Ambipolar diffusion in the magnetorotational instability

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ABSTRACT

The effects of ambipolar diffusion on the linear stability of weakly ionised accretion discs are examined. Earlier work on this topic has focused on axial magnetic fields and perturbation wavenumbers. We consider here more general field and wavenumber geometries, and find that qualitatively new results are obtained. Provided a radial wavenumber and azimuthal field are present along with their axial counterparts, ambipolar diffusion will always be destabilising, with unstable local modes appearing at well-defined wavenumber bands. The wavenumber corresponding to the maximum growth rate need not, in general, lie along the vertical axis. Growth rates become small relative to the local angular velocity when the ion-neutral collision time exceeds the orbital time. In common with Hall electromotive forces, ambipolar diffusion destabilises both positive and negative angular velocity gradients. In at least some cases, therefore, uniformly rotating molecular cloud cores may reflect the marginally stable state of the ambipolar magnetorotational instability.

Key words: accretion, accretion discs - instabilities - MHD - turbulence, ambipolar diffusion.

1 INTRODUCTION

The behaviour of the magnetorotational instability (MRI) in weakly ionised gas is a problem of considerable importance, bearing directly on our understanding of the dynamics of protostellar discs. The low ionisation regime is characterized not only by increased Ohmic dissipative losses (Jin 1996), but by Hall (Balbus & Terquem 2001; Wardle 1999) and ambipolar diffusion (Blaes & Balbus 1994) effects as well. The latter two effects are of particular interest, since they can be destabilising under certain circumstances.

The Hall and ambipolar diffusion regimes are generally important in distinct regions of parameter space (Balbus & Terquem 2001): the former is dominant (along with Ohmic losses) at the high densities typical of inner regions of protostellar discs, whereas the latter dominates in the outer regions of such discs (where they are typically observed), and under interstellar conditions. In this paper, except for a brief excursion in the Appendix, we confine ourselves to the ambipolar regime, and study the general axisymmetric behaviour of the MRI in these environments.

The plan of the paper is as follows: In §2, we discuss the basic formulation of the problem, including a presentation of temperature and number density regimes associated with non-ideal MHD effects. Section 3 is a linear analysis of differentially rotating discs for axial wavenumber and field configurations. This serves to unite this paper with previous work on ambipolar diffusion. Section 4 extends this analysis for arbitrary field geometries and perturbation wavenumbers, and contains the key results of the paper. In §5, we present a brief summary.

2 PRELIMINARIES

The fundamental fluid equations are mass conservation,
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0;
\]
and the equation of motion,
\[
\frac{\partial \rho \mathbf{v}}{\partial t} + (\rho \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \frac{B^2}{8} \mathbf{r} + \frac{B}{4} \mathbf{r} \cdot B;
\]

Our notation is standard: \(\rho\) the mass density, \(\mathbf{v}\) the velocity, \(P\) the gas pressure, \(B\) the magnetic field, and \(\nabla\) the gravitational potential. The density, velocity, and pressure all refer to the dominant neutral species.
The induction equation is (see e.g., Balbus & Terquem 2001):

$$\frac{\partial B}{\partial t} = r \cdot v \cdot B \cdot \frac{4}{c} \cdot \frac{J}{n_e e} + \frac{(J \cdot B) \cdot B}{c^2}$$  \hspace{1cm} (3)

Here $\rho$ is the resistivity, $c$ the speed of light, $J$ the current density, $n_e$ the electron density, $e$ the magnitude of the electron charge, $\gamma$ the electron-ion drag coefficient, and $m_i$ the ion mass density. Numerical values of $\gamma$ and $m_i$ are given in [Draine et al. 1983; Balbus & Terquem 2001]:

$$n_e \approx 2.34 \times 10^{-1} \text{cm}^{-2}, \quad T \approx 10^{3} \text{K}, \quad \gamma \approx 0.1 \text{g cm}^{-1}$$

where $n$ is the number density of the neutrals and $T$ is the temperature. Finally, for future reference, we define the Alfvén velocity in the usual way:

$$v_A = \frac{B}{\sqrt{\rho \mu_0}}$$  \hspace{1cm} (4)

