ON THE DIMENSIONS OF THE OSCILLATOR ALGEBRAS
INDUCED BY ORTHOGONAL POLYNOMIALS

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Abstract. There is a generalized oscillator algebra associated with every class of orthogonal polynomials \(\{\Psi_n(x)\}_{n=0}^{\infty}\), on the real line, satisfying a three term recurrence relation \(x\Psi_n(x) = b_n\Psi_{n+1}(x) + b_{n-1}\Psi_{n-1}(x)\), \(\Psi_0(x) = 1, b_{-1} = 0\). This note presents necessary and sufficient conditions on \(b_n\) for such algebras to be of finite dimension. As examples, we discuss the dimensions of oscillator algebras associated with Hermite, Legendre and Gegenbauer polynomials. In addition we shall also discuss the dimensions of some generalized deformed oscillator algebras. Some remarks on the dimensions of oscillator algebras associated with multi-boson systems are also presented.

1. Introduction

The connection between classical orthogonal polynomials and the classical groups as well as with the quantum groups is well known \([21, 14, 9]\). It is also well known that \(e^{-x^2}H_n(x)\), where \(H_n(x)\) are the Hermite polynomials, are the eigensolutions of the quantum harmonic oscillator Hamiltonian \([11]\). The connection of orthogonal polynomials with the Heisenberg algebra of generalized oscillator is discussed, for example, in \([2, 12, 8, 7]\). In \([3]\), a preassigned Hilbert space with an orthogonal polynomials system as a basis is considered as a Fock space. As usual the ladder operators (annihilation) \(a^-\) and (creation) \(a^+\) as well as the number operator \(N\) are defined in this space. By a standard manner the author used these operators to build the selfadjoint operators, the position operator \(X\), the momentum operator \(P\) as well as the energy operator \(H = X^2 + P^2\). By analogy with the usual Heisenberg algebra these operators generate an algebra, which is called a generalized oscillator algebra. Further, in this manner, the operator \(H\) has a simple discrete spectrum and the initial orthogonal polynomial system is an eigenfunction system of \(H\). As applications, using this algebra, spectrum and the polynomials one can also obtain coherent states of quantum optics \([4, 7]\).

In this note we shall consider generalized oscillator algebras associated with orthogonal polynomials \(\{\Psi_n(x)\}_{n=0}^{\infty}\), on the real line, satisfying a three term recurrence relation \(x\Psi_n(x) = b_n\Psi_{n+1}(x) + b_{n-1}\Psi_{n-1}(x)\), \(\Psi_0(x) = 1, b_{-1} = 0\). In fact we shall provide necessary and sufficient conditions on
b_n for these algebras to be of finite dimension. As applications, we shall also discuss the dimensions of some generalized deformed algebras and algebras associated with multi-boson systems. This note, in a manner, provides a dimension-wise classification for the algebras generated by orthogonal polynomials satisfying three term recurrence relations.

2. Preliminaries and formulation of the problem

In this section, we shall extract the preliminary materials as needed here from [3]. Let μ be a positive Borel measure on the real line, \( \mathbb{R} \), such that

\[
\int_{-\infty}^{\infty} \mu(dx) = 1, \quad \text{and} \quad \mu_{2k+1} = \int_{-\infty}^{\infty} x^{2k+1} \mu(dx) = 0; \quad k = 0, 1, \ldots
\]

The measure μ is called a symmetric probability measure. By \( \mathcal{H} \) we denote the Hilbert space \( L^2(\mathbb{R}, \mu) \).

Let (2.1)

\[
\{b_n\}_{n=0}^{\infty}; \quad b_n > 0; \quad n = 0, 1, \ldots
\]

be a positive sequence defined by the algebraic equations system

\[
\sum_{m=0}^{n} \sum_{s=0}^{n} (-1)^{m+s} \alpha_{2m-1,n-1} \alpha_{2s-1,n-1} \frac{\mu_{2n-2m-2s+2}}{(b_{n-1}^2)!} = b_{n-1}^2 + b_n^2; \quad n = 1, 2, \ldots,
\]

where \((b_{n-1}^2)! = b_0^2 b_1^2 \cdots b_{n-1}^2\), the integral part of \( a \) is denoted by \([a]\), and the coefficients \( \alpha_{ij} \) are given by

\[
\alpha_{2p-1,n-1} = \sum_{k_1=2p-1}^{n-1} b_{k_1}^2 \sum_{k_2=2p-3}^{k_1-2} b_{k_2}^2 \cdots \sum_{k_p=1}^{k_{p-1}-2} b_{k_p}^2.
\]

Let us consider a system \( \{\Psi_n(x)\}_{n=0}^{\infty} \) of polynomials defined by the recurrence relation (2.4):

\[
x \Psi_n(x) = b_n \Psi_{n+1}(x) + b_{n-1} \Psi_{n-1}(x), \quad \Psi_0(x) = 1, \quad b_{-1} = 0,
\]

where \( \{b_n\}_{n=0}^{\infty} \) is a given positive sequence satisfying the relation (2.2). The following theorem was proved in [3].

**Theorem 2.1.** The polynomials system \( \{\Psi_n(x)\}_{n=0}^{\infty} \) is orthonormal in the Hilbert space \( \mathcal{H} \) if and only if the coefficients \( b_n \) and the moments \( \mu_{2k} \) are connected by relations (2.2).

2.1. The generalized oscillator algebra. Let \( \{\Psi_n(x)\}_{n=0}^{\infty} \) be an orthonormal basis of the Fock space \( \mathcal{H} \) which satisfy the recurrence relation (2.4). That is

\[
\mathcal{H} = \text{span} \{\Psi_n(x) \mid n = 0, 1, 2, \ldots\},
\]

where the bar stands for the closure of the linear span. In this subsection, we present the definition of the generalized oscillator algebra corresponding...
to the system \( \{ \Psi_n(x) \}_{n=0}^{\infty} \). Define the ladder operators \( A^\dagger \) and \( A \) in the Fock space, \( \mathcal{H} \) by the usual formulas:

\[
A^\dagger \Psi_n(x) = \sqrt{2} b_n \Psi_{n+1}(x), \quad A \Psi_n(x) = \sqrt{2} b_{n-1} \Psi_{n-1}(x).
\]

**Definition 2.2.** An operator \( N \) in the Fock space \( \mathcal{H} \) equiped with the basis \( \{ \Psi_n(x) \}_{n=0}^{\infty} \) is called a number operator if it acts on the basis vectors as \( N \Psi_n(x) = n \Psi_n(x) \quad n \geq 0 \).

