Relative Turán Problems for Uniform Hypergraphs

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Abstract

For two graphs $F$ and $H$, the relative Turán number $\text{ex}(H, F)$ is the maximum number of edges in an $F$-free subgraph of $H$. Foucaud, Krivelevich, and Perarnau [12] and Perarnau and Reed [24] studied these quantities as a function of the maximum degree of $H$.

In this paper, we study a generalization for uniform hypergraphs. If $F$ is a complete $r$-partite $r$-uniform hypergraph with parts of sizes $s_1, s_2, \ldots, s_r$ with each $s_{i+1}$ sufficiently large relative to $s_i$, then with $1/\beta = \sum_{i=2}^{r} \prod_{j=1}^{i-1} s_j$ we prove that for any $r$-uniform hypergraph $H$ with maximum degree $\Delta$,

$$\text{ex}(H, F) \geq \Delta^{1-\beta-o(1)} \cdot e(H).$$

This is tight as $\Delta \to \infty$ up to the $o(1)$ term in the exponent, since we show there exists a $\Delta$-regular $r$-graph $H$ such that $\text{ex}(H, F) = O(\Delta^{-\beta}) \cdot e(H)$. Similar tight results are obtained when $H$ is the random $n$-vertex $r$-graph $H_{n,p}$ with edge-probability $p$, extending results of Balogh and Samotij [3] and Morris and Saxton [21]. General lower bounds for a wider class of $F$ are also obtained.

1 Introduction

The Turán number $\text{ex}(n, F)$ of a graph $F$ is the maximum number of edges in an $F$-free $n$-vertex graph. The Turán numbers are a central object of study in extremal graph theory, dating back to Mantel’s Theorem [20] and Turán’s Theorem [27]. Given a host graph $H$, we define the relative Turán number $\text{ex}(H, F)$ to be the maximum number of edges in an $F$-free subgraph of $H$, and this is precisely $\text{ex}(n, F)$ when $H = K_n$. The study of $\text{ex}(H, F)$ for various graphs $F$ and $H$ has attracted considerable attention in the literature. One observes that if $F$ has chromatic number $k \geq 3$, then by taking a maximum $(k-1)$-partite subgraph we find for all $H$

$$\text{ex}(H, F) \geq \left( 1 - \frac{1}{k-1} \right) \cdot e(H),$$

which is best possible by the Erdős-Stone Theorem, which shows $\text{ex}(K_n, F) \sim (1 - \frac{1}{k})e(K_n)$.

The case $F$ is bipartite was studied at length by Foucaud, Krivelevich, and Perarnau [12], who conjectured that if $F$ and $H$ are graphs such that $H$ has minimum degree $\delta$ and maximum degree...
Δ, then H has a spanning F-free subgraph of minimum degree \(\Omega(\delta\text{ex}(\Delta, F)/\Delta^2)\) as \(\Delta \to \infty\), and more generally that this holds for any family of graphs \(\mathcal{F}\). This conjecture is true if F has chromatic number \(k \geq 3\), since a maximum \((k - 1)\)-partite subgraph of a graph of minimum degree \(\delta\) can be chosen to have minimum degree at least \((1 - \frac{1}{k-1})\delta\). The conjecture was proved up to a logarithmic factor for \(\mathcal{F} = \{C_3, C_4, \ldots, C_{2r}\}\) by Foucaud, Krivelevich, and Perarnau [12], and later Perarnau and Reed [24] proved the conjecture for this \(\mathcal{F}\) along with other cases such as all bipartite graphs \(F\) of diameter at most three. The following conjecture appears to be at the heart of the above conjecture and of the same level of difficulty:

**Conjecture 1.1.** If \(F\) and \(H\) are graphs such that \(H\) has maximum degree \(\Delta\), then as \(\Delta \to \infty\),

\[
\text{ex}(H, F) = \Omega\left(\frac{\text{ex}(\Delta, F)}{\Delta^2}\right) \cdot e(H).
\]

It is generally an open problem to find such an \(F\)-free subgraph of \(H\) when \(F\) is bipartite, contains a cycle, and has diameter larger than three. We note that one reason may suspect that Conjecture 1.1 holds is that the clique \(K_{\Delta+1}\) is the densest graph of maximum degree \(\Delta\). This means that it should be relatively hard to delete copies of \(F\) from \(K_{\Delta+1}\), which suggests that \(\text{ex}(K_{\Delta+1}, F) = \text{ex}(\Delta+1, F)\) should be relatively small compared to any other graph \(H\) with maximum degree \(\Delta\).

In this paper we generalize the relative Turán number to \(r\)-uniform hypergraphs, which we call \(r\)-graphs for short. For two \(r\)-graphs \(H, F\) we define the relative Turán number \(\text{ex}(H, F)\) to be the maximum number of edges in an \(F\)-free subgraph of \(H\). It follows from results of Katona, Nemetz, and Simonovits [14] that if \(F\) is not \(r\)-partite, then for any \(r\)-graph \(H\) we have \(\text{ex}(H, F) \geq (c(F) - o(1)) \cdot e(H)\), where \(c(F) = \lim_{\Delta \to \infty} \text{ex}(n, F)/(\binom{n}{\ell})\) is the Turán density of \(F\). In particular, equality holds when \(H = K_r^n\), the complete \(r\)-graph on \(n\) vertices. The problem of determining \(c(F)\) when \(F\) is not \(r\)-partite is a famous open problem in extremal hypergraph theory, and the notorious conjecture \(c(K_3^3) = 5/9\) is known as Turán’s conjecture – see Keevash [15] for a survey of hypergraph Turán problems.

### 1.1 Complete \(r\)-partite \(r\)-graphs

For positive integers \(2 \leq s_1 \leq \cdots \leq s_r\), define \(K_{s_1,\ldots,s_r}\) to be the complete \(r\)-partite \(r\)-graph, which has vertex set \(U_1 \cup \cdots \cup U_r\) with \(|U_i| = s_i\) for all \(i\), and which has all edges of the form \(\{u_1, \ldots, u_r\}\) with \(u_i \in U_i\) for all \(i\). We prove the following almost tight theorem on relative Turán numbers for complete \(r\)-partite \(r\)-graphs:

**Theorem 1.2.** For \(r \geq 2\), let \(2 \leq s_1 \leq \cdots \leq s_r\) be integers and \(a_i = \prod_{j<i}s_j\) for \(1 \leq i \leq r\).

1. For any (sufficiently large) \(\Delta\), there exists an \(r\)-graph \(H\) which is \(\Delta\)-regular such that as \(\Delta \to \infty\),

\[
\text{ex}(H, K_{s_1,\ldots,s_r}) = O\left(\Delta^{\frac{1}{\sum_{i=2}^{r}s_i}}\right) \cdot e(H).
\]

2. There exist functions \(f_i : \mathbb{Z}^+ \to \mathbb{Z}^+\) for \(1 < i \leq r\) such that if \(s_i \geq f_i(s_{i-1})\) for \(1 < i \leq r\), then for any \(r\)-graph \(H\) with maximum degree \(\Delta\), as \(\Delta \to \infty\),

\[
\text{ex}(H, K_{s_1,\ldots,s_r}) \geq \Delta^{\frac{1}{\sum_{i=2}^{r}s_i}} - o(1) \cdot e(H).
\]
We note that the second part of Theorem 1.2 will more or less follow from the more general results Theorems 1.4 and 1.5 stated below. We also note that this theorem extends results of Perarnau and Reed [24] for relative Turán numbers of complete bipartite graphs $K_{s_1,s_2}$. In this setting, they proved the second half of Theorem 1.2 without a $o(1)$ term, and it would be of interest to determine whether this error term can be removed for $r \geq 3$ as well.

According to Theorem 1.2, if $s_1, s_2, \ldots, s_r$ grow sufficiently fast, then

$$
\beta(K_{s_1,\ldots,s_r}) := \limsup_{\Delta \to \infty} \frac{\log e(H)/\text{ex}(H, K_{s_1,\ldots,s_r})}{\log \Delta} = \frac{1}{\sum_{i=2}^r s_1 s_2 \cdots s_{i-1}},
$$

(1)

where the supremum ranges over all $H$ with maximum degree at most $\Delta$. The functions $f_i$ for $1 \leq i < r$ in Theorem 1.2 are based on the current state of knowledge of the hypergraph Turán numbers\footnote{By using recent results of Pohoata and Zakharov [25], it can be shown that we can take the functions to be $f_i(t) = \frac{(r-1)!}{(r-1)^{i-1}} - 1 + o(1)$.} $\text{ex}(n, K_{s_1,\ldots,s_r})$, and this condition is unnecessary if the following conjecture attributed to Erdős [9] is true: if $s_1 \leq s_2 \leq \cdots \leq s_r$ then

$$
\text{ex}(n, K_{s_1,\ldots,s_r}) = \Theta\left(n^{r-s_1-s_2-\cdots-s_{r-1}}\right).
$$

(2)

This conjecture was first stated explicitly by Mubayi [22], and at $r = 2$ constitutes the notorious Zarankiewicz problem [30]. When $r = 2$ and $s_2 > (s_1 - 1)!$, the conjecture was solved by Alon, Kollár, Rönyai and Szabó [1, 18]. By adapting the random polynomial constructions introduced by Bukh and Conlon [5], this conjecture was proved by Ma, Yuan, and Zhang [19] when $s_r$ is large enough relative to $s_{r-1}$. Thus in this setting where (2) holds,

$$
\alpha(K_{s_1,\ldots,s_r}) := \lim_{n \to \infty} \frac{\log \binom{n}{r}/\text{ex}(n, K_{s_1,\ldots,s_r})}{\log \binom{n-1}{r-1}} = \frac{1}{(r-1)s_1 s_2 \cdots s_{r-1}}.
$$

(3)

This is strictly less than $\beta(K_{s_1,\ldots,s_r})$ when $r \geq 3$ and $\alpha(K_{s_1,\ldots,s_r}) = \beta(K_{s_1,\ldots,s_r})$ when $r = 2$, as was originally proven by Perarnau and Reed [24].

This implies that the natural generalization of Conjecture 1.1 for $r$ graphs is false for $r \geq 3$, i.e. there exist $r$-graphs $F$ such that the clique with maximum degree $\Delta$ is not an asymptotic minimizer of $\text{ex}(H, F)$ as $H$ ranges over $r$-graphs with maximum degree $\Delta$. We note that the $H$ we use to prove the second half of Theorem 1.2 is a certain unbalanced complete $r$-partite $r$-graph, and a similar construction was used by Foucaud, Krivelevich and Perarnau [12] for a related problem where $H$ had a fixed number of edges.

