Kleene Theorems for Free Choice Nets Labelled with Distributed Alphabets

Ramchandra Phawade

Indian Institute of Technology Dharwad, India
prb@iitdh.ac.in

Abstract. We provided (PNSE’2014) expressions for free choice nets having distributed choice property which makes the nets direct product representable. In a recent work (PNSE’2016), we gave equivalent syntax for a larger class of free choice nets obtained by dropping distributed choice property.

In both these works, the classes of free choice nets were restricted by a product condition on the set of final markings. In this paper we do away with this restriction and give expressions for the resultant classes of nets which correspond to free choice synchronous products and Zielonka automata. For free choice nets with distributed choice property, we give an alternative characterization using properties checkable in polynomial time.

Free choice nets we consider are 1-bounded, S-coverable, and are labelled with distributed alphabets, where S-components of the associated S-cover respect the given alphabet distribution.

Keywords: Kleene theorems · Petri nets · Distributed automata.

1 Introduction

There are several different notions of acceptance to define languages for labelled place transition Petri nets, depending on restrictions on labelling and final markings [13]. The language of a place transition net with an initial marking and a finite set of final markings, is called L-type language [8]. One goal of this work is to give syntax of expressions for L-type languages for various subclasses of 1-bounded, free choice nets labelled with distributed alphabets. One advantage of using distributed alphabet is that we can see free choice nets as products of automata [12], enabling us to write expressions for the nets using components. This also enables us to compare expressiveness of nets and products of automata. Three kinds of formulations of automata over distributed alphabets, in the increasing order of expressiveness: direct products, synchronous products, and asynchronous products are described in [12]. In the present paper[1], we present a hierarchy of 1-bounded free choice nets like automata over distributed

[1] A preliminary version of this paper appeared at 14th PNSE workshop, held at Bratislava [16].
alphabets, and also introduce a fourth product automata in the current hierarchy which is utilized to get the syntax. In this hierarchy, there are four kinds of free choice nets labelled over distributed alphabets. Two out of these four classes were introduced earlier \[17,18,15\]. Two new classes of systems are given in this work. To understand the complete hierarchy and their relations to other formalisms like expressions and automata over distributed alphabets we invite the reader to read these earlier works \[17,18,15\].

We use product automata to get expressions for the Free choice nets, and give correspondences for all these three formalisms for various classes. This kind of correspondence has been used in concurrent code generation for discrete event systems \[7\].

We construct expressions for \(L\)-type languages of free choice nets via free choice Zielonka automata.

Consider the net \(N\) of Figure 1 with \(G = \{\{r_1, s_1\}, \{r_2, s_2\}\}\) as its set of final markings, with its decomposition into finite state machines in Figure 2. Because this net is decomposable into state machines \[6,3\], its markings can be written in tuple form, where each s-component has a place in the tuple: for example \(G = \{(r_1, s_1), (r_2, s_2)\}\). For the final marking \((r_1, s_1)\), its language can be expressed by \(\text{fsync}((ab+ac)^*, (ad+ae)^*)\) \[17,18\]. Similarly, for the final marking \((r_2, s_2)\) the language equivalent expression can be given by \(\text{fsync}((ab+ac)^*a, (ad+ae)^*a)\). In general, if the places involved in the final markings form a product \[17,18\], then its language is specified by taking product of component expressions, using free choice Zielonka automata with product-acceptance \[15\] as intermediary. Even though \(r_1\) and \(s_2\) participate in final markings, marking \(\{r_1, s_2\}\) does not belong to \(G\), hence set \(G\) do not form a product. The language \(L\) of net system \((N, G)\) can be described by, \(\text{fsync}((ab+ac)^*, (ad+ae)^*) + \text{fsync}((ab+ac)^*a, (ad+ae)^*a)\). The key idea is ability to express the language of a net as the union of languages of nets complying with the product condition on final set of markings. This closure under union may not be always possible for restricted classes of languages defined over a distributed alphabet. For example, the union of direct product languages \(L_1 = \{ca, cb\}\) and \(L_2 = \{caa, cbb\}\) defined over \(\Sigma_1 = \{c, a\}\) and \(\Sigma_2 = \{c, b\}\)
respectively, is not expressible as a direct product language. But this language is accepted by synchronous products: the direct products extended with subset-acceptance [12].

For the restricted class of direct product representable free choice nets, with its set of final markings having product condition-we gave expressions via product systems with matchings (matched states of product system correspond to places of a cluster in net) and product-acceptance [17,18]. As a second goal, we develop syntax for free choice nets with distributed choice, now extended with subset-acceptance. For a net in this class also, its language can be expressed as the union of languages accepted by product system with matchings and product-acceptance. This union is accepted by product systems with matching and extended with subset-acceptance (free choice synchronous products). As a third contribution, we develop an alternate characterization of this class of nets, via free choice Zielonka automata with product-moves.

Language equivalent expressions for 1-bounded nets have been given by Grabowski [5], Garg and Ragunath [4] and other authors [9], where renaming operator has been used in the syntax to disambiguate synchronizations. We have chosen to not use this operator and to exploit the S-decompositions of nets instead. The syntax for smaller subclasses of nets such marked graphs and free choice nets with initial markings as feedback vertex set has been given earlier [10,14].

Organization of paper. In the next section, we begin with preliminaries on distributed alphabets and nets. In Section 3 we define product systems with globals and subset-acceptance, and show that their languages can be expressed as the union of languages accepted by product systems with globals and product-acceptance. These product systems are used as intermediary to get expressions for nets and vice versa. The following section relates these product systems to nets. In Section 5 we develop syntax of expressions for product systems with subset acceptance, and next section establishes the correspondence between various classes of product systems and expressions. In the last section we conclude, with an overview of established correspondences between all three formalisms.

2 Preliminaries

\(\mathbb{N}\) denotes the set of natural numbers including 0. Let \(\Sigma\) be a finite alphabet and \(\Sigma^*\) be the set of all finite words over the alphabet \(\Sigma\), including the empty word \(\varepsilon\). A language over an alphabet \(\Sigma\) is a subset \(L \subseteq \Sigma^*\). The projection of a word \(w \in \Sigma^*\) to a set \(\Delta \subseteq \Sigma\), denoted as \(w|_\Delta\), is defined by: \(\varepsilon|_\Delta = \varepsilon\) and \((a\sigma)|_\Delta = \begin{cases} a(\sigma|_\Delta) & \text{if } a \in \Delta, \\ \sigma|_\Delta & \text{if } a \notin \Delta. \end{cases}\)

Given languages \(L_1, L_2, \ldots, L_m\), their synchronized shuffle \(L = L_1| \cdots |L_m\) is defined as: \(w \in L\) iff for all \(i \in \{1, \ldots, m\}\), \(w|_{\Sigma_i} \in L_i\).

**Definition 1 (Distributed Alphabet).** Let \(\text{Loc}\) denote the set \(\{1, 2, \ldots, k\}\). A **distribution** of \(\Sigma\) over \(\text{Loc}\) is a tuple of nonempty sets \((\Sigma_1, \Sigma_2, \ldots, \Sigma_k)\) with
where

\[ F \sigma M \]

Definition 3. A labelled net system is a tuple \( (N, M_0, G) \) where \( N = (S, T, F, \lambda) \) is a labelled net; \( M_0 \) an initial marking; and a finite set of final markings \( G \).

A transition \( t \) is enabled at a marking \( M \) if all places in its pre-set are marked by \( M \). In such a case, \( t \) can be fired or occurs at \( M \), to produce the new marking \( M' \) which is defined as: for each place \( p \) in \( S \), \( M'(p) = M(p) + F(t, p) - F(p, t) \), where \( F(x, y) = 1 \) if \( (x, y) \in F \) and 0 otherwise. We write this as \( M \xrightarrow{t} M' \) or \( M \xrightarrow{\lambda(t)} M' \).

For some markings \( M_0, M_1, \ldots, M_n \) if we have \( M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \ldots \xrightarrow{t_n} M_n \), then the sequence \( \sigma = t_1 t_2 \ldots t_n \) is called occurrence or firing sequence. We write
$M_0 \xrightarrow{\sigma} M_n$ and call $M_n$ the marking reached by $\sigma$. This includes an empty transition sequence $\varepsilon$. For each marking $M$ we have $M \xrightarrow{\varepsilon} M$. We write $M \xrightarrow{\sigma} M'$ and call $M'$ reachable from $M$ if it is reached by some occurrence sequence $\sigma$ from $M$.

A net system $(N, M_0, G)$ is called 1-bounded if for every place $p$ of the net and every reachable marking $M$, we have $M(p) \leq 1$. Any marking of a 1-bounded net can be alternately represented by the subset of places which are marked at $M$. In this paper, we consider only 1-bounded nets.

We say a net system $(N, M_0)$ is live if, for every reachable marking $M$ and every transition $t$, there exists a marking $M'$ reachable from $M$ which enables $t$.

**Definition 4.** For a labelled net system $(N, M_0, G)$, its language is defined as $\text{Lang}(N, M_0, G) = \{ \lambda(\sigma) \in \Sigma^* | \sigma \in T^* \text{ and } M_0 \xrightarrow{\sigma} M, \text{ for some } M \in G\}$. 

**Net Systems and its components** First we define subnet of a net.

Let $X$ be a set of nodes of net $N = (S, T, F)$. Then the triple $N' = (S \cap X, T \cap X, F \cap (X \times X))$ is a subnet of net $N$. Flow relation $F \cap (X \times X)$ is said to be induced by nodes $X$ and $N'$ is said to be a subnet of $N$ generated by nodes $X$ of $N$.

We follow the convention that if $N'$ is a subnet of $N$ and $z$ is a node of $N'$ then $z^-$ and $z^+$ denote the pre-set and post-set taken in $N$, i.e., $z^- = \{x | (x, z) \in F\}$ and $z^+ = \{x | (z, x) \in F\}$.

**Definition 5.** Subnet $N'$ is called a **component** of $N$ if,

- For each place $s$ of $X$, $s^-, s^+ \subseteq X$,
- $N'$ is an $S$-net,
- $N'$ is connected.

A set $C$ of components of net $N$ is called **$S$-cover** for $N$, if every place of the net belongs to some component of $C$.

Our notion of component does not require strong connectedness and so it is different from notion of $S$-component in [3], and therefore our notion of $S$-cover also differs from theirs.

A net is **covered by components or $S$-coverable** if it has an $S$-cover.

Fix a distribution $(\Sigma_1, \Sigma_2, \ldots, \Sigma_k)$ of $\Sigma$. We define $s$-decomposition [6] of a net into sequential components. Note that $s$-decomposition given here is for labelled nets unlike [6][3] and is different from [17][15][16] also, as it takes into account the initial marking of the net.

**Definition 6.** A labelled net system $(N, M_0, G)$ is called **$S$-decomposable** if, there exists an $S$-cover $C$ for net $N = (S, T, F, \lambda)$, such that for each $T_i = \bigcup_{a \in \Sigma_i} \lambda^{-1}(a)$, there exists $S_i \subseteq S$ and the subnet generated by $S_i \cup T_i$ is a component in $C$, and the initial marking $M_0$ marks only one place of the component.
Now each S-decomposable net $N$ admits an S-cover, since there exist subsets $S_1, S_2, \ldots, S_k$ of places $S$, such that $S = S_1 \cup S_2 \cup \ldots S_k$ and $**S_i \cup S_i^* = T_i$, such that the subnet $(S_i, T_i, F_i)$ generated by $S_i$ and $T_i$ is an S-net, where $F_i$ is the induced flow relation from $S_i$ and $T_i$.

Note that, the initial marking, of a 1-bounded and S-decomposable net system, marks exactly one place in each S-component of the given S-cover $S_1, S_2, \ldots, S_k$. At any reachable markings of such a net, the total number of tokens in an S-component remains constant [13]. Therefore, at any reachable marking $M$, each S-component has only one token, so at that marking only one place of that component is marked. Also, if we collect each place from an S-component we get back the marking of net. Hence, marking $M$ can be written as a $k$-tuple from its component places $S_1 \times S_2 \times \ldots \times S_k$.

We use a product condition [17] on the set of final markings of a net system which is known [10,12] to restrict classes of languages.

**Definition 7.** An S-decomposable labelled net system $(N, M_0, G)$ is said to have product-acceptance if its set of final markings $G$ satisfies product condition: if $\langle q_1, q_2, \ldots, q_k \rangle \in G$ and $\langle q_1', q_2', \ldots, q_k' \rangle \in G$ then $\{q_1, q_1'\} \times \{q_2, q_2'\} \times \ldots \times \{q_k, q_k'\} \subseteq G$.

Let $t$ be a transition in $T_a$. Then by S-decomposability a pre-place and a post-place of $t$ belongs to each $S_i$ for all $i$ in $loc(a)$. Let $t[i]$ denote the tuple $\langle p, a, p' \rangle$ such that $(p, t), (t, p') \in F_i$, and $p, p' \in P_i$ for all $i$ in $loc(a)$.

### 2.2 Free choice nets and their properties

Let $x$ be a node of a net $N$. The cluster of $x$, denoted by $[x]$, is the minimal set of nodes containing $x$ such that

- if a place $s \in [x]$ then $s^*$ is included in $[x]$, and
- if a transition $t \in [x]$ then $t^*$ is included in $[x]$.

For a cluster $C$, we denote its set of places by $S_C$, and its set of transitions by $T_C$.

The set of all $a$-labelled transitions along with places $r_1$ and $s_1$ form a cluster of the net shown in Figure 6.

**Definition 8 (Free choice nets [3].)** A cluster $C$ is called free choice (FC) if all transitions in $C$ have the same pre-set. A net is called free choice if all its clusters are free choice.