We denote by \( B(N) \) a function of operator \( N \) in the space \( \mathcal{H} \) which acts on the basis vectors, \( \{ \Psi_n(x) \}_{n=0}^{\infty} \) as

\[
B(N) \Psi_n(x) = b_{n-1}^2 \Psi_n(x), \quad \text{and} \quad B(N + I) \Psi_n(x) = b_n^2 \Psi_n(x); \quad n \geq 0.
\]

The following result is proved in [3, 4]:

**Theorem 2.3.** The operators \( A, A^\dagger \) and \( N \) obey the following commutation relations

\[
[A, A^\dagger] = 2 (B(N + I) - B(N)), \quad [N, A^\dagger] = A^\dagger, \quad [N, A] = -A.
\]

Moreover if there is a real number \( G \) and a real function \( C(n) \), such that

\[
b_n^2 - b_{n-1}^2 = C(n), \quad n \geq 0, \quad b_{-1} = 0,
\]

then the operators (2.5), (2.6) fulfil the relation

\[
AA^\dagger - GA^\dagger A = 2C(N).
\]

The function \( C(N) \) is defined similar to (2.6) with \( b_n^2 \) instead of \( b_{n-1}^2 \).

**Definition 2.4.** An algebra \( A \) is called a generalized oscillator algebra corresponding to the orthonormal system \( \{ \Psi_n(x) \}_{n=0}^{\infty} \), which satisfy, (2.4) if \( A \) is generated by the generators \( A^\dagger, A, N, I \) which satisfy the relations (2.5) and the ones given in (2.7).

3. **Main results**

In this section we prove the main result of the paper. That is, we prove the necessary and sufficient condition, in terms of \( b_n \), for the finiteness of the dimension of the generalized oscillator algebra defined in Definition (2.4).

For this, let us define the following sequence of operators:

\[
M_0 = [A, A^+], \quad M_1^+ = [M_0, A^+], \cdots, M_j^+ = [M_{j-1}^+, A^+]; \quad j = 2, 3, 
\]

The following are the main results:

**Theorem 3.1.** The generalized oscillator algebra, defined in Definition (2.4), is of finite dimension if and only if \( b_n^2 \) has the following form

\[
b_n^2 = R(n),
\]

where \( R \) is a classical polynomial uniquely determined by \( b_n \).

\[
R(n) = a_0 + a_1 n + a_2 n^2; \quad a_0, a_1, a_2 \in \mathbb{R}.
\]

As a corollary we can state the following result.
Corollary 3.1. If the oscillator algebra defined in Definition (2.4) is of finite dimension, then the dimension of the algebra is four.

In order to give a proof to Theorem (3.1), we need to prove few preliminary results and these results appear in the following lemmas. For this, let us consider the following two variables sequence \( \{ \mathfrak{y}^{(j)}_n \}_{n=0, j=1}^\infty \) defined by

\[
\mathfrak{y}^{(0)}_n = b_n^2 - b_{n-1}^2, \ldots , \mathfrak{y}^{(j)}_n = \mathfrak{y}^{(j-1)}_{n+1} - \mathfrak{y}^{(j-1)}_n; \quad j = 1, 2, \ldots, \quad n = 0, 1, \ldots
\]

Lemma 3.2. If for every \( j > 0 \) fixed, the sequence \( \{ \mathfrak{y}^{(j)}_n \}_{n=0}^\infty \) is not constant, that is \( \mathfrak{y}^{(j)}_n \neq \text{constant} \), \( n = 0, 1, \ldots \), then the generalized oscillator algebra is of infinite dimension.

Proof. It is easy to show that

\[
M_0 \psi_n = 2 \mathfrak{y}^{(0)}_n \psi_n,
M_1^+ \psi_n = 2(\sqrt{2})b_n \mathfrak{y}^{(1)}_n \psi_{n+1},
M_2^+ \psi_n = 2(\sqrt{2})^2 b_n b_{n+1} \mathfrak{y}^{(2)}_n \psi_{n+2},
\vdots \vdots 
M_j^+ \psi_n = 2(\sqrt{2})^j \left( \prod_{i=0}^{j-1} b_{n+i} \right) \mathfrak{y}^{(j)}_n \psi_{n+j}.
\]

Thereby, if for every \( j \), \( \mathfrak{y}^{(j)}_n \neq \text{constant}; \quad n = 0, 1, \ldots \), then \( M_j^+ \) is a new operator in the algebra which raises the levels from \( j \) twice. That is, since \( M_j^+ \neq M_k^+ \) for \( j \neq k \), the algebra is of infinite dimension. \( \square \)

Lemma 3.3. Suppose that there exists \( j > 0 \) such that

\( \mathfrak{y}^{(j)}_n = \text{constant}; \quad n = 0, 1, \ldots \)

Let \( S \) be the set of \( j \) satisfying the above property. Let \( j_0 = \inf S \). Then there exits a classical polynomial \( P(n) \) of degree \( j_0 \) such that

\[
(3.3) \quad b_n^2 = \sum_{i=1}^{n} P(i) + b_0^2, \quad \text{where}
\]

\[
(3.4) \quad P(n) = \sum_{i=0}^{j_0} \alpha_i n^i; \quad n = 0, 1, \ldots, \quad \alpha_i \in \mathbb{R}.
\]
Proof. Since \( \mathfrak{y}^{(j_0-1)}_n - \mathfrak{y}^{(j_0-1)}_n = \mathfrak{y}^{(j_0)}_n = \text{constant} \); \( n = 0, 1, \ldots \), we have the following iteration:

\[
\begin{align*}
\mathfrak{y}^{(j_0-1)}_{n+1} - \mathfrak{y}^{(j_0-1)}_n &= \mathfrak{y}^{(j_0)}_n = \text{constant} \\
\mathfrak{y}^{(j_0-1)}_0 &= a_1n + a_2 \\
\mathfrak{y}^{(j_0-2)}_{n+1} - \mathfrak{y}^{(j_0-2)}_n &= \mathfrak{y}^{(j_0-1)}_n = a_1n + a_2 \\
\mathfrak{y}^{(j_0-2)}_0 &= c_1n^2 + c_2n + c_3 \\
& \vdots \\
\mathfrak{y}^{(0)}_0 = b^2_0 - b^2_{n-1} &= P(n) = \sum_{i=0}^{j_0} \alpha_i n^i, \quad n = 0, 1, \ldots
\end{align*}
\]

which implies that

\[
(3.5) \quad b^2_n = \sum_{i=1}^{n} P(i) + b^2_0 \tag{3.5}
\]

\[ \square \]

