\((1 + \vartheta)\)-CONSTACYCLIC CODES OVER \(\mathbb{Z}_8 + \vartheta \mathbb{Z}_8\)

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Abstract. In this paper, \((1 + \vartheta)\)-constacyclic codes of arbitrary length \(m\) over a non-chain finite local Frobenious ring \(\mathbb{Z}_8 + \vartheta \mathbb{Z}_8\) are introduced. A new Gray map is constructed from \(\mathbb{Z}_8 + \vartheta \mathbb{Z}_8\) to \(\mathbb{Z}_8^8\) and proved that the \(\mathbb{Z}_8\)-Gray image of \((1 + \vartheta)\)-constacyclic codes having prescribed length \(m\) over the ring \(\mathbb{Z}_8 + \vartheta \mathbb{Z}_8\) is a cyclic code of length \(8m\) over the ring \(\mathbb{Z}_8\). Moreover, it has been obtained that the binary image of the \((1 + \vartheta)\)-constacyclic code of length \(m\) over \(\mathbb{Z}_8 + \vartheta \mathbb{Z}_8\) is a distance invariant binary quasi-cyclic code of length \(32m\) with index 16.

Keywords: constacyclic code; gray map; distance invariant; cyclic code; quasi-cyclic code.

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1. BACKGROUND

Many optimal binary linear codes have been studied from codes over several new classes of rings via some Gray map. Over the ring \(F_2 + uF_2 + vF_2 + uvF_2\), linear codes are discussed in [1], self dual codes in [2], cyclic codes in [3] and \((1 + u)\)-constacyclic codes are described in [4] alongwith the construction of many optimal binary linear codes. More generally, cyclic codes over the ring \(R_k\) were investigated in [12]. The rings mentioned above are not finite chain rings, however have rich algebraic structures and produce binary codes with large automorphism

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groups and new binary self-dual codes. This demonstrates that the linear codes over such non-chain rings have been received increasing attention to the authors (see [10]-[12], [14]). More recently, linear codes over the non-chain ring \( \mathbb{Z}_4 + u\mathbb{Z}_4 \), where \( u^2 = 0 \), has been explored in [6]. Also, linear codes over the non-chain ring \( \mathbb{Z}_8 + u\mathbb{Z}_8 \), with \( u^2 = 0 \), were obtained in [14]. \((1+u)\)-constacyclic codes over \( \mathbb{Z}_4 + u\mathbb{Z}_4 \) and a class of constacyclic codes over \( \mathbb{F}_p + u\mathbb{F}_p \) and its gray image were studied in [7] and [13] respectively. Motivated by the work over the ring presented in [7] and [14], we focus on the construction of the constacyclic codes over the ring \( \mathbb{Z}_8 + \vartheta \mathbb{Z}_8 \), with \( \vartheta^2 = 0 \) and intent to establish some good binary codes from such codes.

2. The Ring \( \mathbb{Z}_8 + \vartheta \mathbb{Z}_8 \)

Throughout this paper, the ring \( \mathbb{Z}_8 + \vartheta \mathbb{Z}_8 \) with \( \vartheta^2 = 0 \) is denoted by \( \mathcal{R} \). An arbitrary element \( a + \vartheta b \) is a unit in \( \mathcal{R} \) if and only if \( a \) is a unit in \( \mathbb{Z}_8 \). The ring \( \mathcal{R} \) is a local Frobenius ring and a finite non-chain ring having total of 12 ideals defined as

| S.No. | Ideals |
|-------|--------|
| 1     | \( \mathcal{I}_0 = \{0\} \) |
| 2     | \( \mathcal{I}_1 = \mathbb{Z}_8 + \vartheta \mathbb{Z}_8 \) |
| 3     | \( \mathcal{I}_2 = \{a + \vartheta b : a, b \in \{0, 2, 4, 6\}\} \) |
| 4     | \( \mathcal{I}_4 = \{0, 4, 4\vartheta, 4 + 4\vartheta\} \) |
| 5     | \( \mathcal{I}_{6\vartheta} = \{a\vartheta : a \in \mathbb{Z}_8\} \) |
| 6     | \( \mathcal{I}_{2+2\vartheta} = \{a + b\vartheta : a \in \{0, 4\}, b \in \mathbb{Z}_8\} \) |
| 7     | \( \mathcal{I}_{4+2\vartheta} = \{0, 4\vartheta, 4 + 2\vartheta, 4 + 6\vartheta\} \) |

