ON THE DUALITY OF F-THEORY AND THE CHL STRING IN SEVEN DIMENSIONS

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Abstract. We show that the duality between F-theory and the CHL string in seven dimensions defines algebraic correspondences between K3 surfaces polarized by the rank-ten lattices $H \oplus N$ and $H \oplus E_8(-2)$. In the special case when the F-theory admits an additional anti-symplectic involution or, equivalently, the CHL string admits a symplectic one, both moduli spaces coincide. In this case, we derive an explicit parametrization for the F-theory compactifications dual to the CHL string, using an auxiliary genus-one curve, based on a construction given by André Weil.

1. Introduction

By standard lattice-theoretic observations [65], one has the following lattice isomorphism:

$$H \oplus E_8(-2) \cong H(2) \oplus N. \tag{1.1}$$

Here, $E_8$ is the positive definite root lattices associated with the $E_8$ root systems, $H$ is the unique even unimodular hyperbolic rank-two lattice and $N$ is the negative definite rank-eight Nikulin lattice (see [59, Def. 5.3], for a definition). The notation $L(\lambda)$ refers to the lattice obtained from $L$ after scaling of its bilinear form by $\lambda \in \mathbb{Z}$. Moreover, as proved in [8], the lattice isomorphism (1.1) implies that the lattice $H(2) \oplus E_8(-2)$ admits two special overlattices, namely the lattice $H \oplus N$, necessarily of index four, and $H \oplus E_8(-2)$ of index two.

In the above contexts, we shall consider $\mathcal{M}_{H \oplus N}$ and $\mathcal{M}_{H \oplus E_8(-2)}$, as moduli spaces of complex algebraic K3 surfaces with lattice polarizations of type $H \oplus N$ and $H \oplus E_8(-2)$. Both these moduli spaces are 10-dimensional. And the K3 surfaces classified by them can be described explicitly, via Weierstrass models. The first K3 family was studied by van Geemen and Sarti [73]. These K3 surfaces carry a canonical Jacobian elliptic fibration with an element of order two (or 2-torsion section) in its Mordell-Weil group, which, in turn, determines a special class of K3 involution referred to in the literature as van Geemen-Sarti involution. The K3 surfaces in the second family, associated with $H \oplus E_8(-2)$ polarizations, carry canonical Jacobian elliptic fibrations, but in this case one has compatible anti-symplectic involutions, with the property that the minimal resolution of the associated $\mathbb{Z}/2\mathbb{Z}$ quotient is a rational elliptic surface [8].

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The lattice isomorphism (1.1), via Hodge theory, implies then that the two K3 families from above are related via special algebraic correspondences. These correspondences are governed by markings of even-eight configurations in the Néron-Severi group $\text{NS}(\mathcal{X})$ for $\mathcal{X} \in \mathcal{M}_{H\oplus N}$ and pairs $\{\pm \theta\} \subset \text{Br}(\mathcal{Y})$ of Brauer group elements for $\mathcal{Y} \in \mathcal{M}_{H\oplus D(-1)^{\oplus 2}}$. Here, $\mathcal{Y}$ is the K3 surface obtained from the Nikulin construction using a canonical involution on $\mathcal{Y}$. These algebraic correspondences stem from classical constructions in the (mathematics) literature \cite{8,23,29,42,72}.

The above mentioned construction has a remarkable consequence in string theory—it provides a mathematical framework for a class of string dualities linked to the so-called CHL string, named after Chaudhuri, Hockney, and Lykken \cite{10}. The CHL string is obtained from the more familiar $E_8 \times E_8$ heterotic string on a torus $T^2$, as a certain $\mathbb{Z}/2\mathbb{Z}$ quotient. The Narain construction shows that the physical CHL moduli space is identical with the moduli space $\mathcal{M}_{H\oplus E_8(-2)}$ of K3 surfaces principally polarized by the lattice $H \oplus E_8(-2)$ \cite{1,2}.

F-theory, i.e., compactifications of the type-IIB string theory in which the complex coupling varies over a base, is a powerful tool for analyzing the non-perturbative aspects of heterotic string compactifications \cite{60,61}. The simplest F-theory constructions are K3 surfaces that admit Jacobian elliptic fibrations over $\mathbb{P}^1$. It is well known (see \cite{5}) that, for F-theory backgrounds with non-zero flux given by a $B$-field along the base curve $\mathbb{P}^1$, the value of this flux is quantized and fixed to half the Kähler class of $\mathbb{P}^1$. In geometric language, this structure is equivalent to a Jacobian elliptic fibration supported on the K3 surface, admitting a 2-torsion section, i.e., a van Geemen-Sarti involution. The existence of the van Geemen-Sarti involution then implies that the F-theory on the K3 surface is to be further compactified on a circle $S^1$ yielding a 7-dimensional compactification.

It follows from above that the K3 moduli spaces $\mathcal{M}_{H\oplus N}$ and $\mathcal{M}_{H\oplus E_8(-2)}$ are the moduli space of F-theory models and of the dual of the CHL string in seven dimensions, respectively. In this article, we shall use classical geometrical notions - algebraic correspondences, even-eight configurations, and elements of the Brauer group - to give a precise mathematical framework for the F-theory/CHL string duality.

Particular situations of the above duality are also interesting mathematically. One may study F-theory vacua and CHL string backgrounds in the presence of additional structure. Our framework then allows us to give explicit parameterizations for F-theory vacua and CHL string backgrounds in the presence of an additional involution. For instance, a natural 6-dimensional subspace that is contained simultaneously in both aforementioned physical moduli spaces is the moduli space of K3 surfaces polarized by the lattice $H \oplus N_0(-1)$. Here, $N_0$ is a positive definite lattice of rank 12, determinant $2^8$, and its Gram matrix will be computed explicitly. This is a subspace where the F-theory admits an additional anti-symplectic involution, and, on the CHL string side, one has an additional a symplectic involution. We derive an explicit parametrization for elements of the moduli space using an auxiliary genus-one curve. This parametrization is based on a construction by André Weil \cite{75}, in which the Abel-Jacobi map is used to obtain embeddings of genus-one curves as symmetric divisors of bi-degree $(2,2)$ in $\mathbb{P}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. In fact, we show that the
algebraic correspondences for the F-theory backgrounds and vacua of the CHL string are double-quadrics obtained from double covers of \( \mathbb{F}_0 \), where the branching divisor consists of a symmetric divisor of bi-degree \((2,2)\) and an additional collection of lines.

The F-theory moduli space has another natural 6-dimensional subspace, namely the moduli space of K3 surfaces polarized by the lattice \( \langle 2 \rangle \oplus \langle -2 \rangle \oplus D_4(-1)^{\oplus 3} \). This special situation corresponds to the case when, on the F-theory side, surfaces admit an additional symplectic involution and, on the CHL string side, a special anti-symplectic involution exist. The K3 surfaces associated with the CHL string in the situation above also carry a beautiful geometric description: they are special double-sextic surfaces, i.e., they can be obtained as minimal resolutions from double covers of the projective plane branched over a configuration of three distinct lines coincident in a point and an additional generic cubic. The latter divisor gives rise to an elliptic curve capturing part of the K3 moduli coordinates. This fact relates the current work to pervious work by the authors in [14].

It is important to note that, in the two special examples described above, both involving 6-dimensional subspaces of \( \mathcal{M}_{H@N} \), an elliptic curve naturally emerges. And this is not the elliptic curve from which the CHL string is constructed, but rather a Seiberg-Witten type curve that parameterizes certain moduli of the F-theory/CHL vacua under consideration.

This article is structured as follows: in Section 2 we review the physics of the duality between F-theory and the CHL string. We will then argue that the lattice polarized K3 surfaces under consideration correspond to F-theory models that are dual of the CHL string in seven dimensions. In Section 3 we give a construction for families of lattice polarized K3 surfaces with canonical anti-symplectic and symplectic involution, respectively. In Section 4 we prove that the duality between F-theory with discrete flux and the CHL string determines certain algebraic correspondences between pairs of K3 surfaces polarized by the rank-ten lattices \( H@N \) and \( H@E_8(-2) \), respectively. In Section 5, we restrict our attention to the moduli space of K3 surfaces polarized by the lattice \( H@N_0(-1) \), which is contained simultaneously in both \( \mathcal{M}_{H@N} \) and \( \mathcal{M}_{H@E_8(-2)} \). We derive an explicit classification of these surfaces, using an auxiliary genus-one curve. In Section 6 we investigate a special region of the F-theory moduli space corresponding to K3 surfaces principally polarized by the lattice \( \langle 2 \rangle \oplus \langle -2 \rangle \oplus D_4(-1)^{\oplus 3} \). Some concluding remarks are included in Section 7.

This article is based on several prior papers by the authors and their collaborators [7,12,13,17–24,32,54–57], as well as several other works [14,30,31,38–41,43,44,50–52,58,59,63,67].

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2. The duality between F-theory and the CHL string

Compactifications of the type-IIB string theory in which the complex coupling varies over a base are generically referred to as F-theory [60, 61]. One of the simplest F-theory construction corresponds to K3 surfaces that are elliptically fibered over $\mathbb{P}^1$, in physics equivalent to a type-IIB string theory compactified on $\mathbb{P}^1$ and hence eight-dimensional in the presence of 7-branes [55]. In this way, an elliptically fibered K3 surface with section and fiber $F = C/(\mathbb{Z} \oplus \mathbb{Z} \tau)$ defines an F-theory vacuum in eight dimensions where the complex-valued scalar axio-dilaton field $\tau$ of the type-IIB string theory is allowed to be multi-valued and undergo monodromy transformations in $\text{SL}(2, \mathbb{Z})$ when encircling defects of co-dimension one. Kodaira’s table of singular fibers [48] gives the precise dictionary between characteristics of the elliptic fibration and the content of the 7-branes present in the physical theory and the local monodromy of $\tau$. It is well-known that the moduli space of these F-theory models is isomorphic to the moduli space of the heterotic string compactified on a two-torus $T^2$ – equipped with a complex structure and complexified Kähler modulus – together with a principal $G$-bundle where $G$ is the gauge group of the heterotic string, i.e., $G = E_8 \times E_8 \times \mathbb{Z}_2$ or $\text{Spin}(32)/\mathbb{Z}_2$ or a subgroup of these [9, 33, 34, 70]. In fact, the moduli spaces for these physical theories are given by the same Narain space which is the quotient of the symmetric space for $O(2, 18)$ by a particular arithmetic group [62]. This is the basic form of the F-theory/heterotic string duality.

One can also consider the CHL string [5, 11, 53, 76]. The CHL string is obtained from an $E_8 \times E_8$-heterotic string as a $\mathbb{Z}_2$-quotient. The quotient is obtained from a specific involution, called the CHL involution which is due to Chaudhuri and Polchinski in [11]: the CHL involution $\iota_{\text{CHL}}$ acts by a half-period shift on the elliptic curve (obtained from the two-torus $T^2$ equipped with the given complex structure), acts trivially on the complex Kähler modulus, and permutes the two $E_8$’s of the gauge bundle. The moduli space for the CHL string compactified on the elliptic curve is then obtained as follows: first, the Narain construction yields the moduli space of the $E_8 \times E_8$-heterotic string compactified on $T^2$ as the double coset space

$$O(\text{Nar}) \backslash O(\text{Nar} \otimes \mathbb{R}) / K,$$

where $\text{Nar} = H \oplus H \oplus E_8(-1) \oplus E_8(-1)$ is the Narain lattice and $K \subset O(\text{Nar} \otimes \mathbb{R})$ is a maximal compact subgroup. The CHL involution $\iota_{\text{CHL}}: \text{Nar} \rightarrow \text{Nar}$ acts on the Narain lattice as the identity on $H \oplus H$ while interchanging the two summands of $E_8(-1)$. It follows that the invariant sublattice of the Narain lattice is $\text{Nar}^{(\iota_{\text{CHL}})} \cong H \oplus H \oplus E_8(-2)$. The restriction of (2.1) to the corresponding $\iota_{\text{CHL}}$-invariant symmetric space thus gives rise to a moduli space of K3 surfaces, namely the moduli space of K3 surfaces with a transcendental lattice isometric to $H \oplus H \oplus E_8(-2)$; its complement in the K3 lattice $\Lambda_{K3} \cong H^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ is isomorphic to $H \oplus E_8(-2)$. This suggests, that the moduli space of the CHL string is the 10-dimensional moduli space of K3 surfaces polarized by the lattice $H \oplus E_8(-2)$.

Bershadsky et al. analyzed the F-theory compactifications dual to the CHL string in the case of an isotrivial elliptic fibration on the F-theory background [5]: they found that the CHL string in eight dimensions is dual to an F-theory with non-zero flux of an antisymmetric two-form field, or $B$-field, along the base curve $\mathbb{P}^1$. The value of
this flux is quantized and fixed to half the Kähler class of \( \mathbb{P}^1 \). This picture was then extended to the interior of the Narain moduli space of toroidal compactifications where the elliptic fibrations of the F-theory models are no longer isotrivial. The presence of the flux freezes eight of the moduli in the physical moduli space, leaving a 10-dimensional moduli space \([76]\). The construction of Witten is known today as the frozen phase of F-theory \([6]\).

On this moduli space, the single-valued background of an antisymmetric two-form in the physical theory is not compatible with a monodromy of the half-periods of an elliptic fiber in \( \text{SL}(2, \mathbb{Z}) \) around the defects. Rather it must be contained in a subgroup of \( \text{SL}(2, \mathbb{Z}) \) that keeps the flux of the \( B \)-field invariant. It was argued in \([5]\) that the biggest possible being the congruence subgroup \( \Gamma_0(2) \). Then, for a consistent model one has to start with an elliptically fibered K3 surface with section and a monodromy group contained in \( \Gamma_0(2) \) that keeps one of the three half-periods of the elliptic fibers invariant. The corresponding Weierstrass model indeed has eight fibers of type \( I_2 \) and two sections, the zero-section and a 2-torsion section, arranged in such a way that at each reducible fiber of type \( A_1 \) the two sections pass through two different components of the fiber. In physics this fibration is called the \( \Gamma_0(2) \) elliptic fibration \([5]\).

We employ a different point of view that is based on recent work of the authors in \([15]\). There, all Jacobian elliptic K3 surfaces with finite automorphism group were classified. Concretely, all lattices \( L \) that may occur as Néron-Severi lattice for a Jacobian elliptic K3 surface \( X \) with finite automorphism group were given, and all Jacobian elliptic fibration(s) with the reducible fibers and Mordell-Weil groups supported on a very general \( L \)-polarized K3 surface were determined.

Given a lattice \( L \), one has the discriminant group \( A_L = L^\vee/L \) and its associated discriminant form, denoted \( q_L \). A lattice \( L \) is then called 2-elementary if \( A_L \) is a 2-elementary abelian group, i.e., \( A_L \cong (\mathbb{Z}/2\mathbb{Z})^\ell \) where \( \ell = \ell_L \) is the length of \( L \), the minimal number of generators of the group \( A_L \). One also has the parity \( \delta_L \in \{0, 1\} \). By definition, \( \delta_L = 0 \) if \( q_L(x) \) takes values in \( \mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/2\mathbb{Z} \) for all \( x \in A_L \), and \( \delta_L = 1 \) otherwise. In this context, a result of Nikulin \([65, \text{Thm. 4.3.2}] \) and \([66, \text{Thm. 4.3.2}] \) asserts that even, indefinite, 2-elementary lattices (that are sublattices of the K3 lattice) are uniquely determined by their rank \( \rho_L \), length \( \ell_L \), and parity \( \delta_L \).

One consequence of the classification by the authors in \([15]\) is that a Jacobian elliptic K3 surface with an element of order two (or 2-torsion section) in its Mordell-Weil group and finite automorphism group requires the Néron-Severi lattice \( L \) to be 2-elementary. The minimal rank of \( L \) then is \( \rho_L = 10 \), and the only such lattice is precisely \( L = H \oplus N \). In turn, the only Jacobian elliptic fibration that is supported on a very general \( H \oplus N \)-polarized K3 surface is the aforementioned fibration with the singular fibers \( 8I_2 + 8I_1 \) and the Mordell-Weil group \( \mathbb{Z}/2\mathbb{Z} \). We call this fibration the alternate fibration. In \([15, \text{Table 3.4}] \) all 2-elementary lattices \( L \) were classified such that the very general \( L \)-polarized K3 surface has a finite automorphism group and admits an alternate fibration. For ranks \( \rho_L \) between 10 and 18, every such 2-elementary lattice \( L \) supports at most one alternate fibration. This gives a precise criterion for the existence of an alternate fibration and hence a van Geemen-Sarti involution. Note that for Picard numbers 16, 17, and 18 the corresponding lattices
are
\[ H \oplus E_7(-1) \oplus E_7(-1), \quad H \oplus E_8(-1) \oplus E_7(-1), \quad \text{and} \quad H \oplus E_8(-1) \oplus E_8(-1), \]
respectively, and the corresponding alternate fibrations are well known in string theory. The advantage of this approach is that it includes all admissible sub-varieties of the moduli spaces of \( H \oplus N \)-polarized K3 surfaces. For example, the lattice \( H \oplus N \) admits an embedding into the lattice \( L = H \oplus D_8(-1) \oplus D_3(-1) \) of rank 14. A very general \( L \)-polarized K3 surface admits two distinct Jacobian elliptic fibrations: the first one has the singular fibers \( I_4^* + I_6^* + 8I_1 \) and a trivial Mordell-Weil group, the second fibration is the alternate fibration with the singular fibers \( III^* + 5I_2 + 5I_1 \) and the Mordell-Weil group \( \mathbb{Z}/2\mathbb{Z} \). However, singular fiber of type \( III^* \) have a monodromy group that is not contained in \( \Gamma_0(2) \); see \([55, \text{Table 1}]\). Indeed in \([5]\) a reducible fiber of type \( E_7 \) was not considered problematic (as opposed to reducible fibers of type \( E_6 \) and \( E_8 \) in the alternate fibration). This demonstrates that the existence of a canonical van Geemen-Sarti involution is consistent for extensions of the lattice polarization as one restricts to natural subspaces in the moduli space of \( H \oplus N \)-polarized K3 surfaces whereas the notion of \( \Gamma_0(2) \)-monodromy might not.

To our knowledge, however, the construction of \([5]\) has not been much employed in the physics literature, and there are various criticisms that can be made of it. One such criticism is that, as suggested in the paper, one can also consider fluxes of the B-field that restrict the monodromy group to \( \Gamma_0(3) \), \( \Gamma_0(4) \), or \( \Gamma_0(6) \). This would define other disconnected components in the moduli space of 8d string vacua with 16 supersymmetries and gauge groups of smaller rank. From the point of view of frozen singularities, however, these do not arise. Concretely, Tachikawa \([71]\) has given evidence that in F-theory there exist only frozen singularities corresponding to \( I_4^* \) fibers, so that in eight dimensions one can only construct theories with rank reductions by 8 (one such fiber) and 16 (two such fibers)\(^1\). Note that these cases also appear in the classification results of \([15]\): the very general \( H \oplus D_8(-1) \)-lattice polarized K3 surface has a finite automorphism group and admits exactly one Jacobian elliptic fibration which has one singular fiber of type \( I_4^* \) and a trivial Mordell-Weil group. The second case constitutes the family of \( H \oplus D_8(-1) \oplus D_8(-1) \)-polarized K3 surfaces which is the family of Kummer surfaces associated with two non-isogeneous elliptic curves. In light of the work \([6,71]\) one concludes that \( \mathcal{M}_{H \oplus D_8(-1)} \) is in fact the moduli space of lattice polarized K3 surfaces corresponding to the frozen phase of F-theory.

