Stochastic Representation Decision Theory: How Probabilities and Values are Entangled Dual Characteristics in Cognitive Processes

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**Stochastic representation decision theory:**

**How probabilities and values are entangled dual characteristics in cognitive processes**

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**Abstract:** Humans are notoriously bad at understanding probabilities, exhibiting a host of biases and distortions that are context dependent. This has serious consequences on how we assess risks and make decisions. Several theories have been developed to replace the normative rational expectation theory at the foundation of economics. These approaches essentially assume that (subjective) probabilities weight multiplicatively the utilities of the alternatives offered to the decision maker, although evidence suggest that probability weights and utilities are often not separable in the mind of the decision maker. In this context, we introduce a simple and efficient framework on how to describe the inherently probabilistic human decision-making process, based on a representation of the deliberation activity leading to a choice through stochastic processes, the simplest of which is a random walk. Our model leads naturally to the hypothesis that probabilities and utilities are entangled dual characteristics of the real human decision making process. It derives two previously postulated features of prospect theory (Kahneman and Tversky, 1979): the inverse S-shaped subjective probability as a function of the objective probability and risk-seeking behaviour in the loss domain. It also predicts observed violations of stochastic dominance (Birnbaum and Navarrete, 1998) while it does not when the dominance is “evident”. Our theory, which offers many more predictions for future tests, has strong implications for psychology, economics and artificial intelligence.

**Keywords:** stochastic decision theory, duality of probability and value, subjective probability, risk-seeking behaviour, stochastic dominance

**JEL:** A12, C44, D81

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Introduction

Randomness is a fundamental component in most human affairs, from economics and politics to medicine and sports. Yet, people often make poor and inconsistent decisions when confronted with it. The rational normative recipe of Expected Utility Theory (Morgenstern & Von Neumann, 1953) has shown major limitations in accounting for how strongly people misperceive probabilities and uncertainty (Allais, 1953; Kahneman & Tversky, 1972; Tversky & Kahneman, 1974), leading to the notion of bounded rationality (Simon, 1972) and a long list of behavioral biases and fallacies. Several attempts (Chew, Epstein, Segal, et al., 1991; Hey, 1984; Karmarkar, 1978; Loomes & Sugden, 1982; Luce & Fishburn, 1991; Quiggin, 1982; Tversky & Kahneman, 1992; Yaari, 1987) have been made to explain such fallacies, replacing the objective probabilities of events with “decision weights”, but still retaining a sort of expectation principle, where the attractiveness of an event is decomposed into the product of (subjective) probability and (subjective) value. Numerous evidence (Halpern & Irwin, 1973; Harris, Corner, & Hahn, 2009; Irwin, 1953; Kadane & Winkler, 1988; Pruitt & Hoge, 1965; Slovic, 1966) however suggest that the two are not independent; for example, people tend to overestimate the probability of an event if the associated outcome is bad. Rank-Dependent Theories (Luce & Fishburn, 1991; Quiggin, 1982; Tversky & Kahneman, 1992) partially take into account the effect of value on probability, such that decision makers tend to overweight only events with ‘extreme’ consequences. However, their axiomatic structure prevents them to account for observed violations of stochastic dominance (Birnbaum, Patton, & Lott, 1999). Operationally, estimating the subjective probability and utility as two separate entities is subjected to the joint-hypothesis problem (Shaffer, 1995), leading to severe limitations for real-life applications.

The above-cited frameworks are deterministic in nature, postulating that the best option will always be chosen. When tested against empirical data, a probabilistic component is needed (Manski, 1977) to account for observed “noise” and “inconsistencies” (Agranov & Ortoleva, 2017; Mosteller &
Nogee, 1951). We can distinguish two classes of probabilistic theories of decision-making: random utility maximization (RUM) models and stochastic decision processes. The former, introduced by Thurstone (1927), assumes that the “perceived” utility of an option is a random variable, written as the sum of a “true” fixed utility and a random disturbance, encoding the deviation from rational behaviour. Debreu (1958) proves the existence of a utility function representing a stochastic preference relation with a minimal set of assumptions. McFadden (2001) describes the evolution of RUM models over the past decades, linking it to the Luce choice axiom (LCA) (Luce, 1977), a very useful assumption enforcing desirable properties such as independence from irrelevant alternatives and weak stochastic transitivity. However, several empirical studies (Busemeyer, 1980; Myers & Katz, 1962; Myers, Suydam, & Gambino, 1965; Simonson, 1989; Simonson & Tversky, 1992; Tversky & Simonson, 1993) show how humans do not always conform to such structure, the most famous example being the “red bus/blue bus” problem (Dodwell, 1961).

The second class of models assumes that the utility of alternatives is fixed, but the process leading to a decision is inherently stochastic. Regarding choice in uncertain environment, the most famous model is decision field theory (DFT) (Busemeyer & Townsend, 1993), where a stochastic process (Brownian motion) is assumed to mimic the fuzzy and hesitant deliberation activity of human mind. The theory takes inspiration from the Ratcliff Drift-diffusion models (DDMs) (Gold & Shadlen, 2001; Link & Heath, 1975; Ratcliff, 1978), which have shown to well describe choices and reaction times in perceptual decision-making tasks (for example, discriminating a motion direction). As reported by Tajima, Drugowitsch, and Pouget (2016), although for value-based decisions, the DDMs are driven only by the difference in item values - therefore not depending on the absolute values of the options - they are generally better at explaining human behaviour than other models (e.g. race models (Teodorescu & Usher, 2013; Vickers, 1970)).
In almost all theories, the “true” utility of a gamble or its index of worth is obtained by combining probabilities and outcomes in a (subjective) expectation, irrespective of the probabilistic model adopted (additive random disturbance or drift-diffusion process). As a result, all these frameworks carry some problematic aspects of expected utility theory, the most prominent (and oldest) being embodied in the St. Petersburg paradox (Weiss, 1987), where an infinite expectation value of the gamble would imply infinite willingness to pay, while in reality many people would pay at most a small amount (Martin, 2011).

When shifting from the normative perspective of decision theory (telling what people should do) - where expected utility proves best - to a descriptive perspective (reporting what people actually do), it is worthwhile to investigate alternative mechanisms of value formation in human mind that are different from the class of generalized expectation approaches mentioned above. Indeed, from a semantic perspective, separating probability and outcome seems quite odd, since any probabilistic statement must contain explicitly or implicitly information about “value”. In other words, a probability number quantifies the likelihood of a concrete event that is specified, and this event carries an explicit value or implicit assessment of worth or impact. For instance, when conceiving the likelihood of a natural disaster, one cannot help not thinking of the potential associated destruction and losses of lives, which are therefore implicitly connected to a cost. When thinking of the probability of the election of some political candidate, one cannot avoid envisaging the social, economic and financial consequences, which carry an implicit value judgement. Generally, whatever the event, it carries either a direct value or an indirect value assessment, even if not fully formalized in the mind of the probability assessor. Therefore, the way in which outcomes and probabilities interact in human mind seems to be much more entangled than represented by the simple factorization prevalent in utility theories and their generalizations in behavioral economics and psychology. The intermingled nature of probabilities and values have been reported by Lopes (1981) and is highlighted in the above-cited experiments (Halpern &
Irwin, 1973; Harris et al., 2009; Irwin, 1953; Kadane & Winkler, 1988; Pruitt & Hoge, 1965; Slovic, 1966), which demonstrate the effect of outcomes on perceived probability.

