The equitable presentation for the quantum group $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$

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Abstract

We consider the quantum group $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$. We display a presentation for $U_q(\mathfrak{g})$ that we find attractive; we call this the equitable presentation. For $\mathfrak{g} = \mathfrak{sl}_2$ the equitable presentation has generators $X^{\pm 1}, Y, Z$ and relations $XX^{-1} = X^{-1}X = 1,$

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = 1, \quad \frac{qYZ - q^{-1}ZY}{q - q^{-1}} = 1, \quad \frac{qZX - q^{-1}XZ}{q - q^{-1}} = 1.$$ 

Keywords. Quantum group, quantum algebra, Kac-Moody algebra, equitable presentation, Hopf algebra.

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1 Introduction

In [6] T. Ito, C. Weng, and the present author introduced the equitable presentation for the quantum group $U_q(\mathfrak{sl}_2)$. The purpose of this note is to give an analogous presentation for the quantum group $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra. As we will see, the generators for this presentation are related to Koornwinder’s twisted primitive elements [14], [15].

Throughout, $\mathbb{K}$ will denote a field and $q$ will denote an indeterminate. We will work over the field $\mathbb{K}(q)$.

In order to motivate our main result we first recall some facts about $U_q(\mathfrak{sl}_2)$.

2 The quantum group $U_q(\mathfrak{sl}_2)$

We begin with the definition of $U_q(\mathfrak{sl}_2)$.

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Definition 2.1 [8, p. 9] We let $U_q(\mathfrak{sl}_2)$ denote the unital associative $\mathbb{K}(q)$-algebra with generators $E, F, K, K^{-1}$ and the following relations:

\[
\begin{align*}
    KK^{-1} &= K^{-1}K = 1, \\
    KEK^{-1} &= q^2E, \\
    KFK^{-1} &= q^{-2}F, \\
    EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}.
\end{align*}
\]

We call $E, F, K, K^{-1}$ the Chevalley generators for $U_q(\mathfrak{sl}_2)$.

We refer the reader to [2], [8], [10] for background information on $U_q(\mathfrak{sl}_2)$.

Lemma 2.2 [8, p. 35] The quantum group $U_q(\mathfrak{sl}_2)$ has the following Hopf algebra structure. The comultiplication $\Delta$ satisfies

\[
\begin{align*}
    \Delta(E) &= E \otimes 1 + K \otimes E, \\
    \Delta(F) &= F \otimes K^{-1} + 1 \otimes F, \\
    \Delta(K) &= K \otimes K.
\end{align*}
\]

The counit $\varepsilon$ satisfies

\[
\begin{align*}
    \varepsilon(E) &= 0, \\
    \varepsilon(F) &= 0, \\
    \varepsilon(K) &= 1.
\end{align*}
\]

The antipode $S$ satisfies

\[
\begin{align*}
    S(E) &= -K^{-1}E, \\
    S(F) &= -FK, \\
    S(K) &= K^{-1}.
\end{align*}
\]

3 The equitable presentation for $U_q(\mathfrak{sl}_2)$

In the presentation for $U_q(\mathfrak{sl}_2)$ given in Definition 2.1 the generators $E, F$ and the generators $K, K^{-1}$ play a very different role. We now recall a presentation for $U_q(\mathfrak{sl}_2)$ whose generators are on a more equal footing.

Theorem 3.1 [6, Theorem 2.1] The algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to the unital associative $\mathbb{K}(q)$-algebra with generators $X, X^{-1}, Y, Z$ and the following relations:

\[
\begin{align*}
    XX^{-1} &= X^{-1}X = 1, \\
    qXY - q^{-1}YX &= 1, \\
    \frac{qYZ - q^{-1}ZY}{q - q^{-1}} &= 1, \\
    \frac{qZX - q^{-1}XZ}{q - q^{-1}} &= 1.
\end{align*}
\]
An isomorphism with the presentation in Definition 2.1 is given by:

\[
\begin{align*}
X^\pm & \rightarrow K^\pm, \\
Y & \rightarrow K^{-1} + F(q - q^{-1}), \\
Z & \rightarrow K^{-1} - K^{-1} Eq(q - q^{-1}).
\end{align*}
\]

The inverse of this isomorphism is given by:

\[
\begin{align*}
E & \rightarrow (1 - XZ)q^{-1}(q - q^{-1})^{-1}, \\
F & \rightarrow (Y - X^{-1})(q - q^{-1})^{-1}, \\
K^\pm & \rightarrow X^\pm.
\end{align*}
\]

**Proof:** One readily checks that each map is a homomorphism of \(\mathbb{K}(q)\)-algebras and that the maps are inverses. It follows that each map is an isomorphism of \(\mathbb{K}(q)\)-algebras. \(\square\)

The generators \(X, Y, Z\) from Theorem 3.1 are on an equal footing, more or less. This motivates the following definition.

