Differential dispersion relations and elementary amplitudes in a multiple diffraction model

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We discuss the evaluation of the real part of the elementary amplitudes in the context of a multiple diffraction model for \( pp \) elastic scattering earlier developed. The framework is based on the concepts of analyticity and polynomial boundedness, and the techniques of dispersion relations. Novel results concern the use of derivative dispersion relations at the elementary level (constituent-constituent interactions) and an optimization of these relations in terms of one free parameter. In addition to a theoretical improvement, we achieved a satisfactory description of the physical quantities.

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I. INTRODUCTION

In a previous paper, through a multiple diffraction model, a satisfactory description of \( pp \) elastic scattering data at the highest energies was obtained \[1\]. The approach is based on the Glauber formalism, which is characterized by two essential formulas \[2\]:

\[
F(s, q) = i \int db J_0(qb)[1 - e^{i\chi(s, b)}],
\]

\[
\chi(s, b) = \int qdJ_0(qb)G_A G_B f,
\]

where \( F(s, q) \) is the hadronic amplitude, \( G_{A,B} \) the hadronic form factors and \( f \), the elementary (constituent-constituent) amplitude. With inputs for \( G_A, G_B \) and \( f \), physical quantities may be investigated, such as, the differential cross section

\[
\frac{d\sigma}{dq^2} = \pi |F(s, q)|^2,
\]

the total cross section

\[
\sigma_{\text{tot}}(s) = 4\pi \text{Im} F(s, q = 0),
\]

and the ratio of the forward real and imaginary parts of the hadronic amplitude,

\[
\rho(s) = \frac{\text{Re} F(s, q = 0)}{\text{Im} F(s, q = 0)}.
\]

The model we referred to is characterized by the following parametrizations \[1\]:

\[
G_A = G_B \equiv G(s, q) = \frac{1}{1 + q^2/\alpha^2(s)} \frac{1}{1 + q^2/\beta^2}.
\]

*Deceased

\[ f(s, q) = (\lambda + i)C(s)h(q), \quad (7) \]

\[ h(q) = \frac{1 - q^2/a^2}{1 + q^2/a^4}, \quad (8) \]

with two fixed parameters

\[ a^2 = 8.20 \text{ GeV}^2, \quad \beta^2 = 1.80 \text{ GeV}^2 \quad (9) \]

and three parameters depending on the energy,

\[ \alpha^{-2}(s) = 2.57 - 0.217 \ln(s) + 0.0243 \ln^2(s) \quad (\text{GeV}^{-2}), \quad (10) \]

\[ C(s) = 14.3 - 1.65 \ln(s) + 0.159 \ln^2(s) \quad (\text{GeV}^{-2}), \quad (11) \]

\[ \lambda(s) = \frac{0.0695 \ln(s/s_0)}{1 + 0.118 \ln(s/s_0) + 0.015 \ln^2(s/s_0)}, \quad (12) \]

where \( s_0 = 400 \text{ GeV} \).

With the exception of parametrization (12) for \( \lambda(s) \), all the other choices are physically justified, as explained in [1] and references therein. Although Eq. (12) allows a good description of pp experimental data and has also recently been used in analyses of p-nucleus collisions [3], it does not have a full theoretical foundation, as recalled in what follows.

Physically, from assumption (7), \( \lambda(s) \) plays at the elementary level the same role as \( \rho(s) \) at the hadronic level, with the exception of the constancy in terms of momentum transfer:

\[ \lambda(s) = \frac{\text{Re}f(s, q = 0)}{\text{Im}f(s, q = 0)} = \frac{\text{Re}f(s, q)}{\text{Im}f(s, q)}. \quad (13) \]

As explained in [1], parametrization (12) was inferred from the observed similarities between \( \rho(s) \) and \( \lambda(s) \) and then from experimental information on \( \rho(s) \), including pp scattering. For this reason, from the theoretical point of view, this parametrization may be seen as an ad hoc hypothesis.

In this paper we shall introduce novel parametrizations for \( \lambda(s) \), but now based on general principles of local quantum field theory QFT. The framework concerns the concepts of analyticity and polynomial boundedness and the techniques of dispersion relations. The paper is organized as follows. In Sec. II we review some theoretical aspects concerning dispersion relations in the derivative form and apply these relations to the elementary amplitudes. With this we obtain an analytical expression for \( \lambda(s) \), allowing a new determination of the real part of the hadronic amplitude. We shall also introduce an optimization of these relations in terms of a free parameter. In Sec. III we discuss the results obtained and some aspects of the dispersion relations concerning both elementary and hadronic amplitudes, and also asymptotic limits. The conclusions and some final remarks are the content of Sec. IV.

