Proof of a Conjecture of Hirschhorn and Sellers on Overpartitions

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Abstract. Let $\bar{p}(n)$ denote the number of overpartitions of $n$. It was conjectured by Hirschhorn and Sellers that $\bar{p}(40n + 35) \equiv 0 \pmod{40}$ for $n \geq 0$. Employing 2-dissection formulas of quotients of theta functions due to Ramanujan, and Hirschhorn and Sellers, we obtain a generating function for $\bar{p}(40n + 35)$ modulo 5. Using the $(p, k)$-parametrization of theta functions given by Alaca, Alaca and Williams, we give a proof of the congruence $\bar{p}(40n + 35) \equiv 0 \pmod{5}$. Combining this congruence and the congruence $\bar{p}(4n + 3) \equiv 0 \pmod{8}$ obtained by Hirschhorn and Sellers, and Fortin, Jacob and Mathieu, we give a proof of the conjecture of Hirschhorn and Sellers.

Keywords: overpartition, congruence, theta function, $(p, k)$-parametrization.

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1 Introduction

The objective of this paper is to give a proof of a conjecture of Hirschhorn and Sellers on the number of overpartitons. We shall use the technique of dissections of quotients of theta functions.

Let us begin with some notation and terminology on $q$-series and partitions. We adopt the common notation
\begin{equation}
(a; q)_\infty = \prod_{n=0}^{\infty} (1 - a q^n),
\end{equation}
where $|q| < 1$, and we write
\begin{equation}
(a_1, a_2, \ldots, a_n; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_n; q)_\infty.
\end{equation}
Recall that the Ramanujan theta function $f(a, b)$ is defined by
\begin{equation}
f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2},
\end{equation}
This conclusion is a development of the original paper published in the Journal of Number Theory.
where $|ab| < 1$. The Jacobi triple product identity can be restated as

$$f(a, b) = (-a, -b, ab; ab)_{\infty}. \quad (1.4)$$

Here is a special case of (1.3), namely,

$$f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}. \quad (1.5)$$

For any positive integer $n$, we use $f_n$ to denote $f(-q^n)$, that is,

$$f_n = (q^n; q^n)_{\infty} = \prod_{k=1}^{\infty} (1 - q^{nk}). \quad (1.6)$$

The function $f_n$ is related to the generating function of overpartitions. A partition of a positive integer $n$ is any nonincreasing sequence of positive integers whose sum is $n$. An overpartition of $n$ is a partition in which the first occurrence of a number may be overlined, see Corteel and Lovejoy [4]. For $n \geq 1$, let $\bar{p}(n)$ denote the number of overpartitions of $n$, and we set $\bar{p}(0) = 1$. Corteel and Lovejoy [4] showed that the generating function for $\bar{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \bar{p}(n)q^n = \frac{f_2 f_2}{f_4}, \quad (1.7)$$

Hirschhorn and Sellers [6], and Fortin, Jacob and Mathieu [5] obtained the following Ramanujan-type generating function formulas for $\bar{p}(2n+1)$, $\bar{p}(4n+3)$, and $\bar{p}(8n+7)$:

$$\sum_{n=0}^{\infty} \bar{p}(2n+1)q^n = 2 \frac{f_2^2 f_2^2}{f_4 f_8}, \quad (1.8)$$

$$\sum_{n=0}^{\infty} \bar{p}(4n+3)q^n = 8 \frac{f_2 f_6}{f_8}, \quad (1.9)$$

$$\sum_{n=0}^{\infty} \bar{p}(8n+7)q^n = 64 \frac{f_2^2 f_2^{22}}{f_1^{23}}. \quad (1.10)$$

The above identities lead to congruences modulo 2, 8 and 64 for the overpartition function. Mahlburg [8] proved that $\bar{p}(n)$ is divisible by 64 for almost all $n$ by using relations between $\bar{p}(n)$ and the number of representations of $n$ as a sum of squares. Using the theory of modular forms, Treneer [9] showed that the coefficients of a wide class of weakly holomorphic modular forms have infinitely many congruence relations for powers of every prime $p$ other than 2 and 3. In particular, Treneer [9] proved that $\bar{p}(5m^3n) \equiv 0 \pmod{5}$ for any $n$ that is coprime to $m$, where $m$ is a prime satisfying $m \equiv -1 \pmod{5}$.

