Limits of the Banach spaces associated with positive operator $a$ affiliated with von Neumann algebra, which are neither purely projective nor purely inductive.

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February 12, 2019

Abstract

We consider linear normed spaces of operators dominated by positive operator affiliated with the von Neumann algebra powered by real positive parameter. We consider and define different natural constructions of the limits spaces, based on projective and injective limits of the chains of linear spaces, but that are more complicated.

keywords: inductive limit, projective limit, power parameter, (LB)-space, (LF)-space, Frechet space, locally convex space, order unit base norm, inductive limit, initial topology, final topology, order unit space, measurable functions, Banach space

Introduction

In the article [8] we have defined the space $L_\infty(a)$ associated with positive operator affiliated with the von Neu-
mann algebra. Further in [4, 6] we have considered the commutative constructions of the limits spaces $L_\infty(f^\alpha)$, $L_1(f^\alpha)$ and $L_\infty^*(f^\alpha)$, and found that they are total for each other in the dualities $\langle \lim L_1(f^\alpha), \lim L_\infty(f^\alpha) \rangle$ and $\langle \lim L_\infty(f^\alpha), \lim L_\infty^*(f^\alpha) \rangle$. In this work we start to apply the same methodology for the noncommutative $L_\infty(a)$ spaces.

In the result we get the (LF)-spaces (the inductive limits of Frechet spaces), which are studied for example in [1, 2, 3, 5].

1 Definitions and Notation

Throughout this paper we adhere to the following notation. By $\mathcal{M}$ we denote a von Neumann algebra that acts on a Hilbert space $H$ with the scalar product $\langle \cdot, \cdot \rangle$. We denote its selfadjoint part by $\mathcal{M}^{sa}$, and the set of all projections in $\mathcal{M}$ by $\mathcal{M}^{pr}$. Let $p \in \mathcal{M}^{pr}$ and $x \in \mathcal{M}$, then by $x_p$ we denote the restriction of $pxp$ to $pH$ (i.e. $x_p := pxp|_{pH}$), also we denote the reduction of $\mathcal{M}$ to $pH$ by $\mathcal{M}_p$. By $\mathcal{C}(\mathcal{M})$ we denote the center of $\mathcal{M}$. By $\mathcal{M}^*$ and $\mathcal{M}^{h*}$ we denote the predual of $\mathcal{M}$ and its Hermitian part, respectively. If an operator $x$ is affiliated with $\mathcal{M}$ then we write $x_\eta \mathcal{M}$. We denote the domain of an operator $x$ by $D(x)$. The adjoint operator is denoted by $x^*$. We denote the identity operator, the zero operator and the
zero vector by $1, 0$ and $0$, respectively. We use standard notation for multiplication of a functional $\varphi \in \mathcal{M}^*$ by an operator $x \in \mathcal{M}$, namely, $x \varphi, \varphi x$ and $x \varphi x$ denote the linear functionals $y \mapsto \varphi(xy), y \mapsto \varphi(yx)$ and $y \mapsto \varphi(axy)$, respectively.

We also consider partial order for positive selfadjoint operators affiliated with $\mathcal{M}$. If $x$ is affiliated with $\mathcal{M}$ we denote it as $x \eta \mathcal{M}$. For positive selfadjoint $x, y \eta \mathcal{M}$ we write $x \leq y$ if and only if

$$D(y^{\frac{1}{2}}) \subset D(x^{\frac{1}{2}}) \text{ and } \|x^{\frac{1}{2}} f\|^2 \leq \|y^{\frac{1}{2}} f\|^2 \text{ for all } f \in D(y^{\frac{1}{2}}).$$

If for an increasing net $(x_j)_{j \in J}$ of operators affiliated with $\mathcal{M}$ there exists $x = \sup_{j \in J} x_j$, then we write $x_j \nearrow x$. For a positive selfadjoint operator $x \eta \mathcal{M}$ we use $x_\lambda$ to denote $\lambda x(\lambda + x)^{-1}$ with $\lambda \in \mathbb{R}^+ \setminus \{0\}$. From the Spectral theorem it follows that the mapping $\lambda \mapsto x_\lambda \in \mathcal{M}^+$ is monotone operator-valued function and $\lim_{\lambda \to +\infty} x_\lambda^\frac{1}{2} f = x^\frac{1}{2} f$ for all $f \in D(x^\frac{1}{2})$, therefore $x_\lambda \nearrow x$. For an unbounded $x$ and $\varphi \in \mathcal{M}_*^+$ we define $\varphi(x)$ as $\varphi(x) := \lim_{\lambda \to +\infty} \varphi(x_\lambda)$.

From now on $a$ stands for a positive selfadjoint operator affiliated with $\mathcal{M}$. We consider

$$\mathfrak{D}_a^+ \equiv \{ \varphi \in \mathcal{M}_* \mid \varphi(a) < +\infty \},$$

$$\mathfrak{D}_a^h \equiv \mathfrak{D}_a^+ - \mathfrak{D}_a^+ \text{ and } \mathfrak{D}_a \equiv \operatorname{lin}_{\mathbb{C}} \mathfrak{D}_a^h.$$ Note that if operator $a$ is bounded, then $\mathfrak{D}_a^+ = \mathcal{M}_*^+, \mathfrak{D}_a^h = \mathcal{M}_*^h$. 

\( \mathcal{M}_*^h \) and \( \mathfrak{D}_a = \mathcal{M}_* \). We define a seminorm \( \| \cdot \|_a \) on \( \mathfrak{D}_a^h \) as
\[
\| \varphi \|_a := \inf \{ \varphi_1(a) + \varphi_2(a) \mid \varphi = \varphi_1 - \varphi_2; \varphi_1, \varphi_2 \in \mathfrak{D}^+ \}.
\]

Also, Theorem 2 from [7] states, that if operator \( a \) is bounded, then
\[
\| \varphi \|_a = \| a^{\frac{1}{2}} \varphi a^{\frac{1}{2}} \|.
\]
If \( \| \cdot \|_a \) is a norm, then we call it the \( a \)-norm. Note that the \( 1 \)-norm coincides with the restriction of the standard norm in \( \mathcal{M}_* \) onto \( \mathcal{M}_*^h \).

