Seiberg-Witten theory, matrix model and AGT relation

Tohru Eguchi and Kazunobu Maruyoshi

Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

Abstract

We discuss the Penner-type matrix model which has been proposed to explain the AGT relation between the 2-dimensional Liouville theory and 4-dimensional $\mathcal{N}=2$ superconformal gauge theories. In our previous communication we have obtained the spectral curve of the matrix model and showed that it agrees with that derived from M-theory. We have also discussed the decoupling limit of massive flavors and proposed new matrix models which describe Seiberg-Witten theory with flavors $N_f=2,3$. In this article we explicitly evaluate the free energy of these matrix models and show that they in fact reproduce the amplitudes of Seiberg-Witten theory.
1 Introduction

Recently a very interesting relation between the Nekrasov partition function of \( \mathcal{N} = 2 \) conformal invariant \( SU(2) \) gauge theory and the conformal block of the Liouville field theory was proposed \[1\]. It seems that this is the first example of a precise mathematical relationship between quantum field theories defined at different space-time dimensions. There have been various attempts at checking this AGT relation at lower instanton numbers by direct evaluation of Liouville correlation functions \[2, 3, 4, 5, 6\]. There have also been attempts at proving the relation by comparing the recursion relation satisfied by the descendants of the conformal blocks and Nekrasov’s partition function \[7, 8, 9, 10\].

On the other hand, a Penner type matrix model has been proposed to interpolate between the Liouville theory and gauge theory \[11\] and provide an explanation for the AGT relation. In a previous communication \[12\] we have studied this matrix model and also proposed models for asymptotically free theories obtained by decoupling some of massive flavors. We have shown that the spectral curves of these matrix models reproduce those based on the M-theory construction and their free energies satisfy the scaling identities known in the \( SU(2) \) Seiberg-Witten theory. (See also \[13, 14\] for \( A_r \) quiver matrix model).

In this paper we would like to evaluate the free energies of these matrix models in the large \( N \) limit explicitly and show that they in fact exactly reproduce the amplitudes of \( SU(2) \) Seiberg-Witten theory.

In section 2 we first describe the general properties of matrix models. In section 3 we compute the free energies: we integrate the Seiberg-Witten differential of the matrix model and evaluate the filling fraction in terms of the parameters of the spectral curve. We then invert this relation and derive the free energy. We present the computation for \( SU(2) \) gauge theory with two, three and four flavors and show that they all reproduce the amplitudes of Seiberg-Witten theory. In section 4 we discuss decoupling limits of some quiver gauge theories. Section 5 is devoted to conclusion and discussion.

Note: in our convention the free energy of the matrix model \( F_m \) is off by a factor 4 from that of gauge theory. Thus we will check the agreement \( 4F_m = F_{\text{gauge}} \) throughout this paper.

2 \( SU(2) \) gauge theories and matrix models

It has been proposed that the Nekrasov partition function for \( \mathcal{N} = 2 \), \( SU(2) \) gauge theory with four flavors (summarized in appendix A) coincide with the four-point conformal block of
Liouville theory \[1\]:

\[
Z_{\text{inst}}^{SU(2)} = Z_{\text{CFT}} \equiv \langle V_{\tilde{m}_\infty}(\infty)V_{\tilde{m}_1}(1)V_{\tilde{m}_2}(q)V_{\tilde{m}_0}(0) \rangle. \tag{2.1}
\]

Here \(V_{\tilde{m}}\) is the vertex operator, \(Q = b + 1/b\) and the central charge of the Liouville theory is \(c = 1 + 6Q^2\).

In order to relate the Liouville theory to matrix model, we consider the Dotsenko-Fateev integral representation of the four-point conformal block in terms of the free field \(\phi(z)\) \[15\]:

\[
Z_{\text{DF}} = \left\langle \left( \int d\lambda_I : e^{b\phi(\lambda_I)} : \right)^N V_{\tilde{m}_\infty}(\infty)V_{\tilde{m}_1}(1)V_{\tilde{m}_2}(q)V_{\tilde{m}_0}(0) \right\rangle, \tag{2.2}
\]

where the vertex operator \(V_{\tilde{m}_i}(z_i)\) is given by \(e^{\tilde{m}_i\phi(z_i)}\): and we have introduced the \(N\)-fold integration of screening operators. OPE of the scalar field is given by \(\phi(z)\phi(\omega) \sim -2 \log(z - \omega)\).

Momentum conservation condition relates the external momenta and the number of integrals as

\[
\sum_{i=0}^{2} \tilde{m}_i + \tilde{m}_\infty + 2bN = Q. \tag{2.3}
\]

As pointed out in \[16,17\] and recently in \[11\] in the context of the AGT relation, the Dotsenko-Fateev representation may be identified as the \(\beta\)-deformation of a one matrix integral

\[
Z_{\text{DF}} = q^{\frac{m_0m_2}{2g_s^2}} (1 - q)^{\frac{m_1m_2}{2g_s^2}} \left( \prod_{I=1}^{N} \int d\lambda_I \right) \prod_{I<J}(\lambda_I - \lambda_J)^{-2b^2} e^{\frac{ib}{g_s} \sum_{I} W(\lambda_I)}. \tag{2.4}
\]

In the case of \(b = i\), integrations over \(\{\lambda_I, I = 1, \cdots, N\}\) becomes an integral over a hermitian matrix \(M\) with eigenvalues \(\{\lambda_I\}\) and the action

\[
W(M) = \sum_{i=0}^{2} m_i \log(M - z_i), \quad z_0 = 0, \quad z_1 = 1, \quad z_2 = q. \tag{2.5}
\]

We identify the parameters \(m_i\) with the mass parameters of the corresponding gauge theory. The identification of the parameter \(b\) with the Nekrasov’s deformation parameters is given by

\[
\epsilon_1 = -ibg_s, \quad \epsilon_2 = \frac{-ig_s}{b}. \tag{2.6}
\]

In this paper, we focus on the \(b = i\) case, i.e. the self-dual background \(\epsilon_1 = -\epsilon_2 = g_s\). The momentum conservation condition then reduces to

\[
\sum_{i=0}^{2} m_i + m_\infty + 2g_sN = 0. \tag{2.7}
\]
This matrix model is expected to reproduce the results of $SU(2)$ gauge theory with $N_f = 4$. More precisely, as we will see below, the matrix integral together with the overall factor $(1 - q)^{m_1 m_2} 2 g_s^2$ in (2.4) corresponds to the $SU(2)$ gauge theory. Note that the factor $(1 - q)^{m_1 m_2} 2 g_s^2$ is the inverse of the $U(1)$ factor discussed in [1]. (See appendix A.)

Another point is that the Coulomb moduli parameter $a$ of the gauge theory is identified as the filling fraction $g_s N_i$, where $N_i$ is a number of screening operators inserted into the same contour in Dotsenko-Fateev representation. For the four-point conformal block we introduce $N_1$ and $N_2$. The overall condition (2.7) reduce these two degree of freedom to one which corresponds to the Coulomb modulus of $SU(2)$ theory.

