Reloading in barodesy employing the asymptotic state boundary surface

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Abstract
The reloading behaviour of the soil is dominated by effects of the recent deformation history as well as by the overall stress history, indicated by overconsolidation. The stiffness of an overconsolidated soil is higher than the stiffness of normally consolidated soil. In basic hypoplastic and barodetic models, the stiffness for reloading is underestimated, and thus so-called ratcheting effects appear. In this article, we propose a relation to improve the reloading behaviour of barodesy: we introduce an extension, which increases the stiffness in dependence of the distance to the asymptotic state boundary surface and the current dilatancy. Thus, within an intermediate strain range, the stress-strain behaviour is improved and ratcheting effects are significantly reduced. The behaviour of the proposed extension is verified by simulation of typical element tests and validated by a comparison with experimental results.

KEYWORDS
asymptotic states, barodesy, overconsolidation, reloading, state boundary surface

1 INTRODUCTION

In almost every complex geotechnical problem, several loading and unloading cycles occur. A material model, that is used in numerical simulations of geotechnical problems, should be able to make a realistic prediction of the soil behaviour in the complete construction process.

The soil behaviour is different in primary loading, unloading and reloading. The stiffness during unloading is much higher than for primary loading and for a certain strain range, there is also an increase of the reloading stiffness. This may lead to a hysteretic behaviour in the stress-strain relation. This behaviour is observed for different directions of strain paths, as for example, hydrostatic, oedometric and conventional triaxial compression. The experimental results in Figure 1 show this behaviour for reloading of a triaxial test on reconstituted Kaolin clay\(^1\) and an oedometric test on reconstituted Black clay.\(^2\) When the state of the sample approaches a previous state, the stiffness decreases to the one at the primary loading.

There are several aspects, why there is a stiffness increase at the reloading of the soil, which is inherently linked with its deformation and stress history. One point can be found in the small-strain behaviour, that yields a stiffness increase immediately after a sudden change of the deformation direction.\(^3\) These effects are captured by the short-term deformation history. Another reason is related to the overall stress history of the soil, which is represented by its overconsolidation and yields an increased stiffness compared to the normally consolidated behaviour.\(^4\)

There are different approaches to model the reloading behaviour of the soil within different categories of soil models. Elastoplastic models use (subloading) yield surfaces to distinguish between primary loading, unloading and reloading.
behaviour and for the consideration of the overconsolidation. Hypoplastic\textsuperscript{5,6} and barodetic\textsuperscript{7,8} models are formulated within a single tensorial equation, being able to distinguish primary loading from unloading. A shortcoming in the current versions of those models is the prediction of the reloading and cyclic behaviour of the soil. The cyclic behaviour can be captured to a certain degree by so-called small-strain extensions\textsuperscript{9,10} but for various reloading cycles with different strain amplitudes, those models cannot completely suppress the so-called ratcheting of the original model and thus predict unrealistic strain accumulation.

In this article, we propose an extension to overcome these ratcheting effects and to improve the predicted reloading behaviour within an intermediate strain range. Therefore the distance of the actual state to the asymptotic state boundary surface (ASBS)\textsuperscript{11} is used. This measure is combined with a stiffness-dilatancy relation that allows for a specific stiffness increase of the reloading behaviour. The approach is demonstrated and applied to barodesy for clay,\textsuperscript{8} but could easily also be applied to any other kind of hypoplastic/barodetic model that includes an explicit formulation of the state boundary surface. The application of the proposed approach allows for a better deformation prediction in low cyclic problems, as well as the simulation of boundary value problems with several construction stages. For the simulation of high cyclic and dynamic problems, the use of specific models is recommended.\textsuperscript{12–14}

In this article, stresses are effective ones with compression defined negative. Second order tensors are written in bold capital letters (e.g., $\mathbf{X}$), $||\mathbf{X}|| = \sqrt{\text{tr} \, \mathbf{X}^2}$ is the Euclidean norm of a symmetric tensor $\mathbf{X}$, $\text{tr} \, \mathbf{X}$ is the sum of the diagonal components of $\mathbf{X}$. The superscript 0 marks a normalised tensor, that is, $\mathbf{X}^0 = \mathbf{X}/||\mathbf{X}||$. The effective stress tensor is denoted by $\mathbf{T}$. For principle stresses we employ the subscript $T_i$ with $i = 1, 2, 3$. In triaxial stress space, the Roscoe invariants $p = -\frac{1}{3} \text{tr} \, \mathbf{T}$ and $q = |T_1 - T_3|$, as well as the stress ratio $\eta = \frac{q}{p}$ are used. The objective stress rate is denoted by $\dot{\mathbf{T}}$. The stretching tensor $\mathbf{D}$ is the symmetric part of the velocity gradient. The scalar $\text{tr} \, \mathbf{D}^0$ describes the volumetric behaviour and is used as a measure for the dilatancy. For a conventional triaxial compression or an oedometric compression test, the axial stress is denoted with $T_1$ and the radial stress is denoted with $T_2 = T_3$. The associated strains are $\varepsilon_1$ and $\varepsilon_2 = \varepsilon_3$ as well as $\varepsilon_q = \frac{2}{3}(\varepsilon_1 - \varepsilon_3)$ and $\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. The void ratio $e$ is the ratio of the volume of the voids $V_p$ to the volume of the solids $V_s$.

2 | MODELLING OF THE RELOADING BEHAVIOUR OF SOIL

2.1 | Overconsolidation

Modelling the reloading behaviour is inherently linked with the deformation and stress history of the soil.\textsuperscript{15} The overall stress history of the soil can be described by its overconsolidation. For simple elastoplastic models with only a deviatoric
yield surface, like a linear elastic-ideal plastic model, every state inside this yield surface behaves purely elastic. Using an additional cap type loading surface, as the hardening soil model\textsuperscript{16} or critical state models\textsuperscript{17,18} one or more mobilized yield surfaces define whether a deformation is primary loading or reloading. Thus, the memory of former deformations and stress levels is stored in the mobilization of the yield surfaces. Inside these yield surfaces, basic models often assume an hypoelastic behaviour, using a pressure dependent stiffness. Some more sophisticated models also include subloading yield surfaces for a better description of the overconsolidation behaviour and other aspects as, for example, fabric and anisotropy effects.\textsuperscript{19,20}

Another, well-established approach for constitutive modelling is followed by hypoplastic\textsuperscript{5,6} or barodetic\textsuperscript{8,21} models. This group of models is characterized by the fact, that they can be formulated in a single equation and do not require a decomposition of the strains into an elastic and a plastic part. Sophisticated models of this type are based on the principles of critical state soil mechanics and use the overconsolidation ratio of the soil to distinguish between the behaviour of dense and loose soils with a single set of material parameters. These models perform well for monotonic loading and unloading and are able to predict the shear strength and simulate the undrained behaviour of the soil.\textsuperscript{22} A shortcoming in the basic versions of those models appears at the application for cyclic loading because in its basic versions those models consider only the actual state and not the former deformation or the stress history. This causes an underestimation of the reloading stiffness, which is shown in Figure 2 for the simulation of reloading cycles in an oedometric compression test using a hypoplastic model for sand.

