CHARACTERIZATION OF FINITE DIMENSIONAL NILPOTENT LIE ALGEBRAS BY THE DIMENSION OF THEIR SCHUR MULTIPLIERS, \( s(L) = 5 \)

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Abstract. It is known that the dimension of the Schur multiplier of a non-abelian nilpotent Lie algebra \( L \) of dimension \( n \) is equal to \( \frac{1}{2}(n-1)(n-2) + 1 - s(L) \) for some \( s(L) \geq 0 \). The structure of all nilpotent Lie algebras has been given for \( s(L) \leq 4 \) in several papers. Here, we are going to give the structure of all non-abelian nilpotent Lie algebras for \( s(L) = 5 \).

1. Introduction and motivation

Let \( L \) be a finite dimensional nilpotent Lie algebra such that \( L \cong F/R \) for a free Lie algebra \( F \). Then by [1], the Schur multiplier \( \mathcal{M}(L) \) of \( L \) is isomorphic to \( R \cap F^2/[R, F] \). By a result of Moneyhun in [8], there exists a non-negative integer \( t(L) \) such that \( \dim \mathcal{M}(L) = \frac{1}{2}n(n-1) - t(L) \). It is a classical question to determine the structure of \( L \) by looking at the dimension of its Schur multiplier. The answer to this problem was given for \( t(L) \leq 8 \) in [2, 5, 6] and by putting some conditions on \( L \) for \( t(L) \leq 16 \) in [3] by several authors.

From [9], when \( L \) is a non-abelian nilpotent Lie algebra, the dimension of the Schur multiplier of \( L \) is equal to \( \frac{1}{2}(n-1)(n-2) + 1 - s(L) \) for some \( s(L) \geq 0 \). It not only improves the bound of Moneyhun but also let us ask the same natural question about the characterization of Lie algebras in terms of size \( s(L) \). The answer to this question was given by several papers in [11, 17, 18] for \( s(L) \leq 4 \) and for \( s(L) \leq 15 \) when conditions are put on \( L \) in [19].

On the other hand, looking for instance [11] shows that the characterization of nilpotent Lie algebras by looking \( s(L) \) causes to classification of nilpotent Lie algebras in terms of \( t(L) \) by a simple and a shorter way. This paper is devoted to obtain the structure of all nilpotent Lie algebras \( L \) for \( t(L) = 5 \).

Throughout the paper, we may assume that \( L \) is a Lie algebra over an algebraically closed field of characteristic not equal to 2 and \( A(n) \) and \( H(m) \) are used to denote the abelian Lie algebra of dimension \( n \) and the Heisenberg Lie algebra of dimension \( 2m + 1 \), respectively.

For the sake of convenience of reader some notations and terminology from [4, 5, 6, 7] are listed below.

Key words and phrases. Schur multiplier; nilpotent Lie algebra, capable Lie algebra.
Mathematics Subject Classification 2010. 17B30.
Another notion having relation to the capability is the concept of the exterior.

**Proposition 1.1.** From [16, Definition 1.4], let \( L_{3,2} \cong H(1) \) with a basis \( \{ x_1, x_2, x_3 \} \) and the multiplication

\[ [x_1, x_2] = x_3, \]

\( L_{4,3} \cong L(3, 4, 1, 4) \) with a basis \( \{ x_1, \ldots, x_4 \} \) and the multiplication

\[ [x_1, x_2] = x_3, [x_1, x_3] = x_4, \]

\( L_{5,5} \cong L(4, 5, 1, 6) \) with a basis \( \{ x_1, \ldots, x_5 \} \) and the multiplication

\[ [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5, \]

\( L_{5,6} \cong L'(7, 5, 1, 7) \) with a basis \( \{ x_1, \ldots, x_5 \} \) and the multiplication

\[ [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_2, x_3] = x_5, \]

\( L_{5,7} \cong L(7, 5, 1, 7) \) with a basis \( \{ x_1, \ldots, x_5 \} \) and the multiplication

\[ [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, \]

\( L_{5,8} \cong L(4, 5, 2, 4) \) with a basis \( \{ x_1, \ldots, x_5 \} \) and the multiplication

\[ [x_1, x_2] = x_3, [x_1, x_3] = x_5, \]

\( L_{5,9} \cong L(7, 5, 2, 7) \) with a basis \( \{ x_1, \ldots, x_5 \} \) and the multiplication

\[ [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, \]

\( L_{6,10} \) with a basis \( \{ x_1, \ldots, x_6 \} \) and the multiplication

\[ [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_5, \]

