On two applications of truncated variation

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June 20, 2011

Abstract. We present two applications of truncated variation. The first application is a decomposition of a real-valued stochastic process with càdlàg trajectories into a sum of uniformly close, finite variation process and infinite variation process with arbitrary small amplitude. The second application is the definition of a stochastic integral as a limit of pathwise Riemann-Stieltjes integrals.

1 Introduction

Let \((X)_t\geq 0\) be a real-valued stochastic process with càdlàg trajectories and for \(c > 0\) let us define the process \((X^c)_t\geq 0\) with the following formula

\[ X^c_t = X_0 + UT^c(X, t) - DT^c(X, t). \]

Process \(X^c\) has the following properties (cf. [5]).

1. Its trajectories are uniformly close to the trajectories of the process \(X\), i.e.

\[ |X_t - X^c_t| \leq c \]

for any \(t \geq 0\);

2. it has increments which are uniformly close to the increments of \(X\), i.e.

\[ |(X_t - X_s) - (X^c_t - X^c_s)| \leq c \]

for any \(t \geq 0\); (1)

3. has finite total variation on any interval \([0; t]\) and

\[ TV(X^c, [0; t]) = TV^c(X, [0; t]), \]

which is the smallest total variation possible for the process satisfying (1);

4. has càdlàg trajectories with possible jumps only at the points, where also process \(X_t\) has jumps, moreover, for every \(t \geq 0\),

\[ |\Delta X^c_t| \leq |\Delta X_t|; \]

5. is adapted to the natural filtration \(\mathcal{F}_t = \sigma \left( (X_s)_{0 \leq s \leq t} \right)\).

Remark. In the sequel we will use only properties 1,4,5 and the fact that \(X^c\) has finite total variation. Thus there exist many other examples of the families of processes indexed by parameter \(c > 0\), \(\left( X^c \right)_{c > 0}\), which may be applied to the decomposition of a càdlàg process and to the pathwise stochastic integration, described in the sequel.
2 Decomposition of a càdlàg process with infinite total variation

Let \(0 \leq a < b\) and \(p > 0\). \(p\)-variation of the process \(X\) may be calculated at least in three ways.

1. 
\[
v_p^{(1)} (X, [a; b]) = \lim_{\delta \to 0} \limsup_{n} \sup_{a \leq t_0 < t_1 < \ldots < t_n \leq b} \sup_{\max(t_i - t_{i-1}) \leq \delta, i=1,2,\ldots,n} \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^p,
\]

2. 
\[
v_p^{(2)} (X, [a; b]) = \lim_{n \to \infty} \sup_{a \leq t_0 < t_1 < \ldots < t_n \leq b} \sup_{\max(t_i - t_{i-1}) \leq \delta_n, i=1,2,\ldots,n} \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^p,
\]

where \(\delta_n \to 0\) as \(n \to \infty\),

3. 
\[
v_p^{(3)} (X, [a; b]) = \lim_{n \to \infty} \sup_{\pi^{(n)} \subset \pi^{(n+1)}, n = 1,2,\ldots} \sum_{i=1}^{n} |X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)}|^p,
\]

where \(\pi^{(n)} = \{a \leq t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} \leq b\}\) is an increasing sequence of partitions, \(\pi^{(n)} \subset \pi^{(n+1)}\), \(n = 1,2,\ldots\) with \(\max_{i=1,2,\ldots,n} (t_i^{(n)} - t_{i-1}^{(n)}) \to 0\).

Notice that for \(X\) being a continuous process and \(p = 1\) all three definitions coincide (and we will call this common value total variation).

For \(W\) being a Wiener process with drift we have

\[
v_p^{(1)} (W, [a; b]) = \begin{cases} \infty & \text{if } p \leq 2; \\ 0 & \text{if } p > 2. \end{cases} \quad \text{a.s.}
\]

(cf. [4]), \(v_p^{(2)} (W, [a; b]) \leq v_p^{(1)} (W, [a; b])\) and may be infinite if \(\delta_n = O \left(\ln (n)^{-1}\right)\) (cf. [1]) but is a.s. finite and equal \(b - a\) when \(\delta_n = o \left(\ln (n)^{-1}\right)\) (cf. [2]), and for \(v_p^{(3)}\) one has (cf. [4])

\[
v_p^{(3)} (W, [a; b]) = \begin{cases} \infty & \text{if } p < 2; \\ b - a & \text{if } p = 2; \\ 0 & \text{if } p > 2. \end{cases} \quad \text{a.s.}
\]

