Kelvin-Helmholtz Instability of the Magnetopause of Disc-Accreting Stars

R.V.E. Lovelace,1,2⋆ M. M. Romanova,1†, W.I. Newman,3‡

1 Department of Astronomy, Cornell University, Ithaca, NY 14853, USA
2 Department of Applied and Engineering Physics, Cornell University, Ithaca, NY 14853, USA
3 Departments of Earth and Space Sciences, Physics and Astronomy, and Mathematics, University of California, Los Angeles, CA 90095

19 November 2009

ABSTRACT

This work investigates the short wavelength stability of the magnetopause between a rapidly-rotating, supersonic, dense accretion disc and a slowly-rotating low-density magnetosphere of a magnetized star. The magnetopause is a strong shear layer with rapid changes in the azimuthal velocity, the density, and the magnetic field over a short radial distance and thus the Kelvin-Helmholtz (KH) instability may be important. The plasma dynamics is treated using non-relativistic, compressible (isentropic) magnetohydrodynamics. It is necessary to include the displacement current in order that plasma wave velocities remain less than the speed of light. We focus mainly on the case of a star with an aligned dipole magnetic field so that the magnetic field is axial in the disc midplane and perpendicular to the disc flow velocity. However, we also give results for cases where the magnetic field is at an arbitrary angle to the flow velocity. For the aligned dipole case the magnetopause is most unstable for KH waves propagating in the azimuthal direction perpendicular to the magnetic field which tends to stabilize waves propagating parallel to it. The wave phase velocity is that of the disc matter. A quasilinear theory of the saturation of the instability leads to a wavenumber \( k \) power spectrum \( \propto k^{-1} \) of the density and temperature fluctuations of the magnetopause, and it gives the mass accretion and angular momentum inflow rates across the magnetopause. For self-consistent conditions this mass accretion rate will be equal to the disc accretion rate at large distances from the magnetopause.

Key words: accretion, accretion discs — stars: neutron — X-rays: binaries — magnetohydrodynamics — Instabilities — Waves

1 INTRODUCTION

This work investigates the short wavelength stability of the magnetopause between a rapidly-rotating, supersonic, dense accretion disc and the slowly-rotating, low-density magnetosphere of a magnetized star. The nature of the magnetopause is sketched in Figure 1. The rotating disc matter is “held off” by the star’s magnetosphere where the magnetic field is strong and the density is small. The disc matter rotates at approximately the Keplerian velocity which is typically much larger than the velocity of the magnetospheric plasma which corotates with the angular velocity of the star. Thus the interface involves a strong shear layer as sketched in the bottom part of Figure 1. Understanding the instabilities of the magnetopause is important for understanding both the transport of matter and angular momentum towards the star and the temporal variability of the sources (van der Klis 2006).

The magnetohydrodynamic (MHD) stability of configurations such as in Figure 1 was investigated earlier by Li & Narayan (2003) assuming incompressible flow and perturbations independent of \( z \), but with no restrictions on the azimuthal wavelength. They found both long-wavelength (i.e., \( \lesssim r \) Rayleigh-Taylor (RT) and short wavelength (\( \ll r \) Kelvin-Helmholtz (KH) instabilities for different conditions of the shear layer. Earlier, Burnard, Lea, & Arons (1983) studied the short wavelength KH instability of the magnetopause of a spherically accreting rotating magnetized star. Recently Tsang & Lai (2009) have studied the long wavelength RT stability of the sharp interface including the compressibility of the media. The MHD stability of the magnetopause for cases where the shear layer has appreciable radial width was studied by Lovelace & Romanova (2007) and Lovelace, Turner, & Romanova (2009) for compressible, non-barotropic perturbations independent of \( z \) but no restrictions on the azimuthal wavelength. They found a resonant long-wavelength Rossby Wave Instability (RWI; Lovelace et

⋆ E-mail: RVL1@astro.cornell.edu
† E-mail: romanova@astro.cornell.edu
‡ E-mail: win@ucla.edu
al. 1999) which may contribute to the observed twin kilo-Hertz quasi-periodic oscillations of low-mass X-ray binaries (van der Klis 2006).

Here we consider a thin magnetopause and short wavelengths where the KH instability is expected to be important. There is vast literature on the magnetized KH instability in different astrophysical and space applications, to the magnetopause of rotating planets in the solar wind (e.g., Miura & Pritchett 1982; Roy-Choudhury & Lovelace 1986; Faganello, Califano, & Pegoraro 2008), to the stability of astrophysical jets (e.g., Hardee 2007; Osmanov et al. 2008), to the stability of interfaces of molecular/atomic clouds in the interstellar medium (e.g., Hunter, Whitaker, & Lovelace 1998), and to the interface of an unmagnetized dense plasma blob falling through the strong magnetic field of a neutron star (Arons & Lea 1980).

Section 2 of the paper gives the basic equations where the fluid motion is assumed non-relativistic but the displacement current is retained in Maxwell’s equations in order to keep the wave speeds less than the light speed. This section also describes the assumed equilibrium, the waves in the disc plasma, and the MHD waves in the magnetospheric plasma. The disc flow speed is much larger than the disc sound speed. Section 3 obtains a dispersion relation for the Kelvin-Helmholtz modes and develops an approximate solution for cases where the magnetospheric density ($\rho_2$) is much less than the disc density ($\rho_1$). Section 4 discusses the non-linear saturation of the KH modes and develops a quasi-linear theory for the mass accretion rate across the magnetopause. Section 5 discusses briefly the case where the magnetic field is at an arbitrary angle relative to flow velocity in the disc. Section 5 gives the conclusions of this work.