The parameters $m_i$ are the masses associated with the $SU(2)^4(\subset SO(8))$ flavor symmetry. These are related to the masses of the hypermultiplets as

$$m_1 = \frac{1}{2}(\mu_1 + \mu_2), \quad m_2 = \frac{1}{2}(\mu_3 + \mu_4),$$
$$m_\infty = \frac{1}{2}(\mu_1 - \mu_2), \quad m_0 = \frac{1}{2}(\mu_3 - \mu_4).$$

(2.8)

The matrix models associated with gauge theories with $N_f = 2, 3$ are obtained by taking the decoupling limit of heavy flavors [12]. By taking a limit of $\mu_4 \to \infty$ while keeping $\mu_4 q = \Lambda_3$ fixed, the matrix model action becomes

$$W(z) = \mu_3 \log z + m_1 \log(z - 1) - \frac{\Lambda_3}{2 z}.$$  \hspace{1cm} (2.9)

with the following condition:

$$m_1 + m_\infty + \mu_3 + 2 g_s N = 0.$$  \hspace{1cm} (2.10)

The prefactor in front of the matrix integral [2.4] reduces to $e^{-\frac{m_1 \Lambda_3}{4 g_s^2}}$ in this limit. This is identified with the (inverse of the) $U(1)$ factor of $N_f = 3$ theory (see appendix A.2).

In order to obtain the $N_f = 2$ matrix model, we further take the limit $\mu_2 \to \infty$ while keeping $\mu_2 \Lambda_3 = \Lambda_2^2$ fixed. The dynamical scale of this gauge theory is given by $\Lambda_2$. After rescaling $z \to \frac{\Lambda_3}{\Lambda_2} z$, the action (2.9) becomes

$$W(z) = \mu_3 \log z - \frac{\Lambda_2}{2} \left( z + \frac{1}{z} \right).$$  \hspace{1cm} (2.11)

The mass relation reduces in this case to $\mu_1 + \mu_3 + 2 g_s N = 0$. The prefactor becomes simply $e^{-\frac{\Lambda_2^2}{8 g_s^2}}$.  

3
3 Planar free energy and prepotential

In this section, we will evaluate the planar free energy of the matrix models introduced above. In [12], we have shown that the free energy of these models satisfies the identities known in Seiberg-Witten theory [18, 19, 20]. Here, we will evaluate the free energies explicitly and compare them with the instanton expansions of the prepotentials at lower orders. The computation is a bit simpler than in the Seiberg-Witten theory where both the $A$ and $B$ cycle integrals have to be computed [21, 22]. Here we only have to compute the $A$ integral.

We first consider the matrix model for $SU(2)$ gauge theory with $N_f = 2$ in next subsection. Then, we will analyze the cases of $N_f = 3$ and 4 theories in turn.

3.1 $SU(2)$ gauge theory with $N_f = 2$

The matrix model action corresponding to the $SU(2)$ gauge theory with $N_f = 2$ is given by (2.11). For simplicity, we will omit the subscript 2 of the dynamical scale $\Lambda_2$ below. There are two saddle points determined by the classical equation of motion:

$$W'(z) = \mu_3 z - \frac{\Lambda}{2} \left(1 - \frac{1}{z^2}\right) = 0.$$  (3.1)

These lead to the two-cut spectral curve.

The planar loop equation reads as usual

$$R(z) = -\frac{1}{2} \left(W'(z) - \sqrt{W'(z)^2 + f(z)}\right),$$  (3.2)

where the resolvent is defined by

$$R(z) = \langle \sum_i \frac{g_s}{z - \lambda_i} \rangle.$$  (3.3)

The function $f$ is given by

$$f(z) = 4g_s \left( \sum_i \frac{W'(z) - W'(\lambda_i)}{z - \lambda_i} \right) = \frac{c_1}{z} + \frac{c_2}{z^2}.$$  (3.4)

Coefficients $c_1$ and $c_2$ are defined as

$$c_1 = -4g_s \left( \sum_i \left( \frac{\mu_3}{\lambda_i} + \frac{\Lambda}{2\lambda_i^2} \right) \right) = -2g_s N\Lambda, \quad c_2 = -2g_s \left( \sum_i \frac{\Lambda}{\lambda_i} \right).$$  (3.5)

In the formula for $c_1$ we have used the equations of motion $\langle \sum_i W'(\lambda_i) \rangle = 0$.

Then, the spectral curve $x^2 = (2\langle R(z) \rangle + W'(z))^2 = W'(z)^2 + f(z)$ is given by

$$x^2 = \frac{\Lambda^2}{4z^4} + \frac{\mu_3 \Lambda}{z^3} + \frac{1}{z^2} \left(\mu_3^2 + c_2 - \frac{\Lambda^2}{2}\right) + \frac{\mu_1 \Lambda}{z} + \frac{\Lambda^2}{4}.$$  (3.6)
This is similar to the curve obtained in [23]. The differential one form is identified with \( \lambda_m = xdz \) which has double poles at \( t = 0 \) and \( \infty \) with residues \( \mu_3 \) and \( \mu_1 \). Note that the parameter \( c_2 \) corresponds to the variable \( u \) in Seiberg-Witten theory.

We evaluate the filling fraction as

\[
g_s N_1 = \frac{1}{4\pi i} \oint_{C_1} \lambda_m(c_2),
\]

where \( C_1 \) is a cycle around one of the cuts in the curve. This integral is identified with the Coulomb moduli \( a \) in the gauge theory and we invert the above relation to solve the unknown parameter \( c_2 \).

Let us compute the free energy of our model defined by

\[
e^{F_m/g_s^2} = \left( \prod_{I=1}^{N} \int d\lambda_I \right) \prod_{I<J} (\lambda_I - \lambda_J)^2 e^{\frac{1}{g_s} \sum_I W(\lambda_I)}.
\]

The starting point is the formula for the \( \Lambda \) derivative:

\[
\frac{\partial F_m}{\partial \Lambda} = -\frac{g_s}{2} \left( \sum_I \left( \frac{1}{\lambda_I} + \lambda_I \right) \right) = \frac{c_2}{4\Lambda} - \frac{g_s}{2} (\sum_I \lambda_I).
\]

The expectation value \( \langle \sum_I \lambda_I \rangle = \langle \text{tr} M \rangle \) in the second term can be determined by studying the large \( z \) behavior of the resolvent: \( R(z) = -\frac{1}{2}(W'(z) - x) \approx \frac{g_s N}{z} + \frac{g_s \langle \text{tr} M \rangle}{z^2} + \ldots \)

\[
g_s \langle \text{tr} M \rangle = -\frac{1}{2\Lambda}(c_2 - \mu_1^2 + \mu_3^2).
\]

Therefore, we obtain

\[
\Lambda \frac{\partial F_m}{\partial \Lambda} = \frac{1}{4}(2c_2 - \mu_1^2 + \mu_3^2).
\]

Our remaining task is to determine \( c_2 \) in terms of \( g_s N_1 \) by using (3.7), and this leads to the explicit form of the free energy.