In contrast, we argue that the moduli space of \( H \oplus N \)-polarized K3 surfaces corresponds to the dual of the CHL string in seven dimensions. The reason is that an \( H \oplus N \)-polarization is equivalent with the existence of a canonical van Geemen-Sarti involution as in Figure 2. As explained for example in \([71]\), if F-theory on a K3 surface \( \mathcal{X} \) is further compactified on a circle \( S^1 \), one can orbifold the theory quotienting the \( \mathcal{X} \times S^1 \) by an involution \( \iota_{\mathcal{X}} \) and a half-shift on the \( S^1 \); see Sections 3.1 and 3.3 for the definitions of a Nikulin involution and van Geemen-Sarti involution. Furthermore, this construction is dual to M-theory on K3 with two frozen singularities of type \( I_6^* \),

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\(^1\)The scenario has been recently claimed to be reconstructed in the Swampland Program in \([4,36]\), excluding the other disconnected components mentioned above.
as already shown by Witten [76]. The F-theory construction just mentioned is purely geometric and does not involve any flux; its M-theory dual does involve a 3-form flux, and does not uplift to eight dimensions. In particular, this one can indeed be extended to further rank reductions; see [26] for a full treatment of this problem.

The identification of points in the physical moduli space corresponding to a specific non-abelian gauge group of the CHL string is then based on a second elliptic fibration with two fibers of type $I_0^*$, called the inherited elliptic fibration [5]. As explained in [76, p. 31], the two $I_0^*$ fibers appear when one compactifies further on a circle, i.e., the theory is seven dimensional. The correct F-theory compactification does not admit a section but only a bi-section due to the precise nature of the duality with the CHL string. In fact, Witten argued that the moduli space of F-theory compactifications dual to the CHL string is naturally isomorphic to the moduli space of K3 surfaces obtained as genus-one fibrations with two fibers of type $I_0^*$ and a bi-section [76].

In conclusion, it follows that the K3 moduli spaces $\mathcal{M}_{H@E_8(-2)}$ and $\mathcal{M}_{H@N}$ are the moduli space of the CHL string in seven dimensions and the dual F-theory, respectively. Below we shall use classical geometrical notions - algebraic correspondences, even-eight configurations, and elements of the Brauer group - to give a precise mathematical framework for the F-theory/CHL string duality.

3. Constructions of K3 surfaces

3.1. Notions for lattice polarized K3 surfaces. We will use the following standard notations: $L_1 \oplus L_2$ is orthogonal sum of the two lattices $L_1$ and $L_2$, $L(\lambda)$ is obtained from the lattice $L$ by multiplication of its form by $\lambda \in \mathbb{Z}$, $(R)$ is a lattice with the matrix $R$ in some basis; $A_n$, $D_m$, and $E_k$ are the positive definite root lattices for the corresponding root systems, $H$ is the unique even unimodular hyperbolic rank-two lattice, and $N$ is the negative definite rank-eight Nikulin lattice (for a definition, see [59, Sec. 5]). For a lattice $L$ with a primitive embeddings $\iota : L \hookrightarrow \Lambda_{K3}$ into the K3 lattice $\Lambda_{K3} \cong H^{\oplus 3} \oplus E_8(-1)\oplus^2$, Dolgachev proved that there exists a coarse moduli space $\mathcal{M}_L$ of pseudo-ample $L$-polarized K3 surfaces, i.e., a moduli space of algebraic K3 surfaces $\mathcal{X}$ that are polarized by the lattice $L$ such that $\iota(L)$ contains a numerically effective class of positive self-intersection in the Néron-Severi lattice $\text{NS}(\mathcal{X})$. Dolgachev also established a version of mirror symmetry for the moduli spaces of lattice polarized K3 surfaces in [27]: given a lattice $L$ with the property that for any two primitive embeddings $\iota_1, \iota_2 : L \hookrightarrow \Lambda_{K3}$ there is an isometry $g \in O(\Lambda_{K3})$ such that $\iota_2 = g \circ \iota_1$, the isomorphism class of $\bar{L}$ for a fixed splitting $L^{\perp} = H \oplus \bar{L}$ of the orthogonal complement $L^{\perp} \subset \Lambda_{K3}$ is well defined. The mirror moduli space of $\mathcal{M}_L$ is then taken to be $\mathcal{M}_{\bar{L}}$.

An important tool in relating families of K3 surfaces are Nikulin involutions. Recall that a Nikulin involution [59,64] is an involution $\iota_X : \mathcal{X} \to \mathcal{X}$ on a K3 surface $\mathcal{X}$ that satisfies $\iota_X^* (\omega) = \omega$ for any holomorphic two-form $\omega$ on $\mathcal{X}$. The minimal resolution of the quotient $\mathcal{X}'/(\iota_X)$ is another K3 surface $\mathcal{X}'$, and the quotient map induces a two-to-one rational map $\Phi : \mathcal{X} \dasharrow \mathcal{X}'$ whose branch locus is an even set of eight rational curves on $\mathcal{X}'$. Recall that a set of eight rational curves on a K3 surface is called an even eight if the sum is divisible by two in the Néron-Severi lattice; see [3].
For a Jacobian elliptic surface $\mathcal{X}$ we denote the projection by $\pi_\mathcal{X} : \mathcal{X} \to \mathbb{P}^1$, the zero-section by $\sigma_\mathcal{X}$, and the Mordell-Weil group of sections by $\text{MW}(\mathcal{X}, \pi_\mathcal{X})$ with its torsion subgroup denoted by $\text{MW}(\mathcal{X}, \pi_\mathcal{X})_{\text{tor}}$. A complete list of the possible singular fibers in a Weierstrass model was determined by Kodaira [48]: it encompasses two infinite families $(I_n, I_n^*, n \geq 0)$ and six exceptional cases $(II, III, IV, II^*, III^*, IV^*)$. The span of the cohomology classes associated with the elliptic fiber and the section $\sigma_\mathcal{X}$ induces a sublattice of the Néron-Severi lattice $\text{NS}(\mathcal{X})$ isomorphic to $H$. This sublattice determines uniquely the Jacobian fibration. Conversely, a pseudo-ample lattice embedding $H \hookrightarrow \text{NS}(\mathcal{X})$ determines – up to the action of Hodge isometries of $H^2(\mathcal{X}, \mathbb{Z})$ – a unique isomorphism class of a Jacobian elliptic fibration on $\mathcal{X}$.

3.2. K3 surfaces double covering a rational elliptic surface. For two cubics $p, q$ in $\mathbb{P}^2 = \mathbb{P}(X, Y, Z)$ the cubic pencil

$$\mathcal{R} = \left\{ U p(X, Y, Z) + V q(X, Y, Z) = 0 \right\} \subset \mathbb{P}^2 \times \mathbb{P}^1$$

defines an elliptic surface with section $\pi_\mathcal{R} : \mathcal{R} \to \mathbb{P}^1 = \mathbb{P}(U, V)$ which is isomorphic to $\mathbb{P}^2$ blown up in the nine base points of the pencil. The exceptional divisors are not contained in the fibers but yield independent sections. Selecting one of them as the zero section $\sigma_\mathcal{R}$, the remaining eight generate a lattice of type $E_8$. Here, we are assuming that the two cubics $p, q$ do not have common components. Moreover, the base points of the cubic pencil may only yield independent sections if the pencil has no reducible fibers. The minimal resolution gives rise to a rational elliptic surface (RES). By the Shioda-Tate formula the rank of $\text{MW}(\mathcal{R}, \pi_\mathcal{R})$ drops if there are reducible fibers. It turns out that this characterization is complete [25, Thm. 5.6.1], and every rational elliptic surface $\mathcal{R}$ is isomorphic to the blow-up of $\mathbb{P}^2$ in the base points of a cubic pencil.

The general rational elliptic surface with section $\pi_\mathcal{R} : \mathcal{R} \to \mathbb{P}^1 = \mathbb{P}(U, V)$ has the Weierstrass model

$$\mathcal{R} : \quad Y^2 Z = X^3 + f(U, V) XZ^2 + g(U, V) Z^3,$$

where $f, g$ are homogenous of degree four and six, respectively, such that the fibration has twelve singular fibers of type $I_1$ (that is, $4f^3 + 27g^2 = 0$ has twelve distinct roots), and the Mordell-Weil group is $\text{MW}(\mathcal{R}, \pi_\mathcal{R}) \cong E_8$. Here, the Mordell-Weil group can also read off in the classification of all rational elliptic surfaces given by Oguiso and Shioda in [68]. The rational elliptic surface $\mathcal{R}$ with the singular fibers $12I_1$ and $\text{MW}(\mathcal{R}, \pi_\mathcal{R}) \cong E_8$ is referred to as (1) in Oguiso-Shioda classification.

A family of K3 surfaces is obtained by a base change of order two on the rational elliptic fibration $\pi_\mathcal{R}$. A double cover $h_{(d_0, d_{\infty})} : \mathbb{P}^1 = \mathbb{P}(u, v) \to \mathbb{P}^1 = \mathbb{P}(U, V)$ is determined by choosing two points $[u, v] = [0 : 1], [1, 0]$ as the branch points with images $[U, V] = [d_0 : 1], [1, d_{\infty}]$ with $d_0 d_{\infty} \neq 1$, and an additional point, say $[u, v] = [1 : 1]$ with image $[U, V] = [1 : 1]$. The double cover map is then given by

$$h_{(d_0, d_{\infty})} : \left[ u : v \right] \mapsto \left[ U : V \right] = \left[ (1-d_0)u + d_0(1-d_{\infty})v^2 : d_{\infty}(1-d_0)u^2 + (1-d_{\infty})v^2 \right].$$

Here, we are assuming that the fibers of (3.2) over the points $[d_0 : 1], [1, d_{\infty}], [1 : 1]$ are not singular. Geometrically, $h_{(d_0, d_{\infty})}$ determines a line bundle $\mathcal{L} \to \mathbb{P}^1$ such that
\( \mathcal{L}^{\oplus 2} = \mathcal{O}_\mathbb{P}^1. \) Pulling back the Weierstrass model in Equation (3.2), we obtain a Jacobian elliptic K3 surfaces \( \pi_Y : \mathcal{Y} \to \mathbb{P}^1 = \mathbb{P}(u, v) \) given by
\[
\mathcal{Y} : \quad y^2 z = x^3 + F(u, v) x z^2 + G(u, v) z^3,
\]
where we have set \( F = h_{(d_0, d_\infty)}^*(f) \) and \( G = h_{(d_0, d_\infty)}^*(g) \). We have the following:

**Lemma 3.1.** The very general K3 surface \( \mathcal{Y} \) in Equation (3.4) has 24 singular fibers of type \( I_1 \) and a Mordell-Weil group \( \operatorname{MW}(\mathcal{Y}, \pi_Y) \cong E_8(2) \).

**Proof.** In the formula for the height pairing on an elliptically fibered surface with section all summands – the holomorphic Euler characteristic, the intersection number, the number of singular fibers – double when pulling back the Weierstrass equation via Equation (3.3).

In particular, the very general K3 surface \( \mathcal{Y} \) has \( \operatorname{NS}(\mathcal{Y}) \cong H \oplus E_8(-2) \). Conversely, it is known that every element in \( \mathcal{M}_{H \oplus E_8(-2)} \) is represented by a family of double covers of the plane branched over the union of cubics, i.e., double covers of the rational elliptic surface obtained by the minimal resolution of a reducible double-sixtix blown-up in nine points [27, Sec. 9.1]. To find the moduli from the Weierstrass model in Equation (3.4), we observe that \( f \) and \( g \) depend on \( 5 + 7 = 12 \) parameters, and there are two additional parameters \( d_0, d_\infty \). Using a transformation of the type
\[
\left( (u : v), (x : y : z) \right) \mapsto \left( (\lambda u : \lambda v), (\lambda^2 x : \lambda^3 y : z) \right)
\]
with \( \lambda \in \mathbb{C}^* \) and the automorphism group \( \operatorname{PGL}(2, \mathbb{C}) \) of \( \mathbb{P}^1 \) we obtain \( 12 + 2 - 1 - 3 = 10 \) as the number of moduli. In fact, the following was proved in [8, Prop. 4.17]:

**Proposition 3.2.** The K3 surfaces in Equation (3.4) form the 10-dimensional moduli space \( \mathcal{M}_{H \oplus E_8(-2)} \).

Conversely, the rational elliptic surface \( \mathcal{R} \) is the minimal resolution of the quotient \( \mathcal{Y}/(k_Y) \) where \( k_Y \) is the antisymplectic involution preserving the Jacobian elliptic fibration induced by the deck transformation \( [u : v] \mapsto [-u : v] \). A Nikulin involution \( j_Y = k_Y \circ (-1) \) is given by the composition of \( k_Y \) with the (antisymplectic) hyperelliptic involution acting as \( (-1) \) on the fibers. The minimal resolution of \( \mathcal{Y}/(j_Y) \) then yields another Jacobian elliptic K3 surface \( \pi_{\widetilde{Y}} : \widetilde{\mathcal{Y}} \to \mathbb{P}^1 = \mathbb{P}(U, V) \) with the Weierstrass model
\[
\widetilde{\mathcal{Y}} : \quad Y^2 Z = X^3 + (U - d_0 V)^2 (d_\infty U - V)^2 f(U, V) X Z^2
+ (U - d_0 V)^3 (d_\infty U - V) g(U, V) Z^3.
\]
We have the following:

**Lemma 3.3.** The very general K3 surface \( \widetilde{\mathcal{Y}} \) in Equation (3.6) has 2 singular fibers of type \( I_0^* \), 12 singular fibers of type \( I_1 \), and a trivial Mordell-Weil group.

Conversely, starting with a K3 surface \( \widetilde{\mathcal{Y}} \) with the given singular fibers and Mordell-Weil group, a double cover is obtained as the pull-pack via the degree-2 rational map branched over an obvious even eight, namely the even eight given by the non-central components of the two reducible fibers of type \( D_4 \). We have the following:
Proposition 3.4. The K3 surfaces in Equation (3.6) form the 10-dimensional moduli space $\mathcal{M}_{H \oplus D_4(-1)^{g_2}}$.

The results of Proposition 3.2 and Proposition 3.4 also hold (for suitably modified lattices) if we restrict $\mathcal{R}$ to a rational elliptic surface with reducible fibers given in the classification by Oguiso and Shioda in [68]. Geometrically, these rational elliptic surfaces arise if the base points of the pencil in Equation (3.1) are not distinct. We will focus on the cases when the Mordell-Weil group of the rational elliptic surface has half the rank and is isomorphic to $D_4^\vee$ (up to torsion). That is, we will consider cases (9) or (13) in the Oguiso-Shioda classification [68], i.e., the rational elliptic surfaces with the singular fibers and Mordell-Weil groups given by

$$I_0^* + 6I_1, \ MW(\mathcal{R}, \pi_{\mathcal{R}}) \cong D_4^\vee, \quad 4I_2 + 4I_1, \ MW(\mathcal{R}, \pi_{\mathcal{R}}) \cong \mathbb{Z}/2\mathbb{Z} \oplus D_4^\vee,$$

respectively. We have the following:

Proposition 3.5.

(1) Consider the rational elliptic surfaces $\mathcal{R}$ associated with labels (9) and (13) in the Oguiso-Shioda classification. Let $\mathcal{Y}$ be the K3 surfaces of Equation (3.7) obtained via a base-change of order two from $\mathcal{R}$. Then, in the very general case, the Neron-Severi lattice $\text{NS}(\mathcal{Y})$ is $H \oplus K_0(-1)$ or $H \oplus N_0(-1)$, respectively. Here, $K_0$ is the positive definite lattice of rank 12, determinant $2^6$, and the Gram matrix

$$
\begin{pmatrix}
4 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\
\end{pmatrix},
$$

and $N_0$ is the positive definite lattice of rank 12, determinant $2^8$, and the Gram matrix

$$
\begin{pmatrix}
4 & 1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & -2 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\
1 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & -1 & -1 \\
1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
-2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

In particular the associated moduli spaces for $\mathcal{Y}$ are 6-dimensional, given by $\mathcal{M}_L$ with $L = H \oplus K_0(-1)$ and $L = H \oplus N_0(-1)$, respectively.

(2) The 6-dimensional moduli spaces $\mathcal{M}_L$ for $L = H \oplus D_4(-1)^{g_3}$ and $L = H \oplus D_8(-1) \oplus A_1(-1)^{g_4}$ are given by K3 surfaces $\tilde{\mathcal{Y}}$ in Equation (3.6) obtained as
the minimal resolution of $\mathcal{Y}/\langle j_\mathcal{Y} \rangle$ where $j_\mathcal{Y}$ is the Nikulin involution constructed above and $\mathcal{Y}$ are chosen from the first and second family in (1), respectively. These cases are summarized in Table 1.

**Proof.** The singular fibers and Mordell-Weil groups of the K3 surfaces $\mathcal{Y}$ are $2I_0^+ + 12I_1$ and $D_4^+(2)$ (which is isometric to $D_4$) and $8I_2 + 8I_1$ and $\mathbb{Z}/2\mathbb{Z} \oplus D_4^+(2)$, respectively. Thus, $\text{NS}(\mathcal{Y})$ have determinant $2^6$ and $2^8$, respectively. In the first case, $\text{NS}(\mathcal{Y})$ is isomorphic to the polarizing lattice of the surface $\mathcal{X}'$ given in Table 3, which, in turn, also admits an elliptic fibration with section, singular fibers of type $4I_4+8I_1$, and Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$. This follows from the fact that $\mathcal{G}$ in Table 3 admits sections for the elliptic fibrations induced by the projection both onto $\mathbb{P}(u, v)$ and $\mathbb{P}(s, t)$. It follows that $\text{NS}(\mathcal{Y})$ is of the form $H \oplus K_0(-1)$ and the overlattice of $H \oplus A_3(-1)^{\oplus 4}$ associated with $\bar{v} = (0, 0|1, 0, 1|\ldots|\ldots|1, 0, 1)/2$. The latter is equivalent to the class of the 2-torsion section. A Gram matrix computation yields (3.8).

In the second case, one works in the context of Figure 1, with explicit equations for all surfaces involved given in Table 2. The K3 surface $\mathcal{G}$ is described as a double cover of $\mathbb{P}(s, t) \times \mathbb{P}(u, v)$, with both projections inducing elliptic fibrations with section. The relative Jacobians of the two elliptic fibrations are $\mathcal{Y}$ and $\mathcal{X}'$, respectively. As the two elliptic fibrations have sections and the relative Jacobian map is birational, one obtains that K3 surfaces $\mathcal{G}$, $\mathcal{Y}$ and $\mathcal{X}'$ are in fact mutually isomorphic. In particular, one has $\text{NS}(\mathcal{X}') \cong \text{NS}(\mathcal{G}) \cong \text{NS}(\mathcal{Y})$. By construction, $\mathcal{X}$ and $\mathcal{F}$ are isomorphic surfaces and, in particular, one has $\text{NS}(\mathcal{F}) \cong \text{NS}(\mathcal{X})$. In addition, $\mathcal{X}$ and $\mathcal{X}'$ both carry Jacobian elliptic fibrations with singular fiber type $8I_1 + 8I_2$ and Mordell-Weil group $\mathbb{Z}/2\mathbb{Z} \oplus D_4$. This implies $\text{NS}(\mathcal{X}') \cong \text{NS}(\mathcal{X})$. One obtains therefore that surfaces $\mathcal{X}$, $\mathcal{X}'$, $\mathcal{G}$, $\mathcal{Y}$, $\mathcal{F}$ carry isometric Neron-Severi lattices.

