Complex Interpolation
and
Regular Operators Between Banach Lattices
by
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Abstract. We study certain interpolation and extension properties of the space of regular operators between two Banach lattices. Let $R_p$ be the space of all the regular (or equivalently order bounded) operators on $L_p$ equipped with the regular norm. We prove the isometric identity $R_p = (R_\infty, R_1)^\theta$ if $\theta = 1/p$, which shows that the spaces $(R_p)$ form an interpolation scale relative to Calderón’s interpolation method. We also prove that if $S \subset L_p$ is a subspace, every regular operator $u : S \to L_p$ admits a regular extension $\tilde{u} : L_p \to L_p$ with the same regular norm. This extends a result due to Mireille Lévy in the case $p = 1$. Finally, we apply these ideas to the Hardy space $H^p$ viewed as a subspace of $L_p$ on the circle. We show that the space of regular operators from $H^p$ to $L_p$ possesses a similar interpolation property as the spaces $R_p$ defined above.

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In a recent paper [HeP] we have observed that the real interpolation spaces associated to the couples

\[(B(c_0, \ell_\infty), \ B(\ell_1, \ell_1))\]

and

\[(B(L_\infty, L_\infty), \ B(L_1, L_1))\]

can be described and an equivalent of the \(K_t\)-functional can be given (cf. [HeP]). It is natural to wonder whether analogous results hold for the complex interpolation method and this is the subject of the present paper.

Let \(X_0, X_1\) be Banach lattices of measurable functions defined on some set \(S\) (we are deliberately vague, see e.g. [LT] for a detailed theory). We will denote by

\[X_\theta = X_0^{1-\theta}X_1^\theta\]

the space of all measurable functions \(f\) on the set \(S\) such that there are \(f_0 \in X_0, \ f_1 \in X_1\) satisfying \(|f| \leq |f_0|^{1-\theta}|f_1|^\theta\) and we let

\[\|f\|_{X_\theta} = \inf\{\|f_0\|_{X_0}^{1-\theta}\|f_1\|_{X_1}^\theta\}\]

where the infimum runs over all possible decompositions of \(f\).

We will denote by \((X_0, X_1)_\theta\) and \((X_0, X_1)^\theta\) the complex interpolation spaces as defined for examples in [BL]. Recall the fundamental identity (due to Calderón)

\[X_0^{1-\theta}X_1^\theta = (X_0, X_1)^\theta\]

with identical norms, which is valid under the assumption that the unit ball of \(X_0^{1-\theta}X_1^\theta\) is closed in \(X_0 + X_1\) (see [Ca]). Moreover, if either \(X_0\) or \(X_1\) is reflexive (cf. also [HP] and [B]) we have

\[(X_0, X_1)_\theta = (X_0, X_1)^\theta,\]

with identical norms. In particular when \(X_0, X_1\) are finite dimensional spaces, there is no need to distinguish \((X_0, X_1)_\theta\) and \((X_0, X_1)^\theta\). We will use this fact repeatedly in the sequel. Let \(c_0\) (resp. \(\ell_\infty\)) be the space of all sequences of complex scalars tending to zero.
(resp. bounded) at infinity equipped with the usual norm, and let $\ell_1$ denote the usual dual space of absolutely summable sequences. Recall $\ell_1 = (c_0)^*$ and $\ell_\infty = (\ell_1)^*$. Given Banach spaces $X, Y$ we denote by $B(X, Y)$ the space of all bounded operators $u: X \to Y$ equipped with the usual operator norm. We will always identify an operator on a sequence space with a matrix in the usual way. We will denote by $A_0$ (resp. $A_1$) the space of all complex matrices $(a_{ij})$ such that

$$\sup_i \sum_j |a_{ij}| < \infty \quad \left(\text{resp.} \sup_j \sum_i |a_{ij}| < \infty\right)$$

equipped with the norm

$$\|(a_{ij})\|_{A_0} = \sup_i \sum_j |a_{ij}|.$$

$$\text{(resp.} \|(a_{ij})\|_{A_1} = \sup_j \sum_i |a_{ij}|\text{).}$$

We will need to work with complex spaces, so we recall that if $E$ is a real Banach lattice its complexification $E + iE$ can be naturally equipped with a norm so that for all $f = a + ib$ in $E + iE$ we have $\|a + ib\| = \|(a|^2 + |b|^2)^{1/2}\|_E$. We will call the resulting complex Banach space a complex Banach lattice.