To derive \( c_2 \), let us consider the derivative of (3.7) with respect to \( c_2 \):

\[
4\pi i \frac{\partial (g_s N_1)}{\partial c_2} = \oint_{C_1} \frac{1}{\Lambda} \frac{dz}{\sqrt{P_4(z)}},
\]

where \( P_4 \) is the polynomial of degree 4:

\[
P_4(z) = z^4 + \frac{4\mu_1}{\Lambda} z^3 + \frac{4}{\Lambda^2}(\mu_3^2 + c_2 - \frac{\Lambda^2}{2}) z^2 + \frac{4\mu_3}{\Lambda} z + 1.
\]

It is easy to transform this polynomial so that (3.12) becomes the standard elliptic integral of the first kind. In the following, we set \( A = \mu_3^2 + c_2 - \frac{\Lambda^2}{2} \) and express the result in terms of \( A \).
For simplicity, let us consider the equal mass case: $\mu_1 = \mu_3 = m$ in the following. In this case, by the transformation $z = \frac{1}{t+1}$ and rescaling of $t$, the integrand of the right hand side of (3.12) can be brought to the standard form

$$\frac{\sqrt{2}}{\sqrt{S_+(L^2 + 4m\Lambda + 2A)} \sqrt{(1-t^2)(1-k^2t^2)}} \frac{dt}{(1-t^2)(1-k^2t^2)}$$

(3.14)

where $k^2 = S_-/S_+$ and

$$S_+ = \frac{1}{L^2 + 4m\Lambda + 2A} \left(-3L^2 + 2A \pm \Lambda \sqrt{8L^2 - 16A + 16m^2}\right).$$

(3.15)

Then, we can identify the integral (3.12) in terms of the hypergeometric function:

$$4\pi i \frac{\partial (g_s N_1)}{\partial A} = \frac{2\sqrt{2}}{\sqrt{S_+(L^2 + 4m\Lambda + 2A)}} \int_{1/k}^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$= \frac{\sqrt{2}i}{\sqrt{S_+(L^2 + 4m\Lambda + 2A)}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-k^2\right).$$

(3.16)

where we have used $\int_{1/k}^1 \frac{dt}{(1-t^2)(1-k^2t^2)} = iK'(k) = iK(k')$ with $k^2 = 1 - k^2$. We express the right hand side as a small $\Lambda$ expansion which corresponds to the instanton expansion in gauge theory. (Note that $k^2 = 1 + \mathcal{O}(\Lambda)$.) After integrating over $A$, we obtain

$$2g_s N_1 = \sqrt{A} \left(1 - \frac{m^2}{4A^2} L^2 - \frac{A^2 - 6Am^2 + 15m^4}{64A^4} \Lambda^4 - \frac{5m^2(3A^2 - 14Am^2 + 21m^4)}{256A^6} \Lambda^6\right.$$

$$\left. - \frac{15(A^4 - 28A^3m^2 + 294A^2m^4 - 924Am^6 + 1001m^8)}{16384A^8} \Lambda^8 + \mathcal{O}(\Lambda^{10})\right).$$

(3.17)

Then, we invert this equation and solve for $A$:

$$A = a^2 + \frac{m^2}{2a^2} L^2 + \frac{a^4 - 6m^2a^2 + 5m^4}{32a^6} \Lambda^4 + \frac{m^2(5a^4 - 14m^2a^2 + 9m^4)}{64a^{10}} \Lambda^6$$

$$+ \frac{5a^8 - 252m^2a^6 + 1638m^4a^4 - 2860m^6a^2 + 1469m^8}{8192a^{14}} \Lambda^8 + \mathcal{O}(\Lambda^{10}),$$

(3.18)

where we have introduced $a = 2g_s N_1$. Finally, we substitute this into (3.11) and integrate by $\Lambda$ to obtain

$$4F_m = 2 \left(a^2 - m^2\right) \log \Lambda + \frac{a^2 + m^2}{2a^2} L^2 + \frac{a^4 - 6m^2a^2 + 5m^4}{64a^6} \Lambda^4 + \frac{m^2(5a^4 - 14m^2a^2 + 9m^4)}{192a^{10}} \Lambda^6$$

$$+ \frac{5a^8 - 252m^2a^6 + 1638m^4a^4 - 2860m^6a^2 + 1469m^8}{32768a^{14}} \Lambda^8 + \mathcal{O}(\Lambda^{10}).$$

(3.19)

This agrees with the $U(2)$ gauge theory prepotential with $\tilde{a} = (a, -a)$ obtained from the Nekrasov partition function (A.16) or from the Seiberg-Witten theory [22]. (The first term is the one-loop part and the others are the instanton part.) Together with the prefactor $e^{-\frac{\Lambda^2}{8g_s^2}}$ we see that the full free energy is the same as that of $SU(2)$ gauge theory.
3.2 \( SU(2) \) gauge theory with \( N_f = 3 \)

Next, let us consider the matrix model corresponding to the gauge theory with \( N_f = 3 \). The matrix model action is given by (2.9). We will omit the subscript 3 of the dynamical scale \( \Lambda_3 \) from now on. As in the previous subsection, there are two saddle points in the classical equation of motion. In the planar limit, the loop equation leads to the spectral curve \( x(z)^2 = W'(z)^2 + f(z) \) where \( f(z) \) is written as

\[
f(z) = \frac{c_1}{z} + \frac{c_2}{z-1} + \frac{c_3}{z^2},
\]

with coefficients

\[
c_1 = -4g_s \left( \sum_I \frac{\mu_3}{\Lambda_I} + \frac{\Lambda_{1/2}}{2\Lambda_I} \right), \quad c_2 = -4g_s \left( \sum_I \frac{m_1}{\Lambda_I - 1} \right), \quad c_3 = -2g_s \left( \sum_I \frac{\Lambda}{\Lambda_I} \right).
\]

We can easily see that \( c_1 + c_2 = 0 \) due to the equations of motion.

The one form defined by \( \lambda_m = x(z)dz \) has a double pole at \( z = 0 \) and a simple pole at \( z = 1 \) and \( \infty \) with residues \( \mu_3, \ m_1 \) and \( m_{\infty} \), respectively. The residue at \( z = \infty \) gives a further constraint on \( c_i \):

\[
c_2 + c_3 = m_{\infty}^2 - (\mu_3 + m_1)^2.
\]

This condition together with the relation \( c_1 + c_2 = 0 \) leaves only one of the parameters independent. Let us choose \( c_3 \) to be independent.

It is then related to the filling fraction by the integral

\[
4\pi ig_s N_1 = \oint_{C_1} \lambda_m(c_3).
\]

For completeness, let us write down here the explicit form of the curve \( x^2 = \frac{P_4(z)}{4z^4(z-1)^2} \) with

\[
P_4(z) = 4m_{\infty}^2 z^4 + 4(\mu_3 + m_1)\Lambda + m_1^2 - \mu_3^2 - m_{\infty}^2 - c_3)z^3
\]
\[+ (\Lambda^2 - 8\Lambda\mu_3 + 4\mu_3^2 - 4\Lambda m_1 + 4c_3)z^2 - 2\Lambda(\Lambda - 2\mu_3)z + \Lambda^2.
\]

It is convenient to introduce the notation \( B \) as

\[
B = c_3 - \mu_3\Lambda + \mu_3^2.
\]

