Effective temperature of a dissipative driven mesoscopic system.

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We study the nonequilibrium dynamics of a mesoscopic metallic ring threaded by a time-dependent magnetic field \[ \text{10, 11, 12, 13, 14} \]. In this setting there is an electronic and partial equilibration \[ \text{2, 3} \]. A similar phenomenon was found in the relaxation of quantum spin-glasses \[ \text{3} \] and in the Coulomb glass \[ \text{4} \].

The question then arises as to whether the notion of \( T_{\text{eff}} \) plays a role in understanding some features of transport in quantum mesoscopic systems. To address it we study the modification of the FDR in a very simple, exactly solvable, mesoscopic device that consists in a metallic ring connected by a lead to an external particle and thermal reservoir and driven out of equilibrium by a threading time-dependent magnetic field \[ \text{10, 11, 12, 13, 14} \]. In this setting there is an electronic dc current along the wire and a heat flow towards the environment.

We model the metallic wire with a system of non-interacting spin-less electrons described by a one dimensional periodic tight-binding chain with length \( L = Na \) (\( N \) is the number of sites and \( a \) the lattice spacing), hopping matrix element \( w = W/4 \) and bandwidth \( W \):

\[ H_{\text{ring}} = -w \sum_{i=1}^{N} \left( e^{-i\phi t} c_{i}^\dagger c_{i+1} + e^{i\phi t} c_{i+1}^\dagger c_{i} \right). \]  

The time-dependent phase \( \phi t \) with \( \phi \equiv \Phi / (\Phi_0 N) \) and \( \Phi_0 = h c / e \) accounts for the external magnetic flux that we choose to depend linearly in time, \( \Phi_M(t) = \Phi t \). The contact term between the lead and the ring is \( H_{\text{c}} = -w_c (c_{1}^\dagger c_{1} + c_{1}^\dagger c_{1}^\dagger) \). We model the lead and reservoir with a semi-infinite tight-binding chain with hopping amplitude \( w_c = W_c/4 \), bandwidth \( W_c \), and spectral density \( \rho_c(\omega) = 4 \sqrt{1 - \omega^2/W_c^2} / \pi W_c \). We assume that the reservoir is in equilibrium at temperature \( T \) and chemical potential \( \mu \), and that its properties are not affected by the coupling to the small ring. The Hamiltonian of the full system is then \( H = H_{\text{ring}} + H_{\text{c}} + H^R \). We use a system of units such that \( h = k_B = \Phi_0 = 1 \). All our results have been obtained using a ring with \( N = 20 \) and \( W = 1 \), and a reservoir with \( \mu = -1 \) and \( W_{\alpha} = 4 \).

The dynamics of this problem is amenable to an exact treatment within the real-time nonequilibrium formalism. The retarded and Keldysh Green functions are

\[ G_R^{ij}(t, t') = -i \theta(t - t') \langle [c_i(t), c_j^\dagger(t')] \rangle , \]  \[ G_K^{ij}(t, t') = -i \langle [c_i(t), c_j^\dagger(t')] \rangle . \]

The angular brackets indicate an average computed in the grand-canonical ensemble, \emph{i.e.} using \( H - \mu N \) in the statistical weight with \( N = \sum_{i=1}^{N} c_i^\dagger c_i \). The exact evolution equations read \[ \text{15} \]

\[ -i \frac{\partial}{\partial t} G_R^{ij}(t, t') = G_R^{ij}(t, t') w_{kj}^R(t') - \int dt_1 G_R^{ik}(t, t_1) \Sigma_R^{kj}(t_1, t') = \delta_{ij} \delta(t - t') , \]  \[ -i \frac{\partial}{\partial t} G_K^{ij}(t, t') = G_K^{ij}(t, t') w_{kj}^K(t') = \int dt_1 [G_R^{ik}(t, t_1) \Sigma_K^{kj}(t_1, t') + G_K^{ik}(t, t_1) \Sigma_R^{kj}(t_1, t')] , \]

The chemical potential \( \Phi \) is defined as the parameter replacing the environment.

We analyze the relation between the (non-stationary) real-time Keldysh and retarded Green functions and we find that, in the linear response regime with weak heat transfer to the environment, an effective temperature accounts for the modification of the equilibrium fluctuation-dissipation relation. We discuss possible extensions of this analysis.

