Deflection of charged massive particles by a four-dimensional charged Einstein–Gauss–Bonnet black hole

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Abstract

Based on the Jacobi metric method, this paper studies the deflection of a charged massive particle by a novel four-dimensional charged Einstein–Gauss–Bonnet black hole. We focus on the weak field approximation and consider the deflection angle with finite distance effects. To this end, we use a geometric and topological method, which is to apply the Gauss–Bonnet theorem to the Jacobi space to calculate the deflection angle. We find that the deflection angle contains a pure gravitational contribution $\delta_g$, a pure electrostatic $\delta_c$ and a gravitational–electrostatic coupling term $\delta_{gc}$. We find that the deflection angle increases (decreases) if the Gauss–Bonnet coupling constant $\alpha$ is negative (positive). Furthermore, the effects of the BH charge, the particle charge-to-mass ratio and the particle velocity on the deflection angle are analyzed.

Keywords: deflection angle, gravitational lensing, Gauss–Bonnet theorem, charged massive particle, timelike signal, Einstein–Gauss–Bonnet black hole

(Some figures may appear in colour only in the online journal)

1. Introduction

The gravitational deflection of light is one of the classical tests of general relativity (GR). It was first verified in 1919 [1], and has developed into a very active area of research, namely, the gravitational lensing (GL). GL effect can not only help us to understand the properties of the
messengers, but also that of the lens including the gravitational theory itself. Because of this, it has become an important tool in observational astronomy [2–10].

On the messenger side, traditionally all the GL observations are made using light signals. In addition however, we also know that there exist timelike signals such as supernova neutrinos [11–14], gravitational waves (GW) in some theories beyond GR [15–17], and cosmic rays [18]. They could also experience deflection by gravity and in principle be the messengers in GL. Therefore, their deflection is also important and worthy of study from both the theoretical and observational points of view. Among these signals, the neutrinos and (massive) GWs are massive neutral particles (NMP) while the cosmic rays are often charged. For NMPs, their GL has attracted great attention [19–28]. However, due to the reason that the motion of charged massive particles (CMP) is no longer a geodesic in a charged spacetime and therefore more complicated than the motion of NMP, only a few researchers considered the deflection effect of CMP in previous works [29, 30].

On the gravitational theory side, there are already many kinds of modified theories beyond GR, driven mainly by the observed expansion of the Universe and/or the pursuit of a self-consistent quantum description of gravity. The Einstein–Gauss–Bonnet (EGB) theory is one of the most promising candidates for modified theories of gravity [31]. It can be derived from the low energy limit of string theory [32, 33]. In EGB theory, when spacetime dimension $D = 4$, the Gauss–Bonnet (GB) term is a total derivative, so it does not contribute to the gravitational dynamics. However, Glavan and Lin recently proposed a non-trivial four-dimensional EGB theory [34] which can bypass the Lovelock’s theorem [35–37], avoid Ostrogradsky instability [38, 39] and contain a static and spherically symmetric (SSS) black hole (BH) solution [34].

Since then, a number of other exact solutions have been found in this theory, such as the BH ones [40–46] and the wormhole ones [47, 48] and the EGB theory was further generalized to Einstein–Lovelock gravity [49–51]. Meanwhile, various gravitational properties and effects of the EGB theory were studied by a large number of authors [52–71]. Naturally, the GL in this novel EGB theory was also studied by some authors. Islam et al investigated the GL of SSS BH in the strong and weak deflection limits [72]. Kumar et al studied the deflection of light by a charged BH [73]. Jin et al studied the strong GL of SSS BH surrounded by unmagnetised plasma medium [74]. Heydari-Fard et al calculated the bending angle of light in dS spacetime using Rindler-Ishak method [75]. Panah et al computed the deflection angle of light from the charged AdS BH [76]. Jafarzade et al considered the lensing of Born–Infeld BH [77]. Atamurotov et al studied the plasma effect of particle motion in the weak field limit [78]. In this paper we will extend these works by considering the deflection of CMPs in the charged EGB BH, using the famous GB theorem method.

The method of using the GB theorem to study the weak gravitational bending was first introduced by Gibbons and Werner [79] for light signal and then extended to stationary spacetime by using the technique of Randers-Finsler geometry [80]. These works stimulated a number of works using GB theorem to study the deflection of light signals [81–92] and inspired some authors to consider the finite distance effect in light deflection [93–97]. This method has also been extended to investigate the deflection of NMPs [98–101], and even the finite distance effect of NMPs was considered via GB theorem [102–104] (and also the perturbative method [105, 106]). For CMPs using the GB theorem method, recently Crisnejo et al pointed out the correspondence between the motion of charged particle in an external repulsive field and the motion of light in a dispersive medium. Thus, they can apply the GB theorem to the corresponding optical metric to study the deflection of CMP in Reissner-Nordstöm (RN) spacetime [29]. Later, Jusufi extended this work to Kerr–Newman spacetime [30].

In the present work, we will apply the GB theorem to investigate the deflection of CMP by a charged EGB BH under the influence of both electrostatic and gravitational interaction,
and the finite distance effect will also be considered. Towards this purpose, we will use the Jacobi metric of a CMP. This paper is organized as follows. In section 2, we first review the Jacobi metric for a CMP in static spacetime, then we will use the Jacobi metric to drive the equation of motion of a CMP in an SSS spacetime. In section 3, we obtain the Jacobi metric of charged EGB BH and further study the motion of a CMP in the weak field approximation. In section 4, we calculate the finite distance deflection of CMP using the GB theorem. Moreover, the effects of the BH charge, the signal charge-to-mass ratio, the coupling constant of the GB term, and the particle velocity on deflection angle are analyzed. Finally, we end our paper with a short conclusion in section 5. Throughout the paper, greek and latin indices are used to denote spacetime and spacial coordinates respectively. We use the natural units $G = c = 4\pi\varepsilon_0 = 1$ and the metric signature $(−, +, +, +)$.

