Group-like objects in Poisson geometry and algebra

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Abstract

A group, defined as set with associative multiplication and inverse,
is a natural structure describing the symmetry of a space. The concept of group generalizes to group objects internal to other categories than sets. But there are yet more general objects that can still be thought of as groups in many ways, such as quantum groups. We explain some of the generalizations of groups which arise in Poisson geometry and quantization: the germ of a topological group, Poisson Lie groups, rigid monoidal structures on symplectic realizations, groupoids, 2-groups, stacky Lie groups, and hopfish algebras.

1 Introduction

Every mathematician learns that a group is a set with an associative multiplication admitting an identity and inverses. But there are other objects, such as group germs, Lie groups, Poisson groups, and quantum groups, which qualify as groups in many senses, but are either more or less than simply sets with operations satisfying the group actions. These notes will describe some of these objects, with an emphasis on the role of groupoids, both as examples of group-like objects and as a tool for describing the objects themselves. A

These notes are loosely based on the three lectures given by Weinstein at the School on Poisson Geometry and Related Topics, Keio University, May 31–June 2, 2006. We would like to thank Nathan George for the use of his preliminary notes.
goal toward which we strive (but which we will not reach) is to give a unified “categorical” notion which encompasses all of our examples.

Some of our examples will be explicitly geometric. Others will be algebraic, but may still be considered as geometric from the viewpoint which identifies geometric objects with suitable algebras of functions on them and, more generally, considers algebras, even noncommutative ones, as if they were the functions on a space. Even more abstract is the view of spaces as represented by categories, such as the category of representations of an algebra, or the derived category of coherent sheaves on an algebraic variety.

2 Symmetry groups

The (global) symmetry of a space is described by a set of transformations closed under composition and inversion. Conversely, Cayley’s theorem states that every abstract group is also a group of transformations, acting on itself by left multiplication.

At first, a group was just a set, and its structure morphisms were maps between sets. But it is useful to consider groups which themselves have some additional structure, for example a differentiable structure. The product, unit, and inversion morphisms of the group are required to respect this structure, in which case they have to be smooth maps. This leads us naturally to the concept of a Lie group, which is in turn an example of a notion of group internal to a category, meaning that the group is given by an object and group structure morphisms in that category. In this sense, an ordinary group is a group object in the category of sets, a topological group in the category of topological spaces, a Lie group in the category of manifolds, an algebraic group in the category of algebraic varieties, and so on. This notion works well for many categories in which the objects are spaces with geometric or topological structure.

On the other hand, in categories of spaces with algebraic structure, the group objects often turn out to be surprisingly rare. For example, the group objects in the category of vector spaces are the vector spaces themselves with the underlying abelian group structure, while in the category of groups they are the abelian groups. In the category of rings the only group object is the trivial ring with one element.

But there are other kinds of objects which we can naturally associate to symmetries. For example, to the action of a finite group $G$ on a finite
set $S$, the Gelfand “algebraization” functor associates the commutative algebras of functions $\mathcal{A}(G)$ and $\mathcal{A}(S)$ to both the group and the space. This is a contravariant functor, so the group product, unit, and inverse in the group become the comultiplication, counit, and coinverse (antipode) of a Hopf structure on $\mathcal{A}(G)$. The action of the group on the set becomes a coaction of the Hopf algebra $\mathcal{A}(G)$ on the algebra $\mathcal{A}(S)$. Since we can recover the group $G$ from $\mathcal{A}(G)$ as the set of group-like elements and the space $S$ from $\mathcal{A}(S)$ as the set of characters, the description of the symmetry of $S$ in terms of the Hopf algebra $\mathcal{A}(G)$ is completely equivalent to the description in terms of the group. This is why such Hopf algebras, even noncommutative ones, have been termed quantum groups. But quantum groups are not group objects, at least not in any of the underlying categories of vector spaces, of algebras, or of coalgebras. For instance, the coproduct is a map from $\mathcal{A}(G)$ to $\mathcal{A}(G) \otimes \mathcal{A}(G)$, but the tensor product is not a product in the category of algebras (nor in the dual, with arrows reversed). Our conclusion is that there are yet more general structures which we may associate with the concept of a symmetry group.

In fact, there is an ample collection of structures which are considered to be “groups” in the sense that they encode symmetries: groupoids, inverse semi-groups, hypergroups, $n$-groups, Lie algebras, Hopf algebras, etc. — just to name a few. Our goal in these lectures is to identify some of such group-like structures that arise in Poisson geometry and quantization.

3 Group objects

A group germ is an example of a group object in a category which is not (at least in its usual presentation) a subcategory of the category of sets.

We define a topological germ to be the collection of all pointed topological spaces $(X, x)$ modulo the equivalence relation in which two spaces $(X, x)$ and $(Y, y)$ are identified if $x = y$ and if this common point has open neighborhoods in $X$ and $Y$ which are equal as sets and homeomorphic with the induced topologies. A morphism between topological germs is an equivalence class of continuous maps between representatives of the germs, where two maps are considered equivalent if they agree on some neighborhood of the basepoint. This category admits products defined as products of representatives, and a terminal object 1 consisting of a single point. In any such category, a group object is defined to be an object $U$ together with the
structure morphisms of multiplication $m : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$, unit $e : 1 \to \mathcal{U}$, and inverse $\text{inv} : \mathcal{U} \to \mathcal{U}$, such that the following diagrams are commutative. The diagrams for associativity and the unit are

