Majorana fermions comprise the simplest and likely most experimentally accessible non-Abelian anyon. An unambiguous demonstration of their non-Abelian exchange statistics would be a great triumph for condensed matter physics, as this phenomenon reflects one of the most spectacular manifestations of emergence. Furthermore, non-Abelian excitations provide the foundation behind topologically protected quantum computation [1,2], with Majorana fermions playing a crucial role in prototype devices [3–7]. In the solid-state context, Majorana modes were originally perceived as zero-energy states bound to vortices in p-wave superconductors [8], and therefore are also associated with quasi-particles in states bound to vortices in superconductors bridged by an ordinary superconductor. This new term gives rise to a set of Shapiro steps which are essentially absent without Majorana modes and therefore provides a unique signature for these exotic states.

Josephson effects provide yet another important example of unconventional Josephson signatures of Majorana bound states.

Unconventional Josephson signatures of Majorana bound states

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A junction between two topological superconductors containing a pair of Majorana fermions exhibits a ‘fractional’ Josephson effect, 4π periodic in the superconductors’ phase difference. An additional fractional Josephson effect, however, arises when the Majoranas are spatially separated by a superconducting barrier. This new term gives rise to a set of Shapiro steps which are essentially absent without Majorana modes and therefore provides a unique signature for these exotic states.

\[ I' = \frac{e}{\hbar} J_M \sin \left( \frac{\phi_L - \phi_R}{2} \right), \]  

which exhibits a remarkable 4π periodicity in the superconducting phase difference \( \phi_L - \phi_R \) between the left and right wires. In stark contrast to ordinary Josephson currents, this contribution reflects tunneling of half of a Cooper pair across the junction. Such a ‘fractional’ Josephson effect was later established in other systems supporting Majorana modes [10–12], and in direct junctions between p-wave superconductors [28]. In this manuscript we demonstrate that two topological superconductors bridged by an ordinary superconductor with phase \( \phi_m \) generically support a second kind of unconventional Josephson effect with an associated current

\[ I = \frac{e}{\hbar} J_M \sin \left( \frac{\phi_L + \phi_R}{2} - \phi_m \right), \]  

in the right or left superconductors, and twice that in the middle. This contribution arises solely from the fusion of spatially-separated Majoranas across the junction, and represents processes whereby a Cooper pair in the middle region splinters, with half entering the left and half entering the right topological superconductor. We will derive this emergent term in 1d Majorana-supporting systems, and propose several ways of measuring its effects.

This novel Josephson coupling is derived most simply in a 1d Kitaev chain. Consider a junction with Hamiltonian \( H = H_L + H_R + \delta H \), where the left/right superconductors are described by p-wave-paired spinless fermions \( c_{\alpha,x} \) (\( \alpha = \ell, r \)) hopping on an N-site chain

\[ H_\alpha = -\sum_{x=1}^{N-1} (t^\dagger_{\alpha,x} c_{\alpha,x+1} + \Delta e^{i\phi_{\alpha}} c_{\alpha,x} c_{\alpha,x+1} + h.c.). \]  

Eq. (3) adiabatically connects to realistic Majorana-supporting quantum wire Hamiltonians [18–20,29], and therefore describes their universal properties as well. Following Kitaev, we express the spinless fermions in terms of two Majorana operators via \( c_{\alpha,x} = \frac{1}{2} e^{-i\phi_{\alpha}} (\gamma^\alpha_{B,x} + i\gamma^\alpha_{A,x}) \). When \( t = \Delta \), Eq. (3) maps onto a dimerized Majorana chain: \( H_\alpha = -it \sum_{x=1}^{N-1} \gamma^\alpha_{B,x} \gamma^\alpha_{A,x+1} \). The explicit absence of \( \gamma^\alpha_{A,1} \) and \( \gamma^\alpha_{B,N} \) in the Hamiltonians indicates the presence of zero-energy Majorana modes localized at the ends of each superconductor in the junction.

Let us now couple the two superconductors through

\[ \delta H = -t_m (c_{\ell,N} c_{r,1} + h.c.) - \Delta_m (e^{i\phi_m} c_{\ell,N} c_{r,1} + h.c.), \]  

where the two terms describe tunneling and Cooper pair across the junction. These couplings combine the zero-energy Majorana modes residing at the junction into a finite-energy Andreev bound state. Focusing on these zero-energy modes, one can write \( c_{\ell,N} \rightarrow \frac{1}{2} e^{-i\phi/2} \gamma^\ell_{B,N} \) and \( c_{r,1} \rightarrow i\frac{1}{2} e^{-i\phi/2} \gamma^r_{A,1} \), and define an ordinary fermion
operator $f^\dagger = \frac{1}{2}(\gamma_{B,N}^f + i\gamma_{A,1}^f)$; $\delta H$ then becomes