For an injective operator \( a \) by \( L_1^h(a) \) we denote the completion of the normed space \( (\mathfrak{D}_a^h, \| \cdot \|_a) \). The dual of \( L_1^h(a) \) is \( (L_{\infty}^a(a), \| \cdot \|_a) \), where
\[
L_{\infty}(a) \equiv \{ x \in (\mathfrak{D}_a)^{\text{al}} \mid \lambda \in \mathbb{R}, -\lambda a \leq x \leq \lambda a \}
\]
and \( \| x \|_a \equiv \inf \{ \lambda \in \mathbb{R} \mid -\lambda a \leq x \leq \lambda a \} \). We identify the elements of \( \mathfrak{D}_a^h \) with the corresponding elements in \( L_1^h(a) \). Further for an injective operator \( a \) we always assume that \( L_{\infty}^a(a) \) is equiped with the \( a \)-norm.

For \( x \in \mathcal{M} \) we define the sesquilinear form a \( a^{\frac{1}{2}} xa^{\frac{1}{2}} \) on \( D(a^{\frac{1}{2}}) \times D(a^{\frac{1}{2}}) \) by the equality \( a^{\frac{1}{2}} xa^{\frac{1}{2}}(f,g) := \langle xa^{\frac{1}{2}}f, a^{\frac{1}{2}}g \rangle \).
The set of all such sesquilinear forms is denoted by
\[
S_a(\mathcal{M}) \equiv \{ a^{\frac{1}{2}} xa^{\frac{1}{2}} \mid x \in \mathcal{M} \}.
\]
We consider partial order on \( S_a(\mathcal{M}^{sa}) \), such that \( a^{\frac{1}{2}} xa^{\frac{1}{2}} \leq a^{\frac{1}{2}} ya^{\frac{1}{2}} \) if and only if \( a^{\frac{1}{2}} xa^{\frac{1}{2}}(f,f) \leq a^{\frac{1}{2}} ya^{\frac{1}{2}}(f,f) \) for all \( f \in D(a^{\frac{1}{2}}) \). By \( S_a(\mathcal{M}^{sa}) \) we denote the seminormed space
of sesquilinear forms \( \{a^{\frac{1}{2}}xa^{\frac{1}{2}} \mid x \in \mathcal{M}^{sa}\} \) equipped with the seminorm \( p_a(a^{\frac{1}{2}}xa^{\frac{1}{2}}) := \inf \{\lambda \in \mathbb{R}^+ \mid -\lambda a^{\frac{1}{2}}1a^{\frac{1}{2}} \leq a^{\frac{1}{2}}xa^{\frac{1}{2}} \leq \lambda a^{\frac{1}{2}}1a^{\frac{1}{2}}\} \).

2 Preliminaries

For \( \varphi \in \mathcal{D}_a \) the equality
\[
a^{\frac{1}{2}}\varphi a^{\frac{1}{2}}(x) = \lim_{\lambda \to +\infty} \varphi(a^{\frac{1}{2}}x a^{\frac{1}{2}}) \text{ with } x \in \mathcal{M}
\]
defines the normal functional \( a^{\frac{1}{2}}\varphi a^{\frac{1}{2}} \in \mathcal{M}^* \).

If an operator \( a \) is injective then
\[
\inf \{\lambda \mid -\lambda a^{\frac{1}{2}}1a^{\frac{1}{2}} \leq a^{\frac{1}{2}}xa^{\frac{1}{2}} \leq \lambda a^{\frac{1}{2}}1a^{\frac{1}{2}}\} = \|x\| \text{ for any } x \in \mathcal{M}^{sa}
\]
and the latter implies that the mapping \( u_1 : x \mapsto a^{\frac{1}{2}}xa^{\frac{1}{2}} \) is an isometrical isomorphism of \( \mathcal{M} \) onto \( \mathcal{S}_a(\mathcal{M}) \).

For an injective operator \( a \) the mapping
\[
u : \varphi \in \mathcal{D}_a (\subset L_1(a)) \mapsto a^{\frac{1}{2}}\varphi a^{\frac{1}{2}} \in \mathcal{M}^*
\]
is an isometrical isomorphism of $L_1(a)$ onto $\mathcal{M}_*$.

### 3 Limits of noncommutative $L_\infty$ spaces

We split this section into four parts. Firstly, we consider general properties for $L_\infty(a)$ spaces and its norms. Secondly, we consider case of bounded $a$, in the third case we consider unbounded $a$ such that $a^{-1}$ is bounded, and at last we consider the general case of unbounded $a$.

#### 3.1 Case of bounded operator

**Lemma 1.** Let $a\mathcal{M}$ ($a$ is affiliated with $\mathcal{M}$, $a \geq 0$), then $S_a(\mathcal{M}) = S_{\lambda a}(\mathcal{M})$ for any $\lambda > 0$.

