Abstract

Consider the minimum mean-square error (MMSE) of estimating an arbitrary random variable from its observation contaminated by Gaussian noise. The MMSE can be regarded as a function of the signal-to-noise ratio (SNR) as well as a functional of the input distribution (of the random variable to be estimated). It is shown that the MMSE is concave in the input distribution at any given SNR. For a given input distribution, the MMSE is found to be infinitely differentiable at all positive SNR, and in fact a real analytic function in SNR under mild conditions. The key to these regularity results is that the posterior distribution conditioned on the observation through Gaussian channels always decays at least as quickly as some Gaussian density. Furthermore, simple expressions for the first three derivatives of the MMSE with respect to the SNR are obtained. It is also shown that, as functions of the SNR, the curves for the MMSE of a Gaussian input and that of a non-Gaussian input cross at most once over all SNRs. These properties lead to simple proofs of the facts that Gaussian inputs achieve both the secrecy capacity of scalar Gaussian wiretap channels and the capacity of scalar Gaussian broadcast channels, as well as a simple proof of the entropy power inequality in the special case where one of the variables is Gaussian.

Index Terms: Entropy, estimation, Gaussian noise, Gaussian broadcast channel, Gaussian wiretap channel, minimum mean-square error (MMSE), mutual information.

I. INTRODUCTION

The concept of mean-square error has assumed a central role in the theory and practice of estimation since the time of Gauss and Legendre. In particular, minimization of mean-square error underlies numerous methods in statistical sciences. The focus of this paper is the minimum mean-square error (MMSE) of estimating an arbitrary random variable contaminated by additive Gaussian noise.

Let \((X, Y)\) be random variables with arbitrary joint distribution. Throughout the paper, \(E\{\cdot\}\) denotes the expectation with respect to the joint distribution of all random variables in the braces, and \(E\{X|Y\}\) denotes the conditional mean estimate of \(X\) given \(Y\). The corresponding conditional variance is a function of \(Y\) which is denote by

\[
\text{var} \{X|Y\} = E\left\{ (X - E\{X|Y\})^2 \right\}|Y .
\]

(1)

It is well known that the conditional mean estimate is optimal in the mean-square sense. In fact, the MMSE of estimating \(X\) given \(Y\) is nothing but the average conditional variance:

\[
\text{mmse}(X|Y) = E\{\text{var} \{X|Y\}\} .
\]

(2)
In this paper, we are mainly interested in random variables related through models of the following form:

\[ Y = \sqrt{\text{snr}} X + N \]  

where \( N \sim \mathcal{N}(0,1) \) is standard Gaussian throughout this paper unless otherwise stated. The MMSE of estimating the input \( X \) of the model given the noisy output \( Y \) is alternatively denoted by:

\[
\text{mmse}(X, \text{snr}) = \text{mmse}(X|\sqrt{\text{snr}} X + N) = E \left\{ (X - E\{X|\sqrt{\text{snr}} X + N\})^2 \right\}. 
\]  

The MMSE can be regarded as a function of the signal-to-noise ratio (SNR) for every given distribution \( P_X \), and as a functional of the input distribution \( P_X \) for every given SNR. In particular, for a Gaussian input with mean \( m \) and variance \( \sigma_X^2 \), denoted by \( X \sim \mathcal{N}(m, \sigma_X^2) \),

\[
\text{mmse}(X, \text{snr}) = \frac{\sigma_X^2}{1 + \sigma_X^2 \text{snr}}. 
\]  

If \( X \) is equally likely to take \( \pm 1 \), then

\[
\text{mmse}(X, \text{snr}) = 1 - \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \tanh(\text{snr} - \sqrt{\text{snr}} y) dy. 
\]  

The function \( \text{mmse}(X, \text{snr}) \) is illustrated in Fig. 1 for four special inputs: the standard Gaussian variable, a Gaussian variable with variance \( 1/4 \), as well as symmetric and asymmetric binary random variables, all of zero mean.

Optimal estimation intrinsically underlies many fundamental information theoretic results, which describe the boundary between what is achievable and what is not, given unlimited computational power. Simple quantitative connections between the MMSE and information measures were revealed in [1]. One such result is that, for arbitrary but fixed \( P_X \),

\[
\text{mmse}(X, \text{snr}) = 2 \frac{d}{d\text{snr}} I(X; \sqrt{\text{snr}} X + N). 
\]  

This relationship implies the following integral expression for the mutual information:

\[
I(X; \sqrt{\text{snr}} g(X) + N) = \frac{1}{2} \int_{0}^{\text{snr}} \text{mmse}(g(X), \gamma) d\gamma 
\]
which holds for any one-to-one real-valued function \( g \). By sending \( \text{snr} \to \infty \) in (9), we find the entropy of every discrete random variable \( X \) can be expressed as (see [1], [2]):

\[
H(X) = \frac{1}{2} \int_0^\infty \text{mmse}(g(X), \gamma) d\gamma
\]  

(10)

whereas the differential entropy of any continuous random variable \( X \) can be expressed as:

\[
h(X) = \log \left( \frac{2\pi e}{2} \right) - \frac{1}{2} \int_0^\infty \frac{1}{1+\gamma} - \text{mmse}(g(X), \gamma) d\gamma.
\]  

(11)

The preceding information–estimation relationships have found a number of applications, e.g., in non-linear filtering [1], [3], in multiuser detection [4], in power allocation over parallel Gaussian channels [5], [6], in the proof of Shannon’s entropy power inequality (EPI) and its generalizations [2], [7], [8], and in the treatment of the capacity region of several multiuser channels [9]–[11]. Relationships between relative entropy and mean-square error are also found in [12], [13]. Moreover, many such results have been generalized to vector-valued inputs and multiple-input multiple-output (MIMO) models [1], [7], [14].

Partially motivated by the important role played by the MMSE in information theory, this paper presents a detailed study of the key mathematical properties of \( \text{mmse}(X, \text{snr}) \). The remainder of the paper is organized as follows.

In Section II, we establish bounds on the MMSE as well as on the conditional and unconditional moments of the conditional mean estimation error. In particular, it is shown that the tail of the posterior distribution of the input given the observation vanishes at least as quickly as that of some Gaussian density. Simple properties of input shift and scaling are also shown.

In Section III, \( \text{mmse}(X, \text{snr}) \) is shown to be an infinitely differentiable function of \( \text{snr} \) on \((0, \infty)\) for every input distribution regardless of the existence of its moments (even the mean and variance of the input can be infinite). Furthermore, under certain conditions, the MMSE is found to be real analytic at all positive SNRs, and hence can be arbitrarily well-approximated by its Taylor series expansion.

In Section IV, the first three derivatives of the MMSE with respect to the SNR are expressed in terms of the average central moments of the input conditioned on the output. The result is then extended to the conditional MMSE.

Section V shows that the MMSE is concave in the distribution \( P_X \) at any given SNR. The monotonicity of the MMSE of a partial sum of independent identically distributed (i.i.d.) random variables is also investigated. It is well-known that the MMSE of a non-Gaussian input is dominated by the MMSE of a Gaussian input of the same variance. It is further shown in this paper that the MMSE curve of a non-Gaussian input and that of a Gaussian input cross each other at most once over \( \text{snr} \in (0, \infty) \), regardless of their variances.

In Section VI, properties of the MMSE are used to establish Shannon’s EPI in the special case where one of the variables is Gaussian. Sidestepping the EPI, the properties of the MMSE lead to simple and natural proofs of the fact that Gaussian input is optimal for both the Gaussian wiretap channel and the scalar Gaussian broadcast channel.

II. Basic Properties

A. The MMSE

The input \( X \) and the observation \( Y \) in the model described by \( Y = \sqrt{\text{snr}} X + N \) are tied probabilistically by the conditional Gaussian probability density function:

\[
p_{Y|X}(y|x; \text{snr}) = \varphi \left( y - \sqrt{\text{snr}} x \right)
\]  

(12)
where $\varphi$ stands for the standard Gaussian density:

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}. \quad (13)$$

Let us define for every $a \in \mathbb{R}$ and $i = 0, 1, \ldots$,

$$h_i(y; a) = \mathbb{E}\{X_i \varphi(y - aX)\} \quad (14)$$

which is always well defined because $\varphi(y - ax)$ is bounded and vanishes quadratic exponentially fast as either $x$ or $y$ becomes large with the other variable bounded. In particular, $h_0(y; \sqrt{\text{snr}})$ is nothing but the marginal distribution of the observation $Y$, which is always strictly positive. The conditional mean estimate can be expressed as \cite{1, 4}:

$$\mathbb{E}\{X|Y = y\} = \frac{h_1(y; \sqrt{\text{snr})}}{h_0(y; \sqrt{\text{snr})}} \quad (15)$$

and the MMSE can be calculated as \cite{4}:

$$\text{mmse}(X, \text{snr}) = \iint_{\mathbb{R}} \left( x - \frac{h_1(y; \sqrt{\text{snr})}}{h_0(y; \sqrt{\text{snr})}} \right)^2 \varphi(y - \sqrt{\text{snr}} x)dydP_X(x) \quad (16)$$

which can be simplified if $\mathbb{E}\{X^2\} < \infty$:

$$\text{mmse}(X, \text{snr}) = \mathbb{E}\{X^2\} - \int_{-\infty}^{\infty} \frac{h_1^2(y; \sqrt{\text{snr}})}{h_0(y; \sqrt{\text{snr}})} dy. \quad (17)$$

Note that the estimation error $X - \mathbb{E}\{X|Y\}$ remains the same if $X$ is subject to a constant shift. Hence the following well-known fact:

**Proposition 1:** For every random variable $X$ and $a \in \mathbb{R}$,

$$\text{mmse}(X + a, \text{snr}) = \text{mmse}(X, \text{snr}). \quad (18)$$

The following is also straightforward from the definition of MMSE.

