THE GAMBIER MAPPING

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Abstract
We propose a discrete form for an equation due to Gambier and which belongs to the class of the fifty second order equations that possess the Painlevé property. In the continuous case, the solutions of the Gambier equation is obtained through a system of Riccati equations. The same holds true in the discrete case also. We use the singularity confinement criterion in order to study the integrability of this new mapping.
1. Introduction.

One century ago, Painlevé classified all second order of the form $w'' = f(w', w, z)$, where $f$ is rational in $w'$, algebraic in $w$ and analytic in $z$, characterized by the absence of movable singularities [1]. This property, that later came to be known as the Painlevé property, is considered today as an indication of integrability [2]. Integrability was, of course, what Painlevé had in mind when he undertook his classification. The analysis presented by Painlevé, however, was not complete. Although he obtained most interesting results, discovering the equations that define the "Painlevé transcendent", his analysis was plagued by an oversight. In particular he missed a whole class of equations the dominant part of which reads:

$$x'' = \frac{n - 1}{n} \frac{x'^2}{x} + \ldots$$

As consequence he did not obtain the general forms of the transcendent $P_{IV}$, $P_V$ and $P_{VI}$. The gap in the Painlevé classification was filled shortly afterwards by Gambier who analyzed the problem carefully [3] and presented the complete list of all 50 second order equations that possess the Painlevé property.

Foremost among these new equations obtained by Gambier is equation XXVII in his classification:

$$x'' = \frac{n - 1}{n} \frac{x'^2}{x} + ax' + bx - \frac{n - 2}{n} \frac{x'}{x} + \frac{na^2}{(n + 2)^2} x^3 + \frac{n(a' - ab)}{n + 2} x^2 + fx - b - \frac{1}{nx} \quad (1.1)$$

This equation is probably the most complicated that Gambier had to study. It is not one of the transcendental equations, neither is it integrable by quadratures. Rather, it belongs to the class of equations that can be integrated through linearization. The aim of the present paper is to study the discrete analog of this equation of Gambier. In previous publications [4,5], we have presented detailed studies of the discrete Painlevé equations, while the case of linearizable mappings was studied in [6]. However, in the latter, we have limited ourselves to mappings that are linearizable through the discrete equivalent of the Cole-Hopf transformation. The Gambier equation needs a different treatment. It is in fact represented by two Riccati equations ‘in cascade’. In what follows, we shall present a brief review of the continuous case. This is motivated by the fact that Ince’s book [7] (the standard reference on Painlevé equations) does not treat the Gambier equation in any detail. No indication as to how this equation can be integrated is given, and its particular cases are not even hinted at. Once the continuous equation is properly analyzed, its discretization follows in a most logical way.
The very essence of the Gambier equation is that it is obtained as a cascade of two Riccati equations [3]. One starts from a first Riccati:

\[ y' = -y^2 + by + c \]  

(2.1)

where \( b, c \) are functions of the independent variable \( z \), and couple \( y \) to \( x \) through a second Riccati:

\[ x' = ax^2 + nyx + \sigma \]  

(2.2)

Here, \( a \) is a function of \( z \), \( \sigma \) a constant that can be scaled to 1 unless it happens to be 0, and \( n \) is an integer. (As we shall see in what follows, this last point is a first requirement for the absence of critical singularities). The quantities \( a \) and \( \sigma \) in (2.2) are related through a duality. Indeed, replacing \( x \) by \(-1/x\) we find that the equation retains the same form with \( a \) and \( \sigma \) interchanged (and \( n \to -n \)). Then if \( a \neq 0 \), the new \( \sigma \) can be set to 1 (through \( x \to x/a \), and up to a translation of \( y \): the new \( a \) becomes \( a\sigma \) instead of just \( \sigma \)). In order to study the movable singularities of the coupled Riccati system we start from the observation that from (2.1) the dominant behaviour of \( y \) can only be \( y \approx 1/(z - z_0) \). The next terms in the expansion of \( y \) can be easily obtained, and involve the functions \( b, c \) and their derivatives. In order to study the structure of the singularities of (2.2), we first remark that since the latter is a Riccati, its movable singularities are poles. However, (2.2) also has singularities that are due to the singular behaviour of the coefficients of the r.h.s of (2.2), namely \( y \). Now, the locations of the singularities of the coefficients are ‘fixed’ as far as (2.2) is concerned. However, from the point of view of the full system(2.1-2), these singularities are movable and thus should be studied. The ‘fixed’ character reflects itself in the fact -1 is not a resonance. (The terms ‘resonance’ is used here following the ARS terminology [8] and means the order, in the expansion, where a free coefficient enters. A resonance -1 is related to the arbitrariness of the location of the singularity, and is thus absent when the location of the singularity is determined from the ‘outside’ rather than by the initial conditions). Because of the pole in \( y \), \( x \) has a singular expansion with a resonance different from -1 which may introduce a compatibility condition to be satisfied.