### 1.2 Random Hypergraphs

We recall that $H_{n,p}^r$ is the $r$-graph on $n$ vertices where each edge of $K_n^r$ is added to $H_{n,p}^r$ independently and with probability $p - \sigma$ so $H_{n,p}^r = G_{n,p}$. If $(A_n)_{n \geq 1}$ is a sequence of events in a probability space, then we say $A_n$ holds asymptotically almost surely (abbreviated a.a.s.) if $\lim_{n \to \infty} P(A_n) = 1$. A central conjecture of Kohayakawa, Łuczak and Rödl [17] was resolved independently by Conlon and Gowers [7] and by Schacht [26], which determines $\text{ex}(G_{n,p}, F) = (1 - \frac{1}{\log n} + o(1))p^{1/2}$ a.a.s. whenever $F$ has chromatic number $k \geq 3$ and $p = \Omega(n^{-1/2m_2(F)})$, where $m_2(F)$ is the so-called 2-density of $F$. The case $F = C_4$ was essentially resolved by Füredi [13]. This work was generalized to even cycles by Kohayakawa, Kreuter and Steger [16] and to complete bipartite graphs by Balogh...
Given an induced by the parts excluding is tightly connected. We prove the following theorem for
More precisely, for \( n^{-\beta_1} \leq p \leq n^{-\beta_2}(\log n)^{2\alpha r}/(\alpha r - 1) \), we have a.a.s.
\[
\Omega(n^{r-\beta_1}) = \Theta(H^r_{n,p}, K_{s_1,\ldots,s_r}) = O(n^{r-\beta_1}(\log n)^2).
\]
We note that for \( p < n^{-r/2}\log n \) it is easy to show that \( \mathbb{E} [\text{ex}(H^r_{n,p}, K_{s_1,\ldots,s_r})] = \Theta(pn^r) \), but a slightly different argument than the one we present is needed to show that the result holds a.a.s.

### 1.3 General Results

Given a family of \( r \)-graphs \( F \), we define \( \text{ex}(n, F) \) and \( \text{ex}(H, F) \) to be the maximum number of edges in an \( F \)-free subgraph of \( K^r_n \) and \( H \), respectively. The proof techniques used for Theorem 1.2 generalize to other families of \( r \)-graphs, and at its core it relies on two general results which prove effective lower bounds on \( \text{ex}(H, F) \) depending on if \( H \) has small or large codegrees.

We say that an \( r \)-partite \( r \)-graph \( F \) on \( U_1 \cup \cdots \cup U_r \) is **tightly connected** if for all \( i \) and distinct \( u_1, u_2 \in U_i \) there exist edges \( e_1, e_2 \in E(F) \) with \( u_1 \in e_1 \) and \( e_1 \cap e_2 = r - 1 \). For example, \( K_{s_1,\ldots,s_r} \) is tightly connected. We prove the following theorem for \( r \)-graphs \( H \) with low codegrees. Here the \((r-1)\)-degree of a set \( S \subseteq V(H) \) is defined to be the number of edges in \( H \) containing \( S \).

**Theorem 1.4.** For \( r \geq 2 \), let \( F \) be a family of tightly connected \( r \)-graphs with \( \text{ex}(n, F) = \Omega(n^{r-\gamma}) \) for some \( \gamma > 0 \). If \( H \) is an \( r \)-graph with maximum \((r-1)\)-degree at most \( D \geq 1 \), then
\[
\text{ex}(H, F) = \Omega(D^{-\gamma}) \cdot e(H).
\]

Given an \( r \)-partite \( r \)-graph \( F \) and \( r \)-partition \( U_1 \cup \cdots \cup U_r \), define \( \partial F[\bigcup_{j \neq i} U_j] \) to be the \((r-1)\)-graph on \( \bigcup_{j \neq i} U_j \) with all edges of the form \( e \cap \bigcup_{j \neq i} U_j \) for all \( e \in E(F) \). That is, it is the \((r-1)\)-graph induced by the parts excluding \( U_i \). Define the **projection family**
\[
\pi(F) = \{ \partial F[\bigcup_{j \neq i} U_j] : i \in [r], \ U_1, \ldots, U_r \text{ is an } r \text{-partition of } F \}.
\]
For example, \( \pi(K_{s_1,s_2,s_3}) = \{ K_{s_1,s_2}, K_{s_1,s_3}, K_{s_2,s_3} \} \). For a family of \( r \)-graphs \( F \) we define \( \pi(F) = \bigcup_{F \in \mathcal{F}} \pi(F) \). The following theorem is effective for \( r \)-graphs \( H \) with high codegrees:
**Theorem 1.5.** For \( r \geq 3 \), let \( \mathcal{F} \) be a family of \( r \)-partite \( r \)-graphs and \( \gamma > 0 \) such that for all \( \bar{\Delta} \geq 3 \) and all \( (r - 1) \)-graphs \( G \) with maximum degree at most \( \bar{\Delta} \),
\[
\text{ex}(G, \pi(\mathcal{F})) = \Omega((\bar{\Delta}^{-\gamma}(\log \bar{\Delta})^{3-r}) \cdot e(G)).
\]
If \( H \) is an \( r \)-partite \( r \)-graph with maximum degree at most \( \Delta \geq 2 \) such that at least half of the edges of \( H \) contain an \( (r - 1) \)-set with \( (r - 1) \)-degree at least \( D \), then
\[
\text{ex}(H, \mathcal{F}) = \Omega((\Delta^{-\gamma}D^{\gamma}(\log \Delta)^{2-r}) \cdot e(H)).
\]

The logarithmic terms of Theorem 1.5 are a product of its proof, and we suspect that these terms can be removed.

### 1.4 Tight Cycles

Theorems 1.4 and 1.5 allow us to prove effective bounds on \( \text{ex}(H, \mathcal{F}) \) for a wide variety of \( r \)-graphs, and for simplicity we focus on the case of tight cycles. For integers \( r < k \), the **tight \( k \)-cycle** \( TC_k^r \) is the \( r \)-graph with vertex set \( \{u_0, \ldots, u_{k-1}\} \) consisting of the edges \( \{u_i, u_{i+1}, \ldots, u_{i+r-1}\} \) for \( 0 \leq i < k \) with subscripts written modulo \( k \). For instance, \( TC_{r+1}^r = K_{r+1}^r \) and \( TC_k^r \) is \( r \)-partite if and only if \( k \) is a multiple of \( r \), in which case its unique \( r \)-partition up to relabeling of parts has \( u_j \in U_i \) whenever \( j \equiv i \mod r \). For \( r = 2 \), the tight \( k \)-cycle \( TC_k^2 \) is precisely \( C_k \), the cycle of length \( k \), and a well-known conjecture of Erdős and Simonovits [11] states that for all \( \ell \geq 2 \),
\[
\text{ex}(n, \{C_3, C_4, \ldots, C_{2\ell}\}) = \Theta(n^{1+1/\ell})
\]
as \( n \to \infty \). When \( r = 2 \) and \( \ell \in \{2, 3, 5\} \), (4) is true due to the existence of generalized polygons – see Benson [4] for the first description in terms of extremal graph theory, Wenger [29] for an elementary presentation, and [28] for a survey. We prove the following theorem for relative Turán numbers of tight cycles:

**Theorem 1.6.** Let \( r \geq 2 \) and let \( \ell \in \{2, 3, 5\} \). If \( H \) is any \( r \)-graph with maximum degree \( \Delta \), then as \( \Delta \to \infty \),
\[
\text{ex}(H, \{TC_{r+1}^r, TC_{r+2}^r, \ldots, TC_{\ell r}^r\}) \geq \Delta^{n(\ell+1)-(\ell+1)} \cdot e(H).
\]

The inequality (5) generalizes the results of Foucaud, Krivelevich and Perarnau [12] for short cycles of even length in graphs. The proof of Theorem 1.6 relies on an effective lower bound on the extremal function for tight cycles \( TC_{\ell r}^r \) when \( \ell \in \{2, 3, 5\} \). The first moment method in the random \( r \)-graph \( H_{n,p}^r \) gives \( \text{ex}(n, TC_{\ell r}^r) = \Omega(n^{r-1+(r-1)/(\ell-1)}) \) for any \( \ell \geq r + 1 \). We give a simple proof of a slight improvement as follows:

**Theorem 1.7.** For all \( r \geq 2 \) and \( \ell \in \{2, 3, 5\} \),
\[
\text{ex}(n, \{TC_{r+1}^r, TC_{r+2}^r, \ldots, TC_{\ell r}^r\}) = \Omega(n^{r-1+1/\ell}).
\]

Upper bounds for this extremal function were left as an open problem by Conlon [6] in connection with extremal problems for cycles in hypercubes, and for example the current best upper bound for \( TC_6^3 \) is \( \text{ex}(n, TC_6^3) \leq \text{ex}(n, K_{2,2,2}) = O(n^{11/4}) \).
1.5 Organization and Notation

In Section 2 we prove Theorem 1.4 using random homomorphisms. In Section 3 we prove Theorems 1.5 and 1.7. We then prove our main results for general hosts Theorems 1.2 and 1.6 in Section 4. In Section 5 we prove our main result for random hosts Theorem 1.3. Concluding remarks and open problems are given in Section 6.

We gather some notation and definitions that we use throughout the text. A set of size $k$ will be called a $k$-set. If $H$ is an $r$-graph, then the number of edges containing a $k$-set $S = \{v_1, \ldots, v_k\} \subseteq V(H)$ is called the $k$-degree of $S$ and is denoted by $d_H(S)$ or $d_H(v_1, \ldots, v_k)$, and we omit the subscript wherever $H$ is understood from context. If $\chi$ is a map from vertices of $H$ and $e = \{v_1, \ldots, v_r\} \in E(H)$, we define the set $\chi(e) = \{\chi(v_1), \ldots, \chi(v_r)\}$. We often make use of the following basic fact due to Erdős and Kleitman \cite{EK}: every $r$-graph $H$ has an $r$-partite subgraph with at least $r - r(e(H))$ edges. Throughout the text we omit ceilings and floors for ease of presentation.

2 Hosts with Low Codegrees : Proof of Theorem 1.4

We begin with an informal discussion of the technique for giving lower bounds on $\text{ex}(H, F)$ when $H$ has low codegrees, which is based on techniques of Foucaud, Krivelevich and Perarnau \cite{FKP} and Perarnau and Reed \cite{PR} for graphs. We will try to construct a subgraph $H' \subseteq H$ that “looks like” another $r$-graph $J$ which is $F$-free and has many edges. One way to try and do this is to consider a random map $\chi : V(H) \to V(J)$ and to keep the edges $e \in E(H)$ which have $\chi(e) \in E(J)$. However, one quickly sees that this $H'$ may not be $F$-free. Indeed, if $F$ is $r$-partite, then it is possible for $\chi$ to map every edge of $H$ to a single edge of $J$, giving $H' = H$.

We get around this issue by doing two additional steps. The first is to put constraints on $H'$ to disallow edges $e$ which have $\chi(e) = \chi(f)$ for some $f$ with $|e \cap f| = r - 1$. The second is to choose $J$ so that it avoids not only $F$, but also a family of $r$-graphs related to $F$. More precisely, for $r$-graphs $F, F'$ we say that a map $\phi : V(F) \to V(F')$ is a local isomorphism if

(a) $\phi$ is a homomorphism and

(b) $\phi(e) \neq \phi(f)$ for $e, f \in E(F)$ with $|e \cap f| = r - 1$.

For example, with $F = K_{s_1, \ldots, s_r}$, a local isomorphism $\phi : V(F) \to V(F')$ exists only when $F'$ contains a subgraph isomorphic to $F$, as mapping any two vertices of $F$ to the same vertex in $F'$ would cause two edges intersecting in $r - 1$ spots to map to the same edge. As another example, the figure below illustrates a local isomorphism from $C_8$ to two $C_4$’s sharing an edge (where the map sends the two star/diamond vertices on the left to the star/diamond vertex on the right),

For an $r$-graph $F$ we define the set $H(F)$ to be the set of $F'$ for which there exists a local isomorphism
In general it turns out that we want to choose a $J$ which is not only $F$-free but also $\mathcal{H}(F)$-free, and this gives our main technical result for hosts with low codegree.