2 THEORY

The rotating disc matter is termed region 1, while the magnetosphere matter is termed region 2. In the magnetosphere the magnetic field $B$ is significant whereas in the external medium $B$ is negligible as suggested by MHD simulations (Romanova, Kulkarni, & Lovelace 2008; Kulkarni & Lovelace 2008). Depending on the region, the flow is described by non-relativistic hydrodynamic equations or by MHD equations,

$$\rho \frac{dv}{dt} = -\nabla p + \frac{1}{c} \frac{J \times \mathbf{B}}{\rho} + \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 ,$$

(1)

where $\mathbf{g} = -\nabla \Phi$ is the gravitational acceleration due to the star and $\Phi$ is the gravitational potential. In addition we have Maxwell’s equations including the displacement current,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \frac{\partial E}{\partial t} , \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} ,$$

(2)

and Ohm’s law for infinite conductivity,

$$0 = \mathbf{E} + \mathbf{v} \times \mathbf{B}/c ,$$

(3)

where $\mathbf{v}$ is the flow velocity, $\mathbf{B}$ the magnetic field, $p$ the plasma pressure, and $\nabla \cdot \mathbf{B} = 0$. We assume isentropic flow with $\gamma = 5/3$ in both media: $p = \kappa_1 \rho^\gamma$ in the external medium (region 1) and $p = \kappa_2 \rho^\gamma$ in the magnetospheric plasma (region 2).

One can eliminate $\mathbf{E}$ and $\mathbf{J}$ from the equations to obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) ,$$

(4)

and

$$\rho \frac{dv}{dt} = -\nabla \left( p + \frac{B^2}{8\pi} \right) + \nabla \cdot \left( \frac{\mathbf{B}}{4\pi} \times \frac{\partial}{\partial t} (\mathbf{v} \times \mathbf{B}) \right) + \rho \mathbf{g} .$$

(5)

The less familiar last term of this equation arises from the displacement current in equation (2). It is required in order to have the wave speeds less than the velocity of light (Jackson 1975).

2.1 Equilibrium Flow

We consider an axisymmetric time-independent equilibrium plasma flow. The flow velocity $\mathbf{v} = v_\phi (r) \hat{\phi} = r \Omega_e (r) \hat{\phi}$. That is, the accretion velocity $v_\phi$ and the vertical velocity $u_z$ are assumed negligible compared with $v_\phi$. Initially, we use an inertial cylindrical $(r, \phi, z)$ coordinate system. The equilibrium magnetic field is

![Figure 1](image1.png)

**Figure 1.** Sketch of the inner region of a disc around a rotating magnetized star suggested by MHD simulations of Romanova et al. (2008). Inside of $r_m$ the plasma flows along the magnetic field lines to the surface of the star in what is termed a funnel flow (denoted ff in the figure). The bottom part of the figure shows the midplane profiles of the the density $\rho$, magnetic field $B_z$, and azimuthal velocity $v_\phi$.

![Figure 2](image2.png)

**Figure 2.** Geometry of a small patch of the interface between the star’s magnetosphere and the external disc flow.
where \( \Delta \) is the distance between the particle and the star. The effective radial gravitational acceleration is given by \( g = -r \Omega^2 \). The magnetic field direction is given by \( B \) and \( \hat{z} \). In the case where the plasma properties undergo a rapid change in the magnetopause radius \( r_m \), as sketched in the bottom of Figure 1. The change in values is assumed to occur over a distance \( \Delta r_m \ll r_m \). Inside of \( r_m \), the concept of the magnetic field is well-defined. The density \( \rho \), the sound speed is \( c_2 \), and the flow is \( \rho \Omega \hat{r} \). The wave vector is \( \hat{K} \) and \( \Omega \). The angular orientation is \( c_1 \). For such conditions, the force balance at \( r_m \) requires that \( B^2/8\pi \approx \rho_1 c_1^2 / \gamma \).

The frequency \( K = (K_x, K_y, K_z) \) is the wavevector, \( c_1 = (\partial p / \partial \rho)^{1/2} \) is the sound speed in the external medium. In the reference frame we use, the external medium is moving with uniform velocity \( v \). For the non-propagating disturbances, \( \omega_0 = K \cdot v \). Figure 3 shows a polar plot of the phase velocities \( v \) and \( c_1 \) as functions of \( \phi \) which is the angle between \( K \) and \( r \).

**2.3 MHD Modes in the Magnetosphere**

The dispersion relation for the two compressible MHD modes including the displacement current is

\[
\omega^2 - \omega^2 \left[ K^2 (\hat{v}_x^2 + \hat{v}_y^2) + (K \cdot v_x)^2 c_s^2 / c^2 \right] + K^2 \hat{v}_x^2 (K \cdot v_x)^2 = 0, \tag{7}
\]

where \( c_s^2 = v_s^2 = \frac{c^2}{1 + \gamma \rho / \rho_0} \), and \( \gamma = (\partial p / \partial \rho)^{1/2} \). (e.g., Jackson 1975). Here, \( \omega \) is the angular frequency of the wave and \( K \) is the wavevector. Note that the bold-faced \( v \)'s in equation (9) do not have over-tildes and can be larger than \( c \).

