Generalized Yang-Mills actions from Dirac operator determinants

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Abstract

We consider the quantum effective action of Dirac fermions on four dimensional flat Euclidean space coupled to external vector- and axial Yang-Mills fields, i.e., the logarithm of the (regularized) determinant of a Dirac operator on flat $\mathbb{R}^4$ twisted by generalized Yang-Mills fields. According to physics folklore, the logarithmic divergent part of this effective action in the pure vector case is proportional to the Yang-Mills action. We present an explicit computation proving this fact, generalized to the chiral case. We use an efficient computation method for quantum effective actions which is based on calculation rules for pseudo-differential operators and which yields an expansion of the logarithm of Dirac operators in local and quasi-gauge invariant polynomials of decreasing scaling dimension.

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1 Introduction

Determinants of differential operators arise as (exponentials of) effective actions in quantum field theory. The precise definition and investigation of such objects is an interesting and challenging mathematical problem which has lead to an active and fruitful interplay between mathematics and physics.

In this paper we compute the logarithmic divergent part, $S_{\log}(A)$, of the logarithm of the regularized determinant for Dirac operators $D_A$ describing Dirac fermions coupled to a generalized Yang-Mills field $A$ on four dimensional spacetime. For simplicity we assume spacetime to be flat $\mathbb{R}^4$ with Euclidean signature and the natural spin structure. The Yang-Mills fields we consider contain, besides the vector part $V$, 


also a chiral (axial) part $C$ (for precise definitions see Eq. (7) ff. below); we write $A = (V, C)$. Our definition of $S_{\log}$ is motivated by physical considerations and will be explained further below. To indicate the mathematical significance of our calculation, we note that $S_{\log}(A)$ is (essentially) the noncommutative residue [Wo] of the logarithm of $D_A$ (see Eq. (3) for the precise statement). A main motivation for this work is to present a computation method for effective fermion actions which at the same time is mathematically rigorous, close to standard Feynman diagram computations in quantum field theory (see, e.g., [IZ, We]), and simple to use. We believe that this method is a useful alternative to other methods like the $\zeta$-function regularizations or the heat kernel expansions (see, e.g., [Q, BGV]). We therefore made some effort to present this method in a self-contained way, in the hope that this is useful also for readers who are mainly interested in learning how to compute effective actions.

We now discuss our computation method (parts of this method were used previously by us in [L, LM]). We regard the Dirac operator $D_A$ as a PSDO (pseudo-differential operator) on a Hilbert space of square-integrable functions on $\mathbb{R}^4$. Our starting point is the following definition for the regularized effective fermion action,

$$S_{\Lambda}(A) := \text{Tr}_A \left( \log \left( \frac{D_A + im}{\Lambda_0} \right) - \log \left( \frac{D_0 + im}{\Lambda_0} \right) \right)$$

where $m$ is a real parameter which has the physical interpretation of a fermion mass, and $\Lambda$ is a positive regularization parameter which we call $UV$ (ultra-violet) cutoff. The role of the non-zero and complex parameter $\Lambda_0$ is two-fold. Firstly, it makes the argument of the logarithm dimensionless, and secondly, setting $\Lambda_0 = |\Lambda_0|/(1 + i0^+)$ avoids possible ambiguities due to the branch cuts of the logarithm which otherwise can arise.\footnote{Of course, all results must be independent of $|\Lambda_0|$, and this is a useful check.} This definition above has three ingredients. Firstly, a definition of the log of an operator $a$ as an integral of the resolvent of $a$. Secondly, some basic facts about PSDO which imply a simple and powerful formula for the symbol of the resolvent of the Dirac operator $D_A$. And thirdly, a definition of a regularized Hilbert space trace $\text{Tr}_A$ (where removing the regularization corresponds to the limit $\Lambda \to \infty$). Combining these ingredients we obtain an expansion of $S_{\Lambda}(A)$ in local and quasi-gauge invariant polynomials of decreasing scaling dimension. We find

$$S_{\Lambda}(A) = \Lambda^2 S^{(2)}(A) + \log \left( \frac{\Lambda}{|m|} \right) S_{\log}(A) + S^{(0)}(A) + \mathcal{O}(\Lambda^{-1}),$$

and this provides our definition of $S_{\log}(A)$. Our results for $S_{\log}(A)$ and $S^{(2)}(A)$ will be presented in the next Section. We shall also demonstrate on our way that $S_{\log}(A)$
is proportional to the noncommutative residue $\mathcal{W}_0$ of the logarithm of the Dirac operator $D_A$,
\[ S_{\log}(A) = 4 \text{Res} \left( \log \left( \frac{D_A + im}{\Lambda_0} \right) - \log \left( \frac{D_0 + im}{\Lambda_0} \right) \right). \]

The logarithm of the regularized trace of the determinant of the Dirac operator can then be defined as
\[ S^{(0)}(A) = \text{TR} \left( \log \left( \frac{D_A + im}{\Lambda_0} \right) - \log \left( \frac{D_0 + im}{\Lambda_0} \right) \right) \tag{4} \]
where TR is the renormalized trace which we will define, and we will provide all mathematical tools necessary for computing $S^{(0)}(A)$ explicitly.

We note that our computation method is closely related to methods which have been used in the physics literature for a long time (see, e.g., [dW, IZ, We]). The regularization we use is simple and close to how regularizations are often done in Feynman diagram computations, i.e., by introducing a sharp UV cutoff (see Eq. (32)). We believe, however, that we can offer some improvements in detail which make computations easier, more transparent in structure, but nevertheless such that each step can be easily justified with mathematical rigor.

We now discuss some motivation for our computation from a quantum field theory point of view. As was known already to Schwinger for the Abelian case, the effective action of fermions coupled to a Yang-Mills field $A = V$ (i.e., $C = 0$) contains a logarithmic divergence, $\log(\Lambda/m) S_{\log}(A)$, and $S_{\log}(A)$ (for $C = 0$) is proportional to the usual Yang-Mills action
\[ S_{\text{YM}}(A) = \frac{1}{2g^2} \int_{\mathbb{R}^4} d^4x \text{tr} F_{\mu\nu} F^{\mu\nu} \]
(see, e.g., [IZ], Eq. (12.123) where $1/\epsilon$ corresponds to $\log(\Lambda/m)$). This is important since it implies that a change in the cutoff in the gauge theory, $\Lambda \to \Lambda'$, leads to a finite change of the effective fermion action which can be absorbed by changing the Yang-Mills coupled constant, $g^{-2} \to (g')^{-2} = g^{-2} + \text{const.} \log(\Lambda'/\Lambda)$. The logarithmic dependence of the Yang-Mills coupling constant on the UV cutoff is remarkable and distinguishes four spacetime dimensions from all others.

Our computation is closely related to more recent ideas which have lead to a deeper geometric understanding of the standard model of elementary particle physics (including Higgs sector). This approach is based on Connes’ NCG (noncommutative geometry; textbooks on this subject are, e.g., [C, GVF]). One important ingredient
of this approach is to define a generalized Dirac operator $D_A$, and this Dirac operator not only specifies the fermion part of the action of the model but also the Yang-Mills part $S_{YM}(A)$: there is a definition of $S_{YM}(A)$ in terms of $D_A$ (see [CC] and references therein). Our discussion above suggests a simple physical interpretation of this spectral action principle [CC]: the logarithmic divergence of the fermion effective action is potentially ‘dangerous’ since it can make the model ambiguous: there is no preferred choice for the cut-off, and changing it generates a term proportional to $S_{log}(A)$. However, the fact that $S_{log}(A)$ is proportional to the Yang-Mills action resolves this problem for the standard (purely vector) Yang-Mills theory on $\mathbb{R}^4$, as discussed above. It therefore is natural to require that the Yang-Mills action is proportional to the logarithmic divergent part of the fermion effective action in any gauge theory models. In particular this suggests the following definition of the generalized (vector and chiral) Yang-Mills action in terms of the generalized Dirac operators $D_A$,

$$S_{YM}(A) := \text{const.} \frac{1}{2g^2} S_{log}(A)$$

(for one fermion flavor $\text{const.} = 24\pi^2$). Eq. (3) shows that for flat Euclidean space $\mathbb{R}^4$, this definition is equivalent to the one given in [CC]. We conjecture that this is true for other four dimensional spin manifolds as well.

The plan of this paper is as follows. We summarize our notation and results in Section 2. Section 3 contains a summary of the mathematical prerequisites, i.e., the three ingredients of our method mentioned above. The computations of $S_{log}(A)$ is presented in Section 4 with some computation details deferred to Appendix B. We conclude with some remarks in Section 5. Appendix A contains some discussion on regularized traces and the noncommutative residue.

