On Square Metrics of Scalar Flag Curvature

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Abstract

We consider a special class of Finsler metrics — square metrics which are defined by a Riemannian metric and a 1-form on a manifold. We show that an analogue of the Beltrami Theorem in Riemannian geometry is still true for square metrics in dimension \( n \geq 3 \), namely, an \( n(\geq 3) \)-dimensional square metric is locally projectively flat if and only if it is of scalar flag curvature. Further, we determine the local structure of such metrics and classify closed manifolds with a square metric of scalar flag curvature in dimension \( n \geq 3 \).

Keywords: Square Metric, Scalar Flag Curvature, Projectively Flat, Closed Manifold

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1 Introduction

It is the Hilbert’s Fourth Problem to study and characterize locally projectively flat metrics. The Beltrami Theorem in Riemannian geometry states that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. Thus the Hilbert’s Fourth problem is completely solved for Riemannian metrics. For Finsler metrics, the flag curvature is a natural extension of the sectional curvature. It is known that every locally projectively flat Finsler metric is of scalar flag curvature. However, the converse is not true. There are Finsler metrics of constant flag curvature which are not locally projectively flat (3). Therefore, it is a natural problem to study Finsler metrics of scalar flag curvature. This problem is far from being solved for general Finsler metrics. Thus we shall investigate Finsler metrics in a simple form, such as Randers metrics, square metrics or other \((\alpha, \beta)\)-metrics.

Randers metrics are among the simplest Finsler metrics in the following form

\[ F = \alpha + \beta, \]

where \( \alpha \) is a Riemannian metric and \( \beta \) is a 1-form satisfying \( \|\beta\|_\alpha < 1 \). After many mathematician’s efforts (11 [12] [13] [14] [15] [20]), Bao-Robles-Shen finally classify Randers metrics of constant flag curvature by using the navigation method (3). Further, Shen-Yildirim classify Randers metrics of weakly isotropic flag curvature [15]. There are Randers metrics of scalar flag curvature which are not of weakly isotropic flag curvature or not locally projectively flat (8 [16]). Besides, some relevant researches are refereed to [6] [17] [21], under additional conditions. So far, Randers metrics of scalar flag curvature remains mysterious.

Recently, a special class of Finsler metrics, the so-called square metrics, have been shown to have many special geometric properties. A square metric on a manifold \( M \) is defined in the following form

\[ F = \frac{(\alpha + \beta)^2}{\alpha}, \]

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where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form with $\|\beta\|_\alpha < 1$. L. Berwald first constructed a special projectively flat square metric of zero flag curvature on the unit ball in $\mathbb{R}^n$ ([4]). In [19], Shen-Yildirim determine the local structure of all locally projectively flat square metrics of constant flag curvature. Later on, L. Zhou shows that a square metric of constant flag curvature must be locally projectively flat ([24]).

In this paper, we study square metrics of scalar flag curvature and determine the local and global structures of square metrics of scalar flag curvature in dimension $n \geq 3$. More precisely, we have the following results.

**Theorem 1.1** Let $F = (\alpha + \beta)^2/\alpha$ be a square metric on an $n(\geq 3)$-dimensional manifold $M$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is Riemannian and $\beta = b_i(x)y^i$ is a 1-form. Then $F$ is of scalar flag curvature if and only if $F$ is locally projectively flat.

The scalar flag curvature $K$ in Theorem 1.1 can be determined (see (16) and (53) below). To prove Theorem 1.1 we first characterize square metrics of scalar flag curvature in terms of the covariant derivatives $b_{ij}$ and the Riemann curvature $\bar{R}^k_i$ of $\alpha$ (Theorem 3.1 below). Then Theorem 1.1 follows directly from Theorem 3.1. In Section 4, based on Theorem 3.1, we use a special deformation on $\alpha$ and $\beta$ to obtain the local structure of square metrics of scalar flag curvature in dimension $n \geq 3$ (see Theorem 4.1 below) (cf. [22] [23] introducing some deformations for projectively flat $(\alpha, \beta)$-metrics). In Section 5, we use Theorem 3.1 and Theorem 4.1 below to give the local structure of a square metric which is of constant flag curvature (see Corollary 5.1 below) (cf. [19] [24]).

More important is that, based on Theorem 3.1, we can use the deformation determined by (54) to obtain some rigidity results (see Theorem 1.2 below). We will prove in Section 4 that, under the deformation determined by (54), if $\alpha$ and $\beta$ satisfy (12)–(15) below, then $h = h_\mu$ is a Riemann metric of constant sectional curvature (put as $h$) and $\omega$ is a closed 1-form which is conformal with respect to $h_\mu$. Put $h_\mu = \sqrt{h_{ij}y^i y^j}$, and the covariant derivatives $w_{ij}$ of $\omega = w_i y^i$ with respect to $h_\mu$ satisfy

$$w_{ij} = -2c h_{ij}$$

for some scalar function $c = c(x)$ (see (57) below). Let $\nabla c$ be the gradient of $c$ with respect to $h_\mu$, and then

$$\delta := \sqrt{\|\nabla c\|_{h_\mu}^2 + \mu c^2}, \quad (\mu > 0),$$

is a constant (Lemma 6.1 below). We need $c$ and $\delta$ in the following theorem.

**Theorem 1.2** Let $F = (\alpha + \beta)^2/\alpha$ be a square metric on an $n$-dimensional compact manifold $M$ without boundary.

(i) Suppose $n \geq 2$ and $F$ is of constant flag curvature. Then $F = \alpha$ is Riemannian, or $F$ is locally Minkowskian. In the latter case, $F$ is flat-parallel ($\alpha$ is flat and $\beta$ is parallel with respect to $\alpha$).

(ii) Suppose $n \geq 3$ and $F$ is of scalar flag curvature. Then one of the following cases holds:

(iia) If $\mu < 0$ in (1), then $F = \alpha$ (= $h_\mu$) is Riemannian.

