Superfluid hydrodynamics of polytropic gases: dimensional reduction and sound velocity

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Abstract
Motivated by the fact that two-component confined fermionic gases in Bardeen–Cooper–Schrieffer–Bose–Einstein condensate (BCS–BEC) crossover can be described through a hydrodynamical approach, we study these systems—both in the cigar-shaped configuration and in the disc-shaped one—by using a polytropic Lagrangian density. We start from the Popov Lagrangian density and obtain, after a dimensional reduction process, the equations that control the dynamics of such systems. By solving these equations we study the sound velocity as a function of the density by analyzing how the dimensionality affects this velocity.

Keywords: trapped ultracold fermionic gases, reduced dimensionality systems, hydrodynamics

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(Some figures may appear in colour only in the online journal)

1. Introduction

The crossover from the Bardeen–Cooper–Schrieffer (BCS) state of weakly bound Fermi pairs to the Bose–Einstein condensate (BEC) of molecular dimers and systems with reduced dimensionality are two of the most interesting topics in the ultracold atomic physics.

Over the past several years, the predicted BCS–BEC crossover [1–3] has been observed by several experimental groups with \(^6\)Li and \(^40\)K atoms [4–9]. To study these systems it is necessary to confine them by suitable traps that, in the simplest case, act as harmonic potentials.
Even if harmonically trapped fermions are very dilute systems, their properties are governed by the interaction between particles. This interaction, in the dilute regime and at ultra low temperatures, can be characterized by a single parameter, the \( s \)-wave scattering length \( a_s \). Experimentally, it is possible to control \( a_s \) by using a magnetically tuned Feshbach resonance. This can be described by a simple expression, see \([10]\), that is

\[
a_s = a_{bg}(1 - |\Delta/(B - B_0)|)
\]

with \( a_{bg} \) the so-called background (off-resonant) \( s \)-wave scattering length, \( B_0 \) the magnetic field strength where the Feshbach resonance occurs, and \( \Delta \) the resonance width. Then, the Feshbach resonance technique permits one to vary the magnitude and the sign of \( a_s \). When \( a_s \) is large and negative, the ground state is a BCS superfluid of weakly bound Fermi pairs. For large and positive values of \( a_s \), instead, the Bose–Einstein condensation of molecules, associated to the formation of two-body bound states, is observed. In between these two states, there is a smooth crossover when \( a_s \) changes its sign by passing through \( \pm \infty \) corresponding to \( B \) at the resonance position (see the above reported formula). At zero temperature and only in the case of dilute fermionic systems the only relevant dimensionless parameter in the description of the physical properties is \( y = (k_F a_s)^{-1} \), where \( k_F \) is the Fermi wave vector.

From the experimental point of view, it is possible to change the harmonic frequencies inducing a dimensional reduction of the system \([11–13]\): from three dimensions to one dimension and from 3D to 2D dimensions. The first case takes place with an external trapping realized by superimposing a strong harmonic confinement in the transverse radial plane to a generic shallow potential in the axial direction (cigar-shaped configuration). The second circumstance occurs, instead, when the atoms are trapped by a strong harmonic potential in the axial direction plus a loose potential in the transverse plane (disc-shaped (DS) configuration). Both the effects of reduced dimensions \([14–17]\) and the transition from a dimensional regime to another have been carefully studied \([18, 19]\). In particular, it has been suggested that a reduced dimensionality strongly modifies density profiles \([20–23]\), collective modes \([24, 25]\) and the stability of mixtures \([26, 27]\). Also sound velocities have been theoretically investigated in reduced dimensions for both normal \([25, 28, 29]\) and superfluid Fermi gases \([30–32]\). Very recently, Shanenko and co-workers have carried out an interesting analysis about BCS–BEC crossover induced by quantum-size effects in the cigar-shaped configuration \([33]\).

Both in the cigar-shaped case and in the DS one the dynamics are practically frozen in the direction where the harmonic trap has been generated. The idea is, therefore, to describe the system by eliminating the frozen spatial variables by performing a dimensional reduction process as performed in \([18, 19]\). Salasnich and co-workers have studied the problem by achieving an effective one (two) dimensional (1D(2D)) wave equation, that is a time-dependent 1D(2D) non-polynomial Schrödinger equation, by integrating out the directions where the dynamics are blocked \([18]\). In particular, in \([19]\) the authors have studied the sound velocity for Bose-condensed alkali-metal vapors.