\( L_{6,22}(\varepsilon) \) with a basis \( \{ x_1, \ldots, x_6 \} \) and the multiplication

\[ [x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = \varepsilon x_6, [x_3, x_4] = x_5, \varepsilon \in F, \]

\( L_1 \cong 27B \) with a basis \( \{ x_1, \ldots, x_7 \} \) and the multiplication

\[ [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_2, x_3] = x_7, \]

\( L_2 \cong 27A \) with a basis \( \{ x_1, \ldots, x_7 \} \) and the multiplication

\[ [x_1, x_2] = x_6, [x_1, x_4] = x_7, [x_3, x_5] = x_7, \]

\( 157 \) with a basis \( \{ x_1, \ldots, x_7 \} \) and the multiplication

\[ [x_1, x_2] = x_3, [x_1, x_3] = x_2, [x_4] = x_7, \]

\( 37B \) with a basis \( \{ x_1, \ldots, x_7 \} \) and the multiplication

\[ [x_1, x_2] = x_5, [x_2, x_3] = x_6, [x_3, x_4] = x_7, \]

\( 37C \) with a basis \( \{ x_1, \ldots, x_7 \} \) and the multiplication

\[ [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_3] = x_6, [x_2, x_4] = x_7, \]

\( 37D \) with a basis \( \{ x_1, \ldots, x_7 \} \) and the multiplication

\[ [x_1, x_2] = x_3, [x_3, x_4] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = x_7. \]

**Main Theorem.** Let \( L \) be a non-abelian \( n \)-dimensional nilpotent Lie algebra. Then \( s(L) = 5 \) if and only if \( L \) is isomorphic to one of the Lie algebras \( L(4, 5, 2, 4) \oplus A(4), L(3, 4, 1, 4) \oplus A(3), L(4, 5, 1, 6) \oplus A(2), L_{6,22}(\varepsilon) \oplus A(2), L_{6,26} \oplus A(1), L_{6,10}, L_{6,23}, L_{6,25}, 37B, 37C \) or \( 37D \).

We state some results without proof and refer the reader to see [10, 11, 14, 18].

**Proposition 1.1.** (See [12, Proposition 2.10]) The Schur multiplier of Lie algebras \( L_{6,22}(\varepsilon), L_{5,8}, L_1 \) and \( L_2 \) are abelian Lie algebras of dimension \( 8, 6, 9 \) and \( 10 \), respectively.

A Lie algebra \( L \) is called capable provided that \( L \cong H/Z(H) \) for a Lie algebra \( H. \) From [16, Definition 1.4], \( Z^*(L) \) is used to denote the epicenter of \( L. \) The importance of \( Z^*(L) \) is due to the fact that \( L \) is capable if and only if \( Z^*(L) = 0. \) Another notion having relation to the capability is the concept of the exterior center of a Lie algebra \( Z^>(L) \) which is introduced in [10]. It is known that from [10, Lemma 3.1]), \( Z^*(L) = Z^>(L). \)
Lemma 1.2. (See [18] Corollary 2.3) Let \( L \) be a non-capable nilpotent Lie algebra of dimension \( n \) such that \( \dim L^2 \geq 2 \). Then
\[
n - 3 < s(L).
\]

Let \( \otimes_{\text{mod}} \) be used to denote the operator of usual tensor product of Lie algebras. Then

Theorem 1.3. (See [15] Theorem 2.1) Let \( L \) be a finite dimensional nilpotent Lie algebra non-abelian Lie algebra of class two. Then
\[
0 \to \ker g \to L^2 \otimes_{\text{mod}} L^{ab} \xrightarrow{g} \mathcal{M}(L) \to \mathcal{M}(L^{ab}) \to L^2 \to 0
\]
is exact, in where
\[
g : x \otimes (z + L^2) \in L^2 \otimes_{\text{mod}} L^{ab} \mapsto [\pi, z] + [R, F] \in \mathcal{M}(L) = R \cap F^2/[R, F],
\]
\[
\pi(\pi + R) = x \text{ and } \pi(\pi + R) = z.
\]
Moreover, the subalgebra \( K = \langle [x, y] \otimes (z + L^2) + [z, x] \otimes (y + L^2) + [y, z] \otimes (x + L^2) \mid x, y, z \in L \rangle \) is contained in \( \ker g \).

2. The Proof of Main Theorem

We begin with the following lemma that is easily proven.

Lemma 2.1. There is no \( n \)-dimensional nilpotent Lie algebra with \( s(L) = 5 \), when
(i) \( \dim L^2 \geq 4 \);
(ii) \( \dim L^2 = 1 \).