**Notation:** We write $\text{gl}_N$ for the complex $N \times N$ matrices and $\text{GL}_N$ for the invertible matrices in $\text{gl}_N$. We sometimes write $I_V$ or $I$ for the identity operator on a vector space $V$ but often abuse notation and do not distinguish between $cI$ and $c$ for complex numbers. For $V$, $W$ vector spaces and $a$ an operator on $V$, we often use the same symbols $a$ to also denote the corresponding operator $a \otimes I_W$ and $I_W \otimes a$ on $V \otimes W$ and $W \otimes V$, respectively. The real part of a complex number $c$ is denoted as $\Re c$.

## 2 Definitions and Results

For simplicity we assume spacetime $M^4 = \mathbb{R}^4$ with Euclidean signature (the extension of our calculation to other four–dimensional spin manifolds should be possible using symbol calculus of pseudo–differential operators [H]).
We consider the Hilbert
\[ \mathcal{H} = L^2(\mathbb{R}^4) \otimes \mathbb{C}^{4}_{\text{spin}} \otimes \mathbb{C}^{N}_{\text{color}} \] (6)
which has the physical interpretation as space of the 1–particle states of the fermions. We also introduce the space \( \mathcal{D} \) of functions in \( \mathcal{H} \) which are smooth (i.e., \( C^\infty \)) and \( L^1 \); \( \mathcal{D} \) is a convenient dense domain in \( \mathcal{H} \).

The Dirac operators of interest to us are of the form
\[ \hat{D}_A = \gamma^\nu \left(i\partial_\nu + V_\nu(x) + i\gamma_5 C_\nu(x)\right) \] (7)
where \( A = (V, C) \) (repeated indices \( \nu, \mu \ldots = 1, 2, 3, 4 \) are summed over; \( x = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \)), with \( \partial_\nu = \frac{\partial}{\partial x^\nu} \) and \( \gamma^\nu \) the Dirac spin matrices acting on \( \mathbb{C}^{4}_{\text{spin}} \) and obeying
\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^\mu\nu \] (8)
for \( \mu, \nu = 1, 2, 3, 4 \), where \( \eta^\mu\nu = \eta_{\mu\nu} = \text{diag}(1, 1, 1, 1) \) is the metric tensor, and
\[ \gamma_5 : = \gamma^1 \gamma^2 \gamma^3 \gamma^4 \] (9)
as usual (for the convenience of the reader, explicit formulas for these matrices are given in Appendix A.1).

For simplicity we assume that the functions \( V_\nu \) and \( C_\nu \mathbb{R}^4 \to \mathfrak{gl}_N \) are regular, i.e., they are \( C^\infty \) and vanish like \( \mathcal{O}(|x|^{-4-\varepsilon}) \), for some \( \varepsilon > 0 \), as \( |x| \to \infty \) (the latter condition is to ensure that integrals of regular functions over \( \mathbb{R}^4 \) absolutely converge).

In particular, the free Dirac operator is defined by the differential operator
\[ \hat{D}_0 = -i\gamma^\nu \partial_\nu . \] (10)

We define the gauge group \( \mathcal{G} \) as follows. Let \( \text{GL}_N \) be the group of all invertible matrices in \( \mathfrak{gl}_N \). Then \( \mathcal{G} \) is the group of all \( \text{GL}_N \)-valued functions \( U \) on \( \mathbb{R}^4 \) such that \( U(x) - 1 \) is a regular function. Note that one can write
\[ \hat{D}_A = \frac{1}{2}(1 - \gamma_5)\gamma^\nu \left(-i\partial_\nu + V_\nu + iC_\nu\right) + \frac{1}{2}(1 + \gamma_5)\gamma^\nu \left(-i\partial_\nu + V_\mu - iC_\mu\right) \]
where \( V_\mu \pm iC_\mu \) are the chiral components of the gauge field. This representation shows that it is natural to consider two kinds of gauge transformations,
\[ V_\mu \pm iC_\mu \to (U_\pm)^{-1}(V_\mu \pm iC_\mu)U_\pm - i(U_\pm)^{-1}\partial_\mu U_\pm, \quad U_\pm \in \mathcal{G} . \] (11)
For \( U_+ = U_− = U \) we denote these as vector gauge transformation, otherwise as chiral gauge transformation.
Note that \( \hat{D}_A \) in Eq. (7) is well-defined on the domain \( D \subset \mathcal{H} \), and we find it useful to distinguish this formally self-adjoint differential operator in notation from the corresponding self-adjoint extension on \( \mathcal{H} \) which we denote as \( D_A \), i.e., \( (D_A f)(x) = \hat{D}_A f(x) \) for all \( f(x) \in D \). We also write
\[
D_A = D_0 + A
\]
where \( D_0 \) is the free Dirac operator (i.e., self-adjoint extension of \( \hat{D}_0 \)) and \( A \) the operator defined by multiplication with the generalized Yang-Mills field
\[
\hat{A}(x) = \sum_{\nu=1}^{4} \gamma^\nu (V_\nu(x) + i\gamma_5 C_\nu(x)) .
\]

We will compute the fermion effective action \( S_{\Lambda}(A) \) defined in Eq. (1), and we will show that it can be expanded as in Eq. (2). As discussed, \( \text{Tr}_A \) is a Hilbert space trace with an ultraviolet (UV) cutoff \( \Lambda > 0 \), and \( \Lambda_0 \) is an arbitrary, in general complex, parameter makes the argument of the logarithm dimensionless. Moreover, the real (positive or negative) parameter \( m \) corresponds to a fermion mass and serves as an infrared (IR) regulator in our computation. Our main result is an explicit formula for \( S_{\log}(A) \).

**Proposition:** The logarithmic divergent piece \( S_{\log}(A) \) of the logarithm of the (regularized) determinant of the Dirac operator \( D_A \) equals
\[
S_{\log}(A) = \frac{1}{24\pi^2} \int_{M^4} d^4x \text{tr}_N \left( \frac{1}{2} F_{\mu\nu}^+(F^+)_{\mu\nu} + \frac{1}{2} F_{\mu\nu}^-(F^-)_{\mu\nu} - 6m^2 C^\nu C_\mu \right)
\]
where \( \text{tr}_N \) is the usual matrix trace in \( \mathfrak{gl}_N \) and
\[
F_{\mu\nu}^\pm := \partial_\mu A_\nu^\pm - \partial_\nu A_\mu^\pm + i[A_\mu^\pm, A_\nu^\pm], \quad A_\mu^\pm := V_\mu \pm C_\mu
\]
is the curvature associated with the chiral component \( A^\pm \) of the Yang-Mills field.

*(Proof in Section 4 with some details deferred to Appendix B.)*

For \( C = 0 \) (no chiral field) we obtain
\[
S_{\log}(A) = \frac{1}{24\pi^2} \int_{M^4} d^4x \text{tr}_N F_{\mu\nu} F^{\mu\nu}
\]
with

\[ \mathcal{F}_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + i[V_\mu; V_\nu], \]

which is the standard Yang-Mills action. Note that \( \mathcal{F}_{\mu\nu} = i[D_\mu, D_\nu] \) with

\[ D_\nu := -i\partial_\nu + V_\nu(x) \]

the covariant derivative, and similarly,

\[ \mathcal{F}_{\mu\nu}^\pm = i[D_\mu \pm iC_\mu, D_\nu \pm iC_\nu]. \]

It is important to note that for \( m = 0, S_{\log}(A) \) in Eq. (14) this is manifestly invariant under all gauge transformations Eq. (11). For \( m \neq 0 \), there is also a mass term \( \propto C_\mu C_\mu \) for the chiral gauge field which is only invariant under vector gauge transformations, i.e., only the transformations Eq. (11) with \( U_+ = U_- = U \). The parameter in front of this term is fixed by the fermion mass. There is no similar term for the vector gauge field (note that such a term would spoil vector gauge invariance).

It is interesting to note that the result of our computation in Section 4 suggests that for manifolds \( M^4 \) with boundary \( \partial M^4 \), \( S_{\log}(A) \) has an additional contribution

\[ \Delta S_{\log}(A) = \frac{1}{24\pi^2} \int_{M^4} d^4x \partial^\mu \text{tr}_N J_\mu, \]

with

\[ J_\mu := 2C_\mu [D_\nu, C^\nu] - 2C^\nu i[D_\nu, C_\mu] + 2i[D_\mu, C^\nu C_\nu]. \]

This is a boundary term (by Stokes’s theorem). Note that this term is also invariant under vector gauge transformations, and it vanishes if the axial Yang-Mills field \( C_\mu \) is zero.