Using the Jacobian elliptic fibration on $\mathcal{X}$, with singular fiber type $8I_1 + 8I_2$ and Mordell-Weil group $\mathbb{Z}/2\mathbb{Z} \oplus D_4$, one may construct the Néron-Severi lattice explicitly. Note that the zero section $S_1$ and 2-torsion section $S_1$, which span $\text{MW}(\mathcal{X}, \pi_{\mathcal{X}})_{\text{tor}}$, intersect the neutral and the non-neutral components of each reducible fiber of type $A_1$, respectively. Moreover, via the formulas for $\mathcal{X}$ and $\mathcal{F}$ in Table 2, one obtains explicitly four disjoint non-torsion sections $R_i$ with $i = 1, \ldots, 4$, by cutting out:

$$X = \gamma(s^2, t^2)U_i, \quad Z = V_i, \quad Y^2 = \frac{U_i}{4c(U_i, V_i)}\gamma(s^2, t^2)^2(2c(U_i, V_i)s^2 + a(U_i, V_i)t^2)^2,$$

where $[U_i : V_i]$ are solutions of $a(U_i, V_i)^2 - 4c(U_i, V_i)d(U_i, V_i) = 0$. We note that each section $R_i$ intersects the neutral components of the fibers over $\delta(s^2, t^2) = 0$ and the non-neutral components over $\gamma(s^2, t^2) = 0$. One can similarly construct a new set of sections $\bar{R}_i$ for $i = 1, \ldots, 4$, by choosing an alternate factorization $\gamma(s^2, t^2)\delta(s^2, t^2) = \bar{\gamma}(s^2, t^2)\bar{\delta}(s^2, t^2)$. In this context, note that if $\bar{\gamma}$ (and $\bar{\delta}$) is chosen such that $\bar{\gamma}(s^2, t^2)$ has exactly one pair of roots in common with $\gamma(s^2, t^2)$ and one pair of roots in common with $\delta(s^2, t^2)$, then the four sections $(R_1, R_2, R_3, \bar{R}_4)$ generate the non-torsion part of the Mordell-Weil group and their height-pairing matrix is the Cartan matrix for $D_4$.

The classes

$$\langle F, F + S_0, a_1, \ldots, a_8, R_1 - F - 2S_0, R_2 - F - 2S_0, R_3 - F - 2S_0, \bar{R}_4 - F - 2S_0 \rangle$$
span then a lattice $H \oplus \tilde{N}_0$, where $a_j$ are generators for the non-neutral components of the fibers of type $A_1$ and $F$ is the fiber class. It follows that $\text{NS}(\mathcal{X})$ is the overlattice of $H \oplus \tilde{N}_0$ associated with $\tilde{v} = \langle 0, 0| 1, \ldots, 1| 0, 0, 0 \rangle / 2$. The latter is equivalent to the class of $S_1$. One obtains that $\text{NS}(\mathcal{X})$ is of the form $H \oplus N_0(-1)$, and a Gram matrix computation yields (3.9). □

The importance of the subfamilies above stems from their geometric interpretation: the two families in Proposition 3.5 (1) can be understood as imposing the existence of another compatible involution. We consider the two cases when this additional involution, compatible with the elliptic fibration, is antisymplectic or symplectic.

Let us assume that the Weierstrass model in Equation (3.4) admits a second commuting antisymplectic involution preserving the Jacobian elliptic fibration. We use a transformation in $\text{PGL}(2, \mathbb{C})$ to have $d_0 = d_{\infty} = 0$. We can also assume that the second antisymplectic involution is induced by the deck transformation $[u : v] \leftrightarrow [v : u]$. Thus, without loss of generality, a family of K3 surfaces $\mathcal{Y}'$ admitting two commuting antisymplectic involutions preserving the Jacobian elliptic fibration is given by the Weierstrass model

$$Y' : \quad Y'^2 = X^3 + f((\bar{u}^2 - \bar{v}^2)^2, (\bar{u}^2 + \bar{v}^2)^2)XZ^2 + g((\bar{u}^2 - \bar{v}^2)^2, (\bar{u}^2 + \bar{v}^2)^2)Z^3,$$

where $f$ and $g$ are homogenous of degree 2 and 3, respectively, and we changed the variables from $(u, v)$ to $(\bar{u}, \bar{v})$ to avoid any conflict of notation. To find the moduli from the Weierstrass model in Equation (3.11), we observe that $f$ and $g$ depend on $3 + 4 = 7$ parameters. With only the freedom of a transformation as in Equation (3.5) remaining, we obtain $7 - 1 = 6$ as the number of moduli in Equation (3.11). Thus, we have the following:

**Proposition 3.6.** The K3 surfaces in Equation (3.11) form a 6-dimensional subvariety of $\mathcal{M}_{H\oplus E_8(-2)}$ whose general member admits an additional antisymplectic involution induced by an involution on the base curve.

The minimal resolution of $\mathcal{Y}'/\langle j_1 \rangle$ (where $k_1$ is induced by $[\bar{u} : \bar{v}] \mapsto [\bar{u} : -\bar{v}]$ and $j_1 = k_1 \circ (-1)$) yields the Jacobian elliptic K3 surface $\pi\mathcal{Y} : \mathcal{Y} \to \mathbb{P}^1 = \mathbb{P}(u, v)$ with the Weierstrass model

$$Y : \quad y^2z = x^3 + (u^2 - v^2)^2f(u^2, v^2)xz^2 + (u^2 - v^2)^3g(u^2, v^2)z^3.$$

The image of the (second) involution $k_2$ on $\mathcal{Y}'$ (where $k_2$ is induced by $[\bar{u} : \bar{v}] \mapsto [\bar{u} : \bar{u}]$) is an antisymplectic involution $k_\mathcal{Y}$ preserving the Jacobian elliptic fibration on $\mathcal{Y}$ and is induced by $[u : v] \mapsto [-u : v]$. For the Nikulin involution $j_\mathcal{Y} = k_\mathcal{Y} \circ (-1)$ (which is the image of $j_2 = k_2 \circ (-1)$), the minimal resolution of $\mathcal{Y}/\langle j_\mathcal{Y} \rangle$ yields another Jacobian
elliptic K3 surface \( \pi_Y : \widetilde{Y} \rightarrow \mathbb{P}^1 = \mathbb{P}(U, V) \) with the Weierstrass model

\[ Y^2 Z = X^3 + U^2 V^2(U - V)^2 f(U, V) XZ^2 + U^3 V^3(U - V)^3 g(U, V) Z^3. \]  

Similarly, one checks that the minimal resolution of \( \mathcal{Y}/(k_Y) \) yields the rational elliptic surface with the Weierstrass model

\[ Y^2 Z = X^3 + (U - V)^2 f(U, V) XZ^2 + (U - V)^3 g(U, V) Z^3, \]

which realizes the first case in (3.7). Thus, we have the following:

**Proposition 3.7.** The K3 surfaces in Equation (3.12) and (3.13) are the minimal resolutions of \( \mathcal{Y}/(j_1) \) and \( \mathcal{Y}/(j_1, j_2) \) and form the 6-dimensional moduli spaces \( \mathcal{M}_L \) for \( L = H \oplus K_0(-1) \) (with the Gram matrix of \( K_0 \) given in Equation (3.8)) and \( L = H \oplus D_4(-1)^{\oplus 3} \), respectively.

As for the second case in (3.7), we remark that the existence of a 2-torsion section on \( \mathcal{R} \) guarantees the existence of a 2-torsion section on \( \mathcal{Y} \) in Equation (3.4). In turn, the 2-torsion section yields an additional *symplectic involution* on \( \mathcal{Y} \) known as van Geemen-Sarti involution – we will explain the construction of a van Geemen-Sarti involution in detail in Section 3.3.

### 3.3. K3 surfaces with van Geemen-Sarti involution.

When a K3 surface \( X \) admits a Jacobian elliptic fibration with a 2-torsion section, then it admits a special Nikulin involution, called a *van Geemen-Sarti involution*; see [73]. When quotienting by this involution, denoted by \( j_X \), and blowing up the fixed locus, one obtains a new K3 surface \( X' \) together with a rational double cover map \( \Phi_X : X \rightarrow X' \). In general, a van Geemen-Sarti involution \( j_X \) does not determine a Hodge isometry between the transcendental lattices \( T_X(2) \) and \( T_{X'} \). Van Geemen-Sarti involutions always appear as fiber-wise translation by 2-torsion in a suitable Jacobian elliptic fibration \( \pi_X : X \rightarrow \mathbb{P}^1 \) which we call the *alternate fibration*; see [23] for the nomenclature. In the Mordell-Weil group \( \text{MW}(X, \pi_X) \) with identity element \( \sigma_X \), let \( \tau_X \) denote the 2-torsion section such that translation by \( \tau_X \) is the involution \( j_X \). Moreover, the construction induces a Jacobian elliptic fibration \( \pi_{X'} : X' \rightarrow \mathbb{P}^1 \) on \( X' \) which also admits a 2-torsion section. Thus, we obtain Figure 2. We will refer to the construction of Figure 2 as *van Geemen-Sarti-Nikulin duality*. If the Mordell-Weil group contains \((\mathbb{Z}/2\mathbb{Z})^2\), then

| \( \rho \) | surface | polar. lattice \( L \) | sing. fibers | Mordell-Weil grp. |
|---|---|---|---|---|
| 10 | K3 | \( H \oplus E_8(-2) \) | 24I \(_1\) | \( E_8(2) \) |
| 10 | K3 | \( H \oplus D_4(-1)^{\oplus 2} \) | \( 2I_0^* + 12I_1 \) | \{\} |
| RES: (1) | \( \mathcal{R} \) | | | \( E_8 \) |
| 14 | K3 | \( H \oplus K_0(-1) \) | \( 2I_0^* + 12I_1 \) | \( D_4 \) |
| 14 | K3 | \( H \oplus D_4(-1)^{\oplus 3} \) | \( 3I_0^* + 6I_1 \) | \{\} |
| RES: (9) | \( \mathcal{R} \) | | | \( D_4 \) |
| 14 | K3 | \( H \oplus N_0(-1) \) | \( 8I_2 + 8I_1 \) | \( \mathbb{Z}/2\mathbb{Z} \) |
| RES: (13) | \( \mathcal{R} \) | | | \( \mathbb{Z}/2\mathbb{Z} \) |

**Table 1.** K3 lattices and Jacobian elliptic fibrations in Prop. 3.2-3.4
the fiberwise translations by the different 2-torsion sections constitute commuting van Geemen-Sarti involutions compatible with the given Jacobian elliptic fibration, or, commuting van Geemen-Sarti involutions for short.

The K3 surface $X$ has the Weierstrass equation

\[(3.15) \quad X : Y^2 Z = X \left( X^2 - A(s, t) X Z + B(s, t) Z^2 \right), \]

where $[s : t] \in \mathbb{P}^1$, $[X : Y : Z] \in \mathbb{P}^2$, $A(s, t)$ and $B(s, t)$ are homogeneous polynomials of degree four and eight, respectively, and the sections $\sigma_X, \tau_X$ are given by the section at infinity and $[X : Y : Z] = [0 : 0 : 1]$. To find the moduli from the Weierstrass model in Equation (3.15), we note that $A$ and $B$ depend on $5 + 9 = 14$ parameters. Using transformations of the type $(X, Y) \mapsto (\lambda^2 X, \lambda^3 Y)$ with $\lambda \in \mathbb{C} \times$ and the automorphism group $\text{PGL}(2, \mathbb{C})$ of $\mathbb{P}^1$ we get $14 - 1 - 3 = 10$ moduli. Since we have identified coordinates according to

\[\left( (s, t), (X, Y, Z) \right) \sim \left( (\lambda s, \lambda t), (\lambda^4 X, \lambda^6 Y, Z) \right),\]

Equation (3.15) defines a double cover of the Hirzebruch surface $F_4$. Similarly, the K3 surface $X'$ has the Weierstrass model

\[(3.16) \quad X' : y^2 z = x \left( x^2 + 2A(s, t) x z + (A(s, t)^2 - 4B(s, t))^2 \right).\]

Explicit equations for the rational maps $\Phi_X$ and $\Phi_{X'}$ in Figure 2 were given in [23]. The discriminant functions of the two elliptic fibrations are as follows:

\[(3.17) \quad \Delta_X = B(s, t)^2 \left( A(s, t)^2 - 4B(s, t) \right), \quad \Delta_{X'} = 16B(s, t) \left( A(s, t)^2 - 4B(s, t) \right)^2.\]

**Lemma 3.8.** The very general K3 surface $X$ in Equation (3.15) and $X'$ in Equation (3.16) have 8 fibers of type $I_1$ over the zeroes of $A^2 - 4B = 0$ (resp. $B = 0$) and 8 fibers of type $I_2$ over the zeroes of $B = 0$ (resp. $A^2 - 4B = 0$) with $\text{MW}(X, \pi_X) \cong \text{MW}(X', \pi_{X'}) \cong \mathbb{Z}/2\mathbb{Z}$.

In particular, the very general Jacobian elliptic K3 surfaces $X$ in Equation (3.15) and $X'$ in Equation (3.16) have

\[(3.18) \quad \text{NS}(X) \cong \text{NS}(X') \cong H \oplus N, \quad T_X \cong T_{X'} \cong H^2 \oplus N.\]

Thus, we have the following:

**Proposition 3.9.** The K3 surfaces in Equation (3.15) form the 10-dimensional moduli space $\mathcal{M}_{H \oplus N}$. 
Geometrically, the K3 surface $X'$ is a double cover of $X'$ (via the rational map $\Phi_{X'}$) branched over the even eight that consists of the eight components of the fibers of type $I_2$ that are not met by the zero section $\sigma_{X'}$, i.e., the eight exceptional curves in the corresponding reducible fibers of type $A_1$. The other components, which meet $\sigma_{X'}$ map 2:1 to components in $X$ where they are interchanged and also the two singular points of the fiber are permuted resulting in $I_1$-type fibers on $X$. Similarly, the K3 surface $X'$ is a double cover of $X$ (via the rational map $\Phi_{X'}$). The eight exceptional curves $C_1, C_2, \ldots, C_8$ resulting on $X$ from the singularity resolution form an even-eight configuration $[3]$, i.e.

\begin{equation}
\frac{1}{2} \left( C_1 + C_2 + \cdots + C_8 \right) \in \text{NS}(X).
\end{equation}

This configuration of eight curves whose formal sum is in $2\text{NS}(X)$ is known to determine in full the double cover $\Phi_{X'} : X' \rightarrow X$; see $[3]$. Concretely, each reducible fiber in the Jacobian elliptic fibration $\pi : X \rightarrow \mathbb{P}(s, t)$ consists of two components $F_{i0}$ and $F_{i1}$ such that

\begin{equation}
F_{i0} \circ \sigma_X = 1, \quad F_{i0} \circ \tau_X = 0, \quad F_{i1} \circ \sigma_X = 0, \quad F_{i1} \circ \tau_X = 1,
\end{equation}

for $i = 1, \ldots, 8$. The van Geemen-Sarti involution interchanges $F_{i0}$ and $F_{i1}$ for $i = 1, \ldots, 8$. The eight curves $F_{i1}$ for $i = 1, \ldots, 8$ form an even eight that is not met by the zero section $\sigma_X$; see Figure 3. Thus, the double cover $X'$ obtained from the double cover branched on $F_{11} + \cdots + F_{81}$ is elliptically fibered with section $\sigma_{X'}$ and the two-torsion section $\tau_{X'}$; the two sections form the preimage of $\sigma_X$ under $\Phi_{X'}$.

The results of Proposition 3.9 and Figure 2 also hold for suitable lattice extensions induced by the existence of a second involution:

**Corollary 3.10.**

1. The 6-dimensional moduli space $\mathcal{M}_{(2)@(-2)@D_4(-1)@3} \subset \mathcal{M}_{H@N}$ is given by the K3 surfaces admitting commuting van Geemen-Sarti involutions.

2. The 6-dimensional moduli space $\mathcal{M}_{H@N_0(-1)} \subset \mathcal{M}_{H@N}$ (with the Gram matrix of $N_0$ given in Equation (3.9)) is given by the K3 surfaces admitting a canonical van Geemen-Sarti involution and a commuting antisymplectic involution induced by an involution on the base curve.
induces an involution on the moduli space that the

The key observation is the following isometry of lattices
stood as a relation between certain moduli spaces of lattice polarized K3 surfaces.

action of mirror symmetry on the corresponding F-theory models.

homogenous polynomials of degree four. The symplectic transformation given by

such that then there are now 12 singular fibers of type \(I_2\) on \(X\) and the Mordell-

We argued in Section 2 that the moduli space of the CHL string is the moduli space \(\mathfrak{M}_{H\oplus E_6(-2)}\). We also derived a normal form for these K3 surfaces relating them to certain rational elliptic surfaces. Dolgachev proved that \(L = H(2) \oplus E_8(-2)\) diagonally embeds into \(\Lambda_{K3} = H^3 \oplus E_8(-1)^{\oplus 2}\) such that \(L^\perp = H \oplus H(2) \oplus E_8(-2)\) and a splitting \(L^\perp = H \oplus L\) is admissible [28]. Thus, the moduli space \(\mathfrak{M}_{H(2)\oplus E_8(-2)}\) is a self-mirror with respect to this splitting. However, a second admissible splitting is given by \(L^\perp = H(2) \oplus L'\) with \(L' = H \oplus E_8(-2)\) and yields \(\mathfrak{M}_{H\oplus E_6(-2)}\) as the mirror moduli space.

As we have seen, a polarization by the lattice \(H \oplus N\) is equivalent with the existence of a canonical van Geemen-Sarti involution \(j_X : X \to X\) on the K3 surface \(X\). The construction in Figure 2 induces an involution on the moduli space \(\iota_{VGS} : \mathfrak{M}_{H \oplus N} \to \mathfrak{M}_{H \oplus N}\). In the situation of Equation (3.18) one checks that for \(L = H \oplus N\) and \(L^\perp = H^2 \oplus N\) the splitting \(L^\perp = H \oplus L\) with \(L \cong L\) is admissible in the sense of Dolgachev’s mirror symmetry. Thus, the moduli space of the K3 surfaces given by Equation (3.15) is a self-mirror, i.e., \(\mathfrak{M}_L \cong \mathfrak{M}_l\) and the van Geemen-Sarti-Nikulin duality acts as involution on this moduli space. In fact, the action of \(\iota_{VGS}\) can be identified with the action of mirror symmetry on the corresponding F-theory models.

4. The geometric description of the duality

In this section we will prove that the F-theory/CHL string duality can be understood as a relation between certain moduli spaces of lattice polarized K3 surfaces. The key observation is the following isometry of lattices.

For the K3 surface \(X\) in Equation (3.15) we group the base points of the 8 fibers of type \(I_2\) into two unordered sets of 4 elements. Concretely, we group the fibers of type \(I_2\) over \(B = 0\) into two sets by writing \(B(s,t) = C(s,t)D(s,t)\) where \(C, D\) are homogeneous polynomials of degree four. The symplectic transformation given by

\[
(X,Y,Z) \mapsto (U,V,W) = \left( C(s,t)XZ, \ Z^2, \ C(s,t)Y \right),
\]

yields the equation

\[
\mathcal{F} : \ W^2 = UV \left( C(s,t)U^2 - A(s,t)UV + D(s,t)V^2 \right).
\]
Equation (4.2) puts the (marked) K3 surface into the equivalent form of a *double-quadrics* surface, i.e., a double cover of the Hirzebruch surface $\mathbb{P}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \ni ([s : t], [U : V])$ branched over a curve of bi-degree $(4, 4)$. The ruling given by the projection onto the first factor, denoted by $\pi_F : \mathcal{F} \to \mathbb{P}^1 = \mathbb{P}(s, t)$, recovers the Jacobian elliptic fibration in Equation (3.15).

