ON A CLASS OF CERTAIN NON-UNIVALENT FUNCTIONS

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Abstract. In this paper, we introduce a family of analytic functions given by

\[ \psi_{A,B}(z) := \frac{1}{A - B} \log \frac{1 + Az}{1 + Bz}, \]

which maps univalently the unit disk onto either elliptical or strip domains, where either \( A = -B = \alpha \) or \( A = \alpha e^{i\gamma} \) and \( B = \alpha e^{-i\gamma} \) (\( \alpha \in (0, 1] \) and \( \gamma \in (0, \pi/2] \)). We study a class of non-univalent analytic functions defined by

\[ \mathcal{F}[A,B] := \left\{ f \in \mathcal{A} : \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \psi_{A,B}(z) \right\}. \]

Further, we investigate various characteristic properties of \( \psi_{A,B}(z) \) as well as functions in the class \( \mathcal{F}[A,B] \) and obtain the sharp radius of starlikeness of order \( \delta \) and univalence for the functions in \( \mathcal{F}[A,B] \). Also, we find the sharp radii for functions in \( \mathcal{BS}(\alpha) := \{ f \in \mathcal{A} : zf'(z)/f(z) - 1 \prec z/(1 - \alpha z^2) \}, \alpha \in (0,1) \}, \mathcal{S}_\alpha(\alpha) := \{ f \in \mathcal{A} : zf'(z)/f(z) - 1 \prec z/((1 - z)(1 + \alpha z)) \}, \alpha \in (0,1) \} \) and others to be in the class \( \mathcal{F}[A,B] \).

1. Introduction

Let \( \mathcal{A} \) denote the class of analytic functions \( f \) defined on the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \), such that \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \). The subclass of \( \mathcal{A} \) consisting of all univalent functions is denoted by \( \mathcal{S} \). For two analytic functions \( f \) and \( g \), we say that \( f \) is subordinate (\( \prec \)) to \( g \), if \( f(z) = g(\omega(z)) \) where \( \omega(z) \) is a Schwarz function. Let \( \mathcal{S}^* \) be the class of starlike functions, consists of functions \( f \in \mathcal{S} \) such that \( zf'(z)/f(z) \prec (1 + z)/(1 - z) \). In \([6]\), Ma and Minda unifies all the subclasses of \( \mathcal{S}^* \) as follows:

\[ \mathcal{S}^*(\Phi) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \Phi(z) \right\}, \]

where \( \Phi \) is univalent with \( \Phi'(0) > 0 \) satisfying

A. \( \Phi(z) \prec (1 + z)/(1 - z) \)
B. \( \Phi(\mathbb{D}) \) is symmetric about real axis and starlike with respect to \( \Phi(0) = 1 \).

Further, Kumar and Banga \([3]\), called such \( \Phi \) as \textit{Ma-Minda function} and the class of all Ma-Minda functions is denoted by \( \mathcal{M} \). In \([3]\), authors extensively studied Ma-Minda functions and classified them as \textit{non-Ma-Minda function of type-A} whenever condition A doesn’t hold, the class of all such functions is denoted by \( \mathcal{M}_A \). In past many authors studied the class \([1,1] \) for various choices of \( \Phi \in \mathcal{M} \), such as \( \mathcal{S}^*(2/(1 + e^{-z})) =: S_{SG}^* \), \( \mathcal{S}^*(e^z) =: S_e^* \), \( \mathcal{S}^*(z + \sqrt{1 + z^2}) =: S_{L}^* \), \( \mathcal{S}^*(\sqrt{1 + z}) =: S_L^* \), etc. Also the well known classes \( \mathcal{S}^*(A,B) \) and \( \mathcal{S}\mathcal{S}^*(\alpha) \) are also obtained by taking \( \Phi(z) (\in \mathcal{M}) \) as \((1 + Az)/(1 + Bz) \) (\(-1 \leq B < A \leq 1 \)) and \((1 + z)/(1 - z))^\alpha \) (\( \alpha \in (0,1] \)) respectively. In particular, \( \mathcal{S}^*(1,-1) =: \mathcal{S}^* \), \( \mathcal{S}^*(1 - 2\delta,-1) =: \mathcal{S}^*(\delta) (\delta \in [0,1]) \), the class of starlike functions of order \( \delta \). Robertson \([11]\) introduced and investigated the class of starlike functions.

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functions of order $\delta$, denoted by $S^{*}_{\delta}$ for $\delta \leq 1$. Note that if $\delta < 0$, then the functions in $S^{*}_{\delta}$ may not be univalent, i.e. if $\delta < 0$, then $\Phi(z) = (1 + (1 - 2\delta)z)/(1 - z) \in \widetilde{M}_A$ and $S^{*}_{\delta} \not\subseteq S^{*}$. Later many authors considered the classes associated with non-Ma-Minda functions of type-A such as the class by Urangelo [13]

$$\mathcal{M}(\beta) = \left\{ f \in \mathcal{A} : \frac{zf''(z)}{f'(z)} < 1 + (2\beta - 1)z, \beta > 1 \right\},$$

the classes associated with the Booth Lemniscate and Cissoid of Diocles

$$BS^{*}(\alpha) := \left\{ f \in \mathcal{A} : \left( \frac{zf'(z)}{f(z)} - 1 \right) < \frac{z}{1 - \alpha z^2}, \alpha \in [0, 1) \right\},$$

$$S^{*}_{cs}(\alpha) := \left\{ f \in \mathcal{A} : \left( \frac{zf'(z)}{f(z)} - 1 \right) < \frac{z}{(1 - z)(1 + \alpha z)}, \alpha \in [0, 1) \right\},$$

which were introduced by [2] and [7] respectively. Similarly, Masih and Kanas [8] studied the class

$$ST^{*}_{L}(s) := \left\{ f \in \mathcal{A} : zf'(z)/f(z) \not< \mathbb{L}_{s}(z) \right\}$$

associated with the Limaçon of Pascal $\mathbb{L}_{s}(z) = (1 + sz)^2, s \in [-1, 1] \setminus \{0\}$. Note that $\mathbb{L}_{s}(z) \in \mathcal{M}_A$, whenever $|s| > 1/\sqrt{2}$. Recently, Kumar and Gangania [5] studied the following class:

$$\mathcal{F}(\psi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} - 1 \not< \psi(z) \right\},$$