Let $E, F$ be real or complex Banach lattices. We will denote by $B_r(E, F)$ the space of all operators $u: E \to F$ for which there is a constant $C$ such that for all finite sequences $(x_i)_{i \leq m}$ in $E$ we have

$$\left\| \sup_{i \leq m} |u(x_i)| \right\|_F \leq C \left\| \sup_{i \leq m} |x_i| \right\|_E.$$

We will denote by $\|u\|_r$ the smallest constant $C$ for which this holds, i.e. we set $\|u\|_r = \inf\{C\}$. It is known, under the assumption that $F$ is Dedekind complete in the sense of [MN] (sometimes also called order complete), that in the real (resp. complex) case, every regular operator $u: E \to F$ is of the form $u = u_+ - u_-$ (resp. $u = a_+ - a_- + i(b_+ - b_-)$) where $u_+, u_-$ (resp. $a_+, a_-, b_+, b_-$) are bounded positive operators from $E$ to $F$, cf. [MN] or [S] p.233. Since the converse is obvious this gives a very clear description of the space $B_r(E, F)$. Here of course “positive” means positivity preserving. Let $E, F$ be real (resp.
complex) Banach lattices. Under the same assumption on $F$ (cf. e.g. [MN] p.27) it is known that $B_r(E, F)$ equipped with the usual ordering is a real (resp. complex) Banach lattice in such a way that we have

$$\forall \ T \in B_r(E, F) \quad \|T\|_r = \| |T| \|_{B(E, F)}.$$  

We refer to [MN] for more information on the spaces $B_r(E, F)$. We will only use the following elementary particular cases.

If $A$ is in $B_r(\ell_p, \ell_p)$ with associated matrix $(a_{ij})$, let us denote by $|A|$ the operator admitting $(|a_{ij}|)$ as its associated matrix. Then we have

$$\|A\|_r = \||A|\|_{B(\ell_p, \ell_p)}. \tag{2}$$

By a well known result (going back, I believe, to Grothendieck) for any measure spaces $(\Omega, \mu)$, $(\Omega', \mu')$ we have an isometric identity

$$B(L_1(\mu), L_1(\mu')) = B_r(L_1(\mu), L_1(\mu')). \tag{3}$$

On the other hand, we have trivially (isometrically)

$$B(L_\infty(\mu), L_\infty(\mu')) = B_r(L_\infty(\mu), L_\infty(\mu')). \tag{4}$$

The next result is known to many people in some slightly different form (in particular see [W]), I believe that our formulation is useful and hope to demonstrate this in the rest of this note. Motivated by the results in [HeP], I suspected that this result was known and I asked F. Lust-Piquard whether she knew a reference for this, she did not but she immediately showed me the following proof.

**Theorem 1.** Let $A_0, A_1$ be as above. For any fixed integer $n$, let $A_0^n \subset A_0$, $A_1^n \subset A_1$ be the subspace of all matrices $(a_{ij})$ which are supported by the upper left $n \times n$ corner, so that the elements of $A_0^n$ or $A_1^n$ can be viewed as $n \times n$ matrices. Note the elementary identifications

$$A_0^n = B(\ell_n^\infty, \ell_n^\infty) \quad \text{and} \quad A_1^n = B(\ell_n^1, \ell_n^1)$$

$$A_0 = B(c_0, \ell_\infty) \quad \text{and} \quad A_1 = B(\ell_1, \ell_1).$$
We have then for all $0 < \theta < 1$ the following isometric identities where $p = 1/\theta$:

(i) $(A_0^n, A_1^n)_{\theta} = B_r(\ell_p^n, \ell_p^n)$.

(ii) $A_{\theta} = (A_0, A_1)^{\theta} = B_r(\ell_p, \ell_p)$.

**Remark.** Note that $A_0$ and $A_1$ (resp. $A_0^n$ and $A_1^n$) are isometric as Banach spaces. This is a special case of the fact that the transposition induces an isometric isomorphism between the spaces $B(X,Y^*)$ and $B(Y,X^*)$ when $X$ and $Y$ are Banach spaces.