Our Contribution

Here we propose a new framework for describing human decisions under risk, based on a representative stochastic process – in the same spirit of drift-diffusion models – but with a notable difference: outcomes and probabilities are not merely multiplied to form an index of worth, rather they combine in a non-symmetric and non-separable way, as dual characteristics of an event. The difference will be evident when presenting the model into details, but the core concept is the following. In drift-diffusion models, as in DFT and race models, outcomes and associated probabilities of gambles are combined in a unique entity, a mathematical expectation, that then plays the role of a drift component of the stochastic process representing the decision-maker. The decision is triggered when the process reaches a threshold, called decision criterion, usually related to the time available for making a choice (the closer the threshold to the starting point, the faster the process will reach it). In our framework, probabilities and outcomes play a structurally different role; a decision occurs when the diffusive particle is absorbed at the end-point of an interval associated with a given event, whose distance is solely determined by the event’s probability. The existence of \( n \) events is thus represented by \( n \) absorbing end-points at the end of \( n \) arms in a starfish configuration along which the Brownian particle diffuses. The \( n \) arms have different lengths controlled by the probabilities of the associated events. In this representation, it is natural to conceptualize the values or utilities of the outcomes by adding drifts characterizing each arm of the starfish, the larger the value of an event, the larger the drift that biases the random walk towards the corresponding end-point. Notice that this mapping respects the positivity of the probabilities associated with the arms’ lengths, while the drifts can be attractive or repulsive to reflect gains and losses, respectively.
More concretely, consider several outcomes $A, B, C ...$, each understood to occur with probabilities $p_A, p_B, p_C, ...$. Our key idea is that the mind imagines consciously or unconsciously some bundles of random paths wandering around in some abstract space, where the alternative outcomes $A, B, C ...$ are identified as distinct domains (absorbing boundaries) in this space. The distance between domain representing outcome, say, $A$ and the initial position of the particle is inversely proportional to $p_A$, while the bias, responsible for the attraction of the particle to the boundary, is proportional to the outcome $A$. The probability for the diffusing particle to be absorbed by a particular domain is then primarily interpreted as a measure of attractiveness of the associated event, as in DFT; at the same time, a conditional absorption probability can be interpreted as a subjective value-distorted probability, as we will see below.

Thanks to the mutual interaction between perceived probabilities and perceived value of outcomes embedded in the starfish geometry with drifts, we derive two previously postulated features of prospect theory (Kahneman & Tversky, 1979): the inverse S-shaped subjective probability as a function of the objective probability and risk-seeking behaviour in the loss domain. However, these two entities are not exactly those described by prospect theory, because they are not separable. Rather, they can be inferred and rationalized by studying how the predicted choice probability depends on events’ outcomes and probabilities.

Moreover, without added assumptions, our model conforms naturally to Luce choice axiom (Luce, 1977), enforcing weak stochastic transitivity for pairwise choices. It also predicts violations (as well as observance) of stochastic dominance, in agreement with empirical data (Birnbaum & McIntosh, 1996). While usual drift-diffusion models have non-trivial and somehow artificial generalizations beyond binary choices (Roe, Busemeyer, & Townsend, 2001), our representation remains essentially locally unidimensional for an arbitrary number of available options.
Notwithstanding its predictive power, given its simplicity, the present version of our model has some important limitations. Different from decision field theory, it cannot take into account the role of “time pressure” on decision-making (Wallsten & Barton, 1982; Zur & Breznitz, 1981), because the calculations assume infinite available time by construction. There are a number of ways to introduce time pressure and finite decision times, which will be the object of future reports and are beyond the scope of the present introductory paper. Furthermore, because of LCA, it cannot predict observed violations of transitivity (Busemeyer, 1980; Myers & Katz, 1962; Myers et al., 1965) and independence from irrelevant alternatives (Simonson, 1989; Simonson & Tversky, 1992; Tversky & Simonson, 1993) (similarity effect, attraction effect, compromise effect). Finally, the proposed stochastic representation is more of an allegory that should not be taken at literally meaning that the human brain imagines all possible random paths wandering around in some abstract space for several outcomes $A, B, C \ldots$, for instance as a result of limited human working memory. Our framework is proposed as a first minimal complexity model or null-model of human risky choice, which provides the baseline for further elaboration and improvements. Indeed, our present model is characterized essentially by only one tuning parameter (compared for instance to the seven parameters of DFT). In the future, we will present extensions of the model obtained by relaxing some assumptions.

In summary, motivated by: i) empirical evidence for “interaction” between probability and value (Halpern & Irwin, 1973; Harris et al., 2009; Irwin, 1953; Kadane & Winkler, 1988; Pruitt & Hoge, 1965; Slovic, 1966); ii) empirical evidence for intrinsically probabilistic human choice (Kurtz-David, Persitz, Webb, & Levy, 2019); iii) success of drift-diffusion models in describing human behaviour in several tasks, we present a new probabilistic decision theory that combines probability and value in a non-separable way. Despite its simplicity, it provides straightforward derivations at a more microscopic level of several known structures that have been documented empirically in human decision theory.

**Formulation of the stochastic representation of lotteries**

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In the simplest possible situation, a decision maker (DM) has to make a choice between playing two binary lotteries:

\[ L_1 = \{ o_A, p; o_B, 1-p \} \quad \text{or} \quad L_2 = \{ o_C, q; o_D, 1-q \} \]  

(1)

If the DM chooses lottery \( L_1 \) (resp. \( L_2 \)), she will receive amount \( o_A \) (resp. \( o_C \)) with probability \( p \) (resp. \( q \)), and \( o_B \) (resp. \( o_D \)) with probability \( 1-p \) (resp. \( 1-q \)). The amounts can be negative, corresponding to losses.

As mentioned in the introduction, our model is conceptually analogous to drift-diffusion models, including decision field theory (DFT), i.e. a stochastic process is assumed to represent the human deliberation activity leading to a decision; choice is triggered when the process reaches a certain threshold. Figure 1 shows how the above binary choice is represented in DFT: if the process (Brownian particle in the simplest case) reaches the upper boundary (resp. lower boundary) first, then lottery \( L_1 \) (resp. \( L_2 \)) is chosen. The drift component of the motion is related to the difference \( d \) between expected-like utilities of the lotteries

\[
d = EU(L_1) - EU(L_2)
\]

\[
EU(L_1) = \pi(p)u(o_A) + \pi(1-p)u(o_B)
\]

\[
EU(L_2) = \pi(q)u(o_C) + \pi(1-q)u(o_D)
\]

(2)

where \( u \) and \( \pi \) are the so-called utility and probability functions, respectively.