Before proceeding to the examination of the general case (2.2) let us briefly consider the case \( a = 0 \), whereupon (2.2) becomes linear. If, moreover, we take \( \sigma = 0 \), we find that the behaviour of \( x \) is obtained through \( x'/x = ny \) and thus \( x \approx A(z - z_0)^n \). In
this case, since $n$ is integer, (2.2) always has the Painlevé property and the functions $b$ and $c$ are free. Next, we turn to the case of the general linear (2.2): $x' = nyx + \sigma$ with $\sigma \neq 0$. Let us examine a singularity of the form

$$x \sim \lambda(z - z_0)$$  \hspace{1cm} (2.3)

We find $\lambda = \sigma / (1 - n)$ unless $n = 1$. (This last case is excluded as being of non-Painlevé type. Indeed, one easily finds that for $\sigma = 1$, the singular expansion has a logarithmic branching point of the form $x \sim (z - z_0) \log(z - z_0)$.) Thus, excluding the case $n = 1$, we study just the singular behaviour (2.3). The resonance associated to the singularity (2.3) turns out to be exactly $n - 1$, which explains the requirement that $n$ be an integer. For every value of $n > 1$ we expect a resonance condition (with complexity increasing with $n$). For any $n$, a sufficient condition is $\sigma = 0$ (which, as we have seen above, is also necessary for $n = 1$) and we have a further possibility which, for the first few orders of $n$, writes:

$$n = 2 \hspace{0.5cm} b = 0$$

$$n = 3 \hspace{0.5cm} 2b^2 - b' - c = 0$$ \hspace{1cm} (2.4)

$$n = 4 \hspace{0.5cm} 6b^3 - 7bb' + b'' - 8bc + 2c' = 0$$

For $n < 0$ there is no further constraint. In fact the sufficient condition is $a = 0$ which we have assumed in this paragraph.

We now turn to the case of the full Riccati (2.3) with $a \neq 0$. In this case, thanks to the duality, we can assume $\sigma = 1$ (otherwise interchanging $a$ and $\sigma$ we are back to the previous case). Again, only the singularity due to $y$ can lead to trouble. Rewriting (2.2) as $x'/x = ax + ny + \sigma/x$ for $y = 1/(z - z_0) + \ldots$ we remark that unless $n = \pm 1$ a behaviour of the form $x \sim (z - z_0)^n$ is impossible when $a\sigma \neq 0$. For $n = 1$, a logarithmic leading behaviour will be present for $\sigma \neq 0$, for nonzero $a$ exactly as in the case $a = 0$. Conversely the condition $\sigma = 0$ is sufficient for the absence of a critical singularity for $n = 1$ even for nonvanishing $a$. Similarly, in a dual way, for $n = -1$ the necessary and sufficient condition for the absence of a critical singularity is $a = 0$, irrespective of the value of $\sigma$.

