $L_2$ and $L_\infty$ metrics. Algorithmica, 60(2):335–348, 2011.

[52] George B Mathews. On the partition of numbers. Proceedings of the London Mathematical Society, 1(1):486–490, 1896.

[53] Ralph C. Merkle and Martin E. Hellman. Hiding information and signatures in trapdoor knapsacks. IEEE Trans. Information Theory, 24(5):525–530, 1978.

[54] Stephan Mertens. The easiest hard problem: Number partitioning. Computational Complexity and Statistical Physics, 125(2):125–139, 2006.

[55] Jesper Nederlof, Erik Jan van Leeuwen, and Ruben van der Zwaan. Reducing a target interval to a few exact queries. In Branislav Rovan, Vladimiro Sassone, and Peter Widmayer, editors, Mathematical Foundations of Computer Science 2012 - 37th International Symposium, MFCS 2012, Bratislava, Slovakia, August 27-31, 2012. Proceedings, volume 7464 of Lecture Notes in Computer Science, pages 718–727. Springer, 2012.

[56] Richard Schroeppel and Adi Shamir. A $t=O(2^{n/2})$, $s=O(2^{n/4})$ algorithm for certain np-complete problems. SIAM J. Comput., 10(3):456–464, 1981.

[57] Daniel D. Sleator and Robert Endre Tarjan. A data structure for dynamic trees. J. Comput. Syst. Sci., 26(3):362–391, June 1983. ISSN 0022-0000.

[58] Virginia Vassilevska Williams. Hardness of easy problems: Basing hardness on popular conjectures such as the strong exponential time hypothesis (invited talk). In Thore Husfeldt and Iyad A. Kanj, editors, 10th International Symposium on Parameterized and Exact Computation, IPEC 2015, September 16-18, 2015, Patras, Greece, volume 43 of LIPIcs, pages 17–29. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015.

[59] Gerhard J. Woeginger. When does a dynamic programming formulation guarantee the existence of a fully polynomial time approximation scheme (fptas)? INFORMS Journal on Computing, 12(1):57–74, 2000.

[60] Uri Zwick. All pairs shortest paths in weighted directed graphs – exact and almost exact algorithms. In 39th Annual Symposium on Foundations of Computer Science, FOCS ’98, November 8-11, 1998, Palo Alto, California, USA, pages 310–319. IEEE Computer Society, 1998.
A Proof of Theorem 2.2

Corollary A.1 (restated Observation 2.2). If we can weakly $(1 - \varepsilon)$-approximate SUBSET SUM in time $\tilde{O}(T(n, \varepsilon))$, then we can $(1 - \varepsilon)$-approximate PARTITION in the same $\tilde{O}(T(n, \varepsilon))$ time.

Proof. Let $|Z| = n$ be the initial set of items. We run a weak $(1 - \varepsilon)$-approximation algorithm for SUBSET SUM with target $b = \Sigma(Z)/2$. Let $Z^*$ denote the optimal partition of set $Z$:

$$Z^* = \arg \max_{Z' \subseteq Z, \Sigma(Z') \leq b} \Sigma(Z').$$

By the definition of weak $(1 - \varepsilon)$-approximation for SUBSET SUM we get a solution $Z_W$ such that:

$$(1 - \varepsilon)\Sigma(Z^*) \leq \Sigma(Z_W) \quad \text{and} \quad \Sigma(Z_W) < (1 + \varepsilon)b$$

If $\Sigma(Z_W) \leq b$ then it is a correct solution for PARTITION. Otherwise we take a set $Z'_W = Z \setminus Z_W$. Because $\Sigma(Z)/2 = b$ we know that $\Sigma(Z'_W) < b$. Additionally we know, that $\Sigma(Z_W) < (1 + \varepsilon)b$, so $(1 - \varepsilon)b < \Sigma(Z_W')$. Similarly, because $Z^* \leq b$, we have:

$$(1 - \varepsilon)\Sigma(Z^*) \leq (1 - \varepsilon)b < \Sigma(Z'_W) \leq \Sigma(Z^*) \leq b.$$  

So $\Sigma(Z'_W)$ follows the definition of approximation for PARTITION. The running time follows because $T(n, 1/\varepsilon)$ must be superlinear (algorithm needs to read input at least) and we executed the weak $(1 - \varepsilon)$-approximation SUBSET SUM algorithm only constant number of times. 