$$\delta H \rightarrow (2f^\dagger f - 1)\{J_M \cos[(\phi_\ell - \phi_r)/2] + J_Z \cos[(\phi_\ell + \phi_r)/2 - \phi_m]\}.$$  

with $J_M = \frac{t_m}{\lambda_s}$ and $J_Z = \frac{\Delta_m}{2}$. Since the current in region $s$ is given by $\frac{2e}{h} \frac{\partial \langle \hat{\psi}\rangle}{\partial s}$, the fermion tunneling $t_m$ gives rise to the fractional Josephson effect of Eq. [1], while pairing $\Delta_m$ across the junction produces the Josephson current in Eq. [2]. Note that the sign of either current is dictated by the occupation number for the $f$ fermion, and hence can be used as a readout method for qubit states encoded by the Majoranas [11, 29].

A more quantitative understanding is obtained by considering more realistic models. Let us consider Majoranas localized on a topological insulator edge in proximity to a superconductor and subjected to a magnetic field [10]; a very similar analysis applies to quantum wires. In the Nambu spinor basis $\Psi^T = (\psi_1, \psi_2, \psi^\dagger_1, -\psi^\dagger_2)$, the Bogoliubov-de Gennes Hamiltonian for this system is

$$\hat{H} = v\hat{p}\sigma^\tau - \mu\tau^\sigma + \Delta(\cos \phi \tau^x - \sin \phi \tau^y) + B\sigma^z,$$

with $v$ the edge-state velocity, $\hat{p}$ the momentum, $B$ the Zeeman energy, and $\sigma^\tau$ and $\tau^\sigma$ Pauli matrices acting in the spin and particle-hole sectors, respectively. We allow the chemical potential $\mu$, pairing amplitude $\Delta$, and superconducting phase $\phi$, to vary spatially.

Majorana states arise at interfaces between topological (T) and trivial (S) regions of the edge [10]. With $\mu$, $\Delta$, and $\phi$ uniform the quasi-particle gap is $E_{\text{gap}} = |B - \sqrt{\Delta^2 + \mu^2}|$. When $\sqrt{\Delta^2 + \mu^2} > B$ the edge is gapped by proximity-induced superconductivity and forms a topological phase closely related to that of Kitaev’s model described above [10]. In the trivial phase $\sqrt{\Delta^2 + \mu^2} < B$, and the magnetic field dominates the gap. We will study the T-S-T domain sequence of Fig. 1 which localizes Majoranas $\gamma_1$ at $x = 0$ and $\gamma_2$ at $x = L$. Each of the three regions, $\ell, r, m$, couples to a superconductor imparting proximity strength $\Delta_{/\ell, m, r}$ and phase $\phi_{/\ell, m, r}$, and has a chemical potential $\mu_{/\ell, m, r}$ controlled by separate gates. (The main difference in the quantum wire case is that there creating the T-S-T domain structure needed to observe the unconventional Josephson effects discussed here requires the reversed criteria: $\sqrt{\Delta^2 + \mu^2} < B$ in the outer regions and $\sqrt{\Delta^2 + \mu^2} > B$ in the middle region.)

The Majorana-related Josephson effects result from hybridization between $\gamma_1$ and $\gamma_2$. When $\gamma_1$ and $\gamma_2$ are far apart ($L \to \infty$), they constitute exact zero-energy modes, and their wave functions decay exponentially in region $s = \ell, m, r$ with two characteristic lengths:

$$\lambda_s = \frac{v}{|\Delta_s| + \sqrt{B^2 - \mu^2|}},$$

we assume $\lambda_s < B$). For finite $L$, however, $\gamma_{1,2}$ combine into a finite-energy state with creation operator $f^\dagger = \frac{1}{2}(\gamma_1 + i\gamma_2)$. Roughly, each Majorana perceives the interface localizing the other Majorana as a perturbation, yielding a hybridization which is suppressed as a weighted sum of two decaying exponentials. This hybridization is again described by Eq. [6], with $J_{M/Z} = \frac{1}{2}(J_+e^{-L/\lambda_{s+m}} \pm J_+e^{-L/\lambda_{s-m}})$. An explicit calculation (see supp. material) for the symmetric setup, $\mu_{/\ell} = \mu_{/r} = \mu$, $\Delta_{/\ell} = \Delta_{/r} = \Delta > \Delta_m$ and $\mu_m = 0$ yields

$$J_+ = J_\mp \approx \frac{2\Delta}{\Delta_+ + \mu^2} = \frac{2B\Delta}{B^2 - \mu^2}.$$  