A non-empty subset \( \hat{A} \) over \( \mathcal{R}^m \) of length \( m \) is said to be a linear code, if it is an \( \mathcal{R} \)-submodule of \( \mathcal{R}^m \).
Now, defining the mappings $\sigma$, $\gamma$ and $\zeta$ from $\mathbb{R}^m$ to $\mathbb{R}^m$ as follows:

\[
\sigma(c_0, c_1, \ldots, c_{m-1}) = (c_{m-1}, c_0, c_1, \ldots, c_{m-2}),
\]
\[
\gamma(c_0, c_1, \ldots, c_{m-1}) = (-c_{m-1}, c_0, c_1, \ldots, c_{m-2}),
\]
\[
\zeta(c_0, c_1, \ldots, c_{m-1}) = (\zeta c_{m-1}, c_0, c_1, \ldots, c_{m-2}).
\]

Here, the defined mappings $\sigma$, $\gamma$ and $\zeta$ are known as the cyclic, negacyclic and constacyclic shift respectively. Moreover, $\hat{A}$ is a cyclic code, negacyclic code and $\zeta$-constacyclic code if $\sigma(\hat{A}) = \hat{A}$, $\gamma(\hat{A}) = \hat{A}$ and $\zeta(\hat{A}) = \hat{A}$ respectively.

The polynomial representation of the codeword $c = (c_0, c_1, \ldots, c_{m-1})$ is $c(x) = c_0 + c_1 x + \ldots + c_{m-1} x^{m-1}$ and $x c(x)$ corresponds to a $\zeta$-constacyclic shift of $c(x)$ in the ring $\mathbb{R}[x] / \langle x^m - \zeta \rangle$. Thus, $\zeta$-constacyclic codes of length $m$ over $\mathbb{R}$ can be identified as ideals in the ring $\mathbb{R}[x] / \langle x^m - \zeta \rangle$. Thus, we have the following proposition.

**Proposition 2.1.** A subset $\mathcal{C}$ of $\mathbb{R}^m$ is a linear cyclic code of length $m$ if and only if $\mathcal{C}$ is an ideal of $\mathcal{A}_m = \mathbb{R}[x] / \langle x^m - 1 \rangle$. A subset $\mathcal{C}$ of $\mathbb{R}^m$ is a linear $(1+\vartheta)$-constacyclic code of length $m$ over $\mathbb{R}$ if and only if $\mathcal{C}$ is an ideal of $\mathcal{B}_m = \mathbb{R}[x] / \langle x^m - 1 - \vartheta \rangle$.

A unique set of generators for cyclic codes over $\mathbb{Z}_8$ are discussed in the next lemma.

**Lemma 2.2.** Let $\mathcal{C}$ be a cyclic code of length $m$ over $\mathbb{Z}_8$. Then,

1. If $m$ is odd then, $\mathcal{C} = \langle g(x), 4a(x) \rangle = \langle g(x) + 4a(x) \rangle$, where $g(x)$, $a(x)$ are binary polynomials with $a(x)|g(x)|(x^m - 1) \text{ mod } 2$.

2. If $m$ is even, then
   
   (i): If $g(x) = a(x)$ then, $\mathcal{C} = \langle g(x), 4a(x) \rangle = \langle g(x) + 4a(x) \rangle$, where $g(x)$, $a(x)$ are the binary polynomials with $g(x)|(x^m - 1) \text{ mod } 2$, and $g(x)|p(x)\frac{(x^m - 1)}{g(x)}$.

