A GENERALIZATION OF ERDŐS-RÉNYI
TO $m$-FOLD SUMS AND DIFFERENCES

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Abstract. Let $m \geq 2$ be a positive integer. Given a set $E(\omega) \subseteq \mathbb{N}$ we define $r^{(m)}_N(\omega)$ to be the number of ways to represent $N \in \mathbb{Z}$ as any combination of sums and differences of $m$ distinct elements of $E(\omega)$. In this paper, we prove the existence of a "thick" set $E(\omega)$ and a positive constant $K$ such that $r^{(m)}_N(\omega) < K$ for all $N \in \mathbb{Z}$. This is a generalization of a known theorem by Erdős and Rényi. We also apply our results to harmonic analysis, where we prove the existence of certain thin sets.

1. Introduction

Given a set $S \subseteq \mathbb{N}$, we define $R^m_S(n)$ to be the number of ways to represent $n$ as a sum of $m$ elements of $S$. We say that the set $S$ is of type $B_m(g)$ if

$$R^m_S(n) \leq g$$

for all $n$. In [10], Vu gives a brief history of the topic, which we paraphrase here. In 1932, Sidon, in connection with his work in Fourier analysis, investigated power series of type $\sum_{i=1}^{\infty} z^{a_i}$, when $(\sum_{i=1}^{\infty} z^{a_i})^m$ has bounded coefficients [9]. This leads to the study of sets of type $B_m(g)$. One classical question in this area is the following [7].

"Let $S$ be a set of type $B_m(g)$. How fast can $S(n)$ grow, where $S(n)$ is the number of elements of $S$ not exceeding $n$?"

In [2], Erdős and Rényi gave an answer to this question for the case $m = 2$. This result was discussed in great detail in the monograph of Halberstam and Roth [6].

Theorem 1.1 (Erdős-Rényi). For any $\varepsilon > 0$, there exists $g = g(\varepsilon)$ and a set $S \subseteq \mathbb{N}$ of type $B_2(g)$ such that

$$S(n) > n^{\frac{1}{2} - \varepsilon}$$

for sufficiently large $n$. 

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The result is best possible up to the $\varepsilon$ term. Erdős-Rényi’s proof uses a moment computation and is technical. A simpler proof using the language of probability is presented in [6]. This theorem can be generalized from 2-fold sums to the following theorem for arbitrary $m$-fold sums, as was noted in [4] and [6] (without proof), and also by Vu who observed that it can be deduced as a consequence of a more general result proven in [10].

**Theorem 1.2.** For any positive integer $m \geq 2$ and any $\varepsilon > 0$, there exists $g = g(\varepsilon, m)$ and a set $S \subseteq \mathbb{N}$ of type $B_m(g)$ such that

$$S(n) > n^{\frac{1}{m} - \varepsilon}$$

for sufficiently large $n$.

Given a set $E(\omega) \subseteq \mathbb{N}$ we define $r^{(m)}_N(\omega)$ to be the number of ways to represent $N \in \mathbb{Z}$ as any combination of sums and differences of $m$ distinct elements of $E(\omega)$. In this paper, we prove the existence of a “thick” set $E(\omega)$ and a positive constant $K$ such that $r^{(m)}_N(\omega) < K$ for all $N \in \mathbb{Z}$. Hence, our theorem is a (partial) generalization of Theorem 1.2. The caveat here is that with $r^{(m)}_N(\omega)$ we do not allow repeated elements in the representation, for otherwise every infinite set will admit integers $N$ with $r^{(m)}_N(\omega) = \infty$. (Take $N = 0$ when $m$ is even, for instance.)

Our main result is the following:

**Theorem 1.3.** For any positive integer $m \geq 2$ and $\varepsilon > 0$, there exists $K = K(\varepsilon, m)$ and a set $E(\omega) \subseteq \mathbb{N}$ such that

$$r^{(m)}_N(\omega) < K$$

for all $N \in \mathbb{Z}$, and

$$\text{card} \left\{ E(\omega) \cap \{1, 2, \ldots, n\} \right\} \gg n^{\frac{1}{m} - \varepsilon}$$

for sufficiently large $n$.

We will also prove analogous results for $\bigoplus_0^\infty \mathbb{Z}(q)$, $\bigoplus_0^\infty \mathbb{Z}(q_n)$ for $\{q_n\}$ increasing integers, and $\mathbb{Z}(q^\infty)$ where $q$ is prime.

In section 4 we give an application of our results to harmonic analysis. We prove that for any integer $m \geq 2$ and $\varepsilon > 0$, every infinite discrete abelian group contains a set that is completely bounded $\Lambda(2m)$, but not $\Lambda(2m + \varepsilon)$.

Notation. We use $\ll$ and $\gg$ to denote Vinogradov’s well-known notation.
2. Preliminaries

Let $G$ be any one of $\mathbb{Z}, \bigoplus_0^\infty \mathbb{Z}(q)$ where $q \in \mathbb{N}$, $\mathbb{Z}(q^\infty)$ where $q$ is prime, or $\bigoplus_0^\infty \mathbb{Z}(q_n)$ where $\{q_n\}$ are strictly increasing, odd integers. (The case when not all $q_n$ are odd requires a notational adaptation that we will leave for the reader.) We define $G'$ to be $\mathbb{N}$ if $G = \mathbb{Z}$ and $G' = G \setminus \{0\}$ otherwise.

