A POSITIVE PROPORTION OF CUBIC CURVES OVER \( \mathbb{Q} \) ADMIT LINEAR DETERMINANTAL REPRESENTATIONS

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Abstract. Can a smooth plane cubic be defined by the determinant of a square matrix with entries in linear forms in three variables? If we can, we say that it admits a linear determinantal representation. In this paper, we investigate linear determinantal representations of smooth plane cubics over various fields, and prove that any smooth plane cubic over a large field (or an ample field) admits a linear determinantal representation. Since local fields are large, any smooth plane cubic over a local field always admits a linear determinantal representation. As an application, we prove that a positive proportion of smooth plane cubics over \( \mathbb{Q} \), ordered by height, admit linear determinantal representations. We also prove that, if the conjecture of Bhargava–Kane–Lenstra–Poonen–Rains on the distribution of Selmer groups is true, a positive proportion of smooth plane cubics over \( \mathbb{Q} \) fail the local-global principle for the existence of linear determinantal representations.

1. Introduction

A smooth plane cubic \( C \subset \mathbb{P}^2 \) defined over a field \( k \) is said to admit a linear determinantal representation over \( k \) if there is a triple of square matrices \( (M_0, M_1, M_2) \) of size three with entries in \( k \) such that \( C \) is defined by the cubic equation

\[
\det(X_0M_0 + X_1M_1 + X_2M_2) = 0.
\]

It is a classical problem in algebraic geometry to find linear determinantal representations of plane curves over an algebraically closed field ([19, Chapter 4], [38]). For concrete examples of conics and cubics, see Example 6.3, Example 7.6 and Example 7.7.

Recently, the arithmetic nature of linear determinantal representations has been studied by several people ([24, 25, 26, 27, 28]). It is applied to the computation of the Cassels–Tate pairing of elliptic curves ([21]). For an application to the classification of non-associative algebras motivated by coding theory, see [17, 18].

The aim of this paper is to prove that a positive proportion of smooth plane cubics over \( \mathbb{Q} \), ordered by height, admit linear determinantal representations. We also obtain conditional results on the proportion of smooth plane cubics over \( \mathbb{Q} \) which fail the local-global principle for the existence of linear determinantal representations.

Theorem 1.1 (See Theorem 3.5 and §9.4). We order all smooth plane cubics over \( \mathbb{Q} \) by height (for the definition of height of smooth plane cubics, see Section 3).

(i) A positive proportion of smooth plane cubics over \( \mathbb{Q} \) admit linear determinantal representations over \( \mathbb{Q} \).

(ii) If the conjecture of Bhargava–Kane–Lenstra–Poonen–Rains on the distribution of Selmer groups [5, Conjecture 1.3] is true, a positive proportion of smooth plane cubics over \( \mathbb{Q} \) fail the local-global principle for the existence of linear determinantal representations, i.e., they admit linear determinantal representations over each local field \( \mathbb{Q}_v \) of \( \mathbb{Q} \), whereas they do not admit linear determinantal representations over \( \mathbb{Q} \).

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It seems difficult to prove Theorem 1.1 (ii) unconditionally. See Remark 3.8.

We also obtain unconditional results for linear determinantal representations of elliptic curves over \( \mathbb{Q} \). We regard an elliptic curve \((E, \mathcal{O})\) over \( \mathbb{Q} \) as a smooth plane cubic by the complete linear system \( |3\mathcal{O}| : E \to \mathbb{P}^2 \).

**Theorem 1.2** (See Theorem 2.1). We order all isomorphism classes of elliptic curves over \( \mathbb{Q} \) by height (for the definition of height of elliptic curves over \( \mathbb{Q} \), see Section 2).

(i) At least 20.68% of elliptic curves over \( \mathbb{Q} \) admit linear determinantal representations over \( \mathbb{Q} \).

(ii) At least 16.50% of elliptic curves over \( \mathbb{Q} \) fail the local-global principle for the existence of linear determinantal representations.

These results are based on the global results recently obtained by Bhargava–Skinner–Zhang [9], where they calculate lower bounds of the proportion of elliptic curves over \( \mathbb{Q} \) with Mordell–Weil rank zero (respectively, one).

Also, as a key local result, we prove the existence of linear determinantal representations over large fields (Theorem 1.3). Recall that a field \( k \) is called large (or ample) if any smooth geometrically connected curve over \( k \) with a \( k \)-rational point has infinitely many \( k \)-rational points ([34, 29]).

**Theorem 1.3** (See Theorem 8.2). Any smooth plane cubic over a large field admits a linear determinantal representation.

Since local fields (i.e. finite extensions of \( \mathbb{Q}_p, \mathbb{R} \) and \( \mathbb{F}_p((t)) \)) are large, any smooth plane cubic over a local field always admits a linear determinantal representation. Combining the result of Bhargava–Skinner–Zhang, Theorem 1.3 and an interpretation of linear determinantal representations on smooth plane cubics using line bundles (see Section 5), we prove Theorem 1.2.

Another key result to prove Theorem 1.1 is the following result on proportion of the smooth plane cubics over \( \mathbb{Q} \).

**Theorem 1.4** (See Theorem 3.5). We order all smooth plane cubics over \( \mathbb{Q} \) by height (for the definition of height of smooth plane cubics, see Section 3).

(i) A positive proportion of smooth plane cubics over \( \mathbb{Q} \) have Jacobian varieties with positive Mordell–Weil rank.

(ii) Assume the conjecture of Bhargava–Kane–Lenstra–Poonen–Rains [5, Conjecture 1.3] is true. Then a positive proportion of smooth plane cubics over \( \mathbb{Q} \) have Jacobian varieties with trivial Mordell–Weil group.

We prove Theorem 1.4 using similar methods of Bhargava [3], where he studies the proportion of smooth plane cubics over \( \mathbb{Q} \) which satisfy (resp. fail) the local-global principle for the existence of rational points. We choose a fundamental domain \( F \) in the space of ternary cubics over \( \mathbb{R} \) with respect to the action of \( \text{PGL}_3(\mathbb{Z}) \). Then we prove results similar to Theorem 1.3 with respect to the height of Jacobian varieties (the Jacobian height), not with respect to the height of smooth plane cubics (Proposition 3.9). We find that a positive proportion of smooth plane cubics in \( F \) with integral coefficients are generic with respect to the Jacobian height. Since non-generic smooth plane cubics are negligible with respect to the height of smooth plane cubics, we can obtain a lower bound of the number of generic plane cubics with bounded height whose Jacobian variety have positive Mordell–Weil rank (resp. trivial Mordell–Weil group) using the number of generic plane cubics in \( F \) with bounded Jacobian height whose Jacobian variety have positive Mordell–Weil rank (resp. trivial Mordell–Weil group). Combining Theorem 1.4 and an interpretation of linear determinantal representations using line bundles, we prove Theorem 1.1.

The present paper is organized as follows. In Section 2 we prove Theorem 1.2 using the results proved in later sections. In Section 3 we precisely state Theorem 1.4 and prove it.
using similar methods of [3]. In Section 4, we recall some basic definitions and facts about Picard groups and relative Brauer groups. In Section 5, for smooth plane curves of any degree, we prove a bijection between equivalence classes of linear determinantal representations and isomorphism classes of certain line bundles. As examples, we study the case of lines and smooth conics in Section 6. In Section 7, we obtain some sufficient conditions for a smooth plane cubic over a field to admit a linear determinantal representation. In Section 8, we recall the definition of large fields and prove Theorem 1.3. Then we apply it to study the local-global principle for the existence of linear determinantal representations over global fields and prove Theorem 1.4 in Section 9.

Remark 1.5. These results are in contrast with the case of symmetric determinantal representations studied in [26] and [27]. In [26, Proposition 4.2], it is proved that a smooth plane cubic over a field $k$ admits a symmetric determinantal representation if and only if the Jacobian variety $\text{Jac}(C)$ has a non-trivial $k$-rational 2-torsion point. From this, it is not difficult to prove that

- any smooth plane cubic over a global field satisfies the local-global principle for the existence of symmetric determinantal representations (\cite[Theorem 5.1]{26}, \cite[Theorem 6.1]{27}), and
- $100\%$ of smooth plane cubics over $\mathbb{Q}$ do not admit symmetric determinantal representations, and
- there are smooth plane cubics over any non-archimedean local field without symmetric determinantal representations.

