FAMILIES OF CALABI–YAU ELLIPTIC FIBRATIONS IN
\( P(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B) \)

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ABSTRACT. Let \( B \) be a smooth projective surface, and \( \mathcal{L} \) an ample line bundle on \( B \). The aim of this paper is to study the families of elliptic Calabi–Yau threefolds sitting in the bundle \( P(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B) \) as anticanonical divisors. I will show that the number of such families is finite.

INTRODUCTION

The theory of strings is a physical subject which has a great connection with mathematics: its main object of study are in fact elliptic fibrations on Calabi–Yau manifolds. Not only the theoretical aspects are important, but also the research of families of examples plays a central role: to give two examples, in [EY] the \( E_6 \) and \( E_7 \) family of elliptic Calabi–Yau threefolds are defined, and in [EFY] the authors define the \( D_5 \) family.

A simple way to produce elliptic Calabi–Yau varieties is to consider smooth anticanonical subvarieties of some reasonable ambient space: in fact by adjunction these varieties will automatically be Calabi–Yau. Giving different shades to the word “reasonable”, one has different classes of ambient spaces to try describing its anticanonical subvarieties. In particular the class of toric Fano Gorenstein fourfolds has been deeply studied for the following reasons:

1. Since any anticanonical divisor of a Fano variety is ample, we are sure to find effective divisors in the anticanonical system;
2. Gorenstein varieties may be singular, but in this case they have nice resolutions of the singularities and one can then study the anticanonical subvarieties of the resolution;
3. Toric varieties are simple since most of the problems one may have to solve can be translated into a combinatorial problem, which is simpler to deal with.

To each toric Fano Gorenstein fourfold is associated a reflexive 4-dimensional polyhedron and vice versa, so the first attempt to describe the Calabi–Yau subvarieties in these ambient space is to classify all the reflexive 4-dimensional polyhedra. Such a classification is known, and there are 473, 800, 776 4-dimensional reflexive polyhedra (see e.g. [KS00], [KS02]). Among these, in [Bra13] the 102, 581 elliptic fibrations over \( \mathbb{P}^2 \) are identified.

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The elliptic fibrations I will describe in this paper are anticanonical hypersurfaces in a projective bundle \( Z \) over a surface \( B \) of the form \( Z = \mathbb{P}(\mathcal{L}^a \otimes \mathcal{L}^b \otimes \mathcal{O}_B) \) for \( \mathcal{L} \) an ample line bundle on \( B \). Observe that even in the case where the base \( B \) is toric, e.g. \( B = \mathbb{P}^2 \), the ambient bundle is typically not Fano.

The aim of this paper is to show that once \( B \) and \( \mathcal{L} \) are fixed, then the bundle \( \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B) \) can house a Calabi–Yau elliptic fibration only for a finite number of choices of \((a, b)\):

**Theorem** (Thm. 2.1). Let \( B \) be a smooth projective surface, and \( \mathcal{L} \) an ample line bundle on \( B \). Then only for a finite number of pairs \((a, b)\) the generic anticanonical hypersurface in \( \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B) \) is a Calabi–Yau elliptic fibration over \( B \).

As we will see in Sections 2.2 and 2.4, we may fail to find a Calabi–Yau elliptic fibration for the following reasons: the fibration has no sections or its total space is singular.

The outline of the paper is as follows. In section 1 I will recall the definitions of elliptic fibration and of Calabi–Yau variety. In section 2 I will state the finiteness result (Theorem 2.1), and prove it (Sections from 2.4.1 to 2.4.4). Finally, in Section 3 I will give two concrete examples, and find explicit bounds on the number of different families when the base \( B \) is a del Pezzo surface (and in particular for \( B = \mathbb{P}^2 \)).

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1. **Elliptic fibrations and Calabi–Yau manifolds**

   In this section, I want to recall the definition and main properties of elliptic fibrations (Section 1.1) and Calabi–Yau manifolds (Section 1.2).

   **1.1. Elliptic fibrations.** Elliptic fibrations are the geometric realization of elliptic curves over the function field of a variety. Their study has been encouraged by physics, and in particular string theory: to each elliptic fibration correspond a physical scenario, and the fibration itself determines the number of elementary particles, their charges and masses (see e.g. [Vaf96]).

