Lagrangian Formulations of Self-dual Gauge Theories in Diverse Dimensions

Wei-Ming Chen, Pei-Ming Ho

Department of Physics, Center for Theoretical Sciences and Leung Center for Cosmology and Particle Astrophysics, National Taiwan University, Taipei 10617, Taiwan, R.O.C.

Abstract

In this work, we study Lagrangian formulations for self-dual gauge theories, also known as chiral $n$-form gauge theories, for $n = 2p$ in $D = 4p + 2$ dimensional spacetime. Motivated by a recent formulation of M5-branes derived from the BLG model, we generalize the earlier Lagrangian formulation based on a decomposition of spacetime into $(D - 1)$ dimensions plus a special dimension, to construct Lagrangian formulations based on a generic decomposition of spacetime into $D'$ and $D'' = D - D'$ dimensions. Although the Lorentz symmetry is not manifest, we prove that the action is invariant under modified Lorentz transformations.

*tainist@gmail.com
†pmho@phys.ntu.edu.tw
1 Introduction

Self-dual gauge theories, or chiral theories, are full of both physical and mathematical interests. The goal of this work is to provide new Lagrangian formulations of self-dual gauge theories for spacetime dimensions equal to 2 modulo 4, i.e., \( D = 4p + 2 \), for \( p = 0, 1, 2, 3, \cdots \). This includes chiral bosons in 2D, self-dual 3-form gauge field for M5-brane in M theory and self-dual 5-form gauge field in type IIB superstring theory. For simplicity we will assume that the spacetime is Minkowski space. It should be straightforward to generalize the formulation to curved spacetime with Lorentzian signature.

Let us recall that the dual of a tensor \( T \) in \( D \) dimensions is defined as

\[
\tilde{T}_{\mu_1 \cdots \mu_k} \equiv \frac{1}{(D-k)!} \epsilon_{\mu_1 \cdots \mu_D} T^{\mu_{k+1} \cdots \mu_D}.
\]

The self-duality condition of a gauge field is

\[
\mathcal{F} \equiv F - \tilde{F} = 0.
\]

Apparently the field strength must be a \( D/2 \)-form, and \( D \) must be even. Self-duality conditions are not consistent in \( D = 4p \) \((p = 1, 2, 3, \cdots)\) dimensional Minkowski space, and they will not be considered in this work.

It is well known that, without auxiliary fields, a manifestly Lorentz invariant action cannot be found for self-dual gauge theory. In the literature on self-dual theories \( \text{[1,2]} \), a special (but arbitrary) direction has to be singled out to write down a Lagrangian. Recently, in the study of M theory, a new Lagrangian formulation of M5-brane in large C-field background \( \text{[3,4,5]} \), which is a self-dual gauge field theory in 6 dimensions, was derived from the BLG model \( \text{[6,7]} \) for multiple M2-branes. In this formulation, the 6 coordinates of the base space are divided into two sets of 3 coordinates \( \{x^a\}^3_{a=1} \) and \( \{\hat{x}^a\}^3_{a=1} \), in contrast with the old formulation of M5-branes \( \text{[8]} \), which uses a decomposition of base space coordinates into two sets of 1 and 5 coordinates. While both decompositions 6 = 1+5 and 6 = 3+3 admit Lagrangian formulations of self-dual theories, the natural question is: does there exist a formulation corresponding to the decomposition 6 = 2+4? More generally, for a self-dual gauge theory in \( D = 4p + 2 \) dimensions (so the gauge field strength is a \((2p+1)\)-form), can we find a Lagrangian formulation for all possible decompositions \( D = D' + D'' \)?

The answer to the question above is yes. In the following, we provide new Lagrangian formulations for arbitrary spacetime decompositions. The action is given in section 2, its gauge symmetry in section 3. The proof that the theory is a theory of
self-dual gauge fields is given separately for three classes of decompositions: (i) \( D' = D'' \) (section 4), (ii) \( D'' > D' = \text{odd} \) (section 5) and (iii) \( D'' > D' = \text{even} \) (section 6). We show in section 7 that although the action is no longer manifestly invariant under those Lorentz transformations which mix the two sets of coordinates in the decomposition, the action is invariant under certain modified Lorentz transformation laws. In section 8, we give the interaction term in the action to describe the coupling of the gauge field to a charged \((2p - 1)\)-brane. Explicit examples for \((D', D'') = (1, 1), (1, 5), (2, 4), (3, 3)\) are given in section 9. We point out the relationship between our result and the holographic action of Belov and Moore [9] in section 10 and our conclusion will be given in section 11.

2 Action

We decompose the \(D\)-dimensional Minkowski spacetime as a product space \(\mathcal{M}^D = \mathcal{M}_1^{D'} \times \mathcal{M}_2^{D''}\). Correspondingly, the spacetime coordinates \(\{x^\mu | \mu = 1, \ldots, D\}\) are divided into two sets

\[
\mathcal{M}_1^{D'} : \{x^a | a = 1, \ldots, D'\} \quad \text{and} \quad \mathcal{M}_2^{D''} : \{x^{\dot{a}} | \dot{a} = 1, \ldots, D''\}.
\]

We assume that \(D' \leq D''\), so that \(1 \leq D' \leq D/2\). This is just a convention except that when \(D'\) is even, the signature of \(\mathcal{M}_1^{D'}\) must be Lorentzian, and the signature of \(\mathcal{M}_2^{D''}\) Euclidean. The Lorentzian signature of spacetime can be either \((+ - \cdots -)\) or \((- + \cdots +)\). The expressions below are valid for both conventions.

Due to the decomposition of the coordinates, the gauge potential \(A_{\mu_1 \cdots \mu_{2p}}\) is naturally decomposed into fields of different types depending on the number of dotted or undotted indices as

\[
\{ A_{a_1 \cdots a_j \dot{a}_1 \cdots \dot{a}_{2p-j}} | j = 0, \ldots, D' - \alpha, \}
\]

where

\[
\alpha = \delta_{D', D/2}
\]

so that \(D' - \alpha = \min(D', 2p)\). \(D'\) is greater than \(2p\) only for the special case \((D', D'') = (D/2, D/2)\), and the parameters \(\alpha\) is used to keep track of whether the decomposition \(D = D' + D''\) happens to be the special case \((D', D'') = (D/2, D/2)\). The indices \(a_1 \cdots a_j\) of \(A\) in (4) should be skipped altogether if \(j = 0\), and \(\dot{a}_1 \cdots \dot{a}_{2p-j}\) skipped altogether when \(j = 2p\). All expressions below should also be interpreted in this way.
For an arbitrary tensor \( T_{\mu_1 \cdots \mu_k \nu_1 \cdots \nu_l} \), we will use the notation 
\[
\epsilon_{a_1 \cdots a_D} \rho_{1 \cdots \rho_D} = \epsilon_{a_1 \cdots a_D} \rho_{1 \cdots \rho_D},
\]
where the sum \( \sum_\sigma \) is carried out for all permutations of \( m \) objects and \( \text{sgn}(\sigma) \) is the signature of the permutation \( \sigma \). Note that the antisymmetrized tensor is normalized by \( 1/m! \), and that the indices sandwiched by \( | \cdot | \) are not antisymmetrized. Using this notation, we can express the field strength and its dual as
\[
F_{a_1 \cdots a_j \hat{a}_1 \cdots a_{D/2-j}} = j \partial_{a_1} A_{a_2 \cdots a_j} \hat{a}_1 \cdots a_{D/2-j} + (D/2 - j) \partial_{\hat{a}_{D/2-j}} A_{a_1 \cdots a_j} \hat{a}_1 \cdots \hat{a}_{D/2-j-1},
\]
\[
\tilde{F}_{a_1 \cdots a_j \hat{a}_1 \cdots \hat{a}_{D/2-j}} = \frac{(-) \left( \frac{D}{2} - j \right) \left( D' - j \right)! \epsilon_{a_1 \cdots a_{D-j/2}} \epsilon_{a_1 \cdots a_{D-j}} \partial^a_{a_1} \cdots \partial^a_{a_{D-j}} F_{a_1 \cdots a_{D-j} \hat{a}_1 \cdots \hat{a}_{D-j} D'} \left( \frac{D}{2} - j \right) \left( D' - j \right)!}{(D' - j)! \left( \frac{D}{2} - D' + j \right)!}.
\]
for \( j = 0, 1, \cdots, D' \).

The following two identities,
\[
F_{a_1 \cdots a_j \hat{a}_1 \cdots \hat{a}_{D/2-j}} \tilde{F}^{a_1 \cdots a_j \hat{a}_1 \cdots \hat{a}_{D/2-j}} = - \frac{j! \left( \frac{D}{2} - j \right)!}{(D' - j)! \left( \frac{D}{2} - D' + j \right)!} F_{a_1 \cdots a_{D-j/2} \hat{a}_1 \cdots \hat{a}_{D-j} D'} \tilde{F}^{a_1 \cdots a_{D-j/2} \hat{a}_1 \cdots \hat{a}_{D-j} D'} \left( \frac{D}{2} - j \right) \left( D' - j \right)! \left( \frac{D}{2} - D' + j \right)! \]
\[
\tilde{F}_{a_1 \cdots a_j \hat{a}_1 \cdots \hat{a}_{D/2-j}} \tilde{F}^{a_1 \cdots a_j \hat{a}_1 \cdots \hat{a}_{D/2-j}} = - \frac{j! \left( \frac{D}{2} - j \right)!}{(D' - j)! \left( \frac{D}{2} - D' + j \right)!} F_{a_1 \cdots a_{D-j/2} \hat{a}_1 \cdots \hat{a}_{D-j} D'} \tilde{F}^{a_1 \cdots a_{D-j/2} \hat{a}_1 \cdots \hat{a}_{D-j} D'} \left( \frac{D}{2} - j \right) \left( D' - j \right)! \left( \frac{D}{2} - D' + j \right)! \]
will be useful in the calculation below. Another notation we will use below is
\[
\mathcal{F}_{\mu_1 \cdots \mu_{D/2}} \equiv (F - \tilde{F})_{\mu_1 \cdots \mu_{D/2}}.
\]

