Gröbner-Shirshov bases for some one-relator groups

Yuqun Chen and Chanyan Zhong

School of Mathematical Sciences
South China Normal University
Guangzhou 510631
P. R. China
yqchen@scnu.edu.cn
chanyanzhong@yahoo.com.cn

Abstract: In this paper, we prove that two-generator one-relator groups with depth less than or equal to 3 can be effectively embedded into a tower of HNN-extensions in which each group has the effective standard normal form. We give an example to show how to deal with some general cases for one-relator groups. By using the Magnus’ method and Composition-Diamond lemma, we reprove the G. Higman, B. H. Neumann and H. Neumann’s embedding theorem.

Key words: Group, HNN-extension, Gröbner-Shirshov basis, Standard normal form.

AMS Mathematics Subject Classification(2000): 20E06, 20F05, 16S15, 13P10

1 Introduction

Higman, Neumann, Neumann (1949, [10], see also R. C. Lyndon and P. E. Schupp [13], p.188) proved that any countable group with \( \leq n \) relations can be effectively embedded into a group generated by two elements with \( \leq n \) relations. Even before, this result was known for one-relator groups (W. Magnus [14], see also W. Magnus, A. Karrass and D. Solitar [15], p.259). So, any finitely generated one-relator group can be effectively embedded into one-relator group with two generators,

\[
G = gp\langle x, y | x^{n_1}y^{m_1} \cdots x^{n_k}y^{m_k} = 1 \rangle,
\]

where \( n_i, m_i \neq 0, k \geq 0 \). We call \( k \) the depth of \( G \).

On the other hand, any one-relator group can be effectively embedded into a tower of HNN-extensions essentially by the Magnus’ method (see [13], p.198, and Moldavanskii [16]). There is a conjecture, stated by L. A. Bokut, that each group of the Magnus tower for any one-relator group is a group with the standard normal form in the sense of Bokut [2] (see also [1]). If it is true, it would give another proof for the decidability of the word problem for any one-relator group.

*Supported by the NNSF of China (No.10771077) and the NSF of Guangdong Province (No.06025062).
2 Standard normal form and Gröbner-Shirshov basis

In this section, we will cite some literatures about the definition of groups with the standard normal form and Composition-Diamond lemma on free associative algebra $k\langle X \rangle$.

Definition 2.1 [13] Let $G$ be a group, $A$, $B$ the subgroups of $G$ with $\phi : A \rightarrow B$ an isomorphism. Let

$$H = gp(G, t) | t^{-1}at = b, \ a \in A, \ b = \phi(a)),$$

Then $H$ is called an HNN-extension of $G$ relative to $A$, $B$ and $\phi$.

Definition 2.2 [18, 19, 20] Let $G$ be a group, $t$ a letter, $A_i$, $B_i \in G, \ i \in I$. Let

$$H = gp(G, t) | A_it = tB_i, \ i \in I).$$

Then $H$ is called a group with the stable letter $t$ and the base group $G$.

Generally, we may use groups with (many) stable letters $T = \{t\}$.

Remark: Let $H$ be in Definition 2.2. P. S. Novikov ([18, 19, 20]) called the letter $t$ to be regular if the subgroup $gp(A_i | i \in I)$, $gp(B_i | i \in I)$ of $G$ are isomorphic by $\varphi : A_i \mapsto B_i, \ i \in I$. Thus, Novikov’s group $G$ with a regular stable letter $t$ and the base $G$ is exactly an HNN-extension of $G$.

Define the corresponding words relative to a stable letter $t$ by the above relations:

$$A(t) = A_{i_1}^{\pm 1} \cdots A_{i_k}^{\pm 1}, \ B(t) = B_{i_1}^{\pm 1} \cdots B_{i_k}^{\pm 1}.$$ 

Moreover, for convenience, we put $A(t^{-1}) = B(t)$ and $B(t^{-1}) = A(t)$. Then, it is clear that for any $A(t^\varepsilon) \in G$, $A(t^\varepsilon)t^\varepsilon = t^\varepsilon B(t^\varepsilon)$, where $\varepsilon = \pm 1$.

Let $G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_n$ be a tower of groups, where $G_{i+1}$ is a group with some stable letters and the base group $G_i$ for each $i$. We call such a tower a Novikov tower. Moreover, if each $G_{i+1}$ is an HNN-extension of $G_i$, then we call this tower a tower of HNN-extensions or B-tower (Britton tower, see [7]).

Let $G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_n$ be a Novikov tower with $G_0$ free. If $p$ is a stable letter of $G_{i+1}$, we say the weight of $p$ to be $i + 1$. Taking an arbitrary relation $Ap = pB \ (A, B \in G_i)$ from $G_{i+1}$, we can represent it as follows:

$$A'xA''p = pB'yB'',$$

where $x, \ y$ are some stable letters of the highest weight in the words $A$ and $B$, respectively. We call $x, \ y$ to be distinguishing letters of the relation $Ap = pB$. We associate four types of forbidden subwords in $G_{i+1}$ (see [5], §6.4):

$$xB(x)A''p, \ x^{-1}B(x^{-1})A'^{-1}p, \ yB(y)B''p^{-1}, \ y^{-1}B(y^{-1})B'\ p^{-1}$$

Define the set $C_i$ of words in $G_i$ (0 ≤ $i$ ≤ $n$) as follows (see [6, 11, 12]).
(i) \( C_0 \) is the set of all freely reduced words.

(ii) Suppose that \( C_i \) is defined and the algorithm of reducing a word \( u \in G_i \) to canonical form \( C(u) \) has been defined. For any \( w \in C_{i+1} \), \( w = w_0p_1^{j_1} \cdots p_m^{j_m}w_m \) is freely reduced, where \( m \geq 0 \), \( \varepsilon_j = \pm 1 \), \( w_0, w_j \in C_i \), \( p_i \) is stable letter of weight \( i + 1 \), \( j = 1, 2, \ldots, m \), and \( w \) does not contain the subword as in \( (1) \), to be more precise, the subwords \( xc(B(x)A^n)p \) and so on related to \( (1) \).

It is clear that \( C_0 \subset C_1 \subset \cdots \subset C_n \). The elements of \( C_i \) will be called canonical forms.

We describe the algorithm of reducing a word \( w = w_0p_1^{j_1} \cdots p_m^{j_m}w_m \) to go to canonical form by induction on \( i \) (see \( [6] \)):

(a) Reduce the word \( w_j \) to canonical form in \( G_i \).

(b) Perform all the possible cancelations of letters of weight \( i + 1 \).

(c) Distinguish one of the occurrences of forbidden subwords \( (1) \) which are related to the stable letter \( p = p_i \), where \( i, 1 \leq i \leq m \), is the minimal index, and eliminate this subwords by the following rules:

\[
\begin{align*}
xB(x)A''p &= A(x)A^{-1}pB, \\
x^{-1}A(x)A'^{-1}p &= B(x)A''pB^{-1}, \\
yB(y)B''p^{-1} &= A(y)B^{-1}p^{-1}A, \\
y^{-1}A(y)B'^{-1}p^{-1} &= B(y)B''p^{-1}A^{-1}
\end{align*}
\]

(2)

where \( A(x), B(x), A(y), B(y) \) are corresponding words (more precisely, by \( xC(B(x)A^n)p = A(x)A^{-1}pB \) and so on).

(d) Return to step (a).

**Definition 2.3** \([6]\) The group \( G_n \) is called a group with standard normal form if for any \( w \in G_i \) \((0 \leq i \leq n)\), \( w \) can be reducible to canonical form in a finite steps and the canonical form of \( w \) is unique.

Now we cite the definition of the Gröbner-Shirshov basis for the associative algebra \( k\langle X \rangle \) (see \( [21, 3, 4] \)).

Let \( X \) be a set, \( X^* \) the free monoid generated by \( X \). We denote the empty word by \( 1 \), and the length of a word \( u \) by \( l(u) \). In general, we set \( \text{deg}(u) = l(u) \).

A well order \( < \) on \( X^* \) is monomial if it is compatible with the multiplication of words, that is, for \( u, v \in X^* \), we have

\[ u > v \implies w_1uw_2 > w_1vw_2, \text{ for all } w_1, w_2 \in X^* \]

A standard example of monomial order on \( X^* \) is the deg-lex order to compare two words first by degree and then lexicographically, where \( X \) is a linearly ordered set.

Suppose that \( X^* \) equipped with a monomial order. Let \( f \) and \( g \) be two monic polynomials in \( k\langle X \rangle \). We denote by \( \hat{f} \) the leading word of \( f \). Then, there are two kinds of compositions:

(i) If \( w \) is a word such that \( w = \hat{f}b = a\bar{g} \) for some \( a, b \in X^* \) with \( \text{deg}(\hat{f}) + \text{deg}(\bar{g}) > \text{deg}(w) \), then the polynomial \( (f, g)_w = fb - ag \) is called the intersection composition of \( f \) and \( g \) with respect to \( w \).
(ii) If \( w = f = agb \) for some \( a, b \in X^* \), then the polynomial \((f, g)_w = f - agb\) is called the inclusion composition of \( f \) and \( g \) with respect to \( w \).

In the above case, the transformation \( f \mapsto (f, g)_w = f - agb \) is called the elimination of the leading word (ELW) of \( g \) in \( f \).

