Existence of Solution for an Elliptic Problem with a Sublinear Term

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abstract. In this work we prove the existence of a classical positive solution for an elliptic equation with a sublinear term. We use Galerkin approximations to show existence of such solution on bounded domains in \( \mathbb{R}^N \).

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1 Introduction

In this paper, we study existence of solution for the problem

\[
\begin{aligned}
-\Delta v &= \lambda v^q + f(v), & & \text{in } \Omega, \\
 v &> 0 & & \text{in } \Omega, \\
 v &= 0 & & \text{on } \partial\Omega,
\end{aligned}
\]  
(1)
where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary, $\lambda > 0$ is a parameter, $0 < q < 1$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

\begin{equation}
0 \leq f(s)s \leq C|s|^{p+1},
\end{equation}

where $1 < p \leq \frac{N+2}{N-2}$ if $N \geq 3$ or $1 < p$ if $N = 2$.

Our main result in this paper is the following:

**Theorem 1.1** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (2). Then, there exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ the problem (1) has a positive solution $u \in C^{2, \gamma}(\Omega)$, for some $\gamma \in (0, 1)$.

Elliptic problems of the type

\begin{equation}
\begin{cases}
-\Delta v = g(x, v) & \text{in } \Omega, \\
v = 0 & \text{on } \partial\Omega,
\end{cases}
\end{equation}

where $g(x, v)$ is continuous and behaves like $v^q + v^p$ as $|v| \to +\infty$ have been extensively studied; see for example [2, 3, 4] for a survey. One of the main results with nonlinearity combined effects of concave and convex was introduced in [4], namely, $g(x, u) = \lambda u^q + u^p$ with $0 < q < 1 < p$.

We say that $g$ has sublinear growth at $+\infty$ if for every $\sigma \geq 0$ we have

$$\lim_{|s| \to +\infty} \frac{g(x, s)}{|s|^\sigma} = 0$$

uniformly in $x$

and say that $g$ has superlinear growth at $+\infty$ if for every $\sigma \geq 0$ we have

$$\lim_{|s| \to +\infty} \frac{g(x, s)}{|s|^\sigma} = +\infty$$

uniformly in $x$.

We would like to righlight that the only assumptions which we assume are that $0 < q < 1$ and that $f$ is continuous and satisfies the growth condition (2). This way, the nonlinearity $g(s) = \lambda s^q + f(s)$ of problem (1) can have sublinear or superlinear growth at $+\infty$.

Most papers treat problem (3) by means of variational methods, then it is usually assumed that $g$ has sublinear or superlinear growth and, sometimes, $sg(s) \geq c|s|^p$, where $c > 0$ is a constant and $p > 2$; see for example [11]. Another common assumption on $g$ is the so-called Ambrosetti-Rabinowitz condition that means the following:

$$\exists R > 0 \text{ and } \theta > 2 \text{ such that } 0 < \theta G(x, s) \leq sg(x, s) \forall |s| \geq R \text{ and } x \in \Omega,$$

where $G(x, s) = \int_0^s g(x, \tau)d\tau$.

Even when the Ambrosetti-Rabinowitz condition can be dropped, it has to be assumed some condition to give compactness of Palais-Smale sequences or Cerami sequences. See for instance [6], where they assume

$g : \overline{\Omega} \times \mathbb{R}$ is continuous and $g(x, 0) = 0$;
exists \( t_0 > 0 \) and \( M > 0 \) such that \( 0 < G(x, s) \leq Mg(x, s) \forall|s| \geq t_0 \) and \( x \in \Omega; \)

\[
0 < 2G(x, s) \leq sg(x, s) \forall|s| \geq 0 \text{ and } x \in \Omega.
\]

See also [9].

We are able to solve (1) under weaker assumptions by using the Galerkin method. For that matter we approximate \( f \) by Lipschitz functions in Section 2. In Section 3 we solve approximate problems. In Section 4 we prove a regularity result to approximate problems. Section 5 is devoted to prove Theorem 1.1; in doing so we show that solutions \( v_n \) of approximate problems are bounded away from zero and converge to a positive solution of (1).

At last in this introduction, we would like to emphasize that a similar approach was already used in [1], but different to that, we do not assume that the nonlinearity \( f \) is Lipschitz continuous.

