Intersection exponents for biased random walks on discrete cylinders

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Abstract

We prove existence of intersection exponents $\xi(k, \lambda)$ for biased random walks on $d$-dimensional half-infinite discrete cylinders, and show that, as functions of $\lambda$, these exponents are real analytic. As part of the argument, we prove convergence to stationarity of a time-inhomogeneous Markov chain on half-infinite random paths. Furthermore, we show this convergence takes place at exponential rate, an estimate obtained via a coupling of weighted half-infinite paths.

Keywords: Intersection exponent, biased random walk, coupling

1 Introduction

In this paper, we analyze biased random walks on $d$-dimensional discrete cylinders. The treatment of the model is self contained and it does not use previous results.

Our main motivation in approaching this problem is to understand the arguments and techniques needed in the study of 3-dimensional Brownian intersection exponents. In a series of papers, Lawler, Schramm and Werner studied Brownian intersection exponents in dimensions two and three. They also proved that, in dimension two, these exponents are analytic. We think their analyticity argument will apply in three dimensions, as long as one is able to get good estimates on the coupling rate of weighted Brownian motion paths. In two dimensions, the estimates are obtained using conformal invariance, and so they do not transfer to three dimensions. By analyzing a transient random walk on the cylinder, we will be able to highlight the techniques needed to prove analyticity of intersection exponents for Brownian motion, as well as the type of estimates needed for an exponential coupling rate.

Let us note that analyzing random walks on cylinders has been of great interest in recent years. We direct the reader to the work of Sznitman and Dembo on the disconnection of cylinders by random walks, such as [4] and [16]. The recent work of Windisch [17] on the disconnection time of discrete cylinders by biased random walks, where the drift depends on the size of the base, is also worth noting.

1.1 Brownian intersection exponents

Let us begin with a brief introduction to Brownian intersection exponents. Consider a set of $k$ independent Brownian motion paths started at the origin and a set of $j$ independent Brownian
motion paths started away from the origin, on the ball of radius one. The probability that the two packets reach level $e^n$, without intersecting, decays exponentially in $n$ with exponent $\xi(BM)(k,j)$. Roughly speaking, intersection exponents for Brownian motion are a measure of how likely it is that Brownian motion paths do not intersect. We now proceed to make this precise.

For $d = 2, 3$, $B^1_t, B^2_t, ..., B^k_t$ will denote $k$ independent $d$-dimensional Brownian motions started at the origin. Let $X_t$ be another $d$-dimensional Brownian motion started on the ball of radius 1 and independent of $B^1, ..., B^k$. For $1 \leq i \leq k$ we write $B^i[0,t] := \{ z \in \mathbb{R}^d : B^i_s = z \text{ for some } 0 \leq s \leq t \}$ and similarly $X[0,t] := \{ z \in \mathbb{R}^d : X_s = z \text{ for some } 0 \leq s \leq t \}$.

For $1 \leq j \leq k$, let

\[ \tilde{T}^j_n = \inf \{ t : |B^j_t| \geq e^n \} \]

be the first time $B^j$ reaches the ball of radius $e^n$. Similarly, let

\[ \tilde{T}_n = \inf \{ t : |X_t| \geq e^n \} \]

Given $k$ paths, we let $Z_n^{(BM)}$ be the probability that another path avoids them, up to the first time both sets reach the ball of radius $e^n$, as follows:

\[ Z_n^{(BM)} = \mathbb{P} \left\{ X[0,\tilde{T}_n] \cap (B^1[0,\tilde{T}^1_n] \cup \cdots \cup B^k[0,\tilde{T}^k_n]) = \emptyset \mid B^1[0,\tilde{T}^1_n] \cup \cdots \cup B^k[0,\tilde{T}^k_n] \right\} . \]

Then the intersection exponent $\xi(BM)(k,j)$ is defined as

\[ \xi(BM)(k,j) := - \lim_{n \to \infty} \frac{\log \mathbb{E}[Z_n^{(BM)}]}{n} . \]

One can further define generalized intersection exponents that loosely speaking describe non-intersection probabilities between non-integer numbers of Brownian paths. They were first introduced in [13]. In other words, the discrete sequence of intersection exponents can be replaced in a natural way by a continuous function

\[ \xi(BM)(k,\lambda) := - \lim_{n \to \infty} \frac{\log \mathbb{E}[(Z_n^{(BM)})^\lambda]}{n} . \]

The existence of intersection exponents follows from a subadditivity argument. Alternate but equivalent ways of defining such exponents can be found in [8], [7].

Brownian intersection exponents have been studied extensively. In dimensions four and higher, since two Brownian paths do not intersect almost surely, Brownian intersection exponents in these dimensions equal zero. Presently, all intersection exponents for the planar Brownian motion are known (see [9], [10], [11]): intersection exponents for a wide range of values of $\lambda$ have been computed using the Schramm-Loewner evolution (SLE). Lawler, Schramm and Werner further proved that planar Brownian intersection exponents are analytic [12] and consequently $\xi_2(BM)(k,\lambda)$ were extended by analyticity to $\lambda > 0$. However, not much progress has been made in three dimensions. As of this moment, the only known exponents are $\xi_3(BM)(k,0) = 0$ and $\xi_3(BM)(2,1) = \xi_3(BM)(1,2) = 1$. In [2], it was proved that the exponent $\xi_3(BM)(1,1)$ is between 1 and 1/2 and [7] implies it is strictly between 1 and 1/2. Simulations further suggest this exponent is around .58 (see [3]).
1.2 Summary of results

Let $G$ be the half-infinite discrete cylinder $\mathbb{Z} \times \mathbb{T}_L^{d-1}$, where $\mathbb{T}_L^{d-1}$ is a $(d-1)$-dimensional torus of side $L$. We consider random walks on $G$ that move according to the following transition probabilities:

$$
p(z,w) = \begin{cases} 
p/d & \text{if } w - z = (1,0) \\
(1-p)/d & \text{if } w - z = (-1,0) \\
1/(2d) & \text{if } w - z = (0,\vec{1}) \\
0 & \text{otherwise} \end{cases}$$

where $\vec{1}$ denotes any vector of norm one on $\mathbb{T}_L^{d-1}$. Here the second coordinate is a $d$-dimensional vector and addition is, as usual, mod $L$ on $\mathbb{T}_L^{d-1}$. Then the random walk is symmetric on $\mathbb{T}_L^{d-1}$, and it is a one-dimensional asymmetric random walk on $\mathbb{Z}$ with parameter $p > 1/2$.

The path-valued random variables $Z_n$ are defined for this process in the same manner we defined them for Brownian motion. Roughly speaking, they are the probability that, given a set of $k$ half-infinite paths up to level $n$, another random walk coming from negative infinity will reach level $n$ without hitting the given set of paths. In order to give a precise definition for $Z_n$, we introduce a class of paths that we will call nice; these paths have the property that they can be avoided by a random walk coming from negative infinity and, in particular, they do not disconnect negative infinity and level zero. We note that the definition of nice paths is such that two $h$-processes conditioned to avoid a given path can be coupled (see Section 2.2.1). One can use this to describe the hitting measure on any level of a random walk started at negative infinity and conditioned to avoid a given random walk path. A similar result for 3-dimensional Brownian motion has not yet been proved but it is expected to be true. In particular, one should be able to describe the hitting measure on the ball of radius one for a Brownian motion started close to another Brownian motion and conditioned to avoid it.

The random walk intersection exponent $\xi(k,\lambda)$ is defined as

$$\xi(k,\lambda) := -\lim_{n \to \infty} \frac{\log E[Z^\lambda_n]}{n}.$$ 

In Section 2.2 we use a subadditivity argument to show these exponents exist and are finite, with $E[Z^\lambda_n]$ being logarithmically asymptotic to $e^{-\xi(k,\lambda)n}$. As long as we start with a nice initial configuration, we further prove an important estimate: $E[Z^\lambda_n]$ are within constant multiples of $e^{-\xi(k,\lambda)n}$ (see Proposition 2.7), which we will denote by

$$E[Z^\lambda_n] \asymp e^{-\xi(k,\lambda)n}.$$ 

As mentioned before, one of our main tools is coupling of weighted paths. Starting with an initial configuration $\gamma_0$, we attach to it a random walk started at the endpoint of $\gamma_0$ and stopped when it first reaches level one. Call the resulting path $\gamma_1$. We will fix a large $N$ and condition the path $\gamma_1$ to survive up to level $N$, that is, $N-1$ additional steps. Then we weight the new path by the probability it will survive up to level $N$, and normalize it to obtain a probability measure. This procedure defines a time-inhomogeneous Markov chain $X_n$ on the space of nice paths, depending on the initial configuration $\gamma_0$. One of our main results is the following theorem, whose proof is the content of Section 3.
Theorem 1.1. Let $X_n$ and $X'_n$ be Markov chains with $X_0 = \gamma_0$, and $X'_0 = \gamma'_0$ respectively, induced by the weighting described above. There exist constants $C, \beta > 0$ such that, for all $n \geq 1$, for all $\gamma_0, \gamma'_0 \in \mathcal{A}$, we can define $X_n$ and $X'_n$ on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{P}\{X_n \neq n/2 X'_n\} \leq Ce^{-\beta n},$$

where $X_n \equiv_k X'_n$ means that $X_n$ and $X'_n$ have been coupled for the past $k$ steps.

Let us briefly describe how we will couple two Markov chains started with different initial configurations. The coupling is similar to the coupling used in [1] for a one-dimensional Ising-type model. The maximal coupling is done only on the set of paths that have enough connected cross-sections and the other transition probabilities are then adjusted so that we obtain a probability measure on the set of nice paths. Once the two chains are coupled, they do not necessarily remain coupled; in fact, if at the next level the paths do not have enough connected cross-sections they will decouple. However, if the two chains are coupled, the probability they will remain coupled for an additional step increases with the number of steps for which they have been already coupled. Once the coupling is set up, we can use roughly the same argument as in [1] to prove Theorem 1.1.

As a result of this coupling, one can show that starting with a given measure $\nu$ on initial configurations $\gamma_0$, supported on $\mathcal{A}$, if we let the process evolve, then the measure induced by this process on half-infinite paths converges to an invariant measure $\pi$.

Theorem 1.2. Let $\nu$ be a measure supported on $\mathcal{A}$ and let $\nu^n$ be the measure on paths in $\mathcal{A}$, whose density with respect to $\nu$ is

$$\frac{Z^\lambda_n}{E^\nu[Z^\lambda_n]}.$$ 

Then there exists a measure $\pi$ supported on $\mathcal{A}$ such that $\nu^n$ converges to $\pi$. Furthermore, if $\pi^n$ has density $\frac{Z^\lambda_n}{E^\pi[Z^\lambda_n]}$ with respect to the limiting measure $\pi$, then $\pi^n = \pi$.

The proof of this theorem is the content of Section 3.5. Our main result can be found in Section 4. Using the same ideas and techniques that Lawler, Schramm and Werner used in [12], we prove analyticity of the intersection exponent $\xi(k, \lambda)$.

Theorem 1.3. For all $k \geq 1$, $\xi(k, \lambda)$ is a real analytic function of $\lambda$ in $(0, \infty)$.

The argument has similarities with the proof from [15] that the free energy of a one-dimensional Ising model with exponentially decreasing interactions is an analytic function. The proof follows the structure of the proof for analyticity of planar Brownian intersection exponents, presented in [12], and it differs from [12] only in the estimates we use. In fact, this suggests that, if similar estimates can be obtained for 3-dimensional Brownian exponents, the analyticity proof from [12] would immediately apply.

Here is a brief outline of the proof. We will restrict ourselves to proving analyticity for $\xi(\lambda) := \xi(1, \lambda)$. We define linear functionals $T_\lambda$ on a Banach space of functions, defined on the set of nice paths, functions with the property that they depend very little on how the path looks like far away in the past. $T_\lambda$ and the Banach space are described in Section 4.1.

The norm on the Banach space is a little different than the one in [12], but it follows the same principle. We first show $T_z$ is an analytic function in a neighborhood of the positive real
The existence of the spectral gap is done in Section 4.3 by showing the exponential coupling rate implies $e^{-\xi(\lambda)}$ is an isolated simple eigenvalue of $T_\lambda$. The theorem then follows by a simple argument from operator theory.

### 1.3 A word on notation

Throughout the article, we will say a sequence $a_n$ is *logarithmically asymptotic* to $e^{bn}$, which we will write as $a_n \approx e^{bn}$ if

$$\lim_{n \to \infty} \frac{\log a_n}{n} = b.$$ 

Also we will use the notation $a_n \asymp b_n$ to denote the following: there exist constants $c$ and $C$ such that for all $n$,

$$cb_n \leq a_n \leq Cb_n.$$ 

We will often use the notation $a_n = O(b_n)$ by which we mean that there exist constants $C, N > 0$ such that for all $n \geq N$,

$$a_n \leq Cb_n.$$ 

Constants $c, c'$ and $C$ will denote arbitrary positive constants, independent of all other quantities involved in a given expression. Their value will be allowed to change from line to line. However, other constants, such as $a, \hat{c}, c_1, c_2$ will be fixed.

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## 2 Random walk intersection exponents

### 2.1 Paths on the cylinder

Let $\mathbb{T}^{d-1}_L = (\mathbb{Z}/L\mathbb{Z})^{d-1}$ be a $(d-1)$-dimensional torus and let

$$G := \mathbb{Z} \times \mathbb{T}^{d-1}_L, \quad d \geq 1,$$

be the discrete $d$-dimensional infinite cylinder. The set $\{(i, y) : y \in \mathbb{T}^{d-1}_L\}$ will be called *the level* $i$ of the cylinder, and for a point $z = (i, y) \in G$ we write $|z| = i$ to mean $z$ is on level $i$ of the cylinder. Our main motivation in using the torus as a base for the cylinder is the property that every point on the torus “looks the same.” Therefore, one could equally well consider a finite connected regular graph as a generalization of $\mathbb{T}^{d-1}_L$.

For $i \leq j$, denote by $G_{i,j}$ the cylinder between levels $i$ and $j$,

$$G_{i,j} = \{z \in \mathbb{Z} \times \mathbb{T}^{d-1}_L : i \leq |z| \leq j\}.$$ 

We will write $G_j = G_{-\infty,j}$ for the half-infinite cylinder up to level $j$.

We construct half-infinite paths on $G$ as follows. Let $X$ be the set of all paths $\gamma : [0, t_\gamma] \to G$, starting on level 0 and ending when first reaching level 1:

$$X := \{\gamma : [0, t_\gamma] \to G : |\gamma(0)| = 0, |\gamma(t_\gamma)| = 1 \text{ and } |\gamma(t)| < 1 \text{ for all } t < t_\gamma\}.$$
We will refer to \( t_\gamma \) as the time-duration of the path \( \gamma \) and we will say \( \gamma \) and \( \gamma' \) are equal if \( \gamma' \) is a translation of \( \gamma \) in the \( T_{d-1} \) direction of \( G \). Let \( \mathcal{X}_i \) be the set of all paths \( \gamma_i \) starting on level \((i-1)\) and stopped when they reach level \( i \). Then \( \mathcal{X}_i \) is exactly \( \mathcal{X} \) translated by \((i-1)\) in the \( \mathbb{Z} \) direction of \( G \). It will be convenient to think of elements \( \gamma_i \) of \( \mathcal{X}_i \) as paths in \( \mathcal{X} \), translated accordingly. Let

\[
\mathcal{A} = \{ \gamma_0 = \ldots \gamma_{-1} \gamma_0 : \gamma_j \in \mathcal{X}_j \text{ and } \gamma_j(t_{\gamma_j}) = \gamma_{j+1}(0) \text{ for } j < 0 \}.
\]

That is, \( \mathcal{A} \) is the set of half-infinite paths \( \gamma_0 \) constructed as a sequence of \( \gamma_j \in \mathcal{X}_j \), \( -\infty < j \leq 0 \). We will think of \( \mathcal{A} \) as \( \cdots \times \mathcal{X} \times \mathcal{X} \) along with a position at time zero uniquely determines a path in \( \mathcal{A} \).

