ACTEGORIES FOR THE WORKING AMTHEMATICIAN

MATTEO CAPUCCI AND BRUNO GAVRANOVİĆ

Abstract. Actions of monoidal categories on categories, also known as actegories, have been familiar to category theorists for a long time, and yet a comprehensive overview of this topic seems to be missing from the literature. Recently, actegories have been increasingly employed in applied category theory, thereby encouraging an effort to fill this gap according to the new needs of these applications. This work started as an investigation of the notion of monoidal actegory, a compatible pair of monoidal and actegorical structures, and ended up including a sizable reference on the elementary theory of actegories. We cover basic definitions and results on actegories and biactegories, spelling out explicitly many folkloric definitions. Motivated by the use of actegories in the theory of optics, we focus particularly on the way actegories combine and interact with monoidal structure by describing ‘distributive laws’ between the two. In the last section, we provide three Cayley-like classification results for these situations.

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1. Introduction

Monoid actions are ubiquitous structures in mathematics. They provide a concrete implementation of the abstract compositional laws of a monoid as operations on some other object. Each element \( m \) of a monoid \((M, 1, \cdot)\) acts on a given set (or manifold, space, etc.) \( C \) by specifying an operation \( m \cdot - : C \to C \), and the complex of these operations satisfy laws which are the ‘image’ of those of a monoid:

\[
 n \cdot (m \cdot c) = (n \cdot m) \cdot c, \quad 1 \cdot c = c.
\]

There are many examples of monoid actions (Figure 1): scalar multiplication of vectors is an action of the field of scalars on vectors of a vector space, automata are essentially actions of free monoids on a space of ‘states’, in dynamical systems monoids of time \((\mathbb{N}, \mathbb{R}^+, \text{etc.})\) act on spaces of states by tracing out trajectories.

![Figure 1: Examples of monoid actions from left to right: scalar multiplication of vectors, automata transitions, trajectories in spaces of configurations.](image)

Like monoidal categories promote the concept of monoid to a higher categorical dimension, the protagonists of this work, actegories, promote the concept of a monoid action to its higher-dimensional equivalent (Figure 2). In doing so, we lose the sharpness of equality we have in sets and we gain the expressivity of isomorphisms in categories. Hence monoidal categories and actegories alike satisfies their laws (inherited from monoid and their actions, respectively) only up to isomorphism, which are themselves subjected to a new set of coherence equations.

Actegories are not a new invention\(^1\), and not even a recent one. As we learned from [CP09, Footnote 1], the concept was first entertained by Bénabou in [Bén67], while the

\(^1\)or discovery!
Figure 2: An actegory as a combination of two different generalizations of a monoid. A monoid is to a monoid action what a monoidal category is to an actegory. Likewise, a monoid is to a monoidal category what a monoid action is to an actegory.

name itself (a portmanteau of ‘action’ and ‘category’) originated in the Australian Category Seminar at the end of the last century, and first appeared in print in [McC00b].

Actegories have since appeared in categorical algebra [McC00b, Ško09, Par77] and in other various works spanning different subdisciplines in category theory [Gar18, Str20, Szl09, JK01]; though the lack of consistent terminology makes it quite hard to get a grasp on the extent of the literature on actegories.

Nonetheless, in the past two decades there’s been a surge of interest in actegories in a more ‘applied’ context. Indeed, the reason we are interested in actegories in the first place is their effectiveness in capturing the idea of ‘agency’ in categories of systems. Part of these ideas have been published in [CGHR21] which proposes a mathematical-conceptual framework for thinking about cybernetic systems; but our work doesn’t stand alone.

The work in categorical cybernetics has been borrowing from the flurry of activity regarding mixed optics [Boi20, Ril18, CEG+20], of which actegories, and specifically Tambara modules\(^2\) [PS08, Rom20], form the mathematical backbone.

The use of actegories in categorical cybernetics is also quite close to the work of Pastro and Cockett [CP09, Coc13, Yea12] on the categorical vertex of a Curry–Howard–Lambek correspondence for concurrency. They resort to linear actegories in order model message-passing concurrency, by having a ‘sequential world’ of messages acting on a ‘concurrent

---

\(^2\)Tambara modules are lax equivariant profunctor between actegories on the same monoidal category. More details can be found in [Rom20, §5].
world’ of message-passing processes. The action of messages on processes reifies the first as values for the latter.

The work of Nishiwaki and Asai [NA20] fits in the same pattern. They develop a generalization of Moggi’s calculus of computational effects based on actegories, called *semi-effects calculus*, whose categorical semantics is a linear functor between an actegory of ‘values’ and one of ‘computations’, with the intended meaning of representing a sort of ‘let’ binding.

In a completely different setting, [Ste18] shows how actegories and their morphisms can be fruitfully used to talk about the kind of gadgets topological data analysis produces, namely ‘persistency modules’. The way actegories are used in this work recalls the actions of monoids of time which give rise to dynamical systems, explaining his adoption of the terminology *dynamical categories*.

Related to actegories, *module categories* seemed to have enjoyed more attention. These are actions of *(multi-)fusion categories*, which are categories enriched in vector spaces with some extra properties³. Their existence also motivates the choice of a distinct terminology for the ‘vanilla’ notion. Ostrik gives a systematic account of the theory of module categories in [Ost03], where they indicate [Gra76] as the first appearance of said concepts. We also mention [Gre10b, Gre10a].

Since, ultimately, multi-fusion categories are still monoidal categories, much of the theory of module categories can be reproduced in less structured contexts. What really makes compendia such as Ostrik’s less relevant for the uses of applied or even pure category theory is their focus, which is often strongly biased (and rightly so) on the interaction between the fusion stucture and the actegorical structure. Moreover, much of the structure of multi-fusion categories is not available in contexts such as Kleisli categories of commutative monads on Set (i.e. cats of effectful computations), categories of measurable spaces, or categories of manifolds. Hence the need of compiling a more streamlined version of the theory behind module categories, closer to the sensibility of modern applied category theory.

In this work, and in all the works cited above, actions and actegories are treated in the same way, as dictated by common algebraic wisdom. There is a different way

³Specifically, multifusion categories are ‘rigid semisimple $k$-linear tensor categories with finitely many simple objects and finite dimensional spaces of morphisms’, where $k$ is an algebraically closed field [ENO05, p. 583].
to talk about actions though, which is connected with the ideas of cofunctors [AU16, Cla20] and differential forms (whose pullback is known in computer science as ‘reverse differentiation’). In this guise, monoida actions have been studied by Alvarez-Picallo under the name of change actions [Alv20, APO19], in the context of incremental computation and differentiation. To explore this connection more closely is left for future work, together with many other directions that would deserve to be explored.

1.1. Overview of the paper. This work has born out of the practical need of identifying and understanding the structures involved in the proposed foundations of ‘categorical cybernetics’, as outlined in [CGHR21]. In that work, we singled out the Para and Optic constructions as useful abstractions for modelling systems with agency, and moreover realized actegories provide an interesting conceptual view of the way agents and systems interact. This is remindful of the ideas put forward in [Coc13].

The original motivating question for this work concerned the monoidal structures of Para and related constructions. In [CGHR21], we singled out some structure morphisms necessary for promoting a (symmetric) monoidal product on an actegory \((\mathcal{C}, \bullet)\) to a (symmetric) monoidal product on \(\text{Para}_*(\mathcal{C})\). It remained unclear how that structure arised and most importantly what were the constraints on that. Here we clarify these questions by methodically analyzing the compatibility structures between actegorical and monoidal categories, thereby answering the previous question. In doing so, we ended up compiling a much needed reference on actegories.

In Section 2, we briefly go over some preliminary definitions concerning pseudomonads and their categories of algebras. This is done in order to ground the definitions of Section 3, which introduce actegories and their morphisms, all the way up to the detailed definition of the indexed 2-category of actegories. We also provide several examples and ancillary results. These two sections do not contain much new content, though the detailed exposition and meticulous definitions can be considered a new contribution.

Instead, Section 4 and 5 contain the lion share of original content, and form the central part of the work.

The first is dedicated to the study of ‘composition of actegories’, i.e. ways in which new actegories can be systematically generated from old ones. Some of these ways are proper monoidal structures on categories of actegories (cartesian and cocartesian product, tensor product), but others are mathematically given by distributive laws, interpreted as
decoration on the sequential composition of actegories seen as parametric morphisms. The latter, Section 5, is devoted to the study of compatibility structures between monoidal and actegorical structure. We explore various ways to combine them, introducing two new definitions: that of monoidal actegory and that of distributive algebroidal actegory. We show how the first is instrumental in making categories of optics monoidal, while the latter has connections to the theory of hybrid composition of optics. We conclude the section by analyzing the way braiding interacts with each of these new notions, and by proving three classification results for monoidal actegories, showing they are indeed equivalent to monoidal functors into suitably defined classifying objects.

Definitions and proofs we didn’t deem illuminating enough to appear in the main body of the paper but not trivial enough to being omitted have been moved to Appendix A and Appendix B, respectively.

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1.3. Notation and conventions. Throughout the paper, we use \((\mathcal{M}, j, \otimes)\) to refer to a monoidal category [JY21, Definition 1.2.1], using \(\lambda, \rho, \alpha\) to refer to left and right unitors and the associator respectively. When braiding or symmetry are needed, we point it out explicitly. We reserve the names \(\mathcal{M}, \mathcal{N}\) for acting monoidal categories, and \(\mathcal{C}, \mathcal{D}\) for categories that are being acted on. We make extensive use of diagrammatic notation for composition, using the symbol \(\circ\). Actions are plentiful in this paper, and we use a variety of symbols: \(\bullet, \circ, \otimes\), and so on, depending on the context.

Many of the structures we are going to contemplate are pseudo-somethings (pseudoalgebras, pseudomonoids, etc.), which tends to make prose cumbersome and clumsy. Therefore we sometimes dropped the prefix with the understanding this does not hinder readability as it improves it.
2 PRELIMINARIES

2. Preliminaries

As eloquently explained by [BKP89], if universal algebra can be thought as the theory of 1-monads on \( \textbf{Set} \) (‘0-categories’), universal 1-algebra\(^4\) can be studied as the theory of 2-monads on \( \textbf{Cat} \).

For the algebraic structures we are interested in, this is very much true. Let’s see what’s the algebraic story for actions of monoids (‘0-dimensional actegories’) from this point of view. Every monoid \((M, j, \otimes) : \textbf{Set} \to \textbf{Set}\) induces a monad \(M \times - : \textbf{Set} \to \textbf{Set}\), whose join and unit are constructed from the product and unit of \(M\) itself:

\[
j_C : C \to M \times C \quad m_C : M \times M \times C \to M \times C
\]

\[
\begin{align*}
  j_C & : C \to M \times C \\
  c & \mapsto (j, c) \\
  m_C & : M \times M \times C \to M \times C \\
  (m, n, c) & \mapsto (m \otimes n, c)
\end{align*}
\]

(2.1) (2.2)

Algebras for this monad are monoid actions:

\[
\begin{array}{ccc}
  M \times C & \xleftarrow{j_X} & C \\
  \uparrow & & \downarrow \\
  C & \xRightarrow{M \times i} & M \times C
\end{array}
\]

(2.3) (2.4)

\[
j \cdot c = c, \text{ for all } c \in C. \quad m \cdot (n \cdot c) = (m \otimes n) \cdot c, \text{ for all } m, n \in M, c \in C.
\]

Analogously, \(\mathcal{M}\)-actegories turn out to be pseudoalgebras for the pseudomonad \(\mathcal{M} \times -\). A pseudomonad is the higher equivalent of a monad in the same way monoidal categories are the higher version of a monoid. Indeed, both (pseudomonads and monoidal categories) are instances of the categorified notion of ‘monoids in monoidal categories’, which is ‘pseudomonoid in monoidal bicategories’. A pseudomonoid is a monoid whose associativity and unitality only hold up to coherent isomorphism [McC00a, §2], a structure which is very natural to find in 2-dimensional contexts, since 1-things there tend to be defined only up to higher equivalence.

\(^4\)The reference calls it 2-algebra, but we adopt the same perspective on dimension as HoTT, where 0-types are sets and (directed) 1-types are categories. Therefore n-dimensional structure is structure on (possibly directed) n-types, and universal n-algebra is concerned with the study of algebraic structure on n-types.
\[
\begin{array}{cc}
\text{n = 0} & \text{n = 1} \\
\text{monoid } M & \text{monoidal category } \mathcal{M} \\
\text{monoid homomorphism } f : M \to N & \text{strong monoidal functor } F : \mathcal{M} \to \mathcal{N} \\
\text{free } M\text{-action monad } M \times - & \text{free } \mathcal{M}\text{-actegory pseudomonad } \mathcal{M} \times - \\
\text{action } M \times X \to X & \text{pseudoaction } \mathcal{M} \times \mathcal{C} \to \mathcal{C} \\
(\text{algebra of } M \times -) & (\text{pseudoalgebra of } \mathcal{M} \times -) \\
\text{linear morphisms } g : (C, \bullet) \to (D, \circ) & \text{linear functors } G : (\mathcal{C}, \bullet) \to (\mathcal{D}, \circ) \\
(\text{algebra morphisms}) & (\text{strong pseudoalgebra morphisms})
\end{array}
\]

Therefore monoidal categories are pseudomonoids in \((\text{Cat}, 1, \times)\), whereas pseudomonads on a 2-category \(\mathcal{X}\) are pseudomonoids in \((\mathcal{X}, \mathcal{X}, \text{id}, \circ)\) [Mar99, §3]:

2.0.1. Definition. Let \(\mathcal{X}\) be a 2-category. A pseudomonad on \(\mathcal{X}\) is a strict 2-functor \(T : \mathcal{X} \to \mathcal{X}\), together with 2-morphisms \(j : 1 \Rightarrow T(\text{unit})\) and \(m : TT \Rightarrow T(\text{multiplication})\) and modifications (left unitor \(\lambda\), right unitor \(\rho\), associator \(\alpha\)):

\[
\begin{align*}
& T \xrightarrow{j_T} TT \\
\xymatrix{ & T T \ar[d]^m & \\
T & T T \ar[r]^{T j} & T & \\
T T & T T \ar[l]_{T m} & \\
& T T \ar[d]_m & \\
TT & T & \\
}
\end{align*}
\]

(2.5)

that satisfy the coherence laws reported in Appendix A.

This notion of ‘higher dimensional monad’ is intermediate between that of a ‘fully weak 2-monad’, where \(T, j\) and \(m\) are possibly not strict, and a ‘fully strict 2-monad’, where \(\lambda, \rho\) and \(\alpha\) are assumed to be identities. In this paper, the monads we are interested in have the form \(\mathcal{M} \times -\), where \(\mathcal{M}\) is a monoidal category. Since this functor is strict, we will not concern ourselves with weak 2-monads. More can be done, actually: we can generally replace \(\mathcal{M}\) with an equivalent strict monoidal category, making \(\mathcal{M} \times -\) a strict 2-monad (or simply 2-monad).
2.1 The ‘free \( M \)-actegory’ Pseudomonad

2.1. The ‘free \( M \)-actegory’ Pseudomonad. If \( (M, j, \otimes, \lambda, \rho, \alpha) \) is a monoidal category, all its data and coherence properties map pretty much directly to those of a pseudomonadic structure on \( T = M \times - : \text{Cat} \to \text{Cat} \):

\[
\begin{array}{c}
\begin{array}{ccc}
M \times - & \xrightarrow{j \times M \times -} & M \times M \times - \\
\downarrow{\lambda \times -} & & \downarrow{\otimes \times -} \\
M \times - & & M \times -
\end{array} & & \\
\begin{array}{ccc}
M \times M \times - & \xleftarrow{M \times j \times -} & M \times - \\
\downarrow{\otimes \times -} & & \downarrow{\rho \times -} \\
M \times - & & M \times -
\end{array}
\end{array}
\]

(2.7)

\[
\begin{array}{c}
\begin{array}{ccc}
M \times M \times M \times - & \xrightarrow{M \times \otimes \times -} & M \times M \times - \\
\downarrow{\otimes \times M \times -} & & \downarrow{\alpha \times -} \\
M \times M \times - & & M \times -
\end{array}
\end{array}
\]

(2.8)

Coherence for \( T \) follows directly from the coherence of \( M \) as monoidal category (see Proposition A.0.1).

2.2. Algebras for Pseudomonads. Algebras of higher monads also come in various degrees of weakness. We settle with the following general definition:

2.2.1. Definition. A pseudoalgebra for a pseudomonad \((T, j, m)\) on \( X \) is a morphism

\[ t : T x \to x \quad \text{(2.9)} \]

in \( X \) equipped with invertible 2-cells

\[
\begin{array}{ccc}
x & \xrightarrow{j_x} & T x \\
\downarrow{\eta} & & \downarrow{t} \\
x & \xrightarrow{m_x} & T x
\end{array}
\quad \begin{array}{ccc}
T T x & \xrightarrow{T t} & T x \\
\downarrow{\mu} & & \downarrow{t} \\
T x & \xrightarrow{t} & x
\end{array}
\]

(2.10)

satisfying some coherence equations (see Appendix A).

When \( \eta \) and \( \mu \) are not invertible, we call the algebra lax (or, dually, oplax); if they are identities, we say the pseudoalgebra is strict. For \( T = M \times - \), strict algebras (strict \( M \)-actegories) can be hard to come by. Still, every actegory is equivalent (in a suitable
sense) to a strict one (Lemma 3.4.1), hence we can make use of this fact to greatly simplify some proofs.

Instead, there is no way to avoid working with ‘pseudomorphisms’ of algebras—similarly to the case of monoidal categories, once you have fixed two objects, the strict morphisms are simply not the right collection of morphisms between them to consider.

2.2.2. Definition. Let \((T, j, m)\) be a pseudomonad on \(X\) and let \((t, \eta, \mu) : Tx \to x, (t', \eta', \mu') : Ty \to y\) be pseudoalgebras. A lax morphism \((f, \ell) : t \to t'\) consists of a morphism \(f : x \to y\) and a 2-cell

\[
\begin{array}{ccc}
Tx & \xrightarrow{Tf} & Ty \\
\downarrow t & \xleftarrow{\ell} & \downarrow t' \\
x & \xrightarrow{f} & y
\end{array}
\]

(2.11)

satisfying suitable coherence axioms (see Appendix A).

A lax morphism \((f, \ell)\) is called \textbf{strong} if \(\ell\) is invertible, and \textbf{strict} if \(\ell\) is an identity. When we omit any qualification, we default to strong.

2.2.3. Definition. Let \((f, \ell)\) and \((g, \ell')\) be lax morphisms \(t \to t'\). A transformation of morphisms is a 2-cell \(\xi : f \Rightarrow g\) so that the following axiom holds:

\[
\begin{array}{ccc}
Tx & \xrightarrow{Tf} & Ty \\
\downarrow t & \xleftarrow{T\xi} & \downarrow t' \\
x & \xrightarrow{f} & y
\end{array}
\quad\quad\quad
\begin{array}{ccc}
Tx & \xrightarrow{Tf} & Ty \\
\downarrow t & \xleftarrow{\ell} & \downarrow t' \\
x & \xrightarrow{g} & y
\end{array}
\]

(2.12)

For any given pseudomonad \(T\), pseudoalgebras, algebra morphisms and transformations thereof assemble into various bicategories, depending on the degrees of laxity used:

2.2.4. Definition. Given a pseudomonad \(T\) on \(X\), there is a bicategory \(T\text{-Alg}\) of pseudoalgebras, strong morphisms and transformations. If \(T\) is a 2-monad, we further have the categories \(T\text{-Alg}_s\) of strict algebras, strong morphisms and transformations, and \(T\text{-Alg}^s\) of strict algebras, strict morphisms and transformations. Clearly there are forgetful functors

\[
T\text{-Alg}^s \longrightarrow T\text{-Alg}_s \longrightarrow T\text{-Alg} \longrightarrow X.
\]

(2.13)
2.2 Algebras for pseudomonads

To denote the various combinations of lax/oplax/strong/strict objects and morphisms, we adopt a notation based on subscripts and superscripts:

\[ T-\text{Alg}_{\text{laxity of the morphisms}}^{\text{laxity of the objects}} \quad (2.14) \]

The symbols denoting laxity are described in the following table:

| laxity                  | symbol          |
|-------------------------|-----------------|
| lax                     | lx              |
| oplax                   | ox              |
| strong/pseudo           | ps, (empty string) |
| strict                  | s               |

This notation is compatible with that used in [BKP89], except they denote ‘lax’ simply with \( l \).
3. Actegories and their morphisms

Having defined the necessary pseudoalgebraic machinery in the previous section, we proceed to build on top of it and define $\mathcal{M}$-actegories and their morphisms. We give a number of examples, a strictification lemma, describe the free and cofree adjunction, and lastly, describe the indexed 2-category of all actegories.

3.1. Actegories. As anticipated, we can develop the algebraic theory of $\mathcal{M}$-actegories systematically by instantiating the definitions of pseudoalgebras and their morphisms for $T = \mathcal{M} \times -$.

3.1.1. Definition. Let $\mathcal{M}$ be a monoidal category. A left $\mathcal{M}$-actegory $\mathcal{C}$ (or left $\mathcal{M}$-action) is a category $\mathcal{C}$ equipped with a functor

$$- \cdot : \mathcal{M} \times \mathcal{C} \to \mathcal{C},$$

and two natural isomorphisms

$$\eta_c : c \sim j \cdot c, \quad \mu_{m,n,c} : m \cdot (n \cdot c) \sim (m \otimes n) \cdot c,$$

respectively called unitor and multiplicator, satisfying the following coherence laws:

$$m \cdot (n \cdot (p \cdot c)) \xrightarrow{\mu_{m,n,p,c}} (m \otimes n) \cdot (p \cdot c) \xrightarrow{\mu_{m \otimes n,p,c}} (m \otimes (n \otimes p)) \cdot c$$

$$j \cdot (m \cdot c) \xrightarrow{\mu_{j,m,c}} (j \otimes m) \cdot c \xrightarrow{\lambda_m^{-1} \cdot m} m \cdot c$$

$$m \cdot (j \cdot c) \xrightarrow{\mu_{m,j,n}} (m \otimes j) \cdot c \xrightarrow{\rho_m^{-1} \cdot c} m \cdot c$$
3.1 Actegories

3.1.2. Remark. As it happens for triangle axioms for monoidal categories, any of them is redundant in the presence of the other one and the pentagon. Indeed, by inspecting Kelly’s simplification of MacLane’s coherence conditions for monoidal categories [Kel64], one can realize that part of his proof can be reproduced mutatis mutandis for actegories, so that either of Diagram (3.4) or (3.5) can be omitted. We learned of this fact from [McC00b].

3.1.3. Remark. Analogous to left actegories, it is possible to define right actegories. When the action is on the left side, successive actions combine as follows:

\[ m \bullet (n \bullet c) \cong (m \otimes n) \bullet c. \] (3.6)

On the other hand, when the action is on the right side, actions combine as:

\[ (c \bullet n) \bullet m \cong c \bullet (n \otimes m). \] (3.7)

Comparing the two equations above should make the difference clear: multiplying by \( m \) and then \( n \) is like multiplying by \( m \otimes n \) for left actions, and by \( n \otimes m \) for right actions. Indeed, a right \( \mathcal{M} \)-actegory is exactly the same as a left \( \mathcal{M}^{\text{rev}} \)-actegory, where \( \mathcal{M}^{\text{rev}} \) is the monoidal category obtained by equipping the underlying category of \( \mathcal{M} \) with the product \( m^{\text{rev}} \otimes n := n \otimes m \) and suitably tweaking the rest of the structure [JY21, Example 1.2.9].

Notice that right \( \mathcal{M} \)-actegories can also be conceived as pseudoalgebras for the same endofunctor \( \mathcal{M} \times - \) but equipped with a different monad structure, namely the one where multiplication is precomposed with a symmetry:

\[
\mathcal{M} \times \mathcal{M} \times - \overset{\text{swap}_\times}{\longrightarrow} \mathcal{M} \times \mathcal{M} \times - \overset{\otimes}{\longrightarrow} \mathcal{M} \times -
\] (3.8)

Indeed, \( \tilde{\otimes} := \text{swap}_\times \otimes \), so that the pseudomonad of left \( \mathcal{M}^{\text{rev}} \)-actegories is identical to this one, which we might rightly call the pseudomonad of right \( \mathcal{M} \)-actegories.

As customary in algebra, whenever we do not qualify an action with a handedness we default to left.

3.1.4. Remark. If a category receives both a left and a right action (i.e. a \( \mathcal{M} \)-action and a \( \mathcal{N}^{\text{rev}} \)-action), and these interact ‘nicely’, we have a biactegory, a structure we treat extensively in Section 4.3. Likewise, a category may also receive a ‘forward’ and a ‘backward’ action (i.e. an \( \mathcal{M} \)-action and an \( \mathcal{N}^{\text{op}} \)-action) which interact nicely. These deserve the name of diactegories, but we leave their study for future work.
3.1.5. Remark. Contrary to the praxis in the related field of Tambara theory [PS08, Tam06], in the optics literature [Boi20, CEG20, Rom20] definitions implicitly assume every action to be on the left. This doesn’t make much of a difference when $\mathcal{M}$ is symmetric (a common occurrence in practice), and when the actions are not lax, but this obfuscates a nice correspondence to algebra in practice. In particular, there is a distinction between left and right actions in the algebra of $\text{Para}$ and $\text{Copara}$ [CGHR21, Definition 2].

We must mention an equivalent presentation of the structure of actegory, which is handy to have in mind:

3.1.6. Proposition. The data of a left $\mathcal{M}$-actegory is equivalent to that of a strong monoidal functor:

$$ \{ \mathcal{M} \times \mathcal{C} \to \mathcal{C} \text{ left action} \} \cong \{ \mathcal{M} \to [\mathcal{C}, \mathcal{C}] \text{ strong monoidal} \} \quad (3.9) $$

The equivalence is supported by the tensor-hom adjunction in $\mathcal{C}\text{at}$.

Proof. It’s a routine matter to verify that $\eta$ and $\mu$ for a left action $\bullet$ correspond to a strong monoidal structure on its currying $\text{curr}(\bullet)$, and that Diagrams (3.3)-(3.5) are equivalent to the laws obeyed by such structure (see [Mac71, §XI.2]).

3.1.7. Remark. This presentation is preferred in the literature on graded (co)monads [Fuj19]. In that case, the monoidal functor $\mathcal{M} \to [\mathcal{C}, \mathcal{C}]$ is more often considered to be just lax (for graded monads) or oplax (for graded comonads). These correspond to lax and oplax algebras of $\mathcal{M} \times -$ (considered as a pseudomonad) from our point of view. Much of the definitions we give here still work for these weaker structures, but many of the results do not hold for them. Moreover, in the applications we are interested in this is not a limitation, hence in this work we focus on pseudoalgebras.

3.1.8. Remark. Despite making it easier to state the definition of actegories, the curried presentation of monoidal actions is considerably less elegant when it comes to their morphisms, 2-morphisms and other constructions. This can be directly motivated by observing most of the facts we care about for $\mathcal{M}$-actegories are instances of general algebraic theories, while Proposition 3.1.6 is a very peculiar property of actegories. Therefore we’ll continue to use Definition 3.1.1 as our ‘main one’. Nonetheless, extending this result to more structured settings will be the object on the entirety of Section 5.5.
3.2 Examples

As prescribed by the microcosm principle, actegories can be seen as the structure necessary to define the structure of an ‘action of a monoid’ in a category, in the same way monoidal categories are the minimal setting in which monoids themselves can be defined.