   **Definition 1.1.** We say that \( \pi : X \to B \) is an elliptic fibration over \( B \) if
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(1) $X$ and $B$ are projective varieties of dimension $n$ and $n - 1$ respectively, with $X$ smooth;
(2) $\pi$ is a surjective morphism with connected fibres of dimension 1;
(3) the generic fibre of $\pi$ is a smooth connected curve of genus 1;
(4) a section $\sigma : B \to X$ of $\pi$ is given.

When $\pi : X \to B$ satisfies only the first three requirements above, we say that it is a genus one fibration.

I will denote the fibre over the point $P \in B$ with $X_P$.

Remark 1.2. Let $\pi : X \to B$ be an elliptic fibration, with section $\sigma$. Then each smooth fibre $X_P$ is an elliptic curve, where we choose as origin the point $\sigma(P)$.

A morphism between two elliptic fibrations $\pi : X \to B$ and $\pi' : X' \to B$ is a morphism of varieties over $B$, i.e. a morphism $f : X \to X'$ such that

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\pi & \downarrow & \pi' \\
B & \downarrow & B
\end{array}
$$

commutes.

Not every fibre of $\pi$ needs to be smooth: the discriminant locus of the fibration is the subset of $B$ over which the fibres are singular

$$
\Delta = \{ P \in B \mid X_P \text{ is singular} \} \subseteq B.
$$

A rational section of $\pi$ is a rational map $s : B \dashrightarrow X$ such that $\pi \circ s = \text{id}$ over the domain of $s$. The Mordell–Weil group of the fibration is

$$
\text{MW}(X) = \{ s : B \dashrightarrow X \mid s \text{ is a rational section} \},
$$

where the group law is given by addition fibrewise. Observe that even though the elements of the Mordell–Weil group are rational sections, we require its zero element to be a section.

1.2. Calabi–Yau manifolds. Calabi–Yau manifolds are the higher dimensional analogues of elliptic curves and $K3$ surfaces. The mathematical models of $F$-theory are all examples of Calabi–Yau manifolds: this property is needed on the total space of an elliptic fibration in order to get a physically consistent model (see e.g. [MV96a, MV96b]).

Definition 1.3. A Calabi–Yau manifold is a smooth compact Kähler variety $X$ with

(1) trivial canonical bundle $\omega_X \simeq \mathcal{O}_X$,
(2) $h^{0,q} = 0$ for $q = 1, \ldots, \dim X - 1$, where $h^{p,q} = \dim H^q(X, \Omega^p_X)$.

Example 1.4. If $X$ is a Calabi–Yau variety of dimension 1, then $X$ is a smooth Riemann surface of genus 1.

In the case of dimension 2, the Calabi–Yau surfaces are the $K3$.

In dimension 3, the Fermat quintic in $\mathbb{P}^4$, and in fact any smooth quintic, is a classical example of Calabi–Yau variety (see for instance [GHJ03] and [CK99]). Other Calabi–Yau threefolds which are complete intersections in projective spaces are the complete intersection of two hypersurfaces of degree 3 in $\mathbb{P}^5$, of a hyperquadric and a hypersurface of degree 4 in $\mathbb{P}^5$, of two hyperquadric and a hypercubic in $\mathbb{P}^6$ or...
the complete intersection of four hyperquadrics in \( \mathbb{P}^7 \).
For other examples of Calabi–Yau manifolds, see e.g. [H"ub92].

2. A finiteness result

2.1. Notations and general setting. In this section I will fix the notation I will use through the rest of the paper. Let \( B \) be a smooth projective surface, and \( \mathcal{L} \) an ample line bundle on \( B \). Let \( p : Z \rightarrow B \) be the projective bundle of lines associate to the rank two vector bundle \( \mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B \), i.e. \( Z = \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B) \). Let \( X \in \mid -K_Z \mid \) be an anticanonical subvariety, and \( \pi : X \rightarrow B \) the restriction to \( X \) of the structure map \( p \) of \( Z \); observe that the fibres of \( \pi \) are curves.

