Higher homotopy commutativity and cohomology of finite $H$–spaces

YUTAKA HEMMI
YUSUKE KAWAMOTO

We study connected mod $p$ finite $A_p$–spaces admitting $AC_n$–space structures with $n < p$ for an odd prime $p$. Our result shows that if $n > (p - 1)/2$, then the mod $p$ Steenrod algebra acts on the mod $p$ cohomology of such a space in a systematic way. Moreover, we consider $A_p$–spaces which are mod $p$ homotopy equivalent to product spaces of odd dimensional spheres. Then we determine the largest integer $n$ for which such a space admits an $AC_n$–space structure compatible with the $A_p$–space structure.

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1 Introduction

In this paper, we assume that $p$ is a fixed odd prime and that all spaces are localized at $p$ in the sense of Bousfield–Kan [2].

In the paper [10], we introduced the concept of $AC_n$–space which is an $A_n$–space whose multiplication satisfies the higher homotopy commutativity of the $n$–th order. Then we showed that a mod $p$ finite $AC_n$–space with $n \geq p$ has the homotopy type of a torus. Here by being mod $p$ finite, we mean that the mod $p$ cohomology of the space is finite dimensional. To prove it, we first studied the action of the Steenrod operations on the mod $p$ cohomology of such a space. Then we showed that the possible cohomology generators are concentrated in dimension 1.

In the above argument, the condition $n \geq p$ is essential. In fact, any odd dimensional sphere admits an $AC_{p−1}$–space structure by [10, Proposition 3.8]. This implies that for any given exterior algebra, we can construct a mod $p$ finite $AC_{p−1}$–space such that the mod $p$ cohomology of it is isomorphic to the algebra.

On the other hand, if the $A_{p−1}$–space structure of the $AC_{p−1}$–space is extendable to an $A_p$–space structure, then the situation is different. For example, it is known that an
odd dimensional sphere with an $A_p$–space structure does not admit an $AC_{p-1}$–space structure except for $S^1$. In fact, an odd dimensional sphere $S^{2m-1}$ admits an $A_p$–space structure if and only if $m|(p-1)$, and then it admits an $AC_n$–space structure compatible with the $A_p$–space structure if and only if $nm \leq p$ by [6, Theorem 2.4]. In particular, if $p = 3$, then mod 3 finite $A_3$–space with $AC_2$–space structure means mod 3 finite homotopy associative and homotopy commutative $H$–space. Then by Lin [21], such a space has the homotopy type of a product space of $S^1$s and $Sp(2)$s.

In this paper, we study mod $p$ finite $A_p$–spaces with $AC_n$–space structures for $n < p$. First we consider the case of $n > (p-1)/2$. In this case, we show the following fact on the action of the Steenrod operations:

**Theorem A** Let $p$ be an odd prime. If $X$ is a connected mod $p$ finite $A_p$–space admitting an $AC_n$–space structure with $n > (p-1)/2$, then we have the following:

1. If $a \geq 0$, $b > 0$ and $0 < c < p$, then
   \[ QH^{2p^a(pb+c)-1}(X; \mathbb{Z}/p) = \mathcal{P}^{p^a}QH^{2p^a(p(b-t)+c+t)-1}(X; \mathbb{Z}/p) \]
   for $1 \leq t \leq \min\{b, p-c\}$ and
   \[ \mathcal{P}^{p^a}QH^{2p^a(pb+c)-1}(X; \mathbb{Z}/p) = 0 \]
   for $c \leq t < p$.

2. If $a \geq 0$ and $0 < c < p$, then
   \[ \mathcal{P}^{p^a} : QH^{2p^ac-1}(X; \mathbb{Z}/p) \to QH^{2p^a(p+c-t)-1}(X; \mathbb{Z}/p) \]
   is an isomorphism for $1 \leq t < c$.

In the above theorem, the assumption $n > (p-1)/2$ is necessary. In fact, (2) is not satisfied for the Lie group $S^3$ although $S^3$ admits an $AC_{(p-1)/2}$–space structure for any odd prime $p$ as is proved in [6, Theorem 2.4].

**Theorem A** (1) has been already proved for a special case or under additional hypotheses: for $p = 3$ by Hemmi [7, Theorem 1.1] and for $p \geq 5$ by Lin [19, Theorem B] under the hypotheses that the space admits an $AC_{p-1}$–space structure and the mod $p$ cohomology is $A_p$–primitively generated (see Hemmi [8] and Lin [19]).

In the above theorem, we assume that the prime $p$ is odd. However, if we consider the case $p = 2$, then the condition $p > n > (p-1)/2$ is equivalent to $n = 1$, which means that the space is just an $H$–space. Thus Theorem A can be considered as the...
odd prime version of Thomas [26, Theorem 1.1] or Lin [18, Theorem 1]. (Note that in their theorems they assumed that the mod 2 cohomology of the space is primitively generated, while we do not need such an assumption.)

By using Theorem A, we show the following result:

**Theorem B** Let $p$ be an odd prime. If $X$ is a connected mod $p$ finite $A_p$–space admitting an $AC_n$–space structure with $n > (p - 1)/2$ and the Steenrod operations $P^j$ act on $QH^\ast(X; \mathbb{Z}/p)$ trivially for $j \geq 1$, then $X$ is mod $p$ homotopy equivalent to a torus.

Next we consider the case of $n \leq (p - 1)/2$. This includes the case $n = 1$, which means that the space is just a mod $p$ finite $A_p$–space. For the cohomology of mod $p$ finite $A_p$–spaces, we can show similar facts to Theorem A. For example, the results by Thomas [26, Theorem 1.1] and Lin [18, Theorem 1] mentioned above is for $p = 2$, and for odd prime $p$, many results are known (cf. [1], [5], [20]).

However, for odd primes in particular, those results have some ambiguities. In fact, there are many $A_p$–spaces with $AC_n$–space structures for some $n \leq (p - 1)/2$ such that the Steenrod operations act on the cohomology trivially. In the next theorem, we determine $n$ for which a product space of odd dimensional spheres to be an $A_p$–space with an $AC_n$–space structure.

**Theorem C** Let $X$ be a connected $A_p$–space mod $p$ homotopy equivalent to a product space of odd dimensional spheres $S^{2m_1-1} \times \cdots \times S^{2m_l-1}$ with $1 \leq m_1 \leq \cdots \leq m_l$, where $p$ is an odd prime. Then $X$ admits an $AC_n$–space structure if and only if $nm_l \leq p$.

By the results of Clark–Ewing [4] and Kumpel [17], there are many spaces satisfying the assumption of Theorem C. Moreover, we note that the above result generalizes [6, Theorem 2.4].

