QUANTUM EXTENDED CRYSTAL PDE’S

AGOSTINO PRÁSTARO

Department SBAI - Mathematics, University of Rome "La Sapienza", Via A.Scarpa 16, 00161 Rome, Italy.
E-mail: agostino.prastaro@uniroma1.it; prastaro@dmmm.uniroma1.it

ABSTRACT. Our recent results on extended crystal PDE’s are generalized to PDE’s in the category $\mathcal{Q}_S$ of quantum supermanifolds. Then obstructions to the existence of global quantum smooth solutions for such equations are obtained, by using algebraic topologic techniques. Applications are considered in details to the quantum super Yang-Mills equations. Furthermore, our geometric theory of stability of PDE’s and their solutions, is also generalized to quantum extended crystal PDE’s. In this way we are able to identify quantum equations where their global solutions are stable at finite times. These results, are also extended to quantum singular (super)PDE’s, introducing (quantum extended crystal singular (super) PDE’s).

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1. Introduction

In a previous paper [73] we proved that PDE’s can be considered as extended crystals, in the sense that their integral bordism groups can be seen as crystallographic subgroups extensions. In this paper we aim generalize that result to quantum super PDE’s. This is possible, since we utilize our geometric theory of PDE’s considered in the category $\mathcal{Q}$ of quantum manifolds and in the category $\mathcal{Q}_S$ of quantum supermanifolds [63, 64, 68, 69, 70, 71, 78, 79, 80]. Then we relate integral bordism groups of quantum super PDE’s to crystallographic groups. The main results are Theorem 2.4 and Theorem 2.16. The first relates formal integrability and complete integrability of quantum super PDE’s to crystallographic groups. In this way we can consider quantum super PDE’s as quantum extended crystallographic structures. In the second theorem, we identify an obstruction characterizing existence of global quantum smooth solutions. This is called quantum crystal obstruction of a quantum super PDE. Applications to quantum super Yang-Mills PDE’s are given too. Another characterizing aspect of this paper is a new geometric theory for stability of quantum super PDE’s and their solutions. This is made by extending to the category of quantum super manifolds $\mathcal{Q}_S$ our previous geometric approach

1See also companion paper [81].

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on commutative PDE’s stability [72, 73, 74, 75, 76, 77]. Here a $k$-order quantum (super) PDE is considered as a subset $\hat{E}_k \subset J^k\hat{D}(W)$ of the $k$-jet-derivative space $J^k\hat{D}(W)$, built on some fiber bundle $\pi : W \to M$, in the category of quantum smooth (super)manifolds. Then, to investigate the stability of a regular solution $D^k\hat{s}(M) \equiv V \subset \hat{E}_k$, of $\hat{E}_k \subset J^k\hat{D}(W)$, one considers the linearization of $\hat{E}_k$ at the solution $V$. The integrable solutions of the linearized equation $(D^k)s^{*}\nu T^sE_k \equiv \hat{E}_k[s] \subset J^k\hat{D}(s^{*}\nu T\hat{W}) \equiv J^k\hat{D}(\hat{E}[s])$, represent the infinitesimal admissible perturbations of the original solution. Then, if to an initial Cauchy data for such linearized equation there correspond solutions (perturbations) that oscillate around the zero solution (of the linearized equation), or remain limited around such zero solution, then the solution $s$ is said to be stable, otherwise $s$ is called unstable. Taking into account that the linearized equation $\hat{E}_k[s]$ belongs to a vector neighborhood of $\hat{E}_k$, at the solution $V$, and that integrable solutions of $\hat{E}_k[s]$ are infinitesimal vertical symmetries of $\hat{E}_k$, it follows that such perturbations deform the original solution $V \subset \hat{E}_k$ into solutions $\tilde{V} \subset \hat{E}_k$ such that, if $V$ is stable, remain into suitable neighborhoods of the same $V$. When, instead the perturbations blow-up, then $V$ is unstable. The blowing-up of the perturbation corresponds to the fact that such a solution of the linearized equation $\hat{E}_k[s]$ is not regular in all of its points, but there are present singular points. Then in the cases where $V$ is unstable, between the solutions of above type $\tilde{V}$, there are ones that are also singular and this fact just characterizes unstable solutions of $\hat{E}_k$. This approach to the stability can be related to the Lyapunov concept of stability in functional analysis [34], and it is founded on the assumption that the possible perturbations can influence only the given solution, say $V \subset \hat{E}_k$, but do not have any influence on the same equation $\hat{E}_k$.

On the other hand, we can more generally assume that perturbations can change the same original equation. In such a case we can ask wether a given solution of the original equation can change for ”little” perturbations of the same equation. Then we talk about (un)stable equations. This last approach is, instead, related to the concept of Ulam (un)stability for functional equations [102].

We prove that all above points of view for stability in quantum (super) PDE’s can be unified in the geometric theory of quantum (super) PDE’s on the ground of integral bordism groups. This extends to the category of quantum super PDE’s, our previous results on the stability of commutative PDE’s [72, 73, 74, 75, 76, 77].

In this paper, the main results on the quantum super PDE’s stability are the following. Theorem 3.18 that gives some criteria to recognize functional stability in any quantum (super) PDE’s. Theorem 3.21 that relates functional stability with quantum $(k + 1)$-connections. Theorem 3.24 proving that to a formally integrable and completely integrable quantum (super) PDE, one can canonically associate another quantum (super) PDE $(S)\hat{E}_k$, $stable quantum extended crystal (super) PDE$ of $\hat{E}_k$, having the same regular smooth solutions of $\hat{E}_k$, but in $(S)\hat{E}_k$ these solution are

\footnote{For basic informations on the geometry of PDE’s see also the following refs.[8, 15, 16, 19, 35, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67]. For basic informations on some subjects of differential topology and algebraic topology, related to this paper, see also refs.[9, 22, 36, 45, 88, 93, 94, 97, 104, 105].}
stable.\textsuperscript{3} Theorem 3.30 that gives a criterion to recognize the average asymptotic stability with respect to quantum frames. Applications to quantum super d’Alembert equation and quantum super Navier-Stokes equation are considered too. Finally we extend above results also to quantum singular (super) PDE’s, and we characterize quantum extended crystal singular (super) PDE’s. For such equations we identify algebraic-topological obstructions to the existence of global (smooth) solutions solving boundary value problems and crossing singular points too.

The paper, after the Introduction, contains two more sections. In the first section we relate the integral bordism groups of quantum super PDE’s to crystallographic groups and recognize a topologic algebraic obstruction to the existence of quantum smooth solutions. Applications of these results to the quantum super Yang-Mills equation are considered. In Section 3 we formulate a geometric theory of stability for solutions of quantum super PDE’s. We follow some our previous works devoted to the algebraic topological characterization of PDE’s stability and their solutions stability [72, 73, 74, 75, 76, 77]. Thus, in this paper the stability of quantum (super) PDE’s is studied in the framework of the geometric theory of quantum (super) PDE’s, and in the framework of the bordism groups of quantum (super) PDE’s. In particular we identify criteria to recognize quantum (super) PDE’s that are stable (in extended Ulam sense) and in their regular smooth solutions do not occur unstabilities in finite times. We call such equations stable quantum extended crystal (super) PDE’s. Applications to the quantum super d’Alembert equation and the quantum super Navier-Stokes equation respectively are explicitly considered.

Section 4 is devoted to extend above results also to quantum singular super PDE’s. The main results in this section is Theorem 4.8 that identifies conditions in order to recognize global (smooth) solutions of quantum singular super PDE’s crossing singular points. There we characterize quantum 0-crystal singular super PDE’s, i.e., quantum singular super PDE’s having smooth global solutions crossing singular points, stable at finite times.

2. INTEGRAL BORDISM GROUPS OF QUANTUM SUPER PDE’s vs CRYSTALLOGRAPHIC GROUPS

In this section we extend to quantum super PDE’s our previous results on the algebraic topological crystal characterization of commutative PDE’s.\textsuperscript{4}

\textbf{Remark 2.1.} Here and in the following we shall denote the boundary $\partial V$ of a compact quantum supermanifold $V$, of dimension $m|n$, with respect to a quantum superalgebra $A$, split in the form $\partial V = N_0 \cup P \cup N_1$, where $N_0$ and $N_1$ are two disjoint $(m-1|n-1)$-dimensional quantum sub-supermanifolds of $V$, that are not necessarily closed, and $P$ is another $(m-1|n-1)$-dimensional quantum sub-supermanifold of $V$. For example, if $V = \tilde{D}^{m|n} \times \tilde{D}^{1|1}$, where $\tilde{D}^{r|s} \subset \tilde{S}^{r|s}$ is the $(r|s)$-dimensional quantum superdisk, contained in the $(r|s)$-dimensional quantum supersphere, one has that $\dim V = (m+1|n+1)$ and $N_0 = \tilde{D}^{m|n} \times \{0\},$

\textsuperscript{3}This theorem allows to avoid all the problems present in the applications, related to finite instability of solutions.

\textsuperscript{4}Quantum super PDE’s are PDE’s in the category $\Omega_S$ of quantum supermanifolds, in the sense introduced by A.Prástaro [55, 56, 59, 60, 61, 62, 66, 67, 68, 69, 76, 80].
\[ N_{1} = \hat{D}^{m|n} \times \{ 1 \}, \quad P = \partial \hat{D}^{m|n} \times \hat{D}^{1|1} \cong \hat{S}^{m-1|n-1} \times \hat{D}^{1|1}. \] Therefore \( \dim N_{0} = \dim N_{1} = \dim P = (m|n) \). Note that since \( \hat{D}^{m|n} \times \hat{D}^{1|1} = \hat{D}^{m+1|n+1} \), therefore we can also write \( \partial \hat{V} = \partial \hat{D}^{m+1|n+1} = \hat{S}^{m|n} = \hat{S}^{m-1|n-1} \times \hat{S}^{1|1} \). Since

\[
\partial \hat{V} = \partial (\hat{D}^{m|n} \times \hat{D}^{1|1}) = (\partial \hat{D}^{m|n}) \times \hat{D}^{1|1} \cup \hat{D}^{m|n} \times \partial \hat{D}^{1|1} = \hat{S}^{m-1|n-1} \times \hat{D}^{1|1} \cup \hat{D}^{m|n} \times \hat{S}^{0|0}. \]

Therefore, \( \partial \hat{V} \) is obtained by means of the quantum surgering removing \( \hat{S}^{m-1|n-1} \times \hat{D}^{1|1} \subset \hat{S}^{m|n} \). (For details on quantum surgering see [75].) Of course if \( V = \hat{S}^{m|n} \times \hat{D}^{1|1} \), then \( P = \emptyset \), hence \( \partial \hat{V} = \hat{S}^{m|n} \times \{ 0 \} \cup \hat{S}^{m|n} \times \{ 1 \} \).

This example shows that if \( V \) is a solution of a quantum super PDE, then it can be obtained by propagating an initial Cauchy hypersurface \( X \subset V \), \( \dim X = (m|n) \), by means of an integrable full quantum vector field \( \zeta : V \to TV \cong \text{Hom}_{\mathbb{Z}}(A; TV), \partial \phi = \zeta \), where \( \phi : A \times V \to V \). (See Fig. 1.)

![Figure 1. Quantum solution \( V \), of dimension \( m + 1|n + 1 \) over a quantum superalgebra \( A \), propagating \( X \), \( \dim X = (m|n) \).](image)

Let us emphasize also, that in some cases solutions can be obtained also by flows of integrable vector fields \( \zeta : V \to TV \), i.e., \( \zeta = \partial \psi \), with \( \psi : \mathbb{R} \times V \to V \). For example if \( V = \hat{D}^{m|n} \times I \), with \( I \equiv [0, 1] \subset \mathbb{R} \), then \( \dim V = m + 1|n \), and \( \partial V = M_{0} \cup P \cup M_{1} \), with \( P = \hat{S}^{m-1|n-1} \times I \), \( M_{0} = \hat{D}^{m|n} \times \{ 0 \} \), \( M_{1} = \hat{D}^{m|n} \times \{ 1 \} \). So we get \( \dim P = m|n - 1 \), \( \dim M_{0} = \dim M_{1} = m|n \). Therefore \( \partial V \) has some components (\( M_{0} \) and \( M_{1} \)) that have only the even dimension dropped by 1, with respect to \( V \), and other one (\( P \)) where also the odd dimension drops by 1.

Let us also recall that with the term quantum solutions we mean integral bordisms relating Cauchy quantum hypersurfaces of \( \hat{E}_{k+s} \), contained in \( J_{m|n}^{k+s}(W) \), but not necessarily contained into \( \hat{E}_{k+s} \). (For details see refs. [69, 70, 71, 79].)

**Definition 2.2.** We say that a quantum super PDE \( \hat{E}_{k} \subset J_{m|n}^{k}(W) \) is an quantum extended 0-crystal super PDE, if its weak integral bordism group \( \Omega_{m-1|n-1,w}^{\hat{E}_{k}} \) is zero.

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5Recall that a \((m|n)\)-dimensional quantum supersphere, \( \hat{S}^{m|n} \), over a quantum superalgebra \( A \), is the Alexandrov compactification of \( A^{m|n} \), i.e., \( \hat{S}^{m|n} = A^{m|n} \cup \{ \infty \} \). A \((m|n)\)-dimensional quantum supersphere over a quantum superalgebra \( A \), is a connected compact sub-supersmanifold \( \hat{D}^{m|n} \subset \hat{S}^{m|n} \), of dimension \( m|n \) over \( A \), such that \( \partial \hat{D}^{m|n} \cong \hat{S}^{m-1|n-1} \), i.e., with boundary \( \partial \hat{D}^{m|n} \) diffeomorphic to \( \hat{S}^{m-1|n-1} \). For details on such quantum supermanifolds see [79].

6We denote disjoint union by the symbol \( \bigcup \) or \( \cup \).
Theorem 2.3. (Criterion to recognize quantum extended 0-crystal super PDE’s). Let \( \hat{E}_k \subset J_{m|n}^k(W) \) be a formally quantum integrable and completely quantum superintegrable quantum super PDE such that \( W \) is contractible. If \( m - 1 \neq 0 \) and \( n - 1 \neq 0 \), then \( E_k \) is a quantum extended 0-crystal super PDE.

Proof. In fact, one has the following isomorphisms, (see [70]):

\[
\Omega_{m-1|n-1,w}^{\hat{E}_k} \cong H_{m-1|n-1}(W; A) \\
\cong (A_0 \otimes_K H_{m-1}(W; K)) \bigoplus (A_1 \otimes_K H_{n-1}(W; K)).
\]

Thus, when \( W \) is contractible, and \( m - 1 \neq 0, n - 1 \neq 0 \), one has \( H_{m-1}(W; K) = H_{n-1}(W; K) = 0 \), hence we get \( \Omega_{m-1|n-1,w}^{\hat{E}_k} = 0 \). \( \square \)

Theorem 2.4. (Crystal structure of quantum super PDE’s). Let \( \hat{E}_k \subset J_{m|n}^k(W) \) be a formally quantum superintegrable and completely quantum superintegrable quantum super PDE. Then its integral bordism group \( \Omega_{m-1|n-1}^{\hat{E}_k} \) is an extension of some crystallographic subgroup \( G \triangleleft G(d) \). We call \( d \) the crystal dimension of \( \hat{E}_k \) and \( G(d) \) its crystal structure or crystal group.

Proof. Let us first note that there is a relation between lower dimensions integral bordisms in a commutative PDE. In fact, one has the following lemma.

Lemma 2.5. (Relations between lower dimensions integral bordisms in commutative PDE’s). Let \( E_k \subset J_n^k(W) \) be a PDE on the fiber bundle \( \pi : W \to M \), \( \dim W = m + n \), \( \dim M = n \). Let \( S^pC_p(E_k) \) be the set of all compact \( p \)-dimensional admissible integral smooth manifolds of \( E_k \). The disjoint union gives an addition on \( S^pC_p(E_k) \) with \( \emptyset \) as the zero element. Let us consider the homomorphisms \( \partial_p : S^pC_p(E_k) \to S^{p-1}C_p(E_k) \) that associates to any element \( a \in S^pC_p(E_k) \) its boundary \( \partial a = \partial_p(a) \). So we obtain the following chain complex of abelian groups (integral smooth bordisms chain complex):

\[
S^nC_n(E_k) \xrightarrow{\partial_n} S^{n-1}C_{n-1}(E_k) \xrightarrow{\partial_{n-1}} S^{n-2}C_{n-2}(E_k) \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} S^0C_0(E_k).
\]

Then the \( p \)-bordism groups \( \Omega_p^{\hat{E}_k}, 0 < p < n, \) can be represented by means of the homology of the chain complex (3).
Proof. Let us denote by \(\{S\,C_\bullet(E_k), \partial_\bullet\}\) the chain complex in (3). Then, we can build the following exact commutative diagram:

\[
\begin{array}{cccccc}
0 & 0 & S\,B_\bullet(E_k) & \cdots & S\,Z_\bullet(E_k) & S\,H_\bullet(E_k) & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \Omega^E_k & S\,\text{Bor}_\bullet(E_k) & S\,\text{Cyc}_\bullet(E_k) & 0
\end{array}
\]

where

\[
\begin{align*}
S\,B_\bullet(E_k) &= \ker(\partial_\bullet); S\,Z_\bullet(E_k) = \im(\partial_\bullet); \\
S\,H_\bullet(E_k) &= S\,Z_\bullet(E_k)/S\,B_\bullet(E_k), \\
b \in [a] \in S\,\text{Bor}_\bullet(E_k) &\Rightarrow a - b = \partial c; b \in S\,C_\bullet(E_k); b \in [a] \in S\,\text{Cyc}_\bullet(E_k) \Rightarrow \partial(a - b) = 0; \\
b \in [a] \in \Omega^E_k &\Rightarrow \begin{cases} \\
\partial a = \partial b = 0 \\
am - b = \partial c, c \in S\,C_\bullet(E_k)
\end{cases}.
\end{align*}
\]

Then from (4) it follows directly that \(\Omega^E_k \cong S\,H_p(E_k), 0 < p < n\). □

**Lemma 2.6.** (Relations between integral bordisms groups in commutative PDEs). One has the following canonical isomorphism:

\[
\bigotimes_{\Omega^E_k} \mathbb{Z}^S\,\text{Bor}_\bullet(E_k) \cong \mathbb{Z}^S\,\text{Cyc}_\bullet(E_k).
\]

Proof. Follows directly from the extension of groups given at the bottom of the commutative exact diagram (4) and some properties between extension of groups. (See, e.g., [57].) □

**Lemma 2.7.** (Relations between integral bordisms groups in commutative PDEs-2). If \(H^2(S\,\text{Cyc}_\bullet(E_k), \Omega^E_k) = 0\) one has the following canonical isomorphism:

\[
S\,\text{Bor}_\bullet(E_k) \cong \Omega^E_k \times S\,\text{Cyc}_\bullet(E_k).
\]

Proof. Follows directly from the extension of groups given at the bottom of the commutative exact diagram (4) and some properties between extension of groups. (See, e.g., [57].) □

**Example 2.8.** In particular if \(E_k \subset J^n_k(W)\) is a 0-crystal PDE with \(\Omega^E_{n-1} = 0\), one has: \(S\,\text{Bor}_{n-1}(E_k) \cong S\,\text{Cyc}_{n-1}(E_k)\). Such an example is, e.g., the d’Alembert equation \(u_{xx} - u_{yy} = 0\) on the trivial fiber bundle \(\pi : W \equiv \mathbb{R}^3 \to M \equiv \mathbb{R}^2\), \((x, y, u) \mapsto (x, y)\). In fact, in such a case one has \(\Omega^d_A = 0\) and also \(S\,\text{Bor}_1(d^\prime A) \cong S\,\text{Cyc}_1(d^\prime A) \cong 0\). (For the integral bordism group of this equation, and its generalizations, see Refs.[57, 65, 73, 76, 82].)
Lemma 2.9. (Integral ringoid of PDE). A ringoid is a structure \((A, +, \cdot)\), where \(A\) is a set and \(+\) is a binary operation such that \((A, +)\) is an abelian additive group with zero \(0 \in A\); \(\cdot\) is a partially binary operation, i.e., it is defined only for some couples \((a, b) \in A \times A\), such that it is associative, and distributive with respect to \(+\), i.e., if \(a \cdot b\) and \(a \cdot c\) are defined, then it is defined also \(a \cdot (b + c) = a \cdot b + a \cdot c\). A graded ringoid is a set \(A = \bigoplus A_n\), where each \(A_n\) is an abelian additive group and there is a partial binary operation \(\cdot\), associative, and distributive with respect to \(+\), such that if \(a \in A_n\), \(b \in A_m\) then \(a \cdot b \in A_{m+n}\), whenever it is defined. Let \(E_k \subset J^k_n(W)\) be a PDE, with \(\pi: W \to M\) a fiber bundle, \(\dim W = m + n\), \(\dim M = n\). Then the integral bordism groups \(\Omega^E_k\), \(0 \leq p \leq n-1\), identify a graded ringoid \(\Omega^E_k\), that we call integral ringoid of \(E_k\), that is an extension of a graded ringoid contained in the nonoriented bordism ring \(\Omega\).