The main result of this paper is then this. For the decomposition \( D = D' + D'' \) of
spacetime dimension $D = 4p + 2$, the action of a self-dual field theory is

$$S_{D'+D''} = -\frac{1}{4} \int d^D x \left[ \sum_{j=[D'/2]}^{D'} \frac{F_{\mu_1\mu_2...\mu_{D/2}} F^{\mu_1\mu_2...\mu_{D/2}}}{(D/2)!} \frac{\mathcal{F}_{a_1...a_j\hat{a}_1...\hat{a}_{D/2-j}}}{j!(\frac{D}{2} - j)!} \right]$$

(13)

$$= -\frac{1}{2} \int d^D x \sum_{j=[D'/2]}^{D'} \frac{\tilde{F}_{a_1...a_j\hat{a}_1...\hat{a}_{D/2-j}}}{j!(\frac{D}{2} - j)!^2 / 2} \frac{\mathcal{F}_{a_1...a_j\hat{a}_1...\hat{a}_{D/2-j}}}{2^{\delta_j,D'/2}}$$

(14)

$$= -\frac{1}{2} \int d^D x \sum_{j=[D'/2]}^{D'} \frac{F_{a_1...a_j\hat{a}_1...\hat{a}_{D/2-j}}}{(D' - j)!(\frac{D'-D}{2} + j)!^2 / 2} \frac{\mathcal{F}_{a_1...a_j\hat{a}_1...\hat{a}_{D/2-j}}}{2^{\delta_j,D'/2}}$$

(15)

where we used the two identities (10) and (11) to express the action in 3 different ways for the convenience of the reader. In the 2nd and 3rd expressions of the action, the factor $2^{\delta_j,D'/2}$ in the denominator contributes only when $D'$ is even. In the following we will show that the equations of motion derived from varying this action are equivalent to the condition of self-duality, and that this action is invariant under a hidden Lorentz symmetry although it is not manifestly Lorentz invariant.

It was pointed out in [10] that the quantum theory of a self-dual gauge field is not unique on a generic spacetime. Instead there is a 1-1 correspondence between the spin structure on the base space (spacetime) and the path integral of a self-dual gauge field. The action proposed above is presumably the action suitable for spacetime with trivial topology so that the spin structure is unique. We leave the issue on topology and the definition of quantum theory for generic spacetime manifold to future works.

3 Gauge Symmetry

The self-duality condition

$$\mathcal{F}_{\mu_1...\mu_{D/2}} = 0$$

(16)

can be decomposed into $D' + 1$ different types of equations

$$\mathcal{F}_{a_1...a_j\hat{a}_1...\hat{a}_{D/2-j}} = 0 \quad (j = 0, 1, \cdots, D').$$

(17)

(The quantity $\mathcal{F}$ vanishes identically if $j > D'$ because $\mathcal{F}$ is totally antisymmetrized in all indices.) But not all of them are independent. In fact, only $[(D'+1)/2]$, roughly half,
of them are independent. \( \lceil (D' + 1)/2 \rceil \) denotes the smallest integer not smaller than \((D'+1)/2\). Hence the action, if it is correct, should produce \( \lceil (D' + 1)/2 \rceil \) different types of independent equations of motion that are equivalent to these conditions. On the other hand, the number of different types of gauge potential components is \((D' - \alpha + 1)\) (see eq.(4)). Varying them in the action should give us \((D' - \alpha + 1)\) different types of equations of motion. Except the special case \((D', D'') = (1, 1)\), for which the number of equations of motion equals the number of self-duality conditions, for all other cases, \((D' - \alpha + 1) > \lceil (D' + 1)/2 \rceil\). Thus, in general, the equations of motion cannot be all independent. The number of equations of motion that are trivial should be

\[
(D' - \alpha + 1) - \lceil (D' + 1)/2 \rceil = \lceil D'/2 \rceil - \alpha.
\]

(18)

It turns out that the action (13-15) has the special feature that many components of the gauge potential appear only in total derivative terms. Therefore, in addition to the ordinary gauge symmetry

\[
A \rightarrow A + d\Lambda,
\]

the action is invariant under the following gauge transformations

\[
\begin{align*}
\delta A_{a_1 \cdots a_k \dot{a}_1 \cdots \dot{a}_{2p-k}} &= 0 & (k = 0, 1, \cdots, \lceil D'/2 \rceil - 1), \\
\delta A_{a_1 \cdots a_k \dot{a}_1 \cdots \dot{a}_{2p-k}} &= f_{a_1 \cdots a_k \dot{a}_1 \cdots \dot{a}_{2p-k}}(x^a) & (k = \lceil D'/2 \rceil), \\
\delta A_{a_1 \cdots a_k \dot{a}_1 \cdots \dot{a}_{2p-k}} &= \Phi_{a_1 \cdots a_k \dot{a}_1 \cdots \dot{a}_{2p-k}}(x^a, x^{\dot{a}}) & (k = \lceil D'/2 \rceil + 1, \ldots, D' - \alpha),
\end{align*}
\]

(20)

where the \(f\)'s are functions independent of all \(x^{\dot{a}}\)s, and the \(\Phi\)'s are arbitrary functions. By choosing \(\Phi\) suitably, \(\lceil D'/2 \rceil - \alpha\) of the \((D' - \alpha + 1)\) different types of \(A\) fields can be gauged away. Thus there are only \(\lceil (D' + 1)/2 \rceil\) types of nontrivial equations of motion, as it should be (see (18)). The gauge transformation parametrized by the \(f\)'s in (20) will play an important role when \(D = 2\). The fact that the action is invariant under the transformations (20) will be checked below when we compute the equations of motion.

In the following we will show case by case that, for (1) \(D' = D''\), (2) \(D' < D''\) with \(D'\) odd, and (3) \(D' < D''\) with \(D'\) even, the space of solutions to the equations of motion defined by the action (13) is the same as the space of all self-dual field configurations. More specifically, the equations of motion in our theory (for those gauge potential components that cannot be gauged away) are 2nd order differential equations which can be integrated into 1st order differential equations with arbitrary functions introduced as integral "constants". These additional functions can be identified with those components of the gauge potential that are gauged away, or as a shift of those
fields that can be absorbed by a field redefinition. After this identification, the 1st order differential equations are exactly identical to the self-duality conditions.

Strictly speaking, the derivation given below only implies that every solution to the equations of motion of our theory correspond to a self-dual configuration, but it does not immediately imply that every self-dual configuration can find its counterpart in our theory. To prove that the space of solutions for our theory is really the same as the space of self-dual configurations, one needs to count the number of free components in $A$ and examine the constraints (equations of motion or self-duality conditions). We have carried out this straightforward yet tedious proof case by case but will skip it here.

4 \quad D' = D''

For the special case $D' = D'' = \frac{d}{2} = 2p + 1$ (and so $\alpha = 1$), the gauge transformation laws (20) are

\begin{align*}
\delta A^{a_1 a_2 \cdots a_{2p - k} \dot{a}_1 \cdots \dot{a}_k} &= \Phi^{a_1 a_2 \cdots a_{2p - k} \hat{a}_1 \cdots \hat{a}_k} (x^a, \dot{x}^{\hat{a}}), \quad k = 0, 1, \ldots, p - 1, \\
\delta A^{a_1 a_2 \cdots a_{2p - k} \dot{a}_1 \cdots \dot{a}_k} &= f^{a_1 a_2 \cdots a_{2p - k} \dot{a}_1 \cdots \dot{a}_k} (x^a), \quad k = p, \\
\delta A^{a_1 a_2 \cdots a_{2p - k} \dot{a}_1 \cdots \dot{a}_k} &= 0, \quad k = p + 1, \ldots, 2p,
\end{align*}

in addition to the usual gauge transformation (19). Without loss of generality, here we assume the indices $a_i$ and $\dot{a}_i$ correspond to the $SO(D' - 1, 1)$ and $SO(D'')$ subgroup of the full Lorentz group, respectively.

The action (14) is

$$S = -\frac{1}{2} \int d^D x \sum_{j=p+1}^{2p+1} \frac{\tilde{F}^{a_1 \cdots a_j \dot{a}_1 \cdots \dot{a}_{2p-j+1}} a_1 \cdots a_{2p-j+1}}{j!(2p-j+1)!}. \quad (22)$$

Its variation with respect to $A^{a_1 a_2 \cdots a_{2p - k} \dot{a}_1 \cdots \dot{a}_k}$ for $k = 0, 1, \ldots, p - 1$ can be shown to vanish using the Bianchi identity

$$\partial_{\mu_1} \tilde{F}^{\mu_1 \cdots \mu_{D/2}} = 0. \quad (23)$$

The nontrivial equations of motion are

$$\frac{\delta S}{\delta A^{a_1 a_2 \cdots a_{2p - k} \dot{a}_1 \cdots \dot{a}_k}} = 0 \quad (k = p, \ldots, 2p). \quad (24)$$

For $k = p$,

$$\frac{\delta S}{\delta A^{a_1 a_2 \cdots a_p \dot{a}_1 \cdots \dot{a}_p}} = 0 \quad \Rightarrow \quad \frac{\delta^b \mathcal{F}^{a_1 a_2 \cdots a_{p-1} \dot{a}_b}}{p!p!} = 0, \quad (25)$$
and the solution to \((25)\) is

\[
\mathcal{F}_{a_1 \ldots a_p \dot{a}_1 \ldots \dot{a}_{p+1}} = \frac{(-)^{p^2+1}\epsilon_{a_1 \ldots a_{2p+1}} \epsilon_{\dot{a}_1 \ldots \dot{a}_{2p+1}} \partial^{a_1 \ldots a_{2p+1} \dot{a}_1 \ldots \dot{a}_{2p+1}}}{(p+1)!(p-1)!}
\]

(26)

for some arbitrary functions \(\Phi^{a_1 \ldots a_{2p+1} \dot{a}_1 \ldots \dot{a}_{2p+1}}\). An exception is that when \(p = 0\), the totally antisymmetrized tensor \(\epsilon_1\) is trivial and cannot be used to write down a solution. In this case the general solution is given by \(\mathcal{F}_1 = f(x^1)\) for an arbitrary function independent of \(x^1\).

If \(p > 0\), we can use the gauge symmetry \((21)\) for \(k = p - 1\) to achieve one of the self-duality conditions

\[
\mathcal{F}_{a_1 \ldots a_p \dot{a}_1 \ldots \dot{a}_{p+1}} = 0.
\]

(27)

In case \(p = 0\), we can use the second gauge transformation in \((21)\) to gauge \(f\) away and arrive at the same equation.

