REVISITING DWORK COHOMOLOGY: VISIBILITY AND DIVISIBILITY OF FROBENIUS EIGENVALUES IN RIGID COHOMOLOGY

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ABSTRACT. We study Frobenius eigenvalues of the compactly supported rigid cohomology of a variety defined over a finite field of $q$ elements via Dwork’s method. A couple of arithmetic consequences will be drawn from this study. As the first application, we show that the zeta functions for finitely many related affine varieties are capable of witnessing all Frobenius eigenvalues of the rigid cohomology of the variety up to Tate twist. This result does not seem to be known for $\ell$-adic cohomology. As the second application, we prove several $q$-divisibility lower bounds for Frobenius eigenvalues of the rigid cohomology of the variety in terms of the multi-degrees of the defining equations. These divisibility bounds for rigid cohomology are generally better than what is suggested from the best known divisibility bounds in $\ell$-adic cohomology, both before and after the middle cohomological dimension.

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1. INTRODUCTION

Dwork [12] engineered a cohomology theory in order to study the zeta function of a projective hypersurface over a finite field $F_q$ of $q$ elements with characteristic $p$. In this paper, we revisit his construction, and prove a comparison theorem between an overconvergent version of Dwork cohomology and rigid cohomology. We use the comparison theorem to study some problems on Frobenius eigenvalues of affine varieties in rigid cohomology.

Let us postpone the statement of the comparison theorem until Section 2 and start by stating some of its arithmetic consequences.
Visibility of Frobenius eigenvalues. The first application concerns with the visibility of Frobenius eigenvalues in the zeta function of an affine variety. For an algebraic variety $Z$ over $\mathbb{F}_q$, its zeta function is

$$\zeta_Z(t) = \exp \left\{ \sum_{m=1}^{\infty} \frac{|Z(\mathbb{F}_q^m)|}{m} t^m \right\}.$$ 

Weil conjectured, and Dwork [11] proved, that $\zeta_Z(t)$ is a rational function.

By trace formulae in rigid cohomology (due to Étesse and Le Stum [17]) and $\ell$-adic cohomology (see, e.g., [8, Rapport, 4–6]), the zeta function is an alternating product

$$(*) \quad \zeta_Z(t) = \prod_{i=0}^{2 \dim Z} \det \left( 1 - t \cdot \text{Frob}_q \mid H^i_c(Z) \right)^{(-1)^{i+1}}.$$ 

Here, $H^i_c(Z)$ could mean either Berthelot’s compactly supported rigid cohomology $H^i_{\text{rig},c}(Z)$, or compactly supported $\ell$-adic cohomology $H^i_c(Z_{\mathbb{F}_q}, \mathbb{Q}_\ell)$ ($\mathbb{F}_q$ is a fixed algebraic closure of $\mathbb{F}_q$, $Z_{\mathbb{F}_q} = Z \otimes_{\mathbb{F}_q} \mathbb{F}_q$, $\ell \neq p$ is a prime number). The finite dimensionality of $H^i_c(Z)$ also yields a cohomological proof for the rationality of the zeta function. The above trace formula implies that the reciprocal roots and poles of $\zeta_Z(t)$ are among the Frobenius eigenvalues of these cohomology theories. However, the converse is not true in general.

If $Z$ is smooth and proper over $\mathbb{F}_q$, then the Weil conjecture, proved by Deligne [7 Théorème 1.6] (for rigid cohomology, use Katz–Messing [20]), implies that the Frobenius eigenvalues of $H^i_c(Z)$ are algebraic integers having archimedean length $q^{i/2}$ (with respect to any embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$). Therefore the denominator and numerator of the right hand side of (*) do not have common factors. In this case, the zeta function alone can recover the Frobenius eigenvalues.

Without the smooth proper condition, the linear factors of the determinants in (*) could cancel. If a cancellation happens, $\zeta_Z(t)$ may not be capable of witnessing all the Frobenius eigenvalues, not even up to Tate twist. Consider the following example. Let $X$ be a general nonsingular cubic curve in $\mathbb{A}^3_{\mathbb{F}_q}$, and $Y = \mathbb{A}^2_{\mathbb{F}_q} - X$ its complement. Then the zeta function of the affine variety $Z = X \amalg Y$ equals that of $\mathbb{A}^2_{\mathbb{F}_q}$, namely $(1 - q^2t)^{-1}$. On the other hand, there exist Frobenius eigenvalues of $H^1_c(Z) = H^1_c(X) \oplus H^1_c(Y)$ of length $q^{1/2}$.

Our first theorem asserts that, if we are willing to take the defining equations of an affine variety $Z$ into the consideration, then we can recover all the Frobenius eigenvalues of $Z$ up to Tate twist from zeta functions of finitely many varieties related to $Z$. In order to give the precise statement, let us introduce some terminologies.

Definition. Let $\Gamma = \{ \Gamma_a(t), \Gamma_b(t), \ldots \} \subset 1 + t\mathbb{C}_p[[t]]$ be a collection of $p$-adic meromorphic function on $\mathbb{C}_p$ (i.e., fractions of $p$-adic entire functions). We say a $p$-adic number $\gamma \in \mathbb{C}_p \setminus \{ 0 \}$ is visible in $\Gamma$, if $\Gamma_a(\gamma^{-1}) = 0$ or $\infty$ for some $\Gamma_a \in \Gamma$. We say $\gamma$ is weakly visible in $\Gamma$, if there exists $m \in \mathbb{Z}$ such that $q^m \gamma$ is visible in $\Gamma$.

Now let $f_1, \ldots, f_r \in \mathbb{F}_q[x_1, \ldots, x_n]$ be a collection of polynomials. For every subset $I \subset \{ 1, 2, \ldots, r \}$, set $Z_I = \text{Spec} \mathbb{F}_q[x_1, \ldots, x_n]/(f_i : i \in I) \subset \mathbb{A}^n_{\mathbb{F}_q}$, and $Z^*_I = Z_I \cap \mathbb{G}^n_{\mathbb{A}_m}$. Write $Z = Z_{\{ 1, 2, \ldots, r \}}$. 
Theorem 1.1. Any Frobenius eigenvalue of $H^\bullet_{\mathrm{rig},c}(Z)$ is weakly visible in the finite set $\{\zeta_Z^I(t) : I \subset \{1, 2, \ldots, r\}\}$.

The theorem is already interesting when $Z \subset \mathbb{A}^n_{\mathbb{F}_q}$ is an affine hypersurface. In this special situation, it asserts that if a Frobenius eigenvalue $\lambda$ is canceled in the zeta function, then either it equals $q^m$ for some $m$, or there must exist a reciprocal root or reciprocal pole of $\zeta_{Z;G_m}(t)$ that equals a Tate twist of $\lambda$. Thus the zeta function $\zeta_{Z;G_m}(t)$ alone can recover all the Frobenius eigenvalues of $H^\bullet_{\mathrm{rig},c}(Z)$ up to Tate twist.

In view of the “motivic” philosophy, the same result should also hold for $\ell$-adic cohomology. But our method, $p$-adic in nature, depends upon an explicit chain model of rigid cohomology, which does not work in the $\ell$-adic context.

**Divisibility of Frobenius eigenvalues.** The second part of the applications deals with $q$-divisibility as algebraic integers of Frobenius eigenvalues of affine and projective varieties.

Let $f_1, \ldots, f_r \in \mathbb{F}_q[x_1, \ldots, x_n]$ be a collection of polynomials. Write $d_j = \deg f_j$. Without loss of generality, we will assume that all the degrees $d_j$ are positive. By rearranging we could and will assume $d_1 \geq d_2 \geq \cdots \geq d_r$. Let

$$Z = \mathrm{Spec}\, \mathbb{F}_q[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

be the vanishing scheme of these polynomials in $\mathbb{A}^n_{\mathbb{F}_q}$. For integer $j \geq 0$, we define a non-negative integer

$$\mu_j(n; d_1, \ldots, d_r) = j + \max \left\{ 0, \left\lfloor \frac{n - j - \sum_{i=1}^r d_i}{d_i} \right\rfloor \right\}.$$

Recall that the classical Ax–Katz theorem [4,19] states that all the reciprocal roots and poles of the zeta function of $Z$ are divisible by $q^{\mu_j(n; d_1, \ldots, d_r)}$ as algebraic integers. This divisibility was later upgraded to a divisibility on Frobenius eigenvalues on $\ell$-adic cohomology: Esnault and Katz [15] shows that the Frobenius eigenvalues of $H^\bullet(\mathbb{Z}_{\mathbb{F}_q}; \mathbb{Q}_l)$ are divisible by $q^{\mu_j(n; d_1, \ldots, d_r)}$, furthermore, for $j \geq 0$, the Frobenius eigenvalues of $H^{n-1+j}(\mathbb{Z}_{\mathbb{F}_q}; \mathbb{Q}_l)$ are divisible by $q^{\mu_j(n; d_1, \ldots, d_r)}$.

The theorem of Esnault and Katz does not give the most optimal bound when $r > 1$. Recently, Esnault and the first author [16] obtained a better bound beyond the middle cohomological dimension. They showed that any Frobenius eigenvalue of $H^\dim Z+j(Z_{\mathbb{F}_q}; \mathbb{Q}_l)$ is divisible by $q^{\mu_j(n; d_1, \ldots, d_r)}$, in the ring of algebraic integers, for all integers $j$ satisfying $0 \leq j \leq \dim Z$. This theorem simultaneously improves the results of [15] and Deligne’s integrality theorem [3, Exposé XXI, §5] beyond the middle cohomological dimension. Their proof only uses some formal properties of a cohomology theory, one of which (a certain Gysin lemma) is not yet available in rigid cohomology.

Since Frobenius eigenvalues are supposed to be “motivic”, the rigid cohomological analogue of the above divisibility theorems for $\ell$-adic cohomology should also hold, as already pointed out in Esnault–Katz [15]. In fact, the same axiomatic formal proof works if one assumes the conjectural weak Lefschetz theorem and the related Gysin lemma in rigid cohomology.

Since these results are not available in rigid cohomology yet, we would like to pursue the divisibility for rigid cohomology using a different method, via Dwork’s
p-adic theory, refining and upgrading the chain level approach in [28] to Dwork cohomology, and then by comparison with rigid cohomology.

Somewhat surprisingly, the final bound we obtain is sharper than what is anticipated by the ℓ-adic theorems in [9,15,16]. To state it, let us define

\[ d_i^* = \begin{cases} 
  d_i, & \text{if } 1 \leq i \leq n - \dim Z; \\
  1, & \text{if } i > n - \dim Z, \text{ and } d_i = d_1; \\
  0, & \text{if } i > n - \dim Z, \text{ and } d_i < d_1.
\]

For integer \( j \geq 0 \), define another non-negative integer

\[ \nu_j(n;d_1,\ldots,d_r) = j + \max \left\{ 0, \left\lfloor \frac{n - j - \sum_{i=1}^r d_i^*}{d_1} \right\rfloor \right\}. \]

Note that \( d_i \geq d_i^* \), thus \( \nu_j(n;d_1,\ldots,d_r) \geq \mu_j(n;d_1,\ldots,d_r) \); also the numbers \( \nu_j(n;d_1,\ldots,d_r) \) form an increasing sequence in \( j \). In the complete intersection case, namely when \( n - \dim Z = r \), one checks that \( d_i^* = d_i \) and \( \nu_j(n;d_1,\ldots,d_r) = \mu_j(n;d_1,\ldots,d_r) \).

We state our divisibility results as two theorems, as the bound looks different, depending on whether the cohomological degree is at least \( \dim Z \) or at most \( \dim Z \). Together, they cover all cohomological degrees.

**Theorem 1.2** (Divisibility beyond middle dimension). Let notation be as above. For every \( 0 \leq j \leq \dim Z \),

- the Frobenius eigenvalues of \( H^{\dim Z+j}_{\text{rig},c}(Z) \) are divisible by \( q^{\nu_j(n;d_1,\ldots,d_r)} \) in the ring of algebraic integers;
- the Frobenius eigenvalues of \( H^{\dim Z+1+\nu_j(n;d_1,\ldots,d_r)}_{\text{rig},c}(\mathbb{A}_\mathbb{F}_q^n - Z) \) are divisible by \( q^{\nu_j(n;d_1,\ldots,d_r)} \) in the ring of algebraic integers.

**Remarks.**

(a) The second item is the consequence of the first, thanks to the long exact sequence for compactly supported cohomology.

(b) For any separated variety \( Z \) over \( \mathbb{F}_q \), the Frobenius eigenvalues of \( H^i_{\text{rig},c}(Z) \) are always algebraic integers. When \( Z \) is smooth proper, we use the Weil conjecture and the integrality of the zeta function, see [29] Theorem 1). If \( Z \) is proper but possibly singular, we can produce a proper hypercovering using smooth proper varieties by alteration [9,1], and then apply cohomological descent [27]. If \( Z \) is not proper, we can embed \( Z \) into a proper variety \( \overline{Z} \), and conclude by using the assertion for proper varieties and the long exact sequence

\[ \cdots \to H^i_{\text{rig},c}(Z) \to H^i_{\text{rig}}(\overline{Z}) \to H^i_{\text{rig}}(\overline{Z} - Z) \to \cdots. \]

Our method is capable of seeing this too, see p. 511.

Recall that in the ℓ-adic situation, beyond the middle dimension, the theorem of Esnault and the first author says that the divisibility of Frobenius eigenvalues for \( H^{\dim Z+j}(\mathbb{Z}_\ell, \mathbb{Q}_\ell) \) is by \( q^{\mu_j(n;d_1,\ldots,d_r)} \). Since \( \nu_j(n;d_1,\ldots,d_r) \geq \mu_j(n;d_1,\ldots,d_r) \), Theorem 1.2 establishes an improved \( p \)-adic companion of the ℓ-adic theorem of Esnault and the first author. In the complete intersection case, they are exact analogue of each other. If \( Z \) is not a complete intersection, the divisibility bound in Theorem 1.2 can be strictly better.
What about before the middle cohomological dimension? In this range, the only known divisibility for $\ell$-adic cohomology is the theorem of Esnault–Katz which says that the divisibility is by $q^{\mu_0(n;d_1,\ldots,d_r)}$. We have an improved $p$-adic companion in this range as well.

Since $Z$ is cut out by $r$ equations, $H^i_{\text{rig},c}(Z)$ and $H^i_c(Z_{\mathbb{F}_q},\mathbb{Q}_p)$ all vanish if $i < n-r$ (see Lemma 3.1). So, we will assume that $i \geq n-r$. If $n-r = \dim Z$, i.e., $Z$ is a complete intersection, Theorem 1.2 already covers all the possible cohomological degrees $i$ such that $H^i_{\text{rig},c}(Z)$ is nontrivial. However, if $Z$ is not a complete intersection, $H^i_{\text{rig},c}(Z)$ and $H^i_c(Z_{\mathbb{F}_q},\mathbb{Q}_p)$ could be nonzero for $n-r \leq i < \dim Z$. A novelty of our approach is that we can provide improved divisibility information of Frobenius eigenvalues in these degrees as well, of course, only for rigid cohomology.

