More Insight into Heavy Quark Masses from QCD

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Abstract

Using the non-relativistic version of Borel sum rules, combined to the very accurate experimental information on the Charmonium and Upsilon spectra, we rigorously investigate the Euclidean mass of the Charm and Bottom quarks. Our analysis is performed with an improved expansion of the vector correlator function in the infinite heavy quark mass limit. The optimal estimates, which take into account the most recent world average value of the strong coupling constant, as well as a conservative range of the gluon condensate values, are: $m_{c}^{\text{Eucl}} = 1.20 \pm 0.034 GeV, m_{b}^{\text{Eucl}} = 4.18 \pm 0.037 GeV$. Their conversions to the corresponding pole mass and running mass of $c$ and $b$ quarks give respectively: $M_{c} = 1.49 \pm 0.08 GeV$, $M_{b} = 4.65 \pm 0.06 GeV$ and $m_{c}(\overline{m}_{c}) = 1.21 \pm 0.08 GeV$, $m_{b}(\overline{m}_{b}) = 4.20 \pm 0.06 GeV$ in good agreement with the most recent direct estimates.

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1 Introduction:

Heavy quark masses play a central role in the Standard Model of particle physics. They are often present for the phenomenology of a plethora of processes (weak decays, charm or bottom production, ...). However, they cannot be estimated within the theory. In addition, because of their confinement at large scales, they are invisible to the experimentalists. In this context, different theoretical analyses have been used to the extraction of the charm and bottom quark masses. Besides lattice calculations, QCD sum-rules - à-la Shifman Vainshtein-Zakharov have proved to be an extremely useful framework to learn about these masses[1-5]. However as the quarks are not the physical states of QCD, there is no unique physical definition of quark mass. The most widely used definition is the pole quark mass $M_Q$, defined in perturbation theory as the position of the singularity in the renormalized quark propagator. For the charm or bottom quark, such definition becomes meaningless beyond perturbation theory. Indeed it suffers from an intrinsic ambiguity generated by the so called infrared renormalons[17]. The latter induce an additional uncertainties which are presently estimated to be of the order of the QCD scale parameter $\Lambda_{QCD}$.

On the other hand, the authors of ref.[1,2] observed that the radiative corrections are too large in the case of the heavy two point correlator associated to the vector current $\bar{Q}\gamma_\mu Q$, where Q denotes the quantum field of the charm or bottom quark.

In spite of that, most QCD sum-rules determination of the c or b quark mass have ignored the nonperturbative renormalons effects through the use the pole quark mass definition. Moreover, they have performed QCD sum rules analysis of the heavy correlator, with the perturbative contribution given only up to the first order in the strong coupling constant.\footnote{Large order radiative corrections are still lacking, except a three loop contributions to the heavy quark correlator, calculated in part numerically within the method of Pade approximants[20].}

An alternative definition to the pole quark mass $M_Q$ is provided by the Euclidean mass $m_{Eucl}^Q$, which is defined in Landau gauge at the Euclidean point $p^2 = -M_Q^2$. Although gauge dependent, the Euclidean mass is free from renormalon ambiguities. It first has been introduced by SVZ [2] in
order to reduce the effects of higher multi-loop corrections on heavy quarkonium correlators. Using relativistic Hilbert sum-rules, they have obtained the following estimates: $m_{c}^{Eul} = 1.26 \pm 0.02 GeV$ and $m_{b}^{Eul} = 4.23 \pm 0.05 GeV$. Subsequently, Reinders; Rubeinstein and Yazaki [3] confirmed the predictions of SVZ, but they have advocated the use of the Hilbert Moments at $Q^2 = -q^2 \neq 0$ for heavy quark systems. Later on, Guberina et al. [4] and Reinders [5], following the same strategy, found for the b Euclidean mass: $4.19 \pm 0.06 GeV$ and $4.17 \pm 0.02 GeV$ respectively. The discrepancies between these estimates come from two main sources. First, the evolution of the experimental information on the charmonium and the upsilon families. Indeed, only four resonances from six in the bottomonium were observed in the beginning of eighties when these papers appeared. Moreover, the measured hadronic parameters describing these resonances were contaminated with quite large experimental errors. The second source of discrepancies is related to the theoretical inputs ($\langle \alpha_s G^2 \rangle, \alpha_s, \ldots$) used in each work. Thus, predictions of $m_{b}^{Eul}$ are diverse and run from 4.13 to 4.28 GeV, while most predictions of $m_{c}^{Eul}$ are gathered around the value 1.26 GeV. Since, on the one hand, the present situation of the experimental data on the $J/\Psi$ and $\Upsilon$ systems has enormously improved and, on the other hand, the main input parameters of QCD sum-rules analysis are better under control than the past, a new and independent determination of the Euclidean mass of the charm and bottom quarks is certainly called for.

