Optimal Label Splitting for Embedding an LTS into an arbitrary Petri Net Reachability Graph is NP-complete

Uli Schlachter\textsuperscript{1,2*} and Harro Wimmel\textsuperscript{2**}

\textsuperscript{1} Institute of Networked Energy Systems, German Aerospace Center, Oldenburg, Germany, uli.schlachter@dlr.de
\textsuperscript{2} Department of Computing Science, Carl von Ossietzky Universität Oldenburg, Oldenburg, Germany harro.wimmel@informatik.uni-oldenburg.de

Abstract. For a given labelled transition system (LTS), synthesis is the task to find an unlabelled Petri net with an isomorphic reachability graph. Even when just demanding an embedding into a reachability graph instead of an isomorphism, a solution is not guaranteed. In such a case, label splitting is an option, i.e. relabelling edges of the LTS such that differently labelled edges remain different. With an appropriate label splitting, we can always obtain a solution for the synthesis or embedding problem. Using the label splitting, we can construct a labelled Petri net with the intended behaviour (e.g. embedding the given LTS in its reachability graph). As the labelled Petri net can have a large number of transitions, an optimisation may be desired, limiting the number of labels produced by the label splitting. We show that such a limitation will turn the problem from being solvable in polynomial time into an NP-complete problem.

Keywords: Labelled Transition Systems, Petri Nets, System Synthesis, Regions, Label Splitting, NP-Completeness.

1 Introduction

There are two general approaches to investigate the behaviour of Petri nets \cite{16,14}. \textit{Analysis} is used to construct a variety of descriptions from sets of firing sequences \cite{10} to event structures \cite{15}. One of the most common forms for

\* Supported by DFG (German Research Foundation) through grant Be 1267/15-1 \textbf{ARS} (Algorithms for Reengineering and Synthesis)
\** Supported by DFG (German Research Foundation) through grant Be 1267/16-1 \textbf{ASYST} (Algorithms for Synthesis and Pre-Synthesis Based on Petri Net Structure Theory).
Fig. 1. Left: The sequence $abbaa$ is embedded in this LTS, which is the reachability graph of an unlabelled Petri net. Right: An LTS that cannot be embedded into the reachability graph of any unlabelled Petri net, as every reachability graph will identify the states $s_2$ and $s_5$ due to $ab$ and $ba$ producing the same marking.

describing the sequential behaviour is the reachability graph, containing the reachable markings as states together with edges denoting transitions that fire to reach one marking from another. In the reverse direction, i.e. synthesis, we try to build an unlabelled Petri net that behaves like a given specification, e.g. a labelled transition system (LTS). In Process Mining [1], this can be used to find a small model covering a large set of observable behaviours. While all those behaviours must be allowed by the model, exact synthesis is often difficult to achieve. E.g., there is no unlabelled Petri net that produces the sequence $abbaa$ and nothing else. We might overapproximate the behaviour by the LTS shown on the left of Fig. 1 which can be synthesised into an unlabelled Petri net and embeds the sequence $abbaa$, but this approach is not always possible. The right LTS of Fig. 1 is not embedded in any reachability graph. As $ab$ and $ba$ have the same effect in any Petri net, the states $s_2$ and $s_5$ represent the same marking and will necessarily be identified in its reachability graph. Identifying the two states leads to additional behaviour, i.e. $abb$ and $baa$.

As an alternative, we may split the label $a$ into $a'$ and $a''$, relabel e.g. $abbaa$ to $a'bbaa$, and find a Petri net for the new sequence $a'bbaa$. With a labelling function $a, a' \mapsto a$ and $b \mapsto b$ we obtain a labelled Petri net with the sought behaviour. Label splitting [7, 13, 17], as indicated above, may lead to large Petri nets with as many transitions as there are edges in the LTS. We hope to avoid this by allowing over-approximations of behaviour, i.e. embeddings into reachability graphs, as well as by optimising the label splitting itself by limiting the number of newly introduced labels. Essentially, we are interested in the following problem:

For a given LTS, is there a label splitting producing at most $n$ different labels such that the relabelled LTS is embedded in the reachability graph of some (arbitrary) unlabelled Petri net?

or equivalently

Can we find an arbitrary, labelled Petri net with at most $n$ transitions such that a given LTS is embedded in its reachability graph?
An optimal solution can be determined by incrementing \( n \) (starting at the alphabet size of the LTS) until the problem is solved. While, in general, synthesis and finding an embedding can be done in polynomial time using Region theory \([9,2]\), we will show that finding an appropriate label splitting solving our problem is already \( \text{NP} \)-complete.

In the next section, we introduce the basic concepts around labelled transition systems and Petri nets as well as label splitting and a short description of synthesis. We also show that the state separation property of Region theory, requiring different Petri net markings for different states in the LTS, is equivalent to the existence of a reachability graph embedding the LTS. Section \([3]\) formalises our label splitting problem, shows membership in \( \text{NP} \), and provides the construction necessary to prove its \( \text{NP} \)-hardness. The construction is a generic LTS, consisting of six rather independent parts, the first four of which are auxiliary in nature. We take a closer look at these in Section \([4]\) and explain how region values (i.e. marking changes) of labels are determined. In Section \([5]\) we show that our construction is a polynomial-time reduction, which proves the \( \text{NP} \)-completeness. Finally, we give a summary and an outlook in Section \([6]\).

## 2 Basic concepts

**Definition 1. LTS**

A labelled transition system (LTS) with initial state is a tuple \( TS = (S, \Sigma, E, s_0) \) with nodes \( S \) (a countable set of states), edge labels \( \Sigma \) (a finite set of letters), edges \( E \subseteq (S \times \Sigma \times S) \), and an initial state \( s_0 \in S \). An edge \( (s, t, s') \in E \) may be written as \( s(t)s' \). A walk \( \sigma \in \Sigma^* \) from \( s \) to \( s' \), written as \( s(\sigma)s' \), is given inductively by \( s = s' \) for the empty word \( \varepsilon \) and by \( \exists s'' \in S: s[w]s''[t]s' \) for \( \sigma = wt \) with \( w \in \Sigma^* \) and \( t \in \Sigma \). A walk \( s(\sigma)s' \) is a cycle if and only if \( s = s' \).

The Parikh vector \( \mathcal{P}(\sigma): \Sigma \rightarrow \mathbb{Z} \) of a word \( \sigma \in \Sigma^* \) maps each letter \( t \in \Sigma \) to its number of occurrences in \( \sigma \), it will often be written as an element of the group spanned by \( \Sigma \). The neutral element is written as \( 0 \) and comparisons are done componentwise. We map to \( \mathbb{Z} \) here instead of \( \mathbb{N} \) to be able to extend the notion of a Parikh vector later and to handle differences of Parikh vectors more easily.

