On the structure of repeated-root polycyclic codes over local rings

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Abstract

This paper provides the Generalized Mattson Solomon polynomial for repeated-root polycyclic codes over local rings that gives an explicit decomposition of them in terms of idempotents which completes the single root study in [2]. It also states some structural properties of repeated-root polycyclic codes over finite fields in terms of matrix product codes. Both approaches provide a description of the $\perp_0$-dual code for a given polycyclic code.

Keywords: Polycyclic code, Duality, Repeated-root codes, Mattson-Solomon transform, Matrix-product codes

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1. Introduction

Polycyclic codes over a finite local ring $R$ were introduced in [19] and they are described as ideals on the quotient ring $R[x]/(f(x))$ with $f(x) \in R[x]$. These

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codes generalize the well-known classes of cyclic and constacyclic codes. Poly-
cyclic codes over finite fields have been studied from several points of view, we
will be especially interested in the so called ⊥₀-duality (see [1, 2] and the refer-
ences therein). Polycyclic codes over chain rings have been studied in different
directions, see for example [4, 18, 29, 28]. In [2] the authors made a general-
ization where the ring is a finite commutative local ring and the polynomial
defining the ambient space has simple roots. That paper proposed a transform
approach that generalizes the classical Mattson-Solomon (Fourier) transform in
finite fields.

On the other side, several papers have been devoted to explain the matrix
product code structure of repeated-root cyclic and constacyclic codes over finite
fields, see for example [30, 4], and over some finite chain rings [5].

In this paper, we complete the study on the Mattson-Solomon transform
approach in [2] for polycyclic codes over finite local rings in the case that the
defining polynomial has repeated-roots. We also give a matrix product code
structure that describes repeated-root polycyclic codes over finite fields. In
both cases, we provide expressions for the ⊥₀-dual code of a given polycyclic
code.

The structure of the paper is as follows. In Section 2 some preliminaries
are given on finite commutative local rings, on the Hasse derivative of a poly-
nomial over a finite local ring and on the Generalized Discrete Fourier Trans-
form. Section 3 provides the Generalized Mattson Solomon polynomial(GMS)
for polycyclic codes over local rings that gives an explicit decomposition of them
in terms of idempotents. In Section 5 we state some structural properties of
repeated-root polycyclic codes over finite fields in terms of matrix product codes.
In both Section 3 and Section 5 we give a description of the ⊥₀-dual code of a
given polycyclic code.

2. Preliminaries

Throughout the paper, R will denote a finite local ring of characteristic
q = p^r for a prime p and a positive integer r, m will denote the maximal ideal of
R and \( \mathbb{F}_q = R/m \) the finite residue field of R. It is well-known that R is trivially
complete and thus Hensel, i.e. every element of R is nilpotent or a unit and
m is a nilpotent ideal. We denote by \( \tilde{\cdot} \) the natural polynomial ring morphism
\( \tilde{\cdot} : R \to (R/m) \) and, abusing notation, we will use it also for polynomial rings
acting on the coefficients \( \tilde{\cdot} : R[x] \to (R/m)[x] = \mathbb{F}_q[x] \). Let \( \mathcal{J} \) denote the set of
all polynomials \( f \) in \( R[x] \) such that \( \tilde{f} \) has distinct zeros in the algebraic
closure of \( \mathbb{F}_q \), a polynomial in \( \mathcal{J} \) has distinct zeros in local extensions of R,
\( R_f = R[x]/(f) \) (where \( f \) is monic) is a separable local extension ring if and
only if \( f \) is an irreducible polynomial in \( \mathcal{J} \), and the polynomials in \( \mathcal{J} \) admit a
unique factorization into irreducible polynomials and a polynomial in \( \mathcal{J} \) has no
multiple roots in any local extension of R. In the rest of the paper, unless other
thing is stated, \( f \) will denote a polynomial in \( \mathcal{J} \) and \( F = f^m \) for a non-negative
integer \( m \) (in some sections \( m = p^k \) where \( p \) is the characteristic of R).
2.1. Hasse derivative and Generalized Discrete Fourier Transform

The Generalized Discrete Fourier Transform (GDT) for repeated-root cyclic codes over a finite field \( \mathbb{F}_q \) of characteristic \( p \) (a prime) of length \( N = np^k \), where \( (n, p) = 1 \), was defined by Massey in [24]. After that, the definition is generalized for quasi-cyclic and quasi-twisted codes over finite fields in [13] and \[12\], respectively. In those references, the Hasse derivative of polynomials over finite fields plays an important role. For more information about the Hasse derivative of polynomials over fields we refer the reader to [24, 12].

In this section, let \( R \) denote a commutative finite unitary ring and \( p(x) = \sum_{i=0}^{n} p_i x^i \in R[x] \) be a polynomial. For \( k \in \{0, 1, \ldots, n\} \), the \( k \)th formal derivative of \( p(x) \) is defined as \( p^{(k)}(x) = k! \sum_{i=0}^{n} \binom{i}{k} p_i x^{i-k} \), and the \( k \)th Hasse derivative of \( p(x) \) is defined as \( p^{[k]}(x) = \frac{1}{k!} p^{(k)}(x) \) [17, page 363], i.e.

\[
p^{[k]}(x) = \sum_{i=0}^{n} \binom{i}{k} p_i x^{i-k} = \sum_{i=0}^{n-k} \binom{i+k}{k} p_{i+k} x^i.
\]

The following result holds directly from the definition and straightforward computations.

**Lemma 2.1.** Let \( p(x) \) and \( q(x) \) be two polynomials in \( R[x] \).

1. (Taylor expansion) If \( p(x) \) is of degree \( n \) and \( \lambda \) is an arbitrary element in \( R \), then \( p(x + \lambda) = \sum_{k=0}^{n} p^{[k]}(\lambda) x^k \).
2. (Product rule) \( (pq)^{[k]}(x) = \sum_{i=0}^{k} p^{[i]}(x) q^{[k-i]}(x) \).
3. (Product rule) \( (pq)^{[k]}(x) = \sum_{i=0}^{k} p^{[i]}(x) q^{[k-i]}(x) \).

From Now on, let simple-root polynomial \( f(x) = (x - \alpha_0)(x - \alpha_1) \ldots (x - \alpha_{n-1}) \in \mathcal{J} \) has \( n \) fixed ordering distinct roots \( \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \) in an extension ring \( R' \) of \( R \). Recall that the Discrete Fourier Transform (DFT) of an \( n \)-tuple \( (g_0, g_1, \ldots, g_{n-1}) \) is \( (g(\alpha_0), g(\alpha_1), \ldots, g(\alpha_{n-1})) \), where \( g(x) = g_0 + g_1 x + \ldots + g_{n-1} x^{n-1} \in R[x]/(f(x)) \); see [2].

**Definition 2.1.** Let \( F(x) = ((x - \alpha_0)(x - \alpha_1) \ldots (x - \alpha_{n-1}))^m = (f(x))^m \) be a repeated-root monic polynomial in \( R[x] \) of degree \( N = nm \) and \( g(x) = \sum_{i=0}^{N-1} g_i x^i \in R[x]/(F(x)) \). We define the Generalized Discrete Fourier Transform (GDT) of \( g(x) \) as

\[
\begin{pmatrix}
g(\alpha_0) & g(\alpha_1) & \ldots & g(\alpha_{n-1}) \\
g^{[1]}(\alpha_0) & g^{[1]}(\alpha_1) & \ldots & g^{[1]}(\alpha_{n-1}) \\
\vdots & \vdots & \ddots & \vdots \\
g^{[m-1]}(\alpha_0) & g^{[m-1]}(\alpha_1) & \ldots & g^{[m-1]}(\alpha_{n-1})
\end{pmatrix},
\]

where \( g^{[i]} \) is the \( i \)th-Hasse derivative for all \( 1 \leq i \leq m-1 \).