Hence Theorem 1.1, Theorem 1.2, and Theorem 1.3 do not hold for symmetric determinantal representations. See Remark 5.4 and Remark 7.4. For a similar problem considering whether a binary form can be written as a discriminant form of a pencil of quadrics, see \cite[11]{15}.

Notation 1.6. For a field $k$, we denote a separable closure of $k$ as $k^s$. For a $k$-scheme $X$ and a field extension $L$ of $k$, we denote the base change by $X_L := X \times_{\text{Spec}k} \text{Spec}L$. For a geometrically integral $k$-scheme $X$, $k(X)$ denotes the function field of $X$. For a ring $R$, we denote the multiplicative group as $R^\times$. 

2. Linear determinantal representations on elliptic curves over $\mathbb{Q}$

In this section, we study the proportion of elliptic curves over $\mathbb{Q}$ which admit (resp. do not admit) linear determinantal representations over $\mathbb{Q}$.

Before state the precise statement of Theorem 1.2, let us recall the notion of height of elliptic curves over $\mathbb{Q}$. Any elliptic curve $(E, O)$ over $\mathbb{Q}$ is defined by a unique Weierstrass equation

\begin{equation}
E_{A,B}: (X_2^2 X_2 - X_0^3 - AX_0 X_2^2 - BX_2^3 = 0),
\end{equation}

where $A, B \in \mathbb{Z}$ satisfy $4A^3 + 27B^2 \neq 0$, and, for any prime $p$ with $p^4 \mid A$, we have $p^6 \nmid B$. We define the height $H(E_{A,B})$ of the elliptic curve $E_{A,B}$ by

\[ H(E_{A,B}) := \max\{4|A|^3, 27B^2\}. \]

This is a well-defined invariant for an arbitrary isomorphism class of elliptic curves over $\mathbb{Q}$.

For a constant $X > 0$, we define $\text{Ell}(X)$ as the set of isomorphism classes of elliptic curves over $\mathbb{Q}$ having height less than $X$. Let $\text{Ell}(X)_a$ (resp. $\text{Ell}(X)_na$) be the subset of $\text{Ell}(X)$ of elliptic curves over $\mathbb{Q}$ which admit (resp. do not admit) linear determinantal representations over $\mathbb{Q}$.

Theorem 2.1. We order (isomorphism classes of) elliptic curves over $\mathbb{Q}$ by height.
of degree $12$ called the discriminant

$$\Delta(v) := 4A(v)^3 + 27B(v)^2$$

of degree $12$ called the discriminant, which vanishes if and only if the plane cubic $C(v)$ is not smooth.

Remark 3.1. There are slight differences between our settings and those in [20] and [1]. In [20], Fisher considered three invariants $c_4 = -A(v)/27$, $c_6 = -B(v)/54$ and $\Delta = -\Delta(v)/(1728 \cdot 27^3 \cdot 4)$. In [1], An–Kim–Marshall–Marshall–McCallum–Pelis considered two invariants $S = A(v)/(1296 \cdot 27)$ and $T = -B(v)/(5832 \cdot 54)$. 

Proof. (i) Bhargava–Skinner–Zhang proved that, when ordered by height, at least $20.68\%$ of elliptic curves over $\mathbb{Q}$ have Mordell–Weil rank $1$ ([9, Theorem 3]). By Corollary 7.4, an elliptic curve over $\mathbb{Q}$ with positive Mordell–Weil rank admits a linear determinantal representation over $\mathbb{Q}$ (actually, it has infinitely many equivalence classes of linear determinantal representations over $\mathbb{Q}$). Hence at least $16.50\%$ of elliptic curves over $\mathbb{Q}$ admit linear determinantal representations over $\mathbb{Q}$.

(ii) Bhargava–Skinner–Zhang proved that, when ordered by height, at least $16.50\%$ of elliptic curves over $\mathbb{Q}$ have Mordell–Weil rank $0$ ([9, Theorem 3]). It is well-known that $100\%$ of elliptic curves over $\mathbb{Q}$ have trivial Mordell–Weil group with trivial torsion (for example, see [5, Lemma 5.7]). Combining these two results, we conclude that at least $16.50\%$ of elliptic curves over $\mathbb{Q}$ have trivial Mordell–Weil group. They do not admit linear determinantal representations over $\mathbb{Q}$ by Lemma 9.3.
An element $v \in V(\mathbb{Z})$ is called nondegenerate if $\Delta(v) \neq 0$, and an element $v \in V(\mathbb{Z})$ is called generic if $v$ is smooth and $C(v)$ has no $\mathbb{Q}$-rational flex points (i.e. the $\mathbb{Q}$-rational points where the tangent meets the cubic to order at least three). We note that an elliptic curve $(E, \mathcal{O})$ over $\mathbb{Q}$ embedded in $\mathbb{P}^2$ by the complete linear system $|3\mathcal{O}|$ is always non-generic because the origin $\mathcal{O} \in E(\mathbb{Q})$ is a $\mathbb{Q}$-rational flex point. If $v \in V(\mathbb{Z})$ is nondegenerate, the Jacobian variety $\text{Jac}(C(v))$ of the smooth plane cubic $C(v)$ is isomorphic to the elliptic curve

$$E_{A(v), B(v)} : (X_1^2 X_2 - X_0^3 - A(v) X_0 X_2^2 - B(v) X_2^3 = 0).$$

**Definition 3.2.** Let $v \in V(\mathbb{R})$ be a real ternary cubic form. The height $H(v)$ of $v$ is the maximum of the absolute values of the coefficients of $v$. The Jacobian height $H_J(v)$ of $v$ is defined as

$$H_J(v) := \max\{|4A(v)|^3, 27B(v)^2\}.$$

Note that, the Jacobian height of an integral element $v \in V(\mathbb{Z})$ does not coincide with the height of the elliptic curve $\text{Jac}(C_v) = E_{A(v), B(v)}$ because we do not reduce the constants $A(v)$ and $B(v)$.

**Definition 3.3.** For a constant $X > 0$, we define the subsets $V(X), V(X)_{\text{gen}}, V(X)_{\text{rk} \geq 1}, V(X)_{\text{MW} = 0}$ of $V(\mathbb{Z})$ by

$$V(X) := \{v \in V(\mathbb{Z}) \mid H(v) < X\},$$

$$V(X)_{\text{gen}} := \{v \in V(X) \mid v \text{ is generic}\},$$

$$V(X)_{\text{rk} \geq 1} := \{v \in V(X)_{\text{gen}} \mid \text{rank} \text{Jac}(C(v))(\mathbb{Q}) \geq 1\},$$

$$V(X)_{\text{MW} = 0} := \{v \in V(X)_{\text{gen}} \mid \text{Jac}(C(v))(\mathbb{Q}) = 0\}.$$

We fix a fundamental domain $F \subset V(\mathbb{R})$ for the action of $\text{PGL}_3(\mathbb{Z})$.

**Definition 3.4.** For a constant $X > 0$, we define the subsets $F_J(X), F_J(X)_{\text{gen}}, F_J(X)_{\text{rk} \geq 1}, F_J(X)_{\text{MW} = 0}$

$$F_J(X) := \{v \in F \cap V(\mathbb{Z}) \mid H_J(v) < X\},$$

$$F_J(X)_{\text{gen}} := \{v \in F_J(X) \mid v \text{ is generic}\},$$

$$F_J(X)_{\text{rk} \geq 1} := \{v \in F_J(X)_{\text{gen}} \mid \text{rank} \text{Jac}(C(v))(\mathbb{Q}) \geq 1\},$$

$$F_J(X)_{\text{MW} = 0} := \{v \in F_J(X)_{\text{gen}} \mid \text{Jac}(C(v))(\mathbb{Q}) = 0\}.$$

Now we give the precise statement of the main results of this section.