One has the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom$_{\text{ringoid}}(\overline{K^E_k}, \mathbb{R})$} & \longrightarrow & \text{Hom$_{\text{ringoid}}(\Omega^E_k, \mathbb{R})$} & \longrightarrow & \text{Hom$_{\text{ringoid}}((n-1)\Omega, \mathbb{R})$} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & H_*(E_k) & & H_*(E_k) & & H_*(E_k) & & 0
\end{array}
\]

where \(H_*(E_k) \equiv \bigoplus_{0 \leq p \leq n-1} H_p(E_k)\), that allows us to represent differential \(p\)-conservation laws of order \(k\) by means of ringoid homomorphisms \(\Omega^E_k \to \mathbb{R}\), and as extensions of ringoid homomorphisms \(\overline{K^E_k} \to \mathbb{R}\).

Proof. Set

\[
\Omega^E_k \equiv \bigoplus_{0 \leq p \leq n-1} \Omega^E_p.
\]

Each \(\Omega^E_p\) are additive abelian groups, with addition induced by disjoint union, \(\biguplus\). Furthermore, there is a natural product induced by the cartesian product, i.e., \([X_1] \cdot [X_2] = [X_1 \times X_2] \in \Omega^E_k\), for \([X_i] \in \Omega^E_{p_i}\), \(0 \leq p_i \leq n-1\), \(i = 1, 2\), \(0 \leq p_1 + p_2 \leq n - 1\). This product it is not always defined for any closed admissible integral manifolds \(X_i\), \(i = 1, 2\), but only for ones such that \(X_1 \times X_2\) is a closed integral admissible manifold. Therefore \(\Omega^E_k\) is a graded ringoid. Set \((n-1)\Omega = \bigoplus_{0 \leq p \leq n-1} \Omega_p\). It has in a natural way a graded ringoid structure, with respect the same operations with respect to which \(\Omega\) is a graded ring. Furthermore, for any \(0 \leq p \leq n - 1\), one has the following exact sequence, (see proof of Theorem 3.16 in [71]),

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \overline{K^E_k} & \longrightarrow & \Omega^E_k & \longrightarrow & \Omega_p & \longrightarrow & 0
\end{array}
\]
As a by-product one has also the following exact commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & K^E_k & \longrightarrow & \Omega^E_k & \longrightarrow & (n-1)\Omega^E_k & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \Omega^E_k & & \Omega^E_k & & \Omega^E_k & & \\
\end{array}
\]

where \( K^E_k \equiv \bigoplus_{0 \leq p \leq n-1} K^E_p \).

A full \( p \)-conservation law is any function \( f : \Omega^E_p \rightarrow \mathbb{R}, 0 \leq p \leq n-1 \). These, identify elements of \( H_*(E_k) \equiv \bigoplus_{0 \leq p \leq n-1} H_p(E_k) \) in a natural way. In \( H_*(E_k) \) are contained also ones identified by means of differential conservation laws of order \( k \), identified with \( \mathcal{J}(E_k)^* \equiv \bigoplus_{0 \leq q \leq n-1} \mathcal{J}(E_k)^q \), with

\[
\mathcal{J}(E_k)^q \equiv \Omega^q(E_k) \cap d^{-1}(C\Omega^{q+1}(E_k)) \ inn \ \bigoplus \{ \Omega^q(E_k) \cap d^{-1}(C\Omega^{q+1}(E_k)) \}.
\]

Here, \( \Omega^q(E_k) \) is the space of smooth \( q \)-differential forms on \( E_k \) and \( C\Omega^q(E_k) \) is the space of Cartan \( q \)-forms on \( E_k \), that are zero on the Cartan distribution \( E_k \) of \( E_k \). Therefore, \( \beta \in C\Omega^q(E_k) \) iff \( \beta(\zeta_1, \cdots, \zeta_q) = \sum_{q_i} \zeta_i \) for all \( \zeta_i \in C^\infty(E_k) \). Any \( \alpha \in \mathcal{J}(E_k)^* \) identifies a ringoid homomorphism \( f[\alpha] : \Omega^E_k \rightarrow \mathbb{R} \). More precisely one has \( f[\alpha](X_1 + X_2) = f[\alpha](X_1) + f[\alpha](X_2) \), for \( \alpha \in \mathcal{J}(E_k)^p \cap \Omega^E_k \), and \( f[\alpha](X_1 \cdot X_2) = f[\alpha](X_1) \cdot f[\alpha](X_2) \), for \( \alpha \in \mathcal{J}(E_k)^p \cap \Omega^E_k \), \( X_1 \in \Omega^E_k, X_2 \in \Omega^E_k \).

In the following we extend above results to PDE’s in the category \( \Omega_S \).

**Lemma 2.10.** (Relations between lower order integral bigraded-bordisms in quantum super PDE’s). Let \( \tilde{E}_k \subset j_{m|n}(W) \) be a quantum super PDE on the fiber bundle \( \pi : W \rightarrow M, \ dim_B W = (m|n, r|s), \ dim_A M = m|n, B = A \times E, E \) a quantum superalgebra that is also a \( Z \)-module, with \( Z = Z(A) \) the centre of \( A \). Let \( S C_{p|q}(\tilde{E}_k) \) be the set of all compact \( p|q \)-dimensional, (with respect to \( A \)), admissible integral smooth manifolds of \( \tilde{E}_k \), \( 0 \leq p \leq m, 0 \leq q \leq n \). The disjoint union gives an addition on \( S C_{p|q}(\tilde{E}_k) \) with \( \emptyset \) as the zero element. Let us consider the homomorphisms \( \partial_{p|q} : S C_{p|q}(\tilde{E}_k) \rightarrow S C_{p-1|q-1}(\tilde{E}_k) \) that associates to any element \( \alpha \in S C_{p|q}(\tilde{E}_k) \) its boundary \( \partial \alpha = \partial_{p|q}(\alpha) \). So we obtain the following chain complex of abelian groups (integral smooth bigraded-bordisms chain complex) of \( E_k \subset j_{m|n}(W) \):

\[
\begin{array}{cccccc}
S C_{m|n}(E_k) & \partial_{m|n} & S C_{m-1|n-1}(E_k) & \partial_{m-1|n-1} & S C_{m-2|n-2}(E_k) & \cdots & \partial_{m-r|n-r} & S C_{m-r|n-r}(E_k) \\
& & \ & & \ & & \ & & \\
& & \ & & \ & & \ & & \\
\end{array}
\]

where \( r = \min\{m, n\} \). Then the \( p|q \)-integral bordism groups \( \Omega^E_{p|q}, (m-r) < p < m, (n-r) < q < n \), can be represented by means of the homology of the chain complex (12).

\( \mathcal{J}(E_k)^p \) can be identified with the spectral term \( E^1_{p,n} \) of the spectral sequence associated to the filtration induced in the graded algebra \( \Omega^*(\mathbb{R}^\infty) = \bigoplus_{q \geq 0} \Omega^q(\mathbb{R}^\infty) \), by the subspaces \( C\Omega^q(\mathbb{R}^\infty) \subset \Omega^q(\mathbb{R}^\infty) \). (For abuse of language we shall call "conservation laws of \( k \)-order", characteristic integral \((n-1)\)-forms too. Note that \( C\Omega^p(\mathbb{R}^\infty) = 0 \). See also Refs.[59, 61, 68].)
One has the following canonical isomorphism:

\[ \mathbb{Z} \bigotimes \mathbb{Z}^S \text{Bor}^\bullet (\hat{E}_k) \cong \mathbb{Z}^S \text{Cyc}^\bullet (\hat{E}_k). \]  

Furthermore, if \( H^2(\mathbb{C} \text{yc}^\bullet (\hat{E}_k), \Omega^{\hat{E}_k}) = 0 \) one has the following canonical isomorphism:

\[ S \text{Bor}^\bullet (\hat{E}_k) \cong \Omega^{\hat{E}_k} \times S \text{Cyc}^\bullet (\hat{E}_k). \]

**Proof.** The proof can be conducted similarly to the ones for Lemma 2.5, Lemma 2.6 and Lemma 2.7. \( \Box \)

Similarly we can prove the following lemma concerning the total analogous of the complex (12) too.

**Lemma 2.11.** (Relations between lower order integral total-bordisms in quantum super PDE's). Let \( \hat{E}_k \subset \hat{j}_{m|n}^k (W) \) be a quantum super PDE on the fiber bundle \( \pi : W \to M, \dim_B W = (m|n, r|s), \dim_A M = m|n, B = A \times E, E \) a quantum superalgebra that is also a \( \mathbb{Z} \)-module, with \( Z = Z(A) \) the centre of \( A \). Let \( S C_p(\hat{E}_k) \), \( 0 \leq p \leq m + n \), be the set of all compact \( u|v \)-dimensional, (with respect to \( A \)), admissible integral smooth manifolds of \( \hat{E}_k \), such that \( u + v = p \). The disjoint union gives an addition on \( S C_p(\hat{E}_k) \) with \( \varnothing \) as the zero element. Thus we can write

\[ S C_p(\hat{E}_k) = \bigoplus_{u,v: u + v = p} S C_{u|v}(\hat{E}_k) = \text{Tot} S C_p(\hat{E}_k). \]

Let us consider the homomorphisms \( \partial_p : S C_p(\hat{E}_k) \to S C_{p-1}(\hat{E}_k) \) that associates to any element \( a \in S C_p(\hat{E}_k) \) its boundary \( \partial a = \partial_p(a) \), i.e., one has:

\[ \partial_p(a) = \partial_p(a_0|a_{p-1|1}, a_{p-2|2}, \ldots, a_0|0) \]

\[ = (\partial_p(a_0|0), \partial_p(a_{p-1|1}, \partial_p(a_{p-2|2}, \ldots, \partial_p(a_0|0))) \]

\[ \in \bigoplus_{u,v: u + v = p-1} S C_{u|v}(\hat{E}_k) = S C_{p-1}(\hat{E}_k). \]

One has \( \partial_{p-1} \circ \partial_p = 0 \). So we get the following chain complex of abelian groups (integral smooth bigraded-bordisms chain complex) of \( \hat{E}_k \subset \hat{j}_{m|n}^k (W) \):

\[ S C_n(\hat{E}_k) \xrightarrow{\partial_n} S C_{n-1}(\hat{E}_k) \xrightarrow{\partial_{n-1}} S C_{n-2}(\hat{E}_k) \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} S C_0(\hat{E}_k). \]

Then the \( p \)-integral total bordism groups \( \Omega^{\hat{E}_k}_p, 0 < p < m + n \), can be represented by means of the homology of the chain complex (17).

One has the following canonical isomorphism:

\[ \mathbb{Z} \bigotimes \mathbb{Z}^S \text{Bor}^\bullet (\hat{E}_k) \cong \mathbb{Z}^S \text{Cyc}^\bullet (\hat{E}_k). \]

Furthermore, if \( H^2(\mathbb{C} \text{yc}^\bullet (\hat{E}_k), \Omega^{\hat{E}_k}) = 0 \) one has the following canonical isomorphism:

\[ S \text{Bor}^\bullet (\hat{E}_k) \cong \Omega^{\hat{E}_k} \times S \text{Cyc}^\bullet (\hat{E}_k). \]

**Proof.** The proof is similar to the one of Lemma 2.10. \( \Box \)
Lemma 2.12. (Integral ringoid of PDE’s in $\Omega_S$ and quantum conservation laws). Let $\hat{E}_k \subset \hat{J}^{k,m+n}(W)$ be a PDE in the category $\Omega_S$ as defined in Lemma 2.11. Then, $\Omega^\hat{E}_k \equiv \bigoplus_{0 \leq p \leq m+n} \Omega^\hat{E}_k$, has a natural structure of graded ringoid, with respect to the (partial) binary operations similar to the commutative case. We call $\hat{\Omega}$ that defines a subalgebra $\hat{\Omega}$ of $\Omega^\hat{E}_k$. Furthermore, quantum conservation laws of order $k$, $\hat{f} \in \text{Map}(\hat{\Omega}^\hat{E}_k, \mathbb{K}) = H_{pl}(\hat{E}_k)$, can be projected on their classic limits $\hat{f} \mapsto \hat{f}_C \equiv c \circ \hat{f} \in \text{Map}(\Omega^\hat{E}_k, \mathbb{K}) = H_{pl}(\hat{E}_k)_C$. By passing to the corresponding total spaces, we get the following exact commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
H_{pl}(\hat{E}_k) & \rightarrow & H_{pl}(\hat{E}_k)_C \rightarrow 0 \\
\downarrow & & \downarrow \\
H_*(\hat{E}_k) & \rightarrow & H_*(\hat{E}_k)_C \rightarrow 0
\end{array}
\]

Moreover, graded ringoid homomorphisms $\hat{h} \in \text{Hom}_{\text{ringoid}}(\Omega^\hat{E}_k, \mathbb{K})$, can be identified by means of classic limit quantum conservation laws of $\hat{E}_k$. One has the following exact commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & ^R\text{H}_*(\hat{E}_k) \rightarrow \text{H}_*(\hat{E}_k) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Hom}_{\text{ringoid}}(\Omega^\hat{E}_k, \mathbb{K}) \rightarrow \text{H}_*(\hat{E}_k)_C \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

that defines a subalgebra $^R\text{H}_*(\hat{E}_k)$ of $\text{H}_*(\hat{E}_k)$, whose elements we call rigid quantum conservation laws, and whose classic limit can be identified with ringoid homomorphisms $\Omega^\hat{E}_k \rightarrow \mathbb{K}$. In particular, quantum conservation laws arising by full quantum differential form classes

\[
\left\{ [\alpha] \in \bigoplus_{p,q \geq 0} \hat{\mathcal{H}}(\hat{E}_k)^{p|q} \right\}
\]

\[
\hat{\mathcal{H}}(\hat{E}_k)^{p|q} = \frac{\partial^{p+1|q+1}(\hat{E}_k)}{\partial^{p+1|q+1}(\hat{E}_k) \otimes \{C\hat{\mathcal{O}}^{p+1|q+1}(\hat{E}_k)\}}
\]

belong to $^R\text{H}_*(\hat{E}_k)$.

**Proof.** The proof follows directly from above lemmas. (For details on spaces $\hat{\mathcal{H}}(\hat{E}_k)^{p|q}$ see Refs. [63, 68, 70].) \hfill \Box

Let us, now, denote $\Omega^{\hat{E}_k}_{c,p+q}$ (or $\Omega^{\hat{E}_k}_{c,p+q}$), the classic limit of integral $(p|q)$-bordism group of $\hat{E}_k$, i.e., the $(p+q)$-bordism group of classic limits of integral supermanifolds $N \subset \hat{E}_k$, such that $\text{dim}_A N = p|q$. Furthermore, let us denote by $\Omega^{\hat{E}_k}_{c,p+q}$ the classic limit of total integral $(p+q)$-bordism group of $\hat{E}_k$, i.e., the $(p+q)$-bordism group
of classic limits of integral supermanifolds $N \subset \hat{E}_k$, such that $\dim_A N = u|v$, with $u + v = p + q$. One has the following exact commutative diagram:

\[
\begin{array}{cccc}
0 & \to & \Omega^{\hat{E}_k}_{p|q} & \to & \Omega^{\hat{E}_k}_{\hat{p}+q} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & \Omega^{\hat{E}_k}_{\hat{c}p+q} & \to & \Omega^{\hat{E}_k}_{\hat{c}\hat{p}+q} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
\]

Taking into account Theorem 3.6 in [70] we get a relation between $\Omega^{\hat{E}_k}_{m-1|n-1}$, $\Omega^{\hat{E}_k}_{\hat{c}p|q}$ and the bordism group $\Omega_{m+n-2}$. In fact, we can see that there is a relation between integral bordism groups in quantum super PDEs and Reinhart integral bordism groups of commutative manifolds. More precisely, let $N_0, N_1 \subset \hat{E}_k \subset \hat{J}_{m|n}(W)$ be closed admissible integral quantum supermanifolds of a quantum super PDE $\hat{E}_k$, of dimension $(m-1|n-1)$ over $A$, such that $N_0 \cup N_1 = \partial V$, for some admissible integral quantum supermanifold $V \subset \hat{E}_k$, of dimension $(m|n)$ over $A$. Then $(N_0)_C \cup (N_1)_C = \partial V_C$ iff $(N_0)_C$ and $(N_1)_C$ have the same Stiefel-Whitney and Euler characteristic numbers. In fact, by denoting $\Omega^1_p$ the Reinhart $p$-bordism groups and $\Omega_p$ the $p$-bordism group for closed smooth finite dimensional manifolds respectively, one has the following exact commutative diagram

\[
\begin{array}{cccc}
0 & \to & K_{m-1|n-1; m+n-2} & \to & \Omega^{\hat{E}_k}_{m-1|n-1} & \to & \Omega^{\hat{E}_k}_{\hat{c}m+n-2} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & K_{m+n-2}^\dagger & \to & \Omega_{m+n-2}^\dagger & \to & \Omega_{m+n-2} & \to & 0
\end{array}
\]

This has as a consequence that if $N_0 \cup N_1 = \partial V$, then $(N_0)_C \cup (N_1)_C = \partial V_C$ iff $(N_0)_C$ and $(N_1)_C$ have the same Stiefel-Whitney and Euler characteristic numbers. From above exact commutative diagram one has that $\Omega^{\hat{E}_k}_{m-1|n-1}$ is an extension of a subgroup of $\Omega_{m+n-2}$.

Let us consider, now, the following lemmas.

**Lemma 2.13.** [73] Bordism groups, $\Omega_p$, relative to smooth manifolds can be considered as extensions of some crystallographic subgroup $G \triangleleft G(d)$.

**Lemma 2.14.** If the group $G$ is an extension of $H$, any subgroup $\tilde{G} \triangleleft G$ is an extension of a subgroup $\tilde{H} \triangleleft H$.

**Proof.** In fact $\tilde{G}$ is an extension of $p(\tilde{G}) \triangleleft H$, with respect to the following short exact sequence: $0 \longrightarrow K \longrightarrow G \xrightarrow{p} H \longrightarrow 0$. \[\blacksquare\]

---

\[\text{Note that for } p + q = 3 \text{ one has } K_3^\dagger = 0, \text{ hence one has } \Omega_3^\dagger = \Omega_3.\]
Therefore by using above two lemmas, we get also that $\Omega_{m-1|n-1}^{E_k}$ is an extension of some crystallographic subgroup $G \triangleleft G(d)$. 

The theorem below relates the integrability properties of a quantum super PDE to crystallographic groups. Let us first give the following definition.

**Definition 2.15.** We say that a quantum super PDE $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$ is an extended crystal quantum super PDE, if conditions of Theorem 2.4 are verified. Then, for such a PDE $\hat{E}_k$ are defined its crystal group $G(d)$ and crystal dimension $d$.

In the following we relate crystal structure of quantum super PDE’s to the existence of global smooth solutions for smooth boundary value problems, by identifying an algebraic-topological obstruction.

**Theorem 2.16.** Let $B_k$ be the model quantum superalgebra of $\hat{J}_{m|n}^k(W)$, $k \geq 0$. (See [69, 70].) We denote also by $B_\infty = \lim_k B_k$. \(^9\) Let $\hat{E}_k \subset \hat{J}_{m|n}^k(W)$ be a quantum formally integrable and completely quantum superintegrable quantum super PDE. Then, in the algebra $H_{m-1|n-1}(\hat{E}_k) \equiv \text{Map}(\Omega_{m-1|n-1}^{E_k}; B_k)$, Hopf quantum superalgebra of $\hat{E}_k$, there is a quantum sub-superalgebra, (crystal Hopf superalgebra) of $\hat{E}_k$. \(^10\) On such an algebra we can represent the quantum superalgebra $B^{G(d)}_{\infty}$ associated to the quantum crystal supergroup $G(d)$ of $\hat{E}_k$. (This justifies the name.) We call quantum crystal conservation superlaws of $\hat{E}_k$ the elements of its quantum Hopf crystal superalgebra. Then, the obstruction to find global smooth solutions of $\hat{E}_k$, for integral boundaries with orientable classic limit, can be identified with the quotient $H_{m-1|n-1}(\hat{E}_\infty)/B^{\Omega_{m+n-2}}_\infty$.