This paper is organized as follows: In Section 2, we first recall the modified projective space $M(X)$ of a finite $A_p$–space constructed by Hemmi [8]. Based on the mod $p$ cohomology of $M(X)$, we construct an algebra $A^\ast(X)$ over the mod $p$ Steenrod algebra which is a truncated polynomial algebra at height $p + 1$ (Theorem 2.1). Next we introduce the concept of $D_n$–algebra and show that if $X$ is an $A_p$–space with an $AC_n$–space structure, then $A^\ast(X)$ is a $D_n$–algebra (Theorem 2.6). Finally we prove the theorems in Section 3 by studying the action of the Steenrod algebra on $D_n$–algebras algebraically (Proposition 3.1 and Proposition 3.2).
This paper is dedicated to Professor Goro Nishida on his 60th birthday. The authors appreciate the referee for many useful comments.

2 Modified projective spaces

Stasheff [25] introduced the concept of $A_n$-space which is an $H$-space with multiplication satisfying higher homotopy associativity of the $n$-th order. Let $X$ be a space and $n \geq 2$. An $A_n$-form on $X$ is a family of maps $\{M_i: K_i \times X^i \to X\}_{2 \leq i \leq n}$ with the conditions of [25, I, Theorem 5], where $\{K_i\}_{i \geq 2}$ are polytopes called the associahedra. A space $X$ having an $A_n$-form is called an $A_n$-space. From the definition, an $A_2$-space and an $A_3$-space are the same as an $H$-space and a homotopy associative $H$-space, respectively. Moreover, it is known that an $A_\infty$-space has the homotopy type of a loop space.

Let $X$ be an $A_n$-space. Then by Stasheff [25, I, Theorem 5], there is a family of spaces $\{P_i(X)\}_{1 \leq i \leq n}$ called the projective spaces associated to the $A_n$-form on $X$. From the construction of $P_i(X)$, we have the inclusion $\iota_i: P_{i-1}(X) \to P_i(X)$ for $2 \leq i \leq n$ and the projection $\rho_i: P_i(X) \to P_i(X)/P_{i-1}(X) \simeq (\Sigma X)^{(i)}$ for $1 \leq i \leq n$, where $Z^{(i)}$ denotes the $i$-fold smash product of a space $Z$ for $i \geq 1$.

For the rest of this section, we assume that $X$ is a connected $A_p$-space whose mod $p$ cohomology $H^*(X; \mathbb{Z}/p)$ is an exterior algebra

$$(2-1) \quad H^*(X; \mathbb{Z}/p) \cong \Lambda(x_1, \ldots, x_l) \quad \text{with } \deg x_i = 2m_i - 1$$

for $1 \leq i \leq l$, where $1 \leq m_1 \leq \cdots \leq m_l$.

Iwase [12] studied the mod $p$ cohomology of the projective space $P_n(X)$ for $1 \leq n \leq p$. If $1 \leq n \leq p-1$, then there is an ideal $S_n \subset H^*(P_n(X); \mathbb{Z}/p)$ closed under the action of the mod $p$ Steenrod algebra $A_p^*$ such that

$$(2-2) \quad H^*(P_n(X); \mathbb{Z}/p) \cong T_n \oplus S_n \quad \text{with } T_n = T^{[n+1]}[y_1, \ldots, y_l],$$

where $T^{[n+1]}[y_1, \ldots, y_l]$ denotes the truncated polynomial algebra at height $n+1$ generated by $y_i \in H^{2m_i}(P_n(X); \mathbb{Z}/p)$ with $y_1^* \cdots t_{n-1}^*(y_l) = \sigma(x_i)$ for $1 \leq i \leq l$. He also proved a similar result for the mod $p$ cohomology of $P_p(X)$ under an additional assumption that the generators $\{x_i\}_{1 \leq i \leq l}$ are $A_p$-primitive (see Hemmi [8] and Iwase [12]).
Hemmi [8] modified the construction of the projective space \( P_p(X) \) to get the algebra \( T[y_{p+1}, \ldots, y_l] \) also for \( n = p \) without the assumption of the \( A_p \)-primitivity of the generators. Then he proved the following result:

**Theorem 2.1** (Hemmi [8, Theorem 1.1]) Let \( X \) be a simply connected \( A_p \)-space whose mod \( p \) cohomology \( H^*(X; \mathbb{Z}/p) \) is an exterior algebra in \((2-1)\), where \( p \) is an odd prime. Then we have a space \( M(X) \) and a map \( \epsilon : \Sigma X \to M(X) \) with the following properties:

1. There is a subalgebra \( R^*(X) \) of \( H^*(M(X); \mathbb{Z}/p) \) with \( R^*(X) \cong T[y_{p+1}, \ldots, y_l] \oplus M \), where \( y_i \in H^{2m_i}(M(X); \mathbb{Z}/p) \) are classes with \( \epsilon^*(y_i) = \sigma(x_i) \) for \( 1 \leq i \leq l \) and \( M \subset H^*(M(X); \mathbb{Z}/p) \) is an ideal with \( \epsilon^*(M) = 0 \) and \( M \cdot H^*(M(X); \mathbb{Z}/p) = 0 \).

2. \( R^*(X) \) and \( M \) are closed under the action of \( A_p^* \), and so

\[
A^*(X) = \frac{R^*(X)}{M} \cong T[y_{p+1}, \ldots, y_l]
\]

is an unstable \( A_p^* \)-algebra.

3. \( \epsilon^* \) induces an \( A_p^* \)-module isomorphism:

\[
QA^*(X) \longrightarrow QH^{*-1}(X; \mathbb{Z}/p).
\]

Next we recall the higher homotopy commutativity of \( H \)-spaces.

Kapranov [16] and Reiner–Ziegler [23] constructed special polytopes \( \{ \Gamma_n \}_{n \geq 1} \) called the permuto–associahedra. Let \( n \geq 1 \). A partition of the sequence \( n = (1, \ldots, n) \) of type \((t_1, \ldots, t_m)\) is an ordered sequence \((\alpha_1, \ldots, \alpha_m)\) consisting of disjoint subsequences \( \alpha_i \) of \( n \) of length \( t_i \) with \( \alpha_1 \cup \ldots \cup \alpha_m = n \) (see Hemmi–Kawamoto [10] and Ziegler [28] for the full details of the partitions). By Ziegler [28, Definition 9.13, Example 9.14], the permuto–associahedron \( \Gamma_n \) is an \((n-1)\)-dimensional polytope whose facets (codimension one faces) are represented by the partitions of \( n \) into at least two parts. Let \( \Gamma(\alpha_1, \ldots, \alpha_m) \) denote the facet of \( \Gamma_n \) corresponding to a partition \((\alpha_1, \ldots, \alpha_m)\). Then the boundary of \( \Gamma_n \) is given by

\[
\partial \Gamma_n = \bigcup_{(\alpha_1, \ldots, \alpha_m)} \Gamma(\alpha_1, \ldots, \alpha_m)
\]