*Proof.* Let $a^{\frac{1}{2}}xa^{\frac{1}{2}} \in S_a(\mathcal{M})$. Evidently, $D(a^{\frac{1}{2}}) = D((\lambda a)^{\frac{1}{2}}) = D(\lambda^{\frac{1}{2}}a^{\frac{1}{2}})$, thus

\[
\lambda^{\frac{1}{2}}xa^{\frac{1}{2}}(f, g) = \langle xa^{\frac{1}{2}}f, a^{\frac{1}{2}}g \rangle = \langle x\frac{1}{\lambda^{\frac{1}{2}}}(\lambda a)^{\frac{1}{2}}f, \frac{1}{\lambda^{\frac{1}{2}}}(\lambda a)^{\frac{1}{2}}g \rangle =
\]

\[
= \frac{1}{\lambda}\langle (\lambda a)^{\frac{1}{2}}f, (\lambda a)^{\frac{1}{2}}g \rangle = \frac{1}{\lambda}(\lambda a)^{\frac{1}{2}}x(\lambda a)^{\frac{1}{2}}(f, g) \text{ for any } f, g \in D(a^{\frac{1}{2}}).
\]

Hence, $\lambda^{\frac{1}{2}}xa^{\frac{1}{2}} = (\lambda a)^{\frac{1}{2}}x(\lambda a)^{\frac{1}{2}}$, and since $S_a(\mathcal{M})$, $S_{\lambda a}(\mathcal{M})$ are linear spaces, we have $S_a(\mathcal{M}) = S_{\lambda a}(\mathcal{M})$. \hfill $\Box$

Particularly, if $a$ is bounded, then $S_a(\mathcal{M}) = S_{a_0}(\mathcal{M})$, where $a_0 = a/\|a\|$; if $a$ is such that $a^{-1}$ is bounded, then $S_a(\mathcal{M}) = S_{a_\infty}(\mathcal{M})$, where $a_\infty = \|a^{-1}\|a$. 

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**Proposition 1.** Let $a \in M$, $a \geq 0$, then $\| \cdot \|_a$ in $L_\infty(a)$ is equivalent to $\| \cdot \|_{\lambda a}$, where $\lambda > 0$.

**Proof.** Since, $L_\infty(a)$ is isometrically isomorphic to $S_a(M)$ for $x \in L_\infty(a)$ we may consider $x = a^{\frac{1}{2}}ya^{\frac{1}{2}}$, where $y \in M$, then $\|x\|_a = \|y\|$. On the other hand,

$$
\|x\|_{\lambda a} = \|a^{\frac{1}{2}}ya^{\frac{1}{2}}\|_{\lambda a} = \|\frac{\lambda^{1/2}}{\lambda^{1/2}}a^{\frac{1}{2}}ya^{\frac{1}{2}}\|_{\lambda a} = \frac{1}{\lambda} (\lambda a)^{\frac{1}{2}}(\lambda a)^{\frac{1}{2}}\|\lambda a = \frac{1}{\lambda} \|y\| = \frac{1}{\lambda} \|x\|_a
$$

 Particularly, for the bounded $a \in M^+$ we have $\| \cdot \|_a$ is equivalent to $\| \cdot \|_{a_0}$, where $a_0 = a/\|a\|$. If $a^{-1}$ is bounded and we consider $a_\infty = \|a^{-1}\|a$, then $\| \cdot \|_{a_\infty}$ is equivalent to $\| \cdot \|_a$.

Now, consider $L_\infty(a^\alpha)$.

**Lemma 2.** For a bounded $a \in M^+$ and $\alpha, \beta \in \mathbb{R}^+$, if $\alpha < \beta$, then

$$
L_\infty(a^\alpha) \supset L_\infty(a^\beta).
$$

**Proof.** Let $y \in L_\infty(a^\beta)$, then we consider $y = a^{3/2}xa^{3/2}$, where $x \in M$. Thus,

$$
y(f, g) = a^{3/2}xa^{3/2}(f, g) = \langle xa^{3/2}f, a^{3/2}g \rangle =
$$
\[(a^{(\beta-\alpha)/2}xa^{(\beta-\alpha)/2})a^{\alpha/2}f, a^{\alpha/2}g) \] for any \( f, g \in D(a^{1/2}) \).

Note, that \( x^' := a^{(\beta-\alpha)/2}xa^{(\beta-\alpha)/2} \in \mathcal{M} \), thus
\[ y = a^{\alpha/2}x'a^{\alpha/2} \in L_{\infty}(a^{\alpha}). \]

\[ \square \]

The following definition is standard.

**Definition 1.** Let \( X \supset Y \) be normed spaces with the norms \( \| \cdot \|_X, \| \cdot \|_Y \), respectively. We write \( \| \cdot \|_X \prec \| \cdot \|_Y \) if and only if there exists \( C \in \mathbb{R}^+ \), such that for any \( x \in Y \) the inequality \( \| x \|_X \leq C \| x \|_Y \) holds.

**Lemma 3.** For a bounded \( a \in \mathcal{M}^+ \) and \( \alpha, \beta \in \mathbb{R}^+ \). If \( \alpha < \beta \), then
\[ \| \cdot \|_{a^\alpha} \prec \| \cdot \|_{a^\beta}. \]

**Proof.** Let \( y \in L_{\infty}(a^\beta) \), then \( y = a^{\beta/2}xa^{\beta/2} \), where \( x \in \mathcal{M} \), then
\[ \| y \|_{a^\alpha} = \| a^{\beta/2}xa^{\beta/2} \|_{a^\alpha} = \| a^{(\beta-\alpha)/2}xa^{(\beta-\alpha)/2} \| \leq \]
\[ \leq \| a^{(\beta-\alpha)/2} \|^2 \| x \| = \| a^{(\beta-\alpha)} \| \| x \| = \| a^{(\beta-\alpha)} \| \| y \|_{a^\beta} \]

\[ \square \]

Now, let us consider the limit space.

**Definition 2.** For the bounded \( a \in \mathcal{M}^+ \) we define the topological space \( (L_{\infty}(a), \tau(a)) \) as the limit space for \( L_{\infty}(a^\alpha) \), where \( L_{\infty}(a) := \bigcap_{\alpha > 1} L_{\infty}(a^\alpha) \) and \( \tau(a) := \bigcup_{\alpha > 1} \tau_{\infty}(a^\alpha), \tau_{\infty}(a^\alpha) = \)
\{X \cap \mathcal{L}_\infty(a) \mid X \in \tau(a^\alpha)\}, \tau(a^\alpha) is the topology on \(L_\infty(a^\alpha)\) of the norm \(\| \cdot \|_{a^\alpha}\).

