ALMOST MIXED SEMI-CONTINUOUS PERTURBATION OF MOREAU’S SWEEPING PROCESS

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Abstract. In this work, we introduce a new concept of semi-continuous set-valued mappings, called almost mixed semi-continuity, by taking maps that are upper semi-continuous with almost convex values in some points and lower semi-continuous in remaining points. We generalize earlier results obtained for both mixed semi-continuous maps and almost convex sets. We discuss the existence of solution for evolution problems driven by the so-called sweeping process subject to external forces, known as perturbation to the system, by this type of set-valued mappings. Finally, we give some topological properties of the attainable and solution sets in order to solve an optimal time problem.

1. Introduction. It’s well known that the existence of solution for the Cauchy problem \( \dot{x}(t) \in F(t, x), \ t \in [0, T]; \ x(0) = x_0 \) is obtained in both cases: when the right-hand side is upper semi-continuous with convex values or lower semi-continuous with nonconvex values. The concept of mixed semi-continuous mappings was used by Tolstonogov [15] and Fryskowski and Gorniewicz [10] for maps mixing both lower and upper semi-continuity regularity assumptions. The approach is based on the construction of a set-valued selection with convex values of the right-hand side of the Cauchy problem. These results inspired many authors to study differential inclusions with mixed semi-continuous perturbation, particularly the so-called perturbed sweeping process

\[
(P_F) \begin{cases}
-\dot{x}(t) \in N_{C(t)}(x(t)) + F(t, x(t)) & \text{a.e. in } [T_0, T], \\
x(t) \in C(t), & \forall t \in [T_0, T], \\
x(T_0) = u_0 \in C(T_0),
\end{cases}
\]

where \( C(t) \) is a time dependent subset of \( \mathbb{R}^n \), \( N_{C(t)}(x(t)) \) is the normal cone to \( C(t) \) at the point \( x(t) \), and \( F : [T_0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) is the perturbation. In [11], the authors solved the problem \((P_F)\) when the perturbation is mixed semi-continuous in the sense of [15] and satisfying a linear growth intersection condition. The moving sets \( C(t) \) are nonconvex, proximally smooth (or equivalently uniformly prox-regular). The approach is based on the Kakutani fixed point theorem for set-valued mappings, and the connection of the problem with an unconstrained differential inclusion governed by the subdifferential of the distance function established by

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Thibault in [14]. Another way to obtain the existence of absolutely continuous solution is the use of the mapping \( K_F \) defined on \( C_{\mathbb{R}^n}([T_0, T]) \) by
\[
K_F(s) = \{ x \in L_{\mathbb{R}^n}([T_0, T]) : x(t) \in F(t, s(t)) \text{ a.e. in } [T_0, T] \}
\]
which takes decomposable values.

On another side, Cellina and Ornelas [6] have introduced a new class of right-hand side for which the Cauchy problem admits a solution, namely the almost convex sets (see the definition below). In our previous work [1], we have considered an evolution inclusion governed by the Moreau’s sweeping process subject to almost convex perturbations, that is external forces applied on the system. Our first aim in this paper, is to combine between the almost convexity of sets with the mixed semi-continuity, to provide a new class of nonconvex problems admitting a solution, moreover, we establish the compactness of the attainable set for this class of maps.

The autonomous case of \((P_F)\), when \(F\) does not depend separately on \(t\) and the sets \(C(t) := C\) are fixed, presents an interesting and useful application arising in the study of planning procedures in mathematical economy, see [3]. It was studied by [9] and [12] and takes the following form:
\[
(P_G) \begin{cases}
-\dot{x}(t) \in N_C(x(t)) + G(x(t)) \quad \text{a.e. in } [T_0, T], \\
x(t) \in C, \quad \forall t \in [T_0, T], \\
x(0) = u_0 \in C.
\end{cases}
\]

For recent results, we refer to [1] and [2]. Our second aim is to give an existence result for \((P_G)\) that will be used to obtain a solution of reaching any element of the attainable set in a minimum time which is known as the time optimality problem. Some notations, definitions and preliminary results are formulated in Section 2. In Section 3, considering the linear growth condition satisfied by the element of minimal norm instead of \(F\), we extend the existence result of \((P_F)\) obtained in [11] and we study some topological properties for the solution set, then through an intermediate result we establish the existence of an absolutely continuous solution for a sweeping process perturbed by an almost mixed semi-continuous set-valued mapping. Finally in Section 4, we shall show how those results can be investigated to solve an optimal time problem.

2. Notations and preliminary results. Throughout the paper \(\mathbb{R}^n\) is the \(n\)-dimensional Euclidean space, \(\mathcal{B}\) its closed unit ball and \(\mathcal{B}(x, r)\) the closed ball of center \(x\) and rayon \(r > 0\). We denote by \(\mathcal{L}([T_0, T])\) the Lebesgue \(\sigma\)-field of \([T_0, T]\), \(\mathcal{B}(\mathbb{R}^n)\) the Borel \(\sigma\)-field of \(\mathbb{R}^n\) and by \(C_{\mathbb{R}^n}([T_0, T])\) the Banach space of all continuous mappings \(u : [T_0, T] \rightarrow \mathbb{R}^n\) endowed with the sup-norm. \(L_{\mathbb{R}^n}([T_0, T])\) stands for the space of all Lebesgue integrable \(\mathbb{R}^n\)-valued mappings defined on \([T_0, T]\).

Let \(A \subset \mathbb{R}^n\), we denote by \(\text{co}(A)\) the convex hull of \(A\) and \(\overline{\text{co}}(A)\) its closed convex hull. Following [9], \(A\) is called almost convex if and only if for all \(b \in \text{co}(A)\) there exist \(\lambda_1, \lambda_2, 0 \leq \lambda_1 \leq 1 \leq \lambda_2\) such that
\[
\lambda_1 b \in A \quad \text{and} \quad \lambda_2 b \in A.
\]

An evident example of almost convex sets are convex sets, that is easy to see if we choose \(\lambda_1 = \lambda_2 = 1\). Other examples are the boundary \(\partial \mathcal{P}\) of a convex set \(\mathcal{P}\) if \(0 \notin \mathcal{P}\) or the union \(\partial \mathcal{P} \cup \{0\}\) if \(0 \in \mathcal{P}\).