In the following, we will present an update of the theoretical analysis of the vector channel of charmonium and bottomonium families based on non-relativistic Borel type sum-rules. We start from the expressions given in the original work of Bell and Bertlman [6,7], which we expand, in the infinite quark mass limit ($m_{Q}^{Eul} \rightarrow \infty$), up to $1/m_{Q}^3$ order. Next step will follow from the famous quark-hadron duality, which will permit to confront two representations of the ratio of Borel moments: one based on OPE calculations and the other on the data. An important by-product of this analysis will be the prediction of the Euclidean mass $m_{Q}^{Eul}$.

The key input parameters most relevant to this sum-rules analysis are the strong coupling constant $\alpha_s$ and the gluon condensate $\langle \alpha_s G^2 \rangle$. For the former, the world average central value $\alpha_s (M_Z) = 0.119$ is by now stable and
the experimental error taken by all measurements ranges from 0.003 to 0.005 [8]. In this work, our reference values will be: $\alpha_s = 0.119 \pm 0.005$. These values will be combined to the formulae of the running strong coupling with two loop accuracy, in order to evolve $\alpha_s$ down to the sum-rule energy scale. As for $\langle \alpha_s G^2 \rangle$, first estimated by SVZ from the analysis of the charmonium sum-rules to be 0.038 GeV$^4$[2], there has been a lot of activity and discussion of its value. A number of independent estimates have appeared in the literature [9]. Unlike the previous quark mass determinations which use a fixed value of the gluon condensate, we will take the conservative range of values:

$$\langle \alpha_s G^2 \rangle = 0.055 \pm 0.025 \text{GeV}^4$$ (1)

The plan of this paper is the following. Section 2 will be devoted to review the most common mass definitions and their interrelations. In section 3 we present the quarkonium correlation function and its non-relativistic Borel moments. Special emphasize is put on their expansion in the infinite heavy quark mass limit up to $\frac{1}{M^3}$ order. In section 4, rigorous analysis of sum-rules of the Euclidean mass of charm and bottom quarks is performed and finally, our numerical estimates and discussion of the uncertainties entailed by the sum-rules are given in section 5.

## 2 Quark mass definitions

The quark mass definition is quite confusing. Indeed, several definitions have been proposed and the choice of one among them is often correlated to the physical process of interest: each particular case requires its definition. The most common ones rely on purely perturbative calculations. The pole mass $M_Q$, the running mass in modified minimal subtraction scheme $m_{\overline{MS}}(\mu)$ and the Euclidean mass $m_{\text{Eucl}}^Q$ turn out to be very popular and are interrelated. Indeed, it is straightforward to derive the following relations at-next-to-leading order in QCD coupling constant [10],

$$m_{Q}(\mu) = M_Q \{1 - \frac{4}{3\pi} \alpha_s + O(\alpha_s^2)\}$$ (2)

$$m_{\text{Eucl}}^Q(\mu) = M_Q \{1 - \frac{2\ln 2}{\pi} \alpha_s + O(\alpha_s^2)\}$$ (3)
Eq. (2) relates the pole quark mass to the running quark mass defined in the $\overline{\text{MS}}$ scheme and normalized at the scale $\mu$, while the connection between the Euclidean quark mass in Landau gauge and the pole mass is given by eq. (3).