A spanning tree \( \Theta \) of \( TS \) is a set of edges \( \Theta \subseteq E \) such that for every \( s \in S \) there is a unique walk from \( s_0 \) to \( s \) using edges in \( \Theta \) only. This implies that \( \Theta \) is cycle-free. A walk in \( \Theta \) is a walk that uses edges in \( \Theta \) only (and not any of \( E \setminus \Theta \)). Edges in \( E \setminus \Theta \) are called chords. The Parikh vector of a state \( s \) in a spanning tree \( \Theta \) is \( \mathcal{P}_\Theta(s) = \mathcal{P}(\sigma) \) where \( s_0[\sigma]s \) is the unique walk in \( \Theta \). The Parikh vector of an edge \( s(t)s' \) in \( TS \) is \( \mathcal{P}_\Theta(s(t)s') = \mathcal{P}(s) + t - \mathcal{P}(s') \), see Fig. \([2]\) for an example. Note that Parikh vectors of edges in \( \Theta \) always evaluate to zero; for chords the Parikh vector may even contain negative values. For a chord \( s(t)s' \), \( s \) and \( s' \) have a latest common predecessor \( r \) in \( E \), \( t', t'' \in T \) with \( t' \neq t'' \), \( \sigma, \sigma' \in T^* \) with two walks \( r[t'\sigma]s(t)s' \) and \( r[t''\sigma']s' \) in \( \Theta \). These two walks form a cycle in the LTS’ underlying undirected graph, called
a generalised cycle, with the Parikh vector the same as the chord’s, \( P_\varnothing(s) + 1t - P_\varnothing(s') \). The Parikh vector of a walk \( s_1[t_1]s_2 \ldots s_n[t_n]s_{n+1} \) is defined as \( P_\varnothing(s_1[t_1] \ldots t_n) = \sum_{i=1}^{n} P_\varnothing(s_i[t_i]) \). Obviously, \( P_\varnothing(s_1[t_1] \ldots t_n) = P_\varnothing(s_1) + P(t_1 \ldots t_n) - P_\varnothing(s_{n+1}) \). If the walk is a cycle (with \( s_1 = s_{n+1} \)), we thus find \( P(t_1 \ldots t_n) = \sum_{i=1}^{n} P_\varnothing(s_i[t_i]) \) where all non-zero Parikh vectors in the sum stem from chords. The set \( \{ P_\varnothing(s[t]s') \mid (s,t,s') \in E \setminus \varnothing \} \) is then a generator for all Parikh vectors of cycles (the latter being linear combinations of its elements). By simple linear algebra, we can compute a basis from this generator. This cycle base \( \Gamma \) contains at most \( |\Gamma| \leq |\Sigma| \) different Parikh vectors.

An LTS \( TS = (S, \Sigma, E, s_0) \) is finite if \( S \) is finite and it is deterministic if \( s[t]s' \land s[t]s'' \) implies \( s' = s'' \) for all \( s \in S \) and \( t \in \Sigma \). We call \( TS \) reachable if for every state \( s \in S \) exists some \( \sigma \in \Sigma^* \) with \( s_0[\sigma]s \). Reachability implies the existence of a spanning tree. Two labelled transition systems \( TS_1 = (S_1, \Sigma_1, E_1, s_0) \) and \( TS_2 = (S_2, \Sigma_2, E_2, s_{02}) \) are isomorphic if \( \Sigma_1 = \Sigma_2 \) and there is a bijection \( \zeta : S_1 \to S_2 \) with \( \zeta(s_{01}) = s_{02} \) and \( (s,t,s') \in E_1 \iff (\zeta(s),t,\zeta(s')) \in E_2 \), for all \( s,s' \in S_1 \). \( TS_1 \) can be embedded into \( TS_2 \) if \( \Sigma_1 \subseteq \Sigma_2 \) and there is an injection \( \zeta : S_1 \to S_2 \) with \( s \neq s' \) implies \( \zeta(s) \neq \zeta(s') \), such that \( \zeta(s_{01}) = s_{02} \) and \( (s,t,s') \in E_1 \Rightarrow (\zeta(s),t,\zeta(s')) \in E_2 \) for all \( s,s' \in S_1 \).

Figure 2 shows an LTS an the left that can be embedded into the LTS in the middle but is not isomorphic to it (due to \( s_5 \) and three additional edges). All shown edges, whether solid or dashed, belong to the LTS in question. The solid edges present a spanning tree of the LTS.
**Definition 2. Petri nets**

An (initially marked) Petri net is denoted as \( N = (P, T, W, M_0) \) where \( P \) is a finite set of places, \( T \) is a finite set of transitions, \( W \) is the weight function \( W : ((P \times T) \cup (T \times P)) \to \mathbb{N} \) specifying the arc weights, and \( M_0 \) is the initial marking (where a marking is a mapping \( M : P \to \mathbb{N} \), indicating the number of tokens in each place). A transition \( t \in T \) is enabled at a marking \( M \), denoted by \( M[t] \), if \( \forall p \in P : M(p) \geq W(p, t) \). Firing \( t \) leads from \( M \) to \( M' \), denoted by \( M[t]M' \), if \( M[t] = M(p) \to W(p, t) + W(t, p) \). This can be extended, by induction as usual, to \( M[\sigma]M' \) for words \( \sigma \in T^* \), and \( \{ M' \ | \exists \sigma \in T^* : M[\sigma]M' \} \) denotes the set of markings reachable from \( M \). The reachability graph \( RG(N) \) of a Petri net \( N \) is the labelled transition system with the set of nodes \( \{ M_0 \} \), initial state \( M_0 \), label set \( T \), and set of edges \( \{ (M, t, M') \ | \ M, M' \in \{ M_0 \} \wedge \exists [t]M' \} \).

If a labelled transition system \( TS \) is isomorphic to the reachability graph \( RG(N) \) of a Petri net \( N \) we say that \( N \) PN-solves (or simply solves) \( TS \) or that \( TS \) is synthesisable (to \( N \)). If \( TS \) can be embedded into \( RG(N) \), we say that \( N \) over-approximates \( TS \) or that \( TS \) is PN-embeddable (into \( N \)), and write \( TS \subseteq N \).

The LTS in the middle of Figure 2 is synthesisable to the Petri net on the right, which can easily be seen by playing the token game, e.g. state \( s_1 \) corresponds to the marking with one token on the upper left place and three on the lower one, which can be obtained by firing \( a \). The synthesisability of the middle LTS implies that the left LTS is PN-embeddable into this Petri net via the embedding between the two LTS. If an LTS is not synthesisable or embeddable to any Petri net, we might opt to modify the LTS. One way to do this is label splitting.

**Definition 3. Label Splitting**

Let \( TS = (S, \Sigma', E', \phi, \psi) \) be an LTS. A label splitting for \( TS \) is a quadruple \( (\Sigma', E', \phi, \psi) \) of a finite alphabet \( \Sigma' \supseteq \Sigma \), a set \( E' \subseteq S \times \Sigma' \times S \), a surjective mapping \( \phi : \Sigma' \to \Sigma \), and a bijection \( \psi : E' \to E \) such that \( \psi((s, t, s') \in E') = (s, \phi(t), s') \in E \). The result of the label splitting \( (\Sigma', E', \phi, \psi) \) is the LTS \( (S, \Sigma', E', \phi, \psi) \).

The label splitting \((\Sigma', E', \phi, \psi)\) is called optimal for \( TS = (S, \Sigma, E, s_0) \) if \((S, \Sigma', E', s_0)\) is PN-embeddable (to an arbitrary Petri net) and every label splitting \((\Sigma'', E'', \phi', \psi')\) with a PN-embeddable LTS \((S, \Sigma'', E'', s_0)\) yields \( |\Sigma'| \leq |\Sigma''| \).

If an LTS \((S, \Sigma, E, s_0)\) is synthesisisable or embeddable, an optimal label splitting is \((\Sigma, E, \text{id}, \text{id})\). This is the case for the two LTS in Fig. 2. For the right LTS in Fig. 1 (written as \((S = \{s_0, \ldots, s_6\}, \Sigma = \{a, b\}, E, s_0))\), which is not PN-embeddable, it is necessary to make the Parikh vectors of the paths \( s_0|ab|s_2 \) and \( s_0|ba|s_5 \) distinguishable. One optimal label splitting would be \((\{a, b, c\}, E', \phi, \psi)\) with \( \phi(a) = \phi(c) = a, \phi(b) = b, E' = \{ (s_0, c, s_1), (s_1, b, s_2), (s_2, a, s_3), (s_0, b, s_4), \ldots \} \).
finding a solution for every separation problem solves all three issues, making the synthesis successful with RG.