Let \(\hat{t} \in [T_0, T]\), we denote by
\[
A_{u_0}(\hat{t}) = \{ x(\hat{t}) \mid x(\cdot) \in T_{\mathcal{F}}(u_0) \}
\]
the attainable set of \((P_\nu)\) at the time \(\tilde{t}\), where \(T_{\tilde{t}}(u_0)\) is the set of the trajectories of the differential inclusion \((P_\nu)\) on the interval \([T_n, \tilde{t}].\)

For a nonempty closed subset \(S\) of \(\mathbb{R}^n\), we denote by \(d(\cdot, S)\) the usual distance function associated and \(\text{Proj}_S(u)\) the projection of \(u\) onto \(S\) defined by
\[
\text{Proj}_S(u) = \{ y \in S : \| u - y \| \}.
\]
and \(\delta^*(x', S) = \sup \langle x', y \rangle\) the support function of \(S\) at \(x' \in \mathbb{R}^n\).

Let us recall some necessary definitions related to non smooth analysis. Let \(A\) be an open subset of a Hilbert space \(H\) and \(\varphi : A \to ( -\infty, +\infty]\) be a lower semicontinuous function, the proximal subdifferential \(\partial^P \varphi(x)\), of \(\varphi\) at \(x\) (see \([8]\)) is the set of all proximal subgradients of \(\varphi\) at \(x\), any \(\xi \in H\) is a proximal subgradient of \(\varphi\) at \(x\) if there exist positive numbers \(\eta\) and \(\varsigma\) such that
\[
\varphi(y) - \varphi(x) + \eta\|y - x\|^2 \geq \langle \xi, y - x \rangle, \forall y \in x + \varsigma \mathbb{B}_H.
\]
Let \(x\) be a point of \(S \subset H\), we recall (see \([8]\)) that the proximal normal cone to \(S\) at \(x\) is defined by \(N^P_S(x) = \partial^P \delta_S(x)\), where \(\delta_S\) denotes the indicator function of \(S\), i.e. \(\delta_S(x) = 0\) if \(x \in S\) and \(+\infty\) otherwise. Note that the proximal normal cone is also given by
\[
N^P_S(x) = \{ \xi \in H : \exists \eta > 0 \text{ s. t. } x \in \text{Proj}_S(x + \eta \xi) \}.
\]
If \(\varphi\) is a real-valued locally-Lipschitz function defined on \(H\), the Clarke subdifferential \(\partial^C \varphi(x)\) of \(\varphi\) at \(x\) is the nonempty convex compact subset of \(H\) given by
\[
\partial^C \varphi(x) = \{ \xi \in H : \varphi^\circ(x; v) \geq \langle \xi, v \rangle, \forall v \in H \},
\]
where
\[
\varphi^\circ(x; v) = \lim_{y \to x, \|t\| \to 0} \frac{\varphi(y + tv) - \varphi(y)}{t}
\]
is the generalized directional derivative of \(\varphi\) at \(x\) in the direction \(v\). The Clarke normal cone \(N^C_S(x)\) to \(S\) at \(x \in S\) is defined by polarity with \(T^C_S\), that is,
\[
N^C_S(x) = \{ \xi \in H : \langle \xi, v \rangle \leq 0, \forall v \in T^C_S \},
\]
where \(T^C_S\) denotes the clark tangent cone and is given by
\[
T^C_S = \{ v \in H : d^2(x, S; v) = 0 \}.
\]
Recall now, that for a given \(\rho \in [0, +\infty]\) the subset \(S\) is uniformly \(\rho\)-prox-regular (see \([13]\)) or equivalently \(\rho\)-prox-smooth if every nonzero proximal to \(S\) can be realized by a \(\rho\)-ball, this means that for all \(\pi \in S\) and all \(0 \neq \xi \in N^P_S(\pi)\) one has
\[
\langle \xi, |\|\xi\|, x - \pi \rangle \leq \frac{1}{2\rho} \|x - \pi\|^2,
\]
for all \(x \in S\). We make the convention \(\frac{1}{\rho} = 0\) for \(\rho = +\infty\). Recall that for \(\rho = +\infty\) the uniform \(\rho\)-prox-regularity of \(S\) is equivalent to the convexity of \(S\).

The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel. For the proof of these results we refer the reader to \([4]\) and \([13]\).

**Proposition 1.** Let \(S\) be a nonempty closed subset of \(H\) and \(x \in S\). The following assertions hold:
1) \(\partial^P d(x, S) = N^P_S(x) \cap \overline{B}_H;\)
2) if the subset \(S\) is uniformly \(\rho\)-prox-regular with \(\rho \in [0, +\infty]\), then
i) the proximal subdifferential of $d(\cdot, S)$ coincides with its Clarke subdifferential at all points $x \in H$ satisfying $d(x, S) < \rho$. So, $\partial d(x, S) = \partial^C d(x, S)$ is weakly compact set;

ii) the proximal normal cone to $S$ coincides with all the normal cone contained in the Clarke normal cone at all points $x \in H$, i.e., $N_S(x) = N^S(x) = N^S_C(x)$.

Here and above $\partial^C d(x, S)$ and $N^S_C(x)$ denote respectively the Clarke subdifferential of $d(\cdot, S)$ and the Clarke normal cone to $S$ (see [8]);

iii) for all $x \in H$ with $d(x, S) < \rho$, $\text{Proj}_S(x)$ is a singleton of $H$.

The following is an important closeness property of the subdifferential of the distance function associated with a set-valued mapping (see [4]).

**Proposition 2.** Let $\rho \in ]0, +\infty[$, $\Omega$ be an open subset in Hilbert space $H$, and $K : \Omega \to H$ be a Hausdorff-continuous set-valued mapping. Assume that $K(z)$ is uniformly $\rho$-prox-regular for all $z \in \Omega$. Then for a given $0 < \sigma < \rho$, the following holds: for any $\tilde{z} \in \Omega$, $\tilde{x} \in K(\tilde{z}) + (\rho - \sigma)B_H$, $x_n \to \tilde{x}$, $z_n \to \tilde{z}$ with $z_n \in \Omega$, $x_n$ not necessarily in $K(z_n)$) and $\xi_n \in \partial d(x_n, K(z_n))$ with $\xi_n \to^w \xi$ one has $\xi \in d(\tilde{x}, K(z))$, here $\to^w$ means the weak convergence in $H$.

