AN ALMOST PERIODIC EPIDEMIC MODEL WITH AGE STRUCTURE IN A PATCHY ENVIRONMENT

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Abstract. An almost periodic epidemic model with age structure in a patchy environment is considered. The existence of the almost periodic disease-free solution and the definition of the basic reproduction ratio $R_0$ are given. Based on those, it is shown that a disease dies out if the basic reproduction number $R_0$ is less than unity and persists in the population if it is greater than unity.

1. Introduction. Recently, the study of epidemic models with population dispersal has become increasingly important. Population movements between countries, regions or cities are one of main reasons for the spread of many diseases, such as influenza, measles, tuberculosis (TB) and sexually transmitted diseases. Mathematical models and field observations show that population dispersal has a positive impact for the control of a disease sometimes (see [4]), while it may lead to a disease outbreaks (see [28, 29]). Originally, the epidemic model involving population dispersals is between two patches (see [7, 12]). Hereafter, a transmission epidemic model among $n$ patches is considered (see [28, 32]).

Since the work of McKendrick [16], more and more researchers have been aware that the age-structure of a population affects the dynamics of disease transmission and should be taken into account in the realistic model of some infectious diseases. For example, TB is highly age-dependent (see [5]), and the population who is likely to be infected is the adult individuals since newborns are vaccinated in many countries, but the immunity has decreased with increasing age. As noted in [15], infectious diseases like measles admit the same situation because the babies from their birth to semiprecession have a strong immune system which was passed on from their mothers and older children are immunized during their school time under the supervision of the school and government health organizations. Consequently, when we study some diseases, it is realistic to assume that the adult individuals may be only infected and the childhood has a immunity against the disease. Simultaneously, the adults are responsible for the reproduction of the population.

It is well known that seasonal variations in temperature, rainfall, resource availability, contact rates, the birth and death rates of populations and immune defences are ubiquitous and can exert strong pressures on population dynamics. Consequently, many infectious diseases are significantly affected by seasonal factors. As
noted in [1], Watts et al. [31] showed that dengue haemorrhagic fever has a timing of outbreak during hot-dry and rainy season because rainfall and temperature affect mosquito vector abundance, temperature influences parasite replication in vectors. Fine and Clarkson [9] implied that respiratory-aerosol and contact-borne pathogens Measles increases in fall or spring due to host aggregation during school terms increases transmission. From an applied perspective, a periodic epidemic model where the role of seasonality has been relatively well explored offers important insights into understanding the mechanisms of a disease outbreak. Empirical evidence points to several biologically distinct mechanisms by which the birth rate, the death rate, the recovery rate and the emigration rates are not necessary to share a common period. Especially, if the periods of these periodic coefficients have no common integer multiple, then we can treat such a model as an almost periodic system. In this paper, we consider the dynamics of an almost periodic epidemic model with age-structure in a patchy environment.

We use $M_i$ and $G_i$ to denote the number of juvenile individuals and adult individuals in path $i$, respectively. Refer to [15], we assume that $M_i$ satisfies

$$\frac{dM_i}{dt} = F_i(t, G_i)G_i - \nu_i(t)M_i,$$

where $F_i(t, G_i)$ and $\nu_i(t)$ are the per capita birth rate and the per capita death rate of juveniles, respectively, in patch $i$ at time $t$.

Consider the following epidemic model with age-structure:

$$\begin{cases}
\frac{dM_i}{dt} = F_i(t, G_i)G_i - \nu_i(t)M_i - D_i(t, G_i) + \sum_{j=1}^{n} a_{ij}(t)M_j, \\
\frac{dS_i}{dt} = D_i(t, G_i) - d_i(t)S_i - \varpi_i(t)S_iI_i + \gamma_i(t)I_i + \sum_{j=1}^{n} r_{ij}(t)S_j, \\
\frac{dI_i}{dt} = \varpi_i(t)S_iI_i - (d_i(t) + \gamma_i(t))I_i + \sum_{j=1}^{n} h_{ij}(t)I_j,
\end{cases}$$

(1)

where $S_i, I_i$ are the numbers of susceptible and infectious individuals in patch $i$, respectively. Naturally, $G_i = S_i + I_i$. $D_i(t, G_i)$ is the transition rate of juvenile individuals from juvenile stage to adult stage in patch $i$ at time $t$, $d_i(t)$ is the death rate of the population in the $i$th patch at time $t$, $\varpi_i(t)$ is the contact rate of susceptible individuals with infectious individuals at time $t$, and $\gamma_i(t)$ is the recovery rate of infectious individuals in the $i$th patch at time $t$. $r_{ij}(t), h_{ij}(t)$ and $a_{ij}(t), i \neq j$, represent the immigration rates of susceptible individuals, infectious individuals and juvenile individuals, respectively, from the $j$th patch to the $i$th patch. $-r_{ii}(t) \geq 0$, $-h_{ii}(t) \geq 0$ and $-a_{ii}(t) \geq 0$ are the emigration rates of susceptible individuals, infectious individuals and juvenile individuals, respectively, in the $i$th patch.

During the dispersal process, we assume that the death and birth of individuals are neglected, then we have

$$\sum_{j=1}^{n} r_{ji}(t) = 0, \quad \sum_{j=1}^{n} h_{ji}(t) = 0, \quad \sum_{j=1}^{n} a_{ji}(t) = 0, \text{ for all } i = 1, \ldots, n, \quad t \in \mathbb{R}^+.$$

(2)

Furthermore, we assume that

(A1) Two $n \times n$ matrices $(r_{ij}(t))$ and $(h_{ij}(t))$ are strongly irreducible.

(A2) $F_i(t, G_i) > 0$ for all $G_i > 0, t > 0$ and $i = 1, \ldots, n$.

(A3) The continuously differentiable function $F_i(t, G_i)$ satisfies $\frac{\partial F_i(t, G_i)}{\partial G_i} < 0$, $\forall G_i > 0, t > 0, i = 1, \ldots, n$. 
In the following, applying the similar arguments to those in [15], we simplify model (1) without including the item of juvenile individuals by the way of seeking the expression of $D_i(t, G_i)$. Let $M(t, a) = (M_1(t, a), \ldots, M_n(t, a))^T$, where $M_i(t, a)$ is the number of the juveniles in the $i$th patch at time $t$ with age $a$. Hence, the birth law can be denoted by

$$M(t, 0) = (M_1(t, 0), \ldots, M_n(t, 0))^T = (F_1(t, G_1)G_1, \ldots, F_n(t, G_n)G_n)^T := \Omega(t, G(t)).$$

Assume that $\tau$ is the length of the juvenile period. Define $D(t) := (D_1(t), \ldots, D_n(t)) := M(t, \tau)$. As noted in [15] (see also [25]), $M(t, a)$ satisfies the following differential equation:

$$(\partial_t + \partial_a)M_i(t, a) = \sum_{j=1}^{n} o_{ij}(t)M_j(t, a) - \left( \sum_{j=1}^{n} o_{ji}(t) + \nu_i(t) \right) M_i(t, a)$$

Let $\bar{M}(t, a) = M(t, t - a)$ for all $t \geq a \geq 0$. Then we have

$$\frac{\partial \bar{M}(t, a)}{\partial t} = E_M(t)\bar{M}(t, a), \tag{3}$$

where

$$E_M(t) = \begin{pmatrix}
-\nu_1(t) + o_{11}(t) & \cdots & o_{1n}(t) \\
o_{21}(t) & \cdots & o_{2n}(t) \\
\vdots & \ddots & \vdots \\
o_{n1}(t) & \cdots & -\nu_n(t) + o_{nn}(t)
\end{pmatrix}.$$}

Let $\Phi(t, a), t \geq a \geq 0$, be the fundamental solution matrix of (3), that is,

$$\frac{\partial \Phi(t, a)}{\partial t} = E_M(t)\Phi(t, a) \text{ and } \Phi(a, a) \text{ is an identity matrix. Hence, we conclude that}$$

$$\bar{M}(t, a) = \Phi(a, a)\bar{M}(a, a), \forall t \geq a.$$

Since $M(t, s) = \bar{M}(t, t - s) = \Phi(t, t - s)\bar{M}(t - s, t - s)$, we have

$$M(t, s) = \Phi(t, t - s)M(t - s, 0) = \Phi(t, t - s)\Omega(t - s, G(t - s)), \forall t \geq a.$$}

It then follows that

$$D(t) = M(t, \tau) = \Phi(t, t - \tau)\Omega(t - \tau, G(t - \tau)) := D(t, G(t - \tau)).$$

Let $\Phi(t, t - \tau) := (p_{ij}(t))_{n \times n}$. Since $E_M(t)$ is cooperative, it follows from [2, Lemma 2] that $p_{ij}(t) > 0$ for all $i, j = i, \ldots, n$ and $t > 0$. Obviously, $D(t)$ is independent of the variables of juveniles, and hence, we can rewrite model (1) as follows:

$$\begin{cases}
dS_i = D_i(t, G(t - \tau)) - d_i(t)S_i - \omega_i(t)S_iI_i - \gamma_i(t)I_i + \sum_{j=1}^{n} r_{ij}(t)S_j, \\
dI_i = \omega_i(t)S_iI_i - (d_i(t) + \gamma_i(t))I_i + \sum_{j=1}^{n} h_{ij}(t)I_j, \\
D_i(t, G_i(t - \tau)) = \sum_{j=1}^{n} p_{ij}(t)\Omega_j(t - \tau, G_j(t - \tau)), \\
S(\zeta) = \varphi(\zeta), I(\zeta) = \psi(\zeta), \quad \forall \zeta \in [-\tau, 0], \quad (\varphi, \psi) \in C^2_+.
\end{cases} \tag{4}$$

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where \( C_+ := C(\mathbb{R}, \mathbb{R}_+^n) \). Throughout this paper, we assume that \( F_i(t, G_i) \) and \( D_i(t, G_i) \) are uniformly almost periodic, and \( n_i(t), d_i(t), \bar{w}_i(t), \gamma_i(t), r_{ij}(t), h_{ij}(t) \) and \( a_{ij}(t) \) are almost periodic in \( t \). Hence, a similar argument to system (17) implies that \( p_{ij}(t) \) is also almost periodic. Thus, model (4) is an almost periodic time-delayed differential system. Here, the Poincaré map is invalid, we draw support from the tool of skew-product semiflows and the theory developed in [27].

By [23, Theorem 5.2.1], it follows that for any \( (\varphi, \psi) \in C^2_+ \) there exists a unique solution \( (S(t, \varphi, \psi), I(t, \varphi, \psi)) \) of (4) with \( S(t, \varphi, \psi) \geq 0, I(t, \varphi, \psi) \geq 0 \) for all \( t \geq 0 \). If no confusion, we denote \( S(t) = S(t, \varphi, \psi), I(t) = I(t, \varphi, \psi) \).

The remaining parts of this paper are organized as follows. In Section 2, we present the theory of almost periodic delay differential equations. In Section 3, we prove the existence of an almost periodic disease-free solution and define the basic reproduction ratio \( R_0 \). In Section 4, we establish threshold results on uniform persistence and global extinction.

