Non-linear corrections to the longitudinal structure function $F_L$ from the parametrization of $F_2$: Laplace transform approach

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The non-linear corrections (NLC) to the longitudinal structure function in a limited approach is derived at low values of the Bjorken variable $x$ by using the Laplace transforms technique. The non-linear behavior of the longitudinal structure function is determined with respect to the Gribov-Levin-Ryskin Mueller-Qiu (GLR-MQ) and Altarelli-Martellini (AM) equations. These results show that the non-linear longitudinal structure function can be determined directly in terms of the parametrization of $F_L(x, Q^2)$ and the derivative of the proton structure function with respect to $\ln Q^2$. These corrections improve the behavior of the longitudinal structure function at low values of $Q^2$ in comparison with other parametrization methods.

The recent articles [1,2] revive the longitudinal structure function $F_L(x, Q^2)$ due to the parametrization of $F_2(x, Q^2)$ [3] and its derivative at low values of $x$. In these papers (i.e., [1] and [2]) authors show that the longitudinal structure functions have been obtained in the Mellin and Laplace transform techniques from the parametrization of $F_2(x, Q^2)$ respectively. These results at leading-order approximation lead to the unique results in a wide range of $Q^2$ values. Theoretical analysis of these results at low $x$, in context of fulfillment of the Froissart boundary, is of a great importance in ultra-high energy processes in the future electron-proton colliders. Parametrization of the proton structure function has suggested by authors in Ref.[3], which describe fairly well the experimental data [4] at low $x$ in a wide range of the momentum transfer $Q^2$, $x<0.1$ and $0.15$ GeV$^2 < Q^2 < 3000$ GeV$^2$. Ref.[2] shows that the description of the data can be improved by allowing for phenomenological rescaling variable corrections to the Bjorken scaling variable. They are due to the charm quark mass (for $n_f = 4$) by the following form $\chi = x(1 + \frac{4m_c^2}{Q^2})$, with $m_c = 1.29^{+0.077}_{-0.053}$ GeV [5]. Altogether the results in Refs.[1] and [2] contain good agreement with the experimental data at large $Q^2$ for the longitudinal structure functions, but at low $Q^2$ these results can be improved when take into account the non-linear corrections.

It is known that in the low $x$ and low $Q^2$ regions, the non-linear corrections (or gluon recombination effects) are not negligible and reduce the growth of the gluon distribution function [6-9]. The non-linear corrections of gluon recombination to the parton distributions have been calculated by Gribov-Levin-Ryskin (GLR) and Mueller-Qiu (MQ) in [10] based on the Abramovsky-Gribov-Kancheli (AGK) cutting rules in the double leading logarithmic approximation (DLLA). The study of the GLR-MQ equation may provide important insight into the non-linear corrections of gluon recombination due to the high gluon density at sufficiently low $x$. Indeed the number of partons in a phase space cell $(\Delta \ln(1/x)\Delta \ln Q^2)$ increases through gluon splitting and decreases through gluon recombination. That is, all possible $g+g\rightarrow g$ ladder recombinations are resummed to leading order of the parameter $\alpha_s \ln(1/x)(\ln(Q^2/Q_0^2))$. It leads to saturation of the gluon density at low $Q^2$ with decreasing $x$. Thus the non-linear corrections emerged from the recombination of two gluon ladders, modify the evolution equations of singlet quark distribution. Indeed an extra non-linear term added to the linear DGLAP evolution equation by the following form

$$\frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = \frac{\partial F_2(x, Q^2)}{\partial \ln Q^2}_{DGLAP} + < e^2 > \zeta (xg(x, Q^2))^2 + HT.$$ 

Here, the representation for the gluon distribution $G(x, Q^2) = xg(x, Q^2)$ is used and $< e^2 >$ is the average of the charge $e^2$ for the active quark flavors, $< e^2 > = \frac{1}{n_f} \sum_{i=1}^{n_f} e_i^2$ and $\zeta = \frac{\ln(Q^2)}{\ln(Q_0^2)}$. The correlation length $R$ determines the size of the non-linear terms. This value depends on how the gluon ladders are coupled to the nucleon or on how the gluons are distributed within the nucleon. The $R$ is approximately equal to $\simeq 5$ GeV$^{-1}$ if the gluons are populated across the proton and it is equal to $\simeq 2$ GeV$^{-1}$ if the gluons have hotspot like structure. Here the higher dimensional gluon distribution(i.e., higher twist) is assumed to be zero.

