STOCHASTIC WEAK PASSIVITY BASED STABILIZATION OF
STOCHASTIC SYSTEMS WITH NONVANISHING NOISE

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Abstract. For stochastic systems with nonvanishing noise, i.e., at the desired state the noise port does not vanish, it is impossible to achieve the global stability of the desired state in the sense of probability. This bad property also leads to the loss of stochastic passivity at the desired state if a radially unbounded Lyapunov function is expected as the storage function. To characterize a certain (globally) stable behavior for such a class of systems, the stochastic asymptotic weak stability is proposed in this paper which suggests the transition measure of the state to be convergent and the ergodicity. By defining stochastic weak passivity that admits stochastic passivity only outside a ball centered around the desired state but not in the whole state space, we develop stochastic weak passivity theorems to ensure that the stochastic systems with nonvanishing noise can be globally locally stabilized in weak sense through negative feedback law. Applications are shown to stochastic linear systems and a nonlinear process system, and some simulation are made on the latter further.

Key words. Stochastic differential systems, transition measure, ergodicity, stochastic weak passivity, asymptotic weak stability, stabilization

AMS subject classifications. 60H10, 62E20, 70K20, 93C10, 93D15, 93E15

1. Introduction. Stochastic phenomena have emerged universally in many physical systems due to noise, disturbance and uncertainty. The unpredictability to them leads to it a great challenge to stabilize a stochastic system. During the past decades, the stabilization of nonlinear stochastic systems had constituted one of central problems in stochastic process control both theoretically and practically. A great deal of methods emerge as the times require, among which stochastic passivity based control is a popular one. Rooting in the passivity theory [2,17] and the stochastic version of Lyapunov theorem [6], the stochastic passivity theory [4] was developed for stabilization and control of nonlinear stochastic systems. By means of state feedback laws, the asymptotic stabilization in probability can be achieved for a stochastic affine system provided some rank conditions are fulfilled and the unforced stochastic affine system is Lyapunov stable in probability [4]. Following this study, Lin et. al. [8] explored the relationship between a stochastic passive system and the corresponding zero-output system, and further established the global stabilization results. Parallelizing to the development of stochastic passivity in theory, Satoh et. al. [12] applied this methodology to port-Hamiltonian systems, and the solutions for stabilization of a large class of nonlinear stochastic systems are thus available. There are also some reports that stochastic passivity is applied to $H_{\infty}$ filtering problem [19] and controlling stochastic mechanical systems [10].

Despite the large success achieved, stochastic passivity based control seems to only work under the condition that the noise vanishes at the stationary solution (very often being at the origin) if a radially unbounded Lyapunov function is expected as the storage function. This means that if a stochastic system has nonzero noise port at the stationary solution or has persistent noise port, such a method may be out of action. One of the aims of this paper is to derive the necessary conditions that a
stochastic system is stochastically passive, and further give the sufficient conditions to say a stochastic system losing stochastic passivity. Equivalently, we prove that there does not exist a radially unbounded Lyapunov function rendering the stochastic system to be globally asymptotically stable in probability provided the noise does not vanish at the desired state. The ubiquitousness of such a class of systems in the mechanical [13,14] and biological [3] fields motivates us to define a kind of novel stability, termed as stochastic asymptotic weak stability, to characterize a certain (globally) stable behavior for them. The stochastic asymptotic weak stability requests the system state to be convergent in distribution and ergodic. The former means the state to evolve within a small region around the desired state in a large probability while the latter ensures that the state evolution almost always take place within this region.

On the face of it, the stochastic asymptotic weak stability is somewhat similar to the concept of stochastic bounded stability proposed in [13,14] in that a stochastic system with persistent noise is considered for the same purpose. That concept also means that the state will evolve within a bounded region around a desired state with a large probability which depends on the region radius. Especially when the region radius goes infinite, the probability will be one. However, there is evident difference between these two kinds of stability. Stochastic bounded stability cannot characterize the ergodicity of the state. Namely, once the trajectory of the state runs out of the bounded region with a small probability, the coming evolution will take place in a larger bounded region to reach a “new” stochastic bounded stability with a larger probability. In addition, the stochastic asymptotic weak stability is different from stochastic noise-to-state [1,3] and input-to-state stability [9] too. The latter two kinds of stability also serves for characterizing the stable behavior of stochastic systems with nonvanishing noise. They describe the convergence of the expectation of the state, for which the transition measure is controlled by defining a particular function. Comparatively speaking, they say nothing about the ergodicity of the state, and do not mean either that the state must evolve within a small region around the desired state. Therefore, the stochastic asymptotic weak stability is able to provide more details on characterizing the “stable” evolution of the state.

In the concept of stochastic asymptotic weak stability, the convergence in distribution describes the evolution trend of the probability distribution of the stochastic system under consideration. As one may know, for a stochastic system the probability density function satisfies the Fokker-Planck equation [6]. Hence, a usual way to achieve convergence in distribution often starts from analyzing the properties of the solutions of the Fokker-Planck equation, including the existence, uniqueness and convergence. Based on this equation, Zhu et al. [20,21] studied the exact stationary solution of distribution density function for stochastic Hamiltonian systems. Liberzon et. al. [7] developed a feedback controller to stabilize in distribution a class of nonlinear stochastic systems for which the steady-state distribution density function can be solved from the Fokker-Planck equation. In addition, probability analysis is another way to serve for achieving the weak stability. Zakai [18] presented a Lyapunov criterion to suggest the existence of stationary probability distribution and the convergence of transition probability measure for stochastic systems with globally Lipschitzian coefficients. Stettner [15] pointed out that the strongly feller and irreducible process are stable in distribution. Khasminskii [5] constructed a Markov chain to analyze the convergence of the probability distribution, and further obtained the Markov process to be convergent in distribution [6] if it is “mix sufficiently well”
in an open domain and the recurrent time is finite. The conditions that renders the recurrent time to be finite give us large inspiration on developing the stabilizing ways in weak sense for stochastic systems with nonvanishing noise.

In this paper, we will show that the recurrent property of a stochastic system is highly relevant to the stochastic passivity behavior. Based on this comparison, we define the stochastic passivity not in the whole state space, but only outside a ball centered around the desired state, which is labeled as stochastic weak passivity in the context. Within the framework of stochastic weak passivity, we do not need to care whether the noise port of a stochastic system vanishes at the desired state or not. Therefore, it is suited to handle the stabilization issue of stochastic differential systems with nonvanishing noise. Further, we link the stochastic weak passivity with the stochastic asymptotic weak stability, and develop stabilizing controllers using the stochastic weak passivity to achieve the asymptotic weak stability of stochastic systems. The sufficient conditions for global and local asymptotic stabilization in weak sense are provided by means of negative feedback laws, respectively.

The rest of the paper is organized as follows. Section 2 presents some preliminaries on stochastic passivity. In section 3, the loss of stochastic passivity is analyzed and the problem of interest is formulated. In section 4, we propose the framework of stochastic weak passivity theory and make a link between stochastic weak passivity and asymptotic weak stability. Some basic concepts and the main results (expressed as two stochastic weak passivity theorem and one refined version) for stabilizing stochastic systems in weak sense are given in this section. Section 5 illustrates the efficiency of the stochastic weak passivity theory through two application examples. Finally, section 6 concludes this paper and makes a prospect of future research.

2. Preliminaries of stochastic passivity. In this section, we will give a birds-eye view of mathematical systems theory related to stochastic differential systems.

We begin with a stochastic differential equation written in the sense of Itô

\[ dx = f(x)dt + h(x)d\omega \]  

where \( x \in \mathbb{R}^n \), \( t \in \mathbb{R}_{\geq 0} \), \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( h : \mathbb{R}^n \to \mathbb{R}^{n \times r} \) are locally Lipschitz continuous functions, and \( \omega \in \mathbb{R}^r \) is a standard Wiener process defined on a complete probability space. Assume \( x(t) \) to be the stochastic process solution and \( x^* \) to be the equilibrium solution (if exists) of Eq. \( (2.1) \), then we have

**Definition 2.1 (Transition Measure [6]).** The transition measure of \( x(t) \), denoted by \( \mathcal{P}(\cdot, \cdot, \cdot) \), is a function from \( \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathcal{B} \) to \([0, 1]\) such that

\[ \mathcal{P}(t, x_0, A) = \mathbb{P}(x(t) \in A | x(0) = x_0) \]  

where \( \mathcal{B} \) is the \( \sigma \)-algebra of Borel sets in \( \mathbb{R}^n \), \( A \subset \mathcal{B} \) is a Borel subset, and \( \mathbb{P}(\cdot) \) denotes the probability function.

**Definition 2.2 (Invariant Measure [6]).** Let \( \pi \) be a measure defined on a Borel space \( \mathcal{B} \), then \( \pi \) is an probability invariant measure for a stochastic system of Eq. \( (2.1) \) if \( \pi(\mathbb{R}^n) = 1 \) and

\[ \pi(A) = \int_{\mathbb{R}^n} \mathcal{P}(t, x, \Delta)\pi(dx), \quad \forall \ t > 0 \text{ and } \forall \ A \in \mathcal{B} \]  

**Definition 2.3 (Stable in Probability [6]).** The equilibrium solution \( x^* \) of Eq. \( (2.1) \) is
(1) stable in probability if
\[
\lim_{x(0) \to x^*} P \left( \sup_t \| x(t) - x^* \|_2 < \epsilon \right) = 1, \forall \epsilon > 0;
\]

(2) locally asymptotically stable in probability if
\[
\lim_{x(0) \to x^*} P \left( \lim_{t \to \infty} \| x(t) - x^* \|_2 = 0 \right) = 1;
\]

(3) globally asymptotically stable in probability if
\[
P \left( \lim_{t \to \infty} \| x(t) - x^* \|_2 = 0 \right) = 1, \forall x(0).
\]

In order to analyze the stability of stochastic systems, the stochastic version of
the second Lyapunov theorem and passivity theorem were proposed in succession.

**Theorem 1** (Stochastic Lyapunov Theorem [6]). If there exists a positive definite \( \mathcal{C}^2(\mathbb{D}; \mathbb{R}) \) function \( V(x) \) with respect to \( x-x^* \) such that
\[
\mathcal{L}[V(x)] \leq 0, \forall x \in \mathbb{D}
\]
then the equilibrium solution \( x^* \) of Eq. (2.1) is stable in probability, where \( \mathbb{D} \subseteq \mathbb{R}^n \)
is a bounded open neighborhood of \( x^* \) and \( \mathcal{L}[\cdot] \) is the infinitesimal generator of the
solution of Eq. (2.1), calculated through
\[
\mathcal{L}[\cdot] = \frac{\partial}{\partial x} f + \frac{1}{2} \text{tr} \left\{ \frac{\partial^2}{\partial x^2} h h^\top \right\}
\]

If the equality in Eq. (2.4) holds if and only if \( x = x^* \), then \( x^* \) is locally asymptotically stable in probability.