In a labelled net $N$, for a free choice cluster $C$ define the $a$-labelled transitions $C_a = \{ t \in T_C | \lambda(t) = a \}$. If the net has an S-decomposition then we associate a post-product $\pi(t) = \Pi_{i \in loc(a)} (t^* \cap S_i)$ with every such transition $t$. This is well defined since in S-nets, every transition will have at most one post-place in $S_i$. Let $post(C_a) = \bigcup_{t \in C_a} \pi(t)$. Let $C_a[i] = C_a^* \cap S_i$ and postdecomp($C_a$) = $\Pi_{i \in loc(a)} C_a[i]$. Clearly $post(C_a) \subseteq postdecomp(C_a)$.

Sometimes, we may call
Kleene Theorems for Free Choice Nets Labelled with Distributed Alphabets

$C_a[i]$ as post-projection of the cluster $C$ with respect to label $a$ and location $i$. Also, $\text{postdecomp}(C_a)$ is called post-decomposition of cluster $C$ with respect to label $a$.

The following definition from [18,17] is used to get direct product representability.

**Definition 9 (distributed choice property).** An $S$-decomposable free choice net $N = (S,T,F,\lambda)$ is said to have distributed choice property (DCP) if, for all $a$ in $\Sigma$ and for all clusters $C$ of $N$, $\text{postdecomp}(C_a) \subseteq \text{post}(C_a)$.

![Fig. 3. Labelled Free Choice Net system with distributed choice](image)

![Fig. 4. $S$-cover of the net in Fig. 3](image)

**Example 1 (Free choice net system without distributed choice and with product-acceptance).** Consider a distributed alphabet $\Sigma = (\Sigma_1 = \{a,b,c\}, \Sigma_2 = \{a,d,e\})$ and the net system $N$ shown in Figure 1, labelled over $\Sigma$. Its (only possible) $S$-cover having two $S$-components with sets of places $S_1 = \{r_1, r_2, r_3\}$ and $S_2 = \{s_1, s_2, s_3\}$ respectively, is given in Figure 2. For the cluster $C$ of $r_1$, we have the set of $a$-labelled transitions $C_a = \{t_1, t_2\}$ with $C_a[1] = \{r_2, r_3\}$ and $C_a[2] = \{s_2, s_3\}$. So we get $\text{postdecomp}(C_a) = \{(r_2, s_2), (r_2, s_3), (r_3, s_2), (r_3, s_3)\}$.

As $\pi(t_1) = \{(r_2, s_2)\}$ and $\pi(t_2) = \{(r_3, s_3)\}$ so $\text{post}(C_a) = \{(r_2, s_2), (r_3, s_3)\}$. Since $\text{postdecomp}(C_a) \not\subseteq \text{post}(C_a)$, this cluster does not have distributed choice, so the net system does not have it.

With the set of final markings $\{(r_1, s_1), (r_1, s_2), (r_2, s_1), (r_2, s_2)\}$ satisfying product condition, the language $L_p$ accepted by this net system is $r^* [\varepsilon + a + ab + ad]$ where $r = (a(bd + db) + a(ce + ec))$.

**Example 2 (Free choice net system without distributed choice and not satisfying product condition of the set of final markings).** Consider the net system of Example 1 whose underlying net is shown in Figure 1. With set of final markings $\{(r_1, s_1), (r_2, s_2)\}$, which do not satisfy product condition, the language $L_s$ accepted by this net system is $r^* [\varepsilon + a]$ where $r = (a(bd + db) + a(ce + ec))$. 
Example 3 (A net with distributed choice property and product acceptance condition). Consider the labelled net system \((N, (r_1, s_1), G))\) of Figure 3 defined over distributed alphabet \(\Sigma = (\Sigma_1 = \{a, b, c\}, \Sigma_2 = \{a, d, e\})\), and where \(G = \{(r_1, s_1), (r_1, s_2), (r_2, s_1), (r_2, s_2)\}\) is the set of final markings satisfying product condition. Its two S-components with sets of places \(S_1 = \{r_1, r_2, r_3\}\) and \(S_2 = \{s_1, s_2, s_3\}\), are shown in Figure 3. For cluster \(C\) of \(r_1\), we have \(C_a = \{t_1, t_2, t_3, t_4\}\), \(C_a[1] = \{r_2, r_3\}\) and \(C_a[2] = \{s_2, s_3\}\), hence \(postdecomp(C_a) = \{(r_2, s_2), (r_2, s_3), (r_3, s_2), (r_3, s_3)\}\). We have \(\pi(t_1) = \{(r_1, s_2)\}\), \(\pi(t_2) = \{(r_2, s_3)\}\), \(\pi(t_3) = \{(r_4, s_2)\}\) and \(\pi(t_4) = \{(r_3, s_3)\}\).

So \(post(C_a) = \{(r_2, s_2), (r_2, s_3), (r_3, s_2), (r_3, s_3)\}\). Therefore, \(postdecomp(C_a) = post(C_a)\). For all other clusters this holds trivially, because each of them have only one transition and only one post-place, hence the net has distributed choice.

Language \(L_3\) accepted by the net system is \(r^*[\epsilon + a + a(b + c) + a(d + e)]\) where \(r = (a(bd + db) + a(be + eb) + a(cd + dc) + a(ce + ec))\).

Example 4 (A net with distributed choice and subset-acceptance). Consider the net system of Example 3 with the underlying net shown in Figure 3 with set of final markings \(\{(r_1, s_1), (r_2, s_2)\}\). The language \(L_4\) accepted by this net system is \(r^*[\epsilon + a]\) where \(r = (a(bd + db) + a(be + eb) + a(cd + dc) + a(ce + ec))\).

3 Product systems

We define product systems over a fixed distribution \((\Sigma_1, \Sigma_2, \ldots, \Sigma_k)\) of \(\Sigma\). First we define sequential systems.

Definition 10. A sequential system over a set of actions \(\Sigma_i\) is a finite state automaton \(A_i = (P_i, \rightarrow_i, G_i, p_i^0)\) where \(P_i\) are called states, \(G_i \subseteq P_i\) are final states, \(p_i^0 \in P_i\) is the initial state, and \(\rightarrow_i \subseteq P_i \times \Sigma_i \times P_i\) is a set of local moves.

For a local move \(t = (p, a, p')\) of \(\rightarrow_i\) state \(p\) is called pre-state sometimes denoted by \(pre(t)\) and \(p'\) is called post-state of \(t\), sometimes denoted by \(post(t)\).

Such a move is sometimes called an a-move or an a-labelled move.

Let \(\rightarrow_i^a\) denote the set of all a-labelled moves in the sequential system \(A_i\). The language of a sequential system is defined as usual.

Definition 11. Let \(A_i = (P_i, \rightarrow_i, G_i, p_i^0)\) be a sequential system over alphabet \(\Sigma_i\) for \(1 \leq i \leq k\). A product system \(A\) over the distribution \(\Sigma = (\Sigma_1, \ldots, \Sigma_k)\) sometimes denoted by \(\langle A_1, \ldots, A_k\rangle\) is a tuple \((P, \Rightarrow, R^0, G)\), where : 
\[P = \bigoplus_{i \in Loc} P_i\]
\(R^0 = (p_1^0, \ldots, p_k^0)\) is the initial product state of \(A\); \(G \subseteq \bigoplus_{i \in Loc} G_i\) is the set of final product states of \(A\); and, \(\Rightarrow \subseteq \bigcup_{a \in \Sigma} \Rightarrow_a\), denotes the global moves of \(A\) where \(\Rightarrow_a = \prod_{i \in Loc(a)} \rightarrow_a\).

Elements of \(\Rightarrow_a\) are sometimes called global a-moves. Any global a-move is global within the set of component sequential machines where action \(a\) occurs. For a global a-move \(g\), we define its set of pre-states \(pre(g)\) as the set of pre-states of all its component a-moves; the set of post-states \(post(g)\) as the set of post-states of all its component a-moves; and, use notation \(g[i]\) for its \(i\)-th component–local a-move–belonging to \(A_i\), for all \(i\) in \(loc(a)\). We use \(R[i]\) for the projection of a product state \(R\) in \(A_i\).
3.1 Direct products

With set of global moves $\Rightarrow = \bigcup_{a \in \Sigma} \Rightarrow_a$ and final states $G = \Pi_{i \in \text{Loc}} G_i$, $A$ is called product system with product-acceptance. These systems are called direct products in [12].

With set of global moves $\Rightarrow = \bigcup_{a \in \Sigma} \Rightarrow_a$ and final states $G \subseteq \Pi_{i \in \text{Loc}} G_i$, $A$ is called product system with subset-acceptance. These systems are called synchronous products in [12].

The runs of a product system $A$ over some word $w$ are described by associating product states with prefixes of $w$: the empty word is assigned initial product state $R_0$, and for every prefix $va$ of $w$, if $R$ is the product state reached after $v$ and $Q$ is reached after $va$ where, for all $j \in \text{loc}(a)$, $\langle R[j], a, Q[j] \rangle \in \rightarrow_j$, and for all $j \notin \text{loc}(a), R[j] = Q[j]$. A run of a product system over word $w$ is said to be accepting if the product state reached after $w$ is in $G$. We define the language $\text{Lang}(A)$ of product system $A$, as the set of words on which the product system has an accepting run. The set of languages accepted by direct (resp. synchronous) products is called direct (resp. synchronous) product languages.

We use a characterization from [12] of languages accepted by direct products.

**Proposition 1.** Let $L$ be a language defined over distributed alphabet $\Sigma$. The language $L$ is a direct product language iff $L = \{ w \in \Sigma^* | \forall i \in \{1, \ldots, k\}, \exists u_i \in L$ such that $w \downarrow_{\Sigma_i} = u_i \downarrow_{\Sigma_i} \}$.

If $L = \text{Lang}(A)$ for direct product $A = \langle A_1, \ldots, A_k \rangle$ defined over distributed alphabet $\Sigma$ then $L = \text{Lang}(A_1) \| \ldots \| \text{Lang}(A_k)$.

We also use a characterization of synchronous product languages [12].

**Proposition 2.** A language over distributed alphabet $\Sigma$ is accepted by a product system with subset-acceptance if and only if it can be expressed as a finite union of direct product languages.

The following property of direct products from [17] clubs together the places of product system which correspond to places of a cluster in the net.

**Definition 12 (PS-matchings).** For global $a \in \Sigma$, $\text{matching}(a)$ is a subset of tuples $\Pi_{i \in \text{loc}(a)} P_i$ such that for all $i$ in $\text{loc}(a)$, projection of these tuples is the set of all pre-states of $a$-moves in $\rightarrow_i$, and if a state $p \in P_i$ appears in one tuple, it does not appear in another tuple. We say a product state $R$ is in $\text{matching}(a)$ if its projection $R \downarrow_{\text{loc}(a)}$ is in the matching.

A product system is said to have matching of labels if for all global $a \in \Sigma$, there is a suitable $\text{matching}(a)$. Such a system is denoted by PS-matchings.

We have PS-matchings with product-acceptance, if the set of final product states of it is a product of final states of component machines, or PS-matchings with subset-acceptance, if the set of final product states is a subset of product of final states of individual components.
A run of PS-matchings $A$ is said to be consistent with a matching of labels \[17\] if for all global actions $a$ and every prefix of the run $R^0 \xrightarrow{a} R^1 Q$, the pre-states $R^0 \downarrow \text{loc}(a)$ are in the matching.

Consistency of matchings is a behavioural property and to check if a PS-matchings $A$ has it and can be done in PSPACE \[17\][18].

The following property from \[17\] is used to capture free choice property.

**Definition 13 (conflict-equivalent matchings for PS-matchings).** In a product system, we say the local move $\langle p, a, q_1 \rangle \in \rightarrow_i$ is conflict-equivalent to the local move $\langle p', a, q'_{1} \rangle \in \rightarrow_j$, if for every other local move $\langle p, b, q_2 \rangle \in \rightarrow_i$, there is a local move $\langle p', b, q'_{2} \rangle \in \rightarrow_j$ and, conversely, for moves from $p'$ there are corresponding outgoing moves from $p$. For global action $a$, its matching $(a)$ is called conflict-equivalent matching, if whenever $p, p'$ are related by the matching$(a)$, their outgoing local $a$-moves are conflict-equivalent.

**Example 5 (Product system with matchings).** Consider product system of Figure 5 and relation `matching`$(a) = \{(r_1, s_1)\}$ relation. This matching is conflict-equivalent and the system is consistent with this matching relation.

We have a PS-matchings $A = (A_1, A_2)$ with product acceptance condition, if its set of final states is $G_1 \times G_2$. With the set of final states as $\{(r_1, s_1), (r_2, s_2)\} \subseteq G_1 \times G_2$, we have a PS-matchings $B = (A_1, A_2)$ having subset-acceptance.

**Lemma 1.** The language $L_4 = \{abd + adb + aeb + ace + aec + acd + ade\}^*(\varepsilon + a)$ from Example 2 is not accepted by any direct product.

**Proof.** Consider a word $w = ab$ not in $L$ and, words $u_1 = abd, u_2 = a$ which are in $L$. We have projections, $w_1^\downarrow_{\Sigma_1} = ab = u_1^\downarrow_{\Sigma_1}, w_2^\downarrow_{\Sigma_2} = a = u_2^\downarrow_{\Sigma_2}$. Therefore, by Proposition 1 word $w$ is in $L$, which is a contradiction. \hfill \Box

We know that the class of synchronous product languages is strictly larger than the class of direct product languages \[12\]. With the matching relations this
relationship is preserved. The \textit{PS-matchings} $B$ of Example 3 accepts language $L_4$ which by Lemma 1 is not accepted by any direct product. Hence, the class of languages accepted by \textit{PS-matchings} with subset-acceptance condition, is strictly larger, than the class of languages accepted by \textit{PS-matchings} with product-acceptance.

