Invariance of representation dimension under socle equivalence of selfinjective algebras

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\textit{Dedicated to Idun Reiten on the occasion of her 75th birthday}

\section*{Abstract}

We prove that the representation dimension of finite dimensional selfinjective algebras over a field is invariant under socle equivalence and derive some consequences.

\textit{Keywords:} Representation dimension, Selfinjective algebra, Socle equivalence, Auslander-Reiten quiver

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\section{1. Introduction}

Homological invariants are used to measure how far does an algebra, or a module, deviates from a situation considered to be ideal. From the point of view of representation theory, one of the most interesting and mysterious homological invariants is the representation dimension of an Artin algebra, introduced by Maurice Auslander in the early seventies and meant to measure the complexity of the morphisms in a module category. Interest in this invariant was revived when Igusa and Todorov proved that algebras of
representation dimension three have finite finitistic dimension \[17\]. Iyama proved that the representation dimension of an Artin algebra is always finite \[18\] and Rouquier that there exist algebras of arbitrarily large representation dimension \[19\]. One important question is to identify which algebraic procedures leave the representation dimension invariant. It is known that stable equivalence preserves the representation dimension, a result proved independently by Dugas \[10\] and Guo \[15\]. While derived equivalence does not, in general, preserve the representation dimension, this is the case for selfinjective algebras \[30\].

Our objective in this paper is to prove that the representation dimension is preserved under socle equivalence of selfinjective algebras. We recall that two finite dimensional selfinjective algebras \(A\) and \(A'\) over an arbitrary field \(K\) are called socle equivalent provided the quotient algebras \(A/\text{soc } A\) and \(A'/\text{soc } A'\) are isomorphic. Socle equivalence plays a prominent rôle in the representation theory of selfinjective algebras. Frequently, interesting selfinjective algebras are socle equivalent to others for whom the representation theory and related invariants are well-understood. For some results in this direction we refer the reader to \[6\], \[7\], \[13\], \[11\], \[22\], \[23\] or \[24\]. Our main theorem is then the following.

**Theorem A.** Let \(A, A'\) be basic and connected socle equivalent selfinjective algebras. Then \(A\) and \(A'\) have the same representation dimension.

Our proof is constructive: given an Auslander generator for \(\text{mod } A\), we show how to construct one for \(\text{mod } B\).

Auslander’s expectation was that the representation dimension would provide a reasonable way to measure how far an algebra is from being representation finite. Because an algebra is representation finite if and only if it has representation dimension two \[4\], algebras of representation dimension three present a special interest. In this line of ideas, we apply Theorem [A] to selfinjective algebras of tilted type. We recall that a selfinjective algebra \(A\) is called of tilted type if there exists a tilted algebra \(B\) such that \(A\) is an orbit category of the repetitive category \(\hat{B}\) of \(B\), in the sense of \[16\]. As a first consequence of Theorem [A] and the results of \[2\], \[3\], we show that, if \(A\) is a selfinjective algebra socle equivalent to a representation-infinite selfinjective algebra of tilted type, then the representation dimension of \(A\) equals three.

We next turn to the problem of relating the representation dimension with the shape of Auslander-Reiten components. Using the notation of stable slice introduced in \[28\], we prove our second main result.
Theorem B. Let $A$ be a representation-infinite basic and connected selfinjective algebra admitting a $\tau_A$-rigid stable slice in its Auslander-Reiten quiver. Then the representation dimension of $A$ equals three.

This theorem entails the following interesting corollary.

Corollary C. Let $A$ be a basic and connected selfinjective algebra admitting an acyclic generalised standard Auslander-Reiten component. Then the representation dimension of $A$ equals three.

We now describe the contents of the paper. After a preliminary Section 2 in which we fix the notation and recall facts on the representation dimension and socle equivalence, we prove our Theorem A in Section 3. Section 4 is devoted to the application to the selfinjective algebras of tilted type, including Theorem B and Corollary C. Finally, Section 5 consists of illustrative examples.

2. Representation dimension and socle equivalence

2.1. Notation.

Throughout this paper, $K$ denotes an arbitrary (commutative) field. By an algebra $A$ is meant a basic, connected, associative finite dimensional $K$-algebra. Modules are finitely generated right $A$-modules, and we denote by $\text{mod} A$ their category. For a module $M$, we denote by $\ell(M)$ its composition length. The notation $\text{add} M$ stands for the additive full subcategory of $\text{mod} A$ having as objects the direct sums of direct summands of $M$. Given a full subcategory $\mathcal{C}$ of $\text{mod} A$, we sometimes write $M \in \mathcal{C}$ to express that $M$ is an object in $\mathcal{C}$.

We use freely standard results of representation theory, for which we refer to [1, 5, 27, 29].

2.2. Representation dimension.

The notion of representation dimension was introduced in [4]. It is defined as follows.

Definition. Let $A$ be a non-semisimple algebra. Its representation dimension $\text{rep. dim. } A$ is the infimum of the global dimensions of the algebras $\text{End } M$, where $M$ ranges over all $A$-modules which are at the same time generators and cogenerators of $\text{mod } A$. 

3
If \( M \) is a generator-cogenerator of \( \text{mod} \ A \) for which \( \text{rep. dim} A = \text{gl. dim. End} M \), then \( M \) is called an Auslander generator for \( \text{mod} \ A \).

For instance, if \( A \) is a selfinjective algebra, then the module \( A_A \) is at the same time a generator and a cogenerator of \( \text{mod} \ A \). In fact, in this case, an arbitrary generator-cogenerator \( M \) of \( \text{mod} \ A \) can be assumed to be of the form \( M = A \oplus N \), where the module \( N_A \) has no projective direct summand.

In order to give a criterion allowing to compute the representation dimension, we need to recall a few definitions and facts.

Let \( M \) be a fixed \( A \)-module. Given a module \( X \), a morphism \( f_0 : M_0 \to X \) with \( M_0 \in \text{add} M \) is called a right add \( M \)-approximation of \( X \) provided the induced morphism

\[
\text{Hom}_A(M, f_0) : \text{Hom}_A(M, M_0) \to \text{Hom}_A(M, X)
\]

is surjective. Note that, if \( M \) generates \( X \), then any right add \( M \)-approximation of \( X \) is surjective.

A right add \( M \)-approximation \( f_0 : M_0 \to X \) is called right minimal if any morphism \( g : M_0 \to X \) such that \( f_0g = f_0 \) is an isomorphism

\[
\begin{array}{ccc}
M_0 & \xrightarrow{f_0} & X \\
g \downarrow & & \downarrow \\
M_0 & \xrightarrow{f_0} & X
\end{array}
\]

It is a right minimal add \( M \)-approximation of \( X \) if it is a right add \( M \)-approximation of \( X \) and it is right minimal.

It turns out that any right add \( M \)-approximation admits a direct summand which is a right minimal add \( M \)-approximation.

**Lemma 2.1.** Let \( M, X \) be modules and \( f_1 : M_1 \to X \), with \( M_1 \in \text{add} M \), be a right add \( M \)-approximation of \( X \). Then

(a) There exists a direct summand \( M_0 \) of \( M_1 \) such that \( f_0 = f_1|_{M_0} : M_0 \to X \) is a right minimal add \( M \)-approximation.

