ON CALCULUS WITH A QUATERNIONIC VARIABLE AND ITS CHARACTERISTIC CAUCHY-RIEMANN TYPE EQUATIONS

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ABSTRACT. We show sufficient and necessary conditions, in terms of partial differential equations with variable coefficients, for a quaternionic function to admit a continuous derivative in a open set in the sense of C. Schwartz.

1. INTRODUCTION

Let $\mathbb{H}$ denote the quaternions, let $q = t + xi + yj + zk$ be a quaternion written as $q = t + ri$, where $t$ is the real part of the quaternion, $r \geq 0$ and $i^2 = -1$. The problem of the quaternionic left-derivative starts by considering the limit

$$\lim_{h \to 0} h^{-1}(f(q + h) - f(q) - hA) = 0$$

for some number $A$. It is well known that the only functions left-derivable in $\mathbb{H}$ in the above sense turn out to be $f(q) = qa + b$ for some $a, b \in \mathbb{H}$. A new approach for this problem, which results in a larger class of functions has been introduced in [2].

**Definition 1.** Let $f$ be a quaternionic function that is continuous and real-differentiable at $q$. We say $f$ is $S$-derivable by the left at $q$ if there exists quaternions $A$ or $B$ and $C$ such that the limit

$$\lim_{h \to 0} \begin{cases} h^{-1}(f(q + h) - f(q) - hA) & \text{if } q \in \mathbb{R} \\ h^{-1}(f(q + h) - f(q) - hA - hB - hC) & \text{if } q \in \mathbb{H} \setminus \mathbb{R} \end{cases}$$

vanishes at $q$. Then we say the function $f$ is $S$-derivable at $q$. The numbers $A$ and $B$ we shall call the parallel derivative and $C$ the perpendicular derivative at $q$.

Just like in the complex setting we would like to consider functions that are derivable in this new, broader sense in some open set $\Omega \subseteq \mathbb{H}$.

**Definition 2.** Let $f : \Omega \subseteq \mathbb{H} \to \mathbb{H}$ be a quaternionic function that is continuously real-differentiable. We say $f$ is $S$-derivable by the left in $\Omega$ if there exists continuous quaternionic functions $A(q) : \Omega \cap \mathbb{R} \to \mathbb{H}$ and $B(q), C(q) : \Omega \setminus \mathbb{R} \to \mathbb{H}$ such that

$$\lim_{h \to 0} \begin{cases} h^{-1}(f(q + h) - f(q) - hA(q)) = 0 & \text{if } q \in \mathbb{R} \\ h^{-1}(f(q + h) - f(q) - hB(q) - hC(q)) = 0 & \text{if } q \in \mathbb{H} \setminus \mathbb{R} \end{cases}$$

for all $q \in \Omega \subseteq \mathbb{H}$.
In the complex case one obtains characteristic equations, namely the Cauchy-Riemann equations by evaluation of the derivative with two prescribed directions, the real and the pure imaginary. The result we consider first that this class can be characterized by some partial differential equations, just like in the commutative complex setting. More precisely:

**Theorem 1.** A function quaternionic function $f$ whose partial derivatives are continuous in some open set $\Omega$, then $f$ is continuously $S$-derivable in $\Omega$ if, and only if, it satisfies the following partial differential equations in $\Omega \cap \mathbb{R}$

\[
\frac{\partial f}{\partial x} = i \frac{\partial f}{\partial t} \quad (1)
\]
\[
\frac{\partial f}{\partial y} = j \frac{\partial f}{\partial t} \quad (2)
\]
\[
\frac{\partial f}{\partial z} = k \frac{\partial f}{\partial t} \quad (3)
\]

and the following partial differential equations in $\Omega \setminus \mathbb{R}$

\[
\frac{\partial f}{\partial x} = \left(\frac{i - i\nu}{2}\right) \frac{\partial f}{\partial t} - \left(\frac{i + i\nu}{4}\right)Df \quad (4)
\]
\[
\frac{\partial f}{\partial y} = \left(\frac{j - j\nu}{2}\right) \frac{\partial f}{\partial t} - \left(\frac{j + j\nu}{4}\right)Df \quad (5)
\]
\[
\frac{\partial f}{\partial z} = \left(\frac{k - k\nu}{2}\right) \frac{\partial f}{\partial t} - \left(\frac{k + k\nu}{4}\right)Df \quad (6)
\]

where $D$ denotes the left Fueter operator:

\[
Df := \frac{\partial f}{\partial t} + i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}
\]