for all partitions \((\alpha_1, \ldots, \alpha_m)\) of \( n \) with \( m \geq 2 \). If \((\alpha_1, \ldots, \alpha_m)\) is of type \((t_1, \ldots, t_m)\), then the facet \( \Gamma(\alpha_1, \ldots, \alpha_m) \) is homeomorphic to the product \( K_{t_1} \times \ldots \times K_{t_m} \).
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Figure 1: Permuto–associahedra $\Gamma_2$ and $\Gamma_3$

by the face operator $\epsilon^{(\alpha_1, \ldots, \alpha_m)}: K_m \times \Gamma_{t_1} \times \cdots \times \Gamma_{t_m} \to \Gamma(\alpha_1, \ldots, \alpha_m)$ with the relations of [10, Proposition 2.1]. Moreover, there is a family of degeneracy operators $\{\delta_j: \Gamma_i \to \Gamma_{i-1}\}_{1 \leq j \leq i}$ with the conditions of [10, Proposition 2.3].

By using the permuto–associahedra, Hemmi and Kawamoto [10] introduced the concept of $\text{AC}_n$–form on $A_n$–spaces.

Let $X$ be an $A_n$–space whose $A_n$–form is given by $\{M_i\}_{2 \leq i \leq n}$. An $\text{AC}_n$–form on $X$ is a family of maps $\{Q_i: \Gamma_i \times X^i \to X\}_{1 \leq i \leq n}$ with the following conditions:

(2–5) $Q_1(*, x) = x$.

(2–6) $Q_i(\epsilon^{(\alpha_1, \ldots, \alpha_m)}(\sigma, \tau_1, \ldots, \tau_m), x_1, \ldots, x_i) = M_m(\sigma, Q_{t_1}(\tau_1, x_{\alpha_1(1)}, \ldots, x_{\alpha_1(t_1)}), \ldots, Q_{t_m}(\tau_m, x_{\alpha_m(1)}, \ldots, x_{\alpha_m(t_m)}))$

for a partition $(\alpha_1, \ldots, \alpha_m)$ of $i$ of type $(t_1, \ldots, t_m)$.

(2–7) $Q_i(\tau, x_1, \ldots, x_{j-1}, *, x_{j+1}, \ldots, x_i) = Q_{i-1}(\delta_j(\tau), x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_i)$

for $1 \leq j \leq i$.

By [10, Example 3.2 (1)], an $\text{AC}_2$–form on an $A_2$–space is the same as a homotopy commutative $H$–space structure since $Q_2: \Gamma_2 \times X^2 \to X$ gives a commuting homotopy between $xy$ and $yx$ for $x, y \in X$ (see Figure 2). Let us explain an $\text{AC}_3$–form on an $A_3$–space. Assume that $X$ is an $A_3$–space admitting an $\text{AC}_2$–form. Then by using the associating homotopy $M_3: K_3 \times X^3 \to X$ and the commuting homotopy $Q_2: \Gamma_2 \times X^2 \to X$, we can define a map $\tilde{Q}_3: \partial \Gamma_3 \times X^3 \to X$ which is illustrated.
by the right dodecagon in Figure 2. For example, the uppermost edge represents the commuting homotopy between $xy$ and $yx$, and thus it is given by $Q_2(t, x, y)z$. On the other hand, the next right edge is the associating homotopy between $(xy)z$ and $x(yz)$ which is given by $M_3(t, x, y, z)$. Then $X$ admits an $AC_3$–form if and only if $\tilde{Q}_3$ is extended to a map $Q_3 : \Gamma_3 \times X^3 \to X$. Moreover, if $X$ is an $H$–space, then by [10, Example 3.2 (3)], the multiplication of the loop space $\Omega X$ on $X$ admits an $AC_\infty$–form. Hemmi [6] considered another concept of higher homotopy commutativity of $H$–spaces. Let $X$ be an $A_n$–space with the projective spaces $\{P_i(X)\}_{1 \leq i \leq n}$. Let $J_i(\Sigma X)$ be the $i$-th stage of the James reduced product space of $\Sigma X$ and $\pi_i : J_i(\Sigma X) \to (\Sigma X)^{(i)}$ be the obvious projection for $1 \leq i \leq n$. A quasi $C_n$–form on $X$ is a family of maps $\{\psi : J_i(\Sigma X) \to P_i(X)\}_{1 \leq i \leq n}$ with the following conditions:

1. $\psi_1 = 1_{\Sigma X} : \Sigma X \to \Sigma X$.
2. $\psi_i|_{J_{i-1}(\Sigma X)} = \iota_{i-1}\psi_{i-1}$ for $2 \leq i \leq n$.
3. $\rho_i\psi_i \simeq \left(\sum_{\sigma \in \Sigma_i} \sigma\right)^{\pi_i}$ for $1 \leq i \leq n$.

where the symmetric group $\Sigma_i$ acts on $(\Sigma X)^{(i)}$ by the permutation of the coordinates and the summation on the right hand side is given by using the obvious co–$H$–structure on $(\Sigma X)^{(i)}$ for $1 \leq i \leq n$.

Hemmi and Kawamoto [10] proved the following result:

**Theorem 2.2** (Hemmi–Kawamoto [10, Theorem A]) Let $X$ be an $A_n$–space for $n \geq 2$. Then we have the following:
(1) If \( X \) admits an \( AC_n \)-form, then \( X \) admits a quasi \( C_n \)-form.

(2) If \( X \) is an \( A_{n+1} \)-space admitting a quasi \( C_n \)-form, then \( X \) admits an \( AC_n \)-form.

**Remark 2.3** In the proof of Theorem 2.2 (2), we do not need the condition (2–10). In fact, the proof of Theorem 2.2 (2) shows that if \( X \) is an \( A_{n+1} \)-space and there is a family of maps \( \{ \psi_i \}_{1 \leq i \leq n} \) with the conditions (2–8)–(2–9), then there is a family of maps \( \{ Q_i \}_{1 \leq i \leq n} \) with the conditions (2–5)–(2–7).

Now we give the definition of \( D_n \)-algebra:

**Definition 2.4** Assume that \( A^* \) is an unstable \( A_p^* \)-algebra for a prime \( p \). Let \( n \geq 1 \). Then \( A^* \) is called a \( D_n \)-algebra if for any \( \alpha_i \in A^* \) and \( \theta_i \in A_p^* \) for \( 1 \leq i \leq q \) with

\[
\sum_{i=1}^{q} \theta_i(\alpha_i) \in DA^*,
\]

there are decomposable classes \( \nu_i \in DA^* \) for \( 1 \leq i \leq q \) with

\[
\sum_{i=1}^{q} \theta_i(\alpha_i - \nu_i) \in D^{n+1}A^*,
\]

where \( DA^* \) and \( D^tA^* \) denote the decomposable module and the \( t \)-fold decomposable module of \( A^* \) for \( t > 1 \), respectively.