If $\psi \in \bigoplus_0^\infty \mathbb{Z}(q)$, then $\psi = (\psi_i)_{i=0}^\infty$, where each $\psi_i \in \mathbb{Z}(q)$ and all but finitely many $\psi_i = 0$. Given $\psi = (\psi_i)_{i=0}^\infty$, we call the degree of $\psi$ (or deg $\psi$ for short) the maximum $i$ such that $\psi_i \neq 0$. We also let deg $0 = -\infty$. By choosing the representative $0 \leq \psi_i < q$ for each $i$, we can identify $\psi \neq 0$ with the natural number $\psi_0 + \psi_1q + \cdots + \psi_dq^d$, where $d = \deg \psi$. This gives a one to one correspondence between $\bigoplus_0^\infty \mathbb{Z}(q)$ and $\mathbb{N} \cup \{0\}$. Notice there are $q^{d+1} - q^d$ elements of degree $d$ for each $d \geq 0$.

If $\psi \in \mathbb{Z}(q^\infty)$ and $\psi \neq 0$, then $\psi$ is the argument of a primitive $q^M$-th root of unity for a unique choice of $M$, in other words $\psi = j/q^M$ where $j \in \{1, 2, \ldots, q^M - 1\}$ and $q \nmid j$. We let deg $\psi = M - 1$ and deg $0 = -\infty$. Again, for each $d \geq 0$, there are $q^{d+1} - q^d$ elements of degree $d$. We can identify $\mathbb{Z}(q^\infty)$ with $\mathbb{N} \cup \{0\}$ by assigning $0$ to $0$, and elements of degree $d$ to $\{q^d, q^d + 1, \ldots, q^{d+1} - 1\}$ for each $d \geq 0$ in the natural way.

In the case that $G = \bigoplus_0^\infty \mathbb{Z}(q_n)$ where $q_n$ are strictly increasing, odd integers we choose representatives from $\{-(q_n - 1)/2, \ldots, (q_n - 1)/2\}$ for each $\mathbb{Z}(q_n)$. We define the degree of $\psi = (\psi_i)_{i=0}^\infty$ in the same manner as for $\bigoplus_0^\infty \mathbb{Z}(q)$. We will identify the $2q_0 \cdots q_{d-1}$ characters $\psi = (\psi_i)$ which have degree $d$ and $d$-th coordinate $\psi_d = \pm r$, $r \in \mathbb{N}$, with the integers in the interval $[(2r-1)q_0 \cdots q_{d-1}, (2r+1)q_0 \cdots q_{d-1}]$ (where, if $d = 0$, we understand $q_0 \cdots q_{d-1} = 1$). Hence, the characters of degree $d$ are assigned to integers in $\{q_0 \cdots q_{d-1}, \ldots, q_0 \cdots q_d - 1\}$.

Thus, for any of the four choices of $G$ above, we have $G' = \{\chi_n\}_{n=1}^\infty$ where $\chi_n$ is the non-zero element of $G$ uniquely associated with the integer $n$.

Given real numbers $\alpha_n$ with $0 < \alpha_n < 1$, we let $Y_n$, $n = 1, 2, \ldots$, be independent Bernoulli random variables defined on a probability space $(\Omega, M, P)$, with $P(Y_n = 1) = \alpha_n$ and $P(Y_n = 0) = 1 - \alpha_n$. For each of the groups $G$ we define random subsets

$$E(\omega) = E(\omega, G) = \{\chi_n \in G' : Y_n(\omega) = 1\}.$$ 

Throughout the paper we will be specifying a positive number $s$ and putting

$$\alpha_n = n^{-s} \text{ when } G = \mathbb{Z}, \bigoplus_0^\infty \mathbb{Z}(q) \text{ or } \mathbb{Z}(q^\infty)$$

(2.1)
or (2.2)
\[
\alpha_n = \begin{cases} 
 n^{-s} & \text{if } q_0 \cdots q_{d-1} < n \leq (2 \left\lfloor \frac{q_d}{8m} \right\rfloor + 1)q_0 \cdots q_{d-1} \text{ and } q_d > 8m \\
 0 & \text{else}
\end{cases}
\]

when \( G = \bigoplus_0^\infty \mathbb{Z}(q_n) \).

Note we have \( \alpha_n \leq n^{-s} \) for all \( n \) in this case, as well.

Let \( m \geq 2 \) be a positive integer. For \( t \in \{0, 1, \ldots, m\} \), we define
\[
\rho_{N,t}(\omega) := \text{card} \left\{ (a_1, \ldots, a_m) : \chi_{a_i} \in E(\omega), \sum_{i=1}^t \chi_{a_i} - \sum_{i=t+1}^m \chi_{a_i} = \chi_N, \right. \\
\left. a_1 < \cdots < a_t, a_{t+1} < \cdots < a_m, a_i \neq a_j \text{ for } i \neq j \right\}.
\]

For clarification, we note that when \( t = 0 \) and \( t = m \), we mean to consider the expressions \(-\sum_{i=1}^m \chi_{a_i} = \chi_N \) and \( \sum_{i=1}^m \chi_{a_i} = \chi_N \), respectively, with \( a_1 < \cdots < a_m \). We also define
\[
(2.3) \\
\rho_N(\omega) := \sum_{t=0}^m \rho_{N,t}(\omega).
\]

Lastly, we recall a fact about elementary symmetric functions that will be useful later.

**Lemma 1.** Let \( \{y_k\}_{k>0} \) be a sequence of non-negative numbers. For each \( d \in \mathbb{N} \), we write
\[
\sigma_d = \sum_{k_1<\cdots<k_d} y_{k_1}y_{k_2}\cdots y_{k_d},
\]
in other words, \( \sigma_d \) is the \( d \)-th elementary symmetric function of the \( y_k \).