This equations clearly play here the role of the Cauchy-Riemann equations played in the commutative setting. We then show that if a function is $S$-derivable then is necessarily Cullen-regular (also termed Slide regular) in the sense of G. Gentili and D.C Struppa. Cullen-regular functions have been shown, see [1] to be expressible as absolutely convergent quaternionic unilateral power series assuming the domain of the function is a ball centered at the origin with radius $R$. One consequence of the previous result is that it allows to state the following characterization of a continuously $S$-derivable function in a ball of radius $R$ centered at the origin.

**Theorem 2.** Let $f : B(0, R) \to \mathbb{H}$ be a $C^1$ quaternionic function. Then $f$ is $S$-derivable by the left if and only if it is expressible as a absolutely convergent power series

\[
f(q) = \sum_{k=0}^{\infty} q^k a_k, \; a_k \in \mathbb{H}
\]

and

\[
A(q) = \frac{\partial f}{\partial t}, \; q \in B(0, R) \cap \mathbb{R}
\]
\[
B(q) = \frac{\partial f}{\partial t}, \; q \in B(0, R) \cap \mathbb{H} \setminus \mathbb{R}
\]
Thus recovering the original result of C. Schwartz.

2. Necessary condition for S-differentiability using directions based on 1, i, j and k

In complex calculus the necessity of the Cauchy-Riemann equations is usually obtained by considering the limit that defines the derivative in the real and pure imaginary directions at a point. Proceeding analogously with the quaternionic case one obtains corresponding identities.

**Proposition 1.** Let \( f \) be continuously S-differentiable in \( \Omega \), then

\[
A(q) = \frac{\partial f}{\partial t}(q), q \in \Omega
\]

*Proof.* By the nature of the definition we must check the cases when the base point \( q \) is real or not. Let \( q \) be real and \( f \) S-derivable at \( q \). Choosing \( h \) as a real number and after taking the limit as \( h \to 0 \) we obtain

\[
A(q) = \frac{\partial f}{\partial t}(q)
\]

Now assume \( q \) is not real. Choose again \( h \) a real number. Since \( h \) is real then \( h_\parallel = h \) and \( h_\perp = 0 \), so if \( f \) is S-differentiable we have

\[
B(q) = \frac{\partial f}{\partial t}(q)
\]

for all \( q \in \Omega \setminus \mathbb{R} \). \( \square \)

**Proposition 2.** Let \( f \) be continuously S-differentiable, then \( f \) satisfies equations (1) to (6) and

\[
C(q) = -\left(\frac{1}{2}\right)Df(q)
\]

where \( D \) is the left Fueter operator.

*Proof.* The preceding proposition means \( A(q) \) and \( B(q) \) can be replaced by \( \frac{\partial f}{\partial t} \) without loss of generality. Again we start with \( q \) real. Choosing \( h \) as \( \epsilon i, \epsilon j \) and \( \epsilon k \) where \( \epsilon \) is real and taking the corresponding limit as \( \epsilon \) goes to zero we obtain necessary conditions for the partial derivatives with respect to \( x, y \) and \( z \). If \( q \) is real we obtain immediately equations (1), (2) and (3). If \( q \) is non-real then we obtain the following equations, with \( C(q) \) a function to be determined:

\[
\frac{\partial f}{\partial x} = \left(\frac{i - i\epsilon i}{2}\right)\frac{\partial f}{\partial t} + \left(\frac{i + i\epsilon i}{2}\right)C(q), \quad (7)
\]

\[
\frac{\partial f}{\partial y} = \left(\frac{j - j\epsilon i}{2}\right)\frac{\partial f}{\partial t} + \left(\frac{j + j\epsilon i}{2}\right)C(q), \quad (8)
\]

\[
\frac{\partial f}{\partial z} = \left(\frac{k - k\epsilon i}{2}\right)\frac{\partial f}{\partial t} + \left(\frac{k + k\epsilon i}{2}\right)C(q), \quad (9)
\]