**Theorem 1.3** (Divisibility before middle dimension). Let notation be as above. For every $0 \leq m \leq \dim Z - (n-r)$, the Frobenius eigenvalues of $H^{n-r+m}_{\text{rig},c}(Z)$ are divisible by $q^{m(n;d_1,\ldots,d_r)}$ in the ring of algebraic integers, where

$$
\epsilon_m(n;d_1,\ldots,d_r) = \max \left\{ 0, \left[ \frac{n - (d_1 + \cdots + d_r - m) + d_r + d_{r+1} + \cdots + d_n}{d_1} \right] \right\}.
$$

The Frobenius eigenvalues of $H^{n-r+1+m}_{\text{rig},c}(\mathbb{A}_{\mathbb{F}_q}^n - Z)$ are divisible by $q^{m(n;d_1,\ldots,d_r)}$ in the ring of algebraic integers as well.

The numbers $\epsilon_m(n;d_1,\ldots,d_r)$ ($m = 0, 1, \ldots, \dim Z - (n-r)$) form an increasing sequence in the closed interval $[\mu_0(n;d_1,\ldots,d_r),\nu_0(n;d_1,\ldots,d_r)]$, the smallest one $\epsilon_0(n;d_1,\ldots,d_r) = \mu_0(n;d_1,\ldots,d_r)$ being responsible for the Ax–Katz theorem and the Esnault–Katz theorem; and we have $\epsilon_{\dim Z - (n-r)}(n;d_1,\ldots,d_r) = \nu_0(n;d_1,\ldots,d_r)$.

Theorems 1.2, 1.3 have projective analogues. For a closed subvariety $Z$ of $\mathbb{P}_n$, set $H^{\dim Z}_{\text{rig}}(Z)_{\text{prim}} = \text{Coker}(H^i_{\text{rig}}(\mathbb{P}_n^n) \to H^i_{\text{rig}}(Z))$.

**Theorem 1.4.** Let $f_1,\ldots,f_r \in \mathbb{F}_q[x_0,\ldots,x_n]$ be homogeneous polynomials of positive degrees $d_1 \geq \cdots \geq d_r$. Let $Z$ be the vanishing scheme of $f_1,\ldots,f_r$ in $\mathbb{P}_n^r$. Then, for $0 \leq j \leq \dim Z$,

- the Frobenius eigenvalues of $H^{\dim Z+1+j}_{\text{rig},c}(\mathbb{P}_n^r - Z)$ are divisible, as algebraic integers, by $q^{\epsilon_j(n+1;d_1,\ldots,d_r)}$;
- the Frobenius eigenvalues of $H^{\dim Z+1+j}_{\text{rig}}(\mathbb{P}_n^r - Z)$ are divisible, as algebraic integers, by $q^{\epsilon_j(n+1;d_1,\ldots,d_r)}$.

For $0 \leq m \leq \dim Z - (n-r)$,

- the Frobenius eigenvalues of $H^{n-r+m}_{\text{rig},c}(\mathbb{P}_n^r - Z)$ are divisible, as algebraic integers, by $q^{\epsilon_m(n+1;d_1,\ldots,d_r)}$;
- the Frobenius eigenvalues of $H^{n-r+1+m}_{\text{rig},c}(\mathbb{P}_n^r - Z)$ are divisible, as algebraic integers, by $q^{\epsilon_m(n+1;d_1,\ldots,d_r)}$.

These divisibility theorems in rigid cohomology now raise new questions for $\ell$-adic cohomology and Hodge theory, through the “motivic” philosophy.

- The Frobenius eigenvalues of $\ell$-adic cohomology groups of affine or projective $Z$ should also satisfy the divisibility stated in Theorems 1.2, 1.3 and 1.4.
For complex affine or projective varieties, the numbers $\nu_j(n; d_1, \ldots, d_r)$ and $\epsilon_m(n; d_1, \ldots, d_r)$ should give lower bounds for Hodge levels of compactly supported singular cohomology of complex varieties cut out by a set of polynomial equations of degrees $d_1, \ldots, d_r$.

Our method, purely analytic, is certainly not applicable for $\ell$-adic cohomology. But the chain level considerations may be useful for the problem on Hodge levels.

The paper is organized as follows. Sections 2 and 5 do not contain original contents, we simply state the comparison theorem relating Dwork cohomology and rigid cohomology, and recall some basics of the Dwork crystal. Sections 3 and 4 are devoted to the proof of the comparison theorem. In sections 6–8, we prove Theorems 1.1–1.4.

2. Dwork cohomology

**Notation.** We fix the following notation. Let $k$ be a perfect field of characteristic $p > 0$. Let $V$ be the complete discrete valuation ring $W(k)[\zeta_p]$, $\zeta_p$ being a primitive $p^{th}$ root of unity. Thus there exists a uniformizer $\pi$ satisfying $\pi^{p-1} + p = 0$. Let $K$ be the field of fractions of $V$, which is an ultrametric field. The ring $V$ admits an automorphism $\sigma$ which fixes $\pi$ and satisfies $\pi | (\sigma(c) - c^p)$.

Throughout this paper, we consider rigid cohomology over the base field $K$. For an algebraic variety $X$ over $k$, and an overconvergent F-isocrystal $E$ on $X$, $H^i_{\text{rig}}(X, E)$ means $H^i_{\text{rig}}(X/K, E)$, which is a $K$-vector space. According to Kedlaya [21], all the rigid cohomology groups considered in the sequel are finite dimensional.

**Dwork crystal.** On the structure sheaf of the rigid analytic affine line $\mathbb{A}^1_k$ (with coordinate $z$), we define an integrable connection $\nabla_\pi$ by the formula

$$\nabla_\pi \xi = d\xi + \pi \xi d\pi.$$ 

The connection is overconvergent. It is equipped with a Frobenius structure

$$F: \xi \mapsto \xi^{\sigma}(z^p) \cdot \theta(z)^{-1}$$

where $\theta(z)$ is the Dwork exponential

$$\theta(z) = \exp(\pi z - \pi z^p).$$

It is well-known that the radius of convergence of $\theta$ is $|p|^{-\frac{p-1}{p^2}}$, thus overconvergent. The pair $(\nabla_\pi, F)$ defines an overconvergent F-isocrystal on $\mathbb{A}^1_k$, called the Dwork crystal. See [22] §4.2.1 and §8.3] for more details. The dual isocrystal of $\mathcal{L}_\pi$ is $\mathcal{L}_{-\pi}$.

**Dwork cohomology.** Let $\overline{R}$ be a smooth finite type $k$-algebra, $g \in \overline{R}$. Then by taking inverse image, we get an overconvergent F-isocrystal $g^* \mathcal{L}_\pi$ on $\overline{R}$. The rigid cohomology of $\text{Spec} \overline{R}$ with coefficient in $\mathcal{L}_\pi$ will be called the “overconvergent Dwork cohomology” associated to the function $g$.

Let us present this overconvergent F-isocrystal using the method of Monsky and Washnitzer. According to Elkik [14], there is a smooth $V$-algebra $R_{\text{int}}$, such that $R_{\text{int}} \otimes_V k = \overline{R}$. Let $\tilde{g} \in R_{\text{int}}$ be a lift of $g$. Let $R^l_{\text{int}}$ be the weak completion of $R_{\text{int}}$ ([20] Definition 1.1), and let $R^l = R^l_{\text{int}} \otimes_V K$. Then the integrable connection on $R^l$ given by

$$\tilde{g}^* \nabla_\pi: R^l \to \Omega^1_{R^l/K}, \quad \xi \mapsto d\xi + \pi \xi d\tilde{g}$$
is a presentation of $g^*\mathcal{L}_x$. The cohomology of the de Rham complex of $\tilde{g}^*\nabla_x$ equals $H^\bullet_{\text{rig}}(\text{Spec } \overline{R}, g^*\mathcal{L}_x)$, the overconvergent Dwork cohomology of $g$.

**Dwork cohomology and cohomology with support.** Let us consider the following special situation: $\overline{R} = \mathbb{A}[T_1, \ldots, T_r]$ for some smooth finite type $k$-algebra $\mathbb{A}$, and $g = \sum_{i=1}^r T_i f_i$ for some $f_1, \ldots, f_r \in \mathbb{A}$. In the next two sections, we will prove the following theorem.

**Theorem 2.1.** Let $Z$ be the affine variety $\text{Spec}(\overline{A}/(f_1, \ldots, f_r))$. Then there is a natural isomorphism

$$H^\bullet_{\text{rig}}(\text{Spec}(\overline{A}[T_1, \ldots, T_r]), g^*\mathcal{L}_x) \simeq H^\bullet_{\text{rig}, Z}(\text{Spec}(\overline{A})),$$

which is compatible with Frobenius actions.

The theorem can be formulated in a more general fashion, namely $\text{Spec } \overline{A}$ can be replaced by a smooth $k$-variety $X$, $\mathbb{A}_X$ by a vector bundle over $X$, and $(f_1, \ldots, f_r)$ by a section of the bundle. But we will not need this added generality.

**Remarks.** (a) Dwork cohomology originated in Dwork’s study of zeta functions of projective hypersurfaces [12,13]. Dwork used Banach spaces rather than weakly completed algebras in the sense of Monsky–Washington. His construction was systematically generalized in the context of toric exponential sums by Adolphson and Sperber [2] (still using Banach spaces instead of weakly completed versions). Apparently, the nice properties of overconvergent Dwork cohomology was first studied by Monsky [24].

(b) Dwork and Adolphson–Sperber proved the finiteness of the cohomology spaces they considered, when the function $g$ is “Newton nondegenerate and convenient”. By contrast, their overconvergent variant, the “overconvergent Dwork cohomology”, is always finite dimensional, thanks to the finiteness theorem of Kedlaya [21].

(c) If in the definition of Dwork cohomology one uses finite type rings instead of using weakly completed algebra or Banach algebras, one gets the so-called “algebraic Dwork cohomology”. The algebraic analogue of Theorem 2.1 is well-known: it is proved by N. Katz [18] for hypersurfaces; by Adolphson–Sperber [3] for smooth complete intersections in a smooth affine variety; by Dimca et al. [10] and Baldassarri–D’Agnolo [5] in general.

3. **Comparison theorem: hypersurface case**

We begin by introducing some notation in order to explicate the recipes we need.

- Let $\overline{A}$ be a smooth $k$-algebra of pure dimension $n$. Fix an element $f \in \overline{A}$ that is not constant. Let $g = T f \in \overline{A}[T]$, regarded as a regular function $g: \mathbb{A}^1 \times_k \text{Spec } \overline{A} \to \mathbb{A}^1$.
- Fix a smooth $V$-algebra $A_{\text{int}}$ such that $A_{\text{int}}/\pi A_{\text{int}} = \overline{A}$. Fix $\tilde{f} \in A_{\text{int}}$ such that $\tilde{f} \equiv f$ modulo $\pi$. Fix a presentation $A_{\text{int}} \cong V[x_1, \ldots, x_N]/(u_1, \ldots, u_m)$ of $A_{\text{int}}$.
- For a positive real number $\rho > 1$, let

$$A_{\text{int}, \rho} = V(\frac{x_1}{\rho}, \ldots, \frac{x_N}{\rho})/(u_1, \ldots, u_m),$$
where
\[ V(\frac{x_1}{\rho}, \ldots, \frac{x_n}{\rho}) = \left\{ \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} x^\alpha : |a_{\alpha}| |\rho|^{|\alpha|} \to 0, \text{ as } |\alpha| \to \infty \right\}. \]

For \( \rho_1 > \rho_2 > 1 \) we have a restriction homomorphism \( A_{\text{int}, \rho_1} \to A_{\text{int}, \rho_2} \). Let \( A^\dagger_{\text{int}} = \operatorname{colim}_{\rho \to 1^+} A_{\text{int}, \rho} \) be the weak completion of \( A_{\text{int}} \).

- Let \( A_\rho = A_{\text{int}, \rho} \otimes_V K \), which is a Banach algebra, and \( A^\dagger = A^\dagger_{\text{int}} \otimes_V K = \operatorname{colim}_{\rho \to 1^+} A_\rho \).

Let \( D^+(0; \rho) \) be the rigid analytic space associated to the disk in \( \mathbb{A}^{1, \text{an}}_K \) of radius \( \rho \) centered at 0. Then \( A_\rho \) is the ring of rigid analytic functions on the affinoid subspace \( \{ x \in (\text{Spec } A)^{\text{an}} \cap D^+(0; \rho)^n \} : |\hat{f}(x)| \geq \eta^{-1} \} \).

Its norm is denoted by \( | \cdot | \) or sometimes \( | \cdot |_\rho \), to denote the Banach norm of \( A_\rho \) obtained from the quotient \( V(\frac{x_1}{\rho}, \ldots, \frac{x_n}{\rho}) \to A_{\text{int}, \rho} \).

- For a positive real number \( \eta > 1 \), let \( C^{\geq \eta^{-1}}_\rho \) denote the Banach algebra of rigid analytic functions on the affinoid subspace \( \{ x \in (\text{Spec } A)^{\text{an}} \cap D^+(0; \rho)^n \} : |\hat{f}(x)| \geq \eta^{-1} \} \).

The ring \( C^\dagger \) furnishes as a \( p \)-adic incarnation of \( \mathbb{A}[f^{-1}] \).

As Banach spaces, \( A_\rho \) is a closed subspace of \( C^{\geq \eta^{-1}}_\rho \).

- Let \( B_{\rho, \eta} \) be the quotient space \( C^{\geq \eta^{-1}}_\rho / A_\rho \), which is also a Banach space over \( K \). An element of \( B_{\rho, \eta} \) can be expressed as a formal sum
\[
(3.1) \quad \sum_{m=1}^{\infty} c_m \hat{f}^{-m}, \quad c_m \in A_\rho, \quad \text{such that } |c_m|_\rho \eta^m \to 0 \text{ as } m \to \infty.
\]

Define \( B^\dagger = \operatorname{colim}_{\eta \to 1^+, \rho \to 1^+} B_{\rho, \eta} \). This is the \( p \)-adic analogue of the “algebraic local cohomology”. In the language of rigid geometry, we have
\[ B^\dagger[-1] = R\Gamma_Z(\mathcal{O}). \]

See [22, Proposition 5.2.21].

