Liouville theory, accessory parameters and $2 + 1$ dimensional gravity

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Abstract

We prove a relation between the asymptotic behavior of the conformal factor and the accessory parameters of the $SU(1,1)$ Riemann-Hilbert problem. Such a relation shows the hamiltonian nature of the dynamics of $N$ particles coupled to $2+1$ dimensional gravity. A generalization of such a result is used to prove a connection between the regularized Liouville action and the accessory parameters in presence of general elliptic singularities. This relation had been conjectured by Polyakov in connection with 2-dimensional quantum gravity. An alternative proof, which works also in presence of parabolic singularities, is given by rewriting the regularized Liouville action in term of a background field.

1 Introduction

In this paper we shall consider a problem which arises in connection with the hamiltonian formulation of $2+1$ dimensional gravity in the maximally slicing gauge. It is related to a variant of the Riemann-Hilbert problem in the realm of $SU(1,1)$ monodromies and to a conjecture by Polyakov about the auxiliary parameters occurring in the Riemann-Hilbert problem.

From the viewpoint of $2+1$ dimensional gravity coupled to point particles one starts with the usual ADM hamiltonian formulation in the maximally slicing gauge, solves first the diffeomorphism and then the hamiltonian constraints and obtains the equations of motion of particle, i.e. $\dot{z}_n$ and $\dot{P}_n$ as function of $z_n$, $P_n$. The natural question is whether such equations are canonical. As they are obtained by solving the constraint of a canonical problem one expects and affirmative answer. However as the expression of $\dot{z}_n$ and $\dot{P}_n$ are non trivial functions of $z_n$ and $P_n$ a direct proof is desirable.

The proof of the hamiltonian nature of such equations was given in ref. [1] in the following way. For two particle is trivial, for three particles it involves the exploitation of the Garnier equations related to the isomonodromic transformations. For more than three particle the Garnier system of equations is not sufficient for providing the proof of the hamiltonian
nature. In [1] it was shown as such a result is a consequence of an interesting conjecture due to Polyakov [2] about the so called accessory parameters which occur in the Riemann-Hilbert problem.

A proof of Polyakov’s conjecture was given by Zograf and Takhtajan [3] in the case of parabolic singularities and for elliptic singularities of finite order. 2+1 dimensional gravity requires the validity of Polyakov’s conjecture for generic elliptic singularities.

Actually for proving the hamiltonian nature of the equations of motion we shall not need the full strength of Polyakov conjecture, but a weak form of it will be sufficient. The weak form is of interest in itself because it relates the constant part of the asymptotic behavior of the reduced conformal factor to the derivative of some accessory parameters with respect to the total energy.

Two proofs of Polyakov conjecture for general elliptic singularities were given in [4] and [5]. Another proof was supplied in [6], where also a connection with the Kaehler metric on moduli space has been evidenced.

Proofs [4] and [5] go through a direct calculation of the derivative of the regularized action with respect to the position of the singularities. Proof [6] goes through an intermediate step i.e. a weak form of Polyakov conjecture. This passage still simplifies the proof in the case of elliptic singularities and is based on a simple application of Green’s formula to the linear elliptic equation which arise when taking the derivative of Liouville equation with respect to the position and strength of the singularities. By exploiting later the behavior of the solution as a function of the strength of the singularities one reaches the proof of the full Polyakov conjecture.

In this paper we shall reproduce the proof of [4] and [5] with full details, introducing notable simplifications in them. In this context we shall also show how the technique developed in [4] immediately extends to the case of parabolic singularities.

The plan of the paper is the following: in Section 2 we summarize the role of Liouville and uniformization theory in 2+1 dimensional gravity coupled to point particles.

In Section 3 we derive the canonical form of the conformal factor in terms of the solutions of the linear second order Fuchsian differential equation related to the uniformization
problem, both for elliptic and parabolic singularities.

In Section 4 we discuss how $SU(1, 1)$ monodromies are realized in the context of the Riemann-Hilbert problem arising in connection with the Liouville equation with point-like sources. Following such a development we prove that for general elliptic singularities, the accessory parameters $\beta_n$ are real analytic functions of $z_n$ and $g_n$ in an open everywhere dense set of $R^{3N+1}$.

In Section 5 we derive the hamiltonian nature of 2+1 dimensional gravity by proving a relation between the constant term in the asymptotic behavior of the conformal factor at infinity and the accessory parameters $\beta_n$.

In Section 6 we use the same technique to derive a weak form of Polyakov conjecture from which the full form easily follows.

In Section 7 give a proof of Polyakov conjecture both for elliptic and parabolic singularities by exploiting the technique of \[4\] extended to the general case. The method is that to rewrite Polyakov action in terms of a field $\phi_M$ which is less singular than the original $\phi$; this procedure avoids the writing of the regularized action as a limit of an integral.

In Appendix 1 we prove the boundedness of the accessory parameters $\beta_n$ and of the parameter $k$ which occurs in the expression of the conformal factor. Such result will be used in Section 4.

In Appendix 2 we give the expression of the regularized action in terms of the regular field $\phi_M$.

## 2 2+1 dimensional gravity and Liouville theory

The adoption of the maximally slicing gauge \[7, 8, 9\], defined by $K = 0$, where $K$ is the trace of the extrinsic curvature of the time slice, is practically confined to the case of open universes \[8\]. However in this gauge the solution of the diffeomorphism constraint is very simple and the hamiltonian constraint reduces to an equation well studied by mathematicians. The
metric is parametrized in the standard ADM form

\[ ds^2 = -N^2 dt^2 + e^{2\sigma} (dz + N^z dt)(d\bar{z} + N^{\bar{z}} dt) \]  

where we have adopted the complex coordinate \( z = x + iy \) and the conformal gauge \( g_{zz} = 1/2 e^{2\sigma}, \ g_{\bar{z}\bar{z}} = g_{z\bar{z}} = 0 \) for the space metric.

The solution of the diffeomorphism constraint is given by

\[ \pi_{\bar{z} z} = -\frac{1}{2\pi} \sum_n P_n \frac{1}{z - z_n}, \]  

while the Hamiltonian constraint reduces to

\[ 4\partial_z \partial_{\bar{z}} (2\sigma) = -\pi^a_b \pi^b_a e^{-2\sigma} - \sum_n m_n \delta^2(z - z_n) \]  

being \( z_n \) the particle positions and \( m_n \) their rest masses.

By posing \( \pi^a_b \pi^b_a e^{-2\sigma} = 2\pi^z_{\bar{z}} \pi^{\bar{z}}_z e^{-2\sigma} \equiv e^\phi \text{ eq.}(3) \) reduces to

\[ 4\partial_z \partial_{\bar{z}} \phi = e^\phi + 4\pi \sum_n \delta^2(z - z_n)(\mu_n - 1) + 4\pi \sum_B \delta^2(z - z_B), \]  

where we have defined \( \mu_n = m_n / 4\pi \) and \( z_B \) are the positions of the so called apparent singularities given by the zeros of \( \pi^{\bar{z}}_z \) i.e.

\[ \sum_n \frac{P_n}{z_B - z_n} = 0. \]  

Due to the restriction \( \sum_n P_n = 0 \) the \( z_B \) are \( N - 2 \) in number, being \( N \) the number of particles.

As we shall see in the following section, the solution of eq.(4) in unique once one imposes the asymptotic behavior of \( \phi \) at infinity \( \phi = (\mu - 2) \ln z\bar{z} + O(1) \).

The conformal factor \( \phi \) plays the key role in the theory as all other quantities can be derived algebraically from it.

The lapse and shift function are easily obtained from \( \phi \)

\[ N = \frac{\partial \phi}{\partial M} \]
being $M \equiv 4\pi \mu$ the total energy of the system and

$$N_z = -\frac{2}{\pi z(z)} \partial_z N + g(z)$$  \hspace{1cm} (7)

where $g(z)$ is a meromorphic function which is fixed by the boundary condition and the absence of singularities of $N_z$ for finite $z$.

In Section 3 and 4 we shall give a general discussion of eq.(4), while we shall come back to 2+1 dimensional gravity in Section 5.

### 3 The conformal factor

In a series of papers at the turn of the past century Picard [9] proved that the following equation

$$4\partial_z \partial_{\bar{z}} \phi = e^\phi + 4\pi \sum_n g_n \delta^2(z - z_n)$$  \hspace{1cm} (8)

for real $\phi$ with asymptotic behavior at infinity

$$\phi(z) = -g_\infty \ln(z\bar{z}) + O(1)$$  \hspace{1cm} (9)

and $-1 < g_n$, $1 < g_\infty$ (which excludes the case of punctures) and $\sum_n g_n + g_\infty < 0$ admits one and only one solution (see also [10]). Picard [9] achieved the solution of (8) through an iteration process exploiting Schwarz alternating procedure. The same problem has been reconsidered with variational techniques by Lichtenstein [11] and more recently by Troyanov [12], obtaining results which include Picard’s findings. The interest of such results is that they solve the following variant of the Riemann-Hilbert problem: at $z_1, \ldots, z_n$ we are given not with the monodromies but with the class, characterized by $g_j$, of the elliptic monodromies with the further request that all such monodromies belong to the group $SU(1,1)$. The last requirement is imposed by the fact that the solution of eq.(8) has to be single valued. The inequalities on the values of $g_m = -1 + \mu_m$ and $g_\infty = 2 - \mu$ are satisfied in 2+1 dimensional gravity due to the restriction on the masses of the particles, $0 < \mu_n < 1$ (in rationalized
Planck units) and to the fact that the total energy $\mu$ must satisfy the bound $\sum_n \mu_n < \mu < 1$. In this paper we shall confine ourselves to the Riemann sphere.

