ARQ-based Average Consensus over Directed Network Topologies with Unreliable Communication Links

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Abstract—In this paper, we address the discrete-time average consensus problem in strongly connected directed graphs, where nodes exchange information over unreliable error-prone communication links. We enhance the Robustified Ratio Consensus algorithm by exploiting features of the (Hybrid) Automatic Repeat ReQuest - (H)ARQ protocol used for error control of data transmissions, in order to allow the nodes to reach asymptotic average consensus even when information is exchanged over error-prone directional networks. This strategy, apart from handling time-varying information delays induced by retransmissions of erroneous packets, can also handle packet drops that occur when exceeding a predefined packet retransmission limit. Invoking the (H)ARQ protocol allows nodes to: (a) exploit the incoming error-free acknowledgement feedback to initially acquire or later update their out-degree, (b) know whether a packet has arrived or not, and (c) determine a local upper-bound on the delays imposed by the retransmission limit. By augmenting the network's corresponding weight matrix, we show that nodes utilizing our proposed (H)ARQ Ratio Consensus algorithm can reach asymptotic average consensus over unreliable networks, while improving their convergence speed and maintaining low values in their local buffers compared to the current state-of-the-art.

Index Terms—Average consensus, directed graphs, ratio consensus, ARQ/HARQ feedback, packet errors, packet drops, information delays.

I. Introduction

Robust information exchange between computing nodes (usually referred to as agents or workers) is essential to apply various estimation, control, and optimization algorithms in spatially distributed multi-agent systems [3]–[7]. Such distributed systems consist of multiple agents, each interacting with its immediate neighbors and possessing only local information to achieve network-wide objectives, without the need for a central coordinator. Several distributed estimation and optimization methods [7]–[10] typically involve nodes cooperating to reach consensus on a common value by iteratively exchanging their own values (e.g., measurements, state estimates, decision variables, etc.) and appropriately updating them in order to optimize a network-wide performance index. When the agreed consensus value is the average of the initial values of all nodes in the network we say that nodes reach average consensus. In essence, each node holds an initial value (often called mass) and exchanges it with its neighboring nodes in a distributed and iterative fashion, such that it converges to the average consensus value. For a detailed overview of consensus methods, refer to [11].

In practice, the exchange of information in large-scale networks is restricted to be directional (instead of bidirectional) as a consequence of diverse transmission power, communication ranges [12], and interference levels at each individual agent in the network. In this paradigm, information flow is directional, meaning that while an agent \( v_j \) may receive information from agent \( v_i \), it does not necessarily imply that \( v_j \) can also send information back to agent \( v_i \). Consequently, the network topology is best represented by a directed graph (digraph), where agents are depicted as nodes and the communication links are shown as edges. Under reliable directed network topologies, the results in [13]–[16] have shown how agents can asymptotically reach the average, albeit the underlying network can be directed and possibly unbalanced. Among these works, the Ratio Consensus algorithm proposed in [13] has been demonstrated to be superior in terms of both computation and communication for achieving average consensus [17]. This algorithm achieves average consensus in strongly connected digraphs with reliable communication links by iteratively computing the ratio of two concurrently running linear iterations with properly chosen initial conditions, assuming each node \( v_j \) knows its out-degree (i.e., the number of nodes receiving information from \( v_j \)).

In congested networks of limited-bandwidth communication links, several communication impediments are induced, including information delays and packet drops. These communication impediments negatively affect the performance and reliability of distributed average consensus algorithms, while they can often obstruct them from computing the exact average consensus value. In the presence of computational and/or transmission delays on the information exchange between agents, the authors in [18], [19] proposed the Robustified Ratio Consensus algorithm, able to reach (asymptotic in the former, finite-time in the latter) average consensus in directed communication networks in the presence of bounded time-varying delays. In the presence of packet drops, the authors in [20] proposed an algorithm, hereinafter referred to as Ratio Consensus via Running Sums (RC-RS),...
with which nodes exchange messages containing information of their running sums to handle potential loss of packets over directed networks. To analyze the convergence of RC-RS, they introduced virtual nodes and links that model packet dropping communication scenarios, and using results from the ergodicity of matrix products [21] they proved that nodes asymptotically converge to the exact average consensus value. While RC-RS provides a robust solution for average consensus in unreliable networks, it requires nodes to maintain and exchange running sum variables, leading to increased computational and communication costs.

To address the limitations of previous works, we introduce a novel method that employs acknowledgment feedback and retransmissions via the (Hybrid) Automatic Repeat reQuest – (H)ARQ protocol. This approach ensures fast, yet reliable convergence to the average consensus value in directed graphs, without the need for exchanging variables that grow over time. In this work we make the following key contributions:

- By incorporating (H)ARQ protocols into the ratio consensus algorithm, our approach allows nodes to (a) exploit incoming error-free acknowledgement feedback signals to initially acquire or later update their out-degree, (b) know whether a packet has arrived or not, and (c) determine a local upper-bound on the information delays, imposed by the ARQ retransmission limit.
- Furthermore, our proposed (H)ARQ-RC algorithm not only ensures convergence to the average consensus value in the presence of packet errors, but as supported by our numerical evaluation, it also accelerates convergence when the packet retransmission limit is set appropriately and matches the current channel conditions.
- Finally, we provide the proof of asymptotic convergence of the proposed method to the exact average and we evaluate its performance via simulations for different realistic scenarios. The proof is based on augmenting the weighted adjacency matrix that represents the network topology and the interactions between nodes, to model possible arbitrary and unknown time-varying, yet bounded delays, as well as packet drops on the communication links.

To the best of our knowledge, this is the first time that a realistic communication error correction protocol currently used in many real-life telecommunication systems, is integrated with distributed consensus iterations to compute the exact average consensus value over unreliable networks.

The remainder of this paper is organized as follows. Section II describes the notation used in this paper, introduces the network communication model and summarizes the main ideas behind ratio consensus-based algorithms and (H)ARQ error correction protocols. Section III formulates the probability of packet errors under the (H)ARQ protocol, and describes the proposed (H)ARQ Ratio Consensus algorithm for handling unreliable communication in the presence of packet errors. In Section IV we present the augmented digraph representation needed for the proof of convergence of the proposed method, while in Section V we provide numerical simulations under different channel condition scenarios. Finally, in Section VI we provide concluding remarks and future directions.

II. Background

A. Notation

We denote the set of real (integer) numbers by \( \mathbb{R} (\mathbb{Z}) \) and the set of nonnegative real (integer) numbers by \( \mathbb{R}_+ (\mathbb{N}) \). The set of natural numbers from \( a \in \mathbb{N} \) up to \( b \in \mathbb{N} \) is denoted by \( \mathbb{N}_a^b = \{a, a + 1, \ldots, b - 1, b\} \), e.g., \( \mathbb{N}_1^4 = \{1, 2, \ldots, 4\} \). Vectors are denoted by lowercase letters whereas matrices are denoted by capital letters. The all-ones \( n \)-dimensional column vector is denoted by \( \mathbf{1}_n \). The identity matrix and the zero matrix (of appropriate dimensions) are denoted by \( I \) and \( 0 \), respectively. A nonnegative matrix (with nonnegative elements) is denoted by \( A \geq 0 \), while a positive matrix (with positive elements), is denoted by \( A > 0 \).

B. Graph Theory

Consider a strongly connected\(^1\) directed network captured by a graph \( G = (V, E) \), where \( V = \{v_1, \ldots, v_n\} \) is the set of nodes (representing the \( n \) agents) and \( E \subseteq V \times V \) is the set of edges (representing the communication links between agents). The total number of edges in the network is denoted by \( m = |E| \). A directed edge \( e_{ji} \triangleq (v_j, v_i) \in E \), where \( v_i, v_j \in V \), indicates that node \( v_j \) can receive information from node \( v_i \), i.e., \( v_i \rightarrow v_j \). The nodes that transmit information to node \( v_j \) directly are called in-neighbors of node \( v_j \), and belong to the set \( \mathcal{N}_j^- = \{v_i \in V | e_{ji} \in E\} \). The number of nodes in the in-neighborhood is called in-degree and it is represented by the cardinality of the set of in-neighbors, \( d_j^- = |\mathcal{N}_j^-| \). The nodes that receive information from node \( v_j \) directly are called out-neighbors of node \( v_j \), and belong to the set \( \mathcal{N}_j^+ = \{v_i \in V | e_{ij} \in E\} \). The number of nodes in the out-neighborhood is called out-degree and it is represented by the cardinality of the set of out-neighbors, \( d_j^+ = |\mathcal{N}_j^+| \). Note that when self-loops are included, the number of in-going links of node \( v_j \) is \( (d_j^- + 1) \) and similarly the number of its out-going links is \( (d_j^+ + 1) \).