2. Jacobi metric for a charged particle in static and spherically symmetric spacetimes

The motion of a CMP of mass $m$ and charge $q$ in charged BH spacetime is described by the Lorentz equation [107, 108]

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\nu} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} = \frac{q}{m} F_{\rho\nu} \frac{dx^\nu}{d\tau},$$

(1)

where $\tau$ is the proper time of the particle, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor with $A_\mu$ being the electromagnetic potential. From this we can see that the motion of the particle is no longer a geodesic in the background spacetime.

Jacobi metric is a basic tool in geometric dynamics, which is the subject of applying differential geometry to the dynamics of mechanical systems [109]. Gibbons first established the Jacobi metric for NMP in static spacetime [110]. Das and Ghosh extended this work to study the motion of CMP in RN spacetime [111]. Chanda et al derived the Jacobi metric for a NMP in stationary spacetime [112]. The Jacobi metrics related to the more general Lagrangian can be found in references [113, 114].

Let us review the Jacobi metric for a CMP in static spacetime. For a generic static metric,

$$ds^2 = g_{tt} dt^2 + g_{ij} dx^i dx^j,$$

(2)

the mechanical action of a CMP can be written as

$$S = \int_{t_1}^{t_2} dt \ L$$

$$= \int_{t_1}^{t_2} \left( -m \sqrt{-g_{tt} - g_{ij} \dot{x}^i \dot{x}^j} + q A_\mu \dot{x}^\mu \right) dt,$$

(3)

where a dot denotes derivative with respect to $t$. We suppose that $A_\mu$ is an electrostatic potential, that is $A_\mu \ dx^\mu = A_0 \ dt$, then the Lagrangian becomes

$$L = -m \sqrt{-g_{tt} - g_{ij} \dot{x}^i \dot{x}^j} + q A_0.$$  

(4)

From this, we can write the canonical momenta as follows,

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = \frac{mg_{ij} \dot{x}^j}{\sqrt{-g_{tt} - g_{ij} \dot{x}^i \dot{x}^j}}.$$  

(5)
The Jacobi metric thus can be written as [112]

\[ ds_J = p_i \dot{x}^i = \frac{mg_i \dot{x}^i \dot{x}^j}{\sqrt{-g_{tt} - g_{ij} \dot{x}^i \dot{x}^j}} = \sqrt{J_{ij} \dot{x}^i \dot{x}^j}, \]  

which satisfies

\[ J^{ij} p_i p_j = 1. \]  

In addition, the Hamiltonian is

\[ \mathcal{H} = p_i \dot{x}^i - \mathcal{L} = -\frac{mg_i}{\sqrt{-g_{tt} - g_{ij} \dot{x}^i \dot{x}^j}} - qA_0 \]

\[ = \sqrt{-g_{tt} - g_{ij} \dot{x}^i \dot{x}^j} = \sqrt{-g_{tt} - g_{ij} \dot{x}^i \dot{x}^j} - qA_0 = E, \]  

with \( E \) being the energy of the particle. The above equation leads to

\[ \frac{-g_{ij} g^{ij} p_i p_j}{(E + qA_0)^2 + m^2 g_{tt}} = 1. \]  

Comparing this equation with (7), we can see immediately

\[ J^{ij} = \frac{-g_{ij} g^{ij} p_i p_j}{(E + qA_0)^2 + m^2 g_{tt}}. \]  

Then by \( J_{ij} J^{jk} = \delta^j_i \), the Jacobi metric reads

\[ J_{ij} = [(E + qA_0)^2 + m^2 g_{tt}] \frac{g^{ij}}{-g_{tt}}. \]  

Equation (11) is the starting point for the study of the deflection of CMP using GB theorem, and its importance relies on the fact that the motion of a CMP follows the geodesic in this metric. If there is no electric potential, i.e. \( A_0 = 0 \), Jacobi metric (11) reduces to the NMP case [110],

\[ J_{ij} = \frac{(E^2 + m^2 g_{tt})}{-g_{tt}} g^{ij}, \]  

which has been used to investigate the gravitational deflection angle of NMP in a static spacetime geometry [101].

In the following, the Jacobi metric (11) will be used to derive the equation of motion of a CMP in SSS spacetime whose metric is given by

\[ ds^2 = -A(r) dr^2 + B(r) dr^2 + C(r)(d\theta^2 + \sin^2 \theta d\phi^2). \]  

By (11), the Jacobi metric for a CMP in this background reads

\[ dl^2 = [(E + qA_0)^2 - m^2 A] \left( \frac{B}{A} dr^2 + \frac{C}{A} (d\theta^2 + \sin^2 \theta d\phi^2) \right). \]
Without losing any generality, we assume that the particle moves in the equatorial plane, that is $\theta = \pi/2$, then equation (14) becomes

$$\text{d}l^2 = \left[(E + qA_0)^2 - m^2 A\right] \left(\frac{B}{A} \text{d}r^2 + \frac{C}{A} \text{d}\phi^2\right).$$ (15)

According to spherical symmetry, the angular momentum of the motion will be a constant

$$L = \left[(E + qA_0)^2 - m^2 A\right] \frac{C}{A} \left(\frac{\text{d}\phi}{\text{d}t}\right) = \text{constant.}$$ (16)

Using a new variable $u = 1/r$, and equations (15) and (16), we can obtain the orbit equation of a CMP moving in equatorial plane in terms of $u(\phi)$ as follows,

$$\left(\frac{\text{d}u}{\text{d}\phi}\right)^2 = \frac{C^2 u^4}{ABL^2} \left[(E + qA_0)^2 - A \left(m^2 + \frac{L^2}{C}\right)\right].$$ (17)

In addition, the energy and angular momentum of the particle at infinity for an asymptotic observer satisfy

$$E = \frac{m}{\sqrt{1 - v^2}}, \quad L = \frac{mc b}{\sqrt{1 - v^2}},$$ (18)

where $v$ is the asymptotic velocity of the particle, and $b$ is the impact parameter. Clearly, they have a relation

$$b = \frac{L}{E}.$$ (19)

