On approximate quasi Pareto solutions in nonsmooth semi-infinite interval-valued vector optimization problems

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ABSTRACT
This paper deals with approximate solutions of a nonsmooth semi-infinite programming with multiple interval-valued objective functions. We first introduce four types of approximate quasi Pareto solutions of the considered problem by considering the lower-upper interval order relation and then apply some advanced tools of variational analysis and generalized differentiation to establish necessary optimality conditions for these approximate solutions. Sufficient conditions for approximate quasi Pareto solutions of such a problem are also provided by means of introducing the concepts of approximate (strictly) pseudo-quasi generalized convex functions defined in terms of the limiting subdifferential of locally Lipschitz functions. Finally, a Mond–Weir type dual model in approximate form is formulated, and weak, strong and converse-like duality relations are proposed.

KEYWORDS
KKT optimality conditions; Duality relations; Limiting/Mordukhovich subdifferential; Approximate quasi Pareto solutions; Nonsmooth semi-infinite interval-valued vector optimization

AMS CLASSIFICATION
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1. Introduction

In this paper, we are interested in approximate solutions of the following semi-infinite programming with multiple interval-valued objective functions:

\[
LU - \text{Min } f(x) := (f_1(x), \ldots, f_m(x)) \\
\text{s. t. } x \in \mathcal{F} := \{x \in \Omega : g_t(x) \leq 0, t \in T\},
\]

(SIVP)

where \(f_i : \mathbb{R}^n \to \mathcal{K}_c\), \(i \in I := \{1, \ldots, m\}\), are interval-valued functions defined by \(f_i(x) = [f^L_i(x), f^U_i(x)]\), \(f^L_i, f^U_i : \mathbb{R}^n \to \mathbb{R}\) are locally Lipschitz functions satisfying \(f^L_i(x) \leq f^U_i(x)\) for all \(x \in \mathbb{R}^n\) and \(i \in I\), \(\mathcal{K}_c\) is the class of all closed and bounded intervals in \(\mathbb{R}\), i.e.,

\[
\mathcal{K}_c = \{[a^L, a^U] : a^L, a^U \in \mathbb{R}, a^L \leq a^U\},
\]

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$g_t : \mathbb{R}^n \to \mathbb{R}, t \in T$, are locally Lipschitz functions, $T$ is an arbitrary set (possibly infinite), and $\Omega$ is a nonempty and closed subset of $\mathbb{R}^n$. Set $g_T := (g_t)_{t \in T}$.

An interval-valued optimization problem is one of the deterministic optimization models to deal with the uncertain (incomplete) data. In the literature, there are three main approaches to model constrained optimization with uncertainty, say stochastic programming approach, fuzzy programming approach, and interval-valued programming approach; see, e.g., [1–6]. Many methodologies have been developed to solve these problems. However, it should be noted here that the usual way is to transform stochastic and fuzzy optimization problems into the conventional optimization problems; frequently, these problems are very complicated. Consequently, stochastic and fuzzy optimization problems are not easy to be solved. In interval-valued optimization, the coefficients of objective and constraint functions are taken as closed intervals. Hence, the interval-valued optimization problem will be easier to be solved than a stochastic or fuzzy optimization one. That is the main reason why the interval-valued optimization problems have recently received increasing interest in optimization community; see, e.g., [6–22] and the references therein.

In [8], Ishibuchi and Tanaka introduced the lower-upper (LU) interval order relation and reformulated optimization problems with interval-valued objective functions as vector optimization problems using the order relation. Thereafter, many authors have investigated optimality conditions of Karush–Kuhn-Tucker-type (KKT) and duality for optimization problems with one or multiple interval-valued objective functions and finitely many constraints; see, e.g., [6,7,11,13–16,19–22].

The semi-infinite optimization problems play a very important role in optimization theory and their models cover optimal control, approximation theory, semi-definite programming and numerous engineering problems, etc. However, in contrast with the case of optimization problems with finitely many constraints, there have only been several papers dealing with KKT optimality conditions and duality for semi-infinite interval-valued optimization problems. For papers of this topic, we refer the reader to [9,10,12,17,18] and references given therein. In [10], Kumar et al. established optimality conditions and duality theorems for interval-valued programming problems with infinitely many constraints. Then, in [9,12,17,18], the authors presented optimality conditions and duality theorems for Pareto optimal solutions with respect to LU interval order relation of a semi-infinite interval-valued vector optimization problem under the convexity of the objective functions and constraints. It should be noted here that the study of approximate solutions is very important in optimization because, from the computational point of view, numerical algorithms usually generate only approximate solutions if we stop them after a finite number of steps. Furthermore, the solution set may be empty in the general noncompact case, whereas approximate solutions exist under very weak assumptions; see e.g., [22–31].

Motivated by the above observations, in this paper, we introduce four kinds of approximate quasi Pareto solutions with respect to $LU$ interval order relation for problems of the form (SIVP). Then we employ the Mordukhovich/limiting subdifferential and the Mordukhovich/limiting normal cone (cf. [32]) to examine KKT optimality conditions and duality relations for these approximate solutions of problem (SIVP).

The paper is organized as follows. Section 2 provides some basic definitions from variational analysis, interval analysis and several auxiliary results. In Section 3 we introduce four kinds of approximate quasi Pareto solutions of problem (SIVP) and establish KKT-type necessary conditions for these approximate solutions.
the help of approximate generalized convex functions defined in terms of the Mordukhovich/limiting subdifferential and the Mordukhovich normal cone, we provide sufficient conditions for approximate quasi Pareto solutions of the considered problem. Section 4 is devoted to presenting duality relations for approximate quasi Pareto solutions. The conclusions are presented in the final section.

2. Preliminaries

We use the following notation and terminology. Fix \( n \in \mathbb{N} := \{1, 2, \ldots\} \). The space \( \mathbb{R}^n \) is equipped with the usual scalar product and Euclidean norm. The closed unit ball of \( \mathbb{R}^n \) is denoted by \( \mathbb{B}_{\mathbb{R}^n} \). We denote the nonnegative orthant in \( \mathbb{R}^n \) by \( \mathbb{R}_+^n \). The topological closure is denoted by \( \text{cl} \).

**Definition 2.1** (see [32]). Given \( \bar{x} \in \text{cl} S \). The set

\[
N(\bar{x}; S) := \{ z^* \in \mathbb{R}^n : \exists x_k \overset{S}{\to} \bar{x}, \varepsilon_k \to 0^+, z^*_k \to z^*, z^*_k \in \widehat{N}_{\varepsilon_k}(x^*_k; S), \ \forall k \in \mathbb{N} \},
\]

is called the **Mordukhovich/limiting normal cone** of \( S \) at \( \bar{x} \), where

\[
\widehat{N}_{\varepsilon}(x; S) := \left\{ z^* \in \mathbb{R}^n : \limsup_{u \overset{S}{\to} x} \frac{\langle z^*, u - x \rangle}{\| u - x \|} \leq \varepsilon \right\}
\]

is the set of \( \varepsilon \)-normals of \( S \) at \( x \) and \( u \overset{S}{\to} x \) means that \( u \to x \) and \( u \in S \).