**Example 2.2.** Suppose that \( F(x) = x^6 - 3x^5 + 3x^4 - x^3 \in \mathbb{Z}_4[x] \), which is decomposed over \( \mathbb{Z}_{16} \) as \( F(x) = (x - 1)^3(x - 12)^3 \). If \( g(x) = 1 + 2x^3 + x^4 + 3x^5 \in \mathbb{Z}_{16}[x] \), then...
$\mathbb{Z}_4[x]/(F(x))$, then $g^{[1]} = 2x^2 + 3x^4$ and $g^{[2]} = 2x + 2x^2 + 2x^3$. Therefore, the GDFT of $n$-tuples related to $g(x)$ is

$$\text{GDFT}(g) = \begin{bmatrix} g^{[1]}(1) & g^{[1]}(12) \\ g^{[1]}(1) & g^{[1]}(12) \\ g^{[2]}(1) & g^{[2]}(12) \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 5 & 0 \\ 6 & 8 \end{bmatrix}. $$

### 2.2. Generalized Vandermonde matrices

Let $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ be a fixed ordering of the roots of polynomial $f(x) = (x - \alpha_0)(x - \alpha_1) \ldots (x - \alpha_{n-1}) \in R[x]$ in the extension ring $R'$ of $R$.

For $0 \leq i \leq N - 1$, take $p_i(x) = x^i$ and construct the $N \times m$ matrix

$$R(x) = \begin{bmatrix} p_0(x) & p_0^{[1]}(x) & \ldots & p_0^{[m-1]}(x) \\ p_1(x) & p_1^{[1]}(x) & \ldots & p_1^{[m-1]}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{N-1}(x) & p_{N-1}^{[1]}(x) & \ldots & p_{N-1}^{[m-1]}(x) \end{bmatrix}. $$

In fact, $ij$-entry of $R(x)$ is $\binom{i-1}{j-1} x^{i-j}$ for $i \geq j$ and zero otherwise. The generalized Vandermonde matrix related to the roots $\alpha_0, \alpha_1, \ldots, \alpha_n$ of the repeated-root polynomial $F(x) = (f(x))^m$ of degree $N = nm$ over a local ring $R$ is defined by

$$V = V(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) = [R(\alpha_0) \ R(\alpha_1) \ldots R(\alpha_{n-1})].$$

#### Example 2.3.

If $F(x) = (x - \alpha_0)^3(x - \alpha_1)^3$ then

$$V = [R(\alpha_0) \ R(\alpha_1)] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ \alpha_0 & 1 & 0 & \alpha_1 & 1 & 0 \\ \alpha_0^2 & 2\alpha_0 & 1 & \alpha_1^2 & 2\alpha_1 & 1 \\ \alpha_0^3 & 3\alpha_0^2 & 3\alpha_0 & \alpha_1^3 & 3\alpha_1^2 & 3\alpha_1 \\ \alpha_0^4 & 4\alpha_0^3 & 6\alpha_0^2 & \alpha_1^4 & 4\alpha_1^3 & 6\alpha_1 \\ \alpha_0^5 & 5\alpha_0^4 & 10\alpha_0^3 & \alpha_1^5 & 5\alpha_1^4 & 10\alpha_1 \end{bmatrix}. $$

Note that if $m = 1$, the generalized Vandermonde matrix is compatible with the usual Vandermonde matrix related to $F(x)$. The determinant of $V$ is $\prod_{0 \leq i < j \leq n-1}(\alpha_i - \alpha_j)^{n_i n_j}$; see [14]. Thus $V$ is invertible in the local ring $R$ if and only if $\alpha_i - \alpha_j$ is a unit in $R$ if and only if $\overline{\alpha_i} \neq \overline{\alpha_j}$; see Lemma 2.5 in [25]. Therefore $V$ is invertible if and only if $\overline{\alpha_i} \neq \overline{\alpha_j}$ for all $i \neq j$. Note that since throughout the paper, it is assumed that $f \in \mathcal{J}$, then $f$ has distinct roots $\overline{\alpha_i}$ for $0 \leq i \leq n-1$. Thus $V$ will always be an invertible matrix.

Let $F(x) = x^N - \sum_{i=0}^{N-1} F_i x^i$ and $C_F$ be the Companion matrix related to $F(x)$, i.e.

$$C_F = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ F_0 & F_1 & F_2 & \ldots & F_{N-1} \end{bmatrix}. $$
It is a well-known fact that $F(x)$ is the characteristic polynomial of $F$, since the polynomial $F(x)$ has repeated-roots, the matrix $F$ is not diagonalizable, but it can be reduced to a very simple form by means of the generalized Vandermonde matrix. Let us denote the Jordan form of the companion matrix $F$ by $J$, i.e. a diagonal block matrix with $n \times n$ blocks so that each block has roots on the diagonal, $1$ on the superdiagonal and other entries are zero. If $V$ is invertible, then the Companion matrix is reduced to $F = VJ^{-1}$. 

3. Generalized Mattson Solomon polynomial

Let $V$ be the usual Vandermonde matrix related to the distinct elements $\alpha_0, \ldots, \alpha_{n-1}$ and $f(x) = \prod_{i=0}^{n-1} (x - \alpha_i)$. For a given $g(x) = \sum_{i=0}^{n-1} g_i x^i$ in $R[x]/(f(x))$, the Mattson Solomon polynomial of $g(x)$ is

$$\text{MS}(g) = \sum_{i=0}^{n-1} g(\alpha_i) x^i = [g_0 g_1 \ldots g_{n-1}]V[1 x \ldots x^{n-1}]^T.$$  \hspace{1cm} (1) 

Note that the map $\text{MS}$ is well defined in the quotient space $R[x]/(f(x))$ (see [2] for a complete account on it). Now, let $F(x) = f(x)^m$ be a repeated-root polynomial of degree $N = mn$ over the local ring $R$ and fix an ordering on distinct roots $\alpha_0, \ldots, \alpha_{n-1}$. Let us consider the quotient polynomial ring $R = \left( \frac{R[x]}{(y^m)}, \cdot \right)$, where $\cdot$ is the ordinary polynomial multiplication modulo $y^m$.

**Theorem 3.1.** The map

$$\text{MS} : \left( \frac{R[x]}{(f(x))}, \cdot \right) \rightarrow \left( \frac{R[x]}{(f(x))}, \star \right),
\begin{align*}
g(x) & \mapsto \sum_{j=0}^{m-1} \left( \sum_{i=0}^{n-1} g_i(\alpha_j) x^i \right) x^j
\end{align*}$$

is a ring injective homomorphism, where $\cdot$ denotes ordinary polynomial multiplication modulo $F(x)$ and $\star$ denotes the component-wise multiplication modulo $f(x)$.

**Proof.** First, we will show that the mapping is well-defined. Given two representatives $h(x), g(x)$ of an element in $\left( \frac{R[x]}{(f(x))}, \cdot \right)$, that is $g(x) - h(x) = k(x)f(x)^m$, for $0 \leq i \leq m - 1$. We have by applying the product rule that

$$g^{[i]}(x) - h^{[i]}(x) = \sum_{j=0}^{i} k^{[i]}(x)(f(x)^m)^{[i-j]}.$$ 

But $(f(x)^m)^{[i-j]} = (i-j)! (f(x)^m)^{(i-j)}$ (the usual derivative of $f(x)^m$) which is indeed 0 mod $f$ for $0 \leq i \leq m - 1$. Therefore for $0 \leq i \leq m - 1$ one has that $g^{[i]}(x), h^{[i]}(x)$ provide the same values when evaluated at $\alpha_j$, $j = 0, \ldots, n - 1$.