### 2 Approximating functions

In order to proof Theorem 1.1 we make use of the following approximation result by Lipschitz functions, proved by Strauss in [10].

**Lemma 2.1** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( sf(s) \geq 0 \) for all \( s \in \mathbb{R} \). Then, there exists a sequence \( f_k : \mathbb{R} \to \mathbb{R} \) of continuous functions satisfying \( sf_k(s) \geq 0 \) and

(i) \( \forall k \in \mathbb{N}, \exists c_k > 0 \) such that \( |f_k(\xi) - f_k(\eta)| \leq c_k|\xi - \eta|, \) for all \( \xi, \eta \in \mathbb{R} \).

(ii) \( (f_k) \) converges uniformly to \( f \) in bounded subsets of \( \mathbb{R} \).

The proof consists in considering the following family of approximation functions \( f_k : \mathbb{R} \to \mathbb{R} \) defined by

\[
(4) \quad f_k(s) = \begin{cases} 
-k[G(-k - \frac{1}{k}) - G(-k)], & \text{if } s \leq -k, \\
-k[G(s - \frac{1}{k}) - G(s)], & \text{if } -k \leq s \leq -\frac{1}{k}, \\
k^2s[G(-\frac{2}{k}) - G(-\frac{1}{k})], & \text{if } -\frac{1}{k} \leq s \leq 0, \\
k^2s[G(\frac{2}{k}) - G(\frac{1}{k})], & \text{if } 0 \leq s \leq \frac{1}{k}, \\
k[G(s + \frac{1}{k}) - G(s)], & \text{if } \frac{1}{k} \leq s \leq k, \\
k[G(k + \frac{1}{k}) - G(k)], & \text{if } s \geq k.
\end{cases}
\]

where \( G(s) = \int_{0}^{s} f(\tau)d\tau \).

The sequence \( (f_k) \) of the previous lemma has some additional properties.
Lemma 2.2 Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function such that \( sf(s) \geq 0 \) for all \( s \in \mathbb{R} \). Let us suppose that there exist constants \( C > 0 \) and \( 1 < p \leq \frac{N+2}{N-2} \) such that

\[
(5) \quad sf(s) \leq C|s|^{p+1}, \quad \forall s \in \mathbb{R}.
\]

Then, the sequence \((f_k)_{k \in \mathbb{N}}\) from Lemma 2.1 satisfies

(i) \( 0 \leq sf_k(s) \leq C_1|s|^{p+1} \) for all \( |s| \geq \frac{1}{k} \),

(ii) \( 0 \leq sf_k(s) \leq C_2|s|^2 \) for all \( |s| \leq \frac{1}{k} \),

where \( C_1, C_2 \) do not depend on \( k \).

Proof: Everywhere in this proof, the constant \( C \) is the one given by (2).

First step: Suppose \(-k \leq s \leq -\frac{1}{k}\).

By the mean value theorem, there exists \( \eta \in (s - \frac{1}{k}, s) \) such that

\[
f_k(s) = -k[G(s - \frac{1}{k}) - G(s)] = -kG'(\eta)(s - \frac{1}{k} - s) = f(\eta)
\]

and

\[
sf_k(s) = sf(\eta).
\]

As \( s - \frac{1}{k} < \eta < s < 0 \) and \( f(\eta) < 0 \), we have \( sf(\eta) \leq \eta f(\eta) \). Therefore,

\[
sf_k(s) \leq \eta f(\eta) \leq C|\eta|^{p+1} \leq C|s - \frac{1}{k}|^{p+1} \leq C(|s| + \frac{1}{k})^{p+1} \leq C2^{p+1}|s|^{p+1}.
\]

Second step: Suppose \( \frac{1}{k} \leq s \leq k \).

By the mean value theorem, there exist \( \eta \in (s, s + \frac{1}{k}) \) such that

\[
f_k(s) = k[G(s + \frac{1}{k}) - G(s)] = kG'(\eta)(s + \frac{1}{k} - s) = f(\eta)
\]

and

\[
sf_k(s) = sf(\eta).
\]

As \( 0 < s < \eta < s + \frac{1}{k} \) and \( f(\eta) > 0 \), we have \( sf(\eta) \leq \eta f(\eta) \). Therefore,

\[
sf_k(s) \leq \eta f(\eta) \leq C|\eta|^{p+1} \leq C|s + \frac{1}{k}|^{p+1} = C(|s| + \frac{1}{k})^{p+1} \leq C2^{p+1}|s|^{p+1}.
\]

Third step: Suppose \( |s| \geq k \).