It is also worth noting that, as a by-product, we also obtain the explicit expression for the quadratic divergent part of the effective action,

\[ S^{(2)}(A) = \frac{1}{16\pi^2} \int_{M^4} d^4x \text{tr}_N (-V^\mu V_\mu + C^\mu C_\mu). \]

In contrast to \( S_{\log}(A) \) this term is not gauge invariant (as already mentioned, the term \( \propto V^\mu V_\mu \) spoils vector gauge invariance)! This highlights the fact that the regularization procedure we use it not manifestly gauge invariant but only quasi-gauge invariant. It shows that the vector gauge invariance of our result for \( S_{\log} \) somewhat remarkable. It is also interesting to note that for \( V_\mu = \pm C_\mu, S^{(2)}(A) = 0. \)
3 Calculation tools

In this Section we collect the mathematical prerequisites for our computation. We will explain the three ingredients for our method: Firstly, a definition of the logarithm of operators $a$ in terms of an integral of the resolvent of $a$. Secondly, a few basic definitions for PSDO which imply a simple and elegant formula for the symbol of the resolvent of Dirac operators $D_A$. And finally, a definition of a regularized Hilbert space trace $\text{Tr}_A$ (corresponding to introducing an UV cutoff $\Lambda$). In the next Section we will put these ingredients together and obtain an expansion of the effective action as described in the Introduction.

1. The logarithm of operators. Let $a$ be a bounded operator on a Hilbert space $\mathcal{H}$ with norm less then one. Then ($I = I_{\mathcal{H}}$ is the identity operator)

$$\log(I + a) = \int_0^1 \frac{ds}{s} (I - (I + sa)^{-1}),$$

as can be seen by a Taylor expansion,

$$\log(I + a) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} a^n = -\sum_{n=1}^{\infty} \int_0^1 \frac{ds}{s} (-sa)^n,$$

interchanging summation and integration, and using the geometric series.

We take this as a motivation to define

$$\log \left( \frac{D_A + im}{\Lambda_0} \right) = \int_0^1 \frac{ds}{s} \left( I - \left( I + s \left[ \frac{D_A + im}{\Lambda_0} - I \right] \right)^{-1} \right)$$

where $\Lambda_0$ is a some complex number. This representation of the logarithm as integral of a resolvent will be convenient for us since there is a simple formula for the resolvent of (generalized) Dirac operators, as discussed below.

2.A. Pseudo–differential operators. Generalities. We summarize some basic facts about pseudo–differential operators (PSDO) on $\mathbb{R}^4$ (a discussion for general manifolds can be found, e.g., in [H]). We consider PSDO $a$ on $\mathcal{H}$ which can be represented by their symbol $\sigma[a](p, x)$, i.e., a $\mathfrak{gl}_4 \otimes \mathfrak{gl}_N$–valued functions on phase space $\mathbb{R}^4 \times \mathbb{R}^4$ defined such that [F]

$$(af)(x) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \int_{\mathbb{R}^4} d^4y \ e^{ip \cdot (x-y)} \sigma[a](p, x) f(y)$$

(24)
for all \( f(y) \in D \) (matrix multiplication is understood; \( p \cdot x = x^\nu p_\nu \)). In particular, \( D_0 \) and \( A \) are PSDO with symbols

\[
\sigma[D_0](p, x) = \hat{p} := \gamma^\nu p_\nu, \quad \sigma[A](p, x) = \hat{A}(x).
\] (25)

Note that Eq. (24) implies the following equation which encodes the product of operators in terms of their symbols,

\[
\sigma[ab](p, x) = \int \frac{d^4q}{(2\pi)^4} \int d^4y \ e^{i(x-y) \cdot (p-q)} \sigma[a](q, x) \sigma[b](p, y).
\] (26)

We will encounter PSDO \( a \) which allow an asymptotic expansion

\[
\sigma[a] \sim \sum_{j=0}^{\infty} \sigma_{K-j}[a]
\] (27)

where \( \sigma_{K-j}[a](p, x) \) is homogeneous of degree \( K-j \) in \( p \) and goes to zero like \( |p|^{K-j} \) for \( |p| \to \infty \) (\( |p| := \sqrt{p \cdot \bar{p}} \)). We write

\[
\sigma[a](p, x) = \sum_{j=0}^{n} \sigma_{K-j}[a](p, x) + \mathcal{O}(|p|^{K-j-1})
\] (28)

for all integers \( n \). Eq. (26) implies,

\[
\sigma[ab](p, x) \sim \sum_{n=0}^{\infty} (-i)^n \frac{\partial^n \sigma[a](p, x)}{n!} \frac{\partial^n \sigma[b](p, x)}{\partial p_{i_1} \cdots \partial p_{i_n} \partial x_{i_1} \cdots \partial x_{i_n}}.
\] (29)

This equation allows to determine the asymptotic expansions of \( \sigma[ab] \) and \( \sigma[a^{-1}] \) from the ones of \( \sigma[a] \) and \( \sigma[b] \).

2.B. The symbol of the resolvent. Eq. (23) expresses \( \log(D_A+im) \) as an integral of resolvents of the Dirac operator \( D_A \), i.e., of operators \( (c_1 I + c_2 D_A)^{-1} \) with \( c_{1,2} \) complex numbers. We will therefore need the symbol of such a resolvent. To determine this we note that

\[
\sigma[c_1 I + c_2 D_A](p, x) = c_1 + c_2 [\hat{p} + \hat{A}(x)].
\] (30)

We then could use Eq. (29) to find the expansion for \( \sigma[(c_1 I + c_2 D_A)^{-1}](p, x) \). We now present a useful result summarizing this expansion in a simple formula.

\[\text{i.e., } \sigma_{K-j}[a](sp, x) = s^{K-j} \sigma_{K-j}[a](p, x) \text{ for all } s > 0 \text{ and } |p| > 0\]
Lemma: The following holds for all $c_1, c_2 \in \mathbb{C}$,
\[
\sigma[(c_1 I + c_2 D_A)^{-1} a](p, x) = \left( c_1 + c_2 [\hat{\mathbf{p}} + \hat{D}_A] \right)^{-1} \sigma[a](x, p).
\]  
(31)

Remark: The proper interpretation of this equation is as follows,
\[
\sigma[(c_1 I + c_2 D_A)^{-1} a](p, x) \sim \sum_{n=0}^{\infty} (-1)^n (c_1 + c_2 \hat{\mathbf{p}})^{-1} \left[ \hat{D}_A (c_1 + c_2 \hat{\mathbf{p}})^{-1} \right]^n \sigma[a](x, p)
\]
where the differential operators $\partial_\nu$ in $\hat{D}_A = -i\gamma^\mu \partial_\nu + \hat{A}(x)$ act to the right on the functions $\hat{A}(x)$ according to the Leibniz rule. We note that we will need this equation only for $a = I$.

Proof of the Lemma: One can check Eq. (31) by using Eqs. (27) and (29), taking $c_1 I + c_2 D_A$ for $a$ and $[c_1 I + c_2 D_A]^{-1} a$ for $b$, and inserting Eq. (30). A simpler argument avoiding tedious expansions is as follows: Note that by definition, $(D_A f)(x) = [\hat{D}_0 + \hat{A}(x)] f(x)$ for all $f \in \mathcal{D}$, thus
\[
((c_1 I + c_2 D_A) a f)(x) = \left( c_1 + c_2 [\hat{\mathbf{p}} + \hat{A}(x)] \right) (a f)(x) = \int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi)^4} \int_{\mathbb{R}^4} d^4 y \ e^{i p \cdot (x-y)} \left( c_1 + c_2 [\hat{\mathbf{p}} + \hat{D}_0 + \hat{A}(x)] \right) \sigma[a](p, x) f(y)
\]
where we used Eq. (24) and the Leibniz rule. Replacing $a$ in this equation by $(c_1 I + c_2 D_A)^{-1} a$, we see that this is equivalent to Eq. (31). (Note that this argument implies the interpretation of Eq. (31) as given above!)

Remark: We believe that our expansion in powers of the differential operator $\hat{D}_A$ is very natural for at least two reasons. Firstly, since under a vector gauge transformation, $\hat{D}_A \rightarrow U^{-1} \hat{D}_A U$, such an expansion is close to being manifestly gauge invariant (we will discuss this point in more detail below). Secondly, it is natural from the point of view of power counting: in contrast to an expansion in $\hat{A}(x)$, the $n$-th order term in our expansion includes precisely those local polynomials $P_n$ in $V_\mu$ and $C_\nu$ (and derivatives thereof) which all have the same scaling behavior $P_n \rightarrow \lambda^{4-n} P_n$ under $x \rightarrow \lambda x$.

Remark: Loosely speaking, PSDO are useful since they allow to interpolate between Fourier- and position space: generically in quantum theory one deals with operators
\[ (H_0 f)(p) = E_0(p) \hat{f}(p), \]
and a potential term \( V \) diagonal in position space, \( (V f)(x) = V(x)f(x) \). The symbol of \( \sigma[H](p, x) \) is then simply the sum of \( E_0(p) \) and \( V(x) \), which is an attractive feature. The price one has to pay is that the symbol of ('nice') functions \( F \) of \( H \) are somewhat complicated: in a first approximation, \( \sigma[F(H)](p, x) \sim F(E_0(p) + V(x)) \) . . . , but there are correction terms . . . depending on derivatives. The Lemma above is a special case of the following formula,
\[ \sigma[F(H)](p, x) \sim F(E_0(p - i\partial) + V(x))1 \]
nicely summarizing the systematic derivative expansion of functions of \( H \).