**Lemma 2.1.** Let $\mathcal{F}$ be a family of $r$-graphs and $H$ an $r$-graph with maximum $(r-1)$-degree at most $D$. Then as $D \to \infty$,

$$\text{ex}(H, \mathcal{F}) = \Omega \left( \frac{\text{ex}(D, \mathcal{H}(\mathcal{F}))}{D^r} \right) \cdot e(H).$$

**Proof.** Let $J$ be an extremal $\mathcal{H}(\mathcal{F})$-free $r$-graph on $t := 2r^2D$ vertices and let $\chi : V(H) \to V(J)$ be chosen uniformly at random. Let $H' \subseteq H$ be the (random) subgraph which keeps the edge $e \in E(H)$ if

1. $\chi(e) \in E(J)$, and
2. $\chi(f) \neq \chi(e)$ for any other $f \in E(H)$ with $|e \cap f| = r - 1$.

We claim that $H'$ is $\mathcal{F}$-free. Indeed, assume $H'$ contained some subgraph $F'$ isomorphic to $F \in \mathcal{F}$. Let $F''$ be the subgraph of $J$ with $V(F'') = \{ \chi(v) : v \in V(F') \}$ and $E(F'') = \{ \chi(e) : e \in E(F') \}$, and we note that $F' \subseteq H'$ implies that each edge of $F'$ satisfies (1), so every element of $E(F'')$ is an edge in $J$. By conditions (1) and (2), $\chi$ is a surjective local isomorphism from $F'$ to $F''$, so $F'' \in \mathcal{H}(F)$, a contradiction to our condition on $J$.

It remains to compute how many edges $H'$ has in expectation. Given an edge $e \in E(H)$, let $A$ be the event that $\chi(e) \in E(J)$, let $\hat{E}$ be the set of edges $f \in E(H)$ with $|e \cap f| = r - 1$, and let $B$ be the event that $\chi(f) \subseteq \chi(e)$ for all $f \in \hat{E}$, which in particular implies (2) for the edge $e$. It is easy to see that

$$\Pr[A] = r!e(J)t^{-r}.$$ 

It is also not hard to see for any $f \in \hat{E}$ that $\Pr[\chi(f) \subseteq \chi(e) | A] = \frac{t}{r}$. Because $\overline{B}$ (the complement of $B$) is the union of the events $\chi(f) \subseteq \chi(e)$ for $f \in \hat{E}$, the union bound gives

$$\Pr[B | A] \geq 1 - |\hat{E}| \cdot r^{-1} \geq 1 - r^2Dt^{-1} = \frac{1}{2},$$

where the last inequality used that there are at most $D$ edges $f$ intersecting a given $r-1$ sized subset of $e$. Thus

$$\Pr[e \in E(H')] \geq \Pr[A] \cdot \Pr[B | A] \geq \frac{1}{2}r!e(J)t^{-r},$$

and by linearity of expectation we have $\mathbb{E}[e(H')] \geq \frac{1}{2}r!e(J)t^{-r} \cdot e(H)$. We conclude that there exists some subgraph $H'' \subseteq H$ with at least this many edges which is $\mathcal{F}$-free, and the result follows since $e(J) = \text{ex}(t, \mathcal{H}(\mathcal{F}))$ and $t = 2r^2D$.

Lastly, we observe the following.

**Lemma 2.2.** If $F$ is a tightly connected $r$-graph, then it has an $r$-partition which is unique up to relabeling the parts.
Proof. By definition $F$ has an $r$-partition $U_1 \cup \cdots \cup U_r$ such that for any distinct $u_1, u_2 \in U_i$ there exist edges $e_1, e_2$ with $u_1 \in e_1$ and $|e_1 \cap e_2| = r - 1$. Let $e = \{x_1, \ldots, x_r\} \in E(F)$ with $x_i \in U_i$ for all $i$ and let $U'_1 \cup \cdots \cup U'_r$ be any other $r$-partition of $F$ with $x_i \in U'_i$ for all $i$. Note that if $y_i \in U_i \setminus \{x_i\}$, then there is an edge $e' \ni y_i$ with $|e \cap e'| = r - 1$. Thus for $e'$ to contain exactly one vertex from each $U'_i$ set we must have $y_i \in U'_i$. This implies that $U_i \subseteq U'_i$ for all $i$, and hence $U'_i = U_i$ for all $i$. We conclude that every $r$-partition of $F$ is a relabeling of the partition $U_1 \cup \cdots \cup U_r$. \hfill \square

Proof of Theorem 1.4. Let $F$ be tightly connected with $r$-partition $U_1 \cup \cdots \cup U_r$. We claim that if $\chi : V(F) \to V(F')$ is a local isomorphism, then either $F' \cong F$ or $F'$ is not $r$-partite. If $\chi$ is injective then $F' \cong F$, so assume $\chi(u_1) = \chi(u_2)$ for some $u_1 \neq u_2$ and that $F'$ has some $r$-partition $U'_1 \cup \cdots \cup U'_r$. Observe that $\chi$ being a homomorphism implies $\chi(U'_1)^{-1} \cdots \chi(U'_r)^{-1}$ is an $r$-partition of $F$, and by Lemma 2.2 we can relabel parts so that $\chi(U'_i)^{-1} = U_i$. Assume $\chi(u_1) = \chi(u_2) \in U'_i$, meaning $u_1, u_2 \in U_i$. Then $F$ being tightly connected means there exist edges $e_1, e_2 \in E(F)$ with $|e_1 \cap e_2| = r - 1$ and $u_i \in e_i$. Thus $\chi(e_1) = \chi(e_2)$, contradicting $\chi$ being a local isomorphism.

We conclude that for tightly connected $F$ that $\mathcal{H}(F)$ consists of $F$ together with some $r$-graphs which are not $r$-partite, so if $\mathcal{F}$ is a family of tightly connected $r$-graphs then $\mathcal{H}(\mathcal{F})$ consists of $\mathcal{F}$ together with some $r$-graphs which are not $r$-partite. Thus given any extremal $F$-free $r$-graph $H$ on $n$ vertices, we can consider a maximum $r$-partite subgraph of $H$ which has at least $r^{-r}e(H)$ edges and which is $\mathcal{H}(\mathcal{F})$-free. This implies $\operatorname{ex}(n, \mathcal{H}(\mathcal{F})) \geq r^{-r} \operatorname{ex}(n, F)$ for all such $\mathcal{F}$, and the result follows from Lemma 2.1. \hfill \square

3 Hosts with High Codegrees: Proofs of Theorems 1.5 and 1.7

For the proofs of Theorems 1.5 and 1.7, the central idea will be to find an $r$-partite subgraph $H' \subseteq H$ on $V_1 \cup \cdots \cup V_k$ such that the $k$-graph induced by $V_1 \cup \cdots \cup V_k$ avoids certain $k$-graphs associated to $\mathcal{F}$. For Theorem 1.7 we use $k = 2$, and for Theorem 1.5 we use $k = r - 1$.

Proof of Theorem 1.7. Recall $TC_r^\ell$ is defined on $\{u_0, \ldots, u_{k-1}\}$ with edges $\{u_i, u_{i+1}, \ldots, u_{i+r-1}\}$. In order to later prove Theorem 1.6, it will be convenient to prove the slightly more technical result

$$\operatorname{ex}(n, \mathcal{H}(TC_r^{\ell+1}, \ldots, TC_r^\ell)) = \Omega(n^{r-1+1/\ell})$$

for $r \geq 2$, $\ell \in \{2, 3, 5\}$,

which implies Theorem 1.7 since $\mathcal{F} \subseteq \mathcal{H}(\mathcal{F})$ for all $\mathcal{F}$.

As noted in the introduction, for $\ell \in \{2, 3, 5\}$ it is known [4] that there exists a bipartite graph $G$ on $V_1 \cup V_2$ with $|V_i| = n/r$ which is $\{C_3, \ldots, C_{2\ell}\}$-free and such that $e(G) = \Omega(n^{1+1/\ell})$. Define the $r$-graph $H$ on $V_1 \cup \cdots \cup V_r$ with $|V_i| = n/r$ by including all edges $e$ with $e \cap (V_1 \cup V_2) \in E(G)$. Then $e(H) = (n/r)^{r-2} \cdot e(G) = \Omega(n^{r-1+1/\ell})$, so it suffices to prove that $H$ is $\mathcal{H}(TC_r^{\ell+1}, \ldots, TC_r^\ell)$-free.

Assume there exists $F \subseteq H$ such that there exists a local isomorphism $\chi : V(TC_r^\ell) \to V(F)$ for some $k \leq \ell r$. Since $H$ is $r$-partite, $F$ has an $r$-partition $U_1 \cup \cdots \cup U_r$ given by $U_i = V_i \cap V(F)$. Because $\chi$ is a homomorphism, $\chi^{-1}(U_1) \cup \cdots \cup \chi^{-1}(U_r)$ is an $r$-partition of $TC_r^\ell$, and in particular we must have $k = \ell' r$ for some $\ell' \leq \ell$. Further, because $TC_r^\ell'$ has a unique $r$-partition up to relabeling its parts for all $\ell'$, we can obtain without loss of generality that $W_i := \chi^{-1}(U_i)$ consists of all the $u_j$ vertices with $j \equiv i \pmod{r}$.

Observe that the graph induced by $W_1 \cup W_2$ in $TC_r^\ell'$ is a $C_{2\ell'}$ on $u_{12u_{r+1}u_{r+2}} \cdots u_{(\ell' - 1)r + 2}$. Further note that the restricted map $\chi : W_1 \cup W_2 \to U_1 \cup U_2$ is a local isomorphism of $C_{2\ell'}$, as
any two edges of this $C_{2\ell'}$ which intersect in 1 vertex are contained in two edges of $TC'_{r\ell}$ which intersect in $r - 1$ vertices. Thus the graph induced by $U_1 \cup U_2 = V(F) \cap (V_1 \cup V_2)$ in $F$ is a local isomorphism of $C_{2\ell'}$, and in particular this graph must contain some cycle of length $C_{k'}$ with $k' \leq 2\ell'$ as a subgraph. But the graph induced by $V(F) \cap (V_1 \cup V_2)$ is a subgraph of $G$ (since $G$ is the graph induced by $V_1 \cup V_2$), which is $C_{k'}$-free by construction, a contradiction. We conclude that $H$ contains no element of $\mathcal{H}(TC'_{\ell})$ for $k \leq \ell r$, proving (7).

**Proof of Theorem 1.5.** Assume $H$ has $r$-partition $V_1 \cup \cdots \cup V_r$ and let $E_i \subseteq E(H)$ denote the set of edges $e = \{v_1, \ldots, v_r\}$ with $v_j \in V_j$ for all $j$ such that $e \setminus \{v_i\}$ has $(r - 1)$-degree at least $D$. By hypothesis we have

$$\frac{1}{2}e(H) \leq \left| \bigcup E_i \right| \leq \sum |E_i|,$$

so we can assume without loss of generality that $|E_r| \geq (2r)^{-1}e(H)$. Let $G_k$ denote the $(r-1)$-graph on $V_1 \cup \cdots \cup V_{r-1}$ with

$$E(G_k) = \{v_1, \ldots, v_{r-1}\} : 2^k D \leq d(v_1, \ldots, v_{r-1}) < 2^{k+1} D \}.$$

That is, the $G_k$ roughly partition the edges of $E_r$ into subgraphs which are codegree regular. We say that the edge $e \in E_r$ corresponds to the edge $e' \in E(G_k)$ if $e' \subseteq e$.