Proof.
(i) Let \( L \) be a nilpotent Lie algebra such that \( m = \dim L^2 \geq 4 \) and \( s(L) = 5 \). Then 3 Theorem 3.1] and our assumption imply that
\[
\frac{1}{2}(n-1)(n-2) - 4 = \dim \mathcal{M}(L) \leq \frac{1}{2}(n+m-2)(n-m-1) + 1 \leq \frac{1}{2}(n+2)(n-5) + 1.
\]
It is a contradiction.

(ii) By contrary. Let \( L \) be a Lie algebra such that \( \dim L^2 = 1 \) and \( s(L) = 5 \). Then by using [9] Lemma 2.3, \( L \cong H(m) \oplus A(n-2m-1) \) for some \( m \geq 1 \). Looking [17] Corollary 4 shows that \( s(L) = 0 \) or \( s(L) = 2 \), when \( m = 1 \) or \( m \geq 2 \), respectively. It is a contradiction. Hence the result follows.

By using Lemma 2.1 we may assume that a nilpotent Lie algebras \( L \) with \( s(L) = 5 \) has \( 2 \leq \dim L^2 \leq 3 \). First assume that \( \dim L^2 = 2 \).

Lemma 2.2. Let \( L \) be an \( n \)-dimensional non-capable nilpotent Lie algebra of dimension at most 7 and \( \dim L^2 = 2 \). Then \( L \) is isomorphic to one of the Lie algebras \( L_{6,10} \), \( L_{2} \) or 157. Moreover, \( s(L_{6,10}) = 5 \) and \( s(L_{2}) = s(157) = 6 \).

Proof. The proof is similar to [18] Theorem 2.6.

Theorem 2.3. Let \( L \) be an \( n \)-dimensional nilpotent Lie algebra with \( s(L) = 5 \) and \( \dim L^2 = 2 \). Then \( L \) is isomorphic to one of the Lie algebras \( L(4, 5, 2, 4) \oplus A(4), L(3, 4, 1, 4) \oplus A(3), L(4, 5, 1, 6) \oplus A(2), L_{6,22}(\varepsilon) \oplus A(2) \) or \( L_{6,10} \).

Proof. Sine \( \dim L^2 = 2 \), \( L \) is nilpotent of class two or three. Let \( L \) be a Lie algebra of nilpotency class two. If \( L \) is a capable Lie algebra, then it should be isomorphic to one of the Lie algebras \( L_{6,22}(\varepsilon) \oplus A, L_{5,8} \oplus A \) or \( L_1 \oplus A \), for an abelian Lie algebra \( A \) by using [12] Corollary 2.13.
Case (i). Let \( L \cong L_{6,22}(\varepsilon) \oplus A \). Proposition 3 implies \( \dim M(L_{6,22}(\varepsilon)) = 8 \). Since \( 5 = s(L) = \frac{1}{2}(n-1)(n-2)+1 - \dim M(L) \) and \( \dim M(L) = 8 + \frac{1}{2}(n-6)(n+1) \) by using [2, Theorem 1] and [8, Lemma 23], we have \( n = 8 \). Hence \( L \cong L_{6,22}(\varepsilon) \oplus A(2) \).

Case (ii). Let now \( L \cong L_{5,8} \oplus A \). We know from Proposition 3 that \( \dim M(L_{5,8}) = 6 \). Since \( 5 = s(L) = \frac{1}{2}(n-1)(n-2)+1 - \dim M(L) \) and \( \dim M(L) = 6 + \frac{1}{2}(n-5)n \) by using [2, Theorem 1] and [8, Lemma 23], we have \( n = 9 \). Therefore \( L \cong L_{5,8} \oplus A(4) \cong L(4,5,2,4) \oplus A(4) \).

Case (iii). Let \( L \cong L_1 \oplus A \). We know \( \dim M(L_1) = 9 \) by using Proposition 3. Since \( 5 = s(L) = \frac{1}{2}(n-1)(n-2)+1 - \dim M(L) \) and \( \dim M(L) = 9 + \frac{1}{2}(n-7)(n+2) \) by using [2, Theorem 1] and [8, Lemma 23], we have \( n = 5 \), which is contradiction. Thus \( L \) cannot be isomorphic to \( L_1 \oplus A \).

Now let \( L \) be a Lie algebra of nilpotency class 3. If \( L \) is a capable Lie algebra, then it should be isomorphic to one of the Lie algebras \( L_{4,3} \oplus A(n-4) \) or \( L_{5,5} \oplus A(n-5) \) by using [13, Theorem 5.5].