(1.2)

where $\psi(z) \in S^{*}$. If $1 + \psi(z) < (1 + z)/(1 - z)$ and $1 + \psi(\mathbb{D})$ is symmetric about the real axis with $\psi'(0) > 0$ then $\mathcal{F}(\psi)$ reduces to the class $S^{*}(1 + \psi)$, a Ma-Minda class and if $1 + \psi(z) \not< (1 + z)/(1 - z)$ then the functions in $\mathcal{F}(\psi)$ may not be univalent, thus $\mathcal{F}(\psi) \not\subseteq S^{*}$. Note that all above mentioned classes associated with non-Ma-Minda functions of type-A are the special cases of $\mathcal{F}(\psi)$ such as $\mathcal{F}((1 + (2\beta - 1)z)/(1 - z)) =: \mathcal{M}(\beta), \mathcal{F}(z/(1 - \alpha z^2)) =: BS^{*}(\alpha), \mathcal{F}(z/((1 - z)(1 + \alpha z))) =: S^{*}_{cs}(\alpha)$ and $\mathcal{F}((1 + sz)^2) =: \mathbb{L}_{s}(z)$.

Motivated by the above, we define a family of functions $\psi_{A,B}(z)$ as

$$\psi_{A,B}(z) := \frac{1}{A - B} \log \left( \frac{1 + Az}{1 + Bz} \right), \quad z \in \mathbb{D},$$

(1.3)

where either $A = -B = \alpha$ or $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$, for $0 < \alpha \leq 1$ and $0 < \gamma \leq \pi/2$. The function defined by (1.3) is analytic everywhere except the branchcuts $\{-\infty < \text{Re} Az \leq -1, \text{Im} Az = 0\} \cup \{-\infty < \text{Re} Bz \leq -1, \text{Im} Bz = 0\}$ with $\psi_{A,B}(0) = \psi'_{A,B}(0) - 1 = 0$, i.e. $\psi_{A,B}(z) \in \mathcal{A}$. Now we introduce the following classes associated with $\psi_{A,B}(z)$:

**Definition 1.1.** Let $p \in \mathcal{S}$. Then $p \in \mathcal{L}[A, B]$ if and only if

$$p(z) < \psi_{A,B}(z).$$

**Definition 1.2.** Let $f \in \mathcal{A}$. Then $f \in \mathcal{F}[A, B]$ if and only if

$$\frac{zf'(z)}{f(z)} - 1 < \psi_{A,B}(z).$$

(1.4)

In section 2 we investigate various characteristic properties of functions in the classes $\mathcal{L}[A, B]$ and $\mathcal{F}[A, B]$. We also obtain the extremal function of the class $\mathcal{F}[A, B]$ which is non-univalent and study its various geometric properties. Further, in section 3 we derive the sharp radius of starlikeness of order $\delta$, univalence for the functions in $\mathcal{F}[A, B]$ and also find the sharp radii for functions in $BS^{*}(\alpha), S^{*}_{cs}(\alpha)$ and others to be in the class $\mathcal{F}[A, B]$. 

2. Characteristics of $\mathcal{L}[A, B]$ and $\mathcal{F}[A, B]$

In this section, we aim to find various geometric properties of $\psi_{A,B}(z)$ and obtain certain bounds and inclusion relations for functions in the class $\mathcal{L}[A, B]$. Further, we deduce the extremal function and derive the growth and covering theorems for the class $\mathcal{F}[A, B]$. Note that

$$\psi_{A,B}(z) = \sum_{n=1}^{\infty} (-1)^{n+1} C_n z^n,$$

where

$$C_n = \frac{A^n - B^n}{n(A - B)} = \frac{1}{n} \sum_{k=0}^{n-1} A^k B^{n-1-k} \quad (n = 1, 2, \ldots). \quad (2.1)$$

**Remark 2.1.** From (2.1) we have

(i) Since $C_n$ is real valued for all $n$, $\psi_{A,B}(D)$ is symmetric with respect to real axis.

(ii) $|C_n| = \left| \frac{1}{n} \sum_{k=0}^{n-1} A^k B^{n-1-k} \right| \leq \alpha^{n-1} \leq 1.$

**Lemma 2.2.** The function $\psi_{A,B}(z)$ is convex and univalent on $D$.

**Proof.** Let $H(z) := 1 + z \psi_A''(z)/\psi_A'(z)$. By [12, Corollary 3], it is enough to show that $\text{Re} H(z) > 0$. Since $|A| = |B| = \alpha \leq 1$, using basic calculation with [4, Theorem 1], we obtain

$$\text{Re} H(z) = \text{Re} \left( -1 + \frac{1}{1 + Az} + \frac{1}{1 + Bz} \right) > -1 + \frac{2}{1 - \alpha} > 0.$$ 

Thus the result holds.

**Theorem 2.3.** Let $p \in \mathcal{L}[A, B]$, then in the disk $D_r = \{ z \in \mathbb{C} : |z| \leq r < 1 \}$,

(i) For $A = -B = \alpha$, we have

$$|\text{Re} p(z)| \leq \frac{1}{2\alpha} \log \left( \frac{1 + \alpha r}{1 - \alpha r} \right) \quad (2.2)$$

$$|\text{Im} p(z)| \leq \frac{1}{2\alpha} \sin^{-1} \left( \frac{2\alpha r}{1 + \alpha^2 r^2} \right). \quad (2.3)$$