The terms on the right side of equation (3) correspond respectively to Faraday induction, Ohmic resistivity, Hall electromotive forces (HEMFs), and ambipolar diffusion. The relative ratio of the ambipolar to Hall terms is given by [Balbus & Terquem 2001]:

$$A \approx 1.0 \times 10^{3}, \quad H \approx 1.0 \times 10^{1}, \quad \frac{v_A}{c_s} \approx 1;$$  \hspace{1cm} (5)

where $c_s$ is the isothermal sound speed. Assuming that the final two factors are each about 0.1, we see that a neutral density below about $10^{3} \text{cm}^{-3}$ brings us into the ambipolar diffusion regime (see Fig. 1), and we shall assume that this restriction is satisfied. We retain the resistivity in $\gamma$, however, to illustrate the simple relation of ambipolar diffusion to Ohmic dissipation for the case of axial field geometry, and in the Appendix, where both Hall and Ohmic terms are included in the derivation of a very general dispersion relation.

### 3 AXIAL FIELDS AND WAVENUMBERS

#### 3.1 Stability

We consider first the local stability of a differentially rotating disc threaded by a weak vertical field, $B = B_e z$, in the presence of ambipolar diffusion. We use standard cylindrical coordinates $(r; \phi; z)$ with the origin at the disc centre, and consider plane wave perturbations that depend only on $z$. Linearized quantities (indicated by notation) are proportional to $\exp (-i k z)$, where $k$ is the vertical wave number. In the Boussinesq limit, this corresponds to fluid displacements in the plane of the disc, so vertical structure is unimportant. Under these circumstances, pressure, density, vertical velocity, and vertical magnetic field perturbations all vanish.

The two-fluid version of this problem has been considered by [Blaes & Balbus 1994]; we shall compare the results of our single-fluid treatment with those of these authors.

Stability is most easily assessed by working in the limit $\kappa^2 \ll 0$, assuming that the stable-unstable transition is bridged by a $\kappa^2 = 0$ solution. The linearized equations of motion are:

$$2 \frac{\partial v_A}{\partial t} - \frac{3 k B}{4} B_r = 0;$$  \hspace{1cm} (6)

$$2 \frac{\partial v}{\partial t} - \frac{3 k B}{4} B_r = 0;$$  \hspace{1cm} (7)

where $\nu_A$ is the epicyclic frequency defined by $\kappa^2 = 4 - 4 \nu_A^2 = 4 - \frac{1}{2} \frac{d^2}{d \ln R}$.

The linearized induction equations are:

$$\kappa^2 \frac{d}{d \ln R} B_r \frac{k B}{4} v = 0;$$  \hspace{1cm} (9)

where

$$\kappa^2 = 4 - \frac{1}{2} \frac{d^2}{d \ln R}$$  \hspace{1cm} (8)

is the Ohmic resistivity modified by ambipolar diffusion.

Solving the above system of equations, we find

$$2 \kappa^2 v_A^2 + v_{\nu A}^2 \frac{d^2}{d \ln R} + k^2 v_A^2 = 0;$$  \hspace{1cm} (10)

This is the desired marginal stability condition. It is easily shown that the left hand side should be negative for instability. The instability criterion is then,

$$\kappa^2 v_A^2 < \frac{d^2}{d \ln R} 1 + \frac{2}{v_A^2} = 1;$$  \hspace{1cm} (11)

a more restrictive condition than the classical MRI.

It is often the case that Ohmic resistivity is small compared with ambipolar diffusion, which was the limit assumed by [Blaes & Balbus 1994]. If we take $\kappa^2 \ll 0$, $\nu_A^2 = 1$, the instability criterion becomes

$$\kappa^2 v_A^2 < \frac{d^2}{d \ln R} \frac{2}{2 + 2} = 1;$$  \hspace{1cm} (12)

where we have introduced the ion Alfvén velocity

$$v_{\nu A} = \frac{1}{1 - v_A^2}.$$  \hspace{1cm} (13)

The analogous two-fluid instability criterion obtained by
Blaes & Balbus (1994) was
\[ k^2 v_{\lambda,1}^2 < \frac{d^2}{d \ln R} \left( \frac{v_0}{2} + \frac{1}{2} \right) \quad (\text{B94 instability criterion}) \]  

Clearly, the inequality (14) is consistent with the inequality (16) provided that

\[ \frac{d}{d \ln R} = \frac{1}{2} \]

a quantitative implementation of the assumption of negligible ion inertia.

