Quark-Antiquark and Diquark Condensates in Vacuum in a 3D Two-Flavor Gross–Neveu Model

ZHOU Bang-Rong

College of Physical Sciences, Graduate School of the Chinese Academy of Sciences, Beijing 100049, China
CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China

(Received July 24, 2006)

Abstract The effective potential analysis indicates that, in a 3D two-flavor Gross–Neveu model in vacuum, depending on whether $G_S/H_P$ is less or bigger than the critical value $2/3$, where $G_S$ and $H_P$ are respectively the coupling constants of scalar quark-antiquark channel and pseudoscalar diquark channel, the system will have the ground state with pure diquark condensates or with pure quark-antiquark condensates, but never with coexistence of the two forms of condensates. The similarities and differences in the interplay between the quark-antiquark and the diquark condensates in vacuum in the 2D, 3D and 4D two-flavor four-fermion interaction models are summarized.

PACS numbers: 12.38.Aw, 12.38.Lg, 12.10.Dm, 11.15.Pg

Key words: 3D Gross-Neveu model, quark-antiquark and diquark condensates, effective potential

1 Introduction

It has been shown by effective potential approach that in a two-flavor 4D Nambu–Jona–Lasinio (NJL) model,[1] even when temperature $T = 0$ and quark chemical potential $\mu = 0$, i.e. in vacuum, there could exist mutual competition between the quark-antiquark condensates and the diquark condensates.[2] Similar situation has also emerged from a 2D two-flavor Gross–Neveu (GN) model[3] except for some difference in the details of the results.[4] An interesting question is whether such mutual competition between the two forms of condensates is a general characteristic of this kind of two-flavor four-fermion interaction models. To answer this question, on the basis of research on the 4D NJL model and the 2D GN model, we will continue to examine a 3D two-flavor GN model in similar way. The results will certainly deepen our understanding of the feature of the four-fermion interaction models.

We will use the effective potential in the mean-field approximation which is equivalent to the leading order of $1/N$ expansion. It is indicated that a 3D GN model is renormalizable in $1/N$ expansion.[5]

2 Model and Its Symmetries

The Lagrangian of the model will be expressed by

$$\mathcal{L} = i\bar{q}i\gamma^\mu \partial_\mu q + G_S[(\bar{q}q)^2 + (\bar{q}\tau q)^2] + H_P \sum_{A=2,5,7} (\bar{q}\gamma_2 \lambda_A q^c)(\bar{q}^c \gamma_2 \lambda_A q).$$

All the denotations used in Eq. (1) are the same as the ones in the 2D GN model given in Ref. [4], except that the dimension of space-time is changed from 2 to 3 and the coupling constant $H_S$ of scalar diquark interaction channel is replaced by the coupling constant $H_P$ of pseudoscalar diquark interaction channel. Now the matrices $\gamma^\mu (\mu = 0, 1, 2)$ and the charge conjugate matrix $C$ are taken to be $2 \times 2$ ones and have the explicit forms

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = C.$$

It is emphasized that, in 3D case, no “$\gamma_5$” matrix can be defined, hence the third term on the right-hand side of Eq. (1) will be the only possible color-anti-triplet diquark interaction channel which could lead to Lorentz-invariant diquark condensates, where we note that the matrix $C\gamma_2\lambda_4$ is antisymmetric. Without “$\gamma_5$”, the Lagrangian (1) will have no chiral symmetry. Except for this, it is not difficult to verify that the symmetries of $\mathcal{L}$ include the following.

(i) Continuous flavor and color symmetries $SU_f(2) \otimes SU_c(3) \otimes U_f(1)$;
(ii) discrete symmetry $R$: $q \to -q$;
(iii) Parity $P$: $q(t, \vec{x}) \to \gamma^0 q(t, -\vec{x})$ and $q^c(t, \vec{x}) \to -\gamma^0 q^c(t, -\vec{x})$;
(iv) Time reversal $T$: $q(t, \vec{x}) \to \gamma^2 q(-t, -\vec{x})$ and $q^c(t, \vec{x}) \to -\gamma^2 q^c(-t, -\vec{x})$;
(v) Charge conjugate $C$: $q \to q^c$;
(vi) Special parity $P_1$: $q(t, x^1, x^2) \to \gamma^1 q(t, -x^1, x^2)$ and $q^c(t, x^1, x^2) \to -\gamma^1 q^c(t, -x^1, x^2)$;
(vii) Special parity $P_2$: $q(t, x^1, x^2) \to \gamma^2 q(t, x^1, -x^2)$ and $q^c(t, x^1, x^2) \to -\gamma^2 q^c(t, x^1, -x^2)$.