II. DERIVATIVE DISPERSION RELATIONS AND ELEMENTARY AMPITUDNES

Derivative dispersion relations were introduced nearly thirty years ago [1,5] and have, even recently, been used in the investigation of both pp and p\bar{p} scattering [6]. Its validity and practical applicability have also been extensively discussed [8]. These relations may be extended to an arbitrary number of subtractions, for both cross even and odd amplitudes and near the forward direction [6]. Based on the general principles referred to before, it was shown that for an even amplitude the result for the first and second subtractions are the same and the leading term in the tangent series expansion reads [8]

\[ \frac{\text{Re}F_+(s, q)}{s^2} = \tan \left( \frac{\pi}{2}(\nu - 1) \right) \frac{\text{Im}F_+(s, q)}{s^2} + \frac{\pi}{2} \sec^2 \left( \frac{\pi}{2}(\nu - 1) \right) \frac{d}{d\ln s} \left( \frac{\text{Im}F_+(s, q)}{s^2} \right), \quad (14) \]

where \( \nu \) is a real free parameter constrained to the interval \( 0 < \nu < 2 \). This result is valid for \( s >> m^2 \) and \( q^2 \lesssim m^2 \), where \( m \) is the proton mass.

The smooth increase of the hadronic cross sections, roughly as \( \ln^2 s \), is compatible with the class of functions which verifies Eq. (14). From the optical theorem, Eq. (4), this allows the simultaneous investigation of \( \sigma_{\text{tot}}(s) \) and \( \rho(s) \).
for both \( pp \) and \( \overline{p}p \) interactions \([3]\). The conventional form widely applied to hadronic amplitudes corresponds to the particular case \( \nu = 1 \) and the forward direction:

\[
\frac{\text{Re}F_+(s, q = 0)}{s} = \frac{\pi}{2} \frac{d}{d\ln s} \left[ \frac{\text{Im}F_+(s, q = 0)}{s} \right].
\]

Although all known uses of derivative dispersion relations in elastic and diffractive scattering concern hadronic amplitudes, in what follows we shall investigate its applicability at the elementary level.

In the model described in Sec. I, the imaginary part of the elementary amplitude factorizes in the form

\[
\text{Im}f(s, q) = C(s) h(q).
\]

We connect this quantity with the derivative dispersion relation (14) by defining \([10]\)

\[
\frac{\text{Im}F_+(s, q)}{s} = \text{Im}f(s, q).
\]

From Eqs. (14), (16) and (17), we obtain an analytical expression for \( \lambda(s) \) in terms of \( C(s) \) and \( \nu \)

\[
\lambda(s, \nu) = \frac{\text{Re}f(s, q)}{\text{Im}f(s, q)} = \tan \left( \frac{\pi}{2} (\nu - 1) \right) + \frac{\pi}{2} \sec^2 \left( \frac{\pi}{2} (\nu - 1) \right) \left[ \frac{1}{C(s)} \frac{dC(s)}{d\ln s} + 1 - \nu \right].
\]

With parametrization (11) for \( C(s) \) the above connection is valid, since the increase of this quantity with the energy follows a second degree polynomial in \( \ln s \) (smooth increase). Differently from the ad hoc hypothesis (12), we now have a theoretically justified result for \( \lambda(s) \), given in terms of the parametrization for \( C(s) \). For each \( \nu \) value the model described in Sec. I leads to the physical quantities (3), (4), and (5).

As a first test we took the conventional value \( \nu = 1 \). The corresponding behaviour of \( \lambda(s, \nu = 1) \) is shown in Fig. 1, together with the early parametrization (12). The results for the differential and total cross sections are quite similar to the previous ones, presented in \([1]\). However, the predictions for \( \rho(s) \) overestimate the experimental data, as can be seen in Fig. 2.

Based on this result we attempted to optimize the derivative relation (18) by letting free the parameter \( \nu \). In this case we analyzed only the \( \rho(s) \) data, where disagreements have been found. The best fit through the CERN-minut \([11]\) furnished

\[
\nu = 1.25 \pm 0.01,
\]

with \( \chi^2 = 7.76 \) for 6 degrees of freedom. The behaviour of \( \lambda(s) \) is shown in Fig. 1 and the prediction for the corresponding \( \rho(s) \) is displayed in Fig. 2. As before, the results for the differential and total cross sections are similar to the previous ones.