**Proof.** Let $N_0, N_1 \subset \hat{E}_k$ be two respectively initial and final, closed compact Cauchy data of $\hat{E}_k$. Then there exists a weak, (resp. singular, resp. smooth) solution $V \subset \hat{E}_k$, such that $\partial V = N_0 \cup N_1$, if $X \equiv N_0 \cup N_1 \in [0] \in \Omega_{m-1|n-1,w}^{E_k}$, (resp. $X \in [0] \in \Omega_{m-1|n-1,s}^{E_k}$, resp. $X \in [0] \in \Omega_{m-1|n-1}^{E_k}$). Let $X_C$ be orientable, then $X$ is the boundary of a smooth solution, if $X$ has zero all the integral characteristic quantum supernumbers, i.e., $< \alpha, X > = 0$, $\forall \alpha \in H_{m-1|n-1}(\hat{E}_\infty) = \text{Map}(\Omega_{m-1|n-1}^{E_\infty}; B_\infty)$. Taking into account the following short exact sequence: $0 \rightarrow \Omega_{m-1|n-1,w}^{E_\infty} \rightarrow \Omega_{m+n-2}$, where $\Omega_{m+n-2} \triangleleft G(d)$, for some crystallographic group $G(d)$, we get also the following short exact sequence: $H_{m-1|n-1}(\hat{E}_\infty) \triangleleft B^{\Omega_{m+n-2}}_\infty$ (This justifies the definition above). Then the obstruction to find smooth solutions can be identified with the quotient $H_{m-1|n-1}(\hat{E}_\infty)/B^{\Omega_{m+n-2}}_\infty$. Taking into account that $\Omega_{m+n-2} \triangleleft \Omega_{m+n-2} \triangleleft G(d)$, we can also represent $B^{\Omega_{m+n-2}}_\infty$ with $B^{\Omega_{m+n-2}}_\infty$, or with $B^{G(d)}_\infty$. Thus, it is justified also call $B^{\Omega_{m+n-2}}_\infty$ as crystal quantum superlaws algebra of $\hat{E}_k$. 

\(^9\)We also adopt the notation $B_k(A)$ and $B_\infty(A)$, whether it is necessary to specify the starting original quantum super algebra $A$.

\(^10\)Recall that with the term quantum Hopf superalgebra we mean an extension $A \rightarrow C \equiv A \otimes_k H \rightarrow D \rightarrow D/C \rightarrow 0$, where $H$ is an Hopf $k$-algebra and $A$ is a quantum superalgebra. (For more details on generalized Hopf algebras, associated to PDE’s, see Refs.[58, 59, 70].)
Definition 2.17. We define crystal obstruction of $\hat{E}_k$ the above quotient of algebras, and put: $\text{cry}(\hat{E}_k) \equiv H_{m-1|n-1}(\hat{E}_k)/B_{\infty}^{m+n-2}$. We call quantum 0-crystal super PDE a quantum super PDE $\hat{E}_k \subset \hat{j}_m(W)$ such that $\text{cry}(\hat{E}_k) = 0$.

Remark 2.18. A quantum extended 0-crystal super PDE $\hat{E}_k \subset \hat{j}_m(W)$ does not necessitate to be a quantum 0-crystal super PDE. In fact $\hat{E}_k$ is an extended 0-crystal quantum super PDE if $\Omega_{m-1|n-1,w} = 0$. This does not necessarily implies that $\Omega_{m-1|n-1} = 0$. In fact, the different types of integral bordism groups of PDE’s in the category $\Omega_S$, are related by the following proposition.

Proposition 2.19. (Relations between integral bordism groups).[70] The different types of integral bordism groups for a quantum super PDE, are related by the exact commutative diagram reported in (25).

\begin{equation}
\begin{array}{cccccc}
0 & \longrightarrow & K^\hat{E}_k & \longrightarrow & K^\hat{E}_k & \longrightarrow 0 \\
0 & \longrightarrow & K^\hat{E}_k & \longrightarrow & K^\hat{E}_k & \longrightarrow 0 \\
0 & \longrightarrow & K^\hat{E}_k & \longrightarrow & K^\hat{E}_k & \longrightarrow 0 \\
0 & \longrightarrow & \Omega^\hat{E}_k & \longrightarrow & \Omega^\hat{E}_k & \longrightarrow 0 \\
0 & \longrightarrow & \Omega^\hat{E}_k & \longrightarrow & \Omega^\hat{E}_k & \longrightarrow 0 \\
0 & \longrightarrow & \Omega^\hat{E}_k & \longrightarrow & \Omega^\hat{E}_k & \longrightarrow 0
\end{array}
\end{equation}

One has the canonical isomorphisms:

\begin{equation}
\begin{align}
K^\hat{E}_k & \cong K^\hat{E}_k \\
\Omega^\hat{E}_k & \cong \Omega^\hat{E}_k \\
\Omega^\hat{E}_k & \cong \Omega^\hat{E}_k \\
\Omega^\hat{E}_k & \cong \Omega^\hat{E}_k
\end{align}
\end{equation}

Corollary 2.20. Let $\hat{E}_k \subset \hat{j}_m(W)$ be a quantum 0-crystal super PDE. Let $N_0, N_1 \subset \hat{E}_k$ be two closed initial and final Cauchy data of $\hat{E}_k$ such that $X \equiv N_0 \cup N_1 \in [0] \in \Omega_{m-1|n-1}$, and such that $X_C$ is orientable. Then there exists a smooth solution $V \subset \hat{E}_k$ such that $\partial V = X$.

Example 2.21. (Quantum extended crystal SG-Yang-Mills PDE’s.) Let us introduce some fundamental geometric objects to encode quantum supergravity. (See also our previous works on this subjects that formulate quantum supergravity in the framework of our geometric theory of quantum super PDE’s [61, 66, 69, 76, 78].) The first geometric object to consider is a quantum Riemannian (super)manifold,
induces a quantum verbein, is a section where precisely one has Furthermore, if by means of the isomorphism conditions e \alpha = \delta_\alpha^\beta, let us denote respectively by \hat{g}(a \otimes u, b \otimes v) = ab \ , g\ (u,v) \in A. By using the canonical splitting \text{Hom}_Z(T_0^2(A \otimes_R M_C; A) \cong \text{Hom}_Z(D_0^2(A \otimes_R M_C; A) \oplus \text{Hom}_Z(\Lambda_0^1(A \otimes_R M_C; A), we get also the split representation)} \hat{g} = \hat{g}_s + \hat{g}_a\). More precisely one has

\begin{align*}
\hat{g}_s(a \otimes u, b \otimes v) &= [a,b]_+ g(u,v), \\
\hat{g}_a(a \otimes u, b \otimes v) &= [a,b]_- g(u,v).
\end{align*}

Furthermore, if (e_\alpha) is a basis in M_C, and (e^\beta) is its dual, characterized by the conditions e_\alpha e^\beta = \delta_\alpha^\beta, let us denote respectively by \hat{1} e_\alpha and ((1 \otimes e_\alpha)^+) the induced dual bases on the spaces A \otimes_R M_C and (A \otimes_R M_C)^+ respectively. Then one has the following representations

\begin{align*}
\hat{1} &= \hat{1}_{\alpha\beta}(1 \otimes e^\alpha)^+ \otimes (1 \otimes e^\beta)^+ + \hat{1}_{\alpha\beta} \in \hat{A}^2 \\
\hat{g}_s &= \hat{g}_s(a \otimes u, b \otimes v) = (1 \otimes e_\alpha)^+ \bullet (1 \otimes e^\beta)^+ + \hat{g}_s(a \otimes u, b \otimes v) = (1 \otimes e_\alpha)^+ \Delta (1 \otimes e^\beta)^+, \\
\hat{g}_a &= \hat{g}_a(a \otimes u, b \otimes v) = (1 \otimes e_\alpha)^+ \Delta (1 \otimes e^\beta)^+, \quad \hat{g}_a(a \otimes u, b \otimes v) = (1 \otimes e_\alpha)^+ \Delta (1 \otimes e^\beta)^+.
\end{align*}

By means of the isomorphism \hat{\theta}^\otimes, we can induce on M a quantum metric, i.e., the quantum Minkowskian metric of M, \hat{g} = \hat{g} = \hat{\theta}^\otimes. Conversely any quantum metric \hat{g} on M, induces on the space A \otimes_R M_C, scalar products, for any p \in M: \hat{g}(p) = \hat{g}(p) = (\hat{\theta}^\otimes(p))^{-1}. As a by-product, we get that any quantum metric \hat{g} on M, induces a quantum metric on the fiber bundle \hat{\pi} : E \to M, that we call the deformed quantum metrics of \hat{\pi} : E \to M. Therefore, when we talk about locally Minkowskian quantum manifold M, we mean that on M is defined a Minkowskian quantum metric. Since \text{Hom}_Z(TM; A \otimes_R M_C) \cong A \otimes_R M_C \otimes_A (TM)^+ , we can locally represent a quantum vierbein in the following form:

\begin{align*}
\hat{\theta} &= \hat{1} \otimes e_\beta \otimes \hat{\theta}^\beta_\alpha dx^\alpha,
\end{align*}

where \hat{1} e_\beta \in \text{Hom}_Z(A; A \otimes_R M_C), is the full quantum extension of a basis \hat{e}_\alpha \otimes e_\beta = e_\alpha \otimes e_\beta. Furthermore, \hat{\theta}^\beta_\alpha(p) \in \hat{A}. Then, if \zeta : M \to TM = \text{Hom}_Z(A; TM) is a full quantum vector field on M, locally represented by \zeta = \partial x^\alpha e_\alpha, we get that its local representation by means of quantum vierbein, is given by the following formula:

\begin{align*}
\hat{\theta}(\zeta) &= \hat{1} \otimes e_\beta \hat{\theta}^\beta_\alpha e_\alpha,
\end{align*}
Locally one can write

\[ \ldots \]

where the product is given by composition:

\[
A \xrightarrow{\zeta^\alpha} A \xrightarrow{\hat{e}_\alpha^\beta} A \xrightarrow{\hat{\theta}_\alpha^\beta} A \otimes Z \mathcal{M}_C
\]

\[ \sum_{\alpha,\beta} = \hat{\theta}(\zeta) \]

(For abuse of notation we can also denote \( \hat{\theta}(\zeta) \) by \( \zeta \) yet.) Whether \( \hat{g} = \hat{g}_{\alpha\beta} dx^\alpha \otimes dx^\beta \), is the quantum Minkowskian metric of \( M \), then its local representation by means of the quantum vierbein is the following:

\[
\hat{g}_{\alpha\omega} = \hat{\theta}_\alpha^\beta \otimes \hat{g}_{\beta\omega} = \hat{g}_{\alpha\omega}(p) \in \hat{A}, \quad \forall p \in M.
\]

where \( \hat{\theta}_\alpha^\beta \otimes \hat{g}_{\beta\omega}(p) \), can be identified with \( \hat{\theta}_\alpha^\beta \otimes \hat{g}_{\beta\omega}(p) \in \text{Hom}_Z(\hat{T}_0^2(A); \hat{T}_0^2(A)) \). In fact, one has the following extension \( \hat{\theta}^\circ \) of \( \hat{\theta} \):

\[
\hat{\theta}^\circ \in \text{Hom}_Z(T \otimes Z TM; (A \otimes \mathcal{M}_C) \otimes Z (A \otimes \mathcal{M}_C)) \equiv (A \otimes \mathcal{M}_C) \otimes Z (A \otimes \mathcal{M}_C) \otimes R(TM \otimes Z TM)^+.
\]

Locally one can write

\[
\hat{\theta}^\circ = (1 \otimes e_\gamma) \otimes (1 \otimes e_\omega) \otimes \hat{\theta}_\alpha^\beta \otimes \hat{g}_{\beta\omega} \ dx^\alpha \otimes dx^\beta.
\]

In fact, we have

\[
\hat{g}(\zeta, \xi) = \hat{g}(1 \otimes e_\beta \hat{\theta}_\alpha^\gamma, 1 \otimes e_\gamma \hat{\theta}_\alpha^\omega) = \hat{\theta}_\alpha^\beta \hat{\theta}_\gamma^\omega \hat{g}(e_\beta, e_\gamma) = \hat{\theta}_\alpha^\beta \zeta^\alpha \xi^\omega \ g_{\gamma\gamma}.
\]

In the particular case that \( (e_\beta) \) is an orthonormal basis, then we get the following quantum Minkowskian representation for \( \hat{g} \):

\[
(\hat{g}_{\alpha\omega}) = \hat{\theta}_\alpha^\beta \otimes \hat{\theta}_\omega^\gamma \eta_{\beta\gamma}, \quad \eta_{\beta\gamma} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}.
\]

The splitting in symmetric and skew-symmetric part of \( \hat{g} \), i.e.,

\[
\hat{g} = \hat{g}_{(s)} + \hat{g}_{(a)} = \hat{g}_{\alpha\beta} \ dx^\alpha \bullet dx^\beta + \hat{g}_{\alpha\beta} \ dx^\alpha \triangle dx^\beta
\]

can be written in terms of quantum vierbein in the following way:

\[
\hat{\theta}^\circ = \hat{\theta}^\circ + \hat{\theta}^\wedge
\]

\[
\hat{\theta}^\circ = (1 \otimes e_\gamma) \otimes (1 \otimes e_\omega) \otimes \hat{\theta}_\alpha^\gamma \otimes \hat{\theta}_\alpha^\omega \ dx^\alpha \bullet dx^\beta
\]

\[
\hat{\theta}^\wedge = (1 \otimes e_\gamma) \otimes (1 \otimes e_\omega) \otimes \hat{\theta}_\alpha^\gamma \otimes \hat{\theta}_\alpha^\omega \ dx^\alpha \triangle dx^\beta.
\]

Conversely, the local expression of the quantum deformed metrics on \( \hat{\pi} : E \to M \), induced by a quantum metrics \( \hat{g} \) on \( M \), is given by the following formulas:
In the particular case that \( \hat{g} \) is Minkowskian, then we can use for \( g_{\gamma\omega} \), \( \hat{g}_{(s)\gamma\omega} \) and \( \hat{g}_{(a)\gamma\omega} \) the corresponding expressions in (33), and by using the property that \( \hat{\theta}^\gamma_\alpha = \delta^\gamma_\alpha \), we get \( \hat{g}_{\gamma\omega} = \hat{\delta}_{\gamma\omega} \), \( \hat{g}_{(s)\gamma\omega} = \hat{\delta}_{(s)\gamma\omega} \) and \( \hat{g}_{(a)\gamma\omega} = \hat{\delta}_{(a)\gamma\omega} \). The controvantfull quantum metric \( \hat{g} \) of \( g : M \rightarrow \text{Hom}_Z(T_0^2 M; A) \) is a section \( \hat{g} : M \rightarrow H_{/A}(A; T_0^2 M) \) such that the following conditions are satisfied:

\[
\begin{aligned}
\hat{g}(\alpha \beta) = \hat{g}_{\alpha\beta}(p) \in \text{Hom}_Z(A \otimes_Z A; A), \\
\hat{g}_{\alpha\beta}(p) \hat{g}_{\gamma\omega}(p) = \hat{\delta}_\gamma^\alpha \hat{\delta}_\omega^\beta, \\
\hat{g}_{\alpha\beta}(p) \hat{g}_{\gamma\omega}(p) = \hat{\delta}_\gamma^\alpha \hat{\delta}_\omega^\beta, \\
\hat{g}_{\alpha\beta}(p) = \hat{\delta}_\alpha^\gamma \hat{\delta}_\beta^\omega.
\end{aligned}
\]

The products in (41) are meant by composition:

\[
\begin{aligned}
A \otimes_Z A & \xrightarrow{\hat{g}_{\alpha\beta}(p)} A \\
\delta_\gamma & \xrightarrow{\delta_\gamma^\alpha} A \otimes_Z A \\
\delta_\gamma & \xrightarrow{\delta_\gamma^\alpha} A \otimes_Z A
\end{aligned}
\]

In the following commutative diagram it is shown the pairing working between the fiber bundles \( (T_0^2 M)^+ \) and \( T_0^2 M \) over \( M \).

\[
\begin{array}{ccc}
\text{Hom}_Z(T_0^2 M; T_0^2 M) \cong T_0^2 M \otimes_A (T_0^2 M)^+ \xrightarrow{1^{\text{tr}}} M \times \hat{A} & \xrightarrow{\hat{g}} & A \\
\end{array}
\]

In particular, one has:

\[
\begin{aligned}
\frac{1}{s} \langle \hat{g}, \hat{\gamma} \rangle = \frac{1}{s} \hat{g}^{\alpha\beta} g_{\alpha\beta} = \frac{1}{s} \hat{g}^{\alpha\beta} A = A, \\
s = \begin{cases} 
m, & \text{dim}_A M = m \\
\text{dim}_A M = m+n, & \text{dim}_A M = m|n.
\end{cases}
\end{aligned}
\]

It is direct to verify that \( \hat{g}^{\alpha\beta} = \hat{\delta}_\alpha^\gamma \otimes \hat{\delta}_\beta^\omega \) is the controvant expression of the full quantum metric \( g_{\alpha\beta} = \hat{\delta}_\alpha^\gamma \otimes \hat{\delta}_\beta^\omega \), when \( \hat{g}^{\gamma\omega} = \hat{\delta}_\gamma^\alpha \otimes \hat{\delta}_\omega^\beta \) is the controvant one of \( \hat{g}_{\alpha\beta} \). In other words if \( \hat{g}^{\gamma\omega} = \delta^\gamma_\alpha \otimes \delta^\omega_\beta \), then \( g_{\alpha\beta} \hat{g}^{\gamma\omega} = \delta^\gamma_\alpha \hat{\delta}_\omega^\beta \). This means that the full quantum metric \( \hat{g} \), induced on \( A \otimes \mathbb{R} M_C \) by \( g \), is not degenerate, i.e. one has the following short exact sequence:

\[
\begin{array}{c}
0 \xrightarrow{\hat{g}} A \otimes \mathbb{R} M_C \xrightarrow{\hat{g}} (A \otimes \mathbb{R})^+
\end{array}
\]

In fact, one can see that \( \ker(\hat{g}) = \{0\} \). Really, \( \hat{g}(a \otimes v)(b \otimes u) = abq(v, u) = 0 \), for all \( b \in A \) and \( u \in M_C \); if \( a = 0 \) or \( u = 0 \). In fact we can take \( b = 1 \) and \( u \) any vector of \( M_C \). So, since \( \hat{g} \) is not degenerate, it follows that cannot be \( \hat{g}(v, u) = 0 \), for a
non zero \( v \), and \( \forall u \in M_C \). The nondegeneracy of \( \hat{\mathfrak{g}} \) induces also the following isomorphism \( A \otimes_{\mathbb{R}} M_C \cong (A \otimes_{\mathbb{R}} M_C)^{\ast} \).

We define quantum supergravity Yang-Mills PDE, (quantum SG-Yang-Mills PDE), a quantum super Yang-Mills PDE where the quantum super Lie algebra \( \mathfrak{g} \) in the configuration bundle \( \pi: W \equiv \text{Hom}_Z(TM; \mathfrak{g}) \to M \) is a quantum superextension of the Poincaré Lie algebra and admits the following splitting:

\[
\mathfrak{g} = \mathfrak{g}_\otimes + \mathfrak{g}_\circ + \mathfrak{g}_\bullet
\]

where \( \mathfrak{g}_\otimes = A \otimes_{\mathbb{R}} M_C \), (resp. \( \mathfrak{g}_\circ \) is the quantum superextension of the Lorentz part of the Poincaré algebra). Here \( A \) is a quantum (super)algebra on which is modeled the quantum (super)manifold \( M \), and \( M_C \) is the 4-dimensional Minkowsky vector space. Taking into account the canonical splitting:

\[
\text{Hom}_Z(TM; \mathfrak{g}) \cong \text{Hom}_Z(TM; \mathfrak{g}_\otimes) \times \text{Hom}_Z(TM; \mathfrak{g}_\circ) \times \text{Hom}_Z(TM; \mathfrak{g}_\bullet)
\]

we get that the fundamental field \( \hat{\mu} : M \to W \), in a quantum supergravity Yang-Mills PDE, admits the following canonical splitting:

\[
\hat{\mu} = \hat{\mu}_\otimes + \hat{\mu}_\circ + \hat{\mu}_\bullet.
\]

We say that \( \hat{\mu} \) is nondegenerate if \( \hat{\mu}_\circ \) identifies, for any \( p \in M \), an isomorphism \( \hat{\mu}_\circ(p) : T_p M \cong A \otimes_{\mathbb{R}} M_C \), hence \( \hat{\mu}_\circ \) can be identified with a quantum vierbein on \( M : \hat{\mu}_\circ \cong \hat{\theta} \). Then we define \( \hat{\mu}_\circ \), (resp. \( \hat{\mu}_\otimes \), resp. \( \hat{\mu}_\bullet \), the vierbein-component, (resp. Lorentz-component, resp. deviatory-component), of \( \hat{\mu} \). This property is represented, in local quantum coordinates, by the fact that in the following formula

\[
\hat{\mu} = (\hat{\mu}_\circ^A dx^A) = (\mu_\circ^A dx^A + \hat{\mu}_\otimes^A dx^A + \hat{\mu}_\bullet^A dx^A)
\]

one has \( (\mu_\circ^A dx^A(p)) \in \text{GL}(\hat{A}; A) \).