The polynomial is then rewritten as

\[
P_4(z) = 4m_{\infty}^2 z^4 + 4(\Lambda m_1 + m_1^2 - m_{\infty}^2 - B)z^3 + (\Lambda^2 - 4\Lambda(\mu_3 + m_1) + 4B)z^2
\]
\[- 2\Lambda(\Lambda - 2\mu_3)z + \Lambda^2.
\]
Let us consider the free energy of this matrix model. From the definition, its derivative in \( \Lambda \) is written as
\[
\frac{\partial F_m}{\partial \Lambda} = -\frac{g_s}{2} \left\langle \sum_I \frac{1}{\lambda_I} \right\rangle = \frac{c_3}{4\Lambda} \left( \frac{1}{4\Lambda} (B + \mu_3 \Lambda - \mu_3^2) \right).
\] (3.27)

In order to determine \( B \) we take a derivative of (3.23) with respect to \( B \):
\[
4\pi i \frac{\partial (g_s N_1)}{\partial B} = -\oint_{C_1} \frac{dz}{\sqrt{P_4(z)}}.
\] (3.28)

For simplicity, we consider the case where \( \mu_3 = m \) and \( m_1 = m_{\infty} = 0 \) in what follows. In this case, \( P_4 \) becomes a polynomial of degree 3:
\[
P_3(z) = (z - 1)(-4Bz^2 + (\Lambda^2 - 4\Lambda m)z - \Lambda^2).
\] (3.29)

After a change of variable (first shifting \( z \rightarrow z - p \) and then rescaling as \( z = Qt \)), we obtain
\[
P_3(z) \rightarrow -4BQ^2(1 + p) \times t(1 - t)(1 - k^2t),
\] (3.30)

where
\[
k^2 = \frac{Q}{1 + p}, \quad p = \frac{1}{2} \left( -\frac{\Lambda}{4B}(\Lambda - 4m) + Q \right), \quad Q = \frac{\Lambda}{4B} \sqrt{(\Lambda - 4m)^2 - 16B}.
\] (3.31)

As a result, (3.28) becomes
\[
4\pi i \frac{\partial (g_s N_1)}{\partial B} = -\frac{1}{\sqrt{-B(1 + p)}} \int_0^1 \frac{dt}{\sqrt{t(1 - t)(1 - k^2t)}} = -\frac{\pi}{\sqrt{-B(1 + p)}} F(\frac{1}{2}, \frac{1}{2}; 1; k^2).
\] (3.32)

By expanding the hypergeometric function and then integrating over \( B \), we obtain
\[
2g_s N_1 = \sqrt{B} \left( 1 + \frac{m\Lambda}{4B} - \frac{1}{64B^2} (B + 3m^2) \Lambda^2 + \frac{m}{256B^3} (5m^2 + B) \Lambda^3 
- \frac{1}{16384B^4} (3B^2 + 30m^2B + 175m^4) \Lambda^4 - \frac{m}{65536B^5} (9B^2 + 70m^2B + 441m^4) \Lambda^5 
- \frac{1}{1048576B^6} (5B^3 + 105m^2B^2 + 735m^4B + 4851m^6) \Lambda^6 + O(\Lambda^7) \right).
\] (3.33)

We invert this equation for \( B \),
\[
B = a^2 - \frac{m\Lambda}{2} - \frac{m^2 + a^2}{32a^2} \Lambda^2 + \frac{a^4 - 6a^2m^2 + 5m^4}{8192a^6} \Lambda^4 + \frac{m}{16384a^8} (9a^4 + 70m^2a^2 + 441m^4) \Lambda^5 
+ \frac{m^2}{262144a^{10}} (185a^4 + 1946m^2a^2 + 15885m^4) \Lambda^6 + O(\Lambda^7),
\] (3.34)
where we have defined $a = 2g_sN_1$. Finally, by substituting this into (3.27), we obtain
\[
4F_m = (a^2 - m^2) \log \Lambda + \frac{m \Lambda}{2} + \frac{m^2 + a^2}{24a^2} \Lambda^2 + \frac{a^4 - 6m^2a^2 + 5m^4}{2^{15}a^6} \Lambda^4
+ \frac{m}{2^{14} \times 5} \left( \frac{9a^4 + 70m^2a^2 + 441m^4}{a^8} \right) \Lambda^5 + \frac{m^2}{2^{19} \times 3} \left( \frac{185a^4 + 1946m^2a^2 + 15885m^4}{a^{10}} \right) \Lambda^6 + \mathcal{O}(\Lambda^7).
\]

(3.35)

Term with $\log \Lambda$ is the one-loop contribution. Remaining terms agree precisely with the prepotential obtained from the Nekrasov partition function (A.13).

### 3.3 SU(2) gauge theory with $N_f = 4$

We now consider the matrix model with the original action (2.5). The planar loop equation
\[
R(z) = -\frac{1}{z} \left( W(z) - \sqrt{W(z)^2 + f(z)} \right)
\]
involves a function $f(z)$ which now has a form $f(z) = \sum_{i=0}^{2} c_i \frac{1}{z-q_i}$. Parameters $\{c_i\}$ are given by
\[
c_0 = -4g_s m_0 \left( \sum_{i} \frac{1}{\lambda_i} \right), \quad c_1 = -4g_s m_1 \left( \sum_{i} \frac{1}{\lambda_i - 1} \right), \quad c_2 = -4g_s m_2 \left( \sum_{i} \frac{1}{\lambda_i - q} \right).
\]

(3.36)

By studying the behavior of loop equation at large $z$ we find that the parameters obey
\[
\sum_{i=0}^{2} c_i = 0, \quad c_1 + qc_2 = m_2^2 - (\sum_{i=0}^{2} m_i)^2.
\]

(3.37)

By eliminating $c_1$ and $c_2$, the spectral curve becomes
\[
x^2 = \frac{P_4(z)}{z^2(z-1)^2(z-q)^2},
\]

(3.38)

where $P_4$ is the following polynomial of degree 4
\[
P_4(z) = m_\infty^2 z^4 + \left( -(1 + q)(m_\infty^2 + m_0^2) + (1 - q)(m_1^2 - m_2^2) - 2m_0(qm_1 + m_2) + qc_0 \right) z^3
+ \left( qm_\infty^2 + (1 + 3q + q^2)m_0^2 + (1 - q)(m_2^2 - qm_1^2) + 2(1 + q)m_0(qm_1 + m_2)
- (1 + q)qc_0 \right) z^2 + \left( -2q(1 + q)m_0^2 - 2q^2m_0m_1 - 2qm_0m_2 + q^2c_0 \right) z + q^2m_0^2.
\]

(3.39)

The meromorphic one form $xdz$ has simple poles at $z = 0, 1, q$ and $z = \infty$ with residues $m_0, m_1, m_2$ and $m_\infty$.

Again, we consider the derivative of the free energy:
\[
\frac{\partial F_m}{\partial q} = g_s m_2 \langle \frac{1}{q - M} \rangle = m_2 R(z)|_{z=q}.
\]

(3.40)
This can be easily computed by expanding the resolvent at \( z = q \), \( R(z) = \frac{z^2}{4m_2} + \mathcal{O}(z - q) \). Then, we obtain a simple expression for the free energy

\[
\frac{\partial F_m}{\partial q} = \frac{c_2}{4} - \frac{1}{4(1 - q)} \left( \sum_{i=0}^{2} m_i^2 - m_\infty^2 - c_0 \right).
\]

(3.41)

In the last equality we used the relation (3.37).