Next we assume $n \neq \pm 1$, in which case it suffices to study the singularities $x \approx \lambda(z - z_0)$, ($\lambda = \sigma / (1 - n)$) and $x \approx \mu/(z - z_0)$, ($\mu = -(n + 1)/a$). The first singular
behaviour \((x \approx \lambda(z - z_0))\) has a resonance at \(n - 1\), which is negative for \(n < 1\) and thus does not introduce any further condition. For \(n > 1\), the resonance condition can be studied at least for the first few values of \(n\). We find that \(\sigma = 0\) suffices for the resonance condition to be satisfied even for \(a \neq 0\). However, if we demand \(\sigma \neq 0\), we find the further possibilities:

\[
\begin{align*}
  n = 2 & \quad b = 0 \\
  n = 3 & \quad 2b^2 - b' - c + a\sigma/2 = 0 \\
  n = 4 & \quad 18b^3 - 21bb' + 3b'' - 24bc + 6c' + 8ab\sigma - 2a'\sigma = 0
\end{align*}
\]  

The second singular behaviour, \(x \approx \mu/(z - z_0)\), has a resonance at \(-1 - n\). Thus for \(n > 0\) this resonance is negative and does not introduce any further condition, while for \(n < 0\) a compatibility condition must be satisfied. We find that for every case \(n < 0\), \(a = 0\) is a sufficient condition for the absence of critical singularity. (This is not in the least astonishing given the duality of \(a\) and \(\sigma\)). On the other hand if we demand \(a \neq 0\) then a different resonance condition is obtained, at each value of \(n\). For \(n = -1\), whenever \(a \neq 0\), a logarithmic singularity of the form \((z - z_0)^{-1}\log(z - z_0)\) appears irrespective of the value of \(\sigma\). For \(n < -1\), we find:

\[
\begin{align*}
  n = -2 & \quad a' - ab = 0 \\
  n = -3 & \quad 4ab^2 - 2ab' - 2ac + a^2\sigma - 6a'b + 2a'' = 0 \\
  n = -4 & \quad a(18b^3 - 21bb' + 3b'' - 24bc + 6c' + 8ab\sigma) + a'(12b' + 12c - 33b^2 - 6a\sigma) + 18ba'' !- 3a''' = 0
\end{align*}
\]

A remark is in order at this point. The analysis presented above was based explicitly on the assumption that (2.1) is a Riccati. However, if we take \(n \to \infty\) in (2.2) and rescale \(y\) (and \(c\)) in an appropriate way we find that (2.1) becomes linear: \(y' = by + c\). Thus, while on the one hand \(n \to \infty\) in (2.5) leads to a resonance condition that should be implemented after an infinite number of steps, on the other hand we do not have to consider it. Equation (2.1) being now linear, \(y\) does not have any singularity that could interact with (2.2). The equation for \(x\) reads in that case:

\[
x'' = \frac{x'^2}{x} + \left(ax + b - \frac{\sigma}{x}\right)x' + (a' - ab)x^2 + cx - \sigma b
\]  

which is a form of the equation XIV in the Painlevé-Gambier classification \([3,7]\) (not the canonical one but an equivalent, generic one). We recall that the canonical form of the latter is:

\[
w'' = \frac{w'^2}{w} + (qw + \frac{r}{w})w' + q'w^2 - r'
\]  

\[5\]
Equation (2.7) can be integrated to a Riccati:

\[ x' = ax^2 + qx + \sigma \]  \hspace{1cm} (2.9)

where \( q \) is given by \((q\phi)' = c\phi\) and \( \phi \) is an auxiliary function related to \( b \) through \( b = -\phi'/\phi \).

3. The discrete analog of the Gambier equation.

In order to derive the discrete analog of the Gambier equation we start with the observation that in the continuous case we have two Riccati’s in cascade. It is well known [6] that the discrete equivalent of the Riccati equation is a homographic mapping. Guided by this fact we can propose the following form for the discrete Gambier system:

\[ \bar{y} = \frac{by + c}{y + 1} \]  \hspace{1cm} (3.1)

coupled to

\[ \bar{x} = \frac{xyd + \sigma}{1 - ax} \]  \hspace{1cm} (3.2)

where \( \bar{x} \equiv x_{m+1}, \bar{y} \equiv y_{m+1}, \) and the \( x, y \) are understood as \( x_m, y_m \). Appropriate scalings have been made in the two equations in order to bring them to the forms (3.1-2). All four \( a, b, c, d \) are functions of the discrete variable \( m \) while \( \sigma \) is a constant that can be set to 1 unless it is zero. Let us remark that \( d \) may not vanish identically, since in that case the mapping (3.2) would decouple. (From (3.1-2), it is straightforward to obtain a 3-point mapping for \( x \) alone, but the analysis is simpler if we deal with both \( y \) and \( x \)).

