Holomorphic contact via Pascal map and operator theory

Li Chen

Abstract: We give an intrinsic definition of point-wise contact between holomorphic Hermitian vector bundles by introducing a canonical bundle map, called the Pascal map, on their jet bundles. Combined with a holomorphic gluing condition, point-wise contact is further extended to contact along an analytic hyper-surface and we characterize it by the curvatures and their covariant derivatives on the vector bundles. The Pascal map has a realization in operator theory where our study applies to unitary equivalence of shift operators on function spaces.

Key words: contact; Pascal map; curvature; unitary equivalence

1 Introduction

Contact order between smooths maps is a useful extrinsic invariant, which historically implements a universal criteria for the congruence problem of determining if two maps can be identified up to a rigid motion in the ambient spaces([17]). In this paper we work in the holomorphic setting aiming at a refinement and extension of this classical notion in two directions. The first is an intrinsic formulation of contact which makes sense on vector bundles, and the second is to define and study contact along a sub-manifold which is an extension of the usual point-wise contact. As contact is a local notion, we assume all maps and vector bundles are defined on a domain $\Omega \subseteq \mathbb{C}^m$ throughout this paper.

The intrinsic definition of point-wise contact between holomorphic vector bundles will be given in Section 2.1. For a fixed positive integer $n$, we introduce a so-called Pascal matrix algebra and show that the generator of this algebra represents a well-defined bundle map, which we call the Pascal map, on the $n$-jet bundle $E^n$ of a holomorphic vector bundle $E$. In case $E$ admits an Hermitian metric, the existence of a linear isometry intertwining the Pascal maps turns out to well-define $n$-th order contact between the vector bundles at a point. In Section 2.2 we go beyond point-wise contact and use the Pascal map to define contact between the vector bundles along an analytic sub-manifold $Z$ in $\Omega$. It will be seen that although “contact along $Z$” is void for scaler functions, it makes a nontrivial sense on holomorphic vector bundles as “holomorphic glue” of point-wise contact along $Z$. 

1
Remark 1.1. For point-wise contact, we will mainly focus on one complex variable, that is, $\Omega \subseteq \mathbb{C}$, and the general situation is similar modulo a technical extension which will be discussed briefly in the end of Section 2.1. For contact along $Z$, we will focus on the case that $Z$ is an analytic hyper-surface, and the general case will be sketched in the end of Section 3.3.

Section 3 is devoted to geometric characterization of contact between holomorphic Hermitian vector bundles. As to compute geometric invariants via high order information is routine, it is a nontrivial problem to conversely recover the high order behaviours by geometric tensors. In our setting, it is a matter of capturing analytic information carried by the jet bundle $E^n$ via geometric invariants living on $E$. For point-wise contact this can be solved by an existing trick of working with properly chosen frames that is “normalized” at the point (we will review this solution in Section 3.2), hence in this paper, we mainly focus on contact along a hyper-surface. Compared to point-wise contact, the interesting part is to geometrically characterize the holomorphic gluing condition along the hyper-surface, which poses considerable algebraic complexities and requires a “hard” approach as we will present. Geometric tensors to be involved in this characterization include the curvature and covariant derivatives of the curvature on the vector bundles. Analogous to the role of contact in the congruence problem, the curvature and its covariant derivatives were historically used to determine if two connections on fiber bundles can be identified up to “gauge equivalence” ([5, 12, 24, 25]). Here we focus on the unique canonical connection on the holomorphic Hermitian vector bundles, and an important step as well as byproduct of our argument is to prove a recursive formula representing high order covariant derivative of the curvature in terms of lower order ones.

We take an expository Section 4 to discuss operator-theoretic motivation and application of our study. Let $E$ be a vector bundle associated to a holomorphic map into $Gr(l, \mathcal{H})(l$ dimensional subspaces in a Hilbert space $\mathcal{H}$), we consider a bounded linear operator $T$ on the ambient space $\mathcal{H}$ which extend coordinate multiplication on $E$. The restriction of this operator on $E^n$ trivially induces the Pascal map for every $n$(see Section 4.1), which can be seen as an extrinsic model for our intrinsic formulation in Section 2. Historically $T$ is called a Cowen-Douglas operator due to the seminal work of Cowen and Douglas [6], who introduced a geometric approach to study unitary equivalence for a large class of Hilbert space operators (see [23] for a survey on recent developments). In particular, our result on contact along hyper-surfaces implies a complete geometric reduction for unitary equivalence of certain quotient function spaces equipped with adjoint action of shift operators, refining the recent work by Douglas and the author [4].
2 Holomorphic contact via Pascal map

2.1 Pascal algebra, Pascal map and point-wise contact

In Section 2.1 we mainly focus on the one variable case $\Omega \subseteq \mathbb{C}$, and the next Section 2.2 concerns several complex variables $\Omega \subseteq \mathbb{C}^m$. We adopt the rule that a point in $\Omega \subseteq \mathbb{C}$ is denoted by $z, w, \ldots$, and a point in $\Omega \subseteq \mathbb{C}^m, m > 1$ is denoted by bold letters $\mathbf{z}, \mathbf{w}, \ldots$.

A familiar “extrinsic-intrinsic” transition is to start from a smooth map $f$ into a Grassmannian $Gr(l, \mathcal{H})(l$ dimensional subspaces in a vector space $\mathcal{H})$ and then pass to the associated vector bundle $E$ as the pull back of the tautological bundle by $f$(classifying theorem). Precisely, we assume that $f$ is a map from a domain $\Omega \subseteq \mathbb{C}^m$ to $Gr(l, \mathcal{H})$ which is holomorphic in the sense that for any point $z_0$ in $\Omega$, there exists a neighborhood $\Delta$ of $z_0$ and a holomorphic frame consisting of holomorphic $\mathcal{H}$-valued functions $s_1, \ldots, s_l$ on $\Delta$ such that

$$f(z) = \bigvee\{s_1(z), \ldots, s_l(z)\}$$

for every $z \in \Delta$. The vector bundle

$$E := \{(h, z) \in \mathcal{H} \times \Omega | h \in f(z)\}$$

associated to $f$ is a holomorphic vector bundle over $\Omega$ of rank $l$. To make sense of “rigid motion”, we assume that the ambient space $\mathcal{H}$ admits an inner product(Hilbert space) and correspondingly $E$ admits an Hermitian metric inherited from $\mathcal{H}$. Throughout Section 2.1 and next Section 2.2, “holomorphic vector bundle” or “holomorphic Hermitian vector bundle” will always mean vector bundles associated to holomorphic maps into $Gr(l, \mathcal{H})$, and it will be seen that for general holomorphic vector bundles our discussions still work(see Remark 2.16 below).

We begin with the definition of order $n$ point-wise contact between holomorphic maps into $Gr(l, \mathcal{H})$ over a domain of complex dimension one, which amounts to order $n$ agreement at a point up to a rigid motion(isometry) of the ambient space $\mathcal{H}$(see Sec.5,[17] or Sec.2,[6]):

**Definition 2.1.** Two holomorphic maps $f$ and $\tilde{f}$ from a domain $\Omega \subseteq \mathbb{C}$ into $Gr(l, \mathcal{H})$ are said to have contact of order $n$ at a point $z_0$ if there exist holomorphic frames $s = \{s_1, \ldots, s_l\}$ and $\tilde{s} = \{\tilde{s}_1, \ldots, \tilde{s}_l\}$ for $f$ and $\tilde{f}$ around $z_0$ such that the linear map defined by

$$s_i^{(k)}(z_0) \mapsto \tilde{s}_i^{(k)}(z_0), 1 \leq i \leq l, 0 \leq k \leq n$$

is isometric.

Contact order between holomorphic maps into Grassmannians are related to their fundamental forms which in turn determines local or global agreement of the maps up
For fixed positive integer $n$, the **Pascal algebra**, denoted by $\Lambda^n$, is the set
of all lower triangular \((n + 1) \times (n + 1)\) matrices of the form

\[
\begin{pmatrix}
  a_0 & 0 & 0 & \cdots & 0 \\
a_1 & a_0 & 0 & \cdots & 0 \\
a_2 & 2a_1 & a_0 & \cdots & 0 \\
a_3 & 3a_2 & 3a_1 & a_0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & \binom{n}{1}a_{n-1} & \cdots & \cdots & \binom{n}{1}a_1 & a_0
\end{pmatrix}.
\]

(2.3)

That is, a matrix in \(\Lambda^n\) is determined by \(n + 1\) complex numbers \(a_0, a_1, \ldots, a_n\) lying in the first column, whose \((i, j)\) entry is \(\binom{i-1}{j-1}a_{i-j}\) if \(1 \leq j \leq i \leq n + 1\) and 0 if \(j > i\).

Study of classical Pascal matrix \((a_0 = a_1 = \cdots = a_n = 1)\) starts from Call and Velleman [9] and numerous generalizations ([2, 3, 28, 29, 30]) followed since then, where various of rules were assigned on the dependence of \(a_n\) on \(n\). Here we allow arbitrary first column entries to get a bigger class, and the following lemma asserts that \(\Lambda^n\) is the commutant of a particular matrix in it (so \(\Lambda^n\) is indeed an algebra, which is not obvious from the definition).

**Lemma 2.3.** Let \(P\) be the matrix in \(\Lambda^n\) whose first column is given by \(a_1 = 1\) and \(a_k = 0, k \neq 1\), that is,

\[
P = \begin{pmatrix}
  0 & 0 & 0 & \cdots & 0 \\
  1 & 0 & 0 & \cdots & 0 \\
  0 & 2 & 0 & \cdots & 0 \\
  0 & 0 & 3 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & \cdots & n & 0
\end{pmatrix},
\]

(2.4)

then \(\Lambda^n\) is the commutant of \(P\).

**Remark 2.4.** One can check that the minimal polynomial of \(P\) equals its characteristic polynomial, hence \(\Lambda^n\) as its commutant is the algebra generated by \(P\).

**Proof.** To see \(\Lambda^n \subseteq P'\), let \(Q\) be in \(\Lambda^n\) and \((a_0, a_1, \ldots, a_n)^T\) be its first column, then both \(PQ\) and \(QP\) are lower triangular with zero diagonals, and it remains to compare their \((i, j)\) entries for \(j \leq i - 1\). A direct computation shows that the \((i, j)\) entry of \(QP\) is \(j^{(i-1)}a_{i-j-1}\) and the corresponding entry of \(PQ\) is \((i - 1)^{(i-2)}a_{i-j-1}\), hence the conclusion follows from the elementary identity \(j^{(i-1)} = (i - 1)^{(i-2)}\).

For the other direction, let \(Q\) be any \((n + 1) \times (n + 1)\) matrix such that \(PQ = QP\), and we have to show \(Q \in \Lambda^n\). To this end, we view \(P\) and \(Q\) as linear maps acting on an \(n + 1\) dimensional space with a fixed base \(\{s_0, s_1, \ldots, s_n\}\). Then \(P\) corresponds to the action \(Ps_0 = 0\) and \(Ps_k = ks_{k-1}\), \(k = 1, 2, \ldots, n\). Moreover, if \((a_0, a_1, \ldots, a_n)^T\) is the first column of \(Q\), then for every \(0 \leq k \leq n\), \(Qs_k = a ks_{s_0} + \) other terms involving \(s_1, \ldots s_n\).
Now it suffices to show if $PQ = QP$, then

$$Qs_k = \sum_{i=0}^{k} \binom{k}{i} a_{k-i}s_i, \ k = 0, 1 \cdots n \quad (2.5)$$

Comparing $PQs_0$ and $QP s_0$ one immediately sees that $Qs_0$ only has $s_0$ component, hence (2.5) holds for $k = 0$ and we verify (2.5) by induction on $k$. Precisely, it suffices to show

$$Qs_{k+1} = a_{k+1}s_0 + \sum_{i=1}^{k+1} \binom{k+1}{i} a_{k+1-i}s_i$$

assuming (2.5).

To this end, write

$$Qs_{k+1} = a_{k+1}s_0 + \sum_{i=1}^{n} b_is_i$$

for some coefficients $b_1, \cdots, b_n$, thus $PQs_{k+1} = \sum_{i=1}^{n} ib_is_{i-1}$. While by (2.5) we have

$$QP s_{k+1} = (k+1)Qs_k = (k+1) \sum_{i=0}^{k} \binom{k}{i} a_{k-i}s_i = (k+1) \sum_{i=1}^{k+1} \binom{k}{i-1} a_{k+1-i}s_{i-1}.$$ 

Now $PQ = QP$ implies $b_i = 0$ for $k + 2 \leq i \leq n$ and $b_i = \frac{(k+1)\binom{k}{i} a_{k+1-i}}{i} = \binom{k+1}{i} a_{k+1-i}$ for $1 \leq i \leq k + 1$, which gives (2.6) as desired.

\[ \square \]

**Remark 2.5.** It is easy to see that the block matrix version of Definition 2.2 makes sense and Lemma 2.3 holds as well. Precisely, replacing the scalars $a_0, \cdots, a_n$ in (2.3) by $l \times l$ matrices $A_0, \cdots A_n$, one gets an $(n+1) \times (n+1)$ block matrix whose $(i, j)$ block is $(\binom{i-1}{j-1})A_{i-j}$.

The set of these block matrices is still denoted by $\Lambda^n$, which is the commutant of

$$P = \begin{pmatrix}
0 & I & 0 & \cdots & 0 \\
I & 0 & 2I & \cdots & 0 \\
0 & I & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & nI
\end{pmatrix}. \quad (2.7)$$

In particular, the block matrix $\Lambda^n_A$ appearing in (2.2) lies in $\Lambda^n$ whose first column is $A, A', \cdots A^{(n)}$.

Now we are ready to introduce the promised Pascal map, which is a bundle map on $E^n$. Recall that given a collection of frames $\{s_\alpha\}$ with respect to a local trivialization of a vector bundle, the standard way to construct an unambiguously defined bundle map $\Phi$ is to give
a collection of matrix functions \( \{\Phi(s_\alpha)\} \) with respect to \( \{s_\alpha\} \) such that the compatibility condition

\[
\Phi(s_\alpha) = A_{\alpha\beta} \Phi(s_\beta) A_{\alpha\beta}^{-1}
\]  

(2.8)

holds, where \( A_{\alpha\beta} \) is the transition function between \( s_\alpha \) and \( s_\beta \) on their overlapping domains (so different matrices represent the same linear map). The following proposition asserts that on the \( n \)-jet bundle of a holomorphic vector bundle, a single constant matrix (2.7) will do.

**Proposition 2.6.** For a holomorphic vector bundle \( E \) over \( \Omega \subseteq \mathbb{C} \) and a positive integer \( n \), the constant block matrix \( P \) given by (2.7) represents a well defined bundle map on \( E^n \).

**Proof.** From the construction of \( E^n \) one sees that if a collection of frames \( \{s_\alpha\} \) gives a local trivialization for \( E \), then \( \{s_\alpha, s'_\alpha \cdots, s^{(n)}_\alpha\} \) gives a local trivialization for \( E^n \). If \( s \) and \( t \) are any two overlapping holomorphic frames of \( E \) with transition matrix \( A \), then the transition matrix for \( \{s, s' \cdots s^{(n)}\} \) and \( \{t, t' \cdots t^{(n)}\} \) is \( A^n \) (see (2.2) above). Now it suffices to verify the compatibility condition \( P = A^n P (A^n)^{-1} \), or equivalently, \( P A^n = A^n P \). As \( A^n \) lies in the Pascal algebra, \( P A^n = A^n P \) is guaranteed by the block matrix version of Lemma 2.3. 

\[\square\]

**Definition 2.7.** Let \( E \) be a holomorphic vector bundle over \( \Omega \subseteq \mathbb{C} \) and \( n \) be a positive integer, the bundle map on the \( n \)-jet bundle \( E^n \) well-defined by the constant block matrix

\[
P = \begin{pmatrix}
0 & & & \\
I & 0 & & \\
0 & 2I & 0 & \\
0 & 0 & 3I & 0 \\
& & \vdots & \ddots \\
0 & 0 & \cdots & nI \\
\end{pmatrix}
\]

is called the **Pascal map** on \( E^n \).

Explicitly, for any holomorphic frame \( s = \{s_1, \cdots, s_l\} \) of \( E \), the Pascal map (still denoted by \( P \)) acts on the local frame \( \{s(z), s'(z), \cdots, s^{(n)}(z)\} \) of \( E^n \) by

\[
P s_i^{(k)} = k s_i^{(k-1)}, 1 \leq k \leq n, \quad \text{and} \quad P s_i = 0
\]  

(2.9)

for all \( 1 \leq i \leq l \).

**Proposition 2.8.** Let \( f \) and \( \tilde{f} \) be holomorphic maps from \( \Omega \subseteq \mathbb{C} \) to \( \text{Gr}(l, \mathcal{H}) \) and \( E \) and \( \tilde{E} \) be their associated holomorphic Hermitian vector bundles. Fix a point \( z_0 \) in \( \Omega \), the followings are equivalent:

(i) \( f \) and \( \tilde{f} \) have contact of order \( n \) at \( z_0 \).