We have \((s_0, a, s_1) = (s_0, g(c), s_1) = (s_0, a, s_1)\). Optimality of a label splitting leads to an over-approximating Petri net with a minimal number of transitions, in this case, three. The situation can become much more complicated when the LTS contains cycles.

**Definition 4. Synthesis**

A region \(r = (R, B, F)\) of an LTS \((S, \Sigma, E, s_0)\) consists of three functions \(R: S \rightarrow \mathbb{N}, B: \Sigma \rightarrow \mathbb{N},\) and \(F: \Sigma \rightarrow \mathbb{N}\) such that for all edges \(s(t)s'\) in the LTS we have \(R(s) \geq B(t)\) and \(R(s') = R(s) - B(t) + F(t)\). The difference \(E(t) = F(t) - B(t)\) is called the effect of the label \(t\). The defining conditions of a region mimic the firing rule of Petri nets and make regions essentially equivalent to places, i.e. a place \(p\) can be defined from \(r\) via \(M_0(p) = R(s_0), W(p, t) = B(t),\) and \(W(t, p) = F(t)\) for all \(t \in \Sigma\). When a Petri net is constructed from a set of regions of a reachable LTS, this implies a uniquely defined marking \(M(s)\) for each state \(s\) with \(M(s)(p) = M_0(p) + \sum_{t \in \Sigma} P(\sigma)(t) \cdot (W(t, p) - W(p, t))\) for an arbitrary walk \(s_0[\sigma]s\).

The construction of a Petri net \(N = (P, T, W, M_0)\) with one place in \(P\) for each region of the LTS guarantees \(s[t] = M(s)[t]\), but has three issues: (1) \(P\) might become infinite, (2) \(M\) may not be injective, and (3) \(M(s)[t] = s[t]\) need not hold. Failing (2) means that there are states \(s, s' \in S\) with \(s \neq s'\) that are identified in \(RG(N)\) (leading to non-isomorphism). A state separation problem (SSP) is a pair \((s, s')\) in \(S \times S\) with \(s \neq s'\). A region \(r\) solves an SSP \((s, s')\) if \(R(s) \neq R(s')\) (and thus \(M(s) \neq M(s')\))\(^3\). Failing (3) results in an edge \(M(s)[t]\) in \(RG(N)\) but not in the LTS, \(s[t]\). An event/state separation problem (ESSP) is a pair \((s, t)\) in \(S \times \Sigma\) with \(-s[t]\). A region \(r\) solves an ESSP \((s, t)\) if \(R(s) < B(t)\) (and thus \(-M(s)[t]\))\(^4\). The set of all separation problems, \{\((s, s') \in S \times S | s \neq s'\} \cup \{(s, t) \in S \times \Sigma | -s[t]\}\}, is finite for finite LTS, and finding a solution for every separation problem solves all three issues, making the synthesis successful with \(RG(N)\) isomorphic to the finite LTS.

SSPs and ESSPs can be written as linear inequality systems over variables \(F \geq 0, B \geq 0,\) and \(R(s_0) \geq 0\). These systems are generally polynomial in the size of the LTS, see e.g. [LTS].

If the LTS is finite, the linear inequality systems are also finite, and we may employ standard means, e.g. an ILP- or SMT-solver [8], to solve them. If an LTS is not reachable, it is not structurally isomorphic to a reachability graph, i.e. no Petri net solving it exists. The LTS may still be embeddable, but possibly into

---

\(^3\) An example for an SSP in Fig. 2 (both LTS) is \((s_3, s_7)\). Any region \(r = (R, B, F)\) with \(E(b) \neq 0\) solves this SSP, as this implies \(R(s_3) = R(s_0) + E(a) + E(b) \neq R(s_0) + E(a) + 3E(b) = R(s_7)\). We might choose \(R(s_0) = 0, E(a) = 1 = E(b),\) and \(E(c) = -2,\) for example, to obtain \(R(s_3) = 2 \neq 4 = R(s_7)\).

\(^4\) The left LTS of Fig. 2 has two unsolvable ESSPs, but ESSPs will not be important in this paper.
many different reachability graphs. Depending on how the unreachable parts of
the LTS will be connected after the embedding, states must be mapped to other
markings, so we may not be able to find a unique marking $M(s)$ for every state $s$,
see Fig. 3 for an example. For these reasons, we assume all LTS to be finite
and reachable in the remainder of this paper.

If we just aim at embedding an LTS into a Petri net, dealing with ESSPs is not
necessary. For elementary net systems this is already known [2]. For arbitrary
Petri nets a similar result can be derived.

Lemma 1. Embedding and SSP

Let $TS = (S, \Sigma, E, s_0)$ be a finite, reachable LTS. $TS$ is PN-embeddable if and
only if every SSP $(s, s')$ (with $s, s' \in S$, $s \neq s'$) is solvable.

Proof: $\Rightarrow$: Let $(s, s')$ be an SSP and let $N = (P, T, W, M_0)$ be a Petri net
with $TS \sqsubseteq N$ by an injective morphism $\zeta: S \to [M_0]$. If $s \neq s'$, we also have $\zeta(s) \neq \zeta(s')$, i.e. there is a place $p \in P$ with $\zeta(s)(p) \neq \zeta(s')(p)$. We find a region
$r = (R, B, F)$ for $RG(N)$ with $R(s_0) = M_0(p)$ and $\forall t \in T$: $B(t) = W(p, t)$ and
$F(t) = W(t, p)$ such that $R(\zeta(s)) \neq R(\zeta(s'))$. As $\zeta$ is a morphism, $(\zeta \circ R, B, F)$
is a region for $TS$. Thus, the SSP $(s, s')$ is solved by the region $(\zeta \circ R, B, F)$.

$\Leftarrow$: Assume all SSPs $(s, s')$ are solvable. Each solution forms a region $r =
(R, B, F)$, giving rise to a place $p \in P$ with $M_0(p) = R(s_0)$, $W(t, p) = B(t)$, and
$W(t, p) = F(t)$ for all $t \in \Sigma$. Construct a Petri net $N = (P, \Sigma, W, M_0)$
with all these places. For every state $s \in S$, we then obtain the uniquely de-
defined marking $M(s)$ with $M(s)(p) = M_0(p) + \sum_{\gamma \in \Sigma} P(\gamma)(t) \cdot (W(t, p) - W(p, t))$
for every place $p$ and walk $s_0[\gamma]s$ in $TS$. $M$ is obviously injective, as two dif-
ferent states $s$ and $s'$ are distinguished at least by the region solving the SSP
$(s, s')$ and the place $p$ constructed from it. Let $(s, t, s') \in E$, so $M(s)(p) =
M_0(p) + \sum_{\gamma \in \Sigma} P(\gamma)(s')(t) \cdot (W(t, p) - W(p, t)) = R(s_0) + \sum_{\gamma \in \Sigma} P(\gamma)(s)(t) \cdot E(t) =
R(s_0) + E \cdot P(\gamma)(s) = R(s) \geq B(t)$ and also $M(s')(p) = R(s') = R(s) + E \cdot E(t)$ by the

Fig. 3. The LTS with unreachable states on the left can be embedded into the other two
LTS, which are both synthesizable. The unique marking $M$ for the left LTS (as far as it
can be derived) must distinguish each pair of states, especially the states $s_1$ and $s_2$, and
determines the effect of $a$ as $M(s_3) - M(s_1)$. Then, $M(s_0) = M(s_1) - (M(s_3) - M(s_1))$
must be different from $M(s_0) = M(s_2) - (M(s_3) - M(s_1))$, showing that $M(s_0)$ needs
different values in the middle and right LTS. This contradicts the uniqueness of
$M$ for the left LTS, i.e. we cannot distinguish the states in the left LTS without knowing how
they will be embedded.
definition of the region \( r = (R, B, F) \) corresponding with \( p \). Thus, \( M(s)|t\rangle M(s') \) in \( RG(N) \) and with \( M(s_0) = M_0 \) we conclude that \( M \) is an injective morphism embedding \( TS \) in \( RG(N) \), i.e. \( TS \subseteq N \).