3.1.9. Definition. Let $(\mathcal{C}, \cdot)$ be an $\mathcal{M}$-actegory. An action of $m$ on $c$ in $(\mathcal{C}, \cdot)$ is an arrow $*: m \cdot c \to c$ such that the following commute (compare them with Diagrams (2.3)–(2.4)):

\begin{align*}
\begin{array}{c}
m \cdot c \\
c
\end{array}
& \xleftarrow{\cdot c} 
\begin{array}{c}
j \cdot c \\
c
\end{array}
\quad (m \otimes m) \cdot c & \xleftarrow{\mu_{m,m,c}} 
\begin{array}{c}
m \cdot (m \cdot c) \\
m \cdot c
\end{array}
\end{align*}

(3.10)

In the same fashion we can define coactions of comonoids and biactions of bimonoids, for which actegories remain the natural setting.\(^5\)

3.2. Examples. We describe here some examples, some of which are taken from the literature, although with no pretense to be exhaustive. In particular, we omit most of the examples from categorical algebra (e.g. those described in [Ost03]).

3.2.1. Example. [Trivial actegories] Every category $\mathcal{C}$ is trivially a 1-actegory, where 1 is the one-object category whose monoidal product is defined in the only way possible. The action $\cdot : 1 \times \mathcal{C} \to \mathcal{C}$ is an isomorphism, and the unitor and multiplicator are trivial.

This can be generalised: every category $\mathcal{C}$ is trivially an $\mathcal{M}$-actegory for any monoidal category $\mathcal{M}$, with the action given by the projection $\pi_\mathcal{C} : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$.

3.2.2. Example. [Monoid action] Any action of a monoid $M$ on a set $X$ is a ‘discrete actegory’. That is, it’s a $\text{disc}(M)$-actegory $\text{disc}(X)$, where $\text{disc} : \text{Set} \to \text{Cat}$ casts a set as a discrete category. In [Alv20, APO19] monoid actions are called change actions, although their maps are different.

3.2.3. Example. [Monoidal category] Every monoidal category $\mathcal{M}$ is canonically a left and right $\mathcal{M}$-actegory. The tensor $- \otimes = : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, equipped with the left unitor as counitor and the associator as comultiplicator, defines a left action. On the right, one has the ‘same’ functor but typed as $= \otimes - : \mathcal{M}^{\text{rev}} \times \mathcal{M} \to \mathcal{M}$, equipped with the right unitor as counitor and the inverse of the associator as comultiplicator. We call these actions canonical left and canonical right self-action, respectively.

\(^5\)Though biactions could be further abstracted as to take place in biactegories instead.
3.2.4. Example. [Action of natural numbers] Any monoidal category \((\mathcal{M}, j, \otimes)\) is naturally an \(\mathbb{N}\)-actegory, where \(\mathbb{N}\) is considered as a discrete category with addition as monoidal structure. The action \((\cdot)\otimes^n: \mathbb{N} \times \mathcal{M} \to \mathcal{M}\) takes the pair \((n, c)\) to the \(n\)-fold monoidal product of \(c\) with itself, i.e.
\[
c^{\otimes n} := c \otimes \ldots \otimes c.
\]
(3.11)

Since \(\mathbb{N}\) is equivalent to the free monoidal category on one generator, the above fact can be considered an analogue of the fact every abelian group is a \(\mathbb{Z}\)-module, except now the structure of \(\mathbb{N}\)-actegory is not enough to recover a monoidal structure (this what addition buys you). Put differently, not every \(\mathbb{N}\)-actegory is a monoidal category.

3.2.5. Example. [Evaluation] Every category \(\mathcal{C}\) is a \(((\mathcal{C}, \mathcal{C}), 1_\mathcal{C}, \circ)\)-actegory, under the action given by evaluation. In fact, this is the action corresponding to the identity functor of \([\mathcal{C}, \mathcal{C}]\) under the equivalence described in Proposition 3.1.6. This also means than any monoidal subcategory of endofunctors over \(\mathcal{C}\) acts on \(\mathcal{C}\). For instance, the action of applicative functors (a class of endofunctors definable when \(\mathcal{C}\) is symmetric monoidal) can be used to define a class of optics called *kaleidoscopes* [CEG+20, §3.2.2].

3.2.6. Example. [Copower] When a category \(\mathcal{C}\) has all small coproducts, there is an action \(\otimes: \text{Set} \times \mathcal{C} \to \mathcal{C}\) of \((\text{Set}, 1, \times)\) defined as
\[
A \otimes c := \bigsqcup_{a \in A} c
\]
(3.12)

This action is called \textbf{Set-copower} and enjoys the universal property of being ‘left adjoint to the hom functor’:
\[
\mathcal{C}(A \otimes c, y) \cong \text{Set}(A, \mathcal{C}(c, y)).
\]
(3.13)

In general, we can replace \text{Set} with any closed monoidal category \(\mathcal{V}\), and talk about \(\mathcal{V}\)-copowered (or \(\mathcal{V}\)-tensored) enriched categories [Kel82]. In [JK01], the authors prove that \(\mathcal{V}\)-copowered enriched categories are equivalent to \(\mathcal{V}\)-actegories \((\mathcal{C}, \bullet)\) such that \(- \bullet c\) has a right adjoint for each \(c : \mathcal{C}\) (so called \textit{closed actegories}).

3.2.7. Example. [Power] Dually, if \(\mathcal{C}\) has all products one gets an action \(\triangleright: \text{Set}^{\text{op}} \times \mathcal{C} \to \mathcal{C}\) given by
\[
A \triangleright c := \prod_{a \in A} c
\]
(3.14)
which enjoys the following universal property:

\[ C(d, A \otimes c) \cong \text{Set}(A, C(d, c)) \tag{3.15} \]

In [Woo76, Chapter 1, §7] Wood proves that \( \mathcal{M} \)-actegories are equivalent to ‘\( \mathcal{M} \)-powered’ \([\mathcal{M}^{op}, \text{Set}]\)-categories.\(^6\)

3.2.8. Example. [Day convolaction] A given \( \mathcal{M} \)-action \( \bullet \) on \( C \) can be canonically extended to an action of \([\mathcal{M}, \text{Set}]\) on \([C, \text{Set}]\). First of all, the monoidal product of \( \mathcal{M} \) extends canonically to its category of copresheaves \([\mathcal{M}, \text{Set}]\) to a monoidal structure known as Day convolution [Day70]. The extension of \( \bullet \) is given, mutatis mutandis, in the same way:

\[ (M \bullet_{\text{Day}} P)(c') := \int_{m : \mathcal{M}, c \in C} C(m \bullet c, c') \times M(m) \times P(c). \tag{3.16} \]

We call this action Day convolaction. As for Day convolution, this action is well-defined also when \( \text{Set} \) is replaced with a Bénabou cosmos [Str74], such as any quantale.

3.2.9. Example. [Gar18, Theorem 9] In ibid., Garner proves that tangent categories are a special kind of actegory. Let \( \mathcal{W} \) be the ‘Weil category’ the subcategory of that of commutative rigs spanned by objects isomorphic to tensor products of rigs of the form \( W_k := \mathbb{N}[x_1, \ldots, x_k]/(x_i x_j, i \leq j \leq k) \). A tangent structure on a category \( C \) is then a \( \mathcal{W} \)-action \( \bullet \) such that \( - \bullet c \) preserves a certain class of limits (called ‘tangent limits’) for all \( c : C \).

3.2.10. Example. [Metric dilation] The category \( \text{Met} \) of metric spaces and short maps receives an action of \((\mathbb{R}^+, \geq, 1, \cdot)\), the strict monoidal category of positive real numbers. The action of a scalar \( m : \mathbb{R}^+ \) on a metric space \((X, d)\) is by scaling: the resulting metric space \((X, m \bullet d)\) is equipped with a metric scaled exactly by \( m \). Maps are left unchanged, since their shortness is not affected by scaling both domain and codomain equally.

3.2.11. Example. [Stochastic action] Let \( \text{Msbl} \) be the category of measurable spaces and measurable maps between them, and let \( \text{Prob} \) be the category of probability spaces and measure preserving maps between them. There is an action of \( \text{Prob} \) on \( \text{Msbl} \) given by

\[ (\Omega, \mathcal{F}, P) \bullet (X, \Sigma_X) := (\Omega \times X, \mathcal{F} \otimes \Sigma_X) \]

where on the right we have the product of measurable spaces. This action can be used to model stochastic processes, after being fed to the functor \( \text{Para} \), as described in [Cap21].

\(^6\)We learned of this fact from [Gar18].
3.2.12. Example. [Ste18, Example 2.1.8] Let $\textbf{Top}$ be the category of topological spaces and continuous maps, fix a space $X$ and consider the frame $\mathcal{O}(X)$ of the open sets of its canonical topology. There’s an action of $(\mathcal{O}(X), \cap, X)$ on $\textbf{Top}/X$ given on objects by

$$U \bullet (f : Y \to X) = f^{-1}U$$

with the subspace topology (3.17) and on maps $\varphi : f \to g$, where $f : Y \to X$ and $g : Z \to X$ by

$$\varphi|_U : f^{-1}U \to g^{-1}U.$$ (3.18)

To see this is well-defined, notice that since $\varphi ; g = f$, the preimage of $g^{-1}U$ along $\varphi$ is exactly $f^{-1}U$. Clearly $U \bullet (V \bullet f) = (U \cap V) \bullet f$ and $X \bullet f = f$, so this is a strict $\mathcal{O}(X)$-action.

3.3. Morphisms. We have unpacked how pseudoalgebras of $\mathcal{M} \times -$ give $\mathcal{M}$-actegories. We continue to unpack the definitions given in Section 2 to get to a full-fledged definition of the 2-category of $\mathcal{M}$-actegories. This is the 2-category $T\text{-}\mathbf{Alg}^{lx}$ (Definition 2.2.4) instatiated for $T = \mathcal{M} \times -$.

3.3.1. Definition. For a fixed monoidal category $\mathcal{M}$ we denote the 2-category of $\mathcal{M}$-actegories, lax $\mathcal{M}$-linear functors and $\mathcal{M}$-linear transformations as $\mathcal{M}\text{-}\mathbf{Act}^{lx}$.

To get morphisms of $\mathcal{M}$-actegories, it is sufficient to instantiate Definition 2.2.2 for the pseudomonad $\mathcal{M} \times -$:

3.3.2. Definition. Let $(\mathcal{C}, \bullet, \eta^*, \mu^*)$ and $(\mathcal{D}, \circ, \eta^0, \mu^0)$ be left $\mathcal{M}$-actegories. A lax $\mathcal{M}$-linear functor between them is a functor $F : \mathcal{C} \to \mathcal{D}$ equipped with a natural morphism

$$\ell_{m,c} : m \circ F(c) \longrightarrow F(m \bullet c) \quad \text{for } m : \mathcal{M}, c : \mathcal{C},$$

called lineator satisfying the following coherence laws:

$$m \circ (n \circ F(c)) \xrightarrow{\mu^0_{m,n,F(c)}} m \circ F(n \bullet c) \xrightarrow{\ell_{m,n,c}} m \circ F(n \bullet c)$$

$$\xrightarrow{\ell_{m \otimes n, c}} (m \otimes n) \circ F(c)$$

$$\xleftarrow{\ell_{m \otimes n, c}} F((m \otimes n) \bullet c)$$

$$\xleftarrow{F(\mu^0_{m,n,c})} F(m \bullet (n \bullet c))$$

(3.20)
3.3 Morphisms

\[ j \circ F(c) \xrightarrow{\ell_{j,c}} F(j \bullet c) \]

When \( \ell \) is invertible, we call \((F, \ell)\) simply \( \mathcal{M} \)-linear functor, or strong \( \mathcal{M} \)-linear functor if we want to emphasize that.

Morphisms between actegories on different monoidal categories will be defined in Proposition 3.6.5.

3.3.3. Remark. The coherence diagrams in this definition (and in some of the later ones) can be interpreted as inductive definitions for the structure morphisms (here \( \ell \)). It’s easy to spot this phenomenon in the above triangular diagram, where \( \ell_{j,c} \) is being pinned to be \( \eta_{F(c)}^{-1} \circ F(\eta_c^\bullet) \). The same can be said for the pentagon, though: it’s a definition of \( \ell_{m \otimes n, c} \) in terms of \( \ell_{m, c} \) and \( \ell_{n, c} \):

\[ \ell_{m \otimes n, c} = \mu_{m, n, F(c)}^{-1} \circ (m \circ \ell_{n, c}) \circ \ell_{m, n} \circ F(\mu_{n, m, c}) \]  

(3.22)

This gets particularly interesting when \( C \) and \( D \) are strict actegories, in which case \( \ell_{j,c} \) is forced to be the identity and \( \ell_{m \otimes n, c} \) is simply \( (m \circ \ell_{n, c}) \circ \ell_{m, n} \).

3.3.4. Example. [Monoidal functors are \( \mathbb{N} \)-linear] Following up from Example 3.2.4, let \((F, \epsilon, \mu) : (\mathcal{M}, j, \otimes) \to (\mathcal{N}, i, \odot)\) be a lax monoidal functor. Then we can form a \( \mathbb{N} \)-linear functor \((F, \mu^\#) : (\mathcal{M}, (=) \odot) \to (\mathcal{N}, (=) \odot),\) whose lineator

\[ \mu^\# : F(m) \odot \ldots \odot F(m) \to F(m \odot \ldots \odot m) \]  

(3.23)

is defined by repeated applications of the laxator \( \mu \).

3.3.5. Example. Let \( \text{Meas} \) be the category of measure spaces and non-expanding maps. Analogously to Example 3.2.10, \( \mathbb{R}^+ \) acts by dilation on this category, this time dilating the measure instead of the metric.

For \( d > 0 \), there is a functor \( H^q : \text{Met} \to \text{Meas}, \) mapping each metric space \((X, d)\) to the measure space \((X, \mathcal{B}(X), H^q_X)\) where \( \mathcal{B}(X) \) is the Borel \( \sigma \)-field on \( X \) and \( H^q \) is the \( q \)-dimensional Hausdorff measure defined on a given \( A \in \mathcal{B}(X) \) as:

\[ H^q(A) = \lim_{\delta \to 0} \inf \{ \sum_{i=1}^{\infty} (\text{diam} \ U_i)^q \mid \{ U_i \}_{i \in \mathbb{N}} \text{ covers } A \text{ and } \text{diam} \ U_i < \delta \ \forall i \in \mathbb{N} \}. \]

A map \( f : (X, \Sigma_X, \mu) \to (Y, \Sigma_Y, \nu) \) is non-expanding if \( f_* \mu(A) \leq \nu(A) \) for every \( A \in \Sigma_Y. \)
Since maps in $\text{Met}$ are short, it’s easy to see the induced measurable maps are non-expanding. When $q = 1$, this functor is $\mathbb{R}^+$-linear, that is, there are maps $r \cdot H^q(X, d) \to H^q(X, r \cdot d)$. In fact, on the domain we have the space $X$ with measure $r \cdot H^q$, while on the codomain we have the same space with measure $r^q \cdot H^q$. Since $r \leq r^q$ iff $r \geq 1$, one is forced to choose $q = 1$.

Suppose instead we define an action of $(\mathbb{R}^+, 1, \cdot, \geq)$ on $\text{Msbl}$ to be $r \ast (X, d) := e^r \cdot (X, d)$, and analogously for $\text{Meas}$. Now the functors $H^q$ are all lax $\mathbb{R}^+$-linear since $e^r \geq 1$ as long as $r \geq 0$. $H^1$ remains the only strong $\mathbb{R}^+$-linear functor among them.

One of the main properties of lax linear functors is that they send actions to actions:

**3.3.6. Lemma.** Let $(F, \ell) : (\mathcal{C}, \bullet) \to (\mathcal{D}, \circ)$ be a lax $\mathcal{M}$-linear functor. Given a monoid $(m, i, \cdot)$ in $\mathcal{M}$ and an action $\ast : m \bullet c \to c$ of $m$ on $c : \mathcal{C}$, then

$$m \bullet F(c) \xrightarrow{\ell_{m,c}} F(m \bullet c) \xrightarrow{F(\ast)} F(c) \quad (3.24)$$

is an action of $m$ on $F(c)$ in $\mathcal{D}$.

**Proof.** That $\ell_{m,c} \circ F(\ast)$ is an action of $m$ on $F(c)$ as witnessed by the following diagrams:

$$\begin{array}{cccc}
m \bullet F(c) & \xrightarrow{1 \bullet F(c)} & j \bullet F(c) \\
\ell_{m,c} & & & \\
F(m \bullet c) & \xleftarrow{F(1 \bullet c)} & F(j \bullet c) & (3.21) \\
\ast & (3.10 \text{ left}) & & \\
F(c) & & & F(c) \quad (3.25)
\end{array}$$
3.3.7. Remark. Dually, oplax linear functors preserve coactions of comonoid, hence strong linear functors preserve Hopf modules, i.e. biactions of bimonoids [BW03, 14.1]. Finally, we instance Definition 2.2.3:

3.3.8. Definition. An \( M \)-linear transformation \( \xi : (F, \ell) \Rightarrow (G, \nu) \) between two lax linear functors \( F, G : (\mathcal{C}, \bullet) \to (\mathcal{D}, \circ) \) is a natural transformation between the carrier functors of \( F \) and \( G \) such that

\[
\begin{align*}
\mu_{m,m,F(c)} & \quad m \circ (m \circ F(c)) \\
m \circ (m \circ F(c)) & \quad m \circ m \circ F(c) \\
(3.20) & \\
F((m \otimes m) \bullet c) & \quad F(m \bullet (m \bullet c)) \\
F(m \bullet (m \bullet c)) & \quad m \circ F(m \bullet c) \\
F(\bullet c) & \quad F(m \bullet c) \\
F(m \bullet c) & \quad F(c) \\
(3.10 \text{ right}) & \\
F(c) & \quad F(\bullet c) \\
F(\bullet c) & \quad F(c) \\
(3.10 \text{ right}) & \\
m \circ ((m \bullet c) \otimes m) & \quad m \circ (m \bullet c) \\
(3.27) & \\
m \circ (m \circ F(c)) & \quad m \circ G(c) \\
\xi_{m,c} & \quad \nu_{m,c} \\
\xi_{m,c} & \quad \nu_{m,c} \\
(3.27) &
\end{align*}
\]

commutes for every \( m : M, c : C \).

3.3.9. Example. Following up from Example 3.3.4, a monoidal natural transformation \( \alpha : R \Rightarrow S \) gives rise to a \( N \)-linear transformation whose linearity condition follows from monoidality of \( \alpha \).

3.4. Strictification. General actegories are pseudoalgebras for \( M \times - \). We call strict algebras of \( M \times - \) strict actions or strict actegories. The reader might be glad to learn every actegory can be strictified. This result is the actegorical analogue of MacLane’s coherence theorem for monoidal categories [JY21, Theorem 1.2.23]. Indeed, the proof
is very similar: higher structure morphisms are simply moved from the actegory to the equivalence.

3.4.1. **Lemma.** Let $\mathcal{M}$ be strict monoidal, and suppose $(\mathcal{C}, \bullet)$ is a left $\mathcal{M}$-actegory. Then there exists a strict $\mathcal{M}$-actegory $(\mathcal{C}_s, \bullet_s)$ and an equivalence $S : \mathcal{C}_s \to \mathcal{C}$ in $\mathcal{M}$-$\text{Act}$.

**Proof.** A proof of this fact can be obtained by appealing to the general strictification result for algebras of pseudomonads with rank,\(^8\) namely [BKP89, Theorem 3.13]. $\mathcal{M} \times -$ has rank since it preserves all colimits by virtue of being left adjoint.

Concretely, the objects of $\mathcal{C}_s$ are pairs $(m : \mathcal{M}, c : \mathcal{C})$ and its morphisms $f : (m, c) \to (n, d)$ are given by arrows $m \bullet c \to n \bullet d$ in $\mathcal{C}$. Then $S$ is identity on morphisms and on objects sends $(m, c)$ to $m \bullet c$.

3.4.2. **Remark.** In [BKP89, §4], $(\mathcal{C}_s, \bullet_s)$ is denoted by $(\mathcal{C}, \bullet)'$. It is also noted that strictification defines a functor left adjoint to the inclusion $\mathcal{M}^s$-$\text{Act}_s \to \mathcal{M}_s$-$\text{Act}$.

As a consequence of this theorem, we may in general assume that an actegory is strict, similar to the way we routinely assume that monoidal categories are strict, although one has to be careful about which statements can actually be transported along actegory equivalences, especially if we are considering actions of multiple different monoidal categories. A more precise statement is that any diagram which commutes under the assumption of strictness, automatically commutes.

3.4.3. **Corollary.** There is an equivalence of 2-categories:

$$\mathcal{M}$-$\text{Act}^{lx} \simeq (\mathcal{M}^s \times -)$-$\text{Alg}_s^{lx}.$$(3.28)

where on the left we have the 2-category of $\mathcal{M}$-actegories, lax linear functors and linear transformations and on the right we have the 2-category of strict $\mathcal{M}$-actegories, lax linear functors and linear transformations.

**Proof.**

$$\mathcal{M}$-$\text{Act}^{lx} = (\mathcal{M} \times -)$-$\text{Alg}_{ps}^{lx} \xrightarrow{\text{MacLane}} (\mathcal{M}^s \times -)$-$\text{Alg}_{ps}^{lx} \xrightarrow{\text{Lemma 3.4.1}} (\mathcal{M}^s \times -)$-$\text{Alg}_s^{lx}.$$(3.29)

\(^8\)i.e. preserving $\kappa$-small colimits for some cardinal $\kappa$. 
3.4.4. REMARK. Note that Lemma 3.4.1 allows us to strictify objects but not morphisms: indeed, in the statements of the equivalence of Corollary 3.4.3, we still have non-strict morphisms on both sides! Anyway, we observe Corollary 3.4.3 still holds if we replace ‘lax linear’ morphisms with ‘strong linear’ morphisms (i.e. $lx$ with $ps$ on both sides of the equivalence).

Moreover, we have the following:

3.4.5. LEMMA. Let $(\mathcal{M} \times \mathcal{C}, \bullet)$ be a free $\mathcal{M}$-actegory, for $\mathcal{M}$ strict monoidal, and let $(\mathcal{D}, o)$ be any other strict $\mathcal{M}$-actegory. Then given a strong $\mathcal{M}$-linear functor $(F, \ell) : (\mathcal{M} \times \mathcal{C}, \bullet) \to (\mathcal{D}, o)$, there is a unique strict linear functor $(\tilde{F}, =) : (\mathcal{M} \times \mathcal{C}, \bullet) \to (\mathcal{D}, o)$ linearly and naturally isomorphic to $F$.

PROOF. For existence, define

$$\tilde{F}(m, c) = m \circ F(j, c).$$

(3.30)

This functor is strictly linear:

$$m \circ \tilde{F}(n, c) = m \circ (n \circ F(j, c)) = (m \otimes n) \circ F(j, c) = \tilde{F}(m \otimes n, j) = \tilde{F}(m \bullet (n, c)).$$

(3.31)

The natural isomorphism $\gamma_{m,c} : \tilde{F}(m, c) \Rightarrow F(m, c)$ is given by $\ell$ itself:

$$\gamma_{m,c} := \tilde{F}(m, c) = m \circ F(j, c) \xrightarrow{\ell_{m,(j,c)}} F(m \otimes j, c) = F(m, c).$$

(3.32)

To see it is linear we draw the following:

$$m \circ \tilde{F}(n, c) \xrightarrow{m \circ \ell_{n,c}} m \circ F(n, c) \xrightarrow{\ell_{m,(n,c)}} F(m \otimes n, c) \xrightarrow{\gamma_{m \otimes n, c}} F(m \otimes n, c) \xrightarrow{\ell_{m \otimes n, (j,c)}} (m \otimes n) \circ F(j, c)$$

(3.33)

The external perimeter corresponds to Equation (3.22) for the strict case.

For uniqueness, notice the above diagram (drawn for $m = j$) forces $\gamma$ to track $\ell$. ■
If the reader is interested in a deeper understanding of strictification, in [BKP89, §3] the equivalence of strict and pseudoalgebras is analyzed in detail, and more refined statements of the strictification lemma are provided (e.g. regarding exactly how it looks on morphisms and transformations of pseudoalgebras).

3.5. The Free and Cofree Adjunctions. If the reader is acquainted with the relationships between adjunctions and monads, the following is unsurprising:

3.5.1. Proposition. There is a 2-adjunction

\[
\begin{array}{c}
\mathcal{M}\text{-}\text{Act}^\text{lx} \\
\downarrow \quad \downarrow \\
\mathcal{C}at \quad \mathcal{M}[\dashv] \\
\end{array}
\]

(3.34)

where \(U\) forgets the actegorical structure and \(\mathcal{M}[\dashv]\) is the free \(\mathcal{M}\)-actegory functor.

Proof. By Corollary 3.4.3, \(\mathcal{M}\text{-}\text{Act}^\text{lx}\) is equivalent to \((\mathcal{M}^\text{a} \times \dashv)\text{-}\text{Act}^\text{lx}\). By general results about 2-monad theory ([BKP89, 4]), the latter is involved in an Eilenberg–Moore adjunction with \(\mathcal{C}at\). Pre-composing this adjunction with the strictification equivalence yields the desired 2-adjunction.

In lower dimension, the ‘free \(M\)-module’ over a given set \(I\) is well-known to be given by direct sum of \(I\)-many copies of \(M\). An analogous fact is true for actegories:

3.5.2. Proposition. The free \(\mathcal{M}\)-actegory construction is a \(\mathcal{C}at\)-copower:

\[
\mathcal{M}[C] \simeq \mathcal{C} \otimes \mathcal{M}.
\]

(3.35)

Proof. The \(\mathcal{C}\)-copower of \(\mathcal{M}\) in \(\mathcal{M}\text{-}\text{Act}\) is defined by the \(\mathcal{C}\)-weighted colimit of the one-object diagram at \(\mathcal{M}\) [Kel82, §3.7], thus we have

\[
\mathcal{M}\text{-}\text{Act}(\text{colim}^C \mathcal{M}, \mathcal{X}) \cong \mathcal{C}at(\mathcal{C}, \mathcal{M}\text{-}\text{Act}(\mathcal{M}, \mathcal{X})) \cong \mathcal{C}at(\mathcal{C}, U_M(\mathcal{X})).
\]

(3.36)

This shows that \(\dashv \mathcal{M} \dashv U_M\), implying \(\mathcal{C} \otimes \mathcal{M} \simeq \mathcal{M}[C]\).

We also note there exists a cofree construction for actegories, given by \([\mathcal{M}, \dashv]\):

\[
* : \mathcal{M} \times [\mathcal{M}, \mathcal{C}] \to [\mathcal{M}, \mathcal{C}]
\]

(3.37)

\[
(m, F) \mapsto F(m \otimes -).
\]

Indeed, we have the following:
3.5 The free and cofree adjunctions

3.5.3. Proposition. There is a comonadic 2-adjunction:

\[
\begin{array}{ccc}
\mathcal{M}\mathrm{-}\mathbf{Act}^\mathrm{lx} & \xleftarrow{U_M} & \mathbf{Cat} \\
\downarrow \mathcal{M}, - & \downarrow \ & \downarrow \\
\end{array}
\] (3.38)

exhibiting \(\mathcal{M}\mathrm{-}\mathbf{Act}^\mathrm{lx}\) as the category of pseudocoalgebras of the pseudocomonad \([\mathcal{M}, -]\), whose structure is given by:

\[
k_C : [\mathcal{M}, \mathcal{C}] \to \mathcal{C}
F \mapsto F(j)
\]

\[
w_C : [\mathcal{M}, \mathcal{C}] \to [\mathcal{M}, [\mathcal{M}, \mathcal{C}]]
F \mapsto (m \mapsto F(m \otimes -)),
\]

plus the obvious left/right counit and coassociator.