The running coupling constant is given by,

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln(\frac{\mu^2}{\Lambda_{\text{QCD}}^2})} \left\{ 1 - \frac{\beta_1}{\beta_0^2} \ln\left(\ln\left(\frac{\mu^2}{\Lambda_{\text{QCD}}^2}\right)\right) + \ldots \right\} \quad (4)$$

which defines the QCD scale parameter $\Lambda_{\text{QCD}}$ at two loop accuracy. $\beta_0$ and $\beta_1$ are the first beta functions governing the evolution of $\alpha_s$. They are scheme independent and are equal to [16],

$$\beta_0 = 11 - \frac{2}{3} n_f, \quad \beta_1 = 102 - \frac{38}{3} n_f \quad (5)$$

where $n_f$ is the number of flavors with mass below $\mu$. At this stage, it should be understood that the number of flavors changes as $\mu$ crosses a quark threshold. As $\Lambda_{\text{QCD}}$ depends on $n_f$, value of $\Lambda_{\text{QCD}}$ for different number of active flavors are defined by keeping $\alpha_s$ continuous at the threshold scale $\mu = m_Q$. Therefore, according to this prescription [11], we first extract the value of $\Lambda_{\text{QCD}}^{(n_f=5)}$ from the present world average values of $\alpha_s = 0.119 \pm 0.005$. The corresponding range is,

$$\Lambda_{\text{QCD}}^{(n_f=5)} = (235^{+75}_{-60}) \text{MeV} \quad (6)$$

Then owing to the matching condition $\alpha_s(\mu = m_b)_{n_f=4} = \alpha_s(\mu = m_b)_{n_f=5}$, we relate $\Lambda_{\text{QCD}}^{(n_f=4)}$ to $\Lambda_{\text{QCD}}^{(n_f=5)}$. The corresponding range of values for $n_f = 4$ is the following,

$$\Lambda_{\text{QCD}}^{(n_f=4)} = (340^{+95}_{-80}) \text{MeV} \quad (7)$$

Equations (6) and (7) constitute important inputs in this work.
3 Quarkonium Sum-rules

The basic amplitude considered in this sum-rules is the correlation function \( \Pi_{\mu\nu}(q^2) \), induced by the electromagnetic vector current \( j_\mu = \overline{Q}\gamma_\mu Q \) and defined by,

\[
i \int d^4x \exp(irqx) \left\langle \Omega \mid j_\mu(x)j_\nu^+(0) \mid \Omega \right\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2)
\]

The structure function \( \Pi(q^2) \) is related to its imaginary part through the dispersion relation:

\[
\Pi(Q^2) = \frac{1}{\pi} \int ds \frac{\text{Im}\Pi(s)}{(s + Q^2)} + \text{subtraction}, \quad Q^2 = -q^2
\]

where \( \text{Im}\Pi(s) \) is proportional to the total experimental cross-section of \( e^+e^- \) annihilation into final states with open or hidden \( Q \) flavors,

\[
\text{Im}\Pi(s) = \frac{s}{16\pi^2\alpha_s^2e_Q^2}\sigma(e^+e^- \rightarrow \overline{Q}Q)
\]

\( e_Q \) denotes the electric charge of the heavy quark \( Q \). In QCD sum-rules approach, \( \Pi(Q^2) \) is calculated within Wilson Operator Product Expansion (OPE)[12], which take into account the perturbative contributions as well as the non perturbative effects absorbed in the vacuum matrix elements of operators such as \( G^{\alpha\beta}G_{\alpha\beta} \) or \( \overline{q}q \). In the case at hand, the leading and next-to-leading perturbative contributions for the vector current can be found in Schwinger’s book [15],

\[
\text{Im}\Pi(s) = \frac{1}{8\pi} v(3 - v^2)\{1 + \frac{4\alpha_s}{3\pi}\left[\pi^22v^2 - \frac{3 + v}{4}\left(\frac{\pi^2}{2} - \frac{3}{4}\right)\right]\}\Theta(v^2)
\]

where \( v = (1 - 4m_Q^2/s)^{1/2} \), while the non-perturbative corrections to \( \Pi(s) \) will be represented by the leading contribution which is proportional to the gluon consensate \( \langle \alpha_s G^2 \rangle \) and is given by,

