On non-conformal limit of the AGT relations

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The Seiberg-Witten prepotentials for $\mathcal{N} = 2$ SUSY gauge theories with $N_f < 2N_c$ fundamental multiplets are obtained from conformal $N_f = 2N_c$ theory by decoupling $2N_c - N_f$ multiplets of heavy matter. This procedure can be lifted to the level of Nekrasov functions with arbitrary background parameters $\epsilon_1$ and $\epsilon_2$. The AGT relations imply that similar limit exists for conformal blocks (or, for generic $N_c > 2$, for the blocks in conformal theories with $W_{N_c}$ chiral algebra). We consider the limit of the four-point function explicitly in the Virasoro case of $N_c = 2$, by bringing the dimensions of external states to infinity. The calculation is performed entirely in terms of representation theory for the Virasoro algebra and reproduces the answers conjectured in arXiv:0908.0307 with the help of the brane-compactification analysis and computer simulations. In this limit, the conformal block involving four external primaries, corresponding to the theory with vanishing beta-function, turns either into a 2-point or 3-point function, with certain coherent rather than primary external states.

1. The AGT relations [1]-[11] express generic 2d conformal blocks through the Nekrasov functions [12]-[20] $Z(Y)$, associated with $\mathcal{N} = 2$ SUSY quiver 4d gauge theories with extra fundamental multiplets, generalizing the earlier predictions of [14, 15]. Most commonly these theories have vanishing beta-functions and possess conformal invariance in four dimensions. In the simplest case of the 4-point Virasoro conformal block, this is the conformal $SU(2)$ model with $N_f = 2N_c = 4$ flavors. The masses $\mu_1, \ldots, \mu_{N_f}$ of the four fundamentals are related to the dimensions of four external states operators:

$$
\begin{align*}
\mu_1 &= \alpha_1 - \alpha_2 + \frac{\epsilon}{2}, \\
\mu_2 &= \alpha_1 + \alpha_2 - \frac{\epsilon}{2}, \\
\mu_3 &= \alpha_3 - \alpha_4 + \frac{\epsilon}{2}, \\
\mu_4 &= \alpha_3 + \alpha_4 - \frac{\epsilon}{2},
\end{align*}
$$

and the gauge theory condensate (modulus) $a = a_1 = -a_2$ is related to that of the intermediate state:

$$
a = \alpha - \frac{\epsilon}{2}
$$

For large masses $\mu_k \to \infty$ the fundamental fields in 4d theory decouple, and one gets an asymptotically free pure gauge $\mathcal{N} = 2$ SUSY theory, with prepotential expressed through (the $\epsilon_1 = -\epsilon_2 \to 0$ limit of) the pure gauge Nekrasov functions $Z(Y)$:

$$
Z(Y) \sim \lim_{\mu_k \to \infty} Z(Y)
$$

The AGT relation implies that the associated limit of conformal block corresponds to this $Z(Y)$. A natural question is how does this limit look like from the point of view of 2d conformal theory itself.

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This question was addressed in [3] and an elegant answer has been proposed: the relevant conformal blocks are matrix elements for certain “coherent” states in Verma module of Virasoro algebra. However, in [4] the answer was not derived in a direct way, by taking a particular limit of the 4-point conformal block with generic $\mu$’s. Instead, the conclusion was based on analysis of the underlying 5-brane configurations [21], which was also the original source of the AGT relations in [1]. In this letter, we fill the gap and derive the result of [4] straightforwardly, making use of explicit knowledge of the Virasoro conformal blocks from [3]. A similar analysis is possible for conformal blocks with more external states and for some $W$-algebra blocks $N_c > 2$, in the last case the results of [2] [5] [7] should be used. These generalizations are, however, beyond the scope of this paper.