The remaining equations of motion are,

\[
\frac{\delta S}{\delta A^{a_1 \ldots a_{p-1} \dot{a}_1 \ldots \dot{a}_{p+1}}} = 0 \Rightarrow \frac{\partial^b F_{ba_1 \ldots a_{p-1} \dot{a}_1 \ldots \dot{a}_{p+1}} + \partial^b F_{a_1 \ldots a_{p-1} \dot{a}_1 \ldots \dot{a}_{p+1} \dot{b}}}{(p-1)!(p+1)!} = 0,
\]

(28)

\[
\frac{\delta S}{\delta A^{a_1 \ldots a_{2p-1} \dot{a}_1 \ldots \dot{a}_{2p}}} = 0 \Rightarrow \frac{(2p - k + 1)\partial^k (\mathcal{F} + F)_{ba_1 \ldots a_{2p-1} \dot{a}_1 \ldots \dot{a}_{2p}} + (k + 1)\partial^k (\mathcal{F} + F)_{a_1 \ldots a_{2p-1} \dot{a}_1 \ldots \dot{a}_{2p} \dot{b}}}{2(2p - k + 1)!k!} = 0,
\]

(29)

\[
\frac{\delta S}{\delta A^{\dot{a}_1 \ldots \dot{a}_{2p}}} = 0 \Rightarrow \frac{\partial^b F_{b\dot{a}_1 \ldots \dot{a}_{2p}} + \partial^b F_{\dot{a}_1 \ldots \dot{a}_{2p} \dot{b}}}{(2p)!} = 0.
\]

(30)

Using \((27)\), we can rewrite the first term on the left-hand side of \((28)\), i.e., \(\partial^b F_{b\ldots}\), as \(-\partial^b \tilde{F}_{b\ldots}\) due to Bianchi identity. Together with the 2nd term, the left-hand side of \((28)\) is a total derivative of the form \(\partial^b \mathcal{F}_{b\ldots}\), and thus \((28)\) can be solved in the same way we solved \((25)\). That is, we use the gauge symmetry \((21)\) to absorb the functions arising due to integration, to obtain another self-duality condition.

The same story goes on, and each equation of motion is found to be of the form

\[
\partial^{\dot{a}_k+1} \mathcal{F}_{\dot{a}_1 \ldots \dot{a}_{2p-1} \dot{a}_1 \ldots \dot{a}_{k+1}} = 0, \quad (k = p + 1, \ldots, 2p),
\]

(31)
after plugging in the solution of previous equations of motion. For \( k = p+1, \ldots, 2p-1 \), the solution to the equation above is

\[
\mathcal{F}_{a_1\ldots a_{2p-k} \dot{a}_1\ldots \dot{a}_{k+1}} = (-)^{k^2+1} \epsilon_{a_1\ldots a_{2p+1}} \epsilon_{\dot{a}_1\ldots \dot{a}_{2p+1}} \partial^{\dot{a}_{2p+1}} \Phi a^{a_{2p-k+1} \ldots a_{2p+1} \dot{a}_{k+2} \ldots \dot{a}_{2p}} \quad (32)
\]

for some arbitrary functions \( \Phi \). We then use the gauge symmetries (21) (for \( k = 0, \ldots, p-2 \) in (21)) to absorb the right-hand side of (32).

The case of \( k = 2p \) is special, because \( \mathcal{F}_{\dot{a}_1\ldots \dot{a}_{2p+1}} \) has to be a function times the totally antisymmetrized tensor \( \epsilon_{\dot{a}_1\ldots \dot{a}_{2p+1}} \). The solution to its equation of motion is thus

\[
\mathcal{F}_{\dot{a}_1\ldots \dot{a}_{2p+1}} = \epsilon_{\dot{a}_1\ldots \dot{a}_{2p+1}} f(x),
\]

(33)

where \( f(x) \) is a function depending only on the \( D' \) coordinate \( x^a \), which can be always written as \( f(x) = \partial_a f^a(x) \) for some functions \( f^a(x) \) depending only on \( x^a \). We can then absorb the right-hand side of (33) by a field redefinition \( A_{a_1\ldots a_{2p}} \rightarrow A_{a_1\ldots a_{2p}} + \epsilon_{a_1\ldots a_2p+1} f^{a_{2p+1}} \). It is easy to check that this field redefinition does no spoil any of the self-duality conditions already satisfied. Hence we have shown that all self-duality conditions are satisfied

\[
\mathcal{F}_{a_1\ldots a_{2p-k} \dot{a}_1\ldots \dot{a}_{k+1}} = 0 \quad (k = p, \ldots, 2p).
\]

(34)

More precisely, since we have to make various field redefinitions and gauge transformations (21) in order to obtain the self-duality condition, if the solutions of the our equations of motion are denoted by \( A \), the corresponding self-dual gauge field configuration \( A_{SD} \) is given by

\[
A_{SD}^{a_1\ldots a_k \dot{a}_1\ldots \dot{a}_{2p-k}} = A^{a_1\ldots a_k \dot{a}_1\ldots \dot{a}_{2p-k}} \quad \text{for} \quad k = 0, \ldots, p,
\]

(35)

\[
A_{SD}^{a_1\ldots a_k \dot{a}_1\ldots \dot{a}_{2p-k}} = (A + \Phi)^{a_1\ldots a_k \dot{a}_1\ldots \dot{a}_{2p-k}} \quad \text{for} \quad k = p+1, \ldots, 2p-1,
\]

(36)

\[
A_{SD}^{a_1\ldots a_{2p}} = (A + \Phi)^{a_1\ldots a_{2p}} + \epsilon^{a_1\ldots a_{2p+1}} f_{a_{2p+1}}(x^a).
\]

(37)

5 \( D'' > D' = \text{odd} \)

Consider the case \( D' = 2q + 1 < D'' \) (\( 0 \leq q \leq p - 1 \)). The action is (14)

\[
S = -\frac{1}{2} \int d^D x \sum_{j=q+1}^{2q+1} \tilde{F}_{a_1\ldots a_j \dot{a}_1\ldots \dot{a}_{2p-j+1}} \frac{\mathcal{F}_{a_1\ldots a_j \dot{a}_1\ldots \dot{a}_{2p-j+1}}}{j!(2p - j + 1)!},
\]

(38)
and the gauge transformations are (20)

\[
\delta A^{a_1 \cdots a_{2q-k+1} \hat{a}_1 \cdots \hat{a}_{2p-2q+k+1}} = \Phi^{a_1 \cdots a_{2q-k+1} \hat{a}_1 \cdots \hat{a}_{2p-2q+k+1}} (x^a, x^{\hat{a}}), \quad k = 0, 1, \ldots, q,
\]

\[
\delta A^{a_1 \cdots a_{2q-k+1} \hat{a}_1 \cdots \hat{a}_{2p-2q+k+1}} = f^{a_1 \cdots a_{2q-k+1} \hat{a}_1 \cdots \hat{a}_{2p-2q+k+1}} (x^a), \quad k = q + 1,
\]

\[
\delta A^{a_1 \cdots a_{2q-k+1} \hat{a}_1 \cdots \hat{a}_{2p-2q+k+1}} = 0, \quad k = q + 2, \ldots, 2q + 1,
\]

in addition to the ordinary gauge symmetry.

The nontrivial equations of motion are

\[
\frac{\delta S}{\delta A^{a_1 \cdots a_{2q-k+1} \hat{a}_1 \cdots \hat{a}_{2p-2q+k+1}}} = 0 \quad (k = q + 1, \ldots, 2q + 1).
\]

Starting with \( k = q + 1 \), we have

\[
\frac{\delta S}{\delta A^{a_1 \cdots a_q \hat{a}_1 \cdots \hat{a}_{2p-q}}} = 0 \Rightarrow \frac{\partial^{2p-q+1} F^{a_1 \cdots a_q \hat{a}_1 \cdots \hat{a}_{2p-q+1}}}{q!(2p-q)!} = 0,
\]

and it is solved by

\[
F^{a_1 \cdots a_q \hat{a}_1 \cdots \hat{a}_{2p-q+1}} = \frac{(-)^{q+1} \epsilon_{a_1 \cdots a_{2q+1}} \epsilon_{\hat{a}_1 \cdots \hat{a}_{2p-q+1}} \partial^{a_{2p-q+1}} \Phi^{a_{q+1} \cdots a_{2q+1} \hat{a}_{2p-q+2} \cdots \hat{a}_{2p}}}{(q+1)! (2p-q-1)!}
\]

for some \( \Phi \). The field \( \Phi \) can be eliminated by a gauge transformation (39) with \( k = q \)

\[
A^{a_1 \cdots a_{q+1} \hat{a}_1 \cdots \hat{a}_{2p-q-1}} \rightarrow (A + \Phi)^{a_1 \cdots a_{q+1} \hat{a}_1 \cdots \hat{a}_{2p-q-1}},
\]

so that

\[
F^{a_1 \cdots a_q \hat{a}_1 \cdots \hat{a}_{2p-q+1}} = 0.
\]

For the case \( q = 0 \) (\( D' = 1 \)), eq. (43) is the only self-duality condition. This case was already studied in the literature for spacetime dimensions \( D = 6 \) [8] and \( D = 10 \) [11].

