Exponential Integral Representations of Theta Functions

Andrew Bakan¹ · Håkan Hedenmalm²

Received: 2 December 2019 / Revised: 29 April 2020 / Accepted: 18 May 2020 / Published online: 29 July 2020
© Springer-Verlag GmbH Germany, part of Springer Nature 2020

Abstract
Let \( \Theta_3(z) := \sum_{n \in \mathbb{Z}} \exp(i\pi n^2 z) \) be the standard Jacobi theta function, which is holomorphic and zero-free in the upper half-plane \( \mathbb{H} := \{ z \in \mathbb{C} \mid \text{Im } z > 0 \} \), and takes positive values along \( i\mathbb{R}_{>0} \), the positive imaginary axis, where \( \mathbb{R}_{>0} := (0, +\infty) \).

We define its logarithm \( \log \Theta_3(z) \) which is uniquely determined by the requirements that it should be holomorphic in \( \mathbb{H} \) and real-valued on \( i\mathbb{R}_{>0} \). We derive an integral representation of \( \log \Theta_3(z) \) when \( z \) belongs to the hyperbolic quadrilateral \( \mathcal{F}_\square := \{ z \in \mathbb{C} \mid \text{Im } z > 0, -1 \leq \text{Re } z \leq 1, |2z-1| > 1, |2z+1| > 1 \}. \)

Since every point of \( \mathbb{H} \) is equivalent to at least one point in \( \mathcal{F}_\square \) under the theta subgroup of the modular group on the upper half-plane, this representation carries over in modified form to all of \( \mathbb{H} \) via the identity recorded by Berndt. The logarithms of the related Jacobi theta functions \( \Theta_4 \) and \( \Theta_2 \) may be conveniently expressed in terms of \( \log \Theta_3 \) via functional equations, and hence get controlled as well. Our approach is based on a study of the logarithm of the Gauss hypergeometric function for a specific choice of the parameters. This has connections with the study of the universally starlike mappings introduced by Ruscheweyh, Salinas, and Sugawa.

Keywords  Theta functions · Elliptic modular function · Gauss hypergeometric function · Starlike functions

Mathematics Subject Classification  30C45 · 33C05 · 33E05

Communicated by Valdimir V. Andrievskii.

In memory of Stephan Ruscheweyh.

Andrew Bakan
andrew@bakan.kiev.ua

Håkan Hedenmalm
haakanh@math.kth.se

¹ Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv 01601, Ukraine
² KTH Royal Institute of Technology, SE-10044 Stockholm, Sweden
1 Introduction and Main Results

Let $\mathbb{D} := \{ z \in \mathbb{C} \mid |z| < 1 \}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{R}_{>0} := (0, +\infty)$. We also write $\mathbb{H} := \{ z \in \mathbb{C} \mid \text{Im} \ z > 0 \}$ for the upper half-plane. Given a domain $D \subset \mathbb{C}$, we let $\text{Hol}(D)$ denote the set of all holomorphic functions in $D \subset \mathbb{C}$. We will need the cone $\mathcal{M}^+(\mathbb{R})$ of all non-negative locally finite Borel measures on $\mathbb{R}$, and for a given $\mu \in \mathcal{M}^+(\mathbb{R})$ we let $\text{supp} \ \mu := \{ x \in \mathbb{R} \mid \mu((x - \varepsilon, x + \varepsilon)) > 0 \ \text{for all} \ \varepsilon > 0 \}$ denote the support of $\mu$. For $0 < p < +\infty$, we denote by (see [20, p. 112])

$$
H^p := \left\{ f \in \text{Hol}(\mathbb{H}) \left| \sup_{y > 0} \int_{\mathbb{R}} |f(x + iy)|^p \, dx < +\infty \right. \right\},
$$

the Hardy space of the upper half-plane, $m$ the Lebesgue measure on the real line and $L^\infty(\mathbb{R})$ the space of all Borel measurable real-valued functions $f$ on the real line that are essentially bounded, equipped with the essential supremum norm $\| f \|_{L^\infty(\mathbb{R})} := \inf\{ a > 0 \mid m(\{ x \in \mathbb{R} \mid |f(x)| > a \}) = 0 \}$.

Following the definitions of [34, pp. 6, 40], we denote by $\ln : \mathbb{R}_{>0} \mapsto \mathbb{R}$ the real-valued logarithm defined on $\mathbb{R}_{>0}$, and let $\log (z) = \ln |z| + i \ \text{Arg} \ (z)$ be the principal branch of the logarithm defined for $z \in \mathbb{C} \setminus (-\infty, 0]$ with $\text{Arg} \ (z) \in (-\pi, \pi)$. Furthermore, for a simply connected domain $D \subset \mathbb{C}$, a point $a \in D$, and a function $f \in \text{Hol}(D)$ which is zero-free in $D$ with $f(a) > 0$, we write $\log f(z)$ for the holomorphic function in $D$ such that $\exp(\log f(z)) = f(z)$, $z \in D$, and $\log f(a) = \ln f(a)$ (see [12, p. 94]). Then $\text{Re} \ \log f(z) = \ln |f(z)|$ and $\text{arg} f(z) := \text{Im} \ \log f(z)$ for each $z \in D$.

We also denote by $\text{clos}(A)$ (or $\overline{A}$), $\text{int}(A)$, and $\partial A$ the closure, interior, and boundary of a subset $A \subset \mathbb{C}$, respectively. Moreover, we let $C(A)$ denote the set of all continuous functions $f : A \mapsto \mathbb{C}$.

**Nevanlinna-Pick functions.** We let $\mathcal{P}$ denote the class of Nevanlinna-Pick functions, which are holomorphic functions $\Phi$ in $\mathbb{C} \setminus \mathbb{R}$ with

$$
\text{Im} \ \Phi(z) \geq 0, \quad z \in \mathbb{H}, \quad (1.1)
$$
and the symmetry property
\[ \Phi(z) = \Phi(\overline{z}), \quad z \in \mathbb{C} \setminus \mathbb{R}. \] (1.2)

It is well-known (see [8, p. 31]) that unless \( \Phi \in \mathcal{P} \) is a real constant, the strict inequality \( \text{Im } \Phi(z) > 0 \) holds for all \( z \in \mathbb{H} \). Moreover, each function \( \Phi \in \mathcal{P} \) has a unique canonical representation of the form (see [13, p. 20, Thm. 1], [36, p. 23, Lem. 2.1])
\[ \Phi(z) = \alpha z + \beta + \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sigma(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \] (1.3)
where \( \alpha \geq 0, \beta \in \mathbb{R}, \) and \( \sigma \in \mathcal{M}^+(\mathbb{R}) \) is such that
\[ \int_{\mathbb{R}} \frac{d\sigma(t)}{1 + t^2} < +\infty. \]

In the opposite direction, any function of the given form (1.3) is in \( \mathcal{P} \). In the representation (1.3), \( \sigma \) measures the jump in the imaginary part between the upper and lower half-planes. If we write, for an open interval \( I \subset \mathbb{R} \),
\[ \mathcal{P}(I) := \mathcal{P} \cap \text{Hol}(I \cup (\mathbb{C} \setminus \mathbb{R})) \]
then it is a consequence [13, p. 26] of the Schwarz reflection principle that
\[ \Phi \in \mathcal{P}(I) \iff \text{supp } \sigma \subset \mathbb{R} \setminus I. \] (1.4)

**Logarithms of Nevanlinna-Pick functions.** For an arbitrary \( \Phi \in \mathcal{P} \) with \( \Phi(z) \neq a, a \leq 0 \), \( \log \Phi \) is a holomorphic function which maps the upper half-plane \( \mathbb{H} \) into the strip \( \{ w \in \mathbb{C} \mid 0 \leq \text{Im } w < \pi \} \subset \mathbb{H} \cup \mathbb{R} \), and inherits the symmetry property (1.2) from \( \Phi \), so that, in particular, \( \log \Phi \in \mathcal{P} \). We denote by \( \log \mathcal{P} \) the collection of all such functions \( \log \Phi \), where \( \Phi \in \mathcal{P} \) and \( \Phi(z) \neq a, a \leq 0 \). Then the observation just made gives the inclusion \( \log \mathcal{P} \subset \mathcal{P} \). Such functions \( f \in \log \mathcal{P} \subset \mathcal{P} \) are characterized in terms of a corresponding integral representation (see [13, p. 27])
\[ f(z) = b + \int_{-\infty}^{+\infty} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) a(t) \, dt, \quad z \in \mathbb{C} \setminus \mathbb{R}, \] (1.5)
where \( dt := dm(t), b \in \mathbb{R} \) and \( a \in L^\infty(\mathbb{R}) \) with \( 0 \leq a(x) \leq 1 \) on \( \mathbb{R} \) (almost everywhere with respect to \( m \)). On the other hand, any function of the form (1.5) is in \( \log \mathcal{P} \).

**Universally starlike functions.** We begin with some notation. A domain \( D \) in the complex plane \( \mathbb{C} \) is referred to as circular when it is either an open disk or an open half plane. Moreover, a domain \( \Omega \) is said to be starlike with respect to the origin if \( 0 \in \Omega \) and if for each \( z_0 \in \Omega \setminus \{0\} \) the straight line segment from 0 to \( z_0 \) is contained in \( \Omega \).
Associated with the starlike domains we have the notion of starlike univalent mappings [14, p. 40]. Building on this, Ruscheweyh, Salinas, and Sugawa [32, p. 290] introduced the notion of universal starlikeness in the context of holomorphic functions in the set $\mathbb{C} \setminus [1, +\infty)$.

**Definition 1.1** A function $\Psi$ is said to be universally starlike if $\Psi$ is holomorphic in $\mathbb{C} \setminus [1, +\infty)$, with the normalization $\Psi(0) = 0$, $\Psi'(0) = 1$, and if $\Psi$ maps every circular domain $D \subset \mathbb{C} \setminus [1, +\infty)$ with $0 \in D$ one-to-one onto a domain which is starlike with respect to the origin.

In [32, p. 289, Cor. 1.1], Ruscheweyh, Salinas and Sugawa characterized the universally starlike functions $\Psi$ as functions of the form

$$\Psi(z) = z \exp \left( \int_{[0,1]} \frac{\log \frac{1}{1-tz}}{1-t} \, d\sigma(t) \right), \quad z \in \mathbb{C} \setminus [1, +\infty),$$

where $\sigma \in \mathcal{M}^+(\mathbb{R})$ is uniquely determined by the requirements

$$\supp \sigma \subset [0,1], \quad 0 \leq \sigma(\mathbb{R}) \leq 1, \quad \sigma(\{0\}) = 0.$$  

The formulation in [32, p. 289, Cor. 1.1] was slightly different, but it is easy to see that it is equivalent to the given one (see [7, p. 719]).

**Universal starlikeness associated with the hypergeometric function.** Let $F_{a,b;c}(z) := F(a, b; c; z)$ be the Gauss hypergeometric function given by

$$F_{a,b;c}(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{n!\Gamma(c+n)} z^n, \quad z \in \mathbb{D},$$

where we restrict the parameters $a, b, c$ to be positive. It is well-known that the hypergeometric function $F_{a,b;c}$ extends holomorphically to the set $\mathbb{C}\setminus[1, +\infty)$. If $0 < b < c$, we can see this from Euler’s integral representation [15, p. 59]:

$$F_{a,b;c}(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} \, dt.$$  

It was shown in [32, p. 292, Thm. 1.8] that the function $\Psi(z) = zF_{a,b;c}(z)$ is universally starlike provided that $0 < b \leq c$ and $0 < a \leq \min\{1, c\}$. In particular, it follows from this theorem of Ruscheweyh, Salinas, and Sugawa that for arbitrary real triples $(a, b, c)$ with $0 < b < c$ and $0 < a \leq \min\{1, c\}$, there exists a unique measure $\sigma_{a,b;c} \in \mathcal{M}^+(\mathbb{R})$ with

$$\supp \sigma_{a,b;c} \subset [0,1], \quad 0 < \sigma_{a,b;c}(\mathbb{R}) \leq 1, \quad \sigma_{a,b;c}(\{0\}) = 0,$$

such that (compare with (1.6))

$$F_{a,b;c}(z) = \exp \left( \int_{[0,1]} \frac{1}{1-tz} \, d\sigma_{a,b;c}(t) \right), \quad z \in \mathbb{C} \setminus [1, +\infty).$$
This is an existence result and does not tell us what the measure $\sigma_{a,b,c}$ looks like.

We let $\mathcal{P}_{\log}$ denote the collection of all $f \in \mathcal{P}$ with $f(z) \neq 0$ such that the logarithmic derivative $f'/f \in \mathcal{P}$, i.e.,

$$\mathcal{P}_{\log} := \left\{ f \in \mathcal{P} \setminus \{0\} \mid \frac{f'}{f} \in \mathcal{P} \right\}. \quad (1.12)$$

Moreover, we write $\mathcal{P}_{\log}(-\infty, 1)$ for the subset of $\mathcal{P}_{\log}$ consisting of those functions that extend holomorphically to $\mathbb{C} \setminus [1, +\infty)$. The following result explains the connection with the universally starlike functions (see [6, Cor. 2.3]).

**Theorem A** The function $\Psi(z) = z\psi(z)$ is universally starlike if and only if $\psi(0) = 1$ and $\psi \in \mathcal{P}_{\log}(-\infty, 1)$.

One of our main results is the following theorem.