For fermionic systems it would be possible, in principle, to follow a similar approach, i.e. to use a suitable generalization to the fermionic case of the Gross–Pitaevskii equation (see, for example, \([40, 41]\)). However, here, we study a confined zero-temperature two-component Fermi gas both in cigar-shaped configuration and in the DS one by approaching the problem from another perspective. We perform the reductional dimensional process by starting from the Popov Lagrangian (PL) density \([34]\). This Lagrangian density—via the internal energy of the system—embodies both the possibility to cross the boundary between the BCS side and the BEC region of the crossover (by means of a polytropic state equation \([35]\)) and the effects of the quantum pressure by a gradient correction \([36–40]\). For the density \( n \), appearing in the PL, we use an ansatz which consists of factorizing \( n \) as a Gaussian function \([18]\) in the radial (axial) direction times a generic function in the others. By employing this ansatz we obtain an effective Lagrangian by integrating the PL density in the radial (axial) direction.
and write down the corresponding Euler–Lagrange equations (ELEs). We assume that the system is in the stationary regime and suppose to create a perturbation—with respect to the equilibrium—sufficiently weak so to retain in the ELEs only the first-order perturbation terms. We then numerically determine the behavior of the sound velocity $c_s$ as a function of the equilibrium density in the CS(DS) configuration when $y \ll -1$ (BCS limit) and when $y \gg 1$ (BEC limit). Note that on the BEC side, due to formation of two-body molecular bound states (see the above discussion), the binding energy $\epsilon_B = \hbar^2/ma^2_s$ of each molecular dimer ($m$ being the mass of the fermions) will contribute to the chemical potential $\mu$. The energy $\epsilon_B$ does not depend on the space so it will not affect the properties of the system since, as we shall see in the following, in the ELE corresponding to the equation of motion there are the spatial derivatives of $\mu$.

For each of the above cases, i.e. BCS limit and BEC limit, we investigate the role played by the dimensionality in determining the atomic cloud properties. Therefore we compare the above numerical solutions both with those pertaining to the 3D cases (corresponding to the very large densities limit) and to the pure 1D(2D) case (where the radial (axial) cloud width is equal to the radial (axial) characteristic harmonic length). By this comparison, we determine the density-range validity of our numerical solutions by pointing out that these latter solutions describe both the extreme density regimes and the intermediate regions.

2. The system

We study a superfluid gas of $N$ fermions confined by an external three-dimensional (3D) potential $U(r)$ at zero temperature. To do this we use the superfluid hydrodynamic PL density [34]

$$\mathcal{L} = -\hbar \dot{n} - \frac{\hbar^2}{2m} (\nabla \theta)^2 n - U(r)n - \mathcal{E}(n, \nabla n)n,$$

where $m$ is the mass of the fermions and both the density $n$ and the phase $\theta$ depend on the position $r$ and time $t$, that is $n = n(r, t)$ and $\theta = \theta(r, t)$. The quantity $\mathcal{E}(n, \nabla n)$ represents the internal energy of the system, which we write as a polytropic equation of the state plus a gradient correction which reproduces the effect of the quantum pressure, that is

$$\mathcal{E}(n, \nabla n) = \frac{\alpha}{\gamma} n^{\gamma-1} + \frac{\hbar^2}{8m} \left(\nabla n\right)^2.$$

The parameter $\gamma$, as shown in [35], varies within a specific range during the BCS–BEC crossover. $\gamma = 5/3 \left(\alpha = \frac{3}{2} \frac{\hbar}{m} (3\pi^2)^{2/3}\right)$ when $y = (k_F a_s)^{-1} \ll -1$ (BCS limit), while to $\gamma = 2 \left(\alpha = \frac{4\pi^2 \hbar^2}{m a_s}\right)$ with $a_s$ the $s$-wave scattering length) when $y \gg 1$ (BEC limit). The intensity of the gradient correction is controlled by the parameter $\lambda$ [36–40].