The latter definition essentially means, that \(\tau(a)\) is the initial topology on the \(\mathcal{L}_\infty(a)\) for the family of mapping

\[ \varphi_a : x \in \mathcal{L}_\infty(a) \mapsto x \in (L_\infty(a^\alpha), \| \cdot \|_{a^\alpha}). \]

Also, we can describe this topology as the topology on \(\mathcal{L}_\infty(a)\) defined by the family of the seminorms \(\{\| \cdot \|_{a^\alpha}\}_{\alpha > 1}\). The family of the seminorms \(\{\| \cdot \|_{a^\alpha}\}_{n \in \mathbb{N}}\) describes the same topology, thus we get the following theorem.

**Theorem 1.** For the bounded \(a \in \mathcal{M}^+\) the space \((\mathcal{L}_\infty(a), \tau(a))\) is metrizable locally-convex space (Frechet space).

For the more detailed proof of the latter theorem see [6][Lemma 2].

### 3.2 Case of unbounded operator with bounded inverse

Now, consider case, when \(a \in \mathcal{M}, a \geq 0\), \(a\) is not bounded, but \(a^{-1}\) is bounded.

**Lemma 4.** For a \(a \in \mathcal{M}, a \geq 0\) and \(\alpha, \beta \in \mathbb{R}^+\). If \(\alpha < \beta\), then

\[ L_\infty(a^\alpha) \subset L_\infty(a^\beta) \]

**Proof.** Let \(y \in L_\infty(a^\alpha)\), then we consider \(y = a^{\alpha/2}xa^{\alpha/2}\), where \(x \in \mathcal{M}\). Thus,

\[ y(f, g) = a^{\alpha/2}xa^{\alpha/2}(f, g) = \langle xa^{\alpha/2}f, a^{\alpha/2}g \rangle = \]

\[ = \]

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\[(a^{(a^{-}\beta)/2}xa^{(a^{-}\beta)/2})a^{\beta/2}f, a^{\beta/2}g)\] for any \(f, g \in D(a^{1/2})\).

Note, that \(x' := (a^{-1})^{(\beta-\alpha)/2}x(a^{-1})^{(\beta-\alpha)/2} \in \mathcal{M}\), therefore we have that \(y = a^{\beta/2}x'a^{\beta/2} \in L_\infty(a^\beta)\). \qed

**Lemma 5.** For any \(a \in \mathcal{M}\), \(a \geq 0\) such that \(a^{-1}\) is bounded and \(\alpha, \beta \in \mathbb{R}^+\), such that \(\alpha < \beta\) we have

\[\| \cdot \|_{a^\alpha} \succ \| \cdot \|_{a^\beta}.\]

**Proof.** Let \(y \in L_\infty(a^\alpha)\), then \(y = \hat{a}^{\alpha/2}xa^{\alpha/2}\), where \(x \in \mathcal{M}\), then

\[
\|y\|_{a^\beta} = \|\hat{a}^{\alpha/2}xa^{\alpha/2}\|_{a^\beta} = \|a^{(a^{-}\beta)/2}xa^{(a^{-}\beta)/2}\| \leq \\
\leq \|(a^{-1})^{(\beta-\alpha)/2}\|^2\|x\| = \\
= \|(a^{-1})^{(\beta-\alpha)}\|\|x\| = \|(a^{-1})^{(\beta-\alpha)}\|y\|_{a^\alpha}
\]

\[ \qed \]

**Definition 3.** For the unbounded \(a \in \mathcal{M}^+\) with bounded inverse \(a^{-1}\) we define the topological space \((\mathcal{L}_\infty(a), \tau_\infty(a))\) as the limit space for \(L_\infty(a^\alpha)\), where \(\mathcal{L}_\infty(a) := \bigcup_{\alpha>1} L_\infty(a^\alpha)\) and \(\tau(a)\) is the strongest topology such that for all \(\alpha > 0\) the mappings \(m_\alpha : x \in L_\infty(a^\alpha) \mapsto x \in (\mathcal{L}_\infty(a), \tau_\infty(a))\) are continuous.

Evidently, \((\mathcal{L}_\infty(a), \tau_\infty(a))\) is an (LB)-space.
3.3 Case of unbounded operator with unbounded inverse

Now, consider case, when $a \eta \mathcal{M}$, $a \geq 0$, $a$ and $a^{-1}$ are unbounded, simultaneously. Then we take spectra gecomposition

$$ a = \int_{0}^{+\infty} \lambda dP_{\lambda} $$

and determine

$$ a_{0} := \int_{0}^{1} \lambda dP_{\lambda}, \quad a_{\infty} := \int_{1}^{+\infty} \lambda dP_{\lambda}; $$

$$ p_{0} := \int_{0}^{1} dP_{\lambda}, \quad p_{\infty} := \int_{1}^{+\infty} dP_{\lambda}. $$

Note, that

$$ a_{0} = a p_{0} = p_{0} a = p_{0} a p_{0}, $$

$$ a_{\infty} = p_{\infty} a_{\infty} = a_{\infty} p_{\infty} = p_{\infty} a_{\infty} p_{\infty}, $$

$$ p_{\infty} p_{0} = p_{0} p_{\infty} = 0 = a_{0} a_{\infty} = a_{\infty} a_{0}. $$

Evidently, $H = p_{0} H \oplus p_{\infty} H$, thus we represent $\mathcal{M}$ as a subalgebra of the algebra of matrices $\mathbb{M}_{2}(\mathcal{M})$:

$$ x = \begin{pmatrix} x_{p_{0}} & p_{0} x_{p_{\infty}} \\ p_{\infty} x_{p_{0}} & x_{p_{\infty}} \end{pmatrix} \in \begin{pmatrix} \mathcal{M}_{p_{0}} & p_{0} \mathcal{M} p_{\infty} \\ p_{\infty} \mathcal{M} p_{0} & \mathcal{M}_{p_{\infty}} \end{pmatrix}; $$

meaning that if $x$ acts on $h \in H$, then

$$ xh = x(p_{0} h \oplus p_{\infty} h) = $$

$$ = \begin{pmatrix} x_{p_{0}} & p_{0} x_{p_{\infty}} \\ p_{\infty} x_{p_{0}} & x_{p_{\infty}} \end{pmatrix} \begin{pmatrix} p_{0} h \\ p_{\infty} h \end{pmatrix} = $$