**Theorem 1.1** For $a = b = 1/2$ and $c = 1$ the measure $\sigma_{1/2,1/2,1}$ in the exponential integral representation (1.11) under the conditions (1.10) has the following explicit expression:

$$\sigma_{1/2,1/2,1}([0, x]) = \frac{1}{\pi} \arctan \frac{F_{1/2,1/2,1}(x)}{F_{1/2,1/2,1}(1-x)}, \quad 0 < x < 1. \quad (1.13)$$

In particular, $\sigma_{1/2,1/2,1}$ is absolutely continuous with respect to the Lebesgue measure and has total variation $1/2$.

Clearly,

$$0 < \text{Im} \log \frac{1}{1-tz} = \arg \frac{1}{1-tz} < \pi, \quad 0 < t < 1, \quad z \in \mathbb{H},$$

so that in (1.11) for arbitrary $z \in \mathbb{H}$ we have

$$0 < \arg F_{1/2,1/2,1}(z) = \text{Im} \int_{[0,1]} \log \frac{1}{1-tz} \, d\sigma_{1/2,1/2,1}(t) < \frac{\pi}{2},$$

and correspondingly $-\pi/2 < \arg F_{1/2,1/2,1}(z) < 0$ for $z$ lying in the lower open half-plane while $\arg F_{1/2,1/2,1}(x) = 0$ when $x < 1$. Hence,

$$\arg F_{1/2,1/2,1}(z) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \text{Re} F_{1/2,1/2,1}(z) > 0, \quad z \in \mathbb{C} \setminus [1, +\infty). \quad (1.14)$$

This means that for arbitrary $z \in \mathbb{C} \setminus [1, +\infty)$ the equality (1.11) can be written in the form

$$\log F_{1/2,1/2,1}(z) = \frac{1}{\pi^2} \int_0^1 \frac{1}{t(1-t)} \log \left( \frac{1}{1-tz} \right) \, dt, \quad (1.15)$$
or, alternatively,

$$\log F_{1/2,1/2;1}(z) = \frac{1}{\pi} \int_0^1 \frac{t}{1+t^2} \arctan \left( \frac{F_{1/2,1/2;1}(1-t)}{F_{1/2,1/2;1}(t)} \right) dt$$

$$+ \frac{1}{\pi} \int_1^{\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) \arctan \left( \frac{F_{1/2,1/2;1}(1-t)}{F_{1/2,1/2;1}(1/t)} \right) dt.$$  (1.16)

Next, since for a positive real $\alpha$, $\alpha \sigma_{1/2,1/2;1}$ is a non-negative absolutely continuous measure with total mass $\alpha/2$, the conditions of (1.7) hold with $\sigma = \alpha \sigma_{1/2,1/2;1}$, provided that $0 < \alpha \leq 2$. Given the characterization of the universally starlike functions (1.6), we obtain in Sect. 3 the following assertion which for the case when $\alpha = 1$ is the special case of [32, p. 292, Thm. 1.8] when $a = b = 1/2$ and $c = 1$ (see [5, p. 24]).

**Corollary 1.1** We have that $F_{1/2,1/2;1}(z) \in P_{\log(-\infty,1)}$, so that the function $zF_{1/2,1/2;1}(z)^{\alpha}$ is universally starlike, provided that $0 < \alpha \leq 2$.

The Schwarz triangle function. We write $F_{\Delta}$ for the function $F_{1/2,1/2;1}$.

In 1873, Schwarz [35] established the following fact (see [15, p. 97]).

**Theorem B** The Schwarz triangle function

$$\lambda_{\Delta}(z) := i \cdot \frac{F_{\Delta}(1-z)}{F_{\Delta}(z)}, \quad z \in (0,1) \cup (\mathbb{C} \setminus \mathbb{R}),$$  (1.17)

maps the set $(0,1) \cup (\mathbb{C} \setminus \mathbb{R})$ one-to-one onto the fundamental quadrilateral

$$\mathcal{F}_\square := \left\{ z \in \mathbb{H} \mid -1 < \text{Re} z < 1, \ |2z - 1| > 1, \ |2z + 1| > 1 \right\}.  \quad (1.18)$$

The function $\lambda : \mathcal{F}_\square \to (0,1) \cup (\mathbb{C} \setminus \mathbb{R})$ which is the inverse of $\lambda_{\Delta}$ is called the elliptic modular function and is the subject of a large literature (see [15, p. 99] and [28, p. 579]). In the Poincaré half-plane model of the hyperbolic plane $\mathbb{H}$, $\mathcal{F}_\square$ is an ideal hyperbolic quadrilateral, and it is the set of all interior points of the fundamental domain for the subgroup $\Gamma(2)$ of the modular group $\Gamma$ on the upper half-plane $\mathbb{H}$ (see (1.39), [30, p. 20] and [11, p. 115]).

**Remark.** The related function

$$\mu(r) = \frac{\pi}{2} \frac{K\left(\sqrt{1-r^2}\right)}{K(r)} = \frac{\pi}{2} \frac{F_{\Delta}(1-r^2)}{F_{\Delta}(r^2)} = \frac{\pi}{2i} \lambda_{\Delta}(r^2)$$

for $0 < r < 1$ turns up in the theory of quasiconformal mappings and is called the modulus of the Grötzsch ring [3, Ch. 5]. Here, $K$ is the standard elliptic integral of the first kind (see e.g. [1, p. 569]).
The intention of this paper is to derive an integral representation for the logarithm of the theta function \( \Theta_3 \). Apart from that, we are also motivated by the desire to give an elementary exposition of the basic properties of the theta functions \( \Theta_2, \Theta_3, \Theta_4 \) (see [5, p. 24]).

**Outline of the paper.** We first describe the elementary properties of the functions \( F_\Delta \) and \( \lambda_\Delta \). In Sect. 2, we obtain basic formulas for the hypergeometric function \( F_\Delta \), and show that it is in the Hardy space \( H^p \) of the upper half-plane for any \( p \) with \( 2 < p < \infty \), and has the properties

\[
\begin{align*}
& (a) \quad F_\Delta \in \mathcal{P}(-\infty, 1), \quad (b) \quad F_\Delta(z) \neq 0 \quad \text{for all} \quad z \in \mathbb{C} \setminus [1, +\infty). \tag{1.19}
\end{align*}
\]

We also obtain that

\[
\lambda_\Delta'(z) F_\Delta(z)^2 = \frac{1}{\pi z(1-z)}, \quad z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}). \tag{1.20}
\]

In Sect. 3 we prove Theorem 1.1 and obtain an exponential integral representation for \( \lambda_\Delta / i \).

The Schwarz triangle function \( \lambda_\Delta \) satisfies the functional relation

\[
\lambda_\Delta(z)\lambda_\Delta(1-z) = -1, \quad z \in \Lambda := (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}), \tag{1.21}
\]

and in Sect. 4 we obtain the following relationship between the values of \( \lambda_\Delta \) on two sides of the cut along \(( -\infty, 0 )\):

\[
\lambda_\Delta(-x + i0) = 2 + \lambda_\Delta(-x - i0), \quad x > 0. \tag{1.22}
\]

Correspondingly, along the remaining cut \(( 1, +\infty )\), we obtain that

\[
\lambda_\Delta(1 + x - i0) = \frac{\lambda_\Delta(1 + x + i0)}{1 - 2\lambda_\Delta(1 + x + i0)}, \quad x > 0. \tag{1.23}
\]

We also obtain the equality of sets

\[
\lambda_\Delta((0, 1) \cup (\mathbb{C} \setminus \mathbb{R})) = \mathcal{F}_{\square}, \tag{1.24}
\]

which constitutes part of the assertion of Theorem B.

In Sect. 5 we show that Lemma 4.2 easily implies the following result which may be considered as an analogue of Liouville’s theorem (see [5, p. 24]) for the fundamental quadrilateral \( \mathcal{F}_{\square} \), where \( \mathcal{F}_{\square} = -1/\mathcal{F}_{\square} \) and

\[
\mathbb{H} \cap \partial \mathcal{F}_{\square} = (1 + i \mathbb{R}_{>0}) \cup (-1 + i \mathbb{R}_{>0}) \cup (1 - i \mathbb{R}_{>0})^{-1} \cup (-1 - i \mathbb{R}_{>0})^{-1}.
\]

**Lemma 1.1** Suppose \( f \) is holomorphic on \( \mathcal{F}_{\square} \) and extends continuously to its hyperbolic closure \( \mathbb{H} \cap \text{clos} \mathcal{F}_{\square} \). Suppose in addition that the boundary values satisfy (a)
Then we show that $f(z) = f(z + 2)$ and (b) $f(-1/z) = f\left(-1/(z + 2)\right)$ for each $z \in -1 + i \mathbb{R}_{>0}$.

Finally, suppose there exist non-negative integers $n_\infty$, $n_0$, and $n_1$ such that

\begin{align}
(1) & \quad |f(z)| = o\left(\exp\left(\pi\left(n_\infty + 1\right)|z|\right)\right), \quad \mathcal{F}_0 \ni z \to \infty, \\
(2) & \quad |f(z)| = o\left(\exp\left(\pi\left(n_0 + 1\right)|z|^{-1}\right)\right), \quad \mathcal{F}_0 \ni z \to 0, \\
(3) & \quad |f(z)| = o\left(\exp\left(\pi\left(n_1 + 1\right)|z - \sigma|^{-1}\right)\right), \quad \mathcal{F}_0 \ni z \to \sigma,
\end{align}

where in (3) we consider both $\sigma \in \{1, -1\}$. Then there exists an algebraic polynomial $P$ of degree at most $n_\infty + n_0 + n_1$ such that

$$f\left(\lambda_\triangle(z)\right) = \frac{P(z)}{z^{n_\infty}(1-z)^{n_0}}, \quad z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}).$$

In Sect. 6 we introduce some notation for the standard theta functions $\Theta_2$, $\Theta_3$, $\Theta_4$. In Sect. 7 we explain how to obtain Wirtinger’s identity (see [15, p. 99] with $a = b = 1/2, c = 1$)

$$\Theta_3\left(\lambda_\triangle(z)\right)^2 = F_\triangle(z), \quad z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}). \quad (1.25)$$

Together with non-vanishing property (1.19)(b), the equality of sets (1.24) and the relationships established in Sect. 6, this gives that

$$\Theta_3(z)\Theta_4(z)\Theta_2(z) \neq 0, \quad z \in \mathbb{H}. \quad (1.26)$$

In Sect. 8 we show that $f(z) = (\Theta_2(z)^4 + \Theta_4(z)^4)/\Theta_3(z)^4$ satisfies the conditions of Lemma 1.1 with $n_\infty = n_0 = n_1 = 0$ from which we obtain the Jacobi identity

$$\Theta_2(z)^4 + \Theta_4(z)^4 = \Theta_3(z)^4, \quad z \in \mathbb{H}. \quad (1.27)$$

Then we show that $f(z) = \Theta_2(z)^4/\Theta_3(z)^4$ satisfies the conditions of Lemma 1.1 with $n_1 = 1$ and $n_0 = n_\infty = 0$, from which we deduce the property (see [15, p. 23])

$$\frac{\Theta_2(\lambda_\triangle(z))^4}{\Theta_3(\lambda_\triangle(z))^4} = z, \quad z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}). \quad (1.28)$$

As a side remark, this implies that $\lambda_\triangle$ is univalent in the region $(0, 1) \cup (\mathbb{C} \setminus \mathbb{R})$, which together with the mapping property (1.24) proved in Sect. 4 below, completes the proof of Theorem B.

Since the elliptic modular function $\lambda$ is the inverse of $\lambda_\triangle$, (1.28) is the same as the identity $\lambda(z) = \Theta_2(z)^4/\Theta_3(z)^4$ for $z \in \mathcal{F}_0$, whence it is immediate that the modular function $\lambda$ extends to a zero-free holomorphic function in $\mathbb{H}$ with period 2:

$$\lambda(z) = \frac{\Theta_2(z)^4}{\Theta_3(z)^4}, \quad z \in \mathbb{H}. \quad (1.29)$$
By combining (1.28) with (1.27) and (1.20), we find that

\[
\lambda'(z) = i \pi \lambda(z)(1 - \lambda(z)) \Theta_3(z)^4 = i \pi \frac{\Theta_2(z)^4 \Theta_4(z)^4}{\Theta_3(z)^4}, \ z \in \mathbb{H},
\]

which in its turn leads to the following two identities (see [23, Ex. 16, p. 22]):

\[
\frac{\pi}{4i} \Theta_2(z)^4 = \frac{\Theta'_2(z)}{\Theta_3(z)} - \frac{\Theta'_3(z)}{\Theta_4(z)} , \quad \frac{\pi}{4i} \Theta_4(z)^4 = \frac{\Theta'_3(z)}{\Theta_3(z)} - \frac{\Theta'_2(z)}{\Theta_2(z)}, \ z \in \mathbb{H}.
\]

In addition, we see from (1.28) and (1.25) that

\[
\Theta_2(\lambda_\Delta(z))^4 = z F_\Delta(z)^2, \ z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}),
\]

This is the function which is universally starlike by Corollary 1.1.