C. Problem Setup

The problem of (distributed) average consensus involves a number of nodes in a network, that cooperate by exchanging information to compute the network-wide average of their initial values, for a certain quantity of interest. At each time step \( k \geq 0 \), each node \( v_j \in V \) maintains a state \( x_j[k] \) and updates it at each iteration, based on the values received from its in-neighbors and its own value. In what follows, for simplicity of exposition, we assume that the state of each node \( x_j[k] \) is a scalar, but in more general settings, it could be a vector. The goal of the nodes is to collaboratively compute

\(^1\) A directed graph is strongly connected if each node can be reached by every other node via a sequence of edges, respecting their orientation.
the average of their initial values, hereinafter referred to as average consensus value and denoted by
\[ \bar{x} \triangleq \frac{1}{n} \sum_{j=1}^{n} x_j[0]. \] (1)

In the absence of global knowledge, nodes are required to execute an iterative distributed algorithm to eventually converge to the average consensus value, by updating their states using information received from their in-neighbors.

D. Ratio Consensus over Reliable Directed Graphs

In the Ratio Consensus algorithm [13], each node \( v_j \) maintains a state variable \( x_j[k] \in \mathbb{R} \) and an auxiliary variable \( y_j[k] \in \mathbb{R}_+ \) at each time step \( k \). These variables are initialized at \( x_j[0] = V_j \), with \( V_j \) being an arbitrary initial value or measurement of node \( v_j \), and \( y_j[0] = 1 \), respectively. The iterative scheme of the ratio consensus algorithm, involves each node \( v_j \in \mathcal{V} \) receiving a pre-weighted piece of information from its in-neighbors and updating its states which are then used to compute the ratio \( z_j[k] \equiv x_j[k]/y_j[k] \) according to:
\[
\begin{align*}
  x_j[k+1] &= p_{jj} x_j[k] + \sum_{v_i \in \mathcal{N}^-_j} p_{ij} x_i[k], \quad (2a) \\
  y_j[k+1] &= p_{jj} y_j[k] + \sum_{v_i \in \mathcal{N}^-_j} p_{ij} y_i[k], \quad (2b) \\
  z_j[k+1] &= x_j[k+1]/y_j[k+1], \quad (2c)
\end{align*}
\]
where the collection of weights \( P = \{p_{ij}\} \in \mathbb{R}^{n \times n}_+ \) on links \( \varepsilon_{ji} \in \mathcal{E} \), forms a column-stochastic matrix. Often, each node \( v_j \) assigns the weight \( p_{ij} \) as:
\[
p_{ij} = \begin{cases} 
  1/(1 + \varepsilon^+_j), & v_i \in \mathcal{N}^+_j \cup \{v_j\}, \\
  0, & \text{otherwise},
\end{cases} \quad (3)
\]
which requires each node \( v_j \in \mathcal{V} \) to know its out-degree. Possible zero-valued entries in the nonnegative weighted column-stochastic adjacency matrix \( P \), denote the absence of communication links (edges) between the corresponding nodes in the digraph.

The auxiliary scalar variable \( y[k] \) is used to asymptotically compute the right Perron eigenvector of the primitive column stochastic matrix \( P \), i.e., \( \pi > 0 \) (assumed here, without loss of generality, to be normalized so that its entries sum to unity), which is not a scaled version of \( 1_n \) since \( P \) is not row-stochastic [21]. Forcing the auxiliary variable \( y_j[k] \) to be initialized at value 1 for each node \( v_j \in \mathcal{V} \), we can verify that \( \lim_{k \to \infty} y_j[k] = 1_n^\top y[0] \pi_j = n \pi_j \). Similarly, \( \lim_{k \to \infty} x_j[k] = 1_n^\top x[0] \pi_j \). Hence, the limit of the ratio \( x_j[k] \) over \( y_j[k] \), is the average of the initial values and is given by [13], [22]:
\[
\lim_{k \to \infty} z_j[k] = 1_n^\top x[0] \pi_j/n \pi_j = \frac{1}{n} \sum_{i=1}^{n} x_i[0] = \bar{x}, \quad \forall v_j \in \mathcal{V}. \quad (4)
\]

**Remark 1.** The Ratio Consensus algorithm assumes that communication links within the network are perfectly reliable, and that each node \( v_j \) is aware of its out-degree.

E. Robustified Ratio Consensus over Delayed Directed Graphs

In practice, packet transmissions are often delayed due to poor channel conditions and network congestion. To address this, the authors in [18] proposed a protocol that handles arbitrary time-varying delays, ensuring asymptotic consensus to the exact average despite the delays. Let node \( v_j \) at time step \( k \) undergo an a priori unknown delay, denoted by a bounded positive integer \( \tau_{ji}[k] \leq \tau_{ji} < \infty \). The maximum delay in the network is denoted by \( \tau \equiv \max \{\tau_{ji}\} \). Moreover, the own value of node \( v_j \) is always instantly available without delay, i.e., \( \tau_{jj}[k] = 0, \forall k \). Based on this notation, the strategy proposed in [18] involves each node updating its states for each iteration according to
\[
\begin{align*}
  x_j[k+1] &= p_{jj} x_j[k] + \sum_{v_i \in \mathcal{N}^-_j} \sum_{r=0}^{\tau} p_{ij} x_i[k-r] \varepsilon_{ji}[k-r], \\
  y_j[k+1] &= p_{jj} y_j[k] + \sum_{v_i \in \mathcal{N}^-_j} \sum_{r=0}^{\tau} p_{ij} y_i[k-r] \varepsilon_{ji}[k-r], \\
  z_j[k+1] &= x_j[k+1]/y_j[k+1],
\end{align*}
\]
where \( \varepsilon_{ji} \) and \( \varepsilon_{ji}[k-r] \) are indicator functions that take value 1 if \( \tau_{ji}[k-r] = r \) and 0 otherwise.

Here, it is important to note that a transmission on link \( \varepsilon_{ji} \) undergoes an a priori unknown delay, for which node \( v_j \) is unaware of, but instead, it processes (delayed) packets as soon as they arrive successfully. Clearly, in the absence of delays, this strategy reduces to the Ratio Consensus algorithm in [13] as described in (2).

**Remark 2.** The Robustified Ratio Consensus algorithm handles (possibly) delayed information exchange between nodes, neglecting any packet drops due to erroneous packets, while it assumes that each node \( v_j \) is aware of its out-degree.

F. Ratio Consensus via Running Sums over Packet-Dropping Directed Graphs

To handle the loss of information packets, each node \( v_j \in \mathcal{V} \) executing the RC-RS algorithm [20], maintains the variables \( \sigma^x_j[k] \) and \( \sigma^y_j[k] \) for the \( x \) and \( y \) mass, respectively. These variables correspond to the running sum values that it transmits to its out-neighbors \( v_i \in \mathcal{N}^+_j \) at each time step \( k \). Moreover, each node \( v_j \) keeps track of the running sum values that (eventually) arrive from its in-neighbors by updating the variables \( \chi_{ji}[k] \) and \( \psi_{ji}[k] \). Each node \( v_j \) initializes the aforementioned variables at 0, i.e., \( \sigma^x_j[0] = 0 \), \( \sigma^y_j[0] = 0 \), \( \chi_{ji}[0] = 0 \), and \( \psi_{ji}[0] = 0 \) for all \( v_i \in \mathcal{N}^-_j \). Essentially, at each time step \( k \), node \( v_j \in \mathcal{V} \) transmits its running sum values \( \sigma^x_j[k] + p_{jj} x_j[k] \) and \( \sigma^y_j[k] + p_{jj} y_j[k] \) to its out-neighbors \( v_i \in \mathcal{N}^+_j \). Meanwhile, it receives \( \sigma^x_j[k] + \)}
\(p_{ji}x_i[k]\) and \(\sigma_j^+[k] + p_{ji}y_i[k]\) from all its in-neighbors \(v_i \in \mathcal{N}_j^{-}\) and updates the received masses as follows:

\[
\begin{align*}
\chi_{ji}[k+1] &= \theta_{ji}[k](\sigma_j^+[k] + p_{ji}x_i[k]) + (1 - \theta_{ji}[k])\chi_{ji}[k], \\
\psi_{ji}[k+1] &= \theta_{ji}[k](\sigma_j^+[k] + p_{ji}y_i[k]) + (1 - \theta_{ji}[k])\psi_{ji}[k],
\end{align*}
\]

where \(\theta_{ji}[k] = 1\) if the masses \(\sigma_j^+[k] + p_{ji}x_i[k]\) and \(\sigma_j^+[k] + p_{ji}y_i[k]\), sent by node \(v_i\) at time \(k\), have been successfully received by node \(v_j\); while \(\theta_{ji}[k] = 0\) otherwise. According to the updated mass variables \(\chi_{ji}[k+1]\) and \(\psi_{ji}[k+1]\), node \(v_j\) updates its current state and auxiliary variables as

\[
\begin{align*}
x_j[k+1] &= p_{jj}x_j[k] + \sum_{v_i \in \mathcal{N}_j^{-}} (\chi_{ji}[k+1] - \chi_{ji}[k]), \\
y_j[k+1] &= p_{jj}y_j[k] + \sum_{v_i \in \mathcal{N}_j^{-}} (\psi_{ji}[k+1] - \psi_{ji}[k]), \\
z_j[k+1] &= x_j[k+1]/y_j[k+1].
\end{align*}
\] (7)

**Remark 3.** The RC-RS algorithm handles packets that are dropped during the information exchange between nodes, by having each node transmit its running sum values. Each node is assumed to be aware of its out-degree, while the values of running sums grow in an unbounded manner.

**G. Error Correction Protocols**

In modern message transmission systems such as the Transmission Control Protocol (TCP), the High-Level Data Link protocol, and others, nodes utilize Automatic Repeat reQuest (ARQ) protocols to maintain reliable packet transmissions over unreliable communication channels [23, 24, §6], [25, §5]. ARQ-based techniques have been recently applied in estimation [26] to enhance the reliability and efficiency of data transmission in networked control systems, ensuring accurate state estimation even in the presence of communication errors. Although in these applications maintaining information integrity is crucial for system convergence and performance, one may also face the fundamental trade-off between the reliability and freshness of the information [27], [28].

The ARQ protocol employs error-detection codes, acknowledgment (ACK) or negative acknowledgment (NACK) messages, and retransmissions to ensure reliable data transmissions over error-prone channels. An acknowledgment is a feedback signal, often a single bit, sent by the receiver to notify the transmitter of successful or unsuccessful packet reception. In ARQ, a packet is retransmitted after each NACK until it is successfully received, otherwise it is discarded. If the receiver fails to receive the packet after a predefined number of retransmission trials, the packet is dropped. Each retransmission of a packet can be seen as an information delay, since the receiver will get the intended packet successfully as soon as the transmitted packet arrives without errors. ARQ can be enhanced by combining it with forward error correction (FEC) codes, forming the Hybrid ARQ (HARQ) protocol. In HARQ, transmitters encode messages redundantly using FEC codes, allowing receivers to correct some errors in a packet. Standard ARQ is then used to correct errors not corrected by FEC. This combination reduces the probability of packet errors in subsequent time slots, as some errors can be corrected using FEC, minimizing the need for extra retransmissions.

Here it is important to note that, ACK/NACK is sent over a narrowband error-free feedback channel with negligible probability of error in the reception of this small packet, which usually comprises one bit. However, this feedback channel cannot support regular data transmission due to the high data rate required and, hence, larger bandwidth and higher-order modulation. For this reason, the network topology over which the nodes exchange information data packets could still be assumed directed.

A graphical representation of the (H)ARQ error control feedback mechanism is shown in Fig. 1. The flowchart blocks and arrows in red color denote the combined error correction techniques used in the HARQ, on top of the classical ARQ mechanism.

**III. Average Consensus via (H)ARQ Feedback Mechanisms**

In this section, we present a novel distributed algorithm, referred to as the *(Hybrid) Automatic Repeat reQuest Ratio Consensus* – (H)ARQ-RC algorithm, designed to achieve asymptotic convergence to the exact average consensus value in (1), in the presence of unreliable communication that is prone to packet errors. Each node communicates with its out-neighbors and in-neighbors by transmitting and receiving data packets, respectively, over a strongly connected directed graph \(G = (V, E)\), employing the (H)ARQ protocol with a predetermined retransmission limit. Within this framework, at each consensus iteration, each node transmits its current state, while it may also retransmit some
of its previous states, depending on the packets that arrived in error during previous transmissions. More specifically, each node assigns weights to its out-going links based on its out-degree as in formula (3), and (re)transmits its states via broadcast, multicast, or unicast, depending on the acknowledgement feedback of the packets sent in the previous iteration. The out-degree of each node is initially acquired by broadcasting dummy packets and aggregating the resulting incoming (H)ARQ feedback signals ($\text{ACK}/\text{NACK}$) from its out-neighbors. This count, establishes that each node will acquire its out-degree for assigning the weights and multicasting its packets in the subsequent consensus iterations, even in the presence of packet errors.

### A. Modelling Packet Errors

In this work, we assume that each node $v_j \in V$ employs a (H)ARQ protocol with a retransmission limit on its out-going links $\varepsilon_{ij} \in \mathcal{E}$, denoted by $0 \leq \delta_{ij} < \infty$. This retransmission limit determines the maximum number of retransmissions selected before considering a packet as dropped. Let $f_{ij,k-r}[k]$ denote the acknowledgment feedback signal, i.e., $\text{ACK}/\text{NACK}$, determined by the receiving node $v_l$ during time slot $k$, for a packet initially transmitted by node $v_j$ at time step $k-r$ to be defined as:

$$f_{ij,k-r}[k] = \begin{cases} 0, & \text{if NACK,} \\ 1, & \text{otherwise,} \end{cases} \quad (8)$$

for $r \in \mathbb{N}_0^{\delta_{ij}}$. Each feedback signal $f_{ij,k-r}[k]$ is sent back to the transmitting node $v_j$ during time slot $k$, and becomes available by the beginning of the next time slot. Hence, node $v_j$ is able to decide whether to retransmit a packet or not, at time step $k+1$. Note that, at each time step $k+1$, node $v_j$ transmits exactly $\sum_{r=0}^{\delta_{ij}-1} (1 - f_{ij,k-r}[k])$ packets over the link $\varepsilon_{ij} \in \mathcal{E}$. As mentioned in §II-G, the number of consecutive NACK feedback signals of a packet until its successful reception by the intended recipient $v_l$, translates into the delay experienced by node $v_j$ to obtain the information of that particular packet. Hence, the experienced delay of a packet initially transmitted at iteration $k$ over the link $\varepsilon_{ij}$ is given by the variable:

$$\tau_{ij}[k] \triangleq \min\{r : r \in \mathbb{N}_0^{\delta_{ij}}, f_{ij,k}[k+r] = 1\}. \quad (9)$$

A fresh packet that is initially transmitted over the link $\varepsilon_{ij}$ at iteration $k$, is considered as dropped by node $v_j$ at iteration $k+\tau_{ij}+1$, if the retransmission limit is exceeded. In what follows we denote the packet drop indicator variable for that packet by

$$\delta_{ij}[k] = \begin{cases} 1, & \text{if } f_{ij,k}[k+\tau_{ij}+1] = 0, \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

As it will be clarified later, nodes are not required to know $\tau_{ij}[k]$ and $\delta_{ij}[k]$ a priori to run the (H)ARQ-RC algorithm.

\footnote{Note that this approach works even if one considers that the retransmission limit is imposed either on the packets or on the links.}

Instead, they process the information packets as soon as they arrive, while for the packets they send, they keep retransmission counters to determine whether they should retransmit a packet or consider it as dropped.

Hereafter, for simplicity of exposition and without loss of generality, we assume that $\tau_{ij} = \bar{\tau}$ for all $\varepsilon_{ij} \in \mathcal{E}$. This is a natural assumption when all links (or even packets) have the same retransmission limit $\bar{\tau}$. Hence, the information delays in the network are bounded by the global retransmission limit, i.e., $0 \leq \tau_{ij}[k] \leq \bar{\tau} \leq \bar{\tau}$ for all $k \geq 0$.