(b) For any right minimal add \( M \)-approximation \( f_0 : M_0 \to X \), there exists a section \( s : M_0 \to M_1 \) such that \( f_1s = f_0 \).

**Proof.** (a) This is just [5, Theorem I.2.2].

(b) Because \( M_0, M_1 \) are right add \( M \)-approximations, there exist morphisms \( s : M_0 \to M_1 \) and \( r : M_1 \to M_0 \) making the following diagram
Because $f_0$ is minimal, $rs$ is an isomorphism. Therefore, $s$ is a section.

An exact sequence of the form

$$0 \to M_d \xrightarrow{f_d} M_{d-1} \to \cdots \to M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} X \to 0$$

with all $M_i$ lying in $\text{add} M$ is called a right add $M$-approximation resolution of $X$ with length $d$ provided the induced sequence

$$0 \to \text{Hom}_A(M,M_d) \to \text{Hom}_A(M,M_{d-1}) \to \cdots \to \text{Hom}_A(M,M_0) \to \text{Hom}_A(M,X) \to 0$$

is exact. It is called a right minimal add $M$-approximation resolution of $X$ if moreover each of the morphisms $f_i : M_i \to \text{Im} f_i$, with $i \geq 0$, is right minimal.

The following statement is a consequence of Lemma 2.1.

**Corollary 2.2.** Let $M, X$ be given modules and

$$0 \to N_d \xrightarrow{g_d} N_{d-1} \to \cdots \to N_1 \xrightarrow{g_1} N_0 \xrightarrow{g_0} X \to 0$$

a right add $M$-approximation resolution of $X$. Then there exists a right minimal add $M$-approximation resolution of $X$

$$0 \to M_d \xrightarrow{f_d} M_{d-1} \to \cdots \to M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} X \to 0$$

which is a direct summand of the first one.

**Proof.** We construct the second sequence by induction. Because of Lemma 2.1, there exists a right minimal add $M$-approximation $f_0 : M_0 \to X$ and two morphisms $s_0 : M_0 \to N_0$, $r_0 : N_0 \to M_0$ such that the right squares of the following diagram commute

$$
\begin{align*}
0 & \to Y_0 \xrightarrow{j_0} M_0 \xrightarrow{f_0} X \to 0 \\
0 & \to Z_0 \xrightarrow{i_0} N_0 \xrightarrow{g_0} X \to 0 \\
0 & \to Y_0 \xrightarrow{j_0} M_0 \xrightarrow{f_0} X \to 0.
\end{align*}
$$
Letting $Y_0, Z_0$ be respectively the kernels of $f_0, g_0$ and $i_0, j_0$ the inclusion morphisms, one gets $u_0, v_0$ by passing to the kernels. Because $f_0$ is minimal, $r_0 s_0$ is an isomorphism, hence so is $v_0 u_0$. In particular, $s_0$ and $u_0$ are sections. Also $Z_0 = \text{Im } g_1$, and so there exists an epimorphism $q_1 : N_1 \to Z_0$ such that $g_1 = i_0 q_1$.

Again, Lemma 2.1 yields a right minimal add $M$-approximation $p_1 : M_1 \to Y_0$ and morphisms $s_1 : M_1 \to N_1$, $r_1 : N_1 \to M_1$ and $t_1 : M_1 \to M_1$ such that the right squares of the following diagram commute:

\[
\begin{array}{cccccccccc}
0 & \rightarrow & Y_1 & \xrightarrow{j_1} & M_1 & \xrightarrow{p_1} & Y_0 & \rightarrow & 0 \\
\downarrow{u_1} & & \downarrow{s_1} & & \downarrow{u_0} & & & & \\
0 & \rightarrow & Z_1 & \xrightarrow{i_1} & N_1 & \xrightarrow{q_1} & Y_0 & \rightarrow & 0 \\
\downarrow{v_1} & & \downarrow{r_1} & & \downarrow{v_0} & & & & \\
0 & \rightarrow & Y_1 & \xrightarrow{j_1} & M_1 & \xrightarrow{p_1} & Y_0 & \rightarrow & 0 \\
\downarrow{w_1} & & \downarrow{t_1} & & \downarrow{(v_0 u_0)^{-1}} & & & & \\
0 & \rightarrow & Y_1 & \xrightarrow{j_1} & M_1 & \xrightarrow{p_1} & Y_0 & \rightarrow & 0 \\
\end{array}
\]

Letting $Y_1, Z_1$ be respectively the kernels of $p_1, q_1$ and $i_1, j_1$ the inclusion morphisms, one gets $u_1, v_1, w_1$ by passing to the kernels. Because $p_1$ is right minimal, $t_1 r_1 s_1$ and $w_1 v_1 u_1$ are isomorphisms. Therefore $s_1$ and $w_1$ are sections while $t_1$, $w_1$ are retractions, and hence isomorphisms. Moreover, we have

$$g_1 s_1 = i_0 q_1 s_1 = i_0 u_0 p_1 = s_0 j_0 p_1.$$

Setting $f_1 = j_0 p_1 : M_1 \to M_0$, we obtain the second morphism in the required right minimal add $M$-approximation resolution. Continuing in this way, we construct the wanted right minimal add $M$-approximation resolution and sections $s_i : M_i \to N_i$ such that the following diagram commutes:

\[
\begin{array}{cccccccccc}
0 & \rightarrow & M_d & \xrightarrow{f_d} & M_{d-1} & \xrightarrow{f_{d-1}} & \cdots & \rightarrow & M_1 & \xrightarrow{f_1} & M_0 & \xrightarrow{f_0} & X & \rightarrow & 0 \\
\downarrow{s_d} & & \downarrow{s_{d-1}} & & \cdots & & \downarrow{s_1} & & \downarrow{s_0} & & \downarrow{g_0} & & \downarrow{g_0} & & \downarrow{X} & & \rightarrow & 0 \\
0 & \rightarrow & N_d & \xrightarrow{g_d} & N_{d-1} & \xrightarrow{g_{d-1}} & \cdots & \rightarrow & N_1 & \xrightarrow{g_1} & N_0 & \rightarrow & X & \rightarrow & 0 .
\end{array}
\]

\[\square\]

We need essentially the following characterisation of the representation dimension for which we refer to [9], [11].
Theorem 2.3. Let $A$ be an algebra. Then $\text{rep.dim. } A \leqslant d + 2$ if and only if there exists a generator-cogenerator $M$ such that every $A$-module $X$ has a right add $M$-approximation resolution of length $d$.

Because of Corollary 2.2, we may reformulate this theorem by saying that $\text{rep.dim. } A \leqslant d + 2$ if and only if there exists a generator-cogenerator $M$ such that every $A$-module $X$ has a right minimal add $M$-approximation resolution of length $d$.