Define

\[
f_k(s) = \begin{cases} 
-k[G(-k - \frac{1}{k}) - G(-k)], & \text{if } s \leq -k, \\
(k[G(k + \frac{1}{k}) - G(k)], & \text{if } s \geq k.
\end{cases}
\]
If $s \leq -k$, by the mean value theorem, there exist $\eta \in (-k - \frac{1}{k}, -k)$ such that
$$f_k(s) = k[G(-k - \frac{1}{k}) - G(-k)] = -kG'(\eta)(-k - \frac{1}{k} - (-k)) = f(\eta)$$
and
$$sf_k(s) = sf(\eta).$$
As $-k - \frac{1}{k} < \eta < -k < 0$ and $k < |\eta| < k + \frac{1}{k}$, we have $sf(\eta) = \frac{s}{\eta}sf(\eta)$. Therefore,
$$sf_k(s) = \frac{s}{\eta}sf(\eta) \leq \left| \frac{s}{\eta} \right| C|\eta|^{p+1} =$$
$$= C|s||\eta|^p \leq C|s|(k + \frac{1}{k})^p \leq C|s|(|s| + \frac{1}{k})^p \leq C2^p|s|^{p+1}.$$  

If $s \geq k$, by the mean value theorem, there exist $\eta \in (k, k + \frac{1}{k})$ such that
$$f_k(s) = k[G(k + \frac{1}{k}) - G(k)] = kG'(\eta)(k + \frac{1}{k} - k) = f(\eta)$$
and
$$sf_k(s) = sf(\eta) = \frac{s}{\eta}sf(\eta) \leq \left| \frac{s}{\eta} \right| C|\eta|^{p+1} =$$
$$= C|s||\eta|^p \leq C|s|(k + \frac{1}{k})^p \leq C|s|(|s| + \frac{1}{k})^p \leq C2^p|s|^{p+1}.$$  

**Fourth step:** Suppose $-\frac{1}{k} \leq s \leq \frac{1}{k}$.
Define
$$f_k(s) = \begin{cases} 
  k^2s[G(-\frac{2}{k}) - G(-\frac{1}{k})], & \text{if } -\frac{1}{k} \leq s \leq 0, \\
  k^2s[G(\frac{2}{k}) - G(\frac{1}{k})], & \text{if } 0 \leq s \leq \frac{1}{k}.
\end{cases}$$

If $-\frac{1}{k} \leq s \leq 0$, by the mean value theorem, there exists $\eta \in (-\frac{2}{k}, -\frac{1}{k})$ such that
$$f_k(s) = k^2s[G(-\frac{2}{k}) - G(-\frac{1}{k})] = k^2sG'(\eta)(-\frac{2}{k} - (-\frac{1}{k})) = -ksf(\eta).$$
Therefore,
$$sf_k(s) = -ksf^2(\eta) = -k\frac{s^2}{\eta}sf(\eta) \leq k\frac{s^2}{|\eta|}sf(\eta)$$
$$\leq Ck|s|^2|\eta|^p \leq Ck|s|^2(\frac{2}{k})^p \leq C2^p|s|^2.$$  

If $0 \leq s \leq \frac{1}{k}$, by the mean value theorem, there exist $\eta \in (\frac{1}{k}, \frac{2}{k})$ such that
$$f_k(s) = k^2s[G(\frac{2}{k}) - G(\frac{1}{k})] = k^2sG'(\eta)(\frac{2}{k} - \frac{1}{k}) = ksf(\eta).$$
Therefore,
$$sf_k(s) = ks^2f(\eta) = k\frac{s^2}{|\eta|}sf(\eta)$$
$$\leq Ck|s|^2|\eta|^p \leq Ck|s|^2(\frac{2}{k})^p \leq C2^p|s|^2.$$  

The proof of the lemma follows by taking $C_1 = C2^{p+1}$ and $C_2 = C2^p$, where $C$ is like in (5).
# 3 Approximate problem

In order to prove Theorem 1.1, we first study the auxiliary problem

\[ \begin{cases} -\Delta v = \lambda v^q + f_n(v) + \frac{1}{n} & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial \Omega, \end{cases} \]

where \( 0 < q < 1, \lambda > 0 \) is a parameter and \( f_n : \mathbb{R} \to \mathbb{R} \) is a function of the sequence given by Lemma 2.1 and Lemma 2.2.