An example of quantum Lie superalgebra \( \hat{\mathfrak{g}} \) is characterized by means of the following infinitesimal generators: \( \{Z_k\}_{1 \leq k \leq 19} \equiv \{J_{\alpha\beta}, P_\alpha, \bar{Z}, Q_{\beta}\}_{0 \leq \alpha, \beta \leq 1, 1 \leq k \leq 2} \),

\[
\begin{align*}
J_{\alpha\beta} &= \eta_{\alpha\beta} J_{\gamma\delta} - \eta_{\beta\gamma} J_{\delta\alpha} - \eta_{\gamma\delta} J_{\alpha\beta}, \\
P_\alpha &= 0, \\
J_{\alpha\beta} Q_{\gamma} &= \{\sigma_{\alpha\beta} \} Q_{\gamma}, \\
\{Q_{\beta}, Q_{\alpha}\} &= (C_{\alpha\beta})_{\mu\delta} \delta_{\delta j} P_\mu + C_{\alpha\beta} c_{\gamma j} Z.
\end{align*}
\]

Here \( C_{\alpha\beta} \) is the antisymmetric charge conjugation matrix, \( \sigma_{\beta\mu} = \frac{1}{2} \{\gamma_\beta, \gamma_\mu\} \), with \( \gamma^\mu \) the Dirac matrices. \( \bar{Z} \) commutes with all the other ones. Then, \( \hat{\mu}_\circ^\alpha \hat{\gamma}_\beta = \frac{1}{2} \hat{\omega}^\alpha_{\beta\gamma} \), \( \hat{\mu}_\circ^\alpha = \theta^\alpha_\gamma \), \( \hat{\mu}_\bullet^\alpha = (\hat{\Lambda}_\gamma, \hat{\psi}^\gamma_\alpha) \), where \( \hat{\omega}^\alpha_{\beta\gamma} \) are called quantum Levi-Civita connection coefficients, \( \hat{\theta}^\alpha_\gamma \) are the quantum vierbein components, \( \hat{\Lambda}_\gamma \) are the quantum electromagnetic field components and \( \hat{\psi}^\gamma_\alpha \) are the quantum spin \( \frac{1}{2} \) field components. The curvature, corresponding to \( \hat{\mu} \), can be locally written in the form:

\[
\hat{R} = Z_k \otimes R^K_\alpha dx^\alpha \Delta dx^\beta,
\]

with \( R^K_\alpha = (\partial x^\beta \mu^K_\beta) + C^K_{\beta\gamma} \mu^\beta_\gamma + \mu^K_\beta \). The local expression of the dynamic equation \( \hat{E}_{2k} \in J\hat{D}_{2k}(W) \) is resumed in Tab.2, for some
quantum Lagrangian \( L : J \hat{D}^k(W) \to \hat{A} \) of order \( k \).

| Tab.2 - Dynamic Equation \( \hat{E}_{2k} \subset J \hat{D}^{2k}(W) \) and Bianchi identity. |
|---------------------------------------------------------------|
| Fields equations  | \((\partial \omega^\mu_{\alpha\beta} L) - \partial_\mu (\partial \omega^\mu_{\alpha\beta} L) = 0 \) (curvature equation)  |
|                  | \((\partial \omega^\mu_{\alpha\beta} L) - \partial_\mu (\partial \omega^\mu_{\alpha\beta} L) = 0 \) (torsion equation)  |
|                  | \((\partial \omega^\mu_{\alpha\beta} L) - \partial_\mu (\partial \omega^\mu_{\alpha\beta} L) = 0 \) (gravitino equation)  |
|                  | \((\partial A^\alpha L) - \partial_\alpha (\partial A^\alpha L) = 0 \) (Maxwell’s equation)  |
| Bianchi identity | \((\partial x, R^a_{\mu\nu}) + 2[\omega^a_{\mu\nu}, \omega^a_{\rho\sigma}] = 0 \) (curvature)  |
|                  | \((\partial x, R^a_{\mu\nu}) + (\gamma^a)_{\mu\nu} [\psi^\mu, \rho^\nu_{\mu\nu}] = 0 \) (torsion)  |
|                  | \((\partial x, \rho^a_{\mu\nu}) + (\sigma_{ab})_{ij} [\omega^a_{ij}, \rho^b_{\mu\nu}] = 0 \) (gravitino)  |
|                  | \((\partial x, F_{\mu\nu}) + C_{\mu\nu\rho\sigma} [\psi^\rho, \psi^\sigma] = 0 \) (electromagnetic field)  |

So in a quantum SG-Yang-Mills PDE, the quantum Riemannian metric \( \hat{g} \) is not a fundamental field, but a secondary field, obtained by means of the quantum verbein \( \hat{\theta} = \hat{\mu}_G \), that, instead is a fundamental dynamic field. Of course since there is a relation one-to-one between quantum verbein and quantum metric, on a locally Minkowskian quantum (super)manifold, one can choose also quantum metric as a fundamental field, instead of the quantum verbein. However, in a quantum SG-Yang-Mills PDE it is more natural to adopt quantum verbein as independent field, since it is just enclosed in the fundamental field \( \hat{\mu} \). Furthermore, it is important to emphasize that the so-called quantum Levi-Civita connection coefficients \( \hat{\omega}^{ab} \) are not, in general, metric coefficients, i.e., do not necessitate to be uniquely expressed by means of the quantum metric \( \hat{g} \). The name ”Levi-Civita connection coefficients" is reserved since under suitable dynamic conditions they can be uniquely identified by the quantum metric, similarly to what happens in the commutative differential geometry. However, in general, such property is dynamically relaxed.

By assuming the following first order Lagrangian function: \( L : J \hat{D}(W) \to \hat{A} \), \( L \circ \bar{D} = \frac{1}{2} \hat{R}_{K}^{a\beta} \hat{R}^{a\beta}_{K} \), \forall s \in Q^\infty(W) \), the local expression of \((YM)\) results given by the equations reported in Tab.3. Note that the quantum super Yang-Mills equation is now \((\partial \hat{\mu}^{A}_{K} L) - (\partial_{B} (\partial \hat{\mu}^{A\beta}_{K} L)) = 0 \). Furthermore, it results \((\partial \hat{\mu}^{A}_{K} L) = [\hat{C}_{K}^{H} \hat{R}_{K}^{a\beta}, \hat{R}^{a\beta}_{K}^{[AC]}]_{+} \) and \((\partial \hat{\mu}^{A\beta}_{K} L) = \hat{R}^{A}_{K} \).

| Tab.3 - Local expression of \((YM) \subset J \hat{D}^{2}(W)\) and Bianchi identity \((B) \subset J \hat{D}^{3}(W)\). |
|---------------------------------------------------------------|
| (Field equations) \( F^{K}_{A_{1}A_{2}} \equiv - (\partial_{B} \hat{R}^{A_{1}A_{2}}_{K} + |C^{H}_{K} \hat{R}^{a\beta}_{K}, \hat{R}^{a\beta}_{K}^{[AC]}|)_{+} = 0 \) (YM)  |
| (Fields) \( F^{K}_{A_{1}A_{2}} \equiv R^{K}_{A_{1}A_{2}} - [\partial X_{A_{1}}, \hat{R}^{A_{2}}_{K}] - [\partial X_{A_{2}}, \hat{R}^{A_{1}}_{K}] = 0 \) (B)  |
| (Bianchi identities) \( R^{K}_{A_{1}A_{2}} \equiv \partial X_{A_{1}}, \hat{R}^{A_{2}}_{K} + \hat{C}^{K}_{[J} \hat{R}^{J}_{A_{1}A_{2}]_{+}} = 0 \) (B)  |

In Refs.\(61, 69\) it is proved that \((YM) \subset J \hat{D}^{2}(W)\) is formally quantum superintegrable and also completely quantum superintegrable. That proof works well also in this situation, since it is of local nature, and remains valid also for quantum

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\[\text{11}\]The rising and lowering of indexes is obtained by means of the full quantum metrics \( \hat{g} \) on \( M \) and \( \hat{g} \) on \( \hat{g} \) respectively.
supermanifolds that are only locally quantum super-Minkowskian ones. Then, by using Theorem 2.3 we get
\[ \Omega_{3|3,w} \cong \Omega_{3|3,s} = A_0 \otimes_K H_3(W; \mathbb{K}) \oplus A_1 \otimes_K H_3(W; \mathbb{K}). \]
Since the fiber of \( W \) is contractible, we have \( \Omega_{3|3,w} \cong \Omega_{3|3,s} = A_0 \otimes_K H_3(M; \mathbb{K}) \oplus A_1 \otimes_K H_3(M; \mathbb{K}). \)
Thus, under the condition that \( H_3(M; \mathbb{K}) = 0 \), one has \( \Omega_{3|3,w} \cong \Omega_{3|3,s} = 0 \), hence \( (YM) \) becomes a quantum extended crystal super PDE. This is surely the case when \( M \) is globally quantum super Minkowskian. (See Refs. [62, 66, 69, 76, 78].) In such a case one has \( \Omega_{3|3} = K_{3|3}(YM) \), where
\[
(50) \quad K_{3|3}(YM) \equiv \left\{ [N]_{(YM)} \in \Omega_{3|3}(YM) \middle| N = \partial V, \text{ for some (singular)} \right\}
\[
(4|4)\text{-dimensional quantum supermanifold } V \subset W
\]
So \( (YM) \) is not a quantum 0-crystal super PDE. However, if we consider admissible only integral boundary manifolds, with orientable classic limit, and with zero characteristic quantum supernumbers, (full admissibility hypothesis), one has:
\( \Omega_{3|3} = 0 \), and \( (YM) \) becomes a quantum 0-crystal super PDE. Hence we get existence of global \( Q^\infty_\omega \) solutions for any boundary condition of class \( Q^\infty_\omega \).
With respect to the commutative exact diagram in (24) we get the following exact commutative diagram
\[
(51) \quad \begin{array}{cccccc}
0 & \longrightarrow & K_{3|3;2} & \longrightarrow & \Omega_{3|3} & \longrightarrow & \Omega_{3|3}(YM) & \longrightarrow & 0 \\
0 & \longrightarrow & K_{6} & \longrightarrow & \Omega_{6} & \longrightarrow & \Omega_6 & \longrightarrow & 0
\end{array}
\]
Taking into account the result by Thom on the unoriented cobordism groups [99], we can calculate \( \Omega_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Then, we can represent \( \Omega_6 \) as a subgroup of a 3-dimensional crystallographic group type \( [G(3)] \). In fact, we can consider the amalgamated subgroup \( D_2 \times \mathbb{Z}_2 \ast D_2 D_4 \), and monomorphism \( \Omega_6 \rightarrow D_2 \times \mathbb{Z}_2 \ast D_2 D_4 \), given by \( (a, b, c) \mapsto (a, b, b, c) \). Alternatively we can consider also \( \Omega_6 \rightarrow D_4 \ast D_2 D_4 \). (See Appendix C in [73] for amalgamated subgroups of \([G(3)]\).) In any case the crystallographic dimension of \( (YM) \) is 3 and the crystallographic space group type are \( D_{2d} \) or \( D_{4h} \) belonging to the tetragonal syngony. (See Tab. 4 in [73] and, for further informations, [21].)
Finally, the evaluation of \( (YM) \) on a macroscopic shell \( i(M_C) \subset M \) is given by the equations reported in Tab. 2.

| Tab. 4 - Local expression of \( (YM)[i] \subset JD^2(i^*W) \) and Bianchi identity \( (B)[i] \subset JD^2(i^*W) \). |
|---|---|
| (Field equations) \( \partial_{\alpha}R^{\alpha\beta\gamma\delta} + \frac{1}{2} \varepsilon_{\alpha\beta}^{\gamma\delta} R^{\alpha\beta\gamma\delta} = 0 \) | \( (YM)[i] \) |
| (Fields) \( R^{\alpha\beta\gamma\delta}_{\alpha_1 \alpha_2} = \varepsilon_{\alpha_1 \alpha_2} R^{\alpha_\gamma \beta_\delta}_{\alpha_1 \alpha_2} + \sum_{\nu} \varepsilon_{\alpha_1 \alpha_2} \varepsilon_{\beta_\gamma \beta_\delta} R^{\alpha_\nu \beta_\nu \gamma_\nu \delta_\nu}_{\alpha_1 \alpha_2} \) | \( (BM)[i] \) |
| (Bianchi identities) \( \partial_{(\alpha_1} R^{\beta_\gamma \beta_\delta}_{\alpha_1 \alpha_2)} + \frac{1}{3} \varepsilon_{\beta_\gamma \beta_\delta} R^{\beta_\gamma \beta_\delta}_{\alpha_1 \alpha_2} = 0 \) | \( (B)[i] \) |
This equation is also formally quantum superintegrable and completely quantum superintegrable. Furthermore, the 3-dimensional integral bordism group of $(YM)[i]$ and its infinity prolongation $(YM)[i]_{-\infty}$ are trivial, under the full admissibility hypothesis: $\Omega^3_3(YM)[i] \cong \Omega^3_3(YM)[i]_{-\infty} \cong 0$. So equation $(YM)[i] \subset JD^2(i^*W)$ becomes a quantum 0-crystal super PDE and it admits global (smooth) solutions for any fixed time-like 3-dimensional (smooth) boundary conditions.

### 3. STABILITY IN QUANTUM SUPER PDE’s

In this section we shall consider the stability of quantum super PDE’s in the framework of the geometric theory of quantum super PDE’s. We will follow the line just drawn in some our previous papers on this subject for commutative PDE’s, where we have interpreted stability of PDE’s on the ground of their integral bordism groups and related the quantum bordism of PDE’s to Ulam stability too.

Let us first revise some definitions and results about stability of mappings and their relations with singularities of mappings, adapting them to this new category of more complex mathematical noncommutative objects.

**Definition 3.1.** Let $X$, (resp. $Y$), be a quantum supermanifold of dimension $m|n$, (resp. $r|s$), with respect to a quantum superalgebra $A = A_0 \oplus A_1$, (resp. $B = B_0 \oplus B_1$). We shall assume that the centre $Z = Z(A)$ of $A$, acts on $B$ that becomes a $Z$-module.

Let $f \in Q^\infty_w(X,Y)$. Then $f$ is stable if there is a neighborhood $W_f \subset Q^\infty_w(X,Y)$ of $f$, in the natural Whitney-type topology of $Q^\infty_w(X,Y)$, such that every $W_f$ is contained in the orbit of $f$, via the action of the group $\text{Diff}(X) \times \text{Diff}(Y)$.

This is equivalent to say that for any $f' \in W_f$ there exist quantum diffeomorphisms $g : X \to X$ and $h : Y \to Y$ such that $h \circ f = f' \circ g$. Furthermore, $f$ is called infinitesimally stable if there exist a map $\zeta : X \to TY$, such that $\pi_Y \circ \zeta = f$, where $\pi_Y : TY \to Y$ is the canonical map, and integrable vector fields $\nu : Y \to TY$, $\xi : X \to TX$, such that $\zeta = T(f) \circ \xi + \nu \circ f$. Thus the following diagram is commutative.

![Diagram](https://via.placeholder.com/150)

### Theorem 3.2.

Let $X$ be a compact quantum supermanifold and $f : X \to Y$ be quantum smooth. Then $f$ is stable iff $f$ is infinitesimally stable. Furthermore, if $f$ is a proper mapping, then does not necessitate assume that $X$ is compact.

**Proof.** Note that the infinitesimal stability, requires existence of flows $g_t : X \to X$, $\partial g = \xi$, $h_t : Y \to Y$, $\partial h = \nu$, such that for the infinitesimal variation $\zeta$ of $f_t = h_t \circ f \circ g_t$ one has $\zeta = T(f) \circ \xi + \nu \circ f$. In fact, one has the following lemma.

---

12In the following, whether it is not differently specified, $X$ and $Y$ are such quantum supermanifolds.

13Here $\text{Diff}(X)$ denotes the group of quantum diffeomorphisms of a quantum supermanifold $X$.

14Recall that a map $f : X \to Y$ between topological spaces is a proper map if for every compact subset $K \subset Y$, $f^{-1}(K)$ is a compact subset of $X$. 

Lemma 3.3. Let \((W, V, \pi_W; \mathbb{B})\) be a bundle of geometric objects in the category \(\mathcal{Q}_S\) and in the intrinsic sense \([49]\). Let \(\phi : \mathbb{R} \times V \to V\) be a one-parameter group of \(Q^\infty_w\) transformations of \(V\), \(\xi = \partial \phi\) its infinitesimal generator and \(s : V \to W\) a field of geometric objects, i.e. a section of \(\pi_W\). Then, \(\phi\) induces a deformation \(\tilde{s}\) of \(s\) defined by means of the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{R} \times \mathbb{R} \times V & \xrightarrow{\tilde{s}} & W \\
\downarrow{\text{(id}_k, \phi)\phantom{\text{d}}} \quad & \quad & \sideset{\overline{\phi}_\lambda}{\text{d}} \\
\mathbb{R} \times V & \xrightarrow{(\text{id}_k, s)} & \mathbb{R} \times W \\
\end{array}
\]

where \(\overline{\phi}_\lambda \equiv \mathbb{B}(\phi^{-1}_\lambda)\), \(\forall \lambda \in \mathbb{R}\). One has \(\tilde{s}_{(0,0)} = s\). Then, for the infinitesimal variation of \(\tilde{s}\) (Lie derivative of \(s\) with respect to the integrable field \(\xi\)), \(\partial(\tilde{s} \circ d) : V \to s^*vTW\), one has:

\[
\partial(\tilde{s} \circ d) = \partial(s \circ \phi) + \partial(\phi) \circ s = T(s) \circ \xi + \nu \circ s.
\]

Proof. This lemma can be proved by copying the intrinsic proof for the commutative case given in \([49]\).

In our case we can consider the following situation, with respect to Lemma 3.3, \(W = X \times Y\), \(V = X\), \(\mathbb{B}(g_\lambda) = h_\lambda\) and \(s = (\text{id}_X, f)\).

Furthermore, in the case that \(X\) is compact, the proof follows the same lines of the proof given by Mather for commutative manifolds \([39]\).

Theorem 3.4. Stable maps \(f : X \to Y\) do not necessitate to be dense in \(Q^\infty_w(X, Y)\).

Proof. This is just a corollary of the corresponding theorem for commutative manifolds given by Thom-Levine \([31, 32]\).

Example 3.5. (Submersions and stability). Let \(X\) be a compact quantum supermanifold. Let \(f : X \to Y\) be a quantum differentiable mapping of maximum possible super-rank. If \(m \geq r > 1\), \(n \geq s > 1\), \(f\) is a quantum submersion and it is (infinitesimally) stable.

Example 3.6. (Immersions and stability). Let \(X\) a compact quantum supermanifold. Let \(f : X \to Y\) be a quantum differentiable mapping of maximum possible super-rank. If \(m \leq r\), \(n \leq s\), \(f\) is an immersion and if it is 1 : 1 then it is also stable. (Not all immersions are stable.)

Definition 3.7. (Singular solutions of quantum super PDE’s). Let \(\pi : W \to M\) be a fiber bundle, where \(M\) is a quantum supermanifold of dimension \((m|n)\) on the quantum superalgebra \(A\) and \(W\) is a quantum supermanifold of dimension \((m|n, r|s)\) on the quantum superalgebra \(B \equiv A \times E\), where \(E\) is also a \(Z\)-module, with \(Z = Z(A)\) the centre of \(A\).

Let \(E_k \subset JD^k(W)\) be a quantum super PDE. By using the natural embedding \(JD^k(W) \subset \tilde{J}^k_{m|n}(W)\), we can consider quantum super PDEs \(\tilde{E}_k \subset JD^k(W)\) like quantum super PDEs \(E_k \subset \tilde{J}^k_{m|n}(W)\), hence we can consider solutions of \(\tilde{E}_k\) as \((m|n)\)-dimensional, (over \(A\)), quantum supermanifolds \(V \subset \tilde{E}_k\) such that \(V\) can be

\footnote{See also Refs.\([50, 51, 57]\) for related subjects.}
represented in the neighborhood of any of its points \( q' \in V \), except for a nowhere dense subset \( \Sigma(V) \subset V \), of dimension \( \leq (m-1)n - 1 \), as \( N^{(k)} \), where \( N^{(k)} \) is the \( k \)-quantum prolongation of a \((m|n)\)-dimensional (over \( A \)) quantum supermanifold \( N \subset W \). In the case that \( \Sigma(V) = \emptyset \), we say that \( V \) is a regular solution of \( \hat{E}_k \subset \hat{j}^{k}_{m|n}(W) \). Solutions \( V \) of \( \hat{E}_k \subset \hat{j}^{k}_{m|n}(W) \), even if regular ones, are not, in general diffeomorphic to their projections \( \pi_k(V) \subset M \), hence are not representable by means of sections of \( \pi : W \rightarrow M \). \( \Sigma(V) \subset V \) is the singular points set of \( V \).