**Remark 1.** This property means that for every $\rho \in ]0, +\infty[$, for a given $0 < \sigma < \rho$, and for every set-valued mapping $K : \Omega \to H$ with uniformly $\rho$-prox-regular values, the set-valued mapping $(z, x) \mapsto \partial d(x, K(z))$ is upper semi-continuous from $\{(z, x) \in \Omega \times H : x \in K(z) + (\rho - \sigma)B_H\}$ into $H$, which is equivalent to the upper semi-continuity of the function $(z, x) \mapsto \partial^\sigma(\rho, \partial d(x, K(z)))$, on $\{(z, x) \in \Omega \times H : x \in K(z) + (\rho - \sigma)B_H\}$ for any $\rho \in H$.

The set-valued mapping $F$ is said mixed semi-continuous (see [15]) if for every $t \in [T_0, T]$, at each $x \in \mathbb{R}^n$ such that $F(t, x)$ is convex the set-valued mapping $F(t, \cdot)$ is graphically closed, and whenever $F(t, x)$ is not convex, $F(t, \cdot)$ is lower semi-continuous on some neighborhood of $x$ (see also example 4 in [10]). Let introduce the following definition. We say that $F$ is almost mixed semi-continuous if for every $t \in [T_0, T]$, at each $x \in \mathbb{R}^n$ where $F(t, x)$ is almost convex, $F(t, \cdot)$ is graphically closed, and $F(t, \cdot)$ is lower semi-continuous on some neighborhood of $x$ whenever $F(t, x)$ is not almost convex. Obviously, any mixed semi-continuous set-valued mapping is almost mixed semi-continuous. Let recall the following result which ensures the existence of a closed convex valued set-valued selection for a mixed semi-continuous set-valued mapping.

**Theorem 2.1.** [15] Let $M : [T_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a closed valued set-valued mapping global measurable, mixed semi-continuous, and there exists $f : [T_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ a Carathéodory function which is integrably bounded on bounded subsets of $\mathbb{R}^n$ and satisfying

$$M(t, x) \cap B(0, f(t, x)) \neq \emptyset, \quad \forall (t, x) \in [T_0, T] \times \mathbb{R}^n.$$ 

Then for any $\epsilon > 0$ and any compact set $K \subset C([T_0, T], \mathbb{R}^n)$ there is a non empty closed convex valued set-valued mapping $\Phi : K \to L^1_{\mathbb{R}^n}([T_0, T])$ which has sequentially closed graph with respect to the norm of uniform convergence in $K$ and the weak topology $\sigma(L^1_{\mathbb{R}^n}([T_0, T]), L^\infty_{\mathbb{R}^n}([T_0, T]))$ in $L^1_{\mathbb{R}^n}([T_0, T])$ and such that for any $u \in K$ and $\phi \in \Phi(u)$ one has for a.e. $t \in [T_0, T]$

$$\phi(t) \in M(t, u(t)) \quad \text{and} \quad \|\phi(t)\| \leq f(t, u(t)) + \epsilon,$$

for almost every $t \in [T_0, T]$. 
3. Existence results. Let begin by the following weaker version of Theorem 3.1 in [11], obtained by taking an unbounded perturbation: we replace the linear growth intersection condition by the linear growth condition of only the element with minimal norm. We provide the existence of solution and the compactness of the attainable set.

**Theorem 3.1.** Let \( C : [T_0, T] \to \mathbb{R}^n \) be a set-valued mapping with nonempty closed valued satisfying:

1. there exists some constant \( p \in [0, +\infty] \) such that for each \( t \in [T_0, T] \) the sets \( C(t) \) are uniformly \( p \)-prox regular;
2. \( C(t) \) varies in an absolutely continuous way, that is, there exists an absolutely continuous non negative function \( \eta : [T_0, T] \to \mathbb{R}_+ \) such that for all \( x, y \in \mathbb{R}^n \) and \( s, t \in [T_0, T] \)
   \[
   |d(x, C(t)) - d(y, C(s))| \leq \|x - y\| + |\eta(t) - \eta(s)|.
   \]

Let \( F : [T_0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) be a closed valued set-valued mapping satisfying the following assumptions:

1. \( F \) is \( \mathcal{L}([T_0, T]) \otimes \mathcal{B}(\mathbb{R}^n) \)-measurable;
2. for every \( t \in [T_0, T] \), at each \( x \in \mathbb{R}^n \) such that \( F(t, x) \) is convex the set-valued mapping \( F(t, \cdot) \) is upper semi-continuous, and whenever \( F(t, x) \) is not convex \( F(t, \cdot) \) is lower semi-continuous on some neighborhood of \( x \);
3. there are two nonnegative constants \( p, q \) and for all \( (t, x) \in [T_0, T] \times \mathbb{R}^n \)
   \[
   \text{Proj}_{F(t,x)}(0) \leq p + q\|x\|.
   \]

Then, for each \( u_0 \in C(T_0) \), there is an absolutely continuous solution of \((P_F)\), moreover, for any fixed time \( \tilde{t} \in [T_0, T] \), the attainable set of \((P_F)\) at \( \tilde{t} \) is compact.

**Proof.** **Step 1.** Since the assumption \((H_3)\) in [11] is satisfied by our condition \((iii)\), taking \( f(t, x) = p + q\|x\| \) and applying Theorem 3.1 in [11], \((P_F)\) admits an absolutely continuous solution \( u : [T_0, T] \to \mathbb{R}^n \) satisfying

\[
\|\dot{u}(t)\| \leq \dot{\eta}(t) + 2\dot{\delta}(t) \quad \text{a.e.} \quad t \in [T_0, T],
\]

where \( \delta : [T_0, T] \to \mathbb{R}_+ \) be the absolutely continuous solution of the ordinary differential equation

\[
\begin{cases}
\dot{\delta}(t) = \sigma(t) + 2q\delta(t) \\
\delta(T_0) = 0
\end{cases}
\]

with \( \sigma(t) = \epsilon + p + q\left(\|u_0\| + \int_{T_0}^{t} |\dot{\eta}(s)|ds\right) \) and \( \epsilon > 0 \) fixed.