When $\Delta_m = 0$ and the middle region is normal—which is the setup typically studied [10, 13]—$J_Z = 0$ and hence only the Josephson term in Eq. [4] appears. Turning on $\Delta_m \neq 0$ yields a nonzero $J_Z$, and the second Josephson term in Eq. [2]. Furthermore, since both $J_Z$ and $J_{M/Z}$ are dominated by the slowest decay length, they will generically be of the same order. For a quantitative estimate, consider the parameters $\mu_m = 0$, $\mu_{/\ell} = E$, $\Delta_m = E$, $\Delta_{/\ell} = \sqrt{8E}$, $B = 2E$ with energy scale $E = 0.1\mu\text{eV}$. Assuming an edge velocity $v = 10^4\mu\text{m/s}$, for this choice we obtain $\lambda_{s+m} \approx 22\mu\text{m}$, $\lambda_{s-m} \approx 66\mu\text{m}$, and $J_\pm \approx 0.12\mu\text{eV}$. The effect then peaks at $L \approx 50\mu\text{m}$, which yields $J_\mp \approx 0.022\mu\text{eV}$ and $J_Z \approx 5.3\mu\text{A}$.

These Josephson effects are simplest to understand conceptually when two additional Majoranas, $\gamma_3, \gamma_4$, straddle the T segments of the edge as shown in Fig. 1. Let us define fermion operators $f_A = \frac{1}{2}(\gamma_1 + i\gamma_2)$ and $f_B = \frac{1}{2}(\gamma_2 + i\gamma_4)$, and assume that the corresponding
occupation numbers are initially \( n_A = 1 \) and \( n_B = 0 \).

We will further employ a ‘perturbative’ perspective and promote the superconducting phases to quantum operators conjugate to the Cooper pair number. One can then see that the Majorana operators in the term

\[
J_M (2^f f - 1) \exp (i \gamma_2 - \phi_m) = i J_M \gamma_1 \gamma_2 \exp (i \gamma_2 - \phi_m)
\]

drop a single fermion across the S region, changing the state of the edge from \((n_A, n_B) = (1, 0)\) to \((0, 1)\). At the same time, the exponential passes a charge \( e \) from side to side. The combination of these processes makes the term gauge invariant. The persistent superconducting current limit in this case is apparent when we consider an additional tunneling event which restores the parities of the T segments, while the exponent removes a phase difference

\[
\sin \left( \frac{i \phi_1 + \phi_m}{2} - \phi_m \right)
\]

also change the parity of the two T segments, while the exponent removes a Cooper pair from the middle region and adds charge \( e \) to each T region (see Fig. 1).

Next, we discuss the crucial issue of measuring the new Josephson term in Eq. (2). The first and most direct possibility involves manipulating independently the phase differences \( \phi_1 - \phi_m \equiv \Phi_L \) and \( \phi_m - \phi_r \equiv \Phi_R \), e.g., by inserting different fluxes in the two loops in Fig. 2a (ignoring the voltage sources in the figure). By tuning \( \Phi_L = - \Phi_R \) in a symmetric junction, one can probe the \( J_Z \) Josephson term (driving current \( J_Z \sin \Phi_L \) on the middle electrode) while canceling the \( J_M \) term. Such measurements, however, are highly challenging—they require careful flux control; the Majorana-related Josephson current must be disentangled from the conventional 2\( \pi \) periodic contributions; and the measurement must be concluded before the parity of the two Majorana changes.

A potentially more promising measurement scheme relies on Shapiro steps. In a regular Josephson junction, Shapiro steps arise from a combination of a dc voltage \( V_{dc} \) and an ac voltage \( V_{ac} \sin \omega t \), which together generate a current

\[
I = I_1 \sin (\phi_1 + 2eV_{dc}/\hbar - (2eV_{ac}/\hbar) \cos \omega t) + \text{higher harmonics}
\]

Naively, this current averages to zero because of the constant winding phase. This is not the case, however, when \( 2eV_{dc}/\hbar = n\omega \) for some integer \( n \)—here a dc current component exists, producing a step in the \( V \) vs. \( I \) plot for the junction [23, 24]. For the fractional Josephson term in Eq. (1), the \( 4\pi \) periodicity leads to Shapiro steps when

\[
2eV_{dc}/\hbar = 2n\omega
\]

as illustrated in Fig. 2b, without any odd-harmonic steps. This non-local measurement is insensitive to any parasitic two-phase Josephson terms, and therefore automatically eliminates most competing processes. Furthermore, it bears the advantage of being a fast dynamic measurement (since Josephson frequencies are typically in the GHz regime), which reduces its sensitivity to temporal fluctuations of the Majorana-state occupations.

To verify the approximation methods used and to confirm the prominence of the \( J_Z \) term in the three-leg Shapiro measurement, we also numerically analyzed the Josephson effects in a topological insulator edge. Figure 3 shows that our analytical results [e.g., Eq. (8)] indeed agree very well with the exact numerical calculation. We also explored additional current contributions such as

\[
\delta I \sin (\phi_L + \phi_R - 2\phi_m)
\]

which could obscure the Majorana signature by producing unwanted odd-harmonic Shapiro steps. This term is independent of the Majorana modes, and can instead arise from conventional Bogoliubov states in the junction. In the limit of small pairing and tunneling over the middle segments, such a term reflects a high-order process. Numerically, we find that it is suppressed by at least an order of magnitude compared to the Majorana \( J_Z \) contribution in the regime where \( J_Z \) is substantial, i.e., when \( L \) is of order \( \lambda_{M \pm} \).