**Theorem 3.5.** (i) When ordered by height, a positive proportion of smooth plane cubics over $\mathbb{Q}$ have Jacobian varieties with positive Mordell–Weil rank, i.e., we have

$$\liminf_{X \to \infty} \frac{\#V(X)_{\text{rk} \geq 1}}{\#V(X)} > 0.$$

(ii) Assume that the conjecture of Bhargava–Kane–Lenstra–Poonen–Rains [5 Conjecture 1.3] is true. When ordered by height, a positive proportion of smooth plane cubics over $\mathbb{Q}$ have Jacobian varieties with trivial Mordell–Weil group, i.e., we have

$$\liminf_{X \to \infty} \frac{\#V(X)_{\text{MW} = 0}}{\#V(X)} > 0.$$

### 3.2. The Bhargava–Kane–Lenstra–Poonen–Rains conjecture.

Before we prove Theorem 3.5, we recall the notion of Selmer groups of elliptic curves and some implications of the conjecture of Bhargava–Kane–Lenstra–Poonen–Rains.

Let $k$ be a global field, and $n$ an integer invertible in $k$. The $n$-Selmer group $\text{Sel}_n(E)$ of an elliptic curve $E$ over $k$ is a finite group which parametrizes locally soluble $n$-coverings of $E$ (see [33 Chapter X] and [43]). There exists a short exact sequence of the following form

$$0 \longrightarrow E(k) \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow \text{Sel}_n(E) \longrightarrow \text{III}(E)[n] \longrightarrow 0.$$
where $\Sha(E)$ denotes the Tate–Shafarevich group of $E$. We fix a prime number $p$ invertible in $k$. Varying $n$ over all powers of $p$, we obtain the following short exact sequence
\[
0 \longrightarrow E(k) \otimes \mathbb{Q}_p / \mathbb{Z}_p \longrightarrow \text{Sel}_p^\infty(E) \longrightarrow \Sha(E)[p^\infty] \longrightarrow 0
\]
from the short exact sequence (3.1).

Suppose that we are given a short exact sequence $S$ of $\mathbb{Z}_p$-modules. The conjecture of Bhargava–Kane–Lenstra–Poonen–Rains [5, Conjecture 1.3] concerns the density of elliptic curves $E$ in $\text{Ell}(X)$ for which the short exact sequence (3.2) is isomorphic to $S$. We do not recall the precise statement of the conjecture. Instead, we quote from [5, Section 5.3] and [5, Section 5.5] the following two implications.

**Proposition 3.6.** Assume that the conjecture of Bhargava–Kane–Lenstra–Poonen–Rains [5, Conjecture 1.3] is true.

(i) When ordered by height, 50% of elliptic curves over $\mathbb{Q}$ have Mordell–Weil rank 0 and 50% have Mordell–Weil rank 1 ([5, §5.3]).

(ii) When ordered by height, the proportion of elliptic curves over $\mathbb{Q}$ with trivial $p$-Selmer group is
\[
\prod_{j \geq 0} (1 + p^{-j})^{-1}
\]
for any prime number $p$ ([5, §5.5]; see also [33, Conjecture 1.1(a)]).

We will need the case of Mordell–Weil rank 0 of Proposition 3.6 (i) and the case of $p = 3$ of Proposition 3.6 (ii). These results give the following statement for the proportion of elliptic curves over $\mathbb{Q}$ with trivial Mordell–Weil group and non-trivial 3-Selmer group. Corollary 3.7 will be necessary in the proof of Theorem 1.1 (ii).

**Corollary 3.7.** Assume that the conjecture of Bhargava–Kane–Lenstra–Poonen–Rains [5, Conjecture 1.3] is true. When ordered by height, at least $1/8$ of elliptic curves $E$ over $\mathbb{Q}$ satisfy the following conditions:

- the Mordell–Weil group $E(\mathbb{Q})$ is trivial, and
- the 3-Selmer group $\text{Sel}_3(E)$ is non-trivial.

**Proof.** If the conjecture of Bhargava–Kane–Lenstra–Poonen–Rains is true, by Proposition 3.6 (i), we have
\[
\lim_{X \to \infty} \frac{\# \{ E \in \text{Ell}(X) \mid \text{rank } E(\mathbb{Q}) = 0 \}}{\# \text{Ell}(X)} = \frac{1}{2}.
\]

Since 100% of elliptic curves over $\mathbb{Q}$ have Mordell–Weil group with trivial torsion (cf. [5, Lemma 5.7]), we obtain
\[
\lim_{X \to \infty} \frac{\# \{ E \in \text{Ell}(X) \mid E(\mathbb{Q}) = 0 \}}{\# \text{Ell}(X)} = \frac{1}{2}.
\]

On the other hand, by Proposition 3.6 (ii), we have
\[
\lim_{X \to \infty} \frac{\# \{ E \in \text{Ell}(X) \mid \text{Sel}_3(E) = 0 \}}{\# \text{Ell}(X)} = \prod_{j \geq 0} (1 + 3^{-j})^{-1} = \prod_{j \geq 0} \left(1 + \frac{1}{3^j}\right)^{-1} = 0.319502 \ldots < \frac{3}{8}.
\]

Hence, when ordered by height, at least $1/2 - 3/8 = 1/8$ of elliptic curves over $\mathbb{Q}$ have trivial Mordell–Weil group and non-trivial 3-Selmer group. \qed

**Remark 3.8.** It seems difficult to obtain results like Corollary 3.7 unconditionally. If an elliptic curve $E$ over $\mathbb{Q}$ has trivial Mordell–Weil group and non-trivial 3-Selmer group, its 3-Selmer group $\text{Sel}_3(E)$ is expected to have even rank greater than or equal to 2. For the
moment, it seems difficult to prove that a positive proportion of elliptic curves over \( \mathbb{Q} \) have 3-Selmer rank greater than or equal to 2. It is the main reason why our Theorem 1.1 (ii) is conditional.

3.3. The proportion of smooth plane cubics. First we prove the following proposition concerning the proportion of smooth plane cubics over \( \mathbb{Q} \) with respect to the Jacobian height.

**Proposition 3.9.** (i) There is a constant \( a > 0 \) such that
\[
\#F_J(X)_{rk \geq 1} \geq aX^{5/6} + o(X^{5/6}).
\]
(ii) If the conjecture of Bhargava–Kane–Lenstra–Poonen–Rains [5] Conjecture 1.3] is true, there is a constant \( b > 0 \) such that
\[
\#F_J(X)_{MW=0} \geq bX^{5/6} + o(X^{5/6}).
\]

**Proof.** (i) By [6, Theorem 3.17], we see that
\[
(3.3) \quad \#\{E \in \text{Ell}(X) \mid \text{rank } E(\mathbb{Q}) = 1\} \geq c_2X^{5/6} + o(X^{5/6})
\]
for some constant \( c_2 > 0 \). By the short exact sequence [7,7], each elliptic curve \( E \) over \( \mathbb{Q} \) with Mordell–Weil rank 1 has at least 2 non-trivial elements in the 3-Selmer group \( \text{Sel}_3(E) \). Each non-trivial element of the 3-Selmer groups \( \text{Sel}_3(E) \) gives at least one generic element \( v \) in \( F_J(X)_{rk \geq 1} \). (See [8, Proposition 30], where generic elements are called strongly irreducible.) Hence the number of generic elements \( v \in F_J(X)_{rk \geq 1} \) is at least
\[
2c_2X^{5/6} + o(X^{5/6}).
\]
Putting \( a = 2c_2 \), we obtain the desired inequality.

(ii) By Corollary [8,7, if the conjecture of Bhargava–Kane–Lenstra–Poonen–Rains is true, at least \( 1/8 \) of elliptic curves over \( \mathbb{Q} \) have trivial Mordell–Weil group and non-trivial 3-Selmer group. Such curve has at least 2 non-trivial elements in the 3-Selmer group \( \text{Sel}_3(E) \), and each non-trivial element of the group \( \text{Sel}_3(E) \) gives at least one generic element in \( F_J(X)_{MW=0} \). Thus we obtain
\[
F_J(X)_{MW=0} \geq 2 \cdot \frac{1}{8} \cdot c_1X^{5/6} + o(X^{5/6})
\]
\[
= \frac{c_1}{4}X^{5/6} + o(X^{5/6}),
\]
where \( c_1 \) is the same constant as in (3.3). Putting \( b = c_1/4 \), we obtain the second inequality. \( \Box \)

We are now ready to prove Theorem 3.5.