For \( q \neq 0 \), we consider the equations of motion for \( k > q + 1 \). They are

\[
\frac{\delta S}{\delta A^{a_1 \cdots a_q-1 \hat{a}_1 \cdots \hat{a}_{2p-q+1}}} = 0 \Rightarrow \frac{\partial^b F_{ba_1 \cdots a_q-1 \hat{a}_1 \cdots \hat{a}_{2p-q+1}} + \partial^b F_{ba_1 \cdots a_q-1 \hat{a}_1 \cdots \hat{a}_{2p-q+1}}}{(q-1)!(2p-q+1)!} = 0,
\]

\[
\vdots
\]

\[
\frac{\delta S}{\delta A^{a_1 \cdots a_{2q-k+1} \hat{a}_1 \cdots \hat{a}_{2p-2q+k+1}}} = 0 \Rightarrow \frac{\partial^b F_{ba_1 \cdots a_{2q-k+1} \hat{a}_1 \cdots \hat{a}_{2p-2q+k+1}} + \partial^b F_{ba_1 \cdots a_{2q-k+1} \hat{a}_1 \cdots \hat{a}_{2p-2q+k+1}}}{(2q-k+1)!(2p-2q+k-1)!} = 0,
\]

\[
\vdots
\]

\[
\frac{\delta S}{\delta A^{a_1 \cdots a_{2p}}} = 0 \Rightarrow \frac{\partial^b F_{ba_1 \cdots a_{2p}} + \partial^b F_{ba_1 \cdots a_{2p}}}{(2p)!} = 0.
\]
To solve (44), we substitute (43) into (44) and get

\[ \partial^\alpha_{2p-q+2} \mathcal{F}_{a_1 \ldots a_{q-1} \hat{a}_1 \ldots \hat{a}_{2p-q+2}} = 0. \]  

Its solution is

\[ \mathcal{F}_{a_1 \ldots a_{q-1} \hat{a}_1 \ldots \hat{a}_{2p-q+2}} = \frac{(-)^q \epsilon_{a_1 \ldots a_{2q+1}} \epsilon_{\hat{a}_1 \ldots \hat{a}_{4p-2q+1}} \partial^{\hat{a}_{4p-2q+1}} \Phi_{a_{2q+1} \ldots \hat{a}_{2p-q+2} \ldots \hat{a}_{4p-2q}}}{(q+2)!(2p-q-2)!}. \]

Using a gauge transformation (39) with \( k = q - 1 \),

\[ A^{a_1 \ldots a_{q+2} \hat{a}_1 \ldots \hat{a}_{2p-q-2}} \rightarrow (A + \Phi)^{a_1 \ldots a_{q+2} \hat{a}_1 \ldots \hat{a}_{2p-q-2}}, \]

we obtain another self-duality condition

\[ \mathcal{F}_{a_1 \ldots a_{q-1} \hat{a}_1 \ldots \hat{a}_{2p-q+2}} = 0. \]

Now we can iterate this procedure to solve the equations of motion of \( A^{a_1 \ldots a_{q+2} \hat{a}_1 \ldots \hat{a}_{2p-q+2}} \). In general, for a given value of \( k \) from \( q + 2 \) to \( 2q + 1 \), we solve an equation of motion by introducing arbitrary functions \( \Phi \) as

\[ \mathcal{F}_{a_1 \ldots a_{2q-k+1} \hat{a}_1 \ldots \hat{a}_{2p-2q+k}} = \frac{(-)^k \epsilon_{a_1 \ldots a_{2q+1}} \epsilon_{\hat{a}_1 \ldots \hat{a}_{4p-2q+1}} \partial^{\hat{a}_{4p-2q+1}} \Phi_{a_{2q+1} \ldots \hat{a}_{2p-2q+k+1} \ldots \hat{a}_{4p-2q}}}{k!(2p-k)!}. \]

Then we obtain other self-duality conditions by using the gauge symmetry (39) from \( k = q - 1 \) down to \( k = 0 \) iteratively.

Finally, we have

\[ \mathcal{F}_{a_1 \ldots a_{2q-k+1} \hat{a}_1 \ldots \hat{a}_{2p-2q+k}} = 0 \quad (k = q + 1, \ldots, 2q + 1). \]  

These constitute all independent self-duality conditions for the gauge field. That is, the Hodge dual of these conditions imply the other half of the self-duality conditions

\[ \mathcal{F}_{a_1 \ldots a_{2q-k+1} \hat{a}_1 \ldots \hat{a}_{2p-2q+k}} = 0 \quad (k = 0, \ldots, q). \]

Similar to the case in section 44, the self-dual gauge fields \( A_{SD} \) satisfying all self-duality conditions are related to the gauge potential \( A \) in our theory only after field redefinitions (gauge transformations)

\[ A^{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p-k}}_{SD} = A^{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p-k}} \quad (k = 0, \ldots, q), \]

\[ A^{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p-k}}_{SD} = (A + \Phi)^{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p-k}} \quad (k = q + 1, \ldots, 2q + 1), \]

where the fields \( \Phi \) are introduced when solving the equations of motion (see (51)).
6 $D'' > D' = \text{even}$

The story for even $D'$ proceeds essentially in the same way as the odd $D'$ case, except that here we need to assume $\mathcal{M}_2^{D''}$ to be Euclidean (and so $\mathcal{M}_1^{D''}$ is Lorentzian). Let $D' = 2r + 2 < D''$ for some $r \in \{0, 1 \ldots, p - 1\}$. The action for this case is

$$S = -\frac{1}{2} \int d^D x \sum_{j=r+1}^{2r+2} \frac{\tilde{F}_{a_1 \ldots a_j \hat{a}_1 \ldots \hat{a}_{2p+1-j}} \mathcal{F}_{a_1 \ldots a_j a_1 \ldots \hat{a}_{2p-j+1}}}{j!(2p - j + 1)2^{\delta_{j,r+1}}},$$

(55)

and the gauge transformations are

$$\delta A^{a_1 \ldots a_{2r-k+2} \hat{a}_1 \ldots \hat{a}_{2p-2r+k-2}} = \Phi^{a_1 \ldots a_{2r-k+2} \hat{a}_1 \ldots \hat{a}_{2p-2r+k-2}}(x^a, \hat{x}^\alpha), \quad k = 0, 1, \ldots, r,$$

$$\delta A^{a_1 \ldots a_{2r-k+2} \hat{a}_1 \ldots \hat{a}_{2p-2r+k-2}} = f^{a_1 \ldots a_{2r-k+2} \hat{a}_1 \ldots \hat{a}_{2p-2r+k-2}}(x^a), \quad k = r + 1,$$

$$\delta A^{a_1 \ldots a_{2r-k+2} \hat{a}_1 \ldots \hat{a}_{2p-2r+k-2}} = 0, \quad k = r + 2, \ldots, 2r + 2,$$

(56)

in addition to the ordinary gauge transformations. It is easy to check that $S$ is invariant under these transformations.

The nontrivial equations of motion of the gauge fields are

$$\frac{\delta S}{\delta A^{a_1 \ldots a_{2r-k+2} \hat{a}_1 \ldots \hat{a}_{2p-2r+k-2}}} = 0 \quad (k = r + 1, \ldots, 2r + 2).$$

(57)

For $k = r + 1$, the equation of motion is

$$\partial^{\hat{a}_{2p-r}} \mathcal{F}_{a_1 \ldots a_{r+1} \hat{a}_1 \ldots \hat{a}_{2p-r-1} \hat{a}_{2p-r}} = 0,$$

(58)

which is solved by

$$\mathcal{F}_{a_1 \ldots a_{r+1} \hat{a}_1 \ldots \hat{a}_{2p-r-1} \hat{a}_{2p-r}} = \frac{\epsilon_{a_1 \ldots a_{2r+2} \hat{a}_1 \ldots \hat{a}_{4p-2r}} \partial^{\hat{a}_{2p-r+1}} \Phi^{a_1 \ldots a_{2r+2} \hat{a}_1 \ldots \hat{a}_{2p-r+2} \hat{a}_{4p-2r}}}{(r + 1)!(2p - r)!}.$$  

(59)

Taking the Hodge dual of both sides of this equation, we find

$$\mathcal{F}_{a_1 \ldots a_{r+1} \hat{a}_1 \ldots \hat{a}_{2p-r-1} \hat{a}_{2p-r}} = \partial^{\hat{a}_1} \Phi_{[a_1 \ldots a_{r+1} | \hat{a}_2 \ldots \hat{a}_{2p-r}]}.$$  

(60)

Then the equation of motion (58) implies

$$\partial^{\hat{a}_{2p-r}} \partial^{\hat{a}_1} \Phi_{[a_1 \ldots a_{r+1} | \hat{a}_2 \ldots \hat{a}_{2p-r}]} = 0.$$  

(61)

Now we notice that the field $\Phi$ defined via (59) is only defined up to a total derivative term

$$\Phi_{a_1 \ldots a_{r+1} \hat{a}_2 \ldots \hat{a}_{2p-r}} \rightarrow \Phi_{a_1 \ldots a_{r+1} \hat{a}_2 \ldots \hat{a}_{2p-r}} + \partial^{\hat{a}_2} \Lambda_{[a_1 \ldots a_{r+1} | \hat{a}_3 \ldots \hat{a}_{2p-r}]}.$$  

(62)
which is similar in spirit to a gauge symmetry. (It is the gauge symmetry of a gauge symmetry.) Thus we can choose the Lorentz gauge for $\Phi$

$$\partial^2 \Phi_{a_1...a_{r+1}a_2...a_{2p-r}} = 0. \quad (63)$$

After gauge fixing, eq. (61) becomes

$$\partial^2 \Phi_{a_1...a_{r+1}a_1...a_{2p-r-1}} = 0, \quad (64)$$

where $\partial^2 \equiv \partial^a \partial_a$ is the Laplacian on $\mathcal{M}_{2}^{D''}$. Assuming that $\mathcal{M}_{2}^{D''}$ is of Euclidean signature, the only solution to the Laplace equation (64) is **

$$\Phi_{a_1...a_{r+1}a_1...a_{2p-r-1}} = f_{a_1...a_{r+1}a_1...a_{2p-r-1}}(x^a) \quad (66)$$

for some functions $f$ independent of $x^\dot{a}$. Hence (59) becomes a self-duality condition

$$\mathcal{F}_{a_1...a_{r+1}a_1...a_{2p-r-1}a_{2p-r}} = 0. \quad (67)$$

Next we consider the case $k = r + 2$. The equation of motion (57) for $k = r + 2$ is

$$\partial^b (\mathcal{F} + F)_{a_1...a_r\dot{a}_1...\dot{a}_{2p-r}} \dot{b} + \partial^b F_{a_1...a_r\dot{a}_1...\dot{a}_{2p-r}} = 0. \quad (68)$$