The main tool for the investigation of the integrability of discrete systems is the singularity confinement criterion we introduced in [9]. It is based on a conjecture that the movable (i.e. initial condition dependent) singularities of integrable mappings disappear after some iteration steps. It will be used here to investigate the integrability of the coupled mapping (3.1-2).

A first remark before implementing the singularity confinement algorithm is that the singularities of a Riccati mapping are automatically confined. Indeed, if we start from \( \bar{w} = (\alpha w + \beta)/(\gamma w + \delta) \) and assume that at some step \( w = -\delta/\gamma \), we find that \( \bar{w} \) diverges but \( \bar{w} \) and all subsequent \( w \)’s are finite. Thus, the intrinsic singularities of (3.2) do not play any role. However, the singularities due to \( y \) (obtained from (3.1)) may cause problems at the level of (3.2). Two types of singularities may appear. Either
\( y \) diverges (related to the singularity pattern for \( y: \{\ldots, -1, \infty, b, \ldots\} \) or \( y \) takes the value \( y = -a\sigma/d \). In the latter case, we find \( \pi = \sigma \) irrespective of \( x \), and thus \( x \) loses one degree of freedom. On the other hand, once we enter a singularity there is no way for \( x \) to leave it unless \( y \) assumes again a singular value (after a certain number of steps). Thus the first requirement for confinement would be for \( y \) to become again either \( \infty \) or equal to \(-a\sigma/d\). However, if \( y \) were to take the value \( \infty \) again some steps after first taking it, it would take it periodically and the singularity would then also be periodic. This is contrary to the requirement that the singularity be movable: a periodic singularity (with fixed period) is ‘fixed’ in our terminology. The same applies to the singularity \( y = -a\sigma/d \). Thus the only singularity pattern for \( y \) that are acceptable are: either start with \(-a\sigma/d\) and reach \( \infty \) after \( N \) steps, or start with \( \infty \) and reach \(-a\sigma/d\) after \( N' \) steps. (Clearly, these two singularity patterns are mutually exclusive: one cannot combine them since in that case \( \infty \) would be followed by \( \infty \) after \( N + N' \) steps and thus the singularity would again be periodic). So the first condition is that \( y \) take the value \(-a\sigma/d\) \( N \) steps before (or \( N' \) steps after) taking the value \( \infty \). The equivalent of this requirement in the continuous case is that the resonance be integer. We see here that this condition becomes here more complicated. In fact, the first few read:

\[
\begin{align*}
N &= 1 \quad -a\sigma/d = -1 \\
N &= 2 \quad -a\sigma/d = -\frac{c + 1}{b + 1} \\
N &= 3 \quad -a\sigma/d = -\frac{(b + 1)c + \tau + 1}{(b + 1)b + \tau + 1}
\end{align*}
\] (3.3)

In what follows, we shall label these cases for \( N > 0 \) while the cases where the value \(-a\sigma/d\) appears \( N' \) steps after \( \infty \), such as:

\[
\begin{align*}
N' &= 1 \quad -a\sigma/d = b \\
N' &= 2 \quad -a\sigma/d = \frac{bb + c}{b + 1}
\end{align*}
\] (3.4)

will be labelled by \( N \equiv -N' < 0 \).

While one can compute easily more such conditions, it is not possible to give a general expression, like the “\( n \) must be integer” of the continuous case.