Let \( S \subset k\langle X \rangle \) be monic. Then the composition \((f, g)_w = \sum \alpha_i a_i s_i b_i \) where each \( \alpha_i \in k \), \( a_i, b_i \in X^* \), \( s_i \in S \) and \( a_i s_i b_i < w \). If this is the case, then we write

\[
(f, g)_w \equiv 0 \pmod{(S, w)}.
\]

In general, for \( p, q \in k\langle X \rangle \), we write \( p \equiv q \pmod{(S, w)} \) which means that \( p - q = \sum \alpha_i a_i s_i b_i \) where each \( \alpha_i \in k \), \( a_i, b_i \in X^* \), \( s_i \in S \) and \( a_i s_i b_i < w \).

We call the set \( S \) with respect to the monomial order < a Gröbner-Shirshov basis in \( k\langle X \rangle \) if any composition of polynomials in \( S \) is trivial modulo \( S \).

The following lemma was proved by Shirshov [21] for the free Lie algebras (with deg-lex ordering) in 1962 (see also Bokut [3]). In 1976, Bokut [4] specialized the approach of Shirshov to associative algebras (see also Bergman [1]). For commutative polynomials, this lemma is known as the Buchberger’s Theorem (see [8]), published in [9].

**Lemma 2.4** (4, 5, Composition-Diamond Lemma) Let \( S \subset k\langle X \rangle \) be a non-empty set with each \( s \in S \) monic and < a monomial order on \( X^* \). Then the following statements are equivalent:

(i) \( S \) is a Gröbner-Shirshov basis in \( k\langle X \rangle \).

(ii) \( f \in Id(S) \Rightarrow \overline{f} = asb \) for some \( a, b \in X^* \), \( s \in S \).

(iii) \( \text{Irr}(S) = \{ w \in X^* | w \neq asb, a, b \in X^*, s \in S \} \) is a \( k \)-linear basis for the factor algebra \( k\langle X|S \rangle \).

We now give the definition of the standard Gröbner-Shirshov basis which is associated with the standard normal form on a group (see [4]).

**Definition 2.5** Let \( X = Y \cup Z \), words \( Y^* \) and the letters \( Z \) be well ordered. Suppose that the order on \( Y^* \) is monomial. Then, any word in \( X \) has the form \( u = u_0z_1 \cdots z_k u_k \), where \( k \geq 0 \), \( z_i \in Z \), \( u_i \in Y^* \). Define the weight of the word \( u \in X^* \) by

\[
wt(u) = (k, z_1, \cdots, z_k, u_0, \cdots, u_k).
\]

We order \( X^* \) as follows.

\[
u > v \iff wt(u) > wt(v).
\]

Then we call the above order the first tower order. Clearly, this order is a monomial order on \( X^* \).

**Definition 2.6** Let \( X = Y \cup \{t^{\pm 1}\} \), words \( Y^* \) and \( \{t^{\pm 1}\} \) be well ordered. Suppose that the order on \( Y^* \) is monomial. Then, any word in \( X \) has the form \( u = u_0 t_1^{e_1} \cdots t_k^{e_k} u_k \), where \( k \geq 0 \), \( u_i \in Y^* \), \( t_i^{e_i} \in \{t^{\pm 1}\} \), \( e_i = \pm 1 \). Define the weight of the word \( u \in X^* \) by

\[
wt(u) = (k_1, k_2, t_1^{e_1}, \cdots, t_k^{e_k}, u_0, \cdots, u_k),
\]
where $k_1$ and $k_2$ are the numbers of the occurrence of $t_i^{-1}$ and $t_i$ in $u$, respectively. Now we order $X^*$ as follows.

$$u > v \iff wt(u) > wt(v).$$

Then we call the above order the second tower order. Clearly, this order is also a monomial order on $X^*$.

In case $Y = T \cup U$ and $Y^*$ are endowed with one of the above tower orders, we call the order of words in $X$ the tower order of words relative to the presentation

$$X = (T \cup U) \cup Z.$$

In general, we can define the tower order of $X$-words relative to the presentation

$$X = (\cdots (X^{(n)} \cup X^{(n-1)} \cup \cdots) \cup X^{(0)}),$$

where $X^{(n)}$-words are endowed by a monomial order.

Definition 2.7 Let $G_0 \to G_1 \to \cdots \to G_n$ be a Novikov tower as above. If relations (2) together with trivial relations constitute a Gröbner-Shirshov basis for the group $G_n$ relative to the tower order, then we call $G_n$ to be a group with the standard Gröbner-Shirshov basis.

3 Main result

In this section, we will deal with two-generator one-relator groups with depth $k = 1, 2, 3$, respectively and give some examples. Using Magnus’ method, we show that any two-generator one-relator group with the depth $\leq 3$ is effectively embedded into a Novikov tower. Moreover, each group of this tower has effective standard Gröbner-Shirshov basis. Then, from Composition-Diamond lemma, it follows that each group of this tower has the effective standard normal form. Also, it follows that this Novikov tower is in fact a tower of HNN-extensions.

The following is the main result in this paper.

Theorem 3.1 Any two-generator one-relator group with the depth $\leq 3$ is effectively Magnus embeddable into a tower of HNN-extensions in which each group has the effective standard normal form.

We prove Theorem 3.1 step by step.

3.1 $k = 1$

Let

$$G = gp\langle x, y|x^n y^m = 1 \rangle$$

be a one-relator group with depth 1. We can suppose that $n_1 > 0$. Let

$$C = gp\langle a, b|\langle ab^{-m_1} \rangle^{n_1} b^{n_1 m_1} = 1 \rangle$$
and define a map 
\[ \Psi : G \to C, \quad x \mapsto ab^{-m_1}, \quad y \mapsto b^{n_1}. \]
Then, \( \Psi \) can be extended as a group homomorphism and hence, \( G \) can be embed-
ded into \( C \). For \( i \) an integer, we put \( a_i = b^i ab^{-i} \) and rewrite the defining relation
as \( a_0 a_{-m_1} \cdots a_{-(n_1 - 1)m_1} = 1 \). Let \( A = \{0, -m_1, \ldots, -(n_1 - 1)m_1\} \), \( l = \min A \) and
\( k = \max A \).

If \( m_1 > 0 \), then \( l = -(n_1 - 1)m_1 \) and \( k = 0 \). Let
\[
G_1 = \text{gp}(a_i \mid (l < i \leq 0) \cup \varnothing), \\
G_2 = \text{gp}(G_1, b) a_{i+1} b = b a_i \quad (l + 1 \leq i \leq -1), \quad a_{l+1} b = b(a_0 a_{-m_1} \cdots a_{-(n_1 - 2)m_1})^{-1}).
\]
Then, \( G_2 \) is a group with stable letter \( b \) and the base \( G_1 \), and \( C \cong G_2 \) by \( a \mapsto a_0, \ b \mapsto b \).
This means that \( G_1 \leq G_2 \) is a Novikov tower. Distinguishing letters of defining relations
of \( G_2 \) can be uniquely defined in all cases except for the last relation. In the last relation,
we define \( a_0^{-1}, \ a_{l+1} \) to be distinguishing letters.

Now we can get the forbidden subwords of \( G_2 \) as follows:
\[
a_i^\varepsilon b^\delta \quad (l + 1 \leq i \leq 0), \quad a_i^\varepsilon a_i^{-\varepsilon}, \quad b^\delta b^{-\delta},
\]
where \( \varepsilon, \ \delta = \pm 1 \).

Also, we can get the standard rules which would be used to obtain a Gröbner-Shirshov
basis of \( G_2 \) (see the following relations 3.1-3.2).

Let \( X = \{a_i^{\pm 1} \mid (l < i \leq 0) \cup \{b, b^{-1}\} \}, \quad a_{i+1} > a_{i+1}^{-1} > a_{i+2} > \cdots > a_0 > a_0^{-1} \) and \( b^{-1} > b \).
Then we define the first tower order on \( X^* \) as Definition 2.5

At the end, in \( G_2 \), we have the following standard relations:

1. \( a_i^\varepsilon b^{-1} = b^{-1} a_i^{\varepsilon+1}, \quad a_i b = b a_i^\varepsilon. \)
2. \( a_0^\varepsilon b^{-1} = b^{-1} (a_{-m_1+1} \cdots a_{l+1})^{-\varepsilon}, \quad a_{l+1}^\varepsilon b = b(a_0 a_{-m_1} \cdots a_{-(n_1 - 2)m_1})^{-\varepsilon}. \)

Let \( S \) consist of the above relations and the trivial relations in \( G_2 \). It is easy to check that
all compositions in \( S \) are trivial. Thus, with the tower order as above, \( S \) is an effective
standard Gröbner-Shirshov basis. By Lemma 2.4

\[
\text{Irr} (S) = \{ b^n a_{i_1} \cdots a_{i_m} \mid n \in \mathbb{Z}, \ m \geq 0, \quad a_{i_j} \in \{a_i^{\pm 1} \mid (l + 1 \leq i \leq 0)\}, \quad a_{i_j} a_{i_j+1} \neq a_{i_j} a_{i_j}^{-1}, \ \text{and} \ \varepsilon = \pm 1 \}
\]
is an effective \( k \)-basis of the algebra \( kG_2 = k\langle X | S \rangle \). Since the canonical forms of \( G_2 \) is
\( \text{Irr} (S), \ G_2 \) is a group with the effective standard Gröbner-Shirshov basis and the effective
standard normal form.