We will use the Galerkin method together with the following fixed point theorem, see [10] and [8, Theorem 5.2.5]. A similar approach was already used in [1].

**Proposition 3.1** Let \( F : \mathbb{R}^d \to \mathbb{R}^d \) be a continuous function such that \( \langle F(\xi), \xi \rangle \geq 0 \) for every \( \xi \in \mathbb{R}^d \) with \( |\xi| = r \) for some \( r > 0 \). Then, there exists \( z_0 \) in the closed ball \( B_r(0) \) such that \( F(z_0) = 0 \).

The main result in this section is the following theorem.

**Theorem 3.2** There exists \( \lambda^* > 0 \) and \( n^* \in \mathbb{N} \) such that (6) has a weak positive solution for all \( \lambda \in (0, \lambda^*) \) and \( n \geq n^* \).

**Proof:** Fix \( B = \{w_1, w_2, \ldots, w_m, \ldots\} \) a orthonormal basis of \( H_0^1(\Omega) \) and define

\[ W_m = [w_1, w_2, \ldots, w_m], \]

to be the space generated by \( \{w_1, w_2, \ldots, w_m\} \). Define the function \( F : \mathbb{R}^m \to \mathbb{R}^m \) such that \( F(\xi) = (F_1(\xi), F_2(\xi), \ldots, F_m(\xi)) \), where

\[ F_j(\xi) = \int_{\Omega} \nabla v \nabla w_j - \lambda \int_{\Omega} (v_+)^q w_j - \int_{\Omega} f_n(v_+) w_j - \frac{1}{n} \int_{\Omega} w_j \quad j = 1, 2, \ldots, m \]

and let \( v = \sum_{i=1}^{m} \xi_i w_i \). Therefore,

\[ \langle F(\xi), \xi \rangle = \int_{\Omega} \nabla v^2 - \lambda \int_{\Omega} (v_+)^{q+1} - \int_{\Omega} f_n(v_+) v_+ - \frac{1}{n} \int_{\Omega} v. \]

Given \( v \in W_m \) we define

\[ \Omega_n^+ = \{x \in \Omega : |v(x)| \geq \frac{1}{n} \} \]

and

\[ \Omega_n^- = \{x \in \Omega : |v(x)| < \frac{1}{n} \}. \]

Thus we rewrite (7) as

\[ \langle F(\xi), \xi \rangle = \langle F(\xi), \xi \rangle_P + \langle F(\xi), \xi \rangle_N. \]
where

$$\langle F(\xi), \xi \rangle_P = \int_{\Omega^+} |\nabla v|^2 - \lambda \int_{\Omega^+} (v^+_q)^{q+1} - \int_{\Omega^+} f_n(v^+)_v + \frac{1}{n} \int_{\Omega^+} v$$

and

$$\langle F(\xi), \xi \rangle_N = \int_{\Omega^-} |\nabla v|^2 - \lambda \int_{\Omega^-} (v^+_q)^{q+1} - \int_{\Omega^-} f_n(v^+)_v + \frac{1}{n} \int_{\Omega^-} v.$$ 

**Step 1.** Since \(0 < q < 1\), then

$$\int_{\Omega^+} (v^+_q)^{q+1} \leq \int_{\Omega} |v|^{q+1} = \|v\|^{q+1}_{L^{q+1}(\Omega)} \leq C_1 \|v\|^{q+1}_{H^1_0(\Omega)}.$$ 

By virtue of (i) Lemma 2.2 we get

$$\int_{\Omega^+} f_n(v^+)_v \leq C \int_{\Omega} |v^+_q|^{p+1} dx \leq C_2 \|v\|^{p+1}_{H^1_0(\Omega)}.$$ 

It follows from (8) and (9) that

$$\langle F(\xi), \xi \rangle_P \geq \int_{\Omega^+} |\nabla v|^2 - \lambda C_1 \|v\|^{q+1}_{H^1_0(\Omega)} - C_2 \|v\|^{p+1}_{H^1_0(\Omega)} - \frac{C_3}{n} \|v\|_{H^1_0(\Omega)}^2,$$

where \(C_1, C_2\) and \(C_3\) depends on \(C\) and \(|\Omega|\).