2. We use notations from [3] and refer for details and explanations to that paper. The 4-point conformal block is given by the sum over Young diagrams

$$B_{\Delta_1 \Delta_2; \Delta_3 \Delta_4; \Delta}(x) = \sum_{|Y| = |Y'|} x^{|Y|} \gamma_{\Delta_1 \Delta_2}(Y) Q_\Delta^{-1}(Y, Y') \gamma_{\Delta_3 \Delta_4}(Y')$$

(4)

with the inverse Shapovalov form $Q_\Delta(Y, Y') = \langle \Delta | L_Y L_{-Y} | \Delta \rangle$, where $L_{-Y} = L_{-k_1} \ldots L_{-k_L} L_{-k_1}$ for the Young diagram $Y = \{k_1 \geq k_2 \geq \ldots \geq k_L > 0\}$ are made from the Virasoro operators $L_k$, $k \in \mathbb{Z}$, satisfying

$$[L_m, L_n] = \frac{c}{12} n(n^2 - 1) \delta_{m+n,0} + (m - n)L_{m+n}$$

(5)

and the three-point functions [22, 3] are

$$\gamma_{\Delta_1 \Delta_2}(Y) = \prod_{i=1}^{\ell(Y)} \left( \Delta + k_i \Delta_1 - \Delta_2 + \sum_{j<i} k_j \right)$$

(6)

The Shapovalov matrix $Q_\Delta(Y, Y') = Q_\Delta([k_1 k_2 \ldots], [k'_1 k'_2 \ldots])$ is infinite-dimensional, but has an obvious block form, since the matrix elements are non-vanishing only when $|Y| = |Y'|$. Therefore, for generic $\Delta$ and $c$, it is straightforwardly invertible. The AGT relations [1, 3] state that, under the identification (1) and (2),

$$B_{\Delta_1 \Delta_2; \Delta_3 \Delta_4; \Delta}(x) = \mathcal{Z}(x) = \sum_{Y} x^{|Y|} \mathcal{Z}(Y)$$

(7)

and we are going to turn now to the asymptotically free limit of this relation.

3. We would like to consider first the limit of conformal block [4], when all $\mu_1, \ldots, \mu_4 \to \infty$ independently, and, at the same time, $x \to 0$ so that

$$x \prod_{i=1}^{4} \mu_i = \Lambda^4$$

(8)

which is a scale (like $\Lambda_{QCD}$) parameter, in the pure $\mathcal{N} = 2$ SUSY gauge theory with $N_f = 0$. For this we do not even need an explicit form of the Shapovalov matrix, since it does not depend on external dimensions $\Delta_{1,2,3,4}$.

However, explicit formula (6) is crucially important. The number of factors in the r.h.s. of (6) is equal to the number of rows $\ell(Y)$ (the number of non-vanishing $k$’s) in the Young diagram $Y$, and it is maximal for fixed $|Y|$ when the diagram consists of a single column, i.e. when all $k_i = 1$, $1 \leq i \leq \ell(Y)$ or $\ell(Y) = |Y|$.

Since in our limit $\Delta_i \gg \Delta, 1$, the $\gamma$-factor reduces to

$$\gamma(Y) \sim \prod_{i=1}^{\ell(Y)} (k_i \Delta_1 - \Delta_2)$$

(9)
and of all diagrams of a given size \(|Y|\), the sum in (11) is saturated by the terms, where \(\gamma(Y)\)'s (9) contain maximal possible number of factors, i.e. when \(\ell(Y) = |Y|\), or \(Y\) is a single-column diagram \([1^{[Y]}] = [1, \ldots, 1]^{[Y]}\) times

\[
x^{[Y]/2}\gamma_{\Delta_1, \Delta_2}(Y) \to \left(\sqrt{x} (\Delta_1 - \Delta_2)\right)^{[Y]} \delta(Y, [1^{[Y]}]) = \left(\frac{\Lambda^2}{-\epsilon_1 \epsilon_2}\right)^{[Y]} \delta(Y, [1^{[Y]}])
\]

(10)

In what follows we often omit the powers of \(-\epsilon_1 \epsilon_2\), which can be easily restored from dimensional consideration. Since the Shapovalov form does not depend on \(\Delta\) totally by the scalar products \(C\) and, in order to reproduce the r.h.s. of (11), one should take

\[
B_\Delta(\Lambda) = \lim_{\Delta_i \to \infty} B_{\Delta_1; \Delta_2; \Delta_3; \Delta_4; \Delta}(x) = \sum_{|Y| = |Y'|} \Lambda^2 |Y| Q_\Delta^{-1}(Y, Y') \delta(Y, [1^{[Y]}]) \delta(Y', [1^{[Y']}]) = \sum_n \Lambda^n Q_\Lambda^{-1}([1^n], [1^n])
\]

