Generalized Calogero-Moser systems from rational Cherednik algebras

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Abstract

We consider ideals of polynomials vanishing on the \( W \)-orbits of the intersections of mirrors of a finite reflection group \( W \). When \( W \) is either a real reflection group or the complex reflection group \( G(m, p, N) \) we determine all such ideals which are invariant under the action of the corresponding rational Cherednik algebra at certain values of the parameters hence form submodules in the polynomial module. We show that a quantum integrable system can be defined for every such ideal for a real reflection group \( W \). This leads to known and new integrable systems of Calogero-Moser type which we explicitly specify.

1 Introduction

The usual Calogero-Moser (CM) system describes a pairwise interaction of \( n \) particles on the line with the inverse square potential. This system appeared and was studied in [1], [2], [3]. Olshanetsky and Perelomov introduced CM systems related to the root systems of finite Coxeter groups [4]. An elegant and uniform proof of integrability was proposed by Heckman [5] who used Dunkl operators [6]. After that quantum CM systems related to the root systems became ultimately related to the rational Cherednik algebras \( H_{1,c} \). More exactly, CM operators and their quantum integrals can be thought of as elements of the spherical subalgebra of the rational Cherednik algebra. We refer to the book [8] for the exposition of this and other developments.

Generalized CM systems related to non-symmetric arrangements of hyperplanes appeared in the work of Chalykh, Veselov and the author [9]. These systems were studied in [10], [11] where integrability was established with the
help of Baker-Akhiezer functions. Two families of operators were found corresponding to the deformations of the root systems $A_n(m)$, $C_n(m, l)$. The $A_n(m)$ system describes Calogero-Moser type pairwise interaction where one particle has different mass from the mass of all other particles.

Sergeev considered the generalization of $A_n(m)$ CM system with arbitrary number of particles of each of the two types [12]. He obtained the corresponding Calogero-Moser-Sutherland (CMS) operator at $m = -1/2$ as the radial part of the Laplace-Beltrami operator on a symmetric superspace [12, 13]. Sergeev and Veselov introduced further generalization of the families of CMS operators by defining special deformations of the root systems of contragredient superalgebras Lie [14]. Then integrability for the classical series was established in [14] by rather involved computations.

In the works [15, 16] Sergeev and Veselov gave another proof of integrability of CMS systems related to deformations of classical root systems. They showed that the generalized CMS operators are restrictions to generalized discriminants of the non-deformed CMS operators of $A$ or $BC$ type acting in the space of symmetric functions in infinite number of variables. Special eigenfunctions of these systems were studied by Hallnäs and Langmann in [17, 18].

In the present paper we approach integrability of generalized CM systems making use of Dunkl operators and special representations of rational Cherednik algebras. We show that for special values of parameters the Dunkl operators can be restricted to certain parabolic strata which are the Coxeter orbits of intersections of the mirrors. Equivalently, the ideals of polynomials vanishing on these strata are submodules in the polynomial representation of the corresponding rational Cherednik algebra. Then sum of squares of Dunkl operators takes the form of generalized CM operator when acted on the invariants. In this way we recover CM systems from [9]-[16] at the special values of parameters and also obtain new integrable generalized CM systems. It is simpler to work in the non-symmetric settings and we give complete description of the CM systems which can be obtained in this way. The corresponding parabolic strata are easy to describe. Namely, in the case of constant multiplicity the Coxeter subgraph for the parabolic subgroup defining the stratum should have the same Coxeter number for all its connected components.

We note that the interest in the submodules of the polynomial representation of the rational Cherednik algebras goes back to the pioneering work by Dunkl, de Jeu and Opdam [19] where the set of singular parameters when
the representation is reducible was completely determined. The singular set for the trigonometric degeneration of the Cherednik algebra was found by Etingof in [20], the non-degenerate case was studied by Cherednik in [21]. The actual submodules for the non-degenerate Cherednik algebras were under investigation, in particular, by Kasatani for A and C cases in [22], [23], and by Cherednik in [21]. The representations of the Coxeter group on the singular vectors were determined by Dunkl in the rational A case [24].

The paper is organized as follows. In Section 2 we determine parabolic strata of the finite real reflection groups such that the ideals of polynomials vanishing on these strata are submodules for the polynomial representation of the corresponding rational Cherednik algebra. These strata are rational Coxeter versions of Kasatani’s nonsymmetric vanishing conditions ([22], see also [25, 26]). It follows from [27], [28] that it is necessary to impose vanishing conditions on the parabolic strata in order to get non-trivial invariant ideals.

In Section 3 we show that parabolic strata defining invariant ideals also determine quantum integrable systems of CM type corresponding to the sets of vectors which are obtained by projections of the original Coxeter root system. In Section 4 we explicitly specify these invariant parabolic strata and the corresponding CM systems in the case of classical Coxeter groups. The corresponding generalized CM systems are known to be integrable by [11], [14]. In Section 5 we show that systems found in Section 4 remain integrable if a quadratic term is added to the Hamiltonian. For that we review the proof of integrability of the CM systems for classical series in the external quadratic field through the Dunkl operators [32]. Then our restriction procedure can be applied in this case as well. In Section 6 we explicitly determine generalized CM systems corresponding to the invariant parabolic ideals for the exceptional Coxeter groups. In Section 7 we determine invariant parabolic ideals for the complex reflection group $G(m, p, N)$.

2 Invariant parabolic ideals

Let $W$ be a finite real reflection group acting by orthogonal transformations in its complexified reflection representation $V \simeq \mathbb{C}^n$. Let $\mathcal{R}$ be the corresponding Coxeter root system, and let $\Gamma$ be the corresponding Coxeter graph (see, e.g., [29]). We assume that a positive subsystem $\mathcal{R}_+ \subset \mathcal{R}$ is chosen so the vertices of the graph $\Gamma$ can be identified with the simple roots. Similarly, for a subgraph $\Gamma_0 \subset \Gamma$ we will denote by $\Gamma_0^+$ the set of roots corresponding
to the vertices of $\Gamma_0$.

Let $c(\alpha) = c_\alpha$ be a $W$-invariant function on the set of roots $\mathcal{R}$. The rational Cherednik algebra $H_c = H_c^\mathcal{R}$ is associated with the root system $\mathcal{R}$ and multiplicity $c$ (see [7]; in this paper we assume that parameter $t = 1$).

Also in this paper we will need only the faithful representation of $H_c$ in the space of polynomials $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$. Any element of $H_c$ acts on $p \in \mathbb{C}[x]$ as a composition $r(\nabla)wq(x)p$ where $q(x) \in \mathbb{C}[x]$, $w \in W$, and $r(\nabla) = r(\nabla_1, \ldots, \nabla_n)$ is a polynomial in Dunkl operators corresponding to the bases directions $e_1, \ldots, e_n$. For any direction $\xi \in \mathbb{C}^n$ the Dunkl operator $\nabla_\xi$ is defined as

$$\nabla_\xi = \partial_\xi - \sum_{\alpha \in \mathcal{R}_+} \frac{c_\alpha(\alpha, \xi)}{(\alpha, \alpha)} (1 - s_\alpha),$$

where $(\cdot, \cdot)$ is the standard scalar product in $V$, and $s_\alpha$ is the orthogonal reflection with respect to the hyperplane $(\alpha, x) = 0$. Note that the Dunkl operators satisfy commutativity $[\nabla_\xi, \nabla_\eta] = 0$ ([6]) and it is clear that $\nabla_\xi \mathbb{C}[x] \subset \mathbb{C}[x]$.

Let $\Gamma_0$ be a subgraph of the Coxeter graph $\Gamma$ obtained by specifying some vertices of $\Gamma$ and preserving all edges between these vertices. The subgraph $\Gamma_0$ defines the plane $\pi$ obtained as the intersection of the corresponding mirrors $\pi = \{x \in V | (\beta, x) = 0 \ \forall \beta \in \Gamma_0^v\}$. The associated parabolic stratum is defined as

$$D_{\Gamma_0} = \bigcup_{w \in W} w(\pi).$$

We define the corresponding parabolic ideal $I_{\Gamma_0}$ as a set of polynomials vanishing on the stratum, that is $I_{\Gamma_0} = \{p \in \mathbb{C}[x]|p|_{D_{\Gamma_0}} = 0\}$. It is obvious that $I_{\Gamma_0}$ is an ideal in $\mathbb{C}[x]$ and that it is $W$-invariant. We are going to determine the parabolic strata $D_{\Gamma_0}$ which define ideals $I_{\Gamma_0}$ invariant under the whole rational Cherednik algebra $H_c$.

**Theorem 1** Let $\Gamma_0 = \bigsqcup_{i=1}^k \Gamma_i$ be the decomposition of the subgraph $\Gamma_0 \subset \Gamma$ into the connected components. Then the parabolic ideal $I_{\Gamma_0}$ is invariant under the algebra $H_c$ if and only if for any $i = 1, \ldots, k$ we have

$$\sum_{\alpha \in V_i \cap \mathcal{R}} \frac{c_\alpha(\alpha, u)(\alpha, v)}{(\alpha, \alpha)} = (u, v)$$

for any $u, v \in V_i$, where $V_i$ is a linear space spanned by the roots $\Gamma_i^v$. 

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Proof. Denote by \( V_0 = \bigoplus_{i=1}^k V_i \). Let \( f \in I_{\Gamma_0} \). We are going to analyze the submodule condition \( \nabla_\xi f |_{D\Gamma_0} = 0 \) where \( \nabla_\xi \) is the Dunkl operator \( (1) \). At first we consider the condition \( \nabla_\xi f |_{\pi} = 0 \). For that we recall that \( f |_{\pi} = 0 \) and we can represent polynomial \( f \) in the form

\[
f = \sum_{\beta \in \Gamma_0} f_{\beta}(x)(\beta, x),
\]

where \( f_{\beta} \) are some polynomials. Since \( s_\alpha f |_{\pi} = f |_{s_\alpha \pi} = 0 \) we note that \( \frac{1 - s_\alpha}{(\alpha, x)} f |_{\pi} = 0 \) if \( (\alpha, x)|_{\pi} \neq 0 \). Therefore we rearrange

\[
\nabla_\xi f |_{\pi} = \sum_{\beta \in \Gamma_0} \left( (\beta, \xi) f_{\beta}|_{\pi} - \sum_{\alpha \in R_+ \cap V_0} \frac{2c_\alpha(\alpha, \xi)(\alpha, \beta)}{(\alpha, \alpha)} f_{\beta}|_{\pi} \right).
\]

By collecting the coefficients at \( f_{\beta} \) it follows that

\[
(\beta, \xi) - \sum_{\alpha \in R_+ \cap V_0} \frac{2c_\alpha(\alpha, \xi)(\alpha, \beta)}{(\alpha, \alpha)} = 0,
\]

which is equivalent to the property \( \mathbf{(3)} \) as \( V_i \) are pairwise orthogonal for \( i > 0 \).