The magnetic field in the magnetosphere is strong and the plasma density low so that we expect to have \( \hat{v}_x^2 \gg \hat{c}_1^2 \) or equivalently \( v_x^2 \gg v_1^2 \). In this limit the slow magnetosonic wave branch of equation (7) has \( \omega^2_{\text{slow}} \approx K^2 \hat{v}_x^2 \cos^2(\phi) \), where \( \phi \) is the angle between \( K \) and \( B \). For this wave the velocity perturbations are parallel to \( B \) with the result that the wave speed is independent of \( B \). The other,
Figure 5. Phase velocities $u$ of the Alfvén wave in the magnetosphere as well as the fast and slow sound waves $u_{\pm}$ in the external plasma. In this figure, $c_{a1} = c_{a2} = 0.0391$, $v_a = 0.805$, and $v_t = 0.167$ all in units of $c$. The dashed circle is $v_t \sin(\varphi)$ (for $0 \leq \varphi \leq \pi$) and the dashed lines delimit the wavevector region of the Kelvin-Helmholtz instability discussed in §3.2.

A fast magnetosonic branch has $\omega_{js} = K^2 v_A^2$ with $\omega_{js} \gg \omega_{sm}^2$. This wave has velocity perturbations in the plane formed by $K$ and $B$. Both waves give density variations. There is also a non-propagating wave analogous to that in region 1. Figure 4 shows a polar plot of the phase velocities of the waves obtained from equation (7).

The dispersion relation for the shear Alfvén wave including the displacement current is

$$\omega^2 = (K \vartheta_A)^2,$$

where

$$\vartheta_A = \frac{v_A}{(1 + v_A^2/c^2)^{1/2}},$$

where $v_A = B/(4\pi \rho)^{1/2}$ is the usual, non-relativistic Alfvén velocity which may be larger than $c$ (e.g., Jackson 1975). The velocity perturbation of this wave and the field perturbation $\delta B$ are perpendicular to both $K$ and $B$. Thus the wave does not change the fluid density.

3 LINEARIZED EQUATIONS FOR THE KH MODES

The geometry of a small patch of the interface between the external medium and the magnetosphere is shown in Figure 2. The interface is in pressure equilibrium so that $\rho + B^2/8\pi$ is continuous across it.

Perturbations of all scalar quantities have the form $f(z) \exp(i(k \cdot r - i\omega t))$ for all $z$, where $r = (x, y, 0)$, $k = (k_x, k_y, 0)$ is the wavevector in the plane of the interface, and the $z$-dependence remains to be determined. The linearized continuity equation is

$$-i\Delta \omega \delta \rho + \frac{i}{\rho} \delta \rho - \frac{\partial}{\partial z} (\rho \delta v_z) = 0,$$

where $\delta \rho = (\delta v_x, \delta v_y, 0)$ and $\Delta \omega \equiv \omega - k \cdot v_t$ is the Doppler shifted wave frequency in the frame comoving with the external medium. The step-function dependences of $\rho(z), v_x(z), B_x(z),$ and $B_y(z)$, gives rise to various delta-function terms in the linearized equations. For example, from the top side (region 1, $z > 0$) the vertical displacement of the interface is $\delta z_1$ and the velocity at $z = \epsilon^+$ is

$$\delta v_1 = \left( \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial x} \right) \delta z_1 = -i\Delta \omega \delta z_1,$$

while from the bottom side (region 2) it is

$$\delta v_2 = \frac{\partial}{\partial t} \delta z_2 = -i\omega \delta z_2.$$

We must have $\delta z_1 = \delta z_2$ so that

$$\frac{\delta v_1(z = \epsilon^+)}{\omega - k_x v_t(\epsilon^+)} = -i \delta v_2(\omega - k_x v_t(z = \epsilon^+)),$$

(12) (Chandrasekhar 1961), where $\epsilon^+$ denotes an arbitrarily small positive or negative quantity. Thus $\delta v_x$ is discontinuous across the interface. Considering perturbations proportional to $\exp(ik_{x1}z_c)$ equation (11) can be written as

$$\frac{\delta \rho}{\omega - k_x v_t(z)} = -i \delta v_2 \frac{\partial \rho}{\omega - k_x v_t(z)}.$$

Here, $K = (k_x, k_y, k_z)$ is the three-dimensional wavenumber which also comes up later. The second term of this equation gives a delta function dependence, $-\delta z_1 (\rho - \rho_0) \delta (z)$.

Linearization of the Euler equation (5) gives

$$-iW \rho \delta v_z = -ik \delta \rho + i(k \cdot B) \frac{\delta B_x}{4\pi} - \frac{\omega B_y}{4\pi c^2} + \frac{\partial}{\partial z} \frac{\partial \rho}{\partial z} + \frac{1}{4\pi} \frac{\partial B_x}{\partial z} \delta B_z,$$

(14)

$$-iW \rho \delta v_y = -ik \delta \rho + i(k \cdot B) \frac{\delta B_y}{4\pi} - \frac{\omega B_x}{4\pi c^2} + \frac{\partial}{\partial z} \frac{\partial \rho}{\partial z} + \frac{1}{4\pi} \frac{\partial B_y}{\partial z} \delta B_z,$$

(15)

$$-iW \rho \delta v_x = -i \frac{\partial}{\partial z} \delta \rho + i(k \cdot B) \frac{\delta B_z}{4\pi},$$