(iib) If $\mu = 0$ in (1), then $F$ is flat-parallel.
If \( \mu > 0 \) in (1), then \( \alpha \) and \( \beta \) can be written as
\[
\alpha = 4 \mu^{-1} (\rho^2 - c^2) h_{\mu}, \quad \beta = 4 \mu^{-\frac{3}{2}} \sqrt{\rho^2 - c^2} c_0,
\]
where \( c \) and \( \delta \) are given by (1) and (2), and
\[
c_i := c_{x^i}, \quad c_0 := c_i y^i, \quad \rho^2 := \delta^2 \mu^{-1} + \mu/4.
\]
Further, the scalar flag curvature \( K \) satisfies
\[
K = \frac{\rho^2 \mu^3}{16} \left( 1 + \frac{\beta}{\alpha} (\rho^2 - c^2) \right)^{-3},
\]
\[
\left( \frac{4 \delta^2 + \mu^2 - 2 \delta}{\mu \sqrt{4 \delta^2 + \mu^2}} \right)^3 \leq K \leq \left( \frac{4 \delta^2 + \mu^2 + 2 \delta}{\mu \sqrt{4 \delta^2 + \mu^2}} \right)^3.
\]

The ideas shown in the proofs for Theorem 4.1 below and Theorem 1.2 above can be applied to solve similar problems. For example, we can apply such an idea to give another direct proof to the local and global classifications of Randers metrics which are locally projectively flat with isotropic S-curvature (cf. [5]).

### 2 Preliminaries

In local coordinates, the geodesics of a Finsler metric \( F = F(x, y) \) are characterized by
\[
\frac{d^2 x^i}{dt^2} + 2 G^i(x, \frac{dx^i}{dt}) = 0,
\]
where
\[
G^i := \frac{1}{4} g^{ij} \left\{ [F^2]_{x^j y^k} y^k - [F^2]_{x^i} \right\}.
\]

For a Finsler metric \( F \), the Riemann curvature \( R_y = R^i_{k}(y) \frac{\partial}{\partial y^i} \otimes dx^k \) is defined by
\[
R^i_{k} := 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2 G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^j}{\partial y^j} \frac{\partial G^i}{\partial y^k}.
\]

The Ricci curvature is the trace of the Riemann curvature, \( \text{Ric} := R^m_{m} \). A Finsler metric is called of scalar flag curvature if there is a function \( K = K(x, y) \) such that
\[
R^i_{k} = K F^2 (\delta^i_{k} - y^i y^k), \quad y_k := (1/2 F^2 y^k y^j).
\]

A Finsler metric \( F \) is said to be projectively flat in \( U \), if there is a local coordinate system \((U, x^i)\) such that \( G^i = P y^i \), where \( P = P(x, y) \) is called the projective factor.

In projective geometry, the Weyl curvature and the Douglas curvature play a very important role. Put
\[
A^i_{k} := R^i_{k} - R \delta^i_{k}, \quad R := \frac{R^m_{m}}{n-1}.
\]

Then the Weyl curvature \( W^i_{k} \) are defined by
\[
W^i_{k} := A^i_{k} - \frac{1}{n+1} \frac{\partial A^m_{k}}{\partial y^m} y^i.
\]
The Douglas curvature $D_{h\,jk}^i$ are defined by

$$D_{h\,jk}^i := \frac{\partial^3}{\partial y^k \partial y^j \partial y^i} (G^i - \frac{1}{n+1}G^m_i y^j), \quad G^m_i := \frac{\partial G^m}{\partial y^m}. \tag{9}$$

The Weyl curvature and the Douglas curvature both are projectively invariants. A Finsler metric is called a Douglas metric if $D_{h\,jk}^i = 0$. A Finsler metric is of scalar flag curvature if and only if $W^i_k = 0$. It is known that a Finsler metric in dimension $n \geq 3$ is locally projectively flat if and only if $W^i_k = 0$ and $D_{h\,jk}^i = 0$ \(\square\).

In literature, an $(\alpha, \beta)$-metric $F$ is defined as follows

$$F = \alpha \phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\phi(s)$ is some suitable function, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. If we take $\phi(s) = 1 + s$, then we get the well-known Randers metric $F = \alpha + \beta$. In this paper, we will study a class of special $(\alpha, \beta)$-metrics—square metrics, which are defined by taking $\phi(s) = (1 + s)^2$.

To compute the geometric quantities of square metrics, we first give some notations and conventions. For a Riemannian $\alpha = \sqrt{a_{ij} y^i y^j}$ and a 1-form $\beta = b_i y^i$, let

$$r_{ij} := \frac{1}{2}(b_{ij} + b_{ji}), \quad s_{ij} := \frac{1}{2}(b_{ij} - b_{ji}), \quad r^i_j := a^{ik}r_{kj}, \quad s^i_j := a^{ik}s_{kj}, \tag{10}$$

$$q_{ij} := r_{im}s^m_j, \quad t_{ij} := s_{im}s^m_j, \quad r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}, \quad q_j := b^i q_{ij}, \quad r_j := b^i r_{ij}, \quad t_j := b^i t_{ij},$$

where we define $b^i := a^{ij}b_j$, $(a^{ij})$ is the inverse of $(a_{ij})$, and $\nabla \beta = b_{ij}y^i dx^j$ denotes the covariant derivatives of $\beta$ with respect to $\alpha$. Here are some of our conventions in the whole paper. For a general tensor $T_{ij}$ as an example, we define $T_{i0} := T_{ij}y^j$ and $T_{00} := T_{ij}y^i y^j$, etc. We use $a_{ij}$ to raise or lower the indices of a tensor.

Let $F = (\alpha + \beta)^2 / \alpha$ be a square metric, and then by (7) we have

$$G^i = G^i_\alpha + \frac{2}{1-s} \alpha s_0 \frac{1}{1+2b^2-3s^2} (r_{00} - 2\alpha Q s_0), \tag{11}$$

Now by (9) and (10), we can get the expressions of the Weyl curvature tensor $W^i_k$ for an $n$-dimensional square metric $F = (\alpha + \beta)^2 / \alpha$. Assume $F$ is of scalar flag curvature, and then multiplying $W^i_k = 0$ by

$$(n^2 - 1)(1-s)^4(1+2b^2-3s^2)^5 \alpha^4,$$

we have

$$f_0(s) + f_1(s) \alpha + \cdots + f_6(s) \alpha^6 = 0, \tag{12}$$

where $f_i(s)$’s are polynomials of $s$ with coefficients being homogenous polynomials in $(y^i)$.