(ii) For $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$, we have

$$-\frac{1}{2\alpha \sin \gamma} (\eta - \tau) \leq \text{Re} p(z) \leq \frac{1}{2\alpha \sin \gamma} (\eta + \tau). \quad (2.4)$$

$$\frac{1}{2\alpha \sin \gamma} \log T_2 \leq \text{Im} p(z) \leq \frac{1}{2\alpha \sin \gamma} \log T_1. \quad (2.5)$$

where

$$\eta := \sin^{-1} \left( \frac{2\alpha \sin \gamma}{\sqrt{1 + \alpha^4 r^4 - 2\alpha^2 r^2 \cos 2\gamma}} \right),$$

$$\tau := \tan^{-1} \left( \frac{-\alpha^2 r^2 \sin 2\gamma}{1 - \alpha^2 r^2 \cos 2\gamma} \right)$$

and

$$T_j := \left( \sqrt{1 + \alpha^4 r^4 - 2\alpha^2 r^2 \cos 2\gamma} + (-1)^j 2\alpha \sin \gamma \right)^{-1} \quad (j = 1, 2).$$
Now we shall discuss the image domain of $\psi$ for $0 < \alpha < 1$, obtain \eqref{eq:2} and \eqref{eq:3}. Proof. Let $w$ be the Schwarz function such that $w(0) = 0$ and $|w(z)| \leq |z| = r < 1$ for all $z \in \mathbb{D}$. Define the function $F_w : \mathbb{D} \to \mathbb{C}$ such that $p(z) = \psi_{A,B}(w(z)) = \frac{1}{A-B} \log (F_w(z))$. We have 
\[ F_w(z) := \frac{1 + Aw(z)}{1 + Bw(z)}, \]
which maps unit disk into the disk
\[ F_w(\mathbb{D}) \subset \left\{ \zeta \in \mathbb{C} : \left| \zeta - \frac{1 - AB}{1 - |B|^2} \right| < \frac{|A - B|}{1 - |B|^2} \right\}. \]
As $|A| = \alpha \in (0,1]$, therefore origin $O \notin F_w(\mathbb{D})$ (for more details see \cite{4}). Now using \cite{4, Theorem 1} with $|z| = r < 1$, we have
\[ \frac{|1 - AB|^2 - |A - B|r}{1 - |B|^2} \leq |F_w(z)| \leq \frac{|1 - AB|^2 + |A - B|r}{1 - |B|^2} , \]
\[ \left| \arg F_w(z) - \tan^{-1} \frac{\text{Im}(AB)r^2}{\text{Re}(AB)r^2 - 1} \right| < \sin^{-1} \frac{|A - B|r}{1 - |AB|^2}. \]
Since $\log f(z) = \log|f(z)| + i\arg f(z)$ and $p(z) = \frac{1}{A-B} \log (F_w(z))$, thus for $A = -B = \alpha$ we obtain \eqref{eq:2} and \eqref{eq:3} and for $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$, we have \eqref{eq:4} and \eqref{eq:5}. \hfill \Box

For the sake of computational convenience, we shall make the following assumptions:
\[ h_1 := \frac{1}{2\alpha} \log \left( \frac{1 + \alpha}{1 - \alpha} \right), \quad h_2 := \frac{1}{2\alpha} \sin^{-1} \left( \frac{2\alpha}{1 + \alpha^2} \right), \]
\[ k_1 := \frac{1}{2\alpha \sin \gamma} \sin^{-1} \left( \frac{2\alpha \sin \gamma}{\sqrt{1 + \alpha^2 - 2\alpha^2 \cos 2\gamma}} \right), \]
\[ k_2 := \frac{1}{2\alpha \sin \gamma} \log \frac{\sqrt{1 + \alpha^4 - 2\alpha^2 \cos 2\gamma} + 2\alpha \sin \gamma}{1 - \alpha^2} \]
and
\[ k := \frac{1}{2\alpha \sin \gamma} \tan^{-1} \left( \frac{\alpha^2 \sin 2\gamma}{\alpha^2 \cos 2\gamma - 1} \right). \]

Now we shall discuss the image domain of $\psi_{A,B}$ for different cases of $\alpha$:

For $0 < \alpha < 1$, $\psi_{A,B}(\mathbb{D}) = \{u + iv \in \mathbb{C} : (u,v) \in \Omega_1\}$ where
\[ \Omega_1 := \begin{cases} \frac{u^2 + v^2}{h_1^2} < 1, & \text{when } A = -B = \alpha \\ \frac{(u-k)^2}{k_1^2} + \frac{v^2}{k_2^2} < 1, & \text{when } A = \alpha e^{i\gamma}, B = \alpha e^{-i\gamma} \text{ for } 0 < \gamma \leq \pi/2. \end{cases} \]
Moreover when $\alpha = 1$, the major axis $h_1$ and $k_2$ respectively, tends to infinity, hence $\psi_{A,B}(\mathbb{D}) = \{u + iv \in \mathbb{C} : (u,v) \in \Omega_2\}$ where $\Omega_2$ is the following strip domain
\[ \Omega_2 := \begin{cases} \frac{-\pi}{4} < v < \frac{\pi}{4}, & \text{when } A = -B = 1 \\ \frac{-\pi}{2 \sin \gamma} < u < \frac{\gamma}{2 \sin \gamma}, & \text{when } A = e^{i\gamma}, B = e^{-i\gamma} \text{ for } 0 < \gamma \leq \pi/2. \end{cases} \]

Remark 2.4. \hspace{1em} 
(i) $\max_{|z|=r} \text{Re} \psi_{A,B}(z) = \psi_{A,B}(r)$ and $\min_{|z|=r} \text{Re} \psi_{A,B}(z) = \psi_{A,B}(-r)$.
(ii) If $\alpha < 1$, then $p < \psi_{A,B}(z)$ if and only if $p(\mathbb{D}) \subseteq \Omega_1$ and if $\alpha = 1$, then $p < \psi_{A,B}(z)$ if and only if $p(\mathbb{D}) \subseteq \Omega_2$, where $\Omega_1$ and $\Omega_2$ are as defined in \eqref{eq:6} and \eqref{eq:7} respectively.
(iii) $1 + \psi_{A,B} \in \mathcal{M}_A$. 

Now in the following lemma, we aim to find the radius of the smallest (largest) disk centered at the point \((1, 0)\) that can subscribe (inscribe) the domain \(1 + \psi_{A,B}(\mathbb{D})\).