(11)

This vector can be characterized as being orthogonal to all non single-column states \(|\Delta, Y\rangle = L_{-Y} |\Delta\rangle \in \mathcal{H}_\Delta\) with \(Y \neq [1^{[Y]}]\), since

\[
\langle \Delta | L_Y | \Delta, \Lambda^2 \rangle = \sum_{|Y'|} \Lambda^2 |Y'| Q_\Delta^{-1}([1^{[Y']}], Y') \langle \Delta | L_Y L_{-Y'} | \Delta \rangle = \sum_{|Y'|} \Lambda^2 |Y'| Q_\Delta^{-1}([1^{[Y']}], Y') Q_\Delta(Y', Y) = \Lambda^2 |Y| \delta(Y, [1^{[Y]}])
\]

(14)

This means, in particular, that it is a kind of a “coherent” state, satisfying

\[
L_1 |\Delta, \Lambda^2\rangle = \Lambda^2 |\Delta, \Lambda^2\rangle,
L_k |\Delta, \Lambda^2\rangle = 0, \quad k \geq 2
\]

(15)

The implication (14) \(\Rightarrow\) (15) deserves more detailed explanation. Consider the vector \(L_k |\Delta, \Lambda^2\rangle \in \mathcal{H}_\Delta\) for \(k > 0\). The coefficients of its expansion over the basis \(|\Delta, Y\rangle = L_{-Y} |\Delta\rangle \in \mathcal{H}_\Delta\) are characterized totally by the scalar products

\[
\langle \Delta, Y | L_k | \Delta, \Lambda^2 \rangle = \langle \Delta | L_Y L_k | \Delta, \Lambda^2 \rangle = \sum_{|Y'|} b^{(k)}_{Y'} \langle \Delta | L_Y L_{-Y'} | \Delta, \Lambda^2 \rangle = \sum_{|Y'|} b_{Y'}^{(k)} \Lambda^2 |Y'| \delta(Y, [1^{[Y']}]) = \sum_{\ell'} b_{Y'}^{(k)} \Lambda^2 \delta(Y', [1^{[Y']}]) = \sum_{\ell} b_{Y'}^{(k)} \Lambda^2 \delta(Y', [1^{[Y']}])
\]

(16)

where \(\ell' = \ell(Y') = |Y'|\), i.e. only the Young diagrams \(Y' = [1^{[Y']}], [1^{\ell(Y')}\rangle\) can contribute. It is important, however, that due to the Virasoro commutation relations, the sum in (16) is restricted by \(|Y'| \leq |Y| + k\) and \(\ell(Y') \leq \ell(Y) + 1\), meaning that both the number of boxes in \(Y'\) and the number
of elementary Virasoro generators in $L_{Y'}$ is less or equal to those in $L_Y L_k$; moreover, the structure of Virasoro algebra \(^5\) requires necessarily $k_i(Y') \geq k_j(Y)$, $i, j = 1, \ldots, \ell(Y'), \ell(Y)$. Hence, one gets for \(^6\)

$$
\langle \Delta, Y | L_k | \Delta, \Lambda^2 \rangle = \sum_{\ell'} b^{(k)}_{Y[1^{\ell'}]} \Lambda^{2\ell'} = \delta(Y, [1^{\ell}]) \sum_{\ell' \leq \ell+1} b^{(k)}_{[1^{\ell}][1^{\ell'}]} \Lambda^{2\ell'} = \delta(Y, [1^{\ell}]) \delta_{k,1} \Lambda^{2\ell+2}
$$

and this immediately leads to \(^7\), since for $k > 1$ the vector $L_k | \Delta, \Lambda^2 \rangle$ is orthogonal to all vectors in $\mathcal{H}_\Delta$, while for $k = 1$ it coincides with the vector $| \Delta, \Lambda^2 \rangle$ up to a numerical factor $\Lambda^2$.