Let \( \varphi : \mathbb{R}^n \to \overline{\mathbb{R}} \) be an **extended-real-valued function**. The **epigraph** and **domain** of \( \varphi \) are denoted, respectively, by

\[
epi \varphi := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \alpha \geq \varphi(x)\},
\]

\[
dom \varphi := \{ x \in \mathbb{R}^n : |\varphi(x)| < +\infty \}.
\]

**Definition 2.2** (see [32]). Let \( \bar{x} \in \text{dom } \varphi \). The set

\[
\partial \varphi(\bar{x}) := \{ x^* \in \mathbb{R}^n : (x^*, \varphi(\bar{x})) \in N((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi) \},
\]

is called the **Mordukhovich/limiting subdifferential** of \( \varphi \) at \( \bar{x} \). If \( \bar{x} \notin \text{dom } \varphi \), then we put \( \partial \varphi(\bar{x}) = \emptyset \).

We now summarize some properties of the Mordukhovich subdifferential that will be used in the next section.

**Proposition 2.3** (see [32 Theorem 3.36]). Let \( \varphi_l : \mathbb{R}^n \to \overline{\mathbb{R}}, l = 1, \ldots, p, p \geq 2, \) be lower semicontinuous around \( \bar{x} \) and let all but one of these functions be locally Lipschitz around \( \bar{x} \). Then we have the following inclusion

\[
\partial(\varphi_1 + \ldots + \varphi_p)(\bar{x}) \subset \partial \varphi_1(\bar{x}) + \ldots + \partial \varphi_p(\bar{x}).
\]

**Proposition 2.4** (see [32 Theorem 3.46]). Let \( \varphi_l : \mathbb{R}^n \to \overline{\mathbb{R}}, l = 1, \ldots, p, \) be locally Lipschitz around \( \bar{x} \). Then the function \( \phi(\cdot) := \max\{\varphi_l(\cdot) : l = 1, \ldots, p\} \) is also locally
Lipschitz around $\bar{x}$ and one has
\[
\partial \phi(\bar{x}) \subset \bigcup \left\{ \partial \left( \sum_{i=1}^{p} \lambda_i \varphi_i \right)(\bar{x}) : (\lambda_1, \ldots, \lambda_p) \in \Lambda(\bar{x}) \right\},
\]
where $\Lambda(\bar{x}) := \{ (\lambda_1, \ldots, \lambda_p) : \lambda_i \geq 0, \sum_{i=1}^{p} \lambda_i = 1, \lambda_i [\varphi_i(\bar{x}) - \phi(\bar{x})] = 0 \}$.

**Proposition 2.5** (see [32, Proposition 1.114]). Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be finite at $\bar{x}$. If $\varphi$ has a local minimum at $\bar{x}$, then $0 \in \partial \phi(\bar{x})$.

Next we recall some definitions and facts in interval analysis. Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two intervals in $\mathcal{K}_c$. Then, by definition, we have
(i) $A + B := \{ a + b : a \in A, b \in B \} = [a^L + b^L, a^U + b^U]$;
(ii) $A - B := \{ a - b : a \in A, b \in B \} = [a^L - b^U, a^U - b^L]$;
(iii) For each $k \in \mathbb{R}$,
\[
kA := \{ ka : a \in A \} = \begin{cases} [ka^L, ka^U] & \text{if } k \geq 0, \\ [ka^U, ka^L] & \text{if } k < 0, \end{cases}
\]
see, e.g., [33, 35] for more details. It should be noted that if $a^L = a^U$, then $A = [a, a] = a$ is a real number.

**Definition 2.6** (see [8, 19]). Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two intervals in $\mathcal{K}_c$. We say that:
(i) $A \leq_{LU} B$ if $a^L \leq b^L$ and $a^U \leq b^U$.
(ii) $A <_{LU} B$ if $A \leq_{LU} B$ and $A \neq B$, or, equivalently, $A <_{LU} B$ if
\[
\begin{cases} a^L < b^L, \\ a^U \leq b^U, \end{cases}
\quad \text{or} \quad
\begin{cases} a^L \leq b^L, \\ a^U < b^U, \end{cases}
\quad \text{or} \quad
\begin{cases} a^L < b^L, \\ a^U < b^U. \end{cases}
\]
(iii) $A <_{LU}^s B$ if $a^L < b^L$ and $a^U < b^U$.
We also define $A \leq_{LU} B$ (resp., $A <_{LU} B$, $A <_{LU}^s B$) if and only if $B \geq_{LU} A$ (resp., $B \geq_{LU} A$, $B >_{LU} A$).

3. Approximate optimality conditions

Let $\mathbb{R}^{|T|}_+$ denote the set of all functions $\mu : T \to \mathbb{R}_+$ taking values $\mu_t := \mu(t) = 0$ for all $t \in T$ except for finitely many points. The active constraint multipliers set at $\bar{x} \in \Omega$ is defined by
\[
A(\bar{x}) := \left\{ \mu \in \mathbb{R}^{|T|}_+ : \mu_t g_t(\bar{x}) = 0, \ \forall t \in T \right\}.
\]
For each $\mu \in A(\bar{x})$, put $T(\mu) := \{ t \in T : \mu_t \neq 0 \}$.

We now introduce approximate solutions of (SIVP) with respect to $LU$ interval order relation. Let $\epsilon_i^L, \epsilon_i^U, i \in I$, be real numbers satisfying $0 \leq \epsilon_i^L \leq \epsilon_i^U$ for all $i \in I$ and put $\mathcal{E} := (\mathcal{E}_{1}, \ldots, \mathcal{E}_{m})$, where $\mathcal{E}_i := [\epsilon_i^L, \epsilon_i^U]$.

**Definition 3.1.** Let $\bar{x} \in \mathcal{F}$. We say that:
(i) $\bar{x}$ is a type-1 $E$-quasi Pareto solution of $(SIVP)$, denoted by $\bar{x} \in E_{S_q^1}(SIVP)$, if there is no $x \in F$ such that
\[
\begin{align*}
f_i(x) &\leq LU f_i(\bar{x}) - \mathcal{E}_i \|x - \bar{x}\|, \quad \forall i \in I, \\
f_k(x) &< LU f_k(\bar{x}) - \mathcal{E}_k \|x - \bar{x}\|, \quad \text{for at least one } k \in I.
\end{align*}
\]

(ii) $\bar{x}$ is a type-2 $E$-quasi Pareto solution of $(SIVP)$, denoted by $\bar{x} \in E_{S_q^2}(SIVP)$, if there is no $x \in F$ such that
\[
\begin{align*}
f_i(x) &\leq LU f_i(\bar{x}) - \mathcal{E}_i \|x - \bar{x}\|, \quad \forall i \in I, \\
f_k(x) &< LU f_k(\bar{x}) - \mathcal{E}_k \|x - \bar{x}\|, \quad \text{for at least one } k \in I.
\end{align*}
\]

(iii) $\bar{x}$ is a type-1 $E$-quasi-weakly Pareto solution of $(SIVP)$, denoted by $\bar{x} \in E_{S_{qw}^1}(SIVP)$, if there is no $x \in F$ such that
\[
\begin{align*}
f_i(x) &< LU f_i(\bar{x}) - \mathcal{E}_i \|x - \bar{x}\|, \quad \forall i \in I.
\end{align*}
\]

(iv) $\bar{x}$ is a type-2 $E$-quasi-weakly Pareto solution of $(SIVP)$, denoted by $\bar{x} \in E_{S_{qw}^2}(SIVP)$, if there is no $x \in F$ such that
\[
\begin{align*}
f_i(x) &< s LU f_i(\bar{x}) - \mathcal{E}_i \|x - \bar{x}\|, \quad \forall i \in I.
\end{align*}
\]

Remark 3.2. The following relations are immediate from the definition of approximate optimal solutions.