Starting from (11), we are mainly concerned about the computation of the following terms

$$r_{ij}, \quad r_{ij\,k}, \quad q_{ij}, \quad t_{ij}, \quad s_{ij}.$$

We show some ideas in dealing with the equation (11). The key idea is to choose some suitable polynomials in $s$ to divide our equations, and then get some answers by isolating...
rational and irrational terms. In the proof of Theorem 3.1 below, our polynomials in $s$ singled out are

\[ 1 + 2b^2 - 3s^2, \quad 1 - s. \]

Note that the meaning of the divisibility of an equation by a polynomial in $s$ should be understood in the way as show in the following proof.

## 3 Scalar Flag Curvature

In this section, we study square metrics of scalar flag curvature. We have the following theorem.

**Theorem 3.1** Let $F = (\alpha + \beta)^2/\alpha$ be a square metric on an $n(\geq 3)$-dimensional manifold $M$, where $\alpha = \sqrt{\sigma_{ij}y^iy^j}$ and $\beta = b_iy^i$. Then $F$ is of scalar flag curvature if and only if the Riemann curvature $\bar{R}_{\alpha}$ of $\alpha$ and the covariant derivatives $b_i|_j$ of $\beta$ with respect to $\alpha$ satisfy the following equations

\[
\begin{align*}
  b_i|_j &= \tau\left\{ (1 + 2b^2)a_{ij} - 3b_ib_j \right\}, \\
  \bar{R}_{\alpha} &= \lambda(\alpha^2\delta_k^i - y^iy_k) + 2\eta(\beta^2\delta_k^i + \alpha^2b_i^k - \beta b_k^i), \\
  \tau_{x^i} &= ub_i,
\end{align*}
\]

where $\tau = \tau(x), \lambda = \lambda(x)$ are scalar functions on $M$ and $\eta, u$ are given by

\[
\begin{align*}
  \eta &:= \lambda + 4(2 + b^2)\tau^2, \quad u := -(7 + 4b^2)\tau^2 - \lambda. 
\end{align*}
\]

In this case, $F$ is locally projectively flat, and the scalar flag curvature $K$ is given by

\[
K = \frac{\alpha}{F^2}\left\{ [\lambda + \tau(5 + 4b^2)]\alpha + (\eta - 3\tau^2)\beta \right\}. 
\]

**Proof**: We will deal with the equation $W_{i}^i_k = 0$ step by step. By the method described in the above section, we get a formula for $W_{i}^i_k$, which is expressed in terms of the covariant derivatives of $\beta$ with respect to $\alpha$ and the Riemann curvature of $\alpha$.

**Lemma 3.2** For a scalar function $c = c(x)$, the following holds for some $k$,

\[ \alpha b_k - s y_k \not\equiv 0 \mod (s + c). \]

**Proof**: Suppose $\alpha b_k - s y_k$ can be divided by $s + c$ for all $k$, and then we have

\[ \alpha(\alpha b_k - s y_k) = (f_k + g_k\alpha)\alpha(s + c), \]

where $f_k$ are 1-forms and $g_k = g_k(x)$. Thus we have

\[ (b_k - cg_k)\alpha^2 - (cf_k + \beta g_k)\alpha - \beta(y_k + f_k) = 0, \]

which imply

\[ cf_k + \beta g_k = 0, \quad b_k - cg_k = 0, \quad y_k + f_k = 0. \]

Now it is easy to get a contradiction from the above. Q.E.D.

Now in the following, we start our proof step by step from (11).
Firstly (11) can be written in the following form

\[ E_1 : = 648(n-2)(1+s)^2(1-s)^4s^3(ab_k - sy_k)y^i[(s-1)r_{00} + 4\alpha s_0]^2 \]
\[ + C_k^i(1+2b^2 - 3s^2) = 0, \] (17)

where \( C_k^i \) can be written in the form

\[ f_0(s) + f_1(s)\alpha + \cdots + f_m(s)\alpha^m = 0, \] (18)

for some integer \( m \). Now it follows from

\[ E_1 \equiv 0 \mod (1 + 2b^2 - 3s^2) \]

that

\[ (1+s)^2(1-s)^4s^3(ab_k - sy_k)y^i[(s-1)r_{00} + 4\alpha s_0]^2 \equiv 0 \mod (1 + 2b^2 - 3s^2). \] (19)

**Lemma 3.3** Suppose

\[ (s-1)r_{00} + 4\alpha s_0 \equiv 0 \mod (1 + 2b^2 - 3s^2). \] (20)

Then we have

\[ r_{00} = \tau \alpha^2(1+2b^2-3s^2), \quad s_0 = 0, \] (21)

where \( \tau = \tau(x) \) is a scalar function.

**Proof**: Eq. (20) implies that there are a 1-form \( f \) and a scalar \( \tau = \tau(x) \) satisfying

\[ \alpha[(s-1)r_{00} + 4\alpha s_0] = (f-\tau f)\alpha^2(1+2b^2-3s^2), \]

which can be written as

\[ \tau(1+2b^2)\alpha^3 - (f+2b^2f-4s_0)\alpha^2 - (3\tau\beta^2 + r_{00})\alpha + \beta(3f\beta + r_{00}) = 0. \]

Therefore we have

\[ \tau(1+2b^2)\alpha^3 - (f+2b^2f-4s_0)\alpha^2 - (3\tau\beta^2 + r_{00})\alpha = \beta(3f\beta + r_{00}) = 0, \]

in which we solve \( r_{00} \) from the first, and then plugging it into the second gives

\[ 4\alpha^2 s_0 + [(1+2b^2)\alpha^2 - 3\beta^2](\tau\beta - f) = 0. \]

Clearly we have \( s_0 = 0 \). Thus (21) holds. Q.E.D.

