Satellites in low Earth orbits must accurately conserve their orbital eccentricity, since a decrease in perigee of only 5–10% would cause them to crash. However, these satellites are subject to gravitational perturbations from the Earth’s multipole moments, the Moon, and the Sun that are not spherically symmetric and hence do not conserve angular momentum, especially over the tens of thousands of orbits made by a typical satellite. Why then do satellites not crash? We describe a vector-based analysis of the long-term behavior of satellite orbits and apply this to several toy systems containing a single non-Keplerian perturbing potential. If only the quadrupole or $J_2$ potential from the Earth’s equatorial bulge is present, all near-circular orbits are stable. If only the octupole or $J_3$ potential is present, all such orbits are unstable. If only the lunar or solar potential is present, all near-circular orbits with inclinations to the ecliptic exceeding 39° are unstable. We describe the behavior of satellites in the simultaneous presence of all of these perturbations and show that almost all low Earth orbits are stable because of an accidental property of the dominant quadrupole potential. We also relate these results to the phenomenon of Lidov–Kozai oscillations.

I. INTRODUCTION

The study of the motion of a satellite around its host planet is one of the oldest problems in dynamics. This subject was re-invigorated at the dawn of the Space Age in the late 1950s with a new focus on artificial satellites in low Earth orbits.\[1,2\]

Satellites orbiting a spherically symmetric Earth would have Keplerian orbits, described by their semi-major axis $a$, eccentricity $e$, inclination $\beta$, and other orbital elements. These orbits conserve the energy $E$ and angular momentum $L$ per unit mass, which for a test particle are given by

$$E = -\frac{GM_\oplus}{2a}, \quad |L| = \left\{GM_\oplus a(1-e^2)\right\}^{1/2}$$

where $G$ is the gravitational constant and $M_\oplus$ is the mass of the Earth. The perigee of the satellite orbit is $a(1-e)$, so if the energy and angular momentum are conserved the perigee is conserved as well. This is fortunate, since the typical perigee of satellites in low Earth orbits is only $\sim 800$ km above the Earth. Orbits with perigees less than 200–300 km are short-lived because of atmospheric drag, so a decrease in perigee of less than 10% would cause the satellite orbit to decay quickly (we call this a “crash”).

However, the Earth is not spherically symmetric. A better approximation is an oblate spheroid, and in this potential the energy is conserved but not the total angular momentum: if the polar axis of the Earth is the $z$-axis, then only the $z$-component of the angular momentum is conserved,

$$L_z = \left\{GM_\oplus a(1-e^2)\right\}^{1/2} \cos \beta.$$  \hspace{1cm} (2)

In this case the perigee is not necessarily conserved unless the orbit lies in the Earth’s equator. Why then do satellites in low Earth orbit not crash?

A similar question arises if we treat the Earth as spherical and consider the gravitational tides from the Moon and Sun, which are not spherically symmetric and also do not conserve angular momentum.

Of course, the answer to why satellites do not crash is known, in the sense that aerospace engineers design orbits that are stable over the expected lifetime of the satellite. Nevertheless the calculations in textbooks on astrodynamics and celestial mechanics do not provide a physical explanation for this stability. In asking our colleagues we have received a variety of answers, such as “there are only periodic oscillations in the perigee and these are too small to be important” or “the orbits are unstable but on timescales much longer than the satellite lifetime”. We shall show that neither of these answers captures the most relevant physics.

The primary goal of this paper is to understand physically the nature of the strongest long-term perturbations to satellite orbits caused by each non-spherical component that contributes to the gravitational field around a planet. We will focus on understanding how these components affect the stability of an initially circular orbit. We have two secondary goals: first, we derive our results using a geometric formalism that is simpler and more powerful for this purpose than the usual algebraic methods used in celestial mechanics; second, we show how our results are related to Lidov–Kozai oscillations, a remarkable phenomenon in orbital mechanics that is relevant to a wide variety of astrophysical systems.
II. SECULAR DYNAMICS OF EARTH SATELLITES

A. The geometry of Keplerian orbits

We use a vectorial approach to calculate the behavior of the satellite orbit. First we define a coordinate system whose equatorial plane coincides with the satellite’s orbital plane. The orthogonal unit vectors of this system are \( \hat{n} \) in the direction of the satellite’s angular-momentum vector, \( \hat{u} \) pointing towards the perigee, and \( \hat{v} = \hat{n} \times \hat{u} \). We introduce polar coordinates \((r, \phi)\) in the orbital or \( \hat{u} - \hat{v} \) plane, with \( \phi = 0 \) coinciding with the \( \hat{u} \) axis; then the orbit of a test particle with semi-major axis \( a \) and eccentricity \( e \) is given by

\[
r = \frac{a(1 - e^2)}{1 + e \cos \phi}, \quad r = r(\cos \phi \, \hat{u} + \sin \phi \, \hat{v}). \tag{3}
\]

The orbital period \( P = 2\pi a^{3/2}/(GM_\oplus)^{1/2} \) and the angular momentum per unit mass \( L = r^2 d\phi/dt = [(GM_\oplus a)(1 - e^2)]^{1/2} \). Using these results the average of some function \( \Phi(r) \) over one orbit is

\[
(\Phi(r)) = \frac{1}{P} \int_0^P dt \Phi[r(t)] = \frac{(GM_\oplus)^{1/2}}{2\pi a^{3/2}} \int_0^{2\pi} \cos \phi \, dt \Phi(r, \phi)
= \frac{(GM_\oplus)^{1/2}}{2\pi a^{3/2} L} \int_0^{2\pi} d\phi \, \phi^2 \Phi(r, \phi)
= \frac{(1 - e^2)^{3/2}}{2\pi} \int_0^{2\pi} \frac{d\phi}{(1 + e \cos \phi)^2} \Phi(r, \phi). \tag{4}
\]

The unit vectors \( \hat{n} \) and \( \hat{u} \) are undefined for radial orbits \((e = 1)\) and circular orbits \((e = 0)\) respectively. Therefore it is useful to introduce

\[
\mathbf{j} = (1 - e^2)^{1/2} \hat{n}, \quad \mathbf{e} = e \hat{u} \tag{5}
\]

which are well-defined for all bound orbits. Apart from a factor \((GM_\oplus a)^{1/2}\), the first of these is the angular-momentum vector per unit mass, and the second is the eccentricity or Runge-Lenz vector. In terms of the position and momentum vectors,

\[
j = \frac{1}{(GM_\oplus a)^{1/2}} \mathbf{r} \times \mathbf{p}, \quad \mathbf{e} = \frac{1}{GM_\oplus} \mathbf{p} \times (\mathbf{r} \times \mathbf{p}) - \frac{\mathbf{r}}{r}. \tag{6}
\]

We assume the satellite has unit mass so there is no distinction between momentum and velocity.