Then \( V \setminus \Sigma(V) = \bigcup_r V_r \) is the disjoint union of connected components \( V_r \). For every of such components \( \pi_{k,0} : V_r \rightarrow W \) is an immersion and can be represented by means of \( k \)-prolongation of some quantum supermanifold of dimension \( m|n \) over \( A \), contained in \( W \). Whether we consider \( \hat{E}_k \) as contained in \( J\hat{D}^k(W) \) then regular solutions are locally obtained as image of \( k \)-derivative of sections of \( \pi : W \rightarrow M \). So we can (locally) represent such solutions by means of mapping \( f : M \rightarrow E_k \), such that \( f = D^k s \), for some section \( s : M \rightarrow W \).

We shall also consider solutions of \( \hat{E}_k \subset \hat{j}^{k}_{m|n}(W) \), any subset \( V \subset \hat{E}_k \), that can be obtained as projections of ones of the previous type, but contained in some \( s \)-prolongation \( \hat{E}_{k+s} \subset \hat{j}^{k+s}_{m|n}(W) \), \( s > 0 \).

We define weak solutions, solutions \( V \subset \hat{E}_k \), such that the set \( \Sigma(V) \) of singular points of \( V \), contains also discontinuity points, \( q, q' \in V \), with \( \pi_{k,0}(q) = \pi_{k,0}(q') = a \in W \), or \( \pi_k(q) = \pi_k(q') = p \in M \). We denote such a set by \( \Sigma(V)_S \subset \Sigma(V) \), and, in such cases we shall talk more precisely of singular boundary of \( V \), like \( (\partial V)_S = \partial V \setminus \Sigma(V)_S \). However for abuse of notation we shall denote \( (\partial V)_S \), (resp. \( \Sigma(V)_S \)), simply by \( (\partial V) \), (resp. \( \Sigma(V) \)), also if no confusion can arise.

**Definition 3.8.** (Stable solutions of quantum super PDE’s). Let us consider a quantum super PDE \( \hat{E}_k \subset J\hat{D}^k(W) \), and let us denote \( \text{Sol}(\hat{E}_k) \) the set of regular solutions of \( E_k \). This has a natural structure of locally convex manifold. Let \( f : X \rightarrow E_k \) be a regular solution, where \( X \subset M \) is a smooth \((m|n)\)-dimensional compact manifold with boundary \( \partial X \). Then \( f \) is stable if there is a neighborhood \( W_f \) of \( f \) in \( \text{Sol}(\hat{E}_k) \), such that each \( f' \in W_f \) is equivalent to \( f \), i.e., \( f \) is transformed in \( f' \) by some integrable vertical symmetries of \( \hat{E}_k \).

**Theorem 3.9.** Let \( \hat{E}_k \subset J\hat{D}^k(W) \) be a \( k \)-order quantum super PDE on the fiber bundle \( \pi : W \rightarrow M \) in the category of quantum smooth supermanifolds. Let \( s : M \rightarrow W \) be a section, solution of \( \hat{E}_k \), and let \( \nu : M \rightarrow s^* \nu TW \equiv \hat{E}[s] \) be an integrable solution of the linearized equation \( \hat{E}_k[s] \subset J\hat{D}^k(\hat{E}[s]) \). Then to \( \nu \) it is associated a flow \( \{ \phi_\lambda \}_{\lambda \in J} \), where \( J \subset \mathbb{R} \) is a neighborhood of \( 0 \in \mathbb{R} \), that transforms \( V \) into a new solution \( \hat{V} \subset \hat{E}_k \).

**Proof.** Let \((x^\alpha, y^I)\) be fibered coordinates on \( W \). Let \( \nu = \partial y_I(\nu^I) : M \rightarrow s^* \nu TW \) a vertical vector field on \( W \) along the section \( s : M \rightarrow W \). Then \( \nu \) is a solution of
\[ \hat{E}_k[s] \text{ iff the following diagram is commutative:} \]

\[
\begin{align*}
\xymatrix{
\hat{E}_k & vT\hat{E}_k & (D^k s)^*vT\hat{E}_k \ar@{~>}[l] \ar[r]^-{\sim} & \hat{E}_k[s] \\
J\hat{D}^k(W) & vTJ\hat{D}^k(W) & (D^k s)^*vTJ\hat{D}^k(W) \ar@{~>}[l] \ar[r]^-{\sim} & J\hat{D}^k(\hat{E}[s]) \\
W & vTW & \hat{E}[s] \ar@{~>}[l] \ar@{~>}[u] \ar[r] \ar[l] & M
}
\end{align*}
\]

Then \( D^k \nu(p) \) identifies, for any \( p \in M \), a vertical vector on \( \hat{E}_k \) in the point \( q = D^k s(p) \in V = D^k s(M) \subset \hat{E}_k \). On the other hand infinitesimal vertical symmetries on \( E_k \) are locally written in the form

\[
(55) \quad \begin{cases}
\zeta = \sum_{0 \leq |\alpha| \leq k} \partial y_\alpha^2(Y_\alpha), & 0 = \zeta.F^I = \langle dF^I, \left( \sum_{0 \leq r \leq k} \partial y_\alpha^{\alpha_1\cdots\alpha_r}(Y_\alpha^{\alpha_1\cdots\alpha_r}) \right) \rangle \\
Y_\alpha^j = Z_\alpha^{(0)}(Y_\gamma), & Z^{(0)} = \partial x_\alpha + \partial y_\alpha^a \\
Y_\alpha^{j_1\cdots j_{n-1}} = Z^{(r)}_i(Y_\alpha^{j_1\cdots j_{n-1}}), & Z^{(r)}_i = Z^{(r-1)}_i + \partial y_\alpha^{a_1\cdots a_{r-1}}y_\alpha^{a_r}\gamma_i
\end{cases}
\]

where \( Y_\alpha^j \in Q^\infty_2(U \subset J\hat{D}^k(W); \hat{A}(E)), \hat{A}(E) \equiv \text{Hom}_Z(A \otimes Z \cdots \otimes Z A; E), 0 \leq |\alpha| \leq k, \partial y_\alpha^2(q) \in \text{Hom}_Z(A(E); T_\gamma J\hat{D}^k(W)), y_\alpha^{j_1\cdots j_{n-1}} \in Q^\infty_2(U; \hat{A}(E)) \). Then we can see that solutions of \( \hat{E}_k[s] \) are vertical vector fields \( \nu : M \to s^*vTW \equiv \hat{E}[s] \), such that their prolongations \( D^k \nu = \zeta \circ D^k s \), for some vertical symmetry \( \zeta \) of \( E_k \). Therefore, the flows of above integrable vertical vector fields, transform regular solutions \( V \) of \( \hat{E}_k \) into new solutions of \( \hat{E}_k \). Solutions of the linearized equation \( \hat{E}_k[s] \) give initial conditions for the determination of such vertical flows. \hfill \Box

The following lemmas are also important to understand how the structure of solutions of \( \hat{E}_k[s] \) are related to the vertical symmetries of \( E_k \). (For complementary informations on the contact structure of \( \hat{j}^k_{m|n}(W) \), see [69].)

**Lemma 3.10.** (Symmetries of horizontal \( k \)-order contact ideals). Let \( \hat{j}^k_{m|n}(W) \to \hat{j}^{k+1}_{m|n}(W) \) be a quantum \((k+1)\)-connection on \( W \), i.e., a \( Q^\infty_2 \)-section of \( \pi_{k+1,k} \). (The restriction of \( \hat{j}^k \) to \( \hat{j}^k_{m|n}(W) \subset \hat{j}^{k+1}_{m|n}(W) \) is also called quantum \((k+1)\)-connection.) Let \( \hat{S}_k(\cdot) \) be the quantum horizontal \( k \)-order contact ideal of \( Q^\bullet_2(\hat{j}^k_{m|n}(W)) \) given by \( \hat{S}_k(\cdot) \equiv \cdot|^*\hat{\mathfrak{c}}_{k+1}(W) \), where \( \hat{\mathfrak{c}}_{k+1}(W) \) is the contact ideal of \( \hat{j}^{k+1}_{m|n}(W) \). Locally
one can write \( \mathcal{H}_k(\cdot) = \langle \omega^j, \ldots, \omega^{j_1} \rangle, \mathcal{H}_{\alpha_1, \ldots, \alpha_k} \rangle, \) where

\[
\begin{cases}
\mathcal{H}_{\alpha_1, \ldots, \alpha_k} \equiv \langle \omega^j_{\alpha_1, \ldots, \alpha_k} = \langle (dy^j_{\alpha_1, \alpha_k} - y^j_{\alpha_1, \alpha_k} dx^\beta) \\
\quad = dy^j_{\alpha_1, \alpha_k} - \gamma_{\alpha_1, \alpha_k} dx^\beta \in \Omega^1(\mathcal{H}_{m[n]}(W))
\end{cases}
\]

with \( \gamma_{\alpha_1, \ldots, \alpha_k} \equiv y_{\alpha_1, \ldots, \alpha_k} dx^\beta \in \Omega^0(\mathcal{J}_{m[n]}(W)). \) \( \mathcal{C}_k(W) \) is a subideal of \( \mathcal{H}. \) Then the quantum horizontal \( k \)-order Cartan distribution \( \mathcal{H}_k(\cdot) \subset T \mathcal{J}_{m[n]}(W) \) (identified by a \( (k + 1) \)-connection \( \mathcal{J} \)) is the Cauchy characteristic distribution associated to \( \mathcal{H}_k(\cdot). \) \( \mathfrak{d}(\mathcal{H}_k(\cdot)) \) admits the following local (canonical basis):

\[
\begin{cases}
\zeta_{\alpha} = \partial x_{\alpha} + \partial y_{j_{\alpha}} \\
\quad + \cdots + \partial y_{j_{\alpha_1, \ldots, \alpha_k}} y_{j_{\alpha_{k+1}}} + \partial y_{j_{\alpha_1, \ldots, \alpha_k}} \gamma_{j_{\alpha_{k+1}}}
\end{cases}
\]

For any quantum \( (k + 1) \)-connection \( \mathcal{J} \) on \( W, \) one has the following direct sum decompositions:

\[
\begin{align*}
\mathbf{E}_{m[n]}^k(W)_q & \cong \mathcal{H}_k(\cdot)_q \oplus \text{Hom}_{\mathbb{Z}}(\mathcal{S}^k(T_a N); \nu_a) \\
\Omega^1(\mathcal{J}_{m[n]}^k(W)) & \cong \Omega^1(\mathcal{J}_{m[n]}^{k+1}(W)) \oplus \mathfrak{d}(\mathcal{H}_k(\cdot))
\end{align*}
\]

with \( a \equiv \pi_{k,(q)} \in W, \) \( \mathcal{J} = \mathcal{J}_{m[n]}^k(W), \) \( \mathcal{H}_k(\cdot)_q \equiv \mathcal{T}_{q, \mathcal{N}(k)} \), \( \mathcal{H}_k(\cdot) \equiv \mathcal{H}_k(\cdot) \cap \Omega^1(\mathcal{J}_{m[n]}^k(W)). \) The connection \( \mathcal{J} \) is flat, i.e., with zero curvature, iff the differential ideal \( \mathcal{H}_k(\cdot) \) is closed, or equivalently, iff \( \mathcal{H}_k(\cdot) \) is involutive. If \( \mathcal{J} \) is a flat quantum \( (k + 1) \)-connection on \( W, \) then one has the following:

\[
\begin{align*}
\mathcal{C}_k(W) & \subset \mathfrak{d}(\mathcal{H}_k(\cdot)) \quad \text{as a closed subideal} \\
\mathcal{H}_k(\cdot) & \equiv \mathcal{C}_{\text{char}}(\mathcal{H}_k(\cdot)); \quad \text{char}(\mathcal{H}_k(\cdot)) \subset \mathfrak{d}(\mathcal{H}_k(\cdot)).
\end{align*}
\]

\( \mathcal{J}_{m[n]} \) is foliated by regular solutions \( Z \) such that \( \mathcal{H}_k(\cdot)|_Z = 0. \) The leaves of the foliation are given in implicit form by the following equations: \( \mathcal{J}^I(x^\alpha, y^I, \ldots, y^I_{\alpha_1, \ldots, \alpha_k}) = \kappa^I \in \mathcal{E}_k, \) \( 1 \leq I \leq p + q, \) \( \dim \mathcal{J}_{m[n]}^k(W) = (p, q) = m[n], \) where \( \mathcal{J}^I \) represent a complete independent system of primitive integrals of the linear system of PDEs \( (\zeta_{\alpha}, f) = 0, \) \( 1 \leq \alpha \leq m + n, \) where \( \zeta_{\alpha} \) is a basis (e.g., the canonical basis) of the horizontal distribution \( \mathcal{H}_k(\cdot). \) Any \( \zeta \in \mathfrak{s}(\mathcal{H}_k(\cdot)) \) has the following local representation:

\[
\begin{align*}
\zeta &= \zeta_{\alpha}(X^\alpha) + \partial y_{j^I}(Y^j) + \partial y_{j^I}(\zeta_{\alpha}(Y^j) + \partial y_{j^I}(\zeta_{\alpha}(Y^j)) \\
&\quad + \cdots + \partial y_{j^I}(\zeta_{\alpha_1, \ldots, \alpha_k}(Y^j))
\end{align*}
\]

\( ^{\text{For a distribution } E \subset T X \text{ on a manifold } X, \text{ we denote by } \mathfrak{s}(E) \text{ the vector space of vector fields on } X \text{ belonging to } E.} \)

\( ^{\text{\( \mathcal{C}_{\text{char}}(\mathcal{H}_k(\cdot)) \) denotes the characteristic distribution of } \mathcal{H}_k(\cdot), \text{ and } \text{char}(\mathcal{H}_k(\cdot)) \text{ the corresponding vector space of its vector fields. Furthermore, } \mathfrak{s}(\mathcal{H}_k(\cdot)) \text{ denotes the vector space of infinitesimal symmetries of the ideal } \mathcal{H}_k(\cdot). \text{ It is the vector space of vertical infinitesimal symmetries of the ideal } \mathcal{H}_k(\cdot).} \)

\( ^{\text{\( \Lambda \) (local) section } s \text{ of } \pi \text{ identifies a flat (local) } (k + 1) \text{-connection } \mathcal{J}_{m[n]+1} = (\partial x_{\alpha_1}, \ldots, \partial x_{\alpha_{k+1}}, s^I).} \)
for any choice of \( x^\alpha \in Q^\infty_w(U \subset J^k_{m|n}(W), A) \), \( 1 \leq \alpha \leq m + n \), and \( Y^j \in Q^\infty_w(U \subset J^k_{m|n}(W), E) \), \( 1 \leq j \leq r + s \), such that

\[
(\zeta_1 \cdots \zeta_k, Y^j) = (\partial y_i^j|_{\alpha_1 \cdots \alpha_k}) Y^i + (\partial y_i^j|_{\beta_1 \cdots \beta_s}) (\zeta_1 Y^j) + \cdots + (\partial y_i^j|_{\gamma_1 \cdots \gamma_t}) (\zeta_k Y^j).
\]

The space \( \mathfrak{s}(\mathfrak{h}_k(\{\})) \) admits the following direct sum decomposition:

\[
\mathfrak{s}(\mathfrak{h}_k(\{\})) \cong \mathfrak{d}(\mathfrak{H}_k(\{\})) \oplus \mathfrak{v}_k(\{\}),
\]

where \( \mathfrak{v}_k(\{\}) \) is the collection of all vectors of the form

\[
\xi = \zeta - \zeta_0(X^\alpha) = \partial y_j^a(Y^j) + \partial y_j^{a\beta}(\zeta_1 Y^j) + \partial y_j^{a\beta}(\zeta_2 Y^j) + \cdots + \partial y_j^{a\beta}(\zeta_k Y^j),
\]

for any choice of \( Y^j \in Q^\infty_w(U \subset J^k_{m|n}(W), E) \), \( 1 \leq j \leq r + s \), such that conditions (60) are satisfied. \( \mathfrak{s}(\mathfrak{h}_k(\{\})) \) is a Lie algebra that admits the subalgebra \( \mathfrak{d}(\mathfrak{H}_k(\{\})) \) as an ideal.

The general local expression for the symmetries of the \((m|n)\)-dimensional involutive Cartan distribution \( \mathbf{E}_\infty(W) \subset T J^k_{m|n}(W) \), can be also obtained by equations (59) with all \( k > 0 \), and forgetting conditions (60).\(^{19}\) So we get the following expression for \( \zeta \in \mathfrak{s}(\mathbf{E}^\infty_m(W)) \):

\[
\zeta = \partial_a(X^\alpha) + \sum_{r \geq 0} \partial y_i^{a \cdots a r}(Y^j_{\alpha_1 \cdots \alpha_r}),
\]

\[
\partial_a = \partial x_a + \sum_{r \geq 0} \partial y_i^{a \cdots a r}(y^a_{\alpha_1 \cdots \alpha_r}),
\]

\[
Y^j_{\alpha_1 \cdots \alpha_r} = (\partial_{\alpha_1} \cdots \partial_{\alpha_r} Y^j), \quad Y^j \in Q^\infty_w(U \subset J^k_{m|n}(W), E), 1 \leq j \leq r + s.
\]

Then the canonical splitting \( T_q J^k_{m|n}(W) \cong (\mathbf{E}^\infty_m(W))_q \oplus v T_q J^k_{m|n}(W) \), \( q \in J^k_{m|n}(W) \), gives the following splitting in \( \mathfrak{s}(\mathbf{E}^\infty_m(W)) = \mathfrak{d}(\mathbf{E}^\infty_m(W)) \oplus \mathfrak{v}_\infty \), \( \zeta = \zeta_0 + \zeta_v \), with \( \zeta_0 = \partial_a(X^\alpha) \) and \( \zeta_v = \sum_{r \geq 0} \partial y_i^{a \cdots a r}(Y^j_{\alpha_1 \cdots \alpha_r}) \), where \( Y^j_{\alpha_1 \cdots \alpha_r} \) are given in (62).

**Definition 3.11.** Let \( \hat{E}_k \subset J^k_{m|n}(W) \), where \( \pi : W \to M \) is a fiber bundle, in the category of quantum smooth supermanifolds. We say that \( \hat{E}_k \) is functionally stable if for any compact regular solution \( V \subset \hat{E}_k \), such that \( \partial V = N_0 \cup P \cup \tilde{N}_1 \) one has quantum solutions \( \tilde{V} \subset J^k_{m|n}(W) \), \( s \geq 0 \), such that \( \pi_{k+s,0}(\tilde{N}_0 \cup \tilde{N}_1) = \pi_{k,0}(N_0 \cup \tilde{N}_1) \equiv X \subset W \), where \( \partial \tilde{V} = \tilde{N}_0 \cup \tilde{P} \cup \tilde{N}_1 \).

We call the set \( \Omega[V] \) of such solutions \( \tilde{V} \) the full quantum situs of \( V \). We call also each element \( \tilde{V} \in \Omega[V] \) a quantum fluctuation of \( V \).\(^{20}\)

**Definition 3.12.** We call infinitesimal bordism of a regular solution \( V \subset \hat{E}_k \subset J^k W \) an element \( \tilde{V} \in \Omega[V] \), defined in the proof of Theorem 3.9. We denote by

\(^{19}\)In fact the Cartan distribution on \( J^k_{m|n}(W) \) can be considered an horizontal distribution induced by the canonical connection identified by the local canonical basis \( \zeta_0 = \partial x_\alpha + \sum_{|\beta| \geq 0} \partial y_i^\alpha \partial y_i^\beta \) just generating \( \mathbf{E}^\infty_m(W) \).

\(^{20}\)Let us emphasize that to \( \Omega[V] \) belong also (non necessarily regular) solutions \( V' \subset E_k \) such that \( N_0' \cup N_1' = N_0 \cup N_1 \), where \( \partial V' = N_0' \cup P' \cup N_1' \).
\( \Omega_0[V] \subset \Omega[V] \) the set of infinitesimal bordisms of \( V \). We call \( \Omega_0[V] \) the infinitesimal situs of \( V \).

**Definition 3.13.** Let \( \hat{E}_k \subset \tilde{j}^{k}_{m|n}(W) \), where \( \pi : W \to M \) is a fiber bundle, in the category of quantum smooth supermanifolds. We say that a regular solution \( V \subset \hat{E}_k \), \( \partial V = N_0 \cup P \cup N_1 \), is functionally stable if the infinitesimal situs \( \Omega_0[V] \subset \Omega[V] \) of \( V \) does not contain singular infinitesimal bordisms.