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\begin{align*}
&= \left( p_0 x p_0 h + p_0 x p_\infty h \right) = \left( p_0 x h \right) = \\
&= p_0(x h) \oplus p_\infty(x h).
\end{align*}

Evidently, $a^{\frac{1}{2}}Na^{\frac{1}{2}}$ in this notation is represented as

\[
\left( \begin{array}{cc}
L_\infty(a_0) & S_{a_0^{1/2},a_\infty^{1/2}}(\mathcal{M}) \\
S_{a_\infty^{1/2},a_0^{1/2}}(\mathcal{M}) & L_\infty(a_\infty)
\end{array} \right),
\]

where $S_{a_0^{1/2},a_\infty^{1/2}}(\mathcal{M})$ and $S_{a_\infty^{1/2},a_0^{1/2}}(\mathcal{M})$ are defined by the following definitions

**Definition 4.** Let $a, b \in \mathcal{M}$, $x \in \mathcal{M}$, then the sesquilinear form

\[
\widehat{axb} : (f, g) \in D(b) \times D(a) \mapsto \mathbb{R}
\]

is defined by the equality

\[
\widehat{axb}(f, g) := \langle xbf, ag \rangle.
\]

**Definition 5.** Let $a, b \in \mathcal{M}$. We define the linear space of sesquilinear forms

\[
\mathcal{S}_{a,b}(\mathcal{M}) := \{ \widehat{axb} | x \in \mathcal{M} \}
\]

endowed with the norm $\| \widehat{axb} \|_{a,b} := \| x \|.$

Let $a$ be injective, then $a_0|_{p_0H}$ is injective bounded operator in $\mathcal{M}_{p_0}$ acting on $p_0H$ and $a_\infty|_{p_\infty H}$ is injective operator affiliated with $\mathcal{M}_{p_\infty}$ (acting on $p_\infty H$) with bounded inverse. Moreover, $L_\infty(a_0)$ is isometrically isomorphic to $L_\infty(a_0|_{p_0H})$ as well as $L_\infty(a_\infty)$ is isometrically isomorphic
to $L_{\infty}(a_0, a_\infty|_{p_\infty H})$ by construction. Also, it is evident, that there exists the isomorphism $\overline{axb} \mapsto \overline{bxa}$ between the spaces $S_{a,b}(M)$ and $S_{b,a}(M)$. Thus, we only need to consider the limits for $S_{a_\alpha^0, a_\beta^0}^\alpha (M)$ for $\alpha, \beta \rightarrow +\infty$.

Further we also use the notation

$$S_{0,\infty}^{\alpha,\beta} := S_{a_0^\alpha, a_\infty^\beta} (M)$$

and

$$S_{\infty,0}^{\alpha,\beta} := S_{a_\infty^\alpha, a_0^\beta} (M).$$

Let $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, $\alpha_i, \beta_i \in \mathbb{R}$, then note, that the following schemes

$$S_{0,\infty}^{\alpha_1,\beta_1} \subset S_{0,\infty}^{\alpha_1,\beta_2} \quad S_{\infty,0}^{\alpha_1,\beta_1} \supset S_{\infty,0}^{\alpha_1,\beta_2}$$

$$\bigcup \bigcup \quad \cap \cap \quad \text{and} \quad \cap \cap \quad \text{hold}.$$
where
\[ \tau|_{\alpha_2} \equiv \tau|_{\beta_1} \equiv \tau|_{\alpha_2, \beta_1} := \{ X \cap S^{(\alpha_1, \beta_2)}_{0, \infty} \mid X \in \tau \}. \]

Since there exists the isomorphism between \( S^{\alpha, \beta}_{0, \infty} \) and \( S^{\alpha, \beta}_{\infty, 0} \), we will only consider one case. Consider the limits
\[ S^\alpha_0 = \bigcup_{\beta > 0} S^{\alpha, \beta}_{0, \infty} \text{ and } S^\beta_\infty = \bigcap_{\alpha > 0} S^{\alpha, \beta}_{0, \infty} \]
with the topologies \( \tau^\alpha_0 \) and \( \tau^\beta_\infty \), respectively. Topology \( \tau^\alpha_0 \) is the strongest topology on \( S^\alpha_0 \), such that all the mappings
\[ \phi^\alpha_\beta : x \in (S^{\alpha, \beta}_{0, \infty}, \| \cdot \|_{a_0^\alpha, a_\infty^\beta}) \mapsto x \in S^\alpha_0 \] (1)
are continuous and the topology \( \tau^\beta_\infty \) is the weakest topology on \( S^\beta_\infty \) that all the mappings
\[ \psi^\beta_\alpha : x \in S^\beta_\infty \mapsto x \in (S^{\alpha, \beta}_{0, \infty}, \| \cdot \|_{a_0^\alpha, a_\infty^\beta}) \] (2)
are continuous.

Remark 1. Note, that any \( (S^{\alpha, \beta}_{0, \infty}, \| \cdot \|_{a_0^\alpha, a_\infty^\beta}) \) is Banach space, thus \( (S^\alpha_0, \tau^\alpha_0) \) is an \( (LB) \)-space.

Remark 2. Note, that \( (S^\alpha_\infty, \tau^\alpha_\infty) \) is a Frechet space, since its topology is determined by the countable set of the seminorms.