2.2. Statement of the problem. The aim of the paper is to give an answer to the following question:

Main Question. For how many (and for which) pairs \((a, b)\) is it true that for the generic anticanonical subvariety \( X \) of \( \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B) \), the map \( \pi \) defines a Calabi–Yau elliptic fibration over \( B \)?

At first sight the answer seems to be “almost for all pairs”, for the following reasons:

(1) anticanonical subvarieties are Calabi–Yau by adjunction;
(2) since the generic fibre of \( \pi \) is a plane cubic curve (cfr. (2.5)), we have always a genus 1 fibration.

Nevertheless we are wrong. In fact the map \( \pi \) can have no sections, or the total space \( X \) of the fibration can be singular. This last case can happen for two reasons:

(1) the generic \( X \in \mid -K_Z \mid \) is reducible (see Section 2.4.4);
(2) there is a section of \( \pi \) passing through a singular point of a fibre.

In the second case, if the singularities of \( X \) admit a small crepant resolution we can obtain a Calabi–Yau elliptic fibration, but then the resolved fibration would live in another ambient space, so we exclude them from this paper.

Theorem 2.1. Let \( B \) be a smooth projective surface, and \( \mathcal{L} \) an ample line bundle on \( B \). Then only for a finite number of pairs \((a, b)\) the generic anticanonical hypersurface in \( \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B) \) is a Calabi–Yau elliptic fibration over \( B \).

Remark 2.2. The theorem states only the finiteness, but its proof gives also a sort of algorithm to detect a finite superset of the set of pairs satisfying the main question.

Before proving Theorem 2.1 in Section 2.3 I will take a short digression on the projective bundle \( Z \) and its anticanonical subvarieties.

2.3. Calabi–Yau’s in \( \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}) \). We are interested in studying the anticanonical subvarieties of \( Z = \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}) \). In this section I want first to compute the Chern classes of \( Z \), and then find how an equation for an anticanonical subvariety looks like.
2.3.1. The ambient bundle. The bundle projection $p : Z \rightarrow B$ gives the relative tangent bundle exact sequence

\[ 0 \rightarrow T_{Z/B} \rightarrow T_Z \rightarrow p^*T_B \rightarrow 0 \]

from which we see that

\[ c(Z) = c(T_{Z/B})p^*c(B). \]

To compute the total Chern class of the relative tangent bundle, we exploit the fact that it fits into an Euler-type exact sequence (see [Ful98, p. 435, B.5.8]):

\[ 0 \rightarrow O_Z \rightarrow p^*E \otimes O_Z(1) \rightarrow T_{Z/B} \rightarrow 0, \]

where $E = \mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B$.

An explicit computation leads to the following results

\[ \begin{align*}
    c_1(Z) &= p^*c_1(B) + (a + b)p^*L + 3\xi, \\
    c_2(Z) &= abp^*L^2 + (a + b)p^*Lc_1(B) + 2(a + b)p^*L^2 + 3p^*c_1(B)\xi + p^*c_2(B), \\
    c_3(Z) &= 2(a + b)p^*c_1(B)\xi + 3p^*c_2(B)\xi, \\
    c_4(Z) &= 3p^*c_2(B)\xi^2,
\end{align*} \]

where $L = c_1(\mathcal{L})$ and $\xi = c_1(\mathcal{O}_Z(1))$.

2.3.2. Equations for anticanonical subvarieties. Consider the projective bundle $Z = \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$, and let $x$, $y$ and $z$ denote sections on $Z$ whose vanishing gives the subvariety of $Z$ corresponding to the embeddings

\[ \mathcal{L}^b \oplus \mathcal{O}_B \hookrightarrow E, \quad \mathcal{L}^a \oplus \mathcal{O}_B \hookrightarrow E, \quad \mathcal{L}^a \oplus \mathcal{L}^b \hookrightarrow E \]

respectively. Then

\[ \begin{align*}
    x &\in H^0(Z, p^*\mathcal{L}^a \otimes \mathcal{O}_Z(1)) \\
    y &\in H^0(Z, p^*\mathcal{L}^b \otimes \mathcal{O}_Z(1)) \\
    z &\in H^0(Z, \mathcal{O}_Z(1))
\end{align*} \]

and we can use $(x : y : z)$ as global homogeneous coordinates in $Z$ over $B$.