Each \( \gamma_0 \), as a sequence of paths, is in a one-one correspondence with a path indexed by time \( \gamma_0(t) : (-\infty, 0] \to G \). The construction of \( \gamma_0(t) \) from \( \gamma_0 \) is left as a simple exercise. We will write \( \gamma_0 \) when we look at the path as a sequence of elements of \( \mathcal{X} \), and we will write \( \gamma_0(t) \) when we need to look at \( \gamma_0 \) as a sequence of points in the half-infinite cylinder.

For \( n \in \mathbb{Z} \), let \( \mathcal{A}_n \) be the set \( \mathcal{A} \) shifted by \( n \) in the \( \mathbb{Z} \) coordinate of \( G \), and denote its elements by \( \gamma_n \). Observe that for \( n \geq 1 \), \( \gamma_n \) can be decomposed into \( \gamma_n = \gamma_0 \gamma_1 \ldots \gamma_n \), for some unique \( \gamma_0 \in \mathcal{A} \) and \( \gamma_j \in \mathcal{X}_j \) for \( 1 < j \leq n \). Let \( \gamma_n = \gamma_1 \ldots \gamma_n \).

**Definition 2.1.** Let \( \gamma_n = \ldots \gamma_{n-1} \gamma_n \) and \( \gamma'_n = \ldots \gamma'_{n-1} \gamma'_n \) be paths in \( \mathcal{A}_n \). We write

\[
\gamma_n = \gamma'_n
\]

if \( \gamma_j = \gamma'_j \) for all \( n - k + 1 \leq j \leq n \), and we say \( \gamma_n \) and \( \gamma'_n \) agree for the last \( k \) levels.

The half-infinite paths \( \gamma_0 \) that we will be studying are biased random walks coming from negative infinity.

### 2.2 Intersection exponent - existence

Let us consider \( k \) independent random walks \( S^1, \ldots, S^k \) defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), starting on level zero of \( G \), and evolving according to transition probabilities as in \((I)\). Let \( S \) be another random walk, defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P}_1)\), with the same transition probabilities. We will use \( \mathbb{E} \) and \( \mathbb{E}_1 \) for expectations with respect to \( \mathbb{P} \) and \( \mathbb{P}_1 \), respectively.

**Remark 2.2.** Observe that we define \( S \) on \((\Omega, \mathcal{F}_1, \mathbb{P}_1)\) and \( S^1, \ldots, S^k \) on \((\Omega, \mathcal{F}, \mathbb{P})\). This notation may look unnatural, but it will help simplify notation later in the paper.

Define stopping times: for \( 1 \leq j \leq k \),

\[
T_j^r = \inf\{ t : |S^j_t| = r \}
\]

\[
T_r = \inf\{ t : |S_t| = r \}.
\]

\( S[r, s] \) will denote the random-valued set \( \{ S_l : r \leq l \leq s \} \), the set of points on the cylinder visited by the random walk from time \( r \) to time \( s \). Similarly, for \( 1 \leq j \leq k \), \( S^j[r, s] = \{ S^j_l : r \leq l \leq s \} \).

We start with a \( k \)-tuple \( \Gamma_0 := (\gamma_0^1, \ldots, \gamma_0^k) \) of \( \mathcal{A}^k \), which will be called an initial configuration. For \( 1 \leq j \leq k \), let \( S^j \) be a random walk on the cylinder \( G \), started at the endpoint.
of $\gamma_0^j$, and evolving, independent of $\gamma_0^j$, according to transition probabilities in (1). Take the path $S^j$ stopped when it first reaches level $n$ and attach it to $\gamma_0^j$. This is a path from $-\infty$ to level $n$, and in particular it is an element of $A_n$. We denote it by $\gamma_n^j$. Let $\tilde{\gamma}_n^j$ be the path $\gamma_n^j$ shifted accordingly in the $Z$ direction of $G$ so that it is an element of $A$.

For all $n \in Z$, define $\Gamma_n := (\gamma_n^1, \ldots, \gamma_n^j)$ and $\tilde{\Gamma}_n := (\tilde{\gamma}_n^1, \ldots, \tilde{\gamma}_n^j)$. Let $\mathcal{F}_n$ be the $\sigma$-algebra generated by $\Gamma_0$ and the random walks $S^1, \ldots, S^k$ up to stopping times $T_n^1, \ldots, T_n^k$

$$\mathcal{F}_n = \sigma\{\Gamma_0, S^j_i; t \leq T_n^j, \text{ for } 1 \leq j \leq k\}.$$ 

Then $\Gamma_n$ is an $\mathcal{F}_n$-measurable (path-valued) random variable.

We define functions $Z_n : A^k \rightarrow \mathbb{R}$ by

$$Z_n(\Gamma_0) = P_1\{S(-\infty, T_0] \cap \Gamma_0 = \emptyset | S(-\infty, T_n] \cap \Gamma_n = \emptyset\}.$$ 

Equivalently, one can define $Z_n$ as

$$Z_n(\tilde{\Gamma}_n) = P_1\{S(-\infty, T_n] \cap \Gamma_n = \emptyset | S(-\infty, T_0] \cap \Gamma_0 = \emptyset\}. \quad (2)$$

More precisely,

$$Z_n(\tilde{\Gamma}_n) = \lim_{|x| \to -\infty} P^*_{\tilde{\Gamma}_n} \{S[0, T_n] \cap \Gamma_n = \emptyset | S[0, T_0] \cap \Gamma_0 = \emptyset\}, \quad (3)$$

where the initial configuration $\Gamma_0$ is nice, meaning that $\Gamma_0$ is such that this limit exists. Note also that we write conditioning with respect to the event

$$S(-\infty, T_0] \cap \Gamma_0 = \emptyset,$$

which is a set of probability zero. By this conditioning we mean that on $(-\infty, T_0]$, $S$ is an $h$-process conditioned not to hit $\Gamma_0$. Given $\Gamma_0$ is nice, the conditioning is well-defined and the limit exists, as it will be discussed in Section 2.2.1. For now, let us assume $Z_n$ are well-defined.

Let us also consider a random walk started at $z$, on level zero, and define the following $\mathcal{F}_n$-measurable random variables

$$Z_{n,z} = P^*_z \{S[0, T_n] \cap \Gamma_n = \emptyset\}$$

$$\overline{Z}_n = \sup_{|z|=0} P^*_z \{S[0, T_n] \cap \Gamma_n = \emptyset\}.$$ 

Let $q_n = \sup_{\Gamma_0 \in A^k} E^{\Gamma_0}[Z_n]$ and $\overline{q}_n = \sup_{\Gamma_0 \in A^k} E^{\Gamma_0}[\overline{Z}_n]$. Now,

$$Z_n = P_1\{S(-\infty, T_n] \cap \Gamma_n = \emptyset | S(-\infty, T_0] \cap \Gamma_0 = \emptyset\} \leq \sup_{|z|=0} P^*_z \{S[0, T_n] \cap \Gamma_n = \emptyset\} = \overline{Z}_n$$

and taking expectations, we have $q_n \leq \overline{q}_n$.

**Proposition 2.3.** There exists $\xi(k, \lambda)$ such that $q_n \approx e^{-n\xi(k, \lambda)}$, that is

$$\lim_{n \to \infty} \frac{\log q_n}{n} = -\xi(k, \lambda)$$

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**Proof:** Let $\mathcal{A}^k$ be the set of all *nice* $k$-tuples $\Gamma_0$. For $n \geq 1$, define functions $\Phi_n : \mathcal{A}^k \to \mathbb{R}$ by

$$\Phi_n(\Gamma_0) = -\log Z_n(\Gamma_0).$$

We will use $\Phi$ to denote $\Phi_1$ and use the shorthand $\Phi_m$ for $\Phi_m(\Gamma_0)$. We naturally let $\Phi_0 = 0$. It is easy to see that the functions $\Phi_m$ are additive, more precisely

$$\Phi_n(\tilde{\Gamma}_n) + \Phi_m(\tilde{\Gamma}_m) = \Phi_n(\tilde{\Gamma}_n) + \Phi_m(\tilde{\Gamma}_m).$$

Then we have

$$q_{m+n} = \sup_{\Gamma_0} E^{\Gamma_0}[e^{-\lambda \Phi_{m+n}}]$$

$$= \sup_{\Gamma_0} E[E^{\Gamma_0}[e^{-\lambda \Phi_{m+n}}|\mathcal{F}_n]]$$

$$= \sup_{\Gamma_0} E^{\Gamma_0}[e^{-\lambda \Phi_n} E^{\tilde{\Gamma}_n}[e^{-\lambda \Phi_m}]]$$

$$\leq \sup_{\Gamma_0} E^{\Gamma_0}[e^{-\lambda \Phi_n} q_m]$$

$$\leq q_m q_n.$$

Then $\log q_{m+n} \leq \log q_n + \log q_m$, and using an easy subadditivity argument (see [6]), we get

$$\lim_{n \to \infty} \frac{\log q_n}{n} = \inf_n \frac{\log q_n}{n} = -\xi(k, \lambda),$$

(4)

with $\xi(k, \lambda)$ possibly infinite. To see that $\xi(k, \lambda) < \infty$, suppose $S$ has avoided $\Gamma_0$ up to time $T_0$. Let the paths given by $S[T_0, T_1]$ and $S[0, T_n], \ldots, S[k, T_n]$ be straight lines in the $\mathbb{Z}$ direction of $G$. Then $\Gamma_n \cap S(-\infty, T_n] = \emptyset$. This configuration occurs with probability $(p/d)^{(k+\lambda)n}$, if $k < |T_{L-1}| - 1$ and with probability at least $(p/d)^{|T_{L-1}| - 1 + \lambda)n$ if $k \geq |T_{L-1}| - 1$. Therefore $\xi(k, \lambda) \leq \left((|T_{L-1}| - 1 + \lambda)\log(d/p)\right)$.

From now on we will only consider the case $k = 1$ and analyze the exponent $\xi(\lambda) := \xi(1, \lambda)$. Proofs for $k > 1$ are essentially the same.

### 2.2.1 Nice paths

Recall definition (3) of $Z_n$. In this section we will present the technicalities involved in making sense of this definition. The reader is welcome to skip this section at a first reading.

Let $\overline{\gamma}_0$ be a path in $A$. Let $D(\overline{\gamma}_0)$ be the connected component of $G_0 \setminus \overline{\gamma}_0$ connecting level 0 to negative infinity. Of course, $\overline{\gamma}_0$ might disconnect level 0 from negative infinity, in which case $D(\overline{\gamma}_0) = \emptyset$. If $D(\overline{\gamma}_0)$ is non-empty, then it is unique for the following reason: $\overline{\gamma}_0$ has only one point on level zero, so level zero is connected, and hence there is at most one connected component containing $-\infty$ and level zero minus $\overline{\gamma}_0$. For $j \leq 0$, let $D_j = G_{j,j} \cap D(\overline{\gamma}_0)$ be the set of sites on level $j$ that can be reached by a random walk from $-\infty$, conditioned to avoid $\overline{\gamma}_0$. 

Definition 2.4. \( \gamma_0 \) is nice if \( D(\gamma_0) \neq \emptyset \) and for all \( n \geq 0 \), there exists a \( k > n \) such that
\[
\sum_{j=-k}^{-1} 1\{D_j \text{ is connected}\} \geq n
\]

Let \( \mathcal{A} \) be the set of all such nice paths.

This definition simply says that if \( \gamma_0 \) does not disconnect \( -\infty \) from level zero and \( G \setminus \gamma_0 \) has infinitely many connected levels, then \( \gamma_0 \) is nice path.

Let us now address the obvious question of conditioning on a set of measure zero in (3). Let \( \rho_0 = \inf \{ t > 0 : S_t \in \gamma_0 \} \) and
\[
h(z) = P_{z \uparrow}^\ast \{ T_0 < \rho_0 \}
\]
Denote transition probabilities for the unconditioned random walk on \( G \) by \( p(z,w) \). Then
\[
h(z) = \sum_w p(z,w)h(w),
\]
where we sum over all neighbors of \( z \). In other words, \( h(z) \) is zero on the path and harmonic on the complement of the path. Suppose \( z \) and \( w \) are on \( D_j(\gamma_0) \) and \( D_j(\gamma_0) \) is connected. Then we have the following Harnack-type inequality:
\[
a \leq \frac{h(z)}{h(w)} \leq a^{-1}.
\]
(5)

where \( a \) is the minimum over all possible connected configurations \( D_j \), over all \( z \) and \( w \) in \( D_j \), of the probability that starting at \( z \) the random walk reaches \( w \) before leaving \( D_j \). Since the torus has finitely many sites, \( 0 < a < 1/(2d) \). The constant \( a \) is repeatedly used in this article in estimates. It depends only on the size of the torus, or, if generalized to a regular graph, on the structure of the graph.

We start the random walk at \( z \notin \gamma_0 \) and we condition on the random walk surviving up to level zero \( (T_0 < \rho_0) \) to obtain a process evolving according to the following transition probabilities
\[
\overline{p}(z,w) = \frac{P_{z \uparrow} \{ S_1 = w; T_0 < \rho_0 | S_0 = z \}}{P_{z \uparrow} \{ T_0 < \rho_0 | S_0 = z \}} = \frac{p(z,w)h(w)}{h(z)}.
\]
(6)

Thus conditioning on \( \{ S[0,T_0] \cap \gamma_0 = \emptyset \} \) simply means that \( S \) is an \( h \)-process conditioned to avoid \( \gamma_0 \), and evolving according to transition probabilities given in (3). We can also define the hitting measure of level \( -n \) by a random walk started at \( |z| < -n \) and conditioned to avoid \( \gamma_0 \) up to time \( T_0 \) as
\[
\mu_{-n,z}(w) = P_{z \uparrow}^\ast \{ S(T_\neg n) = w; T_\neg n < \rho_0 \} \frac{h(w)}{h(z)}.
\]

Then we can show that if two \( h \)-processes conditioned on avoiding a nice path \( \gamma_0 \) are started far enough, then they can be coupled by the time they hit a given level with high probability. In fact, the reason for choosing this definition for nice paths was the need for such a coupling result. More general definitions of nice paths can be given, but we use this one in our present work for simplicity. We prove the coupling result in the following lemma.
Lemma 2.5. Let $n \geq 0$ be given. For every $\epsilon > 0$, there exists an $m > n$ such that for all pairs $z, z' \in G_{-m}$, if $S$ and $S'$ are $h$-processes started at $z$, and $z'$ respectively, with transition probabilities as in (6), we can define $S$ and $S'$ on the same probability space $(\Omega_1, \mathcal{F}_1, \mu)$ such that

$$\mathbb{P}\{S(T_{-n}) \neq S'(T'_{-n})\} < \epsilon/2.$$  

Furthermore, $||\mu_{-n,z} - \mu_{-n,z'}|| < \epsilon$, where $|| \cdot ||$ denotes the total variation norm.

Proof: Fix $n \geq 0$ and let $\epsilon > 0$ be given. Let $T_j$ and $T'_j$ be the hitting time of level $j$ by $h$-processes $S$ and $S'$ respectively. If $D_j$ is connected, then the hitting measures on level $j + 1$ for the two $h$-processes are within a constant. More precisely, using (5), one can show

$$\frac{\mu_{j+1,z}(w)}{\mu_{j+1,z'}(w)} \geq a^2.$$  

Then we can maximally couple $S$ and $S'$ on the same probability space $(\Omega_1, \mathcal{F}_1, \mu)$, so that

$$\mathbb{P}\{S(T_{j+1}) \neq S'(T'_{j+1})\} = \frac{1}{2}||\mu_{j+1,z} - \mu_{j+1,z'}|| \leq \frac{1}{2}(1 - a^2).$$  

For a detailed discussion of coupling and a proof for existence of the maximal coupling, we refer the reader to [14]. Once $S$ and $S'$ are coupled, we run them together.