(ii) For certain (any) holomorphic frames \( s = \{s_1, \cdots, s_l\} \) and \( \tilde{s} = \{\tilde{s}_1, \cdots, \tilde{s}_l\} \) of \( E \) and \( \tilde{E} \) around \( z_0 \), there is a block matrix in \( \Lambda^n \) which represents a linear isometric map from \( E^n(z_0) \) to \( \tilde{E}^n(z_0) \) with respect to \( \{s(z_0), s'(z_0), \cdots, s^{(n)}(z_0)\} \) and \( \{\tilde{s}(z_0), \tilde{s}'(z_0), \cdots, \tilde{s}^{(n)}(z_0)\} \).
\( (iii) \) There is a linear isometric map \( \Phi \) from \( E^n(z_0) \) to \( \tilde{E}^n(z_0) \) such that \( \Phi P = \tilde{P} \Phi \), where \( P \) and \( \tilde{P} \) are Pascal maps on \( E^n \) and \( \tilde{E}^n \).

**Proof.** \((i)\Rightarrow (iii)\) Let \( s = \{s_1, \ldots, s_l\} \), \( \bar{s} = \{\bar{s}_1, \ldots, \bar{s}_l\} \) and \( \Phi \) be the frames and linear maps as in Definition 2.1 then \( \Phi \) trivially satisfies \( \Phi P = \tilde{P} \Phi \) since its representing matrix with respect to \( \{s(z_0), s'(z_0) \cdots, s^{(n)}(z_0)\} \) and \( \{\bar{s}(z_0), \bar{s}'(z_0) \cdots, \bar{s}^{(n)}(z_0)\} \) is the identity matrix.

\( (ii)\Rightarrow (i) \) Assuming (ii), there exists matrices \( A_0, A_1, \ldots, A_n \) such that with respect to certain holomorphic frames \( s = \{s_1, \ldots, s_l\} \) and \( \bar{s} = \{\bar{s}_1, \ldots, \bar{s}_l\} \) of \( E \) and \( \tilde{E} \), the linear map defined by

\[
\begin{pmatrix}
  s(z_0) \\
  s'(z_0) \\
  s''(z_0) \\
  \vdots \\
  s^{(n)}(z_0)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  A_0 & A_1 & A_2 & 2A_1 & A_0 \\
  A_1 & A_2 & 2A_1 & A_0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  A_n & \binom{n}{1}A_{n-1} & \cdots & \cdots & (n-1)A_1 & A_0 \\
\end{pmatrix}
\begin{pmatrix}
  \bar{s}(z_0) \\
  \bar{s}'(z_0) \\
  \bar{s}''(z_0) \\
  \vdots \\
  \bar{s}^{(n)}(z_0)
\end{pmatrix}
\]

is isometric. Set \( A(z) = \sum_{k=0}^{n} \frac{1}{k!}(z - z_0)^k A_k \), then \( A(z) \) is a holomorphic matrix function around \( z_0 \) with \( A^{(k)}(z_0) = A_k \), \( k = 0, 1 \cdots n \). Let \( \tilde{t} = A(z)\tilde{s} \), then \( \tilde{t} \) is a holomorphic frame for \( \tilde{E} \) (over a sufficiently small neighborhood of \( z_0 \)) and the frames \( s \) and \( \tilde{t} \) meets Definition 2.1

\( (iii)\Rightarrow (ii) \) For any holomorphic frames \( s \) and \( \tilde{s} \) of \( E \) and \( \tilde{E} \) around \( z_0 \), \( P \) and \( \tilde{P} \) are represented by (2.7) with respect to \( \{s(z_0), s'(z_0) \cdots, s^{(n)}(z_0)\} \) and \( \{\bar{s}(z_0), \bar{s}'(z_0) \cdots, \bar{s}^{(n)}(z_0)\} \), hence \( \Phi P = \tilde{P} \Phi \) holds if and only the representing matrix of \( \Phi \) commutes with (2.7), and this representing matrix has to be in \( A^n \) as a consequence of Lemma 2.3

Now we see that \((iii)\) of Proposition 2.8 as a frame-independent criteria is compatible with the original extrinsic Definition 2.1 hence it is eligible to be the intrinsic condition to well-define point-wise contact between holomorphic Hermitian vector bundles over a domain of complex dimension one.

**Definition 2.9.** Two holomorphic Hermitian vector bundles \( E \) and \( \tilde{E} \) over \( \Omega \subseteq \mathbb{C} \) are said to have contact of order \( n \) at a point \( z_0 \) if there is a linear isometric map \( \Phi \) from \( E^n(z_0) \) to \( \tilde{E}^n(z_0) \) such that \( \Phi P = \tilde{P} \Phi \), where \( P \) and \( \tilde{P} \) are Pascal maps on \( E^n \) and \( \tilde{E}^n \).

In the end of this subsection, we outline how to extend Definition 2.9 to several complex variables \( \Omega \subseteq \mathbb{C}^m \), which is not trivial but is just a technical issue and the details are left to interested readers.

Firstly, the several variable version of Definition 2.1 is straightforward, which now requires that for certain holomorphic frames \( s \) and \( \bar{s} \), the map

\[
\partial^l s_i(z_0) \mapsto \partial^l \bar{s}_i(z_0), 1 \leq i \leq l, 0 \leq |I| \leq n
\]
is isometric, where \( \partial^I = \partial^I z_1 \cdots \partial^I z_m \) for a multi-index \( I = (i_1, \cdots, i_m) \), and \( |I| = i_1 + \cdots + i_m \).

For the intrinsic definition, one will need \( m \) Pascal maps acting on the \( n \)-jet bundle. Precisely, fix a point \( z \in \Omega \) and a holomorphic frame \( s = \{ s_1 \cdots s_l \} \) of \( E \) around \( z \), set

\[
E^n(z) := \bigvee_{1 \leq i \leq l, 0 \leq |I| \leq n} \partial^I s_i(z),
\]

and for any \( 1 \leq k \leq m \), we define a linear map on \( E^n(z) \) by

\[
\partial^I s_i(z) \mapsto \left\{ \begin{array}{ll}
i_k \partial_{z_1}^{i_1} \cdots \partial_{z_k}^{i_k-1} \cdots \partial_{z_m}^{i_m} s_i(z), & i_k \geq 1 \\
0, & i_k = 0
\end{array} \right. \tag{2.11}
\]

where \( 1 \leq i \leq l, I = (i_1, \cdots, i_m) \).

One need to check that if \( t = \{ t_1, \cdots t_l \} \) is another holomorphic frame of \( E \) around \( z \), then \( \bigvee_{1 \leq i \leq l, 0 \leq |I| \leq n} \partial^I s_i(z) \) and \( \bigvee_{1 \leq i \leq l, 0 \leq |I| \leq n} \partial^I t_i(z) \) is the same space so the \( n \)-jet bundle \( E^n \) of \( E \) is well-defined. Moreover, with respect to either \( \{ \partial^I s_i(z), 1 \leq i \leq l, 0 \leq |I| \leq n \} \) or \( \{ \partial^I t_i(z), 1 \leq i \leq l, 0 \leq |I| \leq n \} \), the rule (2.11) represents the same linear map on \( E^n(z) \).

With these issues checked, one sees that for every \( 1 \leq k \leq m \), (2.11) gives a well defined bundle map, called the \textbf{k-th Pascal map} on \( E^n \) (denoted by \( P_k \)). Moreover, one can prove the following several variable version of Proposition 2.8.

**Proposition 2.10.** Let \( E \) and \( \tilde{E} \) be two holomorphic Hermitian vector bundles over \( \Omega \subseteq \mathbb{C}^m \) and \( n \) be a positive integer. Fix a point \( z_0 \in \Omega \), the followings are equivalent:

(i) There exists holomorphic frames \( s = \{ s_1, \cdots s_l \} \) and \( \tilde{s} = \{ \tilde{s}_1, \cdots \tilde{s}_l \} \) of \( E \) and \( \tilde{E} \) around \( z_0 \) such that the linear map defined by

\[
\partial^I s_i(z_0) \mapsto \partial^I \tilde{s}_i(z_0), 1 \leq i \leq l, 0 \leq |I| \leq n
\]

is isometric.

(ii) There is a linear isometric map \( \Phi \) from \( E^n(z_0) \) to \( \tilde{E}^n(z_0) \) such that \( \Phi P_k = \tilde{P}_k \Phi \) for all \( 1 \leq k \leq m \), where \( P_k \) and \( \tilde{P}_k \) are \( k \)-th Pascal maps on \( E^n \) and \( \tilde{E}^n \).

Finally, condition (ii) above implements the several variable extension of Definition 2.9.

**Definition 2.11.** Two holomorphic Hermitian vector bundles \( E \) and \( \tilde{E} \) over \( \Omega \subseteq \mathbb{C}^m \) are said to have order \( n \) contact at \( z_0 \) if there is a linear isometric map \( \Phi \) from \( E^n(z_0) \) to \( \tilde{E}^n(z_0) \) such that \( \Phi P_k = \tilde{P}_k \Phi \) for all \( 1 \leq k \leq m \), where \( P_k \) and \( \tilde{P}_k \) are \( k \)-th Pascal maps on \( E^n \) and \( \tilde{E}^n \).
2.2 Contact along a hyper-surface

In this subsection we define contact between holomorphic Hermitian vector bundles along an analytic hyper-surface $Z$ in $\Omega \subseteq C^m$. As this paper only involves local geometry, without loss of generality we assume throughout this paper that $Z$ is a coordinate slice of the form

$$Z = \{(0, z_2, \cdots, z_m) | (z_2, \cdots, z_m) \in \Omega'\}$$

for some domain $\Omega' \subseteq C^{m-1}$.

Unlike point-wise contact, contact along a hyper-surface is not a standard notion even for functions, hence before working on vector bundles we start naively with holomorphic functions to illustrate the motivation.

Let $f$ and $\tilde{f}$ be two holomorphic functions on $\Omega$, we list two conditions on their high order relation on $Z$ as follows

(i) $$\partial^I f(0, z_2, \cdots, z_m) = \partial^I \tilde{f}(0, z_2, \cdots, z_m), \quad 0 \leq |I| \leq n$$

for all $(z_2, \cdots, z_m) \in \Omega'$, that is, $f$ and $\tilde{f}$ have order $n$ contact at all points on $Z$.

(ii) $$\partial_{z_1}^{i_1} f(0, z_2, \cdots, z_m) = \partial_{z_1}^{i_1} \tilde{f}(0, z_2, \cdots, z_m), \quad 0 \leq i_1 \leq n$$

for all $(z_2, \cdots, z_m) \in \Omega'$.

It is easy to see that the above two conditions are equivalent: $(i) \Rightarrow (ii)$ is trivial, and starting from $(ii)$, one actually gets, by taking derivative with respect to $z_2, \cdots, z_m$ along $Z$, that (2.12) hold for all $I = (i_1, \cdots, i_m)$ provided $i_1 \leq n$. On one hand, we see that for functions, “contact along $Z$” is nothing more than “point-wise contact everywhere on $Z”’ hence is not of much interest. On the other hand, this suggests that to compare high order behaviours along $Z$ is a matter of comparing the derivatives along the transverse direction with respect to $Z$ as indicated by (2.13).

It is not hard to formulate analogues of $(i)$ and $(ii)$ above for two holomorphic Hermitian vector bundles $E$ and $\tilde{E}$. The first is straightforward with Definition 2.11.

$(i')$ For every $z \in Z$, there is a linear isometry $\Phi(z)$ from $E^n(z)$ to $\tilde{E}^n(z)$ such that $\Phi(z)P_k = \tilde{P}_k\Phi(z)$, $1 \leq k \leq m$, where $P_k$ and $\tilde{P}_k$ are $k$-th Pascal maps on $E^n$ and $\tilde{E}^n$. In other words, $E$ and $\tilde{E}$ have order $n$ contact everywhere on $Z$.

For analogue of $(ii)$ where only $z_1$-derivative is involved, we use the $z_1$-jet instead of the full jet. Precisely, let $E^n_{z_1}$ be the $n$-jet bundle with respect to $z_1$ of $E$, whose fiber at a point $z$ is given by

$$E^n_{z_1}(z) := \bigvee_{1 \leq i \leq n, 0 \leq k \leq n} \{\partial_{z_1}^{(k,s_i)}(z)\}$$

(2.14)
where \( \{s_1, \ldots, s_l\} \) is any holomorphic frame of \( E \). The bundle \( E^n_{z_1} \) is a sub-bundle of the full jet bundle \( E^n \) (see (2.10) above) on which the first Pascal map \( P_1 \) acts by

\[
P_1 \partial^{(k)} z_i(z) = k \partial^{(k-1)} z_i(z), \quad 1 \leq k \leq n, \quad \text{and} \quad P_1 s_i(z) = 0
\]  

(2.15)

for all \( 1 \leq i \leq l \), and the remaining Pascal maps \( P_2, \ldots P_m \) vanishes on \( E^n_{z_1} \) according to (2.11). Now the vector bundle version of (ii) goes as follows

\[(ii') \quad \text{For every } z \in Z, \text{ there is a linear isometry } \Phi(z) \text{ from } E^n_{z_1}(z) \text{ to } \tilde{E}^n_{z_1}(z) \text{ such that } \Phi(z)P_1 = \tilde{P}_1 \Phi(z).\]

The relation between (i') and (ii') is much less obvious than their original versions for functions. To see (i') \( \Rightarrow \) (ii'), one need the fact that if \( \Phi(z) \) is as in (i'), then it maps the subspace \( E^n_{z_1}(z) \) into \( \tilde{E}^n_{z_1}(z) \) (hence the restriction of \( \Phi(z) \) on \( E^n_{z_1}(z) \) meets (ii')), the verification of which involves a combination of (2.11) and the intertwining property \( \Phi(z)P_k = \tilde{P}_k \Phi(z) \), \( 1 \leq k \leq m \). As we will not need (i') \( \Rightarrow \) (ii') elsewhere in this paper, the detail is left to interested readers.

On the other hand, the direction (ii') \( \Rightarrow \) (i') does not hold anymore. In fact, as the isometry \( \Phi(z) \) varies from fiber to fiber, the argument of “taking derivative along \( Z \)”, which worked for (ii) \( \Rightarrow \) (i), now fails to yield (ii') \( \Rightarrow \) (i'). This suggests the existence of a nontrivial definition of “contact along \( Z \)” properly stronger than “point-wise contact everywhere on \( Z \)”, where the holomorphic structure of the restriction bundle \( E^n_{z_1}|_Z \) over \( Z \) should be taken into account. Precisely, we give the following definition which strengthens (ii') by assuming that \( \Phi(z) \) varies with \( z \) in a holomorphic way.

**Definition 2.12.** Two holomorphic Hermitian vector bundles \( E \) and \( \tilde{E} \) over \( \Omega \in \mathbb{C}^m \) are said to have contact of order \( n \) along the analytic hyper-surface

\[
Z = \{(0, z_2, \ldots, z_m)|(z_2, \ldots, z_m) \in \Omega'\}
\]

if there exists a holomorphic isometric bundle map \( \Phi \) from \( E^n_{z_1}|_Z \) to \( \tilde{E}^n_{z_1}|_Z \) such that \( \Phi P_1 = \tilde{P}_1 \Phi \).

**Remark 2.13.** Holomorphic bundle maps are represented by holomorphic matrix functions with respect to holomorphic frames. The existence of a holomorphic isometric bundle map between two holomorphic Hermitian vector bundles intertwines both holomorphic and Hermitian structure, so in this case the bundles are said to be equivalent.

The following analogue of Proposition 2.8 and its consequence Corollary 2.15 justifies Definition 2.12.
Proposition 2.14. Let $E$ and $\tilde{E}$ be two holomorphic Hermitian vector bundles over $\Omega \in \mathbb{C}^m$ and $$Z = \{(0, z_2, \cdots, z_m)(z_2, \cdots, z_m) \in \Omega'\}$$ is an analytic hyper-surface in $\Omega$. The followings are equivalent

(i) There exists holomorphic frames $s = \{s_1, \cdots, s_l\}$ and $\tilde{s} = \{\tilde{s}_1, \cdots, \tilde{s}_l\}$ of $E$ and $\tilde{E}$ such that for all $z \in Z$, the linear map defined by

$$\partial_{z_1}^k s_i(z) \mapsto \partial_{z_1}^k \tilde{s}_i(z), 0 \leq k \leq n, 1 \leq i \leq l$$

is isometric.

(ii) For certain (any) holomorphic frames $s = \{s_1, \cdots, s_l\}$ and $\tilde{s} = \{\tilde{s}_1, \cdots, \tilde{s}_l\}$ of $E$ and $\tilde{E}$, there exists holomorphic matrix-valued functions $A_0(z), A_1(z), \cdots, A_n(z)$ over $Z$ such that for all $z \in Z$, the linear map defined by

$$\begin{pmatrix} s(z) \\ \partial_{z_1} s(z) \\ \partial_{z_1}^2 s(z) \\
\vdots \\
\partial_{z_1}^n s(z) \end{pmatrix} \mapsto \begin{pmatrix} A_0(z) \\ A_1(z) \\ \vdots \\ A_n(z) \end{pmatrix}$$

and

$$\begin{pmatrix} \tilde{s}(z) \\ \partial_{z_1} \tilde{s}(z) \\ \partial_{z_1}^2 \tilde{s}(z) \\
\vdots \\
\partial_{z_1}^n \tilde{s}(z) \end{pmatrix}$$

is isometric.

(iii) $E$ and $\tilde{E}$ have contact of order $n$ along $Z$, that is, Definition 2.12 holds.

Proof. Comparing (2.1) and (2.9) with (2.14) and (2.15), it is easy to see that we are now in exactly the same algebraic setting as Section 2.1 so (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) follows in the same way as in Proposition 2.8.