**Step 2.** Since \(0 < q < 1\), then

$$\int_{\Omega^-} (v^+_q)^{q+1} \leq \int_{\Omega^-} |v|^{q+1} = \|v\|^{q+1}_{L^{q+1}(\Omega)} \leq \frac{1}{n^q+1}.$$ 

By virtue of (ii) Lemma 2.2 we get

$$\int_{\Omega^-} f_n(v^+)_v \leq C \int_{\Omega^-} |v^+_q|^2 dx \leq C |\Omega| \frac{1}{n^2}.$$ 

It follows from (11) and (12) that

$$\langle F(\xi), \xi \rangle_N \geq \int_{\Omega^-} |\nabla v|^2 - \lambda |\Omega| \frac{1}{n^q+1} - C |\Omega| \frac{1}{n^2} - |\Omega| \frac{1}{n^2}.$$ 

It follows from (10) and (13) that

$$\langle F(\xi), \xi \rangle \geq \|v\|^{2}_{H^1_0(\Omega)} - \lambda |\Omega| \frac{1}{n^q+1} - C \|v\|^{p+1}_{H^1_0(\Omega)} - \frac{C_3}{n} \|v\|^2_{H^1_0(\Omega)} - \lambda |\Omega| \frac{1}{n^q+1} - C |\Omega| \frac{1}{n^2} - |\Omega| \frac{1}{n^2}.$$ 

Assume now that \(\|v\|_{H^1_0(\Omega)} = r\) for some \(r > 0\) to be fixed later. Hence,

$$\langle F(\xi), \xi \rangle \geq r^2 - \lambda C_1 r^{q+1} - C_2 r^{p+1} - \frac{C_3}{n} r - \lambda |\Omega| \frac{1}{n^q+1} - C |\Omega| \frac{1}{n^2} - |\Omega| \frac{1}{n^2}.$$
We want to choose $r$ such that
\[ r^2 - C_2 r^p + 1 \geq \frac{r^2}{2}, \]
in other words,
\[ r \leq \frac{1}{(2C_2)^{\frac{1}{p-1}}}. \]
Choosing $r = \frac{1}{2(2C_2)^{\frac{1}{p-1}}}$, we obtain
\[ \langle F(\xi), \xi \rangle \geq \frac{r^2}{2} - \lambda C_1 r^{q+1} - \frac{C_3}{n} r - \lambda |\Omega| \frac{1}{n^{q+1}} - C|\Omega| \frac{1}{n^2} - |\Omega| \frac{1}{n^2}. \]

Now, defining $\rho = \frac{r^2}{2} - \lambda C_1 r^{q+1}$, we choose $\lambda^* > 0$ such that $\rho > 0$ for $\lambda < \lambda^*$. Therefore, we choose $\lambda^* = \frac{r^2}{\lambda C_1}$. Now we choose $n^* \in \mathbb{N}$ such that
\[ \frac{C_3}{n} r + \lambda |\Omega| \frac{1}{n^{q+1}} + C|\Omega| \frac{1}{n^2} + |\Omega| \frac{1}{n^2} < \frac{\rho}{2}, \]
for every $n \geq n^*$. Let $\xi \in \mathbb{R}^m$, such that $|\xi| = r$, then for $\lambda < \lambda^*$ and $n \geq n^*$ we obtain
\[ \langle F(\xi), \xi \rangle \geq \frac{\rho}{2} > 0. \]

Since $f_n$ is a Lipschitz continuous function for every $n$, by standard arguments it is shown that $F$ is continuous, that is, give $(x_k)$ in $\mathbb{R}^m$ and $x \in \mathbb{R}^m$ such that $x_k \to x$ we obtain $F(x_k) \to F(x)$.