Note that the $(r-1)$-degree of any set of $r-1$ vertices in $H$ is at most $\Delta$, so we only have to consider $G_k$ with $k = O(\log \Delta)$. As each edge in $E_r$ corresponds to an edge in exactly one $G_k$, by the pigeonhole principle there exists some $K$ in this range such that at least $\Omega((\log \Delta)^{-1})e(H)$ edges of $E_r$ correspond to edges in $G_K$. Because each edge of $G_K$ is corresponds to by at most $2^{k+1}D$ edges of $E_r$, we have $e(G_K) = \Omega((2^K D \log \Delta)^{-1})e(H)$. Because each edge of $G_K$ is corresponds to by at least $2^k D$ edges of $E_r$, the maximum degree of $G_K$ is at most $O((2^K D)^{-1}\Delta)$. Let $\bar{\Delta}$ be three times the maximum degree of $G_K$ (so that $\bar{\Delta} \geq 3$) and let $G \subseteq G_K$ be an $\pi(F)$-free subgraph of $G_K$ with as many edges as possible. By the hypothesis of the theorem, we have

$$e(G) = ex(G_K, \pi(F)) = \Omega(\bar{\Delta}^{-\gamma} \cdot (\log \bar{\Delta})^{3-r}) \cdot e(G_K) \leq \Omega(2^{3K}(\Delta/D)^{-\gamma} \cdot (\log \Delta)^{3-r} \cdot (2^K D \log \Delta)^{-1}) \cdot e(H) \leq 2^{-KD} \cdot \Omega(2^{3K}(\Delta/D)^{-\gamma} \cdot (\log \Delta)^{2-r}) \cdot e(H).$$

Define $H' \subseteq H$ to have the edges which correspond to edges of $G$. As each edge in $G \subseteq G_K$ is corresponded to by at least $2^k D$ edges, $e(H') \geq 2^K D e(G)$. This is sufficiently large to prove the result, so it will be enough to show that $H' \not\in \mathcal{F}$. Indeed, assume $H'$ contained some $F' \cong F \in \mathcal{F}$ with $r$-partition $U_1 \cup \cdots \cup U_r$ given by $U_i = V(F') \cap V_i$. In particular $G$, the $(r-1)$-graph of $H'$ induced by parts $V_1 \cup \cdots \cup V_{r-1}$, contains the $(r-1)$-graph of $F'$ induced by $U_1 \cup \cdots \cup U_{r-1}$ as a subgraph, a contradiction to $G$ being $\pi(F) \not\subseteq \pi(F)$-free.

We note that one can replace the log $\Delta$ terms in Theorem 1.5 with log log $\Delta$ using a slightly more refined argument, namely by partitioning the edge set by $2^{(1+\gamma)k} D \leq d_H(v_1, \ldots, v_{r-1}) < 2^{(1+\gamma)k+1} D$ with $\gamma$ as in the theorem statement.
4 Proofs of Theorems 1.2 and 1.6

4.1 Proof of Theorem 1.6

Let $F_{\ell,r} = \{TC_{\ell+1}^r, \ldots, TC_{\ell r}^r\}$. For $\ell \in \{2, 3, 5\}$, we prove by induction on $r$ the lower bound

$$ex(H, F_{\ell,r}) = \Omega(\Delta^{-1+\frac{1}{2}} \sum_{i=2}^r (\log \Delta)^{2-r}) e(H).$$

The case $r = 2$ was established in [12] (and it can also be proven by Lemma 2.1), so we assume the result holds for $(r-1)$-graphs. Let $H$ be an $r$-graph of maximum degree $\Delta$ and let $H' \subseteq H$ be an $r$-partite subgraph with at least $r-r^r e(H)$ edges. Let $D := \Delta^{1/(r-1)}$, and let $H'' \subseteq H'$ contain all the edges which do not contain an $(r-1)$-set with $(r-1)$-degree at least $D$. If $e(H'') \geq \frac{1}{2} e(H')$, then because $H''$ has maximum $(r-1)$-degree at most $D$, we can apply Lemma 2.1 to $H''$ to conclude by (7) that

$$ex(H, F_{\ell,r}) = \Omega(D^{-1+\frac{1}{2}} r^{-r} e(H)) = \Omega(\Delta^{-1+\frac{1}{2}} e(H)),$$

giving the desired bound.

Thus we can assume that at least half the edges of $H'$ contain an $(r-1)$-set with $(r-1)$-degree at least $D$. Let $F'_{\ell,r} \subseteq F_{\ell,r}$ by all the tight cycles of the form $TC_{\ell+1}^r$, and note that $F_{\ell,r} \setminus F'_{\ell,r}$ consists of $r$-graphs which are not $r$-partite, so $H'$ automatically avoids these. Because $TC_{\ell+1}^r$ has a unique $r$-partition up to relabeling its parts, it is straightforward to check $\pi(TC_{\ell+1}^r) = \{TC_{\ell-1}^{r-1}\}$, and thus $\pi(F'_{\ell,r}) = F'_{\ell,r-1} \subseteq F_{\ell,r-1}$. By Theorem 1.5 and the inductive hypothesis we conclude

$$ex(H', F_{\ell,r}) = ex(H', F'_{\ell,r}) = \Omega(\Delta^{-1+\frac{1}{2}} D^{-1}(\log \Delta)^{2-r}) \cdot r^{-r} e(H) = \Omega(\Delta^{-1+\frac{1}{2}} e(H)).$$

4.2 Proof of Theorem 1.2: Lower Bound

Let $2 \leq s_1 \leq \cdots \leq s_r$ be such that $ex(n, K_{s_1, \ldots, s_r}) = \Omega(n^{-1/2})$ for all $2 \leq i \leq r$, and it is known that this holds provided $s_i \geq f_i(s_{i-1})$ for some suitable function $f_i$ [19, 25]. We prove $ex(H, K_{s_1, \ldots, s_r}) = \Omega(\Delta^{-1/2} \sum_{i=2}^r (\log \Delta)^{2-r})$ by induction on $r$. The case $r = 2$ comes from Theorem 1.4 and the assumption $ex(n, K_{s_1, s_2}) = \Omega(n^{-1/2})$. We proceed to $r > 2$.

Let $H' \subseteq H$ be an $r$-partite subgraph with at least $r^r e(H)$ edges. Let $D := \Delta^{r-1}$, and let $H'' \subseteq H'$ contain all the edges which do not contain an $(r-1)$-set with $(r-1)$-degree at least $D$. If $e(H'') \geq \frac{1}{2} e(H')$, then because $H''$ has maximum $(r-1)$-degree at most $D$, we can apply Theorem 1.4 and the hypothesis $ex(n, K_{s_1, \ldots, s_r}) = \Omega(n^{-1/2})$ to $H''$ and conclude

$$ex(H, K_{s_1, \ldots, s_r}) = \Omega(D^{-1} e(H)) = \Omega\left(\Delta^{-1/2} \sum_{i=2}^r a_i\right) e(H),$$
giving the desired bound.

Thus we can assume that at least half the edges of $H'$ contain an $(r-1)$-set with $(r-1)$-degree at least $D$. Because $\pi(K_{s_1, \ldots, s_r})$ is a set of $(r-1)$-graphs which all contain $K_{s_1, \ldots, s_{r-1}}$ as a subgraph,
we can use Theorem 1.5 and the inductive hypothesis to conclude
\[
\text{ex}(H', K_{s_1, \ldots, s_r}) = \Omega \left( \frac{\Delta^{r-1}}{D} \sum_{i=2}^{r-1} s_i (\log \Delta)^{2-r} \right) e(H)
\]
\[
= \Omega \left( \frac{\Delta^{r-1}}{\sum_{i=2}^{r-1} a_i} (\log \Delta)^{2-r} \right) e(H),
\]
where the equality \((\Delta/D)^{-1/\sum_{i=2}^{r-1} a_i} = \Delta^{-1/\sum_{i=2}^{r-1} a_i}\) used
\[
\log \Delta \left( \frac{\Delta}{D} \right) = 1 - \frac{a_r}{\sum_{i=2}^{r} a_i} = \frac{\sum_{i=2}^{r-1} a_i}{\sum_{i=2}^{r} a_i}.
\]
This proves the desired lower bound. 

4.3 Supersaturation for complete r-partite r-graphs

A closer inspection of the proof of Theorem 1.2's lower bound shows that if this bound were to be sharp, then the corresponding construction must have essentially all of its edges containing a k-set with k-degree \(\Delta^{k-1}(\sum_{i=2}^{r-1} a_i)\). To prove Theorem 1.2's upper bound, we use the following supersaturation result for unbalanced r-graphs.

**Lemma 4.1.** Let \(2 \leq s_1 \leq \ldots \leq s_r\) and define \(a_i = \prod_{j<i} s_j\) for \(2 \leq i \leq r+1\). Let \(H\) be a complete r-partite r-graph on \(V_1 \cup \ldots \cup V_r\) with \(n_i := |V_i|\) defined by \(n_1 = n^{a_2}\) and \(n_i = n^{a_i}\) otherwise. There exists a constant \(\alpha_r \geq 1\) such that if \(H' \subseteq H\) has \(m \geq \alpha_r n^{-1} \prod_{i=1}^{r} n_i\) edges, then \(H'\) contains at least \(\alpha_r^{-1} m^{a_{r+1}} \prod_{i=1}^{r} n_i^{s_i-a_{r+1}}\) copies of \(K_{s_1, \ldots, s_r}\).

Strictly speaking, to prove Theorem 1.2 we only need that there exists at least one copy of \(K_{s_1, \ldots, s_r}\) when \(H\) contains at least this many edges, but it is easier to prove Lemma 4.1 by induction on \(r\) if we add this stronger conclusion.

**Proof.** We prove this by induction on \(r\). For \(r = 2\), we use the following supersaturation result of Erdős and Simonovits [11]: for \(s_1 \leq s_2\), there exist constants \(\alpha, \alpha'\) such that if \(G \subseteq K_{N,N}\) has \(e(G) = m \geq \alpha' N^{2-1/s_1}\), then \(G\) contains at least \(\alpha^{-1} m^{s_1 s_2} N\) copies of \(K_{s_1, s_2}\). Substituting \(N = n_1 = n_2 = n^{a_2}\) and taking \(\alpha_2 = \max\{\alpha', \alpha, 1\}\) gives the result.

Assume the result holds up to but not including \(r\). Let \(Q_i\) denote the set of all subsets of \(V_i\) of size \(s_i\) and let \(Q = Q_1 \times \cdots \times Q_{r-1}\). Let \(P\) denote the set of pairs \((Q, v)\) with \(Q \in Q\) and \(v \in V_r\) such that \(\{v_1, \ldots, v_{r-1}, v\} \in E(H)\) for all \(v_i \in Q_i\). That is, the vertices of \(Q\) induce a \(K_{s_1, \ldots, s_{r-1}}\) in the link graph of \(v\). For each \(Q \in Q\), define \(g_Q\) to be the number of pairs in \(P\) using the set \(Q\), and similarly define \(f_v\) for \(v \in V_r\) to be the number of pairs in \(P\) using \(v\). Observe that the number of \(K_{s_1, \ldots, s_r}\)'s in \(H'\) is at least \(\sum_{Q \subseteq Q} (g_Q)^{s_r}\), so it will be enough to lower bound this sum.