(ii) $\Rightarrow$ (i): For any $z \in \mathbb{C}^m$, we write $z = (z_1, z')$ for simplicity, where $z' = (z_2 \cdots z_m) \in \mathbb{C}^{m-1}$. Assuming (ii), let $A(z) = \sum_{k=0}^{\infty} \frac{1}{k!} A_k(0, z') z_1^k$, then $\partial_{z_1}^k A(0, z') = A_k(0, z')$, for all $k = 0, 1 \cdots, n$ and $(0, z') \in Z$. Let $\tilde{t} = A(z)\tilde{s}$, then $\tilde{t}$ is a holomorphic frame for $\tilde{E}$ and the frames $s$ and $\tilde{t}$ meets (i).

Finally we observe that Definition 2.12 implies point-wise contact (in the sense of Definition 2.11) everywhere on $Z$:

Corollary 2.15. If two holomorphic Hermitian vector bundles $E$ and $\tilde{E}$ over $\Omega \in \mathbb{C}^m$ have contact of order $n$ along the analytic hyper-surface $Z$, then they have contact of order $n$ at all points on $Z$.

Proof. In fact, condition (i) of Proposition 2.14 implies that

$$\langle \partial_{z_1}^k s_i(0, z'), \partial_{z_1}^k s_j(0, z') \rangle = \langle \partial_{z_1}^k \tilde{s}_i(0, z'), \partial_{z_1}^k \tilde{s}_j(0, z') \rangle, 1 \leq i, j \leq l, 0 \leq p, q \leq n$$
for all \((0, z') \in Z\), where \(\langle \cdot, \cdot \rangle\) denotes the Hermitian inner products. Since the frames are holomorphic, for any multi-index \(K = (k_1, k_2, \ldots, k_m)\) and \(L = (l_1, l_2, \ldots, l_m)\) with \(k_1 = l_1 = 0\), it is valid to applying \(\partial^p \overline{\partial}^q\) to the equation above which yields
\[
\langle \partial_{z_1}^p \partial^K s_i(0, z'), \partial_{z_1}^q \partial^L s_j(0, z') \rangle = \langle \partial_{z_1}^p \partial^K \tilde{s}_i(0, z'), \partial_{z_1}^q \partial^L \tilde{s}_j(0, z') \rangle, 1 \leq i, j \leq l, 0 \leq p, q \leq n.
\]
This implies that for any \(z \in Z\), condition \((i)\) of Proposition \ref{pascal_map} actually holds for all multi-index \(I = (i_1, \ldots, i_m)\) provided \(i_1 \leq n\).

**Remark 2.16.** The advantage of working with a vector bundle \(E\) associated to a holomorphic map into \(\text{Gr}(l, \mathcal{H})\) is that frames \(s_1, \ldots, s_i\) of \(E\) are holomorphic \(\mathcal{H}\)-valued functions hence their derivatives \(s_i^{(k)}\) makes sense, and these derivatives constantly appeared throughout Section 2 since the construction of the jet bundle. We observe that for a general holomorphic vector bundle where it is not so straightforward to make sense of “high order derivative” of its sections, discussions in Section 2 still work with a slight modification.

In fact, the standard way to define \(E^n\) for general \(E\) is via transition functions. Precisely, let \(\{U_\alpha\}\) be a local trivialization of \(E\) and \(\{A_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \emptyset}\) be the corresponding set of transition matrices, then one can check that the matrix functions \(\{\Lambda_{\alpha\beta}^n\}_{U_\alpha \cap U_\beta \neq \emptyset}\) satisfy the compatibility condition \(\Lambda_{\alpha\beta}^n \Lambda_{\beta\gamma}^n = \Lambda_{\alpha\gamma}^n\) when \(U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset\) hence well defines a vector bundle which is the \(n\)-jet bundle \(E^n\). Therefore our construction of the Pascal map, which only depends on the particular form of transition matrices, still works.

Moreover, if \(E\) admits an Hermitian metric such that \(\{H_\alpha\}\) is the Gram matrix for the local frame on \(U_\alpha\), then one can check that \(\{\langle \partial_p \overline{\partial}^q H_\alpha \rangle_{0 \leq p, q \leq n}\}\) also glue to a well defined Hermitian form on \(E^n\), and one can avoid the appearance of \(s_i^{(k)}\) in Section 2 by using derivatives of the Gram matrices of the original frame of \(E\). For instance, condition \((i)\) of Proposition \ref{pascal_map} should be replaced by \(\partial_{z_1}^p \overline{\partial}_{z_1}^q H = \partial_{z_1}^p \overline{\partial}_{z_1}^q \tilde{H}, 0 \leq p, q \leq n\) where \(H\) and \(\tilde{H}\) are Gram matrices for \(s\) and \(\tilde{s}\) respectively, and condition \((ii)\) of Proposition \ref{pascal_map} should be replaced by
\[
[\partial_{z_1}^p \overline{\partial}_{z_1}^q H(z)]_{0 \leq i, j, k \leq n} = \Lambda [\partial_{z_1}^p \overline{\partial}_{z_1}^q \tilde{H}(z)]_{0 \leq i, j, k \leq n},
\]
where \(\Lambda\) is the block matrix appearing in \ref{pascal_map}.

## 3 Geometric characterization

This section is devoted to characterizing contact between two holomorphic Hermitian vector bundles along a hyper-surface
\[
Z = \{(0, z_2, \ldots, z_m) | (z_2, \ldots, z_m) \in \Omega'\}
\]
in terms of curvatures and their covariant derivatives on the vector bundles, and the main result of this section is Theorem \ref{main_theorem}. We fix terminologies and notations on differential geometry(extracted from \[\text{[5, 18, 27]}\]) in Section 3.1. In Section 3.2 we state Theorem \ref{main_theorem} and give a series of reductions of it. The complete proof will be finished in Section 3.3.
3.1 Curvature and their covariant derivatives

Let \( E \) be a holomorphic Hermitian vector bundle of rank \( l \) over \( \Omega \subset \mathbb{C}^m \), then there exists a unique canonical connection on \( E \), denoted by \( D \), which is metric-preserving and compatible with the holomorphic structure. The curvature with respect to this canonical connection is of form \((1, 1)\) hence can be expressed as

\[
\mathcal{K} = \sum_{1 \leq i,j \leq m} \mathcal{K}_{ij} dz_i \wedge d\overline{z}_j, \tag{3.1}
\]

where the components \( \{\mathcal{K}_{ij}\} \) are bundle maps on \( E \).

Both \( D \) and \( \mathcal{K} \) acts on sections of \( E \). Given a local holomorphic frame \( s = (s_1, \cdots, s_l) \) of \( E \) with Gram matrix \( H = [(s_p, s_q)]_{1 \leq p,q \leq l} \), then with respect to \( s \), the representing matrix \( D(s) \) of \( D \) is given by

\[
D(s) = \partial H \cdot H^{-1}, \tag{3.2}
\]

and the representing matrix for the curvature component is

\[
\mathcal{K}_{ij}(s) = \partial_{\overline{z}_j}(\partial_{z_i} H \cdot H^{-1}) = (\partial_{z_i} \partial_{\overline{z}_j} H - \partial_{z_i} H \cdot H^{-1} \cdot \partial_{\overline{z}_j} H)H^{-1} \tag{3.3}
\]

for \( 1 \leq i, j \leq m \)

**Remark 3.1.** In this paper we adopt the “left action” convention regarding to representing matrices for linear maps. Precisely, let \( \Phi \) be a linear map on a linear space spanned by \( \gamma = (\gamma_1, \cdots, \gamma_n) \), then a matrix \( A = [a_{ij}] \) represents \( \Phi \) if \( \Phi \gamma_i = \sum a_{ij} \gamma_j \), or \( (\Phi \gamma_1, \cdots, \Phi \gamma_n)^T = A(\gamma_1, \cdots, \gamma_n)^T \). In some literatures the representing matrices for \( D \) and \( \mathcal{K} \) are given by \( H^{-1} \partial H \) and \( \overline{\partial}(H^{-1} \partial H) \) since the opposite “right action” convention applies there.

For a bundle map \( \Phi \) on \( E \), one can define the covariant differentiation of \( \Phi \), denoted by \( D\Phi \), in a unique way so that for any section \( s \) of \( E \), the Leibnitz rule holds:

\[
D(\Phi s) = (D\Phi) s + \Phi Ds. \tag{3.4}
\]

Write

\[
D\Phi = \sum_{1 \leq i \leq m} (\Phi_{z_i} \otimes dz_i + \Phi_{\overline{z}_i} \otimes d\overline{z}_i), \tag{3.5}
\]

the components \( \Phi_{z_i}, \Phi_{\overline{z}_i} \) are called covariant derivatives of \( \Phi \) with respect to the \( z_i \) and \( \overline{z}_i \) direction. Let \( \Phi(s) \) be the representing matrix of \( \Phi \) with respect to a holomorphic frame \( s \) and \( H \) be the Gram matrix of \( s \), then a routine computation gives the following matrix representations of \( \Phi_{z_i} \) and \( \Phi_{\overline{z}_i} \) with respect to \( s \):

\[
(\Phi_{z_i})(s) = \partial_{z_i} \Phi(s) - \partial_{z_i} H \cdot H^{-1} \Phi(s) + \Phi(s) \partial_{\overline{z}_i} H \cdot H^{-1} \quad \text{and} \quad (3.6)
\]
\begin{equation}
(\Phi_{z_i})(s) = \partial_{z_i} \Phi(s). \tag{3.7}
\end{equation}

As covariant derivative of a bundle map is again a bundle map, one can continue the
procedure to define various kinds of high order covariant derivatives and use (3.6) (3.7)
to compute their matrix representations.

\textbf{Remark 3.2.} In general, the value of high order covariant derivative depends on
the order of differentiation (for instance, \((\Phi_{z_i}z_j)_{z_i}\) is not equal to \((\Phi_{z_i})_{z_i}\)
unless \(\Phi\) commutes with the \(i \leftrightarrow j\) components of the curvature). Later we use
\((\Phi)_{z_i}z_j^r\) to denote \(r\)-th order covariant derivative of \(\Phi\) with respect to the
\(z_i\) direction, and the notation \((\Phi)_{z_i}z_j^r\) means \(t\)-th order covariant
derivative of \((\Phi)_{z_i}z_j^r\) with respect to the \(z_j\) direction.

Let \(\Phi^*\) be the adjoint of a given bundle map \(\Phi\) in the sense that
\begin{equation}
\langle \Phi \xi, \eta \rangle = \langle \xi, \Phi^* \eta \rangle \tag{3.8}
\end{equation}
for all \(\xi, \eta\) at the same fiber. We will need (both here and Section 3.3) a standard result
in linear algebra on the relation between adjoint of a linear map and adjoint (conjugate transpose)
of its representing matrix.

\textbf{Lemma 3.3.} Let \(\Phi\) be a linear map on a finite dimensional inner product space and \(\gamma =
\{\gamma_1, \cdots, \gamma_n\}\) be a base whose Gram matrix is \(H\). If \(\Phi\) is represented by
a matrix \(A\) with respect to \(\gamma\), then its adjoint map is represented by \(HA^*H^{-1}\),
where \(A^*\) denotes the conjugate transpose of \(A\).

We will use the upper case “*” to denote adjoint of both bundle maps and matrices,
applying which to bundle maps (such as \(K_{ij}, \Phi \cdot \cdot \cdot\)) means adjoint in the sense of (3.8)
and to matrices (such \(K_{ij}(s), \Phi(s), H \cdot \cdot \cdot\)) means conjugate transpose. We end this subsection with
the following well-known lemma and record a computational proof to reveal the relation
between formula (3.6) and (3.7).

\textbf{Lemma 3.4.} (i) Let \(\Phi\) be a bundle map on \(E\), for any \(1 \leq i \leq m\), \((\Phi_{z_i})^* = (\Phi^*)_{z_i}\) and
\((\Phi_{z_i})^* = (\Phi^*)_{z_i}\).

(ii) For any \(1 \leq i \leq m\), \(K_{ij} = K_{ji}^*\)

\textbf{Proof.} (i) Observing \(\Phi^{**} = \Phi\), it suffices to prove the first identity. By Lemma 3.3, it suffices
to show
\[H(\partial_{z_i} \Phi(s) - \partial_{z_i} H \cdot H^{-1} \Phi(s) + \Phi(s) \partial_{z_i} H \cdot H^{-1})^*H^{-1} = \partial_{z_i}(H \Phi^*(s)H^{-1})\]
in light of (3.6) and (3.7).

Now a straightforward computation yields
\[RHS = (\partial_{z_i} H) \Phi^*(s)H^{-1} + H(\partial_{z_i} \Phi^*(s))H^{-1} - H \Phi^*(s)H^{-1}(\partial_{z_i} H)H^{-1}.\]
Since $(\partial_z \Phi(s))^* = \partial_{\bar{z}} \Phi^*(s)$, $H^* = H$ and $(\partial_{\bar{z}} H)^* = \partial_z H$, it holds that

$$LHS = H(\partial_z \Phi^*(s) - \Phi^*(s) H^{-1} \partial_{\bar{z}} H + H^{-1} \partial_{\bar{z}} H \Phi^*(s)) H^{-1}$$

which exactly equals $RHS$.

(ii) It suffices to show

$$K_{ij}(s) = H K_{ji}^*(s) H^{-1}$$

which is easy to verify by (3.3).

\[\square\]

### 3.2 Statement and reduction of the main result

We are ready to state the main result of Section 3 on geometric characterization of contact between holomorphic Hermitian vector bundles along $Z$. Roughly, the geometric condition involves

(a) agreement of the two bundles on $Z$

(b) agreement of their curvatures and covariant derivatives of the curvatures to certain order on $Z$.

The restrictions $E|_Z$ and $\tilde{E}|_Z$ are holomorphic Hermitian vector bundles on the complex manifold $Z$, hence their “agreement” naturally means equivalence in the sense of Remark 2.13 that is, there exists a holomorphic isometric bundle map, say $\Psi$, from $E|_Z$ to $\tilde{E}|_Z$. Moreover, the curvature and their covariant derivatives are linear bundle maps, so condition (b) should mean that they can be intertwined by $\Psi$.

With notations in Section 3.1((3.1) and Remark 3.2), the main result of this section states as follows

**Theorem 3.5.** Two holomorphic Hermitian vector bundles $E$ and $\tilde{E}$ over $\Omega \in \mathbb{C}^m$ have contact of order $n$ along the analytic hyper-surface

$$Z = \{(0, z_2, \ldots, z_m) | (z_2, \ldots, z_m) \in \Omega'\}$$

if and only if there exists a holomorphic isometric bundle map $\Psi$ from $E|_Z$ to $\tilde{E}|_Z$ such that

(i) for $r, t \leq n - 1$, $\Psi(K_{1r})_{z_1}^{\bar{z}_t} = (\tilde{K}_{1r})_{z_1}^{\bar{z}_t} \Psi$ on $Z$

(ii) for $r \leq n - 1$, $2 \leq j \leq m$, $\Psi(K_{1j})_{z_1} = (\tilde{K}_{1j})_{z_1} \Psi$ on $Z$

Let $\Phi$ and $\Psi$ be as in Definition 2.12 and Theorem 3.5 respectively, then Theorem 3.5 is a matter of the relation between $\Phi$ and $\Psi$. More precisely, if there is a bundle map $\Phi$ satisfying Definition 2.12, then by Proposition 2.14, $\Phi$ has a lower triangular matrix representation (2.16), hence its restriction on $E|_Z$, represented by the left upper block $A_0$, is an isometric holomorphic bundle map from $E|_Z$ to $\tilde{E}|_Z$. This assures the existence of $\Psi$ as a necessary
condition for the existence of $\Phi$, and Theorem 3.5 can be reduced to the following extension theorem.

**Theorem 3.6.** Given a holomorphic isometric bundle map $\Psi$ from $E|_Z$ to $\tilde{E}|_Z$, it can be extended to a holomorphic isometric bundle map $\Phi$ from $\tilde{E}_n|_Z$ to $\tilde{E}_n|_Z$ such that $\Phi P_1 = \tilde{P}_1 \Phi$ if and only if

(i) for $r, t \leq n-1$, $\Psi(K_{17})_{z_1} = (\tilde{K}_{17})_{z_1} \Psi$ on $Z$

(ii) for $r \leq n-1, 2 \leq j \leq m$, $\Psi(K_{17})_{z_1} = (\tilde{K}_{17})_{z_1} \Psi$ on $Z$.

The proof of Theorem 3.6 will be a combination of “point-wise extension” and “holomorphic glue”, which are addressed respectively by condition (i) and (ii). Precisely, Theorem 3.6 can be reduced to the following two propositions.

**Proposition 3.7.** Fix $z \in Z$ and let $\Psi$ be an isometric linear map from $E(z)$ to $\tilde{E}(z)$, then $\Psi$ can be extended to an isometric linear map $\Phi : \tilde{E}_n(z) \mapsto \tilde{E}_n(z)$ such that $\Phi P_1 = \tilde{P}_1 \Phi$ if and only if $\Psi(K_{17})_{z_1} = (\tilde{K}_{17})_{z_1} \Psi$ for $r, t \leq n-1$ at $z$. Moreover, the extension is unique.