In the following, we consider two confinement configurations: the CS configuration and the DS one.

3. Cigar-shaped configuration

We assume that the trapping potential $U(r)$ is given by the superposition of an isotropic harmonic confinement in the radial ($x$–$y$) plane and a generic potential $V(z)$ in the axial ($z$) direction, so that

$$U(r) = \frac{1}{2} m \omega_{\perp}^2 (x^2 + y^2) + V(z),$$

where $\omega_{\perp}$ is the trapping frequency in the $x$–$y$ plane. We make the hypothesis of a strong transverse harmonic confinement and a weak axial potential, so that it is possible to achieve
a CS configuration. Due to the form (3) of the confining potential, we can perform on the density and phase the following ansatz [18]

\[ \rho(t, r, \theta) = n_0(x, y, \sigma(z, t)) n_1(z, t) = \frac{1}{\pi \sigma^2} e^{-\frac{\sigma^2}{\pi \sigma^2} n_1(z, t)} \]

where \( n_0 = 1 \) and \( d\rho n_0 = N \), and \( \sigma \) represents the width of the gas cloud in the \( x \) and \( y \) directions. By using the ansatz (4) at the right-hand side of equation (1) and integrating the resulting Lagrangian density in the transverse directions, i.e. \( \int dx dy \mathcal{L} \), we get the effective Lagrangian density

\[ \mathcal{L}(n_1, \theta, \sigma) = -n_1 \left[ \hbar \frac{\dot{\sigma}}{\sigma} + \frac{\hbar^2}{2m} \left( \frac{\partial \sigma}{\partial z} \right)^2 + \frac{1}{2} \omega_\perp^2 \sigma^2 + V(z) \right] \]

By using the result stated by equation (5) we derive the ELEs. The ELE with respect to \( \theta \) is

\[ \frac{\partial \mathcal{L}}{\partial \theta} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial}{\partial z} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0, \]

which gives rise to

\[ \frac{\partial n_1}{\partial t} + \frac{\partial}{\partial z} (n_1 v_z) = 0. \]

This is the continuity equation, where \( v_z = \frac{\hbar \sigma}{m \pi \sigma^2} \). The ELE with respect to \( n_1 \) is

\[ \frac{\partial \mathcal{L}}{\partial n_1} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{n}_1} - \frac{\partial}{\partial z} \frac{\partial \mathcal{L}}{\partial \dot{n}_1} = 0, \]

which, by neglecting the spatial derivatives of \( \sigma \), provides the equation of motion

\[ m \frac{\partial v_z}{\partial t} + \frac{\partial}{\partial z} \left[ \frac{1}{2} m v_z^2 + V(z) + \mu(n) \right] - \frac{\hbar^2 \partial^2 n_1}{4 mn_1^2 \partial z^2} + \frac{\partial n_1}{\partial z} \left[ \frac{\lambda \hbar^2 \partial^2 n_1}{2mn_1^2 \partial z^2} - \frac{\lambda \hbar^2 \left( \partial n_1 \right)^2}{4 mn_1^2 \partial z^2} \right] = 0, \]

where

\[ \mu(n) = \frac{\alpha}{\gamma} \left( \frac{n_1}{\pi \sigma^2} \right)^{\gamma^{-1}} \]

is the chemical potential.

Finally, let us focus on the role played by \( \sigma \). This quantity can be viewed as a function of \( z \) and \( t \) as well. In the first case, i.e. \( \sigma = \sigma(z, t) \), we can start from the ELE with respect to \( \sigma \)

\[ \frac{\partial \mathcal{L}}{\partial \sigma} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\sigma}} - \frac{\partial}{\partial z} \frac{\partial \mathcal{L}}{\partial \dot{\sigma}} = 0. \]