**Lemma 3.4.** There exists a polynomial \( Q \) of degree \( j_0 + 1 \) such that

\[
(3.6) \quad Q(n) - Q(n - 1) = P(n); \quad n = 1, 2, \ldots
\]

**Proof.** It is sufficient to prove that there exists \( Q \) such that

\[
(3.7) \quad Q(n) - Q(n - 1) = P(n).
\]

Let \( Q(n) = \sum_{i=0}^{j_0} a_i n^i; \quad a_i \in \mathbb{R}, \quad i = 0, 1, \ldots, j_0 + 1 \). Thereby, the equation (3.7) is

\[
\sum_{i=1}^{j_0+1} a_i n^i - \sum_{i=1}^{j_0+1} a_i \left( \sum_{k=0}^{i} \binom{i}{k} n^k (-1)^{i-k} \right) = \sum_{i=0}^{j_0} \alpha_i n^i,
\]

which is equivalent to the linear system

\[
(3.8) \quad AX = Y,
\]

where the \((j_0 + 1) \times (j_0 + 1)\) matrix \( A \) is given by

\[
A = \begin{pmatrix}
C^{j_0+1}_{j_0+1}(-1) & 0 & 0 & 0 & \ldots & 0 \\
C^{j_0-1}_{j_0+1}(-1)^2 & C^{j_0-1}_{j_0}(-1) & 0 & 0 & \ldots & 0 \\
C^{j_0-2}_{j_0+1}(-1)^3 & C^{j_0-2}_{j_0}(-1)^2 & C^{j_0-2}_{j_0-1}(-1) & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
C^{1}_{j_0+1}(-1)^{j_0} & C^{1}_{j_0}(-1)^{j_0-1} & C^{1}_{j_0-1}(-1)^{j_0-2} & C^{1}_{j_0-2}(-1)^{j_0-3} & \ldots & C^{1}_{2}(-1) \\
C^{0}_{j_0+1}(-1)^{j_0} & C^{0}_{j_0}(-1)^{j_0} & C^{0}_{j_0-1}(-1)^{j_0-2} & C^{0}_{j_0-2}(-1)^{j_0-3} & \ldots & C^{0}_{1}(-1)
\end{pmatrix},
\]
\[ X = \begin{pmatrix} a_{j_0+1} & a_{j_0} & \cdots & \cdots & a_2 & a_1 \end{pmatrix}^T \text{ and } \]
\[ Y = \begin{pmatrix} \alpha_{j_0} & \alpha_{j_0-1} & \cdots & \cdots & \alpha_1 & \alpha_0 \end{pmatrix}^T, \]
and where \( C_l^k = -\binom{k}{l} \), the binomial coefficient. Since the determinant, \( \det(A) = \frac{1}{2}(j_0 + 1)(j_0 + 2) \neq 0 \), the reals \( a_i; \quad i = 1, 2, ..., j_0 + 1 \) are uniquely determined and \( a_0 \) can take any real value. This proves the lemma. □

**Remark 3.5.** Since the algebras corresponding to \( b_n^2 = \sum_{i=0}^{p} \theta_i n^i \) \( (\theta_i \in \mathbb{R}) \) and \( b_n^2 = n^p + \sum_{i=0}^{p-1} \gamma_i n^i \) are isomorphic, we shall use either one without reluctance.

### 3.1. Proof of Theorem 3.1

From (3.6), we have

\[
Q(n) = \sum_{i=1}^{n} P(i) + Q(0).
\]

Now (3.9) and (3.10) give

\[
b_n^2 = Q(n) + b_0^2 - Q(0).
\]

Since \( Q(n) - Q(0) \) does not depend on \( a_0 \), the right side of (3.10) does not depend on \( a_0 \). If we take \( R(n) = Q(n) + b_0^2 - Q(0) \), the necessary part of Theorem 3.1 get proved.

To prove the converse part, let \( b_n^2 \) have the following form,

\[
b_n^2 = Q(n) = \sum_{i=0}^{p} \theta_i n^i; \quad \theta_i \in \mathbb{R}, \quad i = 0, 1, ..., p
\]

and let \( \mathcal{A}(p) \) denotes the corresponding algebra. We prove the converse by cases.

**The case p=0:** If \( b_n^2 = \text{constant} \neq 0 \quad n = 0, 1, 2, ... \), then we have

\[
[N, A^\dagger] = A^\dagger, \quad [N, A] = -A, \quad [A, A^\dagger] = 0.
\]

So, the algebra is generated by the set of operators \( \{A^+, A, N, I\} \) is of finite dimension, and the dimension of \( \mathcal{A}(0) \) is four.

**The case p=1:** If \( b_n^2 = n + \alpha \), then we have

\[
[N, A^\dagger] = A^\dagger, \quad [N, A] = -A, \quad [A, A^\dagger] = 2I,
\]

the set of generators of the corresponding algebra is \( \{A^+, A, N, I\} \). Thereby, the dimension of \( \mathcal{A}(1) \) is four. In fact, this algebra is isomorphic to the harmonic oscillator algebra which is obtained when \( b_n^2 = \frac{n+1}{2} \).

**The case p=2:** If \( b_n^2 = n^2 + \alpha n + \beta; \quad n = 0, 1, 2, ... \), then

\[
[N, A^\dagger] = A^\dagger, \quad [N, A] = -A
\]

and

\[
[A, A^\dagger] \Psi_n = 2(2n - 1 + \alpha)\Psi_n \implies [A, A^\dagger] = 4N + 2(\alpha - 1)I \in \mathcal{A}(2).
\]
Thereby the set of generators of the corresponding algebra is \( \{ A^+, A, N, I \} \). Hence, as dimension of \( \mathcal{A}(1) \), the dimension of \( \mathcal{A}(2) \) is also four. Thus the algebra \( \mathcal{A}(2) \) is also isomorphic to the harmonic oscillator algebra.