Recently in Ref.[11] the nonlinear modification of the evolution of the gluon density from the parametrization of $F_2$ in the leading order of perturbation theory is considered. To investigate the role of the non-linear corrections on the behaviour of the longitudinal structure function in the low $Q^2$ region, we consider the GLR-MQ equation for singlet structure function and the AM equation [12] for longitudinal structure function at small $x$. The

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longitudinal structure function in Ref. [12] is defined by
\[ F_L(x, Q^2) = C_{L,ns} \otimes F_2(x, Q^2) \]
where the non-singlet densities become negligibly small in comparison with the singlet densities at small \( x \). The symbol \( \otimes \) denotes convolution according to the usual pre-

\[ < e^2 > C_{L,g} \otimes G(x, Q^2), \]

sition of Eq.(1) to transform of a convolution function is simply ordinary functions can be written as \( \nu \) and \( s \) functions in (1) and (2), one obtains the structure and distribution functions explicitly from \( \ln(1/Q^2) \) is always lower than \( \ln(Q^2) \) in space at leading-order (LO) approximation are given by
\[ \Phi_L(s, Q^2) = \frac{\alpha_s(Q^2)}{\pi} C_F \frac{1}{2 + s} \]
\[ \Theta_L(s, Q^2) = \frac{\alpha_s(Q^2)}{\pi} \left( \frac{1}{2 + s} - \frac{1}{3 + s} \right) \]
\[ \Theta_f(s, Q^2) = \frac{n_f}{\pi} \left( \frac{\alpha_s(Q^2)}{2} \left( \frac{1}{2 + s} - \frac{2}{3 + s} \right) + \frac{2}{3 + s} \right) \]
\[ \Phi_f(s, Q^2) = \frac{\alpha_s(Q^2)}{4\pi} \left[ 4 - \frac{8}{3} \left( \frac{1}{2 + s} \right) + \frac{1}{2 + s} + \psi(s + 1 + \gamma_E) \right] \]

and
\[ Df_2(s, Q^2) = \frac{\partial f_2(s, Q^2)}{\partial \ln Q^2} \]

\[ f_L(s, Q^2) = \Phi_L(s) f_2(s, Q^2) + < e^2 > \Theta_L(s) g(s, Q^2). \]

In Eqs. (5) and (6) we used the fact that the Laplace transform of a convolution function is simply ordinary product of the Laplace transform of that function. Transformation of Eq.(1) to \( s \)-space in Eq.(5) is defined in a limited approach as in \( \nu \)-space we assume that the Laplace transform of the \( L[G^2(\nu, Q^2); s] \) to be less than \( g^2(s, Q^2) = G(s, Q^2) \). Indeed \( L[G^2(\nu, Q^2); s] < L[G(\nu, Q^2); s]^2 \). Therefore, the non-linear corrections to

the longitudinal structure function is defined into the proton structure function and the derivative of the proton structure function with respect to \( \ln Q^2 \) in \( s \)-space by a quadratic equation. Inserting Eq. (5) into Eq. (6) the longitudinal structure function becomes
\[ f_L^2(s, Q^2) - k(s, Q^2) f_L(s, Q^2) + h(s, Q^2) = 0, \]

where
\[ k(s, Q^2) = \eta \Theta_L(s, Q^2) \Theta_f(s, Q^2) + 2\Phi_L(s, Q^2) f_2(s, Q^2) \]
\[ h(s, Q^2) = \eta \Theta_L(s, Q^2) \Theta_f(s, Q^2) \Phi_L(s, Q^2) f_2(s, Q^2) \]
\[ + \Phi_L^2(s, Q^2) f_2^2(s, Q^2) + \eta \Theta_L^2(s, Q^2) D f_2(s, Q^2), \]

with \( \eta = < e^2 > / \zeta \)

and
\[ \hat{F}_L(\nu, Q^2) = F_L(e^{-\nu}, Q^2), \]
\[ \frac{\partial \hat{F}_2(\nu, Q^2)}{\partial \ln Q^2} = \frac{\partial F_2(e^{-\nu}, Q^2)}{\partial \ln Q^2} \]
\[ \hat{G}(\nu, Q^2) = G(e^{-\nu}, Q^2). \]