One, at this point, performs a sort of adiabatic approximation—i.e. one neglects the derivatives of \( \sigma \) with respect to space and time—from which one gets the following simple algebraic equation for \( \sigma \):

\[ \sigma^{2\gamma} - \lambda a_\perp^2 \sigma^{2\gamma-4} = \frac{2\alpha(\gamma - 1)n_1^{\gamma-1}}{2\sigma^{2\gamma-1} m \omega_\perp^2}, \]

where \( a_\perp = \sqrt{\frac{\hbar}{m \omega_\perp}} \) is the transverse characteristic harmonic length.
In the second case, i.e. when $\sigma$ plays the role of a variational parameter, we have to minimize the action $S = \int dz \, dm \tilde{L}$ with respect to $\sigma$. By requiring, therefore, that $\frac{\partial S}{\partial \sigma} = 0$ we get the following equation:

$$\sigma^{2y} - \lambda a_i^2 \sigma^{2y-4} - \frac{2a_i(y - 1)}{\gamma^2 \omega_i} I_0 = 0,$$

(13)

where $I_0 = \int dz \, dm_i$. The equations (12) and (13) provide a constraint to which $\sigma(n)$ has to obey. In conclusion, we observe that in equation (12) appears the dependence on the space $z$ and time $t$, while this is not the case of equation (13) because in $I_0$ (see above), $z$ and $t$ will be integrated out.

4. Disc-shaped configuration

For this case we suppose that the confining potential $U(r)$ derives from the superposition of a harmonic potential in the axial direction $z$ and a generic potential, say $W(x, y)$, in the radial transverse plane $(x - y)$. Then

$$U(r) = \frac{1}{2} m \omega_i^2 z^2 + W(x, y),$$

(14)

where $\omega_i$ is the trapping frequency in the $z$ direction. We suppose a strong confinement in the axial direction, while the potential in the radial plane is assumed to be weak, so that we have a DS configuration. The form (14) of the trapping potential allows us to perform on the density and the phase the following ansatz

$$n(r, t) = n_0(z, \eta(x, y, t)) n_1(x, y, t) = \frac{1}{\sqrt{\pi} \eta} e^{-\frac{r^2}{\eta^2}} n_1(x, y, t)$$

$$\theta(r, t) = \theta(x, y, t),$$

(15)

where $\int dz n_0 = 1$ and $\int d\Omega n(r, t) = N$, and $\eta$ represents the width of the gas cloud in the $z$ direction. By employing the ansatz (15) at the right-hand side of equation (1) and performing the integration of the so obtained Lagrangian density in the axial direction, i.e. $\int dz \tilde{L}$, we get the following effective Lagrangian density

$$\tilde{L}(n_1, \theta, \eta) = -n_1 \left[ \hbar \dot{\theta} + \frac{\hbar^2}{2m} (\nabla_R \theta)^2 + \frac{1}{4} m \omega_i^2 \eta^2 - \frac{1}{4} n_1^2 \omega_i^2 \eta^2 \right]$$

$$- \frac{a n_1^2}{\gamma^2 \pi \omega_i^{2 - 1} \eta^{y - 1}} - \frac{\lambda \hbar^2}{8 m n_1} (\nabla_R n_1)^2 - \frac{\lambda \hbar^2 n_1^2}{4 \eta^4} [1 + (\nabla_R \eta)^2]$$

(16)

with $R = (x, y)$ and $\nabla_R = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$. By using the result stated by equation (16) one derives the corresponding ELE. By following the same procedure followed in the previous section, the continuity equation is achieved from the ELE with respect to $\theta$

$$\frac{\partial \tilde{L}}{\partial \theta} - \frac{\partial}{\partial t} \frac{\partial \tilde{L}}{\partial \theta} - \nabla_R \cdot \frac{\partial \tilde{L}}{\partial \nabla_R \theta} = 0$$

(17)

which becomes

$$\frac{\partial n_1}{\partial t} + \nabla_R \cdot (n_1 \nu) = 0,$$

(18)

where $\nu = \frac{\hbar}{m} \nabla_R \theta$.