Using (18), we can rewrite the Jacobi metric (15) as

$$\text{d}l^2 = J_{ij} \text{d}x^i \text{d}x^j$$

$$= m^2 \left[\left(\frac{1}{\sqrt{1 - v^2}} + \frac{qA_0}{m}\right)^2 - A\right] \left(\frac{B}{A} \text{d}r^2 + \frac{C}{A} \text{d}\phi^2\right),$$ (20)

and the trajectory of the CMP (17) as

$$\left(\frac{\text{d}u}{\text{d}\phi}\right)^2 = \frac{C^2 u^4}{ABL^2} \left[1 + \frac{\sqrt{1 - v^2} qA_0}{m}\right]^2 - A \left(1 - v^2 + \frac{b^2 v^2}{C}\right).$$ (21)

3. The motion of charged particle in the weak field limit

The action of the Einstein–Maxwell–Gauss–Bonnet theory in a $D$-dimensional spacetime reads [40]

$$S = \frac{1}{16\pi} \int d^Dx \sqrt{-g} \left[R + \frac{\alpha}{D - 4} G - F_{\mu\nu}F^{\mu\nu}\right].$$ (22)
where $g$ is the determinant of the spacetime metric, $\alpha$ is the GB term coupling constant, and $G$ is the GB term defined by

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \quad (23)$$

Here, $R$, $R_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ are the Ricci scalar, Ricci tensor and Riemann tensor, respectively.

Note that in action (22) we have rescaled the coupling constant $\alpha$ to $\alpha/(D - 4)$. Considering the limit $D \to 4$, the charged EGB BH solution of action (22) was obtained as [40]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (24)$$

where $f(r)$ and the vector potential are

$$f(r) = 1 + \frac{\ell^2}{2\alpha} \left[ 1 - \sqrt{1 + 4\alpha \left( \frac{2M}{r} - \frac{Q^2}{r^4} \right)} \right], \quad A_\mu dx^\mu = -\frac{Q}{r} dr, \quad (25)$$

with $M$ and $Q$ being the mass and charge of the BH, respectively. The solution (24) reduces to the RN spacetime if we take the limit $\alpha \to 0$. In addition, equation (24) has the same form as solutions in the gravity with a conformal anomaly [115], and in the gravity with quantum corrections [116, 117]. Notice that the GB coupling constant $\alpha$ could take a negative value [68, 118].

For charged BH (24), we have

$$A(r) = B(r)^{-1} = f(r), \quad C(r) = r^2, \quad A_0 = -\frac{Q}{r}. \quad (26)$$

Substituting them into equation (20), we can write the two-dimensional Jacobi metric as follows,

$$dl^2 = J_{ij} dx^i dx^j = m^2 \left[ \left( \frac{1}{\sqrt{1 - v^2}} - \frac{Q Q_m}{mr} \right)^2 - f(r) \right] \left[ \frac{1}{f^2(r)} dr^2 + \frac{r^2}{f(r)} d\phi^2 \right]. \quad (27)$$

In this paper we will only be interested in the weak field deflection, and therefore we can expand the metric components of (27) in the large $r$ limit. In this way, the non-zero components of Jacobi metric are given by the following expressions,

$$J_{rr} = \frac{m^2 (1 + v^2)}{1 - v^2} \left[ \frac{v^2}{1 + v^2} + \frac{2M}{r} - \frac{Q^2}{r^2} + \frac{4 (2 + v^2) M^2}{(1 + v^2) r^2} - \frac{4\alpha M^2}{r^3} \right]$$

$$- \frac{2mq}{\sqrt{1 - v^2}} \left( \frac{Q}{r} + \frac{4MQ}{r^2} \right) + \frac{Q^2}{r^2} + \mathcal{O}\left( M^3, Q^3, MQ^2, M^2 Q, \alpha^2 \right). \quad (28)$$
4.1. Gauss–Bonnet theorem and lens geometry

The Gauss–Bonnet theorem, also known as the Chern–Gauss–Bonnet theorem, relates the total Gaussian curvature of a compact, orientable Riemannian manifold to its Euler characteristic. Mathematically, it states that

$$J_{\phi} = \frac{m^2}{1 - v^2} \left( r^2 v^2 + 2Mr + 4M^2 - Q^2 - \frac{4\alpha M^2}{r^2} \right) - \frac{2mq (Qr + 2MQ)}{\sqrt{1 - v^2}}$$

$$+ q^2Q^2 + O (M^3, Q^3, MQQ, M^2Q, \alpha^2).$$

(29)

We can see that up to the above orders, the coupling constant \( \alpha \) appears in the \( M^2 \alpha \) order and the \( Q^2 \alpha \) term does not show up.

Substituting (26) into (21), the orbit equation becomes

$$\left( \frac{d\mu}{d\phi} \right)^2 = \left( \frac{1}{v} - \frac{\sqrt{1 - v^2} qQu}{mbv} \right)^2 - f(u) \left( 1 - \frac{v^2}{b^2v^2} + u^2 \right).$$

(30)

Usually, solving this equation is difficult. Again, since we are interested in the weak field limit, the problem can be greatly simplified. Using the condition \( \frac{du}{d\phi} \bigg|_{\phi=\pi/2} = 0 \), one can use the iterative method (see reference [119] for details) to obtain the following solution

$$u = \frac{\sin \phi}{b} + \frac{1 + v^2 \cos^2 \phi}{v} \frac{M}{b^2} - \frac{\sqrt{1 - v^2} qQu}{mbv} + O \left( M^2, Q^2 \right).$$

(31)

This solution then can be inverted perturbatively using the Lagrange inversion theorem to express \( \phi \) in terms of \( u \). The result is found to be

$$\phi(u) \approx \begin{cases} \arcsin(bu) - M\phi_1 + qQ\phi_2, & \text{if } |\phi| < \frac{\pi}{2}; \\ \pi - \arcsin(bu) + M\phi_1 - qQ\phi_2, & \text{if } |\phi| > \frac{\pi}{2}. \end{cases}$$

(32)

where

$$\phi_1 = \frac{1 + v^2 - b^2u^2v^2}{b^2\sqrt{1 - b^2u^2v^2}}, \quad \phi_2 = \frac{\sqrt{1 - v^2}}{bm\sqrt{1 - b^2u^2v^2}}.$$

4. Deflection angle of charged massive particle

4.1. Gauss–Bonnet theorem and lens geometry

Let \( D_a \) be a subset of a compact, oriented surface, with Gaussian curvature \( K \) and Euler characteristic number \( \chi(D_a) \), and its boundary \( \partial D_a \) a piecewise smooth curve with geodesic curvature \( k \), the GB formula reads [120]

$$\int_{D_a} K\ dS + \oint_{\partial D_a} k\ d\sigma + \sum_{i=1}^{n} \beta_i = 2\pi\chi(D_a),$$

(33)

where \( dS \) is the area element, \( d\sigma \) is the line element of boundary, and \( \beta_i \) is the jump angle at the \( i \)th vertex of \( \partial D_a \) in the positive sense. The three terms on the left side of equation (33) correspond to surface curvature, line curvature and point curvature, respectively, and the right side is the Euler characteristic number. In other words, the GB theorem reveals the relation between the curvature of a Riemannian metric and the topology of the manifold.