Using (67), we can rewrite the 2nd term in (68) as $\partial^b \tilde{F}_{b...}$, which equals $-\partial^b \tilde{F}_{b...}$ due to Bianchi identity. As a result (68) is equivalent to

$$\partial^{a_{2p-r+1}} \mathcal{F}_{a_1...a_r\dot{a}_1...\dot{a}_{2p-r-1}a_{2p-r+1}} = 0, \quad (69)$$

and the solution is

$$\mathcal{F}_{a_1...a_r\dot{a}_1...\dot{a}_{2p-r-1}a_{2p-r+1}} = \frac{\epsilon_{a_1...a_{2p-r+1}} \epsilon_{\dot{a}_1...\dot{a}_{2p-r+1}} \partial^{a_{2p-r+1}} \Phi_{a_1...a_{2p-r+1}}}{(r+2)!(2p-r-2)!}. \quad (70)$$

One can gauge away $\Phi$ on the right-hand side by using the gauge degree of freedom in (56) with $k = r$, and arrive at the self-duality condition

$$\mathcal{F}_{a_1...a_r\dot{a}_1...\dot{a}_{2p-r-1}} = 0. \quad (71)$$

**Here we assume that the boundary condition of $\Phi$ at infinity is such that the integration $\int d^D x \partial_\alpha (\Phi \partial^\alpha \Phi)$ vanishes, so that, for the inner product of two functions $f$ and $g$ defined by $\langle f | g \rangle \equiv \int d^D x \ f^* g$,

$$0 = -\langle \Phi | \partial^2 \Phi \rangle = \langle \partial_\alpha \Phi | \partial_\alpha \Phi \rangle \geq 0, \quad (65)$$

which implies that $\partial_\alpha \Phi = 0$. \hspace{1cm}
The variation of the action with respect to the remaining gauge fields are

\[
\frac{\delta S}{\delta A^a_{a_1 \cdots a_r - 1 \dot{a}_1 \cdots \dot{a}_{2p - r + 1}}} = 0 \Rightarrow \frac{\partial^b F_{a_1 \cdots a_r - 1 \dot{a}_1 \cdots \dot{a}_{2p - r + 1} b}}{(r - 1)!(2p - r + 1)!} = 0, \quad (72)
\]

\[
\frac{\delta S}{\delta A^a_{a_1 \cdots a_{2r - k + 2 \dot{a}_1 \cdots \dot{a}_{2p - 2r + k - 2}}} = 0 \Rightarrow \frac{\partial^b F_{a_1 \cdots a_{2r - k + 2 \dot{a}_1 \cdots \dot{a}_{2p - 2r + k - 2} b}}}{(2r - k + 2)!(2p - 2r + k - 2)!} = 0, \quad (73)
\]

\[
\frac{\delta S}{\delta A^a_{a_1 \cdots \dot{a}_{2p}}} = 0 \Rightarrow \frac{\partial^b F_{a_1 \cdots \dot{a}_{2p} b}}{(2p)!} = 0. \quad (74)
\]

As we have done many times already, after substituting (71) into (72), solving the differential equation, and then using the gauge symmetry (56) with \( k = r - 1 \), we obtain another self-duality condition. Repeating the same manipulation for higher and higher values of \( k \), one derives all the remaining self-duality conditions

\[
F_{a_1 \cdots a_{2r - k + 2 \dot{a}_1 \cdots \dot{a}_{2p - 2r + k - 1}} = 0 \quad (k = r + 3, \ldots, 2r + 2). \quad (75)
\]

Similar to the previous two sections, one can check that there is a 1-1 correspondence between the gauge potential \( A \) in our theory and self-dual gauge configurations \( A_{SD} \)

\[
A_{SD}^{a_1 \cdots a_k \dot{a}_1 \cdots \dot{a}_{2p - k}} = A^{a_1 \cdots a_k \dot{a}_1 \cdots \dot{a}_{2p - k}} \quad (k = 0, \ldots, r + 1), \quad (76)
\]

\[
A_{SD}^{a_1 \cdots a_k \dot{a}_1 \cdots \dot{a}_{2p - k}} = (A + \Phi)^{a_1 \cdots a_k \dot{a}_1 \cdots \dot{a}_{2p - k}} \quad (k = r + 2, \ldots, 2r + 2), \quad (77)
\]

where \( \Phi \) are the fields arising when we solve the equations of motion of \( A \).

7 Hidden Lorentz Symmetry

The action \( S_{D' + D''} \) is manifestly invariant under the \( SO(D' - 1, 1) \times SO(D'') \), or \( SO(D') \times SO(D'' - 1, 1) \) subgroup of the full Lorentz group \( SO(D - 1, 1) \) of the spacetime \( \mathcal{M}^D = \mathcal{M}_{1}^{D'} \times \mathcal{M}_{2}^{D''} \), depending on the signature of \( \mathcal{M}_{1}^{D'} \) and \( \mathcal{M}_{2}^{D''} \). The Lorentz symmetry transformations which mix \( x^a \) with \( x^\dot{a} \) are no longer manifest. Nevertheless, the action \( S \) is actually fully Lorentz invariant under a modified Lorentz transformation rule for those transformations which mix \( x^a \) with \( x^\dot{a} \). This is the main topic of this section.

For simplicity we impose the gauge fixing condition

\[
A_{a_1 \cdots a_k \dot{a}_1 \cdots \dot{a}_{2p - k}} = 0 \quad (k = \lfloor D'/2 \rfloor + 1, \ldots, D' - \alpha), \quad (78)
\]
where \([s]\) is the largest integer not greater than \(s\), \(\alpha\) is given in (55) so that \(D' - \alpha = \min(D', 2p)\). Thus we only have to consider the transformations of the remaining fields

\[
A_{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p - k}} \quad (k = 0, 1, \ldots, \lfloor D'/2 \rfloor).
\]

Our claim is that

\[
\delta S = \sum_{k=0}^{\lfloor D'/2 \rfloor} \frac{\delta S}{\delta A_{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p - k}}} \delta A_{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p - k}}
\]

vanishes for the modified Lorentz transformation rule

\[
\delta A_{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p - k}} = \delta_1 A_{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p - k}} + \delta_2 A_{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p - k}},
\]

where

\[
\delta_1 A_{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p - k}} = k \lambda^b_{a_k} A_{a_1 \ldots a_k - 1 \hat{a}_1 \ldots \hat{a}_{2p - k}} - (2p - k) \lambda^b_{a_1 \ldots a_k b \hat{a}_2 \ldots \hat{a}_{2p - k - 1}}
\]

\[
+ \lambda^b_{a_k}(x_b \partial_b - x_b \partial_{\hat{b}}) A_{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p - k}},
\]

\[
\delta_2 A_{a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p - k}} = \delta[D'/2]_k \lambda^b_{a_1} \lambda^b_{a_2} \mathcal{F}_{b_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p - k}}
\]

\[
+ \frac{1}{2} \delta_{D'=even} \delta[D'/2]_{1, k} \lambda^b_{a_1} \lambda^b_{a_2} \mathcal{F}_{b_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_{2p - k}}.
\]

In the expressions above, \(k = 0, 1, \ldots, \lfloor D'/2 \rfloor\), and \(\lambda_{a\hat{a}}\) is the Lorentz transformation parameter.

Because of the gauge-fixing (78), some of the field strength components vanish, and thus their dual field strengths also vanish

\[
F_{a_1 \ldots a_{D'-j} \hat{a}_1 \ldots \hat{a}_{D'/2 - D' - j}} = \tilde{F}_{a_1 \ldots a_j \hat{a}_1 \ldots \hat{a}_{D'/2 - j}} = 0
\]

for \(j = 0, 1, \ldots, \lfloor D'/2 \rfloor - 2\). Using the expressions of \(\delta S/\delta A\) we computed earlier when we derived the equations of motion, we check after lengthy calculations that \(\delta_1 S + \delta_2 S = 0\) for all decompositions \(D = D' + D''\).

More specifically, for \(D' = D''\),

\[
\delta_1 S = -\delta_2 S = \frac{1}{2^p!p!} \int d^D x \lambda^b_{a_1} \mathcal{F}_{a_1 \ldots a_p \hat{a}_1 \ldots \hat{a}_p} \mathcal{F}_{b_1 \ldots a_p \hat{a}_1 \ldots \hat{a}_p};
\]

for \(D' = 2q + 1\),

\[
\delta_1 S = -\delta_2 S = \frac{1}{2^q!(2p - q)!} \int d^D x \lambda^b_{a_1} \mathcal{F}_{a_1 \ldots a_q \hat{a}_1 \ldots \hat{a}_{2p - q}} \mathcal{F}_{b_1 \ldots a_q \hat{a}_1 \ldots \hat{a}_{2p - q}};
\]

and for \(D' = 2r + 2\),

\[
\delta_1 S = -\delta_2 S = \frac{1}{2^r!(2p - r)!} \int d^D x \lambda^b_{a_r} \mathcal{F}_{a_r \ldots a_{2p - r}} \mathcal{F}_{b_{a_1} \ldots a_r \hat{a}_1 \ldots \hat{a}_{2p - r}}.
\]
8 Interaction with Charged Branes

A 2p-form gauge potential naturally couples to a (2p − 1)-brane with a 2p-dimensional worldvolume as the electric charge. The interaction between the (2p − 1)-brane and the field should preserve all the gauge symmetry. In particular, the new gauge symmetry (20) are needed in order to have first order differential equations as equations of motion.

We find the interaction term of the action to be given by

$$S_{D' + D''}^{\text{int}} = - \int d^D x \sum_{j = \lfloor \frac{D'}{2} \rfloor}^{D'} F_{\dot{a}_{a_1 \ldots a_{D'-j}} \dot{a}_{D/2-D'+j} a_1 \ldots a_{D/2-D'-D'+j}} Q^{a_1 \ldots a_{D'-j} \dot{a}_{D/2-D'+j} a_1 \ldots a_{D/2-D'-D'+j}} (D' - j)! \left( \frac{D' - D'}{2} + j \right)! 2^{\delta j, (D' - D')},$$

(88)

where

$$Q = \sigma - \tilde{\sigma}$$

(89)

is a (2p+1)-form and $\sigma$ represents the source. Roughly speaking, the source term ($\sigma$ and its Hodge dual $\tilde{\sigma}$) should be determined by a Dirac delta function which specifies the location of the (2p − 1)-brane. We will give explicit examples below.

