Collatz Conjecture: Exposition and Proof through a Structured Approach

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Abstract
A structured approach for the Collatz conjecture is presented using just the odd integers that are, in turn, divided into categories based on the roles they play such as Starter, Intermediary and Terminal. The expression $4x+1$ is used as a tool to expose all the hidden and significant characteristics of the conjecture that lead us to its proof. The mixing properties of the iterates are addressed by showing that the Collatz iterates of half of all the odd integers that are of the form $4m+3$, on the average, increase by three times the value of the odd integer that was used to start with, while the iterates of those of $4m+1$, on the average, decrease by a factor of four. Further, expressions are provided to generate all the sets of odd integers where the Collatz iterate of all the integers in each set is an integer of the form $6m+1$ or $6m+5$. The significance of the Collatz net (tree) is obvious since it encompasses all the Collatz trajectories.

1. Introduction

Let $x$ be a positive integer. As per the Collatz conjecture, whenever $x$ is odd, we replace it with $(3x+1)/2$ and whenever $x$ is even we replace it with $x/2$ and ultimately the value of $x$ becomes 1 at some stage. So, when $x$ is odd, compute $y = (3x + 1)/2^\alpha$, where $\alpha$ is a positive integer and it is chosen such that $y$ becomes an odd integer. If the resulting $y$ is $> 1$ make $x = y$ and repeat the above process. As per the conjecture, the value of $y$ ultimately reaches 1 in a finite number of steps. This is the essence of the Collatz conjecture, also known as the $3x+1$ Problem. The initial exposure to the Collatz conjecture topic was through Wikipedia [1]. Later, the book [2] edited by J. C. Lagarias motivated us to work on this $3x+1$ Problem and we found the book to be a comprehensive resource on this topic. In this study, we consider all the odd positive integers only. Also here, ‘conjecture’ refers to the Collatz conjecture.

In Section 2, we describe, from the conjecture’s perspective, a few categories of the odd integers. In one categorization, we use the same type of classification of odd integers (albeit, with different names) that is discussed by Tao in his recent publication [3]. This helps us to come up with a new type of Collatz graph with a specific set of (i) Starter integers, with which the conjecture process begins and (ii) the Intermediary integers, which form the core and the essence of the process. The Intermediary integers are the odd iterates (i.e., the conjecture results) found in all the trajectories. In addition, we also introduce the Terminal integers, which end the process by producing 1 as their conjecture result.

In Section 3, we start with sample conjecture trajectories, presented in a Table, for a few Starter integers and also in the form of graphs just with the odd iterates. In the graphs, we observe the presence of $4x+1$ that links the iterates from the conjecture trajectories of the different Starter integers.

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Based on this, in Sections 4, we develop a generalized version of $4x+1$ and use it in Section 5 with all the odd integers to explore the hidden characteristics of the conjecture itself. We provide sample Tables where each row has a set of odd integers, all of which produce the same conjecture iterate and, most importantly, we provide expressions to get such sets of odd integers for all of the odd iterates. That is, each row in these extended Tables has a predecessor set (see page 62 in [2]) of the corresponding Intermediary integer in the conjecture iterate column. We also provide the basic logic for constructing the conjecture trajectory for any odd integer in a different way by accessing the results in these Tables using suitable expressions.

In Section 6, we also introduce the Collatz Tree with its root (i.e., End integer 1) above, the branches (i.e., the Intermediary integers) below, and the leaves (i.e., Starter integers) spread out everywhere. As a result, all the odd integers go up towards the root using the conjecture. Here, we start with known integers for the first two layers below the root (i.e., 1) of the Collatz Tree and provide the core logical steps for constructing the remaining lower layers with several segments (one under each iterate in the layer above).

In Section 7, we study the complete behavior of all the odd integers that use just one division by 2 in their conjecture process that results in sequentially increasing iterates. Half of all the odd integers are in this category. We classify, using an expression, each of these integers based on its total number of sequentially increasing Collatz iterates and present all the results in a Table. (We use the term Alpha1 Trajectory Length for such a total number; Alpha1 is: $\alpha = 1$).

In Section 8, based on such detailed studies with all the odd integers and their iterates, we show with realistic (not heuristic) evidence that, for half of all the odd integers, their odd iterates in the corresponding trajectories, on the average, increase (swell) by a factor of 3 while for the other half of all the odd integers the corresponding trajectories reduce (shrink), on the average, by a factor of 4. This clarifies the mixing properties and proves that there cannot be any divergent trajectories. In Section 9, following a brief summary of this study, we emphasize the significance of the Collatz net (tree) since it encompasses all the strings (trajectories) of the net.

2. Types of odd Integers from Collatz Conjecture Perspective

We start with 1 since that is where everything ends in the Collatz conjecture. Applying the conjecture to 1: i.e., $3 \times 1 + 1 = 4$; $4/2 = 2$; and $2/2 = 1$, we note that the process keeps looping. To progress further, we add the result before the division (i.e. 4) to the starting integer (i.e., 1) and apply the conjecture to this sum ($1 + 4 = 5$): $3 \times 5 + 1 = 16 = 4^2$ and $16/2^4 = 1$. Continuing with the same approach (i.e., adding the result before the division to its starting integer), the new sum is: $5 + 16 = 21$; and $3 \times 21 + 1 = 64 = 4^3$ and $64/2^6 = 1$. Generalizing the above approach, let $T_k$, (with $k = 0, 1, 2, 3, \ldots$) be a set of integers generated by:

$$T_k = 1 + 4 + 4^2 + 4^3 + \ldots + 4^k = \sum_{n=0}^{k} 4^n = (4^{k+1} - 1)/3 \quad (1)$$

Applying the conjecture to $T_k$, the final result is 1 as summarized below:

$$3T_k + 1 = 4^{k+1} \text{ and } 4^{k+1}/2^{2(k+1)} = 1 \quad (2)$$

The integers $T_k$'s (for every positive integer $k$) collectively provide the required closure to the conjecture. We call them the Terminal integers (consisting of 1, 5, 21, 85, 341, 1365, ... etc.) since they terminate the Collatz conjecture process by producing 1, the End integer, as their result. These Terminal integers are also generated, using a well-known method [1] (aka a Syracuse Function property) starting with $T_0 = 1$ and using

$$T_{k+1} = 4T_k + 1. \quad (3)$$
Now, let us apply the conjecture for a few odd integers: 3, 9, 15 and 21. Their trajectories with only the odd iterates [using \((3x +1)/2^a\)] are: \(3 \rightarrow 5 \rightarrow 1\); \(9 \rightarrow 7 \rightarrow 11 \rightarrow 17 \rightarrow 13 \rightarrow 5 \rightarrow 1\); \(15 \rightarrow 23 \rightarrow 35 \rightarrow 53 \rightarrow 5 \rightarrow 1\); and \(21 \rightarrow 1\). Both 5 and 21, being Terminal integers, end the trajectory with 1 in one step. Both 13 and 53, as they go through the conjecture, result in 5.

The rationale for this lies in the generalized version of Equation (3) \([53= 4 \times 13+1]\):

\[
y = 4x+1
\]

Equation (4) will be used extensively later on. Since the 3x+1 conjecture is about 3, we first examine – using a Theorem – the significance of the odd integers that are ‘odd multiples of 3’.

THEOREM 2.1. In the conjecture process, an odd iterate cannot be an odd multiple of 3.

Proof. Let \(x\) and \(y\) be odd integers such that \(y = (3x +1)/2^a\), where \(x\), a positive integer, is chosen such that \(y\) is an odd integer. We reverse the process and check if \(y\) could be an odd multiple of 3 (i.e., \(y = 3z\) where \(z\) is an odd integer: 1, 3, 5, …) and examine what type of \(x\) we get as a result of this new restriction on \(y\):

\[
x = (2^a \cdot x \cdot y −1)/3 = (2^a \cdot X \cdot 3z −1)/3 = 2^a \cdot z −1/3
\]

We notice that, for any odd value of \(z\) and for any positive integer value of \(\alpha\), \(x\) could no longer be an integer due to the restriction we placed on \(y\). Hence we realize that, for any odd integer values of \(x\), the result of \((3x +1)/2^a\) can never be an odd multiple of 3. So, the odd multiples of 3 could only be at the starting positions of the Collatz conjecture process (Q.E.D.)