2.3. Socle equivalence.

Let $A$ and $A'$ be basic and connected selfinjective algebras. Then $A$ and $A'$ are called socle equivalent if the quotient algebras $A/\text{soc } A$ and $A'/\text{soc } A'$ are isomorphic. Recall indeed that the socle of a selfinjective algebra is a two-sided ideal, see [27, Corollary IV.6.14].

For simplicity of notation, if $A$, $A'$ are socle equivalent, then we treat the isomorphism $A/\text{soc } A \cong A'/\text{soc } A'$ as an identification, that is, we always assume that $A/\text{soc } A = A'/\text{soc } A'$.

If $A$ and $A'$ are socle equivalent, then they have the same ordinary quiver. They also have very similar Auslander-Reiten quivers. Indeed, the indecomposable nonprojective $A$-modules coincide with the indecomposable nonprojective $A'$-modules. Furthermore, if $P$ is an indecomposable projective $A$-module, then there exists an almost split sequence

$$0 \to \text{rad } P \to P \oplus (\text{rad } P/\text{soc } P) \to P/\text{soc } P \to 0$$

in $\text{mod } A$. The $A$-modules $\text{rad } P$, $\text{rad } P/\text{soc } P$ and $P/\text{soc } P$ are clearly annihilated by the socle of $A$, so they are $A/\text{soc } A$-modules. For the same reason, $P$ is not an $A/\text{soc } A$-module. Moreover, the irreducible morphisms $\text{rad } P \to \text{rad } P/\text{soc } P \to P/\text{soc } P$ in $\text{mod } A$, remain irreducible in $\text{mod } (A/\text{soc } A)$, see [5, p. 186]. As an $A/\text{soc } A$-module, $P/\text{soc } P$ is indecomposable projective, while $\text{rad } P$ is the injective envelope of $\text{soc } P$. Finally, because $A$, $A'$ are socle equivalent and so have the same ordinary quiver, there exists a unique indecomposable projective $A'$-module $P'$ such that $P/\text{soc } P = P'/\text{soc } P'$. The relation between $P$ and $P'$ is described in the following lemma.

Lemma 2.4. Let $A$, $A'$ be socle equivalent selfinjective algebras, and $P_A$, $P'_{A'}$, be indecomposable projective modules such that $P/\text{soc } P = P'/\text{soc } P'$. Then we have:
(a) The almost split sequence in mod $A'$ having $P'$ as summand of the middle term is
\[ 0 \to \text{rad } P \to P' \oplus (\text{rad } P/\text{soc } P) \to P/\text{soc } P \to 0. \]

(b) $\text{rad } P = \text{rad } P'$.
(c) $\text{soc } P = \text{soc } P'$.
(d) $\ell(P) = \ell(P')$.
(e) If $j : \text{rad } P \to P$, $j' : \text{rad } P' \to P'$ and $p : P \to P/\text{soc } P$, $p' : P' \to P'/\text{soc } P'$ are the canonical morphisms, then $pj = p'j'$.

Proof. (a) This follows from the discussion before using the hypothesis that $P/\text{soc } P = P'/\text{soc } P'$.

(b) This follows immediately from (a).

(c) $\text{soc } P = \text{soc}(\text{rad } P) = \text{soc}(\text{rad } P') = \text{soc } P'$.

(d) $\ell(P) = \ell(\text{rad } P) + \ell(P/\text{soc } P) - \ell(\text{rad } P/\text{soc } P) = \ell(P')$.

(e) The morphism $pj$ can be rewritten as the composition of the canonical morphisms $\text{rad } P \to \text{rad } P/\text{soc } P \to P/\text{soc } P$. Similarly, $p'j'$ is the composition of the corresponding morphisms in mod $A'$. Because the modules through which they pass are the same, this implies that $pj = p'j'$.

3. Invariance of the representation dimension

Our objective in this section is to prove that two socle equivalent selfinjective algebras have the same representation dimension. Throughout this section, $A$ and $A'$ denote two selfinjective algebras such that $A/\text{soc } A = A'/\text{soc } A'$.

**Proposition 3.1.** Let $0 \to Y \xrightarrow{(f)} N_0 \oplus P \xrightarrow{(u \; v)} X \to 0$ be a short exact sequence in mod $A$, with $P$ projective and $X, Y, N_0$ having no projective direct summand. Let $P'$ be the projective $A'$-module such that $P'/\text{soc } P' = P/\text{soc } P$. Then there exists a short exact sequence in mod $A'$
\[ 0 \to Y \xrightarrow{(f')} N_0 \oplus P' \xrightarrow{(u' \; v')} X \to 0. \]
Proof. Because the morphism $g : Y \to P$ cannot be surjective, its image lies in $\text{rad} P$, and therefore we have a factorisation

$$
\begin{array}{ccc}
Y & \xrightarrow{(f, g)} & N_0 \oplus P \\
\downarrow{(f, h)} & & \downarrow{(1 0 0 j)} \\
N_0 \oplus \text{rad} P & \xrightarrow{(1 0 0 j)} & P \\
\end{array}
$$

where $j : \text{rad} P \to P$ is the canonical inclusion and $g = jh$.

Because of Lemma 2.4, $\text{rad} P = \text{rad} P'$. Let $j' : \text{rad} P' \to P'$ be the canonical inclusion and set $g' = j'h$. We get a composed morphism

$$(f, g') = \begin{pmatrix} 1 & 0 \\ 0 & j' \end{pmatrix} - \begin{pmatrix} f \\ h \end{pmatrix} : Y \to N_0 \oplus P'$$

in $\text{mod} A'$. Because $(f, g')$ is injective, so is $(f, h)$, hence so is $(f, g')$.

We now construct in a similar way a morphism $N_0 \oplus P' \to X$. Because the morphism $v : P \to X$ cannot be injective, it factors through $P/\text{soc} P$. Therefore, we have a factorisation

$$
\begin{array}{ccc}
N_0 \oplus P & \xrightarrow{(u, v)} & X \\
\downarrow{(1 0 0 p)} & & \downarrow{(u w)} \\
N_0 \oplus P/\text{soc} P & \xrightarrow{(u w)} & X \\
\end{array}
$$

where $p : P \to P/\text{soc} P$ is the canonical projection and $v = wp$.

Now, $P/\text{soc} P = P'/\text{soc} P'$. Thus we get a composed morphism

$$(u, v') = (u, w) \begin{pmatrix} 1 & 0 \\ 0 & p' \end{pmatrix} : N_0 \oplus P' \to X$$

in $\text{mod} A'$, where $p' : P' \to P'/\text{soc} P'$ is the canonical projection and $v' = wp'$. Because $(u, v)$ is surjective, so is $(u, w)$, hence so is $(u, v')$. 