(i) $E_{S_q^1}(SIVP) \subset E_{S_q^2}(SIVP) \subset E_{S_{qw}^2}(SIVP)$.

(ii) $E_{S_{qw}^1}(SIVP) \subset E_{S_{qw}^2}(SIVP)$.

To obtain the necessary optimality conditions of KKT-type for approximate quasi Pareto solutions of $(SIVP)$, we consider the following constraint qualification condition.

Definition 3.3 (see [36,37]). Let $\bar{x} \in F$. We say that $\bar{x}$ satisfies the limiting constraint qualification if the following condition holds
\[
N(\bar{x}; F) \subseteq \bigcup_{\mu \in A(\bar{x})} \left[ \sum_{t \in T} \mu_t \partial g_t(\bar{x}) \right] + N(\bar{x}; \Omega). \quad \text{(LCQ)}
\]

It is worth to mention that the constraint qualification (LCQ) has been widely used in the literature and it covers almost the existing constraint qualifications of the Mangasarian–Fromovitz and the Farkas–Minkowski types; see e.g., [27,32,36,38].

Theorem 3.4. Let $\bar{x} \in F$ and assume that $\bar{x}$ satisfies the (LCQ). If $\bar{x} \in E_{S_{qw}^2}(SIVP)$,
then there exist $\lambda^L, \lambda^U \in \mathbb{R}_+^m$ with $\sum_{i \in I}(\lambda^L_i + \lambda^U_i) = 1$, and $\mu \in A(x)$ such that

\[
0 \in \sum_{i \in I} \left[ \lambda^L_i \partial f^L_i(x) + \lambda^U_i \partial f^U_i(x) \right] + \sum_{i \in I} \mu_i \partial g_i(x) + \sum_{i \in I} \left( \lambda^L_i \epsilon^U_i + \lambda^U_i \epsilon^L_i \right) \mathbb{B}_{\mathbb{R}^n} + N(x; \Omega). \tag{1}
\]

**Proof.** Since $\bar{x} \in \mathcal{E}\mathcal{S}_{2}^{\text{ad}}(\text{SIVP})$, there is no $x \in \mathcal{F}$ such that $f_i(\bar{x}) < \lambda_i^U f_i(x) - \epsilon_i$, $\forall i \in I$, or, equivalently,

\[
f_i^L(x) < f_i^L(\bar{x}) - \epsilon_i^L \left\| x - \bar{x} \right\| \quad \text{and} \quad f_i^U(x) < f_i^U(\bar{x}) - \epsilon_i^U \left\| x - \bar{x} \right\|, \quad \forall i \in I.
\]

This means that for each $x \in \mathcal{F}$, there exists $i \in I$ such that

\[
f_i^L(x) \geq f_i^L(\bar{x}) - \epsilon_i^L \left\| x - \bar{x} \right\| \quad \text{or} \quad f_i^U(x) \geq f_i^U(\bar{x}) - \epsilon_i^U \left\| x - \bar{x} \right\|. \tag{2}
\]

For each $x \in \mathbb{R}^n$, put

\[
\varphi(x) := \max_{i \in I} \left\{ f_i^L(x) - f_i^L(\bar{x}) + \epsilon_i^L \left\| x - \bar{x} \right\|, f_i^U(x) - f_i^U(\bar{x}) + \epsilon_i^U \left\| x - \bar{x} \right\| \right\}.
\]

It follows from (2) that $\varphi(x) \geq 0$ for all $x \in \mathcal{F}$. Clearly, $\varphi(\bar{x}) = 0$. Hence, $\bar{x}$ is a minimizer of $\varphi$ on $\mathcal{F}$, or, equivalently, $\bar{x}$ is an optimal solution of the following unconstrained optimization problem

\[
\text{minimizer} \quad \varphi(x) + \delta(x; \mathcal{F}), x \in \mathbb{R}^n,
\]

where $\delta(\cdot; \mathcal{F})$ is the indicator function of $\mathcal{F}$ and defined by $\delta(x; \mathcal{F}) = 0$ if $x \in \mathcal{F}$ and $\delta(x; \mathcal{F}) = +\infty$ if $x \notin \mathcal{F}$. By Proposition 2.5, we have

\[
0 \in \partial(\varphi + \delta(\cdot; \mathcal{F}))(\bar{x}).
\]

Since functions $f_i^L$, $f_i^U$, $i \in I$, and $\left\| \cdot - \bar{x} \right\|$ are locally Lipschitz, $\varphi$ is locally Lipschitz. Clearly, $\delta(\cdot; \mathcal{F})$ is lower semicontinuous around $\bar{x}$. Hence it follows from Proposition 2.3 and the fact that $\partial \delta(\cdot; \mathcal{F})(\bar{x}) = N(\bar{x}; \mathcal{F})$ (see, e.g., [32, Proposition 1.19]) that

\[
0 \in \partial \varphi(\bar{x}) + N(\bar{x}; \mathcal{F}). \tag{3}
\]

By Proposition 2.4 and the fact that $\partial(\left\| \cdot - \bar{x} \right\|)(\bar{x}) = \mathbb{B}_{\mathbb{R}^n}$ (see [39, Example 4, p. 198])), we obtain

\[
\partial \varphi(\bar{x}) \subset \left\{ \sum_{i \in I} \lambda^L_i \left[ \partial f^L_i(x) + \epsilon_i^L \mathbb{B}_{\mathbb{R}^n} \right] + \sum_{i \in I} \lambda^U_i \left[ \partial f^U_i(x) + \epsilon_i^L \mathbb{B}_{\mathbb{R}^n} \right] : \lambda^L_i, \lambda^U_i \geq 0, i \in I, \sum_{i \in I}(\lambda^L_i + \lambda^U_i) = 1 \right\}
\]

\[
\subset \left\{ \sum_{i \in I} \left[ \lambda^L_i \partial f^L_i(x) + \lambda^U_i \partial f^U_i(x) \right] + \sum_{i \in I} \left( \lambda^L_i \epsilon^U_i + \lambda^U_i \epsilon^L_i \right) \mathbb{B}_{\mathbb{R}^n} : \lambda^L_i, \lambda^U_i \geq 0, i \in I, \sum_{i \in I}(\lambda^L_i + \lambda^U_i) = 1 \right\}.
\]

Combining this, (3) and the (LCQ), we get the assertion. \qed
The following example is to illustrate Theorem 3.4.

