On the detectability of quantum radiation in Bose-Einstein condensates

Ralf Schützhold
Institut für Theoretische Physik, Technische Universität Dresden, D-01062 Dresden, Germany

Based on doubly detuned Raman transitions between (meta) stable atomic or molecular states and recently developed atom counting techniques, a detection scheme for sound waves in dilute Bose-Einstein condensates is proposed whose accuracy might reach down to the level of a few or even single phonons. This scheme could open up a new range of applications including the experimental observation of quantum radiation phenomena such as the Hawking effect in sonic black-hole analogues or the acoustic analogue of cosmological particle creation.

Gaseous atomic or molecular Bose-Einstein condensates are in several ways superior to other superfluids: Apart from a very good theoretical understanding via the mean-field formalism (in the dilute-gas limit), they offer unprecedented options for experimental manipulation and control. It is possible to influence the shape, density, flow profile, and coupling strength of Bose-Einstein condensates via external electromagnetic fields. Finally, these condensed gases are rather robust against the impact of the environment such that one may reach extremely low temperatures.

In view of all these advantages, the question naturally arises whether it could be possible to measure so far unobserved quantum radiation phenomena in a suitable setup. These exotic quantum effects include cosmological particle creation (due to the amplification of quantum fluctuations in an expanding/contracting universe) as well as the acoustic analogue of Hawking radiation in “dumb holes”.

For wavelengths which are much longer than the healing length $\xi$, the propagation of phonons in Bose-Einstein condensates is analogous to a scalar field in a curved space-time described by the effective metric

$$ g^{\mu\nu} = \frac{1}{\varrho_0 c_s} \left( \frac{1}{v_0} v_0 v_0 \otimes v_0 - c_s^2 \mathbf{1} \right), \quad (1) $$

which is determined by the density $\varrho_0$ and velocity $v_0$ of the background fluid. For example, assuming an effectively one-dimensional stationary flow, the point where the fluid velocity $v_0$ exceeds the local speed of sound $c_s$ corresponds to the sonic analogue of the horizon of a black hole. The corresponding Hawking temperature is determined by the velocity gradient $d v_0 / d r$

$$ T_{\text{Hawking}} = \frac{\hbar}{2 \pi \mathbf{K}_d} \left| \frac{\partial}{\partial r} (v_0 - c_s) \right|, \quad (2) $$

i.e., the characteristic length scale $\lambda$ over which the flow changes. Since this length scale should be large compared to the healing length $\xi$ (typically of order micrometer) for the curved space-time analogy to apply, a speed of sound of order mm/s leads to an upper bound for the typical energy of the Hawking phonons of order $10^{-13}$ eV corresponding to a temperature on the nano-Kelvin level.

Moreover, since the fluid velocity equals the sound speed at the acoustic horizon, only a limited number of these low-energy phonons will be created by the Hawking effect – unless one has a very large reservoir for the condensate flow: Since the Hawking radiation is thermal, the typical distance between two emitted Hawking phonons is given by their characteristic wavelength $\lambda$ and hence it is much larger than the healing length $\xi$. In addition, Bose-Einstein condensates are formed by atoms (or molecules) whose inter-particle distance $a_d$ is far bigger than the $s$-wave scattering length $a_s$ (dilute-gas limit). As a result, a healing length $\xi \propto a_d \sqrt{a_d/a_s}$ contains many atoms $\xi \gg a_d$, i.e., we have a hierarchy of length scales $\lambda \gg \xi \gg a_d \gg a_s$. Consequently, the number of Hawking phonons is extremely small compared to the number of atoms in the condensate $a_d/\lambda \ll 1$.

Similar arguments apply to the analogue of cosmological particle creation, which require a non-stationary setup. Considering the effective metric in Eq. (1), there are basically two possibilities for simulating the cosmic expansion in Bose-Einstein condensates: an expansion of the condensate or a temporal variation of the speed of sound (which can be achieved via varying $a_s$ by means of a Feshbach resonance, for example). For simplicity, we shall focus on the second possibility in the following, but the general ideas apply to both scenarios. The typical wavelength $\lambda$ of the created phonons is determined by the rate of change $\lambda = \mathcal{O}(c_s^2/\dot{c}_s)$ of $c_s$ and should again be large compared to the healing length $\xi$ for the curved space-time analogy to apply. In the absence of amplification mechanisms such as resonances, the number of created phonon per wavelength is again of order one.