The construction is encoded in the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & Y & \overset{(f)}{\rightarrow} & N_0 \oplus P & \overset{(u \ v)}{\rightarrow} & X & \rightarrow & 0 \\
\vert & & \vert & & \vert & & \vert & & \\
0 & \rightarrow & Y & \overset{(f \ h)}{\rightarrow} & N_0 \oplus \text{rad} \ P & \overset{(1 \ 0 \ p)}{\rightarrow} & N_0 \oplus P/\text{soc} \ P & \overset{(u \ w)}{\rightarrow} & 0 \\
\vert & & \vert & & \vert & & \vert & & \\
0 & \rightarrow & Y & \overset{(f \ h)}{\rightarrow} & N_0 \oplus \text{rad} \ P' & \overset{(1 \ 0 \ p')}{\rightarrow} & N_0 \oplus P'/\text{soc} \ P' & \overset{(u \ w)}{\rightarrow} & 0 \\
\end{array}
\]

We know that the upper sequence is exact, and we want to prove that so is the lower sequence.

We have already proven that \((f \ h)\) is injective, and that \((u \ v')\) is surjective, so we only need to check that \(\text{Ker}(u \ v') = \text{Im}(f \ h)\). First, we have

\[
(u \ v') \begin{pmatrix} f \\ g' \end{pmatrix} = uf + v'g' = uf + wp'j'h = uf + wpjh
\]

\[
= uf + vg = (u \ v) \begin{pmatrix} f \\ g \end{pmatrix} = 0,
\]

where we have used Lemma 2.4(e).

In order to prove that \(\text{Ker}(u \ v') \subseteq \text{Im}(f \ h)\), let \(x \in N_0\) and \(a \in P'\) be such that \(\begin{pmatrix} x \\ a \end{pmatrix} \in \text{Ker}(u \ v')\). Because \(p\) and \(p'\) are surjective, there exists \(b \in P\) such that \(p(b) = p'(a)\), so \(\begin{pmatrix} x \\ b \end{pmatrix} \in N_0 \oplus P\). We have

\[
(u \ v) \begin{pmatrix} x \\ b \end{pmatrix} = u(x) + v(b) = u(x) + wp(b) = u(x) + wp'(a) = u(x) + v'(a)
\]

\[
= (u \ v') \begin{pmatrix} x \\ a \end{pmatrix} = 0,
\]

so that \(\begin{pmatrix} x \\ b \end{pmatrix} \in \text{Ker}(u \ v)\). Therefore, there exists \(y_0 \in Y\) such that

\[
\begin{pmatrix} x \\ b \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} (y_0),
\]

that is, such that \(x = f(y_0)\) and \(b = g(y_0)\).
Now, we have
\[
p'(a - g'(y_0)) = p'(a) - p'g'(y_0) = p'(a) - p'j'h(y_0) = p'(a) - pjh(y_0) \\
= p'(a) - pg(y_0) = p'(a) - p(b) = 0.
\]
Therefore \(a - g'(y_0) \in \ker p' = \soc P'.\) Because of Lemma 2.4 (c), \(a - g'(y_0) \in \soc P\) and so \(p(a - g'(y_0)) = 0.\) Therefore
\[
(u \ v) \begin{pmatrix} 0 \\ a - g'(y_0) \end{pmatrix} = (u \ w) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 0 \\ a - g'(y_0) \end{pmatrix} = 0.
\]
Exactness of the upper row yields a \(c \in Y\) such that
\[
\begin{pmatrix} 0 \\ a - g'(y_0) \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}(c).
\]
This means that \(f(c) = 0\) and \(g(c) = a - g'(y_0).\) Because \(a - g'(y_0) \in \soc P \subseteq \rad P,\) we have \(g(c) = jh(c) \in \rad P,\) and therefore \(jh(c) = h(c).\) Hence \(a = g'(y_0) + h(c).\)

Set \(y = y_0 + c \in Y.\) Because \(f(c) = 0,\) we have \(f(y) = f(y_0) + f(c) = f(y_0) = x.\) On the other hand,
\[
g'(y) = g'(y_0) + g'(c) = g'(y_0) + j'h(c).
\]
Because \(h(c) \in \rad P = \rad P',\) we have \(j'h(c) = h(c)\) and so
\[
g'(y) = g'(y_0) + h(c) = a.
\]
We have proved that \((a) = (f \ g')(y) \in \im (f \ g'),\) as required. \(\square\)

Let now \(M\) be an Auslander generator for \(\mod A.\) Because \(A\) is selfinjective, we can assume that \(M\) is of the form
\[
M = N \oplus A,
\]
where \(N\) has no projective summands. But then \(N\) is also an \(A'\)-module. We claim that
\[
M' = N \oplus A'
\]
is an Auslander generator for \(\mod A'.\) The first step in the proof is the following lemma.
Lemma 3.2. Let \( 0 \to Y \xrightarrow{(f,g)} N_0 \oplus P \xrightarrow{(u,v)} X \to 0 \) be an exact sequence in mod \( A \), with \( P \) projective, \( N_0 \in \text{add } N \) and \( X, Y \) having no projective direct summands. Assume that \((u,v)\) is a right add-\(M\)-approximation in mod \( A \). Then, in the corresponding exact sequence

\[ 0 \to Y \xrightarrow{(f,g')} N_0 \oplus P' \xrightarrow{(u,v')} X \to 0 \]

in mod \( A' \), the morphism \((u,v')\) is a right add-\(M'\)-approximation in mod \( A' \).

Proof. Let \( M'_0 \) be an indecomposable direct summand of \( M' \). We claim that any morphism \( f'_0 : M'_0 \to X \) lifts to \( N_0 \oplus P' \). Because \( M'_0 \) is indecomposable, either \( M'_0 \) is projective, in which case the statement is obvious, or else \( M'_0 \in \text{add } N \). In this latter case, \( f'_0 \) is also a morphism in mod \( A \). Therefore there exists \( t'_2 : M'_2 \to P \) such that \( f'_0 = j t'_2 \) where \( j \) is, as before, the canonical inclusion. But \( \text{rad } P = \text{rad } P' \). Letting \( j' : \text{rad } P' \to P' \) be the canonical inclusion, we set \( t'_2 = j't'_2 \). We claim that \((u,v')(t'_1,t'_2) = f'_0 \). Indeed, denoting, as before, by \( p : P \to P/\text{soc } P \) and \( p' : P' \to P'/\text{soc } P' \) the canonical projections, we have

\[
(u,v') \begin{pmatrix} t_1 \\ t'_2 \\ \end{pmatrix} = ut_1 + v't'_2 = ut_1 + wp'j't'_2 = ut_1 + wpjt'_2
\]

where we have used Lemma 2.4 (e). \( \Box \)

Lemma 3.3. In the notation of Lemma 3.2, if \((u,v)\) is minimal, then so is \((u,v')\).