**Definition 3.2** [6, Definition 2.2] By the equitable presentation for \(U_q(\mathfrak{sl}_2)\) we mean the presentation given in Theorem 3.1. We call \(X, X^{-1}, Y, Z\) the equitable generators.

**Definition 3.3** For notational convenience, throughout this paper we identify the copy of \(U_q(\mathfrak{sl}_2)\) given in Definition 2.1 with the copy of \(U_q(\mathfrak{sl}_2)\) given in Theorem 3.1 via the isomorphism given in Theorem 3.1.

The Hopf algebra structure for \(U_q(\mathfrak{sl}_2)\) given in Lemma 2.2 looks as follows in terms of the equitable generators.

**Theorem 3.4** With reference to Lemma 2.2 and Definition 3.3, the comultiplication \(\Delta\) satisfies

\[
\begin{align*}
\Delta(X) & = X \otimes X, \\
\Delta(Y) & = (Y - 1) \otimes X^{-1} + 1 \otimes Y, \\
\Delta(Z) & = (Z - 1) \otimes X^{-1} + 1 \otimes Z.
\end{align*}
\]

The counit \(\varepsilon\) satisfies

\[
\begin{align*}
\varepsilon(X) & = 1, \\
\varepsilon(Y) & = 1, \\
\varepsilon(Z) & = 1.
\end{align*}
\]

The antipode \(S\) satisfies

\[
\begin{align*}
S(X) & = X^{-1}, \\
S(Y) & = 1 + X - Y X, \\
S(Z) & = 1 + X - Z X.
\end{align*}
\]

**Proof:** Routine verification. \(\square\)

We finish this section with a remark.
**Remark 3.5** With reference to Lemma 2.2 and Definition 3.3 for $y = Y - 1$ and $z = Z - 1$ we have

$$\Delta(y) = y \otimes X^{-1} + 1 \otimes y,$$

$$\Delta(z) = z \otimes X^{-1} + 1 \otimes z.$$  

We will discuss these two equations in Section 6.

### 4 The quantum group $U_q(\mathfrak{g})$

We now turn our attention to the quantum group $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra. We will give an “equitable” presentation for $U_q(\mathfrak{g})$. We begin with a comment. In the standard references [1], [2], [7], [8], [9], [10], [11], [16], [17] the definition of $U_q(\mathfrak{g})$ varies. We will give the equitable presentation for one version of $U_q(\mathfrak{g})$; the equitable presentation for the other versions can be obtained from this one by minor modification.

We will use the following notation.

**Definition 4.1** Let $n$ denote a positive integer and let $A$ denote a symmetrizable generalized Cartan matrix [4, p. 1] of order $n$. Since $A$ is symmetrizable there exists relatively prime positive integers $d_1, \ldots, d_n$ such that $d_i A_{ij} = d_j A_{ji}$ for $1 \leq i, j \leq n$. For $1 \leq i \leq n$ we do the following. We define $q_i = q^{d_i}$. Also, for an integer $m$ we define

$$[m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$$

and for $m \geq 0$ we define

$$[m]_i = [m]_i[m - 1]_i \cdots [2]_i[1]_i.$$  

We interpret $[0]_i = 1$. For integers $m \geq r \geq 0$ we define

$$\left[ \begin{array}{c} m \\ r \end{array} \right]_i = \frac{[m]_i}{[r]_i[m - r]_i}.$$  

Let $\mathfrak{g} = \mathfrak{g}'(A)$ denote the Kac-Moody algebra over $\mathbb{C}$ that corresponds to $A$ [41 p. xi].

**Definition 4.2** [2 p. 281] With reference to Definition 4.1 $U_q(\mathfrak{g})$ is the unital associative $\mathbb{K}(q)$-algebra with generators $E_i, F_i, K_i, K_i^{-1}$ ($i = 1, \ldots, n$) and the following relations:

- **(R1)** $K_i K_i^{-1} = K_i^{-1} K_i = 1$
- **(R2)** $K_i K_j = K_j K_i$
- **(R3)** $K_i E_j K_i^{-1} = q_i^{A_{ij}} E_j$
- **(R4)** $K_i F_j K_i^{-1} = q_i^{-A_{ij}} F_j$
\[(R5) \quad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \]

\[(R6) \quad \sum_{r=0}^{1-A_{ij}} (-1)^r \left[ 1 - \frac{A_{ij}}{r} \right] E_i^{1-A_{ij}-r} F_j E_i^r = 0 \quad \text{if} \quad i \neq j \]

\[(R7) \quad \sum_{r=0}^{1-A_{ij}} (-1)^r \left[ 1 - \frac{A_{ij}}{r} \right] F_i^{1-A_{ij}-r} F_j F_i^r = 0 \quad \text{if} \quad i \neq j. \]

The expression \(\delta_{ij}\) in (R5) is the Kronecker delta.