In what follows, we characterize the packet error probabilities for the standard ARQ and the HARQ protocol. Due to the absence of FEC in the standard ARQ protocol, receiving nodes can only decode and correct the received packet by requesting retransmissions and expecting that the retransmitted packet will be successfully decoded without errors. Hence, the standard ARQ protocol itself, cannot influence packet error probability, and at each iteration $k$, the packet error probability is independent of the packet retransmission number. For HARQ protocols, however, several works (see, e.g., [29]–[31]) have shown that the probability of packet error reduces exponentially with the number of retransmissions. A model on the reduction of probability of error for the $r$-th retransmission trial of a packet that was initially transmitted at time step $k-r$, is given by [31]:

$$\mathbb{P}(f_{ij,k-r}[k] = 0) = q_j \lambda^r, \quad (11)$$

for all $k \geq 0$ and $r \in \mathbb{N}_0^{\bar{\tau}}$. Here, $q_j$ denotes the initial packet error probability for a fresh packet that is transmitted for the first time. Parameter $\lambda \in (0,1]$ is the rate of packet error probability reduction for the subsequent retransmissions. In practice, it models the capability of the HARQ protocol to correctly decode the packet using FEC, and without the need for extra retransmission. This implies that, for small $\lambda$, a packet will successfully arrive at its destined node with less retransmissions, while for larger $\lambda$, more retransmissions will be required. In essence, setting $\lambda = 1$, the HARQ protocol reduces to the ARQ, for which the probability of packet error in the subsequent retransmissions equals to the initial packet error probability $q_j$. Based on the packet error probability model in (11), the probability that a packet initially transmitted over the link $\varepsilon_{ij}$ at time step $k$ is delayed by $r = 0, 1, \ldots, \bar{\tau}$ time steps or dropped, is defined as follows:

$$\mathbb{P}(\tau_{ij}[k] = 0) = 1 - q_j, \quad (0 \text{ NACK})$$

$$\mathbb{P}(\tau_{ij}[k] = 1) = (1 - q_j \lambda)q_j, \quad (1 \text{ NACK})$$

$$\vdots$$

$$\mathbb{P}(\tau_{ij}[k] = \bar{\tau}) = (1 - q_j \lambda^\bar{\tau})\prod_{r=0}^{\bar{\tau}-1} q_j \lambda^r, \quad (\bar{\tau} \text{ NACKs})$$

$$\mathbb{P}(\delta_{ij}[k] = 1) = \prod_{r=0}^{\bar{\tau}} q_j \lambda^r, \quad \text{(packet drop)}$$

where $\mathbb{P}(\cdot)$ refers to the probability that the event "\(\cdot\)" will occur. These probabilities can be obtained by the packet error transition model shown in Fig. 2.
B. (H)ARQ Feedback Scheme

In this work, we assume that each transmitting node expects to receive back a separate acknowledgement for each retransmitted packet. In this scheme, each packet received by node $v_j$ is acknowledged individually, and the transmitting node $v_j$ decides whether to retransmit or not each individual packet. Note that, this scheme requires transmissions and (H)ARQ feedback signals to be identified (matched to packets). To get some intuition, consider a simple example where node $v_j$ transmits to node $v_l$, $p_j x_j[k-r]$ for $r \in \{0, 1, 2\}$ (with $\bar{r} = 2$) at each time step $k$, as shown in Fig. 3. Note that, for simplicity of exposition, we focus only on the $x$ mass, although the same logic follows for the $y$ mass for which node $v_j$ transmits $p_j y_j[k-r]$ for $r \in \{0, 1, 2\}$.

Fig. 2: Packet error transition for a packet initially transmitted at iteration $k$ over the link $\varepsilon_{ij}$ with initial probability of error $q_{ij}$.

Fig. 3: An individual feedback signal ACK/NACK $f_{j,k-r}[k]$ is sent from node $v_l$ for each packet $r \in \{0, 1, 2\}$ it receives within each time slot. Black bullets denote a (re)transmission of packet $p_j x_j[k-r]$; red and green bullets denote ACK and NACK, respectively; yellow bullets denote the iteration when node $v_j$ considers packets as dropped.

In this example, node $v_l$ sends back to node $v_j$ the acknowledgement feedback signals $f_{j,k-r}[k]$ for each packet $r \in \{0, 1, 2\}$ it received at time step $k$. Based on this example, we present the variables for the information delays and packet drops, in Table I, and we briefly describe the (H)ARQ feedback mechanism for the first six iterations. To better explain the example, we consider slotted time where during each time slot, node $v_j$ receives the acknowledgement feedback signals from previous transmissions and (re)transmits packets accordingly, and node $v_l$ acknowledges the packets that arrived within this time slot and sends back the corresponding feedback signals.

**Time slot $k = [0, 1]$**: Within this time slot, node $v_j$ transmits a fresh packet $p_j x_j[0]$ and receives its feedback $f_{j,0}[0] = 0$, which indicates that it should retransmit that packet in the next time slot. Node $v_l$ cannot process the packet it received since it was detected in error.

**Time slot $k = [1, 2]$**: Within this time slot, node $v_j$ transmits a fresh packet $p_j x_j[1]$, retransmits $p_j x_j[0]$ and receives the corresponding feedback signals $f_{j,1}[1] = 1$ and $f_{j,0}[1] = 0$, respectively, which indicate that it should retransmit $p_j x_j[0]$ in the next time slot. Node $v_l$ can only process the fresh packet it received from $v_j$, i.e., $p_j x_j[1]$, at the next iteration $k = 2$.

**Time slot $k = [2, 3]$**: Within this time slot, node $v_j$ transmits a fresh packet $p_j x_j[2]$ and retransmits the last time $p_j x_j[0]$; it also receives the corresponding feedback signals $f_{j,2}[2] = 1$ and $f_{j,0}[2] = 0$, respectively. Since feedback $f_{j,0}[2] = 0$, node $v_j$ will consider the packet as dropped during the next time slot and release the dropped information back to its local state. Node $v_l$ can only process the fresh packet it received from $v_j$, i.e., $p_j x_j[2]$, at the next iteration $k = 3$.

**Time slot $k = [3, 4]$**: Within this time slot, node $v_j$ considers the packet $p_j x_j[0]$ as dropped since $f_{j,0}[2] = 0$, i.e., $\delta_{ij}[0] = 1$, and transmits only the fresh packet $p_j x_j[3]$ for which it receives the corresponding feedback $f_{j,3}[3] = 0$, indicating that $p_j x_j[3]$ should be retransmitted in the next time slot. Node $v_l$ cannot process any packet during this time slot since no packet arrived without errors.

**Time slot $k = [4, 5]$**: Within this time slot, node $v_j$ transmits a fresh packet $p_j x_j[4]$, retransmits $p_j x_j[3]$, and receives the corresponding feedback signals $f_{j,4}[4] = 0$ and $f_{j,3}[4] = 0$, respectively, indicating that both packets should be retransmitted in the next time slot.

**Time slot $k = [5, 6]$**: Within this time slot, node $v_j$ transmits a new packet $p_j x_j[5]$, and retransmits $p_j x_j[4]$ and $p_j x_j[3]$. The corresponding feedback signals are all ACKs, and hence at the next time slot only a fresh packet will be transmitted. Node $v_l$ can process the packets it received, i.e., $p_j x_j[3]$, $p_j x_j[4]$, and $p_j x_j[5]$, at the next iteration.

**Remark 4.** An alternative feedback scheme that could be employed by each receiving node $v_l \in \mathcal{V}$ is to collectively acknowledge all packets that arrived within each time slot, and send back a single feedback signal to each of its neighbors $v_j \in \mathcal{N}_l$. Although this scheme does not require the receiving nodes to identify each packet separately, the information exchange is expected to be slower since all packets (except the ones that exceeded the retransmission limit) should be retransmitted at time step $k + 1$ in case there was at least one error for any packet at time step $k$.

| $k$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|-----|-----|-----|-----|-----|-----|-----|
| $\delta_{ij}[k]$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\tau_j[k]$ | $3$ | $0$ | $0$ | $2$ | $1$ | $0$ |

TABLE I: Packet drop indicator variables and information delay variables for the first six iterations of the transmissions over the link $\varepsilon_{ji}$ using the (H)ARQ-RC algorithm.
C. (H)ARQ Ratio Consensus Algorithm - (H)ARQ-RC

Herein, we describe the (H)ARQ-RC algorithm with individual acknowledgements, but one can adjust the algorithm for the collective feedback scheme. Specifically, each node \( v_j \) executes the steps in Algorithm 1, which can be summarized as follows:

- **Input & Initialization**: The initial state \( x_j[0] = V_j \) is determined by the information (often local sensor reading) on which the network of nodes must reach average consensus. As soon as the input has been processed, the auxiliary variable \( y_j[k] \) is initialized at \( y_j[0] = 1 \).