**Corollary 1.2** The function \( \Theta_2(\lambda_\Delta)^4 \) is universally starlike while \( \Theta_3(\lambda_\Delta)^4 \) belongs to the class \( P_{\log}(-\infty, 1) \), and, for every \( z \in \mathcal{F}_\square \setminus \{i \mathbb{R}_{>0}\} \) we have

\[
(a) \quad (\text{Re} \ z) \cdot \text{Im} \frac{\Theta_3(z)^4}{\lambda'(z) \Theta_3(z)} > 0, \quad (b) \quad (\text{Re} \ z) \cdot \text{Im} \frac{\Theta'_3(z)}{\Theta_3(z)} > 0.
\]

In Sect. 9, we introduce the logarithms of the theta functions. Moreover, in Sect. 10, we apply Theorem 1.1 in combination with the Wirtinger identity (1.25) to obtain an integral representation of \( \log \Theta_3 \) on the set

\[
\mathcal{F}_\square^\square := \mathcal{F}_\square \cup (-1 + i \mathbb{R}_{>0}) \cup (1 + i \mathbb{R}_{>0}).
\]

**Corollary 1.3** For arbitrary \( z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}) \) we have \( \lambda_\Delta(z) \in \mathcal{F}_\square \),

\[
\arg \Theta_3(\lambda_\Delta(z)) \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right),
\]

\[
\log \Theta_3(\lambda_\Delta(z)) = \frac{1}{2\pi^2} \int_0^1 \frac{1}{F_\Delta(t)^2 + F_\Delta(1-t)^2} \log \left( \frac{1}{1-itz} \right) \frac{dt}{t(1-t)} ,
\]

where \( \mathcal{F}_\square = \lambda_\Delta((0, 1) \cup (\mathbb{C} \setminus \mathbb{R})) \), in accordance with (1.24). Moreover,

\[
\log \Theta_3(\pm 1 + iy) = \frac{1}{2\pi^2} \int_0^1 \frac{1}{F_\Delta(t)^2 + F_\Delta(1-t)^2} \ln \left( \frac{1}{1+itz} \right) \frac{dt}{t(1-t)} ,
\]

where \( \Theta_3(1+iy) = \Theta_3(-1+iy) \in (0, 1), \ log \Theta_3(\pm 1 + iy) = \ln \Theta_3(\pm 1 + iy), \)

\[
y = y(x) = \frac{F_\Delta(1/(1+x))}{F_\Delta(x/(1+x))}, \quad \begin{cases} y(0) = +\infty, & \frac{dy(x)}{dx} < 0, \ x > 0. \end{cases}
\]
Let us consider the periodized set

\[ \mathcal{F}_\infty := \bigcup_{m \in \mathbb{Z}} \left( 2m + \mathcal{F}_\square \right) . \]  

(1.37)

From Corollary 1.3 we derive an integral formula for \( \log \Theta_3 \),

\[ \log \Theta_3(z) = \frac{1}{2\pi} \int_0^{+\infty} \log \left( \frac{1}{1 - \lambda(i\tau)\lambda(z)} \right) \frac{d\tau}{1 + \tau^2}, \quad z \in \mathcal{F}_\square , \]  

(1.38)

or, equivalently (cf. (1.29)),

\[ \log \Theta_3(z) = \frac{1}{2\pi} \int_0^{+\infty} \log \left( \frac{\Theta_3(i\tau)^4\Theta_3(z)^4}{\Theta_3(i\tau)^4\Theta_3(z)^4 - \Theta_2(i\tau)^4\Theta_2(z)^4} \right) \frac{d\tau}{1 + \tau^2}, \quad z \in \mathcal{F}_\square . \]

We should contrast this integral formula with the classical series representation (see (9.1), (9.2), (9.3) and compare, e.g., with [10, p. 338, (4.2)])

\[ \log \Theta_3(z) = \sum_{n \geq 1} \frac{2}{2n - 1} \frac{e^{i\pi(2n-1)z}}{1 + e^{i\pi(2n-1)z}}, \quad z \in \mathbb{H} . \]

The Berndt formula. Let \( \text{SL}_2(\mathbb{Z}) \) be the multiplicative group of all \( 2 \times 2 \) matrices

\[ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 . \]

To such a matrix we associate a Möbius transformation

\[ \phi_M(z) := \frac{az + b}{cz + d}, \quad z \in \mathbb{H} , \]

and note that the Möbius transformation retains all the information about the matrix except that the matrices \( M \) and \( -M \) give rise to the same Möbius transformation. We consider the following subsets of the group \( \text{SL}_2(\mathbb{Z}) \):

\[ \text{SL}_2(2, \mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2} \right\} , \]

\[ \text{SL}_2(\vartheta, \mathbb{Z}) := \text{SL}_2(2, \mathbb{Z}) \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2} \right\} . \]

We associate with these subsets the induced collections of Möbius transformations:

\[ \Gamma := \{ \phi_M \mid M \in \text{SL}_2(\mathbb{Z}) \} , \]

\[ \Gamma_{\vartheta} := \{ \phi_M \mid M \in \text{SL}_2(\vartheta, \mathbb{Z}) \} , \quad \Gamma(2) := \{ \phi_M \mid M \in \text{SL}_2(2, \mathbb{Z}) \} . \]  

(1.39)

\[ \square \]
It follows from [9, p. 15, Def. 3.3; p. 16, Thm. 3.1] that
\[ \mathbb{H} = \bigcup_{M = (a \ b \ c \ d) \in SL_2(\vartheta, \mathbb{Z})} \phi_M \left( \text{clos}_\mathbb{H} (\mathcal{F}_0) \right), \quad (1.40) \]

where
\[ \text{clos}_\mathbb{H} (\mathcal{F}_0) := \{ z \in \mathbb{H} \mid -1 \leq \text{Re} z \leq 1, \ |z| \geq 1 \} \subset \mathcal{F}_0^\text{||}. \quad (1.41) \]

This means that for arbitrary \( z \in \mathbb{H}, \) there exists at least one matrix \( M \in SL_2(\vartheta, \mathbb{Z}) \) such that \( \phi_M(z) \in \mathcal{F}_0^\text{||}. \) Now, according to the formula of Berndt (see [10, p. 339, Thm. 4.1]), we have that for any \( M = (a \ b \ c \ d) \in SL_2(\vartheta, \mathbb{Z}) \) with \( c > 0 \) and \( z \in \mathbb{H}, \)
\[ \log \Theta_3 \left( \frac{az+b}{cz+d} \right) = \log \Theta_3(z) + \frac{1}{2} \log \frac{cz+d}{i} + \frac{\pi i}{4} \sum_{k=1}^{c} (-1)^{k+1} \left\lfloor \frac{kd}{c} \right\rfloor, \quad (1.42) \]

where, in view of (9.3), the branch of the logarithm \( \log \Theta_3 \) is selected which is real-valued on the positive imaginary axis, and \( \lfloor x \rfloor \) denotes the integer part of \( x \in \mathbb{R}. \)

Hence, (1.35) and (1.36) combined with (1.42) supply an integral representation for \( \log \Theta_3(z) \) when \( z \in \mathbb{H}, \) since we may find a matrix \( M = (a \ b \ c \ d) \in SL_2(\vartheta, \mathbb{Z}) \) with \( c > 0 \) and \( z \in \mathbb{H}, \)

\[ \log \Theta_3 (z) = \log \Theta_3 (z+1), \quad \log \Theta_2 (z) = \log \Theta_3 \left( 1 - \frac{1}{z} \right) - \frac{1}{2} \log \frac{z}{1}. \]

2 Basic Facts About \( F_\Delta := F_{1/2, 1/2; 1} \)

To simplify the notation, we denote by \( \overline{\mathbb{D}} \) and \( \overline{\mathbb{H}} \) the closures of \( \mathbb{D} \) and \( \mathbb{H} \) in the complex plane \( \mathbb{C}, \) respectively. The series (1.8) and the Euler formula (1.9) for the hypergeometric function in (1.17) have the following form
\[ F_\Delta(z) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)^2}{(n!)^2} z^n, \quad z \in \mathbb{D}, \quad (2.1) \]
\[ F_\Delta(z) = \frac{1}{\pi} \int_{0}^{1} \frac{dt}{\sqrt{t(1-t)(1-tz)}}, \quad z \in \mathbb{C} \setminus [1, +\infty). \quad (2.2) \]

In addition, the function \( F_\Delta \) satisfies the Pfaff formula (see [4, p. 79])
\[ F_\Delta(z) = F_\Delta \left( \frac{z}{z-1} \right), \quad z \in \mathbb{C} \setminus [1, +\infty). \quad (2.3) \]
and for \( z \in (1 + \mathbb{D}) \setminus [1, +\infty) = (1 + \mathbb{D}) \setminus [1, 2] \) has the following expansion (see [1, p. 559, 15.3.10], [5, p. 25])

\[
F_\Delta(z) = \frac{1}{\pi} F_\Delta(1 - z) \log \frac{1}{1 - z} + \frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})^2}{(n!)^2} \left[ \sum_{k \geq n} \frac{1}{(k + 1)(2k + 1)} \right] (1 - z)^n ,
\]

where [29, p. 658, 5.1.8.2] \( \sum_{k \geq 0} (k + 1)^{-1}(2k + 1)^{-1} = 2 \ln 2 \) and the summand corresponding to \( n = 0 \) in the series above is equal to \( \pi^{-1} \ln 16 \). Therefore

(a) \( F_\Delta(z) = \frac{1}{\pi} \log \frac{16}{1 - z} + O \left( |1 - z| \ln \frac{1}{|1 - z|} \right) \), \( z \to 1, \ z \notin [1, 2] \),

(b) \( F_\Delta(z) = 1 + O(|z|) \), \( z \to 0, \ z \in \mathbb{D} \),

(c) \( F_\Delta(z) = \frac{\log 16(1 - z)}{\pi \sqrt{1 - z}} + O \left( \frac{\ln |z|}{|z|^{3/2}} \right) \), \( |z| \to +\infty, \ z \notin [2, +\infty) \).

Here, the property (2.5)(b) is immediate from (2.1), while (2.5)(a) results from (2.4) by the asymptotics of (2.5)(b) applied to the term \( F_\Delta(1 - z) \) and from the observation that for \( z \to 1 \) only the term with \( n = 0 \) survives in the series in (2.4). Finally, (2.5)(c) is obtained by substitution of (2.5)(a), with \( z/(z - 1) \) in place of \( z \), in (2.3) under the condition that \( z/(z - 1) \in (1 + \mathbb{D}) \setminus [1, 2] \) which is equivalent to \( z \notin (1 + \mathbb{D}) \cup [2, +\infty) \) (see [5, p. 27]).

It is well-known that the functions \( F_\Delta(z) \) and \( F_\Delta(1 - z) \) are two independent solutions of the Euler hypergeometric differential equation (see [4, p. 75])

\[
z(z - 1)y''(z) + (2z - 1)y'(z) + \frac{1}{4} y(z) = 0 , \quad z \in \mathbb{D} ,
\]

whose linear independence can be easily deduced from the formula (2.1), invariance of (2.6) with respect to the change of the variable \( z \mapsto 1 - z \) and the expansion (2.4).

The constant \( A \) in the formula for the Wronskian in [4, p. 136, Lem. 3.2.6]

\[
W_\Delta(z) := F_\Delta'(z) F_\Delta(1 - z) + F_\Delta(z) F_\Delta'(1 - z) = \frac{A}{z(1 - z)}
\]

where (see [1, p. 557, 15.2.1])

\[
F_\Delta'(z) = \frac{1}{4} F_{3/2,3/2;2}(z) , \quad z \in \mathbb{C} \setminus [1, +\infty) ,
\]

can be calculated by letting \( z \to 1 \) in this formula and by using the relations \( (1 - z) F_\Delta(z) \to 0, \ (1 - z) F_\Delta'(z) \to 1/\pi \) as \( z \to 1 \), which are immediate from the expansion (2.4) and its differentiated form. This gives (cf.[2, p. 6, (2.5)])
Exponential Integral Representations of Theta Functions 603

\[ W_\Delta(z) = \frac{1}{\pi z (1 - z)} , \quad z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}) . \quad (2.7) \]

For arbitrary \( x > 1 \) and \( t \in [0, 1] \) we obviously have \( |1 - t(x \pm i \varepsilon)| \geq |1 - tx|, \) \( \varepsilon > 0, (\varepsilon \downarrow 0 \text{ means that } \varepsilon \to 0 \text{ and } \varepsilon > 0) \)

\[
\lim_{\varepsilon \downarrow 0} \sqrt{1 - t(x \pm i \varepsilon)} = \begin{cases} 
\sqrt{1 - tx} , & \text{if } 0 \leq t < 1/x ; \\
\varepsilon^{\pm i \pi/2} \sqrt{tx - 1} , & \text{if } 1/x < t \leq 1 ;
\end{cases}
\]

and therefore, by (2.2) and the Lebesgue dominated convergence theorem \[31, p. 26, 1.34\], there exist the finite limits \( (F_\Delta(x \pm i 0) := \lim_{\varepsilon \downarrow 0} F_\Delta(x \pm i \varepsilon) \) are called “radial” limits at \( x \))

\[
F_\Delta(x \pm i 0) = \frac{1}{\pi} \int_0^{1/x} \frac{dt}{\sqrt{t(1 - t)(1 - tx)}} \pm \frac{i}{\pi} \int_{1/x}^1 \frac{dt}{\sqrt{t(1 - t)(tx - 1)}} ,
\]

where (see \[5, p. 31\])

\[
\int_0^{1/x} \frac{dt}{\sqrt{t(1 - t)(1 - tx)}} = \frac{1}{\sqrt{x}} \int_0^{1/x} \frac{dt}{\sqrt{t(1 - t)(1/x - t)}} = \frac{\pi}{\sqrt{x}} F_\Delta\left(\frac{1}{x}\right) ,
\]

\[
\int_{1/x}^1 \frac{dt}{\sqrt{t(1 - t)(tx - 1)}} = \frac{1}{\sqrt{x}} \int_0^{1-1/x} \frac{dt}{\sqrt{1 - t(1/x - t)}} = \frac{\pi}{\sqrt{x}} F_\Delta\left(1 - \frac{1}{x}\right) .
\]

Thus (see \[28, p. 491, 19.7.3; p. 490, 19.5.1\]),

\[
F_\Delta(x \pm i 0) = \frac{1}{\sqrt{x}} F_\Delta\left(\frac{1}{x}\right) \pm \frac{i}{\sqrt{x}} F_\Delta\left(1 - \frac{1}{x}\right) , \quad x > 1 ,
\]

which can also be written as

\[
F_\Delta(1 + x \pm i 0) = \frac{1}{\sqrt{x}} F_\Delta\left(-\frac{1}{x}\right) \pm i F_\Delta\left(-x\right) , \quad x > 0 ,
\]

by virtue of the following equivalent forms of (2.3)

\[
F_\Delta(-z) = \frac{1}{\sqrt{1 + z}} F_\Delta\left(\frac{z}{1 + z}\right) , \quad z \in \mathbb{C} \setminus (-\infty, -1) , \quad (2.10)
\]

\[
\frac{1}{\sqrt{z}} F_\Delta\left(-\frac{1}{z}\right) = \frac{1}{\sqrt{1 + z}} F_\Delta\left(\frac{1}{1 + z}\right) , \quad z \in \mathbb{C} \setminus [-1, 0] . \quad (2.11)
\]
We observe that the relation (2.8) can also be obtained from one of Kummer’s transformation rules,

\[ (-z)^{-1/2} F_{\Delta} \left( \frac{1}{z} \right) - i F_{\Delta}(z) \text{sign}(\text{Im } z) = z^{-1/2} F_{\Delta} \left( 1 - \frac{1}{z} \right), \quad z \in \mathbb{C} \setminus \mathbb{R}, \]

where the principal branch of the square root is used and sign(x) is equal to −1 if \( x < 0 \) if \( x = 0 \) and 1 if \( x > 0 \) (cf. [15, p. 106, (27)], [5, p. 33]).

