Group With Maximum Undirected Edges in Directed Power Graph Among All Finite Non-Cyclic Nilpotent Groups

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Abstract

In [Curtin and Pourgholi, A group sum inequality and its application to power graphs, J. Algebraic Combinatorics, 2014], it is proved that among all directed power graphs of groups of a given order \(n\), the directed power graph of cyclic group of order \(n\) has the maximum number of undirected edges. In this paper, we continue their work and we determine a non-cyclic nilpotent group of an odd order \(n\) whose directed power graph has the maximum number of undirected edges among all non-cyclic nilpotent groups of order \(n\).

We next determine non-cyclic \(p\)-groups whose undirected power graphs have the maximum number of edges among all groups of the same order.

1 Introduction

Many authors studied the directed (undirected) power graphs of finite groups (see \[2\] \[3\] [4]).

This is an interesting question in this field to determine groups with maximum edges or maximum directed edges in their directed power graphs or undirected power graphs.

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Isaacs et al. in [1] proved that among all groups of order $n$, the summation of the element orders of the cyclic group of order $n$ is maximum. This is equivalent to this fact that among all groups of order $n$ the number of directed edges in the directed power graph of cyclic group of order $n$ is maximum. Later in [2] the second maximum sum of element orders of finite nilpotent groups was determined.

Curtin et al. in [6] showed that among directed power graphs of finite groups of a given order $n$, the cyclic group of order $n$ has the maximum number of undirected edges. In another studies [5], they showed that the same is true for undirected power graphs. It is a natural question that which non-cyclic groups have the maximum number of undirected edges in their directed power graphs. In this paper we determine a non-cyclic group $G$ of an odd order such that among all non-cyclic nilpotent groups of order $|G|$ has maximum number of undirected edges in its directed power graph. In the rest of the paper, we determine finite non-cyclic $p$-groups whose undirected power graphs have the maximum number of edges.

**Definition 1.** Let $G$ be a finite group. Let $\langle g \rangle$ denote the cyclic subgroup of $G$ generated by $g \in G$.

(i) The directed power graph $\overrightarrow{G}(G)$ of $G$ is the directed graph whose vertex set is elements of $G$ and for two distinct vertices $x, y \in G$ there is an arc from $x$ to $y$ if and only if $y = x^m$, for some positive integer $m$; hence the set of directed edges is $\overrightarrow{E}(G) = \{(g, h) \mid g, h \in G, h \in \langle g \rangle - \{g\}\}$ and the set of undirected edges is $\overleftrightarrow{E}(G) = \{(g, h) \mid h, g \in G, h \in \langle g \rangle - \{g\} \text{ and } g \in \langle h \rangle - \{h\}\}$.

(ii) The undirected power graph (or power graph) $G(G)$ of $G$ is the undirected graph whose vertex set is the elements of $G$ and two vertices being adjacent if one is a power of the other; hence the set of edges is $E(G) = \{(g, h) \mid h, g \in G, g \in \langle h \rangle - \{h\} \text{ or } h \in \langle g \rangle - \{g\}\}$. 

Let $G$ be a group. We denote the order of an element $a$ in a group $G$ by $o(a)$. Let $\phi$ denotes the Euler totient function. Throughout the paper we use $C_m$ to denote the cyclic group of order $m$. Also, if $n \geq 3$ and $p$ is an odd prime number, then we suppose that $M_{n,p} = \langle a, b \mid b^n = 1 = a^{p^n-1}, a^b = a^{1+p^{n-2}} \rangle$.