- **Out-degree acquisition**: The out-degree of node \( v_j \) is acquired by broadcasting a dummy packet and summing the number of received acknowledgement feedback signals from its out-neighbors. Note that, one could improve the out-degree acquisition accuracy by executing this process several times.

- **At each time step** \( k \geq 0 \):
  1. **Receive feedback**: Node \( v_j \) receives the feedback signals \( f_{ij,k-r}[k] \) from its out-neighbors \( v_i \in N^+_j \) for all \( r \in \mathbb{N}_0^+ \) corresponding to the (re)transmissions of packets of the previous time slot.
  2. **Transmit fresh packets**: Node \( v_j \) transmits a fresh packet with its current states \( p_{ij,j}[k] \) (resp. \( p_{ij,j}[k] \)) to its out-neighbors \( v_i \in N^+_j \). Note that this occurs at each time step \( k \) via broadcast.
  3. **Retransmit old packets**: According to the feedback signals received at node \( v_j \) as described in step (a), node \( v_j \) retransmits \( v_j \) previous packets for which \( f_{ij,k-r}[k] = 0 \) to its out-neighbors \( v_i \in N^+_j \) for all \( r \in \mathbb{N}_0^+ \). If the maximum retransmission limit \( \bar{\tau} \) has been exceeded, i.e., \( f_{ij,k-\bar{\tau}+1} = 0 \), then the packet containing the states \( p_{ij,j}[k-\bar{\tau}-1] = 0 \) and \( p_{ij,j}[k-\bar{\tau}-1] = 0 \) is considered as dropped and no further retransmissions for this packet are allowed.
  4. **Drop obsolete packets**: For packets that are considered as dropped, node \( v_j \) releases the information that was intended to arrive at node \( v_j \) back to its local state.
  5. **Receive packets**: Node \( v_j \) receives some packets from its in-neighbors \( v_i \in N^-_j \). If a packet arrived with errors, then \( v_j \) requests a retransmission by sending a NACK. Otherwise, if the packet arrived without errors, then node \( v_j \) sends back an ACK and uses the packet for updating its next state.
  6. **Send feedback**: According to the feedback determined during the packet reception phase, node \( v_j \) feeds back \( f_{ji,k-r}[k] \) for requesting retransmission in case of packet error, or \( f_{ji,k-r}[k] = 1 \) to acknowledge the successful reception of the packet.

(g) **Update local states**: Upon the successful reception of packets that arrived at time step \( k \), i.e., for all \( h \in \mathbb{N}_0^+ \) such that \( f_{ji,k-h}[k] = 1 \), and by releasing the dropped information back to its state if \( f_{ji,k-\bar{\tau}+1} = 0 \), node \( v_j \) updates its states \( x_j[k+1] \) and \( y_j[k+1] \) as in lines (22-25) of Algorithm 1, and then computes the ratio \( z_j[k+1] = x_j[k+1]/y_j[k+1] \).

Algorithm 1 (H)ARQ-RC at node \( v_j \)

1. **Input**: \( x_j[0] = V_j, \bar{\tau} \)
2. **Initialization**: \( y_j[0] = 1 \)
3. **Out-degree acquisition**: \( d_j^+ = |N^+_j| \)
4. **for** \( k \geq 0 : \)
5. **Receive feedback from all** \( v_i \in N^+_j \): \( f_{ij,k-r}[k], \forall r \in \mathbb{N}_0^+ \) as in (8)
6. **Transmit fresh packets to all** \( v_i \in N^+_j \): \( p_{ij,j}[k] \)
7. **Retransmit old packets to all** \( v_i \in N^+_j \): \( p_{ij,j}[k-r] = 0 \) and \( p_{ij,j}[k-h] = 0 \), for all \( r \in \mathbb{N}_0^+ \)
8. **for which** \( f_{ji,k-r} = 0 \), and update
9. **retransmission counters**
10. **Drop obsolete packets for all** \( v_i \in N^+_j \): if retransmission limit \( \bar{\tau} \) exceeded, i.e.,
11. **for** \( f_{ij,k-\bar{\tau}+1} = 0 \)
12. **Receive packets from all** \( v_i \in N^-_j \): \( p_{ij,j}[k-h] = 0 \) and \( p_{ij,j}[k-h] = 0 \), for all \( h \in \mathbb{N}_0^+ \)
13. **for which** \( f_{ji,k-h} = 0 \) (received without errors)
14. **Send feedback to all** \( v_i \in N^-_j \): \( f_{ji,k-h} = 0 \), for all \( h \in \mathbb{N}_0^+ \) as in (8)
15. **Update local states**: \( x_j[k+1] = x_j[k+h] - \sum_{h=0}^{\bar{\tau}-1} \sum_{h=0}^{\bar{\tau}-h} p_{ij,j}[k-r] \) for all \( r \in \mathbb{N}_0^+ \)
16. **Output**: \( z_j[k+1] = x_j[k+1]/y_j[k+1] \)

IV. Augmented Digraph Representation

In this section, we analyse the convergence of Algorithm 1, by first introducing the random weighted adjacency matrix that corresponds to the delayed and possibly packet dropping communication topology. To simplify the analysis, we consider identical ARQ protocols for each node. This means that the maximum retransmission limit \( \bar{\tau} \) is the same for all nodes, which implies that the maximum delay on each link, \( \bar{\tau}_{ji} \), equals \( \bar{\tau} \). Hence, for each node \( v_j \in \mathcal{V} \), we add \( \bar{\tau} \) extra virtual nodes, namely, \( \tilde{v}_j^{(1)}, \tilde{v}_j^{(2)}, \ldots, \tilde{v}_j^{(\bar{\tau})} \), that temporarily hold and propagate the (delayed) information that is destined to arrive at the actual node \( v_j \) after \( r = 1, \ldots, \bar{\tau} \) time steps. Furthermore, for each node \( v_j \in \mathcal{V} \), we add \( \bar{\tau} \) extra virtual nodes, namely, \( \tilde{v}_j^{(1)}, \tilde{v}_j^{(2)}, \ldots, \tilde{v}_j^{(\bar{\tau})} \), that release the information of the packets that failed \( \bar{\tau} \) consecutive times.
to be received at their destined receiving nodes, and are thus considered dropped after $\tau + 1$ time steps, back to its actual source node $v_j$. Finally, one can obtain the augmented digraph, $\tilde{G} = (\tilde{V}, \tilde{E})$, that models the network where nodes utilize (H)ARQ protocols, and consists of $\tilde{n} = n(2\tilde{\tau} + 1)$ nodes, where there are $n$ original nodes, $n\tilde{\tau}$ virtual nodes due to the delays, and $n\tilde{\tau}$ virtual nodes due to packet drops.

To illustrate the concept of digraph augmentation, consider the following example with two nodes exchanging information over an unreliable error-prone directed network, as shown in Fig. 4. In this example, both nodes $v_i$ and $v_j$ set the retransmission limit of their (H)ARQ-RC algorithm to $\tilde{\tau} = 2$. Hence, the total number of nodes (actual and virtual) in the augmented digraph will be $\tilde{n} = n(2\tilde{\tau} + 1) = 10$. In particular, for each actual node (shown in blue circle nodes) we add two virtual nodes (shown in yellow square nodes) corresponding to the delayed information due to packet retransmissions, and two virtual nodes (shown in red square nodes) corresponding to the path followed when releasing the information back to its source node when a packet drop occurs.

In essence, for a packet sent by $v_i$ over the link $\varepsilon_{ij} \in \mathcal{E}$ and detected in error $0 \leq r \leq \tilde{\tau}$ consecutive times ($r$ retransmissions) and then arrived successfully at its destined node $v_j$ without errors, the information in the augmented digraph will follow the path $v_i \rightarrow \tilde{v}_j^{(r)} \rightarrow \tilde{v}_j^{(r-1)} \rightarrow \ldots \rightarrow \tilde{v}_j^{(1)} \rightarrow v_j$. Conversely, if the packet has been detected in error $r = \tilde{\tau} + 1$ consecutive times (retransmission limit exceeded), then the information in the augmented digraph will follow the path $v_i \rightarrow \tilde{v}_i^{(r)} \rightarrow \tilde{v}_i^{(r-1)} \rightarrow \ldots \rightarrow v_i$.