**Step 2.** Proving that the set of the trajectories of the differential inclusion \((P_F)\) on the interval \([T_0, \tilde{t}]\),

\[
\mathcal{T}_\tilde{t}(u_0) = \{u \in C_{\mathbb{R}_+}([T_0, \tilde{t}]) : u \text{ is an absolutely continuous solution of } (P_F)\}
\]

is compact. Let \( (u_n) \) be a sequence in \( \mathcal{T}_\tilde{t}(u_0) \), for every \( n \in \mathbb{N} \) and \( t \in [T_0, \tilde{t}] \), by (1) we have

\[
\|u_n(t)\| \leq \|u_0\| + \|\bar{\eta} + 2\delta\|_{L^\infty([T_0, \tilde{t}])},
\]

then \( (u_n(t)) \) is relatively compact, and for all \( t, t' \in [T_0, \tilde{t}] \) such that \( t \leq t' \) we have

\[
\|u_n(t') - u_n(t)\| \leq \int_{t}^{t'} (\bar{\eta}(s) + 2\delta(s))ds,
\]

\[
\|u_n(t') - u_n(t)\| \leq \int_{t}^{t'} (\bar{\eta}(s) + 2\delta(s))ds,
\]
we get the equicontinuity of the sequence \((u_n)\). We conclude that \((u_n)\) is relatively compact in \(C([T_0, \tilde{t}])\). By Ascoli’s theorem, \((u_n)\) admits a subsequence (again denoted by) \((u_n)\) that converges uniformly to \(u\) such that \((\dot{u}_n)\) converges \(\sigma(L^1_{R^n}([T_0, \tilde{t}]), L^\infty(R^n))\) to \(\dot{u}\), with

\[
\dot{u}(t) = \lim_{n \to \infty} \dot{u}_n(t) = u_0 + \lim_{n \to \infty} \int_{T_0}^{t} \dot{u}_n(s)ds = u_0 + \int_{T_0}^{t} \dot{u}(s)ds.
\]

Consider the set

\[
K = \{ v \in C_{R^n}([T_0, \tilde{t}]) : v(t) = u_0 + \int_{T_0}^{t} \dot{v}(s)ds, \|\dot{v}(t)\| \leq \gamma(t) \text{ a.e. in } [T_0, \tilde{t}] \},
\]

where \(\gamma(t) = \dot{\gamma}(t) + 2\dot{\delta}(t), \) \(K\) is a compact set, by Theorem 2.1 there exists a nonempty closed convex set-valued mapping \(\Phi : K \to L^1_{R^n}([T_0, \tilde{t}]),\) with the properties cited there, such that for every \(\phi_n \in \Phi(u_n)\) and all \(n \in \mathbb{N}\)

\[
\phi_n(t) \in F(t, u_n(t)) \text{ and } \|\phi_n(t)\| \leq p + q\|u_n(t)\| + \epsilon \text{ a.e. } t \in [T_0, \tilde{t}]
\]

such that,

\[
-\dot{u}_n(t) \in N_{C(t)}(u_n(t)) + \phi_n(t) \text{ a.e. } t \in [T_0, \tilde{t}]
\]

which means that \(-\dot{u}_n(t) - \phi_n(t) \in N_{C(t)}(u_n(t)).\) Since

\[
\|\dot{u}_n(t) + \phi_n(t)\| \leq \|\dot{u}_n(t)\| + \|\phi_n(t)\| \leq \gamma(t) + \dot{\delta}(t) = \mu(t)
\]

so,

\[
-\dot{u}_n(t) - \phi_n(t) \in \mu(t)\mathbb{B},
\]

by Proposition 1 we get

\[
\dot{u}_n(t) + \phi_n(t) \in -\mu(t)d\mu(u_n(t), C(t)) \text{ a.e. } t \in [T_0, \tilde{t}].
\]

In the other hand and by (2), we get

\[
\|\phi_n(t)\| \leq p + q\left(\|u_0\| + \|\dot{\gamma} + 2\dot{\delta}\|_{L^1_{R^n}([T_0, T])}\right) + \epsilon,
\]

so that, \((\phi_n)\) is bounded in \(L^\infty_{R^n}([T_0, \tilde{t}]),\) taking a subsequence if necessary \((\phi_n)\) (that we do not relabel) converging \(\sigma(L^1_{R^n}([T_0, \tilde{t}]), L^\infty_{R^n}([T_0, \tilde{t}]))\) to a mapping \(\phi\). So that, \((u_n + \phi_n)\) weakly converges in \(L^1_{R^n}([T_0, \tilde{t}])\) to \(u + \phi.\) As \((u_n)\) converges uniformly to \(u\) and the convex compact valued set-valued mapping \(y \mapsto \mu(t)d\mu(y, C(t))\) is upper semi-continuous on \(R^n\) applying Theorem VI.4 in [5], we obtain

\[
\dot{u}(t) + \phi(t) \in -\mu(t)d\mu(u(t), C(t)).
\]

In addition since \(\Phi\) has a sequentially strongly-weakly closed graph we have from \(\phi_n \in \Phi(u_n)\) and the convergence of \((\phi_n)\) and \((u_n)\), that \(\phi \in \Phi(u)\) which mean that

\[
\phi(t) \in F(t, u(t))
\]

with

\[
\|\phi(t)\| \leq p + q\left(\|u_0\| + \|\dot{\gamma} + 2\dot{\delta}\|_{L^1_{R^n}([T_0, T])}\right) + \epsilon.
\]