It follows from (2.5)(a)–(c) that \( F_{\Delta} \) belongs to the Hardy space \( H^p \) for arbitrary \( 2 < p < \infty \) (see [5, p. 35]). According to the Schwarz integral formula applied to \( i F_{\Delta} \) (see [33, p. 227], [20, p. 128]) we get from (2.8) and from the obvious consequence of (2.2), \( \text{Im } F_{\Delta}(x) = 0, -\infty < x < 1, \) that

\[ F_{\Delta}(z) = \frac{1}{\pi} \int_{1}^{\infty} F_{\Delta} \left( 1 - \frac{1}{t} \right) \frac{dt}{(t-z)\sqrt{t}}, \quad z \in \mathbb{C} \setminus [1, +\infty). \quad (2.12) \]

By using the Cauchy theorem (see [12, p. 89, 6.6]) and (2.5)(a), it can easily be shown that the contour of integration in (2.12) can be changed to \( 1 + i \mathbb{R}_{\geq 0} \) if \( \text{Im } z < 0 \) and to \( 1 - i \mathbb{R}_{\geq 0} \) if \( \text{Im } z > 0 \), where \( \mathbb{R}_{\geq 0} := [0, +\infty) \). So that

\[ F_{\Delta}(z) = e^{-i\pi \sigma/2} \int_{0}^{\infty} F_{\Delta} \left( \frac{t}{t+i\sigma} \right) \frac{dt}{(1-it\sigma-z)\sqrt{1-it\sigma}}, \quad z \in \sigma \cdot \mathbb{H}, \quad \sigma \in \{1, -1\}, \]

and hence the function \( F_{\Delta} \) allows a holomorphic extension from the upper half-plane \( \mathbb{H} \) to \( \mathbb{C} \setminus (1 - i \mathbb{R}_{\geq 0}) \) and from the lower half-plane \( -\mathbb{H} \) to \( \mathbb{C} \setminus (1 + i \mathbb{R}_{\geq 0}) \) (see [5, p. 36]). This means that \( F_{\Delta} \) can be continuously extended from \( \mathbb{H} \) to \( \mathbb{H} \cup (\mathbb{R} \setminus \{1\}) \) and from \( -\mathbb{H} \) to \( (-\mathbb{H}) \cup (\mathbb{R} \setminus \{1\}) \), and that for every point \( x \in \mathbb{R} \setminus \{1\} \) there exist two finite “radial” limits satisfying \( F_{\Delta}(x \pm i) = \lim_{\mathbb{H} \ni z \to 0} F_{\Delta}(x \pm z) \), which can be written in the following form, by virtue of (2.8) and (2.10),

\[ \lim_{\mathbb{H} \ni z \to 0} F_{\Delta}(x \pm z) = \begin{cases} \frac{1}{\sqrt{x}} F_{\Delta} \left( \frac{1}{x} \right) \pm \frac{i}{\sqrt{x}} F_{\Delta} \left( 1 - \frac{1}{x} \right), & \text{if } x > 1; \\ F_{\Delta}(x), & \text{if } 0 \leq x < 1; \\ \frac{1}{\sqrt{1+|x|}} F_{\Delta} \left( \frac{|x|}{1+|x|} \right), & \text{if } x < 0. \end{cases} \quad (2.13) \]

In view of the obvious consequence of (2.1)

\[ F_{\Delta}(x) > 0, \quad 0 \leq x < 1, \quad (2.14) \]

the expressions (2.13) yield

\[ \text{Re } F_{\Delta}(x \pm i 0) > 0, \quad x \in \mathbb{R} \setminus \{1\}; \quad F_{\Delta}(x) > 0, \quad -\infty < x < 1; \]
\[ \text{Im } F_{\Delta}(x \pm i 0) = 0, \quad -\infty < x < 1; \quad \text{Im } F_{\Delta}(x+i 0) > 0, \quad x > 1. \quad (2.15) \]
The validity of (1.19) follows from (2.12), (1.3), (1.4) and (2.15) (see [21, p. 604, Rem. 2.1]), which in turn proves the correctness of the definition (1.17). This allows to obtain (1.20) from (2.7) and the identity $i \lambda'_\Delta(z) F_\Delta(z)^2 = W_\Delta(z)$, which follows from the definition of the Wronskian $W_\Delta$ and that of the Schwarz triangle function $\lambda_\Delta$.

By setting $z = x > 0$ in (2.10) and (2.11) we derive from (2.9) that

$$
\frac{F_\Delta(1 + x \pm i0)}{F_\Delta(-x)} = \pm i + \frac{F_\Delta\left(\frac{1}{1+x}\right)}{F_\Delta\left(1 - \frac{1}{1+x}\right)}, \quad x > 0.
$$

### 3 Exponential Integral Representation of $F_{1/2,1/2;1}$

In the sequel, for $\mu \in \mathcal{M}^+(\mathbb{R})$ we look at the spaces of Borel measurable real-valued functions $L^p(\mathbb{R}, d\mu)$, $1 \leq p < \infty$, and for arbitrary function $v \in L^p(\mathbb{R}, dx)$ with $1 < p < \infty$ we use the notation for the (sign changed) Hilbert transform

$$
\widetilde{v}(x) := \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|t-x|>\varepsilon} \frac{v(t) \, dt}{t-x}, \quad x \in \mathbb{R},
$$

where it is known that $\tilde{v} \in L^p(\mathbb{R}, dx)$ by the M.Riesz theorem (see [20, p. 128]). Instead of applying the canonical factorization theorem (see [18, p. 74, Thm. 5.5], [20, p. 119]) to the function $F_\Delta$ in $H^p$, we use the property (1.19) of $F_\Delta$ being in $P(({-\infty}, 1))$ and the corresponding representation (1.5) in the following improved form established in [6, Thm. 2.6, Thm. 2.7, Thm. 2.8].

**Theorem C** Let $v \in \bigcup_{p>1} L^p(\mathbb{R}, dx)$ be non-zero and satisfy $v(x) = 0$, if $x < 1$, $v(x) \geq 0$, if $x \geq 1$,

$$
\int_1^{+\infty} \frac{v(t) \, dt}{t} = \pi, \quad \left| \frac{\tilde{v}(x_1) \tilde{v}(x_2)}{v(x_1) v(x_2)} \right| \geq 0
$$

for almost all $1 < x_1 < x_2 < +\infty$. Then the function

$$
\Psi(z) := \frac{1}{\pi} \int_1^{+\infty} \frac{v(t) \, dt}{t - z}, \quad z \in \mathbb{H},
$$

belongs to $P_{\log}$, and there exists a non-decreasing right-continuous function $v$ on $\mathbb{R}$ satisfying $0 = v(y) \leq v(x) \leq 1$, $-\infty \leq y < 1 < x < +\infty$, such that for arbitrary $z \in \mathbb{C} \setminus [1, +\infty)$ the following equalities hold

$$
\Log \Psi(z) = \beta + \int_1^{+\infty} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) v(t) \, dt = \int_{[0,1]} \Log \frac{1}{1 - tz} \, d\mu(t),
$$
where the measure \( \mu \in \mathcal{M}^+(\mathbb{R}) \) and the real constant \( \beta \) are defined by

\[
\beta := \int_0^1 t \nu\left(\frac{1}{t}\right) \frac{dt}{1 + t^2},
\]

\[
\begin{align*}
\mu((-\infty, 0)) &:= 0, \\
\mu([0, x)) &:= v(+\infty) - v\left(\frac{1}{x}\right), \quad x > 0.
\end{align*}
\] (3.4)

Denote by the same letter \( \nu \) the Lebesgue-Stieltjes measure induced on \( \mathbb{R} \) by a non-decreasing function \( \nu: \mathbb{R} \mapsto [0, 1] \) in Theorem C (see [26, p. 147, Def. 3.9]). In addition to that theorem, we will need the following relationships proved in [6, Thm. 2.2, Thm. 2.6] and [6, (2.37)].

**Corollary A** Under the conditions of Theorem 3,

\[
supp \mu \subseteq [0, 1], \quad \lim_{x \downarrow 0} \mu([0, x)) = 0, \quad \mu([0, 1]) = v(+\infty) \in (0, 1],
\]

and for almost all \( x \in \mathbb{R} \) we have

(a) \( \Psi(x + i 0) = \tilde{v}(x) + i v(x) \),

(b) \( \ln \frac{\sqrt{1 + t^2}}{|t - x|} \in L^1(\mathbb{R}, d\nu(t)) \),

\[
\tilde{v}(x) = \left[ \cos \pi v(x) \right] \exp \left( \int_{[1, +\infty)} \ln \frac{t}{|x - t|} d\nu(t) \right),
\] (3.6)

\[
v(x) = \left[ \sin \pi v(x) \right] \exp \left( \int_{[1, +\infty)} \ln \frac{t}{|x - t|} d\nu(t) \right).
\] (3.7)

We observe that property (3.5)(a) follows directly from (3.2) for an arbitrary function \( v \in L^p(\mathbb{R}, dx) \), that vanishes on the interval \((-\infty, 1)\) (for \( 1 < p < \infty \)), in view of known consequences of the M. Riesz theorem (see [20, p. 128]). This fact allows us to deduce from (2.12) that (2.13) yields

\[
v(x) := \frac{F_{\Delta}(1 - \frac{1}{x}) \chi_{[1, +\infty)}(x)}{\sqrt{x}}, \quad x \in \mathbb{R} \Rightarrow \tilde{v}(x) = \frac{F_{\Delta}\left(\frac{1}{x}\right)}{\sqrt{x}}, \quad x > 1.
\] (3.8)

For such \( v \) and \( \tilde{v} \) the equality (3.2) for \( \Psi = F_{\Delta} \) coincides with (2.12) and the condition (3.1)(b) holds because it is equivalent to the non-increasing property of the function \( F_{\Delta}(1 - x)/F_{\Delta}(x) \) on the interval \((0, 1)\) which follows readily from the following consequence of (2.7),

\[
\frac{d}{dx} \frac{F_{\Delta}(1 - x)}{F_{\Delta}(x)} = -\frac{1}{\pi x(1 - x)F_{\Delta}(x)^2} < 0, \quad x \in (0, 1).
\] (3.9)

Furthermore, the condition (3.1)(a) also holds in view of the known integral relationship (see [17, p. 399, (4)])

\[
\int_1^{+\infty} \frac{v(t)}{t} dt = \int_1^{+\infty} t^{-3/2} F_{\Delta}\left(1 - \frac{1}{t}\right) dt = \int_0^1 (1 - t)^{-1/2} F_{\Delta}(t) dt = \pi.
\]
Thus, for $\Psi = F_{\Delta}$ and $\nu, \tilde{\nu}$ defined as in (3.8) the conditions of Theorem C hold and we can apply the results of Corollary A to calculate the function $\nu$. Dividing (3.7) by (3.6) for $x > 1$, we obtain, by virtue of (2.14),

$$\tan \pi \nu(x) = \frac{v(x)}{\tilde{v}(x)} = \frac{F_{\Delta}(1 - \frac{1}{x})}{F_{\Delta}(\frac{1}{x})} > 0, \quad x > 1.$$  

Since $v(x) \in [0, 1]$ we conclude that

$$v(x) = \frac{1}{\pi} \arctan \left( \frac{F_{\Delta}(1 - \frac{1}{x})}{F_{\Delta}(\frac{1}{x})} \right), \quad x > 1, \quad \left\{ \begin{array}{l} v(1 + 0) = 0, \\
v(+\infty) = 1/2, \end{array} \right.$$  