2. Almost periodic delay differential equations. A function \( f \in C(\mathbb{R}, \mathbb{R}^k) \) is said to be almost periodic if for any \( \epsilon^* > 0 \), there exists \( l = l(\epsilon^*) > 0 \) such that every interval of \( \mathbb{R} \) of length \( l \) contains at least one point of the set

\[
T(f, \epsilon^*) = \{ s^* \in \mathbb{R} : |f(t + s^*) - f(t)| < \epsilon^*, \forall t \in \mathbb{R}, \}
\]

where \( |\cdot| \) is the usual Euclidean norm in \( \mathbb{R}^k \). A matrix is said to be almost periodic in the sense that every entry is almost periodic. Let \( AP(\mathbb{R}, \mathbb{R}^k) := \{ f \in C(\mathbb{R}, \mathbb{R}^k) : f \text{ is an almost periodic function} \} \). Then \( AP(\mathbb{R}, \mathbb{R}^k) \) is a Banach space equipped with the supremum norm \( \| \cdot \| \). Let \( \mathcal{D} \subset \mathbb{R}^k \). A function \( f \in C(\mathbb{R} \times \mathcal{D}, \mathbb{R}^k) \) is said to be uniformly almost periodic in \( t \) if \( f(\cdot, x) \) is almost periodic for each \( x \in \mathcal{D} \), and for any compact set \( \mathcal{E} \subset \mathcal{D} \), \( f \) is uniformly continuous on \( \mathbb{R} \times \mathcal{E} \) (see [6, 10]).

It follows from [6, 10] that if \( f \) is an almost periodic function, then there exists a Fourier series \( \sum_{q=1}^{\infty} A_q e^{i\lambda_q t} \) associated with the function \( f \), i.e., \( f(t) \sim \sum_{q=1}^{\infty} A_q e^{i\lambda_q t} \).

We call \( \lambda_q, q = 1, 2, \ldots \), the Fourier exponent of \( f(t) \), and \( A_q \) the Fourier coefficient of \( f(t) \). For a function \( f \in AP(\mathbb{R}, \mathbb{R}^k) \), the module of \( f \), \( mod(f) \), is defined as the smallest additive group of real numbers that contains the Fourier exponent of \( f(t) \).

A square matrix \( M \) is said to be cooperative if all off-diagonal entries of \( M \) are nonnegative. If all entries of \( M \) are nonnegative, then we say \( M \) is nonnegative.

A nonnegative square matrix \( M \) is called positive if \( M \) is not the zero matrix.

A square matrix \( \mathcal{M} = (m_{ij}) \) is called quasi-positive if it is not the zero matrix and all off-diagonal entries of \( \mathcal{M} \) are nonnegative, i.e., \( m_{ij} \geq 0 \) for \( i \neq j \).

A \( k \times k \) square matrix \( (m_{ij}) \) is said to be strongly irreducible if there exists a \( \theta_0 > 0 \) such that if two nonempty subsets \( K_1, K_2 \) form a partition of \( K = \{ 1, 2, \ldots, k \} \), then there exist \( i \in K_1 \) and \( j \in K_2 = K \setminus K_1 \) with \( |m_{ij}| \geq \theta_0 \).

For any \( \tau > 0 \), define \( C = C([-\tau, 0], \mathbb{R}^n) \) with the positive cone \( C_+ := C([-\tau, 0], \mathbb{R}^n_+) \), whose interior, \( \text{Int}(C_+) = C([-\tau, 0], \text{Int}(\mathbb{R}^n_+)) \), is nonempty. The ordering on \( \mathbb{R}^n_+ \) is defined as follows \( (x_i, (\text{resp. } y_i)) \) denotes the \( i \)-th component of \( x \) (resp. \( y \)):

\[
\begin{align*}
x \geq y & \iff x_i \geq y_i, \forall i = 1, \ldots, n, \\
x > y & \iff x \geq y, x \neq y, \\
x \gg y & \iff x_i > y_i, \forall i = 1, \ldots, n.
\end{align*}
\]

The relation on \( C_+ \) is defined as follows:

\[
x \geq y \iff x(\zeta) \geq y(\zeta), \forall \zeta \in [-\tau, 0],
\]
Consider a nonlinear delay differential equation

\[
\frac{du}{dt} = f(t, u(t), u(t - \tau)),
\]

which satisfies the initial function \(u(\zeta) = x(\zeta)\), and \(x \in C_+\), where \(u = (u_1, \ldots, u_n)^T\) and \(f = (f_1, f_2, \ldots, f_n)^T : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) is continuous and uniformly almost periodic, \(f(t, u, v)\) is Lipschitz in \((u, v)\). Furthermore, we make the following assumptions on \(f\):

1. For any \(u, v, t \geq 0, i = 1, \ldots, n, \frac{\partial f}{\partial y}(t, u, v)\) is positive and strongly irreducible, \(\frac{\partial f}{\partial y}(t, u, v)\) is positive and strongly irreducible, and for each \(j\) there exist \(\eta_j > 0\) and \(i\) such that \(\frac{\partial f}{\partial y}(t, u, v) > \eta_j\).

2. For any \(\phi \in C_+\) with \(\phi(0) = 0, f_1(t, \phi(0), \phi(-\tau)) \geq 0\) for all \(t \geq 0\).

3. There exists \(U_0 > 0\) such that \(f_i(t, U, U) \leq 0\) for all \(U > U_0, i = 1, \ldots, n\).

4. \(f\) is sublinear in the sense that \(f(t, \beta u, \beta v) \geq \beta f(t, u, v)\) for any \(\beta \in [0, 1]\), \(u \geq 0, v \geq 0\), and strictly sublinear in the sense that \(f(t, \beta u, \beta v) > \beta f(t, u, v)\) for any \(\beta \in (0, 1), u > 0\) and \(v > 0\).

Denote the hull of \(f\) as

\[
H(f) = \text{cls}\{f_s : s \in \mathbb{R}, f_s(t, \cdot, \cdot) = f(s + t, \cdot, \cdot)\},
\]

where the closure is taken under the topology of uniform convergence on compact subsets. It then follows that the translation \(\zeta : \mathbb{R} \times H(f) \to H(f), \zeta_s(g) = g \cdot s = g_s\) with \(g \cdot s(t) = g(s + t)\), defines a continuous, compact, almost periodic minimal and distal flow (see \[21, \text{Lemma VI.C}\]), which is denoted by \((H(f), \zeta, \mathbb{R})\). Then we can define the ordering on \((H(f) \times C_+)\) in the following way:

\[
\begin{align*}
(g, x) &\geq (g, y) \iff y \geq x, \\
(g, x) &> (g, y) \iff y > x, x \neq y, \\
(g, x) &\gg (g, y) \iff y \gg x.
\end{align*}
\]

Two subsets \(A_1, A_2\) of \((H(f) \times C_+)\) are ordered \(A_1 \leq A_2\) if for each \((g, x^1) \in A_1\), there exists \((g, x^2) \in A_2\) such that \(x^1 \leq x^2\). We say \(A_1 < A_2\) if \(A_1 \leq A_2\) and they are different.

We say the subsets \(A_1, A_2\) of \((H(f) \times C_+)\) to be ordered \(A_1 \ll A_2\) if for each \((g, x^1) \in A_1\), there exists \((g, x^2) \in A_2\) such that \(x^1 \ll x^2\).

Two subsets \(A_1, A_2\) of \((H(f) \times C_+)\) are said to be completely strongly ordered \(A_1 \ll C A_2\) if \(x^1 \ll x^2\) holds for all \((g, x^1) \in A_1\) and \((g, x^2) \in A_2\).

Consider the family of delay systems

\[
\frac{du}{dt} = g(t, u(t), u(t - \tau)), \quad g \in H(f).
\]

By the standard theory of delay differential equations (see \[11, 13]\), for each \((g, x) \in H(f) \times C_+\), system (6) locally admits a unique solution \(u(t, g, x)\) satisfying \(u(\zeta, g, x) = x(\zeta), \forall \zeta \in [-\tau, 0]\). Hence, the solution of (6) induces a skew-product semiflow \(\pi_t : H(f) \times C_+ \to H(f) \times C_+\)

\[
(g, x) \mapsto (g \cdot t, v(t, g, x)),
\]

for
where \( v(t, g, x) \in C([-\tau, 0], \mathbb{R}^n) \) and \( v(t, g, x) = u(t + \varsigma, g, x), \forall \varsigma \in [-\tau, 0] \). Let

\[ \omega(g, x) = \{(\bar{g}, \bar{x}) \in H(f) \times C_+: \exists n \uparrow \infty \text{ such that } g \cdot t_n \to \bar{g}, v(t_n, g, x) \to \bar{x}\}, \]

be the omega limit set of \((g, x)\) for \(\pi_t\).

By [22, Lemma 3.6.2], if \( u(t, g, x) \) is a bounded solution of \((6)\) for \(t\) in its interval of existence, then \( v(t, g, x) \) exists for all \(t \geq 0\) and the forward orbit \(\{\pi(t, g, x) : t \geq \tau + \delta\}\) is relatively compact in \(H(f) \times C_+\) for any \(\delta > 0\). By (B2), (B3) and [23, Remark 5.2.1], for any \(U > U_0\), \(\Sigma = H(f) \times [0, \bar{U}]\) is positively invariant for the skew-product semiflow \((7)\). Thus, for any \(x \geq 0\), there exists \(U_x > U_0\) such that \(U_x \geq x\), and hence, \(v(t, g, x)\) exists for all \(t \geq 0\), and the forward orbit \(\{\pi(t, g, x) : t \geq \tau + \delta\}\) is relatively compact in \(H(f) \times C_+\) for any \(\delta > 0\). By [22, Lemma 3.6.1], the skew-product semiflow \((7)\) is strongly monotone in the sense that (i) For any \(y > 0\) and \((g, x) \in H(f) \times C_+, v_x(t, g, x)y > 0\) if \(t > 0\) and \(v_x(t, g, x)y \geq 0\) if \(t \geq 1 + \tau\); (ii) For any \(y \gg 0\) and \((g, x) \in H(f) \times C_+, v_x(t, g, x)y \geq 0\) if \(t \geq 0\).

The following result is basic in the theory of delay differential equations. For reader’s convenience, we also provide elementary proof below.

**Lemma 2.1.** If \((B1)-(B4)\) hold, then the skew-product semiflow \((7)\) is sublinear, i.e., whenever \(x \geq 0\) then

\[ v(t, g, \beta x) \geq \beta v(t, g, x) \]

for \(t \geq 0\) and \(\beta \in [0, 1], g \in H(f)\). Furthermore, \(v(t, f, x)\) is strictly sublinear on \(C_+, i.e., whenever x \gg 0\) then

\[ v(t, f, \beta x) > \beta v(t, f, x) \]

for \(t > \tau\) and \(\beta \in (0, 1)\).