The following conjecture was posed by Hirschhorn and Sellers [6].
Conjecture 1.1 For \( n \geq 0 \), we have
\[
\bar{p}(40n + 35) \equiv 0 \pmod{40}.
\] (1.11)

This paper is organized as follows. In Section 2, using 2-dissection formulas of quotients of theta functions given by Ramanujan [3], and Hirschhorn and Sellers [7], we derive a generating function for \( \bar{p}(40n + 35) \) modulo 5. In Section 3, we use the \((p, k)\)-parametrization of theta functions due to Alaca, Alaca and Williams [1, 2, 10] to show that \( \bar{p}(40n + 35) \equiv 0 \pmod{5} \). This proves the conjecture of Hirschhorn and Sellers resorting to the congruence \( \bar{p}(4n + 3) \equiv 0 \pmod{8} \) independently obtained by Hirschhorn and Sellers [6], and Fortin, Jacob and Mathieu [5].

2 Generating function of \( \bar{p}(40n + 35) \) modulo 5

In this Section, we deduce a generating function of \( \bar{p}(40n + 35) \) modulo 5. We first recall several 2-dissection formulas for quotients of theta functions due to Ramanujan [3], Hirschhorn and Sellers [7].

The following relations are consequences of dissection formulas of Ramanujan collected in Entry 25 in Berndt’s book [3, p. 40].

Lemma 2.1 Let \( f_n \) be given by (1.6). We have
\[
\frac{f_1^2}{f_2 f_4} = \frac{f_2 f_8}{f_2 f_4} - 2q \frac{f_2 f_1^2}{f_8}, \quad (2.1)
\]
\[
\frac{1}{f_1^2} = \frac{f_8^5}{f_2 f_4} + 2q \frac{f_2^2 f_1^2}{f_8 f_4}, \quad (2.2)
\]
\[
f_1^4 = \frac{f_4^{10}}{f_2^4 f_8} - 4q \frac{f_2^2 f_4^4}{f_4^2}, \quad (2.3)
\]
and
\[
\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^4 f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}, \quad (2.4)
\]

Hirschhorn and Sellers [7] established the following 2-dissection formula.

Lemma 2.2 Let \( f_n \) be given by (1.6). We have
\[
\frac{f_5}{f_1} = \frac{f_8 f_5}{f_2 f_4} + q \frac{f_2^3 f_10 f_40}{f_2^3 f_8 f_20}, \quad (2.5)
\]
By Lemmas 2.1 and 2.2, we are led to a generating function of \( \bar{p}(40n + 35) \) modulo 5.

**Theorem 2.3** We have

\[
\sum_{n=0}^{\infty} \bar{p}(40n + 35)q^n \equiv 2f_2^{122}f_1^{68}f_4^{20} + 3f_1^{26}f_2^{26}f_4^8 + 4qf_1^{98}f_2^8f_4^8 + 4q^2f_1^{74}f_2^{40}f_4^8 + 4q^3f_1^{50}f_2^{8}f_4^8 + \quad \text{(mod 5)}.
\]

(2.6)

**Proof.** Recall that the well-known theta function \( \varphi(q) \) is defined by

\[
\varphi(q) = f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2}.
\]

(2.7)

By the Jacobi triple product identity, we find

\[
\varphi(q) = \frac{f_2^5}{f_1^2f_4^2}.
\]

(2.8)

In view of (1.3), (1.4) and (2.7), we see that

\[
\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \sum_{n=-\infty}^{\infty} q^{25n^2} + 2q \sum_{n=-\infty}^{\infty} q^{25n^2+10n} + 2q^4 \sum_{n=-\infty}^{\infty} q^{25n^2+20n}
\]

\[
= \varphi(q^{25}) + 2qD(q^5) + 2q^4E(q^5),
\]