Let $g(x) = \sum_{i=0}^{n-1} g_i x^i \in \left( \frac{R[x]}{(f(x))} \right)$ and $V$ be the generalized Vandermonde matrix related to roots $\alpha_0, \ldots, \alpha_{n-1}$. Consider the column vector

$$u = \left[ 1 x \ldots x^{n-1} x \ xy \ldots xy^{m-1} \ldots x^{n-1} y \ldots x^{n-1}y^{m-1} \right]^T.$$
where $\text{tr}$ denotes the transpose of the vector. Then we have that $\text{MS}(g)$ is given by

$$
\left[ g(\alpha_0) \ g^{[1]}(\alpha_0) \ldots g^{[n-1]}(\alpha_0) \ g(\alpha_{n-1}) \ g^{[1]}(\alpha_{n-1}) \ldots g^{[n-1]}(\alpha_{n-1}) \right] u \\
= [g_0 \ g_1 \ldots g_{n-1}] V u.
$$

Since the matrix $V$ is invertible, then $\text{MS}$ is injective. Now it is enough to show that $\text{MS}(g \cdot h) = \text{MS}(g) \ast \text{MS}(h)$ that follows applying the product rule of the Hasse derivative, we can easily check that $\text{MS}(g) \ast \text{MS}(h)$ can be computed as

$$
\sum_{i=0}^{n-1} \left( \sum_{j=0}^{m-1} g^{[j]}(\alpha_i)y^j \right) \cdot \left( \sum_{j=0}^{m-1} h^{[j]}(\alpha_i)y^j \right) x^i = \sum_{i=0}^{n-1} \left( \sum_{j=0}^{m-1} (gh)^{[j]}(\alpha_i)y^j \right) x^i.
$$

Note that the mapping in the above theorem gives the ordinary Mattson-Solomon transform when applied to a simple-root polynomial. Thus abusing the notation, we will denote both the same. We will call the map $\text{MS}$ in the above theorem the Generalized Mattson Solomon transform mapped associated to $F$.

**Example 3.2.** (Example 2.2 Cont.) Let $m = 3$, $n = 2$, $f(x) = x^2 - x \in \mathbb{Z}_4[x]$ and $R = \mathbb{Z}_{16}[y]/\langle y^3 \rangle$. Then

$$
\text{MS}(g(x)) = (7 + 5y + 6y^2) + (1 + 8y^2)x \in R[x]/\langle f(x) \rangle.
$$

**Remark 3.3.** Theorem 3.1 states that every repeated-root polycyclic code is isomorphic to an ideal in a bivariate polynomial ring, since $\frac{R[x]}{\langle f(x) \rangle} \cong \frac{R'[x,y]}{\langle f(x),y^m \rangle}$.

**Lemma 3.4.** The map $\text{MS}$ in Theorem 3.1 is equivalent to each of the following mappings.

1. $\text{MS} : \left( \frac{R[x]}{\langle f(x) \rangle}, \cdot \right) \rightarrow \left( \frac{R'[x]}{\langle f(x) \rangle}, \cdot \right)$

$$
g(x) \mapsto \sum_{i=0}^{n-1} g(\alpha_i + y)x^i
$$

2. $\text{MS} : \left( \frac{R[x]}{\langle f(x) \rangle}, \cdot \right) \rightarrow \left( \frac{R'[x]}{\langle f(x) \rangle}, \ast \right)$

$$
g(x) \mapsto \sum_{i=0}^{n-1} g(u\alpha_i)x^i
$$

**Proof.** Since $y^m = 0$, by the Taylor expansion for the Hasse derivative, we get $g(\alpha_i + y) = \sum_{j=0}^{m-1} g^{[j]}(\alpha_i)y^j$. Thus $\text{MS}(g) = \sum_{i=0}^{n-1} g(\alpha_i + y)x^i$, which gives the mapping (2). The set $\{\alpha_0(y + 1) - y, \ldots, \alpha_{n-1}(y + 1) - y\}$ are roots of $F(x)$, since $y^m = 0$. Put $u = y + 1$ and use the mapping (2) to find $\text{MS}(g(x))$. Now, $R'[u - 1] \cong R'[u]$ provides the mapping (3).
Remark 3.5. Note that in the case $F(x)$ is a simple-root polynomial (i.e. $m = 1$), we get $y = 0$ and $u = 1$. Hence, the two mappings presented in the previous lemma are compatible with the Mattson Solomon mapping given in [2].

Remark 3.6. In definition 2.1 we define the GDT for polycyclic codes of length $N = mn$ over rings as a generalization of the GDT for repeated-root cyclic codes of length $N = np^k$ over fields presented by Massey in [24]. Now we are able to present other definitions of the GDT associated with the mappings in Lemma 3.4:

\[
\text{GDT : } R^N \rightarrow R^n \\
(g_0, g_1, \ldots, g_N) \mapsto (g(\alpha_0 + y), g(\alpha_1 + y), \ldots, g(\alpha_{n-1} + y))
\]

(4)

and

\[
\text{GDT : } R^N \rightarrow A^n \\
(g_0, g_1, \ldots, g_N) \mapsto (g(u\alpha_0), g(u\alpha_1), \ldots, g(u\alpha_{n-1}))
\]

(5)

where $A = \frac{R[u]}{(u-1)^m}$. Note that (4) is compatible with the definition of the DFT given in [2].

4. The decomposition of the ambient space

We will start by studying the ring $R$ defined in the previous section.

Lemma 4.1 (Corollary 3.8 in [10]). Let $R$ be a local ring and $g \in R[x]$ be a monic irreducible polynomial. Then $R[x]/\langle g(x)^n \rangle$ is a local ring for any positive integer $n$.

Lemma 4.2. Let $S$ be a Galois extension of the local ring $R$. Then

1. $S$ is the unramified local ring, i.e. $R$ and $S$ has the same maximal ideal.
2. If $f \in R[x]$ is square-free, then $f$ has distinct zeros in the local extension $S$.
3. $S$ is an $R$-free module generated by roots of $f$.

Proof. See Theorems 3.15, 3.18, 5.11 in [10].

Corollary 4.3. Let $R'$ be the Galois extension of the local ring $R$ containing $n$ distinct roots of the polynomial $f(x) = \prod_{i=0}^{n-1}(x - \alpha_i)$. Then $R = R'[y]/\langle y^m \rangle$ is a local ring.

The proof of the corollary follows from the fact that $R$ is local, $R'$ is a Galois extension and applying Lemmas 4.1 and 4.2. Then, from the counting argument in [11], if we count the elements in $R$ that are $p^{ns}$, and the number of zero divisors in $R$ is $p^{c+(s-1)m}$ where $p^c$ is the number of zero divisors of $R$, therefore from [11, Theorem 1] $R$, is a local ring. Furthermore, note that $R$ is also a chain ring if and only if $R$ is a finite field. This follows from the fact that the maximal ideal of $R$ is $\langle m, y \rangle$ where $m$ is the generator of the maximal ideal of $R$ and it is principal if $p$ is the characteristic of $R$. 