However, using Proposition 2 we have the following characterization of \textit{PS-matchings} with subset-acceptance.

\textbf{Corollary 1.} A language $L$ is accepted by a product system with subset-acceptance and, having conflict-equivalent and consistent matchings if and only if $L$ can be expressed as a finite union of languages accepted by product system with product-acceptance and, having conflict-equivalent and consistent matchings.

Lemma 2 presents a language not accepted by any synchronous product.

\textbf{Lemma 2.} The language $L_s = \{abd, adh, ace, aec\}^*(\varepsilon + a)$ of Example 2 is not a synchronous product language.

\textbf{Proof.} If $L$ is accepted by any synchronized product then, $L$ can be expressed as a finite union of direct product languages by Proposition 2. Let these direct product languages be $L_1, \ldots, L_K$. Let $0$ for word $abd$ and $1$ for word $ace$, which are in $L$. Let $U = \{00 \ldots 0, 10 \ldots 0, 01 \ldots 0, \ldots, 00 \ldots 1\}$ be the set of $k+1$ words of length $k$ each. By pigeon hole principle, there must be two words of $U$ which belong to same direct product language. Let $u$ and $v$ denote these two words, and $L_i$ be the component language to which $u, v$ belong to, where $i \in \{1, \ldots, k\}$.

Now we compare $u$ and $v$ to see how they are different from each other. Either they differ in one position or in two different positions.

1. If $u$ and $v$ differ in only one position, then $u = 0^k$ i.e. $1$ does not occur in it, and $v = 0^m10^{k-m}$ i.e. $1$ occurs at $m$-th position. Now we consider word $w = (abd)^{m-1}(abe)(abd)^{k-m}$. Clearly this word is not in $L$. We take projection of word $w$ $w \downarrow_{\Sigma_1} = (ab)^{m-1}(ab)(ab)^{k-m} = (ab)^k = u \downarrow_{\Sigma_1}$ and, $w \downarrow_{\Sigma_2} = (ad)^{m-1}(ac)(ad)^{k-m} = v \downarrow_{\Sigma_2}$. By Proposition 2 the word $w$ is in $L_i$. And since $L_i \subseteq L$, we have $w$ in $L$, which is a contradiction.

2. If $u$ and $v$ differ in two positions, then $u$ has a 1, and $v$ also has a 1, but at a different position. Assume that 1 of $u$ occurs at $m$-th position and 1 in $v$ occurs at $m'$-th position. Without loss of generality, we can assume that $m < m'$. Therefore $1 \leq m < m' \leq k$. We consider word $w = (abd)^{m-1}acd(abd)^{m'-m-1}(abe)(abd)^{k-m'}$, which is not in $L$. Now consider $w \downarrow_{\Sigma_1} = (ab)^{m-1}(ac)(ab)^{m'-m-1}(ab)(ab)^{k-m'} = (ab)^{m-1}(ac)(ab)^{k-m} = u \downarrow_{\Sigma_1}$, $w \downarrow_{\Sigma_2} = (ad)^{m-1}(ad)(ad)^{m'-m-1}(ae)(ad)^{k-m'} = (ab)^{m'-1}(ae)(ad)^{k-m'} = v \downarrow_{\Sigma_2}$. By Proposition 2 word $w \in L_i$ and, as $L_i \subseteq L$ we have $w \in L$, which is a contradiction.

So we have language $L_s$ which is not accepted by any \textit{PS-matchings} with subset-acceptance. This motivates the bigger class of automata over distributed alphabets, which we discuss next.
3.2 Product systems with globals

Let $A = \langle A_1, \ldots, A_k \rangle$ be a product system over distribution $\Sigma = (\Sigma_1, \ldots, \Sigma_k)$ and, let $\text{globals}(a)$ be a subset of its global moves $\Rightarrow_a$, and $a$-global denote an element of $\text{globals}(a)$.

**Definition 14.** A product system with globals (PS-globals) is a product system with relations $\text{globals}(a)$, for each global action $a$ in $\Sigma$.

With subset-acceptance condition these systems are called Asynchronous (or Zielonka) automaton [19,12]. Runs of a product system with globals, are defined in the same way as for the direct products, with an additional requirement of $\Pi_{j \in \text{loc}(a)}(\langle R[j], a, Q[j] \rangle \in \text{globals}(a))$, to be satisfied when $R \xrightarrow{a} Q$ is to be taken.

With abuse of notation sometimes we use $\text{pre}(a)$ to denote the set $\{ R \mid \exists Q, R \xrightarrow{a} Q \}$.

The following property from [15], of product systems with globals, relates to free choice property of nets.

**Definition 15 (same source property).** A product system with globals have same source property if, any two global moves share a pre-state then their sets of pre-states are same.

**Example 6 (Product system with globals).** Consider the product system of Figure 5. Let $\text{globals}(a) = \{(r_1 \xrightarrow{a} r_2, (s_1 \xrightarrow{a} s_2)), ((r_1 \xrightarrow{a} r_3, (s_1 \xrightarrow{a} s_3))\}$. This system has same source property.

With the given $\text{globals}(a)$ we have a PS-globals $C = (A_1, A_2)$ with product acceptance condition, if its set of final states is $G_1 \times G_2$. And, for the set of final states $\{(r_1, s_1), (r_2, s_2)\} \subseteq G_1 \times G_2$, we have a PS-globals $D = (A_1, A_2)$ with subset-acceptance.

The language $L_s = \{ abd, adb, ace, aec \}^*(\varepsilon + a)$ of Lemma 2 is accepted by product system with globals $D$ of Example 6 with same source property.

Product systems with globals and product-acceptance are not considered in [12]. This class of systems are strictly more expressive, as shown in Lemma 3. This lemma is new and was not present in [16].

**Lemma 3.** The language $L_p = \{ (abd + adb + ace + acc)^*(\varepsilon + a + ab + ad) \}$ from Example 7 is not accepted by any direct product.

**Proof.** Consider a word $w = abe$ not in $L_p$ and, words $u_1 = abd$, $u_2 = ace$ which are in $L_p$. We have projections, $w\downarrow_{\Sigma_1} = ab = u_1\downarrow_{\Sigma_1}$, $w\downarrow_{\Sigma_2} = ac = u_2\downarrow_{\Sigma_2}$. Therefore, by Proposition 11 word $w$ is in $L_p$, which is a contradiction. □

We give in Lemma 4 a characterization of class of languages accepted by product systems with globals and having subset-acceptance, in terms of PS-globals and product-acceptance.

**Lemma 4.** A language is accepted by a PS-globals with subset-acceptance if and only if it can be expressed as a finite union of languages accepted by PS-globals with product-acceptance.
Proof. ($\Rightarrow$): Let $A = \{A_1, \ldots, A_k\}$ be a PS-globals with subset-acceptance condition, and having $(p_1^0, \ldots, p_k^0)$ as its initial state and set of final states $G \subseteq \Pi_{i \in \text{Loc}} G_i$, where $A_i = (P_i, \rightarrow_i, G_i, p_i^0)$. Then for each final global state $g = (g_1, \ldots, g_k)$ of $G$, we build a PS-globals with product-acceptance condition $A^g = \langle A_1^g, \ldots, A_k^g \rangle$ by taking $A_i^g = \langle P_i, \rightarrow_i, g_i, p_i^0 \rangle$. The set of globals of $A^g$ is the set of globals of $A$. So if a word is accepted by $A$ by traversing a path from initial global state to some final state $g$, then we can traverse the same path in $A^g$ to its only one final global state $g$. And, the reverse direction also holds. Therefore $\text{Lang}(A) = \bigcup_{g \in G} \text{Lang}(A^g)$.

($\Leftarrow$): Let $L = L_1 \cup \ldots \cup L^m$ be a language defined over $\Sigma$ where each $L^j$ is the language accepted by PS-globals $A^j$ with product-acceptance condition, for all $j$ in $\{1, \ldots, m\}$. Let $A^j = \{A_1^j, \ldots, A_k^j\}$ where, its $i$-th sequential-component over $\Sigma_j$ is $A_i^j = (P_i^j, \rightarrow_i^j, G_i^j, p_i^0)$, for all $i$ in Loc.

Now we construct a product system with globals $B = \{B_1, \ldots, B_k\}$ over $\Sigma$ having subset-acceptance condition as follows: Each local component of $B$ over $\Sigma_i$ is given as $B_i = \langle Q_i, \rightarrow_i^B, G_i^B, B_i^0 \rangle$ with local states $Q_i = \bigcup_{j \in \{1, \ldots, m\}} P_i^j$ i.e. disjoint union of local states of $i$-th component of each PS-globals $A^j$. Its set of initial states is taken as union of initial states of $i$-th components of PS-globals $A^j$ i.e., $B_i^0 = \bigcup_{j \in \{1, \ldots, m\}} P_i^0$; and let its set of local moves be the union of local moves of $i$-th component of PS-globals $A^j$; and its set of final states as the union of final states of $i$-th component of PS-globals $A^j$. Now we define final states $G^B \subseteq \Pi_{i \in \text{Loc}} G_i^B$ of PS-globals $B$ as follows: $G^B = \bigcup_{j \in \{1, \ldots, m\}} (G_i^j \times \ldots \times G_i^j)$.

This ensures that any word accepted in $L$ is accepted by $B$, and in the reverse direction, if we have a word $w$ accepted by $B$, then we have an accepting run of some $A_j$ over $w$.

In the construction, transition structure of local components is preserved, and so are the global moves, hence, we have Corollary 2 which is used to get syntax for product systems with subset-acceptance condition.

**Corollary 2.** A language $L$ is accepted by a PS-globals with subset-acceptance and having same source property if and only if $L$ can be expressed as a finite union of languages accepted by PS-globals with product-acceptance and same source property.

In a product system with globals and having same source property, global moves for an action $a$ can be partitioned into different compartments: two global $a$-moves belong to same compartment if they have the same set of pre-states. For any $a$-global $g$ of a same source compartment $\Rightarrow^a SS$, we associate a target-configuration $\pi(g) = \Pi_{i \in \text{loc}(a)} \text{post}(g) \cap P_i$. Let $\text{post}(\Rightarrow^a SS) = \{\pi(g) \mid g \in \Rightarrow^a SS\}$. We define $\Rightarrow^a SS[i] = \text{post}(\Rightarrow^a SS) \cap P_i$ and $\text{post-decomp}(\Rightarrow^a SS) = \Pi_{i \in \text{loc}(a)} \Rightarrow^a SS[i]$. We may call $\Rightarrow^a SS[i]$ as post-projection and $\text{post-decomp}(\Rightarrow^a SS)$ as post-decomposition of a compartment.

The following property relates to distributed choice property of nets.
Definition 16. A product system with globals and having same source property, is said to have product moves property, if for all $a$ in $\Sigma$, and for all same source compartments $\Rightarrow^a_{SS}$ of $a$-globals, postdecomp$(\Rightarrow^a_{SS}) \subseteq \text{post}(\Rightarrow^a_{SS})$.

Product systems $C$ and $D$ of Example 6 do not have product moves property. Product system of Example 7 has product moves property.

Example 7 (Product system with globals and product moves property). Consider product system $A$ of Example 5 where the set of final states is $G_1 \times G_2$. With $\text{globals}(a) = \{ ((r_1 \xrightarrow{a} r_2), (s_1 \xrightarrow{a} s_2)), ((r_1 \xrightarrow{a} r_2), (s_1 \xrightarrow{a} s_3)), ((r_1 \xrightarrow{a} r_3), (s_1 \xrightarrow{a} s_2)) \}$ relation, we have $\text{PS-globals } A'$ which has product moves property. This also has same source property.

Now consider the product system $B$ of Example 5 with $\{ (r_1, s_1), (r_2, s_2) \} \subseteq G_1 \times G_2$ as its final states, and having the $\text{globals}(a)$ relation as above, we get a $\text{PS-globals } B'$ with subset-acceptance condition, having product moves and same source property.

A product system with globals is said to be live, if for any global move $g$ and any reachable product state $R$, there exists a product state $Q$ such that $g$ is enabled at $Q$.

3.3 Relating product systems with matchings and globals

First we show, in Theorem 1, how to construct a product system with consistent and conflict-equivalent matchings from a $\text{PS-globals}$ with same source property.

Theorem 1. Let $\Sigma$ be a distributed alphabet and $A$ be a product system with globals defined over it. Then we can construct a product system $B$ with matchings, linear in the size of product system $A$ with globals such that,

1. if $A$ has same source property then $B$ has conflict-equivalent matchings,
2. in addition, if $A$ is live then
   (a) $B$ is consistent with matchings, and
   (b) $\text{Lang}(A) = \text{Lang}(B)$.

Proof. Let $A = (A_1, \ldots, A_k)$ be a product system with globals. We construct product system $B = (B_1 = A_1, \ldots, B_k = A_k)$ with matchings, where for each label $a$ in $\Sigma$, $\text{matching}(a) = \{ \Pi_{e \in \text{loc}(a)} \text{pre}(g[i]) | g \in \text{globals}(a) \}$. The size of this $\text{matching}(a)$ relation is at most the size of $\text{globals}(a)$ relation for $e$.