Further, if \( \mathbb{D} = \mathbb{R}^n \), \( \lim_{\| x \|_2 \to \infty} V(x) = \infty \) (often said that the Lyapunov function \( V(x) \) is radially unbounded) and \( \mathcal{L}[V(x)] = 0 \Leftrightarrow x = x^* \), then \( x^* \) is globally asymptotically stable in probability.

The stochastic passivity theorem is not handed directly from the literature, but
it may be obtained immediately from the definition of stochastic passivity.

**Definition 2.4** (Stochastic Passivity [4]). An input-output stochastic differential system in the sense of Itô
\[
\Sigma_S : \begin{cases}
\dot{x} &= f(x, u) dt + h(x, u) d\omega \\
y &= s(x, u)
\end{cases}
\]
is said to be stochastically passive if there exists a positive semi-definite \( \mathcal{C}^2(\mathbb{R}^n; \mathbb{R}) \)
function \( S(x) \) such that
\[
\mathcal{L}[S(x)] \leq u^\top y, \forall x \in \mathbb{R}^n
\]
where \( x \) is the state, \( u \in U \subseteq \mathbb{R}^m \) the input, \( y \in \mathbb{R}^m \) the output, the drift term \( f : \mathbb{R}^n \times U \to \mathbb{R}^n \), the diffusion term \( h : \mathbb{R}^n \times U \to \mathbb{R}^{n \times r} \) and \( s : \mathbb{R}^n \times U \to \mathbb{R}^m \) all
satisfy the condition of local Lipschitz continuity, and \( t, \omega \) share the same meaning
with those in Eq. (2.1). The nonnegative real function \( S(x) \) is called the storage
function, the state where \( S(x) = 0 \) is the stochastic passive state and the inner product \( u^\top y \) is called the supply rate.

**Result 1 (Stochastic Passivity Theorem).** The negative feedback connection of two stochastic passive systems is stochastically passive.

**Proof.** Let 1 and 2 in the form of subscripts represent these two stochastic passive systems, respectively, then we have

\[
\mathcal{L}[S_1(x_1)] \leq u_1^\top y_1 \quad \text{and} \quad \mathcal{L}[S_2(x_2)] \leq u_2^\top y_2
\]

Define the storage function of their negative feedback connection by

\[
S(x) = S_1(x_1) + S_2(x_2)
\]

and note the fact that

\[
x = (x_1^\top, x_2^\top)^\top, \quad y = y_1 = u_2, \quad u = u_1 + y_2
\]

then we get

\[
\mathcal{L}[S(x)] = \mathcal{L}[S_1(x_1)] + \mathcal{L}[S_2(x_2)] \leq (u - y_2)^\top y_1 + u_2^\top y_2 = u^\top y
\]

Therefore, the result is true.

**Result 2 (Stochastic Passivity and Stability in Probability).** A stochastic passive system with a positive definite storage function is stable in probability if a stochastic passive controller with a positive definite storage function is connected in negative feedback.

**Proof.** Based on Result 1, the whole negative feedback connection is stochastically passive. As long as the input of the stochastic passive system (labeled by the subscript “1”) is manipulated according to \( u_1 = C(y_1, x_2) \), then \( u = 0 \), which means \( \mathcal{L}[S(x)] \leq 0 \). Here, the operator \( C(\cdot) \) is the stochastic passive controller (labeled by the subscript “2”) defined by \( y_2 = -C(u_2, x_2) \). The stability in probability of \( x(t) \) is immediately from Theorem 1, so is that of \( x_1(t) \).

**Remark 1.** Deterministic passive systems are a kind of special cases of stochastic passive systems. Therefore, the frequently-used passive controllers [16], such as PID Controller, Model predictive Controller, etc., can all serve for stabilizing the stochastic passive systems in probability.

3. **Loss of stochastic passivity and Problem setting.** This section contributes to elaborating that stochastic passivity will vanish either in some stochastic systems or when some control problems are addressed, and further to formulating the problem of interest.

3.1. **Loss of stochastic passivity.** As can be known from Definition 2.4, a key point to capture stochastic passivity lies in finding a storage function. We will derive the necessary condition for stochastic passivity in the following, and then get the sufficient condition to say the loss of stochastic passivity. For this purpose, we go back to the stochastic differential equation of Eq. (2.1).

**Theorem 2.** If a stochastic differential equation given by Eq. (2.1) has a global solution, then it must be not stable in probability at those states that result in the nonzero diffusion term.

**Proof.** Let the set of all states result in the nonzero diffusion term be given by

\[
\mathbb{H}_{\neq 0} := \{ x^\top \in \mathbb{R}^n | h(x^\top) \neq 0 \}
\]
For any $x^t \in \mathbb{H}_{x^0}$ and $\gamma > 0$, it is expected that
\[
\lim_{x(0) \to x^t} P\left( \sup \| x(t) - x^t \|_2 < \gamma \right) = 1
\]

must be not true. Towards this purpose, we assume $x^t = 0_n$ for simplicity but without loss of generality (which means $h(0_n) \neq 0_{n \times r}$), and further construct a real-valued function $\tilde{U} : \mathbb{R} \mapsto \mathbb{R}$ in the form of
\[
\tilde{U}(x) = \begin{cases} 
\frac{x^2}{2} - \frac{3}{2}x^3 + 2x^2 - \frac{1}{2}x + \frac{1}{2} & 0 \leq |x| \leq \frac{1}{2} \\
\frac{1}{3}x^3 - \frac{6}{2}x^2 + \frac{25}{6}x - \frac{79}{24} & \frac{1}{2} \leq |x| \leq 3 \\
\frac{25}{12} & |x| > 3
\end{cases}
\]

Based on this function, a positive definite, twice continuously differentiable and bounded real function mapping $\mathbb{R}^n$ to $\mathbb{R}$ is defined by
\[
U(x) = \frac{12}{23} \times \tilde{U}(|x|_2)
\]

Clearly, $U(x) \in [0, 1]$ and, moreover, $U(x) = \frac{12}{23}x^\top x$ in the $\frac{1}{2}$-neighborhood of $0_n$.

In order to finish the proof, we impose the infinitesimal generator $L[\cdot]$ on $U(x)$, and are only concerned about the result at $x = 0_n$. From Eq. (2.5), we have
\[
L[U](0_n) = \frac{12}{23} \text{tr}\{h(0_n)h^\top(0_n)\} > 0.
\]

On the other hand, from the definition of $L[\cdot]$ [6] we get
\[
L[U](0_n) = \lim_{t \to 0} \frac{E^n[U(x(t))] - U(0_n)}{t}
\]

where $0_n$ appearing in $E^n[U(x(t))]$ indicates that the initial condition is $x(0) = 0_n$.

Since the stochastic differential equation (2.1) has a global solution, there exist a time $\tau > 0$ and a constant $c > 0$ so that $E^n[U(x(\tau))] = c\tau > 0$. Also, since
\[
E^n[U(x(\tau))] = E^n[U(x(\tau)) \mid U(x(\tau)) < \epsilon^2]P\left(U(x(\tau)) < \epsilon^2 \mid x(0) = 0_n\right) + E^n[U(x(\tau)) \mid U(x(\tau)) \geq \epsilon^2]P\left(U(x(\tau)) \geq \epsilon^2 \mid x(0) = 0_n\right)
\]

and
\[
E^n[U(x(\tau)) \mid U(x(t)) < \epsilon^2] < \epsilon^2, \quad P\left(U(x(\tau)) < \epsilon^2 \mid x(0) = 0_n\right) \leq 1
\]

together with the fact that
\[
U(x(\tau)) \in [0, 1] \Rightarrow E^n[U(x(\tau)) \mid U(x(\tau)) \geq \epsilon^2] < 1
\]

where $\epsilon$ is any positive number, we have
\[
P\left(U(x(\tau)) \geq \epsilon^2 \mid x(0) = 0_n\right) \geq c\tau - \epsilon^2
\]
We set $\epsilon$ to be sufficiently small so that
\[
P(\|x(\tau)\|_2 \geq \gamma \mid x(0) = 0_n) \geq c\tau - \epsilon^2 > 0
\]
where $\gamma = \sqrt{\frac{12}{12}}\epsilon$.

From the definition, the leftmost term in the above inequality can be calculated by
\[
P(\|x(\tau)\|_2 \geq \gamma \mid x(0) = 0_n) = 
\int_{\|y\|_2 = \delta} P(x(\tau_\delta) = dy \mid x(0) = 0_n)P(\|x(\tau)\|_2 \geq \gamma \mid x(\tau_\delta) = y)
\]
where
\[
\tau_\delta = \tau \land \inf\{ t \mid \|x(t)\|_2 = \delta < \gamma \}
\]
is a stopping time. Then there exists at least one point, denoted by $y_\delta$, on the surface of the ball $\|x(t)\|_2 = \delta$ such that
\[
P(\|x(\tau)\|_2 \geq \gamma \mid x(\tau_\delta) = y_\delta) \geq c\tau - \epsilon^2.
\]

Note that Eq. (2.1) is autonomous, therefore
\[
P\left(\sup_{t \in [0, \infty)} \|x(t)\|_2 \geq \gamma \mid x(0) = y_\delta\right) \geq P(\|x(\tau)\|_2 \geq \gamma \mid x(\tau_\delta) = y_\delta) \geq c\tau - \epsilon^2
\]
Namely, for $\forall \, \delta \leq \gamma$ there always exist a $y_\delta$ to make the above inequality be true.

Clearly, the above inequality suggests that Eq. (2.1) must be not stable in probability at $x = 0_n$. □

It is straightforward to write the inverse negative proposition of Theorem 2 as a corollary.

**Corollary 1.** For a stochastic differential equation in the form of (2.1) with a global solution, if it is stable in probability at a desired state $x^\dagger$ (may be not the equilibrium $x^\ast$), then $x^\dagger$ must belong to $\mathbb{H}_{=0}$ which is defined by
\[
\mathbb{H}_{=0} := \{x^\dagger \in \mathbb{R}^n \mid h(x^\dagger) = 0_{n \times \tau}\}
\]

Note that the above result depends on the condition that the stochastic differential equation (2.1) has a global solution. However, under the condition of local Lipschitz continuity, Eq. (2.1) has a unique solution only before explosion time. Based on this result, we will reveal that there is no explosion for some stochastic passive systems, so it must have a global solution. To this task, attention is turned to the Non-explosion condition of a stochastic differential equation proposed by Narita [11].