Changing the ruling one recovers another elliptic fibration $\pi'_F : \mathcal{F} \to \mathbb{P}^1 = \mathbb{P}(U, V)$ without section. Concretely, the elliptic fibration $\pi'_F$ is obtained by re-writing Equation (4.2) in the form

$$
\mathcal{F} : \quad W^2 = UV \left(a_4(U, V) s^4 + a_3(U, V) s^3 t + \cdots + a_0(U, V) t^4\right),
$$

where $a_i(U, V)$ for $0 \leq i \leq 4$ are homogeneous polynomial of degree two, obtained by requiring that we have

$$
C(s, t) U^2 - A(s, t) UV + D(s, t) V^2 = a_4(U, V) s^4 + a_3(U, V) s^3 t + \cdots + a_0(U, V) t^4
$$

for all $s, t, U, V$. The relative Jacobian (fibration) $\text{Jac}^0(\pi_F)$ is the Jacobian elliptic K3 surface $\pi_\mathcal{Y} : \mathcal{Y} \to \mathbb{P}^1 = \mathbb{P}(U, V)$ with the Weierstrass model

$$
\mathcal{Y} : \quad y^2 z = x^3 + U^2 V^2 f(U, V) x z^2 + U^3 V^3 g(U, V) z^3,
$$

where $f, g$ are the homogeneous polynomials of degree four and six, respectively, given by

$$
f = -4a_0a_4 + a_1a_3 - \frac{1}{3}a_2^2, \quad g = -\frac{8}{3}a_0a_2a_4 + a_0a_3^2 + a_1^2a_4 - \frac{1}{3}a_1a_2a_3 + \frac{2}{27}a_2^3.
$$

For a review and details of these classical formulas; see [23, 72, 75]. The very general Jacobian elliptic K3 surface $\mathcal{Y}$ in Equation (4.5) has two singular fibers of type $I_0^*$, twelve singular fibers of type $I_1$, and a trivial Mordell-Weil group.

We also construct a K3 surface $\mathcal{G}$ as the double cover of $\mathcal{X}$ branched over the even eight that consists of eight components in the reducible fibers over the zeros of $B(s, t) = C(s, t) D(s, t) = 0$. Using the same notation as the one used in Figure 3, we denote by $F_{i0}$ and $F_{i1}$ for $i = 1, \ldots, 4$ the components over $C(s, t) = 0$ and by $F_{i0}$ and $F_{i1}$ for $i = 5, \ldots, 8$ the components over $D(s, t) = 0$. The even eight we choose this time is different from the one in (3.19) used in the construction of $\mathcal{X}'$ in Equation (3.16). Over the zeros of $D(s, t) = 0$ we choose the components $\{F_{11}, \ldots, F_{41}\}$ of the reducible fibers that are not met by the zero section $\sigma_\mathcal{X}$. Over the zeros of $C(s, t) = 0$ we choose the components $\{F_{50}, \ldots, F_{80}\}$ that are met by the zero section $\sigma_\mathcal{X}$. In this way, the double cover branched on $F_{11} + \cdots + F_{41} + F_{50} + \cdots + F_{80}$ is an elliptic fibration $\pi_\mathcal{G} : \mathcal{G} \to \mathbb{P}^1 = \mathbb{P}(s, t)$ with the same singular fibers as $\mathcal{X}'$, but it does not admit a section since both $\sigma_\mathcal{X}$ and $\sigma_\mathcal{X}'$ now intersect the branch locus. Thus, the total spaces of $\mathcal{G}$ and $\mathcal{X}'$ are not isomorphic, but the relative Jacobian satisfies $\text{Jac}^0(\pi_\mathcal{G}) = \pi_{\mathcal{X}'}^*$.\(^2\)

Concretely, the new K3 surface is obtained as the minimal resolution of the double-quadrics surface

$$
\mathcal{G} : \quad w^2 = C(s, t) u^4 - A(s, t) u^2 v^2 + D(s, t) v^4,
$$

\(^2\)Here and in the following, several elliptic fibrations are stated to admit no sections, which holds only for the very general K3 surfaces. When the relative Jacobian of this fibration is then considered, we do so by direct computation using the general formulae, displayed as (4.6) applied to (4.4).
which is branched over a curve bi-degree \((4,4)\) (which has genus 9) in \(\mathbb{P}_0 = \mathbb{P}(s,t) \times \mathbb{P}(u,v)\). Moreover, acting upon the even eight \(F_{11} + \cdots + F_{41} + F_{50} + \cdots + F_{80} \in 2\text{NS}(X)\) with the van Geemen-Sarti involution on \(X\) yields the even eight \(F_{10} + \cdots + F_{40} + F_{51} + \cdots + F_{81}\). The double cover branched on the latter yields a genus-one fibration isomorphic to Equation \((4.7)\), as it simply corresponds to the interchange \(u \leftrightarrow v\). We call the two combinations of eight exceptional curves selected from the components of the reducible fibers, namely \(\{F_{10}, \ldots, F_{40}, F_{51}, \ldots, F_{81}\}\) and \(\{F_{11}, \ldots, F_{41}, F_{50}, \ldots, F_{80}\}\), whose induced double covers yield isomorphic genus-one fibrations a marked pair of even eights on \(X\).

A change of ruling yields a second elliptic fibration \(\pi'_G : G \to \mathbb{P}^1 = \mathbb{P}(u,v)\) given by
\[
G : \quad w^2 = a_4(u^2, v^2) s^4 + a_3(u^2, v^2) s^3 t + \cdots + a_0(u^2, v^2) t^4,
\]
where we assumed Equation \((4.4)\). It follows that the relative Jacobian \(\text{Jac}^0(\pi'_G)\) is the Jacobian elliptic K3 surface \(\pi_Y : Y \to \mathbb{P}^1 = \mathbb{P}(u,v)\) with the Weierstrass model
\[
Y : \quad Y^2 Z = X^3 + f(u^2, v^2) X Z^2 + g(u^2, v^2) Z^3,
\]
where the polynomials \(f, g\) are the same polynomials as in Equation \((4.6)\). Thus, the K3 surfaces in Equation \((4.9)\) and \((4.5)\) coincides with K3 surfaces in Equation \((3.4)\) and \((3.6)\), respectively, obtained from the general rational elliptic surface \(R\) in Equation \((3.2)\), normalized so that \(d_0 = 0\) and \(d_\infty = 0\). We have the following:

**Lemma 4.1.** The elliptic fibration \(\pi'_G : G \to \mathbb{P}(u,v)\) admits sections.

**Proof.** The double cover \(G \to X\) is constructed using an even eight that consists of 4 neutral components and 4 non-neutral components. It follows that the double cover has an elliptic fibration with respect to the projection onto \(\mathbb{P}(s,t)\), but the sections of \(X\) do not lift to sections as they intersect the branch locus. The preimages of the 8 fibers of type \(I_1\) then determine in the very general case 16 sections of the fibration with respect to the projection onto \(\mathbb{P}(u,v)\). The sections can be computed explicitly: they are the solutions given by the 8 fixed constants \([s : t]\) obtained from solving \(A(s,t)^2 - 4C(s,t) D(s,t) = 0\) and then solving for \(w\) in the equation \(4C(s,t) w^2 = (2C(s,t) u^2 - A(s,t) v^2)^2\). \(\Box\)

One can also construct a third kind of double covers from \(X\) by grouping the eight fibers of type \(I_2\) over \(B = 0\) into sets of two and six elements:

**Remark 4.2.** If we group the eight fibers of type \(I_2\) over \(B = 0\) into sets of 2 and 6 elements by writing \(B(s,t) = C'(s,t) D'(s,t)\) where \(C', D'\) are homogenous polynomials of degree 2 and 6, respectively, we obtain a K3 double cover as the minimal resolution of the double-sextic surface
\[
w^2 = C'(s,t) u^4 - A(s,t) u^2 + D'(s,t),
\]
which is branched over the strict transform of a sextic in \(\mathbb{F}_1 = \mathbb{P}^2 = \mathbb{P}(s,t,u)\).

Returning to the K3 surface \(G\), note that the Jacobian elliptic fibration on \(Y\) was obtained by taking the relative Jacobian of the elliptic fibration on \(G\) induced by the projection onto \(\mathbb{P}(u,v)\). However, the existence of a section implies that the construction of the relative Jacobian is realized as a birational map. This is a classical
result by Hermite and explicit formulas are given in Section 5.1. It follows that K3 surfaces $\mathcal{G}$ and $\mathcal{Y}$ are isomorphic and carry the same lattice polarization $H \oplus E_8(-2)$. We then have the following:\footnote{For a definition of the multi-section index of a genus-one fibration; see \cite{25}.}

**Proposition 4.3.** Let $\mathcal{G}$ be a K3 surface, as in Equation (4.7). Then, in the very general case, the Néron-Severi lattice $\text{NS}(\mathcal{G})$ is $H \oplus E_8(-2)$. In particular, the elliptic fibration $\pi_{\mathcal{G}} : \mathcal{G} \to \mathbb{P}(s,t)$ admits a bi-section and the elliptic fibration $\pi'_{\mathcal{G}} : \mathcal{G} \to \mathbb{P}(u,v)$ admits a section.

**Proof.** Lemma 4.1 shows that for $\pi'_{\mathcal{G}}$ one has sections whence $\mathcal{G} \cong \mathcal{Y}$. For $\pi_{\mathcal{G}}$ there clearly exist bisectors. Moreover, the Néron-Severi lattice $\text{NS}(\mathcal{G})$ is a sublattice of the lattice $\text{NS}(\mathcal{X}')$ of index $l_1$ such that $\det q_{\text{NS}(\mathcal{G})} = l_1^2 \det q_{\text{NS}(\mathcal{X}')}$ with $l_1 = 2$. The number $l_1$ is the multi-section index of fibration (4.7); see [45, Lemma 2.1]. \hfill \square

Note that the defining Jacobian elliptic fibration on $\mathcal{Y}$ may be identified with the relative Jacobian of the elliptic fibration on $\mathcal{G}$ induced by the projection onto $\mathbb{P}(u,v)$. The presence of sections on the latter implies, however, that the relative Jacobian map is birational. This is a classical result by Hermite and explicit formulas realizing the identification are given in Section 5.1. The two K3 surfaces $\mathcal{G}$ and $\mathcal{Y}$ are isomorphic and carry the same Néron-Severi lattice: $H \oplus E_8(-2)$.

The decomposition $H \oplus E_8(-2)$ in $\text{NS}(\mathcal{G})$ may be described explicitly. To see this, note that the double cover $\mathcal{G} \to \mathcal{X}'$ is branched over a specific even-eight curve configuration. This configuration consists of eight rational curves on $\mathcal{X}'$, four of which are neutral components of four $I_2$ fibers and additional four are non-neutral components of the remaining $I_2$ fibers; see Section 3.3. Furthermore, the preimages of the section and 2-torsion section if the Jacobian elliptic fibration on $\mathcal{X}'$ give two disjoint elliptic curves on $\mathcal{G}$ that are smooth members of an elliptic pencil $[F]$. In the context of this elliptic fibration, one has eight pairs of sections $L_i, R_i$, $1 \leq i \leq 8$, obtained as preimages of the eight $I_1$ fibers on $\mathcal{X}$. Consider $R_1$ as the zero-section. Then $(R_1, F)$ is isomorphic to $H$. Let us examine the orthogonal complement $(R_1, F)^\perp$ in $\text{NS}(\mathcal{G})$. Denote $B_i = L_i - R_i - 4F$ and $B_i = L_i - R_i - 2F$, $2 \leq i \leq 8$. One obtains that $(1/2) \sum B_i \in (R_1, F)^\perp$ and furthermore $(R_1, F)^\perp = (B_1, B_2, \ldots, B_8, (1/2) \sum B_i)$. Upon checking the intersection matrix, one obtains that $(R_1, F)^\perp = Q(-2)$ where $Q$ is a unimodular, even positive-definite lattice of rank eight. It follows that $(R_1, F)^\perp$ is isomorphic to $E_8(-2)$.

**Remark 4.4.** The situation of Proposition 4.3 is depicted in Figure 4. The rational maps $\mathcal{F} \to \mathcal{Y}$ and $\mathcal{G} \to \mathcal{X}'$ are reflected by the lattice-theoretic identities $H \oplus N \cong H(2) \oplus D_4(-1)^{\otimes 2}$ and $H(2) \oplus N \cong H \oplus E_8(-2)$, respectively.

As explained before, the K3 surface $\mathcal{F}$ was obtained as double covers of $\mathbb{F}_0$ branched over the vanishing divisor of a section in the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{F}_0}(4, 4)$. This branch locus is reducible: it decomposes as $L_0 \cup L_\infty \cup E$ into two lines $L_0, L_\infty$ and a curve $C$. The latter is a curve of genus three in the linear system $|\mathcal{L}|$. Let $\mathcal{W}$ be the moduli space of K3 surfaces that can be obtained as the minimal resolution of double-quads over $\mathbb{F}_0$ whose branch locus is the union of divisors of type $(1,0)$, $(1,0)$, and
A comparison with the discriminant of \( X(4.13) \) \( \Delta \) with the discriminant \( (4.12) \) compactifications considered in \( t \). It turns out that the relation between every marked pair of even eights on a K3 surface

Proof. Every marked pair of even eights on a K3 surface \( X \in \mathcal{M}_H \otimes \mathbb{N} \) allows to construct a K3 surface \( G \in \mathcal{W} \) using Equation (4.7). Conversely, for every K3 surface \( G \in \mathcal{W} \), Equation (3.15) then determines a double cover of the Hirzebruch surface \( \mathbb{F}_4 \) and an element in \( \mathcal{W}_H \otimes \mathbb{N} \). The K3 surfaces \( G \in \mathcal{W} \) obtained as the double coverings of an element in \( \mathcal{W}_H \otimes \mathbb{N} \) branched along all possible marked pairs of even eights all project on the same element \( X \).

We also have the equation for a two-parameter family of double-quadrics surfaces given by

\[
\mathcal{X}_{(d_0, d_\infty)}: \quad W^2 = \left( U - d_0 V \right) \left( d_\infty U - V \right) \left( C(s, t) U^2 - A(s, t) UV + D(s, t) V^2 \right),
\]

where the two fibers of type \( I_0^* \) are now at \( [U : V] = [d_0 : 1] \) and \( [1 : d_\infty] \). A base transform, similar to Equation (4.1), converts Equation (4.11) into a double cover of the Hirzebruch surface \( \mathbb{F}_4 \) by forgetting the marking, given by

\[
\mathcal{X}_{(d_0, d_\infty)}: \quad Y^2 Z = X^3 - \left( 2d_0 C + 2d_\infty D - (1 + d_0 d_\infty) A \right) X^2 Z \\
+ \left( C + d_\infty^2 D - d_\infty A \right) \left( d_0^2 C + D - d_0 A \right) XZ^2,
\]

with the discriminant

\[
\Delta_{\mathcal{X}_{(d_0, d_\infty)}} = (d_0 d_\infty - 1)^2 \left( C + d_\infty^2 D - d_\infty A \right)^2 \left( d_0^2 C + D - d_0 A \right)^2 (A^2 - 4CD).
\]

A comparison with the discriminant of \( \mathcal{X} \) in Equation (3.17) shows that only the location of the eight fibers of type \( I_2 \) has changed whereas the location of the fibers of type \( I_1 \) has remained fixed. Moreover, for \( d_0 = d_\infty = 0 \) the surfaces \( \mathcal{F}_{(d_0, d_\infty)} \) and \( \mathcal{F} \) coincide. It turns out that the relation between \( \mathcal{F} \) and \( \mathcal{F}_{(d_0, d_\infty)} \) is symmetric. That

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Geometric construction of the F-theory/CHL string duality}
\end{figure}
is, replacing polynomials as follows

\[(4.14) \quad \iota_{(d_0, d_{\infty})} : \begin{pmatrix} A \\ C \\ D \end{pmatrix} \mapsto \begin{pmatrix} A' \\ C' \\ D' \end{pmatrix} = \begin{pmatrix} 2d_0C + 2d_{\infty}D - (1 + d_0d_{\infty})A \\ C + d_{\infty}^2D - d_{\infty}A \\ d_{\infty}^2C + D - d_0A \end{pmatrix},\]

maps \(F\) in Equation (4.2) to \(F_{(d_0, d_{\infty})}\) and vice versa (up to a trivial rescaling of the equation defined over \(\mathbb{Q}[d_0d_{\infty} - 1]\)). Thus, for \(d_0d_{\infty} \neq 1\) Equation (4.14) defines a two-parameter family of involutions \(\iota_{(d_0, d_{\infty})}\) on the moduli space \(\mathcal{M}\).

4.1. Connection to the Brauer group. As explained above, the Néron-Severi lattice \(\text{NS}(F) \cong H \oplus N \cong H(2) \oplus D_4(-1)^{\otimes 2}\) is a sublattice of the lattice \(\text{NS}(\tilde{Y}) = H \oplus D_4(-1)^{\otimes 2}\) of index two. We now consider the problem of reconstructing \(F\) from \(\tilde{Y}\), that is, constructing a polarized Hodge substructure of the transcendental lattice \(T_{\tilde{Y}}\) that is the transcendental lattice of another K3 surface. One checks that

\[(4.15) \quad \Gamma := T_{\tilde{Y}} = H \oplus H \oplus D_4(-1)^{\otimes 2}.\]

Recall that an element \(\theta\) of order \(n\) in the Brauer group \(\text{Br}(\tilde{Y}) = \text{Hom}(T_{\tilde{Y}}, \mathbb{Q}/\mathbb{Z})\) is a surjective homomorphism \(\theta : T_{\tilde{Y}} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}\) and defines a sublattice of index \(n\) of \(\Gamma\). This sublattice is given by \(T_{(\theta)} = \ker(\theta : \Gamma \rightarrow \mathbb{Z}/n\mathbb{Z})\) where \((\theta)\) denotes the cyclic subgroup generated by \(\theta\). Conversely, any sublattice \(\Gamma'\) of index \(n\) in \(\Gamma\) with cyclic quotient group \(\Gamma/\Gamma'\) determines a subgroup of order \(n\) in \(\text{Br}(\tilde{Y})\). If there exists a primitive lattice embedding of \(T_{(\theta)}\) into the K3 lattice \(\Lambda_{K3}\), then the Hodge structure \(T_{(\theta)}\) is guaranteed to be the transcendental lattice of another K3 surface \(F\). Since the lattice embedding is in general not unique, neither is the surface \(F\); two K3 surfaces with isomorphic transcendental lattice \(T_{(\theta)}\) are Fourier-Mukai partners.

In our situation, we are considering elements \(\theta \in \text{Br}(\tilde{Y})_2\) of order two such that the sublattice \(T_{(\theta)} = \ker(\theta : \Gamma \rightarrow \mathbb{Z}/2\mathbb{Z})\) has index two in \(\Gamma\). The existence of a primitive lattice embedding \(T_{(\theta)} \hookrightarrow \Lambda_{K3}\) is a-priori guaranteed since the K3 surface \(\tilde{Y}\) already admits an elliptic fibration with a section; see [72].