1. For a label $a$ in $\Sigma$, without loss of generality, we assume that $\text{loc}(a) = \{1, 2\}$. Let $(p, q)$ is in $\text{matching}(a)$. Let $(p, a, p_1)$ be a local $a$-move and $(p, b, p_2)$ be a local $b$-move in $B_1$. Also let $(q, a, q_1)$ be a local $a$-move of $B_2$. To prove conflict-equivalence of matchings, we have to show existence of a local $b$-move from state $q$ of $B_2$. Since, $(p, q)$ is in $\text{matching}(a)$ then we have a global $a$-move $g = ((p, a, p_1), (q, a, q_1))$ of $A$. And, since $p$ has an outgoing $b$-move there must be some other global $b$-move $g'$ in $A$, with $g'[1] = (p, b, p_2)$. But, we have $\{p\} \subseteq \text{pre}(g') \cap \text{pre}(g)$. Therefore, by same source property of $A$, we get $\text{pre}(g') = \text{pre}(g)$. Hence, $q$ is in $\text{pre}(g')$ implying existence of a local $b$-move from state $q$, as required.
2. Now in addition, we assume that $A$ is live.
(a) Consider a run $R_0 \xrightarrow{\pi} R \xrightarrow{\pi} Q$ of product system $B$, where $R_0$ is a initial global state of $B$. Inductively, we assume that the run $R_0 \xrightarrow{\pi} R$ is consistent with the constructed matchings. Without loss of generality, we assume that $loc(a) = \{1, 2\}$. At global state $R$ some global $a$-move $g = \langle (p, a, p_1), (q, a, q_1), \ldots, (r_n, a, r_n') \rangle$ of $B$ is enabled to reach $Q$. Other component moves of $g$ from $A_3$ to $A_4$ do not take part in this step.

To show that this run is consistent with matchings, we have to prove, tuple $R_{\text{loc(a)}} = \langle p, q \rangle$ is in $\text{matching}(a)$. If global move of $g$ of $B$ is an $a$-global in system $A$, then clearly by construction $\langle p, q \rangle$ being in $\text{pre}(g)$ also appears in $\text{matching}(a)$. If global move $g$ of $B$ is not an $a$-global in system $A$, then we must be having $a$-globals $g$ and $g'$ of $A$, such that $g[1] = \langle p, a, p_1 \rangle$, and $g'[2] = \langle q, a, q_1 \rangle$. At global state $R$ of $A$, local state of $A_1$ is $p$ and local state of $A_2$ is $q$. Because of same source property of $A$, there is no global move on any other label having $p$ and $q$ in their preset, which is enabled i.e., in these two components control will not be able to move forward from $p$ and $q$, which contradicts the fact that $A$ is live.

(b) We show language equivalence of $A$ and $B$ by showing a stronger property that graphs of reachable states of $A$ and $B$ are isomorphic. States of $A$ are mapped to themselves in $B$ and vice versa. As base step, initial states of $A$ and $B$ are isomorphic.

To prove that $\text{Lang}(A) \subseteq \text{Lang}(B)$, we show that if at any reachable state $R$ of $A$, some $a$-global $g$ is taken to reach $Q$ then there exists a global move on label $a$ in $B$ which is enabled at $R$ and when taken we reach global state $Q$. Since $g$ is enabled at $R$ in system $A$, for all $i$ in $\text{loc(a)}$, we have $g[i] = \langle p_i, a, q_i \rangle$ where $p_i$ a locate state of $A_i$ is in $\text{pre}(g)$ and part of $R$ and similarly, $q_i$ a local state of $A_i$ is in $\text{post}(g)$ and part of $Q$. Therefore, we can take $a$-global $g$ of $A$ itself as the required global move of $B$ taken at $R$ to reach $Q$.

In the reverse direction, we assume that we have reached state $R$ in system $B$, after taking an $a$-labelled global move $h$ to reach state $Q$. Inductively, we assume that we are at state $R$ in $A$. Now we have to show that there exists an $a$-global $g$ in $A$, which is enabled at $R$ and $R \xrightarrow{\pi} Q$ in $A$. Let $\text{loc(a)} = \{1, \ldots, m\}$. We have $\text{pre}(h)$ appearing in $R$ is also in $\text{matching(a)}$ of $B$, due to consistency of matchings for $B$, proved above using liveness of $A$. Therefore, by construction of $B$, we must have some $a$-global $g$ in $A$ such that $\text{pre}(g) = \text{pre}(h)$. Let $\Rightarrow_a^{SS}$ be the set of global moves of $A$ which have same set of pre-states as $g$.

For all $i$ in $\text{loc(a)}$, $h[i] = \langle p_i, a, q_i \rangle$ in each component $B_i$ of $B$ and hence in each $A_i$ of $A$. Therefore, we must have global moves $g_1, g_2, \ldots, g_m$ in $\Rightarrow_a^{SS}$ such that $g[i] = h[i]$, for all $i$ in $\text{loc(a)}$. Hence $\Pi_{i \in \text{loc(a)}} q_i$ is in $\text{postdecomp}(\Rightarrow_a^{SS})$ in $A$. This tuple $\Pi_{i \in \text{loc(a)}} q_i$ is also in $\text{post}(\Rightarrow_a^{SS})$ because $A$ has product moves property. So there exists a global move $g'$ in $\Rightarrow_a^{SS}$ with $\pi(g') = \Pi_{i \in \text{loc(a)}} q_i$. Therefore, this global $g'$ in $A$ can be fired at $R$ to reach $Q$, as required, to complete induction.
Now, from a PS-matchings with consistent and conflict-equivalent matchings we construct a product system with globals having same source property.

**Theorem 2.** Let $\Sigma$ be a distributed alphabet and let $B$ be a product system with conflict equivalent and consistent matchings. Then for the language of $B$ we can construct a product system $A$ with globals over $\Sigma$ having same source and product moves property. The constructed product system $A$ with globals is exponential in the size of system $B$ having matching of labels.

**Proof.** Let $B = (B_1, B_2, \ldots, B_n)$ be the given product system with matchings. We construct a PS-globals $A = (A_1, A_2, \ldots, A_n)$ by taking component systems $A_j = B_j$. It remains to construct the global moves of $A$. We build the set of global moves $\text{globals}(a)$ for $A$ from a given matching relation $\text{matching}(a)$ of $B$ as follows. Let $\text{loc}(a) = \{1, \ldots, m\}$.

$$\text{globals}(a) = \{(p_1, \ldots, p_m), (q_1, \ldots, q_m) \mid (p_1, \ldots, p_m) \in \text{matching}(a)$$

where $(p_k \rightarrow_k^q q_k)$ for all $k \in \text{loc}(a)$.

For tuple $(p_1, \ldots, p_m)$ in $\text{matching}(a)$ of $B$, let us assume that $p_1$ has $k_1$ outgoing local $a$-moves in $B_1$, $p_2$ has $k_2$ outgoing local $a$-moves in $B_2$, and so on. Let $k$ be the minimum of $\{k_1, \ldots, k_m\}$. Then we have $k^m$ number of global $a$-moves, which is exponential in the number of locations.

**Proof of $A$ having same source property:**

Assume not, then we have two global moves $g$ and $g'$ such that intersection of their sets of pre-states is not empty and their sets of pre-states are not equal also. Let $g$ is an $a$-global move and $g'$ be a $b$-global move of $A$. For the sake of simplifying this discussion, we take $m = 2$, hence $\text{loc}(a) = \{1, 2\}$. We have $\{p, q\} \subseteq \text{pre}(g)$ and $\{p, q'\} \subseteq \text{pre}(g')$, and $q \neq q'$, where states $p$ is a local state of $A_1$ and $q, q'$ are local states of $A_2$. Therefore, $g = ((p, a, p_1), (q, a, q_1))$ and $g' = ((p, b, p_2), (q', b, q_2))$. It means that $(p, q') \in \text{matching}(b)$ and $(p, q) \in \text{matching}(a)$ in $B$. Since $B$ has conflict-equivalent matching, it implies existence of a local $b$-move with source state $q$ i.e., $(q, b, q_3)$ in $B_2$, for some state $q_3$ in $B_2$.

Now at any global state $R$ of $B$ where $g$ is enabled, tuple $(p, q)$ which is in $\text{matching}(a)$ relation, also appears in state $R$. Therefore, the global $b$-move $h = ((p, b, p_2), (q, b, q_3))$ is also enabled in $B$. But, $B$ has consistent matching of labels, so $(p, q) \in \text{matching}(b)$. This is a contradiction, as now we have two tuples $(p, q)$ and $(p, q')$ in $\text{matching}(b)$ relation in which state $p$ appears. Therefore, $q = q'$ to get a contradiction.

**Proof of $A$ having product moves property:**

Let $\Rightarrow^a_S$ be a same source compartment of global $a$-moves of the constructed product system $A$ and $\text{loc}(a) = \{1, \ldots, m\}$. Now we have to prove $\text{postdecomp}(\Rightarrow^a_S) \subseteq \text{post}(\Rightarrow^a_S)$. Consider a tuple of states $(p_1, p_2, \ldots, p_m)$ in $\text{postdecomp}(\Rightarrow^a_S)$. Then there exist global moves $g_1, g_2, \ldots, g_m$ in $\Rightarrow^a_S$ such that $\text{post}(g_i[j]) = p_i$, for all $i \in \text{loc}(a)$, implying that we have local $a$-moves $g_j[j] = (q_j, a, p_j)$ in $A_j$, for all $j \in \text{loc}(a)$. Since $(q_1, \ldots, q_m) = \Pi_{j \in \text{loc}(a)}(\text{pre}(g_j[j]))$, we have
tuple \((q_1, \ldots, q_n)\) in \(\text{matching}(a)\) of product system \(B\). Therefore, we have \(((I_{i \in \text{loc}(a)} q_i), (I_{j \in \text{loc}(a)} p_j))\) as a global \(a\)-move in the constructed system \(A\). Hence, tuple \(I_{j \in \text{loc}(a)} p_j\) being a post-configuration of this global \(a\)-move is also in \(\text{post}(()) \Rightarrow^a S\), as required.

(Proof of \(\text{Lang}(B) = \text{Lang}(A)\):

We prove this by showing isomorphism of state reachability graphs of \(A\) and \(B\). For states it is identity mapping. Proving \(\text{Lang}(A) \subseteq \text{Lang}(B)\) is straightforward, as global moves of \(A\) are also global moves of \(B\).

In the reverse direction, we assume that we have some global move \(h\) of \(B\) taken at reachable state \(R\), to reach \(Q\). Inductively, we have reached global state \(R\) in \(A\). We have \(\text{pre}(h)\) in \(R\) and \(\text{post}(h)\) in target state \(Q\). As \(B\) is consistent with matching of labels, we have \(\text{pre}(h)\) in \(\text{matching}(a)\). Since \(A\) has product moves property, proved above using consistency of matching, for each tuple of \(\text{matching}(a)\), we have an \(a\)-global, for this fixed pre-states and each possible post-configuration. So we have an \(a\)-global in \(A\), consisting of \(\text{post}(h)\) as its set of post-states and \(\text{pre}(h)\) as its set of pre-states. We can take this at \(R\) to reach \(Q\) in \(A\) as required.

\[\square\]

4 Nets and Product systems

We first present a generic construction of a 1-bounded S-decomposable labelled net systems, from product systems with globals.

**Definition 17 (PS-globals to nets).** Given a PS-globals \(A = (A_1, \ldots, A_k)\) over distribution \(\Sigma\), a net system \((N = (S, T, F, \lambda), M_0, \mathcal{G})\) is constructed as follows: The set of places is \(S = \bigcup_i P_i\), the set of transitions is \(T = \bigcup_{a \in \Sigma} \text{globals}(a)\). Define \(T_i = \{\lambda^{-1}(a) \mid a \in \Sigma\}\). The labelling function \(\lambda\) labels by action \(a\) the transitions in \(\text{globals}(a)\). The flow relation is \(F = \{(p, g), (g, q) \mid g \in T_a, g[i] = (p, a, q), \text{ for all } i \in \text{loc}(a)\}\), define \(F_i\) as its restriction to the transitions \(T_i\) for \(i \in \text{loc}(a)\). See that \(F\) is union of all \(F_i\)s. Let \(M_0 = \{p_0^1, \ldots, p_0^k\}\), be the initial product state and \(\mathcal{G} = G\) as the set of final global states.

We get one to one correspondence between reachable states of product system and reachable markings of nets because the set of transitions of resultant net is same as the set of global moves in the product system, and construction preserves pre as well as post places.

**Lemma 5.** The constructed net system \(N\) from a PS-globals \(A\), as in Definition 17, is S-decomposable and \(\text{Lang}(N, M_0, \mathcal{G}) = \text{Lang}(A)\). The size of constructed net is linear in the size of product system.

Applying the generic construction above to product systems with same source property, we get a free choice net, because any two global moves having same set of pre-places are put into one cluster.

**Theorem 3.** Let \((N, M_0, \mathcal{G})\) be the net system constructed from PS-globals \(A\) as in Definition 17.
– If $A$ has same source property then $N$ is a free choice net.
– In addition if $A$ has product moves property, then $N$ has distributed choice.

In the construction, if the product system has subset-acceptance then we get a net with a set of final markings, which may not have product condition. Since $A$ has subset-acceptance, we generalize the results obtained in [15].

For the product system $D$ of Example 6 accepting language $L_s$ we can construct the net system of Example 2.

In the case that the product system $A$ has matchings, transitions of net constructed are reachable global moves of system $A$ [17,18].

Now we describe a linear-size construction of a product system from a net which is $S$-decomposable.