**Proof** (Proof of Theorem 3.5). (i) Bhargava–Shankar proved in [9, Theorem 8] that
\[
(3.4) \quad \#F_J(X)_{\text{gen}} = c_3X^{5/6} + o(X^{5/6})
\]
for some constant \( c_3 > 0 \). Hence, by Proposition 3.9 (i), the number of generic elements \( v \in F_J(X)_{\text{gen}} \) with rank \( \text{Jac}(C(v))(\mathbb{Q}) = 0 \) is at most
\[
(3.5) \quad (c_3 - a)X^{5/6} + o(X^{5/6}).
\]
In other words, at most \( 1 - a/c_3 \) of elements in \( F_J(X)_{\text{gen}} \) have Jacobian varieties with Mordell–Weil rank 0.

Next we consider two regions \( D(X), D_J(X) \) of \( V(\mathbb{R}) \cong \mathbb{R}^{10} \) defined by
\[
D(X) := \{ v \in V(\mathbb{R}) \mid H(v) < X^{1/12}\},
\]
\[
D_J(X) := \{ v \in F \mid H_J(v) < X \}.
\]
Note that $D(X)$ is a bounded region. On the other hand, the region $D_J(X)$ is not bounded but has finite volume ([2] §2.3). In fact, we have

\[(3.6)\quad \text{vol}(D_J(X)) = c_3X^{5/6} + o(X^{5/6}),\]

where $c_3 > 0$ is the same constant as in [3.4] (see [2] §2, (11) and [2] Proposition 16; see also the paragraph after Theorem 16 in [3]). Since the region $D_J(X)$ has finite volume and $H_J(Yv) = Y^{12}H_J(v)$ and $H(Yv) = YH(v)$ for $Y > 0$ and $v \in V(\mathbb{R})$, we can take a constant $r > 0$ independently of $X$ such that

\[
\frac{\text{vol}(D(rX) \cap D_J(X))}{\text{vol}(D_J(X))} > 1 - \frac{a}{c_3}.
\]

We fix such a constant $r > 0$, and put

\[
\delta := \frac{\text{vol}(D(rX) \cap D_J(X))}{\text{vol}(D_J(X))} - \left(1 - \frac{a}{c_3}\right) > 0.
\]

The constant $\delta > 0$ does not depend on $X$. Note that $F_J(X) = D_J(X) \cap V(\mathbb{Z})$ while $F(X^{1/12}) = D(X) \cap V(\mathbb{Z})$.

The region $D(rX) \cap D_J(X)$ is bounded and non-generic elements in $D(rX) \cap F_J(X)$ are negligible with respect to $X^{5/6}$ ([2] §2.5]). By [3.9], we have

\[
\#(D(rX) \cap F_J(X)_{\text{gen}}) = \text{vol}(D(rX) \cap D_J(X)) + o(X^{5/6})
\]

\[= \left(1 - \frac{a}{c_3} + \delta \right)\text{vol}(D_J(X)) + o(X^{5/6})
\]

\[= (c_3 - a + \delta c_3)X^{5/6} + o(X^{5/6}).
\]

In combination with [3.9], then we obtain

\[
\#(D(rX) \cap F_J(X)_{rk \geq 1}) \geq \delta c_3X^{5/6} + o(X^{5/6}).
\]

Replacing $X$ by $r^{-1}X^{12}$, we obtain

\[
\#(D(X^{12}) \cap F_J(r^{-1}X^{12})_{rk \geq 1}) \geq \frac{\delta c_3}{r^{5/6}}X^{10} + o(X^{10}).
\]

Since $V$ is a 10-dimensional affine space, we have

\[
\#V(X) = \#(D(X^{12}) \cap V(\mathbb{Z}))
\]

\[= c_4X^{10} + o(X^{10})
\]

for some constant $c_4 > 0$. Combining these results, we conclude

\[
\liminf_{X \to \infty} \frac{\#V_{rk \geq 1}}{\#V(X)} \geq \liminf_{X \to \infty} \frac{\#(D(X^{12}) \cap F_J(r^{-1}X^{12})_{rk \geq 1})}{\#V(X)}
\]

\[\geq \frac{\delta c_3}{c_4 r^{5/6}} > 0.
\]

(ii) We briefly sketch the proof because the proof of (ii) is similar to the proof of (i). First we count the number of generic elements in $F_J(X)_{\text{gen}}$ with $\text{Jac}(C(v))(\mathbb{Q}) \neq 0$. Then we take a constant $r' > 0$ independently of $X$ so that

\[
\frac{\text{vol}(D(r'X) \cap D_J(X))}{\text{vol}(D_J(X))} > 1 - \frac{b}{c_3}.
\]

From this and Proposition [3.2] (ii), we obtain

\[
\#(F_J(r'^{-1}X^{12})_{\text{MW}=0} \cap D(X^{12})) \geq \frac{\delta' c_3}{r'^{5/6}}X^{10} + o(X^{10})
\]
for the constant
\[ \delta' := \frac{\text{vol}(D(r'X) \cap D_J(X))}{\text{vol}(D_J(X))} - \left(1 - \frac{b}{c_3}\right) > 0.\]

Note that the constant \( \delta' > 0 \) does not depend on \( X \). Thus we conclude
\[ \liminf_{X \to \infty} \frac{\#V(X)_{MW=0}}{\#V(X)} \geq \frac{\delta' c_3}{c_4 r^{5/6}} > 0. \]

This completes the proof of Theorem \[?\] \( \square \)

4. Preliminaries on Picard groups and relative Brauer groups

In this section, we recall definitions and basic properties of Picard groups and relative Brauer groups, which will be needed in the proof of the main results of this paper. For details, see [10, Chapter 8], [12, §1] and references therein.

4.1. Picard functors and Picard schemes. Let \( C \) be a smooth projective geometrically connected curve over a field \( k \). We denote the Picard group of \( C \) by \( \text{Pic}(C) = H^1(C, \mathcal{O}_C^\times) \).

The sheafification of the fppf-presheaf
\[(k\text{-Schemes})^{\text{op}} \to (\text{Sets}), \quad T \mapsto \text{Pic}(C_T) = \text{Pic}(C \times \text{Spec} \, k, T).\]
is called the relative Picard functor \( \mathcal{P}ic_{C/k} \) ([10] §8.1). This functor is representable by a \( k \)-group scheme \( \text{Pic}_{C/k} \) locally of finite type called the Picard scheme ([10] §8.2, Theorem 3]). The Picard scheme \( \text{Pic}_{C/k} \) decomposes to the connected components as
\[ \text{Pic}_{C/k} = \bigsqcup_{d \in \mathbb{Z}} \text{Pic}^d_{C/k}. \]

Each component \( \text{Pic}^d_{C/k} \) represents the line bundles on \( C \) of degree \( d \) ([10] §9.3, Theorem 1]). Especially, its identity component
\[ \text{Jac}(C) := \text{Pic}^0_{C/k} \subset \text{Pic}_{C/k} \]
is an abelian variety over \( k \) called the Jacobian variety. Its dimension is equal to the genus \( g(C) \) of \( C \) ([10] §9.2, Proposition 3]). Other components \( \text{Pic}^d_{C/k} \) are torsors under \( \text{Jac}(C) \).

4.2. Relative Brauer groups. For a scheme \( X \), let \( \text{Br}(X) \) denote the (absolute) Brauer group, the group of equivalence classes of Azumaya algebras over \( X \). For an affine scheme \( X = \text{Spec} \, R \), we write
\[ \text{Br}(R) := \text{Br}(\text{Spec} \, R). \]

If \( X \) is a quasi-projective scheme over a field \( k \), the Brauer group \( \text{Br}(X) \) is identified with the cohomological Brauer group, i.e., the torsion part of \( H^2_{\text{fppf}}(X, \mathbb{G}_m) = H^2_{\text{ét}}(X, \mathbb{G}_m) \). Since schemes appearing in this paper are all smooth and projective over a field \( k \), we identify these two groups. We note that, for a global field \( k \), the restriction morphisms yield an injective homomorphism
\[ \text{Br}(k) \to \bigoplus_v \text{Br}(k_v), \]
where \( v \) runs over all places of \( k \), by class field theory ([37, §9, §10]).