**Proposition 2:** For every random variable $X$ and $a \in \mathbb{R}$,

$$\text{mmse}(aX, \text{snr}) = a^2 \text{mmse}(X, a^2 \text{snr}). \quad (19)$$

### B. The Conditional MMSE and SNR Increment

For any pair of jointly distributed variables $(X, U)$, the conditional MMSE of estimating $X$ at SNR $\gamma \geq 0$ given $U$ is defined as:

$$\text{mmse}(X, \gamma|U) = \mathbb{E}\{ (X - \mathbb{E}\{X|\sqrt{\gamma} X + N, U\})^2 \} \quad (20)$$

where $N \sim \mathcal{N}(0, 1)$ is independent of $(X, U)$. It can be regarded as the MMSE achieved with side information $U$ available to the estimator. For every $u$, let $X_u$ denote a random variable indexed by $u$ with distribution $P_{X|U=u}$. Then the conditional MMSE can be seen as an average:

$$\text{mmse}(X, \text{snr}|U) = \int \text{mmse}(X_u, \text{snr})P_U(du). \quad (21)$$

A special type of conditional MMSE is obtained when the side information is itself a noisy observation of $X$ through an independent additive Gaussian noise channel. It has long been noticed that two independent looks through Gaussian channels is equivalent to a single look at the sum SNR, e.g., in the context of maximum-ratio combining. As far as the MMSE is concerned, the SNRs of the direct observation and the side information simply add up.
Proposition 3: For every $X$ and every $\text{snr}, \gamma \geq 0$,
\begin{equation}
\text{mmse}(X, \gamma | \sqrt{\text{snr}} X + N) = \text{mmse}(X, \text{snr} + \gamma)
\end{equation}
where $N \sim N(0,1)$ is independent of $X$.

Proposition 3 enables translation of the MMSE at any given SNR to a conditional MMSE at a smaller SNR. This result was first shown in [1] using the incremental channel technique, and has been instrumental in the proof of information–estimation relationships such as (8). Proposition 3 is also the key to the regularity properties and the derivatives of the MMSE presented in subsequent sections. A brief proof of the result is included here for completeness.

Proof of Proposition 3: Consider a cascade of two Gaussian channels as depicted in Fig. 2:
\begin{align}
Y_{\text{snr}+\gamma} &= X + \sigma_1 N_1 \tag{23a} \\
Y_{\text{snr}} &= Y_{\text{snr}+\gamma} + \sigma_2 N_2 \tag{23b}
\end{align}
where $X$ is the input, $N_1$ and $N_2$ are independent standard Gaussian random variables. A subscript is used to explicitly denote the SNR at which each observation is made. Let $\sigma_1, \sigma_2 > 0$ satisfy $\sigma_1^2 = 1/(\text{snr} + \gamma)$ and $\sigma_1^2 + \sigma_2^2 = 1/\text{snr}$ so that the SNR of the first channel (23a) is $\text{snr} + \gamma$ and that of the composite channel is $\text{snr}$. A linear combination of (23a) and (23b) yields
\begin{equation}
(\text{snr} + \gamma) Y_{\text{snr}+\gamma} = \text{snr} Y_{\text{snr}} + \gamma X + \sqrt{\gamma} W \tag{24}
\end{equation}
where we have defined $W = (\gamma \sigma_1 N_1 - \text{snr} \sigma_2 N_2)/\sqrt{\gamma}$. Clearly, the input–output relationship defined by the incremental channel (23) is equivalently described by (24) paired with (23b). Due to mutual independence of $(X, N_1, N_2)$, it is easy to see that $W$ is standard Gaussian and $(X, W, \sigma_1 N_1 + \sigma_2 N_2)$ are mutually independent. Thus $W$ is independent of $(X, Y_{\text{snr}})$ by (23). Based on the above observations, the relationship of $X$ and $Y_{\text{snr}+\gamma}$ conditioned on $Y_{\text{snr}} = y$ is exactly the input–output relationship of a Gaussian channel with SNR equal to $\gamma$ described by (24) with $Y_{\text{snr}} = y$. Because $Y_{\text{snr}}$ is a physical degradation of $Y_{\text{snr}+\gamma}$, providing $Y_{\text{snr}}$ as the side information does not change the overall MMSE, that is, $\text{mmse}(X|Y_{\text{snr}+\gamma}) = \text{mmse}(X, \gamma | Y_{\text{snr}})$, which proves (22).

C. Bounds

The input to a Gaussian model with nonzero SNR can always be estimated with finite mean-square error based on the output, regardless of the input distribution. In fact, $\hat{X} = Y/\sqrt{\text{snr}}$ achieves mean-square error of $1/\text{snr}$, even if $\mathbb{E}\{X\}$ does not exist. Moreover, the trivial zero estimate achieves mean-square error of $\mathbb{E}\{X^2\}$.

Proposition 4: For every input $X$,
\begin{equation}
\text{mmse}(X, \text{snr}) \leq \frac{1}{\text{snr}} \tag{25}
\end{equation}
and in case the input variance $\text{var}\{X\}$ is finite,
\begin{equation}
\text{mmse}(X, \text{snr}) \leq \min\left\{ \text{var}\{X\}, \frac{1}{\text{snr}} \right\}. \tag{26}
\end{equation}
Proposition 4 can also be established using the fact that $\text{snr} \cdot \text{mmse}(X, \text{snr}) = \text{mmse}(N|\sqrt{\text{snr}} X + N) \leq 1$, which is simply because the estimation error of the input is proportional to the estimation error of the noise [7]:

$$\sqrt{\text{snr}}(X - \mathbb{E}\{X|Y\}) = \mathbb{E}\{N|Y\} - N.$$  \hspace{1cm} (27)

Using (27) and known moments of the Gaussian density, higher moments of the estimation errors can also be bounded as shown in Appendix A:

**Proposition 5:** For every random variable $X$ and $\text{snr} > 0$,

$$\mathbb{E}\{|X - \mathbb{E}\{X|\sqrt{\text{snr}} X + N\}|^n\} \leq \left(\frac{2}{\sqrt{\text{snr}}}\right)^n \sqrt{n!}$$  \hspace{1cm} (28)

for every $n = 0, 1, \ldots$, where $N \sim \mathcal{N}(0, 1)$ is independent of $X$.

In order to show some useful characteristics of the posterior input distribution, it is instructive to introduce the notion of sub-Gaussianity. A random variable $X$ is called sub-Gaussian if the tail of its distribution is dominated by that of some Gaussian [15, Theorem 2].

**Lemma 1:** The following statements are equivalent:

1) $X$ is sub-Gaussian;
2) There exists $C > 0$ such that for every $k = 1, 2, \ldots$,

$$\mathbb{E}\{|X|^k\} \leq C^k \sqrt{k!} ;$$  \hspace{1cm} (30)

3) There exist $c, C > 0$ such that for all $t > 0$,

$$\mathbb{E}\{e^{tX}\} \leq Ce^{ct^2} .$$  \hspace{1cm} (31)

Regardless of the prior input distribution, the posterior distribution of the input given the noisy observation through a Gaussian channel is always sub-Gaussian, and the posterior moments can be upper bounded. This is formalized in the following result proved in Appendix B:

**Proposition 6:** Let $X_y$ be distributed according to $P_{X|Y=y}$ where $Y = aX + N$, $N \sim \mathcal{N}(0, 1)$ is independent of $X$, and $a \neq 0$. Then $X_y$ is sub-Gaussian for every $y \in \mathbb{R}$. Moreover,

$$\mathbb{P}\{|X_y| \geq x\} \leq \frac{\sqrt{2}}{\pi} \frac{e^{\frac{x^2}{\pi h_0(y;a)}}}{h_0(y;a)}e^{-\frac{x^2}{4}}$$  \hspace{1cm} (32)

and, for every $n = 1, 2, \ldots$,

$$\mathbb{E}\{|X_y|^n\} \leq \frac{ne^{\frac{x^2}{2}}}{h_0(y;a)} \left(\frac{\sqrt{2}}{|a|}\right)^n \sqrt{(n-1)!}$$  \hspace{1cm} (33)

and

$$\mathbb{E}\{|X_y - \mathbb{E}\{X_y\}|^n\} \leq 2^n \mathbb{E}\{|X_y|^n\} .$$  \hspace{1cm} (34)
III. SMOOTHNESS AND ANALYTICITY

This section studies the regularity of the MMSE as a function of the SNR, where the input distribution is arbitrary but fixed. In particular, it is shown that mmse($X, \text{snr}$) is a smooth function of snr on $(0, \infty)$ for every $P_X$. This conclusion clears the way towards calculating its derivatives in Section IV. Under certain technical conditions, the MMSE is also found to be real analytic in snr. This implies that the MMSE can be reconstructed from its local derivatives. As we shall see, the regularity of the MMSE at the point of zero SNR requires additional conditions.

A. Smoothness

Proposition 7: For every $X$, mmse($X, \text{snr}$) is infinitely differentiable at every snr $> 0$. If $E\{X^k\} < \infty$ for all $k = 1, 2, \ldots$, then mmse($X, \text{snr}$) is $k$ right-differentiable at snr $= 0$. Consequently, mmse($X, \text{snr}$) is infinitely right differentiable at snr $= 0$ if all moments of $X$ are finite.

Proof: The proof is divided into two parts. In the first part we first establish the smoothness assuming that all input moments are finite, i.e., $E\{X^k\} < \infty$ for all $k = 1, 2, \ldots$.

For convenience, let $Y = aX + N$ where $a^2 = \text{snr}$. For every $i = 0, 1, \ldots$, denote
\[
g_i(y; a) = \frac{\partial^i}{\partial a^i} \left( \frac{h_1}{h_0} \right)(y; a) \tag{35}
\]
and
\[
m_i(a) = \int_{-\infty}^{\infty} g_i(y; a)dy \tag{36}
\]
where $h_i$ is given by (14). By (17), we have
\[
\text{mmse}(X, a^2) = E\{X^2\} - m_0(a). \tag{37}
\]
We denote by $H_n$ the $n$-th Hermite polynomial [16, Section 5.5]:
\[
H_n(x) = \frac{(-1)^n}{\varphi(x)} \frac{d^n \varphi(x)}{dx^n} \tag{38}
\]
\[
= n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{1}{k!(n-2k)!} (2x)^{n-2k}. \tag{39}
\]
Denote $h_i^{(n)}(y; a) = \partial^n h_i(y; a)/\partial a^n$ throughout the paper. Then
\[
\frac{h_i^{(n)}(y; a)}{h_0(y; a)} = \frac{1}{h_0(y; a)} E\{X^{i+n}H_n(y-aX)\varphi(y-aX)\} \tag{40}
\]
\[
= E\{X^{i+n}H_n(N) \mid Y = y\} \tag{41}
\]
where the derivative and expectation can be exchanged to obtain (40) because the product of any polynomial and the Gaussian density is bounded.

