Asymptotic characterization of destabilizing switching signals for switched linear systems

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Abstract—This paper deals with destabilizing switching signals for continuous-time switched linear systems. Our contributions are twofold: Firstly, we propose a class of switching signals under which a switched system is unstable. Our characterization of instability depends solely on the asymptotic behaviour of frequency of switching, frequency of transition between subsystems, and fraction of activation of subsystems. Secondly, we show that our class of destabilizing switching signals is a strict subset of the class of switching signals that does not satisfy asymptotic characterization of stability recently proposed in the literature. This observation identifies a gap between asymptotic characterizations of stabilizing and destabilizing switching signals for switched linear systems.

I. INTRODUCTION

Characterization of classes of purely time-dependent switching signals that preserve stability of a switched system, has attracted a considerable research attention in the past few decades. Research on this topic can be broadly classified into two directions: stability characterization based on point-wise properties of switching signals [8], [2], [9], [6] and stability characterization based on asymptotic properties of switching signals [4], [5]. In case of the former, stabilizing switching signals obey certain upper bounds on the number of switches and duration of activation of unstable subsystems on every interval of time, while in case of the latter, stability is characterized based solely on the asymptotic properties of the switching signals. Recently in [5] it was shown that if a switching signal satisfies any of the existing point-wise or asymptotic characterizations of stability, then it satisfies certain conditions on the asymptotic behaviour of switching frequency, frequency of transitions between subsystems, and fraction of activation of subsystems. The above body of results is derived by employing multiple Lyapunov-like functions, and is only sufficient in nature. Consequently, if a switching signal does not obey these stability conditions, we cannot conclude that the resulting switched system is unstable. This fact motivates the current work. We are interested in instability characterizations of switched systems using multiple Lyapunov-like functions.

Instability is an important concept in stability theory. This concept is often useful in studying behaviour of a switched system under failures in system components, adversarial attacks, etc. It is known that if a family of asymptotically stable systems does not admit a common Lyapunov function, then the family admits at least one switching signal that is destabilizing. In [10] a sufficient condition for existence of a stabilizing switching signal was proposed using matrix pencils. Several classes of subsystems that admit stabilizing switching signals were identified. This set of results was employed to study the connection between existence of a stabilizing switching signal and non-existence of a common quadratic Lyapunov function in [3]. In [11] a necessary but not sufficient condition for instability of a planar switched linear system was proposed by employing flow relations of the constituent subsystems and construction of invariant sets. Instability of stochastic switched systems under arbitrary switching was addressed in [12]. The authors proposed sufficient conditions for instability in a probabilistic sense. In this paper we address the problem of characterizing switching signals that are destabilizing.

Multiple Lyapunov-like functions are a widely used tool for studying stability of switched systems [7, Chapter 3]. The underlying idea is that the maximum increase in these functions caused by activation of unstable subsystems and occurrence of switches is compensated by the minimum decrease caused by activation of asymptotically stable subsystems. We utilize minimum increase and maximum decrease of these functions to characterize instability of a switched system. Our characterization of destabilizing switching signals involves asymptotic behaviour of the following properties of these signals: frequency of switching, frequency of transition between subsystems, and fraction of activation of subsystems. It does not involve nor imply conditions on a switching signal on every interval of time.

Earlier in [5] asymptotic properties of switching signals were used to characterize stability of a switched system. Our class of destabilizing switching signals is a strict subset of the class of switching signals that does not obey the stability condition proposed in [5, Theorem 5]. (De)stabilizing properties of the class of switching signals that satisfies neither the stability condition of [5] nor the instability condition proposed in this paper, remains undetermined. To the best of our knowledge, this is the first instance in the literature when multiple Lyapunov-like functions are employed to characterize instability of a switched system, and the gap between asymptotic characterization of stabilizing and destabilizing switching signals, is addressed.