Then,
\[
\sigma_d \leq \frac{\sigma_1^d}{d!}.
\]

**Proof.** See [6, p 147, Lem 13]. \( \square \)

3. \( m \)-fold sums and differences

Our main result, Theorem 1.3, follows fairly easily from Theorem 3.3. This will be proven by induction, with the base case taken care of in Cor 3.2 (following Prop 3.1). Unless we specify otherwise in the statement of the result, in this section \( G \) can be considered to be any one of \( \mathbb{Z}, \bigoplus_0^\infty \mathbb{Z}(q), \mathbb{Z}(q^\infty) \) or \( \bigoplus_0^\infty \mathbb{Z}(q_n) \).

We begin with some observations utilized in the proof of the next lemma: Suppose \( \chi_n = \sum_{i=1}^m \varepsilon_i \chi_{n_i} \), where \( \varepsilon_i = \pm 1 \), \( n_i \) are distinct and all \( \alpha_{n_i} \neq 0 \).
When \( G = \mathbb{Z} \), \( \max_{1 \leq i \leq m} |n_i| \geq |n| / m \).

When \( G = \bigoplus_0^\infty \mathbb{Z}(q), \mathbb{Z}(q^n) \) (or \( \bigoplus_0^\infty \mathbb{Z}(q_n) \)) and \( \deg \chi_n = d \), then \( q^d \leq n \leq q^{d+1} \) (or \( q_0 \cdots q_{d-1} \leq n \leq q_0 \cdots q_d \)). Since \( \deg(\chi_a \pm \chi_b) \leq \max\{\deg \chi_a, \deg \chi_b\} \), it follows that \( \max_{1 \leq i \leq m} \deg \chi_{n_i} \geq \deg \chi_n \).

When \( G = \bigoplus_0^\infty \mathbb{Z}(q) \) and \( \chi_n = (\chi_{n,j})_j=1^\infty \) with degree \( d \geq 0 \), then
\[
 n \leq (2 \lfloor \chi_{n,d} / m \rfloor + 1)q_0 \cdots q_{d-1}.
\]

If \( \max_{1 \leq i \leq m} \deg \chi_{n_i} = \deg \chi_n \), (which can occur only if \( q_d > 8m \) as we are assuming all \( \alpha_{n_i} \neq 0 \)) then the modulus of the \( d \)-th coordinate of \( \chi_{n_i} \) is at least \( \lfloor |\chi_{n,d}| / m \rfloor \) for some \( \chi_{n_i} \) of maximal degree. This is because addition in the \( d \)-th coordinate on the terms where \( \alpha_{n_i} \neq 0 \) is the same as in \( \mathbb{Z} \). Thus, (whether \( \max_{1 \leq i \leq m} \deg \chi_{n_i} > \deg \chi_n \) or \( \max_{1 \leq i \leq m} \deg \chi_{n_i} = \deg \chi_n \)) there must be some \( i \) such that
\[
 n_i \geq (2 \lfloor |\chi_{n,d}| / m \rfloor - 1)q_0 \cdots q_{d-1} \geq \frac{|\chi_{n,d}|}{m}q_0 \cdots q_{d-1}.
\]

**Lemma 2.** Given \( m \geq 2 \), we let \( s = \frac{m-1}{m} + \theta \) where \( 0 < \theta < \frac{1}{m} \), and let \( \alpha_n \) be as in (2.1) and (2.2). Fix \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \in \{\pm 1\} \). We have
\[
(3.1) \sum_{(n_1, \ldots, n_m) \in \mathbb{N}^m} \alpha_{n_1} \cdots \alpha_{n_m} \leq \begin{cases} C_m, & \text{if } n = 0, \\ C_m \frac{1}{|q|^{kn}}, & \text{if } n \neq 0, \end{cases}
\]
where \( C_m \) is a positive constant dependent only on \( G, m \) and \( s \).

**Proof.** We only give the proof for \( G = \bigoplus_0^\infty \mathbb{Z}(q) \) and \( \bigoplus_0^\infty \mathbb{Z}(q_n) \) as the other cases can be obtained by similar calculations. In both cases we use the fact that \( \sum_{n \leq A} n^{-s} \ll A^{1-s} \).

Case \( G = \bigoplus_0^\infty \mathbb{Z}(q) \): If \( n \neq 0 \), let \( t_0 \geq 0 \) be such that \( n \in [q^{t_0}, q^{t_0+1}) \). If \( n = 0 \), we let \( t_0 = 0 \). For either case, we have \( \max_{1 \leq i \leq m} \deg \chi_{n_i} \geq t_0 \) for any choice of \( (n_1, \ldots, n_m) \) in the summand of (3.1). Therefore, we can simplify and bound the sum (3.1) as follows,
\[
(3.2) \sum_{(n_1, \ldots, n_m) \in \mathbb{N}^m} \alpha_{n_1} \cdots \alpha_{n_m} \leq \sum_{t = t_0}^{\infty} \sum_{\sum_{i=1}^m \varepsilon_i n_i = \chi_n} \alpha_{n_1} \cdots \alpha_{n_m} \leq \sum_{t = t_0}^{\infty} \frac{1}{q^{ts}} \sum_{\sum_{i=1}^m \varepsilon_i n_i = \chi_n} \alpha_{n_1} \cdots \alpha_{n_{m-1}} \leq \sum_{t = t_0}^{\infty} \frac{1}{q^{ts}} \left( \sum_{n < q^{t+1}} \alpha_n \right)^{m-1}.
\]
As \( \alpha_n \leq n^{-s} \) we deduce that
\[
\sum_{(n_1, \ldots, n_m) \in \mathbb{N}^m} \alpha_{n_1} \cdots \alpha_{n_m} \leq C_m q^{-t_0 m \theta},
\]
which is equivalent to the inequality we desired to show.