Definition 4.1. For a morphism of schemes \( X \to Y \), the relative Brauer group \( \text{Br}(X/Y) \) is defined by the kernel of the pullback homomorphism \( \text{Br}(Y) \to \text{Br}(X) \). We write
\[ \text{Br}(X/R) := \text{Br}(X/\text{Spec} \, R). \]
Let \( X \) be a smooth projective geometrically connected scheme over a field \( k \). By the five-term exact sequence of the Leray spectral sequence, we obtain the following exact sequence:

\[
0 \longrightarrow \text{Pic}(X) \longrightarrow \text{Pic}_{X/k}(k) \longrightarrow \delta_X \text{Br}(k) \longrightarrow \text{Br}(X).
\]

The map \( \text{Br}(k) \to \text{Br}(X) \) is injective if \( X \) has a \( k \)-rational point. By the definition of the relative Brauer group, we have the following short exact sequence:

\[
0 \longrightarrow \text{Pic}(X) \longrightarrow \text{Pic}_{X/k}(k) \longrightarrow \delta_X \text{Br}(X/k) \longrightarrow \text{Br}(X/k) \longrightarrow 0.
\]

The sequence (4.1) implies that, for a \( k \)-rational divisor class \( \alpha \in \text{Pic}_{X/k}(k) \), its image \( \delta_X(\alpha) \) in \( \text{Br}(X/k) \) is the obstruction to contain a \( k \)-rational divisor.

### 4.3. Exponents of relative Brauer groups

In this subsection, we treat exponents of relative Brauer groups. Let \( X \) be a smooth projective geometrically connected scheme over a field \( k \).

**Lemma 4.2.** Let \( L \) be a finite separable extension of \( k \), and assume that \( X \) has an \( L \)-rational point. Then the extension degree \( [L : k] \) annihilates the relative Brauer group \( \text{Br}(X/k) \); in other words, for an element \( \alpha \in \text{Br}(X/k) \), we have \( [L : k]\alpha = 0 \).

**Proof.** Let us consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Br}(k) & \xrightarrow{i} & \text{Br}(X) \\
\text{Res}_{L/k} & & \text{Res}_{X_L/X} \\
\text{Br}(L) & \xrightarrow{i_L} & \text{Br}(X_L) \\
\text{Cor}_{L/k} & & \text{Cor}_{X_L/X} \\
\text{Br}(k) & \xrightarrow{i} & \text{Br}(X).
\end{array}
\]

The horizontal map \( i_L \) in the second line is injective since \( X \) has an \( L \)-rational point. Take \( \alpha \in \text{Br}(k) \) and assume \( i_L(\alpha) = 0 \). Then we have

\[
i_L \circ \text{Res}_{L/k}(\alpha) = \text{Res}_{X_L/X} \circ i(\alpha) = 0,
\]

so \( \text{Res}_{L/k}(\alpha) = 0 \) since \( i_L \) is injective. This shows

\[
[L : k]\alpha = \text{Cor}_{L/k} \circ \text{Res}_{L/k}(\alpha) = 0.
\]

As an easy consequence of the above lemma, we obtain a sufficient condition under which the relative Brauer group vanishes.

**Corollary 4.3.** Assume that one of the following conditions is satisfied:

- \( \text{Br}(k) = 0 \), or
- \( X \) has a \( k \)-rational point, or
- \( k \) is a global field and \( X \) has a \( k_v \)-rational point for each place \( v \) of \( k \).

Then we have \( \text{Br}(X/k) = 0 \).

For our purposes, we specialize to the case of smooth plane cubics.

**Corollary 4.4.** Let \( C \subset \mathbb{P}^2 \) be a smooth plane cubic over a field \( k \). Then each element of \( \text{Br}(C/k) \) is killed by 3; in other words, for an element \( \alpha \in \text{Br}(C/k) \), we have \( 3\alpha = 0 \).

**Proof.** When \( k \) is a finite field, the Brauer group \( \text{Br}(k) \) vanishes. Hence \( \text{Br}(C/k) = 0 \). When \( k \) is infinite, there exists a line \( H \subset \mathbb{P}^2 \) such that \( C \cap H \) is an étale \( k \)-scheme of order 3 by Bertini’s theorem [30]. Since \( C \cap H \) is defined by a separable polynomial of degree 3 over \( k \), there is a closed point \( x \in C \cap H \) with \( [k(x) : k] = 1 \) or 3. Thus each element in \( \text{Br}(C/k) \) is killed by 3 by Lemma 4.3. \( \square \)
5. Linear determinantal representations and line bundles

In this section, we recall the definition of linear determinantal representations of smooth plane curves and their equivalences. Then we interpret the equivalence classes with isomorphism classes of certain line bundles.

Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d$ over a field $k$. The genus of the curve $C$ is equal to $g(C) = (d-1)(d-2)/2$. We fix projective coordinates $X_0, X_1, X_2$ of the projective plane $\mathbb{P}^2$.

**Definition 5.1.**

- A linear determinantal representation of $C$ is a square matrix $M$ of size $d$ with entries in $k$-linear forms in $X_0, X_1, X_2$ such that the equation
  $$(\det(M) = 0)$$
  is a defining equation of $C$.
- Two linear determinantal representations $M, M'$ of $C$ are said to be equivalent if we can write
  $$M' = AMB$$
  for some $A, B \in \text{GL}_d(k)$.

We say a line bundle on $C$ is non-effective if its only global section is the zero section. The following theorem gives an interpretation of equivalence classes of linear determinantal representations in terms of isomorphism classes of non-effective line bundles. It is well-known at least when $k$ is an algebraically closed field of characteristic 0 (\cite{25, 13, 24, 19, 25}).

**Theorem 5.2.** There is a natural bijection between the following sets:

- the set of equivalence classes of linear determinantal representations of $C$ over $k$, and
- the set of isomorphism classes of non-effective line bundles of degree $g(C) - 1$ on $C$.

**Proof.** We briefly sketch the proof. The following argument works over arbitrary fields $k$. We follow arguments of Beauville (\cite{2, Proposition 1.11}); see also \cite{25, Proposition 2.6}.

Recall that, for a coherent sheaf $F$ on $\mathbb{P}^2$, the following two conditions are equivalent (see \cite{25, Proposition 2.6}).

1. The sheaf $F$ has a minimal locally free resolution of the form
   $$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^r \to M \to \mathcal{O}_{\mathbb{P}^2}(-1)^r \to F \to 0$$
   for some $r \geq 1$.
2. The sheaf $F$ is arithmetically Cohen–Macaulay, and pure of dimension 1. It also satisfies
   $$H^0(\mathbb{P}^2, F) = H^1(\mathbb{P}^2, F) = 0.$$

When these conditions are satisfied, the sheaf $F$ is the push-forward of a coherent sheaf on the one-dimensional closed subscheme of $\mathbb{P}^2$ defined by $(\det(M) = 0)$.

We go back to the proof of Theorem 5.2. Let $\pi: C \hookrightarrow \mathbb{P}^2$ be a smooth plane curve of degree $d$, and $\mathcal{L}$ a line bundle on $C$. Then the push-forward
$$F := \pi_* \mathcal{L}$$
is arithmetically Cohen–Macaulay, pure of dimension 1 and $\text{Supp} F = C$. If $\mathcal{L}$ is non-effective of degree $g(C) - 1$, we have
$$H^0(\mathbb{P}^2, F) = H^1(\mathbb{P}^2, F) = 0$$
by the Riemann–Roch theorem for $C$. Hence $F$ satisfies \cite{2}, and $F$ has a minimal locally free resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^r \to M \to \mathcal{O}_{\mathbb{P}^2}(-1)^r \to F \to 0$$
for some \( r \geq 1 \). Therefore the equation \((\det(M) = 0)\) defines \( C \) and we have \( r = d \). Hence \( M \) gives a linear determinantal representation of \( C \) over \( k \). A straightforward calculation shows that isomorphic line bundles give equivalent \( M \)'s, and non-isomorphic line bundles give non-equivalent \( M \)'s.