If \( m_1 < 0 \), then \( k = -(n_1 - 1)m_1 \) and \( l = 0 \). We can use the same method to get the
effective standard normal form of \( G_2 \).

Example 3.2 Let \( G = \text{gp}(x, y | x^2 y^2 = 1) \) and \( C = \text{gp}(a, b | (ab^{-1})^2 b^{-1} = 1) \). Clearly,
\( C \cong G \) by \( x \mapsto a b^{-1}, \ y \mapsto b \). Let \( a_i = b^i ab^{-i}, \ i = 1, 2, \cdots, \ H = \text{gp}(a_0 | \varnothing) \) and \( G_1 = \text{gp}(H, b | a_0 b = b a_0^{-1}). \) Then \( H \leq G_1 \) is a Novikov tower and \( C \cong G_1 \) by \( a \mapsto a_0, \ b \mapsto b \).
The forbidden subwords for group \( G_1 \) are as follows:
\[
a_0^\varepsilon b^\delta, \quad a_0 a_0^{-\varepsilon}, \quad b^\delta b^{-\delta}, \quad (\varepsilon, \ \delta = \pm 1).
\]
Thus, the effective standard normal words of $G_1$ is \( \{b^n a_0^m | n, m \in \mathbb{Z} \} \) and $x = a_0 b^{-1}$, $y = b$. Hence, the effective standard normal words of $G$ is \( \{y^n(x y)^m, y^n(x^{-1} y^{-1})^m x^{-1} | n \in \mathbb{Z}, m \geq 0 \} \).

**Example 3.3** Let $G = gp(x, y | x^2 y^3 = 1)$ and $C = gp(a, b | (ab^{-1})^2 b^6 = 1)$. Clearly, $G \hookrightarrow C$ by $x \mapsto ab^{-1}$, $y \mapsto b^2$ is a monomorphism. Let $a_i = b^i a b^{-i}$, $i = 1, 2, \ldots$, $H = gp(a_i, \, -2 \leq i \leq 0 | \emptyset)$ and $G_1 = gp(H, b | a_0 b = b a_{-1}, a_{-1} b = b a_{-2}, a_{-2} b = b a_0^{-1})$. Then $H \leq G_1$ is a Novikov tower and $C \cong G_1$ by $a \mapsto a_0$, $b \mapsto b$. The forbidden subwords for group $G_1$ are as follows:

\[ a_i^\varepsilon b^\delta, \quad a_i^\varepsilon a_i^{-\varepsilon}, \quad b^\delta b^{-\delta} \quad (-2 \leq i \leq 0, \, \varepsilon, \, \delta = \pm 1). \]

Thus, the effective standard normal words of $G_1$ is \( \{b^n a_1^\varepsilon \cdots a_{m}^\varepsilon | n \in \mathbb{Z}, \, -2 \leq i_j \leq 0, \, \varepsilon_j = \pm 1, \, \text{and} \, a_i^\varepsilon a_{i+1}^\varepsilon \neq a_i^\varepsilon a_i^{-\varepsilon} \} \).

### 3.2 $k = 2$

Let

\[ G = gp(x, y | x^{n_1} y^{m_1} x^{n_2} y^{m_2} = 1) \]

be a one-relator group with depth 2. Suppose that $n_1, \, m_i > 0$. There are two cases to consider:

**Case 1.** $n_1 + n_2 = 0$.

Let $y_i = x^i y x^{-i}$ and rewrite the defining relation as $y_i^{m_1} y_0^{m_2} = 1$. Then we have

\[
H_0 = gp(y_i \mid 0 \leq i \leq n_1, \, y_i^{m_1} y_0^{m_2} = 1), \quad H_1 = gp(H_0, \, x \mid y_{i+1} x = xy_i \mid 0 \leq i \leq n_1 - 1),
\]

where $H_1 \cong G$ and $H_0 \leq H_1$ is a Novikov tower. Since $n_1, \, m_1, \, m_2 \neq 0$, similar to the case of depth 1, there exists a Novikov tower $G_0 \leq G_1$ such that $H_0$ can be effectively embedded into $G_1$, where

\[
G_0 = gp(y_i \mid 0 < i < n_1, \, a_j \mid l < j \leq 0 | \emptyset), \quad G_1 = gp(G_0, b \mid a_{j+1} b = b a_j \mid l \leq j < 0, \, a_{j+1} b = b(a_0 a_{m_2} \cdots a_{-(m_1 - 2)m_2}^{-1})^{-1}),
\]

and $l = -(m_1 - 1)m_2$. It is clear that $G_0 \leq G_1 \leq G_2$ is a Novikov tower and $H_1$ can be embedded into $G_2$. The distinguishing letters of defining relations of $G_2$ can uniquely defined in all cases except for the last two relations. Here, we define $y_1$ and the first $b$ the distinguishing letters in $y_1 x = x b^{m_1}$, and $y_{n_1 - 1}$ and the last $b^{-1}$ the distinguishing letters in $a_0 b^{-m_2} x = x y_{n_1 - 1}$. Clearly, $G$ can be embedded into $G_2$. Now we prove that $G_2$ is a group with effective standard normal form. For $G_1$ and $G_2$, we have the forbidden subwords:

\[
G_1: \quad a_j^\varepsilon b^\delta, \quad a_j^\varepsilon a_j^{-\varepsilon}, \quad b^\delta b^{-\delta}, \quad y_i^\varepsilon y_i^{-\varepsilon}
\]

\[
G_2: \quad y_i^\varepsilon x^\delta, \quad b^{m_1} V(a_j) x^{-1}, \quad b^{m_2} V(a_j) a_0^{-1} x, \quad b^{-1} V(a_j) x^\varepsilon, \quad x^\varepsilon x^{-\varepsilon}
\]
where \( l < j \leq 0, 1 \leq i \leq n_1 - 1, \varepsilon, \delta = \pm 1 \) and \( V(X) \) means any group word which is generated by \( X \). Similar to the case of depth 1, \( G_1 \) is a group with effective standard normal form.

Let \( X = \{ y_i^{\pm 1}, a_j^{\pm 1} \} \cup \{ b^{\pm 1} \} \cup \{ x^{\pm 1} \} \). Then we define a tower order on \( X^* \). We first define deg-lex order on \( \{ y_i^{\pm 1}, a_j^{\pm 1} \}^* \) and the second tower order on \( \{ y_i^{\pm 1}, a_j^{\pm 1} \}^* \cup \{ b^{\pm 1} \}^* \), and then, the first tower order on \( X^* \). Clearly, this order is a monomial order on \( X^* \).

In \( G_2 \), we have the following standard relations:

\[(3.3)\] \( y_i^j x^{-1} = x^{-1} y_i^{j+1} \), \( y_i^j x = x y_i^{j+1} \), \( (1 \leq i \leq n_1 - 2) \),

\[(3.4)\] \( y_i^j x = x (b^m y_i)^\varepsilon \), \( y_{n_1-1}^j x^{-1} = x^{-1} (a_0 b^{-m})^\varepsilon \),

\[(3.5)\] \( b^m V(a_j) a_0^{-1} x = V(m_2) (a_j) x y_{n_1-1}^{-1} \), \( b^{-1} V(a_j) x = V(-1)(a_j) b^{m_2-1} a_0^{-1} x y_{n_1-1} \),

\[(3.6)\] \( b^m V(a_j) x^{-1} = V(m_1)(a_j) x^{-1} y_1 \), \( b^{-1} V(a_j) x^{-1} = V(-1)(a_j) b^{m_1-1} a_0^{-1} x^{-1} y_1 \),

where \( V(+1) \) is the result of shifting in \( V \) all indices of all letters with \( a_j^\varepsilon \mapsto a_j^{\varepsilon+1} (l + 1 \leq j \leq n_1 - 1) \), \( a_0^\varepsilon \mapsto (a_{-m_2+1} \cdots a_{1-1})^{-\varepsilon} \), and \( V(-1) \) with \( a_j^{\varepsilon+1} \mapsto a_j^\varepsilon (l + 2 \leq j \leq n_1 - 1) \), \( a_{l+1} \mapsto (a_{-m_2} \cdots a_{-m_2+1})^{-\varepsilon} \). It is clear that \( u \in C(V(a_j)) \) (the canonical form of \( V(a_j) \)) if and only if \( u \) is a freely reduced word on \( a_j \).

Let \( S \) consist of relations (3.1)–(3.6) and the trivial relations in \( G_2 \), where we substitute the index set \( \{ -(n_1-1)m_1 \leq i \leq 0 \} \) with \( \{ -(m_1-1)m_2 \leq i \leq 0 \} \) in the relation (3.1)–(3.2). Clearly, with the tower order as above, \( S \) is an effective standard Gröbner-Shirshov basis. By Lemma 4.2, \( IRR(S) \) is an effective \( k \)-basis of the algebra \( k G_2 = \mathbb{R}[X] \).