**Example 3.5.** Let \( f := (f_1, f_2) \), where the functions \( f_1, f_2 : \mathbb{R}^2 \to K_c \) are defined by
\[
f_1(x) = f_2(x) = [x_1^2 + (x_1x_2 - 1)^2, 2x_1^2 + (x_1x_2 - 1)^2], \quad \forall x = (x_1, x_2) \in \mathbb{R}^2,
\]
and let \( g_i : \mathbb{R}^2 \to \mathbb{R} \) be given by
\[
g_i(x) = -t|x_1| - t|x_2|, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2, t \in T := [0, 1].
\]
Consider problem (SIVP) with \( \Omega = \mathbb{R}^2 \). Clearly, the feasible set of (SIVP) is \( F = \mathbb{R}^2 \).

Let \( E = (E_1, E_2) \), where \( E_1 = E_2 = [1, 2] \). We see that \( \bar{x} = (0, 0) \in E - S_1^T (SIVP) \) and so by Remark 3.2, \( \bar{x} \in E - S_2^T (SIVP) \). Indeed, suppose on the contrary that \( \bar{x} \notin E - S_1^T (SIVP) \), then there exists \( x \in \mathbb{R}^2 \) such that
\[
f_1(x) <_{LU} f_1(\bar{x}) - E_1\|x - \bar{x}\|,
\]
or, equivalently,
\[
\begin{align*}
&\left\{ \begin{array}{l}
x_1^2 + (x_1x_2 - 1)^2 \leq 1 - 2\sqrt{x_1^2 + x_2^2}, \\
2x_1^2 + (x_1x_2 - 1)^2 \leq 1 - \sqrt{x_1^2 + x_2^2},
\end{array} \right.
\end{align*}
\]
with at least one strict inequality. Since (4), we have
\[
\begin{align*}
&\left\{ \begin{array}{l}
x_1^2 + x_1^2x_2^2 + 2\sqrt{x_1^2 + x_2^2} - 2x_1x_2 \leq 0 \\
2x_1^2 + x_1^2x_2^2 + \sqrt{x_1^2 + x_2^2} - 2x_1x_2 \leq 0
\end{array} \right. 
\implies \left\{ \begin{array}{l}
x_1 = 0 \\
x_2 = 0.
\end{array} \right.
\end{align*}
\]
Hence, there is no strict inequality in (4), a contradiction.

Since \( N(x; F) = N(x; \Omega) = \partial g_0(x) = \{0\} \), the (LCQ) is satisfied at every \( x \in \mathbb{R}^2 \).

Hence, by Theorem 3.4 there exist \( \lambda^L, \lambda^U \in \mathbb{R}_+^2 \) with \( \sum_{i \in I}(\lambda^L_i + \lambda^U_i) = 1 \), and \( \mu \in A(\bar{x}) \) satisfying condition (I).

We note here that the exact solution set of the problem is empty. Indeed, let \( x^* \) be an arbitrary point in \( \mathbb{R}^2 \). Then, \( 0 < f_i^L(x^*) \leq f_i^U(x^*) \) for all \( i \in I \). Let \( x^k = (\frac{1}{k}, k), k \in \mathbb{N} \). Then, for all \( i \in I \), we have
\[
\lim_{k \to \infty} f_i^L(x^k) = \lim_{k \to \infty} \frac{1}{k^2} = 0 < f_i^L(x^*),
\]
\[
\lim_{k \to \infty} f_i^U(x^k) = \lim_{k \to \infty} \frac{2}{k^2} = 0 < f_i^U(x^*).
\]
Hence, there exists \( k_0 \in \mathbb{N} \) such that
\[
f_i^L(x^k) < f_i^L(x^*),
\]
\[
f_i^U(x^k) < f_i^U(x^*),
\]
for all \( k \geq k_0 \) and \( i \in I \). This means that \( x^* \) is not a type-2 weakly Pareto solution of (SIVP), as required.
Figure 1. Plot of $f_L^i$ and $f_U^i$ in Example 3.5.

Next we present sufficient conditions for approximate quasi Pareto solutions of (SIVP). In order to obtain these sufficient conditions, we need to introduce concepts of (strictly) generalized convexity at a given point for a family of locally Lipschitz functions. The first definition is inspired from [37], while the second one is motivated from [40].

**Definition 3.6.** (i) We say that $(f, g_T)$ is generalized convex on $\Omega$ at $\bar{x} \in \Omega$ if for any $x \in \Omega$, $z_i^{*L} \in \partial f^L_i(\bar{x})$, $z_i^{*U} \in \partial f^U_i(\bar{x})$, $i \in I$, and $x_t^* \in \partial g_t(\bar{x})$, $t \in T$, there exists $\nu \in [N(x; \Omega)]^0$ satisfying

\[
\begin{align*}
&f^L_i(x) - f^L_i(\bar{x}) \geq \langle z_i^{*L}, \nu \rangle, \quad \forall i \in I, \\
&f^U_i(x) - f^U_i(\bar{x}) \geq \langle z_i^{*U}, \nu \rangle, \quad \forall i \in I, \\
g_t(x) - g_t(\bar{x}) \geq \langle x_t^*, \nu \rangle, \quad \forall t \in T \\
\text{and} \quad \langle b^*, \nu \rangle \leq \|x - \bar{x}\|, \quad \forall b^* \in \mathbb{B}_{\mathbb{R}^n}.
\end{align*}
\]

(ii) We say that $(f, g_T)$ is strictly generalized convex on $\Omega$ at $\bar{x} \in \Omega$ if for any $x \in \Omega \setminus \{\bar{x}\}$, $z_i^{*L} \in \partial f^L_i(\bar{x})$, $z_i^{*U} \in \partial f^U_i(\bar{x})$, $i \in I$, and $x_t^* \in \partial g_t(\bar{x})$, $t \in T$, there exists $\nu \in [N(x; \Omega)]^0$ satisfying

\[
\begin{align*}
&f^L_i(x) - f^L_i(\bar{x}) > \langle z_i^{*L}, \nu \rangle, \quad \forall i \in I, \\
&f^U_i(x) - f^U_i(\bar{x}) > \langle z_i^{*U}, \nu \rangle, \quad \forall i \in I, \\
g_t(x) - g_t(\bar{x}) \geq \langle x_t^*, \nu \rangle, \quad \forall t \in T \\
\text{and} \quad \langle b^*, \nu \rangle \leq \|x - \bar{x}\|, \quad \forall b^* \in \mathbb{B}_{\mathbb{R}^n}.
\end{align*}
\]

**Remark 3.7.** We see that if $\Omega$ is convex and $f^L_i$, $f^U_i$, $i \in I$, and $g_t$, $t \in T$, are convex (resp. strictly convex), then $(f, g_T)$ is generalized convex (resp. strictly generalized convex) on $\Omega$ at any $\bar{x} \in \Omega$ with $\nu = x - \bar{x}$. Moreover, there exist examples that show the class of generalized convex functions is properly larger than the one of convex functions; see, e.g., [37, Example 3.2] and [27, Example 3.12].