**Definition 18 (nets to PS-globals).** Given a 1-bounded labelled and an $S$-decomposable net system $(N, M_0, G)$, with $N = (S, T, F, \lambda)$ the underlying net and $N_i = (S_i, T_i, F_i)$ the components in the $S$-cover, for $i$ in $\{1, 2, \ldots, k\}$, we define a product system $A = \langle A_1, \ldots, A_k \rangle$, as follows. Take $P_i = S_i$, and $p_0^i$ the unique state in $M_0 \cap P_i$. Define local moves $\rightarrow_i = \{ \langle p, \lambda(t), p' \rangle \mid t \in T_i \text{ and } (p, t), (t, p') \in F_i \}$, for $p, p' \in P_i$. So we get sequential systems $A_i = \langle P_i, \rightarrow_i, p_0^i \rangle$, and the product system $A = \langle A_1, A_2, \ldots, A_k \rangle$ over alphabet $\Sigma$. Global moves are $a$-moves $\Pi_{i \in \text{loc}(a)} [i] \mid t \in T_a$. And, the set of final states is $G = G$.

**Lemma 6.** From net system $N$ with a final set of markings, the construction of the PS-globals $A$ in Definition 18 above preserves language. The product system $A$ is linear in the size of net, and product system has subset-acceptance.

For each $a$-labelled transition of the net we get one global $a$-move in the product system having same set of pre-places and post-places. And, for each global $a$-move in product system we have an $a$-labelled transition in the net having same pre and post-places. We get one to one correspondence between reachable states of product system and reachable markings of the net we started with. Therefore, if we begin with a free choice net, we get same source property in the obtained product system. And, for each transition in the net we have a global transition hence, $A$ has product moves property if the net has distributed choice.

**Theorem 4.** Let $(N, M_0, G)$ be a 1-bounded, and an $S$-decomposable labelled net with a set of final markings $G$. Then

– if $N$ has free choice property, then constructed product system $A$ with globals, has same source property,
– in addition, if the net has distributed choice, then $A$ has product moves.

In construction of Definition 18 we start with a net having distributed choice and a final set of markings then we get product system with matching with subset-acceptance condition. Note that in this case we do not have to construct globals [17,18].

For the net system of Example 2 and accepting language $L_s$ we can construct the product system $D$ of Example 6.

Therefore, we generalize the results from [17,18].
Theorem 5. For a 1-bounded, S-decomposable labelled net having distributed choice and given with a set of final markings. Then one can construct a product system with conflict-equivalent and consistent matchings and having subset-acceptance.

Given below is the converse result.

Theorem 6. For a product system with conflict-equivalent, consistent matchings, and subset-acceptance, we get language equivalent free choice net with distributed choice and having a set of final markings.

5 Expressions

First we define regular expressions and its derivatives.

5.1 Regular expressions and their properties

A regular expression over alphabet $\Sigma_i$ such that constants 0 and 1 are not in $\Sigma_i$ is given by:

$$s ::= 0 \mid 1 \mid a \in \Sigma_i \mid s_1 \cdot s_2 \mid s_1 + s_2 \mid s_1^*$$

The language of constant 0 is $\emptyset$ and that of 1 is $\{\varepsilon\}$. For a symbol $a \in \Sigma_i$, its language is $\text{Lang}(a) = \{a\}$. For regular expressions $s_1 + s_2, s_1 \cdot s_2$ and $s_1^*$, its languages are defined inductively as union, concatenation and Kleene star of the component languages respectively.

As a measure of the size of an expression we will use $wd(s)$ for its alphabetic width—the total number of occurrences of letters of $\Sigma$ in $s$.

For each regular expression $s$ over $\Sigma_i$, let $\text{Lang}(s)$ be its language and its initial actions form the set $\text{Init}(s) = \{a \mid \exists v \in \Sigma_i^* \text{ and } av \in \text{Lang}(s)\}$ which can be defined syntactically. We can syntactically check whether the empty word $\varepsilon \in \text{Lang}(s)$.

We use derivatives of regular expressions which are known since the time of Brzozowski [2], Mirkin [11] and Antimirov [1].

Definition 19 (Antimirov derivatives [1]). Given regular expression $s$ and symbol $a$, the set of partial derivatives of $s$ with respect to $a$, written $\text{Der}_a(s)$ are defined as follows.

$$\text{Der}_a(0) = \emptyset$$
$$\text{Der}_a(1) = \emptyset$$
$$\text{Der}_a(b) = \{\varepsilon\} \text{ if } b = a, \emptyset \text{ otherwise}$$

$$\text{Der}_a(s_1 + s_2) = \text{Der}_a(s_1) \cup \text{Der}_a(s_2)$$

$$\text{Der}_a(s_1^*) = \text{Der}_a(s_1) \cdot s_1^*$$

$$\text{Der}_a(s_1 \cdot s_2) = \begin{cases} 
\text{Der}_a(s_1) \cdot s_2 \cup \text{Der}_a(s_2), & \text{if } \varepsilon \in \text{Lang}(s_1) \\
\text{Der}_a(s_1) \cdot s_2, & \text{otherwise}
\end{cases}$$
Inductively $\text{Der}_w(s) = \text{Der}_w(\text{Der}_w(s))$.

The set of all partial derivatives $\text{Der}(s) = \bigcup_{w \in \Sigma^*} \text{Der}_w(s)$, where $\text{Der}_e(s) = \{s\}$.

We have derivatives $\text{Der}_a(ab + ac) = \{b, c\}$ and $\text{Der}_a(a(b + c)) = \{b + c\}$.

A derivative $d$ of $s$ with action $a \in \text{Init}(d)$ is called an $a$-site of $s$. An expression is said to have equal choice if for all $a$, its $a$-sites have the same set of initial actions. For a set $D$ of derivatives, we collect all initial actions to form $\text{Init}(D)$. Two sets of derivatives have equal choice if their $\text{Init}$ sets are same.

As in [17], we put together derivatives which may correspond to the same state in a finite automaton.

**Definition 20 ([17]).** Let $s$ be a regular expression and $L = \text{Lang}(s)$. For a set $D$ of $a$-sites of regular expression $s$ and an action $a$, we define the relativized language $L^D_a = \{xay \mid xay \in L, \exists d \in \text{Der}_x(s) \cap D, \exists d' \in \text{Der}_y(d) \text{ with } \varepsilon \in \text{Lang}(d')\}$, and the prefixes $\text{Pref}^{D}_a(L) = \{x \mid xay \in L^D_a\}$, and the suffixes $\text{Suf}^{D}_a(L) = \{y \mid xay \in L^D_a\}$. We say that the derivatives in set $D$ $a$-bifurcate $L$ if $L^D_a = \text{Pref}^{D}_a(L) \cup \text{Suf}^{D}_a(L)$.

We use partitions of the $a$-sites of $s$ into blocks such that each block (that is, element of the partition) $a$-bifurcates $L$ [17].

**Definition 21 ([17]).** Let $X_1$ be a partition of $a$-sites of $s_1$ and $X_2$ be a partition of $a$-sites of $s_2$, where regular expression $s = s_1 \cdot s_2$ or $s = s_1 + s_2$. For partitions $X_1, X_2$ with blocks $D_1, D_2$ containing elements $d_1, d_2$ respectively, we use the notation $(X_1 \cup X_2)[d/d_1, d_2]$ for the modified partition $((X_1 \setminus \{D_1\}) \cup (X_2 \setminus \{D_2\})) \cup ((D_1 \cup D_2 \cup \{d\}) \setminus \{d_1, d_2\})$. And, for partition $X$ with block $D_1$ in it, having $d_1$ in it, $X[d/d_1]$ is the modified partition $X \setminus \{D_1\} \cup ((D_1 \setminus \{d_1\}) \cup \{d\})$.

\[
\begin{align*}
\text{Part}_a(b) &= \emptyset \text{ if } a \neq b \\
\text{Part}_a(a) &= \{\{a\}\} \\
\text{Part}_a(s_1^*a) &= (\text{Part}_a(s_1) \cdot s_1^*[s_1^*/s_1 \cdot s_1^*]) \\
\text{Part}_a(s_1 + s_2) &= Z_1 \cup Z_2 \cup \{s_1 + s_2\} \text{ if } a \in \text{Init}(s_1 + s_2) \\
\text{Part}_a(s_1 \cdot s_2) &= \begin{cases} \\
\text{Part}_a(s_1) \cdot s_2 \cup \text{Part}_a(s_2)[s_1 \cdot s_2/s_2] & \text{if } a \in \text{Init}(s_1 + s_2) \\
\text{Part}_a(s_1) \cdot s_2 \cup \text{Part}_a(s_2) & \text{otherwise} \\
\end{cases} \\
& \text{ where,} \\
Z_1 &= \text{Part}_a(s_1) \setminus \{s_1\} \text{ if } s_1 \notin \text{Der}_a(s_1 + s_2), \text{ Part}_a(s_1) \text{ otherwise} \\
Z_2 &= \text{Part}_a(s_2) \setminus \{s_2\} \text{ if } s_2 \notin \text{Der}_a(s_1 + s_2), \text{ Part}_a(s_2) \text{ otherwise}.
\end{align*}
\]

For an action $a$, let $\text{Part}_a(s)$ denote such a partition. In addition to thinking of blocks of the partition as places of an automaton, we can think of pairs of blocks and their effects as local moves.

**Definition 22 ([15]).** Given an action $a$, and a set of $a$-sites $B$ of regular expression $s$, and a specified set of $a$-effects $E \subseteq \text{Der}_a(B)$, we define the relativized languages
\[ L_a^{(B, E)} = \{ xay \in L \mid \exists d \in \text{Der}_x(s) \cap B, \exists d' \in \text{Der}_a(d) \cap E, \text{ and } \exists d'' \in \text{Der}_y(d') \text{ with } \varepsilon \in \text{Lang}(d'') \}. \]

We define the prefixes \( \text{Pref}_a^{(B, E)}(L) = \{ x \mid xay \in L_a^{(B, E)} \} \) and the suffixes \( \text{Suf}_a^{(B, E)}(L) = \{ y \mid xay \in L_a^{(B, E)} \} \). We say that a tuple \((B, E)\) \(a\)-funnels \(L\) if \(L_a^{(B, E)} = \text{Pref}_a^{B}(L) \cdot a \cdot \text{Suf}_a^{(B, E)}(L)\). In such a pair \((B, E)\), if \(B\) is a block in the \(\text{Part}_a(s)\) and \(E\) is a nonempty subset of \(\text{a-effects of } B\), then it is called as an \(a\)-duct.

For an \(a\)-duct \((B, E)\), we define its set of initial actions \(\text{Init}(B, E) = \text{Init}(B)\), call \(B\) as its pre-block and call \(E\) as its post-effect. For all \(i\) in \(\text{loc}(a)\) let \(a\)-ducts \((s_i)\) denote the set of all \(a\)-ducts of regular expression \(s_i\). For any two \(a\)-ducts \((B, E)\) and \((B', E')\) in \(a\)-ducts \((s_i)\), define \((B, E) = (B', E')\) if \(B = B'\) and \(E = E'\). Given an \(a\)-duct \(d = (B, E)\) its post-effect \(E\) is sometimes denoted by \(d^*\) and its pre-block \(B\) can be denoted as \(\cdot d\). For a collection of ducts \(z\), the set of all their post-effects (resp. pre-blocks) is denoted as \(z^*\) (resp. \(z^\cdot\)). In a similar way, we define the set of post-effects of an \(a\)-cable \(D\), as \(D^* = \{ D[i]^* \mid i \in \text{loc}(a) \}\) and its set of pre-blocks as \(\cdot D = \{ \cdot D[i] \mid i \in \text{loc}(a) \}\).

### 5.2 Connected expressions over a distributed alphabets

The syntax of connected expressions defined over a distribution \((\Sigma_1, \Sigma_2, \ldots, \Sigma_k)\) of alphabet \(\Sigma\) is given below.

\[ e ::\ = \emptyset | \text{fsync}(s_1, s_2, \ldots, s_k), \text{ where } s_i \text{ is a regular expression over } \Sigma_i \]

When \(e = \text{fsync}(s_1, s_2, \ldots, s_k)\) and \(I \subseteq \text{Loc}\), let the projection \(e[I] = \Pi_{i \in I} s_i\).

A connected expression \(e = \text{fsync}(s_1, s_2, \ldots, s_k)\) over \(\Sigma\), is said to have equal choice if, for all global actions \(a\) in \(\Sigma\) and for any \(i, j\) in \(\text{loc}(a)\), any \(a\)-site of \(s_i\) have same \(\text{Init}\) set as of any \(a\)-site of \(s_j\).

For a connected expression defined over distributed alphabet its derivatives and semantics were given in [17], and are given as follows. For the connected expression \(0\), we have \(\text{Lang}(0) = \emptyset\). For the connected expression \(e = \text{fsync}(s_1, \ldots, s_k)\), its language is \(\text{Lang}(e) = \text{Lang}(s_1) || \text{Lang}(s_2) || \ldots || \text{Lang}(s_k)\).

The definitions of derivatives extended to connected expressions [17] is as follows. The expression \(0\) has no derivatives on any action. Given an expression \(e = \text{fsync}(s_1, s_2, \ldots, s_k)\), its derivatives are defined by induction using the derivatives of the \(s_i\) on action \(a\):

\[ \text{Der}_a(e) = \{ \text{fsync}(r_1, \ldots, r_k) \mid \forall i \in \text{loc}(a), r_i \in \text{Der}_a(s_i); \text{ otherwise } r_j = s_j \}. \]

### 5.3 Connected expressions with pairings

We recall some properties of connected expressions over a distribution, which were, useful in construction of free choice nets. This property relates to matchings of direct products [17].
Definition 23 ([17]). Let $e = \text{fsync}(s_1, s_2, \ldots, s_k)$ be a connected expression over $\Sigma$. For a global action $a$, $\text{pairing}(a)$ is a subset of tuples $\Pi_{i \in \text{loc}(a)} \text{Part}_a(s_i)$ such that the projection of these tuples includes all the blocks of $\text{Part}_a(s_i)$, and if a block of $\text{Part}_a(s_j), j \in \text{loc}(a)$ appears in one tuple of the pairing, it does not appear in another tuple. (For convenience we also write $\text{pairing}(a)$ as a subset of $\Pi_{i \in \text{loc}(a)} \text{Der}(s_i)$ which respects the partition.) We call $\text{pairing}(a)$ equal choice if for every tuple in the pairing, the blocks of derivatives in the tuple have equal choice.