Since the canonical forms of \( G_2 \) is \( IRR(S) \), \( G_2 \) is a group with the effective standard Gröbner-Shirshov basis and the effective standard normal form.

Case 2. \( 0 \neq n_1 + n_2 = \alpha, 0 \neq m_1 + m_2 = \beta \).

Let

\[ C = gp(a, b)(ab^{-\beta})^{n_1} b^{\alpha m_1}(ab^{-\beta})^{n_2} b^{\alpha m_2} = 1. \]

Then \( G \) can be embedded into \( C \) by \( x \mapsto ab^{-\beta}, y \mapsto b^\alpha \). In \( C \), the exponent sum of \( b \) occurred in the defining relation is 0. Let \( a_i = b^i a^{-i} \).

If \( n_2 > 0 \), we rewrite the defining relation as

\[ r = a_0 a_{-\beta} \cdots a_{-(n_1-1)\beta} a_{-n_1 \beta + n_1 \alpha} \cdots a_{-(n_1+n_2-1)\beta + n_1 \alpha} = 1. \]

Let \( A = \{ 0, -\beta, \cdots, -(n_1-1)\beta \}, B = \{ -(n_1 \beta + n_1 \alpha), \cdots, -(n_1 + n_2 - 1)\beta + n_1 \alpha \} \), \( l = \min \{ A, B \} \) and \( k = \max \{ A, B \} \). Then

\[ G_1 = gp(b, a_i (l \leq i \leq k) \mid r = 1, a_{i+1} b = b a_i (l \leq i \leq k-1)) \]

and \( C \cong G_1 \). Let \( G_2 = gp(a_i (l \leq i \leq k) \mid r = 1) \). Then \( G_2 \cong G_1 \) is a Novikov tower.

If \( A \neq B \), there are seven cases to consider:

\[ r = a_0 a_{-\beta} \cdots (a_{-n_2 \beta} \cdots a_{-(n_1-1)\beta})^2 \cdots a_{-(n_1+n_2-1)\beta + n_1 \alpha}, \]

\[ a_0 a_{-\beta} \cdots a_{-i \beta} \cdots a_{-(n_1-1)\beta} a_{-i \beta} \cdots a_{-j \beta}, \]

\[ a_0 a_{-\beta} \cdots (a_{-i \beta} \cdots a_{-(n_1-1)\beta})^2, \]

\[ a_0 a_{-\beta} \cdots a_{-i \beta} \cdots a_{-(n_1-1)\beta} a_{-n_1 \beta + n_1 \alpha} \cdots a_0 \cdots a_{-i \beta}, \]

\[ a_0 a_{-\beta} \cdots a_{-(n_1-1)\beta} a_{-n_1 \beta + n_1 \alpha} \cdots a_0 a_{-\beta} \cdots a_{-(n_1-1)\beta} \cdots a_{-(n_1+n_2-1)\beta + n_1 \alpha}, \]

\[ a_0 a_{-\beta} \cdots a_{-i \beta} \cdots a_{-(n_1-1)\beta} a_{-n_1 \beta + n_1 \alpha} \cdots a_0 \cdots a_{-i \beta}, \]

\[ a_0 a_{-\beta} \cdots a_{-(n_1-1)\beta} a_{-(n_1 \beta) + n_1 \alpha} \cdots a_{-(n_1+n_2-1)\beta + n_1 \alpha}. \]
If this is the case, it is clear that $G_2$ can be viewed as a free group. Then, we can use the same method similar to the depth 1 to get the result.

If $A = B$, then $(a_0a_{-\beta} \cdots a_{-(n_1-1)\beta})^2 = 1$. Let

\[ C_0 = gp(c, d_i (1 \leq i \leq n_1 - 1), a_j (j \notin A) | c^2 = 1), \]
\[ G_0 = gp(C, b | a_{i+1}b = b_{ai} (i, i + 1 \notin A), cd_{n_1-1} \cdots d_1^{-1}b = ba_{-1}, d_ib = ba_{-i\beta-1} (i \neq n_1 - 1), a_{-i\beta+1}b = bd_i). \]

Then $C_0 \leq G_0$ is a Novikov tower and $G_1$ can be embedded into $G_0$ by $a_0 \mapsto cd_{n_1-1}^{-1} \cdots d_1^{-1}, a_{-i\beta} \mapsto d_i$, $b \mapsto b$. Here, we define $c$ and $a_{-1}$ the distinguishing letters in $cd_{n_1-1} \cdots d_1^{-1}b = ba_{-1}$.

Hence, $G$ can be embedded into $G_0$. The forbidden subwords for $G_0$ are as follows:

\[ a_1^\pm b^{-1}, a_{i+1}^\pm b (i, i + 1 \notin A), a_{-1}^\pm b^{-1}, a_{-i\beta-1}^\pm b^{-1}, d_i^\pm b_{i+1} (i \neq n_1 - 1), a_{-i\beta+1}^\pm b, cd_{n_1-1}^{-1}b, cb^{-1}. \]

Let $X = \{c, a_i^\pm, a_i^\pm | 1 \leq i \leq n_1 - 1, j \notin A\} \cup \{b^\pm\}$ and $c > d_{n_1-1}^\pm > d_{n_1-2}^\pm > \cdots$.

Then define the first tower order on $X^*$. In $G_0$, we have the following standard relations:

\[(3.7)\] $a_i^\pm b^{-1} = b^{-1}a_i^\pm$, $a_i^\pm b = ba_i^\pm$, $i, i + 1 \notin A)$, \(a_{-1}^\pm b^{-1} = b^{-1}(cd_{n_1-1} \cdots d_1^{-1})^c), \]
\[(3.8)\] $c^2 = 1$, $cd_{n_1-1}^{-1}b = b(a_{-1} \cdots a_{-(n_1-2)\beta-1})^{-1}$, $cb = d_{n_1-1}^{-1}b(a_{-1} \cdots a_{-(n_1-2)\beta-1})^{-1}$,
\[(3.9)\] $a_{-i\beta-1}^\pm b^{-1} = b^{-1}a_{-i\beta-1}^\pm$, $d_i^\pm b = ba_{-i\beta-1}^\pm (i \neq n_1 - 1)$, $a_{-i\beta+1}^\pm b = bd_i^\pm$, $d_i^\pm b^{-1} = b^{-1}a_{-i\beta+1}^\pm$.

Let $S$ consist of relations (3.7) – (3.9) and the trivial relations in $G_0$. Clearly, with the tower order, $S$ is an effective standard Gröbner-Shirshov basis. By Lemma 2.4, $Irr(S)$ is an effective $k$-basis of the algebra $kG_0 = k\langle X \rangle$. Since the canonical forms of $G_0$ is $Irr(S)$, $G_0$ is a group with the effective standard Gröbner-Shirshov basis and the effective standard normal form.

If $n_2 < 0$, we use the same method as above.

**Remark:** From the above proof, we know that if there exists an $a_i$ such that $a_i$ occurs in the relation $r$ only once, then we can get a Novikov tower of groups with the first group free.

### 3.3 $k = 3$

Let

\[ G = gp(x, y | x^{n_1}y^{m_1}x^{n_2}y^{m_2}x^{n_3}y^{m_3} = 1) \]

be a one-relator group with depth 3. Suppose that $n_1 > 0$. There are two cases to consider:

Case 1. $n_1 + n_2 + n_3 = 0$.

Let $y_i = x^iyx^{-i}$ and rewrite the defining relation as $y^{m_1}y^{m_2}y^{m_3} = 1$. Let $D = \{0, n_1, n_1 + n_2\}$ and suppose that $l = \min D = 0$, $k = \max D = n_1 + n_2$. By Magnus’ method, we can get a Novikov tower of groups:

\[ H_1 = gp(y_i | l \leq i \leq k - 1), \] $a_j (-m_2 - 1)m_3 < j \leq l) | \emptyset\]
\[ H_2 = gp(H_1, b | ay_{j+1}b = ba_j (j \neq 0), a_{-m_2 - 1)m_3+1}b = b(y^{m_1}a_0 \cdots a_{-m_2 - 2)m_3}^{-1}), \]
\[ H_3 = gp(H_2, x | y_{i+1}x = xy_i (i \neq k - 1), a_0b^{-m_3}x = xy_{k-1}, y_1x = x_2b^{m_2}). \]
where $G$ can be embedded into $H_3$ by $x \mapsto x$, $y \mapsto y_0$. Here, in $H_2$, we define $a_{-(m_2-1)m_3+1}$ and $a_0^{-1}$ the distinguishing letters in $a_{-(m_2-1)m_3+1}b = b(y_n^{m_1}a_0 \cdots a_{-(m_2-2)m_3})^{-1}$; in $H_3$, $y_{k-1}$ and the last $b^{-1}$ the distinguishing letters in $a_0 b^{-m_3}x = xy_{k-1}$, and $y_1$ and the first $b$ the distinguishing letters in $y_1 x = xb^{m_2}$. The forbidden subwords for the above groups $H_2, H_3$ are as follows:

$$H_2: a_j^\pm b^{-1}, a_{j+1}^\pm b \ (j \neq 0), a_{(m_2-1)m_3+1}^\pm b, a_0 y_{-m_1}^{-1}b^{-1}, a_0 b^{-1},$$