**Definition 3.8.** (i) We say that $(f, g_T)$ is $E$-pseudo-quasi generalized convex on $\Omega$ at $\bar{x} \in \Omega$ if for any $x \in \Omega$, $z_i^{*L} \in \partial f^L_i(\bar{x})$, $z_i^{*U} \in \partial f^U_i(\bar{x})$, $i \in I$, and $x_t^* \in \partial g_t(\bar{x})$,
Remark 3.9. By definition, it is easy to see that if \((\text{convex on } \Omega \text{ at } \bar{x})\), consider the following example. To see this, let 

Furthermore, the class of (strictly) \(\varepsilon\)-pseudo-quasi generalized convex at this point. We have 

Indeed, it is easy to see that 

\[ \langle z_i^L, \nu \rangle + \varepsilon_i^L \| x - \bar{x} \| \geq 0 \Rightarrow f_i^L(x) \geq f_i^L(\bar{x}) - \varepsilon_i^L \| x - \bar{x} \|, \quad \forall i \in I, \]

\[ g_i(x) \leq g_i(\bar{x}) \Rightarrow \langle x_i^*, \nu \rangle \leq 0, \quad \forall t \in T, \]

and \( \langle b^*, \nu \rangle \leq \| x - \bar{x} \|, \quad \forall b^* \in \mathbb{B}_R^n. \)

(ii) We say that \((f, g_T)\) is strictly \(\mathcal{E}\)-pseudo-quasi generalized convex on \(\Omega\) at \(\bar{x} \in \Omega\) if for any \(x \in \Omega \setminus \{\bar{x}\}\), \(z_i^L \in \partial f_i^L(\bar{x}), z_i^U \in \partial f_i^U(\bar{x}), i \in I, \) and \(x_i^* \in \partial g_i(\bar{x}), t \in T, \) there exists \(\nu \in [N(\bar{x}; \Omega)]^\circ\) satisfying 

\[ \langle z_i^L, \nu \rangle + \varepsilon_i^L \| x - \bar{x} \| \geq 0 \Rightarrow f_i^L(x) > f_i^L(\bar{x}) - \varepsilon_i^L \| x - \bar{x} \|, \quad \forall i \in I, \]

\[ g_i(x) \leq g_i(\bar{x}) \Rightarrow \langle x_i^*, \nu \rangle \leq 0, \quad \forall t \in T, \]

and \( \langle b^*, \nu \rangle \leq \| x - \bar{x} \|, \quad \forall b^* \in \mathbb{B}_R^n. \)

Remark 3.9. By definition, it is easy to see that if \((f, g_T)\) is (strictly) generalized convex on \(\Omega\) at \(\bar{x} \in \Omega\), then for any \(\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_m)\), where \(\mathcal{E}_i = [\varepsilon_i^L, \varepsilon_i^U], 0 \leq \varepsilon_i^L \leq \varepsilon_i^U, i \in I, (f, g_T)\) is (strictly) \(\mathcal{E}\)-pseudo-quasi generalized convex on \(\Omega\) at \(\bar{x} \in \Omega\). Furthermore, the class of (strictly) \(\mathcal{E}\)-pseudo-quasi generalized convex functions is properly wider than the one of (strictly) generalized convex functions. To see this, let us consider the following example.

Example 3.10. Let \(m = 1, f : \mathbb{R} \to K_c\) be defined by \(f(x) = [f^L(x), f^U(x)],\) where 

\[ f^L(x) = \begin{cases} \frac{x}{2}, & x \geq 0, \\ \frac{2x}{3}, & x < 0, \end{cases} \quad f^U(x) = \begin{cases} \frac{2x}{3}, & x \geq 0, \\ \frac{x}{2}, & x < 0, \end{cases} \]

and \(g_t(x) = -tx\) for all \(x \in \mathbb{R}\) and \(t \in T := [0, 1].\) Take \(\Omega = \mathbb{R}\) and \(\mathcal{E} = \left[\frac{2}{3}, \frac{3}{4}\right].\) We show that \((f, g_T)\) is strictly \(\mathcal{E}\)-pseudo-quasi generalized convex at \(x = 0\) but not generalized convex at this point. Indeed, it is easy to see that 

\[ f^L(x) > f^L(\bar{x}) - \frac{3}{4}|x|, f^U(x) > f^U(\bar{x}) - \frac{2}{3}|x|, \quad \forall x \in \mathbb{R} \setminus \{0\}. \]

Hence, by letting \(\nu = 0,\) we see that all conditions in (7) are satisfied for all \(z_i^L \in \partial f^L(\bar{x}), z_i^U \in \partial f^U(\bar{x}), x_i^* \in \partial g_t(\bar{x}), t \in T,\) and \(b^* \in \mathbb{B}_R^n.\)

We now show that \((f, g_T)\) is not generalized convex at \(\bar{x}\) and so it is not strictly generalized convex at this point. We have \(\partial f^L(\bar{x}) = \partial f^U(\bar{x}) = \left\{\frac{1}{2}, \frac{2}{3}\right\}\) and \(\partial g_t(\bar{x}) = \{-t\}, \forall t \in T.\) Let \(x = 1, z_i^L = z_i^U = \frac{2}{3}, x_i^* = -t,\) and assume that there exists \(\nu \in \mathbb{R}\) satisfying (6). Then 

\[ \frac{1}{2} \geq \frac{2\nu}{3}, \quad \frac{2}{3} \geq \frac{2\nu}{3}, \quad -t \geq -t\nu, \quad \forall t \in [0, 1]. \]
Hence, \(1 \leq \nu \leq \frac{3}{4}\), a contradiction.

\[ f^L(x) \]
\[ f^U(x) \]

**Figure 2.** Plot of \(f^L\) and \(f^U\) in Example 3.10

**Theorem 3.11.** Let \(\bar{x} \in F\) and assume that there exist \(\lambda^L, \lambda^U \in \mathbb{R}^m_+\) with \(\sum_{i \in I} (\lambda^L_i + \lambda^U_i) = 1\), and \(\mu \in A(\bar{x})\) satisfying (1).

(i) If \((f, g_T)\) is \(E\)-pseudo-quasi generalized convex on \(\Omega\) at \(\bar{x}\), then \(\bar{x} \in E-S^{qw}_2(SIVP)\).

(ii) If \((f, g_T)\) is strictly \(E\)-pseudo-quasi generalized convex on \(\Omega\) at \(\bar{x}\), then \(\bar{x} \in E-S^{q}_1(SIVP)\) and we therefore get \(\bar{x} \in E-S^{qw}_1(SIVP)\).