$$
\begin{array}{c}
\mathcal{U} \times \mathcal{U} \times \mathcal{U} \xrightarrow{m \times \text{id}} \mathcal{U} \times \mathcal{U} \\
\downarrow \text{id} \times m \quad \downarrow m \quad \downarrow \text{id} \times \text{id} \\
\mathcal{U} \times \mathcal{U} \xrightarrow{m} \mathcal{U}
\end{array} \quad \begin{array}{c}
1 \times \mathcal{U} \xleftarrow{\cong} \mathcal{U} \xrightarrow{\cong} \mathcal{U} \times 1 \\
\downarrow e \times \text{id} \quad \downarrow \text{id} \quad \downarrow \text{id} \times e \\
\mathcal{U} \times \mathcal{U} \xrightarrow{m} \mathcal{U} \quad \mathcal{U} \xleftarrow{m} \mathcal{U} \times \mathcal{U}
\end{array} \quad (1)
$$

while that for the inverse is

$$
\begin{array}{c}
\mathcal{U} \times \mathcal{U} \xrightarrow{\text{diag}} \mathcal{U} \\
\downarrow \text{inv} \times \text{id} \quad \downarrow \text{id} \times \text{inv} \\
\mathcal{U} \times \mathcal{U} \xrightarrow{\eta} \mathcal{U} \times \mathcal{U} \\
\downarrow 1 \quad \downarrow e \\
\mathcal{U} \times \mathcal{U} \xrightarrow{m} \mathcal{U} \quad \mathcal{U} \xleftarrow{m} \mathcal{U} \times \mathcal{U}
\end{array} \quad (2)
$$

where diag is the diagonal morphism, that is, the unique morphism which lifts the identity on $\mathcal{U}$ to the product $\mathcal{U} \times \mathcal{U}$:

$$
\begin{array}{c}
\mathcal{U} \times \mathcal{U} \\
\downarrow \text{pr}_1 \quad \downarrow \text{pr}_2
\end{array} \quad \begin{array}{c}
\exists! \quad \text{diag} \quad \text{diag} \\
\downarrow \text{id} \quad \downarrow \text{id}
\end{array} \quad \mathcal{U} \quad (3)
$$

Restricting the group structure of a Lie $G$ group to its germ at the unit element $e$ yields such a group object in the category of topological germs which is called the germ of the group at $g$. This is no longer a group in the usual sense, because “it has only one point”. In a similar way, one can define manifold germs and group objects in the category thereof. This is useful, for instance, when one tries to integrate a Banach Lie algebra to a group. The global object does not always exist [9] [10], but its germ does (and is unique up to isomorphism).
4 Poisson Lie groups

What is a Poisson Lie group? It is usually defined as a Lie group $G$ with a Poisson structure such that the multiplication morphism $G \times G \to G$ is a Poisson map. It follows from this that the unit map from a point to $G$ is a Poisson map, and the inversion map is anti-Poisson.

If we try to define a Poisson Lie group as a group object in the category of Poisson manifolds, a first problem is that the category of Poisson manifolds does not admit categorical products. The cartesian product $X \times Y$ does not work, since Poisson maps $A \to X$ and $A \to Y$ yield a Poisson map $A \to X \times Y$ only when the images in $C^\infty(A)$ of $C^\infty(X)$ and $C^\infty(Y)$ Poisson commute.

But let’s forget this for a moment and admit $X \times X$ as some kind of product, if not a categorical one, and assume that $X$ is a Poisson Lie semigroup, i.e. a Poisson manifold with an associative multiplication map $m : X \times X \to X$, which is a Poisson map, and a Poisson unit map, $e : 1 \to X$. If $X$ is a group, we know that inversion is an anti-Poisson map, i.e. a Poisson map $\text{inv} : X^\text{op} \to X$. The categorical defining property of inversion is commutativity of the diagram (2), but we must put in the opposite Poisson structure to get

\[
\begin{array}{ccc}
X^\text{op} \times X & \xrightarrow{\text{diag}} & X \\
\downarrow \text{inv} \times \text{id} & & \downarrow \text{id} \times \text{inv} \\
X \times X & \xrightarrow{\text{diag}} & X \times X^\text{op} \\
\end{array}
\]

But the diagonal map is not a Poisson map from $X$ to $X^\text{op} \times X$ or $X \times X^\text{op}$ unless the Poisson bivector is zero. So the inverse axiom does not have an evident interpretation in the Poisson category.

Another approach is to analyze the inversion map via its graph, which is

\[
\text{graph}(\text{inv}) = \{(x, x^{-1}) \mid x \in X\} = \{(x, y) \in X \times X \mid xy = e\},
\]

i.e., the pull-back, $(X \times X) \times_{m, X, e} 1$. This can be expressed in terms of the graph of multiplication and the opposite of the graph of the unit, $\text{graph}(e)^\text{op} = \cdots$
\{(e, 1)\}, which are coisotropic submanifolds,

\[
\text{graph}(m) \in \text{Cois}((X \times X) \times X^{\text{op}}), \quad \text{graph}(e)^{\text{op}} \in \text{Cois}(X \times 1^{\text{op}}).
\]

(6)

The composition of these two coisotropic submanifolds, viewed as Poisson relations, is again a coisotropic submanifold,

\[
\text{graph}(\text{inv}) = \text{graph}(m) \times_X \text{graph}(e)^{\text{op}} \in \text{Cois}(X \times X),
\]

(7)

where we have used that \(\text{Cois}((X \times X) \times 1^{\text{op}}) \cong \text{Cois}(X \times X)\).

This makes it possible to define a Poisson Lie group in the following way. Starting with any Poisson semigroup, we may first require that \(m\) be transverse to \(e\), so that the fiber product \(\text{graph}(m) \times_X \text{graph}(e)^{\text{op}} \in \text{Cois}(X \times X)\) is a coisotropic submanifold, and then require that the projection of the manifold to one of the factors of \(X \times X\) be a diffeomorphism. This is still not completely “categorical,” but we will see below that it has a useful algebraic analogue.