If $\Psi$ is an isometric bundle map from $E|_Z$ to $\tilde{E}|_Z$, then applying Proposition 3.7 point by point on $Z$, one sees that $\Psi$ admits an isometric extension $\Phi$ on $E^n$ satisfying $\Phi P_1 = \tilde{P}_1 \Phi$ if and only if condition (i) of Theorem 3.6 holds. So to complete the proof of Theorem 3.6, it remains to prove the following Proposition 3.8 which asserts that with the additional condition that $\Psi$ is holomorphic, its unique extension $\Phi$ is holomorphic if and only if condition (ii) of Theorem 3.6 holds.

**Proposition 3.8.** Let $\Psi$ be a holomorphic isometric bundle map from $E|_Z$ to $\tilde{E}|_Z$ which admits an isometric extension $\Phi$ from $\tilde{E}_n|_Z$ to $\tilde{E}_n|_Z$ satisfying $\Phi P_1 = \tilde{P}_1 \Phi$. Then $\Phi$ is holomorphic if and only if $\Psi(K_{17})_{z_1} = (\tilde{K}_{17})_{z_1} \Psi$ for $r \leq n-1, 2 \leq j \leq m$ on $Z$.

Proposition 3.7 as a “point-wise” result is not new (see Proposition 2.18 [6] or Proposition 24 [4]) which can be proved by a trick of working with a “normalized frame” that behaves well at the point, and the “hard” one is Proposition 3.8 which will be proved in Section 3.3. We end this subsection by recording a proof of Proposition 3.7 for completeness. In particular, the proof of uniqueness in Proposition 3.7 is a preparation for the proof of Proposition 3.8.

**Proof of Proposition 3.7**

We first prove uniqueness of extension, that is, the extension $\Phi$, if exists, is uniquely determined by $\Psi$. Fix holomorphic frames $s$ and $\tilde{s}$ of $E$ and $\tilde{E}$, then since $\Phi$ intertwines $P_1$ and $\tilde{P}_1$, it has the following matrix representation.
where left upper block $A_0$ represents $\Phi$ with respect to $s(z)$ and $\tilde{s}(z)$. Now it suffices to show that the remaining $n$ matrices $A_1, \ldots, A_n$ at the first column are uniquely determined by $A_0$.

To this end, let $H$ and $\tilde{H}$ be Gram matrices for $s$ and $\tilde{s}$ and let $\Lambda$ denote the block matrix in (3.9), then since $\Phi$ is an isometry, it holds that

$$[\partial^i_{z_1} \bar{\partial}^j_{z_1} H(z)]_{0 \leq i, j \leq n} = \Lambda[\partial^i_{z_1} \bar{\partial}^j_{z_1} \tilde{H}(z)]_{0 \leq i, j \leq n} \Lambda^*$$

(3.10)

where both sides are $(n + 1) \times (n + 1)$ block matrices. In particular, comparing blocks at the first column of the two side of (3.10) gives

$$H = A_0 \tilde{H} A_0^*$$

(3.11)

$$\partial^i_{z_1} H(z) = \sum_{i=0}^{l} \left( \begin{array}{c} l \\ i \end{array} \right) A_{l-i}(\partial^i_{z_1} \tilde{H}(z)) A_0^*$$

(3.12)

which together implies

$$A_l = \partial^i_{z_1} H(z) \cdot H^{-1}(z) A_0 - \sum_{i=1}^{l} \left( \begin{array}{c} l \\ i \end{array} \right) A_{l-i} \partial^i_{z_1} \tilde{H}(z) \cdot \tilde{H}^{-1}(z).$$

(3.13)

This is a recursive formula representing $A_l$ in terms of $A_0, \ldots, A_{l-1}$, so all blocks are determined by $A_0$.

Now we turn to existence of extension and the following proof can be found in Proposition 2.18 [6] or Proposition 24 [3].

The given linear map $\Psi$ corresponds to a matrix $A_0$ with respect to $s(z)$ and $\tilde{s}(z)$ which fits into the left upper block of (3.9), and to construct the extension $\Phi$ amounts to finding $n$ matrices $A_1, \ldots, A_n$ such that (3.10) holds.

From the proof of uniqueness, one sees that $A_1, \ldots, A_n$ must satisfy the recursive formula (3.13) which guarantees that the two sides of (3.10) have the same blocks on the first column and it remains to show the other blocks are also equal by the geometric condition $\Psi(K_{1T})_{z_1}^{-1} = (K_{1T})_{z_1}^{-1} \Psi$. To this end, a standard trick is to work with “normalized frames” at $z$.

Precisely, whenever $z$ is fixed, there exists(see Lemma 2.4 [6]) holomorphic frames $s = (s_1, \ldots, s_l)$ and $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_l)$ for $E$ and $\tilde{E}$ (called normalized frame at $z$) such that $\langle s_i(w), s_j(z) \rangle = \langle \tilde{s}_i(w), \tilde{s}_j(z) \rangle = \delta_{ij}$ for all $w$ in a neighborhood of $z$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. This in turn implies that $H(z) = \tilde{H}(z) = I$ and $\partial^k_{z_1} H(z) = \partial^k_{z_1} \tilde{H}(z) = 0$ for all $k \geq 1$, inserting which into (3.13) yields $A_1 = \cdots = A_n = 0$. In other words, with respect to the normalized frames, the extension, if exists, must be represented by a diagonal block matrix where all diagonal blocks are $A_0$, and the isometric condition (3.10) becomes
\[ \partial^i_{z_1} \overline{\partial}^j_{z_1} H(z) = A_0 \partial^i_{z_1} \overline{\partial}^j_{z_1} \tilde{H}(z) A_0^* \]  
(3.14)

for all \( i, j \leq n \).

On the other hand, the geometric condition \( \Psi((K_{1\tau})_{z_1^{\tau}}) = (\tilde{K}_{1\tau})_{z_1^{\tau}} \Psi \) amounts to the validity of the following matrix identity

\[ (K_{1\tau})_{z_1^{\tau}}(s) A_0 = A_0 (\tilde{K}_{1\tau})_{z_1^{\tau}}(\tilde{s}) \]  
(3.15)

for \( r, t \leq n - 1 \) at \( z \). As \( \Psi \) is an isometry and \( s(z) \) and \( \tilde{s}(z) \) are orthonormal \((H(z) = \tilde{H}(z) = I)\), its representing matrix \( A_0 \) is a unitary matrix, thus the above equation is equivalent to

\[ (K_{1\tau})_{z_1^{\tau}}(s) = A_0 (\tilde{K}_{1\tau})_{z_1^{\tau}}(\tilde{s}) A_0^* \]  
(3.16)

for \( r, t \leq n - 1 \) at \( z \).

Now it remains to show that \( (3.14) \iff (3.16) \). That \( (3.14) \Rightarrow (3.16) \) is straightforward, since for any fixed \( r, t \), \( (K_{1\tau})_{z_1^{\tau}}(s) \) can be expressed in terms of \( \partial^i_{z_1} \overline{\partial}^j_{z_1} H(z) \) for \( i \leq r + 1, j \leq t + 1 \) by (3.6) and (3.7). For the other direction \( (3.16) \Rightarrow (3.14) \), one need the fact that \( \partial^i_{z_1} \overline{\partial}^j_{z_1} H(z) \) can be expressed in terms of \( (K_{1\tau})_{z_1^{\tau}}(s) \) for \( r \leq i - 1, t \leq j - 1 \). This is not so straightforward but can be verified by a careful induction argument which we omit here (see Lemma 20 [4] or Proposition 2.18 [6] for the detail).

### 3.3 Completion of proof

In this subsection we prove Proposition 3.8 which will complete the proof of Theorem 3.5. With the assumption of Proposition 3.8 fix \( s \) and \( \tilde{s} \) as holomorphic frames for \( E \) and \( \tilde{E} \) whose Gram matrices are \( H \) and \( \tilde{H} \), the holomorphic isometric bundle map \( \Psi \) corresponds to a holomorphic matrix-valued function \( A_0 \) over \( Z \) such that

\[ H = A_0 \tilde{H} A_0^* \]

From the proof of the uniqueness part in Proposition 3.7, the isometric extension \( \Phi \) corresponds to another \( n \) matrix-valued functions \( A_1, \ldots, A_n \) which are uniquely determined by \( A_0 \) via the recursive formula

\[ A_l = \partial^l_{z_1} H \cdot H^{-1} A_0 - \sum_{i=1}^l \binom{l}{i} A_{l-i} \partial^i_{z_1} \tilde{H} \cdot \tilde{H}^{-1} A_0, \quad l = 1, 2, \ldots, n \]  
(3.17)

Now to prove Proposition 3.8 it suffices to show that the followings are equivalent

\begin{enumerate}
  \item[A.] the matrix functions \( A_1, A_2, \ldots, A_n \) recursively defined by (3.17) are holomorphic
  \item[B.] \( (K_{1\tau})_{z_1^{\tau-1}}(s) A_0 = A_0 (\tilde{K}_{1\tau})_{z_1^{\tau-1}}(\tilde{s}), \quad 1 \leq l \leq n \) \label{B}
\end{enumerate}

holds for all \( 2 \leq j \leq m \).
Our proof of $A \Leftrightarrow B$ involves several steps which are arranged into Lemma 3.9, 3.10, 3.12 and 3.13 below. Precisely, Lemma 3.9 will reduce condition A into a collection of algebraic identities, and Lemma 3.13 asserts that these algebraic identities is equivalent to condition B, thus completes the proof. Lemma 3.10 and its consequence Lemma 3.12 give recursive formulas which represent high order covariant derivatives of the curvature in terms of lower order ones, and will be needed to justify the use of Lemma 3.13.

**Lemma 3.9.** Let $s$ and $	ilde{s}$ be holomorphic frames for $E$ and $	ilde{E}$ whose Gram matrices are $H$ and $\tilde{H}$ and $A_0$ be a holomorphic matrix-valued function on $Z$ such that $H = A_0 \tilde{H} A_0^*$. Let $A_1, A_2, \ldots$ be a sequence of matrix functions on $Z$ defined recursively by

$$A_l = \partial_{z_l} H \cdot H^{-1} A_0 - \sum_{i=1}^l \binom{l}{i} A_{l-i} \partial_{z_l} \tilde{H} A_0^{-1}, \quad l = 1, 2, \ldots$$  

(3.19)

Moreover, for any positive integer $l$ and $1 \leq j \leq m$, set $L_j^l := (\partial_{z_l} \partial_{\bar{z}_j} H - \partial_{\bar{z}_j} \partial_{z_l} H \cdot H^{-1} \cdot \partial_{\bar{z}_j} H) H^{-1}$ and $\tilde{L}_j^l := (\partial_{z_l} \partial_{\bar{z}_j} \tilde{H} - \partial_{\bar{z}_j} \partial_{z_l} \tilde{H} \cdot \tilde{H}^{-1} \cdot \partial_{\bar{z}_j} \tilde{H}) \tilde{H}^{-1}$. Then for every positive integer $n$, the followings are equivalent

(i) $A_l$ is holomorphic for all $1 \leq l \leq n$.
(ii) For all $1 \leq l \leq n$, $2 \leq j \leq m$, it holds that

$$L_j^l = \sum_{i=1}^l \binom{l}{i} A_{l-i} \tilde{L}_j^0 A_0^{-1},$$  

(3.20)

Note that for a single $l$, holomorphicity of $A_l$ is not equivalent to $L_j^l = \sum_{i=1}^l \binom{l}{i} A_{l-i} \tilde{L}_j^0 A_0^{-1}, 2 \leq j \leq m$, and what this lemma asserts is the equivalence between two collections of statements.

**Proof.** The lemma will be proved by induction on $n$. First we prove the case $n = 1$, that is, $A_1$ is holomorphic if and only if $L_j^1 = A_0 \tilde{L}_j^0 A_0^{-1}, j = 2, 3, \ldots, m$.

Specifying the formula (3.19) with $k = 1$ yields

$$A_1 = \partial_{z_1} H \cdot H^{-1} A_0 - A_0 \partial_{z_1} \tilde{H} \cdot \tilde{H}^{-1}.$$  

(3.21)

Combing it with $H = A_0 \tilde{H} A_0^*$ yields

$$\partial_{z_1} H = (A_1 \tilde{H} + A_0 \partial_{z_1} \tilde{H}) A_0^*$$  

(3.22)

Note that $A_0$ is holomorphic in $z_2, \ldots, z_m$, applying $\partial_{z_j}, 2 \leq j \leq m$, to $H = A_0 \tilde{H} A_0^*$ yields

$$\partial_{z_j} H = A_0 (\partial_{z_j} \tilde{H}) A_0^* + A_0 \tilde{H} \partial_{z_j} A_0^*$$  

(3.23)

Applying $\partial_{z_j}$ to (3.22) yields

$$\partial_{z_1} \partial_{z_j} H = (\partial_{z_j} A_1) \tilde{H} A_0^* + A_1 (\partial_{z_j} \tilde{H}) A_0^* + A_1 \tilde{H} \partial_{z_j} A_0^* + A_0 (\partial_{z_1} \partial_{z_j} \tilde{H}) A_0^* + A_0 (\partial_{z_1} \tilde{H}) \partial_{z_j} A_0^*$$  

(3.24)

Since $A_0$ and $\tilde{H}$ are invertible, (3.24) implies that $A_1$ is holomorphic (that is, $\partial_{z_j} A_1 = 0$) if and only if

$$\partial_{z_1} \partial_{z_j} H = A_1 (\partial_{z_j} \tilde{H}) A_0^* + A_1 \tilde{H} \partial_{z_j} A_0^* + A_0 (\partial_{z_1} \partial_{z_j} \tilde{H}) A_0^* + A_0 (\partial_{z_1} \tilde{H}) \partial_{z_j} A_0^*$$  

(3.25)
holds for \( i = 2, \ldots, m \).

Inserting (3.21), one reduces (3.25) into

\[
\partial_{z_i} \partial_{\bar{z}} H = (\partial_{z_i} H \cdot H^{-1} A_0 - A_0 \partial_{z_i} \bar{H} \cdot \bar{H}^{-1}) \partial_{\bar{z}} \bar{H} \cdot A_0^* + (\partial_{z_i} H \cdot H^{-1} A_0 - A_0 \partial_{z_i} \bar{H} \cdot \bar{H}^{-1} \bar{H}) \partial_{\bar{z}} A_0^* + A_0 (\partial_{z_i} \partial_{\bar{z}} \bar{H}) A_0^* + A_0 \partial_{z_i} \bar{H} \partial_{\bar{z}} A_0^*
\]

where the third identity follows from (3.23), and (3.25) is reduced to

\[
\partial_{z_i} \partial_{\bar{z}} H = \partial_{z_i} H \cdot H^{-1} \partial_{\bar{z}} H = A_0 (\partial_{z_i} \partial_{\bar{z}} \bar{H} - \partial_{z_i} \bar{H} \cdot \bar{H}^{-1} \partial_{\bar{z}} \bar{H}) A_0^*.
\]

Invoking \( H = A_0 \bar{H} A_0^* \), the above equation becomes

\[
(\partial_{z_i} \partial_{\bar{z}} H - \partial_{z_i} H \cdot H^{-1} \partial_{\bar{z}} H) H^{-1} = A_0 (\partial_{z_i} \partial_{\bar{z}} \bar{H} - \partial_{z_i} \bar{H} \cdot \bar{H}^{-1} \partial_{\bar{z}} \bar{H}) \bar{H}^{-1} A_0^{-1},
\]

which is exactly \( \mathcal{L}_j^1 = A_0 \mathcal{L}_j^1 A_0^{-1} \). This completes the proof for \( n = 1 \).

Now we suppose the conclusion holds for \( n - 1 \), then to prove the conclusion for \( n \), it suffices to show the followings are equivalent:

(i)’ \( A_n \) is holomorphic

(ii)’ \( \mathcal{L}_j^n = \sum_{i=1}^n \binom{n}{i} A_{n-i} \tilde{\mathcal{L}}_j^i A_0^{-1} \)

with the additional assumption guaranteed by the induction hypothesis that \( A_i \) is holomorphic along \( Z \) for \( l = 1, 2, \ldots, n - 1 \).