3. Regularized traces and the noncommutative residue. We now define the regularized trace which we will use. We first note that due to our technical assumptions on the gauge fields all operators \( a \) which we will encounter are PSDO which have symbols \( \sigma[a](p, x) \) which go at least like \( O(|x|^{-4-\varepsilon}) \), some \( \varepsilon > 0 \), for fixed \( p \) and \( |x| \to \infty \), and are finite for finite \( p \). Thus
\[
\Tr_{\Lambda}(a) := \int_{|p| \leq \Lambda} \frac{d^4p}{(2\pi)^4} \int_{\mathbb{R}^4} d^4x \ tr[\sigma[a](x, p)]
\]
where \( tr \) is the full matrix trace is well-defined for \( \Lambda < \infty \), and this defines a regularized Hilbert space trace: If \( a \) is a trace–class operator then \( \Tr_{\Lambda}(a) \) has a well–defined limit \( \Lambda \to \infty \) which is equal to the Hilbert space trace of \( a [H] \). More generally one can consider PSDO \( a \) for which \( \Tr_{\Lambda}(a) \) can be expanded as
\[
\Tr_{\Lambda}(a) = c^{(K)}(a)\Lambda^K + c^{(K-1)}(a)\Lambda^{K-1} + \\
+ \ldots + c^{(1)}(a)\Lambda + c_{\log}(a) \log \left( \frac{\Lambda}{|m|} \right) + c^{(0)}(a) + O(\Lambda^{-1})
\]
with \( K \) some non–negative integer.

We recall that the noncommutative residue \([Wo]\) of a PSDO \( a \) with an asymptotic expansion as in Eq. (27) can be defined as (see, e.g., Eq. (2.7) in Ref. [VG])
\[
\Res(a) := \frac{1}{4} \int_{R^4} \frac{d^4p}{(2\pi)^4} \delta(|p| - 1) \int_{R^4} d^4x \ tr[\sigma_{-4}[a](x, p)],
\]
\[ \hat{f}(p) = \int_{\mathbb{R}^n} d^n x e^{ip \cdot x} f(x) \] denotes the Fourier transform.
\[ ^5 \text{including the trace } tr_{\nu} \text{ in } gl_4 \text{ and the trace } tr_{N} \text{ in } gl_N. \]
and for PSDO $a$ as above,

$$\text{Res}(a) = \frac{1}{4} c_{\log}(a),$$  \hspace{1cm} (35)

i.e., the residue is equal, up to a constant, to the logarithmic divergent part of the regularized trace of $a$. (An elementary proof of this latter fact is outlined in Appendix A.)

**Remark:** In our definition Eq. (35) of $\text{Tr}_\Lambda$ we use a sharp cutoff, i.e.,

$$\text{Tr}_\Lambda(a) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} f\left(\frac{|p|}{\Lambda}\right) \int_{|p| \leq \Lambda} d^4x \, \text{tr}\sigma[a](x,p)$$  \hspace{1cm} (36)

where $f(t)$ equals the Heaviside step function $\theta(1 - t)$. In principle one could define a regularized trace using Eq. (36) and choosing any non-negative, piece-wise smooth, function $f(t)$ which vanishes exponentially fast for $|t| \to \infty$ and is such that $f(0) = 1$. For example, the choice $f(t) = \exp(-t^2)$ would correspond to the standard heat kernel regularization.

We will show in Appendix A that $c_{\log}(a)$ is in fact independent of $f$.

Using any such regularization one can define the *renormalized trace* as the finite part of the regularized trace,

$$\text{TR}(a) := c_0(x),$$  \hspace{1cm} (37)

but this is not quite independent of the regularization: as also discussed in Appendix A, changing the regularization function $f \to \tilde{f}$ amounts to changing

$$\text{TR}(a) \to \text{TR}(a) + \log(s) \, c_{\log}(a)$$  \hspace{1cm} (38)

with some constant $s > 0$ depending on $f$ and $\tilde{f}$: the logarithmic divergent piece accounts for the regularization dependence of the renormalized trace, and this is the reason for our interest in it, as discussed in the Introduction.

**Remark:** We note Eq. (35) is equivalent to

$$\text{Tr}_\Lambda(a) = \text{Tr}(P_\Lambda a), \quad P_\Lambda := f(|D_0|/\Lambda)$$  \hspace{1cm} (39)

(using the spectral theorem for self-adjoint operators). This naturally extends the definition of $\text{Tr}_\Lambda$ from PSDO to a large class of operators on $\mathcal{H}$. More generally, one could change the regularization by changing $D_0 \to D_B$ in the definition of $P_\Lambda$, for some fixed Yang-Mills field $B$. One can show that this would change $\text{TR}(a)$ by a term proportional to $\text{Res}(\log(D_B) - \log(D_0))a$ (see, e.g., Eq. (1.6) in \cite{CDMP}). It would be interesting to explore this possibility in more detail.
4 Computation of effective fermion action

In this Section we present the explicit computation of the effective fermion action and thus prove the proposition in Section 2. Our computation amounts to a quasi-gauge invariant gradient expansion, which is essentially an expansion in powers of the UV cutoff \( \Lambda \). This allows us to extract, in a simple manner, the quadratic and logarithmic divergent pieces which is what we are interested in.

1. Quasi-gauge covariant expansion. We write

\[
S_\Lambda(A) = \int_{|p| \leq \Lambda} \frac{d^4 p}{(2\pi)^4} \int_{\mathbb{R}^4} d^4x \text{tr} S(x, p) 1
\]

where \( S(x, p) \) is obtained by computing the symbol of the operator \( \log(D_A + im)/\Lambda_0 - \log(D_0 + im)/\Lambda_0 \) as explained in the last Section, i.e.,

\[
S(x, p) = \int_0^1 \frac{ds}{s} \left( [1 - s + s\hat{p} + s\phi]^{-1} - [1 - s + s\hat{p}]^{-1} \right) = \int_0^\infty \frac{du}{u} \left( [1 + u\hat{p} + u\phi]^{-1} - [1 + u\hat{p}]^{-1} \right) ;
\]

we used Eqs. (22) and (31), introduced the convenient short-hand notion,

\[
\hat{p} : = \frac{\hat{p} + im}{\Lambda_0}, \quad \phi : = \frac{-i\phi + \hat{A}}{\Lambda_0},
\]

and changed integration variables, \( s = u/(1 + u) \). The 1 on the r.h.s. of Eq. (40) is the symbol of the identity operator. As explained in more detail below, \( S \) here is to be regarded as a differential operators acting on 1. It is straightforward to expand the integrand in this equation in powers of \( \phi \),

\[
S = \sum_{n=1}^L (-1)^{n-1} S_n + \mathcal{R}_{L+1}
\]

where

\[
S_n = (-1)^{n-1} \int_0^\infty \frac{du}{u} \frac{1}{1 + u\hat{p}} \left( u\phi - \frac{1}{1 + u\hat{p}} \right)^n
\]

and

\[
\mathcal{R}_{L+1} = \int_0^\infty \frac{du}{u} \frac{1}{1 + u\hat{p}} \left( u\phi - \frac{1}{1 + u\hat{p}} \right)^L u\phi[1 + u(\hat{p} + \phi)]^{-1}
\]
is a remainder term.

In the following we find it convenient to use the short-hand notation and write

\[ \hat{D}_A = \sum_{s=0,5} \gamma_s^\nu D^s_\nu \]  

(46)

where

\[ D^0_\nu := D_\nu \quad D^5_\nu := C_\nu \]  

(47)

and

\[ \gamma_0^\nu := \gamma^\nu, \quad \gamma^5 := i\gamma^\nu \gamma_5. \]  

(48)

We then define

\[ M^{\nu_1,...,\nu_n}_{s_1,...,s_n} := \int_{|p| \leq \Lambda} (2\pi)^4 \text{tr}_D \int_0^\infty du \int_0^\infty \frac{d^4u}{(2\pi)^4} \int d^4x \text{tr} S_n(x, p) = \sum \mathcal{M}_{n \leq 2}^{\nu_1,...,\nu_n} \int d^4x \text{tr} N D^{s_1}_{\nu_1} \cdots D^{s_n}_{\nu_n}. \]  

(49)

(50)

here and in the following, \( s \) is short for \((s_1, \ldots, s_n)\).

The following Lemma simplifies the computation significantly: it implies that the \( S_n \) for odd integers \( n \) all vanish, and that an series expansion in the mass \( m \) only has non-zero even powers.

**Lemma:** The coefficients \( \mathcal{M}_{n \leq 2}^{\nu_1,...,\nu_n} \) in Eq. (49) are non-zero only for even integers \( n \), and they are invariant under \( m \to -m \), i.e., they are independent of the sign of the mass.