### 5 Poisson Lie groups and symplectic realizations

It is sometimes possible to describe a group structure on a mathematical object as an extra structure on a category of “representations” of the object.

For a Poisson manifold \(X\), one such category is that of the symplectic realizations, in which the objects are the Poisson maps from symplectic manifolds to \(X\) \([20]\), and the morphisms are symplectic maps forming commutative diagrams with these. We may think of these as symplectic “points” of \(X\), or as geometric representations of \(X\) in the sense that a Poisson map \(J : S \to X\) induces a representation of the Poisson Lie algebra of functions on \(X\) by that of \(S\), or by hamiltonian vector fields:

\[
C^\infty(X) \longrightarrow \mathcal{X}(S), \quad f \longmapsto -X_{J^*(f)}.
\]

(8)

There always exists a symplectic realization which is a surjective submersion \([8] [11]\), for which (up to locally constant functions) this representation is faithful. Therefore, the collection of symplectic realizations encodes all the structural information of \(X\).
Given two symplectic realizations $J_1 : S_1 \to X$ and $J_2 : S_2 \to X$ we can use a Poisson Lie structure on $X$ to construct a product realization $J_1 \otimes J_2 : S_1 \otimes S_2 := S_1 \times S_2 :\to X$ by

$$
\begin{array}{c}
S_1 \times S_2 \\
\downarrow J_1 \times J_2 \\
X \times X \\
\downarrow m \\
\downarrow J_1 \otimes J_2 \\
X
\end{array}
$$

This multiplication of symplectic realizations is associative because the multiplication on $X$ is. Furthermore, viewing the terminal object in the category of Poisson manifolds $1 = \{\text{pt}\}$ as a zero-dimensional symplectic manifold, the unit $e : 1 \to X$ can also be viewed as symplectic realization. It is the identity for the product of realizations $e \otimes J = J = J \otimes e$, where we identify $1 \times S = S = S \times 1$. In this way, the monoidal structure on $X$ naturally equips the category of symplectic realizations with a monoidal structure.

What structure is induced on the category of symplectic realizations by the inverse on $X$? From the analogous algebraic situation, we might expect that the inverse leads to a rigid monoidal structure [19]. We can try to define a dual symplectic realization by

$$J^\vee : S^\vee \equiv S^{\text{op}} \xrightarrow{J \text{ op}} X^{\text{op inv}} \xrightarrow{\text{inv}} X.
$$

But in the category of symplectic realizations of $X$ the evaluation map would have to make the diagram

$$
\begin{array}{c}
S \otimes S^\vee \\
\downarrow J \otimes J^\vee \\
X
\end{array}
\xrightarrow{\text{ev}}
\begin{array}{c}
1 \\
\downarrow e \\
X
\end{array}
$$

commutative, which is not possible unless $J$ maps all of $S$ to a single point in $X$. If we want to equip the category of symplectic realizations with a rigid structure, we will need a more general notion of morphism.

6 Generalized morphisms

In many categories, morphisms are given by set theoretic maps, but we may allow relations instead of just maps, especially when maps of a certain type
are characterized by properties of their graphs. For instance, a smooth map $f : A \to B$ between symplectic manifolds is symplectic if and only if its graph is a lagrangian submanifold of $A \times B^{\text{op}}$. Moreover, for the graph of the composition of $f$ with another map $g : B \to C$ we have

$$\text{graph}(g \circ f) = \text{graph}(f) \circ \text{graph}(g) := \text{graph}(f) \times_B \text{graph}(g).$$  \hspace{1cm} (12)

This suggests allowing arbitrary lagrangian submanifolds of products as morphisms, i.e., defining

$$\text{Hom}(A, B) := \text{Lag}(A \times B^{\text{op}}),$$  \hspace{1cm} (13)

the symplectic $A$-$B$ relations. Note that, for the morphisms in $\text{Lag}(A, B)$, there is no natural distinction between source and target, as there is for maps. A lagrangian submanifold $L$ of $A \times B^{\text{op}}$ is the same as a lagrangian submanifold of $(A \times B^{\text{op}})^{\text{op}} = A^{\text{op}} \times B \cong B \times A^{\text{op}}$. So $L$ can be equivalently viewed as a morphism from $B$ to $A$. This is why we prefer to denote the source and target maps of a category by $l$ and $r$, because everyone agrees what is left and right.

Using this generalized notion of morphisms we return to the symplectic realizations of a Poisson Lie group $X$. A generalized morphism between two symplectic realizations $J_1 : S_1 \to X$ and $J_2 : S_2 \to X$ is given by a lagrangian submanifold $L \in \text{Lag}(S_1 \times S_2^{\text{op}})$ such that the following diagram commutes:

