Advances in Automatic Hexahedral Meshing

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Abstract. The problem of Automated hexahedral meshing has been a puzzle of mesh generation research for years. A great deal of effort has been made on how to generate high-quality hexahedral mesh automatically. The conflicts between different requirements of the mesh provide the challenge. The paper systematically reviews the requirements of the hexahedral meshing challenge, sums up the various approaches to solve the problem and their respective characteristics, analyzes the breakthroughs in recent research, especially in multi-subdomain and strebel differential method. It finally points out that the development of geometric aspects and optimal transmission theory will be the main topics for the research on the organizational creativity in the future.

1. Introduction
Before the actual production of a CAD model, it needs to be simulated on computer to calculate its mechanical properties. The simulation needs to establish a physical model, which is often a partial differential equation, and then use the finite element method to solve the equation.

The finite element method requires that the input entity is refined into a grid with high quality so that the partial differential equation is converted into an algebraic equation. The quality and quantity of the grid will fundamentally affect the accuracy and speed of the calculation result. There are three main types of meshing methods: unstructured tetrahedral meshes; unstructured hexahedral meshes and structured hexahedral meshes.

Compared with tetrahedral meshes, hexahedral meshes have many advantages[1]: The most important benefit is that hexahedral meshes have higher numerical accuracy, lower space complexity, and higher efficiency.

● Non-uniform scale hexahedral meshes have higher numerical accuracy than tetrahedron[2].
● The number of elements in the hexahedral grid is 4-10 times that of the tetrahedral grid, and the complexity of the input grid is constant[2].
● Compared with the tetrahedron, the numerical calculation of hexahedral mesh saves 75% of memory and time.

The algorithm for automatically generating tetrahedral meshes is relatively mature, and there are reliable tools to automatically generate high-quality tetrahedral meshes[3]. On the contrary, the automatic generation of hexahedral mesh is still a huge challenge, which is the so-called "Holy grid" problem[2].

In this paper, the state-of-the-art review of the automatic hexahedral remeshing is presented. The paper starts with the appraisal of the Hex Meshing Constraints. Then, various existing approaches of hexahedral meshing are critically analyzed along with their attributes. Finally, the major challenges and future works are articulated.
2. Hex Meshing Constrains

In the practice of computer-aided engineering (CAE) field, people often decompose an entity into many parts, divide each part separately, and then integrate them. This requires that the splits are consistent with each other at the interface where the parts intersect. A common method is to first design a quadrilateral mesh on each part surface to be consistent with each other, and then expand the quadrilateral mesh of the surface into a hexahedral mesh into the body while keeping the surface quadrilateral mesh unchanged during the process. This naturally caused following problems:

What are the necessary and sufficient conditions for the existence of this hexahedral mesh: A connected region in a given space $\Omega$, Its surface is a curved surface with complex topology $\partial \Omega$, Given quadrilateral mesh $Q$ on the surface, a hexahedral mesh $H$ in the body, make the edge of the hexahedron equal to the quadrilateral mesh: $\partial H = Q$.

In the past decades, so many scholars and engineers are persistently seeking answers to this question. Jeff Erickson [4] use the homology theory to gave us the most precise and concise answer to this basic question: The necessary and sufficient condition that the quadrilateral grid $Q$ can be extended to the hexahedral grid $H$ is that the duality of $Q$ is the edge of the duality of $H$.

However, this theory mainly focuses on unstructured hexahedral meshes. In engineering practice, people care about structured hexahedral meshes. Structured hexahedral grids have more demanding conditions and require deeper insights and more complex theoretical tools.

3. Existing Approach

At present, the representative hexahedral mesh generation methods are mapping method, sweeping method and so on. The following mainly introduces several types related to research.

3.1. Mapping Method

The mapping approach[5] is one of the most classical mesh generation methods. The construction of the mapping function is the key to this method. The mapping function maps the geometric model from the physical space to the parameter space, then generate the grid in the parameter space and reflect them to the physical space. The Common mapping functions are the transfinite mapping method[6], PDE-based method[7], and conformal mapping method[8].

3.2. Sweeping Method

The sweeping approach[9] is considered as a 2.5-D grid generation method. In this method, the quadrilateral is connected into a grid after a series of geometric transformations. The complexity of this approach depends on the number of sources and target faces[10-12]. For complex geometry, the idea of dividing is usually adopted. Generate grid through sweeping each divided part, then automatically combine each part of the grid to generate a whole grid.

3.3. Whisker Weaving Method

The quadrilateral grid $Q$ on the boundary $\partial \Omega$ can be expanded into a hexahedral grid $H$ inside $\Omega$. The necessary and sufficient condition is that $Q$ has an even number of faces. Based on this method, Mitchell developed the whisker weaving method.