Case (i). Let \( L \cong L_{4,3} \oplus A(n-4) \). Since \( \dim M(L_{4,3}) = 2 \) by using [5, Section 2], we have \( \dim M(L) = 2 + \frac{1}{2}(n-4)(n-1) \) by using [2, Theorem 1] and [8, Lemma 23]. Since \( 5 = s(L) = \frac{1}{2}(n-1)(n-2)+1 - \dim M(L) \) and \( \dim M(L) = 2 + \frac{1}{2}(n-4)(n-1) \), we have \( n = 7 \). Hence \( L \cong L_{4,3} \oplus A(3) \cong L(3,4,1,4) \oplus A(3) \).

Case (ii). Suppose \( L \cong L_{5,5} \oplus A(n-5) \). [5, Section 3] shows that \( \dim M(L_{5,5}) = 4 \). Now [2, Theorem 1] and [8, Lemma 23] imply that \( \dim M(L) = 4 + \frac{1}{2}n(n-5) \). Since \( 5 = s(L) = \frac{1}{2}(n-1)(n-2)+1 - \dim M(L) \) and \( \dim M(L) = 4 + \frac{1}{2}n(n-5) \), we have \( n = 7 \). Therefore \( L \cong L_{5,5} \oplus A(2) \cong L(4,5,1,6) \oplus A(2) \).

If \( L \) is a non-capable Lie algebra of nilpotency class 2 or 3, then by using Lemma 1.2, we have \( n \leq 7 \). Therefore \( L \cong L_{6,10} \) by using Lemma 2.2. This completes the proof.

We now consider the case that \( \dim L^2 = 3 \). By looking all nilpotent Lie algebras listed in [4], we may choose all \( n \)-dimensional nilpotent Lie algebras \( L \) such that \( \dim L^2 = 3 \) for \( n = 5 \) or 6 in the Table 1.

**Table 1.**

| Name  | Nonzero multiplication |
|-------|------------------------|
| \( L_{5,6} \) | \( [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = [x_2,x_3] = x_5 \) |
| \( L_{5,7} \) | \( [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = x_5 \) |
| \( L_{5,9} \) | \( [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_2,x_3] = x_5 \) |
| \( L_{6,6} \) | \( [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = [x_2,x_3] = x_5 \) |
| \( L_{6,7} \) | \( [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = x_5 \) |
| \( L_{6,9} \) | \( [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_2,x_3] = x_5 \) |
| \( L_{6,11} \) | \( [x_1,x_2] = x_3, [x_1,x_3] = x_4, [x_1,x_4] = [x_2,x_3] = [x_2,x_5] = x_6 \) |

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Assume $L$ is nilpotent Lie algebra of dimension 7 such that $\dim L^2 = 3$. By looking the classification of all nilpotent Lie algebras in [7], $L$ must be isomorphic to one of the Lie algebras listed in Table 2 and 3.

### Table 2. 7-dimensional indecomposable nilpotent Lie algebras

| Name  | Nonzero multiplication |
|-------|------------------------|
| 3TA   | $[x_1, x_2] = x_5, [x_2, x_3] = x_6, [x_2, x_4] = x_7$ |
| 37B   | $[x_1, x_2] = x_5, [x_2, x_3] = x_6, [x_3, x_4] = x_7$ |
| 37C   | $[x_1, x_2] = [x_3, x_4] = x_5, [x_2, x_3] = x_6, [x_2, x_4] = x_7$ |
| 37D   | $[x_1, x_2] = [x_3, x_4] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = x_7$ |
| 257A  | $[x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_6, [x_1, x_5] = x_7$ |
| 257B  | $[x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_1, x_4] = [x_2, x_5] = x_7$ |
| 257C  | $[x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_2, x_4] = x_7$ |
| 257D  | $[x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_6, [x_1, x_4] = [x_2, x_5] = x_7$ |
| 257E  | $[x_1, x_2] = x_3, [x_1, x_3] = [x_4, x_5] = x_6, [x_2, x_4] = x_7$ |
| 257F  | $[x_1, x_2] = x_3, [x_2, x_3] = [x_4, x_5] = x_6, [x_2, x_4] = x_7$ |

continued on the next page
Table 2. 7-dimensional indecomposable nilpotent Lie algebras