B. The multipole expansion of Earth’s gravitational potential

The Earth’s gravitational field can be represented as a multipole expansion, of which the first several thousand terms have been measured. Since our goal is physical understanding, for simplicity we shall restrict ourselves to the axisymmetric components of this field (the largest non-axisymmetric components are comparable to the \( J_3 \) and \( J_4 \) axisymmetric components in Eq. 7). In this case the potential can be written

\[
\Phi_\oplus(r, \theta) = -\frac{GM_\oplus}{r} \left[ 1 - \sum_{n=2}^{\infty} J_n \left( \frac{R_\oplus}{r} \right)^n P_n(\cos \theta) \right]. \tag{7}
\]

Here \( r \) and \( \theta \) are spherical polar coordinates relative to the Earth’s spin axis \( \hat{\mathbf{n}}_\oplus \), \( R_\oplus \) is the Earth radius, and \( P_n \) is a Legendre polynomial. The term with \( n = 1 \) is zero if the origin of the coordinates coincides with the Earth’s center of mass, as we shall assume. The values of these parameters, and of the first few multipole moments \( J_n \), are given in Table I. The first non-zero moment, \( J_2 \), is several orders of magnitude larger than the others because of the equatorial bulge of the Earth caused by its spin.

We shall restrict our attention to the first three non-zero moments, \( J_2 \), \( J_3 \), and \( J_4 \), since these are sufficient to illustrate how a non-spherical potential affects satellite orbits. The methods we describe are straightforward to extend to higher multipoles.

Since we are interested in small effects that accumulate over many orbital times, we can average the gravitational potential over the Keplerian orbit of a satellite with semi-major axis \( a \) and eccentricity \( e \). We write the potential associated with \( J_n \) in Eq. 7 as \( \Phi_n \). Then, for example,

\[
\Phi_2(r, \theta) = \frac{GM_\oplus J_2 R_\oplus^2}{r^3} P_2(\cos \theta)
= \frac{GM_\oplus J_2 R_\oplus^2}{2} \left[ 3 \left( \frac{r \cdot \hat{n}_\oplus}{r^3} \right)^2 - \frac{1}{r^3} \right]. \tag{8}
\]

| constant               | value                      |
|------------------------|----------------------------|
| Earth mass             | \( GM_\oplus \times 3.9860 \times 10^{14} \text{ m}^3\text{ s}^{-2} \) |
| Earth radius           | \( R_\oplus \times 6.3781 \times 10^6 \text{ m} \) |
| Multipole moments      | \( J_2 \times +1.0826 \times 10^{-3} \) |
|                        | \( J_3 \times -2.5327 \times 10^{-6} \) |
|                        | \( J_4 \times -1.6196 \times 10^{-6} \) |
|                        | \( J_5 \times -2.2730 \times 10^{-7} \) |
|                        | \( J_6 \times +5.4068 \times 10^{-7} \) |
|                        | \( J_7 \times -3.5236 \times 10^{-7} \) |
|                        | \( J_8 \times -2.0480 \times 10^{-7} \) |
| solar mass             | \( GM_\oplus \times 1.327 \times 10^{20} \text{ m}^3\text{ s}^{-2} \) |
| Earth-Sun semi-major axis | \( a_\odot \times 1.496 \times 10^{11} \text{ m} \) |
| lunar mass             | \( GM_\oplus \times 4.903 \times 10^{12} \text{ m}^3\text{ s}^{-2} \) |
| Earth-Moon semi-major axis | \( a_\oplus \times 3.844 \times 10^8 \text{ m} \) |
Using Eq. (3), the orbit average of $\Phi_2$ is
\begin{equation}
\langle \Phi_2 \rangle = \frac{GM_\odot J_3 R_\odot^3}{2} \left[ 3(\hat{u} \cdot \hat{n}_\oplus)^2 \left\langle \frac{\cos^2 \phi}{r^3} \right\rangle + 6(\hat{u} \cdot \hat{n}_\oplus) \right],
\end{equation}
\begin{equation}
\times (\hat{v} \cdot \hat{n}_\oplus) \left\langle \frac{\cos \phi \sin \phi}{r^3} \right\rangle + 3(\hat{v} \cdot \hat{n}_\oplus)^2 \left\langle \frac{\sin^2 \phi}{r^3} \right\rangle - \left\langle \frac{1}{r^3} \right\rangle.
\end{equation}
(9)

Using Eq. (4) we have
\begin{equation}
\left\langle \frac{\cos^2 \phi}{r^3} \right\rangle = \left\langle \frac{\sin^2 \phi}{r^3} \right\rangle = \frac{1}{2a^3(1 - e^2)^{3/2}},
\end{equation}
\begin{equation}
\left\langle \frac{\cos \phi \sin \phi}{r^3} \right\rangle = 0, \quad \left\langle \frac{1}{r^3} \right\rangle = \frac{1}{a^3(1 - e^2)^{3/2}}.
\end{equation}
(10)

Finally, since $(\hat{u}, \hat{v}, \hat{n})$ is an orthonormal triad, $(\hat{u} \cdot \hat{n}_\oplus)^2 + (\hat{v} \cdot \hat{n}_\oplus)^2 + (\hat{n} \cdot \hat{n}_\oplus)^2 = 1$, and this can be used to eliminate $(\hat{u} \cdot \hat{n}_\oplus)^2$ and $(\hat{v} \cdot \hat{n}_\oplus)^2$. Thus
\begin{equation}
\langle \Phi_2 \rangle = \frac{GM_\odot J_3 R_\odot^3}{4a^3(1 - e^2)^{3/2}} \left[ 1 - 3(\hat{n} \cdot \hat{n}_\oplus)^2 \right].
\end{equation}
(11)

In terms of the vectors $j$ and $e$ (Eq. 5)
\begin{equation}
\langle \Phi_2 \rangle = \frac{GM_\odot J_3 R_\odot^3}{8a^3(1 - e^2)^{3/2}} (e \cdot \hat{n}_\oplus) \left[ 1 - e^2 - 3(j \cdot \hat{n}_\oplus)^2 \right].
\end{equation}
(12)

Similarly,
\begin{equation}
\langle \Phi_3 \rangle = \frac{3GM_\odot J_3 R_\odot^4}{8a^4(1 - e^2)^{7/2}} (e \cdot \hat{n}_\oplus) \left[ 1 - e^2 - 5(j \cdot \hat{n}_\oplus)^2 \right];
\end{equation}
\begin{equation}
\langle \Phi_4 \rangle = \frac{3GM_\odot J_3 R_\odot^4}{128a^6(1 - e^2)^{11/2}} \left\{ \begin{array}{l}
6(e^2 - e^2) \left[ 1 - e^2 \right]^2 \\
-10(6 + e^2)(1 - e^2)[j \cdot \hat{n}_\oplus]^2 + 35(2 + e^2)(j \cdot \hat{n}_\oplus)^4 \\
+20(e \cdot \hat{n}_\oplus)^2(1 - e^2) \left[ 1 - e^2 - 7(j \cdot \hat{n}_\oplus)^2 \right]
\end{array} \right\}.
\end{equation}
(13)

Since $j^2 + e^2 = 1$ any terms involving $e^2$ can be replaced by $1 - j^2$.