Specifying (3.19) to \( l = n \) yields

\[
A_n = \partial_{z_i} H \cdot H^{-1} A_0 - \sum_{i=1}^n \binom{n}{i} A_{n-i} \partial_{z_i} \bar{H} \cdot \bar{H}^{-1}
\]

which combined with \( H = A_0 \bar{H} A_0^* \) gives

\[
\partial_{z_i}^n H = A_n \bar{H} A_0^* + \sum_{i=1}^n \binom{n}{i} A_{n-i} (\partial_{z_i}^i \bar{H}) A_0^*
\]

As \( A_0, \ldots A_{n-1} \) are holomorphic, applying \( \partial_{z_i} \) to (3.27) yields

\[
\partial_{z_i}^n \partial_{\bar{z}} H = (\partial_{z_i} A_n) \bar{H} A_0^* + A_n (\partial_{z_i} \bar{H}) A_0^* + A_n \bar{H} \partial_{z_i} A_0^* + \sum_{i=1}^n \binom{n}{i} A_{n-i} (\partial_{z_i}^i \partial_{\bar{z}} \bar{H}) A_0^* + \sum_{i=1}^n \binom{n}{i} A_{n-i} (\partial_{z_i}^i \bar{H}) \partial_{\bar{z}} A_0^*
\]

As \( \bar{H} \) and \( A_0^* \) are both invertible, (i)’ holds (that is, \( \partial_{\bar{z}} A_n = 0 \)), if and only if

\[
\partial_{z_i}^n \partial_{\bar{z}} H = A_n (\partial_{z_i} \bar{H}) A_0^* + A_n \bar{H} \partial_{z_i} A_0^* + \sum_{i=1}^n \binom{n}{i} A_{n-i} (\partial_{z_i}^i \partial_{\bar{z}} \bar{H}) A_0^* + \sum_{i=1}^n \binom{n}{i} A_{n-i} (\partial_{z_i}^i \bar{H}) \partial_{\bar{z}} A_0^*
\]

holds.
By (3.26),
\[ A_n(\partial_{\bar{z}_j} \tilde{H}) A_*^0 = \partial^n_{z_1} H \cdot H^{-1} A_0(\partial_{\bar{z}_j} \tilde{H}) A_0 - \sum_{i=1}^{n} \binom{n}{i} A_{n-i} \partial^i_{z_1} \tilde{H} \cdot \tilde{H}^{-1}(\partial_{\bar{z}_j} \tilde{H}) A_0^* \quad (3.30) \]
and
\[ A_n \tilde{H} \partial_{\bar{z}_j} A_*^0 = \partial^n_{z_1} H \cdot H^{-1} A_0 \tilde{H} \partial_{\bar{z}_j} A_0^* - \sum_{i=1}^{n} \binom{n}{i} A_{n-i} \partial^i_{z_1} \tilde{H} \cdot \tilde{H}^{-1} \partial_{\bar{z}_j} \tilde{H} A_0^* \quad (3.31) \]
Inserting (3.30) (3.31) we reduce (3.29) into
\[ \partial^n_{z_1} \partial_{\bar{z}_j} H = \partial^n_{z_1} H \cdot H^{-1} (A_0(\partial_{\bar{z}_j} \tilde{H}) A_0^* + A_0 \tilde{H} \partial_{\bar{z}_j} A_0^*) + \sum_{i=1}^{n} \binom{n}{i} A_{n-i} (\partial^i_{z_1} \partial_{\bar{z}_j} \tilde{H} - \partial^i_{z_1} \tilde{H} \cdot \tilde{H}^{-1} \partial_{\bar{z}_j} \tilde{H}) A_0^* \quad (3.32) \]
Now identity (3.23) reduces (3.32) into
\[ \partial^n_{z_1} \partial_{\bar{z}_j} H = \partial^n_{z_1} H \cdot H^{-1} \partial_{\bar{z}_j} H + \sum_{i=1}^{n} \binom{n}{i} A_{n-i} (\partial^i_{z_1} \partial_{\bar{z}_j} \tilde{H} - \partial^i_{z_1} \tilde{H} \cdot \tilde{H}^{-1} \partial_{\bar{z}_j} \tilde{H}) A_0^* \quad (3.33) \]
and \( H = A_0 \tilde{H} A_0^* \) further reduces it into
\[ (\partial^n_{z_1} \partial_{\bar{z}_j} H - \partial^n_{z_1} H \cdot H^{-1} \partial_{\bar{z}_j} H) H^{-1} = \sum_{i=1}^{n} \binom{n}{i} A_{n-i} (\partial^i_{z_1} \partial_{\bar{z}_j} \tilde{H} - \partial^i_{z_1} \tilde{H} \cdot \tilde{H}^{-1} \partial_{\bar{z}_j} \tilde{H}) H^{-1} A_0^* \quad (3.34) \]
which is exactly (ii)', giving the equivalence of (i)' and (ii)' as desired. \( \square \)

**Lemma 3.10.** Let \( E \) be a holomorphic Hermitian vector bundle and \( H \) be the Gram matrix for a holomorphic frame of \( E \). Fix \( 2 \leq j \leq m \), let \( Q_j = \partial_{\bar{z}_j}(\partial_{z_j} H \cdot H^{-1}) \) and \( Q_j^n = (\partial^n_{z_1} \partial_{z_j} H \cdot \partial_{z_j} H \cdot H^{-1} \cdot \partial^n_{z_1} H) H^{-1} \) for \( n = 1, 2, \ldots \). Then for every positive integer \( n \), it holds that
\[ \partial^n_{z_1} Q_j = Q_j^{n+1} - \sum_{i=1}^{n} \binom{n+1}{i} (\partial^n_{z_1} Q_j) \cdot \partial^n_{z_1} H \cdot H^{-1} \quad (3.35) \]

**Remark 3.11.** One immediately sees that \( Q_j \) is just \( Q_j^1 \) and here we adopt the surplus notation \( Q_j \) to emphasis its geometric meaning as the matrix representation for \( \mathcal{K}_{j\bar{z}_j} \) (see (3.3)). By (3.7), \( \partial^n_{z_1} Q_j \) is the matrix representation for \( (\mathcal{K}_{j\bar{z}_j})^n_{\bar{z}_j} \), and \( \partial^n_{z_1} Q_j \) is that for \( (\mathcal{K}_{j\bar{z}_j})_{\bar{z}_j}^{n+1} \), so Lemma (3.10) is a recursive formula representing high order covariant derivative of the curvature in terms of lower order ones.

**Proof.** We prove (3.35) by induction on \( n \). The case \( n = 1 \) can be verified by a straightforward computation. Precisely,
\[ \partial_{\bar{z}_j} Q_j = \partial^2_{z_1}(\partial_{z_j} H \cdot H^{-1}) = \partial^2_{z_1} \partial_{z_j} H \cdot H^{-1} + 2 \partial_{z_1} \partial_{z_j} H \cdot \partial_{\bar{z}_j} H^{-1} + \partial_{z_j} H \cdot \partial^2_{z_1} H^{-1}. \]
Inserting \( \partial_i H^{-1} = -H^{-1} \cdot \partial_i H \cdot H^{-1} \)
and
\[
\partial_i^2 H^{-1} = \partial_i (-H^{-1} \cdot \partial_i H \cdot H^{-1}) = -H^{-1} \cdot \partial_i^2 H \cdot H^{-1} + 2H^{-1} \partial_i H \cdot H^{-1} \partial_i H \cdot H^{-1}
\]
yields
\[
\partial_i \partial_j Q_j = \partial_i^2 \partial_j H \cdot H^{-1} - 2\partial_i \partial_j \partial_j H \cdot H^{-1} \cdot \partial_i H \cdot H^{-1} + \partial_j H (-H^{-1} \cdot \partial_i^2 H \cdot H^{-1} + 2H^{-1} \partial_i H \cdot H^{-1} \partial_i H \cdot H^{-1})
\]
\[
= (\partial_i^2 \partial_j H - \partial_j \partial_i H \cdot H^{-1} \cdot \partial_i^2 H \cdot H^{-1} - 2(\partial_i \partial_j H - \partial_j H \cdot H^{-1} \partial_i H)H^{-1} \partial_i H \cdot H^{-1})
\]
\[
= Q_j^2 - 2Q_j \cdot \partial_i H \cdot H^{-1},
\]
completing the proof for \( n = 1 \).

Now we assume (3.35) and it remains to show that
\[
\partial_i^{n+1} Q_j = Q_j^{n+2} - \sum_{i=1}^{n+1} \binom{n+2}{i} (\partial_i^{n+1-i} Q_j) \cdot \partial_i^i H \cdot H^{-1}.
\]
(3.36)

With (3.35), it suffices to show that
\[
\partial_i \left( Q_j^{n+1} - \sum_{i=1}^{n} \binom{n+1}{i} (\partial_i^{n+1-i} Q_j) \cdot \partial_i^i H \cdot H^{-1} \right)
\]
equals the right hand side of (3.36).

Explicitly computing terms in (3.37) gives
\[
\partial_i Q_j^{n+1} = \partial_i (\partial_i^{n+1} \partial_j H - \partial_j \partial_i H \cdot H^{-1} \cdot \partial_i^i H \cdot H^{-1} \cdot \partial_i^i H \cdot H^{-1}) \cdot H^{-1} + (\partial_i^{n+1} \partial_j H - \partial_j \partial_i H \cdot H^{-1} \cdot \partial_i^i H) \partial_i H \cdot H^{-1}
\]
\[
= (\partial_i^{n+2} \partial_j H - \partial_i^{n+1} \partial_j H \cdot H^{-1} \cdot \partial_i^i H + \partial_j H \cdot H^{-1} \partial_i H \cdot H^{-1} \partial_i H \cdot H^{-1} \cdot \partial_i^i H) \partial_i H \cdot H^{-1}
\]
\[
= Q_j^{n+2} - Q_j^{n+1} \partial_i H \cdot H^{-1} - Q_j \partial_i^{n+1} H \cdot H^{-1}
\]
Moreover, for any fixed \( i \),
\[
\partial_i \left( (\partial_i^{n+1-i} Q_j) \cdot \partial_i^i H \cdot H^{-1} \right) = (\partial_i^{n+1-i} Q_j) \cdot \partial_i^i H \cdot H^{-1} + (\partial_i^{n+1-i} Q_j) \cdot \partial_i^i H \cdot H^{-1} - (\partial_i^{n+1-i} Q_j) \cdot \partial_i^i H \cdot H^{-1} \cdot \partial_i H \cdot H^{-1}.
\]
Now we have
\[
(3.37) = Q_j^{n+2} - Q_j^{n+1} \partial_i H \cdot H^{-1} - Q_j \partial_i^{n+1} H \cdot H^{-1}
\]
\[
- \binom{n+1}{1} ((\partial_i^{n+1} Q_j) \cdot \partial_i H \cdot H^{-1} + (\partial_i^{n+1} Q_j) \cdot \partial_i^2 H \cdot H^{-1} - (\partial_i^{n+1} Q_j) \cdot \partial_i H \cdot H^{-1} \cdot \partial_i H \cdot H^{-1})
\]
\[
- \binom{n+1}{2} ((\partial_i^{n+2} Q_j) \cdot \partial_i^2 H \cdot H^{-1} + (\partial_i^{n+2} Q_j) \cdot \partial_i^3 H \cdot H^{-1} - (\partial_i^{n+2} Q_j) \cdot \partial_i^2 H \cdot H^{-1} \cdot \partial_i H \cdot H^{-1})
\]
\[
- \cdots
\]
\[
- \binom{n+1}{i} ((\partial_i^{n+1-i} Q_j) \cdot \partial_i^i H \cdot H^{-1} + (\partial_i^{n+1-i} Q_j) \cdot \partial_i^i H \cdot H^{-1} \cdot \partial_i H \cdot H^{-1})
\]
\[
- \cdots
\]
\[
- \binom{n+1}{n} ((\partial_i^i Q_j) \cdot \partial_i^i H \cdot H^{-1} + Q_j \cdot \partial_i^i H \cdot H^{-1} - Q_j \cdot \partial_i^i H \cdot H^{-1} \cdot \partial_i H \cdot H^{-1})
\]
Let us re-arrange terms in the above expression of (3.37). Set
\[
I_1 = Q_j^{n+2}, I_2 = -Q_j^{n+1} \partial_i H \cdot H^{-1}, I_3 = -Q_j \partial_i^{n+1} H \cdot H^{-1}, I_4 = -\binom{n+1}{n} Q_j \cdot \partial_i^{n+1} H \cdot H^{-1},
\]
\[ I_5 = -\binom{n+1}{1}(\partial_{z_1}^n Q_j) \cdot \partial_{z_1} H \cdot H^{-1}, \]
\[ I_6 = \sum_{i=1}^{n} \binom{n+1}{i} (\partial_{z_i}^{n-i} Q_j) \cdot \partial_{z_i}^i H \cdot H^{-1} \cdot \partial_{z_1} H \cdot H^{-1}, \]
\[ I_7 = -\sum_{i=1}^{n-1} \binom{n+1}{i} (\partial_{z_i}^{n-i} Q_j) \cdot \partial_{z_1}^{i+1} H \cdot H^{-1}, \]
\[ I_8 = -\sum_{i=2}^{n} \binom{n+1}{i} (\partial_{z_i}^{n-i+1} Q_j) \cdot \partial_{z_1}^i H \cdot H^{-1}, \]

then (3.37) equals \( I_1 + \cdots + I_8 \).

Now
\[ I_2 + I_6 = -(Q_j^{n+1} - \sum_{i=1}^{n} \binom{n+1}{i} (\partial_{z_i}^{n-i} Q_j) \cdot \partial_{z_1}^{i+1} H \cdot H^{-1}) \partial_{z_1} H \cdot H^{-1} \]

which by (3.35) equals
\[ -\partial_{z_1}^n Q_j \cdot \partial_{z_1} H \cdot H^{-1}. \]

So
\[ I_2 + I_5 + I_6 = -(n + 2) \binom{n+1}{1} \partial_{z_1}^n Q_j \cdot \partial_{z_1} H \cdot H^{-1}. \]

Moreover,
\[ I_7 + I_8 = -\sum_{i=1}^{n-1} \binom{n+1}{i} (\partial_{z_i}^{n-i} Q_j) \cdot \partial_{z_1}^{i+1} H \cdot H^{-1} - \sum_{i=2}^{n} \binom{n+1}{i} (\partial_{z_i}^{n-i} Q_j) \cdot \partial_{z_1}^i H \cdot H^{-1} \]
\[ = -\sum_{i=1}^{n-1} \binom{n+1}{i} (\partial_{z_i}^{n-i} Q_j) \cdot \partial_{z_1}^{i+1} H \cdot H^{-1} - \sum_{i=1}^{n-1} \binom{n+1}{i+1} (\partial_{z_i}^{n-i} Q_j) \cdot \partial_{z_1}^{i+1} H \cdot H^{-1} \]
\[ = -\sum_{i=1}^{n-1} \binom{n+2}{i+1} (\partial_{z_i}^{n-i} Q_j) \cdot \partial_{z_1}^{i+1} H \cdot H^{-1} = -\sum_{i=2}^{n} \binom{n+1}{i} (\partial_{z_i}^{n-i} Q_j) \cdot \partial_{z_1}^i H \cdot H^{-1} \]

and
\[ I_3 + I_4 = -Q_j \partial_{z_1}^{n+1} H \cdot H^{-1} - \binom{n+1}{n} Q_j \cdot \partial_{z_1}^{n+1} H \cdot H^{-1} = -\binom{n+2}{n+1} Q_j \partial_{z_1}^{n+1} H \cdot H^{-1}. \]

Finally
\[ I_1 + \cdots + I_8 = Q_j^{n+2} - \binom{n+2}{1} \partial_{z_1}^n Q_j \partial_{z_1} H \cdot H^{-1} - \sum_{i=2}^{n} \binom{n+2}{i} (\partial_{z_i}^{n-i+1} Q_j) \cdot \partial_{z_1}^i H \cdot H^{-1} - \binom{n+2}{n+1} Q_j \partial_{z_1}^{n+1} H \cdot H^{-1} \]

which is exactly the right hand side of (3.36), completing the induction.

\[ \square \]

**Lemma 3.12.** Let \( E \) be a holomorphic Hermitian vector bundle and \( H \) be the Gram matrix for a holomorphic frame \( s \) of \( E \). Fix \( 2 \leq j \leq m \), set \( L_j^n = (\partial_{z_1}^n H - \partial_{z_1}^n H \cdot H^{-1} \cdot \partial_{z_1} H)H^{-1} \) for every positive integer \( n \), and let \( \{J_n\}_{n=1}^{\infty} \) be a sequence of matrix functions defined recursively by
\[ J_1 = L_j^1, J_n = L_j^n - \sum_{i=1}^{n-1} \binom{n}{i} \partial_{z_i}^i H \cdot H^{-1} J_{n-i}, n = 2, 3 \ldots \]

then for every \( n \geq 1 \), \( J_n \) represents \((K_{1j})_{z_1}^{(n-1)}\) with respect to \( s \).
Proof. An application of Lemma 3.4 gives \((\mathcal{K}_\pi)_{z_i^{(n-1)}} = ((\mathcal{K}_\pi)_{z_i^{(n-1)}})^*\). By Lemma 3.3 and (3.7), it suffices to show that
\[
J_n = H(\partial_{z_1}^{n-1}Q_j)^*H^{-1}
\]
for all \(n \geq 1\) where \(Q_j\) is as in Lemma 3.10.

We prove (3.38) by induction on \(n\). First note that for any \(n \geq 1\), it holds that
\[
H(Q_j^n)^*H^{-1} = H[(\partial_{z_1}^n\partial_{z_1}H - \partial_{z_1}H \cdot H^{-1} \cdot \partial_{z_1}H)^*]H^{-1} = H[H^{-1}(\partial_{z_1}^n\partial_{z_1}H - \partial_{z_1}H \cdot H^{-1} \cdot \partial_{z_1}H)]H^{-1} = L_j^n
\]
where \(Q_j^n\) is as in Lemma 3.10.

Specifying (3.39) to \(n = 1\) gives
\[
H(Q_j)^*H^{-1} = J_1,
\]
verifying (3.38) in this case.