Based on the augmented digraph, and for analysis purposes, we can rewrite the consensus iterations performed by Algorithm 1 in compact form as

$$\begin{align*}
\dot{x}[k+1] &= \Xi[k]\dot{x}[k], \\
\dot{y}[k+1] &= \Xi[k]\dot{y}[k],
\end{align*}$$

(12a)

(12b)

where $\dot{x}[k] = (x[k]^T, \tilde{x}[k]^T, \tilde{x}[k]^T) \in \mathbb{R}^n$ (respectively $\dot{y}[k]$) is a vector containing the variables of the actual and virtual nodes (propagating the delayed and dropped information) at time step $k$ as

$$\begin{align*}
x[k] &= (x_1[k], \ldots, x_n[k])^T \in \mathbb{R}^n, \\
x[k] &= (x_1^{(r)}[k], \ldots, x_n^{(r)}[k])^T \in \mathbb{R}^{n\tilde{\tau}}, \\
x[k] &= (x_1^{(r)}[k], \ldots, x_n^{(r)}[k], \ldots, x_n^{(r)}[k])^T \in \mathbb{R}^{n\tilde{\tau}}.
\end{align*}$$

Matrix $\Xi[k] \in \mathbb{R}^{n \times n}$ is a nonnegative random matrix associated with the augmented digraph at iteration $k$:

$$\Xi[k] \equiv \begin{pmatrix}
P(0)[k] & I & 0 & \cdots & 0 \[12pt]
P(1)[k] & 0 & I & \cdots & 0 \[12pt] & \vdots & \vdots & \ddots & \vdots \[12pt] & P(n-1)[k] & 0 & \cdots & I \[12pt] & 0 & 0 & \cdots & 0 & I \[12pt] & \vdots & \vdots & \ddots & \vdots & \vdots \[12pt] & 0 & 0 & \cdots & 0 & 0
\end{pmatrix},$$

(13)

where each element of $P(r)[k]$, $r = 0, 1, 2, \ldots, \tilde{\tau}$, is determined by

$$P(r)[k](l, j) = \begin{cases}
P(l, j), & \text{if } \varepsilon_{lj} \in \mathcal{E}, \\
0, & \text{otherwise}. 
\end{cases}$$

(14)

In other words, if the index $r$ that corresponds to the block matrix $P(r)$ is equal to the packet retransmission number $\tau_{lj}[k]$, then the link $\varepsilon_{lj}$ will be weighted by the actual (original) weight $p_{lj}$; otherwise, its weight will be zero. Moreover, $D(r)[k] \in \mathbb{R}^{n \times n}$ is a diagonal matrix that corresponds to the activation of the packet drop mechanism when the retransmission limit of a packet has been exceeded. In particular, each diagonal element $D(r)[k](j, j)$ corresponds to the virtual node $\tilde{v}_j^{(r)}$ of node $v_j$ and is given by

$$D(r)[k](j, j) = \sum_{v_i \in \mathcal{N}_j^+} \delta_{lj}[k] P(l, j), \quad \text{for all } v_j \in \mathcal{V}. \quad (15)$$

Block matrix $D(r)[k]$ activates the virtual links that release the information that is dropped back to its transmitting node via the virtual nodes that correspond to packet drops. In essence, this means that node $v_j$ will update its local buffer $\tilde{v}_j^{(r)}$ by summing the dropped information that was intended to send to out-neighbor $v_i \in \mathcal{N}_j^+$ from which it received $\tilde{\tau} + 1$ consecutive NACKs, i.e., for which the packet drop indicator variable $\delta_{lj}[k] = 1$, and by releasing this sum back to itself through the virtual nodes $\tilde{v}_j^{(r-1)} \rightarrow \ldots \rightarrow \tilde{v}_j^{(1)} \rightarrow v_j$. Note that, 0 and $I$ in (13) denote the zero and identity matrices, respectively, and they are of dimension $n \times n$. Based on the structure of $\Xi[k]$, which depends on the realized number of retransmissions and packet drops, it is clear that $\Xi[k]$ maintains column-stochasticity, although the links that establish the transmissions between nodes might be unreliable.
A. Convergence Analysis

Prior to establishing the convergence of each node to the network-wide average of the initial values, we first establish some properties of matrices $\Xi[k]$, similar to the ones established in [20] for packet dropping links. Matrix $\Xi[k]$ denotes a particular instance from the set of all possible instances of realized delays induced from packet retransmissions, and packet drops. All these possible instances are in the set $\mathcal{X}$, which consists of a finite number of matrices of identical dimensions $\bar{n} \times \bar{n}$, where each matrix in $\mathcal{X}$ corresponds to a distinct instantiation of the packet drop indicator variable $\delta_{ji}[k]$ and the delay on the links $\tau_{ji}[k]$.

Moreover, each nonzero element of any matrix $\Xi[k]$ is lower bounded by a positive constant $c \equiv \min_{j \in \mathcal{V}} 1/(1 + d_j^*)$, since the $(i, j)$ element of any $\Xi[k] \in \mathcal{X}$ can take values of either 0, 1, or $1/(1 + d_j^*)$. Let $\rho$ be a finite positive integer. Then, a sequence of $\rho$ matrices in $\mathcal{X}$, possibly with repetitions and in a certain order, materializes with a strictly positive probability. The product of a particular choice of these $\rho$ matrices (in the given order) forms a column stochastic matrix, where the entries that correspond to the actual network nodes, contain strictly positive values. This can be written as:

$$
P(\Xi[k + \rho] = \Xi_\rho, \Xi[k + \rho - 1] = \Xi_{\rho - 1}, \ldots, \Xi[k + 1] = \Xi_1) \geq p_{\min} > 0,
$$

where $p_{\min}$ is the strictly positive probability with which this sequence of $\rho$ matrices materializes. Note that matrices $\Xi_{\rho - 1}, \ldots, \Xi_1$ can be chosen so that the top $n$ rows of $L = \Xi_{\rho - 1}, \ldots, \Xi_1$ satisfy: $L(j, i) > 0$ for all $j, i \in \mathcal{V}$, (e.g., $\rho = n$ and $\tau_{ji}[k] = 0$ for all links and all $k$). These properties can be exploited for the proof of the convergence of the (H)ARQ-RC algorithm, stated in Theorem 1 below.

Recall that each node in the network aims at obtaining consensus to the network-wide average of the initial values:

$$
z^* = \frac{\sum_{i \in \mathcal{V}} x_i[0]}{\sum_{i \in \mathcal{V}} y_i[0]} = \frac{1}{n} \sum_{i \in \mathcal{V}} x_i[0],
$$

by having network actual nodes calculate the ratio $z_j[k] = \bar{x}_j[k]/\bar{y}_j[k]$, whenever $\bar{y}_j[k] \geq c^\rho$. To establish the convergence of the proposed algorithm to the exact average of the network-wide initial values, we need to show that the event that the augmented auxiliary variable satisfies $\bar{y}_j[k] \geq c^\rho$ occurs infinitely often, at actual nodes, and hence as $k$ goes to infinity, the sequence of ratio computations $z_j[k]$ converges to the value in (17) almost surely.

**Theorem 1.** Consider a strongly connected digraph $G(V, E)$, where each node $v_j \in \mathcal{V}$ has some initial value $x_j[0]$, and $y_j[0] = 1$. Let $z_j[k]$ (for all $v_j \in \mathcal{V}$ and $k = 0, 1, 2, \ldots$) be the output of the iterations in Algorithm 1. Then, the solution to the average consensus can be obtained by each node, with probability 1, by computing:

$$
\lim_{k \to \infty} z_j[k] = \frac{x_j[k]}{y_j[k]} = \frac{\sum_{v_i \in \mathcal{V}} x_i[0]}{n}, \quad \forall v_j \in \mathcal{V}.
$$

**Proof.** See Appendix B.

V. Numerical Evaluation

Consider the strongly connected directed network $G = (\mathcal{V}, E)$ shown in Fig. 5 consisting of five nodes, i.e., $n = |\mathcal{V}| = 5$, with each node $v_j$ choosing the weights on its outgoing links (including its self-loop link) as in (3). Each node has immediate access to its current states, although the self-loop links are not shown for simplicity. Thus, the digraph corresponds to the following column-stochastic weighted adjacency matrix:

$$
P = \begin{pmatrix}
1/3 & 0 & 0 & 1/2 & 0 \\
1/3 & 1/3 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/2 & 0 & 1/3 \\
0 & 0 & 0 & 1/2 & 1/3 \\
0 & 1/3 & 1/2 & 0 & 1/3
\end{pmatrix}
$$

