Progress in Small $x$ Resummation

Stefano Forte

Dipartimento di Fisica, Università di Milano and INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy

Abstract

I review recent theoretical progress in the resummation of small $x$ contributions to the evolution of parton distributions, in view of its potential significance for accurate phenomenology at future colliders. I show that a consistent perturbative resummation of collinear and energy logs is now possible, and necessary if one wishes to use recent NNLO results in the HERA kinematic region.

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PROGRESS IN SMALL $x$ RESUMMATION

STEFANO FORTE

Dipartimento di Fisica, Università di Milano and
INFN, Sezione di Milano, Via Celoria 16, I-20133 Milan, Italy

I review recent theoretical progress in the resummation of small $x$ contributions to the evolution of parton distributions, in view of its potential significance for accurate phenomenology at future colliders. I show that a consistent perturbative resummation of collinear and energy logs is now possible, and necessary if one wishes to use recent NNLO results in the HERA kinematic region.

1. The dangerous success of NLO calculations

There is a general attitude in the collider physics community that small $x$ resummation is impossible to understand, but there is no need to worry since the data can be described very well without it. It is certainly true that (for instance) standard global parton fits are based on fixed next-to-leading (NLO) order calculations, and that they manage to describe the data very accurately, in particular throughout the HERA kinematic region. The lack of experimental evidence for higher-order perturbative corrections at small $x$, despite theoretical arguments which suggest that they should be large, has been the motivation for a substantial amount of theoretical effort over the last decade. This activity has become less of an academic exercise since the recent determination of the full set of NNLO splitting functions and of the full set of $O(\alpha_s)$ perturbative corrections to deep-inelastic scattering. Indeed, parton fits that include these NNLO terms appear to be unstable at small $x$: when $x \lesssim 10^{-3}$ the difference in results e.g. for the $P_{gg}$ splitting function or for the gluon-dominated structure function $F_L$ when going from NNLO and NLO starts becoming as large or larger than the difference between NLO and LO. Interestingly, it also turns out that at small $x$ there is a significant difference between the full NNLO result and the nominally leading small $x$ contributions: subleading terms are crucial. This means that, whatever the reason for the success of NLO fits, small $x$ resummation is mandatory beyond NLLO, at least for $x < 10^{-3}$.

Thanks to recent theoretical progress, there is now a consistent theory which enables small $x$ resummation, and which is close to being amenable to realistic phenomenology. This theory requires several ingredients, which were developed by various people over the last decade, and have recently led to two approaches (ABF and CCSS) which incorporate similar basic principles, and which arrive at
stable and consistent resummed results. Here we will mainly review the ABF approach, while also comparing with the results of the CCSS approach and briefly discussing differences between the two approaches.

The main problems in small $x$ resummation of the standard GLAP evolution equations is first, that leading logarithmic resummation corrections are to be too large — they would lead to a powerlike small $x$ rise of splitting functions and thus of structure functions which is incompatible with the data, and second, that they are perturbatively unstable — the next-to-leading logarithmic resummation corrections are larger than the leading ones.

These problems are solved thanks to three main ingredients: duality, which makes the joint resummation of collinear and small $x$ logs possible, running coupling small $x$ resummation, and factorization, which softens the resummed small $x$ behaviour, and gluon exchange symmetry, which stabilizes the resummed perturbative expansion.

2. Duality

As is well known, singlet splitting functions at small $x$ receive contributions of the form $\alpha_s \ln(\frac{1}{x})^n (c_n^0 + \alpha_s c_n^1 + \ldots)$ to all perturbative orders. These contributions can be extracted from the BFKL equation, which for a parton distribution $G(x, Q^2)$ (in fact, an eigenvector of the singlet evolution matrix), and its Mellin transforms

$$ G(N, t) = \int_0^{\infty} d\xi \ e^{-N\xi} \ G(\xi, t) $$

$$ G(\xi, M) = \int_{-\infty}^{\infty} dt \ e^{-Mt} \ G(\xi, t) $$

takes the form

$$ \frac{d}{d\xi} G(\xi, M) = \chi(M, \alpha_s) \ G(\xi, M); \quad \xi \equiv \frac{1}{x}, $$
Fig. 2. Various expansions of the dual $\chi$ kernel, determined with $n_f = 4$, $\alpha_s = 0.2$. Note that the associate dual Eq. (5) anomalous dimension is simply the inverse function of $\chi(M)$, as per the axis labelling.

of a standard renormalization-group (or GLAP) equation,
\begin{equation}
\frac{d}{dt} G(N, t) = \gamma(N, \alpha_s) G(N, t); \quad t \equiv \frac{Q^2}{\mu^2}
\end{equation}
but with the roles of the variables $x$ and $Q^2$ interchanged. Upon Mellin transformation, $\frac{1}{x} k^{-1} \leftrightarrow (\ln \frac{1}{N}) k^k$ and $\frac{Q^2}{\mu^2} k^{-1} \leftrightarrow (\ln \frac{1}{M}) k^k$.