| Name | Nonzero multiplication |
|------|------------------------|
| 257G | $[x_1, x_2] = x_3, [x_1, x_3] = [x_4, x_5] = x_6,$  
|      | $[x_1, x_5] = [x_2, x_4] = x_7$ |
| 257H | $[x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_6, [x_4, x_5] = x_7$ |
| 257I | $[x_1, x_2] = x_3, [x_1, x_3] = [x_1, x_4] = x_6,$  
|      | $[x_1, x_5] = [x_2, x_3] = x_7$ |
| 257J | $[x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_6, [x_1, x_5] = [x_2, x_3] = x_7$ |
| 257K | $[x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_2, x_3] = [x_4, x_5] = x_7$ |
| 257L | $[x_1, x_2] = x_3, [x_1, x_3] = [x_2, x_4] = x_6,$  
|      | $[x_2, x_3] = [x_4, x_5] = x_7$ |
| 147A | $[x_1, x_2] = x_4, [x_1, x_3] = x_5,$  
|      | $[x_1, x_6] = [x_2, x_5] = [x_3, x_4] = x_7$ |
| 147B | $[x_1, x_2] = x_4, [x_1, x_3] = x_5,$  
|      | $[x_1, x_4] = [x_2, x_6] = [x_3, x_5] = x_7$ |
| 1457A | $[x_1, x_i] = x_{i+1} i = 2, 3,$  
|      | $[x_1, x_4] = [x_5, x_6] = x_7$ |
| 1457B | $[x_1, x_i] = x_{i+1} i = 2, 3,$  
|      | $[x_1, x_4] = [x_2, x_3] = [x_5, x_6] = x_7$ |
| 137A | $[x_1, x_2] = x_5, [x_1, x_5] = [x_3, x_6] = x_7, [x_3, x_4] = x_6$ |
| 137B | $[x_1, x_2] = x_5, [x_3, x_4] = x_6,$  
|      | $[x_1, x_5] = [x_2, x_4] = [x_3, x_6] = x_7$ |
| 137C | $[x_1, x_2] = x_5, [x_1, x_4] = [x_2, x_3] = x_6,$  
|      | $[x_1, x_6] = [x_2, x_4] = x_7, [x_3, x_5] = -x_7$ |
| 137D | $[x_1, x_2] = x_5, [x_1, x_4] = [x_2, x_3] = x_6,$  
|      | $[x_1, x_6] = [x_2, x_4] = x_7, [x_3, x_5] = -x_7$ |
| 1357A | $[x_1, x_2] = x_4, [x_1, x_4] = [x_2, x_3] = x_5,$  
|      | $[x_1, x_5] = [x_2, x_6] = x_7, [x_3, x_4] = -x_7$ |
| 1357B | $[x_1, x_2] = x_4, [x_1, x_4] = [x_2, x_3] = x_5,$  
|      | $[x_1, x_5] = [x_3, x_6] = x_7, [x_3, x_4] = -x_7$ |
| 1357C | $[x_1, x_2] = x_4, [x_1, x_4] = [x_2, x_3] = x_5,$  
|      | continued on the next page |
Table 2. 7-dimensional indecomposable nilpotent Lie algebras

| Name | Nonzero multiplication |
|------|------------------------|
|      | $[x_1, x_5] = [x_2, x_4] = x_7, [x_3, x_4] = -x_7$ |

Table 3. 7-dimensional decomposable nilpotent Lie algebras

| Name       | Name                     |
|------------|--------------------------|
| $L_{4,3} \oplus H(1)$ | $L_{6,19}(\epsilon) \oplus A(1)$ |
| $L_{5,6} \oplus A(2)$  | $L_{6,20} \oplus A(1)$   |
| $L_{5,7} \oplus A(2)$  | $L_{6,23} \oplus A(1)$   |
| $L_{5,9} \oplus A(2)$  | $L_{6,24}(\epsilon) \oplus A(1)$ |
| $L_{6,11} \oplus A(1)$ | $L_{6,25} \oplus A(1)$   |
| $L_{6,12} \oplus A(1)$ | $L_{6,26} \oplus A(1)$   |
| $L_{6,13} \oplus A(1)$ |                         |

We need the following lemma from [13, Lemma 2.7] for the proof of the Main Theorem.