Derivatives for connected expressions with pairing are defined as follows. A derivative $\text{fsync}(r_1, \ldots, r_k)$ is in $\text{pairing}(a)$ if there is a tuple $D \in \text{pairing}(a)$ such that $r_i \in D[i]$ for all $i \in \text{loc}(a)$. For convenience we may write a derivative as an element of $\text{pairing}(a)$. Expression $e$ is said to have (equal choice) pairing of actions if for all global actions $a$, there exists an (equal choice) pairing of $a$. Expression $e$ is said to be consistent with a pairing of actions if every reachable $a$-site $d \in \text{Der}(e)$ is in $\text{pairing}(a)$. Expression $e$ is said to have equal choice property if it has equal choice pairing of actions for all global actions $a$ in $\Sigma$.

Given a connected expression $e$ with pairings, checking if it is consistent with pairing of actions can be done in PSPACE [17,18].

5.4 Connected expression with cables (CE-cables)

We give some properties of connected expressions over a distribution, which extend the notion of pairing, and have been related to product systems with globals [15]. The notion of cables corresponds to notion of globals of product systems, and hence it corresponds to transitions of a net.

Definition 24 ([15]). Let $e = \text{fsync}(s_1, s_2, \ldots, s_k)$ be a connected expression over $\Sigma$. For each action $a$ in $\Sigma$, we define $a$-cables($e$) = $\Pi_{i \in \text{loc}(a)} a$-ducts($s_i$). For an action $a$, an $a$-cable is an element of the set $a$-cables($e$). We say that a block $B$ of $\text{Part}_a(s_i)$ appears in an $a$-cable $D$ if there exists $j$ in $\text{loc}(a)$ and there exists $Y \subseteq \text{Der}_a(B)$ such that $D[j] = (B, Y)$, i.e. if $B$ is a pre-block of a component $a$-duct of $D$. For any $a$-cable $D$, its set of pre-blocks $\cdot D = \cup_{i \in \text{loc}(a)} \{B_i \mid B_i \text{ appears in } D\}$, i.e. the set of pre-blocks of all the of its component $a$-ducts.

For expression $e$, let $\text{cables}(a) \subseteq a$-cables($e$), such that for all $i$ in $\text{loc}(a)$

1. Each block $B$ in $\text{Part}_a(s_i)$, appears in at least one $a$-cable of it.
2. for all $(B, E)$ and $(B', E')$ in $a$-ducts($s_i$) with $(B, E) \neq (B', E')$, if $B = B'$ $\implies$ $E \cap E' = \emptyset$, i.e. if any two distinct $a$-ducts of $s_i$ appearing in it have same pre-block then, they must have disjoint post-effects.

Connected expressions with cables were defined in [15], as follows.

A connected expression with cables (CE-cables) is a connected expression with relations cables($a$) of it, for each global action $a$ in $\Sigma$. 

Derivatives of a connected expression with cables are defined as follows. The CE-cables 0 has no derivatives on any action. For expression \( e = \mathsf{fsync}(s_1, s_2, \ldots, s_k) \), we define its derivatives on action \( a \), by induction, using \( a \)-ducts and the derivatives of \( s_j \) as:

\[
\text{Der}_a(e) = \{ \mathsf{fsync}(r_1, r_2, \ldots, r_k) \mid r_j \in \text{Der}_a(s_j) \text{ if there exists an } a\text{-cable } D \in \text{cables}(a) \text{ such that, for all } j \text{ in } \text{loc}(a), s_j \text{ is in pre-block } B_j \text{ and } r_j \text{ is in } X_j \text{ of } a\text{-duct } D[j] = (B_j, X_j) \text{ of } s_j \}, \text{ otherwise } r_j = s_j \}.
\]

We use the word \textit{derivative} for expressions such as \( d = \mathsf{fsync}(r_1, \ldots, r_k) \) given above. The \textit{reachable} derivatives are \( \text{Der}(e) = \{ d \mid d \in \text{Der}_x(e), x \in \Sigma^* \} \). A CE-cables is said to have \textit{equal source property} if for any pair of two cables sharing a common pre-block have same set of pre-blocks. This property corresponds to same source property of product systems and relates to transitions belonging to same cluster of nets.

Language of \( e \) is the set of words over \( \Sigma \) defined using derivatives as below.

\[
\text{Lang}(e) = \{ w \in \Sigma^* \mid \exists e' \in \text{Der}_w(e) \text{ such that } e \in \text{Lang}(r_i), \text{ where } e'[i] = r_i \}.
\]

So we can have next derivative on action \( a \), if it is allowed by the cables(\( a \)) relation. The number of derivatives may be exponential in \( k \). Let \( \Sigma = (\Sigma_1 = \{a, b, c\}, \Sigma_2 = \{a, d, e\}) \) be a distributed alphabet.

Example 8 (CE-pairings and CE-cables). Let \( e = \mathsf{fsync}((ab + ac)^*, (ad + ae)^*) \) be a connected expression defined over \( \Sigma \). Here, \( r_1 = (ab + ac)^* \) and \( s_1 = (ad + ae)^* \). The set of derivatives of \( r_1 \) is \( \text{Der}(r_1) = \{ r_1, r_2 = br_1, r_3 = cr_1 \} \) and for \( s_1 \) it is \( \text{Der}(s_1) = \{ s_1, s_2 = ds_1, s_3 = es_1 \} \). We have a-sites(\( r_1 \)) = \( r_1 \) and \( \text{Part}_a(r_1) = \{ D = r_1 \} \). Similarly, a-sites(\( s_1 \)) = \( r_1 \) and \( \text{Part}_a(s_1) = \{ D = s_1 \} \).

The only possible pairing relation is \textit{pairing}(\( a \)) = \{ (D, D') \}. We have \( \text{Der}_a(e) = \{ \mathsf{fsync}(r_1, s_j) \mid i, j \in \{2, 3\} \} \). Expression \( e \) satisfies equal choice property.

Now we associate a cable relation with \( e \). The set of \( a \)-effects of \( D \) is \( \text{Der}_a(D) = \{ r_2, r_3 \} \). The set of \( a \)-ducts of \( r_1 \) is \( \{ (D, r_2), (D, r_3), (D, \{r_2, r_3\}) \} \). The set of \( a \)-effects of \( D' \) is \( \text{Der}_a(D') = \{ s_2, s_3 \} \) and the set of \( a \)-ducts of component expression \( s_1 \) is \( \{ (D', s_2), (D', s_3), (D', \{s_2, s_3\}) \} \). A possible cables(\( a \)) relations for expression \( e \) is \( \{ (D, r_2), (D', s_2), (D, r_3), (D', s_3) \} \). See that each block in the \( \text{Part}_a(r_1) \) and \( \text{Part}_a(s_1) \) appears at least once in the cables(\( a \)) relation. And two \( a \)-ducts of \( r_1 \) appearing in this relation, have same pre-block \( D \), so their set of post-effects \( r_2 \) and \( r_3 \) are disjoint. This condition also holds for \( a \)-ducts of \( s_1 \).

For both \( a \)-cables set of pre-blocks is identical, therefore cables(\( a \)) satisfies equal source property. We have \( \text{Der}_a(e) = \{ \mathsf{fsync}(r_2, s_2), \mathsf{fsync}(r_3, s_3) \} \), but expression \( \mathsf{fsync}(r_2, s_3) \) is not in \( \text{Der}_a(e) \), because only post-effect of \( D \) containing \( r_2 \) is the set \( \{ r_2 \} \) and similarly, only post-effect of \( D' \) containing \( s_3 \) is the set \( \{ s_3 \} \) and there does not exist an \( a \)-cable with \( (D, \{r_2\}) \) and \( (D', \{s_3\}) \) as its components. We have \( \text{Der}(e) = \{ e, (r_2, s_2), (r_3, s_3), (r_1, s_3), (r_1, s_3), (r_2, s_1), (r_3, s_1) \} \).

Another such example of connected expression with pairings (resp. cables) is given below.
and product-derivatives property.

Example 9 (CE-pairings). Let \( e = \text{fsync}((ab+ae)^*a, (ad+ae)^*a) \) be a connected expression defined over \( \Sigma \). Let \( p_1 = (ab + ae)^*a \) with language \( L_1 \) and \( q_1 = (ad + ae)^*a \) with language \( L_2 \). The set of derivatives are \( \text{Der}(p_1) = \{ p_1, p_2 = bp_1, p_3 = cp_1, p_4 = \varepsilon \} \) and \( \text{Der}(q_1) = \{ q_1, q_2 = dq_1, q_3 = eq_1, q_4 = \varepsilon \} \). The partitions of \( a \)-sites are \( \text{Part}_a(p_1) = \{ B = \{ p_1 \} \} \) and \( \text{Part}_a(q_1) = \{ B' = \{ q_1 \} \} \). A pairing relation is \( \text{pairing}(a) = \{ (B, B') \} \) and with respect to that \( \text{Der}_a(e) = \{ \text{fsync}(p_i, q_j) \mid i, j \in \{ 2, 3, 4 \} \} \). Expression \( e \) has equal choice property.

Now we associate a cabling relation with \( e \). The set of \( a \)-effects of \( B \) is \( \text{Der}_a(B) = \{ p_2, p_3, p_4 \} \) and \( \text{Der}_a(B') = \{ q_2, q_3, q_4 \} \). The set of \( a \)-ducts for \( p_1 \) is \( \{ (B, \{ p_2 \}), (B, \{ p_3 \}), (B, \{ p_2, p_4 \}), (B, \{ p_3, p_4 \}) \} \) and for \( q_1 \) is \( \{ (B', q_2), (B', q_3), (B', \{ q_2, q_4 \}), (B', \{ q_3, q_4 \}) \} \). A cables\( (a) \) relation is \( \{ ((B, p_2), (B', q_2)), ((B, p_3), (B', q_3)), ((B, p_2), (B', q_2)) \} \). Expression \( e \) has equal source property and its set of derivatives with respect to letter \( a \) is \( \text{Der}_a(e) = \{ \text{fsync}(p_2, q_2), \text{fsync}(p_3, q_3), \text{fsync}(p_4, q_4) \} \).

Now we give two new properties of connected expressions. In a connected expression with cables and having equal source property, cables for a global action \( a \) can be partitioned into different compartments: two \( a \)-cables belong to the same compartment if they have equal source. To any such \( a \)-cable \( D \) belonging to an equal source compartment \( ES_a \) of \( a \)-cables, we can associate a set of its post-blocks listed in some order as \( \pi(D) = \prod_{s \in \text{loc}(a)} (D^* \cap \chi_s) \), where \( \chi_s = \{ \text{Part}_a(s_a) \mid a \in \Sigma \} \). Let \( \text{post}(ES_a) = \{ \pi(D) \mid D \in ES_a \} \). Then we define \( ES_a[i] = \text{post}(ES_a[i]) \cap \chi_s \) and \( \text{postdecomp}(ES_a) = \prod_{s \in \text{loc}(a)} ES_a[i] \). We call \( ES_a[i] \) as post-projection and \( \text{postdecomp}(ES_a) \) as post-decomposition of the compartment \( ES_a \).

The following property of connected expressions will later be related to product-moves property of direct products and hence to distributed-choice of nets.

Definition 25 (product-derivatives property). A connected expression with cables and having equal source property, is said to have product-derivatives property, if for all \( a \) in \( \Sigma \), and for all equal source compartments \( ES_a \) of \( a \)-cables \( \text{postdecomp}(ES_a) \subseteq \text{post}(ES_a) \).

Example 10. The connected expression \( e = \text{fsync}((ab + ae)^*a, (ad + ae)^*a) \) of Example 8 does not have product-derivative property with the given cabling relation \( \{ ((D, r_2), (D', s_2)), ((D, r_3), (D', s_3)) \} \). If we associate the cabling relation \( \{ ((D, r_2), (D', s_2)), ((D, r_3), (D', s_3)), ((D, r_2), (D', s_3)), ((D, r_3), (D', s_2)) \} \) with \( e \) then it has product-derivative property.

Definition 26. A connected expression \( e \) is action-live if for all actions \( a \) in \( \Sigma \), from any reachable derivative of \( e \), we can reach an \( a \)-derivative of \( e \).

5.5 Relating connected-expressions with pairings and with cables

First, we show how connected expressions with equal choice and consistent pairings can be seen as connected expression with cables and having equal source and product-derivatives property.
Theorem 7. Let $\Sigma$ be a distributed alphabet and $e$ be a connected expression having equal choice and consistent pairing of actions, defined over $\Sigma$. Then for the language of $e$, we can construct a connected expression $e'$ with cables having equal source and product-derivatives property. The constructed expression $e'$ with cables is exponential in the size of expression $e$ with pairings.