If the quark-antiquark condensates $\langle \bar{q}q \rangle$ could be formed, then the time reversal $T$ and the special parities $P_1$ and $P_2$ will be spontaneously broken.[6] If the diquark condensates $\langle \bar{q}^c\gamma_2\lambda_4 q \rangle$ could be formed, then the color
symmetry $SU_c(3)$ will be spontaneously broken down to $SU_c(2)$ and the flavor number $U_f(1)$ will be spontaneously broken but a "rotated" electric charge $U_g(1)$ and a "rotated" quark number $U_q(1)$ leave unbroken.\[^7\] In addition, the parity $P$ will be spontaneously broken, though all the other discrete symmetries survive. This implies that the diquark condensates $\langle \bar{q}q \rangle$ will be a pseudoscalar. In this paper we will neglect discussions of the Goldstone bosons induced by breakdown of the continuous symmetries and pay our main attention to the problem of interplay between the above two forms of condensates.

3 Effective Potential in Mean-Field Approximation

Define the order parameters in the 3D GN model by

$$\sigma = -2G_S \langle \bar{q}q \rangle, \quad \Delta = -2H_P \langle \bar{q}^c \tau_2 \lambda_2 q \rangle,$$

then in the mean-field approximation, the Lagrangian (1) can be rewritten by

$$\mathcal{L} = \bar{\Psi}(x)S^{-1}(x)\Psi(x) - \frac{\sigma^2}{4G_S} - \frac{\Delta^2}{4H_P},$$

where

$$V(\sigma, |\Delta|) = \frac{\sigma^2}{4G_S} + \frac{|\Delta|^2}{4H_P} - \frac{1}{2\pi^2}(3\sigma^2 + 2|\Delta|^2)\Lambda$$

$$+ \frac{1}{3\pi} [6\sigma^2|\Delta| + 2|\Delta|^3 + \sigma^3]$$

$$+ 2\theta(\sigma - |\Delta|)(\sigma - |\Delta|)^2.$$ (9)

4 Ground States

Equation (9) provides the possibility to discuss the ground states of the model analytically. The extreme value conditions $\partial V(\sigma, |\Delta|)/\partial \sigma = 0$ and $\partial V(\sigma, |\Delta|)/\partial |\Delta| = 0$ will lead to the equations

$$\sigma \left( \frac{1}{2G_S} - \frac{6\Lambda}{\pi^2} + \frac{4|\Delta|}{\pi} + \frac{\sigma}{\pi} \right) + \frac{2}{\pi} \theta(\sigma - |\Delta|)(\sigma - |\Delta|)^2 = 0,$$ (10)

$$|\Delta| \left( \frac{1}{2H_P} - \frac{4\Lambda}{\pi^2} + \frac{2|\Delta|}{\pi} \right) + \frac{2}{\pi} [\sigma^2 - \theta(\sigma - |\Delta|)(\sigma - |\Delta|)^2] = 0.$$ (11)

where

$$\Psi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} q(x) \\ q^c(x) \end{pmatrix}, \quad \bar{\Psi}(x) = \frac{1}{\sqrt{2}} \left( \bar{q}(x) \quad \bar{q}^c(x) \right)$$

are the expressions of the quark fields in the Nambu–Gorkov basis.\[^8\] In the momentum space, the inverse propagator $S^{-1}(x)$ for the quark fields may be expressed by

$$S^{-1}(p) = \begin{pmatrix} p - \sigma & -\tau_2 \lambda_2 \Delta^* \\ -\tau_2 \lambda_2 \Delta & -p - \sigma \end{pmatrix}, \quad p = \gamma^\mu p_\mu.$$ (5)

The effective potential corresponding to $\mathcal{L}$ given by Eq. (4) becomes

$$V(\sigma, |\Delta|) = \frac{\sigma^2}{4G_S} + \frac{|\Delta|^2}{4H_P} + \frac{1}{2\pi^2} \int \frac{d^3p}{(2\pi)^3} \text{Tr} \ln S^{-1}(p) S_0(p).$$ (6)