Let us now assume that for any \(p > 0\), \(p\)-variation \(v_p\) is defined by one of the above three formulas.
For a càdlàg process \((X_t)_{t \geq 0}\) with infinite \(1\)-variation one may look at the decomposition

\[
X_t = X_t^c + (X_t - X_t^c)
\]
as a decomposition of the process \(X_t\) into a sum of a process with locally bounded total variation, uniformly approximating \(X_t\), and a "noise" \(X_t - X_t^c\) with small amplitude (smaller than \(c\)) but with infinite \(1\)-variation, i.e.

\[
v_1 (X^c, [0; t]) = TV^c (X, t) < +\infty,
\]

\[
v_1 (X, [0; t]) = v_1 (X - X^c, [0; t]) = +\infty.
\]
Indeed, let
$$ v_p (X^c, [0; t]) = 0, \quad v_p (X, [0; t]) = v_p (X - X^c, [0; t]). \tag{2} $$

The equality (2) follows from continuity of $X$ and from finiteness of its total variation. For any partition $\pi = \{a = t_0 < t_1 < \ldots < t_n = b\}$ we have

$$ \sum_{i=1}^n \left| X_{t_i} - X_{t_{i-1}} \right|^p \leq \max_{i=1,2,\ldots,n} \left| X_{t_i}^c - X_{t_{i-1}}^c \right|^{p-1} \left( \sum_{i=1}^n \left| X_{t_i}^c - X_{t_{i-1}}^c \right| \right) $$

$$ \leq \omega (mesh (\pi), X^c)^{p-1} TV^c (X, [0; t]), $$

where

$$ \omega (h, f) := \sup_{|x-y| \leq h} |f(x) - f(y)| $$

denotes modulus of continuity of the function $f$ and

$$ mesh (\pi) := \max_{i=1,2,\ldots,n} (t_i - t_{i-1}). $$

Hence, by the above inequalities and by the definition of $p$--variation,

$$ v_p (X^c, [0; t]) \leq v_p^{(1)} (X^c, [0; t]) $$

$$ \leq \lim_{\delta \to 0} \omega (\delta, X^c)^{p-1} TV^c (X, [0; 1]) $$

$$ = 0. $$

The equality (3) follows from (2) and the fact that for any deterministic functions $f$ and $g$,

$$ v_p (f + g, [0; t])^{1/p} \leq v_p (f, [0; t])^{1/p} + v_p (g, [0; t])^{1/p}. $$

Indeed, let $\pi^{(n)} = \{a = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = b\}$ be a sequence of partitions such that

$$ v_p (f + g, [0; t]) = \lim_{n \to \infty} \sum_{i=1}^n \left| (f + g) \left( t_i^{(n)} \right) - (f + g) \left( t_{i-1}^{(n)} \right) \right|^p. $$

By the Minkowski inequality,

$$ v_p (f + g, [0; t])^{1/p} \leq \limsup_{n \to \infty} \left( \sum_{i=1}^n \left| f \left( t_i^{(n)} \right) - f \left( t_{i-1}^{(n)} \right) \right|^p \right)^{1/p} $$

$$ \leq \limsup_{n \to \infty} \left( \sum_{i=1}^n \left| g \left( t_i^{(n)} \right) - g \left( t_{i-1}^{(n)} \right) \right|^p \right)^{1/p} $$

$$ \leq v_p (f, [0; t])^{1/p} + v_p (g, [0; t])^{1/p}, $$

$$ \lim_{a \to -\infty} f \left( t_i^{(n)} \right) = f(a), $$

$$ \lim_{c \to \infty} f \left( t_{i-1}^{(n)} \right) = f(c). $$

Notice, that for continuous $X$ and $p > 1$ we also have

$$ v_p (X^c, [0; t]) = 0, \quad v_p (X, [0; t]) = v_p (X - X^c, [0; t]). \tag{3} $$
3 Pathwise Riemann-Stieltjes integration with respect to the process $X^c$