**Proof.** For any \(\beta \in (0, 1)\), let \(z_\beta(t) = \beta u(t, g, x)\). We assume that all solutions of \((6)\) are defined for \(t \geq 0\). Hence, we have

\[ \frac{dz_\beta}{dt} = \beta g(t, u(t, g, x), x(t - \tau)), \forall t \in [0, \tau], g \in H(f). \]

Since \(x \geq 0\) and \(f\) is sublinear, we conclude that

\[ \frac{dz_\beta}{dt} \leq g(t, z_\beta(t), \beta x(t - \tau)), \forall t \in [0, \tau], g \in H(f). \]

It then follows from comparison theorems for cooperative system that \(z_\beta(t) \leq u(t, g, \beta x)\) for each \(t \in [0, \tau]\). Analogous arguments for each interval \([k_*, k_* + \tau]\) show inductively that

\[ \beta u(t, g, x) \leq u(t, g, \beta x), \forall t \geq 0, g \in H(f), \]

that is, \(\beta v(t, g, x) \leq v(t, g, \beta x)\), and hence, the skew-product semiflow is sublinear.

In the following, if \(x \gg 0\), the strict sublinearity implies that

\[ \frac{dz_f}{dt} < f(t, z_f(t), \beta x(t - \tau)), \forall t \in [0, \tau], \beta \in (0, 1). \]

The strong irreducibility and comparison theorems imply that \(z_f(t) < u(t, f, \beta x)\) for each \(t \in (0, \tau]\). Furthermore, for each \(t \in [1, 1 + \tau]\),

\[ \frac{dz_f}{dt} \leq f(t, z_f(t), \beta x(t - \tau)), \forall \beta \in (0, 1). \]
Hence, from $z_f(1) < u(1,f,\beta x)$ and comparison theorems yield to $z_f(t) < u(t,f,\beta x)$ for each $t \in [1,1+\tau]$. Analogous arguments for each interval $[k_*,k_*+\tau]$ show inductively that

$$\beta u(t,f,x) < u(t,f,\beta x)$$

for each $t > 0$, that is, $\beta v(t,f,x) < v(t,f,\beta x)$ for each $t > \tau$, as required. \(\square\)

**Remark 1.** Indeed, for any give $x \gg 0$ and $\beta \in (0,1)$, the strict sublinearity implies that there exists $t_1 > \tau$ such that $v(t_1,f,\beta x) > \beta v(t_1,f,x)$. It then follows from the strong monotonicity of $v(t,\zeta_t(f),\cdot)$ that there exists $t_2 > 1 + \tau$ such that

$$v(t_2,\zeta_{t_1}(f),v(t_1,f,\beta x)) \gg v(t_2,\zeta_{t_1}(f),\beta v(t_1,f,x)) \geq \beta v(t_2,\zeta_{t_1}(f),v(t_1,f,x)).$$

Since $\pi_{t_1} \circ \pi_{t_2} = \pi_{t_1+t_2}$, we conclude that $v(t,f,\beta x) \gg \beta v(t,f,x)$, $\forall t \geq t_1 + t_2$, that is, the skew-product semiflow is eventually strongly sublinear.

**Definition 2.2.** Let $K \subset H(f) \times C_+$ be a compact, positively invariant subset of the skew-product semiflow $(\cdot)$. For $(g,x) \in K$, we define the *Lyapunov exponent* $\lambda(g,x)$ as

$$\lambda(g,x) = \limsup_{t \to \infty} \frac{\ln \|v_x(t,g,x)\|}{t}. $$

The number $\lambda_K = \sup_{(g,x) \in K} \lambda(g,x)$ is called the *upper Lyapunov exponent* over $K$. If $\lambda_K \leq 0$, then $K$ is said to be linearly stable.

The next result shows that the Lyapunov exponent of a point of $K$ can be obtained at discrete times.

**Lemma 2.3.** [17, Lemma 4.4] For any $(g,x) \in K$,

$$\lambda(g,x) = \limsup_{n \to \infty} \frac{\ln \|v_x(n,g,x)\|}{n}.$$ 

In order to obtain the main result of this section, we have the following preparation.

**Proposition 1.** Let (B1)-(B4) hold. If $f(t,0,0) = 0$, $O_g = \{\eta \times 0 : \eta \in H(f)\}$ and $O_+ \subset H(f) \times C_+$ is a compact and minimal invariant set with $O_g \ll C O_+$, then $\lambda_{O_g} > 0$.

**Proof.** Since Lemma 2.1 implies that the skew-product is sublinear, a simple calculation implies that

$$v_x(t,g,0)x \geq v(t,g,x), \forall t \geq 0, (g,x) \in H(f) \times C_+.$$ 

It then follows from the compactness and positive invariance of $O_+$ that there exists a positive constant $c_*$ such that

$$\|v_x(t,g,0)\| \geq c_*, \forall t \geq 0, (g,0) \in O_g.$$ 

Hence, by Definition 2.2, we conclude that $\lambda_{O_g} \geq 0$. Thus, it is only sufficient to prove $\lambda_{O_g} \neq 0$.

On the contrary, we assume that $\lambda_{O_g} = 0$. Since $f(t,0,0) = 0$ implies that $g(t,0,0) = 0$ for all $g \in H(f)$, a completely similar argument to one in [18, Proposition 6.2] for the concave case implies that there exists the subset $O^*$ of $H(f) \times C_+$ such that $O_g \ll C O^* \subset O_+$. By the structure theorem of skew-product semiflows (see [20, Theorem 1]), as applied to the flow $\pi : O_+ \to O_+$, we conclude that...
Proof. (i) Since $0 = b(g_0)$, where $\tilde{b} : (\pi, O_*) \to (\pi, H(f))$ is a flow isomorphism. Hence, there exists a continuous map $b : H(f) \to C_+$ such that $O_* = \{(g_0, b(g_0)) : g_0 \in H(f)\}$. Let $(g_0, b(g_0)) \in O_*$, then there exists $(g_0, \tilde{z}) \in O^*$ such that

$$\tilde{0} \ll \tilde{z} < b(g_0).$$

For $e \gg 0$, we define $e$-norm by

$$\|x\|_e := \inf\{\theta > 0 | \theta e \leq x \leq \theta e\}. \quad (8)$$

Let $e = b(g_0) \gg 0$ in (8) and define

$$\vartheta = \inf\{\|b(g_0) - x^*\|_e : (g_0, x^*) \in O^*\}.$$ 

Obviously, the infimum is attained, i.e., there exists $(g_0, x^*) \in O^*$ such that $\vartheta = \|b(g_0) - x^*\|_e$ with $0 < \vartheta < 1$.

Since $\text{card}(O_\ast \cap \tilde{b}^{-1}(g)) = 1$, from the monotonicity and the eventually strongly sublinearity of the skew-product semiflow at $f$, there exists $0 < \vartheta_0 < \vartheta$ and $t_0 > \tau$ (depending on $g_0, x(g_0)$ and $x^*$) such that

$$u(t, g_0, x^*) \gg (1 - \vartheta_0)u(t, g_0, b(g_0)) = b(g_0) \cdot t$$

for each $t > t_0$.

Hence, we can find $(g_0, \tilde{y}) \in O^*$ such that

$$\tilde{y} \geq (1 - \vartheta_0)b(g_0),$$

i.e., $0 \leq b(g_0) - \tilde{y} \leq \vartheta_0b(g_0) = \vartheta_0 e$, which implies that $\|b(g_0) - \tilde{y}\|_e \leq \vartheta_0 < \vartheta$ and contradicts the definition of $\vartheta$. Consequently, $\lambda_{O_*} > 0$ as stated. \hfill $\Box$

Theorem 2.4. Let (B1)-(B4) hold. Then the following statements are valid:

(i) If $g(t, 0, 0) > 0$, $\forall t \geq 0$, then there exists a positive almost periodic solution of (6) which is globally attractive in $H(f) \times C_+$.

(ii) If $g(t, 0, 0) = 0$, $\forall t \geq 0$, and $\lambda_{O_g} > 0$, then there exists a positive almost periodic solution of (6) which is globally attractive in $H(f) \times (C_+ \setminus \{\tilde{0}\})$.

(iii) If $g(t, 0, 0) = 0$, $\forall t \geq 0$, and $\lambda_{O_g} \leq 0$, then $O_g$ is globally attractive in $H(f) \times C_+$.

Proof. (i) Since $g(t, 0, 0) > 0$, we have $v(t, g, \tilde{0}) \geq \tilde{0}$. We now show $v(t, g, \tilde{0}) > \tilde{0}$. On the contrary, assume that $v(t, g, \tilde{0}) = \tilde{0}$, and hence $u(t + \varsigma, g, \tilde{0}) = \varsigma \in [-\tau, 0]$. Thus, $0 = u'(t + \varsigma, g, \tilde{0})$. The contradiction follows from the strong monotonicity and the invariant of the omega limit set that $\omega(g, \tilde{0}) \subset H(f) \times \text{Int}(C_+)$. Furthermore, $\omega(g, x) \subset H(f) \times \text{Int}(C_+)$, $\forall (g, x) \in H(f) \times C_+$. Hence, by Lemma 2.1 and Remark 1, [33, Theorem 2.1, Remark 2.1, 2.2] imply that for any $(g, x) \in H(f) \times C_+$, there exists a positive almost periodic solution of (6) which is globally attractive in $H(f) \times C_+$.

(ii) Similar to the above, for any $(g, x) \in H(f) \times (C_+ \setminus \{\tilde{0}\})$, we have either $\omega(g, x) \subset H(f) \times \text{Int}(C_+)$ or $\omega(g, x) \subset H(f) \times (C_+ \setminus \text{Int}(C_+))$. It then follows from the strong monotonicity of the skew-product semiflow and the compactness and invariance of the omega limit set that either $\omega(g, x) \subset H(f) \times \text{Int}(C_+)$ or $\omega(g, x) \subset H(f) \times \{\tilde{0}\}$. In the following, we claim that $\omega(g, x) \subset H(f) \times \text{Int}(C_+)$, $\forall (g, x) \in H(f) \times (C_+ \setminus \{\tilde{0}\})$.

On the contrary, if for some $(g_0, x_0) \in H(f) \times (C_+ \setminus \{\tilde{0}\})$, we have $\omega(g_0, x_0) \subset H(f) \times \{\tilde{0}\}$, which implies $O_g = \omega(g_0, x_0)$. Thus, we can prove that for any sequence
$t_n \to \infty$ as $n \to \infty$, \( \lim_{n \to \infty} \| u(t_n, g_0, x_0) \| = 0 \) holds. Otherwise, there exists a sequence $t_k \to \infty$ as $k \to \infty$, the strong monotonicity implies that $\omega(g_0, x_0) \subset H(f) \times \text{Int}(C_+)$, which contradicts to $O_g = \omega(g_0, x_0)$. Hence, for any $\epsilon > 0$ there exists a $n_0$ such that for $n \geq n_0$,

$$
\| u(t_n, g_0, x_0) \| \leq \epsilon.
$$

Since $u(t, g, 0) = 0$, a simple computation and Lemma 2.3 imply that $\lambda_{O_g} \leq 0$, a contradiction. We have proved the claim. It then follows from the similar arguments to the above to that there exists a positive almost periodic solution of (6) which is globally attractive in $H(f) \times (C_+ \setminus \{0\})$.

