TRANSFORMATIONS GENERALIZING THE LEVI-CIVITA, KUSTAANHEIMO-STIEFEL, AND FOCK TRANSFORMATIONS

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ABSTRACT

Preliminary results concerning non-quadratic (and non-bijective) transformations that exhibit a degree of parentage with the well known Levi-Civita, Kustaanheimo-Stiefel, and Fock transformations are reported in this article. Some of the new transformations are applied to non-relativistic quantum dynamical systems in two dimensions.

1. Introduction and Preliminaries

Non-bijective (canonical) transformations have received a great deal of attention in the recent years. In particular, quadratic transformations generalizing the so-called Levi-Civita\textsuperscript{1} and Kustaanheimo-Stiefel\textsuperscript{2} transformations have been studied from algebraic, geometrical and Lie-like viewpoints.\textsuperscript{3–12}

It is the aim of the present paper to extend the study in Ref. 9 to non-quadratic transformations. We shall see that the Fock\textsuperscript{13} (stereographic) transformation belongs to the set of the transformations introduced in this work.

Following the work of Lambert and Kibler\textsuperscript{9} on quadratic transformations, we shall use Cayley-Dickson algebras as a general framework for defining non-quadratic transformations. Indeed, we shall restrict ourselves to \(2m\)-dimensional Cayley-Dickson algebras \(A(c) \equiv A(c_1, c_2, \ldots, c_p), c_i = \pm 1\) with \(i = 1, 2, \ldots, p\), for which \(2^p (= 2m) \leq 8\). The cases \(2m = 2, 4, 8\) correspond to \(A(c_1) = C\) or \(\Omega\), \(A(c_1, c_2) = H\) or \(N_1\), and \(A(c_1, c_2, c_3) = O\) or \(O'\), i.e., to the algebras of complex numbers or hyperbolic complex numbers, quaternions or hyperbolic quaternions, and octonions or hyperbolic octonions, respectively.
A basic ingredient for generating non-bijective transformations depends on the fact that the product $x = uv$ of two hypercomplex numbers $u$ and $v$ in $A(c)$ can be written in a matrix form as $x = A(u)v$, where $A(u)$ is a $2m \times 2m$ matrix generalizing the Hurwitz matrix.\(^9\) An important property of $A(u)$ for what follows is $\tilde{A}(u)^N A(u)^N = (\tilde{u} \eta u)^N$ for $N \in \mathbb{Z}$, where the metric $\eta$ reads $\eta = \text{diag}(1, -c_1, c_1c_2, -c_3, c_1c_3, c_2c_3, -c_1c_2c_3)$ in the case $2m = 8$. (The cases $2m = 4$ and $2$ may be deduced from the case $2m = 8$ by simple dimensional reduction.)

Another ingredient is provided by the anti-involutions described at length in Ref. 9. Let us recall that, in a $2m$-dimensional Cayley-Dickson algebra $A(c)$, it is possible to construct $2m - \delta(m,1)$ anti-involutions including the complex conjugation. We shall use $j$ to denote such anti-involutions. In matrix form, the product $x = uj(v)$ of the two hypercomplex numbers $u$ and $j(v)$ in $A(c)$ is given by $x = A(u)\epsilon v$, where $\epsilon$ is a $2m \times 2m$ diagonal matrix associated with the anti-involution $j$ of $A(c)$.

We are now in a position to define transformations that we shall classify according to the nomenclature $A_N$, $B_N$, and $C_N$. These will be defined in sections 2, 3, and 4, respectively. Section 5 will deal with applications of some of the new transformations to various $R^2$ potentials. Work on transformations in higher dimensions and on applications to other potentials (e.g., the Hénon-Heiles potential) is in progress.

### 2. Transformations of Type $A_N$

The map $A(c) \rightarrow A(c) : u \mapsto x = u^{N+1}$ with $N \in \mathbb{Z}$ gives rise to a $R^{2m} \rightarrow R^{2m}$ transformation defined through

$$x = A(u)^N u$$

(1)

This is a transformation of magnitude $N + 1$ since $r^2 = \rho^{2(N+1)}$, where $r^2 = \tilde{x} \eta x$ and $\rho^2 = \tilde{u} \eta u$.