Case \( G = \bigoplus_0^\infty \mathbb{Z}(q_n) \): Since \( \alpha_{n_1} \cdots \alpha_{n_m} \neq 0 \) if and only if all \( \alpha_{n_j} \neq 0 \), the comments preceding the statement of the lemma imply that if \( \deg \chi_n = d \geq 0 \), (the case \( n = 0 \) will be left for the reader) we may rewrite the sum as
\[
\sum_{(n_1, \ldots, n_m) \in \mathbb{N}^m} \alpha_{n_1} \cdots \alpha_{n_m} \sum_{i=1}^m \varepsilon_i \chi_{n_i} = \chi_n \]
\[
\leq \sum_{k=0}^\infty \left( \sum_{j \leq 2k+1} \varepsilon_i \chi_{n_i} \chi_n \right)^{-s} \left( \sum_{j \leq 2k+1} \alpha_j \right)^{-m-1} \]
\[
\ll \sum_{k=0}^\infty \left( \sum_{j \leq 2k+1} \varepsilon_i \chi_{n_i} \chi_n \right)^{-s} \left( \sum_{j \leq 2k+1} \alpha_j \right)^{-1} \ll \sum_{k=0}^\infty \left( \sum_{j \leq 2k+1} \varepsilon_i \chi_{n_i} \chi_n \right)^{-\theta m}.
\]
The final expression is comparable to \( n^{-\theta m} \). □

We will first study \( r_N^{(m)}(\omega) \) with \( m = 2 \).

**Proposition 3.1.** Let \( s = \frac{1}{2} + \theta \) where \( 0 < \theta < \frac{1}{2} \). Then, for any \( \varepsilon > 0 \), there exists \( K = K(G, s) \) such that
\[
\sum_N P\{ \omega \in \Omega : r_N^{(2)}(\omega) \geq K \} < \varepsilon.
\]

**Proof.** By definition, we have
\[
r_N^{(2)}(\omega) = \text{card}\{ (a, b) : \chi_a, \chi_b \in E(\omega), \}
\]
\[
\pm (\chi_a + \chi_b) = \chi_N, a < b, \text{ or } \chi_a - \chi_b = \chi_N, a \neq b \}.
\]
If \( \chi_{a'} + \chi_{b'} = \chi_N \) and \( \chi_{a'} + \chi_{c'} = \chi_N \), then \( b' = c' \), and similarly, if we consider subtraction. Thus, given any \( \chi_{a'} \) there are at most four ways in which \( \chi_{a'} \) could appear as one of \( \chi_a \) or \( \chi_b \) in the three equations considered in (3.3). Hence, if \( r_N^{(2)}(\omega) \geq K \), then there exist at least \( |K/4| \) pairs \( (a_i, b_i), 1 \leq i \leq |K/4| \), counted in (3.3), such that every element of the
set \{a_i,b_j\}_{1 \leq i,j \leq \lfloor K/4 \rfloor} is distinct. By the pigeon hole principle, one of the three equations considered in (3.3) must be satisfied by at least one third of these \([K/4]\) pairs. Without loss of generality, we suppose it is the equation \(\chi_a + \chi_b = \chi_N\) with \(a < b\), as the other two cases can be treated in a similar manner. Let \(L = \lfloor K/12 \rfloor\). By independence, we have

\[
P\left(\{\omega \in \Omega : r_N^{(2)}(\omega) \geq K\}\right) \leq P\left(\{\omega \in \Omega : \exists L \text{ pairs } (a_i,b_i) \text{ such that } \chi_{a_i}, \chi_{b_i} \in E(\omega), \chi_{a_i} + \chi_{b_i} = \chi_N, a_i < b_i, \text{ and } a_i, b_j \text{ all distinct}\}\right)
\]

\[
\leq \sum_{S(L)} \prod_{i=1}^{L} P\left(\{\omega \in \Omega_m : \chi_{a_i}, \chi_{b_i} \in E(\omega)\}\right),
\]

where \(S(L)\) is the collection of all \(L\) distinct pairs, \((a_i,b_i), 1 \leq i \leq L\), such that \(\chi_{a_i} + \chi_{b_i} = \chi_N\) and \(a_i < b_i\).

Since the last inequality of (3.4) gives us an \(L\)-th elementary symmetric function, we can bound it by Lemmas 1 and 2,

\[
P\left(\{\omega \in \Omega : r_N^{(2)}(\omega) \geq K\}\right) \leq \frac{1}{L!} \left(\sum_{S(1)} \alpha_a \alpha_b\right)^L \leq \begin{cases} \frac{1}{L!} C^L, & \text{if } N = 0 \\ \frac{1}{L!} C^L \frac{1}{|N|^{2\theta L}}, & \text{if } N \neq 0, \end{cases}
\]

for some positive constant \(C\). As \(\theta > 0\), we obtain

\[
\sum_{N=-\infty}^{\infty} P\left(r_N^{(2)}(\omega) \geq K\right) \leq 2 \sum_{N=2}^{\infty} \frac{1}{L!} C^L \frac{1}{|N|^{2\theta L}} + 3 \frac{1}{L!} C^L < \varepsilon,
\]

for \(L\) large enough. Notice that if \(G = \bigoplus_0^\infty \mathbb{Z}(q)\) or \(\mathbb{Z}(q^\infty)\) or \(\bigoplus_0^\infty \mathbb{Z}(q_n)\), then we only need to take the sum on the left side of (3.5) from \(N = 0\) to \(\infty\).

**Corollary 3.2.** Let \(s = \frac{1}{2} + \theta\) and \(0 < \theta < \frac{1}{2}\). Given any \(\varepsilon > 0\), there exists \(K = K(G,s)\) and \(\Omega_2 \subseteq \Omega\) such that \(P(\Omega_2) \geq 1 - \varepsilon\) and \(r_N^{(2)}(\omega) < K\) for all \(N\) and for all \(\omega \in \Omega_2\).