### Small-strain reloading

Additionally to the overall stress history, the soil behaviour depends on the recent deformation history.\textsuperscript{3} A sudden change of the deformation direction increases the stiffness, which falls to the former value by continuous monotonic deformation. This behaviour is described by the so-called small-strain stiffness. There are several approaches to capture the small-strain effects with material models, by introducing additional state variables or history surfaces that serve as a memory of the former deformations.\textsuperscript{9,10,23} A first approach to improve the cyclic behaviour of hypoplastic models was the intergranular strain concept (IGS) by Niemunis and Herle.\textsuperscript{9} Within the IGS, the former deformation is stored in an additional tensorial state variable, the so-called intergranular strain. The IGS is able to capture the small-strain stiffness of the soil and models a stiffness degradation during monotonic deformation. This approach is well suited to predict the cyclic behaviour and the accumulation of pore pressure during undrained cyclic loading.\textsuperscript{22} However, the strain range of the stiffness degradation is fixed by the material parameters in the model and thus it is not possible to make correct predictions for reloading cycles with a large variation of the strain size.\textsuperscript{13} This shortcoming is shown in Figure 2 for the simulation of oedometric compression tests with different stress amplitudes. The basic hypoplastic model shows a large volumetric deformation for reloading. The IGS predicts a realistic result for the first loop, but for the second and the third loop there are ratcheting and overshooting effects, respectively. It is possible to fit the behaviour to a specific range of strain cycles that may fit to the current problem,\textsuperscript{13} but it is not possible to capture the complete reloading process for
various strain ranges. A simple procedure to prevent excessive overshooting is the combination of the IGS with the limit states of the basic model, as proposed by Bode et al.\textsuperscript{24} In this approach the ASBS of barodesy is used as a limit condition for the IGS. Modelling the complete reloading behaviour within the hypoplastic and barodetic framework is still unsolved.

A reduction of the ratcheting effects can be obtained, when the model is able to predict a proper reloading stiffness for intermediate strains. For this purpose, we introduce an approach in the following sections to improve the reloading stiffness of hypoplastic and barodetic models using the overconsolidation ratio of the soil. The effects of overshooting can not be solved by application of the here presented approach.

3 | BARODESY

We use barodesy for clay\textsuperscript{8} as basic model in this article. Barodesy\textsuperscript{7,8} is based on the asymptotic behaviour of the soil and uses the principles of critical state soil mechanics (CSSM). The stress rate $\dot{T}$ is a function of the actual stress state $T$, the stretching $D$ and the void ratio $e$. Critical states in barodesy almost coincide with the failure criteria of Matsuoka-Nakai.\textsuperscript{25} The model consists of one single tensorial equation and does not distinguish between elastic and plastic strains and has thus similarities to hypoplastic models.

Four material parameters have to be calibrated by standard laboratory tests and are defined as shown in Figure 3. The parameters $N$ and $\lambda^*$ define the position and the inclination of the isotropic normal compression line (isoNCL), which is a straight line in the $\ln(1+e)-\ln p$ plane. The isotropic unloading stiffness is controlled by the swelling parameter $\kappa^*$. The stress ratio at the critical state is defined by the critical friction angle $\varphi_c$, that defines the inclination $M_c$ of the critical state line (CSL) in the $p-q$ plane.

Using the principles of CSSM, barodesy is able to model the effects of pyknontropy, barotropy and to distinguish between slightly and highly overconsolidated soil. For highly overconsolidated soils, the model is able to predict peak states with mobilized friction angles $\varphi_{mob} > \varphi_c$. Possible states, that can be reached by proportional stretching are defined by the so-called asymptotic state boundary surface (ASBS).\textsuperscript{11} The barodetic framework, in its basic version, is well suited to predict loading and unloading and to describe the asymptotic behaviour of the soils. For reloading and overconsolidated states, the same shortcomings as in basic hypoplastic models appear and the model underpredicts the reloading stiffness and produces ratcheting effects. To improve the reloading behaviour and to include small-strain effects, the intergranular strain concept is applied,\textsuperscript{26} which yields to the same improvements and shortcomings as mentioned in the previous section. For strains exceeding the small-strain range, the basic version of barodesy also underpredicts the reloading stiffness, according to Figure 2. Barodesy for clay, without any extensions, is denoted as the basic model in this article. The equations of the basic model are given in Appendix B.

4 | STIFFNESS INCREASE USING THE ASYMPOTOTIC STATE BOUNDARY SURFACE

To capture the stiffness increase for overconsolidated soils and to reduce ratcheting effects at reloading we propose an approach that uses the distance of the actual state to the asymptotic state boundary surface (ASBS).\textsuperscript{11} Reaching the ASBS, the soil has forgotten its former stress and deformation history, why this surface is sometimes also named swept-out-memory surface.\textsuperscript{27} Thus, the distance of the current state to the ASBS offers as a measure for the fading of memory. We therefore need to employ an additional and appropriate definition for the overconsolidation of the soil.
4.1 Isotropic overconsolidation

There are several definitions for the overconsolidation, that can be found in the literature. A typically used measure in elastoplastic models is using a preconsolidation stress, that defines the maximum former reached stress level.\(^4,28\) This preconsolidation pressure defines the size of a current yield surface and thus allows for an distinction between different overconsolidation ratios.