Similar to the case of the 2D NG model,\[^4\] the calculations of $\text{Tr}$ for (red, green) and blue color degrees of freedom can be made separately thus equation (6) will be reduced to

$$V(\sigma, |\Delta|) = \frac{\sigma^2}{4G_S} + \frac{|\Delta|^2}{4H_P} + \frac{1}{2\pi^2} \int \frac{d^3p}{(2\pi)^3} \text{Tr} \ln S^{-1}(p) S_0(p).$$ (6)

Define the expressions

$$K = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2,$$

where $A, B,$ and $C$ represent the second order derivatives of $V(\sigma, |\Delta|)$ with the explicit expressions

$$A = \frac{\partial^2 V}{\partial \sigma^2} = \frac{1}{2G_S} - \frac{6\lambda}{\pi^2} + \frac{4|\Delta|}{\pi} + \frac{2\sigma}{\pi} + \frac{1}{\pi} \theta(\sigma - |\Delta|)(\sigma - |\Delta|),$$

$$B = \frac{\partial^2 V}{\partial \sigma \partial |\Delta|} = \frac{\partial^2 V}{\partial |\Delta| \partial \sigma} = \frac{1}{2\pi^2} [\sigma^2 - \theta(\sigma - |\Delta|)(\sigma - |\Delta|)],$$

$$C = \frac{\partial^2 V}{\partial |\Delta|^2} = \frac{1}{2H_P} - \frac{4\lambda}{\pi^2} + \frac{4|\Delta|}{\pi} + \frac{1}{\pi} \theta(\sigma - |\Delta|)(\sigma - |\Delta|).$$ (12)

Equations (10) and (11) have the four different solutions which will be discussed in proper order as follows.

(i) $(\sigma, |\Delta|) = (0, 0)$. It is a maximum point of $V(\sigma, |\Delta|)$, since in this case we have

$$A = \frac{1}{2G_S} - \frac{6\lambda}{\pi^2} < 0, \quad K = A \left( \frac{1}{2H_P} - \frac{4\lambda}{\pi^2} \right) > 0,$$

assuming that equations (10) and (11) have solutions of non-zero $\sigma$ and $|\Delta|$.

(ii) $(\sigma, |\Delta|) = (\sigma_1, 0)$, where the non-zero $\sigma_1$ satisfies the equation

$$\frac{1}{2G_S} - \frac{6\lambda}{\pi^2} + \frac{3\sigma_1}{\pi} = 0.$$ (13)
When equation (13) is used, we obtain
\[
A = \frac{3\sigma_1}{\pi},
\]
\[
K = A\left(\frac{1}{2H_P} - \frac{1}{3G_S} + \frac{2\sigma_1}{\pi}\right) > 0,
\]
if \(\frac{G_S}{H_P} > \frac{2}{3}\).

Hence \((\sigma_1,0)\) will be a minimum point of \(V(\sigma,|\Delta|)\) when \(G_S/H_P > 2/3\).

(iii) \((\sigma_1,|\Delta|) = (0,\Delta_1)\), where non-zero \(\Delta_1\) obeys the equation
\[
\frac{1}{2H_P} - \frac{4\Delta_1}{\pi^2} + \frac{2\Delta_1}{\pi} = 0.
\]
By using Eq. (14) we may get
\[
A = \frac{1}{2G_S} - \frac{3}{4H_P} + \frac{\Delta_1}{\pi}, \quad K = A\frac{2\Delta_1}{\pi}.
\]
Obviously, \((0,\Delta_1)\) will be a minimum point of \(V(\sigma,|\Delta|)\) when \(G_S/H_P < 2/3\).

(iv) \((\sigma,|\Delta|) = (\sigma_2,\Delta_2)\). In view of existence of the function \(\theta(\sigma - |\Delta|)\) in Eqs. (10) and (11), we have to consider the case of \(\sigma_2 > \Delta_2\) and \(\sigma_2 < \Delta_2\) respectively.

(a) \(\sigma_2 > \Delta_2\). In this case, equations (10) and (11) will become
\[
\sigma_2\left(\frac{6\sigma_2}{\pi^2} + \frac{1}{\pi}(3\sigma_2^2 + 2\Delta_2^2) = 0, \quad K = \frac{2\Delta_1}{\pi} > 0. \quad (14)
\]
\[
\frac{1}{2H_P} - \frac{4\Delta_2}{\pi^2} + \frac{4\Delta_2}{\pi} = 0.
\]
From them we can get
\[
A = \frac{3\sigma_2^2 - 2\Delta_2^2}{\pi^2\sigma_2} > 0, \quad K = -\frac{2\Delta_2}{\pi^2} < 0. \quad (15)
\]
Thus it is turned out that \((\sigma_2,\Delta_2)\) will be neither a maximum nor a minimum point of \(V(\sigma,|\Delta|)\) if \(\sigma_2 > \Delta_2\).