2.1. Starter and Intermediary Integers. In view of Theorem-2.1, the odd integers are split into two types: (i) the odd multiples of 3: \(3(2m +1)\) for \(m = 0, 1, 2, 3, \ldots\), (we name them as the Starter integers) and (ii) all the rest of the odd integers: \(3(2m +1)\pm2\), with \(m = 0, 1, 2, 3, \ldots\); we name them as the Intermediary integers and they are of two types: \(6m\pm1\) and \(6m\pm5\). In the Collatz conjecture, we begin the process with each of the Starter integers and continue applying the conjecture sequentially to the resulting odd integers until we reach a Terminal integer which then leads to the End integer 1. All those resulting odd integers occurring in between the Starter and End integers (including the Terminal integer) are found to be the Intermediary integers. [Here is an example to illustrate the classification of odd integers from the Collatz conjecture perspective: Earlier on, we saw the Collatz conjecture trajectory for integer 9: \(9 \rightarrow 7 \rightarrow 11 \rightarrow 17 \rightarrow 13 \rightarrow 5 \rightarrow 1\). Here, 9 is a Starter integer, 5 is a Terminal integer and 1 is the End integer. The other in-between (i.e., 7, 11, 17, 13) are the Intermediary integers. Here 7 and 13 are of type \(6m\pm1\); and 11, and 17 are of type \(6m\pm5\)] We now introduce the idea of a Conjecture Table using a set of Terminal integers for illustration.

2.2. Collatz Conjecture Table for Terminal Integers. We present in Table-1 with the columns showing the iterates of the conjecture applied to odd integers beginning with the corresponding Starter integers for a set of initial Terminal integers (5, 21, 85, 341, 1365, …). We note that, starting with 21, every third Terminal integer is also a Starter integer (odd multiple of 3). In Table-1, the Terminal integers are underlined and they all converge to 1 directly with no in-between iterates.

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Using a reverse process – i.e., going up the trajectory – we can find the Starter integer for a given Intermediary-cum-Terminal integer like 85. We consider, for \( n \geq 1 \), \( 2^n X 85 \) and subtract 1: i.e., \( 2X85 – 1 = 169, 4X85 – 1 = 339, 8X85 – 1 = 679 \); and pick the first result that is a multiple of 3. Here, we see that 339 (= 3X113) is a multiple of 3. However, we see that 113 is not a Starter integer and so we repeat the process now with 113: \( 2X113 – 1 \) = 225 (= 3X75). Since 75 is a Starter integer, the reverse process ends here.

| Starter | 3  | 21 | 75 | 201 | 1365 | 7281 | 17019 | 87381 | 245481 | 932067 | 5592405 |
|---------|----|----|----|-----|------|------|-------|-------|---------|---------|-----------|
| Intermediary | 113 | 151 | 25529 | 184111 |
| Intermediary | 227 | 19147 | 276167 |
| Intermediary | 14563 | 414251 |
| Intermediary | 621377 |
| Intermediary | 466033 |
| Terminal | 5  | 21 | 85 | 341 | 1365 | 5461 | 21845 | 87381 | 349525 | 1398101 | 5592405 |
| End | 1  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**TABLE 1.** Collatz Conjecture Table with Sample Terminal Integers with Their Starter Integers

In the following, we present a Table that shows the complete Collatz trajectory for a few smaller Starter integers to understand the behavior of the iterates. We also present the Collatz graphs, drawn using the results in this Table. The objective is to identify, through observation and analysis of these results, any hidden intrinsic characteristics of the Collatz conjecture.

### 3. Collatz Conjecture Tables and Graphs Beginning with A Few Starter Integers

Having defined, from the conjecture perspective, the three main types of integers (Starter, Intermediary and Terminal), we now apply the conjecture to a set of sample Starter integers (from 3 to 255). We show the results consisting of only the odd integers in Table 2. The first column shows the Starter integer from 3 to 255; here, some of the odd integers with shorter Collatz trajectories are presented in the same row. All the Starter integers, as they go through the conjecture, end with one of the Terminal integers followed by the End integer 1. The Intermediary integers, which are the iterates in the trajectory, are listed in the second column in the order in which they occur. For convenience, we use 27 as the reference since some Starter integers (> 27) have several common Intermediary integers that are in 27’s conjecture trajectory. In the following we present a few observations from Table 2:

(i) In several of these Collatz trajectories, there are continuous steady increase in the values of conjecture iterates by about \( 1\frac{1}{2} \) times the previous value, since they all use \( (3x+1) / 2 \) i.e., with \( \alpha = 1 \). For instance, in the Collatz trajectory for the Starter integer 255, there are seven continuous increases before a reduction occurs:

\[
255 \rightarrow 383 \rightarrow 575 \rightarrow 863 \rightarrow 1295 \rightarrow 1943 \rightarrow 2915 \rightarrow 4373 \rightarrow 205 \rightarrow \ldots \rightarrow 1
\]

However, in this case, this steady increase is followed by a drastic reduction with \( \alpha = 6 \), taking the iterate value lower than the one we started with (i.e., 255 to 205). It appears that there is a need to study the number of such continuous iterate increases for all such odd integers..
The Intermediary integers occur only once in each trajectory. The lengths of the Collatz trajectories vary considerably. The Starter integers use different values of $\alpha$ in computing their first Collatz conjecture iterate. We note that $\alpha = 1$ is used for every alternate odd multiples of 3 starting from 3 (i.e., 3, 15, 27, …); $\alpha = 2$ for every fourth starting from 9 (i.e., 9, 33, 57, …); and so on.

Next, we present a new type of Collatz Conjecture graph using the results in Table 2.

3.1. Collatz Conjecture Graphs. Based on the results in the Table 2, we present a new type of Collatz Graphs (different to those presented in [1] and [2]) and we use MS-Visio to draw these graphs. In these flow-chart form of graphs, we use different graphic symbols to place the different types of odd integers. In our Tables, there are just 43 Starter (3 to 255) integers. For the sake of convenience in presenting the conjecture graphs, we split them into three groups. First, we collected all those Starter integers that have 5 as their Terminal integer and put them into two categories: (i) those that reach 5 from Pre-Terminal 13 (13 $\rightarrow$ 5) and (ii) the rest that reach 5 from Pre-Terminal 53 (53 $\rightarrow$ 5). (The Pre-terminals of 5 are: 3, 13, 53, 213, … $\rightarrow$ 5.) Also, we put all those integers that do not have 5 as their Terminal integers into category (iii).

In these graphs, the flow is assumed to be downwards, when there are no arrows.

Figure 1A shows the Collatz Graph for categories (i) and (iii) and Figure 1B shows the graph for category (ii). Also, the symbols used for the different types of integers are explained under the graphs. In these graphs, some of the iterates are put together in a single block mainly for convenience and also the graphs (i) and (ii) can be combined into one with just a single Terminal integer box for 5. In these Collatz Graphs, every odd integer occurs only once. This is expected since these graphs are based on the results presented in Table 2 where there are no duplicate iterates within any specific Collatz trajectory. In the following, we discuss some significant observations made from these Collatz Graphs.

3.2 Observations and Presence of $4x + 1$. We see the role of Equation (4) in the two Collatz Graphs (Figures 1A and 1B). For instance in Fig. 1B, the integers 27, 109, and 437 produce the same iterate 41 as they go through the conjecture process: $(3 \times 27 + 1)/2 = (3 \times 109 + 1)/8 = (3 \times 437 + 1)/32 = 41$. Here we note the presence of Equation (4): $4 \times 27 + 1 = 109$ and $4 \times 109 + 1 = 437$. In the two main graphs (Fig. 1A and 1B), we see the presence of Equation (4) at some 34 intersections. In Table 3, we list the two or three integers at these intersections along with their Collatz conjecture iterate in the Result-columns.

Observation: We considered only 43 Starter integers in Table-2. In that small sample, among the 34 Intermediary integers in the two Result-columns in Table-3, we note that 11 are of the type $6m + 1$ and 23 are of the type $6m + 5$ (roughly, a ratio of 1:2). Even though $6m + 1$ and $6m + 5$ are equal in number (each with one-third of all the odd integers), it appears that relatively more odd integers, possibly $2/3$rd of them, generate iterates of the type $6m + 5$ and the remaining $1/3$rd only generate the iterates of the type $6m + 1$.

This could be an important observation concerning the iterates and the odd integers that produce them. Next we examine the importance of $4x + 1$ in this study. To begin with, we look at odd integers, expressed in binary, that are related by $4x + 1$. 