(Proof in Appendix B.)

**Remark:** We now can explain why we denote our expansion quasi-gauge invariant. This is because the operators \( D^s_\nu \) transform gauge covariantly under a vector gauge transformation \( U, D^s_\nu \to U^{-1} D^s_\nu U \). This implies that the differential operators defined in Eq. (50) all are gauge invariant. However, the action is a polynomial which is obtained by applying these differentiation operators to 1 (cf. Eqs. (40)) using Leibniz rule and \( \partial_\nu 1 = 0 \), e.g.,

\[ D_\nu 1 = V_\nu(x) \]

\[ D_{\nu_1} C_{\nu_2} = -i[\partial_{\nu_1} C_{\nu_2}(x)] + V_{\nu_1}(x) C_{\nu_2}(x) \]

\[ D_{\nu_1} D_{\nu_2} = -i[\partial_{\nu_1} V_{\nu_2}(x)] + V_{\nu_1}(x) V_{\nu_2}(x) \]  

(51)
etc. The result will be gauge invariant only if the differential operator in Eq. (50) is already a polynomial. This happens, e.g., if the differential operators $D_\nu$ only appear in combinations $[D_\nu, D_\mu]$ and $[D_\nu, C_\mu]$. This is not obvious. However, we will see below that this happens for the terms leading to logarithmic divergent part.

2. Expansion in powers of the UV cutoff. We now show that our expansion above is essentially an expansion in powers of the UV cutoff $\Lambda$. Our computation can be simplified by the following argument (this argument is refined and justified in detail in Appendix B). As mentioned, $m$ serves as a particular IR cutoff for momentum integrals. We expect that our result is independent of the precise form the IR regularization. Thus we use instead the following, simpler one: we set $m = 0$ in $\tilde{p}$ but restrict integrations over $p$ to $m \leq |p| \leq \Lambda$. We stress that we use this simplification in the main text only to ease our presentation, and that it is appropriate only for computing the diverging contributions to the regularized determinant: the computation of the finite part should be done with the method explained in Appendix B. Below we shall see that this simplified procedure gives a IR regularization provided we also set $\Lambda_0 = |\Lambda_0|/(1 + i0^+)$. (a justification of this can be also found in the Appendix B).

Using then

$$\frac{1}{1 + u\tilde{p}} = \frac{1}{1 - u^2|p|^2(\Lambda_0)^{-2}[1 - u\tilde{p}(\Lambda_0)^{-1}]}$$

and rescaling $u|p|(\Lambda_0)^{-1} \rightarrow u(1 + i0^+)$ we see that $\hat{\mathcal{M}}_n$ in Eq. (51) becomes

$$\hat{\mathcal{M}}_{n/2}^\nu = \frac{1}{8\pi^2} \int_{|m|}^{\Lambda} d|p||p|^{3-n} \mathcal{J}_{n/2}^\nu(p)$$  \hfill (52)

where

$$\mathcal{J}_{n/2}^\nu := \int_0^\infty du u^{n-1} \left( \frac{1}{1 - [u(1 + i0^+)]^2} \right)^{n+1} \times \langle \text{tr}_\nu (1 - u\xi)\gamma_{a_1}^{\nu_1}(1 - u\xi)\cdots\gamma_{a_n}^{\nu_n}(1 - u\xi) \rangle;$$  \hfill (53)

we used $(2\pi)^{-4} \int_{m \leq |p| \leq \Lambda} d^4p g(p) = (8\pi^2)^{-1} \int_{|m|}^{\Lambda} d|p||p|^3 \langle g(|p|\xi) \rangle$ with

$$\langle g(\xi) \rangle := \frac{1}{2\pi^2} \int_{\mathbb{R}^4} \frac{d^4\xi}{(2\pi)^4} \delta(|\xi| - 1) g(\xi)$$  \hfill (54)

\footnote{We use the different symbol $\hat{\mathcal{M}}_n$ to indicate that these numbers are obtained with a simplified IR regularization.}
the angular average (i.e., integration over the unit sphere in \( \mathbb{R}^4 \)). We now see that \( \Lambda_0 = |\Lambda_0|/(1 + i0^+) \) is needed to specify how to treat the singularity in the \( u \)-integral. These \( u \)-integrals are then finite (see Eqs. (53) and (57) below). The result we get is independent of \( |\Lambda_0| \), as expected. It shows explicitly that our expansion leads to an expansion of the action in powers of \( \Lambda \). We are interested in \( \Lambda \to \infty \). In this limit, \( \mathcal{M}_n \propto |\Lambda|^{n-4} \) for \( n < 4 \) and \( \propto |m|^{4-n} \) for \( n > 4 \): the former terms are divergent in the UV (i.e., for \( \Lambda \to \infty \)), the latter in the IR (i.e., for \( m \to 0 \)). It is precisely the ‘boundary case’ \( n = 4 \) which gives rise to the logarithmic divergence.

This result is obtained with the simplified IR treatment is correct only in leading order in \( \Lambda \). In Appendix B we show how to do the computation without this simplification, and that

\[
\mathcal{M}_n = \tilde{\mathcal{M}}_n + \mathcal{O} \left( m^2 \Lambda^{2-n} \right), \quad n > 2
\]

showing that the simplified IR treatment gives the correct result for the diverging terms for all \( n \) but \( n = 2 \). For \( n = 2 \) there are corrections \( \propto m^2 \log(\Lambda/m) \) which contribute to \( S_{\log} \) and which we therefore have to compute exactly.

3. Computation of diverging parts of the effective action. We now proceed to compute the coefficients \( J_n \), Eq. (53) for those terms we are interested in, i.e., for \( n = 1, 2, 3, 4 \). Using Eq. (52) this is straightforward: one only needs to evaluate the integrals

\[
\mathcal{N}_{n,k} = \int_0^\infty duu^{n+k-1} (1 - [u(1 + i0^+)]^2)^{-n-1}, \quad (56)
\]

the angular averages \( \langle \xi_{\nu_1} \cdots \xi_{\nu_k} \rangle \), and traces of products of Dirac matrices. The integrals in Eq. (56) are (cf., e.g., Eq. 3.251(11.) in [GR])

\[
\mathcal{N}_{n,k} = -(-1)^{(n-k)/2}B \left( \frac{n+k}{2}, \frac{n-k}{2} + 1 \right), \quad (57)
\]

where \( B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y) \). We only need

\[
\mathcal{N}_{2,0} = \frac{1}{4}, \quad \mathcal{N}_{2,2} = -\frac{1}{4},
\]

\[
\mathcal{N}_{4,0} = \frac{1}{24}, \quad \mathcal{N}_{4,2} = -\frac{1}{24}, \quad \mathcal{N}_{4,4} = \frac{1}{8}.
\]

The computation of the traces of Dirac matrices is simplified using the following relations

\[
(1 - u\xi)\gamma^\nu = \gamma^\nu(1 + u\xi) - 2u\xi^\nu, \quad \nu = 1, 2, 3, 4
\]

\[
\gamma_5\xi = -\xi, \quad \xi^2 = |\xi|^2
\]

(58)
which follow from Eq. (8). We also need
\[ \langle 1 \rangle = 1, \quad \langle \xi_{\mu_1} \xi_{\mu_2} \rangle = \frac{1}{4} \eta_{\mu_1 \mu_2} \]
\[ \langle \xi_{\mu_1} \xi_{\mu_2} \xi_{\mu_3} \xi_{\mu_4} \rangle = \frac{1}{24} (\eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} + \eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4} + \eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3}) \] (59)
and that the angular average for a product of an odd number of components \( \xi_{\mu_j} \) is zero. Moreover,
\[ \text{tr}_N(\gamma^{\mu_1} \gamma^{\mu_2}) = 4 \eta^{\mu_1 \mu_2} \]
\[ \text{tr}_N(\gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4}) = 4 (\eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4} - \eta^{\mu_1 \mu_3} \eta^{\mu_2 \mu_4} + \eta^{\mu_1 \mu_4} \eta^{\mu_2 \mu_3}) \]
\[ \text{tr}_N(\gamma_5 \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4}) = 4 \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \] (60)
where \( \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \) is the completely antisymmetric symbol with \( \epsilon^{1234} = 1 \). Note that
\[ \text{tr}_N(\gamma_{\mu}) = \text{tr}_N(\gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4}) = 0 \] always.
It is now easy to see that \( J^{\mu}_{1,s} = J^{\mu_1 \mu_2 \mu_3}_{3; s_1 s_2 s_3} = 0 \), thus
\[ S_1 = S_3 = 0. \] (61)
The simplest non-zero terms are for \( n = 2 \). Combining the formulas given above it is easy to see that
\[ J^{\mu_1 \mu_2}_{2; s_1 s_2} = \frac{1}{16 \pi^2} A_{s_1 s_2} \eta^{\mu_1 \mu_2} \]
\[ A_{55} = -A_{00} = 1, \quad A_{50} = A_{05} = 0. \] (62)
Thus
\[ \tilde{S}_2 = \Lambda^2 \frac{1}{16 \pi^2} \int d^4 x \text{tr}_N (-D^\mu D_\mu + C^\mu C_\mu). \] (63)
This is a gauge invariant differential operator. When acting on 1 (cf. Eq. (51)) we obtain the quadratic divergent part of the effective action Eq. (21) which is not gauge invariant.
As mentioned, \( S^{(2)} \) is only the leading order contribution to \( S^{(2)} \). A more careful computation without the simplified IR regularization gives (see Appendix B),
\[ S_2 = \tilde{S}_2 - m^2 \log \left( \frac{\Lambda}{|m|} \right) \frac{1}{8 \pi^2} \int d^4 x \text{tr}_N (C^\mu C_\mu) + \ldots \] (64)
where ‘\ldots’ are terms which remain finite for \( \Lambda \to \infty \). We see that the subleading term which was missed by the naive IR regularization contributes to \( S_{\text{log}} \). As discussed in Section 2, this term is gauge invariant.
Table 1: Parameters in Eq. (65) where $s = (s_1, s_2, s_3, s_4)$.

