Nonlinear Unknown Input and State Estimation Algorithm in Mobile Robots

Technical Report No. Cyber-Security-Lab-2018-001

Abstract—This technical report provides the description and the derivation of a novel nonlinear unknown input and state estimation algorithm (NUISE) for mobile robots. The algorithm is designed for real-world robots with nonlinear dynamic models and subject to stochastic noises on sensing and actuation. Leveraging sensor readings and planned control commands, the algorithm detects and quantifies anomalies on both sensors and actuators. Later, we elaborate the dynamic models of two distinctive mobile robots for the purpose of demonstrating the application of NUISE. This report serves as a supplementary document for [1].

Keywords: robotics, estimation theory, anomaly detection, dynamic model

I. NUISE ALGORITHM AND ITS DERIVATION

Minimum variance unbiased state and unknown input estimation is first introduced in [2] with indirect feedthrough[†] unknown input. The method has been extended by many research studies. A general parameterized gain matrix is derived in [3]. Estimation with direct feedthrough unknown input is proposed in [4], [5]. Young et al. [6] analyze the stability of systems with direct and indirect feedthrough unknown input. Estimators with indirect feedthrough unknown input has been applied to the fault detection in systems without noise [7] and with noises [8], [9]. An estimator with both direct and indirect feedthrough unknown input is proposed [10] for the attack detection in systems with noises, where the attack location is unknown.

One limitation of the aforementioned works is that the proposed methods are limited to handle linear systems. An estimator that can handle nonlinear systems is unexplored. In this work, we propose the nonlinear unknown input and state estimation algorithm (NUISE) as an extension of the above references for nonlinear systems. The algorithm can also be viewed as an extension of the extended Kalman filters [11] for state estimation of nonlinear systems by integrating unknown input estimation. It is the first time to study the state and unknown input estimation problem in stochastic nonlinear systems. Leveraging the reference sensor readings and planned control commands from the last iteration, NUISE estimates new robot states, corruptions in testing sensor readings, corruptions in control commands, and a likelihood for each mode.

Algorithm [1] describes the complete NUISE algorithm. We first present the definition of optimal estimates in an estimation problem. Optimal estimates contain two properties. Firstly, the estimates are unbiased, i.e., its expected value is equal to the targeted value. Secondly, the estimates have a minimum error covariance matrix, i.e., the estimation error variances are minimized with the given information.

We derive the NUISE algorithm in 4 steps: 1) actuator anomaly vector estimation, 2) state prediction, 3) state estimation, and 4) testing sensor anomaly vector estimation. In each intermediate step, the estimation errors and covariance matrices are calculated accordingly in order to find the optimal estimates.

Consider a particular mode $m$ of the dynamic model (2) in [1] with potential robot misbehaviors

\[
\begin{align*}
x_{k+1} &= f_k^m(x_k, u_k + d_k^{a,m}) + \xi_k^m \\
z_{1,k}^m &= h_{1,k}^m(x_k) + d_k^{a,m} + \xi_{1,k}^m \\
z_{2,k}^m &= h_{2,k}^m(x_k) + \xi_{2,k}^m
\end{align*}
\]

where vector $d_k^{a,m}$ and $d_k^{a,m}$ represent sensor anomaly vector and actuator anomaly vector, respectively. In mode $m$, testing sensor readings $z_{1,k}^m$ might be modified by anomaly vector $d_k^{a,m}$. Reference sensor readings $z_{1,k}^m$ are assumed to be clean. We omit mode index $m$ in the remaining part of the NUISE derivation for the ease of presentations. The dynamic system (1) can be linearized into

\[
\begin{align*}
x_{k+1} &\simeq A_k x_k + B_k u_k + G_k d_k^a + \xi_k \\
z_{1,k} &\simeq C_{1,k} x_k + d_k^a + \xi_{1,k} \\
z_{2,k} &\simeq C_{2,k} x_k + \xi_{2,k}
\end{align*}
\]

†Indirect feedthrough suggests that the input of a system indirectly influences the output through system states change. Direct feedthrough suggests that the input of a system is directly connected/fed to the output.