For this we multiply Eq. (7) by the left by \( i \), Eq. (8) by the left by \( j \) and Eq. (9) by the left by \( k \).
by the left by \( k \) and sum all three resulting equations. For the left hand side one thus obtains:

\[
\frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial y}.
\]

The right hand side is further simplified with the following identities:

\[
i \left( i - \frac{\imath i}{2} \right) + j \left( j - \frac{\imath j}{2} \right) + k \left( k - \frac{\imath k}{2} \right) = -1,
\]

and

\[
i \left( i + \frac{\imath i}{2} \right) + j \left( j + \frac{\imath j}{2} \right) + k \left( k + \frac{\imath k}{2} \right) = -2.
\]

Thus one obtains

\[
\frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial y} = -\frac{\partial f}{\partial t} - 2C(q)
\]

or

\[
\frac{\partial f}{\partial t} + i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} = -2C(q)
\]

\[\square\]

3. Sufficient conditions for the S-derivability

**Proposition 3.** Let \( f : \Omega \to \mathbb{H} \) be continuously real-differentiable and suppose that \( f \) satisfies Eq. (1) to (6) in \( \Omega \), then \( f \) is S-derivable in \( \Omega \).

**Proof.** Because \( f \) is continuously real differentiable in \( \Omega \) then:

\[
\lim_{h \to 0} \frac{|f(q+h) - f(q) - M_fh|}{|h|} = 0
\]

where \( M_f \) is the Jacobi matrix of \( f \) at \( q \). Now writing \( h = h_0 + ih_1 + jh_2 + kh_3 \), \( h_n \in \mathbb{R} \) we can also write

\[
M_fh = \frac{\partial f}{\partial t} h_0 + \frac{\partial f}{\partial y} h_1 + \frac{\partial f}{\partial y} h_2 + \frac{\partial f}{\partial z} h_3.
\]

If \( q \) is real one starts with and use Eq. (1), (2) and (3), to obtain

\[
M_fh = h_0 \frac{\partial f}{\partial t}.
\]

For \( q \) non real, and if \( f \) satisfies equations (4),(5) and (6) we have

\[
\frac{\partial f}{\partial t} h_0 + \frac{\partial f}{\partial y} h_1 + \frac{\partial f}{\partial y} h_2 + \frac{\partial f}{\partial z} h_3 =
\]

\[
\frac{\partial f}{\partial t} h_0 + \left( \frac{ih_1 - i\imath h_1\imath}{2} \right) \frac{\partial f}{\partial t} - \left( \frac{ih_1 + i\imath h_1\imath}{4} \right) Df
\]

\[
+ \left( \frac{jh_2 - \imath j h_2\imath}{2} \right) \frac{\partial f}{\partial t} - \left( \frac{jh_2 + \imath j h_2\imath}{4} \right) Df
\]

\[
+ \left( \frac{kh_3 - \imath k h_3\imath}{2} \right) \frac{\partial f}{\partial t} - \left( \frac{kh_3 + \imath k h_3\imath}{4} \right) Df.
\]

which can be rewritten in the compact form

\[
M_fh = h_0 \frac{\partial f}{\partial t} + h_\perp \left( \frac{-1}{2} Df \right)
\]

therefore we conclude that for \( q \) real:

\[
\lim_{h \to 0} \frac{h^{-1}|f(q+h) - f(q) - \frac{\partial f}{\partial t}|}{|h|} = \lim_{h \to 0} \frac{|f(q+h) - f(q) - M_fh|}{|h|} = 0
\]
which implies
\[ \lim_{h \to 0} h^{-1}(f(q+h) - f(q) - h \frac{\partial f}{\partial t}) = 0 \]
and similarly, for a \( q \) non-real:
\[ \lim_{h \to 0} \left| h^{-1}(f(q+h) - f(q) - h \frac{\partial f}{\partial t} + h \frac{1}{2} Df) \right| = \lim_{h \to 0} \frac{|h(f(q+h) - f(q) - M_f h)|}{|h|} = 0 \]
which implies
\[ \lim_{h \to 0} h^{-1}(f(q+h) - f(q) - h \frac{\partial f}{\partial t} + h \frac{1}{2} Df) = 0 \]
and therefore we have found the \( A(q), B(q) \) and \( C(q) \) required for this function \( f \) to be continuously S-derivable. \( \square \)

4. Cullen-regularity and commuting directional derivatives

The above section shows that the four directions \( 1, i, j \) and \( k \) are sufficient and necessary for finding characteristic equations. However \( i, j \) and \( k \) are mere examples of \( \sqrt{-1} \) in \( \mathbb{H} \). Let \( \mathbb{S}^2 \subseteq \mathbb{H} \) be the set of roots of \( -1 \), and fix \( I \in \mathbb{S}^2 \). Since \( I^2 = -1 \) the set \( \mathbb{R} + I\mathbb{R} \) is an isomorphic copy of \( \mathbb{C} \). Now let \( f \) be continuously S-derivable in some \( \Omega \) whose intersection with the real numbers is non-empty. For a real \( q \) we consider
\[ \lim_{\epsilon \to 0} (I\epsilon)^{-1}(f(q + \epsilon I) - f(q)) = \frac{\partial f}{\partial t} \]
on the other side, interpreting the left-hand side as a directional derivative:
\[ \lim_{\epsilon \to 0} \frac{f(q + \epsilon I) - f(q)}{\epsilon} = \frac{\partial f}{\partial r} \]
when restricted to \( \mathbb{R} + I\mathbb{R} \). Thus we conclude that for \( q \) a real number the expression
\[ \frac{\partial f}{\partial t} + I \frac{\partial f}{\partial r} = 0 \]
vanishes in \( \mathbb{R} + I\mathbb{R} \). If \( q \in \mathbb{H} \setminus \mathbb{R} \), write it as \( q = t + rI \), for some \( I^2 = -1 \) and \( r \geq 0 \). Consider increments only in the imaginary component of \( \mathbb{R} + I\mathbb{R} \): \( h = \epsilon I \), then \( h\parallel = \epsilon I \), \( h\perp = 0 \). Then
\[ \frac{\partial f}{\partial r} = \lim_{\epsilon \to 0} \frac{f(q + \epsilon I) - f(q)}{\epsilon} = I \frac{\partial f}{\partial t} \]
and, as \( I^2 = -1 \):
\[ \frac{\partial f}{\partial t} + I \frac{\partial f}{\partial r} = 0 \]
at \( q \). Let \( \iota : \mathbb{H} \setminus \mathbb{R} \to \mathbb{S}^2 \subseteq \mathbb{H} \setminus \mathbb{R} \) be the function that sends a quaternion \( q = t + rI \) to \( I \). Then the above means that \( f \) satisfies
\[ \left( \frac{\partial}{\partial t} + (\iota \frac{\partial}{\partial r}) \right) f = 0. \]
This discussion shows that S-derivable functions are necessarily Cullen-regular in the sense of G. Gentili and D.C Struppa.
5. The iota-derivative and the perpendicular derivative

The previous section only considered increments that commuted with the variable \( q \). Start parametrizing the function \( \iota \) as \( \cos \alpha \sin \beta i + \sin \alpha \sin \beta j + \cos \beta \) writing the left Fueter operator as

\[
Df = \frac{\partial f}{\partial t} + i \frac{\partial f}{\partial r} - \frac{1}{r} \frac{\partial f}{\partial \iota}
\]

where

\[
\frac{\partial f}{\partial \iota} := (\frac{\partial \iota}{\partial \alpha})^{-1} \frac{\partial}{\partial \alpha} + (\frac{\partial \iota}{\partial \beta})^{-1} \frac{\partial}{\partial \beta}
\]