The following lemma is established in Appendix C.

Lemma 2: For every $i = 0, 1, \ldots$ and all $w > v$, $(y, a) \mapsto g_i(y; a)$ is integrable on $\mathbb{R} \times [v, w]$.

Using Lemma 2 and (36), we have
\[
\int_v^w m_{i+1}(a)da = \int_v^w \int_{-\infty}^{\infty} g_{i+1}(y; a)dyda \tag{42}
\]
\[
= \int_{-\infty}^{\infty} g_i(y; w) - g_i(y; v)dy \tag{43}
\]
\[
= m_i(w) - m_i(v) \tag{44}
\]
where (43) is due to (55) and Fubini's theorem. Therefore for every \( i \geq 0 \), \( m_i \) is continuous. Hence for each \( a \in \mathbb{R} \),

\[
\frac{dm_i(a)}{da} = m_{i+1}(a)
\]  

(45)

follows from the fundamental theorem of calculus \[17\], p. 97. In view of (37), we have

\[
\frac{d^i \text{mmse}(X, a^2)}{da^i} = -m_i(a).
\]  

(46)

This proves that \( a \mapsto \text{mmse}(X, a^2) \in C^\infty(\mathbb{R}) \), which implies that \( \text{mmse}(X, \text{snr}) \) is infinitely differentiable in \( \text{snr} \) on \((0, \infty)\).

In the second part of this proof, we eliminate the requirement that all moments of the input exist by resorting to the incremental-SNR result, Proposition 3. Fix arbitrary \( \gamma > 0 \) and let \( Y_\gamma = \sqrt{\gamma} X + N \). For every \( u \in \mathbb{R} \), let \( X_{u;\gamma} \sim P_{X|Y_\gamma = u} \). By (17), (21) and Proposition 3, we have

\[
\text{mmse}(X, \gamma + a^2) = \int \text{mmse}(X_{u;\gamma}, a^2) P_{Y_\gamma}(du)
\]  

(47)

\[
= \mathbb{E} \{ X^2 \} - \tilde{m}_0(a)
\]  

(48)

where

\[
h_i(y; a|u; \gamma) = \mathbb{E} \left\{ X^i \varphi(y - aX) \big| Y_\gamma = u \right\}
\]  

(49)

\[
g_i(y; a|u; \gamma) = \frac{\partial^i}{\partial a^i} \left( \frac{h_i}{h_0} \right)(y; a|u; \gamma)
\]  

(50)

and

\[
\tilde{m}_i(a) = \int \int \left( g_i(y; a|u; \gamma) \right) dy h_0(u; \gamma) du
\]  

(51)

for \( i = 0, 1, \ldots \). By Proposition 5, for each \( u \), all moments of \( X_{u;\gamma} \) are finite. Each \( \tilde{m}_i \) is a well-defined real-valued function on \( \mathbb{R} \). Repeating the first part of this proof with \( h_i(y; a) \) replaced by \( h_i(y; a|u; \gamma) \), we conclude that \( a \mapsto \text{mmse}(X, \gamma + a^2) \in C^\infty \) in \( a \) at least on \( |a| \geq \sqrt{\gamma} \), which further implies that \( a \mapsto \text{mmse}(X, a^2) \in C^\infty(\mathbb{R}\backslash[-\sqrt{2\gamma}, \sqrt{2\gamma}]) \) because \( a \mapsto \sqrt{a^2 - \gamma} \) has bounded derivatives of all order when \( |a| > \sqrt{2\gamma} \). By the arbitrariness of \( \gamma \), we have \( a \mapsto \text{mmse}(X, a^2) \in C^\infty(\mathbb{R}\backslash\{0\}) \), hence \( \text{mmse}(X, \cdot) \in C^\infty((0, \infty)) \).

Finally, we address the case of zero SNR. It follows from (41) and the independence of \( X \) and \( Y \) at zero SNR that

\[
\frac{1}{h_0} \frac{\partial^n h_i}{\partial a^n}(y; 0) = \mathbb{E} \left\{ X^{i+n} \right\} H_n(y).
\]  

(52)

Since \( \mathbb{E} \{ |H_n(N)| \} \leq \sqrt{\mathbb{E} \{ H_n^2(N) \}} = \sqrt{n!} \) is always finite, induction reveals that the \( n \)-th derivative of \( m_0 \) at 0 depends on the first \( n + 1 \) moments of \( X \). By Taylor's theorem and the fact that \( m_0(a) \) is an even function of \( a \), we have

\[
m_0(a) = \sum_{j=0}^{i} \frac{m_{2j}(0)}{(2j)!} a^{2j} + O \left( |a|^{2i+2} \right)
\]  

(53)

in the vicinity of \( a = 0 \), which implies that \( m_0 \) is \( i \) differentiable with respect to \( a^2 \) at 0, with \( d^i m_0(0+)/d(a^2)^i = m_{2i}(0) \), as long as \( \mathbb{E} \{ X^{i+1} \} < \infty \).  

\[ \blacksquare \]
B. Real Analyticity

A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is said to be **real analytic** at \( x_0 \) if it can be represented by a convergence power series in some neighborhood of \( x_0 \), i.e., there exists \( \delta > 0 \) such that

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n
\]  

(54)

for every \( x \in (x_0 - \delta, x_0 + \delta) \). One necessary and sufficient condition for \( f \) to be real analytic is that \( f \) can be extended to some open disk \( D(x_0, \delta) \triangleq \{ z \in \mathbb{C} : |z - x_0| < \delta \} \) in the complex plane by the power series \( (54) \) \([18]\).

**Proposition 8:** As a function of \( a \), \( \text{mmse}(X,a^2) \) is real analytic at \( a_0 \in \mathbb{R} \) if either one of the following two sets of conditions holds:

1) \( X \) is sub-Gaussian, and there exist \( c > 0 \) and \( r > 0 \) such that for every \( y \in \mathbb{R} \),

\[
\inf_{z \in D(a_0,r)} |h_0(y; z)| > 0
\]  

(55)

and

\[
\liminf_{|y| \to \infty} \inf_{z \in D(a_0,r)} \frac{|h_0(y; z)|}{h_0(y; \Re(z))} > c
\]  

(56)

2) \( a_0 \neq 0 \), and there exist \( c > 0 \), \( r > 0 \) and \( \delta \in (0, a_0^2) \) such that for every \( y, u \in \mathbb{R} \),

\[
\inf_{z \in D(a_0,r)} |h_0(y; z; u, \delta)| > 0
\]  

(57)

and

\[
\liminf_{|y| \to \infty} \inf_{z \in D(a_0,r)} \frac{|h_0(y; z; u, \delta)|}{h_0(y; \Re(z)|u, \delta)} > c.
\]  

(58)

Moreover, whenever \( \text{mmse}(X,a^2) \) is real analytic at \( a \in \mathbb{R} \), the function \( \text{mmse}(X, \text{snr}) \) is also analytic at \( \text{snr} = a^2 \).

The last statement in Proposition [8] is because of the following. The Taylor series expansion of \( \text{mmse}(X,a^2) \) at \( a = 0 \) is an even function, so that the analyticity of \( \text{mmse}(X,a^2) \) at \( a = 0 \) implies the analyticity of \( \text{mmse}(X, \text{snr}) \) at \( \text{snr} = 0 \). If \( \text{mmse}(X,a^2) \) is analytic at \( a \neq 0 \), then \( \text{mmse}(X, \text{snr}) \) is also analytic at \( \text{snr} = a^2 \) because \( \text{snr} \mapsto \sqrt{\text{snr}} \) is real analytic at \( \text{snr} > 0 \), and composition of analytic functions is analytic \([19]\). It remains to establish the analyticity of \( a \mapsto \text{mmse}(X,a^2) \), which is relegated to Appendix [D].

Conditions \( (55) \) and \( (56) \) can be understood as follows. Recall that \( h_0(y; a) \) denotes the density of \( Y = aX + N \). The function \( h_0(y; a) \) stays positive for all \( a \in \mathbb{R} \), and decays no faster than the Gaussian density. However, \( h_0(y; a) \) may vanish for some \( a \in \mathbb{C} \), so that the MMSE may not be extendable to the convex plane. Hence the purpose of \( (55) \) and \( (56) \) is to ensure that the imaginary part of \( a \) has limited impact on \( |h_0| \).

As an example, consider the case where \( X \) is equiprobable on \( \{ \pm 1 \} \). Then

\[
h_0(y; a) = \varphi(y) \exp(-a^2/2) \cosh(ay).
\]  

(59)

Letting \( a = jt \) yields \( h_0(y; jt) = \varphi(\sqrt{y^2 - t^2}) \cos(ty) \), which has infinitely many zeros. In fact, in this case the MMSE is given by \((7)\), or in an equivalent form:

\[
\text{mmse}(X,a^2) = 1 - \int_{-\infty}^{\infty} \varphi(y) \tanh(a^2 - ay) dy.
\]  

(60)

Then for any \( r > 0 \), there exists \( |a_0| < r \) and \( y_0 \in \mathbb{R} \), such that \( a_0^2 - a_0y_0 = j\frac{\pi}{2} \) and the integral in \( (60) \) diverges near \( y_0 \). Therefore \( \text{mmse}(X,a^2) \) cannot be extended to any point on the imaginary
axis, hence it is not real analytic at \( a = 0 \). Nevertheless, when \( \text{Re}(a) \neq 0 \), condition (56) is satisfied. Hence \( \text{mmse}(X, a^2) \) is real analytic on the real line except zero, which can be shown from (60) directly. Similarly, for any finite-alphabet, exponential or Gaussian distributed \( X \), (57) and (58) can be verified for all \( a \neq 0 \), hence the corresponding MMSE is real analytic at all positive SNR.