**Lemma 2.5.** Let \(\alpha \in (0, 1)\) and \(\gamma \in (0, \pi/2]\), then we have

(i) for \(A = -B = \alpha\),
\[
\{ w \in \mathbb{C} : |w - 1| < h_2 \} \subset 1 + \psi_{A,B}(\mathbb{D}) \subset \{ w \in \mathbb{C} : |w - 1| < h_1 \}.
\]

(ii) for \(A = \alpha e^{i\gamma}\) and \(B = \alpha e^{-i\gamma}\),
\[
\{ w \in \mathbb{C} : |w - 1| < k_1 + k \} \subset 1 + \psi_{A,B}(\mathbb{D}) \subset \{ w \in \mathbb{C} : |w - 1| < k_2 \}.
\]

Further for \(\alpha = 1\), we have

(iii) for \(A = -B = 1\),
\[
\{ w \in \mathbb{C} : |w - 1| < \frac{\pi}{4} \} \subset 1 + \psi_{A,B}(\mathbb{D}).
\]

(iv) for \(A = e^{i\gamma}\) and \(B = e^{-i\gamma}\),
\[
\{ w \in \mathbb{C} : |w - 1| < \frac{\gamma}{2 \sin \gamma} \} \subset 1 + \psi_{A,B}(\mathbb{D}).
\]

**Proof.** Let \(w = 1 + \psi_{A,B}(z)\), then we have
\[
|w - 1| = |\psi_{A,B}(z)| = \frac{1}{|A - B|} \left| \log \left( \frac{1 + Az}{1 + Bz} \right) \right|.
\]

It can be easily seen that for \(\alpha \in (0, 1)\), \(A = -B = \alpha\)
\[
\min_{|z|=1} |\psi_{A,B}(z)| = h_2 \quad \text{and} \quad \max_{|z|=1} |\psi_{A,B}(z)| = h_1
\]

and for \(A = \alpha e^{i\gamma}\) and \(B = \alpha e^{-i\gamma}\)
\[
\min_{|z|=1} |\psi_{A,B}(z)| = k_1 + k \quad \text{and} \quad \max_{|z|=1} |\psi_{A,B}(z)| = k_2.
\]

Further for \(\alpha = 1\), \(h_1\) and \(k_2\) tends to \(\infty\), \(h_2 = \pi/4\) and \(k_1 + k = \gamma/(2 \sin \gamma)\). This completes the proof. \(\square\)

**Remark 2.6.** In general, if we consider a disc \(D(a, r) := \{ w \in \mathbb{C} : |w - a| < r \}\) where \(a \in \mathbb{R}\) and \(1 \in D(a, r)\), then we observe that \(D(a, r) \subset 1 + \psi_{A,B}(\mathbb{D})\) if and only if
\[
D(a, r) \subset \begin{cases} 
D(1, h_2), & \text{for } A = -B = \alpha, \\
D(1 + k, k_1), & \text{for } A = \alpha e^{i\gamma}, B = \alpha e^{-i\gamma}. 
\end{cases}
\]

Thus we arrive at the following:

**Lemma 2.7.** Let \(-1 < D < C \leq 1\) and \(p(z) := (1 + Cz)/(1 + Dz)\). Then \(p(z) \prec L[A, B]\) if and only if

(i) for \(A = -B = \alpha\)
\[
C \leq \begin{cases} 
h_2 + (1 - h_2)D, & \text{when } (1 - CD)/(1 - D^2) \leq 1, \\
h_2 + (1 + h_2)D, & \text{when } (1 - CD)/(1 - D^2) \geq 1.
\end{cases}
\]

(ii) for \(A = \alpha e^{i\gamma}\) and \(B = \alpha e^{-i\gamma}\),
\[
C \leq \begin{cases} 
k_1 + k + (1 - k_1 + k)D, & \text{when } (1 - CD)/(1 - D^2) \leq 1 + k, \\
k_1 + k + (1 + k_1 + k)D, & \text{when } (1 - CD)/(1 - D^2) \geq 1 + k.
\end{cases}
\]
Proof. Since \( p(z) = (1 + Cz)/(1 + Dz) \) maps \( \mathbb{D} \) onto \( \mathcal{D}(a, r) \), where
\[
a = \frac{1 - CD}{1 - D^2} \quad \text{and} \quad r = \frac{C - D}{1 - D^2},
\]
the result follows at once from (2.8).

From Lemma 2.7 we deduce the following corollary, which ensures that \( \mathcal{F}[A, B] \) is nonempty.

**Corollary 2.8.** Let \(-1 < D < C \leq 1\) and \( f \in \mathcal{A} \) be such that \( z f'(z)/f(z) = (1 + Cz)/(1 + Dz) \), then \( f \in \mathcal{F}[A, B] \) if and only if, the condition \( (2.9) \) or \( (2.10) \) holds.

Now from (1.4), we have \( f \in \mathcal{F}[A, B] \) if and only if there exists an analytic function \( p(z) \prec \psi_{A,B}(z) \) such that
\[
f(z) = z \exp \left( \int_0^z \frac{p(t)}{t} \, dt \right).
\] (2.11)

If we take \( p(z) = \psi_{A,B}(z) \), then we obtain from (2.11)
\[
f_{A,B}(z) := z \exp \frac{L_{i2}(-Bz) - L_{i2}(-Az)}{A - B},
\] (2.12)
where \( L_{i2}(x) = \sum_{n=1}^{\infty} x^n/n^2 \), denotes the Spence’s (or dilogarithm) function. The function \( f_{A,B}(z) \) is nonunivalent but have the extremal properties for many problems for the class \( \mathcal{F}[A, B] \) (see Fig.1). Kumar and Gangania [5] studied the class \( \mathcal{F}(\psi) \) as defined in (1.2) and obtained growth and covering theorem for the case when \( 1 + \psi(z) \not\in (1 + z)/(1 - z) \). As a consequence of the same and Remark 2.4, we obtain the following result:

**Lemma 2.9.** Let \( f \in \mathcal{F}[A, B] \) and \( f_{A,B} \) be defined as in (2.12), then for \( |z| = r \)

(i) Growth Theorem: \(-f_{A,B}(-r) \leq |f(z)| \leq f_{A,B}(r)\).