C. The gravitational potential from the Moon

The effects on satellite orbits of the tides from the Sun and the Moon are qualitatively similar, and since the lunar tide is stronger by a factor of about 2.2 we consider only the Moon.

The gravitational potential from the Moon is $\Phi_3 (r, r_\odot) = -GM_\odot / |r - r_\odot|$. Since $|r| \ll |r_\odot|$ (by a factor of about 60), we expand this potential as a Taylor series,
\begin{equation}
\Phi_3 (r, r_\odot) = -\frac{GM_\odot}{r_\odot} \left[ 1 + \frac{r \cdot r_\odot}{r_\odot^2} - \frac{r^2}{2r_\odot^2} + \frac{3(r \cdot r_\odot)^2}{2r_\odot^4} \right] + \text{higher order terms}.
\end{equation}
(14)

by the fictitious potential due to the acceleration of the Earth’s center of mass by the Moon. Then averaging over the satellite orbit using equation (3) we have
\begin{equation}
\langle \Phi_3 \rangle = \frac{GM_\odot}{2r_\odot^3} \left[ \left\langle r^2 \right\rangle - \frac{3(r \cdot r_\odot)^2}{r_\odot^2} \left\langle r^2 \cos^2 \phi \right\rangle - \frac{6(\hat{u} \cdot r_\odot)(\hat{v} \cdot r_\odot)}{r_\odot^2} \left\langle r^2 \cos \phi \sin \phi \right\rangle - \frac{3(\hat{v} \cdot r_\odot)^2}{r_\odot^2} \left\langle r^2 \sin^2 \phi \right\rangle \right] .
\end{equation}
(15)

Using Eq. (4) we have
\begin{equation}
\left\langle r^2 \cos^2 \phi \right\rangle = \frac{a^2}{2} (1 + 4e^2), \quad \left\langle r^2 \cos \phi \sin \phi \right\rangle = 0,
\end{equation}
\begin{equation}
\left\langle r^2 \sin^2 \phi \right\rangle = \frac{a^2}{2} (1 - e^2), \quad \left\langle r^2 \right\rangle = \frac{a^2}{2} (2 + 3e^2).
\end{equation}
(16)

Finally, we eliminate $(\hat{v} \cdot r_\odot)^2$ using the relation $r_\odot^2 = (\hat{u} \cdot r_\odot)^2 + (\hat{v} \cdot r_\odot)^2 + (\hat{n} \cdot r_\odot)^2$ and replace $\hat{u}$ and $\hat{n}$ with $j$ and $e$ using equation (5):
\begin{equation}
\langle \Phi_3 \rangle = \frac{GM_\odot a^2}{4r_\odot^3} \left[ -1 + 6e^2 + \frac{3(j \cdot r_\odot)^2}{r_\odot^2} - \frac{15(e \cdot r_\odot)^2}{r_\odot^2} \right].
\end{equation}
(17)

For simplicity we assume that the Moon’s orbit is circular, with semi-major axis $a_\odot$ and a fixed normal $\hat{n}_\oplus$. Averaging any fixed vector $c$ over the Moon’s orbit, we have $\langle (c \cdot r_\odot)^2 \rangle = \frac{1}{2a_\odot^2} [c^2 - (c \cdot \hat{n}_\oplus)^2]$. Thus
\begin{equation}
\langle \langle \Phi_3 \rangle \rangle = \frac{GM_\odot a^2}{8a_\odot^3} \left[ 15(e \cdot \hat{n}_\oplus)^2 - 3(j \cdot \hat{n}_\oplus)^2 + 1 - 6e^2 \right].
\end{equation}
(18)

D. Rotating reference frame

In some cases it will be useful to work in a reference frame that rotates about the Earth’s spin axis with angular speed $\omega$. In this frame the Lagrangian for a particle of unit mass in a potential $\Phi(r)$ is $L = \frac{1}{2} (\dot{\mathbf{r}} + \omega \hat{n}_\oplus \times \dot{\mathbf{r}})^2 - \Phi(\mathbf{r})$. The canonical momentum is $\mathbf{p} = \partial L / \partial \dot{\mathbf{r}} = \dot{\mathbf{r}} + \omega \hat{n}_\oplus \times \dot{\mathbf{r}}$. The Hamiltonian is $H = \mathbf{p} \cdot \dot{\mathbf{r}} - L = \frac{1}{2} \mathbf{p}^2 + \Phi(\mathbf{r}) - \omega \mathbf{p} \cdot \hat{n}_\oplus \times \mathbf{r} = \frac{1}{2} \mathbf{p}^2 + \Phi(\mathbf{r}) - \omega (GM_\odot a)^{1/2} j \cdot \hat{n}_\oplus$, where the last equality follows from (6). Thus transforming to a rotating frame simply adds a term
\begin{equation}
\Phi_{\text{rot}} = -\omega (GM_\odot a)^{1/2} j \cdot \hat{n}_\oplus
\end{equation}
(19)
to the Hamiltonian.

E. Equations of motion

Having derived the orbit-averaged perturbing potentials from the Earth’s multipole moments and the lunar tidal field, we must now determine how the satellite orbit responds to these perturbations. Once again we use a vectorial method.
The evolution of the orbit is determined by a Hamiltonian $H = H_{\text{Kep}} + \Phi(\mathbf{r})$, where $H_{\text{Kep}} = \frac{1}{2} \mathbf{v}^2 - G M_\oplus / r$ is the Kepler Hamiltonian. If $\Phi(\mathbf{r}) = 0$ the energy $E_{\text{Kep}} = -\frac{1}{2} G M_\oplus / a$ is conserved, where as usual $a$ is the semi-major axis. If the perturbing potential is non-zero, $dE_{\text{Kep}}/dt = \mathbf{p} \cdot \nabla \Phi$. If we orbit-average, as we did for the potentials, $\langle \mathbf{p} \cdot \nabla \Phi \rangle = 0$ so $E_{\text{Kep}}$ and the semi-major axis are conserved. Then the orbit’s shape and orientation are determined entirely by the vectors $\mathbf{j}$ and $\mathbf{e}$ (Eq. (5)).