‡Notations † and ‡ refer pseudoinverse and pseudodeterminant, respectively. $n$ refers to the rank of $F_{k|k-1}$. 
\textbf{Algorithm 1 Nonlinear Unknown Input and State Estimation Algorithm (NUISE)}

\textbf{Input:} \( u_{k-1}, \mathbf{x}_{k-1|k-1}, \mathbf{z}_k, \mathbf{z}_{2,k} \)

\textbf{Output:} \( \hat{\mathbf{x}}_{k|k}, \hat{d}_k, \hat{d}_{k-1}, \mathcal{N}_k \)

1: Initialize;

\textbf{Actuator anomaly vector} \( \mathbf{d}_{a-1} \) estimation
2: \( \hat{P}_{k-1} \leftarrow A_k \hat{P}_{k-1} (A_k)^T + Q_{k-1} \); \( \mathbf{R}_k \leftarrow C_2_k \hat{P}_{k-1} (C_2_k)^T \) \( + R_{2,k} \);
3: \( \mathbf{M}_{2,k} \leftarrow ( (G_{k-1})^T (C_{2,k})^T (\hat{R}_{2,k})^{-1} C_{2,k} G_{k-1})^{-1} (G_{k-1})^T (C_{2,k}) \hat{R}_{2,k}^{-1} \); \( \mathbf{d}_{k-1} \leftarrow M_{2,k} (z_{2,k} - C_{2,k} (\hat{\mathbf{x}}_{k-1|k-1}, u_{k-1})) \);
4: \( \hat{P}_{k} \leftarrow M_{2,k} \hat{P}_{k-1} (M_{2,k})^T \);

\textbf{State prediction}
5: \( \mathbf{x}_{k|k-1} \leftarrow \hat{f}(\mathbf{x}_{k-1|k-1}, u_{k-1} + \mathbf{d}_{k-1}) \);
6: \( \hat{A}_{k-1} \leftarrow I - G_{k-1} M_{2,k} C_{2,k} A_{k-1} \);
7: \( \hat{Q}_{k-1} \leftarrow (I - G_{k-1} M_{2,k} C_{2,k}) \hat{Q}_{k-1} (I - G_{k-1} M_{2,k} C_{2,k})^T + G_{k-1} M_{2,k} R_{2,k} (M_{2,k})^T (G_{k-1})^{-1} \);
8: \( \hat{P}_{k} \leftarrow \hat{A}_{k-1} \hat{P}_{k-1} (\hat{A}_{k-1})^T + \hat{Q}_{k-1} \);