In our framework, an alternative way of value formation is assumed, keeping in mind the numerous evidence (Halpern & Irwin, 1973; Harris et al., 2009; Irwin, 1953; Kadane & Winkler, 1988; Pruitt & Hoge, 1965; Slovic, 1966) showing relevant interaction between probability and value perception. We start from a plausible representation of the lotteries’ objective probabilities, as perceived by the decision maker. Typically, humans find easier to understand probabilities in terms of frequencies (Batanero, Chernoff, Engel, Lee, & Sánchez, 2016). Therefore, we propose to model their cognition via the occurrence of favorable random walk paths that hit some target, an absorbing
boundary in this case. In other words, we view the cognitive processes leading to the “feeling” or “understanding” of probability as imagining a bundle of random walkers wandering about, and the perception of the actual occurrence of the event as the arrival of random walkers in some boundaries or some domains. This representation allows one to give substance and meaning to what is the perception of probability, equal to the fraction of “successful” paths, in the standard frequentist approach of probability theory (Ellis, 1843).

Once the lotteries’ objective probabilities are encoded into some absorption probabilities, we introduce lotteries’ outcomes and account for: i) their intrinsic utility; ii) their effect on perceived probability. The simplest incarnation of this twofold effect is to introduce an outcome-dependent force (derived from a potential energy) that biases the random walk, producing a value-distorted understanding of probability. This construction leads to an effective influence between probabilities and outcomes; such reciprocal interaction will result in a distorted perception of these two entities by the decision-maker, that in turn determines her decision preferences.

Put differently, instead of compressing all the lottery information into an expectation-like index of worth, we “unpack” a lottery by introducing an absorbing branch for each of its outcome-probability pairs. As a consequence, the topology of the space where the stochastic process wanders will depend on the specific choice setup. This condition resonates with the fact that, in many situations, utility maximization is computationally intractable (Bossaerts, Yadav, & Murawski, 2018).

Operationally, we represent choosing between $L_1$ and $L_2$ with a Brownian particle undergoing a continuous random walk (Feller, 2008) that starts at the crossing (taken as the origin) between 4 segments, 2 per lottery, as shown in Figure 2 (to be compared with one segment used in DFT, as shown in Figure 1 along the $y$-axis, while the $x$-axis is the time of deliberation). Pictorially, the decision-maker is identified with the Brownian particle itself, whose stochastic path simulates the deliberation act taking place while evaluating the possible alternatives. Each branch encodes information about one lottery.
outcome - through a potential energy tilting the branch - and its associated probability of occurrence, through the branch length ending with an absorbing boundary.

When the process is restricted to represent only one lottery, the probability to be absorbed at the end of one branch can be interpreted as the value-distorted subjective probability of the associated outcome (see section “Subjective Probability”). In the presence of two (or more) lotteries, the probability to be absorbed at the end of one branch of a given lottery gives a contribution to the total probability that this lottery is chosen. The probability of choosing lottery \( L_1 \) (resp. \( L_2 \)) - denoted by \( P(L_1) \) (resp. \( P(L_2) \)) - is thus given by

\[
P(L_1) = P(1_a) + P(1_b), \quad P(L_2) = 1 - P(L_1) = P(2_c) + P(2_d),
\]

where \( P(k_{\eta_k}) \) denotes the probability for the particle to be absorbed by the wall located at distance \( \eta_k \) on branch \( k_{\eta_k} \), for \( k = 1, 2 \) with \( \eta_{k=1} \in \{a, b\} \) and \( \eta_{k=2} \in \{c, d\} \). In words, the probability of choosing, say, lottery \( L_1 \) is given by the sum of two terms: the probability of being absorbed along branch \( 1_a \) - representing \((a_1, p)\) - plus the probability of being absorbed along branch \( 1_b \), representing \((a_2, 1 - p)\).

To quantify the meaning of an outcome \( o_a \), we assume the existence of a preference or value function \( u(o_a) \), endowed with the minimal standard properties of being non-decreasing and concave on the gain side to represent risk aversion (see section “Risk-seeking behavior for losses” for the loss side). Then, the form of the potential energy acting on the Brownian particle along a branch with outcome \( o_a \) is taken as linear, with a slope proportional to \( u(o_a) \), as represented by dashed lines in Figure 2. This corresponds to a constant force acting on the Brownian particle along each segment. The sign of the energy potential is such that the greater is an outcome, the higher is the attraction toward the corresponding branch end point.
This representation has the advantage of remaining essentially one-dimensional, the motion on each segment being governed by a simple partial differential equation. For example, the probability density $p(x,t)$ of the particle at position $x$ and time $t$ on branch $L_a$ (of length $a$) evolves according to the following Fokker-Planck equation (Fokker, 1914; Planck, 1917)

$$\begin{align*}
\frac{\partial p(x,t)}{\partial t} &= u(o_a)\frac{\partial p(x,t)}{\partial x} + D\frac{\partial^2 p(x,t)}{\partial x^2} \\
p(a,t) &= 0 \forall t \quad \text{(absorbing boundary)} \\
p(0,t) &= f(t) \quad \text{(probability mass from other branches)}
\end{align*}$$

(4)

where $u(o_a)$ is the constant drift acting on the particle, $D$ is the so-called diffusion coefficient and the two boundary conditions account respectively for the absorbing wall at distance $a$ from the origin, and $f(t)$ represents the probability of the random walker incoming at the origin from other branches.

Simple dimension analysis of equation (4) shows that $D$ sets the scales for the impact of the outcome values compared with the probabilities in the value formation process: (i) taking very large $D$’s amounts to neglecting the influence of outcome values; (ii) small $D$’s make outcome values dominant in the construction of preferences.

**Explicit expressions for the decision probabilities**

As shown in Figure 2, the probability $P(L_1)$ (resp. $P(L_2)$) for the decision maker to choose lottery $L_1$ (resp. $L_2$) is represented by solving equation (4) for each of the four branches with the matching condition of the conservation of the probability of presence of the Brownian particle when crossing the junction point at the origin. Using the theory of random walks and diffusion processes (Gardiner, 2009), we obtain (see Appendix for derivation)

$$P(L_1) = \frac{U(L_1)}{U(L_1) + U(L_2)}, \quad P(L_2) = 1 - P(L_1) = \frac{U(L_2)}{U(L_1) + U(L_2)}$$

(5)

with
Expression (5) recovers the ratio scale representation of Luce’s choice axiom for binary choice (Luce, 1977), with effective utilities given by (6) and (7). This implies the desirable property of stochastic transitivity for pairwise choices: \( P(L_1) \geq P(L_2) \) and \( P(L_2) \geq P(L_3) \) implies that \( P(L_1) \geq P(L_3) \). Note that the solution of equation (4) for \( N \) alternatives generalizes into

\[
P_N(L_j) = \frac{\mathcal{U}(L_j)}{\sum_{i=1}^{N} \mathcal{U}(L_i)}
\]

where \( P_N(L_j) \) is the probability of choosing lottery \( L_j \) among the \( N \) available lotteries and the \( \mathcal{U}(L_i) \)'s are the generalized utilities given by expressions of the form (6) and (7).