The spaces \( A_\rho, C^{\geq \eta^{-1}}_\rho, A^\dagger, C^\dagger, B_{\rho, \eta} \) and \( B^\dagger \) are equipped with actions from differential operators (of finite order). Locally, supposing we have a fixed étale coordinate of \( A_{\text{int}} \) given by \( x_1, \ldots, x_n \), then the spaces above are equipped with mutually commuting derivations
\[ E_i = \frac{\partial}{\partial x_i}, \quad i = 1, 2, \ldots, n. \]

In this situation, the \((i-n)^{\text{th}}\) cohomology of the Koszul complex \( \text{Kos}(B^\dagger; E_1, \ldots, E_n) \)
\[ \cdots \to (B^\dagger)^{(2)} \to (B^\dagger)^{(1)} \to B^\dagger \]
(here the dot is indicating the item is placed in cohomological degree 0) is by definition the \( i^{\text{th}} \) rigid cohomology of \( \text{Spec } \mathbb{A} \) with supports in the hypersurface \( Z \).
The rigid cohomology of Spec $\mathcal{A}$ is computed by the cohomology of the shifted Koszul complex $\text{Kos}(A^1; E_1, \ldots, E_n)[-n]$; and the rigid cohomology of Spec $\mathcal{A} - Z$ is computed by the shifted Koszul complex $\text{Kos}(C^1; E_1, \ldots, E_n)[-n]$.

For a positive real number $\eta > 1$, let $A_\rho \langle T_\eta \rangle$ be the Banach algebra

$$\left\{ \sum_{m \in \mathbb{N}} a_m T^m : |a_m|_\rho \eta^m \to 0 \text{ as } m \to \infty, \ a_m \in A_\rho \right\}$$

equipped with the $\eta$-Gauss norm. Define $A_\rho \langle T_\eta \rangle^\dagger = \colim_{\rho \to 1^+} A_\rho \langle T_\eta \rangle$.

The algebras $A_\rho \langle T_\eta \rangle$ and $A_\rho \langle T_\eta \rangle^\dagger$ both underlie the integrable connection $d + \pi \hat{d}$.

When expressed in terms of the étale coordinate $(x_1, \ldots, x_n, T)$ the connection is determined by $(d + 1)$ mutually commuting differential operators

$$D_i = \frac{\partial}{\partial x_i} - \pi T \frac{\partial}{\partial x_i} \hat{f} = \exp(\pi T \hat{f}) \circ \frac{\partial}{\partial x_i} \circ \exp(-\pi T \hat{f}), \ i = 1, 2, \ldots, n;$$
$$D = \frac{\partial}{\partial T} - \pi \hat{f} = \exp(\pi T \hat{f}) \circ \frac{\partial}{\partial T} \circ \exp(-\pi T \hat{f}).$$

Under an étale coordinate, the $i$th overconvergent Dwork cohomology is then the $(i - (n + 1))$th cohomology of the Koszul complex $\text{Kos}(A^1 \langle T \rangle^\dagger; D_1, \ldots, D_n, D)$;

$$\cdots \to (A^1 \langle T \rangle^\dagger)^{\oplus(n+1)} \to (A^1 \langle T \rangle^\dagger)^{\oplus(n+1)} \to A^1 \langle T \rangle^\dagger.$$

Now we can state the main theorem of this section.

**Theorem 3.3.** There is a natural quasi-isomorphism

$$B^\dagger \to \left[ A^1 \langle T \rangle^\dagger \xrightarrow{D} A^1 \langle T \rangle^\dagger \right]$$

Moreover, in any étale coordinate system, the residual actions of the differential operators $D_1, \ldots, D_n$ on the right hand side and the actions $E_1, \ldots, E_n$ on the left hand side agree.

**Proof of Theorem 3.3.** For any pair of numbers $\rho > 1$ and $\eta > 1$, form the following commutative ladder of exact sequences

$$0 \xrightarrow{D} \quad A^1 \langle T_\eta \rangle^\dagger \quad \xrightarrow{D_\rho} \quad C^\geq 0_{\rho} \langle T_\eta \rangle^\dagger \quad \xrightarrow{D_B} \quad C^\geq 0_{\rho} \langle T_\eta \rangle^\dagger \quad \xrightarrow{D_B} \quad 0$$

In the diagram, $B_{\rho, \eta} \langle T_\eta \rangle$ is defined to be the Banach space of formal sums

$$\left\{ \sum_{m=0}^{\infty} b_m T^m : b_m \in B_{\rho, \eta}, |b_m| \eta^m \to 0 \text{ as } m \to \infty \right\},$$
equipped with the $\eta$-norm; the middle differential $\delta$ is still defined as in (3.2); the vertical map $D_B$ is induced by the commutativity of the left square.
Taking colimit with respect to $\rho \to 1^+$ and $\eta \to 1^+$ gives an “overconvergent” version of the ladder \[(3.4')\]

\[
\begin{array}{cccccc}
0 & \rightarrow & A^\dagger(T)^\dagger & \rightarrow & C^\dagger(T)^\dagger & \rightarrow & B^\dagger(T)^\dagger & \rightarrow & 0 \\
\downarrow D & & \downarrow \tilde{D} & & \downarrow \sigma_h & & & \\
0 & \rightarrow & A^\dagger(T)^\dagger & \rightarrow & C^\dagger(T)^\dagger & \rightarrow & B^\dagger(T)^\dagger & \rightarrow & 0 \\
\end{array}
\]

The idea of the proof of Theorem 3.3 can be explained as follows. First, we show the middle column of \[(3.4')\] is exact. We will do so by proving the middle column of \[(3.4)\] is exact, see Lemma 3.7. Second, we show that the kernel of the right column is isomorphic to $B^\dagger$, see Lemma 3.13. The theorem will follow from the snake lemma.

**Lemma 3.5.** For any $\rho > 1$, $\eta > 1$, the map $D: A_\rho(T) \rightarrow A_\rho(T)$ is injective.

**Proof.** It suffices to show

$$
\exp(\pi \hat{T} \hat{f}) = \sum_{m=0}^{\infty} \frac{\pi^m \hat{f}^m}{m!} \notin A_\rho(T)
$$

Since the 1-norm of $\hat{f}$ is 1 (the reduction of $\hat{f}$ is nonzero), $|\hat{f}|_1 \geq 1$. Since $|\pi|^m \sim |m!|$, and $\eta > 1$, we have

\[
(3.6) \quad \frac{|\pi|^m}{|m!|} \cdot |\hat{f}^m|_1 \cdot \eta^m \rightarrow \infty.
\]

The lemma is proved. \hfill \Box

**Lemma 3.7.** For any $\rho \geq 1, \eta > 1$, the map $\tilde{D}$ in \[(3.4)\] is bijective.

**Proof.** The same reasoning as in the proof of Lemma 3.5 shows that the map $\tilde{D}$ is injective. Write $\tilde{D} = \frac{\partial}{\partial T} - \pi \hat{f}$. It is tempting to prove the surjectivity of $\tilde{D}$ by showing the operator norm of $\partial/\partial T$ is less than the operator norm of the multiplication-by-$\pi \hat{f}$ map, so by perturbation the surjectivity of $\tilde{D}$ will be a consequence of that of multiplication-by-$\pi \hat{f}$. Unfortunately, this will only work for the specific radius $\eta = |\pi|^{-1}$. To fix this, we will prove the surjectivity of $\frac{1}{\eta^e} \tilde{D}^e$ for some $e > 1$, which will imply the surjectivity of $\tilde{D}$. For $e$ large, the perturbation argument will work. This technique is learned from [26, Proposition 10.1.3].

For a linear operator $\Psi$ on $C_{\rho}^{\geq \eta^{-1}}(T)$, we denote by $||\Psi||$ the operator norm of $\Psi$. For a series $u \in C_{\rho}^{\geq \eta^{-1}}(T)$, we still denote by $u$ the multiplication operator $v \mapsto uv$. In this case, $||u||$ equals $|u|$.
Recall formally \( \tilde{D} = \exp(\pi T \hat{f}) \circ \frac{\partial}{\partial T} \circ \exp(-\pi T \hat{f}) \). Then
\[
\frac{1}{e!} \tilde{D}^e = \exp(\pi T \hat{f}) \circ \frac{1}{e!} \left( \frac{\partial}{\partial T} \right)^m \circ \exp(-\pi T \hat{f})
\]
\[
= \sum_{i=0}^{e} \frac{1}{(e-i)!} \frac{\partial^{e-i} \exp(-\pi T \hat{f})}{\partial T^{e-i}} \frac{1}{i!} \left( \frac{\partial}{\partial T} \right)^i.
\]
\[
= \sum_{i=0}^{e} \frac{1}{(e-i)!} (-\pi \hat{f})^{e-i} \cdot \frac{1}{i!} \left( \frac{\partial}{\partial T} \right)^i
\]
\[
= \frac{1}{e!} (-\pi \hat{f})^e + R_e.
\]

**Claim.** If \( e \) is sufficiently large, the operator norm of \( R_e \) is strictly smaller than \( \left\| \frac{1}{e!} (-\pi \hat{f})^e \right\| \) on the Banach space \( C^{\geq \eta^{-1}}(\mathbb{T}_n) \).

Granting the claim, let us prove the lemma. Noticing that the operator \( \frac{1}{e!} (-\pi \hat{f})^e \) is invertible, and the operator norm of its inverse is precisely the inverse of its operator norm, we apply the perturbation theorem in functional analysis [26, Proposition 7.2.3] to conclude that the operator \( \frac{1}{e!} \tilde{D}^e \) is also surjective. This proves the theorem.

Let us now prove \( \left\| R_e \right\| < \left\| \frac{1}{e!} (-\pi \hat{f})^e \right\| \) for \( e \) large. Using equation (3.8), for any \( \eta > 1 \), there exists \( e \) such that
\[
\left( \frac{\pi^e}{e!} \right)^{\hat{f}} \bigg| \bigg|_{\rho} \eta^e \geq \max_{1 \leq i \leq e} \left| \frac{\pi^{e-i}}{(e-i)!} \hat{f}^{e-i} \right|_{\rho} \eta^{e-i}.
\]
Moreover, the norm of \( \hat{f} \) in \( C^{\geq \eta^{-1}}(\mathbb{T}_n) \), which equals the supremum of \( \hat{f} \) on the “annulus” \( \{ x \in D^+(0; \rho)^n : |\hat{f}(x)| \geq \eta^{-1} \} \), also equals the \( \rho \)-norm of \( \hat{f} \). On the other hand, for any \( u \in C^{\geq \eta^{-1}}(\mathbb{T}_n) \), we have
\[
\left| \frac{1}{i!} \left( \frac{\partial}{\partial T} \right)^i u \right| \leq |u| \eta^{-i}.
\]
Combining (3.8) and (3.9), we see
\[
\left\| R_e \right\| \leq \sup_{1 \leq i \leq e} \left| \frac{\pi^{e-i}}{(e-i)!} \right| \cdot \left| \hat{f}^{e-i} \right|_{\rho} \cdot \eta^{-i} < \left\| \frac{1}{e!} (\pi \hat{f})^e \right\|.
\]

The claim is proved.

**Lemma 3.10.** For any \( \rho > 1, \eta > 1 \), we have \( \text{Coker } D \simeq \text{Ker } DB \) in (3.4).

**Proof.** Apply the snake lemma to (3.4) and use Lemma 3.7.

**Lemma 3.11.** In (3.4), we have \( \text{Ker } DB \simeq B_{\rho,\eta} \) for any \( \rho > 1, \eta > 1 \).

**Proof.** For an element \( b \in B_{\rho,\eta} \), denote by \( |b|_{B_{\rho,\eta}} \) its Banach norm obtained from the quotient structure. See (3.1).

Let \( \sum_{m=0}^{\infty} b_m T^m \) be an element in the kernel of \( DB \). Then, by induction, we can write
\[
b_m = \frac{(\hat{f} \pi)^m}{m!} b_0.
\]
In other words, the series is completely determined by \( b_0 \in B_{\rho,\eta} \).
It remains to show that for any $b \in B_{\rho,\eta}$, the sum $\sum (\hat{f}_m / m!) bT^m$ makes sense in $B_{\rho,\eta}(\mathbb{L}_{\eta})$. We need to show:

$$\frac{|\pi|^m}{|m!|} \cdot \hat{f}^m \cdot b \bigg|_{B_{\rho,\eta}} \eta^m \to 0.$$ 

By definition, we may represent any element $b$ of $B_{\rho,\eta}$ as a sum $\sum_{j=1}^{\infty} c_j \hat{f}^{-j}$ as in (3.1), such that $|c_m|_{\rho} \eta^m$ converges to zero. The product $\hat{f}^m b$ is then represented by the formal sum $\sum_{j=m+1}^{\infty} c_j \hat{f}^{m-j}$.

By definition of the quotient norm, we have

$$|b|_{B_{\rho,\eta}} \leq \sup_{m} |c_m|_{\rho} \eta^m.$$ 

Since $|\hat{f}^{-1}|_{C^{\infty}_{\rho,\eta}} \leq \eta$, the property of quotient norm yields $|(\hat{f}^{-1})^j m|_{B_{\rho,\eta}} \leq \eta^{j-m}$. Thus

$$|\hat{f}^m \cdot b|_{B_{\rho,\eta}} \cdot \eta^m \leq \sup_{j>m} |c_j|_{\rho} \eta^j.$$ 

Since $|c_m|_{\rho} \eta^m$ converges to zero, $\sup_{j>m} |c_j|_{\rho} \eta^j$ converges to zero as $m \to \infty$ as well. We win by using the asymptotic property $|\pi|^m \sim |m|!$. □

Let $B^\dagger(T)^\dagger$ be the space obtained from $B_{\rho,\eta}(\mathbb{L}_{\eta})$ by taking colimit with respect to $\eta \to 1^{+}$ and $\rho \to 1^{+}$. The derivations $D_B$ extend to a derivation

$$D^\dagger_B: B^\dagger(T)^\dagger \to B^\dagger(T)^\dagger.$$ 

Since taking filtered colimit with respect to vector spaces is an exact operator, Lemma 3.11 implies that

$$\text{Coker}(D: A^\dagger(T)^\dagger \to A^\dagger(T)^\dagger) \simeq \text{Ker} \ D^\dagger_B.$$ 

Taking colimit $\rho \to 1^{+}$, $\eta \to 1^{+}$ in Lemma 3.11 gives the following lemma.

**Lemma 3.13.** There is a natural isomorphism $B^\dagger \simeq \text{Ker} \ D^\dagger_B$. □

The module $B^\dagger(T)^\dagger$ also receives the action of the differential operators $D_i$ defined as in (3.2), which commute with $D^\dagger_B$. These operators induce an action on the space $B^\dagger$ via the isomorphism provided by Lemma 3.13. We now verify that these actions agree with the usual action by $E_i$. This will complete the proof of Theorem 3.3.