By considering the full edge spectrum (including the Andreev bound states and continuum states exactly), we obtained the total Josephson energy of the domain configuration in Fig. 1

\[
E_{\text{tot}} \approx J_L \cos (\phi_1 - \phi_m) + J_R \cos (\phi_r - \phi_m)
\]
FIG. 3: Numerically determined coefficients of conventional Josephson couplings ($J_{L/R}$), Majorana-induced terms ($J_{M/Z}$), and second harmonic of the $J_Z$ term ($J_{Z,2}$). Our analytical estimates of $J_M^3$ and $J_Z^3$ agree well with numerics. The parameters are $\mu_k = E, \mu_m = 0, \Delta_t, r = \sqrt{\xi E}, \Delta_m = E$, and $B_t, r = B_m = 2E$. The characteristic lengths are $\lambda_m = \xi/3$ and $\lambda_m = \xi$. For $E = 0.1$meV and $v = 10^4$m/s, the length unit is $\xi = 66$nm and the maximum current is $I_Z = \frac{e}{\hbar} I$ $\approx 5.3$mA.

\[
\begin{align*}
&+ J_M \cos[(\phi_t - \phi_r)/2] + J_Z \cos[(\phi_t + \phi_r)/2 - \phi_m] \\
&+ \sum_{n=2}^{\infty} J_{Z,n} \cos[n(\phi_t + \phi_r)/2 - \phi_m)] + \cdots (10)
\end{align*}
\]

Here $J_{L/R}$ are conventional Josephson terms (to which the three-leg measurement is insensitive), $J_{M/Z}$ are the Majorana-induced contributions, and $J_{Z,n}$ denote the (unwanted) higher harmonics of the $J_Z$ term. As Fig. 3 illustrates, $J_M$ dominates for $L \ll \lambda_m$, while for $\lambda_m \gtrsim L \gtrsim \lambda_m$ the $J_Z$ term becomes comparable, enabling the three-leg Shapiro-step measurement. The higher harmonics $J_{Z,n}$ are at least an order of magnitude smaller than $J_Z$ in this regime and can be neglected. For $L \gg \lambda_m$ the Majorana signatures are strongly suppressed as expected.

In this manuscript, we explored a new Josephson effect that arises when a pair of Majorana fermions fuse across a junction formed by two topological superconductors separated by an ordinary superconductor. The Majoranas in this setup enable Cooper pairs injected into the barrier superconductor to ‘splitter’ into the left and right legs of the junction—a process which would ordinarily be prohibited at low energies. While Majorana modes can also give rise to a novel fractional Josephson effect in T-normal-T junctions, we argued that an important advantage of our setup is that here one can more readily isolate the Majorana-mediated Josephson current through Shapiro-step measurements. The experiments we proposed could provide a relatively simple and unambiguous detection scheme for Majorana fermions, and may also serve as a practical readout mechanism for qubit states encoded by these particles.

It is a pleasure to thank M. P. A. Fisher, L. Glazman, J. Preskill, A. Kitaev, A. Stern, J. Meyer, K. Shtengel, C. Marcus, L. Kouwenhoven, and B. Halperin for useful discussions, and the Aspen Center for Physics for hospitality. We are also grateful for support from BSF, SPP1285 (DFG), the Packard foundation, the Sherman-Fairchild foundation, the Moore-Foundation funded CEQS, and the NSF through grant DMR-1055522, and IQI grant number: PHY-0456720 and PHY-0803371.

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with \( \exp(i\zeta_s) = \frac{\mu_s + i\phi_s B^2 - \phi_m^2}{\mu_s - i\phi_s B^2} \), for \( s = \ell, r, m \). These solutions are the building blocks for each Majorana state. In order to obtain what the wave function becomes when \( \phi_s \) (the phases of the superconducting electrodes) deviate from zero, we can apply the rotations: \( \hat{U}_\phi = \exp\left(i\frac{\phi_s}{2}\tau_z\right) \) such that:

\[
\psi^{(\phi)}_{s\pm} = \hat{U}_\phi \psi_{s\pm}.
\]

Obtaining the Majorana solution follows from matching the boundary condition of the solutions, and from them finding the coefficients \( R_{s\pm} \).