Since $c_1(Z) = p^*c_1(B) + (a + b)p^*L + 3\xi$ by (2.3), an equation $F$ defining an anticanonical hypersurface must be cubic in $(x : y : z)$, of the form

\[ F = \sum_{i+j+k=3} \alpha_{ijk} x^i y^j z^k, \]

and the coefficient $\alpha_{ijk}$ of the monomial $x^i y^j z^k$ must be a section of a suitable line bundle, according to Table 1.

| Monomial | Weight of the coefficient |
|----------|--------------------------|
| $x^3$    | $c_1(B) - 2aL + bL$      |
| $x^2y$   | $c_1(B) - aL$            |
| $xy^2$   | $c_1(B) - bL$            |
| $y^3$    | $c_1(B) + aL - 2bL$      |
| $x^2z$   | $c_1(B) - aL + bL$       |
| $xyz$    | $c_1(B)$                |
| $y^2z$   | $c_1(B) + aL - bL$       |
| $xz^2$   | $c_1(B) + bL$            |
2.3.3. Chern classes of anticanonical subvarieties. We want to compute the Chern classes of a smooth $X \in | - K_Z |$. We have

\[
\begin{array}{|c|c|}
\hline
yz^2 & c_1(B) + aL \\
\hline
z^3 & c_1(B) + aL + bL \\
\hline
\end{array}
\]

and the normal bundle sequence of $X$ in $Z$

(2.6) \hspace{1cm} 0 \rightarrow T_X \rightarrow i^*T_Z \rightarrow N_{X|Z} \rightarrow 0,

which gives the following relation between the total Chern classes

(2.7) \hspace{1cm} i^*c(Z) = c(X)c(N_{X|Z}) = c(X)i^*(1 - K_Z).

Since we know $c(Z)$ from Section 2.3.1, and $1 - K_Z$ is a unit in the Chow ring of $Z$, we deduce the following formulae for the Chern classes of $X$:

(2.8)\hspace{1cm} c_1(X) = 0,
\hspace{1cm} c_2(X) = 3\xi^2_{1X} + \pi^*(2(a + b)L + 3c_1(B))\xi_{1X} + 
\hspace{1cm} +\pi^*((a + b)Lc_1(B) + abL^2 + c_2(B)),
\hspace{1cm} c_3(X) = -9\pi^*c_1(B)\xi^2_{1X} - \pi^*(2(a^2 - ab + b^2)L^2 + 6(a + b)Lc_1(B) + 3c_1(B)^2)\xi_{1X}.

Remark 2.3. In particular, we have a formula for the Euler–Poincaré characteristic of our varieties:

(2.9) \hspace{1cm} \chi_{\text{top}}(X) = \deg c_3(X) = -6(a^2 - ab + b^2)L^2 - 18c_1(B)^2.

2.4. Proof of Theorem 2.1. I will split the proof of Theorem 2.1 in several steps to make it clearer. In the first step (Section 2.4.1) I will show that with the exception of a finite number of pairs $(a, b)$, the genus one fibrations $X$ in $Z = \mathbb{P}(\mathcal{L}^a \oplus \mathcal{L}^b \oplus \mathcal{O}_B)$ admit a section. In the second step (Section 2.4.2) I will concentrate on such pairs, and use the presence of the section to reduce the problem of the smoothness of the generic anticanonical hypersurface to a problem concerning only the intersection form on the base. In the third step (Sections 2.4.3 and 2.4.4) I will show that this last problem has solution only for a finite number of pairs $(a, b)$, and this will be done in two different ways according to whether $\mathcal{L}$ is a rational multiple of $\omega_B^{-1}$ or not.

2.4.1. Step 1. We can always assume that $a \geq b \geq 0$. Since $L$ is an ample divisor, there exists a suitable integer $n_0$ such that $nL + K_B$ is ample for any $n \geq n_0$. There is only a finite number of pairs $(a, b)$ in the octant $a \geq b \geq 0$ such that $2a - b < n_0$. For such pairs the generic anticanonical hypersurface of $Z$ is a genus 1 fibration, but since the equation $F$ defining the variety is general, it is difficult to see if there are sections or not.