Let $g$ and $h$ be distinct elements of $G$. There is an undirected edge between $g$ and $h$ when they generate the same subgroup of $G$; hence the number of undirected edges which involve the vertex $g$ is precisely $\phi(o(g)) - 1$. Thus

$$|\overleftrightarrow{E}(G)| = \frac{1}{2} \sum_{g \in G}(\phi(o(g)) - 1),$$

Now let $\phi(G) = \Sigma_{g \in G}\phi(o(g))$. In order to find non-cyclic nilpotent groups of a given order with the maximum value of $|\overleftrightarrow{E}(G)|$, we should find non-cyclic nilpotent groups with maximum value of $\phi$. It was shown in [6] that among all groups of a given order, the cyclic group has the maximum value of $\phi$. In this paper, among all non-cyclic nilpotent groups of an odd order, we determine a group with the maximum value of $\phi$. We note that if the order of a nilpotent group $G$ is free square, then $G$ is cyclic. Hence, our main result is the following theorem.
Main Theorem. Let \( n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) be a positive odd integer which is not square free and \( p_1 < p_2 < \cdots < p_k \) are primes. Set \( s = \min\{1, \ldots, k\} \) such that \( \alpha_s > 1 \). Suppose that \( G \) is a non-cyclic nilpotent group of order \( n \). Then \( \phi(G) \leq \phi(C^{n}_{p_s} \times C_{p_s}) \). Therefore

\[
|\overrightarrow{E}(C^{n}_{p_s} \times C_{p_s})| \geq |\overrightarrow{E}(G)|,
\]

i.e., the directed power graph of \( C^{n}_{p_s} \times C_{p_s} \) has the maximum number of undirected edges among all non-cyclic nilpotent groups of order \( n \).

2 Proof of Main Theorem

For the proof of the main result, we use the following lemma.

Lemma 1. [5, Lemma 3.1] Let \( U \) and \( T \) be finite groups with \((|U|, |T|) = 1\), and let \( G = U \times T \) be the direct product of \( U \) and \( T \). Then \( \phi(G) = \phi(U)\phi(T) \).

Proposition 1. Among all finite non-cyclic groups of order \( p^n \) where \( p \) is an odd prime number, the groups \( C_{p^{n-1}} \times C_p \) and \( M_{n,p} \) have the maximum value of \( \phi \).

Proof. Let \( G \) be a non-cyclic \( p \)-group of order \( p^n \). For every non-identity element \( g \in G \), we have \( \phi(o(g)) = o(g) - \frac{o(g)}{p} \). Therefore

\[
\phi(G) = \sum_{g \in G} \phi(o(g)) = \sum_{g \in G - \{e\}} (o(g) - \frac{o(g)}{p}) + 1
\]

\[
= \sum_{g \in G} o(g) - \sum_{g \in G} \frac{o(g)}{p} + \frac{1}{p}
\]

\[
= (1 - \frac{1}{p}) \sum_{g \in G} o(g) + \frac{1}{p}.
\]

Hence, if \( \sum_{g \in G} o(g) \) has the maximum value, \( \phi(G) \) has the maximum value, too. It follows from Proposition 2.3 in [7] that \( \sum_{g \in G} o(g) \) has the maximum value when \( G \cong C_{p^{n-1}} \times C_p \) or \( G \cong M_{n,p} \). The proof is complete.

Corollary 1. Let \( G \) be a non-cyclic group of order \( p^n \) where \( p \) is an odd prime. Thus

\[
|\overrightarrow{E}(C_{p^{n-1}} \times C_p)| = |\overrightarrow{E}(M_{n,p})| \geq |\overrightarrow{E}(G)|.
\]

Equality holds when \( G \cong C_{p^{n-1}} \times C_p \) or \( G \cong M_{n,p} \).

Lemma 2. Let \( p \) be a prime number and \( m \geq 2 \). Then

(i) \( \phi(C_{p^n}) = \phi(C_{p^{n-1}}) + \phi(p^m)^2 \);

(ii) \( \phi(C_{p^{n-1}} \times C_p) = p\phi(C_{p^{n-1}}) + (p-1)(p-2) \).
Proof. (i) It is clear that \( \phi(C_{p^m}) = \sum_{g \in C_{p^m}} \phi(o(g)) = \sum_{g \in C_{p^{m-1}}} \phi(o(g)) + (p^m - p^{m-1})^2 \). The proof of the first part is complete.