**Proof of Theorem 1.** (The main point was shown to me by F. Lust-Piquard.) We will prove (i) only. The second part (ii) follows easily from (i) by a weak-$*$ compactness argument which we leave to the reader.

Now let $(X_0, X_1)$ and $(Y_0, Y_1)$ be compatible couples of finite dimensional complex Banach spaces and let $X_{\theta} = (X_0, X_1)_{\theta}$, $Y_{\theta} = (Y_0, Y_1)_{\theta}$. It is well known and easy to check from the definitions that we have a norm 1 inclusion

$$(B(X_0, Y_0), B(X_1, Y_1))_{\theta} \subset B(X_{\theta}, Y_{\theta}).$$

Applying this to the spaces $X_0 = Y_0 = \ell^m_\infty(\ell^m_\infty)$ and $X_1 = Y_1 = \ell^n_\infty(\ell^n_\infty)$ with $m$ an arbitrary integer, we obtain the norm 1 inclusion

$$(A_0^n, A_1^n)_{\theta} \to B_r(\ell_p^n, \ell_p^n).$$

To show the converse, consider the spaces

$$B_0^n = A_0^{n*} \quad \text{and} \quad B_1^n = A_1^{n*}.$$

For any $n \times n$ matrix $(b_{ij})$ we have

$$\|(b_{ij})\|_{B_0^n} = \sum_{i=1}^n \sup_{j \leq n} |b_{ij}| \quad \text{and} \quad \|(b_{ij})\|_{B_1^n} = \sum_{j=1}^n \sup_{i \leq n} |b_{ij}|.$$

We claim that for all $(b_{ij})$ in the unit ball of $B_\theta^n = (B_0^n)^{1-\theta}(B_1^n)^{\theta} = (B_0^n, B_1^n)_{\theta}$ we have

$$(5) \quad \forall \ A \in B_r(\ell_p^n, \ell_p^n) \quad \sum_{i,j} |a_{ij}b_{ij}| \leq \|A\|_r.$$
With this claim, we conclude easily since (5) yields a norm one inclusion $B_r(\ell_p^\mu, \ell_p^\mu) \subset (B_0^\mu)^* = (A_0^\mu, A_1^\mu)^* = A_0^\mu$. Therefore it suffices to prove the claim (5). Since $(b_{ij})$ is assumed in the unit ball of $(B_0^\mu)^{1-\theta}(B_1^\mu)^\theta$ there are $n \times n$ matrices $(b_{0ij})$ and $(b_{1ij})$ such that $|b_{ij}| = |b_{0ij}| \cdot |b_{1ij}|$ and such that $\sum_i \sup_j |b_{0ij}|^{p'} \leq 1$, $\sum_j \sup_i |b_{1ij}|^{p} \leq 1$, with $p = 1/\theta$ and $p' = 1/(1 - \theta)$.

Now let $\beta_i = \sup_j |b_{0ij}|$ and $\alpha_j = \sup_i |b_{1ij}|$. Then

$$\sum |a_{ij} b_{ij}| \leq \sum \beta_i |a_{ij}| \alpha_j$$

hence by (2)

$$\leq \|A\|_r.$$ 

This proves our claim and concludes the proof. ■

It is then routine to deduce the following extension.

**Corollary 2.** Let $(\Omega, \mu)$ and $(\Omega', \mu')$ be arbitrary measure spaces. Consider the couple

$$X_0 = B(L_\infty(\mu), L_\infty(\mu')) , \quad X_1 = B(L_1(\mu), L_1(\mu')).$$

We will identify (for the purpose of interpolation) elements in $X_0$ or $X_1$ with linear operators from the space of integrable step functions into $L_1(\mu') + L_\infty(\mu')$. We have then isometrically

$$X_0^{1-\theta} X_1^\theta = (X_0, X_1)^\theta = B_r(L_p(\mu), L_p(\mu')).$$

**Remark.** Recalling (3) and (4), we can rewrite (6) as follows

$$(B_r(L_\infty(\mu), L_\infty(\mu')), B_r(L_1(\mu), L_1(\mu')))^{\theta} = B_r(L_p(\mu), L_p(\mu')).$$

By [Be], it follows that the space $(X_0, X_1)^\theta$ coincides with the closure in $B_r(L_p(\mu), L_p(\mu'))$ of the subspace of all the operators which are simultaneously bounded from $L_1(\mu)$ to $L_1(\mu')$ and from $L_\infty(\mu)$ to $L_\infty(\mu')$.