(2.9)

where

\[
D(q) = \sum_{n=-\infty}^{\infty} q^{5n^2+2n} = (-q^3, -q^7, q^{10}; q^{10})_\infty
\]

(2.10)

and

\[
E(q) = \sum_{n=-\infty}^{\infty} q^{5n^2+4n} = (-q, -q^9, q^{10}; q^{10})_\infty.
\]

(2.11)

It is easily checked that

\[
D(q)E(q) = \frac{f_2^{2}f_5f_{20}}{f_1f_4}.
\]

(2.12)

Replacing \( q \) by \(-q\) in (2.8), and using the fact that

\[
f(q) = (-q; -q)_\infty = \frac{f_2^3}{f_1f_4},
\]

(2.13)
we deduce that

\[ \varphi(-q) = \frac{f_2^2}{f_2}. \quad (2.14) \]

Because of (2.14), the generating function (1.7) of \( \bar{p}(n) \) can be rewritten as

\[ \sum_{n=0}^{\infty} \bar{p}(n)(-q)^n = \frac{1}{\varphi(q)}. \quad (2.15) \]

It follows that

\[ \sum_{n=0}^{\infty} \bar{p}(n)(-q)^n = \frac{\varphi^4(q)}{\varphi^5(q)}. \quad (2.16) \]

By the binomial theorem, it is easily seen that for \( k \geq 1 \),

\[ (1 - q^k)^5 \equiv (1 - q^{5k}) \pmod{5}, \quad (2.17) \]

which implies that

\[ \varphi^5(q) \equiv \varphi(q^5) \pmod{5}. \quad (2.18) \]

Combining (2.16) and (2.18), we find that

\[ \sum_{n=0}^{\infty} \bar{p}(n)(-q)^n \equiv \frac{\varphi^4(q)}{\varphi(q^5)} \pmod{5}. \quad (2.19) \]

Substituting (2.9) into (2.19), we get

\[
\sum_{n=0}^{\infty} \bar{p}(n)(-q)^n \equiv \frac{(\varphi(q^{25}) + 2qD(q^5) + 2q^4E(q^5))^4}{\varphi(q^5)}
\]

\[
\equiv \frac{1}{\varphi(q^5)}(\varphi^4(q^{25}) + 3q\varphi^3(q^{25})D(q^5) + 4q^2\varphi^2(q^{25})D^2(q^5) + 2q^3\varphi(q^{25})D^3(q^5)
\]

\[
+ 3q^4\varphi^3(q^{25})E(q^5) + q^4D^4(q^5) + 3q^5\varphi^2(q^{25})D(q^5)D(q^5)
\]

\[
+ q^6\varphi(q^{25})D^2(q^5)E(q^5) + 4q^7D^3(q^5)E(q^5) + 4q^8\varphi^2(q^{25})E^2(q^5)
\]

\[
+ q^9\varphi(q^{25})D(q^5)E^2(q^5) + q^{10}D^2(q^5)E^2(q^5) + 2q^{12}\varphi(q^{25})E^3(q^5)
\]

\[
+ 4q^{13}D(q^5)E^3(q^5) + q^{16}E^4(q^5)) \pmod{5}. \quad (2.20)
\]

Extracting those terms associated with powers \( q^{5n} \) on both sides of (2.20) and replacing \( q^5 \) by \( q \), we obtain

\[
\sum_{n=0}^{\infty} \bar{p}(5n)(-q)^n \equiv \frac{\varphi^4(q^5) + 3q\varphi^2(q^5)D(q)E(q) + q^2D^2(q)E^2(q)}{\varphi(q)} \pmod{5}. \quad (2.21)
\]
By (2.8) and (2.12), we can rewrite (2.21) as follows

\[
\sum_{n=0}^{\infty} \bar{p}(5n)(-q)^n \equiv \frac{f_2^2 f_{10}^5 f_{20}^5}{f_2^2 f_{10}^5 f_{20}^5 f_8^5 f_8^5} + 3q f_1 f_4 f_{10}^5 f_{20}^5 f_8^5 f_8^5 + q^2 \frac{f_2^5 f_{20}^5}{f_2^2 f_{10}^5 f_{20}^5} \quad (\text{mod } 5). \quad (2.22)
\]