(ii) Let \( A = \bigcup_{x \in C_{p^{m-1}}} (x, 0) \), \( B = \bigcup_{0 \neq a \in C_{p^m}} C_p \), and \( C = \bigcup_{0 \neq x \in C_{p^{m-1}}} (x, a) \). Obviously they are a partition for \( C_{p^{m-1}} \times C_p \) and so \( C_{p^{m-1}} \times C_p = A \cup B \cup C \). It is clear that \( \phi(A) = \phi(C_{p^m}) \) and \( \phi(B) = \phi(C_p) \). If \( (x, a) \in C \), then \( o((x, a)) = o(x) \). Since for each \( a \in C_p \) we have \( \phi(\bigcup_{x \neq x \in C_{p^{m-1}}} (x, a)) = \phi(C_{p^{m-1}}) - 1 \), this yields that \( \phi(C) = \sum_{0 \neq a \in C_p} \sum_{x \neq x \in C_{p^{m-1}}} \phi(o(x)) = (p-1)\phi(C_{p^{m-1}}) - 1 \). Thus we have \( \phi(C_{p^{m-1}} \times C_p) = \phi(A) + \phi(B) + \phi(C) = \phi(C_{p^{m-1}}) + \phi(C_p) - 1 + (p-1)(\phi(C_{p^{m-1}}) - 1) \). We also have \( \phi(C_p) = \sum_{x \in C_p} \phi(o(x)) = (p-1)^2 + 1 \); hence, \( \phi(C_{p^{m-1}} \times C_p) = p\phi(C_{p^{m-1}}) + (p-1)(p-2) \).

**Lemma 3.** Let \( p \) be a prime number and \( m \geq 2 \). Then

\[ (p-2)\phi(C_{p^{m-1}} \times C_p) \leq \phi(C_{p^m}) < p\phi(C_{p^{m-1}} \times C_p). \]

**Proof.** It follows from Lemma 2 that

\[ \phi(C_{p^m}) = \phi(C_{p^{m-1}}) + p^{2m-2}(p-1)^2. \]

According to Lemma 2.5 in [5], we have \( \phi(C_{p^{m-1}}) = \frac{p^{2m-2}(p-1) + 2}{p+1} \). Therefore

\begin{align*}
\phi(C_{p^m}) &= \phi(C_{p^{m-1}}) + p^{2m-2}(p-1)^2 \\
&= \phi(C_{p^{m-1}}) + (p-1)^2 \phi(C_{p^{m-1}}) + \frac{(p^{2m-2}(p-1) + 2)(p^2 - 1)}{p+1} \\
&= \phi(C_{p^{m-1}}) + \phi(C_{p^{m-1}})(p^2 - 1) = p^2\phi(C_{p^{m-1}}).
\end{align*}

By Lemma 2 we have

\[ p^2\phi(C_{p^{m-1}}) \leq p\phi(C_{p^{m-1}} \times C_p), \]

and it completes the proof of the right side of the inequality.

By Lemma 2 we have

\begin{align*}
\frac{\phi(C_{p^m})}{\phi(C_{p^{m-1}} \times C_p)} &= \frac{\phi(C_{p^{m-1}}) + \phi(p^m)^2}{\phi(C_{p^{m-1}}) + (p-1)(p-2)} > \frac{\phi(p^m)^2}{p\phi(C_{p^{m-1}}) + p} \\
&= \frac{p^{2m-3}(p-1)^2}{\phi(C_{p^{m-1}}) + p} > \frac{p^{2m-3}(p-1)^2}{p^{m-1}(p^{m-1} - p^{m-2}) + p^{2m-3}} \\
&= \frac{(p-1)^2}{p} > p - 2,
\end{align*}

and we get the result. \( \square \)

**Corollary 2.** Let \( p \) and \( q \) be two prime numbers where \( p < q \) and \( t, m \geq 2 \). Then

\[ \frac{\phi(C_{p^m})}{\phi(C_{p^{m-1}} \times C_p)} < \frac{\phi(C_{q^t})}{\phi(C_{q^{t-1}} \times C_q)}. \]
Proof. The right side of the inequality is greater than $q - 2$ by previous lemma and since $p < q$ are odd primes, we have $p \leq q - 2$. According to previous lemma, the left side is less than $p$. This completes the proof. \hfill \blacksquare