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### Starter Collatz Conjecture Trajectories $[(3x+1)/2^\alpha]$ for Some Initial Starter Integers

| Starter Integers | Trajectories |
|------------------|--------------|
| 3; 9; 15; 21     | 3 -> 5, 1; 9-> 7, 11, 17, 13, 5, 1; 15->23, 35, 53, 5, 1; 21-> 1 |
| 27               | 27 -> 41, 31, 47, 71, 107, 161, 121, 91, 137, 103, 155, 233, 175, 263, 395, 593, 445, 167, 251, 377, 283, 425, 319, 479, 719, 1079, 1619, 2429, 911, 1367, 2051, 3077, 577, 433, 325, 61, 23, 35, 53, 5, 1 |
| 33; 39           | 33 -> 25, 19, 29, 11, 17, 13, 5, 1; 39 -> 59, 89, 67, 101, 19, 29, 11, 17, 13, 5, 1. |
| 45; 51; 57       | 45 -> 17, 13, 5, 1; 51 -> 77, 29, 11, 17, 13, 5, 1; 57 -> 43, 65, 49, 37, 7, 11, 17, 13, 5, 1. |
| 63               | 63 -> 95, 143, 215, 323, 485, 91, ... see under 27 for the rest starting in its first line |
| 69, 75, 81       | 69 -> 13, 5, 1; 75 -> 113, 85, 1; 81 -> 61, 23, 35, 53, 5, 1; |
| 87, 93, 99       | 87 -> 131, 197, 37, 7, 11, 17, 13, 5, 1; 93 -> 35, 53, 5, 1; 99 -> 149, 7, 11, 17, 13, 5, 1. |
| 105              | 105 -> 79, 119, 179, 269, 101, 19, 29, 11, 17, 13, 5, 1; |
| 111              | 111 -> 167, 251, ... see under 27 for the rest starting in its second line |
| 117, 123         | 117 -> 11, 17, 13, 5, 1; 123 -> 185, 139, 209, 157, 59, 89, 67, 101, 19, 29, 11, 17, 13, 5, 1. |
| 129              | 129 -> 97, 73, 55, 83, 125, 47, ... see under 27 for the rest starting in its first line |
| 135, 141         | 135 -> 203, 305, 229, 43, 65, 49, 37, 7, 11, 17, 13, 5, 1; 141 -> 53, 5, 1. |
| 147              | 147 -> 221, 83, ... see under 129 and see under 27 for the rest starting in its first line |
| 153              | 153 -> 115, 173, 65, 49, 37, 7, 11, 17, 13, 5, 1; |
| 159              | 159 -> 239, 359, 539, 809, 607, 911, ... see under 27 for the rest starting in its second line |
| 165              | 165 -> 31, ... see under 27 for the rest starting in its first line |
| 171              | 171 -> 257, 193, 145, 109, 41, ... see under 27 for the rest starting in its first line |
| 177              | 177 -> 133, 25, 19, 29, 11, 17, 13, 5, 1. |
| 183              | 183 -> 275, 413, 155, ... see under 27 for the rest starting in its first line |
| 189              | 189 -> 71, ... see under 27 for the rest starting in its first line |
| 195              | 195 -> 293, 55, ... see under 129 and see under 27 for the rest starting in its first line |
| 201              | 201 -> 151, 227, 341, 1. |
| 207              | 207 -> 311, 467, 701, 263, ... see under 27 for the rest starting in its first line |
| 213, 219         | 213 -> 5, 1; 219 -> 329, 247, 371, 557, 209, 157, 59, 89, 67, 101, 19, 29, 11, 17, 13, 5, 1. |
| 225              | 225 -> 169, 127, 191, 287, 431, 647, 971, 1457, 1093, 205, 77, 29, 11, 17, 13, 5, 1. |
| 231              | 231 -> 347, 521, 391, 437, 41, ... see under 27 for the rest starting in its first line |
| 237              | 237 -> 89, 67, 101, 19, 29, 11, 17, 13, 5, 1. |
| 243              | 243 -> 365, 137, ... see under 27 for the rest starting in its first line |
| 249              | 249 -> 187, 281, 211, 317, 119, 179, 269, 101, 19, 29, 11, 17, 13, 5, 1. |
| 255              | 255 -> 383, 575, 863, 1295, 1943, 2915, 4373, 205, 77, 29, 11, 17, 13, 5, 1. |

**TABLE 2.** Collatz Conjecture Trajectory Table for Odd Multiples of 3 from 3 to 255
Figure-1A: Example Collatz Structured Graphs
Figure-1B: Example Collatz Structured Graphs
TABLE 3. Presence of $4x+1$ extracted from the two graphs in Fig. 1A and 1B.

4. Significance of $4x+1$ expression

The Terminal integers, $T_k$s, in binary form are: 1, 101, 10101, … (alternate 1 and 0 with 1s at both ends). Multiplying the Terminal integers by 3, we get: 11, 1111, 111111, … and adding 1 to this result, we get: 100, 10000, 1000000, …; hence, it is easy to visualize that all $T_k$s – that are linked by the $4x+1$ expression – converge to 1 as they go through the Collatz conjecture. While Equation (1) is intended for Terminal Integers, Equations (4) is applicable to all odd integers. Let $x = 27$ (in binary, 11011), its Collatz iterate is 41 (in binary 101001). Also $4x + 1 = 109$ (in binary, 1101101); as 109 goes through the conjecture, it becomes, in binary, 10100100, resulting in 101001 (due to division by $2^2$) which is 41. So, in $4x+1$, the binary form of $x$ is preserved and just shifted by two positions to the left.
In the case of $3x+1$, we saw a similar pattern with Terminal integers (due to their unique binary format). However, when we compute $3x+1$ for non-Terminal odd integers, the original binary representation of $x$ is distorted. This unique characteristics of $4x+1$ leads us to Theorem 4.1 for generalizing an earlier observation in Section 2 (the odd integers in every trajectory occur only once) so as to state that there are no duplicate iterate within any Collatz trajectory.

THEOREM 4.1. As the conjecture process begins with a non-Terminal Starter integer and reaches its Terminal integer; no duplicates are found among the resulting Collatz iterates in the Collatz trajectory. (This implies that there are no loops in the trajectories as well.)

Proof: Let $S$ be a Starter integer and $T$ be its Terminal integer in its Collatz trajectory: $S \to a \to b \to c \to d \to e \to f \to g \to h \to ... \to m \to n \to o \to p \to q \to r \to s \to ..., \to T \to 1$. The entries in between $S$ and $T$ are a set of Collatz iterates (i.e., Intermediary integers). To prove Theorem 4.1 by negation, we assume that there is a duplication (first one) among these iterates say, $g = q$. This would mean that, by Equation (4) and by assuming that $p = 4^k f + 1$ so that both $f$ and $p$, as they go through the conjecture, produce the same iterate ($g = q$). This would require, from our earlier discussion, $p$ to retain the binary form of $f$ shifted to the left by two digits. However, we also realize that as the $3x+1$ conjecture continues from $f$, it changes the binary form of $f$ in the results that follow and fails to retain the original binary contents as it goes through the Collatz trajectory to reach $p$: i.e., $p$ will not have the binary structure of $f$ and, as a result, $p \neq 4^k f + 1$, invalidating our assumption that $g = q$. (Q. E. D.)

4.1. Generalization of $4x+1$ Expression. Now, let us consider a sequential version of Equation (4) with an intention to generalize Equation (1) that is applicable for Terminal integers, to other odd integers. Let $C_0$ and $y$ be odd integers such that $y = (3C_0 + 1)/2^\alpha$. Also, let $C_k$ for $k = 0, 1, 2, 3,...$ be such that $C_{k+1} = 4C_k + 1$. We present below an expression that lets all the $C_k$s, as they go through the Collatz conjecture, end up with the same odd integer $y = (3C_k + 1)/2^\alpha$, each $C_k$ using an appropriate value for $\alpha$. This turns out to be the generalized version of Equation (1) that resembles Equation (4), (and here for $k > 0$):

$$C_k = 4^k C_0 + \sum_{i=0}^{k-1} (4^i) = 4^k C_0 + T_{k-1} \quad (6)$$

Even though we mainly consider odd integers in this study, $C_0$ could be an even integer in Equation (6) as long as the application of Collatz conjecture is considered for $C_k$ with $k > 0$. Hence, the expression for $C_k$ in Equation (6) can be used for generating sets of odd integers with the $4x+1$ characteristic. In the following, we use Equation (6) to generate all the sets of odd integers with $4x+1$ characteristic so that each set’s Collatz iterate is an Intermediary integer.