Therefore, by Proposition 3.1 for all $m \in \mathbb{N}$ there exists $y \in \mathbb{R}^m$ with $|y| \leq r$ such that $F(y) = 0$, that is, there exists $v_m \in W_m$ verifying $\|v_m\|_{H^1_0(\Omega)} \leq r$, for every $m \in \mathbb{N}$ and such that
\[ \int_\Omega \nabla v_m \nabla w = \lambda \int_\Omega (v_m^q) w + \int_\Omega f_n(v_m^q) w + \frac{1}{n} \int_\Omega w, \forall w \in W_m. \]

Since $W_m \subset H^1_0(\Omega)$, $\forall m \in \mathbb{N}$, and $r$ does not depend on $m$, then $(v_m)$ is a bounded sequence of $H^1_0(\Omega)$. Then, for some subsequence, there exists $v = v_n \in H^1_0(\Omega)$ such that
\[ v_m \rightharpoonup v \text{ weakly in } H^1_0(\Omega) \]
and
\[ v_m \to v \text{ in } L^2(\Omega) \text{ and a.e. in } \Omega. \]

Fixing $k \in \mathbb{N}$ and for every $m$ such that $m \geq k$ we obtain
\[ \int_\Omega \nabla v_m \nabla w_k = \lambda \int_\Omega (v_m^q) w_k + \int_\Omega f_n(v_m^q) w_k + \frac{1}{n} \int_\Omega w_k, \forall w_k \in W_k. \]
Now, as \( g : H^1_0(\Omega) \to \mathbb{R} \) defined by \( g(u) = \int_\Omega \nabla u \nabla w_k \), for every \( u \in H^1_0(\Omega) \), we have that \( g \) is a continuous linear functional. It follows from (14) that

\[
\int_\Omega \nabla v_m \nabla w_k \to \int_\Omega \nabla v \nabla w_k \quad \text{as} \quad m \to \infty
\]

and by (15), we obtain

\[
\int_\Omega f_n(v_m + w_k) \to \int_\Omega f_n(v + w_k) \quad \text{as} \quad m \to \infty.
\]

Indeed, by Lemma 2.1 (ii) it follows that \( |f_n(v_m + w_k) - f_n(v + w_k)| \leq c_n|v_m + w_k| \), hence

\[
\left| \int_\Omega f_n(v_m + w_k) - \int_\Omega f_n(v + w_k) \right| \leq c_n \| w_k \|_{L^2(\Omega)} \| v_m - v \|_{L^2(\Omega)} \quad \text{as} \quad m \to \infty,
\]

and then, (15) implies (18). By (14), (18) and Sobolev compact embedding, letting \( m \to \infty \), we obtain

\[
\lambda \int_\Omega (v_m + w_k)^2 \to \int_\Omega (v + w_k)^2 \quad \text{as} \quad m \to \infty.
\]

By (14), (17), (19) and by the uniqueness of the limit, we obtain

\[
\int_\Omega \nabla v \nabla w_k = \lambda \int_\Omega (v + w_k)^2 + \int_\Omega f_n(v + w_k) \quad \forall \ w_k \in W_k.
\]

For density of \([W_k]_{k \in \mathbb{N}}\) in \( H^1_0(\Omega) \) and by linearity, we conclude that

\[
\int_\Omega \nabla v \nabla w = \lambda \int_\Omega (v + w)^2 + \int_\Omega f_n(v + w) \quad \forall \ w \in H^1_0(\Omega).
\]

Furthermore, \( v \geq 0 \) in \( \Omega \). In fact, as \( v_+ \in H^1_0(\Omega) \), we obtain from (20) that

\[
\int_\Omega \nabla v \nabla v_+ = \lambda \int_\Omega (v_+)^2 + \int_\Omega f_n(v_+) \quad \forall \ w \in H^1_0(\Omega).
\]

Hence, we have from Lemma 2.1 that

\[
0 \geq -\|v_+\|^2_{H^1_0(\Omega)} = \int_\Omega \nabla v \nabla v_+ = \int_\Omega f_n(v_+) \quad \forall \ w \in H^1_0(\Omega)
\]

with the result that \( \|v_+\|_{H^1_0(\Omega)} = 0 \), that is, \( v_+(x) = 0 \) a.e. in \( \Omega \). Therefore, \( v(x) = v_+(x) \geq 0 \) a.e. in \( \Omega \) and we conclude the proof of the theorem.
4 Regularity of Solution of the Approximate Problem