Differently, expanding $| \Delta, \Lambda^2 \rangle = \sum_{n \geq 0} \Lambda^{2n} | \Delta, n \rangle$, one gets for

$$
| \Delta, n \rangle = \sum_{|Y| = n} Q^{-1}_\Delta \left( [1^n], Y \right) L_{-Y} | \Delta \rangle
$$

that

$$
L_1 | \Delta, n \rangle = | \Delta, n - 1 \rangle, \quad n \geq 0
$$

$$
L_k | \Delta, n \rangle = 0, \quad \forall \quad k \geq 2, \quad n \geq 0
$$

which is exactly the claim of \(^8\). Here we have derived and proved it directly, taking the limit of the 4-point conformal block with arbitrary dimensions. Note that the whole reasoning is valid for any $\epsilon_1, \epsilon_2$ and $\epsilon$, i.e. conformal theory with arbitrary central charge $c$ (the central charge dependence arises in $| \Delta, \Lambda^2 \rangle$ through the inverse matrix of the Shapovalov form).

5. In a similar way, one can consider partial decoupling of the fundamental matter, corresponding to the models with $N_f = 1, 2, 3$, when remaining masses (and related combinations of conformal dimensions) are preserved as free parameters. Let us start with the case of $N_f = 1$. In such a limit, $\mu_{2,3,4} \to \infty$ with finite $x \prod_{I=2,3,4} \mu_I = \Lambda_1^4$, but $\mu_1$ remains finite itself. According to \(^9\), this means that $\alpha_1$ and $\alpha_2$ go to infinity, but not independently: their difference remains finite. In terms of conformal dimensions it means that

$$
\Delta_1 - \Delta_2 = \frac{(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2 - \epsilon)}{-\epsilon_1 \epsilon_2} \sim \frac{(2\mu_1 - \epsilon)\sqrt{\Delta_1}}{\sqrt{-\epsilon_1 \epsilon_2}}
$$

i.e. all dimensions are infinite, but $\frac{\Delta_1 - \Delta_2}{\sqrt{\Delta_1}}$ remains finite. Hence, in this limit the single-column diagrams still dominate to contribute into $\gamma_{\Delta_1,\Delta_2}(Y)$, like it has been considered above in the case of flow into pure gauge theory, but the factor $\gamma_{\Delta_1,\Delta_2}(Y)$ is now dominated by a different sort of Young diagrams. The reason is that for $k_i = 1$ the factor $k_i \Delta_1 - \Delta_2$ turns into $\Delta_1 - \Delta_2$ and grows not as fast as $\Delta_1$ and $\Delta_2$ themselves. Instead, the dominant contribution comes now from the Young diagrams of the form $Y = [2^p, 1^q]$ with $|Y| = 2p + q$, $\ell(Y) = p + q$, since for all of them

$$
\gamma_{\Delta_1,\Delta_2}(Y) \sim (2\Delta_1 - \Delta_2)^p (\Delta_1 - \Delta_2)^q \sim \left( \frac{2\mu_1 - \epsilon}{\sqrt{-\epsilon_1 \epsilon_2}} \right)^q \Delta_1^{p+q/2} \sim \frac{(2\mu_1 - \epsilon) (\mu_2/2)^{|Y|}}{(-\epsilon_1 \epsilon_2)^{p+q}}
$$

Instead of \(^10\), the limit of conformal block is now given by (we again omit the powers of $-\epsilon_1 \epsilon_2$)

$$
B_{\Delta}^N_{\ell_1, \ell_2}(\Lambda_1, m) = \lim_{\Delta \to \infty} \lim_{\Delta_1,\Delta_2;\Delta_3;\Delta_4;\ell(x) = \Delta - \Delta_3 \to 2m \sqrt{\Delta_1}} B_{\Delta_1,\Delta_2;\Delta_3;\Delta_4;\ell(x)} = \sum_{|Y| = |Y'|} \sum_p (2m)^{|Y| - 2p} \left( \frac{x\mu_2 \mu_3 \mu_4}{2} \right)^{|Y|} Q^{-1}_\Delta(Y, Y') \delta(Y; [2^p, 1^{1^n}]) \delta(Y', [1^{1^n}]) = \langle \Delta, \Lambda_1/2, 2m | \Delta, \Lambda^2_1 \rangle
$$
where \( m = \mu_1 - \frac{x}{2} \), \( \Lambda_1^2 = x \mu_2 \mu_3 \mu_4 \) to be fixed when taking the limit of \( x \to 0 \) and \( \mu_I \to \infty \), \( I = 2, 3, 4, \) and

\[
|\Delta, \Lambda, m\rangle = \sum_Y \sum_p m^{[Y]-2p} A^{[Y]} Q_\Delta^{-1} \left( [2p, 1^{[Y]-2p}], Y \right) L_{-Y}|\Delta\rangle
\]