We will now consider operators defined on a subspace $S$ of a Banach lattice $E$ and taking values in a Banach lattice $F$. Let $u : S \rightarrow F$ be such an operator. We will again
say that $u$ is regular if there is a constant $C$ such that $u$ satisfies (1) for all finite sequences $x_1, ..., x_m$ in $E$. We again denote by $\|u\|_r$ the smallest constant $C$ for which this holds. Clearly the restriction to $S$ of a regular operator defined on $E$ is regular. Conversely, in general a regular operator on $S$ is not necessarily the restriction of a regular operator on $E$: for instance if $E$ is $L_1$, if $S$ is the closed span of a sequence of standard independent Gaussian random variables and if $u : S \to L_2$ is the natural inclusion map, then $u$ is regular (this is a well known result of Fernique, see e.g. [LeT] p. 60) but does not extend to any bounded map from $L_1$ into $L_2$ since by a weakening of Grothendieck’s theorem (cf. [P4] p. 57), the identity of $S$ would then be 2-absolutely summing, which is absurd since $S$ is infinite dimensional (cf. e.g. [P4] p. 14).

Nevertheless, it turns out that in several interesting cases, conversely every regular operator on $S$ is the restriction of a regular operator on $E$ with the same regular norm. In particular, the next statement is an extension theorem for regular operators which generalizes a result due to M. Lévy [Lé] in the case $p = 1$. We will prove

**Theorem 3.** Let $1 \leq p \leq \infty$. Let $(\Omega, \mu)$, $(\Omega', \mu')$ be arbitrary measure spaces. Let $S \subset L_p(\mu)$ be any closed subspace. Then every regular operator $u : S \to L_p(\mu')$ admits a regular extension $\tilde{u} : L_p(\mu) \to L_p(\mu')$ such that $\|\tilde{u}\|_r = \|u\|_r$.

Actually this will be a consequence of the following more general result. (We refer the reader to [LT] for the notions of $p$-convexity and $p$-concavity.)

**Theorem 4.** Let $L, \Lambda$ be Banach lattices and let $S \subset \Lambda$ be a closed subspace. Assume that $L$ is a dual space, or merely that there is a regular projection $P : L^{\ast\ast} \to L$ with $\|P\|_r \leq 1$. Assume moreover that for some $1 \leq p \leq \infty$ $\Lambda$ is $p$-convex and $L$ $p$-concave. Then every regular operator $u : S \to L$ extends to a regular operator $\tilde{u} : \Lambda \to L$ with $\|\tilde{u}\|_r = \|u\|_r$.

**Remark.** Note that by known results (cf. [K] or [LT]) in the above situation every positive operator $u : \Lambda \to L$ factors through an $L_p$-space, i.e. there is a measure space $(\Omega, \mu)$ and operators $B : \Lambda \to L_p(\mu)$ and $A : L_p(\mu) \to L$ such that $U = AB$ and $\|A\| \cdot \|B\| = \|u\|$. Actually for this conclusion to hold, it suffices to assume that $u$ can be written as the composition of first a $p$-convex operator with constant $\leq 1$ followed by a $p$-concave
operator with constant \( \leq 1 \). Therefore, since every regular operator with regular norm \( \leq 1 \) on a \( p \)-convex Banach lattice clearly is itself \( p \)-convex with constant \( \leq 1 \), every regular \( u: \Lambda \to L \) factors through an \( L_p \)-space with factorization constant at most 1. Actually it is easy to modify Krivine’s argument to prove that, in the same situation as in Theorem 4, every regular \( u: \Lambda \to L \) can be written as \( u = AB \) as above but with \( A, B \) regular and such that \( \|A\|_r \|B\|_r = \|u\|_r \).