For non-negative \(x\) define \((\frac{x}{s_r}) = 0\) if \(x \leq s_r - 1\) and \((\frac{x}{s_r}) = \frac{x(x-1)\ldots(x-k-1)}{s_r!}\) otherwise. This makes \((\frac{x}{s_r})\) a convex function, so we have
\[
\sum_{Q \subseteq Q} \left( \frac{g_Q}{s_r!} \right)^{s_r} \geq \left| Q \right| \left( \frac{|Q|^{-1} \sum_{Q \subseteq Q} g_Q}{s_r!} \right)^{s_r} = \left| Q \right| \left( \frac{|Q|^{-1} \sum_{v \subseteq V_r} f_v}{s_r!} \right)^{s_r} \geq \left( \frac{|Q|^{s_r}}{|Q| s_r!} \right)^{s_r},
\]

(8)
Assume $m \geq \alpha_r n^{-1} \prod_{i \leq r} n_i$ where
\[
\alpha_r := 4^{s_r} \alpha_{r-1}^{s_r} s_r! \geq \max \{ 2^{1/a_r} \alpha_{r-1}, (4s_r \alpha_{r-1})^{1/a_r} \}.
\] (9)

For non-negative $x$ define
\[
h(x) = \alpha^{-1}_{r-1} x^{a_r} \prod_{i < r} n_i^{s_i-a_r}.
\]

Let $d_v$ denote the degree of $v$ in $H'$, and define $m' = \alpha_r n^{-1} \prod_{i < r} n_i$. By the inductive hypothesis, the number of copies of $K_{s_1, \ldots, s_{r-1}}$ in the link graph of $v \in V_r$ will be at least $h(d_v)$ whenever $d_v \geq m'$, and since $h(x)$ is an increasing function we have
\[
f_v \geq h(d_v) - h(m').
\]

From (9) we have
\[
m/n_r \geq 2^{1/a_r} m'.
\]

Using these two observations and the convexity of $h$, we find
\[
\sum_{v \in V_r} f_v \geq \sum_{v \in V_r} (h(d_v) - h(m')) \geq n_r \cdot (h(m/n_r) - h(m')) \geq \frac{1}{2} n_r \cdot h(m/n_r).
\] (10)

Since $|Q| \leq \prod_{i < r} n_i^{s_i}$ and $n_r = n^{a_r}$ for $r \geq 3$, we have by (9)
\[
\frac{1}{2} n_r \cdot h(m/n_r)|Q|^{-1} \geq \frac{1}{2} n_r^{a_r} \cdot h\left(\alpha_r n^{-1} \prod_{i < r} n_i^{s_i} \right) \prod_{i < r} n_i^{-s_i} = \frac{1}{2} \alpha_{r-1}^{-1} \alpha_r^{a_r} \geq 2s_r.
\]

Combining this with (10) gives
\[
|Q|^{-1} \sum_{v \in V_r} f_v - s_r \geq \frac{1}{2} n_r \cdot h(m/n_r)|Q|^{-1} - s_r
\]
\[
\geq \frac{1}{4} n_r \cdot h(m/n_r) \cdot |Q|^{-1 + 1/s_r} |Q|^{-1/s_r}
\]
\[
\geq \frac{1}{4} n_r \alpha_{r-1}^{-1} n_r^{a_r} \prod_{i < r} n_i^{s_i-a_r} \cdot \prod_{i < r} n_i^{-s_i+1} |Q|^{-1/s_r}
\]
\[
= (4\alpha_{r-1})^{-1} m^{a_r} \prod_{i=1}^{r} n_i^{s_i/s_r-a_r} |Q|^{-1/s_r},
\] (11)

where this last inequality again used $|Q| \leq \prod n_i^{s_i}$ and that $-1 + 1/s_r \leq 0$ since $s_r \geq 1$.

Plugging (11) into (8) gives
\[
\sum_{Q \in \mathcal{Q}} \binom{g_Q}{s_r} \geq \frac{|Q|}{s_r} (4\alpha_{r-1})^{-s_r} m_r^{a_r+1} \prod_{i=1}^{r} n_i^{s_i-a_r+1} |Q|^{-1} = \alpha_r^{-1} m_r^{a_r+1} \prod_{i=1}^{r} n_i^{s_i-a_r+1}.
\]

Thus $H'$ contains at least this many copies of $K_{s_1, \ldots, s_r}$ as desired. \qed
4.4 Proof of Theorem 1.2: Upper Bound

We define the $r$-graph $\tilde{H}(n; a_2, \ldots, a_r)$ as follows. $\tilde{H}(n; a_2)$ is the complete balanced bipartite graph with parts of size $n^{a_2}$. If $\tilde{H}(n; a_2, \ldots, a_r-1)$ has been defined, then we construct $\tilde{H}(n; a_2, \ldots, a_r)$ as follows. Take a set $\{H_1, H_2, \ldots\}$ containing $n^{a_r-a_{r-1}}$ disjoint copies of $\tilde{H}(n; a_2, \ldots, a_{r-1})$ and let $V_r$ consist of $n^{a_r}$ new vertices. Define $\tilde{H}(n; a_2, \ldots, a_r)$ to have vertex set $V$ and edge set $E$, where

$$V = V_r \cup \bigcup V(H_i),$$

$$E = \{e \cup \{v\} : e \in \bigcup E(H_i), v \in V_r\}.$$

Let us examine some basic properties of $H_r := \tilde{H}(n; a_2, \ldots, a_r)$. By construction we see that $e(H_r) = n^{a_r} \cdot n^{a_{r-1} - a_r} \cdot e(\tilde{H}(n; a_2, \ldots, a_{r-1}))$, and thus inductively one can prove that $e(H_r) = n^{2a_r + a_{r-1} + \cdots + a_2}$ for $r \geq 3$. Similarly for $r \geq 2$ each vertex in $V_r$ has degree $n^{a_r + \cdots + a_2} := \Delta$, and it is easily seen that $H_r$ is $\Delta$-regular. To prove the upper bound of Theorem 1.2, it suffices to show

$$\text{ex}(H_r, K_{s_1, \ldots, s_r}) = O(\Delta^{-1/\sum(i=2 a_i)} \cdot e(H_r)) = O(n^{2a_r + a_{r-1} + \cdots + a_2 - 1}).$$

Let $\alpha_r$ be as in Lemma 4.1, and let $H'_r \subseteq H_r$ be a subgraph with $e(H'_r) \geq \alpha_r n^{2a_r - a_{r-1} + \cdots + a_2}$ which contains no $K_{s_1, s_2, \ldots, s_r}$. By the pigeonhole principle, one of the copies of $\tilde{H}(n; a_2, \ldots, a_{r-1})$ making up $H_r$ is involved with at least $n^{a_{r-1} - a_r} e(H'_r)$ edges of $H'_r$. Let $H'_r-1$ denote such a copy, and let $H'_r-1$ be $H'_r$ after deleting every copy of $\tilde{H}(n; a_2, \ldots, a_{r-1})$ that is not $H_{r-1}$. Now again in $H'_r-1$ there exists some copy of $\tilde{H}(n; a_2, \ldots, a_{r-2})$ involved with at least $n^{a_{r-2} - a_{r-1}} e(H'_r-1)$ edges of $H'_r-1$. Call this copy $H_{r-2}$, and let $H'_r-2$ be $H'_r-1$ after deleting every copy that is not $H_{r-2}$. Continue this way until one arrives at $H'_2$.

Observe that $H'_2$ is $r$-partite with the $i$th part having size $n_i := n^{a_i}$ for $i \geq 2$ and $n_1 = n^{a_2}$. Because $e(H'_r) \geq e(H'_r) n^{a_2 - a_r}$ for all $i$, we have

$$e(H'_2) \geq e(H'_2) n^{a_2 - a_r} \geq \alpha_r n^{a_r + \cdots + a_3 + 2a_2 - 1} = \alpha_r n^{-1} \prod_{i \leq r} n_i.$$

Then $H'_2$ contains a $K_{s_1, \ldots, s_r}$ by Lemma 4.1, contradicting the assumption that $H'_r$ is $K_{s_1, \ldots, s_r}$-free. Thus there exists no such subgraph of $H_r$ with this many edges, giving the desired result.

5 Random Hosts

Throughout this section we fix integers $2 \leq s_1 \leq \cdots \leq s_r$. Recall that $a_r = \prod_{i \leq r} s_i$, $a_{r+1} = \prod_{i \leq r} s_i$, and

$$\beta_1 = \frac{1}{a_{r+1} - 1} \quad \text{and} \quad \beta_2 = \frac{a_r (\sum_{i=1}^{r-1} s_i - r) + 1}{(a_r - 1)(a_{r+1} - 1)}.$$

5.1 Proof of Theorem 1.3 : Lower Bounds

An ingredient of the proof of Theorem 1.3 is the following version of Azuma’s inequality, a proof of which can be found in [2], for example.
Lemma 5.1. Let $X_1, X_2, \ldots, X_N$ be independent Bernoulli random variables and let $f : \{0, 1\}^N \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq 1$ whenever $x$ and $y$ differ only on the $i$th co-ordinate. Then with $Z = f(X_1, X_2, \ldots, X_N)$, we have for all $\lambda > 0$
\[
\mathbb{P}(|Z - \mathbb{E}(Z)| > \lambda) \leq 2e^{-2\lambda^2/N}.
\]

For example, if $f(X_1, X_2, \ldots, X_N) = X_1 + X_2 + \cdots + X_N$ then we obtain a form of the Chernoff Bound for binomial random variables. In all of our applications the $X_i$ will be the indicator function for whether the $i$th edge of $K^r_n$ is in $H^r_{n,p}$.

We first consider $n^{-r/2} \log n \leq p \leq cn^{-\beta_1}$ for some small constant $c$. Observe that $\mathbb{E}[e(H_{n,p})] = \Theta(pn^r)$, and for $c$ sufficiently small this is at least twice the expected number of copies of $K^r_{s_1, \ldots, s_r}$ in $H^r_{n,p}$ which is $\Theta(p^{a_r+1}n^{\sum_{i=1}^r s_i})$. Thus by deleting an edge from each copy of $K^r_{s_1, \ldots, s_r}$ in $H^r_{n,p}$ we see that $\mathbb{E}[\text{ex}(H^r_{n,p}, K^r_{s_1, \ldots, s_r})] = \Omega(pn^r)$. Observe that $Z = \text{ex}(H^r_{n,p}, K^r_{s_1, \ldots, s_r})$ satisfies the conditions$^2$ of Lemma 5.1, so taking $\lambda = c'n^{-r/2}\sqrt{\log n}$ for some small $c'$ (and using $\mathbb{E}[Z] \geq \Omega(n^{r/2}\log n)$) since $p \geq n^{-r/2}\log n$ shows that the probability that $Z$ is within a constant factor of its expectation tends to 1, giving the a.a.s. result. Moreover, the bound $\text{ex}(H^r_{n,p}, K^r_{s_1, \ldots, s_r}) = \Omega(n^{r-\beta_1})$ continues to hold a.a.s. for $p = cn^{-\beta_1}$ by the monotonicity of the relative Turán number as a function of $p$.

Next we show $\text{ex}(H^r_{n,p}, K^r_{s_1, \ldots, s_r}) = \Omega(p^{1-1/a_r}n^{-1/a_r})$ when $p \geq n^{-\beta_2}$. We observe that the $(r-1)$-degree of any $(r-1)$-set is a binomial random variable with $n-r+1$ trials and probability $p$. Because $\beta_2 < 1$ (which can be proven using the inequality of arithmetic and geometric means), one can show using Lemma 5.1 that a.a.s. $H^r_{n,p}$ has maximum $(r-1)$-degree at most $O(pn)$, and also that a.a.s. $e(H^r_{n,p}) = \Omega(pn^r)$. Thus if $\text{ex}(n, K^r_{s_1, \ldots, s_r}) = \Omega(n^{r-1/a_r})$, by Theorem 1.4 we have a.a.s.
\[
\text{ex}(H^r_{n,p}, K^r_{s_1, \ldots, s_r}) = \Omega((pn)^{1-1/a_r}pn^r) = \Omega(p^{1-1/a_r}n^{-1/a_r}).
\]

This proves the lower bounds of Theorem 1.3.