With the interaction term (88) added, the modified equations of motion can again be reduced to first order differential equation using the additional gauge symmetries eq.(20) as

$$(F + Q)_{\dot{a}_{a_1 \ldots a_k \dot{a}_{2p-k}}} = 0,$$

(90)

saying that $F$ is no longer self-dual but $F + \sigma$ is. At the same time the equations of motion imply

$$\partial^\dot{b} (F + Q)_{\dot{a}_{a_1 \ldots a_k \dot{a}_{2p-k}}} = 0,$$

(91)

By means of the Bianchi identity, we can rewrite eq.(91) in following form which was obtained earlier in [9]:

$$\partial^\dot{b} (F + \sigma)_{\dot{a}_{a_1 \ldots a_k \dot{a}_{2p-k}}} + \partial^\dot{b} (\tilde{F} + \tilde{\sigma})_{ba_1 \ldots a_k \dot{a}_{2p-k}} = \partial^\mu \tilde{\sigma}_{\mu a_1 \ldots a_k \dot{a}_{2p-k}}.$$

(92)

Although it appears that the effect of the source term is merely to replace $F$ by $F + \sigma$ in both equations of motion (90) and (91), one cannot redefine $F$ to absorb $\sigma$. The reason is that, although $F$ and $\sigma$ are both (2p + 1)-forms, $F$ is a closed differential form while $\sigma$ is not. This also explains why there is an additional term on the right-hand side of (92).

To be more explicit, let us give some examples for each type of spacetime decomposition:
1. $D' = D''$

$$\partial^\mu \tilde{\sigma}_{x_1 \cdots x_k \hat{x}_1 \cdots \hat{x}_{2p-k} \mu} = \prod_{\ell=k+1}^{2p+1} \delta(x_\ell) \prod_{\ell=2p-k+1}^{2p+1} \delta(\hat{x}_\ell) \quad (0 \leq k \leq p). \quad (93)$$

2. $D' = \text{odd}$

$$\partial^\mu \tilde{\sigma}_{x_1 \cdots x_k \hat{x}_1 \cdots \hat{x}_{2p-k} \mu} = \prod_{\ell=k+1}^{2q+1} \delta(x_\ell) \prod_{\ell=2p-k+1}^{4p-2q+1} \delta(\hat{x}_\ell) \quad (0 \leq k \leq q). \quad (94)$$

3. $D' = \text{even}$

$$\partial^\mu \tilde{\sigma}_{x_1 \cdots x_k \hat{x}_1 \cdots \hat{x}_{2p-k} \mu} = \prod_{\ell=k+1}^{2r+2} \delta(x_\ell) \prod_{\ell=2p-k+1}^{4p-2r} \delta(\hat{x}_\ell) \quad (1 \leq k \leq r+1). \quad (95)$$

Here the value of $k$ is a fixed number used to specify how the $2p$-dimensional world-volume of the source brane is divided into $\mathcal{M}^{D'}_1 \times \mathcal{M}^{D''}_2$. We use the divergence of $\tilde{\sigma}$ to define $\sigma$, with all other components of the divergence of $\tilde{\sigma}$ vanishing. These divergence relations do not determine $\sigma$ uniquely but different solutions of $\sigma$ are equivalent. The $x_\ell$’s and $\hat{x}_\ell$’s appearing in the delta functions (there are $2p+2$ of them) are the coordinates transverse to the brane.

In order for the brane to be time-like, one should choose the non-vanishing components of $\partial \cdot \tilde{\sigma}$ to have a time-like index. This imposes a constraint on the choice of $\partial \cdot \tilde{\sigma}$ for the $D' = \text{even}$ case because the coordinate space $\{x^a\}$ of $\mathcal{M}_1^{D'}$ is of Lorentzian signature.

For the source given above, the equation of motion (92) becomes

$$\partial^j(F + \sigma)_{x_1 \cdots x_k \hat{x}_1 \cdots \hat{x}_{2p-k} b} + \partial^b(F + \sigma)_{b x_1 \cdots x_k \hat{x}_1 \cdots \hat{x}_{2p-k}} = \begin{cases} \prod_{\ell=k+1}^{2p+1} \delta(x_\ell) \prod_{\ell=2p-k+1}^{2p+1} \delta(\hat{x}_\ell), & (D' = D''), \\ \prod_{\ell=k+1}^{2q+1} \delta(x_\ell) \prod_{\ell=2p-k+1}^{4p-2q+1} \delta(\hat{x}_\ell), & (D' = \text{odd}), \\ \prod_{\ell=k+1}^{2r+2} \delta(x_\ell) \prod_{\ell=2p-k+1}^{4p-2r} \delta(\hat{x}_\ell), & (D' = \text{even}). \end{cases} \quad (96)$$

* A more familiar expression of the interaction term is of the form $A \cdot j$ for some current $j$. In (95), the interaction term is of the form $-F \cdot Q$, and since $F = dA$, it is equivalent to $A \cdot dQ$ up to a total derivative. Thus $dQ$ is the current $j$ coupled to $A$. 

16
9 Examples

9.1 \( D = 2, \ (D', D'') = (1, 1) \)

The simplest example of self-dual field theory is the two-dimensional theory of chiral bosons. The only possible decomposition of the 2D Minkowski space is

\[
\mathbb{R}^{1+1} = \mathbb{R} \times \mathbb{R}.
\]

We use \( x^1 \) and \( x^1 \) to denote the coordinates on each factor of \( \mathbb{R} \). Either \( x^1 \) or \( x^1 \) can be time-like, and the other one space-like.

The gauge potential has no index and has only one component which will be denoted as \( \phi \). The action is

\[
S_{1+1} = -\frac{1}{2} \int d^2 x F^1_{\dot{1}} F^1_{\dot{1}} ,
\]

where \( F_{\dot{1}} = \partial_{\dot{1}} \phi \) and \( \tilde{F}^1 = \partial_1 \phi \). It is the same as the action given in [2]. Note that this action is invariant under the gauge transformation (the second line in (20))

\[
\phi \rightarrow \phi + f(x^1)
\]

for an arbitrary function \( f(x^1) \) that is independent of \( x^\dot{1} \). This is a larger gauge symmetry than the usual gauge symmetry for a 0-form gauge potential, which is restricted to constant \( f \).

The action is not manifestly Lorentz invariant. Define the modified Lorentz transformation law

\[
\delta \phi = \lambda^{\dot{a}} a (x_a \partial_\dot{a} - x_\dot{a} \partial_a) \phi + \lambda_a^a x^{\dot{a}} F_a \equiv \delta_1 \phi + \delta_2 \phi.
\]

where

\[
\delta_1 \phi = \lambda^{\dot{a}} a (x_a \partial_\dot{a} - x_\dot{a} \partial_a) \phi, \quad \delta_2 \phi = \lambda_a^a x^{\dot{a}} F_a.
\]

When \( \mathcal{F} = 0 \), \( \delta_2 \phi = 0 \) and \( \delta_1 \phi \) is the usual Lorentz transformation law. Then one can check that

\[
\delta_1 S = -\delta_2 S = \frac{1}{2} \int d^2 x \lambda_a^a F_a \mathcal{F}^a
\]

and so the action \( S \) is invariant under this modified Lorentz transformation law.

The equation of motion is

\[
\frac{\delta S_{1+1}}{\delta \phi} = 0 \Rightarrow \partial_{\dot{1}} \mathcal{F}_{\dot{1}} = 0,
\]

whose solution is

\[
\mathcal{F}_{\dot{1}} = g(x^1),
\]
where \( g(x^1) \) is an arbitrary function depending only on \( x^1 \). The function \( g(x^1) \) can be absorbed by the gauge transformation (99) with \( f(x^1) = - \int x^1 dy g(y) \), and then we have the self-duality condition
\[
F_1 = 0. 
\] (105)

### 9.2 \( D = 6, (D', D'') = (1, 5) \)

The decomposition \( (D', D'') = (1, 5) \) was used previously for the formulation of M5-brane [8] and type IIB supergravity [11]. The gauge potential has the following components
\[
A_{\dot{1} \dot{\alpha}}, \quad A_{\dot{\alpha} \dot{\beta}} \quad (\dot{\alpha}, \dot{\beta} = \dot{1}, \ldots, \dot{5}). 
\] (106)

The action is
\[
S_{1+5} = - \frac{1}{4} \int d^6 x \tilde{F}_{1\dot{a}\dot{b}} F^{1\dot{a}\dot{b}}. 
\] (107)

The equation of motion obtained by varying \( S_{1+5} \) with respect to \( A_{1\dot{a}} \) is
\[
\frac{\delta S_{1+5}}{\delta A_{1\dot{a}}} = 0 \Rightarrow \partial^{\dot{a}} \tilde{F}_{1\dot{a}\dot{b}} \equiv 0, 
\] (108)
which is identically zero. This means that \( S_{1+5} \) depends on \( A_{1\dot{a}} \) only through total derivative terms. Thus the theory has the gauge symmetry
\[
A_{1\dot{a}} \rightarrow \Phi^{1\dot{a}} 
\] (109)
for arbitrary functions \( \Phi^{1\dot{a}} \).

The equation of motion for \( A_{\dot{a} \dot{b}} \) is
\[
\frac{\delta S_{1+5}}{\delta A_{\dot{a} \dot{b}}} = 0 \Rightarrow \partial^{\dot{a}} \mathcal{F}_{\dot{a} \dot{b} \dot{c}} = 0, 
\] (110)
whose solution is
\[
\mathcal{F}_{\dot{a} \dot{b} \dot{c}} = \epsilon_{\dot{1} \dot{a} \dot{b} \dot{c} \dot{d} \dot{e}} \partial^{\dot{d}} \Phi^{1\dot{e}}, 
\] (111)
where \( \Phi^{1\dot{e}} \) are arbitrary functions. Using the gauge transformation of \( A_{1\dot{a}} \) (109), we can make a gauge transformation to absorb \( \Phi^{1\dot{e}} \) and arrive at the self-duality condition
\[
\mathcal{F}_{\dot{a} \dot{b} \dot{c}} = 0, 
\] (112)
which is the only one self-duality condition in the special case.