To avoid the complicated expression that could arise in the most general case of Majorana coupling, we concentrate on the case where \( \Delta_r = \Delta_\ell > \Delta_m \) and \( \mu_r = \mu_\ell = \mu \) and \( \mu_m = 0 \). This choice does not constitute a substantial loss of generality, and is useful for grasping the results of our calculations. A straightforward but rather tedious calculation leads to the following solution for the amplitudes of the decaying waves of the right Majorana state under the above assumptions:

\[
\begin{pmatrix}
R_{m+} \\
R_{m-}
\end{pmatrix} = \frac{2\sin \zeta_m}{1 + e^{-i\zeta_m}} \begin{pmatrix}
i \sin \left(\frac{\phi_m - \phi_r}{2}\right) \\cos \left(\frac{\phi_m - \phi_r}{2}\right)
\end{pmatrix},
\]

\[
\begin{pmatrix}
R_{r+} \\
R_{r-}
\end{pmatrix} = \frac{-1 + i e^{i\zeta_r}}{1 - i e^{i\zeta_r}} \begin{pmatrix}
i \sin \left(\frac{\phi_r - \phi_m}{2}\right) \\cos \left(\frac{\phi_r - \phi_m}{2}\right)
\end{pmatrix}.
\]

By symmetry, we can infer the structure of the left Majorana, which is localized about \( x = 0 \):

\[
|L\rangle = \Psi_a(L)(x) = L_\ell + \psi_\ell e^{-\frac{|x|}{\lambda_\ell}} + L_m + \psi_m e^{-\frac{|x|}{\lambda_m}},
\]

The amplitudes \( L_{\ell\pm} \) also depend on the phases on the left and middle segment of the wire in a similar way:

\[
\begin{pmatrix}
L_{m+} \\
L_{m-}
\end{pmatrix} = \frac{2\sin \zeta_m}{1 + e^{-i\zeta_m}} \begin{pmatrix}
i \sin \left(\frac{\phi_m - \phi_r}{2}\right) \\cos \left(\frac{\phi_m - \phi_r}{2}\right)
\end{pmatrix},
\]

\[
\begin{pmatrix}
L_{\ell+} \\
L_{\ell-}
\end{pmatrix} = \frac{-1 + i e^{i\zeta_\ell}}{1 - i e^{i\zeta_\ell}} \begin{pmatrix}
i \sin \left(\frac{\phi_r - \phi_m}{2}\right) \\cos \left(\frac{\phi_r - \phi_m}{2}\right)
\end{pmatrix}.
\]

From the above results, and under the symmetric choice of parameters, we can compute the overlap matrix, \( M_{s\beta} = \langle s|\beta \rangle \). Neglecting exponentially suppressed corrections, we obtain the following form:

\[
M_{s\beta} = v \delta_{s\beta} \left[ \frac{B^2 + \Delta_m \cos(\phi_m - \phi_r)}{2B^2 (B^2 - \Delta_m)} + \frac{\Delta_r \Delta_m}{2\Delta_m (B^2 - \Delta_r)^2} \right],
\]

with \( v \) being the spin-orbit velocity.

The coupling between the Majoranas could be calculated perturbatively by considering the two domain walls juxtaposed. For instance, while the left Majorana is an exact zero-energy eigenstate of the Hamiltonian

\[
\mathcal{H}_L = \mathcal{H}_0 \Theta(-x) + \mathcal{H}_m \Theta(x),
\]

\[
\mathcal{H}_L = \frac{i e^{i \zeta_\ell}}{1 - i e^{i \zeta_\ell}} \begin{pmatrix}
i \sin \left(\frac{\phi_r - \phi_m}{2}\right) \\cos \left(\frac{\phi_r - \phi_m}{2}\right)
\end{pmatrix}.
\]
the existence of the right segment of the wire perturbs this wave function, with the perturbation potential being

\[ V_r = (\mathcal{H}_r - \mathcal{H}_m) \theta(x - L). \]

Similarly, we can write \( \mathcal{H} = \mathcal{H}_R + V_l \) with \( V_l = (\mathcal{H}_l - \mathcal{H}_m) \theta(-x) \). This perturbation induces a hybridization matrix between the left Majorana and the right Majorana:

\[ \mathcal{H} = \begin{pmatrix} (L) V_r | L \rangle \langle R | V_r | R \rangle \langle R | V_e | R \rangle \end{pmatrix} = 0. \] (20)

In our case,

\[ V_r = \left[ \left( \Delta_r \cos \phi_r - \Delta_m \cos \phi_m \right) \tau^x - \left( \Delta_r \sin \phi_r - \Delta_m \sin \phi_m \right) \tau^y - \mu_r \tau_z \right] \Theta(x - L). \]

The perturbation matrix we obtain is:

\[ \mathcal{H}_{JM} = \left( e^{-L/\lambda^*} \sin \frac{\phi_r - \phi_m}{2} \sin \frac{\phi_r - \phi_m}{2} + \right. \]

\[ + \left. e^{-L/\lambda^*} \cos \frac{\phi_r - \phi_m}{2} \cos \frac{\phi_r - \phi_m}{2} \right) \]

\[ = (2f \Gamma - 1) \left( J_M \cos \frac{\phi_r - \phi_m}{2} + J_Z \cos \left( \phi_r + \phi_m \right) - \phi_m \right) \]

where \( J_M \) being:

\[ J_+ = J_- \approx \frac{v}{M_{rr}}. \] (23)

with \( \mu = \) an unimportant phase.