**Lemma 1 (Non-explosion Condition [11]).** Given a stochastic differential equation represented by Eq. (2.1), if for $\forall \, T > 0$, there exist two positive numbers $c_T > 0$ and $R_T > 0$, and a scalar function $U_T \in C^2([0, T] \times \mathbb{R}^n; \mathbb{R})$ such that
\[
\mathcal{L}[U_T(t, x)] \leq c_T
\]
holds for all $t \leq T$ and $\|x\|_2 \geq R_T$, and moreover,
\[
\lim_{\|x\|_2 \to \infty} \inf_{0 \leq t \leq T} U_T(t, x) = \infty
\]
then the solutions of Eq. (2.1) are of non-explosion, i.e., the explosion time beginning at any \( t_0 > 0 \) and \( x_0 \in \mathbb{R}^n \), denoted by \( t_c(t_0, x_0) \), satisfying
\[
P(t_c(t_0, x_0) = \infty) = 1
\]

In the following, that Lemma 1 is applied to a stochastic passive system yields

**Proposition 1.** For a stochastic differential system \( \Sigma_S \) governed by Eq. (2.6), if there exists a radially unbounded Lyapunov function so that \( \Sigma_S \) is stochastically passive, then the unforced version of Eq. (2.6) has a global solution.

**Proof.** Assume \( V(x) \) to be the radially unbounded Lyapunov function that suggests \( \Sigma_S \) to be stochastically passive, then by designating the zero controller to \( \Sigma_S \), i.e., \( u = 0_m \), we have \( \mathcal{L}[V(x)] \leq 0 \). It is naturally to observe that \( V(x) \) satisfies Eq. (3.3). Note that the state evolution of this unforced version of \( \Sigma_S \) is just the same as Eq. (2.1). Hence, the solutions of \( \Sigma_S \) are of non-explosion based on Lemma 1. Namely, Eq. (2.6) has a global solution. \[QED\]

From Proposition 1, one can know that some stochastic passive systems must have a global solution without force. Combining this result with Corollary 1, we get the necessary condition for saying \( \Sigma_S \) to be of stochastic passivity, which is expressed as follows.

**Theorem 3 (Necessary Condition for Stochastic Passivity).** If there exists a radially unbounded Lyapunov function that can render a stochastic differential system \( \Sigma_S \) described by Eq. (2.6) to be stochastically passive, then the unforced diffusion term must vanish at the stochastic passive state.

**Proof.** From Theorem 1 of Stochastic Lyapunov theorem, \( \Sigma_S \) is stable in probability at the stochastic passive state with the zero controller. This together with Proposition 1 and Corollary 1 yields the result to be true. \[QED\]

We further express the inverse negative proposition of Theorem 3 to get the sufficient condition for loss of stochastic passivity.

**Corollary 2 (Sufficient Condition for Loss of Stochastic Passivity).** If the unforced diffusion term \( h(x^1, 0_m) \) in a stochastic differential system \( \Sigma_S \) in the form of (2.6) does not vanish at any state \( x \in \mathbb{R}^n \), then there does not exist any radially unbounded Lyapunov function to ensure \( \Sigma_S \) to be stochastically passive.

**Remark 2.** Corollary 2 implies that stochastic passivity will lose when the desired state \( x^1 \) makes \( h(x^1, 0_m) \neq 0_{n \times r} \) and the storage function is expected to be a radially unbounded Lyapunov function, so it is impossible for one to use stochastic passivity theory, and further, stochastic Lyapunov theorem to analyze the globally asymptotical stability of \( \Sigma_S \) at \( x^1 \) in the sense of probability.

### 3.2. Problem setting.

The above analysis reveals that when \( h(x^1, 0_m) \neq 0_{n \times r} \), stochastic passivity will fail to capture the globally asymptotical stability (in the probability sense) of a stochastic differential system \( \Sigma_S \) at the desired state \( x^1 \), often set as the equilibrium state \( x^* \) (if exists) in many control problems. In fact, the nonzero diffusion term is frequently encountered in many real stochastic systems, such as chemical reaction networks, tracking systems, etc. One case is that the noise is persistent in quite a few stochastic systems, which means that for \( \forall x \in \mathbb{R}^n \) and \( \forall u \in U, h(x, u) \neq 0_{n \times r} \) and thus \( x^* \) does not exist at all; the other case is that some special control purposes are served for, such as the desired state \( x^1 \) being not \( x^* \) so that \( h(x^1, 0_m) \neq 0_{n \times r} \), even if \( x^* \) exists.

Apparently, the nonzero diffusion term in real stochastic systems restricts greatly the applications of stochastic passivity theory, a powerful tool for stabilization. However, what is even worse is that it may lead to the system under consideration being
not stable at all in probability at the desired state, as stated in Theorem 2. These two awkward situations motivate us to find a new solution for stabilizing those stochastic systems with nonzero diffusion term at the desired state. On the one hand, it is impossible to stabilize some stochastic systems at any state in probability, on the other hand, the excellent performance of stochastic passivity is hoped to be used. Thus, we take a hack at the next best way to address the current control problem, including seeking the convergence in distribution and ergodicity instead of the convergence in probability, and finding the stochastic passivity behavior only outside a certain neighborhood of the desired state instead of in the whole state domain.

4. Stochastic weak passivity theory. The objective in this section is to present the theory of stochastic weak passivity with which some stochastic systems with nonzero diffusion term can be analyzed concerning the convergence of the transition measure and ergodicity. This theoretical framework includes some basic concepts related to stochastic weak passivity, properties of invariant measure, and results for stabilization which are parallel to those appearing in the stochastic passivity theory.

4.1. Basic concepts. We firstly give the definitions of convergence in distribution and of ergodicity.

**Definition 4.1 (Convergence in distribution and Ergodicity).** Assume a stochastic differential equation described by Eq. (2.1) to have an invariant measure $\pi$. If there exists a subset of $\mathbb{R}^n$, denoted by $\mathbb{R}^n_\pi$, such that for any Borel subset $A$ with zero $\pi$-measure boundary the equation

$$\lim_{t \to \infty} P(t, x(0), A) = \pi(A), \quad \forall x(0) \in \mathbb{R}^n_\pi$$

is true, then the stochastic process is said to be locally convergent in distribution. If $\mathbb{R}^n_\pi = \mathbb{R}^n$, then the convergence in distribution is globally.

If for any Borel subset $B$ the state $x(t)$ satisfies

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{x(t) \in B\}} dt = \pi(B), \quad \text{a.s.} \quad \forall x(0) \in \mathbb{R}^n_\pi$$

where “a.s.” represents “almost surely” and

$$\mathbb{1}_{\{x(t) \in B\}} = \begin{cases} 1 & x(t) \in B, \\ 0 & x(t) \notin B \end{cases}$$

then the stochastic process is said to be locally ergodic. Especially, when $\mathbb{R}^n_\pi = \mathbb{R}^n$, the ergodicity is global.

Here, analogous to the definition of stability in probability, we also distinguish the local and global notations to emphasize the importance of the initial condition.

**Remark 3.** In the control viewpoint, the convergence of the transition measure and ergodicity both describe certain senses of stable behaviors for stochastic systems. The former means that the distribution of the state will converge to an invariant measure as time goes infinite. Therefore, as long as the invariant measure is shaped to fasten on a small region around the desired state, then the state will evolve within this region with a large probability, i.e., not to deviate from the desired point too far with a large probability. The latter implies that the state evolution almost always take place within the mentioned region. Even if the trajectory sometimes run from the region, it will come back into the region immediately.
Clearly, the convergence of the transition measure and ergodicity reveal that the state of a stochastic system almost always evolves near the desired state if the invariant measure is assigned properly. We define this behavior as stochastic asymptotic weak stability.

**Definition 4.2 (Stochastic Asymptotic Weak Stability).** A stochastic differential equation of Eq. (2.1) is of local\,\,global stochastic asymptotic weak stability if its distribution locally\,globally converges to an invariant measure and its process is of local\,global ergodicity.

Next, we define the stochastic weak passivity that serves for stabilizing a stochastic differential system in weak sense. Note that the loss of stochastic passivity mainly originates from the nonzero diffusion term at the desired state, which further results in some unexpected behaviors appearing around it. Thus, a naive idea is to give up the stochastic passivity near the desired state, but only to suggest it outside a neighborhood of the desired state.

**Definition 4.3 (Stochastic Weak Passivity).** A stochastic differential system $\Sigma_S$, as described by Eq. (2.6), is said to be of stochastic weak passivity if there exist a $C^2(\mathbb{R}^n;\mathbb{R}^n)$ function $V(x)$, i.e., the storage function, such that for $\forall x \in \mathbb{R}^n$ and $\|x - x_R\|_2 \geq R$ the following inequality holds
\[
\mathcal{L}[V(x)] \leq u^T y
\]
where the state $x_R \in \mathbb{R}^n$ is the sole minimum point for $V(x)$ and $R \geq 0$ is called the stochastic passive radius.

Similar to the concept of the strict passivity, we may further define strict stochastic weak passivity.

**Definition 4.4 (Strict Stochastic Weak Passivity).** Consider a stochastic weak passive system. Suppose that there exists a positive constant $\delta$ such that for $\forall x \in \mathbb{R}^n$ and $\|x - x_R\|_2 \geq R$
\[
\mathcal{L}[V(x)] \leq u^T y - \delta \|\xi\|_2^2
\]
The system is
- strictly state stochastic weak passive if $\xi = x - x_R$.
- strictly input stochastic weak passive if $\xi = u$.
- strictly output stochastic weak passive if $\xi = y$.

4.2. Properties of invariant measure. Definition 4.2 reveals that the stochastic asymptotic weak stability is concerned with the convergence in distribution of the state and ergodic behavior. However, for a stochastic system, unlike its equilibrium it is not quite obvious to know something about its invariant measure, such as the existence, uniqueness, etc. We separate this subsection to analyze the properties of the invariant measure of the stochastic differential equation under consideration.

In fact, it is not a new research issue to analyze the properties of the invariant measure of a stochastic system [5, 15]. A sufficient condition to say it convergent in distribution was reported as follows.

**Theorem 4 (cf. [15]).** If a right Markov process on $\mathbb{R}^n$ is strongly Feller, i.e., $\forall t>0$ the transition semigroup $\mathcal{P}(t, \cdot, \cdot)$ transforms bounded Borel functions into $C(\mathbb{R}^n)$, and moreover $\mathcal{P}$ is irreducible, i.e., $\forall t>0$, $\forall x \in \mathbb{R}^n$ and any open set $O \neq \emptyset$ there is $\mathcal{P}(t, x, O) > 0$, then any probability measure converges to the invariant measure (if exists). Moreover the invariant measure (if exists) is equivalent to each transition measure $\mathcal{P}(t, x, \cdot)$, $t > 0$, $x \in \mathbb{R}^n$. 