3.2 Axial Wavenumber Dispersion Relation

We next obtain the full dispersion relation for finite \( v \). The linearized radial and azimuthal equations of motion are now

\[ v_t + \frac{d}{d \ln R} \left( \frac{v}{2} + \frac{1}{2} \right) B_k = 0; \]

\[ v + \frac{1}{2} v \frac{\partial B}{\partial R} B = 0; \]

The same components of the linearized induction equation are

\[ (1 + k^2 \partial) B_k = \frac{\partial B}{\partial R} v = 0; \]

\[ (1 + k^2 \partial) B = \frac{d}{d \ln R} B_k - \frac{\partial B}{\partial R} v = 0; \]

The resulting dispersion relation is

\[ 4 + 2k^2 v_0^2 + 2B_k B + 2k^2 v_0^2 + B_0 = 0; \]

where the constants \( B_2 \) and \( B_0 \) are given by

\[ B_2 = \frac{d}{d \ln R} B_k \]

\[ B_0 = k^2 v_0^2 \frac{d}{d \ln R} + k^2 v_0^2 + 2k^2 v_0^2; \]

Ambipolar diffusion acts on axial wavenumber disturbance only as though it were a field-dependent additive resistivity. \( v \). Adding other field components while retaining \( k = k_{\text{e}} \), or adding other wavenumber components while retaining \( B = B_{\text{e}} \), does not change the dispersion relation. Qualitative changes are introduced only if nonaxial wavenumbers as well as nonaxial field components are considered.

4 GENERAL ANALYSIS

4.1 Preliminary Assumptions

We now consider the axisymmetric behaviour of the instability with more general field geometries and wavenumbers. We ignore buoyancy, so our analysis holds either either for a barotropic disc, or locally at the midplane. The perturbation wavevector is

\[ k = k_0 e_k + k_\| e_\|; \]

and disturbances have space-time dependence \( \exp (ik \cdot r + t) \). In the presence of shear, a toroidal magnetic field \( B \) would grow linearly with time if a radial field component \( B_k \) were also present:

\[ B(t) = B(0) + t B_k \frac{d}{d \ln R}; \]

(34)

4.2 Dispersion Relation

The linearized mass conservation equation in the Boussinesq limit is:

\[ \frac{k}{k} v + k \frac{B}{d \ln R} B = 0; \]

Using this to eliminate \( v \) from the equations of motion leads to the radial and azimuthal forms

\[ v + \frac{k}{k} \frac{B}{d \ln R} B = 0; \]

\[ v + \frac{k}{k} \frac{B}{d \ln R} B = 0; \]

By similarly eliminating \( B_k \) via a vanishing divergence condition, only the \( B \)- and \( \| \)-components of the induction equation are needed. The linearized induction equations are

\[ + \frac{k}{k} \frac{B}{d \ln R} B = 0; \]

\[ + \frac{k}{k} \frac{B}{d \ln R} B = 0; \]

The resulting dispersion relation is

\[ 4 + C_3 + C_2 + C_1 + C_0 = 0; \]

with

\[ C_3 = \frac{k}{k} \frac{B}{d \ln R} B; \]

\[ C_2 = \frac{k}{k} \frac{B}{d \ln R} B; \]

\[ C_1 = k v^2 + \frac{k}{k} \frac{B}{d \ln R} B; \]

\[ C_0 = k v^2 + \frac{k}{k} \frac{B}{d \ln R} B; \]