5. Intermediary Integers and their Predecessor sets

The main focus here is on the two types of Intermediary integers ($6m+1$ and $6m+5$). In Section 1, we noted that the Terminal integers (1, 5, 21, 85, 341, ...) that are linked by $4x+1$ produce 1 as they go through the Collatz conjecture. Also, in Section 3, we noted that the Pre-Terminal integers of 5 consist of the following: 3, 13, 53, 213, ... and these are also linked by $4x+1$ with their Collatz iterate as 5. Here, we use these Terminal integers and Pre-Terminal integers (of 5) to start with and adapt Equation (6) suitably to find the Collatz iterates for all the odd integers.
The iterate for every Terminal integer is 1, which is also the first integer in the $6m+1$ type iterates (Intermediary integers). Also, the iterate for the Pre-Terminal integers 3, 13, 53, … is 5, which is also the first integer in the $6m+5$ type iterates (Intermediary integers). Related to the $4x+1$ expression, we made an Observation in sub-section 3.2 that states: relatively more odd integers, possibly $2/3$rd of them, generate iterates of the type $6m+5$ and the remaining $1/3$rd only generate the iterates of the type $6m+1$. Our objective is to generate – with suitable $C_0$ in Equation 6 – all the remaining sets of odd integers (that are related by $4x+1$) whose Collatz iterates are the remaining integers of both $6m+1$ and $6m+5$ ($7, 13, 19, \ldots$ and $11, 17, 23, \ldots$). To reflect the ratio of odd integers involved in producing the iterates of the two types, we could let $C_0 = n$ for $6m+5$ and $C_0 = 2n$ for $6m+1$, where $n > 0$. This could be one way of ensuring that $6m+5$ will be the iterates for double the number of odd integers compared with the number of odd integers that $6m+1$ will be the iterates for.

For illustration, in Table 4A, we start with just the first five Terminal integers 1, 5, 21, 85, and 341 (related by $4x+1$) that are listed in the top row ($n = 0$) along with the first $6m+1$ integer i.e., 1 (for $m = 0$) in the last column. Also, in Table-4B, we start with only the first six Pre-Terminal integers of 5, which are: 3, 13, 53, 213, 853, and 3413 (related by $4x+1$) that are listed in the top row ($n = 0$) along with the first $6m+5$ integer i.e., 5 (for $m = 0$) in the last column. In Table 4A, we create a sequence of odd integers for each of these five top row Terminal integers, listed under their respective columns, using the expression in Equation (6) with $C_0 = 2n$: $T_{k-1} + 4^k (2n)$; here $n$ is the row (1, 2, 3, …) and $k$ is the column (1, 2, 3, 4, 5). Since all the five odd integers in each row are related by $4x+1$, they all produce, using the respective $\alpha$ values shown in the Title row, the same Collatz iterate (listed in the last column of Table 4A). In Table 4B, we start with the first six Pre-Terminal integers for 5 and follow the same procedure except with $P_k$ in place of $T_k$ and $C_0 = n$ in Equation (6): $P_k + 4^k n$, where $n$ is the row (1, 2, 3, …) and $k$ is the column (1, 2, 3, 4, 5, 6). The six integers in each row produce, using the respective $\alpha$ values shown in the Title row, the same Collatz iterate (shown in the last column of Table 4B). We show only a few columns and rows in the Tables 4A & 4B; however $n$ and $k$ in the two expressions take all the positive integer values: $T_{k-1} + 4^k (2n)$ and $P_k + 4^k n$.

5.1. Main Observations. We list observations from the “Extended” Tables 4A and 4B.

Observation 1: The last column in Table 4A lists the iterates that are the Intermediary integers of the form $6m+1$ for all $m = 0, 1, 2, 3, \ldots$; and the last column in Table 4B lists the iterates that are the Intermediary integers of the form $6m+5$ for all $m = 0, 1, 2, 3, \ldots$. Hence each of these two last columns (listing $6m+1$ and $6m+5$) have one-third of all the odd integers.

Observation-2: Examining the columns other than the two iterate ones (discussed above), we note that all the odd integers (including the Starter and the Intermediary integers) are present in the two Extended Tables 4A and 4B (extended in both directions – down and right). Thus all the odd integers collectively produce all the Intermediary integers as their Collatz iterates. Most importantly, there is no duplication of any odd integer in the two Tables.

Observation-3: Half of the odd integers are of the form $4m+3$ ( $m = 0, 1, 2, 3,\ldots$) and they are all under the second column of Table 4B, while $4m+1$, the remaining half, are spread out in all the remaining non-iterate columns of both the Tables 4A and 4B. We note that one-third of all the odd integers are in Table 4A: i.e., adding all the fractions of odd integers in the various columns of Table 4A, we get $\sum_{i=1}^{\infty}(1/4)^i = 1/3$. As a result, two-thirds of all the odd integers (i.e., all the $4m+3$ and the remaining $4m+1$) are in Table 4B.
| $\alpha$ in $2^\alpha$ | 2 | 4 | 6 | 8 | 10 | $(3x+1)/2^\alpha$ |
|-----------------------|---|---|---|---|----|-----------------|
| Integer n             | 1+8*n | 5+32*n | 21+128*n | 85+512*n | 341+2048*n |
| 0                     | 1   | 5   | 21   | 85   | 341 | 1               |
| 1                     | 9   | 37  | 149  | 597  | 2389 | 7               |
| 2                     | 17  | 69  | 277  | 1109 | 4437 | 13              |
| 3                     | 25  | 101 | 405  | 1621 | 6485 | 19              |
| 4                     | 33  | 133 | 533  | 2133 | 8533 | 25              |
| 5                     | 41  | 165 | 661  | 2645 | 10581| 31              |
| 6                     | 49  | 197 | 789  | 3157 | 12629| 37              |
| 7                     | 57  | 229 | 917  | 3669 | 14677| 43              |
| 8                     | 65  | 261 | 1045 | 4181 | 16725| 49              |
| 9                     | 73  | 293 | 1173 | 4693 | 18773| 55              |
| 10                    | 81  | 325 | 1301 | 5205 | 20821| 61              |
| 11                    | 89  | 357 | 1429 | 5717 | 22869| 67              |
| 12                    | 97  | 389 | 1557 | 6229 | 24917| 73              |
| 13                    | 105 | 421 | 1685 | 6741 | 26965| 79              |
| 14                    | 113 | 453 | 1813 | 7253 | 29013| 85              |
| 15                    | 121 | 485 | 1941 | 7765 | 31061| 91              |
| 16                    | 129 | 517 | 2069 | 8277 | 33109| 97              |
| 17                    | 137 | 549 | 2197 | 8789 | 35157| 103             |
| 18                    | 145 | 581 | 2325 | 9301 | 37205| 109             |
| 19                    | 153 | 613 | 2453 | 9813 | 39253| 115             |
| 20                    | 161 | 645 | 2581 | 10325| 41301| 121             |
| 21                    | 169 | 677 | 2709 | 10837| 43349| 127             |
| 22                    | 177 | 709 | 2837 | 11349| 45397| 133             |
| 23                    | 185 | 741 | 2965 | 11861| 47445| 139             |
| 24                    | 193 | 773 | 3093 | 12373| 49493| 145             |
| 25                    | 201 | 805 | 3221 | 12885| 51541| 151             |
| 26                    | 209 | 837 | 3349 | 13397| 53589| 157             |
| 27                    | 217 | 869 | 3477 | 13909| 55637| 163             |
| 28                    | 225 | 901 | 3605 | 14421| 57685| 169             |
| 29                    | 233 | 933 | 3733 | 14933| 59733| 175             |
| 30                    | 241 | 965 | 3861 | 15445| 61781| 181             |
| 31                    | 249 | 997 | 3989 | 15957| 63829| 187             |
| 32                    | 257 | 1029| 4117 | 16469| 65877| 193             |
| 33                    | 265 | 1061| 4245 | 16981| 67925| 199             |
| 34                    | 273 | 1093| 4373 | 17493| 69973| 205             |
| 35                    | 281 | 1125| 4501 | 18005| 72021| 211             |