Once \( y \) hits a second singularity there is a possibility for \( x \) to recover its lost degree of freedom through some indeterminate form (like \( 0/\infty \) or \( 0/0 \)). Thus when \( y \) becomes
infinite \((N\) steps after assuming the value \(-a\sigma/d\)) we must simultaneously have \(x = 0\). Similarly when \(y\) assumes the value \(-a\sigma/d\) \((N'\) steps after being infinite) \(x\) must be equal to \(1/a\) so as to lead to \(0/0\) for \(\pi\). We can study the first few cases and obtain the condition for the confinement of the singularity. Let us start with \(N < 0\). For \(N = -1\), we enter the singularity through \(y = \infty\), and thus \(\pi = \infty\), \(\eta = b\). We ask next that \(-\pi\sigma/d = \eta = b\) and moreover that this lead to an indeterminate form in the calculation of \(\pi\). The condition in that case is just \(a = 0\) (and this implies \(b = 0\) also, as \(d\) identically zero is not allowed).

For \(N = -2\) we obtain \(\pi = -db/\pi\) and ask that

\[
\frac{\pi}{\eta} = \frac{bb + c}{b + 1} = -\frac{a\sigma}{d} \tag{3.5}
\]

The condition for \(\pi\) to result from an indeterminate form is \(1 - \pi\pi = 0\) i.e. \(\pi^2 db + \pi = 0\). This last condition is automatically satisfied when \(a = 0\).

In fact, it turns out that for every \(N < 0\), \(a = 0\) is a sufficient condition for confinement. In this case, the second equation of the mapping becomes linear. Obtaining the conditions for the next higher \(N\) is straightforward. For \(N = -3\), for instance, we find:

\[
\frac{\pi}{\eta} = \frac{bb + b\overline{c} + \overline{b}c + \overline{c}}{bb + b + \overline{c} + 1} = -\frac{a\sigma}{d}
\]

leading to

\[
(b + 1)(\pi\pi\sigma - \pi - \pi db) - \pi d db (bb + \overline{c}) = 0 \tag{3.6}
\]

The second possibility is to enter the singularity through \(y = -a\sigma/d\), i.e. the \(N > 0\) cases. For \(N = 1\), we enter the singularity through \(y = -a\sigma/d\) which gives \(\pi = \sigma\) and we exit through \(\eta = \infty\), which implies \(a\sigma/d = 1\). In order to have an indeterminate form at the level of \(\pi\) we must have \(\pi\eta = 0\infty\) and thus \(\sigma = 0\) a condition clearly incompatible with \(a\sigma/d = 1\). Thus (3.1-2) as written cannot confine for \(N = 1\). In fact, the confinement condition would be \(\sigma = 0\) provided the denominator of (3.1) is just \(y\) instead of \(y + 1\) (which is dual to \(N = -1\) with \(a = b = 0\)). However, for \(N > 1\) we will find that it is possible to recover the lost degree of freedom. For \(N = 2\), we ask that \(\pi\) be zero and it turns out that this is possible if either \(\sigma = 0\) or \(\overline{d} = 1\). In fact, the condition \(\sigma = 0\) is sufficient for all cases \(N > 1\) (provided, of course, that the condition for \(N\), i.e. “\(y\) becomes infinite after \(N\) steps”, holds). For \(N = 3\), we write the condition (based on the assumption \(\sigma \neq 0\)):

\[
(b + 1)(a\sigma + \overline{d} - 1) - (1 + c)d\overline{d} = 0 \tag{3.7}
\]
Conditions for the next higher $N$’s can easily be obtained.

A question that naturally comes to the mind is whether the continuous limit of system (3.1-2) is indeed the coupled-Riccati original system of Gambier. The answer appears trivially affirmative at first sight: in fact, system (3.1-2) was constructed as a direct discretization of Gambier’s system. However, a subtle question remains. In equation (2.2) of Gambier, $n$ appears explicitly: what is its equivalent in the discrete case? On the one hand, it is always possible to present a formal limit of (3.1-2) where $n$ is related to $d$; in fact we have $d = n - 1$. However, in the previous paragraph we have argued that what plays the role of the position of the resonance in the discrete case is a complicated relation. This condition alone is not sufficient in order to ensure a form like (2.2) at the continuous limit with $n$ appearing naturally. (In particular, it does not prevent $d$ from being non constant, which at the continuous limit would have given a $n$ function of $z$). Only when one takes into account both confinement conditions does system (3.1-2) turn out to be of the precise form (2.1-2) at the continuous limit for the corresponding $n$. Moreover the ‘continuous’ $n$ is precisely the ‘discrete’ $N$ introduced by the confinement. (Note, for instance, that the second confinement condition for $N = 2$ is, precisely, $d = 1$ i.e. $n = 2$).