   (ii): $\mathcal{C} = \langle g(x) + 4p(x), 4a(x) \rangle$, where $g(x)$, $a(x)$ and $p(x)$ are the binary polynomials with $a(x)|g(x)|(x^m - 1) \text{ mod } 2$, $a(x)|p(x)\frac{(x^m - 1)}{g(x)}$ and $\text{deg}(g(x)) > \text{deg}(a(x)) > \text{deg}(p(x))$. 
For a linear code $C$ of length $m$ over $R$, the two linear codes: Torsion code, Tor($C$) and Residue code, Res($C$) of length $m$ over $Z_8$ are defined as:

$$\text{Tor}(C) = \{x \in Z_8^m \mid \vartheta x \in C\},$$

$$\text{Res}(C) = \{x \in Z_8^m \mid \exists y \in Z_8^m : x + \vartheta y \in C\}.$$

The homomorphism $\varphi : R \to Z_8$ as $\varphi(a + \vartheta b) = a$, extends naturally to a ring homomorphism $\varphi : R^m \to \frac{Z_8[x]}{<x^m - 1>}$ defined as

$$\varphi(c_0 + c_1x + \ldots + c_{m-1}x^{m-1}) = \varphi(c_0) + \varphi(c_1)x + \ldots + \varphi(c_{m-1})x^{m-1}.$$

Acting $\varphi$ on $C$ over $R$, define a ring homomorphism $\varphi : C \to \text{Res}(C)$ as $\varphi(a + \vartheta b) = a$, where $a, b \in Z_8$ with $\text{Ker}\varphi \cong \text{Tor}(C)$ and $\varphi(C) = \text{Res}(C)$.

By the application of first isomorphism theorem of finite groups, $|C| = |\text{Tor}(C)||\text{Res}(C)|$. Also, the image of $C$ under the map $\varphi$ is a cyclic code of length $m$ over $Z_8$.

Combining the above result with lemma 2.2, the set of generators for cyclic code of length $m$ over $R$ can be obtained as provided in following theorem.

**Theorem 2.3.** Let $C$ be a $(1 + \vartheta)$-constacyclic code of length $m$ over $R$. Then

1. If $m$ is odd then, $C = <g_1(x), 4a_1(x) + \vartheta b(x), \vartheta (g_2(x) + 4a_2(x))>$, where $b(x)$ is a polynomial in $Z_8[x]$ and for $i = 1, 2$, $g_i(x), a_i(x)$ are the binary polynomials with $a_i(x) \mid g_i(x) \mid (x^m - 1) \mod 2$.

2. If $m$ is even then,
   (i) If $g_i(x) = a_i(x)$ then, $C = <g_1(x) + 4p_1(x) + \vartheta d_1, \vartheta (g_2(x) + 4p_2(x))>$, where $b(x)$ is a polynomial in $Z_8[x]$, and for $i = 1, 2$, $g_i(x), a_i(x)$ are the binary polynomial with $g_i(x) \mid (x^m - 1) \mod 2$, and $g_i(x) \mid p_i(x) \frac{(x^m - 1)}{g_i(x)}$.
   (ii) $C = <g_1(x) + 4p_1(x) + \vartheta e_1(x), 4a_1(x) + \vartheta (g_2(x) + 4p_2(x), 4a_2(x))>$, where $g(x), a(x)$ and $p(x)$ are the binary polynomials with $a(x) \mid g(x) \mid (x^m - 1) \mod 2$, $a(x) \mid p(x) \frac{(x^m - 1)}{g(x)}$ and $\deg(g(x)) > \deg(a(x)) > \deg(p(x))$. 
3. **Gray Maps**