**TABLE 4-A.** Influence of $4x+1$ in Collatz Conjecture Starting with Terminal Integers
\[
\alpha \text{ in } 2^\alpha
\]

| Integer n | 1 | 3 | 5 | 7 | 9 | 11 | \((3x+1)/2^\alpha\) for all columns |
|-----------|---|---|---|---|---|----|----------------------------------|
| 0         | 3 | 13| 53| 213|853|3413|5|
| 1         | 7 | 29|117| 469|1877|7509|11|
| 2         | 11| 45|181| 725|2901|11605|17|
| 3         | 15| 61|245| 981|3925|15701|23|
| 4         | 19| 77|309|1237|4949|19797|29|
| 5         | 23| 93|373|1493|5973|23893|35|
| 6         | 27|109|437|1749|6997|27989|41|
| 7         | 31|125|501|2005|8021|32085|47|
| 8         | 35|141|565|2261|9045|36181|53|
| 9         | 39|157|629|2517|10069|40277|59|
| 10        | 43|173|693|2773|11093|44373|65|
| 11        | 47|189|757|3029|12117|48469|71|
| 12        | 51|205|821|3285|13141|52565|77|
| 13        | 55|221|885|3541|14165|56661|83|
| 14        | 59|237|949|3797|15189|60757|89|
| 15        | 63|253|1013|4053|16213|64853|95|
| 16        | 67|269|1077|4309|17237|68949|101|
| 17        | 71|285|1141|4565|18261|73045|107|
| 18        | 75|301|1205|4821|19285|77141|113|
| 19        | 79|317|1269|5077|20309|81237|119|
| 20        | 83|333|1333|5333|21333|85333|125|
| 21        | 87|349|1397|5589|22357|89429|131|
| 22        | 91|365|1461|5845|23381|93525|137|
| 23        | 95|381|1525|6101|24405|97621|143|
| 24        | 99|397|1589|6357|25429|101717|149|
| 25        |103|413|1653|6613|26453|105813|155|
| 26        |107|429|1717|6869|27477|109909|161|
| 27        |111|445|1781|7125|28501|114005|167|
| 28        |115|461|1845|7381|29525|118101|173|
| 29        |119|477|1909|7637|30549|122197|179|
| 30        |123|493|1973|7893|31573|126293|185|
| 31        |127|509|2037|8149|32597|130389|191|
| 32        |131|525|2101|8405|33621|134485|197|
| 33        |135|541|2165|8661|34645|138581|203|
| 34        |139|557|2229|8917|35669|142677|209|
| 35        |143|573|2293|9173|36693|146773|215|

**TABLE 4-B. Influence of 4x+1 in Collatz Conjecture Starting with Pre-Terminal Integers for 5**
Observation -4: We also note that the Starter integers are evenly spread out in the two Tables 4A and 4B: i.e., every third integer in all the columns (except the two iterate columns) and in all the rows is an odd multiple of 3. Since the Starter integers are evenly spread out, this explains why they use different values of $\alpha$ as they go through the Collatz conjecture process, as noted under Observation (ii) in Section 2. Here we clearly see how uniformly the Starter integers mix, in the Collatz conjecture process, with the Intermediary integers. It appears that the pending issue in [3] is to do with proving the kind of mixing between the two different categories of odd integers (i.e., Starter and Intermediary). We find this mixing to be uniform.

[Note: We now look at the significance of Tao’s explanation for the “irregularities” in the behavior of these $6m+1$ and $6m+5$ odd integers (i.e., $\text{Syr}(N)$ given below). Tao’s statement from the first URL-link listed under [3] (that reflects Observation-3): When viewed $3$-adically, we soon see that iterations of the Syracuse map become somewhat irregular. Most obviously, $\text{Syr}(N)$ is never divisible by $3$. A little less obviously, $\text{Syr}(N)$ is twice as likely to equal $2 \mod 3$ as it is to equal $1 \mod 3$. This is because for a randomly chosen odd $N$, the number of times $\alpha$ that $2$ divides $3N + 1$ can be seen to have a geometric distribution of mean $2$ – it equals any given value $\alpha \in N + 1$ with probability $2^{-\alpha}$. Such a geometric random variable is twice as likely to be odd as to be even, which is what gives the above irregularity.]

Observation -5 In each row of the two Tables, there is a set of odd integers (related by $4x+1$) whose Collatz iterate – i.e., $(3x+1)/2^\alpha$ using different values for $\alpha$ – match with the Intermediary integer in the last column of that row. Thus each row displays the entire predecessor set of the corresponding Intermediary integer in the last column, addressing the issues discussed on pages 62-63 in [2] regarding the size of predecessor set.

Observation-6: The Tables 4A and 4B provide the actual evidence for the heuristic probabilities discussed on page 34 in [2] (and also the URL listed under [2]) that the expected growth in size between two consecutive odd integers in such a trajectory is the multiplicative factor $\frac{3}{4} < 1$: i.e., on the average, the iterates in a trajectory tend to shrink in size. We see the following evidence in the two Tables. In the second column of Table-4B, all the alternate odd integers (i.e., $\frac{1}{2}$ of all the odd integers that are of the form $4m+3$, for all $m = 0, 1, 2, 3, \ldots$) are listed and they all use $\alpha = 1$ in $(3x+1)/2^\alpha$. Hence each of these integers in this column produces a conjecture result that is larger (by about $1\frac{1}{2}$ times) than itself. (In Section 7, we conduct an in depth analysis of the $4m+3$ type integers.) In the two Tables, all the odd integers in all the other columns, that are of type $4m+1$, use $\alpha \geq 2$ in $(3x+1)/2^\alpha$ to produce the results shown in their respective last column. Specifically, looking at all the columns in the two Tables, we note that, as the value of $\alpha$ increases by 1, the number of odd integers in their respective column reduces by half (i.e., higher the value of $\alpha$, proportionately fewer the number of odd integers). We will have more discussion on these observations in Section 8. Now, we show, in the next sub-section, that the Starter integers are like the leaves on the Collatz tree.

5.2. Constructing Collatz Tree and Conjecture Trajectory. We transform the Collatz Graphs (Figures 1A & 1B) into a single Collatz Tree, shown in Figure 2, with its root (i.e., End integer 1) above, the branches (i.e., Intermediary integers) below and the leaves (i.e., Starter integers) spread out everywhere. Here, we also use some results from the Extended Tables 4A and 4B and also the $4x+1$ expression to include additional odd integers so that it is symmetrical. Also in Figure 3, we show a Collatz Tree with just the iterates (i.e., the Intermediary integers). In Section 6, we discuss a method for systematically generating all the odd integers for the orderly placement in different lower layers of the Collatz Tree with 1 at the top.
Here is a simple logic to build the Collatz trajectory for a given odd integer $x > 1$. Using the rows of odd integers along with their conjecture iterate that are in Tables 4A and 4B, it is possible to construct the Conjecture Trajectory for any given odd integer in a simpler way. Given an odd integer $x$ (other than 1) we first find where $x$ is positioned in the Tables 4A and 4B: i.e., (i) in which of the two Tables and (ii) in which row in that Table so that we can get its Collatz iterate (say $y$) from the last column of the corresponding Table.
We begin by checking if it is an integer of the form $4m+3$ (i.e., half of the odd integers) and these are all in the first column of Table 4B. Among the $4m+1$ type integers, half of them are in the first column of Table 4A and rest are distributed. Hence, as the second possibility, we check if it is in the first column of Table 4A. If the $4m+1$ integer is in some other column, we look for the integer in the previous column of the row where $x$ is and make that integer as the new $x$ and repeat the process until $x$ is in one of the first columns in the two Tables. First we print the given $x$ and as we go through the process, we only print all those iterates (values of $y$) when the $x$ is in one of the first columns. Finally when $x$ becomes 1, we print 1 and end the process. The simple logical steps are presented below:

Print $x$;
Begin: print ‘→’ if $[x = 1]$
  then print $x$; Stop
  else if $[x \text{mod} 4 = 3]$
    then $[m = \lfloor x/4 \rfloor; y = 6m+5; \text{print } y; \quad x = y; \text{Goto Begin;} ]$ \hspace{1cm} // first column of Table 4B
    else if $[x \text{mod} 8 = 1]$
      then $[m = \lfloor x/8 \rfloor; y = 6m+1; \text{print } y; \quad x = y; \text{Goto Begin;} ]$ \hspace{1cm} // first column of 4A
      else $x = (x-1)/4; \text{Goto Begin;}$ \hspace{1cm} // in some column other than the first in 4A or 4B

6. Mechanism for Constructing a Collatz Tree

In the Collatz Tree shown in Figure 3, the branches are the Intermediary integers which constitute the complete set of solutions $(3x+1)/2^a$ samples presented in Tables 4A and 4B where $x$ represents every odd integer and these branches lead up to the root at the top (i.e., End integer 1). We skip the Starter integers since, as they go through the conjecture process, they get attached as leaves (by not being iterates for any odd integers). Hence they are ignored when creating the next lower layer.