We arrive at the final answer for the Josephson coupling using Eq. (11). The result indeed coincides with Eq. (12):

\[ \mathcal{H}_{JM} = \left( e^{-L/\lambda^*} \sin \frac{\phi_r - \phi_m}{2} \sin \frac{\phi_r - \phi_m}{2} + \right. \]

\[ + \left. e^{-L/\lambda^*} \cos \frac{\phi_r - \phi_m}{2} \cos \frac{\phi_r - \phi_m}{2} \right) \]

\[ = (2f \Gamma - 1) \left( J_M \cos \frac{\phi_r - \phi_m}{2} + J_Z \cos \left( \phi_r + \phi_m \right) - \phi_m \right) \]

NUMERICAL CALCULATION

We now detail the procedure of our numerical calculation. In the Nambu spinor basis \( \Psi^T = (\psi_1, \psi_2, \psi_3^*, -\psi_1^*) \), the Bogoliubov-de Gennes Hamiltonian for this system is

\[ \mathcal{H} = v \hat{p} \sigma^z \tau^z - \mu \tau^z + \Delta (\cos \phi \tau^x \sin \phi \tau^y) + B \sigma^z, \]

with \( v \) the edge-state velocity, \( \hat{p} \) the momentum, \( B \) the Zeeman energy, and \( \sigma^a \) and \( \tau^a \) Pauli matrices acting in the spin and particle-hole sectors, respectively. We allow the chemical potential \( \mu \), pairing amplitude \( \Delta \), and superconducting phase \( \phi \) to vary spatially. In region \( s \) (with \( s = l, m, r \)), the parameters \( (\mu, \Delta, \phi) = (\mu_s, \Delta_s, \phi_s) \) are constant. Without loss of generality, we assume \( \phi_m = 0 \) to be a reference of superconducting phase.

The Josephson effects in the TST junction has both bound states and continuum contributions. In the following, we first present the procedure to compute the exact interaction energy \( E \) between two Majoranas, and then provide the formalism to calculate the energy contribution from the continuum.

Bound state energy

For TST configuration, there are two Majoranas at interfaces between topological and trivial regions. The finite separation leads to a finite interaction energy \( E = E_{int} \) between these two Majoranas, with spatial-dependent wave function \( \Psi = \Psi(x) \) satisfying the equation

\[ \mathcal{H} \Psi = E \Psi. \] (25)

We will solve the interaction energy \( E = E_{int} \) by matching the boundary condition of the wave function.

First, we replace the momentum operator \( \hat{p} \) with \( -i \partial \), and obtain the linear differential equation associated with energy \( E \)

\[ \partial_x \Psi(x) = G_E \Psi(x), \]

with 4 × 4 matrix

\[ G_E = i \frac{v}{\sigma^z} + \frac{\Delta}{v} (\cos \phi \tau^y \sin \phi \tau^x) - \frac{B}{v} \sigma^y \tau^z + i \frac{E}{v} \sigma^z \tau^z. \] (27)

In region \( s \) (with \( s = l, m, r \)), the parameters \((\mu, \Delta, \phi) = (\mu_s, \Delta_s, \phi_s)\) are constant, and the matrix \( G_E^{(s)} \) has eigen-system

\[ \mathcal{G}_E^{(s)} \mathbf{u}_j = \kappa_j^{(s)} \mathbf{u}_j \]

with eigenvalues \( \kappa_j^{(s)} \) and eigenvectors \( \mathbf{u}_j^{(s)} \) for \( j = 1, \ldots, 4 \) and \( s = l, m, r \).

Then, we expand the four-component wave function \( \Psi(x) \) in terms of eigenvectors \( \mathbf{u}_j^{(s)} \). We are interested in the localized state with \( E < E_{gap}^{(r)} \). In the left region, there are two localized modes \( (\text{Re} \kappa_1^{(r)} > 0) \) and the two divergent modes \( (\text{Re} \kappa_2^{(r)} < 0) \). Similarly, in the right region, there are two localized modes \( (\text{Re} \kappa_1^{(r)} < 0) \) and the two divergent modes \( (\text{Re} \kappa_2^{(r)} > 0) \). The wave function with two localized Majoranas consists of localized modes