The discussion presented above was based on the form (2.1) for the first Riccati. However one should also consider another case that cannot be obtained as a limit of (2.1) with the scaling we chose, namely:

$$
\bar{y} = b + \frac{c}{y}
$$

(3.8)

Once the denominator has been simplified a new scaling is possible and we rewrite (3.8) as:

$$
\bar{y} = b + \frac{1}{y}
$$

(3.9)

The analysis of the coupled system (3.9) and (2.2) proceeds along the same lines as before. One obtains simpler discrete forms of the Gambier equation with only one free function instead of two in the generic case. As we have already remarked, this is the only case that confines for $N = 1$, the condition being still $\sigma = 0$. 

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4. DISCRETE SYSTEMS RELATED TO THE GAMBIER MAPPING.

In the previous section, we have examined the mappings consisting in two Riccati’s in cascade. Moreover, the particular case where the second Riccati becomes a linear equation was included in our study through \( a = 0 \) and, because of the duality of \( a \) and \( \sigma \), through \( \sigma = 0 \) as well. Finally the case where the first Riccati reduces to \( \overline{y} = b + \frac{c}{y} \) was briefly considered in section 3 since it does not alter qualitatively the conclusions reached with the full (3.1). In this section, we are going to consider two particular cases of the Gambier mapping. The first case corresponds to equation (3.1) becoming linear. We thus have the mapping:

\[
\begin{align*}
\overline{y} &= y + c \\
\overline{x} &= \frac{xyd + \sigma}{1 - ax}
\end{align*}
\]  

(4.1) (4.2)

(where by the appropriate scaling we have taken \( b = 1 \)).

Clearly, this mapping is a discrete form of the equation \( P_{14} \) encountered in section 2. This can be readily seen when we consider its continuous limit. Putting \( x = w/\epsilon, d = \epsilon q(z), a = \epsilon^2 p(z), c = -s + \epsilon r(z)/\epsilon \) where \( s = q'/q \), we obtain indeed, at the limit \( \epsilon \to 0 \):

\[
w'' = \frac{w'^2}{w} + \left( pw + s - \frac{\sigma}{w} \right) w' + (p' - sp)w^2 + (r + s^2)qw - \sigma b
\]

(4.3)
i.e. exactly (2.7), i.e. \( P_{14} \), in noncanonical form.

We thus expect system (4.1-2) to be integrable without constraint. What does singularity confinement tell us on this point? Before proceeding to the actual calculation, let us try to find the answer by analogy. The case of a linear+Riccati system was obtained in the continuous case by the limit \( n \to \infty \). Since the number \( N \) of steps before confinement is related to the position \( n \) of the resonance in the continuous case (in fact \( n = N \)) we expect confinement to be obtained only after an infinite number of steps! This turns out indeed to be the case. Given the structure of the mapping (4.1-2) the only singularity can be \( y = -a\sigma/d \) (where (4.2) loses one degree of freedom). However, the lost degree of freedom cannot be recovered in a finite number of steps since we need \( y = \infty \). Thus the mapping has a very peculiar behaviour: it can only confine after an infinite number of iterations steps. More on this subtle point will be given in the next section.
Let us now turn to a second particular case of the Gambier mapping. Suppose now that the second equation (3.2) of the mapping has a denominator equal to \(-ax\) instead of \(1 - ax\). To avoid a trivial reduction we must here take \(\sigma \neq 0\). Then (after a rescaling and a redefinition of the parameters) the system becomes:

\[
\begin{align*}
\overline{y} &= \frac{by + c}{y + 1} \\
\overline{x} &= \frac{xyd - 1}{x}
\end{align*}
\tag{4.4}
\]

Here also we have a peculiar behaviour with respect to singularity confinement. Clearly, (4.5) loses one degree of freedom when \(y = \infty\). Moreover, there is no possibility to recover this lost degree of freedom for any value of \(y\). Thus the singularity never confines.