### 3 A Reduction for Near-Optimal Label Splits

For finding an optimal label splitting, we must determine the minimal number of labels required to make an LTS PN-embeddable, i.e. solve an optimisation problem. If we turn this number into an input parameter of our problem, we can convert the latter into a decision problem:

**Definition 5. A Decision Problem**

Let \( TS = (S, \Sigma, E, s_0) \) be a finite, reachable LTS and \( q \in \mathbb{N} \) a number. The near-optimal label splitting problem \((TS, q)\) is the question whether there exists a label splitting \((\Sigma', E', q, \varphi)\) with \( |\Sigma'| \leq q \) such that \( TS' = (S, \Sigma', E', s_0) \) is PN-embeddable, i.e. all SSPs in it are solvable (cf. Lemma 1).

If the decision problem is decidable, the optimisation problem is also decidable with only polynomial overhead. Take \( TS \) and check for \( |\Sigma| \leq q \leq |E| \) whether \((TS, q)\) can be positively decided, starting with \( q = |\Sigma| \) and incrementing. When we get a positive answer (which must happen with \( q = |E| \) at the latest), we have found an optimal label splitting. We need to make at most \( |E| - |\Sigma| + 1 \) decisions, a number polynomial in the size of \( TS \).

**Theorem 1. Near-optimal Label Splitting is in NP**

For finite, reachable LTS \( TS = (S, \Sigma, E, s_0) \) and \( q \in \mathbb{N} \), the near-optimal label splitting problem \((TS, q)\) is in \( \text{NP} \).

**Proof:** Guess \( \Sigma' \) with \( |\Sigma'| \leq q \) and mappings \( q \) and \( \varphi \) for a label splitting \((\Sigma', E', q, \varphi)\). This takes \( O(q + |E|) \) time, which is polynomial in the size of \( TS \). To check if the guess is correct, the at most \(|S|^2\) SSPs of \( TS \) need to be solved according to Lemma 1. An SSP is represented by an inequality system polynomial in the size of \( TS \), which in turn can be solved by Karmarkar’s algorithm in polynomial time \[12\].

We now want to show that such a near-optimal label splitting is also \( \text{NP} \)-hard.

---

5 Karmarkar’s algorithm finds a rational solution of an inequality system (if it exists) in \( O(|S|^{3.5} \cdot L^2 \cdot \log L \cdot \log \log L) \) time with \( L = |S| \cdot |\Sigma'| \cdot \log |c| \), where \( c \) is the largest coefficient. Our inequality systems do not contain constant terms, allowing for multiplication with the common denominator to find an integer solution. The inequality system for an SSP contains one true inequality relation (\( \neq \)), which can be avoided to apply Karmarkar’s algorithm by solving two separate systems, replacing \( \neq \) by \( < \) and \( > \). Alternatively, dual systems without \( \neq \) may be solved in the rationals \[5\].
Theorem 2. NEAR-OPTIMAL LABEL SPLITTING IS NP-HARD

For finite, reachable LTS $TS = (S, \Sigma, E, s_0)$ and $q \in \mathbb{N}$, the near-optimal label splitting problem $(TS, q)$ is NP-hard.

To prove this theorem, we construct a polynomial-time reduction from some NP-complete problem to our problem. There are various NP-complete problems on graphs known, but even when edge labels occur in such problems, they are typically cost functions and the optimisation problem usually is to minimise the cost for some walk. A direct translation to markings and regions does not look easy, especially since we need to distinguish markings for an SSP and not minimise them. We chose to start our reduction at the problem of subset sums instead, which allows us to construct our own graph rather freely:

Definition 6. NP-COMPLETE SUBSET SUM PROBLEM $\text{[11]}$

Decide for $n \in \mathbb{N}$, $b \in \mathbb{N}_{\neq 0}$, and $C = \{c_1, \ldots, c_n\} \subseteq \mathbb{N}_{\neq 0}$, whether there is an index set $I \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in I} c_i = b$.

We make a reduction from the subset sum problem to our near-optimal label splitting problem. For an input $(n, b, \{c_1, \ldots, c_n\})$ of a subset sum problem we determine two parameters: the unique $k \in \mathbb{N}$ such that $2^k \leq 1 + 2b + 2\sum_{i=1}^{n} c_i < 2^{k+1}$ and $q = 2n + k + 11$. With the parameter $k$, we construct the LTS shown in Fig. 4. The value $q$ is the parameter for the label splitting problem.

4 Units and Region Values

Let us assume a fixed subset sum problem with constructed parameters $k$, $q$, and the LTS $TS = (S, \Sigma, E, s_0)$ from Fig. 4. At the initial state $s_0$, there are six edges with labels $h_i$ ($1 \leq i \leq 6$). Everything following such an edge $h_i$ (but without the $h_i$-edge) will be called the strand $h_i$. The strand $h_1$ defines a unit $u_0$ and some multiples, in the following sense:

Lemma 2. UNITS

Let $TS'$ be any reachable LTS embedding the strand $h_1$ of $TS$ in Fig. 4. Let $r = (R, B, F)$ be a region of $TS'$ and let $s(v)s'$ by any walk in $TS'$ with $v \in \{u_0, u_1, \ldots, u_k\}^*$. Then, $\mathbb{E}(v) = R(s') - R(s) = \sum_{i=0}^{k} \mathcal{P}(v)(u_i) \cdot 2^i \cdot \mathbb{E}(u_0)$.

---

$^6$ This problem is called Knapsack in [11] and defined with $\mathbb{Z}$ instead of $\mathbb{N}_{\neq 0}$, but the construction for the NP-hardness proof produces non-negative numbers only. Without loss of generality, we may even assume that all $c_i$ and $b$ are non-zero. If some $c_i$ is zero, it can simply be excluded from $C$, and if $b = 0$ the problem is trivially solvable (and we can replace it e.g. with $n = 1$, $c_1 = 2$, $b = 2$).
Fig. 4. The LTS constructed for a subset sum problem \((n, b, \{c_1, \ldots, c_n\})\) consists of six strands starting with \(h_1, \ldots, h_6\). The strands do not interact, they are only connected by the fact that the same label must have the same effect (on a potential marking) even in different strands. A term \(u(x)\) denotes a binary encoding of some \(x \in \mathbb{N}\) where each bit with value \(2^j\) \((0 \leq j \leq k)\) is expressed by the presence or absence of \(u_j\) in the word \(u(x)\). The value of \(k\) is chosen (just) high enough such that for every occurring value \(x\) we get some valid binary encoding \(u(x)\). Dotted lines denote a canonical enumeration (up to \(k\) or \(n\) or \(n+1\)), dashed lines, together with the edges before and after them, denote a sequence of edges inscribed with some \(u(x)\). The first four strands are auxiliary, they define units of measurement \(u_i\), assert distances via \(o\) and \(O\), and allow to express the two values \(\sum_{i \in I} c_i\) and \(b\) to be compared for the subset sum problem via \(\alpha\) and \(\beta\). The strand \(h_5\) will force the selection of some \(c_i\) for an index set \(I\) and strand \(h_6\) will assert the correct choice for the index set \(I\), guaranteeing \(\sum_{i \in I} c_i = b\).
Proof: Let us name the states of strand \( h_1 \) as \( s_i, s_i' \) for \( 1 \leq i \leq k \) with a last state \( s_{k+1} \) such that \( s_i[u_i]s_{i+1} \) (the lower, curved edges) and \( s_i[u_i-1]s_i'[u_i-1]s_{i+1} \) (the upper edges) for \( 1 \leq i \leq k \). Clearly, \( E(u_0) = F(u_0) - B(u_0) = R(s_1') - R(s_1) \).