\[ \Psi(x) = \begin{cases} \sum_{j=1,2} c_j^{(r)} e^{i \kappa_j^{(r)} (x - L)} \mathbf{u}_j^{(r)} & \text{for } x \leq L \\ \sum_{j=1,2} c_j^{(r)} e^{i \kappa_j^{(r)} (x - L)} \mathbf{u}_j^{(r)} & \text{for } x \geq L \end{cases}. \] (29)
In order to match the coefficients associated with left and right regions, we integrate the wavefunction over the middle region and obtain the condition
\[ \sum_{j=1,2} c_j^{(r)} \overrightarrow{u}_j^{(r)} = \Psi (L) = e^{G_E^{(m)} L} \Psi (0) = \sum_{j=1,2} c_j^{(l)} e^{G_E^{(m)} L} \overrightarrow{u}_j^{(l)}, \]
which can be written as
\[ \mathbf{M}_E \begin{pmatrix} c_1^{(l)} \\ c_2^{(l)} \\ c_1^{(r)} \\ c_2^{(r)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]
with $4 \times 4$ matrix
\[ \mathbf{M}_E = \begin{bmatrix} e^{G_E^{(m)} L} \overrightarrow{u}_1^{(l)} & e^{G_E^{(m)} L} \overrightarrow{u}_1^{(r)} \\ e^{G_E^{(m)} L} \overrightarrow{u}_2^{(l)} & e^{G_E^{(m)} L} \overrightarrow{u}_2^{(r)} \end{bmatrix}, \]
\[ \text{(32)} \]

The necessary condition for non-zero solution is
\[ \det \mathbf{M}_E = 0, \]
\[ \text{(33)} \]
which can be used to numerically determine the interaction energy $E_{\text{int}}$. As illustrated in Fig. 4, the function $\det \mathbf{M}_E$ vanishes at $E = \pm E_{\text{int}}$. (There is a technical subtlety associated with the fact that $G_E^{(l,r)}$ is not a Hermitian matrix. For some fixed values of $E$, the eigenvalues of $G_E^{(l,r)}$ have multiplicity larger than one, and the eigenvector $\overrightarrow{u}_j^{(l,r)}$ might be a zero vector, which may also lead to spurious solutions with vanishing $\det \mathbf{M}_E$. This issue can be resolved by using a polynomial discriminant to identify and remove these spurious solutions.)

**Continuum contribution**

We now consider the energy contribution from the continuum. The continuum states can be characterized by the scattering matrix $S_E = S_E (\phi, \phi_r)$, which can be computed by matching the boundary conditions for all incoming and outgoing modes. Once we know the scattering matrix, we can use the Fumi’s sum rule to compute the continuum contribution to the system energy.

\[ W (\phi, \phi_r) = \int_{E_{\text{gap}}}^{\infty} \frac{dE}{2\pi} \ln |\det [S_E (\phi, \phi_r)]|. \]

The continuum contribution consists of many Fourier components

\[ W (\phi, \phi_r) = \frac{1}{2} \sum_{n_l, n_r} W_{n_l, n_r} \cos(n_l \phi + n_r \phi_r) \]
\[ \text{(35)} \]
with $W_{n_l, n_r} = W_{-n_l, -n_r}$. Then conventional Josephson terms are $J_{L/R} = W_{1,0}$ and $W_{0,1}$, and the even harmonics of the zipper terms are $J_{Z,2n} = W_{n,n}$. In the following, we provide the formalism to compute the scattering matrix $S_E (\phi, \phi_r)$.

For energy $E > E_{\text{gap}}^{(l,r)}$, there are the propagating modes ($\Phi_j^{(l,r)} = 0$), with momentum $p_j^{(l,r)} = \text{Im} \kappa_j^{(l,r)}$. Suppose there are four incoming modes $(\overrightarrow{u}_1^{(l)}, \overrightarrow{u}_2^{(l)}, \overrightarrow{u}_1^{(r)}, \overrightarrow{u}_2^{(r)})$ with $p_{1,2} > 0$ and $p_{1,2}^{(r)} < 0$, and four outgoing modes $(\overrightarrow{u}_3^{(l)}, \overrightarrow{u}_3^{(r)}, \overrightarrow{u}_4^{(l)}, \overrightarrow{u}_4^{(r)})$ with $p_{3,4}^{(l)} < 0$ and $p_{3,4}^{(l)} > 0$. The wave function can be written as a linear combination of all these modes
\[ \Psi (x) = \begin{cases} \sum_{j=1,2} c_j^{(l)} e^{\kappa_j^{(l)} x} \overrightarrow{u}_j^{(l)} & \text{for } x \leq 0 \\ \sum_{j=1,2} c_j^{(r)} e^{\kappa_j^{(r)} (x-L)} \overrightarrow{u}_j^{(r)} & \text{for } x \geq L \end{cases}. \]
\[ \text{(36)} \]