**The case \( p=3 \):** If \( b_n^2 = n^3 + \alpha n^2 + \beta n + \gamma \ \forall n \in \mathbb{N} \), then we have
\[
[N, A^+] = A^+, \ [N, A] = -A \quad \text{and} \quad [A, A^+] = 6N^2 + 2(2\alpha - 3)N + 2(1 + \beta - \alpha)I.
\]
Since \( N \) and \( I \) are elements of the algebra, \( N^2 \) is also in the algebra. Let us prove that for any \( m \in \mathbb{N} \), \( (A^+)^m \) is in the algebra. For, since
\[
[A^+, A^+] = 2A^+N + A^+,
\]
and \( N^2, A^+ \in \mathcal{A}(3) \), then \( A^+N \) is in the algebra. Let us prove, by induction, that for any \( m \in \mathbb{N} \),
\[
[A^{+m}, A^+] = -mA^{+m+1}.
\]
For \( m = 1 \), we have
\[
[A^+, A^+]N = A^{+2}N - A^+NA^+ = -A^+[N, A^+] = -A^{+2}.
\]

Thereby \( A^{+2} \in \mathcal{A}(3) \). Suppose that for \( m \geq 1 \), we have
\[
[A^{+m}, A^+] = -mA^{+m+1} \implies A^{+m+1}N - A^+NA^{+m} = -mA^{+m+1}.
\]
Now for \( m + 1 \), we have
\[
[A^{+m+1}, A^+] = A^{+m+2}N - A^+NA^{+m+1} = A^{+m+2}N - A^{+m+1}NA^+ + A^{+m+1}NA^+ - A^+NA^{+m+1} = A^{+m+1} \left( A^+N - NA^+ \right) + \left( A^{+m+1}N - A^+NA^{+m} \right) A^+ = -A^{+m+2} - mA^{+m+1}A^+ = -(m+1)A^{+m+2}.
\]

Therefore by induction \( \{ A^{+m} : m \geq 1 \} \subset \mathcal{A}(3) \). Thus, the algebra, \( \mathcal{A}(3) \) is of infinite dimension.

**General case:** In general, let us prove that \( \mathcal{A}(p) \) is of infinite dimension for \( p \geq 3 \). Let \( b_n \) have the following form
\[
(3.12) \quad b_n^2 = Q(n) = \sum_{i=0}^{p} \alpha_i n^i; \quad \alpha_i \in \mathbb{R}, \quad i = 0, 1, \ldots, p.
\]

It is easy to see that
\[
(3.13) \quad [A, A^+] = \sum_{i=0}^{p-1} \theta_i N^i = \theta_{p-1} N^{p-1} + \ldots + \theta_2 N^2 + \theta_1 N + \theta_0 I,
\]
where \( \theta_i \in \mathbb{R}, \ i = 0, 1, \ldots, p - 1 \). Since \( N, I \in \mathcal{A}(p) \),
\[(\text{3.14}) \quad W^0 = N^{p-1} + \gamma_{p-2}N^{p-2} + \ldots + \gamma_2N^2 \in \mathcal{A}(p),\]

where \(\gamma_{p-1} = 1\), and \(\gamma_i = \frac{\delta_i}{\sigma_{p-1}}\); \(i = 2, \ldots, p-2\). The following commutation relations can easily be computed:

\[(\text{3.15}) \quad [N^2, A^\dagger] = 2A^\dagger N + A^\dagger\]

\[(\text{3.16}) \quad [N^3, A^\dagger] = 3A^\dagger N^2 + 3A^\dagger N + A^\dagger\]

\[(\text{3.17}) \quad [N^4, A^\dagger] = 4A^\dagger N^3 + 6A^\dagger N^2 + 4A^\dagger N + A^\dagger.\]

That is, in general we have

\[(\text{3.18}) \quad [N^k, A^\dagger] = \sum_{i=1}^{k} c^i_k A^\dagger N^{k-i}, \quad \text{where } c^i_k \in \mathbb{R}.\]

Using (3.14) and (3.18), we have

\[(\text{3.19}) \quad W^\dagger := [W^0, A^\dagger] = \sum_{i=2}^{p-1} \sum_{j=1}^{i} \gamma^i_j c^j_i A^\dagger N^{i-j},\]

Since \(W^0 \in \mathcal{A}(p)\), we have \(W^\dagger \in \mathcal{A}(p)\). After \((p - 3)\)-iterations it can be seen that

\[(\text{3.20}) \quad W^{\dagger}_{(p-2)} := \left[A^\dagger ... \left[A^\dagger, \left[A^\dagger, W^\dagger \right] \right] ... \right] = (-1)^{p-1}(p - 1)!A^\dagger^{(p-2)}N + f(p)A^\dagger^{(p-2)},\]

where \(f(p)\) is some function of \(p\). Since \(W^\dagger \in \mathcal{A}(p)\), we get \(W^{\dagger}_{(p-2)} \in \mathcal{A}(p)\). Further, the following commutation relation can easily be verified by induction

\[(\text{3.21}) \quad \left[A^\dagger m, A^\dagger^{(p-2)}N \right] = -mA^\dagger^{(p-2+m)}, \quad m \geq 1.\]

Now, (3.20) and (3.21) implies that

\[(\text{3.22}) \quad \left[A^\dagger, W^{\dagger}_{(p-2)} \right] = (-1)^{p-1}(p - 1)! \left[A^\dagger, A^\dagger^{(p-2)}N \right] = (-1)^{p}(p - 1)!A^\dagger^{(p-1)}.\]

Thereby, since \(A^\dagger, W^{\dagger}_{(p-2)} \in \mathcal{A}(p)\), we see \(A^\dagger^{(p-1)} \in \mathcal{A}(p)\). Again using the relation (3.21), we get

\[(\text{3.23}) \quad \left[A^\dagger^{(p-1)}, W^{\dagger}_{(p-2)} \right] = (-1)^{p-1}(p - 1)! \left[A^\dagger^{(p-1)}, A^\dagger^{(p-2)}N \right] = (-1)^{p}(p - 1)!A^\dagger^{(2p-3)}, \quad m \geq 1;\]

Thereby, \(A^\dagger^{(2p-3)} = A^\dagger^{(p-1)+(p-2)} \in \mathcal{A}(p)\). By iteration, we can prove that \(A^\dagger^{(p-1)+m(p-2)} \in \mathcal{A}(p)\) for every \(m \geq 1\). Further, for \(p \geq 3\), the operators \(A^\dagger^{(p-1)+m(p-2)}\) are new elements of \(\mathcal{A}(p)\) for every \(m \geq 1\). Thereby \(\mathcal{A}(p)\) is of infinite dimension.
3.2. Proof of Corollary 3.1. The proof follows from the cases $p = 0, 1, 2$.