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in equilibrium. The effect of the lead and reservoir is then captured by a correction at the contact site $i = 1$

$$\Sigma_{kj}^{R,K}(t, t') = |w_{1\alpha}|^2 g_{\alpha}^{R,K}(t - t') \delta_{k1} \delta_{j1}. \quad (8)$$

If one assumes that the magnetic field and coupling to the reservoir had been switched on in the far past in such a way that any transient behaviour dependent on the initial conditions died out at the working times $(t, t')$, Eq. (6) reduces to

$$G_{ij}^{K}(t, t') = \int dt_1 dt_2 G_{ik}^{R}(t, t_1) \Sigma_{kl}^{K}(t_1, t_2) G_{lj}^{A}(t_2, t')$$

that yields $G_{ij}^{K}$ once $G_{ij}^{R}$ has been computed from Eq. (6)

In equilibrium a model-independent fluctuation-dissipation relation (FDR) between the retarded and Keldysh Green functions holds. Calling $G_{ij}^{R,K}(t, \omega)$ the Fourier transform with respect to $\tau$ the FDR reads

$$G_{ij}^{K}(t, \omega) = \tanh(\beta \omega/2) \left( G_{ij}^{R}(t, \omega) - G_{ij}^{A}(t, \omega) \right) \quad (9)$$

where, for later convenience, we kept explicit a dependence on the observation time $t$ that does not exist in equilibrium. Out of equilibrium there is no reason why such a relation should hold, and the temperature and chemical potential of the system are not even defined. Our aim is to determine how the relation (9) should hold, and the temperature and various parameters. Upper-left panel: the $dc$ current $J_{dc}$ against the coupling to the lead $V_L$ for $\Phi = 0.6$ (open blue symbols) and $\Phi = 0.1$ (filled magenta symbols). Different symbols correspond to $T = 0.05$ (circles), $T = 1$ (squares) and $T = 10$ (diamonds). The large circle and square indicate the values of $J_{dc}$ that correspond to the data shown in Figs. 2 and 3. Upper-right panel: $J_{dc}$ against $\Phi$ for $T = 0.05$. Open symbols with dashed blue line and solid symbols with dashed black line correspond to $V_L = 0.1$ and $V_L = 1$, respectively. Lower panel: $J_{dc}(t)$ for $\Phi = 0.6$, $T = 0.05$, $V_L = 0.1$ (solid black line) and $V_L = 0.9$ (dashed blue line). The period $\tau_B \approx 10$ is indicated with an arrow.

$T$ the lower $J_{dc}$, as can be checked by comparing the curves with different symbols.

The upper-right panel in Fig. 4 shows that $J_{dc}$ is in approximate linear relation with $\Phi$ for small biases; deviations appear at a value $\Phi_m$ that increases with $V_L$, e.g. $\Phi_m \approx 0.075$ for $V_L = 0.1$ and $\Phi_m \approx 0.2$ for $V_L = 1$.

We now turn to the detailed study of the Green functions that we parametrize as $G_{ij}^{R,K}(t, \tau - \tau')$. In Fig. 2 we illustrate the non-trivial dependence on $t$ and $\tau = t - \tau'$ by tracing, with solid black lines, $\text{Re} G_{ij}^{R}(\tau)$ for three values of the total time $t$ equally spaced within $\tau_B$. The two panels show the evolution on the intervals $0 \leq \tau \leq 8$ (upper) and $30 \leq \tau \leq 38$ (lower). Interestingly enough, all curves fall on top of each other for $\tau \leq \tau^*(\Phi, \beta, V_L) \approx 1.5$, and they later deviate demonstrating the breakdown of stationarity. This trend can be understood by noting that $G_{ij}^{R}(t, \omega)$ contains a time dependent structure within the spectral range of the free ring, i.e. in the interval $|\omega + \mu| \leq W/2$, while the high frequency part is dominated by the stationary spectral features of the reservoir which set the quick response in the time domain.

Figure 2 also shows a qualitative study of the FDR. The dashed lines are the inverse Fourier transform of the
environmental temperature 'classical' regime (loosely) identified as the values of the
In Fig. 3 we show a quantitative test of the FDR of the first minimum in the upper panel, the Fourier transform of the rhs of Eq. (9) (dashed magenta lines) as a function of $\tau$ for three total times $t$ equally spaced in the interval $[0, \tau_B]$. $V_L = 1$, $\Phi = 0.1$, $T = 0.05$ and $J_{dc} \approx 0.03$ (red square in Fig. 1).

rhs of Eq. (9) for the same values of $t$. The companion curves do not match and the FDR does not hold. We note, however, that for $\tau \leq \tau^* \approx 1.5$ the behaviour is not only stationary but the FDR holds as well.