**Proof.** Since \(\bar{x}\) satisfies (1) with respect to \((\lambda^L, \lambda^U, \mu)\), there exist \(z^*_i \in \partial f^L_i(\bar{x})\), \(z^*_i \in \partial f^U_i(\bar{x})\), \(i \in I\), \(x^*_t \in \partial g_t(\bar{x})\), \(t \in T\), \(b^* \in \mathbb{B}_{\mathbb{R}^n}\), and \(\omega^* \in N(\bar{x}; \Omega)\) such that

\[
\sum_{i \in I} [\lambda^L_i z^*_i + \lambda^U_i z^*_i] + \sum_{t \in T} \mu_t x^*_t + \sum_{i \in I} (\lambda^L_i \epsilon_i + \lambda^U_i \epsilon^*_i) b^* + \omega^* = 0,
\]

or, equivalently,

\[
\sum_{i \in I} [\lambda^L_i z^*_i + \lambda^U_i z^*_i] + \sum_{t \in T} \mu_t x^*_t + \sum_{i \in I} (\lambda^L_i \epsilon_i + \lambda^U_i \epsilon^*_i) b^* = -\omega^*. \tag{8}
\]

We first justify (i). Suppose on the contrary that \(\bar{x}\) is not a type-2 \(E\)-quasi-weakly Pareto solution of (SIVP), then there exists \(x \in F\) such that

\[
f_i(x) <_{LU} f_i(\bar{x}) - \mathcal{E}_i\|x - \bar{x}\|, \quad \forall i \in I,
\]

or, equivalently,

\[
f^L_i(x) < f^L_i(\bar{x}) - \epsilon_i^L \|x - \bar{x}\| \quad \text{and} \quad f^U_i(x) < f^U_i(\bar{x}) - \epsilon_i^L \|x - \bar{x}\|, \quad \forall i \in I. \tag{9}
\]

By the \(E\)-pseudo-quasi generalized convexity of \((f, g_T)\), there is \(\nu \in [N(\bar{x}; \Omega)]^0\) satisfying (6). Thus, we deduce from (6) and (9) that

\[
\langle z^*_i, \nu \rangle + \epsilon_i^L \|x - \bar{x}\| < 0 \quad \text{and} \quad \langle z^*_i, \nu \rangle + \epsilon_i^L \|x - \bar{x}\| < 0.
\]
For each \( t \in T(\mu) \), we have \( g_t(\bar{x}) = 0 \). Hence, \( g_t(x) \leq g_t(\bar{x}) \) for all \( t \in T(\mu) \). This, together with (6), gives

\[
\langle x^*_t, \nu \rangle \leq 0 \quad \text{for all } t \in T(\mu).
\]

Since \( \nu \in [N(\bar{x}; \Omega)]^\circ \), (6), and (8), we obtain

\[
0 \leq \langle -\omega^*, \nu \rangle = \sum_{i \in I} [\lambda^L_t \langle z^L_i, \nu \rangle + \lambda^U_t \langle z^U_i, \nu \rangle] + \sum_{t \in T(\mu)} \mu_t \langle x^*_T, \nu \rangle + \sum_{i \in I} (\lambda^L_t \epsilon^L_i + \lambda^U_t \epsilon^U_i) \langle b^*, \nu \rangle
\]

\[
= \sum_{i \in I} [\lambda^L_t \langle z^L_i, \nu \rangle + \lambda^U_t \langle z^U_i, \nu \rangle] + \sum_{t \in T(\mu)} \mu_t \langle x^*_T, \nu \rangle + \sum_{i \in I} (\lambda^L_t \epsilon^L_i + \lambda^U_t \epsilon^U_i) \langle b^*, \nu \rangle
\]

\[
< -\sum_{i \in I} (\lambda^L_t \epsilon^L_i \|x - \bar{x}\| + \lambda^U_t \epsilon^U_i \|x - \bar{x}\|) + \sum_{i \in I} (\lambda^L_t \epsilon^L_i + \lambda^U_t \epsilon^U_i) \|x - \bar{x}\| = 0,
\]

a contradiction. The proof of (i) is completed.

We now prove (ii). Suppose on the contrary that \( \bar{x} \) is not a type-1 \( \mathcal{E} \)-quasi Pareto solution of (SIVP), then there exists \( x \in \mathcal{F} \) such that

\[
f_i(x) \leq LU f_i(\bar{x}) - \mathcal{E}_i \|x - \bar{x}\|, \quad \forall i \in I,
\]

where at least one of the inequalities is strict. This implies that \( x \neq \bar{x} \). Therefore, by the strictly \( \mathcal{E} \)-pseudo-quasi generalized convexity of \( (f, g_T) \) at \( \bar{x} \), (8), and by the same argument as in the proof of part (i), we can deduce the contradiction. Hence, \( \bar{x} \) is a type-1 \( \mathcal{E} \)-quasi Pareto solution of (SIVP). The proof is completed. \( \square \)

**Remark 3.12.**

(i) By Remark 3.9 and Example 3.2, the condition (3.4) alone is not sufficient to guarantee that \( \bar{x} \) is a \( \mathcal{E} \) (weakly) Pareto solution of (SIVP) if the (strict) \( \mathcal{E} \)-pseudo generalized convexity of \( (f, g_T) \) on \( \Omega \) at \( \bar{x} \) is violated.

(ii) If \( (f, g_T) \) is \( \mathcal{E} \)-pseudo-quasi generalized convex on \( \Omega \) at \( \bar{x} \in \mathcal{F} \) and there exist \( \lambda^L, \lambda^U \in \mathbb{R}^m_+ \) with \( \lambda^L_i > 0, \lambda^U_i > 0, \forall i \in I, \sum_{i \in I} (\lambda^L_i + \lambda^U_i) = 1 \), and \( \mu \in A(\bar{x}) \) satisfying (1), then \( \bar{x} \in \mathcal{E}-S^*_1 \) (SIVP).

(iii) Since the class of (strictly) \( \mathcal{E} \)-pseudo-quasi generalized convex functions is properly wider than the class of (strictly) generalized convex functions, our results in Theorem 3.4 generalize and improve the corresponding results in [22, 27, 30, 38].

To see this, let us consider the following simple example.

**Example 3.13.** Let \( m = 1 \), \( f : \mathbb{R} \rightarrow \mathcal{K}_c \) be defined by \( f(x) = [f^L(x), f^U(x)] \), where

\[
f^L(x) = f^U(x) = \begin{cases} 
\frac{x}{2}, & x \geq 0, \\
\frac{2x}{3}, & x < 0.
\end{cases}
\]

Consider problem (SIVP) with \( g_t(x) = -tx \) for all \( x \in \mathbb{R} \) and \( t \in T := [0, 1] \), \( \Omega = \mathbb{R} \) and \( \mathcal{E} = [\frac{1}{2}, \frac{3}{2}] \). Actually, in this case problem (SIVP) is a semi-infinite programming problem. Analysis similar to that in Example 3.10 shows that \( (f, g_T) \) is strictly \( \mathcal{E} \)-pseudo-quasi convex at \( \bar{x} = 0 \) but not generalized convex at this point. It is easy to see that \( \bar{x} \) satisfies condition (1); e.g., \( \lambda^L = \lambda^U = \frac{1}{2}, \mu^1 = 1, \) and \( \mu_t = 0 \) for all \( t \in T \setminus \{\frac{1}{2}\} \). Hence, by Theorem 3.11 \( \bar{x} \in \mathcal{E}-S^*_1 \) (SIVP). However, since \( (f, g_T) \) is not
lem in the sense of Mond–Weir (stated in an approximate form):

\[ 22, \text{Theorem 6} \] cannot be applied for this example.