**Proof.** By Proposition 3.1, we can find \(K\) such that \(\sum_N P\left(r_N^{(2)}(\omega) \geq K\right) < \varepsilon\). Let \(\Omega_2 = \{\omega \in \Omega : r_N^{(2)}(\omega) < K \text{ for all } N\}\). Then, we have

\[
P(\Omega_2^c) = P\left(\{\omega \in \Omega : \exists N \text{ such that } r_N^{(2)}(\omega) \geq K\}\right) \leq \sum_N P\left(r_N^{(2)}(\omega) \geq K\right) < \varepsilon.
\]

\[\square\]
We complete the induction argument in the following proof. The argument is similar to the base case, but slightly more involved due to the larger value of \( m \). We will then prove the required density property of the subsets in Cor. \( 3.6 \).

**Theorem 3.3.** Let \( m \geq 2 \) be a positive integer. If \( s = \frac{m-1}{m} + \theta \) for \( 0 < \theta < \frac{1}{m} \), then, for all \( \varepsilon > 0 \), there exists \( K_m = K_m(G, \varepsilon, s) \) and \( \Omega_m \subseteq \Omega \) such that \( P(\Omega_m) \geq 1 - \varepsilon \) and \( r_N^{(m)}(\omega) < K_m \) for all \( N \) and for all \( \omega \in \Omega_m \).

**Proof.** We proceed by induction. Corollary \( 3.2 \) gives us the base case. Suppose the statement holds for \( m_0 < m \). Fix \( \varepsilon > 0 \). We may rewrite \( s \) as \( s = \frac{m_0-1}{m_0} + \theta' \) with \( 0 < \theta' < \frac{1}{m_0} \). Thus, by the inductive hypothesis there exist \( K_{m_0} \) and \( \Omega_{m_0} \) such that \( P(\Omega_{m_0}) \geq 1 - \frac{\varepsilon}{2(m_0+2)} \) and \( r_N^{(m_0)}(\omega) < K_{m_0} \) for all \( N \) and for all \( \omega \in \Omega_{m_0} \).

Let \( \omega \in \Omega_{m_0} \) and fix \( t \in \{0, 1, \ldots, m_0 + 1\} \) and integer \( N \). Suppose for each \( i = 1, \ldots, K \), we have

\[
\chi_{a_1^{(i)}} + \cdots + \chi_{a_t^{(i)}} = \left( \chi_{s_{t+1}^{(i)}} + \cdots + \chi_{s_{m_0+1}^{(i)}} \right) = \chi_N,
\]

with

\[
a_1^{(i)} < \cdots < a_t^{(i)} \quad \text{and} \quad a_{t+1}^{(i)} < \cdots < a_{m_0+1}^{(i)} \quad \text{and} \quad a_u^{(i)} \neq a_u^{(i')} \quad \text{if} \quad u \neq u'.
\]

Assume there exist some \( i_1, \ldots, i_r \) and \( s_1, \ldots, s_r \) such that \( a_{s_1}^{(i_1)} = a_{s_j}^{(i_j)} \) for all \( j \in \{1, \ldots, r\} \). Then, for each \( j \in \{1, \ldots, r\} \), we have

\[
\chi_{a_1^{(i_j)}} + \cdots + \chi_{a_t^{(i_j)}} = \left( \chi_{s_{t+1}^{(i)}} + \cdots + \chi_{s_{m_0+1}^{(i)}} \right) + \varepsilon_j \chi_{a_{s_j}^{(i_j)}} = \chi_N + \varepsilon_j \chi_{a_{s_j}^{(i_j)}},
\]

where \( \varepsilon_j = -1 \) if \( s_j \leq t \) and \( +1 \) if \( s_j > t \), making the left hand side of \( (3.8) \) into a combination of \( m_0 \) terms. This gives us \( r \) representations for \( \chi_N + \chi_{a_{s_1}^{(i_1)}} \) and \( \chi_N - \chi_{a_{s_1}^{(i_1)}} \) as a combination of \( m_0 \) terms. By the inductive hypothesis, we have \( r \leq 2K_{m_0} \). Therefore, it follows that each \( (m_0+1) \)-tuple, \( (a_1^{(i)}, \ldots, a_{m_0+1}^{(i)}) \), \( 1 \leq i \leq K \), has at most \( 2(m_0+1)K_{m_0} \), other \( (m_0+1) \)-tuples, which it may possibly share an entry with. Hence, by reordering if necessary, there exists a subset of \( L = \left\lfloor \frac{K}{2(m_0+1)K_{m_0}} \right\rfloor \) \((m_0+1) \)-tuples, \( (a_1^{(l)}, \ldots, a_{m_0+1}^{(l)}) \), \( 1 \leq l \leq L \), which satisfy \( (3.6) \) and \( (3.7) \), and the additional condition that every element of the set \( \{a_j^{(l)}\}_{1 \leq j \leq m_0+1, 1 \leq l \leq L} \) are distinct.
From the discussion above, and by independence, we have

\[(3.9) \quad P\{\omega \in \Omega_{m_0}: r_{N,t}^{(m_0+1)}(\omega) \geq K\} \leq P\left\{\omega \in \Omega_{m_0} : \exists L (m_0 + 1) \text{-tuples } (a_1^{(l)}, \ldots, a_{m_0+1}^{(l)}), \right. \\
1 \leq l \leq L, \text{ such that } \sum_{s=1}^{t} \chi_{a_1^{(l)}} - \sum_{s=t+1}^{m_0+1} \chi_{a_s^{(l)}} = \chi_N, \right. \\
\left. \text{all } a_j^{(l)} \text{ are distinct, and } \chi_{a_j^{(l)}} \in E(\omega)\right\} \leq \sum \prod_{L(L) \rightarrow 1} P(\{\omega \in \Omega_{m_0} : \chi_{a_j^{(l)}} \in E(\omega), 1 \leq j \leq m_0 + 1\}), \]

where \(S(L)\) is the collection of all \(L\) distinct \((m_0+1)\)-tuples, \((a_1^{(l)}, \ldots, a_{m_0+1}^{(l)})\), \(1 \leq l \leq L\), such that

\[
\sum_{s=1}^{t} \chi_{a_1^{(l)}} - \sum_{s=t+1}^{m_0+1} \chi_{a_s^{(l)}} = \chi_N, \\
\]

and \(a_i^{(l)} \neq a_j^{(m)}\) if \(i \neq j\).