In what follows, we consider the simple case where all the hypermultiplet masses are equal to \( m \): i.e. \( m_0 = m_\infty = 0 \) and \( m_1 = m_2 = m \). In this case, the polynomial is reduced to degree 3: \( P_3(z) = C z(z - z_+)(z - z_-) \), where we have introduced \( C \equiv c_0 q \) and

\[
z_{\pm} = \frac{1}{2} \left( 1 + q - (1 - q)^2 \frac{m^2}{C} \pm (1 - q) \sqrt{1 - 2(1 + q) \frac{m^2}{C} + (1 - q)^2 \frac{m^4}{C^2}} \right).
\]

(3.42)

By taking the \( C \) derivative of \( x dz \), the holomorphic one form becomes

\[
\frac{\partial}{\partial C} x dz = \frac{1}{2\sqrt{C}z_+} \frac{dz}{\sqrt{z(1 - z)(1 - k^2 z)}}, \quad k^2 = \frac{q^2}{C}.
\]

(3.43)

The remaining calculation is similar to those considered in the previous subsections. That is, we first evaluate the period integral of the above one form. Then by expanding in \( \frac{m^2}{C} \) and integrating over \( C \), we obtain

\[
2ig_s N_1 = \sqrt{C} \left( h_0(q) - h_1(q) \frac{m^2}{C} - h_2(q) \frac{m^4}{3C^2} - h_3(q) \frac{m^6}{5C^3} + \mathcal{O}(\frac{m^8}{C^4}) \right),
\]

(3.44)

where \( h_i(q) \) are the expansion coefficients of the period integral in \( \frac{m^2}{C} \) and depend only on \( q \). \( h_0(q) \) is for the theory with massless flavors:

\[
h_0(q) = 1 + \frac{1}{4} q + \frac{9}{64} q^2 + \frac{25}{256} q^3 + \frac{1225}{16384} q^4 + \mathcal{O}(q^5).
\]

(3.45)

Lower order expansions of \( h_1, h_2 \) and \( h_3 \) are given by

\[
\begin{align*}
    h_1(q) &= \frac{1}{2} + \frac{1}{8} q + \frac{1125}{576} q^2 + \frac{1225}{16384} q^3 + \mathcal{O}(q^5), \\
    h_2(q) &= \frac{3}{8} q + \frac{1}{32} q^2 + \frac{3}{2048} q^3 + \frac{27}{131072} q^4 + \mathcal{O}(q^5), \\
    h_3(q) &= \frac{5}{16} q + \frac{125}{1024} q^2 + \frac{125}{4096} q^3 + \mathcal{O}(q^5).
\end{align*}
\]

(3.46)

Solving for \( C \), we obtain

\[
C = a^2 \left( \frac{1}{h_0(q)^2} + \frac{2h_1(q) m^2}{h_0(q)} a^2 + \frac{2h_0(q) h_2(q) - 3h_1(q)^2 m^4}{3 a^4} \right.

\[
+ \left. \frac{10h_0(q) h_1(q)^3 - 10h_0(q)^2 h_1(q) h_2(q) + 2h_0(q)^3 h_3(q) m^6}{5 a^6} + \ldots \right),
\]

(3.47)
where \( a = 2ig_sN_1 \). By substituting the above expression into (3.41) and integrating over \( q \), we finally obtain the \( N_f = 4 \) free energy

\[
4F_m = (a^2 - m^2) \log q + \frac{a^4 + 6a^2m^2 + m^4}{2a^2} q + \frac{13a^8 + 100m^2a^6 + 22m^4a^4 - 12m^6a^2 + 5m^8}{64a^6} q^2 \\
+ \frac{23a^{12} + 204m^2a^{10} + 51m^4a^8 - 48m^6a^6 + 45m^8a^4 - 28m^{10}a^2 + 9m^{12}}{192a^{10}} q^3 + \mathcal{O}(q^4). \tag{3.48}
\]

This perfectly agrees with the instanton partition function (A.9).

Finally, we make a brief comment on the massless theory. In this case, the expression for \( C \) simplifies and becomes

\[
C = \frac{a^2}{h_0^2(q)} \tag{3.45}
\]

Thus, it is easy to derive

\[
4F_m = a^2 \left( \log q - \log 16 + \frac{1}{2} q + \frac{13}{64} q^2 + \frac{23}{192} q^3 + \frac{2701}{32768} q^4 + \frac{5057}{81920} q^5 + \mathcal{O}(q^6) \right), \tag{3.49}
\]

where we have added the one-loop contribution \(-a^2 \log 16\). Note that this can be written as

\[
4F_m = a^2 \log q_{\text{IR}} \quad \text{where} \quad q_{\text{IR}} = e^{2\pi i \tau_{\text{IR}}}
\]

as already discussed in [25, 26, 1, 27, 7, 12]. Thus the theory appears classical in terms of IR coupling constant \( \tau_{\text{IR}} \).

### 4 Matrix model and Quiver gauge theories

In this section, we study matrix models which describe \( N = 2 \) SU(2) quiver gauge theories. First of all, we consider a matrix model describing SU(2) linear quiver gauge theory where each gauge group has a vanishing beta function [28]. Then by taking its decoupling limit, we propose models for asymptotically free gauge theories in subsection 4.1.

According to the AGT conjecture, SU(2)\(^n-3\) linear quiver gauge theory is related to the \( n \)-point conformal block of the Liouville theory, which is represented by the trivalent graph [29] as in Fig 1. As seen in section 2, the Dotsenko-Fateev representation of the conformal block suggests a matrix model with the following action [11]:

\[
W(M) = \sum_{i=0}^{n-2} m_i \log(M - t_i), \tag{4.1}
\]

where \( t_0 = 0 \) and \( t_1 = 1 \). Other parameters \( t_i = \prod_{k=1}^{i-1} q_k \) \((i = 2, \ldots, n - 2)\) describe complex structure of the \( n \)-punctured sphere. Note that we also have the prefactor as in (2.4).

From the gauge theory perspective, the parameters \( q_k \) are related to the gauge coupling constants \( q_k = e^{2\pi i \tau_k} \) of the gauge group SU(2)\(^n-3\). For \( n = 4 \), this reduces to the matrix
model which we studied in subsection 3.3. Parameters $m_0$ and $m_{n-2}$ are related to the mass parameters of two hypermultiplets in the fundamental representation of the $SU(2)$ at one end of the quiver: $m_{n-2} = \frac{1}{2} (\mu_3 + \mu_4)$ and $m_0 = \frac{1}{2} (\mu_3 - \mu_4)$. Also, the masses of the bifundamental hypermultiplets are identified with $m_i$ ($i = 2, \ldots, n-3$). Finally, the masses of two fundamental hypermultiplets of the other end of the quiver are related to $m_1$ and $m_\infty$ as $m_1 = \frac{1}{2} (\mu_1 + \mu_2)$ and $m_\infty = \frac{1}{2} (\mu_1 - \mu_2)$. The mass parameter $m_\infty$ is introduced by the following condition:

$$\sum_{i=0}^{n-2} m_i + m_\infty + 2g_s N = 0. \quad (4.2)$$

The critical points are determined by the equation of motion

$$\sum_{i=0}^{n-2} \frac{m_i}{\lambda_I - t_i} + 2g_s \sum_{j \neq t} \frac{1}{\lambda_I - \lambda_J} = 0. \quad (4.3)$$

If we ignore the second term, we obtain $n-2$ critical points $e_p$ ($p = 1, 2, \ldots, n-2$). Let each $N_p$ ($p = 1, 2, \ldots, n-2$) be the number of the matrix eigenvalues which are at the critical point $e_p$.