As can be seen from (6) and (7), the utility \( \mathcal{U}(L) \) of a given lottery is given by the sum of two terms, each representing the attractiveness of an outcome-probability pair, which cannot be decomposed in a simple product of utility and subjective probability, as in expected utility theories. In contrast, probabilities and utilities combine and interact in a non-trivial way, with \( D \) quantifying the relative importance of value with respect to probability assigned by the DM. This becomes evident when taking the asymptotic limits of, e.g., \( P(L_1) \) (in the presence of another lottery \( L_2 \) offered as the second option):
In our framework, a decision maker characterized by \( D \to 0 \) (resp. \( D \to \infty \)) is influenced only by outcome values (resp. probabilities), while for finite \( D \) his decision derives from an entangled mixture of both.

**Subjective probability**

Equation (5), together with (6) and (7), show the resulting form of the decision probabilities for the binary risky choice (1). From here, we now focus on studying the predicted probability perception of the DM, say, of outcome \( o_A \) of lottery \( L_1 \). A convenient way to extract such information is to look at the probability of absorption along branch \( 1_a \), conditional on being absorbed along any branch pertaining to \( L_1 \)

\[
\pi(p) := P(1_a \mid L_1) = \frac{\tilde{U}_p(o_A)}{U_p(o_A) + U_{1-p}(o_B)}
\]

\[
\tilde{U}_p(o_A) := \frac{u(o_A)}{1 - e^{-\frac{2u(o_A)}{pD}}} , \tilde{U}_{1-p}(o_B) := \frac{u(o_B)}{1 - e^{-\frac{2u(o_B)}{(1-p)D}}}
\]

\( \pi(p) \) can be interpreted as the amount of attention devoted to outcome \( o_A \) when the DM is looking at lottery \( L_1 \). Several authors (Diecidue & Wakker, 2001; Schkade & Johnson, 1989) have established connections between subjective probability and similar psychological notions. Indeed, the fact that \( \pi(p) \) defined in (10) represents a meaningful measure of subjective probability is supported by its asymptotic limits as a function of \( D \):
For $D \to +\infty$, outcome values (potential energies) become negligible compared with the stochastic component and the probability perception is unaltered, so that the subjective probability is equal to the objective one. In contrast, for $D \to 0$, the decision maker does not pay attention to the probabilities and focuses solely on the payoffs, interpreting their likelihood only as a function of their magnitude.

For finite non-zero $D$, an interesting value-distortion of probability perception arises: Figures 3a-b shows $\pi(p)$ vs $p$ for different $u(o_B)/u(o_A)$ and $D$ values. Our theory thus derives the empirical inverse S-shape of subjective probability as a function of objective probability, for instance used in standard Prospect Theory by Tversky and Kahneman (1992), indicating that human beings tend to overestimate rare events and underestimate high probability events. More specifically, $\pi(p) \geq p$ (resp. $\pi(p) < p$) for $p \leq p^*$ (resp. $p > p^*$), where $p^*$ is the inflection point of $\pi(p)$ given by

$$p^* = \frac{u(o_A)}{u(o_A) + u(o_B)} \left( \frac{\partial^2 \pi(p)}{\partial p^2} \bigg|_{p=p^*} = 0 \right).$$

Our theory predicts that the asymmetry in the distortion of $\pi(p)$ for $p \to 0$ and $p \to 1$ is controlled by $u(o_A)/u(o_B)$: the larger this ratio is, the larger is the subjective distortion for small $p$’s compared with large $p$’s.

While deriving or recovering the empirical inverse S-shape, our formulation of the subjective probability is fundamentally different from those used in existing decision theories, such as the Prelec II weighting function (Prelec, 1998), which are parametrised to account for some assumed probability distortion, which is supposed to be intrinsic to the DM and can be determined by calibration of the results of a number of standard tests and questions presented to the DM (Bachmann, de Giorgi, & Hens, 2018). These subjective probabilities are considered independent of the values of the outcomes to which the probabilities are associated. We have previously argued and also referred to empirical
evidence that there is no such thing as an outcome without value. Even a question as far from every life on the probability of life on Mars, say, carries, depending on the DM, religious, scientific, and cultural values and possibly more. In our framework, the subjective probability (10) is influenced by the outcome values and represent the contribution of each outcome in a lottery to the choice of that lottery by the DM. Thus, our theory suggests that it is ill-conceived to attempt characterizing the subjective probabilities of DM. Our approach allows us to formulate a general hypothesis that subjective probabilities are value-dependent, which deserve empirical investigations. Because, in existing decision theories, the subjective probabilities are multiplied by the utilities of the associated events to form a measure of worth and then the choice probability “layer” is added on top, while in our theory subjective probabilities are instead encapsulated into decision probabilities, the former determining the latter, our theory can be viewed as a natural generalisation beyond the standard factorisation of probabilities and values to form value preferences.

**Risk-seeking behavior for losses**

In the previous section, we have studied how the outcome values alter the probability perception of the DM. Here, we study the effect of probability on value perception. Equation (5) has introduced the effective utilities \( \tilde{U}_p(.) \), which are transformed from the utilities \( u(.) \) via a non-trivial nonlinear operation involving the outcome probabilities. This corresponds to the dual of the value-distorted probability \( \pi(p) \) given in expression (10) in the form of a value perception \( \tilde{U}_p(.) \) influenced by probability. Figure 4 shows the transformed utility function \( \tilde{U}_p(.) \) as a function of the original one \( u(.) \) for different values of \( D_p := \frac{pD}{2} \).

The interaction between probabilities and values transforms an initially risk-averse (concave) utility function \( u(.) \) into a convex risk-seeking utility on a sub-interval of the loss domain, *predicting* the existence of a reference point to discriminate between behavior toward gains and behavior toward
losses, as postulated in prospect theory. We stress here that this comes as another prediction of the theory without any parameter adjustment or added ingredients. In particular, it is not a phenomenological assumption put in the theory, for instance as in Prospect Theory.