\textbf{State estimation}
9: \( \hat{Z}_{k} \leftarrow C_{2,k} \hat{P}_{k-1} (C_{2,k})^T + R_{2,k} C_{2,k} G_{k-1} M_{2,k} R_{2,k} + R_{2,k} (M_{2,k})^T (G_{k-1})^{-1} \);
10: \( \mathbf{u}_{k} \leftarrow (C_{2,k} \hat{P}_{k-1} (C_{2,k})^T (G_{k-1})^{-1} (\hat{R}_{2,k})^{-1} \); \( \hat{x}_k \leftarrow \hat{x}_{k|k-1} + L_{k} (\mathbf{z}_k - \hat{h}(\hat{x}_{k|k-1}) ; \)
11: \( \hat{P}_{k} \leftarrow C_{2,k} \hat{P}_{k-1} (C_{2,k})^T + R_{2,k} \);
12: \( \hat{M}_{2,k} \leftarrow (I - L_{k} C_{2,k} R_{2,k} (L_{k})^T + L_{k} R_{2,k} (L_{k})^T (I - L_{k} C_{2,k} G_{k-1} M_{2,k} R_{2,k} (L_{k})^T - L_{k} R_{2,k} (M_{2,k})^T (G_{k-1})^{-1} (I - L_{k} C_{2,k})^{-1} \); \( \hat{d}_{k-1} \leftarrow \hat{M}_{2,k} (z_{2,k} - C_{2,k} (\hat{\mathbf{x}}_{k-1|k-1}, u_{k-1})) \);
13: \( \hat{d}_{k} \leftarrow \hat{x}_{k|k-1} - \hat{h}(\hat{x}_{k|k-1}) \);
14: \( \hat{d}_{k} \leftarrow \hat{M}_{2,k} (z_{2,k} - C_{2,k} (\hat{\mathbf{x}}_{k|k-1}, u_{k-1})) \);
15: \( \hat{d}_{k} \leftarrow \hat{x}_{k|k} - \hat{h}(\hat{x}_{k|k}) \);
16: \( \hat{d}_{k} \leftarrow \hat{M}_{2,k} (z_{2,k} - C_{2,k} (\hat{\mathbf{x}}_{k|k-1}, u_{k-1})) \);
17: \( \hat{d}_{k} \leftarrow \hat{x}_{k|k-1} - \hat{h}(\hat{x}_{k|k-1}) \);
18: \( \hat{d}_{k} \leftarrow \hat{M}_{2,k} (z_{2,k} - C_{2,k} (\hat{\mathbf{x}}_{k|k-1}, u_{k-1})) \);
19: \( \hat{n} \leftarrow \text{rank} (\hat{P}_{k|k-1}) \);
20: \( \mathcal{N}_k \leftarrow \frac{1}{(2\pi)^{n/2} |\hat{P}_{k|k-1}|^{n/2}} \exp \left(-\frac{1}{2} \mathbf{d}^T_{k-1} \hat{P}_{k|k-1}^{-1} \mathbf{d}_{k-1} \right) \)

\textbf{Actuator anomaly vector} \( \mathbf{d}_{a-1} \) estimation:

\( A_k \triangleq \frac{\partial f_k}{\partial x} |_{x_{k-1|k-1}, u_{k-1} + \mathbf{d}_{a-1}} \), \( B_k \triangleq \frac{\partial f_k}{\partial u} |_{x_{k-1|k-1}, u_{k-1} + \mathbf{d}_{a-1}} \), \( C_{1,k} \triangleq \frac{\partial h_{1,k}}{\partial x} |_{x_{k-1|k-1}} \), \( C_{2,k} \triangleq \frac{\partial h_{2,k}}{\partial x} |_{x_{k-1|k-1}} \), \( G_k \triangleq \frac{\partial f_k}{\partial \mathbf{d}_{a}} |_{x_{k-1|k-1}, u_{k-1} + \mathbf{d}_{a-1}} \).

\textbf{State prediction}:

\[ \hat{x}_{k|k-1} = f_{k-1}(\hat{x}_{k-1|k-1}, u_{k-1} + \hat{d}_{a-1}) \]

The state estimates are now unbiased, i.e., \( \mathbb{E} [\hat{x}_{k|k-1}] = x_{k|k-1} \), since \( \mathbb{E} [\hat{\mathbf{d}}_{a}] = \mathbf{d}_{a} \). Now we find the state prediction error covariance matrix:

\[ P_{k} = \hat{A}_{k-1} \hat{P}_{k-1} \hat{A}_{k-1}^T + \hat{Q}_{k-1} \]

where \( \hat{A}_{k-1} = (I - G_{k-1} M_{2,k} C_{2,k}) A_{k-1} \) and \( \hat{Q}_{k-1} = (I - G_{k-1} M_{2,k} C_{2,k}) Q_{k-1} (I - G_{k-1} M_{2,k} C_{2,k})^T + G_{k-1} M_{2,k} R_{2,k} M_{2,k}^T G_{k-1} \).