We shall refer to the transformations (1) as transformations of type $A_N$. The transformations of type $A_1$ (or quasiHurwitz transformations) have been investigated in Ref. 9. They correspond to fibrations on spheres ($S^{2m-1} \rightarrow S^{2m-1}/Z_2$) or to fibrations on hyperboloids ($H^{2m-1}(m, m) \rightarrow H^{2m-1}(m, m)/Z_2$). As a typical example of a transformation of type $A_1$, for $2m = 2$, $c_1 = -1$, and $N = 1$, we obtain the Levi-Civita ($R^2 \rightarrow R^2$) transformation associated with the fibration $S^1 \rightarrow S^1/Z_2 = RP^1$.

Still for $N = 1$, the line element $dS^2 = d\tilde{x} \eta d\tilde{x}$ has been given in Ref. 9 for an arbitrary transformation of type $A_1$. As a consequence, the Laplace-Beltrami
operator may be obtained in a straightforward way for such a transformation. By way of illustration, for the transformation introduced in Ref. 5 and corresponding to \(2m = 4\) and \(c_1 = c_2 = -1\), we get

\[
\Delta_x = (1/4\rho^2)\left\{ \Delta_u \right\} \\
+ (1/u_1^2)(u_2^2 + u_3^2)\partial_z u_4 + (u_3^2 + u_4^2)\partial_z u_2 + (u_2^2 + u_4^2)\partial_z u_3 \\
- (2/u_1^2)(u_2 u_3 \partial_z u_3 + u_3 u_4 \partial_z u_4 + u_4 u_2 \partial_z u_2) \\
+ (2/u_1^2)(u_1 \partial_z u_1 - u_2 \partial_z u_2 - u_3 \partial_z u_3 - u_4 \partial_z u_4) \\
\Delta_x = \sum_{\alpha=1}^4 \partial_x \omega_{\alpha}, \quad \Delta_u = \sum_{\alpha=1}^4 \partial_u \omega_{\alpha}, \quad \rho^2 = \sum_{\alpha=1}^4 u_\alpha^2 
\]

General properties of the transformations of type \(A_N\) with \(N \in Z - \{-1\}\) may be easily derived in the case \(2m = 2\). We have the 5 following properties

\[
x_1^2 - c_1 x_2^2 = (u_1^2 - c_1 u_2^2)^{N+1} \\
dx = (N+1)A(u)^N du \\
\nabla \cdot dx = (N+1)^{-1}(u_1^2 - c_1 u_2^2)^{-N} \eta A(u)^N \eta \nabla u \\
dx_1^2 - c_1 dx_2^2 = (N+1)^2(u_1^2 - c_1 u_2^2)^N(du_1^2 - c_1 du_2^2) \\
\partial_{x_1 x_1} - c_1 \partial_{x_2 x_2} = (N+1)^{-2}(u_1^2 - c_1 u_2^2)^{-N}(\partial_{u_1 u_1} - c_1 \partial_{u_2 u_2}) 
\]

The situation where \(N = -1\) deserves a special treatment. Indeed, the transformations of type \(A_{-1}\) are not interesting since they correspond to \(A(c) \rightarrow A(c) : u \mapsto x = 1\). In the particular case \(2m = 2\), we define transformations of the type \(A'_{-1}\) in the following way. We start with the column vector \(\omega = A(u)^{-1} du\). The elements of \(\omega\) are one-forms which turn out to be complete differentials. A direct integration leads to the \(R^2 \rightarrow R^2\) transformations \((u_1, u_2) \mapsto (x_1, x_2)\) with

\[
x_1 = (1/2) \ln(u_1^2 - c_1 u_2^2) \quad \text{for} \quad u_1^2 - c_1 u_2^2 > 0 \\
tan x_2 = u_2/u_1 \quad \text{for} \quad c_1 = -1 \quad \text{or} \quad \tanh x_2 = u_2/u_1 \quad \text{for} \quad c_1 = +1
\]