Whereas the way to extract the coefficients of the leading singular $\left(\frac{Q^2}{\mu^2}\right)^n$ contributions to the anomalous dimension $\gamma(N)$ Eq. (4) from the (BFKL) kernel $\chi(M)$ of Eq. (3) has been known since a long time, it has been realized only more recently that in fact, up to higher twist corrections, the BFKL and GLAP equations admit the same solution, provided the boundary conditions are suitably matched, and the corresponding kernels satisfy the duality relations
\begin{align}
\chi(\gamma(N, \alpha_s), \alpha_s) &= N \\
\gamma(\chi(M, \alpha_s), \alpha_s) &= M.
\end{align}
These relations hold at fixed coupling, whereas in the running case they are corrected by terms which may be computed order by order in perturbation theory.

Duality maps the expansion of $\gamma$ in powers of $\alpha_s$ at fixed $N$ (in Fig. 1 leading $\gamma_0$, next-to-leading $\gamma_1$ etc.) into the expansion of $\chi$ in powers of of $\alpha_s$ at fixed $\alpha_s/M$ (in Fig. 1 leading $\chi_s$, next-to-leading $\chi_{ss}$ etc.): hence $\gamma_0$ or $\chi_s$ sum leading logs of $Q^2$ (collinear logs) while $\gamma_s$ or $\chi_0$ sum leading logs of $\frac{1}{x}$ (energy logs) and so on.
A joint (double-leading) resummation can be constructed by simply combining the
two expansions and subtracting the terms which are in common: so the LO DL anomalous dimension is \(\chi_{DL,LO}(N) = \alpha_s \gamma_0(N) + \gamma_s (\alpha_s/N) - \frac{3\alpha_s}{\pi N} + \ldots\) and so on. A nontrivial property of the DL expansion is that the dual of LO DL \(\chi\) is LO DL \(\gamma\) (up to subleading terms), and similarly at next-to-leading order and so on.

The DL expansion allows one to understand and cure the notorious problem of the large size of subleading \(\chi_1\) in comparison to leading \(\chi_0\), seen in Fig. 2, where BFKL LO denotes \(\alpha_s \chi_0\) and BFKL NLO denotes \(\alpha_s \chi_0 + \alpha_s^2 \chi\). Indeed, momentum conservation implies that to all perturbative orders \(\gamma(1) = 0\). By duality this also implies \(\chi(0) = 1\). But \(\chi\) is a polynomial in \(\alpha_s\), so \(\chi(0) = 1\) means that as \(M \to 0\) \(\chi\) must behave as

\[
\chi_s(M) \sim \frac{1}{M^3} \frac{\alpha}{\alpha + M} = \frac{1}{M^3} - \frac{\alpha^2}{M^2} + \frac{\alpha^3}{M^3} + \ldots
\]

up to subleading corrections. Hence the expansion of \(\chi\) in powers of \(\alpha_s\) has alternating-sign poles at \(M = 0\), which are resummed into \(\chi_s\) and thus absent in double-leading \(\chi\). Indeed, Fig. 2 also displays the DL LO and NLO curves, which agree respectively with LO and NLO GLAP when \(M \approx 0\), and with LO and NLO BFKL when \(M \approx \frac{1}{2}\). Hence, the DL expansion is stable for all \(M \lesssim \frac{1}{2}\).

3. Running coupling

The running of the coupling \(\alpha(t) = \alpha_\mu [1 - \beta_0 \alpha_\mu t + \ldots]\) is a leading log \(Q^2\), but next-to-leading log \(\frac{1}{2}\) effect. As a consequence, beyond LLx the fixed-coupling duality relations (5) get corrected: for instance \(\gamma_{ss}\) determined from \(\chi_0\) and \(\chi_1\) according to Eq. (5) must be supplemented by a running-coupling correction \(\Delta \gamma_{ss} = -\frac{\beta_0 \chi_0^\prime \chi_0}{2(\chi_0)^2}\).
One can view these running coupling corrections as a contribution to an “effective” \( \chi \) which then respects the duality relation even at the running coupling level.

The good thing about these running coupling correction is that one can show that through their inclusion duality between the BFKL and GLAP equations holds to all perturbative orders. This means that the running-coupling BFKL equation, just like the GLAP equation, admits a factorized solution, whose scale dependence is determined by the kernel independent of the boundary condition. The bad thing is that the associate effective \( \chi \) is singular at the minimum of the fixed-coupling LO \( \chi \), as shown in Fig. 3. This singularity implies that the associate splitting function grows as a power of \( \xi \) in comparison to the leading-order one as \( x \to 0 \):

\[
\Delta P_s^{\xi}(\alpha_s, x)/P(\alpha_s, x) \sim (\beta_0 \alpha_s \xi)^k.
\]

These singularities must therefore be resummed to all orders if one wishes the \( x \to 0 \) limit to be stable.

The resummation can be accomplished in an asymptotic expansion, it leads to the nonsingular result of Fig. 3, and it has a further interesting consequence. Namely, if \( \chi \) has a minimum, duality implies that \( \gamma \) has a cut: e.g. if \( \chi \) is quadratic, then \( \gamma \) has a square-root cut. After running coupling resummation, however, the cut in \( \gamma \) is replaced by a simple pole (Fig. 3, right). Interestingly, the pole is located on the real axis to the left of the cut, as seen in Fig. 3 where a cut at \( N_0 \sim 0 \).