### III. DISCUSSION

A central point in this work concerns the introduction of Eq. (18) for the ratio of the real and imaginary parts of the elementary amplitude. In this section we discuss some consequences of this formula related with the predictions of physical quantities (descriptions of experimental data) and asymptotic behaviors.

#### A. Real part of the hadronic amplitude and differential cross sections

An essential aspect of the multiple diffraction models is the connection between hadronic and elementary amplitudes. In applying the derivative dispersion relation at the elementary level we obtain both the real and imaginary parts of the hadronic amplitude. As illustration and for comparison, we calculate the real part of the hadronic amplitude through the parametrizations presented in Sec. I. The contribution of this part to the differential cross section, for \( pp \) elastic scattering at \( \sqrt{s} = 52.8 \) GeV is shown in Fig. 3, together with the differential cross section data and for the three cases investigated, namely, \( \lambda(s) \) from Eq. (12) and \( \lambda(s, \nu = 1.0) \), \( \lambda(s, \nu = 1.25) \) from Eq. (18). We observe that in all cases the real part of the amplitude presents two changes of signal and at the same positions. As showed in Ref. \([1]\), similar results are obtained, for example, through the Martin’s formula applied directly to the hadronic amplitude.

Since the imaginary part of the amplitude presents a zero at the dip position, our results overestimate the experimental data at this region, as can be seen in Fig. 3. This is the case even for the best result, corresponding to \( \nu = 1.25 \). However, as commented in Ref. \([1]\), simultaneous and complete descriptions of total/differential cross
sections and the $\rho$ parameter still remain an open problem in geometrical and multiple diffraction models. In our case, as in the previous approach, the limitations concern only the dip region at the highest ISR energies (see Fig. 4 in Ref. [1]). This seems to be consequence of the naive assumption of factorization of the imaginary part of the elementary amplitude, Eq. (16), leading to constant $\lambda$ in terms of the momentum transfer. Recent results presented evidence for eikonal zeros in the momentum transfer space and this may bring new insights on the search for a more suitable assumption. We are presently investigating this point.

B. Derivative dispersion relation and asymptotic limits

A novel aspect of this work concerns the optimization of the derivative dispersion relation by letting free the parameter $\nu$. Contrary to some authors, which understand that “...the choice of a $\nu$ different from 1 has no practical advantage” we showed that in the context of our model, the best results were obtained with $\nu = 1.25$ and this demands further discussion.

We shall analyze some consequences of Eqs. (14) and (18) for $\lambda(s, \nu)$ when $\nu \neq 1$. It is important to note that despite our analyses treat the elementary level, similar results may be inferred at the hadronic level. In fact, as shown in Figs. 1 and 2 (and commented in Ref. [1]), $\lambda(s)$ and $\rho(s)$ are strongly correlated: if $\lambda$ increases (decreases) also $\rho$ increases (decreases), $\lambda = 0$ at the same energy value where $\rho = 0$, and the same is true for the position of their maxima. Moreover, our parametrization for $C(s)$, Eq. (11), has the same functional form as the usual total cross sections parametrizations at sufficiently high energies.

We first recall that the two-subtracted differential dispersion relation (14) corresponds to the first term in a tangent series expansion, involving the derivative in the variable $\ln s$. Since this is an odd series and the parametrization for $C(s)$ is a second degree polynomial in $\ln s$, Eq. (14) is an “exact” result, that is, no other terms in the expansion must be taken into account. Obviously the same is true in the case of total cross sections parametrizations.

According to the approach of Ref. the $\nu$ parameter is constrained to the interval $(0, 2)$ and formula (18) diverges at these extremes. The behavior of $\lambda(s, \nu)$ for $\nu = 0.75$, 1.00, and 1.25, is shown in Fig. 4, in the wide energy interval $10^1 - 10^{15}$ GeV. Based on the strong correlation between $\rho(s)$ and $\lambda(s)$ (see Figs. 1 and 2) it is phenomenologically evident that, in order to reproduce the experimental $\rho$ data, we should expect values of $\nu$ near to 1. However a crucial point concerns the asymptotic behavior $\lambda(s \to \infty, \nu)$. From our parametrization for $C(s)$ the term involving this quantity in Eq. (18) vanishes at sufficiently high energies. The asymptotic form of $\lambda(\nu)$, from eq. (18), is displayed in Fig 5 in the above interval of the parameter $\nu$. We see that $\nu = 1$ is a point of inflection and the asymptotic values of $\lambda$ are positive for $\nu < 1$ and negative for $\nu > 1$.