(iii) On the contrary, if $O_g$ is not globally attractive in $H(f) \times C_+$, then the similar arguments to the above imply that there exists a compact and minimal invariant subset $K^*$ with $K^* \subset H(f) \times \text{Int}(C_+)$, that is, $O_g \ll C K^*$. It then follows from Proposition 1 that $\lambda_{O_g} > 0$, which is a contradiction.

3. The disease-free solution and basic reproduction ratio. From now on, we embed system (4) into a skew-product semiflow, and draw support from the conclusions in Section 2 to consider the existence of the almost periodic disease-free solution. Furthermore, we give the definitions of the exponential growth bound and the basic reproduction ratio of system (4).

Define the hull of $D_i(t, \cdot, \cdot)$ by

$$
H(D_i) = \text{cls}\{(D_i)s : s \in \mathbb{R}, (D_i)s(t, \cdot) = D_i(s + t, \cdot)\},
$$

where the closure is taken under the compact open topology. Similarly, we can define the hulls of $d_i(t), \varpi_i(t), \gamma_i(t), r_{ij}(t), h_{ij}(t)$ and $p_{ij}(t)$, denoted by $H(d_i), H(\varpi_i), H(\gamma_i), H(r_{ij}), H(h_{ij})$ and $H(p_{ij})$, respectively. Let

$$
\xi_i(t) = (D_i(t, \cdot), d_i(t), \varpi_i(t), \gamma_i(t), r_{ij}(t), h_{ij}(t))\in H(D_i)\times H(\varpi_i)\times H(\gamma_i)\times H(r_{ij})\times H(h_{ij})\times H(p_{ij}).
$$

Then $H(\xi_i) = H(D_i) \times H(d_i) \times H(\varpi_i) \times H(\gamma_i) \times H(r_{ij}) \times \cdots \times H(h_{in}) \times H(p_{in}), i = 1, \ldots, n$. Let $\eta = (\eta_1, \ldots, \eta_n) \in H(\xi_i) \times \cdots \times H(\xi_n) := H(\xi)$, where $\eta_i = (D_i, 0, \alpha_i, \theta_i, k_i, \ldots, k_i^{(n-1)}, l_i^{(n-1)}, \ldots, l_i, q_i, q_i, \ldots, q_i) \in H(\xi_i)$, $i = 1, \ldots, n$. The translation $\zeta : \mathbb{R} \times H(\xi) \to H(\xi), (t, \eta) \mapsto \eta + \zeta$ with $\eta \cdot s(t) = \eta(s + t)$ defines a compact, almost periodic minimal and distal flow (see [21, Lemma VI.C]), denoted by $(H(\xi), \zeta, \mathbb{R})$.

Consider the family system

$$
\left\{
\begin{array}{l}
\frac{dS_i}{dt} = D_i(t, G(t - \tau)) - c_i(t)S_i - a_i(t)I_i + \theta_i(t)I_i + \sum_{j=1}^{n} k_{ij}(t)S_j, \\
\frac{dI_i}{dt} = a_i(t)S_i - (c_i(t) + \theta_i(t))I_i + \sum_{j=1}^{n} l_{ij}(t)I_j, \\
D_i(t, G_i(t - \tau)) = \sum_{j=1}^{n} q_i(t)G_j(t - \tau, G_j(t - \tau)), \\
S_i(r, \eta) = \varphi(r), I_i(r, \eta) = \psi(r), \quad \forall r \in [-\tau, 0], (\varphi, \psi) \in C_+^2, \eta \in H(\xi),
\end{array}
\right.
$$

(9)

where $C_+ := C([-\tau, 0], \mathbb{R}_+^n)$.

Let $(S(t, \eta, x_0), \ldots, S_n(t, \eta, x_0), I_1(t, \eta, x_0), \ldots, I_n(t, \eta, x_0)) = (S(t, x_0, 0), I(t, x_0, \eta)) = \Gamma(t, x_0, \eta)$ be the solution of (9) satisfying the initial value $x_0$, i.e.,

$$
\Gamma_1(s, \eta, x_0) = \zeta(s) \Rightarrow (\varphi(s), \psi(s)) \quad \forall s \in [-\tau, 0].
$$

The solution deduces a skew-product semiflow

$$
\Pi_t : H(\xi) \times C_+^2 \to H(\xi) \times C_+^2,
$$

$$(\eta, x_0) \mapsto (\eta_t, \Gamma_t(\eta, x_0)),
$$

(10)
where \( \eta_i = ((\eta_1), \ldots, (\eta_n)) \) and \( \Gamma_t(\eta, x^0)(\varsigma) = \Gamma(t + \varsigma, \eta, x^0) \) for \( \varsigma \in [-\tau, 0] \).

If no confusion, we delete the initial value and write \((S(t, \eta, x^0), I(t, \eta, x^0))\) as \((S(t, \eta), I(t, \eta))\). Similarly, we can define the omega limit set of \( \Pi_t \), denoted by \( \omega(\eta, x) \).

Let \( c_{ij} = \max\{k_{ij}, l_{ij}\}, i \neq j \), and \( c_{ii} = \min\{k_{ii}, l_{ii}\} \).

Furthermore, we have the following assumption.

\( (H1) \quad \frac{\partial \Gamma_t(\xi, G)}{\partial t} > 0, \forall (t, G_i) \in \mathbb{R}^+ \times (0, \infty) \) and \( i = 1, \ldots, n; \)

\( (H2) \quad \exists \chi_0 > 0 \) such that

\[
\sum_{j=1}^{n} q_{ij}(t) \Omega_j(t - \tau, \chi) \leq c_i(t) - \sum_{j=1}^{n} c_{ij}(t), \forall t \geq \tau, \chi \geq \chi_0 \text{ and } i = 1, \ldots, n.
\]

Note that \((H1)\) implies that the greater adult individuals, the higher birth rate in each patch; \((H2)\) assures that the number of susceptible individuals dose not admit a unlimited growth.

Under the assumptions above, a family of compact, positively invariant sets hold.

**Theorem 3.1.** Assume that \((H1)-(H2)\) hold. Then system \((9)\) admits a unique nonnegative skew-product solution. Furthermore, all skew-product solutions \((10)\) are ultimately bounded and uniformly bounded.

**Proof.** The existence of the nonnegative skew-product solution can be obtained straight from [15, Theorem 2.2].

In view of \( G_i = S_i + I_i, i = 1, \ldots, n \), we conclude that

\[
\frac{dG_i}{dt} \leq D_i(t, G(t - \tau)) - e_i(t)G_i + \sum_{j=1}^{n} c_{ij}(t)G_j, \quad i = 1, \ldots, n.
\]

Consider the auxiliary system

\[
\frac{d\bar{G}_i}{dt} = D_i(t, \bar{G}(t - \tau)) - e_i(t)\bar{G}_i + \sum_{j=1}^{n} c_{ij}(t)\bar{G}_j := Q_i(t, \bar{G}(t), \bar{G}(t - \tau)), \quad i = 1, \ldots, n.
\]

It is easy to verify that \( Q(t, \bar{G}(t), \bar{G}(t - \tau)) = (Q_1(t, \bar{G}(t), \bar{G}(t - \tau)), \ldots, Q_n(t, \bar{G}(t), \bar{G}(t - \tau))) \) satisfy \((B1)-(B4)\) (see also [15, Theorem 2.2]). Let \( \lambda_{O_\alpha} \) be the upper Lyapunov exponent over \( O_\alpha = \{ \eta \times 0 : \eta \in H(\xi) \} \) associated with system \((11)\).

By Theorem 2.4, system \((11)\) admits a unique positive almost periodic solution \( \bar{G}^*(t, \eta) = (\bar{G}^*_1(t, \eta), \ldots, \bar{G}^*_n(t, \eta)) \) if \( \Omega_i(t, 0) > 0 \) or \( \Omega_i(t, 0) = 0 \) and \( \lambda_{O_\alpha} > 0 \), which is globally attractive in \( \mathbb{R}_+^n \setminus \{0\} \). If \( \lambda_{O_\alpha} \leq 0 \), the zero almost periodic solution is globally attractive in \( \mathbb{R}_+^n \). The standard comparison implies that

\[
\limsup_{t \to \infty} \{S_i(t, \eta) + I_i(t, \eta)\} \leq \max_{t \in (0, \infty)} \bar{G}^*_i(t, \eta), \quad \forall \eta \in H(\xi), \quad i = 1, \ldots, n.
\]

It then follows from the boundedness of an almost periodic function (see [6, Theorem 1.1.2]) that all skew-product solutions \((10)\) are ultimately bounded. Furthermore, the condition \((H2)\) shows that all skew-product solutions are uniformly bounded.

In the following, we consider the disease-free almost periodic solution. Let \( I_i(t, \eta) = \bar{I}_i, i = 1, \ldots, n \), in \((9)\), then

\[
\frac{dS_i}{dt} = D_i(t, S(t - \tau)) - e_i(t)S_i + \sum_{j=1}^{n} k_{ij}(t)S_j := J_i(t, S(t), S(t - \tau)), \quad i = 1, \ldots, n.
\]
Especially, let \( I_i(t) = 0, \ i = 1, \ldots, n \), in (4), then we have
\[
\frac{dS_i}{dt} = D_i(t, S(t - \tau)) - d_i(t)S_i + \sum_{j=1}^{n} r_{ij}(t)S_j := J_i(t, S(t), S(t - \tau)), \ i = 1, \ldots, n.
\]
\[
(13)
\]

In order to assure the global attractivity of the positive almost periodic solution of system (12), we assume that the trivial solution is unstable as \( J(t, 0, 0) = 0 \). According to the conclusion of the above section, we have the following assumption.

\((H3)\) If \( D_i(t, \cdot) > 0 \), then \( D_i(t, \cdot) > 0 \) for all \( D_i \in H(D_i) \).

\((H4)\) \( \lambda_{O_n} > 0 \) if \( J(t, 0, 0) \equiv 0 \), where \( \lambda_{O_n} \) is the upper Lyapunov exponent associated with system (12).

**Theorem 3.2.** Let \((H1)-(H4)\) hold. Then (12) has a unique positive almost periodic solution \( S^*(t, \eta) = (S^*_1(t, \eta), ..., S^*_n(t, \eta)) \) which is globally attractive in \( H(\xi) \times (C_+ \setminus \{0\}) \). Furthermore, \( (S^*(t, \eta), 0) \) is a disease-free almost periodic solution of (9).

**Proof.** We can clarify that \( J(t, S(t), S(t - \tau)) = (J_1(t, S(t), S(t - \tau)), ..., J_n(t, S(t), S(t - \tau))) \) satisfies the conditions \((B1)-(B4)\). Then the results can be obtained from Theorem 2.4 straightforwardly.

\(\square\)

**Corollary 1.** If \((H1)-(H4)\) are satisfied, then (13) admits a unique globally attractive and positive almost periodic solution \( S^*(t) = (S^*_1(t), ..., S^*_n(t)) \) in \( C_+ \setminus \{0\} \). Furthermore, \( (S^*(t), 0) \) is a disease-free almost periodic solution of (4).