For a small number of phonons with an energy of order $10^{-13}$ eV, the usual detection mechanisms for sound via mechanical motion are extremely difficult to apply since the kinetic energy of a single atom with a velocity of order mm/s already yields this amount. Usually, these measurements involve many atoms and thus many phonons (limit of classical waves). For example, it was possible to excite phonon modes via light scattering and to map out the dispersion relation etc. An indirect observation of the phonon number was achieved in ultra-
sensitive temperature measurements \[\hat{O}\], which reached very low energies.

Fortunately, a non-mechanical detection mechanism may circumvent these obstacles. For example, the ion-trap quantum computer in \[\boxed{10}\] is based on optically induced transitions which require the simultaneous absorption of a phonon in a given mode (due to the detuning of the Laser). Using the occurrence of the transition and the photon emitted during the decay back to the ground state as an indicator for the existence of the phonon yields an energy amplification over many orders of magnitude; and the detection of single photons in the optical range is difficult but feasible (in principle). A further amplification is possible if the phonon-assisted transition mediates between (meta) stable atomic states which can be separated or addressed individually: The detection of a small number of atoms can be achieved via fluorescence measurements involving many photons \[\boxed{11}\],

\[\text{FIG. 1: Sketch (not to scale) of the three-level (Λ) system and the doubly detuned Raman transitions denoted by } \Omega_{1,2}.\]

In the following, a scheme for the transformation of low-energy phonons in a given mode into an equal number of atoms in a different atomic state with controlled energy and momentum based on doubly detuned Raman transitions is presented. Let us consider atoms which can be described by a three-level (Λ) system consisting of two (meta) stable states \(\Psi_1\) and \(\Psi_2\) together with a third excited level \(\Psi_3\) with the energies \(\omega_1 < \omega_2 < \omega_3\). This three-level system is illuminated by two optical Laser beams which consist of many photons and can therefore be treated as rapidly oscillating classical fields described by the effective Rabi frequencies \(\Omega_1(t)\) and \(\Omega_2(t)\). Within the rotating wave and dipole approximation, the Lagrangian reads (\(\hbar = 1\))

\[
L = i \dot{\psi}_1 \hat{\Psi}_1 + i \dot{\psi}_2 \hat{\Psi}_2 + i \dot{\psi}_3 \hat{\Psi}_3 - \omega_1 |\psi_1|^2 - |\omega_2| |\psi_2|^2 - \omega_3 |\psi_3|^2 + [\Omega_1(t) \hat{\Psi}_1 \Psi_3 + \Omega_2(t) \Psi_2 \hat{\Psi}_3 + \text{H.c.}] .
\] (3)

The frequencies of the two doubly detuned Laser beams are chosen according to (see Fig. 1)

\[
\begin{align*}
\Omega_1(t) &= \Omega_1 \exp\{i(\omega_2 - \omega_1 + \Delta)t\} , \\
\Omega_2(t) &= \Omega_2 \exp\{i(\omega_3 - \omega_2 + \Delta + \delta)t\} ,
\end{align*}
\] (4)

with a large detuning \(\Delta\) and a small detuning \(\delta\) (which will later determine the phonon energy). Introducing the slowly varying variables \(\psi_1\), \(\psi_2\), and \(\psi_3\) via \(\Psi_1(t) = \psi_1(t) \exp\{-i\omega_1 t\}\), \(\Psi_2(t) = \psi_2(t) \exp\{-i\omega_2 t\}\), and \(\psi_3(t) \exp\{-i(\omega_3 + \Delta)t\}\), we may solve the equation for the upper level \(\psi_3\) approximately for large detuning \(\Delta\) via \(\psi_3 = -(\Omega_1^2 \psi_1 + \Omega_2^* e^{-i\delta t} \psi_2)/\Delta + O(1/\Delta^2)\). Insertion into Eq. (3) yields the Lagrangian for the remaining two levels in the adiabatic approximation \(|\psi_3| \ll \Delta\)

\[
L_{\text{eff}} = \frac{i}{\hbar} \dot{\psi}_1 + i \dot{\psi}_2 \psi_3 - |\Omega_1^2|/\Delta |\psi_1|^2 - |\Omega_2^*|/\Delta |\psi_2|^2 - \left[\frac{\Omega_1 \Omega_2^*}{\Delta} e^{-i\delta t} \psi_1 \psi_2 + \text{H.c.}\right] .
\] (5)

Assuming \(|\Omega_1| = |\Omega_2| = \Omega\), both levels acquire the same additional shift \(\Omega^2/\Delta\); otherwise we would obtain an effective detuning \(\delta \to \delta'\) shifted by \((|\Omega_1^2| - |\Omega_2^2|)/\Delta\).