Conversely, a linear determinantal representation \( M \) of \( C \) gives a coherent sheaf \( \mathcal{F} \) on \( \mathbb{P}^2 \) by the short exact sequence \((5.1)\). Hence \( \mathcal{F} \) satisfies the condition \((I)\), and we can identify \( \mathcal{F} \) with the push-forward of a coherent sheaf \( \mathcal{L} \) on \( C \). Since \( \mathcal{F} \) is arithmetically Cohen–Macaulay, \( \mathcal{L} \) is a vector bundle on \( C \). By computing the Hilbert polynomial of \( \mathcal{L} \), we have

\[
\text{length}_{\mathcal{O}_{C,\eta}}(\mathcal{L}_\eta) = 1
\]

for the generic point \( \eta \) of \( C \). Hence \( \mathcal{L} \) is a line bundle on \( C \). By the vanishing of cohomology

\[
H^0(C, \mathcal{L}) = H^1(C, \mathcal{L}) = 0,
\]

we obtain that \( \mathcal{L} \) is non-effective of degree \( g(C) - 1 \) by the Riemann–Roch theorem. It is straightforward to prove that equivalent linear determinantal representations give isomorphic line bundles. This finishes the proof of Theorem 5.2. \( \square \)

6. Linear determinantal representations of lines and conics

As a simple application of Theorem 5.2 let us treat the case of lines and conics.

6.1. Lines. Let \( C \subset \mathbb{P}^2 \) be a line defined over a field \( k \). It is defined by a \( k \)-linear form \( l(X_0, X_1, X_2) \). The linear form \( l \) considered as a square matrix of size one gives a unique equivalence class of linear determinantal representations of \( C \).

Let us reconsider this from the viewpoint of Theorem 5.2. We shall find a non-effective line bundle of degree \( g(C) - 1 = -1 \) on \( C \). The degree homomorphism gives an isomorphism

\[
\text{deg}: \text{Pic}(C) \xrightarrow{\sim} \mathbb{Z}; \ [\mathcal{L}] \mapsto \text{deg} \mathcal{L}.
\]

Let \( P \in C(k) \) be a \( k \)-rational point. The line bundle \( \mathcal{O}_C(-P) \) is non-effective of degree \(-1\), so there is a unique equivalence class of linear determinantal representations over \( k \) corresponding to \( \mathcal{O}_C(-P) \).

6.2. Conics. Next we treat the case of smooth conics. Compare the following proposition with [26, Proposition 4.1], where similar results are proved for symmetric determinantal representations.

**Proposition 6.1.** Let \( C \subset \mathbb{P}^2 \) be a smooth plane conic defined over a field \( k \). The following conditions are equivalent.

(i) The conic \( C \) admits a linear determinantal representation over \( k \).

(ii) The conic \( C \) has a \( k \)-rational point.

(iii) The conic \( C \) is isomorphic to \( \mathbb{P}^1 \) over \( k \).

(iv) The relative Brauer group \( \text{Br}(C/k) \) vanishes.

When these conditions are satisfied, the conic \( C \) has a unique equivalence class of linear determinantal representations over \( k \).

**Proof.** We first assume \( (i) \). By Theorem 5.2, there exists a non-effective line bundle \( \mathcal{L} \) on \( C \) of degree \(-1\). Then \( \mathcal{L}^{-1} \) is a line bundle of degree one. The complete linear system \( |\mathcal{L}^{-1}| \) gives a \( k \)-isomorphism \( C \xrightarrow{\sim} \mathbb{P}^1 \). This shows \( (i) \Rightarrow (iii) \).

The implication \( (iii) \Rightarrow (ii) \) is obvious, and \( (ii) \Rightarrow (iii) \) follows from Corollary 6.3.
Since the action of the absolute Galois group $\text{Gal}(k^s/k)$ does not change the degree of line bundles, we have

$$\text{Pic}_{C/k}(k) = \text{Pic}_{C/k}(k^s)^{\text{Gal}(k^s/k)}$$

$$= \text{Pic}_{C/k}(k^s)$$

$$\cong \mathbb{Z}.$$

In particular, the degree minus one part $\text{Pic}^{-1}_{C/k}(k)$ is a singleton. Hence if we assume (iii), we obtain that $\text{Pic}^{-1}(C) \cong \text{Pic}^{-1}_{C/k}(k)$ is a singleton and $C$ has a unique equivalence class of linear determinantal representations over $k$ by Theorem 7.2. This shows the implication (iii) $\Rightarrow$ (i) and the last assertion. 

As seen in the above proof, the relative Brauer group $\text{Br}(C/k)$ is generated by a single element in $\delta_C([\text{Pic}^1_{C/k}(k)])$ where $\delta_C : \text{Pic}_{C/k}(k) \to \text{Br}(C/k)$ is the map appearing in (11). By class field theory, we have the following corollary. Compare it with [26, Theorem 5.1 (1)].

**Corollary 6.2.** Let $C \subset \mathbb{P}^2$ be a smooth plane conic over a global field $k$. Then $C$ admits a linear determinantal representation over $k$ if and only if $C$ admits a linear determinantal representation over each local field $k_v$ of $k$.

**Proof.** By Proposition 6.1, $C$ admits a linear determinantal representation over $k$ if and only if the relative Brauer group $\text{Br}(C/k)$ vanishes. By the above observation, $\text{Br}(C/k)$ is generated by a single element

$$\alpha_C \in \delta_C([\text{Pic}^1_{C/k}(k)]).$$

If $C$ admits a linear determinantal representation over each local field $k_v$ of $k$, the image of $\alpha_C$ in $\text{Br}(k_v)$ is zero for each place $v$ of $k$. By class field theory, $\alpha_C$ vanishes in $\text{Br}(k)$ ([37 §9, §10]). Hence $C$ admits a linear determinantal representation over $k$. 

**Example 6.3 (See [25]).** Let $C \subset \mathbb{P}^2$ be a smooth conic over a field $k$ with a $k$-rational point $P \in C(k)$. By changing the projective coordinates, we may assume $P = [0:0:1]$ and

$$C = (X_0X_2 - X_1^2 = 0).$$

The conic $C$ has a unique equivalence class of linear determinantal representations over $k$ given by the matrix

$$M = \begin{pmatrix} X_0 & X_1 \\ X_1 & X_2 \end{pmatrix}.$$ 

Since $M$ is a symmetric matrix, we find any linear determinantal representation of $C \subset \mathbb{P}^2$ is equivalent to a symmetric determinantal representation.

**7. LINEAR DETERMINANTAL REPRESENTATIONS OF CUBICS**

In this section, we study linear determinantal representations of smooth plane cubics over arbitrary fields.

**Proposition 7.1.** Let $C \subset \mathbb{P}^2$ be a smooth plane cubic over a field $k$.

(i) If $C$ admits a linear determinantal representation over $k$, then $\text{Jac}(C)(k) \neq 0$.

(ii) If the relative Brauer group $\text{Br}(C/k)$ vanishes and $\text{Jac}(C)(k) \neq 0$, then $C$ admits a linear determinantal representation over $k$.

**Proof.** (i) Let $\mathcal{L}$ be a line bundle on $C$ corresponding to a linear determinantal representation of $C$ over $k$ by Theorem 7.2. The line bundle $\mathcal{L}$ is non-effective of degree 0. The image $[\mathcal{L}]$ of $\mathcal{L}$ in $\text{Jac}(C)(k)$ is a non-trivial $k$-rational point on $\text{Jac}(C)$.

(ii) Since $\text{Br}(C/k) = 0$, by the short exact sequence ([4, 7]), we have an isomorphism

$$\text{Pic}(C) \cong \text{Pic}_{C/k}(k).$$
Hence each non-trivial \( k \)-rational point of \( \text{Jac}(C) \) corresponds to a non-trivial line bundle \( \mathcal{L} \) on \( C \) of degree 0 defined over \( k \). The line bundle \( \mathcal{L} \) is non-effective, hence it gives a linear determinantal representation of \( C \) by Theorem 7.2.

**Corollary 7.2.** Assume that one of the following conditions is satisfied:
- \( \text{Br}(k) = 0 \), or
- \( C \) has a \( k \)-rational point, or
- \( k \) is a global field and \( X \) has a \( k_v \)-rational point for each place \( v \) of \( k \).

Then \( C \) admits a linear determinantal representation over \( k \) if and only if \( \text{Jac}(C)(k) \neq \emptyset \).