To complete the proof, it remains to show that \(\dot{u}(t) + \phi(t) \in -N_{C(t)}(u(t))\) a.e. \(t \in [T_0, \tilde{t}].\) Consider the classical transformation, for each \(t \in [T_0, \tilde{t}],\)

\[
y(t) = \int_{T_0}^{t} \phi(s)ds, \quad J(t) = C(t) + y(t) \quad \text{and} \quad z(t) = u(t) + y(t).
\]
It is clear that the set-valued mapping $J$ has closed, uniformly $\rho$-prox-regular values, and satisfies $(H_2)$ with an absolutely continuous function $V(\cdot)$ such that $V(t) = \int_{T_0}^t \mu(s)\,ds$. Hence, we obtain the inclusion
\[-\dot{z}(t) \in \mu(t)\partial d(z(t), J(t)) \text{ a.e. } t \in [T_0, \tilde{t}],\]
with
\[z(T_0) = u_0 \in J(T_0) = C(T_0).\]

By Theorem 3.2 in [14], $z(\cdot)$ satisfies
\[
\begin{cases}
-\dot{z}(t) \in N_{J(t)}(z(t)) & \text{a.e. } t \in [T_0, \tilde{t}]; \\
z(T_0) = u_0 \in J(T_0),
\end{cases}
\]
so that
\[
\begin{cases}
\dot{u}(t) \in N_{C(t)}(u(t)) + \phi(t) & \text{a.e. in } [T_0, \tilde{t}]; \\
u(T_0) = u_0 \in C(T_0),
\end{cases}
\]
since $\phi(t) \in F(t, u(t))$, we get
\[
\begin{cases}
\dot{u}(t) \in N_{C(t)}(u(t)) + F(t, u(t)) & \text{a.e. in } [T_0, \tilde{t}]; \\
u(T_0) = u_0 \in C(T_0).
\end{cases}
\]

This shows that $T_u(u_0)$ is compact. From the compactness of $T_u(u_0)$ we deduce that of $A_{u_0}(\tilde{t})$. \qed

Now, we address the particular case of a fixed set $C(t) := C$, and an autonomous perturbation $F(t, x) := G(x)$. Such problems arises in the analysis of resource allocation mechanisms in economics and crowd motion modeling variational inequalities. The following result is crucial in the statement of our next result.

**Proposition 3.** Let $C$ be a closed subset of $\mathbb{R}^n$ uniformly $\rho$-prox-regular, and $G : \mathbb{R}^n \to \mathbb{R}^n$ be a measurable set-valued mapping satisfying the following assumptions:

(i) at each $x \in \mathbb{R}^n$ such that $G(x)$ is compact and almost convex the set-valued mapping $G(\cdot)$ is upper semi-continuous, and whenever $G(x)$ is not almost convex $G(\cdot)$ is lower semi-continuous on some neighborhood of $x$;

(ii) there are two nonnegative constants $p, q$ and for all $x \in \mathbb{R}^n$ $Proj_{G(x)}(0) \leq p + q\|x\|$.

Let $u : [T_0, T] \to \mathbb{R}^n$ be an absolutely continuous solution of the differential inclusion
\[
(P_{co}) \begin{cases}
-\dot{x}(t) \in N_C(x(t)) + F(x(t)) & \text{a.e. in } [T_0, T]; \\
x(t) \in C, & \forall t \in [T_0, T]; \\
x(T_0) = u_0 \in C.
\end{cases}
\]

where
\[F(u(t)) = \begin{cases}
\text{co}(G(u(t))) & \text{if } t \in D(u); \\
G(u(t)) & \text{if } t \in [T_0, T] \setminus D(u),
\end{cases}\]
and $D : \mathbb{R}^n \to \mathcal{L}([T_0, T])$ is given by
\[D(u) = \{t \in [T_0, T] : G(u(\cdot)) \text{ is upper semi-continuous}\},\]
and let $g(\cdot) : D(u) \to \mathbb{R}^n$ be a measurable selection of $\text{co}(G(u(\cdot)))$ such that $-\dot{u}(t) \in N_C(u(t)) + g(t)$ a.e. $t \in D(u)$. Then,

1) there exist two integrable functions $\lambda_1(\cdot), \lambda_2(\cdot)$ defined on $D(u)$ satisfying $0 \leq \lambda_1(t) \leq 1 \leq \lambda_2(t)$, and
\[
\lambda_1(t) g(t) \in G(u(t)) \quad \text{and} \quad \lambda_2(t) g(t) \in G(u(t)), \forall t \in D(u);
\]
2) there exists a non decreasing absolutely continuous function \( \theta : D(u) \to D(u) \) such that the mapping \( \tilde{u} : [T_0, T] \to \mathbb{R}^n \) given by
\[
\tilde{u}(t) = \begin{cases} 
  u(\theta(t)) & \text{if } t \in D(u), \\
  u(t) & \text{if } t \in [T_0, T] \setminus D(u),
\end{cases}
\]
is solution of the problem \( (P_G) \) with \( \tilde{u}(T) = u(T) \) and \( \tilde{u}(T_0) = u(T_0) \).

Proof. 1) Since, for all \( t \in D(u), \) \( G(u(t)) \) is almost convex, then there exist two nonempty set-valued mappings \( \Delta_1 : D(u) \to [0, 1] \) and \( \Delta_2 : D(u) \to [1, +\infty] \) defined by
\[
\Delta_1(t) = \{ \lambda_1 \in [0, 1] : \lambda_1 g(t) \in G(u(t)) \}
\]
and
\[
\Delta_2(t) = \{ \lambda_2 \in [1, +\infty] : \lambda_2 g(t) \in G(u(t)) \}.
\]
Let \( Z = \{ t \in D(u) : g(t) = 0 \} \). There is no loss of generality in assuming that, for \( t \in Z, \Delta_1(t) = \Delta_2(t) = \{1\} \). Let show that \( \Delta_1 \) is measurable on \( J = D(u) \setminus Z \).

Consider its graph
\[
Gph\Delta_1 = \{(t, \lambda_1) \in J \times [0, 1] : \lambda_1 g(t) \in G(u(t)) \}
\]
\[
= \{(t, \lambda_1) \in J \times [0, 1] : d(\lambda_1 g(t), G(u(t))) = 0 \}
\]
\[
= (J \times [0, 1]) \cap \sigma^{-1}(\{0\}),
\]
with \( \sigma : (t, \lambda_1) \to d(\lambda_1 g(t), G(u(t))) \), then \( Gph\Delta_1 \) is measurable as the intersection of two measurable subsets. In addition, its values are closed subsets of \([0, 1]\) because the values of \( G \) are closed. Then, we conclude that \( \Delta_1 \) is measurable on \( D(u) \).