Conditions \( \nabla_\xi f |_{w(\pi)} = 0 \) for non-trivial \( w \in W \) are obtained from conditions \( \mathbf{(3)} \) by the \( W \)-action, they are equivalent to the properties \( \mathbf{(3)} \) hence the theorem is proven.

**Theorem 2** Assume that the multiplicity function \( c \) is constant on the roots \( \Gamma_0 \) of the subgraph \( \Gamma_0 \subset \Gamma \). Then the ideal \( I_{\Gamma_0} \) is \( H_c \)-invariant if and only if all the connected components of \( \Gamma_0 \) have same Coxeter number \( h = 1/c \).

This theorem is a direct corollary of Theorem \( \mathbf{1} \) and of the following lemma which can be easily checked by direct case by case calculation for all finite reflection groups.

**Lemma 1** For any irreducible Coxeter root system \( R \) in a Euclidean space \( V \), for any \( u, v \in V \)

\[
\sum_{\alpha \in R} \frac{(\alpha, u)(\alpha, v)}{(\alpha, \alpha)} = h(u, v),
\]

where \( h \) is the Coxeter number of \( R \).

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For the case of different multiplicities we define the generalized Coxeter number $h^c = h^c_\mathcal{R}$ for the irreducible Coxeter root system $\mathcal{R}$ as the coefficient of proportionality between the following two $W$-invariant inner products

$$\sum_{\alpha \in \mathcal{R}} c_\alpha (\alpha, u) (\alpha, v) \left(\frac{\alpha}{\alpha, \alpha}\right) = h^c(u, v).$$

In the case $c = 1$ we have $h^1 = h$ is the usual Coxeter number. Then Theorem 1 has the following reformulation.

**Theorem 3** Let $\Gamma_0 = \coprod_{i=1}^k \Gamma_i$ be the decomposition of the subgraph $\Gamma_0 \subset \Gamma$ into the connected components. Then the parabolic ideal $I_{\Gamma_0}$ is invariant under the algebra $H_c$ if and only if the generalized Coxeter numbers $h^c_i$ for the Coxeter root systems determined by subgraphs $\Gamma_i$ satisfy $h^c_i = 1$ for all $i = 1, \ldots, k$.

The submodules in the polynomial representation appearing in Theorems 1, 2 correspond to particular singular values only. The description of all singular values of parameters, that is the values when polynomial representation has non-trivial submodules, is given in the work [19]. For instance, all values $c = (2m - 1)/2$ are singular when $m \in \mathbb{Z}_+$. The next proposition describes submodules corresponding to these singular values. When $m = 1$ the proposition is a particular case of Theorem 2 when the subgraph $\Gamma_0$ is a Coxeter graph $A_1$, that is consists of one vertex, and the proof is different.

**Proposition 1** Let $\mathcal{R}$ be a Coxeter root system, let $c$ be invariant multiplicity function on $\mathcal{R}$. Let $S_1$ be an orbit of the corresponding Coxeter group acting on $\mathcal{R}$ with the multiplicity $c_1 = c(S_1)$. Let $I$ be the ideal of polynomials having zero of order $2m - 1$ on the hyperplanes $(\alpha, x) = 0$ for any $\alpha \in S_1$. Then $I$ is $H_c$-invariant if and only if $c_1 = (2m - 1)/2$.

**Proof.** We denote by $D_{S_1}$ the parabolic stratum $\cup_{\alpha \in S_1} \{ x : (\alpha, x) = 0 \}$. An arbitrary polynomial $p(x)$ vanishing on $D_{S_1}$ with order $2m - 1$ has the form

$$p(x) = \prod_{\alpha \in S_1} (\alpha, x)^{2m-1} f(x)$$

for some polynomial $f$. Let $S_2 = \mathcal{R} \setminus S_1$ be (possibly empty) set of roots not contained in $S_1$. 

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Using invariance of $\prod_{\alpha \in S_1}(\alpha, x)^{2m-1}$ with respect to reflections $s_\beta$ for $\beta \in S_2$ and its anti-invariance with respect to $s_\beta$ for $\beta \in S_1$ we rearrange $\nabla_\xi p(x)$ as

$$
\sum_{\beta \in S_1} \left( \frac{(2m-1)(\beta, \xi)}{(\beta, x)} \prod_{\alpha \in S_1} (\alpha, x)^{2m-1} f(x) - 2c_1 (\beta, \xi) \prod_{\alpha \in S_1} (\alpha, x)^{2m-1} f(x) \right) - \sum_{\beta \in S_2} c_\beta (\beta, \xi) \prod_{\alpha \in S_1} (\alpha, x)^{2m-1} \frac{f(x) - s_\beta f(x)}{(\beta, x)}
$$

modulo elements of the ideal $I$. The last sum in (4) belongs to the ideal $I$. The first sum in (4) belongs to the ideal if and only if $c_1 = (2m - 1)/2$. Proposition is proven.

### 3 Restricted Calogero-Moser systems

In this section we explain how $H_c$-invariant parabolic ideals lead to the quantum integrable systems of Calogero-Moser type. We say that a differential operator $L$ acting in $N$-dimensional space is *quantum integrable* if there exist $N$ pairwise commuting differential operators $L_1 = L, \ldots, L_N$ so that $L_i$ are algebraically independent.

Consider the parabolic ideal $I$ consisting of polynomials vanishing on the parabolic stratum $D$ which is the $W$-orbit of the subspace $\pi$. Assume that ideal $I$ is a submodule for the rational Cherednik algebra $H_c$. Let $p = \bar{p}|_D$ be a restriction to the stratum of a polynomial $\bar{p}$ defined in the whole space $V$. It follows from the invariance of the ideal $I$ that for any Dunkl operator $\nabla_\xi$ the result of restriction $\nabla_\xi \bar{p}|_D$ does not depend on the extension $\bar{p}$ but depends on $p$ only. Therefore the restricted Dunkl operators $\nabla_\xi|_D$ are correctly defined.

Moreover, the analysis of the property of the ideal $I$ to be invariant in the proof of Theorem implies that the restricted operators $\nabla_\xi|_D$ are defined correctly in the locally analytic settings. Namely, let point $x_0 \in \pi$ be generic, let $U \ni x_0$ be its small neighborhood, $U \subset \pi$. Consider its $W$-orbit $U^W = \cup_{w \in W} w(U)$. Let $f$ be a union of analytic germs defined in the neighborhoods $U^W$, let $\tilde{f}$ be analytic extension of these germs to $\tilde{U}^W = \cup_{w \in W} w(\tilde{U})$ where $\tilde{U} \supset U$ is a small neighborhood of $x_0$ in the space $V$. Then the result of the restriction $\nabla_\xi \tilde{f}|_D$ does not depend on the locally analytic extension $\tilde{f}$ but depends on $f$ only.
Consider now the space $L$ of $W$-invariant union of germs $f$ defined in $U^W$. So $f$ is determined by its values $f_\pi = f|_U$ in the neighborhood $U \subset \pi$. The invariant combinations of Dunkl operators $\sigma(\nabla) = \sigma(\nabla_1, \ldots, \nabla_N)$, where $\sigma(x_1, \ldots, x_N) \in \mathbb{C}[x_1, \ldots, x_N]^W$, act in the space $L$. We denote by $\sigma(\nabla)^{\text{Res}_\pi}$ the operator which maps $f_\pi$ to the result of the restriction $\sigma(\nabla)f|_U$ on to the neighborhood $U \subset \pi$ of the application of the operator $\sigma(\nabla)$ to any $W$-invariant extension $\bar{f}$ of $f$ from $U^W$ to $\tilde{U}^W$ (c.f. [5]).

**Theorem 4** Assume that a stratum $D$ defines $H_c$-invariant parabolic ideal. Then the operator $\sum_{i=1}^N \nabla_i^2$ restricted to the $W$-invariant functions on $D$ has the generalized Calogero-Moser form

$$\left(\sum_{i=1}^N \nabla_i^2\right)^{\text{Res}_\pi} = \Delta_y - \sum_{\alpha \in \mathbb{R}_+} \frac{2c_\alpha}{(\hat{\alpha}, y)} \partial_{\hat{\alpha}},$$

(5)

where $y = (y_1, \ldots, y_n)$ are orthonormal coordinates on the plane $\pi$, $\Delta_y = \frac{\partial^2}{\partial y_1^2} + \ldots + \frac{\partial^2}{\partial y_n^2}$, vector $\hat{\alpha}$ is orthogonal projection of vector $\alpha$ onto $\pi$.

For any polynomials $\sigma, \tau \in \mathbb{C}[x]^W$ the restrictions $\sigma(\nabla)^{\text{Res}_\pi}, \tau(\nabla)^{\text{Res}_\pi}$ are commuting differential operators in the space $\pi$, in particular, operator (5) is quantum integrable.

**Proof.** The operator

$$H = \sum_{i=1}^N \nabla_i^2$$

can be expanded as

$$H = \Delta - \sum_{\alpha \in \mathbb{R}_+} \frac{2c_\alpha}{(\alpha, x)} \partial_\alpha + \sum_{\alpha \in \mathbb{R}_+} \frac{2c_\alpha(1 - s_\alpha)}{(\alpha, x)^2}.$$  

(6)

Consider $f$ which is a $W$-invariant analytic function defined in the neighborhoods $w(U) \subset w(\pi)$ of the $W$-orbit of the generic point $x_0 \in \pi$. Consider now invariant analytic extension $\bar{f}$ of the function $f$ to the union of neighborhoods $w(\tilde{U})$ where $\tilde{U} \supset U$ is a neighborhood of $x_0$ in $\mathbb{C}^N$, $w \in W$. We are going to apply the operator $H$ to the function $\bar{f}$. The assumption of the theorem says that the result of the restriction of $H\bar{f}$ onto $\pi$ does not depend...
on the extension $\bar{f}$ and it is determined by $f_x = \bar{f}|_U$ only. So we may choose $\bar{f}$ to be constant along the normal directions to $\pi$. Then

$$\partial_\alpha \bar{f} = \partial_\alpha \bar{f}, \quad \Delta \bar{f} = \Delta_y \bar{f},$$

and also $(\alpha, x) = (\hat{\alpha}, x)$ when $x \in \pi$. Since function $\bar{f}$ is $W$-invariant, the last sum in (5) disappears and the Calogero-Moser operator (6) takes the form (5). This proves the first part of the theorem.

The second statement follows from the commutativity of the Dunkl operators $[\nabla_\xi, \nabla_\eta] = 0$ and from the fact that operators $\sigma(\nabla)$ preserve the space $L$ of $W$-invariant germs, so the theorem is proven.

We note that the CM system itself can also be restricted to the stratum $D$ considered inside the orbit space if the corresponding parabolic ideal is $H_\epsilon$-invariant. Also it is known that in the orbit space the CM system becomes algebraic [31] so we have restriction of the non-singular differential operator to a subvariety.