**Remark 2.5** It is clear from Definition 2.4 that any unstable \( A_p^* \)-algebra is a \( D_1 \)-algebra. On the other hand, for an \( A_p \)-space \( X \) which satisfies the assumption of Theorem 2.1 with \( l \geq 1 \), the unstable \( A_p^* \)-algebra \( A^*(X) \) given in (2–3) cannot be a \( D_p \)-algebra since \( P^m(\alpha) = \alpha^p \neq 0 \) for \( \alpha \in QA^{2m}(X) \) from the unstable condition of \( A_p^* \) and \( D^{p+1}A^*(X) = 0 \) in (2–12).

To prove Theorem A and Theorem B, we need the following theorem:

**Theorem 2.6** Let \( p \) be an odd prime and \( 1 \leq n \leq p - 1 \). Assume that \( X \) is a simply connected \( A_p \)-space whose mod \( p \) cohomology \( H^*(X; \mathbb{Z}/p) \) is an exterior algebra in (2–1). If the multiplication of \( X \) admits a quasi \( C_n \)-form, then \( A^*(X) \) is a \( D_n \)-algebra.

We need the following result which is a generalization of Hemmi [6]:

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Lemma 2.7 Assume that \( X \) satisfies the same assumptions as Theorem 2.6. If \( \alpha_i \in H^*(P_\alpha(X); \mathbb{Z}/p) \) and \( \theta_i \in A^p_\alpha \) for \( 1 \leq i \leq q \) satisfy
\[
\sum_{i=1}^{q} \theta_i(\alpha_i) = a + b \quad \text{with} \quad a \in DH^*(P_\alpha(X); \mathbb{Z}/p) \quad \text{and} \quad b \in S_n,
\]
then there are decomposable classes \( \nu_i \in DH^*(P_\alpha(X); \mathbb{Z}/p) \) for \( 1 \leq i \leq q \) with
\[
\sum_{i=1}^{q} \theta_i(\alpha_i - \nu_i) = b.
\]

Proof We give an outline of the proof since the argument is similar to Hemmi [6, Lemma 4.8]. It is clear for \( n = 1 \). If the result is proved for \( n - 1 \), then by the same reason as [6, Lemma 4.8], we can assume \( a \in D^pH^*(P_\alpha(X); \mathbb{Z}/p) \).

Put \( U_n = \tilde{H}^*(\Sigma X)^n; \mathbb{Z}/p \), \( V_n = QH^*(X; \mathbb{Z}/p)^\otimes n \) and
\[
W_n = \bigoplus_{i=1}^{n} \tilde{H}^*(X; \mathbb{Z}/p)^\otimes i - 1 \otimes DH^*(X; \mathbb{Z}/p) \otimes \tilde{H}^*(X; \mathbb{Z}/p)^\otimes n - i,
\]
where \( Z[n] \) denotes the \( n \)-fold fat wedge of a space \( Z \) given by
\[
Z[n] = \{(z_1, \ldots, z_n) \in Z^n \mid z_j = * \text{ for some } 1 \leq j \leq n \}.
\]

Then we have a splitting as an \( A^p_\alpha \)-module
\[
(2-13) \quad \tilde{H}^*(\Sigma X)^n; \mathbb{Z}/p) \cong U_n \oplus V_n \oplus W_n.
\]

Let \( \tilde{K}_n : \tilde{H}^*(X; \mathbb{Z}/p)^\otimes n \to H^*(P_\alpha(X); \mathbb{Z}/p) \) denote the following composite:
\[
\tilde{H}^*(X; \mathbb{Z}/p)^\otimes n \xrightarrow{\sigma^n} H^*(\Sigma X; \mathbb{Z}/p)^\otimes n \cong H^*((\Sigma X)^n; \mathbb{Z}/p) \xrightarrow{\psi_n} H^*(P_\alpha(X); \mathbb{Z}/p).
\]

Then by [6, Theorem 3.5], there are \( \tilde{a} \in V_n \) and \( \tilde{b} \in W_n \) with \( a = \tilde{K}_n(\tilde{a}) \) and \( b = \tilde{K}_n(\tilde{b}) \).

Now we set \( \lambda^*_n(\alpha_i) = c_i + d_i + e_i \) with respect to the splitting (2–13) for \( 1 \leq i \leq q \), where \( \lambda_n : (\Sigma X)^n \to P_\alpha(X) \) denotes the composite of \( \psi_n \) with the obvious projection \( \omega_n : (\Sigma X)^n \to J_n(\Sigma X) \). From the same reason as Hemmi [6, Lemma 4.8], we have
\[
\sum_{i=1}^{q} \theta_i(d_i) = \sum_{\tau \in \Sigma_n} \tau(\tilde{a}) = \lambda^*_n(a),
\]
and so
\[
\lambda^*_n \left( \sum_{i=1}^{q} \theta_i(\tilde{K}_n(d_i)) \right) = \sum_{\tau \in \Sigma_n} \tau \left( \sum_{i=1}^{q} \theta_i(d_i) \right) = n! \sum_{\tau \in \Sigma_n} \tau(\tilde{a}) = n!(\lambda^*_n(a)),
\]
which implies

\[
(2–14) \quad a = \frac{1}{n!} \sum_{i=1}^{q} \theta_i(K_n(d_i))
\]

by [6, Lemma 4.7]. If we put

\[
\nu_i = \frac{1}{n!}K_n(d_i) \in D^nH^*(P_n(X); \mathbb{Z}/p)
\]

for \(1 \leq i \leq q\), then by (2–14),

\[
\sum_{i=1}^{q} \theta_i(\alpha_i - \nu_i) = b,
\]

which completes the proof. \(\square\)

**Proof of Theorem 2.6** From the construction of the space \(\mathcal{M}(X)\) in Hemmi [8, Section 2], we have a space \(\mathcal{N}(X)\) and the following homotopy commutative diagram:

\[
\begin{array}{ccccccc}
P_{p-2}(X) & \xrightarrow{\xi} & \mathcal{N}(X) & \xrightarrow{\eta} & \mathcal{M}(X) \\
| & | & \downarrow{\zeta} & & \downarrow{\iota} \\
P_n(X) & \xrightarrow{\iota_n} & \cdots & \xrightarrow{\iota_{p-2}} & P_{p-2}(X) & \xrightarrow{\iota_{p-2}} & P_{p-1}(X) & \xrightarrow{\iota_{p-1}} & P_p(X).
\end{array}
\]