**Proof of Theorem 4.** By a standard ultraproduct argument it is enough to consider the case when \( L \) is finite dimensional with an unconditional basis \((e_1, \ldots, e_n)\). As usual in extension problems, we will use the Hahn-Banach theorem. We need to introduce a Banach space \( X \) such that \( X^* = B_r(\Lambda, L) \). The space \( X \) is defined as the tensor product \( L^* \otimes \Lambda \) equipped with the following norm, for all \( v = \sum_1^n \alpha_k e_k^* \otimes x_k \) with \( \alpha_i \) scalar and \( x_i \in \Lambda \) we define

\[
\|v\|_X = \inf \left\{ \left\| \sum_1^n \alpha_i e_i \right\|_{L^*} \left\| \sup_{i \leq n} |x_i| \right\|_\Lambda \right\}.
\]

The only assumption needed for our extension theorem is that \( \| \|_X \) is a norm (see the remark below). This follows from the \( p' \)-convexity of \( L^* \) and the \( p \)-convexity of \( \Lambda \).

To check this we assume as we may that \( \Lambda \) is included in a space of measurable functions \( L_0(\mu) \) on some measure space. Let \( Y_0 \) be the space of \( n \)-tuples of measurable functions \( y_1, \ldots, y_n \) in \( L_0(\mu) \) equipped with the norm

\[
\|(y_i)\|_{Y_0} = \left\| \sum_1^n \|y_i\|^{\frac{1}{p'}} e_i^* \right\|_{L^*}^{p'}.
\]

That this is indeed a norm follows from the \( p' \)-convexity of \( L^* \).

Let \( Y_1 \) be the space of \( n \)-tuples of measurable functions \( y_1, \ldots, y_n \) in \( L_0(\mu) \) such that \( |y_i|^{\frac{1}{p}} \in \Lambda \) equipped with the norm

\[
\|(y_i)\|_{Y_1} = \left\| \sup_{i \leq n} |y_i|^{\frac{1}{p}} \right\|_\Lambda^p.
\]

Again this is a norm by the \( p \)-convexity of \( \Lambda \). But now if we consider the unit ball of the space

\[
Y_{0}^{1-\theta}Y_{1}^{\theta} \quad \text{with} \quad \theta = \frac{1}{p},
\]
we find exactly the set $C$. This shows that $C$ is convex as claimed above. We will now check that $X^* = B_r(\Lambda, L)$ isometrically.

Consider $u: \Lambda \to L$. We have

$$(7) \quad \|u\|_r = \sup \left\{ \left\| \sum_{k=1}^n \sup_{i \leq m} \langle u(x_i), e_k^* \rangle e_k \right\|_L \right\}$$

where the supremum runs over all $m$ and all $m$-tuples $(x_1, \ldots, x_m)$ in $\Lambda$ such that $\| \sup_{i \leq m} |x_i| \|_{\Lambda} \leq 1$. Let us denote by $\beta$ the unit ball of $L^*$. Then (7) can be rewritten

$$(8) \quad \|u\|_r = \sup \left\{ \left| \sum_{k=1}^n \alpha_k \langle u(x_{i_k}), e_k^* \rangle \right| \right\}$$

where the supremum runs over all integers $m$, all choices $i_1, \ldots, i_n$ in $\{1, \ldots, m\}$, all elements $\alpha = \sum_{k=1}^n \alpha_k e_k^* \in \beta$ and all $m$-tuples $x_1, \ldots, x_m$ in $\Lambda$ with $\| \sup_{i \leq m} |x_i| \|_{\Lambda} \leq 1$. But for such elements clearly $v = \sum_{k=1}^n \alpha_k e_k^* \otimes x_{i_k}$ is in the set $C$ which is the unit ball of $X$, hence (8) yields

$$\|u\|_r = \sup \{ |\langle u, v \rangle| \mid v \in B_X \}.$$  

This proves the announced claim that $X^* = B_r(\Lambda, L)$ isometrically.

We can then complete the proof by a well known application of the Hahn-Banach theorem.