So far we were using the “radial normalization” (5) of the generalized CM systems. The restricted operators are also gauge equivalent to the operators in the “potential normalization”. More exactly, we have the following property of the generalized CM systems related to arbitrary parabolic stratum.

**Proposition 2** Let $\pi \subset V$ be an intersection of mirrors

$$\pi = \{ x \in V | (\beta, x) = 0 \quad \forall \beta \in \Gamma_0^\alpha \},$$

(7)

corresponding to a Coxeter subgraph $\Gamma_0 \subset \Gamma$. Let $\hat{u}$ denote the orthogonal projection of a vector $u \in V$ onto the space $\pi$. Then

$$f^{-1}(\Delta - \sum_{\alpha \in R_+, \hat{\alpha} \neq 0} \frac{2c_\alpha}{(\hat{\alpha}, x)} \partial_{\hat{\alpha}})f = \Delta - \sum_{\alpha \in R_+, \hat{\alpha} \neq 0} \frac{c_\alpha(c_\alpha + 1)(\hat{\alpha}, \hat{\alpha})}{(\hat{\alpha}, x)^2}$$

(8)

where $f = \prod_{\alpha \in R_+, \hat{\alpha} \neq 0} (\hat{\alpha}, x)^{c_\alpha}$, and $\Delta$ is Laplacian on $\pi$.

**Proof.** The gauge property (8) is equivalent to the following series of identities for all $\alpha \in R$:

$$\sum_{\beta \in R \atop \beta = \hat{\alpha}} c_\beta(\hat{\alpha}, \hat{\beta}) = 0$$

(9)
when $x \in \pi$ and $(\hat{\alpha}, x) = 0$. To establish identity (9) consider the set $S \subset R$ of the roots $\beta$ such that $\hat{\beta}$ is not proportional to $\hat{\alpha}$, we also assume that $\hat{\alpha} \neq 0$. Consider the action on $S$ of the group $W_0$ which is generated by reflections at the simple roots $\Gamma_\nu^0$ and by the reflection $s_\alpha$. We have decomposition of the set $S$ as the union of $W_0$-orbits $S = O_1 \cup \ldots \cup O_k$. For each $i = 1, \ldots, k$ we claim that

$$\sum_{\beta \in O_i} \frac{(\hat{\alpha}, \hat{\beta})}{(\hat{\beta}, x)} = 0$$

(10)

when $x \in \pi$ and $(\hat{\alpha}, x) = 0$. Indeed, for vectors $x$ under consideration, we have $(\hat{\beta}_1, x) = (\hat{\beta}_2, x)$ for any roots $\beta_1, \beta_2 \in O_i$. Also vector $b := \sum_{\beta \in O_i} \beta$ satisfies

$$(\alpha, b) = (\gamma, b) = 0$$

(11)

for any $\gamma \in \Gamma^0$ because of invariancy $s_\alpha b = b = s_\gamma b$. Property (11) implies $\hat{b} = b$ and then the identities (10) hold which in turn imply (9). The Proposition is proven.

4 Parabolic ideals and CM systems from classical Coxeter groups

We are going now to apply the described method of restriction to derive particular integrable systems of Calogero-Moser type. In this section we deal with the rational Cherednik algebras for the classical root systems, exceptional Coxeter root systems are dealt with in Section 6.

4.1 A-series

Consider the root system $A_{N-1} \subset C^N$ given by the collection of vectors $e_i - e_j$, $1 \leq i < j \leq N$. Theorems 1-3 remain true when the rank of the root system is less than the dimension of the ambient space $V$. As it follows from Theorem 2 the parabolic strata which define invariant ideals for the rational Cherednik algebra for the $A_{N-1}$ root system must have the form

$$D_{m,k} = \bigcup_{w \in S_N} w(\pi_{m,k})$$

(12)
where the plane $\pi_{m,k}$ is given by the equations

$$x_1 = x_2 = \ldots = x_k, \quad x_{k+1} = x_{k+2} = \ldots = x_{2k},$$

$$\ldots$$

$$x_{(m-1)k+1} = x_{(m-1)k+2} = \ldots = x_{mk}.$$  \hfill (13)

Here $m \geq 1, k > 1$ are integer such that $mk \leq N$, the corresponding parameter $c = 1/k$.

**Proposition 3**  For the root system $\mathcal{A}_{N-1}$, the parabolic stratum (12), (13) and the multiplicity $c = 1/k$ the restricted Calogero-Moser operator takes the form

$$H = \left( \sum_{i=1}^{N} \nabla_{t_i}^2 \right)_{Res_{\pi_{m,k}}} = \Delta - 2k \sum_{1 \leq i < j \leq m} \frac{\partial_i - \partial_j}{y_i - y_j}$$

$$- \frac{2}{k} \sum_{m+1 \leq i < j \leq m+n} \frac{\partial_i - \partial_j}{y_i - y_j} - 2 \sum_{m+1 \leq j \leq m+n} \frac{\partial_i - \sqrt{k} \partial_j}{y_i - \sqrt{k} y_j},$$ \hfill (14)

where $\Delta = \sum_{i=1}^{m+n} \partial_i^2$, $\partial_i = \frac{\partial}{\partial y_i}$, and $n = N - mk$.

**Proof.** One way to deduce this statement is to consider the projection of the root system $\mathcal{A}_{N-1}$ to the plane $\pi_{m,k}$ as Theorem 4 directs. Alternatively, we may introduce the following change of coordinates. For every block of colliding coordinates $x_{jk+1} = \ldots = x_{jk+k}$ we define new coordinates $z_{jk+1}, \ldots, z_{jk+k}$ such that

$$z_{jk+1} = \frac{x_{jk+1} + \ldots + x_{jk+k}}{\sqrt{k}}, \quad z_{jk+2} = \frac{x_{jk+1} - x_{jk+2}}{\sqrt{k}}, \quad z_{jk+3} = \frac{x_{jk+2} - x_{jk+3}}{\sqrt{k}}, \ldots,$$

$$z_{jk+k} = \frac{x_{jk+k-1} - x_{jk+k}}{\sqrt{k}}$$

where $j = 0, \ldots, m - 1$. The remaining coordinates are not changed: $z_i = x_i$ for $i > mk$. Then on the plane $\pi_{m,k}$ we have $x_{jk+s} = \frac{1}{\sqrt{k}} z_{jk+1}$, $s = 1, \ldots, k$.

Also $\frac{\partial}{\partial z_{jk+s}} = \frac{1}{\sqrt{k}} \frac{\partial}{\partial x_{jk+s}}$ when acting on functions $f$ which are constant along the directions orthogonal to the plane $\pi_{m,k}$. One gets operator (14) after renaming surviving $z$-coordinates into $y$-coordinates.
The operator (14) in the case $n = 1$ appeared first in the work [9] where its (algebraic) integrability was established. For arbitrary $m, n, k$ the integrability of the trigonometric version of the operator (14) was established in [14] using explicit calculations. In the work [15] integrability for arbitrary $m, n, k$ was established by obtaining the operator as a restriction of the trigonometric Calogero-Moser operator acting in the space of symmetric functions of infinitely many variables to the generalized discriminants.

4.2 $B$-series

Consider the group $B_N$ generated by reflections at $x_i = \pm x_j$ for $1 \leq i < j \leq N$, and at $x_i = 0$ for $i = 1, \ldots, N$. Let $c_2$ be the multiplicity of the roots $e_i$, and let $c_1$ be the multiplicity of the roots $e_i \pm e_j$. There are parabolic strata corresponding to the Coxeter subgraphs of the type $A_{k-1} \times \ldots \times A_{k-1}$, $k > 1$, where the number of subsystems $A_{k-1}$ is $m$, $mk \leq N$. The corresponding plane $\pi_{m,k} \subset \mathbb{C}^N$ is given by the equations

$$x_1 = x_2 = \ldots = x_k, \quad x_{k+1} = x_{k+2} = \ldots = x_{2k},$$

$$\ldots$$

$$x_{(m-1)k+1} = x_{(m-1)k+2} = \ldots = x_{mk}. \quad (15)$$

The corresponding parabolic stratum $D_{m,k} \subset \mathbb{C}^N$ is the orbit of the plane $\pi_{m,k}$ under the group $B_N$:

$$D_{m,k} = \bigcup_{w \in B_N} w(\pi_{m,k}). \quad (16)$$

Theorem 2 and easy calculations imply the following

**Proposition 4** For the root system $B_N$ with the multiplicity $c_1 = 1/k$ the parabolic ideal

$$I_{m,k} = \{ f \in \mathbb{C}[x_1, \ldots, x_N] \mid f|_{D_{m,k}} = 0 \}$$

corresponding to the stratum (15), (16) is invariant under the rational Cherednik algebra $H^\mathbb{C}_{c}B_N$. The restricted Calogero-Moser operator takes the form
\[ H = \Delta - \sum_{i,j=1, i \neq j}^{m} \left( \frac{2k(\partial_i - \partial_j)}{y_i - y_j} + \frac{2k(\partial_i + \partial_j)}{y_i + y_j} \right) - \sum_{i=1}^{m} \sum_{j=m+1}^{m+n} \left( \frac{2(\partial_i - \sqrt{k}\partial_j)}{y_i - \sqrt{k}y_j} + \frac{2(\partial_i + \sqrt{k}\partial_j)}{y_i + \sqrt{k}y_j} \right) \]
\[ - \sum_{i,j=m+1}^{m+n} \left( \frac{2k^{-1}(\partial_i - \partial_j)}{y_i - y_j} + \frac{2k^{-1}(\partial_i + \partial_j)}{y_i + y_j} \right) - \sum_{i=1}^{m} \left( kq + k - 1 \right) \partial_i \]
\[ - \sum_{i=m+1}^{m+n} q \partial_i. \]

(17)

where \( \Delta = \sum_{i=1}^{m+n} \partial_i^2 \), \( q = 2c_2 \in \mathbb{C} \), and \( n = N - mk \).

For \( n = 1 \) the first sum in the second line in (17) disappears and the integrability of this operator was established in [11] where it corresponded to the configuration \( C_{m+1}(\frac{kq+k-1}{2}, \frac{q}{2}) \). In the case \( n > 1 \) the integrability of the trigonometric version of (17) was shown in [14],[16].

Consider the plane \( \pi_l \subset \mathbb{C}^N \) given by the equations
\[ x_1 = x_2 = \ldots = x_l = 0. \]

Consider the corresponding parabolic stratum \( D_l \) which is the orbit of the plane \( \pi_l \) under the group \( B_N \):
\[ D_l = \bigcup_{i_1 < \ldots < i_l} \{ x | x_{i_1} = \ldots = x_{i_l} = 0 \}. \]

(18)

Define the ideal \( I_l = \{ f \in \mathbb{C}[x_1, \ldots, x_N] | f|_{D_l} = 0 \} \).