In material models that does not use a preconsolidation pressure and yield surfaces to describe the actual state,\(^6,8\) an often used measure to describe the overconsolidation is the so-called Hvorslev equivalent pressure \(p_e = \exp\left(\frac{N-\ln(1+e)}{\lambda^*}\right)\). The equivalent pressure \(p_e\) defines the stress level at the isotropic normal compression line (isoNCL) for the current void ratio. Using the ratio \(R_e = \frac{p}{p_e}\), we obtain a definition of the overconsolidation ratio OCR = \(\frac{1}{R_e}\).

Under isotropic compression at a normally consolidated state, the soil follows the isoNCL, line i in Figure 4A and we obtain \(p = p_e\) and thus OCR = 1. In a continuous monotonic undrained deformation with \(tr\ D^0 = 0\) the soil reaches the critical state (line c) that is defined by the stress ratio \(\eta = \frac{q}{p} = M_c\), as well as the critical void ratio \(e = e_c\). The critical state line (CSL) is defined for OCR = 2, which also serves as a limit between slightly and highly overconsolidated soils.

For each continuous monotonic stretching, the soil follows a respective normal compression line with a constant OCR. Every volume decreasing stretching with \(tr\ D^0 < 0\) yields an according normal compression line with an overconsolidation \(1 \leq OCR < 2\). An example is the oedometric compression (line o; \(tr\ D = -1\)), that follows the oedometric normal compression line (oedNCL) in Figure 4A, which lies between the isoNCL and the CSL. Volume increasing constant stretching (e.g., line a in Figure 4) yields an according normal extension line with OCR > 2.\(^{29,30}\) The labeled lines (a,c,o,i) in Figure 4 represent possible asymptotic states that can be reached by the according specific monotonic stretching.

A shortcoming in the OCR definition using the equivalent pressure \(p_e\) is, that it does not take into account the deviatoric stress of the soil. To describe the whole reloading process also for shearing tests, the information of the isotropic overconsolidation information is thus not sufficient. For example the information OCR = 2 does not sufficiently define a critical state. Only if, for example, the stress ratio \(\eta = M_c\) for triaxial compression, the actual state is also a critical state.

4.2 Overconsolidation with respect to the ASBS

In this article we use an additional measure for the overconsolidation, that is based on ideas of the bounding surface theory\(^19,31\) and uses the distance of the actual state to the ASBS. A similar definition of the overconsolidation is also employed by Niemunis\(^12\) to define the rate effects within the visco-hypoplastic model, where an additional elliptical surface is used as reference bounding surface. In the here proposed extension, we directly use the information of the ASBS of barodesy.

Within the asymptotic state framework, every direction of proportional stretching yields an specific asymptotic stress ratio.\(^29,32\) Thus, additionally to a certain respective OCR, each asymptotic state has its certain stress ratio, which is defined by a stress-dilatancy relation.\(^33\) An isotropic compression yields a stress state on the hydrostatic axis, an
isochoric deformation will reach a stress ratio \( \eta = M_c \) and thus follows the CSL in the \( p-q \) space. An oedometric compression yields an oedometric normally consolidated state, following the oedNCL in the \( e-p \) plane and also reaching the stress ratio \( M_{\text{oord}} \), with, for example, the relation of Jáky with \( K_0 = 1 - \sin \varphi_c \). This information is stored in the asymptotic state boundary surface, which is represented by a closed line in Figure 4B for a specific void ratio within the \( p-q \) plane. The information of the different normal compression lines within the \( p-e \) plane (Figure 4A) and in the \( p-q \) plane (Figure 4B) can be combined to plot this surface in the \( p-q-e \) space in Figure 4C. Thus the curves in Figures 4A and 4B are projections of the asymptotic states shown in Figure 4C and do not necessarily represent stress paths.

Here we can see that every monotonic stretching, yields an state lying on the asymptotic state boundary surface, that is defined with its respective normal compression/extension line,\(^{29}\) for example, line a. These states are denoted as asymptotic compression and extension states.

With a normalization of the stress state by the Hvorslev equivalent pressure \( p_e \), the ASBS in Figure 4C is projected into a single curve in the \( \frac{p}{p_e} - \frac{q}{p_e} \) plane, Figure 5, as well as a closed surface in the principle stress space.\(^{11}\) We introduce a state surface, that is similar to the ASBS and intersects the actual stress state \( T \), Figure 5. The state surface meets the hydrostatic axis at the pressure \( p_{e,s} \). The scalar measure

\[
R_a = \frac{p_{e,s}}{p_e} \tag{1}
\]

describes the size of the state surface in relation to the ASBS and is thus a measure for the distance of the current state to the ASBS. For \( R_a = 1 \) the actual state is an asymptotic state related to a specific stretching. Every state with \( R_a < 1 \) lies inside the ASBS, which we denote, in this sense, as overconsolidated with respect to the ASBS.

At critical state, the OCR is defined as OCR = 2. With the stress ratio \( \eta = M_c \) it is also an asymptotic state and we obtain \( R_a = 1 \). A state with the pressure dependent critical void ratio \( e_c \) but lying on the hydrostatic axis has also OCR = 2, but \( R_a = 0.5 \) and is thus not an asymptotic state. For an isotropic stress state \( p_{e,s} = p \) holds \( R_a = R_e \).

The here presented definitions describe some different point of views of overconsolidation. While \( R_e \) describes the distance of the actual stress level to the isonCL, using the Hvorslev equivalent pressure \( p_e \) as reference, the measure \( R_a \) describes the distance of the actual state to its respective state on the ASBS. We use both definitions in this article. The Hvorslev overconsolidateion ratio OCR is used to define the current state within the \( p-e \) space, for example, for the initial void ratio and the distinction between slightly and highly overconsolidated soils. The here additionally introduced quantity \( R_a \) is used to describe the distance to the ASBS as a measure to identify non-asymptotic states, relevant for reloading.

To obtain the state surface in Figure 5, we use an fictive stress state \( \hat{T} \) with the same stress ratio \( \eta \) as the actual stress state, but lying on the asymptotic state boundary surface. We use the OCR of \( \hat{T} \) and use the similarity between the the ASBS and the state surface

\[
\frac{p_{e,s}}{p} = \frac{p_e}{\hat{p}} \tag{2}
\]

to calculate the interception of the state surface with the hydrostatic axis \( p_{e,s} = p \frac{p_e}{\hat{p}} \).