Proof. Assume that \((u,v')\) is not minimal. Then there exists a minimal approximation \((u'_1,v'_1) : N_1 \oplus P'_1 \to X \) with \( N_1 \in \text{add } N, P'_1 \) projective, and a commutative diagram

\[
\begin{array}{ccc}
N_1 \oplus P'_1 & \xrightarrow{(u,v'_1)} & X \\
\downarrow s & & \\
N_0 \oplus P' & \xrightarrow{(u,v')} & \\
\end{array}
\]
where $s$ is a proper section, see Lemma 2.1. Let $P_1$ be the projective $A$-module such that $P_1/\text{soc} P_1 = P'_1/\text{soc} P'_1$. Exchanging the rôles of $A$ and $A'$ in Lemma 3.2 we get a right add $M$-approximation

$$(u_1 \ v_1) : N_1 \oplus P_1 \to X$$

in mod $A$. Now, we have $\ell(P_1) = \ell(P'_1)$ because of Lemma 2.4 (d), and therefore

$$\ell(N_1 \oplus P_1) = \ell(N_1) + \ell(P_1) = \ell(N_1) + \ell(P'_1)$$

$$= \ell(N_1 \oplus P'_1) < \ell(N_0 \oplus P'_1) = \ell(N_0 \oplus P),$$

and this contradicts the minimality of $(u \ v) : N_0 \oplus P \to X$. Therefore, $s$ is not proper and so $(u_1 \ v'_1) : N_1 \oplus P'_1 \to X$ is a right minimal add $M'$-approximation.

We are now able to prove our Theorem A.

**Theorem 3.4.** Let $A, A'$ be socle equivalent basic and connected selfinjective algebras. Then $\text{rep-dim} \ A = \text{rep-dim} \ A'$. Furthermore, if $M = N \oplus A$ is an Auslander generator for mod $A$, with $N$ having no projective direct summands, then $M' = N \oplus A'$ is an Auslander generator for mod $A'$.

**Proof.** For simplicity, we may assume $A/\text{soc} \ A = A'/\text{soc} \ A'$. Let $X$ be an indecomposable nonprojective $A$-module, and let

$$0 \to N_d \to N_{d-1} \oplus P_{d-1} \to \cdots \to N_0 \oplus P_0 \to X \to 0$$

be a right minimal add $M$-approximation resolution, with $N_i \in \text{add} \ N$ and the $P_i$ projective for all $i$. Notice that the minimality of this sequence implies that the last nonzero term on the left has no projective direct summand, and therefore belongs to add $N$.

For each $i$, let $P'_i$ be the projective $A'$-module such that $P'_i/\text{soc} P'_i = P_i/\text{soc} P_i$. We claim that the corresponding sequence

$$0 \to N_d \to N_{d-1} \oplus P'_{d-1} \to \cdots \to N_0 \oplus P'_0 \to X \to 0$$

is a right minimal add $M'$-approximation resolution in mod $A'$.

We prove this claim by induction. Let first $(u \ v) : N_0 \oplus P_0 \to X$ be a right minimal add $M$-approximation. Because $M$ is a generator of mod $A$,
the morphism \((u \ v)\) is injective. Letting \(Y = \text{Ker}(u \ v)\), we have a short exact sequence

\[
0 \to Y \to N_0 \oplus P_0 \xrightarrow{(u \ v)} X \to 0
\]

in \(\text{mod}\ A\). If \(Y\) has a projective (= injective) direct summand, then this summand splits off and we have a contradiction to the minimality of \((u \ v)\). Therefore \(Y\) has no projective direct summand. Applying Proposition 3.1, we get a short exact sequence

\[
0 \to Y \to N_0 \oplus P'_0 \xrightarrow{(u \ v')} X \to 0
\]

in \(\text{mod}\ A'\). Because of Lemmata 3.2 and 3.3, \((u \ v') : N_0 \oplus P'_0 \to X\) is a right minimal add \(M\)-approximation in \(\text{mod}\ A'\).

Now, we have a right minimal add \(M\)-approximation resolution of \(Y\) in \(\text{mod}\ A\)

\[
0 \to N_d \to N_{d-1} \oplus P_{d-1} \to \cdots \to N_1 \oplus P_1 \to Y \to 0.
\]

The induction hypothesis yields a right minimal add \(M'\)-approximation resolution of \(Y\) in \(\text{mod}\ A'\)

\[
0 \to N_d \to N_{d-1} \oplus P'_{d-1} \to \cdots \to N_1 \oplus P'_1 \to Y \to 0. \quad (**)
\]

Splicing the sequences (*) and (**) yields the desired right minimal add \(M'\)-approximation resolution of \(X\) in \(\text{mod}\ A'\). This establishes our claim.

The statement of the theorem now follows easily from the claim and Theorem 2.3.

4. Selfinjective algebras of tilted type

In this section, we present some applications of the main result of the paper to selfinjective algebras, which are socle equivalent to selfinjective algebras of tilted type. For background on hereditary and tilted algebras over arbitrary fields we refer to [29, Chapters VII and VIII]. We also refer to [26] for general results on selfinjective algebras of tilted type.

Let \(B\) be a basic finite dimensional \(K\)-algebra and \(1 = e_1 + \cdots + e_n\) be a decomposition of the identity of \(B\) into a complete sum of primitive
orthogonal idempotents. We associate to $B$ a selfinjective locally bounded $K$-category $\widehat{B}$, called its repetitive category $[16]$. The objects of $\widehat{B}$ are the $e_{m,i}$, with $m \in \mathbb{Z}$ and $i \in \{1, \ldots, n\}$, and the morphism spaces are defined by

$$\widehat{B}(e_{k,i}, e_{s,j}) = \begin{cases} 
  e_jBe_i & \text{if } k = s \\
  D(e_iBe_j) & \text{if } k = s - 1 \\
  0 & \text{otherwise.}
\end{cases}$$

We denote by $\nu_{\widehat{B}}$ the so-called Nakayama automorphism of $\widehat{B}$ defined by

$$\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i}$$

for all $(m, i)$. A group $G$ of $K$-linear automorphisms of the category $\widehat{B}$ is said to be admissible if $G$ acts freely on the objects of $\widehat{B}$ and has finitely many orbits. Then we may consider the orbit category $\widehat{B}/G$ defined as follows, see [14]. The objects of $\widehat{B}/G$ are the $G$-orbits of objects of $\widehat{B}$ and the morphism spaces are given by

$$(\widehat{B}/G)(a,b) = \left\{ f_{y,x} \in \prod_{(x,y) \in (a,b)} \widehat{B}(x,y) \mid gf_{y,x} = f_{gy,gx} \text{ for all } g \in G, x \in a, y \in b \right\}$$

for all objects $a, b$ of $\widehat{B}/G$. Then $\widehat{B}/G$ is a bounded selfinjective $K$-category which we identify with the associated finite dimensional selfinjective $K$-algebra.

An automorphism $\varphi$ of the $K$-category $\widehat{B}$ is called:

- **positive** if, for every $(m, i)$, we have $\varphi(e_{m,i}) = e_{p,j}$ for some $p \geq m$ and $j \in \{1, \ldots, n\}$;
- **rigid** if, for every $(m, i)$, we have $\varphi(e_{m,i}) = e_{m,j}$ for some $j \in \{1, \ldots, n\}$;
- **strictly positive** if it is positive and not rigid.

Thus, for instance, the automorphisms $\nu_{\widehat{B}}^n$ with $n \geq 1$, are strictly positive automorphisms of $\widehat{B}$.

We recall that an algebra $B$ is called tilted if there exists a basic and connected hereditary $K$-algebra $H$ and a multiplicity-free tilting $H$-module $T$ such that $B = \text{End} T$. Moreover, $B$ is said to be of Dynkin, Euclidean or wild type according as the valued quiver of $H$ is a Dynkin, Euclidean or wild quiver, respectively.

