NEW EXAMPLES OF BIHARMONIC SUBMANIFOLDS
IN $CP^n$ AND $S^{2n+1}$

WEI ZHANG

ABSTRACT. We construct biharmonic real hypersurfaces and Lagrangian submanifolds of Clifford torus type in $CP^n$ via the Hopf fibration; and get new examples of biharmonic submanifolds in $S^{2n+1}$ as byproducts.

1. INTRODUCTION

Let $\phi : (M, g) \rightarrow (N, h)$ be a smooth map between two Riemannian manifolds and define its first tension field as $\tau(\phi) = \text{trace}\nabla d\phi$. $\phi$ is called harmonic if it is a critical point of the energy functional:

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g$$

This amounts to $\tau(\phi) = 0$. In the isometric immersion case, harmonic is equivalent to minimal.

Considering the bienergy $E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g$, its critical points are defined as biharmonic maps ([BB]). The associated Euler-Lagrange equation is given by the vanishing of the bitension field:

$$\tau_2(\phi) = -\Delta^\phi \tau(\phi) - \text{trace} R^N(d\phi, \tau(\phi))d\phi$$

Obviously, any harmonic map is biharmonic. We call the non-harmonic one proper biharmonic. A submanifold is called biharmonic if the inclusion map is biharmonic.

Concerning the proper biharmonic map, there are several non-existence results for the non-positive sectional curvature codomains ([J1, BB]), for instance:

**Theorem 1** ([J1, J2]). Let $\phi : (M, g) \rightarrow (N, h)$ be a smooth map. If $M$ is compact, orientable and $\text{Riem}_N \leq 0$, then $\phi$ is biharmonic if and only if it is harmonic.

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Suggested by this kind of results:

**Generalized Chen’s Conjecture.** *Biharmonic submanifolds of a manifold N with Riem⁰ ≤ 0 are minimal.*

So it is sensible to focus on the proper biharmonic submanifolds in sphere or other non-negatively curved space. Although some partial classification results had been obtained ([BMO]), the known examples are still relatively rare. In this article, we will construct series new examples in $S^{2n+1}$ and $CP^n$.

**Theorem 2.** Denote $M_{p,q}^C(r,s)$ be the image of the generalized Clifford torus $M_{p,q}(r, s) = S^{2p+1}(r) \times S^{2q+1}(s)$ in $S^{2n+1}$ via the Hopf map, where $p+q=n-1$. Then it is biharmonic in $CP^n$ if and only if:

1. \[
\left( \frac{s}{r} \right)^2(2p+1) + \left( \frac{r}{s} \right)^2(2q+1) = 2(n+2)
\]

And

**Theorem 3.** Identify $R^{2n+2}$ with $C^{n+1}$. Define the Clifford torus $T^{n+1} = \{|z_i| = a_i| \sum_{i=1}^{n+1} a_i^2 = 1\}$ in $S^{2n+1}$. Then it is biharmonic if and only if:

2. \[
a_i d - \frac{1}{a_i^3} = 2(n+1)((n+1)a_i - \frac{1}{a_i})
\]

where $d = \sum \frac{1}{a_i^2}$, $i = 1, \ldots, n+1$.

Followed by:

**Theorem 4.** Let $T^n_C$ be the quotient of $T^{n+1}$ by the $S^1$ action. Then it is biharmonic in $CP^n$ if and only if:

3. \[
a_i d - \frac{1}{a_i^3} = 2(n+3)((n+1)a_i - \frac{1}{a_i})
\]

The explicit expressions of $r, s$ and $a_i$ will be solved out in section 3. Among these torus, we can pick out plenty proper ones. As the construction is very routinely, we would emphasis on the method rather than the concrete examples.

2. Preliminary

2.1. Hopf fibration. There are already many known examples of biharmonic submanifolds in sphere. To find examples in $CP^n$, the Hopf
fibration: \( \pi : S^{2n+1} \rightarrow CP^n \) is a natural candidate, where \( \pi \) is a Riemann submersion with totally geodesic fibres \( S^1 \) and \( CP^n \) has constant holomorphic sectional curvature 4.

This submersion has lots of good properties, such as:

**Lemma 5 ([WL]).** If \( \tilde{M}, M \) are submanifolds respect this Riemann submersion, i.e. the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\pi} & S^{2n+1} \\
\downarrow \pi & & \downarrow \pi \\
M & \xrightarrow{\pi} & CP^n
\end{array}
\]

then \( \tilde{M} \) minimal(totally geodesic) is equivalent to \( M \) minimal(totally geodesic). More precisely, the mean curvature \( \tilde{H} \) of \( \tilde{M} \) is the horizontal lift of \( H \) of \( M \).