If $G_{-m,-n} \cap D(\gamma_0)$ has at least $k$ connected cross-sections, then the two $h$-processes do not couple by the time they reach level $-n$ only if they do not couple at any of the connected cross-sections:

$$\mathbb{P}\{S(T_{-n}) \neq S'(T'_{-n})\} \leq \mathbb{P}\{S(T_j) \neq S'(T'_j)\text{ for all } j \in [-m, -n]\} \leq \left(\frac{1 - a^2}{2}\right)^k.$$  

Moreover, from the standard coupling inequality we obtain

$$||\mu_{-n,z} - \mu_{-n,z'}|| \leq 2\mathbb{P}\{S(T_{-n}) \neq S'(T'_{-n})\} \leq 2\left(\frac{1 - a^2}{2}\right)^k.$$  

Choose $k$ large enough such that $(\frac{1 - a^2}{2})^k \leq \epsilon/2$. Then let $m$ be so that $G_{-m,-n} \cap D(\gamma_0)$ has at least $k$ connected levels. Since $\gamma_0$’s complement has infinitely many connected cross-sections, $m$ is finite. \qed

We can now show that given a nice path $\gamma_0$, conditioning on avoiding this path makes sense in (3). We start with the following lemma which basically says that given a nice path $\gamma_0$, we can define a hitting measure on level $-n$ of the $h$-process induced by this conditioning.

Lemma 2.6. If $\gamma_0$ is nice, then for each $n \geq 0$, there exists a unique limiting measure

$$\mu_{-n} = \lim_{z \to -\infty} \mu_{-n,z}$$

Proof: Fix $n$. Let $(z_k)_{k=1}^{\infty}$ be a sequence in $G_{-n} \cap D(\gamma_0)$, with the property $\lim_{k \to \infty} |z_k| = -\infty$. Let $\epsilon > 0$. By Lemma 2.5, there exists $m > 0$ such that for all $z, z' \in G_{-m} \cap D(\gamma_0)$,

$$||\mu_{-n,z} - \mu_{-n,z'}|| < \epsilon$$

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Let \( n_1 \) be the smallest integer so that \( (z_k)_{k=n_1}^{\infty} \) is in \( G_{-m} \cap D(\overline{\eta}_0) \). Then for all \( i, j \geq n_1 \), \( \|\mu_{-n,z_i} - \mu_{-n,z_j}\| < \epsilon \) and so for every \( w \) on level \( -n \), \( \{\mu_{-n,z_k}(w)\}_{k=1}^{\infty} \) is a Cauchy sequence converging to some \( \mu_{-n}(w) \). Then clearly \( \mu_{-n,z} \Rightarrow \mu_{-n} \) as \( z \to -\infty \).

If one can define a probability measure on conditioned paths coming from \(-\infty\), then the random function \( Z_n \) in (8) is well defined. Let \( \overline{\eta}_0 \) be a half infinite path. We define \( v_k \) to be the measure on \( \overline{\eta}_0[k = \eta_{-k+1}\eta_k \ldots \eta_0] \), the restriction of \( \overline{\eta}_0 \) to the last \( k \) elements of the path, in the following way:

\[
v_k(\overline{\eta}_0|k) = \mathcal{M}(\overline{\eta}_0|k)1_{\{\overline{\eta}_0|k \cap \overline{\eta}_0 = \emptyset\}} \frac{\mu_{-k}(w_k)}{h(w_k)} ,
\]

where \( \mathcal{M} \) denotes the unconditioned random walk measure on paths and \( w_k \) is the first site on level \(-k\) reached by \( \overline{\eta}_0 \). Note that \( \{v_k\}_{k=1}^{\infty} \) is a consistent sequence of measures, and hence by Kolmogorov Extension Theorem, it can be extended to a measure on half-infinite paths \( v := \lim_{k \to \infty} v_{-n,k} \), which depends on the initial configuration \( \overline{\eta}_0 \). Thus, when we condition on avoiding \( \overline{\eta}_0 \) up to level zero, we mean that the measure induced by the \( h \)-process on half-infinite paths is given by \( v \).

### 2.3 Exponent estimate

From the definition of the intersection exponent, we know that \( q_n \approx e^{-n\xi(\lambda)} \). However, for \( \lambda \) restricted to a closed interval, away from zero, we will show that \( q_n \) and \( \overline{\sigma}_n \) are within multiplicative constants of \( e^{-n\xi(\lambda)} \). Moreover, this will also hold for \( E_{\overline{\eta}_0}[Z^\lambda_n] \) for all \( \overline{\eta}_0 \in \overline{A} \).

We fix \( \lambda_1 > 0 \) and \( \lambda_2 < \infty \) and restrict \( \lambda \) to \( [\lambda_1, \lambda_2] \).

**Proposition 2.7.** For every \( 0 < \lambda_1 < \lambda_2 < \infty \), there exist positive constants \( c_1 \) and \( c_2 \) such that for all \( n \geq 0 \) and all \( \lambda \in [\lambda_1, \lambda_2] \),

\[
c_1 e^{-\xi(\lambda)n} \leq q_n \leq c_2 e^{-\xi(\lambda)n}. \tag{7}
\]

Note that \( c_1 \) and \( c_2 \) can be chosen so they are independent of \( \lambda \in [\lambda_1, \lambda_2] \).

We will use the notation \( q_n \approx e^{-\xi(\lambda)n} \), to mean that \( q_n \) is bounded as in (7). The reason to restrict \( \lambda \) between two values \( \lambda_1, \lambda_2 \) is to get constants uniform in \( \lambda \). We proceed to prove the proposition, but we will first need a couple of estimates and technical lemmas.

#### 2.3.1 Preparation lemmas

We define the following stopping times:

- for \( j \leq 0 \), let \( \eta_j = \min\{t > T_0 : |S_t| = j\} \) and \( \eta_1^j = \min\{t > T_0^1 : |S_t^1| = j\} \)
- for \( j > 0 \), let \( \eta_j = \min\{t > T_j : |S_t| = j\} \) and \( \eta_1^j = \min\{t > T_j^1 : |S_t^1| = j\} \)

In other words, for non-positive \( j \), \( \eta_j \) is the first time after reaching level zero that \( S_t \) returns to level \( j \) and, for positive \( j \), \( \eta_j \) is the second time \( S_t \) reaches level \( j \).

**Lemma 2.8.** For any \( z \) with \( |z| = 0 \),

\[
P_1^z(T_n < \eta_0) \geq \frac{2p - 1}{d}. \tag{8}
\]
Proof: Let $\psi(x) = [(1 - p)/p]^x$. If $a < 0 < b$, one can easily solve
\[
P_1 \{ T_b < T_a \} = \frac{\psi(0) - \psi(a)}{\psi(b) - \psi(a)}.
\]
See [5] for a standard proof of such a result. The random walk started on level zero will reach level $n$ before returning to level zero if and only if it first goes one step forward and then from level 1 it reaches level $n$ before reaching level 0. But, because the state space is a cylinder, starting from 1 and reaching $n$ before 0 is equivalent to starting at 0 and reaching $n - 1$ before -1. Hence we can use the above expression with $a = -1$ and $b = n - 1$:
\[
P_1^a \{ T_n < \eta_0 \} = \frac{p}{d} P_1 ^{\{ z^* | = 1 \}} \{ T_n < T_0 \} = \frac{p}{d} \frac{\psi(0) - \psi(-1)}{\psi(n-1) - \psi(-1)}.
\]
Now plugging in $\psi$, and recalling that $p > 1/2$ and hence $p > 1 - p$, we obtain
\[
\frac{\psi(0) - \psi(-1)}{\psi(n-1) - \psi(-1)} = 2p - 1 \cdot \frac{1}{1 - [(1 - p)/p]^n} \geq 2p - 1 \frac{1}{p},
\]
and the lemma follows immediately. Similarly, $P \{ T_n^1 < \eta_1 \} \geq 2p - 1 \frac{1}{d}$. □

Lemma 2.9. For all $k$, all $\gamma_1 \in A_1$, and all $z$ on level zero of $G$,
\[
P_1^z \{ S[0, T_1] \cap \gamma_1 = \emptyset | S[0, T_1] \cap G_{-k} \neq \emptyset \} \leq \frac{d}{p} P_1^z \{ S[0, T_1] \cap \gamma_1 = \emptyset \}
\]
Proof: First, on the event $\{ z \notin \gamma_1 \}$, since we can avoid both $\gamma_0$ and $\gamma_1$ by simply taking one step in the $Z$ direction of $G$, we have $P_1^z \{ S[0, T_1] \cap \gamma_1 = \emptyset \} \geq p/d$ and so,
\[
P_1^z \{ S[0, T_1] \cap \gamma_1 = \emptyset | S[0, T_1] \cap G_{-k} \neq \emptyset \} \leq \frac{d}{p} \cdot \frac{p}{d} \leq \frac{d}{p} \cdot \frac{p}{d} P_1^z \{ S[0, T_1] \cap \gamma_1 = \emptyset \}
\]
Secondly, on the event $\{ z \in \gamma_1 \}$, we also get
\[
P_1^z \{ S[0, T_1] \cap \gamma_1 = \emptyset | S[0, T_1] \cap G_{-k} \neq \emptyset \} = 0 = \frac{d}{p} P_1^z \{ S[0, T_1] \cap \gamma_1 = \emptyset \}
\]
Furthermore, if $A$ is any set in the $\sigma$-algebra generated by $S$ up to time $T_1$, conditioning on this event $A$ instead of $\{ S[0, T_1] \cap G_{-k} \neq \emptyset \}$, yields the same inequality as [9]. □

From the proof of Lemma 2.9, also note that if $z$ and $z'$ are on level zero of $G$ and $z, z' \notin \gamma_1$,  
\[
P_1^z \{ S[0, T_1] \cap \gamma_1 = \emptyset \} \leq \frac{d}{p} P_1^{z'} \{ S[0, T_1] \cap \gamma_1 = \emptyset \}
\]
In order to show $q_n \propto e^{-\xi(\lambda)n}$, we need some preliminary results which we summarize in the two lemmas below. First we consider the random walk started at $z$, $|z| = 0$, that reaches level $n$ without hitting $\gamma_n$, while staying above level $-1$. We define random variables
\[
\hat{Z}_{n,z} = P_1^z \{ S[0, T_n] \cap \gamma_n = \emptyset; T_n < \eta_{-1} \},
\]
\[
\hat{Z}_n = \sup_{|z|=0} \hat{Z}_{n,z}.
\]
Lemma 2.10. There exists a constant $c$ such that for all $n$, $\hat{Z}_n \geq c\overline{Z}_n$. In particular,

$$\sup_{\overline{Z}_n} \mathbb{E}^{\gamma_0}[\hat{Z}^\lambda_n] \geq c\overline{Z}_n.$$  

Proof: Suppose we start $S$ at $z$, with $|z| = 0$. We prove the lemma by first looking at the random walk on the event it falls below level zero before time $T_n$. Let

$$Y_{n,z} = \mathcal{P}_1^s\{S[0,T_n] \cap \gamma_n = \emptyset; T_n > \eta_1\}.$$  

Let $A_n$ be the event that the random walk $S$, up to time $T_n$, hits every site on level zero. Since the cross-section of the cylinder has a finite state space, for all $n$, $\mathcal{P}_1(A_n) \geq a$, for some same constant $a > 0$ that can be easily computed. Similarly, given $T_n > \eta_1$, let $A_{n-1}$ be the event that the random walk $S$, up to time $\eta_1$, hits every site on level zero, with $\mathcal{P}_1(A_{\eta_1} \mid T_n > \eta_1) \geq a$. Here $A_{n-1}$ is a subset of $\{S[0,\eta_1] \cap \gamma_n = \emptyset\}$ which implies

$$Y_{n,z} \leq \mathcal{P}_1\{S[0,\eta_1] \cap \gamma_n = \emptyset; T_n > \eta_1\} \mathcal{P}_1\{T_n > \eta_1\} \times \mathcal{P}_1\{S[0,T_n] \cap \gamma_n = \emptyset; S[0,\eta_1] \cap \gamma_n = \emptyset; T_n > \eta_1\} \leq (1 - a) \sup_{|z'| = -1} \mathcal{P}_1\{S[0,T_n] \cap \gamma_n = \emptyset\} \leq (1 - a)\overline{Z}_n.$$  

Taking the supremum over all $z$ yields $Y_n = \sup_{|z| = 0} Y_{n,z} \leq (1 - a)\overline{Z}_n$. Since $\overline{Z}_n \leq \hat{Z}_n + Y_n$, this implies $\hat{Z}_n \geq a\overline{Z}_n$, which proves the lemma, with a constant $c = a^\lambda$, uniform in $\lambda$.  

Lemma 2.11. There exists a constant $c$ such that for all $n$, all histories $\gamma_m$, and all $z \notin \gamma_m$ with $|z| = m$,

$$\mathbb{E}^{\gamma_m}[\hat{Z}^\lambda_n 1\{T^*_n < \eta_0\} \geq c\overline{Z}_n.$$  

Proof: Without loss of generality, assume $m = 0$. Recall that $S$ is started on level zero of $G$. Let $x_0$ be the endpoint of $\gamma_0$, where we attach $\hat{\gamma}_n$. Consider another starting configuration $\gamma'_0$ with endpoint at $x'_0$ to which we attach $\hat{\gamma}_n$ (translated accordingly in the $t_{L}^{d-1}$ direction of $G$). We let $\gamma_n = \gamma_0 \hat{\gamma}_n$ and $\gamma_n = \gamma_0 \hat{\gamma}_n$. First note that if $x_0 = x'_0$, and if the random walk $S$ does not fall below level zero, $S$ can only hit $\hat{\gamma}_n$, and so $\hat{Z}_{n,z}(\gamma_n) = \hat{Z}_{n,z}(\gamma'_n)$. For the case when $\gamma_0$ and $\gamma'_0$ do not have the same endpoint, we start one random walk at $z$ and consider $\hat{Z}_{n,z}(\gamma_n)$. We can find a point $z'$ on level zero of $G$ such that the “relative position” of $z'$ to $x'_0$ is the same as the “relative position” of $z$ to $x_0$. And again, if $S$ does not fall below level zero, we get

$$\hat{Z}_{n,z}(\gamma_n) = \hat{Z}_{n,z}(\gamma'_n) \quad (11)$$

Now, starting a random walk anywhere on level zero, away from $\gamma_0$, say at $z$, we claim that for all $z'$ on level zero, $z' \notin \gamma_0$,

$$\overline{Z}_{n,z}1\{T^*_n < \eta_0\} \geq a\overline{Z}_{n,z'}1\{T^*_n < \eta_0\} \quad (12)$$

This is easy to see: starting the random walk at $z$, with probability greater than $a$ it will reach $z'$ before hitting level 1, or returning to level 0, without hitting the starting point of $\hat{\gamma}_n$. Note that in this case, the random walk from $z$ to $z'$ will not hit $\hat{\gamma}_n$ and the claim follows. Furthermore, by the same argument,

$$\hat{Z}_{n,z}1\{T^*_n < \eta_0\} \geq a\hat{Z}_{n,z'}1\{T^*_n < \eta_0\} \quad (13)$$
Since (13) holds for all \( z' \) not on \( \gamma_0 \), taking the supremum over all \( z' \), along with equation (11),
\[
\hat{Z}_{n,z}(\gamma_0)1_{\{T_n^1<\eta_0^1\}} \geq a\hat{Z}_n(\gamma_0)1_{\{T_n^1<\eta_0^1\}},
\]
so averaging over all \( \gamma_n \), we get
\[
E^{\gamma_0}[\hat{Z}_{n,z}^11_{\{T_n^1<\eta_0^1\}}] \geq a^{\lambda_2}E^{\gamma_0}[\hat{Z}_n^11_{\{T_n^1<\eta_0^1\}}],
\]
Observe that this inequality holds for any pair \( \gamma_0, \gamma_0 \), and thus,
\[
E^{\gamma_0}[\hat{Z}_{n,z}^11_{\{T_n^1<\eta_0^1\}}] \geq a^{\lambda_2}\sup_{\gamma_0}E^{\gamma_0}[\hat{Z}_n^11_{\{T_n^1<\eta_0^1\}}].
\]
Next we want to show there exists a constant \( c' < 1 \) such that
\[
\sup_{\gamma_0}E^{\gamma_0}[\hat{Z}_n^11_{\{T_n^1<\eta_0^1\}}] \leq c' \sup_{\gamma_0}E^{\gamma_0}[\hat{Z}_n^1].
\]
This, along with Lemma 2.10 will finish the proof with a constant equal to \((1-c')a^{2\lambda_2}\). In order to prove (14), we condition on the event that \( \hat{\gamma}_n \) returns to level zero before reaching level \( n \). Then a random walk started on level zero will avoid \( \gamma_n \) only if it avoids the part of \( \hat{\gamma}_n \) from the first return to level zero up to the hitting time of level \( n \). If we denote this part of \( \hat{\gamma}_n \) by \( S^1[\eta_0^1, T_n^1] \),
\[
E^{\gamma_0}[\hat{Z}_n^1|T_n^1>\eta_0^1] \leq E^{\gamma_0}\sup_xP_1^x\{S[0, T_n^1] \cap S^1[\eta_0^1, T_n^1] = \emptyset; T_n < \eta_{n-1}\}^\lambda \leq \sup_{\gamma_0}E^{\gamma_0}[\hat{Z}_n^1]
\]
where the second inequality follows from observing that \( \hat{Z}_{n,z} \) depends on \( \gamma_0 \) only in terms of the endpoint of \( \gamma_0 \). Recall that \( P\{T_n^1 > \eta_0^1\} \leq 1 - (2p-1)/d \), hence taking \( c' = 1 - (2p-1)/d > 0 \) completes our proof.