In Figure 1 it is shown this fact in the particular case where $B = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$.
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There is an infinite number of pairs $(a, b)$ satisfying $2a - b \geq n_0$: the divisor $(2a - b)L + K_B$ is ample, hence

$$H^0(B, (b - 2a)L - K_B) = H^0(B, -(2a - b)L + K_B)) = 0,$$

and so the coefficient of $x^3$ in (2.5) is identically 0 (cfr. Table I). Equation (2.5) looks then like

$$F = \alpha_{200} x^3 + \alpha_{210} x^2 y + \alpha_{201} x^2 z + \ldots$$

and so $\pi : X \to B$ has a distinguished section, given by

$$P \mapsto (1 : 0 : 0) \in X_P.$$  

2.4.2. **Step 2.** We now focus on the infinitely many cases where $2a - b \geq n_0$, so that we can exploit the presence of the section (2.11).

Let $\partial$ be any (local) derivation on $B$, and observe that

$$(\partial F)_{|(1:0:0)\in X_P} = 0.$$

Since in a smooth fibration with a section the singularities in the fibres cannot lie on the section (for otherwise the total space of the fibration would be singular), the following system must have no solutions

$$\begin{cases}
\frac{\partial F}{\partial x}_{|(1:0:0)\in X_P} = 0 \\
\frac{\partial F}{\partial y}_{|(1:0:0)\in X_P} = \alpha_{210}(P) = 0 \\
\frac{\partial F}{\partial z}_{|(1:0:0)\in X_P} = \alpha_{201}(P) = 0.
\end{cases}$$

(2.12)

This is equivalent to require that the curves defined by $\alpha_{210} = 0$ and $\alpha_{201} = 0$ in $B$ do not intersect, i.e. that

$$(c_1(B) - aL)(c_1(B) - (a - b)L) = a(a - b)L^2 + (b - 2a)c_1(B)L + c_1(B)^2 = 0.$$  

(2.13)

Observe that now we have a problem concerning only the base and its intersection theoretic properties. Thinking to $(a, b) \in \mathbb{R}^2$, equation (2.13) defines a plane conic, which is reducible if and only if

$$L^2 = 0 \quad \text{or} \quad (c_1(B)L)^2 = L^2 c_1(B)^2.$$
The first case is impossible since we are assuming that $L$ is ample. By the Hodge index theorem, $(c_1(B)L)^2 \geq L^2 c_1(B)^2$ and
\[ (c_1(B)L)^2 = L^2 c_1(B)^2 \iff rL = sc_1(B) \]
for suitable integers $r$ and $s$. So equation (2.13) defines a reducible conic if and only if $L$ is a rational multiple of $\omega_B^{-1}$.

Our next step is to study the conic defined in (2.13) when it is irreducible (Section 2.4.3) and when it is reducible (Section 2.4.4), and to show that in each of these two cases we have only a finite number of integral points $(a, b)$ in the octant $a \geq b \geq 0$ on the conic (2.13).

2.4.3. Step 3: (2.13) is irreducible. Let’s concentrate first on the case when the conic (2.13) is irreducible.

It is a hyperbola, with asymptotes
\[ a = \frac{c_1(B)L}{L^2} \quad \text{and} \quad b = a - \frac{c_1(B)L}{L^2}. \]
The change of variables
\[
\left\{ \begin{array}{l}
  a = a' + 2b' \\
  b = b'
\end{array} \right.
\]
is represented by a matrix in $\text{SL}(2, \mathbb{Z})$, hence preserves the integral lattice in $\mathbb{R}^2$ and the integral points on the hyperbola we are studying. In these new coordinates, the equation of the hyperbola is
\[ L^2 a'^2 + 3L^2 a'b' + 2L^2 b'^2 - 2c_1(B)La' - 3c_1(B)Lb' + c_1(B)^2 = 0; \]
it satisfies the hypothesis of [Zel] and so we have both that the number of integral points on the conic is finite and a way to compute them. Using the previous transformation and the (simple) algorithm in [Zel], the integral points $(a_i, b_i)$ of the conic (2.13) are among the following:
\[
a_i = \pm \frac{2L^2(c_1(B)L)^2 + 2(L^2)^2 c_1(B)^2 + d_i c_1(B)L}{2},
b_i = \pm \frac{4(L^2)^2 c_1(B)L - 4(L^2)^2 c_1(B)^2 - a^2}{2d_i (L^2)},
\]
where $d_i$ runs through the positive divisors of $4(L^2)^2((c_1(B)L)^2 - c_1(B)^2 L^2)$.