**Lemma 2.4.** Let $L$ be an $n$-dimensional nilpotent Lie algebra such that $n = 5, 6$ or $7$, $\dim L^2 = \dim Z(L) = 3$ and $Z(L) = L^2$. Then the structure and the Schur multiplier of $L$ are given in the following table.

| Name | dim $\mathcal{M}(L)$ | s($L$) | Name | dim $\mathcal{M}(L)$ | s($L$) |
|------|-----------------------|--------|------|-----------------------|--------|
| $L_{6,26}$ | 8                      | 3      | $37C$ | 11                     | 5      |
| $37A$ | 12                     | 4      | $37D$ | 11                     | 5      |
| $37B$ | 11                     | 5      |      |                       |        |

**Lemma 2.5.** Let $L$ be a nilpotent Lie algebra of dimension at most 7 such that $\dim L^2 = 3$, $\dim Z(L) = 2$ and $Z(L) \subset L^2$. Then the structure and the Schur multiplier of $L$ are given in the following table.

| Name | dim $\mathcal{M}(L)$ | s($L$) | Name | dim $\mathcal{M}(L)$ | s($L$) |
|------|-----------------------|--------|------|-----------------------|--------|
| $L_{6,9}$ | 3                      | 4      | $257E$ | 8                     | 8      |
| $L_{6,23}$ | 6                      | 5      | $257F$ | 9                     | 7      |
| $L_{6,24}(\epsilon)$ | 5          | 6      | $257G$ | 8                     | 8      |
| $L_{6,25}$ | 6                      | 5      | $257H$ | 8                     | 8      |
| $257A$ | 9                      | 7      | $257I$ | 8                     | 8      |

*continued on the next page*
Table 5.

| Name   | $\dim \mathcal{M}(L)$ | $s(L)$ |
|--------|------------------------|--------|
| 257B   | 8                      | 8      |
| 257C   | 9                      | 7      |
| 257D   | 8                      | 8      |

257J 8 8 257K 6 10
257L 6 10

Proof. The proof is similar to [18, Lemma 2.5]. \qed

Lemma 2.6. Let $L$ be an $n$-dimensional nilpotent Lie algebra such that $n = 7$, $\dim L^2 = 3$ and $\dim Z(L) = 4$. Then the structure and the Schur multiplier of $L$ are given in the following table.

Table 6.

| Name     | $\dim \mathcal{M}(L)$ | $s(L)$ |
|----------|------------------------|--------|
| $L_{5,9} \oplus A(2)$ | 8 | 8 |
| $L_{6,26} \oplus A(1)$ | 11 | 5 |

Proof. Since $\dim Z(L) = 4$, $L$ is isomorphic to $L_{5,9} \oplus A(2)$ or $L_{6,26} \oplus A(1)$ by searching in Tables 2 and 3. Let $L \cong L_{6,26} \oplus A(1)$. Since $\dim \mathcal{M}(L_{6,26}) = 8$ by using Table 4, we have $\dim \mathcal{M}(L) = 11$ by using [2, Theorem 1] and [8, Lemma 23]. Hence $s(L) = 5$. Also by using similar method, we can see $\dim \mathcal{M}(L_{5,9} \oplus A(2)) = 8$ and $s(L) = 8$. \qed

Lemma 2.7. [18, Lemma 2.9] Let $L$ be an $n$-dimensional nilpotent Lie algebra such that $n = 5, 6$ or $7$, $\dim L^2 = 3$ and $\dim Z(L) = 1$. Then the structure and the Schur multiplier of $L$ are given in the following table.

Table 7.

| Name     | $\dim \mathcal{M}(L)$ | $s(L)$ |
|----------|------------------------|--------|
| $L_{5,6}$ | 3                      | 4      |
| $L_{5,7}$ | 3                      | 4      |
| $L_{6,11}$| 5                      | 6      |
| $L_{6,12}$| 5                      | 6      |
| $L_{6,13}$| 4                      | 7      |
| $L_{6,19}(\epsilon)$ | 5 | 6 |
| $L_{6,20}$ | 5                      | 6      |

1457A 6 10
1457B 6 10
137A 7 9
137B 7 9
137C 7 9
137D 7 9

1357A 7 9

continued on the next page
Lemma 2.9. There is no nilpotent Lie algebra $s$.

Proof. $s$ and $\dim \ker \lambda_{cl} M$ hand, $\dim g$.

Let $\dim \ker \lambda g$ be of rank $n$. Then $\dim L^2 = 3$. Therefore the assumption is false and the result follows.

□

Recall that a Lie algebra $L$ is called generalized Heisenberg of rank $n$ if $L^2 = Z(L)$ and $\dim L^2 = n$.

Lemma 2.8. Let $L$ be an $n$-dimensional generalized Heisenberg of rank 3 with $s(L) = 5$, then $n \leq 7$.

Proof. By Theorem 1.3 we have $\dim \ker g = \dim \mathcal{M}(L) - \dim L^2 + \dim L^{ab} \otimes_{mod} L^2 = \dim \mathcal{M}(L)$. Since $\dim \mathcal{M}(L) = \frac{1}{2}(n-1)(n-2) - 4$ and $\dim L^{ab} = n - 3$, we have $\dim \ker g = n - 3$.