| $s_1$ | 1 | 1 | 1 | 1 | 5 | 5 | 5 | 5 | 1 | 1 | 1 | 1 | 5 | 5 | 5 | 5 |
|-------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $s_2$ | 1 | 1 | 5 | 5 | 1 | 1 | 5 | 5 | 1 | 1 | 5 | 5 | 1 | 1 | 5 | 5 |
| $s_3$ | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 | 1 | 5 |
| $s_4$ | 1 | 5 | 5 | 1 | 5 | 1 | 1 | 5 | 5 | 1 | 1 | 5 | 1 | 5 | 1 | 5 |

$A_s = 0 -2 -2 2 -2 -2 -2 0 0 0 0 0 0 0 0 0 0$

$B_s = -2 2 2 0 4 2 2 -2 0 0 0 0 0 0 0 0 0$

$C_s = 2 -2 0 2 -2 0 -2 2 0 0 0 0 0 0 0 0 0$

$D_s = 0 0 0 0 0 0 0 0 -i -i i i i i -i -i$

We now turn to the case $n = 4$ which leads to the logarithmic divergence. All relations needed to compute the $J_{2}^{\nu_1 \nu_2 \nu_3 \nu_4}$ were listed above. The result can be written as follows

$$J_{2}^{\nu_1 \nu_2 \nu_3 \nu_4} = \frac{1}{3} (A_s \eta^{\nu_1 \nu_2} \eta^{\nu_3 \nu_4} + B_s \eta^{\nu_1 \nu_3} \eta^{\nu_2 \nu_4} + C_s \eta^{\nu_1 \nu_4} \eta^{\nu_2 \nu_3} + D_s \epsilon^{\nu_1 \nu_2 \nu_3 \nu_4})$$

(65)

where $s = (s_1, s_2, s_3, s_4)$. The numbers $A_s, B_s, C_s, D_s$ are all given in Table 1. (We have checked this result extensively using the symbolic programming language MAPLE.) We note that the numbers $A_s, B_s, C_s (D_s)$ all are real (purely imaginary) and non-zero only if an even (odd) number of the $s_j$ equal 5.

Combining these results we find

$$\tilde{S}_4 = \log \left( \frac{\Lambda}{m_l} \right) \frac{1}{24\pi^2} \int_{\mathbb{R}^4} d^4x \tr N [\mathcal{P}_R + \mathcal{P}_I],$$

(66)

where

$$\mathcal{P}_R = \sum_{\pm} (A_s \eta^{\nu_1 \nu_2} \eta^{\nu_3 \nu_4} + B_s \eta^{\nu_1 \nu_3} \eta^{\nu_2 \nu_4} + C_s \eta^{\nu_1 \nu_4} \eta^{\nu_2 \nu_3} + D_s \epsilon^{\nu_1 \nu_2 \nu_3 \nu_4}) D_{\nu_1}^{s_1} D_{\nu_2}^{s_2} D_{\nu_3}^{s_3} D_{\nu_4}^{s_4}$$

$$\mathcal{P}_I = \sum_{\pm} D_s \epsilon^{\nu_1 \nu_2 \nu_3 \nu_4} D_{\nu_1}^{s_1} D_{\nu_2}^{s_2} D_{\nu_3}^{s_3} D_{\nu_4}^{s_4}$$

(67)

with the coefficients given in Table 1. $\mathcal{P}_R$ is a sum of 19 non-zero terms. We now claim that it is possible to write $\mathcal{P}_R = \mathcal{P}_{R,1} + \mathcal{P}_{R,2}$ where

$$\mathcal{P}_{R,1} = -[D^\mu, D^\nu][D_\mu, D_\nu] - [C^\mu, C^\nu][C_\mu, C_\nu] + [D^\mu, D^\nu][C_\mu, C_\nu] + [D^\mu, C^\nu][D_\mu, C_\nu] + 2[D^\mu, C^\nu][D_\mu, C_\nu]$$

$$+ [C^\mu, C^\nu][D_\mu, D_\nu] + 2[D^\mu, C^\nu][D_\mu, C_\nu] + 2[D^\mu, C^\nu][C_\mu, D_\nu]$$

(68)
and
\[ P_{R,2} = i[D^\mu, J_\mu] + [[D^\mu, D^\nu], [C_\mu, C_\nu]] - 2[C_\mu, [D_\mu, D_\nu]C_\nu] \] (69)
with \( J_\mu \) given in Eq. (20). Similarly,
\[ P_I = \frac{i}{2} \epsilon_{\nu_1\nu_2\nu_3\nu_4} [[D_{\nu_1}, D_{\nu_2}] + [C_{\nu_1}, C_{\nu_2}], [D_{\nu_3}, C_{\nu_4}]]. \] (70)
(The proof of Eqs. (68)–(70) are straightforward calculation which we skip.)

We see that, \( P_{R,1} \) equals \( \frac{1}{2}(F^+(F^+)\mu\nu + F^- (F^-)\mu\nu) \) with \( F^\pm \), defined in Eq. (15).

The remaining terms are linear combinations of commutators! Using the cyclicity of
the matrix trace we thus obtain
\[ \text{tr}_N P_{R,2} = \partial^\mu \text{tr}_N J_\mu, \quad \text{tr}_N P_I = 0. \] (71)
This implies Eqs. (14)–(20) and completes our computation.

Remark: Note that \( P_R \) and \( P_I \) are not differential operators but polynomials (i.e.,
there are no terms \((\cdots)D_\mu\)). This implies that both these terms are gauge covariant
which, as we believe, is remarkable.

5 Conclusions

The regularization which we used was simple but not manifestly gauge invariant.
For the result computed in this paper the latter property is irrelevant: since the
logarithmic divergence is regularization dependent one can compute it using any
regularization. However, we believe that our method is useful even for computing
the finite part of the effective action, i.e., \( S^{(0)}(A) \) in Eq. (4). We stress again that
the simplified IR regularization used in the main text is not appropriate in this
computation but the formulas given in Appendix B should be used. We conjecture
that \( S^{(0)}(A) \) computed in this way is gauge invariant.

As mentioned in the Remark at the end of Section 3, we defined a renormalized
trace \( \text{TR}_{[D_0]} \) using the free Dirac operator \( D_0 \). More general we could use the Dirac
operator \( D_B \) with some fixed non-trivial Yang-Mills field \( B \). In particular, we expect
that the standard \( \zeta \)-function regularization of the logarithm of the determinant of
\( D_A \) should be identical with
\[ \text{TR}_{[D_A]} \log \left( \frac{D_A + im}{\Lambda_0} \right) \]
were the regularization function is \( f(t) = \exp(-t^2) \). The latter definition has the advantage that it is manifestly gauge invariant, but it seems less easy to use for explicit computations as ours. It is natural to expect that the difference between the latter definition and \( S(0)(A) \) in Eq. (34) is also proportional to \( S_{\log}(A) \).

Effective action computations are used in many applications of quantum field theory. We believe that the methods which we presented should be useful in other such contexts as well.

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Appendix A: More on regularized traces

In this Appendix we outline elementary proofs of some facts about regularized traces stated in the main text.