Proposition 5 The Dunkl operators \( \nabla_i \) preserve the ideal \( I_l \) if and only if \( 2(l-1)c_1 + 2c_2 = 1 \). The corresponding restricted Calogero-Moser operator is the CM operator constructed by the root system \( B_{N-l} \).

Proof. It is easy to check that the generalized Coxeter number \( h_c \) for the root system \( B_l \) with multiplicities \( c(e_i \pm e_j) = c_1, c(e_i) = c_2 \) equals \( h_c = 2c_1(l-1) + 2c_2 \). The statement follows from Theorem 3.

Theorem 5 All possible parabolic strata defining \( H_{B_N}^{B_N} \)-invariant ideals are either the strata \( D_{m,k} \) defined by (16) (with \( c_1 = 1/k \)) or strata \( D_l \) defined by
(18) (with $2(l − 1)c_1 + 2c_2 = 1$) or their intersection. In the latter case the stratum is the $B_N$-orbit of the subspace

$$x_1 = x_2 = \ldots = x_k, \quad x_{k+1} = x_{k+2} = \ldots = x_{2k},$$

$$\ldots$$

$$x_{(m-1)k+1} = x_{(m-1)k+2} = \ldots = x_{mk},$$

$$x_{mk+1} = \ldots = x_{mk+l} = 0,$$  \hspace{1cm} (19)

with $c_1 = 1/k, c_2 = \frac{1}{2} - \frac{l-1}{k}$, and $k > 1, l \geq 1, mk + l \leq N$. In this case the corresponding restricted Calogero-Moser operator is operator (17) where $q = \frac{k+2l+2}{k}$.

The proof follows from the general structure of the parabolic strata for the $B_N$ group, Theorem 3 and Propositions 4, 5.

4.3 $D$-series

Consider the group $D_N$ generated by reflections at $x_i = \pm x_j$ for $1 \leq i < j \leq N$. Take two integer parameters $m > 0, k > 1$ such that $mk \leq N$. Consider the plane $\pi_{m,k}^{\varepsilon} \subset \mathbb{C}^N$ given by the equations

$$x_1 = x_2 = \ldots = x_k, \quad x_{k+1} = x_{k+2} = \ldots = x_{2k},$$

$$\ldots$$

$$x_{(m-1)k+1} = x_{(m-1)k+2} = \ldots = \varepsilon x_{mk}. $$  \hspace{1cm} (20)

Here $\varepsilon = 1$ except the case when $k$ is even and $N = mk$. In the latter case $\varepsilon = \pm 1$. Consider the corresponding parabolic stratum $D_{m,k}^{\varepsilon} \subset \mathbb{C}^N$ which is the orbit of the plane $\pi_{m,k}^{\varepsilon}$ under the group $D_N$:

$$D_{m,k}^{\varepsilon} = \bigcup_{w \in D_N} w(\pi_{m,k}^{\varepsilon}).$$  \hspace{1cm} (21)

This describes all possible strata in $D_N$ with the Coxeter graphs $A_{k-1} \times \ldots \times A_{k-1}$ (see [30]).

Theorem 2 and easy calculations imply the following

Proposition 6 For the root system $D_N$, the parabolic stratum (27), (21) and the multiplicity $c = 1/k$ the corresponding ideal

$$I_{m,k}^{\varepsilon} = \{ f \in \mathbb{C}[x_1, \ldots, x_N] \mid f|_{D_{m,k}^{\varepsilon}} = 0 \}$$
is invariant under the corresponding rational Cherednik algebra $H_c$. The restricted Calogero-Moser operator has the form (17) with $q = 0$.

Another type of the parabolic strata for the $D_N$ group has the Coxeter graph of type $D_p$, $1 < p < N$. Consider the plane $\pi_p \subset \mathbb{C}^N$ given by the equations

$$x_1 = x_2 = \ldots = x_p = 0.$$ \hfill (17)

The corresponding parabolic stratum $D_p$ is the orbit of the plane $\pi_p$ under the group $D_N$:

$$D_p = \bigcup_{1 \leq i_1 < \ldots < i_p} \{ x | x_{i_1} = \ldots = x_{i_p} = 0 \}. \hfill (22)$$

Theorem 2 implies that the ideal $I_p = \{ f \in \mathbb{C}[x_1, \ldots, x_N] | f|_{D_p} = 0 \}$ is invariant under the rational Cherednik algebra if and only if $c = \frac{1}{2(p-1)}$. The corresponding restricted Calogero-Moser operator is the CM operator constructed by the root system $B_{N-p}$.

The next theorem is a corollary of Theorem 2, previous considerations and calculations of the restricted root systems.

**Theorem 6** For the root system $D_N$ all possible parabolic strata defining $H^D_N$-invariant ideals are either the strata $D^e_{m,k}$ defined by (21) (with $c = 1/k$) or strata $D_p$ defined by (22) (with $c = \frac{1}{2(p-1)}$) or their intersection. In the latter case the stratum is the $D_N$-orbit of the plane

$$x_1 = x_2 = \ldots = x_k, \quad x_{k+1} = x_{k+2} = \ldots = x_{2k},$$

$$\ldots$$

$$x_{(m-1)k+1} = x_{(m-1)k+2} = \ldots = x_{mk},$$

$$x_{mk+1} = \ldots = x_{mk+\frac{k}{2}+1} = 0, \hfill (23)$$

where $k \geq 2$ is even, and $c = 1/k$. In this case the restricted Calogero-Moser operator is the operator (17) with $q = \frac{2(k+2)}{k}$.

5 Generalized CM systems with quadratic potential

In this section we show that generalized CM operators (14), (17) obtained from the parabolic strata for the classical Coxeter groups remain integrable when the term $\omega^2 \sum_{i=1}^{m+n} y_i^2$ is added to the Hamiltonians $H$. 

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First we recall integrability of the Calogero-Moser system with square potential following [32]. Let $\nabla_i$ be Dunkl operator in the basis direction $e_i$ for the symmetric group $S_N$,

$$\nabla_i = \partial_i - c \sum_{j=1, j \neq i}^{N} \frac{1}{x_i - x_j}(1 - s_{ij}).$$

Define now the operators $\nabla_i^+ , \nabla_i^-$ as

$$\nabla_i^\pm = \nabla_i \pm \omega x_i,$$

where $\omega \in \mathbb{C}$ is a parameter. A combination of operators $\nabla_i^\pm$ gives CM system with square potential. Namely, because of commutation relations

$$[x_i, \nabla_i] = -1 + c \sum_{j=1, j \neq i}^{N} s_{ij}, \quad (24)$$

we have

$$\sum_{i=1}^{N} \nabla_i^+ \nabla_i^- = \sum_{i=1}^{N} \nabla_i^2 - \omega^2 \sum_{i=1}^{N} x_i^2 - \omega N + 2\omega c \sum_{i<j}^{N} s_{ij}.$$ The last expression becomes CM Hamiltonian with the added quadratic terms and constant when restricted to the space of symmetric functions. Denoting the operation of restriction to invariants by $\text{Res}$ (c.f. [5]) we recall the following result.

**Theorem 7** [32] The CM Hamiltonian with quadratic potential

$$H = \Delta - \sum_{i<j}^{N} 2c(\partial_i - \partial_j) \frac{1}{(x_i - x_j)} - \omega^2 \sum_{i=1}^{N} x_i^2$$

can be obtained as

$$H = \left( \sum_{i=1}^{N} \nabla_i^+ \nabla_i^- \right)^{\text{Res}} + \omega N - \omega cN(N - 1).$$

The differential operators $\left( \sum_{i=1}^{N} (\nabla_i^+ \nabla_i^-)^k \right)^{\text{Res}}$ for $k \in \mathbb{Z}_+$ pairwise commute, so $H$ is quantum integrable.
Proof. Commutativity of the differential operators follows from the fact that the combinations $\sum_{i=1}^{N} (\nabla_i^+ \nabla_i^-)^k$ preserve the space of symmetric polynomials and from the following commutativity of the combinations of Dunkl operators

$$[\sum_{i=1}^{N} h_i^k, \sum_{i=1}^{N} h_i^l] = 0,$$

(25)

where $h_i = \nabla_i^+ \nabla_i^-$ and $k, l \in \mathbb{Z}_+$. To establish (25) we note first that from the relations $[x_i, \nabla_j] = -c s_{ij}$ valid for $i \neq j$ it follows that $[h_i, h_j] = 2\omega c(h_i - h_j)s_{ij}$. Then by induction in $k$ it is easy to deduce that

$$[h_i^k, h_j] = 2\omega c(h_i^k - h_j^k)s_{ij},$$

and then by induction in $l$ one obtains that

$$[h_i^k, h_j^l] = 2\omega c \sum_{t=1}^{l} (h_j^{t-1} h_i^{k+l-t} - h_j^{k+l-t} h_i^{t-1}) s_{ij}.$$

The last formula implies by induction in $l$ that $[h_i^k, h_j^l]$ is antisymmetric with respect to permutation $h_i$ and $h_j$. Therefore $[h_i^k, h_j^l] + [h_i^l, h_j^k] = 0$, hence commutativity of the operators (25) holds, and the theorem follows.

Similar arguments allow to establish quantum integrability of the Calogero-Moser systems with quadratic potential in the case of other classical Coxeter root systems [33]. Indeed, let now $\nabla_i^\pm = \nabla_i \pm \omega x_i$, where $\nabla_i$ is the Dunkl operator in the direction $e_i$ for the root system $B_N$ or $D_N$, $i = 1, \ldots, N$. The operators $\sum_{i=1}^{N} h_i^k$ where $h_i = \nabla_i^+ \nabla_i^-$ preserve the spaces of the corresponding $B_N$ or $D_N$ invariants. Commutativity of these combinations for different $k$ can be established similar to the proof of commutativity (25) in the proof of Theorem 7. Indeed, the commutation relations need to be modified as follows

$$[x_i, \nabla_j] = -c(s_{ij} - s_{ij}^+), [h_i, h_j] = 2\omega c(h_i - h_j)(s_{ij} + s_{ij}^+), [h_i^k, h_j] = 2\omega c(h_i^k - h_j^k)(s_{ij} + s_{ij}^+),$$

where $1 \leq i < j \leq N$, and then

$$[h_i^k, h_j^l] = 2\omega c \sum_{t=1}^{l} (h_j^{t-1} h_i^{k+l-t} - h_j^{k+l-t} h_i^{t-1})(s_{ij} + s_{ij}^+),$$

where $s_{ij}^+$ is the reflection at the hyperplane $x_i + x_j = 0$.  

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Finally, the differential operators \((\sum_{i=1}^{N} \nabla_i^+ \nabla_i^-)^{Res}\) coincides with the corresponding \(B_N\) or \(D_N\) CM operator with quadratic potential up to a constant, if \(Res\) denotes the restriction to \(B_N\) or \(D_N\)-invariants respectively (c.f. [5]).