II. PRELIMINARIES

We consider a family of continuous-time linear systems

\[ \dot{x}(t) = A_p x(t), \quad x(0) = x_0, \quad p \in \mathcal{P}, \quad t \in [0, +\infty[, \quad (1) \]

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where $x(t) \in \mathbb{R}^d$ is the vector of states at time $t$, and $\mathcal{P} = \{1, 2, \ldots, N\}$ is an index set. We assume that for each $p \in \mathcal{P}$ the matrix $A_p \in \mathbb{R}^{d \times d}$ has full rank; consequently, $0 \in \mathbb{R}^d$ is the unique equilibrium point for each system in (1). Let $\sigma : [0, +\infty[ \to \mathcal{P}$ be a switching signal; it is a piecewise constant function that specifies, at each time $t$, the subsystem $A_{\sigma(t)}$, that is active at $t$. By convention, $\sigma$ is assumed to be continuous from right and having limits from the left everywhere. A switched system generated by the family of systems (1) and a switching signal $\sigma$ is given by

$$
\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0, \quad t \in [0, +\infty[.
$$

(2)

Let $\mathcal{P}_S$ and $\mathcal{P}_U$ denote the sets of indices of asymptotically stable and unstable subsystems, respectively, $\mathcal{P} = \mathcal{P}_S \sqcup \mathcal{P}_U$, and $E(\mathcal{P})$ denote the set of all ordered pairs $(p, q)$ such that $p$ can switch to subsystem $q$ is admissible, $p, q \in \mathcal{P}$. We let $0 =: \tau_0 < \tau_1 < \tau_2 < \cdots$ be the switching instants; these are the points in time where $\sigma$ jumps. We call a switching signal $\sigma$ admissible if it satisfies

$$(\sigma(\tau_i), \sigma(\tau_{i+1})) \in E(\mathcal{P}), \quad i = 0, 1, \ldots.$$ 

Let $\mathcal{S}$ denote the set of all admissible switching signals. For $t > 0$, let $N(t)$ denote the number of switches on $[0, t]$. The solution $(x(t))_{t \geq 0}$ to the switched system (2) corresponding to an admissible switching signal $\sigma \in \mathcal{S}$ is the map $x : [0, +\infty[ \to \mathbb{R}^d$ defined by $x(t) = e^{A_{\sigma(t)}(t - \tau_0)} \cdots e^{A_{\sigma(0)}(\tau_1 - \tau_0)} x_0$, where the dependence of $x$ on $\sigma$ is suppressed for notational simplicity.

**Definition 1:** The switched system (2) is globally asymptotically stable (GAS) for a given switching signal $\sigma$ if (2) is Lyapunov stable, and globally asymptotically convergent, i.e., for all $x(0)$, $\|x(t)\| \to 0$ as $t \to +\infty$.

**Definition 2:** The switched system (2) is unstable for a given switching signal $\sigma$ if for all $x(0) \neq 0$, $\|x(t)\| \to +\infty$ as $t \to +\infty$.

Given a family of systems (1) containing both asymptotically stable and unstable subsystems, and a set of admissible transitions $E(\mathcal{P})$, our objective is to characterize a class of switching signals $\mathcal{S}_s \subset \mathcal{S}$ under which the switched linear system (2) is unstable.

**Remark 1:** Since $\mathcal{P}_U \neq \emptyset$, a switching signal $\sigma$ satisfying $\sigma(t) = p$ for all $t$ with a fixed $p \in \mathcal{P}_U$ is destabilizing. However, we are seeking for a (possibly large) class of destabilizing switching signals not restricted to the class of constant switching signals.

**Fact 1:** For each $p \in \mathcal{P}$ there exists a pair $(P_p, \lambda_p)$, where $P_p \in \mathbb{R}^{d \times d}$ is a symmetric and positive definite matrix, and $\lambda_p > 0$, $p \in \mathcal{P}_S$, $\lambda_p \leq 0$, $p \in \mathcal{P}_U$, such that, with the function $V_p : \mathbb{R}^d \to [0, +\infty[$ defined as

$$
\mathbb{R}^d \ni \xi \mapsto V_p(\xi) := \langle P_p \xi, \xi \rangle \geq 0, \quad t \in [0, +\infty[,
$$

we have for all $\gamma_p(0) \in \mathbb{R}^d$, $t \in [0, +\infty[,$

$$
V_p(\gamma_p(t)) \geq \exp(-\lambda_p t) V_p(\gamma_p(0)),
$$

(4)

and $\gamma_p(\cdot)$ solves the $p$-th system dynamics in (1), $p \in \mathcal{P}$.



1. By asymptotically stable subsystems, we mean that the matrices $A_p$’s are Hurwitz, and for the unstable subsystems, $A_p$’s are not Hurwitz.