By Theorem 2.1 (2) and [8, page 593], we have that \(M \subset R^*(X)\) is closed under the action of \(A_p^*\) with \(\eta^*(M) = 0\), which implies that \(\eta^*|_{R^*(X)}: R^*(X) \to H^*(\mathcal{N}(X); \mathbb{Z}/p)\) induces an \(A_p^*-\)homomorphism \(F: A^*(X) = R^*(X)/M \to H^*(\mathcal{N}(X); \mathbb{Z}/p)\). Then by applying the mod \(p\) cohomology to the diagram (2–15), we have the following commutative diagram of unstable \(A_p^*\)-algebras and \(A_p^*\)-homomorphisms:

\[
\begin{array}{ccccccc}
A^*(X) & \xrightarrow{\mathcal{F}} & H^*(\mathcal{N}(X); \mathbb{Z}/p) & \xleftarrow{\zeta^*} & H^*(P_{p-1}(X); \mathbb{Z}/p) \\
| & \eta_* & & \downarrow{\iota_{p-2}^*} & \\
H^*(P_{p-2}(X); \mathbb{Z}/p) & \xrightarrow{\iota_{p-3}^*} & H^*(P_{p-2}(X); \mathbb{Z}/p) & & \cdots & \downarrow{\iota_{p-2}^*} & \\
& & & & & & H^*(P_n(X); \mathbb{Z}/p).
\end{array}
\]
First we assume $1 \leq n \leq p - 2$. Put $G_n(\alpha_i) = \beta_i$ for $1 \leq i \leq q$, where $G_n: A^*(X) \rightarrow H^*(P_n(X); \mathbb{Z}/p)$ is the composite given by $G_n = \iota_n^* \cdots \iota_{p-3}^* \xi^* \mathcal{F}$. Then by applying $G_n$ to (2–11), we have

$$
\sum_{i=1}^{q} \theta_i(\beta_i) \in DH^*(P_n(X); \mathbb{Z}/p),
$$

and so by Lemma 2.7, there are decomposable classes $\tilde{\nu}_i \in DH^*(P_n(X); \mathbb{Z}/p)$ for $1 \leq i \leq q$ with

$$
(2–16) \quad \sum_{i=1}^{q} \theta_i(\tilde{\alpha}_i - \tilde{\nu}_i) = 0.
$$

If we choose decomposable classes $\nu_i \in DA^*(X)$ to satisfy $G_n(\nu_i) = \tilde{\nu}_i$ for $1 \leq i \leq q$, then by (2–16),

$$
\sum_{i=1}^{q} \theta_i(\alpha_i - \nu_i) \in D^{a+1}A^*(X),
$$

which completes the proof in the case of $1 \leq n \leq p - 2$.

Next let us consider the case of $n = p - 1$. Put $\mathcal{F}(\alpha_i) = \bar{\alpha}_i \in H^*(\mathcal{N}(X); \mathbb{Z}/p)$ for $1 \leq i \leq q$. Then we have

$$
\sum_{i=1}^{q} \theta_i(\bar{\alpha}_i) \in DH^*(\mathcal{N}(X); \mathbb{Z}/p).
$$

By [8, Proposition 5.2], we see that $\mathcal{F}(A^*(X))$ is contained in $\zeta^*(H^*(P_{p-1}(X); \mathbb{Z}/p))$, and so we can choose $\beta_i \in H^*(P_{p-1}(X); \mathbb{Z}/p)$ and $a \in DH^*(P_{p-1}(X); \mathbb{Z}/p)$ with $\zeta^*(\beta_i) = \bar{\alpha}_i$ and

$$
\zeta^*(a) = \sum_{i=1}^{q} \theta_i(\bar{\alpha}_i)
$$

for $1 \leq i \leq q$. Then we can set

$$
\sum_{i=1}^{q} \theta_i(\beta_i) = a + b
$$

with $\zeta^*(b) = 0$, and by [8, Lemma 5.1], we have $b \in S_{p-1}$. By Lemma 2.7, there are decomposable classes $\mu_i \in DH^*(P_{p-1}(X); \mathbb{Z}/p)$ for $1 \leq i \leq q$ with

$$
\sum_{i=1}^{q} \theta_i(\beta_i - \mu_i) = b.
$$
Let \( \nu_i \in DA^*(X) \) with \( F(\nu_i) = \zeta^*(\mu_i) \) for \( 1 \leq i \leq q \). Then we have
\[
\sum_{i=1}^{q} \theta_i(\alpha_i - \nu_i) \in DA^*(X),
\]
which implies the required conclusion. This completes the proof of Theorem 2.6. \( \square \)

### 3 Proofs of Theorem A and Theorem B

In this section, we assume that \( A^* \) is an unstable \( A^*_p \)-algebra which is the truncated polynomial algebra at height \( p + 1 \) given by
\[
A^* = T^{[p+1]}[y_1, \ldots, y_l] \quad \text{with \( \deg y_i = 2m_i \)}
\]
for \( 1 \leq i \leq l \), where \( 1 \leq m_1 \leq \cdots \leq m_l \). Moreover, we choose the generators \( \{y_i\} \) to satisfy
\[
P^1(y_i) \in DA^* \text{ or } P^1(y_i) = y_j \quad \text{for some } 1 \leq j \leq l.
\]
The above is possible by the same argument as Hemmi [5, Section 4].

First we prove the following result:

**Proposition 3.1** Suppose that \( A^* \) is a \( D_n \)-algebra and \( 1 \leq i \leq l \). If \( P^1(y_i) \) contains the term \( y^t_j \) for some \( 1 \leq j \leq l \) and \( 1 \leq t \leq n \), then \( y_j = P^1(y_k) \) for some \( 1 \leq k \leq l \).

**Proof** If \( t = 1 \), then by (3–2), the result is clear. Let \( t \) be the smallest integer with \( 1 < t \leq n \) such that the term \( y^t_j \) is contained in \( P^1(y_{i'}) \) for some \( 1 \leq i' \leq l \). Then by (3–2), we have \( P^1(y_{i'}) \in DA^* \). Since \( A^* \) is a \( D_n \)-algebra, there is a decomposable class \( \nu \in DA^* \) with \( P^1(y_{i''} - \nu) \in D^{n+1}A^* \). This implies that \( P^1(\nu) \) contains the term \( y^t_j \), and so there is one of the generators \( y^t_{i''} \) of (3–1) for \( 1 \leq i'' \leq l \) such that \( P^1(y^t_{i''}) \) contains the term \( y^s_j \) for some \( 1 \leq s < t \). Then we have a contradiction, and so \( t = 1 \). This completes the proof. \( \square \)

In the proof of Theorem A, we need the following result:

**Proposition 3.2** Let \( p \) be an odd prime. If \( A^* \) is a \( D_n \)-algebra with \( n > (p - 1)/2 \), then the indecomposable module \( QA^* \) of \( A^* \) satisfies the following:

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1. If \( a \geq 0 \), \( b > 0 \) and \( 0 < c < p \), then
\[
QA^{2^{p^{(pb+c)}}} = \mathcal{P}^{p^t} QA^{2^{p^{(p(b-t)+c+t)}}}
\]
for \( 1 \leq t \leq \min \{b, p-c\} \) and
\[
\mathcal{P}^{p^t} QA^{2^{p^{(pb+c)}}} = 0 \quad \text{in} \quad QA^{2^{p^{(p(b+t)+c-t)}}}
\]
for \( c \leq t < p \).