Now we are ready to obtain generalized quantum Calogero-Moser systems with the additional quadratic potential.

**Theorem 8** The operators \(H - \omega^2 \sum_{i=1}^{m+n} y_i^2\) are quantum integrable if \(H\) is given either by the formula (14) or by the formula (17).

In this theorem we assume that \(k \in \mathbb{Z}\) as it also happens in Propositions 3, 4 although we expect Theorem 8 to be true for any \(k\). When \(H\) is given by (14) with \(n = 1\) Theorem 8 is already established in [10] for any \(k \in \mathbb{C}\). We also note that some eigenfunctions of the operators from the theorem were already investigated in [17, 18].

The proof of the theorem follows from the fact that if a parabolic ideal \(I\) corresponding to the Coxeter orbit of a linear subspace \(\pi\) is preserved under the action of the Dunkl operators \(\nabla_i\), then \(I\) is also preserved under the action of the operators \(\nabla_i^\pm\). Indeed, consider the combinations

\[ L_k = \sum_{i=1}^{N} h_i^k \]

where \(h_i = \nabla_i^+ \nabla_i^-\) is a product of Dunkl operators for the Coxeter groups \(A_{N-1}\) or \(B_N\). The operators \(L_k\) are invariant under the action of the corresponding classical Coxeter group. The restrictions \(L_k^{Res}\) are commuting differential operators on the subspace \(\pi\). For the Coxeter group \(A_{N-1}\) with multiplicity \(c = k^{-1}\) and the subspace \(\pi\) given by the equations (13) the operator

\[ L_1^{Res} = H - \omega^2 \sum_{i=1}^{m+n} y_i^2 - \omega N + \omega k^{-1}N(N - 1), \]

where \(H\) is given by (14). For the Coxeter group \(B_N\) with multiplicities \(c(e_i \pm e_j) = k^{-1}\), \(c(e_i) = q/2\) and the subspace \(\pi\) given by the equations (15) the operator

\[ L_1^{Res} = H - \omega^2 \sum_{i=1}^{m+n} y_i^2 - \omega N + 2\omega k^{-1}N(N - 1) + \omega qN, \]

where \(H\) is given by (17).
6 Generalized CM systems from exceptional Coxeter groups

First we consider invariant ideals for the Coxeter root system $F_4$. We use Theorem 3 and the fact that the parabolic strata are given by non-isomorphic Coxeter subgraphs and by isomorphic subgraphs if they are different as Dynkin subgraphs of the root system $F_4$ ([30]). We also use that for the root system $A_n$ with multiplicity $c$ we have $h^c = (n + 1)c$, and for the root system $B_n$ with multiplicities $c(e_i \pm e_j) = c_1$, $c(e_i) = c_2$ we have $h^c = 2(n - 1)c_1 + 2c_2$.

The action of the group on the root system $F_4$ has two orbits with multiplicities $c_1, c_2$. There are two strata corresponding to the subgraphs of type $A_1$, the corresponding ideals are invariant when the corresponding multiplicity $c_1$ or $c_2$ equals $1/2$. There are two strata corresponding to subgraphs of type $A_2$. They are invariant iff the corresponding multiplicity $c|A_2 = 1/3$. There is one stratum of the type $A_1 \times A_1$, it is invariant iff $c_1 = c_2 = 1/2$. The stratum of type $B_2$ is invariant iff $c_1 + c_2 = 1/2$. There are two strata of type $B_3$, they are invariant iff the corresponding generalized Coxeter number $h_{B_3} = 1$ which gives $4c_1 + 2c_2 = 1$ or $4c_2 + 2c_1 = 1$ respectively. There are two strata of type $A_1 \times A_2$, they are invariant iff $c|A_2 = 1/3$ and $c|A_1 = 1/2$.

The restricted Calogero-Moser systems correspond to the root system $G_2$ for the strata $A_2$, and the restricted system corresponds to the root system $B_2$ for the stratum $B_2$.

The restricted Calogero-Moser systems corresponding to the strata $A_1$ give equivalent new non-Coxeter one-parametric families of integrable systems in dimension 3.

**Theorem 9** The restricted $F_4$ CM system for the stratum $A_1$ has the form

$$H = \Delta - \sum_{i=1}^{3} \frac{(4c + 1)\partial_i}{x_i} - \sum_{i,j=1}^{3} \frac{2c(\partial_i \pm \partial_j)}{x_i \pm x_j} - \sum \frac{2(\partial_1 \pm \partial_2 \pm \partial_3)}{x_1 \pm x_2 \pm x_3},$$

(26)

where $c \in \mathbb{C}$ and summations run over arbitrary choices of signs. In particular, operator (26) is quantum integrable.

In the case of the $A_1 \times A_1$ strata the restricted integrable CM system has
the Hamiltonian

\[
H = \partial^2_x + \partial^2_y - \frac{2m}{x} \partial_x - \frac{2m}{y} \partial_y - \frac{2n}{x+y} (\partial_x + \partial_y) - \frac{2n}{x-y} (\partial_x - \partial_y) - \frac{2(\partial_x + \alpha \partial_y)}{x + \alpha y} - \frac{2(\partial_x - \alpha \partial_y)}{x - \alpha y} - \frac{2(\partial_x + \alpha^{-1} \partial_y)}{x + \alpha^{-1} y} - \frac{2(\partial_x - \alpha^{-1} \partial_y)}{x - \alpha^{-1} y}
\]

where \(m = 7/2, n = 0\) and \(\alpha = \sqrt{2}\).

**Proposition 7** The system (27) is quantum integrable for any \(m, n\) and \(\alpha = \pm (\sqrt{2n+1} \pm \sqrt{2(m+n+1)})/\sqrt{2m+1}\).

Proof follows from the fact that operator (27) in the potential form satisfies the locus conditions ([11]) when \(m, n \in \mathbb{Z}\). There is a commuting operator to (27) with the highest symbol \((\alpha^2 \xi_1^2 - \xi_2^3)(\xi_1^2 - \alpha^2 \xi_2^3)^3\).

Now we give a complete list of the sets \(A\) of vectors \(\alpha\) with multiplicities \(m_\alpha\) for the discrete generalized CM systems, which are obtained as restrictions of the exceptional Coxeter root systems of \(\mathcal{E}\) and \(\mathcal{H}\) type. The corresponding operators

\[
\Delta - \sum_{\alpha \in A} \frac{2m_\alpha}{(\alpha, x)} \partial_\alpha, \quad \Delta - \sum_{\alpha \in A} \frac{m_\alpha(m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}
\]

are quantum integrable. These systems are labeled by a pair \((\Gamma, \Gamma_0)\), where \(\Gamma\) is a Coxeter graph of \(\mathcal{E}\) or \(\mathcal{H}\) type, and \(\Gamma_0\) is its subgraph satisfying conditions in Theorem 3 so the parabolic ideal \(I_{\Gamma_0}\) is \(H_c\)-invariant. We will assume that there are at least two vertices in \(\Gamma \setminus \Gamma_0\) so that the restricted CM system [43] is at least two-dimensional hence non-trivial. Then in all the cases below the parabolic strata are in one to one correspondence with the isomorphism classes of the Coxeter subgraphs \(\Gamma_0\) except the following cases for the \(\mathcal{E}_7\) root system ([30]). Namely, there are two different \(\mathcal{A}_1^3\) strata denoted as \((\mathcal{E}_7, \mathcal{A}_1^3)_{1,2}\) and two different \(\mathcal{A}_5\) strata denoted as \((\mathcal{E}_7, \mathcal{A}_5)_{1,2}\).

\[
\mathcal{E}_7: \quad \begin{array}{c}
\circ \\
1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6
\end{array}
\]

The stratum \((\mathcal{E}_7, \mathcal{A}_1^3)_{2}\) corresponds to the choice of subgraph \(\Gamma_0\) corresponding to the roots numbered by 4,6,7 in the diagram. The stratum \((\mathcal{E}_7, \mathcal{A}_1^3)_{1}\) corresponds to any other choice of the subgraph of type \(\mathcal{A}_1^3\). Also the stratum \((\mathcal{E}_7, \mathcal{A}_5)_{2}\) corresponds to the choice of the subgraph with the
vertices 3,4,5,6,7. And the stratum \((E_7, A_5)\) corresponds to any of the two remaining embeddings of the subgraph \(A_5\) into \(E_7\).

In the following table for every system \((\Gamma, \Gamma_0)\) we specify vectors \(A\) in this system, their multiplicities, dimension of the linear space spanned by \(A\), and number of vectors in \(A\).