Now suppose \(J_k = H(\partial_{z_1}^{k-1}Q_j)^*H^{-1}\) holds for all \(1 \leq k \leq n\), then combining Lemma 3.10 and (3.39) yields
\[
H(\partial_{z_1}^nQ_j)^*H^{-1} = H\left(Q_j^n - \sum_{i=1}^{n} \binom{n+1}{i} (\partial_{z_1}^{n-i}Q_j) \cdot \partial_{z_1}H \cdot H^{-1}\right)^*H^{-1}
\]
\[
= H(Q_j^{n+1})^*H^{-1} - \sum_{i=1}^{n} \binom{n+1}{i} H[(\partial_{z_1}^{n-i}Q_j) \cdot \partial_{z_1}H \cdot H^{-1}]H^{-1}
\]
\[
= L_j^{n+1} - \sum_{i=1}^{n} \binom{n+1}{i} H[(\partial_{z_1}H \cdot H^{-1}) \cdot (\partial_{z_1}^{n-i}Q_j)^*H^{-1}]
\]
\[
= L_j^{n+1} - \sum_{i=1}^{n} \binom{n+1}{i} (\partial_{z_1}H \cdot H^{-1})J_{n-i+1}
\]
completing the induction.

\[\square\]

Lemma 3.13. Let \(\{F_1\}_{l=1}^{\infty}, \{G_1\}_{l=1}^{\infty}, \{\tilde{F}_1\}_{l=1}^{\infty}, \{\tilde{G}_1\}_{l=1}^{\infty}\) be four sequences of square matrices and let \(\{H_1\}_{l=1}^{\infty}\) and \(\{\tilde{H}_1\}_{l=1}^{\infty}\) be another two matrix sequences defined recursively by
\[
H_1 = F_1, H_l = F_l - \sum_{i=1}^{l-1} \binom{l}{i} G_i H_{l-i}, l = 2, 3 \ldots
\]
and
\[
\tilde{H}_1 = \tilde{F}_1, \tilde{H}_l = \tilde{F}_l - \sum_{i=1}^{l-1} \binom{l}{i} \tilde{G}_i \tilde{H}_{l-i}, l = 2, 3 \ldots
\]
respectively. Moreover, let $Z_0$ be a fixed invertible matrix and $\{Z_i\}_{i=1}^\infty$ be a matrix sequence defined recursively by

$$Z_l = G_l Z_0 - \sum_{i=1}^l \binom{l}{i} Z_{l-i} \tilde{G}_i, l = 1, 2, \cdots.$$  \hfill (3.42)

Then for every positive integer $n$, the followings are equivalent

(i) \[ F_l = \sum_{i=1}^l \binom{l}{i} Z_{l-i} \tilde{F}_i Z_0^{-1}, \quad \text{for all} \quad 1 \leq l \leq n \] \hfill (3.43)

(ii) \[ H_l = Z_0 \tilde{H}_l Z_0^{-1}, \quad \text{for all} \quad 1 \leq l \leq n. \]

Lemma 3.13 is totally elementary but it absorbs the algebraic complexities underlying Proposition 3.8. As its proof takes several pages, here we directly use it to complete the proof of Proposition 3.8 and the proof of Lemma 3.13 is given in the Appendix.

Proof of Proposition 3.8

We adopt the notations in the beginning of Section 3.3 and prove the equivalence of statement $A$ and $B$ there. By Lemma 3.9, $A_1, \cdots A_n$ are holomorphic if and only if for all $2 \leq j \leq m,$

$$(\partial_{z_j}^1 \partial_{\overline{z}_j} \overline{H} - \partial_{z_j}^1 H \cdot H^{-1} \cdot \partial_{\overline{z}_j} \overline{H}) H^{-1} = \sum_{i=1}^l \binom{l}{i} A_{l-i}(\partial_{z_j}^i \partial_{\overline{z}_j} \overline{H} - \partial_{z_j}^i \overline{H} \cdot H^{-1} \cdot \partial_{\overline{z}_j} \overline{H}) H^{-1} A_0^{-1}, 1 \leq l \leq n$$ \hfill (3.44)

so it suffices to show $(3.18) \iff (3.44).$

To this end, we resort to Lemma 3.13 which involves 7 matrix sequences. Precisely, for $l \geq 1,$ set

$$F_l := (\partial_{z_j}^l \partial_{\overline{z}_j} \overline{H} - \partial_{z_j}^l H \cdot H^{-1} \cdot \partial_{\overline{z}_j} \overline{H}) H^{-1}, G_l := \partial_{z_j}^l H \cdot H^{-1}$$

$$\tilde{F}_l := (\partial_{z_j}^l \partial_{\overline{z}_j} \overline{H} - \partial_{z_j}^l \overline{H} \cdot H^{-1} \cdot \partial_{\overline{z}_j} \overline{H}) \overline{H}^{-1}, \tilde{G}_l := \partial_{z_j}^l \overline{H} \cdot \overline{H}^{-1}$$

$$H_l := (K_{1j})_{z_j}^{l-1}(s), \quad \overline{H}_l := (\overline{K}_{1j})_{\overline{z}_j}^{l-1}(\overline{s})$$

and for $l \geq 0,$ set

$$Z_l := A_l.$$  

Then $(3.18) \iff (3.44)$ is exactly what Lemma 3.13 asserts, so it remains to check the identities $(3.42)-(3.46)$ assumed in Lemma 3.13. Validity of $(3.42)$ is straightforward which is just $(3.17),$ while Lemma 3.12 gives

$$(K_{1j})_{z_j}^{l-1}(s) = (\partial_{z_j}^l \partial_{\overline{z}_j} \overline{H} - \partial_{z_j}^l H \cdot H^{-1} \cdot \partial_{\overline{z}_j} \overline{H}) H^{-1} - \sum_{i=1}^{l-1} \binom{l}{i} \partial_{z_j}^i H \cdot H^{-1} (K_{1j})_{z_j}^{i-1}(s), l = 2, 3, \cdots$$ \hfill (3.45)

$$(\overline{K}_{1j})_{\overline{z}_j}^{l-1}(\overline{s}) = (\partial_{z_j}^l \partial_{\overline{z}_j} \overline{H} - \partial_{z_j}^l \overline{H} \cdot H^{-1} \cdot \partial_{\overline{z}_j} \overline{H}) \overline{H}^{-1} - \sum_{i=1}^{l-1} \binom{l}{i} \partial_{z_j}^i \overline{H} \cdot \overline{H}^{-1} (\overline{K}_{1j})_{\overline{z}_j}^{i-1}(\overline{s}), l = 2, 3, \cdots$$ \hfill (3.46)
which assures (3.40) (3.41). The proof is complete.

In the end of this section we take a few words on contact along sub-manifolds with lower complex dimension, that is,

\[ Z = \{(0, \cdots, 0, z_{d+1}, \cdots, z_m)|(z_{d+1}, \cdots, z_m) \in \Omega'\} \]

for a positive integer \(d\) and a domain \(\Omega' \subseteq \mathbb{C}^{n-d}\).

Analogous to the hyper-surface case, both definition and geometric characterization of contact should involve jet bundles and Pascal maps with respect to the transverse direction as well as the “holomorphic gluing” condition along the tangent direction. Precisely, set

\[ N_d = \{I = (i_1, \cdots, i_m) | i_{d+1} = \cdots = i_m = 0\} \]

and let \(E^n_d\) be the holomorphic vector bundle defined by

\[ E^n_d(z) := \text{span}\{\partial^l s_i(z), i = 1, 2 \cdots, l, I \in N^d, |I| \leq n\} \]

where \(\{s_1, \cdots, s_l\}\) is any holomorphic frame for \(E\). According to the rule (2.11), only the first \(d\) Pascal maps \(P_1, \cdots, P_d\) acts non-trivially on \(E^n_d\) which justifies the following definition

**Definition 3.14.** Two holomorphic Hermitian vector bundles \(E\) and \(\tilde{E}\) are said to have order \(n\) contact along \(Z\) if there exists an isometric holomorphic bundle map \(\Phi\) from \(E^n_d|_Z\) to \(\tilde{E}^n_d|_Z\) such that \(\Phi P_k = \tilde{P}_k \Phi\) for all \(1 \leq k \leq d\).

One immediately sees that Definition 3.14 extends Definition 2.12 by “adding more Pascal maps” (the same situation happened in Section 2.1 where we pass from Definition 2.9 to Definition 2.11). Correspondingly, geometric characterization for Definition 3.14 amounts to extending Theorem 3.5 by including more transverse components of the curvatures. Such an extension will conceivably involve more technical details and will not be further addressed in this paper.

**4 Pascal map induced by bounded operators and unitary equivalence of quotient spaces**

Our discussions in previous sections are related to unitary equivalence of Hilbert space operators. Two operators \(T\) and \(\tilde{T}\) acting respectively on Hilbert spaces \(\mathcal{H}\) and \(\tilde{\mathcal{H}}\) are said to be **unitarily equivalent** if there exists a unitary operator \(U : \mathcal{H} \leftrightarrow \tilde{\mathcal{H}}\) such that \(UT = \tilde{T}U\). Unitary equivalence of operators is a fundamental topic in operator theory and has some extensively studied variations such as

(i) **Unitary equivalence of operator tuples:** given two operator tuples \(T = (T_1, T_2, \cdots, T_m)\) and \(\tilde{T} = (\tilde{T}_1, \tilde{T}_2, \cdots, \tilde{T}_m)\), determine if there is a unitary operator \(U\) such that \(UT_i = \tilde{T}_i U, 1 \leq i \leq m\)

(ii) **Unitary equivalence of invariant subspaces:** if \(\mathcal{M}\) and \(\tilde{\mathcal{M}}\) are invariant subspaces for \(T\) and \(\tilde{T}\), determine if the restrictions \(T|_\mathcal{M}\) and \(\tilde{T}|_{\tilde{\mathcal{M}}}\) are unitarily equivalent
(iii) Unitary equivalence of quotient subspaces: if $M$ and $\tilde{M}$ are invariant subspaces for $T$ and $\tilde{T}$, determine if the adjoint restrictions $T^*|_{M^\perp}$ and $\tilde{T}^*|_{\tilde{M}^\perp}$ are unitarily equivalent.

Cowen and Douglas [6] initiated an extensive study of “geometric operator theory” where the key idea is to reduce the study of unitary equivalence of operators to the study of geometric invariants on vector bundles. Instead of recalling the full theory of Cowen and Douglas (see [23] for a nice survey), we focus on unitary equivalence for a kind of quotient function spaces which has been studied in a series of works [4, 11, 13, 14, 15] since the beginning of this century. The following exposition is mostly self-contained and we refer readers to ([4, 6, 14, 15]) for more details.

4.1 Pascal induced by bounded extension of coordinate multiplication

This subsection concerns the extrinsic model of the Pascal map as operators on Hilbert spaces, and we begin with one complex variable.

**Definition 4.1.** Let $E$ be a holomorphic vector bundle over a domain $\Omega \subseteq \mathbb{C}$. The bundle map defined by the rule

$$s_i(z) \mapsto z s_i(z), 1 \leq i \leq l$$

(4.1)

where $s_1, \ldots, s_l$ is any holomorphic frame, is called the **coordinate multiplication** on $E$.

This bundle map is obviously well-defined since it is represented by a scaler matrix $zI$ which trivially commutes with all transition matrices between different frames. If $E$ is associated to a holomorphic map $f$ into $Gr(l, H)$, then all fibers of $E$ are subspaces of $H$ and we consider operators on $H$ which extend the coordinate multiplication. Here we say a linear operator $T$ on $H$ extends a bundle map $\Phi$ on $E$ if for every point $z$, $T$ maps the $z$-fiber of $E$ into itself and coincides with the action of $\Phi$ there.

**Proposition 4.2.** Let $E$ be a holomorphic vector bundle associated to a holomorphic map from $\Omega \subseteq \mathbb{C}$ to $Gr(l, H)$. Suppose there is a bounded linear operator $T$ on $H$ which extends the coordinate multiplication on $E$, then for every positive integer $n$ and every $z \in \Omega$, $T - z$ extends the Pascal map on $E^n(z)$.

**Proof.** Fix any holomorphic frame $s_1(z), \ldots, s_l(z)$, the assumption on $T$ gives

$$Ts_i(z) = z s_i(z), 1 \leq i \leq l.$$  

(4.2)

Since $T$ is bounded and $s_i$ is a holomorphic $H$-valued function, for any positive integer $k$ one can differentiate (4.2) $k$ times to get

$$T^{(k)} s_i(z) = z^{(k)} s_i(z) + k s_i^{(k-1)}(z)$$  

(4.3)

or

$$(T - z) s_i^{(k)}(z) = k s_i^{(k-1)}(z)$$  

(4.4)

which exactly coincides with (2.9) as desired. \qed
The several variable case $\Omega \subseteq \mathbb{C}^m$ is similar. Now there are $m$ coordinate multiplications where the $j$-th coordinate multiplication, $1 \leq j \leq m$ is given by

$$s_i(z) \mapsto z_j s_i(z), 1 \leq i \leq l, z = (z_1, \cdots z_m)$$

(4.5)

where $s_1, \cdots, s_l$ is any holomorphic frame.

Correspondingly, Proposition 4.2 has the following analogue.

**Proposition 4.3.** Let $E$ be a holomorphic vector bundle associated to a holomorphic map from $\Omega \subseteq \mathbb{C}^m$ to $\text{Gr}(l, \mathcal{H})$. For fixed $1 \leq j \leq m$, if there is a bounded linear operator $T_j$ on $\mathcal{H}$ which extends the $j$-th coordinate multiplication on $E$, then for every positive integer $n$ and every point $z \in \Omega$, $T_j - z_j$ extends the $j$-th Pascal map on $E^n(z)$ where $z_j$ is the $j$-th coordinate of $z$.

**Proof.** Fix any holomorphic frame $s_1(z), \cdots, s_l(z)$, the assumption on $T_j$ gives

$$T_j s_i(z) = z_j s_i(z), 1 \leq i \leq l, z = (z_1, \cdots z_m).$$

As $T_j$ is bounded, applying $\partial^l = \partial_{z_1} \cdots \partial_{z_m}$ yields

$$T_j \partial^l s_i(z) = \partial^l (z_j s_i(z)) = \begin{cases} z_j \partial^l s_i(z) + i_j \partial_{z_1}^{i_1} \cdots \partial_{z_j}^{i_j-1} \cdots \partial_{z_m}^{i_m} s_i(z), & i_j \geq 1 \\ z_j \partial^l s_i(z), & i_j = 0 \end{cases}$$

(4.6)

So $T_j - z_j$ coincides with the $j$-th Pascal map given by (2.11).

Bounded-ness of $T$(and $T_j$) on the entire space $\mathcal{H}$ is critical in above arguments, and existence of such extension is a nontrivial problem in general. Next we consider holomorphic vector bundles constructed from reproducing kernels of function spaces, where the existence of bounded extension comes from bounded-ness of shift operators on the function spaces. Precisely, we now assume $\mathcal{H}$ is a Hilbert space consisting of holomorphic $\mathbb{C}^l$-valued functions over $\Omega \subseteq \mathbb{C}^m$ such that

(a) for every $1 \leq j \leq m$, the shift operator $S_j$ defined by $(S_j f)(z) = z_j f(z), z = (z_1, \cdots z_j \cdots z_m), f \in \mathcal{H}$, is bounded on $\mathcal{H}$.

(b) for every $z \in \Omega$, the point-wise evaluation $e_z : f \mapsto f(z)$ is a bounded functional from $\mathcal{H}$ to $\mathbb{C}^l$.

This kind of function spaces are conventionally called reproducing kernel Hilbert space, including well-studied function spaces such as Hardy, Bergman or Dirichlet spaces, and the study of shift operators on these spaces constitutes an important chapter in operator theory since last century (11, 7, 16, 20, 26).

For a vector $\xi \in \mathbb{C}^l$ and $z \in \Omega$, $e_z^* \xi$ is a function in $\mathcal{H}$ which is conventionally denoted by $K(\cdot, z)\xi$. The two variable function $K(\cdot, \cdot)$ takes values in the linear operators on $\mathbb{C}^l$ such that $K(w, z)\xi = (e_z^* \xi)(w)$ and the following reproducing property holds

$$\langle f, K(\cdot, z)\xi \rangle_{\mathcal{H}} = \langle f(z), \xi \rangle_{\mathbb{C}^l}.$$  

(4.7)
Since $H$ consists of holomorphic functions,\
\[ z \mapsto K(\cdot, z)\xi \]
as a $H$-valued function is anti-holomorphic in $z$ and a differentiation gives\
\[ \langle f, \overline{\partial}^j K(\cdot, z)\xi \rangle_H = \langle \partial^j f(z), \xi \rangle_{C^l}. \] (4.8)\

Let \( \{e_1, \ldots, e_l\} \) be the standard orthonormal basis for $C^l$, then \[ z \mapsto \bigvee \{K(\cdot, \overline{z})e_1, \ldots, K(\cdot, \overline{z})e_l\}, z \in \Omega^* \]
defines a holomorphic map from $\Omega^*$ to $Gr(l, H)$ where $\Omega^*$ is the conjugate domain of $\Omega$, and we denote the associated holomorphic vector bundle by $E_H$.

**Proposition 4.4.** For $1 \leq j \leq m$, the operator $S_j^*$ extends the $j$-th coordination multiplication on $E_H$.

**Proof.** From (4.7) one sees that for every $f \in H$ and $z \in \Omega^*$,\
\[ \langle f, S^* K(\cdot, \overline{z})e_i \rangle_H = \langle (Sf)(\overline{z}), e_i \rangle_{C^l} = \langle \overline{z}f(\overline{z}), e_i \rangle_{C^l} = \langle f, z_j K(\cdot, \overline{z})e_i \rangle_H \]
which implies\
\[ S_j^* K(\cdot, \overline{z})e_i = z_j K(\cdot, \overline{z})e_i. \]

If we set $s_i(z) := K(\cdot, \overline{z})e_i$, then $s_1, \ldots, s_l$ is a holomorphic frame and $S_j^* s_i(z) = z_j s_i(z)$, that is, $S_j^*$ extends the coordinate multiplication on $E_H$. \qed

Adjoint of shift operator on reproducing kernel Hilbert spaces implements a universal model for Cowen-Douglas operators([S]). In Section 4.2 below, we relate its operator theory to contact theory on $E_H$.