As Fig. 2 demonstrates, the total time dependence is very complicated. Instead of studying the modification of the FDR for each value of $t$ we found it natural to work with the averaged Green functions:

$$
\langle G_{ij}^{K,R}(\tau) \rangle = \frac{1}{\tau_B} \int_0^{\tau_B} dt \ G_{ij}^{K,R}(t, \tau) .
$$

(10)

In Fig. 3 we show a quantitative test of the FDR in a 'classical' regime (loosely) identified as the values of the environmental temperature $T$ such that the classical limit of the FDR, $g_{\alpha}^{K}(\tau) = -i/(2T) \partial_{\tau} g_{\alpha}^{R}(\tau)$, $\tau \geq 0$, holds for the bath. We compare the left and right hand sides of the inverse Fourier transform of Eq. (9) averaged over $\tau$ as defined in Eq. (10). In both intervals shown the accord between the two curves is rather good, proving that the FDR approximately holds when the driving force is weak and the temperature of the environment is high in such a way that $P$ is extremely low.

In Fig. 2 we test the FDR in the 'quantum' regime, $T = 0.05$. The symbols represent a fit using the functional form in the rhs of Eq. (9) with $T$ replaced by $T_{eff} = 0.143$. For short time-differences, say before the first minimum in the upper panel, the FDR holds – and the fit also falls on top of the original curves. Indeed, if $\beta_{eff} \omega \gg 1$, tanh($\beta_{eff} \omega/2 \approx 1$ and the Keldysh and retarded Green functions are in linear relation. For longer time differences the deviations are clear but the fit accounts well for the data.

Within the accuracy of our numerical solution, the effective temperature describes rather correctly the modification of the FDR whenever the dc-current is in linear relation with the emf $\Phi$ (see Fig. 1). Out of the linear response regime one cannot match the two sides of Eq. (9) by simply using a single-valued $T_{eff}$. This limitation parallels the one observed in classical strongly driven glassy systems. This is reasonable since the notion of an effective temperature was proposed to apply to a regime of small entropy production only and this, in our case, corresponds to low $P = J_{dc} \Phi$ values.

| $V_L$ | $\Phi$ | $T$ | $T_{eff}$ | $(T_{eff} - T)/T$ | $P(\times 10^{-3})$ |
|-------|--------|-----|-----------|----------------|------------------|
| 1     | 0.1    | 0.05 | 0.143     | 1.84           | 2.7              |
| 1     | 0.1    | 0.1  | 0.167     | 0.67           | 2.6              |
| 1     | 0.1    | 0.2  | 0.264     | 0.32           | 2.5              |
| 1     | 0.1    | 1    | 1.052     | 0.052          | 1.7              |
| 1     | 0.1    | 10   | 10.02     | 0.002          | 0.3              |

| $V_L$ | $\Phi$ | $T$ | $T_{eff}$ | $(T_{eff} - T)/T$ | $P(\times 10^{-3})$ |
|-------|--------|-----|-----------|----------------|------------------|
| 0.1   | 0.06   | 0.05 | 0.769     | 14.38          | 7.1              |
| 0.1   | 0.06   | 0.1  | 0.8       | 7              | 7.0              |
| 0.1   | 0.06   | 0.2  | 0.833     | 3.16           | 6.9              |
| 0.1   | 0.06   | 1    | 1.43      | 0.43           | 4.6              |

TABLE I: Estimates of the effective temperature and dissipated power within the linear response regime.

In all cases $T_{eff} > T$, as also found in glassy systems relaxing from a disordered initial configuration and in driven classical systems. An heuristic argument that explains this result is the following. The equations that govern the dynamics of the Hamiltonian $H$ are the same as those of an effective Hamiltonian where the elec-
The right-hand side of Eq. (9) (solid black line) averaged over a period $\tau_0$ against $T$. The symbols correspond to the latter functional form with a fitting parameter $T_{\text{eff}} = 0.143$. The $\Phi = 0.1$, $V_L = 1$ and $J_{dc} \approx 0.03$ (red square in Fig. 1).

We did not need to use a $\mu_{\text{eff}}$ since $\mu$ basically controls the period of the oscillations of the inverse Fourier transform of the RHS of Eq. (9) that coincides with the period of $\text{Re}(G_{11}^K) (\tau)$, see Figs. 3 and 4. In other nonequilibrium settings an upgrading of $\mu$ to $\mu_{\text{eff}}$ might be necessary.

We defined $T_{\text{eff}}$ via the Green functions that are not, however, directly accessible experimentally. A similar description should apply to the current fluctuations that can be measured in noise experiments.

Our problem bears some resemblance with a classical ratchet for which a $T_{\text{eff}}$ could be defined in the strongly driven limit for long wave-length observables. It would be interesting to check whether this holds for quantum mesoscopic devices too. Even if we have not found a site-dependent $T_{\text{eff}}$ for weak forcings we cannot exclude such a dependence in a more general setting.

Finally, it would be interesting to check the modifications of the FDR using different time-dependent magnetic fields, including disorder in the hopping rates, considering different dissipative mechanisms, etc.

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