(ii) Covering Theorem: Either \( f \) is a rotation of \( f_{A,B} \) or \( \{ w \in \mathbb{C} : |w| \leq f_{A,B}(-1) \} \subset f(\mathbb{D}) \).

![Figure 1](image_url)

**Figure 1.** (A) The images of \( \partial \mathbb{D}_{f_{A,B}(1)} \), \( f_{A,B}(\partial \mathbb{D}) \) and \( \partial \mathbb{D}_{f_{A,B}(-1)} \), for \( A = 0.5 e^{i \pi/3} \) and \( B = 0.5 e^{-i \pi/3} \) and (B) zoomed image of cusp.

For \( z = r e^{i \theta} \), where \( \theta \) is fixed but arbitrary, as a consequence of growth theorem and \( \psi_{A,B}(-r) \leq \text{Re} \psi_{A,B}(re^{i \theta}) \leq \psi_{A,B}(r) \), we obtain
\[
\log \frac{f(z)}{z} = \int_0^r \frac{p(te^{i \theta})}{t} \, dt,
\]
where \( p(z) := \psi_{A,B}(w(z)) \) and \( w \) is a Schwarz function. Further
\[
\frac{f(z)}{z} = \exp \int_0^r \frac{p(te^{i\theta})}{t} dt = \exp \left( \int_0^r \text{Re} \frac{p(te^{i\theta})}{t} dt + i \int_0^r \text{Im} \frac{p(te^{i\theta})}{t} dt \right) \left| \frac{f(z)}{z} \right| \leq \exp \left( \int_0^r \psi_{A,B}(te^{i\theta}) dt \right) \left| \frac{f(z)}{z} \right| \leq \exp \left( \int_0^r \psi_{A,B}(t) dt \right) dt.
\]
Here \( L(f, r) := \int_0^{2\pi} |z f'(z)| d\theta \) is the length of the boundary curve \( f(|z|= r) \). Now we obtain the following result:

**Corollary 2.10.** Let \( f \in \mathcal{F}[A, B] \) and \( M(r) = \exp \int_0^r \frac{\psi_{A,B}(t)}{t} dt \), then for \( |z|= r \), we have
(i) \( M(-r) \leq \left| \frac{f(z)}{z} \right| \leq M(r), \)
(ii) \( (1 + \max_{|z| \leq r} |\psi_{A,B}(z)|) M(-r) \leq |f'(z)| \leq (1 + \max_{|z| \leq r} |\psi_{A,B}(z)|) M(r) \),
(iii) \( 2\pi \rho (1 + \max_{|z| \leq r} |\psi_{A,B}(z)|) M(-r) \leq L(f, r) \leq 2\pi \rho (1 + \max_{|z| \leq r} |\psi_{A,B}(z)|) M(r) \),
(iv) \( f(z)/z \prec f_{A,B}(z)/z \).

### 3. Radius Estimates

In this section, we obtain some radii estimates. From Theorem 2.3, we conclude that \( f \in \mathcal{S}[A, B] \) is starlike of order \( 1 + \frac{1}{A-B} (\varphi - \vartheta) < 0 \), hence \( f \) may not be univalent in \( \mathbb{D} \), thus \( \mathcal{F}[A, B] \not\subseteq \mathcal{S}^* \).

Therefore, in the following result, we find the radius of starlikeness of order \( \delta \), where \( \delta \in [0, 1) \) for functions in the class \( \mathcal{F}[A, B] \).

**Theorem 3.1.** Let \( \alpha \in (0, 1) \), \( \gamma \in (0, \gamma_0) \), where \( \gamma_0 \approx 1.2461 \ldots \) and \( \delta \in [0, 1) \) be given numbers. If \( f \in \mathcal{F}[A, B] \), then \( f \) is starlike of order \( \delta \) in the disc \( |z| < r(\delta) \), where
(i) For \( A = -B = \alpha \),
\[
r(\delta) = \frac{\exp(2\alpha(1-\delta)) - 1}{\alpha \exp(2\alpha(1-\delta)) + 1}.
\]
(ii) For \( A = \alpha e^{i\gamma} \) and \( B = \alpha e^{-i\gamma} \), \( r(\delta) \) is the smallest positive root of
\[
\tan(2\alpha \sin \gamma (\delta - 1)) = \frac{\alpha^4 r^4 \sin 2\gamma + 2\alpha^3 r^3 \sin \gamma \cos 2\gamma (\alpha^2 r^2 - 2\alpha r \sin \gamma)}{\alpha^4 r^4 \cos 2\gamma - 2\alpha^3 r^3 \sin \gamma \sin 2\gamma (\alpha^2 r^2 - 2\alpha r^2 + 1)}.
\]

The result is sharp.