B Proofs of the Rounding Lemmas

Lemma B.1. For $k$ natural numbers $x_1, x_2, \ldots, x_k$ and positive $q, \varepsilon$ such that $q \leq \sum_{i=1}^{k} x_i$ and $0 < \varepsilon < 1$, it holds:

$$\sum_{i=1}^{k} x_i \leq \frac{q\varepsilon}{k} \sum_{i=1}^{k} \left\lceil \frac{kx_i}{q\varepsilon} \right\rceil < (1 + \varepsilon)\sum_{i=1}^{k} x_i$$

Proof. Let $x_i = \frac{q\varepsilon}{k}c_i + d_i$ where $0 < d_i \leq \frac{q\varepsilon}{k}$. If some $x_i = 0$ then we set $c_i = d_i = 0$, however we know there is at least one positive $d_i$ (we will use this fact later). We have $\left\lceil \frac{kx_i}{q\varepsilon} \right\rceil = c_i + 1$. First, note that:

$$\sum_{i=1}^{k} x_i = \frac{q\varepsilon}{k} \sum_{i=1}^{k} c_i + \sum_{i=1}^{k} d_i \leq \frac{q\varepsilon}{k} \sum_{i=1}^{k} c_i + q\varepsilon = \frac{q\varepsilon}{k} \sum_{i=1}^{k} (c_i + 1),$$

what proves the left inequality. To handle the right inequality we take advantage of the assumption $\sum_{i=1}^{k} x_i \geq q$ and get:

$$(1 + \varepsilon)\sum_{i=1}^{k} x_i = \varepsilon \sum_{i=1}^{k} x_i + \sum_{i=1}^{k} x_i \geq q\varepsilon + \sum_{i=1}^{k} x_i = q\varepsilon + \frac{q\varepsilon}{k} \sum_{i=1}^{k} c_i + \sum_{i=1}^{k} d_i =$$

$$= \frac{q\varepsilon}{k} \sum_{i=1}^{k} (c_i + 1) + \sum_{i=1}^{k} d_i > \frac{q\varepsilon}{k} \sum_{i=1}^{k} (c_i + 1).$$

\qed
Lemma B.2. For \( k \) natural numbers \( x_1, x_2, \ldots, x_k \) and positive \( q, \varepsilon \) such that \( q \leq \sum_{i=1}^{k} x_i \) and \( 0 < \varepsilon < 1 \), it holds:

\[
(1 - \varepsilon) \sum_{i=1}^{k} x_i < \frac{q\varepsilon}{k} \sum_{i=1}^{k} \left\lfloor \frac{kx_i}{q\varepsilon} \right\rfloor \leq \sum_{i=1}^{k} x_i.
\]

Proof. The proof is very similar to the proof of Lemma B.1, however now we represent \( x_i \) as \( \frac{q\varepsilon}{k} c_i + d_i \) where \( 0 \leq d_i < \frac{q\varepsilon}{k} \). We have \( \left\lfloor \frac{kx_i}{q\varepsilon} \right\rfloor = c_i \). The right inequality holds because:

\[
\sum_{i=1}^{k} x_i = \frac{q\varepsilon}{k} \sum_{i=1}^{k} c_i + \sum_{i=1}^{k} d_i \geq \frac{q\varepsilon}{k} \sum_{i=1}^{k} c_i
\]

and the left one can be proven as follows:

\[
(1 - \varepsilon) \sum_{i=1}^{k} x_i = \sum_{i=1}^{k} x_i - \varepsilon \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} x_i - q\varepsilon = \left( \sum_{i=1}^{k} \frac{q\varepsilon}{k} c_i + d_i \right) - q\varepsilon \leq \frac{q\varepsilon}{k} \sum_{i=1}^{k} c_i.
\]

\[\blacksquare\]

C Problems Definitions

C.1 Exact problems

**KNAPSACK**

**Input:** A set of \( n \) items \( \{(v_1, w_1), \ldots, (v_n, w_n)\} \)

**Task:** Find \( x_1, \ldots, x_n \) such that:

\[
\text{maximize } \sum_{j=1}^{n} v_j x_j
\]

subject to \( \sum_{j=1}^{n} w_j x_j \leq t \), \( x_j \in \{0, 1\}^n, \ j = 1, \ldots, n. \)

Sometimes, instead of exact solution \( x_1, \ldots, x_n \) in KNAPSACK-type problems one needs to return the value of such solution. In decision version of such problems we are given capacity \( t \) and value \( v \) and ask if there is a subset of items with the total capacity not exceeding \( t \) and total value exactly \( v \) (e.g., see discussion in [24, 47]).
### Unbounded Knapsack

**Input:** A set of \( n \) items \( \{(v_1, w_1), \ldots, (v_n, w_n)\} \)

**Task:** Find \( x_1, \ldots, x_n \) such that:

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{n} v_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} w_j x_j \leq t,
\end{align*}
\]

\( x_j \in \mathbb{N} \cup \{0\}, \ j = 1, \ldots n. \)

### Subset Sum

**Input:** A set of \( n \) integers \( \{w_1, \ldots, w_n\} \)

**Task:** Find \( x_1, \ldots, x_n \) such that:

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{n} w_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} w_j x_j \leq t,
\end{align*}
\]

\( x_j \in \{0,1\}^n, \ j = 1, \ldots n. \)