**Lemma 3.14.** The isomorphism provided by Lemma 3.13 takes the operator $D_i$ to $E_i$.

**Proof.** Each $b \in B^\dagger$ corresponds to the formal sum $\sum \frac{1}{m!}(\pi \hat{f})^m bT^m$, which will be denoted by $\text{Init}(b)$. It suffices to prove $\text{Init}(\partial b/\partial x_i)$ equals the constant term of $D_i(\text{Init}(b))$. But this is a rather simple computation. □

## 4. Comparison theorem: general case

Generalizing Theorem 3.3 to the general case will only use general nonsense. Let us first set up the notation.
Notation. Let $\overline{A}$ be a smooth $k$-algebra. Let $f_1, \ldots, f_r$ be a collection of elements in $\overline{A}$. Let $Z_i$ be the hypersurface cut out by $f_i$ in Spec $\overline{A}$, and $Z = \bigcap_{i=1}^r Z_i$. We do not assume $Z$ is a complete intersection of $r$ equations; its codimension is usually smaller than $r$. Let $g(x,T) = \sum_{i=1}^r T_i f_i(x)$.

For each $i$, choose a lift $\hat{f}_i \in A\text{int}$ of $f_i$. In the algebra $A\langle T_1, \ldots, T_r \rangle^\dagger$, we have a lift $\hat{g} = \sum_{i=1}^r T_i \hat{f}_i$ of $g$.

Let us explain how to compute the overconvergent Dwork cohomology $H^{\cdot}_{\rig}(\text{Spec} \overline{A} \times_k A, g^*L - \pi)$. The algebra $A\langle T_1, \ldots, T_r \rangle^\dagger$ is equipped with an integrable connection given by $d + \pi \hat{\partial}_{\hat{g}}$.

When an étale coordinate is given, the de Rham complex of this integrable connection can be expressed as a Koszul complex of the following commuting differential operators:

$$D_i = \frac{\partial}{\partial x_i} - \pi \sum_{j=1}^r T_j \frac{\partial}{\partial x_i} \hat{f}_j, \quad i = 1, 2, \ldots, n \quad (4.1)$$

as well as

$$(4.2) \quad D^{(j)} = \frac{\partial}{\partial T_j} - \pi \hat{f}_j, \quad j = 1, 2, \ldots, r.$$ 

Then the overconvergent Dwork cohomology $H^{i}_{\rig}(\text{Spec} \overline{A} \times_k A, g^*L - \pi)$ is just the $(i - (n + r))^{th}$ cohomology of the Koszul complex

$$\text{Kos}(A\langle T_1, \ldots, T_r \rangle^\dagger; D^{(1)}, \ldots, D^{(r)}, D_1, \ldots, D_n).$$

Temporarily ignoring the operators $D_i$, let us compute the partial Koszul complex

(4.2) \quad \text{Kos}(A\langle T_1, \ldots, T_r \rangle^\dagger; D^{(1)}, \ldots, D^{(r)}).

Note that this is the weakly completed tensor product of the Koszul complexes:

$$\text{Kos}(A\langle T_1, \ldots, T_r \rangle^\dagger; D^{(1)}, \ldots, D^{(r)}) = \text{Kos}(A\langle T_1 \rangle^\dagger, D^{(1)}) \otimes_{A^\dagger} L_{r+1}^\dagger \otimes_{A^\dagger} \text{Kos}(A\langle T_r \rangle^\dagger, D^{(r)}).$$

Recall that if $R = \colim_{\rho \to 1} R_{\rho}$ is a weakly completed, finitely generated algebra, and $M = \colim_{\rho \to 1} M_{\rho}, N = \colim_{\rho \to 1} N_{\rho}$ are finitely generated $R$-modules which are “overconvergent”, then $M \otimes_R^\dagger N_{\rho}$ is defined as the colimit of the completed tensor products

$$\colim_{\rho \to 1} M_{\rho} \otimes_R^\dagger N_{\rho}.$$ 

When $M$ and $N$ are $R$-algebras, the weakly completed product is also an $R$-algebra.

Now we can state the theorem that generalizes Theorem 3.3 to a general subvariety $Z$.

Theorem 4.3. There is a natural quasi-isomorphism between the partial Koszul complex

$$\text{Kos}(A\langle T_1, \ldots, T_r \rangle^\dagger; D^{(1)}, \ldots, D^{(r)})$$
and $\text{RT}_{Z_1 \cap \cdots \cap Z_r}(\mathcal{O})[r]$, the “derived incarnation” of the rigid cohomology of $\text{Spec } \overline{A}$ supported in $Z$. In addition, this quasi-isomorphism is compatible with the derivations on $A^\dagger$.

**Proof.** By Lemmas 3.5, 3.12, and 3.13, we conclude that this Koszul complex is computed by the weakly completed tensor product

$$\tilde{B}^\dagger = B^\dagger_{A^\dagger} \otimes_{A^\dagger} \cdots \otimes_{A^\dagger} B^\dagger_{r}.$$  

Here $B^\dagger_{i}$ is the space of overconvergent hyperfunctions along $Z_i$. Indeed, for two closed subvarieties $Y_1$ and $Y_2$ of a variety $X$, one has [22, Proposition 5.2.4(iii) and projection formula]

$$\text{RT}_{Y_1} \otimes \text{RT}_{Y_2} = \text{RT}_{Y_1 \cap Y_2}.$$  

Ergo, $\tilde{B}^\dagger \simeq \text{RT}_{Z_1 \cap \cdots \cap Z_r}(\mathcal{O})[r]$.

Then, it remains to check, as in Lemma 3.14, that the derivations induced by $D_1, \ldots, D_n$ on this tensor product via the isomorphism agree with $E_1, \ldots, E_n$ respectively. By definition, the shifted Koszul complex of $\tilde{B}^\dagger$ with respect to $E_1, \ldots, E_n$ is the rigid cohomology of $\text{Spec } \overline{A}$ supported on $Z = Z_1 \cap \cdots \cap Z_r$. See [22, Definition 5.2.3] for the definition of cohomology with support (sheaf level); and [22, Proposition 5.2.21] for the result on rigid cohomology (in our circumstance, $E = (\mathcal{O}, d)$ is the trivial connection). □

Taking the de Rham cohomology functor on $A^\dagger$, with respect to the $\text{RT}_{Z}(\mathcal{O})$ and the partial Koszul complex $\text{Kos}(A^\dagger(T_1, \ldots, T_r)^\dagger; D^{(1)}, \ldots, D^{(r)})$ gives the following comparison theorem of rigid cohomology spaces.

**Corollary 4.4.** There is a natural isomorphism

$$u: H^i_{\text{rig}}(\text{Spec } \overline{A} \times_k \mathbb{A}^r_k; g^*\mathcal{L}_{-\pi}) \xrightarrow{\sim} H^i_{\text{rig}, Z}(\text{Spec } \overline{A}).$$  

Taking Poincaré duality yields an isomorphism

$$v: H^i_{\text{rig}, c}(Z) \xrightarrow{\sim} H^{i+2r}_{\text{rig}, c}(\text{Spec } \overline{A} \times_k \mathbb{A}^r_k; g^*\mathcal{L}_{\pi}).$$  

Our next objective is to discuss the Frobenius actions on the two rigid cohomology spaces. To this end, recall $\sigma: V \to V$ is a ring homomorphism such that $\pi$ divides $\sigma(e) - e^\sigma$ in $V$, and $\sigma(\pi) = \pi$. Denote also by $\sigma$ its extension to $K$. Both the rigid cohomology with support in $Z$ and the overconvergent Dwork cohomology admit $\sigma$-semilinear Frobenius actions.

**Theorem 4.5.** Then the isomorphism $u$ is an isomorphism of $K$-vector spaces with $\sigma$-linear semilinear maps.

**Proof.** Let $\phi: A^\dagger \to A^\dagger$ be a lift of the Frobenius. It induces a Frobenius operator on the spaces $B^\dagger_{i}$, $i = 1, 2, \ldots, r$, and on the tensor product $\tilde{B}^\dagger$. The operator on $\tilde{B}^\dagger$ induces the Frobenius on the rigid cohomology $H^i_{\text{rig}, Z}(\text{Spec } \overline{A})$.

Extend the operator $\phi$ to $A^\dagger(T_1, \ldots, T_r)^\dagger$ by

$$\phi \left( \sum_{I \subseteq \mathbb{N}^r} a_I(x)T^I \right) = \sum_{I \subseteq \mathbb{N}^r} \phi(a_I(x))T^I.$$  

Taking Poincaré duality yields an isomorphism

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Our next objective is to discuss the Frobenius actions on the two rigid cohomology spaces. To this end, recall $\sigma: V \to V$ is a ring homomorphism such that $\pi$ divides $\sigma(e) - e^\sigma$ in $V$, and $\sigma(\pi) = \pi$. Denote also by $\sigma$ its extension to $K$. Both the rigid cohomology with support in $Z$ and the overconvergent Dwork cohomology admit $\sigma$-semilinear Frobenius actions.

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Extend the operator $\phi$ to $A^\dagger(T_1, \ldots, T_r)^\dagger$ by

$$\phi \left( \sum_{I \subseteq \mathbb{N}^r} a_I(x)T^I \right) = \sum_{I \subseteq \mathbb{N}^r} \phi(a_I(x))T^I.$$
The Frobenius operator on the (inverse image of the) Dwork crystal $g^*\mathcal{L}$ is presented by a $\phi$-semilinear map

$$\varphi: A^1(T_1)^\dagger \otimes_{A^1} \cdots \otimes_{A^1} A^1(T_r)^\dagger \to A^1(T_1)^\dagger \otimes_{A^1} \cdots \otimes_{A^1} A^1(T_r)^\dagger,$$

which can be chosen as $\varphi(\xi) = \phi(\xi) \cdot \bigotimes_{i=1}^r \theta_i(x, T_i)$, where $\theta_i(x, T_i)$ is the overconvergent series $\exp(\pi T_1 f_i - \pi T_i^p \phi(f_i))$. We must check that under the isomorphism given in Lemma 3.13 the operator $\varphi$ is transformed to the action of $\phi$. It suffices to check this on each individual factor.

Recall the map $b \mapsto \text{Init}(b)$ defined in the proof of Lemma 3.14. For $b \in B_1^\dagger$, $\text{Init}(b) = \sum b \cdot \frac{\pi^m f_m}{m!} T_i^m \in B^\dagger(T_i)$ fall in the kernel of $D_1^\dagger$. Then

$$\varphi(\text{Init}(b)) = \theta_i(x, T_i) \cdot \phi(b) \cdot \sum \frac{\pi^m \phi(f_i)^m}{m!} T_i^m.$$

Since the constant term of $\theta_i(x, T_i)$ is 1, the constant term of $\varphi(\text{Init}(b))$ equals $\phi(b)$. Since it is still horizontal, it has to be equal to $\text{Init}(\phi(b))$. This completes the verification.

The theorem also implies a result on cohomology with compact supports by taking Poincaré duality. Before stating it let us recall some standard notation about Tate twists.

For an integer $w$, let $K(w)$ be the 1-dimensional vector space over $K$ equipped with a $\sigma$-semilinear map $x \mapsto p^{-w} \sigma(x)$. For a finite dimensional $K$-vector space $M$ equipped with a $\sigma$-semilinear map, we use $M(w)$ to denote the tensor product $M \otimes_K K(w)$, whose $\sigma$-semilinear map is twisted by $p^{-w}$. Note that $(M(w))^* \simeq M^*(-w)$ as spaces equipped with $\sigma$-semilinear maps.

**Corollary 4.6.** Then the isomorphism $\nu$ of Corollary 4.4 induces an isomorphism

$$H^i_{\text{rig,c}}(Z)(-r) \simeq H^{i+2r}_{\text{rig,c}}(A^r_{/X} g^*\mathcal{L}_x).$$

**Proof.** Poincaré duality in rigid cohomology (see [22 Corollary 8.3.14], and [21 Theorem 1.2.3]; note the latter article ignores the Tate twist) asserts that, for a nonsingular, connected variety $X$ of pure dimension $d$ over $k$, and an overconvergent $F$-isocrystal $E$ on $X$, the pairing

$$\text{H}^i_{\text{rig,c}}(X, E) \otimes_K \text{H}^{2d-i}_{\text{rig,c}}(X, E^\vee) \to \text{H}^{2d}_{\text{rig,c}}(X) \simeq K(-d)$$

is a perfect pairing. In other words,

$$\text{H}^i_{\text{rig,c}}(X, E^\vee) \simeq (\text{H}^{2d-i}_{\text{rig,c}}(X, E)(d))^*.$$

Similarly, for any closed subvariety $Z$ of $X$, we have a perfect pairing

$$\text{H}^{i}_{\text{rig,c}}(X) \otimes_K \text{H}^{2d-i}_{\text{rig,c}}(Z) \to K(-d).$$

Applying these facts to the overconvergent Dwork cohomology yields

$$\text{H}^{i+2r}_{\text{rig,c}}(A^r_{/X} g^*\mathcal{L}_x) \simeq (\text{H}^{2n-i}_{\text{rig,c}}(A^r_{/X} g^*\mathcal{L}_x)(n + r))^*$$

(by Theorem 4.5) \simeq (\text{H}^{2n-i}_{\text{rig,c}}(\text{Spec} \mathcal{X})(n + r))^* = (\text{H}^{2n-i}_{\text{rig,c}}(\text{Spec} \mathcal{X})(n))^*(-r) \simeq \text{H}^{i}_{\text{rig,c}}(Z)(-r).

This completes the proof. \qed
5. The Artin–Hasse presentation

From this section onward, we assume in addition that \( k = \mathbb{F}_q \) is a finite field, where \( q = p^a \) is a power of \( p \). Thus \( K = W(\mathbb{F}_q)(\zeta_p) \) is the unramified extension of \( \mathbb{Q}_p(\zeta_p) \) of degree \( a \).

Let \( g \in k[x_1, \ldots, x_N] \) be a polynomial. The main purpose of this section is to explain another presentation of the Dwork crystal \( g^* \mathcal{L}_g \) in terms of the Artin–Hasse exponentials.

To begin with, we define a set of operators which will play the role of “inverse Frobenii”. The reader is referred to Dwork’s article [12, §4] for the assertions made in this paragraph. Recall that the Artin–Hasse exponential is defined by

\[
E(z) = \exp \left\{ \sum_{m=0}^{\infty} \frac{z^p^m}{p^m} \right\}.
\]

Then \( E(z) \in \mathbb{Z}_p[[z]] \). Let \( \gamma \) be a root of the power series \( \sum_{m=0}^{\infty} z^p^m \) satisfying \( |\gamma| = |\pi| \) (the existence of \( \gamma \) is proven by examining the Newton polygon, for example). Let \( \vartheta(z) = E(\gamma t) \). Since \( E \in \mathbb{Z}_p[[z]] \), \( \vartheta(z) \) defines a rigid analytic function on the open disk \( \{|z| < |p|^{\frac{1}{p-1}}\} \) bounded by 1. For each choice of \( \gamma \), it is well-known that

\[
(5.1) \quad \Psi(t) = \vartheta(1)^T_{\mathbb{F}_q/p_\mathbb{F}_q}(t)
\]

is a nontrivial additive character on \( \mathbb{F}_q \).