We call the generators \(E_i, F_i, K_i, K_i^{-1}\) from Definition 4.2 the Chevalley generators for \(U_q(\mathfrak{g})\).

We now recall a Hopf algebra structure for \(U_q(\mathfrak{g})\).

**Lemma 4.3** [8, p. 55] The quantum group \(U_q(\mathfrak{g})\) has the following Hopf algebra structure. The comultiplication \(\Delta\) satisfies

\[
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,
\]

\[
\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,
\]

\[
\Delta(K_i) = K_i \otimes K_i.
\]

The counit \(\varepsilon\) satisfies

\[
\varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.
\]

The antipode \(S\) satisfies

\[
S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1}.
\]

5. **The equitable presentation for \(U_q(\mathfrak{g})\)**

In this section we give the equitable presentation for \(U_q(\mathfrak{g})\). The following is our main result.

**Theorem 5.1** The \(\mathbb{K}(q)\)-algebra \(U_q(\mathfrak{g})\) from Definition 4.2 is isomorphic to the unital associative \(\mathbb{K}(q)\)-algebra with generators \(X_i, X_i^{-1}, Y_i, Z_i (i = 1, \ldots, n)\) and the following relations:

\[(E1) \quad X_i X_i^{-1} = X_i^{-1} X_i = 1\]

\[(E2) \quad X_i X_j = X_j X_i\]

\[(E3) \quad Y_i X_j - q_i^{A_{ij}} X_j Y_i = X_j^{-1} X_i (1 - q_i^{A_{ij}})\]

\[(E4) \quad X_i Z_j - q_i^{A_{ij}} Z_j X_i = X_i X_j^{-1} (1 - q_i^{A_{ij}})\]

\[(E5) \quad Z_i Y_i - q_i^2 Y_i Z_i = 1 - q_i^2\]
The following identity is a special case of the binomial theorem [3, p. 236]. For an indeterminate \( \lambda \) and for an integer \( m \geq 0 \),

\[
\sum_{r=0}^{m} (-1)^r \binom{m}{r} \lambda^r = \prod_{s=0}^{m-1} (1 - \lambda q_i^{1-m+2s}) \quad (1 \leq i \leq n).
\]  

Let \( U \) denote the unital associative \( \mathbb{K}(q) \)-algebra with generators \( X_i, X_i^{-1}, Y_i, Z_i \) \((i = 1, \ldots, n)\) and relations (E1)–(E8). Our goal is to show that [11]–[13] gives an isomorphism of \( \mathbb{K}(q) \)-algebras from \( U \) to \( U_q(\mathfrak{g}) \), and that the inverse of this isomorphism satisfies (14)–(6). To this end, we first show that [11]–[13] gives a homomorphism of \( \mathbb{K}(q) \)-algebras from \( U \) to \( U_q(\mathfrak{g}) \). In order to do this we let \( \hat{X}_i^{\pm 1}, \hat{Y}_i, \hat{Z}_i \) denote the expressions on the right in (11)–(13) respectively and show the following hold in \( U_q(\mathfrak{g}) \):

\[
\hat{X}_i \hat{X}_j^{-1} = \hat{X}_j^{-1} \hat{X}_i = 1, \quad (9)
\]

\[
\hat{X}_i \hat{X}_j = \hat{X}_j \hat{X}_i, \quad (10)
\]

\[
\hat{Y}_i \hat{X}_j - q_i^{A_{ij}} \hat{X}_j \hat{Y}_i = \hat{X}_j^{-1} \hat{X}_j (1 - q_i^{A_{ij}}), \quad (11)
\]

\[
\hat{X}_i \hat{Z}_j - q_i^{A_{ij}} \hat{Z}_j \hat{X}_i = \hat{X}_i \hat{X}_j^{-1} (1 - q_i^{A_{ij}}), \quad (12)
\]

\[
\hat{Z}_i \hat{Y}_j - q_i^{2} \hat{Y}_j \hat{Z}_i = 1 - q_i^{2}, \quad (13)
\]

\[
\hat{Z}_i \hat{Y}_j - q_i^{A_{ij}} \hat{Y}_j \hat{Z}_i = \hat{X}_i^{-1} \hat{X}_j^{-1} (1 - q_i^{A_{ij}}) \quad \text{if} \quad i \neq j, \quad (14)
\]
We obtain the result is plus $q$ for all integers $m$.