(3.10)

and the differentiation of this equality, by taking account of (3.9), gives

$$v'(x) = \frac{1}{\pi^2(x - 1)} \frac{1}{F_{\Delta}(\frac{1}{x})^2 + F_{\Delta}(1 - \frac{1}{x})^2} > 0, \quad x > 1,$$

and $v' \in L^1([0, 1], dx)$, in view of (2.5)(a). This means that the formulas (3.4) (see also (3.10)) for the measure $\mu$ can be written as follows

$$\mu([0, x)) = v(+\infty) - v\left( \frac{1}{x} \right) = \frac{1}{2} - \frac{1}{\pi} \arctan \left( \frac{F_{\Delta}(1 - x)}{F_{\Delta}(x)} \right), \quad x \in (0, 1);$$

$$\mu([0, x)) = \frac{1}{2}, \quad x \geq 1; \quad \mu(\{0\}) = 0.$$  

(3.11)

This proves the validity of (1.13) and shows that $\mu$ is absolutely continuous with respect to the Lebesgue measure $m$ on $[0, 1]$ and it follows from (3.9) and (2.5)(a) that the Radon-Nikodym derivative $d\mu/dx$ of $\mu$ with respect to $m$ (see [26, p. 214]) has the following form

$$\frac{d\mu(x)}{dx} = \frac{1}{\pi^2x(1-x)} \left( \frac{1}{F_{\Delta}(x)^2 + F_{\Delta}(1-x)^2} \right), \quad x \in (0, 1),$$  

(3.12)

and $d\mu/dx \in L^1([0, 1], dx)$. Therefore for arbitrary $z \in \mathbb{C} \setminus [1, +\infty)$ we can write exponential integral representations (3.3) in the forms (1.15) and (1.16) where $\beta := \int_0^1(tv(1/t)/(1+t^2))dt$. It follows from (3.11), (3.12) and (1.15) that

$$\text{Arg} F_{\Delta}(z) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), \quad z \in \mathbb{C} \setminus [1, +\infty),$$  

which shows that (1.14) holds, and we have

$$\log F_{\Delta}(1-z) - \log F_{\Delta}(z) = \log \frac{F_{\Delta}(1-z)}{F_{\Delta}(z)}, \quad z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}),$$

$$\log F_{\Delta}(z)^{\alpha} = \alpha \log F_{\Delta}(z), \quad 0 < \alpha \leq 2, \quad z \in \mathbb{C} \setminus [1, +\infty).$$
Then the representation (1.15) written for \( z \in \mathbb{C} \setminus [1, +\infty) \) in the form

\[
\frac{zF_{\Delta}(z)^{\alpha}}{z} = \exp \left( \int_{[0,1]} \frac{1}{1 - tz} \log (\alpha \mu(t)) \, dt \right), \quad \alpha \mu([0, 1]) = \frac{\alpha}{2} \in (0, 1),
\]

gives Corollary 1.1 because, by virtue of (3.11), (1.6) and (1.7) hold for \( \Psi = zF_{\Delta}(z)^{\alpha} \) and \( \alpha \mu \) in place of \( \sigma \). In addition, the representations (1.15) and (1.16) for arbitrary \( z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}) \) yield (see [5, p. 38])

\[
\log \frac{F_{\Delta}(1-z)}{F_{\Delta}(z)} = \frac{1}{\pi^2} \int_{0}^{1} \frac{\log \frac{1-tz}{1-t+tz}}{t(1-t)} \left( F_{\Delta}(t)^2 + F_{\Delta}(1-t)^2 \right) \, dt, \tag{3.13}
\]

\[
\log \frac{F_{\Delta}(1-z)}{F_{\Delta}(z)} = \frac{1-2z}{\pi} \int_{0}^{1} \frac{\arctan \frac{F_{\Delta}(1-t)}{F_{\Delta}(t)}}{(1-tz)(1-t+tz)} \, dt. \tag{3.14}
\]

To obtain (3.13), it is sufficient to use the fact that for arbitrary points \( z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}) \) and \( t \in (0, 1) \), the two numbers \( 1-tz \) and \( 1-t+tz \) lie in the open half-plane \( \{ a + sz \mid s \in \mathbb{R}, \ a > 0 \} \) which implies that

\[
| \arg (1-tz) - \arg (1-t+tz) | < \pi, \quad t \in (0, 1), \ z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}), \tag{3.15}
\]

and therefore we have

\[
\log \frac{1}{1-tz} - \log \frac{1}{1-t+tz} = \log \frac{1-tz}{1-t+tz}, \quad t \in (0, 1), \ z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}).
\]

Together with (3.11) and (3.12), the inequality (3.15) allows us to deduce from (3.13) that

\[
\arg \frac{F_{\Delta}(1-z)}{F_{\Delta}(z)} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}). \tag{3.16}
\]

By (1.17), the formulas (3.13) and (3.14) give the exponential representation of \( \lambda_{\Delta}/i \) for all \( z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}) \). Moreover, for arbitrary \( t \in (0, 1) \) and \( z \in \mathbb{C} \setminus \mathbb{R} \) the sign of \( \arg (1-tz) - \arg (1-t+tz) \) is obviously equal to \( -\text{sign}(\text{Im} \, z) \) and therefore it follows from (3.13), (3.11) and (1.17) that

\[
\arg \lambda_{\Delta}(z) \in \left[0, \frac{\pi}{2}\right) \cdot \text{sign}(\text{Im} \, z), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.17}
\]
4 Special Properties of \( \lambda_\Delta \)

By (2.16), we have

\[
\lambda_\Delta(-x \pm i 0) = \pm 1 + \lambda_\Delta \left( \frac{x}{1 + x} \right) = \pm 1 + i \frac{F_\Delta \left( \frac{x}{1 + x} \right)}{F_\Delta \left( \frac{1}{1 + x} \right)} , \quad x > 0 . \tag{4.1}
\]

**Proof of (1.21), (1.22) and (1.23).** The properties (1.19) and definition (1.17) imply the validity of (1.21). It follows from (4.1) that (1.22) holds. Setting \( z = -x \pm i 0 \) and 
\( z = 1 + x \pm i 0 \) in (1.21) and combining these with (1.22), we obtain that (1.23) holds for arbitrary \( x > 0 \) (see [5, p. 41]). \( \square \)

For the function \( \lambda_\Delta \) we introduce its *remainders from singularities*

\[
\lambda_\Delta(\infty; z) := \frac{1}{\lambda_\Delta(z)} - \text{sign}(\text{Im} z) + \frac{i}{\pi} \ln |z| ; \quad \lambda_\Delta(0; z) := -\lambda_\Delta(z) + \frac{i}{\pi} \ln \frac{1}{|z|} ;
\]

\[
\lambda_\Delta(1; z) := \frac{1}{\lambda_\Delta(z)} + \frac{i}{\pi} \ln \frac{1}{|1 - z|} , \quad z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}) . \tag{4.2}
\]

A direct calculation of ratios of asymptotic expansions (2.5) according to formula (1.24) and combining these with (1.22), we obtain that (1.23) holds for arbitrary \( x > 0 \) (see [5, p. 41]). \( \square \)

**Lemma 4.1** Let \( \Lambda := (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}) \). Then the following asymptotic formulas hold:

\[
\lambda_\Delta(\infty; z) = \frac{\ln 16}{i \pi} + \frac{\text{Arg} (1 - z)}{\pi} + O \left( \frac{\ln^2 |z|}{|z|} \right) , \quad \Lambda \ni z \to \infty , \tag{4.3}
\]

\[
\lambda_\Delta(1; z) = \frac{\ln 16}{i \pi} + \frac{\text{Arg} \left( \frac{1}{1 - z} \right)}{\pi} + O \left( |1 - z| \ln \frac{1}{|1 - z|} \right) , \quad \Lambda \ni z \to 1 , \tag{4.4}
\]

\[
\lambda_\Delta(0; z) = \frac{\ln 16}{i \pi} + \frac{\text{Arg} \left( \frac{1}{z} \right)}{\pi} + O \left( |z| \ln \frac{1}{|z|} \right) , \quad \Lambda \ni z \to 0 . \tag{4.5}
\]

**Here,** \( 1 - z, 1/(1 - z), 1/z \in \mathbb{C} \setminus (-\infty, 0] \) for any \( z \in \Lambda \).

The following crucial properties of the remainders from singularities hold. As for notation, let \( \text{sign}(x) \) be equal to \(-1 \) if \( x < 0 \), \( 0 \) if \( x = 0 \) and \( 1 \) if \( x > 0 \).

**Lemma 4.2** Let \( \sigma(z) := \text{sign}(\text{Im} z) , \quad z \in \mathbb{C} \setminus \mathbb{R} \). Then

\[
\lim_{\Lambda \ni z \to 0} \left| \text{Re} \lambda_\Delta(z) \right| \leq 1 , \quad \lim_{\Lambda \ni z \to 1} \lambda_\Delta(z) = \lim_{\Lambda \ni z \to \infty} \left| \lambda_\Delta(z) - \sigma(z) \right| = 0 , \tag{4.6}
\]

and there exists a finite positive number \( \varepsilon_\Delta \) such that

\[
(a) \quad |\lambda_\Delta(\infty; z)| \leq 2 , \quad |z| \geq \frac{1}{\varepsilon_\Delta} ; \quad (b) \quad |\lambda_\Delta(0; z)| \leq 2 , \quad |z| \leq \varepsilon_\Delta ;
\]

\[
(c) \quad |\lambda_\Delta(1; z)| \leq 2 , \quad |z - 1| \leq \varepsilon_\Delta ; \quad z \in (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}) . \tag{4.7}
\]
Remark. The known property $\lambda(z \pm 1) = \lambda(z)/(\lambda(z) - 1), z \in \mathbb{H}$, of the elliptic modular function $\lambda$ (see [11, p. 111]) which is immediate from (1.29) and (6.8) established below, together with the identity (1.28) imply that (see [5, p. 46])

$$
\lambda_{\Delta}(z) = \lambda_{\Delta}\left(\frac{z}{z - 1}\right) + \sigma(z), \quad z \in \mathbb{C} \setminus \mathbb{R}.
$$

By letting here $\Lambda \ni z \to \infty$ and by using the asymptotic formula (4.4), we can improve the remainder in the asymptotic formula (4.3) to $O(|z|^{-1}\ln|z|)$.

Proof of Lemma 4.2 The formulas (4.3), (4.4), (4.5), the estimate $(1/\pi) \ln 16 < 0, 883$ and the fact that $\arg y \in (-\pi, \pi), y \in \mathbb{C} \setminus (-\infty, 0]$, together imply that (4.7) and two limit identities in (4.6) hold. Regarding the inequality (4.6), it is obtained by separation of the real and imaginary parts in (4.5). This completes the proof of Lemma 4.2.

Proof of (1.24). By (4.1), we have for $x > 0$ that $|\Re\lambda_{\Delta}(-x \pm i 0)| = 1$, while after substitution of (4.1) in (1.21) we obtain $|\Re\lambda_{\Delta}(1 + x \pm i 0)| < 1$. Together with the following obvious consequence of (2.13) and (2.14),

$$
\lambda_{\Delta}(x \pm i 0) = \lim_{\mathbb{H} \ni z \to 0} \lambda_{\Delta}(x \pm z), \quad x \in \mathbb{R} \setminus [0, 1), \quad (4.8)
$$

and (4.6) this entails that

$$
\lim_{\Lambda \ni z \to a} \pm \Re\lambda_{\Delta}(z) \leq \lim_{\Lambda \ni z \to a} |\Re\lambda_{\Delta}(z)| \leq 1, \quad a \in [\infty] \cup \partial\Lambda.
$$

By the maximum principle applied to the two harmonic functions $-1 \pm \Re\lambda_{\Delta}(z)$ in $\Lambda$, we obtain that (see [12, pp. 254, 129, 40], [19, p. 47], [5, p. 48])

$$
|\Re\lambda_{\Delta}(z)| < 1, \quad z \in \Lambda. \quad (4.9)
$$

Next, if we assume that there exists a point $\zeta \in \lambda_{\Delta}(\Lambda) \setminus \mathcal{F}_{\square}$ with $\Re\zeta \in (-1, 1)$, we appeal to (1.21) and the reflection property $1 - \Lambda = \Lambda$ to obtain that $-1/\zeta \in \lambda_{\Delta}(\Lambda)$. On the other hand, as $\mathcal{F}_{\square}$ is invariant under inversion, the facts that $\zeta \notin \mathcal{F}_{\square}$ and $-1 < \Re \zeta < 1$ entail that $|\Re(-1/\zeta)| > 1$. This, however, is impossible because as we just observed $-1/\zeta \notin \lambda_{\Delta}(\Lambda)$, which would yield by (4.9) the opposite inequality. Thus the inclusion $\lambda_{\Delta}(\Lambda) \subset \mathcal{F}_{\square}$ is immediate.