**Gray images of \((1 + \vartheta)\)-constacyclic codes over \(\mathcal{R}\)**

The gray map \(\rho_1\) from \(\mathbb{Z}_8\) to \(\mathbb{Z}_2^4\) defined as
\[
\rho_1(z) = (q + r, p + r, q + r),
\]
where \(z = p + 2q + 4r\) with \(p, q, r \in \mathbb{Z}_2\), is a distance preserving map from \(\mathbb{Z}_8^m\) (Lee distance) to \(\mathbb{Z}_2^{4m}\) (Hamming distance) and can be extended to \(\mathbb{Z}_8^m\) as:
\[
\rho_1(z_0, z_1, \ldots, z_{m-1}) = (q_0 + r_0, q_1 + r_1, \ldots, q_{m-1} + r_{m-1}).
\]
Now, defining a new gray map \(\rho_2\) from \(\mathbb{Z}_8^m\) to \(\mathbb{Z}_8^n\) as
\[
\rho_2(c) = (b + 7a, b + 6a, b + 5a, b + 4a, b + 3a, b + 2a, b + a, b),
\]
where \(c = a + ub\) and \(a, b \in \mathbb{Z}_8\) and can also be extended from \(\mathbb{Z}_8^m\) to \(\mathbb{Z}_8^n\) as
\[
\rho_2(c_0, c_1, \ldots, c_{m-1}) = (b_0 + 7a_0, \ldots, b_{m-1} + 7a_{m-1}, b_0 + 6a_0, \ldots, b_{m-1} + 6a_{m-1}, b_0 + 5a_0, \ldots, b_0 + a_0, \ldots, b_{m-1} + a_{m-1}, b_0, \ldots, b_0).
\]
where \(c_i = a_i + \vartheta b_i\) and \(a_i, b_i \in \mathbb{Z}_8\).

It is well known that the homogeneous weight has many applications for codes over finite rings and provides a good metric for the underlying ring in constructing superior codes. Next, a homogeneous weight on \(\mathcal{R}\) is defined after defining of the homogeneous weight on arbitrary finite ring \(\mathcal{X}\).

**Definition 3.1.** A real valued function \(w\) on the finite ring \(\mathcal{X}\) is called a left homogeneous weight if \(w(0) = 0\) and the following holds:

1. For all \(x, y \in \mathcal{X}\), \(\mathcal{X} x = \mathcal{X} y\) implies \(w(x) = w(y)\).
(2) There exists a real number $\gamma$ such that
\[
\sum_{y \in \mathcal{K}(x)} w(y) = \gamma |\mathcal{K}| \quad \text{for all } x \in \mathcal{K} \setminus \{0\}.
\]

The Right homogeneous weight can be defined in a similar manner and if weight is both left homogeneous and right homogeneous, it is known as a homogeneous weight. For any element $c = a + \vartheta b \in \mathcal{R}$; the homogeneous weight denoted by $w_{\text{hom}}(c)$, as $w_L(b + 7a, b + 6a, ..., b + a, b)$. By simple calculations the weight of any element $c = a + \vartheta b \in \mathcal{R}$ is:

\[
w_{\text{hom}}(x) = \begin{cases} 
0 & \text{if } c = 0, \\
8 & \text{if } c = \vartheta, 7\vartheta, \\
24 & \text{if } c = 3\vartheta, 5\vartheta, \\
32 & \text{if } c = 4\vartheta, \\
16 & \text{if otherwise}.
\end{cases}
\]

It is easy to verify that, the above defined weight meets the conditions of the Definition 3.1, hence it is actually a homogeneous weight on $\mathcal{R}$. The homogeneous distance of a linear code $C$ over $\mathcal{R}$, denoted by $d_{\text{hom}}(C)$, is defined as the minimum homogeneous weight of the non-zero codewords of $C$.

The map $\rho_2$ is a distance preserving map from $\mathcal{R}^m$ (homogeneous distance) to $\mathbb{Z}_{8^m}$ (Lee distance). Thus, we have the following three distance preserving maps:

\[
\rho_1 : (\mathbb{Z}_{8^m}, \text{Lee Distance}) \rightarrow (\mathbb{Z}_2^m, \text{Hamming Distance})
\]

\[
\rho_2 : (\mathcal{R}^m, \text{Homogeneous Distance}) \rightarrow (\mathbb{Z}_{8^m}, \text{Lee Distance})
\]

\[
\rho = \rho_1 \rho_2 : (\mathcal{R}^m, \text{Homogeneous Distance}) \rightarrow (\mathbb{Z}_2^{32m}, \text{Hamming Distance})
\]

4. $(1 + \theta)$-CONSTACYCLIC CODES

The following theorem defined a result on the above defined map $\rho_2$.