The coefficient functions \( \Phi \) and \( \Theta \) in \( s \)-space at leading- 

\[ \Phi_L(s, Q^2) = \frac{\alpha_s(Q^2)}{\pi} C_F \frac{1}{2 + s}, \]

\[ \Theta_L(s, Q^2) = \frac{\alpha_s(Q^2)}{\pi} \left( \frac{1}{2 + s} - \frac{1}{3 + s} \right), \]

\[ \Theta_f(s, Q^2) = \frac{n_f}{\pi} \left( \frac{\alpha_s(Q^2)}{2} \left( \frac{1}{2 + s} - \frac{2}{3 + s} \right) + \frac{2}{3 + s} \right), \]

\[ \Phi_f(s, Q^2) = \frac{\alpha_s(Q^2)}{4\pi} \left[ 4 - \frac{8}{3} \left( \frac{1}{2 + s} \right) + \frac{1}{2 + s} + \psi(s + 1 + \gamma_E) \right], \]

where \( \psi(x) \) is the digamma function and \( \gamma_E = 0.5772156 \) is Euler constant. For the SU(N) gauge

at low \( x \) introduced by
\[ G(x, Q^2) = f(Q^2)x^{-\delta} \]
where the low \( x \) behavior could well be more singular. By considering the variable change \( \nu = \ln(1/x) \), one can rewrite the gluon distribution in \( s \)-space as
\[ L[G^2(\nu, Q^2); s] = \frac{f(Q^2)^2}{(s - 25)}, \]
\[ L[G(\nu, Q^2); s] = \frac{f(Q^2)^2}{(s - \delta^2)}. \]

We observe that the function \( L[G^2(\nu, Q^2); s] \) is always lower than \( L[G(\nu, Q^2); s]^2 \) for low \( s \) values in a wide range of \( Q^2 \) values. According to this result, we use from this limited approach for solving the quadratic equation in \( s \)-space.

\[ 1 \text{ The standard parametrization of the gluon distribution function} \]
group, $C_F = 4/3$ is the color Cassimir operator in QCD. The quadratic equation (7) has two roots. Consequently the longitudinal structure function in $s$-space is given by

$$f_L(s, Q^2) = \frac{1}{2} k(s, Q^2)[1 \pm (1 - \frac{4h(s, Q^2)}{k^2(s, Q^2)})^{1/2}].$$

The above equation (i.e., Eq.(10)) can be solving by a Taylor series expansion around a particular choice of a point of expansion, as the series is convergent when $\beta = \frac{4h}{k^2} < 1$. Therefore, the non-linear longitudinal structure function has the following forms:

$$f_L(s, Q^2) = \frac{1}{2} k(s, Q^2) \begin{cases} M(\beta) & \text{for positive root} \quad \text{(11)} \\ N(\beta) & \text{for negative root} \end{cases}$$

where

$$M(\beta) = 2 - \frac{1}{2} \beta - \frac{1}{16} \beta^3 - \frac{5}{128} \beta^4.$$  

and

$$N(\beta) = \frac{1}{2} \beta + \frac{1}{8} \beta^2 + \frac{1}{16} \beta^3 + \frac{5}{128} \beta^4.$$  

In Fig.1 the behavior of $M(\beta)$ and $N(\beta)$ with respect to the $\beta$ function are shown. In this figure the shaded area is show the convergence region for the functions $M(\beta)$ and $N(\beta)$. Thus, we rewrite Eq.(11) until the fourth sentence as

$$f_L(s, Q^2) = \begin{cases} k - \frac{h}{k} - \frac{k^3}{3\pi} - 2 \frac{k^3}{27\pi^3} - 5 \frac{k^3}{128\pi^3} & \text{for positive root} \\ \frac{h}{k} + \frac{k^3}{3\pi} + 2 \frac{k^3}{27\pi^3} + 5 \frac{k^3}{128\pi^3} & \text{for negative root} \end{cases}$$

The calculation of the above sentences in Eq.(12) and the inverse Laplace transforms of those are straightforward, and we find that

$$\tilde{J}_0(\nu) = L^{-1}[k(s, Q^2); \nu] = (\frac{16\alpha_s^2}{9\pi^2} - \frac{4\pi}{3\pi}) \tilde{F}_2(\nu, Q^2)^{\delta(\nu)}$$

$$\tilde{J}_1(\nu) = L^{-1}[\frac{h\tilde{F}_2(\nu, Q^2)}{k(s, Q^2)}; \nu] = (\frac{16\alpha_s^2}{9\pi^2} - \frac{4\pi}{3\pi}) \tilde{F}_2(\nu, Q^2)^{\delta(\nu)} \\ \times (\frac{16\alpha_s^2}{9\pi^2} - \frac{4\pi}{3\pi}) \tilde{F}_2(\nu, Q^2)^{-1} \delta(\nu),$$