4.1. Decomposition of the codes

In this section, we are going to find a decomposition of the ambient space $\mathbb{R}[x]/(p(x))$. Recall that the ring $\mathbb{R}[x]/(f(x))$ is equipped with the component-wise product and the ring $\mathbb{R}[x]/(f(x))$ is equipped with the ordinary polynomial product. Let us denote $\mathbb{R}[x]/(f(x))$ by $R_F$. Let $f = f_1f_2\ldots f_r$, where $f_1, f_2, \ldots, f_r$ are distinct monic irreducible polynomials. We will define a relation on the set of indices $I = \{0, 1, \ldots, n-1\}$ as follows: $i \sim j$ if and only if $\alpha_i, \alpha_j$ are roots of the same polynomial $f_k$, i.e. $f_k(\alpha_i) = f_k(\alpha_j) = 0$. Therefore $I$ will be partitioned into the disjoint classes $I_k$ related to $f_k$.

From now on in Subsection 4.1.1 we will consider the MS-map in Theorem 3.1 extended to $R'$

$$MS : \left( R'_F = \frac{R'[x]}{(f(x))}, \star \right) \rightarrow \left( \frac{R[x]}{(f(x))}, \ast \right)$$

or, what is the same, consider a polynomial $f(x)$ which completely splits in linear factors in the ring we are working on.

It is easy to see that, again for cardinality reasons, it is now an isomorphism and we can define $E_i = MS^{-1}\left( \sum_{j \in I_i} x^j \right)$. The pre-images $\{E_1, \ldots, E_r\}$ will provide us the primitive idempotents, more precisely:

**Proposition 4.4.**

1. Each $E_i$ is a primitive idempotent.
2. $E_iE_j = 0$ for $i \neq j$, and $\sum_{i=1}^r E_i = 1$
3. The only idempotents in $R_F$ are in the form $\sum_{j \in S} E_j$ for some $S \subseteq \{1, 2, \ldots, r\}$.
4. $R'_F \cong \bigoplus_{i=1}^r (E_i) \cong \bigoplus_{i=1}^r \frac{R_F}{1-E_i}$

**Proof.**

1. Note that $x^j \ast x^j = x^j$ for all $0 \leq j \leq n-1$, thus $E_i^2 = MS^{-1}\left( \sum_{j \in I_i} x^j \right) = E_i$. To check that $E_i$ is primitive, let $E_i = A(x) + B(x)$, where $A(x)$ and $B(x)$ are primitive idempotents in $R_F$. Denote $MS(A(x)) = \sum_{k=0}^{n-1} a_kx^k = a(x)$ and $MS(B(x)) = \sum_{k=0}^{n-1} b_kx^k = b(x)$. Then $\sum_{j \in I_i} x^j = a(x) + b(x) = \sum_{i=0}^{n-1} (a_i + b_i)x^i$, and hence $a_k + b_k = 0$ for $k \notin I_i$ and $a_k + b_k = 1$ otherwise. Since $A(x), B(x)$ are idempotent elements, $a(x), b(x)$ are also idempotent elements in $\mathbb{R}[x]$. According to component-wise multiplication in $\mathbb{R}[x]$, we conclude that $a_i$ and $b_i$ are idempotent elements in $\mathbb{R}$ for all $0 \leq i \leq n-1$. Now since $R_y$ is local, $a_k, b_k \in \{0, 1\}$ for all $0 \leq k \leq n-1$. So if we let $a(x)$ and $b(x)$ be equal to the unit element $\sum_{j=0}^{n-1} x^j$ in $R_y[x]$, then $a_k = b_k = 1$ for all $0 \leq k \leq n-1$, which is a contradiction with $a_k + b_k = 1$ for $k \in I_i$. Therefore, $a(x)$ and $b(x)$ are not the unit element in $R_y[x]$. On the other hand, we have $a(x) = 0$ or $b(x) = 0$, that is $A(x) = 0$ or $B(x) = 0$. 

8
2. For $i \neq j$, $I_i$ and $I_j$ are disjoint and hence $E_iE_j = 0$. Moreover, since
\[ \sum_{i=0}^{n-1} x^i \] is the unit element of $R[x]$ we get
\[ 1 = MS^{-1}(\sum_{i=0}^{n-1} x^i). \]

3. Clearly, to obtain the idempotents in $R_F'$, it is necessary to study idempotents in $MS(R_F') = \frac{R_F[x]}{(f(x))}$. Let $a(x) = \sum_{k=0}^{n-1} a_k x^k$ be an idempotent element in $\frac{R_y[x]}{(f(x))}$. We get $\sum_{k=0}^{n-1} a_k x^k = a(x) = a(x)^2 = \sum_{k=0}^{n-1} a_k^2 x^k$. Thus $a_k = a_k^2$ for all $0 \leq k \leq n - 1$ and since $R_y$ is local, we have $a_k \in \{0, 1\}$ for all $0 \leq k \leq n - 1$. If we let $S = \{i \mid a_i \neq 0\}$, then $a(x) = \sum_{i \in S} x^i$ and
\[ A(x) = MS^{-1}(a(x)) = \sum_{i \in S} E_i. \]

4. The first isomorphism follows from the fact that $\{E_1, \ldots, E_r\}$ is the set of pairwise primitive orthogonal idempotents. To prove the second isomorphism we define $\theta : R_F \to \langle E_i \rangle$ via $g \mapsto gE_i$. Let $gE_i = 0$. Then $g = g(1 - E_i) + gE_i = g(1 - E_i)$, and hence $\ker \theta = (1 - E_i)$, which gives the result.

This provides the following description of the codes in terms of the idempotents in the case of a ring of prime characteristic.

**Proposition 4.5.** Let $R'$ be a local ring with prime characteristic $p$ and $N = np^k$. Then

1. If $f_i(x) = \prod_{j \in I_i} (x - \alpha_j)$ then $(f_i(x))^p = 1 - E_i$.
2. If the ideal $C$ of $R_F'$ has an idempotent generator, then $C$ is generated by
\[ \prod_{i \in S} (f_i(x))^p \] for some $S \subseteq \{1, 2, \ldots, r\}$.

**Proof.**

1. $1 - E_i = MS^{-1}(\sum_{i=0}^{n-1} x^i) - MS^{-1}(\sum_{i=0}^{n-1} d_i x^i) = MS^{-1}(\sum_{i=0}^{n-1} e_i x^i)$ such that $e_i \notin I_i$. On the other hand, recall that $\alpha_i - \alpha_j$ is a unit in $R'$ if and only if $\overline{e_i} \neq \overline{e_j}$. Since $f \in \mathcal{J}$, $\tilde{f}$ has distinct roots $\overline{e_i}$ for $0 \leq i \leq n - 1$, and we get

\[
(f_i(\alpha_j + y))^p = (\alpha_j + y - \alpha_{i_1})^p \ldots (\alpha_j + y - \alpha_{i_t})^p \\
= ((\alpha_j - \alpha_{i_1})^p + y^p) \ldots ((\alpha_j - \alpha_{i_t})^p + y^p) \\
= (\alpha_j - \alpha_{i_1})^p \ldots (\alpha_j - \alpha_{i_t})^p \\
= \begin{cases} 
0 & j \in I_i, \\
\text{unit} & j \notin I_i. 
\end{cases}
\]

Thus $MS((f_i(x))^p) = \sum_{j=0}^{n-1} (f_i(\alpha_j + y))^p x^j \in \langle \sum_{j \notin I_i} x^i \rangle = MS(1 - E_i)$. Now since $MS$ is injective, the result holds.
2. The only idempotent elements in $R'_F$ are in the form $\sum_{i \in K} E_i$ for some subset $K$ of $\{1, 2, \ldots, r\}$. By the fact that $E_i$’s are orthogonal we have

$$\sum_{i \in K} E_i = 1 - \sum_{i \notin K} E_i = \prod_{i \notin K} (1 - E_i) = \prod_{i \notin K} (f_i(x))^{p^k}.$$ 

Now it is enough to take $S = K^c$.

\[\square\]

**Corollary 4.6.** Let $R'$ be a local ring with prime characteristic $p$ and $N = np^k$ where $f$ completely splits. Then

$$\frac{R'[x]}{\langle F(x) \rangle} = \frac{R'[x]}{\langle (f(x))^{p^k} \rangle} \cong \bigoplus_{i=1}^r \frac{R'[x]}{\langle (f_i(x))^{p^k} \rangle}.$$ 

**Proof.** Part (4) of Proposition 4.4, part (1) of Proposition 4.5 and the Third Isomorphism Theorem give the proof. \[\square\]

**Remark 4.7.** Note that in this section (Section 4.1) we have considered codes over the ring $R'_F$, if we want to restrict ourselves to $R_F$ we must consider subring subcodes that behave as subfield subcodes in the field case, for a reference on they, their Galois closure and a Delsarte’s like theorem in the chain ring case see [20].