The motion equation is provided from the ELE with respect to $n_1$:

$$\frac{\partial \tilde{L}}{\partial n_1} - \frac{\partial}{\partial t} \frac{\partial \tilde{L}}{\partial n_1} - \nabla_R \cdot \frac{\partial \tilde{L}}{\partial \nabla_R n_1} = 0,$$

(19)
which, by neglecting the spatial derivatives of \( \eta \), reads
\[
m \frac{\partial v}{\partial t} + \nabla R \left[ \frac{mv^2}{2} + W(x, y) + \mu(n) \right] - \frac{\lambda h^2}{4mn_1} \nabla R \left( \nabla^2 n_1 \right) + \nabla R n_1 \left[ \frac{\lambda h^2}{2mn_1} \nabla R \nabla^2 n_1 - \frac{\lambda h^2}{4mn_1} (\nabla R n_1)^2 \right] = 0
\]
(20)

with
\[
\mu(n) = \frac{\alpha}{\sqrt{\gamma}} \left( \frac{n_1}{\sqrt{\pi \eta}} \right)^{\gamma - 1}
\]
(21)
the chemical potential.

Finally, \( \eta \). As in the CS case, \( \eta \) can be interpreted as a function of space and time, i.e \( \eta = \eta(x, y, t) \), and also as a variational parameter. In the first case, we start from the ELE with respect to \( \eta \)
\[
\frac{\partial \mathcal{L}}{\partial \eta} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\eta}} - \nabla R \cdot \frac{\partial \mathcal{L}}{\partial \nabla R \eta} = 0.
\]
(22)
This equation, by neglecting the derivatives of \( \eta \) with respect to space and time, becomes
\[
\eta^{\gamma + 1} - \lambda \alpha^4 \eta^{\gamma - 3} = \frac{2\alpha (\gamma - 1)n_1^{\gamma - 1}}{\gamma^2 \pi^{\frac{3\gamma - 2}{2}} m o_c^2} = 0,
\]
(23)
where \( \alpha_c = \sqrt{\frac{\pi}{m o_c}} \) is the axial harmonic characteristic length.

If \( \eta \) is, instead, interpreted as a variational parameter, then we minimize the action with respect to \( \eta \) and get
\[
\eta^{\gamma + 1} - \lambda \alpha^4 \eta^{\gamma - 3} = \frac{2\alpha (\gamma - 1) I_1}{\gamma^2 \pi^{\frac{3\gamma - 2}{2}} m o_c^2} = 0
\]
(24)
with \( I_1 = \int \mathrm{d}x \mathrm{d}y \mathrm{d}t n_1^4 \). Due to this latter integral, the space-time dependence does not appear in equation (24).

5. Atomic cloud properties

To gain a deeper physical insight into our system, we focus on the study of the sound velocity \( c_s \). This component is strictly related to the propagation of a perturbation, with respect to the equilibrium, created at a given point of the system at a given time. We analyze \( c_s \) as a function of the spatial density, both in the CS configuration and in the DS one. As previously, we derive the sound velocity in a general scenario, that is, starting from a general external potential \( U(r) \).

5.1. Sound velocity: general treatment

We start from the Lagrangian density (1). Through the related ELEs we achieve two equations for the spatial density \( n(r, t) \) and the velocity field \( v(r, t) \):
\[
\frac{\partial n}{\partial t} + \nabla \cdot (n v) = 0,
\]
(25)
and
\[
m \frac{\partial v}{\partial t} + \nabla \left[ \frac{1}{2} m v^2 + U(r) + X(n, \nabla n) \right] = 0,
\]
(26)
where \( X(n, \nabla n) = \alpha n^{\gamma - 1} - \lambda \frac{\hbar^2}{2m} \nabla^2 n \). We assume to be in the stationary regime, i.e. \( v = 0 \), and in the uniform case, i.e. \( U(r) = 0 \). At this point, let us suppose to perturbate the system with respect to the equilibrium configuration characterized by \( n(r, t) = n_{eq} \) and \( v = 0 \), that is
\[
\begin{align*}
    n(r, t) &= n_{eq} + \tilde{n}(r, t) \\
    v(r, t) &= 0 + \tilde{v}(r, t).
\end{align*}
\]  
(27)