$$H_3: b^{-1}V(a_j)x^\pm -1 (-m_2 - 1)m_3 < j \leq 0, Vb_1(a_j, y_n^{m_1}a_0) \cdots bV_m(a_j, y_n^{m_1}a_0)a_0^{-1}x, bV_1(a_j, y_n^{m_1}a_0) \cdots bV_m(a_j, y_n^{m_1}a_0)x^{-1} (j \neq 0)$$

where for any $1 \leq s \leq m_2, m_3$, $V_s^{(s)}$ exists (see following). Similar to the depth 1, $H_2$ is a group with the effective standard normal form. Let $X = ((\{y_i^{\pm 1} | l \leq i \leq k - 1\} \cup \{a_j^{\pm 1} | \ - (m_2 - 1)m_3 < j \leq 0\}) \cup \{b^{\pm 1}\}) \cup \{x^{\pm 1}\}$. Define the first tower order on $((\{y_i^{\pm 1} | l \leq i \leq k - 1\} \cup \{a_j^{\pm 1} | \ - (m_2 - 1)m_3 < j \leq 0\})\},$ the second tower order on $((\{y_i^{\pm 1} | l \leq i \leq k - 1\} \cup \{a_j^{\pm 1} | \ - (m_2 - 1)m_3 < j \leq 0\})\}$ and then the first tower order on $X^*$. In $H_3$, we have the following standard relations:

$$y_i^j x^{-1} = x^{-1}y_i^{j+1}, y_i^{j+1}x = xy_i^{j}, (1 \leq i \leq k - 2),$$

$$y_i^j x = x(b^{m_2})^\varepsilon, y_i^{j-1}x^{\varepsilon} = x^{-1}(a_0 b^{-m_3})^\varepsilon, a_0 y_{n_1}^{-1}a_0^{-1}b^{-1} = b^{-1}(a_{-m_3+1} \cdots a_{-(m_2-1)m_3+1}),$$

$$b^{-1}V(a_j)x = V(\varepsilon)(a_{j+1})b^{m_3-1}a_0^{-1}x y_{k-1},$$

$$b^{-1}V(a_j+1)x^{-1} = V^{-1}(a_{j+1})b^{m_3-1}a_0^{-1}x^{-1}y_1^{-1},$$

$$bV_1(a_j, y_n^{m_1}a_0) \cdots bV_m(a_j, y_n^{m_1}a_0)a_0^{-1}x = V_1^{(1)}(a_j, y_n^{m_1}a_0) \cdots V_m^{(m)}(a_j, y_n^{m_1}a_0)xy_{k-1}^{-1},$$

$$bV_1(a_j, y_n^{m_1}a_0) \cdots bV_m(a_j, y_n^{m_1}a_0)x^{-1} = V_1^{(1)}(a_j, y_n^{m_1}a_0) \cdots V_m^{(m)}(a_j, y_n^{m_1}a_0)x^{-1}y_1,$$

$$a_0^\varepsilon b^{-1} = b^{-1}a_0^\varepsilon, a_{j+1}^\varepsilon b = ba_j^\varepsilon, (j \neq 0), a_0 b^{-1} = y_{n_1}^{-1}a_0^{-1}(a_{-m_3+1} \cdots a_{-(m_2-1)m_3+1})^{-\varepsilon},$$

$$a_{-(m_2-1)m_3+1}^\varepsilon b = b(y_n^{m_1}a_0 \cdots a_{-(m_2-2)m_3})^{-\varepsilon},$$

where the relations (3.14) – (3.15) hold only if $V_1^{(s)}$ exists. Here, $V(\varepsilon)$ is the result of shifting in $V$ all indices of all letters with $a_j^\varepsilon \mapsto a_{j+1}^\varepsilon (j \neq 0), y_n^{m_1}a_0 \mapsto (a_{-(m_3-1)m_3+1})^{-\varepsilon}$ and $V^{-1}$ with $a_j^\varepsilon \mapsto a_{j+1}^\varepsilon (j \neq -(m_2 - 1)m_3 + 1), a_{(m_2-1)m_3+1}^\varepsilon \mapsto (y_{n_1}^{-1}a_0 a_{-m_3} \cdots a_{-(m_2-2)m_3})^{-\varepsilon}$. Since $\{a_{j+1}\}$ ($\{a_j, y_n^{m_1}a_0\}$) freely generate a subgroup of $H_1$, $C(V(a_{j+1}))$ ($C(V(a_j, y_n^{m_1}a_0)$) is the freely reduced word on $\{a_{j+1}\}$ ($\{a_j, y_n^{m_1}a_0\}$).

Let $S$ consist of relations (3.10) – (3.17) and the trivial relations in $H_3$. Clearly, with the tower order mentioned as above, $S$ is an effective standard Gröbner-Shirshov basis. By Lemma 2.2.1 $\text{Irr}(S)$ is an effective $k$-basis of the algebra $kH_3 = k\langle X | S \rangle$. Since the canonical forms of $H_3$ is $\text{Irr}(S)$, $H_3$ is a group with the effective standard Gröbner-Shirshov basis and the effective standard normal form.

Case 2. $0 \neq n_1 + n_2 + n_3 = \alpha, 0 \neq m_1 + m_2 + m_3 = \beta$.

We may assume that all the powers are positive. For other cases, we use a similar way to prove the result. Let

$$G_1 = gp(a, b | (ab^{-\beta})^{n_1}b^{\alpha_m}(ab^{-\beta})^{n_2}b^{\alpha_m}(ab^{-\beta})^{n_3}b^{\alpha_m} = 1),$$

$$G_2 = gp(b, a_i (l \leq i \leq k) | r = 1, a_{i+1}b = ba_i (l \leq i < k)).$$
where \( l = \min\{A, B, C\} \), \( k = \max\{A, B, C\} \),

\[
\begin{align*}
r &= a_0 a_{-\beta} \cdots a_{-(n_1-1)\beta} a_{-\beta} a_{-(n_1+1)\beta} m_1 \cdots a_{-(n_1+n_2)\beta} a_{-\beta} a_{-(n_1+1)\beta} \alpha \cdot \\
A &= \{0, -\beta, \ldots, -(n_1 - 1)\beta\}, \\
B &= \{-n_1 \beta + m_1 \alpha, \ldots, -(n_1 + n_2 - 1) \beta + m_1 \alpha\}, \\
C &= \{-n_1 + n_2\beta + (m_1 + m_2)\alpha, \ldots, -(n_1 + n_2 + n_3 - 1 \beta + (m_1 + m_2)\alpha\}.
\end{align*}
\]

Then \( G_1 \cong G_2 \). Let \( H_1 = gp(a_i (l \leq i \leq k) \mid r = 1) \), \( H_2 = gp([H_1, b] a_{i+1} b = b a_i (l \leq i < k)) \). It is clear that \( H_1 \preceq H_2 \) is a Novikov tower and \( G_2 \cong H_2 \).

If there exists an \( i \) such that \( i \) is only in one of the sets \( A, B, \) or \( C \), then we get the result by the Remark in 3.2.

If \( A = B \cup C \), then \( r \) has four possibilities:

\[
\begin{align*}
r &= a_0 a_{-\beta} \cdots a_{-(n_1-1)\beta} a_{-s\beta} \cdots a_{-(n_1-1)\beta} a_0 \cdots a_{-(s-1)\beta}, \\
&\quad (a_0 a_{-\beta} \cdots a_{-(n_1-1)\beta})^2, \\
&\quad a_0 a_{-\beta} \cdots a_{-(n_1-1)\beta} a_{-s\beta} \cdots a_{-(n_1-1)\beta} a_0 \cdots a_{-(t-1)\beta} (0 \leq s \leq t \leq n_1 - 1), \\
&\quad a_0 a_{-\beta} \cdots a_{-(n_1-1)\beta} a_{-s\beta} \cdots a_{-(n_1-1)\beta} (0 \leq s \leq t \leq n_1 - 1).
\end{align*}
\]

We only consider the first case. Other cases can be similarly proved.