Proof. Let $e = (s_1, \ldots, s_k)$ be a connected expression with pairings. We take $e' = (s'_1 = s_1, \ldots, s'_k = s_k)$ as our connected expression with cables. We construct $\text{cables}(a)$ relation for each action $a$ in $\text{loc}(a) = \{1, \ldots, m\}$. In each $s_i$ and for all $a$ in $\Sigma$, for each block $B$ in $\text{Part}_a(s_i)$ we consider the set $X$-post-effects($B$) as the set of post-effects of $B$ which are mutually disjoint. We build the set of $a$-cables $\text{cables}(a)$ for $e'$ from the given pairing relation $\text{pairing}(a)$ of $e$ as follows.

$$\text{cables}(a) = \{((B_1, \ldots, B_m), (E_1, \ldots, E_m)) | (B_1, \ldots, B_m) \in \text{pairing}(a)\}.$$ 

See that each block $B$ appears in at least one $a$-cable as it appears in at least one tuple of $\text{pairing}(a)$, and when $B$ appears more than once i.e., it appears in two distinct $a$-ducts then the post-effects are disjoint. Therefore, the relation constructed above satisfies definition of cables relation. If we have $k_i$ mutually exclusive post-effects for $B_i$, for each $i$ in $\text{loc}(a)$, then we have $k_1 \times k_2 \times k_m$ cables in $\text{cables}(a)$ having $B_1, \ldots, B_m$ as their set of pre-blocks, which is exponential in the number of locations.

(Proof of $e'$ having equal source property): Assume not i.e., we have an $a$-cable $c$ and a $b$-cable $c'$ having at least one common pre-block but their sets of pre-blocks are not equal. Without loss of generality we assume that $\text{loc}(a) = \{1, 2\}$. Let $c = ((B_1, E_1), (B_2, E_2))$ and $c' = ((B_1, E'_1), (B_3, E_3))$, where $B_2 \neq B_3$. So we must have had $(B_1, B_2)$ in $\text{pairing}(a)$ and $(B_1, B_3)$ in $\text{pairing}(b)$ of expression $e$ with pairings. Therefore, init actions of $B_1$ includes $a$ and $b$. We know that $e$ has equal choice pairing, therefore the sets of init actions of $B_1$ and $B_2$ are equal, hence $b$ is also an init action of $B_2$, which means that $B_2$ is a block in $\text{Part}_b(s_2)$. Let $E'_2$ be one of the post-effects of $B_2$ with respect to action $b$.

Now consider a reachable derivative $e'' = (r_1, r_2, \ldots, r_n)$ of $e$ in which blocks $B_1$ and $B_2$ appear i.e., $r_1$ is in $B_1$ and $r_2$ is in $B_2$. We take an $a$-derivative $r'_1$ of $r_1$ and $r'_2$ of $r_2$ respectively, so that $r'_1$ is in $E_1$ and $r'_2$ is in $E_2$. We also have a $b$-derivative of $r'_1$ in $E'_1$. But because of equal choice of $B_1$ and $B_2$ we must also have a $b$-derivative $r''_2$ of $r_2$ which reside in $E'_2$. Since connected expression $e$ has consistent pairing of actions, $(B_1, B_2)$ must appear in $\text{pairing}(b)$. This is a contradiction as we have $B_1$ appears twice in $\text{pairing}(b)$ relation, and is paired with distinct blocks of $\text{Part}_b(s_2)$.

(Proof of $e'$ having product-derivatives property): Let ES-cables($a$) be a set of $a$-cables having equal source property and let $(e_1, \ldots, e_m)$ be a tuple in postdecomp(ES-cables($a$)). To show that $e'$ has product-derivatives property we have to show that $(e_1, \ldots, e_m)$ is in postdecomp($\text{ES-cables}(a)$) i.e., we have to prove existence of some $a$-cable $c$ in ES-cables($a$) such that $\pi(e) = (e_1, \ldots, e_m)$. Since $(e_1, \ldots, e_m)$ is in postdecomp(ES-cables($a$)), there exist $a$-cables $c_1, \ldots, c_m$ such
that \( c[i] \ast e_i \) for all \( i \) in \( loc(a) \). Hence we must have \( a\)-ducks \( c[i] = (B_i, E_i) \) and \( (B_1, \ldots, B_m) = \Pi_{i \in loc(a)} c[i] \), the tuple of source blocks of all cables in ES-cables(\( a \)). By construction, it must be the case that \( (B_1, \ldots, B_m) \) is in \( pairing(a) \) of expression \( e \) with pairings and \( E_i \) are \( X \)-post-effects of \( b_i \). Therefore, we must have constructed an \( a \)-cable \( \Pi_{i \in loc(a)}(B_i, E_i) \), whose set of post-effects is \( \Pi_{i \in loc(a)}(E_i) \) as required.

(Proof of \( \text{Lang}(e) = \text{Lang}(e') \)): Let \( e = (s_1, \ldots, s_k) \) and \( w \in \text{Lang}(e) \). We prove that \( w \in \text{Lang}(e') \) by doing induction on length of \( w \). For each prefix of \( w \) prove that if each \( w \)-derivative of \( e \) is also a \( w \)-derivative of \( e' \). Let \( f = (u_1, \ldots, u_k) \) be \( w \)-derivative of \( e \) and \( f' = (v_1, \ldots, v_k) \) is in \( \text{Der}_a(f) \) therefore, \( f' \) is in \( \text{Der}_a(e) \). Inductively, we assume that \( f \) is also a valid \( w \)-derivative of \( e' \) and now we have to prove that \( f' \) is a valid \( a \)-derivative of \( f \) to show that \( f' \) is in \( \text{Der}_a(e') \). Therefore we have to show that there exits an \( a \)-cable whose pre-blocks appear in \( f \) and whose post-effects appear in \( f' \).

Since \( e \) is consistent with pairing of actions, there exist blocks \( B_i \) in \( \text{Part}_a(s_i) \) where each regular expression \( u_i \) is in \( B_i \) for all \( i \) in \( loc(a) \), and \( (B_1, \ldots, B_m) \) is a tuple in \( pairing(a) \). We have \( v_i \in \text{Der}_a(u_i) \) for \( i \) in \( loc(a) \). Therefore, we must have \( X \)-post-effects \( E_i \) of \( B_i \) containing \( u_i \) in it. So, \( (B_i, E_i) \) is an \( a \)-duct of \( s_i \) for all \( i \) in \( loc(a) \). But \( e' \) has product-derivatives property as proved above, therefore, for \( B_1, \ldots, B_m \) appearing in \( pairing(a) \) and \( E_1, \ldots, E_m \) their \( X \)-post-effects respectively, we have an \( a \)-cable \( \Pi_{i \in loc(a)}(B_i, E_i) \). Since each \( u_i \) is a component expression of \( f' \), each \( E_i \) appears in \( f' \). Therefore, \( f' \) is a valid \( a \)-derivative of connected expression \( f \) which completes the induction.

Now we give a language preserving construction of connected expression with equal choice and consistent pairings from a CE-cables having equal source and product-derivatives property.

**Theorem 8.** Let \( \Sigma \) be a distributed alphabet and \( e' \) be a connected expression with cables defined over it. Then we can construct a connected expression \( e \) with pairings linear in the size of \( e' \). And, if \( e' \) has equal source then \( e \) has equal choice property. In addition, if \( e' \) is action-live then if \( e' \) has product moves property then \( e \) has consistency of pairing. \( \text{Lang}(e) = \text{Lang}(e') \).

**Proof.** Let \( e' = (s'_1, \ldots, s'_k) \) be a connected expression with cables. We construct an expression with pairings \( e = (s'_1 = s_1, \ldots, s'_k = s_k) \) by taking \( pairing(a) = \{ \Pi_{i \in loc(a)} \ast c[i] \mid c \in \text{cables}(a) \} \), for each \( a \) in \( \Sigma \). The size of constructed \( pairing(a) \) relation is at most the size of \( \text{cables}(a) \) relation for \( e \). Without loss of generality we assume that \( loc(a) = \{1, 2\} \).

(Proof of \( e \) having equal choice property): Let \( (B_1, B_2) \) be in \( pairing(a) \), and actions \( a, b \) are in \( \text{Init}(B_1) \), and action \( a \) is in \( \text{Init}(B_2) \). Now we have to prove that \( b \) is also present in \( \text{Init}(B_2) \). Since \( (B_1, B_2) \) is in \( pairing(a) \) we must have had an \( a \)-cable \( c \) with \( B_1, B_2 \) as its set of pre-blocks, so let \( c = ((B_1, E_1), (B_2, E_2)) \). And since \( b \) is in \( \text{Init}(B_1) \), block \( B_1 \) is also present in the \( \text{Part}_b(s_1) \), the set of partitions of \( b \)-derivatives of \( s_1 \). Let \( E'_1 \) be post-effects(\( B_1 \)) of \( b \)-derivatives of \( B_1 \). Then \( (B_1, E'_1) \) is a \( b \)-duct of \( s_1 \). Therefore, we should have \( b \)-cable \( e' \) having \( e'[1] = (B_1, E'_1) \). Hence, we have two cables \( c \) and \( e' \) of expression \( e' \) sharing a
pre-block $B_1$. But $\epsilon'$ has equal source property, therefore, $B_2$ is also a pre-block of some $b$-cable $\epsilon'$ i.e., $B_2$ is a partition of $b$-derivatives of $s_2$ and hence it belongs to $\text{Part}_a(s_2)$, implying that $b$ is present in $\text{Init}(B_2)$ as required.

(Proof of $e$ having consistent pairing of actions): Let $e = (s_1, s_2, \ldots, s_k)$ with $f$ in $\text{Der}_w(e)$ and $f'$ in $\text{Der}_a(f)$ i.e., $f' \in \text{Der}_w(e)$. Inductively we assume that all intermediate derivatives excluding $f$ are valid, i.e., their projections on loc$(a)$ appear in pairing$(a)$ which we have constructed from $e$.

Let $f = (r_1, r_2, \ldots, r_k)$ and $f' = (r'_1, r'_2, \ldots, r'_k)$, so that $r'_k$ is in $\text{Der}_a(r_i)$ for all $i$ in loc$(a)$. Let $B_j$ be the block in $\text{Part}_a(s_1)$ containing $r_j$ and, $E_j$s be the post-effects of $B_j$ containing $r'_j$, for all $j$ in $\{1, \ldots, k\}$.

To complete the induction step, we have to prove that $f +^{\text{loc(a)}} (B_1, B_2)$ appears in pairing$(a)$. In the case that $\epsilon' = ((B_1, E_1), (B_2, E_2))$ is already an a-cable in cables$(a)$ relation of $\epsilon'$ then we are done, because then by construction of pairing$(a)$ relation we have $(B_1, B_2)$ in pairing$(a)$.

Now suppose that $\epsilon'$ is not an a-cable. Since $B_1$ is block of partition of $a$-derivatives of $s_1$, it must appear in at least one a-cable of $\epsilon'$ and similarly it holds for $B_2$ also. Therefore, there must exist $a$-cables $c = ((B_1, E_1), (B_2, E_2))$ and $d = ((B'_1, E'_1), (B_2, E_2))$ in which $B_1$ and $B_2$ appear separately. At this point one could argue that why $(B_1, E_1)$ is the chosen a-duct to participate in cables$(a)$ relation? We could have some $E'' \epsilon'$ a post-effect of $B_1$ containing $r'_1$ and that $(B_1, E'' \epsilon')$ could be part of a-cable $c$ and not $(B_1, E_1)$. If $r'_1 \in E_1 \cap E'' \epsilon'$ then both a-ducts $(B_1, E_1)$ and $(B_1, E'' \epsilon')$ can not appear in cables$(a)$ relation simultaneously. So at this point we could have chosen $(B_1, E'' \epsilon')$ and argument does not change, because all we require is $B_1$ being a pre-block of some a-cable, which is guaranteed by definition of cables$(a)$ relation.

If $\epsilon'$ is not a-cable then we have $B_2 \neq B'_2$, and hence $r_2 \notin B'_2$ and therefore, $f' \notin \text{Der}_a(f)$. It also implies that $B_1 \neq B'_1$, and hence $r_1 \notin B'_1$ and therefore, $f' \notin \text{Der}_a(f)$. Hence $\text{Der}_a(f) = \emptyset$. Importantly it implies that for any $u'$ containing action $a$, Der$_w(u') = \emptyset$. Therefore, we can not reach an a-derivative of $\epsilon'$ from $f$, which is in contradiction with the fact that $\epsilon'$ is action-live.

(Proof of $\text{Lang}(e) = \text{Lang}(\epsilon')$): The direction of proving $\text{Lang}(\epsilon') \subseteq \text{Lang}(e)$ is easy because each reachable derivative of $\epsilon'$ is also valid derivative of $e$, since pairing$(a)$ relation is constructed from pre-blocks of cables in cables$(a)$ relation, for all $a \in \Sigma$.

To prove that $\text{Lang}(e) \subseteq \text{Lang}(\epsilon')$, consider a word $w$ in language of $e$, i.e., we reach some expression $e''$ in Der$_w(e)$ having empty word in the language of its each component expression. Let $w = wa$. Let $f$ be a $u$-derivative of $\epsilon'$ i.e., it is in Der$_w(\epsilon')$. Inductively we assume that $f' \in \text{Der}_w(e')$, i.e., $f' \in \text{Der}_a(f)$. For induction step, we have to prove that $f' \in \text{Der}_w(e')$, i.e., $f' \in \text{Der}_a(f)$. Let $f = (r_1, r_2, \ldots, r_k)$, and $f' = (r'_1, r'_2, \ldots, r'_k)$, where $r'_1$ (resp. $r'_2$) is an a-derivative of $r_1$ (resp. of $r_2$). Since $e$ is consistent with pairing$(a)$, as proved above, $f' +^{\text{loc(a)}}$ appears in pairing$(a)$. Let $f +^{\text{loc(a)}} (B_1, B_2)$. So by construction $\epsilon'$ must have cables $c_1$ and $c_2$ such that $\bullet c_1[1] = B_1$ and $\bullet c_2[2] = B_2$, but since they have equal source, $\bullet c_1 = \bullet c_2 = (B_1, B_2)$. Let $E_1$ (resp. $E_2$) be the post-effect of $B_1$ (resp. $B_2$)
containing \( r'_1 \) (resp. \( r'_2 \)). Since \( e' \) have product-derivatives property, there exist an \( a \)-cable \( c \) with \( B_1, B_2 \) as its pre-blocks and \( e^* = (E_1, E_2) \), as required.