In this paper, to simplify the notations, we take the convention that \( H \) is non-normalized, i.e \( H = \tau(i) \).

Unfortunately, this submersion do not preserve the biharmonicity. See [LO], consider the Hopf map \( S^3 \rightarrow S^2(\frac{1}{2}) \). Lifting the biharmonic submanifold \( S^1(\frac{\sqrt{2}}{4}) \), we get the Clifford torus \( S^1(\frac{\sqrt{2}+\sqrt{2}}{2}) \times S^1(\frac{\sqrt{2}-\sqrt{2}}{2}) \), which is not biharmonic in \( S^3 \).

But it provides us the stereotype of constructing biharmonic submanifolds in \( CP^n \): If we modify the radius of the standard Clifford torus properly, its image would be biharmonic in \( CP^n \).

### 2.2. The biharmonic equations in \( S^n \) and \( CP^n \)

When the submanifold lies in \( S^n \), it is convenient to split the bitension field in its normal and tangent components.

**Theorem 6 ([CMO]).** \( M^m \) is a submanifold of \( S^n \), then it is biharmonic if and only if

\[
\begin{cases}
-\Delta H - \text{trace} B(\cdot, A_H \cdot) + mH = 0 \\
2\text{trace} A_{\nabla^H f} H(\cdot) + \frac{1}{2} \text{grad}(|H|^2) = 0
\end{cases}
\]

Moreover, if we assume \( M \) has parallel mean curvature, the equation turns to:

\[
\sum_{i,j} B_{ij} < B_{ij}, H >= mH
\]
Where $A$ denotes the Weingarten operator, $B$ the second fundamental form, $\nabla^\perp$ and $\Delta^\perp$ the connection and the Laplacian in the normal bundle.

When the ambient space is $CP^n$, considering the real hypersurfaces or the Lagrangian submanifolds, the biharmonic equation has similar form:

**Proposition 7.** $M^{2n-1}$ is a real hypersurface in $CP^n$, then it is biharmonic if and only if

$$\begin{aligned}
-\Delta^\perp H - trace B(\cdot, A_H \cdot) + 2(n + 1)H &= 0 \\
2trace A\nabla^\perp_{\cdot, H}(\cdot) + \frac{1}{2} grad(|H|^2) &= 0
\end{aligned}$$

If $M$ has parallel mean curvature in addition, the equation becomes:

$$||B||^2 = 2(n + 1)$$

And

**Proposition 8.** $M^n$ is a Lagrangian submanifold of $CP^n$, then it is biharmonic if and only if

$$\begin{aligned}
-\Delta^\perp H - trace B(\cdot, A_H \cdot) + (n + 3)H &= 0 \\
2trace A\nabla^\perp_{\cdot, H}(\cdot) + \frac{1}{2} grad(|H|^2) &= 0
\end{aligned}$$

When $M$ with parallel mean curvature, it is simplified to:

$$\sum_{i,j} B_{ij} < B_{ij}, H >= (n + 3)H$$

*Proof of Proposition 7:* Denote the canonical inclusion map as $i$, then $\tau(i) = H$.

Since $trace R^{CP^n}(di, \tau(i))di = -2(n + 1)\tau(i) = -2(n + 1)H$, $i$ biharmonic $\iff$

$$\tau_2(i) = trace \nabla dH + 2(n + 1)H = 0$$

Choose normal frame $\{e_\alpha\}$ on $M$, where $1 \leq \alpha \leq 2n - 1$.