**Proof.** Let \( f \in \mathcal{F}[A, B] \), then using Theorem 2.3 for \( |z| < r \), we have
**Case 1.** when \( A = -B = \alpha \),
\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 1 - \frac{1}{2\alpha} \log \left( \frac{1 + \alpha r}{1 - \alpha r} \right) =: t_1(r),
\]
Now as \( t_1(r) < 0 \) for all choices of \( \alpha \), thus \( t_1(r) \) is a strictly decreasing function from 1 to \( 1 - 1/(2\alpha) \log((1 + \alpha)/(1 - \alpha)) < 0 \). Hence the root \( r(\delta) \), given in (3.1), of the equation \( t_1(r) = \delta \), is the radius of starlikeness of order \( \delta \) of \( \mathcal{F}[A, B] \).
**Case 2.** when \( A = \alpha e^{i\gamma} \) and \( B = \alpha e^{-i\gamma} \),
\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 1 - \frac{1}{2\alpha} \sin \gamma (\eta - \tau) =: t_2(r),
\]
where
where $\eta$ and $\tau$ are given in Theorem 2.3. Since $t_2(r) = \delta$ can be simplified to the equation (3.2) by a simple computation, thus it is enough to find the smallest positive root of (3.2). Further
\[
 t'_2(r) = -\frac{1 - \alpha^4 r^4 + 2\alpha r \cos \gamma(1 - \alpha^2 r^2)}{(1 - \alpha^2 r^2)^2 \sqrt{1 + \alpha^4 r^4 - 2\alpha^2 r^2 \cos 2\gamma}} < 0
\]
for all choices of $\alpha$ and $\gamma$, $t_2(r)$ is a strictly decreasing function from 1 to $g(\alpha, \gamma)$, where
\[
g(\alpha, \gamma) := 1 + \frac{1}{2\alpha \sin \gamma} \left( \tan^{-1} \left( \frac{\alpha^2 \sin 2\gamma}{\alpha^2 \cos 2\gamma - 1} \right) - \sin^{-1} \left( \frac{2\alpha \sin \gamma}{\sqrt{1 + \alpha^4 - 2\alpha^2 \cos 2\gamma}} \right) \right).
\]
Note that $g(\alpha, \gamma)$ is real valued, non-constant analytic function on a bounded domain $R := (0, 1) \times (0, \pi/2)$ and is bounded above on $R$. Thus by Maximum Modulus Theorem, $g(\alpha, \gamma)$ attains its maximum value on the boundary of $R$ and the four boundaries are enlisted below:

1. $g(0, \gamma) = 0$.
2. $g(1, \gamma) = 1 + (\gamma - \pi)/(2 \sin \gamma)$. Now as
   \[
g'(1, \gamma) = (1 + (\pi - \gamma) \cot \gamma)/(2 \sin \gamma) > 0,
   \]
   therefore $g(1, \gamma)$ is strictly increasing from $-\infty$ to $1 - \pi/4 \simeq 0.2146 \ldots$ and $\gamma_0$ is the zero of it.
3. $g(\alpha, 0) = 1 - 1/(1 - \alpha)$. As $g'(\alpha, 0) = \log(1 - \alpha) < 0$, therefore $g(\alpha, 0)$ is strictly decreasing from 0 to $-\infty$.
4. $g(\alpha, \pi/2) = 1 - (\sin^{-1} (2\alpha/(1 + \alpha^2))) / (2\alpha)$. As
   \[
g'(\alpha, \pi/2) = \left( (2\alpha/(1 + \alpha^2)) + \sin^{-1} (2\alpha/(1 + \alpha^2)) \right) / (2\alpha^2) > 0,
   \]
   therefore $g(\alpha, \pi/2)$ is also strictly increasing from 0 to $1 - \pi/4 \simeq 0.2146$.

From Fig. 2 and all four boundaries, we observe that $g(\alpha, \gamma) < 0$ for all choices of $\alpha \in (0, 1], \gamma \in (0, \gamma_0)$, where $\gamma_0$ is the root of the equation $g(1, \gamma) = 0$. Hence $r(\delta)$ is the radius of starlikeness of order $\delta$ of $F[A, B]$.

**Theorem 3.2.** Let $\alpha \in (0, 1]$ and $\delta \in [0, 1]$. If $f \in F[A, B]$, for the case when $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$, then there is an unique $\gamma' \in (\gamma_0, \pi/2)$ such that

(i) when $\gamma \in (\gamma_0, \gamma')$, $f$ is starlike of order $\delta$ in $D_{r(\delta)}$, where $r(\delta)$ the root of the equation (3.2),
(ii) when $\gamma \in (\gamma', \pi/2)$, $f$ is starlike of order $\delta \in [0, g(\alpha, \pi/2))$ in $D$ and starlike of order $\delta \in [g(\alpha, \pi/2), 1)$ in $D_{r(\delta)}$. 

![Image of g(α, γ) on R := (0, 1) × (0, π/2).](image-url)
Proof. By observing the Fig. 2 and the nature of all boundary curves discussed in Theorem 3.1, we arrive at $g(\alpha, \gamma) < 0$ and $g(\alpha, \pi/2) > 0$ for all $\alpha \in (0, 1)$. Thus by IVP for any choice of $\alpha \in (0, 1)$, there is an unique $\gamma' \in (\gamma, \pi/2)$ such that $g(\alpha, \gamma') = 0$, $g(\alpha, \gamma) < 0$ for all $\gamma \in (\gamma, \gamma')$ and $g(\alpha, \gamma) > 0$ for all $\gamma \in (\gamma', \pi/2)$. Hence the result follows.

Taking $\delta = 0$ in Theorems 3.1 and 3.2 we obtain the following result:

**Corollary 3.3.** Let $\alpha \in (0, 1]$ and $\gamma \in (0, \pi/2]$. If $f \in F[A, B]$ then

(i) For $A = -B = \alpha$, $f$ is starlike univalent in the disc $|z| < r_0$, where

$$ r_0 = \frac{\exp(2\alpha) - 1}{\alpha(\exp(2\alpha) + 1)}. $$

(ii) For $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$, $f \in S^*$, whenever $\gamma \in (\gamma', \pi/2)$, and $f$ is starlike univalent in the disc $|z| < r_0$, whenever $\gamma \in (0, \gamma_0) \cup (\gamma_0, \gamma')$, where $\gamma_0, \gamma'$ are given in Theorems 3.1 and 3.2 respectively and $r_0$ is the smallest positive root of

$$ \tan(2\alpha \sin \gamma) + \frac{\alpha^4 r^4 \sin 2\gamma + 2\alpha^3 r^3 \sin \gamma \cos 2\gamma - \alpha^2 r^2 \sin 2\gamma - 2\alpha r \sin \gamma}{\alpha^4 r^4 \cos 2\gamma - 2\alpha^3 r^3 \sin \gamma \sin 2\gamma - \alpha^2 r^2 \cos 2\gamma - \alpha^2 r^2 + 1} = 0. $$

The result is sharp.