We take the large $N$ limit with mass parameters $\{m_i\}$ and filling fractions $\{\nu_p \equiv g_s N_p\}$ being kept fixed. Since this is one matrix model, the loop equation is still the same as in the previous cases (3.2)

$$f(z) \equiv 4g_s \text{tr} \left( \frac{W'(z) - W'(M)}{z - M} \right) = \sum_{i=0}^{n-2} c_i \frac{z - t_i}{z - t_i} \equiv \frac{Z(t)}{\prod_{i=0}^{n-2} (z - t_i)}. \quad (4.4)$$

We note that a polynomial $Z(t)$ is of degree $n-3$, since the leading term vanishes due to equations of motion.

Finally, we define the meromorphic one form $\lambda = x(z) dz$ as

$$x(z)^2 \equiv (2\langle R(z) \rangle + W'(z))^2 = W'(z)^2 + f(z). \quad (4.5)$$
4.1 Matrix model for asymptotically free quiver gauge theory

The matrix model corresponding to asymptotically free quiver gauge theory can be obtained by taking the decoupling limit as in section 2. Only possible limits which does not spoil the condition (4.2) is the case where $\mu_2(= m_1 - m_\infty)$ or $\mu_4(= m_{n-2} - m_0)$ is taken to infinity.

For the sake of illustration, let us consider the $n=5$ case with the action

$$W(z) = \sum_{i=0}^{3} m_i \log(z - t_i), \quad (4.6)$$

where $t_2 = q_1$ and $t_3 = q_1 q_2$. This corresponds to $SU(2)_1 \times SU(2)_2$ quiver gauge theory whose gauge coupling constants are $q_1$ and $q_2$. We first take a limit $\mu_4 \to \infty$ with $\mu_4 q_2 = \tilde{\Lambda}$ fixed. In this limit, we obtain

$$W(z) \to \mu_3 \log z + \sum_{i=1,2} m_i \log(z - t_i) - \frac{q_1 \tilde{\Lambda}}{2z}, \quad (4.7)$$

It is natural to anticipate that this matrix model corresponds to the quiver theory of one fundamental matter coupled to the second gauge group $SU(2)_2$ and two fundamental multiplets are coupled to the first gauge group. The relation of the mass parameters (4.2) becomes $\mu_3 + \sum_{i=1,2} m_i + m_\infty + 2g_s N = 0$ in this limit.

By further taking the limit $\mu_2 \to \infty$ with $\mu_2 q_1 = \Lambda$ fixed, we obtain from (4.7)

$$W(z) \to \mu_3 \log z + m_2 \log(z - 1) - \frac{\Lambda z}{2} - \frac{\tilde{\Lambda}}{2z}, \quad (4.8)$$

where we have also rescaled $z \to q_1 z$. The relation of the mass parameters (4.2) becomes

$$\mu_3 + m_2 + \mu_1 + 2g_s N = 0. \quad (4.9)$$

This matrix model is expected to describe $SU(2)_1 \times SU(2)_2$ quiver gauge theory with each gauge factor coupled to one hypermultiplet. Both of the gauge factors have nonvanishing beta functions and the theory is asymptotically free.

It is possible to generalize this construction to the case with $n > 5$. A decoupling limit of a hypermultiplet at the last end of the quiver is $\mu_4 \to \infty$ with $\mu_4 q_{n-3} = \tilde{\Lambda}$ fixed. Also, another decoupling limit of a hypermultiplet at the first end of the quiver is $\mu_2 \to \infty$ with $\mu_2 q_1 = \Lambda$ fixed. By taking these limits, we finally obtain

$$W(z) = \mu_3 \log z + m_2 \log(z - 1) + \sum_{i=3}^{n-3} m_i \log \left( z - \prod_{k=2}^{i-1} q_k \right) - \frac{\Lambda z}{2} - \frac{\tilde{\Lambda}}{2z} \left( \prod_{k=2}^{n-4} q_k \right), \quad (4.10)$$

with the following relation for the mass parameters:

$$\mu_3 + \sum_{i=2}^{n-3} m_i + \mu_1 + 2g_s N = 0. \quad (4.11)$$
5 Conclusion and discussion

In this paper we have studied the matrix model proposed to explain the AGT relation and interpolate the Liouville and $\mathcal{N} = 2$ $SU(2)$ gauge theories. We have explicitly evaluated the free energy of the matrix models describing $SU(2)$ gauge theory with $N_f = 2, 3, 4$ flavors and have shown that they in fact reproduce the amplitudes of Seiberg-Witten theory. Our analysis is limited to the large $N$ limit and it is very important to see if our results can be generalized and reproduce full Nekrasov partition functions. There is already an interesting work in this direction [30, 31] and we hope that we can report further results in future publications.

Acknowledgements

K.M. would like to thank K. Hosomichi, H. Itoyama and F. Yagi for discussions and comments. He also would like to thank Ecole normale Superieure, SISSA and University of Amsterdam for warm hospitality during part of this project. Research of T.E. is supported in part by the project 19GS0219 of the Japan Ministry of Education, Culture, Sports, Science and Technology. Research of K.M. is supported in part by JSPS Bilateral Joint Projects (JSPS-RFBR collaboration).

Appendix

A Nekrasov partition function

The instanton partition function of $\mathcal{N} = 2$ $U(2)$ gauge theory with $N_f = 4$ is expressed as a sum over all possible Young tableaus parametrized as $Y = (\lambda_1 \geq \lambda_2 \geq \ldots)$ where $\lambda_\ell$ is the height of the $\ell$-th column [24, 1]:

$$Z_{\text{inst}} = \sum_{(Y_1,Y_2)} q^{|\vec{Y}|} Z_{\text{vector}}(\vec{a}, \vec{Y})Z_{\text{antifund}}(\vec{a}, \vec{Y}, \mu_1)Z_{\text{antifund}}(\vec{a}, \vec{Y}, \mu_2)Z_{\text{fund}}(\vec{a}, \vec{Y}, -\mu_3)Z_{\text{fund}}(\vec{a}, \vec{Y}, -\mu_4).$$

(A.1)
Here
\[
Z_{\text{vector}}(\vec{a}, \vec{Y}) = \prod_{i,j=1,2} \prod_{s \in Y_i} (a_{ij} - \epsilon_1 L_{Y_j}(s) + \epsilon_2 (A_{Y_i}(s) + 1))^{-1} \times \prod_{t \in Y_j} (a_{ji} + \epsilon_1 L_{Y_j}(t) - \epsilon_2 (A_{Y_i}(t) + 1) + \epsilon_+)^{-1},
\]
\[
Z_{\text{fund}}(\vec{a}, \vec{Y}, \mu) = \prod_{i=1,2} \prod_{s \in Y_i} (a_i + \epsilon_1 (\ell - 1) + \epsilon_2 (m - 1) - \mu + \epsilon_+),
\]
\[
Z_{\text{antifund}}(\vec{a}, \vec{Y}, \mu) = \prod_{i=1,2} \prod_{s \in Y_i} (a_i + \epsilon_1 (\ell - 1) + \epsilon_2 (m - 1) + \mu),
\]
\[
(A.2)
\]
and \( \epsilon_+ = \epsilon_1 + \epsilon_2 \) and \( a_{ij} = a_i - a_j \). For a box \( s \) at the coordinate \((\ell, m)\), the leg-length \( L_Y(s) = \lambda'_m - \ell \) and the arm-length \( A_Y(s) = \lambda_e - m \) where \( \lambda'_m \) is the length of the \( m \)-th row. The minus signs of the masses in \( Z_{\text{fund}} \) are due to the convention.