The latter transformations, referred to as transformations of type \(A'_{-1}\), correspond to the map \(A(c_1) \rightarrow A(c_1) : u \mapsto x = \log u\). These transformations satisfy the properties

\[
\nabla \cdot dx = \tilde{A}(u) \nabla u
\]
\[
\begin{align*}
    &dx_1^2 - c_1 dx_2^2 = (u_1^2 - c_1 u_2^2)^{-1}(du_1^2 - c_1 du_2^2) \\
    &\partial x_{11} - c_1 \partial x_{22} = (u_1^2 - c_1 u_2^2)(\partial u_1 u_1 - c_1 \partial u_2 u_2)
\end{align*}
\]

(5)

3. Transformations of Type \(B_N\)

We now write Eq. (1) in a slightly modified form: the relation

\[
x = A(u)^N u, \quad N \in \mathbb{Z}
\]

(6)

defines a \(R^{2m} \to R^{2m-n}\) transformation \((n \geq 0)\) that we shall call a transformation of type \(B_N\). This transformation (of magnitude \(N+1\)) corresponds to the map \(A(c) \to A(c) : u \mapsto x = u^N j(u)\).

As a first typical example, we take \(2m = 4, c_1 = c_2 = -1, N = 1, \) and \(\epsilon = \text{diag}(1,1,1,-1)\). Then, we obtain a \(R^4 \to R^3\) surjection that is nothing but the Kustaanheimo-Stiefel transformation (associated with the Hopf fibration \(S^3 \to S^2\) of compact fiber \(S^1\)). More generally, the \(R^{2m} \to R^{2m-n}\) transformations of type \(B_1\) (or Hurwitz transformations) correspond to \(n = 2m-1\) or \(n = m-1 + \delta(m,1)\). They have been studied by Lambert and Kibler.\(^9\) For \(2m\) fixed and \(n = m-1 + \delta(m,1)\), there are 3 classes of transformations of type \(B_1\) associated with (i) Hopf fibrations on spheres \((S^{2m-1} \to S^{m-\delta(m,1)}\) with compact fiber \(S^{m-1+\delta(m,1)}\)), (ii) fibrations on hyperboloids with compact fibers, and (iii) fibrations on hyperboloids with non-compact fibers.

As a second typical example, let us set \(2m = 4, N = -1, \) and \(\epsilon = \text{diag}(1,1,1,-1)\). Thus, Eq. (6) yields the \(R^4 \to R^4\) transformation

\[
\begin{align*}
    x_1 &= \frac{1}{\rho^2}(u_1^2 - c_1 u_2^2 - c_2 u_3^2 - c_1 c_2 u_4^2) \\
    x_2 &= -\frac{2}{\rho^2} c_2 u_3 u_4, \quad x_3 = \frac{2}{\rho^2} c_1 u_2 u_4, \quad x_4 = -\frac{2}{\rho^2} u_1 u_4
\end{align*}
\]

(7)

where \(\rho^2 = u_1^2 - c_1 u_2^2 - c_2 u_3^2 + c_1 c_2 u_4^2\). For \(c_1 = c_2 = -1\), the latter transformation particularizes to the \(R^4 \to S^3\) stereographic projection known as the Fock\(^13\) projection. Similar results may be obtained in the cases \(2m = 2\) and \(8\). In particular, for \(2m = 8, c_1 = c_2 = c_3 = -1, N = -1, \) and \(\epsilon = \epsilon_k (k = 1, 2, \cdots, 7), \) see Ref. 9, we get a \(R^8 \to S^7\) projection.

4. Transformations of Type \(C_N\)

A last class of transformations is obtained when the matrix \(\epsilon\) in Eq. (6) is replaced by an (arbitrary) matrix which is neither the identity matrix (yielding
transformations of type $A_N$) nor a matrix associated with an anti-involution $j$ of $A(c)$ (yielding transformations of type $B_N$). Some transformations of type $C_1$ (or pseudoHurwitz transformations) have been described elsewhere.\textsuperscript{9,10} In the case $2m = 8$, the transformations of type $C_1$ corresponding to diagonal matrices $\epsilon$ having $\pm 1$ for matrix elements have been classified in Ref. 9.