2. If \( a \geq 0 \) and \( 0 < c < p \), then
\[
\mathcal{P}^{p^t} : \ QA^{2^{p^{pc}}} \longrightarrow QA^{2^{p^{(p+c-t)}}}
\]
is an isomorphism for \( 1 \leq t < c \).

**Proof** First we consider the case of \( a = 0 \). Let us prove (1) by downward induction on \( b \). If \( b \) is large enough, then the result is clear since \( QA^{2^{p^{(pb+c)}}} = 0 \). Assume that \( y_j \) is one of the generators of (3–1) for \( 1 \leq j \leq l \) and \( \deg y_j = 2(pb + c) \) with \( b > 0 \) and \( 0 < c < p \). By inductive hypothesis, we can assume that if \( f > b \) and \( 0 < g < p \), then
\[
QA^{2^{p^{(pf+g)}}} = \mathcal{P}^{p^t} QA^{2^{p^{(p(f-1)+g+1)}}}
\]
for \( 1 \leq t \leq \min \{f, p-g\} \). If we put
\[
\beta = \frac{1}{pb+c} \mathcal{P}^{p^{pb+c-1} y_j} \in A^{2^{p^{(pb+c-1)+1}}},
\]
then by (3–6), we have
\[
\beta - \mathcal{P}^{p-1}(\gamma) \in DA^*
\]
for some \( \gamma \in QA^{2^{p^{(b-1)+1}}} \). Since \( A^* \) is a \( D_n \)-algebra,
\[
\beta - \frac{1}{pb+c} \mathcal{P}^{p^{pb+c-1}(\mu)} - \mathcal{P}^{p-1}(\gamma - \nu) \in D^{n+1} A^*
\]
for some decomposable classes \( \mu \in DA^{2^{p^{(b+c)}}} \) and \( \nu \in DA^{2^{p^{(b-1)+1}}} \). If we apply \( \mathcal{P}^1 \) to (3–7), then \( y_j^\beta = \mathcal{P}^1(\xi) \) for some \( \xi \in D^{n+1} A^* \) since \( \mathcal{P}^{p^{pb+c}}(\mu) = \mu^p = 0 \) in \( A^* \) and \( \mathcal{P}^{p-1} p^{p-1} = p \mathcal{P}^p = 0 \). Then for some generator \( y_i \), \( \mathcal{P}^1(y_i) \) must contain some \( y'_j \) with \( 1 \leq t \leq p \) and \( t + n = p \). By the assumption of \( n > (p-1)/2 \), we have \( 1 \leq t \leq n \), which implies that \( y_j = \mathcal{P}^1(y_k) \) for some \( 1 \leq k \leq l \) by Proposition 3.1. By iterating this argument, we have (3–3).

Now (3–4) follows from (3–3). In fact, if \( y_j \) is a generator in (3–1) with \( \deg y_j = 2(pb+c) \) for some \( b > 0 \) and \( 0 < c < p \), then we show that \( \mathcal{P}^{c}(y_j) = 0 \). If \( b + c < p \), then by (3–3), we have \( y_j = \mathcal{P}^{b}(\kappa) \) for \( \kappa \in QA^{2^{(b+c)}} \), which implies that
\[
\mathcal{P}^{c}(y_j) = \mathcal{P}^{c} \mathcal{P}^{b}(\kappa) = \left( \begin{array}{c} b+c \\ b \end{array} \right) \kappa^p = 0
\]
in $QA^{2p(b+c)}$. On the other hand, if $p \leq b + c$, then by (3–3), we have $y_j = P^p(y_j)$ for $j \in QA^{2p(b+c-p+1)}$, and so

$$(3-9) \quad P^p(y_j) = P^pP^{p-c}(\zeta) = \left(\begin{array}{c} p \\ c \end{array}\right) P^p(\zeta) = 0.$$  

Next we show (2) with $a = 0$. We only have to show that $P^{c-1}$ is a monomorphism on $QA^{2c}$. Let $y_j$ be a generator in (3–1) such that $\deg y_j = 2c$ with $0 < c < p$. Suppose contrarily that $P^{c-1}(y_j) = 0$ in $QA^{2(c-1)p+1}$. Since $A^*$ is a $D_n$–algebra, we have that

$$P^{c-1}(y_j - \mu) \in D^{p+1}A^{2(c-1)p+1}$$

for some decomposable class $\mu \in DA^{2c}$. Then by a similar argument to the proof of (1), we have that $y_j = P^1(y_k)$ for some $1 \leq k \leq l$ with $\deg y_k = 2(c - p + 1)$, which is impossible for dimensional reasons. This completes the proof of Proposition 3.2 in the case of $a = 0$.

Let $I$ denote the ideal of $A^*$ generated by $y_i$ with $m_i \not\equiv 0 \mod p$. Then for dimensional reasons and by (3–8) and (3–9), we see that $I$ is closed under the action of $\mathcal{A}^*_p$, which implies that $A^*/I$ is an unstable $\mathcal{A}^*_p$–algebra given by

$$A^*/I = T^{[p+1]}[y_{i_1}, \ldots, y_{i_k}] \quad \text{with} \quad m_{i_d} \equiv 0 \mod p$$

for $1 \leq d \leq q$. Set $m_{i_d} = ph_d$ with $h_d \geq 1$ for $1 \leq d \leq q$. Let $B^*$ denote the truncated polynomial algebra at height $p + 1$ given by

$$B^* = T^{[p+1]}[z_1, \ldots, z_q] \quad \text{with} \quad \deg z_d = h_d$$

for $1 \leq d \leq q$. If we define a map $\tilde{\mathcal{L}}: \{y_{i_1}, \ldots, y_{i_k}\} \to B^*$ by $\tilde{\mathcal{L}}(y_{i_d}) = z_d$ for $1 \leq d \leq q$, then $\tilde{\mathcal{L}}$ is extended to an isomorphism $\mathcal{L}: A^*/I \to B^*$. Moreover, $B^*$ admits an unstable $\mathcal{A}^*_p$–algebra structure by the action $P^r(z_d) = \mathcal{L}(P^{pr}(y_{i_d}))$ for $r \geq 1$. Then we can show that $B^*$ is a $D_n$–algebra concerning this structure since so is $A^*$.