(16)

where

$$W \equiv \Delta \omega + \omega^2,$$

and

$$\delta \rho \equiv \frac{1}{4\pi} (B_x \delta B_x + B_y \delta B_y).$$

The terms $\ll 1/c^2$ are due to the displacement current and they must in general be retained. Recall that $B = B_x \mathbf{x} + B_y \mathbf{y} + B_z \mathbf{z}$ so that $K \cdot B = k \cdot \mathbf{B}$. Linearization of the induction equation (4) for the magnetic field gives

$$\delta B_x = B_x \frac{\partial \rho}{\rho} = \frac{k \cdot \mathbf{B}}{\Delta \omega} \delta v_x + \frac{k \cdot \mathbf{B}}{\Delta \omega} \frac{\delta v_y}{\delta z} + \frac{B_x}{\rho} \frac{d \delta B_x}{d z} + \frac{B_x}{\rho} \frac{d \delta B_y}{d z} + \frac{B_x}{\rho} \frac{d \delta B_z}{d z},$$

(17)

$$\delta B_y = B_y \frac{\partial \rho}{\rho} = \frac{k \cdot \mathbf{B}}{\Delta \omega} \delta v_y - \frac{k \cdot \mathbf{B}}{\Delta \omega} \frac{\delta v_x}{\delta z} + \frac{B_y}{\rho} \frac{d \delta B_x}{d z} + \frac{B_y}{\rho} \frac{d \delta B_y}{d z},$$

(18)

$$\delta B_z = i(k \cdot \mathbf{B}) \delta z = -i \frac{k \cdot \mathbf{B}}{\Delta \omega} \delta z.$$
\[ \rho \mathbf{B} \cdot \partial \mathbf{v} = \frac{\Delta \omega}{F} (\mathbf{k} \cdot \mathbf{B}) (\delta \rho - v_A^2 \delta \rho'), \]  
(20)

where

\[ F = \Delta \omega^2 - (\mathbf{k} \cdot \mathbf{v}_A)^2, \]

and \( \mathbf{v}_A \equiv \mathbf{B}/\sqrt{\rho \mu_0}. \) Note \( \mathbf{k} \cdot \mathbf{v}_A = \Delta \omega (\delta \rho/\rho) \) and that \( \mathbf{k} \cdot \mathbf{B} = 0. \)

As mentioned we use the reference frame comoving with the magnetosphere so that in region 2 \( \Delta \omega = \omega. \) From equations (14) and (15) we have

\[ \omega \left( 1 + \frac{v_A^2}{c^2} \right) \rho \mathbf{B} \cdot \partial \mathbf{v} = \mathbf{K}^2 \delta \rho + \frac{\omega}{4\pi \mu_0} \mathbf{B} \cdot \partial \mathbf{v}. \]  
(21)

From equations (17) and (18) we have

\[ \mathbf{B} \cdot \partial \mathbf{B} = B^2 \frac{\partial \rho}{\rho} - \frac{\mathbf{k} \cdot \mathbf{B}}{\Delta \omega} \mathbf{B} \cdot \partial \mathbf{v}. \]  
(22)

Using the above relation between \( \delta \rho \) and \( \delta \tilde{\rho} \), and the relation for isentropic perturbations \( \delta \rho = c_s^2 \delta \tilde{\rho} \), we obtain

\[ \delta \tilde{\rho} = \left( \frac{c_s^2 + v_A^2}{\Delta \omega^2} \right) \delta \rho. \]  
(23)

Equations (21) - (23) can be readily combined to give the dispersion relation for the magnetosonic modes.

Away from the interface equations (11) and (15) give

\[ \frac{\partial^2 \delta \rho}{\partial z^2} = -k_t^2 \delta \rho. \]  
(24)

For the external disc medium, \( z > 0 \), we must have \( \delta \rho \propto \exp(ik_z z) \) with \( \Im(k_{i1}) > 0 \) so that the perturbation decays as \( z \) increases. In this region we have

\[ k_{i1}^2 = -k_t^2 + \frac{(\omega - k_z v_A)^2}{c_s^2}, \]  
(25)

which is identical to equation (6) with \( \mathbf{K}^2 = k_t^2 + k_z^2 \) as it should be. Dividing this equation by \( k_t^2 \) and taking the square root gives

\[ k_{i1}(u) = \pm \left( \frac{|u - \sin(\varphi)v_A|^2}{c_s^2} - 1 \right)^{1/2}, \]  
(26)

where \( \tilde{k}_{i1} \equiv k_{i1}/|k|, \varphi \) is the angle between \( \mathbf{k} \) and \( \mathbf{B} \), and \( u \equiv \omega/|k| \) is the phase velocity of the perturbation which is complex with a positive imaginary part for an unstable perturbation. The choice of the sign in this expression is determined by the condition \( \Im(k_{i1}) > 0. \)

For the magnetospheric plasma, \( z < 0 \), we must have \( \delta \rho \propto \exp(ik_z z) \) with \( \Im(k_{i2}) < 0 \) so that the perturbation decays as \( -z \) increases. We find

\[ k_{i2}^2 = -k_t^2 + \frac{\omega^2 |\mathbf{v}_A - (\mathbf{k} \cdot \mathbf{v}_A)c_s^2/|c_s^2|}{\omega^2 c_t^2 - (\mathbf{k} \cdot \mathbf{v}_A)^2 c_t^2}, \]  
(27)

where \( \mathbf{v}_A \) is given by equation (8), \( \bar{c}_t \) is given by equation (10), and \( c_t^2 = \bar{c}_t^2 + v_A^2 \) is the fast magnetosonic wave speed. Note that equation (27) with \( \mathbf{K}^2 = k_t^2 + k_z^2 \) is identical to equation (9) as it should be. Dividing this equation by \( k_t^2 \) and taking the square root gives

\[ k_{i2}(u) = \pm \left( \frac{|u - \cos(\varphi)v_A|^2 c_t^2/|c_t^2| - 1}{u^2 c_t^2 - |\cos(\varphi)v_A|^2 c_t^2} \right)^{1/2}. \]  
(28)