Now we have (20) by (19) and Lemma 3.2, and then (21) holds by Lemma 3.3. Then by (21) we can obtain the expressions of the following quantities:

\[ r_{00}, r_i, r_{m}^m, r, r_{00|0}, r_{0|0}, r_{00|k}, r_{k0|m}, r_{k|0}, s_{k|m}, t_k, q_{km}, q_k, b^m q_{0m}, etc. \]

For example we have

\[ r_{00|0} = (1+2b^2-3s^2)r_0\alpha^2 - 2s(1+8b^2-9s^2)\tau^2\alpha^3, \quad \tau_i := \tau_{m^i}, \]

and

\[ s_{k|m} = 0, \quad t_k = 0, \quad q_{km} = \tau(1+2b^2)s_{km}, \quad q_{00} = 0, \quad q_k = 0, \quad b^m q_{0m} = 0. \]
Plug all the above quantities into (17) and then multiplied by $1/(1 + 2b^2 - 3s^2)^5$ the equation (17) is written as

$$Eq_2 := D^i_k(1 - s) + 12(n + 1)\alpha^2(\alpha b_k - y_k)y^i t_{00} = 0,$$  

(22)

where $D^i_k$ can be written in the form of the left hand side of (18). It follows from

$$Eq_2 \equiv 0 \mod (1 - s)$$

that

$$(n + 1)(\alpha b_k - y_k)y^i t_{00} \equiv 0 \mod (1 - s).$$  

(23)

**Lemma 3.4** Suppose

$$t_{00} \equiv 0 \mod (1 - s).$$  

(24)

Then we have

$$t_{00} = \gamma(\alpha^2 - \beta^2),$$  

(25)

where $\gamma = \gamma(x)$ is a scalar function.

**Proof**: Assume (24) hold. We have

$$t_{00} = (f + \gamma \alpha)\alpha(1 - s),$$

where $f$ is a 1-form and $\gamma = \gamma(x)$ is a scalar. Then we can easily get (25). Q.E.D.

Now by (23) we have (24). So we get (25) by Lemma 3.4. Then we have

$$t_{00} = \gamma(y_i - \beta b_i), \quad t^m_m = \gamma(n - b^2).$$  

(26)

Plug (26) and (26) into (22) and then multiplied by $1/(1 - s)$ the equation (22) is written as

$$Eq_3 = 0.$$  

(27)

It follows from

$$Eq_3 \equiv 0 \mod (1 - s)$$

that

$$3(n - 1)s_{i0}s_k0 + \gamma(\alpha b_k - y_k)[(n - 1)\alpha b_i - (3 + b^2)y_i] \equiv 0 \mod (1 - s).$$  

(28)

**Lemma 3.5** Suppose (26) holds for some scalar $\gamma = \gamma(x)$. Then $\beta$ is closed.

**Proof**: Equation (28) implies

$$3(n - 1)s_{i0}s_k0 + \gamma(\alpha b_k - y_k)[(n - 1)\alpha b_i - (3 + b^2)y_i] = (f_{ik} + \sigma_{ik}\alpha)\alpha(1 - s),$$  

(29)

where $f_{ik}$ are 1-forms and $\sigma_{ik} = \sigma_{ik}(x)$ are scalar functions. Eq. (29) can be rewritten as

$$[(n - 1)\gamma b_i b_k - \sigma_{ik}]\alpha^2 - [(\gamma(3 + b^2)y_i b_k + (n - 1)\gamma b_k y_k + f_{ik} - \beta \sigma_{ik})\alpha

+ (3 + b^2)\gamma y_i y_k + 3(n - 1)s_{i0}s_k0 + f_{ik}\beta = 0.$$  

(30)

It shows that (30) is equivalent to

$$\gamma(3 + b^2)y_i b_k + (n - 1)\gamma b_k y_k + f_{ik} - \beta \sigma_{ik} = 0,$$  

(31)
\[
[(n - 1)\gamma b_k b_k - \sigma_{ik}] \alpha^2 + (3 + b^2)\gamma y_i y_k + 3(n - 1)s_i s_k + f_{ik}\beta = 0.
\] (32)

Solve \( f_{ik} \) from (31) and plug them into (32), and then we obtain
\[
(\beta^2 - \alpha^2)\sigma_{ik} - \gamma \beta [(n - 1)b_k y_k + (3 + b^2)b_k y_k] + 3(n - 1)s_i s_k = 0.
\]
(33)

Exchanging the indices \( i \) and \( k \) in (33) gives
\[
(\beta^2 - \alpha^2)\sigma_{ki} - \gamma \beta [(n - 1)b_k y_i + (3 + b^2)b_k y_i] + 3(n - 1)s_i s_k = 0.
\]
(34)

From (33) and (34) we get
\[
(n - 4 - b^2)\gamma \beta (b_k y_i - b_i y_k) - (\alpha^2 - \beta^2)(\sigma_{ik} - \sigma_{ki}) = 0.
\]
(35)

Since \( 0 < b^2 < 1 \), we get \( n - 4 - b^2 \neq 0 \). Thus by (35) we easily get \( \gamma = 0 \). Plugging \( \gamma = 0 \) into (34) we have
\[
(\beta^2 - \alpha^2)\sigma_{ki} + 3(n - 1)s_i s_k = 0.
\]
(36)

Now we see clearly that \( s_i s_k = 0 \) from (36), that is, \( \beta \) is closed. Q.E.D.

Now by Lemma 3.5, \( \beta \) is closed. Then we see that (12) holds by (21) and the fact that \( \beta \) is closed.

Further, since \( \beta \) is closed by Lemma 3.5, the terms \( s_{ij}, s_{ijkl}, \tau_{ij}, \) etc. all vanish. By this fact, we see (27) has become very simple, and we can get the Weyl curvature \( \tilde{W}_{ik} := a_{im}\bar{W}^m_k \) of \( \alpha \) given as follows
\[
\bar{W}_{ik} = \frac{2}{n - 1} h^m \omega_m (\alpha^2 a_{ik} - y_i y_k) - \frac{2}{n - 1}\beta \omega_0 a_{ik}
\]
\[
+ \frac{2}{n^2 - 1} [(2n - 1)\beta \omega_k - (n - 2)\omega_0 b_k] y_i + 2\omega_0 b_i y_k - 2\alpha^2 b_i \omega_k,
\]
(37)

where \( \tau_i := \tau_x \), and
\[
\omega_i := \tau_i - \tau^2 b_i.
\]

Lemma 3.6 (37) \( \iff \) (13) and (14).