With the same technique as above, \( \Delta_2 \) is a measurable selection on \( D(u) \), with the difference that the values of \( \Delta_2 \) need not be bounded. In this case, we write \( J \) as the countable union of the sets \( B_n = \{ t : \|g(t)\| \geq \frac{1}{n} \} \). On each \( B_n \), and for all \( \lambda_2 \in \Delta_2(t) \) we have \( \lambda_2 g(t) \in G(u(t)) \). So, \( \Delta_2 \) has an upper bound on \( B_n \), since the values of \( G \) are bounded and the same reasoning as in the previous point can be applied. Consequently, there are measurable selections \( \lambda_1(\cdot), \lambda_2(\cdot) \) defined on \( D(u) \) satisfying \( 0 \leq \lambda_1(t) \leq 1 \leq \lambda_2(t) \), and
\[
\lambda_1(t) g(t) \in G(u(t)) \quad \text{and} \quad \lambda_2(t) g(t) \in G(u(t)), \quad \forall t \in D(u).
\]

2) Step 1. By the upper semi-continuity of \( G \) and continuity of \( u(\cdot) \), the set-valued mapping \( D(\cdot) \) has closed values on \([T_0, T]\). Also we can write \( D(u) \) as the countable union of open intervals so, \( D(u) = \bigcup_{k \in I} [a_k, b_k] \) where for all \( k \in I, \) \( [a_k, b_k] \) are open intervals of \( D(u) \).

Step 2. Let \([a, b] \subset D(t)\) be any interval, assume that on this interval there exists two integrable functions \( \lambda_1(\cdot) \) and \( \lambda_2(\cdot) \), such that \( 0 \leq \lambda_1(t) \leq 1 \leq \lambda_2(t) \). In addition, assume that \( \lambda_1(\cdot) > 0 \) a.e., using the same technique as in [2] and [6], there exist two measurable subsets of \([a, b]\), having characteristics functions \( \chi_1 \) and \( \chi_2 \) such that \( \chi_1 + \chi_2 = \chi_{[a,b]} \) and an absolutely continuous function \( \gamma : [a, b] \to [a, b] \) such that
\[
\dot{\gamma}(t) = \frac{1}{\lambda_1(t)} \chi_1(t) + \frac{1}{\lambda_2(t)} \chi_2(t) \quad \text{and} \quad \gamma(b) - \gamma(a) = b - a.
\]

Step 3. Consider \( \Omega = \{ t \in D(u) : 0 \in G(u(t)) \} \), since \( G(\cdot) \) has closed values and upper semi-continuous and \( u(\cdot) \) is continuous for every \( t \in D(u) \), then \( \Omega \) is closed relative to \( D(u) \).
a) Suppose that \( \Omega \) is empty, in this case \( \lambda_1(\cdot) > 0 \) on \( D(u) \), since \( D(u) = \bigcup_{k \in I} [a_k, b_k] \), so the step 2 can be applied on each subinterval \([a_k, b_k]\). Setting \( \gamma(t) = a_k + \int_a^t \gamma(s)ds \), \( \gamma \) is increasing and we have \( \gamma(a_k) = a_k \), and \( \gamma(b_k) = b_k \) that is \( \gamma \) maps \([a_k, b_k]\) to itself. Let \( \theta_k : [a_k, b_k] \rightarrow (a_k, b_k) \) be the inverse of \( \gamma \) on \([a_k, b_k]\), then \( \theta(a_k) = a_k, \theta(b_k) = b_k \) and

\[
\dot{\theta}_k(t) = \frac{1}{\gamma'(\theta_k(t))} = \lambda_1(\theta_k(t)) \chi_1^k(\theta_k(t)) + \lambda_2(\theta_k(t)) \chi_2^k(\theta_k(t)).
\]

Let \( \theta(t) = \sum_{k \in I} \theta_k(t) \), with

\[
\dot{\theta}(t) = \sum_{k \in I} \dot{\theta}_k(t) = \sum_{k \in I} \left( \lambda_1(\theta_k(t)) \chi_1^k(\theta_k(t)) + \lambda_2(\theta_k(t)) \chi_2^k(\theta_k(t)) \right).
\]

Defined \( \tilde{u} : [T_0, T] \rightarrow \mathbb{R}^n \) as

\[
\tilde{u}(t) = \begin{cases} 
  u(\theta(t)) & \text{if } t \in D(u), \\
  u(t) & \text{if } t \in [T_0, T] \setminus D(u),
\end{cases}
\]

then, for all \( t \in D(u) \)

\[
-\frac{d}{dt} \tilde{u}(t) = -\dot{\theta}(t) \frac{d}{dt} u(\theta(t)) \in \dot{\theta}(t) \left( N_C(u(\theta(t))) + g(\theta(t)) \right),
\]

by the properties of the normal cone and the assumption on \( g \), we get, for all \( t \in D(u) \)

\[
-\frac{d}{dt} \tilde{u}(t) \in N_C(u(\theta(t))) + \dot{\theta}(t) g(\theta(t))
\]

\[
\in N_C(u(\theta(t))) + G(u(\theta(t))) + G(\tilde{u}(t)).
\]

b) Suppose that \( \Omega \neq \emptyset \), setting \( \tau = \sup \Omega \), since \( \Omega \) is closed relative to \( D(u) \) then \( \tau \in \Omega \), here the complement of \( \Omega \) is open relative to \( D(u) \), then it consists of at most a countably many non-overlapping open intervals \([a_i, b_i]\), with the possible exception of one of the form \([\tau, b_i]\). For each \( i \), apply step 2 to the interval \([a_i, b_i]\) to infer the existence of two measurable subsets of \([a_i, b_i]\) with characteristic functions \( \chi_1^i(\cdot) \) and \( \chi_2^i(\cdot) \) such that \( \chi_1^i(\cdot) + \chi_2^i(\cdot) = \chi_{[a_i, b_i]}(\cdot) \). Setting

\[
\gamma(t) = \chi_1^i(t) \frac{1}{\lambda_1(t)} + \chi_2^i(t) \frac{1}{\lambda_2(t)} \quad \text{and} \quad \int_{a_i}^{b_i} \gamma(s)ds = b_i - a_i.
\]