By definition of a region, edges with the same label have the same effect, e.g., \( R(s_2) - R(s_1') = E(u_0) \). Thus, \( R(s_2) - R(s_1) = 2E(u_0) = E(u_1) \) as \( s_1[u_1]s_2 \), so \( u_1 \) has the effect \( 2E(u_0) \). The same reasoning for \( s_2[u_1]s_2'[u_1]s_3 \) and \( s_2[u_2]s_3 \) yields the effect \( 4E(u_0) \) for \( u_2 \). Recursively, we obtain \( 2^iE(u_0) \) as the effect \( E(u_i) \). As the effects are added up over a walk, \( E(v) = R(s') - R(s) = \sum_{i=0}^{k} p(v)(u_i) \cdot 2^i \cdot E(u_0) \).

We define now a function allowing us to compute the effect on region values for most edges in \( TS \).

**Definition 7. Unit Mapping**

Let \( x \in \mathbb{N} \) with \( x < 2^{k+1} \). Let \( x = \sum_{i=0}^{k} m_i \cdot 2^i \) with \( m_i \in \{0,1\} \) be a binary encoding of \( x \). Define \( u(x) = x_k \ldots x_1 x_0 \in \{u_0, \ldots, u_k\}^* \) to be the word with \( x_i = u_i \) if \( m_i = 1 \) and \( x_i = \varepsilon \) if \( m_i = 0 \).

As an example, \( u(25) = u_4u_3u_0 \) since 25 is written 11001 as a binary number. When we annotate a combination of an edge, a dashed line, and another edge in \( TS \) by a word \( u(x) \), this shall denote a sequence of states and edges where the edges have the letters of \( u(x) \) as labels. This occurs once in each strand \( h_2 \), \( h_3 \), and \( h_4 \) (with \( u(1 + 2b + 2 \sum_{i=0}^{n} c_i) \), \( u(\sum_{i=1}^{n} c_i) \), and \( u(2b) \) as inscription, respectively), as well as \( n \) times in the strand \( h_5 \) (with \( u(c_i) \) for \( 1 \leq i \leq n \)). We can now determine the effects on region values for most of the functional labels occurring in \( TS \).

**Lemma 3. Region Values**

Let \( TS' \) be any reachable LTS embedding the strands \( h_1 \) through \( h_5 \) of \( TS \) in Fig. 4. Let \( r = (R, B, F) \) be a region of \( TS' \) and \( E(u_0) \) be the effect of \( u_0 \). Then, \( E(\alpha) = \sum_{i=1}^{n} c_i \cdot E(u_0) \), \( E(\beta) = 2b \cdot E(u_0), E(\alpha) = (1 + 2b + 2 \sum_{i=1}^{n} c_i) \cdot E(u_0) \), and \( E(\gamma) = -(n + 1) \cdot E(u_0) \).

Proof: As walks between the same two states yield the same effect on the region value, the first part of strand \( h_2 \) enforces directly \( E(\alpha) = (1 + 2b + 2 \sum_{i=1}^{n} c_i) \cdot E(u_0) \) when applying Definition 7 and Lemma 2. The remaining parts of the lemma are derived analogously.

Let us try to determine the region effect of the labels \( \gamma_i \) in strand \( h_5 \) now. We find that there are pairs of \( \gamma_i \) forming cycles (asserting \( E(\gamma_i) = 0 \)) and \( \gamma_i \) can also be expressed in units, yielding \( E(u_0) = 0 \). This collapses our choice of regions and makes any pair of states from the same strand indistinguishable. The only remedy is to relabel one of the \( \gamma_i \)’s:
Lemma 4. Required Label Splitting for $\gamma_i$

Let $s, s' \in S$ be two states in strand $h_5$ of $TS$ such that $s[\gamma_i)s'[\gamma_i)s$ is fulfilled for some $i \in \mathbb{N}$. Then, the SSP $(s, s')$ is unsolvable. The only way to make this SSP solvable is to relabel one of the two $\gamma_i$-edges via label splitting.

Proof: The lower $\gamma_i$-label in strand $h_5$ lets us determine $E(\gamma_i) = c_i \cdot E(u_0)$, while the upper $\gamma_i$-label yields $E(\gamma_i) + c_i \cdot E(u_0) = 0$ due to the cycle via $u(c_i)$. The only solution for both equations is $E(\gamma_i) = E(u_0) = 0$ (since we assumed $c_i \neq 0$ in the beginning). So, in every region $r = (R, B, F)$ of $TS$, $R(s') = R(s) + E(\gamma_i) = R(s)$, i.e. the two states cannot be distinguished. Therefore, the SSP $(s, s')$ is unsolvable.

Since no other edges are involved in the cycle $s[\gamma_i\gamma_i)s$, the SSP $(s, s')$ can only become solvable by relabelling one (or both) of these two edges, giving the edges different labels. The only way to relabel an edge is via a label splitting. Relabelling exactly one of the two $\gamma_i$-edges in strand $h_5$ to a new label $\bar{\gamma_i}$ will potentially allow regions with $E(\gamma_i) \neq 0 \neq E(u_0)$.

5 Proving the polynomial-time reduction

Let us count the number of different labels in $TS$, i.e. the alphabet size. We have $k + 1$ labels $u_i$, $n$ labels $\gamma_i$, and one each of: $\alpha, \beta, O, h_1, h_2, h_3, h_4, h_5, h_6$. Summing these up, we come to $|\Sigma| = n + k + 11$. We will show now the following statements:

1. The construction of $TS$ can be done in polynomial time.
2. If a label splitting applied to $TS$ introduces less than $q = 2n + k + 11$ labels, some SSP instances will be unsolvable. To be more precise, each label $\gamma_i$ ($1 \leq i \leq n$) needs a new, “opposite” label $\bar{\gamma_i}$. This also means that all other labels must remain “unsplit”, as we are not allowed to have more than $q$ labels.
3. There is at least one label splitting with exactly $q$ labels making all SSPs solvable if and only if the corresponding subset sum problem has a solution.

This will show that our construction is a polynomial-time reduction, concluding the proof of Theorem 2.

Lemma 5. Polynomial-Time Construction

The construction of $TS$ can be done in polynomial time (of the size of the input subset sum problem $(n, b, \{c_1, \ldots, c_n\}$)).
Proof: Note first that the value of numbers occurring in the input, i.e. \( b \) and the \( c_i \), can be exponentially larger than the size of the input in its binary encoding. The function \( u : \mathbb{N} \to \{u_0, \ldots, u_k\}^* \) can be directly applied to the binary encoding of the input and uses only linear time. Using this function in the strands \( h_2 \) to \( h_6 \) reduces the values \( b \) and \( c_i \) to a logarithmic number of states and edges, matching it with the size of the input. The parameter \( k \) is logarithmic in the value \( 1 + 2b + 2\sum_{i=1}^{k} c_i \), which is the largest occurring number. Strands \( h_2 \) and \( h_6 \) contain sequences of states and edges of length \( O(n) \), this can be matched with the \( n \) coefficients \( c_i \) of the input. Overall, \( TS \) has a size linear in the input and can be constructed in polynomial time.