Proof. This is a consequence of the tensor-hom adjunction in \(\mathbf{Cat}\):

\[
\{ \bullet : \mathcal{M} \times \mathcal{C} \to \mathcal{C} \} \cong \{ \bullet : \mathcal{C} \to [\mathcal{M}, \mathcal{C}] \}
\] (3.40)

One easily verifies the rest of a pseudoalgebra structure for \(\mathcal{M} \times -\) is carried along to a pseudocoalgebra structure of \([\mathcal{M}, -]\), and the same happens to morphisms and 2-cells. ■

Hence actegories also admit a presentation as coalgebras. The usefulness of such presentation can be seen, for instance, in Garner’s result we reported in Example 3.2.9. In fact that’s saying that tangent categories are coalgebras of the pseudocomonad \(\mathbf{Tang}(\mathcal{W}, -)\), i.e. the subcomonad of \([\mathcal{W}, -]\) obtained by restricting to functors preserving tangent limits.

3.5.4. Remark. The coalgebraic presentation of actegories has a relative one dimension down. In fact coalgebras of \(\mathcal{M} \times -\), for \(\mathcal{M} = \Sigma^*\) a free monoid on a given (usually finite) alphabet \(\Sigma = \{a, b, \ldots\}\) are ‘half of an automaton’, lacking a choice of output map. Nevertheless, this ‘dynamical’ interpretation of actegories seems worthy to explore. In particular we might ask: what’s a 1-automaton, i.e. an automaton whose underlying states form a category instead of a mere set, and likewise does its alphabet?

Finally, knowing \(\mathcal{M}\mathrm{-}\mathbf{Act}^\mathrm{lx}\) is monadic and comonadic has various technical advantages, among which:

3.5.5. Corollary. \(\mathcal{M}\mathrm{-}\mathbf{Act}^\mathrm{lx}\) admits all pseudo and lax limits and colimits.

Proof. [BKP89, §2] proves that categories of pseudoalgebras admit all pseudo and lax limits. By duality we obtain the rest. ■
3.6. The indexed 2-category $\mathbf{Act}$. So far we have been focusing on $\mathcal{M}$-actegories, for a single monoidal base $\mathcal{M}$. In the below proposition we show that actegories admit a canonical notion of ‘change of base’, or, as it is known in algebra, ‘restriction of scalars’.

3.6.1. Proposition. Let $(\mathcal{M}, j, \otimes)$ and $(\mathcal{N}, i, \odot)$ be monoidal categories. Then every strong monoidal functor $R : \mathcal{M} \to \mathcal{N}$ induces a 2-functor $R^* : \mathcal{N}^{\mathbf{Act}^{lax}} \to \mathcal{M}^{\mathbf{Act}^{lax}}$ given by restriction of scalars:

$$R^*(C, \bullet) := (C, \mathcal{M} \times C \xrightarrow{R \times C} \mathcal{N} \times C \xrightarrow{\bullet} C). \quad (3.41)$$

Proof. First of all, let’s complete the definition on objects. Denote by $(\epsilon, \varpi)$ the monoidal structure of $R$. Then the unitor of $R^*(C, \bullet)$ is given by

$$\eta_{R^* \bullet}^C : c \mapsto i \bullet c \xrightarrow{\epsilon \bullet c} R(j) \bullet c \quad (3.42)$$

while the multiplicator is defined as

$$\mu_{R^* \bullet}^{m,m',c} : R(m) \bullet (R(m') \bullet c) \xrightarrow{\mu_{R(m),R(m'),c}^R} (R(m) \odot R(m')) \bullet c \xrightarrow{\varpi_{m,m',c}} R(m \odot m') \bullet c. \quad (3.43)$$

Now let $(C, \bullet)$ and $(D, \circ)$ be $\mathcal{N}$-actegories and $(F, \ell)$ a lax $\mathcal{N}$-linear functor between them. We define the action of $R^*$ on this to be $R^*(F, \ell) := (F, R^* \ell)$ where $R^* \ell$ denotes the 2-cell:

$$\begin{array}{ccc}
\mathcal{M} \times C & \xrightarrow{\mathcal{M} \times F} & \mathcal{M} \times D \\
R \times C \downarrow & & \downarrow R \times D \\
\mathcal{N} \times C & \xrightarrow{\mathcal{N} \times F} & \mathcal{N} \times D \\
\bullet & \xrightarrow{\ell} & \circ \\
C & \xrightarrow{F} & D
\end{array} \quad (3.44)$$

defined by whiskering along $R \times C$, hence whose components are

$$R^* \ell_{n,c} : R(n) \circ F(c) \xrightarrow{\sim} F(R(n) \bullet c). \quad (3.45)$$

Finally, let $\varphi : (F, \ell) \Rightarrow (G, \nu)$ be an $\mathcal{M}$-linear natural transformation. Since $F$ and $G$ are mapped to themselves, $\varphi$ is still a well-defined natural transformation. Its $\mathcal{N}$-linearity can be concluded by observing that the squares which have to commute (3.27) are a subset of those that commute by assumption of $\mathcal{M}$-linearity (since every action of a scalar $n : \mathcal{N}$ is mediated by the $\mathcal{M}$-action).

Functoriality and 2-functoriality can be verified by routine. \hfill $\blacksquare$
3.6 The indexed 2-category $\text{Act}$

3.6.2. Remark. In Proposition 4.4.9, we prove each of these functors $R^*$ is actually a right adjoint, with its left adjoint $R_!$ being known as *extension of scalars*.

3.6.3. Example. In Example 3.2.12 we described an action of $\mathcal{O}(X)$ on $\text{Top}/X$, for a given topological space $X$. Observe this action can be obtained from the canonical left self-action (Example 3.2.3) of $(\text{Top}/X, 1, \times_X)$ by restriction along the evident monoidal functor $\mathcal{O}(X) \to \text{Top}/X$.

Furthermore, a monoidal natural transformation between monoidal functors induces a natural transformation between the corresponding restriction functors:

3.6.4. Proposition. Given strong monoidal functors $R, S : \mathcal{M} \to \mathcal{N}$ and a monoidal natural transformation $\alpha : R \Rightarrow S$, there is a strictly 2-natural transformation (notice the change in direction)

$$\alpha^* : S^* \Rightarrow R^* \quad (3.46)$$

between the corresponding restrictions $S^*, R^* : \mathcal{N}\text{-}\text{Act}^{\text{lx}} \to \mathcal{M}\text{-}\text{Act}^{\text{lx}}$.

**Proof.** For each category $(D, \circ)$ in $\mathcal{N}\text{-}\text{Act}^{\text{lx}}$ the natural transformation $\alpha^*$ picks out a lax $\mathcal{M}$-linear morphism $(1_D, \ell^\alpha_{(D, \circ)}) : S^*(D, \circ) \to R^*(D, \circ)$ defined as

$$\ell^\alpha_{(D, \circ)} := \alpha_m \circ d : R(m) \circ d \to S(m) \circ d. \quad (3.47)$$

It’s evident $\ell^\alpha_{(D, \circ)}$ is strictly natural because is the whiskering of strictly natural transformations, and monoidality of $\alpha$ makes $\ell^\alpha_{(D, \circ)}$ a well-defined lineator (i.e. its coherence as monoidal transformation maps to coherence for $\ell^0_{(D, \circ)}$). The fact $\ell^\alpha$ is natural amounts to check that for a given lax $\mathcal{N}$-linear functor $(F, \nu) : (D, \circ) \to (C, \bullet)$, the following commutes:

$$\begin{align*}
R(m) \bullet F(d) & \xrightarrow{\alpha_m \bullet F(d)} S(m) \bullet F(d) \\
\downarrow^{\nu_{R(m), d}} & \downarrow^{\nu_{S(m), d}} \\
F(R(m) \circ d) & \xrightarrow{F(\alpha_m \circ d)} F(S(m) \circ d)
\end{align*} \quad (3.48)$$

But this is a naturality square for $\ell$, hence commutes by assumption. ■
3.6.5. Proposition. The assignment $\mathcal{M} \mapsto \mathbf{M-Act}^{\text{lx}}$ extends to an indexed 2-category

$$(-)\text{-}\mathbf{Act}^{\text{lx}} : \text{MonCat}^{\text{coop}} \to 2\text{Cat};$$

whose action on a strong monoidal functor $R : \mathcal{M} \to \mathcal{N}$ is defined in Proposition 3.6.1 and whose action on a monoidal natural transformation $\alpha : R \Rightarrow S$ is defined in Proposition 3.6.4.

By effecting a bicategorical Grothendieck construction [JY21, Definition 10.7.2], we obtain a 2-category of actegories over an arbitrary base that we denote by $\mathbf{Act}$.

The 2-category $\mathbf{Act}$ has pairs $(\mathcal{M} : \text{MonCat}, (\mathcal{C}, \bullet) : \mathcal{M}\text{-}\mathbf{Act}^{\text{lx}})$ as objects. A morphism from an $\mathcal{M}$-actegory $(\mathcal{C}, \bullet)$ to an $\mathcal{N}$-actegory $(\mathcal{D}, \circ)$ consists of a strong monoidal functor $R : \mathcal{M} \to \mathcal{N}$, and a lax $\mathcal{M}$-linear functor between between actegories $(\mathcal{C}, \bullet)$ and $(\mathcal{D}, (R \times \mathcal{D}) \circ \circ)$. This lax linear functor itself consists of a functor $R^\sharp : \mathcal{C} \to \mathcal{D}$ and a lineator $\ell_{m,c} : R(m) \circ R^\sharp(c) \rightsquigarrow R^\sharp(m \bullet c)$. All in all, the data of a morphism in $\mathbf{Act}$ consist of a triple $(R, R^\sharp, \ell)$ that can be neatly arranged in a lax commutative square:

$$
\begin{array}{ccc}
\mathcal{M} \times \mathcal{C} & \xrightarrow{\mathcal{M} \times R^\sharp} & \mathcal{M} \times \mathcal{D} \\
\downarrow & & \downarrow R \times \mathcal{D} \\
\mathcal{C} & \xrightarrow{R^\sharp} & \mathcal{D}
\end{array}
$$

A 2-cell in $\mathbf{Act}$ between $(R, R^\sharp, \ell)$ and $(S, S^\sharp, \nu)$ consists of a pair $(\alpha, \alpha^\sharp)$, where $\alpha : R \Rightarrow S$ is a monoidal natural transformation and $\alpha^\sharp$ is an $\mathcal{M}$-linear transformation filling the following lax triangle in $\mathcal{M}\text{-}\mathbf{Act}^{\text{lx}}$:

$$
\begin{array}{ccc}
(\mathcal{C}, \bullet) & \xrightarrow{(R^\sharp, \ell)} & R^\sharp(\mathcal{D}, \circ) \\
\downarrow \alpha^\sharp & & \downarrow \alpha^\sharp(\mathcal{D}, \circ) \\
(S^\sharp, \nu) & \xrightarrow{(S^\sharp, \nu)} & S^\sharp(\mathcal{D}, \circ)
\end{array}
$$
3.6 The indexed 2-category $\mathbb{A}ct$

3.6.6. Remark. There is a cartesian factorization system on $\mathbb{A}ct$ deriving from the indexing it was born from. Its left maps are vertical maps, i.e. lax linear functors as described in Definition 3.3.2, living inside a given fibre. Right maps are cartesian maps, which turn out to be maps between different actegorical structures on a fixed category $\mathcal{C}$.

\[(R, 1_{R^*(\mathcal{C}, \bullet)}, =) : (\mathcal{M}, (\mathcal{C}, \bullet)) \rightarrow (\mathcal{N}, (\mathcal{C}, \circ)).\] (3.52)

Additionally, 2-cells between such morphisms are 2-cells $(\alpha, =)$ in $\mathbb{A}ct$, i.e. 2-cells whose vertical part is trivial. We already have a name for the ‘vertical subcategories’ of $\mathbb{A}ct$, namely its fibres $\mathcal{M}-\mathbb{A}ct^x$ as $\mathcal{M}$ varies. Correspondingly, its cartesian subcategories (the ‘transverse fibres’) will be denoted by $\mathbb{A}ct^{\text{cart}}(\mathcal{C})$, as $\mathcal{C}$ varies, not to be confused with $\mathcal{C}-\mathbb{A}ct$, which would denote actions of $\mathcal{C}$.

3.6.7. Example. In Proposition 3.1.6, we’ve seen how every action can be curried in order to get a strong monoidal functor into $[\mathcal{C}, \mathcal{C}]$. We can now rephrase this observation as the fact that the evaluation action

\[
\text{eval} : [\mathcal{C}, \mathcal{C}] \times \mathcal{C} \rightarrow \mathcal{C}
\]

\[(F, c) \mapsto F(c)\] (3.53)

is (pseudo)terminal in $\mathbb{A}ct^{\text{cart}}(\mathcal{C})$, meaning every other action $\bullet$ of a monoidal category $\mathcal{M}$ factors uniquely through it:

\[
\begin{array}{ccc}
[\mathcal{C}, \mathcal{C}] \times \mathcal{C} & \xrightarrow{\text{eval}} & \mathcal{C} \\
\downarrow & & \\
\mathcal{M} \times \mathcal{C} & \xrightarrow{\exists \text{curr(\bullet) } \times \mathcal{C}} & \bullet
\end{array}
\] (3.54)

It is somehow relevant that every monoidal category gives an example of actegory, despite this being quite a trivial fact. Whence, we record the following:

3.6.8. Proposition. The canonical fibration $\pi : \mathbb{A}ct^x \rightarrow \mathbb{M}on\mathbb{C}at$ has a section $\Upsilon : \mathbb{M}on\mathbb{C}at \rightarrow \mathbb{A}ct^x$ that maps each monoidal category to left self-action.

Proof. On objects, this has been defined in Example 3.2.3. On morphisms, a strong monoidal functor $(R, \epsilon, \mu) : (\mathcal{M}, j, \otimes) \rightarrow (\mathcal{N}, i, \odot)$ is sent to the triple $(R, R, \mu)$. The fact that the functors between the scalars and the underlying categories of the actegories are
actually the same strong monoidal functor $R$ allows us to use the laxator $\mu$ as lineator. Indeed, $(R, \mu)$ forms a lax $\mathcal{M}$-linear morphism of actegories $(\mathcal{M}, \otimes) \to R^*(\mathcal{N}, \emptyset)$. Lastly, a monoidal natural transformation $\alpha : R \Rightarrow S$ is sent to the pair $(\alpha, \alpha)$, where the $\mathcal{M}$-linearity of the second component follows from the monoidality of $\alpha$. \hfill \blacksquare
4. Composing actegories

In this section we explore some ways actegories can be combined together to give rise to new ones. Although all of the following can be obtained by general abstract considerations, we deem interesting to explain the way we got some of these definitions, and the way we think about them, coming from operations on parametric morphisms. This will yield the cartesian and cocartesian products (Proposition 4.2.1 and 4.2.3) and an ‘hybrid’ we call external choice (Definition 4.2.2) by analogy with the same operation on open games and servers [CGLF21, VC22]. Besides those, we are going to consider other ways to combine actegories, notably their tensor product (Definition 4.4.3) and the corresponding internal hom construction (Proposition 4.4.6), which are suggested by the algebra of the situation. Throughout this section, let $(\mathcal{M}, j, \otimes)$ and $(\mathcal{N}, i, \odot)$ be two monoidal categories.

4.1. Actegories as parametric morphisms. An action $\bullet : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ can be seen as an $\mathcal{M}$-parametric morphism $\mathcal{C} \to \mathcal{C}$, and as such it can be conceptualized as a process which consumes $\mathcal{C}$, produces $\mathcal{C}$, and whose execution is commanded by a choice of $\mathcal{M}$, which we picture as chosen by an agent ‘guiding’ the execution of the process. Therefore, actions of monoidal categories can be effectively represented as morphisms in $\text{Para}(\text{Cat})$, the bicategory of parametric functors in $\text{Cat}$ (for a definition of $\text{Para}$, see [CGHR21, Definition 2]).

Since $\text{Cat}$ is rich in structure, $\text{Para}(\text{Cat})$ becomes a monoidal category in two ways:

1. By parallel product, that is, by putting parametric morphisms in parallel, thereby taking the product of all three boundaries:

Concretely, we can do this thanks to the symmetry of $\times$:

$$(\mathcal{M}, \bullet) \otimes (\mathcal{N}, \circ) := \mathcal{M} \times \mathcal{N} \times \mathcal{C} \times \mathcal{D} \xrightarrow{\mathcal{M} \times \text{swap} \times \mathcal{D}} \mathcal{M} \times \mathcal{C} \times \mathcal{N} \times \mathcal{D} \xrightarrow{\bullet \otimes \circ} \mathcal{C} \times \mathcal{D}. \quad (4.1)$$

9This ‘game semantics’ of parametric morphisms is discussed in detail in [CGHR21, Cap21].
2. By **external choice** (terminology borrowed from [CGLF21]), hence by summing the boundaries except for the top ones which are multiplied: Concretely, we can do

\[
(M, \bullet) \& (N, \circ) := M \times N \times (C + D) \rightarrow M \times N \times C + M \times N \times D
\]

\[
\xrightarrow{\pi_{M \times C + N \times D}} M \times C + N \times D
\]

\[
\xrightarrow{\bullet \times \circ} C + D.
\]

Moreover, \textbf{Para}(\textbf{Cat}) is a bicategory, so we can also:

3. **Sequentially compose** two parametric morphisms:

\[
(N, \circ) \circ (M, \bullet) := N \times M \times C \xrightarrow{N \times \bullet} N \times C \rightarrow C.
\]

4. **Reparameterise** a given morphism:
Obtaining the morphism

\[ R^*(\mathcal{M}, \bullet) := \mathcal{N} \times \mathcal{C} \xrightarrow{R \times \mathbb{C}} \mathcal{M} \times \mathcal{C} \xrightarrow{\bullet} \mathcal{C}. \] (4.4)

When \( R \) is a strong monoidal functor, this last operation can be recognized as being restriction of scalars along \( R \), as defined in Proposition 3.6.5 (even better, it represents a cartesian morphism in \( \text{Act} \), as observed in Remark 3.6.6). In particular, we know that doing so yields again an action. So actions are stable under reparameterisation along strong monoidal functors.

Can we say the same thing for operations (1)–(3)? We anticipate the answer is yes for parallel product and external choice. For sequential composition, however, we’ll see that to get another action we have to supply additional structure in the form of a compatibility morphism, i.e. a kind of distributive law between the two actions. This will be a wonderful excuse to introduce the two-sided analogue of actegories, biactegories. However, as we’ll see in Section 5, this will not be the only form of sequential composition.

4.2. PRODUCT, COPRODUCT AND EXTERNAL CHOICE OF ACTEGORIES. In what follows, we are going to need a description of the product and coproduct of monoidal categories (we’ve learned about the latter from [Rom20, §6.3]). The cartesian product is easy: the product of \( \mathcal{M} \) and \( \mathcal{N} \) is supported by their product as categories and equipped with the expected componentwise monoidal structure: \( (\mathcal{M} \times \mathcal{N}, (j, i), \otimes \times \otimes) \). Projections are given exactly as in \( \text{Cat} \).

However, their coproduct looks a bit more bizarre. Its construction can be carried out by analogy with that of (non-commutative!) monoids. The objects of \( \mathcal{M} + \mathcal{N} \) consists of
finite words of objects from both \(\mathcal{M}\) and \(\mathcal{N}\), quotiented by the equivalences

\[
\begin{align*}
m_1 m_2 &\sim m_1 \otimes m_2, \quad j \sim e, \\
n_1 n_2 &\sim n_1 \odot n_2, \quad i \sim e,
\end{align*}
\]

where \(e\) denotes the empty word. An analogous construction is carried out on the morphisms. The monoidal product on \(\mathcal{M} + \mathcal{N}\) is given by juxtaposition of words, and the unit is given by \(e\). It’s easy to realize that under the given equivalences, every object (and similarly for morphisms) in \(\mathcal{M} + \mathcal{N}\) can be written uniquely as \(m_1 n_2 \cdots m_k n_k\) for \(m_1, \ldots, m_k : \mathcal{M}\) and \(n_1, \ldots, n_k : \mathcal{N}\). The injections \(\varepsilon_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} + \mathcal{N}\) is simply defined as \(m \mapsto m\), and likewise for \(\varepsilon_{\mathcal{N}}\). Observe these are monoidal on the nose because of the quotients defining \(\mathcal{M} + \mathcal{N}\).

The coproduct of two monoidal categories falls shortly of being a biproduct because the words \(mn\) and \(nm\) are kept distinct. Still, there is a universal morphism \(\Gamma : \mathcal{M} + \mathcal{N} \to \mathcal{M} \times \mathcal{N}\) given by

\[
\Gamma(m_1 n_1 \cdots m_k n_k) := (m_1 \otimes \cdots \otimes m_k, \ n_1 \odot \cdots \odot n_k)
\]

This functor is strict monoidal because of the quotient imposed on the objects of \(\mathcal{M} + \mathcal{N}\).

Having defined the product and coproduct of monoidal categories, we can now describe the product, coproduct, and external choice in \(\textbf{Act}\).

4.2.1. PROPOSITION. [Cartesian product in \(\textbf{Act}\)] The cartesian product of an \(\mathcal{M}\)-actegory \((\mathcal{C}, \bullet, \eta^\bullet, \mu^\bullet)\) and an \(\mathcal{N}\)-actegory \((\mathcal{D}, \circ, \eta^\circ, \mu^\circ)\) is an \(\mathcal{M} \times \mathcal{N}\)-actegory \((\mathcal{C} \times \mathcal{D}, \bullet^\times)\), where \(\bullet^\times\) is the parallel product of \(\bullet\) and \(\circ\) described in Equation (4.1):

\[
\bullet^\times := \mathcal{M} \times \mathcal{N} \times \mathcal{C} \times \mathcal{D} \xrightarrow{\mathcal{M} \times \text{swap} \times \mathcal{D}} \mathcal{M} \times \mathcal{C} \times \mathcal{N} \times \mathcal{D} \xrightarrow{\bullet^\circ} \mathcal{C} \times \mathcal{D}.
\]

PROOF. First of all, let’s be explicit on the structure morphisms \(\bullet^\times\) comes with: its unitor is given by the pair of unitors

\[
\eta^\bullet_{c,d} : (c, d) \xrightarrow{(\eta_\bullet^c, \eta_\circ^d)} (j \bullet c, i \circ d) = (j, i) \bullet^\times (c, d)
\]
and the multiplicator as a pair of underlying multiplicators

\[ \mu_{(c,d)}^\times : (m, n) \mapsto (m', n') \mapsto (c, d) \]

\[ = (m, n) \mu^\times (m' \bullet c, n' \circ d) \]

\[ = (m \bullet (m' \circ c), n \circ (n' \circ d)) \]

\[ \frac{(\mu^\times, \mu'^\times)}{(m \otimes m', n \otimes n')} \mapsto ((m \otimes m') \circ c, (n \otimes n') \circ d) \]

\[ = ((m \otimes m', n \otimes n')) \mu^\times (c, d) \] \hspace{1cm} (4.9)

It’s easy to convince oneself these make \( C \times D \) a well-defined actegory.

To say this is the cartesian product we need to supply projections, and these are built componentwise by the projections in \( \textbf{MonCat} \) and \( \textbf{Cat} \), and happen to be linear on the nose. Finally, consider a \( \mathcal{P} \)-actegory \( (\mathcal{E}, \ast) \) and two morphisms \( (F, F^\sharp, \ell) : (\mathcal{P}, (\mathcal{E}, \ast)) \rightarrow (\mathcal{M}, (\mathcal{C}, \bullet)) \) and \( (G, G^\sharp, \nu) : (\mathcal{P}, (\mathcal{E}, \ast)) \rightarrow (\mathcal{N}, (\mathcal{D}, \circ)) \). Then there is a unique morphism \( ((F, G), (F^\sharp, G^\sharp), (\ell, \nu)) : (\mathcal{P}, (\mathcal{E}, \ast)) \rightarrow (\mathcal{M} \times \mathcal{N}, (\mathcal{C} \times \mathcal{D}, \mu^\times)) \) from the \( \mathcal{P} \)-actegory \( \mathcal{E} \) to the product, given by universal pairing in each component. It’s trivial to check \( (\ell, \nu) \) is indeed a well-defined lineator for \( (F^\sharp, G^\sharp) \), since coherence can be proven componentwise.

We now proceed to describe the external choice in \( \textbf{Act} \), which will be a crucial ingredient in defining the coproduct in \( \textbf{Act} \).

4.2.2. Definition. [External choice product in \( \textbf{Act} \)] The external choice product of an \( \mathcal{M} \)-actegory \( (\mathcal{C}, \bullet, \eta^\bullet, \mu^\bullet) \) and an \( \mathcal{N} \)-actegory \( (\mathcal{D}, \circ, \eta^\circ, \mu^\circ) \) is a \( \mathcal{M} \times \mathcal{N} \)-actegory \( (\mathcal{C} + \mathcal{D}, \mu^\times) \), where \( \mu^\times \) is defined as the external choice of \( \bullet \) and \( \circ \) described in Equation (4.2):

\[ \mu^\times := \mathcal{M} \times \mathcal{N} \times (\mathcal{C} + \mathcal{D}) \rightarrow \mathcal{M} \times \mathcal{N} \times \mathcal{C} + \mathcal{M} \times \mathcal{N} \times \mathcal{D} \]

\[ \xrightarrow{\pi_{\mathcal{M}, \mathcal{N} : \mathcal{C} + \mathcal{D}}} \mathcal{M} \times \mathcal{C} + \mathcal{N} \times \mathcal{D} \]

\[ \xrightarrow{\times, \circ} \mathcal{C} + \mathcal{D}. \] \hspace{1cm} (4.10)

The unitor \( \eta^\times : c \rightarrow (j, i)\mathcal{E}^\times c \) and the multiplicator \( \mu^\times : (m, n)\mathcal{E}^\times ((m', n')\mathcal{E}^\times c) \rightarrow ((m \otimes m', n \otimes n'))\mathcal{E}^\times c \) are defined by cases to coincide with \( \eta^\bullet \) and \( \mu^\bullet \) on \( \mathcal{C} \) and \( \eta^\circ \) and \( \mu^\circ \) on \( \mathcal{D} \).

Restricting scalars (or ‘reparametrising’ if we want to use the language of parametric morphisms) of the above defined external choice product along the natural morphism \( \Gamma \) defined in Remark 4.6, we obtain the coproduct of actegories:
4.2.3. Proposition. [Coproduct in \( \mathbf{Act} \)] The coproduct of an \( \mathcal{M} \)-actegory \( (\mathcal{C}, \circ, \eta^*, \mu^*) \) and an \( \mathcal{N} \)-actegory \( (\mathcal{D}, \circ, \eta^0, \mu^0) \) is an \( \mathcal{M} + \mathcal{N} \)-actegory \( \mathcal{C} + \mathcal{D} \) whose underlying functor \( \mathcal{C} + \mathcal{D} \) is defined as:

\[
\mathcal{C} + \mathcal{D} := \Gamma^*(\mathcal{C} + \mathcal{D}) = (\mathcal{M} + \mathcal{N}) \times (\mathcal{C} + \mathcal{D}) \xrightarrow{\Gamma \times (\mathcal{C} + \mathcal{D})} \mathcal{M} \times \mathcal{N} \times (\mathcal{C} + \mathcal{D}) \xrightarrow{\Gamma^k} \mathcal{C} + \mathcal{D}.
\]

Proof. Injections out of \( (\mathcal{M} + \mathcal{N}, (\mathcal{C} + \mathcal{D}, \mathcal{C}^+)) \) are built out of those of \( \mathbf{MonCat} \) and \( \mathbf{Cat} \), and happen to be linear on the nose. Consider a \( \mathcal{P} \)-actegory \( (\mathcal{E}, \circ) \) and two morphisms \( (F, F^\sharp, \ell) : (\mathcal{M}, (\mathcal{C}, \circ)) \to (\mathcal{P}, (\mathcal{E}, \circ)) \) and \( (G, G^\sharp, \nu) : (\mathcal{N}, (\mathcal{D}, \circ)) \to (\mathcal{P}, (\mathcal{E}, \circ)) \). Then there is a unique morphism \( ((F, G), (F^\sharp, G^\sharp), (\ell, \nu)) : (\mathcal{M} + \mathcal{N}, (\mathcal{C} + \mathcal{D}, \mathcal{C}^+)) \to (\mathcal{P}, (\mathcal{E}, \circ)) \) from the coproduct to \( \mathcal{E} \), given by universal copairing in each component. It’s trivial to check \( (\ell, \nu) \) is indeed a well-defined lineator for \( (F^\sharp, G^\sharp) \), since coherence can be proven by cases.