Since the last inequality of (3.9) gives us an \(L\)-th elementary symmetric function, we can bound it by Lemmas 1 and 2,

\[
P(\omega \in \Omega_{m_0} : r_{N,t}^{(m_0+1)}(\omega) \geq K) \leq \frac{1}{L!} \left( \sum_{S(1)} \alpha_{a_1} \ldots \alpha_{a_{m_0+1}} \right)^L \leq \left\{ \begin{array}{ll} 
\frac{1}{L!} \frac{L!}{C_{m_0+1}} , & \text{if } N = 0, \\
\frac{1}{L!} \left( \frac{1}{C_{m_0+1} |N|^{-|m_0+1|}} \right)^L, & \text{if } N \neq 0,
\end{array} \right.
\]

for some positive constant \(C_{m_0+1}\).

For each \(t \in \{0, 1, \ldots, m_0 + 1\}\), let \(\tilde{\Omega}_t = \{\omega \in \Omega_{m_0} : r_{N,t}^{(m_0+1)}(\omega) < K\} \text{ for all } N\}.\) We can then follow the arguments of Proposition 3.1 and Corollary 3.2 and deduce the existence of \(K\) such that for any \(t \in \{0, 1, \ldots, m_0 + 1\},\)

\[(3.10) \quad P(\tilde{\Omega}_t^c) \leq P(\{\omega \in \Omega_{m_0} : \exists N \text{ such that } r_{N,t}^{(m_0+1)}(\omega) \geq K\}) + P(\Omega/\Omega_{m_0}) \leq \sum_{N} P(\omega \in \Omega_{m_0} : r_{N,t}^{(m_0+1)}(\omega) \geq K\}) + \frac{\varepsilon}{2(m_0 + 2)} \leq \frac{\varepsilon}{m_0 + 2}.
\]

If we let \(K_{m_0+1} = (m_0 + 2)K\) and \(\Omega_{m_0+1} = \bigcap_{t=0}^{m_0+1} \tilde{\Omega}_t\), the result follows by (2.3). \(\square\)
Corollary 3.4. Let $m \geq 2$ be a positive integer. If $s = \frac{m-1}{m} + \theta$ for $0 < \theta < \frac{1}{m}$, then for a.e. $\omega$

$$\sup_N r_N^{(m)}(\omega) < \infty.$$ 

Proof. This follows easily from Theorem 3.3. □

To prove Theorem 1.3 we make use of the following variant of the Strong law of large numbers (c.f. [6, p 140, Thm 11]).

Theorem 3.5. Let $\{Y_i\}$ be simple, independent random variables and $S_N = \sum_{i=1}^{N} Y_i$. Assume $\text{Exp}(Y_i) > 0$, $\lim_{N \to \infty} S_N = \infty$ and

$$\sum_i \frac{\text{Var} Y_i}{(\text{Exp}(S_i))^2} < \infty.$$ 

Then $S_N/\text{Exp}(S_N) \to 1$ as $N \to \infty$ a.e.

Corollary 3.6. Let $m \geq 2$ be a positive integer. Given any $\varepsilon > 0$, there exists $K_m = K_m(G, \varepsilon)$ and a set $E(\omega) \subseteq G'$ such that

$$r_N^{(m)}(\omega) < K_m$$

for all $N$ and there is a constant $c = c(G, m, \varepsilon)$ such that

$$(3.11) \quad \text{card} \left( \{\chi_1, \ldots, \chi_n\} \cap E(\omega) \right) \geq cn^{\frac{1}{m}-\varepsilon}$$

for all $n$ when $G = \mathbb{Z}, \bigoplus_0^\infty \mathbb{Z}(q)$ or $\mathbb{Z}(q^\infty)$, and for infinitely many $n$ when $G = \bigoplus_0^\infty \mathbb{Z}(q)$. 

Proof. Let $s = \frac{m-1}{m} + \varepsilon$. The result follows easily from Theorem 3.3 when $G = \mathbb{Z}, \bigoplus_0^\infty \mathbb{Z}(q)$ or $\mathbb{Z}(q^\infty)$.

In the case that $G = \bigoplus_0^\infty \mathbb{Z}(q_n)$, let $\{j_i\}$ be the indices such that $\alpha_{j_i} \neq 0$. Put $Z_i = Y_{j_i}$ and $S_N = \sum_{i=1}^{N} Z_i$. Then $\text{Var} Z_i \leq \text{Exp}(Y_{j_i}) \leq j_i^{-s}$ and $\text{Exp}(S_N) = \text{Exp} \left( \sum_{i=1}^{N} Y_i \right)$. If we suppose $j_N \in [q_0 \cdots q_d, q_0 \cdots q_{d+1}]$ (which implies $j_N \leq (2 \left\lfloor \frac{q_d}{s_m} \right\rfloor + 1)q_0 \cdots q_d$), then

$$(3.12) \quad \text{Exp}(S_N) = \sum_{j < q_0 \cdots q_d} j^N \alpha_j + \sum_{j = q_0 \cdots q_d} j_N^N \alpha_j.$$ 

Provided $d$ is suitably large, the first sum is at least

$$\sum_{j = q_0 \cdots q_d} j^N \alpha_j \geq \sum_{j = q_0 \cdots q_d} j^{-s} \geq (q_0 \cdots q_d/4m)^{1-s} - (q_0 \cdots q_{d-1})^{1-s} > (q_0 \cdots q_d)^{1-s}.$$
If \( q_0 \cdots q_d < j \leq j_N \), then we must also have \( \alpha_j = j^{-s} \). Hence the second sum in (3.12) is equal to \( \sum_{j=q_0 \cdots q_d}^{j_N} j^{-s} \) and that is comparable to \( j_N^{1-s} - (q_0 \cdots q_d)^{1-s} \). Putting these together, it follows that \( \text{Exp}(S_N) \gg j_N^{1-s} \).