Now we can try again to find an evaluation morphism from $S \otimes S^\vee$ to $1$ as in Eq. (11). What we need is a Lagrangian submanifold $L_{\text{ev}} \in \text{Lag}(S \times S^{\text{op}}) \cong \text{Lag}((S \times S^{\text{op}}) \times \{\text{pt}\})$ such that for all $(s, s') \in L_{\text{ev}}$ we have $J(s)J(s')^{-1} = e$. The natural lagrangian submanifold satisfying these requirements is the diagonal $L_{\text{ev}} := \Delta_S = \{(s, s) \mid s \in S\}$. The same reasoning leads us to define the coevaluation morphism also by $L_{\text{cv}} := \Delta_S$. Moreover, we have the same evaluation and coevaluation morphisms for $S^\vee \otimes S$. It is easy to see that the morphism of symplectic realizations

$$S \xrightarrow{\cong} S \otimes 1 \xrightarrow{\text{Id}_S \otimes L_{\text{ev}}} S \otimes (S^\vee \otimes S) \xrightarrow{\cong} (S \otimes S^\vee) \otimes S \xrightarrow{L_{\text{ev}} \otimes \text{Id}_S} 1 \otimes S \xrightarrow{\cong} S$$  \hspace{1cm} (15)
is the identity morphism, which is also given by the diagonal \( \text{Id}_S = \Delta_S \). Going in an analogous way from \( S^\vee \) via \( S^\vee \otimes S \otimes S^\vee \) to \( S^\vee \) we obtain the identity on \( S^\vee \) which is also given by the diagonal. We are tempted to conclude that the category of symplectic realizations and generalized morphisms is rigid monoidal. However, there is a catch:

Unfortunately, the symplectic relations are not really the morphisms of a category; when the projections to \( B \) of elements in \( \text{Lag}(A, B) \) and \( \text{Lag}(B, C) \) intersect badly, their composition as defined in Eq. (12) is not a manifold. To avoid this, we can define \( \text{Hom}(A, B) \) to be \( A \times B^{\text{op}} \) itself, rather than the set of lagrangian submanifolds therein. The price we pay is that the composition operation

\[
\text{Hom}(A, B) \times \text{Hom}(B, C) = (A \times B^{\text{op}}) \times (B \times C^{\text{op}}) \rightarrow A \times C^{\text{op}} = \text{Hom}(A, C) \quad (16)
\]

is not a mapping of sets, but a relation, namely the lagrangian submanifold of

\[
\left((A \times B^{\text{op}}) \times (B \times C^{\text{op}})\right) \times (A \times C^{\text{op}})^{\text{op}} \quad (17)
\]

consisting of the product \( \{(a, b, b, c, a, c) | a \in A, b \in B, c \in C\} \) of three diagonals. The result is what we have called in Section 5.2 of [6] a “symplectic category,” i.e. a category internal to the “category” of symplectic relations.

### 7 Groupoids and stacks

Even more general than a relation between the sets \( X \) and \( Y \) is a “multi-relation”, i.e. a map from a set \( M \) to \( X \times Y \), which might not be injective. The best theory of such generalized morphisms comes about when \( M \) is acted upon by groupoids over \( X \) and \( Y \). The result is the theory of stacks, which we will describe in its smooth version [2] [3] [18]. (See [14] for the topological case.)

Roughly speaking, a stack is a device to describe a “bad” quotient. Here is a simple example. Let \( \mathbb{Z}_2 \) act on an open disc of unit radius in \( \mathbb{R}^2 \) by reflection at the origin, \( 1 \cdot (x, y) = (-x, -y) \). The \( \mathbb{Z}_2 \) action is not free, because the origin is a fixed point. Taking the quotient amounts to cutting

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\(^{1}\)It may not always be necessary to bring in the groupoids from the beginning; see for instance the use of “bisubmersions” in [1].
the disc along, say, the positive $x$-axis and rolling it up until you have two layers at every point with the exception of the origin.

\[ \text{\includegraphics[width=0.3\textwidth]{cone.png}} \]

The result is the surface of a cone, which is no longer a smooth manifold. You can do this for other finite cyclic groups, say, $\mathbb{Z}_3$ where now $1 \in \mathbb{Z}_3$ acts by rotation by a third of the full circle. Now you obtain a cone with a smaller opening angle.

\[ \text{\includegraphics[width=0.3\textwidth]{smaller_cone.png}} \]

We can smoothly glue together two of these cones along an outer annulus of each disk to obtain a “Christmas tree ornament”:

\[ \text{\includegraphics[width=0.2\textwidth]{christmas_tree.png}} \]

This is an example for an orbifold, a manifold with lower dimensional singularities which look locally like the quotient of $\mathbb{R}^n$ by a finite group. (The example above is still a manifold, topologically, but this is not true for more general orbifolds.) Like a manifold, an orbifold can be described by charts with these quotient spaces as local models. A drawback of this description is that there is no natural tangent bundle of an orbifold which is itself an orbifold, and the definition of morphism is rather complicated and unnatural-looking.

A more effective way to describe a bad quotient space is to remember all the gluings. This leads us to the concept of a groupoid. As a set, the gluing groupoid for $D^2/\mathbb{Z}_2$ is $G_1 := \mathbb{Z}_2 \times D^2$. There are two maps to the base $G_0 = D^2$, which map each element $g$ of the groupoid to the two points which $g$ glues together. We denote these maps by $l$ and $r$,

\begin{align}
  l(0, (x, y)) &= (x, y), & r(0, (x, y)) &= (x, y), \\
  l(1, (x, y)) &= (-x, -y), & r(1, (x, y)) &= (x, y),
\end{align}
that is, \( l(a, p) = a \cdot p \), \( r(a, p) = p \) for \( a \in \mathbb{Z}_2 \) and \( p \in D^2 \). We can compose two elements of the groupoid \((a, p)(a', p') = (a + a', p')\) whenever \( r(a, p) = p = a' \cdot p' = l(a', p')\). This construction can be extended to arbitrary group actions, but but there are more general groupoids. For example, the groupoid presenting the Christmas ornament orbifold looks like this.

The \( l \) and \( r \) maps on the annuli are given by the embeddings into the disks.