Furthermore, for all \( n \) and for all \( \gamma_0 \), we have \( E^{\gamma_0}[\hat{Z}_n^11_{\{T_n^1<\eta_0^1\}}] \geq c\bar{q}_n \).

2.3.2 Proof of Proposition 2.7
We prove the proposition by showing that for all \( n \geq 1 \), both \( q_n \sim e^{-\xi(\lambda)n} \) and \( \bar{q}_n \sim e^{-\xi(\lambda)n} \).

By subadditivity of \( \log(q_n) \), we have
\[
\lim_{n \to \infty} \frac{\log(q_n)}{n} = \inf_n \frac{\log(q_n)}{n} = -\xi(\lambda)
\]
and thus, for all \( n \),
\[
q_n \geq e^{-\xi(\lambda)n},
\]
Note that this means \( c_1 = 1 \).

We claim there exists a constant \( \hat{c} \) such that for all \( n \), \( q_n \geq \hat{c}\bar{q}_n \). Then, along with the trivial inequality \( q_n \leq \bar{q}_n \), this implies
\[
q_n \sim \bar{q}_n.
\]
On the event where \( \bar{q}_n \) does not reach level zero after time 0, the random walk \( S \), coming from \(-\infty\) and conditioned to avoid \( \gamma_0 \), will avoid \( \gamma_n \) if and only if \( S[T_0, T_n] \) avoids \( \gamma_n \). This
is the same as starting a random walk at \( S(T_0) \) and avoiding \( \gamma_n \). But, from (12) we know this is bounded below by \( a Z_{n,z} \), for all \( z \notin \gamma_0 \). Taking the supremum over all \( z \), we have \( Z_n \{ T_n^1 < \eta_0 \} \geq a Z_n \{ T_n^1 < \eta_0 \} \). From Lemma 2.11

\[
E_{\gamma_0}^\gamma[Z_n \{ T_n^1 < \eta_0 \} \geq a^\gamma E_{\gamma_0}^\gamma[Z_n \{ T_n^1 < \eta_0 \}] \geq a^\gamma c_\gamma n.
\]

We let \( c = a^\lambda \), \( c = a^{\lambda z} (2p - 1) / d \) and then \( q_n \geq c \gamma_n \), with a constant uniform in \( \lambda \). Furthermore,

\[
E_{\gamma_0}^\gamma[Z_n \{ T_n^1 < \eta_0 \}] \geq c E_{\gamma_0}^\gamma[Z_n^\lambda].
\] (17)

In the last part of the proof we will show there exists a constant \( c_2 \), uniform in \( \lambda \) such that

\[
\gamma_n \leq c_2 e^{-\xi(\lambda)n}
\] (18)

Then the proposition follows immediately from inequalities (16), (15) and (18). To prove inequality (18), we bound \( Z_{n+m,z} \) by the probability \( S \) avoids \( \gamma_{n+m} \) while not going below level \( n \) between times \( T_n \) and \( T_{n+m} \). We are considering this probability only on the event where \( S^1 \) reaches level \( n + m \) before returning to level \( n \). Intuitively, we want \( S \) and \( S^1 \) to have a nice behavior from level \( n \) onward, so we can “separate” what happens up to level \( n \) from what happens from level \( n \) to \( m + n \). Then for every pair \( n, m \) we have the following relation between \( \gamma_{n+m}, \gamma_n \) and \( \gamma_m \):

\[
\gamma_{n+m} = \sup_{\gamma_0} E_{\gamma_0}^\gamma[\sup_{|z|=0} P_1^z \{ S[0, T_{n+m}] \cap \gamma_{m+n} = \emptyset \}^\lambda]
\geq \sup_{\gamma_0} E_{\gamma_0}^\gamma[\sup_{|z|=0} P_1^z \{ S[0, T_{n+m}] \cap \gamma_{m+n} = \emptyset; S(T_n, T_{n+m}] \cap G_{n-1} = \emptyset \}^\lambda 1_{\{T_{n+m} < \eta_0 \}}]
\geq \sup_{\gamma_0} E_{\gamma_0}^\gamma[\sup_{|z|=0} Z_{n,z}^\lambda E_{\gamma_0}^\gamma[\gamma_{m+n} = \emptyset; S(T_n, T_{n+m}] \cap G_{n-1} = \emptyset; S[0, T_n] \cap \gamma_n = \emptyset \}^\lambda 1_{\{T_{n+m} < \eta_0 \}|F_n}]
\geq \sup_{\gamma_0} E_{\gamma_0}^\gamma[\sup_{|z|=0} Z_{n,z}^\lambda c \gamma_m]
\geq c \gamma_m \gamma_n
\]

with the second to last step following from Lemma 2.11. Note that \( \log(c \gamma_n) \) is a super-additive function, and then using (16)

\[
\sup_n \frac{\log(c \gamma_n)}{n} = \lim_{n \to \infty} \frac{\log(c \gamma_n)}{n} = \lim_{n \to \infty} \frac{\log(q_n)}{n} = -\xi(\lambda)
\]

and so, for all \( n, \gamma_n \leq c_2 e^{-\xi(\lambda)n} \), where \( c_2 = [a^{2\lambda \gamma} (2p - 1) / d]^{-1} \) and the proof is complete.

From the proof to Proposition 2.7 we also get the following result:

**Corollary 2.12.** For all \( n \) and all \( \gamma_0 \in \mathcal{A} \), \( E_{\gamma_0}^\gamma[Z_n^\lambda] \sim e^{-\xi(\lambda)n} \).
3 Exponential convergence of Markov chains

In this section we will first define a Markov process on pairs of non-disconnecting random walk paths on $G$. Each $\xi(\lambda) := \xi(1, \lambda)$ will then be associated to such a Markov process, along with a weighting that corresponds to the value of $\lambda$. We will fix $\lambda$ and start two Markov chains $X$ and $X'$, with different initial configurations $\gamma_0$ and $\gamma_0$ respectively. The goal is to show that the two chains can be coupled at exponential rate, and as a result we will be able to describe an invariant limiting measure on half-infinite paths.

3.1 Markov process on random walk paths

Fix $N$ large and start with an initial configuration $\gamma_0$ from $A$. Attaching a random walk started at the endpoint of $\gamma_0$ and run until it hits level $N$ gives a path $\gamma_N$, which, translated accordingly, produces $\gamma_N$. If $\gamma_N$ does not disconnect $-\infty$ from level zero, then it is a nice path. (This follows from the fact that $\gamma_N$ is finite almost surely and hence one can find a level $m$ such that $\gamma_N$ does not fall below level $m$; but $G_{m-n} \cap D(\gamma_n) = G_m \cap D(\gamma_0)$ must have infinitely many connected levels, hence $\gamma_N$ is nice.) Each $\gamma_N$ is weighted by $e^{-\lambda \Phi_N}$ and then normalized by $E^{\gamma_0}[e^{-\lambda \Phi_N}]$ to get a probability measure $Q_N = Q_{\gamma_0}^N$. Clearly, paths $\gamma_N$ that cannot be avoided by an $h$-process from $-\infty$ to level $N$ will be assigned measure zero and we will say that these paths ”do not survive” up to level $N$. For a given $\gamma_0$, we denote the expectation with respect to the measure $Q_0$ by $E^{\gamma_0}_{Q_0}$. Let

$$K_N(\gamma_0) = e^{\xi(\lambda) N}E^{\gamma_0}[e^{-\lambda \Phi_N}].$$

From Corollary 2, we get $K_N(\gamma_0) \approx 1$, i.e. $K_N(\gamma_0)$ are bounded above and below by positive constants, uniform in $\gamma_0$ and $N$. Furthermore, we can show that $\lim_{N \to \infty} K_N(\gamma_0)$ exists. A proof of this result can be found in Section 2.3, as a part of the proof of analyticity of $\xi(\lambda)$.

If $m < N$, we have

$$E^{\gamma_0}[e^{-\lambda \Phi_N} | F_m] = e^{-\lambda \Phi_m}E^{\gamma_m}[e^{-\lambda \Phi_N-m}].$$

Let $\mathcal{M}$ be the unconditioned random walk measure on paths. We let the path $\gamma_n$ evolve, starting with $\gamma_0$ and conditioned to survive up to level $N$. Then, for this conditioned random walk we write transition probabilities:

$$Q_N(\gamma_1 | \gamma_0) = e^{-\lambda \Phi(\gamma_1)}e^{\xi(\lambda) \frac{K_{N-1}(\gamma_1)}{K_N(\gamma_0)}}\mathcal{M}(\gamma_1), \quad (19)$$

The $m$-step transition probabilities, for $m < N$ are given by

$$Q_N(\gamma_m | \gamma_0) = e^{-\lambda \Phi_m}e^{\xi(\lambda) m} \frac{K_{N-m}(\gamma_m)}{K_N(\gamma_0)}\mathcal{M}(\gamma_m).$$

Here we have used the fact that $\gamma_0 = \gamma_0$ and we will use the two notations interchangeably. It is trivial to check that if $n+m \leq N$, the distribution of $\gamma_{m+n}$ under $Q_N^{\gamma_0}$ is the same as the distribution of $\gamma_{m+n}$ under $Q_N^{\gamma_0}$. Intuitively, this should be the case since we condition on paths surviving up to level $N$ and once they have reached level $n$ they only have to survive another $N-n$ steps. Then transition probabilities from (19) describe a time-inhomogeneous
Markov chain \( X_n \) taking values in \( \overline{A} \) and conditioned to survive for a total of \( N \) levels, with \( X_0 = \overline{\gamma}_0 \) and \( X_n := \overline{\gamma}_n \) for \( n \geq 1 \).

Starting with a different nice initial configuration \( \overline{\gamma}_0 \), we construct a Markov chain \( X'_n \) with transition probabilities given by a formula similar to (19) and \( X'_0 = \overline{\gamma}'_0 \). Then we will show that we can couple \( X_n \) and \( X'_n \) as in Theorem 1.1. Let \( \eta|_k \) denote the restriction to the last \( k \) levels of a half-infinite path \( \eta \) from \( \overline{A} \). Then Theorem 1.1 implies the following corollary.

**Corollary 3.1.** For all \( \overline{\gamma}_0, \overline{\gamma}'_0, \eta \) in \( \overline{A} \) and all \( 0 \leq k \leq N/2 \),

\[
\sum_{\eta|_k} \left| \frac{E_{\overline{\gamma}_0}[e^{-\lambda \Phi_N \gamma_N}; \overline{\gamma}_N = k \eta]}{E_{\overline{\gamma}_0}[e^{-\lambda \Phi_N \gamma_N}]} - \frac{E_{\overline{\gamma}'_0}[e^{-\lambda \Phi_N \gamma'_N}; \overline{\gamma}'_N = k \eta]}{E_{\overline{\gamma}'_0}[e^{-\lambda \Phi_N \gamma'_N}]} \right| = O(e^{-\beta_1 N}).
\]

**Proof:** Let \( X_n \) and \( X'_n \) be Markov chains given by initial configuration \( \overline{\gamma}_0 \) and \( \overline{\gamma}'_0 \), respectively, and evolving according to transition probabilities given in (19). Observe that

\[
\frac{E_{\overline{\gamma}_0}[e^{-\lambda \Phi_N \gamma_N}; \overline{\gamma}_N = k \eta]}{E_{\overline{\gamma}_0}[e^{-\lambda \Phi_N \gamma_N}]} = Q_{\overline{\gamma}_0}^0 \{ X_N = k \eta \}.
\]

Then for all \( k \leq N/2 \), we have

\[
\sum_{\eta|_k} \left| Q_{\overline{\gamma}_0}^0 \{ X_N = k \eta \} - Q_{\overline{\gamma}_0}^0 \{ X'_N = k \eta \} \right| \leq 2 \tilde{P} \{ X_N \not\equiv k X'_N \},
\]

and the corollary follows from noting that for all \( k \leq N/2 \),

\[
\tilde{P} \{ X_N \not\equiv k X'_N \} \leq \tilde{P} \{ X_N \not\equiv N/2 X'_N \} \leq C e^{-\beta_1 N}.
\]

This result implies existence of an invariant measure on paths \( \overline{\gamma}_0 \in A \) which we will prove in Section 3.5.

\[ \square \]

The rest of Section 3 is devoted to proving Theorem 1.1. First we present some technical lemmas, followed by the description of the coupling in Section 3.3 and the proof of the theorem.

### 3.2 Preliminary estimates

For all \( i \leq 0 \), and \( \overline{\gamma} \in \overline{A} \), let \( J_i \) be the random variable that takes the value 1 if the \( i \)-th cross-section of \( G \setminus \overline{\gamma} \) is connected, and is zero otherwise. Then for \( k \geq j \geq 0 \), the sum \( \sum_{i=-k}^{-j} J_i \) represents the number of connected cross-sections between levels \( -k \) and \( -j \) in \( G \setminus \overline{\gamma} \). For all \( k \geq 0 \) and \( j > 0 \), we define the sets

\[
V_{k,j}(\delta) = \left\{ \overline{\gamma} \in \overline{A} : \sum_{i=-k}^{-k+j-1} J_i > \delta j \right\}
\]

\[
W_{k,j} = \{ \gamma \in X_k : \gamma \cap G_j = \emptyset \}.
\]

Then \( V_{k,j}(\delta) \) is the set of all \( \overline{\gamma} \in \overline{A} \) with the property that it has at least \( \delta j \) connected cross-sections between levels \( -k \) and \( -k + j - 1 \), and \( W_{k,j} \) is the set of all \( \gamma \) started on level \( k - 1 \) and reaching level \( k \) before reaching level \( j \).
The set $V_{k,j}$ is large for $j$ large enough, that is, for an appropriate $\delta$, the $Q_N$-probability that $\tilde{\gamma}_k$ is not in $V_{k,j}(\delta)$ decays exponentially in $j$, and this is independent of $\tilde{\gamma}_0$. Therefore, the weighted measure put on paths that are not in $V_{k,k/2}$ is exponentially small. Moreover, we will show that the set $V_k$ of all paths that have enough connected cross-sections between levels $-k$ and $-k + j - 1$ for all integers $j$ in $[k/2,k]$, is also large. Note that

$$V_k = \{ \tilde{\gamma} \in \mathcal{A} : \tilde{\gamma} \in \bigcap_{j=k/2}^k V_{k,j} \},$$

and the set $V_k$ is a subset of $V_{k,k/2}$.