**Proof.** The assertion follows from Corollary 4.3 and Proposition 7.1 (ii).

**Remark 7.3.** If \( C \) has no \( k \)-rational point, there can be a difference between the set of equivalence classes of linear determinantal representations over \( k \) and the set \( \text{Jac}(C)(k) \backslash \{0\} \). It turns out that there are distinct smooth plane cubics \( C, C' \) over \( \mathbb{Q} \) satisfying the following conditions:
- \( \text{Jac}(C) \cong \text{Jac}(C') \), and
- \( C \) admits a linear determinantal representation over \( \mathbb{Q} \), and
- \( C' \) does not admit a linear determinantal representation over \( \mathbb{Q} \).

For explicit examples, see Example 7.5.

**Corollary 7.4.** Let \( C \subset \mathbb{P}^2 \) be a smooth plane cubic over a field \( k \). If there is a \( k \)-rational point \( P \in \text{Jac}(C)(k) \) with \([3]P \neq 0\), then the cubic \( C \) admits a linear determinantal representation over \( k \).

**Proof.** We have
\[
\delta_C([3]P) = 3\delta_C(P) = 0
\]
in the relative Brauer group \( \text{Br}(C/k) \) by Corollary 4.4. Hence \([3]P \) comes from a non-trivial line bundle \( \mathcal{L} \) on \( C \) of degree 0 by the short exact sequence 4.7. The line bundle \( \mathcal{L} \) corresponds to a linear determinantal representation of \( C \) over \( k \) by Theorem 5.2.

**Corollary 7.5.** Let \( C \subset \mathbb{P}^2 \) be a smooth plane cubic over a field \( k \) with \( \#C(k) \geq 2 \). Then the cubic \( C \) admits a linear determinantal representation over \( k \).

**Proof.** Since \( C \) has a \( k \)-rational point, the existence of a linear determinantal representation of \( C \) over \( k \) is equivalent to \( \text{Jac}(C)(k) \neq \emptyset \) by Corollary 7.2. For two distinct \( k \)-rational points \( P \neq Q \) on \( C \), the line bundle \( \mathcal{O}_C(P - Q) \) gives a non-zero \( k \)-rational point on \( \text{Jac}(C) \).

In the rest of this section, we exhibit some examples of linear determinantal representations of smooth plane cubics.

**Example 7.6.** Let \( k \) be a field of characteristic not equal to 2 nor 3 and
\[
E: (X_1^2X_2 - X_0^3 - aX_0X_2^2 - bX_2^3 = 0) \subset \mathbb{P}^2
\]
an elliptic curve over \( k \) with origin \( O = [0 : 1 : 0] \) defined by a Weierstrass equation. By Theorem 5.2 and Proposition 7.1, there is a bijection between \( E(k) \backslash \{O\} \) and the set of equivalence classes of linear determinantal representations of \( E \) over \( k \). This can be given explicitly as follows. For a \( k \)-rational point
\[
P = [\lambda : \mu : 1] \in E(k) \backslash \{O\},
\]
the matrix
\[
M_P := \begin{pmatrix}
X_0 - \lambda X_2 & 0 & -X_1 - \mu X_2 \\
-X_1 + \mu X_2 & X_0 + \lambda X_2 & (a + \lambda^2)X_2 \\
0 & X_2 & -X_0
\end{pmatrix}
\]
gives a linear determinantal representation of \( E \) over \( k \). Galinat proved in [22] that they give all distinct representatives of equivalence classes of linear determinantal representations of \( E \).
over \( k \). See \cite{ES} for other representatives when \( k \) is algebraically closed of characteristic not equal to 2 nor 3.

**Example 7.7.** Let \( k \) be a field of characteristic not equal to 2 nor 3 and
\[
C: (X_0^3 + X_1^3 + X_2^3 + \lambda X_0 X_1 X_2 = 0) \subset \mathbb{P}^2
\]
a smooth plane cubic over \( k \) defined by Hesse’s normal form. Let
\[
P = [a_0 : a_1 : a_2] \in C(k)
\]
be a \( k \)-rational point with \( a_0 a_1 a_2 \neq 0 \). Then the Moore matrix
\[
M_P := \begin{pmatrix}
a_0 X_0 & a_1 X_2 & a_2 X_1 \\
a_1 X_1 & a_2 X_0 & a_0 X_2 \\
a_2 X_2 & a_0 X_1 & a_1 X_0
\end{pmatrix}
\]
gives a linear determinantal representation of \( C \) over \( k \). Buchweitz and Pavlov proved in \cite{BuP} Theorem A] that two Moore matrices \( M_P, M_{P'} \) are equivalent if and only if two divisors \( 3P \) and \( 3P' \) on \( C \) are linearly equivalent. Moreover, if \( k \) is algebraically closed, any equivalence class of linear determinantal representations of \( C \) over \( k \) can be represented by a Moore matrix \( M_P \) corresponding to a point \( P = [a_0 : a_1 : a_2] \in C(k) \) with \( a_0 a_1 a_2 \neq 0 \).

8. Linear Determinantal Representations of Plane Cubics over Large Fields

In this section, we prove that any smooth cubic over a large field admits a linear determinantal representation. Let us recall the notion of large fields introduced by F. Pop.

**Definition 8.1** \((\cite{P1} \cite{P2})\). A large field \( k \) is a field having the following property: for any smooth algebraic curve \( C \) over \( k \) with a \( k \)-rational point, \( C \) has infinitely many \( k \)-rational points.

Large fields are also called ample fields \((\cite{P2})\). It is known that the class of large fields is quite rich, and it contains many interesting fields. For example, local fields, i.e. finite extensions of \( \mathbb{Q}_p, \mathbb{F}_p((t)) \) or \( \mathbb{R} \), are known to be large as well as \( k((x,y)) = \text{Frac} \; k[[x,y]] \) for any field \( k \).

**Theorem 8.2.** Let \( C \subset \mathbb{P}^2 \) be a smooth plane cubic over a large field \( k \). Then \( C \) has infinitely many equivalence classes of linear determinantal representations over \( k \).

**Proof.** Since \( k \) is large, the Jacobian variety \( \text{Jac}(C) \) has infinitely many \( k \)-rational points. Hence \cite{C} \( \text{Jac}(C)(k) \) is an infinite set. By Corollary \cite{C} each element in \cite{C} \( \text{Jac}(C)(k) \) has the vanishing image under the homomorphism \( \delta_C \), so any element in \cite{C} \( \text{Jac}(C)(k) \) comes from a line bundle of degree 0 on \( C \) defined over \( k \). By Corollary \cite{C} any element \( \alpha \in \text{Jac}(C)(k) \) with \( \alpha \neq 0 \) gives an equivalence class of linear determinantal representations of \( C \) over \( k \).

**Corollary 8.3.** Any smooth plane cubic over a local field \( k \) has infinitely many equivalence classes of linear determinantal representations over \( k \).

When \( k \) is a \( p \)-adic field, Corollary \cite{C} was proved by Deajim–Grant (see \cite{DG} Proposition 4.1]).

**Remark 8.4.** For a local field \( k \), it is possible to prove \( \# \text{Jac}(C)(k) = \infty \) directly. It is obvious for \( k = \mathbb{R} \) or \( \mathbb{C} \). When \( k \) is non-archimedean, it can be proved using formal group laws (see \cite{C} Chapter VII, Proposition 2.2] and \cite{DG} Proposition 4.1]).

**Remark 8.5.** If the degree of a smooth plane curve \( C \subset \mathbb{P}^2 \) is different from 3, Theorem \cite{C} need not be true; in fact, it does not hold for smooth conics. The Severi–Brauer variety of a quaternion division algebra over a local field \( k \) is a smooth conic without \( k \)-rational points.
Hence it does not admit a linear determinantal representation by Proposition 6.1. For a concrete example, the conic

\[ C = (X_0^2 + X_1^2 + X_2^2 = 0) \]

over \( \mathbb{R} \) does not have an \( \mathbb{R} \)-rational point. Hence it does not admit a linear determinantal representation over \( \mathbb{R} \) by Proposition 6.1.

**Remark 8.6.** Smooth plane cubics over a large field need not admit symmetric determinantal representations. In fact, they correspond to a special class of line bundles called non-effective theta characteristics (see [2, Proposition 4.2] and [25, Theorem 1.1]). They consist a finite subscheme of \( \text{Jac}(C) \) over \( k \). Thus the largeness of the base field \( k \) does not assure the existence of non-effective theta characteristics over \( k \).