In order to derive the expression for \( SU(2) \) gauge theory, we set the Coulomb moduli as \( \vec{a} = (a, -a) \) which gives
\[
Z_{\text{vector}}(a, \vec{Y}) = \prod_{i=1,2} \prod_{s \in Y_i} (2a \delta_{ij} - \epsilon_1 L_{Y_j}(s) + \epsilon_2 (A_{Y_i}(s) + 1))^{-1} \times \prod_{t \in Y_j} (-2a \delta_{ij} + \epsilon_1 L_{Y_j}(t) - \epsilon_2 (A_{Y_i}(t) + 1) + \epsilon_+)^{-1},
\]
\[
Z_{\text{fund}}(a, \vec{Y}, \mu) = \prod_{i=1,2} \prod_{s \in Y_i} (a \delta_i + \epsilon_1 (\ell - 1) + \epsilon_2 (m - 1) - \mu + \epsilon_+),
\]
\[
Z_{\text{antifund}}(a, \vec{Y}, \mu) = \prod_{i=1,2} \prod_{s \in Y_i} (a \delta_i + \epsilon_1 (\ell - 1) + \epsilon_2 (m - 1) + \mu),
\]
\[
(A.3)
\]
where we define \( \delta_1 = +1 \) and \( \delta_2 = -1 \), and
\[
\delta_{ij} = \begin{cases} 
0 & \text{for } i = j, \\
1 & \text{for } i = 1 \text{ and } j = 2, \\
-1 & \text{for } i = 2 \text{ and } j = 1.
\end{cases}
\]
\[
(A.4)
\]
Then, the \( SU(2) \) and \( U(2) \) partition functions are related by the \( U(1) \) factor as pointed out in [1]:
\[
Z_{\text{inst}}|_{\vec{a} = (a,-a)} = f^{SU(2)} Z_{\text{inst}}^{SU(2)}, \quad f^{U(1)} = (1 - q)^{\frac{\mu_1 + \mu_2}{2} (\epsilon_+ + \epsilon_+)}. 
\]
\[
(A.5)
\]
Note that this expression differs by a minus sign in front of \((\mu_3 + \mu_4)\) from the one of [1]. As argued in [1], the \( SU(2) \) partition function is invariant under “flips”. These flips are reduced in the self-dual case \( \epsilon_1 = -\epsilon_2 = h \) to
\[
a \to -a, \quad \mu_1 \pm \mu_2 \to -(\mu_1 \pm \mu_2), \quad \mu_3 \pm \mu_4 \to -(\mu_3 \pm \mu_4).
\]
\[
(A.6)
\]
Gauge theory prepotential can be obtained in the limit where the deformation parameters go to zero (with a fixed ratio $\epsilon_1/\epsilon_2$):

$$F_{\text{inst}} = \lim_{\epsilon_1, \epsilon_2 \to 0} (-\epsilon_1 \epsilon_2) \log Z_{\text{inst}}.$$ \hspace{1cm} (A.7)

In the self-dual case, $SU(2)$ gauge theory prepotential is written as

$$F_{\text{inst}}^{SU(2)} = \lim_{\hbar \to 0} \hbar^2 (\log Z_{\text{inst}} - \log f^{U(1)}_{\text{inst}})$$

$$= F_{\text{inst}} + \frac{1}{2} (\mu_1 + \mu_2) (\mu_3 + \mu_4) \log(1 - q).$$ \hspace{1cm} (A.8)

To compare with the free energy of the matrix model, we present an expansion of $F_{\text{inst}}$ for the equal mass case $\mu_i = m$

$$F_{\text{inst}} = \frac{a^4 + 6m^2a^2 + m^4}{2a^2} q + \frac{13a^8 + 100m^2a^6 + 22m^4a^4 - 12m^6a^2 + 5m^8}{64a^6} q^2$$

$$+ \frac{23a^{12} + 204m^2a^{10} + 51m^4a^8 - 48m^6a^6 + 45m^8a^4 - 28m^{10}a^2 + 9m^{12}}{192a^{10}} q^3 + \mathcal{O}(q^4).$$ \hspace{1cm} (A.9)

A.1 $U(2)$ gauge theory with $N_f = 3$

Let us consider Nekrasov partition function of the theory with $N_f = 3$. This can be obtained from the above partition function by taking a limit $\mu_4 \to \infty$ with $\mu_4q \equiv \Lambda_3$ fixed. In the $k$-instanton part the only factor which contains $\mu_4$ is

$$Z_{\text{fund}}(a, \bar{Y}, -\mu_4) = \prod_{i=1,2} \prod_{s \in Y_i} (a\delta_i + \epsilon_1(\ell - 1) + \epsilon_2(m - 1) + \mu_4 + \epsilon_+).$$ \hspace{1cm} (A.10)

When combined with $k$-instanton factor $q^k$, this gives the leading contribution $\Lambda_3^k$ and the other contributions are suppressed in the limit. Therefore, we obtain

$$Z_{\text{inst}}^{N_f=3} = \sum_{(Y_1,Y_2)} \Lambda_3^{\vert Y \vert} Z_{\text{vector}}(a, \bar{Y}) Z_{\text{antifund}}(a, \bar{Y}, \mu_1) Z_{\text{antifund}}(a, \bar{Y}, \mu_2) Z_{\text{fund}}(a, \bar{Y}, -\mu_3).$$ \hspace{1cm} (A.11)

The $U(1)$ factor reduces to

$$f^{U(1)} \to f^{U(1),N_f=3} = \exp \left( -\frac{(\mu_1 + \mu_2)\Lambda_3}{2\epsilon_1 \epsilon_2} \right).$$ \hspace{1cm} (A.12)

In the simple case of $\mu_3 = m$ and $\mu_1 = \mu_2 = 0$ which we considered in subsection 3.2 the prepotential of the gauge theory is given by

$$F_{\text{inst}}^{N_f=3} = \frac{1}{2} m\Lambda_3 + \frac{a^2 + m^2}{64a^2} \Lambda_3^2 + \frac{a^4 - 6m^2a^2 + 5m^4}{2^{15}a^6} \Lambda_3^4 + \mathcal{O}(\Lambda_3^5).$$ \hspace{1cm} (A.13)
A.2 $U(2)$ gauge theory with $N_f = 2$

We can further take a limit where $\mu_2 \to \infty$ while keeping $\mu_2 \Lambda_3 \equiv \Lambda_2^2$ fixed. In this limit, the partition function becomes:

$$Z^\text{inst}_{N_f=2} = \sum_{(Y_1, Y_2)} \Lambda_2^{2|Y|} Z\text{vector}(a, \vec{Y}) Z\text{antifund}(a, \vec{Y}, \mu_1) Z\text{fund}(a, \vec{Y}, -\mu_3), \quad (A.14)$$

and the $U(1)$ factor is reduced to $f^{U(1)} \to \exp \left(-\frac{\Lambda_2^2}{2\epsilon_1\epsilon_2}\right)$. $SU(2)$ prepotential is given by

$$F^{SU(2), N_f=2}_\text{inst} = F^{N_f=2}_\text{inst} - \frac{\Lambda_2^2}{2}. \quad (A.15)$$