**Lemma 3.2.** There exist constants $\alpha' > 0$ and $\delta > 0$ such that for all $\tilde{\gamma}_0 \in \mathcal{A}$, all $n \leq N$ and $k \leq n$,

$$Q_N^0 \{ \tilde{\gamma}_k \notin V_k(\delta) \} \leq c e^{-\alpha' k/2}.$$

**Proof:** Using Chebyshev’s inequality, for all integers $j \in [k/2,k]$

$$Q_N^0 \{ \tilde{\gamma}_k \notin V_{k,j}(\delta) \} = Q_N^0 \left\{ \exp\{-t \sum_{i=-k}^{-k+j+1} J_i \} \geq e^{-t\delta j} \right\} \leq e^{\delta j} E_Q^{0} \left[ \exp\{-t \sum_{i=-k}^{-k+j+1} J_i \} \right].$$

Suppose we know the path $\tilde{\gamma}_k$ up to level $i-1$ (note that $i < 0$ and we have information from $\mathcal{F}_{k+i-1}$). What is the $Q_N^0$-probability that the chain evolved so that level $i-1$ is connected? If, starting from level $i-1$, the path moved forward in the Z direction of $G$ and it did not return to this level, then level $i-1$ remained connected. It follows from (17) that $Q_N^0 \{ J_i = 1 | \mathcal{F}_{k+i-1} \} \geq \hat{c}$, and so

$$E_Q^{0} \left[ \exp\{-t \cdot J_i \} | \mathcal{F}_{k+i-1} \right] = e^{-t} Q_N^0 \{ J_i = 1 | \mathcal{F}_{k+i-1} \} + Q_N^0 \{ J_i = 0 | \mathcal{F}_{k+i-1} \} \leq e^{-t} + 1 - \hat{c}.$$}

Using this, one can check that

$$E_Q^{0} \left[ \exp\{-t \sum_{i=-k}^{-k+j+1} J_i \} \right] \leq (e^{-t} + 1 - \hat{c})^j.$$

Therefore, $Q_N^0 \{ \tilde{\gamma}_k \notin V_{k,j}(\delta) \} \leq e^{\delta j} (e^{-t} + 1 - \hat{c})^j$. Fix $t$ large enough such that $e^{-t} + 1 - \hat{c} < 1$, then let $\alpha' = -\delta t - \log(e^{-t} + 1 - \hat{c})$, and choose $\delta \leq 1/4$, so that $\alpha' > 0$. We now let $V_{k,j} = V_{k,j}(\delta)$ for this particular choice of $\delta$.

To get an estimate on the size of $V_k$, we need only consider $j \in [k/2,k]$. A path $\tilde{\gamma}_k$ is not in $V_k$ if it is not in at least one of the sets $V_{k,j}$ for $j$ between $k/2$ and $k$:

$$Q_N^0 \{ \tilde{\gamma}_k \notin V_k \} \leq \sum_{j=k/2}^k Q_N^0 \{ \tilde{\gamma}_k \notin V_{k,j} \} \leq \frac{e^{-\alpha' k/2}}{1 - e^{-\alpha'}}.$$
and the lemma follows. □

We start with $\tilde{\gamma}_0 = k \tilde{\gamma}'_0$ satisfying $\tilde{\gamma}_0 \in V_k$ (this implies $\tilde{\gamma}'_0$ also in $V_k$). To $\tilde{\gamma}_0$ and $\tilde{\gamma}'_0$ we attach the same $\tilde{\gamma}_1 \in W_{1-k/2}$. We want to show $e^{-\lambda \Phi(\tilde{\gamma}_1)}$ is close to $e^{-\lambda \Phi(\tilde{\gamma}'_1)}$, and $K_N(\tilde{\gamma}_0)$ is close to $K_N(\tilde{\gamma}'_0)$. The estimate on how close these quantities are relies on a coupling of $h$-processes. Let $S$ and $S'$ be $h$-processes given by random walks conditioned to avoid $\overline{\gamma}_0$, and $\overline{\gamma}'_0$ respectively. If $\overline{\gamma}_0 = k \overline{\gamma}'_0 \in V_k$, then one can show the $h$-processes $S$ and $S'$ can be coupled by the time they first hit level $-k/2$ with high probability.

Lemma 3.3. There exist constants $c, \alpha'' > 0$ such that for all $k \geq 0$, for all $\overline{\gamma}_0, \overline{\gamma}'_0 \in V_k$, with $\overline{\gamma}_0 = k \overline{\gamma}'_0$, if $S$ and $S'$ are $h$-processes described as above, then $S$ and $S'$ can be defined on the same probability space $(\Omega_1, \mathcal{F}_1, \overline{\mu})$ such that

$$\overline{\mu}(S(T_{-k/2}) \neq S'(T'_{-k/2})) \leq c e^{-\alpha'' k}.$$ 

Proof: For $-k < j < -k/2$, let $\mu_j(x)$ be the hitting measure on level $j$ of the $h$-process $S$ conditioned to avoid the path $\overline{\gamma}_0$, and $\mu'_j(x)$ be the hitting measure on level $j$ of the $h$-process $S'$ conditioned to avoid the path $\overline{\gamma}'_0$. Then, if level $j$ of $\overline{\gamma}_0$ is connected, we claim there exists a constant $c > 0$ such that

$$\frac{\mu_{j+1}(w)}{\mu'_{j+1}(w)} \geq c.$$  

(20)

Let $h(x) = P^+_1 \{S[0, T_0] \cap \overline{\gamma}_0 = \emptyset \}$ and $h'(x) = P^+_1 \{S'[0, T'_0] \cap \overline{\gamma}'_0 = \emptyset \}$. Using (5), for all $x, x' \notin \overline{\gamma}_0$, with $|x| = |x'| = j$, we have

$$\frac{\mu_{j+1, x}(w)}{\mu'_{j+1, x'}(w)} \geq \left(\frac{p_d}{d} \right) \frac{h(w)}{h(x)} \frac{h'(x')}{h'(w)} \geq \left(\frac{p_d}{d} \right) \frac{p^3 h(w)}{h(z)} \frac{h'(z)}{h'(w)},$$

where $z = w - (1, 0)$ is the last point on level $j$ touched by $S$ and $S'$ before reaching level $j + 1$ at $w$. Furthermore, $\frac{1-p}{d} h(z) \leq h(w) \leq \frac{d}{p} h(z)$ and a similar inequality holds for $h'(w)$. Thus, for all $x, x' \notin \overline{\gamma}_0$, with $|x| = |x'| = j$,

$$\frac{\mu_{j+1, x}(w)}{\mu'_{j+1, x'}(w)} \geq p^2 (1 - p) d^3 > 0.$$ 

Then inequality (20) follows from noting that $\mu_{j+1}(w) = \sum_{|x|=j} \mu_j(x) \mu_{j+1, x}(w)$.

Recall that on $V_k$, both $D(\overline{\gamma}_0)$ and $D(\overline{\gamma}'_0)$ have $k\delta$ connected cross-sections between levels $-k$ and $-k/2$. Using the same coupling as in Lemma 2.5, we can define $S$ and $S'$ on the same probability space $(\Omega_1, \mathcal{F}_1, \overline{\mu})$, with

$$\overline{\mu}(S(T_{-k/2}) \neq S'(T'_{-k/2})) \leq 2 \left(\frac{1-c}{2}\right)^{4k},$$

and then let $\alpha'' = -\delta \log \left(\frac{1-c}{2}\right)$ to complete the proof. □

We are now ready to estimate $K_N(\tilde{\gamma}_0)/K_N(\tilde{\gamma}'_0)$.
Proposition 3.4. There exists $\beta > 0$ such that for all $n \leq N$, all $k \geq 0$, and all histories $\gamma_0, \gamma_0$ in $V_k$ with $\gamma_0 = k \gamma_0$, $K_n(\gamma_0) = K_n(\gamma_0)[1 + O(e^{-\beta k})]$.

Proof: To simplify notation, let

$$Z_n(\gamma_n) = P_1\{S(-\infty, T_n) \cap \gamma_n = \emptyset|S(-\infty, T_0) \cap \gamma_0 = \emptyset; S[T_0, T_n] \cap G_{-k} \neq \emptyset\}.$$ 

Let $\mathcal{U}$ be the event $\{\tilde{\gamma}_n \cap G_{-k} = \emptyset\}$. Then on $\mathcal{U}$, using the coupling result from Lemma 3.3 and an estimate similar to (8),

$$Z_n(\tilde{\gamma}_n) \leq Z_n(\tilde{\gamma}_n') + \left(\frac{1-p}{p}\right)^k Z_n(\gamma_n) + (c e^{-\alpha'k}) Z_n(\gamma_n) \quad (21)$$

Using Lemma 2.11 it is easy to show that

$$E[\lambda^{\lambda} 1_\mathcal{U}] \leq c E[\lambda^{\lambda} 1_\mathcal{U}],$$

for some constant $c$ uniform over all $\lambda$, and not depending on $k$. We will show

$$E[\lambda^{\lambda} 1_\mathcal{U}] \geq E[\lambda^{\lambda} 1_\mathcal{U}] [1 - O(e^{-\beta k})],$$

where $\beta = \lambda_1 \min\{\alpha', \alpha''\} \frac{1}{2} \log \frac{p}{1-p}$. There are two cases to consider. If $\lambda \leq 1$, raising (21) to $\lambda$ and taking expectations,

$$E[\lambda^{\lambda} 1_\mathcal{U}] \leq E[\lambda^{\lambda} 1_\mathcal{U}] + c \left(\frac{1-p}{p}\right)^{\lambda k} + e^{-\alpha''k} E[\lambda^{\lambda} 1_\mathcal{U}]$$

When $\lambda > 1$, using the Minkowski inequality,

$$E[\lambda^{\lambda} 1_\mathcal{U}]^{1/\lambda} \leq E[\lambda^{\lambda} 1_\mathcal{U}]^{1/\lambda} + c^{1/\lambda} \left(\frac{1-p}{p}\right)^{k} + e^{-\alpha''k} E[\lambda^{\lambda} 1_\mathcal{U}]^{1/\lambda}$$

Collecting terms and raising to $\lambda$,

$$E[\lambda^{\lambda} 1_\mathcal{U}] \geq E[\lambda^{\lambda} 1_\mathcal{U}] \left[1 - c^{1/\lambda} \left(\frac{1-p}{p}\right)^{k} - e^{-\alpha''k}\right] \geq E[\lambda^{\lambda} 1_\mathcal{U}] [1 - O(e^{-\beta k})].$$

We conclude that

$$E[\lambda^{\lambda} 1_\mathcal{U}] \leq E[\lambda^{\lambda}] [1 + O(e^{-\beta k})] \quad (22)$$

Now, conditioning on paths satisfying $\tilde{\gamma}_n \cap G_{-k} \neq \emptyset$, and using Lemma 2.11 we can find a constant $c$, uniform over $\lambda$ and independent of $k$, such that

$$E[\lambda^{\lambda} | \tilde{\gamma}_n \cap G_{-k} \neq \emptyset] \leq c E[\lambda^{\lambda}] \quad (23)$$
Using equations 22 and 23, we get
\[
E^\tau_0[Z_n^\lambda] = E^\tau_0[Z_n^\lambda 1_{\mathcal{U}}] + E^\tau_0[Z_n^\lambda; \hat{\gamma}_n \cap G_{-k} \neq \emptyset]
\]
\[
\leq [1 + O(e^{-\beta k})]E^\tau_0[Z_n^\lambda] + c \left( \frac{1 - p}{p} \right)^k E^\tau_0[Z_n^\lambda]
\]
\[
= [1 + O(e^{-\beta k})]E^\tau_0[Z_n^\lambda],
\]
with the last step coming from \( \beta \leq \frac{1}{2} \log \left( \frac{p}{1-p} \right) \). The proposition follows immediately from the definition of \( K_n(\tau_0) \). \( \square \)

Corollary 3.5. For all \( k \) and all \( \tau, \tau_0 \in V_k \) with \( \tau = \tau_0 \), and \( \gamma_1 \in W_{1-k/2} \),
\[
e^{-\lambda \Phi(\gamma'_1)} \geq e^{-\lambda \Phi(\gamma_1)}[1 - O(e^{-\beta k})].
\]

Proof: Since \( \gamma_0 \) and \( \gamma'_0 \) have the same endpoint, we can attach to both of them the same element of \( \mathcal{X} \), namely \( \gamma_1 \). Let \( \hat{\gamma}_1 \) and \( \gamma'_1 \) be the resulting paths translated accordingly. From 21, using Lemma 2.9 in the case \( n = 1 \) we get:
\[
Z_1(\hat{\gamma}_1) \geq Z_1(\gamma_1) \left[ 1 - e^{-\alpha' k} - \frac{d}{p} \left( \frac{1 - p}{p} \right)^k \right]^\lambda
\]
Taking \( \beta \) as in the previous proposition, it is easy to see that for all \( \lambda \in [\lambda_1, \lambda_2] \),
\[
\left[ 1 - e^{-\alpha' k} - \frac{d}{p} \left( \frac{1 - p}{p} \right)^k \right]^\lambda \geq 1 - O(e^{-\beta k})
\]
and the corollary follows. \( \square \)

Then, using our results above, for all \( \gamma_1 \in W_{1-k/2} \), the chain has the following property: the infimum over all \( \tau = \tau_0 \) in \( V_k \),
\[
\inf Q_N(\hat{\gamma}_1|\gamma_0) = \inf \left[ \frac{e^{-\lambda \Phi(\gamma_1)}}{e^{-\lambda \Phi(\gamma_0)}} \cdot \frac{K_{N-1}(\gamma_1)}{K_{N-1}(\gamma_0)} \cdot \frac{K_N(\gamma'_0)}{K_N(\gamma_0)} \right] = 1 - O(e^{-\beta k}) \geq 1 - c e^{-\beta k},
\]
for some constant \( c \) uniform over \( \lambda \). This will be the main estimate used in the proof of Theorem LI.

3.3 Coupling of weighted paths

Given a double history \( (\gamma_0, \gamma'_0) \), we let \( X_0 = X'_0 \) and \( X'_n = X'_0 \) be the starting configurations of our time-inhomogeneous Markov chains. In this section, we define \( X_n \) and \( X'_n \) on the same probability space \( \hat{\Omega}, \hat{\mathcal{F}}, \hat{P} \), we will show there is a good chance the paths will couple in two steps, and once coupled, they will remain coupled for another step with positive probability. Furthermore, if the paths are coupled for \( k \) steps, we will show the paths will decouple at a rate that decays exponentially in \( k \). The key tool used in proving the exponential rate of decay is our main estimate from the previous section, equation (24).

On \( V_k \), we will use a maximal coupling for \( X_n \) and \( X'_n \). It is essentially the same coupling as the one described in [1]. For \( n \geq k + 2 \), if \( \gamma_n = k \gamma'_n \), we say the chains \( X \) and \( X' \) have been coupled for \( k + 1 \) steps by level \( n + 1 \), denoted by \( X_{n+1} \equiv_{k+1} X'_{n+1} \), if:
\begin{itemize}
  \item $X_n \equiv_k X'_n$, that is, $X$ and $X'$ have been coupled for $k$ steps by level $n$,
  \item $\gamma_{n+1} = \gamma'_{n+1}$,
  \item $\tilde{\gamma}_n, \tilde{\gamma}'_n \in V_k$
\end{itemize}

We think of this coupling in the following way: suppose $X_n$ and $X'_n$ have been coupled for $k$ steps; if $\tilde{\gamma}_n$ does not have enough connected cross-sections, we decouple, otherwise, the chains couple for an additional step if $\gamma_{n+1} = \gamma'_{n+1}$.

\textbf{Remark 3.6.} Note that when we say $X_j$ mean that $\gamma_j$, and thus, given two histories $\tilde{\gamma}_n$, $\tilde{\gamma}'_n$, and  $\tilde{\gamma}_n$ to be coupled for the last $k$ steps, we simply need $\gamma_j = \gamma'_j$ for $n - k + 1 \leq j \leq n$. An equivalent way to set up this problem is to construct a chain $\tilde{X}_n$ with history $\tilde{\gamma}_0$, on one-level paths from $X$: for $1 \leq n$, let $\tilde{X}_j = \gamma_j$, with transition probabilities as in $[19]$. These chains would be non-Markovian, time-inhomogeneous and dependent on the initial configurations, but they would couple in the classical sense, that is, we would say $\tilde{X}_n$ and $\tilde{X}'_n$ are coupled if $\tilde{X}_n = \tilde{X}'_n$. We prefer our setup for notation purposes only and the reader is welcome to think of the coupling in terms of $\tilde{X}_n$ if so wishes.