From the above arguments, we have the required results for $B^*$ in the case of $a = 0$, which implies that $A^*$ satisfies the required results for $a = 1$. By repeating these arguments, we can show that $A^*$ satisfies the desired conclusions of Proposition 3.2 for any $a \geq 0$. This completes the proof.

Now we prove Theorem A as follows:

**Proof of Theorem A**  By Browder [3, Theorem 8.6], $H^*(X; \mathbb{Z}/p)$, the mod $p$ cohomology, is an exterior algebra in (2–1). Let $\tilde{X}$ be the universal cover of $X$. From the

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By Kane [14, page 24]. By Theorem 2.2 and Theorem 2.6, we have that the composite

where $QH$ has

To show Theorem C, we need the following definition:

By using Theorem A, we prove Theorem B as follows:

**Proof of Theorem B** We proceed by using a similar way to the proof of [6, Theorem 1.1]. Since $X \simeq \widetilde{X} \times T$ for a torus $T$ as in the proof of Theorem A, the Steenrod operations $\mathcal{P}^i$ act trivially on $QH^*(\widetilde{X}; \mathbb{Z}/p)$ for $j \geq 1$. By Theorem A, if $QH^{2m-1}(\widetilde{X}; \mathbb{Z}/p) \neq 0$, then $m = p^a$ for some $a \geq 1$, and so the mod $p$ cohomology of $\widetilde{X}$ is an exterior algebra in (2–1), where $m_i = p^{a_i}$ with $a_i \geq 1$ for $1 \leq i \leq l$.

Let $P_{p-1}(\widetilde{X})$ be the $(p-1)$-th projective space of $\widetilde{X}$. Then by (2–2), there is an ideal $S_{p-1} \subset H^*(P_{p-1}(\widetilde{X}); \mathbb{Z}/p)$ closed under the action of $\mathcal{A}_p^*$ with

$$H^*(P_{p-1}(\widetilde{X}); \mathbb{Z}/p)/S_{p-1} \cong T[p][y_1, \ldots, y_l],$$

where $T[p][y_1, \ldots, y_l]$ is the truncated polynomial algebra at height $p$ generated by $y_i \in H^{2p^{a_i}}(P_{p-1}(\widetilde{X}); \mathbb{Z}/p)$ with $\iota^*_{p-1} \cdot \iota^*_{p-2}(y_i) = \sigma(x_i) \in H^{2p^{a_i}}(\Sigma \widetilde{X}; \mathbb{Z}/p)$. Moreover, we have that the composite

(3–10) \[ H^*(P_{p-1}(\widetilde{X}); \mathbb{Z}/p) \xrightarrow{\iota^*_{p-1}} H^*(P_{p-1}(\widetilde{X}); \mathbb{Z}/p) \xrightarrow{\iota^*_{p-2}} T[p][y_1, \ldots, y_l] \]

is an isomorphism for $t < 2p^{a_1+1}$ and an epimorphism for $t < 2(p^{a_1+1} + p^{a_1} - 1)$ by [6, page 106, (4.11)]. As in [6, page 106, (4.11)], we can show

(3–11) \[ \text{Im } \mathcal{P}^t \cap H^*(P_{p}(\widetilde{X}); \mathbb{Z}/p) = 0 \]

for $t \leq 2p^{a_1+1}$. In fact, by (3–10) and for dimensional reasons, we have

$$\text{Im } \beta \cap H^*(P_{p}(\widetilde{X}); \mathbb{Z}/p) = 0$$

$$\text{Im } \mathcal{P}^t \cap H^*(P_{p}(\widetilde{X}); \mathbb{Z}/p) = 0$$

for $t \leq 2p^{a_1+1}$, which implies (3–11) by Liulevicius [22] or Shimada–Yamanoshiba [24].

Now we can choose $w_1 \in H^{2p^{a_1}}(P_{p}(\widetilde{X}); \mathbb{Z}/p)$ with $\iota^*_{p-1}(w_1) = y_1$ by (3–10), and we have $w_1^p = \mathcal{P}^{t_{p-1}}(w_1) = 0$ by (3–11). Then we have a contradiction by using the same argument as the proof of [6, Theorem 1.1], and so $\widetilde{X}$ is contractible, which implies that $X$ is a torus. This completes the proof of Theorem B. \qed

To show Theorem C, we need the following definition:
Definition 3.3 Assume that $X$ is an $A_n$–space and $Y$ is a space.

(1) An $AC_n$–form on a map $\phi: Y \to X$ is a family of maps $\{R_i: \Gamma_i \times Y^i \to X\}_{1 \leq i \leq n}$ with the conditions $R_i(\ast, y) = \phi(y)$ for $y \in Y$ and (2–6)–(2–7).

(2) A quasi $C_n$–form on a map $\kappa: \Sigma Y \to \Sigma X$ is a family of maps $\{\zeta_i: J_i(\Sigma Y) \to P_i(\Sigma X)\}_{1 \leq i \leq n}$ with the conditions $\zeta_1 = \kappa$ and (2–9).

By using the same argument as the proof of Theorem 2.2 (1), we can prove the following result:

Theorem 3.4 Assume that $X$ is an $A_n$–space, $Y$ is a space and $\phi: Y \to X$ is a map. Then any $AC_n$–form on $\phi$ induces a quasi $C_n$–form on $\Sigma \phi$.

Now we prove Theorem C as follows:

Proof of Theorem C First we show that if $X$ admits an $AC_n$–form, then $nm_l \leq p$.

We prove by induction on $n$. If $n = 1$, then the result is proved by Hubbuck–Mimura [11] and Iwase [13, Proposition 0.7]. Assume that the result is true for $n - 1$. Then by inductive hypothesis, we have $(n - 1)m_l \leq p$. Now we assume that $X$ admits an $AC_n$–form with

\[(3–12) \quad (n - 1)m_l \leq p < nm_l.\]

Then we show a contradiction.

Let $\tilde{X}$ be the universal covering space of $X$. Then $\tilde{X}$ is a simply connected $A_p$–space mod $p$ homotopy equivalent to

\[(3–13) \quad S^{2m_{l-1}} \times \cdots \times S^{2m_{l-1}} \quad \text{with} \quad 1 < m_1 \leq \cdots \leq m_l\]

and the multiplication of $\tilde{X}$ admits an $AC_n$–form by [10, Lemma 3.9]. Now we can set that

$$A^*(\tilde{X}) = T^{[p+1]}[y_1, \ldots, y_l] \quad \text{with} \deg y_i = 2m_i$$

for $1 \leq i \leq l$, where $1 < m_1 \leq \cdots \leq m_l \leq p$. By Theorem 2.2 and Theorem 2.6, $A^*(\tilde{X})$ is a $D_n$–algebra.