From eq. (8) one can easily prove [10, 13] that the function $Q(z)$ defined by

$$e^{\frac{\phi}{2}} \partial_z^2 e^{-\frac{\phi}{2}} = -Q(z) \tag{10}$$

is analytic i.e. as pointed out in [13] $Q(z)$ is given by the analytic component of the energy momentum tensor of a Liouville theory. $Q(z)$ is meromorphic with poles up to the second order [14] i.e. of the form

$$Q(z) = \sum_n -\frac{g_n(g_n+2)}{4(z-z_n)^2} + \frac{\beta_n}{2(z-z_n)}. \tag{11}$$

All solutions of eq. (8) can be put in the form

$$e^\phi = \frac{8f^t \bar{f}^t}{(1 - ff)^2} = \frac{8|w_{12}|^2}{(y_2 \bar{y}_2 - y_1 \bar{y}_1)^2}, \quad f(z) = \frac{y_1}{y_2} \tag{12}$$

being $y_1, y_2$ two properly chosen, linearly independent solutions of the fuchsian equation

$$y'' + Q(z)y = 0. \tag{13}$$

$w_{12}$ is the constant wronskian. In fact following [10, 13] as $e^{-\phi/2}$ solves the fuchsian equation (13) it can be put in the form

$$e^{-\frac{\phi}{2}} = \frac{1}{\sqrt{8}}[\psi_2(z)\bar{\chi}_2(\bar{z}) - \psi_1(z)\bar{\chi}_1(\bar{z})] \tag{14}$$

with $\psi_j(z)$ solutions of eq. (13) with wronskian 1 and $\chi_j(z)$ also solutions of eq. (13) with wronskian 1. The solution of eq. (8) ($\phi = \text{real}$) with the stated behavior at infinity is unique [14, 11, 12]. Exploiting the reality of $e^\phi$ it is possible by an $SL(2C)$ transformation to reduce eq. (14) to the form eq. (12). In fact, being $\chi_j$ linear combinations of the $\psi_j$, the reality of $e^\phi$ imposes

$$\psi_2(z)\bar{\chi}_2(\bar{z}) - \psi_1(z)\bar{\chi}_1(\bar{z}) = \sum_{jk} \bar{\psi}_j H_{jk} \psi_k \tag{15}$$
with the $2 \times 2$ matrix $H_{jk}$ hermitean and $\det H = -1$. By means of a unitary transformation, which belongs to $SL(2\mathbb{C})$ we can reduce $H$ to diagonal form $\text{diag}(-\lambda, \lambda^{-1})$ and with a subsequent $SL(2\mathbb{C})$ transformation we can reduce it to the form $\text{diag}(-1, 1)$ i.e. to the form $(12)$ where we relaxed the condition on the wronskian $w_{12} = 1$. Through eq.(10) $\phi$ contains the full information about the accessory parameters $\beta_n$ defined in eq.(11). It is important to notice that being all of our monodromies elliptic, we can by means of an $SU(1, 1)$ transformation and by multiplying $y_1$ and $y_2$ by a common factor, choose around a given singularity $z_n$ (not around all singularities simultaneously) $y_1$ and $y_2$ with the following canonical behavior

$$y_1(\zeta) = k_n \zeta^{\frac{g_n}{2}+1} A(\zeta), \quad y_2(\zeta) = \zeta^{-\frac{g_n}{2}} B(\zeta)$$

(16)

with $\zeta = z - z_n$ and $A$ and $B$ analytic functions of $\zeta$ in a neighborhood of 0 with $A(0) = B(0) = 1$. So the most general form for $e^\phi$, compatible with the reality condition and the monodromy condition around $z_n$ is

$$e^\phi = \frac{8k_n^2(g_n+1)^2(\zeta\bar{\zeta})^{g_n}}{|BB - k_n^2(\zeta\bar{\zeta})^{g_n+1}AA|^2}$$

(17)

and the real parameter $k_n$ is fixed by the imposition of the monodromy condition around the other singularities.

If we are in presence of parabolic singularities (i.e. punctures) the Liouville equation has the form

$$4\partial_z \partial_{\bar{z}} \phi = e^\phi - 4\pi \sum_{n=1}^{N} \delta^2(z - z_n),$$

(18)

with asymptotic behavior at infinity

$$\phi = -2 \ln(z\bar{z}) + O(1).$$

(19)

We recall that Picard’s theorem about existence and uniqueness of the solution of Liouville equation, requires the following condition on the parameters $g_n, g_\infty$

$$g_n > -1, \quad g_\infty > 1, \quad \sum_{n=1}^{N} g_n + g_\infty < 0.$$
Now \( g_n = -1 \), so we cannot appeal to Picard’s findings in order to assure existence and uniqueness of solution of eq.\( \text{(18)} \); however the problem of solving Liouville equation with parabolic singularities is equivalent to the problem of uniformization of Riemann surfaces, which has been solved by Poincarè and Koebe at the beginning of the last century \cite{15}. Again one can write

\[
e^{-\frac{\phi}{2}} = \frac{1}{\sqrt{8}}[\psi_2 \bar{\psi}_2 - \psi_1 \bar{\psi}_1].
\] (20)

The most important difference between the case of elliptic and that of parabolic singularities is that, while eq.\( \text{(13)} \) in the elliptic case leads to an indicial equation, which admits two different solution, in the parabolic case the indicial equation is

\[
\alpha(\alpha - 1) = -\frac{1}{4}
\]

i.e. we have a double solution \( \alpha = \frac{1}{2} \). In this way around the singularity \( z_n \) we have a solution with expansion

\[
\zeta^{\frac{1}{2}} B(\zeta) = \zeta^{\frac{1}{2}}(1 + c_2 \zeta + \ldots), \quad \text{with} \quad \zeta = z - z_n, \quad \text{and} \quad c_2 = -\frac{\beta_n}{2}
\] (21)

and a solution which contains a singularity of logarithmic type

\[
\zeta^{\frac{1}{2}} B(\zeta) \ln(\zeta) + \zeta^{\frac{1}{2}} A_0(\zeta)
\] (22)

where \( A_0(\zeta) \) is the solution of the inhomogeneous fuchsian differential equation

\[
\partial_\zeta^2 (\zeta^{\frac{1}{2}} A_0) + Q(\zeta)(\zeta^{\frac{1}{2}} A_0) = \frac{1}{\zeta^2}(\zeta^{\frac{1}{2}} B) - \frac{2}{\zeta} \partial_\zeta(\zeta^{\frac{1}{2}} B)
\] (23)

and expansion

\[
A_0(\zeta) = (a_1 \zeta + a_2 \zeta^2 + a_3 \zeta^3 + \ldots).
\] (24)

In the following we shall call

\[
Y_1(\zeta) = \zeta^{\frac{1}{2}} B(\zeta), \quad Y_2(\zeta) = -\zeta^{\frac{1}{2}}[B(\zeta) \ln \zeta + A_0(\zeta)]
\] (25)
whose wronskian is \( w_{12} = 1 \). In general a pair of linearly independent solutions of eq. (13) with wronskian equal to 1 is

\[
\psi_1 = aY_1 + bY_2, \quad \psi_2 = cY_1 + dY_2 \text{ with } ad - bc = 1.
\]

The most general form for \( e^{-\hat{\phi}/2} \) compatible with the reality condition will be

\[
e^{-\hat{\phi}/2} = \frac{|\zeta|}{\sqrt{8}} \left[ (cB(\zeta) - dB(\zeta) \ln \zeta - dA_0(\zeta)) \times (c.c.) - (aB(\zeta) - bB(\zeta) \ln \zeta - bA_0(\zeta)) \times (c.c.) \right].
\]

If we impose the monodromy condition around \( z_n \), then we reach the conditions \( \text{Im}(ab - cd) = 0, |b| = |d| \) which together with the condition \( ad - bc = 1 \) imply \( b = \pm d \) and \( ab - cd = \pm 1 \). The positivity of \( e^{-\hat{\phi}/2} \) chooses the solution with the minus sign. Thus the most general form for \( e^{-\hat{\phi}/2} \) compatible with the monodromy condition around \( z_n \) is

\[
e^{-\hat{\phi}/2} = -\frac{|\zeta|}{\sqrt{8}} \left[ B\bar{A}_0 + \bar{B}A_0 + B\bar{B} \ln \frac{\zeta}{k_n^2} \right].
\]

Eq. (13) can be interpreted as the differential equation related to a variant of the Riemann-Hilbert problem, in which at \( z_1, \ldots z_N \) we are given not with the monodromy but with the class of the monodromy, with the further request that all such monodromies belong to the group \( SU(1,1) \). The last requirement, which is equivalent to the requirement that \( e^\phi \) is a monodromic function, fixes in principle not only \( k_n^2 \) but also the values of the accessory parameters \( \beta_n \).

4 The realization of fuchsian SU(1,1) monodromies

The results of Picard assure us that given the position of the singularities \( z_n \) and the classes of monodromies characterized by the real numbers \( g_n \) and \( g_\infty \) subject to the restrictions \(-1 < g_n, 1 < g_\infty \) and \( \sum_n g_n + g_\infty < 0 \), there exists a unique fuchsian equation which realizes \( SU(1,1) \) monodromies of the prescribed classes. In particular the uniqueness of the solution of Liouville equation tells us that the accessory parameters \( \beta_i \) are single valued functions of
the parameter $z_n$ and $g_n$. We shall examine in this section how such dependence arises from the viewpoint of the imposition of the $SU(1, 1)$ condition on the monodromies in order to understand the nature of the dependence of the $\beta_i$ on the $g_n$ and on the $z_n$. The (non trivial) proof of the real analytic dependence of the accessory parameters on the $z_n$ for finite order elliptic singularities has been given by Kra [16].