IV. DERIVATIVES

A. Derivatives of the MMSE

With the smoothness of the MMSE established in Proposition 7, its first few derivatives with respect to the SNR are explicitly calculated in this section. Consider first the Taylor series expansion of the MMSE around \( \text{snr} = 0 \) to the third order:

\[
\text{mmse}(X, \text{snr}) = 1 - \text{snr} + \left[ 2 - (EX^3)^2 \right] \frac{\text{snr}^2}{2} - \left[ 15 - 12(EX^3)^2 - 6EX^4 + (EX^4)^2 \right] \frac{\text{snr}^3}{6} + O(\text{snr}^4)
\]

(61)

where \( X \) is assumed to have zero mean and unit variance. The first three derivatives of the MMSE at \( \text{snr} = 0 \) are thus evident from (61). The technique for obtaining (61) is to expand (12) in terms of the small signal \( \sqrt{\text{snr}} X \), evaluate \( h_i(y; \sqrt{\text{snr}}) \) given by (14) at the vicinity of \( \text{snr} = 0 \) using the moments of \( X \) (see equation (90) in [1]), and then calculate (16), where the integral over \( y \) can be evaluated as a Gaussian integral.

The preceding expansion of the MMSE at \( \text{snr} = 0 \) can be lifted to arbitrary SNR using the SNR-incremental result, Proposition 3. Finiteness of the input moments is not required for \( \text{snr} > 0 \) because the conditional moments are always finite due to Proposition 5.

For notational convenience, we define the following random variables:

\[
M_i = \mathbb{E}\left\{ (X - \mathbb{E}\{X|Y\})^i \right| Y \right\}, \quad i = 1, 2, \ldots
\]

(62)

which, according to Proposition 5, are well-defined in case \( \text{snr} > 0 \), and reduces to the unconditional moments of \( X \) in case \( \text{snr} = 0 \). Evidently, \( M_1 = 0 \), \( M_2 = \text{var}\{X|\sqrt{\text{snr}} X + N\} \) and

\[
\mathbb{E}\{M_2\} = \text{mmse}(X, \text{snr})
\]

(63)

If the input distribution \( P_X \) is symmetric, then the distribution of \( M_i \) is also symmetric for all odd \( i \).

The derivatives of the MMSE are found to be the expected value of polynomials of \( M_i \), whose existence is guaranteed by Proposition 5.

**Proposition 9**: For every random variable \( X \) and every \( \text{snr} > 0 \),

\[
\frac{d}{d \text{snr}} \text{mmse}(X, \text{snr}) = -\mathbb{E}\{M_2^2\}
\]

(64)

\[
\frac{d^2}{d \text{snr}^2} \text{mmse}(X, \text{snr}) = \mathbb{E}\{2M_2^3 - M_3^2\}
\]

(65)

and

\[
\frac{d^3}{d \text{snr}^3} \text{mmse}(X, \text{snr}) = \mathbb{E}\{6M_4M_2^2 - M_4^2 + 12M_3^2M_2 - 15M_4^2\}
\]

(66)

The previous result for the expansion of \( \text{mmse}(\text{snr}) \) around \( \text{snr} = 0^+ \), given by equation (91) in [1] is mistaken in the coefficient corresponding to \( \text{snr}^2 \). The expansion of the mutual information given by (92) in [1] should also be corrected accordingly. The second derivative of the MMSE is mistaken in [20] and corrected in Proposition 9 in this paper. The function \( \text{mmse}(X, \text{snr}) \) is not always convex in \( \text{snr} \) as claimed in [20], as illustrated using an example in Fig. 1.
The three derivatives are also valid at \( \text{snr} = 0^+ \) if \( X \) has finite second, third and fourth moment, respectively.

We relegate the proof of Proposition 9 to Appendix E. It is easy to check that the derivatives found in Proposition 9 are consistent with the Taylor series expansion (61) at zero SNR.

In light of the proof of Proposition 7 (and (46)), the Taylor series expansion of the MMSE can be carried out to arbitrary orders, so that all derivatives of the MMSE can be obtained as the expectation of some polynomials of the conditional moments, although the resulting expressions become increasingly complicated.

Proposition 9 is easily verified in the special case of standard Gaussian input \( X \sim \mathcal{N}(0, 1) \), where

\[
X \sim \mathcal{N}\left(\sqrt{\frac{\text{snr}}{1 + \text{snr}}} y, \frac{1}{1 + \text{snr}}\right).
\]

(67)

In this case \( M_2 = (1 + \text{snr})^{-1} \), \( M_3 = 0 \) and \( M_4 = 3(1 + \text{snr})^{-2} \) are constants, and (64), (65) and (66) are straightforward.

**B. Derivatives of the Mutual Information**

Based on Proposition 8 and 9, the following derivatives of the mutual information are extensions of the key information-estimation relationship (8).

**Corollary 1:** For every distribution \( P_X \) and \( \text{snr} > 0 \),

\[
\frac{d^i}{d\text{snr}^i} I(X ; \sqrt{\text{snr}} X + N) = \frac{(-1)^i - 1}{2} E\{M_i^2\} \quad (68)
\]

for \( i = 1, 2 \),

\[
\frac{d^3}{d\text{snr}^3} I(X ; \sqrt{\text{snr}} X + N) = E\left\{M_3^2 - \frac{1}{2} M_3^2\right\} \quad (69)
\]

and

\[
\frac{d^4}{d\text{snr}^4} I(X ; \sqrt{\text{snr}} X + N)
\]

\[
= \frac{1}{2} E \left\{-M_2^4 + 6M_4M_2^2 + 2M_3^2M_2 - 15M_4^2\right\} \quad (70)
\]

as long as the corresponding expectation on the right hand side exists. In case one of the two set of conditions in Proposition 8 holds, \( \sqrt{\text{snr}} \mapsto I(\sqrt{\text{snr}} X + N; X) \) is also real analytic.

Corollary 1 is a generalization of previous results on the small SNR expansion of the mutual information such as in [21]. Note that (68) with \( i = 1 \) is exactly the original relationship of the mutual information and the MMSE given by (8) in light of (63).

**C. Derivatives of the Conditional MMSE**

The derivatives in Proposition 9 can be generalized to the conditional MMSE defined in (20). The following is a straightforward extension of (64).

**Corollary 2:** For every jointly distributed \((X, U)\) and \( \text{snr} > 0 \),

\[
\frac{d}{d\text{snr}} \text{mmse}(X, \text{snr}|U) = -E\{M_2^2(U)\} \quad (71)
\]

where for every \( u \) and \( i = 1, 2, \ldots \),

\[
M_i(u) = E\left\{[X_u - E\{X_u|Y\}]^i \right| Y = \sqrt{\text{snr}} X_u + N\right\} \quad (72)
\]

is a random variable dependent on \( u \).
V. PROPERTIES OF THE MMSE FUNCTIONAL

For any fixed snr, \( \text{mmse}(X, \text{snr}) \) can be regarded as a functional of the input distribution \( P_X \). Meanwhile, the MMSE curve, \{\(\text{mmse}(X, \text{snr}), \text{snr} \in [0, \infty)\}\}, can be regarded as a “transform” of the input distribution.

A. Concavity in Input Distribution

**Proposition 10:** The functional \( \text{mmse}(X, \text{snr}) \) is concave in \( P_X \) for every \( \text{snr} \geq 0 \).

**Proof:** Let \( B \) be a Bernoulli variable with probability \( \alpha \) to be 0. Consider any random variables \( X_0, X_1 \) independent of \( B \). Let \( Z = X_B \), whose distribution is \( \alpha P_{X_0} + (1 - \alpha) P_{X_1} \). Consider the problem of estimating \( Z \) given \( \sqrt{\text{snr}} Z + N \) where \( N \) is standard Gaussian. Note that if \( B \) is revealed, one can choose either the optimal estimator for \( P_{X_0} \) or \( P_{X_1} \) depending on the value of \( B \), so that the average MMSE can be improved. Therefore,

\[
\text{mmse}(Z, \text{snr}) \geq \alpha \text{mmse}(X_0, \text{snr}) + (1 - \alpha) \text{mmse}(X_1, \text{snr})
\]

which proves the desired concavity. \( \square \)

B. Conditioning Reduces the MMSE

As a fundamental measure of uncertainty, the MMSE decreases with additional side information available to the estimator. This is because an informed optimal estimator performs no worse than any uninformed estimator by simply discarding the side information.

**Proposition 11:** For any jointly distributed \((X, U)\) and \( \text{snr} \geq 0 \),

\[
\text{mmse}(X, \text{snr}|U) \leq \text{mmse}(X, \text{snr}).
\]

For fixed \( \text{snr} > 0 \), the equality holds if and only if \( X \) is independent of \( U \).

**Proof:** The inequality (75) is straightforward by the concavity established in Proposition 10. In case the equality holds, \( P_{X|U=u} \) must be identical for \( P_U \)-almost every \( u \) due to strict concavity \( \square \), that is, \( X \) and \( U \) are independent. \( \square \)

C. Monotonicity

Propositions 10 and 11 suggest that a mixture of random variables is harder to estimate than the individual variables in average. A related result in [2] states that a linear combination of two random variables \( X_1 \) and \( X_2 \) is also harder to estimate than the individual variables in some average:

**Proposition 12 ([2]):** For every \( \text{snr} \geq 0 \) and \( \alpha \in [0, 2\pi] \),

\[
\text{mmse}(\cos \alpha X_1 + \sin \alpha X_2, \text{snr}) \geq \cos^2 \alpha \text{mmse}(X_1, \text{snr}) + \sin^2 \alpha \text{mmse}(X_2, \text{snr})
\]

A generalization of Proposition 12 concerns the MMSE of estimating a normalized sum of independent random variables. Let \( X_1, X_2, \ldots \) be i.i.d. with finite variance and \( S_n = (X_1 + \cdots + X_n)/\sqrt{n} \). It has been shown that the entropy of \( S_n \) increases monotonically to that of a Gaussian random variable of the same variance [8], [23]. The following monotonicity result of the MMSE of estimating \( S_n \) in Gaussian noise can be established.

**Proposition 13:** Let \( X_1, X_2, \ldots \) be i.i.d. with finite variance. Let \( S_n = (X_1 + \cdots + X_n)/\sqrt{n} \). Then for every \( \text{snr} \geq 0 \),

\[
\text{mmse}(S_{n+1}, \text{snr}) \geq \text{mmse}(S_n, \text{snr}).
\]

\(^2\)Strict concavity is shown in [22].
Because of the central limit theorem, as \( n \to \infty \) the MMSE converges to the MMSE of estimating a Gaussian random variable with the same variance as that of \( X \).

Proposition 13 is a simple corollary of the following general result in [8].