An equivalent formulation of this theory is to start with a theory without \( A_{1\dot{a}} \). The only fields of this alternative formulation are \( A_{\dot{a} \dot{b}} \). The action is taken to be \( S_{1+5} \) with all \( A_{1\dot{a}} \) set to zero, so that \( F_{1\dot{a} \dot{d} \dot{e}} \), which appears in \( \mathcal{F}_{\dot{a} \dot{b} \dot{c}} \), is now replaced by
\[
f_{1\dot{d} \dot{e}} = \partial_{\dot{1}} A_{\dot{d} \dot{e}}. 
\] (113)
The equation of motion of $A^{\dot{a}\dot{b}}$ is still of the form (110), but with $\mathcal{F}_{\dot{a}\dot{b}\dot{c}}$ replaced by
\[ F_{\dot{a}\dot{b}} - \frac{1}{2} \epsilon_{\dot{a}\dot{b}\dot{c}\dot{d}} F^{\dot{1}\dot{d}} = F_{\dot{a}\dot{b}} + \frac{1}{2} \epsilon_{\dot{a}\dot{b}\dot{c}\dot{d}} \partial^1 A^{\dot{d}} = \epsilon_{\dot{1}\dot{a}\dot{b}\dot{c}} \partial^d \Phi^{\dot{1}\dot{e}}. \] (114)

This can be rewritten as the self-duality condition (112) if we define
\[ F_{1\dot{d}e} \equiv \partial_1 A_{\dot{d}e} - \partial_{\dot{d}} \Phi_{1\dot{e}} + \partial_{\dot{e}} \Phi_{1\dot{d}}. \] (115)

Hence the functions $\Phi^{1\dot{a}}$ appearing in the solution of the equations of motion should be interpreted as the components $A^{1\dot{a}}$ of a self-dual gauge potential together with $A^{\dot{a}\dot{b}}$.

9.3 $D = 6$, $(D', D'') = (2, 4)$

Instead of decomposing the 6D Minkowski space $\mathbb{R}^{1+5}$ as $\mathbb{R}^1 \times \mathbb{R}^5$ or $\mathbb{R}^1 \times \mathbb{R}^{1+4}$ like we did in the previous subsection, here we consider another decomposition
\[ \mathbb{R}^{1+5} = \mathbb{R}^{1+1} \times \mathbb{R}^4. \] (116)

To our knowledge this decomposition was never discussed in the literature. The 2-form gauge potential has the following components
\[ A_{a\dot{b}}, \quad A_{a\dot{a}}, \quad A_{\dot{a}\dot{b}} \quad (a, \dot{b} = 1, 2; \quad \dot{a}, \dot{b} = \dot{1}, \ldots, \dot{4}). \] (117)

The action is
\[ S_{2+4} = -\frac{1}{8} \int d^6 x \left( 2 F_{ab} F^{ab} + F_{\dot{a}\dot{b}} F^{\dot{a}\dot{b}} \right). \] (118)

The variation of the action with respect to $A_{ab}$,
\[ \frac{\delta S_{2+4}}{\delta A_{ab}} = 0 \Rightarrow \frac{1}{4} \partial^a \tilde{F}_{a\dot{b}} = 0, \] (119)
vanishes identically, and thus the gauge potential $A_{ab}$ can perform the gauge transformation
\[ A_{ab} \to A_{ab} + \Omega_{ab}. \] (120)

The equations of motion of $A_{a\dot{a}}$ and $A_{\dot{a}\dot{b}}$ are,
\[ \frac{\delta S_{2+4}}{\delta A_{a\dot{a}}} = 0 \Rightarrow \frac{1}{2} \partial^b \tilde{F}_{a\dot{a}b} = 0, \] (121)
\[ \frac{\delta S_{2+4}}{\delta A_{\dot{a}\dot{b}}} = 0 \Rightarrow \frac{1}{4} \left[ \partial^\dot{e} (F + F)_{\dot{a}\dot{b}} + \partial^\dot{e} F_{a\dot{a}\dot{b}} \right] = 0. \] (122)

The solution to eq. (121) is
\[ F_{a\dot{a}b} = \epsilon_{ab} \epsilon_{\dot{a}\dot{b}\dot{c}\dot{d}} \partial^\dot{c} \Phi^{\dot{1}\dot{d}} \] (123)
for some arbitrary functions $\Phi^{\dot{a}\dot{b}}$. Taking the Hodge-dual of both sides of eq. (123), we have another solution to eq. (121)

$$F_{a\dot{a}} = \partial_a \Phi_{\dot{a}} - \partial_{\dot{a}} \Phi_a.$$  

(124)

Identifying these two solutions of eq. (121) leads to

$$\partial_a \Phi_{\dot{a}} - \partial_{\dot{a}} \Phi_a = \epsilon_{ab} \epsilon_{\dot{a}\dot{b}cd} \partial^c \Phi^{\dot{b}},$$  

(125)

and then we act $\partial^a$ on both sides of the equivalence relation,

$$\partial^a \partial_a \Phi_{\dot{a}} - \partial_{\dot{a}} \partial^a \Phi_a = 0.$$  

(126)

There are two gauge degrees of freedom in the field $\Phi_{a\dot{a}}$ because the content of the equation (123) is unchanged under the transformation

$$\Phi_{a\dot{a}} \rightarrow \Phi_{a\dot{a}} + \partial_a \Lambda_{\dot{a}}.$$  

(127)

As a result we can choose the Lorentz gauge $\partial^a \Phi_{a\dot{a}} = 0$ and then (126) reduces to

$$\partial^a \Phi_{a\dot{a}} = 0,$$  

where $$\partial^a \equiv \partial_a.$$  

(128)

Imposing the boundary condition that $\Phi_{a\dot{a}}$ vanishes at infinities of the 4D Euclidean space $\mathbb{R}^4$ such that (128) has the unique solution $\Phi_{a\dot{a}} = 0$, we arrive at one of the self-duality conditions

$$F_{a\dot{a}} = 0.$$  

(129)

Next, plugging (129) into (122) implies that

$$F_{a\dot{b}\dot{c}} = \frac{1}{2} \epsilon_{ab} \epsilon_{\dot{a}\dot{b}cd} \partial^c \Omega_{\dot{a}}^b$$  

(130)

for some arbitrary functions $\Omega_{\dot{a}}^b$. After performing the gauge transformation $\delta A_{ab} = \Omega_{ab}$, the equation above becomes the other self-duality condition

$$F_{a\dot{b}\dot{c}} = 0.$$  

(131)

Eqs. (129) and (131) are sufficient to guarantee the self-duality conditions for all components of the field strength.

To summarize, solutions of $A$ to the equations of motion derived from the action $S_{2+4}$ (118) can be identified with a self-dual gauge potentials $A_{SD}$

$$A_{SD}^{ab} = (A + \Omega)^{ab}, \quad A_{SD}^{a\dot{b}} = A^{a\dot{b}}, \quad A_{SD}^{\dot{a}\dot{b}} = A^{\dot{a}\dot{b}}.$$  

(132)
9.4 $D = 6, \ (D', D'') = (3, 3)$

Now we introduce the 3rd formulation for 6D self-dual gauge theory based on the decomposition of 6D Minkowski space

$$\mathbb{R}^{1+5} = \mathbb{R}^{1+2} \times \mathbb{R}^3.$$ (133)

This formulation was first introduced in [3] as a linearized version of the M5-brane theory. It was later extended to the full interacting M5-brane theory in [4], and the self-duality conditions in the nonlinear version of the pure gauge theory were explicitly examined in [12], and the full theory including interactions between gauge fields and matter fields is analyzed in [13].

The gauge potential has the following components

$$A_{ab}, \ A_{a\dot{a}}, \ A_{\dot{a}b} \quad (a, b = 1, 2, 3; \ \dot{a}, \dot{b} = \dot{1}, \dot{2}, \dot{3}),$$ (134)

where the indices $a = (1, 2, 3)$ and $\dot{a} = (\dot{1}, \dot{2}, \dot{3})$, correspond, respectively, to the $SO(2, 1)$ and $SO(3)$ subgroup of the full $D = 6$ Lorentz group. The action is

$$S_{3+3} = -\frac{1}{12} \int d^6x (\tilde{F}_{abc} F^{abc} + 3 \tilde{F}_{a\dot{b}\dot{a}} F^{a\dot{b}\dot{a}}).$$ (135)

The terms involving $A_{ab}$ is a total derivative, and thus the equation of motion of $A_{ab}$ is trivial

$$\frac{\delta S_{3+3}}{\delta A_{ab}} = 0 \Rightarrow \frac{1}{4} \left( \partial^c \tilde{F}_{abc} + \partial^\dot{c} \tilde{F}_{ab\dot{c}} \right) = 0.$$ (136)

This means that

$$A_{ab} \rightarrow A_{ab} + \Phi_{ab}$$ (137)

is a gauge symmetry.

The equations of motion of $A_{a\dot{a}}$ and $A_{\dot{a}b}$ are

$$\frac{\delta S_{3+3}}{\delta A_{a\dot{a}}} = 0 \Rightarrow \partial^b \tilde{F}_{a\dot{a}b} = 0,$$ (138)

$$\frac{\delta S_{3+3}}{\delta A_{\dot{a}b}} = 0 \Rightarrow \frac{1}{2} \left( \partial^a \tilde{F}_{a\dot{a}b} + \partial^{\dot{a}} \tilde{F}_{\dot{a}b} \right) = 0.$$ (139)

The solution to (138) is

$$\tilde{F}_{a\dot{a}b} = \frac{1}{2} \epsilon_{abc} \epsilon_{\dot{a}\dot{b}\dot{c}} \partial^c \Phi^{bc}$$ (140)

for some arbitrary functions $\Phi^{bc}$. Through a suitable gauge transformation (137), it can be reduced to

$$\tilde{F}_{a\dot{a}b} = 0.$$ (141)
Substituting (141) into (139), we solve the latter as

\[ F_{\dot{a}\dot{b}\dot{c}} = \epsilon_{\dot{a}\dot{b}\dot{c}} f(x^a), \]  

(142)

where \( f(x) \) is a function depending only on the coordinates \( x^a \) but not on \( x^\dot{a} \). Since \( f(x) \) can always be written as \( \partial^a f_a \) for some fields \( f_a(x^b) \), it can be absorbed by a field redefinition \( A_{ab} \rightarrow A_{ab} + \epsilon_{abc} f^c \). One can check that this does not spoil the self-duality condition (141) we obtained earlier. Hence we finally have the other self-duality condition,

\[ F_{\dot{a}\dot{b}\dot{c}} = 0. \]  

(143)

Together with (141), they constitute the full self-duality conditions for the 6D self-dual gauge field \( A \).