The extended Tables 4A and 4B together list all the Intermediary integers as the Collatz conjecture iterates for sets of connected odd integers shown in their respective rows. Here we provide a procedure that could be mechanized for drawing a Collatz Tree, starting with 1 at the top (Layer-0). The next layer, Layer-1, consists of all the Terminal integers (5, 21, 85, 341, 1365, 5461, …) whose Collatz conjecture result is 1 and hence they are all linked upwards to 1. The layer below this consists of the sets of Pre-terminal integers for each of these Terminal integers (except for the Starter integers like 21, 1365, …). So, Layer-2 consists of the Pre-terminal integers under each of the non-Starter Terminal integers. For 5 they are: (3), 13, 53, (213), 853, 3413, …; for 85: 113, (453), 1813, 7253, …; for 341 are: 227, (909), 3637, 14549, …; and so on (those in parenthesis are Starter integers). We see that the Collatz Tree spreads out as the process goes down the layers. The initial set up is given here with $i = 1, 2, 3$, indicating the sequence: L0 = 1; $L1(i) = 5, 85, 341, 5461, …$; $L2(1, i) = 13, 53, 853, …$; $L2(2, i) = 113, 1813, 7253, …$; $L2(3, i) = 227, 3637, 14549, …$; and so on. We note that the results in the first row of the two extended Tables 4A and 4B appear in layers 1 and 2 respectively. For the remaining lower layers we use, given a Collatz iterate, a simple mechanism for finding the corresponding lower layer odd integers. This mechanism consists of a few steps and we start with these results in the third layer: L2 and use that as the new $y$ to continue the process. The mechanism that could automatically get such next $y$ is described next.
6.1 Collatz Tree Construction Mechanism. Let \( y \) be an odd integers from Layer-2, say \( y = L(n, j) \). We first find where \( y \) is positioned in the Tables 4A and 4B: i.e., (i) in which of the two result-columns and (ii) in which row of that column so that we can get the set of odd integers whose Collatz iterate is \( y \). We place all these odd integers in the layer just under \( y \) and continue the process by picking the first odd integer in the list (that is not a Starter integer) and make that as the new \( y \) and repeat the process. The first entries in the first columns of the two Tables are different (1 for Table 4A and 3 for Table B). Since we need these integers for later use, we also use them to identify the Tables. Also, the row can be identified by computing the integer part of \( y/6 \). This logic is presented below:

If \( [y \mod 6] = 1 \) then \( s = 1 \) else \( s = 3 \)
   // \( s = 1 \) top entry in the first column. in Table 4A & \( s = 3 \) in the first column in Table 4B
   // We note, 13 is in Table 4A; 53 is in Table 4B which are used as examples below
   \( r = \lfloor y/6 \rfloor \)
   // Here \( r \) represents the row
   // We note that 13 is in row 2 (of Table 4A) and 53 is in row 8 (of Table B)
   // At this stage we know the Table and the row where the Collatz conjecture result \( y \) is
   // If the current \( y \) is in Table 4A we use the following;
   // also make the first non-Starter integer as the next \( y \)
   \( z = 8^r + 1; \ y = z; \) if \( [y \mod 3] = 0 \) then \( y = 4^z + 1 \) // \( z \) is in Table 4A
   // Now, create an array \( e(n) \) to store the first \( n \) odd integers that are in row \( r \) of Table 4A
   \( e(1) = z; \) do for all \( i = 1 \) to \( n \) \( [e(i+1) = 4^* e(i) + 1] \)
   // If the current \( y \) is in Table 4B, we use the following;
   // also make the first non-Starter integer as the next \( y \)
   \( z = 4^r + 3; \ y = z; \) if \( [y \mod 3] = 0 \) then \( y = 4^z + 1 \) // \( z \) is in Table 4B
   // Now, create an array \( e(n) \) to store the first \( n \) odd integers in row \( r \) of Table 4B
   \( e(1) = z; \) do for all \( i = 1 \) to \( n \) \( [e(i+1) = 4^* e(i) + 1] \)
   // Store or use these results and move on to the next layer with the new value of \( y \).

In the above, assuming we started with 13, we note \( z = 17 \) and hence the new \( y = 17 \); \( e(n) \) will contain 17, (69), 277;...; \( z \) and \( e(n) \) will go in the layer below 13 in the Collatz tree. Similarly with 53, \( z = 35 \), the new \( y = 35 \); \( e(n) \) will contain 35, (141), 565;... As a result, Layer-3 will have several segments linking up to 13: we have L3(1,1,i) = 17, (69), 277, ...; and also segments linking up to 53: L3(1,2,i) = 35, (141), 565;... Obviously, for drawing the Collatz Tree, the above may not be the most efficient approach and certainly it is not comprehensive. However, our objective here is just to indicate how the results in the Tables 4A and 4B can be interpreted for generating the entire Collatz Tree leading up to the root integer 1 at the top.

7. Collatz Trajectory Characteristics of 4m+3 Integers

Each of the 4m+3 integers uses \( \alpha = 1 \) at least once as it starts going through the Collatz conjecture process, producing an iterate that is larger (by about 1 ½ times ) than itself. Some of such Collatz iterates may also use \( \alpha = 1 \) in their Collatz conjecture process and this pattern could continue. For instance, we see in the Collatz trajectory for 63 (which is a 4m+3 integer) presented in Table 2A: \( 63 \rightarrow 95 \rightarrow 143 \rightarrow 215 \rightarrow 323 \rightarrow 485 \rightarrow 91 \rightarrow ... \rightarrow 1 \). There are five increases before a reduction since 63, 95, 143, 215, and 323 are all integers of type 4m+3 (and use \( \alpha = 1 \)) while 485, an integer of type 4m+1 using \( \alpha = 4 \), produces a lower iterate.
As a result, there is a need to classify the \(4m+3\) integers listed under the second column in Table 4B based on their length of continuously increasing Collatz trajectory (for which we use the term: Alpha1 Trajectory Length, since we are considering all the integers using \(\alpha = 1\) in the trajectory). For instance, the Alpha1 Trajectory Length for 63 is 5 as each of these five integers use \(\alpha = 1\): \(63 \rightarrow 95 \rightarrow 143 \rightarrow 215 \rightarrow 323\). Next, we present a method for categorizing the entire \(4m+3\) integers based on their Alpha1 Trajectory Lengths.

7.1 Alpha1 Trajectory Lengths of \(4m+3\) Integers. Let \(y_1 = (3x+1)/2\), where \(x\) and \(y\) are odd integers and \(y\) is \(x\)'s first Collatz iterate (using \(\alpha = 1\)). When \(x\) happens to have a higher Alpha1 Trajectory Length, we will use, depending on its trajectory length, one of these equations: \(y_2 = (3y_1+1)/2 = (9x+5)/4\); or \(y_3 = (27x+19)/8\); or \(y_4 = (81x+65)/16\); or \(y_5 = (243x+211)/32\); and so on. Let \(h\) denote the Alpha1 Trajectory Length. First, find a set of odd integers with \(h = 1\) using \(y_1 = (3x+1)/2\). To ensure that no odd integers with \(h > 1\) are included, one approach would be to consider integers of \(4m+1\) whose Collatz conjecture, we know, will use \(\alpha \geq 2\) (seen in Figures 1A and 1B). Hence, if \(y_1 = 4z + 1\) where \(z\) is an odd integer, we note that \((3y_1+1)/2 = [3(4z+1) +1]/2 = (12z + 4)/2\), indicating that the conjecture result for \(y_1\) will be using \(\alpha \geq 2\) and hence this conjecture result cannot belong to the category of odd integers with \(\alpha = 1\) (ensuring that \(x\) belongs to the category of \(h = 1\) only and not to any higher values of \(h\)).

The above being a desired result for what we are looking for, the odd integers with \(h = 1\) could belong to the group where \(y_1 = (3x+1)/2 = 4z+1\). So, \(z = (3x – 1)/8\), which can be used to find all the odd integers \(x\) with \(h = 1\) while ensuring that \(z\) remains an odd integer. Such odd integers of \(x\) are: 3, 11, 19, …; i.e., \(x = 3 + 8(n-1)\), where \(n = 1, 2, 3, …\) Following this approach, it is possible to get the set of all odd integers with \(h = 2\) only by using \(y_2 = (9x+5)/4 = 4z+1\). The odd integers for \(h = 2\) are chosen so that the integer \(z = (9x+1)/16\) and they are: 7, 23, 39, 55, …; i.e., \(x = 7 + 16(n-1)\), where \(n = 1, 2, 3, …\); further, for \(h = 3\), \(z = (27x + 11)/32\) and the respective odd integers are: 15, 47, 79, …; i.e., \(x = 15 + 32(n-1)\), where \(n = 1, 2, 3, …\); and this process could be continued, using the appropriate expression for any \(y_h\), to find the odd integers with higher values of \(h\).