5.2 Proof of Theorem 1.3 : Upper Bounds

The upper bound of Theorem 1.3 for small $p$ follows since a.a.s. $H^r_{n,p}$ has at most $O(pn^r)$ edges, and the result for $p$ in the middle range will follow from the large range since $\mathbb{E}[\text{ex}(H^r_{n,p}, K^r_{s_1, \ldots, s_r})]$ is non-decreasing in $p$. Thus we can assume $p \geq n^{-\beta_2}(\log n)^{2a_r/(a_r-1)}$.

Our approach for the upper bounds in this range borrows heavily from the argument used by Morris and Saxton [21] for the case $r = 2$. To this end, we let $\mathcal{I} = \mathcal{I}(n)$ denote the collection of $K^r_{s_1, \ldots, s_r}$-free $r$-graphs on $n$ vertices, and let $\mathcal{G} = \mathcal{G}(n, k)$ denote the collection of all $r$-graphs with $n$ vertices and at most $knr^{-1/a_r}$ edges. The following lemma is the main technical result we need to prove our upper bounds, where by a colored $r$-graph we mean an $r$-graph together with an arbitrary labeled partition of its edge set.

Lemma 5.2. For any $2 \leq s_1 \leq \cdots \leq s_r$, there exists a constant $c > 0$ such that the following holds for sufficiently large $n,k$ with $k \leq n^{\beta_2/(a_r)}(\log n)^{2/(a_r-1)}$. There exists a collection $\mathcal{S}$ of colored $r$-graphs with $n$ vertices and at most $ckk^{1-a_r}n^{-1/a_r}$ edges and functions $g : \mathcal{I} \to \mathcal{S}$ and $h : \mathcal{S} \to \mathcal{G}(n, k)$ with the following properties:

$^2$To be somewhat more explicit, we let $X_i = 1$ if the $i$th edge of $K^r_n$ is in $H^r_{n,p}$ and $X_i = 0$ otherwise. Then $Z$ is a function of the $X_i$, and changing one value of $X_i$ (i.e. adding or deleting an edge in $H^r_{n,p}$) changes the value of $Z$ (i.e. the size of a largest $K^r_{s_1, \ldots, s_r}$-free subgraph of $H^r_{n,p}$) by at most 1.
(a) For every \(s \geq 0\), the number of colored \(r\)-graphs in \(\mathcal{S}\) with \(s\) edges is at most
\[
\left( \frac{cn^{r-1/a_r}}{s} \right)^{a_r s/(a_r-1)} \exp(ck^{1-a_r}n^{r-1/a_r}).
\]

(b) For every \(I \in \mathcal{I}\), we have \(g(I) \subseteq I \subseteq g(I) \cup h(g(I))\).

Before proving Lemma 5.2, we first illustrate how it implies the upper bound of Theorem 1.3 when \(p \geq n^{-\beta_2}(\log n)^{2a_r/(a_r-1)}\).

Define \(k = p^{-1/a_r}\), which means \(k \leq n^{\beta_2/a_r}(\log n)^{-2/(a_r-1)}\), and thus we can define \(S, g, h\) as in Lemma 5.2. If there exists a \(K_{s_1,\ldots,s_r}\)-free subgraph \(I \subseteq H_{n,p}^r\) with \(m\) edges, then in particular \(g(I) \subseteq H_{n,p}^r\) and \(H_{n,p}^r\) contains at least \(m - e(g(I))\) edges of \(h(g(I))\). Thus for \(m \geq 2k^{1-a_r}n^{r-1/a_r}\), the probability of \(I \subseteq H_{n,p}^r\) for some \(K_{s_1,\ldots,s_r}\)-free \(I\) with \(m\) edges is at most
\[
\sum_{S \in \mathcal{S}} p^m \left( \frac{kn^{r-1/a_r}}{m - e(S)} \right) \leq \sum_{S \in \mathcal{S}, e(S) = s} \sum_{s=0}^{ck^{1-a_r}n^{r-1/a_r}} p^s \cdot \left( \frac{epkn^{r-1/a_r}}{m - s} \right)^{m-s} \exp(ck^{1-a_r}n^{r-1/a_r} \cdot \left( \frac{epkn^{r-1/a_r}}{m - s} \right)^{m-s}).
\]

where the second inequality used Lemma 5.2(a), to get the last inequality we used that \((d/s)^s \leq e^{d/e}\) and that \(m - s \geq 1/2m\) for \(m \geq 2k^{1-a_r}n^{r-1/a_r}\), and the last equality used \(k = p^{-1/a_r}\). This quantity will tend to 0 as \(n\) towards infinity provided \(m \geq c'p^{1-1/a_r}n^{r-1/a_r}\) for some sufficiently large constant \(c'\), proving the result. \(\square\)

It remains to prove Lemma 5.2, and for ease of presentation we do this in the following two subsections.

### 5.3 Balanced Supersaturation

To adapt the proof of Theorem 6.1 of Morris and Saxton [21], we require a balanced supersaturation result from Corsten and Tran [8] which roughly says that if \(H\) is an \(r\)-graph with significantly more than \(n^{r-1/a_r}\) edges, then one can find a large collection \(\mathcal{H}\) of copies of \(K_{s_1,\ldots,s_r}\) in \(H\) which are relatively spread apart.

More precisely, given an \(r\)-graph \(H\), we identify copies of \(K_{s_1,\ldots,s_r}\) in \(H\) by ordered tuples \((S_1,\ldots,S_r)\) with \(|S_i| = s_i\) such that these sets induce a \(K_{s_1,\ldots,s_r}\) in \(H\). If \(\mathcal{H}\) is a collection of copies of \(K_{s_1,\ldots,s_r}\) in \(H\) and \((T_1,\ldots,T_r)\) is an ordered tuple with \(1 \leq |T_i| \leq s_i\), we define \(d_H(T_1,\ldots,T_r)\) to be the number of \((S_1,\ldots,S_r) \in \mathcal{H}\) such that \(T_i \subseteq S_i\) for all \(i\). If \(t_1,\ldots,t_r\) are such that \(1 \leq t_i \leq s_i\) for all \(i\), then for \(\delta, \ell, n > 0\) we define the functions
\[
D^{t_1,\ldots,t_r}(\delta, \ell, n) = \ell \sum_{i=1}^{r} a_i(s_i-t_i) (\delta n) \sum_{a=1}^{r} (s_i-t_i).
\]
where we recall \( a_i = \prod_{j<i} s_j \); and whenever \( \delta, \ell, n \) are understood we simply denote this function by \( D^{s_1, \ldots, s_r} \). For example, when \( r = 3 \) we have

\[
D^{t_1, t_2, t_3} = \ell^{(s_1 - t_1) + s_1 (s_2 - t_2) + s_1 s_2 (s_3 - t_3)} (\delta n)^{s_1 - t_1 + s_2 - t_2 + s_3 - t_3}.
\]

We note that in [8] this function was defined in terms of \( k = \ell n^{-1/a_r} \), but our intermediate computations will be greatly simplified by using this change in variables. In particular, we can rephrase Theorem 3.1 of Corsten and Tran [8] as follows.

**Proposition 5.3** ([8]). For every \( 2 \leq s_1 \leq \cdots \leq s_r \) there exist constants \( \delta, \ell_0 > 0 \) such that the following holds for every \( \ell \geq \ell_0 n^{-1/a_r} \) and every \( n \in \mathbb{N} \). Given an \( r \)-graph \( H \) with \( n \) vertices and \( \ell n^r \) edges, there exists a collection \( \mathcal{H} \) of copies of \( K_{s_1, \ldots, s_r} \) in \( H \) such that

(a) \( |H| \geq \delta^{\ell a_r + 1} n^\sum_{i=1}^r s_i \), and

(b) \( d_H(T_1, \ldots, T_r) \leq D_{|T_1|, \ldots, |T_r|}^{T_1, \ldots, T_r} \) for all \( T_i \subseteq V(G) \) with \( 1 \leq |T_i| \leq s_i \).

We can treat \( \mathcal{H} \) from Proposition 5.3 as an \( a_{r+1} \)-uniform hypergraph with \( V(\mathcal{H}) = E(H) \) and where edges \( E \in E(\mathcal{H}) \) correspond to copies of \( K_{s_1, \ldots, s_r} \) in \( H \) which use all of the edges \( e \in E \).

For any \( \sigma \subseteq E(H) \), let \( d_H(\sigma) \) denote the number of edges in \( \mathcal{H} \) containing \( \sigma \). To apply the method of hypergraph containers, we show that \( d_H(\sigma) \) is relatively small for all \( \sigma \).

For \( \sigma \subseteq V(\mathcal{H}) = E(H) \), define \( V(\sigma) := \bigcup_{e \in \sigma} e \) to be the vertices involving edges of \( \sigma \). Observe that if \( (S_1, \ldots, S_r) \) corresponds to some \( K_{s_1, \ldots, s_r} \) in \( \mathcal{H} \) containing \( \sigma \), then this copy contributes to \( d_H(T_1, \ldots, T_r) \) where \( T_i = S_i \cap V(\sigma) \). Thus if \( \mathcal{P} \) is the set of all \( r \)-partitions of \( V(\sigma) \), we have

\[
d_H(\sigma) \leq \sum_{(T_1, \ldots, T_r) \in \mathcal{P}} d_H(T_1, \ldots, T_r) = O(\max D^{s_1, \ldots, s_r}), \tag{12}
\]

where the maximum ranges over all integers \( 1 \leq t_i \leq s_i \) with \( \prod t_i \geq |\sigma| \) (because the \( r \)-graph induced by any element of \( \mathcal{P} \) must be a complete \( r \)-partite \( r \)-graph to have positive degree in \( \mathcal{H} \)).

To simplify computations, we extend the definition of \( D^{s_1, \ldots, s_r} \) and the maximum of (12) to all real \( t_i \) in this range. Observe that \( D^{s_1, \ldots, s_r} \) is a decreasing function in each of the variables \( t_i \) provided \( \ell a_r \geq (\delta n)^{-1} \), so possibly by setting \( t_0 = \delta^{-1/a_r} \) we can assume for our range of \( \ell \) that we have \( \prod t_i = |\sigma| \) exactly.