In order to match the coefficients associated with left and right regions, we integrate the wavefunction over the middle region and obtain the condition $\Psi (L) = e^{G_E^{(m)} L} \Psi (0)$. The relation between the amplitudes of incoming and outgoing modes is

\[ \mathbf{M}_{E, \text{in}} \begin{pmatrix} c_1^{(l)} \\ c_2^{(l)} \\ c_1^{(r)} \\ c_2^{(r)} \end{pmatrix} = \mathbf{M}_{E, \text{out}} \begin{pmatrix} c_3^{(l)} \\ c_4^{(l)} \\ c_3^{(r)} \\ c_4^{(r)} \end{pmatrix}, \]
\[ \text{(37)} \]
with $4 \times 4$ matrices
\[ \mathbf{M}_{E, \text{in}} = \begin{bmatrix} e^{G_E^{(m)} L} \overrightarrow{u}_1^{(l)} & e^{G_E^{(m)} L} \overrightarrow{u}_1^{(r)} \\ e^{G_E^{(m)} L} \overrightarrow{u}_2^{(l)} & e^{G_E^{(m)} L} \overrightarrow{u}_2^{(r)} \end{bmatrix}, \]
\[ \text{(38)} \]
\[ \mathbf{M}_{E, \text{out}} = \begin{bmatrix} -e^{G_E^{(m)} L} \overrightarrow{u}_3^{(l)} & -e^{G_E^{(m)} L} \overrightarrow{u}_3^{(r)} \\ -e^{G_E^{(m)} L} \overrightarrow{u}_4^{(l)} & -e^{G_E^{(m)} L} \overrightarrow{u}_4^{(r)} \end{bmatrix}. \]
\[ \text{(39)} \]
The scattering relation is

\[ S_E = \begin{pmatrix}
    (p_1^{(l)})^{1/2} & c_1^{(l)} \\
    (p_2^{(l)})^{1/2} & c_2^{(l)} \\
    (p_1^{(r)})^{1/2} & c_1^{(r)} \\
    (p_2^{(r)})^{1/2} & c_2^{(r)}
\end{pmatrix} = \begin{pmatrix}
    (-p_3^{(l)})^{1/2} & c_3^{(l)} \\
    (-p_4^{(l)})^{1/2} & c_4^{(l)} \\
    (-p_3^{(r)})^{1/2} & c_3^{(r)} \\
    (-p_4^{(r)})^{1/2} & c_4^{(r)}
\end{pmatrix} \] (40)

with scattering matrix

\[ S_E = P_{out}^{1/2} \cdot M_{out}^{-1} \cdot M_{in} \cdot P_{in}^{-1/2}, \] (41)

where \( P_{in} = \text{Diag} \left[ p_1^{(l)}, p_2^{(l)}, p_1^{(r)}, p_2^{(r)} \right] \) and \( P_{out} = -\text{Diag} \left[ p_3^{(l)}, p_4^{(l)}, p_3^{(r)}, p_4^{(r)} \right] \).

The requirement of conservation of current is

\[ \sum_j p_j^{(l)} \left| c_j^{(l)} \right|^2 = \sum_j p_j^{(r)} \left| c_j^{(r)} \right|^2, \] (42)

which ensures the unitarity of the scattering matrix

\[ S_E^\dagger S_E = I. \] (43)

Hence, \( \det[S_E] = e^{i2\delta_E} \) and \( \frac{1}{2\pi} \ln[\det[S_E(\phi_l, \phi_r)]] = \frac{1}{2\pi} \delta_E(\phi_l, \phi_r) \).

Numerically, we just need to compute the quantity \( \delta_E(\phi_l, \phi_r) \) and the integral

\[ W(\phi_l, \phi_r) = \int_{E_{gap}}^\infty \frac{dE}{\pi} \delta_E(\phi_l, \phi_r). \] (44)

The continuum contribution \( W(\phi_l, \phi_r) \) has 2\( \pi \) periodicity in both \( \phi_l \) and \( \phi_r \), with Fourier decomposition of \( W(\phi_l, \phi_r) = \frac{1}{2} \sum_{n,n=-\infty}^{\infty} W_{n,n} \cos(n_l \phi_l + n_r \phi_r) \), with Fourier coefficients of \( W_{n,n} \). The relevant Fourier components are \( J_{L/R} = W_{1,0} = W_{0,1} \), and \( J_{Z,2n} = W_{n,n} \) for \( n = 1, 2, \ldots \).

There is one subtle issue in the computation of the scattering matrix. There are four propagating modes for \( E > |B^{(l,r)} + \sqrt{(\mu^{(l,r)})^2 + (\Delta^{(l,r)})^2}| \), but there are two propagating modes and two localized modes for

\[ |B^{(l,r)} + \sqrt{(\mu^{(l,r)})^2 + (\Delta^{(l,r)})^2}| > E > E_{gap}^{(l,r)} = |B^{(l,r)} - \sqrt{(\mu^{(l,r)})^2 + (\Delta^{(l,r)})^2}|. \]

In the latter case, we need to compute the effective scattering matrix that are projected to the subspace spanned by the propagating modes.