We note that the first integer in each of the above three series show a pattern: \(2^{h+1} - 1\). This is not a new pattern in the study of \(3x+1\), since we see reported results on page 34 in [2] from the earlier studies on odd integers such as \(2^5 - 1\) and \(2^500 - 1\). Since the current study is about Alpha1 Trajectory Lengths we also provide, for different values of \(h\), the set of integers that follow each of the first integers in \(2^{h+1} - 1\). The list of odd integers for the increasing number of Alpha1 Trajectory Lengths (i.e., \(h = 1, 2, 3, …\)) are presented in Table-5, and this Table, just as Tables 4A and 4B, could be extended both to the right and down so as to include all the integers of the type \(4m+3\). The extended entries in Table-5 (i.e., for all positive integer values of \(h\) and \(n\)) is generated from the following comprehensive and generalized expression that captures the earlier observations, where \(n \geq 1\) indicates the row:

\[
x_{h, n} = 2^{h+1}(2n - 1) - 1
\]

(7)

Main Observation: In the above, the focus was on generating comprehensive results concerning Alpha1 Trajectory Lengths of \(4m+3\). In Table-5, we note that the first column (with Alpha1 trajectory length =1) has half of these \(4m+3\) integers and, as the length increases, the number of entries in each of the subsequent columns reduces by half (a similar pattern we saw in Tables 4A and 4B), indicating that these three Tables contain comprehensive information regarding the actual behavior of the Collatz conjecture.
| Length | \(x_{1,n}\) | \(x_{2,n}\) | \(x_{3,n}\) | \(x_{4,n}\) | \(x_{5,n}\) | \(x_{6,n}\) | \(x_{7,n}\) | \(x_{8,n}\) | \(x_{9,n}\) | \(x_{10,n}\) |
|--------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1      | 3           | 7           | 15          | 31          | 63          | 127         | 255         | 511         | 1023        | 2047        |
| 2      | 11          | 23          | 47          | 95          | 191         | 383         | 767         | 1535        | 3071        | 6143        |
| 3      | 19          | 39          | 79          | 159         | 319         | 639         | 1279        | 2559        | 5119        | 10239       |
| 4      | 27          | 55          | 111         | 223         | 447         | 895         | 1791        | 3583        | 7167        | 14335       |
| 5      | 35          | 71          | 143         | 287         | 575         | 1151        | 2303        | 4607        | 9215        | 18431       |
| 6      | 43          | 87          | 175         | 351         | 703         | 1407        | 2815        | 5631        | 11263       | 22527       |
| 7      | 51          | 103         | 207         | 415         | 831         | 1663        | 3327        | 6655        | 13311       | 26623       |
| 8      | 59          | 119         | 239         | 479         | 959         | 1919        | 3839        | 7679        | 15359       | 30719       |
| 9      | 67          | 135         | 271         | 543         | 1087        | 2175        | 4351        | 8703        | 17407       | 34815       |
| 10     | 75          | 151         | 303         | 607         | 1215        | 2431        | 4863        | 9727        | 19455       | 38911       |
| 11     | 83          | 167         | 335         | 671         | 1343        | 2687        | 5375        | 10751       | 21503       | 43007       |
| 12     | 91          | 183         | 367         | 735         | 1471        | 2943        | 5887        | 11775       | 23551       | 47103       |
| 13     | 99          | 199         | 399         | 799         | 1599        | 3199        | 6399        | 12799       | 25599       | 51199       |
| 14     | 107         | 215         | 431         | 863         | 1727        | 3455        | 6911        | 13823       | 27647       | 55295       |
| 15     | 115         | 231         | 463         | 927         | 1855        | 3711        | 7423        | 14847       | 29695       | 59391       |
| 16     | 123         | 247         | 495         | 991         | 1983        | 3967        | 7935        | 15871       | 31743       | 63487       |
| 17     | 131         | 263         | 527         | 1055        | 2111        | 4223        | 8447        | 16895       | 33791       | 67583       |
| 18     | 139         | 279         | 559         | 1119        | 2239        | 4479        | 8959        | 17919       | 35839       | 71679       |
| 19     | 147         | 295         | 591         | 1183        | 2367        | 4735        | 9471        | 18943       | 37887       | 75775       |
| 20     | 155         | 311         | 623         | 1247        | 2495        | 4991        | 9983        | 19967       | 39935       | 79871       |
| 21     | 163         | 327         | 655         | 1311        | 2623        | 5247        | 10495       | 20991       | 41983       | 83967       |
| 22     | 171         | 343         | 687         | 1375        | 2751        | 5503        | 11007       | 22015       | 44031       | 88063       |
| 23     | 179         | 359         | 719         | 1439        | 2879        | 5759        | 11519       | 23039       | 46079       | 92159       |
| 24     | 187         | 375         | 751         | 1503        | 3007        | 6015        | 12031       | 24063       | 48127       | 96255       |
| 25     | 195         | 391         | 783         | 1567        | 3135        | 6271        | 12543       | 25087       | 50175       | 100351      |
| 26     | 203         | 407         | 815         | 1631        | 3263        | 6527        | 13055       | 26111       | 52223       | 104447      |
| 27     | 211         | 423         | 847         | 1695        | 3391        | 6783        | 13567       | 27135       | 54271       | 108543      |
| 28     | 219         | 439         | 879         | 1759        | 3519        | 7039        | 14079       | 28159       | 56319       | 112639      |
| 29     | 227         | 455         | 911         | 1823        | 3647        | 7295        | 14591       | 29183       | 58367       | 116735      |
| 30     | 235         | 471         | 943         | 1887        | 3775        | 7551        | 15103       | 30207       | 60415       | 120831      |
| 31     | 243         | 487         | 975         | 1951        | 3903        | 7807        | 15615       | 31231       | 62463       | 124927      |
| 32     | 251         | 503         | 1007        | 2015        | 4031        | 8063        | 16127       | 32255       | 64511       | 129023      |
| 33     | 259         | 519         | 1039        | 2079        | 4159        | 8319        | 16639       | 33279       | 66559       | 133119      |
| 34     | 267         | 535         | 1071        | 2143        | 4287        | 8575        | 17151       | 34303       | 68607       | 137215      |
| 35     | 275         | 551         | 1103        | 2207        | 4415        | 8831        | 17663       | 35327       | 70655       | 141311      |
| 36     | 283         | 567         | 1135        | 2271        | 4543        | 9087        | 18175       | 36351       | 72703       | 145407      |

**TABLE 5** Integers Producing Collatz Results that Continuously Increase (First 10 \(\alpha_1\) Chain Length)

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We present, in the following Section, the evidence for on-the-average convergence of all the odd integers’ Collatz conjecture trajectories using the results presented in Tables 4A, 4B and 5.

8. Validity of the Collatz Conjecture

Earlier in Section 5 under Observation 6, we used a quote from page 34 in [2]. By expanding the idea in that quote, we now present a Theorem that captures the essence of the Collatz conjecture. Here we consider the mixing properties of the iterates of the two types of odd integers: (i) 4m+3 whose iterates keep increasing and 4m+1 whose iterates keep decreasing.

THEOREM 8.1 The Collatz conjecture results of half of all the odd integers that are of the form 4m+3, on the average, increase by a factor of 3 and the results of the other half of odd integers that are of the form 4m+1, on the average, decrease by a factor 4 (i.e., a multiplication factor of $\frac{1}{4}$). As a result, considering all the odd integers, there are no divergent Collatz conjecture trajectories.