From Eq. (6) the Poisson brackets of $\mathbf{j}$ and $\mathbf{e}$ are

$$\{ j_i, j_j \} = \frac{1}{\sqrt{GM_\oplus a}} \epsilon_{ijk} j_k, \quad \{ e_i, e_j \} = \frac{1}{\sqrt{GM_\oplus a}} \epsilon_{ijk} e_k,$$

$$\{ j_i, e_j \} = \frac{1}{\sqrt{GM_\oplus a}} \epsilon_{ijk} e_k.$$

The time evolution under the Hamiltonian $H$ of any function $f$ of the phase-space coordinates is

$$\frac{df}{dt} = \{ f, H \}.$$

Then by the chain rule

$$\frac{df}{dt} = \{ f, j \} \nabla_j H + \{ f, e \} \nabla_e H,$$

where $\nabla_j$ is the vector $(\partial / \partial j_1, \partial / \partial j_2, \partial / \partial j_3)$, and similarly for $\nabla_e$. Replacing $f$ by $j_i$ and $e_i$ and using relations (20) we have

$$\frac{dj}{dt} = -\frac{1}{\sqrt{GM_\oplus a}} \left( \mathbf{j} \times \nabla_j H + \mathbf{e} \times \nabla_e H \right),$$

$$\frac{de}{dt} = -\frac{1}{\sqrt{GM_\oplus a}} \left( \mathbf{j} \times \nabla_e H + \mathbf{e} \times \nabla_j H \right).$$

Since $\mathbf{e}$ and $\mathbf{j}$ are constants of motion for the Kepler Hamiltonian $H_{\text{Kep}}$, we can replace $H = H_{\text{Kep}} + \langle \Phi \rangle$ by $(\Phi)$.

By definition, $\mathbf{j}$ and $\mathbf{e}$ must satisfy:

$$\mathbf{j} \cdot \mathbf{e} = 0, \quad j^2 + e^2 = 1.$$

It is straightforward to show that these constraints are conserved by the equations of motion (23). There is a gauge freedom in the Hamiltonian $H$ since it can be replaced by $H + F$ where $F(j, e)$ is any function that is constant on the manifold (24); however, adding this function has no effect on the equations of motion (26).

III. STABILITY OF SATELLITES ON CIRCULAR ORBITS

We now investigate the effect of each of the perturbations we have discussed—the multipole moments $J_2$, $J_3$, and $J_4$ and the lunar tide—on the stability of a satellite on a circular low Earth orbit. We have verified most of these conclusions by numerical integrations of test-particle orbits.

A. Quadrupole ($J_2$) potential

If the only perturbation is the quadrupole term (12) we have

$$\nabla_j \langle \Phi_2 \rangle = -\frac{3GM_\oplus J_2 R_\oplus^2}{2a^3(1-e^2)^{3/2}} (\mathbf{j} \cdot \mathbf{n}_\oplus) \mathbf{n}_\oplus,$$

$$\nabla_e \langle \Phi_2 \rangle = \frac{3GM_\oplus J_2 R_\oplus^2}{4a^3(1-e^2)^{3/2}} \left[ 1 - e^2 - 5(\mathbf{j} \cdot \mathbf{n}_\oplus)^2 \right] \mathbf{e}.$$

Substituting these results into the equations of motion (23) and dropping all terms higher than linear order in eccentricity,

$$\frac{dj}{dt} = \frac{3(GM_\oplus)^{1/2} J_2 R_\oplus^2}{2a^{7/2}} (\mathbf{j} \cdot \mathbf{n}_\oplus) \mathbf{j} \times \mathbf{n}_\oplus,$$

$$\frac{de}{dt} = \frac{3(GM_\oplus)^{1/2} J_2 R_\oplus^2}{4a^{7/2}} \left\{ [1 - 5(\mathbf{j} \cdot \mathbf{n}_\oplus)^2] \mathbf{e} \times \mathbf{j} + 2(\mathbf{j} \cdot \mathbf{n}_\oplus) \mathbf{e} \times \mathbf{n}_\oplus \right\}.$$

The first of these equations describes precession of the satellite’s orbital angular momentum $\mathbf{j}$ around the symmetry axis of the Earth $\mathbf{n}_\oplus$. The angular frequency of the precession is

$$\omega = -\frac{3(GM_\oplus)^{1/2} J_2 R_\oplus^2}{2a^{7/2}} (\mathbf{j} \cdot \mathbf{n}_\oplus).$$

where $\mathbf{j} \cdot \mathbf{n}_\oplus = \cos \beta$ and $\beta$ is the constant inclination of the orbit to the Earth’s equator. The minus sign indicates retrograde precession of the angular momentum vector around the symmetry axis.

The second equation is best analyzed by transforming to the frame rotating with the precession of the orbit. In this frame $\mathbf{j}$ is a constant; the Hamiltonian is modified according to equation (19); and the equation of motion for the eccentricity is modified to

$$\frac{de}{dt} = \frac{\omega}{2 \cos \beta} (5 \cos^2 \beta - 1) \mathbf{e} \times \mathbf{j}.$$

We now switch to Cartesian coordinates with the positive $z$-axis along $\mathbf{n}_\oplus$ and with $\mathbf{j}$ in the $x$-$z$ plane, so $\mathbf{j} = (\sin \beta, 0, \cos \beta)$. We look for a solution of the form $\mathbf{e} = e_0 \exp(i \lambda t)$; solving this simple eigenvalue problem yields either $\lambda = 0$ or

$$\lambda = \pm i \frac{\omega}{2 \cos \beta} (1 - 5 \cos^2 \beta)$$

$$= \mp i \frac{3(GM_\oplus)^{1/2} J_2 R_\oplus^2}{4a^{7/2}} (1 - 5 \cos^2 \beta).$$

Since the eigenvalues are zero or purely imaginary, we conclude that nearly circular low Earth orbits of all inclinations are stable under the influence of the quadrupole moment of the Earth. Note that if the inclination $\beta$ satisfies $1 - 5 \cos^2 \beta = 0$, or $\beta = \beta_{\text{crit}} = \cos^{-1} \sqrt{1/5} = 63.43^\circ$, all three eigenvalues $\lambda$ are zero in Eq. (29). This is the...
so-called critical inclination, and the motion of satellites in the vicinity of this inclination has been the subject of many studies.