To obtain the reverse inclusion as well, we observe that the set $\lambda_{\Delta}(\Lambda)$ is open, by the open mapping theorem (see [12, p. 99]). Suppose to the contrary that $\mathcal{F}_{\square} \setminus \lambda_{\Delta}(\Lambda) \neq \emptyset$. Then one may find at least one point $\zeta$ in the intersection $\mathcal{F}_{\square} \cap \partial(\lambda_{\Delta}(\Lambda))$, as otherwise for each point $\xi \in \mathcal{F}_{\square} \setminus \lambda_{\Delta}(\Lambda)$ there would exist $\varepsilon = \varepsilon(\xi) > 0$ such that $(\xi + \varepsilon\mathbb{D}) \cap \lambda_{\Delta}(\Lambda) = \emptyset$. It would then follow that $\mathcal{F}_{\square} \setminus \lambda_{\Delta}(\Lambda)$ is open and that $\mathcal{F}_{\square}$ may be represented as the union of the two non-empty disjoint open subsets $\lambda_{\Delta}(\Lambda)$ and $\mathcal{F}_{\square} \setminus \lambda_{\Delta}(\Lambda)$, in contradiction with the connectivity of $\mathcal{F}_{\square}$ (see [22, p. 92, Thm. 1.6]). For this point $\zeta \in \mathcal{F}_{\square} \cap \partial(\lambda_{\Delta}(\Lambda))$ there must exist a sequence $\{y_n\}_{n \geq 1} \subset \Lambda$ such that $\zeta = \lim_{n \to \infty} \lambda_{\Delta}(y_n)$ and, by replacing this sequence by a suitable subsequence, we may assume that $y_n$ converges either to $\infty$ or to some boundary point $y_\infty \in \mathbb{H}$. Springer
It follows from (4.1) and (1.21) that
\[ \zeta \in \{ \lambda_\Delta(y_\infty \pm i0) \} \subset (\pm 1 + i\mathbb{R}_{>0}) \cup (\pm 1 - i\mathbb{R}_{>0})^{-1} \subset \partial \mathcal{F}_\Box. \]

Finally, if \( y_\infty \in \{ 0, 1 \} \) then according to (4.2), (4.4) and (4.5) we have either \( \zeta = \infty \) or \( \zeta = 0 \in \partial \mathcal{F}_\Box \). Thus for all cases we obtain \( \zeta \notin \mathcal{F}_\Box \) which contradicts the assumption \( \zeta \in \mathcal{F}_\Box \cap \partial \lambda_\Delta(A) \). This proves \( \mathcal{F}_\Box \subset \lambda_\Delta(A) \) and completes the proof of (1.24). \( \Box \)

## 5 Proof of Lemma 1.1

We introduce the function (see [5, p. 50])
\[ \Phi(z) := f(\lambda_\Delta(z)), \quad z \in \Lambda := (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}). \]

It follows from (4.1) and (1.21) that \( \lambda_\Delta(1 + x + i0) \in 1/(1 - i\mathbb{R}_{>0}) \) for arbitrary \( x > 0 \), and therefore, by (1.23) and the second invariance property Lemma 1.1(b) of \( f \), which can be written as \( f(z) = f(z/(1 - 2z)) \) for all \( z \in 1/(1 - i\mathbb{R}_{>0}) \), we deduce that \( \Phi(1 + x - i0) = \Phi(1 + x + i0), x > 0 \). While the first invariance property Lemma 1.1(a) of \( f \) together with (4.1) and (1.22) yields \( \Phi(-x - i0) = \Phi(-x + i0), x > 0 \). By Morera’s theorem (see [24, p. 96]) we get \( \Phi \in \text{Hol}(\mathbb{C}\setminus\{0, 1\}) \). In the notation for the remainders from singularities, the properties (4.7) entail that for arbitrary \( z \in \Lambda \) we have
\[
\begin{align*}
(a) \quad & \frac{1}{|\lambda_\Delta(z) - \sigma(z)|} \leq 2 + \frac{\ln |z|}{\pi}, \quad |z| \geq \frac{1}{\varepsilon_\Delta}; \\
(b) \quad & |\lambda_\Delta(z)| \leq 2 + \frac{1}{\pi} \ln \frac{1}{|z|}, \quad |z| \leq \varepsilon_\Delta; \\
(c) \quad & 1/|\lambda_\Delta(z)| \leq 2 + \frac{1}{\pi} \ln \left( \frac{1}{|1 - z|} \right), \quad |z - 1| \leq \varepsilon_\Delta.
\end{align*}
\]

Here \( \sigma(z) := \text{sign} \text{(Im} z) \), and, in view of (1.24) and (4.6), we obtain
\[
\begin{align*}
(a) \lambda_\Delta(z) - \sigma(z) & \to 0, \quad A \ni z \to \infty, \\
(b) \lambda_\Delta(z) & \to \infty, \quad A \ni z \to 0, \\
(c) \lambda_\Delta(z) & \to 0, \quad A \ni z \to 1, \\
(d) \lambda_\Delta(z) & \in \mathcal{F}_\Box, \quad z \in \Lambda.
\end{align*}
\]

By substituting \( \lambda_\Delta(z) \) in place of \( z \) in Lemma 1.1, (1)-(3), and letting \( z \to 0 \) in (1), \( z \to 1 \) in (2), and \( z \to \infty \) in (3), and in addition by using (5.2) and the inequalities (5.1), we find that
\[
\begin{align*}
(1) \quad & |z|^{n_0+1} |\Phi(z)| \to 0, \quad A \ni z \to 0, \\
(2) \quad & |1 - z|^{n_0+1} |\Phi(z)| \to 0, \quad A \ni z \to 1, \\
(3) \quad & |z|^{-n_1-1} |\Phi(z)| \to 0, \quad A \ni z \to \infty.
\end{align*}
\]
From the Riemann theorem about removable singularities (see [12, p. 103]) it follows that the function \( \Phi_1(z) := z^{n_0} (1 - z)^{n_1} \Phi(z) \) is holomorphic at the points 0 and 1 and is hence an entire function which, by the property (5.3)(3) above, has the asymptotics

\[
\Phi_1(z) = o \left( |z|^{n_0 + n_1 + n + 1} \right) \quad \text{as} \quad \Lambda \ni z \to \infty .
\] (5.4)

The continuity of \( \Phi_1 \) ensures the existence of \( C \in \mathbb{R}_{>0} \) such that \( |\Phi_1(z)| \leq C (1 + |z|)^{n_0 + n_1 + n + 1} \), \( z \in \mathbb{C} \), which by the extended version of the Liouville theorem (see [25, p. 2, Thm. 1]) yields that \( \Phi_1(z) \) is an algebraic polynomial of degree at most \( n_\infty + n_0 + n_1 + 1 \). But the relationship (5.4) proves that actually its degree cannot exceed \( n_\infty + n_0 + n_1 \). Lemma 1.1 follows.

6 Definitions of the Theta Functions

We introduce the functions

\[
\theta_3(u) := 1 + 2 \sum_{n \geq 1} u^{n^2} , \quad \theta_2(u) := 1 + \sum_{n \geq 1} u^{n^2 + n} , \quad \theta_4(u) := 1 + 2 \sum_{n \geq 1} (-1)^n u^{n^2} , \quad u \in \mathbb{D} ,
\] (6.1)

which are obviously holomorphic in the unit disk \( \mathbb{D} \) and satisfy

\[
\theta_4(u) = \theta_3(-u) , \quad \theta_2(-u) = \theta_2(u) , \quad u \in \mathbb{D} .
\] (6.2)

In order to apply the Poisson summation formula to the series in (6.1) we assume \( u \in (0, 1) \) and replace \( u \) by \( \exp(-\pi x) \) with \( x > 0 \). Then the well-known integrals [16, p. 146, (27)],

\[
\sqrt{\frac{\pi}{x}} e^{-2a \sqrt{x}} = \int_0^\infty e^{-xt} \frac{e^{-a^2/t}}{\sqrt{t}} \, dt , \quad \sqrt{\frac{\pi}{x}} e^{-a \sqrt{x}} = \int_0^\infty e^{-xt} \frac{e^{-a^2/(4t)}}{\sqrt{t}} \, dt , \quad a, x > 0 ,
\]

allow us to derive from the known formulas

\[
\coth \sqrt{\pi x} = 1 + 2 \sum_{n \geq 1} e^{-2n \sqrt{\pi x}} , \quad \frac{1}{\sinh \sqrt{\pi x}} = 2 \sum_{n \geq 0} e^{-(2n+1) \sqrt{\pi x}} ,
\]

that for arbitrary \( x > 0 \) we have

\[ \square \] Springer
\[
\sqrt{\pi} \coth \sqrt{\pi} x = \int_0^\infty e^{-x t} \frac{1}{\sqrt{t}} \left(1 + 2 \sum_{n \geq 1} e^{-\pi n^2 / t}\right) \, dt, \quad (6.3)
\]

\[
\sqrt{\pi} \sinh \sqrt{\pi} x = \int_0^\infty e^{-x t} \frac{2 e^{-\pi / (4 t)}}{\sqrt{t}} \left(\sum_{n \geq 0} e^{-\pi (n^2 + n) / t}\right) \, dt. \quad (6.4)
\]

The known expansions into the series of the simple fractions [28, p. 126]

\[
\sqrt{\pi} \coth \sqrt{\pi} x \sqrt{x} = \frac{1}{x} + 2 \sum_{n \geq 1} \frac{1}{x + \pi n^2}, \quad \sqrt{\pi} \sinh \sqrt{\pi} x = \frac{1}{x} + 2 \sum_{n \geq 1} \frac{(-1)^n}{x + \pi n^2},
\]

for arbitrary \( x > 0 \) yield readily that

\[
\sqrt{\pi} \coth \sqrt{\pi} x = \int_0^\infty e^{-x t} \left(1 + 2 \sum_{n \geq 1} e^{-n^2 \pi t}\right) \, dt, \quad (6.5)
\]

Comparing these equalities with (6.3) and (6.4) we conclude, by the uniqueness theorem for the Laplace transform (see [38, p. 63, Thm. 6.3]), that for any \( t > 0 \) we have

\[
\theta_3 \left( e^{-\pi t} \right) = t^{-1/2} \theta_3 \left( e^{-\pi / t} \right), \quad \theta_4 \left( e^{-\pi t} \right) = 2 t^{-1/2} e^{-\pi / (4 t)} \theta_2 \left( e^{-\pi / t} \right), \quad (6.5)
\]

and the replacement of \( t \) by \( 1/t \) in the latter equality gives (cf. Exercise 20 in [23, p. 23])

\[
2 e^{-\pi t / 4} \theta_2 \left( e^{-\pi t} \right) = t^{-1/2} \theta_4 \left( e^{-\pi / t} \right), \quad t > 0. \quad (6.6)
\]

To get a simpler form for writing the relationships (6.2), (6.5) and (6.6) between \( \theta_k(u), \) \( 2 \leq k \leq 4, \) the following analytic functions in \( \mathbb{H} \) are introduced

\[
\Theta_3(z) := \theta_3 \left( e^{i \pi z} \right), \quad \Theta_4(z) := \theta_4 \left( e^{i \pi z} \right), \quad \Theta_2(z) := 2 e^{i \pi z / 4} \theta_2 \left( e^{i \pi z} \right), \quad (6.7)
\]

where \( z \in \mathbb{H}. \) Regarding these functions, the main relationships can be written for arbitrary \( z \in \mathbb{H} \) as follows, by using the principal branch of the square root (see [28, p. 531, 20.7.27-29,31-33]).
(a) \( \Theta_2(-\frac{1}{z}) = \left( \frac{z}{i} \right)^{1/2} \Theta_4(z) \),
(b) \( \Theta_3(-\frac{1}{z}) = \left( \frac{z}{i} \right)^{1/2} \Theta_3(z) \),
(c) \( \Theta_4(-\frac{1}{z}) = \left( \frac{z}{i} \right)^{1/2} \Theta_2(z) \),
(d) \( \Theta_2(z + 1) = e^{i\pi/4} \Theta_2(z) \),
(e) \( \Theta_3(z + 1) = \Theta_4(z) \),
(g) \( \Theta_4(z + 1) = \Theta_3(z) \),

(6.8)

where (6.8)(d),(e),(g) follow readily from (6.2), while (6.8)(a),(b),(c) follow from (6.5) and (6.6) because according to these relations the three functions

\[
\Theta_2\left(-\frac{1}{z}\right) - \Theta_3\left(-\frac{1}{z}\right)
\]

are all holomorphic on \( \mathbb{H} \) and vanish for all \( z \in i \mathbb{R}_{>0} \), so that by the uniqueness theorem for analytic functions (see [12, p. 78, Thm. 3.7(c)]), they all vanish identically on \( \mathbb{H} \) (see [5, p. 56]).