For \( b > 0 \), let \( B(b) \) be the Banach space of analytic functions on the closed polydisk \( \mathbb{D}^+(0;|p|^{-b})^N = \{|z_i| \leq |p|^{-b}\} \subset A_K^{N,\text{an}} \). Then \( B = \colim_{b \to 0^+} B(b) \) equals the Monsky–Washnitzer algebra in \( N \) variables.

For each subset \( I \) of \( S = \{1, 2, \ldots, N\} \), let \( x^I \) be the product \( \prod_{i \in I} x_i \), and let \( B_I = x^I B \), which is a closed subspace of \( B \). Similarly one can define \( B_I(b) \).

Next we define operators which operate on \( B_I \). Write \( g(x) = \sum u a_u x^u \). Let \( A_u \) be the Teichmüller lift of \( a_u \). Set \( \hat{g}(x) = \sum A_u x^u \). Define

\[
G(x) = \prod_u \vartheta(A_u x^u).
\]

Then it is easy to see that \( G \) falls in \( B(b) \) for any \( b < \frac{1}{d(p-1)} \), where \( d \) is the degree of the polynomial \( g \).

Define an operator \( \psi : B(b) \to B(pb) \) by

\[
\psi \left( \sum a_u x^u \right) = \sum a_{pu} x^{wu}.
\]

For \( 0 < b < \frac{p}{d(p-1)} \), we define an endomorphism \( \alpha_0 \) of \( B(b) \) by

\[
(5.2) \quad \alpha_0 : B(b) \xrightarrow{G} B(p^{-1}b) \xrightarrow{\sigma^{-1}\psi} B(b).
\]

Then \( \alpha_0 \) is \( \mathbb{Q}_p(\zeta_p) \)-linear, and is \( \sigma^{-1} \)-semilinear over \( K \). Its \( a^k \)th iteration \( \alpha_0^a \), which is \( K \)-linear, is denoted by \( \alpha \).

The operators \( \alpha_0 \) and \( \alpha \) both preserve the spaces \( B_I(b) \). Passing to colimit, \( \alpha_0 \) and \( \alpha \) induce \( \sigma^{-1} \)-semilinear resp. linear endomorphisms on the spaces \( B_I \). We will
informally refer to \( \alpha \) as the “inverse Frobenius” operator, although it is, on the chain level, only a left inverse.

**Remark 5.3.** For suitable \( b \), the decomposition \((5.2)\) implies that operators \( \alpha|_{B_I(b)} \) are completely continuous. By Monsky’s theorem [24, Theorem 2.1], each \( \alpha|_{B_I} \) is then a nuclear operator, and its Fredholm determinant \( \det(1 - t \cdot \alpha|_{B_I}) \) is a \( p \)-adic entire function. Moreover, if \( b \) is not too large, the infinite matrices of \( \alpha \) on all the spaces \( B_I(b) \) are similar (one sees this by choosing suitable scalar multiples of monomials as orthonormal basis). It follows that the Fredholm determinant \( \det(1 - t \cdot \alpha|_{B_I}) \) equals \( \det(1 - t \cdot \alpha|_{B_I(b)}) \).

Next, let us define another integrable connection similar to the integrable connection presenting the overconvergent \( F \)-isocrystal \( g^*\mathcal{L}_\pi \). We call this complex the “Artin–Hasse presentation”.

As the reader will see, compared with the usual connection presenting \( g^*\mathcal{L}_\pi \), the connection operator of the Artin–Hasse presentation are transcendental; and is only defined on a proper subspace of entire rigid analytic affine line. However, it has the virtue that its “inverse Frobenius” operator has better convergence property, which is crucial for the chain level argument of §8.

The reader is referred to the articles [1, 2, 12] for more discussions (including motivations) about the following definitions.

Write \( \gamma^\ell = \sum_{i=0}^N c_i^\ell \), \( h(t) = \sum_{\ell=0}^\infty \gamma_\ell t^\ell \), and

\[
H(x) = \sum_{\ell=0}^\infty \gamma_\ell \gamma_\ell^\ell (x^\ell).
\]

Then \( H \in B(b) \) for any \( b < \frac{1}{a(d(p-1))} \), and \( x_i \frac{\partial}{\partial x_i} H \in B(b) \) for any \( 0 < b < \frac{p}{a(d(p-1))} \).

Let

\[
D_i = x_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_i} H.
\]

Then one checks that \( D_i \circ D_j = D_j \circ D_i \), and

\[
\alpha \circ D_i = qD_i \circ \alpha.
\]

Let \( \nabla_g = \sum_{i=1}^N D_i \cdot \frac{dx_i}{x_i} \). Then \( \nabla_g \) is an integrable connection on the rigid analytic closed disk \( D^+(0; |p|^{-b})^N = \{ |x| \leq |p|^{-b} \} \).

The space of \( i \)-forms on the rigid analytic closed disk can be written as a direct sum

\[
\Omega_{D^+(0; |p|^{-b})^N}^i \cong \bigoplus_{|I|=i} B_I(b) \frac{dx^I}{x^I}
\]

where \( \frac{dx^I}{x^I} \) the wedge power of \( \frac{dx_i}{x_i} \), in the increasing order. Thus the de Rham complex of \( \nabla_g \) can be written as:

\[
B(b) \to \bigoplus_{|I|=1} B_I(b) \frac{dx^I}{x^I} \to \bigoplus_{|I|=2} B_I(b) \frac{dx^I}{x^I} \to \cdots \to B_{\{1, 2, \ldots, N\}}(b) \frac{dx_1 \wedge \cdots \wedge dx_N}{x_1 \cdots x_N}.
\]
Regarding the monomial forms $dx^I/x^I$ as a “placeholders”, and suppressing them in (5.7) above, we can express the de Rham complex of $\nabla g$ in a simpler form:

$$(5.7') \quad B(b) \to \bigoplus_{|I|=1} B_I(b) \to \bigoplus_{|I|=2} B_I(b) \to \cdots.$$ 

Remark 5.8. The operators $D_i$ (5.5) are defined using the “toric convention”. Their “affine” counterparts are $x^{-1}D_i$. Using the affine convention, the complex (5.7′) is simply the shifted Koszul complex $\text{Kos}(B(b); x_1^{-1}D_1, \ldots, x_N^{-1}D_N)[-N]$. The reason that we use the “toric” convention is because the following “inverse Frobenius operator” can be defined more naturally.

On (5.7′), the operator

$$\bigoplus_{|I|=i} q^{-i} \alpha|_{B_I(b)} : \bigoplus_{|I|=i} B_I(b) \to \bigoplus_{|I|=i} B_I(b)$$

acts (thus, in terms of (5.7), $\alpha(dx^I/x^I) = q^{-i}dx^I/x^I$). Thanks to the relation (5.6), these operators form a chain map of the de Rham complex (5.7′).

Letting $b \to 0^+$ in (5.7′) yields an overconvergent de Rham complex

$$(5.9) \quad B \to \bigoplus_{|I|=1} B_I \to \bigoplus_{|I|=2} B_I \to \cdots$$

equipped with “inverse Frobenii”.

The following lemma of Peigen Li shows that the Artin–Hasse presentation is indeed a presentation of the Dwork crystal.

Lemma 5.10 ([23, Proposition 2.1]). The cohomology of the complex (5.9) is isomorphic to the overconvergent Dwork cohomology of $g^\ast L$. The induced action of $q^{-\ast} \sum_{|I|=\ast} \alpha|_{B_I}$ on cohomology equals the inverse of the Frobenius on the overconvergent Dwork cohomology.

Indication of proof. The de Rham complex of the ordinary presentation of the Dwork crystal also looks like (5.9), except the latter has a different set of derivations. The upshot is that the differentials of the de Rham complex of the ordinary presentation are the exponential twists

$$\exp(\pi g) \circ d \circ \exp(-\pi g),$$

whereas the differentials of (5.9) are given by different exponential twists

$$\exp(\sum_u h(A_u x^u)) \circ d \circ \exp(-\sum_u h(A_u)x^u)$$

(the series $h$ was defined in (5.4)). The same remark applies to the respective “inverse Frobenii”.

Formally, multiplying the “untwist-then-retwist” function

$$(5.11) \quad R = \exp(\sum_u h(A_u x^u) - \pi g)$$
gives an isomorphism from “usual” de Rham complex to and it commutes with the inverse Frobenius. In loc. cit., Peigen Li showed is indeed overconvergent. Therefore, the “untwist-then-retwist” operation is indeed well-defined.

□

6. Visibility of Frobenius eigenvalues

Notation. • Let be the nontrivial additive character defined by 
\[ \Psi(x) = \vartheta(1)^{T_{F_q/F_p}(x)}, \]
where 
\[ T_{F_q/F_p}(x) = \text{Tr}_{F_q/F_p}(x), \]
see (5.1). • Let \( f_1, \ldots, f_r \in F_q[x_1, \ldots, x_n] \) be a collection of polynomials. • For every subset \( I \subset \{1, 2, \ldots, r\} \), set \( Z_I = \text{Spec } F_q[x_1, \ldots, x_n]/(f_i : i \in I) \), and \( Z_I^* = Z_I \cap G_m^n \). Write \( Z = Z_{1, 2, \ldots, r} \). • In Section 5, set \( N = n + r \), and let \( g = \sum_{i=1}^r x_{n+i}f_i \in F_q[x_1, \ldots, x_{n+r}] \).

The purpose of this section is to prove Theorem 1.1, that is, we want to show that the Frobenius eigenvalues of \( Z \) are weakly visible in the zeta functions of \( \zeta_{Z_I^*} \).

The idea of the proof of Theorem 1.1 can be explained as follows:

(1) Renaming \( T_i \) by \( x_{n+i} \), Corollary 4.6, applying to \( g \), shows that the over-convergent Dwork cohomology associated to \( g \) computes the rigid cohomology of \( Z \). Hence the Frobenius eigenvalues of \( Z \) can be manifestly computed using the operator \( \alpha \). This is our starting point of the proof of Theorem 1.1.

(2) As we have seen, the over-convergent Dwork cohomology of \( g \) admits an explicit chain model \((B_I, \alpha_I)\). The chain level operators \( \alpha|_{B_I} \) are “nuclear operators”. Since cohomology spaces are subquotients of \( \bigoplus B_I \), Monsky’s spectral theory (see Lemmas 6.1, 6.2) implies that the cohomological eigenvalues are also “eigenvalues” of \( \bigoplus \alpha|_{B_I} \). In the latter context, “eigenvalue” should be interpreted as the reciprocal roots of the Fredholm determinant of \( \alpha \). Thus, Frobenius eigenvalues are visible in the Fredholm determinants \( \det(1 - t\alpha | B_I) \).

(3) The spaces \( B_I \) are all subspaces of \( B \), on which \( \alpha \) also operates. Thus \( \det(1 - t\alpha | B) \) witnesses all the “chain level eigenvalues” of \( \alpha|_{B_I} \). The Dwork trace formula, applying to \( \alpha|_B \), equates an alternating product of \( \det(1 - t \cdot q^n\alpha | B) \) with an alternating product of zeta functions (see Theorem 6.3 and Lemma 6.9). Möbius inversion (see the formulae in Definition 6.5) then allows us to represent \( \det(1 - t\alpha | B) \) as an infinite product of zeta functions. The Frobenius eigenvalues, which are visible in the Fredholm determinant, are thereby weakly visible in the zeta functions.

Let us carry out the details. To show the Fredholm determinants \( \det(1 - t\alpha | B_I) \) contain all the information of Frobenius eigenvalues, we need the following lemma, extracted from [24 Theorem 1.4].

Lemma 6.1. Let \( X \) be a variable. Let \( M' \xrightarrow{f} M \xrightarrow{g} M'' \) be a complex of nuclear \( K[X]\)-modules in the sense of [24 Definition 1.4]. Then \( H = \ker g/\im f \) is also nuclear, and \( \det(1 - t \cdot X | H) \) is a factor of \( \det(1 - t \cdot X | M) \).

Proof. We follow Monsky’s notation [24 §1]. By [24 Theorem 1.4], \( H \) is a nuclear \( K[X]\)-module. It follows that for any bounded subset (24 Definition 1.3)
Moreover, the polynomial \( \det(1 - t \cdot q^{-i} \alpha \mid B_I) \) contains \( \det(1 - tF^{-1} \mid H^i_{\text{rig}}(\mathbb{A}^{n+r}, g^* \mathcal{L}_{\pi})) \) as a factor.

**Proof.** By [24, Theorem 1.6] and the discussions in (5.3), all summands \( B_I \) of (5.9) are nuclear \( K[X] \)-modules, where the action of \( X \) coming from that of \( \alpha \). The lemma then follows from Lemma 6.1. □

**Lemma 6.3.** For each \( I \subset S \), \( \det(1 - t \cdot q^{-i} \alpha \mid B_I) \) is a \( p \)-adic entire function. Moreover, the polynomial \( \det(1 - tF \mid H^{n-r+j}_{\text{rig},c}(Z)) \) is a factor of

\[
\prod_{|I| = n+r-j} \det(1 - t \cdot q^{j-r} \alpha \mid B_I).
\]

**Proof.** As \( \alpha \) is nuclear, the series \( \det(1 - t \cdot q^{-i} \alpha \mid B_I) \) is a \( p \)-adic entire function by Monsky’s trace formula. Now we have the following rather straightforward computation:

\[
det(1 - tF^{-1} \mid H^i_{\text{rig}}(\mathbb{A}^{n+r}, g^* \mathcal{L}_{\pi})) = det(1 - tF^{-1} \mid H^{2n+2r-2i}_{\text{rig},c}(\mathbb{A}^{n+r}, g^* \mathcal{L}_{\pi})^*(-n-r)) = det(1 - tq^{n+r}F^{-1} \mid H^{2n+2r-2i}_{\text{rig},c}(\mathbb{A}^{n+r}, g^* \mathcal{L}_{\pi})) = det(1 - tq^{-n-r}F \mid H^{2n+2r-2i}_{\text{rig},c}(\mathbb{A}^{n+r}, g^* \mathcal{L}_{\pi})) = det(1 - tq^{-n-r}F \mid H^{2n+2r-2i}_{\text{rig},c}(Z)).
\]

The last step used Corollary 1.16. Making a change of variable \( 2n - i = n - r + j \), Lemma 6.2 implies that \( \det(1 - tF \mid H^{n-r+j}_{\text{rig},c}(Z)) \) is a factor of \( \prod_{|I| = n+r-j} \det(1 - t \cdot q^{j-r} \alpha \mid B_I) \). □

**Corollary 6.4.** Any Frobenius eigenvalue of \( H^*_{\text{rig},c}(Z) \) is weakly visible in the \( p \)-adic entire function \( \det(1 - t \alpha \mid B) \).