For the equal mass case with $\mu_1 = \mu_3 = m$, lower terms of instanton expansion are given by

$$F^{N_f=2}_\text{inst} = \frac{a^2 + m^2}{2a^2} \Lambda_2^2 + \frac{a^4 - 6a^2m^2 + 5m^4}{64a^6} \Lambda_2^4 + \frac{m^2(5a^4 - 14a^2m^2 + 9m^4)}{192a^{10}} \Lambda_2^6 + O(\Lambda_2^8). \quad (A.16)$$

References

[1] L. F. Alday, D. Gaiotto and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” Lett. Math. Phys. 91, 167 (2010) [arXiv:0906.3219 [hep-th]].

[2] A. Mironov, A. Morozov and S. Shakirov, “Matrix Model Conjecture for Exact BS Periods and Nekrasov Functions,” JHEP 1002, 030 (2010) [arXiv:0911.5721 [hep-th]].

[3] A. Mironov, A. Morozov and S. Shakirov, “Conformal blocks as Dotsenko-Fateev Integral Discriminants,” arXiv:1001.0563 [hep-th].

[4] H. Itoyama and T. Oota, “Method of Generating q-Expansion Coefficients for Conformal Block and N=2 Nekrasov Function by beta-Deformed Matrix Model,” arXiv:1003.2929 [hep-th].

[5] A. Mironov, A. Morozov and A. Morozov, “Matrix model version of AGT conjecture and generalized Selberg integrals,” arXiv:1003.5752 [hep-th].

[6] A. Morozov and S. Shakirov, “The matrix model version of AGT conjecture and CIV-DV prepotential,” arXiv:1004.2917 [hep-th].

[7] R. Poghossian, “Recursion relations in CFT and N=2 SYM theory,” JHEP 0912, 038 (2009) [arXiv:0909.3412 [hep-th]].
[8] L. Hadasz, Z. Jaskolski and P. Suchanek, “Recursive representation of the torus 1-point conformal block,” arXiv:0911.2353 [hep-th].

[9] V. A. Fateev and A. V. Litvinov, “On AGT conjecture,” JHEP 1002, 014 (2010) arXiv:0912.0504 [hep-th].

[10] L. Hadasz, Z. Jaskolski and P. Suchanek, “Proving the AGT relation for $N_f = 0, 1, 2$ antifundamentals,” arXiv:1004.1841 [hep-th].

[11] R. Dijkgraaf and C. Vafa, “Toda Theories, Matrix Models, Topological Strings, and N=2 Gauge Systems,” arXiv:0909.2453 [hep-th].

[12] T. Eguchi and K. Maruyoshi, “Penner Type Matrix Model and Seiberg-Witten Theory,” JHEP 1002, 022 (2010) arXiv:0911.4797 [hep-th].

[13] H. Itoyama, K. Maruyoshi and T. Oota, “Notes on the Quiver Matrix Model and 2d-4d Conformal Connection,” arXiv:0911.4244 [hep-th].

[14] R. Schiappa and N. Wyllard, “An $A_r$ threesome: Matrix models, 2d CFTs and 4d N=2 gauge theories,” arXiv:0911.5337 [hep-th].

[15] V. S. Dotsenko and V. A. Fateev, “Conformal algebra and multipoint correlation functions in 2D statistical models,” Nucl. Phys. B 240, 312 (1984);
V. S. Dotsenko and V. A. Fateev, “Four Point Correlation Functions And The Operator Algebra In The Two-Dimensional Conformal Invariant Theories With The Central Charge C<1,” Nucl. Phys. B 251, 691 (1985).

[16] A. Marshakov, A. Mironov and A. Morozov, “Generalized matrix models as conformal field theories: Discrete case,” Phys. Lett. B 265, 99 (1991).

[17] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and S. Pakuliak, “Conformal Matrix Models As An Alternative To Conventional Multimatrix Models,” Nucl. Phys. B 404, 717 (1993) arXiv:hep-th/9208044.

[18] M. Matone, “Instantons and recursion relations in N=2 SUSY gauge theory,” Phys. Lett. B 357, 342 (1995) arXiv:hep-th/9506102.

[19] J. Sonnenschein, S. Theisen and S. Yankielowicz, “On the Relation Between the Holomorphic Prepotential and the Quantum Moduli in SUSY Gauge Theories,” Phys. Lett. B 367, 145 (1996) arXiv:hep-th/9510129.
[20] T. Eguchi and S. K. Yang, “Prepotentials of $N = 2$ Supersymmetric Gauge Theories and Soliton Equations,” Mod. Phys. Lett. A 11, 131 (1996) [arXiv:hep-th/9510183].

[21] A. Klemm, W. Lerche and S. Theisen, “Nonperturbative effective actions of N=2 supersymmetric gauge theories,” Int. J. Mod. Phys. A 11, 1929 (1996) [arXiv:hep-th/9505150].

[22] K. Ito and S. K. Yang, “Prepotentials in N=2 SU(2) supersymmetric Yang-Mills theory with massless hypermultiplets,” Phys. Lett. B 366, 165 (1996) [arXiv:hep-th/9507144].

[23] D. Gaiotto, G. W. Moore and A. Neitzke, “Wall-crossing, Hitchin Systems, and the WKB Approximation,” [arXiv:0907.3987 [hep-th]].

[24] N. A. Nekrasov, “Seiberg-Witten Prepotential From Instanton Counting,” Adv. Theor. Math. Phys. 7, 831 (2004) [arXiv:hep-th/0206161].

[25] N. Dorey, V. V. Khoze and M. P. Mattis, “On $N = 2$ supersymmetric QCD with 4 flavors,” Nucl. Phys. B 492, 607 (1997) [arXiv:hep-th/9611016].

[26] T. W. Grimm, A. Klemm, M. Marino and M. Weiss, “Direct integration of the topological string,” JHEP 0708, 058 (2007) [arXiv:hep-th/0702187].

[27] A. Marshakov, A. Mironov and A. Morozov, “Zamolodchikov asymptotic formula and instanton expansion in $\mathcal{N}=2$ SUSY $\mathcal{N}=2$ QCD,” [arXiv:0909.3338 [hep-th]].

[28] D. Gaiotto, “N=2 dualities,” [arXiv:0904.2715 [hep-th]].

[29] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, “Infinite conformal symmetry in two-dimensional quantum field theory,” Nucl. Phys. B 241, 333 (1984).

[30] M. Fujita, Y. Hatsuda and T. S. Tai, “Genus-one correction to asymptotically free Seiberg-Witten prepotential from Dijkgraaf-Vafa matrix model,” JHEP 1003, 046 (2010) [arXiv:0912.2988 [hep-th]].

[31] C. Kozcaz, S. Pasquetti and N. Wyllard, “A & B model approaches to surface operators and Toda theories,” [arXiv:1004.2025 [hep-th]].