The choice of the sign in this expression is determined by the condition \( \Im(k_{i2}) < 0. \)

### 3.1 Fundamental Dispersion Relation

Equation (16) can be rewritten as

\[ \frac{\partial}{\partial z} \delta \tilde{\rho} = -\left[ W \Delta \omega - (\mathbf{k} \cdot \mathbf{v}_A)^2 \right] \delta \zeta, \]  
(29)

where the terms on the right-hand-side are finite. Thus we have \( \delta \zeta = e^* \) so that \( \delta \tilde{\rho} \propto \exp(ik_z z). \) Using the fact that \( \delta \tilde{\rho}_{1,2} \propto \exp(ik_{i1,2} z) \), we find \( ik_{i2} \delta \tilde{\rho} = -[W \Delta \omega - (\mathbf{k} \cdot \mathbf{v}_A)^2] \delta \zeta_{i2} \) and \( ik_{i1} \delta \tilde{\rho} = -\Delta \omega \rho \delta \zeta_{i1}. \) Taking the ratio of these equations gives

\[ k_{i2}/k_{i1} = (\rho_2/\rho_1)(\omega^2(1 + v_A^2/c_s^2) - (\mathbf{k} \cdot \mathbf{v}_A)^2)/(\Delta \omega)^2. \]  
(30)

This is the fundamental dispersion relation. It agrees with the result of HWL in the limit \( c \to \infty. \) Equation (30) can be rewritten in terms of \( u = \omega/|k| \) as

\[ F(u) \equiv \left[ u - \sin(\varphi)v_A \right]^2 - g^2 \left[ \frac{k_{i1}(u)}{k_{i2}(u)} \right] \left[ u^2 - \cos(\varphi)v_A^2 \right] = 0, \]  
(31)

where \( k_{i1}(u) \) is given by equation (26) and \( k_{i2}(u) \) by equation (28), and

\[ g^2 \equiv \frac{\rho_2}{\rho_1} + \frac{B^2}{4\pi \mu_0 c_s^2}. \]  
(32)

The choice of signs for \( k_{i1} \) (equation 26) and \( k_{i2} \) (equation 28) is fixed by the above mentioned requirements that \( \Im(k_{i1}) > 0 \) and \( \Im(k_{i2}) < 0. \)

The pressure balance across the interface gives

\[ \rho_1 c_s^2 = \frac{B^2}{8\pi} + \frac{\rho_2 c_t^2}{\gamma}, \]  
(33)

where \( c_s \) is the sound speed in the magnetosphere. This is the same as

\[ v_A^2 + 2c_t^2 = \frac{\rho_1}{\rho_2} 2c_t^2. \]  
(34)

Hence equation (32) can be written as

\[ g^2 = \frac{\rho_2}{\rho_1} \left( 1 - \frac{2c_t^2}{\gamma c_s^2} \right) + \frac{2c_t^2}{\gamma c^2} = \frac{\rho_2}{\rho_1}, \]  
(35)

where the last equality takes into account that the plasma motion is assumed to be non-relativistic.

### 3.2 Approximate Instability Solution for \( g^2 \ll 1 \)

We are interested in conditions where \( g^2 \ll 1 \) where an approximate solution to equation (31) can be developed as follows. The two terms of equation (31) are written as

\[ F = F_0(u) + g^2 F_1(u) = 0. \]  
(36)

We develop a perturbation expansion for this equation based on the small parameter \( g^2. \) Thus, we take \( u = u_0 + \delta u \) with \( u_0 \) is chosen to give \( F_0(u_0) = 0 \) and \( |\delta u| \ll |u_0| \) assumed. Hence

\[ 0 = F_0(u_0) + \frac{dF_0}{du_0} \delta u + \frac{1}{2} \frac{d^2F_0}{du_0^2} (\delta u)^2 + g^2 F_1(u_0) + O(g^3|\delta u|). \]  
(37)

We choose \( u_0 \) such that \( F_0(u_0) = 0 \) which implies that \( u_0 = \sin(\varphi)v_A \). Then we have \( dF_0/du_0 = 0 \) and \( d^2F_0/du_0^2 = 2. \) Equation (37) then gives
\[ \delta u = \pm \left[ \frac{\tilde{k}_1(u_0)}{\tilde{k}_2(u_0)} \right]^{1/2} \left[ \sin^2(\varphi) \nu^2 - \cos^2(\varphi) \nu^2 \right]^{1/2}. \] (38)

For \( u_0 = \sin(\varphi) v_x \), equation (26) gives \( \tilde{k}_1(u_0) = \pm i \). Also, for \( u_0 = \sin(\varphi) v_x \) and for \( v_x^2 \gg c_s^2 \) and \( v_x^2 \ll c_s^2 \), equation (28) gives \( \tilde{k}_2(u_0) = \mp i \). The mentioned conditions on the imaginary parts of the \( k \)'s then implies that \( \tilde{k}_1(\mp i) = -1 \). Therefore, equation (38) implies instability for \( v_x > \tilde{v}_A(\tan \varphi)^{-1} \) which is the condition for the Kelvin-Helmholtz instability for the considered equilibrium. The real part \( \mathfrak{R}(u) = \mathfrak{R}(\omega/k) \) corresponds to the \( x \)-component of the phase velocity of the perturbation matching flow speed of the external medium \( v_x \).