| \((\Gamma, \Gamma_0)\) | Vectors \(A\) of the restricted CM system                                                                 | Mult | Dim | \(|A|\) |
|-----------------------|--------------------------------------------------------------------------------------------------------|------|-----|--------|
| 1 \((E_8, A_1)\)      | \(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 + \sqrt{2} e_7\) (even \(\xi\) of minuses); \(e_i \pm e_j(1 \leq i < j \leq 6)\); \(e_7\) \(e_i \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6\) (odd \(\xi\) of minuses); \(2(e_i \pm e_7)(1 \leq i \leq 6)\) | 1/2  | 7   | 91     |
| 2 \((E_6, A_1^2)\)    | \(e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6\) \(e_1, e_2, e_3, e_4, e_5, e_6\) \(e_i \pm e_j(1 \leq i < j \leq 6)\) | 1    | 2   | 6      |
| 3 \((E_8, A_2)\)      | \(e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm \sqrt{3} e_6\) (even \(\xi\) of minuses); \(e_i \pm e_j(1 \leq i < j \leq 6)\); \(e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm \frac{1}{\sqrt{3}} e_6\) (odd \(\xi\) of minuses); \(\sqrt{3} e_i \pm e_6(1 \leq i \leq 5)\); \(e_6\) | 1/3  | 6   | 63     |
| 4 \((E_8, A_1^3)\)    | \(e_1 \pm e_2 \pm e_3 \pm e_4 \pm \sqrt{2} e_5; \sqrt{2} e_i \pm e_5(1 \leq i \leq 4)\) \(e_1 \pm e_2 \pm e_3 \pm e_4; e_1, e_2, e_3, e_4\) \(e_i \pm e_j(1 \leq i < j \leq 4)\) \(e_5\) | 1    | 2   | 5      |
| 5 \((E_8, A_3)\)      | \(e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5\) \(e_i \pm e_j(1 \leq i < j \leq 5)\) \(e_1, e_2, e_3, e_4, e_5\) | 1    | 1/4 | 5      |
| 6 \((E_8, A_1^4)\)    | \(e_1 \pm e_2 \pm e_3; e_1 \pm e_2 \pm e_4; e_1 \pm e_3 \pm e_4; e_2 \pm e_3 \pm e_4\) \(e_1 \pm e_2; e_1 \pm e_3; e_1 \pm e_4; e_2 \pm e_3; e_2 \pm e_4; e_3 \pm e_4\) \(e_1, e_2, e_3, e_4\) | 1    | 2   | 4      |
| 7 \((E_8, A_2^2)\)    | \(\pm e_1 \pm e_2 \pm \frac{1}{\sqrt{3}} e_3 \pm \sqrt{3} e_4\) (odd \(\xi\) of minuses); \(\pm e_1 \pm e_2 \pm \sqrt{3} e_3 + \frac{1}{\sqrt{3}} e_4\) (odd \(\xi\) of minuses); \(\sqrt{3} e_1 \pm e_3; \sqrt{3} e_2 \pm e_3; \sqrt{3} e_1 \pm e_4; \sqrt{3} e_2 \pm e_4; e_3, e_4\) \(e_1 \pm e_2 \pm \sqrt{3} e_3 \pm \sqrt{3} e_4\) (even \(\xi\) of minuses); \(e_1 \pm e_2\) \(e_1 \pm e_2 \pm \frac{1}{\sqrt{3}} (e_3 + e_4); e_1 - e_2 \pm \frac{1}{\sqrt{3}} (e_3 - e_4); e_3 \pm e_4\) | 1    | 4   | 30     |
| 8 | $(\mathcal{E}_8, A_4)$ | $e_1 \pm e_2 \pm e_3 \pm \sqrt{5} e_4$ (even $\frac{5}{3}$ of minuses); $e_1 \pm e_2$; $e_1 \pm e_3$; $e_2 \pm e_3$; $e_1 \pm e_2 \pm e_3 \pm \frac{1}{\sqrt{5}} e_4$ (even $\frac{5}{3}$ of minuses); $e_5$ $e_1 \pm e_2 \pm e_3 \pm \frac{2}{\sqrt{5}} e_4$ (odd $\frac{5}{3}$ of minuses); $\sqrt{5} e_1 \pm e_4$; $\sqrt{5} e_2 \pm e_4$; $\sqrt{5} e_3 \pm e_4$ | 1/5 | 4 | 25 |
|---|---|---|---|---|---|
| 9 | $(\mathcal{E}_8, D_4)$ | $\mathcal{F}_4$ | 4/3, 1/6 | 4 | 24 |
| 10 | $(\mathcal{E}_8, A_5)$ | $e_1 + e_2 \pm \sqrt{6} e_3$; $e_1 + e_2$ | 1/6 | 3 | 13 |
| | | $e_1 - e_2 \pm \frac{2 \sqrt{6}}{3} e_3$; $\sqrt{6} e_1 \pm e_3$; $\sqrt{6} e_2 \pm e_3$ | 1/5/2 | 7/2 | 13 |
| 11 | $(\mathcal{E}_8, D_5)$ | $e_1 \pm e_2 \pm e_3$ | 2 | 3 | 13 |
| | | $e_1 \pm e_3$; $e_1 \pm e_2$; $e_2 \pm e_3$; $e_1, e_2, e_3$ | 1/8 | 5/4 | 13 |
| 12 | $(\mathcal{E}_8, A_3)$ | $e_1, e_2$ | 15/2 | 2 | 8 |
| | | $e_1 \pm 2 e_2$; $e_1 \pm \frac{1}{2} e_2$ | 1 | 4 |
| | | $e_1 \pm e_2$ | 1 | 4 |
| 13 | $(\mathcal{E}_8, A_6)$ | $e_1$ | 1/7 | 2 | 6 |
| | | $e_2$ | 6 | 6 |
| | | $\sqrt{7} e_1 \pm 3 e_2$ | 1 | 3 |
| | | $\sqrt{7} e_1 \pm e_2$ | 3 | 4 |
| 14 | $(\mathcal{E}_8, D_6)$ | $B_2$ | 33/10, 6/5 | 2 | 4 |
| 15 | $(\mathcal{E}_8, G_2)$ | $G_2$ | 23/12, 1/12 | 2 | 6 |
| 16 | $(\mathcal{E}_7, A_1)$ | $\sqrt{2} e_6 \pm \sqrt{2} e_5 \pm e_1 \pm e_2 \pm e_3 \pm e_4$ (odd $\frac{5}{3}$ of minuses in the last four terms); $e_1 \pm e_j$ ($1 \leq i < j \leq 4$); $e_5, e_6$ $\sqrt{2} e_6 \pm e_1 \pm e_2 \pm e_3 \pm e_4$ (even $\frac{5}{3}$ of minuses); $\sqrt{2} e_1 \pm e_5$ ($1 \leq i \leq 4$) | 1/2 | 6 | 46 |
| 17 | $(\mathcal{E}_7, A_1)$ | $\sqrt{2} e_5 \pm e_1 \pm e_2 \pm e_3 \pm e_4$ $e_1 \pm e_j$ ($1 \leq i < j \leq 4$); $e_5$ $e_1, e_2, e_3, e_4$ | 1 | 1/2 | 5 | 33 |
| 18 | $(\mathcal{E}_7, A_2)$ | $\pm e_1 \pm e_2 \pm e_3 \pm \sqrt{3} e_4 + \sqrt{2} e_5$ (odd $\frac{5}{3}$ of minuses); $e_1 \pm e_2$; $e_1 \pm e_3$; $e_2 \pm e_3$; $e_5$ | 1/3 | 5 | 30 |
| n | \(\mathbb{E}_7, \mathbb{A}^3\) | \(\mathbb{F}_4\) | 2, 1/2 | 4 | 24 |
|---|---|---|---|---|---|
| 19 | \(\mathbb{E}_7, \mathbb{A}^3\) \(_1\) | \(\mathbb{A}^1, \mathbb{A}^3\) | \(\pm e_1 \pm e_2 \pm e_3 \pm \sqrt[3]{e_1} + \sqrt{2}e_2\) (even \(\eta\) of minuses); \(\sqrt{3}e_1 \pm e_4; \sqrt{3}e_2 \pm e_4; \sqrt{3}e_3 \pm e_4; e_4\) | 1 | 2, 1/2 | 4 | 24 |
| 20 | \(\mathbb{E}_7, \mathbb{A}^3\) \(_2\) | \(\mathbb{A}^1, \mathbb{A}^3\) | \(e_1 \pm e_3 \pm e_4; e_1 \pm e_2 \pm e_4; e_2 \pm e_3 \pm e_4\) | 1 | 2 | 1/2 | 9/2 |
| 21 | \(\mathbb{E}_7, \mathbb{A}^3\) | \(\mathbb{A}^3\) | \(e_1 \pm e_2 \pm e_3 \pm \sqrt{2}e_4\) | 1 | 1/4 | 3/2 | 4 | 18 |
| 22 | \(\mathbb{E}_7, \mathbb{A}^3\) | \(\mathbb{A}^1\) | \(e_1 \pm e_2 \pm e_3\) | 1 | 2 | 9/2 | 3 | 13 |
| 23 | \(\mathbb{E}_7, \mathbb{A}^3\) \(_2\) | \(\mathbb{A}^1\) | \(\sqrt{3}(e_1 - e_2) \pm \sqrt{2}e_3; e_3\) \(e_1 + 3e_2 \pm \sqrt{6}e_3; \sqrt{3}e_1 \pm e_2\) | 1/3 | 1 | 3 | 13 |
| 24 | \(\mathbb{E}_7, \mathbb{A}^3\) \(_1\) | \(\mathbb{A}^1\) | \(\sqrt{2}e_2 \pm \sqrt{2}e_3; e_3\) \(e_1 + \sqrt{2}e_2 \pm \sqrt{2}e_3; 3e_1 \pm e_2\) \(e_1 - \frac{3}{\sqrt{2}}e_2 \pm \sqrt{2}e_3; e_2\) | 1/5 | 1 | 2 | 10 |
| 25 | \(\mathbb{E}_7, \mathbb{D}^1\) | \(\mathbb{B}_4\) | \(4/3, 1/6\) | 3 | 9 |
| 26 | \(\mathbb{E}_7, \mathbb{D}^1\) | \(\mathbb{E}_6\) | \(e_1 \pm \sqrt{2}e_2\) | 2 | 5/4 | 1/8 | 2 | 4 |
| 27 | \(\mathbb{E}_7, \mathbb{A}^3\) \(_1\) | \(\mathbb{C}_2\) | \(e_1 \pm \sqrt{2}e_2\) | 1, 5/2, 7/2 | 2 | 4 |
| 28 | \(\mathbb{E}_7, \mathbb{A}^3\) \(_2\) | \(\mathbb{D}_2\) | \(5/2, 1/6\) | 2 | 6 |
| 29 | \(\mathbb{E}_6, \mathbb{A}^1\) | \(\mathbb{A}^1\) | \(\pm e_1 \pm e_2 \pm e_3 \pm \sqrt{2}e_1 \pm \sqrt{3}e_5\) (even \(\eta\) of minuses); \(e_1 \pm e_2; e_1 \pm e_3; e_2 \pm e_3; e_4\) \(\pm e_1 \pm e_2 \pm e_3 \pm \sqrt{3}e_5\) (odd \(\eta\) of minuses); \(\sqrt{2}e_1 \pm e_4; \sqrt{2}e_2 \pm e_4; \sqrt{2}e_3 \pm e_4\) | 1/2 | 5 | 25 |
| 30 | \(\mathbb{E}_6, \mathbb{A}^1\) | \(\mathbb{A}^3\) \(_1\) | \(e_1 \pm e_2 \pm e_3 \pm \sqrt{3}e_4\) | 1 | 1/2 | 2 | 4 | 17 |
| 31 | \(\mathbb{E}_6, \mathbb{A}^3\) | \(\mathbb{A}^1\) \(_2\) | \(e_1 \pm e_2 \pm \sqrt{3}e_3 + \sqrt{3}e_4\) (even \(\eta\) of minuses); \(e_1 \pm e_2\) | 1/3 | 4 | 15 |
The restrictions of Coxeter root systems appear also, in particular, in the context of \(\vee\)-systems [34]. We note that the number of vectors in the \(\vee\)-system \((\mathcal{H}_4, A_1)\) is stated inaccurately in [34] as some of the vectors listed there are actually proportional. We also refer to [36] where, in particular, bases of the restricted Weyl root systems are discussed.

The CM systems corresponding to the Coxeter restrictions \((\mathcal{E}_7, D_6), (\mathcal{E}_6, A_4)\)
belong to the one-parametric family of two-dimensional integrable CM systems introduced and studied in [35]. The system 12 belongs to the family from Proposition 7 and the system 22 belongs to the family from Theorem 9.

We note that in the table above in all the cases there are non-integer multiplicities. So all integrable systems with integer multiplicities which are restrictions of the CM systems with Coxeter root systems already appeared in [11]. Also according to the result from [37] the group generated by reflections along the hyperplanes with non-integer multiplicities is finite. This agrees with the table above.