\textbf{State prediction}:

\[ \hat{x}_{k|k-1} = \hat{x}_{k|k-1} + \hat{d}_{a-1} \]

The state estimates are not perfect because of process and measurement noises. In order to obtain the estimates accurately considering noises, we do corrections on the state estimates using sensor readings. We utilize the discrepancy between the newly predicted outputs \( C_{2,k} \hat{x}_{k|k-1} \) and the reference sensor outputs \( \mathbf{z}_{2,k} \) as an indication of the impact of unknown noises:

\[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k (\mathbf{z}_{2,k} - \hat{h}(\hat{x}_{k|k-1})) \]
We approximate the output error
\[
\tilde{x}_{k|k} = x_k - \hat{x}_{k|k} = (I - L_k C_{2,k}) \tilde{x}_{k|k-1} - L_k \xi_{2,k}
\]
and
\[
P_k^x = (I - L_k C_{2,k})P_{k|k-1}^x (I - L_k C_{2,k})^T + L_k R_{2,k} L_k^T
\]
\[- (I - L_k C_{2,k})G_{k-1} M_{2,k} R_{2,k} L_k^T
\]- \(L_k R_{2,k} M_{2,k} G_{k-1}^T (I - L_k C_{2,k})^T\).

To achieve optimal estimation, we solve the variance mini-
mization problem: \(\min_{L_k} \text{tr}(P_k^x)\). We take the derivative of
the objective function with respect to the decision variable \(L_k\)
and set it as zero
\[
L_k = (C_{2,k} P_{k|k-1} + R_{2,k} M_{2,k} G_{k-1}^T + L_k R_{2,k} L_k^T (I - L_k C_{2,k})^T) R_{2,k}^T
\]
where \(\hat{R}_{2,k} = C_{2,k} P_{k|k-1}^x C_{2,k}^T + R_{2,k} + C_{2,k} G_{k-1} M_{2,k} R_{2,k} + R_{2,k} M_{2,k} G_{k-1}^T C_{2,k}^T\).

**Testing sensor anomaly vector \(d^*\) estimation:** Given \(x_{k|k}\), the linear estimation for unknown sensor anomaly vector \(d_k\) can be
\[
\hat{d}_k = M_{1,k}(z_{1,k} - h_{1,k}(\hat{x}_{k|k})) = M_{1,k}(C_{1,k} \hat{x}_{k|k} + d_k^* + \xi_{1,k})
\]
where the estimates are unbiased, i.e., \(E[d_k^*] = d_k^*\), providing that \(M_{1,k} = I\). This also can be found by Gauss Markov
theorem. By the theorem, the optimal estimates are
\[
M_{1,k} \triangleq (\hat{R}_{1,k}^{-1})^{-1} \hat{R}_{1,k} = I
\]
where \(\hat{R}_{1,k} = C_{1,k} P_{k|k}^x C_{1,k}^T + R_{1,k}^T\). The covariance matrices can be obtained by
\[
P_k^x = \hat{R}_{1,k}
\]

**Likelihood of a mode:** In order to determine the ground
truth condition of a robot, i.e., mode, we calculate a likelihood
that reflects the discrepancy between the predicted output and
the measured output of a mode. For \(\forall m\), we quantify the discrepancy between the predicted output and the measured output as follows
\[
\nu_k^m = z_{2,k} - h_{2,k}(\hat{x}_{k|k})
\]
where we approximate the output error \(\nu_k^m\) as a multivariate Gaussian random variable. Then, the likelihood function is given by
\[
N_k^m \triangleq \mathcal{P}(y_k|m = \text{true}) = N(v_k^m; 0, \bar{P}_k^m)
\]
\[
= \exp(-\nu_k^m)^T \bar{P}_k^m \nu_k^m / 2 / (2\pi)^{n_m/2} |\bar{P}_k^m|^{1/2}
\]
where \(\bar{P}_k^m = C_{2,k} P_{k|k-1}^x C_{2,k}^T + R_{2,k} - C_{2,k} G_{k-1} M_{2,k} R_{2,k} C_{2,k}^T\) is the error covariance matrix of \(\nu_k^m\) and \(n_m = \text{Rank}(P_{k|k-1})\).