For \( u_0 = \sin(\varphi) v_x \) and \( \delta u \) given by equation (38), one has in region 1, \( \rho \partial v_{13} = k z \delta \rho / \omega \) from equation (14). From the following equation following (11), we have \( \delta z_1 = \delta v_{13}(\epsilon^e)/\omega \), where \( \delta z_1 \) is the displacement of the interface. Consequently, \( \delta \rho_{13}(\epsilon^e)/\rho_1 = \omega^2 \delta z_1(k c_s^2) \), which shows that the perturbation in region 1 involves a change in the density. The perturbation is a sound wave evanescent in the \( z \)-direction with amplitude \( \propto \exp(-k z) / |k z - i \omega t| \) where \( \omega = k v_x \), and \( |k| = |k| [1 + (\nu_1/c_s^2)]^{1/2} \) from equation (25). The instability results from the interaction of the wave with the Alfvén wave \( (\Delta z^2) \) in region 2 which has \( \delta \rho_2 = 0 \) and does not change the magnitude of the magnetic field. Perturbations which change the magnitude of \( B \) are suppressed because they increase the magnetic energy of the system. For this reason the slow and fast magnetosonic waves are not excited.

Having the wavevector along the magnetic field corresponds to bending the field line which requires energy and is stabilizing. Thus the maximum growth rate occurs for \( \varphi = 90^\circ \) (where \( k = [k]_1 \)) with no field line bending and is

\[ \max(\omega_i) = \mathfrak{R}(k \delta u) = g k v_x \left( \frac{\delta z_1}{\rho_1} \right)^{1/2} \sqrt{v_x} \cdot \] (39)

This formula for \( \omega_i \) applies only for a restricted range of \( k \) for say \( k > 3 \Delta \nu^2 \) where the planar description of the interface is valid, and for \( k < (\Delta \nu)^{-1} \), where \( \Delta \nu \) is the radial thickness of the interface. The maximum growth rate does not depend explicitly on the value of the magnetic field. However, it depends implicitly on \( B \) since the field allows conditions with \( \rho_2/\rho_1 \ll 1 \) and the field enters the expression for \( \tilde{k}_2(u) \). The same formula for the growth rate is found by Li & Narayan (2004) who assume incompressible fluid motion in both media. In our treatment the response in the low-density magnetized region 2 is an Alfvén wave which is incompressible, but the response in the high-density unmagnetized region 1 is incompressible only in the zeroth approximation where \( \nu_2 = 0 \) in equation (36) and \( F_{\nu}(u) = 0 \). In the first approximation including the \( g^2 \) term the medium is compressible. As discussed in §4 the compressibility of the region 1 medium gives rise to observable fluctuations in the emissions from the interface.

For comparison with the results of Arons & Lea (1980) who took into account the displacement current (important for \( v_j/c \geq 1 \), we need to give the correspondence of our variables with theirs.

We find that our \( \rho_2/\rho_1 \) corresponds with their \( \rho_2/\rho_1 \ll 1 \), that their \( v_j \) is the same as theirs, \( v_A = |B|/(4\pi \rho_0)^{1/2} \leq 1 \) and that their \( a = v_A(\rho_2/\rho_1)^{1/2} \ll v_A \). In their equation (A48) we can neglect terms involving \( \rho_2/\rho_1 \) in comparison with unity. In the limit where \( a^2 \ll 1 \), their equation gives our equation (38) multiplied by \( (1 + v_x^2/c_s^2)^{1/2} \). Thus the two results agree only in the limit \( v_x^2/c_s^2 \ll 1 \). For \( a \gg 1 \) the two expressions are also different.

As a numerical example consider \( c_{s_1} = c_{s_2} = 0.0391 \), \( v_A = 1.36 \), \( c_\parallel = 0.805 \), \( v_x = 0.167 \) (all in units of \( c_\parallel \)), \( g^2 = \rho_2/\rho_1 = 0.00283 \) and \( g = 0.0532 \), Mach number \( M_1 = v_x/c_{s_1} = 4.26 \), we find \( \omega_i = 0.00887k_c \) for propagation in the \( x \)-direction. There is instability for \( \varphi > \arctan(\tilde{v}_A/v_x) = 78.3^\circ \) up to \( \varphi = 90^\circ \). The range propagation directions of the unstable waves is shown in Figure 5 as marked KH. The saturation of the exponential growth is discussed in §4.

We can express equation (39) as \( \omega_i = (\rho_2/\rho_1)^{1/2} k \nu_c M_1 \) where \( M_1 \) is the Mach number of the external plasma flow. In the commonly considered case of Kelvin-Helmholtz instability between equal density media there is instability for waves propagating parallel to the flow only for \( M < 2^{1/2} \) (e.g., Hunter & Whitaker 1989). In the present case there is no similar limit on the Mach number for the growth of waves propagating parallel to the flow.

We of course have the limit \( v_x^2 \ll c_s^2 \) because of our assumption of non-relativistic fluid motion.

## 4 NONLINEAR EFFECT OF UNSTABLE KH MODES

A large number of computer simulation studies have been done on the nonlinear evolution of the KH instability for different initial configurations (e.g., Zhang, MacFadyen, & Wang 2009; Keppens et al. 1999; Frank et al. 1996). The studies do not address the configuration considered here, but they suggest the approximate treatment discussed below. A comparison is made with the nonlinear model of Burnett et al. (1983).