Lemma 6. Let \( \alpha < \beta, \alpha, \beta \in \mathbb{R}^+ \), then

(i) \( S^\alpha_0 \supset S^\beta_0 \);

(ii) \( S^\alpha_\infty \subset S^\beta_\infty \).
Proof. (i) Since 
\[ \mathcal{G}_0^\beta = \bigcup_{\gamma > 0} S_{0,\infty}^{\beta,\gamma}, \quad \mathcal{G}_0^\alpha = \bigcup_{\gamma > 0} S_{0,\infty}^{\alpha,\gamma} \quad \text{and} \quad \mathcal{G}_0^\beta,\gamma \subset \mathcal{G}_0^{\alpha,\gamma}, \]

it follows that if \( x \in \mathcal{G}_0^\beta \), then there exists \( \gamma_0 \) such that \( x \in S_{0,\infty}^{\beta,\gamma_0} \subset \mathcal{G}_0^{\alpha,\gamma_0} \), thus \( x \in \mathcal{G}_0^\alpha \).

(ii) If \( x \in \mathcal{G}_\infty^\alpha \), then for all \( \gamma > 0 \) \( x \in S_{0,\infty}^{\gamma,\alpha} \subset S_{0,\infty}^{\gamma,\beta} \), thus
\[ x \in \bigcap_{\gamma > 0} S_{0,\infty}^{\gamma,\beta} = \mathcal{G}_\infty^\beta. \]

\[ \square \]

We consider the following constructions:
\[ \mathcal{L}_{0,\infty} = \bigcap_{\alpha > 0} \mathcal{G}_0^\alpha, \quad \mathcal{L}_{0,\infty} = \bigcup_{\beta > 0} \mathcal{G}_\infty^\beta. \]

Note that
\[ \mathcal{L}_{0,\infty} = \bigcup_{\beta > 0} \mathcal{G}_\infty^\beta = \bigcup_{\beta > 0} \bigcap_{\alpha > 0} S_{0,\infty}^{\alpha,\beta} \subset \bigcap_{\alpha > 0} \bigcup_{\beta > 0} S_{0,\infty}^{\alpha,\beta} = \bigcap_{\alpha > 0} \mathcal{G}_0^\alpha = \overline{\mathcal{L}_{0,\infty}}. \]

**Proposition 2.**
\[ \mathcal{L}_{0,\infty} \supset \lim_{\alpha \to 0} S_{0,\infty}^{\alpha,\alpha} \equiv \bigcup_{\alpha > 0} \bigcap_{\beta \geq \alpha} S_{0,\infty}^{\beta,\beta} \supset \bigcup_{\alpha > 0} \bigcap_{\beta \geq \alpha} S_{0,\infty}^{\beta,\beta} \equiv \lim_{\alpha \to 0} S_{0,\infty}^{\alpha,\alpha} \supset \mathcal{L}_{0,\infty}. \]

Proof. Note, that if \( \alpha_1 < \alpha_2 \), then \( S_{0,\infty}^{\alpha_1,\beta} \supset S_{0,\infty}^{\alpha_2,\beta} \), thus
\[
\mathcal{L}_{0,\infty} = \bigcap_{\alpha > 0} \bigcup_{\beta > 0} \mathcal{S}_{0,\infty}^{\alpha,\beta} = \bigcap_{\alpha > 0} \bigcup_{\beta \geq \alpha} \mathcal{S}_{0,\infty}^{\alpha,\beta} \supset \bigcap_{\alpha > 0} \bigcup_{\beta > 0} \mathcal{S}_{0,\infty}^{\beta,\beta}
\]

Note, that if \( \beta_1 < \beta_2 \), then \( \mathcal{S}_{0,\infty}^{\alpha,\beta_1} \subset \mathcal{S}_{0,\infty}^{\alpha,\beta_2} \), thus

\[
\bigcup_{\alpha > 0} \bigcap_{\beta \geq \alpha} \mathcal{S}_{0,\infty}^{\beta,\beta} \supset \bigcup_{\alpha > 0} \bigcap_{\beta > 0} \mathcal{S}_{0,\infty}^{\beta,\alpha} = \bigcup_{\alpha > 0} \bigcap_{\beta > 0} \mathcal{S}_{0,\infty}^{\beta,\alpha} = \mathcal{L}_{0,\infty}. \]

\[\square\]

**Definition 6.** We call the family \( \mathcal{S}_{0,\infty}^{\alpha,\beta} \) converging if \( \mathcal{L}_{0,\infty} = \mathcal{L}_{0,\infty} \) and denote it as \( \mathcal{S}_{0,\infty}^{\alpha,\beta} \rightarrow \mathcal{L}_{0,\infty} \).

**Corollary 1.** If the family \( \mathcal{S}_{0,\infty}^{\alpha,\beta} \) is converging, then

\[
\overline{\lim} \mathcal{S}_{0,\infty}^{\alpha,\alpha} = \overline{\lim} \mathcal{S}_{0,\infty}^{\alpha,\alpha} = \mathcal{L}_{0,\infty}.
\]

**Lemma 7.** Let \( \alpha < \beta, \alpha, \beta \in \mathbb{R}^+ \) and

\[
\tau_0^{\alpha}|_{\beta} := \{ X \cap \mathcal{S}_{0}^{\beta} \mid X \in \tau_0^{\alpha} \}
\]

i.e. the topology induced by \( \tau_0^{\alpha} \) on \( \mathcal{S}_{0}^{\beta} \), then

\[
\tau_0^{\alpha}|_{\beta} \subset \tau_0^{\beta}.
\]

**Proof.** Let \( X_0 \in \tau_0^{\alpha}|_{\beta} \), i.e. \( X_0 = X \cap \mathcal{S}_{0}^{\beta} \) with \( X \in \tau_0^{\alpha} \).