4.3 Stability

We assume that the transition from stability to instability proceeds through \( = 0 \), a point that can be confirmed numerically. In the limit, \( \neq 0 \).
settings, the primary interest is not outwardly increasing angular
sation of outwardly increasing rotation profiles. In astrophysical
spite the presence of a resistive term in the dispersion relation, there
do not follow fluid elements. One consequence is the destabil-
ing only in the presence of a nonaxial wavenumber component, it
always be chosen to be destabilising. Since this term is nonvanish-
This has important consequences. Let us view the marginal
stability transition \( C_0 = 0 \) as an equation for
\[
\frac{\delta}{2} = \frac{\delta^2}{2}
\]
as a function of \( k_i = k_2 \), for a given set of disc parameters. Transi-
tion to instability is present only if there are solutions to this equation with \( \delta \) real and positive. With
\[
\delta^2 = \delta^2
\]
one finds that the \( C_0 = 0 \) requirement is satisfied when
\[
\delta = \frac{2}{k_i} \frac{\ln R}{k_i} \frac{6}{k_i} \frac{\ln R}{k_i} 
\]
The denominator vanishes at a wavenumber ratio
\[
\frac{k_i}{k_2} = \frac{B}{B} \frac{\ln R}{k_i} \frac{1}{k_i} + \frac{B}{B} \frac{1}{k_i} : \quad (38)
\]
It is straightforward to show that there will always be solutions
with large positive values of \( \delta \) near this selected value of \( k_i = k_2 \).
These large wavenumber modes are well localised in a WKB sense.
In other words, ambipolar diffusion always destabilises differential
rotation, whether the profile is increasing inward or outward. De-
spite the presence of a resistive term in the dispersion relation, there
are unstable modes at large nonaxial wavenumbers. Moreover, be-
cause fluid motions do not simply bend field lines locally when
ambipolar effects dominate (loss of flux-freezing), some of the sta-
bulising effects of large wavenumber magnetic tension seen in the
standard MRI are lost.

The modification of the MRI in the presence of ambipolar dif-
fusion does not change the fundamental cause of the instability,
which is based on rotational dynamics: angular momentum is re-
moved from fluid elements with less angular momentum and given
to fluid elements with more angular momentum, an intrinsically
destabilising process. What is complicated by the presence of am-
bigular diffusion is the fluid element “tethering,” now no longer a
simple, spring-like magnetic tension based on flux-freezing. The
radial component of the disturbed magnetic field, for example, de-
ponds both upon azimuthal as well as radial motions, and field lines
do not follow fluid elements. One consequence is the destabil-
sation of outwardly increasing rotation profiles. In astrophysical
settings, the primary interest is not outwardly increasing angular
velocity profiles of course, but the additional destabilisation that
attends the usual outwardly decreasing profiles. Ambipolar dif-
sion does not, however, lead to growth rates in excess of \( 0.75 \)
(Balbus & Hawley 1992). Indeed, as shown in the following sec-
tion, growth rates become a small fraction of when the ambipolar
parameter \( = \) significantly exceeds unity.

4.4 Growth Rates
It is of interest to calculate explicitly some representative growth
rates in the two-dimensional phase space described by the
wavenumber parameters \( (k, \gamma) \) and \( k=k_2 \). In Figure 2, we
present three-dimensional plots of the growth rate for representa-
tive cases with \( k=k_2 \). While the general tendency of ambipol-
diffusion is to lower the maximum growth rate below \( 0.75 \),
there is also a significant extension of the domain of unstable non-
axial wavenumbers.

As \( \gamma \) approaches and exceeds unity, the most unstable
modes are associated with a near cancellation of the resistive and nonresistive (destabilising) ambipolar terms in the dispersion relation. The modes appear at progressively larger values of $k_R = k_z$. The maximum growth rate decreases sharply from $0.01$ at $1$ to $0.001$ at $10$ (see Fig. 3). Once the rotation frequency drops below the ion-neutral collision frequency, the growth time becomes very long compared with an orbital time. The instability remains viable, however, if its growth rate remains shorter than other evolutionary timescales.

5 SUMMARY

In this paper we have examined the axisymmetric magnetorotational instability in the presence of ambipolar diffusion. Our results should find their primary applicability in molecular discs on interstellar or galactic scales, as well as the low density outermost regions of protostellar discs. We have allowed radial and axial wavenumber geometries, and azimuthal and axial magnetic field components. The inclusion of a radial magnetic field component would induce a linear time dependence in the azimuthal field component. Since $B_{\phi}$ appears explicitly in the dispersion relation, we considered only the case of a vanishing background radial field component.