\( \square \)

Expressions given in the following subsection correspond to product systems with subset acceptance condition.

### 5.6 Sum of Connected Expressions (SCE)

We give syntax for sum of connected expressions (SCE) defined over a distribution \((\Sigma_1, \Sigma_2, \ldots, \Sigma_k)\) of alphabet \(\Sigma\).

\[
e := 0|e_1 + \cdots + e_m, \text{ where } e_i \text{ is a connected expression (CE) over } \Sigma
\]

For an SCE \( e \) its semantics is given as follows: For the SCE \( 0 \), we have \( \text{Lang}(0) = \emptyset \). For the SCE \( e = e_1 + \cdots + e_m \), its language is given as \( \text{Lang}(e) = \text{Lang}(e_1) \cup \text{Lang}(e_2) \cup \cdots \cup \text{Lang}(s_m) \). The definitions of derivatives extended to SCEs is as given below. The expression \( 0 \) has no derivatives on any action. Derivative of an SCE with respect to a letter \( a \) is defined as: \( \text{Der}_a(e) = \text{Der}_a(e_1) \cup \text{Der}_a(e_2) \cup \cdots \cup \text{Der}_a(s_m) \). Inductively \( \text{Der}_{aw}(e) = \text{Der}_{aw}(\text{Der}_a(e)) \). The set of all derivatives \( \text{Der}(e) \) is union of sets of derivatives of \( e \) over all words \( w \) in \( \Sigma_i^* \) for all \( i \in \{1, \ldots, m\} \).

**Definition 27 (SCE with pairings)).** An SCE \( e = e_1 + \cdots + e_m \) where each \( e_i \) is a CE-pairings, is called a sum of connected expressions with pairing (SCE-pairings). An SCE-pairings \( e \) is said to have equal choice property if each component CE-pairings \( e_i \) of the sum also has it.

**Example 11.** The expression \( e = e_1 + e_2 \) where \( e_1 = \text{fsync}((ab + ac)^*, (ad + ae)^*) \) is the CE-pairings of Example 8 and \( e_2 = \text{fsync}((ab + ac)^*a, (ad + ae)^*a) \) is the CE-pairings of Example 9 is an SCE-pairings with equal choice property, as both \( e_1 \) and \( e_2 \) have it.

**Definition 28 (SCE with cables)).** An SCE \( e = e_1 + \cdots + e_m \) where each \( e_i \) is a CE-cables, then \( e \) is called a sum of connected expressions with cables (SCE-cables). An SCE-cables \( e \) is said to have equal source property if each component CE-cables \( e_i \) of the sum also has it. An SCE-cables \( e \) has product-derivatives property if each component CE-cables \( e_i \) of the sum has it.

**Example 12.** The expression \( e' = e_3 + e_4 \) where \( e_3 = \text{fsync}((ab + ac)^*, (ad + ae)^*) \) is the CE-cables of Example 8 and \( e_4 = \text{fsync}((ab + ac)^*a, (ad + ae)^*a) \) is the CE-cables of Example 9.

As an example of how derivatives of SCE-cables from derivatives of SCE-pairings differ even while having identical components, see that \( \text{fsync}(r_2, s_3) \in \text{Der}_a(e) \) (of Example 11) but it does not belong to \( \text{Der}_a(e') \).
6 Connected Expressions and Product Systems

To get a product system with globals having subset-acceptance, from a sum of connected expression, we use an earlier result from [15], where construction of PS-globals with product-acceptance was given from a CE-cables.

For each set of derivatives (pre-blocks and after-effects), of a component regular expression, we constructed an unique state of a local component, which gives us product moves property in the constructed product system, if we have product-derivatives property for given CE-cables.

Lemma 7 (CE-cables to PS-globals with product-acceptance [15]). Let \( e \) be a CE-cables, defined over a distribution \( \Sigma \). Then for the language of \( e \), we can compute a PS-globals with product-acceptance linear in the size of expression \( e \). Further, if \( e \) had equal-source, then system \( A \) has same source property; and, if \( e \) had product-derivatives then \( A \) has product moves property.

Using Lemma 7 we get PS-globals with subset-acceptance, from sum of connected expressions.

Theorem 9 (SCE-cables to PS-globals with subset-acceptance). Let \( e \) be an sum of connected expression defined over \( \Sigma \). Then we can construct a PS-globals \( A \) with subset-acceptance for the language of \( e \). If \( e \) had equal source property, then has same source property. In addition, if \( e \) has product-derivatives property then \( A \) has product moves property.

Proof. Let \( e = e_1 + \ldots + e_m \) be an sum of CE-cables having equal source and product-derivatives property. Language of expression \( e \) is \( \text{Lang}(e) = \text{Lang}(e_1) \cup \ldots \cup \text{Lang}(e_m) \).

Using Lemma 7 we construct PS-globals \( A_i \) with product-acceptance condition, for the language of each CE-cables \( e_i \) having the same source and product-derivatives property. That is \( \text{Lang}(A_i) = \text{Lang}(e_i) \) for all \( i \in \{1, \ldots, m\} \).

Using language characterization of PS-globals with subset-acceptance conditions given in Corollary 2 we get an PS-globals \( A \) with subset-acceptance condition, over \( \Sigma \) such that \( \text{Lang}(A) = \text{Lang}(A_1) \cup \ldots \cup \text{Lang}(A_m) \). Since Corollary 2 preserves global moves of component PS-globals and, as underlying CE-cables had equal source and product-derivatives property, we get same source property and product moves property for each of the component \( A_i \). Hence \( A \) has both these properties as required.

Example 13. For SCE-cables \( e' \) of Example 12 we can construct PS-globals with same source property and subset-acceptance \( D \) of Example 6 accepting language \( L_s \), using Theorem 9.

A language preserving construction of connected expressions with equal source property, from a PS-globals with same source property and product-acceptance, was given in [15]. Since each local state–either a source state or a target state of local move–is mapped uniquely to a set of derivatives of component regular expression, we have product-derivatives property for the expression, if the product system had product-moves property.
Lemma 8 (PS-globals with product-acceptance to CE-cables \cite{15}). Let \( \Sigma \) be a distributed alphabet and, \( A \) be a product system with globals and product-acceptance, defined over \( \Sigma \). For the language of \( A \), we can construct a connected expression \( e \) with cables, exponential in the size of the given product system. Furthermore, if product system has same source property then connected expression with cables has equal source property, in addition, if it has product-moves property then connected expression with cables has product-derivatives property.

Now using Lemma \( 8 \) we get a sum of connected expressions for PS-globals with subset-acceptance.

Theorem 10 (PS-globals with subset-acceptance to SCE-cables). Let \( A \) be a product system with globals and subset-acceptance, defined over distribution \( \Sigma \). For the language of \( A \), we can construct a sum of connected expression \( e \) with cables. And, if product system has same source property then \( e \) has equal source property. Also, in addition, if \( A \) has product moves property then \( e \) has product-derivatives property.

Proof. Let \( A \) be an PS-globals with subset-acceptance condition and same source property. Then using Corollary \( 2 \) there exist PS-globals \( A_1, \ldots, A_m \) with product-acceptance conditions such that \( \text{Lang}(A) = \text{Lang}(A_1) \cup \ldots \cup \text{Lang}(A_m) \). Note that each \( A_i \) has same source property.

For the language of each PS-globals \( A_i \) with product-acceptance condition, we can construct \( CE \)-cables \( e_i \) with equal source property, using Lemma \( 8 \). From these we construct a sum of \( CE \)-cables \( e = e_1 + \ldots + e_m \) which has equal source property and language \( \text{Lang}(e_1) \cup \ldots \cup \text{Lang}(e_m) \) which is \( \text{Lang}(A) \).

Example 14. For a PS-globals with same source property and subset-acceptance \( D \) of Example 6 accepting language \( L_s \), we can construct an \( SCE \)-cables \( e' \) of Example 12 using Theorem 10.

Using equivalence of \( PS \)-matchings with product-acceptance and \( CE \)-cables, from \cite{17,13}, and Corollary \( 1 \) we get language equivalent \( SCE \)-pairings for \( PS \)-matchings with subset-acceptance, and vice-versa.

Theorem 11 (PS-matchings with subset-acceptance to SCE-pairings). Let \( A \) be a product system with conflict-equivalent and consistent matchings, having subset-acceptance. For the language of \( A \), we can construct a sum of connected expression \( e \) with equal choice and consistent pairings.

The converse result follows.

Theorem 12 (SCE-pairings to PS-matchings with subset-acceptance). Let \( \Sigma \) be a distributed alphabet and a sum of connected expression \( e \) defined over it, with equal choice and consistent pairings. Then for its language we can construct a product system with conflict-equivalent and consistent matchings, having subset-acceptance.
7 Conclusion

In this paper, we have given a language \((L_s \text{ of Example 2})\) which can be accepted by free choice Zielonka automata. This language is not accepted by any synchronous product or direct product, using Lemma 2 proof of which is presented here, and was not given in [16]. We have also given a language \((L_4 \text{ of Example 4})\) which can be accepted by a free choice synchronous product and not by any direct product. A language which can be accepted by free choice direct product \((L_3 \text{ of Example 3})\) was given in [17]. With this we have a hierarchy of labelled free choice nets similar to automata over distributed alphabets. In addition we have defined Zielonka automata with product acceptance condition and its free choice restriction. We have given language \((L_p \text{ of Example 1})\) of this class. We used this intermediate automata to obtain Kleene theorem for free choice Zielonka automata. Lemma 3 shows that this class is strictly more expressive than direct products with matching. In addition, ZA with same source has product moves property (class is not shown in the figure) then it is equivalent to SP with matching.

We give below the summary of correspondences established for the nets, automata over distributed alphabets and expressions. To get an expression, for the language of a labelled 1-bounded and S-coverable free choice net (with or without distributed choice) having a finite set of final markings, we use Theorem 4 and Theorem 10. In the reverse direction, we use Theorem 9 to get product system.
with subset acceptance from expressions and then Theorem 3 to get a language equivalent free choice net system.

If the labelled free choice net has distributed choice and has a finite set of final markings, we have an alternate syntax for it. We first use Theorem 5 to get equivalent product system, and then Theorem 11 to get equivalent expressions for the product system constructed. In the reverse direction, we use Theorem 12 and then Theorem 13. All these correspondences and hierarchy of classes is shown in Figure 6.

References

1. Antimirov, V.: Partial derivatives of regular expressions and finite automaton constructions. Theoret. Comp. Sci. 155(2), 291–319 (1996)
2. Brzozowski, J.A.: Derivatives of regular expressions. J. ACM 11(4), 481–494 (1964)
3. Desel, J., Esparza, J.: Free choice Petri nets. Cambridge University Press, New York, USA (1995)
4. Garg, V.K., Ragunath, M.: Concurrent regular expressions and their relationship to petri nets. Theoret. Comp. Sci. 96(2), 285–304 (1992)
5. Grabowski, J.: On partial languages. Fundam. Inform. 4(2), 427–498 (1981)
6. Hack, M.H.T.: Analysis of production schemata by Petri nets. Project Mac Report TR-94, MIT (1972)
7. Iordache, M.V., Antsaklis, P.J.: The ACTS software and its supervisory control framework. In: Proceedings Conference on Decision and Control, CDC, pp. 7238–7243. IEEE (2012)
8. Jantzen, M.: Language theory of petri nets. In: ACPT, LNCS, vol. 254 (1987)
9. Lodaya, K.: Product automata and process algebra. In: SEFM. IEEE (2006)
10. Lodaya, K., Mukund, M., Phawade, R.: Kleene theorems for product systems. In: DCFS, Proceedings. LNCS, vol. 6808. Springer (2011)
11. Mirkin, B.G.: An algorithm for constructing a base in a language of regular expressions. Engg. Cybern. 5, 110–116 (1966)
12. Mukund, M.: Automata on distributed alphabets. In: D’Souza, D., Shankar, P. (eds.) Modern Applications of Automata Theory. World Scientific (2011)
13. Petersen, J.L.: Computation sequence sets. Journal of Computing and Systems Science 13(1), 1–24 (1976)
14. Phawade, R.: Labelled Free Choice Nets, finite Product Automata, and Expressions. Ph.D. thesis, Homi Bhabha National Institute (2015)
15. Phawade, R.: Kleene theorems for labelled free choice nets without distributed choice. In: Cabac, L., Kristensen, L.M., Rölke, H. (eds.) Proc. PNSE. CEUR Workshop Proceedings, vol. 1591, pp. 132–152. CEUR-WS.org (2016)
16. Phawade, R.: Kleene theorems free choice nets labelled with distributed alphabets. In: Daniel Moldt, E.K., Rölke, H. (eds.) Proc. PNSE. CEUR Workshop Proceedings, vol. 2138, pp. 77–98. CEUR-WS.org (2018)
17. Phawade, R., Lodaya, K.: Kleene theorems for labelled free choice nets. In: Moldt, D., Rölke, H. (eds.) Proc. PNSE. CEUR Workshop Proceedings, vol. 1160, pp. 75–89. CEUR-WS.org (2014)
18. Phawade, R., Lodaya, K.: Kleene theorems for synchronous products with matching. Transactions on Petri nets and other models of concurrency X, 84–108 (2015)
19. Zielonka, W.: Notes on finite asynchronous automata. Inform. Theor. Appl. 21(2), 99–135 (1987)