The logarithmic divergence. We compute the regularized trace in Eq. (36) for an operator \( a \) with a symbol allowing for an asymptotic expansion as in Eq. (27). It is easy to see that the contribution of \( \sigma_k[a](p, x) \) to \( \text{Tr}_A(a) \) is

\[
\int_0^\infty d|p| |p|^{k+3} f(\frac{|p|}{\Lambda}) \int_{\mathbb{R}^4} \frac{d^4\xi}{(2\pi)^4} \delta(|\xi| - 1) \int_{\mathbb{R}^4} d^4x \text{tr} \sigma_k[a](\xi, x)
\]

where we used the homogeneity of \( \sigma_k[a] \). Changing variables, \( |p| \to u = |p|/L \), and comparing with Eq. (33) we see that for all \( k \geq -3 \),

\[
c_k(a) = N_k \int_{\mathbb{R}^4} \frac{d^4\xi}{(2\pi)^4} \delta(|\xi| - 1) \text{tr} \sigma_k[a](\xi, x)
\]

with \( N_k = \int_0^\infty du u^{k+3} f(u) \) constants depending on \( f \). For \( k = -4 \) the computation above does not make sense (the constant \( N_{-4} \) diverges), but we can compute \( c_{\log}(a) \) as follows. We first subtract from the symbol of \( a \) the diverging part which we already accounted for and define,

\[
\sigma_{-3}^{-}[a](p, x) := \sigma[a](p, x) - \sum_{j=0}^{K+3} \sigma_{-j}^{-}[a](p, x) = \sigma_{-4}[a](p, x) + O(|p|^{-5}) \quad (A2)
\]
Eq. (33) then suggests that

\[
c_{\log}(a) = \lim_{\Lambda \to \infty} \frac{1}{\log(\Lambda)} \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} f\left(\frac{|p|}{\Lambda}\right) \int_{\mathbb{R}^4} d^4x \, \text{tr} \sigma[a]_{-3}^+(p, x) .
\]

Computing this using L’Hospital’s rule we obtain

\[
c_{\log}(a) = \lim_{\Lambda \to \infty} \Lambda \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} f'(\frac{|p|}{\Lambda})(-\frac{|p|}{\Lambda^2}) \int_{\mathbb{R}^4} d^4x \, \text{tr} \sigma[a]_{-3}^+(p, x)
\]

\[
= \lim_{\Lambda \to \infty} \left( \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} f'(\frac{|p|}{\Lambda}) \frac{|p|}{\Lambda} \int_{\mathbb{R}^4} d^4x \, \text{tr} \sigma[a]_{-4}(p, x) + O(\Lambda^{-1}) \right) .
\]

Changing variables etc. as above and using \(\int_0^\infty du(-f'(u)) = f(0) = 1\) (independent of \(f\!\!\)) we obtain

\[
c_{\log}(a) = \int_{\mathbb{R}^4} \frac{d^4\xi}{(2\pi)^4} \delta(|\xi| - 1) \int_{\mathbb{R}^4} d^4x \, \text{tr} \sigma_{-4}(a)(\xi, x).
\]

(A3)

Recalling Eq. (34) we obtain Eq. (35). \(\square\)

**Renormalized traces.** It is obvious that changing the regularization functions \(f(t) \to \tilde{f}(t) = f(t/s)\) for some fixed \(s > 0\), amounts to changing \(\Lambda \to s\Lambda\), and thus changes \(c(0) \to c(0) + \log(s) c_{\log}\). Thus (33) is obvious for this special case. For more general changes \(f(t) \to \tilde{f}(t)\) of the regularization function, Eq. (33) can be shown using

\[
\int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} f\left(\frac{|p|}{\Lambda}\right) \int_{\mathbb{R}^4} d^4x \, \text{tr} \sigma[a]_{-3}^+(p, x) = c_{\log}(a) \log\left(\frac{\Lambda}{m}\right) + c(0)(a) + O(\Lambda^{-1})
\]

which follows from our discussion above.

**Appendix B: Computation details**

In this Appendix we present some details concerning our computations discussed in the main text. In particular, we give explicit formulas for the Dirac matrices, and we also show how to compute the structure constants \(\mathcal{M}_a\) in Eq. (19) exactly, i.e., without the simplified IR regularization. We also prove the Lemma in Section 4.1 and Eq. (33), and we give some details about the computation yielding Eq. (64).
B.1. Dirac matrices

A convenient representation for the Dirac matrices is as follows,

\[ \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3, \quad \gamma^4 = \begin{pmatrix} 0 & i1 \\ -i1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  

(B1)

where 1 and 0 are the 2×2 unit- and zero matrices and

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

the Pauli sigma matrices as usual.

B.2. Details about the gradient expansion

We start by rewriting the \( M_n \) in a convenient form. We define

\[ P_\varepsilon := \frac{1}{2} \left( 1 + \varepsilon \frac{\hat{p}}{|p|} \right), \quad \varepsilon = \pm \]  

(B2)

which are orthogonal projections, \( P_+ P_- = 0 \) and \( P_+^2 = P_+ \), satisfying \( P_+ + P_- = 1 \).

We then can write

\[ (1 - u\hat{p})^{-1} = \sum_{\varepsilon = \pm} \frac{1}{P_\varepsilon} \frac{1}{1 + u\varepsilon|p| + im\Lambda_0} \]

which we insert \( n+1 \) times in Eq. (49),

\[ M_{\nu_1...\nu_n}^{\nu_1...\nu_n} = (\Lambda_0)^{-n} \int_{|p| \leq \Lambda} \frac{d^4p}{(2\pi)^4} \int_0^\infty du u^{n-1} \]

\[ \times \sum_{\varepsilon_1,...,\varepsilon_{n+1} = \pm} \left( \prod_{j=1}^{n+1} \frac{1}{1 + u\varepsilon_j|p| + im\Lambda_0} \right) \text{tr}_\nu (P_{\varepsilon_1} \gamma_{s_1}^{\nu_1} P_{\varepsilon_2} \gamma_{s_2}^{\nu_2} ... \gamma_{s_n}^{\nu_n} P_{\varepsilon_{n+1}}) . \]

We thus obtain

\[ M_{\nu_1...\nu_n}^{\nu_1...\nu_n} = \sum_{\varepsilon_1,...,\varepsilon_{n+1} = \pm} I_{\varepsilon_1,...,\varepsilon_{n+1}} \text{tr}_\nu \langle P_{\varepsilon_1} \gamma_{s_1}^{\nu_1} P_{\varepsilon_2} \gamma_{s_2}^{\nu_2} ... \gamma_{s_n}^{\nu_n} P_{\varepsilon_{n+1}} \rangle \]  

(B3)

with

\[ I_{\varepsilon_1,...,\varepsilon_{n+1}} = I_{n;k}, \quad k \text{ such that } \sum_{j=1}^{n+1} \varepsilon_j = n + 1 - 2k \]  

(B4)
and

$$I_{n;k} = (\Lambda_0)^{-n} \frac{1}{8\pi^2} \int_0^\Lambda d|p||p|^3 \int_0^\infty \frac{du u^{n-1}}{1 + u \frac{|p| + i m}{\Lambda_0}} \left( \frac{1}{1 + u \frac{|p| + i m}{\Lambda_0}} \right)^{n+1-k} \left( \frac{1}{1 + u \frac{|p| + i m}{\Lambda_0}} \right)^k.$$

Rescaling $u \Lambda/\Lambda_0 \to u$ and introducing $\xi = \frac{|p|}{\Lambda}$ yields

$$I_{n;k} = (\Lambda)^{4-n} \frac{1}{8\pi^2} \int_0^1 d\xi \xi^3 \int_0^\infty \frac{du u^{n-1}}{1 + u[\xi + i \frac{m}{\Lambda}]} \left( \frac{1}{1 + u[\xi + i \frac{m}{\Lambda}]} \right)^{n+1-k} \left( \frac{1}{1 + u[\xi + i \frac{m}{\Lambda}]} \right)^k.$$

(B5)

Proof of the Lemma in Section 4.1. We note that

$$\text{tr}_\nu \langle P_{\varepsilon_1} \gamma_5 \cdot \gamma_{\varepsilon_2} \cdots \gamma_5 \cdot \gamma_{s_{n+1}} \rangle = \mathcal{T}_{\varepsilon_1 \varepsilon_2}^{\nu}$$

is invariant under $\varepsilon_j \to -\varepsilon_j$ (since the latter transformation amounts to the variable change $\xi \to -\xi$ in the integral Eq. (54) defining the angular average). Moreover, the cyclicity of trace and $\gamma_2^5 = 1$ implies that $\mathcal{T}_{\varepsilon_1 \varepsilon_2}^{\nu}$ does not change if we replace all $P_{\varepsilon_j}$ and $\gamma_{s_j}$ by $\gamma_5 P_{\varepsilon_j} \gamma_5$ and $\gamma_5 \gamma_{s_j} \gamma_5$, respectively. Using $\gamma_5 P_{\varepsilon_j} \gamma_5 = P_{-\varepsilon_j}$ and $\gamma_5 \gamma_{s_j} \gamma_5 = -\gamma_{s_j}$, we obtain $\mathcal{T}_{\varepsilon_1 \varepsilon_2}^{\nu} = (-1)^n \mathcal{T}_{-\varepsilon_1 \varepsilon_2}^{\nu}$, and using $\mathcal{T}_{-\varepsilon_1 \varepsilon_2}^{\nu} = \mathcal{T}_{\varepsilon_1 \varepsilon_2}^{\nu}$ this proves that $\mathcal{T}_{\varepsilon_1 \varepsilon_2}^{\nu}$ — and thus $\mathcal{M}_n$ in Eq. (49) — is non-zero only for even $n$.