### 4.2. $\perp_0$ duality

Consider the following inner product over the ring $R_F = \frac{R[x]}{\langle F(x) \rangle}$

$$\langle g_1(x), g_2(x) \rangle_{(0)} = g_1g_2(0), \quad g_1(x), g_2(x) \in R_F. \quad (6)$$

We will denote the dual of the polycyclic code $C \subseteq R_F$ associated with this inner product by $C^{\perp_0}$ given by

$$C^{\perp_0} = \{g(x) \in R_F \mid \langle g(x), h(x) \rangle_{(0)} = 0, \text{for all } h(x) \in C\}.$$ 

**Theorem 4.8.** Let $C$ be a polycyclic code of length $N = np^k$ in $R_F$. If $F_0$ is an invertible element in $R$, then

1. The inner product $\langle , \rangle_{(0)}$ is non-degenerate.
2. $C^{\perp_0} = \text{Ann}(C)$, where Ann stands for the annihilator ideal.
3. $C^{\perp_0}$ is a polycyclic code.

**Proof.**

1. We must show that the orthogonal of $R_F$ is zero. Let $g = g_0 + g_1x + \ldots + g_{N-1}x^{N-1} \in R_F$ and $\langle g, x^i \rangle_{(0)} = 0$ for all $0 \leq i \leq N-1$. From $\langle g, 1 \rangle_{(0)} = 0$ we conclude $g_0 = 0$. Also, by considering $0 = \langle g, x^i \rangle_{(0)} = g_{N-i}F_0$ for all $1 \leq i \leq N - 1$ and invertibility $F_0$ we obtain $g_{N-i} = 0$, i.e. $g = 0$. 

10
2. Let \( h(x) \in \text{Ann}(C) \), therefore \( h(x)g(x) = 0 \) for all \( g(x) \in C \) and hence \( hg(0) = 0 \), i.e. \( h(x) \in C^{\perp_0} \). Thus \( \text{Ann}(C) \subseteq C^{\perp_0} \). Conversely, let \( h \in C^{\perp_0} \) and \( g \in C \) be an arbitrary element. Hypothesis \( 0 = \langle g, h \rangle(0) = hg(0) \) implies that \( x^i hg(0) = 0 \) for all \( 0 \leq i \leq N - 1 \). Now by part (1) we have \( hg = 0 \), which gives the result.

3. It is obvious by part (2).

\[\square\]

**Remark 4.9.** In the literature of simple-root polycyclic codes over \( R[x]/\langle f(x) \rangle \), it is always assume that \( f_0 \) is a unit in the ring \( R \), see [19, 9]. Because this assumption is guaranteed that every left polycyclic code is right polycyclic and as a result we get ride of studying left and right at the same time. In this paper, we always assume that \( F(0) = F_0 \), the constant term of the polynomial \( F \), is a unit in \( R \). Because this assumption is guaranteed that the dual of every polycyclic code \( (C^{\perp_0}) \) is again polycyclic (also in our previous paper in simple-root case [2] we have assumed that \( f_0 \) is a unit in order to have a polycyclic dual).

We now define another inner product over \( R_F \):

\[ \langle g_1(x), g_2(x) \rangle_{\text{MS}} = \text{MS}(g_1) \star \text{MS}(g_2), \quad g_1(x), g_2(x) \in R_F, \quad (7) \]

As usual we will denote the dual of the polycyclic code \( C \subseteq R_F \) associated with this inner product by \( C^{\perp_{\text{MS}}} \), which is naturally defined as

\[ C^{\perp_{\text{MS}}} = \{ g \in R_F \mid \text{MS}(g) \star \text{MS}(c) = 0 \text{ for all } c \in C \}. \quad (8) \]

The following result shows how one can check the annihilator duality in terms of the Mattson Solomon transform.

**Theorem 4.10.** For the polycyclic code \( C \) over \( R_F \), we have \( \text{Ann}(C) = C^{\perp_{\text{MS}}} \).

**Proof.** Since the Mattson-Solomon mapping is an injective morphism we have

\[ gc = 0 \iff \text{MS}(gc) = 0 \iff \text{MS}(g) \star \text{MS}(c) = 0 \]

which implies \( \text{Ann}(C) = C^{\perp_{\text{MS}}} \). \[\square\]

**Remark 4.11.** Note that all the results in Subsection 4.2 are given in the ring \( R_F \) since only injectivity of the MS map is needed, so we do not need to consider the ring \( R_F' \).

4.3. A note on multivariable codes

In [2], the Mattson Solomon map for several variable serial codes over chain rings presented. That construction was based on the decomposition of the tensor product of the two \( R \)-modules \( R[x_1]/\langle f_1(x_1) \rangle \) and \( R[x_2]/\langle f_2(x_2) \rangle \) in terms of the tensor product of powers of their related companion matrices \( E_f \) and \( E_g \) and their simultaneous diagonalization by the matrix \( V_f \otimes V_g \) where \( V_f \), is
Vandermonde matrix corresponding to $f_i$, $i = 1, 2$. In the principal ideal case, one of the defining polynomials is a repeated-root one, say $f_1(x) = f(x_1)^m$, and the remainder ones should be non-repeated-root polynomials and $R$ is a Galois ring, see [21]. In that later is the case, we can provide a Mattson Solomon transform in terms of the Generalized Vandermonde matrices in the same fashion as in [2].

Multivariable codes over the ring $R$ are ideals of the quotient ring $\mathcal{R} = R[x_1, \ldots, x_w]/\langle t_1(x_1), \ldots, t_w(x_w) \rangle$. If all polynomials $t_1(x_1), \ldots, t_w(x_w)$ are simple-roots, then these codes are called serial multivariate codes, and otherwise they are called modular multivariate codes. The transform approach to the serial case over local rings was studied in [2]. Note that serial multivariate codes are well-behaved because they can be regarded as principal ideals in $\mathcal{R}$. This property is not generally true in the modular case. In the case $r > 2$, $\mathcal{R}$ is principal ideal ring if and only if $R$ is a Galois ring and the number of polynomials for which $t_j(x_i)$ is not square-free is at most one, see [21, Theorem 1].