All structure maps of the groupoid are smooth (and \( l \) and \( r \) submersions) so we have a Lie groupoid. In order to see what is special about the tips of the Christmas ornament, we have to look at the isotropy group \( \text{Iso}(p) := l^{-1}(p) \cap r^{-1}(p) \) for \( p \in G_0 \), which for an action groupoid is the stabilizer of \( p \). The origins of the two disks of the base \( G_0 = D^2 \cup D^2 \) are the only points which have non-trivial isotropy, \( \mathbb{Z}_2 \) for the left disk and \( \mathbb{Z}_3 \) for the right disk.

For orbifolds, the stabilizers are by definition finite groups. For the groupoid it means that it is proper étale, i.e., the anchor map \((l, r) : G_1 \rightarrow G_0 \times G_0, g \mapsto (l(g), r(g))\) is proper étale. Indeed, we can give the following definition of an orbifold [13]:

**Definition 1.** An orbifold is a differentiable stack presented by a proper étale Lie groupoid.

Now we have to explain how we can associate a stack to a Lie groupoid. We can think of a Lie groupoid as a generalized equivalence relation describing the quotient space of equivalence classes. As we have seen, the actual quotient space \( G_0/G_1 \) of \( G_1 \)-orbits in \( G_0 \) is usually not a nice space, so we write \( G_0//G_1 \) for the as-if quotient the groupoid is thought to describe. The usual notion of isomorphism of Lie groupoids is that of a diffeomorphism which is compatible with all the structure maps. However, now two groupoids should be considered to be equivalent if they present the same quotient.
Given two groupoids $G$ and $H$ we must find a smooth way to associate the $G_1$-orbits on $G_0$ with the $H_1$-orbits on $H_0$. This cannot be just a map $G_0/G_1 \to H_0/H_1$, because these quotient spaces are in general not manifolds, and so there is no notion of smoothness. Our description of a bad quotient as a groupoid which is a generalized equivalence relation suggests defining morphisms between them in an analogous manner. Now the two spaces we want to relate are the bases $G_0$ and $H_0$ of the groupoids. A generalized relation consists of triples $(x \xrightarrow{m} y)$ where we say that $x \in G_0$ is related via $m$ to $y \in H_0$. We denote the set of all such triples by $M$. The projections of the elements of $M$ on the elements of the groupoid bases they relate, $l_M(x \xrightarrow{m} y) := x$ and $r_M(x \xrightarrow{m} y) := y$, are called the moment maps of $M$. When no confusion can arise, we will drop the subscripts of the moment maps.

We do not require $M$ to be a map from $G_0$ to $H_0$. For example, a single pair $x$ and $y$ can be related by several elements of $M$. But we want the relation $M$ to descend to a well-defined relation on the orbits. For notational reasons it is convenient to use left orbits in $G_0$ and right orbits in $H_0$. (Note that the left orbits for a groupoids acting on its base are the same as the right orbits.) For the generalized relation $M$ between elements $x \in G_0$ and $y \in H_0$, to descend to a well-defined relation on the orbits $G \cdot x$ and $y \cdot H$ we have to require that for all $g$ acting on $x$ and $h$ acting on $y$ we have elements $g \cdot m$ and $m \cdot h$ of $M$ such that

$$x \xrightarrow{m} y \Rightarrow (g \cdot x) \xrightarrow{gm} y \quad \text{and} \quad x \xrightarrow{m} y \Rightarrow x \xrightarrow{m \cdot h} (y \cdot h).$$

(20)

We want to be able to chose $g \cdot m$ and $m \cdot h$ in a consistent way, such that we get maps $m \mapsto g \cdot m$ and $m \mapsto m \cdot h$ which are compatible with the groupoid structures, $g \cdot (g' \cdot m) = gg' \cdot m$ and $(m \cdot h) \cdot h' = m \cdot hh'$, as well as, $(g \cdot m) \cdot h = g \cdot (m \cdot h)$ whenever defined, i.e., we have two commuting groupoid actions on $M$. Such an object $M$ is called a groupoid bibundle.

So far, $M$ only descends to a relation on $G_0/G_1 \times H_0/H_1$. To obtain a function from $G_0/G_1$ to $H_0/H_1$, we first have to require that $l_M$ is surjective, so that the function will be defined on all of $G_0/G_1$. Second, for the relation to be a function, a given $x \in G_0$ has to be related only to elements of one orbit in $H_0$;

$$x \xrightarrow{m} y \quad \text{and} \quad x \xrightarrow{m'} y' \Rightarrow y \cdot h = y' \Rightarrow x \xrightarrow{m \cdot h} y'$$

(21)

for some groupoid element $h \in H_1$. Again, we want to be able to chose $h$ in a nice way, requiring that there is a unique $h$ such that $m \cdot h = m'$. This
gives us a right principal bibundle. Finally, we require all the structures to be smooth, that is, $M$ is a manifold, the moment maps are smooth, and the groupoid actions are smooth.

We can depict the situation by the diagram

Now let $G, H, K$ be Lie groupoids, $M$ a smooth right principal $G$-$H$ bibundle and $N$ a smooth right principal $H$-$K$ bibundle. The composition of the induced functions on the quotient spaces can be lifted to a smooth composition of the bibundles:

Here, the right $H$-action on $M \times_{H_0} N$ is given by $(m, n) \cdot h = (m \cdot h, h^{-1} \cdot n)$. The $G$-action and the $K$-action descend to actions on the quotient, since they both commute with the $H$-action. The conclusion is that $M \circ N$ is a $G$-$K$ bibundle, which we call the composition of $M$ and $N$. This composition is only associative up to a biequivariant diffeomorphism of bibundles. This
means that we should really be working in the weak 2-category having Lie groupoids as objects, smooth right principal bibundles as 1-morphisms, and smooth biequivariant maps of bibundles as 2-morphism. We denote this category by $\text{LieGrpdPrBibu}$.