We have the following general result, see [26, Theorem 7.1].
Proposition 4.1. Let $B$ be a tilted algebra and $G$ an admissible torsion-free automorphism group of $\hat{B}$. Then $G$ is an infinite cyclic group generated by a strictly positive automorphism $\varphi$ of $\hat{B}$.

By selfinjective algebra of tilted type, we mean an orbit algebra $\hat{B}/G$, where $B$ is a tilted algebra and $G$ is an admissible infinite cyclic group of automorphisms of $\hat{B}$. Moreover, a selfinjective algebra $A = \hat{B}/G$ of tilted type is said to be of Dynkin, Euclidean or wild type according as the tilted algebra is of Dynkin, Euclidean or wild type, respectively.

We note that $A$ is representation-finite if and only if $B$ is of Dynkin type.

The following corollary is the first application of our Theorem [A] and the results of [2], [3].

Corollary 4.2. Let $A$ be a selfinjective algebra socle equivalent to a representation infinite selfinjective algebra of tilted type. Then $\text{rep. dim. } A = 3$.

Proof. Assume $A$ is socle equivalent to a representation-infinite selfinjective algebra $A' = \hat{B}/G$ of tilted type. It follows from [2, Theorem] and [3, Theorem A] that $\text{rep. dim. } A' = 3$ if the ground field $K$ is algebraically closed. In fact, we gave an explicit construction of an Auslander generator for $\text{mod } A'$, applying the canonical Galois covering functor $\hat{B} \to \hat{B}/G = A'$. But the arguments used in [2] and [3] remain valid for algebras over an arbitrary field $K$, thanks to general results on selfinjective algebras of tilted type, presented in [26].

Let $A$ be a selfinjective algebra. We denote by $\tau_A$ the Auslander-Reiten translation in $\text{mod } A$ and by $\Gamma(\text{mod } A)$ its Auslander-Reiten quiver. A full valued subquiver $\Delta$ of $\Gamma(\text{mod } A)$ is called a stable slice [28] if the following conditions are satisfied:

1. $\Delta$ is connected, acyclic and without projective modules.
2. For any valued arrow $V \xrightarrow{(d,d')} U$ in $\Gamma(\text{mod } A)$ with $U$ in $\Delta$ and $V$ non-projective, $V$ belongs to $\Delta$ or to $\tau_A \Delta$.
3. For any valued arrow $V \xrightarrow{(d,d')} U$ in $\Gamma(\text{mod } A)$ with $V$ in $\Delta$ and $U$ noninjective, $U$ belongs to $\Delta$ or to $\tau_A^{-1} \Delta$.

A stable slice $\Delta$ of $\Gamma(\text{mod } A)$ is called regular if $\Delta$ contains neither the socle factor $P/\text{soc } P$ nor the radical $\text{rad } P$ of an indecomposable projective $A$-module $P$. A stable slice $\Delta$ of $\Gamma(\text{mod } A)$ is called $\tau_A$-rigid if $\text{Hom}_A(X, \tau_A Y) = \ldots$
0 for all indecomposable modules $X, Y$ from $\Delta$. Because of a result proved in \cite{20}, a $\tau_A$-rigid stable slice $\Delta$ of $\Gamma(\text{mod } A)$ is always finite.

Our Theorem B is the second application of Theorem A and the main result of \cite{28}.

**Theorem 4.3.** Let $A$ be a representation-infinite selfinjective algebra admitting a $\tau_A$-rigid stable slice in $\Gamma(\text{mod } A)$. Then $\text{rep. dim. } A = 3$.

**Proof.** Assume $\Delta$ is a $\tau_A$-rigid stable slice in $\Gamma(\text{mod } A)$. Let $M$ be the direct sum of all the indecomposable $A$-modules lying on $\Delta$, $I = \{a \in A \mid Ma = 0\}$ the right annihilator of $M$ and $B = A/I$. We claim that there exist, for some $r, s \geq 1$, a monomorphism $\tau_A M \to M^r$ and an epimorphism $M^s \to \tau_A^{-1}M$ in $\text{mod } A$, and hence in $\text{mod } B$.

Because $\Delta$ is a regular stable slice in $\Gamma(\text{mod } A)$, an injective envelope $f : \tau_A M \to I(\tau_A M)$ of $\tau_A M$ and a projective cover $g : P(\tau_A^{-1}M) \to \tau_A^{-1}M$ of $\tau_A^{-1}M$ in $\text{mod } A$ factor through $M^r$ and $M^s$, respectively, for some $r; s \geq 1$. This establishes the claim.

In particular, $\text{Hom}_A(M, \tau_A M) = 0$ implies that also $\text{Hom}_A(\tau_A^{-1}M, M) = 0$, so $\Delta$ is a double $\tau_A$-rigid stable slice of $\Gamma(\text{mod } A)$. Then, because of \cite{28, Proposition 3.8}, the following statements hold:

(a) $M$ is a tilting $B$-module,
(b) $H = \text{End}_B M$ is a hereditary algebra,
(c) $T = D(M)$ is a tilting $H$-module, and
(d) $B = \text{End}_H T$.

Applying \cite{28, Theorem 2}, we conclude that $A$ is socle equivalent to an orbit algebra $A' = \hat{B}/(\varphi \nu_{\hat{B}})$ for some positive automorphism $\varphi$ of $\hat{B}$. Moreover, $B$ is not of Dynkin type. Therefore $A'$ is a representation-infinite selfinjective algebra of tilted type. Applying now Corollary 4.2, we get that $\text{rep. dim. } A = 3$. We note that the algebras $A$ and $A'$ are not necessarily isomorphic (see Example 5.4).

Recall that a connected component $C$ of an Auslander-Reiten quiver $\Gamma(\text{mod } A)$ is called *generalised standard* \cite{21} whenever, for two modules $X, Y$ in $C$, we have $\text{rad}^\infty_A(X, Y) = 0$. Here, $\text{rad}^\infty_A$ denotes the infinite radical of $\text{mod } A$.

We have the following consequence of Theorem 4.3 extending \cite[Theorem B]{3} to algebras over an arbitrary field.
Corollary 4.4. Let $A$ be a connected selfinjective algebra admitting an acyclic generalised standard Auslander-Reiten component. Then $\text{rep} \dim A = 3$.

Proof. Let $C$ be an acyclic generalised standard component in $\Gamma(\text{mod} A)$. Then $C$ is an infinite component admitting a $\tau_A$-rigid regular stable slice, because it contains only finitely many projective modules. In particular, $A$ is representation-infinite. Applying Theorem B yields $\text{rep} \dim A = 3$. \hfill $\square$

We refer to [24], [25] for the structure of module categories of selfinjective algebras admitting generalised standard acyclic Auslander-Reiten components.

5. Examples

The aim of this section is to present illustrative examples. The first two describe selfinjective algebras over an algebraically closed field which are socle equivalent but not isomorphic to selfinjective algebras of Euclidean and wild types.