Starting from the singularity in $z_1$ we can consider the canonical pair of solutions around $z_1$ i.e. those solutions which behave as a single fractional power multiplied by an analytic function with leading coefficient 1. We shall call such pair of solutions $(y_1^1, y_2^1)$ and let $(y_1, y_2)$ the solution which realize $SU(1, 1)$ around all singularities. Obviously all conjugations with any element of $SU(1, 1)$ is still an equivalent solution in the sense that they provide the same conformal factor $\phi$. The canonical pairs around different singularities are linearly related i.e. $(y_1^1, y_2^1) = (y_1^2, y_2^2)C_{21}$. We fix the conjugation class by setting

$$(y_1, y_2) = (y_1^1, y_2^1)K$$

with $K = \text{diag}(k, 1)$ being the overall constant irrelevant in determining $\phi$. Moreover if the solution $(y_1, y_2)$ realizes $SU(1, 1)$ monodromies around all singularities also $(y_1, y_2) \times \text{diag}(e^{i\alpha}, e^{-i\alpha})$ accomplishes the same purpose being $\text{diag}(e^{i\alpha}, e^{-i\alpha})$ an element of $SU(1, 1)$. Thus the phase of the number $k$ is irrelevant and so we can consider it real and positive. This choice of the canonical pairs is always possible in our case. In fact the roots of the indicial equation are $-\frac{g_m}{2}$ and $\frac{g_m}{2} + 1$ and thus the monodromy matrix has eigenvalues $e^{-i\pi g_m}$ and $e^{i\pi g_m}$ which are different when $g_m$ is not an integer. If $g_m$ is an integer in general in the solution of the fuchsian equation the less singular solution possesses a logarithmic term which however has to be absent in our case (no logarithm condition [17]) in order to have a single valued $\phi$. In this case the monodromy matrix is simply the identity or minus the identity. The monodromy around $z_1$ thus belongs to $SU(1, 1)$ for any choice of $K$. If $D_n$ denote the diagonal monodromy matrices around $z_n$, we have that the monodromy around $z_1$ is $D_1$ and the one around $z_2$ is

$$M_2 = K^{-1}C_{12}D_2C_{21}K$$

(30)
where with $C_{12}$ we have denoted the inverse of the $2 \times 2$ matrix $C_{21}$.

In the case of three singularities (one of them at infinity) the counting of the degrees of freedom is the following: by using the freedom on $K$ we can reduce $M_2$ to the form

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
$$

with $\text{Re } b = \text{Re } c$, or if either $\text{Re } b$ or $\text{Re } c$ is zero we can obtain $\text{Im } b = -\text{Im } c$. Then we use the fact that $D_1 M_2 = C D_\infty C^{-1}$ and thus in addition to $a + d = \text{real}$ we have also $ae^{i\pi g_1} + de^{-i\pi g_1} = \text{real}$, which gives $d = \bar{a}$ and thus using $a\bar{a} - bc = 1$ we have $c = \bar{b}$. The fact that a real $k$ is sufficient to perform the described reduction of the matrix $M_2$ is assured by Picard’s result on the solubility of the problem and in this simple case also by the explicit solution in terms of hypergeometric functions \cite{7, 8}.

We give now a qualitative discussion of the case with four singularities and then give the analytic treatment of it. The case with more than four singularities is a trivial extension of the four singularity case. The following treatment relies heavily on Picard’s result about the existence and uniqueness of the solution of eq.\cite{8}. We recall that the accessory parameters $\beta_n$ are bound by two algebraic relations known as Fuchs relations \cite{L}. Thus after choosing $M_1$ of the form $M_1 = D_1$, in imposing the $SU(1,1)$ nature of the remaining monodromies we have at our disposal three real parameters i.e. $k$, $\text{Re } \beta_3$ which we shall denote by $\beta_R$ and $\text{Im } \beta_3$ which we shall denote by $\beta_I$. It is sufficient to impose the $SU(1,1)$ nature of $M_2$ and $M_3$ as the $SU(1,1)$ nature of $M_\infty$ is a consequence of them. As the matrices $M_n = K^{-1} C_{1n} D_n C_{n1} K$ satisfy identically $\det M_n = 1$ and $\text{Tr } M_n = 2 \cos \pi g_n$ we need to impose generically on $M_2$ only two real conditions e.g. $\text{Re } b_2 = \text{Re } c_2$ and $\text{Im } b_2 = -\text{Im } c_2$. The same for $M_3$. Thus it appears that we need to satisfy four real relations when we can vary only three real parameters. The reason why we need only three and not four is that for any solution of the fuchsian problem the following relation among the monodromy matrices is identically satisfied

$$
D_1 M_2 M_3 M_\infty = 1. 
$$

Rigorously the conditions for realizing $SU(1,1)$ monodromies are

$$
\text{Re } a_i = \text{Re } d_i, \quad \text{Im } a_i = -\text{Im } d_i, \quad \text{Re } b_i = \text{Re } c_i, \quad \text{Im } b_i = -\text{Im } c_i \quad (i = 2, 3)
$$

(32)
Through a projective transformation we can bring three singularities in 0, 1 and \( \infty \) and we shall call \( z_1 \) the position of the remaining singularity. We shall denote the above eight relations by \( \Delta^{(i)}(\beta_R, \beta_I, k, z_1) = 0 \) \( (i = 1 \ldots 8) \). Satisfying the eight above equations is a sufficient (and necessary) condition to realize the \( SU(1,1) \) monodromies.

The matrices \( A_n = C_{n1} K \) which give the solution of the problem in terms of the canonical solutions around the singularities are completely determined by the two equations

\[
(y_1, y_2) = (y_1^{(n)}, y_2^{(n)})A_n; \quad (y_1', y_2') = (y_1^{(n)'}, y_2^{(n)'} )A_n
\]

(33)

due to the non vanishing of the wronskian of \( y_1^{(n)}, y_2^{(n)} \). Being \( (y_1, y_2) \) solutions of a fuchsian equation, \( A_n \) depend analytically on \( \beta_R, \beta_I, z_1 \). It follows that \( \Delta^{(i)} \) are analytic functions of the independent variables \( \beta_R, \beta_I, z_1 \). Thus the equations \( \Delta^{(i)} = 0 \) which determine implicitly \( k, \beta_R \) and \( \beta_I \) state the vanishing of the real analytic functions \( \Delta^{(i)} \).

In Appendix 1 it is shown that for \( z_1 \) varying in a disk lying in the domain \( X_\epsilon \), given by the complex plane from which small disks of radius \( \epsilon \) around the singularities \( z_n \) in addition to the region \( |z| > 1/\epsilon \) have been removed, \( \beta_R, \beta_I \) and \( k \) are bounded function of \( z_1 \). From the uniqueness of Picard’s solution and the continuity of the \( \Delta^{(i)} \) it follows that the \( \beta_R, \beta_I, k \) are continuous functions of \( z_1 \) in \( X_\epsilon \). We come now the existence of the derivative of \( \phi \) with respect to \( z_n \). Due to the real analytic dependence of \( \phi \) on \( \beta_R, \beta_I, k \) it is enough to prove the existence of the derivative of such quantities with respect to \( \text{Re} z_1 \) and \( \text{Im} z_1 \). Given a value \( z_{10} \) in \( X_\epsilon \), let us consider the Picard solution \( \beta_R(z_{10}), \beta_I(z_{10}), k(z_{10}) \). We have

\[
\Delta^{(i)}(\beta_R(z_{10}), \beta_I(z_{10}), k(z_{10}), z_{10}) = 0.
\]

The functions of \( \beta_R \Delta^{(i)}(\beta_R, \beta_I(z_{10}), k(z_{10}), z_{10}) \) cannot all be identically zero in a neighborhood of \( \beta_R(z_{10}) \) otherwise Picard’s solution would be not unique. Let be \( \Delta^{(1)}(\beta_R, \beta_I(z_{10}), k(z_{10}), z_{10}) \) not identically zero in \( \beta_R \) in a neighborhood of \( \beta_R(z_{10}) \). Then we can apply to it Weierstrass preparation theorem \[18\] and write in a neighborhood of \( \beta_R(z_{10}), \beta_I(z_{10}), k(z_{10}), z_{10} \)

\[
\Delta^{(1)}(\beta_R, \beta_I, k, z_1) = u P(\beta_R|\beta_I, k, \text{Re}(z_1), \text{Im}(z_1))
\]

(34)

where \( u \) is a unit and \( P \) is a monic polynomial in \( \beta_R \) and coefficients analytic functions of the remaining variables. In the Weierstrass neighborhood all the zeros of \( \Delta^{(1)} \) are zeros of \( P \) and viceversa. In addition we have \( P(\beta_R(z_{10})|\beta_I(z_{10}), k(z_{10}), \text{Re}(z_{10}), \text{Im}(z_{10})) = 0. \)
a) If \( P'(\beta R(z_{10}), \beta_I(z_{10}), k(z_{10}), \Re(z_{10}), \Im(z_{10})) \neq 0 \) we can solve for \( \beta_R \) and obtain \( \beta_R = \beta_R(\beta_I, k, \Re(z_1), \Im(z_1)) \) in the Weierstrass neighborhood of \( \beta_R(z_{10}), \beta_I(z_{10}), k(z_{10}), z_{10} \).

b) If \( P'(\beta R(z_{10}), \beta_I(z_{10}), k(z_{10}), \Re(z_{10}), \Im(z_{10})) = 0 \) but \( P'(\beta R(z_{1}), \beta_I(z_{1}), k(z_{1}), \Re(z_{1}), \Im(z_{1})) \) not identically zero in a neighborhood of \( z_{10} \) then we can solve \( \beta_R = \beta_R(\beta_I, k, \Re(z_1), \Im(z_1)) \) in a dense open set around \( z_{10} \).

c) If \( P'(\beta R(z_1), \beta_I(z_1), k(z_1), \Re(z_1), \Im(z_1)) \) is identically zero in a neighborhood of \( z_{10} \) then we consider \( P''(\beta_R|\beta_I, k, \Re(z_1), \Im(z_1)) \) and proceed as above.