In the following, we introduce the basic reproduction ratio of (4). In epidemiology, the basic reproduction number \( R_0 \) (sometimes called basic reproductive rate, basic reproductive ratio) of an infection is defined as the expected number of secondary cases produced by a single (typical) infection in a completely susceptible population (see [8]). It is used to measure the transmission potential of a disease, i.e., it helps determine whether or not an infectious disease can spread through a population. The basic reproduction ratio for periodic cases can be found in [3]. A generalized case for periodic cases is showed in [30]. Almost periodic functions are a generalization of periodic functions. Here, we will apply the theory developed in [27].

Linearizing system (4) at \( A_0(t) = (S^*(t), 0) \), a simple computation implies that the infectious classes variable \( z = (I_1, ..., I_n) \) admits the following linear almost periodic system:
\[
\frac{dz}{dt} = [-Z(t) + Y(t)]z, \ i = 1, ..., n,
\]
\[
(14)
\]

where
\[
-Z(t) = \begin{pmatrix}
-(d_1(t) + \gamma_1(t)) + h_{11}(t) & \cdots & h_{1n}(t) \\
 h_{21}(t) & \cdots & h_{2n}(t) \\
 \vdots & \ddots & \vdots \\
 h_{n1}(t) & \cdots & -(d_n(t) + \gamma_n(t)) + h_{nn}(t)
\end{pmatrix},
\]
\[
(15)
\]

\[
Y(t) = \begin{pmatrix}
\varpi_1(t)S^*_1(t) \\
\vdots \\
\varpi_n(t)S^*_n(t)
\end{pmatrix}.
\]
\[
(16)
\]
Let $\Psi_{-Z}(t, s) \ (t \geq s, \ s \in \mathbb{R})$ be the evolution operator of the linear almost periodic system

$$\frac{dy}{dt} = -Z(t)y,$$

that is, the $n \times n$ matrix $\Psi_{-Z}(t, s)$ satisfies

$$\frac{d}{dt}\Psi_{-Z}(t, s) = -Z(t)\Psi_{-Z}(t, s), \ \forall t \geq s, \ s \in \mathbb{R},$$

and

$$\Psi_{-Z}(s, s) = i_d, \ \forall s \in \mathbb{R},$$

where $i_d$ is the $n \times n$ identity matrix. Then the fundamental solution matrix $\Phi_{-Z}(t)$ of (17) equals $\Psi_{-Z}(t, 0), \ t \geq 0$.

We define the exponential growth bound of the evolution operator $\Psi_{-Z}(t, s)$ as

$$\omega(\Psi_{-Z}) = \inf\{\tilde{\omega} \in \mathbb{R} : \exists K_0 \geq 1, \|\Psi_{-Z}(t + s, s)\| \leq K_0 e^{\tilde{\omega}t}, \ \forall s \in \mathbb{R}, \ t \geq 0\}.$$  (18)

Note that $\omega(\Psi_{-Z})$ is finite because $Z(t)$, as an almost periodic function, is bounded on $\mathbb{R}$.

Note that the internal evolution of individuals in the infectious patches due to deaths, recoveries and movements among the patches is dissipative, and exponentially decays in many cases because of the loss of infective members from natural mortalities and disease-induced mortalities. According to (2) and applying mathematical inductions, we can verify that $\omega(\Psi_{-Z}) < 0$. It then follows from (18) that there exist $K_1 > 0$ and $\sigma_1 > 0$ such that

$$\|\Psi_{-Z}(t, s)\| \leq K_1 e^{-\sigma_1(t-s)}, \ \forall t \geq s, \ s \in \mathbb{R}. $$  (19)

Furthermore, the existence of a unique almost periodic solution (see [10, Theorem 7.7, Section 11.4]) implies that the fundamental solution matrix $\Phi_{-Z}(t)$ of (17) is almost periodic.

Define

$$AP_{(Y, Z)} := \{\phi : \phi \in AP(\mathbb{R}, \mathbb{R}^n), mod(\phi) \subset mod(Y, Z)\}.$$  

By [27, Lemma 2.1], $AP_{(Y, Z)}$ is a Banach space with the supremum norm $\| \cdot \|$, and the positive cone $AP_{(Y, Z)}^+ = \{\phi \in AP_{(Y, Z)} : \phi(t) \geq 0, \forall t \in \mathbb{R}\}$ has a nonempty interior Int$(AP_{(Y, Z)}^+)$. For each $\phi(s) \in AP_{(Y, Z)}$, define

$$\mathcal{V}(t) := \int_{-\infty}^t \Psi_{-Z}(t, s)Y(s)\phi(s)ds = \int_0^\infty \Psi_{-Z}(t, t-\hat{c})Y(t-\hat{c})\phi(t-\hat{c})d\hat{c},$$

which induces a linear map $L$ by

$$(L\phi)(t) = \int_0^\infty \Psi_{-Z}(t, t-\hat{c})Y(t-\hat{c})\phi(t-\hat{c})d\hat{c}, \ \forall t \in \mathbb{R}, \ \phi \in AP_{(Y, Z)}.$$  

It then follows from [27, Lemma 3.1], $L$ is a positive and continuous linear operator from $AP_{(Y, Z)}$ to $AP_{(Y, Z)}$.

Using the ideas in [30], in an epidemic model, assume that $\phi(s) \in AP_{(Y, Z)}$ is the initial distribution of infectious individuals. Then, $Y(s)\phi(s)$ is the distribution of new infections produced by the infected individuals who were introduced at time $s$. Given $t \geq s$, then $\Psi_{-Z}(t, s)Y(s)\phi(s)$ denotes the distribution of those infected individuals who were newly infected at time $s$ and remain in the infected compartments at time $t$. $\mathcal{V}(t)$ is the distribution of accumulative new infections at time $t$ produced by all those infected individuals $\phi(s)$ introduced at previous time.
to $t$. Following the appellation in [8, 26, 30], $L$ is called the next infection operator, and define the spectral radius of $L$

$$R_0 := \rho(L)$$

as the basic reproduction ratio of model (4).

Let $\Psi_{-Z+Y}(t,s)$ ($t \geq s, s \in \mathbb{R}$) be the evolution operator of (14). Similarly, we can define the exponential growth bound $\omega(\Psi_{-Z+Y})$ of $\Psi_{-Z+Y}(t,s)$.

**Lemma 3.3.** [27, Theorem 3.2] Suppose that (H1)-(H4) hold. Then the following statements are valid:

(i) If $R_0 = 1$ if and only if $\omega(\Psi_{-Z+Y}) = 0$.

(ii) If $R_0 > 1$ if and only if $\omega(\Psi_{-Z+Y}) > 0$.

(iii) If $R_0 < 1$ if and only if $\omega(\Psi_{-Z+Y}) < 0$.

4. Threshold dynamics. In this section, we consider the uniform persistence and the global extinction of system (4) in terms of its basic reproduction ratio.

**Theorem 4.1.** Suppose that (H1)-(H4) are satisfied. If $R_0 > 1$, then there exists $\rho > 0$ such that for any $x^0 = (\varphi, \psi) \in C^2_+$ with $\psi(0) > 0$, the solution $(S(t), I(t))$ of (4) satisfies

$$\liminf_{t \to \infty} (S(t), I(t)) \geq (\rho, \rho) \text{ for all } i = 1, \ldots, n.$$  

**Proof.** We use the skew-product semiflows approach to prove the desired uniform persistence and practical uniform persistence. (see, e.g., [15, Theorem 3.3] and [27, Theorem 4.1 (ii)]).

Define

$$\Lambda = C^2_+, \quad \Lambda_0 = \{(\varphi, \psi) \in \Lambda : \psi(0) > 0, i = 1, \ldots, n\}, \quad \partial \Lambda_0 = \Lambda \setminus \Lambda_0,$$

$$P = H(\xi) \times \Lambda, \quad P_0 = H(\xi) \times \Lambda_0, \quad \partial P_0 = P \setminus P_0 = H(\xi) \times \partial \Lambda_0.$$  

Then $P_0$ and $\partial P_0$ are relatively open and closed in $P$ for $\Pi_i$, respectively. From the second system of system (9) for any $(\eta, x^0) := (\eta, \varphi, \psi) \in P_0$, $I(t, \eta) \in \text{Int}(\mathbb{R}^n_+)$, $\forall t \geq 0$. Hence, $P_0$ is positively invariant for $\Pi_i$, that is, $\Pi_i P_0 \subset P_0$, $\forall i \geq 0$.

Let

$$M_\beta := \{(\eta, x^0) \in \partial P_0 : \Pi_i(\eta, x^0) \in \partial P_0, \forall t \geq 0\}.$$  

We now show that

$$M_\beta = \{(\eta, \varphi, \psi) \in P : \psi(0) = 0\}. \quad (20)$$

It is sufficient to show that $I(t, \eta, \varphi, \psi) = 0$ for all $(\eta, \varphi, \psi) \in M_\beta$ and $t \geq 0$. On the contrary, if there exist some $i_*$, $1 \leq i_* \leq n$, and $t_* \geq 0$ such that $I_i(t_*, \eta, \varphi, \psi) > 0$. Assume that $\Sigma_1$ and $\Sigma_2$ is a partition of $\{1, 2, \ldots, n\}$ such that

$$I_i(t_*, \eta, \varphi, \psi) > 0, \quad \forall i \in \Sigma_1,$$

$$I_j(t_*, \eta, \varphi, \psi) > 0, \quad \forall j \in \Sigma_2.$$  

Obviously, $\Sigma_2$ is not empty because $I_i(t_*, \eta, \varphi, \psi) > 0$. If $\Sigma_1 = \emptyset$ holds, then we conclude that $I(t, \eta, \varphi, \psi) > 0$, $\forall t \geq t_*$, by the invariance of $P_0$. Hence, we obtain a contradiction. If $\Sigma_1 \neq \emptyset$, let $\tilde{l}$ be a constant associated with strong irreducible matrix $(l_{ij}(t))$, it then follows from the second equations of (9) that there exist $i_* \in \Sigma_1$ and $j_* \in \Sigma_2$ such that

$$\frac{dI_{i_*}}{dt} |_{t=t_*} \geq l_{ij}(t_*) I_{j_*}(t_*, \eta, \varphi, \psi) > \tilde{l} I_{j_*}(t_*, \eta, \varphi, \psi) > 0. \quad (21)$$

Consequently, there exists an $\epsilon_* > 0$ such that $I_{i_*}(t, \eta, \varphi, \psi) > 0$ for all $t_* < t < t_* + 2\epsilon_*$. For $j \in \Sigma_2$, the continuity implies that we can choose $\epsilon_*$ sufficiently small such that $I_j(t, \eta, \varphi, \psi) > 0$ for all $t_* < t < t_* + 2\epsilon_*$. If $I_i(t + \epsilon_*, \eta, \varphi, \psi) > 0$,
Claim. For open topology. Furthermore, the following claim holds.

\[ i = 1, \ldots, n, \text{ by the invariance of } P_0, \text{ we have } I_i(t, \eta, \varphi, \psi) > 0, \ i = 1, \ldots, n \text{ for } t \geq t_\epsilon + \epsilon. \] Otherwise, we can find another partition \( \Sigma_1 \) and \( \Sigma_2 \) of \( \{1, 2, \ldots, n\} \) such that

\[ I_i(t_\epsilon + \epsilon, \eta, \varphi, \psi) = 0, \quad \forall i \in \Sigma_1 \subset (\Sigma_1 \setminus \{i_\epsilon\}), \]

\[ I_j(t_\epsilon + \epsilon, \eta, \varphi, \psi) > 0, \quad \forall j \in \Sigma_2 \supset (\Sigma_1 \setminus \{i_\epsilon\}). \]

Repeat the above argument until \( I_i(t + \epsilon, \eta, \varphi, \psi) > 0, \ i = 1, \ldots, n, \) for some \( \epsilon > 0, \) and hence, \( I_i(t, \eta, \varphi, \psi) > 0, \ i = 1, \ldots, n, \) for all \( t \geq t + \epsilon, \) which contradicts the definition of \( M_\beta. \) This proves (20).