7 Invariant parabolic ideals for the group $G(m, p, N)$

Consider the complex reflection group $G(m, p, N)$ and its natural action in $\mathbb{C}^N$. Recall that the group $G(m, p, N)$ defined when $p|m$ is generated by the elements $s_{ij}^k$ for $1 \leq i < j \leq N$, $k = 0, \ldots, m - 1$ and the elements $\tau_i$ for $i = 1, \ldots, N$. The element $\tau_i$ acts on the basic vectors as $\tau_i(e_i) = \eta e_i$, where $\eta = e^{2\pi i p/m}$ and $\tau_i(e_j) = e_j$ for $j \neq i$. The elements $s_{ij}^k$ defined for $i \neq j$ act as $s_{ij}^k(e_j) = \xi^{-k}e_i$, $s_{ij}^k(e_i) = \xi^k e_j$, where $\xi = e^{2\pi i/m}$, and $s_{ij}^k(e_s) = e_s$ for $s \neq i, j$.

The complex reflections $s_{ij}^k$ are reflections of order 2 at the hyperplanes $x_i - \xi^k x_j = 0$. The complex reflections $\tau_i$ are reflections of order $m/p$ at the hyperplanes $x_i = 0$.

The Dunkl operators for the complex reflection groups were introduced in [38]. In the case of the group $G(m, p, N)$ they depend on $m/p$ complex parameters $c_0, \ldots, c_{m-1}$ and have the form

$$\nabla_i = \partial_i - c_0 \sum_{j=1, j \neq i}^{N} \sum_{k=0}^{m-1} \frac{1 - s_{ij}^k}{x_i - \xi^k x_j} - \sum_{t=1}^{m-1} c_t \sum_{s=0}^{m-1} \frac{\eta^{-st} \tau_i^s}{x_i},$$

(28)

$i = 1, \ldots, N$. The commutativity $[\nabla_i, \nabla_j] = 0$ holds.

By a parabolic stratum for the group $G(m, p, N)$ we will mean the $G(m, p, N)$-orbit of an intersection of the reflection hyperplanes $x_i - \xi^k x_j = 0$, $x_s = 0$. We are interested in the parabolic strata such that the ideals of polynomials vanishing on these strata (parabolic ideals) are invariant under the Dunkl operators (28).
Consider the plane \( \pi_{q,r}^\varepsilon \subset \mathbb{C}^N \) given by the equations
\[
\begin{align*}
x_1 = x_2 = \ldots = x_r, & \quad x_{r+1} = x_{r+2} = \ldots = x_{2r}, \\
\vdots & \\
\varepsilon x_{(q-1)r+1} = x_{(q-1)r+2} = \ldots = x_{qr},
\end{align*}
\]
where \( \varepsilon^n = 1 \). We may assume that \( \varepsilon = 1 \) unless \( qr = N \). The corresponding parabolic stratum \( D_{q,r}^\varepsilon \subset \mathbb{C}^N \) is the orbit of the plane \( \pi_{q,r}^\varepsilon \) under the group \( G(m, p, N) \):
\[
D_{q,r}^\varepsilon = \bigcup_{w \in G(m,p,N)} w(\pi_{q,r}^\varepsilon).
\]

**Proposition 8** The parabolic ideal
\[
I_{q,r}^\varepsilon = \{ f \in \mathbb{C}[x_1, \ldots, x_N] \mid f|_{D_{q,r}^\varepsilon} = 0 \}
\]
corresponding to the stratum (29), (30) is invariant under the group \( G(m, p, N) \) Dunkl operators (28) if and only if \( c_0 = 1/r \).

**Proof.** Let \( f \in I_{q,r}^\varepsilon \). We analyze first the condition \( \nabla_1 f \in I_{q,r}^\varepsilon \). We have
\[
\nabla_1 f|_{\pi_{q,r}^\varepsilon} = \partial_1 f|_{\pi_{q,r}^\varepsilon} - c_0 \sum_{i=2}^r \frac{(1 - s_{1i})f}{x_1 - x_i}|_{\pi_{q,r}^\varepsilon}.
\]
Since \( f \) vanishes on \( \pi_{q,r}^\varepsilon \), we can represent polynomial \( f \) as
\[
f = \sum_{i=2}^r (x_1 - x_i)f_i + \sum_{i=r+2}^{r+2} (x_{r+1} - x_i)f_i + \ldots + \sum_{i=(q-1)r+2}^{qr} (\varepsilon x_{(q-1)r+1} - x_i)f_i
\]
for some polynomials \( f_i \). Then we have
\[
\nabla_1 f|_{\pi_{q,r}^\varepsilon} = \sum_{i=2}^r f_i|_{\pi_{q,r}^\varepsilon} - c_0 \sum_{i=2}^r \frac{1 - s_{1i}}{x_1 - x_i} \sum_{j=2}^r (x_1 - x_j)f_j|_{\pi_{q,r}^\varepsilon}.
\]
Notice that on the subspace \( \pi_{q,r}^\varepsilon \) we have
\[
\frac{1 - s_{1i}}{x_1 - x_i}(x_1 - x_j)f_j|_{\pi_{q,r}^\varepsilon} = f_j|_{\pi_{q,r}^\varepsilon}, \quad \frac{1 - s_{1i}}{x_1 - x_i}(x_1 - x_i)f_i|_{\pi_{q,r}^\varepsilon} = 2f_i|_{\pi_{q,r}^\varepsilon}
\]
\( (31) \)}
where $2 \leq i \neq j \leq r$. Therefore

$$\nabla_1 f|_{\pi_{q,r}} = (1 - c_0 r) \sum_{i=2}^{r} f_i|_{\pi_{q,r}},$$

and the last expression vanishes if and only if $c_0 = 1/r$.

We also need to verify that $\nabla_1 f = 0$ on other subspaces $\pi \neq \pi_{q,r}$ from $D_{q,r}$ when $c_0 = 1/r$. In the case when $e_1 \notin \pi$ the linear equations defining $\pi$ involve coordinate $x_1$ and the arguments are similar to the considered case. The only difference is in the relations (31) which in general take the form

$$\frac{1 - s^k_{ij}}{x_1 - \xi^k x_i} (x_1 - \xi^a x_j) f_j|_{\pi} = f_j|_{\pi}, \quad \frac{1 - s^k_{ij}}{x_1 - \xi^k x_i} (x_1 - \xi^k x_i) f_i|_{\pi} = 2 f_i|_{\pi}$$

where $i \neq j$ and coordinates of the vectors in $\pi$ satisfy the equations $x_1 = \xi^k x_i = \xi^a x_j$. If $e_1 \in \pi$ then it is easy to see that $\nabla_1 f|_{\pi} = 0$.

Invariance of the ideal $I_{q,r}$ under the Dunkl operators $\nabla_i$ with $i > 1$ is analyzed similarly and it leads to the same condition $c_0 = 1/r$. The statement is proven.

Consider now the plane $\pi_l \subset \mathbb{C}^N$ given by the equations $x_1 = x_2 = \ldots = x_l = 0$. Consider the corresponding parabolic stratum $D_l$ which is the orbit of the plane $\pi_l$ under the group $G(m, p, N)$:

$$D_l = \bigcup_{i_1 < \ldots < i_l} \{x | x_{i_1} = \ldots = x_{i_l} = 0\}. \quad (33)$$

Define the ideal $I_l = \{f \in \mathbb{C}[x_1, \ldots, x_N] | f|_{D_l} = 0\}$.

**Proposition 9** The Dunkl operators (28) for the group $G(m, m, N)$ preserve the ideal $I_l$ if and only if $c_0 = \frac{1}{m(l-1)}$. The Dunkl operators (28) for the group $G(m, p, N)$ with $p < m$ preserve the ideal $I_l$ if and only if $(l-1)c_0 + cp^{-1} = m^{-1}$.

**Proof.** Let $f \in I_l$. We analyze first the condition $\nabla_1 f \in I_l$. Since polynomial $f$ vanishes on the plane $\pi_l : x_1 = x_2 = \ldots = x_l = 0$ we can represent $f$ as $f = \sum_{i=1}^{l} x_i f_i$ for some polynomials $f_i$. In order to compute $\nabla_1 f$ we note at first that for $2 \leq j \leq l$

$$\left(\frac{1 - s^k_{ij}}{x_1 - \xi^k x_j} \sum_{i=1}^{l} x_i f_i\right)|_{\pi_l} = f_1|_{\pi_l} - \xi^{-k} f_j|_{\pi_l},$$
and that \( \sum_{k=0}^{m-1} \xi^{-k} = 0 \), \( \sum_{s=0}^{m-1} \eta^{-st} = 0 \) for \( m \) \( - \) \( 1 \) \( > \) \( t \) \( \geq 1 \). Then it follows that
\[
\nabla_1 f|_{\pi_l} = (1 - c_0 m(l - 1) - c_1 \frac{m}{p}) f_1|_{\pi_l},
\]
where we assume that \( c_1 = 0 \) for the group \( G(m, m, N) \). Therefore \( \nabla_1 f|_{\pi_l} = 0 \) iff \( m^{-1} = c_0(l - 1) + c_1 p^{-1} \). Property that \( \nabla_1 f \) vanishes on other subspaces \( \pi \) from \( D_l \) is either satisfied for all values of parameters (when \( e_1 \in \pi \)) or satisfied under the same relation among the parameters as for the plane \( \pi_l \) (when vector \( e_1 \) is orthogonal to the subspace \( \pi \)). This shows that invariance of the ideal \( I_l \) under the Dunkl operator \( \nabla_1 \), as well as under all operators \( \nabla_i \), is equivalent to the condition \( m^{-1} = c_0(l - 1) + c_1 p^{-1} \) as stated.

Propositions \( \text{x} \) \( \text{[9]} \) and their proofs imply the following

**Theorem 10** All possible parabolic strata defining parabolic ideals invariant under the \( G(m, p, N) \) Dunkl operators \( \text{(28)} \) are either the strata \( D_{q,r} \) defined by \( \text{(10)} \) with \( c_0 = 1/r \) or strata \( D_l \) defined by \( \text{(18)} \) with parameters values specified in Proposition \( \text{x} \) or the strata \( D_{q,r,l} \). The latter stratum is the \( G(m,p, N) \) - orbit of the subspace
\[
x_1 = x_2 = \ldots = x_r, \quad x_{r+1} = x_{r+2} = \ldots = x_{2r}, \\
\ldots \\
x_{(q-1)r+1} = x_{(q-1)r+2} = \ldots = x_{qr}, \\
x_{qr+1} = \ldots = x_{qr+l} = 0, \quad (34)
\]
where \( r > 1 \), \( l \geq 1 \), \( qr + l \leq N \). The corresponding parameters satisfy \( c_0 = 1/r \), \( (l - 1)c_0 + \frac{m}{p} = 1/m \), where it is assumed that \( c_1 = 0 \) in the case of the group \( G(m, m, N) \).