3.1 Continuous case

Let $X_t$ and $Y_t$ be two continuous semimartingales. We will be interested in Itô’s integral with respect to the semimartingale $X^c_t$ (martingale part of which vanishes)

$$\int_0^T Y_t dX^c_t$$

and in the limit (in probability)

$$\lim_{c \downarrow 0} \int_0^T Y_t dX^c_t.$$

By the integration by parts formula (cf. [6, Proposition 3.1, Chapt. IV]) we have

$$Y_T X^c_T = Y_0 X^c_0 + \int_0^T Y_t dX^c_t + \int_0^T X^c_t dY_t + \langle Y, X^c \rangle_T.$$ 

Since $X^c$ is continuous and has locally finite total variation,

$$\langle Y, X^c \rangle_T = 0.$$

Hence,

$$\int_0^T Y_t dX^c_t = Y_T X^c_T - Y_0 X^c_0 - \int_0^T X^c_t dY_t.$$

Since $X^c_t \Rightarrow X_t$ as $c \downarrow 0$, we get (cf. [6, Theorem 2.12, Chap. IV])

$$P - \lim_{c \downarrow 0} \int_0^T X^c_t dY_t = \int_0^T X_t dY_t.$$

Again, by the integration by parts formula, we have

$$\int_0^T X_t dY_t = Y_T X_T - Y_0 X_0 - \int_0^T Y_t dX_t - \langle Y, X \rangle_T.$$

Finally

$$P - \lim_{c \downarrow 0} \int_0^T Y_t dX^c_t = \lim_{c \downarrow 0} \left\{ Y_T X^c_T - Y_0 X^c_0 - \int_0^T X^c_t dY_t \right\} = \int_0^T Y_t dX_t + \langle Y, X \rangle_T.$$

In particular, for $Y = X$ we get

$$P - \lim_{c \downarrow 0} \int_0^T X_t dX^c_t = \int_0^T X_t dX_t + \langle X, X \rangle_T = \frac{1}{2} \left( X^2_T - \langle X, X \rangle_T \right) + \langle X, X \rangle_T$$

$$= \frac{1}{2} X^2_T + \frac{1}{2} \langle X, X \rangle_T.$$
3.2 General case of semimartingales

Now let us look closer for the case when $X$ and $Y$ are general (not necessarily continuous) semimartingales.

We use again integration by parts formula (cf. [3, formula (1), page 438]) and we get

$$ Y_T X_T = \int_0^T Y_t dX_t^c + \int_0^T X_t Y_t - [Y, X^c]_T. $$

Again, by the uniform convergence, $X_t^c \Rightarrow X_t$ as $c \downarrow 0$, we get convergence in probability

$$ \mathbb{P} - \lim_{c \downarrow 0} \int_0^T X_{t-} dY_t = \int_0^T X_t dY_t. $$

Let us decompose $Y_t = Y_{t}^{cont} + \sum_{s \leq t} \Delta Y_s$, where $Y_{t}^{cont}$ is a continuous part of $Y$. Since $X^c$ has locally bounded variation (cf. [3, Theorem 23.6 (viii)]), we have

$$ [Y, X^c]_T = \sum_{s \leq T} \Delta Y_s \Delta X_s^c $$

(notice that for any $s$, $|\Delta X_s^c| \leq |\Delta X_s|$ thus the above sum is convergent). We calculate the limit

$$ \mathbb{P} - \lim_{c \downarrow 0} [Y, X^c]_T = \lim_{c \downarrow 0} \sum_{s \leq T} \Delta Y_s \Delta X_s^c = \sum_{s \leq T} \Delta Y_s \Delta X_s $$

and finally obtain

$$ \mathbb{P} - \lim_{c \downarrow 0} \int_0^T Y_t dX_t^c = \lim_{c \downarrow 0} \left\{ Y_T X_T - \int_0^T X_{t-} dY_t - [Y, X^c]_T \right\} $$

$$ = Y_T X_T - \int_0^T X_{t-} dY_t - \sum_{s \leq T} \Delta Y_s \Delta X_s. $$

Again, by integration by parts formula, we obtain

$$ \int_0^T X_{t-} dY_t = Y_T X_T - \int_0^T Y_{t-} dX_t - [Y, X]_T. $$

Finally, by [3, Corollary 23.15],

$$ \mathbb{P} - \lim_{c \downarrow 0} \int_0^T Y_t dX_t^c = \int_0^T Y_{t-} dX_t + [X, Y]_T - \sum_{s \leq T} \Delta Y_s \Delta X_s $$

$$ = \int_0^T Y_{t-} dX_t + [X^{cont}, Y^{cont}]_T. $$

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