**Corollary 3.4.** Let $f \in F[A, B]$. Then for the disc $|z| < r \leq r_0$, where $r_0$ is given in Corollary 3.3, we have

(i) For $A = -B = \alpha$,

$$ \left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \tan^{-1} \left( \frac{\sin^{-1} \left( \frac{2\alpha r}{1 + \alpha^2 r^2} \right)}{2\alpha - \log \left( \frac{1 + \alpha r}{1 - \alpha r} \right)} \right), $$

(ii) For $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$,

$$ \left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \tan^{-1} \left( \frac{\log T_2}{2\alpha \sin \gamma - \eta + \tau} \right), $$

where $T_2, \eta$ and $\tau$ are given in Theorem 2.3

**Corollary 3.5.** Let $f \in F[A, B]$. Then for the disc $|z| < r_s \leq r_0$, where $r_0$ is given in Corollary 3.3, $f \in SS^*(\beta)$, where $r_s \in (0, r_0]$ is the smallest positive root of the following equation:

(i) For $A = -B = \alpha$,

$$ \sin^{-1} \left( \frac{2\alpha r}{1 + \alpha^2 r^2} \right) = \tan \left( \frac{\beta \pi}{2} \right) \left( 2\alpha - \log \left( \frac{1 + \alpha r}{1 - \alpha r} \right) \right), $$

(ii) For $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$,

$$ \frac{\log T_2}{2\alpha \sin \gamma - \eta + \tau} = \tan \left( \frac{\beta \pi}{2} \right), $$

where $T_2, \eta$ and $\tau$ are given in Theorem 2.3

We now discuss the radius estimates for the classes $BS^*(\alpha)$ and $S_{cs}^*$, which are not contained in $S^*$. Further we derive radii for functions in $BS^*(\alpha)$ and $S_{cs}^*$ to be in $F[A, B]$.

**Theorem 3.6.** Let $\alpha \in (0, 1)$, $r_0 = (-1 + \sqrt{1 + 4\alpha h_1^2})/(2\alpha h_1)$ and $\alpha_0 \in (0, 1)$ is the smallest solution of

$$ 1 - \frac{r_0^2 \cos^2 \theta (1 - \alpha r_0^2)^2}{(1 + \alpha^2 r_0^2 - 2\alpha r_0^2 \cos 2\theta)^2 h_1^4} - \frac{r_0^2 \sin^2 \theta (1 + \alpha r_0^2)^2}{(1 + \alpha^2 r_0^2 - 2\alpha r_0^2 \cos 2\theta)^2 h_2^4} = 0, $$
where \( \theta \in (0, \pi/2) \). If \( f \in \mathcal{BS}^s(\alpha) \) then for the disc \(|z| \leq r_b < 1\), \( f \in \mathcal{F}[A,B] \), where

\[
 r_b = \begin{cases} 
 r_0, & \text{for } A = -B = \alpha \leq \alpha_0 \\
 r_1 := \frac{-1 + \sqrt{1 + 4\alpha h_2^2}}{2\alpha h_2}, & \text{for } A = -B = \alpha > \alpha_0 \\
 r_2 := \frac{-1 + \sqrt{1 + 4\alpha(k_1 + k)^2}}{2\alpha(k_1 + k)}, & \text{for } A = \alpha e^{i\gamma} \text{ and } B = \alpha e^{-i\gamma}.
\end{cases}
\]

Moreover the radii \( r_0 \) and \( r_2 \) are sharp.

**Proof.** Let \( f \in \mathcal{BS}^s(\alpha) \). Since for \( A = -B = \alpha \), \( \max_{|z|=1} \Re \psi_{A,B}(z) = h_1 \). Thus to find such \( r < 1 \) for which the image of \( zf'(z)/f(z) - 1 \) under the disc \(|z| < r \) lies inside \( \psi_{A,B}(\mathbb{D}) \), it is necessary that

\[
\max_{|z|=r < 1} \Re \left( \frac{z}{1 - \alpha z^2} \right) = \frac{r}{1 - \alpha r^2} \leq h_1
\]

must hold. Clearly for \(|z| \leq r_0\), the inequality \((3.3)\) holds. Now to see that for \( \alpha \leq \alpha_0 \), radius \( r_0 \) is also sufficient for \( zf'(z)/f(z) - 1 < z/(1 - \alpha z^2) \in \psi_{A,B}(\mathbb{D}) \) in the disc \(|z| \leq r_0\). For \( \zeta = re^{i\theta} \) \((\theta \in [0,2\pi))\), we have

\[
B_r(\theta) := \frac{\zeta}{1 - \alpha \zeta^2} = \frac{r \cos \theta(1 - \alpha r^2)}{1 + \alpha^2 r^4 - 2\alpha r^2 \cos 2\theta} + i\frac{r \sin \theta(1 + \alpha r^2)}{1 + \alpha^2 r^4 - 2\alpha r^2 \cos 2\theta}.
\]

Since \( \Re B_r(\theta) = \Re B_r(-\theta) \), \( \Re B_r(\theta) = -\Re B_r(\pi - \theta) \) and \( \Im B_r(\theta) = \Im B_r(\pi - \theta) \), therefore the curve \( B_r(\theta) \) is symmetric about real and imaginary axis thus it is sufficient to consider for \( \theta \in [0,\pi/2] \). Now for \( r = r_0 \), the square of the distance from the origin to the points of \( B_{r_0}(\theta) \) is given by

\[
\text{Dist}(0; B_{r_0}(\theta)) := \frac{r_0^2}{1 + \alpha^2 r_0^4 - 2\alpha r_0^2 \cos 2\theta}.
\]

Since \( \text{Dist}(0; B_{r_0}(\theta))' < 0 \), thus \( \text{Dist}(0; B_{r_0}(\theta)) \) is a decreasing function of \( \theta \). Hence the farthest point of \( B_{r_0}(\theta) \) from origin is \((r_0/(1 - \alpha r_0^2), 0)\), which lies on the boundary of \( \Omega_1 \). Now for \( A = -B = \alpha > \alpha_0 \) and \( A = \alpha e^{i\gamma}, B = \alpha e^{-i\gamma} \) with \(|z|=r\), we have

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{r}{1 - \alpha r^2}.
\]

Therefore, by Lemma 2.5, we see that \( \mathcal{F}[A,B] \)−radius for the class \( \mathcal{BS}^s(\alpha) \) are the smallest positive roots \( r_1 \) and \( r_2 \) of the equations \( r/(1 - \alpha r^2) = h_2 \) and \( r/(1 - \alpha r^2) = h_1 + k \), respectively.