5. Applications

First, let us consider the two-dimensional Schrödinger equation

$$(-\frac{1}{2}\Delta_x - \frac{Z}{r^\alpha})\psi = E\psi, \quad \alpha \in R$$

(8)

where $r = (x_1^2 + x_2^2)^{1/2}$. The application to Eq. (8) of a ($R^2 \to R^2$) transformation of type $A_N$ with $2m = 2$, $c_1 = -1$, and $N \in Z - \{-1\}$ leads to the partial differential equation

$$[-\frac{1}{2}\Delta_u - (N + 1)^2 E\rho^{2N}]\hat{\psi} = (N + 1)^2 Z\rho^{2N - \alpha(N+1)}\hat{\psi}$$

(9)

where $\rho = (u_1^2 + u_2^2)^{1/2}$ and $\hat{\psi} \equiv \hat{\psi}(u)$ is the transform of $\psi \equiv \psi(x)$ under the considered transformation. Furthermore, let us impose that $2N - \alpha(N + 1) = 0$. Then, the transformation of type $A_N$ allows to transform the $R^2$ Schrödinger equation for the potential $-Z(x_1^2 + x_2^2)^{-N/(N+1)}$ and the energy $E$ into the $R^2$ Schrödinger equation for the potential $-(N + 1)^2 E(u_1^2 + u_2^2)^N$ and the energy $(N + 1)^2 Z$. (Note that in such a transformation the roles of the energy $E$ and the coupling constant $Z$ are interchanged.) The solutions for $\alpha (= 2N/(N + 1)) \in Z$ and $N \in Z - \{-1\}$ correspond to $(\alpha, N) = (1, 1), (3, -3), \text{ and } (4, -2)$. (The solution $(0, 0)$ is trivial!) In other words, the Schrödinger equations for the potentials $1/r$ (Coulomb), $1/r^3$, and $1/r^4$ are transformed into Schrödinger equations for the potentials $\rho^2$ (harmonic oscillator), $1/\rho^6$, and $1/\rho^4$, respectively.

Second, we consider the Schrödinger equation

$$[-\frac{1}{2}\Delta_u - \frac{Z}{(u_1^2 + u_2^2)^{1/2}}]\hat{\psi} = E\hat{\psi}$$

(10)

for a two-dimensional hydrogen atom. By using the transformation of type $A'_{-1}$ with $c_1 = -1$, Eq. (10) may be converted into

$$(-\frac{1}{2}\Delta_x - Ze^{x_1} - Ee^{2x_1})\psi = 0$$

(11)
Since $x_2$ is a cyclical coordinate, we can set $-(1/2)\partial_{x_2 x_2} \psi = -K \psi$ so that we arrive at

$$(-\frac{1}{2} \partial_{x_1 x_1} - Z e^{x_1} - E e^{2x_1}) \psi = K \psi$$

(12)

which may be recognized as the Schrödinger equation for an one-dimensional Morse potential (provided $Z > 0$ and $E < 0$). (The usual Morse\textsuperscript{14} potential is reverted in the $x_1$ variable.)

Third, we close with the $R^2$ Schrödinger equation

$$[- \frac{1}{2} \Delta_u + V_0 \ln(u_1^2 + u_2^2)] \hat{\psi} = E \hat{\psi}$$

(13)

The transformation of type $A'_{-1}$ with $c_1 = -1$ makes it possible to change Eq. (13) into

$$(-\frac{1}{2} \Delta_x - E e^{2x_1} + 2V_0 x_1 e^{2x_1}) \psi = 0$$

(14)

and the separation of variables $-(1/2)\partial_{x_2 x_2} \psi = -K \psi$ leads to

$$[-\frac{1}{2} \partial_{x_1 x_1} - (E - 2V_0 x_1) e^{2x_1}] \psi = K \psi$$

(15)

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