Since \( D(u) \) is closed, then \( m = \inf D(u) \) and \( M = \sup D(u) \) are in \( D(u) \) and we get the following cases:

1) On \([m, \tau]\), set

\[
\gamma(t) = \frac{1}{\lambda_2(t)} \chi_0(t) + \sum_i \left( \chi_1^i(t) \frac{1}{\lambda_1(t)} + \chi_2^i(t) \frac{1}{\lambda_2(t)} \right),
\]

where the sum is over all intervals contained in the \([m, \tau]\), since \( \lambda_2(t) \geq 1 \) and \( \int_{a_i}^{b_i} \gamma(s)ds = b_i - a_i \), we get

\[
\int_m^\tau \gamma(t)dt = l \leq \tau - m.
\]

Then \( \gamma(\cdot) \) given by \( \gamma(t) = \int_m^t \gamma(s)ds \) is an invertible map from \([m, \tau]\) to \([m, \bar{l}]\).

Define \( \theta : [m, \bar{l}] \rightarrow [m, \tau] \) to be the inverse of \( \gamma \), extend \( \theta \) as an absolutely continuous function \( \tilde{\theta} \) on \([m, \tau]\), setting \( \tilde{\theta}(t) = 0 \) for all \( t \in [\tau, \bar{l}] \), to prove that the mapping \( \tilde{u}(t) = u(\tilde{\theta}(t)) \) is solution of \((P_C')\) on \([m, \tau]\), satisfying \( \tilde{u}(\tau) = u(\tau) \).
• For $t \in [m,n]$ : we have $\hat{\theta}(t) = \theta(t)$ is invertible and
  \[
  \hat{\theta}(t) = \lambda_2(\theta(t))\chi_{\Omega}(\theta(t)) + \sum_i \left( \chi_{1}(\theta(t))\lambda_1(\theta(t)) + \chi_{2}(\theta(t))\lambda_2(\theta(t)) \right).
  \]
  As $\frac{d}{dt} \hat{u}(t) = \hat{\theta}(t)\hat{u}(\theta(t))$, we get
  \[
  -\frac{d}{dt} \hat{u}(t) \in \hat{\theta}(t) \left( N_C(u(\theta(t))) + g(\theta(t)) \right),
  \]
  by the properties of the normal cone and the assumption on $g$ we get
  \[
  -\frac{d}{dt} \hat{u}(t) \in N_C(u(\theta(t))) + G(u(\theta(t))) = N_C(\hat{u}(t)) + G(\hat{u}(t)).
  \]

• For $t \in [l,n]$ : we have $\hat{\theta}(l) = \tau$ and $\hat{\theta}(\tau) = 0$, then we get
  \[
  \hat{\theta}(t) = \hat{\theta}(l) = \theta(l), \ \forall t \in [l,\tau],
  \]
  we obtain
  \[
  \hat{u}(l) = u(\hat{\theta}(l)) = u(\theta(l)) = \hat{u}(\theta(t)) = \hat{u}(t), \ \forall t \in [l,\tau],
  \]
  so, $u(\hat{\theta}(\tau)) = u(\tau)$ and $\hat{u}$ is constant on $[l,\tau]$ so that
  \[
  -\frac{d}{dt} \hat{u}(t) = 0 \in G(u(\tau)) = G(\hat{u}(t)),
  \]
  in addition $0 \in N_C(\hat{u}(t))$, we conclude that
  \[
  -\frac{d}{dt} \hat{u}(t) = 0 \in N_C(\hat{u}(t)) + G(\hat{u}(t))
  \]

2) On the set $[\tau,n]$, $\Omega$ is empty, then $\lambda_1(t) > 0$ so step 1 and step 3 part a) can be repeated to solve the problem $(P_{\hat{c}})$ on that set. This finishes the proof. \(\square\)

We can now state and prove our main theorem.

**Theorem 3.2.** Let $C$ be a closed subset of $\mathbb{R}^n$ uniformly $\rho$-prox-regular and $G : \mathbb{R}^n \to \mathbb{R}^n$ be a measurable set-valued mapping, satisfying the assumptions (i) and (ii). Then for all $u_0 \in C$,

1) the problem $(P_{\hat{c}})$ has at least an absolutely continuous solution;

2) for all $t \in [T_0,T]$ the attainable set at $t$, $A_{u_0}(t)$ coincides with $A_{u_0}^c(t)$, the attainable set at $t$ of the convexified problem $(P_{\hat{c}})$.

**Proof.** 1) Observing that with $D(\cdot)$ defined as above, the set-valued mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ given by
  \[
  F(x(t)) = \begin{cases}
  \text{co}G(x(t)) & \text{if } t \in D(x),
  \\
  G(x(t)) & \text{if } t \in [T_0,T] \setminus D(u).
  \end{cases}
  \]
  is mixed semi-continuous. Since $G(x) \subset F(x)$ for all $x \in \mathbb{R}^n$, then
  \[
  \text{Proj}_{F(x)}(0) \leq \text{Proj}_{G(x)}(0) \leq p + q\|x\|, \ \forall x \in \mathbb{R}^n.
  \]

Consequently and by Theorem 3.1, there exists an absolutely continuous function $u : [T_0,T] \to \mathbb{R}^n$ solution of the problem $(P_{\hat{c}})$. And by Proposition 3, the problem $(P_{\hat{c}})$ admits at least an absolutely continuous solution $\hat{u} : [T_0,T] \to \mathbb{R}^n$ such that $\hat{u}(T) = u(T)$.

2) For every $t \in [T_0,T]$, the attainable set at $t$, $A_{u_0}(t)$, is contained in the attainable set at $t$ of the convexified problem, $A_{u_0}^c(t)$, it is enough to show that $A_{u_0}^c(t) \subset A_{u_0}(t)$. 