Whisker weaving[13-16] is a method built based on spatial twist continuum[17]. This method first generates quad meshes of the model boundary surface, and takes the boundary surface meshes as input, and uses the frontier advancement method to establish the mesh dual of the hexahedral mesh within the geometric model.

3.4. Grid-Based Method

The grid-based method divides the model into a combination of regular grids through hierarchical recursive decomposition. To effectively organize the grid, the most commonly used spatial decomposition data structure - octree, so it is also commonly called the octree method. A typical hexahedral mesh generation process based on the octree method is as follows:[18,19]

1. Build octree
2. Raster subdivision
3. Remeshing
4. Laplace optimization

The result of the Grid-based method is related to the direction of the coordinate axis. After the same model undergoes an affine transformation, applying this method will result in different grids, and all the singular lines are on the surface. The advantage of the grid method is that it is robust, efficient, can adapt to any shape, and does not need to generate a curved grid in advance.

4. Recent Research

The algorithm for automatic generation of hexahedral meshes has made breakthroughs in recent years, and a variety of effective algorithms have been formed. However, the problem of automatic generation of full hexahedral meshes for arbitrary complex three-dimensional shapes has not been solved. In recent years, the new research direction of hexahedral mesh generation algorithm mainly includes the following aspects.

4.1. Multi-Subdomain Methods

Multi-subdomain methods are a large class of methods based on the idea of "divide and conquer", specifically, the multi-subregion method is divided into three main steps:

First, the complex target area is decomposed into simple sub-areas that can be divided by existing algorithms, then each sub-area is divided. Then combine the meshing results to form the overall mesh of the target area. So the problem is decomposed into three small problems: one is the automatic decomposition of complex target areas, the other is the meshing of simple subregions, and the third is the assembly of subregional meshes that meet the requirements of the finite element mesh compatibility. All methods described above can be used for meshing sub-regions; the assembly of sub-region meshes is closely related to the meshing of sub-regions. The automatic decomposition of complex three-dimensional entities is the main difficulty in the multi-subregion method.

The research on automatic decomposition technology is quite active, and the representative work is the mid-surface method and the automatic decomposition method of 3D solids based on feature recognition technology. Lu et al.[20] proposed an automatic decomposition method for three-dimensional entities based on feature recognition technology. This method can be divided into three relatively independent steps: the feature recognition technology is used to extract the decomposed features of the model, and the cutting surface and cutting target domain are generated to generate separate three-dimensional Subregions. The basic steps of Tam et al.'s[21] mid-surface method are: first, the mid-surface extraction algorithm is used to calculate the mid-surface of the three-dimensional target domain, and secondly, the three-dimensional target domain is decomposed into predefined 13 types of simple sub-regions based on the mid-surface; The decomposition technique divides each sub-region into a hexahedral mesh, and finally assembles it. If there is no grid variable density requirement, the midpoint decomposition technique can automatically meet the grid compatibility requirements between sub-regions. However, if there is a requirement for the variable density of the grid, the integer programming technique needs to be used to determine the number of divisions of each edge, to achieve the density control of the grid under the condition of meeting the compatibility requirements[23]. This multi-subregion method based on automatic decomposition technology often introduces singular lines with many singular points.

4.2. Strebel Differential Method

Lei [22] have proved the equivalence among three fundamental concepts:
   \{Colorable Quad-Mesh\} ↔ \{Finite Measured Foliation\} ↔ \{Strebel Differential\}

And this equivalence is named the Trinity, based on the theory of trinity, the automatic generation algorithm of hexahedral mesh was designed.

For the surfaces with different genres, they give a specific hexahedral meshing algorithm. In short, by generating Strebel differentiation on the surface. A quadrilateral mesh on the surface is obtained, and then the surface is divided into several cylindrical surfaces while extending inward will divide the internal body of the surface into several cylinders. A hexahedral mesh was generated inside each cylinder, and finally glued them together, and finally got a three-dimensional solid hexahedral mesh.
Algorithm pipeline 1: The user inputs the admissible curve system (a); the cylindric decomposition graph is constructed (b); the harmonic map induces a Strebel differential (c), where the horizontal and vertical trajectories are illustrated; the cylindric decomposition induced by the Strebel differential. (d)

Algorithm pipeline 2: Each volumetric cylinder (the first and the third rows) is mapped onto a canonical cuboid by a homeomorphism (the second and the fourth rows).