A good way to study the generalized space described by a groupoid $G$ is the Grothendieck approach of considering all the morphisms from ordinary manifolds to $G$, where a manifold $X$ is described by the groupoid $X_1 = X_0 = X$. The collection of all such morphisms to $G$, more concretely described as

$$\{M \mid M \text{ a right principal } X-G \text{ bibundle, } X \text{ a manifold}\}$$

is denoted by $BG$ and called the classifying space of the groupoid $G$. This is a smooth stack in the usual sense which is presented by the groupoid $G$. Two stacks $BG$ and $BH$ are isomorphic if and only if the groupoids $G$ and $H$ are Morita equivalent (Theorem 2.24 in [3]). We thus get a 1-to-1 correspondence of isomorphism classes of presentable stacks and Morita equivalence classes of groupoids. This is often stated as “a stack is a groupoid up to Morita equivalence”. But beware that the actual functor between the category of stacks and the weak 2-category of groupoids and bibundles is a weak 2-equivalence of 2-categories.

8 Stacky Lie groups

It can be shown that, in the weak 2-category $\text{LieGrpdPrBibu}$ of Lie groupoids and smooth right principal bibundles, all products exist and the one-element groupoid 1 is a terminal object. (Note that, if the bibundles are not required to be principal, this is no longer true.) Since we are dealing with a weak 2-category, the categorical product is associative only up to weak 1-isomorphisms, that is, up to Morita equivalence of groupoids.

If we have products and a terminal objects we also have the notion of group objects, which we will call stacky Lie groups [4, 22]. The question is whether in $\text{LieGrpdPrBibu}$ the notion of group objects is useful, as for differentiable spaces, or uninteresting as for groups or algebras. This is not easy to see because, on the one hand we think of a groupoid as a generalized quotient space, but on the other hand a Lie groupoid is itself an algebraic structure internal to the category of manifolds. It turns out that while not many examples of truly stacky groups have been studied until now, there are some particularly interesting ones.
Consider the group $S^1$ and the dense subgroup which is the image of the embedding $\mathbb{Z} \to S^1 \cong U(1), \ k \mapsto e^{i\lambda k}$ where, $\lambda/2\pi$ is an irrational number. By abuse of notation, we will also denote the subgroup itself by $\mathbb{Z}$. The quotient $S^1/\mathbb{Z}$ is an abelian group, in which we will denote the multiplication by $m: S^1/\mathbb{Z} \times S^1/\mathbb{Z} \to S^1/\mathbb{Z}$. Because the subgroup $\mathbb{Z}$ is dense, the quotient topology is trivial. This suggests that we should work with the stack $S^1//\mathbb{Z}$ rather than with the actual quotient. We then try to lift the multiplication map $m$ on the bottom level of the diagram of the form (22) to the smooth top levels:

The inner pull-back $(S^1/\mathbb{Z} \times S^1/\mathbb{Z}) \times_{S^1/\mathbb{Z}} S^1$ is the graph of the group multiplication $m$, the pull-back projections being the range and image maps. Any object in the top left upper corner which makes the diagram commute can be viewed as a lift of the graph of $m$ to $S^1$, the pull-back being the universal lift. The diagonal arrow is the unique map which exists by the universal property of the inner pull-back. Explicitly, the pull-back is

$$(S^1 \times S^1) \times_{S^1/\mathbb{Z}} S^1 = \{(\theta_1, \theta_2, \theta) \in (S^1 \times S^1) \times S^1 \mid \theta_1 + \theta_2 = \theta \mod 2\pi \lambda\}$$

where $\theta$ is the angle representing $e^{i\theta} \in U(1) \cong S^1$. This set can be identified with $S^1 \times S^1 \times \mathbb{Z}$ and viewed as the graph of a multi-valued multiplication on $S^1$. It has the structure of a smooth manifold and inherits smooth actions of the action groupoid $G := S^1 \times \mathbb{Z} \rightrightarrows S^1$ presenting the stack $S^1//\mathbb{Z}$. We thus obtain the smooth right principal $(G \times G)$-$G$ bibundle $E_m$ of multiplication. In an analogous manner we construct the 1-$G$ bibundle $E_e$ of the identity element, and the $G$-$G$ bibundle of the inverse $E_{inv}$. It can be checked that the bibundles $E_m$, $E_e$, $E_{inv}$ equip the groupoid $G$ with the structure of a stacky Lie group.
9 Hopfish algebras

The Gelfand Theorem tells us that a locally compact topological space and its commutative algebra of continuous functions vanishing at infinity contain the same structural information. In our example, $S^1/\mathbb{Z}$, the quotient topology is trivial, so the only continuous functions are constant. But we are working instead with the groupoid presenting the stack $S^1//\mathbb{Z}$. What is the algebra of continuous “functions” on $S^1//\mathbb{Z}$? It is one of the main ideas of noncommutative geometry that we should consider the convolution algebra of the groupoid $G := S \rtimes \mathbb{Z}$ to be the analogue of the algebra of functions [15] [7]. For two compactly supported functions $a$ and $b$ on $G$, the convolution product is

\[
(a * b)(g) = \int a(h)b(h^{-1}g)dh,
\]

which looks just like the convolution algebra of a group. The difference is that on a groupoid the product $h^{-1}g$ is only defined if $l(h) = l(g)$, so for a given $g$ we have to integrate over the left groupoid fiber of $r(g) = x$. Since we have to do this for all $g$, we need a whole family of Haar measures $dh = dxh$. Alternatively, we can work with half-densities instead of functions, as in [7].