As we shall argue in the next section this is a genuinely nonconfined singularity. Our argument can be further strengthened by considering the continuous limit of (4.4-5). We put \(x = 1 + \epsilon w\), \(b = -1 - \epsilon - \epsilon^2(q - 1/2)\), \(c = b + \epsilon^2(-1/4 + \epsilon(2q - 1)/4 + \epsilon^2 p)\) and take just \(d = -2 + \epsilon\), the latter being clearly a restriction which is sufficient in order to prove our point. We find

\[
w'' = -2w'w - (w' + w^2)^2 - 2q(w' + w^2) - q + 4p + \frac{1}{4}
\tag{4.6}
\]

This last equation is manifestly of non-Painlevé type (because of the \((w' + w^2)^2\) term).

Although (4.4-5) is not confining in general, there exists a particular case where one can recover confinement, namely when (4.4) becomes linear:

\[
\begin{align*}
\overline{y} &= y + c \\
\overline{x} &= \frac{xyd - 1}{x}
\end{align*}
\tag{4.7}
\]

In this case, ther is no way for \(y\) to become singular and thus the mapping has only confined singularities. The continuous limit is compatible with our conclusion. Indeed, putting \(x = 1 + \epsilon w\), \(d = d(z)\), \(c = -2ed'/d^2 + \epsilon^3 q/d\) we find at the continuous limit \(\epsilon \to 0\) the equation:

\[
w'' = -2w'w - \frac{d'}{d}(w' + w^2) - q
\tag{4.9}
\]

The latter is (a non canonical form of) equation P5 in the Painlevé-Gambier list [3,7] and is obtained precisely as the derivative of a Riccati. (Note that by P5 we mean here the fifth equation in the list and not the fifth Painlevé transcendent, which we usually denote by PV).
Thus, the confining subcases of the Gambier mapping obtained in this section are all variants of the ‘discrete derivative’ (with respect to one of the parameters) of a homographic mapping. This means that we can start with a mapping of the form \( \overline{w} = \frac{\alpha w + \beta}{\gamma w + \delta} \), solve for one of the parameters, take the discrete derivative of this parameter and assign an \( m \)-dependent value to it. Given the duality of the homographic mapping under the transformation \( w \to 1/w \), only two different types of such mappings, corresponding to derivatives with respect to either \( \alpha \) or \( \beta \), exist.

5. A DISCUSSION OF THE CONFINEMENT PROPERTY.

In the preceding sections, we have repeatedly used the criterion of the confinement of singularities in order to characterize the integrable or not character of the Gambier mapping under its various forms. Since the behaviours of the systems of section 4 were rather atypical as far as the confinement is concerned, we feel that some detailed discussion is necessary at this point.

The confinement property was initially introduced (and related to integrability) for rational mappings, the structure of which would lead to vanishing denominators and the appearance of divergences [9]. However it was rapidly realized that this is not the only manifestation of a singular behaviour. Thus, in [6], we introduced the notion of singularity so as to include all cases where the system loses one degree of freedom. Confinement of the singularity in this case means recovery of the lost degree of freedom through the appearance of an indeterminate form (0/0, 0∞, etc.). One important condition (in perfect analogy to the continuous case) is that the singularity be ‘movable’ i.e. initial condition dependent. Let us illustrate this last point in the case of the Riccati mapping:

\[
    w_{m+1} = \frac{\alpha w_m + \beta}{\gamma w_m + \delta} \tag{5.1}
\]

where \( \alpha, \beta, \gamma \) and \( \delta \) depend in general on \( m \). The mapping variable \( w \) can very well become infinite for some \( m \), but this (movable) singularity is immediately confined: infinity does not play a particular role in a homographic mapping. However, if at certain iteration the determinant \( \alpha \delta - \beta \gamma \) of the coefficient matrix vanishes the mapping loses one degree of freedom and moreover there is no way that it can recover it in its subsequent evolution. However this non-confined singularity is perfectly acceptable since it is fixed i.e. its location does not depend on the initial conditions.