Similar to the depth 2, we can get a Novikov tower of groups:

\[
\begin{align*}
C_0 &= gp(c, a_i (i \notin A), d_j (1 \leq j \leq n_1 - 2)|\emptyset), \\
C_1 &= gp(C_0, d cd = dc^{-1}), \\
C_2 &= gp(C_1, b) a_{i+1} b = ba_i (i, i + 1 \notin A), cd^{-1} d_{s-1} \cdots d_{1} b = ba_{-1}, \\
&\quad a_{-i\beta+1} b = bd_i, d_i b = ba_{-i\beta+1} (1 \leq i \leq s - 1), \\
&\quad a_{-j\beta+1} b = bd_{j-1} (s + 1 \leq j \leq n_1 - 1), d_{j-1} b = ba_{-j\beta-1} (s + 1 \leq j \leq n_1 - 2), \\
&\quad cd_{n-1} \cdots d_{1} b = ba_{-1} \cdots a_{-s\beta-1}, a_{-s\beta+1} b = bdd_{n-1} \cdots d_{1}.
\end{align*}
\]

where \( G_2 \cong C_2 \) by \( a_i \mapsto a_i (i \notin A), a_0 \mapsto cd^{-1} d_{s-1} \cdots d_{1}, a_{-i\beta} \mapsto d_i (1 \leq i \leq s - 1), a_{-j\beta} \mapsto d_{j-1} (s + 1 \leq j \leq n_1 - 1), a_{-s\beta} \mapsto dd_{n-1} \cdots d_{1} \). Here, in \( C_2 \), we define \( a_{-1} \) and \( d^{-1} \) in \( cd^{-1} d_{s-1} \cdots d_{1} b = ba_{-1}, a_{-s\beta} \) and \( d_{n-1} \cdots d_{1} b = ba_{-1} \cdots a_{-s\beta+1}, a_{-s\beta+1} b = bdd_{n-1} \cdots d_{1} \) the distinguishing letters. The forbidden subwords for \( C_1 \) and \( C_2 \) are:

\[
\begin{align*}
C_1 &: c^e d^f, \\
C_2 &: a_i^e b^f, d_i^e b^f (i \neq n_1 - 2), d_{n_1-2}^e d_{1}^e, d_{n_1-2}^e c^{-1} b, d c^{-1} d_{n_1-2}^e b, d^{-1} c^{-1} b, d c^{-1} b.
\end{align*}
\]

Let \( X = \{(c^e, a_i^e, d_i^e) \mid i \notin A, 1 \leq j \leq n_1 \cup \{d^e\}\} \cup \{b^e\} \) and \( d_{n-1}^e > d_{n-2}^e > \cdots > a_i > a_i^e > c > c^{-1} \). Then define the first tower order on \( X^\ast \). In \( C_2 \), we have the following standard relations:

\[
\begin{align*}
(3.18) &\quad c^e d^f = d^e c^{-1}, a_{i+1}^e b = ba_i^e, a_i^e b^{-1} = b^{-1} a_{i+1}^e (i, i + 1 \notin A), a_{i-1}^e b^{-1} = b^{-1} (cd^{-1} d_{s} \cdots d_{1})^e, \\
(3.19) &\quad d^{-1} c^m b = c^{-m} b a_{-1} \cdots a_{-(s-1)\beta} b, (dc^{-1}) c^{-1} b = c^{-m} b (a_{-1} \cdots a_{-(s-1)\beta})^{-1}, \\
(3.20) &\quad d_i^e b^{-1} = b^{-1} a_{i+1}^e, a_{i+1}^e b = bd_i^e, d_i^e b = ba_{-i-1} b^{-1} = b^{-1} d_i^e (1 \leq i \leq s - 1),
\end{align*}
\]
(3.21) \( d_{j-1}^x b^{-1} = b^{-1} a_{j-1}^x, \quad a_{j-1} b = b a_{j-1} \) (s + 1 \leq j \leq n_1 - 1),

(3.22) \( d_{j-1}^x b = b a_{j-1}^x, \quad a_{j-1} b^{-1} = b^{-1} d_{j-1}^x \) (s + 1 \leq j \leq n_1 - 2),

(3.23) \( d_{n_1-2} c^{-1} b = b(a_{-1} \cdots a_{-(n_1-2)} b^{-1})^{-1}, \quad dc^{m} d_{n_1-2} b^{-1} = c^{-m} b^{-1}(a_{-s+1} \cdots a_{-(n_1-2)} b^{-1}),

(3.24) \( d^{-1} c^{m} b^{-1} = c^{-m} d_{n_1-2} b^{-1}(a_{-s+1} \cdots a_{-(n_1-2)} b^{-1})^{-1}, \quad d_{n_1-2}^{-1} b = c^{-1} b(a_{-1} \cdots a_{-(n_1-2)} b^{-1}),

(3.25) \( a_{-s+b}^{-1} b^{-1} = b^{-1}(dd_{n_1-2}^{-1} \cdots d_{s}^{-1})^x, \quad a_{-s+b+1} b = b(dd_{n_1-2}^{-1} \cdots d_{s}^{-1})^x.

Let \( S \) consist of relations (3.18) – (3.25) and the trivial relations in \( C_2 \). Clearly, with the tower order as above, \( S \) is an effective standard Gröbner-Shirshov basis. By Lemma 2.4 \( I r r(S) \) is an effective \( k \)-basis of the algebra \( kC_2 = k\langle X | S \rangle \). Since the canonical forms of \( C_2 \) is \( I r r(S) \), \( C_2 \) is a group with the effective standard Gröbner-Shirshov basis and the effective standard normal form.

When we replace \( A \) with \( B \) or \( C \), using the same method, we can get the result.

By the above proof, each group in the Novikov tower has effective standard normal form. From this it follows that each Novikov tower is exactly a tower of HNN-extensions. Thus, the proof of Theorem [5.1] is completed.

### 4 An example

In this section, we will give an example to show how to deal with some general cases for one-relator groups.

**Example 4.1** \( G = gp(a, b, c, t_1 | (a^2 b t_1 c^{-1} b c t_1^{-1} b^{-1} a^{-3} b^{-2} t_1 c^{-1} t_1^{-1})^r = 1) \) \( (r \geq 2) \).

Let \( a_{(i)}^0 = t_1^i a_{(i)}^{-1}, \quad b_{(i)} = t_1^i b_{(i)}^{-1}, \quad c_{(i)} = t_1^i c_{(i)}^{-1} \) and rewrite the defining relation as

\[
\left( (a_{(0)}^0)^2 c_{(1)}^2 t_1^{-1} b_{(1)}^2 c_{(0)}^0 (a_{(-2)}^0)^{-3} b_{(-2)}^2 (a_{(0)}^0)^2 b_{(0)}^2 c_{(1)}^{-2} \right)^r = 1.
\]

Then, we get

\[
G_2 = gp(a_{(i)}^0, b_{(i)}, c_{(i)}, t_1 | a_{(i)}^0 t_1 = t_1 a_{(i-1)}^0, b_{(j)} t_1 = t_1 b_{(j-1)}, c_{(j)} t_1 = t_1 c_{(j)},

(2 \leq i \leq 0, -2 \leq j \leq 1) \}
\]

Clearly, \( G \cong G_1 \). Let \( a_{(0)} = a_{(0)}^{(0)}, t_2, a_{(-2)} = t_2 \) and rewrite the defining relation of \( G_2 \) as

\[
(a_{(1)}^0 t_2 a_{(0)}^{(0)} t_2 c_{(1)}^2 t_1 a_{(-2)}^0 c_{(0)}^2 t_2^{-1} b_{(-2)}^2 (a_{(0)}^0)^2 b_{(0)}^2 c_{(1)}^{-2} \right)^r = 1.
\]

Let \( a_{(0)}^{(0)} = t_2^i a_{(0)}^{(1)} t_2^{-1}, b_{(j)} = t_2^i b_{(j)} t_2^{-1}, c_{(j)} = t_2^i c_{(j)} t_2^{-1}, a_{(j)} = a_{(j)}^{(0)} \).

Let

\[
G_3 = gp(a_{(j)}^{(0)}, b_{(j)}, c_{(0)}, t_1 | a_{(j)}^{(0)} t_2 = t_2 a_{(j-1)}^{(0)}, a_{(j)}^{(0)} t_2 = t_2 a_{(j)}^{(1)}, b_{(-2)} t_2 = t_2 b_{(-2)} t_2, b_{(j)} t_2 = t_2 b_{(j)} t_2, c_{(j)} t_2 = t_2 c_{(j)} t_2, c_{(j)} t_2 = t_2 c_{(j)} t_2,

(a_{(j)}^{(0)} a_{(j)}^{(0)} c_{(j)}^2 t_1 a_{(-j)}^{(1)} b_{(-2)}^2 (a_{(0)}^{(1)})^2 b_{(0)}^2 c_{(1)}^{-2} \right)^r = 1).
\]
Let $a_{(00)}^{(01)} = a_{(00)}^{(01)} t_3^{-1}$, $a_{(01)}^{(01)} = t_3$ and rewrite the last defining relation of $G_3$ as

$$\left(a_{(000)}^{(011)} c_{(020)}^{(01)} c_{(120)}^{(1)} b_{(120)}^{(2)} b_{(2-10)}^{(0)} a_{(0-10)}^{(01)} b_{(000)}^{(0)} c_{(100)}^{(2)} \right)^r = 1.$$  

Then, just like $G_3$, we have $G_4$. Let $a_{(000)}^{(011)} = a_{(000)}^{(011)} t_4^{-1}$, $c_{(020)}^{(01)} = t_4$ and rewrite above relation as

$$\left(a_{(000)}^{(011)} t_4^{-2} c_{(120)}^{(1)} b_{(120)}^{(2)} t_4 b_{(2-10)}^{(0)} a_{(0-10)}^{(01)} b_{(000)}^{(0)} c_{(100)}^{(2)} \right)^r = 1.$$  