Using the arguments of [72] it follows that there are isomorphism classes for elements in the Brauer group \(\text{Br}(\tilde{Y})_2\), with representatives \(\Gamma_{2,c} = \Lambda_{2,c} \oplus H \oplus D_4(-1)^{\otimes 2}\) for \(c = 0, 1\). Here, we are using the notation \(\Lambda_{b,c} = (b, 2c)\) for the indefinite lattice of rank two such that \(\Lambda_{2,0} = H(2)\) and \(\Lambda_{2,1} = (2) \oplus (-2)\). Notice that there are more sublattices of \(\Gamma\) of index 2. However, we are only considering elements with the isomorphism classes \(\Gamma_{2,c}\). One can then construct projective models for the corresponding K3 surfaces \(F_{2,c}\) with the Néron-Severi lattice \(\Lambda_{2,c} \oplus D_4(-1)^{\otimes 2}\). The construction of the projective models is based on the well-developed theory of linear series on K3 surfaces; see [69]. It follows that the a projective model for the K3 surface \(F_{2,0}\) is always that of a double cover of \(\mathbb{P}_0\) branched over a curve of bi-degree \((4,4)\), necessarily with two elliptic fibrations, \(\pi_F\) and \(\pi'_F\), associated with the two elliptic pencils; see [72, Sec. 5.5]. Similarly, \(F_{2,1}\) is obtained as the desingularization of a double-sextic surface; see [72, Sec. 5.6].

From Ogg-Shafarevich theory it follows that \(F_{2,c}\) with \(c = 0, 1\) admits a genus-one fibration \(\pi'_{F_{2,c}} : F_{2,c} \rightarrow \mathbb{P}^1\) with a bisection such that the relative Jacobian fibration recovers \(\pi_Y : Y \rightarrow \mathbb{P}^1\). (For \(F_{2,0}\) we always take this elliptic fibration to coincide
with the elliptic fibration $\pi'_F$ introduced before.) Moreover, every pair of elements $\pm \theta \in \text{Br}(\widetilde{\mathcal{Y}})_2$ uniquely determines such a genus-one fibration on $\mathcal{F}_{2,0}$; see [72, Sec. 4.1]. We then have the following:

**Proposition 4.6.** $\mathcal{W}$ is a finite covering space of $\mathcal{M}_{H \oplus D_4(-1)^{\oplus 2}}$. In particular, for a K3 surface $\widetilde{\mathcal{Y}} \in \mathcal{M}_{H \oplus D_4(-1)^{\oplus 2}}$ and every pair $\pm \theta \in \text{Br}(\widetilde{\mathcal{Y}})_2$ in the isomorphism class $\Gamma_{2,0}$, the Hodge substructure $T(\theta)$ matches the one on the transcendental lattice of a K3 surface $\mathcal{F}$ in Equation (4.2).

**Proof.** Every pair of elements $\pm \theta \in \text{Br}(\widetilde{\mathcal{Y}})_2$ in the isomorphism class $\Gamma_{2,0}$ determines an equation for $\mathcal{F}_{2,0}$ as a double cover of $\mathbb{P}_0$ branched over a curve of bi-degree $(4, 4)$, with a genus-one fibration $\pi'_{\mathcal{F}_{2,0}} : \mathcal{F}_{2,0} \to \mathbb{P}^1 = \mathbb{P}(u, v)$ with multi-section index 2 such that the relative Jacobian fibration is $\pi_{\mathcal{F}} : \widetilde{\mathcal{Y}} \to \mathbb{P}(u, v)$.

Because of the lattice isomorphism $H(2) \oplus D_4(-1)^{\oplus 2} \cong H \oplus N$, the second elliptic fibration $\pi_{\mathcal{F}_{2,0}}$ must admit a section and a 2-torsion section. In fact, it was proved in [15] that there is a unique Jacobian elliptic fibration on the K3 surface with Néron-Severi lattice $H \oplus N$, i.e., the Jacobian elliptic fibration with singular fibers $8I_2 + 8I_1$ and Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$. A fractional linear transformation on $\mathbb{P}(U, V)$ then allows us to bring $\mathcal{F}_{2,0}$ in the form of Equation (4.2). Conversely, taking the relative Jacobian $\text{Jac}^0(\pi'_{\mathcal{F}_{2,0}})$ we always obtain a Jacobian elliptic K3 surface $\widetilde{\mathcal{Y}} \in \mathcal{M}_{H \oplus D_4(-1)^{\oplus 2}}$ such that K3 surfaces $\mathcal{F}_{2,0} \in \mathcal{W}$ obtained from different factorization $B(s, t) = C(s, t)D(s, t)$ yield the same Jacobian elliptic K3 surface $\widetilde{\mathcal{Y}}$. □

We have the following result:

**Theorem 4.7.** In Figure 4 we have the following:

1. every K3 surface $\mathcal{X} \in \mathcal{M}_{H \oplus N}$ has an algebraic correspondence with an element $\mathcal{Y} \in \mathcal{M}_{H \oplus E_8(-2)}$ and vice versa,

2. every K3 surface $\mathcal{X} \in \mathcal{M}_{H \oplus N}$ has an algebraic correspondence with an element $\widetilde{\mathcal{Y}} \in \mathcal{M}_{H \oplus D_4(-1)^{\oplus 2}}$ and vice versa.

A correspondence is a double-quadric $\mathcal{G} \in \mathcal{W}$ in (1) and $\mathcal{F} \in \mathcal{W}$ in (2).

**Proof.** The proof follows from Propositions 4.3, 4.5, 4.6. □

The discrete choices involved in Theorem 4.7 are as follows:

**Remark 4.8.** One has:

1. for every $\mathcal{X} \in \mathcal{M}_{H \oplus N}$, an element $\mathcal{G} \in \mathcal{M}_{H \oplus E_8(-2)}$ is determined by a marked pair of even eight configurations $F_{i_1} + \cdots + F_{i_4} + F_{j_5} + \cdots + F_{j_8} \in 2\text{NS}(\mathcal{X})$, for $(i, j) = (0, 1), (1, 0)$,

2. for every $\widetilde{\mathcal{Y}} \in \mathcal{M}_{H \oplus D_4(-1)^{\oplus 2}}$, an element $\mathcal{F} \in \mathcal{M}_{H \oplus N}$ is determined by a pair $\pm \theta \in \text{Br}(\widetilde{\mathcal{Y}})_2$ in the isomorphism class $\Gamma_{2,0}$.

5. **A natural subspace of dimension six**

The construction of the K3 surface $\mathcal{Y}$ in Equation (3.4), considered a CHL string background, is based on a rational elliptic surface $\mathcal{R}$; see Proposition 3.2. We recall
that in the general case the existence of an antisymplectic involution $k_Y$ on $Y$ (such that the minimal resolution of $Y/(k_Y)$ is isomorphic to $R$) is equivalent to $\text{NS}(Y) \cong H \oplus E_8(-2)$. One can ask what happens when there also exists a 2-torsion section $\tau_R \in \text{MW}(R, \pi_R)$ on the rational elliptic surface. If there is such a section, then the K3 surface $Y$ will inherit this property. In fact, one easily checks that there exists a 2-torsion section $\tau_R \in \text{MW}(R, \pi_R)$ if and only if there exists a 2-torsion section $\tau_Y \in \text{MW}(Y, \pi_Y)$. The physical relevance of this requirement for the CHL string was discussed in [23]. The classification of rational elliptic surfaces in [68] provides an answer for what this rational elliptic surface $R$ with a 2-torsion section is: it must satisfy

\begin{equation}
\text{MW}(R, \pi_R) \cong \mathbb{Z}/2\mathbb{Z} \oplus D_4'.
\end{equation}

Equation (5.1) describes the rational elliptic surface of highest rank, admitting a 2-torsion section. Moreover, there are no other rational elliptic surfaces with section and rank($\text{MW}$) $\geq 4$ that contain a 2-torsion section. We note that Equation (5.1) is precisely the second case considered earlier in (3.7). It follows that now there is also a (canonical) van Geemen-Sarti involution on $Y$, i.e., a symplectic involution obtained as the translation by the 2-torsion section. Thus, we have the following:

**Corollary 5.1.** The moduli space of F-theory models with discrete flux admitting an antisymplectic involution induced by an involution on the base curve whose quotient is a rational elliptic surface is naturally isomorphic to the moduli space of the CHL string backgrounds admitting a (symplectic) van Geemen-Sarti involution.

For a general CHL string background $Y$ we also constructed the Jacobian elliptic K3 surface $\mathcal{Y}$ in Equation (3.6) as the minimal resolution of $Y/(j_Y)$ using the Nikulin involution $j_Y = k_Y \circ (-1)$. The K3 surfaces $\mathcal{Y}$ form the 10-dimensional moduli space $\mathcal{M}_{H \oplus D_4(-1)^{\oplus 2}}$; see Proposition 3.4. Theorem 4.7 proves that the algebraic correspondences between the K3 surfaces $X$ and $\mathcal{Y}$ form a moduli space $\mathcal{W}$. Here, $\mathcal{W}$ is the moduli space of K3 surfaces that can be obtained as the minimal resolution of double-quadrics surfaces over $F_0$ whose branch locus is the union of divisors of type $(1,0)$, $(1,0)$, and $(2,4)$.

Proposition 3.5 proved that – as the lattice polarization of $Y$ extends from $H \oplus E_8(-2)$ to $H \oplus N_0(-1)$ – the lattice polarization of $\mathcal{Y}$ extends from $H \oplus D_4(-1)^{\oplus 2}$ to $H \oplus D_8(-1) \oplus A_1(-1)^{\oplus 4}$. Figure 4 can then be extended into Figure 5 where now the Jacobian elliptic K3 surfaces $X, X'$, considered F-theory backgrounds, also admit antisymplectic involutions (with rational quotient surfaces $\mathcal{R}, \mathcal{R}'$ and K3 quotients $\mathcal{X}, \mathcal{X}'$) and the Jacobian elliptic K3 surface $Y$, considered a CHL background, also admits a van Geemen-Sarti-Nikulin dual $Y'$ which in turn inherits an antisymplectic involution with rational quotient surface $R'$. Antisymplectic involutions also lift to the double-quadrics surfaces $\mathcal{F}$ and $\mathcal{G}$ whose quotients (after composing with the hyperelliptic involution) are $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$, respectively. For the K3 surface $\mathcal{F}$ in Equation (4.2) the existence of a compatible antisymplectic involutions implies that the polynomials $A, C, D$ satisfy

\begin{equation}
A(s, t) = -\alpha(s^2, t^2), \quad C(s, t) = \gamma(s^2, t^2), \quad D(s, t) = \delta(s^2, t^2),
\end{equation}
for some homogeneous polynomials $\alpha, \gamma, \delta$ of degree two. (Here, we introduced a minus sign for convenience.) We have the following:

**Proposition 5.2.** In the situation of Corollary 5.1 there exist the algebraic correspondences in Figure 5. The defining equations for the surfaces are given in Table 2. Here, $\alpha, \gamma, \delta$ are general homogeneous polynomials of degree 2, and the polynomials $a, c, d$ of degree 2 are defined by requiring

$$\gamma(S,T)U^2 + \alpha(S,T)UV + \delta(S,T)V^2 = c(U,V)S^2 + a(U,V)ST + d(U,V)T^2$$

for all $S, T, U, V$. In particular, we have:

1. $X, X', Y, Y, G, F, \tilde{G}, \in \mathcal{M}_{H\oplus N_0(-1)}$,
2. $X, X', Y, Y, F, \tilde{F}, \in \mathcal{M}_{H\oplus D_4(-1)\oplus A_1(-1)\oplus}$,
3. $R, R', \tilde{R}, \tilde{R}'$ are rational elliptic surfaces with $MW(R, \pi_R) \cong \mathbb{Z}/2\mathbb{Z} \oplus D_4$,
4. $G, F, \tilde{G}, \tilde{F}$ are double-quadrics surfaces over $\mathbb{F}_0$.

**Proof.** One uses the explicit constructions for double covers in Section 3.2 and Section 3.3 to construct all surfaces in Figure 5 explicitly. The Gram matrix of $N_0$ was given in Equation (3.9). The existence of sections that imply isomorphisms $X' \cong \tilde{G} \cong Y$ and $\tilde{G} \cong Y'$ is proved as in Lemma 4.1. The existence of sections implies $X \cong \tilde{F} \cong Y'$ and follows immediately from the equation for $\tilde{F}$ in Table 2. \qed

We make the following:

**Remark 5.3.** A marking of $X$, given by the factorization $\gamma(s^2, t^2) \cdot \delta(s^2, t^2)$, determines the double cover $\tilde{G}$. The marking also induces a canonical marking of $Y'$, given by the factorization $c(u^2, v^2) \cdot d(u^2, v^2)$, with the same double cover $G$.

**Remark 5.4.** The fundamental object in Figure 5 is the double-quadric $\tilde{F}$. All other $K3$ surfaces can be obtained from $\tilde{F}$ using various the double covers.

We will now determine the space of correspondences between $X$ and $Y'$. We consider the general double-quadrics given by

$$W^2 = (S-c_0T)(c_\infty S-T)(d_0V)(d_\infty U-V)(\gamma(U,V)S^2 + a(U,V)ST + d(U,V)T^2),$$

Figure 5. Extension of the F-theory/CHL string duality (Case I)
The correspondences \( \tilde{\mathcal{M}} \) in Figure 5 form the moduli space \( \mathfrak{W} \).

We now derive an explicit parametrization for the K3 surfaces in \( \mathfrak{W} \) using an auxiliary genus-one curve. This parametrization of the curves in the branch locus is based on a construction given by André Weil [75] using the Abel-Jacobi map for genus-one curves.

\[\\]

| label | defining equation |
|-------|-------------------|
| \( \chi \) | \( Y^2Z = X^3 + \alpha(s^2, t^2) X^2Z + \gamma(s^2, t^2) \delta(s^2, t^2) XZ^2 \) |
| \( \chi' \) | \( y^2z = x^3 - 2 \alpha(s^2, t^2) x^2z + (\alpha(s^2, t^2)^2 - 4 \gamma(s^2, t^2) \delta(s^2, t^2)) xz^2 \) |
| \( \gamma \) | \( Y^2Z = X^3 - 2 a(u^2, v^2) X^2Z + (a(u^2, v^2)^2 - 4 c(u^2, v^2) d(u^2, v^2)) XZ^2 \) |
| \( \gamma' \) | \( y^2z = x^3 + a(u^2, v^2) x^2z + c(u^2, v^2) d(u^2, v^2) xz^2 \) |
| \( \tilde{\chi} \) | \( y^2z = x^3 + ST \alpha(S, T) x^2z + S^2T^2 \gamma(S, T) \delta(S, T) xz^2 \) |
| \( \tilde{\chi}' \) | \( Y^2Z = X^3 - 2 ST \alpha(S, T) X^2Z + S^2T^2 (\alpha(S, T)^2 - 4 \gamma(S, T) \delta(S, T)) XZ^2 \) |
| \( \tilde{\gamma} \) | \( y^2z = x^3 - 2 UV a(U, V) x^2z + U^2V^2 (a(U, V)^2 - 4 c(U, V) d(U, V)) xz^2 \) |
| \( \tilde{\gamma}' \) | \( Y^2Z = X^3 + UV a(U, V) X^2Z + U^2V^2 c(U, V) d(U, V) XZ^2 \) |
| \( \tilde{\mathcal{G}} \) | \( w^2 = \left\{ \begin{array}{l} \gamma(s^2, t^2) u^4 + \alpha(s^2, t^2) u^2v^2 + \delta(s^2, t^2) v^4 \\ c(u^2, v^2) s^2 + a(u^2, v^2) s^2t^2 + d(u^2, v^2) t^4 \end{array} \right. \) |
| \( \mathcal{F} \) | \( W^2 = \left\{ \begin{array}{l} UV (\gamma(s^2, t^2) U^2 + \alpha(s^2, t^2) UV + \delta(s^2, t^2) V^2) \\ UV (c(U, V)^2 s^2 + a(U, V) s^2t^2 + d(U, V) t^4) \end{array} \right. \) |
| \( \tilde{\mathcal{G}} \) | \( W^2 = \left\{ \begin{array}{l} ST (\gamma(S, T) u^4 + \alpha(S, T) u^2v^2 + \delta(S, T) v^4) \\ ST (c(u^2, v^2) S^2 + a(u^2, v^2) ST + d(u^2, v^2) T^2) \end{array} \right. \) |
| \( \tilde{\mathcal{F}} \) | \( w^2 = \left\{ \begin{array}{l} STUV (\gamma(S, T) U^2 + \alpha(S, T) UV + \delta(S, T) V^2) \\ STUV (c(U, V)^2 S^2 + a(U, V) ST + d(U, V) T^2) \end{array} \right. \) |

Table 2. Defining equations for surfaces in Figure 5
5.1. The Abel-Jacobi map. Let $H$ be a smooth curve of genus one given by $w^2 = P(x) = \sum_{i=0}^{4} a_i x^i$, using the affine coordinates $(x, w) \in \mathbb{C}^2$. For a point $(x_0, -w_0) \in H$ we consider the Abel-Jacobi map $J_{(x_0, -w_0)} : H \to \text{Jac}(H)$ which relates the algebraic curve $H$ to its Jacobian variety $\text{Jac}(H)$, i.e., an elliptic curve. A classical result due to Hermite states that $\text{Jac}(H) \cong E$ where $E$ is the elliptic curve given by

$$
\text{E: } \eta^2 = S(\xi) = \xi^3 + f \xi + g.
$$

Here, we are using the affine coordinates $(\xi, \eta) \in \mathbb{C}^2$ and Equations (4.6), i.e.,

$$
f = -4a_0 a_4 + a_1 a_3 - \frac{1}{3} a_2^2, \quad g = \frac{8}{3} a_0 a_2 a_4 + a_0 a_3^2 + a_1^2 a_4 - \frac{1}{3} a_1 a_2 a_3 + \frac{2}{27} a_2^3.
$$

We introduce the polynomial

$$
R(x, x_0) = a_4 x^2 x_0^2 + \frac{a_3}{2} x x_0 (x + x_0) + \frac{a_2}{6} (x^2 + x_0^2) + \frac{2 a_2}{3} x x_0 + \frac{a_1}{2} (x + x_0) + a_0
$$

such that $R(x, x) = P(x)$. It turns out that the polynomial $P(x) P(x_0) - R(x, x_0)^2$ factors. There is a polynomial $R_1(x, x_0)$ of bi-degree $(2, 2)$ such that

$$
\forall x, x_0 : R(x, x_0)^2 + R_1(x, x_0) (x - x_0)^2 - P(x) P(x_0) = 0.
$$

We obtain

$$
R_1(x, x_0) = \frac{8 a_2 - 3 a_3^2}{12} x_0^2 + \frac{6 a_1 - a_2 a_3}{6} x x_0 (x + x_0) + \frac{36 a_0 - a_2^2}{36} (x^2 + x_0^2)
$$

$$
+ \frac{36 a_0 + 9 a_1 a_3 - 5 a_2^2}{18} x x_0 + \frac{6 a_0 a_3 - a_1 a_2}{6} (x + x_0) + \frac{8 a_0 a_2 - 3 a_1^2}{12},
$$

and set $Q(x) = R_1(x, x)$. In particular, we have

$$
Q(x) = \frac{1}{3} P(x) P''(x) - \frac{1}{4} P'(x)^2.
$$

We denote the discriminants of $H$ and $E$ by $\Delta_H = \text{Disc}_x(P)$ and $\Delta_E = \text{Disc}_\xi(S)$, respectively, such that $\Delta_H = \Delta_E$ by construction. One also checks $\text{Disc}_x(Q) = S(0)^2 \text{Disc}_x(P)$. From now on, we will assume that

$$
\text{Disc}_x(Q) = S(0)^2 \text{Disc}_x(P) \neq 0.
$$

We also set $[P, Q] = \partial_x P \cdot Q - P \cdot \partial_x Q$. A tedious but straight-forward computation yields the following:

**Lemma 5.6.** For a smooth curve $H$ of genus one given by $w^2 = \sum_{i=0}^{4} a_i x^{4-i}$, the Abel-Jacobi map $J_{(x_0, -w_0)} : H \to E \cong \text{Jac}(H)$ maps $(x, y) \mapsto (\xi, \eta)$ with

$$
\xi = \frac{2 R(x, x_0) - w w_0}{(x - x_0)^2}, \quad \eta = \frac{4 w w_0 (w - w_0)}{(x - x_0)^3} - \frac{P'(x) w_0 + P'(x_0) w}{(x - x_0)^2}
$$

for $x \neq x_0$,

the point $(x_0, -w_0) \in H$ to the point at infinity on $E$, and $(x_0, w_0)$ to the point with

$$
\xi = -Q(x_0) / P(x_0), \quad \eta = [P, Q]_{x_0} / (2w_0^3)
$$

if $w_0 \neq 0$.