$$\tilde{J}_2(\nu) = L^{-1}[\frac{h^2\tilde{F}_2(\nu, Q^2)}{k(s, Q^2)}; \nu] = (\frac{16\alpha_s^2}{9\pi^2} - \frac{4\pi}{3\pi}) \tilde{F}_2(\nu, Q^2)^{\delta(\nu)} \\ \times (\frac{16\alpha_s^2}{9\pi^2} - \frac{4\pi}{3\pi}) \tilde{F}_2(\nu, Q^2)^{-1} \delta(\nu),$$

$$\tilde{J}_3(\nu) = L^{-1}[\frac{2h^3\tilde{F}_2(\nu, Q^2)}{k(s, Q^2)}; \nu] = (\frac{16\alpha_s^2}{9\pi^2} - \frac{4\pi}{3\pi}) \tilde{F}_2(\nu, Q^2)^{\delta(\nu)} \\ \times (\frac{16\alpha_s^2}{9\pi^2} - \frac{4\pi}{3\pi}) \tilde{F}_2(\nu, Q^2)^{-1} \delta(\nu),$$

$$\tilde{J}_4(\nu) = L^{-1}[\frac{5h^4\tilde{F}_2(\nu, Q^2)}{k(s, Q^2)}; \nu] = (\frac{16\alpha_s^2}{9\pi^2} - \frac{4\pi}{3\pi}) \tilde{F}_2(\nu, Q^2)^{\delta(\nu)} \\ \times (\frac{16\alpha_s^2}{9\pi^2} - \frac{4\pi}{3\pi}) \tilde{F}_2(\nu, Q^2)^{-1} \delta(\nu).$$

FIG. 1: The shaded area represents the convergence of positive ($M(\beta)$- solid curve) and negative ($N(\beta)$-dashed curve) roots in Eq.(11).
Therefore the non-linear corrections to the longitudinal structure function due to the Laplace-transform method is defined by the parametrization of \(F_2(x, Q^2)\) and its derivative \(DF_2(x, Q^2)\) at low \(x\) as we have

\[
F_L(x, Q^2) = J_0(x, Q^2) + \sum_{n=0}^{\infty} J_{n+1}(x, Q^2),
\]

where

\[
J_0(x, Q^2) = \left(\frac{16\alpha_s^2}{9\pi^2}\eta + \frac{4\alpha_s}{3\pi} F_2(x, Q^2)\right),
\]

\[
J_{n+1}(x, Q^2) = \left(\frac{16\alpha_s^2}{9\pi^2}\eta DF_2(x, Q^2) + \frac{32\alpha_s^3}{27\pi^3}\eta F_2(x, Q^2) + \frac{24\alpha_s^3}{27\pi^3}\eta ((-1/2)F_2(x, Q^2) + \frac{1}{2}F_2(x, Q^2)^2)\right)^{n+1} \times \left(\frac{16\alpha_s^2}{9\pi^2}\eta + \frac{4\alpha_s}{3\pi} F_2(x, Q^2)^2\right)^{-(2n+1)},
\]

where

\[
F_2(x, Q^2) = D(Q^2)(1-x)^2 \sum_{m=0}^{\infty} A_m(Q^2) L^m,
\]

and

\[
DF_2(x, Q^2) = \frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = F_2(x, Q^2) \left[\frac{\partial \ln D(Q^2)}{\partial \ln Q^2} + \frac{\partial \ln \left(\sum_{m=0}^{\infty} A_m(Q^2) L^m\right)}{\partial \ln Q^2}\right].
\]

The explicit expression for the proton structure function and effective parameters are defined in Appendix A and Table I. Now, with the explicit form of the proton structure function, we can proceed to extract the non-linear corrections to the longitudinal structure function \(F_L(x, Q^2)\) from data mediated by the parametrization of \(F_2(x, Q^2)\) and its derivative. In Fig.2, we show the \(Q^2\)-dependence of the non-linear corrections to the longitudinal structure function at low \(x\). Results of calculations and comparison with the H1 collaboration data [14] are presented in this figure (i.e., Fig.2), where the charm quark mass effects are considered in the rescaling variable. These results have been performed at fixed value of the invariant mass \(W\) as \(W = 230\) GeV. The extracted non-linear longitudinal structure functions are in good agreement in comparison with the H1 collaboration data over a wide range of \(Q^2\) values. Also this behavior improved at low values of \(Q^2\) in comparison with the other parametrization models. We observe that at low \(Q^2\), the behavior of the non-linear longitudinal structure functions is reduced in comparison with the behavior of the linear. In Fig.3 we obtained our results with respect to the only negative roots and compared with other results as defined in Fig.2. In these figures (i.e., Fig.2 and Fig.3) we observe that the positive roots decrease the growth rate of \(F_L\) as \(Q^2\) decreases. However, the high-order corrections are important for comparing these results with experimental data at low \(Q^2\).