In this section, we show that all weak solutions of the problem (6) are regular. Let \( v \in H^1_0(\Omega) \) be a weak solution of the problem (6) and define

\[
g(x) := \lambda v^q(x) + f_n(v(x)) + \frac{1}{n}.
\]

We have that

\[
|g| \leq \lambda |v|^q + |f_n(v)| + \frac{1}{n}, \tag{21}
\]

Notice that

\[
|v|^q \leq 1 + |v|^{t-1}, \tag{22}
\]

where \(2 \leq t \leq 2^*\). Here, \(2^*\) is the critical Sobolev exponent, that is,

\[
2^* = \frac{2N}{N-2}.
\]

Furthermore, since \(f_n : \mathbb{R} \to \mathbb{R}\) is a Lipschitz continuous function and \(f_n(0) = 0\), we have for each \(n \in \mathbb{N}\) that

\[
|f_n(v)| \leq C_n|v|,
\]

and consequently,

\[
|f_n(v)| \leq C_n(1 + |v|^{t-1}), \tag{23}
\]

where \(2 \leq t \leq 2^*\). This way, by combining (21), (22) and (23), we obtain

\[
|g| \leq C_1 + C_2|v|^{t-1}, \tag{24}
\]

where

\[
C_1 := \lambda + C_n + \frac{1}{n}
\]

and

\[
C_2 := \lambda + C_n.
\]

Then, using (24) and well-known Bootstrap arguments, similar to those found in [7], we conclude that \(v \in C^{2,\gamma}(\Omega)\), for some \(\gamma \in (0,1)\).
5 Proof of the Theorem 1.1

In this section, we demonstrate Theorem 1.1. The following lemma of [10, Theorem 1.1] is used to show that \( v_n \) converges to a solution \( v \) of (1).

**Lemma 5.1** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^N \), \( u_k : \Omega \to \mathbb{R} \) be a sequence of functions and \( g_k : \mathbb{R} \to \mathbb{R} \) be a sequence of functions such that \( g_k(u_k) \) are measurable in \( \Omega \) for every \( k \in \mathbb{N} \). Assume that \( g_k(u_k) \to v \) a.e. in \( \Omega \) and \( \int_{\Omega} |g_k(u_k)u_k| \, dx < C \) for a constant \( C \) independent of \( k \). Suppose that for every bounded set \( B \subset \mathbb{R} \) there is a constant \( C_B \) depending only on \( B \) such that \( |g_k(x)| \leq C_B \), for all \( x \in B \) and \( k \in \mathbb{N} \). Then \( v \in L^1(\Omega) \) and \( g_k(u_k) \to v \) in \( L^1(\Omega) \).

Since \( v \in C^{2,\gamma}(\Omega) \), \( \gamma \in (0,1) \), satisfies \( v \geq 0 \) and

\[-\Delta v = \lambda v^q + f_n(v) + \frac{1}{n},\]

it follows by assumptions on \( f_n \) that

\[-\Delta v \geq 0.\]

Then, by Maximum Principle, we have \( v > 0 \) in \( \Omega \), that is, \( v \) is a solution of the problem (6). For each \( n \in \mathbb{N} \), let us denote by \( v_n \) the solution of (6). It follows from (14) that

\[ v_m^{(n)} \rightharpoonup v_n \text{ weakly in } H^1_0(\Omega) \text{ as } m \to \infty, \]

where, for each \( n \in \mathbb{N} \), \( (v_m^{(n)})_{m \in \mathbb{N}} \) is a sequence in \( H^1_0(\Omega) \) satisfying

\[ ||v_m^{(n)}|| \leq r, \quad \forall m \in \mathbb{N}. \]

Then,

\[ ||v_n|| \leq \liminf_{m \to \infty} ||v_m^{(n)}|| \leq r, \quad \forall n \in \mathbb{N}. \]

Since \( r \) does not depend on \( n \), there exists \( v \in H^1_0(\Omega) \) such that

\[ v_n \rightharpoonup v \text{ weakly in } H^1_0(\Omega). \]