**Proposition 14 ([8]):** Let \( X_1, \ldots, X_n \) be independent. For any \( \lambda_1, \ldots, \lambda_n \geq 0 \) which sum up to one and any \( \gamma \geq 0 \),

\[
\text{mmse} \left( \sum_{i=1}^{n} X_i, \gamma \right) \geq \sum_{i=1}^{n} \lambda_i \text{mmse} \left( \frac{X_i}{\sqrt{(n-1)\lambda_i}}, \gamma \right)
\]

(78)

where \( X_{\setminus i} = \sum_{j=1, j \neq i}^{n} X_j \).

Setting \( \lambda_i = 1/n \) in (78) yields Proposition 13.

In view of the representation of the entropy or differential entropy using the MMSE in Section I, integrating both sides of (77) proves a monotonicity result of the entropy or differential entropy of \( S_n \) whichever is well-defined. More generally, [8] applies (11) and Proposition 14 to prove a more general result, originally given in [23].

**D. Gaussian Inputs Are the Hardest to Estimate**

Any non-Gaussian input achieves strictly smaller MMSE than Gaussian input of the same variance. This well-known result is illustrated in Fig. I and stated as follows.

**Proposition 15:** For every \( \text{snr} \geq 0 \) and random variable \( X \) with variance no greater than \( \sigma^2 \),

\[
\text{mmse}(X, \text{snr}) \leq \frac{\sigma^2}{1 + \text{snr} \sigma^2}.
\]

(79)

The equality of (79) is achieved if and only if the distribution of \( X \) is Gaussian with variance \( \sigma^2 \).

**Proof:** Due to Propositions 1 and 2, it is enough to prove the result assuming that \( \text{E}\{X\} = 0 \) and \( \text{var}\{X\} = \sigma^2 \). Consider the linear estimator for the channel (3):

\[
\hat{X}^l = \frac{\sqrt{\text{snr}}}{\sqrt{\text{snr} \sigma_X^2} + 1} Y
\]

(80)

which achieves the least mean-square error among all linear estimators, which is exactly the right hand side of (79), regardless of the input distribution. The inequality (79) is evident due to the suboptimality of the linearity restriction on the estimator. The strict inequality is established as follows: If the linear estimator is optimal, then \( \text{E}\{Y^k(X - \hat{X}^l)\} = 0 \) for every \( k = 1, 2, \ldots \), due to the orthogonality principle. It is not difficult to check that all moments of \( X \) have to coincide with those of \( N(0, \sigma^2) \). By Carleman’s Theorem [24], the distribution is uniquely determined by the moments to be Gaussian.

Note that in case the variance of \( X \) is infinity, (79) reduces to (25).

**E. The Single-Crossing Property**

In view of Proposition 15 and the scaling property of the MMSE, at any given SNR, the MMSE of a non-Gaussian input is equal to the MMSE of some Gaussian input with reduced variance. The following result suggests that there is some additional simple ordering of the MMSEs due to Gaussian and non-Gaussian inputs.

**Proposition 16 (Single-crossing Property):** For any given random variable \( X \), the curve of \( \text{mmse}(X, \gamma) \) crosses the curve of \( (1 + \gamma)^{-1} \), which is the MMSE function of the standard Gaussian distribution, at most once on \( (0, \infty) \). Precisely, define

\[
f(\gamma) = (1 + \gamma)^{-1} - \text{mmse}(X, \gamma)
\]

(81)

on \( [0, \infty) \). Then
1) $f(\gamma)$ is strictly increasing at every $\gamma$ with $f(\gamma) < 0$;
2) If $f(\text{snr}_0) = 0$, then $f(\gamma) \geq 0$ at every $\gamma > \text{snr}_0$;
3) $\lim_{\gamma \to \infty} f(\gamma) = 0$.

Furthermore, all three statements hold if the term $(1 + \gamma)^{-1}$ in (81) is replaced by $\sigma^2/(1 + \sigma^2 \gamma)$ with any $\sigma$, which is the MMSE function of a Gaussian variable with variance $\sigma^2$.

Fig. 3. An example of the difference between the MMSE for standard Gaussian input and that of a binary input equally likely to be $\pm \sqrt{2}$. The difference crosses the horizontal axis only once.

Proof: The last of the three statements, $\lim_{\gamma \to \infty} f(\gamma) = 0$ always holds because of Proposition 4.

If $\text{var}\{X\} \leq 1$, then $f(\gamma) \geq 0$ at all $\gamma$ due to Proposition 15, so that the proposition holds. We suppose in the following $\text{var}\{X\} > 1$. An instance of the function $f(\gamma)$ with $X$ equally likely to be $\pm \sqrt{2}$ is shown in Fig. 3. Evidently $f(0) = 1 - \text{var}\{X\} < 0$. Consider the derivative of the difference (81) at any $\gamma$ with $f(\gamma) < 0$, which by Proposition 9, can be written as

$$f'(\gamma) = \mathbb{E}\{M_2^2\} - (1 + \gamma)^{-2}$$

(82)

$$> \mathbb{E}\{M_2^2\} - (\text{mmse}(X, \gamma))^2$$

(83)

$$= \mathbb{E}\{M_2^2\} - (\mathbb{E}M_2)^2$$

(84)

$$\geq 0$$

(85)

where (84) is due to (63), and (85) is due to Jensen’s inequality. That is, $f'(\gamma) > 0$ as long as $f(\gamma) < 0$, i.e., the function $f$ can only be strictly increasing at every point it is strictly negative. This further implies that if $f(\text{snr}_0) = 0$ for some $\text{snr}_0$, the function $f$, which is smooth, cannot dip to below zero for any $\gamma > \text{snr}_0$. Therefore, the function $f$ has no more than one zero crossing.

For any $\sigma$, the above arguments can be repeated with $\sigma^2 \gamma$ treated as the SNR. It is straightforward to show that the proposition holds with the standard Gaussian MMSE replaced by the MMSE of a Gaussian variable with variance $\sigma^2$.

The single-crossing property can be generalized to the conditional MMSE defined in (20).

Proposition 17: Let $X$ and $U$ be jointly distributed variables. All statements in Proposition 16 hold literally if the function $f(\cdot)$ is replaced by

$$f(\gamma) = (1 + \gamma)^{-1} - \text{mmse}(X, \gamma|U).$$

(86)

Proof: For every $u$, let $X_u$ denote a random variable indexed by $u$ with distribution $P_{X|U=u}$. Define also a random variable for every $u$,

$$M(u, \gamma) = M_2(X_u, \gamma)$$

(87)

$$= \text{var}\{X_u|\sqrt{\text{snr}} X_u + N\}$$

(88)

The single-crossing property has also been extended to the parallel degraded MIMO scenario [25].
where $N \sim \mathcal{N}(0,1)$. Evidently, $E\{M(u, \gamma)\} = \text{mmse}(X_u, \gamma)$ and hence
\[
f(\gamma) = \frac{1}{1+\gamma} - E\{E\{M(U, \gamma)|U\}\}
\]

\[
= \frac{1}{1+\gamma} - E\{M(U, \gamma)\}.
\]

Clearly,
\[
f'(\gamma) = -\frac{1}{(1+\gamma)^2} - E\left\{\frac{d}{d\gamma} M(U, \gamma)\right\}
\]

\[
= E\{M^2(U, \gamma)\} - \frac{1}{(1+\gamma)^2}
\]

by Proposition 9. In view of (90), for all $\gamma$ such that $f(\gamma) < 0$, we have
\[
f'(\gamma) > E\{M^2(U, \gamma)\} - (E\{M(U, \gamma)\})^2
\]

\[
\geq 0
\]

by (92) and Jensen’s inequality. The remaining argument is essentially the same as in the proof of Proposition 16.

\[\boxed{\text{F. The High-SNR Asymptotics}}\]

The asymptotics of $\text{mmse}(X, \gamma)$ as $\gamma \to \infty$ can be further characterized as follows. It is upper bounded by $1/\gamma$ due to Propositions 4 and 15. Moreover, the MMSE can vanish faster than exponentially in $\gamma$ with arbitrary rate, under for instance a sufficiently skewed binary input [26]. On the other hand, the decay of the MMSE of a non-Gaussian random variable need not be faster than the MMSE of a Gaussian variable. For example, let $X = Z + \sqrt{\sigma_X^2 - 1} B$ where $\sigma_X > 1$, $Z \sim \mathcal{N}(0,1)$ and the Bernoulli variable $B$ are independent. Clearly, $X$ is harder to estimate than $Z$ but no harder than $\sigma_X Z$, i.e.,
\[
\frac{1}{1+\gamma} < \text{mmse}(X, \gamma) < \frac{\sigma_X^2}{1+\sigma_X^2 \gamma}
\]

where the difference between the upper and lower bounds is $O(\gamma^{-2})$. As a consequence, the function $f$ defined in (51) may not have any zero even if $f(0) = 1 - \sigma_X^2 < 0$ and $\lim_{\gamma \to \infty} f(\gamma) = 0$. A meticulous study of the high-SNR asymptotics of the MMSE is found in [22], where the limit of the product $\text{snr} \cdot \text{mmse}(X, \text{snr})$, called the MMSE dimension, has been determined for input distributions without singular components.

VI. APPLICATIONS TO CHANNEL CAPACITY

A. Secrecy Capacity of the Gaussian Wiretap Channel

This section makes use of the MMSE as an instrument to show that the secrecy capacity of the Gaussian wiretap channel is achieved by Gaussian inputs. The wiretap channel was introduced by Wyner in [27] in the context of discrete memoryless channels. Let $X$ denote the input, and let $Y$ and $Z$ denote the output of the main channel and the wiretapper’s channel respectively. The problem is to find the rate at which reliable communication is possible through the main channel, while keeping the mutual information between the message and the wiretapper’s observation as small as possible. Assuming that

\[\text{In case the input is equally likely to be } \pm 1, \text{ the MMSE decays as } e^{-\frac{1}{2} \text{snr}}, \text{ not } e^{-2 \text{snr}} \text{ as stated in [1, 26].}\]
the wiretapper sees a degraded output of the main channel, Wyner showed that secure communication can achieve any rate up to the secrecy capacity

$$C_s = \max_{X} [I(X;Y) - I(X;Z)]$$

(96)

where the supremum is taken over all admissible choices of the input distribution. Wyner also derived the achievable rate-equivocation region.