(b) \(\sigma_2 < \Delta_2\). Now equations (10) and (11) are changed into
\[
\frac{1}{2G_S} - \frac{6\Delta_2}{\pi^2} + \frac{4\Delta_2 + \sigma_2}{\pi} = 0, \quad K = \frac{2\Delta_1}{\pi} > 0. \quad (16)
\]
Hence we will have the results that
\[
A = \frac{\sigma_2}{\pi}, \quad K = \frac{2\sigma_2}{\pi^2\Delta_2}(\Delta_2^2 - \sigma_2^2 - 8\sigma_2\Delta_2),
\]
from which it may be deduced that only if
\[
\sigma_2 < (\sqrt{17} - 4)\Delta_2, \quad (17)
\]
\((\sigma_2,\Delta_2)\) is just a minimum point of \(V(\sigma,|\Delta|)\). On the other hand, from Eqs. (15) and (16) obeyed by \(\sigma_2\) and \(\Delta_2\) we may get
\[
\frac{1}{2H_P} - \frac{1}{3G_S} = \frac{2}{\pi\Delta_2}\left(\frac{\sqrt{13} - 1}{6}\Delta_2 + \sigma_2\right) \times \left(\frac{\sqrt{13} + 1}{6}\Delta_2 - \sigma_2\right). \quad (18)
\]
Equation (18) indicates that for the minimum point \((\sigma_2,\Delta_2)\) satisfying Eq. (17) one will certainly have \(G_S/H_P > 2/3\). Taking this and the result obtained in case (ii) into account we see that if \(G_S/H_P > 2/3\) the effective potential \(V(\sigma,|\Delta|)\) will have two possible minimum points \((\sigma_1,0)\) and \((\sigma_2,\Delta_2)\). To determine which one of the two minimum points is the least value point of \(V\), we must make a comparison between \(V(\sigma_1,0)\) and \(V(\sigma_2,\Delta_2)\) with the constraint given by Eq. (17). In fact, it is easy to find out that when equation (13) is used,
\[
V(\sigma_1,0) = -\frac{\sigma_1^3}{2\pi}, \quad (19)
\]
and that when equations (15) and (16) are used,
\[
V(\sigma_2,\Delta_2) = -\frac{1}{3\pi}\left(\Delta_2^2 + \frac{3\sigma_2^2}{2}\right). \quad (20)
\]
By comparing Eq. (13) with Eq. (15) we may obtain the relation
\[
3\sigma_1 = \sigma_2 + 4\Delta_2. \quad (21)
\]
By means of Eqs. (19) \sim (21) it is easy to verify that
\[
V(\sigma_1,0) - V(\sigma_2,\Delta_2) \leq \frac{1}{3\pi}\left(\frac{2(3\Delta_2^2 - 4\sigma_2^2) + \sigma_2^2}{3}(8\Delta_2 - 7\sigma_2)\right) < 0,
\]
when equation (17) is satisfied. This result indicates that when \(G_S/H_P > 2/3\), the least value point of \(V(\sigma,|\Delta|)\) will be \((\sigma_1,0)\) but not \((\sigma_2,\Delta_2)\).

In summary, if the necessary conditions \(G_S\Lambda > \pi^2/12\) and \(H_P\Lambda > \pi^2/8\) for non-zero \(\sigma\) and \(\Delta\) are satisfied, then the least value points of the effective potential \(V(\sigma,|\Delta|)\) will be at
\[
(\sigma,|\Delta|) = \begin{cases} (0,\Delta_1), & \text{if } 0 \leq \frac{G_S}{H_P} < \frac{2}{3}, \\ (\sigma_1,0), & \text{if } \frac{G_S}{H_P} > \frac{2}{3}. \end{cases} \quad (22)
\]
As a result, in the ground state of the 3D two-flavor GN model, depending on whether the ratio \(G_S/H_P\) is bigger or less than \(2/3\), one will have either pure quark-antiquark condensates or pure diquark condensates, but no coexistence of the two forms of condensates could happen.