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This theorem provides a solution to capture the convergence in distribution for a right Markov process. However, it is not easy to verify the conditions of “strongly Feller” and “irreducible” in practical applications. As an alternative, Khasminskii [6] proposed a more practical way to say a stochastic system to be convergent in distribution, which works if a Markov process is “mix sufficiently well” in an open domain $O$ and the recurrent time is finite (cf. Theorems 4.1, 4.3 and Corollary 4.4 in [6]). Here, we will combine this practical way with Zakai’s work [18], and give a Lyapunov criterion to say stochastic asymptotic weak stability. However, the drift and diffusion terms of the stochastic system are set to have local Lipschitz continuity instead of global Lipschitz continuity in [18].

**Lemma 2 (Finite Mean Recurrent Time [18]).** For a stochastic differential equation (2.1) having a global solution $x(t)$, if there exist a function $V(x) \in C^2(\mathbb{R}^n; \mathbb{R}_{\geq 0})$, a state $x_{\tilde{R}}$, and two positive numbers $\tilde{R}$ and $k$ such that
\[
L[V(x)] < -k, \ \forall \|x(t) - x_{\tilde{R}}\|_2 \geq \tilde{R},
\]
then for all $x(0) \in \mathbb{R}^n$ the first passage time from $x(0)$ to the sphere $\|x(t) - x_{\tilde{R}}\|_2 \leq \tilde{R}$, denoted by $\tau$, satisfies
\[
E[\tau|x(0)] \leq \frac{V(x(0))}{k}
\]

**Proof.** At the time of $t \land \tau$, by Dynkin’s formula we have
\[
E\left[V\left(x(t \land \tau) \mid x(0)\right) \right] = V(x(0)) + E\left[\int_0^{t \land \tau} L[V(x(s))] ds\right]
\]
\[
\leq V(x(0)) - E\left[\int_0^{t \land \tau} k ds\right]
\]
Note that $V(x) \geq 0$, so $E[t \land \tau|x(0)] \leq \frac{V(x(0))}{k}$. The inequality (4.4) naturally holds due to the monotone convergence. □

**Theorem 5.** For a stochastic equation in the form of (2.1), if there exists a nonnegative function $V(x) \in C^2(\mathbb{R}^n; \mathbb{R}_{\geq 0})$ satisfying the following conditions:

- $\lim_{\|x\|_2 \to \infty} V(x) = \infty$;
- $\exists x_{\tilde{R}} \in \mathbb{R}^n$, $\tilde{R}>0$ and $k>0$, if $\|x - x_{\tilde{R}}\|_2 \geq \tilde{R}$, then $L[V(x)] < -k$;
- $\exists \epsilon>0$, if $\|x - x_{\tilde{R}}\|_2 < \tilde{R} + \epsilon$, then $\operatorname{rank}(h(x)h^+(x)) = n$.

Then there is a unique finite invariant measure $\pi$ such that for any Borel subset $A$ with zero $\pi$-measure boundary
\[
\lim_{t \to \infty} P(t, x(0), A) = \pi(A), \ \forall x(0) \in \mathbb{R}^n
\]

and for any Borel subset $B$
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{x(t) \in B} dt = \pi(B), \ \text{a.s.} \ \forall x(0) \in \mathbb{R}^n
\]
i.e., Eq. (2.1) being globally asymptotical stable in weak sense.

**Proof.** According to Lemmas 1 and 2, the first two conditions could suggest that Eq. (2.1) has a unique global solution and for any initial state $\tilde{x}$ satisfying $\tilde{x} \in \{x \mid \|x - x_{\tilde{R}}\|_2 \geq \tilde{R}\}$, we have
\[
E[\tau|\tilde{x}] \leq \frac{V(\tilde{x})}{k}
\]
Hence, for any compact subset \( K \subset \mathbb{R}^n \) we get

\[
\sup_{\tilde{x} \in K} \mathbb{E}[\tau_{\tilde{x}}] \leq \sup_{\tilde{x} \in K} \frac{V(\tilde{x})}{k} < \infty.
\]

Further based on the strong maximum principle for solutions of elliptic equations, the third condition implies the system (2.1) to be irreducible (cf. Lemma 4.1 in [6]), which combining the above inequality suggests that an ergodic Markov chain can be induced for this stochastic process by constructing a circle. The ergodic property of the Markov chain will ensure that there exists a sole invariant measure to which the transition measure converges (cf. Theorems 4.1 and 4.3, and Corollary 4.4 in [6]), and the ergodicity of the system under consideration is true (cf. Theorem 4.2 in [6]). Namely, Eqs. (4.5) and (4.6) hold. □

The theorem provides a Lyapunov function based method to address the issues of the existence and uniqueness of the invariant measure together with the convergence of the transition probability measure and ergodicity for a stochastic differential equation, so it can be associated with the Lyapunov stability theory conveniently.

Remark 4. There are two differences between the above theorem and the corresponding result in [18]. One is that the non-singularity of \( h(x)h^T(x) \) is not necessary in the whole state space but only holds in an open ball \( \|x - \tilde{x}_R\|_2 < \tilde{R} + \varepsilon \). The latter is believed to be achieved more easily in practice. The other is that the storage function must be radially unbounded here. In fact, this is not a necessary condition, which can be removed if the drift term and diffusion term in the stochastic equation are assumed to be globally Lipschitz continuous.

For a stochastic asymptotic weak stable system, to ensure the state to evolve within a small region around the desired point, the invariant measure needs to be assignable or at least partially shaped by the control to fasten on this region. In the sequel, we will prove that the invariant measure can be shaped purposefully by controlling the change rates of the nonnegative function \( V(x) \) and the radius \( \tilde{R} \) of the ball \( \|x - \tilde{x}_R\|_2 \geq \tilde{R} \).

Lemma 3. For a stochastic differential equation (2.1) admitting a global solution \( x(t) \), if \( \exists V(x) \in C^2(\mathbb{R}^n, \mathbb{R}_\geq 0) \) and \( k, C \in \mathbb{R}_>0 \) such that

\[
LV(x) \leq \begin{cases} 
-k & \forall x \in \{x \mid V(x) \geq V_1\}, \\
C & \forall x \in \mathbb{R}^n
\end{cases}
\]

and

(4.7) \hspace{1cm} P(\tau_{2i - \tau_{2i - 1}} = \infty) = 0 \quad \forall i \in \mathbb{Z}_{>0}

then for any \( i \geq 1, \ i \in \mathbb{Z}_{>0} \) we have

1. \( \mathbb{E} [\tau_{2i} - \tau_{2i-1}] \geq \frac{(V_2 - V_1)^2}{2CV_2} \); 2. \( \mathbb{E} [\tau_{2i-1} - \tau_{2i-2}] \leq \frac{V_2 - V_1}{k} \); and

3. \( \mathbb{E} \left[ \liminf_{T \to \infty} \frac{1}{T} \int_0^T 1_{\{V(x(t)) \geq V_2\}} dt \right] \leq \frac{2CV_2}{2CV_2 + k(V_2 - V_1)} \)

where \( V_1, V_2 \in \mathbb{R}_{>0} \) satisfying \( V_2 > V_1 \), \( \tau_{2i-2} \) represents the first time at which the state hits the region \( \{V(x) \geq V_2\} \) after \( \tau_{2i-3} \), \( \tau_{2i-1} \) is the first time at which the
trajectory reaches the surface of \( \{V(x) \leq V_1\} \) after \( \tau_{2i-2} \), and \( \tau_1 \) means the initial time.

**Proof.** (1) According to Dynkin’s formula, for any \( i \geq 1, \ i \in \mathbb{Z}_{>0} \)

\[
E[V(x(t \land \tau_{2i}))] = V(x(\tau_{2i-1})) + E \left[ \int_{\tau_{2i-1}}^{t \land \tau_{2i}} \mathcal{L}V(x(s)) \, ds \right]
\]

\[
\leq V(x(\tau_{2i-1})) + E \left[ \int_{\tau_{2i-1}}^{t \land \tau_{2i}} C \, ds \right]
\]

\[
\leq V_1 + C(t - \tau_{2i-1})
\]

Also, since

\[
E[V(x(t \land \tau_{2i}))] = E[V(x(t))] P(\tau_{2i} > t) + E[V(x(\tau_{2i}))] P(\tau_{2i} \leq t) \geq V_1 P(\tau_{2i} \leq t)
\]

we have

\[
P(\tau_{2i} \leq t) \leq \frac{E[V(x(t \land \tau_{2i})] \rightleftharpoons}{V_2} \leq \frac{V_1 + C(t - \tau_{2i-1})}{V_2}
\]

Therefore, we get

\[
E[\tau_{2i} - \tau_{2i-1}] = \int_0^\infty P(\tau_{2i} - \tau_{2i-1} > s) \, ds \geq \int_0^{V_2 - V_1} P(\tau_{2i} > s + \tau_{2i-1}) \, ds
\]

\[
\geq \int_0^{V_2 - V_1} (1 - \frac{V_1 + Cs}{V_2}) \, ds = \frac{(V_2 - V_1^2)}{2CV_2}
\]

(2) On the other side, for any \( i > 1 \) we have

\[
E[V(x(t \land \tau_{2i-1}))] = V(x(\tau_{2i-2})) + E \left[ \int_{\tau_{2i-2}}^{t \land \tau_{2i-1}} \mathcal{L}V(x(s)) \, ds \right]
\]

\[
\leq V(x(\tau_{2i-2})) + E \left[ \int_{\tau_{2i-2}}^{t \land \tau_{2i-1}} -k \, ds \right]
\]

\[
= V_2 - kE[t \land \tau_{2i-1} - \tau_{2i-2}]
\]

i.e.,

\[
E[t \land \tau_{2i-1} - \tau_{2i-2}] \leq \frac{V_2 - E[V(x(t \land \tau_{2i-1})] \rightleftharpoons}{k} \leq \frac{V_2 - V_1}{k}
\]

By monotone convergence theorem, the inequality \( E[\tau_{2i-1} - \tau_{2i-2}] \leq \frac{V_2 - V_1}{k} \) is true.