An ideal quantum gas containing many of these atoms with mass \(m\) can be described by the many-particle field operator \(\hat{\Sigma}_r\) with the dynamics (Heisenberg picture)

\[
i \frac{\partial}{\partial t} \hat{\Sigma}_r = \left(\frac{\nabla^2}{2m} + V_r\right) \hat{\Sigma}_r + \Xi_{rs} \hat{\Sigma}_s ,
\] (6)

where \(r, s = 1, 2\) are labels for the remaining two levels and \(V_r\) the corresponding potentials. The anti-hermitian space-time dependent transition amplitude \(\Xi_{12}(t, x) = \exp\{-i\delta t + i|\kappa| \cdot r\} \Omega^2/\Delta\) represents the mode-coupling in Eq. (5), where \(\kappa\) arises from a small angle between the Raman beams and the resulting wavenumber mismatch \(\kappa = k_1 - k_2\).

An expansion into single-particle energy-eigenstates

\[
\hat{\Sigma}_r(t, r) = \sum_{\alpha} \hat{a}_{\alpha r}(t) f_{\alpha r}(r) \exp\{-iE_{\alpha r} t\} ,
\] (7)

diagonalizes Eq. (6) apart from the transitions, which are (in the rotating wave approximation) only relevant for \(E_{1,\alpha} - E_{2,\beta} = \delta\) (energy conservation) and if the spatial matrix element \(f_{1\alpha} \Xi_{12} f_{2\beta}\) is large enough. For nearly homogeneous potentials \(V_r \approx \text{const}\), the eigenfunctions are plane waves \(\alpha \to k\) with \(E_{r, k} = k^2/2m\) + \(V_r\) and the latter condition represents momentum conservation \(\kappa = k_1 - k_2\). Hence, for a given frequency and wavenumber mismatch of the Lasers \((\delta, \kappa)\), these energy and momentum conservation conditions determine \(k_1\) and \(k_2\) up to a contribution perpendicular to \(\kappa\). For effectively one-dimensional condensates, therefore, we can address single modes \(k_1 \to k_1 e_x\) and \(k_2 \to k_2 e_x\) by adjusting the Lasers.

Now let us consider the following gedanken experiment: Initially all atoms are in the state \(r = 1\) and form a nearly homogeneous and (quasi) one-dimensional condensate, which is not in its ground state but contains a single phonon with a given wavenumber \(k_p \to k_p e_x\). In contrast to Eq. (6), this requires a non-vanishing coupling \(g\). However, if we switch off this interaction \(g\) adiabatically (e.g., via a Feshbach resonance), the system stays in this first excited state and finally contains a single atom with the momentum \(k_p\) of the original
phonon. After applying a Raman $\pi$-pulse (with the duration $T = \pi \Delta / \Omega^2$) adapted to this wavenumber, e.g., $\kappa = k_p$ and $\delta = k_p^2 / (2m) + V_1 - V_2$, exactly this single atom will be transferred to the other state $r = 2$, while all the condensate atoms are not affected (assuming that rotating wave approximation applies).

If we can separate the two species $r = 1$ and $r = 2$ or address them individually, the number of atoms in the state $r = 2$ can be counted via fluorescence measurements \cite{11, 12} and yields (in the ideal case) the number of phonons in a given mode $k$ present initially, i.e., one. For example, a beam with a frequency just between the phonon frequencies $\omega$ and $\omega + \delta$ (for the two species otherwise we would only have the energy-momentum balance (14) is a bit more complicated than in the previous case without interactions, but exhibits a similar direction-degeneracy, which can again be eliminated by considering effectively one-dimensional condensates. In the phonon regime (for $k \gg \kappa$) or an even higher energy $\omega_k \geq \delta$.

If there are $n$ phonons to annihilate ($\hat{a}_k$), $n$ atoms can be transferred to the state $r = 2$ state ($\hat{\zeta}_{k+\kappa}$) such that the final number of these transferred atoms measures the initial number of phonons. Vice versa, if the component 2 is not empty initially, the Raman beams transfer atoms ($\hat{\zeta}_{k+\kappa}$) from the state 2 to the level 1 with simultaneous emission of an equal number of phonons ($\hat{a}_k$).