Which is defined outside the singular subplane \( \mathbb{R} + k\mathbb{R} \). observe that \( (\frac{\partial \iota}{\partial \alpha})^{-1} \) and \( (\frac{\partial \iota}{\partial \beta})^{-1} \) anti-commute with the function \( \iota \), so they are geometrically perpendicular to \( \iota \). For any continuously real-derivable Cullen-regular function \( f \) we can write

\[
Df = \frac{-1}{r} \frac{\partial f}{\partial \iota}
\]

so the perpendicular derivative can also be written, without loss of generality as

\[
\frac{1}{2r} \frac{\partial f}{\partial \iota}.
\]

6. A criterium for the S-derivability

**Theorem 3.** Let \( f : \Omega \to \mathbb{H} \) be a real-derivable quaternionic function such that

- \( f \) is S-derivable in \( \Omega \cap \mathbb{R} \).
- \( f = u + iv \) for some \( C^1 \) quaternionic functions \( u, v \).
- \( u \) and \( v \) are independent of \( \alpha \) and \( \beta \).
- \( u \) and \( v \) satisfy:
  \[
  \frac{\partial u}{\partial t} - \frac{\partial v}{\partial r} = \frac{\partial u}{\partial r} + \frac{\partial v}{\partial t} = 0
  \]

then with this conditions \( f \) is continuously left S-derivable in \( \Omega \) and its perpendicular derivative is \( \frac{v}{r} \).

**Proof.** Let \( f \) satisfy the hypothesis. We show \( f \) is continuously S-derivable in \( \Omega \setminus \mathbb{R} \).

Computing the partial derivatives:

\[
\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i_x v + i \frac{\partial v}{\partial x}
\]

\[
\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i_y v + i \frac{\partial v}{\partial y}
\]

\[
\frac{\partial f}{\partial z} = \frac{\partial u}{\partial z} + i_z v + i \frac{\partial v}{\partial z}
\]

on the other hand, since neither \( u \) nor \( v \) depend on \( \alpha \) and \( \beta \) implies that

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial x}{\partial r}
\]

\[
\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} = \frac{\partial u}{\partial r} \frac{\partial y}{\partial r}
\]

\[
\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial z} = \frac{\partial u}{\partial r} \frac{\partial z}{\partial r}
\]

\[
\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r} \frac{\partial r}{\partial x} = \frac{\partial v}{\partial r} \frac{\partial x}{\partial r}
\]

\[
\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r} \frac{\partial r}{\partial y} = \frac{\partial v}{\partial r} \frac{\partial y}{\partial r}
\]

\[
\frac{\partial v}{\partial z} = \frac{\partial v}{\partial r} \frac{\partial r}{\partial z} = \frac{\partial v}{\partial r} \frac{\partial z}{\partial r}
\]
thus we can write
\[ \frac{\partial f}{\partial x} = \frac{x \partial f}{r \partial r} + txv \]
\[ \frac{\partial f}{\partial y} = \frac{y \partial f}{r \partial r} + tyv \]
\[ \frac{\partial f}{\partial z} = \frac{z \partial f}{r \partial r} + tzv \]
the hypothesis on \( f \) imply that we can replace \( \frac{\partial f}{\partial r} \) with \( i \frac{\partial f}{\partial t} \).
\[ \frac{\partial f}{\partial x} = \frac{x}{r} \frac{\partial f}{\partial t} + txv \]
\[ \frac{\partial f}{\partial y} = \frac{y}{r} \frac{\partial f}{\partial t} + tyv \]
\[ \frac{\partial f}{\partial z} = \frac{z}{r} \frac{\partial f}{\partial t} + tzv \]
finally we need the following identities:
\[ \frac{x}{r} = (\frac{i-ii}{2}), \quad \frac{y}{r} = (\frac{j-ij}{2}), \quad \frac{z}{r} = (\frac{k-ik}{2}) \]
and
\[ \frac{\partial x}{r} = \frac{1}{r} (\frac{i+ii}{2}), \quad \frac{\partial y}{r} = \frac{1}{r} (\frac{j+ij}{2}), \quad \frac{\partial z}{r} = \frac{1}{r} (\frac{k+ik}{2}) \]
Since the function is Cullen-regular and neither \( u \) nor \( v \) depend of \( \alpha \) and \( \beta \) we conclude that
\[ Df = -2v \]
so \( f \) satisfies equations (4),(5) and (6).
\[ \square \]