In the above expression we pull the terms $K_i^{\eta}$.

For $i \neq j$ we show the expressions (18), (20) are equal and the expression (19) is zero. In (18) we evaluate the terms $(K_i^{-1} + F_i(q_i - q_i^{-1}))^{-A_{ij}}$ and $(K_i^{-1} + F_i(q_i - q_i^{-1}))^{r}$ using (17); the result is
\[
\begin{align*}
&\sum_{r=0}^{1-A_{ij}} (-1)^r \left[ 1 - A_{ij} \right] \sum_{\eta=0}^{1-A_{ij}-r} \sum_{t=0}^{r} (-1)^r \left[ 1 - A_{ij} \right] \left[ 1 - A_{ij} - r \right] \left[ \eta \right] \left[ t \right] q_i^{t(r-t)} (q_i - q_i^{-1})^{\eta+r} F_i^\eta K_i^{A_{ij}+\eta+r-1} K_j^{-1} F_i^t K_j^{-r}.
\end{align*}
\]

In the above expression we pull the terms $K_i^{A_{ij}+\eta+r-1} K_j^{-1}$ to the right past the $F_i$ using (R4). We obtain
\[
\begin{align*}
&\sum_{r=0}^{1-A_{ij}} (-1)^r \left[ 1 - A_{ij} \right] \left[ 1 - A_{ij} - r \right] \left[ r \right] q_i^{t(r-t)} (q_i - q_i^{-1})^{\eta+r} F_i^\eta K_i^{A_{ij}+\eta+r-1} K_j^{-1}.
\end{align*}
\]
In (21) we change variables by substituting $u = r - t$ and $v = \eta + t$. We find that for $0 \leq v \leq 1 - A_{ij}$ the coefficient of $F_i^v K_i^{A_{ij} + v - 1} K_j^{-1}$ in (21) is equal to

$$\left[ 1 - A_{ij} \right]_v q_i^{v(1 - A_{ij} - v)} (q_i - q_i^{-1})^v$$

(22)
times

$$\sum_{t=0}^v (-1)^t \left[ \begin{array}{c} v \\ t \end{array} \right] q_i^{t(1 - v)}$$

(23)
times

$$\sum_{u=0}^{1 - A_{ij} - v} (-1)^u \left[ 1 - A_{ij} - v \right]_u q_i^{u(1 - A_{ij} - u)}.$$  

(24)

The sum (23) is equal to $\delta_{u,0}$ in view of (8). For $v = 0$ the expression (22) is equal to 1 and the sum (24) is equal to

$$\prod_{s=0}^{-A_{ij}} (1 - q_i^{A_{ij} + 2s})$$

(25)
in view of (8). Therefore (21) is equal to (20). From our above comments (18) is equal to (20). In (19) we evaluate the terms $(K_i^{-1} + F_i(q_i - q_i^{-1}))^{1 - A_{ij} - r}$ and $(K_i^{-1} + F_i(q_i - q_i^{-1}))^r$ using (17); the result is

$$\sum_{r=0}^{1 - A_{ij} - r} \sum_{\eta=0}^{1 - A_{ij} - r} \sum_{t=0}^{r} (-1)^r \left[ 1 - A_{ij} \right]_r \left[ 1 - A_{ij} - r \right]_\eta \left[ r \right]_t q_i^{r(1 - A_{ij} - r)} q_i^{t(r-t)} (q_i - q_i^{-1})^{\eta+1} F_i^\eta F_i^{(A_{ij} + \eta)} F_i^{1} K_i^{A_{ij} + \eta + r - 1 - A_{ij} - r}.$$  

(26)

In the above expression we pull the terms $K_i^{A_{ij} + \eta + r - 1}$ to the right past the $F_j, F_i$ using (R4). We obtain

$$\sum_{r=0}^{1 - A_{ij} - r} \sum_{\eta=0}^{1 - A_{ij} - r} \sum_{t=0}^{r} (-1)^r \left[ 1 - A_{ij} \right]_r \left[ 1 - A_{ij} - r \right]_\eta \left[ r \right]_t q_i^{r(1 - A_{ij} - r)} q_i^{t(r-t)} (q_i - q_i^{-1})^{\eta+1} F_i^\eta F_i^{1} K_i^{A_{ij} + \eta + r - 1}.$$  