Recalling that Morita equivalent groupoids have Morita equivalent (but generally not isomorphic) convolution algebras, we also conclude that Morita equivalent algebras should be thought of as representing the “same noncommutative space”[3].

In our example, the Lie groupoid $G = S^1 \rtimes \mathbb{Z}$ is étale, so the Haar measures are merely counting measures. Thus,

\[
(a * b)(\theta, k) := \sum_{k' \in \mathbb{Z}} a(\theta + \lambda k', k - k')b(\theta, k'),
\]

for all compactly supported functions $a$ and $b$ on $S^1 \times \mathbb{Z}$. Among such functions are the standard Fourier basis functions on the circle times the

\[16\]

\[2\]One must use this identification with some care. For instance, there are plenty of examples of nonisomorphic groups with isomorphic group algebras (e.g. pairs of finite abelian groups with the same number of elements); thinking of these groups as representing stacks of the form $BG$, one finds different stacks with the same “algebra of functions”. Even more different, it seems to us, are the stacks presented by the group $\mathbb{Z}_2$ and the trivial groupoid over a set with two elements, yet again they have isomorphic group algebras. These examples suggest that it takes more than an algebra to make a “noncommutative space,” but it is not clear to us exactly what that “more” should be.
characteristic functions of integers, \( a_n(k, \theta) := e^{ink} \delta_k \), for which we obtain
\[
a_{n_1 l_1} * a_{n_2 l_2} = e^{i \lambda_{n_1} l_2} a_{n_1 + n_2, l_1 + l_2}.
\]
(27)

Let us denote the algebra of finite linear combinations of these functions by \( \mathcal{A} \). The closure of \( \mathcal{A} \) with respect to a suitable norm is known as the algebra continuous functions on a noncommutative torus\(^3\) which is a deformation of the usual algebra of continuous functions on the 2-torus [16].

Now the quotient space \( S^1 / \mathbb{Z} \) is also a quotient group. Since \( \mathcal{A} \) plays the role of an algebra of functions on this quotient, we might expect the group structure to translate into a Hopf structure on \( \mathcal{A} \). But this is not the case. In fact, \( \mathcal{A} \) is simple, so it does not even possess a counit. So what happened to the group structure? The answer lies with the stacky group structure.

The space of functions on every \( G-H \) bibundle naturally acquires the structure of an \( \mathcal{A}(G)-\mathcal{A}(H) \) bimodule. Composition of bibundles corresponds to tensor product of bimodules, so, under suitable technical assumptions, we get a functor from \( \text{LieGrpdPrBibu} \) to a category in which the objects are algebras and the morphisms are bimodules. By analogy with the usual Gelfand functor, we view it as contravariant. Since this functor takes categorical products of groupoids to tensor products of algebras, which are not categorical products, it certainly does not take group objects to group objects. In fact, it does not even take them to Hopf algebras, as we saw already for the example of the noncommutative torus. (There, the functor takes the unit of the group to a \( \mathbb{C}-\mathcal{A} \) bimodule rather than to a homomorphism \( \mathbb{C} \leftarrow \mathcal{A} \).) Instead, the image of a stacky group is another structure which we call a hopfish algebra [5, 17].

A space of functions on the bibundle \( E_m \) of multiplication for a stacky group with function algebra \( \mathcal{A} \) becomes an \( (\mathcal{A} \otimes \mathcal{A})-\mathcal{A} \) bimodule, which we denote by \( \Delta \) and view as the bimodule of comultiplication. Analogously, the bibundle \( E_e \) of the unit element, already mentioned above, becomes the \( \mathbb{C}-\mathcal{A} \) bimodule \( e \) of the counit. By functoriality these bibundles satisfy coassociativity and counitality relations
\[
(\mathcal{A} \otimes \Delta) \otimes_{\mathcal{A} \otimes \mathcal{A}} \Delta \cong (\Delta \otimes \mathcal{A}) \otimes_{\mathcal{A} \otimes \mathcal{A}} \Delta,
\]
(28)
\[
(e \otimes \mathcal{A}) \otimes_{\mathcal{A} \otimes \mathcal{A}} \Delta \cong \mathcal{A} \cong (\mathcal{A} \otimes e) \otimes_{\mathcal{A} \otimes \mathcal{A}} \Delta,
\]
(29)
quoth up to isomorphism of bimodules. We thus obtain a weak comonoidal object

\(^3\)By abuse of terminology, the algebra itself is often known as a noncommutative torus.
in the weak 2-category of algebras and bimodules, which is also called a sesquilinear sesquialgebra.

The algebraic image of the bibundle $E_{\text{inv}}$ of the inverse is more difficult to interpret. Because $E_{\text{inv}}$ is a $G$-$G$ bibundle, the space of functions on $E_{\text{inv}}$ is an $\mathcal{A}$-$\mathcal{A}$ bimodule, which we would expect to become the bimodule of an antipode. But a Hopf antipode is an algebra antihomomorphism, so the hopfish antipode should be an $\mathcal{A}$-$\mathcal{A}^{\text{op}}$ bimodule. A way to accomplish this is to use the star structure on $\mathcal{A}$ which comes from the groupoid inverse of $G$, $a^*(g) = a(g^{-1})$, in order to convert the $\mathcal{A}$-$\mathcal{A}$ bimodule of functions on $E_{\text{inv}}$ into the $\mathcal{A}$-$\mathcal{A}^{\text{op}}$ bimodule $\mathcal{S}$ of the hopfish antipode. Just as we illustrated above in the case of Poisson groups with the diagram [4], it is not easy to formulate what is required of the antipode in a hopfish algebra. But a definition is given in [17], with a modification in [5] to cover the case of noncommutative torus algebras, and the bimodule described above does satisfy the definition.