Let $a_{(000)}^{(011)} = t_4^i a_{(000)}^{(011)} t_4^{-i}$, $b_{(jki)} = t_4^i b_{(jki)} t_4^{-i}$, $c_{(jki)} = t_4^i c_{(jki)} t_4^{-i}$. Then, we have

$$G_5 = gp\langle X, t_4 | b_{(120)} t_4 = t_4 b_{(120)}, b_{(120)-1} t_4 = t_4 b_{(120)-2}, b_{(120)-3} t_4 = t_4 b_{(120)-4}, c_{(120)-1} t_4 = t_4 c_{(120)-2}, \rangle \left(a_{(000)}^{(011)} t_4^{-2} c_{(120)}^{(1)} b_{(120)}^{(2)} t_4 b_{(2-10)}^{(0)} a_{(0-10)}^{(01)} b_{(000)}^{(0)} c_{(100)}^{(2)} \right)^r = 1,$$

where $X = \{a_{(000)}^{(001)}, c_{(0-100)}^{(000)}, c_{(120)}^{(000)}, b_{(0120)}, b_{(10120)}, b_{(120)}, b_{(120)-1}, b_{(120)-2}, c_{(120)}, c_{(120)-1} \} \leq i \leq 0, j, k = 0, 1 \}$. Let $a_{(000)}^{(011)} = a_{(0000)}^{(002)} t_5^j$, $c_{(120)-2} = t_5$ and rewrite above relation as

$$\left(a_{(0000)}^{(0111)} b_{(120)-2}^{(0000)} b_{(2-1000)}^{(00000)} b_{(00000)}^{(00000)} c_{(10000)}^{(2)} \right)^r = 1.$$  

Let $a_{(000)}^{(014)} = a_{(000)}^{(015)} t_6^{-3}$, $b_{(120)-20} = t_6$ and rewrite above relation as

$$\left(a_{(000)}^{(015)} b_{(2-1000)}^{(0100)} t_6 b_{(000)}^{(00000)} c_{(10000)}^{(2)} \right)^r = 1,$$

where, for example, $(01^4) = (01111)$. Let $a_{(000)}^{(015)} = a_{(07^6)}^{(016)} t_7^2$, $b_{(2-1000)} = t_7$ and rewrite above relation as

$$\left(a_{(07^6)}^{(016)} t_7^2 t_6 c_{(10000)}^{(2)} \right)^r = 1.$$  

Repeating this process, we can get a Novikov tower:

$$H_1 = \text{gp}\langle X_1, a_{(001)}^{(015)} \rangle = 1,$$

$$H_2 = \text{gp}\langle H_1, t_3, t_4, t_5 | a_{(120)}^{(010)} b_{(120)} t_4 = t_4 b_{(120)}, b_{(120)-10} t_4 = t_4 b_{(120)-10}, c_{(120)} t_4 = t_4 c_{(120)-10} \rangle,$$

$$H_3 = \text{gp}\langle H_2, t_2 | b_{(2-10)} t_2 = t_2 b_{(2-10)} \rangle \left(a_{(000)}^{(010)} t_2^2 t_6 t_7 t_8 t_9 \right)^r = 1,$$

$$H_4 = \text{gp}\langle H_3, t_1 | t_1 a_{(100)}^{(001)} = a_{(100)}^{(001)} t_1 = t_1 b_{(100)} t_1 = t_1 b_{(100)} \rangle,$$

where $X_1 = \{a_{(000)}^{(010)}, a_{(100)}^{(010)}, b_{(100)}, b_{(110)}, b_{(120)}, b_{(120)-1}, b_{(120)-2}, c_{(110)}, c_{(100)}, c_{(120)}, c_{(120)-1} \}$.

The standard relations for groups $H_1, H_2, H_3, H_4$ are as follows:

$$\langle a_{(001)}^{(010)} \rangle \langle r \rightarrow i \varepsilon \rangle = \langle a_{(001)}^{(010)} \rangle \langle i \varepsilon \rangle , \quad (0 \leq i \leq [r/2]),$$

$$b_{(120)}^{(01)} t_4 = t_4 b_{(120)}, b_{(120)-10}^{(01)} t_4 = t_4 b_{(120)}, b_{(120)-10}^{(01)} t_4 = t_4 t_6^5.$$
where

\[ Y = \{ b_t^{(120-10^6)}, c_t^{(120-10^6)}, t_5, t_6 \}, \]

\[ Y' = \{ b_t^{(120-10^6)}, c_t^{(120-10^6)}, b_t^{(120)} \}, \]

\[ Y_1 = \{ t_7, t_8, t_{10}, b_t^{(110^6)}, b_t^{(10^9)}, c_t^{(110^6)}, c_t^{(10^9)}, a_t^{(10^5)} t_{10}^{t_9 - 2} t_8 t_7^2 t_6^2 t_5^4 t_3^3 c_t^{(110^6)} \}, \]

\[ Y'_1 = \{ t_3, t_4, b_t^{(-20^9)}, b_t^{(120)}, b_t^{(110^6)}, c_t^{(120)}, c_t^{(110^6)}, a_t^{(10^5)} t_{10}^{t_9 - 2} t_8 t_7^2 t_6^2 t_5^4 t_3^3 \}, \]

and \( V(Y) \leftrightarrow V'(Y') \) by \( b_t^{(120-10^6)} \leftrightarrow b_t^{(120)}, c_t^{(120-10^6)} \leftrightarrow c_t^{(120)}, t_5 \leftrightarrow c_t^{(120-10^6)}, t_6 \leftrightarrow b_t^{(120-10^6)}, V(Y_t) \leftrightarrow V'(Y'_t) \) by \( t_7 \leftrightarrow b_t^{(-20^9)}, t_8 \leftrightarrow (a_t^{(10^5)} t_{10}^{t_9 - 2} t_8 t_7^2 t_6^2 t_5^4 t_3^3 c_t^{(110^6)} - 1)^{-1}, t_{10} \leftrightarrow c_t^{(110^6)}, b_t^{(110^6)} \leftrightarrow b_t^{(120)}, b_t^{(10^9)} \leftrightarrow b_t^{(110^6)}, c_t^{(10^9)} \leftrightarrow c_t^{(120)}, c_t^{(10^9)} \leftrightarrow t_4, c_t^{(10^9)} \leftrightarrow c_t^{(10^8)}, a_t^{(10^5)} t_{10}^{t_9 - 2} t_8 t_7^2 t_6^2 t_5^4 t_3^3 c_t^{(110^6)} \leftrightarrow \)
In this section, we present a question related to the one-relator groups.

**Question:** Let \(G_0 \to G_1 \to \cdots \to G_n\) be a Novikov tower. For any \(G_i\) \((0 \leq i \leq n)\) and relations \(A_i p = p B_i\) in \(G_i\), assume that the following conditions hold:

(i) \(\{A_i\}, \{B_i\}\) freely generate two subgroups of \(G_{i-1}\) respectively;

(ii) \(A_i p = p B_i\) can be presented as \(A_i' x A_i'' p = p B'_i y B''_i\), where \(x, y\) are two distinguishing letters of the highest weight in the words \(A_i\) and \(B_i\) \((x\) is relative to \(p\) and \(y\) to \(p^{-1}\)) respectively;

(iii) All the distinguishing letters relative to \(p\) are different (also to \(p^{-1}\) are different).

Can we get an effective standard normal form for \(G_n\)?

We can see that all the one-relator groups mentioned in this paper can be embedded into Novikov towers which satisfy the above conditions and have the effective standard normal form.

**6 Regression to embedding into two-generator groups**

Let

\[
G = gp(X|S),
\]

\[
G_1 = gp(X, a, b|S, x_i = a^{-1} b^{-2i+1} a^{-1} b^{2i-1} a b^{-2i+1} a b^{2i-1}),
\]
where $X = \{x_i | 1 \leq i \leq n\}$ ($n$ may be infinite).

B. H. Neumann and H. Neumann \cite{17} proved that $G$ can be embedded into $G_1$.

Now, we reprove this result by using also the Magnus’ method and Composition-Diamond lemma to reprove the B. H. Neumann and H. Neumann’s embedding theorem.

We may assume that $S$ is a Gröbner-Shirshov basis for $G$.

Let $b = t$, $a_i = b^{-i}ab^i$ and rewrite the last defining relation as $x_i = a_0^{-1}a_{-2i+1}a_0a_{-2i+1}$. Then

$$H_1 = gp(X, t, a_j (-2n + 1 \leq j \leq 0)|S, x_i = a_0^{-1}a_{-2i+1}a_0a_{-2i+1}, a_j t = ta_{j-1} (j - 1 \neq 0))$$

and $G_1 \cong H_1$ by $x \mapsto x$, $a \mapsto a_0$, $b \mapsto t$. Let $a_{-2i+1} = t_i$. Then

$$H_2 = gp(X, t, t_i, a_{-2i+2} (1 \leq i \leq n)|S, a_0t_i = t_i a_0x_i, a_{-2i+2}t = tt_i, t_it = ta_{-2i}(i \neq n))$$

and $H_1 \cong H_2$.

Let $Y = \{X^{\pm 1}, t_i^{\pm 1}, a_{-2i+2}^{\pm 1} | 1 \leq i \leq n\}$. Define the tower order on $Y^*$ as Definition \ref{2.5} with $t > t^{-1}$ and $t_i > t_i^{-1} > a_{-2j+2} > a_{-1} > x > x^{-1}$.