By contrary let $n \geq 8$. Then $d = \dim L^{ab} = n - 3 \geq 5$. Since $\dim L^2 = 3$, we can choose a basis $\{x_1 + L^2, \ldots, x_d + L^2\}$ for $L^{ab}$ such that $[x_1, x_2], [x_2, x_3]$ and $[x_3, x_4]$ are non-trivial in $L^2$. Thus

$$L^{ab} \otimes_{mod} L^2 \cong \bigoplus_{i=1}^{d} ((x_i + L^2)) \otimes_{mod} L^2).$$

Hence all elements of

$$\{[x_1, x_2] \otimes x_i + L^2 \otimes [x_1, x_1] \otimes x_2 + L^2 \otimes [x_2, x_i] \otimes x_1 + L^2, | 3 \leq i \leq d, i \neq 1, 2\}$$

and

$$\{[x_2, x_3] \otimes x_i + L^2 \otimes [x_1, x_3] \otimes x_2 + L^2 \otimes [x_3, x_i] \otimes x_2 + L^2, | 3 \leq i \leq d, i \neq 1, 2, 3\}$$

$$\{[x_3, x_4] \otimes x_i + L^2 \otimes [x_1, x_3] \otimes x_4 + L^2 \otimes [x_4, x_i] \otimes x_3 + L^2, | 3 \leq i \leq d, i \neq 2, 3, 4\}$$

are linearly independent and so $2(n-6) + n - 5 \leq \ker g$. That is a contradiction for $n \geq 8$. Therefore the assumption is false and the result follows.

□

Let $c(L)$ be used to show the nilpotency class of $L$. Then

Lemma 2.9. There is no nilpotent Lie algebra $L$ with $\dim L^2 = 3$, $\dim Z(L) = 1$ and $s(L) = 5$ such that $L/Z(L) \cong L_{5,8} \oplus A(2)$.

Proof. By contrary, let $L$ be a nilpotent Lie algebra $L$ with $\dim L^2 = 3$, $\dim Z(L) = 1$ and $s(L) = 5$ such that $L/Z(L) \cong L_{5,8} \oplus A(2)$. Then $\dim L = 8$ and $c(L) = 3$. Since $c(L) = 3$ and $\dim Z(L) = 1$, we have $L^3 = Z(L)$. On the other hand, $\dim \mathcal{M}(L) = \dim \mathcal{M}(L/Z(L)) + (\dim L/L^2 - 1) \dim Z(L) - \dim \ker \lambda_3$ and $\dim \ker \lambda_3 \geq 2$ by using proof [1], Theorem 1.1]. Thus

$$\dim \mathcal{M}(L) \leq \dim \mathcal{M}(L/Z(L)) + (\dim L/L^2 - 1) \dim Z(L) - 2$$

It is a contradiction.

□

Theorem 2.10. Let $L$ be an $n$-dimensional nilpotent Lie algebra with $s(L) = 5$ and $\dim L^2 = 3$. Then $L$ is isomorphic to one of the Lie algebras $L_{6,23}, L_{6,25}, 37B, 37C$ or $37D$. 

\begin{table}[h]
\centering
\caption{Table 7.}
\begin{tabular}{cccccc}
\hline
Name & $\dim \mathcal{M}(L)$ & $s(L)$ & Name & $\dim \mathcal{M}(L)$ & $s(L)$ \\
\hline
147A & 8 & 8 & 1357B & 6 & 10 \\
147B & 8 & 8 & 1357C & 6 & 10 \\
\hline
\end{tabular}
\end{table}
Proof: First assume that $\dim Z(L) \geq 5$, or $\dim Z(L) = 3$ and $Z(L) \neq L^2$, or $\dim Z(L) = 2$ and $Z(L) \not\subset L^2$. We show that in these cases, there is no such a Lie algebra $L$ of dimension $n$ with $s(L) = 5$.

Let $I$ be a central ideal of $L$ of dimension one such that $L^2 \cap I = 0$. Since $\dim(L/I)^2 = 3$, by using [11, Theorem 3.1], we have

$$\dim M(L/I) \leq \frac{1}{2} n(n-5) + 1.$$ 

If the equality holds, then

$$\frac{1}{2} (n-2)(n-3) + 1 - s(L/I) = \dim M(L/I) = \frac{1}{2} n(n-5) + 1.$$

Therefore $s(L/I) = 3$ and by using [17, Theorem 3.2], there is no Lie algebra satisfying in $\dim(L/I)^2 = 3$. Thus [11, corollary 2.3] and our assumption imply

$$\dim M(L) = \frac{1}{2} (n-1)(n-2) - 4 \leq \frac{1}{2} n(n-5) + (n-4),$$

which is a contradiction. Therefore we may assume that $\dim Z(L) = 4$, or $\dim Z(L) = 3$ and $L^2 = Z(L)$, or $\dim Z(L) = 2$ and $Z(L) \subset L^2$, or $\dim Z(L) = 1$.