**Proof.** By Lemma 6.3 The Frobenius eigenvalue of \( H^{2n-i}_{\text{rig},c}(Z) \) is weakly visible in the product \( \prod_{|I| = i} \det(1 - t \alpha \mid B_I) \). The corollary follows since \( B_I \) is a nuclear submodule of \( B \). □

Next, let us explain how to read off the Fredholm determinants from the zeta functions.

**Definition 6.5.** Following Dwork, we introduce an operation \( \delta \) on the set \( 1 + tC_p[t] \) of formal power series with constant term one:

\[
\delta(\Gamma(t)) \overset{\text{def}}{=} \frac{\Gamma(t)}{\Gamma(qt)}.
\]
The endomorphism $\delta$ is invertible, and its inverse reads

$$\delta^{-1}(\Gamma(t)) = \prod_{i=0}^{\infty} \Gamma(q^it).$$

**Lemma 6.6.** If $\Gamma(t) \in 1 + t\mathbb{C}_p[t]$ is an entire function, then $\delta^{-1}\Gamma(t)$ is also an entire function.

**Proof.** Let $\Gamma(t) = \prod_{i=1}^{\infty} (1 - \gamma_it)$ be its infinite product expansion, where the reciprocal zero $\gamma_j$ approaches zero as $j$ goes to infinity. Then

$$\delta^{-1}(\Gamma(t)) = \prod_{i=0}^{\infty} \prod_{j=1}^{\infty} (1 - q^i\gamma_jt)$$

is also such an infinity product whose reciprocal zero approaches zero. \hfill $\square$

**Lemma 6.7.** Assume that $\Gamma \in 1 + t\mathbb{C}_p[[t]]$ is a $p$-adic meromorphic function on $\mathbb{C}_p$, and $\lambda \in \mathbb{C}_p$ is weakly visible in $\Gamma$. Then $\lambda$ is weakly visible in $\delta(\Gamma)$.

**Proof.** Let $Z(t) = \delta(\Gamma)$. Then $\Gamma(t) = \prod_{i=0}^{\infty} Z(q^it)$. Write $Z(t) = u(t)/v(t)$, where $u, v$ are entire functions, without common zeros. Then

$$\Gamma(t) = \prod_{i=0}^{\infty} \frac{u(q^it)}{v(q^it)} = \frac{\delta^{-1}(u(t))}{\delta^{-1}(v(t))}.$$ 

By Lemma 6.6, $\delta^{-1}u(t)$ and $\delta^{-1}v(t)$ are entire. If $q^m\lambda$ is a reciprocal zero of $\Gamma(t)$. Then $1-q^m\lambda t$ must be a factor of the infinite product $\delta^{-1}(u(t)) = \prod_i u(q^i t)$. Hence $q^m\lambda$ is a reciprocal zero of $u(q^i t)$ for some $i$, i.e., $q^{m-1}\lambda$ is a reciprocal zero of $Z(t)$. Thus $\lambda$ is weakly visible in $Z(t)$. The polar case is similar. \hfill $\square$

At this point, we recall the Dwork trace formula [11, Lemma 2, p. 637]. The following overconvergent version is due to Monsky [24, Theorem 5.3].

**Theorem 6.8 (Dwork trace formula).** For each positive integer $m$, define the $m$-th toric exponential sum

$$S_m^*(g) = \sum_{x \in G_{m,v}^{\text{toric}}(\mathbb{F}_q^m)} (\Psi \circ \text{Tr}_{\mathbb{F}_q^m/\mathbb{F}_q})(g(x)).$$

Then

$$L^*(t) \overset{\text{def}}{=} \exp\left\{ \sum_{m=1}^{\infty} \frac{S_m^*(g) t^m}{m} \right\} = \left\{ \delta^n\left(\text{det}(1-t\alpha | B)\right)\right\}^{(-1)^{n-1}}.$$

Combining Corollary 6.4, Lemma 6.7, and Theorem 6.8, we infer that any Frobenius eigenvalue of $\Omega_{\text{rig,c}}^*(Z)$ is weakly visible in $L^*(t)$.

Our next objective is to relate the function $L^*(t)$ to the zeta functions of $Z_I$.

**Lemma 6.9.** We have

$$L^*(t) = \prod_{J \subset \{1,2,\ldots,r\}} \zeta_{Z^J_I}(q^{|J|}t)^{(-1)^{|J|}}.$$

Then, there are constants \( c \) weakly visible in \( \det(1) \). We deduce from the inclusion-exclusion that \( A \) is an algebraically closed field. The common zero locus, in the proof of Theorem 1.2. Throughout this section, we assume that \( \{ f \} \) are polynomials. Lemma 7.1. Suppose that \( Z(f_{1}, \ldots , f_{r}) \) is a set-theoretic complete intersection in \( A^{n} \). Then, there are constants \( c_{m+1}, \ldots , c_{r} \) in \( k \) such that

- if \( f_{m+1} \) does not vanish on \( Z(g_{1}, \ldots , g_{m}) \), \( c_{m+1} = 1 \),
- if \( f_{m+1} \) vanishes identically on \( Z(g_{1}, \ldots , g_{m}) \), \( c_{m+1} = 0 \), and
- \( Z(g_{1}, \ldots , g_{m}, c_{m+1}f_{m+1} + \cdots + c_{r}f_{r}) \) is a set-theoretic complete intersection in \( A^{n} \).
Proof. Denote the irreducible components of the variety $Z(g_1,\ldots,g_m)$ by $D_1,\ldots,D_h$. These components have the same dimension by the unmixedness theorem. If $f_{m+1}$ vanishes (identically) on all components $D_1,\ldots,D_h$, then $\dim Z(g_1,\ldots,g_m,f_{m+1}) = \dim Z(g_1,\ldots,g_m)$. In this case, we set $c_{m+1} = 0$, drop $f_{m+1}$ from our list, and consider the shorter list $\{g_1,\ldots,g_m,f_{m+2},\ldots,f_r\}$ which still satisfies the condition (7.2).

Hence, without loss of generality, we may assume that $f_{m+1}$ does not identically vanish on $D_1,\ldots,D_{h_1}$ with $h_1 = h$, but vanishes on $D_{h_1+1},\ldots,D_h$. If $h_1 = h$, then

$$\dim Z(g_1,\ldots,g_m,f_{m+1}) < \dim Z(g_1,\ldots,g_m).$$

Since the variety $Z(g_1,\ldots,g_m,f_{m+1})$ is non-empty, this forces that

$$\dim Z(g_1,\ldots,g_m,f_{m+1}) = \dim Z(g_1,\ldots,g_m) - 1,$$

that is, $Z(g_1,\ldots,g_m,f_{m+1})$ is a set theoretic complete intersection.

Assume now $h_1 < h$. By condition (7.2), there is another polynomial among \{f_{m+2},\ldots,f_r\}, say $f_{m+2}$, which does not vanish identically on all of $D_{h_1+1},\ldots,D_h$. Without loss of generality, we may assume that $f_{m+2}$ does not vanish on $D_{h_1+1},\ldots,D_{h_1+h_2}$ with $h_2 > 0$, but vanishes on $D_{h_1+h_2+1},\ldots,D_h$.

Claim. There is a non-zero constant $c$ in $k$ such that $f_{m+1}+cf_{m+2}$ does not vanish on $D_1,\ldots,D_{h_1},\ldots,D_{h_1+h_2}$.

Proof of Claim. Because $f_{m+1}$ vanishes on $D_{h_1+1},\ldots,D_{h_1+h_2}$, and $f_{m+2}$ does not, for any non-zero constant $c$ in $k$, the polynomial $f_{m+1}+cf_{m+2}$ does not vanish on $D_{h_1+1},\ldots,D_{h_1+h_2}$. For each $i = 1,\ldots,h_1$, we can choose $x_i$ in $D_i$ such that $f_{m+1}(x_i)$ is non-zero as $f_{m+1}$ is not identically zero on $D_i$. Choose non-zero constant $c$ in $k$ such that none of the $h_1$ numbers

$$f_{m+1}(x_i) + cf_{m+2}(x_i), i = 1,\ldots,h_1$$

is zero: one simply chooses any non-zero $c$ in $k$ such that $c$ is not among the $h_1$ numbers \{-$f_{m+1}(x_i)/f_{m+2}(x_i), i = 1,\ldots,h_1\}$, which is possible since $k$ is an infinite field. The claim is proved.

Repeating the above procedure, we see there are constants $c_{m+1},\ldots,c_r$ in $k$ such that the linear combination

$$g_{m+1} = c_{m+1}f_{m+1} + c_{m+2}f_{m+2} + \cdots + c_rf_r$$

does not vanish identically on the component $D_i$ for $i = 1,\ldots,h$. It follows that $Z(g_1,\ldots,g_m,g_{m+1})$ is a set-theoretical complete intersection. \hfill $\square$

Lemma 7.3. Let $f_1,\ldots,f_r \in k[x_1,\ldots,x_n]$ be a collection of polynomials. Set $d_i = \deg f_i$. Assume that $d_1 \geq d_2 \geq \cdots \geq d_r$. Let $Z = Z(f_1,\ldots,f_r)$. Then there exists a new sequence of polynomials $g_1,\ldots,g_r \in k[x_1,\ldots,x_r]$ such that

1. $Z(g_1,\ldots,g_r) = Z$,
2. $\deg g_i \leq d_i$,
3. $Z(g_1,\ldots,g_{n-\dim Z})$ is a set-theoretic complete intersection.

Proof. Applying Lemma 7.1 repeatedly gives rise to a new sequence of polynomials $g_1,\ldots,g_r \in k[x_1,\ldots,x_n]$, satisfying the following:

- $g_1 = f_1$, $g_m = f_m$ if $m > n - \dim Z$;
there exists an upper-triangular square matrix \( B = (b_{\alpha \beta})_{1 \leq \alpha, \beta \leq r} \) with entries in \( k \), whose diagonal entries are either 0 or 1, such that
\[
\begin{bmatrix}
g_1 \\
\vdots \\
g_r
\end{bmatrix} = B \cdot 
\begin{bmatrix}
f_1 \\
\vdots \\
f_r
\end{bmatrix}
\]

and
\[
\text{• } Z(g_1, \ldots, g_{n - \dim Z}) \text{ is a set-theoretic complete intersection.}
\]

Thus the condition (3) is ensured. By construction, \( \deg g_i \leq d_i \) for any \( i = 1, 2, \ldots, r \). The condition (2) is checked.

Proof of (1). Since \( g_1, \ldots, g_r \) are \( k \)-linear combinations of \( f_1, \ldots, f_r \), \( Z \) is contained in \( Z(g_1, \ldots, g_r) \). Let us prove \( Z(g_1, \ldots, g_r) \subset Z \).

If the \( j \)th diagonal entry of \( B \) is zero, we say \( j \) is a “jumping” index. By Lemma 7.1, for each jumping \( j \), \( f_j \) vanishes identically on \( Z(g_1, \ldots, g_{j-1}) \); hence \( f_j \) vanishes identically on \( Z(g_1, \ldots, g_r) \) as well.

It remains to show that if \( \beta \) is not a jumping index, \( f_\beta(Q) = 0 \) for any \( Q \in Z(g_1, \ldots, g_r) \). For each jumping \( j \), remove the \( j \)th row and \( j \)th column from the matrix \( B \). The resulting matrix \( C \) is upper triangular, and its diagonal entries are all 1. In particular, \( C \) is invertible. Evaluating (7.4) at \( Q \), using the vanishing of jumping \( f_j \) at \( Q \), we see that, for any non-jumping index \( \alpha \), we have
\[
0 = g_\alpha(Q) = \sum_{\beta \text{ non-jumping}} b_{\alpha \beta} f_\beta(Q),
\]
The matrix associated with the above system of linear equations is the invertible matrix \( C \). Thus \( f_\beta(Q) = 0 \) for any non-jumping \( \beta \). This concludes the proof. □

8. Divisibility of Frobenius eigenvalues

We retain the notation made in Section 6. Recall the situation: we are given a collection of polynomials \( f_1, \ldots, f_r \in \mathbf{F}_q[x_1, \ldots, x_n] \), and denote by
\[
Z = \text{Spec } \mathbf{F}_q[x_1, \ldots, x_n]/(f_1, \ldots, f_r)
\]
the vanishing scheme of \( f_1, \ldots, f_r \). By rearranging the order, we shall assume \( d_1 \geq \cdots \geq d_r \), where \( d_i = \deg f_i \). The codimension \( n - \dim Z \) of \( Z \) is denoted by \( c \).

The following lemma should be well-known. It shows that the vanishing of compactly supported cohomology of \( Z \) can be controlled by the number of defining equations of \( Z \). So Theorems 1.2 and 1.3 cover all nontrivial cohomology degrees. In its statement, \( H_c^i(Z) \) could either be \( H_{rig,c}^i(Z) \) or \( H_c^i(Z_{\mathbf{F}_q}, \mathbf{Q}_l) \).

Lemma 8.1. Let \( Y \) be a nonsingular affine variety of dimension \( n \). Let \( f_1, \ldots, f_r \in \Gamma(Y, \mathcal{O}_Y) \) be regular functions on \( Y \). Let \( Z \) be the common zero locus of \( f_1, \ldots, f_r \) in \( Y \). Then \( H_c^i(Z) = 0 \) for \( i < n - r \).

Proof. We have a long exact sequence
\[
\cdots \rightarrow H_c^i(Y) \rightarrow H_c^i(Z) \rightarrow H_c^{i+1}(Y - Z) \rightarrow H_c^{i+1}(Y) \rightarrow \cdots.
\]
If \( i < n - 1 \), then \( H_c^i(Y) = 0 \) by smoothness of \( Y \), Poincaré duality, and Artin vanishing. Thus, it suffices to prove \( H_c^i(Y - Z) = 0 \) for \( i < n - r + 1 \).
Write $Y \Rightarrow Z = \bigcup_{i \in I} U_i$, where $U_i = Y \Rightarrow \{ f_i = 0 \}$. Then for $I \subset \{ 1, 2, \ldots, r \}$, $igcap_{i \in I} U_i = Y \Rightarrow \{ \prod_{i \in I} f_i = 0 \}$. By Mayer–Vietoris, we have a spectral sequence

$$E_1^{-a,b} = \bigoplus_{\text{Card } I = a+1} H^b_c(U_I) \Rightarrow H^{b-a}_c(Y \Rightarrow Z).$$

Since each $U_I$ is a smooth affine variety of dimension $n$, by Poincaré duality and Artin vanishing again, $H_i^{n+1}(Y \Rightarrow U_I) = 0$ if $i < n-1$. It follows that

$$E_1^{-a,b} \neq 0 \implies \begin{cases} b \geq n, \\ a \leq r-1, \end{cases} \implies b-a \geq n-r+1.$$ 

Ergo, $H^n_c(Y \Rightarrow Z) = 0$ if $i < n-r+1$. □

**Remarks.** (a) The above same argument also works for the Betti cohomology of an algebraic variety $Z$ defined by the vanishing of $r$ regular functions on a smooth affine variety $Y$ over $C$. 