For the sake of simplicity, all results in this section will be proved for $w = 2$ and can be straightforwardly worked out for $w > 2$. Let $R$ be a Galois ring, $f(x_1)$ a polynomial of degree $n$ over $R$ with distinct simple-roots $\alpha_0, \ldots, \alpha_{n-1}$ in an extension ring $R_{1}^{1}$, and $F(x_1) = (f(x_1))^m$ a polynomial of degree $N = nm$. Moreover, let $g(x_2)$ be a polynomial of degree $M$ over $R$ with distinct simple-roots $\beta_0, \ldots, \beta_{M-1}$ in an extension ring $R_{2}^{1}$. Let $V$ be the generalized Vandermonde matrix related to $\alpha_0, \ldots, \alpha_{n-1}$ and $v$ be the usual Vandermonde matrix related to $\beta_0, \ldots, \beta_{M-1}$. Consider the tensor product

\[ v \otimes V = \begin{bmatrix} V & \cdots & V \\ \beta_0V & \cdots & \beta_{M-1}V \\ \vdots & \cdots & \vdots \\ \beta_0^{M-1} & \cdots & \beta_{M-1}^{M-1}V \end{bmatrix}. \]

Since $det(v \otimes V) = det(v)^M det(V)^N$ and $v, V$ are invertible, then $v \otimes V$ is invertible. A polynomial $p(x_1, x_2) \in R[x_1, x_2]/\langle F(x_1), g(x_2) \rangle$ can be written as $p(x_1, x_2) = \sum_{j=0}^{M-1} p_j(x_1)x_2^j$, where $p_j(x_1) = \sum_{i=0}^{N-1} b_{i,j}x_1^i$. Relate the vector

\[ p = (p_0, 0, p_{1,0}, \ldots, p_{N-1,0}, p_{0,1}, p_{1,1}, \ldots, p_{N-1,1}, \ldots, p_0, M-1, p_1, M-1, \ldots, p_{N-1}, M-1) \]

to the polynomial $p(x_1, x_2)$. It can be easily seen that the product of the vector $p$ and matrix $v \otimes V$ is as follows:

\[ p(v \otimes V) = (p(\alpha_0 + y, \beta_0), \ldots, p(\alpha_{n-1} + y, \beta_0), p(\alpha_0 + y, \beta_1), \ldots, p(\alpha_{n-1} + y, \beta_1), \ldots, p(\alpha_0 + y, \beta_{M-1}), \ldots, p(\alpha_{n-1} + y, \beta_{M-1})). \]

Take $R'' = R_1^{1} + R_2^{1}$. Clearly, $p(\alpha_i + y, \beta_j) \in R''$ for all $0 \leq i \leq n - 1$ and $0 \leq j \leq M - 1$. Define the multivariable Mattson-Solomon transform for modular
multivariable codes as

\[ \text{MS} : \left( \frac{R[x_1, x_2]}{(f(x_1), g(x_2))}, \bullet \right) \rightarrow \left( \frac{R''[x_1, x_2]}{(f(x_1), g(x_2))}, \ast \right) \]

\[ p(x_1, x_2) \quad \mapsto \quad \sum_{i=0}^{n-1} \sum_{j=0}^{M-1} p(\alpha_i + y, \beta_j) x_1^i x_2^j \]

where \( \bullet \) denotes ordinary polynomial multiplication modulo \( F(x_1), g(x_2) \) and \( \ast \) denotes the component-wise multiplication modulo \( f(x_1), g(x_2) \). Obviously, the mapping MS is a ring homomorphism and since \( v \otimes V \) is invertible, MS is also injective.

5. Matrix-Product Structure of Certain Polycyclic Codes

We prove the structure of some repeated-root polycyclic codes with the help of matrix-product codes in the paper [30]. From now on, we will consider repeated-root polynomials just over the finite field \( F_q \), where \( q = p^r \) where \( p \) is a prime number. Let \( f(x) \in F_p[x] \) be a simple-root polynomial of degree \( n \) and of order \( e \), i.e. \( e \) is the smallest integer for which \( f(x)|x^e - 1 \) and \( \text{gcd}(p, e) = 1 \). Let \( f(x) = \prod_{i=1}^s f_i(x) \) be the unique factorization of \( f(x) \) into distinct irreducible polynomials over \( F_p[x] \). Then, we have \( f(x^k) = \prod_{i=1}^s f_i(x^k) \) and for each \( 1 \leq i \leq s \), there exists an irreducible polynomial \( g_i(x) \) in \( F_p[x] \) such that \( f_i(x^k) = g_i(x)^{p^k} \). From now on, we will assume that \( R \) is the ring

\[ R = \frac{F_{p^r}[x]}{(f(x^k))} = \frac{F_{p^r}[x]}{\left( \prod_{i=1}^s g_i(x) \right)^{p^k}} \]

and we will have that \( N = np^k \). One can write any element \( a(x) \in R \) as \( a_0(x) + a_1(x)x^{p^k} + \ldots + a_{n-1}(x)x^{(n-1)p^k} \), where \( a_i(x) \in F_{p^r}[x] \). Let \( S \) be the ring \( F_{p^r}[x, y]/(x^{p^k} - y, f(y)) \). We have the following straightforward results.

Lemma 5.1. Any ideal of the ring \( R \) is principally generated by a divisor of \( f(x^k) \). In fact, it is of the form \( (G(x)) \), where \( G(x) = \prod_{j=1}^s g_i(x)^{i_j} \) and \( 0 \leq i_j \leq p^k \).

Remark 5.2. Note that in the case of cyclic codes, the above ideals give us the so-called monomial like codes in [23].

Lemma 5.3. The map \( \varphi : R \rightarrow S \) given by \( \varphi \left( \sum_{i=0}^{n-1} a_i(x)x^{ip^k} \right) = a(x, y) = \sum_{i=0}^{n-1} a_i(x)y^i \) is a ring isomorphism.

Now we will consider the ring

\[ T = \frac{F_{p^r}[x, y]/(x^{p^k} - 1, f(y)) = \left( \frac{F_{p^r}[x]}{(x^{p^k} - 1)} \right)[y]/(f(y))}, \]

and denote as \( W \) the ring \( W = F_{p^r}[x]/(x^{p^k} - 1) \). Note that \( W \) is a finite chain ring whose maximal ideal is \( (x - 1) \).
Lemma 5.4. The map \( \psi : S \to T \) defined by \( \psi(a(x, y)) = a(y' x, y) \) is a ring isomorphism, where \( e' \) is the inverse of \( p^k \) in \( \mathbb{Z}_e \).

As an easy corollary we have the following.

Corollary 5.5. The code \( C \) is a polycyclic code in \( \mathbb{F}_p[r]/\langle f(x^p) \rangle \) if and only if \( \mu(C) = \psi(\varphi(C)) \) is a polycyclic code in \( W[y]/\langle f(y) \rangle \).

Therefore, since \( W \) is a chain ring we can apply \( \mathbb{F}_p[r]/\langle f(x^p) \rangle \) and get the following unique \( (x - 1) \)-adic expansion of the code \( C \) (Note that we have also a description of a system of generators of a polycyclic code over a chain ring in \( [26] \) Theorem 4.4) and its generalization in \( [22] \) Theorem 3.13).

Proposition 5.6. Any polycyclic code \( C \) in \( W[y]/\langle f(y) \rangle \) is of the form

\[ C = \langle h_0(y), (x - 1)h_1(y), \ldots, (x - 1)p^k - 1h_{p^k - 1}(y) \rangle, \]

where \( h_{p^k - 1}(y) \mid h_{p^k - 2}(y) \mid \cdots \mid h_0(y) \mid f(y) \) over \( \mathbb{F}_p[r] \). Moreover, we have

\[ C = \bigoplus_{i=0}^{p^k - 1} (x - 1)^i C_i, \]

where for \( 0 \leq i \leq p^k - 1 \), \( C_i = \langle h_i(y) \rangle \) is a polycyclic code in \( \mathbb{F}_p[y]/\langle f(y) \rangle \) and \( C_0 \subseteq C_1 \subseteq \cdots \subseteq C_{p^k - 1} \).

Note that the ideal defining \( C \) over the ring \( W \) is a single generated and the generator can be derived from the polynomials \( h_i(x) \) in the above expression (see the proof of \( [22] \) Theorem 3.13). The following theorem follows directly.