\[
\Pi_{NP}(s) = \frac{1}{48s^2}\left\{\frac{3(v^2 + 1)(1 - v^2)^2}{2v^5}\ln\left(\frac{1 + v}{1 - v}\right) - \frac{3v^4 - 2v^2 + 3}{v^4}\right\} < \alpha_s G^2 >
\]

\[ (12) \]
In the sum-rules analysis, the subtracted dispersion relation in eq. (9) is improved by the Borel transform [2] leading to the exponential moments,

\[ M(\sigma) = \frac{1}{\pi} \int_{4m_Q^2}^{\infty} ds \exp(-s\sigma) \text{Im}\Pi(s) \quad (13) \]

The exponential weight in (13) has the merit to cut off large contributions and to take under control the effects of high dimension vacuum condensates. This means an improvement of the convergence of the OPE as well as a better enhancement of the quark-hadron duality.

In the present work, we shall be concerned with the non-relativistic version of Borel sum-rules defined as,

\[ M(\tau) = \frac{1}{\pi} \int_{0}^{\infty} dE \exp(-E\tau) \text{Im}\Pi(E) \quad (14) \]

where we have introduced the new sum-rule parameter \( \tau = 4m_Q^{\text{Eucl}}\sigma \) and the heavy quark energy \( E \), through the relation \( s = (2m_Q^{\text{Eucl}} + E)^2 \). Moments present two important features: On the one hand, they can be evaluated by using the detailed experimental data for \( \text{Im}\Pi(s)(\sim \text{cross-section}) \). On the other hand, the moments in eqs. (13) and (14) are very sensitive to the heavy quark mass. Therefore, by equating the theoretical moments based on OPE calculations to the corresponding ones that use highly accurate experimental information in the heavy channel associated to the vector current, one can make a reliable prediction of the heavy quark mass. However, we will consider the ratio of the (Exponential / Borel) moments instead of the moments themselves: since in contrast to the moments, the ratio has weaker dependence on higher order radiative corrections to the unit operator of the OPE. As shown in ref.[7], the non-relativistic ratio of Borel sum-rules is given by,

\[ R_{NR}(\tau) = 2m_Q^{\text{Eucl}} - \frac{d}{d\tau} \left[ \ln M_{NR}(\tau) \right] \quad (15) \]

where at \( \tau \)-stability region the ground state mass is, in principal provided by
the minimum of $R_{NR}$.

The complete analytic expressions of the moments have been calculated by Bell and Bertlmann in terms of Whittaker functions [6]. Owing to their asymptotic properties, we expand them in powers of $\frac{1}{m_{Q\text{Eucl}}}$ up to the next-to-next-to-leading order. The non-relativistic ratio is then given by,

$$R_{NR}(\tau) = 2m_{Q\text{Eucl}} \left\{ 1 + \frac{3}{4}\omega[1 - \frac{5}{6}\omega + \frac{10}{3}\omega^2] - \frac{\sqrt{\pi}\omega}{3}\alpha_s(\omega)[1 - (\frac{2}{3} + \frac{3}{8\pi^2})\omega + (\frac{107}{32} + \frac{51}{32\pi^2})\omega^2] \ight.$$ 

$$+ \frac{2\ln 2}{\pi}\alpha_s(\omega)[1 + \frac{5}{4}\omega^2] + \frac{\pi^2}{3} < \frac{\alpha_s}{\pi} G^2 > \frac{1}{(4m_{Q\text{Eucl}}^2\omega)^2}[1 + \frac{4}{3}\omega - \frac{5}{12}\omega^2] \right\}$$

where $\omega = \frac{1}{m_{Q\text{Eucl}}^{\text{exp}}}$. Our expression in (16) agrees with the one given by ref.[13] which take into account the mass corrections to the perturbative part of the ratio up to-the-first order. However, we disagree with the expansion done in [14]. At this stage some remarks are in order: The ratio $R_{NR}$ is a sign alternating serie, therefore, it is not always fully justified to keep only the next-to-leading mass corrections (NLO) and to ignore the next-to-next-to-leading ones (NNLO). Indeed, for comparison, we show in Table3 the results of the sum rules analysis for the c and b-Euclidean mass, with the ratio (16) expanded up to the leading order, NLO and NNLO respectively. As we see, in contrast to the case of the b-quark mass, the effects of the higher order mass corrections are important for the c-quark mass determination which is sensitive to the order of the $\frac{1}{m_{Q\text{Eucl}}}$ expansion.