We can represent the coproduct of actegories with sheet diagrams of parametric morphisms:

4.2.4. Remark. These two monoidal structures on \( \mathbf{Act} \) specialize to each of its fibres thanks to the (co)cartesian structure of \( \mathbf{MonCat} \) [Shu08, Theorem 12.7]. Given two \( \mathcal{M} \)-actegories \( (\mathcal{C}, \circ) \) and \( (\mathcal{D}, \circ) \), we obtain another \( \mathcal{M} \)-actegory by first taking their product in \( \mathbf{Act} \) and then restricting along the copy functor \( \Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M} \) to turn the product \( \mathcal{M} \times \mathcal{M} \)-actegory into an \( \mathcal{M} \)-actegory. Concretely, we get an action supported by the functor:

\[
m_{\mathcal{M}}^\times(c, d) := (m \circ c, m \circ d).
\]

The case for the coproduct is analogous. Notably, this restriction of the product of \( \mathbf{Act} \) to
4.3 Biactegories

a fibre $\mathcal{M}$-$\text{Act}^\times$ yields the product in the latter category. That is, $\Delta^*(\mathcal{C} \times \mathcal{D}, \bullet \times \circ)$ is the product of $(\mathcal{C}, \bullet)$ and $(\mathcal{D}, \circ)$ in $\mathcal{M}$-$\text{Act}^\times$; and likewise applies to $\oplus$.

4.2.5. Remark. The ‘transverse fibres’ $\text{Act}^\text{cart}(\mathcal{C})$ (Remark 3.6.6) also have products and coproducts. These are easily described by deploying the fact illustrated in Example 3.6.7 and Proposition 3.1.6 that actions in $\mathcal{C}$ are exactly given by monoidal functors in $[\mathcal{C}, \mathcal{C}]$.

The coproduct action is thus induced by universal copairing (as shown in [Rom20, §6.3]):

\[
\begin{align*}
\mathcal{M} \xrightarrow{\epsilon_M} \mathcal{M} + \mathcal{N} & \xrightarrow{\epsilon_N} \mathcal{N} \\
\text{curr}(\bullet) & \downarrow \exists \text{curr}(\epsilon^+_\text{cart}) \\
[\mathcal{C}, \mathcal{C}] & \xleftarrow{\text{curr}(\circ)}
\end{align*}
\]

(4.13)

Coproducts in $\text{Act}^\text{cart}(\mathcal{C})$ have particular relevance in the theory of optics since they provide the mathematical basis for hybrid composition of optics, that is, for composing different classes of optics (the original motivation behind the profunctor encoding [PWG17]). Indeed, both optics for $\bullet$ and for $\circ$ can be embedded in the category of optics for $\epsilon^+_{\text{cart}}$, and by universal property of $\oplus$, the latter is the smallest category of optics with this property. For instance, lenses (stemming from the self-action of a cartesian monoidal category) and prisms (stemming from the self-action of a cocartesian monoidal category) ‘compose’ to affine traversals. A much more detailed discussion of this can be found in ibid.

We can also define a product action, by pullback:

\[
\begin{align*}
\mathcal{M} \times_{[\mathcal{C}, \mathcal{C}]} \mathcal{N} \xrightarrow{p_M} \mathcal{M} & \xleftarrow{p_N} \mathcal{N} \\
\text{curr}(\bullet) & \downarrow \exists \text{curr}(\epsilon^\times_{\text{cart}}) \\
[\mathcal{C}, \mathcal{C}] & \xleftarrow{\text{curr}(\circ)}
\end{align*}
\]

(4.14)

The monoidal category $\mathcal{M} \times_{[\mathcal{C}, \mathcal{C}]} \mathcal{N}$ contains only those pairs $(m, n)$ such that $m \bullet - \simeq n \circ -$. Hence optics for $\epsilon^\times_{\text{cart}}$ are the maximal category of optics which can be embedded both in optics for $\bullet$ and optics for $\circ$. Using again lenses and prisms as an example, this category would be that of adapters, since the product of $\times$ and $\oplus$ (as actions) is trivial (indeed, $a \times - \simeq a + - \text{ iff } a = 1$).

4.3 Biactegories. When we have a category $\mathcal{C}$ equipped with two actions, we might wonder whether these actions can be ‘sequentially composed’. That is, starting with a
\(\mathcal{M}\)-action \((\bullet, \eta^\bullet, \mu^\bullet)\) and a \(\mathcal{N}\)-action \((\circ, \eta^\circ, \mu^\circ)\) on \(\mathcal{C}\) we can form the functor

\[
\mathcal{N} \times \mathcal{M} \times \mathcal{C} \xrightarrow{\times \bullet} \mathcal{N} \times \mathcal{C} \xrightarrow{\circ} \mathcal{C}
\]

and conjecture it is the underlying functor of an \(\mathcal{N} \times \mathcal{M}\)-actegory.

What kind of actegory is it?

To answer the question, it’s better to approach the situation more formally, borrowing from the algebraic intuition on actegories. We are contemplating a category receiving two actions, i.e. carrying two pseudoalgebra structures, one for \(\mathcal{M} \times -\) and one for \(\mathcal{N} \times -\). In other words, we can multiply objects (and morphisms) of \(\mathcal{C}\) with scalars (and morphisms) from both \(\mathcal{M}\) and \(\mathcal{N}\). It seems natural to ask for these two actions to be compatible with each other in some way. This is even more tempting when the action of \(\mathcal{N}\) is actually a right action, in which case it feels very natural to ask that

\[
m \bullet (c \circ n) \cong (m \bullet c) \circ n.
\]

This isomorphism is all that’s needed to make \(\mathcal{C}\) into an \(\mathcal{N} \times \mathcal{M}\)-actegory. A pair of a left and a right action with a compatibility isomorphism as above is called a biactegory [Ško06, Ško09].

The ‘bi’ in biactegory does not refer to some additional dimension added to an actegory (as in bicategory): if actegories are analogue to modules over monoids, biactegories are analogue to bimodules.

Even though the intuition of ‘sequential composition’ of actegories as parametric morphisms invites to consider situations in which a category is equipped with two left actions, following [Ško06] we define biactegories as equipped with a left and a right action. Since left \(\mathcal{N}\)-actegories are equivalently right \(\mathcal{N}^{\text{rev}}\)-actegories, this choice does not make any mathematical difference, and it’s simply a matter of convenience: having a left and a right action makes it easier to work with the tensor product we are introducing later.

Coming back to our initial question, we are going to prove (Theorem 4.3.6) that algebras of \(\mathcal{N}^{\text{rev}} \times \mathcal{M} \times -\) (hence of its strictification) are equivalently given by one algebra structure for \(\mathcal{M} \times -\), one for \(\mathcal{N}^{\text{rev}} \times -\), and the aforementioned ‘compatibility structure’
that we call *bimodulator*:

\[
\begin{array}{c}
M \times N^{\text{rev}} \times C \xleftarrow{\text{swap} \times C} \ N^{\text{rev}} \times M \times C \\
\xrightarrow{\ast} \\
\xrightarrow{\zeta} \\
\xrightarrow{\ast} \\
M \times C \xrightarrow{\ast} C
\end{array}
\]

(4.17)

This morphism is a kind of ‘distributive law’ between each action of a scalar in \(M\) and each action of a scalar in \(N\). Indeed, we are going to come back to this idea in Section 5.

**4.3.1. Definition.** An \((M, N)\)-biactegory is a category \(C\) equipped with a left \(M\)-action \((\ast, \eta^\ast, \mu^\ast)\), a right \(N\)-action \((\circ, \eta^\circ, \mu^\circ)\) and a natural isomorphism called bimodulator:

\[
\zeta_{m,c,n} : m \ast (c \circ n) \xrightarrow{\sim} (m \ast c) \circ n.
\]

(4.18)

The bimodulator needs to satisfy the following coherence axioms:

\[
\begin{align*}
(m \otimes n) \ast (c \circ p) & \xrightarrow{\zeta_{m \otimes n, c, p}} ((m \otimes n) \ast c) \circ p \\
& \quad \xrightarrow{\mu^\ast_{m,n,c,p}} \quad \xrightarrow{\mu^\ast_{m,n,c,p}} \\
& \xrightarrow{\zeta_{m,n \circ c,p}} \quad \xrightarrow{\zeta_{m,n,c,p}} \quad \xrightarrow{\zeta_{m,n,c,p}} \\
& m \ast ((n \ast c) \circ p)
\end{align*}
\]

(4.19)

\[
\begin{align*}
p \ast ((c \circ m) \otimes n) & \xrightarrow{\zeta_{p,c,m \otimes n}} (p \ast c) \circ (m \otimes n) \\
& \quad \xrightarrow{\mu^\ast_{p,c,m,n}} \quad \xrightarrow{\mu^\ast_{p,c,m,n}} \\
& \xrightarrow{\zeta_{p,c,m \circ n}} \quad \xrightarrow{\zeta_{p,c,m \circ n}} \quad \xrightarrow{\zeta_{p,c,m \circ n}} \\
& (p \ast (c \circ m) \circ n)
\end{align*}
\]

(4.20)

\[
\begin{align*}
j \ast (c \circ m) & \xrightarrow{\zeta_{j,c,m}} (j \ast c) \circ m \\
& \quad \xrightarrow{\eta^\ast_{j \circ c,m}} \quad \xrightarrow{\eta^\ast_{j \circ c,m}} \\
& (c \circ m)
\end{align*}
\]

(4.21)
Like one-sided \( \mathcal{M} \)-actegories, \( \mathcal{M} \)-biactegories form a 2-category:

4.3.2. Definition. A **lax** \((\mathcal{M}, \mathcal{N})\)-bilinear functor between \((\mathcal{M}, \mathcal{N})\)-biactegories \((\mathcal{C}, \bullet, \circ, \zeta^C)\) and \((\mathcal{D}, \bullet, \circ, \zeta^D)\) is a functor \( F : \mathcal{C} \to \mathcal{D} \) equipped with a left lax \( \mathcal{M} \)-linear structure \( \ell \) and a right lax \( \mathcal{N} \)-linear structure \( r \), obeying the following compatibility axiom:

\[
\begin{align*}
\eta^c_m \circ \ell_{m, c} &\sim \zeta^D_{m, F(c), n} \circ (m \bullet F(c)) \circ n \\
\ell_{m, c} \circ (m \bullet (c \circ n)) &\sim \zeta^C_{m, c, n} \circ F((m \bullet c) \circ n)
\end{align*}
\] (4.23)

4.3.3. Remark. Observe bilinear functors are not maps which are ‘linear in each argument separately’—those are balanced maps, the subject of the next section (Definition 4.4.1). The two are sometimes confused since it is often assumed the left and right structures of a biactegory (or a bimodule) ‘coincide’ (see Proposition 4.3.7), in which case balance and bilinearity are related.

4.3.4. Definition. A **bilinear transformation** \( \xi : (F, \ell, r) \Rightarrow (G, \nu, \varpi) : (\mathcal{C}, \bullet, \circ, \zeta^C) \to (\mathcal{D}, \bullet, \circ, \zeta^D) \) is a natural transformation \( F \Rightarrow G \) which is both a linear transformation (Definition 3.3.8) \( (F, \ell) \Rightarrow (G, \nu) \) (‘on the left’) and \( (F, r) \Rightarrow (G, \varpi) \) (‘on the right’).

4.3.5. Definition. The 2-category of \((\mathcal{M}, \mathcal{N})\)-biactegories, lax bilinear functors and bilinear natural transformation is denoted by \( \mathcal{M}\text{-Act}\text{-}\mathcal{N}^{lx} \).

Consider the composite pseudomonad of left \( \mathcal{M} \)- and right \( \mathcal{N} \)-actegories. This pseudomonad exists because there is a distributive law of pseudomonads (as defined in [Mar99, §4])

\[
\text{swap} \times - : \mathcal{N}^{rev} \times \mathcal{M} \times - \Rightarrow \mathcal{M} \times \mathcal{N}^{rev} \times -
\] (4.24)
4.3 Biactegories

given by symmetry of $\times$ in $\text{Cat}$. The four structure morphisms such a distributive law should be endowed with [Mar99, §4] are all trivial in this simple case. Moreover, we can strictify $\mathcal{M}$ and $\mathcal{N}$ to make the coherence laws hold on the nose [Mar99, (coh 1)–(coh 9)], thus making the pseudomonad structure of $\mathcal{N}^{\text{rev}} \times \mathcal{M} \times -$ [Mar99, §5] more manageable.

Ultimately, as anticipated, we have the following result which equates the definition of biactegories as pseudoalgebras of this composite monad to the definition we gave in terms of pairs of compatible actions:

4.3.6. **Theorem.** There is an equivalence of 2-categories

$$\mathcal{N}\text{-}\text{Act}\text{-}\mathcal{M}^\text{lx} \simeq (\mathcal{N}^{\text{rev}} \times \mathcal{M})\text{-}\text{Act}^\text{lx}. \quad (4.25)$$

**Proof.** See Appendix B. 

A very important fact for the following section (and also for the ‘zoology’ of biactegories) is that left or right $\mathcal{M}$-actegories can be canonically promoted to $\mathcal{M}$-biactegories every time $\mathcal{M}$ is braided, since in that case there is an isomorphism $\mathcal{M}^{\text{rev}} \cong \mathcal{M}$. This can also be interpreted as saying that actegories for braided categories are biactegories in the sense of ‘sequential composition’, that is, for two left actions (which in this case happen to coincide).

4.3.7. **Proposition.** [Mirroring] Let $(\mathcal{C}, \bullet)$ be a left $\mathcal{M}$-actegory, where $(\mathcal{M}, j, \otimes, \beta)$ is a braided monoidal category. Then there is a canonical right $\mathcal{M}$-actegorical structure on $\mathcal{C}$ and a canonical $\mathcal{M}$-biactegorical structure induced by the braiding. We call the resulting biactegory the **mirroring** of $(\mathcal{C}, \bullet)$.

**Proof.** Consider a braiding $\beta$ on $\mathcal{M}$ as a strong monoidal structure on the identity functor $\mathcal{M}^{\text{rev}} \to \mathcal{M}$. We can pullback $\bullet$ along this morphism (Proposition 3.6.5) to obtain a left $\mathcal{M}^{\text{rev}}$-action $\bullet^{\text{rev}}$ on $\mathcal{C}$. Concretely, this is given by the same functor $- \bullet : \mathcal{M}^{\text{rev}} \times \mathcal{C} \to \mathcal{C}$, now equipped with

$$\eta_c^{\text{rev}} := c^{\text{rev}} \bullet j \Rightarrow j \bullet c \xrightarrow{\eta_c} c, \quad (4.26)$$

$$\mu_{c,m,n}^{\text{rev}} := (c^{\text{rev}} \bullet n)^{\text{rev}} m \Rightarrow m \bullet (n \bullet c)$$

$$\xrightarrow{\beta_{c,m,n}} (m \otimes n) \bullet c$$

$$\xrightarrow{\beta_{c,m,n}} (n \otimes m) \bullet c$$

$$\Rightarrow (m \otimes n) \bullet c.$$  

$$(4.27)$$
To make \((\mathcal{C}, \bullet, \bullet^\text{rev})\) into an \(\mathcal{M}\)-biactegory, hence a natural isomorphism
\[
\zeta_{m,c,n} : m \bullet (c^\text{rev} \bullet n) \xrightarrow{\sim} n \bullet (m \bullet c) = (m \bullet c)^\text{rev} \bullet n
\] (4.28)
that we obtain by conjugating \(\beta\) (or better, the action thereof) with \(\mu\):
\[
m \bullet (n \bullet c) \xrightarrow{\mu_{m,n,c}^{-1}} (m \otimes n) \bullet c \xrightarrow{\beta_{m,n,c}} (n \otimes m) \bullet c \xrightarrow{\mu_{n,m,c}} n \bullet (m \bullet c).
\] (4.29)
To prove this is a well-defined bimodulator, we appeal to strictification (Lemma 3.4.1) and move to a setting where both the monoidal structure of \(\mathcal{M}\) and the structure of \(\bullet\) are strict. Note this doesn’t apply to the braiding of \(\mathcal{M}\), which can’t be strictified. Anyway, in this setting we notice immediately that (4.19) and (4.20) collapse to the two hexagonal axioms for a braiding [JY21, Definition 1.2.34]. The two triangular axioms (4.21)–(4.22) hold for analogously reasons.

4.3.8. Corollary. Since \((\mathcal{M}^{\text{rev}})^{\text{rev}} = \mathcal{M}\), any right \(\mathcal{M}\)-actegory, for \(\mathcal{M}\) braided, can be canonically turned into an \(\mathcal{M}\)-biactegory.

4.3.9. Remark. The above proves that \(\mathcal{M}\text{-}\text{A}ct^\text{lx}\) embeds into \(\mathcal{M}\text{-}\text{A}ct^\text{-}\mathcal{M}^\text{lx}\), but this embedding is not wide, even when \(\mathcal{M}\) is braided. There still are biactegories whose left and right action are genuinely distinct and not simply the symmetrization of each other. Despite this, often we abuse notation denoting the left and right structure of \(\mathcal{M}\)-biactegory with the same symbol.

4.3.10. Example. [Monoidal categories] As observed in Example 3.2.3, every monoidal category \((\mathcal{C}, i, \otimes)\) is canonically a left and right self-actegory (both supported by \(\otimes\)). It is trivial to observe that these two actions are compatible in the sense of biactegories:
\[
\zeta_{a,b,c} := \alpha_{a,b,c}^{-1} : a \otimes (b \otimes c) \xrightarrow{\sim} (a \otimes b) \otimes c.
\] (4.30)
We call \((\mathcal{C}, \otimes, \otimes)\) the canonical self-biactegory associated to \(\mathcal{C}\).

4.3.11. Example. [Braided monoidal categories] When \((\mathcal{C}, i, \otimes)\) comes with a braiding \(\beta\), there is an additional way to make it into a self-biactegory, given by the mirroring construction described above (Proposition 4.3.7) applied to the left action. This yields the
bicategory \((\mathcal{C}, \boxtimes, \boxtimes)\), where the bimodulator is given by braiding:

\[
\begin{align*}
m \boxtimes (c \boxtimes n) &= m \boxtimes (n \boxtimes c) \\
&\xrightarrow{\alpha_{m,n,c}^{-1}} (m \boxtimes n) \boxtimes c \\
&\xrightarrow{\beta_{m,n} \boxtimes c} (n \boxtimes m) \boxtimes c \\
&\xrightarrow{\alpha_{n,m,c}} c \boxtimes (m \boxtimes c) \\
&= (m \boxtimes c) \boxtimes n.
\end{align*}
\]

(4.31)

The identity map \(1_\mathcal{C} : \mathcal{C} \to \mathcal{C}\) is a bilinear functor from \((\mathcal{C}, \boxtimes, \boxtimes)\) (the canonical self-biactegory) to \((\mathcal{C}, \boxtimes, \boxtimes)\), equipped with the linear structures

\[
\begin{align*}
\ell_{c,d} : c \boxtimes d &= c \boxtimes d, \\
&\xrightarrow{r_{c,d}} c \boxtimes d \\
\beta_{c,d} : c \boxtimes d &= d \boxtimes c \\
&\xrightarrow{\alpha_{c,d}^{-1}} c \boxtimes d.
\end{align*}
\]

(4.32)

The hexagonal coherence for these then reproduces the first hexagonal identity for \(\beta\). We recover the second one from repeating the same construction for \(\mathcal{C}^{\text{rev}}\).

### 4.4. Tensor product of actegories

Like any category of ‘modules’, actegories possess a natural notion of monoidal product, namely that of tensor product. This notion is indeed totally analogous to the eponymous product on vector spaces, modules, and the closely related cousins of actegories, module categories [Ost03]. Also, let us remind the reader we are not the first to observe the existence of such a product, nor to define it explicitly. See, for instance, [Ško06, Ško09].

The tensor operation is defined between actegories of different handedness: the tensor \(\mathcal{D} \otimes \mathcal{M} \mathcal{C}\) requires \(\mathcal{D}\) to be a right \(\mathcal{M}\)-actegory and \(\mathcal{C}\) to be a left one. As usual, these actions are then ‘used up’ and are not available on \(\mathcal{D} \otimes \mathcal{M} \mathcal{C}\) anymore. However, if further actegorical structures are present, then these are not affected. Therefore if \(\mathcal{D}\) is a left \(\mathcal{N}\)-actegory or \(\mathcal{C}\) is a right \(\mathcal{P}\)-actegory, so will be \(\mathcal{D} \otimes \mathcal{M} \mathcal{C}\).

Here we define the tensor product of actegories using its universal properties of representing balanced functors, i.e. functors \(F : \mathcal{D} \times \mathcal{C} \to \mathcal{E}\) such that \(F(d \circ m, c) \cong F(d, m \bullet c)\), which is of course the categorification of the notion of balanced map of modules, where the isomorphism is simply an equality. That isomorphism is now structure obeying its own set of coherence laws:
4.4.1. Definition. Let \((\mathcal{C}, \bullet, \eta^\bullet, \mu^\bullet)\) be a left \(\mathcal{M}\)-actegory, \((\mathcal{D}, \circ, \eta^\circ, \mu^\circ)\) a right \(\mathcal{M}\)-actegory and \(\mathcal{E}\) a category. An \(\mathcal{M}\)-balanced functor is a functor \(F : \mathcal{D} \times \mathcal{C} \to \mathcal{E}\) together with a natural isomorphism
\[
\varepsilon_{d,m,c} : F(d \circ m, c) \xrightarrow{\sim} F(d, m \bullet c)
\] (4.33)
called equilibrator, obeying the following coherence laws:
\[
\begin{align*}
F(d \circ (m \otimes n), c) & \xrightarrow{\varepsilon_{d,m\otimes n,c}} F(d, (m \otimes n) \bullet c) \\
F((d \circ m) \circ n, c) & \xrightarrow{\varepsilon_{d,m,n,c}} F(d, m \bullet (n \bullet c)) \\
F(d \circ j, c) & \xrightarrow{\varepsilon_{d,j,c}} F(d, j \bullet c)
\end{align*}
\] (4.34)
\[
\begin{align*}
F(\eta^\circ, c) & \xrightarrow{\varepsilon_{\eta^\circ,c}} F(d, \eta^\bullet) \\
F(d, c) & \xrightarrow{\varepsilon_{d,c}} F(d, m \bullet c)
\end{align*}
\] (4.35)

4.4.2. Definition. Let \((F, \varepsilon), (G, \vartheta) : \mathcal{D} \times \mathcal{C} \to \mathcal{E}\) be \(\mathcal{M}\)-balanced functors. An \(\mathcal{M}\)-balanced transformation is a natural transformation \(\xi : F \Rightarrow G\) for which all the following squares commute:
\[
\begin{array}{ccc}
F(d \circ m, c) & \xrightarrow{\varepsilon_{d,m,c}} & F(d, m \bullet c) \\
\varepsilon_{d,m,c} \downarrow & & \downarrow \varepsilon_{d,m \bullet c} \\
G(d \circ m, c) & \xrightarrow{\vartheta_{d,m,c}} & G(d, m \bullet c)
\end{array}
\] (4.36)

Evidently, balanced functors \(\mathcal{D} \times \mathcal{C} \to \mathcal{E}\) and balanced transformations thereof gather in a category \(\text{Bal}(\mathcal{D} \times \mathcal{C}, \mathcal{E})\), and this construction is 2-functorial in \(\mathcal{E}\).

4.4.3. Definition. Let \((\mathcal{C}, \bullet)\) be a left \(\mathcal{M}\)-actegory and \((\mathcal{D}, \circ)\) a right \(\mathcal{M}\)-actegory. Their tensor product is the corepresenting object \(\mathcal{D} \otimes_{\mathcal{M}} \mathcal{C}\) of the 2-functor \(\text{Bal}(\mathcal{D} \times \mathcal{C}, -)\). That is, it’s a category \(\mathcal{D} \otimes_{\mathcal{M}} \mathcal{C}\) together with the structure of an equivalence
\[
\text{Bal}(\mathcal{D} \times \mathcal{C}, \mathcal{E}) \simeq \text{Cat}(\mathcal{D} \otimes_{\mathcal{M}} \mathcal{C}, \mathcal{E}), \quad \text{for all} \; \mathcal{E} : \text{Cat}.
\] (4.37)
4.4 Tensor product of actegories

One can ‘read off’ a concrete construction of the tensor product from this universal property. Saying every functor out of $\mathcal{D} \otimes_{\mathcal{M}} \mathcal{C}$ is automatically balanced implies such a category must have the same objects as $\mathcal{D} \times \mathcal{C}$, but more isomorphisms, namely all those of the form $(d \circ m, c) \cong (d, m \bullet c)$.

4.4.4. Proposition. Let $(\mathcal{C}, (\bullet, \eta^*, \mu^*))$ be a left $\mathcal{M}$-actegory and $(\mathcal{D}, (\circ, \eta^\circ, \mu^\circ))$ a right $\mathcal{M}$-actegory. Their tensor product is the category $\mathcal{D} \otimes_{\mathcal{M}} \mathcal{C}$ presented by the category $\mathcal{D} \times \mathcal{C}$ together with a family of isomorphisms

$$\tau_{d,m,c} : (d \circ m, c) \sim (d, m \bullet c), \quad \text{for all } m : \mathcal{M}, d : \mathcal{D}, c : \mathcal{C}. \quad (4.38)$$

and the following relations:

I) For every morphism $f : d \to d'$ in $\mathcal{D}$, $g : c \to c'$ in $\mathcal{C}$ and $\alpha : m \to n$ in $\mathcal{M}$, the following square must commute:

$$
\begin{array}{ccc}
(d \circ m, c) & \xrightarrow{(f \circ \alpha, g)} & (d' \circ n, c') \\
\tau_{d,m,c} & & \tau_{d',n,c'} \\
(d, m \bullet c) & \xrightarrow{(f, \alpha \bullet g)} & (d', n \bullet c')
\end{array}
\quad (4.39)
$$

II) For every $d : \mathcal{D}, m, n : \mathcal{M}, c : \mathcal{C}$, the following diagrams must commute:

$$
\begin{array}{ccc}
(d \circ (m \otimes n), c) & \xrightarrow{(d \bullet m \otimes n, c)} & (d, (m \otimes n) \bullet c) \\
& \xrightarrow{(d \mu^*, m \otimes n, c)} & (d, m \bullet (n \bullet c)) \\
((d \circ m) \circ n, c) & \xrightarrow{\tau_{d \circ m, n, c}} & (d \circ m, n \bullet c) \\
& \xrightarrow{\tau_{d, m \bullet n, c}} & (d \circ m, n \bullet c)
\end{array}
\quad (4.40)
$$
Proof. There is an obvious functor

\[ Q : \mathcal{D} \times \mathcal{C} \longrightarrow \mathcal{D} \otimes_{\mathcal{M}} \mathcal{C} \]  

(4.42)

which sends a pair of maps \((f, g)\) to ‘itself’. This is a ‘pseudoquotient map’, in the sense that the addition of the family \(\tau\) and the relations it satisfies amounts to impose an ‘identification up to isomorphism’ of \((d \circ m, c)\) and \((d, m \bullet c)\). Notice \(Q\) is itself balanced, with \(\varepsilon := \tau\).