Let $\sigma_n = \sigma(\tilde{\gamma}_n, \tilde{\gamma}'_n)$, be the minimum number of steps backward that are needed to find a difference in coupling. On the event $\{\sigma_n = k\}$, either the paths decouple or they couple for an additional step, and thus $\sigma_{n+1} \in \{0, k + 1\}$. We define the following family of transition probabilities $q_n$ for the triple $(X_n, X'_n; \sigma_n)$: if $\gamma_{n+1} = \gamma'_{n+1},$

$$q_{n+1}(\tilde{\gamma}_{n+1}, \tilde{\gamma}'_{n+1}; k + 1|\tilde{\gamma}_n, \tilde{\gamma}'_n; k) = Q_N(\tilde{\gamma}_{n+1}|\tilde{\gamma}_n) \land Q_N(\tilde{\gamma}'_{n+1}|\tilde{\gamma}'_n)1_{\{\tilde{\gamma}_n, \tilde{\gamma}'_n \in V_k\}}$$

If $\gamma_{n+1} \neq \gamma'_{n+1}$, the transition probability $q_{n+1}(\tilde{\gamma}_{n+1}, \tilde{\gamma}'_{n+1}; 0|\tilde{\gamma}_n, \tilde{\gamma}'_n; k)$ could also be positive. One can give a formula for this case, but since we will not use it, we refer the reader to the maximal coupling presented in $[1]$ for details. Then these transition probabilities define a coupling of $X_n$ and $X'_n$. We use $\tilde{P}$ as shorthand for $\tilde{P}^{\tilde{\gamma}_0 \tilde{\gamma}'_0}$. The two chains decouple at the $(n + 1)$-th step if either we attach different elements of $X$ to $\tilde{\gamma}_n$, and $\tilde{\gamma}'_n$ respectively, or if $\tilde{\gamma}_n$ and implicitly $\tilde{\gamma}'_n$ do not have enough connected cross-sections.

We would like to estimate $\tilde{P}\{\sigma_{n+1} = k + 1|\sigma_n = k\}$. First observe that given $\tilde{\gamma}_n \in V_k$,

$$\sum_{\gamma_{n+1} \notin W_{n+1,n-k/2}} Q_N(\tilde{\gamma}_{n+1}|\tilde{\gamma}_n) \leq O\left(\left(\frac{1 - p}{p}\right)^{k/2}\right) = O(e^{-\beta k}), \quad (25)$$

and thus, given two histories $\tilde{\gamma}_n$ and $\tilde{\gamma}'_n$ in $V_k$, and using (24), and (25),

$$\tilde{P}\{\sigma_{n+1} = k + 1|\tilde{\gamma}_n, \tilde{\gamma}'_n; \sigma_n = k\} \geq \sum Q_N(\tilde{\gamma}_{n+1}|\tilde{\gamma}_n) \land Q_N(\tilde{\gamma}'_{n+1}|\tilde{\gamma}'_n)$$

$$\geq \sum Q_N(\tilde{\gamma}_{n+1}|\tilde{\gamma}_n) \left[1 - O(e^{-\beta k})\right]$$

$$\geq 1 - O(e^{-\beta k}),$$

where the summation above is taken over all $\gamma_{n+1} \in W_{n+1,n-k/2}$. Hence, on $V_k$, the chains will decouple with probability less than $O(e^{-\beta k})$. Taking expectations over $\tilde{\gamma}_n$, and recalling our earlier result from Lemma 3.2, we can find a constant $c$ such that

$$\tilde{P}\{\sigma_{n+1} = 0|\sigma_n = k\} \leq O(e^{-\beta k}) + O(e^{-\alpha' k}) \leq ce^{-\beta k}. \quad (26)$$

Suppose \( k \geq 1 \) and \( \tilde{\gamma}_n = k \tilde{\gamma} \), and the chains have been coupled for \( k \) steps. First assume \( \tilde{\gamma}_n \in V_k \) and let \( \gamma_{n+1} \) be a step forward in the \( Z \) direction of \( G \). Since \( \tau_n \) and \( \tau'_n \) have the same endpoint, we attach \( \gamma_{n+1} \) to both. Consider the \( h \)-process with the property that it reaches level \( n+1 \) in exactly one step from level \( n \). Then one can show \( Q_N(\tilde{\gamma}_{n+1}|\tilde{\gamma}_n) \geq c(p/d)^{1+\lambda} \) and thus there exists a constant \( c > 0 \) such that for all \( k \geq 1 \), for all \( \tilde{\gamma}_n = k \tilde{\gamma}_n \in V_k \),

\[
\hat{P}\{\sigma_{n+1} = k + 1|\tilde{\gamma}_n, \tilde{\gamma}'; \sigma_n = k\} \geq Q_N(\tilde{\gamma}_{n+1}|\tilde{\gamma}_n) \land Q_N(\tilde{\gamma}'_{n+1}|\tilde{\gamma}_n') \geq c \left( \frac{p}{d} \right)^{1+\lambda}
\]

Taking expectations over all pairs \( \tilde{\gamma}_n = k \tilde{\gamma}_n \), for all \( k \geq 1 \),

\[
\hat{P}\{\sigma_{n+1} = k + 1|\sigma_n = k\} \geq c \left( \frac{p}{d} \right)^{1+\lambda} Q_N^0(\tilde{\gamma}_n) \land V_k|\tilde{\gamma}_n - 1 \in V_{k-1}
\]

Reasoning as above, if we start with a half-infinite path with enough connected cross-sections, and we attach an additional step, with \( Q_N \)-probability greater than \( c(p/d)^{\lambda+1} \) we get a half-infinite path that has enough connected cross-sections. Therefore, for all \( k \geq 1 \)

\[
\hat{P}\{\sigma_{n+1} = k + 1|\sigma_n = k\} \geq c \left( \frac{p}{d} \right)^{1+\lambda} \hat{P}\{\tilde{\gamma}_n, \tilde{\gamma}'_n \in V_k|\tilde{\gamma}_n - 1 \in V_{k-1}\}
\]

Observe that even when \( k = 0 \), if \( \tau_n \) and \( \tau'_n \) have the same endpoint, the paths will get coupled on the next step with probability greater than \( c \left( \frac{d}{p} \right)^{1+\lambda} \). Suppose \( n \geq 2 \). Then

\[
\hat{P}\{\sigma_{n+1} = 1|\sigma_n = 0\} \geq c \left( \frac{p}{d} \right)^{1+\lambda} \hat{P}\{(n,0) \in \tau_n \cap \tau'_n\}
\]

We proceed to find a positive lower bound for \( \hat{P}\{(n,0) \in \tau_n \cap \tau'_n\} \). Suppose the chains have evolved up to level \( n-2 \). Let \( \gamma_{n-1} \) be given by taking one step forward in the \( Z \) direction of \( G \). Define \( \gamma'_{n-1} \) in the same way. Let \( \gamma_n \) be the following path: take the shortest path to \( (n-1,0) \) by moving only on level \( n-1 \) of the cylinder, then take a step forward in the \( Z \) direction of \( G \). Define \( \gamma'_n \) in the same way. Then the 2-step paths \( \gamma_{n-1} \gamma_n \) and \( \gamma'_{n-1} \gamma'_n \) each have a random walk measure bounded from below by \( \left( \frac{d}{p} \right)^2 \). Attach them to \( \tau_{n-2} \) and \( \tau'_{n-2} \) respectively. Now we have \( \tau_n \) and \( \tau'_n \) ending at the same point, namely \( (n,0) \).

Observe that for any choice of \( \gamma_{n-2} \), if we attach paths \( \gamma_{n-1} \gamma_n \) as described above, \( Q_N(\tilde{\gamma}_n|\tilde{\gamma}_n) \times e^{-\Phi_2(\tilde{\gamma}_2)} \). It is easy to see that this quantity is bounded from below by a positive constant: let \( x \) be the point on the torus with coordinates \( \lfloor L/2 \rfloor, 0, \ldots, 0 \) and \( x' \) be the point on \( T \) with coordinates \( \lfloor L/2 \rfloor + 1, 0, \ldots, 0 \). Consider the following path: if the endpoint of \( \tau_{n-2} \) is equal to \( (n-2, x) \), then \( S \) takes the shortest path to \( (n-2, x') \), conditioned to avoid \( (n-2, x) \), otherwise \( S \) takes the shortest path to \( (n-2, x) \), conditioned to avoid the the endpoint of \( \tau_{n-2} \); then \( S \) moves two steps forward in the radial direction. Observe that our choices for \( x \) and \( x' \) assure \( S[T_{n-2}, T_n] \) avoids \( \tau_n \). This occurs with probability greater than \( a \left( \frac{d}{p} \right)^2 \), so we can bound \( Q_N(\tilde{\gamma}_n|\tilde{\gamma}_{n-2}) \) from below by a constant which we can choose uniformly over \( \lambda \). Call this constant \( c' \). Then

\[
\hat{P}\{\sigma_{n+1} = 1|\sigma_n = 0\} \geq c \left( \frac{d}{p} \right)^{1+\lambda} \left( c' \right)^2.
\]

Let \( b = \min \left\{ c^2 \left( \frac{d}{p} \right)^{2(1+\lambda)}, c \left( \frac{d}{p} \right)^{1+\lambda} \left( c' \right)^2 \right\} \). From (27) and (28), for all \( n \geq 2 \), for all \( k \geq 0 \)

\[
\hat{P}\{\sigma_{n+1} = k + 1|\sigma_n = k\} \geq b.
\]
3.4 Proof of Theorem 1.1

Using the setup from the coupling above, Theorem 1.1 can be restated in the following way:

**Theorem 3.1.** Suppose \(\sigma_1, \sigma_2, \ldots\) are non-negative integer valued random variables adapted to a filtration \(\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots\). Suppose there exist positive constants \(c, \alpha, b\) such that

- on the event \(\{\sigma_n = k\}\), \(\sigma_{n+1} \in \{0, k + 1\}\),
- for all \(k\) and all \(n \geq 2\), \(\tilde{P}\{\sigma_{n+1} = k + 1|\mathcal{F}_n\} \geq b1_{\{\sigma_n = k\}}\),
- for all \(k\) and all \(n\), \(\tilde{P}\{\sigma_{n+1} = k + 1|\mathcal{F}_n\} \leq (1 - ce^{-\beta k}) 1_{\{\sigma_n = k\}}\).

Then there exist constants \(C\) and \(\beta_1\) such that for all \(n\)

\[
\tilde{P}\{\sigma_{2n} < n\} \leq Ce^{-\beta_1 n}.
\]

**Proof:** Let \(k_0\) be large enough so that \(1 - ce^{-\beta k} > 0\) for \(k \geq k_0\). Then we choose \(\alpha\) small enough so that \(1 - b \leq e^{-\alpha k_0}\) and \(ce^{-\beta k} \leq e^{-\alpha(k+1)}\) for \(k \geq k_0\). Note that \((\sigma_n)\) is not a Markov chain, so consider the following setup: let \(s_n\) be a non-negative integer valued Markov chain with \(s_0 = 0\), and whose transition probabilities are given by:

\[
p_{k,k+1} = 1 - e^{-\alpha(k+1)} \quad p_{k,0} = e^{-\alpha(k+1)}
\]

\(s_n\) stochastically dominates \(\sigma_{n+2}\). We claim that for each \(n \geq 0\),

\[
\tilde{P}\{s_n \geq k\} \leq \tilde{P}\{\sigma_{n+2} \geq k\}, \text{ for all } k \geq 0.
\]  \hspace{1cm} (29)

The proof is done by induction on \(n\). Note that (29) holds for all \(n\), when \(k = 0\). We assume \(k \geq 1\). For \(n = 0\), since \(\tilde{P}\{s_0 = 0\} = 1\), it follows that \(\tilde{P}\{s_0 \geq k\} = 0 \leq \tilde{P}\{\sigma_2 \geq k\}\). Assume \(\tilde{P}\{s_n \geq k\} \leq \tilde{P}\{\sigma_{n+2} \geq k\}\) for all \(k \geq 1\). Then

\[
\tilde{P}\{\sigma_{n+3} \geq k\} = \sum_{j=k-1}^{\infty} \tilde{P}\{\sigma_{n+3} = j + 1\}
\]

\[
= \sum_{j=k-1}^{\infty} \tilde{P}\{\sigma_{n+3} = j + 1|\sigma_{n+2} = j\} \tilde{P}\{\sigma_{n+2} = j\}
\]

\[
\geq \sum_{j=k-1}^{\infty} (1 - e^{-\alpha(j+1)}) \left( \tilde{P}\{\sigma_{n+2} \geq j\} - \tilde{P}\{\sigma_{n+2} \geq j + 1\} \right)
\]

\[
= (1 - e^{-\alpha k}) \tilde{P}\{\sigma_{n+2} \geq k - 1\} + \sum_{j=k}^{\infty} (e^{-\alpha j} - e^{-\alpha(j+1)}) \tilde{P}\{\sigma_{n+2} \geq j\}
\]

Using our inductive assumption, and then taking the same steps as above, but backwards,

\[
\tilde{P}\{\sigma_{n+3} \geq k\} \geq (1 - e^{-\alpha k}) \tilde{P}\{s_n \geq k - 1\} + \sum_{j=k}^{\infty} (e^{-\alpha j} - e^{-\alpha(j+1)}) \tilde{P}\{s_n \geq j\}
\]

\[
= \sum_{j=k-1}^{\infty} (1 - e^{-\alpha(j+1)}) \left( \tilde{P}\{s_n \geq j\} - \tilde{P}\{s_n \geq j + 1\} \right)
\]

\[
= \tilde{P}\{s_{n+1} \geq k\}.
\]

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We prove the theorem by showing there exist constants $C$ and $\beta_1$ such that for all $n \in \mathbb{N}$,
\[
\mathbb{P}\{s_{2n-2} \geq n\} \geq 1 - Ce^{-\beta_1 n}.
\]
Let $\tau = \inf\{n \geq 1 : s_n = 0\}$. Then $\mathbb{P}\{\tau = 1\} = e^{-\alpha}$, $\mathbb{P}\{\tau = k + 1\} = e^{-\alpha(k+1)} \prod_{j=1}^{k}(1 - e^{-j\alpha})$ for $k \geq 1$, and $\mathbb{P}\{\tau = +\infty\} = \prod_{j=1}^{\infty}(1 - e^{-j\alpha})$. Relating these stopping times to $s_n$, for $n \geq 2$ we have
\[
\mathbb{P}\{s_n = 0\} = \sum_{k=1}^{n} \mathbb{P}\{\tau = k\} \mathbb{P}\{s_{n-k} = 0\}.
\]
Define the following function
\[
F(s) = \sum_{n=1}^{\infty} \mathbb{P}\{\tau = n\} s^n.
\]
The radius of convergence of $F(s)$ is $e^\alpha$, and since $F(1) = \mathbb{P}\{\tau < \infty\} < 1$, by continuity of $F$, we can find some $s^* \in (1, e^\alpha)$ for which $F(s^*) = 1$. Then for all $s \in (1, s^*)$, $F(s) < 1$ and
\[
\sum_{n=0}^{\infty} \mathbb{P}\{s_n = 0\} s^n = \frac{1}{1 - F(s)} < \infty.
\]
Thus, for all $s \in (1, s^*)$ and $n$ large enough $\mathbb{P}\{s_n = 0\} \leq s^{-n}$. Then clearly we can find constants $c$ and $\beta_1$ such that $\mathbb{P}\{s_n = 0\} \leq ce^{-\beta_1 n}$ for all $n$. So,
\[
\sum_{k=0}^{n-1} \mathbb{P}\{s_{2n-2} = k\} \leq \sum_{k=0}^{n-1} \mathbb{P}\{s_{2n-2-k} = 0\} \leq \sum_{k=0}^{n-1} ce^{-\beta_1(2n-2-k)} = ce^{-\beta_1(2n-2)} \left( \frac{e^{\beta_1 n} - 1}{e^{\beta_1} - 1} \right)
\]
and the theorem follows, with a constant $C = ce^{2\beta_1}(e^{\beta_1} - 1)^{-1}$. \(\square\)