In the context of our model and parametrizations, the result $\nu = 1.25$ leads to the asymptotic value $\lambda \sim -0.05$. From Fig. 2 it is evident that $\nu \leq 1$ cannot reproduce the experimental $\rho$ data. Therefore, a second change of signal is predicted and the same is inferred for $\rho(s)$ at sufficiently high energies.

IV. CONCLUSIONS AND FINAL REMARKS

In this work we discussed the evaluation of the real part of the elementary amplitude in the context of a multiple diffraction model earlier developed. The novel results are (a) application for the first time of differential dispersion relations at the elementary level, allowing the introduction of a theoretically justified analytical parametrization for $\lambda(s)$, Eq. (18); (b) optimization of the derivative dispersion relation by letting free the parameter $\nu$.

Although we still have the parameter $\nu$ to be determined by fit, we stress that the behavior of $\rho(s)$ beyond the region with data available comes directly from the dispersion relation applied to the elementary amplitude. That is, this behavior is not imposed as in the previous parametrization for $\lambda(s)$, Eq. (12). From Fig. 2 we also observe a faster decrease of $\rho$ at the highest energies than in the previous approach. In particular we predict for $pp$ elastic scattering: $\rho_{\text{max}} = 0.14$ at $\sqrt{s} \sim 580$ GeV and $\rho = 0.11$ at $\sqrt{s} = 16$ TeV (LHC). We also presented indication that $\rho$ may become negative at sufficiently high energies: That is the case for $\nu > 1.0$.

This change of sign is obviously an unconventional result, bringing some resemblance only with some kinds of Odderon analyses. Concerning this point, it should be stressed that Eq. (14) is the general formula for the derivative relation, as introduced by Bronzan, Kane and Sukhatme. The $\rho$ parameter and total cross section should then be correlated through Eq. (18). As we have shown, to assume $\nu = 1$ means to impose that $\rho$ goes asymptotically to zero through positive values, if in this region, $\sigma_{\text{tot}} \sim \ln^2 s$. For this reason, it should be interesting to investigate the $pp$ and $p\bar{p}$ available data on $\rho$ and $\sigma_{\text{tot}}$, through Eq. (18), with adequate cross symmetry conditions and treating $\nu$ as a free parameter. This could bring new information on the asymptotic behavior of $\rho(s)$ and of total cross sections.

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FIG. 1. Ratio of the real to imaginary parts of the elementary amplitude, $\lambda(s)$: previous parametrization, Eq. (12) (dotted); results through the derivative dispersion relation, Eq. (18), with $\nu = 1.0$ (dashed) and $\nu = 1.25$ (dot-dashed).

FIG. 2. Predictions for $\rho(s)$ with $\lambda(s)$ from Fig. 1 (same legend) and the model described in Sec. I, together with experimental data (See Ref. [1]).

FIG. 3. Contributions of the real part of the hadronic amplitude to the differential cross section, for $pp$ elastic scattering at $\sqrt{s} = 52.8$ GeV and experimental data (see Ref. [1]). The curves were obtained with $\lambda(s)$ from Fig. 1 (same legend) and the model described in Sec. I.

FIG. 4. Results for $\lambda(s, \nu)$ from Eq. (18) and parametrization (11) for $C(s)$.

FIG. 5. Asymptotic behavior of $\lambda(\nu)$ from Eq. (18).
Fig. 2
Fig. 3

\[ \frac{d\sigma}{dq^2} \ (\text{mb/GeV}^2) \]

\[ q^2 \ (\text{GeV}^2) \]

\[ 10^{-13} \quad 10^{-8} \quad 10^{-3} \quad 10^{0} \quad 10^{2} \]

\[ 0.0 \quad 2.0 \quad 4.0 \quad 6.0 \quad 8.0 \quad 10.0 \]
Fig. 4

\[ \lambda(\nu, \sqrt{s}) \]

- $\nu = 0.75$
- $\nu = 1.0$
- $\nu = 1.25$