9. Applications to the Local-Global Principle and Proof of Theorem 1.1

As applications of the results in the previous section, we give a sufficient condition for a smooth plane cubic over a global field to fail the local-global principle for the existence of linear determinantal representations.

**Definition 9.1.** A smooth plane cubic \( C \subset \mathbb{P}^2 \) over a global field \( k \) is said to fail the local-global principle for the existence of linear determinantal representations if it satisfies the following two conditions:

- the cubic \( C \) admits a linear determinantal representation over each local field \( k_v \) of \( k \), but
- the cubic \( C \) does not admit a linear determinantal representation over \( k \).

If \( C \) does not fail the local-global principle, we say that \( C \) satisfies the local-global principle for the existence of linear determinantal representations.

Note that any smooth plane cubic \( C \subset \mathbb{P}^2 \) over a global field \( k \) always admits a linear determinantal representation over each local field \( k_v \) by Corollary 8.3. Hence we have the following lemma.

**Lemma 9.2.** A smooth plane cubic \( C \subset \mathbb{P}^2 \) over a global field \( k \) satisfies (resp. fails) the local-global principle for the existence of linear determinantal representations if and only if \( C \) admits (resp. does not admit) a linear determinantal representation over \( k \).

**Lemma 9.3.** Let \( k \) be a global field. Any smooth plane cubic \( C \) over \( k \) with \( \text{Jac}(C)(k) = 0 \) fails the local-global principle for the existence of linear determinantal representations.

**Proof.** By Proposition 7.1 (i), the cubic \( C \) does not admit a linear determinantal representation over \( k \). Hence it fails the local-global principle for the existence of linear determinantal representations by Lemma 9.2. \( \square \)

**Remark 9.4.** Lemma 9.3 is in contrast with the results in [26] and [27]. In those papers, it is proved that any smooth plane cubic over a global field satisfies the local-global principle for the existence of symmetric determinantal representations (see [26, Theorem 1.1] for the case of characteristic not equal to two and [27, Theorem 1.2] for the case of characteristic two).

By Lemma 9.3, it is easy to construct examples of smooth plane cubics which fail the local-global principle as follows. There are infinitely many explicit examples of elliptic curves over \( \mathbb{Q} \) with trivial Mordell–Weil group; for example, see [32]. Since elliptic curves can be considered as smooth plane cubics, we can construct infinitely many smooth plane cubics over \( \mathbb{Q} \) which fail the local-global principle for the existence of linear determinantal representations.

In general, the condition “\( \text{Jac}(C)(k) = 0 \)” is not a necessary condition for a smooth plane cubic \( C \) over a global field \( k \) to fail the local-global principle for the existence of linear determinantal representations. We conclude this section with examples of smooth plane cubics \( C \) over \( \mathbb{Q} \) such that \( \text{Jac}(C)(\mathbb{Q}) \neq 0 \) but \( C \) does not admit a linear determinantal representation over \( \mathbb{Q} \).
Example 9.5. Let us give infinitely many examples of smooth plane cubics \( C \) over \( \mathbb{Q} \) such that

- the cubic \( C \) does not admit a linear determinantal representation over \( \mathbb{Q} \), and
- the Mordell–Weil group \( \text{Jac}(C)(\mathbb{Q}) \) is non-trivial, and
- the relative Brauer group \( \text{Br}(C/\mathbb{Q}) \) is non-trivial.

Let \( \mathbb{Q}(u) \) be a cyclic extension of \( \mathbb{Q} \) of degree 3 generated by a root \( u \) of the equation
\[
u^3 - 9u + 9 = 0,
\]
and \( \tau \) a generator of the Galois group \( \text{Gal}(\mathbb{Q}(u)/\mathbb{Q}) \). Let us take a prime number \( p \) which is inert in \( \mathbb{Q}(u) \). For example, we can take \( p = 2 \) or 5. There are infinitely many such primes \( p \) by Chebotarev’s density theorem [31, Theorem 13.4]. Consider the following smooth plane cubic over \( \mathbb{Q} \):
\[
C_p: (pX_0^3 + p^2X_1^3 - X_2^3 = 0) \subset \mathbb{P}^2.
\]
The Jacobian variety \( E_p = \text{Jac}(C_p) \) of \( C_p \) has a Weierstrass equation
\[
E_p: \left( X_1^2X_2 - X_3^3 + \frac{27p^6}{4}X_2^3 = 0 \right)
\]
([23 §4]), and is isomorphic to the elliptic curve of Cremona label 27a1 ([13])
\[
E: \left( X_1^2X_2 - X_3^3 + \frac{27}{4}X_2^3 = 0 \right)
\]
independently of the choice of \( p \). The Mordell–Weil group \( E_p(\mathbb{Q}) \) is isomorphic to \( \mathbb{Z}/3\mathbb{Z} \), and generated by \( [6p^2 : -9p^3 : 2] \in E_p(\mathbb{Q}) \). Let us calculate the relative Brauer group \( \text{Br}(C_p/\mathbb{Q}) \) of \( C_p \). We apply [23 Proposition 4.4] with constants \( a = p, b = p^2, c = -27p^6/4, r = 3p^2, s = -9p^3/2 \) and
\[
\xi = -\frac{9p^3}{2} + \sqrt{\frac{-27p^6}{4}} = -9 + 3\sqrt{-3} \cdot p^3.
\]
Then the relative Brauer group \( \text{Br}(C_p/\mathbb{Q}) \) is generated by the class
\[
\delta_{C_p}([6p^2 : -9p^3 : 2]) = [(\mathbb{Q}(u)/\mathbb{Q}, \tau, p)] \in \text{Br}(\mathbb{Q})[3].
\]
Since the prime \( p \) is inert in \( \mathbb{Q}(u) \), the ideal \( p\mathcal{O}_{\mathbb{Q}(u)} \) is a prime ideal, and the central simple algebra \( (\mathbb{Q}(u)/\mathbb{Q}, \tau, p) \) is a division algebra over \( \mathbb{Q} \) ramified at \( p \). Hence the class
\[
[(\mathbb{Q}(u)/\mathbb{Q}, \tau, p)] \in \text{Br}(\mathbb{Q})
\]
is non-trivial. We conclude
\[
\text{Br}(C_p/\mathbb{Q}) = \langle \delta_{C_p}([6p^2 : -9p^3 : 2]) \rangle = \langle [(\mathbb{Q}(u)/\mathbb{Q}, \tau, p)] \rangle \cong \mathbb{Z}/3\mathbb{Z}.
\]
By the short exact sequence [4.7], we have \( \text{Pic}^0(C_p) = 0 \) and there is no non-effective line bundle of degree 0 on \( C_p \) defined over \( \mathbb{Q} \). By Theorem 6.2, the cubic \( C_p \) does not admit a linear determinantal representation over \( \mathbb{Q} \). Because the elements
\[
[(\mathbb{Q}(u)/\mathbb{Q}, \tau, p)] \in \text{Br}(\mathbb{Q})
\]
are distinct for different choices of primes \( p \), the cubics \( C_p \) are not isomorphic to each other over \( \mathbb{Q} \).
Remark 9.6. The smooth plane cubics $C_p$ in Example 9.5 show another phenomenon stated in Remark 7.3. Smooth plane cubics $C_p$ and the elliptic curve $E$ have isomorphic Jacobian varieties, the cubics $C_p$ do not admit linear determinantal representations over $\mathbb{Q}$. But the elliptic curve $E$ admits a linear determinantal representation over $\mathbb{Q}$ by Corollary 7.4.

Finally, we prove Theorem 1.1 using Theorem 3.5.

Proof (Proof of Theorem 1.1). (i) By Theorem 3.5 (i), there are a positive proportion of smooth plane cubics over $\mathbb{Q}$ whose Jacobian varieties have Mordell–Weil rank 1. By Corollary 7.4, they admit linear determinantal representations over $\mathbb{Q}$.

(ii) By Theorem 3.5 (ii), there are a positive proportion of smooth plane cubics over $\mathbb{Q}$ whose Jacobian varieties have trivial Mordell–Weil group. By Lemma 9.3, they fail the local-global principle for the existence of linear determinantal representations.

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