### 4.2 Contact and unitary equivalence of quotient function spaces

We identify a class of invariant subspaces of shift operators given by vanishing conditions, and it turns out that unitary equivalence of their quotient spaces coincides with contact between the vector bundles. We begin with finite dimensional quotient spaces which corresponds to point-wise contact.

Fix $z_0 \in \Omega^*$, set\n\[ H_{z_0}^n = \{ f \in H | \partial^j f(z_0) = 0, |I| \leq n \} \]
then it is easy to see that $H_{z_0}^n$ is invariant under $S_1, \ldots, S_m$. By (4.8), we have\n\[ H \ominus H_{z_0}^n = \bigvee \{ \overline{\partial}^l K(\cdot, \overline{z_0})e_i, 1 \leq i \leq l, 0 \leq |I| \leq n \} \]
where $H \ominus H_{z_0}^n$ is the orthogonal complement (quotient space) of $H_{z_0}^n$.

Write $s_i(z) = K(\cdot, \overline{z})e_i$, then $s = \{s_1, \ldots, s_l\}$ is a holomorphic frame of $E_H$ and $\partial^l s_i(z) = \overline{\partial}^l K(\cdot, \overline{z})e_i$, which in turn gives\n\[ H \ominus H_{z_0}^n = \bigvee_{1 \leq i \leq l, 0 \leq |I| \leq n} \{ \partial^l s_i(z_0) \}, \]
and this is just the $z_0$ fiber the $n$-jet bundle $E_{z_0}^n$ of $E_H$. 30
Proposition 4.5. Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be two reproducing kernel Hilbert spaces over $\Omega \subseteq \mathbb{C}^m$. Then the quotient spaces $\mathcal{H} \ominus \mathcal{H}^n_{z_0}$ and $\tilde{\mathcal{H}} \ominus \tilde{\mathcal{H}}^n_{z_0}$ are unitarily equivalent (that is, $(S_1^n, \cdots, S_m^n)|_{\mathcal{H} \ominus \mathcal{H}^n_{z_0}}$ and $(\tilde{S}_1^n, \cdots, \tilde{S}_m^n)|_{\tilde{\mathcal{H}} \ominus \tilde{\mathcal{H}}^n_{z_0}}$ are unitarily equivalent) if and only if $E^n_{\mathcal{H}}$ and $E^n_{\tilde{\mathcal{H}}}$ have order $n$ contact at $z_0$.

Proof. By Definition 2.11, Proposition 4.3 and Proposition 4.4, $E_{\mathcal{H}}$ and $E_{\tilde{\mathcal{H}}}$ have order $n$ contact at $z_0$ if and only if there is a linear isometric map $\Phi$ from $E^n_{\mathcal{H}}(z_0)$ to $E^n_{\tilde{\mathcal{H}}}(z_0)$ such that for every $1 \leq j \leq m$, $\Phi(S_j^n - z_j) = (\tilde{S}_j^n - z_j)\Phi$, or equivalently, $\Phi S_j^n = \tilde{S}_j^n \Phi$. As $E^n_{\mathcal{H}}(z_0) = \mathcal{H} \ominus \mathcal{H}^n_{z_0}$ and $E^n_{\tilde{\mathcal{H}}}(z_0) = \tilde{\mathcal{H}} \ominus \tilde{\mathcal{H}}^n_{z_0}$, this is just unitary equivalence between $S_j^n|_{\mathcal{H} \ominus \mathcal{H}^n_{z_0}}$ and $\tilde{S}_j^n|_{\tilde{\mathcal{H}} \ominus \tilde{\mathcal{H}}^n_{z_0}}$.

Next we turn to invariant subspaces defined by vanishing condition on a hyper-surface

$$Z = \{(0, z_2, \cdots, z_m)|(z_2, \cdots, z_m) \in \Omega'\}$$

lying in $\Omega'$. This time we set

$$\mathcal{H}^n_Z = \{f \in \mathcal{H}, f(\overline{z}) = \partial_{z_1} f(\overline{z}) = \cdots = \partial_{z_m} f(\overline{z}) = 0, \text{ for all } z \in Z\}.$$ 

Then $\mathcal{H}^n_Z$ is just functions in $\mathcal{H}$ vanishing to order $n$ on $Z'$ (see [19]) which is easily seen to be invariant under the shift operators $S_1, \cdots, S_m$.

The quotient space $\mathcal{H} \ominus \mathcal{H}^n_Z$ is of infinite dimension and to determine unitary equivalence of $(S_1^n, \cdots, S_m^n)|_{\mathcal{H} \ominus \mathcal{H}^n_Z}$ is a nontrivial problem. In particular, $Z$ admits a holomorphic structure hence one can apply the Cowen-Douglas theory [6] to reduce unitary equivalence of the operator tuple $(S_1^n, \cdots, S_m^n)|_{\mathcal{H} \ominus \mathcal{H}^n_Z}$ to geometric conditions on $E^n_{\mathcal{H}}|Z$, and this geometric reduction has been the theme in a series of works [4, 11, 13, 14, 15]. In particular, Douglas and Misra settled the problem assuming $l = 1$ and $n = 1$ (see Sec 5, [14]), which was later extended by Douglas and the author allowing arbitrary $l$ while still assuming $n = 1$ (Theorem 21, [4]). To solve the problem in full generality (allowing arbitrary $n$ and $l$) had also been attempted first by Douglas and the author [4] and later by Deb [11], but the results obtained were not “fully geometric” (see Remark 4.9 below).

Here we use Theorems 4.5 to conclude this problem both in full generality and geometric completeness, and we first record two preparatory propositions.

Proposition 4.6. (Theorem 10, [4]) Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be two reproducing kernel Hilbert spaces over $\Omega \subseteq \mathbb{C}^m$ and

$$Z = \{(0, z_2, \cdots, z_m)|(z_2, \cdots, z_m) \in \Omega'\}$$

be an analytic hyper-surface in $\Omega'$. Then $\mathcal{H} \ominus \mathcal{H}^n_Z$ and $\tilde{\mathcal{H}} \ominus \tilde{\mathcal{H}}^n_Z$ are unitarily equivalent (that is, $(S_1^n, \cdots, S_m^n)|_{\mathcal{H} \ominus \mathcal{H}^n_Z}$ and $(\tilde{S}_1^n, \cdots, \tilde{S}_m^n)|_{\tilde{\mathcal{H}} \ominus \tilde{\mathcal{H}}^n_Z}$ are unitarily equivalent) if and only if there exists an isometric holomorphic bundle map $\Phi$ from $(E^n_{\mathcal{H}})^n_{z_1}|Z$ to $(E^n_{\tilde{\mathcal{H}}})^n_{z_1}|Z$ such that $\Phi S_1^n = \tilde{S}_1^n \Phi$.

Proposition 4.6 is a non-trivial result whose proof relies on an important theorem called Rigidity Theorem (Theorem 2.2, [6]) in the seminal work of Cowen and Douglas. Combing Proposition 4.6, Proposition 4.3 and Definition 2.12 gives the following analogue of Proposition 4.5.
Proposition 4.7. Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be two reproducing kernel Hilbert spaces over $\Omega \subseteq \mathbb{C}^m$ and 

$$Z = \{(0,z_2,\cdots,z_m)| (z_2,\cdots,z_m) \in \Omega' \}$$

be an analytic hyper-surface in $\Omega'$. Then $\mathcal{H} \oplus \mathcal{H}_Z^n$ and $\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}_Z^n$ are unitarily equivalent if and only if $E_{\mathcal{H}}$ and $E_{\tilde{\mathcal{H}}}$ have contact of order $n$ along $Z$.

Finally, combining Proposition 4.7 with Theorem 3.5 yields the following complete geometric reduction for unitary equivalence of $\mathcal{H} \oplus \mathcal{H}^Z_2$:

Theorem 4.8. The quotient spaces $\mathcal{H} \oplus \mathcal{H}_Z^n$ and $\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}_Z^n$ are unitarily equivalent if and only if there exists a holomorphic isometric bundle map $\Psi$ from $E_{\mathcal{H}}|_Z$ to $E_{\tilde{\mathcal{H}}}|_Z$ such that

(i) for $r,t \leq n-1$, $\Psi(K_{1t})_{zt} = (\tilde{K}_{1t})_{zt} \Psi$ on $Z$

(ii) for $r \leq n-1, 2 \leq j \leq m, \Psi(K_{1j})_{z_j} = (\tilde{K}_{1j})_{z_j} \Psi$ on $Z$

Remark 4.9. In the earlier work by Douglas and the author, our characterization(Theorem 22 [4]) was stated in terms of matrix functions $\partial_{z'_j} (\partial_{z'_1} H \cdot H^{-1}), r \leq n-1$, where $H$ is the Gram matrix for a holomorphic frame $s$ normalized at a fixed point on $Z$(also see Theorem 5.12 [11] by Deb where $Z$ is of arbitrary complex dimension). It is easy to check that for another holomorphic frame $t$ whose Gram matrix is $G$ and the transition matrix with $s$ is $X$, the compatibility condition

$$\partial_{z'_j} (\partial_{z'_1} G \cdot G^{-1}) = X \partial_{z'_j} (\partial_{z'_1} H \cdot H^{-1}) X^{-1},$$

does not hold(unless $t$ is also normalized at the same point, see Remark 23, [4]) hence $\partial_{z'_j} (\partial_{z'_1} H \cdot H^{-1})$ does not represent a geometric tensor. Now Theorem 4.8 refines Theorem 22 [4] by identifying $(K_{1j})_{z_j}$ as the “tensorial correction” of $\partial_{z'_j} (\partial_{z'_1} H \cdot H^{-1})$ in condition (ii) of Theorem 4.8. Precisely, $\partial_{z'_j} (\partial_{z'_1} H \cdot H^{-1})$ is actually the leading term $L_{z'_j}$ in the recursive matrix representation of $(K_{1j})_{z'_j}$ given by Lemma 3.12.

5 Appendix

This appendix is devoted to the proof of Lemma 3.13. We begin with some preparatory results.

Lemma 5.1. Let $\{F_l\}_{l=1}^\infty$ and $\{G_l\}_{l=1}^\infty$ be two sequences of square matrices(all matrices are of the same size). Let $\{H_l\}_{l=1}^\infty$ and $\{K_l\}_{l=1}^\infty$ be another two matrix sequences defined recursively by

$$H_1 = F_1, H_l = F_l - \sum_{i=1}^{l-1} \binom{l}{i} G_i H_{l-i}, l = 2,3,\cdots$$

$$K_1 = -G_1, K_l = -G_l - \sum_{i=1}^{l-1} \binom{l}{i} G_{l-i} K_i, l = 2,3,\cdots$$

Then for any positive integer $n$, it holds that

$$F_n + \sum_{i=1}^{n-1} \binom{n}{i} K_i F_{n-i} = H_n.$$
Proof. From (5.1) one easily sees that either side of (5.3) is a “linear combination” of \( F_1, F_2, \cdots, F_n \) whose coefficients are polynomials (linear sums of words) in \( G_1, G_2, \cdots, G_n \), hence it suffices to show that the coefficients for \( F_1, F_2, \cdots, F_{n-1} \) on both sides of (5.3) are equal (\( F_n \) appears in both sides with coefficient 1).

That (5.3) holds for \( n = 1 \) is trivial. Fix \( n \) and suppose

\[
H_m = F_m + \sum_{i=1}^{m-1} \binom{m}{i} K_i F_{m-i}
\]

for all \( m \leq n \) as an induction hypothesis. It remains to show that

\[
H_{n+1} = F_{n+1} + \sum_{i=1}^{n} \binom{n+1}{i} K_i F_{n+1-i},
\]

(5.5)

Now fix any \( 0 \leq k \leq n-1 \), we show that both sides of (5.5) have the same \( F_{n-k} \) coefficient which completes the induction.

For the left hand side, we compute the coefficient by combining the recursive representation

\[
H_{n+1} = F_{n+1} - \sum_{i=1}^{n} \binom{n+1}{i} G_i H_{n+1-i}
\]

and the induction hypothesis (5.4). Precisely, for \( n-k \leq m \leq n \) as an induction hypothesis. It remains to show that

\[
\begin{align*}
&- \binom{n+1}{k+1} G_{k+1} L_{n-k} - \binom{n+1}{k} G_k L_{n-k+1} - \binom{n+1}{k-1} G_{k-1} L_{n-k+2} - \cdots - \binom{n+1}{1} G_1 L_n \\
&\text{(5.7)}
\end{align*}
\]

Specifying (5.4) to every \( n-k \leq m \leq n \) and tracing the \( F_{n-k} \) term therein gives

\[
L_{n-k} = I, L_{n-k+i} = \binom{n-k+i}{i} K_i, i = 1, 2 \cdots k,
\]

inserting which into (5.1) we see that the coefficient of \( F_{n-k} \) in \( H_{n+1} \) is

\[
- \binom{n+1}{k+1} G_{k+1} - \sum_{i=1}^{k} \binom{n+1}{k+1-i} \binom{n-k+i}{i} G_{k+1-i} K_i
\]

(5.8)

On the other hand, the coefficient for \( F_{n-k} \) in the right hand side of (5.3) is \( \binom{n+1}{k+1} K_{k+1} \), hence it remains to show that

\[
\binom{n+1}{k+1} K_{k+1} = - \binom{n+1}{k+1} G_{k+1} - \sum_{i=1}^{k} \binom{n+1}{k+1-i} \binom{n-k+i}{i} G_{k+1-i} K_i
\]

(5.9)

To see this, specifying (5.2) with \( l = k+1 \) gives

\[
\binom{n+1}{k+1} K_{k+1} = - \binom{n+1}{k+1} G_{k+1} - \sum_{i=1}^{k} \binom{n+1}{k+1-i} \binom{k+1}{i} G_{k+1-i} K_i
\]

(5.10)
where the product of binomial coefficients

\[
\binom{n+1}{k+1-i} \binom{n-k+i}{i} = \binom{n+1}{k+1} \binom{k+1}{i}.
\]  

(5.11)

\[\square\]

**Corollary 5.2.** Let \( \{F_l\}_{l=1}^\infty \), \( \{G_l\}_{l=1}^\infty \) and \( \{H_l\}_{l=1}^\infty \) be as in Lemma 5.1. Let \( \{K_l\}_{l=1}^\infty \) be another matrix sequence defined recursively by

\[K_1 = -G_1, K_l = -G_l - \sum_{i=1}^{l-1} \binom{l}{i} K_i G_{l-i}, l = 2, 3 \ldots\]

(5.12)

Then for any positive integer \( n \), it holds that

\[F_n + \sum_{i=1}^{n-1} \binom{n}{i} K_i F_{n-i} = H_n.\]

(5.13)

**Proof.** This corollary differs from Corollary 5.1 by changing the recursive rule (5.2) into (5.12), hence it suffices to show that the formally different recursive rules (5.2) and (5.12) actually define the same sequence \( \{K_l\}_{l=1}^\infty \).

It is easy to see that with either (5.2) or (5.12), a word \( G_{i_1} G_{i_2} \cdots G_{i_k} \) appears in \( K_l \) if and only if \( i_1 + i_2 + \cdots + i_k = l \) (for instance, \( K_4 \) is a linear sum of \( G_4 \), \( G_1 G_3 \), \( G_3 G_1 \), \( G_2 G_2 \), \( G_1 G_1 G_2 \), \( G_1 G_2 G_1 \), \( G_2 G_1 G_1 \) and \( G_1 G_1 G_1 G_1 \)), so in the sequel, by coefficient of \( G_{i_1} G_{i_2} \cdots G_{i_k} \) we mean the coefficient of this word in \( K_{i_1 + i_2 + \cdots + i_k} \). We show that with either (5.2) or (5.12), the coefficient of \( G_{i_1} G_{i_2} \cdots G_{i_k} \) assume the same value, so these two recursive rules define the same sequence \( \{K_l\}_{l=1}^\infty \).

We first assume \( \{K_l\}_{l=1}^\infty \) is defined by (5.2) and compute the coefficient \( c_{i_1, i_2, \ldots, i_k} \) of \( G_{i_1} G_{i_2} \cdots G_{i_k} \) by “tracing the leading letters”. Precisely, let \( l = i_1 + i_2 + \cdots + i_k \), then \( G_{i_1} G_{i_2} \cdots G_{i_k} \) lies in \( K_l \) with leading letter \( G_{i_1} \), hence by (5.2) it must happen in the term \( -\binom{l}{i_1} G_{i_1} K_{l-i_1} \) therein, which gives

\[c_{i_1, i_2, \ldots, i_k} = -\binom{l}{l-i_1} c_{i_2, \ldots, i_k},\]

where \( c_{i_2, \ldots, i_k} \) is the coefficient of \( G_{i_2} G_{i_3} \cdots G_{i_k} \). Now the word \( G_{i_2} G_{i_3} \cdots G_{i_k} \) appears in \( K_{l-i_1} \) with leading letter \( G_{i_2} \), specifying (5.2) for \( K_{l-i_1} \) gives

\[c_{i_2, \ldots, i_k} = -\binom{l-i_1}{l-i_1-i_2} c_{i_3, i_4, \ldots, i_k},\]

where \( c_{i_3, \ldots, i_k} \) is the coefficient of \( G_{i_3} G_{i_4} \cdots G_{i_k} \).