4. Some Examples

4.1. Hermite Polynomials: The Hermite polynomials $H_n(x)$ are given by [13, 15, 16].

\[
H_n(x) = n! \sum_{\nu=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^\nu (2x)^{n-2\nu}}{\nu!(n-2\nu)!}.
\]

These polynomials are orthogonal in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}; \frac{1}{\sqrt{\pi}} \exp(-x^2)dx)$. The normalized polynomials $\{\Psi_n(x)\}_{n=0}^{\infty}$ takes the form

\[
\Psi_n(x) = \pi^{\frac{1}{4}} d_n^{-1} H_n(x),
\]
\[
d_n = \left( 2^n n! \sqrt{\pi} \right)^{\frac{1}{2}}, \quad n \geq 0.
\]

The three term recurrence relations for the Hermite polynomials (for example see [13]), that is the formula (2.4), is satisfied with

\[
b_n = \frac{1}{2} \left( \frac{d_{n+1}}{d_n} \right) = \frac{\sqrt{n+1}}{2}.
\]

It is well known that the oscillator algebra generalized by this family of orthogonal polynomials is of finite dimension and we can see that $b_n^2$ has the form (3.1).

4.2. The Legendre Polynomials: The Legendre polynomials are defined by

\[
P_n(x) = {}_2F_1 \left( -n, n+1; 1; \frac{1-x}{2} \right).
\]

and these polynomials are orthogonal in the Hilbert space $\mathcal{H} = L^2([-1, 1]; 2^{-1}dx)$. The normalized polynomials $\{\Psi_n(x)\}_{n=0}^{\infty}$ are given by

\[
\Psi_n(x) = \sqrt{2} \Phi_n(x), \quad \Phi_n(x) = \sqrt{\frac{2n+1}{2}} P_n(x), \quad n \geq 0.
\]

Taking into account the recurrence relations for the Legendre polynomials (see [2]) we obtain the three term recurrence relation (2.4) with

\[
b_n = \sqrt{\frac{(n+1)^2}{(2n+1)(2n+3)}}, \quad n \geq 0.
\]

In this case, $b_n^2$ does not take the from (3.1). Thereby the generalized oscillator algebra generated by the Legendre polynomials is of infinite dimension.
4.3. The Gegenbauer Polynomials. These polynomials are orthogonal in the Hilbert space $H = L^2([-1, 1]; (d_0(\alpha))^{-2}(1-x^2)^\alpha dx)$ where

$$ (d_0(\alpha))^2 = 2^{2\alpha+1} \frac{(\Gamma(\alpha + 1))^2}{(\Gamma(2\alpha + 2))}. $$

The ultraspherical polynomials are defined by the hypergeometric function

$$ P_n^{(\alpha,\alpha)}(x) = \frac{(\alpha + 1)_n}{n!} _2F_1 \left( -n, n + 2\alpha + 1; \alpha + 1; \frac{1-x}{2} \right), $$

where Pochhammer-symbol $(\beta)_n$ is defined by $(\beta)_0 = 1$, and $(\beta)_n = \beta(\beta + 1)(\beta + 2)...(\beta + n - 1)$; $n \geq 1$. For $\alpha > -1$ the following orthogonal relation is valid

$$ \int_{-1}^{1} P_n^{(\alpha,\alpha)}(x) P_m^{(\alpha,\alpha)}(x) (1-x^2)^\alpha dx = d_n^2 \delta_{mn}, \quad n, m \geq 0, $$

with the constant of normalization $d_n$ given by

$$ d_n^2 = \frac{2^{2\alpha+1} (\Gamma(n + 2\alpha + 1))^2}{(2n + 2\alpha + 1)n!\Gamma(n + \alpha + 1)}, \quad n \geq 0. $$

For $\alpha = \lambda - 2^{-1}, \lambda > -2^{-1}$ and $n \geq 0$, the Gegenbauer polynomials are defined as

$$ P_n^{(\lambda)}(x) = \frac{\Gamma(n + 1)\Gamma(n + 2\alpha + 1)}{\Gamma(2\alpha + 1)\Gamma(n + \alpha + 1)} P_n^{(\alpha,\alpha)}(x). $$

The normalized polynomials are given by

$$ \Psi_n(x) = d_0 d_n^{-1} P_n^{(\alpha,\alpha)}(x) $$

The functions $\Psi_n(x)$ satisfy the relations (2.4) with

$$ b_n^2 = \frac{(n+1)(n+2\alpha+1)}{(2n+2\alpha+1)(2n+2\alpha+3)}; \quad n \geq 0, \quad b_{-1} = 0. $$

It is clear that this $b_n^2$ is strictly different from (3.1). Thereby, the oscillator algebra associated with this family of orthogonal polynomials is of infinite dimension.

5. Application to the Dimensions of Generalized Deformed Oscillator Algebras $\mathcal{W}^{\gamma}_{\alpha,\beta}(q)$

In this section all the necessary information are extracted from [5, 7] as needed here. For an enhanced explanation the reader can consult [5, 7] and the many references cited there in. Consider the generalized deformed oscillator algebra $\mathcal{W}^{\gamma}_{\alpha,\beta}(q)$ generated by the operators $\{a, a^+, N, I\}$ with defining relations

$$ aa^+ - q^\gamma a^+ a = q^{\alpha N + \beta}, $$

$$ [N, a] = -a, \quad [N, a^+] = a^+, $$

$$ [a, a^+] = N. $$
where parameters $\alpha, \beta, \gamma \in \mathbb{R}$ and $q$ is an arbitrary positive number with $q \neq 1$. Let operators $a$, $a^+$, $N$ act on orthonormal basis elements of a separable Hilbert space $\mathcal{H}$ as

$$a\Psi_n = b_{n-1}\Psi_{n-1}, \quad a^+\Psi_n = b_n\Psi_{n+1}, \quad N\Psi_n = n\Psi_n,$$

and fulfill natural hermiticity conditions:

$$\tag{5.3} (a^+)^\dagger = a, \quad (a)^\dagger = a^+.$$ 

According to (5.1), $b_n$ satisfy the following recurrence relation

$$\tag{5.4} b_n^2 - q^\gamma b_{n-1}^2 = q^{\alpha + \beta}; \quad n = 1, 2, \ldots$$