One can also easily check that

\[
\sum_{j \leq q_0 \cdots q_d} \alpha_j \ll j_N^{1-s},
\]

so that \( \text{Exp}(S_N) \) is comparable to \( j_N^{1-s} \). Thus Theorem 3.5 can again be applied to deduce that for a.e. \( \omega \),

\[
S_N(\omega) = \text{card} \left( E(\omega) \cap \{ \chi_1, \ldots, \chi_{j_N} \} \right) \gg j_N^{1-s}.
\]

In particular, for large \( d \),

\[
\text{card} \left( E(\omega) \cap \prod_{n=0}^{d} \mathbb{Z}(q_n) \right) = \text{card} \left( E(\omega) \cap \{ \chi_1, \ldots, \chi_{2[q_d/8m]+1} q_0 \cdots q_{d-1} \} \right)
\]

\[
\gg (2 \lfloor q_d/8m \rfloor + 1) q_0 \cdots q_{d-1})^{1-s} \gg (q_0 \cdots q_d)^{1-s}.
\]

\( \Box \)

4. Application to the existence of thin sets in harmonic analysis

In this section, \( G \) will denote any discrete abelian group with compact dual group \( X \). The groups \( \mathbb{Z}, \bigoplus_0^\infty \mathbb{Z}(q), \bigoplus_0^\infty \mathbb{Z}(q_n), \) and \( \mathbb{Z}(q^\infty) \) are examples of such discrete groups. The notation \( \hat{f} \) denotes the Fourier transform of the integrable function \( f \) defined on \( X \). A subset \( E \) of \( G \) is said to be a \( \Lambda(p) \) set for \( p > 2 \) if there is a constant \( C_p \) such that \( \| f \|_p \leq C_p \| f \|_2 \) whenever \( f \) is an \( E \)-trigonometric polynomial, meaning \( \hat{f} \) is non-zero only on \( E \). As \( L^p(X) \subseteq L^q(X) \) if \( p \geq q \), it follows that if \( E \) is \( \Lambda(p) \), then \( E \) is \( \Lambda(q) \) for all \( q \leq p \).

This notion was introduced for subsets of \( \mathbb{Z} \) by Rudin \((8)\), who proved many important facts about \( \Lambda(p) \) sets. In particular, he showed that if \( E \subseteq \mathbb{Z} \) is \( \Lambda(p) \), then for all integers \( a, d \),

\[
(4.1) \quad \text{card} \left( E \cap \{ a \pm d, \ldots, a \pm Nd \} \right) \ll N^{2/p}.
\]

He also showed that if for some integer \( m \geq 2 \),

\[
(4.2) \quad \sup_{n \in \mathbb{Z}} \text{card} \left\{ (n_1, \ldots, n_m) \in E^m : n = n_1 + \cdots + n_m \right\} < \infty
\]

then \( E \) is \( \Lambda(2m) \). Rudin used these properties to construct examples of subsets of \( \mathbb{Z} \) which were \( \Lambda(2m) \) for a specified integer \( m \geq 2 \), but not \( \Lambda(2m + \varepsilon) \) for any \( \varepsilon > 0 \). Hajela in \([4]\) extended Rudin’s properties and constructions
to various other discrete abelian groups, although only achieving the existence of ‘exact’ \( \Lambda(2m) \) sets for \( m < q \) when \( G = \bigoplus_0^\infty \mathbb{Z}(q) \) for \( q \) prime, and for \( m = 2 \) when \( G = \mathbb{Z}(q^\infty) \). Later, Bourgain \cite{11} completely settled this problem by using sophisticated probabilistic methods to prove the existence of exact sets \( \Lambda(p) \) for all \( p > 2 \) and for all infinite, discrete abelian groups \( G \).

Using Pisier’s operator space complex interpolation, Harcharras in \cite{5} introduced the notion of completely bounded \( \Lambda(p) \) sets: Let \( p > 2 \). The set \( E \subseteq G \) is called completely bounded \( \Lambda(p) \) (or \( \text{cb} \Lambda(p) \) for short) if there is a constant \( C_p \) such that

\[
\|f\|_{L^p(X,S_p)} \leq C_p \max \left[ \left\| \sum_{\gamma \in E} \hat{f}(\gamma)^* \hat{f}(\gamma) \right\|_{S_p}^{1/2}, \left\| \sum_{\gamma \in E} \hat{f}(\gamma) \hat{f}(\gamma)^* \right\|_{S_p}^{1/2} \right]
\]

for all \( S_p \)-valued, \( E \)-trigonometric polynomials defined on \( X \). Here \( S_p \) denotes the Schatten \( p \)-class with \( \|T\|_{S_p} = (\text{tr} |T|^p)^{1/p} \) and

\[
\|f\|_{L^p(X,S_p)} = \left( \int_X \|f(x)\|_{S_p}^p dx \right)^{1/p}.
\]