4. Time optimal problem. In this section we prove the existence of solution to the minimum time problem for the differential inclusion

\[
(P_h) \begin{cases} 
\dot{u}(t) \in -N_C(u(t)) + h(u(t), z(t)), & \text{a.e. in } [T_0, T], \\
z(t) \in Z(u(t)), & \forall t \in [T_0, T], \\
u(t) \in C, & \forall t \in [T_0, T], \\
u(0) = u_0,
\end{cases}
\]

under the almost convexity assumption on the set \( G(x) = h(x, Z(x)) \).

**Corollary 1.** Let \( C \) be a closed subset of \( \mathbb{R}^n \), uniformly \( p \)-prox-regular. Let \( D \) be a subset of \( \mathbb{R}^n \) and \( Z : \mathbb{R}^n \to \mathbb{R}^n \) be a nonempty measurable set-valued mapping such that at each \( x \in D \), \( Z(\cdot) \) is upper semi-continuous at \( x \) with compact values and whenever \( x \notin D \), \( Z(\cdot) \) is lower semi-continuous on some neighborhood of \( x \), and let a continuous single-valued map \( h : \text{Gph}(Z) \to \mathbb{R}^n \) satisfying the following assumption:

\( H_h \) there are nonnegative constants \( p \) and \( q \) such that

\[ ||h(x, z)|| \leq p + q||x||, \forall (x, z) \in \text{Gph}(Z). \]

We associate with these data the set-valued mapping \( G : \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[ \forall x \in \mathbb{R}^n, \quad G(x) = \{ h(x, z) \}_{z \in Z(x)}. \]

Assume that \( G(\cdot) \) is compact and almost convex for every \( x \in D \).

Let \( u_{0}, u_1 \) be given in \( \mathbb{R}^n \), and assume that for some \( T_0 \leq t \leq T \), \( u_1 \in A_{u_0}(t) \). Then, the problem of reaching \( u_1 \) from \( u_0 \) in a minimum time admits a solution.

**Proof.** Under the hypotheses on \( h \) and \( Z \), the set-valued mapping \( G \) is almost mixed semi-continuous and

\[ \text{Proj}_{G(x)}(0) \leq p + q||x||, \forall (x, z) \in \text{Gph}(Z). \]

Let \( t_1 = \inf\{ \tau \in [T_0, t] : u_1 \in A_{u_0}(\tau) \} \). Let \((t_n)\) be decreasing to \( t_1 \) and for each \( n \) let \( u_n(\cdot) \) be a solution of the problem

\[
\begin{cases} 
\dot{u}(t) \in -N_C(u(t)) + G(u(t)), & \text{a.e. in } [T_0, t_n], \\
u(t) \in C, & \forall t \in [T_0, t_n], \\
u(T_0) = u_0
\end{cases}
\]

such that \( u_n(t_n) = u_1 \). We define the sequence \((\bar{u}_n(\cdot))\) by \( \bar{u}_n(\tau) = u_n(\tau) \), for all \( \tau \in [T_0, t_1] \). Then

\[ (\bar{u}_n(\tau)) \subset A_{u_0}(\tau) = A_{u_0}^{co}(\tau). \]

Since \( A_{u_0}^{co}(\tau) \) is compact, by extracting a subsequence if necessary we may conclude that \((\bar{u}_n(\cdot))\) converges to \( \bar{u}(\tau) \in A_{u_0}^{co}(\tau) \), and we have \( \bar{u}(t_1) = u_1 \in A_{u_0}^{co}(t_1) \). Using Theorem 3.2, we get \( A_{u_0}^{co}(t_1) = A_{u_0}(t_1) \). So that, \( \bar{u} \) is the solution of the problem \((P_h)\) that reaches \( u_1 \) in the minimum time, and \( t_1 \) is the value of the minimum time. This completes the proof. \( \square \)
REFERENCES

[1] D. Affane, M. Aissous and M. F. Yarou, Existence results for sweeping process with almost convex perturbation, Bull. Math. Soc. Sci. Math. Roumanie, 61 (2018), 119–134.
[2] D. Affane and D. Azzam-Laouir, Almost convex valued perturbation to time optimal control sweeping processes, Essaim: Control, Optim. Calcul. Variat., 23 (2017), 1–12.
[3] H. Attouch and A. Damlamian, On multivalued evolution equations in Hilbert spaces, Israel J. Math., 12 (1972), 373–390.
[4] M. Bounkhel and L. Thibault, Nonconvex sweeping process and prox-regularity in Hilbert space, J. Nonlin. Convex Anal., 6 (2005), 359–374.
[5] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lectures Notes in Mathematics, Vol. 580. Springer-Verlag, Berlin-New York, 1977.
[6] A. Cellina and A. Ornelas, Existence of solution to differential inclusion and optimal control problems in the autonomous case, SIAM J. Control Optim., 42 (2003), 260–265.
[7] F. H. Clarke, Y. S. Ledyaev, R. J. Stern and P. R. Wolenski, Nonsmooth Analysis and Control Theory, Graduate Texts in Mathematics, 178. Springer-Verlag, New York, 1998.
[8] F. H. Clarke, R. J. Stern and P. R. Wolenski, Proximal smoothness and the lower-C^2 property, J. Convex Anal., 2 (1995), 117–144.
[9] B. Cornet, Existence of slow solutions for a class of differential inclusions, J. Math. Anal. Appl., 96 (1983), 130–147.
[10] A. Fryszkowski and L. Gorniewicz, Mixed semi-continuous mappings and their applications to differential inclusions, Set-Valued Anal., 8 (2000), 203–217.
[11] T. Haddad and L. Thibault, Mixed semi-continuous perturbation of nonconvex sweeping process, Math. Program. Ser. B, 123 (2010), 225–240.
[12] C. Henry, An existence theorem for a class of differential equation with multi-valued right hand side, J. Math. Anal. Appl., 41 (1973), 179–186.
[13] R. A. Poliquin, R. T. Rockafellar and L. Thibault, Local differentiability of distance functions, Trans. Math. Soc., 352 (2000), 5231–5249.
[14] L. Thibault, Sweeping process with regular and nonregular sets, J. Diff. Eqs., 193 (2003), 1–26.
[15] A. A. Tolstonogov, Solutions of a differential inclusion with unbounded right-hand side (Russian), Sib. Math. Zh., 29 (1988), 212–225.

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