First we consider the case of $m_l < p$. Let $J$ be the ideal of $A^*(\tilde{X})$ generated by $y_i$ for $1 \leq i \leq l - 1$. Then we see that

\[(3–14) \quad \mathcal{P}^1(y_i) \notin J \quad \text{for some} \quad 1 \leq i \leq l.\]
In fact, if we assume that $P^1(y_i) \in J$ for any $1 \leq i \leq l$, then $P^1(y_i) \in J$ and $P^1(J) \subset J$. This implies that
\[ y_i^p = P^m(y_i) = \frac{1}{m!}(P^1)^m(y_i) \in J, \]
which is a contradiction, and so we have (3–14). Then for dimensional reasons and by (3–12),
\[ 2(n - 1)m_l < \deg P^1(y_i) < 2(n + 1)m_l, \]
which implies that $P^1(y_i)$ contains the term $ay_i^p$ with $a \neq 0$ in $\mathbb{Z}/p$ by (3–14). By Proposition 3.1, we have $y_i \in P^1QA^{2(m_l - p + 1)}(X)$, which causes a contradiction since $m_l < p$.

Next let us consider the case of $m_l = p$. In this case, (3–12) is equivalent to $n = 2$, and so $X$ is assumed to have an $AC_2$–form. Then from the same arguments as above, we have that $A^*(\tilde{X})$ is a $D_2$–algebra. By Kanemoto [15, Lemma 3], there is a generator $y_k \in QA^{2(p - 1)}(\tilde{X})$ for some $1 \leq k < l$. Let $K$ be the ideal of $A^*(\tilde{X})$ generated by $y_i$ with $i \neq k$. From the same reason as (3–14), we see that $P^1(y_i) \notin K$ for some $1 \leq i \leq l$. Then for dimensional reasons, we see that $P^1(y_i)$ contains the term $by_k^2$ with $b \neq 0$ in $\mathbb{Z}/p$. By Proposition 3.1, we have a contradiction, and so $\tilde{X}$ does not admit an $AC_2$–form.

Next we show that if $nm_l \leq p$, then $X$ admits an $AC_n$–form. Since it is clear for $n = 1$ or $m_l = 1$, we can assume that $nm_l < p$. Let $Y$ denote the wedge sum of spheres given by
\[ Y = (S^{2m_l - 1} \lor \ldots \lor S^{2m_l - 1})_p \]
with the inclusion $\phi : Y \to X$. First we construct an $AC_n$–form $\{R_i : \Gamma_i \times Y^i \to X\}_{1 \leq i \leq n}$ on $\phi : Y \to X$.

Suppose inductively that $\{R_i\}_{1 \leq i \leq t}$ are constructed for some $t \leq n$. Then the obstructions for the existence of $R_t$ belong to the following cohomology groups for $j \geq 1$:
\[ H^{j+1}(\Gamma_t \times Y^t, \partial \Gamma_t \times Y^t \cup \Gamma_t \times Y^t; \pi_j(X)) \cong \tilde{H}^{j+2}((\Sigma Y)^{(i)}; \pi_j(X)) \]
since $\Gamma_t \times Y^t/(\partial \Gamma_t \times Y^t \cup \Gamma_t \times Y^t) \cong \Sigma^{-1}Y^{(i)}$. This implies that (3–15) is non-trivial only if $j$ is an even integer with $j < 2p - 2$ since
\[ \Sigma Y \cong (S^{2m_1} \lor \ldots \lor S^{2m_l})_p \]
and $tm_l \leq nm_l < p$. On the other hand, according to Toda [27, Theorem 13.4], $\pi_j(X) = 0$ for any even integer $j$ with $j < 2p - 2$ since $X$ is given by (3–13). Thus

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(3–15) is trivial for all $j$, and we have a map $R_j$. This completes the induction, and we have an $AC_n$–form $\{R_i\}_{1 \leq i \leq n}$ on $\phi: Y \to X$.

Since $X$ is an $H$–space, there is a map $\beta: \Omega \Sigma X \to X$ with $\beta \alpha \simeq 1_X$, where $\alpha: X \to \Omega \Sigma X$ denotes the adjoint of $1_{\Sigma X}: \Sigma X \to \Sigma X$. Moreover, $\Sigma Y$ is a retract of $\Sigma X$, and so we have a map $\nu: \Sigma X \to \Sigma Y$ with $\nu(\Sigma \phi) \simeq 1_{\Sigma Y}$. Put $\lambda = \beta \theta: X \to X$, where $\theta: X \to \Omega \Sigma X$ denotes the adjoint of $(\Sigma \phi)\nu$. Then we see that $\lambda$ induces an isomorphism on the mod $p$ cohomology, and so $\lambda$ is a mod $p$ homotopy equivalence.

By Theorem 3.4, there is a quasi $C_n$–form $\{\zeta_i: J_i(\Sigma Y) \to P_i(X)\}_{1 \leq i \leq n}$ on $\Sigma \phi: \Sigma Y \to \Sigma X$. Let $\xi_i: J_i(\Sigma X) \to P_i(X)$ be the map defined by $\xi_i = \zeta_i J_i(\nu(\Sigma \lambda^{-1}))$ for $1 \leq i \leq n$, where $\lambda^{-1}: X \to X$ denotes the homotopy inverse of $\lambda$. Then the family $\{\xi_i\}_{1 \leq i \leq n}$ satisfies $\xi_i|_{J_{i-1}(\Sigma X)} = \iota_{i-1} \xi_{i-1}$ for $2 \leq i \leq n$ and $\xi_1 = (\Sigma \phi)\nu(\Sigma \lambda^{-1}) = \chi(\Sigma \theta)(\Sigma \lambda^{-1})$, where $\chi: \Sigma \Omega \Sigma X \to \Sigma X$ is the evaluation map. Since $\iota_1(\Sigma \beta) \simeq \iota_1 \chi: \Sigma \Omega \Sigma X \to P_2(X)$ by Hemmi [9, Lemma 2.1], we have $\xi_2|_{\Sigma X} = \iota_1 \xi_1 \simeq \iota_1$. Let $\psi_i: J_i(\Sigma X) \to P_i(X)$ be the map defined by $\psi_1 = 1_{\Sigma X}$ and $\psi_i = \xi_i$ for $2 \leq i \leq n$. Then the family $\{\psi_i\}_{1 \leq i \leq n}$ satisfies (2–8)–(2–9). By Theorem 2.2 (2) and Remark 2.3, we have an $AC_n$–form $\{Q_i: \Gamma_i \times X^i \to X\}_{1 \leq i \leq n}$ on $X$ with (2–5)–(2–7). This completes the proof of Theorem C. \hfill \Box

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*Department of Mathematics, Faculty of Science, Kochi University
Kochi 780-8520, Japan*

*Department of Mathematics, National Defence Academy
Yokosuka 239-8686, Japan*

hemmi@math.kochi-u.ac.jp, yusuke@nda.ac.jp

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