We consider the following Gaussian wiretap channel studied in [28]:

$$Y = \sqrt{\text{snr}_1} X + N_1$$

(97a)

$$Z = \sqrt{\text{snr}_2} X + N_2$$

(97b)

where \(\text{snr}_1 \geq \text{snr}_2\) and \(N_1, N_2 \sim \mathcal{N}(0,1)\) are independent. Let the energy of every codeword of length \(n\) be constrained by \(\frac{1}{n} \sum_{i=1}^{n} x_i^2 \leq 1\). Reference [28] showed that the optimal input which achieves the supremum in (96) is standard Gaussian and that the secrecy capacity is

$$C_s = \frac{1}{2} \log \left( \frac{1 + \text{snr}_1}{1 + \text{snr}_2} \right).$$

(98)

In contrast to [28] which appeals to Shannon’s EPI, we proceed to give a simple proof of the same result using (9), which enables us to write for any \(X\):

$$I(X;Y) - I(X;Z) = \frac{1}{2} \int_{\text{snr}_2}^{\text{snr}_1} \text{mmse}(X, \gamma) d\gamma.$$  

(99)

Under the constraint \(\mathbb{E}\{X^2\} \leq 1\), the maximum of (99) over \(X\) is achieved by standard Gaussian input because it maximizes the MMSE for every SNR under the power constraint. Plugging \(\text{mmse}(X, \gamma) = (1+\gamma)^{-1}\) into (99) yields the secrecy capacity given in (98). In fact the whole rate-equivocation region can be obtained using the same techniques. Note that the MIMO wiretap channel can be treated similarly [11].

B. The Gaussian Broadcast Channel

In this section, we use the single-crossing property to show that Gaussian input achieves the capacity region of scalar Gaussian broadcast channels. Consider a degraded Gaussian broadcast channel also described by the same model (97). Note that the formulation of the Gaussian broadcast channel is statistically identical to that of the Gaussian wiretap channel, except for a different goal: The rates between the sender and both receivers are to be maximized, rather than minimizing the rate between the sender and the (degraded) wiretapper. The capacity region of degraded broadcast channels under a unit input power constraint is given by [29]:

$$\bigcup_{P_{U:X}: \mathbb{E}\{X^2\} \leq 1} \left\{ \begin{array}{l} R_1 \leq I(X;Y|U) \\ R_2 \leq I(U;Z) \end{array} \right\}$$

(100)

where \(U\) is an auxiliary random variable with \(U-X-(Y,Z)\) being a Markov chain. It has long been recognized that Gaussian \(P_{U,X}\) with standard Gaussian marginals and correlation coefficient \(\mathbb{E}\{UX\} = \sqrt{1-\alpha}\) achieves the capacity. The resulting capacity region of the Gaussian broadcast channel is

$$\bigcup_{\alpha \in [0,1]} \left\{ \begin{array}{l} R_1 \leq \frac{1}{2} \log \left( 1 + \alpha \text{snr}_1 \right) \\ R_2 \leq \frac{1}{2} \log \left( \frac{1 + \text{snr}_2}{1 + \alpha \text{snr}_2} \right) \end{array} \right\}.$$  

(101)

The conventional proof of the optimality of Gaussian inputs relies on the EPI in conjunction with Fano’s inequality [30]. The converse can also be proved directly from (100) using only the EPI [31], [32]. In the following we show a simple alternative proof using the single-crossing property of MMSE.
Due to the power constraint on $X$, there must exist $\alpha \in [0, 1]$ (dependent on the distribution of $X$) such that
\[
I(X; Z|U) = \frac{1}{2} \log (1 + \alpha \text{snr}_2) \tag{102}
\]
\[
= \frac{1}{2} \int_0^{\text{snr}_2} \frac{\alpha}{\alpha \gamma + 1} d\gamma. \tag{103}
\]
By the chain rule,
\[
I(U; Z) = I(U, X; Z) - I(X; Z|U) \tag{104}
\]
\[
= I(X; Z) - I(X; Z|U). \tag{105}
\]
By (100) and (102), the desired bound on $R_2$ is established:
\[
R_2 \leq \frac{1}{2} \log (1 + \text{snr}_2) - \frac{1}{2} \log (1 + \alpha \text{snr}_2) \tag{106}
\]
\[
= \frac{1}{2} \log \left( \frac{1 + \text{snr}_2}{1 + \alpha \text{snr}_2} \right). \tag{107}
\]

It remains to establish the desired bound for $R_1$. The idea is illustrated in Fig. 4 where crossing of the MMSE curves imply some ordering of the corresponding mutual informations. Note that
\[
I(X; Z|U = u) = \frac{1}{2} \int_0^{\text{snr}_2} \text{mmse}(X_u, \gamma)d\gamma \tag{108}
\]
and hence
\[
I(X; Z|U) = \frac{1}{2} \int_0^{\text{snr}_2} \mathbb{E}\{\text{mmse}(X_U, \gamma|U)\} d\gamma. \tag{109}
\]
Comparing (109) with (103), there must exist $0 \leq \text{snr}_0 \leq \text{snr}_2$ such that
\[
\mathbb{E}\{\text{mmse}(X_U, \text{snr}_0|U)\} = \frac{\alpha}{\alpha \text{snr}_0 + 1}. \tag{110}
\]
By Proposition [17] this implies that for all $\gamma \geq \text{snr}_2 \geq \text{snr}_0$,
\[
\mathbb{E}\{\text{mmse}(X_U, \gamma|U)\} \leq \frac{\alpha}{\alpha \gamma + 1}. \tag{111}
\]
Consequently,

\[ R_1 \leq I(X; Y|U) \]
\[ = \frac{1}{2} \int_0^{\text{snr}_1} \mathbb{E}\{\text{mmse}(X_U, \gamma|U)\} \, d\gamma \]
\[ = \frac{1}{2} \left( \int_0^{\text{snr}_2} + \int_{\text{snr}_2}^{\text{snr}_1} \right) \mathbb{E}\{\text{mmse}(X_U, \gamma|U)\} \, d\gamma \]
\[ \leq \frac{1}{2} \log (1 + \alpha \text{snr}_2) + \frac{1}{2} \int_{\text{snr}_2}^{\text{snr}_1} \frac{\alpha}{\alpha \gamma + 1} \, d\gamma \]
\[ = \frac{1}{2} \log (1 + \alpha \text{snr}_1) \]

where the inequality (115) is due to (102), (109) and (111).

C. Proof of a Special Case of EPI

As another simple application of the single-crossing property, we show in the following that

\[ e^{2h(X+Z)} \geq e^{2h(X)} + 2\pi e \sigma_Z^2 \]

for any independent \( X \) and \( Z \) as long as the differential entropy of \( X \) is well-defined and \( Z \) is Gaussian with variance \( \sigma_Z^2 \). This is in fact a special case of Shannon’s entropy power inequality. Let \( W \sim \mathcal{N}(0, 1) \) and \( a^2 \) be the ratio of the entropy powers of \( X \) and \( W \), so that

\[ h(X) = h(aW) = \frac{1}{2} \log (2\pi e a^2). \]

Consider the difference

\[ h(\sqrt{\text{snr}} X + N) - h(\sqrt{\text{snr}} aW + N) \]
\[ = \frac{1}{2} \int_0^{\text{snr}} \text{mmse}(X, \gamma) - \text{mmse}(aW, \gamma) \, d\gamma \]

where \( N \) is standard Gaussian independent of \( X \) and \( W \). In the limit of \( \text{snr} \to \infty \), the left hand side of (119) vanishes due to (118). By Proposition 16, the integrand in (119) as a function of \( \gamma \) crosses zero only once, which implies that the integrand is initially positive, and then becomes negative after the zero crossing (cf. Fig. 3). Consequently, the integral (119) is positive and increasing for small \( \text{snr} \), and starts to monotonically decrease after the zero crossing. If the integral crosses zero it will not be able to cross zero again. Hence the integral in (119) must remain positive for all \( \text{snr} \) (otherwise it has to be strictly negative as \( \text{snr} \to \infty \)). Therefore,

\[ \exp \left( 2h(\sqrt{\text{snr}} X + N) \right) \geq \exp \left( h(\sqrt{\text{snr}} W + N) \right) \]
\[ = 2\pi e \left( a^2 \text{snr} + 1 \right) \]
\[ = \exp \left( 2h(\sqrt{\text{snr}} X) \right) + 2\pi e \]

which is equivalent to (117) by choosing \( \text{snr} = \sigma_Z^{-2} \) and appropriate scaling.

The preceding proof technique also applies to conditional EPI, which concerns \( h(X|U) \) and \( h(X + Z|U) \), where \( Z \) is Gaussian independent of \( U \). The conditional EPI can be used to establish the capacity region of the scalar broadcast channel in [30], [31].
VII. CONCLUDING REMARKS

This paper has established a number of basic properties of the MMSE in Gaussian noise as a transform of the input distribution and function of the SNR. Because of the intimate relationship MMSE has with information measures, its properties find direct use in a number of problems in information theory.

The MMSE can be viewed as a transform from the input distribution to a function of the SNR: 
\[ P_X \mapsto \{ \text{mmse}(P_X, \gamma), \gamma \in [0, \infty) \} \]. An interesting question remains to be answered: Is this transform one-to-one? We have the following conjecture:

Conjecture 1: For any zero-mean random variables \( X \) and \( Z \), 
\[ \text{mmse}(X, \text{snr}) \equiv \text{mmse}(Z, \text{snr}) \] for all \( \text{snr} \in [0, \infty) \) if and only if \( X \) is identically distributed as either \( Z \) or \( -Z \).

There is an intimate relationship between the real analyticity of MMSE and Conjecture 1. In particular, MMSE being real-analytic at zero SNR for all input and MMSE being an injective transform on the set of all random variables (with shift and reflection identified) cannot both hold. This is because given the real analyticity at zero SNR, MMSE can be extended to an open disk \( D \) centered at zero via the power series expansion, where the coefficients depend only on the moments of \( X \). Since solution to the Hamburger moment problem is not unique in general, there may exist different \( X \) and \( X' \) with the same moments, and hence their MMSE function coincide in \( D \). By the identity theorem of analytic functions, they coincide everywhere, hence on the real line. Nonetheless, if one is restricted to the class of sub-Gaussian random variables, the moments determine the distribution uniquely by Carleman’s condition [24].