In the case of \( J(t, 0, 0) = 0, \) it is easy to see that \( (0, 0) \) and \( (S^*(t, \eta, 0)) \) are the solution of (9). Let \( O_0 = \{(\eta, 0, 0) : \eta \in H(\xi)\} \) and \( O_1 = \{(\eta, S^*_0(\eta), 0) : \eta \in H(\xi)\}. \) For simplicity, we define \( O := \{(\eta, 0, 0), (\eta, S^*_0(\eta), 0) : \eta \in H(\xi)\}. \) Since \( S^*_0(\eta) \) is continuous in \( \xi \) and \( H(\xi) \) is compact, \( S^*_0(\eta) \) is uniformly continuous in \( \eta \in H(\xi). \) This implies that \( S^*_t(\eta) = S^*_0(\eta), \) and hence, \( O \) is compact and invariant sets for \( \Pi_t \) in \( \partial P_0. \)

For any given \( (\eta, x^0) \in M_\beta, \) let \( (\hat{\eta}, \hat{x}^0) \in \omega(\eta, x^0), \) then there exists a sequence \( t_n \to \infty \) such that \( \lim \Pi_{t_n}(\eta, x^0) = (\hat{\eta}, \hat{x}^0). \) Clearly, \( \omega(\eta, 0, 0) \subset O_0. \) Note that \( \Pi_{t_n}(\eta, x^0) = (\Gamma_{t_n}(\eta, x^0), \eta_{t_n}), \) and \( \lim_{n \to \infty} \|\Gamma_{t_n}(\eta, x^0) - (S^*_n(\eta), 0)\| = 0. \) Since \( S^*_n(\eta) = S^*_0(\eta_{t_n}) \to S^*_0(\hat{\eta}) \) as \( n \to \infty, \) we get \( (\hat{\eta}, \hat{x}^0) = (\hat{\eta}, S^*_0(\hat{\eta}), 0) \in O_1, \) and hence, \( \omega(\eta, x^0) \subset O_1. \) Then it follows that \( O \) is disjoint, compact and isolate invariant sets for \( \Pi_t : \partial P_0 \to \partial P_0, \cup_{(\eta, x^0) \in M_\beta} \omega(\eta, x^0) \subset O, \) and no subset of \( \Omega_k. \)

Let \( \Psi_{-Z-Y-\epsilon Y}(t, s), \ t, s \geq 0, \) be the evolution operator of (14) when we replace \( -Z(t) + Y(t) \) by \( -Z(t) + Y(t) - \epsilon Y(t), \) where \( Y(t) = \text{diag}(\varpi_1(t), \ldots, \varpi_n(t)). \) Then we can define the exponential growth bound of \( \Psi_{-Z-Y-\epsilon Y}(t, s), \) denoted by \( \omega(\Psi_{-Z-Y-\epsilon Y}). \) Since \( R_0 > 1, \) Lemma 3.3 implies that \( \omega(\Psi_{-Z-Y}) > 0. \) Hence, we can restrict \( \epsilon \) sufficiently small such that \( \omega(\Psi_{-Z-Y-\epsilon Y}) > 0. \) Define the hulls \( H(Z) \) and \( H(Y), \) of \( Z(t) \) and \( Y(t), \) respectively. Let \( W_k(t) := \text{diag}(\alpha_1(t), \ldots, \alpha_n(t)) S^*_t(\eta) \) and \( \hat{Y}(t) = \text{diag}(\alpha_1(t), \ldots, \alpha_n(t)). \) Similarly, we can define the evolution operator and the exponential growth bound associated with

\[ \frac{d\hat{Y}}{dt} = W_{\epsilon}(t) \hat{Y}, \quad (22) \]

denoted by \( \Psi_{W_{\epsilon}}(t, s) \) and \( \omega(\Psi_{W_{\epsilon}}) \) respectively. By [27, Lemma 2.4], we know that \( \omega(\Psi_{W_{\epsilon}}) > 0. \) The uniform continuity of \( S^*_0(\eta) \) in \( \eta \in H(\xi) \) implies that we can fix a positive \( \delta < \epsilon/2 \) such that \( ||S^*_0(\eta) - S^*_0(\hat{\eta})|| \leq \epsilon/2 \) for any \( \eta, \hat{\eta} \in H(\xi) \) with \( d(\eta, \hat{\eta}) < \delta, \) where \( d \) is the metric on \( C(\mathbb{R}, \mathbb{R}^{3n^3 + 4n}) \) equipped with the compact open topology. Furthermore, the following claim holds.

Claim. For \( (\eta, x^0) = (\eta, S^0(\eta), 0) \in P_0, \) we have

\[ \limsup_{t \to \infty} d(\Pi_t(\eta, x^0), O_{k*}) \geq \delta, \quad k* = 0, 1. \]
First, we consider $k^* = 1$. On the contradiction, for some $(\bar{\eta}, \bar{x}^0) \in P_0$, there holds $\limsup_{t \to \infty} d(\Pi_t(\bar{x}^0, \bar{\eta}), O_1) < \delta$. Then there exists $t_0 > 0$ such that $d(\Pi_t(\bar{\eta}, \bar{x}^0), O_1) < \delta$ for all $t \geq t_0$, and hence, $d(\Pi_t(\eta, x^0), O_1) < \delta$ for all $t \geq 0$, where $(\eta, x^0) = \Pi_{t_0}(\bar{\eta}, \bar{x}^0) \in P_0$. It then follows that for each $t \geq 0$, there exists $\tilde{\eta} \in H(\xi)$ such that $\|\Gamma_t(\eta, x^0) - (S_0(\tilde{\eta}), 0)\| < \delta$ and $\bar{d}(\tilde{\eta}(\eta), \tilde{\eta}) < \delta$. Thus, we conclude that

$$
\|\Gamma_t(\eta, x^0) - (S_0^*(\eta), 0)\| = \|\Gamma_t(\eta, x^0) - (S_0^*(\tilde{\eta}), 0)\| \\
\leq \|\Gamma_t(\eta, x^0) - (S_0^*(\tilde{\eta}), 0)\| + \|\Gamma_t(\eta, x^0) - (S_0^*(\tilde{\eta}), 0)\| \\
< \delta + \epsilon/2 < \epsilon, \ \forall t \geq 0,
$$

which implies that $I_i(t, \eta) := I_i(t, \eta, x^0) < \epsilon, \ \forall t \geq \tau, \ i = 1, \ldots, n$. From the first equations of (9), we have

$$
\frac{dS_i}{dt} > D_i(t, S(t - \tau)) - (e_i(t)S_i + \epsilon \alpha_i(t))S_i + \sum_{j=1}^{n} k_{ij}(t)S_j. \quad (23)
$$

Consider the perturbed system of (12)

$$
\frac{dS_i}{dt} = D_i(t, G(t - \tau)) - (e_i(t)S_i + \epsilon \alpha_i(t))S_i + \sum_{j=1}^{n} k_{ij}(t)S_j, \quad i = 1, \ldots, n. \quad (24)
$$

It then follows that we can restrict $\epsilon$ sufficiently small such that system (24) admits a unique positive almost periodic solution $S_i^*(t, \eta) = (S_i^*(t, \eta), \ldots, S_n^*(t, \eta))$ which is globally attractive in $C_+ \setminus \{0\}$, and there exist $t_1 > 0$ and $\epsilon_0 > 0$ such that $S_i^*(t, \eta) \geq S_i^*(t, \eta) - \epsilon \geq \epsilon_0, \ \forall 1 \leq i \leq n, \ t \geq t_1$. Applying the comparison principle, (23) implies that $S(t, \eta) = S(t, \eta, x^0) \geq S_i^*(t, \eta), \ \forall t \geq t_1$. Hence, we have

$$
S_i(t, \eta) \geq S_i^*(t, \eta) - \epsilon \geq \epsilon_0, \ \forall i = 1, \ldots, n, \ t \geq t_1. \quad (25)
$$

Consequently, from the second equations of (9) there holds

$$
\frac{dI_i}{dt} \geq \alpha_i(t)(S_i^*(t, \eta) - e)I_i - (e_i(t) + \theta_i(t))I_i + \sum_{j=1}^{n} l_{ij}(t)I_j, \quad \forall i = 1, \ldots, n, \ t \geq t_1. \quad (26)
$$

It is easy to see that $W_i(t)$ is the coefficient matrix of the right equations of (26) and $\omega(\Psi_{W_i}) > 0$. Since $-Z$ is cooperative and strongly irreducible, [27, Lemma 2.4, Theorem 2.5] tell us that there exist two functions $a_{W_i}(t, \eta)$ and $\tilde{x}_{W_i}(t, \eta) \in \text{Int}(C_+)$ such that $V_{W_i}(t, \eta) = e^{\int_{0}^{t} a_{W_i}(\tau, \eta)d\tau} \tilde{x}_{W_i}(t, \eta)$ is the solution of (22), and

$$
\lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} a_{W_i}(\tau, \eta)d\tau = \omega(\Psi_{W_i}) > 0.
$$

Since $(\eta, x^0) \in P_0$ implies that $I(t_2, \eta) \in \text{Int}(C_+)$, where $t_2 = \max\{t_0, t_1\}$, we can choose $\vartheta > 0$ to be small enough such that $I(t_2, \eta) \geq \vartheta \tilde{x}_{W_i}(t_2, \eta)$. By the comparison principle, as applied to system (26), it then follows that

$$
I(t, \eta) \geq \vartheta^* V_{W_i}(t, \eta) = \vartheta^* e^{\int_{0}^{t} a_{W_i}(\tau, \eta)d\tau} \tilde{x}_{W_i}(t, \eta), \ \forall t \geq t_2.
$$

Since $\tilde{x}_{W_i}(t, \eta)$ is almost periodic in $t$, and

$$
\lim_{t \to \infty} e^{\int_{0}^{t} a_{W_i}(\tau, \eta)d\tau} = \lim_{t \to \infty} \left( e^{t \int_{0}^{t} a_{W_i}(\tau, \eta)d\tau} \right)^t = \infty,
$$

we get $\lim_{t \to \infty} I_i(t, \eta) = \infty, \ i = 1, \ldots, 2$, for all $\eta \in H(\xi)$, a contradiction.
Next, we prove the case of $k^* = 0$. Assume, for the sake of contradiction, that 
\[
\limsup_{t \to \infty} d(\Pi_t(\eta, x^0), O_0) < \delta.
\]
It then follows that there exists $t_3 > 0$ such that $S_i(t, \eta) < \delta$ and $I_i(t, \eta) < \delta$ for all $t \geq t_3$. Hence, we have 
\[
\frac{dS_i}{dt} > D_i(t, S(t - \tau)) - (\epsilon_i(t)S_i + \delta \alpha_i(t))S_i + \sum_{j=1}^{n} k_{ij}(t)S_j, \quad \forall t \geq t_3.
\]
Replacing $\epsilon$ by $\delta$ in (23), a similar result to (25) holds. That is $S_i(t, \eta) \geq S_i^*(t, \eta) - \delta \geq \epsilon_0, \forall 1 \leq i \leq n, t \geq t_3$, which contradicts to $S_i(t, \eta) < \delta < \epsilon_0$ for all $t \geq t_3$ and $i = 1, \ldots, n$.