(3) Based on the results of (1) and (2), we have that for any \( j \in \mathbb{Z}_{>0} \)

\[
E \left[ \sum_{i=1}^j (\tau_{2i} - \tau_{2i-1}) \right] \geq \frac{k(V_2 - V_1)}{2CV_2}
\]

Besides, Eq. (4.7) and the result (2) imply there’re almost surely infinite many \( \tau_i \), so the notations “\( \limsup \)” and “\( \liminf \)” in the following are not in vain. Applying Fatou’s lemma yields

\[
(4.8) \quad E \left[ \limsup_{j \to \infty} \frac{\sum_{i=1}^j (\tau_{2i} - \tau_{2i-1})}{\sum_{i=1}^j (\tau_{2i+1} - \tau_{2i})} \right] \geq \frac{k(V_2 - V_1)}{2CV_2}
\]
Let \( i(T) = \max\{i \mid \tau_{2i} \leq T\} \), utilizing which we have

\[
\frac{1}{T} \int_0^T \mathbb{1}_{\{V(x(t)) \geq V_2\}} dt = \frac{\sum_{i=0}^{i(T)-1} \frac{\tau_{2i+2} - \tau_{2i+1}}{\tau_{2i+1} - \tau_{2i}} \mathbb{1}_{\{V(x(t)) > V_2\}} dt + \int_{\tau_{2i(T)}}^T \mathbb{1}_{\{V(x(t)) > V_2\}} dt}{T} \\
\leq \frac{\tau_{i(T)-1} - \tau_{0}}{T} \mathbb{1}_{\{i(T) \geq 1\}} + \frac{\sum_{i=0}^{i(T)-1} (\tau_{2i+1} - \tau_{2i})}{\tau_{2i+1} - \tau_{2i}} \left( \tau_0 + \sum_{i=0}^{i(T)-1} (\tau_{2i+2} - \tau_{2i}) + (T - \tau_{2i(T)}) \right) \\
+ \left( \tau_{2i(T)+1} - \tau_{2i(T)} \right) \mathbb{1}_{\{T > \tau_{2i(T)+1}\}} + (T - \tau_{2i(T)}) \mathbb{1}_{\{T < \tau_{2i(T)+1}\}} \\
= \frac{\tau_{i(T)-1} - \tau_{0}}{T} \mathbb{1}_{\{i(T) \geq 1\}} + \frac{\sum_{i=1}^{i(T)} (\tau_{2i+1} - \tau_{2i})}{\tau_{2i+1} - \tau_{2i}} \left( \tau_0 + \sum_{i=0}^{i(T)-1} (\tau_{2i+2} - \tau_{2i}) + (T - \tau_{2i(T)}) \right) \\
\leq \frac{\tau_{i(T)-1} - \tau_{0}}{T} \mathbb{1}_{\{i(T) \geq 1\}} + \frac{1}{1 + \frac{\sum_{i=1}^{i(T)} (\tau_{2i+1} - \tau_{2i})}{\tau_{2i+1} - \tau_{2i}}}
\]

Hence,

\[
\liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{V(x(t)) \geq V_2\}} dt \leq \liminf_{T \to \infty} \left[ \frac{\tau_{i(T)-1} - \tau_{0}}{T} \mathbb{1}_{\{i(T) \geq 1\}} + \frac{1}{1 + \frac{\sum_{i=1}^{i(T)} (\tau_{2i+1} - \tau_{2i})}{\tau_{2i+1} - \tau_{2i}}} \right]
\]

\[
= \liminf_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{V(x(t)) \geq V_2\}} dt
\]

By taking the expectation of both sides and further combining Eq. (4.8), we will get the result of (3).

**Theorem 6.** For a stochastic equation in the form of (2.1), if there exists a nonnegative function \( V(x) \in C^2(\mathbb{R}^n; \mathbb{R} \geq 0) \) satisfying the following conditions:

- \( \lim_{\|x\| \to \infty} V(x) = \infty \);
- \( \exists x_{\tilde{R}} \in \mathbb{R}^n, \tilde{R} > 0 \) and \( k > 0, \) if \( \|x - x_{\tilde{R}}\|_2 \geq \tilde{R}, \) then \( \mathcal{L}[V(x)] < -k; \)
- \( \exists \epsilon > 0, \) if \( \|x - x_{\tilde{R}}\|_2 < \tilde{R} + \epsilon, \) then \( \text{rank}(h(x)h^\top(x)) = n; \)

then for any Borel subset \( \mathcal{B} \) satisfying \( V_3 > V_0, \) we have

\[
\pi(\mathcal{B}) \geq \frac{k (V_3 - V_0)}{2CV_3 + k (V_3 - V_0)}.
\]

where \( C = \sup_{x \in \mathbb{R}^n} \mathcal{L}[V(x)], V_3 = \inf_{x \notin \mathcal{B}} V(x) \) and \( V_0 = \sup_{\|x - x_{\tilde{R}}\|_2 \leq \tilde{R}} V(x). \)

**Proof.** If \( \pi(\mathcal{B}) = 1, \) then the result holds true automatically. So we only prove the result in the case of \( \pi(\mathcal{B}) < 1 \) which implies \( \pi(\{V(x) > V_3\}) > 0. \)

From \( \lim_{\|x\| \to \infty} V(x) = \infty \) and \( V_0 = \sup_{\|x - x_{\tilde{R}}\|_2 \leq \tilde{R}} V(x), \) we obtain

\[
\{x \mid V(x) \geq V_0\} \subseteq \{x \mid \|x - x_{\tilde{R}}\|_2 > \tilde{R}\}
\]

Therefore, when \( V(x) \geq V_0, \) \( \mathcal{L}[V(x)] < -k. \) Let \( V_1 \) and \( V_2 \) mentioned in Lemma 3 be chosen as \( V_0 \) and \( V_3, \) then, by ergodicity in Theorem 5, Eq. (4.7) is achieved.
According to Theorem 5 and Eq. (4.6) we have
\[ \mathbb{E} \left[ \liminf_{T \to \infty} \frac{1}{T} \int_0^T 1_{\{V(x(t)) \geq V_B\}} \, dt \right] = \liminf_{T \to \infty} \frac{1}{T} \int_0^T 1_{\{V(x(t)) \geq V_B\}} \, dt = \pi (V(x) \geq V_B) \]
By applying the result (3) of Lemma 3 we can get
\[ \pi (V(x) \geq V_B) \leq \frac{2CV_B}{2CV_B + k(V_B - V_0)} \]
Note that \( \{x | V(x) < V_B\} \subseteq \mathbb{B} \), we thus have
\[ \pi(\mathbb{B}) \geq \pi (V(x) < V_B) = 1 - \pi (V(x) \geq V_B) \geq \frac{k(V_B - V_0)}{2CV_B + k(V_B - V_0)} \]

**Remark 5.** The above theorem reveals that the invariant measure can be shaped by controlling the change rates \( k \) and \( C \) of \( V(x) \) together with the ball radius \( \tilde{R} \) that takes effect through affecting \( V_0 \) in terms of \( V_0 = \sup_{\|x - x_R\|_2 \leq \tilde{R}} V(x) \). When \( \tilde{R} \) is fixed, the invariant measure will become larger if \( k \) increases and \( C \) decreases. The larger the invariant measure is, the greater possibility the trajectory of the state fastens on the region near the desired state, i.e., the more stable the stochastic system is in the weak sense.

### 4.3. Stochastic weak passivity theorems.

Now, we are able to tackle the problem of stabilizing the stochastic systems in weak sense based on the stochastic weak passivity. We name the main result as stochastic weak passivity theorems in the context. Here, concern is only given to the implicit negative proportional controller \( u = -Ky(x, u) \) for the purpose of stabilization, where \( K \) is a positive definite matrix with suitable dimension.

**Theorem 7 (Stochastic Weak Passivity Theorem 1).** For a stochastic differential system \( \Sigma_S \) in the form of Eq. (2.6), assume that there exists a radially unbounded storage function \( V(x) \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R} \geq 0) \) suggesting it to be stochastically weakly passive with the stochastic passive radius \( \tilde{R} \) and the desired state \( x^\dagger \). Also, we suppose that there exists a negative proportional controller \( u(x) = -Ky(x, u) \) connected with \( \Sigma_S \) in feedback so that
- \( \|y(x, u(x))\|_2 \) be bounded away from zero when \( \|x - x^\dagger\|_2 > R \);
- \( \exists \epsilon > 0, \text{rank}(h(x, u(x))h^T(x, u(x))) = n \) when \( \|x - x^\dagger\|_2 < R + \epsilon \).

Then there exists an unique finite invariant measure \( \pi \), and moreover, for any Borel subset \( \mathcal{A} \) with zero \( \pi \)-measure boundary
\[ \lim_{t \to \infty} \mathcal{P}(t, x(0), \mathcal{A}) = \pi(\mathcal{A}), \quad \forall x(0) \in \mathbb{R}^n \]
and for any Borel subset \( \mathbb{B} \)
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T 1_{\{x(t) \in \mathbb{B}\}} \, dt = \pi(\mathbb{B}), \quad \text{a.s.} \quad \forall x(0) \in \mathbb{R}^n \]
That is to say \( \Sigma_S \) being globally asymptotical stable in weak sense.
Proof. From the given conditions and the definition of $\mathcal{L}[\cdot]$ in Eq. (2.5), for any state outside the ball $\|x - x^\dagger\|_2 \geq R$ we have

$$\mathcal{L}[V(x)] \leq y^\top u = -y^\top (x, u)Ky(x, u) \leq -\lambda_{\text{min}}\|y(x, u(x))\|_2^2$$

where $\lambda_{\text{min}} > 0$ is the minimum eigenvalue of $K$. Since $\|y(x, u(x))\|_2$ is bounded away from zero when $\|x - x^\dagger\|_2 > R$, we could find a positive number $k$ such that for any $\|x - x^\dagger\|_2 > R$

$$\mathcal{L}[V(x)] < -k$$

Combining it with the non-singularity of $h(x, u(x))h^\top (x, u(x))$ when $\|x - x^\dagger\|_2 < R + \epsilon$ yields the results (based on Theorem 5). \qed

Note that the above theorem works when the output norm is bounded way from zero outside a ball. In fact, this condition is not necessary. It may be replaced by setting the system to be strictly state stochastic weak passive.

Theorem 8 (Stochastic Weak Passivity Theorem 2). Consider a strictly state stochastic weak passive system under the conditions of radially unbounded $V(x)$, $R$, $x^\dagger$ and $\delta$. Suppose that there exists a negative proportional controller $u(x) = -Ky(x, u)$ connecting the system with the system in feedback so that

- $\exists \epsilon > 0, \text{rank}(h(x, u(x))h^\top (x, u(x))) = n$ when $\|x - x^\dagger\|_2 < R + \epsilon$.

Then there exists an unique finite invariant measure $\pi$, and moreover, for any Borel subset $A$ with zero $\pi$-measure boundary

$$\lim_{t \to \infty} \mathcal{P}(t, x(0), A) = \pi(A), \quad \forall x(0) \in \mathbb{R}^n$$

and for any Borel subset $B$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{x(t) \in B\}} dt = \pi(B), \quad \text{a.s.} \quad \forall x(0) \in \mathbb{R}^n$$

That is to say $\Sigma_S$ being globally asymptotical stable in weak sense.