Proof: \( \implies \): By (37) we have
\[
\bar{W}_{ik} - \bar{W}_{ki} = \frac{2}{n^2 - 1} [(2n - 1)\beta \omega_k - (n^2 + 3)\omega_0 b_k] y_i - \frac{2}{n^2 - 1} [(2n - 1)\beta \omega_i - (n^2 + 3)\omega_0 b_i] y_k - 2(\omega_0 b_k - \omega_0 b_i)\alpha^2.
\]
(38)

On the other hand, by the definition of the Weyl curvature \( \bar{W}_{ik} \) of \( \alpha \) we have
\[
\bar{W}_{ik} = \bar{R}_{ik} - \frac{1}{n - 1} \bar{R} c_{0i} a_{ik} + \frac{1}{n - 1} \bar{R} c_{ki} y_i,
\]
(39)

where \( \bar{R}_{ik} := a_{im}\bar{R}^m_k \) and \( \bar{R} c_{i} \) denote the Ricci tensor of \( \alpha \). Using the fact \( \bar{R}_{ik} = \bar{R}_{ki} \) we get from (39)
\[
\bar{W}_{ik} - \bar{W}_{ki} = \frac{1}{n - 1}(\bar{R} c_{ki} y_i - \bar{R} c_{ki} y_k).
\]
(40)
Then by (38) and (40) we obtain
\[ T_i y_k - T_k y_i + 2(n^2 - 1)(\omega_i b_k - \omega_k b_i)\alpha^2 = 0, \] (41)
where we define
\[ T_i := (n + 1)\bar{\text{Ric}}_{i0} - 2(2n - 1)\beta\omega_i + 2(n^2 + n - 3)\omega_0 b_i. \]
Contracting (41) by \( y^k \) we get
\[ [T_i + 2(n^2 - 1)(\omega_i \beta - \omega_0 b_i)]\alpha^2 - T_0 y_i = 0. \] (42)
So there is some scalar function \( \bar{\eta} = \bar{\eta}(x) \) such that
\[ T_0 = -(n + 1)\bar{\eta}\alpha^2. \] (43)
Then by the definition of \( T_i \) and (43) we have
\[ \bar{\text{Ric}}_{00} = -\bar{\eta}\alpha^2 - 2(n - 2)\beta\omega_0, \quad \bar{\text{Ric}}_{i0} = -\bar{\eta} y_i - (n - 2)(\beta\omega_i + b_i\omega_0). \] (44)
Plugging (44) into (42) we get
\[ (n - 1)(n - 2)\alpha^2(\beta\omega_i - b_i\omega_0) = 0, \]
which imply that there is some scalar function \( u = u(x) \) satisfying
\[ \omega_i = (u - \tau^2)b_i. \] (45)
Now plugging (44) and (45) into (37) and (39) we obtain
\[ \bar{R}_{ik} = \lambda(\alpha^2 a_{ik} - y^i y_k) + 2\eta(\beta^2 a_{ik} + \alpha^2 b_k b_k - \beta b_i y_k - \beta b_k y_i), \] (46)
where we define
\[ \lambda := \frac{2(u - \tau^2)b^2 - \bar{\eta}}{n - 1}, \quad \eta := \tau^2 - u. \] (47)
Clearly, (46) is just (13). We get (14) for some \( u = u(x) \) by (45) and the definition of \( \omega_i \). It follows from (47) that \( u = \tau^2 - \eta \). In the following, we will further determine \( \eta \) and \( u \) given by (15).

\[ \iff \quad \text{We verify that both sides of (37) are equal. By (13) we have (14). Since (39) naturally holds, we plug (44) and (45) into (39) and then we obtain the left side of (37). By (14) we get (45). Then plugging (45) into the right side of (37) we obtain the result equal to the left side of (37). Q.E.D.} \]
where the expression of $K = K(x,y)$ is omitted. Since $F$ is of scalar flag curvature and $n \geq 3$, by (48) we must have

$$\eta - \lambda - 4(2 + b^2)\tau^2 = 0.$$  (49)

Thus we get $\eta$ given by (15). Plug $\eta$ given by (15) into $K$, and then we obtain the scalar flag curvature $K = K$ given by (16). By $u = \tau^2 - \eta$ and $\eta$ in (15), we get $u$ given by (15).

So far we have completed the proof of Theorem 3.1. Q.E.D.

4 A deformation and local structures

In this section, we give the local structure of a square metric of scalar flag curvature based on Theorem 3.1. In [22] [23], C. Yu introduced metric deformations for projectively flat $(\alpha, \beta)$-metrics $F = \alpha \phi(\beta/\alpha)$. In particular, he determines the local structure of locally projectively flat square metrics for the dimension $n \geq 3$ in a different way.

**Theorem 4.1** Let $F = (\alpha + \beta)^2/\alpha$ be a square metric on an $n(\geq 3)$-dimensional manifold $M$. Suppose $F$ is of scalar flag curvature. Then we can express $\alpha$ and $\beta$ as

$$\alpha = \frac{h_\mu}{1 - b^2}, \quad \beta = \frac{\omega}{\sqrt{1 - b^2}},$$  (50)

where $\omega$ is a closed 1-form which is conformal with respect to $h_\mu$. If $h_\mu$ takes the local form (62) below, then $\alpha$ and $\beta$ can be locally expressed as

$$\alpha = \frac{\sigma^2}{1 + \mu|x|^2} h_\mu, \quad \beta = \frac{\sigma}{1 + \mu|x|^2} \left[ (a, y) + \frac{k - \mu(a, x)}{1 + \mu|x|^2} (x, y) \right].$$  (51)

where $\sigma = \sigma(x)$ is defined as

$$\sigma := \sqrt{k^2 + (1 + |a|^2)\mu} |x|^2 + (2k - \mu(a, x))(a, x) + |a|^2 + 1,$$  (52)

and $k$ is a constant and $a = (a^i) \in \mathbb{R}^n$ is a constant vector. In this case, $K$ in (16) can be rewritten as

$$K = \frac{(k^2 + \mu + \mu|a|^2)(1 + \mu|x|^2)^3}{\sigma^6} \frac{\alpha^3}{(\alpha + \beta)^3}.$$  (53)

**Proof:** Here we will give a direct proof using (12), (13) and (14). Let

$$h := (1 - b^2)\alpha, \quad \omega := \sqrt{1 - b^2}\beta,$$  (54)

where $b := ||\beta||_\alpha$. For the metric deformation in (54), we will prove that $h$ is of constant sectional curvature and $\omega$ is a closed 1-form which is conformal with respect to $h$.