(23)

while the vector \(|\Delta, \Lambda^2\rangle\) has been already defined in (13). Considering the matrix elements

\[
\langle \Delta|L_Y|\Delta, \Lambda, m\rangle = \sum_p m^{[Y]-2p} A^{[Y]} \delta \left( Y, [2p, 1^{[Y]-2p}] \right)
\]

(24)

and

\[
\langle \Delta|L_Y L_k|\Delta, \Lambda, m\rangle = \sum_{Y'} b_{YY'}^{(k)} \langle \Delta|L_{Y'}|\Delta, \Lambda, m\rangle = \sum_{Y'} b_{YY'}^{(k)} \sum_p m^{[Y']-2p} A^{[Y']} \delta \left( Y', [2p, 1^{[Y']-2p}] \right) =
\]

\[
= \delta_{k,1} b_{[2p,1^{[Y]-2p}][2p,1^{[Y]+1-2p}]}^{(1)} m^{[Y]+1-2p} A^{[Y]+1} + \delta_{k,2} b_{[2p,1^{[Y]-2p}][2p+1,1^{[Y]+2-2(p+1)}]}^{(2)} m^{[Y]+2-2(p+1)} A^{[Y]+2}
\]

(25)

one proves exactly in the same way as before that

\[
L_1|\Delta, \Lambda, m\rangle = m\Lambda|\Delta, \Lambda, m\rangle,
\]

\[
L_2|\Delta, \Lambda, m\rangle = \Lambda^2|\Delta, \Lambda, m\rangle,
\]

\[
L_k|\Delta, \Lambda, m\rangle = 0 \quad \text{for} \quad k \geq 3
\]

(26)

again in agreement with the claim of [4].

Note also that, in the limit when \( m \to \infty \) together with \( \Lambda \to 0 \) so that \( m\Lambda = \Lambda^2_{N_f=0} \), only the term with \( p = 0 \) survives in the sum (23) and this state turns into (13): \(|\Delta, m, \Lambda\rangle \to |\Delta, \Lambda^2_{N_f=0}\rangle\), while constraints (26) turn into (13). It deserves mentioning that, due to separation of powers of \( \Lambda_1 \) in (22), between two vectors in the scalar product (which is, of course, ambiguous), this limit is a little bit different from the conventional “physical” limit in Seiberg-Witten theory, \( \mu_1 \Lambda_1^2 \to \Lambda^2_{N_f=0} \).

The calculation is very similar in the case of \( N_f = 2 \), if keeping finite the masses \( \mu_1 \) and \( \mu_3 \). Then, both the factors \( \gamma_{\Delta_1 \Delta_2} \) and \( \gamma_{\Delta_3 \Delta_4} \) behave according to (24), when taking \( \mu_2 \to \infty \) and \( \mu_4 \to \infty \), and one gets that the conformal block (11)

\[
B_{\Delta}^{N_f=2}(\Lambda_2, m_1, m_3) = \lim_{\Delta_1 \to \infty} B_{\Delta_1 \Delta_2; \Delta_1 \Delta_4; \Delta}(x) = \sum_{|Y|=|Y'|} \sum_{p,p'} (2m_1)^{[Y]-2p} (2m_3)^{[Y]-2p'} \left( \frac{\Lambda_2}{2} \right)^{|Y|} Q_\Delta^{-1} \left( Y, [2p, 1^{[Y]-2p}] \right) \delta \left( Y, [2p, 1^{[Y]-2p}] \right) = \sum_{n,p,p'} (2\mu_1)^{n-2p} (2\mu_3)^{n-2p'} \left( \frac{\Lambda_2}{2} \right)^{2n} Q_\Delta^{-1} \left( [2p, 1^{n-2p}], [2p', 1^{[Y']-2p'}] \right) =
\]

\[
= \langle \Delta, \Lambda_2/2, 2m_1|\Delta, \Lambda_2/2, 2m_3\rangle
\]

(27)

in this limit is a scalar product of two states (23), where \( m_{1,3} = \mu_{1,3} - \frac{x}{2} \), \( \Lambda_2^2 = x \mu_2 \mu_4 \), are again to be fixed finite in the limit of \( x \to 0 \) and \( \mu_{2,4} \to \infty \).