Being the Weierstrass polynomial monic the process in c) ends in a finite number of steps with the result that we are able to solve \( \beta_R \) as an analytic function of \( \beta_I, k \) and \( z_1 \) for \( z_1 \) in a dense open set around \( z_{10} \).

One can substitute the obtained \( \beta_R \) in the \( \Delta^{(i)} \) which become analytic functions of only \( \beta_I, k, \Re(z_1), \Im(z_1) \). Then one eliminates by the same procedure \( \beta_I \) and then \( k \) with the final result that \( \beta_R, \beta_I \) and \( k \) are real analytic functions of \( \Re(z_1), \Im(z_1) \) in a everywhere dense open set of \( R^2 \).

The procedure is immediately extended to any number of singularities and also to the case of parabolic singularities. Our result is not as strong as the one given by Kra \[16\] in the special case of elliptic singularities of finite order and the one given by Zograf and Takhtajan \[3\] in the case of parabolic singularities.

Similarly one can deal with the dependence of the \( \beta_n \) and \( k^2 \) as functions of the \( g_n \).

5 Proof of the hamiltonian nature of 2+1 dimensional gravity

In this section we shall analyze the asymptotic behavior of the solution of the Liouville equation \( \Phi \) in the case of elliptic singularities, proving several useful relations, in particular we shall prove that the derivative of the constant term of the expansion of \( \phi \) at infinity with respect of the position of a singularity \( z_n \) equals the value of the derivative of the auxiliary parameters \( \beta_n \) related with the same singularity with respect to the parameter \( g_\infty \).
First of all we examine the expansion of \( \phi \) around the singularity \( z_n \)

\[
\phi(z) = g_n \ln(z - z_n)(\bar{z} - \bar{z}_n) + r_n(z)
\]

being \( r_n(z) \) a continuous function in a finite neighborhood of \( z_n \) and at infinity

\[
\phi\left(\frac{1}{w}\right) = -g_\infty \ln\left(\frac{1}{w\bar{w}}\right) + r_\infty(w)
\]

being \( r_\infty(w) \) a continuous function in a finite neighborhood of \( w = 0 \).

To make explicit the form of \( r_n(z) \) we can compute the coefficients in the expansions

\[
A = 1 + c_1 \zeta + O(\zeta^2) \quad \text{and} \quad B = 1 + c_2 \zeta + O(\zeta^2),
\]

with \( \zeta = z - z_n \), which are known from the fuchsian differential equation

\[
c_1 = -\frac{\beta_n}{2(2 + g_n)} \quad \text{and} \quad c_2 = \frac{\beta_n}{2g_n}.
\]

Then we can substitute the expansion of \( A \) and \( B \) into eq. (17), where \( k_n \) is fixed by the global requirement of the \( SU(1, 1) \) nature of the monodromies.

By taking the logarithm of (17) we have

\[
\phi = g_n \ln \zeta \bar{\zeta} + \ln 8k_n^2(g_n + 1)^2 - 2 \ln \left(1 + c_2 \zeta + \cdots |^2 - k_n^2(\zeta \bar{\zeta})^g_{n+1}1 + c_1 \zeta + \cdots |^2\right) =
\]

\[
= g_n \ln \zeta \bar{\zeta} - \ln s_n^2 - 2(c_2 \zeta + \bar{c}_2 \bar{\zeta}) + O(|\zeta|^2) +
\]

\[
+ 2k_n^2(\zeta \bar{\zeta})^{g_n+1}(1 + O(|\zeta|)) + O(|\zeta|^{4(g_n+1)})
\]

where we designed by \(- \ln s_n^2\) the constant term in the expansion and \( s_n^2 = 1/8k_n^2(g_n + 1)^2 \).

Similarly at infinity we have

\[
\phi = -g_\infty \ln z \bar{z} - \ln s_\infty^2 - 2 \ln \left(1 + \frac{c_2}{\bar{z}} + \cdots |^2 - k_\infty^2(zz)^{1-g_\infty}|1 + \frac{c_1}{z} + \cdots |^2\right) =
\]

\[
= -g_\infty \ln z \bar{z} - \ln s_\infty^2 - 2\left(\frac{c_2}{\bar{z}} + \frac{\bar{c}_2}{\bar{z}}\right) + O\left(\frac{1}{|z|^2}\right) +
\]

\[
+ 2k_\infty^2(z \bar{z})^{1-g_\infty} \left(1 + O\left(\frac{1}{|z|}\right)\right) + O((z \bar{z})^{2-2g_\infty}).
\]
We shall now prove the following result

\[ \frac{\partial \ln s_\infty^2}{\partial z_n} = \frac{\partial \beta_n}{\partial g_\infty}. \]  

(40)

In order to do so let us consider an elliptic equation of the form

\[ \partial_z \partial_{\bar{z}} \phi = F(\phi) \]  

(41)

and let \( \phi(z,v_1,\ldots,v_K) \) be a family of solutions of eq.(41) depending on the parameters \( v_1, \ldots, v_K \) in some given domain \( D \) of the complex plane. By taking the derivative of eq.(41) with respect to \( v_i \) we have

\[ \partial_z \partial_{\bar{z}} \frac{\partial \phi}{\partial v_i} = F'(\phi) \frac{\partial \phi}{\partial v_i}. \]  

(42)

Then from eq.(42) we have in \( D \)

\[ \partial_z \left( \frac{\partial \phi}{\partial v_j} \partial_{\bar{z}} \left( \frac{\partial \phi}{\partial v_i} \right) \right) - \partial_{\bar{z}} \left( \frac{\partial \phi}{\partial v_i} \partial_z \left( \frac{\partial \phi}{\partial v_j} \right) \right) = 0. \]  

(43)

We shall now apply eq.(43) to the inhomogeneous Liouville equation

\[ 4\partial_z \partial_{\bar{z}} \phi = e^\phi + 4\pi \sum_n g_n \delta^2(z - z_n) \]  

(44)

in a domain which excludes the sources. The solutions of eq.(44) depend on the parameters \( z_n, g_n, g_\infty \) which now play the role of the parameters \( v_i \). We apply eq.(43) choosing as first parameter \( g_\infty \) and second parameter \( z_n \), specifying the domain \( X_\epsilon \) as a disk of radius \( R \) which includes all singularities, from which disks of radius \( \epsilon \) have been removed. Using Stokes’ theorem we obtain

\[
0 = \oint_R \left( \frac{\partial \phi}{\partial z_n} \partial_z \left( \frac{\partial \phi}{\partial g_\infty} \right) \right) \frac{i}{2} dz + \left( \frac{\partial \phi}{\partial g_\infty} \partial_z \left( \frac{\partial \phi}{\partial z_n} \right) \right) \frac{i}{2} d\bar{z} - \\
\sum_l \oint_{\gamma_l} \left( \frac{\partial \phi}{\partial z_n} \partial_z \left( \frac{\partial \phi}{\partial g_\infty} \right) \right) \frac{i}{2} dz + \left( \frac{\partial \phi}{\partial g_\infty} \partial_z \left( \frac{\partial \phi}{\partial z_n} \right) \right) \frac{i}{2} d\bar{z}.
\]  

(45)
Let us consider the first integral; the first term gives in the limit $R \to \infty$

$$-\pi \frac{\partial \ln s^2}{\partial z_n} \tag{46}$$

while the second

$$R \ln R^2 \ O\left(\frac{1}{R^2}\right) \to 0. \tag{47}$$

We consider now the contribution of the integral around the circle of center $z_n$ and radius $\epsilon_n$; we have for the first term

$$- \oint_{\gamma_n} \left( \frac{\partial \phi}{\partial z_n} \frac{\partial \phi}{\partial g_\infty} \right) \frac{i}{2} dz = \tag{48}$$

$$- \oint i \frac{d\zeta}{2} \left( -\frac{g_n}{\zeta} + \frac{\beta_n}{g_n} - \frac{\partial \ln s^2}{\partial z_n} + O(|\zeta|) + \frac{O(|\zeta| g + 1)}{\zeta} \right) \times$$

$$\left( - \frac{1}{g_n} \frac{\partial \beta_n}{\partial g_\infty} + 2(g_n + 1) \frac{\partial k_n^2}{\partial g_\infty} \frac{(\zeta \bar{\zeta})^{g_n + 1}}{\zeta} + \ldots \right) \to \pi \frac{\partial \beta_n}{\partial g_\infty}$$

where the terms like

$$\oint i \frac{d\zeta}{2} \left( -\frac{g_n}{\zeta} \right) \times 2(g_n + 1) \frac{\partial k_n^2}{\partial g_\infty} \frac{(\zeta \bar{\zeta})^{g_n + 1}}{\zeta} \tag{49}$$

which do not vanish by power counting are identically zero by the phase integration.