It follows from Equation (5.12) that the coordinates $x$ and $\xi$ in the Abel-Jacobi map $(\xi, \eta) = J_{(x_0, -w_0)}(x, y)$ are related by the bi-quadratic polynomial

$$
\xi^2 (x - x_0)^2 - 4 \xi R(x, x_0) - 4 R_1(x, x_0) = 0.
$$

The equation defines an algebraic correspondence between points of the two projective lines with affine coordinates $\xi$ and $x$, respectively, where $-\xi$ given a point $x$ - there
are two solutions for $\xi$ in Equation (5.13) and vice versa. Equivalently, we consider Equation (5.13) an affine equation of bi-degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ with the affine variables $(x, x_0)$ (which has genus one). A direct computation yields the following:

Lemma 5.7. The $j$-invariants of the following genus-one curves are identical:

\begin{equation}
\eta^2 = S(\xi), \quad w^2 = P(x) \quad \xi^2 (x - x_0)^2 - 4 \xi R(x, x_0) - 4R_1(x, x_0) = 0.
\end{equation}

In particular, Lemma 5.7 shows that the $j$-invariant of the curve in Equation (5.13) is independent of the variable $\xi$. We have the following:

Proposition 5.8. Equation (5.13) is an embedding $\iota_\xi : H \leftrightarrow H' \subset \mathbb{P}^1 \times \mathbb{P}^1$ of a genus-one curve $H$ as a curve of bi-degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, given by

\begin{equation}
H' : \quad \xi^2 (x - x_0)^2 - 4 \xi R(x, x_0) - 4R_1(x, x_0) = 0.
\end{equation}

In particular, it is a symmetric affine equation in the variables $(x, x_0)$.

Based on results in [37, 74, 75] we have the following:

Theorem 5.9. Let

\begin{equation}
0 = \gamma(S, T) \ U^2 + \alpha(S, T) \ U V + \delta(S, T) \ V^2
\end{equation}

be a curve of genus one in the linear system $|\mathcal{O}_{\mathbb{P}^1}(2, 2)|$ over $\mathbb{F}_0 \ni ([S : T], [U, V])$ with the same $j$-invariant as $H : w^2 = P(x)$. Then, there is a point in the Jacobian $\xi \in \text{Jac}(H)$ such that Equation (5.16) is isomorphic to $\iota_\xi(H)$.

Proof. Equation (5.16) depends on eight parameters since one has

\begin{equation}
\mathbb{P}H^0(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(2, 2)) = 3 \cdot 3 - 1 = 8.
\end{equation}

Since $\mathbb{F}_0$ has a 6-dimensional group of automorphisms, we have two moduli. The first one is the $j$-invariant defining an elliptic curve $E$. It follows that $E \cong \text{Jac}(H)$. On the hand, the points in $\text{Jac}(H)$ parameterize the automorphisms of $H$. Thus, there is a point $\xi \in \text{Jac}(H)$ such that Equation (5.16) is isomorphic to $\iota_\xi(H)$. $\square$

5.2. A parametrization of K3 surfaces. We derive an explicit parametrization for the K3 surfaces in $\mathfrak{M}$. The central idea is that genus-one curves in the branch locus of K3 surfaces $\mathcal{F} \subset \mathfrak{M}$ can be represented by equations of bi-degree $(2, 2)$ that are symmetric under the interchange of $(U, V)$ and $(S, T)$ in $\mathbb{F}_0 = \mathbb{P}(S, T) \times \mathbb{P}(U, V)$.

We start by using transformations in $\text{PGL}(2, \mathbb{C})$ to turn the first terms of Equation (5.4) into the monomial $STUV$. We then use the transformation

\begin{equation}
(U, V) \mapsto (\mu U, \lambda V), \quad (S, T) \mapsto (\lambda(T - \lambda^{-1}c_\infty S), S - \lambda c_0 T),
\end{equation}

with parameters $\lambda, \mu \in \mathbb{C}^\times$ and $c_0, c_\infty$ with $c_0 c_\infty \neq 1$, to transform an equation of the form

\begin{equation}
W^2 = STUV\left(\gamma'(U, V) S^2 + \alpha'(U, V) ST + \delta'(U, V) T^2\right)
\end{equation}

for polynomials $\alpha', \gamma', \delta'$ with coefficients $\alpha_2, \ldots, \gamma_0'$ into the equation

\begin{equation}
W^2 = (S - \lambda c_0 T)(T - \lambda^{-1}c_\infty S) UV \left(\gamma_2 S^2 + \lambda \alpha_2 ST + \lambda^2 \gamma_0 T^2\right)U^2
\end{equation}

\begin{equation}
+ \left(\lambda \alpha_2 S^2 + \lambda^2 \alpha_1 ST + \lambda^3 \alpha_0 T^2\right)UV
+ \left(\lambda^2 \gamma_0 S^2 + \lambda^3 \alpha_0 ST + \lambda^4 \delta_0 T^2\right)V^2,
\end{equation}
where $\mu, c_0, c_\infty$ have been chosen such that the new coefficients $\alpha_2, \ldots, \gamma_0$ do not depend on $\lambda$ and satisfy $\gamma_1 = \alpha_2, \delta_2 = \gamma_0, \delta_1 = \alpha_0$. Thus, we have proved the following:

Proposition 5.10. The moduli space $\overline{\mathcal{M}}$ is given by the K3 surfaces obtained from the minimal resolution of the double-quadrics

\[(5.21) \quad W^2 = (S - c_0T)(T - c_\infty S)^2 \left( \gamma(U, V) S^2 + \alpha(U, V) ST + \delta(U, V) T^2 \right).\]

Here, $c_0, c_\infty \in \mathbb{C}$ with $c_0c_\infty \neq 1$ and $\alpha, \gamma, \delta$ are given by

\[(5.22) \quad \alpha = \alpha_2 x^2 + \alpha_1 xy + \alpha_0 y^2, \quad \gamma = \gamma_2 x^2 + \gamma_1 xy + \gamma_0 y^2, \quad \delta = \delta_0 x^2 + \delta_1 xy + \delta_0 y^2.\]

We make the following:

Remark 5.11. Proposition 5.10 implies that we can assume

\[(5.23) \quad \gamma(S, T) U^2 + \alpha(S, T) UV + \delta(S, T) V^2 = \gamma(U, V) S^2 + \alpha(U, V) ST + \delta(U, V) T^2\]

for all $S, T, U, V$.

By a slight abuse of notation we will also denote by $\overline{\mathcal{F}}(\delta_0, \alpha_0, \gamma_0, \alpha_1, \alpha_2, \gamma_0; c_0, c_\infty)$ the smooth complex surface obtained as the minimal resolution of Equation (5.21).

We have the following symmetries:

Lemma 5.12. One has the following isomorphisms of K3 surfaces:

(a) $\overline{\mathcal{F}}(\delta_0, \alpha_0, \gamma_0, \alpha_1, \alpha_2, \gamma_2; c_0, c_\infty) \simeq \overline{\mathcal{F}}(\lambda \delta_0, \lambda \alpha_0, \lambda \gamma_0, \lambda \alpha_1, \lambda \alpha_2, \lambda \gamma_2; c_0, c_\infty)$,

(b) $\overline{\mathcal{F}}(\delta_0, \alpha_0, \gamma_0, \alpha_1, \alpha_2, \gamma_2; c_0, c_\infty) \simeq \overline{\mathcal{F}}(\mu^4 \delta_0, \mu^3 \alpha_0, \mu^2 \gamma_0, \mu \alpha_1, \mu \alpha_2, \gamma_2; \mu c_0, \mu^{-1} c_\infty)$,

(c) $\overline{\mathcal{F}}(\delta_0, \alpha_0, \gamma_0, \alpha_1, \alpha_2, \gamma_2; c_0, c_\infty) \simeq \overline{\mathcal{F}}(\gamma_2, \alpha_2, \gamma_0, \alpha_1, \alpha_0, \delta_0; c_0, c_\infty)$,

(d) $\overline{\mathcal{F}}(\delta_0, \alpha_0, \gamma_0, \alpha_1, \alpha_2, \gamma_2; c_0, c_\infty) \simeq \overline{\mathcal{F}}(\delta_0, \alpha_0, \gamma_0, \alpha_1, \alpha_2, \gamma_2; -c_0, -c_\infty)$,

for $\lambda, \mu \in \mathbb{C}^\times$ and

\[
\begin{align*}
\delta'_0 &= \gamma_2 c_0^4 + 2 \alpha_2 c_0^3 + (\alpha_1 + 2 \gamma_0) c_0^2 + 2 \alpha_0 c_0 + \delta_0, \\
\alpha'_0 &= (\alpha_2 c_0^2 + 2 \gamma_2) c_0^3 + ((\alpha_1 + 2 \gamma_0) c_0 + 3 \alpha_2) c_0^2 + (3 \alpha_0 c_0 + \alpha_1 + 2 \gamma_0) c_0 + 2 \delta_0 c_\infty + \alpha_0, \\
\gamma'_0 &= (\gamma_0 c_0^2 + 2 \alpha_2 c_0 + \gamma_2) c_0^2 + (\alpha_0 c_0^2 + 2 \alpha_2 c_0 + \gamma_2) c_0 + \delta_0 c_\infty + \alpha_0 c_\infty + \gamma_0, \\
\alpha'_1 &= (\alpha_1 c_0^4 + 2 \alpha_2 c_0^2 + 4 \gamma_2) c_0^2 + (4 \alpha_0 c_\infty + 2 \alpha_1 + 4 \gamma_0) c_\infty + 2 \alpha_2 c_0, \\
&\quad + 4 \delta_0 c_\infty^2 + 4 \alpha_0 c_\infty + \alpha_1, \\
\alpha'_2 &= (3 \alpha_0 c_0 + 2 \delta_0) c_\infty^3 + (3 \alpha_1 + 2 \gamma_0) c_\infty^2 + 3 \alpha_2 c_\infty + (4 \alpha_0 c_\infty^2 + 2 \alpha_1 + 2 \gamma_0) c_\infty + 2 \gamma_2 c_0 + \alpha_2, \\
\gamma'_2 &= \delta_0 c_\infty^4 + 2 \alpha_0 c_\infty^3 + (\alpha_1 + 2 \gamma_0) c_\infty^2 + 2 \alpha_2 c_\infty + \gamma_2.
\end{align*}
\]

Proof. Part (1) follows from rescaling $W \mapsto W/\sqrt{\lambda}$, similarly part (2) follows from the rescaling $(T, V) \mapsto (\mu T, \mu V)$. For the remaining parts, the proof follows by constructing a suitable automorphism of $F_0$ in Equation (5.21) that gives to the change of parameters. For part (3) this is given by interchanging $S \leftrightarrow T$ and $U \leftrightarrow V$. For part (4) the transformation is given by

\[(5.25) \quad U = S' + c_0 T', \quad V = c_\infty S' + T', \quad S = U' + c_0 V', \quad T = c_\infty U' + V'.\]

\[\square\]

We now give an explicit parametrization for the K3 surfaces in $\overline{\mathcal{M}}$ using the Abel-Jacobi map from Section 5.1. We have the following:
Theorem 5.13. The moduli space \( \mathfrak{M} \) of correspondences \( \bar{F} \) in Figure 5 is given by the K3 surfaces obtained as the minimal resolution of the double-quadrics

\[
W^2 = (S - c_0 T)(T - c_{\infty} S) UV (\gamma(U, V) S^2 + \alpha(U, V) ST + \delta(U, V) T^2).
\]

Here, \( c_0, c_{\infty} \in \mathbb{C} \) with \( c_0 c_{\infty} \neq 1 \), the polynomials \( \alpha, \gamma, \delta \) are given by

\[
\forall x, x_0 : \gamma(x, 1) x_0^2 + \alpha(x, 1) x_0 + \delta(x, 1) = \xi^2 (x - x_0)^2 - 4 \eta R(x, x_0) - 4 R_1(x, x_0),
\]

and the polynomials \( R(x, x_0) \) and \( R_1(x, x_0) \) are

\[
R(x, x_0) = x^2 x_0^2 + \frac{a_2}{6} (x^2 + x_0^2) + \frac{2 a_2}{3} x x_0 + \frac{a_1}{2} (x + x_0) + a_0,
\]

\[
R_1(x, x_0) = \frac{2 a_2}{3} x^2 x_0^2 + a_1 x x_0 (x + x_0) + \frac{36 a_0 - a_2^2}{36} (x^2 + x_0^2)
\]

\[
+ \frac{36 a_0 - 5 a_2^2}{18} x x_0 - \frac{a_1 a_2}{6} (x + x_0) + \frac{8 a_0 a_2 - 3 a_2^2}{12},
\]

for the smooth genus-1 curve \( H : w^2 = x^4 + \sum_{i=0}^2 a_i x^i \) and \( \xi \in \text{Jac}(H) \).

Remark 5.14. The parameterization in Theorem 5.13 is given by the six parameters \( c_0, c_{\infty}, a_0, a_1, a_2, \xi \) such that \( c_0 c_{\infty} \neq 1 \), the curve \( H : w^2 = x^4 + \sum_{i=0}^2 a_i x^i \) is smooth, and \( \xi \in \text{Jac}(H) \). The curve \( H \) is smooth if its discriminant does not vanish, i.e.,

\[
16 a_0 a_1^2 - 4 a_1^2 a_2^2 - 128 a_0^2 a_2^2 + 144 a_0 a_1 a_2 - 27 a_1^4 + 256 a_0^3 \neq 0.
\]

The elliptic curve \( E = \text{Jac}(H) \) in \( \mathbb{P}^2 = \mathbb{P}(Z_1, Z_2, Z_3) \) is given by

\[
E : Z_2^2 Z_3 = Z_1^3 - \left( 4 a_0 + \frac{1}{3} a_2^2 \right) Z_1^2 Z_3 + \left( a_1^2 + \frac{2}{27} a_2^3 - \frac{8}{3} a_0 a_2 \right) Z_3^2.
\]

Proof. We apply Proposition 5.10, Theorem 5.9, and Proposition 5.8. One then checks that in terms of the curve \( H \) given by \( w^2 = P(x) = \sum_{i=0}^4 a_i x^i \) and the Jacobian \( \text{Jac}(H) \) given by Equation (5.5) one has

\[
2 a_0 \gamma_2 - a_1 \alpha_2 + 2 a_2 \gamma_0 = -8 a_3 \eta^2.
\]

Using Lemma 5.12 (4) one checks that the equation

\[
2 a_4 \gamma_2' - a_1' \alpha_2' + 2 a_2' \gamma_0' = 0.
\]

is linear in \( c_0 \) and can be solved such that \( c_0 \) is a rational function in \( c_{\infty} \) where both the numerator and denominator have degree 3 in \( c_{\infty} \) with coefficients quadratic in \( \mathbb{Z} [\delta_0, a_0, \gamma_0, a_1, \alpha_2, \gamma_2] \). Thus, for a smooth curve \( H \) of genus one we can assume \( a_4 \neq 0 \) and then shift the coordinate \( x \) to obtain \( a_3 = 0 \). Moreover, Lemma 5.12 (1) provides an overall scaling that can be used to be have \( a_4 = 1 \).

6. A second subspace of dimension six

The Jacobian elliptic K3 surfaces \( X \) with \( \text{MW}(X, \pi_X) \cong \mathbb{Z}/2\mathbb{Z} \) in Equation (3.15) represent F-theory backgrounds with discrete flux, or, equivalently, elements of the moduli space \( \mathfrak{M}_{H \oplus N} \). Based on Corollary 3.10, imposing the existence of a second (commuting) van Geemen-Sarti involution on \( X \) implies that the Jacobian elliptic fibration has 12 singular fibers of type \( I_2 \), a Mordell-Weil group \( (\mathbb{Z}/2\mathbb{Z})^2 \), and a lattice polarization that extends to

\[
H \oplus D_4(-1)^\oplus 2 \oplus A_1(-1)^\oplus 4 \cong H \oplus D_0(-1) \oplus A_1(-1)^\oplus 6 \cong (2) \oplus (-2) \oplus D_4(-1)^\oplus 3.
\]
The equivalence of the lattices was proven in [15]. The existence of commuting van Geemen-Sarti involutions means that there are two independent 2-torsion sections in \( \text{MW}(\mathcal{X}, \pi_\mathcal{X}) \) and implies that the polynomial \( B(s, t) \) in Equation (3.15) satisfies \( B = A^2 - C^2 \) for some homogeneous polynomial \( C(s, t) \) of degree 4; see proof of Corollary 3.10. In turn, the factorization \( 4B = (A + C)(A - C) \) determines a marking of the eight fibers of type \( I_2 \) over \( B = 0 \) on \( \mathcal{X} \) with the double cover \( \mathcal{G} \). Observing that the coefficient of \( xz^2 \) in Equation (3.16) then becomes a perfect square, we obtain a canonical marking of \( \mathcal{X}' \) with another double cover \( \mathcal{G}' \). The double-quadrics \( \mathcal{G}' \) has a particular simple form which we will determine presently. Changing the ruling and computing the relative Jacobian fibration we obtain the K3 surface \( \mathcal{Y}' \) admitting two commuting antisymplectic involutions due to the symmetry of \( \mathcal{G}' \). We already constructed the K3 surfaces \( \mathcal{Y}' \) and their quotients \( \mathcal{Y}, \tilde{\mathcal{Y}} \) in Propositions 3.6 and 3.7. Thus, we have the following:

**Corollary 6.1.** The moduli space of F-theory models with discrete flux admitting commuting van Geemen-Sarti involutions is dual to the moduli space of the CHL string admitting 2 antisymplectic involutions induced by involutions on the base curve.