Clearly, the Riccati mapping is too simple to exhibit any interesting behaviour as far as the singularity confinement is concerned. As soon as we enlarge our scope and
consider systems of two Riccati’s in cascade (as in the case of the Gambier mapping) the behaviour becomes considerably richer. A first point must be made about periodic singularities. They are assimillated to fixed ones since they do not have a specific location, that could depend on the initial conditions, but rather they extend everywhere.

The main difficulty of the Gambier mapping comes from the fact that the distinction between fixed and movable singularities is somewhat artificial [10]. A movable singularity in the first equation of the system enters in the second one through a coefficient and should thus be considered as ‘fixed’, as far as the second equation is concerned. (Let us recall again that the ‘movable’ singularities of a homographic mapping (5.1) do not cause any difficulties). Still its location depends on the initial conditions of the system, considered globally, and must, thus, be treated as a movable singularity.

In section 4 we have encountered two pathological cases. In one of them, we have argued that confinement was possible (although after an infinite number of steps) and thus that the equation was in principle confining. The second equation was rejected as non-confining since there was no possibility whatsoever to confine. We are aware that the distinction is subtle and lies on grounds that are not very firm. In the cases at hand, we were guided by the underlying structure of the system in order to make our claims on the confining or not character of the mapping. However, if we are given a general 3-point mapping, there is a priori no easy, algorithmic, way to decide whether the mapping confines after an infinite number of steps, or not at all. We hope that just as in the continuous case, the occurrence of a resonance at $n = \infty$, compatible with the Painlevé property, is an exceptional occurrence, and that the 3-point mappings confining after an infinite number of iterations will be restricted to those of the Gambier family.

A useful guide as to the confining or not character of the mappings (in such tricky cases) is the study of all possible continuous limits. Our conjecture is that confining mappings have as continuous limits differential equations that possess the Painlevé property. Thus if for some mapping we are able to find some non-Painlevé continuous limit, this is a clear indication as to its non-confining character. However, the absence of such non-Painlevé limits does not allow us to draw clear conclusions: it may only mean that non-Painlevé continuous limits do not exist (or that we were not able to find them!) Still, the continuous limit can be a ‘last stand’ argument for some mapping that is difficult to interpret.

Finally, let us point out that all our discussion of Gambier’s mapping and, in fact,
of Gambier’s equation as well, is a discussion of the Painlevé property rather than plain integrability. In a system such as (2.1-2) or (3.1-2) of Riccati equations in cascade, we can always solve the first equation for $y$, obtain $y(t)$ or $y_m$, and inject it into the second equation. The latter can always be written as a linear second order differential equation for $x$, or a linear 3-point mapping. So, in principle, the problem can always be solved formally. The difficulty comes when one wishes to actually compute $x$, in terms of contour integrals in the continuous case while $y$ has bad analytic properties. The Painlevé property guarantees that one can perform the integration and obtain indeed the solution of $x(t)$ over the complex $t$-plane. In the discrete case the difficulty arises when one tries to compute $x_m$ in terms of matrix products (the elements of which contain $y_m$). When $y_m$ has the wrong properties some of these matrices are singular in such a way that degrees of freedom are irretrievably lost. Singularity confinement means that these lost degrees of freedom are recovered at some later stage.

6. Conclusion.

In this paper we have analysed a mapping consisting in two discrete Riccati’s in cascade. This mapping is the discrete analogue to Gambier’s equation (number XXVII of the Painlevé-Gambier list). The interesting feature of this discrete system is the fact that the singularities can be confined after an arbitrary number of steps. This is quite unusual, since, in most cases encountered up to now, confinement was obtained after a small number of iteration steps. This new behaviour, encountered for the first time in the Gambier mapping has made possible the refinement of the whole notion of singularity confinement.

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