To illustrate this effect quantitatively, let us consider the utility function $u(o) = \frac{1-e^{-ro}}{r}$ for $o \in R$ with constant absolute risk aversion (CARA) $-u''/u'$ equal to the constant $r$, which is everywhere concave, continuous and differentiable. The corresponding transformed utility function $\tilde{U}_p(o)$ given by expression (10) reads

$$\tilde{U}_p(o) = \frac{(1-e^{-ro})/r}{1-\exp\left(-\frac{1}{rD_p}(1-e^{-ro})\right)}, \text{ with } D_p := \frac{pD}{2}. \quad (12)$$

Figure 5 shows $\tilde{U}_p(o)$ for different values of $D_p$ ($r = 1$). The presence of an inflection point $o_*(D_p)$ implies a risk-averse concave portion for $o > o_*(D_p)$ and a convex risk-taking behavior for $o < o_*(D_p)$ (in particular for losses). Therefore, transformation (10) and (12) predict a risk taking behavior on the loss side even when starting with a utility function that is everywhere concave. Let us illustrate this point with the following standard examples:

$$(A) \ L_1 = \{2\epsilon, 1; 0\epsilon; 0\} \text{ or } L_2 = \{4\epsilon, 0.5; 0\epsilon; 0.5\}$$

$$(B) \ L_3 = \{-2\epsilon, 1; 0\epsilon; 0\} \text{ or } L_4 = \{-4\epsilon, 0.5; 0\epsilon; 0.5\} \quad (13)$$

Within Expected Utility theory, for instance using the standard CARA utility function $u(o) = \frac{1-e^{-ro}}{r}$, one would predict that, for $r > 0$, the decision maker, being risk-averse, chooses $L_1$ in (A) and $L_3$ in (B). Kahneman and Tversky (1979) showed that most people choose instead $L_4$ in (B), exhibiting risk-seeking behavior when faced with losses. Our theory predicts this behavior for typical values of the parameters, say $r = 1$ (risk-averse in EU framework), since
\[ P(L_1) = \frac{\tilde{\mathcal{U}}_{p=1}(2) + \tilde{\mathcal{U}}_{p=0}(0)}{\mathcal{U}_{p=1}(2) + \mathcal{U}_{p=0}(0) + \mathcal{U}_{p=1/2}(4) + \mathcal{U}_{p=1/2}(0)} > P(L_2) \quad \forall D \geq D^* \approx 0.42 \quad (14) \]

and

\[ P(L_3) = \frac{\tilde{\mathcal{U}}_{p=1}(-2) + \tilde{\mathcal{U}}_{p=0}(0)}{\mathcal{U}_{p=1}(-2) + \mathcal{U}_{p=0}(0) + \mathcal{U}_{p=1/2}(-4) + \mathcal{U}_{p=1/2}(0)} < P(L_4) \quad \forall D \geq \tilde{D} \approx 0.57 \quad (15) \]

where \( \tilde{\mathcal{U}}_p(o) \) is given by expression (12). Note that only for a tiny interval of values of \( D \) the converse behavior is predicted.

Decision (A) in (13) is a classic example where the weak risk-aversion relation, denoted by \( R_w \), can be applied:

\[ L_2 R_w L_1 \iff E[L_2] = E[L_1] \text{ and } L_1 \text{ is a degenerate lottery} \quad (16) \]

meaning that \( L_2 \) is riskier than \( L_1 \). More general relations (Rothschild & Stiglitz, 1970) have been suggested to formalize risk-aversion, such as the so-called strong risk-aversion (or second-order stochastic dominance) \( R_s \):

\[ LR_s L' \iff L = L' + \hat{U} \text{ where } \hat{U} \text{ is a white noise} \quad (17) \]

Within expected utility, these two definitions of risk-aversion coincide (Pratt, 1978; Rothschild & Stiglitz, 1970), but, in general, when departing from the expectation structure, the two relations differ (Cohen, 1995) and need to be studied separately. An example for strong risk-aversion is the following:

\[ L_5 = \{0€, 0.75€; 2€; 0.25€\} \quad \text{or} \quad L_6 = \{0€, 0.5€; 1€; 0.5€\} \quad (18) \]

According to (17), \( L_5 \) is riskier than \( L_6 \); our framework, using for example \( r = 1 \) as before, predicts the correct pattern for the majority of \( D \)'s:

\[ P(L_5) \geq P(L_6) \quad \forall D \geq 1 \quad (19) \]

In summary, without added assumptions, our theory predicts what has been postulated for instance.
by prospect theory, with a concave part of the value function for gains and a convex part for losses.
These properties derive naturally from the stochastic representation of probabilities in the presence of
values.

However, the way in which the above-presented transformed utility determines choice
preferences is different from usual decision models; in expected utility, the concavity of utility function
implies risk aversion through Jensen’s inequality (Jensen et al., 1906). Here, due to the non-linear form
of $\tilde{u}_p(.)$, it is in general not easy to derive analogous simple constraints for the model parameters.

**Stochastic Dominance**

First order Stochastic Dominance (Hadar & Russell, 1969) is a property that decision theorists
usually are not willing to give up, as it essentially encodes the reasonable behaviour that “more is
better”. A random variable (gamble) $L_1$ has first-order stochastic dominance over gamble $L_2$ if
$P(L_1 \geq o) \geq P(L_2 \geq o) \, \forall \, o$ and for some $o \, \, P(L_1 \geq o) > P(L_2 \geq o)$, where \( \{o\} \) is the set of possible
outcomes. However, people often violate it when presented with choices like

\[
L_1 = \{96€, 0.90; 14€; 0.05; 12€; 0.05\} \text{ or } L_2 = \{96€, 0.85; 90€; 0.05; 12€; 0.10\}
\]  \hspace{1cm} (20)

Even if $L_1$ stochastically dominates $L_2$, most people choose $L_2$. Popular decision models like rank-
dependent utility theory (Quiggin, 1982) and cumulative prospect theory (Tversky & Kahneman, 1992)
cannot account for this pattern. Within our framework, this is explained when DM exhibit relatively low
values of $D$, such that the decision is "value-oriented" and the DM does not pay sufficient attention to
the probabilities. For this particular gamble, assuming linear utility function, $P(L_2) \approx 0.65$ for small
values of $D$, quite close to the fraction 70% of people choosing $L_2$ experimentally found by Birnbaum
and Navarrete (1998). Note that our model does not predict any violation when the dominance is
"evident", as in the following examples
It is clear that $L_1$ dominates $L_2$ in (A) and $L_3$ dominates $L_4$ in (B), and people choose accordingly.

However, several descriptive theories (Handa, 1977; Karmarkar, 1978) allowing violations in cases like (20) predicted unreasonable behaviour in tasks like (21), essentially because two outcomes with the same objective probability were forced to have the same subjective one (Fishburn, 1978). Our framework, thanks to the non-separable form of the lotteries' attractiveness, avoids this problem and confirms its significant predictive power.

Discussion

We have presented a simple and efficient “stochastic representation” framework that describes the human decision-making process as inherently probabilistic. It is based on a representation of the deliberation process leading to a choice through stochastic processes, the simplest of which is a random walk. Differently from random utility theory (external noise added to the rational utility and probability representation as a calibration procedure), our stochastic representation framework relies on a plausible description of the (assumed) intrinsic stochasticity of the human choice process. Our proposed approach does not disentangle probability and value as in expected utility theories, rather it allows interaction between them in a non-trivial way. Despite its simplicity, the model provides straightforward derivations at a more microscopic level of several known structures that have been documented empirically in human decision theory. Our theory also provide a number of novel predictions.