Because the location \( N_0 \) of the rightmost singularity in \( \gamma \) implies a small \( x \) behaviour of parton distributions \( \sim x^{-N_0} \) this means that running coupling corrections considerably soften the resummed small \( x \) behaviour, a result first obtained in Ref. 12.

4. Gluon exchange symmetry

The double-leading perturbative expansion of \( \chi(M) \) is stable for \( M \lesssim \frac{1}{2} \), but still unstable in the vicinity of \( M = 1 \), where it has alternating-sign poles. In particular,
it will lack a minimum at even perturbative orders. This is problematic for the running coupling resummation discussed in the previous section, which relies on the existence of a minimum. The instability at $M = 1$ can be understood and cured on the basis of the observation that in fact, due to the symmetry of the underlying Feynman diagrams upon interchange of incoming and outgoing gluons, the BFKL kernel is symmetric about $M = \frac{1}{2}$, i.e. $\chi(M) = \chi(1 - M)$. This symmetry is broken by the DIS choice of kinematical variables, which treats asymmetrically the initial and final parton virtualities $\mu^2$ and $Q^2$, but it can be restored by choosing e.g. $x = \frac{Q^2}{s}$, where $s$ is the center-of-mass energy of the partonic process. It is also broken by the asymmetric choice of scale in running of the coupling $\alpha_s(Q^2)$.

These symmetry breaking effects are computable and can be undone once the symmetry is restored, one can symmetrize the DL expansion of $\chi$, thereby obtaining a kernel which is perturbatively stable and free of poles at both $M = 0$ and $M = 1$. In fact, the momentum conservation constraints implies that symmetrized $\chi$ is an entire function of $M$, and has a minimum to all orders. The symmetrized DL expansion of $\chi$ which ensues is displayed in Fig. 4, where one sees that the powerful combination of duality and gluon symmetry leads to a stable expansion: the LO and NLO approximations are quite close. One can then revert to DIS variables (Fig. 4, right), thus obtaining the kernel which lead through duality and running coupling resummation to an anomalous dimension and splitting function. Interestingly, symmetrization implies a further softening of the kernel: the minimum of $\chi$, hence the branch cut of the corresponding $\gamma$, is moved (for $\alpha_s = 0.2$) from $N_0 \sim 0.5$ to $N_0 \sim 0.3$. Note that the NLO resummed behaviour is now harder than the LO one.

The combination of symmetrization and running-coupling duality can be exploited to obtain powerful analytic results. For instance, one can use the knowledge of $\gamma$ up to NNLO to determine all singular contributions to NNLO $\chi$, as shown in Fig. 6. This requires a treatment of running-coupling corrections up to NLO, and of various interference terms, which is feasible if running coupling duality equations are solved in an operator approach.

5. Results

Various approximations to the splitting function are displayed in Fig. 6, determined with $n_f = 0$ in order to avoid problems related to the diagonalization of the splitting function.
function matrix. The GLAP LO and NLO results are very close to each other and coincide at small $x$ (their small difference is proportional to $n_f$), but the NNLO GLAP result is seen to be unstable at small $x$ because of an unresummed $\frac{\alpha_3}{\pi} \ln \frac{1}{x}$ term (with negative coefficient). The simple (LO) fixed-coupling double leading result (‘DL fix’ in Fig. 6) displays a dramatic rise $\sim x^{-0.5}$ which is certainly not seen in the data. This is considerably softened by the running coupling resummation (‘DL run’ in Fig. 6). A yet softer behaviour is obtained if the running coupling resummation is applied to the symmetrized result (‘res LO’ and ‘res NLO’ in Fig. 6), which also leads to a stable perturbative expansion.

For comparison the NLO curve (i.e. including NLO resummation of $\xi$ and $t$) obtained by CCSS $^{18,29}$ is also shown. The CCSS approach starts from the BFKL equation, with the aim of determining the off-shell gluon density, not just the anomalous dimensions of parton distributions. This is then improved by enforcing symmetry, matching to GLAP (analogous to duality), and running coupling. The main advantages of this approach are its somewhat wider scope, and the fact that, being based on the BFKL equation, it allows for an exact treatment of the running coupling at small $x$ (unlike the ABF approach, where it is determined through an asymptotic expansion). The main disadvantages are that, due to the latter feature, anomalous dimensions can only be obtained through a numerical deconvolution procedure, and due to the former feature the perturbative expansion of the anomalous dimension is not naturally organized as a leading log series (unlike the DL approach), which implies for instance that the full NLO $n_f = 4$ result in this approach is not yet available.

The NLO ABF and CCSS results turn out to be quite close: their difference can be taken as an estimate of the intrinsic uncertainty in the resummation procedure.
These results are now essentially ready for a phenomenological implementation, though this will require the further development of suitable tools, such as matched resummed coefficient functions (for which all the required theoretical information is available), and interpolation of the resummed results for their efficient numerical implementation.

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