Remark. In the notations \( \theta_k(z|\tau) = \theta_k(z, q) \), \( 2 \leq k \leq 4, z \in \mathbb{C}, \tau \in \mathbb{H}, q = e^{i\pi \tau} \in \mathbb{D} \) of [28, p. 524, 20.2.2-4], we have \( \theta_3(0, u) = \theta_3(u), \theta_4(0, u) = \theta_4(u), u \in \mathbb{D} \), and

\[
\Theta_k(\tau) = \theta_k \left( 0, e^{i\pi \tau} \right), \quad 2 \leq k \leq 4, \quad \tau \in \mathbb{H}.
\]

### 7 Wirtinger’s Identity

**Proof of the identity** (1.25). By (1.19), the function \( F_\Delta(z) \) does not vanish on \( \mathbb{C} \setminus [1, +\infty) \) and therefore we can introduce the function (see [5, p. 58])

\[
\Phi(z) = \frac{\Theta_3\left(\lambda_\Delta(z)\right)^2}{F_\Delta(z)}, \quad z \in \Lambda := (0, 1) \cup (\mathbb{C} \setminus \mathbb{R}), \quad \Phi \in \text{Hol}(\Lambda).
\]

The formulas (6.8)(b) and (1.21) for any \( z \in \Lambda \) yield that

\[
\Phi(z) = \frac{i}{\lambda_\Delta(z)} \frac{\Theta_3\left(-1/\lambda_\Delta(z)\right)^2}{F_\Delta(z)} = \frac{\Theta_3\left(\lambda_\Delta(1-z)\right)^2}{F_\Delta(1-z)} = \Phi(1-z).
\]

(7.1)

By using (6.8)(e),(g) and (1.22), for arbitrary \( x > 0 \) we deduce that

\[
\Phi(-x+i0) = \frac{\Theta_3\left(2+\lambda_\Delta(-x-i0)\right)^2}{F_\Delta(-x)} = \frac{\Theta_3\left(\lambda_\Delta(-x-i0)\right)^2}{F_\Delta(-x)} = \Phi(-x-i0).
\]

(7.2)

For arbitrary \( z \in \mathbb{H} \) and \( x > 0 \) it follows from (6.8)(b),(e),(g) and (1.23) that

\[
\Theta_3\left(\frac{z}{1-2z}\right)^2 = (1-2z)\Theta_3(z)^2, \quad 1-2\lambda_\Delta(1+x+i0) = \frac{F_\Delta(1+x-i0)}{F_\Delta(1+x+i0)}.
\]
from which for arbitrary \( x > 0 \) we derive

\[
\Phi(1 + x - i 0) = \frac{(1 - 2\lambda_△(1 + x + i 0)) \Theta_3(\lambda_△(1 + x + i 0))}{F_△(1 + x - i 0)} = \Phi(1 + x + i 0). \quad (7.3)
\]

By applying Morera’s theorem to the equations (7.2) and (7.3) we obtain that \( \Phi \in \text{Hol}(\mathbb{C} \setminus \{0, 1\}) \) (see [24, p. 96]).

When \( \Lambda \ni z \to 0 \), by (5.2)(b), (4.7)(b) and (6.7), (6.1), (1.18), we have \( F_△(z) \to 1, \lambda_△(z) = (i/\pi) \ln(1/|z|) - \lambda_△(0; z), |\lambda_△(0; z)| \leq 2 \), and \( \Theta_3(\lambda_△(z))^2 = \theta_3(\exp(-\ln(1/|z|) - i \pi \lambda_△(0; z)))^2 \) tends to 1, correspondingly. Next, by the symmetry property (7.1), we obtain the existence of the two limits

\[
\lim_{\Lambda \ni z \to 0} \Phi(z) = \lim_{\Lambda \ni z \to 1} \Phi(z) = 1. \quad (7.4)
\]

By the Riemann theorem about removable singularities (see [12, p. 103]) we deduce that \( \Phi \) is an entire function satisfying \( \Phi(0) = \Phi(1) = 1 \).

But if \( z \to \infty \) lying in the one of the half-planes \( \sigma := \text{sign}(\text{Im} \ z) \in \{1, -1\} \) then by (6.8)(c),(e) and (5.2)(a) we have \( \Theta_3(z)^2 = i(z-\sigma)^{-1} \Theta_2(-1/(z-\sigma))^2 \) and \( \lambda_△(z) \to \sigma \), respectively. In the notation of (4.2) and in view of (4.7)(a), by denoting \( \delta(z) := i \pi \lambda_△(\infty; z) \), we deduce from (6.7) and (2.5)(c) that

\[
\Phi(z) = \frac{i \theta_2\left(\exp\left(-\frac{i\pi}{(\lambda_△(z)-\sigma)}\right)\right)^2 \exp\left(-\frac{i\pi}{(\lambda_△(z)-\sigma)}\right)}{\left(\lambda_△(z)-\sigma\right)F_△(z)} = \frac{\delta(z) + \ln|z|}{4\sqrt{1-z}} + O\left(\frac{\ln|z|}{|z|^{3/2}}\right) = \frac{e^{-\delta(z)/2}}{\sqrt{|z|}} \theta_2\left(e^{-\delta(z)-\ln|z|}\right)^2 = O(1),
\]

as \( \Lambda \ni z \to \infty \). As a consequence, the entire function \( \Phi \) is bounded and by Liouville’s theorem [12, p. 77] it is a constant, which must equal 1, by (7.4). This establishes the Wirtinger identity (1.25).

\( \square \)

Proof of (1.26). In view of (1.24) and (1.19), (1.25) yields that \( \Theta_3(z) \neq 0 \) for all \( z \in \mathcal{F}_△ \), and \( \Theta_3(z) \neq 0 \) for all \( z \in \pm 1 + i \mathbb{R}_{>0} \), by (4.1). The relations (1.40), (1.41) and the equality (1.42) (see also [27, p. 32, Thm. 7.1], [5, p. 62]), which is the result of successive applications of the transformations \( z \mapsto z + 2 \) and \( z \mapsto -1/z \) in (6.8)(b),(e),(g) (see [11, p. 112, Lem. 2]), prove that \( \Theta_3(z) \neq 0 \) for all \( z \in \mathbb{H} \). Then the following consequences of (6.8), \( \Theta_4(z) = \Theta_3(z+1), \Theta_2(z) = (i/z)^{1/2} \Theta_3(1-1/z) \), for \( z \in \mathbb{H} \), complete the proof of (1.26). \( \square \)
8 Identity for the Elliptic Modular Function

Proof of (1.27) and (1.28). We prove that two holomorphic functions in \( \mathbb{H} \)

\[ f_1(z) := \frac{\Theta_2(z)^4 + \Theta_4(z)^4}{\Theta_3(z)^4}, \quad f_2(z) := \frac{\Theta_2(z)^4}{\Theta_3(z)^4}, \quad z \in \mathbb{H}, \]

satisfy the conditions of Lemma 1.1 with \( n_\infty = n_0 = n_1 = 0 \) and \( n_0 = n_\infty = 0 \), \( n_1 = 1 \), respectively. It follows from (6.8) that for arbitrary \( z \in \mathbb{H} \) and \( 2 \leq k \leq 4 \) we have: \( \Theta_k(z)^4 = \Theta_k(z + 2)^4 \), \( \Theta_k(z/(1 - 2z))^4 = (1 - 2z)^2 \Theta_k(z)^4 \) and \( \Theta_k(z)^4 = -\Theta_{6-k}(-1/z)^4/z^2 \), while if \( \sigma \in \{1, -1\} \) then (see [5, p. 63])

\[ \Theta_k(z)^4 = \frac{(-1)^{m+k}}{(z-\sigma)^2} \Theta_m \left( \frac{1}{z-\sigma} \right)^4, \quad \left( \frac{k}{m} \right) \in \left\{ \left( \frac{2}{4} \right), \left( \frac{3}{2} \right), \left( \frac{4}{3} \right) \right\}, \]

from which for any \( z \in \mathbb{H} \) and \( \sigma \in \{1, -1\} \) we get

(a) \( f_k(z + 2) = f_k(z) \), \quad (b) \( f_k \left( \frac{z}{1 - 2z} \right) = f_k(z), \quad k \in \{1, 2\} \);

(c) \( f_1(z) = f_1 \left( \frac{-1}{z} \right) \), \quad (d) \( f_2(z) = \frac{\Theta_4 \left( -\frac{1}{z} \right)^4}{\Theta_3 \left( -\frac{1}{z} \right)^4} \),

(e) \( f_1(z) = \frac{\Theta_3(y)^4 - \Theta_4(y)^4}{\Theta_2(y)^4} \), \quad (f) \( f_2(z) = \frac{\Theta_4(y)^4}{\Theta_2(y)^4}, \quad y := \frac{1}{z - \sigma} \). \hspace{1cm} (8.1)

The two conditions of invariance (a) and (b) in Lemma 1.1 hold for \( f_1 \) and \( f_2 \), in view of (8.1)(a),(b) with \(-1/z\) in place of \( z \). It follows from (6.7) that

(a) \( \Theta_3(z)^4 = 1 + 8 e^{i \pi z} + O \left( e^{2i \pi z} \right) \), \quad (b) \( \Theta_4(z)^4 = 1 - 8 e^{i \pi z} + O \left( e^{2i \pi z} \right) \),

(c) \( \Theta_2(z)^4 = 16 e^{i \pi z} + O \left( e^{3i \pi z} \right) \), \quad \mathcal{F}_{\square} \ni z \to \infty . \hspace{1cm} (8.2)

which together with (8.1)(c),(d) show that

\[ \lim_{\mathcal{F}_{\square} \ni z \to 0} f_k(z) = 1, \quad \lim_{\mathcal{F}_{\square} \ni z \to \infty} f_k(z) = 2 - k, \quad k \in \{1, 2\} , \hspace{1cm} (8.3) \]

and hence the conditions (1) and (2) in Lemma 1.1 with \( n_0 = n_\infty = 0 \) hold for \( f_1 \) and \( f_2 \). The condition (3) in Lemma 1.1 with \( n_1 = 0 \) also holds for \( f_1 \) as follows from (8.1)(a),(e) and (8.2). Applying the result of Lemma 1.1 to \( f_1 \) we obtain that \( f_1(\lambda_\Delta(z)) = a \) holds on \( \Lambda \) for some constant \( a \in \mathbb{C} \). By letting \( \Lambda \ni z \to 0 \) we obtain from (5.2)(b) and (8.3) that \( a = 1 \) and therefore, by virtue of (1.24), \( f_1(z) = 1 \) holds for all \( z \in \mathcal{F}_{\square} \). Since \( f_1 \) is holomorphic on \( \mathbb{H} \) we get that \( f_1(z) = 1 \) throughout \( \mathbb{H} \), which is the same as (1.27).
We now show that the condition (3) in Lemma 1.1 holds for \( f_2 \) with \( n_1 = 1 \). In view of (8.1)(a), it is sufficient to verify this when \( \mathcal{F} \ni z \to 1 \). For this case it follows from (8.1)(e) and (8.2)(b),(c) that

\[
f_2(z) = -\frac{1}{16} e^{i\pi/(z-1)} + O(1) \quad \text{as} \quad \mathcal{F} \ni z \to 1.
\]

This shows that the condition (3) of Lemma 1.1 holds with \( n_1 = 1 \), since

\[
\text{Re}\left( \frac{i\pi}{z-1} \right) - \frac{2\pi}{|z-1|} < -\frac{\pi}{|z-1|} \to -\infty \quad \text{as} \quad \mathcal{F} \ni z \to 1.
\]

Applying the result of Lemma 1.1, we obtain the existence of \( a, b \in \mathbb{C} \) such that \( f_2(\lambda_\Delta(z)) = az + b \). By letting \( \Lambda \ni z \to 0 \) we obtain by (5.2)(b) and by (8.3) that \( f_2(\lambda_\Delta(z)) \to 0 \) which yields \( b = 0 \) and therefore \( f_2(\lambda_\Delta(z)) = az \). At the same time, by letting \( \Lambda \ni z \to 1 \) we obtain from (5.2)(c) and (8.3) that \( f_2(\lambda_\Delta(z)) \to 1 \) and therefore \( a = 1 \). It follows that \( f_2(\lambda_\Delta(z)) = z \) for all \( z \in \Lambda \), which gives (1.28).

**Proof of (1.30) and (1.31).** It follows from (1.20), (1.25) and (1.28) written in the form \( \lambda_\Delta(\lambda(z)) = z, \ z \in \mathcal{F} \), that (see [5, p. 66])

\[
\lambda_\Delta'(\lambda(z))\lambda_\Delta'(z) = 1, \quad \lambda_\Delta(\lambda(z)) = \frac{1}{\pi \lambda(z)(1 - \lambda(z))}, \quad z \in \mathcal{F},
\]

from which, by using (1.29) and (1.27), we obtain (1.30) for \( z \in \Lambda \). Since all functions in (1.30) are holomorphic in \( \mathbb{H} \) we obtain (1.30) for all \( z \in \mathbb{H} \). The equalities (1.31) follow from (1.30) and from the following forms of writing \( \lambda'(z) \), taking account of (1.27),

\[
\lambda'(z) = \frac{d}{dz} \Theta_2(z)^4 = 4 \Theta_2(z)^4 \left( \frac{\Theta_2'(z)}{\Theta_2(z)} - \frac{\Theta_3'(z)}{\Theta_3(z)} \right),
\]

\[
\lambda'(z) = -\frac{d}{dz} (1 - \lambda(z)) = -\frac{d}{dz} \Theta_4(z)^4 = 4 \Theta_4(z)^4 \left( \frac{\Theta_3'(z)}{\Theta_3(z)} - \frac{\Theta_4'(z)}{\Theta_4(z)} \right).
\]

**Proof of Corollary 1.2** By combining Corollary 1.1 for \( \alpha = 2 \) with (1.32) and (1.25) we find that \( \Theta_2(\lambda_\Delta)^4 \) is universally starlike and \( \Theta_3(\lambda_\Delta)^4 \in \mathcal{P}_{\log}(-\infty, 1) \). According to the definition (1.12) of the class \( \mathcal{P}_{\log} \) the latter property means that \( \Theta_3(\lambda_\Delta)^4 \in \mathcal{P} \) and (see [5, p. 68])

\[
\mathcal{P} \ni \frac{d}{dz} \Theta_3(\lambda_\Delta(z))^4 = \frac{\Theta_3'(\lambda_\Delta(z))\lambda_\Delta'(z)}{\Theta_3(\lambda_\Delta(z))} = \frac{\Theta_3'(\lambda_\Delta(z))}{\lambda'(\lambda_\Delta(z))\Theta_3(\lambda_\Delta(z))},
\]

because \( \lambda'(\lambda_\Delta(z))\lambda_\Delta'(z) = 1 \) for each \( z \in \Lambda \ni (\mathbb{C} \setminus \mathbb{R}) \), as follows from (1.28). In the following we need three facts: (a) by (1.1) and [8, p. 31], every non-constant
function $f$ in $\mathcal{P}$ satisfies $(\text{Im } z)f(z) > 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$; (b) the relation (3.17) yields that the numbers $\text{Im } z$ and $\text{Re } \lambda_\Delta(z)$ have the same sign for every $z \in \mathbb{C} \setminus \mathbb{R}$; (c) we have $\lambda_\Delta(\mathbb{C} \setminus \mathbb{R}) = \mathcal{F}[\mathbb{C} \setminus \{i \mathbb{R}_{>0}\}]$, in view of Theorem 1 and the equality $\mathbb{R}_{>0} = \{ F_\Delta(1 - x)/F_\Delta(x) \mid x \in (0, 1) \}$ (see Corollary 1.3). By applying these facts to the properties $\Theta_3(\lambda_\Delta)^4 \in \mathcal{P}$ and (8.4) we obtain that (1.33)(a) and (1.33)(b) hold. □