When $\gamma = \alpha$, from (5.4) we get

$$\tag{5.5} b_n^2 = q^{\gamma n} (nq^\beta + b_0^2),$$

in this case, according to (3.1) the algebra, $\mathcal{W}_{\alpha,\beta}^\gamma(q)$ is of finite dimension if and only if $\gamma = 0$. In addition, for $\gamma = 0$, the algebra is isomorphic to the harmonic oscillator algebra. When $\gamma \neq \alpha$, from (5.4) we obtain

$$\tag{5.6} b_n^2 = \frac{q^{\gamma n} [q^{-\alpha + \beta} b_0^2 - b_0^2 + q^\beta] - q^{\alpha n + \beta}}{q^{-\alpha + \gamma} - 1}.$$ 

In this case, according to the assumptions ($q \neq 1$ and $\gamma \neq \alpha$) and (3.1), we can see that the algebra is of infinite dimension.

In the following few subsections, according to the results (5.5) and (5.6), we shall look at the dimensions of some well known algebras. For detailed information and the physical relevance of these algebras one may consult [5, 7].

5.1. **Tamm-Dancoff oscillator algebra** $\mathcal{W}_{1,0}^1(q)$. For this algebra, since $\gamma = 1$, $\beta = 0$ from (5.5) we get

$$b_n^2 = q^n (n + b_0^2).$$

Thereby, according to (3.1) this algebra is of infinite dimension.

5.2. **Arik-Coon-Kuryskin oscillator algebra** $\mathcal{W}_{0,0}^1(q)$. For this algebra, since $\gamma = 1$, $\alpha = 0$, from (5.6) we get

$$b_n^2 = \frac{q^n (q b_0^2 - b_0^2 + 1)}{q - 1}.$$ 

Thereby, according to (3.1) this algebra is of infinite dimension.

5.3. **Quantum deformed oscillator algebra** $\mathcal{W}_{1,0}^{-1}(q)$. For this algebra, since $\gamma = 1$, $\alpha = -1$, $\beta = 0$, from (5.6) we get

$$b_n^2 = \frac{q^n (q^2 b_0^2 - b_0^2 + 1) - q^{-n}}{q^2 - 1}.$$ 

Thereby, according to (3.1) this algebra is of infinite dimension.
5.4. **Feinsilver oscillator algebra** \( W_{0,0}^0(q) \). For this algebra, since \( \gamma = 0, \alpha = -2, \beta = 0 \), from \((5.6)\) we get

\[
b^2_n = \frac{(q^2 - 1) b_0^2 + 1 - q^{-2n}}{q^2 - 1}.
\]

Thereby, according to \((3.1)\) this algebra is of infinite dimension.

5.5. **Quesne oscillator algebra** \( W_{-1,-1}^0(q) \). For this algebra, since \( \gamma = 0, \alpha = -1, \beta = -1 \), from \((5.6)\) we get

\[
b^2_n = \frac{(q^2 - q) b_0^2 + 1 - q^{-n}}{q^2 - q}.
\]

Thereby, according to \((3.1)\) this algebra is of infinite dimension.

5.6. **The case** \( aa^+ - qa^+a = C(N) \). Here we study some special cases of q-deformed algebras satisfying the following commutation relations:

\[
(5.7) \quad aa^+ - qa^+a = C(N), \quad [N, a] = -a, \quad [N, a^+] = a^+.
\]

where \( C(N) \) is some function of the number operator. The relations \((5.7)\) imply the following recurrence relation

\[
(5.8) \quad b^2_n - qb^2_{n-1} = C(n),
\]

and thereby, we have

\[
(5.9) \quad b^2_n = q^n b_0^2 + \sum_{i=1}^{n} q^{n-i} C(i).
\]

Let us see some special cases of \( C(N) \).

- **Case 1**: If \( C(n) = c_0; \) \( n = 1, 2, 3, ... \), then

\[
(5.10) \quad b^2_n = \frac{c_0}{1 - q} + q^n \left( b_0^2 - \frac{c_0}{1 - q} \right).
\]

According to \((3.1)\) the corresponding algebra is of finite dimension if and only if \( q = 1 - \frac{c_0}{b_0} \).

- **Case 2**: If \( C(n) = c_0n; \) \( n = 1, 2, 3, ... \), then

\[
(5.11) \quad b^2_n = \frac{c_0}{(q - 1)^2} \left[ n - (n + 1)q + q^{n+1} \right] + b_0^2 q^n.
\]

According to \((3.1)\) this algebra is of infinite dimension for all \( c_0 \in \mathbb{R} \).

- **Case 3**: If \( C(n) = c_0n^2; \) \( n = 1, 2, 3, ... \), then

\[
b^2_n = \frac{c_0 \left[ q^{n+1}(1 + q) + 2n + 1 - (2n + 3)q - (n + 1)^2(q - 1)^2 \right]}{(q - 1)^2} + q^n b_0^2.
\]
According to (3.1) this algebra is also of infinite dimension for any \( c_0 \in \mathbb{R} \).

- From the previous cases it is clear that the algebra will be of infinite dimension for

\[
C(n) = \sum_{k=0}^{m} c_k n^k; \quad n = 1, 2, 3..., \quad m = 1, 2, 3,...
\]

5.7. Some other interesting algebras. For a detailed treatment of the algebras presented in this subsection one can consult [7, 6].

- The \( \nu \)-modified oscillator algebra of Calogero model. The commutations relations of this algebra are the following [6]

\[
a a^+ - a^+ a = 1 + 2 \nu K, \quad [N, a] = -a, \quad [N, a^+] = a^+
\]

\[
a K = -K a, \quad a^+ = -K a^+, \quad K^2 = 1; \quad \nu \in \mathbb{R},
\]

where \( K = (-1)^N \) is the Klein operator. In this case,

\[
b_n^2 = \begin{cases} 
    n + b_0^2 & \text{if } n \text{ is even} \\
    n - 2 \nu + b_0^2 & \text{if } n \text{ is odd}
\end{cases}
\]

Thereby, according to (3.1) this algebra is of finite dimension.