Theorem 5.7. Let \( C = \langle g_1(x)^{i_1} g_2(x)^{i_2} \cdots g_r(x)^{i_r} \rangle \). Then we have

\[ \mu(C) = \bigoplus_{i=0}^{p^k - 1} (x - 1)^i C_i \]

where \( C_i \) is a simple-root polycyclic code with respect to \( f(y) \) over \( \mathbb{F}_p[r] \). In fact we have \( C_i = \langle k_i(y) \rangle \), where \( k_i(y) = \prod_{j \in A_i} g_j(y) \) and \( A_i = \{ 1 \leq j \leq r \mid i_j > i \} \).

The following definition introduces matrix product codes in this work. Matrix-product codes over some classes of rings have been studied in several works, see for example \( [5, 7, 8, 10] \), but they did not consider the \( \perp_0 \)-orthogonality.

Definition 5.1. Let \( A = [a_{ij}] \) be an \( \alpha \times \beta \) matrix with entries in \( \mathbb{F}_p[r] \) and let \( C_1, \ldots, C_\alpha \) be codes of length \( n \) over \( \mathbb{F}_p[r] \). The matrix-product code \( [C_1, \ldots, C_\alpha], A \) is the set of all matrix products \( [c_1, \ldots, c_\alpha] A \), where \( c_i \in C_i \), defined by

\[
[c_1, \ldots, c_\alpha] \cdot A = [c_1, \ldots, c_\alpha]
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1\beta} \\
a_{21} & a_{22} & \cdots & a_{2\beta} \\
\vdots & \vdots & \ddots & \vdots \\
a_{\alpha 1} & a_{\alpha 2} & \cdots & a_{\alpha \beta}
\end{bmatrix}
\]

(11)

\[
= [a_{11} c_1 + a_{21} c_2 + \cdots + a_{\alpha 1} c_\alpha, a_{12} c_1 + a_{22} c_2 + \cdots + a_{\alpha 2} c_\alpha, \]
\[
\ldots, a_{1\beta} c_1 + a_{2\beta} c_2 + \cdots + a_{\alpha \beta} c_\alpha].
\]
Lemma 5.8 (Proposition 2.9 \[3\]). If a matrix consisting of some α columns of A is non-singular and C = [C₁, ..., C_α] · A, then |C| = |C₁| · ... · |C_α|.

Definition 5.2 (Definitions 1 and 2 in \[30\]).

- J to be the p^k × p^k matrix whose (i, p^k − i + 1)-th entry (1 ≤ i ≤ p^k) is equal to 1 and other entries are equal to zero, P to be the p^k × p^k matrix whose (i, j)-th entry (1 ≤ i, j ≤ p^k) is equal to \((\binom{j−1}{i−1})\) mod p, Q to be the p^k × p^k matrix whose (i, j)-th entry is equal to \((-1)^{(i+j)}(i−1)\binom{j−1}{i−1}\) mod p and CYC(p, k) to be JQJ.

- For 0 ≤ i ≤ N − 1 we will write i = ap^k + j where 0 ≤ a ≤ n − 1, 0 ≤ j ≤ p^k − 1. We define the permutation σ on \(\{0, 1, \ldots, N − 1\}\) as σ(i) = jn + a.

Lemma 5.9. \(A = \text{CYC}(p, k)\) is a non-singular matrix.

Proof. Matrix CYC(p, 1) is upper triangular with exactly p − i zeros in the column i and ones in the diagonal, and \(A = \text{CYC}(p, k) = \bigotimes_{i=1}^{k} \text{CYC}(p, 1)\) by \[30\]. Since the tensor product of two upper triangular matrices is again upper triangular the result follows.

Theorem 5.10. Let C be a polycyclic code in \(\mathbb{F}_p[x]/(f(x))^p\) and \(μ(C) = \bigoplus_{i=0}^{p^k−1}(x−1)^iC_i\), then we have that

\[\sigma(C) = [C_{p^k−1}, C_{p^k−2}, \ldots, C_0] \cdot \text{CYC}(p, k)\].

Proof. Assume \(a(x) = \sum_{i=0}^{n−1} a_i(x)x^i \in C\), then \(φ(a(x)) = \sum_{i=0}^{n−1} a_i(x)y^i\) and hence \(ψ(φ(a(x))) = \sum_{i=0}^{n−1} a_i(y^e x)y^i\). If \(σ(a(x)) = b(x) = \sum_{i=0}^{p^k−1} x^ib_i(x^p^e)\) then we have \(ψ(φ(a(x))) = \sum_{i=0}^{p^k−1}(y^e x)^i b_i(y)\).

On the other hand, we can write

\[b_0(y) + (y^e)x b_1(y) + \cdots + y^{(p^k−1)e}x^{(p^k−1)} b_{p^k−1}(y) = b_0(y) + (y^e)(x−1+1)b_1(y) + \cdots + y^{(p^k−1)e}(x−1+1)^{(p^k−1)} b_{p^k−1}(y) = \sum_{i=0}^{p^k−1} \left(\sum_{j=0}^{i} \binom{i}{j} (x−1)^j\right) y^{ie} b_i(y) = \sum_{j=0}^{p^k−1} y^{ie} \left(\sum_{i=j}^{p^k−1} \binom{i}{j} b_i(y)\right) (x−1)^j\].

For 0 ≤ j ≤ p^k − 1, let us denote by \(c_j'(y) = y^{je} \sum_{i=j}^{p^k−1} \binom{i}{j} b_i(y)\), and \(c_j(y) := \sum_{i=j}^{p^k−1} \binom{i}{j} b_i(y)\). Hence \(\sum_{j=0}^{p^k−1} c_j'(y)(x−1)^j \in ψ(φ(C))\) and since \(ψ(φ(C)) = \bigoplus_{i=0}^{p^k−1}(x−1)^iC_i\), we have \(c_j'(y) \in C_j\). But \(C_j\) is a polycyclic code and \(y\) is
a unit element because we assume that \( f_0 \) is a unit (see Remark 4.9). Hence \( c_j(y) \in C_j \) as well. Now we have

\[
[c_0(y), c_1(y), \ldots, c_{p^k-1}(y)] = [b_0(y), b_1(y), \ldots, b_{p^k-1}(y)] \cdot P,
\]

where \( P \) is an invertible matrix whose inverse is the matrix \( Q \). Therefore we have

\[
[c_0(y), c_1(y), \ldots, c_{p^k-1}(y)] \cdot Q = [b_0(y), b_1(y), \ldots, b_{p^k-1}(y)],
\]

and it follows \( \sigma(C) \subseteq \{C_0, C_1, \ldots, C_{p^k-1}\} \cdot Q \) and since both of the sets have the same size, we have \( \sigma(C) = \{C_0, C_1, \ldots, C_{p^k-1}\} \cdot Q \). Using similar arguments as those used in [30], we get \( \sigma(C) = \{C_{p^k-1}, C_{p^k-2}, \ldots, C_0\} \cdot CYC(p, k) \), and the proof is now completed.

**Remark 5.11.** Note that if we consider \( C \) as a cyclic code of length \( np^k \) over the field \( F_{p^m} \) in [30], a permutation \( \pi \) is provided such that \( \pi(C) = \{C_{p^k-1}, C_{p^k-2}, \ldots, C_0\} \cdot CYC(p, k) \).