Note that, similar to hypoplasticity,\(^{27}\) it is possible that certain paths can slightly exceed the ASBS of barodesy. These cases are not considered here, and we get \( R_a \leq 1 \). A summary of the equations to calculate \( R_a \) is given in Appendix A.
FIGURE 6 Influence of the parameter $\alpha_s$ and the direction of stretching on the exponent $c_s$ and the stiffness factor $f_s$ for several dilatancies: $i$ isotropic, $o$ oedometric, $c$ isochoric deformation, $-i$ and $-o$ define the respective volume increasing deformation.

4.3 Stiffness-dilatancy relation

We use the overconsolidation with respect to the ASBS to improve the reloading stiffness in barodesy. Some other recent developments in hypoplasticity use similar definitions for the overconsolidation with respect to the ASBS, which helps to improve the loading behaviour of overconsolidated clays by increasing the stiffness related to the overconsolidation. Only the information of $R_a$ is not sufficient to fix the reloading behaviour, because the unloading stiffness and the reloading stiffness would be increased by the same factor. To overcome this problem, we couple this information with a stiffness-dilatancy relation.

The stiffness of the basic model is increased by a scalar factor $f_s$

$$\dot{T} = f_s \dot{T}_m,$$

where $\dot{T}_m$ is the stress rate of the basic model and

$$f_s = 1 + (1 - R_a^c) \cdot (m_s - 1).$$

The function $f_s$ depends on the material parameter $m_s$, the distance to the ASBS $R_a$ and an exponent $c_s$. For $R_a < 1$, the stiffness is increased up to $f_s = m_s$. For asymptotic states ($R_a = 1$) we obtain $f_s = 1$ and using Eq.(3) yields $\dot{T} = \dot{T}_m$.

The interpolation of the stiffness increase between $f_s = m_s$ and $f_s = 1$, depending on the ratio $R_a$, is controlled by the exponent $c_s$. The stiffness increase should not be the same for all deformation directions. For example, for isotropic unloading ($\text{tr} D^0 = 1$), in barodesy, the volumetric stiffness is already defined by the material parameter $\kappa^*$ that can be directly calibrated to fit the experimental results. For isotropic compression ($\text{tr} D^0 = -\sqrt{3}$) starting at an overconsolidated state, the stiffness is underestimated by the basic model and needs to be increased. To fit the function $f_s$ to the behaviour of the basic model, we propose a stiffness-dilatancy relation with

$$c_s = \alpha_s \left( \sin \left( \frac{\pi - \text{tr} D^0}{2\sqrt{3}} \right) + 1 \right).$$

The stiffness-dilatancy relation depends on the dilatancy $\text{tr} D^0$ and the material parameter $\alpha_s$. The influence of $\alpha_s$ is shown in Figure 6A. Using the exponent $c_s$ from Equation (5) in Equation (4), we obtain the influence of the ratio $R_a$ on the stiffness factor $f_s$ which depends on the current dilatancy and the parameter $\alpha_s$, Figure 6B and 6C. According to Equation (3), $f_s \geq 1$ increases the stiffness by a scalar factor and does not change the direction of the stress path. Thus the proposed extension has no effect on the uniqueness of the solution of the basic model. Within the possible range of $\text{tr} D^0$, $f_s$ is a monotonic function with respect to $D$, which preserves the convexity of the response envelopes. A mathematical discussion of the stiffness function can be found in Appendix D.

The maximum stiffness increase is obtained for an isotropic compression test, (denoted as $i$ in Figure 6). The stiffness is increased with $R_a$ and decreases approaching the ASBS. The stiffness of isotropic unloading ($-i$) is not affected. Other tests as oedometric compression ($o; \text{tr} D^0 = -1$) and extension ($-o; \text{tr} D^0 = 1$) or isochoric/undrained deformation ($c; \text{tr} D^0 = 0$)
Influence of the parameter $\alpha_s$ on the reloading cycle of an isotropic compression test. The IGS is used for all simulations lie in between. In general, the stiffness increase of volume decreasing paths is higher than for volume increasing paths for the same value of $R_a$.

The parameter $m_s$ controls the maximum stiffness increase at $R_a = 0$. The difference of the volumetric stiffness between unloading and primary loading is controlled by the relation of the parameters $\lambda^*$ and $\kappa^*$. An estimation of this parameter can be $m_s = \frac{\lambda^*}{\kappa^*}$. Another possible approximation for this parameter is the assumption, that the maximum stiffness of the soil is limited by the small strain shear stiffness $G_0$, which can be related to the stiffness factor $m_R$ from the intergranular strain concept.\textsuperscript{26} Thus, $m_R$ should be an upper limit for the stiffness factor $m_s$. Typical $m_R$ values for clays lie between 5 and 20.\textsuperscript{36}

### 4.4 Combination with the intergranular strain concept

The proposed ASBS stiffness extension improves the stress-strain behaviour for reloading of overconsolidated soils at an intermediate strain range until asymptotic states are reached. For a realistic simulation of small load cycles and abrupt changes in the deformation direction, the recent deformation history needs to be considered. Additionally to the proposed ASBS extension, we use the intergranular strain concept (IGS) to capture the small-strain effects with barodesy.\textsuperscript{26} For the behaviour within the elastic region, we define an hypoelastic model with a pressure dependent shear modulus $G$ and a fixed Poisson’s ratio $\nu$. The model parameters $G$ and $\nu$ are directly calculated from the material parameters of barodesy. The intergranular strain stiffness factor $m_R = \frac{G_0}{G}$ is calibrated to fit the small-strain shear modulus $G_0$ and is a fixed material parameter in this model. Another possible way for clays is using a pressure dependent $m_R$.\textsuperscript{37}

Within the small strain range, only the stress rate of the elastic model is used. For an intermediate strain range, the stiffness decreases, and there is an interpolation between the elastic model and barodesy enhanced by the ASBS stiffness extension. For asymptotic states, only the basic version of barodesy is employed. The whole set of equations for the IGS as used in this article and for the combination with barodesy is given in Appendix C. Note that for small closed strain cycles, the hypoelastic model, which is here used for the IGS, may cause a stress accumulation.\textsuperscript{38} For the simulation of high cycle problems, an appropriate hyperelastic formulation should be used.