$$\begin{aligned}
trace \nabla dH &= \nabla_{e_\alpha}^{CP^n} \nabla_{e_\alpha}^{CP^n} H = \nabla_{e_\alpha}^{CP^n} (\nabla_{e_\alpha}^\perp H - A_H(e_\alpha)) \\
&= \nabla_{e_\alpha}^\perp \nabla_{e_\alpha} H - A_{\nabla_{e_\alpha}^\perp H}(e_\alpha) - \nabla_{e_\alpha} A_H(e_\alpha) - B(e_\alpha, A_H(e_\alpha)) \\
&= -\Delta^\perp H - trace B(\cdot, A_H \cdot) - (A_{\nabla_{e_\alpha}^\perp H}(e_\alpha) + \nabla_{e_\alpha} A_H(e_\alpha))
\end{aligned}$$
Rewrite $A_H$ and use Codazzi equation:
\[
\nabla_{e_\alpha}A_H(e_\alpha) = (\nabla_{e_\alpha} < B(e_\beta, e_\alpha), H >) e_\beta
\]
\[
= < B(e_\beta, e_\alpha), \nabla^\perp_{e_\alpha} H > e_\beta + (\nabla^\perp_{e_\alpha} B)(e_\beta, e_\alpha), H >) e_\beta
\]
\[
= A_{\nabla^\perp_{e_\alpha} H}(e_\alpha) + (\nabla^\perp_{e_\alpha} B)(e_\beta, e_\alpha), H >) e_\beta + R^{C^n}(e_\alpha, e_\beta) e_\alpha, H > e_\beta
\]
\[
= A_{\nabla^\perp_{e_\alpha} H}(e_\alpha) + \frac{1}{2} \text{grad}(|H|^2)
\]
where $< R^{C^n}(e_\alpha, e_\beta) e_\alpha, H > = \text{Ric}^{C^n}(e_\beta, H) = 0$ for any $\beta$ by $C^n$ is Einstein.

Replacing $\text{trace} \nabla dH$ in the identity and arranging the terms in tangent and normal components, we get what desired.

\[\square\]

Proof of Proposition 8 is similar.

Our general modus operandi is adjusting the radius of the Clifford torus until their second fundamental forms satisfying the biharmonic equations, and the fact that the Clifford type torus we concern about all have parallel mean curvature simplifies the computation.

3. The examples

3.1. Generalized circles $M^C_{p,q}(r, s)$ in $C^p$. In [L], Lawson introduced the concept of generalized equator $M^C_{p,q}$ which is minimal in $C^n$. Thus we call $M^C_{p,q}(r, s)$ generalized circle which is not always minimal.

For the generalized Clifford torus in $S^{2n+1}$ have constant mean curvature, so does $M^C_{p,q}(r, s)$.

By (7), $M^C_{p,q}(r, s)$ is biharmonic if $||B||^2 = 2(n + 1)$.

While form [L], we know $||B||^2 = ||\tilde{B}||^2 - 2$, where $\tilde{B}$ is the second fundamental form of the generalized Clifford torus in $S^{2n+1}$. Combining the fact $||\tilde{B}||^2 = (\zeta)^2(2p + 1) + (\xi)^2(2q + 1)$ leads to (1). It is easy to check the solution is never minimal.

This is a table of biharmonic real hypersurfaces in $C^p$ (We only list the case $n=5$).

| n  | p  | q  | r                        | s                        |
|-----|----|----|--------------------------|--------------------------|
| 5   | 0  | 4  | $\sqrt{(3 \times (5 - \sqrt{3})}/22$ | $\sqrt{(7 + (3 \times \sqrt{3})}/22$ |
| 5   | 0  | 4  | $\sqrt{(3 \times (5 + \sqrt{3})}/22$ | $\sqrt{(7 - (3 \times \sqrt{3})}/22$ |
| 5   | 1  | 3  | $\sqrt{(13 - \sqrt{15})}/22$       | $\sqrt{(9 + \sqrt{15})}/22$ |
| 5   | 1  | 3  | $\sqrt{(13 + \sqrt{15})}/22$       | $\sqrt{(9 - \sqrt{15})}/22$ |
| 5   | 2  | 2  | $\sqrt{(11 + \sqrt{11})}/22$       | $\sqrt{(11 - \sqrt{11})}/22$ |
3.2. **Clifford torus** $T^{n+1}$ in $S^{2n+1}$. Denote the position vector by $x = a_i x_i$, where $x_i$ is the unit vector. Choosing normal frame $\{e_i\}$ of the tangent space s.t $J e_i = x_i$ where $J$ is the complex structure in $C^{n+1}$. Direct computation show that $B_{ij} = \delta_{ij}(-\frac{a_i}{a_j} + x)$ and $H = \sum(a_i(n+1) - \frac{1}{a_i})x_i$.

Thus

$$\sum B_{ij} < B_{ij}, H >_e (\sum (a_i d - \frac{1}{a_i^2})x_i) - (n+1)H$$

where $d = \sum_{i=1}^{n+1} \frac{1}{a_i}$.