**Theorem 4.1.** If $\zeta$ is a $(1 + \theta)$ - constacyclic shift on $\mathcal{R}^m$, $\varpi$ is a cyclic shift on $\mathbb{Z}_{8^m}$ and $\rho_2$ be a map defined as above, then $\rho_2 \zeta = \varpi \rho_2$. 

Proof. If \( c = (c_0, c_1, \ldots, c_{m-1}) \in \mathbb{R}^m \) where \( c_i = a_i + \vartheta b_i \) and \( a_i, b_i \in \mathbb{Z}_8 \) for \( 0 \leq i \leq m - 1 \).

The definition of the map \( \rho_2 \), implies

\[
\rho_2(c) = (b_0 + 7a_0, \ldots, b_{m-1} + 7a_{m-1}, b_0 + 6a_0, \ldots, b_{m-1} + 6a_{m-1}, b_0 + 5a_0, \ldots, b_{m-1} + 5a_{m-1}, b_0 + 4a_0, \ldots, b_{m-1} + 4a_{m-1}, b_0 + 3a_0, \ldots, b_{m-1} + 3a_{m-1}, b_0 + 2a_0, \ldots, b_{m-1} + 2a_{m-1}, b_0 + a_0, \ldots, b_{m-1} + a_{m-1}, b_0, \ldots, b_{m-1}),
\]

and

\[
\varpi \rho_2(c) = (b_{m-1}, b_0 + 7a_0, \ldots, b_{m-1} + 7a_{m-1}, b_0 + 6a_0, \ldots, b_{m-1} + 6a_{m-1}, b_0 + 5a_0, \ldots, b_{m-1} + 5a_{m-1}, b_0 + 4a_0, \ldots, b_{m-1} + 4a_{m-1}, b_0 + 3a_0, \ldots, b_{m-1} + 3a_{m-1}, b_0 + 2a_0, \ldots, b_{m-1} + 2a_{m-1}, b_0 + a_0, \ldots, b_{m-1} + a_{m-1}, b_0, \ldots, b_{m-2})).
\]

On the other hand,

\[
\zeta (c) = ((1 + \vartheta)c_{m-1}, c_0, c_1, \ldots, c_{m-2})
= ((1 + \vartheta)(a_{m-1} + \vartheta b_{m-1}), a_0 + \vartheta b_0, a_1 + \vartheta b_1, \ldots, a_{m-2} + \vartheta b_{m-2}),
\]

and therefore,

\[
\rho_2 \zeta (c) = (b_{m-1} + a_{m-1} + 7a_{m-1}, b_0 + 7a_0, \ldots, b_{m-1} + 7a_{m-1}, b_0 + 6a_0, \ldots, b_{m-1} + 6a_{m-1}, b_0 + 5a_0, \ldots, b_{m-1} + 5a_{m-1}, b_0 + 4a_0, \ldots, b_{m-1} + 4a_{m-1}, b_0 + 3a_0, \ldots, b_{m-1} + 3a_{m-1}, b_0 + 2a_0, \ldots, b_{m-1} + 2a_{m-1}, b_0 + a_0, \ldots, b_{m-1} + a_{m-1}, b_0, \ldots, b_{m-2})).
\]

Hence, the result follows. \( \square \)

**Theorem 4.2.** A linear code \( \mathcal{C} \) of length \( m \) over \( \mathbb{R} \) is a \((1 + \vartheta)\)-constacyclic code if and only if \( \rho_2(\mathcal{C}) \) is a cyclic code of length \( 8m \) over \( \mathbb{Z}_8 \).