In physical units, the timescale of these oscillations is:

$$|\lambda|^{-1} = \frac{11.5 d}{1 - 5 \cos^2 \beta} \left( \frac{a}{R_\oplus} \right)^{7/2}.$$  \hspace{1cm} (30)

### B. Octupole (J₃) potential

If J₃ is the only non-zero multipole moment then Eq. (13) gives

$$\nabla_j (\Phi_3) = -\frac{15GM_\oplus J_3 R_\oplus^3}{4a^4(1 - e^2)^{3/2}} \left( j \cdot \hat{n}_\oplus \right) (e \cdot \hat{n}_\oplus) \hat{n}_\oplus,$$

$$\nabla_e (\Phi_3) = \frac{3GM_\oplus J_3 R_\oplus^3}{8a^4(1 - e^2)^{7/2}} \left[ \hat{n}_\oplus \left( 1 - e^2 - 5(j \cdot \hat{n}_\oplus)^2 \right) \right] + 5\frac{e}{1 - e^2} \left( e \cdot \hat{n}_\oplus \right) \left[ 1 - e^2 - 7(j \cdot \hat{n}_\oplus)^2 \right].$$  \hspace{1cm} (31)

We substitute these results into the equations of motion (23); in this case stability can be determined by dropping all terms linear or higher in the eccentricity from the right-hand side and we have

$$\frac{dj}{dt} = 0,$$

$$\frac{de}{dt} = -\frac{3(GM_\oplus)^{1/2} J_3 R_\oplus^3}{8a^{9/2}} \left[ 1 - 5(j \cdot \hat{n}_\oplus)^2 \right] j \times \hat{n}_\oplus. $$  \hspace{1cm} (32)

Thus the eccentricity vector grows linearly with time, e(t) = e₀ + ct where

$$|c| = \frac{3(GM_\oplus)^{1/2} |J_3| R_\oplus^3}{8a^{9/2}} \sin \beta \left( 1 - 5 \cos^2 \beta \right).$$  \hspace{1cm} (33)

We conclude that the J₃ term causes instability for all inclinations except \( \beta = 0 \) (equatorial orbit) and \( \beta = \beta_{\text{crit}} \). Hence if only the J₃ multipole were non-zero and there were no other external forces, Earth satellites on all orbits except these two would crash. The survival timescale would be much smaller than

$$|c|^{-1} = \frac{26.9 \text{yr}}{\sin \beta \left( 1 - 5 \cos^2 \beta \right)} \left( \frac{a}{R_\oplus} \right)^{9/2}.$$  \hspace{1cm} (34)

### C. J₄ potential

The analysis of orbital stability for higher multipoles proceeds in the same way as for J₂ and J₃, although the algebra rapidly becomes more complicated. For J₄, we substitute the averaged potential \( \langle \Phi_4 \rangle \) (Eq. 13) into the equations of motion (23). Keeping terms up to first order in eccentricity on the right-hand side we find

$$\frac{dj}{dt} = \frac{15(GM_\oplus)^{1/2} J_4 R_\oplus^4}{16a^{11/2}} \left[ 3(j \cdot \hat{n}_\oplus) - 7(j \cdot \hat{n}_\oplus)^3 \right] j \times \hat{n}_\oplus,$$

$$\frac{de}{dt} = \frac{15(GM_\oplus)^{1/2} J_4 R_\oplus^4}{16a^{11/2}} \left\{ \left[ 3(j \cdot \hat{n}_\oplus) - 7(j \cdot \hat{n}_\oplus)^3 \right] e \times \hat{n}_\oplus - [1 - 7(j \cdot \hat{n}_\oplus)^2] (e \cdot \hat{n}_\oplus) j \times \hat{n}_\oplus \ight.$$  \hspace{1cm} continued on next page
D. The lunar potential

We substitute the doubly averaged potential \(\langle \Phi_b \rangle\) from Eq. (18) into the equations of motion (23):

\[
\frac{d\vec{j}}{dt} = \frac{3G^{1/2}M_p a^{3/2}}{4M_{\oplus}^{1/2} a_0^3}\left[ (\vec{j} \cdot \hat{n}_b) \vec{j} \times \hat{n}_b - 5(\vec{e} \cdot \hat{n}_b) \vec{e} \times \hat{n}_b \right],
\]

\[
\frac{d\vec{e}}{dt} = \frac{3G^{1/2}M_p a^{3/2}}{4M_{\oplus}^{1/2} a_0^3}\left[ (\vec{j} \cdot \hat{n}_b) \vec{e} \times \hat{n}_b - 5(\vec{e} \cdot \hat{n}_b) \vec{j} \times \hat{n}_b \right]
+ 2j \times \vec{e}. \tag{40}
\]

To first order in the eccentricity, the first of these describes uniform precession of \(\vec{j}\) around \(\hat{n}_b\) with frequency

\[
\omega = -\frac{3(G M_{\oplus})^{1/2} a^{3/2}}{4a_0^3} M_b \left( \vec{j} \cdot \hat{n}_b \right), \tag{41}
\]

where \(\vec{j} \cdot \hat{n}_b = \cos \beta_b\) and \(\beta_b\) is the inclination of the satellite orbit to the lunar orbit, which is nearly in the ecliptic plane. We then transform to the frame rotating at this frequency and the equation of motion for the eccentricity simplifies to

\[
\frac{d\vec{e}}{dt} = \frac{3(G M_{\oplus})^{1/2} a^{3/2}}{4a_0^3} \frac{M_b}{M_b} [2j \times \vec{e} - 5(\vec{e} \cdot \hat{n}_b) \vec{j} \times \hat{n}_b]. \tag{42}
\]

Assuming \(\vec{e} = e_0 \exp(\lambda t)\) and solving the eigenvalue equation, we find \(\lambda = 0\) or

\[
\lambda = \pm \frac{3(G M_{\oplus})^{1/2} a^{3/2}}{2^{3/2} a_0^3} \frac{M_b}{M_b} (3 - 5 \cos^2 \beta_b)^{1/2}. \tag{43}
\]

The solution is stable if and only if the inclination \(\beta_b < \beta_L = \cos^{-1} \sqrt{3/5} = 39.23^\circ\) or \(\beta_b > 180^\circ - \beta_L\). In other words, all nearly circular orbits with an inclination between 39.23° and 140.77° with respect to the lunar orbit will have an exponentially increasing eccentricity.