**Proof:** We begin with asymptotically stable subsystems $p \in \mathcal{P}_S$. Let $\mathbb{R}^d \ni \xi \mapsto V_p(\xi) := \langle P_p \xi, \xi \rangle$, where $P_p \in \mathbb{R}^{d \times d}$ is a symmetric and positive definite solution to the Lyapunov equation

$$
A_p^T P_p + P_p A_p = -Q_p
$$

(5)

for some pre-selected symmetric and positive definite matrix $Q_p \in \mathbb{R}^{d \times d}$ [1, Corollary 11.9.1]. Recall that [1, Lemma 8.4.3] any symmetric and positive definite matrix $Z \in \mathbb{R}^{d \times d}$ satisfies

$$
\lambda_{\min}(Z) \|z\|^2 \leq z^T Z z \leq \lambda_{\max}(Z) \|z\|^2.
$$

Consequently, for all $\xi \in \mathbb{R}^d$, we have

$$
\xi^T Q_p \xi \geq -\lambda_{\max}(Q_p) \xi^T P_p \xi.
$$

Let $\lambda_p = \frac{\lambda_{\max}(Q_p)}{\lambda_{\min}(P_p)}$. We have

$$
\frac{d}{dt} V_p(\gamma_p(t)) \geq -\lambda_p V_p(\gamma_p(t))
$$

leading to (4) with $\lambda_p > 0$.

We now move on to unstable subsystems $p \in \mathcal{P}_U$. There exist $\varepsilon_p > 0$ such that $A_p - \varepsilon_p I$ is asymptotically stable. Select the Lyapunov-like function

$$
\mathbb{R}^d \ni \xi \mapsto V_p(\xi) := \langle P_p \xi, \xi \rangle,
$$

where $P_p \in \mathbb{R}^{d \times d}$ is a symmetric and positive definite solution to the Lyapunov equation (5) with $A_p = A_p - \varepsilon_p I$ and a pre-selected symmetric and positive definite matrix $Q_p \in \mathbb{R}^{d \times d}$. Following the set of arguments for asymptotically stable subsystems, we have that

$$
\frac{d}{dt} V_p(\gamma_p(t)) \geq -\lambda_p V_p(\gamma_p(t)),
$$

where $\lambda_p = \left(2\varepsilon_p - \frac{\lambda_{\max}(Q_p)}{\lambda_{\min}(P_p)}\right)$. Notice that $\varepsilon_p$ is any scalar strictly bigger than $Re(\lambda_{\max}(A_p))$. One needs to choose $\varepsilon_p$ and $Q_p$ such that the term

$$
2\varepsilon_p - \frac{\lambda_{\max}(Q_p)}{\lambda_{\min}(P_p)} \geq 0.
$$

Consequently, (4) follows with $\lambda_p < 0$.

The functions $V_p, p \in \mathcal{P}$ are called Lyapunov-like functions. Fact 1 captures the maximum rate of decay and minimum rate of growth of $V_p$ for $p \in \mathcal{P}_S$ and $p \in \mathcal{P}_U$, respectively. The scalar $\lambda_p$, $p \in \mathcal{P}$ gives a quantitative measure of (in)stability of the $p$-th subsystem in the above sense. We also require a measure of the minimum increase between the Lyapunov-like functions $V_p$ and $V_q$ caused by a transition from subsystem $p$ to subsystem $q$.

**Fact 2:** For each $(p, q) \in E(\mathcal{P})$, the respective Lyapunov-like functions are related as follows: there exists $\mu_{pq} > 0$ such that

$$
V_q(\xi) \geq \mu_{pq} V_p(\xi) \quad \text{for all } \xi \in \mathbb{R}^d.
$$

(6)

In the spirit of [4, Proposition 4] we propose a tight estimate of $\mu_{pq}$, $(p, q) \in E(\mathcal{P})$ as follows:
Proposition 1: Let the Lyapunov-like functions be defined as in (3) with each $P_p$ symmetric and positive definite, $p \in \mathcal{P}$. Then $\bar{\mu}_{pq}$ in (6) can be computed as

$$\bar{\mu}_{pq} = \lambda_{\min}(P_q P_p^{-1}), \quad (p,q) \in E(\mathcal{P}).$$