Put

$$w := ||\omega||_h.$$

By (54) we have

$$w^2 = \frac{b^2}{1 - b^2}.$$  (55)

By (12), a direct computation gives

$$G^i_k = G^i_\alpha - 2\tau \beta y^i + \tau \alpha^2 b^i.$$  (56)
Then by (12) and (56) we get

\[ w_{ij} = \frac{\tau}{(1 - b^2)^{\frac{3}{2}}} h_{ij} \]  \hspace{1cm} (57)

where the covariant derivatives are taken with respect to \( h \), and the scalar function \( c = c(x) \) can be determined (see (84) below). Now (57) implies that \( \omega \) is a closed 1-form which is conformal with respect to \( h \).

By (56) and (12), a direct computation shows

\[ \tilde{R}^i_k = \frac{\lambda + 4(1 + b^2)\tau^2}{(1 - b^2)^2} (\alpha^2 \delta^i_k - y^i \bar{y}_k) \]  \hspace{1cm} (58)

where \( \tilde{R}^i_k \) and \( \tilde{R}^i_k \) are the Riemann curvatures of \( h \) and \( \alpha \) respectively, and \( \bar{y}_k := a_{kn}y^n \). Then plugging (14) into (58) we obtain

\[ \tilde{R}^i_k = \tilde{R}^i_k + 4\tau^2(1 + b^2)(\alpha^2 \delta^i_k - y^i \bar{y}_k) - 2\eta(\beta^2 \delta^i_k + \alpha^2 b^i \bar{b}_k - \beta b^i \bar{y}_k - \beta b^i y^i). \]  \hspace{1cm} (59)

Then by (13) and (59) we get

\[ \tilde{R}^i_k = \text{a constant}. \]  \hspace{1cm} (60)

where \( \bar{y}_k := h_{km}y^m \). It follows from (60) that \( h \) is of constant sectional curvature. We put it as \( \mu \), and then we obtain

\[ \lambda = \mu(1 - b^2)^2 - 4(1 + b^2)\tau^2. \]  \hspace{1cm} (61)

So far we have proved that \( h \) is of constant sectional curvature and \( \omega \) is a closed conformal 1-form under the change (54). Thus in some local coordinate system we may put \( h = h_\mu \) as follows

\[ h_\mu = \sqrt{(1 + \mu |x|^2)|y|^2 - \mu \langle x, y \rangle^2 1 + \mu |x|^2.} \]  \hspace{1cm} (62)

Meanwhile, by (57) we obtain the 1-form \( \omega = w_i y^i \) given by

\[ w_i = \frac{(k - \mu \langle a, x \rangle)x^i + (1 + \mu |x|^2)a^i}{(1 + \mu |x|^2)^\frac{3}{2}}, \hspace{1cm} w^i = \sqrt{1 + \mu |x|^2}(kx^i + a^i). \]  \hspace{1cm} (63)

where \( k \) is a constant and \( a = (a^i) \) is a constant vector, and \( w_i = h_{im}w^m \). In this case, the function \( \tau \) in (57) is given by

\[ \tau = \frac{(1 + \mu |x|^2)(k - \mu \langle a, x \rangle)}{\sigma^6}, \]  \hspace{1cm} (64)

where \( \sigma \) is defined by (52). By (63) we have

\[ w^2 = ||\omega||^2_h = |a|^2 + \frac{k^2|x|^2 + 2k\langle a, x \rangle - \mu \langle a, x \rangle^2}{1 + \mu |x|^2}. \]  \hspace{1cm} (65)

By (64) and (65) we get

\[ \alpha = \frac{h}{1 - b^2} = (1 + w^2)h, \hspace{1cm} \beta = \frac{\omega}{\sqrt{1 - b^2}} = \sqrt{1 + w^2} \omega. \]  \hspace{1cm} (66)
Then by (62), (63), (65) and (66) we get (51) for $\alpha$ and $\beta$.

Finally, we show the expression of $K$ given in (53). Differentiate $\tau$ in (64) by $x^i$ and then we can get the function $u$ in (14) given by

$$u = \frac{(1 + \mu |x|^2)^2}{-\sigma^6} \left\{ \left[ (1 + |a|^2)\mu^2 + k^2 \mu \right] |x|^2 + 2\mu^2 (a, x)^2 + (1 + |a|^2 - 4k(a, x))\mu + 3k^2 \right\}. \quad (67)$$

Now by (14), (61), (64), (65) and (67), we can rewrite (16) in the form (53). Q.E.D.

### 5 Constant flag curvature

**Corollary 5.1** Let $F = (\alpha + \beta)^2/\alpha$ be a non-Riemannian square metric on an $n(\geq 2)$-dimensional manifold $M$. Then $F$ is of constant flag curvature if and only if (12)–(15) hold with

$$\lambda = -(5 + 4b^2)r^2. \quad (68)$$

In this case, the constant flag curvature $K = 0$, and further, either $\alpha$ is flat and $\beta$ is parallel with respect to $\alpha$, or up to a scaling on $F$, $\alpha$ and $\beta$ can be locally expressed in the following forms

$$\alpha = (1 + \langle a, x \rangle)^2 \frac{\sqrt{(1 - |x|^2)|y|^2 + (x, y)^2}}{1 - |x|^2}, \quad (69)$$

$$\beta = \pm \frac{(1 + \langle a, x \rangle)^2}{1 - |x|^2} \left\{ \frac{\langle a, y \rangle}{1 + \langle a, x \rangle} + \frac{(x, y)}{1 - |x|^2} \right\}, \quad (70)$$

where $a = (a^i) \in \mathbb{R}^n$ is a constant vector.