In the Jacobi space \( (M, J_i) \), a CMP follows the geodesic from source (S) to observer (O) and is deflected by a lens (see figure 1 for the illustration). In the following, the method of reference [93] will be used to study the deflection angle \( \delta \), which is defined as

$$\delta \equiv \Psi_O - \Psi_S + \phi_{OS}.$$

(34)
Figure 1. A region $D \subset (\mathcal{M}, J_{ij})$. Notice that $\beta_S = \pi - \Psi_S$ and $\beta_O = \Psi_O$. The radial and angular coordinates of $O$ and $S$ are $(r_O, \phi_O)$ and $(r_S, \phi_S)$.

where $\Psi_O$ and $\Psi_S$ are angles between the tangent of the particle ray and the radial direction from the lens to observer and source, respectively, and the change of the angular coordinate $\phi_{OS} \equiv \phi_O - \phi_S$.

Let us choose $D_s$ to be the region $D \subset (\mathcal{M}, J_{ij})$ bounded by four curves, as shown in figure 1. Three of these four curves are geodesics, including the particle ray $\gamma_g$ and two spatial geodesics of outgoing radial lines passing through $O$ and $S$ respectively, and one is a non-geodesic circular arc segment $C_\infty$, i.e. $C_r$ in the $r \to \infty$ limit. Notice that only $C_\infty$ has a non-zero geodesic curvature, and two jump angles where $C_\infty$ and the radial curves intersect are both $\pi/2$. In addition, Euler characteristic number $\chi (D) = 1$ because $D$ is a non-singular region.

Applying GB theorem (33) to region $D$, we have

$$\int \int_D K dS + \int_{\phi_S}^{\phi_O} \left[ \left( k \frac{dr}{d\phi} \right) (C_\infty) \right] d\phi + \beta_S + \beta_O = \pi. \quad (35)$$

Here, we have (see equation (23) of reference [101])

$$\left( k \frac{dr}{d\phi} \right) (C_\infty) = \lim_{r_c \to \infty} \sqrt{\frac{B(r_c)}{C(r_c)}} \left( \Gamma_{\phi\phi}^r (r_c) \right)^2. \quad (36)$$

with $\Gamma_{ij}^k$ being the Christoffel symbols of $J_{ij}$. Using equations (26) and (27), we find $\left( k \frac{dr}{d\phi} \right) (C_\infty) = 1$, which shows that our Jacobi metric is asymptotically Euclidean. Then considering $\beta_O = \Psi_O$ and $\beta_S = \pi - \Psi_S$, and using the definition (34), equation (35) leads to

$$\delta = -\int \int_D K dS. \quad (37)$$

This two dimensional integral can be expressed explicitly as

$$\delta = -\int_{\phi_S}^{\phi_O} \int_{r(\phi) = 1/u(\phi)}^{\infty} K \sqrt{J} dr d\phi, \quad (38)$$
where $J$ is the determinant of $J_{ij}$. The Gaussian curvature $K$ can be calculated by [80]
\[
K = \frac{1}{\sqrt{J}} \left[ \frac{\partial}{\partial \phi} \left( \frac{\sqrt{J}}{J_{rr} r} \right) - \frac{\partial}{\partial r} \left( \frac{\sqrt{J}}{J_{rr} r} \phi \right) \right].
\] (39)
Therefore the task of finding the deflection angle in an SSS spacetime reduces to the integration of equation (38).

### 4.2. Deflection angle of CMP in the charged EGB BH

While the above procedure is general, i.e. applicable to arbitrary SSS spacetimes, to proceed we will have to go to a specific spacetime, in this case the EGB BH described by equations (24) and (25). We first calculate the Gaussian curvature $K$ in equation (38). Substituting (28) and (29) into (39), the result is found to be
\[
K = \frac{1 - v^2}{m^2 v^4} \left[ \frac{(1 + v^2) M}{r^3} + \frac{3 (2 - v^2) M^2}{r^6} + \frac{8 (4 + v^2) (M^2 \alpha)}{r^6} + \frac{(2 + v^2) Q^2}{r^6} \right] + \frac{qQ (1 - v^2)^{3/2}}{m^3 v^6} \left[ \frac{v^2}{r^3} - \frac{3 M (4 - v^2)}{r^4} \right] + \frac{2}{m^4 v^6} \frac{(3 - v^2) (1 - v^2)^2 q^2 Q^2}{r^4} + \mathcal{O} \left( M^3, Q^3, MQ^2, M^2 Q, \alpha^3 \right). \tag{40}
\]