In order to prove \(\mathcal{D} \otimes_{\mathcal{M}} \mathcal{C}\) satisfies the universal property of the tensor, we claim every balanced functor \((F, \varepsilon) : \mathcal{D} \times \mathcal{C} \rightarrow \mathcal{E}\) factors uniquely through the tensor product.

\[ \begin{array}{ccc}
\mathcal{D} \times \mathcal{C} & \overset{\varepsilon}{\longrightarrow} & \mathcal{E} \\
\downarrow Q & & \
\mathcal{D} \otimes_{\mathcal{M}} \mathcal{C} & \longrightarrow & \mathcal{E} \\
\end{array} \]  

(4.43)

The definition of \(\tilde{F}\) is pinned by the factorization condition: we need to have \(\tilde{F}(d, c) = F(d, c)\) and same on morphisms from \(\mathcal{D} \times \mathcal{C}\). Its value on the adjoined family \(\tau\) is given by the equilibrator of \(F\), so that \(\tilde{F}(\tau_{d,m,c}) = \varepsilon_{d,m,c}\). It’s straightforward to see that this is functorial since Diagrams (4.34)–(4.35) are the image of Diagrams (4.40)–(4.41) under this definition. We deem evident that \(\tilde{F}\) is uniquely determined by this definition.

Viceversa, every functor \(G : \mathcal{D} \otimes_{\mathcal{M}} \mathcal{C} \rightarrow \mathcal{E}\) yields a balanced functor \(Q \circ G : \mathcal{D} \times \mathcal{C} \rightarrow \mathcal{E}\) whose equilibrator \(\varepsilon\) is given by \(G(\tau)\). Naturality of \(\varepsilon\) is guaranteed by Diagrams (4.39).

Similarly, every balanced natural transformation \(\xi : (F, \varepsilon) \Rightarrow (F', \vartheta) : \mathcal{D} \times \mathcal{C} \rightarrow \mathcal{E}\) induces a natural transformation \(\tilde{\xi} : \tilde{F} \Rightarrow \tilde{F}'\), since Diagrams (4.36) supply the missing naturality diagrams for the family of morphisms \(\tau\), and viceversa.

4.4.5. Proposition. Let \((\mathcal{C}, \bullet, \bullet)\) be an \((\mathcal{N}, \mathcal{M})\)-biactegory and \((\mathcal{D}, \circ, \circ)\) be an \((\mathcal{M}, \mathcal{P})\)-biactegory. Then \(\mathcal{D} \otimes_{\mathcal{M}} \mathcal{C}\) is an \((\mathcal{N}, \mathcal{P})\)-biactegory, inheriting the left action from \(\mathcal{D}\) and the right action from \(\mathcal{C}\).

Proof. On \(\mathcal{D} \otimes_{\mathcal{M}} \mathcal{C}\), we define

\[ m \circ (d, c) := (m \circ d, c) \]  

(4.44)

\[ ^{10} \text{This sentence can be made precise by defining } Q \text{ and } \mathcal{D} \otimes_{\mathcal{M}} \mathcal{C} \text{ as the data of a bicoequalizer. The parallel arrows } Q \text{ coequalizes can be obtained from the relations specified in Proposition 4.4.4.} \]
and keep \( \eta \) and \( \mu \) of the left action \( \circ \) of \( \mathcal{D} \). The same applies on the right, and the bimodulator for \( \mathcal{D} \otimes_{\mathcal{M}} \mathcal{C} \) is trivial:

\[
\zeta_{m,(d,c),n} : m \circ ((d, c) \bullet n) = (m \bullet d, c \bullet n) = (m \circ (d, c)) \bullet n.
\] (4.45)

The fact the tensor of actegories is a more natural choice of monoidal product between actegories can also be motivated by the following—which sometimes is used as definition:

4.4.6. Proposition. Let \( \mathcal{C} \) be an \((\mathcal{N}, \mathcal{M})\)-biactegory, \( \mathcal{D} \) be a \((\mathcal{M}, \mathcal{P})\)-biactegory and \( \mathcal{E} \) be a \((\mathcal{N}, \mathcal{P})\)-biactegory, let \( \bullet \) denote all actions by abuse of notation. Then the category of (right) \( \mathcal{P} \)-linear maps \( \mathcal{D} \rightarrow \mathcal{E} \), which we denote by \([\mathcal{D}, \mathcal{E}]_\mathcal{P}\) has a canonical \((\mathcal{N}, \mathcal{M})\)-biactegory structure, and is such that:

\[
\mathcal{N} \text{-}\mathcal{A}ct \text{-}\mathcal{P} \text{lx}(\mathcal{C} \otimes_{\mathcal{M}} \mathcal{D}, \mathcal{E}) \cong \mathcal{N} \text{-}\mathcal{A}ct \text{-}\mathcal{M} \text{lx}(\mathcal{C}, [\mathcal{D}, \mathcal{E}]_\mathcal{P}).
\] (4.46)

We refer to this bracket as the **internal hom** \((\mathcal{N}, \mathcal{M})\)-biactegory of \( \mathcal{D} \) and \( \mathcal{E} \).

**Proof.** See Appendix B.

By duality, we also have (note the switch between \( \mathcal{C} \) and \( \mathcal{D} \) during transposition):

4.4.7. Corollary. Let \( \mathcal{C} \) be an \((\mathcal{M}, \mathcal{N})\)-biactegory, \( \mathcal{D} \) be a \((\mathcal{P}, \mathcal{M})\)-biactegory and \( \mathcal{E} \) be a \((\mathcal{P}, \mathcal{N})\)-biactegory. Then the category of (left) \( \mathcal{P} \)-linear maps \( \mathcal{D} \rightarrow \mathcal{E} \), which we denote by \([\mathcal{D}, \mathcal{E}]_\mathcal{P}\) has a canonical \((\mathcal{M}, \mathcal{N})\)-biactegory structure, and is such that:

\[
\mathcal{P} \text{-}\mathcal{A}ct \text{-}\mathcal{N} \text{lx}(\mathcal{D} \otimes_{\mathcal{M}} \mathcal{C}, \mathcal{E}) \cong \mathcal{M} \text{-}\mathcal{A}ct \text{-}\mathcal{N} \text{lx}(\mathcal{C}, [\mathcal{D}, \mathcal{E}]_\mathcal{P}).
\] (4.47)

4.4.8. Corollary. The tensor product of \( \mathcal{M} \)-actegories \( \otimes_{\mathcal{M}} \) can be equipped with the structure of a monoidal product on \( \mathcal{M} \text{-}\mathcal{A}ct \text{-}\mathcal{M} \text{lx} \), and the internal hom \([\cdot, =]_\mathcal{M}\) defines a pseudoclosed structure for it.

**Proof.** The extra structure required to obtain a monoidal structure on the 2-category \( \mathcal{M} \text{-}\mathcal{A}ct \text{-}\mathcal{M} \text{lx} \) is described in Appendix B. The fact \([\cdot, =]_\mathcal{M}\) is a pseudoclosed structure has been proven in Proposition 4.4.6.
We can now prove the following:

4.4.9. **Proposition.** Let \( R : \mathcal{M} \to \mathcal{N} \) be a strong monoidal functor, and let \( R^* : \mathcal{N}\text{-}\text{Act}^{\text{lx}} \to \mathcal{M}\text{-}\text{Act}^{\text{lx}} \) be the associated restriction of scalars, as defined in Proposition 3.6.1. Then \( R^* \) has a left adjoint \( R! \), called extension of scalars, given by

\[
R!(\mathcal{C}, \bullet) \cong \mathcal{N}_R \otimes_{\mathcal{M}} \mathcal{C}.
\]  

(4.48)

where \( \mathcal{N}_R \) is the canonical biactegory associated to \( \mathcal{N} \) except the right action has been restricted by \( R \).

**Proof.** Notice if \( \mathcal{D} \) is a left \( \mathcal{N} \)-actegory, we have \( R^*(\mathcal{D}) \cong [\mathcal{N}_R, \mathcal{D}]_\mathcal{N} \), since \( [\mathcal{N}, \mathcal{D}]_\mathcal{N} \cong \mathcal{D} \) as categories, and the right \( \mathcal{M} \)-action on \( \mathcal{N}_R \) permeates through the hom (Proposition 4.4.6) to give the same left \( \mathcal{M} \)-action as \( R^*(\mathcal{D}) \). Therefore, using the tensor-hom adjunction described in Corollary 4.4.7, we obtain the desired transposition isomorphism:

\[
\mathcal{M}\text{-}\text{Act}^{\text{lx}}(\mathcal{C}, R^*(\mathcal{D})) \cong \mathcal{M}\text{-}\text{Act}^{\text{lx}}(\mathcal{C}, [\mathcal{N}_R, \mathcal{D}]_\mathcal{N}) \cong \mathcal{N}\text{-}\text{Act}^{\text{lx}}(\mathcal{N}_R \otimes_{\mathcal{M}} \mathcal{C}, \mathcal{D}).
\]  

(4.49)

4.4.10. **Proposition.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be categories, then

\[
\mathcal{M}[\mathcal{C}] \otimes_{\mathcal{M}} \mathcal{M}[\mathcal{D}] \simeq \mathcal{M}[\mathcal{C} \times \mathcal{D}].
\]  

(4.50)

**Proof.** Recall from Proposition 3.5.2 that \( \mathcal{M}[-] \simeq - \otimes \mathcal{M} \), where \( \otimes \) denotes the \( \text{Cat} \)-copower operation (as described, for instance, in Example 3.2.6 or [Kel82, §3.7]) on \( \mathcal{M}\text{-}\text{Act} \). We proved \( \otimes_{\mathcal{M}} \) is left adjoint (on both sides), so it preserves colimits. Hence we have

\[
\mathcal{M}[\mathcal{C}] \otimes_{\mathcal{M}} \mathcal{M}[\mathcal{D}] \cong (\mathcal{C} \otimes \mathcal{M}) \otimes_{\mathcal{M}} (\mathcal{D} \otimes \mathcal{M}) \cong \mathcal{C} \otimes (\mathcal{D} \otimes \mathcal{M}).
\]  

(4.51)

If we can prove \( \mathcal{C} \otimes (\mathcal{D} \otimes \mathcal{M}) \cong (\mathcal{C} \times \mathcal{D}) \otimes \mathcal{M} \), we are done. Indeed, we can prove the first satisfies the universal property of the latter:

\[
\mathcal{M}\text{-}\text{Act}(\mathcal{C} \otimes (\mathcal{D} \otimes \mathcal{M}), -) \cong \text{Cat}(\mathcal{C}, \mathcal{M}\text{-}\text{Act}(\mathcal{D} \otimes \mathcal{M}, -)) \\
\cong \text{Cat}(\mathcal{C}, \text{Cat}(\mathcal{D}, \mathcal{M}\text{-}\text{Act}(\mathcal{M}, -))) \\
\cong \text{Cat}(\mathcal{C} \times \mathcal{D}, \mathcal{M}\text{-}\text{Act}(\mathcal{M}, -)) \\
\cong \mathcal{M}\text{-}\text{Act}((\mathcal{C} \times \mathcal{D}) \otimes \mathcal{M}, -).
\]  

(4.52)
5. Interaction of monoidal and actegorical structures

For the purposes of categorical cybernetics, it’s almost impossible to avoid pondering situations in which an $\mathcal{M}$-actegory $\mathcal{C}$ is also equipped with a monoidal structure $(i, \otimes)$. When this happens, it’s natural to ask whether the two structures are compatible in any way. In practice, this compatibility turns out to be a requirement for making some constructions run smoothly, chiefly the Para construction [CGHR21]. In fact for the latter construction to be monoidal,\(^\text{11}\) we need exactly the data of an $\mathcal{M}$-actegory $\mathcal{C}$ equipped with a monoidal structure and a natural morphism

$$\iota_{m,n,c,d} : (m \otimes n) \bullet (c \boxtimes d) \to (m \bullet c) \boxtimes (n \bullet d)$$

(5.1)

In [CGHR21, §2.1] this was called *mixed interchanger*, and it was observed it can be decomposed in strength-like morphisms

$$\kappa_{m,c,d} : m \bullet (c \boxtimes d) \to c \boxtimes (m \bullet d),$$

$$\chi_{m,c,d} : m \bullet (c \boxtimes d) \to (m \bullet c) \boxtimes d.$$  

(5.2)

The three were related by making heavy use of symmetric structure on $\mathcal{M}$, and it wasn’t clear which properties of $\mathcal{M}$, $\mathcal{C}$ and $\bullet$ were actually relevant.

In this section, we expound the situation by framing those sparse observations in the context of the theory of actegories we have built so far. We are going to investigate multiple ways to combine actegorical and monoidal structure, as summarized later in Table 1. In Section 5.1 we define ‘monoidal actegories’ as pseudoactions in MonCat and show these are those for which the Para and Optic constructions are monoidal. Later, in Section 5.2, we study pseudomonoids in $\mathcal{M}$-Act, taken with respect to either its tensor or its cartesian monoidal structures. We show these turn out to be equivalent to monoidal actegories in the first case, and explore some connections with hybrid optics in the second.

In Section 5.4 we extend the previous definitions to the braided and symmetric cases. Finally, in Section 5.5 we prove three Cayley-type classification theorems for monoidal, braided and symmetric actegories.

5.1. Monoidal actegories. The easiest approach to describe actions on monoidal categories is to just have the action of $\mathcal{M}$ to take place in MonCat. To even talk about ‘actions’ of $\mathcal{M}$, $\mathcal{M}$ has to be a (pseudo)monoid in MonCat, i.e. a braided monoidal

\(^{11}\)In the bicategorical sense, see [JY21, Explanation 12.1.3].
category. Hence, from now on, unless explicitly specified, we will assume \( \mathcal{M} \) is equipped with a braiding \( \beta_{m,n} : m \otimes n \to n \otimes m \).

5.1.1. Definition. An **oplax monoidal left \( \mathcal{M} \)-actegory** is a monoidal category \((\mathcal{C}, i, \boxtimes)\) equipped with a lax monoidal functor \( \bullet : \mathcal{M} \times \mathcal{C} \to \mathcal{C} \) and two monoidal natural transformations \( \mu \) and \( \eta \) defined analogously to Definition 3.1.1. When \( \bullet \) is strong, we call it simply **monoidal actegory**.

5.1.2. Remark. Explicitly, the monoidal structure on \( \bullet \) is given by natural morphisms

\[
\begin{align*}
v : i & \longrightarrow j \bullet i, \\
\iota_{m,c,n,d} : (m \otimes n) \bullet (c \boxtimes d) & \longrightarrow (m \bullet c) \boxtimes (n \bullet d).
\end{align*}
\]

where we still call \( \iota \) **mixed interchanger**. Note one of the monoidality axioms for \( \eta \) is

\[
\begin{align*}
v & \quad (5.3) \\
i & \quad \eta
\end{align*}
\]

which makes \( v \) redundant. We’ve drawn the remaining coherence laws satisfied by \( \mu, \eta \) and \( \iota \) in Appendix A.

5.1.3. Remark. Diagram \((A.11)\) (monoidality of \( \mu \)) shows why \( \mathcal{M} \) needs to be braided to even spell this definition out: we need to be able to talk about the monoidal structure of \( \otimes \) itself, which is indeed given by the braiding.

5.1.4. Remark. From an interchanger, both \( \kappa \) and \( \chi \) (as in \((5.2)\)) can be derived:

\[
\begin{align*}
\kappa_{m,c,d} & := m \bullet (c \boxtimes d) \xrightarrow{\lambda_{m,c,d}} (j \otimes m) \bullet (c \boxtimes d) \\
& \xrightarrow{\iota_{j,c,m,d}} (j \bullet c) \boxtimes (m \bullet d) \\
& \xrightarrow{\eta_{c,i}(m \bullet d)} c \boxtimes (m \bullet d),
\end{align*}
\]

\[
\begin{align*}
\chi_{m,c,d} & := m \bullet (c \boxtimes d) \xrightarrow{\rho_{m,c,d}} (m \otimes j) \bullet (c \boxtimes d) \\
& \xrightarrow{\iota_{m,j,c,d}} (m \bullet c) \boxtimes (j \bullet d) \\
& \xrightarrow{(m \bullet c) \boxtimes \eta_d} (m \bullet c) \boxtimes d.
\end{align*}
\]

Viceversa, \( \kappa \) and \( \chi \) determine \( \iota \):

\[
\begin{align*}
\iota_{m,n,c,d} & := (m \otimes n) \bullet (c \boxtimes d) \xrightarrow{\mu_{m,n,c,d}^{-1}} m \bullet (n \bullet (c \boxtimes d)) \\
& \xrightarrow{m \bullet \kappa_{m,c,d}} m \bullet (c \boxtimes (n \bullet d)) \\
& \xrightarrow{\chi_{n,c,d}} (m \bullet c) \boxtimes (n \bullet d).
\end{align*}
\]
Observe that monoidal actegories could be presented as a biactegorical structure between $\Upsilon(C, i, \boxtimes)$ (where $\Upsilon$ is actegorification of a monoidal category from Proposition 3.6.8) and $(C, \bullet)$: in a sense, $\kappa$ and $\chi$ are bimodulators for $\bullet$ and $\boxtimes$, where the latter is used both as a left action (hence $\kappa$) and a right action (hence $\chi$), as in Example 4.3.10.

It’s evident the definition of lax linear functors and linear natural transformation can be adapted too to this case, by asking the underlying functors and natural transformations, respectively, to be monoidal. Therefore there is an indexed 2-category $\textbf{Act}_{\text{MonCat}}^{\text{lax}}$ analogous to the one described in Proposition 3.6.5, giving rise to a 2-category $\textbf{Act}_{\text{MonCat}}^{\text{lax}}$. The same applies to all the other kinds of actegories we are going to define in this section, so we won’t bother the reader with a similar remark again and we will tacitly consider analogous 2-categories to have been defined.

5.1.5. Example. The most trivial example of monoidal actegory is given by the self-action (Example 3.2.3) of any braided monoidal category.

5.1.6. Example. The canonical $\mathbb{N}$-action on any monoidal category, defined in Example 3.2.4, is monoidal since $m^\otimes(n + n') = m^\otimes n \otimes m^\otimes n'$ for any $m : \mathcal{M}$.

5.1.7. Example. In [CP09], the central structure is that of linear actegory. Using our terminology, it’s an $\mathcal{M}$-diactegory (Remark 3.1.4) supported by a category $\mathcal{C}$ equipped with two monoidal products $\boxtimes$, $\boxplus$ and isomorphisms $a \boxtimes (b \boxplus c) \cong (a \boxtimes b) \boxplus c \cong b \boxplus (a \boxtimes c)$. Most importantly, the $\mathcal{M}$-action $\bullet$ and the $\mathcal{M}^{\text{op}}$-action $\circ$ are required to be monoidal with respect to both $\boxtimes$ and $\boxplus$. Moreover, $m \bullet - \dashv m \circ -$ for every $m : \mathcal{M}$.

5.1.8. Example. Similarly as above, suppose $\mathcal{C}$ is cartesian and cocartesian monoidal. Then there always is an oplax monoidal structure on the self-action induced by $\times$ on $(\mathcal{C}, 0, +)$:

$$(a \times b) \times (c + d) \xrightarrow{\text{exists}} a \times b \times c + a \times b \times d \xrightarrow{\pi_{a,c} + \pi_{b,d}} a \times c + b \times d.$$ 

As anticipated, we investigated the notion of monoidal actegory in order to capture the structure needed to make the results of Copara and Para constructions monoidal bicategories. For the sake of clarity, we remind the reader of [CGHR21, Definition 2]. Given an $\mathcal{M}$-actegory $(\mathcal{C}, \bullet)$, the bicategory $\text{Para}_{\bullet}(\mathcal{C})$ has the same objects of $\mathcal{C}$ but a morphism $(m, f) : c \to d$ is now given by a choice of scalar $m : \mathcal{M}$ and a morphism $f : m \bullet c \to d$. Composition and identities are built from the multiplicator and unitor of $\bullet$. 

Suppose now we additionally equip $\mathcal{C}$ with the monoidal structure $(i, \boxtimes)$. The object part of $\boxtimes$ trivially extends to $\operatorname{Para}_\bullet(\mathcal{C})$, but in order to extend it to morphisms, we need to assume $\bullet$ is an oplax monoidal $\mathcal{M}$-action. In fact if $(m, f) : c \to d$ and $(n, g) : c' \to d'$ are morphisms, we can define (denoting by $\boxtimes_\bullet$ the candidate monoidal product on $\operatorname{Para}_\bullet(\mathcal{C})$):

$$f \boxtimes_\bullet g := (m \otimes n, (m \otimes n) \bullet (c \boxtimes c') \xrightarrow{\iota_{m,n,c,c'}} (m \bullet c) \boxtimes (n \bullet c') \xrightarrow{f \boxtimes g} d \boxtimes d').$$  \hspace{1cm} (5.7)

Ultimately, bifunctoriality follows from the braidedness of $\mathcal{M}$, the bifunctoriality of $\boxtimes$, and the monoidality of $\mu^\bullet$ (see Diagram A.11). Conversely, given a monoidal product on $\operatorname{Para}_\bullet(\mathcal{C})$, we can recover $\iota_{m,n,c,c'}$ by setting $f = (m, 1_{m \bullet c})$ and $g = (n, 1_{n \bullet c'})$. The same can be shown for $\operatorname{Copara}_\bullet(\mathcal{C})$.

5.1.9. **Remark.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{M}$-actegories. In [BCG+21] it is shown how $\operatorname{Optic}_\bullet \circ \circ$ arises as the (local discretization of) the pullback of $\operatorname{Para}_\bullet(\mathcal{C})$ coop $\mathcal{B} \mathcal{M}$ op $\operatorname{Copara}_\circ(\mathcal{D})$, where $P$ and $C$ respectively project out the parameter and coparameter of a given (co)parametric morphism. Therefore, in case $\mathcal{C}$ and $\mathcal{D}$ are actually oplax monoidal actegories, the induced monoidal structure on $\operatorname{Para}_\bullet(\mathcal{C})$ and $\operatorname{Copara}_\circ(\mathcal{D})$ would automatically yield a monoidal structure on $\operatorname{Optic}_\bullet \circ \circ$, defined ‘componentwise’ on the pullback.

5.1.10. **Remark.** Observe the product defined in Equation 5.7 yields both the parallel product and external choice mentioned in Section 4.1. Indeed, when $\mathcal{C}$ is in the situation outlined in Example 5.1.8, we get parallel product as $\times$ and external choice as $+\times$.

5.2. **Algebroidal actegories.** Monoidal actegories are thus actions in $\operatorname{MonCat}$, but nothing forbids going the other way: instead of expressing the $\mathcal{M}$-action on $\mathcal{C}$ internally to $\operatorname{MonCat}$, we might express the monoidal structure internally to $\mathcal{M}$-$\operatorname{Act}$.

Monoidal objects in categories of modules have been traditionally called ‘algebras’, thus we are going to call their categorified version ‘algebroidal actegories’.[12] In order to talk about these ‘algebroidal structures’, we have to choose a monoidal structure for in $\mathcal{M}$-$\operatorname{Act}$. As proven in Remark 4.2.4, this category is cartesian monoidal, but that’s not the only useful monoidal product around. We’ve proven in Proposition 4.3.7 that when $\mathcal{M}$ is braided, every left $\mathcal{M}$-actegory is canonically a $\mathcal{M}$-biactegory therefore we can restrict to $\mathcal{M}$-$\operatorname{Act}$ the monoidal structure of $\mathcal{M}$-$\operatorname{Act}$-$\mathcal{M}$ given by tensor product.

[12]Quantum algebra literature seems to use ‘algebra’ anyway for this kind of objects, like in [BZFN10].
5.2 Algebroidal actegories

We start by analyzing the notion of algebroidal actegory we get from the tensor product of actegories:

5.2.1. Definition. [Balanced algebroidal \( \mathcal{M} \)-actegory] A \textit{balanced algebroidal \( \mathcal{M} \)-actegory} is a pseudomonoid in \((\mathcal{M}\text{-}\text{Act}, \mathcal{M}, \otimes_{\mathcal{M}})\), amounting to a left \( \mathcal{M} \)-actegory \((\mathcal{C}, \bullet)\) together with \( \mathcal{M} \)-linear functors

\[
I : \mathcal{M} \to \mathcal{C}, \\
\boxtimes : \mathcal{C} \otimes_{\mathcal{M}} \mathcal{C} \to \mathcal{C}
\]

(5.9)

and \( \mathcal{M} \)-linear natural transformations \( \lambda, \rho, \alpha \) satisfying the usual axioms for unitors and associators in pseudomonoids.

By appealing to Lemma 3.4.5, most of the data pertaining \( I \) (i.e. its left and right lineators) can be safely discarded by agreeing that \( I(m) = m \bullet I(j) \), leaving just the choice of \( I(j) : \mathcal{C} \) as data, that from now on we simply denote by \( i \).

By Proposition 4.4.4 and Corollary 4.4.8, the product map is equivalently given by a \textit{balanced} (and still linear) functor out of the cartesian product:

\[
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{\exists! \boxtimes} & \mathcal{C} \\
\downarrow Q & & \\
\mathcal{C} \otimes_{\mathcal{M}} \mathcal{C} & \xrightarrow{\boxtimes} & \mathcal{C}
\end{array}
\]  

(5.10)

Therefore a balanced \( \mathcal{M} \)-algebroidal structure amounts to a monoidal structure \((i, \boxtimes, \lambda, \rho, \alpha)\) on \( \mathcal{C} \), plus an equilibrator (Definition 4.4.1) for \( \boxtimes \):

\[
\varepsilon_{m,c,d} : (m \bullet c) \boxtimes d \sim c \boxtimes (m \bullet d);
\]

(5.11)

and a left lineator (Definition 3.3.2) for \( \boxtimes \):

\[
\ell_{m,c,d} : m \bullet (c \boxtimes d) \sim (m \bullet c) \boxtimes d.
\]

(5.12)

We spelled out the coherence laws obeyed but these in Appendix A.

5.2.2. Proposition. For a braided monoidal category \( \mathcal{M} \), every balanced algebroidal \( \mathcal{M} \)-actegory can be presented as a (strong) monoidal \( \mathcal{M} \)-actegory and viceversa.
Proof. Given a balanced algebroidal \( \mathcal{M} \)-actegory, we get the data needed to define a monoidal \( \mathcal{M} \)-actegory by setting \( \chi := \ell \) and \( \kappa := \ell \frac{\varepsilon}{\varepsilon} \). Viceversa, we set \( \ell := \chi \) and \( \varepsilon = \chi^{-1} \frac{\varepsilon}{\kappa} \). Regarding coherence, it’s a bit hard to eyeball exactly the correspondence between the two set of diagrams, since those for monoidal actegories are naturally spelled in terms of \( \ell \) instead of \( \kappa \) and \( \chi \). In broad terms, Diagram (A.8) corresponds to Diagram (A.17) and Diagram (A.18), Diagram (A.9) to Diagram (A.16), Diagram (A.10) to Diagram (A.13) and Diagram (A.15), and finally Diagram (A.11) to Diagram (A.12) and Diagram (A.14).

5.2.3. Remark. Even though balanced \( \mathcal{M} \)-algebroidal actegories ‘trivialize’ to monoidal actegories, if we let Definition 5.2.1 take place in bona fide \( \mathcal{M} \)-biactegories with their tensor product, we obtain something genuinely different. Indeed, the equivalence we just sketched hinges crucially on \( \mathcal{C} \) being used as a ‘mirrored’ biactegory, i.e. one for which left and right action only differ by a braiding twist in the multiplicator (Proposition 4.3.7). When \( \mathcal{C} \) doesn’t have this property, a morphism like \( \kappa \), that changes the order of symbols in an expression like \( m \bullet (c \boxtimes d) \), does not emerge anymore. Therefore a generic pseudomonoid object in \( (\mathcal{M}\text{-}\mathbf{Act}, \mathcal{M}, \otimes) \), even when \( \mathcal{M} \) is braided, is not ‘just’ a ‘monoidal \( \mathcal{M} \)-biactegory’ (i.e. Definition 4.3.1 internalized to \( \mathcal{M}\text{-}\mathbf{Act} \), as we did for Definition 3.1.1 to get Definition 5.1.1), whereas the converse is true: every monoidal \( \mathcal{M} \)-biactegory defines an \( \mathcal{M} \)-algebroidal biactegory (by forgetting \( \kappa \)).