The variables of each node $x_j$, $y_j$ are updated at each time step using Algorithm 1 with initial values $x[0] = (x_1[0] \ldots x_5[0])^\top = (4 \ 5 \ 6 \ 3 \ 2)^\top$, and $y[0] = (y_1[0] \ldots y_5[0])^\top = (1 \ 1 \ 1 \ 1 \ 1)^\top$, respectively. In the examples that follow we consider that each packet initially transmitted over any link $e_{ji} \in E$ may arrive in error with initial probability of packet error $q = q_{ji}$. Similarly, without loss of generality, we assume that all nodes $v_i \in \mathcal{V}$ utilize an (H)ARQ-RC algorithm of which the retransmission limit is set to $\tau_{ji} = \bar{\tau}$ for all $v_i \in \mathcal{X}^+$. First, to gain insight into the operation of the (H)ARQ-RC algorithm in terms of the state updates, consider the aforementioned network where each packet follows the error transition in (11) with initial probability of error $q = 0.6$. In this example, we assume that each node is employed with the ARQ-RC algorithm with retransmission limit set to $\bar{\tau} = 2$. To keep the example simple, consider that nodes acknowledge the received packets collectively (as described in Remark 4), that is, they send an ACK if all packets arrived without errors, while they send a NACK if there exists at least one packet that arrived with errors. Fig. 6 shows the evolution of the variables and states at node $v_1$, which receives packets only from node $v_4$, during the first time steps of the execution of the ARQ-RC algorithm. As shown in Fig. 6, the state variables $x_1[k]$ and $y_1[k]$ (and hence the ratio $z_1[k]$) are driven by the reception of packets $p_{14}x_4[k]$ that arrive without errors (when $f_{14}[k] = 1$) from node $v_4$, and its own weighted state $p_{11}x_1[k]$ which is delay-free at all $k \geq 0$. Note that, the packets (re)transmitted between time steps 4 – 7 are...
To quantify the performance of the proposed algorithm with respect to different retransmission limits \( \bar{\tau} \), we consider three different channel conditions captured by different initial probability of error, i.e., \( q_{ji} = q \in \{0.1, 0.5, 0.9\} \). In particular, to obtain an overall assessment of the convergence over all time steps of the (H)ARQ-RC algorithm, we record the mean residual, denoted by \( \bar{c} \), over the total duration of 200 iterations for 100 simulations. We obtain the mean residual \( \bar{c} \) for different retransmission limits, as shown in Fig. 11. This figure suggests that when channel conditions are good, i.e., \( q = 0.1 \), then it is preferable to set a lower retransmission limit, since, according to (11), most of the packets arrive at the receiving node successfully without many retransmission trials, for both \( \lambda = 0.5 \) and \( \lambda = 1 \). Here it is worth mentioning that, for \( \bar{\tau} \geq 3 \), there is no improvement in terms of convergence speed. Similar behavior is observed for \( q = 0.5 \), although the saturation of the mean residual occurs for higher retransmission limits, i.e., around \( \bar{\tau} \geq 7 \). In contrast, for bad channel conditions, i.e., \( q = 0.9 \), (H)ARQ-RC performs better in terms of convergence speed for \( \bar{\tau} = 4 \) with \( \lambda = 1 \), while with \( \lambda = 0.5 \), the convergence is slightly faster for \( \bar{\tau} = 1 \), and saturates for \( \bar{\tau} \geq 3 \). This is due to the fact that the trade-off between transmitting fresh information, rather than retransmitting old information, diminishes.

**B. Convergence Speed vs. Communication Cost**

To get intuition regarding the convergence speed and communication cost of the proposed algorithm, we consider the same setup as before for the digraph shown in Fig. 5. Specifically, we run 500 simulations, in which we examine the mean average consensus error \( \bar{\parallel x[k] - x \parallel} \), the mean number of (re)transmissions sent over the network, and the mean values held in the local buffers for compensating the dropped information, at each time step \( k \). Fig. 12 and Fig. 13 depict the aforementioned metrics for different channel conditions captured by the initial probability of error \( q \in \{0.1, 0.5, 0.9\} \). In particular, Fig. 12 compares the performance of the state-of-the-art algorithm RC-RS with the ARQ-RC algorithm with \( \lambda = 1 \), while Fig. 13 compares it with the HARQ-RC algorithm with \( \lambda = 0.5 \). Under this setup, we additionally examine convergence speed and communication cost for different number of retransmission limits \( \bar{\tau} \).

Clearly, for higher probability of initial packet error, i.e., \( q = \{0.5, 0.9\} \), our proposed algorithm (H)ARQ-RC (with any \( \lambda = \{0.5, 1\} \) and any retransmission limit \( \bar{\tau} = \{2, 5, 10\} \)) outperforms the RC-RS algorithm in terms of convergence speed, as shown on the left plots of Fig. 12 and Fig. 13. Of course, this is in exchange of a higher number of transmitted packets as shown in the middle plots of Fig. 12 and Fig. 13. However, one can notice the convergence speed-up when using the HARQ-RC (with \( \lambda = 0.5 \)), especially in the poor channel conditions where \( q = 0.9 \). This significant improvement in terms of convergence speed is due to the capability of the HARQ-RC algorithm to reduce the erroneous, and since the retransmission limit is exceeded \( \tau_{ji} = 3 \), the corresponding packets will be dropped and released back to node \( v_4 \) at time step \( k = 7 \).

Next, we evaluate the performance of the ARQ-RC (with \( \lambda = 1 \)) and HARQ-RC (with \( \lambda = 0.5 \)) algorithms for three different initial packet error probabilities, i.e., \( q = \{0.1, 0.5, 0.9\} \), and for two different configurations on the retransmission limit, i.e., \( \bar{\tau} = \{1, 3\} \). In Fig. 7-10 we present the ratio \( z_j[k] \) of each node \( v_j \in V \) using the ARQ-RC and HARQ-RC algorithms, for \( \bar{\tau} = 1 \) and \( \bar{\tau} = 3 \). For relatively good channel conditions, i.e., \( q = 0.1 \), the convergence speed of the nodes is similar for both algorithms and both retransmission limits since most of the packets will arrive at their destined nodes successfully without errors. For \( q = 0.5 \), we can observe a slightly better performance using the ARQ-RC with retransmission limit set to \( \bar{\tau} = 3 \) rather than \( \bar{\tau} = 1 \). Now, focusing on the case where \( q = 0.9 \), it is more obvious that when setting a higher retransmission limit, i.e., \( \bar{\tau} = 3 \), for both algorithms, the convergence to the exact average consensus value is faster. The convergence speed-up becomes more apparent for the HARQ-RC when \( \bar{\tau} = 3 \). This suggests that, when the channel conditions are bad, setting an appropriately chosen higher retransmission limit would allow the nodes to achieve faster convergence to the average consensus value.

![Fig. 6: Packet transmissions from node v4 to node v1 using the (H)ARQ-RC algorithm and q_{ji} = 0.6, τ = 2.](image)
probability that a packet is received with errors in the subsequent transmissions as previously discussed in (11).

Interestingly, the mean number of sent packets within the network at each iteration is quite close to the one achieved with ARQ-RC, i.e., around 18 – 20 transmissions per iteration.

As aforementioned, implementing RC-RS algorithm would require nodes to maintain extra variables to store run sum values that correspond to the information that is lost due to packet errors and drops. These values are accumulated at each iteration, and as discussed in [20], they grow linearly in an unbounded manner with the number of iterations. This is shown in Fig. 12 and Fig. 13 for different packet error probabilities. Although this would not be an issue for nodes’ memory capabilities (assuming that there exists a distributed termination mechanism), it would require nodes to transmit packets of increasing length, which could induce additional delays and packet drops due to the increased probability of packet errors. However, when using the ARQ-RC and HARQ-RC algorithm shown in Fig. 12 and Fig. 13, respectively, nodes are not required to keep such variables as they have an acknowledgement feedback at each iteration that allows them to reset their local buffers immediately after releasing the dropped information back to their current local states. This allows nodes to keep their local buffers (the values of which
are not transmitted, as opposed to the RC-RS algorithm, almost empty at each time step.