6. In the case of “asymmetric limit”, i.e. if instead of taking \( \mu_{2,4} \to \infty \), one decouples, say, \( \mu_3,4 \to \infty \), no simplification occurs in the factor \( \gamma_{\Delta_1 \Delta_2}(Y) \) in (11), while the second factor degenerates according to (10), i.e. \( x^{[Y']} \gamma_{\Delta_3 \Delta_4}(Y') \to \Lambda^2 [Y'] \delta \left( Y', [1^{[Y']}\right) \). This means that the conformal block
simplifies, though not as drastically as in the symmetric limit:

\[
\tilde{B}_\Delta^{N_f=2}(\Lambda_2, \mu_1, \mu_2) = \lim_{\Delta_3 \to \infty} B_{\Delta_1 \Delta_2;\Delta_3;\Delta}(x) = \sum_{Y} A^{2|Y|} \gamma_{\Delta_1 \Delta_2}(Y) Q^{-1}_{\Delta}(Y, [1|Y]) = \langle \Delta, \Lambda^2|V_{\Delta_1}(1)V_{\Delta_2}(0)\rangle
\]

(28)

since \[3\]

\[
\gamma_{\Delta_1 \Delta_2}(Y) = \langle L_{-Y} V_{\Delta_1}(1)V_{\Delta_2}(0)\rangle
\]

(29)

Thus, the 4-point conformal block in this limit reduces to a triple vertex, as was conjectured in [4]. It depends on \(\mu_1\) and \(\mu_2\) through \(\Delta_1\) and \(\Delta_2\).

Similarly, if only one mass, say, \(\mu_4 \to \infty\), one obtains

\[
B_\Delta^{N_f=3}(\Lambda_3, \mu_1, \mu_2, \mu_3) = \lim_{\Delta_3 \to \infty} B_{\Delta_1 \Delta_2;\Delta_3;\Delta}(x) = \sum_{Y} \sum_{p} (2\mu_3 - \epsilon)^{|Y|} 2p \left(\frac{\Lambda}{2}\right)^{|Y|} \gamma_{\Delta_1 \Delta_2}(Y) Q^{-1}_{\Delta}(Y, [2p, 1|Y| - 2p]) = \langle \Delta, \Lambda/2, 2\mu_3 - \epsilon|V_{\Delta_1}(1)V_{\Delta_2}(0)\rangle
\]

(30)

which is again a reduction from the 4-point function to a 3-point one.

7. To conclude, in this paper we have studied the non-conformal limits (in the sense of 4d supersymmetric gauge theory) of conformal blocks related to Nekrasov partition functions by the AGT correspondence. We have derived directly from 2d CFT analysis the results, conjectured in [4] from brane considerations and confirmed by computer simulations, for the asymptotically free limit of conformal blocks. The proof holds at the level of Nekrasov functions for arbitrary values of \(\epsilon_1, \epsilon_2\) and \(\epsilon = \epsilon_1 + \epsilon_2\), the result for the Seiberg-Witten prepotentials [23] follows [15, 16] after taking the limit of \(\epsilon_1, \epsilon_2 \to 0\). The proof is self-consistent within 2d CFT, and, in application to Nekrasov functions, it assumes that the original AGT relation is correct. After numerous checks in [1]-[11] this looks indisputably true, though so far has been proven exactly [6, 7] only in the hypergeometric case for the \(W\)-algebra blocks with one special, one fully-degenerate external state and a free field theory like selection rule imposed on the intermediate state.

There is a number of other interesting limits, which are natural and well understood from the point of view of 2d CFT (e.g. large intermediate dimension \(\Delta\) or the central charge \(c\)). It can be interesting to find their interpretation in terms of the Nekrasov functions and/or instanton expansions in 4d SUSY models.

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