We now turn to the second possible definition of algebroidal actegory, using the cartesian monoidal structure on \( \mathcal{M}\text{-}\mathbf{Act} \).

5.2.4. Definition. [Distributive algebroidal \( \mathcal{M} \)-actegory] A distributive algebroidal \( \mathcal{M} \)-actegory is a pseudomonoid in \( (\mathcal{M}\text{-}\mathbf{Act}, 1, \times) \), i.e. a left \( \mathcal{M} \)-actegory \((\mathcal{C}, \bullet)\) together with \( \mathcal{M} \)-linear functors:

\[
\begin{align*}
o & : 1 \longrightarrow \mathcal{C}, \\
\boxtimes & : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}
\end{align*}
\]  

and \( \mathcal{M} \)-linear natural transformations \( \lambda, \rho, \alpha \) satisfying the usual axioms for unitors and associators in pseudomonoids, and natural \( \mathcal{M} \)-linear isomorphisms

\[
\begin{align*}
\gamma_m &: m \bullet o \overset{\sim}{\longrightarrow} o, \\
\delta_{m,c,d} &: m \bullet (c \boxtimes d) \overset{\sim}{\longrightarrow} (m \bullet c) \boxtimes (m \bullet d),
\end{align*}
\]

called absorber and distributor, respectively, satisfying the usual laws of strong monoidal structures.
The coherence axioms for this structure are found in Appendix A. Notably, Diagram (A.19) shows $\gamma_j = \eta_o^{-1}$.

We are not aware of this definition being in the literature already, except for the case in which $\bullet$ descends from a monoidal structure $(i, \boxtimes)$ already present on $C$, i.e. $C$ is an actegory of the kind of Example 3.2.3. In that case, the above definition captures a fragment of that of Laplaza [Lap72], including coherences.

5.2.5. **Example.** In [CP09, Alc14], they consider the structure of a category being acted upon in two adjoint ways, that is, a category $C$ receiving an $\mathcal{M}$-action $\circ$ and an $\mathcal{M}^{\text{op}}$-action $\bullet$ such that for each $m : \mathcal{M}$, $m \circ - = \mathcal{M} \bullet -$. If $C$ has coproducts, then $m \circ -$ preserves them by virtue of being left adjoint, therefore $(C, \bullet, 0, +)$ is a distributive algebroidal $\mathcal{M}$-actegory.

5.2.6. **Example.** Let $(\text{Vec}, \mathbb{R}, \otimes_\mathbb{R})$ be the category of finite dimensional real vector spaces and linear maps thereof, and let $(\text{Euc}, \mathbb{R}^0, \times)$ be the category of smooth euclidean manifolds and smooth maps thereof. There is an action $\{-, =\}$ of $\text{Vec}^{\text{op}}$ on $\text{Euc}$ given by

$$\{V, \mathbb{R}^m\} = \text{Lin}(V, \mathbb{R}^m)$$

where the latter is the (Euclidean) space of linear maps from $V$ to $\mathbb{R}^m$. It’s easy to verify this makes $\text{Euc}$ into a distributive algebroidal $\text{Vec}^{\text{op}}$-actegory.

Most importantly for our interests, this kind of distributivity is used in the optics literature to define a class of mixed optics called ‘affine traversals’ [Rom20, CEG+20].

In fact, suppose $C$ is a distributive algebroidal $\mathcal{M}$-actegory. We can form (at least) two kinds of optics from this data. The first kind is ‘homogeneous’ optics using up the data of the $\mathcal{M}$-action on $C$, namely $\text{Optic}_{\bullet, \bullet}$. The second kind is obtained using the monoidal structure on $C$ instead, yielding $\text{Optic}_{\boxplus, \boxplus}$. More often than not, $\boxplus$ is a cocartesian monoidal structure, so these latter optics can be considered ‘prism-like’.

Affine traversals arise when one tries to compose optics from $\text{Optic}_{\bullet, \bullet}$ with the prism-like optics $\text{Optic}_{\boxplus, \boxplus}$. Such ‘chimeric’ optics are usually obtained by combining the actegorical data used for each of the parts making up the chimera to yield a ‘minimal’ composite actegory (we’ve seen this in Remark 4.2.5). In the case at hand, this means combining $\bullet$ and $\boxplus$ in some way, and if they form a distributive algebroidal $\mathcal{M}$-actegory then we have the data required to do so:
5.2.7. Proposition. [Waff product] Let \((\mathcal{C}, \bullet, o, \boxplus)\) be a distributive algebroidal \(\mathcal{M}\)-actegory. Then the product category \(\mathcal{C} \times \mathcal{M}\) can be equipped with a monoidal product \(\otimes\), whose unit is \((o, j)\) and whose product law is so defined:

\[(c, m) \otimes (d, n) := (c \boxplus (m \bullet d), m \otimes n) \quad (5.15)\]

We call this \textit{waff product}, for ‘Weird AFFine’ product, and we denote such monoidal category by \(\mathcal{W}(\mathcal{C}, \mathcal{M})\).

Proof. See Appendix B. \(\blacksquare\)

5.2.8. Remark. This product might seem puzzling, but it’s actually very natural. One has to think about pairs \((c, m)\) as standing for the \textit{affine endofunctors} \(c \boxplus (m \bullet -)\) on \(\mathcal{C}\). Then \(\otimes\) is abstracting the composition product of such functors, meaning that \((c, m) \otimes (d, n)\) are the coefficients of \(c \boxplus (m \bullet -) \circ d \boxplus (n \bullet -)\). Notice, however, that the inclusion \((c, m) \mapsto c \boxplus (m \bullet -)\) is not full, that is, we can’t say in general that every natural transformation \(c \boxplus (m \bullet -) \Rightarrow d \boxplus (n \bullet -)\) arises from a pair of morphisms \(c \to d\) in \(\mathcal{C}\) and \(m \to n\) in \(\mathcal{M}\).

5.2.9. Example. A dual situation to that of Example 5.2.5 arises when \(\mathcal{C}\) is cartesian closed, in which case \((\mathcal{C}, [-, =], 1, \times)\) is a distributive algebroidal \(\mathcal{C}^{\text{op}}\)-actegory, where \([- , =] : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}\) denotes the internal hom. The waff product in this case is given by

\[(c, m) \otimes (d, n) := (c \times [m, d], m \times n)\]

and the associated class of optics is known as \textit{glasses} \cite[§3.3.2]{CEG+20}.

Naturally, affine endofunctors on \(\mathcal{C}\) act on \(\mathcal{C}\):

\[(c, m) \otimes d = c \boxplus (m \bullet d). \quad (5.16)\]

The unitor and multiplicator of this action are routine to define (see Corollary B.0.1).

5.2.10. Remark. One might notice we already have the means to obtain affine traversals (or, in general, of hybrid optics) using the coproduct of actions described in Remark 4.2.5. Indeed, there is a ‘projection functor’ \(\pi : \mathcal{C} + \mathcal{M} \to \mathcal{W}(\mathcal{C}, \mathcal{M})\) defined as

\[c_1 m_1 \cdots c_k m_k \mapsto (c_1 \boxplus (m_1 \bullet c_2) \boxplus \cdots \boxplus (m_1 \otimes \cdots \otimes m_{k-1} \bullet c_k), m_1 \otimes \cdots \otimes m_k). \quad (5.17)\]
This is a morphism in $\text{Act}^{\text{cart}}(C)$, meaning

$$\pi(c_1 m_1 \cdots c_k m_k) \otimes - \cong (c_1 m_1 \cdots c_k m_k) \cdot_{\text{cart}} - .$$

Combining this observation with Remark 5.2.8 above, we realize $\mathcal{W}(C, M)$ is a direct presentation of the replete image\(^{13}\) of $\text{curr}(e_{\text{cart}}^+)$ in $[C, C]$; that is, it captures the essential way $C + M$ acts on $C$.

The difference between the action of $\mathcal{W}(C, M)$ and that of $C + M$ is a guiding example in [Rom20, §6.4], where clear optics are studied, i.e. optics associated to actions whose currying is pseudomonic. Róman notes that the Optic construction is invariant under repletion, that is, forming categories of optics only depends on the data contained in the replete image of a given action:

$$\begin{array}{ccc}
\text{Act}^{\text{cart}}(C) & \xrightarrow{\text{repl}} & \text{Act}^{\text{cart}}(C) \\
\downarrow & & \downarrow \\
\text{Optic} & \cong & \text{Optic} \\
\downarrow & & \downarrow \\
\text{Cat} & & \text{Cat}
\end{array}$$

Therefore any given class of optics can be presented by multiple, inequivalent actions (indeed, $\otimes \not\cong e_{\text{cart}}^+$), as long as they all have the same replete image. In practice, one would like to use the ‘least redundant’ among these actions, that is, the one for which the map $M \to [C, C]$ is pseudomonic, which is indeed what the repletion gives. Thus comparing the waff action and the coproduct action, we see that the albeit the latter is more general, the former directly yields such maximally efficient action, by exploiting the structure of the situation.

This also happens when the structure of biactegory is around. Suppose in fact $C$ is an $(\mathcal{M}, \mathcal{N})$-actegory. Then the repletion of $\text{curr}(e_{\text{cart}}^+)$ coincides with $\text{curr}(\langle \bullet, \circ \rangle)$, since the bimodulator provides a way to commute the actions of scalars in $\mathcal{M}$ past the actions of scalars in $\mathcal{N}$. In other words, the functor $\Gamma : \mathcal{M} \times \mathcal{N} \to \mathcal{M} + \mathcal{N}$ described in Equation (4.6) becomes a morphism in $\text{Act}^{\text{cart}}(C)$, specifically the repletion projection.

The common theme here is that both distributive algebroidal actegories and biactegories describe structure to ‘sequentially compose’ (recall the beginning of Section 4) two actegories. One might then speculate that such distributive laws provide general ways to

---

\(^{13}\) The replete image of a functor $F : A \to B$ is the universal pseudomonic functor $F$ factors through. In other words, it’s the smallest subcategory of $B$ closed under isomorphisms that contains the image of $F$. 

reduce the coproduct action: distributive algebroidal structure allows to reduce $\mathcal{C} + \mathcal{M}$ to
$\mathcal{W}(\mathcal{C}, \mathcal{M})$ whereas biactegorical structure reduces $\mathcal{N} + \mathcal{M}$ to to $\mathcal{N} \times \mathcal{M}$.

5.3. DISTRIBUTIVE LAWS. As remarked in Section 4.3, bimodulators are akin to distribu-
tivel laws for the endofunctors induced by scalars in $\mathcal{M}$ and $\mathcal{N}$ respectively. Proposition 5.2.7
shows, in some cases, different kind of distributive laws might be around, thus yielding a
different ‘sequential composition’ of the actions.

All in all, we have the following classification:

| (\mathcal{C}, i, \boxtimes) in MonCat, 1, \times | monoidal structure \((a, \boxtimes)\) | \(\mathcal{N}\)-actegorical structure \(\circ\) |
|---|---|---|
| braided monoidal category | \((a \boxtimes c) \boxtimes (b \boxtimes d) \cong (a \boxtimes b) \boxtimes (c \boxtimes d)\) | monoidal \(\mathcal{N}\)-actegory |
| | | \(\mathcal{N}\) braided |
| | | \((n \circ c) \boxtimes (n' \circ d) \cong (n \circ n') \circ (c \boxtimes d)\) |

| (\mathcal{C}, \bullet) in (\mathcal{M}\text{-}\text{Act}, 1, \times) | distributive algebroidal \(\mathcal{M}\)-actegory | \((\mathcal{M}, \mathcal{N}^{\text{rev}})\)-biactegory |
|---|---|---|
| | \(m \bullet o \cong o\) | \(m \bullet (n \circ c) \cong n \circ (m \bullet c)\) |
| | \(m \bullet (c \boxtimes d) \cong (m \bullet c) \boxtimes (m \bullet d)\) | |

| (\mathcal{C}, \bullet) in (\mathcal{M}\text{-}\text{Act}, \mathcal{M}, \otimes, \mathcal{M}) | balanced algebroidal \(\mathcal{M}\)-actegory | \((\mathcal{M}, \mathcal{N}^{\text{rev}})\)-biactegory |
|---|---|---|
| | \(\mathcal{M}\) braided | \(\mathcal{M}\) braided |
| | \((m \bullet c) \boxtimes (m' \bullet d) \cong (m \otimes m') \bullet (c \boxtimes d)\) | \(m \bullet (n \circ c) \cong n \circ (m \bullet c)\) |

**Table 1:** Each cell contains the result of equipping the object in the corresponding row
with the structure of the corresponding column. In the above, \((\mathcal{M}, j, \otimes)\) and \((\mathcal{N}, e, \oslash)\) are
monoidal categories.

From this point of view we recover all the ‘hybrid’ structures we contemplated so
far, including an additional cameo of braided categories whose equivalence to ‘monoidal
monoidal categories’ is well-known [JS93].

Perhaps the only entries of the table which are not completely obvious are the ones
pertaining biactegories. In fact we didn’t present \((\mathcal{M}, \mathcal{N})\)-biactegories as ‘\(\mathcal{N}\)-actions
in \(\mathcal{M}\text{-}\text{Act}\)’, but as bialgebras for the \(\mathcal{M} \times -\) and \(\mathcal{N} \times -\) (for sake of simplicity, let’s
5.4 Braided and symmetric monoidal actegories

temporarily ignore the fact we should technically consider $\mathcal{N}$ acting on the right).

This gives us the opportunity to brush up something we briefly mentioned at the beginning of the paper, in Definition 3.1.9, where we defined the structure of ‘action of a monoid’ in any category $\mathcal{C}$ which is itself subject to an action of a monoidal category $\mathcal{M}$.

It’s easy to convince oneself that this statement stays true if we categorify one more time. Hence to talk about ‘$\mathcal{N}$-actions in $\mathcal{M}$-Act’ it is enough for $\mathcal{C}$ to act on $\mathcal{M}$-Act, so that $\mathcal{N}$ (which is indeed a monoid in $\mathbf{Cat}$) can act on objects of $\mathcal{M}$-Act. One can prove (the characterization theorems in the next section are useful in this regard) that there are two such actions:

$$\begin{align*}
\mathcal{D} \ltimes (\mathcal{C}, \bullet) &:= (\mathcal{D} \times \mathcal{C}, \mathcal{D} \times \bullet), \\
\mathcal{D} \lhd (\mathcal{C}, \bullet) &:= \mathcal{M}[\mathcal{D}] \otimes \mathcal{M} \mathcal{C}.
\end{align*}$$

Therefore an $\mathcal{N}$-action in $\mathcal{M}$-Act can be interpreted in two different ways:

1. Using $\ltimes$, we get an $\mathcal{M}$-actegory $(\mathcal{C}, \bullet)$ together with an $\mathcal{M}$-linear morphism $\circ : \mathcal{N} \times \mathcal{C} \to \mathcal{C}$, and it is indeed this linear structure that gives rise to the bimodulator morphism $n \circ (m \bullet c) \cong m \bullet (n \circ c)$.

2. Using $\lhd$, we get an $\mathcal{M}$-actegory $(\mathcal{C}, \bullet)$ together with an $\mathcal{M}$-linear morphism $\circ : \mathcal{M}[\mathcal{N}] \otimes \mathcal{M} \mathcal{C} \to \mathcal{C}$. But we can easily prove that $\mathcal{M}[\mathcal{N}] \otimes \mathcal{M} \mathcal{C} \cong \mathcal{N} \times \mathcal{C}$, so this structure coincides with the one above.

5.4. Braided and symmetric monoidal actegories. So far, we’ve been focusing on the case where a braided monoidal category $\mathcal{M}(\mathcal{M}, j, \otimes, \beta)$ acts on a monoidal category $\mathcal{C}(\mathcal{C}, i, \boxtimes)$. What happens if instead $\mathcal{M}$ is symmetric? And what if $\mathcal{C}$ is braided or symmetric?

We frame this question in the same way we did before, i.e. by formulating the definition of action not in $\mathbf{MonCat}$ anymore, but internally to $\mathbf{BrMonCat}$ and $\mathbf{SymMonCat}$. Conversely we formulate the notion of braided and symmetric pseudomonoid inside $\mathcal{M}$-Act, with either choice of monoidal product.

In the first case, recall that $\mathbf{BrMonCat} \cong \mathbf{PsdMon}(\mathbf{MonCat})$, $\mathbf{SymMonCat} \cong \mathbf{PsdMon}(\mathbf{BrMonCat})$ (and $\mathbf{PsdMon}(\mathbf{SymMonCat}) \cong \mathbf{SymMonCat}$), defining a column in the periodic table of higher categories [BD95]. This means that actions on braided and symmetric monoidal categories, by definition, come from symmetric monoidal categories. Thus we have:
5.4.1. Definition. [Braided monoidal actegory] Let \((\mathcal{M}, j, \otimes, \sigma)\) be a symmetric monoidal category. A braided monoidal left \(\mathcal{M}\)-actegory is a braided monoidal category \((\mathcal{C}, i, \boxtimes, \beta)\) equipped with a braided monoidal functor \(\bullet : \mathcal{M} \times \mathcal{C} \to \mathcal{C}\) and two monoidal natural transformations \(\mu\) and \(\eta\) defined analogously to Definition 3.1.1. The strong monoidal structure on \(\bullet\) is given as in Definition 5.1.1, and additionally the following axiom is satisfied:

\[
\begin{align*}
(m \otimes n) \bullet (c \boxtimes d) &\xrightarrow{\kappa_{m,n,c,d}} (m \bullet c) \boxtimes (n \bullet d) \\
\sigma_{m,n} \bullet \beta_{c,d} &\quad \beta_{m,c,n,d}\end{align*}
\]  

(5.21)

5.4.2. Remark. In terms of \(\kappa\) and \(\chi\) (Remark 5.1.4), the last axiom splits in two:

\[
\begin{align*}
m \bullet (c \boxtimes d) &\xrightarrow{\chi_{m,c,d}} (m \bullet c) \boxtimes d \\
m \bullet \beta_{c,d} &\quad \beta_{m,c,d} \\
m \bullet (d \boxtimes c) &\xrightarrow{\kappa_{m,d,c}} d \boxtimes (m \bullet c) \\
m \bullet \beta_{c,d} &\quad \beta_{c,m,d}
\end{align*}
\]  

(5.22)

5.4.3. Remark. [Symmetric monoidal actegory] Since ‘symmetry’ is a property of braided structures, and braided monoidal functors automatically preserve it, ‘symmetric monoidal actegories’ amount to braided monoidal actegories such that \(\beta\) (the braiding on \(\mathcal{C}\)) happens to be symmetric.

We can still wonder what happens if we take an algebroidal actegory and ask the pseudomonoidal structure to be braided or symmetric (as defined in [McC00a, §3]).

For a balanced algebroidal actegory \((\mathcal{C}, \bullet, i, \boxtimes)\) (Definition 5.2.1), a braiding in this sense amounts to a braiding \(\beta\) on the underlying monoidal category \((\mathcal{C}, i, \boxtimes)\) which happens to also be a linear and balanced transformation in \(\mathcal{M}\text{-}\text{Act}\):

\[
\begin{align*}
\mathcal{C} \times \mathcal{C} &\xrightarrow{\text{swap}} \mathcal{C} \times \mathcal{C} \quad \mathcal{C} \times \mathcal{C} &\xrightarrow{\text{swap}} \mathcal{C} \times \mathcal{C} \\
\boxtimes &\quad \beta \quad \boxtimes &\quad \beta\end{align*}
\]

(5.23)
5.5 The classifying objects of actions

An interesting fact to observe is that $\varepsilon$, the equilibrator of $\circlearrowright$, becomes part of the left linear structure of swap $\circlearrowright\circlearrowleft$, so that linearity for $\beta$ looks like this:

\[
\begin{align*}
 m \bullet (c \circlearrowright d) & \quad \xrightarrow{\ell_{m,c,d}} \quad (m \bullet c) \circlearrowright d \\
 m \bullet \beta_{c,d} & \quad \downarrow \beta_{m \bullet c,d} \\
 m \bullet (d \circlearrowleft c) & \quad \xrightarrow{\ell_{m,d,c}} \quad (m \bullet d) \circlearrowleft c \quad \xrightarrow{\varepsilon_{m,d,c}} \quad d \circlearrowleft (m \bullet c)
\end{align*}
\]

(5.24)

Recall now the proof of Proposition 5.2.2, where we have shown monoidal actegories and balanced algebroidal categories coincide. We set $\chi = \ell$ and $\kappa = \ell \circlearrowright \varepsilon$, a substitution under which Diagram (5.22) is transmuted to the one just drawn. Viceversa, given $\chi$ and $\kappa$ we set $\ell = \chi$ and $\varepsilon = \chi^{-1} \circlearrowright \kappa$, and the transformation of diagrams reverses. Therefore, we conclude the equivalence of balanced algebroidal and monoidal actegories extends to the braided (and thus, the symmetric) case.

When it comes to distributive algebroidal actegories (Definition 5.2.4), a braiding for $(\mathcal{C}, \bullet, o, \circlearrowright)$ is still a braiding $\beta$ for the underlying monoidal category satisfying an additional linearity axiom, namely

\[
\begin{align*}
 m \bullet (c \circlearrowright d) & \quad \xrightarrow{m \bullet \beta_{c,d}} \quad m \bullet (d \circlearrowright c) \\
 \delta_{m,c,d} & \quad \downarrow \delta_{m,d,c} \\
 (m \bullet c) \quad \quad \xrightarrow{\beta_{m,c,m \bullet d}} \quad (m \bullet d) \quad \quad \xrightarrow{\delta_{m,c,d}} \quad (m \bullet c)
\end{align*}
\]

(5.25)

5.5. The classifying objects of actions. We now have a pretty clear picture of the possible ways monoidal and actegorical structures can interact. In particular, we’ve seen how this interaction amounts to specific ‘distributive laws’ (Table 1). One way to think of those laws is as constraints on the effect of the action on products. Take for example a monoidal $\mathcal{M}$-actegory $(\mathcal{C}, i, \circlearrowright, \bullet)$. We know that in that case

\[
m \bullet (c \circlearrowright d) \cong c \circlearrowright (m \bullet d).
\]

(5.26)
However, *every element of* \( \mathcal{C} \) *can be written as a monoidal product*, in at least two ways:

\[
c \boxtimes i \cong c \cong i \boxtimes c.
\]  

(5.27)

As a consequence, \( \bullet \) is completely determined by its effect on the monoidal unit \( i \):

\[
m \bullet c \cong m \bullet (c \boxtimes i) \cong c \boxtimes (m \bullet i).
\]  

(5.28)

In particular, \( m \mapsto m \bullet i \) defines functor \( - \bullet i : \mathcal{M} \to \mathcal{C} \) which is strong monoidal:

\[
j \bullet i \cong i,
\]

\[
(m \otimes m') \bullet i \cong (m \boxtimes m') \bullet (i \boxtimes i) \cong (m \bullet i) \boxtimes (m' \bullet i).
\]  

(5.29)

Thus one might speculate an equivalence between action on \( \mathcal{C} \) and strong monoidal functors into \( \mathcal{C} \). After all, we already stumbled upon a similar fact in Example 3.6.7, where we observed how actions on \( \mathcal{C} \) are classified by \( [\mathcal{C}, \mathcal{C}] \).

There is a problem though: while every action on \( \mathcal{C} \) does indeed produce a monoidal functor into \( \mathcal{C} \), the converse is not true. In fact given \( F : \mathcal{M} \to \mathcal{C} \), and defining \( m \bullet c := F(m) \boxtimes c \), we do get a well defined action but when it comes to defining the mixed interchanger (i.e. the monoidal structure of \( \bullet \)), we are stuck:

\[
(m \otimes m') \bullet (c \boxtimes d) = F(m \otimes m') \boxtimes c \boxtimes d
\]

\[

\cong F(m) \boxtimes F(m') \boxtimes c \boxtimes d
\]

\[

\cong F(m) \boxtimes c \boxtimes F(m') \boxtimes d
\]

\[

= (m \bullet c) \boxtimes (m' \bullet d).
\]  

(5.30)

Note the missing isomorphism is something of the form \( F(m') \boxtimes c \to c \boxtimes F(m') \).

Thus, the takeaway is that monoidal actions on \( \mathcal{C} \) *encode more information* than a monoidal functor into \( \mathcal{C} \) can provide, namely that of a choice of ‘braiding’ \( F(m) \boxtimes - \to - \boxtimes F(m) \) for each scalar \( m \).

To fix this problem, we are going to introduce two objects, the *Drinfeld center* (Definition 5.5.2) and the *symmetric center* (Definition 5.5.8), that play the role of \( [\mathcal{C}, \mathcal{C}] \) for, respectively, monoidal (Definition 5.1.1) and braided actions (Definition 5.4.1).

The main results of this section are classifying result: they tell us that monoidal actions on a monoidal category \( \mathcal{C} \) correspond to strong monoidal functors into \( \mathcal{Z}(\mathcal{C}) \) and braided actions on a braided category \( \mathcal{C} \) correspond to braided functors into \( \Sigma(\mathcal{C}) \), formalizing the idea we illustrated above. When \( \mathcal{C} \) is symmetric, such a correspondence has the simplest form: braided \( \mathcal{M} \)-actions on \( \mathcal{C} \) correspond to braided functors \( \mathcal{M} \to \mathcal{C} \).
5.5 The classifying objects of actions

5.5.1. Classifying monoidal actions. The Drinfeld center construction is a ‘proof-relevant’ version of the usual center construction $Z(M)$ of a monoid $M$. In fact, the latter is the monoid of those elements $m \in M$ with the property that $\forall n \in M, m \cdot n = n \cdot m$, while the first is a monoidal category of those elements $c : C$ equipped with the structure of a natural isomorphism $c \boxtimes - \xrightarrow{\sim} - \boxtimes c$.

Inasmuch as a monoid is commutative if the inclusion $Z(M) \hookrightarrow M$ admits a section, i.e. if it is surjective; a braiding on a monoidal category $(C, i, \boxtimes)$ correspond to a braided section of the evident forgetful functor $Z(C) \to C$ [Kas95, Corollary XIII.4.4].

The exact definition of $Z(C)$ goes as follows:

5.5.2. Definition. [JS91, Definition 3] Let $(C, i, \boxtimes)$ be a monoidal category. Its Drinfeld center $Z(C)$ is the braided monoidal category defined as follows. Objects are pairs $(c : C, \nu : c \boxtimes - \xrightarrow{\sim} - \boxtimes c)$ such that for every $d, e : C$ (up to associators):

\[
\begin{align*}
    c \boxtimes d \boxtimes e & \xrightarrow{\nu_{d\boxtimes e}} d \boxtimes e \boxtimes c \\
    & \xrightarrow{\nu_{d\boxtimes c}} d \boxtimes c \boxtimes e
\end{align*}
\]

(5.31)

Morphisms $(c, \nu) \to (d, \tau)$ are arrows $f : c \to d$ in $C$ such that for every $b : C$:

\[
\begin{align*}
    c \boxtimes b & \xrightarrow{f \boxtimes b} d \boxtimes b \\
    b \boxtimes c & \xrightarrow{b \boxtimes f} b \boxtimes y
\end{align*}
\]

(5.32)

The monoidal product is given by

\[
(c, \nu) \boxtimes (d, \tau) := (c \boxtimes d, (\nu \boxtimes d) \circ (c \boxtimes \tau)),
\]

(5.33)

with unit $(i, \lambda, \rho^{-1})$. Finally, the braiding is defined as $\beta(c, \nu), (d, \tau) := \nu_d$.