5 Concluding Remarks

The result (22) in the 3D GN model can be compared with the ones in the 4D NJL model and in the 2D GN model. The minimal points of the effective potential \(V(\sigma,|\Delta|)\) for the latter models have been obtained and are located respectively at
\[
(\sigma,|\Delta|) = \begin{cases} (0,\Delta_1), & \text{if } 0 \leq \frac{G_S}{H_S} < \frac{2}{3(1+C)}, \\ (\sigma_2,\Delta_2), & \text{if } \frac{2}{3(1+C)} < \frac{G_S}{H_S} < \frac{2}{3}, \\ (\sigma_1,0), & \text{if } \frac{G_S}{H_S} > \frac{2}{3}, \end{cases} \quad (23)
\]
with \(C = (2H_S\Lambda_4^2/\pi^2 - 1)/3\) and \(\Lambda_4\) denoting the 4D Euclidean momentum cutoff in the 4D two-flavor NJL.
model, if the necessary conditions $G_S \Lambda_4^2 > \pi^2/3$ and $H_S \Lambda_4^2 > \pi^2/2$ for non-zero $\sigma$ and $\Delta$ are satisfied,\cite{Zhou07} and

$$
(\sigma, |\Delta|) = \begin{cases} 
(0, \Delta_1) , & \text{if } \frac{G_S}{H_S} = 0 , \\
(\sigma_2, \Delta_2) , & \text{if } 0 < \frac{G_S}{H_S} < \frac{2}{3} , \\
(\sigma_1, 0) , & \text{if } \frac{G_S}{H_S} > \frac{2}{3} ,
\end{cases}
$$

(24)
in the 2D two-flavor GN model.\cite{Zhou07} In Eqs. (23) and (24), $G_S$ and $H_S$ always represent the coupling constants in scalar quark-antiquark channel and scalar diquark channel separately.

By a comparison among Eqs. (22) $\sim$ (24) it may be found that the three models lead to very similar results. In all the three models, the interplay between the quark-antiquark and the diquark condensates in vacuum depends on the ratio $G_S/H_D$ ($D = S$ for the 4D and 2D model and $D = P$ for the 3D model). In particular, the diquark condensates could emerge (in separate or coexistent pattern) only if $G_S/H_D < 2/3$. This is probably a general characteristic of the considered two-flavor four-fermion models, since in these models the color number of the quarks participating in the diquark condensates and in the quark-antiquark condensates is just 2 and 3 respectively. However, there are also some differences in the pattern realizing the diquark condensates among the three models, though the pure quark-antiquark condensates arise only if $G_S/H_D > 2/3$ in all of them. In the 2D GN model, the pure diquark condensates emerge only if $G_S/H_S = 0$ and this is different from the 4D NJL model where the pure diquark condensates may arise if $G_S/H_S$ is in a finite region below 2/3. Another difference is that in the 3D GN model, there is no coexistence of the quark-antiquark condensates and the diquark condensates but such coexistence is clearly displayed in the 4D and 2D model. This implies that in the 3D GN model, $G_S/H_P = 2/3$ becomes the critical value which distinguishes between the ground states with the pure diquark condensates and with the pure quark-antiquark condensates.

It is also indicated that if the two-flavor four-fermion interaction models are assumed to be simulations of QCD (of course, only the 4D NJL model is just the true one) and the four-fermion interactions are supposed to come from the heavy color gluon exchange interactions $-g(\bar{q}i\gamma^\mu\lambda_a q)^2$ ($a = 1,\ldots,8; \mu = 0,\ldots,D-1$) via the Fierz transformation,\cite{Buballa05} then one will find that in all the three models, for the case of two flavors and three colors the ratio $G_S/H_D$ is always equal to 4/3, which is larger than the above critical value 2/3. From this we can conclude that there will be only the pure quark-antiquark condensates and no diquark condensates in the ground states of all these models in vacuum.

References

[1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345; 124 (1961) 246.
[2] Zhou Bang-Rong, Commun. Theor. Phys. (Beijing, China) 47 (2007) 95.
[3] D.J. Gross and A. Neveu, Phys. Rev. D 10 (1974) 3235.
[4] Zhou Bang-Rong, Commun. Theor. Phys. (Beijing, China) 47 (2007) 520.
[5] B. Rosenstein, B.J. Warr, and S.H. Park, Phys. Rep. 205 (1991) 59.
[6] Bang-Rong Zhou, Phys. Lett. B 444 (1998) 455.
[7] M. Buballa, Phys. Rep. 407 (2005) 205.
[8] Y. Nambu, Phys. Rev. 117 (1960) 648; L.P. Gorkov, JETP 7 (1958) 993.