Proof: Recall that by definition of $V_p$, $p \in \mathcal{P}$ in (3), each $P_p$, $p \in \mathcal{P}$ is symmetric and positive definite. Hence, $P_p^{-1}$, $p \in \mathcal{P}$ exist. Also, $P_q P_p^{-1}$ is similar to the matrix $P_p^{1/2} (P_q P_p^{-1}) P_p^{1/2}$, and the latter is symmetric and positive definite. Since the spectrum of a matrix is invariant under similarity transformations, the eigenvalues of $P_q P_p^{-1}$ are the same as the eigenvalues of $P_p^{1/2} P_q P_p^{-1/2}$, and consequently, the eigenvalues of $P_q P_p^{-1}$ are real numbers. Now,

$$\inf_{\mathbb{R}^d} \frac{V_q(\xi)}{V_p(\xi)} = \inf_{\mathbb{R}^d} \frac{< P_q \xi, \xi >}{< P_p \xi, \xi >} = \inf_{\mathbb{R}^d} \frac{< P_q^{-1/2} P_p^{-1/2} \xi, \xi >}{< P_q^{1/2} P_p^{1/2} \xi, \xi >}.$$

Let $y = P_p^{-1/2} \xi$. Then the right-hand side of the above equality is same as

$$\inf_{\mathbb{R}^d} \frac{< P_q (P_p^{-1/2} y), P_p^{-1/2} y >}{< y, y >} = \inf_{\mathbb{R}^d} \frac{< P_q^{-1/2} P_p^{-1/2} y, y >}{< y, y >} = \lambda_{\min}(P_q^{-1/2} P_p^{-1/2}) = \lambda_{\min}(P_q P_p^{-1}).$$

Since $V_q(\xi) \geq \bar{\mu}_{pq} V_p(\xi)$ for all $\xi \in \mathbb{R}^d$, the constant $\bar{\mu}_{pq}$ satisfies (7).

Remark 2: Notice that there exist scalars $\bar{\lambda}_p > 0$, $p \in \mathcal{P}_S$ and $\bar{\lambda}_p > 0$, $p \in \mathcal{P}_U$ such that the Lyapunov-like functions defined in (3) satisfies

$$V_p(\gamma_p(t)) \leq \exp(-\bar{\lambda}_p t) V_p(\gamma_p(0)), \quad (8)$$

where $\gamma_p(\cdot)$ solves the $p$-th system dynamics in (1), $p \in \mathcal{P}$. Also, there exist scalars $\bar{\mu}_{pq} \geq 1$, $(p,q) \in E(\mathcal{P})$ such that the following inequality holds:

$$V_q(\xi) \leq \bar{\mu}_{pq} V_p(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^d. \quad (9)$$

Inequality (8) captures the minimum rate of decay and maximum rate of growth of $V_p$ for the asymptotically stable and unstable subsystems, respectively, while inequality (9) captures the maximum increase of the Lyapunov-like functions caused by a transition from subsystem $p$ to subsystem $q$. Inequalities (8) and (9) are utilized to provide stability guarantee for switched systems, see e.g., [5] and the references therein. We will rely on inequalities (4) and (6) to address instability of the switched system (2).

Fix $t > 0$. We let $N(t)$ denote the number of switches before (and including) $t$. Let

$$v(t) := \frac{N(t)}{t} \quad (10)$$

be the frequency of switching at $t$.

We let $N_{pq}(t)$ denote the number of times a switch from subsystem $p$ to subsystem $q$ has occurred before (and including) time $t$. It follows that

$$N(t) = \sum_{(p,q) \in E(\mathcal{P})} N_{pq}(t).$$

Let

$$\rho_{pq}(t) := \frac{N_{pq}(t)}{N(t)} \quad (11)$$

be the transition frequency from subsystem $p$ to subsystem $q$ on $[0,t]$, $(p,q) \in E(\mathcal{P})$.

We let $T_p(t)$ denote the total duration of activation of subsystem $p$ on $[0,t]$, $p \in \mathcal{P}$. Let

$$\eta_p(t) := \frac{T_p(t)}{t} \quad (12)$$

denote the fraction of activation of subsystem $p$ on the interval $[0,t]$, $p \in \mathcal{P}$.