## 5 SIMULATIONS

### 5.1 Application of the ASBS extension

For the verification of the proposed extension we simulate standard laboratory tests using a default set of parameters for barodesy with $\varphi_c = 25^\circ$, $N = 1$, $\lambda^* = 0.1$, $\kappa^* = 0.01$ and for the IGS with $m_R = m_T = 10$, $R = 5 \cdot 10^{-5}$, $\beta_R = 0.2$, $\chi = 2$. For the stiffness factor we choose $m_s = \lambda^* / \kappa^* = 10$. The influence of the parameter $\alpha_s$ is presented in Figure 7 with the results of an isotropic compression test. The IGS is used for the simulations, so also the basic model shows a small hysteresis.
within the reloading behaviour. All simulations show the same unloading behaviour, as the isotropic extension is not effected. The proposed extension yields an increased reloading stiffness and the parameter $\alpha_s$ controls the degradation of the stiffness with the distance to the isoNCL. For further simulations we use $\alpha_s = 0.75$.

To distinguish between the influence of the ASBS and the IGS, an initially normally consolidated drained triaxial test is simulated, performing several reloading loops, Figure 8. For primary loading, all simulations show the same behaviour. The improvement of the stiffness increase due to the IGS can be seen in Figure 8B. Due to the change of the deformation direction, there is a stiffness increase and a hysteretic behaviour within the stress strain relation. After a certain strain range, the stiffness reduces to the stiffness of the basic model. With the additional application of the ASBS extension, there is an additional stiffness increase also for an intermediate strain range, that yields a reduction of the strain accumulation in each cycle.

The simulation of an undrained triaxial test shows the influence of the proposed extension and the combination with the IGS on the stiffness degradation after a change in the deformation direction. Figure 9 shows the results of an undrained triaxial test that starts with OCR = 3 and $p = 100$ kPa after a complete reversal of the deformation direction. The IGS predicts an increase of the shear modulus to $G_0$ due to the reversal of the deformation direction which decreases to the stiffness of the basic model at $\varepsilon_q = 10^{-3}$. Employing only the ASBS stiffness there is also a higher stiffness than for the basic model, because of the overconsolidation of the soil with respect to the ASBS. This stiffness increase holds over a larger strain range until the ASBS is reached. The combination of the ASBS extension and the IGS yields the small-strain shear modulus $G_0$ for the small-strain range and an interpolation with the overconsolidation stiffness at an intermediate strain range. Closer to the asymptotic state, the stiffness decreases to the stiffness predicted by the basic model.

The results of simulated undrained triaxial tests with a different initial overconsolidations in Figure 10 show the influence of the extensions on the predicted stress paths. The basic barodetic model predicts a different behaviour for slightly
and heavily overconsolidated soils. Employing the IGS yields a modification of the stress path. For normally consolidated and slightly overconsolidated soils, barodesy predicts an decreasing pressure $p$ which yields the development of an additional pore pressure. Using the IGS, the initial degradation of the pressure is reduced, and the undrained stress paths start perpendicular to the hydrostatic axis. The here proposed additional ASBS extension has only a small effect on the stress path due to the increased stiffness. All models reach the same asymptotic state lying on the CSL. The results of a cyclic undrained triaxial tests on normally consolidated soil in Figure 10B show the accumulated decrease of the pressure $p$ each cycle, which means an increasing pore pressure. The IGS yields a reduction of the pore pressure accumulation each cycle and the additional use of the ASBS has only a small effect on the stress paths. Similar effects can be seen on the cyclic undrained test on heavily overconsolidated soil in Figure 10C, where the pressure $p$ increases each load cycle.

Figure 11 shows the results of a simulated oedometric compression test with one reloading loop. Starting at an overconsolidated state, all simulations approach the oedNCL in Figure 11A. Due to its initial overconsolidation, the simulation using IGS + ASBS shows a stiffer response until the oedNCL is reached. Starting from the hydrostatic axis, the stress path in Figure 11B approaches the $K_0$-line asymptotically. The $K_0$-line is defined by the stress ratio $M_{oed}$. A state lying on the oedNCL in Figure 11A, as well as on the $K_0$-line in Figure 11B, is an asymptotic state. These asymptotic states are represented by a point lying on the normalized ASBS in Figure 11C. Unloading yields an overconsolidation of the soil, resulting in states inside of the ASBS. Using the ASBS extension, the stiffness increases due to the increasing distance from the ASBS. The stress paths from the simulations using IGS and IGS + ASBS almost coincide in Figure 11B and differ from the simulations using only the basic model. In the normalized plot in Figure 11C, the paths of IGS and IGS + ASBS differ slightly, because of the different void ratio at the same stress level, which yields an different equivalent pressure $p_e$. 

**FIGURE 10** (A) Stress paths of undrained triaxial tests with different initial states: OCR = 1.5, OCR = 2, OCR = 4 (from left to right). (B) Initially normally consolidated undrained triaxial test with several deviatoric load cycles starting at $p = 100$ kPa. (C) Initially overconsolidated (OCR = 4) undrained triaxial test with several deviatoric load cycles starting at $p = 100$ kPa.

**FIGURE 11** Oedometric compression test with one reloading cycle.


TABLE 1  Employed material parameters for Newfield clay with the reference stress $\sigma^* = 1$ kPa

| $\varphi_c$ | N  | $\lambda^*$ | $\kappa^*$ | $m_R$ | $m_T$ | $R$  | $\beta_r$ | $\chi$ | $\alpha_s$ | $m_s$ |
|------------|----|-------------|------------|-------|-------|------|-----------|-------|------------|-------|
| 28.85°     | 0.615 | 0.032       | 0.009      | 5.0   | 5.0   | $5 \cdot 10^{-5}$ | 0.2   | 6          | 0.75  | 3.54 |

| basic model | IGS | ASBS |

FIGURE 12  Simulation of the isotropic compression test ISO6 and comparison with experimental results (A) Comparison of the proposed model with the basic model and the IGS for two reloading loops. (B) Simulation of all reloading loops using the proposed model

The reloading stiffness of the oedometric compression test is clearly increased using the ASBS extension compared to the other simulations.

5.2  Comparison with experimental data

We use experimental results from laboratory tests on reconstituted Newfield clay to validate the results of the proposed reloading extension. We simulate monotonic and low cyclic experiments with different boundary conditions. The material parameters for barodesy are given in Table 1. The parameters $\lambda^*$, $\kappa^*$ and $N$ are calibrated using test results from isotropic loading and unloading. The critical friction angle $\varphi_c$ is obtained from the results of several triaxial tests and is directly taken from Namy.