From Eq. (B4) it is obvious that $\varepsilon_j \to -\varepsilon_j$ corresponds to $k \to n + 1 - k$, and thus $\mathcal{T}_{\varepsilon_1 \varepsilon_2}^{\nu} = \mathcal{T}_{-\varepsilon_1 \varepsilon_2}^{\nu}$ implies that we can replace $I_{n;k}$ by $[I_{n;k} + I_{n;n+1-k}]/2$ in Eq. (B3). We can write the latter as a sum of the terms which are even and odd under the change the sign of the mass $m \to -m$. A simple change of variables shows that $u$-integrals in the odd term

$$\frac{1}{4} \left[ I_{n;k}(m) + I_{n;n+1-k}(m) - I_{n;k}(-m) - I_{n;n+1-k}(-m) \right]$$

can be written as follows ($n$ even),

$$\frac{1}{4} \int_{-\infty}^\infty du u^{n-1} \left( \frac{1}{1 + u[\xi + i \frac{m}{\Lambda}]} \right)^{n+1-k} \left( \frac{1}{1 + u[-\xi + i \frac{m}{\Lambda}]} \right)^k$$

plus the same integral but with $k$ and $n + 1 - k$ interchanged. The latter integrals can be computed using Cauchy’s theorem: the poles of the integrand are in $u = -1/(\xi + i m/\Lambda)$ and $u = 1/(\xi - i m/\Lambda)$ and thus both always in the same half of the complex $u$-plane (upper or lower, depending on the sign of $m$). Computing
the integral by closing the integration path in the half plane where the integrand is analytic (which is possible since the integrand vanishes like $O(|u|^{-2})$ for $|u| \to \infty$) one sees that the integral is zero. This implies $M_n(-m) = M_n(m)$.

**Proof of Eq. (55):** Our discussion above implies that we can replace $I_{n;k}$ in Eq. (B3) by $\Re I_{n;k} = [I_{n;k}(m) + I_{n;k}(-m)]/2$.

We are interested in the terms which diverge for $\Lambda \to \infty$. To isolate them it is convenient to determine $\partial M_n/\partial \Lambda$. We thus compute

$$
\frac{\partial}{\partial \Lambda} \Re I_{n;k} = \Lambda^{3-n} \frac{1}{8\pi^2} I_{n;k}(\frac{m}{\Lambda})
$$

where we introduced the functions

$$
I_{n,k}(\eta) = \Re \int_0^\infty ds u^{n-1} \left( \frac{1}{1 + u[1+i\eta]} \right)^{n+1-k} \left( \frac{1}{1 + u[-1+i\eta]} \right)^k.
$$

(B7)

Note that the functions $I_{n,k}(\eta)$ are well-defined for all real $\eta \neq 0$, have a finite limit $I_{n,k}(0^+)$ as $\eta \to 0$, and they have series expansions in $\eta^2$.

It is easy to see that with the simplified regularization used in the main text we can obtain a formula for $\tilde{M}_{n,s_1...s_n}$ as in Eqs. (B3)–(B4) but with $I_{n;k}$ replaced by

$$
\tilde{I}_{n;k} = \frac{1}{8\pi^2} \int_0^\Lambda d|p||p|^{3-n} I_{n;k}(0^+).
$$

(B8)

We thus get

$$
\frac{\partial}{\partial \Lambda} \left( I_{n;k} - \tilde{I}_{n;k} \right) = \frac{1}{8\pi^2} \Lambda^{3-n} \left( I_{n;k}(\frac{m}{\Lambda}) - I_{n;k}(0^+) \right) = O(m^2 \Lambda^{1-n}),
$$

(B9)

which proves Eq. (B5).

**Remark:** We now can explain the reason for our choice $\Lambda_0 = |\Lambda_0|/(1 + i0^+)$ in the main text: this yields a regularization specifying the otherwise undefined integrals $I_{n,k}(0)$, and from Eq. (B9) it is clear that this is the regularization yielding a result identical with the one obtained with the proper regularization, up to lower order terms.

**Computation of $S_2.$** For $n = 2$ we need compute $M_2$ in Eq. (B9) exactly, using the formulas given above.

7To see this note that $I_{n,k}(\eta) = \Re \int_0^\infty ds \left( \frac{1}{s+1+i\eta} \right)^{n+1-k} \left( \frac{1}{s-1+i\eta} \right)^k$. 

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Similarly as explained in the main text we compute (cf. Eq. (B3))
\[
\text{tr}_{\nu} \left< P_{\varepsilon_1} \gamma_{s_1} P_{\varepsilon_2} \cdots \gamma_{s_2} P_{\varepsilon_3} \right> = \delta_{\varepsilon_1,\varepsilon_3} \eta^\nu \eta^\nu_2 (1 - \varepsilon_1 \varepsilon_2 (-1)^{s_1}) .
\]
Moreover, the integrals defined in Eq. (B7) for \( n = 2 \) and \( k = 0, 1 \) are,
\[
I_{2,0}(\eta) = \mathcal{R} \frac{1}{2(1 + i\eta)^2} = \frac{1}{2} - \frac{3}{2} \eta^2 + \mathcal{O}(\eta^4)
\]
\[
I_{2,1}(\eta) = \mathcal{R} \frac{1}{4(1 + i\eta)} \left( (1 + i\eta) \log \left( \frac{1 + i\eta}{1 - i\eta} \right) - 2 \right) = \frac{1}{2} + \frac{1}{2} \eta^2 + \mathcal{O}(\eta^4),
\]
and with Eqs. (B6), (B3) and (B4) we can compute \( \partial \mathcal{M}_2 / \partial \Lambda \). Straightforward computations yield
\[
\mathcal{M}_{2; s_1 s_2}^{\nu_1 \nu_2} = \delta_{s_1 s_2} \eta^\nu \eta^\nu_2 \frac{1}{16\pi^2} \left( A^2 A_{s_1 s_2} + m^2 \log \left( \frac{\Lambda}{|m|} \right) A^{(0)}_{s_1 s_2} + \mathcal{O}(\Lambda^0) \right)
\]
\[
A_{s_5} = -A_{00} = 1, \quad A^{(0)}_{s_5} = -2, \quad A^{(0)}_{00} = 0, \quad (B10)
\]
and with Eq. (50) we obtain Eqs. (63)–(64).

References

[BGV] Berline N., Getzler E., and Vergne M.: *Heat kernels and Dirac operators*, Grundl. math. Wiss. 298. Springer Verlag, Berlin (1992)

[C] Connes A.: *Noncommutative Geometry*, Academic Press, San Diego (1994)

[CC] Chamseddine A.H. and Connes A.: The spectral action principle, *Comm. Math. Phys.* 186 731 (1997)

[dW] DeWitt B.S.: *Dynamical Theory of groups and fields*, Gordon and Breach, New York (1965)

[G] Gilkey P.B.: *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, Publish or Perish, Dilmington (1995)

[GVF] Gracia-Bondia J.M., Varilly J.C., and Figueroa H: *Elements of Noncommutative Geometry*, Birkhäuser, Boston (2001)
[CDMP] Cardona A., Ducourtioux C., Mognot J.P., and Paycha S.: Weighted traces on algebras of pseudo-differential operators and geometry of loop groups, math.OA/0001117

[GR] Gradshteyn I.S. and Ryzhik I.M.: *Table of integrals, series, and products*, Academic Press (1980)

[H] Hörmander L.: *The Analysis of Linear Partial Differential Operators III*, Grundl. math. Wiss. 274 Springer-Verlag, Berlin (1985)

[IZ] Itzykson C., Zuber J.-B.: *Quantum field theory*, McGraw-Hill, New York (1985)

[L] Langmann E.: Noncommutative Integration Calculus, *J. Math. Phys.* **36**, 3822 (1995)

[LM] Langmann E., and Mickelsson J.: Elementary derivation of the chiral anomaly, *Lett. Math. Phys.* **36**, 45 (1996)

[VG] Várilly J.C., and Gracia–Bondía J. M.: Connes’ noncommutative differential geometry and the standard model, *J. Geom. Phys.* **12**, 223 (1993)

[We] Weinberg S.: *The quantum theory of fields. Vol. II. Modern applications*, Cambridge University Press, Cambridge (1996)

[Wo] Wodzicki M.: ‘Noncommutative Residue,’ in: *K–theory, arithmetic and geometry*, Yu. I. Manin (ed.), Lecture notes in Mathematics 1289, Springer-Verlag, Berlin (1985)