Notations \(\dagger\) and \(| \cdot |_+\) refer to pseudoinverse and
pseudodeterminant, respectively. By the Bayes’ theorem, the
a posteriori probability is
\[
\mu_k^m \triangleq \mathcal{P}(m = \text{true}| y_k, \cdots, y_0) = \frac{\sum_{i=1}^M N_{k,i} \mu_{k-1,i}}{\sum_{i=1}^M N_{k,i}}
\]
However, such updates might cause the \(\mu_k^m\) of certain modes to converge to zero. To prevent this, we modify the posterior
probability update to the following
\[
\mu_k^m = \frac{\mu_k^m}{\sum_{i=1}^M \mu_k^i}
\]
where \(\mu_k^m = \max\{N_{k,m}^m, \mu_{k-1,m}, \epsilon\}\), and \(\epsilon > 0\) is a pre-
selected small constant preventing the vanishment of the mode
probability. The last step is to generate estimates of states and
anomaly vector estimates of the maximum a posteriori mode.

**II. KHEPERA DYNAMIC MODEL**

**Kinematic model** The kinematic model of Khepera includes
three states: \((x, y, \theta)\) is the robot location at a 2-D plane, and \(\theta\) is its heading. The control commands are specified by
two variables: \(v_L\) and \(v_R\), which are the speeds of the left and
right wheels, respectively. Considering actuator misbehaviors
with anomaly vector \(\theta_{k-1} = [a_{k-1}^L, a_{k-1}^R]^T\) on the left and
right wheel, the kinematic model can be presented as
\[
x_k = x_{k-1} + T \cos \theta_{k-1}(v_L + a_{k-1}^L + v_R + a_{k-1}^R)/2 + \xi_k^x
\]
\[y_k = y_{k-1} + T \sin \theta_{k-1}(v_L + a_{k-1}^L + v_R + a_{k-1}^R)/2 + \xi_k^y
\]
\[\theta_k = \theta_{k-1} + T(v_R + a_{k-1}^R - v_L - a_{k-1}^L)/D + \xi_k^\theta
\]
where \(\xi_k^x = [\xi_k^x, \xi_k^y, \xi_k^\theta]^T\) is assumed to be zero mean
gaussian process noises, and \(D\) is the distance between the
left and right wheel on the chassis of Khepera.

**Measurement model** The sensor readings include sensing
data from three sensors: \(z_k = [z_{k,I}, z_{k,W}, z_{k,L}]^T\) where \(z_{k,I}\)
is from the IPS, \(z_{k,W}\) is from the wheel encoder, and \(z_{k,L}\) is
from the LiDAR.

IPS sensor directly measures the states of Khepera, hence, the
measurement model can be directly specified by
\[
z_{k,I} = x_k + d_{k,I}^* + \xi_{k,I}
\]
where \(\xi_{k,I} = [\xi_{k,I}^x, \xi_{k,I}^y, \xi_{k,I}^\theta]^T\) refers to measurement noises
from the IPS sensor, and \(d_{k,I}^* = [d_{k,I}^x, d_{k,I}^y, d_{k,I}^\theta]^T\) refers to
the sensor anomaly vector on IPS.

The raw data measured by the wheel encoder are the
distances traveled by each wheel \((l_L, l_R)\) in a control iteration.
the map information. Using noises from LiDAR. The distance where the car is presented in Figure 2. The states of the vehicle also include the location and the orientation \((x, y, \theta)\) in a 2D plane. The control includes the longitudinal velocity and the steering \((v, \phi)\). The kinematic model of the vehicle can be described as

\[
\begin{align*}
x_k &= x_{k-1} + T(v_k - 1 + d_k^{v}) \cos \theta_{k-1} + \xi_k^{x} \\
y_k &= y_{k-1} + T(v_k - 1 + d_k^{v}) \sin \theta_{k-1} + \xi_k^{y} \\
\theta_k &= \theta_{k-1} + T(v_k - 1 + d_k^{v}) \tan(\theta_{k-1} + d_k^{v}) + \xi_k^{\theta}
\end{align*}
\]

where \(\xi_{k-1} = [\xi_k^{x}, \xi_k^{y}, \xi_k^{\theta}]^T\) is assumed to be a zero mean Gaussian process noise vector, \(d_k^{v} = [d_k^{v}, d_k^{v}, d_k^{v}]^T\) is the actuator anomaly vector, \(L\) is the wheelbase, and \(T\) is the control iteration interval.