To give a functor $M \to Z(C)$, therefore, is to specify a functor in $C$ together with a way to commute each $F(m)$ past any other object in $C$, exactly what we need to get unstuck in Equation (5.30). In fact, we have the following:

5.5.3. Theorem. [Monoidal actions classifier] $Z(C)$ classifies monoidal actions on $C$, meaning $Z(C)$ acts monoidally on $C$ and every other monoidal action $\bullet$ of a braided
monoidal category \( \mathcal{M} \) factors uniquely through it:

\[
\begin{array}{c}
\mathcal{Z}(\mathcal{C}) \times \mathcal{C} \\
\uparrow \exists ! F \times \mathcal{C} \\
\mathcal{M} \times \mathcal{C}
\end{array} \xrightarrow{\exists !} \mathcal{C}
\]  

(5.34)

In other words, \( \exists !_1 \) is terminal in the category \( \text{Act}_{\text{MonCat}}^\text{cart}(\mathcal{C}) \) of monoidal actions on \( \mathcal{C} \).

5.5.4. Remark. In the above, \( \text{Act}_{\text{MonCat}} \) is the (2-Grothendieck construction of) indexed 2-category of monoidal actions, i.e. the functor \( \text{BrMonCat}^{\text{coop}} \rightarrow \text{2Cat} \) assigning to each braided monoidal category the 2-category of monoidal actegories over it. This is defined analogously to that in Proposition 3.6.5, included the notation \( \text{Act}_{\text{MonCat}}^\text{cart}(\mathcal{C}) \) to denote the cartesian subcategory of \( \text{Act}_{\text{MonCat}}(\mathcal{C}) \) at \( \mathcal{C} \), as defined in Remark 3.6.6.

To swiftly prove Theorem 5.5.3, we’ll rely on the following:

5.5.5. Lemma. [Characterization of the Drinfeld center] Consider the canonical \( \mathcal{C} \)-biactegory \( (\mathcal{C}, \boxtimes, \boxtimes) \) (as defined in Example 4.3.10) associated to \( \mathcal{C} \), that we still denote by \( \mathcal{C} \). Then

\[
\mathcal{Z}(\mathcal{C}) \simeq [\mathcal{C}, \mathcal{C}]_{\mathcal{C}-\text{Act-C}}
\]  

(5.35)

where on the right we have the category of maps \( \varphi : \mathcal{C} \rightarrow \mathcal{C} \) equipped with \( \mathcal{C} \)-linear structures

\[
\ell_{c,d} : \varphi(c) \boxtimes d \xrightarrow{\sim} \varphi(c \boxtimes d), \\
\kappa_{c,d} : c \boxtimes \varphi(d) \xrightarrow{\sim} \varphi(c \boxtimes d).
\]

(5.36)

satisfying the laws stated in Definition 4.3.2.

Proof. See Appendix B. \( \blacksquare \)

The object \( [\mathcal{C}, \mathcal{C}]_{\mathcal{C}-\text{Act-C}} \) is indeed called center in \([BZFN10, \text{Definition 5.1}]\).

Proof of Theorem 5.5.3. By Lemma 5.5.5, we can replace \( \mathcal{Z}(\mathcal{C}) \) with \( [\mathcal{C}, \mathcal{C}]_{\mathcal{C}-\text{Act-C}} \). The latter acts on \( \mathcal{C} \) by evaluation, so we denote the action as eval. Moreover, to give an \( \mathcal{M} \)-action on \( \mathcal{C} \) in \( \text{MonCat} \) is to give isomorphisms (see Remark 5.1.4)

\[
\chi_{m,c,d} : m \bullet (c \boxtimes d) \xrightarrow{\sim} (m \bullet c) \boxtimes d, \\
\kappa_{m,c,d} : m \bullet (c \boxtimes d) \xrightarrow{\sim} c \boxtimes (m \bullet d).
\]

(5.37)
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Hence eval factors ⋅ through the functor

\[ F_\bullet(m) = (m \bullet -, \chi_-\pi, \kappa_-\pi). \]  

(5.38)

Such a factorization is manifestly unique.

5.5.6. COROLLARY. There is an equivalence

\[ \mathcal{BrMonCat}_{\mathcal{MonCat}} \downarrow \mathcal{Z}(C) \simeq \mathcal{Act}^\text{cart}_{\mathcal{MonCat}}(C) \]

(5.39)

where the 2-category on the left amounts to the subcategory of \( \mathcal{MonCat}/\mathcal{Z}(C) \) given by braided monoidal categories whose braiding has been forgotten.

PROOF. Observe that the action eval of \( [C, C]_{\mathcal{C}, \mathcal{Act}_C} \) translates back to the action \( \boxtimes_1 \) of \( \mathcal{Z}(C) \), which is simply monoidal product after forgetting the braiding:

\[ (c, v) \boxtimes_1 d := c \boxtimes d. \]

(5.40)

Running the proof of Lemma 5.5.5 backwards, we thus get that the correspondence is given as follows. Given a monoidal functor \( F : \mathcal{M} \to \mathcal{Z}(C) \), where \( \mathcal{M} \) is braided, we get the \( \mathcal{M} \)-action

\[ m \bullet^F c := F_1(m) \boxtimes c, \]

(5.41)

while given a monoidal \( \mathcal{M} \)-action \( \bullet \) on \( C \), we get the monoidal functor \( \mathcal{M} \to \mathcal{Z}(C) \) defined as

\[ F_\bullet(m) := (m \bullet i, \chi_{m,i}). \]

(5.42)

That these are well-defined objects follows from the main theorem.

5.5.7. CLASSIFYING BRAIDED ACTIONS. Now the same can be done for actions of symmetric monoidal categories on braided and symmetric monoidal categories but, again, the classifying object will be different.

5.5.8. DEFINITION. Let \( (C, i, \boxtimes, \beta) \) be a braided monoidal category. Its symmetric center \( \Sigma(C) \) is the full subcategory of \( C \) determined by those objects \( c : C \) so that, for all \( d : C \), the following commutes:

\[ \beta_{c,d} \quad d \boxtimes c \quad \beta_{d,c} \]

\[ c \boxtimes d \quad \boxtimes \quad c \boxtimes d \]

(5.43)
5.5.9. **Theorem.** [Braided actions classifier] Let $(\mathcal{C}, i, \boxtimes, \beta)$ be a braided monoidal category. $\Sigma(\mathcal{C})$ classifies braided actions on $\mathcal{C}$, meaning $\Sigma(\mathcal{C})$ acts braidedly on $\mathcal{C}$ and every other braided action $\bullet$ of a symmetric monoidal category $\mathcal{M}$ factors uniquely through it:

$$\Sigma(\mathcal{C}) \times \mathcal{C} \xrightarrow{\boxtimes} \mathcal{C}$$

$$\exists F \times \mathcal{C}$$

$\mathcal{M} \times \mathcal{C}$

$$\text{(5.44)}$$

In other words, $\boxtimes$ is terminal in the category $\text{Act}_{\text{cart MonCat}}(\mathcal{C})$ of braided actions on $\mathcal{C}$.

As before, in order to prove this we rely on the following characterization:

5.5.10. **Lemma.** [Characterization of the symmetric center] The symmetric center of $(\mathcal{C}, i, \boxtimes, \beta)$ is equivalent to the full subcategory of $\mathcal{Z}(\mathcal{C})$ spanned by those $\varphi$ for which the following diagrams commute:

$$\varphi(c) \boxtimes d \xrightarrow{r_{c,d}} \varphi(c \boxtimes d) \quad \quad c \boxtimes \varphi(d) \xrightarrow{\ell_{c,d}} \varphi(c \boxtimes d)$$

$$\beta_{\varphi(c),d} \downarrow \quad \quad \varphi(\beta_{c,d}) \downarrow \quad \quad \varphi(\beta_{c,d})$$

$$d \boxtimes \varphi(c) \xrightarrow{\ell_{d,c}} \varphi(d \boxtimes c) \quad \quad \varphi(d) \boxtimes c \xrightarrow{r_{d,c}} \varphi(d \boxtimes c)$$

$$\text{(5.45)}$$

We denote this subcategory by $[\mathcal{C}, \mathcal{C}]_{\mathcal{C} \text{-Act-}\mathcal{C}}^{\text{br}}$.

**Proof.** The proof amounts to checking that bilinear endomorphisms which satisfy (5.45) are given, up to isomorphism, by tensoring with a symmetric object. Assume $\mathcal{C}$ is strict monoidal, and let $(\varphi, r)$ be a bilinear endomorphism strict on the left (we use the same trick employed in Lemma 5.5.5). Drawing (5.45) for $c = j$ proves that $\beta_{d,\varphi(j)}$ and $\beta_{\varphi(j),d}$ are both identities. In fact, when $\mathcal{C}$ is strict, $r_{j,d} = 1$ as observed in Remark 3.3.3. Viceversa, given a symmetric object, we readily prove tensoring with it gives a functor satisfying (5.45).

**Proof of Theorem 5.5.9.** By Lemma 5.5.10, we can replace $\Sigma(\mathcal{C})$ with $[\mathcal{C}, \mathcal{C}]_{\mathcal{C} \text{-Act-}\mathcal{C}}^{\text{br}}$. With this setup, everything works very much like the proof of Theorem 5.5.3. One only has to be sure that the universal morphism $F_\bullet$ actually lands in $[\mathcal{C}, \mathcal{C}]_{\mathcal{C} \text{-Act-}\mathcal{C}}^{\text{br}}$ when $\bullet$ is braided, and viceversa. But Diagram (5.22) (sealing $\bullet$ braidedness) looks exactly like Diagram (5.45), concluding the proof.
5.5 The classifying objects of actions

5.5.11. Corollary. There is an equivalence

\[
\text{SymMonCat} \downarrow_{\text{BrMonCat}} \Sigma(C) \simeq \text{Act}^{\text{cart}}_{\text{BrMonCat}}(C) \tag{5.46}
\]

where the 2-category on the left amounts to the subcategory of $\text{BrMonCat}/\Sigma(C)$ given by symmetric monoidal categories.

**Proof.** To spell explicitly the correspondence on objects, given a braided functor $F : \mathcal{M} \to \Sigma(C)$, where $\mathcal{M}$ is symmetric, we get the $\mathcal{M}$-action

\[
m \cdot^F c := F(m) \boxtimes c, \tag{5.47}
\]

while given a braided $\mathcal{M}$-action $\bullet$ on $\mathcal{C}$, we get the braided functor $\mathcal{M} \to \Sigma(C)$ defined as

\[
F \bullet (m) := m \cdot i. \tag{5.48}
\]

The well-definedness of these objects follows from the main theorem. \qed

Now suppose $\mathcal{C}$ is symmetric. In that case, $\Sigma(C) = \mathcal{C}$, therefore we automatically have the following:

5.5.12. Corollary. [Symmetric actions classifier] Let $(\mathcal{C}, i, \boxtimes, \sigma)$ be a symmetric monoidal category. $\mathcal{C}$ classifies symmetric actions on $\mathcal{C}$, meaning $\mathcal{C}$ acts symmetrically on $\mathcal{C}$ and every other symmetric action $\bullet$ of a symmetric monoidal category $\mathcal{M}$ factors uniquely through it:

\[
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{\boxtimes} & \mathcal{C} \\
\downarrow_{\exists F \times \mathcal{C}} & & \uparrow \\
\mathcal{M} \times \mathcal{C}
\end{array}
\tag{5.49}
\]

In other words, $\boxtimes$ is terminal in the category $\text{Act}^{\text{cart}}_{\text{SymMonCat}}(\mathcal{C})$ of braided actions on $\mathcal{C}$, thus inducing an equivalence

\[
\text{SymMonCat}/\mathcal{C} \simeq \text{Act}^{\text{cart}}_{\text{SymMonCat}}(\mathcal{C}). \tag{5.50}
\]

5.5.13. Example. Let $(\text{Msbl}, \bullet)$ be the $\text{Prob}$-actegory of Example 3.2.11. This is a monoidal actegory, since the action of a probability space $(\Omega, \mathcal{F}, P)$ corresponds to the forgetful functor $\text{Prob} \to \text{Msbl}$ which is evidently braided monoidal, like $(\text{Msbl}, 1, \times)$.
5.5.14. **EXAMPLE.** In Equation (5.20), we have exhibited an action $\triangleleft$ of $\mathbf{Cat}$ on $(\mathcal{M}\text{-}\mathbf{Act}, \mathcal{M}, \otimes_{\mathcal{M}})$, given by $\mathcal{M}[-] \otimes_{\mathcal{M}} =$. We can conclude this action is symmetric by noting the functor $\mathcal{M}[-] \otimes_{\mathcal{M}} \mathcal{M} \cong \mathcal{M}[-]$ is indeed braided monoidal. In fact, we proved in Proposition 4.4.10 that $\mathcal{M}[\mathcal{C}] \otimes_{\mathcal{M}} \mathcal{M}[\mathcal{D}] \simeq \mathcal{M}[\mathcal{C} \times \mathcal{D}]$ and this makes $\mathcal{M}[-]$ evidently monoidal and braided.

5.5.15. **REMARK.** In Remark 4.2.5, we observed how the classification result of Example 3.6.7 could be used to construct coproducts and products in $\mathbf{Act}^{\text{cart}}(\mathcal{C})$. We observe that totally analogous definitions can be repeated for monoidal, braided and symmetric actions. This has repercussions on the theory of hybrid composition of optics by providing us with guarantees on the monoidal structure of the resulting category of hybrid optics (remember optics generated by monoidal actions are monoidal, as shown in Remark 5.1.9).
6. Conclusion

In the past pages, we’ve been on a journey through ‘actegory theory’. We charted a territory sparsely surveyed before, and managed to uncover some new corners of this vast subject. Specifically, we contemplated the role of monoidal structure in the different ways it can present itself, and described the links between these phenomena and those in the theory of optics.

Large swaths of actegory theory didn’t make it in our maps. Perhaps the largest and most relevant to our eyes is Tambara theory [Tam06, PS08, Rom20] and the study of the proarrow equipment of actegories and Tambara modules.

There’s also dark corners, still to be visited, like the exact nature of the relation between braiding and linear structures. It seems it has never been contemplated the idea of a structure $\rho_{c,m} : c \circ m \xrightarrow{\sim} m \bullet c$ on a biactegory $(\mathcal{C}, \bullet, \circ)$, linking the left and right actions. It is, in some sense, the actegorical equivalent of a braiding, and in fact a braiding on the scalars induces one canonically (we have seen this in Proposition 4.3.7). Indeed, a ‘reflection structure’ such as $\rho$ seems to arise anytime there is a strong monoidal functor $\mathcal{M} \to \mathcal{M}^{\text{rev}}$. A shadow of these structures can be observed in Lemma 5.5.10, where we characterized the symmetric center of a monoidal category in terms of endomorphisms of $(\mathcal{C}, \boxtimes, \boxtimes)$ that somehow commute with $\beta$. In terms of reflection structures, this constitutes a natural notion of morphism.

In conclusion, let us mention a major perspective we didn’t cover here—that of actegories as presheaves. The data of a left $\mathcal{M}$-action, in fact, can be encoded ‘externally’ as a functor $\mathbb{B}\mathcal{M} \to \text{Cat}$, whereas a right actegory would be given by $\mathbb{B}\mathcal{M}^{\text{op}} \to \text{Cat}$ since $\mathbb{B}(\mathcal{M}^{\text{rev}}) = \mathbb{B}\mathcal{M}^{\text{op}}$. The upshot is, actegories can be seen as very special indexed categories, opening up the extension of all the theory we sketched here to actions of categories, bicategories and ultimately double categories [Bak08]. This extension seems to be warranted if we aim to extend the theory of optics to ‘dependent’ or ‘indexed optics’, i.e. if we desire to deploy dependent types in the current constructions. The idea that actions of bicategories are instrumental to this objective has been advanced in [BCG+21] and then later in [Mil22].
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A. Definitions

2.0.1. Definition. Let \( \mathcal{X} \) be a 2-category. A \textbf{pseudomonad} on \( \mathcal{X} \) is a strict 2-functor \( T: \mathcal{X} \to \mathcal{X} \), together with 2-morphisms \( j: 1 \Rightarrow T \) (\textit{unit}) and \( m: TT \Rightarrow T \) (\textit{multiplication}) and modifications (left unitor \( \lambda \), right unitor \( \rho \), associator \( \alpha \)):

\[
\begin{align*}
T & \xrightarrow{j_T} TT & TT & \xleftarrow{T_j} T \\
& \quad \downarrow^{\lambda} \downarrow^{m} \quad & \quad \downarrow^{m} \quad & \quad \downarrow^{\rho} \\
T & & T & \\
\end{align*}
\]  

(2.5)

\[
\begin{align*}
TTT & \xrightarrow{T_m} TT & TT & \xrightarrow{m} T \\
& \quad \downarrow^{\alpha} \downarrow^{m} \quad & \quad \downarrow^{m} \quad & \quad \downarrow^{\lambda} \\
TT & & T & \\
\end{align*}
\]  

(2.6)

satisfying the following coherence laws:

I) Coherence of the associator:

\[
\begin{align*}
TTTT & \xrightarrow{T_{TTm}} TTT & TTTT & \xrightarrow{TTm} TTT \\
& \quad \downarrow^{\alpha} \downarrow^{m} \quad & \quad \downarrow^{m} \quad & \quad \downarrow^{\alpha} \\
TTT & \xrightarrow{mT} TT & TTT & \xrightarrow{Tm} TT \\
& \quad \downarrow^{\alpha} \downarrow^{m} \quad & \quad \downarrow^{m} \quad & \quad \downarrow^{\alpha} \\
TT & \xrightarrow{m} T & TT & \xrightarrow{TTm} T \\
& \quad \downarrow^{\alpha} \downarrow^{m} \quad & \quad \downarrow^{m} \quad & \quad \downarrow^{\alpha} \\
T & & T & \\
\end{align*}
\]  

(A.1)

II) Coherence of the unitors:

\[
\begin{align*}
TT & \xrightarrow{Tj_T} TT & TT & \xrightarrow{T}\nu
\\
\end{align*}
\]  

(A.2)
A.0.1. **Proposition.** The endofunctor $\mathcal{M} \times - : \mathbf{Cat} \to \mathbf{Cat}$ admits a pseudomonad structure given by $(j \times -, \otimes \times -, \lambda \times -, \rho \times -, \alpha \times -)$.

**Proof.** Notice Equation (A.1) depicts two halves of a cube. The fillings of the faces allow to map between the six different ways to move from the top left vertex to the bottom right one. The two sides of the equation correspond to the two ways to map from the lower outer edge of the cube to the upper outer edge. Notice these maps are themselves obtained by applying the modifications $A$ in succession. There are exactly five such modifications appearing in the axiom, thus forming a pentagon. Such a pentagon has a composite of length two, given by the left hand side of (A.1), and a composite of length three, given by the right hand side. Therefore asking (A.1) to hold for $\mathcal{M} \times -$ is equivalent to the pentagonal coherence of the associator $\alpha$ of $\mathcal{M}$ [JY21, Equation 1.2.5].

A similar reasoning applies to the Equation (A.2), which is equivalent to the triangular coherence involving the unitors $\lambda, \rho$ and the associator $\alpha$ [JY21, Equation 1.2.4].

2.2.1. **Definition.** A **pseudoalgebra** for a pseudomonad $(T, j, m)$ on $\mathbb{X}$ is a morphism

$$t : Tx \to x$$

in $\mathbb{X}$ equipped with invertible 2-cells

$$
\begin{array}{c}
  x \\ \\
  \downarrow \eta \\ \\
  x \\
\end{array}
\quad
\begin{array}{c}
  TTx \\ \\
  \downarrow m_x \\ \\
  TTx \\
\end{array} 
\quad
\begin{array}{c}
  TTx \\ \\
  \downarrow m_x \\ \\
  TTx \\
\end{array} 
\quad
\begin{array}{c}
  TTx \\ \\
  \downarrow m_x \\ \\
  Tx \\
\end{array} 
\quad
\begin{array}{c}
  Ttx \\ \\
  \downarrow \mu_x \\ \\
  Ttx \\
\end{array}
$$

satisfying the following coherence axioms:

I) Compatibility with the unit:
II) Compatibility with the multiplication:

\[ T_x \xrightarrow{T_j} TT_x = \xymatrix{T_x \ar[r]^{T_j} & TT_x \ar[r] & T_x \ar[r]^t & x} \quad (A.4) \]

2.2.2. **Definition.** Let \((T, j, m)\) be a pseudomonad on \(X\) and let \((t, \eta, \mu) : T x \to x\), \((t', \eta', \mu') : T y \to y\) be pseudoalgebras. A **lax morphism** \((f, \ell) : t \to t'\) consists of a morphism \(f : x \to y\) and a 2-cell

\[ \xymatrix{T_x \ar[r]^{Tf} & Ty \ar[d]_{\ell} \ar[r]^{Tf} & Ty \ar[d]_{\ell} \ar[r] & Ty \ar[d]_{\ell} \ar[r] & Ty} \]

\[ \xymatrix{x \ar[r]^f & y \ar[d]_\ell \ar[r]^f & y \ar[d]_\ell \ar[r] & y \ar[d]_\ell \ar[r] & y} \]

obeying the following coherence laws:

I) Compatibility with the unitor:

\[ \xymatrix{T_x \ar[r]^{Tf} & Ty \ar[r]_{Tj_y} & T_x \ar[r]^t & x} = \xymatrix{x \ar[r]^{j_x} & T_x \ar[r]_{\eta_x} & T_x \ar[r]^t & x} \quad (A.5) \]

II) Compatibility with the multiplicator:

\[ \xymatrix{TTx \ar[r]^{TTf} & TTx \ar[r]_{TTf} & TTx \ar[r] & Ty} = \xymatrix{TTx \ar[r]^{TTf} & TTy \ar[r]_{TTf} & TTy \ar[r] & Ty} \quad (A.6) \]
5.1.1. Definition. An oplax monoidal left $\mathcal{M}$-actegory is a monoidal category $(\mathcal{C}, i, \boxtimes)$ equipped with a lax monoidal functor $\bullet : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ and two monoidal natural transformations $\mu$ and $\eta$ defined analogously to Definition 3.1.1. When $\bullet$ is strong, we call it simply monoidal actegory.

As observed in Remark 5.1.2, the only extra structure needed is the mixed interchanger:

$$\iota_{m,n,d} : (m \otimes n) \bullet (c \boxtimes d) \xrightarrow{\sim} (m \bullet c) \boxtimes (n \bullet d).$$ (A.7)

All in all, this structure needs to satisfy, on top of the two laws of actegories (Definition 3.1.1), the following coherence laws regarding $\bullet$, $\mu$ and $\eta$ monoidality.

I) Compatibility with the associator, for every $a, b, c : \mathcal{C}$ and $m, n, p : \mathcal{M}$:

$$((m \bullet a) \boxtimes (n \bullet b)) \boxtimes (p \bullet c) \xrightarrow{\alpha^\mathcal{C}_{m \bullet a, n \bullet b, p \bullet c}} (m \bullet a) \boxtimes ((n \bullet b) \boxtimes (p \bullet c))$$  

$$\iota_{m,n,a,b,c} : (m \otimes n) \bullet (a \boxtimes b) \boxtimes (p \bullet c) \xrightarrow{\iota_{m,n \otimes p, a \boxtimes b, c}} (m \bullet (n \otimes p)) \bullet (a \boxtimes (b \boxtimes c))$$  

II) Compatibility with unitors, for every $c : \mathcal{C}$ and $m : \mathcal{M}$:

$$i \boxtimes (m \bullet x) \xleftarrow{\eta_i \boxtimes (m \bullet x)} (j \bullet i) \boxtimes (m \bullet x) \xrightarrow{(m \bullet x) \boxtimes i} (m \bullet x) \boxtimes (j \bullet i)$$  

$$\lambda^\mathcal{M}_{m \bullet x} : m \bullet x \xrightarrow{\lambda^\mathcal{M}_{m \bullet x}} (j \otimes m) \bullet (i \boxtimes x) \xrightarrow{\iota_{j,m,i,x}} m \bullet x \xleftarrow{\rho^\mathcal{M}_{m \bullet x}} (m \otimes j) \bullet (x \boxtimes i)$$  

III) Monoidality of $\eta$, for every $c, d : \mathcal{C}$:

$$j \bullet c \boxtimes (j \bullet d) \xrightarrow{\eta_{j \bullet c \boxtimes j \bullet d}} c \boxtimes d$$  

$$\lambda^\mathcal{M}_{j \bullet c \boxtimes j \bullet d} : (j \otimes j) \bullet (c \boxtimes d) \xrightarrow{\lambda^\mathcal{M}_{j \bullet c \boxtimes j \bullet d}} j \bullet (c \boxtimes d)$$  

(A.10)
IV) Monoidality for $\mu$, for every $c, d : C$ and $m, n, m', n' : M$:

\[
\begin{array}{c}
(m \bullet (m' \bullet c)) \boxtimes (n \bullet (n' \bullet d)) \\
\downarrow \ell_{m, n, m', n', c, n', d} \\
(m \otimes n) \bullet ((m' \bullet c) \boxtimes (n' \bullet d)) \\
\downarrow \ell_{m \otimes m', n \otimes n', c, d} \\
(m \otimes n) \bullet ((m' \otimes n') \bullet (c \boxtimes d)) \\
\downarrow \ell_{m \otimes m', n \otimes n', c, d} \\
((m \otimes m') \bullet (n \otimes n')) \bullet (c \boxtimes d)
\end{array}
\]

(A.11)

5.2.1. Definition. [Balanced algebroidal $\mathcal{M}$-actegory] A balanced algebroidal $\mathcal{M}$-actegory is a pseudomonoid in $(\mathcal{M} \text{-Act}, \mathcal{M}, \otimes_{\mathcal{M}})$, amounting to a left $\mathcal{M}$-actegory $(\mathcal{C}, \bullet)$ together with $\mathcal{M}$-linear functors

\[
I : \mathcal{M} \to \mathcal{C},
\]

\[
\boxtimes : \mathcal{C} \otimes_{\mathcal{M}} \mathcal{C} \to \mathcal{C}
\]

and $\mathcal{M}$-linear natural transformations $\lambda, \rho, \alpha$ satisfying the usual axioms for unitors and associators in pseudomonoids. The extra coherence laws satisfied by $\lambda, \rho, \alpha, \ell$ and $\varepsilon$ are the following:

I) Coherence for the left lineator $\ell$:

\[
\begin{array}{c}
m \bullet (n \bullet (c \boxtimes d)) \\
\downarrow \mu_{m, n, c \boxtimes d} \\
(m \otimes n) \bullet (c \boxtimes d) \\
\downarrow \ell_{m \otimes n, c, d} \\
((m \otimes n) \bullet c) \boxtimes d
\end{array}
\]

(A.12)

\[
\begin{array}{c}
j \bullet (c \boxtimes d) \\
\downarrow \eta_{c \boxtimes d} \\
c \boxtimes d
\end{array}
\]

(A.13)
II) Coherence for the equilibrator $\varepsilon$:

\[
\begin{array}{c}
\begin{array}{ll}
(n \bullet (m \bullet c)) \boxtimes d \\
(m \bullet c) \boxtimes (n \bullet d) \\
((n \otimes m) \bullet c) \boxtimes d
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ll}
\varepsilon_{n,m,c,d} & \mu_{n,m,c,d} \\
\varepsilon_{m,c,n,d} & \beta_{n,m,c,d} \\
\varepsilon_{m\otimes n,c,d} & \mu_{m\otimes n,c,d}
\end{array}
\end{array}
\]

\[\text{(A.14)}\]

III) Linearity of $\lambda$ and $\rho$:

\[
\begin{array}{c}
\begin{array}{ll}
m \bullet (i \boxtimes c) & m \bullet c \\
(m \bullet i) \boxtimes c & i \boxtimes (m \bullet c)
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ll}
m \bullet (c \boxtimes i) & m \bullet c \\
(m \bullet c) \boxtimes i
\end{array}
\end{array}
\]

\[\text{(A.16)}\]