VI. CONCLUSIONS AND FUTURE DIRECTIONS

A. Conclusions

In this paper, we proposed a distributed algorithm to achieve discrete-time asymptotic average consensus, in the presence of unreliable directed networks that induce packet errors. By incorporating (H)ARQ protocols into an iterative consensus-based algorithm, nodes are guaranteed to achieve average consensus despite the packet errors that may induce information delays due to retransmissions and packet drops. Moreover, by utilizing the acknowledgement feedback of the proposed algorithm, nodes are able to determine their out-degree, and know whether and when their transmitted packets arrived at their destination successfully. The convergence analysis of our proposed algorithm shows that nodes converge to the average consensus value with probability 1 despite the packet errors. Finally, in the numerical evaluation we emphasize the convergence speed-up that can be achieved under different settings of the proposed (H)ARQ-RC algorithm, and discuss the trade-off between convergence speed and communication cost, under different channel conditions, while also comparing against the current state-of-the-art algorithm.

B. Future Directions

As it is evident from the results of this work, there exists a trade-off between transmitting new information with a lower success probability and retransmitting previously failed information with a higher success probability. This is a well-known inherent tradeoff between freshness and reliability in networked systems [31], [32]. Thus, a promising extension of this work to achieve faster convergence to the average consensus value, as well as to operate under specific communication and computation resource limitations, is to derive a packet transmission policy to dynamically select the retransmission limit of the (H)ARQ-RC algorithm. Such a policy could be derived based on the quality of the channel that is implicitly inferred from the acknowledgement feedback.

APPENDIX A

Auxiliary Lemmas

Lemma 1 implies that the event $\tilde{y}_j[k] \geq c\rho$, where $c$ and $\rho$ were defined in §IV, occurs infinitely often and hence each node computes the ratio consensus value $y_j[k] = \ddot{x}_j[k]/\tilde{y}_j[k]$ at infinitely many time steps with probability 1. Lemma 2...
implies that as $k$ goes to infinity, the sequence of ratio computations $z_j[k]$ converges to the network-wide average of the initial values $z^*$ in (17) almost surely (with probability 1).

Lemma 1. Hadjicostis et al. [20]. Let $K_j = \{k_1, k_2, \ldots\}$ denote the sequence of time steps when $\tilde{y}_j[k] \geq e^\rho$ and thus node $v_j \in \mathcal{V}$ updates its ratio consensus estimate using $z_j[k] = \tilde{x}_j[k] / \tilde{y}_j[k]$, where $k_t^j < k_t^{j+1}$, $t \geq 1$. The sequence $K_j$ contains infinitely many elements with probability 1.

Lemma 2. Coefficient of Ergodicity, Hadjicostis et al. [20]. Let $L_k = \prod_{\ell=0}^{k-1} \Xi[\ell]$ be the resulting column stochastic matrix from the product of column stochastic matrices $\Xi[\ell]$. Then the coefficient of ergodicity $\delta(L_k) \triangleq \max_{j} \max_{i_1, i_2} |L_k(j, i_1) - L_k(j, i_2)|$ converges almost surely to zero.

Appendix B
Proof of Theorem 1

Proof. Theorem 1 is established following similar analysis as in [20] mutatis mutandis, by replacing the matrix that corresponds to the network topology of the non-delayed case with the augmented matrix that incorporates the ARQ-based communication protocol defined in (13). First, it is clear to see that the evolution of the augmented variables in (12) can be written in the following form:

\[
\tilde{x}[k] = \Xi[k-1] \Xi[k-2] \cdots \Xi[L \Xi[0] \tilde{x}[0] = (\Pi_{\ell=0}^{k-1} \Xi[\ell]) \tilde{x}[0] = L_k \tilde{x}[0], \\
\tilde{y}[k] = \Xi[k-1] \Xi[k-2] \cdots \Xi[L \Xi[0] \tilde{y}[0] = (\Pi_{\ell=0}^{k-1} \Xi[\ell]) \tilde{y}[0] = L_k \tilde{y}[0],
\]

where $L_k = \prod_{\ell=0}^{k-1} \Xi[\ell]$ is the forward product of column stochastic matrices $\Xi[\ell] \in \mathcal{X}$, $\forall \ell = 0, 1, \ldots, k - 1$. From the above notation, it is clear that for each node $v_j \in \mathcal{V}$, we have $\tilde{x}_j[k] = L_k(j, i) \tilde{x}[0]$ and $\tilde{y}_j[k] = L_k(j, i) \tilde{y}[0]$, where $L_k(j, i)$ denotes the $j$-th row of the forward product matrix $L_k$. From Lemma 2 we can infer that $\delta(L_k) < \psi$ with probability $1 - \varepsilon$ among chosen realizations for all $k \geq k_\psi$, where $0 < \psi \leq 1$ and $\varepsilon > 0$. Considering any $k \geq k_\psi$ such that $\tilde{y}_j[k] \geq e^\rho$, we can further infer that the maximum entry of $L_k(j, i)$ is at least $e^{\rho / n}$ since $\tilde{y}_j[k] = L_k(j, i) \tilde{y}[0] \geq e^\rho$ and $\sum_i \tilde{y}_i[0] = n$. Moreover, since $\delta(L_k) < \psi$, the columns of matrix $L_k$ are "within $\psi$" of each other. Hence, the $j$-th row of $L_k$ can be written as $L_k(j, i) = c_j \{1^T + e^T[k]\}$, where $e^T[k] = [e_1[k], e_2[k], \ldots, e_m[k]]$ is an $m$-dimensional row vector that satisfies $e_{max}[k] := \max_i |e_i[k]| < \frac{\psi}{2 \sqrt{m} \psi}$, and $c_j \geq e^\rho / n - \frac{\psi}{2}$ (assume without loss of generality that $\psi < e^\rho / n$), and $1^T$ is an all-ones row vector of appropriate dimension. Hence, at each node $v_j$, the ratio is obtained by:

\[
z_j[k] = \frac{\tilde{x}_j[k]}{\tilde{y}_j[k]} = \frac{L_k(j, i) \tilde{x}[0]}{L_k(j, i) \tilde{y}[0]} = \frac{c_j \{1^T + e^T[k]\} \tilde{x}[0]}{c_j \{1^T + e^T[k]\} \tilde{y}[0]} = \frac{\{1^T + e^T[k]\} \tilde{x}[0]}{\{1^T + e^T[k]\} \tilde{y}[0]}.
\]

First, we establish the bounds on the numerator of the above expression as

\[
\Sigma_x - e_{max}[k] | \Sigma_x| \leq \{1^T + e^T[k]\} \tilde{x}[0] \leq \Sigma_x + e_{max}[k] | \Sigma_x|,
\]

where $\Sigma_x = \sum_i x_i[0]$, and $| \Sigma_x| = \sum_i |x_i[0]|$, for $i = 1, \ldots, n$. Now, for the bounds on the denominator, we know that $\sum_i y_i[0] = n$, since $y_i[0] = 1$ for $i = 1, \ldots, n$. Hence, we can bound the denominator as

\[
n(1 - e_{max}[k]) \leq \{1^T + e^T[k]\} \tilde{y}[0] \leq n(1 + e_{max}[k]).
\]

Then, the ratio of the bounded numerator and denominator gives:

\[
\frac{\Sigma_x - e_{max}[k] | \Sigma_x|}{n(1 + e_{max}[k])} \leq \frac{\{1^T + e^T[k]\} \tilde{x}[0]}{\{1^T + e^T[k]\} \tilde{y}[0]} \leq \frac{\Sigma_x + e_{max}[k] | \Sigma_x|}{n(1 - e_{max}[k])}.
\]

From (17) we have $z^* = \frac{1}{n} \sum_i x_i[0] = \frac{\Sigma_x}{n}$, and thus:

\[
z^* - E[k] \leq z_j[k] \leq z^* + E[k],
\]

where $E[k] = z^* (\Sigma_x + | \Sigma_x|) e_{max}[k] / \{1 - e_{max}[k]\}$. To ensure that $z^* - \epsilon \leq z_j[k] \leq z^* + \epsilon$ holds whenever $k \geq k_\psi$ and $k \in K_j$, we can choose $\psi < (2e^\rho / (\Sigma_x + | \Sigma_x| + 2n\epsilon))$, such that for any desirable $\epsilon > 0$, $e_{max}[k] < (n\epsilon / (\Sigma_x + | \Sigma_x| + n\epsilon))$, holds. Hence, the ratio at each node in (20) converges with probability 1 to $z^*$ as $k$ goes to infinity.

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