If $\dim Z(L) = 4$, then there is a central ideal of $L$ of dimension one such that $L^2 \cap I = 0$. Since $\dim(L/I)^2 = 3$, by using [11, Theorem 3.1], we have

$$\dim M(L/I) \leq \frac{1}{2} n(n-5) + 1.$$ 

If the equality holds, then

$$\frac{1}{2} (n-2)(n-3) + 1 - s(L/I) = \dim M(L/I) = \frac{1}{2} n(n-5) + 1.$$

Therefore $s(L/I) = 3$ and by using Table 4, $L/I \cong L_{6.26}$. Since $\dim Z(L) = 4$ and $\dim L = 7$, we have $L \cong L_{6.26} \oplus A(1)$ by using Lemma 2.6. Now let $\dim M(L) \leq \frac{1}{2} n(n-5)$. Thus [11, corollary 2.3] and our assumption imply

$$\dim M(L) = \frac{1}{2} (n-1)(n-2) - 4 \leq \frac{1}{2} n(n-5) + (n-4),$$

which is a contradiction.

If $\dim Z(L) = 3$ and $L^2 = Z(L)$, then $L$ is isomorphic to one of the Lie algebras $37B$, $37C$ or $37D$ by using Lemmas 2.4 and 2.8.

Assume that $\dim Z(L) = 2$ and $Z(L) \subset L^2$. Then $\dim(L/Z(L))^2 = 1$. Since $L/Z(L)$ capable, by using [10, Theorem 3.5] and [9, Lemma 3.3], we have $L/Z(L) \cong H(1) \oplus A(n-5)$. Hence $L$ is nilpotent of class 3. Therefore, by using [17, Theorem 2.6] for $c = 3$, we have

$$\dim L^3 + \frac{1}{2} (n-1)(n-2) - 3 \leq \dim M(L/L^3) + \dim(L/Z_2(L) \otimes L^3).$$

Now since $1 \leq \dim L^3 \leq 2$, we can obtain that $n \leq 7$. Hence Lemma 2.6 implies that $L \cong L_{6.23}$ or $L \cong L_{6.25}$.

Finally, assume that $\dim Z(L) = 1$. Then $\dim(L/Z(L))^2 = 2$. By using [11, corollary 2.3], we have

$$\frac{1}{2} (n-2)(n-3) + 1 - s(L/Z(L)) + n - 3.$$

Thus $s(L/Z(L)) \leq 3$.

If $s(L/Z(L)) = 0$, then $L \cong H(1) \oplus A(n-4)$ by [11, Theorem 3.1]. This case cannot
Hence Theorem 3.9] and Lemma 2.9.

If $s(L/Z(L)) = 1$, then $[9] \text{Theorem } 3.9\] implies that $L \cong L(4,5,2,4)$. Therefore $n = 6$.

If $s(L/Z(L)) = 2$, then $L/Z(L)$ is isomorphic to one of the Lie algebras $L(3,4,1,4)$, $L(4,5,2,4) \oplus A(1)$ or $H(m) \oplus A(n-2m-1)(m \geq 2)$ by using $[9] \text{Theorem } 4.5\]$. In the case $L(3,4,1,4)$ or $L(4,5,2,4) \oplus A(1)$, we have $n = 5$ or 7.

In the case $L/Z(L) \cong H(m) \oplus A(n-2m-1)(m \geq 2)$, then we have a contradiction, since dim$(L/Z(L))^2 = 2$.

If $s(L/Z(L)) = 3$, $L/Z(L)$ is isomorphic to one of the Lie algebras $L(4,5,1,6)$, $L(5,6,2,7)$, $L'(5,6,2,7)$, $L(7,6,2,7)$, $L'(7,6,2,7)$ or $L(3,4,1,4) \oplus A(1)$ by $[17] \text{Main Theorem}$] and Lemma $2.7$.

Hence $n = 5, 6$ or 7 when dim $Z(L) = 1$. But there is no such Lie algebra by Lemma $2.7$. This completes proof.

\begin{thebibliography}{99}

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