APPENDIX A
PROOF OF PROPOSITION 5

Proof: Let \( Y = \sqrt{\text{snr}} X + N \) with \( \text{snr} > 0 \). Using (27) and then Jensen’s inequality twice, we have

\[
E \{|X - E\{X|Y\}|^n\} = \text{snr}^{-\frac{n}{2}} 2^n E \{2^{-n}|E\{N|Y\} - N|^n\} \\
\leq \text{snr}^{-\frac{n}{2}} 2^{n-1} E \{|E\{N|Y\}|^n + |N|^n\} \\
\leq \text{snr}^{-\frac{n}{2}} 2^n E \{|N|^n\}
\]

which leads to (28) because

\[
E \{|N|^n\} = \sqrt{\frac{2n}{\pi}} \Gamma \left( \frac{n+1}{2} \right) \\
\leq \sqrt{n!}.
\]

APPENDIX B
PROOF OF PROPOSITION 6

Proof: We use the characterization by moment generating function in Lemma 1

\[
E \{e^{tX_a}\} = \frac{1}{h_0(y;a)} E \{e^{tX} \varphi(y - aX)\} \\
= \frac{\varphi(y)}{h_0(y;a)} E \left\{ \exp \left( (t + ay)X - \frac{a^2 X^2}{2} \right) \right\} \\
\leq \frac{\varphi(y)}{h_0(y;a)} \exp \left( \frac{(t + ay)^2}{2a^2} \right) \\
\leq \frac{\varphi(y)}{h_0(y;a)} \exp \left( \frac{t^2}{a^2} + y^2 \right)
\]
where (130) and (131) are due to elementary inequalities. Using Chernoff’s bound and (131), we have
\[
P \{ X_y \geq x \} \leq \mathbb{E} \left\{ e^{t(X_y-x)} \right\} \leq \frac{\varphi(y) e^{y^2/2}}{h_0(y; a)} \exp \left( \frac{t^2}{2a^2} - tx \right)
\]
for all \( x, t > 0 \). Choosing \( t = \frac{a^2 x^2}{2} \) yields
\[
P \{ X_y \geq x \} \leq \frac{e^{y^2/2}}{h_0(y; a)} \varphi \left( \frac{ax}{\sqrt{2}} \right).
\]
Similarly, \( P \{ X_y \leq -x \} \) admits the same bound as above, and (32) follows from the union bound. Then, using an alternative formula for moments [33, p. 319]:
\[
\mathbb{E} \left\{ |X|^{n} \right\} = n \int_{0}^{\infty} x^{n-1} P \left\{ |X| \geq x \right\} \, dx
\]
\[
\leq 2ne^{y^2/2} \int_{0}^{\infty} x^{n-1} \varphi \left( \frac{ax}{\sqrt{2}} \right) \, dx
\]
\[
\leq ne^{y^2/2} \left( \frac{\sqrt{2}}{|a|} \right)^n \mathbb{E} \{ \mathcal{N}^{n-1} \}
\]
where \( \mathcal{N} \sim \mathcal{N}(0, 1) \) and (136) is due to (32). The inequality (33) is thus established by also noting (127). Conditioned on \( Y = y \), using similar techniques leading to (125), we have
\[
\mathbb{E} \{ |X - \mathbb{E} \{ X \mid Y \}|^{n} \mid Y = y \}
\leq 2^{n-1}(\mathbb{E} \{ |X|^{n} \mid Y = y \} + |\mathbb{E} \{ X \mid Y = y \}|^{n})
\]
\[
\leq 2^{n} \mathbb{E} \{ |X|^{n} \mid Y = y \}
\]
which is (34).

**APPENDIX C**

**PROOF OF LEMMA 2**

We first make the following observation:

Lemma 3: For every \( i = 0, 1, \ldots \), the function \( g_i \) is a finite weighted sum of functions of the following form:
\[
\frac{1}{h_0^{k-1}} \prod_{j=1}^{k} h_{n_j}^{(m_j)}
\]
(140)

where \( n_j, m_j, k = 0, 1, \ldots \).

*Proof:* We proceed by induction on \( i \): The lemma holds for \( i = 0 \) by definition of \( g_0 \). Assume the induction hypothesis holds for \( i \). Then
\[
\frac{\partial}{\partial a} \left( \frac{1}{h_0^{k-1}} \prod_{j=1}^{k} h_{n_j}^{(m_j)} \right) = \frac{-(k-1)}{h_0^{k-1}} h_0' \prod_{j=1}^{k} h_{n_j}^{(m_j)}
\]
\[
+ \frac{1}{h_0^{k-1}} \sum_{l=1}^{k} h_{n_l+1}^{(m_l+1)} \prod_{j \neq l} h_{n_j}^{(m_j)}
\]
(141)

which proves the lemma.
To show the absolutely integrability of \( g_i \), it suffices to show the function in (140) is integrable:

\[
\int_{-\infty}^{\infty} \left| \frac{1}{h_0^{k-1}(y; a)} \prod_{j=1}^{k} \frac{\partial^m_i h_{n_j}(y; a)}{\partial a_{m_j}} \right| dy
\]

\[= E \left\{ \prod_{j=1}^{k} \left| \frac{1}{h_0(Y; a)} \frac{\partial^m_i h_{n_j}(Y - aX)}{\partial a_{m_j}} \right| \right\} \tag{142}
\]

\[= E \left\{ \prod_{j=1}^{k} \left| E \left\{ X^{n_j + m_j} H_{m_j}(Y - aX) \right| Y \right\} \right\} \tag{143}
\]

\[\leq \prod_{j=1}^{k} \left[ E \left\{ \left( E \left\{ |X^{n_j + m_j} H_{m_j}(Y - aX)| | Y \right\} \right)^k \right\} \right]^{\frac{1}{2}} \tag{144}
\]

\[\leq \prod_{j=1}^{k} \left[ E \left\{ |X|^{k(n_j + m_j)} \right\} E \left\{ |H_{m_j}(N)|^k \right\} \right]^{\frac{1}{2}} \tag{145}
\]

\[< \infty \tag{146}
\]

where (143) is by (41), (144) is by the generalized Hölder inequality [34, p. 46], and (145) is due to Jensen’s inequality and the independence of \( X \) and \( N = Y - aX \).

APPENDIX D

PROOF OF PROPOSITION 8 ON THE ANALYTICITY

We first assume that \( X \) is sub-Gaussian.

Note that \( \varphi \) is real analytic everywhere with infinite radius of convergence, because \( \varphi^{(n)}(y) = (-1)^n H_n(y) \varphi(y) \) and Hermite polynomials admits the following bound [35, p. 997]:

\[|H_n(y)| \leq \kappa \sqrt{n!} e^{y^2} \tag{147}
\]

where \( \kappa \) is an absolute constant. Hence

\[\lim_{n \to \infty} \left| \varphi^{(n)}(y) \right|^{\frac{1}{n!}} = 0 \tag{148}
\]

and the radius of convergence is infinite at all \( y \). Then

\[\varphi(y - a'x) = \sum_{n=0}^{\infty} \frac{H_n(y - ax) \varphi(y - ax) x^n}{n!} (a' - a)^n \tag{149}
\]

holds for all \( a, x \in \mathbb{R} \). By Lemma 1, there exists \( c > 0 \), such that \( E \left\{ |X|^n \right\} \leq c^n \sqrt{n!} \) for all \( n = 1, 2, \ldots \).

By (147), it is easy to see that \( |H_n(y)\varphi(y)| \leq \kappa \sqrt{n!} \) for every \( y \). Hence

\[E \left\{ |H_n(y - aX)\varphi(y - aX)X^n| \right\} \leq \kappa c^n n! \tag{150}
\]

Thus for every \( |a' - a| < R \triangleq \frac{1}{c} \),

\[\sum_{n=0}^{\infty} \frac{|a' - a^n|}{n!} E \left\{ |(H_n \cdot \varphi)(y - aX)X^n| \right\} < \infty. \tag{151}
\]

Applying Fubini’s theorem to (149) yields

\[h_0(y; a') = \sum_{n=0}^{\infty} \frac{(a' - a)^n}{n!} E \left\{ (H_n \cdot \varphi)(y - aX)X^n \right\} \tag{152}
\]
Therefore, \( h_0(y; a) \) is real analytic at \( a \) and the radius of convergence is lower bounded by \( R \) independent of \( y \). Similar conclusions also apply to \( h_1(y; a) \) and

\[
h_1(y; a') = \sum_{n=0}^{\infty} \frac{(a' - a)^n}{n!} E \left\{ (H_n \cdot \varphi)(y - aX)X^{n+1} \right\}
\]

(153)

holds for all \( y \in \mathbb{R} \) and all \( |a' - a| < R \). Extend \( h_0(y; a) \) and \( h_1(y; a) \) to the complex disk \( D(a, R) \) by the power series (152) and (153). By (55), there exists \( 0 < r < R/2 \), such that \( h_0(y; z) \) does not vanishes on the disk \( D(a, r) \). By [19, Proposition 1.1.5], for all \( y \in \mathbb{R} \),

\[
g_0(y; z) = \frac{h_1^2(y; z)}{h_0(y; z)}
\]

(154)

is analytic in \( z \) on \( D(a, r) \).

By assumption (56), there exist \( B, c > 0 \), such that

\[
|h_0(y; z)| \geq c h_0(y; \text{Re}(z))
\]

(155)

for all \( z \in D(a, r) \) and all \( |y| \geq B \). Define

\[
m_0^B(z) = \int_{-B}^{B} g_0(y; z) dy.
\]

(156)

Since \( (y, z) \mapsto g_0(y; z) \) is continuous, for every closed curve \( \gamma \) in \( D(a, r) \), we have

\[
\oint_{\gamma} \int_{-B}^{B} |g_0(y; z)| dy dz < \infty.
\]

By Fubini’s theorem,

\[
\oint_{\gamma} \int_{-B}^{B} g_0(y; a) dy dz = \int_{-B}^{B} \oint_{\gamma} g_0(y; a) dz dy = 0
\]

(157)

where the last equality follows from the analyticity of \( g_0(y; \cdot) \). By Morera’s theorem [36, Theorem 3.1.4], \( m_0^B \) is analytic on \( D(a, r) \).