Secondly, the integral limits of equation (38) should be clarified. Our lensing setup in figure 1 supposed that $\phi_0 > \frac{\pi}{2} > \phi_s$, and then according to equation (32) the coordinate angle of the source and the observer becomes,
\[
\phi_s = \arcsin \left( bu_s \right) - \left( \Delta u_s + \frac{1}{v^2 \Delta u_s} \right) \frac{M}{b} + \frac{\sqrt{1 - v^2} qQ}{mv^2 \Delta u_s} + \mathcal{O} \left( M^2, Q^2 \right), \tag{41}
\]
\[
\phi_0 = \pi - \arcsin \left( bu_0 \right) + \left( \Delta u_0 + \frac{1}{v^2 \Delta u_0} \right) \frac{M}{b} - \frac{\sqrt{1 - v^2} qQ}{mv^2 \Delta u_0} + \mathcal{O} \left( M^2, Q^2 \right), \tag{42}
\]
where $u_i = 1/r(i = O, S)$ and $\Delta u_i = \sqrt{1 - b^2 u_i^2}$. After obtaining Gaussian curvature (40), particle orbit (31), and coordinate angles (41) and (42), now we can calculate the finite distance deflection angle of CMP by a four-dimensional charged EGB BH according to equation (38). Fortunately, the integral is only tedious but not difficult. The result is given by
\[
\delta = \delta_1 \frac{M}{b} + \delta_2 \frac{g_m Q}{b^2} + \delta_3 \frac{q_m Q^2}{b^2} + \delta_4 \frac{g_m Q M}{b^2} + \delta_5 \frac{g_m Q M^2}{b^2} + \delta_6 \frac{M^2}{b^2} + \delta_7 \frac{M^2 \alpha}{b^4} + \mathcal{O} \left( M^3, Q^3, MQ^2, M^2 Q, \alpha^3 \right), \tag{43}
\]
where
\[
\delta_1 = (\Delta u_O + \Delta u_S) \left( 1 + \frac{1}{v^2} \right), \quad \\
\delta_2 = - (\Delta u_O + \Delta u_S) \frac{\sqrt{1 - v^2}}{v^2}, \quad \\
\delta_3 = - (\Xi + \Gamma) \left( \frac{1}{4} + \frac{1}{2v^2} \right),
\]
\[
\delta_4 = \frac{(1 - v^2)}{2v^4} \left[ \Xi \nu^2 + 2b\Omega - (2 - v^2) \Gamma \right],
\]
\[
\delta_5 = \frac{\sqrt{1 - v^2}}{v^2} \left[ -3\Xi - \frac{2(1 + v^2)}{v^2} \Omega + \frac{2 - v^2}{v^2} \Gamma + \frac{b^3u_0^3}{\Delta u_0} + \frac{b^3u_S^3}{\Delta u_S} \right],
\]
\[
\delta_6 = \frac{3}{4} \left( 1 + \frac{4}{v^2} \right) (\Xi + \Omega) + \frac{b^3(4 - 8v^2 - 3v^4)}{4v^4} \left( \frac{u_0^3}{\Delta u_0} + \frac{u_S^3}{\Delta u_S} \right),
\]
\[
\delta_7 = -\frac{(4 + v^2)}{4v^2} \left[ 3\Xi + 4\Gamma + b^3 \left( u_0^3 \Delta u_0 + u_S^3 \Delta u_S \right) - bu_0(\Delta u_o)^3 - bus(\Delta u_S)^3 \right],
\]

with
\[
q_m = q/m,
\]
\[
\Xi = \pi - \arcsin(bu_0) - \arcsin(bus),
\]
\[
\Gamma = b(u_0\Delta u_0 + u_S\Delta u_S),
\]
\[
\Omega = b \left( \frac{u_0}{\Delta u_0} + \frac{u_S}{\Delta u_S} \right),
\]
\[
\Delta u_i = \sqrt{1 - b^2u_i^2} \quad (i = O, S).
\]

We emphasis that the equation (43) is a very comprehensive result because it contains dependence on both the spacetime parameters, such as \(M, Q, \alpha\), and signal parameters such as \(v, q, m\). And moreover, the effects of geometrical parameters \(r_5 = 1/u_S\) and \(r_0 = 1/u_0\) are also included. Therefore there is a quite large parameter space we can study, and a few limits we can take to compare with previously known results in simpler cases.

Among the spacetime parameters, \(M\) provides a basic length scale against which all other quantities with the same dimension, i.e. \(Q, \sqrt{\alpha}, q, m, r_5, r_0\) can be compared with. Therefore, it can be interpreted as the main (although not the sole) source of gravity in this spacetime. \(Q\) on the other hand, provides not only extra gravitational effect, but also a direct source to the electrostatic interaction with the charged signal (see figure 2). Finally the GB coupling \(\alpha\) controls the amount of deviation of the spacetime from the RN solution. Now for the geometrical parameters \(r_5, r_0\) as well as the signal parameter \(v\), they are actually determined by the initial and final conditions and therefore quite free. For the other signal parameters \(m\) and \(q\), due to general equivalence principle, \(m\) will not appear in the deflection if there was only gravity, which is the case if \(q = 0\). In other words, \(q/m\) always appears in this fixed form in the deflection angle.

Observing the above, we find there are the following limits one can take to compare with known literature. In the RN and infinite source/observer distance limits, setting \(\alpha = 0\) and \(r_5 = r_0 = \infty\), equation (43) reduces to equation (183) of reference [29] or equation (48) of reference [30]. On the other hand, taking the RN and NMP limits by letting \(\alpha = 0\) and \(q_m = 0\), equation (43) agrees with equation (5.1) of reference [106] after setting its spacetime spin \(a\) to zero. If one further sets \(r_5, r_0\) to infinity, then the equation (52) in reference [28] is recovered. Moreover, if we set \(Q = \alpha = 0\) in equation (43) but keep \(r_5, r_0\) finite, then the charge of the signal would automatically become ineffective to the deflection angle, and equation (42) of reference [102] after setting its spacetime spin \(a\) to zero, and equation (B4) of reference [106] after setting high order metric coefficients to zero, are obtained.
Figure 2. The deflection of charged signal by the charged lens $L$ from source $S$ to observer $O$. The gravitational interaction always attracts the signal particle (dot-dashed line) while the electrostatic interaction might attract or repel (dotted line) the CMP depending on the sign of $Q$ and $q$. The true trajectory is the combined one plotted using solid curve.