III. Main results

Theorem 1: Consider a family of systems (1). The switched system (2) is unstable for every switching signal $\sigma \in \mathcal{S}$ that satisfies

$$\lim_{t \to +\infty} \left( v(t) \sum_{(p,q) \in E(\mathcal{P})} (\ln \bar{\mu}_{pq}) \rho_{pq}(t) - \sum_{p \in \mathcal{P}_S} |\bar{\lambda}_p| \eta_p(t) + \sum_{p \in \mathcal{P}_U} |\bar{\lambda}_p| \eta_p(t) \right) > 0, \quad (13)$$

where $\bar{\lambda}_p$, $p \in \mathcal{P}$ and $\bar{\mu}_{pq}$, $(p,q) \in E(\mathcal{P})$ are as in Facts 1 and 2, respectively, and $v(t)$, $\rho_{pq}(t)$, $(p,q) \in E(\mathcal{P})$ and $\eta_p(t)$, $p \in \mathcal{P}$ are as defined in (10), (11) and (12), respectively.

Proof: Recall that $0 = t_0 < \tau_1 < \cdots < \tau_{N(t)}$ are the switching instants before (and including) $t > 0$. In view of (4), we have

$$V_{\sigma(t)}(x(t)) \geq \exp(-\bar{\lambda}_{\sigma(\tau_{N(t)})}(t - \tau_{N(t)})) V_{\sigma(t)}(x(\tau_{N(t)})). \quad (14)$$

A straightforward iteration of (14) applying (4) and (6), we obtain

$$V_{\sigma(t)}(x(t)) \geq \exp \left( - \sum_{\tau_{N(t)} = \tau}^{\tau_{N(t) + 1} = t} \bar{\lambda}_{\sigma(\tau)}(\tau_{i+1} - \tau_i) \right. \times \left. \prod_{i=0}^{N(t)-1} \bar{\mu}_{\sigma(\tau_{i}) \sigma(\tau_{i+1})} \right) V_{\sigma(0)}(x(0)). \quad (15)$$

Now, from [5, Proof of Theorem 5], we have

$$\prod_{i=0}^{N(t)-1} \bar{\mu}_{\sigma(\tau_{i}) \sigma(\tau_{i+1})} = \exp \left( - \sum_{\tau_{N(t)} = \tau}^{\tau_{N(t) + 1} = t} \bar{\lambda}_{\sigma(\tau)}(\tau_{i+1} - \tau_i) \right) \quad (16)$$

and

$$\exp \left( - \sum_{\tau_{N(t)} = \tau}^{\tau_{N(t) + 1} = t} \bar{\lambda}_{\sigma(\tau)}(\tau_{i+1} - \tau_i) \right)$$
where transitions are unstable. We will call the set of all conditions such that completes our proof of Theorem 1. Clearly, condition (13) is sufficient to guarantee (23).

For \( t > 0 \), the right-hand side above is equal to

\[
\left( \psi(t) - \sum_{(p,q) \in \mathcal{E}(\mathcal{P})} (\ln \bar{\lambda}_{pq}) \rho_{pq}(t) - \sum_{p \in \mathcal{P}_S} |\bar{\lambda}_p| \eta_p(t) + \sum_{p \in \mathcal{P}_U} |\bar{\lambda}_p| \eta_p(t) \right),
\]

where \( \psi(t) \) and \( \eta_p(t) \), \( p \in \mathcal{P} \) are as defined in (10) and (12), respectively.

From the definition of \( V_p, p \in \mathcal{P} \) in (3), we have

\[
\alpha(\|\xi\|) \leq V_p(\xi) \leq \alpha(\|\xi\|) \quad \text{for all } \xi \in \mathbb{R}^d,
\]

where

\[
\alpha(r) := \min_{p \in \mathcal{P}} \left( \lambda_{\min}(P_p) \right) r^2
\]

and

\[
\overline{\alpha}(r) := \max_{p \in \mathcal{P}} \left( \lambda_{\min}(P_p) \right) r^2.
\]

It follows that

\[
\overline{\alpha}(\|x(t)\|) \geq \exp(\psi(t)\alpha(\|x(0)\|)).
\]

Armed with (22), to verify instability of the switched linear system (2) (according to Definition 2), we need to find conditions such that

\[
\lim_{t \to +\infty} \exp(\psi(t)) = +\infty.
\]

Clearly, condition (13) is sufficient to guarantee (23). This completes our proof of Theorem 1.