**Measurement model** At each instant of time, sensor readings include data from three sensors: \(z_k = [z_{k,x}, z_{k,y}, z_{k,M}]^T\), where each vector refers to the sensor readings from IPS, LiDAR, and IMU, respectively. The measurement models for IPS and LiDAR are similar to those in Khepera (see Section II). The IMU sensor generates a quaternion \([q_0, q_1, q_2, q_3]^T\), a 3-D acceleration \(a_{k,M}^{local}\), and a 3-D rotational speed \(\omega_{k,M}^{local}\) on a body-fixed coordinate. We first obtain the coordinate transformation matrix \(C(q)\) from the body-fixed coordinate to the global coordinate \([15]\).

\[
C(q) = \begin{bmatrix}
q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_3q_0) & 2(q_1q_3 + q_2q_0) \\
2(q_1q_2 + q_3q_0) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_1q_0) \\
2(q_1q_3 - q_2q_0) & 2(q_2q_3 + q_1q_0) & q_0^2 - q_1^2 - q_2^2 + q_3^2
\end{bmatrix}
\]

The acceleration vector and the rotation speed on the global coordinate system can be obtained as \(C(q)a_{k,M}^{local}\) and \(C(q)\omega_{k,M}^{local}\), respectively. The vehicle velocity vector can be updated by:

\[
v_k = [v_{k,x}^{x}, v_{k,y}^{y}, v_{k,M}^{z}]^T = v_{k-1} + a_{k,M}^{global} T.
\]

Then the state vector can be calculated by integration as follows

\[
\begin{align*}
x_k &= x_{k-1} + v_{k,M}^{x} T + \frac{1}{2} a_{k,M}^{x} T^2 \\
y_k &= y_{k-1} + v_{k,M}^{y} T + \frac{1}{2} a_{k,M}^{y} T^2 \\
\theta_k &= \theta_{k-1} + w_{k,M}^{z} T
\end{align*}
\]

**IV. SEPARATING ACTUATOR ANOMALY VECTOR**

In Section IV.D. of [1], we mention that RoboADS only checks the aggregate test statistics instead of each individual actuator. This section explains the reason in detail.

At a high level, the actuator anomaly vectors are statistically correlated. Without loss of generosity, we consider a robot with two actuators such as Khepera. During actuator anomaly vector estimation, we obtain \(d_k = [d_k^{a}, d_k^{b}]^T\), with error covariances \(P_k^{a}\). In Algorithm 1 line 20, we test

\[
(d_k^{a})^T (P_k^{a})^{-1} d_k^{a} \geq \chi_{p=2}(\alpha)
\]

to determine the existence of actuator misbehaviors. The threshold \(\chi_{p=2}(\alpha)\) is a Chi-square test value with the degree of freedom \(p = 2\) and the confidence level \(\alpha\).

**III. TAMIA RC CAR DYNAMIC MODEL**

**Kinematic model** The kinematic model of a Tamiya RC car is presented in Figure 2. The states of the vehicle also

For convenience reasons, we convert them into robot states using previous states \(x_{k-1}\) before we feed the data to the planner

\[
\begin{align*}
x_k &= x_{k-1} + (l_L + l_R) \cos \theta_k / 2 \\
y_k &= y_{k-1} + (l_L + l_R) \sin \theta_k / 2 \\
\theta_k &= \theta_{k-1} + (l_R - l_L) / r
\end{align*}
\]

Analogously with IPS, the measurement model for the wheel encoder can be specified as

\[
z_{k,W} = x_k + d_{k,W} + \xi_{k,W}
\]

after the conversion, where \(\xi_{k,W} = [\xi_{k,x}, \xi_{k,y}, \xi_{k,M}]^T\) refers to measurement noises from the wheel encoder and \(d_{k,W} = [d_{k,x}, d_{k,y}, d_{k,M}]^T\) refers to the sensor anomaly vector on the wheel encoder.