Since $O$ is isolated for $\Pi_t$ in $\partial P_0$, the claim above implies that $O$ is isolated for $\Pi_t$ in $P$. The claim above also implies that $W^s(O) \cap P_0 = \emptyset$, where 
\[
W^s(O) = \{(y^0, x^0) \in P : \omega(y^0, x^0) \neq \emptyset, \omega(y^0, x^0) < O\}
\]
is the stable set of $O$ for the semiflow $\Pi_t : P \to P$.

In the case of $J(t, 0, 0) > 0$, then $O_1$ is the unique compact invariant set for $\Pi_t$ in $M_0$. A similar result can be obtained to refer to the case of $k^* = 1$.

By the continuous-time version of [34, Theorem 1.3.1 and Remark 1.3.1], the skew-product semiflow $\Pi_t : P \to P$ is uniformly persistent with respect to $(P_0, \partial P_0)$.

In the following, we prove the practical uniform persistence. By Theorem 3.1, system (9) is dissipative. It then follows from [14, Theorem 4.5] with $d = d(\eta, x^0), \partial P_0)$ that $\Pi_t : P_0 \to P_0$ has a globally compact attractor $\mathcal{A}$. Since $\mathcal{A} = \Pi_t(\mathcal{A})$, we conclude that $\psi_i(0) > 0$, and hence, $\varphi_i(0) > 0$ for all $i = 1, \ldots, n$. Obviously, $\mathcal{A} \subset P_0$ and $\lim_{t \to \infty} d(\Pi_t(\varphi, \psi), \mathcal{A}) = 0$ for all $(\varphi, \psi) \in P_0$. Define a continuous function $\hat{d} : P \to \mathbb{R}_+$ by 
\[
\hat{d}(\eta, \varphi, \psi) = \min_{1 \leq i \leq n} \{\varphi_i(0), \psi_i(0)\}.
\]
It is easy to see that $\hat{d}(\eta, \varphi, \psi) > 0$ for all $(\eta, \varphi, \psi) \in \mathcal{A}$. The compactness of $\mathcal{A}$ implies that $\min_{(\eta, \varphi, \psi) \in \mathcal{A}} \hat{d}(\eta, \varphi, \psi) > 0$. Consequently, we conclude that there exists $\varrho > 0$ such that 
\[
\liminf_{t \to \infty} \min_{t \geq 0} (S(t, \eta, \varphi, \psi), I(t, \eta, \varphi, \psi)) = \liminf_{t \to \infty} \hat{d}(\Pi_t(\eta, \varphi, \psi)) \geq (\varrho, \varrho), \quad \forall (\varphi, \psi) \in P_0.
\]
This completes the proof. \qed

In the case of $R_0 < 1$, we show the global extinction whenever the initial value of infective individuals is sufficiently small or susceptible and infective individuals have the same dispersal rate.

**Theorem 4.2.** Suppose that (H1)-(H4) hold and $R_0 < 1$. If there exists an $i > 0$ such that every $(\varphi, \psi) \in C^+_k$ admits $\varphi \neq 0$ and $\psi_i(0) < i$, $i = 1, \ldots, n$, then the solution $(S(t), I(t))$ of system (4) satisfies $\lim_{t \to \infty} (S(t) - S^*(t)) = 0, \lim_{t \to \infty} I(t) = 0$.

**Proof.** It suffices to prove that the solution of (9) satisfies $\lim_{t \to \infty} (S(t, \eta) - S^*(t, \eta)) = 0$, $\lim_{t \to \infty} I(t, \eta) = 0$, and some ideas come from [15, Theorem 3.4]. For $U > 0$, define $\mathcal{X}_U = H(\xi) \times C([-\tau, 0], [0, U]^n) \times C([-\tau, 0], [0, U]^n)$. It then follows from Theorem 3.1 that $\mathcal{X}_U$ is positively invariant for the skew-product semiflow (10). That is, $\Pi_t(\eta, \varphi, \psi) \in \mathcal{X}_U$ for all $t \geq 0$ and $(\eta, \varphi, \psi) \in \mathcal{X}_U$. 


Consider the perturbed system:
\[
\frac{dS_i}{dt} = D_i(t, S(t - \tau) + \varepsilon) - e_i(t)S_i + \theta_i(t)\varepsilon + \sum_{j=1}^{n} k_{ij}(t)S_j, \quad i = 1, \ldots, n. \tag{27}
\]

By Theorem 3.2, we can restrict \( \varepsilon \) sufficiently small such that (27) admits a globally attractive and positive almost periodic solution \( S^*(t, \varepsilon, \eta) \). Obviously, \( \lim_{\varepsilon \to 0^+} S^*(t, \varepsilon, \eta) = S^*(t, \eta) \). Let \( \varepsilon \) be sufficiently small, (H2) implies that there exists \( \sigma_0 > 0 \) such that
\[
\sum_{j=1}^{n} q_{ij}(t)\Omega_j(t - \tau, \sigma)(\sigma + \varepsilon) \leq [c_i(t) - \sum_{j=1}^{n} c_{ij}(t)]\sigma - \alpha_i\varepsilon, \quad \forall t, \sigma \geq \sigma_0, i = 1, \ldots, n.
\]

Let \( S_\varepsilon(t, \eta, \varphi) \) be the solution of system (27) satisfying the initial value \( \varphi \). Since (27) is the perturbation of (12), for \( \varepsilon \) above, there exists \( t_0 > 0 \) such that
\[
S_\varepsilon(t, \eta, \varphi) < S^*(t, \varepsilon, \eta) + \bar{\varepsilon}, \quad \forall t \geq t_0,
\]
where \( \bar{\varepsilon} = (\varepsilon, \ldots, \varepsilon) \).

Define
\[
Y_\varepsilon(t) = Y(t, \varepsilon) - Z(t) + \varepsilon\bar{Y}(t),
\]
where \( Y(t, \varepsilon) = \text{diag}(\alpha_1(t)S_1^*(t, \varepsilon, \eta), \ldots, \alpha_n(t)S_n^*(t, \varepsilon, \eta)) \). Let \( \Psi_{Y_\varepsilon}(t, s), \quad t \geq s, \quad s \in \mathbb{R} \), be the evolution operator of the linear almost periodic system
\[
\frac{dC}{dt} = Y_\varepsilon(t)C. \tag{28}
\]

Similarly, we can define the exponential growth bound, \( \omega(\Psi_{Y_\varepsilon}) \), of \( \Psi_{Y_\varepsilon}(t, s) \) as (18). The continuous dependence of solutions on parameters implies that \( \lim_{\varepsilon \to 0^+} \Psi_{Y_\varepsilon}(t, s) = \Psi_{-Z+Y}(t, s) \), where \( \Psi_{-Z+Y}(t, s) \) is the evolution operator when we replace \(-Z(t) + Y(t)\) by \(-Z(t) + Y(t)\) in (14). Recall that Lemma 3.3 implies that \( R_0 < 1 \) if and only if \( \omega(\Psi_{-Z+Y}) < 0 \). Hence, if we use \( \omega(\Psi_{-Z+Y}) \) to denote the exponential growth bound of \( \Psi_{-Z+Y}(t, s) \), then [27, Lemma 2.4] implies that \( \omega(\Psi_{-Z+Y}) < 0 \). Thus, we can restrict \( \varepsilon \) sufficiently small such that \( \omega(\Psi_{Y_\varepsilon}) < 0 \). By [27, Lemma 2.4 and Theorem2.5], we conclude that there exist two almost periodic functions \( a(t, Y_\varepsilon) \) and \( \tilde{x}(t, Y_\varepsilon) \) in \( \mathbb{R}^n \) such that \( C(t, Y_\varepsilon(t)) = e^{\int_0^t a(\tau, Y_\varepsilon) d\tau} \tilde{x}(t, Y_\varepsilon(t)) \) is a solution of (28) and
\[
\omega(\Psi_{Y_\varepsilon}) = \lim_{t \to \infty} \frac{1}{t} \int_0^t a(\tau, Y_\varepsilon(\tau)) d\tau < 0.
\]

Choose \( \mu^* > 0 \) sufficiently small such that \( \mu^* \tilde{x}(t, Y_\varepsilon(t)) < \bar{\varepsilon}, \quad \forall t \in [0, t_0] \), where \( \bar{\varepsilon} = (\varepsilon, \ldots, \varepsilon) \). Consider another perturbed system:
\[
\frac{d\hat{I}_i}{dt} = \alpha_i(t)(S_i^*(t, \varepsilon, \eta) + \varepsilon)\hat{I}_i - (e_i(t) + \theta_i(t))\hat{I}_i + \sum_{j=1}^{n} l_{ij}(t)\hat{I}_j, \quad i = 1, \ldots, n. \tag{29}
\]

Let \( \hat{I}(t, \eta, \xi) \) be the solution of (29) satisfying \( \hat{I}(0, \eta, \xi) = (\xi, \ldots, \xi) \in \mathbb{R}^n, \forall \xi \in H(\xi) \). We restrict \( t > 0 \) sufficiently small such that
\[
\hat{I}(t, \eta, \xi) < \mu^* e^{\int_0^t a(\tau, Y_\varepsilon(\tau)) d\tau} \tilde{x}(t, Y_\varepsilon(t)) \leq \mu^* \tilde{x}(t, Y_\varepsilon(t)) < \varepsilon, \forall t \in [0, t_0]. \tag{30}
\]