Given a family of systems (1) containing both asymptotically stable and unstable subsystems, and a set of admissible transitions \( \mathcal{E}(\mathcal{P}) \), in Theorem 1 we characterize a class of switching signals under which the switched system (2) is unstable. We will call the set of all \( \sigma \) that satisfy condition (13) as \( \mathcal{S}_\sigma \). Our characterization of instability of (2) relies on the asymptotic behavior of frequency of switching, frequency of transition between subsystems, and fraction of activation of subsystems; the properties of \( \sigma \) on every interval of time is not considered.

In (13), the fractions of activation of subsystems, \( \eta_p(t), p \in \mathcal{P} \), are weighted by the maximum rate of decay and minimum rate of growth of the corresponding Lyapunov-like function \( V_p, p \in \mathcal{P} \) for asymptotically stable and unstable subsystems, respectively. The weighing factors for frequency of transition between subsystems, \( \rho_{pq}(t), (p,q) \in \mathcal{E}(\mathcal{P}) \) are the scalars \( \bar{\lambda}_{pq}, (p,q) \in \mathcal{E}(\mathcal{P}) \) that give a measure of the minimum “jump” in the corresponding Lyapunov-like functions \( V_p \) and \( V_q \) caused by a transition from subsystem \( p \) to subsystem \( q \).

Remark 3: Let \( \mathcal{P}_U = \emptyset \). It is known that if all subsystems are linear and asymptotically stable, then a sufficiently “slow” switching signal between these subsystems preserves stability of the resulting switched system. On the one hand, known estimates of “slowness” [8], [2] are sufficient in the sense that switching at a faster rate does not necessarily guarantee instability of the resulting switched system. On the other hand, condition (13) provides a measure of how “fast” one needs to switch between asymptotically stable subsystems such that the resulting switched system loses stability.

Example 1: Consider a family of systems (1) with \( \mathcal{P} = \{1,2,3,4\} \) and \( \mathcal{E}(\mathcal{P}) = \{(1,2),(1,3),(2,1),(2,4),(3,1),(3,4), (4,2),(4,3)\} \).
\( N_{pq}(t) = \frac{N(t)}{\sigma}, \quad (p,q) \in E(P), \)
\( T_{p}(t) = \frac{t}{\gamma}, \quad p \in \{1,2\} \) and \( T_{p}(t) = \frac{3t}{\gamma}, \quad p \in \{3,4\}. \)
Consequently,
\( \nu(t) = \frac{1}{20}, \)
\( \rho_{pq}(t) = \nu(t) - \eta(t), \quad (p,q) \in E(P), \)
\( \eta_{p}(t) = \frac{1}{2}, \quad p \in \{1,2\} \) and \( \eta_{p}(t) = \frac{3}{2}, \quad p \in \{3,4\}. \)

We have that condition (13) holds.

In Figures 1 we illustrate \((\sigma(t))_{t \geq 0}\) and \((||x(t)||)_{t \geq 0}\) till \(t = 200\) units of time, respectively. For plotting \((||x(t)||)_{t \geq 0}\), ten different initial conditions are chosen from \([-10,10]^2\) uniformly at random. Divergence of \(||x(t)||\) is observed in each case.

![Figure 1: Plot of \((\sigma(t))_{t \geq 0}\) and \((||x(t)||)_{t \geq 0}\) for Example 1](image)

IV. Discussion

Given a family of systems (1) and a set of admissible transitions \(E(P)\), earlier in [5] the authors employed asymptotic behaviour of \(\nu(t), \eta_{p}(t), p \in P\) and \(\rho_{pq}(t), (p,q) \in E(P)\) to characterize a class of switching signals that ensures GAS of the resulting switched system (2). We will show that our class of destabilizing switching signals is a subset of the class of switching signals that does not satisfy the stability condition of [5]. Below we briefly recall the stability condition from [5] to set the stage.

**Theorem 2:** [5, Theorem 5] Consider a family of systems (1). The switched system (2) is globally asymptotically stable (GAS) for every switching signal \(\sigma \in S\) that satisfies

\[
\lim_{t \to +\infty} \left( \nu(t) \sum_{(p,q) \in E(P)} (\ln \hat{\mu}_{pq}) \rho_{pq}(t) - \sum_{p \in P \setminus \{S\}} |\hat{\lambda}_p| \eta_p(t) + \sum_{p \in P \setminus \{U\}} |\hat{\lambda}_p| \eta_p(t) \right) < 0, \quad (24)
\]

where \(\hat{\lambda}_p, p \in P\) and \(\hat{\mu}_{pq}, (p,q) \in E(P)\) are as in (8) and (9), respectively, and \(\nu(t), \rho_{pq}(t), (p,q) \in E(P)\) and \(\eta_p(t), p \in P\) are as defined in (10), (11) and (12), respectively.