**Theorem 3.14.** Let \( \hat{E}_k \subset \tilde{j}^{k}_{m|n}(W) \), where \( \pi : W \to M \) is a fiber bundle, in the category of quantum smooth supermanifolds. If \( \hat{E}_k \) is quantum formally integrable and completely quantum superintegrable, then it is functionally stable as well as Ulam-extended superstable.

A regular solution \( V \subset \hat{E}_k \) is stable iff it is functionally stable.

**Proof.** In fact, if \( \hat{E}_k \) is quantum formally integrable and completely quantum superintegrable, we can consider, for any compact regular solution \( V \subset \hat{E}_k \), its \( s \)-th prolongation \( V^{(s)} \subset (\hat{E}_k)^{+}_s \subset \tilde{j}^{k+s}_{m|n}(W) \). Since one has the following short exact sequence

\[
\begin{array}{c}
\Omega_{m-1|n-1}^{(E_k)^{+}_s} \\
\rightarrow \\
\Omega_{m-1|n-1}(\hat{E}_k)^{+}_s \\
\rightarrow \\
0
\end{array}
\]

where \( \Omega_{m-1|n-1}^{(E_k)^{+}_s} \) (resp. \( \Omega_{m-1|n-1}(\hat{E}_k)^{+}_s \)), is the integral bordism group, (resp. quantum bordism group),\(^{21}\) we get that there exists a solution \( \bar{V} \subset \tilde{j}^{k+s}_{m|n}(W) \) such that

\[
\begin{array}{l}
\partial \bar{V} = \tilde{N}_0 \cup \tilde{P} \cup \tilde{N}_1; \\
\partial V^{(s)} = \tilde{N}_0^{(s)} \cup \tilde{P}^{(s)} \cup \tilde{N}_1^{(s)}; \\
\tilde{N}_0^{(s)} = \tilde{N}_0^{(s)}; \\
\tilde{N}_1^{(s)} = \tilde{N}_1^{(s)}.
\end{array}
\]

Then, as a by-product we get also: \( \pi_{k+s,0}(\tilde{N}_0 \cup \tilde{N}_1) = \pi_{k,0}(N_0 \cup N_1) \subset W \). Therefore, \( \hat{E}_k \) is functionally stable. Furthermore, \( \hat{E}_k \) is also Ulam-extended superstable, since the integral bordism group \( \Omega_{m-1|n-1}^{\tilde{E}_k} \) for smooth solutions and the integral bordism group \( \Omega_{m-1|n-1,s}^{\tilde{E}_k} \) for singular solutions, are related by the following short exact sequence:

\[
\begin{array}{c}
0 \\
\rightarrow \\
\tilde{E}_k^{m-1|n-1,s} \\
\rightarrow \\
\tilde{E}_k^{m-1|n-1} \\
\rightarrow \\
\tilde{E}_k^{m-1|n-1,s} \\
\rightarrow \\
0
\end{array}
\]

This implies that in the neighborhood of each smooth solution there are singular solutions.

Finally a regular solution \( V \subset \hat{E}_k \) is stable iff the set of solutions of the corresponding linearized equation \( \tilde{E}_k[V] \) does not contains singular solutions. But this is just the requirement that \( \Omega_0[V] \) does not contains singular solutions. Therefore, \( V \) is stable if it is functionally stable and vice versa. More precisely if \( f = D^{k,s} : X \to \hat{E}_k \) is a stable solution of \( \hat{E}_k \), then there exists an open set \( W_s \subset \text{Sol}(\hat{E}_k) \) such that for any \( s' \in W_s \), \( s' \) is equivalent to \( s \).\(^{22}\) Let us consider the tangent space \( T_s\text{Sol}(\hat{E}_k) \).

---

\(^{21}\)Here the considered bordism groups are for admissible non-necessarily closed Cauchy hypersurfaces.

\(^{22}\)Recall that \( Q^\infty(W) \) has a natural structure of quantum smooth supermanifold modeled on locally convex topological vector fields. \( \text{Sol}(\hat{E}_k) \) is a closed submanifold of \( \text{Sol}(\hat{E}_k) \subset Q^\infty(W) \). (For details see ref. [57].)
One has the following isomorphism
\begin{equation}
T_s \text{Sol}(\hat{E}_k) \cong \left\{ \xi \in (Q^\infty_w)_0((D^k s)^*vT \hat{E}_k) \mid \exists \xi \in T_s Q^\infty_w(W), \zeta = |k \circ D^k \xi| \right\} \cong \Omega_0[V]
\end{equation}
where \(|k|\) is the canonical isomorphism \(J \hat{D}^k(s^*vT W) \cong (D^k s)^*vT J \hat{D}^k(W)\), and \(V = D^k s(X) \subset \hat{E}_k\). Since \(W_s\) is open in \(\text{Sol}(\hat{E}_k)\), one has also the following isomorphism \(T_s W_s \cong T_s \text{Sol}(\hat{E}_k)\). Thus also to \(s'\) there correspond vector fields \(\zeta \in T_s W_s\) that must be regular ones, i.e., without singular points. Therefore \(\Omega_0[V]\) cannot contain singular solutions, hence \(V\) is functionally stable. Vice versa, if \(V\) is functionally stable, then we can find an open neighborhood \(W_s \subset \text{Sol}(\hat{E}_k)\) built by perturbing \(V\) with all the flows induced by the regular vector fields belonging to \(\Omega_0[V]\). This set is an open set of \(W_s \subset \text{Sol}(\hat{E}_k)\) since its tangent space at any of its point \(s'\) is isomorphic to \(T_s \text{Sol}(\hat{E}_k)\), since this last is isomorphic to \(\Omega_0[V]\). Furthermore, any two of such points of such an open set are equivalent since they can be related both to \(s\) by local diffeomorphisms. Therefore \(V\) that is functionally stable is also stable. 

\begin{remark}
Let us emphasize that the definition of functionally stable quantum super PDE interprets in pure geometric way the definition of Ulam superstable functional equation just adapted to PDE’s.\footnote{Let us recall the concept of Ulam stability. Let \(F\) be a functional space, i.e., a space of suitable applications \(f : X \to Y\) between finite dimensional Riemannian manifolds \(X\) and \(Y\). Let \(E\) be a Banach space and \(S\) a subset of \(X^n\). Let us consider a functional equation:
\begin{equation}
G(f, q^1, \cdots, q^n) = 0, \quad \forall(q^1, \cdots, q^n) \in S \subset X^n
\end{equation}
defined by means of a mapping \(G : F \times X^n \to E, (f(q^1, \cdots, q^n)) \to G(f(q^1, \cdots, q^n))\). We say that such a functional equation is \emph{Ulam-extended stable} if for any function \(f \in F\), satisfying the inequality
\begin{equation}
\|G(\tilde{f}, (q^1, \cdots, q^n))\| \leq \varphi(q^1, \cdots, q^n), \quad \forall(q^1, \cdots, q^n) \in X^n
\end{equation}
with \(\varphi : X^n \to [0, \infty)\) fixed, there exists a solution \(f\) of \(67\) such that \(d_Y(\tilde{f}(q), f(q)) \leq \Phi(q), \forall q \in X\), for suitable \(\Phi : X \to [0, \infty)\). Here \(d_Y(\cdot, \cdot)\) is the metric induced by the Riemannian structure of \(Y\). More precisely, \(d_Y(a, b) = \inf_{[0, 1]} \sqrt{g_Y(\gamma'(t), \gamma'(t))} dt\), \(\forall a, b \in Y\), where \(\gamma \in C^1([0, 1], Y)\), with \(\gamma(0) = a, \gamma(1) = b\) and \(g_Y\) the Riemannian metric on \(Y\). One has the following important propositions: (i) If \(\gamma \in C^1([0, 1], Y)\), such that \(\gamma(0) = a, \gamma(1) = b\), and \(\int_{[0, 1]} \sqrt{g_Y(\gamma'(t), \gamma'(t))} dt = d_Y(a, b)\), then \(\gamma([0, 1]) \subset Y\) is a geodesic. If in addition, \(\gamma\) has constant speed, then \(\gamma\) is a \(C^\infty\) geodesic. (ii) (de Rham) When \(Y\) is geodesically complete, then given \(a, b \in Y\), there exists at least one geodesic \(\gamma\) connecting \(a\) to \(b\) with \(\int_{[0, 1]} \sqrt{g_Y(\gamma'(t), \gamma'(t))} dt = d_Y(a, b)\). (iii) (Hopf, Rinov) The geodesic completeness of \(Y\) is equivalent to the completeness of \(Y\) as a metric space, which is equivalent to the statement that a subset of \(Y\) is compact if it is closed and bounded. (iv) \(Y\) is complete whenever it is compact. If each solution \(f \in F\) of the inequality \(68\) is either a solution of the functional equation \(67\) or satisfies some stronger conditions, then we say that equation \(67\) is \emph{Ulam-extended-superstable}.}
\end{remark}
We have the following criteria for functional stability of solutions of quantum super PDE’s and to identify stable extended crystal quantum super PDE’s.

**Theorem 3.18.** (Functional stability criteria). Let $\hat{E}_k \subset J\hat{D}^k(W)$ be a $k$-order quantum formally integrable and completely quantum superintegrable quantum super PDE on the fiber bundle $\pi : W \to M$.

1) If the symbol $\hat{g}_k = 0$, then all the quantum smooth regular solutions $V \subset \hat{E}_k \subset J\hat{D}^k(W)$ are functionally stable, with respect to any non-weak perturbation. So $\hat{E}_k$ is a stable extended crystal.

2) If $\hat{E}_k$ is of finite type, i.e., $\hat{g}_{k+r} = 0$, for $r > 0$, then all the quantum smooth regular solutions $V \subset \hat{E}_{k+r} \subset J\hat{D}^{k+r}(W)$ are functionally stable, with respect to any non-weak perturbation. So $\hat{E}_k$ is a stabilizable extended crystal with stable extended crystal $(S)\hat{E}_k = \hat{E}_{k+r}$.

3) If $V \subset (\hat{E}_k)_{+\infty} \subset J\hat{D}^\infty(W)$ is a smooth regular solution, then $V$ is functionally stable, with respect to any non-weak perturbation. So any quantum formally integrable and completely quantum superintegrable quantum super PDE $\hat{E}_k \subset J\hat{D}^k(W)$, is a stabilizable extended crystal, with stable extended crystal $(S)\hat{E}_k = (\hat{E}_k)_{+\infty}$.

**Proof.** We shall use the following lemmas.

**Lemma 3.19.** Let $\hat{E}_k \subset J\hat{D}^k(W)$ be a quantum formally integrable and completely quantum superintegrable quantum super PDE the fiber bundle $\pi : W \to M$. Then for any quantum smooth regular solution $s : M \to W$, one has the following canonical isomorphism: $(\hat{E}_k[s])_{+h} \cong (\hat{E}_k)_{+h}[s], \forall h \geq 1, \infty$.

**Proof.** In fact one has the following commutative diagram.

\[
\begin{align*}
(\hat{E}_k[s])_{+h} &= J\hat{D}^h((D^k s)^*vT\hat{E}_k) \cap J\hat{D}^{k+h}(s^*vTW) \\
&\cong (D^{k+h}s)^*vT(J\hat{D}^h(\hat{E}_k) \cap J\hat{D}^{k+h}(W)) \\
&\cong (D^{k+h}s)^*vT((\hat{E}_k)_{+h}) = ((\hat{E}_k)_{+h})[s].
\end{align*}
\]  

**Lemma 3.20.** Let $\hat{E}_k \subset J\hat{D}^k(W)$ be a formally integrable and completely integrable PDE the fiber bundle $\pi : W \to M$. Let $\hat{g}_k = 0$. Then also the prolonged equations $(\hat{E}_k)_{+r}, \forall r \geq 1, \infty$, have their symbols zero: $(\hat{g}_k)_{+r} = 0, \forall r \geq 1, \infty$.

**Proof.** In fact, from the definition of symbol and prolonged symbols, it follows that the prolonged symbols coincide with the symbols of the corresponding prolonged equations.

1) This follows from Lemma 3.19 and from the fact that if $\hat{g}_k = 0$ is also $\hat{g}_k[s] = 0$. This excludes that $\hat{E}_k[s]$ could have singular solutions. Furthermore, Lemma 3.20 excludes also that there are singular (nonweak) solutions in the prolonged equations $\hat{E}_k[s]_{+r}, \forall r \geq 1, \infty$.

2) If $\hat{E}_k$ is of finite type, with $\hat{g}_{k+r} = 0$, then it is also $\hat{g}_{k+r}[s] = 0$. Then $\hat{E}_{k+r}[s]$ cannot have singular (nonweak) solutions.

3) $\hat{E}_\infty$ has zero symbol, hence also $\hat{E}_\infty[s]$ has zero symbol and cannot have singular (nonweak) solutions.

(So the proof follows the same lines drawn for commutative PDE’s.)
Theorem 3.21. (Functional stable solutions and \((k+1)\)-connections). Let \(\hat{E}_k \subset \hat{J}^k_{m|n}(\hat{W})\) be a quantum formally integrable and completely quantum superintegrable quantum super PDE. Let \(\hat{\gamma}\) be a quantum flat \((k+1)\)-connection, such that \(\hat{\gamma}|_{\hat{E}_k}\) is a \(Q^\infty_w\)-section of the affine fiber bundle \(\pi_{k+1,k} : (\hat{E}_k)_{+1} \rightarrow \hat{E}_k\). Then, the subequation \(\hat{E}_k \subset \hat{E}_k\) identified, by means of the ideal \(\mathcal{J}(\hat{\gamma})|_{\hat{E}_k}\), is formally integrable and completely quantum superintegrable sub-equation with zero symbol \(\hat{\gamma}_k\). Then \(\hat{E}_k \subset \hat{E}_k\) is functionally stable and Ulam-extended superstable. Furthermore any regular quantum smooth solution \(V \subset \hat{E}_k\) is also functionally stable in \(\hat{E}_k\), with respect to any non-weak perturbation.

Proof. In fact, one has the following commutative diagram of exact lines.

\[
\begin{array}{c}
\Omega_{m-1|n-1}^{(\hat{E}_k)_+}, & \Omega_{m-1|n-1}^{((\hat{E}_k)_+)} & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\Omega_{m-1|n-1}^{(\hat{E}_k)_+}, & \Omega_{m-1|n-1}^{((\hat{E}_k)_+)} & \rightarrow & 0 \\
0 & 0 &
\end{array}
\]

Furthermore, since \(\hat{\gamma}_k = 0\), \(\hat{E}_k\) is of finite type, hence its smooth regular solutions are functionally stable.

Taking into account the meaning that connections assume in any physical theory, we can give the following definition.

Definition 3.22. Let \(\hat{E}_k \subset \hat{J}^k_{m|n}(\hat{W})\) be a quantum formally integrable and completely quantum superintegrable quantum super PDE. Let \(\hat{\gamma}\) be a flat quantum \((k+1)\)-connection, such that \(\hat{\gamma}|_{\hat{E}_k}\) is a \(Q^\infty_w\)-section of the affine fiber bundle \(\pi_{k+1,k} : (\hat{E}_k)_{+1} \rightarrow \hat{E}_k\). We call the couple \((\hat{E}_k, \hat{\gamma})\) a polarized quantum super PDE. We call also polarized quantum super PDE, a couple \((\hat{E}_k, \hat{E}_k)\), where \(\hat{E}_k \subset \hat{E}_k\), is defined in Theorem 3.21. We call \(\hat{E}_k\) a polarization of \(\hat{E}_k\).

Corollary 3.23. Any quantum smooth regular solutions of a polarization of a polarized couple \((\hat{E}_k, \hat{E}_k)\), is functionally stable, with respect to any non-weak perturbation.

Theorem 3.24. (Finite stable extended crystal PDE) Let \(\hat{E}_k \subset J\hat{D}^k(\hat{W})\) be a quantum formally integrable and completely quantum superintegrable quantum super PDE, such that the centre \(Z(A)\) of the quantum superalgebra \(A\), model for \(M\), is Noetherian. Then, under suitable finite ellipticity conditions, there exists a stable extended crystal quantum super PDE \((S)^{\hat{E}_k}\) canonically associated to \(\hat{E}_k\), i.e., \(\hat{E}_k\) is a stabilizable extended crystal.

Proof. In fact, we can use the following lemma.

Lemma 3.25. (Finite stability criterion). Let \(\hat{E}_k \subset J\hat{D}^k(\hat{W})\) be a quantum formally integrable and completely quantum superintegrable quantum super PDE, such that the centre \(Z(A)\) of the quantum superalgebra \(A\), model for \(M\), is Noetherian.
Then there exists an integer \( s_0 \) such that, under suitable finite ellipticity conditions, any regular quantum smooth solution \( V \subset (\hat{E}_k)_{+s_0} \) is functionally stable.

**Proof.** Under the hypotheses that \( Z(A) \) is Noetherian, the proof follows the same line of the commutative case. \( \square \)

Let us, now, use the hypothesis that \( \hat{E}_k \) is quantum formally integrable and completely quantum superintegrable. Then all its regular quantum smooth solutions are all that of \((\hat{E}_k)_{+s_0}\). In fact, these are all the solutions of \((\hat{E}_k)_{+\infty} \subset JD^\infty(W)\). However, even if a smooth regular solution \( V \subset \hat{E}_k \), and their \( s_0 \)-prolongations, \( V^{(s_0)} \subset (\hat{E}_k)_{+s_0} \), are equivalent as solutions, they cannot be considered equivalent from the stability point of view!!! In fact, \( \hat{E}_k \) can admit singular solutions, instead for \((\hat{E}_k)_{+s_0}\) these are forbidden. Therefore, for \( \hat{E}_k[s] \) singular perturbations are possible, i.e. are possible infinitesimal vertical symmetries of \( \hat{E}_k \), in a neighborhood of the solution \( s \), having singular points. Instead for \((\hat{E}_k)_{+s_0}[s] \) all solutions are without singular points, hence \( s \) considered as solution of \((\hat{E}_k)_{+s_0} \) necessitates to be functionally stable.

By conclusions, \( \hat{E}_k \), under the finite ellipticity conditions is a stabilizable extended crystal quantum super PDE, and its stable extended crystal quantum super PDE is \((S)\hat{E}_k = (\hat{E}_k)_{+s_0} \), for a suitable finite number \( s_0 \). \( \square \)

**Remark 3.26.** With respect to a quantum frame \([63, 69, 70]\), we can consider the perturbation behaviours of global solutions for \( t \to \infty \), where \( t \) is the proper time of the quantum frame. Then, we can talk about asymptotic stability by reproducing similar situations for commutative PDE’s. (See Refs.[72, 77].) In particular we can consider the concept of ”averaged stability” also for solutions of quantum (super) PDE’s. With this respect, let us recall the following definition and properties of quantum (pseudo)Riemannian supermanifold given in [63, 71].

**Definition 3.27.** \([63, 71]\) A quantum (pseudo)Riemannian supermanifold \((M, \hat{A})\) is a quantum supermanifold \( M \) of dimension \((m|n)\) over a quantum superalgebra \( A \), endowed with a \( Q^\infty_\omega \) section \( \hat{g} : M \to \text{Hom}_Z(TM \otimes_Z TM; A) \) such that the induced homomorphisms \( T_pM \to (T_pM)^+, \forall p \in M \), are injective.

**Proposition 3.28.** \([63, 71]\) In quantum coordinates \( \hat{g}(p) \) is represented by a matrix

\[
\hat{g}_{\alpha\beta}(p) \in \hat{A}_{00}(A) \times \hat{A}_{10}(A) \times \hat{A}_{01}(A) \times \hat{A}_{11}(A).
\]

The corresponding dual quantum metric gives \( \hat{g}^{\alpha\beta}(p) \in \hat{A}^{00}(A) \times \hat{A}^{10}(A) \times \hat{A}^{01}(A) \times \hat{A}^{11}(A) \), with \( \hat{A}^{ij}(A) \equiv \text{Hom}_Z(A; \hat{A}_i \otimes_Z \hat{A}_j) \), \( i, j \in \mathbb{Z}_2 \), such that \( \hat{g}_{\gamma\beta}(p)\hat{g}^{\alpha\beta}(p) = \delta^\beta_\gamma \in \hat{A}, \hat{g}^{\alpha\beta}(p)\hat{g}_{\gamma\beta}(p) = \delta^\gamma_\alpha \in \text{Hom}_Z(A \otimes_Z A; A \otimes_Z A) \).

In fact we have the following definition.