In conclusion, we have presented a certain theoretical model to describe the non-linear corrections to the longitudinal structure function based on the Laplace transform method at low values of \(x\) in a limited approach. A detailed analysis has been performed to find an analytical solution of the non-linear longitudinal structure function in s-space into the proton structure function and its derivative. The nonlinear corrections improved the behavior of the longitudinal structure function at low values of \(Q^2\) in comparison with the H1 collaboration data, but the high-order corrections are important in the region of low \(Q^2\).

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Appendix A

The proton structure function parameterized in Ref.[3] provide good fits to the HERA data at low \(x\) and large \(Q^2\) values. The explicit expression for the proton structure function, with respect to the Block-Halzen fit, in a range of the kinematical variables \(x\) and \(Q^2\), \(x \leq 0.1\) and \(0.15\text{ GeV}^2 < Q^2 < 3000\text{ GeV}^2\), is defined by the following form

\[
F_{2p}(x, Q^2) = D(Q^2)(1-x)^n[C(Q^2) + A(Q^2) \ln(\frac{Q^2}{x Q^2 + \mu^2})]
\]

\[
+ B(Q^2) \ln^2(\frac{Q^2}{x Q^2 + \mu^2})]
\]

where

\[
A(Q^2) = a_0 + a_1 \ln(1 + \frac{Q^2}{\mu^2}) + a_2 \ln^2(1 + \frac{Q^2}{\mu^2}),
\]

\[
B(Q^2) = b_0 + b_1 \ln(1 + \frac{Q^2}{\mu^2}) + b_2 \ln^2(1 + \frac{Q^2}{\mu^2}),
\]

\[
C(Q^2) = c_0 + c_1 \ln(1 + \frac{Q^2}{\mu^2}),
\]

\[
D(Q^2) = \frac{Q^2(Q^2 + \lambda M^2)}{(Q^2 + M^2)^2}.
\]

Here \(M\) is the effective mass and \(\mu^2\) is a scale factor. The additional parameters with their statistical errors
are given in Table I.

TABLE I: The effective parameters at low $x$ for $0.15 \text{ GeV}^2 < Q^2 < 3000 \text{ GeV}^2$ provided by the following values. The fixed parameters are defined by the Block-Halzen fit to the real photon-proton cross section as $M^2 = 0.753 \pm 0.068 \text{ GeV}^2$, $\mu^2 = 2.82 \pm 0.290 \text{ GeV}^2$ and $c_0 = 0.255 \pm 0.016$ [3].

| parameter | value         |
|-----------|---------------|
| $a_0$     | $8.205 \times 10^{-4} \pm 4.62 \times 10^{-4}$ |
| $a_1$     | $-5.148 \times 10^{-2} \pm 8.19 \times 10^{-3}$ |
| $a_2$     | $-4.725 \times 10^{-3} \pm 1.01 \times 10^{-3}$ |
| $b_0$     | $2.217 \times 10^{-3} \pm 1.42 \times 10^{-4}$ |
| $b_1$     | $1.244 \times 10^{-2} \pm 8.56 \times 10^{-4}$ |
| $b_2$     | $5.958 \times 10^{-4} \pm 2.32 \times 10^{-4}$ |
| $c_3$     | $1.475 \times 10^{-1} \pm 3.025 \times 10^{-2}$ |
| $n$       | $11.49 \pm 0.99$ |
| $\lambda$ | $2.430 \pm 0.153$ |
| $\chi^2$  | 0.95          |

FIG. 2: Non-linear corrections to the $Q^2$ dependence of the extracted longitudinal structure function at fixed value of the invariant mass $W = 230 \text{ GeV}$ (short-dashed curve) compared with the Mellin transform method [1](dashed curve) at the LO approximation and also the Laplace transform method [2](solid curve) with the rescaling variable at the LO approximation. Experimental data by the H1 Collaboration are taken from Ref. [14] as accompanied with total errors.

FIG. 3: Non-linear corrections to the $Q^2$ dependence of the extracted longitudinal structure function, with respect to the only negative roots, at fixed value of the invariant mass $W = 230 \text{ GeV}$ (short-dashed curve) compared with the Mellin transform method [1](dashed curve) at the LO approximation and also the Laplace transform method [2](solid curve) with the rescaling variable at the LO approximation. Experimental data by the H1 Collaboration are taken from Ref. [14] as accompanied with total errors.

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