Like an ordinary coproduct, the hopfish coproduct bimodule $\Delta$ can be used to multiply two right $\mathcal{A}$-modules $T, T' \in \text{Mod}_{\mathcal{A}}$ by

$$T \otimes \Delta T' := (T \otimes T') \otimes_{\mathcal{A} \otimes \mathcal{A}} \Delta. \quad (30)$$

In [5], we have tried out this new multiplication on certain modules over the hopfish algebra associated to the stacky group $S^1/\mathbb{Z}$. For $\alpha \in \mathbb{R}$, $p, q \in \mathbb{Z}$ relatively prime, we have shown that

$$T^\alpha_{pq} := \mathcal{A}/(e^{-i\alpha}a_{pq} - 1)\mathcal{A},$$

is a simple module generated by $\xi := [1]$ with $a_{pq} \cdot \xi = e^{i\alpha}\xi$, and that adding any integer multiple of $\lambda$ to $\alpha$ results in an isomorphic module. (For $p, q$ not relatively prime the situation is slightly more complicated.) Some calculations lead to the following result.

**Theorem 1.** For $p_1 \neq 0$ or $p_2 \neq 0$ we have:

$$T^{\alpha_1}_{p_1 q_1} \otimes \Delta T^{\alpha_2}_{p_2 q_2} \cong \gcd(p_1, p_2) T^{\alpha}_{pq},$$

where

$$p := \text{lcm}(p_1, p_2), \quad q := \frac{p_1 q_2 + p_2 q_1}{\gcd(p_1, p_2)}, \quad \alpha := \frac{\alpha_1 p_2 + \alpha_2 p_1}{\gcd(p_1, p_2)}. \quad (31)$$

For $p_1 = 0$ and $p_2 = 0$ we have:

$$T^{\alpha_1}_{0, q_1} \otimes \Delta T^{\alpha_2}_{0, q_2} \cong \begin{cases} T^{\alpha}_{0, q} & \text{for } \frac{\alpha_1 q_2 - \alpha_2 q_1}{\lambda \gcd(q_1, q_2)} \in \mathbb{Z} \mod \text{lcm}(q_1, q_2)^2 \pi \lambda, \\ 0 & \text{otherwise} \end{cases}.$$
where
\[ q := \gcd(q_1, q_2), \quad \alpha := s_1\alpha_2 - s_2\alpha_1, \quad s_1, s_2 \in \mathbb{Z} : \quad \frac{s_1q_2 - s_2q_1}{\gcd(q_1, q_2)} = 1. \] (32)

Observe that \( q/p = q_1/p_1 + q_2/p_2 \) and \( \alpha/p = \alpha_1/p_1 + \alpha_1/p_2 \). Surprisingly, we did not notice these simple relations until we found the following geometric interpretation of them.

When we identify \( \mathcal{A} \) with an algebra of functions on the torus \( T^2 \), with coordinates \((\theta_1, \theta_2)\), the algebra element \( e^{-i\alpha} a_{pq} - 1 \) becomes the function \( e^{-i\alpha} e^{i(p\theta_1 + q\theta_2)} - 1 \), and so the classical analogue of the quotient module \( T_{pq}^\alpha = \mathcal{A}/(e^{-i\alpha} a_{pq} - 1) \mathcal{A} \) appears to be the functions on the embedded circle in \( T^2 \) defined by the equation \( p\theta_1 + q\theta_2 - \alpha = 0 \), or \( \theta_1 = -q/p \theta_2 + \alpha/p \). The classical analogue of the hopfish structure on \( \mathcal{A} \) (and hence the symplectic model of the group structure on \( S^1/\mathbb{Z} \)) turns out to be the symplectic groupoid structure \( T^2 \rightrightarrows T^1 \) for which the source and target maps are the projection in the \( \theta_2 \) direction, and the composition law is addition in \( \theta_1 \). The tensor product operation on modules corresponds to the application of the composition law on the groupoid to the embedded circles which represent them. This results in the addition of fractions mentioned above.

Note that, if we “unwrap” the \( \theta_2 \) circle to a line, we obtain the cotangent bundle \( T^1 \times \mathbb{R} \cong T^*T^1 \), the symplectic groupoid \( \Gamma(T^1) \) of \( T^1 \) with the zero Poisson structure. The groupoid structure described in the preceding paragraph is the second symplectic groupoid structure on \( \Gamma(T^1) \) which is obtained by lifting the (Poisson) Lie group structure on \( T^1 \) given by addition in \( \theta \). This is an instance of the double symplectic groupoid structures attached to general Poisson Lie groups [12].

What has become of the parameter \( \lambda \)? In fact, it seems that the curve \( p\theta_1 + q\theta_2 - \alpha \) represents only one generator of the module in question. The others are obtained by applying the unitary basis elements of the algebra itself. These correspond to constant “integer” bisections of \( \Gamma(T^2) \cong T^*T^2 \); the bisection \( nd\theta_1 + nd\theta_2 \) acts via the source and target maps, determined by the Poisson structure (see [21]), as translation by \((-\lambda n, \lambda m)\). This yields the collection of all circles of the form \( p\theta_1 + q\theta_2 - (\alpha + \lambda(np - mq)) = 0 \). Under the assumption \( \gcd(p, q) = 1 \), all the integer multiples of \( \lambda \) occur, so that this collection of circles, like the isomorphism class of \( T_{pq}^\alpha \), depends only on the image of \( \alpha \) in \( S^1/\mathbb{Z} \).
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