It can be easily checked that, in general, \( \pi \neq \sigma \), where \( \sigma \) is the permutation defined above, while the codes \( C_i, 0 \leq i \leq p^k - 1 \), are the same. Therefore we have two permutations for which \( \pi(C) = \sigma(C) \) or equivalently \( \pi^{-1} \circ \sigma \in \text{Aut}(C) \), the group of automorphism of the code \( C \). The reason for getting different permutation in this case is related to the different kinds of isomorphisms we have considered. In fact, in [30] the mapping considered was

\[
\frac{F_{p^m}[x]}{(x^{np^k} - 1)} \sim \frac{F_{p^m}[x, y]}{(x^n - y, y^{pn^k} - 1)},
\]

while in this paper we have considered the isomorphism

\[
\frac{F_{p^m}[x]}{(x^{np^k} - 1)} \sim \frac{F_{p^m}[x, y]}{(x^{p^k} - y, y^{n} - 1)}.
\]

Since the matrix \( CYC(p, 1) \) is a Non-Singular by Columns matrix (NSC matrix) (see [30] for a definition), Proposition 2 in [31] implies the following corollary involving the minimum Hamming distance \( d_i \) of each of the component codes \( C_i \) and the distance of the code \( d(C) \).

**Corollary 5.12.** Let \( C \) be a polycyclic code in \( F_{p^r}[x]/\langle (f(x))^{p^k} \rangle \) such that \( \mu(C) = \bigoplus_{i=0}^{p^k-1}(x - 1)^i C_i \), then we have

\[
d(C) = \min\{p^k d_{p^k-1}, (p^k - 1)d_{p^k-2}, \ldots, d_0\},
\]

where \( d_t = d(C_t) \) and \( t = 0, 1, \ldots, p^k - 1 \).
5.1. Duality

The annihilator dual of a matrix-product code can be also explicitly described in terms of matrix-product codes. First we will introduce the following auxiliary result.

Lemma 5.13. The isomorphism $\mu$ introduced in Corollary 5.5 is a $\perp_0$-duality preserving map, i.e., $\mu(C^{\perp_0}) = (\mu(C))^{\perp_0}$.

Proof. For all $p(x)$ and $q(x)$ in $F_p[x]/(f(x^{p^k}))$, it is easy to see that

$$\langle p(x), q(x) \rangle_0 = 0 \iff \langle \mu(p(x)), \mu(q(x)) \rangle_0 = 0.$$  \hspace{1cm} (12)

Let $p(x) \in C^{\perp_0}$. By Equation (12), we have $\langle \mu(p(x)), \mu(q(x)) \rangle_0 = 0$ for all $q(x) \in C$, i.e $\mu(p(x)) \in (\mu(C))^{\perp_0}$, which gives $\mu(C^{\perp_0}) \subseteq (\mu(C))^{\perp_0}$. Conversely, let $z \in (\mu(C))^{\perp_0}$. Then $\langle z, \mu(p(x)) \rangle_0 = 0$ for all $p(x) \in C$. Using Equation (12), we get $\langle \mu^{-1}(z), p(x) \rangle_0 = 0$ for all $p(x) \in C$, which implies $\mu^{-1}(z) \in C^{\perp_0}$, i.e., $z \in \mu(C^{\perp_0})$.

We will need the following theorem to prove some results relating the $\perp_0$-dual of the matrix product code in terms of the of the $\perp_0$-duals of their constituent codes. For a matrix $A$ we will denote its transpose as $A^\text{tr}$.

Theorem 5.14. Let $D_0, \ldots, D_{p^k-1}$ be polycyclic codes over $F_p[x]/(f(x^{p^k}))$. Then

$$([D_{p^k-1}, \ldots, D_1, D_0] \cdot A)^{\perp_0} = [D_{p^k-1}^{\perp_0}, \ldots, D_1^{\perp_0}, D_0^{\perp_0}] \cdot (A^{-1})^\text{tr}. $$

Proof. We claim that

$$[\text{Ann}(D_{p^k-1}), \ldots, \text{Ann}(D_0)] \cdot (A^{-1})^\text{tr} \subseteq \text{Ann}([D_{p^k-1}, \ldots, D_0] \cdot A).$$  \hspace{1cm} (13)

Indeed, let $z = [z_{p^k-1}, \ldots, z_0] \cdot (A^{-1})^\text{tr} \in [\text{Ann}(D_{p^k-1}), \ldots, \text{Ann}(D_0)] \cdot (A^{-1})^\text{tr}$. Note that $z$ is a row vector. If we consider the product of two row vectors $v, w$ as $v.w = v^\text{tr}w$, then for an arbitrary element $x = [x_{p^k-1}, \ldots, x_0] \cdot A \in [D_{p^k-1}, \ldots, D_0] \cdot A$ we have

$$z.x = ([z_{p^k-1}, \ldots, z_0] \cdot (A^{-1})^\text{tr}) \cdot (A^\text{tr}, [x_{p^k-1}, \ldots, x_0]^\text{tr})$$

$$= [z_{p^k-1}, \ldots, z_0] \cdot [x_{p^k-1}, \ldots, x_0]^\text{tr} = 0.$$

Using the above claim, we get

$$[D_{p^k-1}^{\perp_0}, \ldots, D_0^{\perp_0}] \cdot (A^{-1})^\text{tr} \subseteq ([D_{p^k-1}, \ldots, D_0] \cdot A)^{\perp_0}.$$
By Lemmas 5.8, 5.9 it follows
\[
\left| \left( D_{p^k-1} \right)^{1.o}, \ldots, (D_0)^{1.o} \right| \cdot (A^{-1})^{tr} = \left| \left( D_{p^k-1} \right)^{1.o}, \ldots, (D_0)^{1.o} \right|
\]
\[
= \left| \frac{\mathbb{F}_{p^l}^{|n|}}{D_{p^k-1}} \right| \cdots \left| D_0 \right|
\]
\[
= \frac{\mathbb{F}_{p^k}}{|p^k n|} \left| D_{p^k-1} \right| \cdots \left| D_0 \right|
\]
\[
= \left| \left( D_{p^k-1}, \ldots, D_0 \right) \cdot A \right|
\]
\[
= \left| \left( D_{p^k-1}, \ldots, D_0 \right) \cdot A \right|^{1.o}
\]
which gives the proof. \(\square\)

**Corollary 5.15.**

\[
\left( D_{p^k-1}, \ldots, D_1, D_0 \right) \cdot CYC(p, k) \]^{1.o} = \left[ D_{p^k-2}^{1.o}, D_0^{1.o} \right] \cdot CYC(p, k).
\]

**Proof.**

\[
\left( D_{p^k-1}, \ldots, D_1, D_0 \right) \cdot CYC(p, k) \]^{1.o} = \left[ D_{p^k-1}^{1.o}, \ldots, D_1^{1.o}, D_0^{1.o} \right] \cdot (CYC(p, k))^{-1}\]^{tr}
\[
= \left[ D_{p^k-1}^{1.o}, \ldots, D_1^{1.o}, D_0^{1.o} \right] \cdot Q
\]
\[
= \left[ D_0^{1.o}, \ldots, D_{p^k-2}^{1.o}, D_{p^k-1}^{1.o} \right] \cdot JQ
\]
\[
= \left[ D_0^{1.o}, \ldots, D_{p^k-2}^{1.o}, D_{p^k-1}^{1.o} \right] \cdot CYC(p, k).
\]

Now, combining Theorem 5.14 and Corollary 5.15 we get the following result

**Corollary 5.16.** Let C be a polycyclic code in \( \mathbb{F}_{p^r}[x]/\langle f(x^{p^k}) \rangle \) of such that \( \sigma(C) = \left[ C_{p^k-1}, C_{p^k-2}, \ldots, C_0 \right] \cdot CYC(p, k) \) as in Theorem 5.7. Then

\[
\sigma(C^1.o) = \left[ C_{p^k-2}^{1.o}, C_{p^k-1}^{1.o} \right] \cdot CYC(p, k).
\]

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