We now take a look at the conditions a label splitting must fulfill to make SSPs solvable. We partition the SSPs into three sets, pairs of states in strand \( h_i \) and can be constructed in polynomial time. The parameter \( k \) is logarithmic in the value \( 1 + 2b + 2\sum_{i=1}^{k} c_i \), which is the largest occurring number. Strands \( h_2 \) and \( h_6 \) contain sequences of states and edges of length \( O(n) \), this can be matched with the \( n \) coefficients \( c_i \) of the input. Overall, \( TS \) has a size linear in the input and can be constructed in polynomial time.

We now take a look at the conditions a label splitting must fulfill to make SSPs solvable. We partition the SSPs into three sets, pairs of states in strand \( h_i \) and can be constructed in polynomial time. The parameter \( k \) is logarithmic in the value \( 1 + 2b + 2\sum_{i=1}^{k} c_i \), which is the largest occurring number. Strands \( h_2 \) and \( h_6 \) contain sequences of states and edges of length \( O(n) \), this can be matched with the \( n \) coefficients \( c_i \) of the input. Overall, \( TS \) has a size linear in the input and can be constructed in polynomial time.

\[ \text{Lemma 6. Inter-strand SSPs are always Solvable} \]

Let \( s, s' \in S \) be two states that do not lie in the same strand of \( TS \). Let \( TS' \) be the result of some label splitting of \( TS \). Then, the SSP \( (s, s') \) is solvable in \( TS' \).

Proof: Since the events \( h_i \) each appear only once in \( TS \), we can without loss of generality assume that they still appear in \( TS' \) and were not split/renamed. We define a region \( r = (R, B, F) \) that assigns \( R(s') = i \) to all states \( s' \) in strand \( h_i \) and \( R(s_0) = 0 \). This already determines \( R \) and guarantees that SSP \( (s, s') \) is solvable. To extend this into a region, it suffices to set \( B(t) = 0 \) for all \( t \), \( F \) is \( F(h_i) = i \) for \( i = 1, \ldots, 6 \), and \( F(t) = 0 \) otherwise.

\( r \) satisfies the requirements of a region, because \( B(t) = 0 \leq R(s) \in \mathbb{N} \) for all \( t \) and \( s \), and the only events with non-zero effect each appear on a single edge and satisfy \( R(s') = R(s) + E(t) \), thus this holds for all \( s[t]s' \).

\[ \text{Lemma 7. Effects in Strand Five} \]

Let \( TS' \) be the result of some label splitting of \( TS \) that relabels exactly one instance of each \( \gamma_i \) in strand \( h_5 \) and nothing else. Let \( r = (R, B, F) \) be a region of \( TS' \) and \( E(u_0) \) be the effect of \( u_0 \). Then, \( E(\gamma_i) = c_i \cdot E(u_0) \) if the upper edge with label \( \gamma_i \) was relabelled and otherwise \( E(\gamma_i) = -c_i \cdot E(u_0) \).

Proof: Assume that the upper edge was relabelled. Then there are two walks \( s[u(c_i)]s' \) and \( s[\gamma_i]s' \) in strand five. These two walks must have the same effect, so \( E(\gamma_i) = E(u(c_i)) = c_i \cdot E(u_0) \) when applying Definition \( \ref{def:effect} \) and Lemma \( \ref{lem:single-effect} \).

In the other case, there is a cycle \( s[u(c_i)\gamma_i]s \), thus \( E(\gamma_i) = -E(u(c_i)) = -c_i \cdot E(u_0) \).

\[ \text{For all occurring } u(x) \text{ in } TS, x \text{ is less or equal to this value.} \]
Lemma 8. SSPs in the First Five Strands

Let $TS'$ be the result of some label splitting of $TS$ that relabels exactly one instance of each $\gamma_i$ in strand $h_5$ and nothing else. Assume a region $r = (R, B, F)$ of $TS'$ with $\mathbb{E}(u_0) \neq 0$. For every pair $(s, s')$ of states with $s \neq s'$ inside one of the strands $h_i$ with $1 \leq i \leq 5$ we obtain $R(s) \neq R(s')$.

Proof: Since $s$ and $s'$ are in the same strand, there is a walk $v \in \{u_0, \ldots, u_k, o\}^*$ (cf. Fig. 4 and Definition 2) with $s[v]s'$ or $s'[v]s$. W.l.o.g. assume $s[v]s'$. We compute $R(s') = R(s) + \mathbb{E}(v)$ where the effect of each letter in $\{u_0, \ldots, u_k, o\}$ is a positive multiple of $\mathbb{E}(u_0)$ (see Lemma 2 and 3). Thus, we find some $m \in \mathbb{N}_{\neq 0}$ with $R(s') = R(s) + m \cdot \mathbb{E}(u_0)$ and due to $\mathbb{E}(u_0) \neq 0$, also $R(s) \neq R(s')$ is true. $\square$

Lemma 9. SSPs in the Strand $h_6$

Let $TS'$ be the result of some label splitting of $TS$ that relabels exactly one instance of each $\gamma_i$ in strand $h_5$ and nothing else. Assume a region $r = (R, B, F)$ of $TS'$ with $\mathbb{E}(u_0) \neq 0$. For every pair of states $(s, s')$ with $s \neq s'$ inside the strand $h_6$ we get $R(s) \neq R(s')$.

Proof: Without loss of generality, assume that $\mathbb{E}(u_0) > 0$ (otherwise negate all values to obtain a region “$-r$”). Note that $\mathbb{E}(o) > 2\mathbb{E}(\alpha) + \mathbb{E}(\beta)$ by Lemma 3 as well as $\mathbb{E}(o) > 2\mathbb{E}(\gamma_i)$ for $1 \leq i \leq n$ by Lemma 7. In any walk $s_1[x]s_2[o]s_3[y]s_4$ with $x, y \in \{\alpha, \gamma_1, \ldots, \gamma_n\}$, the difference $R(s_3) - R(s_2) = \mathbb{E}(o)$ is so big that all four states must have pairwise different region values, with $R(s_3)$ and $R(s_4)$ being greater than both $R(s_1)$ and $R(s_2)$. The same holds with $s_2[\beta]s_1$ and $s_2[o]s_3[\alpha]s_4$ and also for the case $s_4[\gamma_n]s_3[O]s_2$ and $s_1[\beta]s_2$ where $\mathbb{E}(O) < -\mathbb{E}(o)$. If we order the states of strand $h_6$ into pairs connected by an edge label from $\{\alpha, \beta, \gamma_1, \ldots, \gamma_n\}$, the pair at $\beta$ has the lowest region values, and the values increase through the $\alpha$ and $\gamma_1$ pairs up to the states adjacent to $\gamma_n$. The two states inside a pair also have different values since $\mathbb{E}(u_0) \neq 0$ (see Lemmas 5 and 7). $\square$

To be a region, a triple of mappings $r = (R, B, F)$ must fulfill the two conditions $R(s) \geq B(t)$ and $R(s') = R(s) - B(t) + F(t)$ for every edge $s[t]s'$. If $R(s) < B(t)$ for some $s[t]$, we can determine $c := B(t) - R(s)$ and modify the mappings to $r' = (R + c, B, F)$. The new triple $r'$ distinguishes the same pairs of states as $r$, so it solves the same SSPs. We can easily see that the second condition holds in the first five strands if $\mathbb{E}$ satisfies the requirements of Lemmas 2, 3, and 7.