Example 5.1. Let $K$ be an algebraically closed field and $Q$ be the quiver

$$
\alpha \quad 1 \quad \gamma \\
\beta \\
$$

Consider the quotient algebras $A = KQ/I$ and $A' = KQ/I'$, where $I$ and $I'$ are the ideals

$I = (\alpha^2 - \alpha \gamma \beta, \alpha \gamma \beta + \gamma \beta \alpha, \beta \gamma, \beta \alpha \gamma \beta),$

$I' = (\alpha^2, \alpha \gamma \beta + \gamma \beta \alpha, \beta \gamma, \beta \alpha \gamma \beta).$

Then $A' \cong \hat{B}/(\varphi)$, where $B = K\Delta/J$ is the tilted algebra of Euclidean type $\hat{A}_3$ given by the quiver $\Delta$

$$
1 \quad \alpha \\
\beta \quad 2 \quad \gamma \\
3 \quad \sigma \\
4$

and the ideal $J = (\sigma \gamma)$, and $\varphi$ is a strictly positive automorphism of $\hat{B}$ such that there exists a rigid automorphism $\rho$ with $\varphi^2 = \rho \nu_{\hat{B}}$. Moreover, $A/\text{soc}A$ and $A'/\text{soc}A'$ are isomorphic to the algebra $A^* = KQ/I^*$, where
I^* = (α^2, αγβ, γβα, βγ). Hence A and A' are socle equivalent, and therefore rep. dim. A = rep. dim. A' = 3, because of Theorem B. On the other hand, it is easily seen that A and A' are not isomorphic. We refer to [7] for a general construction of such socle equivalent algebras. We also note that A and A' are not stably equivalent, see [8, Theorem 1.2].

**Example 5.2.** Let K be an algebraically closed field and Q be the quiver

\[ \begin{array}{c}
3 & \overset{\eta}{\rightarrow} & 1 & \overset{\gamma}{\rightarrow} & 2
\end{array} \]

Consider the quotient algebras A = KQ/I and A' = KQ/I', where I and I' are the ideals

\[ I = (\alpha^2 - \alpha\gamma\beta, \alpha\gamma\beta + \gamma\beta\alpha, \beta\gamma, \beta\alpha\gamma\beta, \alpha\gamma\beta - \delta\eta, \beta\delta, \eta\alpha, \eta\gamma, \alpha\delta), \]

\[ I' = (\alpha^2, \alpha\gamma\beta + \gamma\beta\alpha, \beta\gamma, \beta\alpha\gamma\beta, \alpha\gamma\beta - \delta\eta, \beta\delta, \eta\alpha, \eta\gamma, \alpha\delta). \]

Let C = K\Delta/J be the quotient algebra of the path algebra K\Delta of the quiver \(\Delta\)

\[ \begin{array}{c}
1 & \overset{\alpha}{\rightarrow} & 3 & \overset{\sigma}{\rightarrow} & 4
\end{array} \]

by the ideal J = (σγ, σδ). Then the Auslander-Reiten quiver Γ(mod C) of C admits a unique preinjective component having a section of the form

\[ \begin{array}{c}
I_6 & \overset{I_2}{\rightarrow} & S_3 & \overset{I_3}{\rightarrow} & I_5
\end{array} \]
It follows from [1, Theorem 5.6] that $C$ is a tilted algebra of wild type

\[ \begin{array}{cccccc}
1 & 2 & 4 & 5 & 3 \\
& 6 & & &
\end{array} \]

Moreover, a simple checking shows that $A'$ is isomorphic to the orbit algebra $\widehat{C}/(\psi)$ where $\psi$ is a strictly positive automorphism of $\widehat{C}$ such that there exists a rigid automorphism $\rho$ of $\widehat{C}$ with $\psi^2 = \rho \nu_{\widehat{C}}$. Hence $A'$ is a selfinjective algebra of wild tilted type. Further, $A/\text{soc} A$ and $A'/\text{soc} A'$ are both isomorphic to the quotient algebra $A^* = KQ/I^*$, where

\[ I^* = (\alpha^2, \alpha \gamma \beta, \gamma \beta \alpha, \beta \gamma, \delta \eta, \eta \gamma, \alpha \delta). \]

Therefore, $A$ and $A'$ are socle equivalent, while they are clearly not isomorphic. Applying Theorem we obtain $\text{rep. dim. } A = \text{rep. dim. } A' = 3$.

**Example 5.3.** Let $K$ be an algebraically closed field. To each nonzero element $\lambda \in K$, we associate the four-dimensional local selfinjective algebra

\[ A(\lambda) = K\langle x, y \rangle/(x^2, y^2, xy - \lambda yx). \]

For any nonzero elements $\lambda, \mu$, the algebras $A(\lambda)$ and $A(\mu)$ are socle equivalent. On the other hand, it was shown by Rickard that the algebras $A(\lambda)$ and $A(\mu)$ are stably equivalent if and only if $\mu = \lambda$ or $\mu = \lambda^{-1}$, in which case $A(\lambda)$ and $A(\mu)$ are also isomorphic. This was done by a careful analysis of actions of the syzygy operator on the indecomposable 2-dimensional modules forming the mouth of the stable tubes of rank 1 in the stable Auslander-Reiten quiver of the algebra $A(\lambda)$, see [27, Example IV.10.7] for a description of these actions.

We note that $A(1)$ is a selfinjective algebra of the form $A(1) = \widehat{H}/(\varphi)$, where $H$ is the path algebra of the Kronecker quiver

\[ 1 \xrightarrow{\alpha} 2 \]

and $\varphi$ is an automorphism of $\widehat{H}$ with $\varphi^2 = \nu_{\widehat{H}}$. Because of [2, Theorem], we have $\text{rep. dim. } A(1) = 3$. Applying now our Theorem we get that $\text{rep. dim. } A(\lambda) = 3$ for any $\lambda \in K \setminus \{0\}$. 

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We also note that $A(-1)$ is the exterior algebra $\Lambda(K^2)$. It follows from [19, Theorem 4.1] that, for any integer $n \geq 2$, the exterior algebra $\Lambda(K^n)$ of the $n$-dimensional vector space $K^n$ has representation dimension $n + 1$. It is thus very natural to expect that there are many selfinjective algebras $A$ socle equivalent but not isomorphic (even not stably equivalent) to the exterior algebra $\Lambda(K^n)$, and then such that $\text{rep. dim. } A = n + 1$.

The next example shows that socle equivalences exist naturally for Hochschild extensions of hereditary algebras by duality bimodules. We refer to [29, Chapter X] for the general theory of Hochschild extensions.

**Example 5.4.** Let $K$ be a field of characteristic 2, and $L$ be a finite field extension of $K$ such that the Hochschild cohomology group $H^2(L, L)$, where $L$ is considered as a $K$-algebra, is nonzero. We refer to [29, Section X.5] for such field extensions.