**Definition 3.29.** Let \( E_k \subset JD^k(W) \) be a formally integrable and completely integrable PDE the fiber bundle \( \pi : W \to M \), and let \( V = D^k s(M) \subset E_k \) be a regular smooth solution of \( E_k \). Let \( \xi : M \to E_k[s] \) be the general solution of \( E_k[s] \). Let us assume that there is an Euclidean structure on the fiber of \( E[s] \to M \). Let \( (\psi : \mathbb{R} \times N \to N; i : N \to M) \) be a quantum frame \([63, 69, 70]\). Then, we say that \( V \) is average asymptotic stable, with respect to the quantum frame, if the function
of time \( p[i](t) \) defined by the formula:

\[
(71) \quad p[i](t) = \frac{1}{2 \text{vol}(B_t)} \int_{B_t} i^* \xi^2 \eta
\]

has the following behaviour: \( p[i](t) = p[i](0) e^{-ct} \) for some real number \( c > 0 \). Here \( B_t \equiv N_t \cap \text{supp}(i^* \xi^2) \), where \( N = \bigcup_{t \in T} N_t \) is the fiber structure of \( N \), over the proper-time of the quantum frame. We call \( \tau_0 = 1/c_0 \) the characteristic stability time of the solution \( V \). If \( \tau_0 = \infty \) it means that \( V \) is average unstable.\(^{24}\)

We have the following criterion of average asymptotic stability.

**Theorem 3.30.** (Criterion of average asymptotic stability). A regular global smooth solution \( s \) of \( E_k \) is average stable, with respect to the quantum frame \( (\psi : \mathbb{R} \times N \to N; i : N \to M) \), if the following conditions are satisfied.\(^{25}\)

\[
(72) \quad \langle \dot{p}[i](t) \rangle \leq c \langle p[i](t) \rangle, \quad c \in \mathbb{R}^+, \forall t.
\]

where

\[
(73) \quad p[i](t) = \frac{1}{2 \text{vol}(B_t)} \int_{B_t} i^* \xi^2 \eta
\]

and

\[
(74) \quad \dot{p}[i](t) = \frac{1}{2 \text{vol}(B_t)} \int_{B_t} \left( \frac{\delta i^* \xi^2}{\delta t} \right) \eta = \frac{1}{\text{vol}(B_t)} \int_{B_t} \left( \frac{\delta i^* \xi}{\delta t} \dot{i}^* \xi \right) \eta.
\]

Here \( i^* \xi \) represents the integrable general solution of the linearized equation \( E_k[s][i] \) of \( E_k \) at the solution \( s \), and with respect to the quantum frame. Let us denote by \( c_0 \) the infimum of the positive constants \( c \) such that inequality (72) is satisfied. Then we call \( \tau_0 = 1/c_0 \) the characteristic stability time of the solution \( V \). If \( \tau_0 = \infty \) means that \( V \) is unstable.\(^{26}\)

Furthermore, Let \( s \) be a smooth regular solution of a formally quantum integrable and completely quantum superintegrable quantum super PDE \( E_k \subset \hat{J}^k \hat{D}(W) \), where \( \pi : W \to M \). There exists a differential operator \( \mathcal{P}[s][i](\xi) \), on \( \hat{\pi} : \hat{E}[s][i] \equiv i^* (s^* vTW) \to N \), canonically associated to the solution \( s \), and with respect to the quantum frame, such that \( s \) is average stable in \( \hat{E}_k \), or in some suitable prolongation \((\hat{E}_k)_+ \), \( k + h = 2s \geq k \), if the following conditions are verified:

(i) \( \mathcal{P}[s][i](\xi) \) is self-adjoint (or symmetric) on the constraint

\[
(75) \quad (\hat{E}_k)_{r+h}[s][i] \subset \hat{J}^k \hat{D}^{k+r}(\hat{E}[s][i]),
\]

for some \( r \geq 0 \).

(ii) The smallest eigenvalue \( \lambda_1 = \lambda_1(t) \) of \( \mathcal{P}[s][i](\xi) \) is positive for any \( t \in T \) and lower bounded: \( \lambda_1 \geq \lambda_1 > 0 \). Furthermore, average stability can be also translated into a variational problem constrained by \((\hat{E}_k)_{r+h}[s] \), for some \( h \geq 0 \), such that \( k + h = 2s \).

**Proof.** We shall use Theorem 3.9 and the following lemma.

\(^{24}\)In the following, if there are not reasons of confusion, we shall call also stable solution a smooth regular solution of a PDE \( E_k \subset \hat{J}^k \hat{D}(W) \) that is average asymptotic stable.

\(^{25}\)The large cuspidated brackets \( \langle, \rangle \) denote expectation value.

\(^{26}\)\( \tau_0 \) has just the physical dimension of a time.
Lemma 3.31. (Grönwall’s lemma)\textsuperscript{20} Suppose \( f(t) \) is a real function whose derivative is bounded according to the following inequality: \( \frac{df}{dt} \leq g(t)f + h(t) \), for some real functions \( g(t) \) and \( h(t) \). Then, \( f(t) \) is bounded pointwise in time according to \( f(t) \leq f(0)e^{G(t)} + \int_{[0,t]} e^{G(t-s)}h(s)ds \), where \( G(t) = \int_{[0,t]} g(r)dr \).

Then a sufficient condition for the solution \( V \) stability, with respect to the quantum frame, is that inequality (72) should be satisfied. In fact it is enough to use Lemma 3.31 with \( g(t) = -c \), \( h(t) = 0 \), to have \( p[i](t) = p[i](0)e^{-ct} \). Furthermore, condition (72) is satisfied iff

\[
I[\xi | i] = \left< -2 \int_{B_t} \frac{\delta i^*\xi}{\delta t} + \alpha i^*\xi, i^*\xi > \eta \right> \geq 0,
\]

for some constant \( c > 0 \) and for any integrable solution \( i^*\xi \) of \( \hat{E}_k [s|i] \). So the problem is converted to study the spectrum of the differential operator, \( \mathcal{P}[s|i](\xi) = \delta i^*\xi \), on \( \hat{\pi} : \hat{E}[s|i] \to N \), constrained by \( (\hat{E}_k + r)[s] \), for some \( r \geq 0 \), since \( \mathcal{P}[s|i](\xi) \) is of order \( \geq k \). If this is self-adjoint, (or symmetric), it follows that it has real spectrum and the stability of the solution is related to the sign of the smallest eigenvalue.\textsuperscript{27} If such an eigenvalue \( \lambda_1(t) \) is positive, \( \forall t \in T \), and \( \lambda_1 = \inf_{t \in T} > 0 \), then the ratio \( \frac{-\dot{p}[i](t)}{p[i](t)} > \frac{p[i](t)}{p[i](t)} \) is higher than a positive constant, hence the solution \( s \) is average stable. In fact, we get

\[
\begin{align*}
-\dot{p}[i](t) - \lambda_1 p[i](t) &= \int_{B_t} \left( \mathcal{P}[s|i]\xi, \xi \right) - \lambda_1 i^*\xi^2 \eta \\
&= \int_{B_t} (\lambda_1(t) - \lambda_1) i^*\xi^2 \eta = (\lambda_1(t) - \lambda_1) \int_{B_t} i^*\xi^2 \eta \geq 0,
\end{align*}
\]

for any \( t \in T \). Thus we have also

\[
\frac{-\dot{p}[i](t)}{p[i](t)} \geq \lambda_1 > 0, \quad \forall t \in T.
\]

So condition (72) is satisfied, hence the solution \( s \) is average stable. In order to complete the proof of Theorem 3.30, let us emphasize that in general \( \mathcal{P}[s|i](\xi) - \alpha i^*\xi \) is not identified with the Euler-Lagrange operator for some Lagrangian. In fact, in general, the differential order of such an operator does not necessitate to be even. By the way, since \( \hat{E}_k \) is assumed formally quantum integrable and completely quantum superintegrable, we can identify any smooth solution \( V \subset \hat{E}_k \), with its \( h \)-prolongation \( V^{(h)} \subset JD^{k+h}(W) \), such that \( k + h = 2s \). Thus the problem of average stability can be translated in a variational problem, constrained by solutions of \( (\hat{E}_k + (h))[s|i] \).

\[
\delta i^*\xi \left( \frac{\delta}{\delta t} \right) = 2\lambda(t)i^*\xi, \quad F_{\alpha}^I[s|i] = 0, \quad 0 \leq |\alpha| \leq h, \quad k + h = 2s
\]

on the fiber bundle \( \hat{\pi} : \hat{E}[s|i] \equiv i^*(s^*\nu T \mathcal{W}) \to N \). Here \( F_{\alpha}^I[s|i] = 0 \) are the equations encoding \( \hat{E}_k [s|i] \). This can be made not only locally but also globally. In

\textsuperscript{27}Really it should be enough to require that \( \mathcal{P}[s|i] \) is a symmetric operator in the Hilbert space \( \mathcal{H}_t \), canonically associated to \( \hat{E}[s|i] \). In fact the point spectrum \( \mathcal{S}(\mathcal{P}(A)) \) of a symmetric linear operator \( A \) on \( \mathcal{H}_t \) is real \( \mathcal{S}(\mathcal{P}(A)) \subset \mathbb{R} \). (This is true also for its continuous spectrum: \( \mathcal{S}(\mathcal{P}(A)) \subset \mathbb{R} \).) In our case it is enough that \( \mathcal{P}[s|i] \) should symmetric on the space of \( \hat{E}_k [s|i] \) solutions. However, it is well known in functional analysis that every symmetric operator has a self-adjoint extension, on a possibly larger space \([13]\).
Lemma 3.32. [80] Let $\pi : W \to M$, a fiber bundle in the category $\Omega_S$, $\dim_A M = m|n$, $\dim_B W = (m|n, r|s)$. Let $L : \hat{J}^k_{m|n}(W) \to \hat{A}$ be a $k$-order quantum Lagrangian function and $\theta \equiv L\eta \in \hat{\Omega}^{m+n}(\hat{J}^k_{m|n}(W))$, locally given by $\theta = Ldx^1\triangle\cdots\triangle dx^{m+n} = l\mu_* \circ dx^1\triangle\cdots\triangle dx^{m+n}$, where $(x^n, y^p)$ are fibered quantum coordinates on $W$, and $\hat{\mu}_* : \hat{T}^{m+n}_0(A) \to A$ is the $Z$-homomorphism induced by the product on $A$. Then, extremals for $\theta$, constrained by $\hat{E}_k$, are solutions $f : X \to \hat{E}_k$, with $X$ a quantum supermanifold of dimension $m|n$ with respect to $A$, such that the following condition is satisfied:

$$
(80) \quad \left\langle \sum_{1 \leq j \leq r+s} \nu^j \sum_{0 \leq |i| \leq k} (-1)^{|i|} \partial_i \left( \frac{\partial L}{\partial y_i^j} \right) \right\rangle \eta, X = 0,
$$

for any $\nu = \nu^j \partial y_j$, solution of the linearized equation of $\hat{E}_k$ at the solution $s$. In particular, if $\hat{E}_k = \hat{J}^k_{m|n}(W)$, then extremals are solutions of the following equation (Euler-Lagrange equation):

$$
(81) \quad \hat{E}[\theta] \subset \hat{J}^{2k}_{m|n}(W) : \left\{ \sum_{0 \leq |i| \leq k} (-1)^{|i|} \partial_i \left( \frac{\partial L}{\partial y_i^j} \right) = 0 \right\} _{1 \leq j \leq r+s}.
$$

This completes the proof. \qed

Example 3.33. (Quantum super d’Alembert equation). Let $A = A_0 \oplus A_1$ be a quantum superalgebra with $Z = Z(A)$ Noetherian and $\mathbb{K} = \mathbb{R}$. Let us consider the following trivial fiber bundle $\pi : W \equiv A^3 \to A^2 \equiv M$ with quantum coordinates $(x, y, u) \to (x, y)$. Then, the quantum super d’Alembert equation, $(d^A) \subset J\hat{\Delta}^2(W) \subset \hat{J}^2_{2|2}(W)$, is defined by means of the following $A$-valued $Q_w^\infty$-function $F \equiv uu_{xy} - u_yu_x : J\hat{\Delta}^2(W) \to A \subset \hat{A}$. By forgetting the $\mathbb{Z}_2$-gradation of $A$, we get the same situation just considered in [63]. So, $(d^A) \subset J\hat{\Delta}^2(W)$ is just a formally quantum integrable quantum super PDE of dimension $(3, 2, 2)$ over the quantum algebra $B \equiv A \times \hat{A} \times \hat{A}$, in the open quantum submanifold $u \neq 0$, and also completely quantum integrable there. On the other hand, by considering that $M$ has a natural structure of quantum supermanifold of dimension $(2|2)$ over $A$, it follows also that for any initial condition, i.e., any point $q \in (d^A) \setminus u^{-1}(0)$, passes a quantum supermanifold of dimension $(2|2)$ over $A$, solution of $(d^A)$. Therefore, $(d^A) \subset \hat{J}^2_{2|2}(W)$ is completely quantum superintegrable in the quantum supermanifold $u \neq 0$ too. We can also state that $(d^A)$ is completely quantum superintegrable, as it is algebraic in the open set $(d^A) \setminus u^{-1}(0)$. Thus $(d^A)$ is an extended crystal PDE. With respect to the commutative exact diagram in (24) we get the following exact commutative diagram
With respect to the commutative exact diagram in (24) we get the following exact commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K_{1,1:2}^{\hat{d}A} & \longrightarrow & \Omega_{1,1}^{\hat{d}A} & \longrightarrow & \Omega_{2}^{\hat{d}A} & \longrightarrow & 0 \\
0 & \longrightarrow & K_{2}^{\hat{d}A} & \longrightarrow & \Omega_{2}^{\hat{d}A} & \longrightarrow & \mathbb{Z}_{2} & \longrightarrow & 0
\end{array}
\]

Therefore, the crystal group of \((d'A)\) is \(G(2) = \mathbb{Z}^2 \times \mathbb{Z}_2 \equiv p2\), \((p2)\) is its usual crystallographic notation, and its crystal dimension is 2.

Furthermore, according to Theorem 3.14 we get that \((d'A) \setminus \{u = 0\}\) is functionally stable. From Theorem 3.24 we get that \((d'A)\) is a stabilizable quantum extended crystal PDE with associated stable quantum extended crystal PDE \(\hat{d}A = (\hat{d}A)_{+\infty}\). (The symbol of \((d'A)\) is not zero, thus smooth global solutions are in general unstable into finite times in \((d'A)\).) Moreover, we get that the weak integral \((1|1)\)-bordism group of \((d'A)\) is trivial. In fact, we have: \(\Omega_{1,1,\infty}^{\hat{d}A} \equiv A_{\Omega_{1,1}}(W) \equiv (Z \otimes_{\mathbb{K}} H_{1}(W; \mathbb{K})) \oplus (A_{1} \otimes_{\mathbb{K}} H_{1}(W; \mathbb{K})) = 0\). Thus \((d'A)\) is an extended 0-crystal.

Under the full-admissibility hypothesis, i.e., by considering admissible closed smooth integral quantum supermanifolds of dimension \((m-1|n-1)\), on the which all integral characteristic quantum supernumbers are zero, we can consider \((d'A)\) a 0-crystal quantum super PDE, hence for such fully admissible Cauchy data, the existence of global smooth solutions of \((d'A)\) is assured. Finally, applying Theorem 3.30 we get further informations on the asymptotic average stability of \((d'A)\) solutions.

**Example 3.34.** (Quantum super Navier-Stokes equation). In some previous works we have considered the Navier-Stokes equation for quantum (super)fluids as a quantum (super)PDE. (See Refs. [62, 63, 67].) Now, we can extend such considerations to stability of such equations. By using results in [63, 70] we can prove that when \(A\) has Noetherian centre \(Z = Z(A)\), \((NS)\) contains a formally quantum integrable quantum super PDE \((\hat{NS})\) that is completely quantum superintegrable. So \((NS)\) is not functionally stable, but \((\hat{NS})\) is so. This last equation is also an extended crystal quantum super PDE with its infinity prolongation \((\hat{NS})_{+\infty}\) as stable extended crystal quantum super PDE. Furthermore, one can prove that the weak integral \((3|3)\)-bordism group \(\Omega_{3,3,\infty}^{\hat{NS}} = 0 = \Omega_{3,3,\infty}^{\hat{NS}}\). This means that \((\hat{NS})\) is an extended 0-crystal quantum super PDE. However, it is not a 0-crystal quantum super PDE. By the way, whether we adopt the full admissibility hypothesis, then \((\hat{NS})\) becomes a 0-crystal quantum super PDE and this is enough to state the existence of global smooth solutions of the quantum super PDE \((NS)\) for such admissible smooth boundary condition contained int \((\hat{NS})\).

With respect to the commutative exact diagram in (24) we get the following exact commutative diagram
Therefore, the crystal group and the crystal dimension of \( (NS) \) are the same ones of \( (YM) \).

Finally, applying Theorem 3.30 we get further informations on the asymptotic average stability of \((NS)\) solutions.

4. QUANTUM EXTENDED CRYSTAL SINGULAR PDE’S

In this section we shall consider singular quantum super PDE’s extending our previous theory of singular PDE’s,\(^{28}\) i.e., by considering singular quantum (super) PDE’s as singular quantum sub-(super)manifolds of jet-derivative spaces in the category \( \Omega \) or \( \Omega_S \). In fact, our previous formal theory of quantum (super) PDE’s works well on quantum smooth or quantum analytic submanifolds, since these regularity conditions are necessary to develop such a theory. However, in many mathematical problems and physical applications, it is necessary to work with less regular structures, so it is useful to formulate a general geometric theory for such more general quantum PDE’s in the category \( \Omega_S \). Therefore, we shall assume that quantum singular super PDE’s are subsets of jet-derivative spaces where are presents regular subsets, but also other ones where the conditions of regularity are not satisfied. So the crucial point to investigate is to obtain criteria that allow us to find existence theorems for solutions crossing ”singular points” and study their stability properties.

The main result of this section is Theorem 4.8 that relates singular integral bordism groups of singular quantum PDE’s to global solutions passing through singular points. Some example are explicitly considered.

Let us, now, first begin with a generalization of algebraic formulation of quantum super PDE’s, starting with the following definitions. (See also Refs.[61, 67, 68, 69].)

**Definition 4.1.** The general category of quantum superdifferential equations, \( \Omega^S \), is defined by the following: 1) \( \mathfrak{B} \in \text{Ob}(\Omega^S) \) iff \( \mathfrak{B} \) is a filtered quantum superalgebra \( \mathfrak{B} \equiv \{ \mathfrak{B}_i \}, \mathfrak{B}_i \subset \mathfrak{B}_{i+1}, \) such that in the differential calculus in the category \( \Omega^S_G(\mathfrak{B}) \) over \( \mathfrak{B} \) is defined a natural operation \( C \) that satisfies \( C\hat{\Omega}^1 \wedge \hat{\Omega}^* = C\hat{\Omega}^* \), where \( \hat{\Omega}^i \equiv \mathfrak{B} \wedge \cdots \wedge \mathfrak{B} \) are the representative objects of the functor \( \bar{D}_i \) in the category \( \Omega^S_G(\mathfrak{B}) \) over \( \mathfrak{B} \), where \( \bar{D}_i \equiv D \cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \), being \( D(P) \) the \( \mathfrak{B} \)-module of all quantum superdifferentiations of algebra \( \mathfrak{B} \) with values in module \( P \). Furthermore, \( \hat{\Omega}^* = \bigoplus_{i \geq 0} \hat{\Omega}^i, \hat{\Omega}^0 \equiv A \). 2) \( f \in \text{Hom}(\Omega^S) \) iff \( f \) is a homomorphism of filtered quantum superalgebras preserving operation \( C \).

\(^{28}\)See Refs.[61, 71]. See also [4] where some interesting applications are considered.
Remark 4.2. In practice we shall take $\mathfrak{B} \equiv \{ \mathfrak{B}_i \equiv Q^\infty(M_i; A) \}$, where $M_i$ is a quantum supermanifold and $A$ is a quantum superalgebra. Then, we have a canonical inclusion: $j_i : M_i \rightarrow Sp(\mathfrak{B}_i)$, $x \mapsto j_i(x) \equiv e_x \equiv$ evaluation map at $x \in M_i$. To the inclusion $\mathfrak{B}_i \subset \mathfrak{B}_{i+1}$ corresponds the quantum smooth map $M_{i+1} \rightarrow M_i$. So we set $M_\infty = \lim_{\rightarrow} M_i$. One has $\mathcal{M}_\infty = Sp(\mathfrak{B}_\infty)$. However, as $M_\infty$ contains all the "nice" points of $Sp(\mathfrak{B}_\infty)$, we shall use the space $M_\infty$ to denote an object of the category of quantum superdifferential equations.