Consider the subspace $M \subset X$ formed by all the $v = \sum_{i=1}^n \alpha_k e_k^* \otimes x_k$ such that $\alpha_k x_k \in S$ for all $k = 1, \ldots, n$. If $u: S \to L$ is regular we clearly have for all $v$ in $M$

$$|\langle u, v \rangle| = \left| \sum_{i=1}^n \langle u(\alpha_k x_k), e_k^* \rangle \right| \leq \|u\|_r \|v\|_X$$

hence we can find a Hahn-Banach extension of the linear form $v \in M \to \langle u, v \rangle$ defined on the whole of $X$ and still with norm $\leq \|u\|_r$. Clearly we can write the extension in the form $v \in X \to \langle \tilde{u}, v \rangle$ for some operator $\tilde{u}: \Lambda \to L$ and since $\sup_{\|v\|_X \leq 1} |\langle \tilde{u}, v \rangle| \leq \|u\|_r$, we have $\|\tilde{u}\|_r \leq \|u\|_r$ as announced.  

\[ \square \]

**Remark.** Assume again $L$ finite dimensional as above. The assumption “$\Lambda$ $p$-convex, $L$ $p'$-concave” can be replaced by the property that in $L^* \otimes \Lambda$ the set

$$C = \left\{ \sum_{i=1}^n \alpha_i e_i^* \otimes x_i \mid \alpha_i \in C \quad x_i \in \Lambda \right\}, \quad \left\| \sum_{i=1}^n \alpha_i e_i^* \right\|_{L^*} \leq 1, \quad \| \sup_{i \leq m} |x_i| \|_{\Lambda} \leq 1 \}$$
is a convex set.

As the preceding proof shows this is true is $L^*$ is $p'$-convex and $\Lambda$ $p$-convex. However, it is clearly true also in other cases. For instance if $L^* = \ell^n_\infty$ then $C$ is just the unit ball of $\Lambda(\ell^n_\infty)$ which is clearly convex for all $\Lambda$. Moreover if $\Lambda = L_\infty(\mu)$ for some measure $\mu$ then $C$ is the unit ball of $L^*(L_\infty(\mu))$ which is convex for all $L$. More generally, what we really use (and which is then equivalent to the extension theorem, by a reasoning well known to many Banach space specialists) is that the closed convex hull of the set $C$, satisfies

$$\overline{\text{conv}}(C) \cap M = \text{conv}(C \cap M),$$

where $M$ denotes as above the subspace $M = L^* \otimes S \subset L^* \otimes \Lambda$.

We now give some applications to $H^p$-spaces, mainly motivated by our paper [P2]. Let $1 \leq p \leq \infty$. Let $H^p$ be the usual $H^p$-space of functions on the torus $T$ equipped with its normalized Haar measure $m(dt) = \frac{dt}{2\pi}$. We denote simply $L^p = L^p(T, m)$. Given a finite dimensional normed space $E$ we denote

$$H^p(E) = \{ f \in L^p(m; E) \mid \hat{f}(n) = 0 \quad \forall \ n < 0 \}.$$

By a result of P. Jones [J] (see also [BX, P1, X] for a discussion of the vector valued case) we have isomorphically and with isomorphism constants independent of $E$

$$(9) \quad H^p(E) = (H^\infty(E), H^1(E))_\theta \quad \text{if} \quad \theta = 1/p. $$

More precisely, there is a constant $C_p$ such that for all $f$ in $H^p(E)$ we have

$$(10) \quad \| f \|_{(H^\infty(E), H^1(E))_\theta} \leq C_p \| f \|_{H^p(E)}. $$

We will prove the following extension of Corollary 2.

**Theorem 5.** Let $(\Omega, \mu)$ be an arbitrary measure space. Let

$$B_0 = B_r(H^\infty, L_\infty(\mu)) \quad B_1 = B_r(H^1, L_1(\mu)).$$

Then (isomorphically) $(B_0, B_1)^\theta = B_r(H^p, L_p(\mu))$ with $\theta = 1/p$.

**Proof.** By Theorem 4, if $u: H^p \to L_p(\mu)$ is such that $\|u\|_r < 1$, then $\exists \hat{u}: L_p \to L_p(\mu)$ extending $u$ such that $\|\hat{u}\|_r < 1$. By Corollary 2, $\hat{u}$ is of norm $< 1$ in the space $C = \overline{\text{conv}}(C \cap M).$
\((B_r(L_\infty, L_\infty(\mu)), B_r(L_1, L_1(\mu)))^\theta\) hence by restriction \(u\) is of norm \(< 1\) in \((B_0, B_1)^\theta\). Conversely, assume that \(u\) is in the unit ball of \((B_0, B_1)^\theta\). Consider then \(f\) in the unit ball of \(H^p(\ell_\infty^n)\), or equivalently consider an \(n\)-tuple \((f_1, \ldots, f_1)\) in \(H^p\) such that \(\int \sup_{k \leq n} |f_k|^p dm \leq 1\).