9 Definitions for the Logarithms of the Theta Functions

By virtue of (1.26) and (6.7), for every $2 \leq k \leq 4$ the function $\theta_k$ in (6.1) does not vanish on $\mathbb{D}$ and consequently, by [12, p. 94, Cor. 6.17], there exists a holomorphic function $\log \theta_k$ in $\mathbb{D}$ such that $\exp(\log \theta_k) = \theta_k$ on $\mathbb{D}$, and $\log \theta_k(0) = \ln \theta_k(0) = 0$. In addition, we know from (6.7) with $z \in \mathbb{R}_{>0}$ and (1.26) that $\theta_k(x) > 0$ for $x \in [0, 1)$. Since for arbitrary $x \in (0, 1)$ all other solutions $y$ of the equation $\exp(y) = \theta_k(x)$ differ from $\ln \theta_k(x)$ by $2\pi i n$ for some $n \in \mathbb{Z} \setminus \{0\}$, and both functions $\log \theta_k(x)$ and $\ln \theta_k(x)$ are continuous on $[0, 1)$ we obtain that $\log \theta_k(x) = \ln \theta_k(x)$, $x \in [0, 1)$. To obtain the Maclaurin series for $\log \theta_k$ we use the following classical Jacobi’s expansions into infinite products for arbitrary $u \in \mathbb{D}$ (see [37, pp. 469, 470], [28, p. 529, 20.4.3, 20.4.4])

$$\theta_2(u) = \prod_{n \geq 1} \left(1 - u^{2n}\right)^2 \left(1 + u^{2n}\right)^2, \quad \theta_4(-u) = \theta_3(u) = \prod_{n \geq 1} \left(1 - u^{2n}\right)^2 \left(1 + u^{2n-1}\right)^2.$$

Taking the real-valued logarithm of these products for $u \in (0, 1)$, we see that

$$\ln \theta_3(u) = \sum_{n \geq 1} \ln \left(1 - u^{2n}\right) + 2 \sum_{n \geq 1} \ln \left(1 + u^{2n-1}\right),$$

$$\ln \theta_2(u) = \sum_{n \geq 1} \ln \left(1 - u^{2n}\right) + 2 \sum_{n \geq 1} \ln \left(1 + u^{2n}\right).$$

Next, by expanding $\ln(1 \pm x)$ in its Maclaurin series [1, p. 68, 4.1.24] after several algebraic manipulations we obtain (compare, e.g., with [10, p. 338, (4.2)]) that (see [5, p. 69])

$$\ln \theta_3(u) = \sum_{n \geq 1} \frac{2}{2n - 1} \frac{u^{2n-1}}{1 + u^{2n-1}}, \quad \ln \theta_2(u) = \sum_{n \geq 1} \frac{1}{n} \frac{u^{2n}}{1 + u^{2n}}, \quad \ln \theta_4(u) = \ln \theta_3(-u) = -\sum_{n \geq 1} \frac{2}{2n - 1} \frac{u^{2n-1}}{1 - u^{2n-1}}, \quad u \in (0, 1). \quad (9.1)$$

As the three series on the right-hand sides of these equalities converge absolutely and uniformly on compact subsets of the unit disk, they represent holomorphic functions in $\mathbb{D}$, which by the standard uniqueness theorem (see [12, p. 78, Thm. 3.7(c)]) shows that the identities (9.1) and (9.2) hold throughout $\mathbb{D}$.
As we recall the relationships (6.7) connecting \( \theta_k \) with \( \Theta_k \), we see that this allows us to define the logarithms \( \log \Theta_k \) via

\[
\log \Theta_k(z) := \log \theta_k \left( e^{i \pi z} \right), \quad k \in \{3, 4\},
\]

\[
\log \Theta_2(z) := \frac{i \pi z}{4} + \ln 2 + \log \Theta_2 \left( e^{i \pi z} \right), \quad z \in \mathbb{H}.
\]

The counterpart of the functional relationships (6.8) reads

\[
\log \Theta_k(2m + z) = \log \Theta_k(z), \quad 3 \leq k \leq 4, \quad \log \Theta_2(2m + z) = \frac{i \pi m}{2} + \log \Theta_2(z),
\]

\[
\log \Theta_k \left( \frac{-1}{z} \right) = \log \Theta_{6-k}(z) + \frac{1}{2} \log \frac{z}{i}, \quad 2 \leq k \leq 4,
\]

\[
\log \Theta_k(z - 1) = \log \Theta_{7-k}(z), \quad 3 \leq k \leq 4, \quad \log \Theta_2(z - 1) = \log \Theta_2(z) - \frac{i \pi}{4}.
\]

10 Exponential Integral Representation of \( \Theta_3 \)

**Proof of Corollary 1.3** The Wirtinger identity (1.25) and the integral representation (1.15), by taking into account the notation (9.3), allow one to write down (1.35) and (1.36) in the set (1.34) because according to (9.4) we have \( \Theta_3(-1 + z) = \Theta_3(1 + z), \) \( z \in \mathbb{H} \). Moreover, it follows from (1.25), (1.14) and (2.15) that \( \arg \Theta_3(z) \in (-\pi/4, \pi/4) \) for all \( z \in \mathcal{F}_{\square} \) (see [5, p. 70]). The expression for \( y = y(x) \) follows from (4.1) and (3.9).

**Proof of (1.38).** By (9.4), we have \( \Theta_3(2m + z) = \Theta_3(z) \) for arbitrary \( m \in \mathbb{Z} \) and \( z \in \mathbb{H} \), from which it follows that the left-hand sides of the equalities (1.35) and (1.36) can be equivalently replaced by \( \log \Theta_3 \left( 2m + \lambda_{\triangle}(z) \right) \) and \( \log \Theta_3 \left( 2m + 1 + i y \right) \), respectively, for an arbitrary integer \( m \). This gives the integral representation of \( \log \Theta_3(z) \) for all \( z \) in the set (1.37), which with the help of (1.28) can be written in the form

\[
\log \Theta_3(z) = \frac{1}{2\pi^2} \int_{0}^{1} \frac{1}{t(1-t)} \frac{\log \frac{1}{1-t} \Theta_2(z)^4 \Theta_3(z)^{-4}}{F_{\triangle}(t)^2 + F_{\triangle}(1-t)^2} \, dt, \quad z \in \mathcal{F}_{\square}^\infty.
\]

The representation (1.38) is obtained from (10.1) with the help of making the change of variable \( \tau = \lambda_{\triangle}(t)/i, \) \( t \in (0, 1) \), which yields \( t = \lambda(i \tau), \) \( \tau \in (0, +\infty) \), because in view of (2.5)(a),(b) and (3.9) we have \( \lambda_{\triangle}(t)/i \to +\infty \), as \( t \downarrow 0, \lambda_{\triangle}(1)/i = 0 \) and \( \lambda_{\triangle}'(t)/i < 0, \) \( t \in (0, 1) \) (see [5, p. 72]).

**Acknowledgements** We thank Danylo Radchenko for sharing his insides with us. We also thank the referees for positive feedback and constructive criticism. In particular, we thank one of the referees for telling us about the connection with quasiconformal mappings and the Grötzsch ring.
References

1. Abramowitz, M., Stegun, I.: Handbook of Mathematical Functions, National Bureau of Standards. Appl. Math. Ser. 55, (1964)
2. Agard, S.: Distortion theorems for quasiconformal mappings. Ann. Acad. Sci. Fenn. Ser. A I No. 413, pp. 12 (1968)
3. Anderson, G. D., Vamanamurthy, M. K., Vuorinen, M. K.: Conformal invariants, inequalities, and quasiconformal maps. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, (1997)
4. Andrews G., Askey, R., Roy, R.: Special functions. Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge (1999)
5. Bakan A., Hedenmalm H.: Exponential integral representations of theta functions, arXiv preprint, arXiv: 1912.10568v7 (2020)
6. Bakan A., Ruscheweyh St., Salinas L.: Universally starlike and Pick functions. J. Anal. Math. (to appear), ArXiv preprint, arXiv:1804.03931 (2018)
7. Bakan, A., Ruscheweyh, St, Salinas, L.: Universal convexity and universal starlikeness of polylogarithms. Proc. Am. Math. Soc. 143(2), 717–729 (2015)
8. Berg, Ch., Pedersen, H.: Nevanlinna matrices of entire functions. Math. Nachr. 171, 29–52 (1995)
9. Berndt B, Knopp, M.: Hecke’s theory of modular forms and Dirichlet series, Monographs in Number Theory, 5. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2008)
10. Berndt, B.: Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan. J. Reine Angew. Math. 303, 332–365 (1978)
11. Chandrasekharan, K.: Elliptic functions, Fundamental Principles of Mathematical Sciences, vol. 281. Springer-Verlag, Berlin (1985)
12. Conway J.: Functions of one complex variable. Second edition. Graduate Texts in Mathematics, 11, Springer-Verlag, New York-New York, (1978)
13. Donoghue, W.: Monotone Matrix Functions and Analytic Continuation. Springer-Verlag, New York, Heidelberg, Berlin (1974)
14. Duren P.: Univalent functions. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 259. Springer-Verlag, New York, Heidelberg, Berlin (1983)
15. Erdelyi A., Magnus W., Oberhettinger F., Tricomi F.: Higher transcendental functions, Vol. I., McGraw-Hill Book Company (1985)
16. Erdelyi A., Magnus W., Oberhettinger F., Tricomi F.: Tables of integral transforms. Vol. I., McGraw-Hill Book Company, Inc., New York-Toronto-London, (1954)
17. Erdelyi A., Magnus W., Oberhettinger F., Tricomi F.: Tables of integral transforms. Vol. II. McGraw-Hill Book Company, Inc., New York-Toronto-London, (1954)
18. Garnett, J.: Bounded analytic functions. Academic Press, New York (1981)
19. Hayman W., Kennedy P.: Subharmonic functions, Vol. I, London Mathematical Society Monographs, No. 9. Academic Press, London-New York (1976)
20. Koosis, P.: Introduction to $\mathcal{H}^p$ spaces, Cambridge Tracts in Mathematics, vol. 115. Cambridge University Press, Cambridge (1998)
21. Küstner, R.: Mapping properties of hypergeometric functions and convolutions of starlike or convex functions of order $\alpha$. Comput. Methods Funct. Theory 2(2), 597–610 (2002)
22. Lang, S.: Complex analysis, Fourth edition. Graduate Texts in Mathematics, 103. Springer-Verlag, New York (1999)
23. Lawden D.: Elliptic Functions and Applications, Applied Math. Sci. 80. Springer-Verlag, New York (1980)
24. Lawrentjew, M., Schabat, B.: Methoden der komplexen Funktionentheorie, Mathematik für Naturwissenschaft und Technik, vol. 13. VEB Deutscher Verlag der Wissenschaften, Berlin (1967)
25. Levin B. Ja.: Distribution of zeros of entire functions, Translations of Mathematical Monographs, 5. American Mathematical Society, Providence, R.I. (1980)
26. Mukherjea, A., Pothoven, K.: Real and functional analysis. Plenum Press, New York (1978)
27. Mumford, D.: Tata lectures on theta. I, Progress in Mathematics, 28. Birkhäuser Boston, Inc., Boston, MA, (1983)
28. Olver F., Lozier D., Boisvert R., Clark Ch.: NIST handbook of mathematical functions, U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge (2010)
29. Prudnikov, A., Brychkov, Yu., Marichev, O.: Integrals and series, vol. 1. Gordon & Breach Science Publishers, New York, Elementary functions (1986)
30. Rankin, R.: Modular forms and functions. Cambridge University Press, Cambridge-New York-Melbourne (1977)
31. Rudin, W.: Real and complex analysis, 2nd edn. McGraw-Hill Book Co., New York (1974)
32. Ruscheweyh, S., Salinas, L., Sugawa, T.: Completely monotone sequences and universally prestarlike functions. Israel J. Math. 171, 285–304 (2009)
33. Saff, E., Snider, A.: Fundamentals of complex analysis with applications to engineering and science, 3rd edn. Prentice Hall, Pearson Education (2003)
34. Sarason, D.: Complex function theory, 2nd edn. American Mathematical Society, Providence (2007)
35. Schwarz, H.A.: Über diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt. J. Reine Angew. Math. 75, 292–335 (1873)
36. Shohat, J., Tamarkin, J.: The problem of moments, A M S. RI, rev.ed, Providence (1950)
37. Whittaker E., Watson G.: A course of modern analysis, Cambridge, (1950)
38. Widder, D.: The Laplace Transform, vol. 1. Princeton University, Princeton (1946)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.