Here, only structural properties have been presented through simple examples, which are not sufficient to falsify the theory. At this stage, the parsimony of its formulation and the wealth of obtained properties, which are in qualitative or semi-quantitative agreement with empirically observations, makes our theory interesting to further explore. We plan to use more sophisticated procedures to test our model against the major decision theories, based on cross-validation methods: parameters are first
estimated from one part of an experiment, and then these same parameters are applied to a separate part of the experiment and the predictions are evaluated. Note that we cannot use the usual Wilks likelihood-ratio test (Wilks, 1938), because in general the models will not be nested, but other methods are possible, such as the Vuong test (Vuong, 1989) and Information Criteria (AIC: Akaike, 1974 and BIC: Schwarz et al., 1978).

The current formulation is not meant to be “the” definitive framework (if it would exist) - since as already mentioned it presents some limitations, such as those deriving from Luce choice axiom and the inability to account for “time pressure” - but a baseline to construct more elaborate models, keeping in mind the trade-off between parsimony and explanatory power.

In general, we are aware that testing alternative ways of value formation is very difficult, because of the measurement problem in economics (Mata, Frey, Richter, Schupp, & Hertwig, 2018). Indeed, we cannot really measure the “degree of happiness” of the decision-maker, but we have to infer it – adopting one particular model - through her choices. This adds an additional layer of complexity with respect to other hard sciences, such as physics or chemistry. On the other hand, contribution of this type may help to devise more effective ways to elicit preferences, deepening our understanding of decision processes. In addition, further theoretical and empirical work may lead to modifications of the presented theory, where the expected utility hypothesis (separability of probability and value) can be seen as a particular case of a more complex structure, where probability and value do interact to some extent in the decision maker’s mind.
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Figure 1. DFT-representation of binary choice (Busemeyer & Townsend, 1993): if the process reaches the upper boundary (resp. lower boundary) first, then lottery $L_1$ (resp. $L_2$) is chosen. The drift component of the motion is related to the difference between expected-like utilities of the lotteries. Time elapsed along the $x$-axis ("number of sample" denotes "time"), leading to directed paths along it.
Figure 2. Stochastic representation of the decision process between lotteries \( L_1 = \{o_A, p; o_B, 1-p\} \) and \( L_2 = \{o_C, q; o_D, 1-q\} \). Branches \( 1_{a/b} \) (resp. \( 2_{c/d} \)) represent the outcomes of \( L_1 \) (resp. \( L_2 \)) and their related probabilities. The difference between the continuous and dashed lines represent the energy potential associated with the constant forces \( \{u(o_A), u(o_B), u(o_C), u(o_D)\} \) exerted on the Brownian particle along each segment \( \{1a, 1b, 1c, 1d\} \) respectively. The thick bars at the end of each branch depict the absorption boundary conditions. The segment lengths \( \{a, b, c, d\} \) are determined by the objective probabilities \( \{p, 1-p, q, 1-q\} \). The probability of choosing lottery \( L_1 \) (resp. \( L_2 \)) is given by the probability of being absorbed along branch \( 1_a \) or \( 1_b \) (resp. \( 2_c \) or \( 2_d \)).
Figure 3. value-distorted subjective probability $\pi(p)$ given by expression (10): (a) fixed $u(o_A) = u(o_B) = 30$, $D$ is varied. As $D$ grows, $\pi(p) \to p$; (b) fixed $D = 20$ and $u(o_B) = 30$, $u(o_A)$ is varied. As $u(o_A)$ grows, the inflection point $p^* = \frac{u(o_A)}{u(o_A) + u(o_B)}$ shifts toward the right while the curve shifts upward. Varying $u(o_B)$ gives symmetrical behaviors.
Figure 4. Transformed utility function $\tilde{U}_p(.)$ (10) as a function of the original one $u(.)$ for different values of $D_p := \frac{pD}{2}$. The distortion is significant for negative $u$ and for small positive $u$ (i.e. small $u$ compared to $D_p$) and is all the stronger, the smaller is $\left| \frac{u}{D_p} \right|$. The transformed utility flattens out for large negative $u$. $\tilde{U}_p(.)$ approaches $u$ asymptotically for large positive values $\left( \tilde{U}_p(.) \rightarrow u \right)$.
Figure 5. Transformed utility \( \tilde{U}_p(o) \) (12) for different values of \( D_p := \frac{pD}{2} \ (r = 1) \). Black star points indicate the location of the inflection points \( o_r^*(D_p) \), where the function changes the sign of its concavity.
Appendix

Analytical derivation of absorption probabilities

In this appendix, we show how to derive the absorption probabilities corresponding respectively to subjective probability perception (10) and choice probability (5). The setup for the former is a Brownian particle on a line in presence of two absorbing boundaries, while for the latter the particle wanders on a “starfish” graph, where each leg represents a lottery outcome (see below). Note that the one-dimensional configuration is just a particular case of the general starfish.

Absorption probabilities in one-dimensional case

For the derivation of the absorption probabilities of a one-dimensional drifted Brownian motion in presence of two absorbing walls, we follow the procedure outlined by Gardiner (2009). For convenience, we rewrite here our Fokker-Planck equation with arbitrary initial position $x_0$:

$$\begin{align}
\frac{\partial p(x,t)}{\partial t} &= -\frac{\partial}{\partial x}[-V'(x)p(x,t)] + \frac{D}{2} \frac{\partial^2 p(x,t)}{\partial x^2} \\
p(a,t) &= p(b,t) = 0 \quad \forall t \quad \text{(absorbing boundaries)} \\
p(x,t = 0) &= \delta(x - x_0) \quad \text{(initial condition)}
\end{align}$$ (22)

Given the delta-function initial condition, we can focus on the conditional probability density $p(x,t \mid x_0, t = 0)$ that satisfies the same Fokker-Planck equation. We then define the conditional probability current $J(x,t \mid x_0, 0)$ as

$$J(x,t \mid x_0, 0) = -V'(x)p(x,t \mid x_0, 0) - \frac{D}{2} \frac{\partial p(x,t \mid x_0, 0)}{\partial x}$$ (23)

so that the Fokker-Planck equation can be rewritten as a continuity equation

$$\frac{\partial p(x,t \mid x_0, 0)}{\partial t} = -\frac{\partial J(x,t \mid x_0, 0)}{\partial x}$$ (24)
By looking at equation (24), we understand that the probability current describes the flow of probability, as if it was a fluid. Therefore, the probability \( g_a(x_0,t) \) (resp. \( g_b(x_0,t) \)) that the particle gets absorbed at \( a \) (resp. \( b \)) after time \( t \), given it is in \( x_0 \) at \( t = 0 \), is

\[
\begin{align*}
g_a(x_0,t) &= -\int_t^\infty dt' J(a,t'|x_0,0) \left. + \frac{D}{2} \frac{\partial p(a,t|x_0,0)}{\partial x} \right|_{x=a} \\
g_b(x_0,t) &= +\int_t^\infty dt' J(b,t'|x_0,0) \left. - \frac{D}{2} \frac{\partial p(b,t|x_0,0)}{\partial x} \right|_{x=b}
\end{align*}
\]  
(25)