By P. Jones’s theorem (10) we have

\[ \|f\|_{(H^\infty(\ell_\infty^n), H^1(\ell_\infty^n))^\theta} \leq C_p, \]

hence since \(\|u\|_{(B_0, B_1)^\theta} \leq 1\) by assumption, it is easy to deduce

\[ \|u(f)\|_{(L_\infty(\ell_\infty^n), L_1(\ell_\infty^n))^\theta} \leq C_p \]

or equivalently since \(L_p(\ell_\infty^n) = (L_\infty(\ell_\infty^n), L_1(\ell_\infty^n))^\theta\)

\[ \int \sup_{k \leq n} |u(f_k)|^p dm \leq C_p^p. \]

By homogeneity we conclude that \(\|u\|_{B_r(H^p, L_p(\mu))} \leq C_p. \)

**Remark.** Once again by [Be], the space \((B_0, B_1)_\theta\) coincides with the closure in \(B_r(H^p, L_p(\mu))\) of the operators which are simultaneously regular from \(H^1\) to \(L_1(\mu)\) and from \(H^\infty\) to \(L_\infty(\mu)\).

**Remarks.**

(i) For a version of Theorem 1 and Corollary 2 in the case of noncommutative \(L_p\)-spaces, we refer the reader to [P3].

(ii) Of course Theorem 5 and its proof remain valid with the couple \((H^\infty, H^1)\) replaced by any couple of subspaces of \((L^\infty, L^1)\) for which (10) holds.

**References**

[B] J. Bergh. On the relation between the two complex methods of interpolation. Indiana Univ. Math. Journal 28 (1979) 775-777.

[BL] J. Bergh and J. Löfström. Interpolation spaces. An introduction. Springer Verlag, New York. 1976.

[BX] O. Blasco and Q. Xu. Interpolation between vector valued Hardy spaces, J. Funct. Anal. 102 (1991) 331-359.
[Ca] A. Calderón. Intermediate spaces and interpolation, the complex method. Studia Math. 24 (1964) 113-190.

[HP] U. Haagerup and G. Pisier. Factorization of analytic functions with values in non-commutative \( L^1 \)-spaces and applications. Canadian J. Math. 41 (1989) 882-906.

[HeP] A. Hess and G. Pisier, On the \( K_t \)-functional for the couple \( B(L_1, L_1), B(L_\infty, L_\infty) \).

[K] J.L. Krivine. Théorèmes de factorisation dans les espaces de Banach réticulés. Séminaire Maurey-Schwartz 73/74, Exposé 22, Ecole Polytechnique, Paris.

[Lé] M. Lévy. Prolongement d’un opérateur d’un sous-espace de \( L^1(\mu) \) dans \( L^1(\nu) \). Séminaire d’Analyse Fonctionnelle 1979-1980. Exposé 5. Ecole Polytechnique, Palaiseau.

[LeT] M. Ledoux and M. Talagrand. Probability in Banach spaces. Springer-Verlag 1991.

[LT] J. Lindendrauss and L. Tzafriri. Classical Banach spaces II, Function spaces, Springer-Verlag, 1979.

[MN] P. Meyer-Nieberg. Banach Lattices. Universitext, Springer-Verlag, 1991.

[P1] G. Pisier. Interpolation of \( H^p \)-spaces and noncommutative generalizations I. Pacific J. Math. 155 (1992) 341-368.

[P2] Interpolation of \( H^p \)-spaces and noncommutative generalizations II. Revista Mat. Iberoamericana. To appear.

[P3] The Operator Hilbert space \( OH \), Complex Interpolation and Tensor Norms. To appear.

[P4] Factorization of linear operators and the Geometry of Banach spaces. CBMS (Regional conferences of the A.M.S.) 60, (1986), Reprinted with corrections 1987.

[S] H.H. Schaefer. Banach lattices and positive operators. Springer-Verlag, Berlin Heidelberg New-York, 1974.

[W] L. Weiss. Integral operators and changes of density. Indiana University Math. Journal 31 (1982) 83-96.

[X] Q. Xu. Notes on interpolation of Hardy spaces. Ann. Inst. Fourier. To appear.