Then \((\varphi_\gamma^{\alpha})^{-1}(X)\) is open in \((\mathcal{S}_{0,\infty}^{\alpha,\gamma}, \| \cdot \|_{a_0^\alpha, a_\infty^\gamma})\). Note, that the embedding

\[
m_\gamma^{\beta,\alpha} : x \in (\mathcal{S}_{0,\infty}^{\beta,\gamma}, \| \cdot \|_{a_0^\beta, a_\infty^\gamma}) \mapsto x \in (\mathcal{S}_{0,\infty}^{\alpha,\gamma}, \| \cdot \|_{a_0^\alpha, a_\infty^\gamma})
\]

is also continuous, thus

\[
(\varphi_\gamma^{\alpha} m_\gamma^{\beta,\alpha})^{-1}(X) = (m_\gamma^{\beta,\alpha})^{-1}(\varphi_\gamma^{\alpha})^{-1}(X)
\]
is open in $(S^{\beta,\gamma}_{0,\infty}, \| \cdot \|_{a_0^\beta, a_\infty^\beta})$ for any $\gamma > 0$.

The topology $\tau_0^\beta$ is the strongest topology, such that all of the embeddings $\varphi_\gamma^\beta$ are continuous. If $X_0 \notin \tau_0^\beta$, then there exists the topology

$$
\tau = \tau_0^\beta \cup \{X_0 \cap Y \mid Y \in \tau_0^\beta\} \cup \{X_0 \cup Y \mid Y \in \tau_0^\beta\}
$$

such that it is stronger, then $\tau_0^\beta$, $X_0 \in \tau$ and for any $A \in \tau$ the preimages $(\varphi_\gamma^\beta)^{-1}(A)$ are open. Further we explain why it is open.

Consider three cases $A \in \tau_0^\beta$, $A = X_0 \cap Y$, $A = X_0 \cup Y$ ($Y \in \tau_0^\beta$).

1) Let $A \in \tau_0^\beta$, then evidently $(\varphi_\gamma^\beta)^{-1}(A)$ is open.

2) Let $A = X_0 \cap Y$, then

$$
(\varphi_\gamma^\beta)^{-1}(X \cap S^{\beta,\gamma}_0 \cap Y) =
$$

$$
= (\varphi_\gamma^\beta)^{-1} \left( X \cap \bigcup_{\gamma > 0} S^{\beta,\gamma}_{0,\infty} \right) \cap (\varphi_\gamma^\beta)^{-1}(Y) =
$$

$$
= \left( \bigcup_{\gamma > 0} (\varphi_\gamma^\beta)^{-1} \left( X \cap S^{\beta,\gamma}_{0,\infty} \right) \right) \cap (\varphi_\gamma^\beta)^{-1}(Y) =
$$

$$
= \left( \bigcup_{\gamma > 0} (\varphi_\gamma^\alpha m_{\gamma,\alpha}^\beta)^{-1}(X) \right) \cap (\varphi_\gamma^\beta)^{-1}(Y).
$$

is open, since $(\varphi_\gamma^\alpha m_{\gamma,\alpha}^\beta)^{-1}(X)$ is open and $(\varphi_\gamma^\beta)^{-1}(Y)$ is open.

3) Let $A = X_0 \cup Y$, then

$$
(\varphi_\gamma^\beta)^{-1}\left( (X \cap \mathcal{G}_0^\beta) \cup Y \right) =
$$
\[(\varphi_\gamma^\beta)^{-1} \left( X \cap \bigcup_{\gamma>0} S_{0,\infty}^{\beta,\gamma} \right) \cup (\varphi_\gamma^\beta)^{-1}(Y) = \]

\[= \left( \bigcup_{\gamma>0} (\varphi_\gamma^\beta)^{-1} \left( X \cap S_{0,\infty}^{\beta,\gamma} \right) \right) \cup (\varphi_\gamma^\beta)^{-1}(Y) = \]

\[= \left( \bigcup_{\gamma>0} (\varphi_\gamma^\beta m_{\gamma}^{\beta,\alpha})^{-1}(X) \right) \cup (\varphi_\gamma^\beta)^{-1}(Y). \]

is open, since \((\varphi_\gamma^\beta m_{\gamma}^{\beta,\alpha})^{-1}(X)\) is open and \((\varphi_\gamma^\beta)^{-1}(Y)\) is open.

Thus, we get a contradiction with the maximality of \(\tau_0^\beta\), therefore \(X_0 \in \tau_0^\beta\). \(\Box\)

**Lemma 8.** Let \(\alpha < \beta, \alpha, \beta \in \mathbb{R}^+\) and

\[\tau_\infty^{\beta|\alpha} := \{ X \cap \mathcal{S}_\infty^{\alpha} \mid X \in \tau_\infty^{\beta}\}\]

i.e. the topology induced by \(\tau_\infty^{\beta}\) on \(\mathcal{S}_\infty^{\alpha}\), then

\[\tau_\infty^{\alpha} \supset \tau_\infty^{\beta|\alpha}.\]

**Proof.** The topology \(\tau_\infty^{\alpha}\) is determined with the set of the seminorms \(\{\| \cdot \|_{a_0, a_\infty}^{\gamma=1}\}_{\gamma=1}\) and the topology \(\tau_\infty^{\beta|\alpha}\) is determined by the system \(\{\| \cdot \|_{a_0, a_\infty}^{\gamma=1}\}_{\gamma=1}\). It is sufficient to note, that \(\| \cdot \|_{a_0, a_\infty}^{\gamma} < \| \cdot \|_{a_0, a_\infty}^{\beta}\). \(\Box\)

For \(\mathcal{L}_{0,\infty}\) it is natural to define the topology \(\tau_{0,\infty}\) which is determined as the strongest topology such that all of the embeddings

\[\Phi_\alpha : x \in (\mathcal{S}_\infty^{\alpha}, \tau_\infty^{\alpha}) \mapsto x \in (\mathcal{L}_{0,\infty}, \tau_{0,\infty}) \quad (3)\]
are continuous.

On the other side, it is natural to define the topology \( \tau_{0,\infty} \) which would be the weakest topology on \( \underline{\Sigma}_{0,\infty} \) such that all of the embeddings

\[
\Psi_\alpha : x \in (\underline{\Sigma}_{0,\infty}, \tau_{0,\infty}) \mapsto x \in (\underline{\Sigma}_0^\alpha, \tau_0^\alpha) \quad (4)
\]

are continuous.