Harcharras showed that completely bounded \( \Lambda(p) \) sets are always \( \Lambda(p) \), but not the converse. She also improved upon Rudin’s condition (4.2) by establishing that \( E \subseteq G \) is \( \text{cb} \Lambda(2m) \) for integer \( m \geq 2 \) if

\[
\sup_{\chi \in G} \left( \text{card} \left\{ (\chi_1, \ldots, \chi_m) \in E^m : \chi = \sum_{j=1}^m (-1)^j \chi_j \text{ with } \chi_j \text{ distinct} \right\} \right) < \infty.
\]

We remark that this condition was new even for \( \Lambda(2m) \) sets. In \cite{3}, Harcharras used this property to construct examples of subsets of \( \mathbb{Z} \) that were \( \text{cb} \Lambda(2m) \) but not \( \Lambda(2m + \varepsilon) \) for any \( \varepsilon > 0 \). Here we will partially improve upon this result by using (1.3) to show that every infinite, discrete abelian group \( G \) admits a set that is \( \text{cb} \Lambda(2m) \), but not \( \Lambda(2m + \varepsilon) \) for any given \( \varepsilon > 0 \). This will use the work of the previous part of the paper, as well as the following known generalization of (1.1).

**Lemma 3.** \cite{3} If \( E \subseteq G \) is a \( \Lambda(p) \) set for some \( p > 2 \), then there is a constant \( C \) such that \( \text{card} (E \cap Y) \leq CN^{2/p} \) whenever \( Y \subseteq G \) is either an arithmetic progression or a finite subgroup of cardinality \( N \).

Fix an integer \( m \geq 2 \) and \( \varepsilon > 0 \). We will first consider \( G = \mathbb{Z}, \mathbb{Z}(q^\infty), \bigoplus_0^\infty \mathbb{Z}(q) \) or \( \bigoplus_0^\infty \mathbb{Z}(q_n) \), where the \( q_n \) are strictly increasing, odd primes. We denote the elements of \( G' \) as \( \{\chi_n\}_{n=1}^\infty \), as described in the previous section.
and let \( E(\omega) \) be the random sets defined previously, with \( s = (m-1)/m + \theta \) where \( \theta > 0 \) is chosen so that \( 1 - s > 2/(2m + \varepsilon) \).

**Proposition 4.1.** For a.e. \( \omega \), \( E(\omega) \) is cb\( \Lambda(2m) \) but not \( \Lambda(2m + \varepsilon) \).

*Proof.* In the notation of (2.3), Haracharras’ condition could be expressed as

\[
E(\omega) \text{ is cb} \Lambda(2m) \quad \text{if} \quad \sup_N r_N^{(m)}(\omega) < \infty
\]

and we have already seen that \( \sup_N r_N^{(m)}(\omega) \) is finite for a.e. \( \omega \) by Cor. 3.4. Also Cor. 3.6 shows that for a.e. \( \omega \),

\[
\text{card} \left( E(\omega) \cap \{\chi_{1}, \ldots, \chi_N\} \right) \gg N^{1-s}
\]

when \( G = \mathbb{Z}, \mathbb{Z}(q^{\infty}) \) or \( \bigoplus_0^\infty \mathbb{Z}(q) \), and

\[
\text{card} \left( E(\omega) \cap \{\chi_{1}, \ldots, \chi_{q_0-q_N}\} \right) \gg (q_0 \cdots q_N)^{1-s}
\]

for sufficiently large \( N \) when \( G = \bigoplus_0^\infty \mathbb{Z}(q_n) \).

When \( G = \mathbb{Z}, \{\chi_{1}, \ldots, \chi_N\} \) is an arithmetic progression of length \( N \). When \( G = \mathbb{Z}(q^{\infty}) \) or \( \bigoplus_0^\infty \mathbb{Z}(q) \), \( \{\chi_{1}, \ldots, \chi_{q^{N-1}}\} \) is a subset of a subgroup of cardinality \( q^N \), and similarly for \( G = \bigoplus_0^\infty \mathbb{Z}(q_n) \), but with \( q^N \) replaced by \( q_0 \cdots q_N \). In all cases, the choice of \( s \) together with Lemma 3 implies that \( E(\omega) \) is not \( \Lambda(2m + \varepsilon) \) for a.e. \( \omega \). \( \square \)

**Theorem 4.2.** Let \( m \geq 2 \) be an integer and \( \varepsilon > 0 \). Every infinite discrete abelian group \( G \) contains a set \( E \) that is cb\( \Lambda(2m) \), but not \( \Lambda(2m + \varepsilon) \).

*Proof.* As observed in [3], any such group \( G \) contains a subgroup isomorphic to one of \( \mathbb{Z}, \mathbb{Z}(q^{\infty}), \bigoplus_0^\infty \mathbb{Z}(q) \) for \( q \) prime, or \( \bigoplus_0^\infty \mathbb{Z}(q_n) \) where \( q_n \) are strictly increasing, odd primes.

Observe that if \( G_0 \) is a subgroup of \( G \) and \( f \) is a \( S_p \)-valued \( G_0 \)-polynomial, then \( f \) is constant on the cosets of \( G_0^\perp \), the annihilator of \( G_0 \). The same is true for \( \|f\|_{S_{2m}} \), for any integer \( m \). It follows from this that if \( E \subseteq G_0 \) is a cb\( \Lambda(2m) \) set, then \( E \) viewed as a subset of \( G \) is also cb\( \Lambda(2m) \) and that \( E \subseteq G_0 \) is a \( \Lambda(2m + \varepsilon) \) set if and only if \( E \) viewed as a subset of \( G \) is \( \Lambda(2m + \varepsilon) \).

Hence it suffices to prove the theorem for the four subgroups listed above, and this was done in the previous proposition. \( \square \)

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