(b) The lemma for rigid cohomology also follows from Corollary 4.4 and Poincaré duality with coefficients.

The remainder of this section is devoted to the proofs of Theorems 1.2 and 1.3. The upshot is that, by Lemma 6.2 and Lemma 6.3, we can reduce the “cohomological divisibility problem” to a “chain level divisibility problem”. Thus we will only need to study the divisibility of the reciprocal roots of the Fredholm determinants $\det(1 - t\alpha | B_I)$. Then, by some “yoga”, we could further reduce ourselves to estimating the slopes (i.e., $q$-orders) of these reciprocal roots, which we achieve in Lemma 8.3.

Unlike Section 6, where only a qualitative understanding to the Dwork complex and the operator $\alpha$ is needed, Lemma 8.3 requires some quantitative analysis for $\alpha$, due to Adolphson–Sperber [1]. In the work of Adolphson and Sperber, the operator $\alpha$ acts on some Banach spaces smaller than $B_I$; we shall use the overconvergent variants $B'_I$ of these spaces (which will have the same effect, see Remark 8.5 below).

**Definition.** Let $\Delta$ be the Newton polyhedron of $g$ at infinity, that is, the convex closure of $0 \in \mathbb{R}^{n+r}$ and $\{ u \in \mathbb{N}^{n+r} : \text{the coefficient of } x^u \text{ in } g \text{ is nonzero} \}$. Let $C(\Delta)$ be the smallest conical region spanned by $\Delta$. For any subset $I \subset \{ 1, 2, \ldots, n+r \}$, set

$$B'_I = \{ \sum A_u x^u \in B_I : A_u = 0 \text{ if } u \notin C(\Delta) \cap \mathbb{Z}^{n+r} \}.$$ 

It is clear that $B'_I$ is an $\alpha$-stable linear subspace of $B_I$.

The theorem of Adolphson and Sperber concerns with the lower bound of slopes of reciprocal roots of $\det(1 - t\alpha | B'_I)$. However, we need the information about $\det(1 - t\alpha | B_I)$. Fortunately, the following lemma shows that the Fredholm determinant of $\alpha|_{B_I}$ and that of $\alpha|_{B'_I}$ are the same.

**Lemma 8.2.** The Fredholm determinant of the induced endomorphism on the quotient space $B_I/B'_I$ is 1.

**Proof.** By Lemma 6.1, $\alpha$ induces a nuclear operator on the quotient space $B_I/B'_I$. It is enough to prove that $\text{Tr}(\alpha^m | B_I/B'_I) = 0$ for all positive integers $m$. A monomial basis for $B_I/B'_I$ consists of $x^u$, where $u$ runs over all lattice points in
which are not in $C(\Delta)$. In particular, $u$ is non-zero. By definition of $\alpha$, one computes that
\[ \alpha^m(x^u) = \sum_{v} b_v x^{(u+v)/q^m}, \]
where $v$ runs over $C(\Delta) \cap \mathbb{Z}^{n+r}$, and $b_v = 0$ if $q^m \nmid u + v$. If $u = (u+v)/q^m$, then $v = (q^m - 1)u$ will not be in the cone generated by $\Delta$, a contradiction. This proves that the diagonal entries of $\alpha^m$ with respect to the monomial basis $\{x^u\}$ are all zero. Consequently, $\text{Tr}(\alpha^m | B_I/B_I') = 0$.

We introduce the following

**Notation.** For a subset $I$ of $\{1,2,\ldots,n+r\}$, let $I' = I \cap \{1,2,\ldots,n\}$, and $I'' = I \cap \{n+1,\ldots,n+r\}$.

**Lemma 8.3.** Let $\lambda$ be a reciprocal root of the $p$-adic entire function $\text{det}(1-t\alpha | B_I)$ Then
\[ \text{ord}_q \lambda \geq \frac{1}{d_1} \left( |I'| + \sum_{i \in I''} (d_1 - d_i) \right) = \frac{1}{d_1} \left( |I| + \sum_{i \in I'} (d_1 - d_i - 1) \right), \]
where $\text{ord}_q$ is the $p$-adic valuation normalized so that $\text{ord}_q(q) = 1$.

**Proof.** Since the problem only concerns with the Fredholm determinant, we can replace $B_I$ by $B_I'$ thanks to Lemma 8.2.

As we shall explain in Remark 8.5, the computation made in [1] works equally well with their Banach spaces $L_I(b)$ (to be recalled in Remark 8.5] replaced by their overconvergent cousins $B_I'$. Thus, by [1, Proposition 4.2], a lower bound of $\text{ord}_q\lambda$ is given by the quantity
\[ w_I = \min \{ w(u) : u \in C(\Delta) \cap \mathbb{N}^{n+r}, \text{u} > 0, \forall i \in I \}, \]
where $w(u)$ is Adolphson–Sperber’s weight function on the cone $C(\Delta)$ which in the present case is given by $w(u_1,\ldots,u_{n+r}) = u_{n+1} + \cdots + u_{n+r}$.

Thus, it suffices to show
\[ w_I \geq \frac{1}{d} \left( |I'| + \sum_{i \in I''} (d - d_i) \right). \]
The numbers $y_1,\ldots,y_{n+r}$ are subject to the following constraints:
\[ \begin{cases} y_i \geq 1, \forall i \in I, \\ y_1 + \cdots + y_n \leq d_1 y_{n+1} + \cdots + d_r y_{n+r}. \end{cases} \]
Write
\[ \xi_i = \begin{cases} y_i - 1, & i \in I; \\ y_i, & i \notin I. \end{cases} \]
Then (8.4) is equivalent to
\[ \begin{cases} \xi_i \geq 0, \forall i = 1,2,\ldots,n+r; \\ \sum_{i=1}^r d_i \xi_i \geq \sum_{i=1}^n \xi_i + |I'| - \sum_{i \in I''} d_i. \end{cases} \]
It follows that
\[ y_{n+1} + \cdots + y_{n+r} = \sum_{i=1}^{r} \xi_{i+r} + |I''| \]
\[ = \frac{1}{d_1} \left( \sum_{i=1}^{r} (d_1 - d_i) \xi_{i+r} + \sum_{i=1}^{r} d_i \xi_{i+r} \right) + |I''| \]
\[ \geq \frac{1}{d_1} \left( |I'| - \sum_{i \in I''} d_i \right) + |I''| = \frac{1}{d_1} \left( |I'| - \sum_{i \in I''} (d_1 - d_i) \right). \]

This completes the proof. \qed

**Remark 8.5.** Let us indicate how to transplant Adolphson–Sperber’s result to the spaces $B'_I$. In \cite{1}, all the calculations took place in some $p$-adic Banach spaces $L(b)$ and $L_I(b)$, where

\[ L(b) = \left\{ \sum_{u \in C(\Delta)^0 \mathbb{Z}^{n+r}} A_u x^u : \exists c \in \mathbb{R}, \text{ord}_p(A_u) \geq bw(u) + c \right\} \]

(ord$_p$ being the $p$-adic valuation normalized by ord$_p(p) = 1$), and $L_I(b) = x^I L(b)$.

The estimate for ord$_q \lambda$ can be obtained from the estimate for the slopes of the Fredholm determinant $\det(1 - t \alpha_0 | L_I(b))$ of the semilinear operantor $\alpha_0$ by a standard argument. For $b = \frac{p}{p-1}$, \textit{loc. cit.} proved that the first slope of $\det(1 - t \alpha_0 | L(b))$ is at least $w_I$. For all $0 < b \leq \frac{p}{p-1}$, we have the factorization of $\alpha_0$ as in \cite{5.2}

\[ L_I(b) \xrightarrow{\gamma} L_I(b/p) \xrightarrow{G} L_I(b/p) \xrightarrow{\sigma^{-1} \psi} L_I(b). \]

Thus $\alpha_0|_{L_I(b)}$ is nuclear for any such $b$. Moreover, the matrices of $\alpha_0|_{L_I(b)}$ with respect to the $\mathbb{Q}_p(\zeta_p)$-bases $\{x^u\}$ of $L_I(b)$ are all similar. Hence the Fredholm determinants for all $b \leq \frac{p}{p-1}$ are the same. Since $B'_I = \text{colim}_{b \to 0^+} L(b)$, the Fredholm determinant for $\alpha_0|_{B'_I}$ also equals that of $\alpha_0|_{L_I(b/p)}$. This gives the desired bound ord$_q \lambda \geq w_I$.

With these preparations handy, we can now step into the proof of Theorems \cite{1.2} and \cite{1.3}

**Step 1: Reduction.** Recall that $n - \dim Z$ is denoted by $c$. By Lemma \cite{7.3} there exists a finite extension $k'$ of $\mathbf{F}_q$, and a collection of polynomials $g_1, \ldots, g_r$, such that

- $\deg g_1 = d_1, \deg g_i \leq d_i$;
- $Z$ is the common zero locus of $g_1, \ldots, g_r$, and
- Spec $k'[x_1, \ldots, x_n]/(g_1, \ldots, g_r)$ is a set-theoretic complete intersection of dimension $\dim Z$.

Since the conclusion of Theorem \cite{1.2} is not sensitive to the base field, and since we have,

\[ \nu_j(n; \deg g_1, \ldots, \deg g_r) \geq \nu_j(n; d_1, \ldots, d_r) \]
\[ \epsilon_m(n; \deg g_1, \ldots, \deg g_r) \geq \epsilon_m(n; d_1, \ldots, d_r), \]
it suffices to prove the theorems with \( F_q \) replaced by \( k' \) and \( f_i \) replaced by \( g_i \). Thus, it suffices to prove Theorems 1.2 and 1.3 under the following additional hypothesis:

(8.6) The scheme \( \text{Spec} F_q[x_1, \ldots, x_n]/(f_1, \ldots, f_c) \) has dimension equal to \( \dim Z \).

**Step 2: Slope estimates.** Let us first prove that the numbers \( \nu_j(n; d_1, \ldots, d_r) \) and \( \epsilon_m(n; d_1, \ldots, d_r) \) provide lower bounds of the \( q \)-order of the Frobenius eigenvalues of \( H^*_{\text{rig}, c}(Z) \). Later we will bootstrap this bound to a bound of \( q \)-divisibility of algebraic numbers.

Rewrite the overconvergent Dwork complex (5.9) as the total complex of the following double complex (in order to save ink, we have omitted the monomials \( d x^I/x^I \) in the expression):

\[
\bigoplus_{|I'|=n} B_I \to \cdots \to \bigoplus_{|I'|=n} B_I \to \cdots \to \bigoplus_{|I'|=n} B_I
\]

(8.7)

\[
\bigoplus_{|I'|=0} B_I \to \cdots \to \bigoplus_{|I'|=0} B_I \to \cdots \to \bigoplus_{|I'|=0} B_I
\]

In the diagram, the horizontal differentials are induced by \( D_{r+1}, \ldots, D_n \), and the vertical ones are induced by \( D_1, \ldots, D_n \) (see (5.5)).

The following lemma shows that the 0th, 1st, ..., and \((c - 1)\)st columns of the \( E_1 \)-page of the spectral sequence associated to the double complex (8.7) are all zero.

**Lemma 8.8.** For each \( 0 \leq i \leq n \), the \( i \)th row of (8.7) is exact in cohomology degree 0, 1, ..., \( c - 1 \).

**Proof.** For the proof of the lemma, the “affine convention” mentioned in Remark 5.8 is more convenient.

Let \( A^\dagger = K\langle x_1, \ldots, x_n \rangle^\dagger \). Then the zeroth row of the double complex, up to a shift, is a tensor product (over \( A^\dagger \)) of

(I) \( \text{Kos}(A^\dagger \langle x_{n+1}, \ldots, x_{n+c} \rangle^\dagger; D^{(1)}, \ldots, D^{(n+c)}) \)

and

(II) \( \text{Kos}(A^\dagger \langle x_{n+c+1}, \ldots, x_{n+r} \rangle^\dagger; D^{(n+c+1)}, \ldots, D^{(n+r)}) \),

where \( D^{(j)} \) is the differential operator

\[
D^{(j)} = \frac{\partial}{\partial x_{n+j}} - \frac{\partial H}{\partial x_{n+j}}, \quad j = 1, 2, \ldots, r,
\]

that \( x_{n+j}D^{(j)} = D_{n+j} \), see Remark 5.8.

By the Künneth formula, it suffices to prove the Koszul complex \( \text{I} \) is acyclic except in top degree. Using Li’s lemma, Lemma 5.10 (specifically, use the untwist-then-retwist series (5.11) in the proof), we see it does not matter whether we use the differentials above or the operators in (4.4). Ergo, the complex \( \text{II} \) computes
\[ \Gamma_f^{(f_1, \ldots, f_c = 0)}(\mathcal{O})[c] \] by Theorem 4.3. Under Hypothesis (8.6), \( \{f_1 = \cdots = f_c = 0\} \) is a set-theoretic complete intersection. It is a standard fact in rigid cohomology that this complex is acyclic except in degree 0, the top degree (we are unable to find a reference; a proof is provided below, see Lemma 8.9). The desired acyclicity is proved.

The \( i \)th row is equal to a direct sum of \( \binom{n}{i} \) copies of the 0th row. By the argument above, we see this row computes \( \Gamma_f^{(f_1, \ldots, f_c = 0)}(\Omega^{n_{T_{\mathcal{A}}}}) \). Thus the acyclicity of the zeroth row implies the acyclicity of the \( i \)th row. \( \square \)

The following lemma was used in the above proof.

**Lemma 8.9.** Let \( k \) be a perfect field of characteristic \( p > 0 \). Let \( A \) be a smooth \( k \)-algebra, lifting smoothly to a smooth \( V \)-algebra \( A_{\text{int}} \). Let \( F_1, \ldots, F_r \) be a regular sequence in \( A_{\text{int}} \). Let \( Z_i \) be the vanishing locus of \( F_i \) modulo the uniformizer. Then \( \Gamma_f^{(Z_1 \cap \cdots \cap Z_r)}(\mathcal{O}) \) is acyclic except in degree \( r \).