Proof. If \( \mathcal{C} \) is a \((1 + \vartheta)\)-constacyclic code, then Theorem 4.1 implies

\[
\varpi (\rho_2(\mathcal{C})) = \rho_2(\zeta(\mathcal{C})) = \rho_2(\mathcal{C}),
\]
Hence, $\rho_2(\mathcal{C})$ is a cyclic code of length $8m$ over $\mathbb{Z}_8$. Further, if $\rho_2(c)$ is a cyclic code of length $8m$ over $\mathbb{Z}_8$, then use Theorem 4.1 to obtain

$$\rho_2(\zeta(\mathcal{C})) = \sigma(\rho_2(\mathcal{C})) = \rho_2(\mathcal{C})$$

Since, $\rho_2$ is an injective mapping, therefore $\zeta(\mathcal{C}) = \mathcal{C}$ and hence, the result holds. \qed

The following corollary is an immediate consequence of above theorem.

**Corollary 4.3.** The image of $(1 + \vartheta)$-constacyclic code of length $m$ over $\mathbb{R}$ under the map $\rho_2$ is a distance invariant cyclic code of length $8m$ over $\mathbb{Z}_8$.

If $\sigma$ is a cyclic shift, then for a positive integer $s$, the quasi-shift $\sigma_s$ is given by

$$\sigma_s(a^{(1)}|a^{(2)}|...|a^{(s)}) = (\sigma(a^{(1)})|\sigma(a^{(2)})|...|\sigma(a^{(s)})),$$

where $a^{(1)}$, $a^{(2)}$, ..., $a^{(s)} \in \mathbb{F}_2^{(2m)}$ and " | " represents the usual vector concatenation. A binary quasi-cyclic code $\mathcal{C}$ of index $s$ and length $2ms$ is a subset of $(\mathbb{Z}_2^{2m})^s$ such that $\sigma_s(\mathcal{C}) = \mathcal{C}$.

**Lemma 4.4.** If $\zeta$ is a $(1 + \vartheta)$-constacyclic shift on $\mathbb{R}^m$ and $\rho$ be a mapping defined as above, then $\rho \zeta = \sigma_{16} \rho$.

**Proof.** For $r = (r_0, r_1, ..., r_{m-1}) \in \mathbb{R}^m$, where $r_i = a_i + 2b_i + 4c_i + \vartheta d_i + 2\vartheta e_i + 4\vartheta f_i$, $a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{Z}_2$, for $0 \leq i \leq m - 1$. Then,
\[ \rho(r) = (c_0 + e_0 + f_0, \ldots, c_{m-1} + e_{m-1} + f_{m-1}, b_0 + e_0 + f_0, \ldots, b_{m-1} + e_{m-1} + f_{m-1}, a_0 \\
+ b_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} + b_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, a_0 + e_0 + f_0, \ldots, \\
+ b_0 + e_0 + f_0, \ldots, a_{m-1} + b_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, b_0 + c_0 + e_0 + f_0, \ldots, b_{m-1} \\
+ c_{m-1} + e_{m-1} + f_{m-1}, e_0 + f_0, \ldots, e_{m-1} + f_{m-1}, a_0 + b_0 + c_0 + f_0, \ldots, a_{m-1} \\
+ b_{m-1} + c_{m-1} + f_{m-1}, a_0 + b_0 + f_0, \ldots, a_{m-1} + b_{m-1} + f_{m-1}, a_0 + c_0 + f_0, \\
\ldots, a_{m-1} + c_{m-1} + f_{m-1}, a_0 + f_0, \ldots, a_{m-1} + f_{m-1}, b_0 + c_0 + f_0, \ldots, b_{m-1} + c_{m-1} \\
+ f_{m-1}, b_0 + f_0, \ldots, b_{m-1} + f_{m-1}, c_0 + f_0, \ldots, c_{m-1} + f_{m-1}, f_0, \ldots, f_{m-1}, b_0 + c_0 \\
+ d_0 + f_0, \ldots, b_{m-1} + c_{m-1} + d_{m-1} + f_{m-1}, a_0 + b_0 + d_0 + f_0, \ldots, a_{m-1} + b_{m-1} \\
+ d_{m-1} + f_{m-1}, c_0 + d_0 + f_0, \ldots, c_{m-1} + d_{m-1} + f_{m-1}, a_0 + d_0 + f_0, \ldots, a_{m-1} \\
+ d_{m-1} + f_{m-1}, a_0 + b_0 + c_0 + d_0 + f_0, \ldots, a_{m-1} + b_{m-1} + c_{m-1} + d_{m-1} + f_{m-1}, \\
b_0 + d_0 + f_0, \ldots, b_{m-1} + d_{m-1} + f_{m-1}, a_0 + c_0 + d_0 + f_0, \ldots, a_{m-1} + c_{m-1} + d_{m-1} \\
+ f_{m-1}, d_0 + f_0, \ldots, d_{m-1} + f_{m-1}, c_0 + e_0 + f_0, \ldots, c_{m-1} + e_{m-1} + f_{m-1}, b_0 + e_0 \\
+ f_0, \ldots, b_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} + b_{m-1} + c_{m-1} \\
+ e_{m-1} + f_{m-1}, a_0 + e_0 + f_0, \ldots, a_{m-1} + e_{m-1} + f_{m-1}, a_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} \\
+ c_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + e_0 + f_0, \ldots, a_{m-1} + b_{m-1} + e_{m-1} + f_{m-1}, b_0 \\
+ c_0 + e_0 + f_0, \ldots, b_{m-1} + c_{m-1} + e_{m-1} + f_{m-1}, e_0 + f_0, \ldots, e_{m-1} + f_{m-1}) \]
and therefore,