Replacing \( q \) by \(-q\) in (2.22), we get

\[
\sum_{n=0}^{\infty} \bar{p}(5n)q^n \equiv \frac{f_2^2 f_{10}^5 f_{20}^5 f_8^5 f_8^5}{f_2^2 f_{10}^5 f_{20}^5 f_8^5 f_8^5} - 3q f_1 f_4 f_{10}^5 f_{20}^5 f_8^5 f_8^5 + q^2 \frac{f_2^5 f_{20}^5}{f_2^2 f_{10}^5 f_{20}^5} f_8^5 f_8^5 \quad (\text{mod } 5). \quad (2.23)
\]

Substituting (2.13) into (2.23), we arrive at

\[
\sum_{n=0}^{\infty} \bar{p}(5n)q^n \equiv \frac{f_2^2 f_{10}^5 f_{20}^5}{f_2^2 f_{10}^5 f_{20}^5} - 3q f_1 f_4 f_{10}^5 f_{20}^5 + q^2 \frac{f_2^5 f_{20}^5}{f_2^2 f_{10}^5 f_{20}^5} \quad (\text{mod } 5). \quad (2.24)
\]

According to 2-dissection formulas (2.1), (2.2), (2.3), (2.5) and congruence (2.24), we obtain

\[
\sum_{n=0}^{\infty} \bar{p}(5n)q^n \equiv \frac{f_2^2 f_{10}^5 f_{20}^5}{f_2^2 f_{10}^5 f_{20}^5} \left( \frac{f_2^2 f_{10}^5 f_{20}^5}{f_2^2 f_{10}^5 f_{20}^5} + 2q f_4 f_6 f_{10}^5 f_{20}^5 f_8^5 f_8^5 \right) \left( \frac{f_2^5 f_{10}^5 f_{20}^5}{f_2^5 f_{10}^5 f_{20}^5} - 4q^5 \frac{f_2^5 f_{10}^5 f_{20}^5}{f_2^5 f_{10}^5 f_{20}^5} \right)^2
\]

\[
- 3q f_{10} f_{10} f_{20}^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5 - 2q^5 \frac{f_2^5 f_{10}^5 f_{20}^5}{f_2^5 f_{10}^5 f_{20}^5} \left( f_8 f_{20}^5 f_8 f_{20}^5 + q f_2^3 f_{10} f_{20}^5 \right)
\]

\[
\quad + q^2 \frac{f_2^5 f_{10}^5 f_{20}^5}{f_2^5 f_{10}^5 f_{20}^5} \left( f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5 + 2q^5 \frac{f_2^5 f_{10}^5 f_{20}^5}{f_2^5 f_{10}^5 f_{20}^5} \right)
\]

\[
\equiv \frac{f_2^5 f_{10}^5 f_{20}^5}{f_2^5 f_{10}^5 f_{20}^5} \left( 2q^2 f_4 f_6 f_{10}^5 f_{20}^5 f_8^5 f_8^5 - 3q f_4 f_6 f_{10}^5 f_{20}^5 f_8^5 f_8^5 - 3q^2 f_2^3 f_{10} f_{20}^5 f_8^5 f_8^5 \right)
\]

\[
+ q^2 \frac{f_2^5 f_{10}^5 f_{20}^5}{f_2^5 f_{10}^5 f_{20}^5} \left( 3q^5 f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5 + q^6 \frac{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5}{f_2^5 f_{10}^5 f_{20}^5} - q^7 \frac{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5}{f_2^5 f_{10}^5 f_{20}^5} \right)
\]

\[
\quad + 2q^7 \frac{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5}{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5} + q^{11} \frac{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5}{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5} \quad (\text{mod } 5). \quad (2.25)
\]