- The \((q, l, \lambda)\)-deformed Heisenberg algebra. The commutations relations of this algebra are the following (5.12)

\[
a a^+ - a^+ a = l^2 q^{\lambda - N - 1}, \quad [N, a] = -a, \quad [N, a^+] = a^+,
\]

where \( l, \lambda \in \mathbb{R} \) and \( l \neq 0 \). In this case

\[
b_n^2 = l^2 q^{\lambda - \frac{n - 1}{q^{n+1}(q-1)}} + b_0^2.
\]

Thereby, according to (3.1) this algebra is of infinite dimension.

- \((p, q, \alpha, \beta, l)\)-deformed oscillator algebra. This algebra is generated by the operators \( I, a, a^+, N \) with the commutations relations

\[
a a^+ - q^l a^+ a = p^{-\alpha N - \beta}, \quad a a^+ - p^{-1} a^+ a = q^{\alpha N + \beta},
\]

\[
[N, a] = -l a, \quad [N, a^+] = l a^+; \quad \alpha, \beta, l \in \mathbb{R}.
\]

In this case

\[
b_n^2 = \begin{cases} 
    q^{nl} b_0^2 + \frac{p^n q^{nl - 1}}{q^{nl + p^{\alpha N} (p q - 1)}} & \text{if } p^n q^l \neq 1 \\
    q^{nl} b_0^2 + \frac{p^n q^{nl - 1}}{p^n p^{\alpha N}} & \text{if } p^n q^l = 1.
\end{cases}
\]

Thereby, according to (3.1) this algebra is of finite dimension for \( \alpha = l = 0 \) and of infinite dimension otherwise. Note that, for \( l = 1, \alpha = 1 \) and \( \beta = 0 \), the algebra becomes the infinite dimensional \((p, q)\)-Chakrabarti-Jagannathan algebra [7].
6. Reduced algebras in multi-boson systems

The dynamics of \((N+1)\)-boson systems is assumed to be governed by a Hamiltonian operator of the form \([17, 18]\):

\[
H = h_0(a_0^*a_0, \ldots, a_N^*a_N) + g_0(a_0^*a_0, \ldots, a_N^*a_N)a_0^{k_0}\cdots a_N^{k_N} + a_0^{-k_0}\cdots a_N^{-k_N}\bar{g}_0(a_0^*a_0, \ldots, a_N^*a_N),
\]

where \((a_0, a_N)\) and \((a_0^*, \ldots, a_N^*)\) are bosonic annihilation and respectively creation operators with standard Heisenberg commutation relations \([17]\). The operators \(g_0(a_0^*a_0, \ldots, a_N^*a_N)\) is a kind of generalization of the coupling constant. The operator \(h_0(a_0^*a_0, \ldots, a_N^*a_N)\) can be chosen as a free Hamiltonian being a weighted sum of the occupation number operators of the elementary modes \(a_0^*a_0, \ldots, a_N^*a_N\).

The reduced algebra associated to this system in a reduced Hilbert space \(\mathcal{H}_{\lambda_1, \ldots, \lambda_N}\), which is formed by the orthonormal vectors \(|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle\), is denoted by \(A_{\text{red}}\). In order to analyse the quantum system described by the Hamiltonian \((6.1)\), the following operators were introduced in \([17]\).

\[
A := g_0(a_0^*a_0, \ldots, a_N^*a_N)a_0^{k_0}\cdots a_N^{k_N},
\]

\[
A_i := A_i^* := \sum_{j=0}^{N} \alpha_{ij}a_j^*a_j.
\]

and

\[
A^*A = g_0(a_0^*a_0 - k_0, \ldots, a_N^*a_N - k_N)\mathcal{P}_{k_0}(a_0^*a_0 - k_0)\cdots\mathcal{P}_{k_N}(a_N^*a_N - k_N),
\]

\[
AA^* = g_0(a_0^*a_0, \ldots, a_N^*a_N)\mathcal{P}_{k_0}(a_0^*a_0)\cdots\mathcal{P}_{k_N}(a_N^*a_N),
\]

where

\[
\mathcal{P}_k(a^*a) = a^k a^{-k} = \begin{cases} a^k (a^*a)^k = (a^*a + 1)\cdots(a^*a + k) & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ (a^*a)^{-k} a^{-k} = a^*a (a^*a - 1)\cdots(a^*a - k + 1) & \text{if } k < 0 \end{cases}.
\]

The operators \(A_0, A\) and \(A^*\) satisfy the commutation relations

\[
[A_0, A] = -A, \quad [A_0, A^*] = A^*
\]

and act on the basis vectors as

\[
(6.4) \quad A_0|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle = \lambda_0|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle
\]

\[
(6.5) \quad A|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle = \sqrt{\mathcal{G}(\lambda_0 - 1, \lambda_1, \ldots, \lambda_N)}|\lambda_0 - 1, \lambda_1, \ldots, \lambda_N\rangle
\]

\[
(6.6) \quad A^*|\lambda_0, \lambda_1, \ldots, \lambda_N\rangle = \sqrt{\mathcal{G}(\lambda_0, \lambda_1, \ldots, \lambda_N)}|\lambda_0 + 1, \lambda_1, \ldots, \lambda_N\rangle,
\]

and replacing the occupation numbers \(a_0^*a_0, \ldots, a_N^*a_N\) by the operators \(A_0, A_1, \ldots, A_N\) one obtains

\[
(6.7) \quad A^*A = \mathcal{G}(A_0 - 1, A_1, \ldots, A_N)
\]

\[
(6.8) \quad AA^* = \mathcal{G}(A_0, A_1, \ldots, A_N),
\]
where \( \mathcal{G} \) is uniquely determined by \( g_0 \), the polynomials \( P_{k_0}, \ldots, P_{k_N} \) and the linear map \( (6.3) \). The reduced algebra \( A_{\text{red}} \) is generated by the operators \( A_0, A, A^* \) and \( I \). The operators \( A^* A \) and \( AA^* \) are diagonal in the standard Fock basis,

\[
|n_0, n_1, \ldots, n_N\rangle = \frac{1}{\sqrt{n_0! \cdots n_N!}}(a_0^*)^{n_0} \cdots (a_N^*)^{n_N} |0\rangle.
\]

The maximal system of commuting observables is diagonalized in the Fock basis and the eigenvalues of \( A_0, A_1, \ldots, A_N \) on \( |n_0, n_1, \ldots, n_N\rangle \) are given by

\[
\lambda_i = \sum_{j=0}^{\infty} \alpha_{ij} n_j; \quad i = 0, 1, \ldots, N.
\]