Lemma 10. $u_0$ provides Region Effects in the First Five Strands

Take only the first five strands of $TS$, apply a label splitting that relabels exactly one instance of each $\gamma_i$ in strand $h_5$ (and nothing else), and call the resulting LTS $TS'$. Let $\mathbb{E}$ be a mapping $\mathbb{E}: \Sigma \rightarrow \mathbb{Z}$ that fulfills the equations given in Lemma 2, 3, and 4. There is a region $(R, B, F)$ of $TS'$ such that $F - B = \mathbb{E}$.
Proof: Assume an arbitrary, fixed spanning tree of $T S'$. Remember that this assigns a unique walk $s_0|\sigma_s)$ to each state $s$. Define $R(s_0) = \max\{0\} \cup \{-E(\sigma_s) \mid s \in S\}$ and extend this via $R(s) = R(s_0) + E(\sigma_s)$ to all states. The choice of $R(s_0)$ guarantees $R(s) \geq 0$ for all states. Next, define $F(t) = E(t)$ and $B(t) = 0$ for all $t$ with $E(t) \geq 0$, and $F(t) = 0$ and $B(t) = -E(t)$ otherwise. We can now see that $(R, B, F)$ is a region, i.e. that for all $s(t)s'$ we have $R(s) \geq B(t)$ and $R(s') = R(s) + E(t)$. Both conditions follow trivially for edges in the spanning tree. Assume $s[t]s'$ to be a chord. Every chord completes a (generalised) cycle, therefore the effects defined via such cycles in Lemmas 2, 3, and 7 ensure $R(s') = R(s) + E(t)$.

As a consequence, all SSPs becoming solvable depends on only two points now: That we use a label splitting that relabels one of each $\gamma_i$ for $1 \leq i \leq n$ in strand $h_5$ and that the two walks in strand $h_6$ from the leftmost to the rightmost state have the same region effect, i.e. that the (generalised) cycle formed by these two walks has effect zero. We will see next that this is equivalent to finding a solution to the subset sum problem.

**Theorem 3. Subset Sum Solution is Equivalent to Solvable SSPs**

Let $S = \{n, b, \{c_1, \ldots, c_n\}\}$ be a subset sum problem and $T S$ (with the computed $k \in \mathbb{N}$) be the LTS constructed from it as per Figure 4. $S$ has a solution if and only if there is a PN-embeddable LTS $T S'$ (i.e. all SSPs are solvable) resulting from $T S'$ by a label splitting at most $q = 2n + k + 11$ labels.

Proof: Let $I$ be an index set of our subset sum problem $S$ (not necessarily a solution, though). Define a label splitting $(\Sigma', E', g, \varphi)$ with $\Sigma' = \Sigma \cup \{\overline{\gamma_i} \mid 1 \leq i \leq n\}$, $g(t) = t$ for $t \in \Sigma$, and $g(\overline{\gamma_i}) = \gamma_i$ for $1 \leq i \leq n$. Relabel in strand $h_5$ of $T S$ the upper $\gamma_i$ to $\overline{\gamma_i}$ if $i \in I$, and relabel the lower $\gamma_i$ to $\overline{\gamma_i}$ if $i \notin I$. Note how this changes the region effect of $\gamma_i$: If $i \in I$, $\mathbb{E}(\gamma_i) = c_i \cdot \mathbb{E}(u_0)$, and if $i \notin I$, then $\mathbb{E}(\gamma_i) = -c_i \cdot \mathbb{E}(u_0)$ (see Lemma 7). The relabelled LTS has now exactly $q = 2n + k + 11$ different labels.

Let us take a look at the strand $h_6$ now. The upper walk from the leftmost to the rightmost state contains exactly $n + 1$ $a$’s and one $O$. By Lemma 3, $\mathbb{E}(O) = -(n + 1) \cdot \mathbb{E}(a)$, thus the region effects of these labels cancel each other out. The remaining effect of the upper walk is then $\mathbb{E}(\alpha \gamma_1 \ldots \gamma_n)$ with $\mathbb{E}(\alpha) = \sum_{i=1}^{n} c_i \cdot \mathbb{E}(u_0)$. Therefore, if $\mathbb{E}(\gamma_i) = -c_i \cdot \mathbb{E}(u_0)$ it will cancel out the $c_i \cdot \mathbb{E}(u_0)$ in $\mathbb{E}(\alpha)$, while a positive $\mathbb{E}(\gamma_i) = c_i \cdot \mathbb{E}(u_0)$ will double the effect. Overall, we get $\mathbb{E}(\alpha \gamma_1 \ldots \gamma_n) = 2 \sum_{i \in I} c_i \cdot \mathbb{E}(u_0)$.

If $I$ is a solution of the subset sum problem, we have $\sum_{i \in I} c_i = b$ and we obtain $\mathbb{E}(\alpha \gamma_1 \ldots \gamma_n) = 2 \sum_{i \in I} c_i \cdot \mathbb{E}(u_0) = 2b \cdot \mathbb{E}(u_0) = \mathbb{E}(\beta)$. By defining $\mathbb{E}(u_0) = 1$ and $\mathbb{E}(h_i) = 0$ and extending this to a mapping $\Sigma \rightarrow \mathbb{Z}$ via the requirements from Lemma 2 (for $u_1$), Lemma 3 (for $\alpha$, $\beta$, $o$, and $O$), and Lemma 7 (for $\gamma_i$), the preconditions of Lemma 10 are fulfilled. This provides us with a region $r$ for our effects $\mathbb{E}$. For strand $h_6$, we can derive the effect of $(\beta)\cdot s'$ via the upper chain of edges as $R(s') = R(s) + \mathbb{E}(\alpha \gamma_1 \ldots \gamma_n) = R(s) + \mathbb{E}(\beta)$ as required. Thus,
\( r = (R, B, F) \) is a region with \( E(u_0) \neq 0 \), and according to the Lemmas it solves all SSPs inside of the same strand. SSPs between strands are always solvable by Lemma. We conclude that the LTS is PN-embeddable.

Assume for the other direction that we find a label splitting with at most \( q = 2n + k + 11 \) different labels that leads to a PN-embeddable, relabelled LTS, i.e. all SSPs are solvable. Any label splitting yielding less than \( q \) different labels has a guaranteed unsolvable SSP by Lemma. Therefore, the alphabet size is exactly \( q \), which means that one of each pair of \( \gamma_i \) has been relabelled, and nothing else. Thus, we have used a label splitting as constructed in the first paragraph of this proof, which stems from some arbitrary index set \( I \). Let \( s, s' \) be the first two states in strand \( h_1 \), with \( s(u_0)s' \). There is a region \( r = (R, B, F) \) with \( R(s') - R(s) = E(u_0) \neq 0 \) solving the SSP \( (s, s') \). Strand \( h_2 \) provides again \( E(O) = -(n + 1) \cdot E(o) \), so that in strand \( h_6 \) we find \( 2 \sum_{i \in I} c_i \cdot E(u_0) = E(\alpha \gamma_1 \ldots \gamma_n) = E(\beta) = 2b \cdot E(u_0) \). Dividing by \( 2E(u_0) \) we get \( \sum_{i \in I} c_i = b \), therefore the index set \( I \) is a solution of the subset sum problem.