Take a 2-cocycle $\alpha : L \times L \to L$ corresponding to a nonsplit extension $0 \to L \to M \to L \to 0$. For example, we may take $K = \mathbb{Z}_2(u)$, the field of rational functions in one variable $u$ over the field $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, and $L = K[X]/(X^2 - u)$, where $K[X]$ is the polynomial algebra in one variable $X$ over $K$. Denoting by $x$ the residual class $X + (X^2 - u)$, we see that $L$ has $\{1, x\}$ as a $K$-basis, and a nonsplit 2-cocycle $\alpha : L \times L \to L$ as required above is given by

$$\alpha(x^l, x^m) = x^{l+m}$$

for $l, m \in \{0, 1\}$, see [29, Example X.5.4].

Let $Q = (Q_0, Q_1)$ be a finite connected acyclic quiver without double arrows, and $H = LQ$ be its path algebra over $L$. For each point $i \in Q_0$, choose a primitive idempotent $e_i$ of $H$ and for each path from $i$ to $j$, choose an element $h_{ji} = e_j h_{ji} e_i$ of $H$. Then $DH = \text{Hom}_L(LQ, L) \cong \text{Hom}_K(LQ, K)$ has a dual basis $e_i^*, h_{ji}^*$ over $L$. Let $\tilde{H} = H \oplus DH$ be the direct sum of $H$ and $DH$ considered as $K$-spaces, and define a multiplication on $\tilde{H}$ in the following way

$$(a, v)(b, w) = (ab, aw + vb + \sum_{i \in Q_0} \alpha(a_i, b_i)e_i^*)$$

for $a, b \in H$, $v, w \in DH$, where $a_i, b_i \in L$ are such that

$$a = \sum a_i e_i + \sum r_{ji} h_{ji}, \quad b = \sum b_i e_i + \sum s_{ji} h_{ji},$$

$$21$$
for $r_{ji}, s_{ji} \in L$ are the basis presentations of $a$ and $b$.

Letting $\rho : \bar{H} \to H$ denote the canonical epimorphism and $\omega : DH \to \bar{H}$ the embedding, we have a nonsplit Hochschild extension

$$0 \to DH \xrightarrow{\omega} \bar{H} \xrightarrow{\rho} H \to 0,$$

see [29, Theorem X.6.7]. Moreover, $\bar{H}$ is selfinjective, and even weakly symmetric, and the elements $\bar{e}_i = (e_i, -\alpha(1,1)e_i^*) \in \bar{H}$ form a complete set of orthogonal primitive idempotents of $\bar{H}$. Because of [23, Corollary 4.2], $\bar{H}$ is socle equivalent to the trivial extension $T(H) = H \ltimes DH$. We also know that $\bar{H}$ is not isomorphic to an orbit algebra $\overline{B}/(\varphi \nu \overline{B})$ where $B$ is a $K$-algebra and $\varphi$ is a positive automorphism of $\overline{B}$ (see [25, Proposition 4]). Clearly, $\bar{H}$ and $T(H)$ are not isomorphic.

We end with an example of socle equivalence of symmetric algebras arising from triangulated surfaces.

**Example 5.5.** Let $K$ be an algebraically closed field, $m$ a positive natural number, $c$ a nonzero scalar from $K$, and $b : \{1, 2, 3\} \to K$ a function.

Consider the surface $S$ of the triangle $T$

$$\begin{array}{c}
1 \\
\downarrow \\
3 \\
\downarrow \\
2
\end{array}$$

where 1, 2, 3 are boundary edges, and let $\overline{T}$ be the clockwise orientation $(1 2 3)$ of the edges of $T$.

According to [12, Section 4], we may associate to the pair $(S, \overline{T})$ the triangulation quiver $(Q(S, \overline{T}), f)$ of the form

$$\begin{array}{c}
1 \\
\downarrow \alpha \\
\downarrow \gamma \\
3 \\
\downarrow \beta \\
\downarrow \mu \\
2
\end{array}$$

where $f$ is the permutation of arrows defined as follows

$$f(\alpha) = \beta, \quad f(\beta) = \gamma, \quad f(\gamma) = \alpha, \quad f(\varepsilon) = \varepsilon, \quad f(\eta) = \eta, \quad f(\mu) = \mu$$

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Consider the bound quiver algebra
\[ \Lambda(S, \vec{T}, m, c, b_\bullet) = KQ(S, \vec{T})/I(Q(S, \vec{T}), f, m, c, b_\bullet) \]
where \( I(Q(S, \vec{T}), f, m, c, b_\bullet) \) is the admissible ideal in the path algebra \( KQ(S, \vec{T}) \) generated by the elements
\[
\begin{align*}
\alpha \beta - c(\varepsilon \alpha \eta \beta \mu \gamma)^{m-1} \varepsilon \alpha \eta \beta \mu, & \quad \varepsilon^2 - c(\alpha \eta \beta \mu \varepsilon \gamma)^{m-1} \alpha \eta \beta \mu \gamma - b_1(\alpha \eta \beta \mu \varepsilon)^{m}, \\
\beta \gamma - c(\eta \beta \mu \varepsilon \alpha)^{m-1} \eta \beta \mu \varepsilon, & \quad \eta^2 - c(\beta \mu \varepsilon \alpha \eta)^{m-1} \beta \mu \varepsilon \alpha - b_2(\beta \mu \varepsilon \alpha \eta)^{m}, \\
\gamma \alpha - c(\mu \varepsilon \alpha \eta \beta)^{m-1} \mu \varepsilon \alpha \eta, & \quad \mu^2 - c(\varepsilon \alpha \eta \beta \mu \gamma)^{m-1} \varepsilon \alpha \eta \beta \mu - b_3(\varepsilon \alpha \eta \beta \mu)^{m}, \\
\alpha \beta \mu, & \quad \varepsilon^2 \alpha, \quad \beta \gamma \varepsilon, \quad \eta^2 \beta, \quad \gamma \alpha \eta, \quad \mu^2 \gamma.
\end{align*}
\]

We also denote by 0 the zero function from \( \{1, 2, 3\} \) to \( K \) and set \( \Lambda(S, \vec{T}, m, c) = \Lambda(S, \vec{T}, m, c, 0) \). Following \cite{[12]}, \( \Lambda(S, \vec{T}, m, c) \) is called a weighted surface algebra of \( (S, \vec{T}) \). Because of \cite{[12]}, Propositions 8.1 and 8.2, \( \Lambda(S, \vec{T}, m, c) \) and \( \Lambda(S, \vec{T}, m, c, b_\bullet) \) are socle equivalent representation-infinite tame symmetric algebras of dimension \( 36m \), which are isomorphic if the characteristic of \( K \) is different from 2. On the other hand, it was shown in \cite{[12]}, Example 8.4] that, if char \( K = 2, m = 1 \) and \( b_\bullet \) is nonzero, then the algebras \( \Lambda(S, \vec{T}, m, c) \) and \( \Lambda(S, \vec{T}, m, c, b_\bullet) \) are not isomorphic. We refer to \cite{[12]}, Section 8] and \cite{[13]}, Section 6] for socle equivalence of representation-infinite tame symmetric algebras associated to arbitrary triangulated surfaces with nonempty boundary.

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