The growth rate is

\[
|\lambda|^{-1} = \frac{248 \text{ yr}}{\sqrt{1 - \frac{e^2}{3}}} \left( \frac{R_{\oplus}}{a} \right)^{3/2}. \tag{44}
\]

E. Power-law quadrupole potential

It is striking that both the \(J_2\) potential and the lunar potential are quadrupoles [angular dependence \(\propto P_2(\cos \theta)\)], yet circular orbits are stable in one of these perturbing potentials at all inclinations and unstable in the other for a wide range of inclinations. Some insight into this behavior comes from examining an artificial axisymmetric perturbing potential \(\Phi_b = c b^b P_2(\cos \theta)\) with symmetry axis \(\hat{n}_c\); the \(J_2\) potential corresponds to \(b = -3\) and the lunar potential to \(b = +2\). After orbit-averaging and dropping all terms of order higher than \(e^2\),

\[
\langle \Phi_c \rangle = \frac{ca^b}{32} \left\{ 6(2 + b)(3 + b)(\vec{e} \cdot \hat{n}_c)^2 + 8 - (18 + 13b + b^2)e^2 - 3[8 + (2 - 3b + b^2)e^2](\vec{j} \cdot \hat{n}_c)^2 \right\}. \tag{45}
\]

We substitute this potential into the equations of motion (23) and find \(\vec{e} = e_0 \exp(\lambda t)\) with

\[
\lambda = \pm \frac{ca^b}{16\sqrt{GM_{\oplus}a}} \times \left\{ [18 + 13b + b^2 + 3(2 - 3b + b^2) \cos^2 \beta] \times \left( [18 + 17b + 5b^2 - 3(14 + 7b + 3b^2) \cos^2 \beta] \right)^{1/2} \right\}. \tag{46}
\]

where \(\cos \beta = \hat{n} \cdot \hat{n}_c\). The satellite orbit is unstable if the quantity in braces is positive.

FIG. 1 Stability of nearly circular orbits in a potential of the form \(c b^b P_2(\cos \theta)\). The horizontal axis is the exponent \(b\) and the vertical axis is the orbit inclination relative to the symmetry plane of the potential. Unstable regions of parameter space are stippled, and bounded by red and green lines. Vacuum solutions of Poisson’s equations require \(b = -3\) or 2, and are marked by vertical dashed lines.

Figure 1 shows the unstable inclinations (stippled) as a function of the inclination \(\beta\) and the exponent of the potential, \(b\). Remarkably, the only exponents for which all inclinations are stable are \(b = -3\), corresponding to the potential arising from the planetary quadrupole \(J_2\),
and $b = -2$, which is unphysical. We conclude that the stability of orbits of all inclinations circling an oblate planet is an accidental consequence of the properties of the quadrupole potential.

**F. General axisymmetric potential**

Now consider the more general case of an arbitrary axisymmetric perturbation $\Phi(r)$ to the Kepler potential. Let $\hat{z}$ be the symmetry axis of the potential. The orbit-averaged potential $\langle \Phi \rangle$ can only depend on the semimajor axis $a$ and the dimensionless eccentricity and angular-momentum vectors $\mathbf{e}$, $\mathbf{j}$. Since the semimajor axis is fixed in the orbit-averaged equations of motion (HE) we do not need to consider the dependence of $\langle \Phi \rangle$ on it. We shall assume that the potential $\Phi(r)$ is a smooth function of position $r$, so $\langle \Phi \rangle$ must be a smooth function of $\mathbf{e}$ and $\mathbf{j}$. Moreover because of axisymmetry the dependence on these vectors can only occur through the scalars $e^2$, $j^2$, $\mathbf{e} \cdot \hat{z}$ and $\mathbf{j} \cdot \hat{z}$; and since $j^2 + e^2 = 1$ we need only one of the first two in this list. Finally, since the shape of the orbit is unchanged if $j \rightarrow -j$ the potential can only depend on $(\mathbf{j} \cdot \hat{z})^2$. Thus the orbit-averaged potential can be written $\langle \Phi \rangle [e^2, \mathbf{e} \cdot \hat{z}, (\mathbf{j} \cdot \hat{z})^2]$. Denoting the derivative with respect to argument $i$ by $\Phi_i$, the equations of motion (23) become

$$\begin{align*}
\frac{d\mathbf{j}}{dt} &= -\frac{1}{\sqrt{GMa}} \left[ 2 \langle \Phi \rangle_{,j} \mathbf{e} \times \hat{z} + 2 \langle \Phi \rangle_{,j} (\mathbf{j} \cdot \hat{z}) \mathbf{j} \times \hat{z} \right] \\
\frac{d\mathbf{e}}{dt} &= -\frac{1}{\sqrt{GMa}} \left[ 2 \langle \Phi \rangle_{,j} \mathbf{j} \times \hat{z} + 2 \langle \Phi \rangle_{,j} (\mathbf{j} \cdot \hat{z}) \mathbf{e} \times \hat{z} \right].
\end{align*}$$

(47)

The first of these shows that $\mathbf{j} \cdot \hat{z}$ is constant, i.e., the $z$-component of angular momentum is conserved.

Now assume $|\mathbf{e}| \ll 1$ and keep only the terms on the right side that are independent of $\mathbf{e}$ in the first equation and up to linear in $\mathbf{e}$ in the second:

$$\begin{align*}
\frac{d\mathbf{j}}{dt} &= -\frac{2}{\sqrt{GMa}} \langle \Phi \rangle_{,j} (\mathbf{j} \cdot \hat{z}) \mathbf{j} \times \hat{z} \\
\frac{d\mathbf{e}}{dt} &= -\frac{1}{\sqrt{GMa}} \left[ 2 \langle \Phi \rangle_{,j} \mathbf{e} \times \hat{z} + 2 \langle \Phi \rangle_{,j} (\mathbf{j} \cdot \hat{z}) \mathbf{e} \times \hat{z} \right].
\end{align*}$$

(48)

where the derivatives of $\langle \Phi \rangle$ are evaluated at $e = 0$. Since $\mathbf{j} \cdot \hat{z}$ is constant, these derivatives can be taken to be constants. The first equation describes uniform precession of $\mathbf{j}$ around $\hat{z}$ at angular frequency

$$\omega = \frac{2 \langle \Phi \rangle_{,j}}{\sqrt{GMa}} (\mathbf{j} \cdot \hat{z}).$$

(49)

In the frame rotating at $\omega$, $\frac{d\mathbf{j}}{dt} = 0$; the Hamiltonian is modified according to Eq. (19); and the equation of motion for the eccentricity is modified to

$$\frac{de}{dt} = -\frac{1}{\sqrt{GMa}} \left[ 2 \langle \Phi \rangle_{,j} \mathbf{j} \times \mathbf{e} + \langle \Phi \rangle_{,j} (\mathbf{e} \cdot \mathbf{j}) \mathbf{j} \times \hat{z} \right].$$

(50)

All quantities other than $e$ on the right side are constants, so the general solution is of the form $e = e_0 + \sum_{i=1}^{2} c_i \exp(\lambda_i t)$ where

$$\lambda = \pm \frac{2i}{\sqrt{GMa}} \left[ \langle \Phi \rangle_{,j} \left( \langle \Phi \rangle_{,j} + \frac{1}{2} \langle \Phi \rangle_{,22} \sin^2 \beta \right) \right]^{1/2};$$

(51)

$$2 \langle \Phi \rangle_{,j} \mathbf{j} \times e_0 + \langle \Phi \rangle_{,j} (\mathbf{e} \cdot \mathbf{j}) \mathbf{j} \times \hat{z} = -\langle \Phi \rangle_{,j} \mathbf{j} \times \hat{z},$$

(52)

and $\cos \beta = \mathbf{j} \cdot \hat{z}$ for circular orbits.