(26)

In (26) we change variables by substituting $u = r - t$ and $v = \eta + t$. We find that for $0 \leq v \leq 1 - A_{ij}$ and for $0 \leq t \leq v$ the coefficient of $F_i^{v-t} F_j^{1} K_i^{A_{ij} + v - 1}$ in (26) is equal to

$$(-1)^t \left[ 1 - A_{ij} \right]_t \left[ v \right]_t q_i^{t(v - t)} (q_i - q_i^{-1})^t (q_i - q_i^{-1})^v$$

(27)
times

$$\sum_{u=0}^{1 - A_{ij} - v} (-1)^u \left[ 1 - A_{ij} - v \right]_u q_i^{(A_{ij} + v + t)(1 - A_{ij} - v)} (q_i - q_i^{-1})^v.$$  

(28)
The sum \( (28) \) is equal to \( \delta_{v,1-A_{ij}} \) in view of \( (8) \). For \( v = 1 - A_{ij} \) the expression \( (27) \) is equal to
\[
(-1)^t \left[ \frac{1}{t} \right] (q_i - q_i^{-1})^{1-A_{ij}}.
\]
(29)
Therefore \( (20) \) is equal to
\[
(q_i - q_i^{-1})^{1-A_{ij}} \sum_{t=0}^{1-A_{ij}} (-1)^t \left[ \frac{1}{t} \right] F_i^{1-A_{ij}-t} F_j F_i^t
\]
(30)
and this is zero in view of \( (R7) \). From these comments we find \( (19) \) is equal to zero. We have

\[
\text{Line (33) holds by (E1) and line (34) holds by (E2). Line (35) follows from (E1), (E2), (E4). Line (36) follows from (E1), (E2), (E3). Line (37) follows from (E1)–(E6). Before we verify}
\]
we make some comments. In the equation of (E4) we multiply each term on the left by $X_j$ and simplify the result using (E1), (E2) to obtain
\[ X_i(1 - X_j Z_j) = q_i^{A_{ij}} (1 - X_j Z_j) X_i. \]
(40)
In (40) we set $i = j$ and $A_{ii} = 2$ to obtain
\[ X_i(1 - X_i Z_i) = q_i^2 (1 - X_i Z_i) X_i. \]
(41)
Using (7), (41) and induction we obtain
\[ (1 - X_i Z_i)^m = \sum_{r=0}^{m} (-1)^r \binom{m}{r} q_i^{r(1 - m)} X_i^r Z_i^r \]
(42)
for all integers $m \geq 0$. The expression on the left in (38) is a scalar multiple of
\[ \sum_{r=0}^{1 - A_{ij}} (-1)^r \binom{1 - A_{ij}}{r} (1 - X_i Z_i)^{1 - A_{ij} - r}(1 - X_j Z_j)(1 - X_i Z_i)^r. \]
(43)
We assume $i \neq j$ and show that (43) is zero. In (43) we evaluate the terms $(1 - X_i Z_i)^r$ using (42) and obtain
\[ \sum_{r=0}^{1 - A_{ij}} \sum_{\eta=0}^{r} (-1)^{r+\eta} \binom{1 - A_{ij}}{r} \binom{r}{\eta} q_i^{\eta(1 - r - A_{ij})} (1 - X_i Z_i)^{1 - A_{ij} - r} X_i^\eta (1 - X_j Z_j) Z_i^\eta. \]
(44)
In (45) we evaluate the terms $(1 - X_j Z_j) X_i^\eta$ using (40) and obtain
\[ \sum_{r=0}^{1 - A_{ij}} \sum_{\eta=0}^{r} (-1)^{r+\eta} \binom{1 - A_{ij}}{r} \binom{r}{\eta} q_i^{\eta(1 - r - A_{ij})} (1 - X_i Z_i)^{1 - A_{ij} - r} X_i^\eta (1 - X_j Z_j) Z_i^\eta. \]
(45)
In (46) we evaluate the terms $(1 - X_i Z_i)^{1 - A_{ij} - r} X_i^\eta$ using (41) and obtain
\[ \sum_{r=0}^{1 - A_{ij}} \sum_{\eta=0}^{r} (-1)^{r+\eta} \binom{1 - A_{ij}}{r} \binom{r}{\eta} q_i^{\eta(1 - r - A_{ij})} (1 - X_i Z_i)^{1 - A_{ij} - r} X_i^\eta (1 - X_j Z_j) Z_i^\eta. \]
(46)
Observe (46) is equal to
\[ \sum_{r=0}^{1 - A_{ij}} \sum_{\eta=0}^{r} (-1)^{r+\eta} \binom{1 - A_{ij}}{r} \binom{r}{\eta} q_i^{\eta(A_{ij} + r - 1)} (1 - X_i Z_i)^{1 - A_{ij} - r} X_i^\eta (1 - X_j Z_j) Z_i^\eta. \]
(47)
minus
\[ \sum_{r=0}^{1 - A_{ij}} \sum_{\eta=0}^{r} (-1)^{r+\eta} \binom{1 - A_{ij}}{r} \binom{r}{\eta} q_i^{\eta(A_{ij} + r - 1)} (1 - X_i Z_i)^{1 - A_{ij} - r} X_j Z_i Z_i^\eta. \]
(48)
So far we have shown that (43) is equal to (47) minus (48). We evaluate (47), (48) separately. In (47) we eliminate the terms \((1 - X_i Z_i)^{1 - A_{ij} - r}\) using (42) and obtain
\[
\sum_{r=0}^{1-A_{ij}} \sum_{\eta=0}^{1-A_{ij}-r} \sum_{t=0}^{1-A_{ij}-r} (-1)^{r+\eta+t} \left[ 1 - A_{ij} \right] \left[ r \right] \left[ 1 - A_{ij} - r \right] q_i^{(A_{ij}+\eta)(A_{ij}+\eta-1)} t^{(A_{ij}+r)} X_i^{\eta+t} Z_i^{\eta+t}. \tag{49}
\]
In (49) we change variables by substituting \(r = u + \eta\) and \(t = v - \eta\). We find that for \(0 \leq v \leq 1 - A_{ij}\) the coefficient of \(X_i^v Z_i^v\) in (49) is equal to
\[
(-1)^v \left[ 1 - A_{ij} \right] \left[ \sum_{\eta=0}^{v} (-1)^\eta \left[ \sum_{r=0}^{1-A_{ij}-v} (-1)^u \left[ 1 - A_{ij} - v \right] \eta(v-1) \right] q_i^{u \eta(v-1)}. \tag{50}
\]