The LiDAR sensor is placed on top of the robot with a shift of \((r, \phi)\) from the origin \(O'\) as shown in the left plot of Figure 1. Raw sensor readings returned from LiDAR are the distances between LiDAR and the surrounding walls (see the right plot of Figure 1). Given the LiDAR readings, we process the raw data into the perpendicular distance \(d_{k,j}\) from each boundary wall \(j \in \{1, 2, 3, 4\}\) and the orientation \(\theta_{k}\) of Khepera. Specifically, we recognize the straight line segments using raw distances from all direction, and calculate the distances to each wall as follows

\[
l_k = r - (x_k + x' \sin \theta_k + y' \cos \theta_k) \cos \phi_k - (y_k - x' \cos \theta_k + y' \sin \theta_k) \sin \phi_k + d_{k,L}^x + \xi_{k,L}^x
\]

where \(\xi_{k,L} = [\xi_{k,L}]^T, j \in \{1, 2, 3, 4\}\) refers to measurement noises from LiDAR. The distance \(r\) and the angle \(\phi\) of each wall in the global coordinate is known in advance as the map information. Using \(\phi\) of each wall and the 240 degrees of range, we can also infer the angle of the robot. We use the distance and the angle to each wall as the sensor readings from LiDAR: \(z_{k,L} = [l_k, \theta_k]^T, j \in \{1, 2, 3, 4\}\). In outdoor environments, LiDAR measurement model can be obtained using more complicated simultaneous localization and mapping (SLAM) algorithms [13]. For demonstration purposes, we apply a simple transformation in the indoor environment [14].
In order to confirm actuator misbehaviors on each actuator, we need to separately conduct Chi-square test \( \hat{d}_k^a \) and \( \hat{d}_k^R \) with corresponding marginal variances \( P_k^a(1, 1) \) and \( P_k^a(2, 2) \):

\[
(\hat{d}_k^a)^T (P_k^a(1, 1))^{-1} \hat{d}_k^a \geq \chi^2_{p=1}(\alpha) \\
(\hat{d}_k^R)^T (P_k^a(2, 2))^{-1} \hat{d}_k^R \geq \chi^2_{p=1}(\alpha). \tag{10}
\]

However, a positive testing result in (9) does not guarantee a positive testing result in (10) because the off-diagonal terms of matrix \( P_k^a \) are neglected in (10). The explanation is shown as follows:

\[
(\hat{d}_k^a)^T (P_k^a(1, 1))^{-1} \hat{d}_k^a = (\hat{d}_k^a)^T (P_k^a(1, 1))^{-1} (1, 1) \hat{d}_k^a + (\hat{d}_k^a)^T (P_k^a(1, 1))^{-1} (1, 2) \hat{d}_k^R \\
+ (\hat{d}_k^R)^T (P_k^a(2, 2))^{-1} (2, 1) \hat{d}_k^a + (\hat{d}_k^R)^T (P_k^a(2, 2))^{-1} (2, 2) \hat{d}_k^R
\]

\[
(\hat{d}_k^R)^T (P_k^a(2, 2))^{-1} \hat{d}_k^R = \hat{d}_k^R
\]

Note that \( (\hat{d}_k^a)^T (P_k^a(1, 1))^{-1} \hat{d}_k^a + (\hat{d}_k^R)^T (P_k^a(2, 2))^{-1} \hat{d}_k^R \) if \( P_k^a \) is a diagonal matrix.

Another problem for the separation is that the Chi-square test threshold is nonlinear. For instance, \( \chi^2_{p=1}(0.01) = 6.635 \) and \( \chi^2_{p=2}(0.01) = 9.210 \). Suppose \( P_k^a \) is a diagonal matrix and the test scores after separation are \( (\hat{d}_k^a)^T (P_k^a(1, 1))^{-1} \hat{d}_k^a = 5 \) and \( (\hat{d}_k^R)^T (P_k^a(2, 2))^{-1} \hat{d}_k^R = 5 \). The actuator misbehaviors would be detected by (9) but not by (10).

Therefore, we conduct the Chi-square test on the aggregate actuator anomaly vector instead of the separated vector components. The decision results from the hypothesis tests indicate whether the robot has actuator misbehaviors with a certain level of confidence, yet no decision is made on whether a particular actuator is misbehaving.

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