Based on Theorem 3.1 we can only prove Corollary 5.1 for the case $n \geq 3$, and the case $n \geq 2$ has been verified in [19] [24]. In a different way, L. Zhou gives the characterization of square metrics with constant flag curvature ([24]), which is just the former part of Corollary 5.1. The latter part for the local structure in Corollary 5.1 has been solved in another way by Shen-Yildirim ([19]).

**Proof of Corollary 5.1**

We use Theorem 3.1 to prove the first part of Corollary 5.1. In this case we only require that the scalar flag curvature given by (10) be a constant $K = K$. Then by (16) we have

$$[\lambda + (5 + 4b^2)r^2 - K] \alpha^4 + (\eta - 3r^2 - 4K)\beta \alpha^3 - 6K \beta^2 \alpha^2 - 4K \beta^3 \alpha - K \beta^4 = 0. \quad (71)$$

It is easy to see that (71) is equivalent to

$$(\eta - 3r^2 - 4K)\alpha^2 - 4K \beta^2 = 0, \quad (72)$$

$$[\lambda + (5 + 4b^2)r^2 - K] \alpha^4 - 6K \beta^2 \alpha^2 - K \beta^4 = 0. \quad (73)$$

Now by (72) and (73) we easily get

$$K = 0, \quad \lambda = -(5 + 4b^2)r^2, \quad \eta = 3r^2.$$ 

Now the first part of Corollary 5.1 has been proved.
Next we use Theorem 4.1 to prove the second part of Corollary 5.1. We only need to simplify the conditions (68) and (61).

By (61) and the first formula in (68) we get

\[ \mu (1 - b^2)^2 - 4(1 + b^2) \tau^2 + (5 + 4b^2) \tau^2 = 0. \]  

(74)

Plug (55), (64) and (65) into (74) and then we get

\[ \frac{(1 + \mu |x|^2)^3(k^2 + \mu + \mu |a|^2)}{\sigma^6} = 0, \]

(75)

where \( \sigma \) is defined by (52). Therefore by (75) we get

\[ \mu = -\frac{k^2}{1 + |a|^2}. \]

(76)

If \( k = 0 \), then \( \mu = 0 \) by (76). In this case, we see from (61) and (52) that \( \alpha \) is flat and \( \beta \) is parallel. If \( k \neq 0 \), we plug (76) into (61) and then put

\[ k = \delta d, \quad a = \frac{\bar{a}}{d}, \quad 1 + |a|^2 = \delta^2, \]

and next put

\[ \delta = k, \quad d^2 = -\mu, \quad \bar{a} = a, \]

and finally we get

\[ \alpha = \frac{(k + \langle a, x \rangle)^2}{1 + \mu |x|^2} h_\mu, \quad \mu < 0, \]

(77)

\[ \beta = \pm \frac{1}{\sqrt{-\mu}} \frac{(k + \langle a, x \rangle)^2}{1 + \mu |x|^2} \left\{ \frac{\langle a, y \rangle}{k + \langle a, x \rangle} - \frac{\mu \langle x, y \rangle}{1 + \mu |x|^2} \right\}. \]

(78)

Thus by choosing another system \( \bar{x}^i = \sqrt{-\mu} x^i \) and a scaling on \( F \) we obtain (69) and (70).

Q.E.D.

6 Proof of Theorem 1.2

To prove Theorem 1.2 we need the following lemma.

Lemma 6.1 Let \( \alpha = \sqrt{a_{ij} y^i y^j} \) be an n-dimensional Riemannian metric of constant sectional curvature \( \mu \) and \( \beta = b_i y^i \) is a 1-form on \( M \). If \( \beta \) satisfies

\[ r_{ij} = -2c a_{ij}, \]

(79)

where \( c = c(x) \) is a scalar function on \( M \), then

\[ f := |\nabla c|^2_\alpha + \mu c^2 \]

is a constant in case of \( n \geq 3 \), where \( \nabla c \) is the gradient of \( c \) with respect to \( \alpha \). If \( n \geq 2 \) and \( M \) is compact without boundary, then \( c = 0 \) if \( \mu < 0 \) and \( c = \text{constant} \) if \( \mu = 0 \).
Proof: It has been proved in [17] that for \( n \geq 3 \), \( c \) in (79) satisfies
\[
c_{ij} + \mu c_{ij} = 0,
\]
where \( c_i := c_{x^i} \), and the covariant derivatives \( c_{ij} \) are taken with respect to \( \alpha \), and for \( n \geq 2 \), \( c \) in (79) satisfies
\[
\Delta c + n\mu c = 0,
\]
where \( \Delta \) is the Laplacian of \( \alpha \). Now by (81) we have
\[
(c^i c_{i} + \mu c^2)_{j} = 2(c^i c_{i} + \mu c_{i}) = 2(-\mu c_{i} + \mu c_{i}) = 0,
\]
where \( c^i := a^i c_j \). So \( f \) in (80) is a constant. Next by (80) we have
\[
\|\nabla c\|_\alpha^2 = \text{div}(\nabla c) - c\Delta c = \text{div}(\nabla c) + n\mu c^2.
\]
Therefore if \( M \) is compact without boundary, we get
\[
\int_M \|\nabla c\|_\alpha^2\, dV_\alpha = n\mu \int_M c^2\, dV_\alpha.
\]
(83)
Then it follows from (83) that \( c = 0 \) if \( \mu < 0 \) and \( c = \text{constant} \) if \( \mu = 0 \).

Now we begin to prove Theorem 1.2 as follows.

Case I: Assume \( F \) is of constant flag curvature.
We have two ways to prove Theorem 1.2(i). One way is to use Corollary 5.1. Since the square metric determined by (69) and (70) is incomplete (also see [19]), Theorem 1.2(i) naturally holds. The other way is to use the proofs of Theorem 4.1 and Corollary 5.1. Since for a square metric of constant flag curvature \((n \geq 2)\), we have (12), (13), (14) and (68).
Therefore, by the proofs of Theorem 4.1 and Corollary 5.1 we have (76). Thus \( \mu \leq 0 \), and so Theorem 1.2(ii)(iic) does not occur. Then Theorem 1.2(ii) implies Theorem 1.2(i).