The energy-momentum balance (14) is a bit more complicated than in the previous case without interactions, but exhibits a similar direction-degeneracy, which can again be eliminated by considering effectively one-dimensional condensates. In the phonon limit ($\lambda \gg \xi$ and $\delta \ll \mu$), we obtain a unique solution for the phonon energy $\omega_k \approx \delta$ which allows us to address single modes with suitably tuned Lasers. If we choose $\delta$ and $\kappa$ to lie on the phonon dispersion curve $\delta = \omega(\kappa)$, we annihilate one phonon with energy $\delta$ and momentum $\kappa$ and create one particle in the component $r = 2$ in the ground state.
With sufficiently long pulses leading to a good energy resolution, it should be possible to “see” the discrete nature of the phonon spectrum, i.e., to address single (or a few) phonon modes. In order to annihilate all phonons in the \( r = 1 \) condensate with a given energy/momentum and to transfer the same number of atoms to the \( r = 2 \) component, we apply an effective Raman \( \pi \)-pulse with the duration \[ T = \frac{\pi \Delta}{\Omega^2} \sqrt{\frac{2 \omega k}{k^2} \left( 1 + \frac{\omega k}{k^2} \right)^{-1}} \approx \frac{\pi \Delta}{\Omega^2} \sqrt{\frac{\delta}{\mu}}, \] where the \( \approx \) sign applies to the phonon limit.

Of course, the approximations used in the presented derivations must be checked for a potentially realistic set of experimental parameters. Let us assume a speed of sound of a few millimeters per second and a heating length around one micrometer. In this case, the wavelength \( \lambda \) of the phonons to be detected would typically be several micrometers \( 1/\kappa = \mathcal{O}(10 \mu \text{m}) \) and their frequency a few hundred Hertz \( \delta = \mathcal{O}(100 \text{Hz}) \).

Using Lasers in the optical range \( \mathcal{O}(10^{15} \text{Hz}) \), the large detuning \( \Delta \) depends on the atomic level structure and would be a little bit below this value, say \( \Delta = \mathcal{O}(10^{13} - 10^{14} \text{Hz}) \). With quite moderate Rabi frequencies \( \Omega = \mathcal{O}(10^4 - 10^7 \text{Hz}) \), we can achieve an effective Raman transition rate \( \sqrt{\mu/\delta \Omega^2/\Delta} \) of a few tens of Hertz. Consequently, the duration of the effective Raman \( \pi \)-pulse in Eq. (16) would be of the order of hundred milliseconds \( T = \mathcal{O}(100 \text{ms}) \) leading to an energy resolution of circa ten Hertz. In view of the aforementioned parameters, the assumptions and approximations (e.g., the adiabaticity \( \Delta \gg \Omega \)) used in the derivation are reasonably well justified. The major constraint is given by the energy resolution of the effective Raman \( \pi \)-pulse peaked around \( \delta = \mathcal{O}(100 \text{Hz}) \pm \mathcal{O}(10 \text{Hz}) \). Apart from a few excitations (i.e., phonons), the beams illuminate many atoms in the ground state (zero energy) and one has to make sure that the probability of transferring an atom from the ground state of the condensate in component \( r = 1 \) into the state \( r = 2 \) is small enough. Thus the negative-frequency tail of the Fourier transform of the pulse (which is peaked around \( \delta \) in frequency space) must be suppressed accordingly.

With the ability of measuring a few low-energy phonons, it might become possible to observe some of the exotic quantum effects mentioned in the Introduction. The analogue of cosmological particle production is probably easier to realize experimentally than Hawking radiation since it can be done with a condensate at rest and a practically unlimited measurement time (after varying \( c_s \) via a Feshbach resonance, for example). The same advantage applies to a small wiggling stirrer in the condensate, which would act as a point-like non-inertial scatterer and generate the analogue of moving-mirror radiation [1], which can be interpreted as a signature of the Unruh effect. In contrast, the detection of the Hawking radiation requires either a flowing condensate or a motion of the horizon via a space-time dependent sound velocity \( c_s(t, x) \), cf. [14]. Apart from measuring this striking effect, these experiments may also shed light onto the trans-Planckian problem, i.e., impact of the short-range physics on the long-wavelength Hawking radiation: Even though the Hawking effect seems to be quite robust against modifications of the dispersion relation at short wavelengths (such as the Bogoliubov dispersion in Bose-Einstein condensates) only very little is known about the impact of interactions, see, e.g., [12].

Acknowledgments The idea to the presented detection scheme was developed together with Mark Raizen [12] and emerged during the workshop “Low dimensional Systems in Quantum Optics” in September 2005 at the Centro Internacional de Ciencias in Cuernavaca (Mexico), which was supported by the Alexander von Humboldt foundation. This work was supported by the Emmy-Noether Programme of the German Research Foundation (DFG) under grant No. SCHU 1557/1-2. Further support by the ESF-COSLAB and the EU-ULTI programmes is also gratefully acknowledged.