Starting from a sharp interface, exponential growth of the KH ceases at some time when typically one or more “Kelvin’s Cat Eyes” form. Subsequently the interiors of the Eyes may become highly irregular but their widths remain roughly constant. For simplicity, we consider the region of the magnetopause outside of \( r_m \) or equivalently \( z > 0 \) (region 1). Also we assume that the wave propagates in the \( x \)-direction where the growth rate is a maximum. From §3 it is clear that the unstable wave initially corrugates the interface displacing it by the amount \( \delta z = \tilde{k} v (\epsilon^e)/\Delta \omega \). The exponential growth will cease when the width of the Cat Eyes is of the order of the reduced wavelength. That is, \( |\delta z| = \varphi \Delta z \), where \( \varphi = \text{const} \) is a number of the order unity and \( \lambda = 1/k \). At later times the initially sharp interface is effectively smoothed over a distance \( \sim \delta z \) giving the interface a radial thickness \( \Delta \nu \) (Frank et al. 1996). This finite thickness acts to stabilize the KH instability for a given \( k = \lambda^{-1} \). At saturation \( \nu_{\nu_1} = q \omega_i k \) since \( \Delta \omega \). From equation (14) we have \( \rho_1 v_{\nu_1} = |k| \omega_i / \Delta \omega \). This gives \( \delta \rho/\rho_1 = q (\omega_i/k)^2 \) and \( \delta \rho/\rho_1 = q (\omega_i/k)^2 \) at saturation. The use of the linear relations between the fluid variables is plausible in the nonlinear regime if \( |\delta \rho/k|/\rho_1 = q^2 (\nu_1/c_s)^2 \ll 1 \).

In general, there is a spectrum of saturated KH waves with the contribution to \( \langle |\delta \rho|^2 \rangle \) from wavenumbers of order \( k \) equal to \( k (\omega_i/k)^2 \), where \( |\delta \rho|^2 \) is the wavenumber power spectrum of \( \delta \rho \). From the previous paragraph we have

\[ |\delta \rho|^2 = \frac{\rho_2^2 q^2 g^4}{k} \left( \frac{v_A}{c_s} \right)^4. \] (40)

The total mean-squared density fluctuation is therefore \( \langle |\delta \rho|^2 \rangle = \frac{k (\omega_i/k)^2}{k} \) which can give spatial variations in the radiation from the optically thin regions of the magnetopause.
Connecting the spectrum of waves with observed noise spectra of X-ray sources (van der Klis 2006) is, however, beyond the scope of this work since it involves integration over the entire magnetopause as well a treatment of the radiation transfer.

Figure 6 shows a sample realization of the interface with the magnetopause of \( u_{\|} \) viewed in the reference frame comoving with the disc plasma \( (\rho_1) \). The Fourier amplitudes \( \hat{\delta}\rho \) are generated with a random phase and magnitude with rms magnitude \( \propto k^{-1/2} \).

5 GENERAL ORIENTATION OF THE MAGNETIC FIELD

For the general case the star’s magnetic moment \( \mu \) is not aligned with \( \Omega \). However, we assume the rotation axis of the disc is aligned with \( \Omega \). The shear layer between the disc and star will be inherently time dependent due to the star’s rotation. For example, for an orthogonal rotator where \( \mu \) is in the equatorial plane, the plasma in the disc sees a magnetic field reversing its direction with an angular frequency \( \omega_B = v_\|/r_m = (v_K - \Omega_c)_r/v_m \). This frequency may be larger or smaller than the growth rate of the KH instability. From equation (39) we have \( \omega_\|/\omega_B > 1 \) significant wave growth can occur before the moves from a given field region. In the other limit the wave growth may be recurrent as a given plasma region returns to a given field region. In the following we consider \( B \) to be time-independent.

For a general field orientation, \( B \) is still in the plane of the shear layer, that is, the \((x,y)\) plane of Figure 2. An equilibrium with a \( B_\| \) component is not possible. Note however that the \( z\)-direction is not necessarily in the \( \tilde{r}\)-direction. For this case we discuss the needed modification of Figure 5 where it is appropriate to use the angle \( \phi \) rather than \( \varphi \). The flow velocity remains in the \( x\)-direction so that the fast and slow sound wave curves remain unchanged as does the dashed circle. The dashed circle can be written as \( v_\| \cos(\phi) \) for \( -\pi/2 < \phi \leq \phi \). What changes in Figure 5 is that the “figure-eighth” curve for the Alfvén wave is rotated by angle \( \theta \) say in the counter-clockwise direction. The figure-eight curve is give by \( v_{\|}, \sin(\phi - \theta) \). In place of equation (38) we find

\[
\delta u = -\frac{\mu}{\rho v_i} \frac{1}{k_{\text{max}}(\mu_{\text{disc}})} \left[ \cos^2(\phi) \nu_{\|} - \sin^2(\phi - \theta) \nu_{\|} \right].
\]

The limits on the directions of the unstable Kelvin-Helmholtz modes are given again by the intersection of the dashed circle in Figure 5 and the rotated figure-eight curve. These limits are easily found to be

\[
\phi_{\|,2} = \arctan \left( \tan(\theta) - \frac{v_\|}{v_{\|,1}} \cos(\theta) \right).
\]

Figure 7 shows the dependence of the real and imaginary parts of the KH most unstable mode on the tilt angle of the field \( \theta \).