The sum (51) is equal to \(\delta_{0,v}\) in view of (8). For \(v = 0\) the expression (50) is equal to 1 and the sum (52) is equal to
\[
\prod_{s=0}^{-A_{ij}} (1 - q_i^{A_{ij}+2s}) \tag{53}
\]
in view of (8). Therefore (47) is equal to (53). In (48) we evaluate the terms \((1 - X_i Z_i)^{1 - A_{ij} - r} X_j\) using (40) and simplify the result using \(q_i^{A_{ij}} = q_j^{A_{ij}}\). We obtain
\[
\sum_{r=0}^{1-A_{ij}} \sum_{\eta=0}^{1-A_{ij}-r} (-1)^{r+\eta+t} \left[ 1 - A_{ij} \right] \left[ r \right] \left[ 1 - A_{ij} - r \right] q_i^{(A_{ij}+\eta)(A_{ij}+\eta-1)} t^{(A_{ij}+r)} X_i^{\eta+t} Z_i^{\eta+t}. \tag{54}
\]
In (54) we evaluate the terms \((1 - X_i Z_i)^{1 - A_{ij} - r}\) using (42) and obtain
\[
\sum_{r=0}^{1-A_{ij}} \sum_{\eta=0}^{1-A_{ij}-r} \sum_{t=0}^{1-A_{ij}-r} (-1)^{r+\eta+t} \left[ 1 - A_{ij} \right] \left[ r \right] \left[ 1 - A_{ij} - r \right] q_i^{(A_{ij}+\eta)(A_{ij}+\eta-1)} t^{(A_{ij}+r)} X_i^{\eta+t} Z_i^{\eta+t}. \tag{55}
\]

In (55) we change variables by substituting \(r = u + \eta\) and \(t = v - \eta\). We find that for \(0 \leq \eta \leq 1 - A_{ij}\) and for \(\eta \leq v \leq 1 - A_{ij}\) the coefficient of \(X_i^v Z_j^v X_j^{v-\eta} Z_i^{\eta}\) in (55) is equal to
\[
(-1)^{v+\eta} \left[ 1 - A_{ij} \right] \left[ \sum_{\eta=0}^{v} (-1)^\eta \left[ \sum_{r=0}^{1-A_{ij}-v} (-1)^u \left[ 1 - A_{ij} - v \right] \eta(v-1) \right] q_i^{(A_{ij}+\eta)(A_{ij}+v-1)} \tag{56}
\]