2.4.4. Step 3: (2.13) is reducible. We concentrate now on the case where the conic (2.13) is reducible, i.e. the case where $(c_1(B)L)^2 = L^2 c_1(B)^2$.

The equation for the conic (2.13) is
\[ (L^2 a - c_1(B)L)(L^2 a - L^2 b - c_1(B)L) = 0. \]

By (2.14), $rL = sc_1(B)$ implies $\frac{c_1(B)L}{L^2} = \frac{r}{s}$, we have two further subcases according to whether $\frac{r}{s}$ is a positive integer or not.

If $\frac{r}{s} \notin \mathbb{N}$, the two lines
\[ a = \frac{c_1(B)L}{L^2} \quad \text{and} \quad b = a - \frac{c_1(B)L}{L^2} \]
have no integral points at all. This means that we have no new smooth Calabi–Yau fibrations.

If instead $\frac{r}{s} \in \mathbb{N}$, then in the range $a \geq b \geq 0$ we have a finite number of pairs $(a, b)$ on the line $a = \frac{c_1(B)L}{L^2}$, namely $\frac{c_1(B)L}{L^2} + 1 = \frac{r}{s} + 1$, and an infinite number of $(a, b)$
on the line \( b = a - \frac{c_1(B)L}{L^2} \). To give a limitation on the number of these last, we look at the coefficients of the first monomials in equation (2.5) (Table 2).

Table 2: Weight of \( \alpha_{ij0} \) on the line \( b = a - \frac{c_1(B)L}{L^2} \)

| Monomial | Weight of the coefficient |
|----------|--------------------------|
| \( x^3 \) | \(-(b + \frac{r}{s})L\) |
| \( x^2y \) | \(-bL\) |
| \( xy^2 \) | \((\frac{r}{s} - b)L\) |
| \( y^3 \) | \((2\frac{r}{s} - b)L\) |

If \( b - 2\frac{r}{s} > 0 \), i.e. \( b > 2\frac{r}{s} \), we have that \((b - 2\frac{r}{s})L\) is ample, hence

\[
H^0 \left( B, \left( 2\frac{r}{s} - b \right) L \right) = H^0 \left( B, \left( b - 2\frac{r}{s} \right) L \right) = 0.
\]

The same argument applies to the other cases in Table 2 since

\[
b - 2\frac{r}{s} < b - \frac{r}{s} < b < b + \frac{r}{s}.
\]

Hence the coefficients of \( x^3, x^2y, xy^2 \) and \( y^3 \) in (2.5) are necessarily identically zero, and so the equation \( F \) for the variety factors as \( F(x, y, z) = z \cdot f(x, y, z) \). Then \( F = 0 \) can’t define a smooth variety.

Observe that \( z = 0 \) defines a divisor whose class is \( \xi \), while \( f(x, y, z) = 0 \) defines a divisor of class \( p^*c_1(B) + (a + b)p^*L + 2\xi \), which is neither a Calabi–Yau variety nor an elliptic fibration.

In particular, we have only a finite number of pairs \((a, b)\) on the line \( b = a - \frac{c_1(B)L}{L^2} = a - \frac{r}{s} \) such that the generic anticanonical hypersurface in \( \mathbb{P}(L^a \oplus L^b \oplus O_B) \) could define a Calabi–Yau elliptic fibration over \( B \), and a limitation is

\[
(2.15) \quad \frac{r}{s} \leq a \leq 3\frac{r}{s}, \quad 0 \leq b \leq 2\frac{r}{s}.
\]

**Figure 2.** If \( B = \mathbb{P}^2 \) and \( L \) is the class of a line, then we are in the case described in Section 2.4.4 and this is the corresponding picture.
If \( \frac{r}{s} \in \mathbb{N} \) we have then at most

\[
3 \frac{r}{s} + 1 = \left( \frac{r}{s} + 1 \right) + \left( 2 \frac{r}{s} + 1 \right) - \frac{1}{r}
\]

Pairs on the line \( a = \frac{c_1(B)L}{L} \)  
Pairs on the line \( b = a - \frac{c_1(B)L}{L^2} \)  
The common case \((a, b) = \left( \frac{c_1(B)L}{L^2}, 0 \right)\)

such pairs \((a, b)\).