In this section our aim is to give the necessary and sufficient form of the Hamiltonian \( (6.1) \) for the associated reduced algebra to be of finite dimension. According to the theorem \( (3.1) \), the necessary and sufficient condition for the reduced algebra to be of finite dimension is that the function \( \mathcal{G} \) must have the following form:

\[
\mathcal{G}(\lambda_0, \lambda_1, \ldots, \lambda_N) = \lambda_0^2 U_0(\lambda_1, \ldots, \lambda_N) + \lambda_0 U_1(\lambda_1, \ldots, \lambda_N) + U_2(\lambda_1, \ldots, \lambda_N),
\]

where \( U_i; \quad i = 0, 1, 2 \) are \( N \) variables real valued functions. Thereby, the complex valued function \( g \) defined in \( (6.1) \) must satisfy the following condition (we consider it as a function of variables \( x_1, \ldots, x_N \))

\[
|g_0(x_0, \ldots, x_N)|^2 P_{k_0}(x_0) \cdots P_{k_N}(x_N) = \lambda_0^2 U_0(\lambda_1, \ldots, \lambda_N) + \lambda_0 U_1(\lambda_1, \ldots, \lambda_N) + U_2(\lambda_1, \ldots, \lambda_N),
\]

where the polynomials \( P_k \) are defined as follows

\[
P_k(x) = \begin{cases} (x + 1) \cdots (x + k), & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ x(x - 1) \cdots (x - k + 1), & \text{if } k < 0 \end{cases}
\]

and

\[
\lambda_i = \sum_{j=0}^{N} \alpha_{ij} x_j; \quad i = 0, 1, 2, \ldots, N.
\]

As an example, let us work with the two modes Hamiltonian

\[
H = h_0(a_0^* a_0, a_1^* a_1) + g_0(a_0^* a_0, a_1^* a_1) a_0^{k_0} a_1^{k_1} + a_0^{-k_0} a_1^{-k_1} g_0(a_0^* a_0, a_1^* a_1).
\]

For this Hamiltonian, the condition \( (6.10) \) is reduced to

\[
|g_0(x_0, x_1)|^2 P_{k_0}(x_0) P_{k_1}(x_1) = (a_{0,0} x_0 + a_{0,1} x_1)^2 U_0(a_{1,0} x_0 + a_{1,1} x_1) + (a_{0,0} x_0 + a_{0,1} x_1) U_1(a_{1,0} x_0 + a_{1,1} x_1) + U_2(a_{1,0} x_0 + a_{1,1} x_1).
\]
Thereby, the necessary and sufficient conditions for the reduction algebra to be isomorphic to the harmonic oscillator algebra is the following:

\[ U_0 := 0. \]

In the case, \( k_0, k_1 \geq 0 \), we have

\[ |g_0(x_0, x_1)|^2 = \frac{(\alpha_{0,0} x_0 + \alpha_{0,1} x_1) \cdot U_1 (\alpha_{1,0} x_0 + \alpha_{1,1} x_1) + U_2 (\alpha_{1,0} x_0 + \alpha_{1,1} x_1)}{P_{k_0}(x_0)P_{k_1}(x_1)}. \]

Actually, we suppose \( g \) to be null function in the set where the expression (6.15) is not defined. For simplicity in (6.15) we may take \( U_1 = 1 \) and \( U_2 = c \), some constant, then the necessary and sufficient conditions for the reduction algebra to be isomorphic to the harmonic oscillator algebra becomes

\[ |g_0(x_0, x_1)|^2 = \frac{(\alpha_{0,0} x_0 + \alpha_{0,1} x_1) + c}{P_{k_0}(x_0)P_{k_1}(x_1)}, \]

which implies that

\[ g_0(x, y) = \exp(i\theta) \left[ \frac{(\alpha_{0,0} x + \alpha_{0,1} y) + c}{P_{k_0}(x)P_{k_1}(y)} \right]^{\frac{1}{2}}. \]

Now let us look at a particular example given in [18], where

\[ H = h_0(a_0^*a_0, a_1^*a_1) + g_0(a_0^*a_0, a_1^*a_1)a_0^{k_0}a_1^{k_1} + a_0^{*k_0}a_1^{*k_1}g_0(a_0^*a_0, a_1^*a_1), \]

with the matrix elements

\[ \alpha_{0,0} = \frac{1}{k_0}, \quad \alpha_{0,1} = 0, \quad \alpha_{1,0} = k_1, \quad \alpha_{1,1} = k_0 \]

is given. In this case, according to (6.17), the necessary and sufficient condition for the algebra, \( A_{\text{red}} \) is to be isomorphic to the harmonical oscillator algebra is that the function \( g \) must have the following form:

\[ g_0(x, y) = \exp(i\theta) \left[ \frac{x + k_0 c}{k_0 \prod_{i=1}^{k_0}(x + i) \prod_{j=0}^{k_1-1}(y - j)} \right]^{\frac{1}{2}}. \]

7. CONCLUSION

In this paper, we have only discussed orthogonal polynomials and some of their deformed versions satisfying three terms recurrence relations. There are many other orthogonal polynomials and their deformations satisfying four terms recurrence relations, for example, Laguerre polynomials, Jacobi polynomials, etc. Studying their dimension-wise classification, as we have done here, will be an interesting and rather challenging problem to consider. Further, the normalized 2D-Hermite polynomials \( H_{n,m}(z, \overline{z}) \) satisfy the three term recurrence relation [22] [10]

\[ zH_{m,n}(z, \overline{z}) = \sqrt{m+1}H_{m+1,n}(z, \overline{z}) + \sqrt{n}H_{m,n-1}(z, \overline{z}), \]
and also their quaternionic extension, introduced in [19, 20], $H_{m,n}(q,\overline{q})$ satisfy the same recurrence relation with the complex number $z$ replaced by a quaternion $q$ in (7.1). The 2D-Zernike polynomials also satisfy a three term recurrence relation of the form [23]

$$zP_{m,n}^\alpha(z,\overline{z}) = a_{m,n}P_{m+1,n}^\alpha(z,\overline{z}) + b_{m,n}P_{m,n-1}^\alpha(z,\overline{z}).$$

The recurrence relations of these 2D polynomials do not apt to (2.4), and thereby the theory developed in this note does not fit to these polynomials.

In this regard, for the algebras and their dimension-wise classification, a separate theory needs to be established for the 2D polynomials. We shall consider these issues in our future work.

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