Similarly

$$\oint_{\gamma_n} \left( \frac{\partial \phi}{\partial g_\infty} \frac{\partial \phi}{\partial z_n} \right) \frac{i}{2} d\bar{z} = \tag{50}$$

$$\oint i \frac{d\zeta}{2} \left( -\frac{\partial \ln s^2}{\partial g_\infty} - \frac{1}{g_n} \frac{\partial \beta_n}{\partial g_\infty} \zeta - \frac{1}{g_n} \frac{\partial \beta_n}{\partial g_\infty} \bar{\zeta} + \ldots \right) \left( -2 \frac{\partial \bar{c}_2}{\partial z_n} + \ldots \right) \to 0$$

thus obtaining

$$\frac{\partial \ln s^2}{\partial z_n} = \frac{\partial \beta_n}{\partial g_\infty}. \tag{51}$$

Relation (51) is of fundamental importance in canonical 2+1 dimensional gravity; in fact the hamiltonian nature of particles dynamics can be proved by means of this relation, and
the hamiltonian is just \( \ln s^2_\infty \). In fact we recall that the equations of motions in the rotating frame are (see [1])

\[
\dot{z}'_n = -\sum_B \frac{\partial \beta_B}{\partial \mu} \frac{\partial z'_B}{\partial P'_n} \quad n = 2, \ldots \mathcal{N}
\]

and

\[
\dot{P}'_n = \frac{\partial \beta_n}{\partial \mu} + \sum_B \frac{\partial \beta_B}{\partial \mu} \frac{\partial z'_B}{\partial z'_n} \quad n = 2, \ldots \mathcal{N},
\]

where the indices \( B \) label the so-called apparent singularities, \( n \) label the particles and \( z'_n = z_n - z_1 \) and \( P'_n = P_n (n = 2, \ldots, \mathcal{N}) \). The previous equations take the form

\[
\dot{z}'_n = \sum_B \frac{\partial \ln s^2_\infty}{\partial z_B} \frac{\partial z'_B}{\partial P'_n} = \frac{\partial \ln s^2_\infty}{\partial P'_n} \quad n = 2, \ldots \mathcal{N}
\]

and

\[
\dot{P}'_n = -\frac{\partial \ln s^2_\infty}{\partial z'_n} |_{z'_n} - \sum_B \frac{\partial \ln s^2_\infty}{\partial z'_B} \frac{\partial z'_B}{\partial z'_n} = -\frac{\partial \ln s^2_\infty}{\partial z'_n} \quad n = 2, \ldots \mathcal{N}.
\]

So \( \ln s^2_\infty \) is the hamiltonian for the \( \mathcal{N} \) particle system. As \( \ln s^2_\infty \) is a single valued function of the arguments we have that the hamiltonian is a single valued function on the phase space and not a section.

### 6 Proof of Polyakov conjecture for general elliptic singularities

In this section we shall give a proof of Polyakov conjecture for general elliptic singularities following the path outlined in the previous paragraph, generalizing relation (51) to all \( g_n \) and \( \ln s^2_m \). This results in a weak form of Polyakov relation. From the weak form the strong form is immediately obtained. This method provides the simplest proof of Polyakov conjecture for elliptic singularities.
We exploit the relations that can be found applying eq.(45) choosing the couples of parameters \((z_n, g_m), (g_n, g_m), (z_n, z_m)\). Let us start with \((z_n, g_m)\) (and \(n \neq m\)). The only surviving contributions are those at \(z_n\) and \(z_m\), the first giving

\[
\oint_{\gamma_n} \left( \frac{\partial \phi}{\partial z_n} \frac{\partial \phi}{\partial g_m} \right) i \frac{dz}{2} = \oint_{\gamma} \frac{i}{2} d\zeta \left( -\frac{g_n}{\zeta} + \ldots \right) \left( -\frac{1}{g_n} \frac{\partial \beta_n}{\partial g_m} + \ldots \right) \to -\pi \frac{\partial \beta_n}{\partial g_m}
\]

and

\[
\oint_{\gamma_m} \left( \frac{\partial \phi}{\partial g_m} \frac{\partial \phi}{\partial z_n} \right) i \frac{dz}{2} \to 0.
\]

Around the circle \(z_m\) the integration gives

\[
\oint_{\gamma_m} \left( \frac{\partial \phi}{\partial z_n} \frac{\partial \phi}{\partial g_m} \right) i \frac{dz}{2} = \oint_{\gamma} \frac{i}{2} dz \left( -\frac{\partial \ln s_m^2}{\partial z_n} + \ldots \right) \left( \frac{1}{\zeta} + \ldots \right) \to \pi \frac{\partial \ln s_m^2}{\partial z_n}
\]

while

\[
\oint_{R} \left( \frac{\partial \phi}{\partial g_m} \frac{\partial \phi}{\partial z_n} \right) i \frac{dz}{2} \to 0.
\]

In this way we have reached

\[
\frac{\partial \ln s_m^2}{\partial z_n} = \frac{\partial \beta_n}{\partial g_m}.
\]

For \(n = m\) both contributions come from the circle of center \(z_m\) but the result is the same, that is

\[
\frac{\partial \ln s_m^2}{\partial z_m} = \frac{\partial \beta_m}{\partial g_m}.
\]

When in eq.(45) we take the derivative with respect to \(g_m\) and \(g_n\) of \(\phi\) we reach

\[
\frac{\partial \ln s_m^2}{\partial g_n} = \frac{\partial \ln s_n^2}{\partial g_m}.
\]
and when the chosen parameters are \( z_n \) and \( z_m \) we reach the relation
\[
\frac{\partial \beta_m}{\partial z_n} = \frac{\partial \beta_n}{\partial z_m}.
\] (63)

We can summarize the previous results by stating that the form \( \omega \) defined by
\[
\omega = \sum_n \beta_n dz_n + \sum_n \ln s_n^2 dg_n + c.c.
\] (64)
is closed. Finally we observe that even the derivative \( \frac{\partial \phi}{\partial z} \) satisfies in \( D \) the linear equation
\[
4 \partial_x \partial_z \frac{\partial \phi}{\partial z} = F'(\phi) \frac{\partial \phi}{\partial z}
\] (65)
but if we study the relation
\[
0 = \oint_R \left( \frac{\partial \phi}{\partial z} \frac{\partial \phi}{\partial v} \right) i \frac{1}{2} dz + \left( \frac{\partial \phi}{\partial v} \frac{\partial \phi}{\partial z} \right) i \frac{1}{2} d\bar{z}
\]
\[
- \sum_i \oint_{\gamma_i} \left( \frac{\partial \phi}{\partial z} \frac{\partial \phi}{\partial v} \right) i \frac{1}{2} dz + \left( \frac{\partial \phi}{\partial v} \frac{\partial \phi}{\partial z} \right) i \frac{1}{2} d\bar{z}
\] (66)
with \( \nu = g_\infty, g_n, z_n \) we find simply the vanishing of the derivative of the well-known first Fuchs relation \( \sum_n \beta_n = 0 \), with respect to \( \nu \).

We shall now relate the previous results to the regularized Liouville action [19]
\[
S_P[\phi] = \lim_{\epsilon \to 0} S_\epsilon[\phi]
\] (67)
where
\[
S_\epsilon[\phi] = \frac{i}{2} \int_{X_\epsilon} (\partial_z \phi \partial_{\bar{z}} \phi + \frac{\epsilon \phi}{2}) dz \wedge d\bar{z} + \frac{i}{2} \sum_n g_n \oint_n \phi \left( \frac{d\bar{z}}{\bar{z} - \bar{z}_n} - \frac{dz}{z - z_n} \right)
\]
\[
+ \frac{i}{2} g_\infty \oint_\infty \phi \left( \frac{d\bar{z}}{\bar{z}} - \frac{dz}{z} \right) - \pi \sum_n g^2_n \ln \epsilon^2 - \pi g^2_\infty \ln \epsilon^2.
\] (68)

\( X_\epsilon \) is a disk of radius \( 1/\epsilon \) which includes all singularities, from which disks of radius \( \epsilon \) around each singularity have been removed. The variation of such action provides the inhomogeneous Liouville equation [3]. The contour terms impose the correct behavior of the \( \phi \) on the
singularities and at infinity. We shall now compute the derivative with respect to \( g_m \) of \( S_\epsilon \) calculated on the solutions of eq.(8). As the action is stationary on the solution of eq.(8) the only contribution is due to the term in eq.(68) which explicitly depend on \( g_m \). A simple computation gives

\[
\frac{\partial S_P}{\partial g_m} = -2\pi \ln s_m^2.
\]  

(69)

Putting together the results of eqs.(60,69) we have

\[
-\frac{1}{2\pi} dS_P [\phi] = \sum_n \frac{\partial \beta_n}{\partial g_m} dz_n + c.c.
\]  

(70)

from which

\[
-\frac{1}{2\pi} dS_P [\phi] = \sum_n \beta_n dz_n + c.c. + F
\]  

(71)

where

\[
F = \sum_n f_n dz_n + c.c.
\]  

(72)

with \( f_n \) dependent on \( z_n \) but not on \( g_n \).

We prove now that \( F = 0 \). To this end we shall take one of the parameters \( g_n \) e.g. \( g_1 \) to zero. Recalling the expansion of \( \phi \) around \( z_n \)

\[
\phi = g_1 \ln \zeta \bar{\zeta} + O(1) - 2c_2 \zeta - 2\bar{c}_2 \bar{\zeta} + 2k_1^2 (\zeta \bar{\zeta})^{g_1+1} + \cdots
\]  

(73)

we can compute \( Q(z) \)

\[
Q(z) = \frac{1}{2} \partial_z^2 \phi - \frac{1}{4} (\partial_z \phi)^2 = -\frac{g_1 (g_1 + 2)}{4\zeta^2} + \frac{g_1 c_2}{\zeta} + \cdots
\]  

(74)

from which \( \beta_1 = 2g_1 c_2 \). For \( g_1 \to 0 \) both terms disappear from \( Q(z) \) and \( g_1 \) disappears from the r.h.s. of eq.(8) as well (cfr. also [14]). Now \( S_P [\phi] \) depends on \( z_2, \cdots z_N \) and

\[
-\frac{1}{2\pi} dS_P [\phi] = \beta_2 dz_2 + \cdots + \beta_N dz_N + f_1 dz_1 + f_2 dz_2 + \cdots f_N dz_N + c.c.
\]  

(75)

from which it follows \( f_1 \equiv 0 \). Similarly we reason with the other singularities obtaining \( F = 0 \).
7 Direct proof, elliptic and parabolic case

The previous proof goes through the weak form of Polyakov relation as an intermediate step. We think this is the simplest way to prove Polyakov conjecture. On the other hand, as one has to take the derivative with respect to $g_m$, this path cannot be followed in the case of parabolic singularities. In [4] we gave a direct proof of Polyakov conjecture by rewriting the action in terms of some background fields. In the following we shall follow this path introducing some shortcuts with respect to the proof in [4] (see also [6] for a different approach). Even if the treatment is not as simple as the one given in Section 4 it has the advantage of being immediately extendible to the case of parabolic singularities. The case of parabolic singularities was first solved in [3]; on the other hand we think worthwhile to see how the present method applies to both cases.