**Proof.** Let \( A \) be the weak completion of \( A_{\text{int}} \). Let \( J_e \) be the ideal of \( A \) generated by \( F_e^1, \ldots, F_e^r \). The system \( (A/J_e)^{\infty}_{\leftarrow} \) admits a system of Koszul resolutions

\[
\cdots \rightarrow A^{(2)} \rightarrow A^r \rightarrow A \rightarrow \cdots \rightarrow A \rightarrow A/J_2 \rightarrow \cdots \rightarrow A/J_1 \rightarrow \cdots
\]

Since \( F_1, \ldots, F_r \) is a regular sequence, each row is exact except at the right most entry. Taking \( R\text{Hom}_A(-, A) \), we get an inductive system of Koszul complexes. Hence each row is still exact except at the right most entry. Taking colimits with respect to the columns, we see that the only nonzero cohomology of complex \( \text{colim}_e R\text{Hom}_A(A/J_e, A) \) is the zeroth one. Because

\[
\text{colim} \left[ A \xrightarrow{a} A \xrightarrow{a} A \xrightarrow{} \cdots \right] = A[a^{-1}] \quad (\forall a \in A),
\]

the complex \( \text{colim}_e R\text{Hom}_A(A/J_e, A) \) is the same as

\[
A \rightarrow \bigoplus_i A[F_i^{-1}] \rightarrow \bigoplus_{i,j} A[F_i^{-1}, F_j^{-1}] \rightarrow \cdots.
\]

We conclude by taking weak completion to the above complex, inverting the uniformizer — both are exact in the present context — and noting that the resulting complex computes \( \Gamma_f^{(Z_1 \cap \cdots \cap Z_r)}(\mathcal{O}) \). \( \square \)

Since Lemma 8.8 implies the spectral sequence associated to the double complex (8.7) satisfies

\[ E_1^{i,j} = 0, \quad \forall i < c, \]

in Figure 1 only the gray part contributes to the final abutment of the spectral sequence.
For this reason, $H^{n+r-j}_{\text{rig}}(A^{n+r}_F, \mathcal{L}_x)$ is in fact a subquotient of

\[(8.10) \bigoplus_{\begin{subarray}{l} |I| = n + r - j \\ |I''| \geq c \end{subarray}} B_I;
\]

and Lemma [13] can be refined: \(\det(1 - t \cdot F | H^{n-r+j}_{\text{rig},c}(Z))\) is a factor of

\[\prod_{\begin{subarray}{l} |I| = n + r - j \\ |I''| \geq c \end{subarray}} \det(1 - t \cdot q^{i-r} \alpha | B_I).\]

Hence every Frobenius eigenvalue of $H^{n-r+j}_{\text{rig},c}(Z)$ is a reciprocal root of $\det(1 - t \cdot q^{i-r} \alpha | B_I)$ for some $I$ with $|I| = n + r - j$, $|I''| \geq c$.

To proceed, there are two cases.

**First case:** $j \geq r - c$. This case corresponds to Theorem 1.2. All the relevant spaces $B_I$ lie on the blue line of Figure 1. Let $\gamma$ be one reciprocal root of $\det(1 - t \cdot q^{i-r} \alpha | B_I)$. By Lemma [23] the $q$-order of any reciprocal root $\gamma$ of $\det(1 - t \cdot \alpha | B_I)$ is at least $\frac{1}{d_1} (n + r - j + \sum_{i \in I''} (d_1 - d_i - 1))$. Since $d_1 \geq \cdots \geq d_r$, and $|I''| \geq c$, we have

\[
\sum_{i \in I''} (d - d_i - 1) \geq \sum_{i=1}^{c} (d - d_i - 1) - \sum_{i=c+1}^{|I''|-c} d_i \\
\geq \sum_{i=1}^{c} (d - d_i - 1) - \sum_{i=c+1}^{r} d_i,
\]

where, recall that

\[
d_i^* = \begin{cases} 
  d_i, & \text{if } 1 \leq i \leq c; \\
  1, & \text{if } i > c, \text{ and } d_i = d_1; \\
  0, & \text{if } i > c, \text{ and } d_i < d_1.
\end{cases}
\]

Moreover, by [28] Lemma 3.1, the $q$-order of every reciprocal root of $\det(1 - t \cdot \alpha | B_I)$ is at least $|I''| \geq c$. 

**Second case**.
Remark 8.11. It should be noted that the cited lemma was stated for a certain Banach space denoted by $B^{J_1,J_2}$ in loc. cit. In our context, its role is subsumed by the overconvergent space $B_I$, where $J_1 = I'$, $J_2 = I''$. The proof of the cited lemma only uses Dwork trace formula, which is applicable to $B_I$ as well.

In fact, the cited lemma actually says that the reciprocal roots of $\det(1-t\cdot\alpha)_{B_I}$ are algebraic integers, and are divisible by $q^{\nu''}$ in the ring of algebraic integers.

Hence, Lemma 6.3 implies that the $q$-order of every Frobenius eigenvalue of $H_{\text{rig},e}^{n-r+\gamma}(Z)$ is at least
\begin{equation}
(8.12) \quad j - (r - c) + \max \left\{ 0, \left\lfloor \frac{n - j + (r - c) - \sum_{i=1}^{r} d_i^*}{d_1} \right\rfloor \right\}.
\end{equation}

Making change of variable $j - (r - c) \to j$, the above argument implies that the $q$-order of Frobenius eigenvalues of $H_{\text{rig},e}^{\dim Z + j}(Z)$ are at least $\nu_j(n; d_1, \ldots, d_r)$.

Second case: $0 \leq j < r - c$, which corresponds to Theorem 1.3. In this case, the spaces $B_I$ appeared in (8.10) all lie on the red line of Figure 1. Thus, $r \geq |I''| \geq r - j > c$. It follows from Lemma 6.3 that every reciprocal root $\gamma$ of $\det(1-t\cdot\alpha)_{B_I}$ satisfies $\text{ord}_q \gamma \geq \frac{1}{d_1} \left( n + r - j + \sum_{i\in I''} (d_1 - d_i - 1) \right)$. Since $|I''| \geq r - j$, arguing as in the first case, we have
\begin{align}
\frac{1}{d_1} & \left( n + r - j + \sum_{i\in I''} (d_1 - d_i - 1) \right) \\
\geq & \frac{1}{d_1} \left( n + r - j + \sum_{i=1}^{r-j} (d_1 - d_i - 1) - \sum_{i=r-j+1}^{r} d_i^* \right) \\
= & \frac{1}{d_1} \left( n - \sum_{i=1}^{r-j} d_i - \sum_{i=r-j+1}^{r} d_i^* \right) + r - j.
\end{align}

Again, by [28, Lemma 3.1], we have $\text{ord}_q \gamma \geq |I''| \geq r - j$. Hence, by Lemma 6.3 the $q$-order of every Frobenius eigenvalue of $H_{\text{rig},e}^{n-r+\gamma}(Z)$ is at least
\begin{equation}
\text{max} \left\{ 0, \left\lfloor \frac{n - \sum_{i=1}^{r-j} d_i - \sum_{i=r-j+1}^{r} d_i^*}{d_1} \right\rfloor \right\} = \epsilon_j(n; d_1, \ldots, d_r).
\end{equation}

Step 3. Bootstrap. To finish the proof of Theorems 1.2 and 1.3 it remains to explain why the $a\text{ priori}$ weaker slope estimate given above implies the integrality as well as divisibility in the ring of algebraic integers. All we need is this (see Lemma 8.17 below):

Lemma 8.14. If $\gamma$ is a reciprocal root of the Fredholm determinant $\det(1-t\cdot\alpha)_{B_I}$, then $\gamma$ is an algebraic integer, and any Galois conjugate of it is still a reciprocal root of $\det(1-t\cdot\alpha)_{B_I}$.

That $\gamma$ is an algebraic integer had been shown by the first author in [28, Lemma 3.1], but it will naturally come up again in the argument below.

Before proving Lemma 8.14 let us take a tour through the Dwork theory of exponential sums. Let $N$ be a positive integer, and $g \in k[x_1, \ldots, x_N]$ be a polynomial...
in $N$ variables. In our situation, $N = n + r$, and this $g$ is our $g = \sum x_{n+i}f_i$. For $J \subset \{1, 2, \ldots, N\}$, let $X(J)$ be the linear subspace defined by the vanishing of the variables $(x_j)_{j \in J}$, and $X^*_J$ its standard embedded torus. Let $g_J$ be the restriction of $g$ to $X(J)$. For a nontrivial additive character $\Psi$, consider the exponential sum

$$S^*_J (m) = \sum_{x \in X^*_J (\mathbb{F}_q^m)} (\Psi \circ \text{Tr}_{\mathbb{F}_q^m/\mathbb{F}_q})(g_J(x)).$$

Let again $B = K\langle x_1, \ldots, x_N \rangle^{\dagger}$ be the Monsky–Washnitzer algebra in $N$ variables, $B^J = B/\sum_{j \in J} x_jB$. Applying Dwork trace formula (Theorem 6.8) to $g_J$ gives

$$(q^m - 1)^{|J|} \text{Tr}(a^m \mid B^J) = S^*_J (m),$$

where, as before, $\alpha$ is the nuclear operator defined in Section 5.

For $I \subset \{1, 2, \ldots, N\}$, let $B_I = (\prod_{i \in I} x_i) \cdot B$. Then $B/B_I$ should be thought of as a dagger algebra lifting the divisor $D_I = \{\prod_{i \in I} x_i = 0\}$. The divisor $D_I$, being of strict normal crossings, has a standard semisimplicial resolution

$$\cdots \rightarrow \prod_{|J| = 2} X(J) \rightarrow \prod_{|J| = 1} X(J) \rightarrow D_I.$$

By inclusion-exclusion, $\text{Tr}(a^m \mid B_I) = \sum_{J \subset I} (-1)^{|J|} \text{Tr}(a^m \mid B^J)$. Exponentiating, we get

$$(8.16) \quad \det(1 - ta \mid B_I) = \prod_{J \subset I} \det(1 - ta \mid B^J)^{(-1)^{|J|}}$$

After the digression, we may now go back to our problem, namely when $N = n + r$, and $g(x_1, \ldots, x_{n+r}) = \sum_{i=1}^{n+r} x_{n+i}f_i(x_1, \ldots, x_n)$.

**Proof of Lemma 8.14.** The proof of the assertion is based on a similar, but more precise, argument used in the visibility proof.

For each $J \subset \{1, 2, \ldots, n + r\}$, by (8.15), we have

$$\det(1 - ta \mid B^J)^{\delta_{n+r, |J|}} = L^*_J (t)^{\delta_{n+r-|J|, 1}},$$

where

$$L^*_J (t) = \exp \left\{ \sum_{m=1}^{\infty} S^*_J (m) \frac{tm}{m} \right\}.$$

But by Lemma 6.9 (applying to the lower dimensional affine space $X(J)$), $L^*_J$ is an alternating product of zeta functions of $\zeta_{2^r \cap X_j}(q^{E^{d_j}}t)$, where $E'$ is the intersection of a subset $E$ of $J$ with $\{1, 2, \ldots, r\}$. Thus by the second equation in Definition 8.5 as well as (8.16), $\det(1 - ta \mid B_I)$ is an infinite alternating product of “shifted” zeta functions $\zeta_{2^r \cap X_j}(q^{M}t)$, where $M \in \mathbb{N}$.

Therefore, any reciprocal root of $\det(1 - ta \mid B_I)$, say $\gamma$, is a reciprocal zero or reciprocal pole of some zeta function $\zeta_{2^r \cap X_j}(q^{M}t)$, not being canceled in the infinite product, for some natural number $M$. Therefore, $\gamma$ is an algebraic integer. Since such a shifted zeta function is a ratio of integral polynomials of constant term one, the (reciprocal) minimal polynomial of $\gamma$ must not be canceled either. \qed
By Lemma [8.14] all the conjugates of $\gamma$ are still reciprocal roots of $\det(1-t\alpha | B_I)$. Thus their $q$-orders are bounded by $[8.12]$ or $[8.13]$ depending on $|I|$. The proof of Theorems [1.2] and [1.3] is then concluded thanks to the following elementary lemma.

**Lemma 8.17.** Fix an algebraic closure $\overline{Q}_p$ of $Q_p$. Let $\gamma \in \overline{Q}_p$ be an algebraic integer. Suppose that for any automorphism $\sigma$ of $\overline{Q}_p$, $\text{ord}_q(\sigma(\gamma)) \geq m$. Then $q^m | \gamma$ in the ring of algebraic integers.

**Proof.** Let $P(T) = T^e - a_1T^{e-1} + \cdots + (-1)^{e}a_e$ be the minimal polynomial of $\gamma$. Then for every $i$, $a_i \in \mathbb{Z}$, and is the $i$th elementary symmetric polynomial of $\{\sigma(\gamma)\}$. It follows that $a_i = q^{im}b_i$ for some $b_i \in \mathbb{Z}$. The numbers $\sigma(\gamma) \cdot q^{-m}$ all satisfy the polynomial $T^e - b_1T^{e-1} + \cdots + (-1)^e b_e$, thus are all algebraic integers. The lemma is proved. \hfill \Box

**Proof of Theorem 1.4** Using the long exact sequence of the pair $(P^n, Z)$, the assertions about $P^n - Z$ follow from those about $Z$.

The previous works [14, 10] on $\ell$-adic cohomology treated affine and projective cases separately (although using similar strategies). But in fact the projective case is a formal consequence of the affine case. Let $\mathcal{Z}$ be the affine cone of $Z$. Let $L \to P^n_{F_q}$ be the geometric line bundle associated to $O_{P^n_{F_q}}(-1)$. Then $\mathcal{Z} = \{0\}$ naturally embeds into $L|_Z \overset{\text{def}}{=} L \times_{P^n_{F_q}} Z$ as the complement of the zero section. Identifying $Z$ with the zero section of $L|_Z$, the relative cohomology sequence associated to the pair $(L|_Z, \mathcal{Z} = \{0\})$ reads:

$$\cdots \to H^i_{\text{rig}}(L|_Z) \overset{u}{\to} H^i_{\text{rig}}(Z) \to H^{i+1}_{\text{rig}}(\mathcal{Z} \to \{0\}) \to H^{i+1}_{\text{rig}}(L|_Z) \to \cdots.$$ 

Note that the restriction map $u$ can be factored as

$$\begin{array}{ccc}
H^i_{\text{rig}}(L|_Z) & \overset{u}{\longrightarrow} & H^i_{\text{rig}}(Z) \\
\downarrow & & \uparrow \\
H^i_{\text{rig}}(L) & \longrightarrow & H^i_{\text{rig}}(P^n_{F_q})
\end{array}.$$

It follows that the image of $u$ falls into the “non-primitive part” of the cohomology of $Z$. Hence the above exact sequence gives rise to a Frobenius equivariant embedding $H^i_{\text{rig}}(Z)_{\text{prim}} \hookrightarrow H^{i+1}_{\text{rig}}(\mathcal{Z} \to \{0\})$. If $i > 0$, applying the relative cohomology sequence for compactly supported cohomology, we know the restriction map $H^i_{\text{rig}}(\mathcal{Z} \to \{0\})$ is bijective. Thus for $n - r + j > 0$, the Frobenius eigenvalues of $H^{n-r+j}_{\text{rig}}(Z)_{\text{prim}}$ are also Frobenius eigenvalues of $H^{n+1-r+j}_{\text{rig}}(\mathcal{Z})$. We then conclude by applying Theorems 1.2 and 1.3.

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