Now we return back to some limits in the case that GB coupling constant $\alpha$ is nonzero. If we consider the asymptotic case that both the observer and source are at infinite distance, then letting $r_s = r_o = \infty$, result (43) leads to the asymptotic deflection angle,

$$\delta_\infty = \left(1 + \frac{1}{v^2}\right) \frac{2M}{b} - \sqrt{1 - v^2} \frac{2q_m Q}{b} + \left(1 + \frac{4}{v^2}\right) \frac{3\pi M^2}{4b^2} - \left(1 + \frac{2}{v^2}\right) \frac{\pi Q^2}{4b^2}$$

$$- \frac{\sqrt{1 - v^2}}{v^2} \frac{3\pi q_m MQ}{b^2} + \frac{1 - v^2}{v^2} \frac{\pi q_m^2 Q^2}{2b^2} - \left(1 + \frac{4}{v^2}\right) \frac{3\pi M^2 \alpha}{4b^4} + O\left(M^3, Q^3, MQ^2, M^2 Q, \alpha^2\right).$$

Furthermore, for a NMP, setting $q_m = 0$, the above expression becomes

$$\delta_{\infty, g} = \left(1 + \frac{1}{v^2}\right) \frac{2M}{b} - \left(1 + \frac{2}{v^2}\right) \frac{\pi Q^2}{4b^2} + \left(1 + \frac{4}{v^2}\right) \frac{3\pi M^2}{4b^2} \left(1 - \frac{\alpha}{b^2}\right)$$

$$+ O\left(M^3, Q^3, MQ^2, M^2 Q, \alpha^2\right).$$

On the other hand, if the signal is null and neutral like light or GW, then further setting $v = 1$ in equation (45) yields

$$\delta_{\infty, g, \text{null}} = \frac{4M}{b} - \frac{3\pi Q^2}{4b^2} + \frac{15\pi M^2}{4b^2} \left(1 - \frac{\alpha}{b^2}\right) + O\left(M^3, Q^3, MQ^2, M^2 Q, \alpha^2\right).$$

When we neglect the terms containing charge $Q$, this expression is consistent with equation (18) of reference [75]. If the EGB coupling constant $\alpha$ is set to zero, equations (45) and (46) agree with equations (53) and (54) of reference [28] respectively.

In the above limits of the deflection angle, we largely focused on the effect of gravitational interaction. However, since there are two kinds of interactions in this situation, it would also be very inspiring to consider the electrostatic limit of the deflection. This limit can be approached by setting $M \to 0$ in equation (43) and $Q \to 0$ in some but not all terms of it. The point is that, the charge $Q$ is responsible for both gravitational (although only partially) and electrostatic interactions, through respectively the $Q^2/r^2$ term in the metric and $-Q/r$ term in the Coulomb potential $A_0$, and we would like to turn off the gravitational part only and keep the Coulomb interaction. Therefore, we see that in equation (43), the terms containing $\delta_1, \delta_3, \delta_6$ and $\delta_7$ are
of gravitational origin and can be collectively denoted as $\delta_g$, i.e.

$$\delta_g = \delta_1 \frac{M}{b} + \delta_3 \frac{Q^2}{b^2} + \delta_6 \frac{M^2}{b^2} + \delta_7 \frac{M^2 \alpha}{b^2}. \quad (47)$$

While the term involving $\delta_g$ in equation (43) is proportional to both $q_m$ and $M$ and therefore represents a gravitational–electrostatic coupling, and it will be denoted as $\delta_{gC}$. This interaction is repulsive (or attractive) when $q_m$ and $Q$ has the same (or opposite) signs.

Lastly, the terms $\delta_2$ and $\delta_4$ are the ones purely due to electrostatic interaction and they are denoted as

$$\delta_e = -\frac{(\Delta u_O + \Delta u_S)}{v^2} \sqrt{1 - v^2} q_m Q \left[ (\pi - \arcsin(b u_O) - \arcsin(b u_S)) v^2 \right.$$ 

$$- (2 - v^2) b (u_O \Delta u_O + u_S \Delta u_S) + \frac{2 b u_O}{\Delta u_O} + \frac{2 b u_S}{\Delta u_S} \left( 1 - v^2 \right) \frac{Q^2}{b^2} \left. + \mathcal{O}(Q^3). \quad (48) \right.$$ 

The infinite distance limit, $u_O \rightarrow 0$ and $u_S \rightarrow 0$, of equation (48) is then

$$\delta_{\text{infty}, e} = -\frac{\sqrt{1 - v^2}}{v^2} 2 q_m Q \frac{1}{b^2} + \frac{1}{v^2} \frac{\pi q_m^2 Q^2}{2 b^2} + \mathcal{O}(Q^3). \quad (49)$$

The first order term of this equation was given in reference [121] for classical Coulomb scattering in the relativistic case. Its non-relativistic limit $v \rightarrow 0$ is the well-known Rutherford scattering formula. In general, equation (48) provides for Coulomb scattering a deflection angle formula with finite distance correction.

In the derivation process and the results in prior sections and subsections, we have used the conserved energy $E$ and asymptotic velocity $v$ as kinetic variable of the signal. One might hope to express these quantities using local variables, given that our result has already allowed freely varying observer radius $r_O$. If the observer is static, then this is relatively simple. Since the energy measured by the local observer is $E_i = -g_{\mu \nu} u'_{\mu} p'_{\nu}$, where $u'_{\mu} = (1/\sqrt{A(r_O)}, 0, 0, 0)$ is the normalized-four velocity of the observer and $p' = m \left( \frac{d \theta}{d \tau}, \frac{d \phi}{d \tau}, \frac{d \phi}{d \tau}, \frac{d \varphi}{d \tau} \right)$ is the four momentum of the signal (we use $m = 1$ for unit mass). Now since in the SSS spacetime it is known that $\frac{d}{d \tau} = \frac{d}{A(r)} [122]$, the local energy measured by the observer is then

$$E_i = \frac{E}{\sqrt{A(r_O)}}, \quad \text{i.e.}, \quad E = E_i \sqrt{A(r_O)}. \quad (50)$$

Translating into the local velocity $v_l$ using $E_i = \frac{1}{\sqrt{1 - v_l^2}}$ and also using equation (18), we have the following relation between $v_l, v$ and $E$

$$v_l = \frac{\sqrt{E^2 - A(r_O)}}{E} = \sqrt{1 - A(r_O)(1 - v^2)}, \quad (51)$$

or

$$v = \frac{\sqrt{v_l^2 + A(r_O)} - 1}{\sqrt{A(r_O)}}. \quad (52)$$

Therefore, if one is interested in the results expressed using observables of a static observer, we can in principle replace every $E$ in the results above by equation (50) and every $v$ by equation (52).
4.3. Effects of charges, parameter $\alpha$ and velocity $v$