Figure 4 can then be extended. We obtain Figure 6 where now the rational elliptic surface \( \mathcal{R} \) – due to the extended symmetry of \( \mathcal{Y} \) – must satisfy

\[
\text{MW}(\mathcal{R}, \pi_\mathcal{R}) \cong D_4'.
\]

We note that Equation (6.2) is precisely the first case considered earlier in (3.7). We have the following:

**Proposition 6.2.** In the situation of Corollary 6.1 there exist the algebraic correspondences in Figure 6. The defining equations for the surfaces in Figure 6 are given by Table 2. Here, \( A, C \) are general homogeneous polynomials of degree four, and \( a_0, \ldots, a_4 \) of degree 1 are defined by requiring

\[
(6.3) \quad A(s, t) \frac{U - V}{2} - C(s, t) \frac{U + V}{2} = a_0(U, V) s^4 + a_1(U, V) s^3 t + \cdots + a_4(U, V) t^4
\]

for all \( s, t, U, V \). The polynomials \( f, g \) are given by

\[
(6.4) \quad f = -4a_0a_4 + a_1a_3 - \frac{1}{3} a_2^2, \quad g = \frac{8}{3} a_0a_2a_4 + a_0a_3^2 + a_1^2a_4 - \frac{1}{3} a_1a_2a_3 + \frac{2}{27} a_2^3.
\]

In particular, we have:

1. \( \mathcal{X}, \mathcal{F} \in \mathcal{M}_{(2)\oplus(2)\oplus D_4(-1)\oplus3}, \mathcal{X}', \mathcal{G}, \mathcal{Y} \in \mathcal{M}_{H\oplus K_0(-1)}, \tilde{\mathcal{Y}} \in \mathcal{M}_{H\oplus D_4(-1)\oplus3}, \)
2. \( \mathcal{G}', \mathcal{G}, \mathcal{F} \) are double-quadrics over \( \mathbb{F}_0 \),
3. \( \mathcal{R} \) is a rational elliptic surface with \( \text{MW}(\mathcal{R}, \pi_\mathcal{R}) \cong D_4' \).

**Proof.** One uses the explicit constructions for double covers in Section 3.2 and Section 3.3 to construct all surfaces in Figure 6 explicitly. The Gram matrix of \( K_0 \) was given in Equation (3.8). The existence of sections that imply isomorphisms \( \mathcal{G}' \cong \mathcal{Y}' \) and \( \mathcal{G} \cong \mathcal{Y} \) is proved as in Lemma 4.1. The existence of sections that implies \( \mathcal{X}' \cong \mathcal{G} \) follows immediately from the equation for \( \mathcal{G} \) in Table 3. \( \square \)
Figure 6. Extension of the F-theory/CHL string duality (Case II)

| label | defining equation |
|-------|-------------------|
| $\mathcal{X}$ | $Y^2Z = X^3 - A(s, t) X^2 Z + \left(A(s, t)^2 - C(s, t)^2\right) X Z^2 / 4$ |
| $\mathcal{X}'$ | $y^2 z = x^3 + 2 A(s, t) x^2 z + C(s, t)^2 x z^2$ |
| $\mathcal{G}'$ | $W^2 = \left\{ C(s, t) \tilde{u}^4 + 2 A(s, t) \tilde{u}^2 \tilde{v}^2 + C(s, t) \tilde{v}^4 \right\} a_4 \left( (\tilde{u}^2 - \tilde{v}^2)^2, (\tilde{u}^2 + \tilde{v}^2)^2 \right) s^4 + \cdots + a_0 \left( (\tilde{u}^2 - \tilde{v}^2)^2, (\tilde{u}^2 + \tilde{v}^2)^2 \right) t^4$ |
| $\mathcal{G}$ | $w^2 = \left\{ (u^2 - v^2) \left( \frac{A(s, t) - C(s, t)}{2} u^2 - \frac{A(s, t) + C(s, t)}{2} v^2 \right) \right\} a_4 (u^2, v^2) s^4 + \cdots + a_0 (u^2, v^2) t^4$ |
| $\mathcal{F}$ | $W^2 = \left\{ U V (U - V) \left( \frac{A(s, t) - C(s, t)}{2} U - \frac{A(s, t) + C(s, t)}{2} V \right) \right\} a_4 (U, V) s^4 + \cdots + a_0 (U, V) t^4$ |
| $\mathcal{Y}'$ | $Y^2 Z = X^3 + f(\tilde{u}^2 - \tilde{v}^2)^2, (\tilde{u}^2 + \tilde{v}^2)^2) X Z^2 + g((\tilde{u}^2 - \tilde{v}^2)^2, (\tilde{u}^2 + \tilde{v}^2)^2) Z^3$ |
| $\mathcal{Y}$ | $y^2 z = x^3 + (u^2 - v^2)^2 f(u^2, v^2) x z^2 + (u^2 - v^2)^3 g(u^2, v^2) z^3$ |
| $\tilde{\mathcal{Y}}$ | $Y^2 Z = X^3 + U^2 V^2 (U - V)^2 f(U, V) X Z^2 + U^3 V^3 (U - V)^3 g(U, V) Z^3$ |
| $\mathcal{R}'$ | $y^2 z = x^3 + f(u^2, v^2) x z^2 + g(u^2, v^2) z^3$ |
| $\mathcal{R}$ | $Y^2 Z = X^3 + (U - V)^2 f(U, V) X Z^2 + (U - V)^3 g(U, V) Z^3$ |

Table 3. Defining equations for surfaces in Figure 6

6.1. The double 4$\mathcal{H}$-surfaces. We will now determine the space of correspondences $\mathcal{F}$ for the K3 surfaces $\mathcal{X}$ and $\tilde{\mathcal{Y}}$ in Figure 6.

Let $\mathcal{F}$ be a double cover of the Hirzebruch surface $\mathbb{P}^1 \times \mathbb{P}^1$ branched over the union of four bi-degree (1, 1) curves satisfying a certain generality condition. Such a surface $\mathcal{F}$ is also called a double 4$\mathcal{H}$-surface. We construct a geometric model as follows: using the coordinates $U$ and $s$ on $\mathbb{P}^1 \left( \frac{U}{(U)} \right) \times \mathbb{P}^1 \left( \frac{s}{(s)} \right)$ a double cover is given by

$$
(6.5) \quad y^2 = \prod_{k=1}^{4} \left( \rho_1^{(k)} s U + \rho_2^{(k)} U + \rho_3^{(k)} s + \rho_4^{(k)} \right),
$$

for complex parameters $\rho_j^{(i)}$ with $i, j \in \{1, 2, 3, 4\}$. We denote by $H_1, \ldots, H_4$ the four different rational curves of bi-degree (1, 1) and impose the following genericity conditions: (1) every $H_i$ is irreducible, (2) $H_i \cap H_j$ consists of two different points for all $i \neq j$, and (3) for any three different indices $i, j, k$ we have $H_i \cap H_j \cap H_k = \emptyset$. This is precisely the family considered in [49], and it was proven there that under the above
conditions $\mathcal{F}$ is a K3 surface. If we define the quadratic polynomials
\begin{equation}
P^{(i,j)} = \left(\rho^{(i)}_1 \rho^{(j)}_3 - \rho^{(i)}_3 \rho^{(j)}_1\right) s^2 + \left(\rho^{(i)}_2 \rho^{(j)}_4 - \rho^{(i)}_4 \rho^{(j)}_2\right) s + \left(\rho^{(i)}_1 \rho^{(j)}_4 + \rho^{(i)}_2 \rho^{(j)}_3 - \rho^{(i)}_3 \rho^{(j)}_2 - \rho^{(i)}_4 \rho^{(j)}_1\right),
\end{equation}
then Equation (6.5) can be brought into the Weierstrass form
\begin{equation}
Y^2Z = X \left( X - P^{(1,2)} P^{(3,4)} Z \right) \left( X - P^{(1,3)} P^{(2,4)} Z \right),
\end{equation}
which coincides with the equation for $X$ in Table 3 in the affine chart $t = 1$ with
\[ p^{(1,2)} p^{(3,4)} = \frac{A + C}{2}, \quad p^{(1,3)} p^{(2,4)} = \frac{A - C}{2}. \]
The following is then immediate:

**Lemma 6.3.** The generic K3 surface $\mathcal{F}$ admits a Jacobian elliptic fibration with 12 singular fibers of type $I_2$ and a Mordell-Weil group of sections $(\mathbb{Z}/2\mathbb{Z})^2$.

**Proof.** Given the Weierstrass model in Equation (6.7) the statement is checked by explicit computation. \hfill \square

Given any two distinct complex parameters $\mu, \nu \in \mathbb{C}$ with $\mu \neq \nu$, Equation (6.7) can be brought into the form
\[ y^2 = \left( \xi + \mu \right) \left( \xi + \nu \right) \left( \left( p^{(1,2)} p^{(3,4)} - p^{(1,3)} p^{(2,4)} \right) \xi + \left( \mu p^{(1,2)} p^{(3,4)} - \nu p^{(1,3)} p^{(2,4)} \right) \right), \]
which in turn can be re-written as
\begin{equation}
Y^2 = \sum_{i=0}^{4} (\xi + \mu) (\xi + \nu) A_i(\xi, \mu, \nu) u^i,
\end{equation}
with $A_i(\xi, \mu, \nu) = a_{i,1} \xi + a_{i,2} \mu + a_{i,3} \nu$ for $0 \leq i \leq 4$. The coefficients are of the form
\begin{equation}
a_{i,j} = \sum_{k_1, \ldots, k_4} \alpha_{i,j}(\vec{k}) \rho^{(1)}_{k_1} \rho^{(2)}_{k_2} \rho^{(3)}_{k_3} \rho^{(4)}_{k_4},
\end{equation}
with $\alpha_{i,j}(\vec{k}) \in \{0, \pm 1\}$. Considering $\xi$ the affine coordinate of a projective line and $(u, y)$ the affine coordinates of a genus-one fiber in $\mathbb{P}^2$, it follows that Equation (6.8) induces a genus-one fibration on the K3 surface $\mathcal{F}$. Using [23, Prop. 3.3] it follows immediately:

**Lemma 6.4.** The very general K3 surface $\mathcal{F}$ admits a genus-one fibration (without section and) with three singular fibers of type $I_0^*$ and six singular fibers of type $I_1$.

The existence of such a genus-one fibration with three singular fibers of type $I_0^*$ on the K3 surface $\mathcal{F}$ allowed the authors in [49, Thm. 1] to compute the lattice polarization of the family. We have:

**Proposition 6.5.** For generic parameters $\rho^{(i)}_j$ with $i, j \in \{1, 2, 3, 4\}$ the K3 surface $\mathcal{F}$ is endowed with a canonical polarization given by the rank-fourteen lattice
\begin{equation}
(2) \oplus (-2) \oplus D_4(-1)^{\oplus 3}.
\end{equation}
Moreover, the transcendental lattice is $H(2)^{\oplus 2} \oplus (-2)^{\oplus 4}$ of signature $(2, 6)$. 
We have the immediate:

**Corollary 6.6.** The double 4H-surfaces $\mathcal{F}$ form the moduli space of correspondences between the K3 surfaces $\mathcal{X}$ and $\mathcal{Y}$ in Figure 6.

### 6.2. Double covers of a cubic and three lines.

In this section we describe the geometry of the K3 surfaces $\mathcal{Y}$ in Figure 6. They represent the CHL string backgrounds dual to the F-theory with additional symplectic involution.

Let $\mathcal{Y}$ be the double cover of the projective plane $\mathbb{P}^2 = \mathbb{P}(Z_1, Z_2, Z_3)$ branched over the union of three lines $\ell_1, \ell_2, \ell_3$ coincident in a point and a cubic $E$. We call such a configuration *generic* if the cubic is smooth and meets the three lines in nine distinct points. In particular, the cubic does not meet the point of coincidence of the three lines. We construct a geometric model as follows: we use a suitable projective transformation to move the line $\ell_3$ to $\ell_3 = V(Z_3)$. We then mark three distinct points $q_0, q_1,$ and $q_\infty$ on $\ell_3$ and use a M"obius transformation to move these points to $[Z_1 : Z_2 : Z_3] = [0 : 1 : 0], [1 : 1 : 0], \text{ and } [1 : 0 : 0]$. Let $q_1 : [1 : 1 : 0]$ be the point of coincidence of the three lines. Up to scaling, the three lines, coincident in $q_1$, are then given by

\begin{equation}
\ell_1 = V(Z_1 - Z_2 + \mu Z_3), \quad \ell_2 = V(Z_1 - Z_2 + \nu Z_3), \quad \ell_3 = V(Z_3),
\end{equation}

for some parameters $\mu, \nu$ with $\mu \neq \nu$. Let the cubic $E = V(C(Z_1, Z_2, Z_3))$ intersect the line $\ell_3$ at $q_0, q_\infty$, and at the point $[-d_2 : c_1 : 0] \neq [1 : 1 : 0]$. Thus, we have

\begin{equation}
C = e_3 Z_3^2 + \left(d_0 Z_1 + e_1 Z_2\right) Z_3^2 + \left(c_0 Z_1^2 + d_1 Z_1 Z_2 + e_2 Z_2^2\right) Z_3 + Z_1 Z_2 \left(c_1 Z_1 + d_2 Z_2\right).
\end{equation}

This can be written as

\begin{equation}
C = \left(c_1 Z_2 + c_0 Z_3\right) Z_1^2 + \left(d_2 Z_2^2 + d_1 Z_2 Z_3 + d_0 Z_3^2\right) Z_1 + \left(e_2 Z_2^2 + e_1 Z_2 Z_3 + e_0 Z_3^2\right) Z_3,
\end{equation}

such that in $\mathbb{P}^\mathbb{H}_{(1,1,1,3)} = \mathbb{P}(Z_1, Z_2, Z_3, Y)$ the surface $\mathcal{Y}$ is given by

\begin{equation}
Y^2 = \left(Z_1 - Z_2 + \mu Z_3\right) \left(Z_1 - Z_2 + \nu Z_3\right) Z_3 C(Z_1, Z_2 Z_3),
\end{equation}

for parameters $\mu, \nu, c_0, c_1, d_0, d_1, d_2, e_0, e_1, e_2$ with $c_1 \neq 0, c_1 + d_2 \neq 0, \mu \neq \nu$, and a smooth cubic $E$ that intersects each line $\ell_1, \ell_2, \ell_3$ in three distinct points. We have the following:

**Lemma 6.7.** The cubic $E$ is tangent to the line $\ell_3$ at $q_0$ if and only if $d_2 = 0$ and the remaining parameters are generic. The cubic $E$ is singular at $q_0$ if and only if $d_2 = e_2 = 0$ and the remaining parameters are generic.

In [14] the authors proved that the coefficients of the curves in the branch locus can be normalized as follows:

**Lemma 6.8.** Let $\mathcal{Y}$ be the double cover of the projective plane $\mathbb{P}^2 = \mathbb{P}(Z_1, Z_2, Z_3)$ branched over three lines coincident in a point and a generic cubic. There are affine parameters $(d_2, \mu, c_0, e_2, d_0, e_1, e_0) \in \mathbb{C}^7$, unique up to the action given by

\begin{equation}
(d_2, \mu, c_0, e_2, d_0, e_1, e_0) \mapsto (d_2, \Lambda \mu, \Lambda c_0, \Lambda e_2, \Lambda^2 d_0, \Lambda^2 e_1, \Lambda^3 e_0)
\end{equation}
with $\Lambda \in \mathbb{C}^*$, such that $\tilde{Y}$ in $\mathbb{P}(1,1,1,3) = \mathbb{P}(Z_1, Z_2, Z_3, Y)$ is obtained by
\[
Y^2 = (Z_1 - Z_2 + \mu Z_3)(Z_1 - Z_2 + \nu Z_3)Z_3
\]
\[
\times \left( \left( Z_2 + c_0 Z_3 \right)^2 + d_2 Z_2^2 + d_0 Z_3 \right) Z_1 + \left( e_2 Z_2^2 + e_1 Z_2 Z_3 + e_0 Z_3 \right) Z_3, \tag{6.16} \]
with $\mu + \nu = (1 + d_2/2)(c_0 + e_2)$ and $d_2 \neq -1$.

We denote by $\tilde{Y}$ the surface obtained as the minimal resolution of $\tilde{Y}$. Since $\tilde{Y}$ is the resolution of a double-sixfold surface, it is a K3 surface. We will now construct a Jacobian elliptic fibration on it to establish the connection with $\tilde{Y}$ in Figure 6:

**Lemma 6.9.** A generic K3 surface $\tilde{Y}$ admits a Jacobian elliptic fibration with the singular fibers $3I_0^* + 6I_1$ and a trivial Mordell-Weil group.

**Proof.** The pencil of lines $(Z_1 - Z_2) - tZ_3 = 0$ for $t \in \mathbb{C}$ through the point $q_1 = [1:1:0]$ induces an elliptic fibration on $\tilde{Y}$. We refer to this fibration as the standard fibration. When substituting $Z_1 = X$, $Z_2 = X - (c_1 + d_2)(t + \mu)(t + \nu)t$, and $Z_3 = (c_1 + d_2)(t + \mu)(t + \nu)$ into Equation (6.14) we obtain the Weierstrass model
\[
Y^2 = X^3 - (t + \mu)(t + \nu) \left( (c_1 + d_2)t - (c_0 + d_1 + e_2) \right) X^2
\]
\[
+ \left( c_1 + d_2 \right)(t + \mu)^2(t + \nu)^2 \left( d_2 t^2 - (d_1 + 2e_2)t + (d_0 + e_1) \right) X
\]
\[
+ \left( c_1 + d_2 \right)^2(t + \mu)^3(t + \nu)^3 \left( e_2 t^2 - e_1 t + e_0 \right), \tag{6.17} \]
with a discriminant function of the elliptic fibration $\Delta = (t + \mu)^6(t + \nu)^6(c_1 + d_2)^2p(t)$, where $p(t) = c_1^2d_2^2t^6 + \ldots$ is a polynomial of degree six. Given the Weierstrass model in Equation (6.17) the statement is checked by explicit computation.

Since we always assume $c_1 \neq 0$ we have:

**Corollary 6.10.** The fibration in Lemma 6.9 has the singular fibers $I_1^* + 2I_0^* + 5I_1$ if and only if $d_2 = 0$ and the remaining parameters are generic. It has the singular fibers $I_2^* + 2I_0^* + 4I_1$ if and only if $d_2 = e_2 = 0$ and the remaining parameters are generic, and the singular fibers $I_3^* + 2I_0^* + 3I_1$ if and only if $d_2 = e_2 = e_1 = 0$ and the remaining parameters are generic.

We also have the converse statement of Lemma 6.9:

**Proposition 6.11.** A K3 surface admitting a Jacobian elliptic fibration with the singular fibers $3I_0^* + 6I_1$ and a trivial Mordell-Weil group arises as the double cover of the projective plane branched over three lines coincident in a point and a cubic.

**Proof.** Using a Möbius transformation we can move the base points of the three singular fibers of type $I_0^*$ to $\mu, \nu, \infty$. An elliptic surface admitting the given Jacobian elliptic fibration then has a Weierstrass model of the form
\[
Y^2 = X^3 + (t + \mu)(t + \nu) \left( \tilde{c}_1 t + \tilde{d}_0 \right) X^2 + (t + \mu)^2(t + \nu)^2 \left( \tilde{d}_2 t^2 + \tilde{d}_1 t + \tilde{d}_0 \right) X
\]
\[
+ (t + \mu)^3(t + \nu)^3 \left( \tilde{c}_3 t^3 + \tilde{c}_2 t^2 + \tilde{c}_1 t + \tilde{c}_0 \right). \tag{6.18} \]
A shift \( X \mapsto X + \rho t(t + \mu)(t + \nu) \) eliminates the coefficient \( \hat{c}_3 \) in Equation (6.18) if \( \rho \) is a solution of \( \rho^3 + \hat{c}_1 \rho^2 + \hat{d}_2 \rho + \hat{c}_3 = 0 \). Thus, we can assume \( \hat{c}_3 = 0 \). Next, let \( c_1 \) be a root of \( c_1^2 = \hat{c}_1^2 - 4\hat{d}_2 \). Then substituting

\[
\begin{align*}
    c_0 &= \frac{2\hat{d}_1}{c_1 - \hat{c}_1} + \frac{4\hat{e}_2}{(c_1 - \hat{c}_1)^2} + \hat{c}_0, \\
    d_0 &= \frac{2\hat{d}_0}{c_1 - \hat{c}_1} + \frac{4\hat{e}_1}{(c_1 - \hat{c}_1)^2}, \\
    e_0 &= \frac{4\hat{e}_0}{(c_1 - \hat{c}_1)^2}, \\
    d_1 &= -\frac{2\hat{d}_2}{c_1 - \hat{c}_1} - \frac{8\hat{e}_1}{(c_1 - \hat{c}_1)^2}, \\
    e_1 &= -\frac{4\hat{e}_1}{(c_1 - \hat{c}_1)^2}, \\
    d_2 &= -\frac{c_1 + \hat{c}_1}{2}, \\
    e_2 &= \frac{4\hat{e}_2}{(c_1 - \hat{c}_1)^2},
\end{align*}
\]

into Equation (6.17) recovers Equation (6.18).