By compact embedding, up to a subsequence, we have

\[ v_n \to v \text{ in } L^s(\Omega), \quad \text{for } 1 \leq s < 2^* \text{ if } N \geq 3 \text{ or for } 1 \leq s < +\infty \text{ if } N = 2, \]

and then, up to a subsequence,

i) \( v_n(x) \to v(x) \) a.e. in \( \Omega \);

ii) \( |v_n(x)| \leq h(x), \quad \forall n \in \mathbb{N} \) a.e. in \( \Omega \), for some \( h \in L^s(\Omega) \).
Notice that the following inequality holds:
\[
\begin{cases}
-\Delta v_n \geq \lambda v_n^q, & \text{in } \Omega, \\
v_n > 0 & \text{in } \Omega, \\
v_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]
This way, considering \( w_n = \lambda^{\frac{1}{q-1}} v_n \), we obtain
\[
-\Delta \left( \frac{w_n}{\lambda^{\frac{1}{q-1}}} \right) \geq \lambda \left( \frac{w_n}{\lambda^{\frac{1}{q-1}}} \right)^q,
\]
and consequently,
\[
-\Delta w_n \geq w_n^q.
\]
Let us denote by \( \tilde{w} \) the unique solution of the problem
\[
\begin{cases}
-\Delta \tilde{w} = \tilde{w}^q, & \text{in } \Omega, \\
\tilde{w} > 0 & \text{in } \Omega, \\
\tilde{w} = 0 & \text{on } \partial \Omega.
\end{cases}
\]
The existence and uniqueness of such solution is proved in [5]. By Lemma 3.3 of [4], it follows that \( w_n \geq \tilde{w}, \forall n \in \mathbb{N} \), that is,
\[
(25) \quad v_n(x) \geq \lambda^{\frac{1}{q-1}} \tilde{w}(x), \text{ a.e. in } \Omega, \forall n \in \mathbb{N}.
\]
Taking the limit as \( n \to +\infty \) in \((25)\), we obtain
\[
(26) \quad v(x) \geq \lambda^{\frac{1}{q-1}} \tilde{w}(x), \text{ a.e. in } \Omega
\]
and hence \( v > 0 \text{ a.e. in } \Omega \).

Recall that, from \(20\),
\[
\int_\Omega \nabla v_n \nabla w = \lambda \int_\Omega (v_n)^q w + \int_\Omega f_n(v_n)w + \frac{1}{n} \int_\Omega w, \quad \forall w \in H^1_0(\Omega),
\]
and using that \( v_n \) is a classical solution we have
\[
(26) \quad -\Delta v_n = \lambda (v_n)^q + f_n(v_n) + \frac{1}{n} \text{ in } L^2(\Omega).
\]
Since
\[
v_n \to v \text{ a.e. in } \Omega,
\]
we have
\[
(27) \quad f_n(v_n(x)) \to f(v(x)) \text{ a.e. in } \Omega
\]
by the uniform convergence of Lemma \(2.1\) (ii).

Multiplying the equation (26) by \(w = v_n\) and since \(v_n\) is bounded in \(H^1_0(\Omega)\) we obtain

\[
\int_{\Omega} f_n(v_n)v_n dx \leq C,
\]

for every \(n \in \mathbb{N}\), where \(C > 0\) is a constant independent of \(n\). By (27), (28) and by the expression of \(f_n\) defined in (4), the assumptions of Lemma 5.1 are satisfied implying

\[
f_n(v_n) \to f(v) \text{ strongly in } L^1(\Omega).
\]

Multiplying (26) by \(w \in \mathcal{D}(\Omega)\), integrating on \(\Omega\) and using the previous convergences, we have

\[
-\Delta v = \lambda v^q + f(v) \text{ in } \mathcal{D}'(\Omega).
\]

Since \(f(v) \in L^{\frac{p+1}{p}}(\Omega)\) and \(\lambda v^q \in L^{\frac{p+1}{p}}(\Omega)\), we conclude from (29) that \(v \in H^1_0(\Omega) \cap W^{2, \frac{p+1}{p}}(\Omega)\) and

\[
-\Delta v = \lambda v^q + f(v)
\]

in the strong sense. Notice that the assumption (2) implies that

\[
|f(s)| \leq C|s|^{t-1},
\]

where \(2 \leq t \leq 2^*\). Thus, using well-known Bootstrap arguments, we conclude that \(v \in C^{2, \gamma}(\Omega)\), for some \(\gamma \in (0, 1)\), and it is a classical positive solution of problem (1).

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