**Proof of the Main Theorem.** Let $G$ be a non-cyclic nilpotent group of order $n$. Since $G$ is a nilpotent group of order $n$, we have $G \cong P_1 \times P_2 \times \cdots \times P_k$, where $P_i$ is the unique Sylow $p_i$-subgroup of $G$ of order $p_i^{a_i}$. Therefore $\phi(G) = \phi(P_1)\phi(P_2)\cdots\phi(P_k)$. Because we assume that $G$ is not cyclic, at least one of the Sylow subgroups, say $P_j$, is not cyclic. Hence we have $\phi(P_j) \leq \phi(C_{p_j}^{a_j-1} \times C_{p_j})$ by Proposition 1. It follows from Lemma 3.6 in [6] that $\phi(P_j) \leq \phi(C_{p_j}^{a_j})$, for $1 \leq j \leq k$. Therefore, we have

$$\phi(G) \leq \phi(C_{p_1}^{a_1}) \cdots \phi(C_{p_j}^{a_j-1})\phi(C_{p_j}^{a_j})\phi(C_{p_{j+1}}^{a_{j+1}}) \cdots \phi(C_{p_k}^{a_k})$$

where the last inequality holds by Corollary 2. So the proof is complete.

**Corollary 3.** Let $n = p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$ be a positive odd integer which is not square free and $p_1 < p_2 < \cdots < p_k$ are primes. Set $s = \min\{1, \ldots, k\}$ such that $\alpha_s > 1$. Then among directed power graph of non-cyclic nilpotent groups of order $n$, the directed power graph $\overrightarrow{G}(C_{p_s} \times C_{p_s})$ has the maximum number of undirected edges.

**Proposition 2.** Let $G$ be a non-cyclic $p$-group of order $p^n$ whose undirected power graph has the maximum number of edges among all non-cyclic $p$-groups of order $p^n$. Then

(i) if $p$ is odd, then $G \cong C_p^{n-1} \times C_p$ or $G \cong M_{n,p}$;

(ii) if $p = 2$ and $n \neq 3$, then $G \cong C_{2n-1} \times C_2$;

(iii) if $p^n = 8$, then $G \cong Q_8$.

**Proof.** It follows from Theorem 4.2 in [4] that

$$|E(G)| = \frac{1}{2}\sum_{g \in G}(2o(g) - \phi(o(g)) - 1).$$

Since $G$ is a finite $p$-group, for every non-identity element $g \in G$ we have $\phi(o(g)) = o(g) - \frac{o(g)}{p}$. Therefore

$$|E(G)| = \frac{1}{2}\left[\sum_{g \in G\setminus\{e\}}\left(2o(g) - (o(g) - \frac{o(g)}{p})\right) - |G| + 1\right]$$

$$= \frac{1}{2}\sum_{g \in G}o(g) + \frac{1}{p}\sum_{g \in G}o(g) - |G| - \frac{1}{p}.$$
If $p$ is an odd prime number, then $E(G)$ has the maximum value when $\Sigma_{g \in G} o(g)$ has its maximum value, too. By Proposition 3.2 in [7], we know that if a non-cyclic $p$-group $G$ of order $p^n$ has the maximum value of $\Sigma_{g \in G} o(g)$, then $G \cong C_{p^{n-1}} \times C_p$ or $G \cong M_{n,p}$. It completes the proof of (i).

If $p = 2$ and $n \neq 3$, then according to Proposition 3.2 in [7], $\Sigma_{g \in G} o(g)$ has the maximum value when $G \cong C_{2^{m-1}} \times C_2$.

If $p^n = 8$, then by Proposition 3.2 in [7], $\Sigma_{g \in G} o(g)$ has the maximum value when $G \cong Q_8$.

\[ \square \]

**Question 1.** Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be a positive odd integer which is not square free and $p_1 < p_2 < \cdots < p_k$ are primes. Set $s = \min\{1, \ldots, k\}$ such that $\alpha_s > 1$. Among non-cyclic nilpotent groups of order $n$, does the undirected power graph $G(C_{p_s^{\alpha_s}} \times C_{p_s})$ have the maximum number of edges?

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