## 10 Comparison with the holographic action

The results presented above are closely related to the work of Belov and Moore [9]. They defined an action, which is called the holographic action, for self-dual gauge fields, based on a decomposition of the space of \((2p+1)\)-forms \( V_\mathbb{R} \). In their notation, \( V_\mathbb{R} = V_2 \oplus V_2^\perp \), where \( V_2^\perp = \ast E V_2 \) is the orthogonal complement with respect to the Hodge metric. (The subscript \( E \) stands for Euclidean, as the formulation starts with a Wick-rotated space with Euclidean signature.) Thus, in particular, a field strength \( F \) can be decomposed as \( F = F_2 + F_2^\perp \). (\( F \) is denoted as \( R \) in [9].)

The decomposition of spacetime \( \mathcal{M}^D = \mathcal{M}^D_1 \times \mathcal{M}^D_2 \) we considered in this paper also leads to a decomposition of the space of differential forms. For \((2p+1)\)-forms, we can choose

\[ V_2^\perp = \{ dx^{a_1} \cdots dx^{a_k} dx^{\dot{a}_1} \cdots dx^{\dot{a}_{2p-k+1}} | k = 0, 1, \cdots, \lfloor D'/2 \rfloor \}. \]  

(144)

For a given field strength \( F \), the part of \( F_2^\perp \) depends only on the components

\[ \{ A_{a_1 \cdots a_k \dot{a}_1 \cdots \dot{a}_{2p-k}} | k = 0, 1, \cdots, \lfloor D'/2 \rfloor \} \]  

(145)

of the gauge potential, and these are exactly the components of the gauge potential that cannot be gauged away in our formulation (see, e.g, (20)). The decomposing the space of the field strength \( F \) as in the work of Belov and Moore is thus implied by our classification of the gauge potential components \( A \).

---

† We thank Yuji Tachikawa for pointing out this reference to us after the first version of this paper appeared on arXiv.
Another technical difference between these two approaches is that we used the equations of motion to prove the self-duality condition, while Belov and Moore used the equations of motion to prove that the \((2p + 1)\)-form \(F^+ \equiv F_2^{\perp} + \ast F_2^{\perp}\), which is by definition self-dual, is closed. Nevertheless, since the self-dual field strength is completely fixed by the part \(F_2^{\perp}\) in \(V_2^{\perp}\), the full field strength is the same independent of whether the rest of the components are obtained by definition as \(\ast F_2^{\perp}\), or obtained by solving equations of motion which lead to self-duality conditions.

Apart from these technical details, our work can be viewed as an application of the holographic action [9] with a special choice of the decomposition of \(V\). The work of Belov and Moore [9] is more general and has the advantage of being able to deal with topological issues more properly. They have also shown how to couple D-branes to the self-dual fields. Our work is on the other hand focused on a special realization of their general formulation, giving explicit expressions of the action in terms of gauge potential components, and explicit expressions for Lorentz transformation laws. This is achieved by realizing that the construction of a self-dual field theory is naturally connected with the decomposition of space-time, when it is a product space.

11 Conclusion

In this work we constructed Lagrange formulations for self-dual gauge theories in \(D = 4p + 2\) dimensions \((p = 0, 1, 2, \cdots)\) for arbitrary decomposition \(D = D' + D''\). In addition to the ordinary gauge symmetry

\[
A^{(2p)}(x) \rightarrow A^{(2p)}(x) + d\Lambda^{(2p-1)},
\]

the action (13)-(15) is invariant under additional gauge transformations, which allow us to gauge away some of the components of \(A\), leaving the following components

\[
A_{a_1 \cdots a_{k-1} \bar{a}_1 \cdots \bar{a}_{2p-k}} \quad (k = 0, 1, \cdots, \lfloor D'/2 \rfloor) \tag{147}
\]
as dynamical variables satisfying 2nd order differential equations. In the process of solving the equations of motion for these remaining components of \(A\), additional fields \(\Phi\) are introduced when 2nd order differential equations are integrated to 1st order differential equations. Together with those components of \(A\) in (147), the fields \(\Phi\) are to be identified with the rest of the components of the self-dual configuration. Despite the fact that the action is not manifestly Lorentz invariant, it enjoys the full Lorentz symmetry with a modified transformation law (81).
Potential applications of our new formulations of self-dual gauge theories include rewriting a self-dual gauge theory, say, the action of type IIB supergravity for decompositions other than \((D', D'') = (1, 9)\), which was already constructed in [11]. Other decompositions \((D', D'')\) may be more convenient when specific backgrounds are considered, e.g. when the spacetime is naturally viewed as a product space. For example, the decomposition \((D', D'') = (5, 5)\) will be especially convenient when we try to construct new solutions of IIB supergravity as deformations of the \(AdS_5 \times S^5\) background.

Acknowledgments

The authors thank Chien-Ho Chen, Kazuyuki Furuuchi, Yu-Tin Huang, Sheng-Lan Ko, Sangmin Lee, Paolo Pasti, Dmitri Sorokin, Yuji Tachikawa, Tomohisa Takimi, Daniel Thompson and Chi-Hsien Yeh. This work is supported in part by the National Science Council, and the National Center for Theoretical Sciences, Taiwan, R.O.C.

References

[1] D. Zwanziger, “Local Lagrangian quantum field theory of electric and magnetic charges,” Phys. Rev. D 3, 880 (1971). S. Deser and C. Teitelboim, “Duality Transformations Of Abelian And Nonabelian Gauge Fields,” Phys. Rev. D 13, 1592 (1976). M. Henneaux and C. Teitelboim, “Dynamics of chiral (self-dual) p Forms,” Phys. Lett. B 206, 650 (1988). A. A. Tseytlin, “Duality Symmetric Formulation of String World Sheet Dynamics,” Phys. Lett. B 242, 163 (1990). A. A. Tseytlin, “Duality Symmetric Closed String Theory And Interacting Chiral Scalars,” Nucl. Phys. B 350, 395 (1991). J. H. Schwarz and A. Sen, “Duality symmetric actions,” Nucl. Phys. B 411, 35 (1994) [arXiv:hep-th/9304154].

[2] R. Floreanini and R. Jackiw, “Selfdual Fields As Charge Density Solitons,” Phys. Rev. Lett. 59, 1873 (1987).

[3] P. M. Ho and Y. Matsuo, “M5 from M2,” JHEP 0806, 105 (2008) [arXiv:0804.3629 [hep-th]].

[4] P. M. Ho, Y. Imamura, Y. Matsuo and S. Shiba, “M5-brane in three-form flux and multiple M2-branes,” JHEP 0808, 014 (2008) [arXiv:0805.2898 [hep-th]].

[5] P. M. Ho, “A Concise Review on M5-brane in Large C-Field Background,” arXiv:0912.0445 [hep-th].
[6] J. Bagger and N. Lambert, “Modeling multiple M2’s,” Phys. Rev. D 75, 045020 (2007) [arXiv:hep-th/0611108]; J. Bagger and N. Lambert, “Gauge Symmetry and Supersymmetry of Multiple M2-Branes,” Phys. Rev. D 77, 065008 (2008) [arXiv:0711.0955 [hep-th]]; J. Bagger and N. Lambert, “Comments On Multiple M2-branes,” JHEP 0802, 105 (2008) [arXiv:0712.3738 [hep-th]].

[7] A. Gustavsson, “Algebraic structures on parallel M2-branes,” arXiv:0709.1260 [hep-th]; A. Gustavsson, “Selfdual strings and loop space Nahm equations,” arXiv:0802.3456 [hep-th].

[8] P. S. Howe and E. Sezgin, “D = 11, p = 5,” Phys. Lett. B 394, 62 (1997) [arXiv:hep-th/9611008]. P. Pasti, D. P. Sorokin and M. Tonin, “Covariant action for a D = 11 five-brane with the chiral field,” Phys. Lett. B 398, 41 (1997) [arXiv:hep-th/9701037]. I. A. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. P. Sorokin and M. Tonin, “Covariant action for the super-five-brane of M-theory,” Phys. Rev. Lett. 78, 4332 (1997) [arXiv:hep-th/9701149]. M. Aganagic, J. Park, C. Popescu and J. H. Schwarz, “World-volume action of the M-theory five-brane,” Nucl. Phys. B 496, 191 (1997) [arXiv:hep-th/9701166]. P. S. Howe, E. Sezgin and P. C. West, “Covariant field equations of the M-theory five-brane,” Phys. Lett. B 399, 49 (1997) [arXiv:hep-th/9702008]. I. A. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. P. Sorokin and M. Tonin, “On the equivalence of different formulations of the M theory five-brane,” Phys. Lett. B 408, 135 (1997) [arXiv:hep-th/9703127].

[9] D. Belov and G. W. Moore, “Holographic action for the self-dual field,” arXiv:hep-th/0605038.

[10] E. Witten, “Five-brane effective action in M-theory,” J. Geom. Phys. 22, 103 (1997) [arXiv:hep-th/9610234].

[11] G. Dall’Agata, K. Lechner and D. P. Sorokin, “Covariant actions for the bosonic sector of D = 10 IIB supergravity,” Class. Quant. Grav. 14, L195 (1997) [arXiv:hep-th/9707044]. G. Dall’Agata, K. Lechner and M. Tonin, “D = 10, N = IIB supergravity: Lorentz-invariant actions and duality,” JHEP 9807, 017 (1998) [arXiv:hep-th/9806140]. G. Dall’Agata, K. Lechner and M. Tonin, “Action for IIB supergravity in 10 dimensions,” arXiv:hep-th/9812170.
[12] P. Pasti, I. Samsonov, D. Sorokin and M. Tonin, “BLG-motivated Lagrangian formulation for the chiral two-form gauge field in D=6 and M5-branes,” Phys. Rev. D 80, 086008 (2009) [arXiv:0907.4596 [hep-th]].

[13] K. Furuuchi, “Non-Linearly Extended Self-Dual Relations From The Nambu-Bracket Description Of M5-Brane In A Constant C-Field Background,” arXiv:1001.2300 [hep-th].