Definition 4.3. The category of quantum superdifferential equations $\mathcal{Q}_S^\infty$ is defined by the Frobenius full quantum superdistribution $\tilde{C}(X) \subset \tilde{T}X \equiv \text{Hom}_Z(A;TX)$, which is locally the same as $\tilde{E}_\infty$, i.e., the Cartan quantum superdistribution of $\tilde{E}_\infty$ for some quantum super PDE $\tilde{E}_k \subset JD^k(W)$. We set: $s \dim X \equiv \dim \tilde{C}(X) = (m + n|m + n)$, i.e., the Cartan quantum superdimension of $X \in \text{Ob}(\mathcal{Q}_S^\infty)$. $f \in \text{Hom}(\mathcal{Q}_S^\infty)$ iff it is a quantum smooth map $f : X \rightarrow Y$, where $X,Y \in \text{Ob}(\mathcal{Q}_S^\infty)$, such that conserves the corresponding Frobenius full superdistributions: $\tilde{T}(f) : \tilde{C}(X) \rightarrow \tilde{C}(Y)$, $f \in \text{Hom}(\mathcal{Q}_S^\infty(X,Y)$, $s \dim X = (m + n|m + n)$, $s \dim Y = (m'|n'|m' + n')$, $\text{rank} f = (r|s) = \dim(\tilde{T}(f)_x(\tilde{C}(X)_x))$, $x \in X$. Then the fibers $f^{-1}(y)$, $y \in \text{im}(f) \subset Y$, are $(m + n - r|m + n - s)$-quantum superdimensional objects of $\mathcal{Q}_S^\infty$. Isomorphisms of $\mathcal{Q}_S^\infty$: quantum supermorphisms with fibres consisting of separate points. Covering maps of $\mathcal{Q}_S^\infty$: quantum supermorphisms with zero-quantum superdimensional fibres.

Example 4.4. (Some quantum singular PDE’s).

| Name | Singular PDE |
|------|-------------|
| PDE with node and triple point | $p_1 \equiv (u_2^3)^2 - (u_1^4)^2 - (u_2^4)^2 = 0$ |
| $\tilde{R}_1 \subset JD(E)$ | $p_2 \equiv (u_2^3)^2 - (u_2^4)^4 - u_2 u_1^4 = 0$ |
| PDE with cusp and tacnode | $q_1 \equiv (u_2^3)^2 - (u_2^4)^4 - (u_2^4)^2 = 0$ |
| $\tilde{S}_1 \subset JD(E)$ | $q_2 \equiv (u_2^3)^2 - (u_2^4)^4 - (u_2^4)^2 (u_2^4)(u_2^4)^2 = 0$ |
| PDE with conical double point, double line and pinch point | $r_1 \equiv (u_2^3)^2 - (u_2^4)^3 = 0$ |
| $\tilde{T}_1 \subset JD(F)$ | $r_2 \equiv (u_2^3)^2 - (u_2^4)^3 = 0$ |
| $\tilde{T}_1 \subset JD(F)$ | $r_3 \equiv (u_3)^3 + (u_2^4)(u_2^4) = 0$ |

In Tab.5 we report some quantum singular PDE’s having some algebraic singularities. For the first two equations these are quantum singular PDE’s of first order defined on the quantum fiber bundle $\pi : E \rightarrow M$, with $E \equiv A^4$, $M \equiv A^2$, where $A$ is a quantum algebra. Then $JD(E) \cong B_{1,4}^1 = A^4 \times \tilde{A}^4$. Furthermore, for the third equation one has the quantum fiber bundle $\tilde{F} : F \rightarrow M$, with $F \equiv A^5$, $M \equiv A^2$, and $JD(F) \cong B_{1,6}^8 = A^5 \times \tilde{A}^6$. We follow our usual notation introduced in some previous works on the same subject. In particular for a given quantum (super)algebra.
A, we put

$$\begin{align*}
\hat{T}_0^r(H) &\equiv H \otimes \mathbb{Z} \cdots \otimes \mathbb{Z} H, \quad r \geq 0 \\
A &\equiv \text{Hom}_Z(\hat{T}_0^r(A); A), \quad r \geq 0 \\
\hat{A} &\equiv \text{Hom}_Z(\hat{T}_0^0(A); A \equiv \text{Hom}_Z (A; A) \equiv \hat{A}
\end{align*}$$

with $Z$ the centre of $A$ and $H$ any $Z$-module. Furthermore, we denote also by $\hat{S}_0^r(H)$ and $\hat{\Lambda}_0^r(H)$ the corresponding symmetric and skew-symmetric submodules of $\hat{T}_0^r(H)$.

To the ideals $a \equiv q_1, q_2 > \mathfrak{B}_1$, $b \equiv q_1, q_2 > \mathfrak{B}_1$, and $c \equiv q_1, q_2 > \mathfrak{P}_1$, where $\hat{\mathfrak{B}}_1 \equiv \mathbb{Q}_\infty(\hat{J}D(E), \mathfrak{B}_2)$, with $\mathfrak{B}_2 \equiv A \times \hat{A} \times A$, and $\hat{\mathfrak{P}}_1 \equiv \mathbb{Q}_\infty(\hat{J}D(F), \mathfrak{B}_2)$, one associates the corresponding algebraic sets $\hat{R}_1 = \{ q \in B_1^{4,4} | f(q) = 0, \forall f \in a \} \subset B_1^{4,4}$, $\hat{S}_1 = \{ q \in B_1^{4,4} | f(q) = 0, \forall f \in b \} \subset B_1^{4,4}$, and $\hat{T}_1 = \{ q \in B_1^{5,6} | f(q) = 0, \forall f \in c \} \subset B_1^{5,6}$.

Let us consider in some details, for example, the first equation in Tab. 5. There the node and the triple point refer to the singular points in the planes $(u_x^1, u_y^2)$ and $(u_x^2, u_y^1)$ respectively, with respect to the $\mathbb{R}$-restriction. However, the equation $\hat{R}_1$ has a set $\Sigma(\hat{R}_1) \subset \hat{R}_1$ of singular points that contains:

$$\Sigma(\hat{R}_1) = \{ q_0 = (x, y, u^1_1, u^2_1, 0, 0, 0, 0) \} \cong A^4 \subset \hat{R}_1.$$

$\Sigma(\hat{R}_1)$ is, in general, larger than $\Sigma(\hat{R}_1)_{0}$. In fact the Jacobian $(j(F)_{ij})$, $i = 1, 2$, $j = 1, \cdots , 8$, with $(F_1) \equiv (p_1, p_2) : JD(E) \to B_2$, is given by the following matrix with entries in the quantum algebra $B_2$:

$$j(F)_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 2u_x^2[2(u_x^1)^2 - 1] & 0 & 0 & 0 & 4(u_y^2)^3 \\ 0 & 0 & 0 & 0 & 0 & 6(u_y^1)^5 - u_x^2 & 6(u_x^2)^5 - u_y^1 & 0 & 0 \end{pmatrix}.$$

Since, in general, $A$ has a non-empty set of zero-divisors, in order $\hat{Y}_1 \equiv \hat{R}_1 \setminus \Sigma(\hat{R}_1)$ should represent $\hat{R}_1$ without singular points, i.e., in order to apply the implicit quantum function theorem (see Theorem 1.38 in [63]), it is enough to take the points $q \in \hat{R}_1$, where there are $2 \times 2$ minors in (86) with invertible determinant. This allows us to identify an open submanifold $\hat{X}_1$ in $\hat{J}D(E)$. Then, we get $\hat{Y}_1 = \hat{R}_1 \setminus \hat{X}_1$. Thus we can call $\hat{Y}_1 \subset \hat{R}_1$ the regular component of $\hat{R}_1$. This submanifold is not empty, since it contains the regular part of the $\mathbb{R}$-restriction of $\hat{R}_1$. Let us define the following subsets of $\hat{R}_1$:

$$\begin{align*}
\hat{Y}_1 &\equiv \hat{R}_1 \setminus \Sigma(\hat{R}_1) \subset \hat{R}_1 \\
2\hat{R}_1 &\equiv \left\{ q \in \hat{R}_1 | u_x^1(q) = 0, u_y^2(q) = 0 \right\} \subset \hat{R}_1 \\
3\hat{R}_1 &\equiv \left\{ q \in \hat{R}_1 | u_x^2(q) = 0, u_y^1(q) = 0 \right\} \subset \hat{R}_1.
\end{align*}$$

One has $2\hat{R}_1 \cap 3\hat{R}_1 \neq \varnothing$, $\hat{Y}_1 \cap 2\hat{R}_1 \neq \varnothing$, $\hat{Y}_1 \cap 3\hat{R}_1 \neq \varnothing$. Furthermore the set of singular points $\Sigma(2\hat{R}_1)$ (resp. $\Sigma(3\hat{R}_1)$) of $2\hat{R}_1$ (resp. $3\hat{R}_1$) is contained in $\Sigma(\hat{R}_1)$ and contains $\Sigma(2\hat{R}_1)_{0} \equiv 2\hat{R}_1 \cap \Sigma(\hat{R}_1)_{0} \cong A^4$ (resp. $\Sigma(3\hat{R}_1)_{0} \equiv 3\hat{R}_1 \cap \Sigma(\hat{R}_1)_{0} \cong A^4$).

We can write:

$$\hat{R}_1 = \hat{Y}_1 \cup \Sigma(\hat{R}_1) \cong \hat{2}\hat{R}_1 \times 3\hat{R}_1 \cong [\hat{Y}_2 \cup \Sigma(2\hat{R}_1)] \times [\hat{Y}_3 \cup \Sigma(3\hat{R}_1)] \subset \hat{J}D(E),$$
where $\hat{Y}_2 \equiv 2\hat{R}_1 \setminus \Sigma(2\hat{R}_1)$ (resp. $\hat{Y}_3 \equiv 3\hat{R}_1 \setminus \Sigma(3\hat{R}_1)$). $\hat{Y}_1$ is a formally quantum integrable and completely quantum integrable quantum PDE of first order. (For the theory of formal integrability of quantum PDE’s, see Refs.\[55, 59, 67, 68, 69\].) In fact $\hat{Y}_1$ and its prolongations $(\hat{Y}_1)_{+1+} \subset \hat{J}D^{r+1}(E)$, are subbundles of $\hat{J}D^{r+1}(E) \to \hat{J}D^r(E)$, $r \geq 0$. One can also see that the canonical maps $\pi_{r+1,r}: (\hat{Y}_1)_{+r} \to (\hat{Y}_1)_{+(r-1)}$, are surjective mappings. For example, for $r = 1$, one has the following isomorphisms:

(89)

$$
\begin{align*}
\dim_{B_1} \hat{J}D(E) &= (4, 4) \\
\hat{Y}_1 &\cong A^4 \times \hat{A}^2 \Rightarrow \dim_{B_1} \hat{Y}_1 = (4, 2) \\
\hat{J}D^2(E) &\cong A^4 \times \hat{A}^2 \times (\hat{A})^8 \Rightarrow \dim_{B_2} \hat{J}D^2(E) = (4, 4, 8) \\
(\hat{Y}_1)_{+1} &\cong A^4 \times \hat{A}^2 \times (\hat{A})^8 \Rightarrow \dim_{B_2}(\hat{Y}_1)_{+1} = (4, 2, 4) \\
\text{Hom}_Z(\hat{S}_0^2(T_p M); vT_q E) &\cong (\hat{A})^8 \Rightarrow \dim_{B_2} \text{Hom}_Z(\hat{S}_0^2(T_p M); vT_q E) = (0, 0, 8) \\
Y \setminus \hat{Y}_1 &\cong (\hat{A})^4 \Rightarrow \dim_{B_2}(Y \setminus \hat{Y}_1) = (0, 0, 4) \\
\left[ \dim_{B_2}(\hat{Y}_1)_{+1} \right] &\equiv \left[ \dim_{B_2}(Y \setminus \hat{Y}_1) \right].
\end{align*}
$$

Therefore, $(\hat{Y}_1)_{+1} \to (\hat{Y}_1)$, is surjective, and by iterating this process, we get that also the mappings $(\hat{Y}_1)_{+r} \to (\hat{Y})_{+(r-1)}$, $r \geq 0$, are surjective. We put $(\hat{Y}_1)_{+(r-1)} \equiv E$. Thus $\hat{Y}_1$ is a quantum regular quantum PDE, and under the hypothesis that $A$ has a Noetherian centre, it follows that $\hat{Y}_1$ is quantum $\delta$–regular too. Then, from Theorem 3.4 in \[59\], it follows that $\hat{Y}_1$ is formally quantum integrable. Since it is quantum analytic, it is completely quantum integrable too.

**Definition 4.5.** We define quantum extended crystal singular super PDE, a singular quantum super PDE $E_k \subset J^k_{m,n}(W)$ that splits in irreducible components $\hat{A}_i$, i.e., $\hat{E}_k = \bigcup \hat{A}_i$, where each $\hat{A}_i$ is a quantum extended crystal super PDE. Similarly we define quantum extended 0-crystal singular PDE, (resp. quantum 0-crystal singular PDE), a quantum extended crystal singular PDE where each component $\hat{A}_i$ is a quantum extended 0-crystal PDE, (resp. quantum 0-crystal PDE).

**Definition 4.6.** (Algebraic singular solutions of quantum singular super PDE’s). Let $\hat{E}_k \subset J^k_{m,n}(W)$ be a quantum singular super PDE, that splits in irreducible components $\hat{A}_i$, i.e., $\hat{E}_k = \bigcup \hat{A}_i$. Then, we say that $\hat{E}_k$ admits an algebraic singular solution $V \subset \hat{E}_k$, if $V \cap \hat{A}_r \equiv V_r$ is a solution (in the usual sense) in $\hat{A}_r$ for at least two different components $\hat{A}_r$, say $\hat{A}_i, \hat{A}_j$, $i \neq j$, and such that one of following conditions are satisfied: (a) \((i)_j\) $\hat{E}_k \equiv \hat{A}_i \cap \hat{A}_j \neq \emptyset$; (b) \((i)_j\) $\hat{E}_k \equiv \hat{A}_i \cup \hat{A}_j$ is a connected set, and \((i)_j\) $\hat{E}_k = \emptyset$. Then we say that the algebraic singular solution $V$ is in the case (a), weak, singular or smooth, if it is so with respect to the equation \((i)_j\) $\hat{E}_k$. In the case (b), we can distinguish the following situations: (weak solution): There is a discontinuity in $V$, passing from $V_i$ to $V_j$; (singular solution): there is not discontinuity in $V$, but the corresponding tangent spaces $TV_i$ and $TV_j$ do not belong to a same $n$-dimensional Cartan sub-distribution of $J^k_{m,n}(W)$, or alternatively $TV_i$
and $TV_i$ belong to a same $(m|n)$-dimensional Cartan sub-distribution of $J^k_{m|n}(W)$, but the kernel of the canonical projection $(\pi_{k,0})_* : T\hat{j}^k_{m|n}(W) \to TW$, restricted to $V$ is larger than zero; (smooth solution): there is not discontinuity in $V$ and the tangent spaces $TV_i$ and $TV_j$ belong to a same $(m|n)$-dimensional Cartan sub-distribution of $J^k_{m|n}(W)$ that projects diffeomorphically on $W$ via the canonical projection $(\pi_{k,0})_* : T\hat{j}^k_{m|n}(W) \to TW$. Then, we say that a solution passing through a critical zone bifurcate.\textsuperscript{29}

**Definition 4.7.** (Integral bordism for quantum singular super PDE’s). Let $\hat{E}_k \subset \hat{j}^k_{m|n}(W)$ be a quantum super PDE on the fiber bundle $\pi : W \to M$, $\dim_B W = (m|n,r|s)$, $\dim_A M = (m|n)$, $B = A \times E$, $E$ a quantum superalgebra that is also a $Z$-module, with $Z = Z(A)$ the centre of $A$. Let $N_1, N_2 \subset \hat{E}_k \subset \hat{j}^k_{m|n}(W)$ be two $(m - n - 1)$-dimensional, (with respect to $A$), admissible closed integral quantum supermanifolds. We say that $N_1$ algebraic integral bords with $N_2$, if $N_1$ and $N_2$ belong to two different irreducible components, say $N_1 \subset \hat{A}_i$, $N_2 \subset \hat{A}_j$, $i \neq j$, such that there exists an algebraic singular solution $V \subset \hat{E}_k$ with $\partial V = N_1 \bigcup N_2$. In the integral bordism group $\Omega^{\hat{E}_k}_{m-1|n-1}$ (resp. $\Omega^{\hat{E}_k}_{m-1|n-1,s}$, resp. $\Omega^{\hat{E}_k}_{m-1|n-1,w}$) of a quantum singular super PDE $\hat{E}_k \subset \hat{j}^k_{m|n}(W)$, we call algebraic class a class $[N] \in \Omega^{\hat{E}_k}_{m-1|n-1}$, (resp. $[N] \in \Omega^{\hat{E}_k}_{m-1|n-1,s}$, resp. $[N] \in \Omega^{\hat{E}_k}_{m-1|n-1,w}$), with $N \subset \hat{A}_j$, such that there exists a closed $(m - n - 1)$-dimensional, (with respect to $A$), admissible integral quantum supermanifolds $X \subset \hat{A}_i \subset \hat{E}_k$, algebraic integral bording with $N$, i.e., there exists a smooth (resp. singular, resp. weak) algebraic singular solution $V \subset \hat{E}_k$, with $\partial V = N \bigcup X$.

**Theorem 4.8.** (Singular integral bordism group of quantum singular super PDE). Let $\hat{E}_k \equiv \bigcup \hat{A}_i \subset \hat{j}^k_{m|n}(W)$ be a quantum singular super PDE. Then under suitable conditions, algebraic singular solutions integrability conditions, we can find (smooth) algebraic singular solutions bording assigned admissible closed smooth $(m - n - 1)$-dimensional, (with respect to $A$), integral quantum supermanifolds $N_0$ and $N_1$ contained in some component $\hat{A}_i$ and $\hat{A}_j$, $i \neq j$.

**Proof.** In fact, we have the following lemmas.

**Lemma 4.9.** Let $\hat{E}_k \equiv \bigcup \hat{A}_i \subset \hat{j}^k_{m|n}(W)$ be a quantum singular super PDE with

(i) $\hat{E}_k \equiv \hat{A}_i \cap \hat{A}_j \neq \emptyset$. Let us assume that $\hat{A}_i \subset \hat{j}^k_{m|n}(W)$, $\hat{A}_j \subset \hat{j}^k_{m|n}(W)$ and (ii) $\hat{E}_k \subset \hat{j}^k_{m|n}(W)$ be formally integrable and completely integrable quantum super PDE’s with nontrivial symbols. Then, one has the following isomorphisms:

\[
\Omega^{\hat{A}_i}_{m-1|n-1,w} \cong \Omega^{\hat{A}_j}_{m-1|n-1,w} \cong \Omega^{\hat{E}_k}_{m-1|n-1,w} \cong \Omega^{\hat{A}_i}_{m-1|n-1,s} \cong \Omega^{\hat{A}_j}_{m-1|n-1,s} \cong \Omega^{\hat{E}_k}_{m-1|n-1,s}
\]

So we can find a weak or singular algebraic singular solution $V \subset \hat{E}_k$ such that $\partial V = N_0 \bigcup N_1$, $N_0 \subset \hat{A}_i$, $N_1 \subset \hat{A}_j$, if $N_1 \in [N_0]$.

**Proof.** In fact, under the previous hypotheses one has that we can apply Theorem 2.1 in [58] to each component $\hat{A}_i$, $\hat{A}_j$ and (ii) $\hat{E}_k$ to state that all their weak and

\textsuperscript{29}Note that the bifurcation does not necessarily imply that the tangent planes in the points of $V_{ij} \subset V$ to the components $V_i$ and $V_j$ should be different.
singular integral bordism groups of dimension \((m - 1|n - 1)\) are isomorphic to \(H_{m-1|n-1}(W; A)\).

**Lemma 4.10.** Let \(\hat{E}_k = \bigcup_i \hat{A}_i\) be a quantum 0-crystal singular PDE. Let \((i)\hat{E}_k \equiv \hat{A}_i \cup \hat{A}_j\) be connected, and \((ij)\hat{E}_k \equiv \hat{A}_i \cap \hat{A}_j \neq \emptyset\). Then \(\Omega_{m-1|n-1,s}^{(i)}\hat{E}_k = 0\).

**Proof.** In fact, let \(Y \subset (ij)\hat{E}_k\) be an admissible closed \((m - 1|n - 1)\)-dimensional closed integral quantum supermanifold, then there exists a smooth solution \(V_i \subset \hat{A}_i\) such that \(\partial V_i = N_0 \cup Y\) and a solution \(V_j \subset \hat{A}_j\) such that \(\partial V_j = Y \cup N_1\). Then, \(V = V_i \cup Y \cup V_j\) is an algebraic singular solution of \(\hat{E}_k\). This solution is singular in general.

After above lemmas the proof of the theorem can be considered done besides the algebraic singular solutions integrability conditions.

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\[\text{But, in general, it is } \Omega_{m-1|n-1,s}^{(i)}\hat{E}_k \neq 0.\]
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