Since \( p(a,t|x_0,0) \) satisfies a backward Fokker-Planck equation (with respect to \( x_0 \)), we find an equation for \( g_a(x_0,t) \)

\[
-V'(x_0) \frac{\partial g_a(x_0,t)}{\partial x_0} + \frac{D}{2} \frac{\partial^2 g_a(x_0,t)}{\partial x_0^2} = J(a,t|x_0,0) = \frac{\partial g_a(x_0,t)}{\partial t}.
\]  
(26)

and similarly for \( g_b(x_0,t) \). The absorption probabilities we are interested in correspond to \( g_{a/b}(x_0,0) \), i.e. the probabilities of being absorbed after \( t = 0 \). By taking the limit for \( t \to +\infty \) in (25), the right-hand side of Eq. (26) vanishes, so we have

\[
\begin{align*}
-V'(x_0) \frac{d\pi_a(x_0)}{dx_0} + \frac{D}{2} \frac{d^2\pi_a(x_0)}{dx_0^2} &= 0 \\
\pi_a(a) &= 1, \quad \pi_a(b) = 0 \\
\pi_a(x_0) + \pi_b(x_0) &= 1
\end{align*}
\]  
(27)

where, following the notation in Gardiner (2009), we defined

\[
\begin{align*}
\pi_a(x_0) &= \text{probability of being eventually absorbed at } a \text{ starting at } x_0 = g_a(x_0,0) \\
\pi_b(x_0) &= \text{probability of being eventually absorbed at } b \text{ starting at } x_0 = g_b(x_0,0)
\end{align*}
\]  
(28)

In absence of any potential \( (V(x) = 0) \), solving Eq. (27) provides
\[ \pi_a(x_0) = \frac{x_0 - b}{a - b} = \frac{b - x_0}{a + b} \]

\[ \pi_b(x_0) = 1 - \pi_a(x_0) = \frac{|a| + x_0}{|a| + b} \]  

(29)

Putting \( x_0 = 0 \) above, we retrieve the desired absorption probabilities in the flat potential case.

For the presence of a potential leading to drifting Brownian motions, let us recall the potential energy acting on the particle

\[ V(x) = \begin{cases} 
    u(o_A)x & \text{if } x \leq 0 \\
    -u(o_B)x & \text{if } x > 0 
\end{cases} \]  

(30)

Equation (27) becomes

\[ \begin{cases} 
    -u(o_A) \frac{d\pi_a^{(L)}(x_0)}{dx_0} + \frac{D}{2} \frac{d^2\pi_a^{(L)}(x_0)}{dx_0^2} = 0 & \text{if } x_0 \leq 0 \\
    u(o_B) \frac{d\pi_a^{(R)}(x_0)}{dx_0} + \frac{D}{2} \frac{d^2\pi_a^{(R)}(x_0)}{dx_0^2} = 0 & \text{if } x_0 > 0 \\
    \pi_a^{(L)}(a) = 1, \quad \pi_a^{(R)}(b) = 0 
\end{cases} \]  

(31)

where \( \pi_a^{(L)}(x_0) \) and \( \pi_a^{(R)}(x_0) \) denote \( \pi_a(x_0) \) for negative and positive values of \( x_0 \), respectively.

It is sufficient to calculate \( \pi_{a/b}(0) \), since what matters is the relative position between the starting point and the location of the absorbing walls \( a \) and \( b \). Imposing the following continuity conditions

\[ \frac{\pi_a^{(L)}(0)}{\pi_a^{(R)}(0)} = \frac{d\pi_a^{(L)}(x_0)}{dx_0} \bigg|_{x_0=0} = \frac{d\pi_a^{(R)}(x_0)}{dx_0} \bigg|_{x_0=0} \]  

(32)

we can solve Eq. (31) obtaining
By setting $a = \frac{1}{p}$ and $b = \frac{1}{1-p}$, we get the absorption probabilities reported in expression (10).

**Absorption probabilities on the “starfish” graph**

The suggested stochastic representation of choice (Fig. 2) has the advantage of being easily generalizable beyond binary choices while remaining essentially one-dimensional. Calculating directly the absorption probabilities is however quite cumbersome; therefore, we attack the problem by switching to an analogous discrete representation (random walk) both in time and in space, calculating the corresponding absorption probabilities, and finally taking the proper continuous limit.

Let us first apply this procedure to the already solved one-dimensional case, in order to show that the absorption probabilities correspond to the ones obtained in the previous section. Specifically, consider a biased random walk starting at $x = 0$ - moving by $\Delta x$ each unit time $\Delta t$ - in presence of two absorbing walls located at $-(n_A + 1)\Delta x$ and $(n_B + 1)\Delta x$, as shown in Figure A1. The transition probabilities are site-dependent

$$p_{\text{right}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2 & \text{if } 0 < x < (n_B + 1)\Delta x \\ p_B & \text{if } 0 < x < (n_A + 1)\Delta x < x < (n_B + 1)\Delta x \\ (1-p_A) & \text{if } -(n_A + 1)\Delta x < x < 0 \\ \end{cases}$$

$$p_{\text{left}}(x) = 1 - p_{\text{right}}(x)$$

(34)

where $p_A + p_B \neq 1$ in general. Our aim is to calculate the probability for the random walk to be absorbed either at $x = -(n_A + 1)\Delta x$, denoted by $P_L$, or at $x = (n_B + 1)\Delta x$, denoted by $P_R$. 

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The so-defined random walk is equivalent to a Markov chain with $r = 2$ absorbing states and $t = n_A + n_B + 1$ transient states with the following transition matrix $P$

$$P = \begin{bmatrix} Q & R \\ 0 & I_N \end{bmatrix}$$  \hspace{1cm} \text{(35)}$$

where

- $Q$ is a $t \times t$ matrix containing transition probabilities among transient states.
- $R$ is a $t \times r$ matrix containing transition probabilities from some transient states to absorbing states.
- $0$ is the $r \times t$ zero matrix.
- $I_r$ is the $r \times r$ identity matrix.

For example, with $n_A = n_B = 2$, $P$ is a $7 \times 7$ matrix with $Q$ and $R$ given by

$$Q = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1-p_B & 0 & 0 & p_B & 0 \\ 1-p_A & 0 & 0 & 0 & p_A \\ 0 & 1-p_B & 0 & 0 & 0 \\ 0 & 0 & 1-p_A & 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ p_B & 0 \\ 0 & p_A \end{bmatrix}$$  \hspace{1cm} \text{(36)}$$

The time-evolution of the stochastic process is described by the following equation

$$\vec{x}^{(s)} P = \vec{x}^{(s+1)}$$  \hspace{1cm} \text{(37)}$$

where $\vec{x}^{(s)}$ is the row vector containing the probability distribution of the particle at time $s$. Given the probability distribution $\vec{x}^{(0)}$ at time 0 (in our case the particle will start with probability 1 in $x = 0$), the distribution at time $s$ is given by

$$\vec{x}^{(s)} = \vec{x}^{(0)} P^s$$  \hspace{1cm} \text{(38)}$$

For our purpose, it is sufficient to obtain only the asymptotic absorption probabilities for the 2 absorbing states. Following Kemeny and Snell (1976), the probability of being absorbed in the absorbing
state $i$, given that the particle started in the transient state $l$, is the $(l-i)$-entry of the following matrix

$$B = FR , \quad \text{where} \quad F = \sum_{k=0}^{\infty} Q^k = (I - Q)^{-1} . \quad (39)$$