With this formula several of our earlier results become easy to interpret:

(i) The potential $\langle \Phi_2 \rangle [e^2, \mathbf{e} \cdot \hat{z}, (\mathbf{j} \cdot \hat{z})^2]$ associated with the Earth’s quadrupole moment (Eq. [12]) has no dependence on $\mathbf{e} \cdot \mathbf{n}_{\oplus}$ so $\langle \Phi_2 \rangle_{,22} = 0$ and $\lambda$ is always pure imaginary, which explains why orbits in this perturbing potential are always stable.

(ii) The potential $\langle \Phi_3 \rangle [e^2, \mathbf{e} \cdot \hat{z}, (\mathbf{j} \cdot \hat{z})^2]$ associated with the multipole moment $J_3$ is an odd function of $\mathbf{e} \cdot \hat{z}$ (Eq. [13]). Thus $\langle \Phi_3 \rangle_{,1}$ is also odd, so it is zero when evaluated at $e = 0$. Then Eq. (51) implies that both eigenvalues $\lambda$ are zero. Two degenerate eigenvalues give rise to linear growth in the solution to equations like (50), which is what was seen in the solution to Eq. (22).

(iii) Note also that $\langle \Phi_3 \rangle$ is linear in $\mathbf{e} \cdot \mathbf{z}$ (Eq. [13]) so $\langle \Phi_3 \rangle_{,22} = 0$. Thus for the combined potential $\langle \Phi_2 + \Phi_3 \rangle$, Eq. (51) is simply $\lambda = \pm 2i \langle \Phi_2 \rangle_{,j} / \sqrt{GMa}$, independent of $\Phi_3$. We conclude that any quadrupole potential, no matter how small, will stabilize a circular satellite orbit against the octupole potential $\Phi_3$.

(iv) The quadrupole potential associated with $J_3$ is the strongest non-Keplerian potential for satellites in low Earth orbit by several orders of magnitude. Let us then write the perturbing potential is $\langle \Phi \rangle = \langle \Phi_2 \rangle + \epsilon \langle \phi \rangle$ where $\epsilon \ll 1$ and $\langle \phi \rangle$ represents the potentials from higher order multipoles, the Moon and Sun, etc. Since $\langle \Phi_2 \rangle_{,22} = 0$ (see paragraph [i]) we have to first order in $\epsilon$

$$\lambda = \pm \frac{2i}{\sqrt{GMa}} \times \left[ (\langle \Phi_2 \rangle_{,j}^2 + \epsilon \langle \Phi_2 \rangle_{,j} (2 \langle \phi \rangle_{,j} + \frac{1}{2} \langle \phi \rangle_{,22} \sin^2 \beta) \right]^{1/2};$$

(53)

At most inclinations $\langle \Phi_2 \rangle_{,j}^2$ is much larger than the other term since the latter is multiplied by $\epsilon \ll 1$; thus $\lambda$ is pure imaginary and the orbit is stable. Physically, the rapid precession of the angular-momentum and eccentricity vectors averages out the perturbations from other sources and suppresses the instabilities that they would otherwise induce. However, $\langle \Phi_2 \rangle_{,j}^2$ is proportional
to \((1 - 5 \cos^2 \beta)\) so near the critical inclination \(\beta_{\text{crit}}\) there can be a narrow range of inclinations in which the square brackets in Eq. (53) are negative so the orbit is unstable.

IV. LIDOV–KOZAI OSCILLATIONS

The nonlinear trajectories of the linear instabilities we have described are known as Kozai, Kozai–Lidov, or Lidov–Kozai oscillations. Although Laplace had all of the tools needed to investigate this phenomenon, it was only discovered in the early 1960s by Lidov\(^8\) in the Soviet Union and brought to the West by Kozai\(^9\). The simplest, and most astrophysically relevant, examples of Lidov–Kozai oscillations arise when a distant third body perturbs a binary system, as in the discussion of the lunar potential in III D. The perturbing potential \(\langle \Phi_3 \rangle\) (Eq. 15) depends on the orbital elements of the satellite through the semi-major axis \(a\), the eccentricity \(e\), and the projections of the eccentricity and angular-momentum vectors on the lunar orbit axis, \(\mathbf{e} \cdot \mathbf{n}_l\) and \(\mathbf{j} \cdot \mathbf{n}_l\). The semi-major axis is conserved because we have orbit-averaged (see discussion in II E), \(\mathbf{j} \cdot \mathbf{n}_l\) is conserved because the orbit-averaged potential is axisymmetric, and the Hamiltonian \(\langle \Phi_3 \rangle\) is conserved because it is autonomous. Given these four variables and three conserved quantities, the orbit-averaged oscillation has only one degree of freedom and thus is integrable,\(^10\) although the integrability disappears when higher-order multipole moments are included or the angular momentum in the inner orbit is not small compared to the outer orbit.\(^11\)\(^12\)

Some properties of the oscillations are straightforward to determine. Since \(\langle \Phi_3 \rangle\) and \(\mathbf{j} \cdot \mathbf{n}_l\) are conserved, Eq. 18 tells us that the eccentricity \(e\) and the normal component of the eccentricity \(\mathbf{e} \cdot \mathbf{n}_l\) must evolve along the track \(5(e \cdot \mathbf{n}_l)^2 - 2e^2 = \text{constant}\). If the satellite is initially on a circular orbit, the constant must be zero so \((e \cdot \mathbf{n}_l)^2 = \frac{1}{5}e^2\). Since \(\mathbf{e} \cdot \mathbf{j} = 0\), \((e \cdot \mathbf{n}_l)^2/e^2\) cannot exceed \(\sin^2 \beta\) where \(\beta\) is the inclination, so \(\sin^2 \beta \geq \frac{1}{5}\) if \(e > 0\), which immediately gives the stability criterion for circular orbits, \(\beta < \beta_{\text{L}}\) (Eq. 43). If the inclination \(\beta_{\text{L}}\) of the initial circular orbit exceeds \(\beta_{\text{L}}\), then \((\mathbf{j} \cdot \mathbf{n}_l)^2 = (1 - e^2) \cos^2 \beta = \cos^2 \beta_0\) and we find \(e^2 \leq 1 - \frac{1}{5} \cos^2 \beta_0\) which gives the maximum eccentricity achieved in the Lidov–Kozai oscillation. At the maximum eccentricity the inclination is \(\beta_{\text{L}}\).