**Corollary 1.** The function \( u \) is continuously S-derivable in \( \mathbb{H} \setminus \mathbb{R} \).

**Proof.** Take \( u = 0 \) and \( v = 1 \). \( \square \)

**Corollary 2.** The power function is S-derivable in \( \mathbb{H} \).

**Proof.** Let \( f(q) = q^n \) for some \( n \in \mathbb{N} \), \( f \) is continuously real-derivable in \( \mathbb{H} \). Let \( q \) be a real number. Then
\[ \frac{\partial}{\partial t} q^n = nq^{n-1} \]
\[ \frac{\partial}{\partial x} q^n = iq^{n-1} \]
\[ \frac{\partial}{\partial y} q^n = jq^{n-1} \]
\[ \frac{\partial}{\partial z} q^n = kq^{n-1} \]
and therefore \( q^n \) is S-derivable in \( \mathbb{R} \). Since \( q^n = u_n(t,r) + iv_n(t,r) \), for some real functions \( u_n \) and \( v_n \) satisfying
\[ \frac{\partial u_n}{\partial t} - \frac{\partial v_n}{\partial r} = \frac{\partial u_n}{\partial r} + \frac{\partial v_n}{\partial t} = 0 \]
we obtain the desired result. \( \square \)
Corollary 3. Let
\[ f(q) = \sum_{n=0}^{\infty} q^n a_n \]
be an absolutely and uniformly convergent series in a ball \( B(0, R) \). Then \( f \) is S-derivable in \( B(0, R) \).

Proof. Observe such \( f \) is real-derivable in \( B(0, 1) \), (in fact is real-analytic). Let \( q \in B(0, R) \) be real. Uniform and absolute convergence means the series can be partial-derivated termwise. Thus one obtains that \( f \) satisfies equations (1),(2) and (3) for every \( q \) in \( B(0, R) \cap \mathbb{R} \). On the other side we have
\[
\sum_{n=0}^{\infty} q^n a_n = \sum_{n=0}^{\infty} u_n a_n + i \sum_{n=0}^{\infty} v_n a_n = u(t, r) + i v(t, r)
\]
and such \( u \) and \( v \) satisfy:
\[
\frac{\partial u}{\partial t} - \frac{\partial v}{\partial r} = \frac{\partial u}{\partial r} + \frac{\partial v}{\partial t} = 0
\]

Observe that for functions constructed as in this section one has \( f(\overline{q}) = u(q) - i v(q) \). So they also must also satisfy the following equalities:
\[
\frac{1}{2r} \frac{\partial f}{\partial t} = \frac{v}{r} = (ir)^{-1}(iv) = (q - \overline{q})^{-1}(f(q) - f(\overline{q}))
\]
We are now ready to prove Theorem 2.

Proof. In [1] it is proved that every Cullen-regular functions \( f : B(0, R) \rightarrow \mathbb{H} \) can be expressed as such power series. Since S-derivable functions are in particular Cullen-regular this theorem remains valid. On the other side, functions expressible as such power series have been shown to be S-derivable themselves. The identities \( A(q) = B(q) = \frac{\partial f}{\partial t} \) and \( Df = -2(q - \overline{q})^{-1}(f(q) - f(\overline{q})) \) holds for all functions constructed as in Theorem 3 and such power series are of this form. □

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