2.4.5. **Conclusion.** Only for a finite number of pairs \((a, b)\) the generic anticanonical hypersurface in \( \mathbb{P}(\mathcal{L}^2 \oplus \mathcal{L}^b) \) is a smooth Calabi–Yau elliptic fibration, which completes the proof of Theorem 2.1.

We summarize the results obtained in Table 3.

| \((2a - b)L + K_B\) is not ample | \((2a - b)L + K_B\) is ample |
|----------------------------------|----------------------------------|
| \((K_BL)^2 \neq K^2_BL^2\) | \((K_BL)^2 = K^2_BL^2\) |
| \(\frac{r}{s} \notin \mathbb{N}\) | \(\frac{r}{s} \in \mathbb{N}\) |

- **Finite number of cases, which are a priori only genus one fibrations.** It is not clear if they have at least one section or not.
- The conic (2.13) is irreducible, and we have a finite number of Calabi–Yau elliptic fibrations.
- No pairs.
- Finite number of Calabi–Yau elliptic fibrations, at most \(3 \frac{r}{s} + 1\).

**Remark 2.4.** I want to stress that we proved that the number of genus 1 fibrations whose total space is smooth lie in a finite number of \( \mathbb{P}(\mathcal{L}^2 \oplus \mathcal{L}^b) \), but we don’t know a priori if all of them are elliptic fibrations. In the finite number of cases detected in Section 2.4.1 it is not clear in fact if there is at least a section.

### 3. Examples

I want to run this program in a case of interest: the case where the base \( B \) is a del Pezzo surface and \( L \) is a rational multiple of an anticanonical divisor. The reason why this is interesting is in the following observation.

**Remark 3.1.** Let \( B \) be a surface and \( L \) an ample divisor on \( B \). Assume that at the end of Step 2 (Section 2.4.2), the conic (2.13) is reducible. It follows easily from (2.14) that then \( B \) is a del Pezzo surface and \( L \) is a rational multiple of \( c_1(B) \).

Before dealing with the general case in Section 3.2, it is worthwhile to study apart the subcase \( B = \mathbb{P}^2 \).
3.1. The case of $B = \mathbb{P}^2$. Observe that if $B$ is a smooth surface with $\text{Pic } B \simeq \mathbb{Z}$, then we are necessarily in the case described in Section 2.4.4.

Take $B = \mathbb{P}^2$, and $L = dl$ for $d \in \mathbb{N}$ and $l$ a line in $\mathbb{P}^2$ (Figure 1 and 2 correspond to the choice $d = 1$). Now we compute the least integer $n_0$ such that $n_0L + K_{\mathbb{P}^2}$ is ample:

$$n_0 = \begin{cases} 
4 & \text{if } d = 1 \\
2 & \text{if } d = 2, 3 \\
1 & \text{if } d \geq 4,
\end{cases}$$

so the cases where we can’t apply the Kodaira vanishing theorem (Section 2.4.1), i.e. those satisfying $2a - b < n_0$, are

$$(0, 0), (1, 0), (1, 1), (2, 1), (2, 2), (3, 3) \quad \text{if } d = 1$$

$$(0, 0), (1, 1) \quad \text{if } d = 2, 3$$

$$(0, 0) \quad \text{if } d \geq 4.$$ Since $c_1(\mathbb{P}^2) = 3l$, we have

$$rdl = 3sl \iff rd = 3s \iff \frac{r}{s} = \frac{3}{d}.$$ We have only two cases where the ratio $\frac{r}{s}$ is an integer, which correspond to $d = 1$ and $d = 3$, i.e. $L = l$ or $L = -K_{\mathbb{P}^2}$. For all the other cases, the only possible pair is then $(a, b) = (0, 0)$, with the exception of $L = 2l$, which has also $(a, b) = (1, 1)$.