$F$ is called the fundamental matrix, whose $(i-j)$-entry contains the expected number of times the particle visits state $j$, given the starting point in state $i$. The desired absorption probabilities, starting from $x = 0$, are then given by

$$P_R = B(1,1) = \frac{p_{n_R}^n f_{n_A} p_{n_B}}{p_{n_R}^n f_{n_A} p_{n_B} + p_{n_A}^n f_{n_B} p_{n_B}}$$

$$P_L = B(1,2) = \frac{p_{n_L}^n f_{n_A} p_{n_B}}{p_{n_R}^n f_{n_A} p_{n_B} + p_{n_A}^n f_{n_B} p_{n_B}}$$

with

$$f_n(p) = \sum_{k=0}^{n} p^k (1-p)^{n-k} = \frac{p^{n+1} - (1-p)^{n+1}}{2p - 1} \quad (41)$$

To retrieve expression $(10)$, let us recall that a random walk moving by $\Delta x$ to the right (resp. left) with probability $p$ (resp. $1 - p$) after $\Delta t$ time converges as $\Delta t \to 0$ to a Brownian motion with drift $\mu$ and variance $\sigma^2 = D$ if the following relations hold:

$$\Delta x = \sigma \sqrt{\Delta t}$$

$$p = \frac{1}{2} \left[ 1 + \frac{\mu \sqrt{\Delta t}}{\sigma} \right] n \quad (42)$$

For our present case, given the potential reported in Eq. $(30)$ and the absorbing boundaries located at $a < 0$, $b > 0$, we set

$$n_B = \frac{b}{\Delta x} = \frac{b}{\sigma \sqrt{\Delta t}} , \quad p_B = \frac{1}{2} + \frac{u(o_B) \sqrt{\Delta t}}{2\sigma}$$

$$n_A = \frac{|a|}{\Delta x} = \frac{|a|}{\sigma \sqrt{\Delta t}} , \quad p_A = \frac{1}{2} + \frac{u(o_A) \sqrt{\Delta t}}{2\sigma}$$

$$=$
By taking the limit for \( \Delta t \to 0 \) of, say, \( P_R \) we have

\[
\lim_{\Delta t \to 0} P_R = \frac{u(o_B) \cdot \left( 1 - e^{-\frac{2n(o_B)\Delta t}{D}} \right)}{u(o_B) \cdot \left( 1 - e^{-\frac{2n(o_B)\Delta t}{D}} \right) + u(o_A) \cdot \left( 1 - e^{-\frac{2n(o_A)\Delta t}{D}} \right)} = \pi_b(0)
\]

Again, setting \( |q| = \frac{1}{p} \) and \( b = \frac{1}{1-p} \), we recover the expression reported in (10), showing the equivalence between the continuous and discrete representations.

Let us therefore switch to an equivalent discrete version of Figure 2, shown in Figure A2. The random walk starts in the centre of the starfish (black point) and can be absorbed in any of the absorbing states located at the end of each branch. Note that the transition probabilities are different for each segment, in order to correctly represent the different outcome-dependent potentials, while the distance of the absorbing states from the centre is different for each segment, to represent the different lottery probabilities. In this case, the random walk is equivalent to a Markov chain with \( r = 4 \) absorbing states and \( t = n_A + n_B + n_C + n_D + 1 \) transient states. Recall that the probability of choosing \( L_1 \) (resp. \( L_2 \)), denoted by \( P_w(L_1) \) (resp. \( P_w(L_2) \)), is given by the probability to be absorbed either along branch \( 1_a \) or \( 1_b \) (resp. \( 2_c \) or \( 2_d \))

\[
P_w(L_1) = P_w(1a) + P_w(1b), \quad P_w(L_2) = P_w(2c) + P_w(2d)
\]

where the subscript \( w \) indicates that we are working in the discrete random walk representation. By calculating again the matrix \( B \) defined in Eq. (39), we obtain the desired asymptotic absorption probabilities.
\[ P_w(1a) = B(1,1) = \frac{p_A}{f_{n_a}(p_A)} + \frac{p_B}{f_{n_b}(p_B)} + \frac{p_C}{f_{n_c}(p_C)} + \frac{p_D}{f_{n_d}(p_D)} \]

\[ P_w(1b) = B(1,2) = \frac{p_B}{f_{n_b}(p_B)} \]

\[ P_w(2c) = B(1,3) = \frac{p_C}{f_{n_c}(p_C)} \]

\[ P_w(2d) = B(1,4) = \frac{p_D}{f_{n_d}(p_D)} \]

(46)

with \( f_n(p) \) defined in (41). To ensure the equivalence between continuous and discrete representation we set

\[ n_A = \frac{a}{\Delta x} = \frac{a}{\sigma \sqrt{\Delta t}}, \quad p_A = \frac{1}{2} + \frac{u(o_A) \sqrt{\Delta t}}{2\sigma} \]

\[ n_B = \frac{b}{\Delta x} = \frac{b}{\sigma \sqrt{\Delta t}}, \quad p_B = \frac{1}{2} + \frac{u(o_B) \sqrt{\Delta t}}{2\sigma} \]

\[ n_C = \frac{c}{\Delta x} = \frac{c}{\sigma \sqrt{\Delta t}}, \quad p_C = \frac{1}{2} + \frac{u(o_C) \sqrt{\Delta t}}{2\sigma} \]

\[ n_D = \frac{d}{\Delta x} = \frac{d}{\sigma \sqrt{\Delta t}}, \quad p_D = \frac{1}{2} + \frac{u(o_D) \sqrt{\Delta t}}{2\sigma} \]

(47)

Expression (5) is finally retrieved by taking the continuous time limit of \( P_w(L_t) \), giving

\[ \lim_{\Delta t \to 0} P_w(L_t) = \frac{u(o_A)}{1-e^{\frac{2u(o_A)\Delta t}{D}}} + \frac{u(o_B)}{1-e^{\frac{2u(o_B)\Delta t}{D}} + \frac{u(o_C)}{1-e^{\frac{2u(o_C)\Delta t}{D}}} + \frac{u(o_D)}{1-e^{\frac{2u(o_D)\Delta t}{D}}} \]

(48)
Figure A1. Discrete Random walk analogue of a one-dimensional drifted Brownian motion in presence of two absorbing boundaries.
Figure A2. Random walk analogue of the continuous stochastic representation of choice between two binary lotteries (Figure 2). The transition probabilities are different for each segment, in order to correctly represent the different outcome-dependent potentials, while the distance of the absorbing states from the centre is different in each branch, to encode the different lottery probabilities.
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