Next we show that as \( B \to \infty \), \( m_0^B \) tends to \( m_0 \) uniformly in \( z \in D(a, r) \). Since uniform limit of analytic functions is analytic [37, p. 156], we obtain the analyticity of \( m_0 \). To this end, it is sufficient to show that \( \{|g_0(\cdot; z) : z \in D(a, r)\} \) is uniformly integrable. Let \( z = s + it \). Then

\[
|h_1(y; z)| = |E \{ X \varphi(y - zX) \} |
\]

(158)

\[
\leq E \{|X||\varphi(y - zX)|\}
\]

(159)

\[
= E \left\{ |X|\varphi(y - sX)e^{\frac{1}{2}t^2X^2} \right\}.
\]

(160)

Therefore, for all \( z \in D(a, r) \),

\[
\int_{\mathbb{R}} \int_{-K}^{K} \frac{|g_0(y; z)|^2 dy \cdot \int_{-K}^{K} |g_0(y; z)|^2 dy}{\frac{1}{c^2} \int_{\mathbb{R}} \left| \frac{h_1(y; z)}{h_0(y; s)} \right|^4 h_0^2(y; s) dy}
\]

(161)

\[
\leq \frac{1}{c^2} \int_{\mathbb{R}} \left| \frac{h_1(y; z)}{h_0(y; s)} \right|^4 h_0^2(y; s) dy
\]

(162)

\[
\leq \frac{1}{c^2} E \left\{ \left( \frac{|X|e^{\frac{1}{2}t^2X^2} \varphi(y - sX)}{h_0(y; s)} \right)^4 \right\} h_0(y; s) dy
\]

(163)

\[
\leq \frac{1}{c^2} E \left\{ \left( \frac{|X|e^{\frac{1}{2}t^2X^2} \varphi(y - sX)}{h_0(y; s)} \right)^4 \right\}
\]

(164)

\[
X^4 e^{2t^2X^2}
\]
where (161) is by (56), (162) is by $|h_0(y; s)| \leq 1$, (163) is by (160), and (164) is due to Jensen’s inequality and $|t| \leq r$. Since $X$ is sub-Gaussian satisfying (29) and $r < R/2 = 1/(2c)$,
\[
E \left\{ X^4 e^{2r^2 X^2} \right\} \leq \sum_{n=0}^{\infty} \frac{(2r^2)^n}{n!} E \left\{ |X|^{2n+4} \right\}
\leq \sum_{n=0}^{\infty} \frac{(2r^2)^n}{n!} \sqrt{(2n + 4)} e^{2n+4}
\leq 4c^4 \sum_{n=0}^{\infty} (n^2 + 3n + 2)(2r)^{2n}
< \infty.
\]
Therefore $\{|g_0(\cdot; z) : z \in D(a, r)\}$ is $L^2$-bounded, hence uniformly integrable. We have thus shown that $m_0(a)$, i.e., the MMSE, is real analytic in $a$ on $\mathbb{R}$.

We next consider positive SNR and drop the assumption of sub-Gaussianity of $X$. Let $a_0 > 0$ and fix $\delta$ with $0 < \sqrt{\delta} < a_0/2$. We use the incremental-SNR representation for MMSE in (48). Define $X_\delta$ to be distributed according to $X - E \{ X | Y_\delta = u \}$ conditioned on $Y_\delta = u$ and recall the definition of and $h_1(y; a|u; \delta)$ in (49). In view of Proposition 6, $X_\delta$ is sub-Gaussian whose growth of moments only depends on $\delta$ (the bounds depend on $u$ but the terms varying with $n$ do not depend on $u$). Repeating the arguments from (147) to (153) with $c = \sqrt{2/\delta}$, we conclude that $h_0(y; a|u; \delta)$ and $h_1(y; a|u; \delta)$ are analytic in $a$ and the radius of convergence is lower bounded by $R = \sqrt{\delta}/2$, independent of $u$ and $y$.

Let $r < \sqrt{\delta}/4$. The remaining argument follows as in the first part of this proof, except that (161)–(168) are replaced by the following estimates: Let $\tau = t^2/2$, then
\[
E \left\{ \left( E \left\{ \left| X \right| e^{\tau X^2} \right| Y_{\delta'}, Y_\delta \right\} \right)^4 \right\}
\leq E \left\{ \left( E \left\{ \left| X \right| e^{\tau X^2} \right| Y_\delta \right\} \right)^4 \right\}
\leq E \left\{ \prod_{i=1}^{4} \sum_{n_i=0}^{\infty} \frac{\tau^{n_i}}{n_i!} E \left\{ \left| \bar{X} \right|^{2n_i+1} \right| Y_\delta \right\} \right\}
\leq \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \binom{8\tau}{\delta} \sum_{n_i=0}^{\infty} \binom{32\tau}{\delta^2} n_i
\leq \left( \frac{8\tau}{\delta} \right) \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \binom{32\tau}{\delta^2} n_i
\leq \left( \frac{8\tau}{\delta} \right) \sum_{n=0}^{\infty} \binom{32\tau}{\delta^2} n_i^4
< \infty
\]
where (169) is by Jensen’s inequality, (170) is by Fubini’s theorem, (171) is because $\tau \leq r^2/2 < \delta^2/32$, and (172) is by Lemma 4 to be established next.

Let $M_i$ be defined as in Section IV-A. The following lemma bounds the expectation of products of $|M_i|$:

**Lemma 4:** For any $\text{snr} > 0$, $k$, $i_j$, $n_j \in \mathbb{N}$,
\[
E \left\{ \prod_{j=1}^{k} |M_{i_j}|^{n_j} \right\} \leq \text{snr}^{-\frac{2}{3} 2^n \sqrt{n!}}
\]
where \( n = \sum_{j=1}^{k} i_j n_j \).

**Proof:** In view of Proposition 5, it suffices to establish:

\[
\text{E} \left\{ \prod_{j=1}^{k} |M_{i_j}|^{n_j} \right\} = \text{E} \left\{ \prod_{j=1}^{k} \prod_{l=1}^{n_j} |M_{i_j}|^{\frac{n_j}{n}} \right\} \leq \prod_{j=1}^{k} \prod_{l=1}^{n_j} \left( \text{E} \left\{ |M_{i_j}|^{\frac{n_j}{n}} \right\} \right)^{\frac{n}{n_j}} \leq \prod_{j=1}^{k} \prod_{l=1}^{n_j} \left( \text{E} \left\{ |X - \text{E} \{X \mid Y\}|^{\frac{n}{n_j}} \right\} \right)^{\frac{n}{n_j}} \leq \text{E} \left\{ |X - \text{E} \{X \mid Y\}|^{n} \right\}
\]

where (177) and (178) are due to the generalized Hölder’s inequality and Jensen’s inequality, respectively.

**APPENDIX E**

**PROOF OF PROPOSITION 9 ON THE DERIVATIVES**

The first derivative of the mutual information with respect to the SNR is derived in [1] using the incremental channel technique. The same technique is adequate for the analysis of the derivatives of various other information theoretic and estimation theoretic quantities.

The MMSE of estimating an input with zero mean, unit variance and finite higher-order moments admits the Taylor series expansion at the vicinity of zero SNR given by (61). In general, given a random variable \( X \) with arbitrary mean and variance, we denote its central moments by

\[
m_i = \text{E} \left\{ (X - \text{E} \{X\})^i \right\}, \quad i = 1, 2, \ldots
\]

Suppose all moments of \( X \) are finite, the random variable can be represented as \( X = \text{E} \{X\} + \sqrt{m_2} Z \) where \( Z \) has zero mean and unit variance. Clearly, \( \text{E} Z^i = m_2^{\frac{i}{2}} m_i \). By (61) and Proposition 2,

\[
\text{mmse}(X, \text{snr}) = m_2 \text{mmse}(Z, \text{snr} m_2) = m_2 - m_2^2 \text{snr} + \left( 2m_3^2 - m_3^2 \right) \frac{\text{snr}^2}{2} - \left( m_4^2 - 6m_4 m_2^2 \right) - 12m_3^2 m_2 + 15m_4^2 \frac{\text{snr}^3}{6} + O(\text{snr}^4).
\]

In general, taking into account the input variance, we have:

\[
\text{mmse}'(X, 0) = -m_2^2 \quad (183)
\]

\[
\text{mmse}''(X, 0) = 2m_3^2 - m_3^2 \quad (184)
\]

\[
\text{mmse}'''(X, 0) = -m_4^2 + 6m_4 m_2^2 + 12m_3^2 m_2 - 15m_4^4 \quad (185)
\]

Now that the MMSE at an arbitrary SNR is rewritten as the expectation of MMSEs at zero SNR, we can make use of known derivatives at zero SNR to obtain derivatives at any SNR. Let \( X_{y;\text{snr}} \sim P_{X \mid Y_{\text{snr}} = y} \). Because of (183),

\[
\frac{\text{dmmse}}{\text{d} \gamma} \bigg|_{\gamma=0^+} = - (\text{var} \{ X \mid Y_{\text{snr}} = y \})^2.
\]

24
Thus,

\[
\frac{d\text{mmse}(X, \text{snr})}{d\text{snr}} = \frac{d}{d\gamma} \text{mmse}(X, \text{snr} + \gamma) \bigg|_{\gamma=0^+} = \frac{d}{d\gamma} \text{mmse}(X, \gamma | \text{snr}) \bigg|_{\gamma=0^+} = -E \{ (\text{var} \{ X | \text{snr} \})^2 \} = -E \{ M_2^2 \}
\]

where (188) is due to Proposition 3 and the fact that the distribution of \( Y_{\text{snr}} \) is not dependent on \( \gamma \), and (189) is due to (186) and averaging over \( y \) according to the distribution of \( Y_{\text{snr}} = \sqrt{\text{snr}} X + N \). Hence (64) is proved. Moreover, because of (184),

\[
\frac{d^2\text{mmse}(X_{\gamma;\text{snr}}, \gamma)}{d\gamma^2} \bigg|_{\gamma=0} = 2 (\text{var} \{ X | \text{snr} = y \})^3 - (E \{ (X - E \{ X | \text{snr} \})^3 | \text{snr} = y \})^2
\]

which leads to (65) after averaging over the distribution of \( Y_{\text{snr}} \). Similar arguments, together with (185), lead to the third derivative of the MMSE which is obtained as (66).

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