Proof. From the known conditions and the definition of $\mathcal{L}[\cdot]$ in Eq. (2.5), we have at $\|x - x^\dagger\|_2 \geq R$

$$\mathcal{L}[V(x)] \leq y^\top u - \delta \|x - x^\dagger\|_2^2 = -y^\top (x, u)Ky(x, u) - \delta \|x - x^\dagger\|_2^2 \leq -\delta R^2$$

i.e., $\exists k > 0, \mathcal{L}[V(x)] < -k$. Further applications of Theorem 5 yield the results immediately. \qed

Remark 6. Clearly, the change rate $k$ of the energy function $V(x)$ is closely dependent on the the feedback gain matrix $K$. When $K$ is designed to be stronger, $k$ will be larger. This can lead to the increase of the invariant measure $\pi(B)$, and further the state evolution taking place within a more intensive region around the desired state $x^\dagger$. Therefore, it is an effective way for shaping the invariant measure $\pi(B)$ purposefully to strengthen the feedback gain matrix $K$.

Remark 7. Stochastic weak passivity theorems suggest sufficient conditions to stabilize a stochastic differential system in weak sense. Although some items are difficult to realize, such as the non-singularity of $h(x, u(x))h^\top (x, u(x))$ when $\|x - x^\dagger\|_2 < R + \epsilon$, some ones are relatively weak, e.g., the energy function $V(x)$ only requires to be positive semi-definite, and the simple proportional controller is qualified.

To weaken the harsh condition on the non-singularity of $h(x, u(x))h^\top (x, u(x))$ in the Stochastic weak passivity theorem, we might separate those linear independent
rows from $h(x, u(x))$ to construct a new diffusion term $h_1(\cdot, \cdot) \in \mathbb{R}^{n_1 \times r}$. Clearly, it is much easier to realize $h_1(\cdot, \cdot)h_2^T(\cdot, \cdot)$ to be full-rank than $h(x, u(x))h^T(x, u(x))$ to be nonsingular. For this purpose, we first define a transformation.

**Definition 4.5 (Decomposition Transformation).** A homeomorphism $\Phi(x) \in \mathcal{C}^2(\mathbb{R}^r; \mathbb{R}^{n_2})$, expressed as

$$
\Phi(x) = \begin{pmatrix}
\bar{x}_1 \\
\bar{x}_2
\end{pmatrix}
$$

is called a decomposition transformation of system $\Sigma_S$ if it can transform the stochastic differential system $\Sigma_S$ equipped with Eq. (2.6) into two subsystems: one is a stochastic differential system $\Sigma_{subS}$, the other is a deterministic system $\Sigma_{subD}$. Here, $\bar{x}_1 \in \mathbb{R}^{n_1}$, $\bar{x}_2 \in \mathbb{R}^{n_2}$ and $n_1 + n_2 = n$. These two subsystems are written respectively as

$$
\Sigma_{subS} : \begin{cases}
d\bar{x}_1 &= f_1(\bar{x}_1, u)dt + h_1(\bar{x}_1, u)d\omega \\
y_1 &= s_1(\bar{x}_1, u)
\end{cases}
$$

$$
\Sigma_{subD} : \begin{cases}
d\bar{x}_2 &= 0_{n_2}dt \\
y_2 &= s_2(\bar{x}_2, u)
\end{cases}
$$

where the drift term $f_1(\bar{x}_1, u) \in \mathbb{R}^{n_1}$ and diffusion term $h_1(\bar{x}_1, u) \in \mathbb{R}^{n_1 \times r}$ in $\Sigma_{subS}$ are both locally Lipschitz continuous.

Remark 8. The diffusion term $h_1(\bar{x}_1, u)$ in $\Sigma_{subS}$ can be extracted from $h(x, u(x))$ to the greatest extent according to the rank so that the rank of $h_1(\bar{x}_1, u)$ may reach $n_1$. $\Sigma_{subD}$ is obviously a fixed point, and is certainly stable. Thus, the stabilization of $\Sigma_S$ may be realized by stabilizing $\Sigma_{subS}$.

Remark 9. In practice, the defined decomposition transformation (4.10) is not difficult to be constructed. For many nonlinear stochastic systems, the state evolution really takes place in an invariant manifold, denoted by $\mathcal{M}$, but not in the whole state space $\mathbb{R}^n$. For example, the state of a chemical reaction network will evolve in a positive stoichiometric compatibility class, which is a subset of $\mathbb{R}^n$. Therefore, an immediate idea to construct the decomposition transformation is to decompose the state space $\mathbb{R}^n$ into the invariant manifold $\mathcal{M}$ and its orthogonal complement $\mathcal{M}^\perp$. The projection from $x \in \mathbb{R}^n$ to $\bar{x}_1 \in \mathcal{M}$ leads to the stochastic subsystem $\Sigma_{subS}$ while the projection from $x \in \mathbb{R}^n$ to $\bar{x}_2 \in \mathcal{M}^\perp$ induces the fixed point subsystem $\Sigma_{subD}$.

Utilizing the decomposition transformation, we can give the refined stochastic weak passivity theorem.

**Lemma 4.6.** For a stochastic differential system (2.6) if there exists a decomposition transformation $\Phi(x)$ transform the system into two subsystems $\Sigma_{subS}$ (4.11) and $\Sigma_{subD}$ (4.12), then the state $\bar{x}(t)$ starting from any initial state $\bar{x}(0)$ satisfies

$$
\bar{x}(t) \in \Phi^{-1}(\mathbb{R}^{n_1} \otimes \{\bar{x}_2(0)\}), \quad \forall \ t \in [0, \infty)
$$

**Proof.** Since $\Phi(x(0)) = \bar{x}(0) = \bar{x}_1(0) \otimes \bar{x}_2(0)$, also since $\bar{x}_2(t) = \bar{x}_2(0)$, the result is true. \[ \]

Remark 10. It is clear that $\bar{x}_2(0)$ can be any element in $\mathbb{R}^{n_2}$. This in turn means that for any $z \in \mathbb{R}^{n_2}$ there is a manifold $\Phi^{-1}(\mathbb{R}^{n_1} \otimes \{z\})$ defined. For simplicity of notation, we identify this manifold by $\mathbb{R}^{n_2}$.

**Theorem 9 (Refined Stochastic Weak Passivity Theorem).** Assume that the system $\Sigma_S$ can be decomposed into two subsystems $\Sigma_{subS}$ (4.11) and $\Sigma_{subD}$ (4.12)
using the decomposition transformation \( \Phi(x) \). Further, suppose that there exist a radially unbounded storage function \( V(x_1) \in C^2(\mathbb{R}^{n_1};\mathbb{R}_{\geq 0}) \) suggesting \( \Sigma_{subS} \) to be stochastic weak passivity with the stochastic passive radius \( R \) and the desired state \( \bar{x}_1 \), and a negative implicit proportional controller \( u(x_1) = -Ky_1(x_1, u) \) so that

- \( \|y_1(x_1, u)\|_2 \) is bounded away from 0 when \( \|x_1 - \bar{x}_1\| > R \); Or the system is strictly state stochastic weak passive.

- \( \exists \epsilon > 0, \text{rank}(h_1(x_1, u)h_1^\top(x_1, u)) = n_1 \) when \( \|x_1 - \bar{x}_1\| < R + \epsilon \).

Then there exist an invariant measure \( \pi \) and a corresponding manifold \( \mathbb{R}^n_\pi \) such that for any Borel subset \( A \) with zero \( \pi \)-measure boundary

\[
\lim_{t \to \infty} P(t, x(0), A) = \pi(A), \ \forall x(0) \in \mathbb{R}^n
\]

and for any Borel subset \( B \)

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T 1_{\{x(t) \in B\}} \, dt = \pi(B), \ \text{a.s.} \ \forall x(0) \in \mathbb{R}^n
\]

when the controller \( u(x_1) \) is connected with \( \Sigma_{subS} \) in feedback. That is to say \( \Sigma_S \) being locally asymptotically weakly stable.

**Proof.** First, we consider the existence of invariant measure under the transformed coordinate \( \bar{x} \).

For \( \Sigma_{subS} \) connected by the controller \( u(x_1) \) in feedback, by stochastic weak passivity theorems (Theorem 7 and Theorem 8) there exists an unique finite invariant measure \( \pi_1 \) so that for any Borel subset \( A_1 \subset \mathbb{R}^{n_1} \) with boundary \( \Gamma_1 \), when \( \pi_1(\Gamma_1) = 0 \) we have

\[
\lim_{t \to \infty} P(t, \bar{x}_1(0), A_1) = \pi_1(A_1), \ \forall \bar{x}_1(0) \in \mathbb{R}^{n_1}
\]

and for any Borel subset \( B_1 \subset \mathbb{R}^{n_1} \)

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T 1_{\{\bar{x}_1(t) \in B_1\}} \, dt = \pi_1(B_1), \ \text{a.s.} \ \forall \bar{x}_1(0) \in \mathbb{R}^{n_1}
\]

For \( \Sigma_{subD} \), define \( \pi_2 \) be a measure on \( (\mathbb{R}^{n_2}, \mathcal{B}(\mathbb{R}^{n_2})) \) that satisfies

\[
\pi_2(A_2) = \begin{cases} 
1 & z \in A_2 \\
0 & z \notin A_2
\end{cases}
\]

where \( z \) is a fixed point in \( \mathbb{R}^{n_2} \) and \( A_2 \) is a Borel subset of \( \mathbb{R}^{n_2} \), then for all \( t \) and the initial condition \( z \) we have

\[
P(t, \bar{x}_2(0), A_2) = P(\bar{x}_2(t) \in A_2 | \bar{x}_2(0) = z) = \pi_2(A_2)
\]

Consider the product measure of \( \pi_1 \) and \( \pi_2 \), denoted by \( \pi_3 \), i.e.,

\[
\pi_3(* \otimes *) = \pi_1(*) \times \pi_2(*)
\]

Note that the existence and uniqueness of \( \pi_3 \) are guaranteed by Hahn-Kolmogorov theorem and \( \sigma \)-finite property, respectively. Imposing \( \pi_3 \) on the set \( \mathbb{R}^{n_1} \otimes \{z\}^c \) yields

\[
\pi_3\left(\mathbb{R}^{n_1} \otimes \{z\}^c\right) = \pi_1\left(\mathbb{R}^{n_1}\right) \times \pi_2\left(\{z\}^c\right) = 0.
\]
where \( \{z\}^c \) is the complementary set of \( \{z\} \) in \( \mathbb{R}^n \).