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The sum (57) is equal to \( \delta_{1-A_{ij},v} \) in view of (8). For \( v = 1 - A_{ij} \) the expression (56) is equal to
\[
(-1)^{1-A_{ij}+\eta} \left[ \frac{1 - A_{ij}}{\eta} \right].
\] (58)

From these comments we find (55) is equal to
\[
(-1)^{1-A_{ij}} \prod_{r=0}^{1-A_{ij}} (1 - q_i^{A_{ij} + 2s}).
\] (60)

In line (60) the product is 0 if \( A_{ij} \) is even and of course \((-1)^{1-A_{ij}} = 1\) if \( A_{ij} \) is odd. Therefore (60) is equal to (53). We have now shown that (48) is equal to (53). Since each of (47), (48) is equal to (53), their difference is zero. We showed earlier that (43) is equal to this difference so (43) is equal to zero. We have now verified (38). Before we verify (39) we have a comment. Observe that there exists an antiautomorphism of \( U \) that satisfies
\[
X_i \rightarrow X_i, \quad Y_i \rightarrow Z_i, \quad Z_i \rightarrow Y_i
\] (61)
for \( 1 \leq i \leq n \). The expression on the left in (59) is a scalar multiple of
\[
(-1)^{1-A_{ij}} \sum_{r=0}^{1-A_{ij}} (-1)^r \left[ \frac{1 - A_{ij}}{r} \right] (Y_i - X_i^{-1})^{1-A_{ij}-r} (Y_j - X_j^{-1})(Y_i - X_i^{-1})^r.
\] (62)

We assume \( i \neq j \) and show that (62) is zero. Earlier we showed that (43) is zero. Applying the antiautomorphism (61) to (43) we find
\[
(-1)^r \left[ \frac{1 - A_{ij}}{r} \right] (1 - Y_i X_i)^r (1 - Y_j X_j^r)(1 - Y_i X_i) \equiv 0.
\] (63)

In (63) we change variables by replacing \( r \) by \( 1 - A_{ij} - r \). We also multiply both sides by \(-1\). We find
\[
(-1)^r \left[ \frac{1 - A_{ij}}{r} \right] (Y_i X_i - 1)^{1-A_{ij}-r} (Y_j X_j - 1)(Y_i X_i - 1)^r = 0.
\] (64)
Applying the antiautomorphism (61) to (40), (41) and manipulating the result we obtain
\[(Y_j X_j - 1) X_i^{-1} = q_i^{-A_{ij}} X_i^{-1} (Y_j X_j - 1), \tag{65}\]
\[(Y_i X_i - 1) X_i^{-1} = q_i^{-2} X_i^{-1} (Y_i X_i - 1). \tag{66}\]
Multiplying both sides of (64) on the right by \(X_i A_{ij} X_j^{-1}\) and evaluating the result using (65), (66) we routinely find that (32) is zero. By the above comments (39) holds in \(U\).
We have now shown that (33)–(39) hold in \(U\). Therefore (4)–(6) gives a homomorphism of \(\mathbb{K}(q)\)-algebras from \(U_q(g)\) to \(U\). So far we have shown that (1)–(3) gives a homomorphism of \(\mathbb{K}(q)\)-algebras from \(U\) to \(U_q(g)\) and that (4)–(6) gives a homomorphism of \(\mathbb{K}(q)\)-algebras from \(U_q(g)\) to \(U\). One routinely verifies that these maps are inverses. Therefore, each of these maps is a bijection and hence an isomorphism of \(\mathbb{K}(q)\)-algebras.

**Definition 5.2** By the **equitable presentation** for \(U_q(g)\) we mean the presentation given in Theorem 5.1. We call the generators \(X_i, X_i^{-1}, Y_i, Z_i (i = 1, \ldots, n)\) the **equitable generators**.

**Definition 5.3** For notational convenience, we identify the copy of \(U_q(g)\) given in Definition 4.2 with the copy of \(U_q(g)\) given in Theorem 5.1, via the isomorphism given in Theorem 5.1.

The Hopf algebra structure for \(U_q(g)\) given in Lemma 4.3 looks as follows in terms of the equitable generators.