As an example, Lidov pointed out that if the inclination of the lunar orbit to the ecliptic were changed to 90°, with all other orbital elements kept the same, this “vertical Moon” would collide with the Earth in about four years as a result of a Lidov–Kozai oscillation induced by the gravitational field of the Sun. More recent work shows that Lidov–Kozai oscillations may play a significant role in the formation and evolution of a wide variety of astrophysical systems. These include:

• The giant planets in the solar system are surrounded by over 100 small “irregular” satellites, most of them discovered in the last two decades. These satellites, typically < 100 km in radius, are found at much larger semi-major axes and have more eccentric and inclined orbits than the classical satellites. One of the striking features of their orbital distribution is that no satellites are found with inclinations between 55° and 130° (relative to the ecliptic), although prograde orbits with smaller inclination and retrograde orbits with larger inclination are common (\(\sim 20\%\) and \(\sim 80\%\) of the total population, respectively). This gap is explained naturally by Lidov–Kozai oscillations: at inclinations close to 90° the oscillations are so strong that the satellite either collides with one of the much larger classical satellites or the planet at periapsis, or escapes from the planet’s gravitational field at apapsis.\(^3\)\(^4\)\(^5\)

• Some extrasolar planets have remarkably high eccentricities—the current record-holder has \(e = 0.97\) and it is likely that some or most of the highest eccentricities have been excited by Lidov–Kozai oscillations due to a companion star.\(^14\) In fact, the four planets with the largest known eccentricities all orbit host stars with companions.\(^20\)

• “Hot Jupiters” are giant planets orbiting within 0.1 AU of their host star, several hundred times closer than the giant planets in our own solar system. Such planets cannot form in situ. One plausible formation mechanism is “high-eccentricity migration”, which involves the following steps: the planet forms at 5–10 AU from the host star, like Jupiter and Saturn in our own solar system; it is excited to high eccentricity by gravitational scattering off another planet; Lidov–Kozai oscillations due to a companion star or other giant planets periodically bring the planet so close to the host star that it loses orbital energy through tidal friction; the orbit then decays, at a faster and faster rate, until the planet settles into a circular orbit close to the host star.\(^21\)\(^22\)

• Many stars are found in close binary systems, with separations of only a few stellar radii. Forming such systems is a challenge, since the radius of a star shrinks by a large factor during its early life. It is possible that most or all close binary systems were formed from much wider binaries by the combined effects of Lidov–Kozai oscillations induced by a distant companion star and tidal friction.\(^22\) Supporting this hypothesis is the remarkable observation that almost all close binary systems (orbital period less than 3 days) have a tertiary companion star.\(^23\) The formation rate of Type Ia supernovae may be dominated by a similar process in triple systems containing a white dwarf-white dwarf binary and a distant companion: either Lidov–Kozai oscillations plus energy loss through gravitational radiation.\(^24\)
or Lidov–Kozai oscillations that excite the binary to sufficiently high eccentricity that the two white dwarfs collide.\textsuperscript{20}

- Most galaxies contain supermassive black holes at the centers, and when galaxies merge their black holes spiral towards the center of the merged galaxy through dynamical friction.\textsuperscript{20} However, dynamical friction becomes less effective once the black holes have formed a tightly bound binary, and it is unclear whether the black hole inspiral will “stall” before gravitational radiation becomes effective. Lidov–Kozai oscillations in the binary, induced either by the overall gravitational field of the galaxy or the presence of a third black hole, can pump the binary to high eccentricity where gravitational radiation is more efficient, leading to inspiral and merger of the black holes.\textsuperscript{20}

V. DISCUSSION

Satellites in low Earth orbit are subject to a variety of non-Keplerian perturbations, both from the multipole moments of the Earth’s gravitational field and from tidal forces from the Moon and Sun. For practical purposes in astrodynamics, the effects of these perturbations have been well-understood for decades. Nevertheless, in generic perturbing potentials—even axisymmetric ones—many orbits are likely to be unstable and crash in a short time; a classic example is Lidov’s vertical lunar orbit, described in \textsuperscript{34}. Thus it is worthwhile to understand physically what features the perturbing potential must have so that low Earth orbits are stable.

Our most general result, for an axisymmetric perturbing potential, is that the orbit is stable if (Eq. 31)

\[
\frac{\partial}{\partial e^2} \left( \frac{\partial \langle \Phi \rangle}{\partial e^2} + \frac{1}{2} \frac{\partial^2 \langle \Phi \rangle}{\partial (e \cdot \mathbf{z})^2} \sin^2 \beta \right) > 0,
\]

where \(z\) is the symmetry axis of the potential and \(\langle \Phi \rangle\) is the orbit-averaged potential written as a function of \(e^2, e \cdot \mathbf{z}\), and \((e \cdot \mathbf{z})^2 = \cos^2 \beta\) with \(\beta\) the inclination. It is an accidental property of the potential due to an internal quadrupole moment that the coefficient of \(\sin^2 \beta\) vanishes and the motion is always stable. This statement is not meant to imply that all other potentials are unstable—the \(J_4\) potential analyzed in \textsuperscript{34} is unstable only over a total range of inclinations of about 8\degree, and Figure 1 shows that other potentials can be stable for most inclinations—but the potential associated with \(J_2\) is the only axisymmetric potential arising from a physically plausible mass distribution for which all inclinations are stable.

We have also shown that because the other perturbing potentials are much smaller than the one due to the Earth’s quadrupole, all low Earth orbits are stable except perhaps in a small region near the critical inclination \(\beta_{\text{crit}} = \cos^{-1} \sqrt{1/5} = 63.43\degree\), where \(\langle \Phi \rangle / \partial e^2 = 0\) for the quadrupole potential.

We chose to investigate only azimuthally symmetric terms in the perturbing potential, but we strongly suspect that similar arguments apply to small non-axisymmetric potentials: the strong quadrupole potential of the Earth guarantees stability except for small interval(s) in the inclination \(\beta\). These intervals will be centered on resonances between the precession frequencies of the angular-momentum vector and the eccentricity vector (eqs. \textsuperscript{27}, \textsuperscript{29}), that is, when \(1 - 5 \cos^2 \beta = 2n \cos \beta\) for integer \(n\) (\(\beta = 46.38\degree, 63.43\degree, 73.14\degree, 78.46\degree, \ldots\)).

Finally, we hope that this paper will introduce readers to the use of vector elements to study the long-term evolution of Keplerian orbits. These elements are not new\textsuperscript{38,10} but they deserve to be more widely known.

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