\[ K \geq \text{by} \ (SIVD) \]

It should be noticed that approximate quasi Pareto solutions of the dual problem is a type-2 \( E \)-quasi weakly Pareto solution of \( \text{SIVD}_{MW} \) if there is no \( (y, \lambda^L, \lambda^U, \mu) \in \mathcal{F}_{MW} \) such that

\[ \mathcal{L}_i(y, \lambda^L, \lambda^U, \mu) - \mathcal{E}_i \| y - \bar{y} \| \geq_{LU} \mathcal{L}_i(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}), \quad \forall i \in I, \]

or, equivalently,

\[ \mathcal{L}(\bar{y}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \not<_{LU} \mathcal{L}_i(y, \lambda^L, \lambda^U, \mu) - \mathcal{E}_i \| y - \bar{y} \|, \quad \forall (y, \lambda^L, \lambda^U, \mu) \in \mathcal{F}_{MW}. \]

The following theorem describes weak duality relations for approximate quasi Pareto solutions between the primal problem \( \text{SIVP} \) and the dual problem \( \text{SIVD}_{MW} \).

**Theorem 4.1 (\( \mathcal{E} \)-weak duality).** Let \( x \in \mathcal{F} \) and \( (y, \lambda^L, \lambda^U, \mu) \in \mathcal{F}_{MW} \).
(i) If \((f, g_\top)\) is \(E\)-pseudo-quasi generalized convex on \(\Omega\) at \(y\), then
\[
f(x) \not\lesssim^*_LU \mathcal{L}(y, \lambda^L, \lambda^U, \mu) - E\|x - y\|.
\]

(ii) If \((f, g_\top)\) is strictly \(E\)-pseudo-quasi generalized convex on \(\Omega\) at \(y\), then
\[
f(x) \not\lesssim^*_LU \mathcal{L}(y, \lambda^L, \lambda^U, \mu) - E\|x - y\|.
\]

**Proof.** Since \(x \in \mathcal{F}\) and \((y, \lambda^L, \lambda^U, \mu) \in \mathcal{F}_{MW}\), we have \(x, y \in \Omega\),
\[
g_t(x) \leq 0, \quad \mu_t g_t(y) \geq 0, \quad \forall t \in T, \tag{10}
\]
and
\[
0 \in \sum_{i \in I} [\lambda^L_i \partial f^L_i(y) + \lambda^U_i \partial f^U_i(y)] + \sum_{t \in T} \mu_t \partial g_t(y) + \sum_{i \in I} (\lambda^L_i \epsilon^L_i + \lambda^U_i \epsilon^U_i) \mathbb{B}_{\mathbb{R}^n} + N(y; \Omega).
\]

Hence, there exist \(z^*_i \in \partial f^L_i(y), \; z^*_i \in \partial f^U_i(y), \; i \in I, \; x^*_i \in \partial g_t(y), \; t \in T, \; b^* \in \mathbb{B}_{\mathbb{R}^n}\), and \(\omega^* \in N(y; \Omega)\) such that
\[
\sum_{i \in I} [\lambda^L_i z^*_i + \lambda^U_i z^*_i] + \sum_{t \in T} \mu_t x^*_t + \sum_{i \in I} (\lambda^L_i \epsilon^L_i + \lambda^U_i \epsilon^U_i) b^* = -\omega^*. \tag{11}
\]

We first justify (i). Assume to the contrary that
\[
f(x) \not\lesssim^*_LU \mathcal{L}(y, \lambda^L, \lambda^U, \mu) - E\|x - y\|.
\]
This means that
\[
f_i(x) \not\lesssim^*_LU \mathcal{L}_i(y, \lambda^L, \lambda^U, \mu) - E_i\|x - y\|, \quad \forall i \in I,
\]
or, equivalently,
\[
\begin{cases}
f^L_i(x) < f^L_i(y) - \epsilon^U_i \|x - y\| , \\
f^U_i(x) < f^U_i(y) - \epsilon^L_i \|x - y\|,
\end{cases}
\]
for all \(i \in I\). By (10) and \(\mu_t \geq 0\) for all \(t \in T\), we have
\[
g_t(x) \leq 0 \leq g_t(y)
\]
for all \(t \in T(\mu)\). Since \((f, g_\top)\) is \(E\)-pseudo-quasi generalized convex on \(\Omega\) at \(y\), there exists \(\nu \in [N(y, \Omega)]^*\) such that
\[
\langle z^*_i, \nu \rangle + \epsilon^U_i \|x - y\| < 0, \quad \forall i \in I,
\]
\[
\langle z^*_i, \nu \rangle + \epsilon^L_i \|x - y\| < 0, \quad \forall i \in I,
\]
\[
\langle x^*_i, \nu \rangle \leq 0, \quad \forall t \in T(\mu),
\]
and
\[
\langle b^*, \nu \rangle \leq \|x - y\|, \quad \forall b^* \in \mathbb{B}_{\mathbb{R}^n}.
\]

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Hence, 
\[ x \]
This implies that 
\[ y \]
a contradiction, which completes the proof of (i).

On the other hand, since \( \nu \in [N(y; \Omega)]^\circ \), \( \omega^* \in N(y; \Omega) \), and \( \|U\| \), we have
\[ 0 \leq \langle -\omega^*, \nu \rangle = \sum_{i \in I} \left[ (\lambda_i L z_i^* L + \lambda_i U z_i^* U, \nu) \right] + \sum_{t \in T} \langle x_t^*, \nu \rangle + \sum_{t \in I} \left( \lambda_i L \epsilon_i + \lambda_i U \epsilon_i \right) \langle b^* , \nu \rangle < 0, \]
a contradiction, which completes the proof of (i).

We now prove (ii). Suppose on the contrary that 
\[ f(x) \preceq_{LU} L(y, \lambda^L, \lambda^U, \mu) - E \|x - y\|, \]
This implies that
\[
\begin{cases}
  f_i(x) \leq_{LU} f_i(y) - E_i \|x - y\|, \quad \forall i \in I, \\
  f_k(x) <_{LU} f_k(y) - E_k \|x - y\|, \quad \text{for at least one } k \in I.
\end{cases}
\]
Hence, \( x \neq y \). Therefore, by the strictly \( E \)-pseudo-quasi generalized convexity of \( (f, g_T) \) at \( y \), \( \|U\| \), and by the same argument as in the proof of part (i), we can get the contradiction. The proof is completed.

The following example shows that the approximate pseudo-quasi generalized convexity of \( (f, g_T) \) on \( \Omega \) used in Theorem 4.1 cannot be omitted.