Let \((S(t, \eta), I(t, \eta)) = (S(t, \eta, \varphi, \psi), I(t, \eta, \varphi, \psi))\) be a nonnegative solution of (9) with \((\eta, \varphi, \psi) \in \mathbb{X}_\sigma, \varphi \neq 0 \) and \( \psi_i(0) \leq \xi, i = 1, \ldots, n \). Furthermore, we claim that
\[
I(t, \eta) \leq \mu^* e^{\int_0^t a(\tau, Y_\varepsilon(\tau)) d\tau} \tilde{x}(t, Y_\varepsilon(t)), \forall t \geq t_0.
\]
On the contrary, by the comparison theorem and (30), there exists $k$, $1 \leq k \leq n$, and $T > t_0$ such that
\[ I(t, \eta) \leq \mu^* e^{\int_0^t a(\tau, Y_0^i) d\tau} \tilde{x}(t, Y_0^i), \quad t_0 \leq t \leq T, \]
\[ I_k(T, \eta) = \mu^* (e^{\int_{t_0}^T a(\tau, Y_0^i) d\tau} I(T, Y_0^i)) k, \]
\[ I_k(t, \eta) > \mu^* (e^{\int_0^t a(\tau, Y_0^i) d\tau} \tilde{x}(t, Y_0^i)) k, \quad 0 < t - T \ll 1. \quad (31) \]
From the arguments above and the assumption $(H_2)$, we conclude that for $t_0 \leq t \leq T$,
\[ \frac{dS_i}{dt} < D_i(t, S(t - \tau) + \varepsilon) - e_i(t) S_i + \theta_i(t) \varepsilon + \sum_{j=1}^n k_{ij}(t) S_j, \quad i = 1, \ldots, n. \quad (32) \]
Applying the comparison principle, it then follows that $S(T, \eta) < S^*(T, \varepsilon, \eta) + \varepsilon$. Thus, for $0 < t - T \ll 1$, we have $S(t, \eta) < S^*(t, \varepsilon, \eta) + \varepsilon$, and hence
\[ \frac{dI}{dt} < -\alpha_i(t) (S_i^*(t, \varepsilon, \eta) + \varepsilon) I_i - (e_i(t) + \theta_i(t)) I_i + \sum_{j=1}^n I_{ij}(t) I_j, \quad i = 1, \ldots, n. \quad (33) \]
Since $I(T, \eta) \leq \mu^* e^{\int_0^T a(\tau, Y_0^i) d\tau} \tilde{x}(T, Y_0^i)$, the comparison principle implies that
\[ I(t, \eta) \leq \mu^* e^{\int_0^t a(\tau, Y_0^i) d\tau} \tilde{x}(t, Y_0^i), \quad 0 < t - T \ll 1, \]
Hence,
\[ I_k(t, \eta) \leq \mu^* (e^{\int_0^t a(\tau, Y_0^i) d\tau} \tilde{x}(t, Y_0^i)) k, \quad 0 < t - T \ll 1, \]
which contradicts to (31). The claim is proved.

The claim above implies that (32) holds for all $t \geq 0$. Applying the comparison principle again, we conclude that $S(t, \eta) < S^*(t, \varepsilon, \eta) + \varepsilon$, for all $t \geq t_0$. By a similar argument to the above, it then follows that
\[ I(t, \eta) < \mu^* e^{\int_0^t a(\tau, Y_0^i) d\tau} \tilde{x}(t, Y_0^i), \quad \forall t > T. \]
Since $\tilde{x}(t, Y_0^i)$ is almost periodic, and
\[ \lim_{t \to \infty} e^{\int_0^t a(\tau, Y_0^i) d\tau} = \lim_{t \to \infty} \left( e^{\int_0^t a(\tau, Y_0^i) d\tau} \right)^t = 0, \]
we get $\lim_{t \to \infty} I(t, \eta) = 0$.

For any $(\eta^*, \varphi^*, \psi^*) \in X_\sigma$, $\varphi^* \neq 0$ and $\psi^*(0) < t$, $i = 1, \ldots, n$, let $\omega(\eta^*, \varphi^*, \psi^*)$ be the omega limit set of $(\eta^*, \varphi^*, \psi^*)$ for $\Pi_i$ in (10). The arguments above implies that $\lim_{t \to \infty} I(t, \eta^*, \varphi^*, \psi^*)$ is unique and $0$. Hence, $\omega(\eta^*, \varphi^*, \psi^*) = (\eta^*, (\tilde{\omega}, 0))$. We claim that $\tilde{\omega} \neq \emptyset$. On the contrary, assume that $\tilde{\omega} = \emptyset$, which implies that $\lim_{t \to \infty} S(t, \eta^*, \varphi^*, \psi^*) = 0$.

It is contradictory to the claim in the proof of Theorem 4.1 for the case of $k^* = 0$.

By the form of system (9), the disease-free space
\[ X_\epsilon = \{(S, I) \in \mathbb{R}_+^2 : I_i = 0, i = 1, \ldots, n\} \]
is invariant. Let $\hat{S}(t, \eta, S^0)$ the unique solution of (12) satisfying $\hat{S}(\xi, \eta, S^0) = S^0(\eta)$, $\xi \in [-\tau, 0]$. Then we define the skew-product semiflow
\[ \bar{\Pi}_t : H(\xi) \times C_+ \to H(\xi) \times C_+ \]
\[ (\eta, S^0) \mapsto (\bar{\eta}_t, \hat{S}_t(\eta, S^0)). \quad (34) \]
Let $\bar{\omega}(\eta, S^0)$ be the omega limit set of $\bar{\Pi}_t$.

For simplicity, let
\[ \hat{A} = \{(\eta_t, S_t^*(\eta)) : t \in \mathbb{R}, \eta \in H(\xi)\}, \]
and
\[ W^*(\bar{A}) := \{(\eta, S^0) \in H(\xi) \times C_+ : \bar{\omega}(\eta, S^0) \neq \emptyset, \bar{\omega}(\eta, S^0) \subset \bar{A}\}, \]
which is the stable set of $\bar{A}$ for $\Pi_l$. By the continuous-time version of [34, Lemma 1.2.1], $\omega(\eta^*, \varphi^*, \psi^*)$ is an internal chain transitive set for $\Pi_l$, it then follows that $\bar{\omega}$ is an internal chain transitive set for $\Pi_l$. By Theorem 3.2, $\bar{A}$ is globally attractive for $\Pi_l$ in $H(\xi) \times C_+ \setminus \{0\}$. Hence, we conclude that $\bar{\omega} \cap W^*(\bar{A}) \neq \emptyset$. Similarly, applying continuous-time version of the [34, Theorem 1.2.1], we have $\tilde{\omega}(\eta^*, \varphi^*, \psi^*) = \{(\eta, (S^*_l(\eta)), 0)) : t \in \mathbb{R}, \eta \in H(\xi)\}$. Thus, $\lim_{t \to \infty} (S(t, \eta) - S^*(t, \eta)) = 0$ and $\lim_{t \to \infty} I(t, \eta) = 0$. \hfill \ensuremath{\Box}

**Theorem 4.3.** Suppose that (H1)-(H4) hold and $R_0 < 1$. Let $r_{ij}(t) = h_{ij}(t)$, $\forall t \geq 0, i, j = 1, \ldots, n$, then for all $(\varphi, \psi) \in C^2_+$ with $\varphi \neq 0$, $\lim_{t \to \infty} (S(t) - S^*(t)) = 0$, $\lim_{t \to \infty} I(t) = 0$.

**Proof.** Similar to the above, we use the skew-product semiflows approach to prove the results and the process is motivated by [15, Theorem 3.5]. First, we prove that
\[ \lim_{t \to \infty} I(t, \eta) = 0. \] (35)

Since $r_{ij}(t) = h_{ij}(t)$ implies that $k_{ij}(t) = l_{ij}(t)$, system (9) can be rewritten as:
\[ \frac{dG_i}{dt} = D_i(t, G(t - \tau)) - e_i(t)G_i + \sum_{j=1}^{n} k_{ij}(t)G_j, \quad i = 1, \ldots, n. \] (36)

Let $G(t, \eta) = G(t, \eta, G^0)$ be the solution of (36) satisfying the initial value $G^0$. By Theorem 3.2, system (36) has a unique positive almost periodic solution $S^*(t, \eta) = (S^*_1(t, \eta), \ldots, S^*_n(t, \eta))$ which is globally attractive in $H(\xi) \times C_+ \setminus \{0\}$, that is, $\lim_{t \to \infty} (G(t, \eta) - S^*(t, \eta)) = 0$. Hence, we conclude that for any $\epsilon > 0$, there exist $t_0 > 0$ such that for $t \geq t_0$, $G(t, \eta) = S(t, \eta) + I(t, \eta) < S^*(t, \eta) + \bar{\epsilon}$, where $\bar{\epsilon} = (\epsilon, \ldots, \epsilon)$. Thus, we have
\[ \frac{dI_i}{dt} < \alpha_i(t)(S^*_i(t, \eta) + \epsilon)I_i - (e_i(t) + \theta_i(t))I_i + \sum_{j=1}^{n} k_{ij}(t)I_j, \quad \forall t \geq t_0, \quad i = 1, \ldots, n. \]

Consider the comparison system
\[ \frac{d\hat{I}_i}{dt} = \alpha_i(t)(S^*_i(t, \eta) + \epsilon)\hat{I}_i - (e_i(t) + \theta_i(t))\hat{I}_i + \sum_{j=1}^{n} k_{ij}(t)\hat{I}_j, \quad i = 1, \ldots, n. \] (37)

Obviously, the efficient matrix of (37) is $-Z(t) + \gamma(t) + \epsilon \bar{\gamma}(t) := W^*(t)$. Since $R_0 < 1$, the similar arguments to those in the proof of Theorem 4.1 imply that $\omega(\Psi_{W^*}) < 0$. Furthermore, there exist two almost periodic functions $a(t, W^*)$ and $\hat{I}(t, W^*) \in \text{Int}(R^n_+)$ such that $\hat{I}(t, W^*) = e^{\int_0^t a(\tau, W^*)d\tau}I(t, W^*)$ is a solution of (37), and
\[ \omega(\Psi_{W^*}) = \lim_{t \to \infty} \frac{1}{t} \int_0^t a(\tau, W^*)d\tau < 0. \]

Since $\hat{I}(t, W^*)$ is almost periodic, and
\[ \lim_{t \to \infty} e^{\int_0^t a(\tau, W^*)d\tau} = \lim_{t \to \infty} \left( e^{\frac{1}{t} \int_0^t a(\tau, W^*)d\tau} \right)^t = 0, \]
which implies that \( \lim_{t \to \infty} \tilde{I}(t, W^r) = 0 \). Since \( \tilde{I}(t, W^r) \in \text{Int}(R^+_n) \) for all \( t \geq 0 \), there exists \( \bar{a} > 0 \) such that \( I(t, \eta) \leq \bar{a} \tilde{I}(t, W^r) \), \( \forall t \geq t_0 \), which implies that \( \lim_{t \to \infty} I(t, \eta) = 0 \).

For any \( (\varphi, \psi) \in C^2_+ \) with \( \varphi \neq \hat{0} \), we have \( G^0 = \varphi + \psi \in C_+ \setminus \{0\} \). Since \( \lim_{t \to \infty} (G(t, \eta) - S^*(t, \eta)) = 0 \), it follows from
\[
S(t, \eta) - S^*(t, \eta) = G(t, \eta) - I(t, \eta) - S^*(t, \eta) = (G(t, \eta) - S^*(t, \eta)) - I(t, \eta)
\]
that \( \lim_{t \to \infty} (S(t, \eta) - S^*(t, \eta)) = 0 \). This completes the proof.

**Remark 2.** Here, we consider the threshold dynamics of an almost periodic epidemic model with age structure in a patchy environment in terms of the basic reproduction ratio \( R_0 \). The arguments above implies that \( R_0 \) plays a crucial role in the control of a disease. Refer to [27, Theorem 3.4], we can create a method how to compute \( R_0 \).

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