On the other side, the experimental ratio can be thought of as a sum of zero width resonances, plus a continuum contribution approximated by perturbation theory above a threshold $s_T$. In this approximation $M(\sigma)$ is evaluated through,

$$M_{\text{exp}}(\sigma) = \frac{3}{4d\epsilon_Q^2} \sum_R M_R \Gamma_R \left( e^+ e^- \rightarrow R \right) \exp \left( -\sigma M_R^2 \right) + \frac{1}{\pi} \int_{s_T}^{\infty} ds e^{-\sigma s} \text{Im} \Pi_{\text{pert}}(s)$$

where $\alpha$ is the QED effective coupling constant. Experimental data in $J/\Psi$
and Υ regions is very accurate. Six resonances have been observed in the vector channel of charmonium and bottomonium systems. Their masses and electronic widths are detailed in the Review of Particle Properties[18]. They come endowed with experimental errors $(ΔM_R, ΔΓ_R)$ that we can safely ignore in our analysis. Indeed we show, for instance, that the influence due to the variations of $Γ_R$ on the Euclidean mass estimates is very small: it amounts to 0.1% for the c-quark and to 0.2% for the b-quark as is indicated in Table 2.

As for the continuum threshold, we take it somewhere above the last resonance of the family of interest. Then, we sensibly increase it in order to test the stability of our results with respect to $s_T$. For our analysis, the incertainties induced by the variations of $s_T$ from 4.5 to 8 $GeV^2$ for charmonium and from 11.2 to 15 $GeV^2$ for bottomium, are insignificant.

4 Sum-rules analysis

On the basis of the above discussion, we are now able to extract the Euclidean mass of the charm and bottom quarks by confronting the two representations shown in (16) and (17). Such confrontation should be realized within the context of the duality between the hadron and quark-gluon descriptions, invoked by SVZ. This means that the agreement between the experiment and the theory should be ensured in a compromising region of the Borel parameter $τ$ over which both contributions, perturbative and non-perturbative, to the theoretical ratio of moments are on the same footing. In the other hand, the sum-rules defined through the experimental ratio (17) should be saturated by the ground state within the Borel region: The sum-rules become increasingly sensitive to the ground state contribution, while the effects of the higher mass states are less substantial and should not exceed a given moderate percentage of the full contribution.

In Figure 1, we study the behavior of both the theoretical and the experimental ratio as a function of the Borel scale $σ$ for the Charmonium family. The input parameters are kept fixed to their central values, $<α_sG^2> = 0.55 GeV^4$ and $Λ_{QCD}^{(n_f=4)} = 340 MeV$. As expected, we see that above $σ = 0.65 GeV^{-2}$

\footnote{This is the so called fiducial window in SVZ’ works}
the sum-rule (17) is dominated by the contribution of a single resonance \( J/\Psi \). The analysis of theoretical expression (16) for different values of \( \sigma \) (or equivalently \( \tau \)) shows a (Borel) region of stability where a clear minimum appears at \( \sigma \sim 0.7 \text{GeV}^{-2} \). The best fit of (16) to the experimental curve is realized with \( m_{c}^{\text{Eucl}} = 1.20 \text{GeV} \). Next, we study the effects due to the variations of the inputs on the determination of the Euclidean mass of Charm. With \( <\alpha_{s}G^{2}> = 0.6 \text{GeV}^{4} \), we show from Figure 3 that the changes in \( \Lambda_{QCD}^{(n_{f}=4)} \) within the range given in (7) induce changes in \( m_{c}^{\text{Eucl}} \) from 1.176 to 1.219\text{GeV}. Now, fixing the QCD scale parameter to its central value 340\text{MeV}, the variations of the gluon condensate in the interval 0.03–0.08\text{GeV}^{4} generate an uncertainties on the Charm quark mass of the order \( \Delta m_{c}^{\text{Eucl}} = +18 \text{MeV} \) (Table 1), which is necessary to the restoration of the agreement between the two descriptions. Therefore, our final results is,

\[
m_{c}^{\text{Eucl}} = 1.20 \pm 0.034 \text{GeV}
\]