\[ (1 + \vartheta)\text{-CONSTACYCLIC CODES OVER } \mathbb{Z}_8 + \vartheta \mathbb{Z}_8 \]
On the other hand,
\[ \zeta(r) = ((1 + \theta)r_{m-1}, r_0, r_1, \ldots, r_{m-2}) \]
\[ = ((1 + \theta)(a_{m-1} + 2b_{m-1} + 4c_{m-1} + \vartheta d_{m-1} + 2\vartheta e_{m-1} + 4\vartheta f_{m-1}), r_0, \ldots, r_{m-2}) \]
\[ = (a_{m-1} + 2b_{m-1} + 4c_{m-1} + \vartheta (a_{m-1} + d_{m-1}) + 2\vartheta (b_{m-1} + e_{m-1}) + 4\vartheta (c_{m-1} + f_{m-1}), r_0, \ldots, r_{m-2}) \]
and therefore,
\[ \rho(\zeta(r)) = (b_{m-1} + e_{m-1} + f_{m-1}, c_0 + e_0 + f_0, \ldots, c_{m-1} + e_{m-1} + f_{m-1}, b_0 + e_0 + f_0, \ldots, \]
\[ b_{m-2} + e_{m-2} + f_{m-2}, a_{m-1} + e_{m-1} + f_{m-1}, a_0 + b_0 + c_0 + e_0 + f_0, \ldots, a_{m-1} \]

Hence the result.
Theorem 4.5. A linear code $C$ of length $m$ over $R$ is a $(1 + \vartheta)$-constacyclic code if and only if $\rho(C)$ is a binary quasi-cyclic code of length $32m$ with index 16.

Proof. If $C$ is a $(1 + \vartheta)$-constacyclic code, then use of Theorem 4.4 gives,

$$\varpi_{16}(\rho(C)) = \rho(\zeta(C)) = \rho(C),$$

which implies $\rho(C)$ is a binary quasi-cyclic code of length $32m$ with index 16, and again applying Theorem 4.4 to obtain

$$\rho(\zeta(C)) = \varpi_{16}(\rho(C)) = \rho(C).$$

Further, $\rho$ is an injective mapping and therefore, $\zeta(C) = C$. □

From Theorem 4.5 and the definition of the map $\rho$, the following result holds immediately.

Corollary 4.6. The image of a $(1 + \vartheta)$-constacyclic code of length $m$ over $R$ under the map $\rho$ is a distance invariant binary quasi-cyclic code of length $32m$ with index 16.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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