The combination of nonaxial wavenumbers and magnetic fields greatly enhances the response of a differentially rotating gas to ambipolar diffusion, compared with the axial geometries that are usually analysed. In particular, the wavevector corresponding to the maximum growth rate is nonaxial when ambipolar diffusion is present. Short wavelength unstable wavenumber modes are always present, even for increasing outward angular velocity profiles. In a non-accreting system, solid body rotation is an energy extremum. Differential rotation causes no local linear instabilities in a purely hydrodynamic gas, and even in the presence of a magnetic field, ideal MHD only destabilises outwardly decreasing angular velocity profiles. It is therefore noteworthy that non-ideal MHD processes open paths to destabilise any differentially rotating configuration. A likely consequence of the action of the instability in nonaccreting systems is solid body rotation, which may explain the velocity profiles of molecular cloud cores (Barranco & Goodman 1998). In a disc in which accretion is a possibility, the nonlinear outcome of the instability will not be solid body rotation, but pronounced differential rotation and turbulence.

The parameter that measures the relative importance of ambipolar diffusion is $\gamma$, the ratio of the angular rotation velocity to ion-neutral collision rate. Although there are unstable large wavenumber modes at all values of this ratio, the growth rate of the instability drops rapidly below when this ratio approaches or exceeds unity. The viability of the instability then depends upon whether the growth time remains small compared to the expected lifetime of the system in question. When a background $B_R$ is present, there is the additional uncertainty of whether the dynamical consequences of a steady, linear-in-time growth of $B_R$ will outpace a more leisurely growing exponential instability.

A subsequent paper will address the stability of a system in which Ohmic losses, HEMFs, and ambipolar diffusion are simultaneously present, the dispersion relation of which is given in the Appendix.

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APPENDIX

We present here the generalised dispersion relation including all non-ideal MHD effects: Ohmic diffusion, Hall effect, and ambipolar diffusion for the case $B_R = 0$, as in the text. Although the range of parameter space where all three terms are comparable is very small, it is nevertheless useful to have a general dispersion relation which may be conveniently specialised to a particular gas. Additionally, the final form of the complete dispersion relation has a mathematical form which can be written as a natural generalisation of the (comparatively) simple Hall dispersion relation of Balbus & Terquem (2001). This too is worth noting.

Let us first introduce some preliminary notation. We start with the generalised resistivity parameters, $\gamma_1$ and $\gamma_2$:
\[ 1 = \frac{v^2_A}{c}; \quad 2 = \frac{k^2}{k^2 + v^2_A}; \tag{39} \]

and their respective arithmetic and geometric means, and:

\[ 2 = \frac{1}{2} (1 + 2); \quad 2 = \frac{1}{2} 2; \tag{40} \]

Next, there are two parameters with dimensions of magnetic field divided by frequency, \( B = \) and \( B = \).:

\[ B = \frac{n}{n_e} \frac{B}{B}; \quad B = \frac{n}{n_e} \frac{B}{B}; \tag{41} \]

where \( c_0 \) is the cyclotron frequency of a particle of mass \( m \), the mean mass per neutral particle,

\[ c_0 = \frac{eB}{mc}; \]

and \( eB \) is the unit vector in the radial direction.

With these definitions, the generalised non-ideal MHD induction equations may be written:

\[ + k^2 1 B + 2 + k^2 1 k^2 \left( \frac{B}{B} \right) + \frac{k^2 B}{B}; \tag{42} \]

\[ + k^2 2 + \frac{d}{d ln R} \left( k^2 B \right) + \frac{k^2 B}{B}; \tag{43} \]

\[ + k^2 2 + k^2 2 \left( \frac{B}{B} \right) + \frac{k^2 B}{B}; \tag{44} \]

The dynamical equations (27) and (28) remain unchanged.

The non-ideal MHD dispersion relation may be written in a form similar to that of the axisymmetric dispersion relation of a disc in the Hall regime (Balbus & Terquem 2001):

\[ 4 + 2k^2 3 + K_2^2 2 + 2k^2 \left( \frac{B}{B} \right)^2 + \frac{k^2}{k^2}; \tag{45} \]

with

\[ K_2 = \frac{k^4 2}{k^2} + 2 \left( \frac{B}{B} \right)^2 + \frac{k^2}{k^2} \left( \frac{B}{B} \right) \tag{46} \]

\[ K_2 = \frac{k^4 2}{k^2} \left( \frac{B}{B} \right)^2 + \frac{k^2}{k^2} \left( \frac{B}{B} \right)^2 \tag{47} \]

A detailed analysis of this equation is deferred to a later paper.

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