Extracting the terms with odd powers of \( q \) on both sides of (2.25), we have

\[
\sum_{n=0}^{\infty} \bar{p}(10n + 5)q^{2n+1} \equiv 2q^2 f_2^6 f_{10}^5 f_{20}^5 f_8^5 f_8^5 - 3q f_4 f_6 f_{10}^5 f_{20}^5 f_8^5 f_8^5 - 3q^5 f_2^3 f_{10}^5 f_{20}^5 f_8^5 f_8^5 + q^7 f_2^3 f_{10}^5 f_{20}^5 f_8^5 f_8^5
\]

\[
\quad + 2q^7 \frac{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5}{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5} + q^{11} \frac{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5}{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5} \quad (\text{mod } 5). \quad (2.26)
\]

Dividing \( q \) on both sides of (2.26) and replacing \( q^2 \) by \( q \), we get

\[
\sum_{n=0}^{\infty} \bar{p}(10n + 5)q^n \equiv 2q f_2^5 f_2 f_{10}^5 f_{20}^5 f_8^5 f_8^5 - 3q f_4 f_6 f_{10}^5 f_{20}^5 f_8^5 f_8^5 - 3q^2 f_2^3 f_{10}^5 f_{20}^5 f_8^5 f_8^5
\]

\[
\quad + 2q^7 \frac{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5}{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5} + q^{11} \frac{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5}{f_2^5 f_{10}^5 f_{20}^5 f_8^5 f_8^5} \quad (\text{mod } 5).
\]
employing 2-dissection formulas (2.4), (2.5) and congruence (2.27), we deduce that

\[
\sum_{n=0}^{\infty} \bar{p}(10n + 5)q^n = 2 f_2^2 f_4^2 f_{10} f_{20} \left( \frac{f_{14}^4 f_{14}}{f_{2}^2 f_{10}^4} + 4 q f_{4}^2 f_{8}^4 f_{10} \right) \left( \frac{f_{20}^4}{f_{10}^4 f_{40}} + 4 q^5 f_{20}^2 f_{40}^4 \right) \]  

\[
- 3 f_{4}^2 f_{20}^4 \left( \frac{f_{8}^2 f_{20}^2}{f_{2}^2 f_{40}^2} + q f_{4}^3 f_{10} f_{40} \right)^2 
\]

\[
- 3 q^2 f_{4}^5 f_{10} f_{20}^2 \left( \frac{f_{14}^4 f_{14}}{f_{2}^2 f_{4}^4} + 4 q f_{4}^2 f_{8}^4 f_{10} \right) \left( \frac{f_{20}^4}{f_{10}^4 f_{40}^2} + 4 q^5 f_{20}^2 f_{40}^4 \right) 
\]

\[
+ q^3 f_{2}^2 f_{40}^4 \left( \frac{f_{8}^2 f_{20}^2}{f_{2}^2 f_{40}^2} + q f_{4}^3 f_{10} f_{40} \right) + 2 q^2 f_{10} f_{20}^2 \left( \frac{f_{8}^2 f_{20}^2}{f_{2}^2 f_{40}^2} + q f_{4}^3 f_{10} f_{40} \right)^2 
\]

\[
+ 2 q^5 f_{2}^2 f_{40}^8 f_{20}^8 \left( \frac{f_{14}^4 f_{14}}{f_{2}^2 f_{4}^8 f_{8}^2 f_{20}^2} + 4 q f_{4}^2 f_{8}^4 f_{10} \right) 
\]

\[
\equiv - 3 f_{4}^2 f_{20}^4 f_{8}^6 f_{20}^8 f_{40}^2 + 2 f_{10} f_{20}^4 f_{12}^2 f_{2}^2 f_{8}^6 f_{10} f_{40}^2 + 3 q f_{4}^2 f_{10} f_{20}^6 f_{8}^8 f_{40}^4 - q f_{4}^3 f_{10} f_{20}^5 f_{5}^5 f_{40}^4 
\]

\[
- 3 q^2 f_{10}^2 f_{20}^4 f_{14} f_{8}^6 f_{10} f_{40}^4 - 3 q^2 f_{12}^2 f_{2}^2 f_{8}^6 f_{10} f_{40}^4 - 2 q^3 f_{10}^2 f_{20}^6 f_{8}^3 f_{20}^4 f_{40} + 2 q^3 f_{10}^2 f_{10} f_{20}^6 f_{20} f_{40} 
\]