**Theorem 5.4** With reference to Lemma 4.3 and Definition 5.3, the comultiplication \(\Delta\) satisfies
\[
\Delta(X_i) = X_i \otimes X_i, \\
\Delta(Y_i) = (Y_i - 1) \otimes X_i^{-1} + 1 \otimes Y_i, \\
\Delta(Z_i) = (Z_i - 1) \otimes X_i^{-1} + 1 \otimes Z_i.
\]

The counit \(\varepsilon\) satisfies
\[\varepsilon(X_i) = 1, \quad \varepsilon(Y_i) = 1, \quad \varepsilon(Z_i) = 1.\]

The antipode \(S\) satisfies
\[S(X_i) = X_i^{-1}, \quad S(Y_i) = 1 + X_i - Y_i X_i, \quad S(Z_i) = 1 + X_i - Z_i X_i.\]

**Proof:** Routine verification.

We finish this section with some remarks.

**Remark 5.5** Referring to the relations (E7), (E8) in Theorem 5.1, if \(A_{ij}\) is even then the product on the right-hand side is zero.
Remark 5.6 With reference to Lemma 4.3 and Definition 5.3 for \(1 \leq i \leq n\) and for \(y_i = Y_i - 1, z_i = Z_i - 1\) we have
\[
\Delta(y_i) = y_i \otimes X_i^{-1} + 1 \otimes y_i, \\
\Delta(z_i) = z_i \otimes X_i^{-1} + 1 \otimes z_i.
\]
We will discuss these two equations in the next section.

Remark 5.7 For the quantum group of type \(A_1^{(1)}\) the equitable presentation is essentially the same as the presentation given in [5, Theorem 2.1].

6 The equitable generators and twisted primitive elements

In this section we discuss how the equitable generators are related to Koornwinder’s twisted primitive elements [14], [15]. We begin with a definition.

Definition 6.1 [10, p. 56] For a Hopf algebra \(U\), an element \(u \in U\) is called group-like whenever \(u \neq 0\) and \(\Delta(u) = u \otimes u\). We let \(G(U)\) denote the set of group-like elements of \(U\).

Lemma 6.2 [10, p. 56] With reference to Definition 6.1, for all \(u, v \in G(U)\) we have \(uv \in G(U)\). Also for \(u \in G(U)\), \(u\) has an inverse in \(G(U)\) which is equal to \(S(u)\). Consequently \(G(U)\) is a group.

Example 6.3 [18, Lemma 1] For the Hopf algebra \(U = U_q(sl_2)\) the group \(G(U)\) consists of the elements \(X^i\) \((i \in \mathbb{Z})\), where \(X\) is from Definition 3.2.

Definition 6.4 [10, p. 48] With reference to Definition 6.1 an element \(u \in U\) is called primitive whenever \(\Delta(u) = 1 \otimes u + u \otimes 1\).

Definition 6.5 [15, p. 801] With reference to Definition 6.1, for \(u \in U\) and for \(g \in G(U)\) we say \(u\) is twisted primitive with respect to \(g\) whenever \(\Delta(u) = g \otimes u + u \otimes S(g)\).

Example 6.6 For the Hopf algebra \(U_q(sl_2)\) an element \(u\) is twisted primitive with respect to \(X\) whenever \(\Delta(u) = X \otimes u + u \otimes X^{-1}\).

Comparing Example 6.6 with Remark 3.5 we find that although \(y, z\) are not twisted primitive with respect to \(X\), there is some resemblance. In order to interpret this resemblance we make a definition.

Definition 6.7 With reference to Definition 6.1, for \(u \in U\) and for \(g \in G(U)\) we say \(u\) is quasi twisted primitive with respect to \(g\) whenever \(\Delta(u) = 1 \otimes u + u \otimes S(g)\).

The following lemma shows how quasi twisted primitive elements are related to twisted primitive elements. The proof is routine and omitted.
Lemma 6.8 With reference to Definition 6.1, for \( g \in G(U) \) the map \( u \mapsto uS(g) \) gives a bijection from (i) the set of elements in \( U \) that are twisted primitive with respect to \( g \), to (ii) the set of elements in \( U \) that are quasi twisted primitive with respect to \( g^2 \).

Remark 6.9 With reference to Definition 6.1 in general the set of squares \( \{ g^2 \mid g \in G(U) \} \) is a proper subset of \( G(U) \). By this and in view of Lemma 6.8 the concept of a quasi twisted primitive element is a bit more general than the concept of a twisted primitive element.

Theorem 6.10 Referring to Remark 5.6 each of \( y, z \) is quasi twisted primitive with respect to \( X \). Referring to Remark 5.6 for \( 1 \leq i \leq n \) each of \( y_i, z_i \) is quasi twisted primitive with respect to \( X_i \).

Proof: Immediate from Remark 3.5, Remark 5.6 and Definition 6.7.

We refer the reader to [12], [13], [14], [19], [20], [21], [22], [23], [24] for more information about twisted primitive elements and related topics.

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