Similar analysis is performed for the Upsilon family. In figure 2, we plot the curve of the theoretical ratio versus the experimental data. The stability region shows up around \( \sigma \simeq 0.2 - 0.24 \text{GeV}^{-2} \) where the best agreement between theory and the experiment is reached for \( m_{b}^{\text{Eucl}} = 4.18 \text{GeV} \), with \( \Lambda_{QCD}^{(n_{f}=5)} = 235 \text{MeV} \) and \( <\alpha_{s}G^{2}> = 0.55 \text{GeV}^{4} \). The sensitivity of the b-quark Euclidean mass with respect to the errors of the input quantities is displayed in figure 4 and Table 1, from which we extract our final estimate:

\[
m_{b}^{\text{Eucl}} = 4.18 \pm 0.037 \text{GeV}
\]

where all the errors have been added in quadrature.

5 Conclusions

We have extracted the numerical values of the charm and bottom quark Euclidean masses from a rigorous analysis of the heavy correlator associated to the vector current \( \overline{Q}\gamma_{\mu}Q \), which we combine to the precise experimental data on Charmonium and Upsilon systems. The analysis, performed within the nonrelativistic Borel sum-rules, yields to the optimal results,
\[ m_{c}^{Eucl} = 1.20 \pm 0.034 \text{GeV} \]
\[ m_{b}^{Eucl} = 4.18 \pm 0.037 \text{GeV} \]

From Table 2, we show the origins of the main uncertainties which enters the errors on the c and b-quark masses. Besides the minor uncertainty coming from the experimental data on \( \Gamma_{R} \), they mainly result from the variations of the gluon condensate in a conservative range of values and from the present world average QCD coupling constant at \( M_{Z} \) scale, evolved down to the sum rules scale. Our predictions improve the previous determinations of the Euclidean mass from Hilbert Quarkonium sum-rules\[2-5\]. Moreover, the conversion of the values in (19) to the quark pole mass or \( \overline{\text{MS}} \) mass of Charm and Beauty, through the relations (2) and (3) given to the second order in \( \alpha_{s} \), are in good agreement with the most recent direct estimates\[19\]. Indeed, the corresponding ranges are respectively:

\[ M_{c} = 1.49 \pm 0.08 \text{GeV} \quad (20) \]
\[ M_{b} = 4.65 \pm 0.06 \text{GeV} \quad (21) \]

and,

\[ \overline{m}_{c}(\overline{m}_{c}) = 1.21 \pm 0.08 \text{GeV} \quad (22) \]
\[ \overline{m}_{b}(\overline{m}_{b}) = 4.20 \pm 0.06 \text{GeV} \quad (23) \]

where the errors on the pole masses and on \( \overline{\text{MS}} \) running-masses have increased, due to the additional uncertainty related to \( \alpha_{s}(M_{Q}) \) variations in the formulae (2,3).

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Fig. 1: Charm channel: Theoretical ratio (16) (continuous line) and the Experimental ratio (Dashed line) as a function of the Borel sum-rule parameter \( \sigma \), with \( \langle \alpha_s G^2 \rangle = 0.055 \text{GeV}^4 \), \( \Lambda_{QCD}^{n_f=4} = 235 \text{MeV} \) and \( s_T = 4.5 \text{GeV}^2 \).

Fig. 2: Similar to Fig. 1 in the bottom channel, with \( \Lambda_{QCD}^{n_f=5} = 235 \text{MeV} \) and \( s_T = 11.2 \text{GeV}^2 \).

Fig. 3: Behaviour of the Charm Euclidean mass versus the QCD scale parameter, with \( \alpha_s G^2 \) = 0.055 \text{GeV}^4.

Fig. 4: Similar to Fig. 3 for the bottom quark, with \( \alpha_s G^2 \) = 0.055 \text{GeV}^4.