The intergranular strain parameters are estimated, based on default values. The stiffness factor $m_s = \frac{\lambda^*}{\kappa^*} = 3.54$. The dilatancy exponent $\alpha_s = 0.75$ is chosen to obtain the best overall performance in the isotropic compression test ISO6. The application of the proposed extension allows for a better prediction of the volumetric stiffness for different loops in compression tests. Figure 12A shows the results of the first two load cycles of the isotropic compression test ISO6. Without the ASBS extension, barodesy overpredicts the volumetric deformation. The IGS allows the prediction of a small hysteresis in the stress-strain relation but is limited to a fixed strain range. Using also the distance to the ASBS allows for a prediction of the deformation behaviour with a good accordance to the experimental results in all load cycles, Figure 12B.

The test S1 is initially normally consolidated to $p = 196.2$ kPa followed by a single drained triaxial loading and unloading. The experimental results after the consolidation are presented in Figure 13. The test is simulated with the basic model, the IGS and with the ASBS extension including the IGS. The initial stiffness is overestimated by the basic model and there is almost no influence of the extensions on the stress-strain behaviour. A slight stiffness increase can be seen for the triaxial unloading for the IGS and the ASBS extension.

Test S4 in Figure 14 is a combination of drained triaxial tests of different stress levels with several reloading cycles. Test S4 starts at $p_0 = 49.1$ kPa with an isotropic compression to $p = 588.6$ kPa to point 1. A drained triaxial load cycle is applied up to $\eta = 0.81$ (path 1-2-3). The pressure is reduced to $p = 196.2$ kPa followed by two drained triaxial load cycle up to $\eta = 0.75$ (4-5-6) and reloading to $\eta = 1.1$ (6-7) with subsequent unloading (7-8) and a final loading up to $\eta = 1.19$ (8-9).
FIGURE 13 Results of the normally consolidated drained triaxial test S1 after isotropic consolidation to $p = 196.2$ kPa

FIGURE 14 Test S4 from Namy.39 (A) Sketch of the test program; (B) Experimental results (point 9 lies outside of the plot at $\varepsilon_1 = 0.07$); (C) Simulations with barodesy; Axial strain $\varepsilon_1$ without the initial consolidation

Figures 14B and 14C show the experimental results and the simulations using barodesy. The initial shear stiffness of the normally consolidated soil is overpredicted by the basic model. The first loading is almost the same with both models. For the reloading cycles, the ASBS extension reduces the predicted axial strains compared to the results of the basic models and the results only using the IS concept.

Using the proposed model, a reduction of the strain accumulation can be observed between the first and the last reloading cycles (points 4 and 8). Using the ASBS extension, the model predicts a strain accumulation of $\Delta \varepsilon_1 \approx 0.007$ while the calculation with the basic model predicts $\Delta \varepsilon_1 \approx 0.016$. The results using the ASBS extension are more realistic compared to the experimental results with $\Delta \varepsilon_1 \approx 0.005$.

6 | SUMMARY

Using the distance to the ASBS in combination with a stiffness-dilatancy relation allows for a better prediction of the reloading behaviour with barodesy for an intermediate strain range. The distance to the ASBS is measured by introducing a state surface that defines the overconsolidation with respect to asymptotic states. The stiffness-dilatancy relation enables the distinction between unloading and reloading states. The proposed extension captures the reloading behaviour at an intermediate strain range and can be used as an addition to the intergranular strain concept (IGS), which covers the small-strain stiffness of the soil. The effects of overshooting, produced by the basic IGS, see Figure 2, cannot be reduced by the here presented approach. A simple procedure to overcome this shortcoming is presented by Bode et al.24 performing a hard-cut at the ASBS. Further research will focus on the application of the distance to the ASBS also directly in the IGS. This could improve the prediction of the small-strain behaviour and could help to reduce the overshooting effects. The proposed extension can be calibrated using two additional material parameters, of which one parameter can be estimated using the relation between the isotropic compression and swelling parameters, that are material parameters of the basic model. Finally, one additional parameter needs to be calibrated fitting the reloading behaviour to experimental results. The extension works as an overlay model, which is here applied to barodesy for clay as basic model, employing its information about the ASBS. The approach allows for a direct implementation to other hypoplastic or barodetic models that contain
information of the asymptotic state boundary surface and will help for a better prediction of the reloading behaviour. In this case another stiffness-dilatancy relation may be more suitable.

The performance of the presented approach is validated and verified by the simulations of several tests and the comparison with experimental results from Newfield clay. The reloading behaviour predicted by the proposed model shows a clear improvement compared with the results using the basic model or the basic model extended by the intergranular strain concept, only.

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AUTHOR CONTRIBUTIONS
Manuel Bode, Gertraud Medicus, and Wolfgang Fellin developed the ideas and the main concept of the article together as team. M.B. performed the calculations and analyses. Manuel Bode mainly wrote the manuscript with input from Gertraud Medicus and Wolfgang Fellin.

DATA AVAILABILITY STATEMENT
The data used to support the findings of this study are available from the corresponding author upon request.