6 Concluding remarks

Synthesising a small model like a Petri net from a large LTS or a set of observable processes can be done in more than one way. For Process Mining [1], Badouel and Schlachter [3] have constructed an incremental over-approximation algorithm. This allows behaviour that has not been observed essentially by adding edges to the LTS. Carmona [7] introduced a heuristic label splitting algorithm that can relabel a finite LTS to make it the reachability graph of a Petri net. For neither of the two approaches the time complexity is known. Both could well be exponential. A first polynomial-time label splitting algorithm has been shown in [19], but it will not always generate optimal results.

In this paper, we have investigated a case where both over-approximation (by embedding into a Petri net reachability graph) and label splitting are allowed in the synthesis procedure. Finding a Petri net with a minimal alphabet size or even only limiting the alphabet size makes this problem NP-complete, therefore a small model is not easily obtainable. The NP-completeness proof could be modified in certain ways, e.g. we can ask whether the removal of a certain number of edges (instead of label splitting) can make an LTS embeddable into a Petri net reachability graph. This leads to a model with a fault tolerance, i.e. a few desired behaviours may be missing and additional ones will exist at the same time. A problem instance would be \((TS, n)\), where \( n \) edges may be removed from an LTS \( TS \) to make it PN-embeddable. The construction for our reduction remains the same, but we would need to remove one of each pair of \( \gamma_i \)-edges in strand \( h_5 \) now to allow for a solution (instead of relabelling them). The remainder of the proof will stay the same, so this problem is also NP-complete.

\[ \text{We exclude simple renamings without loss of generality.} \]
We believe that label splitting aiming at exact synthesis, making the LTS isomorphic to a Petri net reachability graph, should not be easier than embedding, since the exact synthesis additionally demands all ESSPs to be solvable. We have no proof for this conjecture, though. In our construction, a label $x$ in any of the six strands of $TS$ has the effect $E(x) = m \cdot E(u_0)$ for some $m \in \mathbb{N}$ and the unit label $u_0$. If an edge $s \langle x \rangle_{s'}$ occurs, the solvability of all ESSPs demands that also $s \langle u_0 \rangle_{s'}$ must be possible. Here, $m$ has essentially the same size as the parameter values of the input, a subset sum problem. As the subset sum problem can be written in a binary encoding, our constructed LTS $TS$ with $m u_0$-edges will be exponential in the size of the input. Therefore, our construction cannot be done in polynomial time anymore.

Finally, observe that the embedding problem for unlabelled Petri nets is a special case of our problem, where the parameter $q$ for an input instance $(TS, q)$ must be set so that no label splitting can occur. Our reduction only works for trivial instances of the subset sum problem in this case, and indeed the unlabelled embedding problem is a subproblem of the exact synthesis problem for unlabelled nets, solvable with a polynomial-time algorithm.

References

1. W.M.P. van der Aalst: *Process Mining: Discovery, Conformance and Enhancement of Business Processes*. Springer, 352 pages, ISBN 978-3642193446 (2011).
2. É. Badouel, L. Bernardinello, P. Darondeau: *Petri Net Synthesis*. Texts in Theoretical Computer Science, Springer, 339 pages, ISBN 978-3-662-47967-4 (2015).
3. É. Badouel, U. Schlachter: Incremental Process Discovery using Petri Net Synthesis. *Fundamenta Informaticae* 154(1-4), 1-13, DOI: 10.3233/FI-2017-1548 (2017).
4. K. Barylska, E. Best, E. Erofeev, L. Mikulski, M. Piątkowski: On Binary Words being Petri Net Solvable. In: ATAED’2015, J. Carmona, R. Berghenthun, W. van der Aalst (eds), 1-15, [http://ceur-ws.org/Vol-1371](http://ceur-ws.org/Vol-1371) (2015).
5. E. Best, R. Devillers, U. Schlachter: A Graph-theoretical Characterisation of State Separation. In: 43th International Conference on Current Trends in Theory and Practice of Computer Science (SOFSEM 2017), 163-175, DOI: 10.1007/978-3-319-51963-0_13 (2017).
6. S. K. L. M. vanden Broucke, J. De Weerdt: Fodina: A robust and flexible heuristic process discovery technique. Decision Support Systems 100, 109-118, DOI: 10.1016/j.dss.2017.04.005 (2017).
7. J. Carmona: The Label Splitting Problem. In: Transactions on Petri Nets and Other Models of Concurrency VI, K. Jensen, W.M.v.d. Aalst, M. Ajmone-Marsan, G. Franceschinis, J. Kleijn, L.M. Kristensen (eds), Lecture Notes in Computer Science 7400, 1-23 (2012).
8. J. Christ, J. Hoenicke, A. Nutz: SMTInterpol: An Interpolating SMT Solver. Proc. of Model Checking Software, A. Donaldson and D. Parker (ed.), Lecture Notes in Computer Science 7385, 248-254 (2012). See also: [https://ultimate.informatik.uni-freiburg.de/smtinterpol/](https://ultimate.informatik.uni-freiburg.de/smtinterpol/)
9. A. Ehrenfeucht, G. Rozenberg: Partial 2-structures, Part I: Basic Notions and the Representation Problem, and Part II: State Spaces of Concurrent Systems. Acta Informatica 27(4), 315-368 (1990).
10. M.H.T. Hack: Petri Net Languages. Computation Structures Memo 124, Project MAC, MIT (1975).
11. R.M. Karp: Reducibility among Combinatorial Problems, Proceedings of an IBM symposium on the Complexity of Computer Computations, Springer, 85-103, DOI: 10.1007/978-1-4684-2001-2_9 (1972).
12. N. Karmarkar: A new polynomial-time algorithm for linear programming. Combinatorica 4, 373-395, DOI: 10.1007/BF02579150 (1984).
13. X. Lu, D. Fahland, F. J. H. M. van den Biggelaar, W. M. P. van der Aalst: Handling Duplicated Tasks in Process Discovery by Refining Event Labels. In: BPM 2016, M. La Rosa, P. Loos, O. Pastor (eds.). Lecture Notes in Computer Science 9850, 90-107, DOI: 10.1007/978-3-319-45348-4_6 (2016).
14. T. Murata. Petri Nets: Properties, Analysis and Applications. Proceedings of the IEEE 77, 541-580 (1989).
15. M. Nielsen, G. Plotkin, G. Winskel. Petri nets, event structures and domains, part I. Theoretical Computer Science 13(1), 85-100, DOI: 10.1016/0304-3975(81)90112-2 (1981).
16. W. Reisig: Petri Nets. EATCS Monographs on Theoretical Computer Science 4, Springer-Verlag (1985).
17. J. de San Pedro, J. Cortadella: Discovering Duplicate Tasks in Transition Systems for the Simplification of Process Models. In: BPM 2016, M. La Rosa, P. Loos, O. Pastor (eds.). Lecture Notes in Computer Science 9850, 108-124. DOI: 10.1007/978-3-319-45348-4_7 (2016).
18. U. Schlachter, H. Wimmel: A Geometric Characterisation of Event/State Separation. In: Application and Theory of Petri Nets and Concurrency 2018, V. Khomenko, O.H. Roux (eds.). Lecture Notes in Computer Science 10877, 99-116, DOI: 10.1007/978-3-319-91268-4_4 (2018).
19. U. Schlachter, H. Wimmel: Relabeling LTS for Petri Net Synthesis via Solving Separation Problems. In: Transactions on Petri Nets and Other Models of Concurrency XIV, M. Koutny, L. Pomello, L.M. Kristensen (eds). Lecture Notes in Computer Science 11790, 222-254 (2019).
20. U. Schlachter: Over-Approximative Petri Net Synthesis for Restricted Subclasses of Nets. Proceedings of LATA’18, S.T. Klein, C. Martin-Vide, D. Shapira (eds), Lecture Notes in Computer Science 10792, 296-307 (2018).