For any Borel subset \( A \subset \mathbb{R}^n \), we could express it as
\[
A = \left[ A \cap \left( \mathbb{R}^{n_1} \otimes \{z\} \right) \right] \cup \left[ A \cap \left( \mathbb{R}^{n_1} \otimes \{z\} \right)^c \right]
\]
which may be further rewritten by defining a map \( \Psi(\bar{x}_1 \otimes \bar{x}_2) \equiv \bar{x}_1 \) as
\[
(4.15) \quad A = \left[ \Psi(A) \otimes \{z\} \right] \cup \left[ A \cap \left( \mathbb{R}^{n_1} \otimes \{z\} \right)^c \right]
\]
Here, \( \Psi(A) = \{ \Psi(\bar{x}_1 \otimes \bar{x}_2) | \bar{x}_1 \otimes \bar{x}_2 \in A \cap (\mathbb{R}^{n_1} \otimes \{z\}) \} \). Then we have
\[
\mathcal{P}(t, \bar{x}, \bar{A}) = \mathcal{P}(t, \bar{x}, [\Psi(A) \otimes \{z\}] \cup [A \cap (\mathbb{R}^{n_1} \otimes \{z\})^c])
\]
\[
(4.16) \quad = \mathcal{P}(t, \bar{x}, \Psi(A) \otimes \{z\}) + \mathcal{P}(t, \bar{x}, A \cap (\mathbb{R}^{n_1} \otimes \{z\})^c)
\]
Note that for all \( \bar{x}(0) \in \mathbb{R}^{n_1} \otimes \{z\} \) the second term in the above equality satisfies
\[
\mathcal{P}(t, \bar{x}(0), A \cap (\mathbb{R}^{n_1} \otimes \{z\})^c) \leq \mathcal{P}(t, \bar{x}(0), \mathbb{R}^{n_1} \otimes \{z\}^c)
\]
\[
= \mathcal{P}(\bar{x}_2(t) \notin \{z\})
\]
\[
(4.17)
\]
then we have
\[
\int_{\bar{x} \in \mathbb{R}^{n_1}} \mathcal{P}(t, \bar{x}, \bar{A}) \pi_3(d\bar{x})
\]
\[
= \int_{\bar{x} \in \mathbb{R}^{n_1} \otimes \{z\}} \mathcal{P}(t, \bar{x}, \Psi(A) \otimes \{z\}) \pi_3(d\bar{x})
\]
\[
= \int_{\bar{x} \in \mathbb{R}^{n_1}} \mathcal{P}(t, \bar{x}, \Psi(A)) \pi_1(d\bar{x}_1)
\]
\[
(\text{Eqs. (4.15) & (4.17))}
\]
\[
\pi_1(\Psi(A)) \]
\[
(\text{Eqs. (4.15) & (4.17))}
\]
Hence, \( \pi_3 \) is invariant under the coordinate \( \bar{x} \).

Next, we discuss the convergency of \( \pi_3 \) under the transformed coordinate \( \bar{x} \).

Let the boundary of \( \Psi(A) \subset \mathbb{R}^{n_1} \) be \( \Gamma_1 \), then for \( \forall \bar{\alpha} \in \Gamma_1 \) there exist two sequences of points \( \{\bar{\beta}_i\}_{i=1}^{\infty} \) \( \{\bar{\gamma}_i\}_{i=1}^{\infty} \) \( \bar{\alpha} \in \Psi(A) \) such that
\[
\lim_{i \to \infty} \bar{\beta}_i = \lim_{i \to \infty} \bar{\gamma}_i = \bar{\alpha}
\]
Hence, for any point \( \bar{\alpha} \otimes z \) in \( \Gamma_1 \otimes \{z\} \), there exist two sequences of points \( \{\bar{\beta}_i \otimes z\}_{i=1}^{\infty} \) \( \{\bar{\gamma}_i \otimes z\}_{i=1}^{\infty} \) \( \bar{\alpha} \otimes z \in A \) such that
\[
\lim_{i \to \infty} \bar{\beta}_i \otimes z = \lim_{i \to \infty} \bar{\gamma}_i \otimes z = \bar{\alpha} \otimes z
\]
Further let $\Gamma$ denote the boundary of $\Lambda$, then we have $\Gamma_1 \otimes \{ z \} \subset \Gamma$ and

$$\pi_3(\Gamma) = \pi_3(\Gamma)$$

Assume $\pi_3(\Gamma) = 0$, i.e., $\pi_1(\Gamma_1) = 0$, with which we get for $\forall \bar{x}(0) = \bar{x}_1(0) \otimes \bar{x}_2(0) \in \mathbb{R}^{n_1} \otimes \{ z \}$

$$\lim_{t \to \infty} \mathcal{P}(t, \bar{x}(0), \Lambda) = \pi_1(\Psi(\Lambda)) = \pi(\Lambda)$$

This complete the proof of the convergence of $\pi_3$.

Finally, we consider the existence and convergence of invariant measure under original coordinate.

Let a measure $\pi$ satisfy $\pi(\Lambda) = \pi_3(\Phi(\Lambda))$ for all $\Lambda \in \mathcal{B}(\mathbb{R}^n)$. We have

$$\int_{x \in \mathbb{R}^n} \mathcal{P}(t, x, \Lambda) \pi(dx) = \int_{x \in \mathbb{R}^n} \mathcal{P}(t, \bar{x}, \Phi(\Lambda)) \pi_3(d\bar{x}) = \pi(\Phi(\Lambda)) = \pi(\Lambda)$$

which means $\pi$ is invariant. Further let $\Gamma_0$ denote the boundary of $\Phi(\Lambda)$, then $\Phi^{-1}(\Gamma_0) \subset \Gamma$ due to the bicontinuity of $\Phi$. Thus, if we assume $\pi(\Gamma) = 0$, then

$$\pi_3(\Gamma_0) = \pi(\Phi^{-1}(\Gamma_0)) = 0$$

Hence, for $\forall x(0) \in \Phi^{-1}(\mathbb{R}^{n_1} \otimes \{ z \}) = \mathbb{R}^n_\pi$ (denoted as $\mathbb{R}^n$)

$$\lim_{t \to \infty} \mathcal{P}(t, x(0), \Lambda) = \lim_{t \to \infty} \mathcal{P}(t, \bar{x}(0), \Phi(\Lambda)) = \pi_3(\Phi(\Lambda)) = \pi(\Lambda)$$

which shows the convergence of the transition measure.

Similarly, we can prove the local ergodicity of the process. □

Remark 11. A point should be noted that the current invariant measure is no longer unique. It is closely dependent on the initial condition $x(0)$. Hence, the refined stochastic weak passivity theorem actually suggests the conditions of local asymptotic weak stability for a stochastic differential system.

5. Applications. In this section, the stochastic weak passivity theory is applied to linear systems and a nonlinear process system.

5.1. Application to linear systems. Consider a representative linear time-invariant system described by

$$\begin{cases}
\dot{x} = (Ax + Bu)dt + \sigma d\omega \\
y = Cx
\end{cases}$$

where $A$, $B$, $C$ and $\sigma \neq 0$ are constant matrices with suitable dimensions. For simplicity let $0_n$ be the desired state.

Since the noise port $\sigma \neq 0$, there does not exist any Lyapunov function that could suggest this linear system to be globally stable in the sense of probability. However, it is possible for this system to reach stochastic asymptotic weak stability.

Theorem 10. For a linear system described by Eq. (5.1), if there exists a positive definite matrix $D$ with suitable dimension such that
\[ C = B^\top D; \]

\[ DA + A^\top D \text{ is negative definite.} \]

then the system is stochastically weak passive.

**Proof.** Let \( V(x) = \frac{1}{2} x^\top D x \), then

\[
\mathcal{L}[V(x)] = x^\top D (Ax + Bu) + \frac{1}{2} \text{tr}\{ D\sigma\sigma^\top \}
\]

\[
= x^\top DBu + x^\top DAx + \frac{1}{2} \text{tr}\{ D\sigma\sigma^\top \}
\]

\[
= y^\top u + \frac{1}{2} x^\top (DA + A^\top D)x + \frac{1}{2} \text{tr}\{ D\sigma\sigma^\top \}
\]

Assume \( \lambda_{\text{max}} \) to be the maximum eigenvalue of \( DA + A^\top D \). Since the matrix is negative definite, we have \( \lambda_{\text{max}} < 0 \) and \( \frac{1}{2} x^\top (DA + A^\top D)x \leq \frac{1}{2} \lambda_{\text{max}} \|x\|_2^2 \). Note that \( \text{tr}\{ D\sigma\sigma^\top \} = \text{tr}\{ \sigma^\top D\sigma \} \geq 0 \). Hence, as long as \( \|x\|_2 \geq \sqrt{\frac{\text{tr}\{ D\sigma\sigma^\top \}}{\lambda_{\text{max}}}} \), we get

\[ \mathcal{L}[V(x)] \leq y^\top u \]

which completes the proof. \( \Box \)

Clearly, the used Lyapunov function \( V(x) = \frac{1}{2} x^\top D x \) is radially unbounded, so if the noise port \( \sigma \) is full row rank and the measurement matrix \( C \) is full column rank, then all conditions required in Theorem 7 are true. Namely, any negative proportional controller \( u(x) = -Ky(x, u) \) can globally stabilize this class of stochastic linear systems in weak sense.

### 5.2. Applications to a nonlinear process system

Next, we manage to analyze the stochastic weak passivity of a nonlinear process system and further stabilize it based on the refined stochastic weak passivity theorem.