### 3.5 Invariant measure: proof of Theorem [1.2]

Let $\Lambda$ denote the space of measures supported on $\mathcal{A}$. Let $L_\lambda : \Lambda \to \Lambda$ denote the following transformation:
\[
L_\lambda \nu(\bar{\gamma}_1) = \nu(\bar{\gamma}_0)e^{-\lambda \Phi(\bar{\gamma}_1)}
\]
$L$ is linear and continuous on $\Lambda$. Furthermore, the $n$-th iterate of $L_\lambda$ is, as expected, given by the expression:
\[
L^n_\lambda \nu(\bar{\gamma}_n) = \nu(\bar{\gamma}_0)e^{-\lambda \Phi_n(\bar{\gamma}_n)}.
\]
Suppose we start with a distribution $\nu$ on $\mathcal{A}$, we attach an $n$-level path according to the probability measure $Q_n$, then re-scale the path accordingly to obtain an element of $\mathcal{A}$. Then the measure of this new path is given by $\nu^n$, whose density with respect to $\nu$ is
\[
\frac{e^{-\lambda \Phi_n}}{E_{\nu}[e^{-\lambda \Phi_n}]} \quad (30)
\]
Note that $\nu^n$ is simply $L^n_\lambda \nu$ normalized to a probability measure and it depends on the starting distribution $\nu$ and on $\lambda$. Let $\nu^n_k$ be restriction of $\nu^n$ to the last $k$ steps of $\eta$. Then
\[
\nu^n_k(\eta) = Q^n_{\nu}\{X_n = k \ \eta\} = \frac{E_{\nu}[e^{-\lambda \Phi_n}; \bar{\gamma}_n = k \ \eta]}{E_{\nu}[e^{-\lambda \Phi_n}]}.
\]

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We are interested in convergence of $\nu^n$ as $n \to \infty$. So let us first consider $\nu$ to be the Dirac measure $\delta_{\sigma_0}$ and define measures $\pi_k$ on $\mathcal{A}$ as follows: for all $\eta$ in $\mathcal{A}$, let

$$\pi_k(\eta) = \lim_{N \to \infty} Q_N^{\gamma_0}(X_N = k \eta).$$

Note that $\pi_k$ is a measure on the restriction of $\eta$ to the last $k$ steps. To show that the measures $\pi_k$ are well-defined and this limit exists, first recall that given $X_j = \gamma_j$, the law of $X_{N+j}$ under $Q_{N+j}$ is the same as the law of $X_N$ under $Q_N^{\gamma_j}$. Then for all $k < N/2$ and all $j \geq 0$, the coupling result from Theorem 1.1 implies

$$|Q_N^{\gamma_0}(X_N = k \eta) - Q_{N+j}^{\gamma_0}(X_{N+j} = k \eta)|$$

$$= \left| \sum_{\gamma_j} Q_N^{\gamma_0}(X_j = \gamma_j) \left( Q_N^{\gamma_0}(X_N = k \eta) - Q_N^{\gamma_j}(X_N = k \eta) \right) \right|$$

$$\leq \sum_{\gamma_j} Q_N^{\gamma_0}(X_j = \gamma_j) \left| Q_N^{\gamma_0}(X_N = k \eta) - Q_N^{\gamma_j}(X_N = k \eta) \right|$$

$$\leq \sum_{\gamma_j} Q_N^{\gamma_0}(X_j = \gamma_j) (Ce^{-\beta_1 N})$$

$$\leq Ce^{-\beta_1 N}$$

(31)

Then clearly, for a given history $\gamma_0 \in \mathcal{A}$, $Q_N^{\gamma_0}(X_N = k \eta)$ converges as $N \to \infty$. Now, if we start with a different history $\gamma'_0$, from our coupling, and more precisely from Corollary 3.1, we get

$$\lim_{N \to \infty} Q_N^{\gamma_0}(X_N = k \eta) = \lim_{N \to \infty} Q_N^{\gamma'_0}(X'_N = k \eta).$$

Therefore, the measures $\pi_k$ are well-defined. Moreover, if we start with some other initial distribution $\nu$ on paths $\gamma_0$ from $\mathcal{A}$, the following holds

$$\sum_{\eta|k} |Q_N^{\nu}(X_N = k \eta) - \pi_k(\eta)| = O(e^{-\beta_1 N}).$$

(32)

The measures $\pi_k$ are consistent and then, by the Kolgomorov Extension Theorem, they converge to a limiting measure on $\mathcal{A}$, which we will denote by $\pi = \lim_{k \to \infty} \pi_k$. It is easy to check that $\pi = \lim_{n \to \infty} \nu^n$.

We claim that $\pi$ is the unique stationary measure for the Markov chains described in this section. The result follows from the next proposition.

**Proposition 3.7.** There exists $\beta_2 > 0$ such that for any starting distribution $\nu$, we can find a constant $c(\nu)$ so that for all $n \geq 0$ the following holds

$$E^\nu[e^{-\lambda \Phi_n}] = c(\nu)e^{-\zeta(\lambda)n}[1 + O(e^{-\beta_2 n})]$$

(33)

**Proof:** Let $a_n = E^\nu[e^{-\lambda \Phi_n}]$. Then is it a quick check that

$$E^\nu[e^{-\lambda \Phi_{n+m}}] = E^\nu[e^{-\lambda \Phi_n}]E^\nu[e^{-\lambda \Phi_m}].$$
Let $\Phi$ depend only on the last $n/2$ steps of the path $\tilde{\gamma}_n$, that is, let $\tilde{\gamma}_n' = \tilde{\gamma}_n|_{n/2}$ be the restriction of $\tilde{\gamma}_n$ to its last $n/2$ steps and then

$$e^{-\lambda \Phi(\gamma_n)} = e^{-\lambda \Phi(\gamma'_n)}.$$

Suppose $\gamma_n$ has enough connected cross-sections above level $-n/2$, more precisely, $\gamma_n, \gamma_n'$ are in $\mathcal{V}_{n/2}$. By Proposition 3.4, $E^{\tilde{\gamma}_n}[e^{-\lambda \Phi}] = E^{\tilde{\gamma}_n}[e^{-\lambda \Phi}][1 + O(e^{-\beta n/2})]$, and so

$$|E^{\nu^n}[e^{-\lambda \Phi}; V_{n/2}] - E^{\pi}[e^{-\lambda \Phi}; V_{n/2}]| \leq \left[1 + O(e^{-\beta n/2})\right] \left|E^{\nu^n}_{n/2}[e^{-\lambda \tilde{\Phi}}; V_{n/2}] - E^{\pi}_{n/2}[e^{-\lambda \tilde{\Phi}}; V_{n/2}]\right|$$

(34)

On the complement of $V_{n/2}$, which we will denote by $V_{n/2}^c$, we will bound $E^{\tilde{\gamma}_n}[e^{-\lambda \Phi}]$ by $1$. From Lemma 3.2, we know that

$$\nu^n(V_{n/2}^c) = \nu^n_{n/2}(V_{n/2}^c) = O(e^{-\alpha' n/4}).$$

(35)

Observe that (32) implies $\|\nu_{n/2}^n - \pi_{n/2}\| = O(e^{-\beta_1 n})$, and thus,

$$\pi(V_{n/2}^c) = \pi_{n/2}(V_{n/2}^c) = O(e^{-\beta_1 n}) + O(e^{-\alpha' n/4}).$$

(36)

Combining estimates (34), (35) and (36),

$$|E^{\nu^n}[e^{-\lambda \Phi}] - E^{\pi}[e^{-\lambda \Phi}]| = O(e^{-\beta_1 n}) + O(e^{-\alpha' n/2}).$$

Let $\beta_2 = \min\{\beta_1, \alpha' / 4\}$, and since $E^{\pi}[e^{-\lambda \Phi}] \asymp e^{-\xi(\lambda)}$ and thus it is bounded from below by a positive constant, we get

$$|E^{\nu^n}[e^{-\lambda \Phi}] - E^{\pi}[e^{-\lambda \Phi}]| = O(e^{-\beta_2 n})E^{\pi}[e^{-\lambda \Phi}].$$

Note that our error term depends on the initial measure on paths, namely $\nu$. Thus,

$$a_n = a_0 \prod_{j=0}^{n-1} \frac{a_{j+1}}{a_j} = \prod_{j=0}^{n-1} E^{\pi}[e^{-\lambda \Phi}][1 + O(e^{-\beta_2 j})] = c(\nu) \left(E^{\pi}[e^{-\lambda \Phi}]\right)^n [1 + O(e^{-\beta_2 n})].$$

We know that $a_n \asymp e^{-\xi(\lambda)n}$, and therefore we must have

$$E^{\pi}[e^{-\lambda \Phi}] = e^{-\xi(\lambda)},$$

which completes the proof of the proposition.

We want to show $\pi$ is invariant under the linear transformation $L_\lambda$. Re-writing (30) in this notation, we obtain

$$\nu^n = \frac{L_\lambda^n \nu}{E^{\nu}[e^{-\lambda \Phi}_n]}.$$

Starting with any distribution $\nu$ as the initial measure on paths, we have seen that

$$L_\lambda \nu^n = \frac{L_\lambda^{n+1} \nu}{E^{\nu}[e^{-\lambda \Phi_{n+1}}]} = \nu^{n+1} \frac{E^{\nu}[e^{-\lambda \Phi_{n+1}}]}{E^{\nu}[e^{-\lambda \Phi_n}]} = e^{-\xi(\lambda)} \nu^{n+1} [1 + O(e^{-\beta_2 n})].$$

Passing to the limit, as $n \to \infty$, from continuity of $L_\lambda$ we have $L_\lambda \pi = e^{-\xi(\lambda)} \pi$ and therefore for all $n \geq 0$, $\pi^n = \pi$. Note that $\pi$ depends on $\lambda$. 

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4 Analyticity of intersection exponents

In this section we prove that $\xi(1, \lambda)$ is a real analytic function of $\lambda$ for $\lambda > 0$. The proof for other values of $k$ is essentially the same. Key to this proof is convergence to the unique invariant measure $\pi$, presented in the previous section, and more significantly the exponential rate at which this convergence takes place. For each $\lambda > 0$, we associate to $L_\lambda$ a linear functional defined as

$$T^*_\lambda f(\tilde{\gamma}_0) = E[f(\tilde{\gamma}_n)Z^\lambda_n],$$

for continuous functions $f$, bounded under an appropriately chosen norm. This norm will be chosen so that on the Banach space of functions with finite norm $\lambda \mapsto T^*_\lambda$ is an analytic operator-valued function. In particular, the functions on this Banach space have the property that their dependence on the behavior of paths far away in the past decays exponentially. Estimates already obtained from convergence to invariant measure will show $\xi(\lambda) := \xi(1, \lambda)$ is an isolated simple eigenvalue for $T^*_\lambda$. Using results from operator theory, for every $\lambda > 0$ one can then extend $x \mapsto \xi(x)$ to an analytic function in a neighborhood of $\lambda$. This immediately proves that intersection exponent $\xi(\lambda)$ is real analytic in $\lambda$.

Remark 4.1. We reiterate that this section follows the notation and proof outlines from [12] in which Lawler, Schramm and Werner prove analyticity of 2-dimensional Brownian exponents. We include here the full proofs, for the sake of completeness, with the understanding that they differ from [12] only in the estimates we use.

4.1 The operator

Let $C$ be the set of continuous functions $f : \mathbb{R} \to \mathbb{C}$, bounded under the uniform norm $\|f\| = \sup_{\tilde{\gamma}_0} |f(\tilde{\gamma}_0)|$. We are interested in functions that depend very little on how $\tilde{\gamma}_0$ looks like near negative infinity. Recall that $\tilde{\gamma} \equiv_k \tilde{\gamma}'$ means the paths $\tilde{\gamma}$ and $\tilde{\gamma}'$ have been coupled for the last $k$ steps, in the sense of Section 3.3. Thus, let us consider the following $u$-norm: for all $f \in C$, and $u > 0$, let

$$\|f\|_u := \max\{\|f\|, \sup_{\tilde{\gamma}} e^{ku}|f(\tilde{\gamma}) - f(\tilde{\gamma}')| : k = 1, 2, \ldots, \tilde{\gamma} \equiv_k \tilde{\gamma}'\}.$$ 

Recall that $\tilde{\gamma} \equiv_k \tilde{\gamma}'$ means the paths $\tilde{\gamma}$ and $\tilde{\gamma}'$ have been coupled for the last $k$ steps, in the sense of the coupling described in Section 3.3. This norm similar to the one used in [12]. Let $C_u := \{f \in C : \|f\|_u < \infty\}$ denote the Banach space of all bounded functions $f$ under the norm $\|f\|_u$. Let $L_u$ be the Banach space of continuous linear operators from $C_u$ to $C_u$ with the usual norm

$$N_u(T) := \sup_{\|f\|_u = 1} \|T(f)\|_u.$$ 

For all $\lambda > 0$, and all $n > 0$, we define the linear operator $T^*_\lambda : C \to \mathbb{R}$ by

$$T^*_\lambda f(\tilde{\gamma}_0) := E[\tilde{\gamma}_0 \left( f(\tilde{\gamma}_n)e^{-\lambda \Phi_n} \right),$$

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where the expectation is over the randomness of \( \tilde{\gamma}_n \). It is easy to see that \( T_\lambda \) is a semigroup of operators:

\[
T_\lambda^{n+m} f(\tilde{\gamma}_0) = E^{\tilde{\gamma}_0} \left[ f(\tilde{\gamma}_{n+m}) e^{-\lambda \Phi_{n+m}} \right] \\
= E^{\tilde{\gamma}_0} \left[ e^{-\lambda \Phi_n} E^{\tilde{\gamma}_n} \left[ f(\tilde{\gamma}_{n+m}) e^{-\lambda \Phi_m} \right] \right] \\
= E^{\tilde{\gamma}_0} \left[ e^{-\lambda \Phi_n} T_\lambda^m f(\tilde{\gamma}_n) \right] \\
= T_\lambda^n T_\lambda^m f(\tilde{\gamma}_0).
\]

We will use \( T_\lambda \) for \( T_\lambda^1 \). One can similarly define \( T_\lambda^n \) for complex \( \lambda \) with \( \Re(\lambda) > 0 \).

### 4.2 Analyticity of operator

If we look at the functional \( T_z \) as a function of \( z \), it is analytic in a small neighborhood of the positive real line. This will be the first step in proving \( e^{-\mathcal{E}(\lambda)} \) is analytic in \( \lambda \).

**Proposition 4.2.** If \( \lambda_1 \leq \lambda \leq \lambda_2 \), there exist \( \epsilon > 0 \) and \( v(\lambda) > 0 \) such that for all \( u \in (0, v) \), \( \lambda \mapsto T_z \) is an analytic function from \( \{ z : |z - \lambda| < \epsilon \} \) into \( \mathcal{L}_u \).