For ease of notation we define

\[
b_i = \prod_{j \geq i} s_j,
\]

where we adopt the convention \( b_r = 1 \). Because \( \ell = O(1) \) (since the \( r \)-graph \( H \) has at most \( O(n^r) \) edges), we see that \( D^{s_1, \ldots, s_r} \) is maximized given \( \prod t_i = |\sigma| \) when \( t_r \) is made as large as possible, with \( t_{r-1} \) as large as possible subject to this, and so on. In particular, if \( |\sigma| \) lies in the interval \([b_i, b_{i-1}]\), then the maximum of (12) occurs when \( t_j = s_j \) for all \( j > i \), \( t_j = 1 \) for all \( j < i \), and \( t_i = |\sigma|/b_j \). Thus

\[
d_H(\sigma) = O\left(\ell^{\sum_{j<i} a_j (s_{j+1} - s_j)} (\delta n)^{\sum_{j<i} s_j - i + |\sigma|/b_j} \right) \quad \text{for } |\sigma| \in [b_i, b_{i-1}],
\]

and because \( a_j s_j = a_{j+1} \) for all \( j \), this can be written more succinctly as

\[
d_H(\sigma) = O\left(\ell^{a_i (s_{i+1} - s_i)} (\delta n)^{\sum_{j<i} s_j - i + |\sigma|/b_j} \right) \quad \text{for } |\sigma| \in [b_i, b_{i-1}],
\]

\[
d_H(\sigma) = O\left(\ell^{a_i - 1} (\delta n)^{\sum_{j<i} s_j - i} \right) \quad \text{for } |\sigma| = b_i. \tag{13}
\]

\[
d_H(\sigma) = O\left(\ell^{a_i - 1} (\delta n)^{\sum_{j<i} s_j - i} \right) \quad \text{for } |\sigma| = b_i. \tag{14}
\]
In particular, (14) implies
\[ d_H(\sigma) = O \left( \ell^{a_r+1-1}(\delta n)^{\sum_{i=1}^r s_i - r} \right), \quad \text{for } |\sigma| = 1. \] (15)

From now on we consider \( \sigma \) with \(|\sigma| \geq 2\) and define
\[ \phi(\sigma) = \left( \frac{e(H)}{|H|} d_H(\sigma) \right)^{\frac{1}{|\sigma|-1}}, \quad \tau' = \max_{\sigma : 2 \leq |\sigma| \leq a_{r+1}} \phi(\sigma). \]

Using (13) and the bound on \(|H|\) from Proposition 5.3, we can bound \( \phi(\sigma) \) by a function of the form
\[ O \left( \ell^{-a_{r+1}}(\delta n)^{\sum_{i=1}^r s_i}/|\sigma|-1 \right), \]
where the constants \( c_i, d_i, c', d' \) depend on which \([b_i, b_{i-1}]\) interval \(|\sigma|\) belongs to. With this formulation and a bit of calculus, one sees that restricted to any \([b_i, b_{i-1}]\) interval this bound on \( \phi(\sigma) \) is either non-decreasing or non-increasing (depending only on the value of \( \ell \) relative to \( \delta n \)). In particular, to upper bound \( \tau' \) it is enough to use these upper bounds for \( \phi(\sigma) \) whenever \(|\sigma|\) is an endpoint of some \([b_i, b_{i-1}]\) interval.

To end this, first observe that for all \( i \),
\[ e(H)/|H| = O \left( \ell^{1-a_{r+1}}(\delta n)^{r-\sum_{j=1}^r s_j} \right) = O \left( \ell^{1-a_{r+1}b_i}(\delta n)^{r-\sum_{j=1}^r s_j} \right). \]

Thus for \( 0 \leq i < r \) we have by (14) that with \(|\sigma| = b_i\),
\[ \phi(\sigma) = O \left( \left( \ell^{a_{r+1} (1-b_i)}(\delta n)^{r-i-\sum_{j=1}^r s_j} \right)^{1/(b_i-1)} \right) = O \left( \ell^{-a_{r+1}}(\delta n)^{r-\sum_{j=1}^r s_j}/b_i \right), \]
and for all relevant \(|\sigma| \in [b_r, b_{r-1}]\) (i.e. those with \( 2 \leq |\sigma| \leq b_{r-1} = s_r \)), we have by (13) that
\[ \phi(\sigma) = O \left( (\ell^{a_r (1-|\sigma|)}(\delta n)^{1-|\sigma|})^{1/(|\sigma|-1)} \right) = O(\ell^{-a_r}(\delta n)^{-1}). \]

Putting all of this together, we find
\[ \tau' = O \left( \max_{0 \leq i \leq r-1} \left\{ \ell^{-a_{r+1}}(\delta n)^{r-\sum_{j=1}^r s_j}/b_i \right\} \right). \] (16)

We claim that only the \( i = 0, r - 1 \) terms of this maximum are relevant. Indeed, observe that the point at which the \((r - 1)\)st term \( \ell^{-a_r}(\delta n)^{-1} \) equals the \( i \)th term is exactly when
\[ \log_{\delta n}(\ell) = \frac{\sum_{j>i} s_j - (r - i) - (b_i - 1)}{(a_r - a_{r+1})(b_i - 1)} = \frac{\sum_{j>i} s_j - (r - i - 1) - b_i}{a_{r+1}(b_i s_r^{i-1} - 1)(b_i - 1)} := \gamma_i. \]

Because \( b_i = b_{i+1} s_{i+1} \) and \( s_{i+1} \geq 1 \), we have
\[ b_i - 1 \geq b_{i+1} - 1, \quad b_i s_r^{i-1} - 1 \geq b_{i+1} s_r^{i-1} - 1. \]

We also have
\[ \sum_{j>i} s_j - (r - i - 1) = s_{i+1} + \sum_{j>i+1} s_j - (r - i - 1) \leq s_{i+1} + s_{i+1} \left( \sum_{j>i+1} s_j - (r - i - 2) \right), \]
where this last step used \( s_{i+1} \geq 1 \) and \( \sum_{j=i+1} s_j \geq (r - i - 2) \) since \( s_j \geq 1 \) for all \( j \). These observations imply

\[
\gamma_i \leq \frac{s_{i+1} + s_{i+1}(\sum_{j=i+1} s_j - (r - i - 2)) - b_i}{a_{i+1} \cdot s_{i+1}(b_{i+1}s_r^{-1} - 1)(b_{i+1} - 1)} = \frac{1 + \sum_{j=i+1} s_j - (r - i - 2) - b_{i+1}}{a_{i+2}(b_{i+1}s_r^{-1} - 1)(b_{i+1} - 1)} = \gamma_{i+1},
\]

where this first equality used \( a_{i+1} s_{i+1} = a_{i+2} \) and \( b_i = b_{i+1}s_{i+1} \). In total, for \( \ell \leq n^{\tau_0} \) the \( i = r - 1 \) term of (16) is the maximum, and at \( \ell = n^{\tau_0} \) the \( i = 0 \) term is equal to the \( i = r - 1 \) term. Because the \( i = 0 \) term has the largest power of \( \ell \), it will continue to be the maximum value for all \( \ell \geq n^{\tau_0} \), proving the claim.

Thus \( \tau' = O\left(\max\{\ell^{-a_r}(\delta n)^{-1}, \ell^{-1}(\delta n)(r-\sum_{i=1} s_i)/(b_0 - 1)\}\right) \). If we let \( k = \ell(\delta n)^{-1/a_r} \), then note that the exponent of \( \delta n \) in this second term of the maximum equals

\[
a_r^{-1} - \frac{\sum_{i=1}^{r-1} s_i - r}{b_0 - 1} = -\frac{a_r(\sum_{i=1}^{r-1} s_i - r) + 1}{a_r(a_{r+1} - 1)} := -\gamma.
\]

Using this and (15), we can reformulate Proposition 5.3 as follows.

**Proposition 5.4.** For every \( 2 \leq s_1 \leq \cdots \leq s_r \) there exist constants \( \delta, k_0 > 0 \) such that the following holds for every \( k \geq k_0 \) and every \( n \in \mathbb{N} \). Given an \( r \)-graph \( H \) with \( n \) vertices and \( kn^{r-1/a_r} \) edges, there exists a collection \( \mathcal{H} \) of copies of \( K_{s_1,\ldots,s_r} \) in \( H \) which we view as an \( a_{r+1} \)-graph such that

(a) \( e(\mathcal{H}) \geq \delta k^{a_{r+1}} n^{\sum_{i=1}^{r-1} s_i} \), and

(b) If \( \Delta_j \) is the maximum \( j \)-degree of \( \mathcal{H} \), then

\[
\Delta_1 \leq k^{a_{r+1}-1}(\delta n)^{\sum_{i=1}^{r-1} s_i - r + 1/a_r},
\]

and for \( j \geq 2 \) we have

\[
\left(\frac{e(H)}{e(\mathcal{H})}\Delta_j\right)^{1/(j-1)} = O(\delta^{-\gamma} \max\{k^{-a_r}, k^{-1/n^{-\gamma}}\}).
\]

### 5.4 Proof of Lemma 5.2

With Proposition 5.4 established, the other technical tool we need to prove Lemma 5.2 is the following container lemma.

**Lemma 5.5 ([21]).** Let \( q \geq 2 \) and \( 0 < \delta < \delta_0(q) \) be sufficiently small. Let \( \mathcal{H} \) be a \( q \)-graph with \( N \) vertices and maximum \( j \)-degree \( \Delta_j \) for all \( j \), and suppose \( \tau > 0 \) is such that

\[
\frac{|V(\mathcal{H})|}{e(\mathcal{H})} \sum_{j=2}^q \frac{\Delta_j}{\tau^{j-1}} \leq \delta.
\]

Then there exists a collection \( \mathcal{C} \) of subsets of \( V(\mathcal{H}) \) and a function \( f \) from subsets of \( V(\mathcal{H}) \) to \( \mathcal{C} \) such that:

(a) for every independent set \( I \subseteq V(\mathcal{H}) \) there exists \( T \subseteq I \) with \( |T| \leq \tau N/\delta \) and \( I \subseteq f(T) \), and
(b) \(e(\mathcal{H}[\mathcal{C}]) \leq (1 - \delta)e(\mathcal{H})\) for every \(C \in \mathcal{C}\).

With these two results we prove the following.

**Lemma 5.6.** For every \(2 \leq s_2 \leq \cdots \leq s_r\), there exist \(\epsilon, k_0 > 0\) such that the following holds for every \(k \geq k_0\) and \(n \in \mathbb{N}\). Set \(\mu = \epsilon^{-1}k^{-1}\max\{k^{-a_r}, n^{-\gamma}\}\). Given an \(r\)-graph \(H\) with \(n\) vertices and \(kn^{r-1/a_r}\) edges, there exists a function \(f_H\) that maps subgraphs of \(H\) to subgraphs of \(H\) such that for every \(K_{s_1,\ldots,s_r}\)-free subgraph \(I \subseteq H\) we have:

(a) There exists a subgraph \(T \subseteq I\) with \(e(T) \leq \mu n^{r-1/a_r}\), and

(b) \(I \subseteq f_H(T)\) with \(e(f_H(T)) \leq (1 - \epsilon)e(H)\).

**Proof.** Let \(\delta, k_0, \) and \(\mathcal{H}\) be as in Proposition 5.4, and assume that \(\delta\) is sufficiently small so that Lemma 5.5 applies with \(g = a_{r+1}\) (otherwise we can take a smaller \(\delta\) and the result of Proposition 5.4 continues to hold). Observe that \(\tau = \delta^{-2\gamma}\max\{k^{-a_r}, k^{-1}n^{-\gamma}\}\) gives

\[
\frac{|V(\mathcal{H})|}{e(\mathcal{H})} \sum_{j=2}^{g} \frac{\Delta_j}{\tau^{j-1}} = O(\delta^2),
\]

and by taking \(\delta\) sufficiently small we can assume this sum is at most \(\delta\).

Let \(c = \max\{3 + \gamma, \sum_{i=1}^{r-1} s_i - r + 1/a_r\}\) and \(\epsilon = \delta^c\). By applying Lemma 5.5, we obtain a collection \(\mathcal{C}\) of subsets of \(V(\mathcal{H})\) and a function \(f_H\) from subsets of \(V(\mathcal{H})\) to \(\mathcal{C}\) such that for every \(K_{s_1,\ldots,s_r}\)-free subgraph \(I \subseteq H\) we have:

(a) There exists a subgraph \(T \subseteq I\) with

\[
e(T) \leq (\delta^{-2\gamma}\max\{k^{-a_r}, k^{-1}n^{-\gamma}\})(kn^{r-1/a_r})/\delta \leq \mu n^{r-1/a_r},
\]

and

(b) \(I \subseteq f_H(T)\) with \(e(f_H(T)) \leq (1 - \delta)e(\mathcal{H})\).