Let \(S^c_g\) denote the set of all switching signals that satisfy condition (24).

**Proposition 2:** Consider a family of switching signals (1). Then the set \(S^c_g\) is a strict subset of \(S^c_u\).

**Proof:** It suffices to show that an element \(\sigma\) of the set \(S^c_g\) is not necessarily an element of the set \(S_u\). Consider a family of systems (1) with \(P = \{1,2\}\), where

\[
A_1 = \begin{pmatrix} -0.2 & -0.4 \\ 3 & -0.2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -0.2 & -3 \\ 0.4 & -0.2 \end{pmatrix}.
\]

Clearly, \(P_S = \{1,2\}\) and \(P_U = \emptyset\). Let \(E(P) = \{(1,2), (2,1)\}\).

We fix the Lyapunov-like functions \(V_p(\xi) = \langle P_p \xi, \xi \rangle\), \(p \in P\) with

\[
P_1 = \begin{pmatrix} 10.3629 \\ 0.5242 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 1.4516 \\ -0.5242 \end{pmatrix}.
\]

The scalars \(\hat{\lambda}_1, \hat{\lambda}_2, \rho_{pq}, \hat{\mu}_{pq}, (p,q) \in E(P)\) are obtained from \(P_p, p \in P\) as follows:

\[
\hat{\lambda}_1 = \hat{\lambda}_2 = 0.0962, \quad \hat{\lambda}_1 = \hat{\lambda}_2 = 0.7038,
\]
\[
\hat{\mu}_{12} = \hat{\mu}_{21} = 7.3149, \quad \hat{\mu}_{12} = \hat{\mu}_{21} = 0.1367.
\]

Now, consider a switching signal \(\sigma\) that satisfies:

\(\nu(t) = \frac{1}{10},\)
\(N_{pq}(t) = \frac{N(t)}{2}, \quad (p,q) \in E(P),\) and
\(T_{p}(t) = \frac{t}{2}, \quad p \in P.\)

Clearly,
\(\nu(t) = \frac{1}{10},\)
\(\rho_{pq}(t) = \frac{1}{2}, \quad (p,q) \in E(P),\) and
\(\eta_{p}(t) = \frac{1}{2}, \quad p \in P.\)

We have

\[
\lim_{t \to +\infty} \left( \nu(t) \sum_{(p,q) \in E(P)} (\ln \hat{\mu}_{pq}) \rho_{pq}(t) - \sum_{p \in P \setminus S} |\hat{\lambda}_p| \eta_p(t) + \sum_{p \in P \setminus U} |\hat{\lambda}_p| \eta_p(t) \right) = 0.1028 > 0,
\]
and

\[
\lim_{t \to +\infty} \left( \nu(t) \sum_{(p,q) \in E(P)} (\ln \hat{\mu}_{pq}) \rho_{pq}(t) - \sum_{p \in P \setminus S} |\hat{\lambda}_p| \eta_p(t) + \sum_{p \in P \setminus U} |\hat{\lambda}_p| \eta_p(t) \right) = -0.9028 < 0,
\]
It follows that \(\sigma \in S^c_g \subset S^c_u\). ■

Proposition 2 implies that there is a gap between the characterization of stabilizing and destabilizing switching signals based solely on the asymptotic behaviour of these...
signals. We use the term “gap” in the following sense: (de)stabilizing properties of the elements of $S^c_g \cap S^c_u$ cannot be determined from conditions (24) and (13). Indeed, for asymptotic characterization of stabilizing switching signals, convergence of $\|x(t)\|$ for all $x(0)$ under the “worst case” switching is ensured, while for asymptotic characterization of destabilizing switching signals divergence of $\|x(t)\|$ for all $x(0)$ under the “best case” switching is ensured. It is, therefore, immediate that conditions (24) and (13) are not sufficient to conclude (in)stability of the switched system (2) under the elements of the set $S^c_g \cap S^c_u$, and additional analysis tools capturing properties of subsystems are required. This problem is currently under investigation, and will be reported elsewhere.

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