IV) Linearity of $\alpha$:

\[
\begin{array}{c}
\begin{array}{ll}
m \bullet ((a \boxtimes b) \boxtimes c) & m \bullet (a \boxtimes (b \boxtimes c)) \\
(m \bullet (a \boxtimes b)) \boxtimes c
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ll}
m \bullet (m \bullet a) \boxtimes (b \boxtimes c)
\end{array}
\end{array}
\]

\[\text{(A.17)}\]
A DEFINITIONS

V) Balance of $\alpha$:

\[
\begin{array}{c}
(m \cdot (a \boxtimes b)) \boxtimes c \\
\downarrow \epsilon_{m,a \boxtimes b,c}
\end{array}
\xrightarrow{\alpha_{a,b,m \cdot c}}
\begin{array}{c}
(a \boxtimes b) \boxtimes (m \cdot c)
\end{array}
\]

(5.18)

5.2.4. Definition. [Distributive algebroidal $\mathcal{M}$-actegory] A **distributive algebroidal $\mathcal{M}$-actegory** is a pseudomonoid in $(\mathcal{M} \text{-Act}, 1, \times)$, i.e. a left $\mathcal{M}$-actegory $(\mathcal{C}, \bullet)$ together with $\mathcal{M}$-linear functors:

\[
o : 1 \to \mathcal{C},
\]

\[
times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}
\]

and $\mathcal{M}$-linear natural transformations $\lambda, \rho, \alpha$ satisfying the usual axioms for unitors and associators in pseudomonoids, and natural $\mathcal{M}$-linear isomorphisms

\[
\gamma_m : m \cdot o \sim o,
\]

\[
\delta_{m,c,d} : m \cdot (c \boxplus d) \sim (m \cdot c) \boxplus (m \cdot d),
\]

(5.14)

called **absorber** and **distributor**, respectively, satisfying the usual laws of strong monoidal structures. The extra coherence laws satisfied by $\lambda, \rho, \alpha, \ell$ and $\varepsilon$ are the following:

I) Coherence for the absorber $\gamma$:

\[
\begin{array}{c}
m \cdot (n \cdot o)
\end{array}
\xrightarrow{m \cdot \gamma_n}
\begin{array}{c}
m \cdot o
\end{array}
\]

\[
\begin{array}{c}
(m \otimes n) \cdot o
\end{array}
\xrightarrow{\gamma_{m \otimes n}}
\begin{array}{c}
o
\end{array}
\]

(5.19)

II) Coherence for the distributor $\delta$:

\[
\begin{array}{c}
m \cdot (n \cdot (c \boxplus d))
\end{array}
\xrightarrow{m \cdot \delta_{n,c,d}}
\begin{array}{c}
m \cdot ((n \cdot c) \boxplus (n \cdot d))
\end{array}
\]

\[
\begin{array}{c}
(m \otimes n) \cdot (c \boxplus d)
\end{array}
\xrightarrow{\delta_{m \otimes n,c,d}}
\begin{array}{c}
((m \otimes n) \cdot c) \boxplus ((m \otimes n) \cdot d)
\end{array}
\]

(5.20)
III) Linearity of \( \lambda \) and \( \rho \):

\[
\begin{align*}
\delta_{m,o,c} & \quad \lambda_{m \cdot c} \\
(m \cdot o) \boxdot (m \cdot c) & \quad \to \quad o \boxdot (m \cdot c) \\
(m \cdot c) \boxdot (m \cdot o) & \quad \to \quad (m \cdot c) \boxdot o
\end{align*}
\]

(A.22)

IV) Linearity of \( \alpha \):

\[
\begin{align*}
\delta_{m,a,b,c} & \quad \alpha_{m \cdot a,m \cdot b,m \cdot c} \\
(m \cdot a) \boxdot (m \cdot b \boxdot c) & \quad \to \quad ((m \cdot a) \boxdot b) \boxdot c \\
(m \cdot a) \boxdot ((m \cdot b) \boxdot (m \cdot c)) & \quad \to \quad ((m \cdot a) \boxdot (m \cdot b) \boxdot (m \cdot c))
\end{align*}
\]

(A.23)

B. Proofs

4.3.6. THEOREM. There is an equivalence of 2-categories

\[
\mathcal{N} \textbf{-Act} \text{-}\mathcal{M}^\text{lx} \simeq (\mathcal{N}^\text{rev} \times \mathcal{M}) \text{-}\textbf{Act}^\text{lx}. \quad (4.25)
\]

PROOF. The most interesting part of the proof concerns obtaining an \( \mathcal{N}^\text{rev} \times \mathcal{M} \)-action from a given \( (\mathcal{M}, \mathcal{N}) \)-biactegory.

Indeed, suppose given a left \( \mathcal{M} \)-action \( (\bullet, \eta^\bullet, \mu^\bullet) \), a right \( \mathcal{N} \)-action \( (\circ, \eta^\circ, \mu^\circ) \) and a bimodulator \( \zeta \). We already know how from Equation (4.3) how to define the carriers of the composite \( \mathcal{N}^\text{rev} \times \mathcal{M} \)-action \( \circ \):

\[
(n,m) \circ c := (m \cdot c) \circ n. \quad (B.1)
\]
The unitor is given canonically by:

\[ \eta_c : c \xrightarrow{j \circ} (j \circ c) \xrightarrow{\eta \circ c} (j \circ c) \circ i \Rightarrow (i, j) \circ c. \]

(B.2)

In defining the multiplicator, it’s crucial to have \( \zeta \):

\[ \mu_{(n, m), (n', m'), c} := (n, m) \circ ((n', m') \circ c) \]
\[ \Rightarrow (m \circ ((m' \circ c) \circ n')) \circ n \]
\[ \xrightarrow{\zeta_{m, m', (n', m')} \circ c \circ n} ((m \circ (m' \circ c)) \circ n') \circ n \]
\[ \xrightarrow{(\mu_{m, m', c} \circ (n', m') \circ c)} ((m \circ m') \circ c) \circ (n' \circ n) \]
\[ \Rightarrow (n \circ n', m \circ m') \circ c. \]

(B.3)

Using strictification and the coherence diagrams of \( \zeta \), this data can be proven well-defined. For instance, from (4.22), we can see \( \zeta_{m, c, j} \) is an identity when \( \circ \) and \( \cdot \) are strict. This immediately entails \( \mu_{(j, j), (n', m'), c} \) is an identity, thereby satisfying (3.4).

Given a \( N \times M \)-action \((\cdot, \eta, \mu)\), we get a left \( M \)-action \((\cdot, \eta', \mu')\) and a right \( N \)-action \((\circ, \eta, \mu)\) as follows:

\[ m \circ c := (i, m) \circ c, \quad \eta_c := \eta_c, \quad \mu_{m, m', c} := \mu_{(j, m), (j, m'), c} \]
\[ c \circ n := (n, j) \circ c, \quad \eta^c := \eta^c, \quad \mu_{n, n', c} := \mu_{(n, j), (n', j), c} \]

(B.4)

The bimodulator \( \zeta \) is then given by

\[ \zeta_{m, c, n} := m \circ (c \circ n) \]
\[ \Rightarrow (j, m) \circ ((n, j) \circ c) \]
\[ \xrightarrow{\mu^c} (j \circ n, m \circ j) \circ c \]
\[ \xrightarrow{(m \circ m, \mu^M)} (n, m) \circ c \]
\[ \xrightarrow{(\chi^N, \chi^M)} (n \circ j, j \circ m) \circ c \]
\[ \xrightarrow{\mu^c} (n, j) \circ ((j, m) \circ c) \]
\[ \Rightarrow (m \circ c) \circ n. \]

(B.5)

By strictification (Lemma 3.4.1), this is a well-defined bimodulator (in fact, substituting the above in (4.19)–(4.22) produces diagrams involving only structure morphisms of \( \cdot, N \) and \( M \), which become identities after the strictification).
On morphisms, it’s easy to observe every lax \((\mathcal{M}, \mathcal{N})\)-bilinear functor (Definition 4.3.2) is also lax \(\mathcal{N}^{\text{rev}} \times \mathcal{M}\)-linear, since this amounts to be lax \(\mathcal{N}\)-linear on the right and lax \(\mathcal{M}\)-linear on the left, and (4.23) corresponds to the fact that

\[ b_{(m \otimes j, j \otimes n), c} = b_{(j \otimes m, n \otimes j), c} \]  

for an \(\mathcal{N}^{\text{rev}} \times \mathcal{M}\)-linear structure \(b\) on a given functor. For the same reasons, the opposite is true too. Finally, \((\mathcal{M}, \mathcal{N})\)-bilinear transformations (Definition 4.3.4) and transformations which are both \(\mathcal{M}\)-linear and \(\mathcal{N}\)-linear are exactly the same thing.

4.4.6. Proposition. Let \(\mathcal{C}\) be an \((\mathcal{N}, \mathcal{M})\)-biactegory, \(\mathcal{D}\) be a \((\mathcal{M}, \mathcal{P})\)-biactegory and \(\mathcal{E}\) be a \((\mathcal{N}, \mathcal{P})\)-biactegory, let \(\bullet\) denote all actions by abuse of notation. Then the category of (right) \(\mathcal{P}\)-linear maps \(\mathcal{D} \to \mathcal{E}\), which we denote by \([\mathcal{D}, \mathcal{E}]_\mathcal{P}\) has a canonical \((\mathcal{N}, \mathcal{M})\)-biactegory structure, and is such that:

\[ \mathcal{N}\text{-}\text{Act-}\mathcal{P}^{\text{lx}}(\mathcal{C} \otimes_\mathcal{M} \mathcal{D}, \mathcal{E}) \cong \mathcal{N}\text{-}\text{Act-}\mathcal{M}^{\text{lx}}(\mathcal{C}, [\mathcal{D}, \mathcal{E}]_\mathcal{P}). \]  

We refer to this bracket as the \textbf{internal hom} \((\mathcal{N}, \mathcal{M})\)-biactegory of \(\mathcal{D}\) and \(\mathcal{E}\).

\textbf{Proof.} We preemptively put ourselves in the situation in which \(\mathcal{C}, \mathcal{D}\) and \(\mathcal{E}\) are all strict, as well as their categories of scalars. Of course this doesn’t mean \([\mathcal{D}, \mathcal{E}]_\mathcal{P}\) only contains strictly linear functors. Let’s start to show it is a well-defined object, i.e. that there are well-defined actions of \(\mathcal{N}\) and \(\mathcal{M}\) and a compatibility between them.

Given a scalar \(n : \mathcal{N}\), one defines its action on a right \(\mathcal{P}\)-linear functor pointwise \(F : \mathcal{D} \to \mathcal{E}\):

\[ (n \bullet F)(d) := n \bullet F(d) \]  

\text{(B.7)}

In this way we inherit all left \(\mathcal{N}\)-structure from \(\mathcal{E}\), including a little help from its bimodulator to provide \(n \bullet F\) with a \(\mathcal{P}\)-linear structure. On the right, a scalar \(m : \mathcal{M}\) acts \textit{before} the functor is applied:

\[ (F \bullet m)(d) := F(m \bullet d). \]  

\text{(B.8)}

Again, all the structure is inherited from \(\mathcal{D}\), including what’s needed for fully defining the \(\mathcal{P}\)-linear structure. Hence the right \(\mathcal{M}\)-structure on \([\mathcal{D}, \mathcal{E}]_\mathcal{P}\) thus defined is as strict as the one we assumed on \(\mathcal{D}\):

\[ (F \bullet j)(d) = F(j \bullet d) = F(d), \]

\[ ((F \bullet m) \bullet n)(d) = F(m \bullet (n \bullet d)) = F((m \otimes n) \bullet d) = (F \bullet (m \otimes n))(d). \]  

\text{(B.9)}
Finally, the bimodulator interchanging these two actions is just the identity:

\[ (n \bullet (F \bullet m))(d) = n \bullet F(m \bullet d) = ((n \bullet F) \bullet m)(d). \] (B.10)

To prove (4.46), we ground ourselves in the usual tensor-hom adjunction for categories. Let \( \lambda \) denote the currying morphism. On objects, we have to prove that the balanced structure of a given \((N, P)\)-bilinear functor \( G : C \otimes_{\mathcal{M}} D \to \mathcal{E} \) corresponds to a right \( \mathcal{M} \)-linear structure on its curried version \( \lambda G \), and similarly for the left \( \mathcal{N} \)-linear structures.

Regarding the first, we have

\[ (\lambda G(c) \bullet m)(d) = \lambda G(c)(m \bullet d) = G(c, m \bullet d) = G(c \bullet m, d) = \lambda G(c \bullet m)(d), \] (B.11)

and likewise for the second:

\[ (n \bullet \lambda G(c))(d) = n \bullet (\lambda G(c)(d)) = n \bullet G(c, d) = G(n \bullet c, d) = \lambda G(n \bullet c)(d). \] (B.12)

Regarding morphisms, given a \((N, P)\)-bilinear transformation \( \xi_{c,d} : F(c, d) \Rightarrow G(c, d) \), then we can curry this one too to get a family \( \lambda \xi_c : \lambda F(c) \Rightarrow \lambda G(c) \) of natural morphisms \( (\lambda \xi_c)_d : \lambda F(c)(d) \Rightarrow \lambda G(c)(d) \), which amount to the same data. By the previous discussion on linearity, the linear constraints also move across the currying without impediment. This concludes the proof.

\[ \square \]

4.4.8. Corollary. The tensor product of \( \mathcal{M} \)-actegories \( \otimes_{\mathcal{M}} \) can be equipped with the structure of a monoidal product on \( \mathcal{M}\text{-}\text{Act}\text{-}\mathcal{M}^\text{lx} \), and the internal hom \([-,-]_\mathcal{M} \) defines a pseudoclosed structure for it.

\textbf{Proof.} We complete the proof by defining the extra structure required to make \( \mathcal{M}\text{-}\text{Act}\text{-}\mathcal{M}^\text{lx} \) into a monoidal 2-category [JY21, Explanation 12.1.3]:

1. The unit for this product is the biactegory \((\mathcal{M}, \otimes, \otimes)\). In fact it can be easily verified that the family \( \tau \) described in Proposition 4.4.4 induces an equivalence

\[
\begin{array}{ccc}
\mathcal{M} \otimes_{\mathcal{M}} C & \xrightarrow{L} & C \\
(m, c) \downarrow \phi & & \downarrow \phi \circ f \\
(n, d) & \xrightarrow{m \bullet c} & n \bullet d
\end{array}
\]
and an analogous equivalence $R$ can be defined from the tensor $\mathcal{C} \otimes_{\mathcal{M}} \mathcal{M}$. This functor is an equivalence since its inverse, $c \mapsto (j, c)$ is essentially surjective thanks to the additional family of isomorphisms $\tau$ in $\mathcal{M} \otimes_{\mathcal{M}} \mathcal{C}$ (see Proposition 4.4.4). The bilinear structure can be obtained by using the multiplicator of $\mathcal{C}$ as a left and right lineator. Therefore, $L$ and $R$ are, respectively, the left and right unitor for $\otimes_{\mathcal{M}}$.

2. The associator $A : (\mathcal{D} \otimes_{\mathcal{M}} \mathcal{C}) \otimes_{\mathcal{M}} \mathcal{B} \rightarrow \mathcal{D} \otimes_{\mathcal{M}} (\mathcal{C} \otimes_{\mathcal{M}} \mathcal{B})$ rebrackets triples on objects and morphisms. Its linear structures are both identities:

$$
\ell_{m,d,c,b} := m \circ (d, (c, b)) = (m \circ d, (c, b)),
$$
$$
r_{m,d,c,b} := (d, (c, b)) \cdot' m = (d, (c, b) \cdot' m) = (d, (c, b \cdot' m))
$$

(B.13)

where $\cdot'$ is the right $\mathcal{M}$-action on $\mathcal{B}$.

3. The pentagonator is trivial.

4. The middle 2-unitor is a natural, bilinear, invertible transformation which makes the following commute:

$$
\begin{array}{ccc}
(D \otimes_{\mathcal{M}} \mathcal{M}) \otimes_{\mathcal{M}} \mathcal{C} & \xrightarrow{A} & D \otimes_{\mathcal{M}} (\mathcal{M} \otimes_{\mathcal{M}} \mathcal{C}) \\
R \otimes_{\mathcal{M}} \mathcal{C} & \downarrow{M} & D \otimes_{\mathcal{M}} \mathcal{C} \\
D \otimes_{\mathcal{M}} \mathcal{C} & \xleftarrow{D \otimes_{\mathcal{M}} L} & D \otimes_{\mathcal{M}} \mathcal{C}
\end{array}
$$

(B.14)

It’s defined as:

$$
(\mathcal{M}_{D,\mathcal{C}})_{((d,m),c)} := (d, m \cdot c) \overset{\tau^{-1}}{\rightarrow} (d \circ m, c),
$$

(B.15)

thus its bilinearity and invertibility are satisfied by definition.

5. The left and right 2-unitors are the natural, bilinear, invertible transformations which make the following commute:

$$
\begin{array}{ccc}
(M \otimes_{\mathcal{M}} D) \otimes_{\mathcal{M}} \mathcal{C} & \xrightarrow{L \otimes_{\mathcal{M}} \mathcal{C}} & D \otimes_{\mathcal{M}} \mathcal{C} \\
A & \xrightarrow{L} & \mathcal{M} \otimes_{\mathcal{M}} (D \otimes_{\mathcal{M}} \mathcal{C}) \\
& \xleftarrow{A} & D \otimes_{\mathcal{M}} \mathcal{C}
\end{array}
$$

(B.16)

$$
\begin{array}{ccc}
D \otimes_{\mathcal{M}} (C \otimes_{\mathcal{M}} M) & \xrightarrow{D \otimes_{\mathcal{M}} R} & D \otimes_{\mathcal{M}} \mathcal{C} \\
A & \xrightarrow{R} & (D \otimes_{\mathcal{M}} \mathcal{C}) \otimes_{\mathcal{M}} \mathcal{M}
\end{array}
$$
and are both just identity cells:

\[
\begin{align*}
(\mathcal{L}_{\mathcal{D}, \mathcal{C}})((m, d), c) &:= (m \circ d, c) = (m \circ d, c), \\
(\mathcal{R}_{\mathcal{D}, \mathcal{C}})((d, c), m) &:= (d, c \cdot m) = (d, c \cdot m).
\end{align*}
\]

(B.17)

Checking the three coherence diagrams for this structure is tedious but routine.

5.2.7. PROPOSITION. [Waff product] Let \((\mathcal{C}, \bullet, o, \boxplus)\) be a distributive algebroidal \(\mathcal{M}\)-actegory. Then the product category \(\mathcal{C} \times \mathcal{M}\) can be equipped with a monoidal product \(\otimes\), whose unit is \((o, j)\) and whose product law is so defined:

\[(c, m) \otimes (d, n) := (c \boxplus (m \bullet d), m \otimes n)\]  \hspace{1cm} (5.15)

We call this waff product, for ‘Weird AFFine’ product, and we denote such monoidal category by \(\mathcal{W}(\mathcal{C}, \mathcal{M})\).

PROOF. Following the idea outlined in Remark 5.2.8, one can verify easily that the repletion\(^{14}\) of the wide subcategory of affine endofunctors on \(\mathcal{C}\) is closed under composition, hence it’s a monoidal subcategory of \(((\mathcal{C}, \mathcal{C}), 1_{\mathcal{C}}, \circ)\). Then \((\mathcal{C} \times \mathcal{M}, (o, j), \otimes)\) can be realized as a monoidal subcategory of this one, thus proving it is a well-defined monoidal category.

Concretely, the associator for \(\otimes\) is given by

\[
\alpha_{(c, m), (d, n), (e, p)}^{\otimes} : ((c, m) \otimes (d, n)) \otimes (e, p)
\]

\[
= (c \boxplus (m \bullet d), m \otimes n) \otimes (e, p)
\]

\[
= (c \boxplus (m \bullet d) \boxplus ((m \otimes n) \boxplus e), (m \otimes n) \bullet p)
\]

\[
\xrightarrow{\alpha_{(c, m), (d, n), (e, p)}^{\otimes}}
\]

\[
(1 \boxplus (m \bullet d) \boxplus (m \bullet e)) \boxplus (m \bullet (n \bullet p))
\]

\[
\xrightarrow{\alpha_{(c, m), (d, n), (e, p)}^{\otimes}}
\]

\[
(c \boxplus (m \bullet (d \boxplus (n \bullet e))), m \bullet (n \bullet p))
\]

\[
= (c, m) \otimes (d \boxplus (n \bullet e), n \bullet p)
\]

\[
= (c, m) \otimes ((d, n) \otimes (e, p)).
\]

Finally, the left and right unitors are defined respectively as

\(^{14}\)Meaning we include not just affine endofunctors, but also all endofunctors naturally isomorphic to those.
\[ \lambda_{(c,m)}^\oplus : (o, j) \otimes (c, m) \]
\[ = (o \boxplus (j \bullet c), j \otimes m) \]
\[ \xrightarrow{(\lambda_{j \bullet c, m}^\oplus)} (j \bullet c, m) \]
\[ \xrightarrow{(\eta_{c,m}^\oplus, 1)} (c, m). \]
\[ \rho_{(c,m)}^\oplus : (c, m) \otimes (o, j) \]
\[ = (c \boxplus (m \bullet o), m \otimes j) \]
\[ \xrightarrow{(c \boxplus m, \rho_{m}^\oplus)} (c \boxplus o, m) \]
\[ \xrightarrow{(\rho_{c,m}^\oplus, 1)} (c, m). \]

Notice the role of the distributor and absorber in defining \( \alpha^\oplus \) and \( \rho^\oplus \).

\[ \text{Corollary.} \quad \text{The monoidal category} (\mathcal{C} \times \mathcal{M}, (o, j), \otimes) \text{ defined in Proposition 5.2.7 acts on} \mathcal{C}. \]

\[ \text{Proof.} \quad \text{We established in the previous proof that} (\mathcal{C} \times \mathcal{M}, (o, j), \otimes) \text{ is a monoidal sub-category of} ([\mathcal{C}, \mathcal{C}], 1_{\mathcal{C}}, \circ), \text{ thus we obtain an action on} \mathcal{C} \text{ from the latter by restriction of scalars along the inclusion} \mathcal{C} \times \mathcal{M} \to [\mathcal{C}, \mathcal{C}]. \]

Concretely, the unitor and multiplicator of this action are given by:

\[ \eta_{c}^\oplus : (o, j) \otimes c \xrightarrow{o \boxplus (j \bullet c)} j \bullet c \xrightarrow{\eta_{c}^\bullet} c, \]

\[ \mu_{(a,m),(b,n),c}^\oplus : (a, m) \otimes ((b, n) \oplus c) \]
\[ = (a, m) \oplus (b \boxplus (n \bullet c)) \]
\[ = a \boxplus (m \bullet (b \boxplus (n \bullet c))) \]
\[ \xrightarrow{\alpha_{b,m, b \boxplus (n \bullet c)}^{-1}} a \boxplus ((m \bullet b) \boxplus (m \bullet (n \bullet c))) \]
\[ \xrightarrow{\alpha_{(m \bullet b) \boxplus (n \bullet c)}} a \boxplus ((m \bullet b) \boxplus ((m \otimes n) \bullet c)) \]
\[ = (a \boxplus (m \bullet b), m \otimes n) \oplus c \]
\[ = ((a, m) \otimes (b, m)) \oplus c. \]

\[ \text{Lemma.} \quad \text{[Characterization of the Drinfeld center]} \quad \text{Consider the the canonical} \mathcal{C}\text{-biactegory} (\mathcal{C}, \boxplus, \boxminus) \text{ (as defined in Example 4.3.10) associated to} \mathcal{C}, \text{ that we still denote by} \mathcal{C}. \text{ Then} \]

\[ \mathcal{Z}(\mathcal{C}) \simeq [\mathcal{C}, \mathcal{C}]_{\mathcal{C}\cdot \text{-act} \cdot \mathcal{C}} \]

\[ (5.35) \]
where on the right we have the category of maps $\varphi : C \rightarrow C$ equipped with $C$-linear structures

$$\ell_{c,d} : \varphi(c) \boxtimes d \xrightarrow{\sim} \varphi(c \boxtimes d),$$

$$r_{c,d} : c \boxtimes \varphi(d) \xrightarrow{\sim} \varphi(c \boxtimes d).$$

satisfying the laws stated in Definition 4.3.2.

**Proof.** Without loss of generality, suppose $C$ is strict. Furthermore, given a bilinear endomorphism $(\varphi, \ell, r)$, we know by Lemma 3.4.5 we can consider $\ell$ to be trivial.\(^{15}\)

So given $(\varphi, r)$ we obtain an object $\varphi(j)$ and a natural isomorphism

$$v_c := \varphi(j) \boxtimes c \xrightarrow{r_{j,c}} \varphi(j \boxtimes c) = \varphi(c \boxtimes j) = c \boxtimes \varphi(j).$$

This satisfies Equation (5.31) by a reduction to Diagram (4.23).

Viceversa, given $(c, v)$, we get a bilinear endomorphism $- \boxtimes c$, whose left linear structure is trivial and right linear structure is given by

$$r_{d,b} : d \boxtimes c \boxtimes b \xrightarrow{\xi_{d,b}} d \boxtimes b \boxtimes c.$$ \hspace{1cm} (B.23)

Well-definedness of this bilinear structure follows from the properties of $v$, as above.

Now given $(\varphi, r)$, going back and forth this correspondence yields $(- \boxtimes \varphi(j), - \boxtimes r)$. Hence we must check that $(- \boxtimes \varphi(j), - \boxtimes r) = (\varphi, r)$ as right $C$-linear functors. The functor parts coincide by strictness of the left $C$-linear structure of $\varphi$. If we check whether the identity is a valid right linear natural isomorphism, we are led to contemplate the commutativity of the following diagram (we omitted some identity morphisms arising from strictification):

$$
\begin{array}{ccc}
    c \boxtimes \varphi(j) \boxtimes d & = & \varphi(c) \boxtimes d \\
    \downarrow c \boxtimes r_{j,d} & & \downarrow r_{c,d} \\
    c \boxtimes \varphi(d) & = & \varphi(c \boxtimes d)
\end{array}
$$

It’s easy to convince oneself this is what’s left of Diagram (4.23) when $\ell$ and $\zeta$ are both identities.

Instead, when $(x, v)$ is given, after a back and forth we get back the pair comprised of $j \boxtimes x = x$ and the natural isomorphism $j \boxtimes v_z = v_z$.\(^{15}\)

\(^{15}\)In other words, we are factoring the equivalence we have to construct through that defined in Lemma 3.4.5.
Finally, let us define this isomorphism on morphisms. Given a right linear transformation \( \xi : (\varphi, r) \Rightarrow (\psi, r') \), we have to prove its image \( \xi_j : (\varphi(j), r_j) \rightarrow (\varphi(j), r'_j) \), we know the top square in the following diagram commutes:

\[
\begin{array}{ccc}
\varphi(j) \boxtimes d & \xrightarrow{\xi_j \boxtimes d} & \psi(j) \boxtimes d \\
\downarrow r_{j,d} & & \downarrow r'_{j,d} \\
\varphi(j \boxtimes d) & \xrightarrow{\xi_j \boxtimes d} & \psi(j \boxtimes d)
\end{array}
\]  

(B.25)

Since the bottom one commutes by assumption on \( \varphi \), the composite square proves \( \xi_j \) satisfies the required condition (namely Diagram (5.32)). Contemplating the same diagram proves that given \( f : (c, \upsilon) \rightarrow (d, \tau) \), we get a well-defined right linear transformation \( \xi : (- \boxtimes c, - \boxtimes \upsilon) \Rightarrow (- \boxtimes d, - \boxtimes \tau) \) whose components are defined as \( \xi_b := b \boxtimes c \xrightarrow{b \boxtimes f} b \boxtimes d \).

Mathematically Structured Programming Group,
Department of Computer and Information Sciences,
University of Strathclyde,
Glasgow, Scotland

Email: matteo.capucci@strath.ac.uk
bruno@brunogavranovic.com