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REFERENCES
1. Roscoe K, Burland J. On the generalized stress-strain behaviour of wet clay. In: J. Heyman, F. Leckie (Eds.) Engineering Plasticity. Cambridge: Cambridge University Press; 1968:535–609.
2. Habibbeygi F, Nikraz H. The effect of unloading and reloading on the compression behaviour of reconstituted clays. Int J Geomate. 2018;15(51):53–59.
3. Atkinson JH, Richardson D, Stallebrass SE. Effect of recent stress history on the stiffness of overconsolidated soil. Géotechnique. 1990;40(4):531–540.
4. Houlsby G, Wroth C. The variation of shear modulus of a clay with pressure and overconsolidation ratio. Soils Found. 1991;31(3):138–143.
5. Wolffersdorff v. P.A. A hypoplastic relation for granular materials with a predefined limit state surface. Mech Cohesive-frictional Mater. 1996;1(3):251–271.
6. Mašín D. Clay hypoplasticity with explicitly defined asymptotic states. Acta Geotech. 2013;8(5):481–496.
7. Kolymbas D. Introduction to barodesy. Géotechnique. 2015;65(1):52–65.
8. Medicus G, Fellin W. An improved version of barodesy for clay. Acta Geotech. 2017;12(2):365–376.
9. Niemunis A, Herle I. Hypoplastic model for cohesionless soils with elastic strain range. Mech Cohesive-frictional Mater. 1997;2(4):279–299.
10. Fuentes W, Triantafyllidis T. ISA model: A constitutive model for soils with yield surface in the intergranular strain space. Int J Numer Anal Methods Geomech. 2015;39(11):1235–1254.
11. Medicus G. Asymptotic states and peak states in barodesy for clay. Géotechnique Lett. 2020;10(2):262–169.
12. Niemunis A. Extended hypoplastic models for soils. Schriftenreihe des Institutes für Grundbau und Bodenmechanik der Ruhr-Universität Bochum; 2003. Heft 34.
13. Wichmann T. Explicit accumulation model for non-cohesive soils under cyclic loading. PhD thesis. Germany: Inst. für Grundbau und Bodenmechanik der Ruhr-Universität Bochum; 2005.
14. Gudehus G. Granular solid dynamics with eutaraxy and hysteresis. Acta Geotech. 2020;15:1–15.
15. Nova R, Hueckel T. An engineering theory of soil behaviour in unloading and reloading. Meccanica. 1981;16(3):136–148.
16. Schanz T, Vermeer P, Bonnier P. The hardening soil model: formulation and verification. Beyond 2000 in computational geotechnics. 1999:281–296.
17. Roscoe KH, Schofied AN, Wroth CP. On The yielding of soils. Géotechnique. 1958;8(1):22–53.
18. Wang ZL, Daifalasis YF, Shen CK. Bounding surface hypoplasticity model for sand. J Eng Mech. 1990;116(5):983–1001.
19. Yao YP, Hou W, Zhou AN. UH model: three-dimensional unified hardening model for overconsolidated clays. Géotechnique. 2009;59(5):451–469.
20. Bergholz K. Extended bounding surface model for general stress paths in practical applications. In: Proceedings of the 9th European Conference on Numerical Methods in Geotechnical Engineering (NUMGE 2018), 2018, Porto: Portugal CRC Press; 2018:265.
21. Kolymbas D. Barodesy as a novel hypoplastic constitutive theory based on the asymptotic behaviour of sand. geotechnik. 2012;35(3):187–197.
APPENDIX A: CALCULATION OF THE DISTANCE TO THE ASBS

The fictive stress state $\hat{T}$ has the same stress ratio $K_{\text{mob}}$ as the actual stress state $T$, according to Figure 5:

$$K_{\text{mob}} = \exp \left( - \frac{3}{2} \left( \ln \left( \frac{-T_1}{\sqrt[3]{-T_1T_2T_3}} \right)^2 + \ln \left( \frac{-T_2}{\sqrt[3]{-T_1T_2T_3}} \right)^2 + \ln \left( \frac{-T_3}{\sqrt[3]{-T_1T_2T_3}} \right)^2 \right) \right)$$

With

$$m = c_2 + \sqrt{\frac{1}{c_1} \left( \frac{1}{1 - K_{\text{mob}}} - 1 \right)}$$

we obtain two solutions for the dilatancy $\text{tr} \, D_a^0$ depending on the stress ratio $K_{\text{mob}}$:

$$\text{tr} \, D_a^0 = \begin{cases} \sqrt{\frac{6m^2}{9+2m^2}} & \text{for } K_{\text{mob}} \leq K_c \\ -\sqrt{\frac{6m^2}{9+2m^2}} & \text{for } K_{\text{mob}} > K_c \end{cases}$$

with the critical stress ratio $K_c = \frac{1 - \sin \varphi_c}{1 + \sin \varphi_c}$. We can calculate the Hvorslev ratio $R_{e,a}$ for the asymptotic state related to $\text{tr} \, D_a^0$

$$R_{e,a} = \frac{1}{2} \left( \frac{-\text{tr} \, D_a^0}{A^* + c_3 - c_3 \, \text{tr} \, D_a^0 \, c_2}{c_3} \right)^{\frac{1}{\sqrt{c_5}}}$$
with

\[ \beta = -\frac{1}{c_3 \Lambda} + \frac{1}{\sqrt{3}} 2^\varepsilon \lambda^* - \frac{1}{\sqrt{3}} \]

and

\[ \Lambda = -\frac{\lambda^* - \kappa^*}{2\sqrt{3}} \text{tr} D_0^0 + \frac{\lambda^* + \kappa^*}{2}. \]

Using the similarity of the ASBS and the state surface and the Hvorslev overconsolidation of the fictive stress state \( R_{e,a} = \frac{p}{p_e} \), we can calculate the intersection of the state surface with the hydrostatic axis

\[ p_{e,s} = \frac{p}{R_{e,a}} \]

and the overconsolidation with respect to the ASBS

\[ R_a = \frac{p_{e,s}}{p_e} \]

**APPENDIX B: BASIC EQUATIONS OF BARODESY**

In the following section the basic equations of barodesy\(^6\) are summarized. The objective stress rate \( \dot{T} \) is a function of the effective stress \( T \), the stretching \( D \) and the void ratio \( e \).

\[ \dot{T} = h \cdot (f R^0 + g T^0) \|D\| \]

\[ R = -\exp(\alpha D^0) \quad \text{mit} \quad \alpha = \frac{\ln K}{\sqrt{3/2 - (\text{tr} D^0)^2 / 2}} \]

\[ K = 1 - \frac{1}{1 + c_1 (m - c_2)^2} \quad \text{mit} \quad m = \frac{-3 \text{tr} D^0}{\sqrt{6 - 2(\text{tr} D^0)^2}} \]

\[ h = c_3 \|T\|^c_4 \]

\[ f = c_6 \beta \text{tr} D^0 - \frac{1}{2} \]

\[ g = (1 - c_6) \beta \text{tr} D^0 + \left( \frac{1 + e}{1 + e_c} \right)^c_5 - \frac{1}{2} \]

\[ e_c = \exp \left( N - \lambda^* \ln \frac{-2 \text{tr} T}{3\sigma^*} \right) - 1 \]

\[ \beta = -\frac{1}{c_3 \Lambda} + \frac{1}{\sqrt{3}} 2^\varepsilon \lambda^* - \frac{1}{\sqrt{3}} \]

\[ \Lambda = -\frac{\lambda^* - \kappa^*}{2\sqrt{3}} \text{tr} D^0 + \frac{\lambda^* + \kappa^*}{2} \]