6 CONCLUSIONS

This work investigated the short wavelength \( (\lambda \ll 2r_m) \) stability of the magnetopause at radius \( r_m \) between a rapidly-rotating, supersonic, dense \( (\rho_2) \) accretion disc and a slowly-rotating low-density magnetosphere \( (\rho_1 \gg \rho_2) \) of a magnetized star. The magnetopause is a strong shear layer with rapid changes in the azimuthal velocity, the density, and the magnetic field over a short radial distance
We found that the maximum growth rate and the minimum wavenumber discussed in §2(δu) are the same as for Figure 5.

(AΔρm ≪ r0), and thus the Kelvin-Helmholtz (KH) instability may be important. This work has focused on the case of a star with an aligned dipole magnetic field so that the magnetic field is axial in the disc midplane and perpendicular to the disc flow velocity. For the aligned dipole case the magnetopause is most unstable for KH waves propagating in the azimuthal direction perpendicular to the magnetic field. Propagation not perpendicular to the magnetic field changes the magnetic field which gives a stabilizing effect. The growth rate of the instability is

$$ω = (ρ_2/ρ_1)^{1/2}k_1v_1,$$

where $v_1$ is the velocity of the disc plasma relative to that of the magnetospheric plasma which corotates with the star. The wave phase velocity is that of the disc matter.

We discussed the non-linear saturation of the instability which we argued occurs for a mode of wavelength $λ = 2π/k$ when the displacement of the interface between the disc and magnetospheric plasma is of the order of $λ/2π$. From this we developed a quasi-linear model which led to a wavenumber power spectrum $k^{-1}$ of the density and temperature fluctuations of the magnetopause. The quasi-linear model gave the mass accretion and angular momentum inflow rates across the magnetopause. The mass accretion rate (per unit area of the magnetopause) was found to be

$$2(ρ_2/ρ_1)^{1/2}ρ_1v_1 ln(k_{max}/k_{min}),$$

where $k_{max,min}$ are the maximum and minimum wavenumbers discussed in §4. For self-consistent conditions this mass accretion rate will be equal to the disc accretion rate at large distances from the magnetopause.

We also considered the case where the magnetic field is not perpendicular to the flow velocity but tilted by an angle $θ$ relative to the $z$-axis, where $θ = 0$ corresponds to the aligned rotator case treated earlier. We found that the maximum growth rate and the associated wave phase velocity decrease monotonically with $θ$ and that both approach zero as $θ → 90°$. That the growth rate goes to zero is a result of the stabilizing effect of the magnetic field. Thus an orthogonal rotator where the star’s magnetic moment $μ$ is perpendicular to the star’s rotation axis $Ω$, is stable to the KH mode.

**REFERENCES**

Arons, J., & Lea, S.M. 1980, ApJ, 235, 1016
Burnard, D.J., Lea, S.M., & Arons, J. 1983, ApJ, 266, 175
Chandrasekhar, S. 1961, *Hydrodynamic and Hydromagnetic Stability* (Oxford Press: London), p. 481
Faganello, M., Califano, F., & Pegoraro, F. 2008, PRL, 101, 175003
Frank, A., Jones, T.W., Ryu, D., & Gaalaas, J.B. 1996, ApJ, 460, 777
Hardee, P.E. 2007, ApJ, 664, 26
Hunter, J.H., & Whitaker, R.W. 1989, ApJS, 71, 777
Hunter, J.H., Whitaker, R.W., & Lovelace, R.V.E. 1998, ApJ, 508, 680
Jackson, J.D. 1975, *Classical Electrodynamics*, Second Edition (John Wiley & Sons: New York), p. 489
Keppens, R., Tøth, Westermann, R.H.J., & Goedbloed, J.P. 1999, J. Plasma Physics, 61, 1
Kulkarni, A.K., & Romanova, M.M. 2008, MNRAS, 386, 673
Landau, L.D., & Lifshitz, E.M. 1987, *Fluid Mechanics*, (Pergamon Press: Oxford), p. 315.
Li, L.-X., & Narayan, R. 2004, ApJ, 601, 414
Lovelace, R.V.E., Li, H., Colgate, S.A., & Nelson, A.F. 1999, ApJ, 513, 805
Lovelace, R.V.E., & Romanova, M.M. 2007, ApJ, 670, L13 (LR07)
Lovelace, R.V.E., Turner, L., & Romanova, M.M. 2009, ApJ, in press (arXiv:0905.1071)
Miura, A., & Pritchett, P.L. 1982, JGR, 87, 7431
Osmanov, Z., Mignone, A., Massaglia, S., Bodo, G., & Ferrari, A. 2008, A&A, 490, 493
Romanova, M.M., Kulkarni, A.K., & Lovelace, R.V.E. 2008, ApJ, 673, L171
Roy-Choudhury, S., & Lovelace, R.V.E. 1986, ApJ, 302, 188
Shakura, N.I., & Sunyaev, R.A. 1973, A&A, 24, 337
Tsang, D., & Lai, D. 2009, MNRAS, 396, 589
van der Klis, M. 2006, in *Compact Stellar X-Ray Sources*, Eds. W.H.G. Lewin & M. van der Klis (Cambridge: Cambridge Univ. Press), p. 39
Zhang, W., MacFadyen, A., & Wang, P. 2009, ApJ, 692, L40

(RVEL and MMR) were supported in part by NASA grant NNX08AH25G and by NSF grants AST-0607135 and AST-0807129.