**Proof:** Fix \( \lambda > 0 \). For all \( k \geq 0 \), and all \( f \in C \), \( \tilde{\gamma}_0 \in A \), let

\[
U_k f(\tilde{\gamma}_0) = E^{\tilde{\gamma}_0} \left[ f(\tilde{\gamma}_1) \frac{\Phi^k}{k!} e^{-\lambda \Phi} \right]
\]

An upper bound for \( U_k f \) is

\[
|U_k f(\tilde{\gamma}_0)| \leq \|f\| E^{\tilde{\gamma}_0} \left[ \frac{(\lambda \Phi)^k}{k!} \lambda^{-k} e^{-\lambda \Phi} \right] \leq \|f\| E^{\tilde{\gamma}_0} \left[ e^{\lambda \Phi} \lambda^{-k} e^{-\lambda \Phi} \right] \leq \|f\| \lambda^{-k}, \tag{37}
\]

and by dominated convergence, for all \( z \in \mathbb{C} \) with \( |z| < \lambda \),

\[
T_{\lambda-z} f(\tilde{\gamma}_0) = \sum_{k=0}^{\infty} U_k f(\tilde{\gamma}_0) z^k.
\]

We need to show there exists a \( v(\lambda) > 0 \) such that for \( u < v \), for all \( k \), the operator norm of \( U_k \) in \( \mathcal{L}_u \) is bounded by \( b^k \) for some \( b > 0 \). Then for \( |z| < b^{-1} \), \( T_{\lambda-z} f \) is an analytic function of \( z \) into \( \mathcal{L}_u \) and the proposition follows. We will now prove this claim. From (37), we have \( \|U_k\| \leq \lambda^{-k} \). Now suppose \( \tilde{\gamma}_0 \equiv_m \tilde{\gamma}_0' \) (that is, \( \tilde{\gamma}_0 \) and \( \tilde{\gamma}_0' \) are coupled for \( m \) steps). We use \( \Phi \) to denote \( \Phi(\tilde{\gamma}_1) \) and \( \Phi' \) for \( \Phi(\tilde{\gamma}_1') \).

\[
|U_k f(\tilde{\gamma}_0) - U_k f(\tilde{\gamma}_0')| \leq E \left[ |f(\tilde{\gamma}_1) - f(\tilde{\gamma}_1')| \frac{\Phi^k}{k!} e^{-\lambda \Phi} \right] \\
+ \|f\| E \left[ \frac{\Phi^k}{k!} e^{-\lambda \Phi} - \frac{\Phi'^k}{k!} e^{-\lambda \Phi'} \right]. \tag{38}
\]

Given \( \tilde{\gamma}_0 \equiv_m \tilde{\gamma}_0' \), on the event the two paths remain coupled for an additional step, we can bound \( |f(\tilde{\gamma}_1) - f(\tilde{\gamma}_1')| \) by \( \|f\| e^{-(m+1)u} \) and otherwise we bound it by \( 2\|f\| \). Using our coupling result in (26), an upper bound for the first term in (38) is

\[
\|f\| u e^{-(m+1)u} \lambda^{-k} + 2\|f\| \lambda^{-k} \tilde{P} \{ \tilde{\gamma}_1 \not\equiv_m \tilde{\gamma}_1' | \tilde{\gamma}_0 \equiv_m \tilde{\gamma}_0' \} \\
\leq \|f\| u \lambda^{-k} \left( e^{-(m+1)u} + ce^{-m\beta} \right). \tag{39}
\]
For the second term in (38), we have
\[
\left(\frac{\lambda}{2}\right)^k E\left[\lambda \Phi_k e^{-\lambda \Phi} - \frac{(\lambda \Phi')^k}{k!} e^{-\lambda \Phi'}\right] = \frac{1}{k!} E\left[\left(\frac{\lambda \Phi}{2}\right)^k e^{-\lambda \Phi} - \left(\frac{\lambda \Phi'}{2}\right)^k e^{-\lambda \Phi'}\right] \leq 3E\left[e^{-\lambda \Phi/2} - e^{-\lambda \Phi'/2}\right].
\]

Suppose that after being coupled for \( m \) steps the paths remain coupled for an additional step. We thus attach the same \( \bar{\gamma}_1 \) to \( \tilde{\gamma}_0 \) and \( \tilde{\gamma}'_0 \), and we consider two cases: if \( \bar{\gamma}_1 \in G_{m/2} \), then using our estimate from (24) we get \( |e^{-\lambda \Phi/2} - e^{-\lambda \Phi'/2}| \leq c e^{-m \beta} \). On the complement of \( G_{m/2} \), as well as when the paths decouple after \( m \) steps, we will bound this difference by 2. Using the inequality \( \bar{\mathbb{P}}\{\bar{\gamma}_1 \notin G_{m/2}\} \leq e^{-\beta m} \), an recalling our result from (26), we obtain
\[
3E\left[e^{-\lambda \Phi/2} - e^{-\lambda \Phi'/2}\right] \leq c e^{-\beta m},
\]
with a different \( c \), uniform in \( \lambda \) and independent of \( k \). Thus, the second term in (38) can be bound by
\[
c \|f\|_u (\frac{2}{\lambda})^k e^{-m \beta}. \tag{40}
\]
From estimates (39) and (40), for all \( u \leq \beta \),
\[
|U_k f(\tilde{\gamma}_0) - U_k f(\tilde{\gamma}'_0)| \leq c \|f\|_u (\frac{2}{\lambda})^k e^{-mu}.
\]
Hence \( N_u(U_k) \leq c (\frac{2}{\lambda})^k \) and the proposition follows with \( v(\lambda) \leq \beta \) and \( \epsilon \leq \lambda/2 \). \( \square \)

4.3 Analyticity of exponent

**Proposition 4.3.** \( e^{-\xi(\lambda)} \) is an isolated simple eigenvalue for \( T_\lambda \).

**Proof:** Fix \( \lambda > 0 \). For ease of notation, we write \( T \) as shorthand for \( T_\lambda \) and \( e^{-\xi} \) for \( e^{-\xi(1,\lambda)} \).

First we will show \( \frac{T^n f(\tilde{\gamma}_0)}{T^n(\tilde{\gamma}_0)} \) converges to a bounded functional \( h \). Recall the result of our coupling: for any \( \tilde{\gamma}_0, \tilde{\gamma}'_0 \in \mathcal{A} \),
\[
\bar{\mathbb{P}}\{\tilde{\gamma}_n \neq n/2 \tilde{\gamma}'_n\} \leq C e^{-\beta_1 n}.
\]
Suppose \( \tilde{\gamma}_0 \) is fixed. If we let \( \tilde{\gamma}'_0 = \tilde{\gamma}_k \) then the law of \( \tilde{\gamma}'_0 \) under \( Q_n \) is the same as the law of \( \tilde{\gamma}_k \) under \( Q_{n+k} \), and from the coupling result, for all \( f \in \mathcal{C}_u \),
\[
\left|\frac{T^{n+k} f(\tilde{\gamma}_0)}{T^{n+k}(\tilde{\gamma}_0)} - \frac{T^n f(\tilde{\gamma}_0)}{T^n(\tilde{\gamma}_0)}\right| \leq \int |f(\tilde{\gamma}_n) - f(\tilde{\gamma}'_n)| d\bar{\mathbb{P}} \leq 2\|f\|_u e^{-\beta_1 n} + \|f\|_u e^{-nu/2} \tag{41}
\]
Therefore, \( \frac{T^n f(\tilde{\gamma}_0)}{T^n(\tilde{\gamma}_0)} \to h(f, \tilde{\gamma}_0) \). Similarly, for any starting configurations \( \tilde{\gamma}_0, \tilde{\gamma}_0' \in \mathcal{A} \) and all \( f \in \mathcal{C}_u \),
\[
\left|\frac{T^n f(\tilde{\gamma}_0)}{T^n(\tilde{\gamma}_0)} - \frac{T^n f(\tilde{\gamma}_0')}{T^n(\tilde{\gamma}_0')}\right| \leq 2\|f\|_u C e^{-\beta_1 n} + \|f\|_u e^{-nu/2}.
\]
This shows \( h \) is independent of \( \tilde{\gamma}_0 \). It is easy to see that \( h \) is a linear on \( \mathcal{C} \) and \( \|h\| \leq \|f\| \). Then \( h \) is a bounded linear functional on \( \mathcal{C}_u \).
Now we want to consider the functional $f \mapsto T^n f - h(f)T^n 1$ and to find an upper bound for its $N_u$ norm. From (11), for $u \leq 2\beta_1$ and all $f \in C_u$ and $\tilde{\gamma}_0 \in \mathcal{A}$,

$$|T^n f(\tilde{\gamma}_0) - h(f)T^n 1(\tilde{\gamma}_0)| \leq (2C + 1)\|f\|_u e^{-nu/2}T^n 1(\tilde{\gamma}_0) \leq c\|f\|_u e^{-n(\xi + u/2)}. \quad (42)$$

Then $\|T^n (\cdot) - h(\cdot)T^n 1\| \leq c e^{-n(\xi + n/2)}$. To find the $N_u$ norm, consider $\tilde{\gamma}_0 \equiv \tilde{\gamma}_0'$. For $k \leq n/4$, the estimate above gives

$$|T^n f(\tilde{\gamma}_0) - h(f)T^n 1(\tilde{\gamma}_0) - T^n f(\tilde{\gamma}_0') + h(f)T^n 1(\tilde{\gamma}_0')| \leq 2c\|f\|_u e^{-n(\xi + u/4)} e^{-ku}. \quad \text{When } k > n/4, \text{ if } \tilde{\gamma}_0 \text{ and } \tilde{\gamma}_0' \text{ are coupled for the last } k \text{ steps, we have shown in Proposition 3.4 that } T^n 1(\tilde{\gamma}_0) = T^n 1(\tilde{\gamma}_0') \left[1 + O(e^{-\beta k})\right]. \text{ From our coupling, the two paths will remain coupled for } n \text{ additional steps with probability greater than } \prod_{j=0}^{n-1} [1 - ce^{-\beta(k+j)}] \text{ which can be shown to be bounded by } 1 - ce^{-\beta k}, \text{ with a different } c, \text{ independent of } n. \text{ Hence for all } u \leq \beta/2 \text{ and all } f \in C_u,$$

$$\frac{|T^n f(\tilde{\gamma}_0) - T^n f(\tilde{\gamma}_0')|}{|T^n 1(\tilde{\gamma}_0) - T^n 1(\tilde{\gamma}_0')|} \leq 2c\|f\|_u e^{-\beta k} + \|f\|_u e^{-(k+n)u} \leq c'\|f\|_u e^{-nu/4} e^{-ku} \quad \text{Multiplying this expression by } T^n 1(\tilde{\gamma}_0) \text{ and recalling that } T^n 1(\tilde{\gamma}_0) \leq ce^{-\xi n}, \text{ we get}

$$|T^n f(\tilde{\gamma}_0) - h(f)T^n 1(\tilde{\gamma}_0) - T^n f(\tilde{\gamma}_0') T^n 1(\tilde{\gamma}_0') + h(f)T^n 1(\tilde{\gamma}_0')| \leq c e^{-\xi n}\|f\|_u e^{-nu/4} e^{-ku} \quad \text{From (42) and Proposition 3.4}

$$|T^n f(\tilde{\gamma}_0) - h(f)T^n 1(\tilde{\gamma}_0) - T^n f(\tilde{\gamma}_0') - h(f)T^n 1(\tilde{\gamma}_0')| \leq c e^{-\xi n}\|f\|_u e^{-nu/4} e^{-ku} + O(e^{-\beta k})\|f\|_u e^{-n(\xi + u/2)} \leq c'\|f\|_u e^{-n(\xi + u/4)} e^{-ku}. \quad \text{We conclude that}

$$N_u(T^n (\cdot) - h(\cdot)T^n 1) \leq ce^{-n(\xi + u/4)}. \quad (43)$$

This will show that $e^{-\xi}$ is a simple eigenvalue of $T$. Since for all $k \geq 1$, $\frac{T^{n+k}}{T^n} \rightarrow h(T^k)$ and for all $\tilde{\gamma}_0$, $T^n 1(\tilde{\gamma}_0) \preceq e^{-\xi n}$, we get $h(T1) = e^{-\xi}$ and $h(T^n 1) = e^{-\xi n}$. From (43),

$$\|T^{n+k} 1 - h(T^k)T^n 1\| \leq ce^{-\xi(n+k)} e^{-nu/4}, \quad \text{Recall that } K_n(\tilde{\gamma}_0) = e^{\xi n}T^n 1(\tilde{\gamma}_0) \text{ and then for all } k \geq 0,$$

$$\|K_{n+k} - K_n\|_u \leq ce^{-nu/4},$$

and therefore $K_n \rightarrow K$, for some function $K : \mathcal{A} \rightarrow \mathbb{R}$. Furthermore, $\|K_n - K\|_u \leq ce^{-nu/4}$. It follows that $N_u(T^n (\cdot) - e^{-\xi n}h(\cdot)K) \leq ce^{-\xi n} e^{-nu/4}$. In particular, for all $f \in C_u,$

$$\|T^n (f) - e^{-\xi n}h(f)K\|_u \leq ce^{-n(\xi + u/4)}\|f\|_u. \quad (44)$$
It is easy to see that for all \( n \geq 1 \),
\[
h(T^n f) = e^{-\xi n} h(f). \tag{45}
\]
Moreover, from continuity of \( T \),
\[
TK = \lim_{n \to \infty} TK_n = \lim_{n \to \infty} e^{\xi n} T^{n+1} = e^{-\xi} \lim_{n \to \infty} K_{n+1} = e^{-\xi} K,
\tag{46}
\]
and hence \( e^{-\xi} \) is an eigenvalue for \( T \). By continuity and linearity of \( h \),
\[
h(K) = \lim_{n \to \infty} h(K_n) = \lim_{n \to \infty} e^{\xi n} h(T^n 1) = 1. \tag{47}
\]

From estimates (45), (46) and (47), one can easily check that \( T^n(\cdot) - e^{-\xi n} h(\cdot)K \) is the \( n \)-th iterate of \( T(\cdot) - e^{-\xi} h(\cdot)K \) and thus \( N_u(T(\cdot) - e^{-\xi} h(\cdot)K) \leq e^{-\xi} e^{-u/4} \).

We claim this implies \( e^{-\xi} \) is an isolated eigenvalue. We will prove that for every \( z \) with \( 1/2(1 + e^{-u/4}) < |z| < 1 \), there exists \( \epsilon > 0 \) such that for all \( \|f\|_u = 1 \), \( \| e^{\xi} T f - z f \|_u \geq \epsilon \), that is, \( z \) is in the resolvent set of \( T = e^{\xi} T \).

Fix \( z \) with \( 1/2(1 + e^{-u/4}) < |z| < 1 \), and for \( \| f \|_u = 1 \) let
\[
\tilde{T} f = zf + g \quad v_n(f) = \tilde{T}^n f - h(f)K.
\]
Note that \( g \in \mathcal{C}_u \) and
\[
\tilde{T}^n f = z^n f + \sum_{j=1}^n z^{n-j} \tilde{T}^{j-1} g.
\tag{48}
\]
Since \( K_n \) converges to \( K \), by Proposition 2.7 we have \( \|K\|_u \leq c_2 \). Recalling that \( \| h(g)\|_u \leq \| g \|_u \) and using (44), we arrive at
\[
\left\| \tilde{T}^n f - z^n f - \frac{1}{1-z} h(g)K \right\|_u \leq \left\| \sum_{j=1}^n z^{n-j} \tilde{T}^{j-1} g - \frac{1}{1-z} h(g)K \right\|_u
\]
\[
= \left\| \sum_{j=1}^n z^{n-j} v_{j-1}(g) \right\|_u + \left\| \left( \sum_{j=1}^n z^{n-j} - \frac{1}{1-z} \right) h(g)K \right\|_u
\]
\[
\leq c|z|^n \| g \|_u,
\]
for some constant \( c > 1 \) that depends on \( z \) and \( u \). Since this bound holds for all \( n \), and so does (44), we must have \( h(g) = (1-z)h(f) \). Therefore, for \( f \) with \( \| f \|_u = 1 \), and for all \( n \),
\[
c|z|^n \| g \|_u \geq \| \tilde{T}^n f - z^n f - h(f)K \|_u \geq |z|^n \| f \|_u - \| v_n(f) \|_u \geq |z|^n - e^{-nu/4}.
\]

For \( |z| > 1/2(1 + e^{-u/4}) \), this implies
\[
\| g \|_u \geq \frac{1 - e^{-u/4}}{c(1 + e^{-u/4})}.
\]

It follows that the spectrum of \( T \) in \( \mathcal{L}_u \) is the union of \( e^{-\xi} \) and a set contained in the ball of radius \( 1/2(1 + e^{-u/4})e^{-\xi} \) and centered at the origin. \( \Box \)
Proof of Theorem 1.3: We prove the theorem for $k = 1$. From Proposition 4.3, $\xi(1, \lambda)$ is an isolated simple eigenvalue for the analytic operator $T_\lambda$. By 4.16 in [18], for all $\lambda > 0$, $x \mapsto \xi(1, x)$ can be extended analytically in a neighborhood of $\lambda$. More precisely, for all $\lambda > 0$, there exists $\epsilon > 0$ such that $z \mapsto \xi(1, z)$ is analytic in $|z - \lambda| < \epsilon$. Therefore, piecing together these $\epsilon$-balls we obtain a neighborhood of the positive real-line $(0, \infty)$ where $z \mapsto \xi(1, z)$ is analytic. □

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