\[
+ q^3 f_{2}^3 f_{6}^6 f_{20}^6 f_{40}^4 + 2 q^4 f_{12}^2 f_{2}^2 f_{8}^6 f_{10} f_{40}^2 + 3 q^4 f_{4}^4 f_{8} f_{10} f_{40}^4 f_{20} f_{40} + 3 q^3 f_{4}^5 f_{10} f_{20}^6 f_{8}^3 f_{40} 
\]

\[
+ 3 q^5 f_{14}^2 f_{8}^6 f_{40}^4 + 2 q^6 f_{12}^2 f_{2}^2 f_{8}^6 f_{10} f_{40}^2 + q^5 f_{4}^8 f_{10}^2 f_{20} f_{40}^4 f_{20} f_{40} - 2 q^7 f_{14}^2 f_{2}^2 f_{8}^6 f_{10} f_{40}^2 
\]

\[
- 3 q^8 f_{14}^2 f_{8}^6 f_{20}^4 f_{40}^4 + 2 q^10 f_{12}^2 f_{2}^2 f_{8}^6 f_{10} f_{40}^4 + 3 q^11 f_{4}^4 f_{8} f_{10} f_{40}^4 f_{20} f_{40} \quad \text{(mod 5). (2.28)} 
\]

extracting the terms with odd powers of q on both sides of (2.28), then dividing by q and replacing q² by q, we find that

\[
\sum_{n=0}^{\infty} \bar{p}(20n + 15)q^n \equiv 3 f_{2}^2 f_{4}^6 f_{10} f_{20}^4 f_{8}^6 f_{8}^8 f_{10} f_{40}^4 - f_{4}^2 f_{5}^5 f_{5}^5 f_{10} f_{20}^6 f_{1} f_{2} f_{20}^2 - 2 q f_{2}^7 f_{2}^2 f_{10} f_{40}^4 f_{20} f_{20} f_{40} + 2 q f_{4}^2 f_{10} f_{20}^4 f_{20} f_{20} \frac{f_{1} f_{4} f_{8} f_{10} f_{20}}{f_{1} f_{2}^2 f_{5} f_{20}^2} 
\]

\[
+ q f_{1} f_{4} f_{2} f_{5} f_{20} + 3 q^2 f_{5}^2 f_{3} f_{5} f_{20}^4 f_{1} f_{4} f_{8} f_{10} f_{20}^2 + 3 q^3 f_{14}^3 f_{8}^8 f_{10} f_{20}^6 f_{20} f_{20} \frac{f_{1} f_{4} f_{8} f_{10} f_{20}}{f_{1} f_{2}^2 f_{5} f_{20}^2} \quad \text{(mod 5). (2.29)} 
\]
By (2.17), we see that 
\[ f_5 \equiv f_1^5 \pmod{5}. \] (2.30)

Substituting (2.30) into (2.29) gives
\[
\sum_{n=0}^{\infty} \bar{p}(20n + 15)q^n \equiv 3 f_{14}^{10} f_8^{10} f_4^{14} f_8^{10} - f_2^{29} f_8^{10} f_4^{14} + 2q f_2^{29} f_8^{10} f_4^{14} + q f_2^{29} f_8^{10} f_4^{14} + q f_2^{29} f_8^{10} f_4^{14}
\]
\[ + 3q^2 f_2^{14} f_4^{14} + 3q^2 f_2^{14} f_4^{14} - 2q^3 f_2^{14} f_4^{14} + 3q^3 f_2^{14} f_4^{14} \pmod{5}. \] (2.31)