**Example 4.2.** Let \( m = 1 \), \( f: \mathbb{R} \to \mathcal{K}_e \) be defined by \( f(x) = [f^L(x), f^U(x)] \), where
\[ f^L(x) = \frac{1}{3} x^3 \quad \text{and} \quad f^U(x) = \frac{1}{3} x^3 + 1. \]
Consider problem \( \text{SIVP} \) with \( g_t(x) = -tx \) for all \( x \in \mathbb{R} \) and \( t \in T := [0, 1] \), \( \Omega = \mathbb{R} \) and \( E = [\frac{3}{5}, \frac{4}{5}] \). Let \( \bar{y} = 1 \), \( \lambda^L = \lambda^U = \frac{1}{2} \), \( \mu_t = 0 \), \( \forall t \in [0, 1] \), and \( \mu_1 = 1 \). It is easy to see that \( (\bar{y}, \lambda^L, \lambda^U, \mu) \in \mathcal{F}_{MW} \). However, for \( \bar{x} = 0 \in \mathcal{F} \), we have
\[ f(\bar{x}) = [0, 1] \preceq_{LU} L(\bar{y}, \lambda^L, \lambda^U, \mu) \]
\[ = [\frac{1}{3} \cdot \frac{4}{3}] = \frac{1}{12} \cdot \frac{17}{15} \cdot \frac{1}{10}. \]
This means that the conclusion of Theorem 4.1 is no longer true. The reason is that the \( E \)-pseudo-quasi generalized convexity of \( (f, g_T) \) on \( \Omega \) at \( \bar{y} \) has been violated.

Next we present a theorem that formulates strong duality relations between the primal problem \( \text{SIVP} \) and the dual problem \( \text{SIVD}_{MW} \).
Figure 3. Plot of $f^L$ and $f^U$ in Example 4.2.

**Theorem 4.3** ($\mathcal{E}$-strong duality). Let $\bar{x}$ be a type-2 $\mathcal{E}$-quasi-weakly Pareto solution of (SIVP) and assume that the (LCQ) holds at this point. Then there exist $\bar{x}^L, \bar{x}^U \in \mathbb{R}_+^m$, and $\bar{\mu} \in A(\bar{x})$ such that $(\bar{x}, \bar{x}^L, \bar{x}^U, \bar{\mu}) \in F_{MW}$, $f(\bar{x}) = \mathcal{L}(\bar{x}, \bar{x}^L, \bar{x}^U, \bar{\mu})$. Furthermore,

(i) If $(f, g^T)$ is $\mathcal{E}$-pseudo-quasi generalized convex on $\Omega$ at $\bar{x}$, then $(\bar{x}, \bar{x}^L, \bar{x}^U, \bar{\mu})$ is a type-2 $\mathcal{E}$-quasi weakly Pareto solution of (SIVD$_{MW}$).

(ii) If $(f, g^T)$ is strictly $\mathcal{E}$-pseudo-quasi generalized convex on $\Omega$ at $\bar{x}$, then $(\bar{x}, \bar{x}^L, \bar{x}^U, \bar{\mu})$ is a type-1 $\mathcal{E}$-quasi Pareto solution of (SIVD$_{MW}$).

**Proof.** By Theorem 3.4 there exist $\bar{\lambda}^L, \bar{\lambda}^U \in \mathbb{R}_+^m$, and $\bar{\mu} \in A(\bar{x})$ satisfying $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in F_{MW}$. Clearly, $f(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$.

(i) If $(f, g^T)$ is $\mathcal{E}$-pseudo-quasi generalized convex on $\Omega$ at $\bar{x}$, then, by invoking now (i) of Theorem 4.1 we obtain

$$f(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \not\in \mathcal{L}(y, \lambda^L, \lambda^U, \mu) - \mathcal{E}\|\bar{x} - y\|$$

for all $(y, \lambda^L, \lambda^U, \mu) \in F_{MW}$. Therefore, $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a type-2 $\mathcal{E}$-quasi weakly Pareto solution of (SIVD$_{MW}$).

The proof of (ii) is similar to that of (i) by using the strictly $\mathcal{E}$-pseudo-quasi generalized convexity of $(f, g^T)$ on $\Omega$ at $\bar{x}$ instead of the $\mathcal{E}$-pseudo-quasi generalized convexity of $(f, g^T)$ on $\Omega$ at the corresponding point.

We close this section by presenting converse-like duality relations for approximate quasi Pareto solutions between the primal problem (SIVP) and the dual problem (SIVD$_{MW}$).

**Theorem 4.4** (Converse-like duality). Let $(\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in F_{MW}$.

(i) If $\bar{x} \in F$ and $(f, g^T)$ is $\mathcal{E}$-pseudo-quasi generalized convex on $\Omega$ at $\bar{x}$, then $\bar{x}$ is a type-2 $\mathcal{E}$-quasi weakly Pareto solution of (SIVP).

(ii) If $\bar{x} \in F$ and $(f, g^T)$ is strictly $\mathcal{E}$-pseudo-quasi generalized convex on $\Omega$ at $\bar{x}$, then $\bar{x}$ is a type-1 $\mathcal{E}$-quasi Pareto solution of (SIVP).
Proof. (i) Since \((\bar{x}, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu}) \in \mathcal{F}_{MW}\), we have

\[
0 \in \sum_{i \in I} [\bar{\lambda}_i^L \partial f_i^L(\bar{x}) + \bar{\lambda}_i^U \partial f_i^U(\bar{x})] + \sum_{t \in T} \bar{\mu}_t \partial g_t(\bar{x}) + \sum_{i \in I} (\bar{\lambda}_i^L \epsilon_i + \bar{\lambda}_i^U \epsilon_i^L) \|B \|_{\mathbb{R}^n} + N(\bar{x}; \Omega),
\]

\[
\sum_{i \in I} (\bar{\lambda}_i^L + \bar{\lambda}_i^U) = 1, \quad \text{and}
\]

\[
\bar{\mu}_t g_t(\bar{x}) \geq 0, \quad \forall t \in T. \tag{12}
\]

It follows from \(\bar{x} \in \mathcal{F}\) and \((12)\) that \(\bar{\mu}_t g_t(\bar{x}) = 0\) for all \(t \in T\), i.e., \(\bar{\mu} \in A(\bar{x})\). By the \(\mathcal{E}\)-pseudo-quasi generalized convexity of \((f, g_T)\) on \(\Omega\) at \(\bar{x}\) and Theorem 3.11(i), \(\bar{x}\) is a type-2 \(\mathcal{E}\)-quasi weakly Pareto solution of \((\text{SIVP})\).

(ii) The proof of (ii) is quite similar to that of (i) by using the strictly \(\mathcal{E}\)-pseudo-quasi generalized convexity of \((f, g_T)\) and Theorem 3.11(ii), so it is omitted.

\[\square\]

5. Conclusions

In this paper, we focus for the first time on studying approximate solutions of semi-infinite interval-valued vector optimization problems. By employing the Mordukhovich/limiting subdifferential and the Mordukhovich/limiting normal cone and introducing some types of approximate pseudo-quasi generalized convexity of a family of locally Lipschitz functions, we establish optimality conditions of KKT-type and duality relations for proposed approximate solutions of the considered problem. Observe that the class of approximate pseudo-quasi generalized convex functions is properly wider than the class of convex functions in the sense of \([27,37,41]\), our results generalize and improve some existing results. Furthermore, the model of problems of the form \((\text{SIVP})\) covers the one of cone-constrained convex vector optimization problems and semidefinite vector optimization problems, and so our approach can be developed to study these problems. We aim to investigate this problem in future work.

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