The constants \( c_1 \cdot c_6 \) are calculated using the material parameters \( \varphi_c, N, \lambda^* \) and \( \kappa^* \):

\[ c_1 = \frac{1 - \sin \varphi_c}{2 c^2 \sin \varphi_c} \]

\[ c_2 = -\frac{3 \sqrt{2} + 3}{2} \]
\[c_3 = \frac{-\sqrt{3}/\lambda^* + \sqrt{3}/\chi^*}{2^{c_3\lambda^* + \left(\frac{1}{500}\right) c_5\lambda^*} - 2}\]
\[c_4 = 1\]
\[c_5 = \frac{1 + \sin \varphi_c}{1 - \sin \varphi_c}\]
\[c_6 = \frac{1}{2\left(-\sqrt{3}/c_3\chi^* + 2c_5\lambda^* - 1\right)}\]

**APPENDIX C: EQUATIONS OF THE INTERGRANULAR STRAIN CONCEPT**

The intergranular strain concept is applied to capture the stiffness increase within the small-strain range. In the following section, the equations for the application of this concept to barodesy according to Bode et al. \(26\) are given. The final stressrate \(\dot{T}\) is obtained by the interpolation between the stressrate of the hypoelastic model for small-strains \(\dot{T}_e\) and the model for intermediate and large strains \(\dot{T}_m\). In the following equations, the stiffness factor \(f_s\) is according to (4) already included within \(\dot{T}_m\).

The intergranular strain concept uses five additional material parameters: \(R, m_R, m_T, \beta_R, \chi\). The final stressrate reads:

\[
\dot{T} = f_A \dot{T}_e + \begin{cases} 
    f_{B1} \eta \|D\| \dot{T}_e^\delta + \rho \xi \eta \cdot (\dot{T}_m - \dot{T}_e) & \text{for } \delta^0 : D^0 > 0 \\
    f_{B2} \eta \|D\| \dot{T}_e^\delta & \text{for } \delta^0 : D^0 \leq 0
\end{cases}
\]

(C1)

using the abbreviations

\[
f_A = [\rho \chi m_T + (1 - \rho \chi)m_R] , \quad f_{B1} = \rho \chi (1 - m_T) , \quad f_{B2} = \rho \chi (m_R - m_T) , \quad \eta = \delta^0 : D^0 .
\]

The information of the deformation history is stored in an additional tensorial state variable \(\delta\), which follows the evolution equation

\[
\dot{\delta} = \begin{cases} 
    (I - \rho \delta \delta^0 \otimes \delta^0) : D & \text{for } \delta^0 : D > 0 \\
    D & \text{for } \delta^0 : D \leq 0
\end{cases}
\]

(C2)

with \(\rho = \|\delta\| / R\). The hypoelastic model used within the small-strain range is defined as \(\dot{T}_e = M_e : D\) and \(\dot{T}_\delta^\delta = M_e : \delta\)

using

\[
M_e = 2G \left( I + \frac{\nu}{1 - 2\nu} I \otimes I \right) .
\]

(C3)

The pressure-dependent additional parameters \(G\) and \(\nu\) are calculated using the parameters of barodesy according to Appendix B:
\[ G = p \frac{c_3(K_c - 1)}{2\sqrt{2}\sqrt{1 + K_c^2}}, \quad \text{with} \quad K_c = \frac{1 - \sin \varphi_c}{1 + \sin \varphi_c}, \]

\[ K = \frac{1}{2} \frac{p}{\kappa^*} \left( \frac{1}{\kappa^*} \frac{2c_3}{\sqrt{3}} (2\lambda^* c_5 - 1) + \frac{1}{\lambda^*} \right) \quad \text{and} \]

\[ \nu = \frac{3K - 2G}{6K + 2G}. \]

**APPENDIX D: MATHEMATICAL INVESTIGATION OF THE STIFFNESS FUNCTION**

The proposed extension can be applied to several constitutive models and does not affect the uniqueness of the solution of the basic model. The stiffness of the basic model is scaled by the factor \( f_s \). As the sign of \( f_s \) does not change, the direction of the stress path does not change and the uniqueness of the basic model is not affected.

The stiffness factor \( f_s \) is a monotonic function with respect to the stretching \( D \). Thus, it preserves the convexity of the response envelopes predicted by the basic model. The stiffness factor depends only on the material parameters (\( \alpha_s \) and \( c_s \)), the current asymptotic ratio \( R_a \) and the dilatancy \( \text{tr}D^0 \). The stiffness function in Equation (4) reads

\[ f_s = 1 + (1 - R_a^{c_s}) \cdot (m_s - 1), \quad (D1) \]

with

\[ c_s = \alpha_s \left( \sin \left( \frac{\pi - \delta}{2\sqrt{3}} \right) + 1 \right), \quad (D2) \]

and the abbreviation \( \delta = \text{tr}D^0 \). Derivation of Equation (D1) with respect to \( \delta \) yields

\[ \frac{\partial f_s}{\partial \delta} = (1 - m_s) - \frac{\alpha_s \pi}{2\sqrt{3}} \cdot \ln (R_a) \cdot R_a^{c_s} \cdot \cos \left( -\delta \frac{\pi}{2\sqrt{3}} \right). \quad (D3) \]

We discuss now each term on the right hand side of Equation (D3). The variables \( m_s > 1 \) and \( \alpha_s \geq 0 \) are material parameters and thus \( (1 - m_s) - \frac{\alpha \pi}{2\sqrt{3}} \geq 0 \). Within the possible range of \( -\sqrt{3} \leq \delta \leq \sqrt{3} \) we obtain \( \cos(-\delta \frac{\pi}{2\sqrt{3}}) \geq 0 \) and \( c_s \geq 0 \). The definition of \( 0 < R_e \leq 1 \) yields \( R_a^{c_s} > 0 \) and \( \ln (R_a) \leq 0 \).

Using these limitations we can show that the derivation

\[ \frac{\partial f_s}{\partial \delta} = (1 - m_s) - \frac{\alpha \pi}{2\sqrt{3}} \begin{cases} \ln (R_a) \geq 0 \\ R_a^{c_s} > 0 \end{cases} \cos \left( -\delta \frac{\pi}{2\sqrt{3}} \right) \leq 0 \quad (D4) \]

does not change the sign with in the possible range of \( \delta \) and thus the stiffness function \( f_s \) is a monotonic function with respect to the stretching \( D \).