Combining 2-dissection formulas (2.3), (2.4) and congruence (2.31), we see that
\[
\sum_{n=0}^{\infty} \bar{p}(20n + 15)q^n \equiv 3 f_{14}^{10} f_8^{10} f_4^{14} f_8^{10} - f_2^{29} f_8^{10} f_4^{14} + 2q f_2^{29} f_8^{10} f_4^{14} + q f_2^{29} f_8^{10} f_4^{14} + q f_2^{29} f_8^{10} f_4^{14}
\]
\[ + 2q^2 f_2^{14} f_4^{14} (f_2^{10} f_4^{14} f_8^{14} f_4^{14} - 4 q f_2^{14} f_4^{14} f_8^{14} f_4^{14})^2 + q^2 f_2^{29} f_8^{10} f_4^{14} + 4q f_2^{29} f_8^{10} f_4^{14}
\]
\[ + 3q^2 f_2^{14} f_4^{14} + 3q^2 f_2^{14} f_4^{14} \left( f_2^{14} f_4^{14} f_8^{14} f_4^{14} + 4q f_2^{14} f_4^{14} f_8^{14} f_4^{14} \right)^2
\]
\[ - 2q^3 f_2^{29} f_4^{14} \left( f_2^{14} f_4^{14} f_8^{14} f_4^{14} + 4 q f_2^{14} f_4^{14} f_8^{14} f_4^{14} \right)^2 + 3q^3 f_2^{14} f_4^{14} f_8^{14} f_4^{14} + 4q f_2^{14} f_4^{14} f_8^{14} f_4^{14}
\]
\[ \equiv 3 f_{14}^{13} f_8^{14} f_4^{14} f_8^{14} f_4^{14} + 2q f_2^{12} f_8^{14} f_4^{14} f_8^{14} f_4^{14} + 3q f_2^{26} f_8^{14} f_4^{14} f_8^{14} f_4^{14} + q^2 f_2^{110} f_8^{14} f_4^{14} f_8^{14} f_4^{14}
\]
\[ + 4q^3 f_2^{98} f_8^{14} f_4^{14} f_8^{14} f_4^{14} + 3q^3 f_2^{98} f_8^{14} f_4^{14} f_8^{14} f_4^{14} + q^4 f_2^{86} f_4^{14} f_8^{14} f_4^{14} f_8^{14} + 4q^5 f_2^{74} f_8^{14} f_4^{14} f_8^{14} f_4^{14}
\]
\[ + 4q^7 f_2^{50} f_8^{14} f_4^{14} f_8^{14} f_4^{14} + 8 f_2^{38} f_4^{14} f_8^{14} f_4^{14} f_8^{14} + 4q^9 f_2^{26} f_8^{14} f_4^{14} f_8^{14} f_4^{14} + q^{10} f_2^{14} f_4^{14} f_8^{14} f_4^{14} f_8^{14}
\]
\[ + 2q^{11} f_2^{18} f_8^{14} f_4^{14} f_8^{14} f_4^{14} + 3q^{12} f_2^{48} f_8^{14} f_4^{14} f_8^{14} f_4^{14} \pmod{5}. \] (2.32)

Extracting the terms with odd powers of $q$ on both sides of (2.32), then dividing by $q$ and replacing $q^2$ by $q$, we reach (2.6). This completes the proof.

\section{3 Proof of Conjecture 1.1}

In this section, we use the $(p,k)$-parametrization of theta functions given by Alaca, Alaca and Williams [1, 2, 10] to represent the generating function of $\bar{p}(40 + 35)$ modulo 5 as a
linear combination of several functions in $p$ and $k$, where $p$ and $k$ are defined in terms of theta function $\varphi(q)$ as given by
\[
p = \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)} \quad (3.1)
\]
and
\[
k = \frac{\varphi^3(q^3)}{\varphi(q)}, \quad (3.2)
\]
see Alaca, Alaca and Williams [1]. Williams [10] proved that
\[
p = 2\frac{f_3^2 f_3^4 f_6^6}{f_1^2 f_2^2 f_9^6}, \quad (3.3)
\]
It turns out that the coefficients of linear combination are divisible by 5. This confirms the conjecture of Hirschhorn and Sellers. More precisely, we have the following congruence.