The relative standard relations in $H_2$ are as follows.

\begin{align*}
(6.1) & \quad a_0 t_i = t_i a_0x_i, a_0^{-1}t_i^{-1} = x_i t_i^{-1}a_0^{-1}, a_0x_i t_i^{-1} = t_i^{-1}a_0, \\
(6.2) & \quad a_{-2i+2}t = tt_i, t_i V(a_0x_i) t^{-1} = V'(a_0)t^{-1}a_{-2i+2}, t_i^{-1}V'(a_0)t^{-1} = V(a_0x_i)t^{-1}a_{-2i+2}, \\
(6.3) & \quad t_i V(a_0x_i)t = V'(a_0)ta_{-2i}, t_i^{-1}V'(a_0)t = V(a_0x_i)ta_{-2i}, a_{-2i}t^{-1} = t^{-1}t_i^{\varepsilon},
\end{align*}

where $V(a_0x_i) \leftrightarrow V'(a_0)$ by $a_0x_i \leftrightarrow a_0$.

Let $R$ consist of the relations (6.1) – (6.3), $S$ and the trivial relations of $H_2$. Since $S$ is a Gröbner-Shirshov basis for $G$ and there is no composition occurred among (6.1) – (6.3) with $S$, $R$ is a Gröbner-Shirshov basis for $H_2$ with the tower order. Therefore, by Composition-Diamond lemma, $Irr(S) \subset Irr(R)$ and so, $G$ can be embedded into $H_2 \cong G_1$.

G. Higman, B. H. Neumann and H. Neumann \cite{10} also proved that $G$ can be embedded into $G_2$, where

$$G_2 = gp(X, a, b|S, x_i = a^{-1}b^{-1}ab^{-i}ab^{-1}a^{-1}b^{i}a^{-1}bab^{-i}ab^{-1}b^{i}).$$

Now, we reprove this result by using also the Magnus’ method and Composition-Diamond lemma.

Let $a = t_1$, $b_j = a^{-j}ba^j$ and rewrite the last defining relation as $x_i = b_{-1}^{-1}b_0^{-1}b_1^{-1}b_0b_{-1}^{-1}b_1b_0$. Then we can get

$$H_1 = gp(X, t_1, b_j (-1 \leq j \leq 0)|S, x_i = b_{-1}^{-1}b_0^{-1}b_1^{-1}b_0b_{-1}^{-1}b_1b_0, b_j t = tb_{j-1} (j - 1 \neq 0))$$

and $G_1 \cong H_1$ by $x \mapsto x$, $a \mapsto t_1$, $b \mapsto b_0$. Let $b_0 = t_2$, $b_{(1j)} = b_0^{-j}b_1b_0$, $b_{(-1j)} = b_0^{-j}b_{-1}b_0$ and rewrite the defining relation as $x_i = b_{-1}^{-1}b_0^{-1}b_1^{-1}b_0b_{-1}^{-1}b_1b_0$. Let $b_{(-10)} = t_3$. At last, we can get

$$H_2 = gp(X, t_1, t_2, t_3 b_{1-j} (1 \leq j \leq n) | S, b_{(-1)}^{-1}t_3 = t_3 x_i b_{(-1-i)}^{-1}, b_{(-1-i+1)}^{-1}t_2 = t_2 b_{(-1-i)}^{-1}, b_{(-10)}t_1 = t_1 t_2, t_2 t_1 = t_1 t_3)$$

16
and $H_1 \cong H_2$.

Let $Y = \{X^\pm 1, \ t_1^\pm 1, \ t_2^\pm 1, \ t_3^\pm 1, \ b_{i-j}^\pm 1 \mid 1 \leq j \leq n\}$. Define the tower order on $Y^*$ as Definition 2.5 with $t_1 > t_1^{-1} > t_2 > t_2^{-1} > t_3 > t_3^{-1} > b_{(1-j)} > b_{(1-j)}^{-1} > x > x^{-1}$.

The relative standard relations in $H_2$ are as follows.

\[(6.4) \ b_{(1-i)-1}^\epsilon t_3 = t_3(x, b_{(1-i)-1}^\epsilon, b_{(1-i)-1}^{-1} t_3)^\epsilon, \ b_{(1-i)-1}^\epsilon t_3^{-1} = x_i^{-1} t_3^{-1} b_{(1-i)-1}, \ b_{(1-i)-1} x_i^{-1} t_3^{-1} = t_3^{-1} b_{(1-i)-1}^\epsilon, \]

\[(6.5) \ b_{(1-i)-1}^\epsilon t_2 = t_2 b_{(1-i)-1}^\epsilon, \ b_{(1-i)-1}^{-1} t_2 = t_2^{-1} b_{(1-i)-1}^\epsilon, \ b_{(10)}^\epsilon t_1 = t_1 t_1^\epsilon, \]

\[(6.6) \ t_2 V(b_{1-i}) t_1^{-1} = V'(b_{1-i}) t_1^{-1} t_2^{-1} b_{(1-i)+1} t_1^{-1} = V(b_{1-i}) t_1^{-1} b_{(10)}, \]

\[(6.7) \ t_2 V(b_{1-i}) t_1 = V'(b_{1-i}) t_1 t_3, \ t_2^{-1} V'(b_{1-i}) t_1 = V(b_{1-i}) t_4 t_3^{-1}, \]

\[(6.8) \ t_3^{-1} W(x, b_{1-i}) t_1^{-1} = W'(b_{1-i}) t_1^{-1} t_2^{-1}, \ t_3^{-1} W'(b_{1-i}) t_1^{-1} = W(x, b_{1-i}) t_4^{-1} t_2^{-1}, \]

where $V(b_{1-i}) \leftrightarrow V'(b_{1-i})$ by $b_{1-i} \leftrightarrow b_{1-(i-1)}$, $W(x, b_{1-i}) \leftrightarrow W'(b_{1-i})$ by $x, b_{1-(i-1)} \leftrightarrow b_{1-i}$.

Let $R$ consist of the relations (6.4) – (6.8), $S$ and the trivial relations of $H_2$. It is clear that, with the tower order, $R$ is a Gröbner-Shirshov basis for $H_2$. Therefore, $Irr(S) \subset Irr(R)$ and so, $G$ can be embedded into $H_2 \cong G_2$.

**Acknowledgement:** The authors would like to express their deepest gratitude to Professor L. A. Bokut for his kind guidance, useful discussions and enthusiastic encouragement.

**References**

[1] G.M. Bergman, The diamond lemma for ring theory, *Adv. Math.* 29 (1978) 178-218.

[2] L.A. Bokut, On a certain property of the Boone group, *Algebra i Logika* 5 (5) (1966) 5-23.

[3] L.A. Bokut, Unsolvability of the word problem, and subalgebras of finitely presented Lie algebras, *Izv. Akad. Nauk. SSSR (Ser. Mat.)* 36 (1972) 1173-1219.

[4] L.A. Bokut, Imbeddings into simple associative algebras, *Algebra i Logika* 15 (1976) 117-142.

[5] L.A. Bokut, Yuqun Chen, Gröbner-Shirshov basis for free Lie algebras: after A. I. Shirshov, *SEA Bull. Math.* 31 (2007) 1057-1076.

[6] L.A. Bokut, G.P. Kukin, *Algorithmic and Combinatorial Algebra*, Kluwer Academic Publishers, 1994.

[7] J.L. Britton, The word problem, *Ann. of Math.* 77 (1963) 16-32.

[8] B. Buchberger, An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal, Ph.D. thesis, University of Innsbruck, Austria, 1965 (in German).
[9] B. Buchberger, An algorithmical criteria for the solvability of algebraic systems of equations, *Aequationes Math.* 4 (1970) 374-383 (in German).

[10] G. Higman, B.H. Neumann, H. Neumann, Embedding theorems for groups, *J. London Math. Soc.* 24 (1949) 247-254.

[11] K. Kalorkoti, Decision problems in group theory, *Proc. LMS* 44 (2) (1982) 312-332.

[12] K. Kalorkoti, Turing degrees and the word and conjugacy problems for finitely presented groups, *SEA Bull. Math.* 40 (5) (2006) 855-887.

[13] R.C. Lyndon, P.E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, 1977.

[14] W. Magnus, Untersuchungen über einige unendliche diskontinuierliche Gruppen, *Journal für Math.* 105 (1931) 52-74.

[15] W. Magnus, A. Karrass, D. Solitar, *Combinatorial Group Theory: Presentations of Groups in terms of Generators and Relations*, 2nd Edition, Dover Publications, 1976.

[16] D.I. Moldavanskii, Certain subgroups of groups with one defining relation, *Sibirsk. Mat. Z.* 8 (1967) 1370-1384.

[17] B.H. Neumann, H. Neumann, Embedding theorems for groups, *J. London Math. Soc.* 34 (1959) 465-479.

[18] P.S. Novikov, On algorithmic undecidability of the word problem, *Dokl. Akad. Nauk SSSR* 85 (1952) 709-712.

[19] P.S. Novikov, Undecidability of the conjugacy problem in group theory, *Izv. Akad. Nauk SSSR* (Ser. Mat.) 18 (1954) 485-524.

[20] P.S. Novikov, On algorithmic undecidability of the word problem in the theory of groups, *Trudy Mat. Inst. Steklov* 44 (1955) 1-144.

[21] A.I. Shirshov, Some algorithmic problem for Lie algebras, *Sibirsk. Mat. Z.* 3 (1962) 292-296 (in Russian). English translation: *SIGSAM Bull.* 33 (2) (1999) 3-6.