Furthermore, $T^{n+1}$ has parallel mean curvature. If it is biharmonic, by (5)

$$(\sum (a_i d - \frac{1}{a_i^2})x_i) - (n+1)H = (n+1)H$$

Compare each components of $H$, we have (2).

This equation system is some how funny. Denote $b_i = a_i^2$, rewrite it as:

$$(10) \quad (2(n+1)^2 - d)b_i^2 - 2(n+1)b_i + 1 = 0$$

where $d = \sum_{i=1}^{n+1} \frac{1}{b_i}$.

There are $(n+2)$ equations and $(n+1)$ variables, while the equations are not independent. It is always solvable. To see this, view $d$ as a known number. Let $r, s$ be the two different roots of the quadratic (10), and $p, q$ be their multiplicities in $b_i$ respectively. Then $pr + qs = 1$, and

$$d = \frac{p}{r} + \frac{q}{s} = \frac{p}{r} + \frac{q-p}{s} = \frac{r+s}{rs} + \frac{q-p}{q-p}$$

Let $t = 2(n+1)^2 - d$, use Vieta’s Theorem:

$$(2q(n+1) - t)(t - 2p(n+1)) = (q - p)^2 t$$

i.e.

$$t^2 - (2(n+1)^2 - (p - q)^2)t + 4pq(n+1)^2 = 0$$

This equation’s discriminant is:

$$\Delta = (p - q)^4 + 4(n+1)^4 - (4(p - q)^2(n+1)^2 + 16pq(n+1)^2)$$
$$= (p - q)^4 + 4(n+1)^4 - 4(p + q)^2(n+1)^2 = (p - q)^4$$

The root $t = (n+1)^2$ leads to minimal torus, so we pick the one $t = (n+1)^2 - (p - q)^2$. The original quadratic becomes:
\(((n+1)^2 - (p-q)^2)b_i^2 - 2(n+1)b_i + 1 = 0\)

Its two roots are \(\frac{1}{(n+1)\pm (p-q)}\) with multiplicities \(p\) and \(q\).

Thus the torus is with elegant form \(T_{p,q}^{n+1}(\frac{1}{\sqrt{2p}}, \frac{1}{\sqrt{2q}})\). It can not be minimal unless \(p=q\).

3.3. **Clifford torus** \(T_C^n\) in \(CP^n\). \(T_C^n\) is the quotient of \(T^{n+1}\) by the \(S^1\) action. It is Lagrangian in \(CP^n\), and has parallel mean curvature fields.

Choose orthonormal frame \(\{f_i\}, i = 1, \ldots, n\). To compute \(\sum_{i,j} B_{i,j} < B_{i,j}, H >\), as the Hopf map preserve the horizontal part of second fundamental([WL]) and the mean curvature, we lift it to \(S^{2n+1}\). Without ambiguity, denote \(\{f_i, \nu\}\) the orthonormal basis of \(T^{n+1}\), where \(\nu = Jx, x\) the position vector, is the vertical vector fields. Notice that \(\sum_{\alpha, \beta} \tilde{B}_{\alpha,\beta} < \tilde{B}_{\alpha,\beta}, \tilde{H} >\) is independent of basis, thus:

\[
\sum_{\alpha, \beta} \tilde{B}_{\alpha,\beta} < \tilde{B}_{\alpha,\beta}, \tilde{H} > - 2\tilde{B}_{i,\nu} < \tilde{B}_{i,\nu}, \tilde{H} > - \tilde{B}_{\nu,\nu} < \tilde{B}_{\nu,\nu}, \tilde{H} > = \sum_{i,j} B_{i,j} < B_{i,j}, H >
\]

It is not hard to show:

\[
\tilde{B}_{i,\nu} < \tilde{B}_{i,\nu}, \tilde{H} > = - J f_i < - J f_i, \tilde{H} > = \tilde{H}
\]

and \(\tilde{B}_{\nu,\nu} = 0\).

Follow the biharmonic equation (9):

\[
\sum_{\alpha, \beta} \tilde{B}_{\alpha,\beta} < \tilde{B}_{\alpha,\beta}, \tilde{H} > = (n+5)\tilde{H}
\]

This is nothing but (3). We solve it in the same way.

\[
d = \frac{p}{r} + \frac{p}{s} + \frac{q-p}{s} = \frac{r}{p} + \frac{s}{r} + \frac{q-p}{q-p}
\]

Let \(d = 2(n+1)(n+3) - d\), we have:

\[
t^2 - 2(n+1)(n+3) - (p-q)^2)t + 4pq(n+3)^2 = 0
\]

Its discriminant is:

\[
\Delta = (p-q)^4 + 8(n+2)(p-q)^2
\]
Both roots of $t$ make the original quadratic about $r$ and $s$ solvable. Same as the generalized circles $M_{p,q}^C(r,s)$ case, the biharmonic torus $T_{C_{p,q}}^n$ could never be minimal.