Case II: Assume \( F \) is of scalar flag curvature.
Under the deformation (54), we have proved that \( h \) is of constant sectional curvature \( \mu \) and \( \omega \) satisfies (57). So we have (11) for some scalar function \( c = c(x) \). Then by (11) and Lemma 6.1 \( \delta \) defined by (2) is a constant, and \( c = 0 \) if \( \mu < 0 \) and \( c = \text{constant} \) if \( \mu = 0 \).

Let \( h \) take the local form (62). All the relevant symbols in the following are the same as that in Section 4. Now \( c \) in (11) can be expressed as
\[
c = \frac{-k + \mu(a, x)}{2\sqrt{1 + \mu|a|^2}}.
\]
(84)

Case IIA: Assume \( \mu < 0 \). We have \( c = 0 \). By (84), we have \( k = 0, a = 0 \). Then by (63) we get \( w_i = 0 \). Thus (66) shows \( \alpha = h, \beta = 0 \). Therefore, \( F \) is Riemannian in this case.

Case IIB: Assume \( \mu = 0 \). We have \( c = -k/2 = \text{constant} \). We will show that \( k = 0 \). Assume \( k \neq 0 \). Note that \( \tau \) in (57) is defined on the whole \( M \) and (64) gives a local representation of \( \tau \). By \( \mu = 0 \) and (64) we have
\[
\tau = \frac{k}{(k^2|x|^2 + 2k(a, x) + |a|^2 + 1)^3} = \frac{k}{(1 + |kx + a|^2)^3}.
\]
So we have \( \tau \neq 0 \) on the whole \( M \). Now define
\[
f := \left( \frac{T}{k} \right)^{-\frac{1}{3}}.
\]
Then \( f \) is also defined on the whole \( M \), and locally we have
\[
f = k^2|x|^2 + 2k\langle a, x \rangle + |a|^2 + 1.
\]
Since \( \mu = 0 \), we have
\[
f|_{x} = f_{x^i} = 2k^2 \delta_{ij} = 2k^2 h_{ij},
\]
where the covariant derivatives are taken with respect to \( h \). By the above we have
\[
\Delta f = 2k^2 n,
\]
where \( \Delta \) is the Laplacian of \( h \). Integrating the above on \( M \) yields
\[
0 = \int_{M} \Delta f \, dV_h = \int_{M} 2k^2 n \, dV_h = 2k^2 n \, Vol_h(M).
\]
Obviously it is a contradiction since \( k \neq 0 \). So we have \( k = 0 \). Thus by (63), (65) and (66) we easily conclude that \( \alpha \) is flat and \( \beta \) is parallel with respect to \( \alpha \).

**Case IIC:** Assume \( \mu > 0 \). By (84), we can rewrite (51) as
\[
\alpha = 4\mu^{-1}(\rho^2 - c^2) h, \quad \beta = 4\mu^{-\frac{3}{2}} \sqrt{\rho^2 - c^2} \, c_0,
\]
where \( \rho \) is defined by
\[
\rho^2 := k^2 + (1 + |a|^2) \mu.
\]
We will prove that \( \rho^2 \) is actually given by (4). By (55), (65) and (85) we get
\[
\|\nabla c\|^2_h = h_{ij} c_i c_j = \frac{\mu(4\rho^2 - 4c^2 - \mu)}{4}.
\]
Then by (2) and (86), we obtain \( \rho^2 \) given by (4). Now using (84), we obtain (5) by (53).

Finally, we prove the inequalities (6) satisfied by \( K \) by aid of (5). Firstly, we evaluate the term \( \beta/\alpha \). By (85) and then by (2) we get
\[
\frac{\beta}{\alpha} = \frac{1}{\sqrt{\mu} \sqrt{\rho^2 - c^2}} \, \frac{c_0}{h},
\]
\[
\leq \frac{\|\nabla c\|_h}{\sqrt{\mu} \sqrt{\rho^2 - c^2}} = \frac{\sqrt{\delta^2 \mu^{-1} - c^2}}{\sqrt{\delta^2 \mu^{-1} + \mu/4 - c^2}}
\]
\[
\leq \frac{\sqrt{\delta^2 \mu^{-1}}}{\sqrt{\delta^2 \mu^{-1} + \mu/4}} = \frac{2\delta}{\sqrt{4\delta^2 + \mu^2}}.
\]
Define
\[
\xi(x) := \text{Sup}_{y \in T_x \mathcal{M}} \left| \frac{\beta}{\alpha} \right|, \quad \xi := \text{Sup}_{x \in \mathcal{M}} \xi(x).
\]
Then by (87) we have
\[
\xi(x) = \frac{\sqrt{\delta^2 \mu^{-1} - c^2}}{\sqrt{\delta^2 \mu^{-1} + \mu/4 - c^2}}, \quad \xi \leq \frac{2\delta}{\sqrt{4\delta^2 + \mu^2}}.
\]
By (88) we can get $c^2$ in terms of $\xi(x)$. Then it follows from (88) that

$$
(1 + \frac{\beta}{\alpha})(\rho^2 - c^2) \leq (1 + \xi(x))(\rho^2 - c^2) = \frac{\mu}{4(1 - \xi(x))} \leq \frac{\mu}{4(1 - \xi)},
$$

(89)

$$
(1 + \frac{\beta}{\alpha})(\rho^2 - c^2) \geq (1 - \xi(x))(\rho^2 - c^2) = \frac{\mu}{4(1 + \xi(x))} \geq \frac{\mu}{4(1 + \xi)}.
$$

(90)

Now by (88), (89) and (90) we get

$$
\frac{4\delta^2 + \mu^2}{\mu}(1 - \xi)^3 \leq K \leq \frac{4\delta^2 + \mu^2}{\mu}(1 + \xi)^3.
$$

(91)

Then by (88) and (91) we immediately obtain (6). Q.E.D.

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