The technique to prove Polyakov conjecture will be to express the original action in terms of a field $\phi_M$ which is less singular than the original conformal field $\phi$. This procedure will give rise to an action $S$ for the field $\phi_M$ which does not involve the $\epsilon \to 0$ process. Despite that, computing the derivative of the new action $S$ is not completely trivial because one cannot take directly the derivative operation under the integral sign. In fact such unwarranted procedure would give rise to an integrand which is not absolutely summable. This does not apply to the simpler case of the derivative with respect to $g_m$ considered in the previous section, as in that case the derivative of the integrand is absolutely summable and bounded by a summable function as $g_m$ varies in an interval.

In the global coordinate system $z$ on $\mathbb{C}$ one writes $\phi = \phi_M + \phi_1 + \alpha_1 \phi_B$ where $\phi_B$ is a background conformal factor which is regular and behaves at infinity like $\phi_B = -2 \ln(z\bar{z}) + c_B + O(1/|z|)$ (a possible choice for $\phi_B$ is the conformal factor of the sphere with constant curvature $e^{\phi_B} = \frac{8}{(1+z\bar{z})^2}$) while $\phi_1$ is defined by

$$\phi_1 = \sum_n g_n \ln |z - z_n|^2 + c_0. \tag{76}$$

Then we have for $\phi_M$

$$4\partial_z \partial_{\bar{z}} \phi_M = e^{\phi_M + \phi_1 + \alpha_1 \phi_B} - \alpha_1 4 \partial_z \partial_{\bar{z}} \phi_B. \tag{77}$$
We shall choose \( \alpha_1 = \frac{\sum_n g_n + g_\infty}{2} \) so that \( \phi_M \) will be a function regular at infinity. The action which generates the above equation is

\[
S = \int \left( \partial_z \phi_M \partial_{\bar{z}} \phi_M + \frac{e^\phi}{2} - 2\alpha_1 \phi_M \partial_z \partial_{\bar{z}} \phi_B \right) \frac{idz \wedge d\bar{z}}{2}
\]

(78)

The fields \( \phi_1 \) and \( \phi_M \) transform under a change of chart like scalars while \( e^{\phi_B} \) transforms as a \((1,1)\) density. This choice is also in agreement with the invariance of eqs.(77,78). Due to the behavior of \( \phi_M \) and \( \phi_1 \) at the singularities and at infinity the integral in eq.(78) converges absolutely.

The regularized Liouville action is given by [19]

\[
S_P[\phi] = \lim_{\epsilon \to 0} S_\epsilon[\phi]
\]

(79)

where

\[
S_\epsilon[\phi] = \frac{i}{2} \int_{X_\epsilon} (\partial_z \phi \partial_{\bar{z}} \phi + \frac{e^\phi}{2}) dz \wedge d\bar{z} + \frac{i}{2} \sum_n g_n \oint \phi \left( \frac{d\bar{z}}{z - z_n} - \frac{dz}{\bar{z} - \bar{z}_n} \right)
\]

\[
+ \frac{i}{2} g_\infty \oint_\infty \phi \left( \frac{d\bar{z}}{z} - \frac{dz}{\bar{z}} \right) - \pi \sum_n g_n^2 \ln \epsilon^2 - \pi g_\infty^2 \ln \epsilon^2
\]

(80)

both for elliptic and parabolic singularities (in which case \( g_n = -1, g_\infty = 2 \), even if the behavior of the solutions of the Liouville equation around parabolic singularities is completely different from that around elliptic singularities. It is proven in Appendix 2 that the action \( S \) computed on a function \( \phi \), with the following asymptotic behavior at the singularities

\[
\begin{align*}
g_m \ln \zeta \bar{\zeta} & \quad \text{around an elliptic singularity} \\
- \ln(\zeta \bar{\zeta}) - \ln(\ln(\zeta \bar{\zeta}))^2 & \quad \text{around a parabolic singularity} \\
- g_\infty \ln z \bar{z} & \quad \text{at infinity}
\end{align*}
\]

is related to the original Liouville action \( S_P \) by

\[
S_P = S + \pi \sum_m \sum_{n \neq m} g_m g_n \ln |z_m - z_n|^2 + 4\pi \alpha_1 c_0 - \alpha_1^2 \int \phi_B \partial_z \partial_{\bar{z}} \phi_B \frac{idz \wedge d\bar{z}}{2} + 2\pi \alpha_1^2 c_B.
\]

(81)
We saw in Section 4 how on an open everywhere dense set of \( C \) there exists the derivative with respect to \( z_n \) of the parameters \( k, \text{Re}\beta_i, \text{Im}\beta_i \) which determine the solutions of the fuchsian equation related by eq. (12) to the conformal factor \( \phi \). Actually as pointed out at the end of Section 4 in that domain such parameters are real analytic functions of \( z_n \). On the other hand the solutions of the fuchsian equation and thus \( \phi_M \) depend analytically on such parameters \(^{20}\).

The procedure to compute the derivative will be to prove that
\[
\frac{\partial S}{\partial z_m} = \lim_{\epsilon \to 0} \int_{X_\epsilon} \frac{\partial F}{\partial z_m} i\, dz \wedge d\bar{z} \tag{82}
\]
where \( X_\epsilon \) has been defined after eq. (68) and \( F \) is given by
\[
F = \partial \phi_M \partial \bar{s} \phi_M + \frac{e \phi}{2} - 2\alpha_1 \phi_M \partial z \phi_B. \tag{83}
\]

First we write the identity
\[
S = \int_{X_\epsilon} F \frac{i}{2} dz \wedge d\bar{z} + \int_{\mathbb{C} \setminus X_\epsilon} F \frac{i}{2} dz \wedge d\bar{z}. \tag{84}
\]
The second integral is a sum of integrals having as domain disks of radius \( \epsilon \) around each singularity. If we take the derivative with respect to \( z_m \) we have
\[
\frac{\partial S}{\partial z_m} = \frac{\partial}{\partial z_m} \int_{X_\epsilon} F \frac{i}{2} dz \wedge d\bar{z} + \frac{\partial}{\partial z_m} \int_{\mathbb{C} \setminus X_\epsilon} F \frac{i}{2} dz \wedge d\bar{z}, \tag{85}
\]
and the second term on the r.h.s. goes to zero as \( \epsilon \to 0 \). It will be sufficient to prove this for the contribution coming from the disk \( D_\epsilon^m \) around the singularity \( z_m \); in the following we shall denote with \( D_\epsilon^0 \) the disk of center 0 and radius \( \epsilon \). In the elliptic case we have the structure
\[
\frac{\partial}{\partial z_m} \int_{D_\epsilon^m} F \frac{i}{2} dz \wedge d\bar{z} =
\]

\[
\frac{\partial}{\partial z_m} \int_{D_\epsilon^m} \left( \frac{2k_m^2(1 + g_m)}{\zeta [BB - k_m^2 AA(\zeta)_{1+g_m}]} + \ldots \right) \left( \frac{2k_m^2(1 + g_m)}{\zeta [BB - k_m^2 AA(\zeta)_{1+g_m}]} + \ldots \right) \frac{i}{2} d\zeta \wedge d\bar{\zeta} +
\]

\[
\frac{\partial}{\partial z_m} \int_{D_\epsilon^0} \frac{4k_m^2(g_m + 1)^2(\zeta_{1+g_m})^i}{[BB - k_m^2(\zeta)_{(g_m+1)AA}^2]} i \frac{i}{2} d\zeta \wedge d\bar{\zeta} +
\]

}\]

\[23\]
\[ \frac{\partial}{\partial z_m} 2\alpha_1 \int_{D_0} d^2 \zeta \left( \ln[BB - k_m^2 A \bar{A} (\zeta \bar{\zeta})^{1+g_m}]^2 + \ldots \right) \partial_z \partial_{\bar{z}} \phi_B (\zeta + z_m) \frac{i}{2} d\zeta \wedge d\bar{\zeta}, \quad (86) \]

and in the parabolic case

\[ \frac{\partial}{\partial z_m} \int_{D_0} d^2 \zeta \left( \ln\left[ \frac{\bar{A}_0}{B} + \frac{A_0}{B} + \ln \frac{\zeta \bar{\zeta}}{k_m^2} \right] + \ldots \right) \left( \ln\left[ \frac{\bar{A}_0}{B} + \frac{A_0}{B} + \ln \frac{\zeta \bar{\zeta}}{k_m^2} \right] + \ldots \right) \frac{i}{2} d\zeta \wedge d\bar{\zeta} + \]

\[ \left( \frac{\partial}{\partial z_m} \int_{D_0} d^2 \zeta \zeta \bar{B} \left[ \frac{\bar{A}_0}{B} + \frac{A_0}{B} + \ln \frac{\zeta \bar{\zeta}}{k_m^2} \right] \right) \frac{4}{2} \frac{i}{2} d\zeta \wedge d\bar{\zeta} + \]

\[ 2\alpha_1 \frac{\partial}{\partial z_m} \int_{D_0} d^2 \zeta \left( \ln\left[ \frac{\bar{A}_0}{B} + \frac{A_0}{B} + \ln \frac{\zeta \bar{\zeta}}{k_m^2} \right] + \ldots \right) \partial_z \partial_{\bar{z}} \phi_B \frac{i}{2} d\zeta \wedge d\bar{\zeta}. \quad (87) \]

It is possible to commute derivative and integral because the derivative of the integrand is an absolutely summable function, bounded by an absolutely summable function independent of \( z_m \), when \( z_m \) varies in a small interval. So, being the integrand absolutely summable, if we take the limit in which the domain of integration vanishes we obtain zero.