**Theorem 3.1** For any nonnegative integer $n$, we have
\[
\bar{p}(40n + 35) \equiv 0 \pmod{5}. \quad (3.4)
\]

**Proof.** The following representations of $q^{\frac{1}{12}} f_1$, $q^{\frac{1}{12}} f_2$ and $q^{\frac{1}{6}} f_4$ in terms of $p$ and $k$ are due to Alaca and Williams [2],
\[
q^{\frac{1}{12}} f_1 = 2^{-\frac{1}{3}} p^{\frac{1}{2}} (1 - p)^{\frac{1}{2}} (1 + p)^{\frac{1}{2}} (1 + 2p)^{\frac{1}{2}} (2 + p)^{\frac{1}{2}} k^{\frac{1}{2}}, \quad (3.5)
\]
\[
q^{\frac{1}{12}} f_2 = 2^{-\frac{1}{3}} p^{\frac{1}{2}} (1 - p)^{\frac{1}{2}} (1 + p)^{\frac{1}{2}} (1 + 2p)^{\frac{1}{2}} (2 + p)^{\frac{1}{2}} k^{\frac{1}{2}} \quad (3.6)
\]
and
\[
q^{\frac{1}{6}} f_4 = 2^{-2/3} p^{\frac{1}{2}} (1 - p)^{\frac{1}{2}} (1 + p)^{\frac{1}{2}} (1 + 2p)^{\frac{1}{2}} (2 + p)^{\frac{1}{2}} k^{\frac{1}{2}}. \quad (3.7)
\]
Substituting (3.5), (3.6) and (3.7) into (2.6), we find that
\[
2^{19} \sum_{n=0}^{\infty} \bar{p}(40n + 35)q^n \equiv \frac{\sqrt{2}p^{7/8}(1 + 2p)^{21/8}(2 + p)^{21/8}}{16q^{7/8}(1 - p)^6(1 + p)^2}\sqrt{k} \quad F(p, k) \pmod{5}, \quad (3.8)
\]
where $F(p, k)$ is defined by
\[
F(p, k) = 2621440 + 30146560p + 443678720p^2 + 4203806720p^3 + 25364889600p^4
+ 112351805440p^5 + 378957086720p^6 + 980173332480p^7
+ 1961928110080p^8 + 3051430471680p^9 + 3658168560640p^{10}
\]
$$+ 3316049272320p^{11} + 2205104730880p^{12} + 1020945279360p^{13}$$
$$+ 295430818880p^{14} + 40648474720p^{15} + 694662000p^{16}$$
$$+ 12386590p^{19} − 82928860p^{18} + 168540920p^{17} + 98305p^{20}. \tag{3.9}$$

By (3.3) and (3.6), we have
$$\frac{f^{22}}{f^{23}} = \frac{\sqrt{2}p^{7/8}(1 + 2p)^{21/8}(2 + p)^{21/8}}{16q^{7/8}(1 − p)^6(1 + p)^2\sqrt{k}}. \tag{3.10}$$

Hence (3.8) can be rewritten as
$$2^{19} \sum_{n=0}^{\infty} \bar{p}(40n + 35)q^n \equiv \frac{f^{22}}{f^{23}} F(p, k) \pmod{5}, \tag{3.11}$$

where \( F(p, k) \) are defined by (3.9). Clearly, \( \frac{f^{22}}{f^{23}} \) is a formal power series in \( q \) with integer coefficients. By (3.2) and (3.3), we see that \( p \) and \( k \) are also formal power series in \( q \) with integer coefficients. It can be seen that the coefficients of \( F(p, k) \) are divisible by 5. So we reach the assertion that for all \( n \geq 0, \bar{p}(40n + 35) \equiv 0 \pmod{5} \) for \( n \geq 0 \). This completes the proof.\]

To complete the proof of Conjecture 1.1 we recall that Hirshhorn and Sellers [6], and Fortin, Jacob and Mathieu [5] independently derived the congruence
$$\bar{p}(4n + 3) \equiv 0 \pmod{8}, \tag{3.12}$$

for \( n \geq 0 \). This yields
$$\bar{p}(40n + 35) \equiv 0 \pmod{8}, \tag{3.13}$$

for \( n \geq 0 \). Combining the above congruence (3.13) and the congruence \( \bar{p}(40n + 35) \equiv 0 \pmod{5} \) for \( n \geq 0 \), we come to the conclusion that \( \bar{p}(40n + 35) \equiv 0 \pmod{40} \) for \( n \geq 0 \).

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