To conclude the consideration of the invariant parabolic ideals for the complex reflection group \( G(m, p, N) \) we note that when \( N = 2 \) and \( p \) is even the roots \( e_1 - \xi^s e_2 \), \( s = 0, \ldots, m - 1 \) form two \( G(m, p, N) \) - orbits. This adds extra parameter \( \tilde{c}_0 \) to the associated Dunkl operators comparing to the general case \( \text{(28)} \). Namely the operators take the form
\[
\nabla_1 = \partial_1 - c_0 \sum_{k=0}^{m-1} \frac{x_1 - \xi^{2k} x_2 \eta^{-st} \tau_1^s}{x_1 - \xi^{2k+1} x_2}, \\
\nabla_2 = \partial_2 - c_0 \sum_{k=0}^{m-1} \frac{x_2 - \xi^{2k} x_1 \eta^{-st} \tau_1^s}{x_1 - \xi^{2k+1} x_2}. \quad (35)
\]
Then we have the following four parabolic ideals

\[ I_1 = \{ f \in \mathbb{C}[x_1, x_2] | f = 0 \text{ if } x_1 = \xi^{2k}x_2 \quad \forall k = 0, \ldots, \frac{m}{2} - 1 \}, \]

\[ I_2 = \{ f \in \mathbb{C}[x_1, x_2] | f = 0 \text{ if } x_1 = \xi^{2k+1}x_2 \quad \forall k = 0, \ldots, \frac{m}{2} - 1 \}, \]

\[ I_3 = \{ f \in \mathbb{C}[x_1, x_2] | f = 0 \text{ if } x_1x_2 = 0 \}, \quad I_4 = \{ f \in \mathbb{C}[x_1, x_2] | f = 0 \text{ if } x_1 = x_2 = 0 \}. \]

The following Proposition can be established similarly to the previous results of this Section.

**Proposition 10** The parabolic ideal \( I_1 \) is invariant under the Dunkl operators (35) iff \( c_0 = 1/2 \), the parabolic ideal \( I_2 \) is invariant under the Dunkl operators (35) iff \( \tilde{c}_0 = 1/2 \), the parabolic ideal \( I_3 \) is invariant under the Dunkl operators (35) iff \( c_1 = p/m \), the parabolic ideal \( I_4 \) is invariant under the Dunkl operators (35) iff \( \frac{1}{2}(c_0 + \tilde{c}_0) + c_1/p = 1/m \).

**Concluding Remarks**

In the paper we were systematically deriving generalized CM systems from the special subrepresentations in the polynomial representation of the rational Cherednik algebra. The natural development would be to extend this approach to obtain integrable generalizations of the Calogero-Moser-Sutherland and Ruijsenaars-Macdonald systems starting from less degenerate Cherednik algebras. Some integrable generalizations of these systems are known from [10], [14], [40], [41].

It would also be interesting to analyze if the approach can be extended to cover matrix integrable systems of the generalized Calogero-Moser type ([42]). A close direction is to investigate possible generalizations of matrix CMS systems associated to the complex reflection group \( G(m, p, N) \) introduced in [43].

Regarding representations of the rational Cherednik algebras, it is clear by Theorem 1 how to form the chains of submodules in the polynomial module of the rational Cherednik algebras for any Coxeter group. It would be interesting to investigate if the corresponding factors are irreducible like in the \( A_N \) case ([22], [39]). Also in the paper we consider submodules corresponding to special singular values only, it would also be interesting to see if
the submodules for other singular values can be described in a natural way (c.f. Proposition [1] for the submodule of polynomial vanishing on the corank 1 spaces).

We are going to return to some of the above questions soon.

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References

[1] F. Calogero Solution of a three-body problem in one dimension, J. Math.Phys., 1969, V.10. N. 12. pp. 2191–2196.

[2] F. Calogero Solution of the one-dimensional n-body problem with quadratic and/or inversely quadratic pair potential, J. Math.Phys., 1971, V. 12. pp. 419–436.

[3] J. Moser Three integrable hamiltonian systems connected with isospectral deformation, Adv. Math., 1975, V. 16, pp. 197–220.

[4] M.A. Olshanetsky, A.M. Perelomov Quantum completely integrable systems connected with semi-simple Lie algebras, Lett. Math. Phys., 1977, V.2, pp. 7–13.

[5] G.J. Heckman A remark on the Dunkl differential-difference operators, Prog. in Math., 1991, V. 101. pp. 181–191.

[6] C.F. Dunkl Differential–difference operators associated to reflection groups, Trans. AMS, 1989, V. 311, pp. 167–183.

[7] P. Etingof, V. Ginzburg Symplectic reflection algebras, Calogero–Moser space, and deformed Harish–Chandra homomorphism, Invent. Math., 2002, V. 147, pp.243–348.
[8] P. Etingof *Calogero-Moser systems and representation theory*, Zurich Lectures in Advanced Mathematics, 2007.

[9] A.P. Veselov, M.V. Feigin, O.A. Chalykh *New integrable deformations of quantum Calogero–Moser problem*, Russian Math Surveys, 1996, V. 51, N.3. pp. 185–186.

[10] O.A. Chalykh, M.V. Feigin, A.P. Veselov *New integrable generalisations of Calogero–Moser quantum problem*, J. of Math. Phys, 1998, V. 39, N.2, pp. 695–703.

[11] O.A. Chalykh, M.V. Feigin, A.P. Veselov *Multidimensional Baker–Akhiezer Functions and Huygens’ Principle*, Commun. Math. Phys., 1999, V. 206, pp. 533–566.

[12] A.N. Sergeev *Superanalogs of the Calogero operators and Jack polynomials*, J. Nonlinear Math. Phys., 2001, V.8, no. 1, pp. 59–64.

[13] A.N. Sergeev *Calogero operator and Lie superalgebras*, Theor. Math. Phys., 2002, V.131 (3), pp. 747–764.

[14] A.N. Sergeev, A.P. Veselov *Deformed quantum Calogero-Moser problems and Lie superalgebras* Comm. Math. Phys., 2004, V. 245, no. 2, 249–278.

[15] A.N. Sergeev, A.P. Veselov *Generalised discriminants, deformed quantum Calogero-Moser system and Jack polynomials*, Adv. Math., 2005, V. 192, no. 2, 341–375.

[16] A.N. Sergeev, A.P. Veselov *BC-infinity Calogero-Moser operator and super Jacobi polynomials* [arXiv:0807.3858]

[17] M. Hallnäs, E. Langmann *Quantum Calogero-Sutherland type models and generalised classical polynomials*, arXiv:math-ph/0703090

[18] M. Hallnäs *A basis for the polynomial eigenfunctions of deformed Calogero-Moser-Sutherland operators*, [arXiv:0712.1496]

[19] C.F. Dunkl, M.F. de Jeu, E.M. Opdam *Singular polynomials for finite reflection groups*, Trans. Amer. Math. Soc., 1994, V. 346, no. 1, pp.237–256.
[20] P. Etingof *Reducibility of the polynomial representation of the degenerate double affine Hecke algebra*, arXiv:0706.4308

[21] I. Cherednik *Non-semisimple Macdonald polynomials*, arXiv:0709.1742

[22] M. Kasatani *Subrepresentations in the Polynomial Representation of the Double Affine Hecke Algebra of type GL_n at t^{k+1}q^{r-1} = 1*, arXiv:math/0501272

[23] M. Kasatani *The polynomial representation of the double affine Hecke algebra of type (C\_n, C\_n) for specialized parameters*, arXiv:0807.2714

[24] C.F. Dunkl *Singular polynomials and modules for the symmetric groups*, Int. Math. Res. Not., 2005, no. 39, pp.2409–2436.

[25] B. Feigin, M. Jimbo, T. Miwa, E. Mukhin *A differential ideal of symmetric polynomials spanned by Jack polynomials at \( \beta = -(r-1)/(k+1) \)*, Intern. Math. Res. Notices, 2002, no.23, pp.1223–1237.

[26] B. Feigin, M. Jimbo, T. Miwa, E. Mukhin *Symmetric polynomials vanishing on the shifted diagonals and Macdonald polynomials*, Intern. Math. Res. Notices, 2003, no.18, pp.1015–1034.

[27] V. Ginzburg *On primitive ideals* Selecta Math. (N.S.), 2003, V. 9 (3), pp.379–407.

[28] R. Bezrukavnikov, P. Etingof *Parabolic induction and restriction functors for rational Cherednik algebras*, arXiv:0803.3639

[29] J.E. Humphreys *Reflection groups and Coxeter groups*, CUP, 1990

[30] P. Orlik, L. Solomon *Coxeter arrangements*, Proc. of Symp. in Pure Mathematics, 1983, V.40 (2), pp.269–291.

[31] W. Rühle, A. Turbiner *Exact solvability of the Calogero and Sutherland models*, Modern. Phys. Lett. A, 1995, V.10 (29), pp.2213–2221.

[32] A.P. Polychronakos *Exchange operator formalism for integrable systems of particles*, Phys. Rev. Letters, 1992, V.69 (5), pp. 703–705.

[33] O.A. Chalykh *Private communication*
[34] M. Feigin, A.P. Veselov *Logarithmic Frobenius structures and Coxeter discriminants*, Adv. Math., 2007, V. 212, no. 1, pp.143–162.

[35] K. Taniguchi *Deformation of two body quantum Calogero-Moser-Sutherland models*, arXiv:math-ph/0607053

[36] J. Brundan, S.M. Goodwin *Good grading polytopes*, Proc. Lond. Math. Soc., 2007, V.94 (3), no. 1, pp.155–180.

[37] K. Taniguchi *On the symmetry of commuting differential operators with singularities along hyperplanes* Int. Math. Res. Not., 2004, no. 36, pp.1845–1867.

[38] C.F. Dunkl, E.M. Opdam *Dunkl operators for complex reflection groups* Proc. London Math. Soc., 2003, V.86, pp.70–108.

[39] N. Enomoto *Composition Factors of Polynomial Representation of DAHA and Crystallized Decomposition Numbers*, arXiv:math/0604368

[40] O.A. Chalykh *Bispectrality for the quantum Ruijsenaars model and its integrable deformation*, J. Math. Phys. 2000. V. 47, pp. 5139–5167.

[41] M. Feigin *Bispectrality for deformed Calogero-Moser-Sutherland systems*, Journal of Nonlin. Math. Phys., 2005, V.12,sup.2, pp. 95-136.

[42] O.A.Chalykh, V.M.Goncharenko, A.P.Veselov *Multidimensional integrable Schrödinger operators with matrix potential*, J. Math. Phys. 1999, V.40, no.11, pp. 5341-5355.

[43] N. Crampe, C. A. S. Young *Sutherland Models for Complex Reflection Groups*, Nucl. Phys. B 2008, V. 797, pp. 499-519.