We come now to the first term of eq.(85). It is possible to rewrite it as

\[ \frac{\partial}{\partial z_m} \int_{X} F \frac{i}{2} d\zeta + \left( \frac{\partial}{\partial z_m} \int_{D_0} d^2 \zeta \left( \ln\left[ \frac{\bar{A}_0}{B} + \frac{A_0}{B} + \ln \frac{\zeta \bar{\zeta}}{k_m^2} \right] + \ldots \right) \right) \partial_z \partial_{\bar{z}} \phi_B \frac{i}{2} d\zeta \wedge d\bar{\zeta}. \quad (88) \]

where the contour term comes from the movement of the domain of integration. In the limit \( \epsilon \to 0 \) this last term vanishes, so we are left with

\[ \frac{\partial S}{\partial z_m} = \lim_{\epsilon \to 0} \int_{X} \frac{\partial}{\partial z_m} \left( \partial_z \phi_M \partial_{\bar{z}} \phi_M + \frac{e^\phi}{2} - 2\alpha_1 \phi_M \partial_z \partial_{\bar{z}} \phi_B \right) \frac{i}{2} d\zeta + d\bar{\zeta}. \quad (89) \]

Using now the equation of motion (77) we obtain

\[ \frac{\partial S}{\partial z_m} = \lim_{\epsilon \to 0} \int_{X} \left( \partial_z \left( \frac{\partial \phi_M}{\partial z_m} \partial_{\bar{z}} \phi_M \right) + \partial_{\bar{z}} \left( \frac{\partial \phi_M}{\partial z_m} \partial_z \phi_M \right) + \frac{\partial \phi_1}{\partial z_m} \frac{e^\phi}{2} \right) \frac{id\zeta + d\bar{\zeta}}{2}. \quad (90) \]

It is easily checked that the only contribution which survives in the limit \( \epsilon \to 0 \) is

\[ \frac{\partial S}{\partial z_m} = \lim_{\epsilon \to 0} \int_{X} \frac{\partial \phi_1}{\partial z_m} \frac{e^\phi}{2} \frac{id\zeta + d\bar{\zeta}}{2}. \quad (91) \]
which can be computed by using eq. (77) and \( \frac{\partial \phi_1}{\partial z_m} = -\frac{g_m}{(z - z_m)} \) to obtain

\[
\frac{\partial S}{\partial z_m} = -ig_m \lim_{\epsilon \to 0} \oint_{\gamma_\epsilon} \frac{1}{z - z_m} \partial_z (\phi_M + \alpha_1 \phi_B) \, dz.
\]

(92)

We use now \( \phi_M + \alpha_1 \phi_B = \phi - \sum_n g_n \ln |z - z_n|^2 - c_0 \). Both in the elliptic and in the parabolic case we have

\[
\partial_z (\phi_M + \alpha_1 \phi_B) = -2c_2 - \sum_{n \neq m} \frac{g_n}{z - z_n} + \rho_n(\zeta)
\]

(93)

where \( c_2 = \beta_m/2g_m \) and the contour integral of \( \rho_n(\zeta) \) in eq. (92) vanishes for \( \epsilon \to 0 \). Finally we have

\[
\frac{\partial S}{\partial z_m} = -2\pi \beta_m - 2\pi \sum_{n \neq m} \frac{g_m g_n}{z_m - z_n}
\]

(94)

equivalent to Polyakov conjecture

\[
-\frac{1}{2\pi} \frac{\partial S_P}{\partial z_m} = \beta_m
\]

(95)

due to the relation (81) between \( S \) and \( S_P \).

**Appendix 1**

In ref. [11] Lichtenstein proves among others, the following result: one can write \( \phi = U + v \) where \( v \) solves the linear equation (in modern notation)

\[
\Delta_{LB} v = \beta + 4\pi e^{-\phi_B} \sum_n g_n \delta^2(z - z_n).
\]

(96)

\( \Delta_{LB} \) is the Laplace- Beltrami operator on the background \( e^{\phi_B} \) i.e. \( 4e^{-\phi_B} \partial_z \partial_{\bar{z}} \) and \( \beta(z) \) is a positive function regular in \( \mathbb{C} \) except at the singularities in a neighbourhood of which it equals in the parabolic case

\[
\frac{8}{e^{\phi_B} \zeta (\ln \zeta)^2}
\]

(97)
and in the elliptic case is subject to the inequality

$$0 < \chi' < \beta(\zeta \bar{\zeta})^{-g_n} < \chi''.$$  \hfill (98)

Moreover

$$\int \beta e^{\phi} \frac{dz \wedge d\bar{z}}{2} = -4\pi \sum g_n.$$  \hfill (99)

Most important, $U$ is a bounded function over all $\mathbb{C}$ with a bound $H$ which depends continuously on the $z_n$ and on the $g_n$. $U$ obviously solves the equation

$$\Delta_{LB} U + \beta = e^{U + v_B}$$  \hfill (100)

and thus we can write

$$U = \Delta_{LB}^{-1}(e^{U + v_B} - \beta)$$  \hfill (101)

where $\Delta_{LB}^{-1}$ is the usual Green function $\Delta_{LB}^{-1} = \frac{1}{4\pi} \ln[(z - z')(\bar{z} - \bar{z}')]$. Eq. (101) allows us to put uniform bounds of $\partial_z \phi$ and $\partial^2_z \phi$ in a domain which excludes the singularities $z_n$. In fact we have

$$\partial_z U = \frac{1}{4\pi} \int \frac{1}{z - z'} [e^{U + v_B} - \beta](z') e^{\phi_B(z')} d^2 z'.$$  \hfill (102)

We can write the integral as the sum over a small disk $D_\eta$ of radius $\eta$ around $z$ and the remaining of the complex plane. The modulus of the first term is bounded by

$$e^H \int_{D_\eta} \frac{1}{|z - z'|} e^{v(z')} d^2 z' + \int_{D_\eta} \frac{1}{|z - z'|} |\beta(z')| e^{\phi_B(z')} d^2 z'$$  \hfill (103)

and as such bounded when e.g. $z_1$ varies in a small neighborhood which does not overlap with the other singularities. The integral over $\mathbb{C} \setminus D_\eta$ is bounded by

$$\frac{1}{\eta} \int e^{U(z) + v(z')} d^2 z' + \frac{1}{\eta} \int |\beta(z')| e^{\phi_B(z')} d^2 z'.$$  \hfill (104)

Both integrals are finite and vary regularly with $z_n$. Similarly one can put a bound on the second derivative.

$$\partial^2_z U = \frac{1}{4\pi} \int_{D_\eta} \frac{1}{z - z'} [e^{U + v_B} \partial_{z'}(U + v) - \partial_{z'} \beta - \beta \partial_{z'} \phi_B](z') e^{\phi_B(z')} d^2 z' -$$  \hfill (105)
\[
\frac{1}{4\pi} \int_{\partial D_n} \frac{1}{z - z'} \left[ e^{U+v-\phi_B(z')} - \beta \right] e^{\phi_B(z')} \frac{idz'}{2} \\
\frac{1}{4\pi} \int_{C \setminus D_n} \frac{1}{(z - z')^2} \left[ e^{U+v-\phi_B(z')} - \beta \right] e^{\phi_B(z')} \, dz' 
\]

The first term is bounded by the sum of the convergent integrals

\[
\frac{1}{4\pi} e^H \int_{D_n} \frac{1}{|z - z'|} e^v |\partial z'(U + v)| \, d^2 z' \tag{106} \\
\frac{1}{4\pi} \int_{D_n} \frac{1}{|z - z'|} |\partial z' \beta - \beta \partial z' \phi_B| \, d^2 z',
\]

the second is bounded due to uniform boundedness of the integrand on the contour and the last is bounded similarly as done for the first derivative.

We recall now that the rational function \(Q(z)\) is related to the analytic component of the energy momentum tensor of Liouville theory by

\[
Q(z) = \sum_{n} \left( \frac{1 - \mu_n^2}{4(z - z_n)^2} + \frac{\beta_n}{2(z - z_n)} \right) = -\frac{1}{2} \left[ \frac{1}{2} \left( \partial_z \phi \right)^2 - \partial_z^2 \phi \right]. \tag{107}
\]

Thus we can extract the \(\beta_n\) as contour integrals of the r.h.s. of eq.\((107)\) choosing contours which enclose the singularities. As \(\partial_z \phi\) and \(\partial^2_z \phi\) are uniformly bounded when the \(z_n\) vary on small disks of \(\mathbb{C}\) which do not overlap, we have that on such domain the \(\beta_n\) remain bounded.

With regard to the boundedness of \(k_n\) we recall that \(\phi\) in the case of elliptic singularities has the following expansion near \(z_n\)

\[
\phi = g_n \ln |z - z_n|^2 - \ln s_n^2 + o(1) \tag{108}
\]

and \(v\)

\[
v = g_n \ln |z - z_n|^2 + c_n + o(1) \tag{109}
\]

where \(c_n\) varies continuously with the \(z_m\).

Taking into account that \(U\) is bounded we have

\[
- \ln s_n^2 = c_n + U(z_n) \tag{110}
\]
and thus $\ln s_n^2$ is bounded when the $z_m$ vary in small non overlapping disks of $\mathbb{C}$. As according to eq. (38)

$$- \ln s_n^2 = \ln k_n^2 + \ln(g_n + 1)^2 + \ln 8 \quad (111)$$

the same can be said about the boundedness of $\ln k_n^2$. Similarly one can deal with the boundedness of $\beta_n$ and of $k_n^2$ as functions of the $g_n$ when $g_n$ vary in small domains respecting Picard’s bounds. One can easily extend the proof of the boundedness of $\beta_n$ and $k_n^2$ to the case of parabolic singularities.