The software "Mathematica" is helpful to get the explicit expressions. The following is the table of $T_{C_{p,q}}^n(r,s)$ when $n=4$:

| $n$ | $p$ | $q$ | $r$ | $s$ |
|-----|-----|-----|-----|-----|
| 4   | 1   | 4   | $\sqrt{(11-\sqrt{65})/7}/2$ | $\sqrt{(17+\sqrt{65})/7}/4$ |
| 4   | 1   | 4   | $\sqrt{(11+\sqrt{65})/7}/2$ | $\sqrt{(17-\sqrt{65})/7}/4$ |
| 4   | 2   | 3   | $\sqrt{(13-\sqrt{57})/14}/2$ | $\sqrt{(15+\sqrt{57})/21}/2$ |
| 4   | 2   | 3   | $\sqrt{(13+\sqrt{57})/14}/2$ | $\sqrt{(15-\sqrt{57})/21}/2$ |

We add a remark on the biharmonic stability of such type torus. A biharmonic submanifold is called stable if for any variation of the inclusion map $\{i_t\}$, its second derivative of $E_2$ is nonnegative. Using $H = \tau(i)$ as the variation vector fields, by the second variation formula in [12]:

$$\frac{1}{2} \frac{d^2}{dt^2} E_2(i_t)|_{t=0} = 4 \int_T R_{C^P}^n(f_k,H) \nabla_{f_k}^{C^P} H, H > dv_g$$

$$= 4 \int_T (\|H\|^4 + 3 < f_k, JH > < \nabla_{f_k}^{C^P} H, JH > ) dv_g$$

Notice that $JH = < JH, f_i > f_i$ and

$$< \nabla_{f_k}^{C^P} H, f_i > = - < \nabla_{f_k}^{C^P} f_i, H > = - < B_{kl}, H >$$

the above expression becomes:

(12) $$- 4 \int_T (\|H\|^4 + 3 < Jf_k, H > < B_{kl}, H > < Jf_l, H > ) dv_g$$

Still lift them to $S^{2n+1} \subset C^{n+1}$ for computation. Using $\{f_i, \nu\}$ and $\{e_\alpha\}$ as frames alternatively, as

$$< Jf_k, H > < B_{kl}, H > < Jf_l, H >$$

and $< J\nu, \tilde{H} > = 0$ where $J$ is the complex structure in $C^{n+1}$, we have:

$$< Jf_k, H > < B_{kl}, H > < Jf_l, H > = < J\tilde{e}_\alpha, \tilde{H} > < \tilde{B}_{\alpha\beta}, \tilde{H} > < J\tilde{e}_\beta, \tilde{H} >$$

$$= < J\tilde{e}_\alpha, \tilde{H} >^2 < \tilde{B}_{\alpha\alpha}, \tilde{H} >$$
Every term in it has explicit expression, thus:

\[
\frac{1}{2} \frac{d^2}{dt^2} E_2(i_t) |_{t=0} = -4 \int [d-(n+1)^2]^2 + 3(2(n+1)^3 + (\sum \frac{1}{a_i}) - 3(n+1)d] dv_g
\]

Recall (11) and \( t = 2(n + 1)(n + 3) - d \), after tedious computation, we get:

\[
\frac{1}{2} \frac{d^2}{dt^2} E_2(i_t) |_{t=0} < 0
\]

i.e. the torus are unstable.

By (12), we easily get:

**Proposition 9.** Let \( M \) be a compact proper biharmonic submanifold of \( CP^n \), if the eigenvalues of its Weingarten operator are all nonnegative, then it can not be stable.

Although it can't apply to our examples.

All the computations in above sections are valid in \( QP^n \) and \( S^{4n+3} \), so the same method can generate lots of examples of biharmonic submanifolds in \( QP^n \) as well.

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Wei Zhang

School of Mathematical Sciences
Fudan University
Shanghai, 200433, P. R. China

Email address: 032018009@fudan.edu.cn