To show (b), it suffices to show that (b') implies \(e(C) \leq (1 - \epsilon)e(H)\) for every \(C \in \mathcal{C}\). Indeed, let

\[
\mathcal{D}(C) = E(\mathcal{H}) \setminus E(\mathcal{H}[\mathcal{C}]) = \{E \in E(\mathcal{H}) : e \in E\text{ for some }e \in E(\mathcal{H}) \setminus C\}.
\]

By definition \(|\mathcal{D}(C)| = e(\mathcal{H}) - e(\mathcal{H}[\mathcal{C}])\), and this is at least \(\delta e(\mathcal{H})\) by (b'). By the bound on the maximum degree of \(\mathcal{H}\) from Proposition 5.4, we find

\[
|\mathcal{D}(C)| \leq \frac{e(\mathcal{H})}{\delta c^{-1}kn^{r-1/a_r}} \cdot |E(\mathcal{H}) \setminus C|.
\]

Combining these two results implies \(|E(\mathcal{H}) \setminus C| \geq ckn^{r-1/a_r}\) as desired.

Lastly, we need the following inequality.

**Lemma 5.7** ([23]). Let \(M, s > 0\), and \(0 < \delta < 1\). If \(b_1, \ldots, b_m \in \mathbb{R}\) satisfy \(s = \sum b_j\) and \(1 \leq b_j \leq (1 - \delta)^j M\) for each \(j \in [m]\), then

\[
s \log s \leq \sum b_j \log b_j + O(M).
\]
Proof of Lemma 5.2. We construct the functions $g,h$ and family $S$ as follows. Given a $K_{s_1,\ldots,s_r}$-free $r$-graph $I \in \mathcal{I}$, we repeatedly apply Lemma 5.6 first to $H_0 = K_n^r$, then to $H_1 = f_{H_0}(T_1) \setminus T_1$, where $T_1 \subseteq I$ is the set guaranteed to exist by (a) of Lemma 5.6; then to $H_2 = f_{H_1}(T_2) \setminus T_2$ where $T_2 \subseteq I \cap H_1 = I \setminus T_1$, and so on. We continue until we arrive at an $r$-graph $H_m$ with at most $k n^{r-1/ar}$ edges and set $g(I) = (T_1,\ldots,T_m)$ and $h(g(I)) = H_m$. Since $H_m$ depends only on the sequence $(T_1,\ldots,T_m)$, the function $h$ is well-defined.

It remains to bound the number of colored graphs in $S$ with $s$ edges. To do this, it suffices to count the number of choices for the sequence of $r$-graphs $(T_1,\ldots,T_m)$ with $\sum e(T_j) = s$. For each $j \geq 1$, define $k(j)$ and $\mu(j)$ by $e(H_{m-j}) = k(j)n^{r-1/ar}$ and $\mu(j) = \epsilon^{-1}\max\{k(j)^{1-ar},n^{-\gamma}\}$, and note that

$$(1-\epsilon)^{-j+1}k \leq k(j) = O(n^{1/ar}), \quad T_{j+1} \subseteq H_j, \quad e(T_{m-j}) \leq \mu(j)n^{r-1/ar}.$$ 

Thus fixing $k,\epsilon,s$ as above, we define

$$\mathcal{K}(m) = \{k = (k(1),\ldots,k(m)) : (1-\epsilon)^{-j+1}k \leq k(j) \leq n^{1/ar}\} \quad \text{for each } m \in \mathbb{N},$$

$$\mathcal{B}(k) = \{b = (b(1),\ldots,b(m)) : b(j) \leq \mu(j)n^{r-1/ar} \quad \text{and} \quad \sum b(j) = s\} \quad \text{for each } k \in \mathcal{K}(m).$$

It follows that the number of colored graphs in $S$ with $s$ edges is at most

$$\sum_{m=1}^{\infty} \sum_{k \in \mathcal{K}(m)} \sum_{b \in \mathcal{B}(k)} \prod_{j=1}^{m} \binom{k(j)n^{r-1/ar}}{b(j)}.$$

Given $m,k,b$, we partition this product over $j$ according to whether or not $\mu(j) = \epsilon^{-1}n^{-\gamma}$. Since $\mathcal{K}(m) = \emptyset$ for $m = \Omega(\log n)$ and $b(j) \leq \mu(j)n^{r-1/ar}$, $\epsilon^{-1}n^{-r-1/ar-\gamma}$ in this case, the product over these terms is at most

$$(n^{r}\sum b(j)) \leq \exp(O(1) \cdot n^{r-1/ar-\gamma}(\log n)^2) \leq \exp(O(1) \cdot k^{1-ar}n^{r-1/ar}),$$

where this last step used the hypothesis of $k \leq n^{\beta_2/ar}(\log n)^{2/(ar-1)}$ and that $\beta_2/ar = \gamma/(ar-1)$ by the way we defined $\gamma$ in (17). On the other hand, if $b(j) \leq \epsilon^{-1}k(j)^{1-ar}n^{r-1/ar}$, then

$$\binom{k(j)n^{r-1/ar}}{b(j)} \leq \left(\frac{ek(j)n^{r-1/ar}}{b(j)}\right)^{b(j)} \leq \left(\frac{n^{r-1/ar}}{eb(j)}\right)^{a,b(j)/(ar-1)}.$$

Thus by Lemma 5.7 the product of these remaining $j$ terms is at most

$$\left(\frac{cn^{r-1/ar}}{s}\right)^{ar,s/(ar-1)} \exp(ek^{1-ar}n^{r-1/ar})$$

for some constant $c$. The result follows after noting $\sum_{m} \sum_{k} |\mathcal{B}(k)| = n^{O(\log n)}$. \qed

6 Concluding Remarks

- Foucaud, Krivelevich, and Perarnau [12] conjectured that if $F$ and $H$ are graphs such that $H$ has minimum degree $\delta$ and maximum degree $\Delta$, then $H$ has a spanning $F$-free subgraph of minimum
degree $\Omega(\delta \text{ex}(\Delta, F)/\Delta^2)$ as $\Delta \to \infty$. This conjecture was proved for bipartite graphs of diameter at most three by Perarnau and Reed [24]. A key part of the proof is to show that for some $c, c' > 0$, every $\Delta$-regular graph $H$ has a spanning subgraph $G$ of minimum degree at least $c\Delta$ with an injective $c'\Delta$-coloring, which is a map $\chi : V(G) \to \{1, 2, \ldots, c'\}$ such that every pair of edges $e_1, e_2$ with $|e_1 \cap e_2| = 1$ has $\chi(e_1) \neq \chi(e_2)$. It is natural to consider a similar framework for $r$-graphs, where now we require that $e_1, e_2$ with $|e_1 \cap e_2| = r - 1$ have distinct color sets.

**Conjecture 6.1.** There exist constants $c, c' > 0$ such that if $H$ is a $\Delta$-regular $r$-graph with maximum $(r - 1)$-degree at most $D$, then $H$ has a spanning subgraph $G$ of minimum degree at least $c\Delta$ with an injective $c'\Delta$-coloring.

Note that the proof of Lemma 2.1 essentially shows that one can find a subgraph of $H$ with at least $\Omega(e(H))$ edges which has an injective $O(D)$-coloring, so the central difficulty is in maintaining the minimum degree.

- The main open question for relative Turán numbers of $r$-graphs is to give bounds on $\text{ex}(H, F)$ when $F$ is $r$-partite. We observe that for each positive integer $\Delta$ there exists an $r$-graph $H$ of maximum degree at most $\Delta$ such that

$$\text{ex}(H, F) = O\left(\frac{\text{ex}(\frac{1}{\Delta - r}, F)}{\Delta - r}\right) \cdot e(H),$$

namely with $H = K_{\Delta + 1/(r - 1)}$. This leads to the question of determining for which $F$ the above upper bound is tight up to constants for all $r$-graphs $H$ of maximum degree $\Delta$ – Conjecture 1.1 states that this holds for all graphs $F$. To this end, we generalize (3) by defining the Turán exponent of an $r$-graph $F$, when it exists, to be

$$\alpha(F) = \lim_{n \to \infty} \frac{\log (n) / \text{ex}(n, F)}{\log \binom{n - 1}{r - 1}}.$$

It seems likely that $\alpha(F)$ exists for every $r$-graph $F$, and the existence of $\alpha(F)$ when $F$ is a graph is a consequence of a conjecture of Erdős and Simonovits [11]. Similarly we generalize (1) by defining the relative Turán exponent of $F$, when it exists, to be

$$\beta(F) = \lim_{\Delta \to \infty} \sup_{H} \frac{\log e(H)/\text{ex}(H, F)}{\log \Delta},$$

where the supremum ranges over all $H$ with maximum degree at most $\Delta$. Theorem 1.2 shows that whenever each $K_{s_1, \ldots, s_i}$ is known to have Turán exponent $(s_1 \cdots s_{i-1})^{-1}$, the relative Turán exponent exists and is given by (1). It is noteworthy that unlike $r = 2$, $\alpha(K_{s_1, \ldots, s_r}) < \beta(K_{s_1, \ldots, s_r})$ for $r \geq 3$.

- Analogous to the conjecture that $\alpha(F)$ exists for all graphs, we conjecture $\beta(F)$ exists for all $F$.

**Conjecture 6.2.** For every $r$-graph $F$, the relative Turán exponent $\beta(F)$ exists.

For $r = 2$, if Conjecture 1.1 were true then the existence of $\beta(F)$ would follow from the existence of $\alpha(F)$, and in fact $\alpha(F) = \beta(F)$ in this case. While we have $\alpha(F) \leq \beta(F)$ for all $r$-graphs $F$, by (18) we see that these quantities may differ sharply in the setting of hypergraphs. For example, Theorem 1.2 shows $\alpha(F) < \beta(F)$ when $F = K_{s_1, s_2, \ldots, s_r}$ and $r \geq 3$. It seems difficult in general to determine whether $\alpha(F) = \beta(F)$ for a given $F$, and we leave this as an open problem.
Problem 6.3. Determine the $r$-partite $r$-graphs for which $\alpha(F) = \beta(F)$.

- In Theorem 1.2 we determined $\beta(F)$ for $F = K_{s_1, \ldots, s_r}$ and certain values of $s_i$, and in this case we showed $\alpha(F) \neq \beta(F)$. Our proof extends to a somewhat wider family of hypergraphs as follows.

Given a graph $F$, we define its $s$-extension to be the 3-graph $F_{+s}$ on $V(F) \cup [s]$ with edge set $E(F_{+s}) = \{e \cup \{i\} : e \in E(F), \ i \in [s]\}$. For example, $(K_{s_1, s_2})_{+s_3} = K_{s_1, s_2, s_3}^{(3)}$. By going through a nearly identical proof as that of Theorem 1.2 and using the method of random polynomials due to Bukh and Conlon [5], it is possible to determine $\beta(F_{+s})$ for $s$ sufficiently large provided $F$ is a non-empty connected bipartite graph of diameter at most 3 which has a supersaturation result analogous to the result of Erdős and Simonovits [11] that was used in the proof of Lemma 4.1. In this setting we further have that $\alpha(F_{+s}) \neq \beta(F_{+s})$.

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