Proof: All the $4m+3$ odd integers are listed under the column $\alpha = 1$ in Table 4B. In Table 5, these $(4m+3)$ odd integers are, based on their value of Aplhal Chain Lengths $(h)$, distributed under different columns of $h$ (i.e., $h = 1, 2, 3, \ldots$)

In Table 5, half of the $4m+3$ type integers are listed under the first column $h = 1$ and the number of integers under each of the subsequent columns reduces by half as the value of $h$ increases by 1. The Collatz iterate for each of the odd integer under the first column increases roughly by $3/2$ – ignoring the 1 in the expression $(3x+1)/2$. Similarly the Collatz iterates in the subsequent columns also increase by roughly $(3/2)^h$. The overall average increase is given by:

$$
(1/2) (3/2) + (1/4) (3/2)^2 + (1/8)(3/2)^3 + \ldots = (3/4) + (3/4)^2 + (3/4)^3 \ldots = \sum_{n=1}^{\infty} (3/4)^n = 3
$$

This could be interpreted as if the Collatz iterates of half of all the odd integers that are of the form $4m+3$, on the average, increase by three times the value of the integer that was used to start with. Now let us examine all the remaining columns ($\alpha \geq 3$) in Table 4B that has one-third of the $4m+1$ odd integers. The Collatz iterate for each of the odd integer under the column $\alpha = 3$ decreases roughly to $(3/8)$ times its original value and there are $1/8$th of all the odd integers here. The average decrease of these one-third of the $4m+1$ integers is given by:

$$
(1/8) (3/8) + (1/32) (3/8) + (1/128) (3/8) + \ldots = (3/8^2) + (3/32^2) + (3/128^2) \ldots = (3/2^2) \sum_{n=1}^{\infty} (1/4^2)^n
$$

We now consider the remaining two-thirds of the $4m+1$ odd integers listed under all the columns in Table 4A with even values of $\alpha$. The Collatz iterate for each of the integer under the column $\alpha = 2$ decreases roughly to $(3/4)$ times its original value and there are $1/4$th of all the odd integers here. The average decrease of the remaining two-thirds of the $4m+1$ integers is given by:

$$
(1/4) (3/4) + (1/16) (3/4) + (1/64) (3/4) + \ldots = [3/(4^2)] + [3/(16^2)] + [3/(64^2)]\ldots = 3 \sum_{n=1}^{\infty} (1/4^2)^n
$$

Combining the results in Equations (9) and (10), we note that the Collatz iterates of the other half of odd integers that are of the form $4m+1$, on the average, decrease and their overall multiplication factor is given by:

$$
(3/2^2) \sum_{n=1}^{\infty} (1/4^2)^n + (3) \sum_{n=1}^{\infty} (1/4^2)^n = (15/4) \sum_{n=1}^{\infty} (1/4^2)^n = (15/4) (1/15) = 1/4
$$
This could be interpreted as if the Collatz iterates of half of the odd integers of the form \(4m+1\), on the average, get smaller by a factor 4 of the integer that was used to start with.

In the above, we have the results for the odd integers of the type \(4m+3\) in Equation(8) that shows an average increase by a factor of 3 and for those of the type \(4m+1\) in Equation(11) that shows an average decrease by a factor of 4. Both have equal number of odd integers in them.

[Note: The Observation-4 under Sub-Section 5.1 is relevant here: The Starter integers are evenly spread out in the two Tables 4A and 4B: i.e., every third integer in all the columns (except the two iterate columns) and in all the rows is an odd multiple of 3. In view of this uniform distribution of the Starter integers, we can assume that they are uniformly excluded in all the Equations (8) to (11) without affecting the final results]. The overall result \(3/4\) matches with the theoretical (heuristic) result 0.75 on page 34 in [2]. Hence, on the average, the odd integer iterates in the Collatz conjecture trajectory shrink and hence there cannot be any divergent Collatz conjecture trajectories. (Q. E. D.)

9. Conclusion

First, we summarize the classification of odd integers used in this study on Collatz conjecture. Terminal integers are those whose Collatz conjecture result is 1, the End integer. We classified all the odd integers in two ways: (i) consisting of three equal categories: odd multiples of 3 (known as Starter integers); \(6m+1\), and \(6m+5\), with \(m = 0, 1, 2, 3, \ldots\) (the last two collectively known as Intermediary integers); and (ii) consisting of two equal categories: \(4m+1\) and \(4m+3\) (where \(m = 0, 1, 2, 3, \ldots\)). The first type of classification is used in [3] without these names. In Section 5, we clearly explain the mixing between these two types of integers (Starter and Intermediary) in Observation 4 that pertains to all the odd integers in Tables 4A and 4B.

We started with the Collatz trajectories for a few Starter integers and, using these results, we presented a new type of Collatz Graph. From this graph, we identified the role that \(4x+1\) expression plays in the analysis of the Collatz conjecture. Further in Theorem 4.1, we proved that a Collatz trajectory cannot have duplicate odd integers in it (hence no looping as well). One-half of the odd integers of type \(x = 4m+1\) use values of \(\alpha \geq 2\) in the expression \((3x +1)/2^\alpha\) while the other half of the odd integers of type \(x = (4m+3)\) just use \((3x +1)/2\) (i.e., \(\alpha = 1\)).

The valuable role that the generalized version of \(4x+1\) expression plays in this study was demonstrated through the creation of Tables 4A and 4B (and also, to some extent, Table 5). The Intermediary integers of type \(6m+1\) (last column in Table 4A) are the Collatz iterates of one third of the odd integers and also those of type \(6m+5\) (last column in Table 4B) are the Collatz iterates of the remaining two thirds of all the odd integers. As can be seen, we have avoided the need for tools such as the Syracuse random variables in order to identify all the iterates that are listed in the last two columns of the Tables 4A and 4B. In Table 4B, the \(4m+3\) integers make half of all odd integers that are listed under the column \(\alpha = 1\); and one-sixth of the odd integers in Table 4B are of the type \(4m+1\) that are listed under the remaining columns \(\alpha \geq 3\). The remaining \(4m+1\) integers are presented in Table 4A shown in columns with even values of \(\alpha\). We also looked into the behavior of these \(4m+3\) integers in terms of their Alpha1 trajectory length and presented these results in Table 5 in the increasing order of Alpha1 trajectory lengths.
From the proof of the Collatz conjecture perspective, Tables 4A and 4B (excluding the column for $\alpha = 1$) show how well, from Collatz conjecture perspective, the integer (iterate) values get reduced, while Table 5 shows how badly, again from Collatz conjecture perspective, the $4m+3$ integer (iterate) values get increased. In Theorem-8.1, we proved that the Collatz conjecture iterate of all the odd integers considered together shrink, on the average, by a factor of $3/4$ (matching with the heuristic result reported in [2]). The ideas used here are rather simple. However, we go into depths when analyzing the conjecture showing the mixing properties of the iterates of the two types of odd integers.

Since it is difficult to trace all the earlier contributions, we used [2] as our main source as it presents all the past information in a condensed manner. The recent contribution by Tao [3], where the conjecture has been almost proved, takes a highly formal approach. Even though one of the categorization of odd integers used here is similar to the one seen in [3], we have taken a structured approach to observe all the hidden intricacies of the conjecture. Here, all the odd integers are organized based on their conjecture iterates. We have described a simple logic involving expressions for locating integers in Tables 4A and 4B [and not using again the conventional $(3x +1)/2^\alpha$] for generating the Collatz trajectory for any given odd integer $x > 1$. Similarly, we have come up with a mechanism, based on the results in Tables 4A and 4B, to draw the complete Conjecture Tree starting with 1 at the top, the Terminal layer (consisting of Terminal Integers) below that, and the appropriate Pre-terminal layer below each of the Terminal integer. This process can be used to construct the Collatz conjecture net (tree) without resorting to further use of $(3x +1)/2^\alpha$ expression. In Theorem 8.1, considering all the odd integers and their conjecture iterates we prove that there cannot be any divergent Collatz conjecture trajectories.

The current practice used in solving the Collatz conjecture seems to be based on a verbatim interpretation, where we start with an integer and produce a trajectory to reach the end integer 1. However, after using the $4x+1$ key to open the $(3x+1)$'s-box and seeing what are all inside (such as samples displayed in Tables 4A, 4B and 5), we may have to come up with a (w)holistic approach. The current interpretation is like looking at strings (trajectories) of a big net instead of recognizing the net itself. The new Theorem statement, perhaps, could be:

All odd integers like $x$ and $y$ that are related by $(3x+1)/2^\alpha = y$, where $\alpha$ is a suitable integer, go up through the Collatz net with $y$ in a layer above $x$ to reach 1.

Acknowledgement: We thank Dr. Jeffrey C. Lagarias for pointing out that (i) the categorization of odd integers into two main types (similar to Starter and Intermediary integers used in this paper) has already been discussed by Dr. Tao in [3] and (ii) there is still a need to prove the kind of mixing between these two categories of odd integers.

References:
[1] https://en.wikipedia.org/wiki/Collatz_conjecture (Section on Syracuse Function for 4x+1)
[2] Jeffery C. Lagarias, The Ultimate Challenge: The 3x+1 Problem, American Mathematical Society (2010) Also: http://www.cecm.sfu.ca/organics/papers/lagarias/paper/html/node3.html
[3] Terence Tao, “Almost All Orbits Of The Collatz Map Attain Almost Bounded Values”, September 2019; Also see https://terrytao.wordpress.com/2019/09/10/almost-all-collatz-orbits-attain-almost-bounded-values/ and https://arxiv.org/pdf/1909.03562.pdf 7 Jun 2020

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