Consider a continuous stirred tank reactor (CSTR) in which a first-order chemical reaction takes place

\[
X_1 \rightarrow X_2
\]

The pure component \( X_1 \) with a fixed concentration \( C_1^{\text{in}} \) is fed at the inlet of the reactor while the fluid mixture of \( X_1 \) and \( X_2 \) is released from the outlet of the reactor. For simplicity this process is assumed to be isothermal and isometric. Furthermore, the volume flow rates of inflow and outflow are regulated to be the same. In addition, the chemical reaction goes on with an unstable reaction rate coefficient. Denote the components concentrations of \( X_1 \) and \( X_2 \) by \( x_1 \) and \( x_2 \), respectively, the volume flow rate of inflow or outflow by \( q \), the reaction rate coefficient by \( k \) and the disturbance on \( k \) by \( \sigma \) (note that \( \sigma > 0 \) and \( \sigma \ll k \)), then we have the dynamical equation

\[
\begin{align*}
\frac{dx_1}{dt} &= [-kx_1 + (C_1^{\text{in}} - x_1)q] \, dt - \sigma x_1 \, d\omega \\
\frac{dx_2}{dt} &= [kx_1 - x_2q] \, dt + \sigma x_1 \, d\omega
\end{align*}
\]

(5.2)

Construct the input-output pair as

\[
\begin{align*}
u &= q - \frac{k x_1}{C_1^{\text{in}} - x_1} \\
y &= (x_1 - x_1^\dagger) (C_1^{\text{in}} - x_1)
\end{align*}
\]

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where \(x_1^\dagger\) is the desired concentration of \(X_1\) constrained by \(0 < x_1^\dagger < C_1^{in}\). Then the input-output representation of the stochastic CSTR model (5.2) can be written as

\[
\begin{align*}
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = \begin{bmatrix} -kx_1 + \frac{kx_1^\dagger(C_1^{in} - x_1)}{C_1^{in} - x_1^\dagger} + (C_1^{in} - x_1)u \\
\sigma x_1 & + kx_1 - \frac{kx_1^\dagger x_2}{C_1^{in} - x_1^\dagger} - x_2u 
\end{bmatrix} dt + \begin{bmatrix} -\sigma x_1 \\
\sigma x_1 \end{bmatrix} d\omega \\
y & = (x_1 - x_1^\dagger)(C_1^{in} - x_1)
\end{align*}
\]

Apparently, at the desired state the diffusion term will not vanish. Therefore, it is impossible to find a radially unbounded Lyapunov function so that the stochastic CSTR system is stable at the desired point in the sense of probability.

According to the conservation law, it is easy to get that the state \((x_1, x_2)^T\) only evolves in the manifold \(\{(x_1, C_1^{in} - x_1), \forall x_1\}\). Therefore, based on Remark 9 the decomposition transformation can be defined as

\[
\Phi(x) = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\
x_1 + x_2 \end{bmatrix}
\]

through which the system is decomposed into

\[
\Sigma_{subS} : \begin{cases} 
\begin{array}{l}
d\bar{x}_1 = \left[-k\bar{x}_1 + \frac{kx_1^\dagger(C_1^{in} - \bar{x}_1)}{C_1^{in} - x_1^\dagger} + (C_1^{in} - \bar{x}_1)u\right] dt - \sigma \bar{x}_1 d\omega \\
y_1 = (\bar{x}_1 - x_1^\dagger)(C_1^{in} - \bar{x}_1)
\end{array}
\end{cases}
\]

and

\[
\Sigma_{subD} : \begin{cases} 
\begin{array}{l}
d\bar{x}_2 = (C_1^{in} - \bar{x}_2)qdt \\
y_2 = s_2(\bar{x}_2, u)
\end{array}
\end{cases}
\]

Note the facts that \(C_1^{in} = x_1(0) + x_2(0) = \bar{x}_2(0)\) and the volume of inflow or outflow is finite within a finite interval, i.e., \(\int_0^t q(\tau)d\tau \leq \infty (\forall \, t > 0)\), so we get

\[
\bar{x}_2(t) = \bar{x}_2(0) + e^{-\int_0^t q(s)ds}\left(\bar{x}_2(0) - \bar{x}_2(0)\right) = \bar{x}_2(0)
\]

which means the subsystem \(\Sigma_{subS}\) following

\[
d\bar{x}_2 = 0dt
\]

Hence, the vector field of \(\Sigma_{subD}\) equals zero and \(\Phi(x)\) is a decomposition transformation of the CSTR process system.

Construct the Lyapunov function to be \(V(\bar{x}_1) = \frac{1}{2} \left(\bar{x}_1 - x_1^\dagger\right)^2\), then we have

\[
\mathcal{L}[V(\bar{x}_1)] = \left(\bar{x}_1 - x_1^\dagger\right) \left[-k\bar{x}_1 + \frac{kx_1^\dagger(C_1^{in} - \bar{x}_1)}{C_1^{in} - x_1^\dagger} + (C_1^{in} - \bar{x}_1)u\right] + \frac{1}{2} \sigma^2 \bar{x}_1^2
\]

\[
= yu - \frac{kC_1^{in}}{C_1^{in} - x_1^\dagger}(\bar{x}_1 - x_1^\dagger)^2 + \frac{1}{2} \sigma^2 \bar{x}_1^2
\]

Let \(\delta = \frac{kC_1^{in}}{2(C_1^{in} - x_1^\dagger)}\), then the above equation changes to be

\[
\mathcal{L}[V(\bar{x}_1)] = yu - 2\delta \left(\bar{x}_1 - x_1^\dagger\right)^2 + \frac{1}{2} \sigma^2 \bar{x}_1^2
\]
Theorem clearly more intensive around the desired state. This information can be also observed
made on the above CSTR process with the initial state designated as $x(0)$ = 5.5, 3 $^T$, and the disturbance as $\sigma = 0.03$. The other parameters are $k = 1$ mole/m$^3$/s, $q(0) = 0.33$ m$^3$/s and $C_{in} = 8.5$ mole/m$^3$. Fig. 1 shows the time evolution of the state $x(t) = (x_1(t), x_2(t))^T$ without control and the corresponding invariant probability density functions. The ranges, 5.0 ± 0.25 for $x_1$ and 3.5 ± 0.25 for $x_2$, bounded by two dotted lines, respectively, represent the areas in which the state evolves with probability 90%. When the controller $u = -y_1$ is implemented on this process, the state evolution and the invariant probability density functions will fasten on the region around the desired state more. Shown in Fig. 2 are the results. We also use two dotted lines to bound the areas in which the state evolves with 90%. Now, they change to be 5.0±0.1 for $x_1$ and 3.5±0.1 for $x_2$, which is clearly more intensive around the desired state. This information can be also observed.

Hence, for any $\|\bar{x} - x_1\| \geq R$ where

$$R = \frac{\sigma^2 + \sqrt{2\delta}}{2\delta - \sigma^2} x_1$$

we have

$$\mathcal{L}[V(\bar{x}_1)] \leq y u - \delta \|\bar{x}_1 - x_1\|^2$$

This means that the subsystem $\Sigma_{subD}$ is strictly state stochastic weak passive with respect to the radially unbounded storage function $V(\bar{x}_1)$. Note that the stochastic passive radius $R$ is quite small due to $\sigma \ll k$.

Additionally, there exists a positive constant $\epsilon$ such that $0 < \epsilon < \min\{x_1 - R, C_{in} - R - x_1\}$ and for any $\|\bar{x}_2 - x_2\| < R + \epsilon$ the noise port $\sigma^2 \bar{x}_2^2$ is nonsingular. Thus, based on Theorem 9, the system can be locally stochastically asymptotically weakly stable under any negative proportional controller.

To better exhibit the stochastic asymptotic weak stability, some simulations are made on the above CSTR process with the initial state designated as $x(0) = (5.5, 3)^T$, the desired state as $x_1 = (5, 3.5)^T$, and the disturbance as $\sigma = 0.03$. The other parameters are $k = 1$ mole/m$^3$/s, $q(0) = 0.33$ m$^3$/s and $C_{in} = 8.5$ mole/m$^3$. Fig. 1 shows the time evolution of the state $x(t) = (x_1(t), x_2(t))^T$ without control and the corresponding invariant probability density functions. The ranges, 5.0 ± 0.25 for $x_1$ and 3.5 ± 0.25 for $x_2$, bounded by two dotted lines, respectively, represent the areas in which the state evolves with probability 90%. When the controller $u = -y_1$ is implemented on this process, the state evolution and the invariant probability density functions will fasten on the region around the desired state more. Shown in Fig. 2 are the results. We also use two dotted lines to bound the areas in which the state evolves with 90%. Now, they change to be 5.0±0.1 for $x_1$ and 3.5±0.1 for $x_2$, which is clearly more intensive around the desired state. This information can be also observed.
Fig. 2. Time evolution (a) of the state of the CSTR process and the corresponding invariant probability density (b) with the controller $u = -y_1$ implemented. The bounded areas in (a) are $5.0 \pm 0.1$ for $x_1$ and $3.5 \pm 0.1$ for $x_2$.

Fig. 3. Convergent behaviors of probability density functions (a) for $X_1$ and (b) for $X_2$ with the controller $u = -y_1$ implemented.

from Fig. 2(b) in which the invariant probability density function changes “thinner” around the desired state than the corresponding one appearing in Fig. 1(b). The convergent behaviors of the invariant probability density functions under the control are exhibited in Fig. 3. To observe more detailed convergent process, we only exhibit the simulation from $t = 0$ s to $t = 3$ s. As can be seen in Fig. 3, the initial probability density functions, at $t = 0$, deviate the desired state too much, but they will converge to the invariant probability density functions that fasten on the small region around the desired state as the controller is put into force.

6. Conclusions and future research. This work has presented a theoretical framework of stochastic weak passivity serving for stabilizing the stochastic differential systems with nonvanishing noise. The main contributions include: i) deriving the necessary conditions to say a stochastic system stochastically passive or the sufficient conditions that a stochastic system must lose stochastic passivity; ii) proving that it
is impossible for some stochastic systems to be stabilized in probability; iii) defining a new concept of stochastic weak passivity to serve for those systems losing stochastic passivity, which captures the stochastic passivity of the system not in the whole state space but only outside a ball centered around the desired state; iv) associating stochastic weak passivity to asymptotic weak stability of systems, and further providing the sufficient conditions for global and local asymptotic weak stabilization of nonlinear stochastic differential systems by means of negative feedback laws.

The stochastic weak passivity provides an alternative way to stabilize the transition measure as well as capturing the ergodicity of the stochastic differential systems with nonvanishing noise. However, there is still a large room for this method to be improved or expanded. An important issue is that the whole theoretical framework works under the assumption that the stochastic term $\omega$ is a standard Wiener process. The motivation of making such an assumption is that the current concept is developed on the stochastic passivity. For the latter, the noise term is assumed as a standard Wiener process [4]. However, the standard Wiener process is just a kind of ideal noise, and is used mainly for the simplicity of analysis. As far as many practical systems are concerned, this ideal noise is not accurate enough to represent the internal modeling uncertainty. Therefore, it is interesting but challenging to use other stochastic processes instead of the standard Wiener process for developing stochastic weak passivity. Towards this task, the infinitesimal generator $\mathcal{L}[\cdot]$ of Eq. (2.5) needs to be redefined accordingly. In addition, other possible points of future research include: i) weakening the condition of nonsingularity of the diffusion matrix $h_1(x, u)h_1^\top(x, u)$; ii) applying the stochastic weak passivity theory to some special stochastic differential systems, such as stochastic affine systems, thermodynamic process systems, and drive the development of these fields in control techniques; iii) developing the determinist version of stochastic weak passivity.

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