In astronomy, it is often assumed implicitly and by default that all astrophysical objects including compact objects (stars, BH, etc) and galaxies or their clusters are electrically neutral because of the possible selective accretion of the opposite charges from the surrounding environment. However, this is indeed an oversimplification. It is known that astrophysical bodies should be slightly positively charged in order for electrons and protons in the stellar atmosphere to maintain quasi-local equilibrium [123, 124]. This will result in a BH charge $Q_{eq}$ proportional to the BH mass with a coefficient about 100 [C] per Sun mass, i.e.

$$Q_{eq} \approx 100 \frac{M}{M_\odot} \text{[C].}$$  \hspace{1cm} (53)

If the charge of the BH is induced by magnetic field $B_{mag}$ around it, then the corresponding charge $Q_{mag}$ can be much larger [125]

$$Q_{mag} \approx 1.46 \times 10^2 \left(\frac{M}{M_\odot}\right)^2 \frac{B_{mag}}{10 \text{ [G]}} \text{[C]},$$  \hspace{1cm} (54)

where $B_{mag}$ is typically of order 10 [G]. Of course, both these two charges are still very far from the extreme RN spacetime limit

$$Q_{ext} \approx 1.72 \times 10^{20} \frac{M}{M_\odot} \text{[C]}$$ \hspace{1cm} (55)

for typical supermassive BHs (SMBH) in Galaxy centers with $M \sim \mathcal{O}(10^7 M_\odot)$. Observing all these, therefore we will limit the charge $Q$ in our analysis of the effect of charge to the deflection angle to a somewhat loosely fixed value $Q_l = \sqrt{Q_{mag} Q_{ext}} \approx 5.4 \times 10^5 Q_{mag} = Q_{ext}/(5.4 \times 10^5)$.

We will choose the Sgr A* SMBH as the lens and then study the deflection angle of ultra-high energy (UHE) protons and other charged particles in the cosmic rays. Using its mass of $4.1 \times 10^6 M_\odot$ and distance of 8.1 [kpc] [126], $10^{19}$ [eV] for proton energy (which fixes its velocity $v$) and an impact parameter $b$ corresponding to an apparent angle of $\mathcal{O}(1')$, the dependence of the deflection angle on $Q$, $\alpha$, and $v$ are plotted in figure 3 to figure 5.

In figure 3(a), the effect of the charge $Q$ on the deflection (43), the gravitational deflection $\delta_g$, the electric deflection $\delta_e$, and the coupling $\delta_{gc}$ are all plotted for UHE protons and electrons. It is seen that for the entire range of $Q$ considered, i.e. $[0, Q_l]$, the gravitational deflection of a proton, $\delta_{g,p}$, is almost (although not exactly) constant as $Q$ increases, because $Q_l/Q_{ext} \ll 1$. In contrast, the magnitude of the electrostatic deflection $\delta_{e,p}$ increases almost linearly as dictated mainly by the first term of equation (48). The gravitational–electric coupling term $\delta_{gc}$ although also increases as $Q$ increases, its magnitude is smaller than that of $\delta_{g}$ by a factor of $M/b$, and therefore negligible in the competition of $\delta_{g}$ and $\delta_{e}$. For protons, the electrostatic repulsion is smaller than the gravitational attraction until $Q/Q_{mag} \approx 5.58 \times 10^5$ or $Q = 1.35 \times 10^{19}$ [C], beyond which the total deflection becomes negative and the proton is pushed away from the central lens. Clearly, although this charge is much larger than the magnetically induced charge of the Sgr A* SMBH $Q_{mag} \approx 2.42 \times 10^{15}$ [C], it is still much smaller than its extreme limit $M$. Therefore observing the deflection or GL of such protons by the Sgr A* BH might offer a chance to constrain its charge. On the other hand, if it was the electron signal that experience such deflections, then the electrostatic and gravitational interaction will cause the same amount of deflection when $Q$ is even smaller, at about $3.04 Q_{mag}$. 


Figure 3. Dependence of the total deflection $\delta$, the gravitational deflection $\delta_g$ and electrostatic deflection $\delta_e$ on (a) $Q$ with $q_m$ of proton and electron and on (b) $q_m$ with $Q = \sqrt{Q_{eq} Q_{mag}}$. In all plots, $\alpha = b^2$, $r_S = r_O = 8.1$ [kpc] and $b = 1'' r_O$ are used.