**Appendix 2**

In this appendix we rewrite Polyakov’s regularized action in terms of the field $\phi_M$ and a background field $\phi_B$. We write

$$\phi = \phi_M + \phi_1 + \alpha_1 \phi_B \quad (112)$$

choosing $\phi_1$

$$\phi_1 = \sum_n g_n \ln(z - z_n)(\bar{z} - \bar{z}_n) + c_0. \quad (113)$$

e$^\phi_B$ is the background conformal factor describing a surface with the topology of the sphere. Thus we have

$$- \int e^{-\phi_B} 4 \partial_z \partial_{\bar{z}} \phi_B d\mu \equiv \int \Delta_{LB} \phi_B d\mu = - \int 4 \partial_z \partial_{\bar{z}} \phi_B \frac{idz \wedge d\bar{z}}{2} = 8\pi. \quad (114)$$

This relation fixes the asymptotic behavior of $\phi_B$

$$\phi_B = -2 \ln z\bar{z} + c_B + o(z). \quad (115)$$

As a consequence $\phi_M$ solves

$$4 \partial_z \partial_{\bar{z}} \phi_M = e^\phi - \alpha_1 4 \partial_z \partial_{\bar{z}} \phi_B. \quad (116)$$
\( \phi_M \) is finite for \( z = z_n \) in the case of elliptic singularities, while for parabolic singularities it diverges like \( \ln \ln (\zeta \bar{\zeta}) \); in order to have \( \phi_M \) regular at infinity we choose \( \alpha_1 = (\sum_n g_n + g_{\infty})/2 \).

We shall write \( \phi_M(\infty) = c_M \). We have

\[
\int_{X_{\epsilon}} \frac{\partial_z \phi \partial_{\bar{z}} \phi}{2} \frac{idz \wedge d\bar{z}}{2} = \int (\partial_z \phi_M \partial_{\bar{z}} \phi_M - 2\alpha_1 \phi_M \partial_{\bar{z}} \phi_B) \frac{idz \wedge d\bar{z}}{2} + \tag{117}
\]

\[
\int_{X_{\epsilon}} \partial_z (\phi_M \partial_{\bar{z}} (\phi_1 + \alpha_1 \phi_B)) \frac{idz \wedge d\bar{z}}{2} + \int_{X_{\epsilon}} \partial_{\bar{z}} (\phi_M \partial_z (\phi_1 + \alpha_1 \phi_B)) \frac{idz \wedge d\bar{z}}{2} +
\]

\[
\int_{X_{\epsilon}} \partial_z (\phi_1 + \alpha_1 \phi_B) \partial_{\bar{z}} (\phi_1 + \alpha_1 \phi_B) \frac{idz \wedge d\bar{z}}{2}.
\]

The second and third integral reduce to

\[
-\frac{i}{2} \sum_n g_n \oint_{\gamma_n} \phi_M \left[ \frac{dz}{z - z_n} - \frac{d\bar{z}}{\bar{z} - \bar{z}_n} \right] - \frac{i}{2} g_{\infty} \oint_{\gamma_\infty} \phi_M \left[ \frac{d\bar{z}}{\bar{z}} - \frac{dz}{z} \right] \tag{118}
\]

while the fourth becomes

\[
-\frac{i}{2} \sum_n \oint_{\gamma_n} \phi (\phi_1 + \alpha_1 \phi_B) \partial_z \phi_1 d\bar{z} +
\]

\[
\frac{i}{2} \oint_{\gamma_\infty} (\phi_1 + \alpha_1 \phi_B) \partial_{\bar{z}} \phi_1 d\bar{z} - \int (\phi_1 + \alpha_1 \phi_B) \alpha_1 \partial_z \partial_{\bar{z}} \phi_B \frac{idz \wedge d\bar{z}}{2}. \tag{119}
\]

The terms \((119)\) combines with \((118)\) as follows: the first two terms in \((119)\) sum to the integrals in \((118)\) to give the contour integrals in the original action \((68)\), leaving the term

\[
-\alpha_1^2 \int \phi_B \partial_z \partial_{\bar{z}} \phi_B \frac{idz \wedge d\bar{z}}{2} \tag{120}
\]

and the terms

\[
-\frac{i}{2} \sum_n g_n \oint_{\gamma_n} (\phi_1 + \alpha_1 \phi_B) \frac{dz}{z - z_n} - \frac{i}{2} g_{\infty} \oint_{\gamma_\infty} (\phi_1 + \alpha_1 \phi_B) \frac{dz}{z} - \alpha_1 \int \phi_1 \partial_z \partial_{\bar{z}} \phi_B \frac{idz \wedge d\bar{z}}{2}. \tag{121}
\]

These terms do not contain the field \( \phi_M \) and can be computed explicitly; they give rise to the divergence \( \pi \sum_n g_n^2 \ln \epsilon^2 + \pi g_{\infty}^2 \ln \epsilon^2 \) which cancel the one in the original action \((68)\) and to the finite terms

\[
\pi \sum_n g_n \sum_{m \neq n} g_m \ln |z_n - z_m|^2 + 2\pi \alpha_1 c_B + 4\pi \alpha_1 c_0. \tag{122}
\]
Thus the final form of the action is

\[ S_P = \int (\partial_z \phi_M \partial_\bar{z} \phi_M + \frac{e^\phi}{2} - 2\alpha_1 \phi_M \partial_z \partial_\bar{z} \phi_B) \frac{idz \wedge d\bar{z}}{2} + \pi \sum_n g_n \sum_{m \neq n} g_m \ln |z_n - z_m|^2 \]  

\[ + 4\pi \alpha_1 c_0 - \alpha_1^2 \int \phi_B \partial_z \partial_\bar{z} \phi_B \frac{idz \wedge d\bar{z}}{2} + 2\pi \alpha_1^2 c_B. \]  

(123)

The same procedure holds in the case when one or more singularities are of the parabolic type yielding the same result. In fact the kinematic field \( \phi_B \) is the same, \( \phi_1 \) is obtained by replacing some \( g_n \) with \( -1 \) and the integral containing the field \( \phi_M \) are still absolutely convergent.
References

[1] L. Cantini, P. Menotti, D. Seminara, Class. Quant. Grav. 18 (2001) 2253.

[2] A.M. Polyakov as reported in refs. [3].

[3] P. G. Zograf, L. A. Takhtajan, Math. USSR Sbornik 60 (1988) 143; Math. USSR Sbornik 60 (1988) 297.

[4] L. Cantini, P. Menotti, D. Seminara, Phys. Lett. B517 (2001) 203.

[5] L. Cantini, P. Menotti, D. Seminara, 25th Johns Hopkins Workshop on Current Problems in Particle Theory, Florence, Italy, 3-5 Sep 2001, hep-th/0112102.

[6] L. A. Takhtajan, P. G. Zograf, math.cv/0112170.

[7] A. Bellini, M. Ciafaloni, P. Valtancoli, Physics Lett. B 357 (1995) 532; Nucl. Phys. B 454 (1995) 449; Nucl. Phys. B 462 (1996) 453; M. Welling, Class. Quantum Grav. 13 (1996) 653; Nucl. Phys. B 515 (1998) 436.

[8] P. Menotti, D. Seminara, Ann. Phys. 279 (2000) 282; Nucl. Phys. (Proc. Suppl.) 88 (2000) 132.

[9] E. Picard, Compt. Rend. 116 (1893) 1015; J. Math. Pures Appl. 4 (1893) 273 and (1898) 313; Bull. Sci. math. XXIV 1 (1900) 196.

[10] H. Poincaré, J. Math. Pures Appl. (5) 4 (1898) 137.

[11] L. Lichtenstein, Acta Mathematica 40 (1915) 1.

[12] M. Trojanov, Trans. Am. Math. Soc. 324 (1991) 793.

[13] A. Bilal, J-L. Gervais, J. Geom. Phys. 5 (1988) 277; Nucl. Phys. B305 (1988) 33.

[14] J. A. Hempel, Bull. London Math. Soc. 20 (1988) 97; S. J. Smith, J. A. Hempel, J. London Math. Soc. (2) 40 (1989) 269; Bull. Austral. Math. Soc. 39 (1989), no. 3, 369.
[15] H. Poincaré, Acta Mathematica 31 (1907) 1, P. Koebe, Math. Ann. 75 (1914) 42.

[16] I. Kra, Trans. AMS 313 (1989) 589.

[17] M. Yoshida, “Fuchsian differential equations”, Fried. Vieweg & Sohn, Braunschweig (1987); K. Okamoto, J. Fac. Sci. Tokio Univ. 33 (1986) 575.

[18] R. C. Gunning, H. Rossi “Analytic functions of several complex variables” Prentice-Hall, Inc., Englewood Cliffs, N.J. 1965; J. P. D’Angelo, “Several complex variables and the geometry of real hypersurfaces”. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1993.

[19] L. A. Takhtajan, Mod. Phys. Lett. A11 (1996) 93; “Topics in quantum geometry of Riemann surfaces: two dimensional quantum gravity”, Proc. Internat. School Phys. Enrico Fermi, 127, IOS, Amsterdam, 1996; “Semi-classical Liouville theory, complex geometry of moduli spaces, and uniformization of Riemann surfaces” in New symmetry principles in quantum field theory (Cargese, 1991), 383 NATO Adv. Sci. Inst. Ser. B Phys. 295, Plenum, New York, 1992.

[20] See e.g. E. Hille “ Ordinary differential equations in the complex domain” Dover Publications, New York (1967).