**Theorem 3.7.** Let \( \alpha \in (0,1) \) and \( \alpha_0 \in (0,1) \) be the smallest solution of

\[
\frac{r_0^2((\alpha - 1) r_0 + \cos \theta(1 - \alpha r_0^2))^2}{h_2^2} + \frac{r_0^2(1 + \alpha r_0^2)^2 \sin^2 \theta}{h_2^2} = N,
\]

where \( \theta \in (0,\pi/2) \) and

\[
N = (1 + r_0^2 - 2r_0 \cos \theta)^2 (1 + \alpha^2 r_0^2 + 2\alpha r_0 \cos \theta)^2
\]

and

\[
r_0 = \frac{-(1 + (1 - \alpha) h_1) + \sqrt{(1 + (1 - \alpha) h_1)^2 + 4\alpha h_1^2}}{2\alpha h_1}.
\]

If \( f \in S_{cs}^s(\alpha) \) then for the disc \(|z| \leq r_{cs} < 1\), \( f \in \mathcal{F}[A,B] \), where

\[
r_{cs} = \begin{cases} 
 r_0, & \text{for } A = -B = \alpha \leq \alpha_0 \\
 r_1 := \frac{-1 + (1 + (1 - \alpha) h_2) + \sqrt{(1 + (1 - \alpha) h_2)^2 + 4\alpha h_2^2}}{2\alpha h_2}, & \text{for } A = -B = \alpha > \alpha_0 \\
 r_2 := \frac{-1 + (1 + (1 - \alpha)(k_1 + k)) + \sqrt{(1 + (1 + (1 - \alpha) (k_1 + k)) + 4\alpha(k_1 + k)^2}}{2\alpha(k_1 + k)}, & \text{for } A = \alpha e^{i\gamma}, B = \alpha e^{-i\gamma}.
\end{cases}
\]
Moreover the radii $r_0$ and $r_2$ are sharp.

**Proof.** Let $f \in \mathcal{S}_e^*(\alpha)$. Since for $A = -B = \alpha$, $\max_{|z|=1} \Re \psi_{A,B}(z) = h_1$. Thus to find such $r < 1$ for which the image of $zf'(z)/f(z) - 1$ under the disc $|z|< r$ lies inside $\psi_{A,B}(\mathbb{D})$, it is necessary that

$$\max_{|z|=r<1} \Re \left( \frac{z}{(1-\overline{z})(1+\alpha z)} \right) = \frac{r}{(1-r)(1+\alpha r)} \leq h_1 \tag{3.4}$$

must hold. Clearly for $|z| \leq r_0$, the equation (3.4) holds. Now to see that for $\alpha \leq \alpha_0$ radius $r_0$ is also sufficient for $zf'(z)/f(z) - 1 < z/((1-z)(1+\alpha z)) \in \psi_{A,B}(\mathbb{D})$ in the disc $|z| \leq r_0$. For $\zeta = re^{i\theta}$ ($\theta \in [0, 2\pi]$), we have

$$CS_r(\theta) := \frac{\zeta}{(1-\overline{\zeta})(1+\alpha \zeta)} = \frac{r((\alpha-1)r + \cos \theta(1-\alpha r^2))}{(1+r^2-2r \cos \theta)(1+\alpha^2 r^2 + 2\alpha r \cos \theta)} + \frac{r(1+\alpha r^2) \sin \theta}{(1+r^2-2r \cos \theta)(1+\alpha^2 r^2 + 2\alpha r \cos \theta)}.$$

Since $\Re CS_r(\theta) = \Re CS_r(-\theta)$, therefore the curve $CS_r(\theta)$ is symmetric about real axis thus it is sufficient to consider for $\theta \in [0, \pi]$. Now for $r = r_0$, the square of the distance from the origin to the points of $CS_{r_0}(\theta)$ is given by

$$\text{Dist}(0; CS_{r_0}(\theta)) := \frac{r_0^2}{(1+r^2-2r \cos \theta)^2(1+\alpha^2 r^2 + 2\alpha r \cos \theta)^2}.$$

Since $\text{Dist}(0; CS_{r_0}(\theta))' = 0$ for $\theta = 0, \theta_0$ and $\pi$ with $\text{Dist}(0; CS_{r_0}(\theta))'< 0$ whenever $\theta \in (0, \theta_0)$ and $\text{Dist}(0; CS_{r_0}(\theta))'> 0$ whenever $\theta \in (\theta_0, \pi)$. And $\text{Dist}(0; CS_{r_0}(0)) - \text{Dist}(0; CS_{r_0}(\pi)) > 0$, hence the farthest point of $CS_{r_0}(\theta)$ from origin is equal to $h_1$ obtained at $\theta = 0$. Now for $A = -B = \alpha > \alpha_0$ and $A = \alpha e^{i\gamma}$, $B = \alpha e^{-i\gamma}$ with $|z|= r$, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{r}{(1-r)(1+\alpha r)}.$$

Therefore, by Lemma 2.5, we see that $\mathcal{F}[A, B]$-radius for the class $\mathcal{S}_e^*(\alpha)$ are the smallest positive roots $r_1$ and $r_2$ of the equations $r/((1-r)(1+\alpha r)) = h_2$ and $r/((1-r)(1+\alpha r)) = k_1 + k$, respectively.

![Figure 3](image-url)

**Figure 3.** $f_0(z) = z/(1-\alpha z^2)$ and $g_0(z) = z/((1-z)(1+\alpha z))$, (A) $f_{0.5}(\mathbb{D}_{r_0=0.77}) \subset \psi_{0.5,-0.5}(\partial \mathbb{D})$ and (B) $g_{0.5}(\mathbb{D}_{r_0=0.59}) \subset \psi_{0.5,-0.5}(\partial \mathbb{D})$. 

[1]
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