Figure 3(b) shows the deflection angle of charges with different $q_m$. Although the most probable signal candidates are UHE protons, for completeness we still allow the $|q_m|$ to range from roughly $|e|/2m_p$ for heavy ion to $|e|/m_p$ for protons/antiprotons and then further to $|e|/m_e$ for electron/positron [127]. Note that to be conservative, we set the charge of the BH to a relatively large value $Q = 100Q_{mag}$ in this plot. It is seen that unlike figure 3(a), the gravitational deflection $\delta_g$ here is exactly a constant with respect to the change of $q_m$. Again, the magnitude of the electrostatic deflection $\delta_e$ grows linearly as $q_m$ increases. For our primary signals, UHE protons and possibly heavier ions (He, C, O etc) [127], it is seen that because their electrostatic interaction with the lens is repulsive, the deflection angles $\delta_e$ are quantitatively negative, as expected. More importantly, even for this relatively large $Q$, the size of $\delta_e$ for these signals are always smaller than their gravitational deflection $\delta_g$ by a factor of about $1/55.8$ for (anti)protons and roughly $1/112$ for heavier ions. For electrons/positrons in cosmic rays, it is generally expected that the intergalactic magnetic field will bend their trajectories more strongly than any of the gravitational deflection $\delta_{g,e}$ or electrostatic deflection $\delta_{e,e}$ considered here. However, if we were to compare $\delta_{g,e}$ and $\delta_{e,e}$, then as seen from the plot, this BH with a charge of $100Q_{mag}$ can even result in a larger $\delta_{e,e}$ than $\delta_{g,e}$, roughly by a factor of $32.9$. Again, for all considered signals, their gravitational–electrostatic coupled deflections $\delta_{g,e}$ are much smaller than the corresponding electrostatic deflection.
For the coupling constant $\alpha$ of the GB term, currently, there is no theoretical constraint on its value except its dimension is of $[M^2]$. Since the current observational uncertainty of solar deflection of light is about 0.0008 times the GR value [128], we can conclude that $\alpha$ should be constrained to about $0.0008b^2/M^2$. Its effect on the deflection angle is only through the fourth order term in $1/b^4$, i.e. the term containing $\delta_7$ in equation (43). It is clear that this term is linear to $\alpha$, with a positive $\alpha$ decreasing the deflection angle while a negative one increasing it. In figure 4, this term is plotted as a function of $\alpha$ for $Q = Q_{\text{eq}}$ and then compared with other terms of equation (43). Comparing to the leading order term containing $\delta_1$, contribution of $\delta_7$ term to the deflection will be weaker if $|\alpha| < b^2/M^2$ (this value is marked by the red dashed vertical line in figure 4). It will even be weaker than the second order term proportional to $M^2/b^2$ if $|\alpha| < b^2$ (blue dashed vertical line). However, since $Q$ itself is very small comparing to $Q_{\text{cut}} = M$, the contribution from $\alpha$ might be larger than that from terms involving $Q$ and $q_{\text{cut}}$ depending on the exact relation between these parameters. Comparison of terms in $\delta_7$ with those in $\delta_2, \delta_4$ and $\delta_5$, one can see that when $\alpha \approx Q_{\text{cut}}\sqrt{1 - v^2}/M^2, Q_{\text{cut}}q_{\text{in}}(1 - v^2)b^2/M^2, Q_{\text{cut}}q_{\text{in}}\sqrt{1 - v^2}b^2/M^2$ (marked by purple, magenta, green dot-dashed vertical lines) respectively, the contribution to the deflection angle from the parameter $\alpha$ would be comparable to the corresponding electrostatic contributions. The first of these three values is close to $bM$ and the second is close to $M^2$ (gray vertical line) while the last is only about $10^{-10}M^2$ for the given choice of other parameters. Therefore although the parameter $\alpha$ only starts to appear from the $1/b^4$ order, its contribution can be comparable to some of the electrostatic deflections, even if $\alpha$ is only at the order of $M^2$ or even much smaller. Lastly, in figure 4 the $\delta_3$ contribution to the deflection was not plotted because for the chosen parameters, this term is much smaller than all terms plotted in this figure, even smaller than the value of the $\alpha$ term at $\alpha = 10^{-10}M^2$.

Finally for the effect of signal velocity, the most prominent feature recognizable from equation (43) is that both the electrostatic deflection $\delta_c$ and the gravitational–electrostatic deflection $\delta_{gc}$ are proportional to $\sqrt{1 - v^2}$ or its positive powers and therefore vanish as $v \to 1$. In other words, the higher the energy of the charged particle, the less electrostatic and gravitational–electrostatic deflections it will experience. In comparison, the gravitational deflection $\delta_g$ is nonzero even for null signals. This feature is illustrated in figure 5, where we plot $\delta_c, \delta_{gc}$ and the total $\delta$ for different velocities/energies of the cosmic ray proton, assuming the charge of the

![Figure 4](image)

**Figure 4.** Absolute value of various terms of equation (43) as $\alpha$ varies (see legend for labeling of the terms). All terms except the one containing $\delta_7$ are independent of $\alpha$ and therefore horizontal in this plot. See text for the vertical lines. $Q = Q_{\text{eq}}$ is used and other parameters are the same as in figure 3. The $\alpha$ is plotted to about $10^{15}M^2$ because this is the theoretical upper limit that the convergence of the series result breaks.
Figure 5. Dependence of $\delta_g$, $\delta_c$ and $\delta$ on $1 - v$. The vertical lines from left to right correspond to the proton energy of $10^{20}$, $10^{19}$, $10^{18}$, $10^{15}$ [eV] respectively. $Q = 100Q_{\text{mag}}$ is used and other parameters are the same as in figure 3.

BH is still given by the value $100Q_{\text{mag}}$. It is seen that for UHE protons with energy greater than $10^{19}$ [eV], $\delta_c$ is smaller than $\delta_g$ by at least a factor of $\sim 1/55.8$. Around $10^{17}$ [eV], the protons’ electrostatic deflection becomes comparable to the gravitational one and they are of opposite directions. If the energy/velocity continue to decrease, then the deflection of the electrostatic force will bend the signal away from the lens. Also note that the gravitational–electrostatic coupling deflection $\delta_{gc}$ in this parameter setting is a factor $M/b$ smaller than $\delta_c$, but their way of dependence on $v$ are roughly the same.

5. Conclusion

In this paper, we have studied the deflection of a CMP by a charged BH in a novel four-dimensional EGB gravity using the GB theorem. In order to use this theorem, we constructed the Jacobi space $(\mathcal{M}, J_{ij})$ as the background space, in which the motion of a CMP follows the geodesic. This motion was solved in the weak field approximation via iterative method.

After integrating the Gaussian curvature, the result of the deflection angle of a CMP in the EGB BH spacetime is given in equation (43), which also takes into account the finite distance effect. Note that the finite distance effect to the deflection angle is at order $O(2b/r_0)$ (see equation (29) of reference [129]). Since the leading term in the infinite distance approximation is $4M/b$, then when $b \gtrsim \sqrt{2MR_0}$, the finite distance contribution becomes larger. From equation (43), various limits (null limit, neutral limit, infinite distance observer/source limit, and RN limit) were obtained and compared with known literature. Moreover, we find that a positive coupling constant $\alpha$ decreases the deflection angle, while a negative $\alpha$ increases it.

Using equation (43), we modeled the Sgr A$^\star$ SMBH as the EGB BH and studied its deflection to charged cosmic ray, primarily to UHE protons. It is found that the electrostatic deflection of proton with energy $10^{19}$ [eV] is smaller (or larger) than the gravitational deflection if the BH charge is smaller (or larger) than $5.58 \times 10^3 Q_{\text{mag}}$. This might offer a new way to constrain the Sgr A$^\star$ SMBH charge if the angular resolution of UHE protons can be significantly improved in the future.
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Data availability statement

No new data were created or analysed in this study.

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