**Theorem 2.** The embedding

\[
\Lambda : x \in (\underline{\Sigma}_{0,\infty}, \tau_{0,\infty}) \mapsto x \in (\underline{\Sigma}_{0,\infty}, \tau_{0,\infty}) \quad (5)
\]

is continuous.

**Proof.** Note, that

\[
\forall \beta_0 > 0 \quad \tau_{0,\infty}|_{\beta_0} = \bigcap_{\beta \geq \beta_0} \tau_\infty^\beta|_{\beta_0} \quad \text{and} \quad \tau_{0,\infty} = \bigcup_{\alpha > 0} \tau_0^\alpha|_{\infty}.
\]

At the same time,

\[
\tau_\infty^\beta = \bigcup_{\alpha > 0} \tau_{0,\infty}^{\alpha,\beta}|_{\infty} \quad \text{and} \quad \forall \beta_0 > 0 \quad \tau_0^\alpha|_{\beta_0} = \bigcap_{\beta \geq \beta_0} \tau_{0,\infty}^{\alpha,\beta}|_{\beta_0}.
\]

It is sufficient to prove that for any \( \beta_0 > 0 \) we have the inclusion

\[
\tau_{0,\infty}|_{\beta_0} \supset \tau_{0,\infty}|_{\beta_0}, \quad \text{where} \quad \tau|_{\beta_0} = \{ X \cap \underline{\Sigma}_\infty^{\beta_0} \mid X \in \tau \}.
\]

Thus,

\[
\tau_{0,\infty}|_{\beta_0} = \bigcap_{\beta \geq \beta_0} \left( \bigcup_{\alpha > 0} \tau_{0,\infty}^{\alpha,\beta}|_{\infty} \right)|_{\beta_0}
\]

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and
\[
\tau_{0,\infty}|_{\beta_0} = \left( \bigcup_{\alpha>0} \tau_{0,\infty}^\alpha \right)|_{\beta_0}.
\]

Note, that the reductions always lead to the space $\mathcal{S}_{\infty}^{\beta_0}$ and may be rewrited as
\[
\tau_{0,\infty}|_{\beta_0} = \bigcap_{\beta \geq \beta_0} \left( \bigcup_{\alpha>0} \tau_{0,\infty}^{\alpha,\beta} \right) \quad \text{and} \quad \tau_{0,\infty}|_{\beta_0} = \bigcup_{\alpha>0} \left( \tau_{0,\infty}^\alpha \right)|_{\beta_0}.
\]

But then $\tau_{0}^{\alpha}|_{\beta_0} = \bigcap_{\beta \geq \beta_0} \tau_{0,\infty}^{\alpha,\beta}|_{\beta_0}$, therefore
\[
\tau_{0,\infty}|_{\beta_0} = \bigcap_{\beta \geq \beta_0} \left( \bigcup_{\alpha>0} \left( \tau_{0,\infty}^{\alpha,\beta} \right) \right) \supset \bigcup_{\alpha>0} \left( \bigcap_{\beta \geq \beta_0} \left( \tau_{0,\infty}^{\alpha,\beta} \right) \right) = \tau_{0,\infty}|_{\beta_0}.
\]

\[\square\]

**Remark 3.** Note, that $\tau_{0,\infty}$ is a topology of $(LF)$-space and $\tau_{0,\infty}$ is a topology of the projective limit of $(LB)$-spaces.

**Remark 4.** All the constructions of this section may be applied to the system $\mathcal{S}_{\infty,0}^{\alpha,\beta}$. We will distinguish such constructions by the notation $\mathcal{L}_{\infty,0}^{\alpha,\beta}$, $\mathcal{L}_{\infty,0}^{\alpha,\beta}$, $\tau_{0,\infty}^{\alpha,\beta}$ and so on.

### 3.4 Bring it all together

Evidently we may consider different limits of $L_{\infty}(a^\alpha)$ spaces, using the constructions above, particularly, we may define
**Definition 7.** Let $a$ be a positive operator affiliated with von Neumann algebra $\mathcal{M}$ acting on the Hilbert space $H$, such that not $a$ nor $a^{-1}$ are necessarily bounded. Then we define the lower limit of the spaces $\lim L_\infty(a^\alpha)$ as a vector space of sesquilinear forms formally written as

$$\left( (\mathfrak{L}_\infty(a_0), \tau_\infty(a_0)) , (\mathfrak{L}_{0,\infty}, \tau_{0,\infty}) \right)$$

**Definition 8.** Let $a$ be a positive operator affiliated with von Neumann algebra $\mathcal{M}$ acting on the Hilbert space $H$, such that not $a$ nor $a^{-1}$ are necessarily bounded. Then we define the upper limit of the spaces $\overline{\lim} L_\infty(a^\alpha)$ as a vector space of sesquilinear forms formally written as

$$\left( (\mathfrak{L}_\infty(a_0), \tau_\infty(a_0)) , (\overline{\mathfrak{L}_{0,\infty}}, \overline{\tau_{0,\infty}}) \right)$$

**Theorem 3.** Let $\mathfrak{L}_{0,\infty} = \overline{\mathfrak{L}_{0,\infty}}$, $\tau_{0,\infty} = \overline{\tau_{0,\infty}}$, then

$$\mathfrak{L}_\infty(a) = \overline{\lim} L_\infty(a^\alpha) = \lim L_\infty(a^\alpha)$$

is $(LF)$-space.

**Corollary 2.** If $a \geq 0$ is affiliated with the center $\mathfrak{C}(\mathcal{M})$ of the von Neumann algebra $\mathcal{M}$, then the limits space $\mathfrak{L}_\infty(a) = \lim_{\alpha} L_\infty(a^\alpha)$ is an $(LF)$-space.

**Acknowledgment**

Research is partially supported by Russian Foundation for Basic Research grant 18-31-00218 (мов a).
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