Continuing this procedure, the value of \( c_{i_1, i_2, \ldots, i_k} \) can be finally expressed as the following product of binomial coefficients

\[(-1)^k \binom{l}{l-i_1} \binom{l-i_1}{l-i_1-i_2} \binom{l-i_1-i_2}{l-i_1-i_2-i_3} \cdots \binom{l-i_1-i_2-\cdots-i_{k-2}}{l-i_1-i_2-\cdots-i_k-1}\]

(5.14)
In the same way, if \( \{K_l\}_{l=1}^\infty \) is defined by (5.12), then by tracing the ending letters of the words, the \( G_{i_1}G_{i_2} \cdots G_{i_k} \) coefficient can be expressed as

\[
(-1)^k \left( \frac{l}{l-i_m} \right) \left( \frac{l-i_m}{l-i_m-i_{m-1}} \right) \cdots \left( \frac{l-i_m-\cdots-i_3}{l-i_m-\cdots-i_2} \right) \quad (5.15)
\]

Finally, an explicit computation shows that both (5.14) and (5.15) equal \((-1)^k\frac{l!}{i_1i_2\cdots i_k}\) (so the coefficient actually does not depend on the order of the letters) which completes the proof.

\[\square\]

**Corollary 5.3.** Let \( \{F_l\}_{l=1}^\infty \), \( \{G_l\}_{l=1}^\infty \) and \( \{H_l\}_{l=1}^\infty \) be as in Lemma 5.1. Let \( n \) and \( k \) be fixed positive integers with \( k \leq n \) and \( \{I_l\}_{l=1}^k \) be a finite matrix sequence defined recursively by

\[
I_1 = I, I_l = - \sum_{i=1}^{l-1} \binom{n-k+l-1}{i} I_{l-i} G_i, 2 \leq l \leq k.
\]

Then it holds that

\[
\sum_{i=1}^{k} \binom{n}{k-i+1} I_i F_{k+1-i} = \binom{n}{k} H_k. \quad (5.16)
\]

**Proof.** First write (5.16) into

\[
H_k = \sum_{i=1}^{k} \binom{n}{k-i+1} I_i F_{k+1-i} = F_k + \sum_{i=2}^{k} \binom{n}{k-i+1} I_i F_{k+1-i}.
\]

By Corollary 5.2, we also have

\[
H_k = F_k + \sum_{i=1}^{k-1} \binom{k}{i} K_i F_{k-i}
\]

where \( K_1, \cdots K_{k-1} \) is given by (5.12).

Comparing the coefficients of \( F_1, \cdots F_{k-1} \) in above two equations, it suffices to show that

\[
K_l = \binom{n-k-l}{n-k} I_{l+1} \quad (5.17)
\]

for all \( 1 \leq l \leq k-1 \). For convenience, we set \( J_l := \binom{n}{k-l+1} I_{l+1} \) and it remains to show that \( K_l = J_l \) for all \( 1 \leq l \leq k-1 \). That \( K_1 = J_1 \) is straightforward:

\[
J_1 = \binom{n-k}{n} I_2 = \binom{n-k}{n} \binom{k}{1} (-1) \binom{n-k+1}{1} I_1 G_1 = -G_1 = K_1
\]

and it suffices to show that \( \{J_l\}_{l=1}^{k-1} \) satisfies the same recursive rule as \( \{K_l\}_{l=1}^{k-1} \), that is,
\[ J_l = -G_l - \sum_{i=1}^{l-1} \binom{l}{i} J_i G_{l-i}, \quad l = 2, 3, \ldots, k - 1 \]  

(5.18)

To this end, we insert \( I_{l+1} = \binom{n}{k-1} \frac{(I_i)}{I_i} \) \( J_l, 1 \leq l \leq k - 1 \) into the recursive representation

\[
I_{l+1} = -\sum_{i=1}^{l} \binom{n-k+l}{i} I_{l+i} G_i = -\binom{n-k+l}{l} G_l - \sum_{i=1}^{l-1} \binom{n-k+l}{i} I_{l+i} G_i,
\]

to get

\[
J_l = -\binom{n-k+l}{l} G_l - \sum_{i=1}^{l-1} \binom{n-k+l}{i} \binom{n-1}{k-1} \frac{(I_i)}{I_i} J_{l+i} G_i.
\]

But this is exactly (5.18) as a consequence of the following easy-to-check identities on binomial coefficients \( n \geq k \geq l \geq i \):

\[
\binom{n}{k} \binom{n-k+i}{l} = 1, \quad \binom{n}{l} = \binom{n-1}{k-1} \frac{(I_i)}{I_i} = \binom{l}{i}.
\]

\[\square\]

**Proof.** We prove the conclusion by induction which trivially holds for \( n = 1 \).

Suppose \( (i) \iff (ii) \) holds for \( n - 1 \). To prove the conclusion for \( n \), it suffices to show

\[
F_n = \sum_{k=1}^{n} \binom{n}{k} Z_{n-k} \widetilde{F}_k Z_0^{-1} \iff H_n = Z_0 \widetilde{H}_n Z_0^{-1},
\]

(5.19)

with an additional condition (by induction hypothesis) that \( H_k = Z_0 \widetilde{H}_k Z_0^{-1} \) for all \( k \leq n - 1 \). We will prove this by appropriately re-arrange terms in the equation \( F_n = \sum_{k=1}^{n} \binom{n}{k} Z_{n-k} \widetilde{F}_k Z_0^{-1} \) so that its two sides become \( H_n \) and \( Z_0 \widetilde{H}_n Z_0^{-1} \) respectively.

From (3.42), it is easy to see that every word appearing in the sum \( \sum_{k=1}^{n} \binom{n}{k} Z_{n-k} \widetilde{F}_k Z_0^{-1} \) (except the last term \( Z_0 \widetilde{F}_n Z_0^{-1} \)) falls into either Class A or Class B defined as follows.

\[ A := \bigcup_{k=1}^{\infty} A_k, \text{ where } A_k := \{ \text{words starting with } G_k Z_0 \}, k = 1, 2, \ldots \]

\[ B := \bigcup_{k=1}^{\infty} B_k, \text{ where } B_k := \{ \text{words starting with } Z_0 \widetilde{G}_k \}, k = 1, 2, \ldots \]

Now we separate the recursive rule (3.42) to define another two recursive sequences \( \{X_l\}_{l=1}^{\infty} \) and \( \{Y_l\}_{l=1}^{\infty} \) by

\[
X_1 = G_1 Z_0, \quad X_l = G_l Z_0 - \sum_{i=1}^{l-1} \binom{l}{i} X_i \widetilde{G}_{l-i}
\]

(5.20)
and

\[ Y_1 = -Z_0\tilde{G}_1, \quad Y_l = -Z_0\tilde{G}_l - \sum_{i=1}^{l-1} \binom{l}{i} Y_i\tilde{G}_{l-i}. \] (5.21)

Comparing (5.20) with (3.42), one sees that for every \( l \geq 1 \), \( Z_l = X_l + Y_l \). Moreover, \( X_l \) only contains Class A words and \( Y_l \) contains Class B words.

Now the equation \( F_n = \sum_{k=1}^{n} \binom{n}{k} Z_{n-k}\tilde{F}_kZ_0^{-1} \) can be written as

\[ F_n = \sum_{k=1}^{n-1} \binom{n}{k} X_{n-k}\tilde{F}_kZ_0^{-1} + \sum_{k=1}^{n-1} \binom{n}{k} Y_{n-k}\tilde{F}_kZ_0^{-1} + Z_0\tilde{F}_nZ_0^{-1}. \]

If we can show

\[ F_n - \sum_{k=1}^{n-1} \binom{n}{k} X_{n-k}\tilde{F}_kZ_0^{-1} = H_n \] (5.22)

and

\[ \sum_{k=1}^{n-1} \binom{n}{k} Y_{n-k}\tilde{F}_kZ_0^{-1} + Z_0\tilde{F}_nZ_0^{-1} = Z_0\tilde{H}_nZ_0^{-1}, \] (5.23)

then \( F_n = \sum_{k=1}^{n} \binom{n}{k} Z_{n-k}\tilde{F}_kZ_0^{-1} \) is just \( H_n = Z_0\tilde{H}_nZ_0^{-1} \) which completes the proof.

To prove (5.22), it suffices to show that

\[ \sum_{k=1}^{n-1} \binom{n}{k} X_{n-k}\tilde{F}_kZ_0^{-1} = \sum_{k=1}^{n-1} \binom{n}{k} G_kH_{n-k}. \]

Since all words in the sum \( \sum_{k=1}^{n-1} \binom{n}{k} X_{n-k}\tilde{F}_kZ_0^{-1} \) lies in \( \cup_{k=1}^{n-1} A_k \), it suffices to show that for any fixed \( 1 \leq k_0 \leq n-1 \), the sum of class \( A_{n-k_0} \) words in \( \sum_{k=1}^{n-1} \binom{n}{k} X_{n-k}\tilde{F}_kZ_0^{-1} \) is \( \binom{n}{k_0} G_{n-k_0}H_{k_0} \).

Observing that by (5.20), words in class \( A_{n-k_0} \) only appear with \( X_{n-k_0}, X_{n-k_0+1}, \ldots, X_{n-1} \) in the sum \( \sum_{k=1}^{n-1} \binom{n}{k} X_{n-k}\tilde{F}_kZ_0^{-1} \). For every \( 1 \leq l \leq k_0 \), we denote the sum of Class \( A_{n-k_0} \) words in \( X_{n-k_0+l-1} \) by \( G_{n-k_0}Z_0I_l \) for some polynomial \( I_l \), then the sum of \( A_{n-k_0} \) words in \( \sum_{k=1}^{n-1} \binom{n}{k} X_{n-k}\tilde{F}_kZ_0^{-1} \) is

\[ G_{n-k_0}Z_0 \left( \binom{n}{k_0} I_1\tilde{F}_{k_0} + \binom{n}{k_0-1} I_2\tilde{F}_{k_0-1} + \cdots + \binom{n}{1} I_{k_0}\tilde{F}_{1} \right) Z_0^{-1}. \] (5.24)

On the other hand, (5.20) applied to \( X_{n-k_0} \) gives

\[ I_1 = I \]

In general,

\[ X_{n-k_0+l-1} = G_{n-k_0+l-1}Z_0 - \sum_{i=1}^{n-k_0+l-1-2} \binom{n-k_0+l-1}{i} X_{n-k_0+l-i-1}\tilde{G}_i. \] (5.25)
where in the right hand side, $X_{n-k_0+l-i-1}$ has no $A_{n-k_0}$ words if $i > l-1$, and if $i \leq l-1$, the $A_{n-k_0}$ contribution of $X_{n-k_0+l-i-1}$ is $\binom{n-k_0+l-1}{i}G_{n-k_0}Z_0I_{l-i}\tilde{G}_i$, hence we have the following recursive formula,

$$I_l = -\sum_{i=1}^{l-1} \binom{n-k_0+l-1}{i}I_{l-i}\tilde{G}_i, 1 \leq l \leq k_0.$$  

Now Lemma 5.3 applies to yield

$$\binom{n}{k_0}I_1\tilde{F}_{k_0} + \binom{n}{k_0-1}I_2\tilde{F}_{k_0-1} + \cdots + \binom{n}{1}I_{k_0}\tilde{F}_1 = \binom{n}{k_0}\tilde{H}_{k_0}. $$

Inserting this into (5.24) we finally see that the sum of $A_{n-k_0}$ terms in $\sum_{k=1}^{n-1} \binom{n}{k}X_{n-k}\tilde{F}_kZ_0^{-1}$ is $\binom{n}{k_0}G_{n-k_0}Z_0\tilde{H}_{k_0}Z_0^{-1} = \binom{n}{k_0}G_{n-k_0}H_{k_0}$ as desired.

The proof of (5.23) is more straightforward. In fact, we define a new sequence $\tilde{K}_1, \tilde{K}_2, \cdots$ by

$$\tilde{K}_1 = -\tilde{G}_1, \tilde{K}_l = -\tilde{G}_l - \sum_{i=1}^{l-1} \binom{l}{i}\tilde{K}_i\tilde{G}_{l-i}$$

then an application of Lemma 5.2 gives

$$\tilde{F}_n + \sum_{i=1}^{n-1} \binom{n}{i}\tilde{K}_i\tilde{F}_{n-i} = \tilde{H}_n, $$

or equivalently,

$$Z_0\tilde{F}_nZ_0^{-1} + \sum_{i=1}^{n-1} \binom{n}{i}Z_0\tilde{K}_i\tilde{F}_{n-i}Z_0^{-1} = Z_0\tilde{H}_nZ_0^{-1}. $$

On the other hand, comparing (5.21) with (5.26), one easily finds that $Y_i = Z_0\tilde{K}_i$ for all $i$, hence the above equation is exactly (5.23), completing the proof. 

\begin{flushright}$\square$\end{flushright}

**References**

[1] W. Arveson, *Subalgebras of C*-algebras. III. Multivariable operator theory*, Acta Math. 181 (1998), 159-228.

[2] R. Brawer, M. Pirovino, *The linear algebra of the Pascal matrix*, Linear Algebra Appl. 174 (1992), 13-23.

[3] M. Bayat, H. Teimoori, *The linear algebra of the generalized Pascal functional matrix*, Linear Algebra Appl. 295 (1999), 81-89.

[4] L. Chen, R. Douglas, *A local theory for operator tuples in the Cowen-Douglas class*, Adv Math. 307 (2017), 754-779.
[5] M. Cowen, R. Douglas, *Equivalence of connections*, Adv. Math. 56(1985), 39-91.

[6] M. Cowen, R. Douglas, *Complex geometry and operator theory*, Acta. Math. 141(1978), 187-261.

[7] X. Chen, K. Guo, *Analytic Hilbert modules*, Chapman Hall/CRC Research Notes in Mathematics, vol. 433, 2003.

[8] R. Curto, N. Salinas, *Generalized Bergman kernels and the Cowen-Douglas theory*, Amer J. Math. 106(1984), 447-488.

[9] G. Call, D. Velleman, *Pascals matrices*, Amer. Math. Monthly 100(1993), 372-376.

[10] Q. Chi, Y. Zheng, *Rigidity of pseudo-holomorphic curves of constant curvature in Grassmann manifolds*, Trans. Amer. Math. Soc. 313(1989), 393-406.

[11] P. Deb, *On unitary invariants of quotient Hilbert modules along smooth complex analytic sets*, arXiv:1708.06964v2.

[12] S. Deser, W. Drechsler, *Generalized gauge field copies*, Phys. Lett. B. 86(1979), 189-192.

[13] R. Douglas, G. Misra, *Equivalence of quotient Hilbert modules*, Proc. Indian Acad. Sci. Math. Sci. 113(2003), 281-291.

[14] R. Douglas, G. Misra, *Equivalence of quotient Hilbert modules. II*, Trans. Amer. Math. Soc. 360 (2008), 2229-2264.

[15] R. Douglas, G. Misra and C. Varughese, *On quotient modules—the case of arbitrary multiplicity*, J. Funct. Anal. 174(2000), 364-398.

[16] R. Douglas, V. Paulsen, *Hilbert modules over function algebras*, Pitman research notes in mathematics, Longman Scientific and Technical,1989.

[17] P. Griffiths, *On Cartan’s method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry*, Duke Math. J. 41(1974), 775-814.

[18] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics. Wiley-Interscience, New York, 1978.

[19] R. Gunning, H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, New York, 1965.

[20] J. Gleason, S. Richter, and C. Sundberg, *On the index of invariant subspaces in spaces of analytic functions of several complex variables*, J. Reine Angew. Math. 587(2005), 49-76.

[21] X. Jiao, J. Peng, *Rigidity of holomorphic curves in complex Grassmann manifolds*, Math. Ann. 327(2003), 481-486.
[22] X. Jiao, J. Peng, Classification of holomorphic spheres of constant curvature in complex Grassmann manifold $G(2, 5)$, Differ. Geom. Appl. 20(2004), 267-277.

[23] G. Misra, Operators in the Cowen-Douglas class and related topics, arXiv:1901.03801.

[24] M. Mostow, The field copy problem: to what extent do curvature (Gauge Field) and its covariant derivatives determine connection (Gauge Potential)?, Commun. Math. Phys. 78(1980), 137-150.

[25] M. Mostow, S. Shnider, Does a Generic Connection Depend Continuously on its Curvature?, Commun. Math. Phys. 90(1983), 417-432.

[26] Sz.-Nagy, C. Foias, H. Bercovici and L.Kerchy, Harmonic analysis of operators on Hilbert space, Universitext, Springer, New York, 2010.

[27] R. Wells, Differential analysis on complex manifolds, Springer, New York, 1973.

[28] Y. Yang, C. Micek, Generalized Pascal functional matrix and its applications, Linear Algebra Appl. 423(2007), 230-245.

[29] Z. Zhang, M. Liu, An extension of the generalized Pascal matrix and its algebraic properties, Linear Algebra Appl. 271(1998), 169-177.

[30] X. Zhao, T. Wang, The algebraic properties of the generalized Pascal functional matrices associated with the exponential families, Linear Algebra Appl. 318(2000), 45-52.

SCHOOL OF MATHEMATICS, SHANDONG UNIVERSITY, JINAN 250100, CHINA.
Email: lchencz@sdu.edu.cn