### Partition

**Input:** A set of \( n \) integers \( \{w_1, \ldots, w_n\} \) and \( b = \frac{1}{2} \sum_{i=1}^{n} w_i \)

**Task:** Find \( x_1, \ldots, x_n \) such that:

\[
\begin{align*}
\text{maximize} & \quad \sum_{j=1}^{n} w_j x_j \\
\text{subject to} & \quad \sum_{j=1}^{n} w_j x_j \leq b,
\end{align*}
\]

\( x_j \in \{0,1\}^n, \ j = 1, \ldots n. \)

### (min,+-)convolution

**Input:** Sequences \( (a[i])_{i=0}^{n-1}, (b[i])_{i=0}^{n-1} \)

**Task:** Output sequence \( (c[i])_{i=0}^{n-1} \), such that \( c[k] = \min_{i+j=k} (a[i] + b[j]) \)

### k-SUM

**Input:** \( k - 1 \) sets \( A_1, A_2, \ldots, A_{k-1} \) and the set \( S \) of integers, each with cardinality at most \( n \).

**Task:** Is there a \( (a_1, \ldots, a_{k-1}, s) \in A_1 \times \ldots \times A_{k-1} \times S \) such that \( a_1 + \ldots + a_{k-1} = s \)

### 3SUM

**Input:** 3 sets \( A, B, C \) of integers, each with cardinality at most \( n \).

**Task:** Is there a triple \( (a, b, c) \in A \times B \times C \) such that \( a + b = c \)
Tree Sparsity

Input: A rooted tree $T$ with a weight function $x: V(T) \rightarrow \mathbb{N}$, parameter $k$

Task: Find the maximal total weight of a rooted subtree of size $k$

C.2 Approximate problems definition

Let $\Sigma(S)$ denote the sum of elements in $S$. The $V(I)$ denotes the total value of items $I$ and $W(I)$ denotes the total weight of items.

$(1 - \varepsilon)$-approximation of Knapsack

Input: A set $S = \{(v_1, w_1), \ldots, (v_n, w_n)\}$ items and a target number $t$

Task: Let $Z^*$ be the optimal solution of exact Knapsack with target $t$. The $(1 - \varepsilon)$-approximate algorithm for Knapsack returns $Z_H$ such that $(1 - \varepsilon)V(Z^*) \leq V(Z_H) \leq V(Z^*)$ and $W(Z_H) \leq t$.

Analogous definition is for Unbounded Knapsack.

$(1 - \varepsilon)$-approximation of Subset Sum

Input: A set $S = \{a_1, \ldots, a_n\}$ of positive integers and a target number $t$

Task: Let $Z^*$ be the optimal solution of exact Subset Sum with target $t$. The $(1 - \varepsilon)$-approximate algorithm returns $Z_H$ such that $(1 - \varepsilon)\Sigma(Z^*) \leq \Sigma(Z_H) \leq \Sigma(Z^*)$.

$(1 - \varepsilon)$-approximation of Partition

Input: A set $S = \{a_1, \ldots, a_n\}$ of positive integers

Task: Let $Z^*$ be the optimal solution of exact Partition. The $(1 - \varepsilon)$-approximate algorithm returns $Z_H$ such that $(1 - \varepsilon)\Sigma(Z^*) \leq \Sigma(Z_H) \leq \Sigma(Z^*)$.

Weak $(1 - \varepsilon)$-approximation of Subset Sum

Input: A set $S = \{a_1, \ldots, a_n\}$ of positive integers and a target number $t$

Task: Let $Z^*$ be the optimal solution of exact Subset Sum with target $t$. The $(1 - \varepsilon)$-approximate algorithm returns $Z_H$ such that $(1 - \varepsilon)\Sigma(Z^*) \leq \Sigma(Z_H) \leq \Sigma(Z^*)$ or $t \leq \Sigma(Z_H) \leq (1 + \varepsilon)t$.

Approximate 3SUM [33]

Input: Three sets $A, B, C$ of positive integers, each with cardinality at most $n$.

Task: The algorithm:

- concludes that no triple $(a, b, c) \in A \times B \times C$ with $a + b = c$ exists, or
- it outputs a triple $(a, b, c) \in A \times B \times C$ with $a + b \in [c/(1 + \varepsilon), c(1 + \varepsilon)]$
Approximate \((\min, +)\)-convolution

**Input:** Sequences \(A[0, \ldots, n-1], B[0, \ldots, n-1]\) of positive integers and approximation parameter \(0 < \varepsilon \leq 1\)

**Task:** Assume that \(\text{OPT}[k] = \min_{0 \leq i \leq k} (A[i] + B[k - i])\) is the exact \((\min, +)\)-convolution of \(A\) and \(B\). The task is to output a sequence \(C[0, \ldots, n-1]\) such that \(\forall i \, \text{OPT}[i] \leq C[i] \leq (1 + \varepsilon) \text{OPT}[i]\)