For the double 4\( \mathcal{H} \)-surface \( \mathcal{F} = \bigsqcup_\xi \mathcal{F}_\xi \) in Proposition 6.4 with the fibers \( \mathcal{F}_\xi \) of genus one, we construct the relative Jacobian fibration \( \bigsqcup_\xi \text{Jac}(\mathcal{F}_\xi) \). We have the following:

**Theorem 6.12.** The relative Jacobian fibration \( \bigsqcup_\xi \text{Jac}(\mathcal{F}_\xi) \) associated with a generic double 4\( \mathcal{H} \)-surface \( \mathcal{F} \) is a K3 surface \( \widetilde{\mathcal{Y}} \) obtained as the minimal resolution of the double-sixth surface for a generic configuration of three lines coincident in a point and a cubic. The latter defines an elliptic curve \( E \) in \( \mathbb{P}^2 = \mathbb{P}(Z_1, Z_2, Z_3) \) given by

\[
E: \quad 0 = Z_1^3 + f(Z_2, Z_3) Z_1 + g(Z_2, Z_3),
\]

where \( f, g \) were given by Equation (6.4).

**Proof.** It was shown in [23] how a Weierstrass model for \( \mathcal{F}' = \bigsqcup_\xi \text{Jac}(\mathcal{F}_\xi) \) is constructed explicitly. Applied to Equation (6.8) we obtain a Weierstrass model for \( \mathcal{F}' \) given by

\[
Y^2 = X^3 + (\xi + \mu)(\xi + \nu) A_2 X^2 \\
+ (\xi + \mu)^2 (\xi + \nu)^2 (A_1 A_3 - 4A_0 A_4) X \\
+ (\xi + \mu)^3 (\xi + \nu)^3 (A_1^2 A_4 + A_0 A_3^2 - 4A_0 A_2 A_4). 
\]

This equation has the form of Equation (6.18) considered in Proposition 6.11. For the Weierstrass model in Equation (6.17) we then reconstruct the cubic in the branch locus by setting \( Y = 0 \), rescaling \( X \mapsto (t + \mu)(t + \nu) X \), and extracting the irreducible cubic part. Using the defining equation for \( \widetilde{\mathcal{Y}} \) in Table 3 yields Equation (6.20). \( \square \)

### 7. Summary of results and discussion

The highly non-trivial connection between families of K3 surfaces and their polarizing lattices appears in string theory as the manifestation of the F-theory/heterotic string duality. This viewpoint has been studied in [16, 22, 35, 46, 47, 55, 56]. We proved in Theorem 4.7 that there are algebraic correspondences between the K3 surfaces polarized by the rank-ten lattices \( H \oplus N \) and \( H \oplus E_8(-2) \). Since the moduli spaces \( \mathfrak{M}_{H \oplus N} \) and \( \mathfrak{M}_{H \oplus E_8(-2)} \) are also the moduli spaces of F-theory models with discrete flux and the CHL string (heterotic string with CHL involution), respectively, Theorem 4.7 and Figure 4 provide a mathematical framework for the duality between he CHL string in seven dimensions and the dual F-theory models.
A natural 6-dimensional subspace that is contained simultaneously in both afore-mentioned moduli spaces is the subspace where the F-theory admits an additional anti-symplectic involution (induced by an involution on the base curve), and on the CHL string side one has an additional symplectic involution (namely, a van Geemen-Sarti involution). The duality diagram in Figure 4 then extends to Figure 5. In Theorem 5.13 we proved an explicit parametrization for elements \( \tilde{\mathcal{F}} \) of the moduli space of correspondences. A general element \( \tilde{\mathcal{F}} \) is a double-quadrics with a branch locus that is reducible and consists of two curves of bi-degree \((1, 0)\) and \((0, 1)\), respectively, and a genus-one curve in the linear system \( |\mathcal{O}_{F_0}(2, 2)| \) such that the curves intersect in ordinary rational double points. The parametrization is then based on a construction of André Weil [75], in which the Abel-Jacobi map is used to obtain embeddings of genus-one curves as symmetric divisors of bi-degree \((2, 2)\) in \( \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \).

We also showed that the F-theory moduli space has another natural 6-dimensional subspace, namely the moduli space of K3 surfaces polarized by the lattice \( (2) \oplus (-2) \oplus D_4 \oplus (-1)^6 \). This special situation corresponds to the case when, on the F-theory side, surfaces admit an additional symplectic involution and, on the CHL string side, an additional anti-symplectic involution exists. The duality diagram in Figure 4 then extends to Figure 6. The correspondences \( \mathcal{F} \) turn out to then be precisely the double \( 4\mathcal{H} \)-surfaces considered in [49]. In Theorem 6.12 we proved that the K3 surfaces \( \tilde{\mathcal{Y}} \) associated with the CHL string also carry a beautiful geometric description: they are special double-sextic surfaces branched over a configuration of three distinct lines coincident in a point and an additional generic cubic. The latter divisor gives rise to an elliptic curve capturing part of the K3 moduli coordinates.

In the two special examples above, both involving 6-dimensional subspaces of \( \mathcal{M}_{H \oplus N} \), an elliptic curve naturally emerges. This is not the elliptic curve upon which the CHL string is constructed. Rather, the elliptic curve underlying the heterotic string arises an an anti-canonical curve (cf. [5]), here, the anti-canonical curve in the rational elliptic surface \( \mathcal{R} \) in Figure 5 and Figure 6, respectively. The role of the elliptic curve underlying the heterotic string and the rational elliptic surface was investigated in previous work of the authors in [23]. In contrast, the elliptic curve that emerges in the parametrization of the 6-dimensional subspaces above is a Seiberg-Witten type curve and parameterizes certain moduli of the F-theory/CHL vacua under consideration. The relation between this Seiberg-Witten type curve and the twisted principal \( E_8 \times E_8 \) bundle over the elliptic curve defining the heterotic string will be investigated in future work by the authors.

References

[1] P. S. Aspinwall, K3 surfaces and string duality, Fields, strings and duality (Boulder, CO, 1996), 1997, pp. 421–540. MR1479699
[2] Paul S. Aspinwall, K3 surfaces and string duality, Surveys in differential geometry: differential geometry inspired by string theory, 1999, pp. 1–95. MR1772271
[3] Wolf Barth, Even sets of eight rational curves on a K3 surface, Complex geometry (Göttingen, 2000), 2002, pp. 1–25. MR1922094
[4] Alek Bedroya, Yuta Hamada, Miguel Montero, and Cumrun Vafa, Compactness of Brane Moduli and the String Lamppost Principle in \( d > 6 \), available at arXiv:2110.10157.
ON THE DUALITY OF F-THEORY AND THE CHL STRING

[5] Michael Bershady, Tony Pantev, and Vladimir Sadov, F-theory with quantized fluxes, Adv. Theor. Math. Phys. 3 (1999), no. 3, 727–773. MR1797021
[6] Lakshya Bhardwaj, David R. Morrison, Yuji Tachikawa, and Alessandro Tomasiello, The frozen phase of F-theory, J. High Energy Phys. 8 (2018), 138, front matter+47. MR3861184
[7] Noah Braeger, Andreas Malmendier, and Yih Sung, Kummer sandwiches and Greene-Plesser construction, J. Geom. Phys. 154 (2020), 103718. MR4099481
[8] Chiara Camere and Alice Garbagnati, On certain isogenies between K3 surfaces, Trans. Amer. Math. Soc. 373 (2020), no. 4, 2913–2931. MR4069236
[9] P. Candelas, Gary T. Horowitz, Andrew Strominger, and Edward Witten, Vacuum configurations for superstrings, Nuclear Phys. B 258 (1985), no. 1, 46–74. MR800347
[10] Shyamoli Chaudhuri, George Hockney, and Joseph Lykken, Maximally supersymmetric string theories in $D < 10$, Phys. Rev. Lett. 75 (1995), no. 12, 2264–2267. MR1351447
[11] Shyamoli Chaudhuri and Joseph Polchinski, Moduli space of Chaudhuri-Hockney-Lykken strings, Phys. Rev. D (3) 52 (1995), no. 2, 7168–7173. MR1375878
[12] A. Clingher, A. Malmendier, and T. Shaska, Six line configurations and string dualities, Comm. Math. Phys. 371 (2019), no. 1, 159–196. MR4015343
[13] , On isogenies among certain abelian surfaces, Michigan Math. J. (2020).
[14] Adiran Clingher and Andreas Malmendier, On K3 surfaces of Picard rank 14, available at arXiv:2009.09635.
[15] , On Picard lattices of Jacobian elliptic K3 surfaces, available at arXiv:2109.01929.
[16] Adrian Clingher, Ron Donagi, and Martijn Wijnholt, The Sen limit, Adv. Theor. Math. Phys. 18 (2014), no. 3, 613–658. MR3274790
[17] Adrian Clingher and Charles F. Doran, Modular invariants for lattice polarized K3 surfaces, Michigan Math. J. 55 (2007), no. 2, 355–393. MR2369941 (2009a:14049)
[18] , Note on a geometric isogeny of K3 surfaces, Int. Math. Res. Not. IMRN 16 (2011), 3657–3687. MR2824841 (2012f:14072)
[19] , Lattice polarized K3 surfaces and Siegel modular forms, Adv. Math. 231 (2012), no. 1, 172–212. MR2935386
[20] Adrian Clingher, Charles F. Doran, and Andreas Malmendier, Special function identities from superelliptic Kummer varieties, Asian J. Math. 21 (2017), no. 5, 909–951. MR3767270
[21] Adrian Clingher, Thomas Hill, and Andreas Malmendier, Jacobian elliptic fibrations on a special family of K3 surfaces of Picard rank sixteen, arXiv:1908.09578 [math.AG] (2019).
[22] , The duality between F-theory and the heterotic string in $D = 8$ with two Wilson lines, Lett. Math. Phys. 110 (2020), no. 11, 3081–3104. MR4160930
[23] Adrian Clingher and Andreas Malmendier, Nikulin involutions and the CHL string, Comm. Math. Phys. 370 (2019), no. 3, 959–994. MR3995925
[24] , Normal forms for Kummer surfaces, Integrable Systems and Algebraic Geometry (London Mathematical Society Lecture Note Series), 2020, pp. 119–174.
[25] François R. Cossec and Igor V. Dolgachev, Enriques surfaces. I, Progress in Mathematics, vol. 76, Birkhäuser Boston, Inc., Boston, MA, 1989. MR986969
[26] Jan de Boer, Robbert Dijkgraaf, Kentaro Hori, Arjan Keurentjes, John Morgan, David R. Morrison, and Savdeep Sethi, Triples, fluxes, and strings, Adv. Theor. Math. Phys. 4 (2000), no. 5, 995–1186 (2001). MR1868756
[27] Igor V. Dolgachev, Mirror symmetry for lattice polarized K3 surfaces, J. Math. Sci. 81 (1996), no. 3, 2599–2630. Algebraic geometry, 4. MR1420220 (97i:14024)
[28] , A brief introduction to Enriques surfaces, Development of moduli theory—Kyoto 2013, 2016, pp. 1–32. MR3586505
[29] Alice Garbagnati and Cecilia Salgado, Linear systems on rational elliptic surfaces and elliptic fibrations on K3 surfaces, J. Pure Appl. Algebra 223 (2019), no. 1, 277–300. MR3833460
[30] Alice Garbagnati and Alessandra Sarti, On symplectic and non-symplectic automorphisms of K3 surfaces, Rev. Mat. Iberoam. 29 (2013), no. 1, 135–162. MR3010125
Kummer surfaces and $K3$ surfaces with $(\mathbb{Z}/2\mathbb{Z})^4$ symplectic action, Rocky Mountain J. Math. 46 (2016), no. 4, 1141–1205. MR3563178

Elise Griffin and Andreas Malmendier, Jacobian elliptic Kummer surfaces and special function identities, Commun. Number Theory Phys. 12 (2018), no. 1, 97–125. MR3798883

David J. Gross, Jeffrey A. Harvey, Emil Martinec, and Ryan Rohm, Heterotic string theory. I. The free heterotic string, Nuclear Phys. B 256 (1985), no. 2, 253–284. MR796086

Heterotic string theory. II. The interacting heterotic string, Nuclear Phys. B 267 (1986), no. 1, 75–124. MR844696

Jie Gu and Hans Jockers, Nongeometric $F$-theory–heterotic duality, Phys. Rev. D 91 (2015), no. 8, 086007, 10. MR3417046

Heterotic string theory. I.

Heterotic string theory. II.

Klaus Hulek and Matthias Schütt, Enriques surfaces and Jacobian elliptic $K3$ surfaces, Math. Z. 268 (2011), no. 3-4, 1025–1056. MR2818742

Jong Hae Keum, Automorphisms of Jacobian Kummer surfaces, Compositio Math. 107 (1997), no. 3, 269–288. MR1458752 (98e:14039)

Automorphisms of a generic Jacobian Kummer surface, Geom. Dedicata 76 (1999), no. 2, 177–181. MR1703212 (2000c:14039)

A note on elliptic $K3$ surfaces, Trans. Amer. Math. Soc. 352 (2000), no. 5, 2077–2086. MR1707196

Yusuke Kimura, Discrete gauge groups in certain $F$-theory models in six dimensions, J. High Energy Phys. 7 (2019), 027, 17. MR3991815

Nongeometric heterotic strings and dual $F$-theory with enhanced gauge groups, J. High Energy Phys. 2 (2019), 036, front matter+38. MR3933163

K. Kodaira, On compact analytic surfaces. II, III, Ann. of Math. (2) 77 (1963), 563–626; ibid. 78 (1963), 1–40. MR0184257

Kenji Koike, Hiroshi Shiga, Nobuki Takayama, and Terumitsu Tsutsui, Study on the family of $K3$ surfaces induced from the lattice $(D_4)^1 \oplus (-2) \oplus (2)$, Internat. J. Math. 12 (2001), no. 9, 1049–1085. MR1871336 (2003d:14049)

Abhinav Kumar, $K3$ surfaces associated with curves of genus two, Int. Math. Res. Not. IMRN 6 (2008), Art. ID rnm165, 26. MR2427457 (2009d:14044)

Elliptic fibrations on a generic Jacobian Kummer surface, J. Algebraic Geom. 23 (2014), no. 4, 599–667. MR3263663

Masato Kuwata and Tetsuji Shioda, Elliptic parameters and defining equations for elliptic fibrations on a $K3$ surface, Algebraic geometry in East Asia—Hanoi 2005, 2008, pp. 177–215. MR2409557

Wolfgang Lerche, Christoph Schweigert, Ruben Minasian, and Stefan Theisen, A note on the geometry of $CHL$ heterotic strings, Phys. Lett. B 424 (1998), no. 1-2, 53–59. MR1621170

Andreas Malmendier, Kummer surfaces associated with Seiberg-Witten curves, J. Geom. Phys. 62 (2012), no. 1, 107–123. MR2854198
[55] Andreas Malmendier and David R. Morrison, \textit{K3 surfaces, modular forms, and non-geometric heterotic compactifications}, Lett. Math. Phys. 105 (2015), no. 8, 1085–1118. MR3366121
[56] Andreas Malmendier and Tony Shaska, \textit{The Satake sextic in F-theory}, J. Geom. Phys. 120 (2017), 290–305. MR3712162
[57] Andreas Malmendier and Yih Sung, \textit{Counting rational points on Kummer surfaces}, Res. Number Theory 5 (2019), no. 3, Paper No. 27, 23. MR3992148
[58] Afsaneh Mehran, \textit{Double covers of Kummer surfaces}, Manuscripta Math. 123 (2007), no. 2, 205–235. MR2306633
[59] David R. Morrison, \textit{On K3 surfaces with large Picard number}, Invent. Math. 75 (1984), no. 1, 105–121. MR728142 (85j:14071)
[60] David R. Morrison and Cumrun Vafa, \textit{Compactifications of F-theory on Calabi-Yau threefolds. I}, Nuclear Phys. B 473 (1996), no. 1-2, 74–92. MR1409284
[61], \textit{Compactifications of F-theory on Calabi-Yau threefolds. II}, Nuclear Phys. B 476 (1996), no. 3, 437–469. MR1412112 (97g:81060b)
[62] K. S. Narain, \textit{New heterotic string theories in uncompactified dimensions < 10}, Phys. Lett. B 169 (1986), no. 1, 41–46. MR834338
[63] V. V. Nikulin, \textit{Kummer surfaces}, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), no. 2, 278–293, 471. MR0429917
[64] , \textit{Finite groups of automorphisms of Kählerian K3 surfaces}, Trudy Moskov. Mat. Obshch. 38 (1979), 75–137. MR544937
[65] , \textit{Quotient-groups of groups of automorphisms of hyperbolic forms by subgroups generated by 2-reflections. Algebra-geometric applications}, Current problems in mathematics, Vol. 18, 1981, pp. 3–114. MR633160
[66] , \textit{Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. Algebra-geometric applications}, Journal of Soviet Mathematics 22 (1983), no. 4, 1401–1475.
[67] Keiji Oguiso, \textit{On Jacobian fibrations on the Kummer surfaces of the product of nonisogenous elliptic curves}, J. Math. Soc. Japan 41 (1989), no. 4, 651–680. MR1013073 (90j:14044)
[68] Keiji Oguiso and Tetsuji Shioda, \textit{The Mordell-Weil lattice of a rational elliptic surface}, Comment. Math. Univ. St. Paul. 40 (1991), no. 1, 83–99. MR1104782
[69] B. Saint-Donat, \textit{Projective models of K3 surfaces}, Amer. J. Math. 96 (1974), 602–639. MR364263
[70] A. N. Schellekens, \textit{Classification of ten-dimensional heterotic strings}, Phys. Lett. B 277 (1992), no. 3, 277–284. MR1154939
[71] Yuji Tachikawa, \textit{Frozen singularities in M and F theory}, J. High Energy Phys. 6 (2016), 128, front matter+8. MR3538766
[72] Bert van Geemen, \textit{Some remarks on Brauer groups of K3 surfaces}, Adv. Math. 197 (2005), no. 1, 222–247. MR2166182
[73] Bert van Geemen and Alessandra Sarti, \textit{Nikulin involutions on K3 surfaces}, Math. Z. 255 (2007), no. 4, 731–753. MR2274533
[74] André Weil, \textit{Remarques sur un mémoire d’Hermite}, Arch. Math. (Basel) 5 (1954), 197–202. MR61857
[75] , \textit{Euler and the Jacobians of elliptic curves}, Arithmetic and geometry, Vol. I, 1983, pp. 353–359. MR717601
[76] Edward Witten, \textit{Toroidal compactification without vector structure}, J. High Energy Phys. 2 (1998), Paper 6, 43. MR1615617