5. Ideas For Future Research
In the new algorithm process, the steps of mapping topological volume cylinders to canonical volume cylinders have not been fully explained. The current algorithm also cannot handle sharp creases. In the future, a combination of surface diffraction can be used to align a series of hexahedral edges with sharp creases. In addition, how to control the density of hexahedral cells by applying the best quality transportation map will be a new application for hexahedral automatic generation.
6. Conclusion
A fully automatic hexahedral mesh generation algorithm for any arbitrary geometry is still way off. In the current situation, the hexahedral mesh generation algorithms mainly focus on the topology and conformal structure of the manifold, while ignoring the geometric information. In practical applications, meshing needs to fully consider the geometric characteristics of the surface, especially the direction of the main curvature of the surface, the characteristic curve of the surface, the sharp angle curve and so on. Many times, we need to add more singular points or singular lines to make meshing better adapt to geometric features. So where to add singular lines can minimize the impact of the calculation results should be given enough attention.

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8. Reference
[1] Shepherd J F, Johnson C R 2008 Hexahedral Mesh Generation Constraints J. Engineering with Computers 24 195
[2] Ted Blacker 2000 Meeting the Challenge for Automated Conformal Hexahedral Meshing J. International Meshing Roundtable 49(5) 11
[3] Si H 2015 TetGen a Delaunay-Based Quality Tetrahedral Mesh Generator J. ACM transactions on mathematical software 41(2) 11
[4] Jeff Erickson 2014 Efficiently Hex-meshing Things with Topology J. Discrete & Computational Geometry 52(3) 427
[5] Cook W A and Oakes W R 1982 Mapped Methods for Generating Three-Dimensional Meshes J. Cine Computers in Mechanical Engineering 1(1) 67
[6] Kadivar M H and Sharifi H 1996 Double Mapping of Isoparametric Mesh Generation J. Computers and Structures 59(3) 471
[7] Spekreijse S P 1995 Elliptic Grid Generation Based On Laplace Equations And Algebraic Transformations J. Journal of Computational Physics 118(1) 38
[8] Petersen S B, Rodrigues J M C, Martins P A F 2000 Automatic Generation of Quadrilateral Meshes for the Finite Element Analysis of Metal Forming Processes J. Finite Elements in Analysis and Design 35(2) 157
[9] Staten M L, Canann S A, Owen S J 1999 BMSweep: Locating Interior Nodes During Sweeping J. Engineering with Computers 15(3) 212
[10] Rypal D 2010 Sweeping of Unstructured Meshes over Generalized Extruded Volumes J. Finite Elements in Analysis & Design 46(1-2) 203
[11] Scott M A, Benzley S E, Owen S J 2006 Improved many-to-one sweeping J. International Journal for Numerical Methods in Engineering 65(3) 332
[12] White D R, Saigal S, Owen S J 2004 CCSweep: Automatic Decomposition of Multi-sweep Volumes J. Engineering with Computers 20(3) 222
[13] T J Tautges, T Blacker, S A.Mitchell. 1996 The Whisker Weaving Algorithm: A Connectivity-Based Method For Constructing All-Hexahedral Finite Element Meshes J. International Journal for Numerical Methods in Engineering 39(19) 3327
[14] Folwell N T, Mitchell S A 1999 Reliable Whisker Weaving via Curve Contraction J. Engineering with Computers 15(3) 292
[15] Ledoux, F and Weill J C 2008 Proc. of the 16th Int. Meshing Roundtable (Seattle, Washington, U.S.A.) p 215
[16] Tautges T J and Mitchel S A 2005 Proc. of the 4th Int. Meshing Roundtable vol 39 (San Jose, CA, USA) p 399
[17] Murdoch P, Benzley S, Blacker T and Mitchell S A 1997 The Spatial Twist Continuum: A Connectivity Based Method for Representing All-hexahedral Finite Element Meshes J. Finite Elements in Analysis & Design 28(2) 137
[18] Schneiders R 1996 A Grid-based Algorithm for the Generation of Hexahedral Element Meshes J. Engineering with Computers 12(3-4) 168
[19] Schneiders R, R Schindler and F Weiler 1996 In Proc. of the 5th Int. Meshing Roundtable (Pittsburgh, Pennsylvania) p 205

[20] Lu Y, Gadh R 2001 Feature Based Hex Meshing Methodology: Feature Recognition and Volume Decomposition J. CAD Computer Aided Design 33(3) 221

[21] Tam T K H and Armstrong C G 1993 Finite Element Mesh Control By Integer Programming J. International Journal for Numerical Methods in Engineering 36(15) 2581

[22] Lei N, Zheng X, Jiang J, Lin Y and Gu D 2017 Quadrilateral and Hexahedral Mesh Generation Based on Surface Foliation Theory J. Computer Methods in Applied Mechanics and Engineering 316(APR.1) 758

[23] Lei N, Zheng X, Luo Z and Gu D 2017 Quadrilateral and Hexahedral Mesh Generation Based on Surface Foliation Theory ii J. Computer Methods in Applied Mechanics & Engineering 321(316) 406