For $d = 3$, there are five possibilities: besides the two we already know, on the reducible conic (2.13) we have also the pairs $(a, b) = (1, 1), (2, 1), (2, 3)$.

Table 4: Summary of cases with $B = \mathbb{P}^2$, $L = dl$ and $d \geq 2$.

| $d$ | Possible $(a, b)$ |
|-----|------------------|
| 2   | $(0, 0), (1, 1)$ |
| 3   | $(0, 0), (1, 0), (1, 1), (2, 1), (2, 3)$ |
| $\geq 4$ | $(0, 0)$ |

The only case left is $d = 1$ in the situation of Section 2.4.4. We have to count the integral points on the conic

$$(a - 3)(a - b - 3) = 0$$

which are in the first octant and have $b \leq 6$ (estimate (2.15)). On the line $a = 3$ we have the points $(3, 2), (3, 1)$ and $(3, 0)$, while on the line $b = a - 3$ the points $(4, 1), (5, 2), (6, 3), (7, 4), (8, 5)$ and $(9, 6)$.

Then the pairs $(a, b)$ such that the generic anticanonical hypersurface in the bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b) \oplus \mathcal{O}_{\mathbb{P}^2})$ could be a smooth Calabi–Yau elliptic fibration are the following 15:

$$(0, 0), (1, 0), (1, 1), (2, 1), (2, 2), (3, 3), (3, 2), (3, 1), (3, 0), (4, 1), (5, 2), (6, 3), (7, 4), (8, 5), (9, 6).$$
3.2. The case of del Pezzo surfaces. Let $B$ denote a del Pezzo surface and $L$ a rational multiple of the anticanonical bundle, say $L = \omega_B^{-s}$ (this is the natural setting by Remark 3.1). Let $n_0 = \left\lceil \frac{s}{r} \right\rceil + 1$, then $nL + K_B$ is ample for all $n \geq n_0$. With the notation of Section 2.4.1 the number of pairs $(a, b)$ for which we cannot ensure the presence of a section, i.e. those satisfying the system

\[
\begin{align*}
    a \geq b & \geq 0 \\
    2a - b & < n_0,
\end{align*}
\]

is

\[
(3.2) \quad n_0(n_0 + 2) \quad \text{for } n_0 \text{ even, } \quad n_0^2 + 4n_0 - 1 \quad \text{for } n_0 \text{ odd.}
\]

If the ratio $\frac{s}{r}$ is not an integer, then these are the only cases among which we can find elliptic fibrations.

Remark 3.2. In particular, for $r < s$ we have only the pair $(a, b) = (0, 0)$.

If the ratio $\frac{s}{r}$ is an integer $m$, then $r = ms$ and so $mL = -K_B$, i.e. $L$ is a submultiple of $-K_B$. In this case $n_0 = m + 1$ and we have to count also the points on the reducible conic (2.13); in view of estimate (2.16) these are $3m$ since the point $(a, b) = (m, m)$ was already taken into account. But then the number of families of elliptic Calabi–Yau threefolds over $B$ is bounded by

\[
(3.3) \quad \frac{m^2 + 18m + 4}{4} \quad \text{for } m \text{ even, } \quad \frac{m^2 + 16m + 3}{4} \quad \text{for } m \text{ odd.}
\]

Remark 3.3. Observe that these results agree with the ones we found in Section 3.1 for the plane $\mathbb{P}^2$. Let $l$ be the class of a line, then:

1. For $L = l$, we have $r = 3, s = 1$ and so we can use (3.3) with $m = 3$: we have 15 cases.
2. For $L = 2l$, we have $r = 3, s = 2$ and so we can use (3.2) with $n_0 = 2$: we have 2 cases.
3. For $L = 3l$, we have $r = s = 1$ and so we can use (3.3) with $m = 1$: we have 5 cases.
4. For $L = kl$, with $k \geq 4$, we have $\frac{s}{r} < 1$ and so we can use (3.2) with $n_0 = 1$: we have only one case.

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