We use these equations in the ELEs (25) and (26). Under the hypothesis that the perturbation is sufficiently weak so as to retain only the first-order perturbation terms in the ELEs, we get
\[
\frac{\partial \tilde{n}}{\partial t} + n_{eq} \nabla \cdot \tilde{v} = 0,
\]  
(28)

and
\[
n_{eq} \frac{\partial \tilde{v}}{\partial t} + \frac{\alpha (\gamma - 1) n_{eq}^{\gamma - 1}}{m} \nabla \tilde{n} - \frac{\hbar^2}{4m^2} \nabla \nabla^2 \tilde{n} = 0.
\]  
(29)

By calculating the time derivative of equation (28) and the divergence of equation (29) and subtracting the former equation from the latter, we get
\[
\left[ \frac{\partial^2}{\partial t^2} - \frac{\alpha (\gamma - 1) n_{eq}^{\gamma - 1}}{m} \nabla^2 + \frac{\hbar^2}{4m^2} \nabla^2 \nabla^2 \right] \tilde{n} = 0.
\]  
(30)

This equation describes the spatio-temporal propagation of the wave associated with the perturbation. The propagation of the wave pressure takes place with velocity \( c_s \) (sound velocity) which is given by the squared root of the coefficient of the Laplacian in equation (30) (that is also the coefficient of \( \nabla \tilde{n} \) in equation (29)), i.e.
\[
c_s = \left( \frac{\alpha (\gamma - 1) n_{eq}^{\gamma - 1}}{m} \right)^{\frac{1}{2}}.
\]  
(31)

Note that equation (30) is different from the well-known equation for the spatio-temporal propagation of the waves. In fact, this equation contains an additional term, i.e. that controlled by the parameter \( \lambda \).

At this point, it is worth observing that if the perturbation is a plane wave, that is \( \tilde{n}(r, t) = A e^{i(k \cdot r - \omega t)} + A^* e^{-i(k \cdot r - \omega t)} \), the relation of dispersion that characterizes the oscillations associated with the sound wave induced by the perturbation is
\[
\omega = c_s k \sqrt{1 + \frac{\lambda \hbar^2}{4m^2 c_s^2} k^2},
\]  
(32)

where \( c_s \) is the sound velocity (31). When \( k \to 0 \), equation (32) returns the usual dispersion relation of the sound wave in the absence of the gradient correction.

### 5.2. Sound velocity in the cigar-shaped configuration

Let us start from the external trapping potential (3). We assume that \( V(z) = 0 \) and the system to be in the stationary regime, i.e. \( v = 0 \). We presume to perturbate the system with respect to the equilibrium configuration where \( n_1(z, t) = n_{eq} \) and \( v(z, t) = 0 \):
\[
\begin{align*}
    n_1(z, t) &= n_{eq} + \tilde{n}(z, t) \\
    v_1(z, t) &= 0 + \tilde{v}(z, t).
\end{align*}
\]  
(33)

Then, we use the two above equations in equation (7) (continuity equation) and in equation (9) (motion equation) that after the linearization (we are supposing that the perturbation is sufficiently weak so as to retain only the first-order perturbation terms) provide
\[
\frac{\partial \tilde{n}}{\partial t} + n_{eq} \frac{\partial \tilde{v}}{\partial z} = 0,
\]  
(34)
\[
\frac{\partial \tilde{v}}{\partial t} + \frac{\alpha (\gamma - 1)n_{eq}^{-1}}{m \gamma \pi^{\gamma - 1} \sigma^{2(\gamma - 1)}} \partial \tilde{n} = -\frac{\hbar^2}{4m^2} \partial^3 \tilde{n} = 0. \tag{35}
\]

Then (see the discussion of the previous subsection after equations (29) and (30)) the sound velocity \( c_s \) is
\[
c_s = \left( \frac{\alpha (\gamma - 1)}{m \gamma^{1/2}} \right)^{1/2} \left( \frac{n_{eq}}{\pi \sigma^2(n_{eq})} \right)^{(\gamma - 1)/2}, \tag{36}
\]
where the radial width \( \sigma \) is constrained according equation (12) which (with \( n_1 \) replaced by \( n_{eq} \)) reads
\[
\sigma^2 \gamma^{-1} + \frac{\lambda^2 a_\perp^2 \sigma^{2\gamma - 4}}{\gamma^2 \pi^{\gamma - 1} \eta m \omega^2} = 0. \tag{37}
\]

### 5.3. Sound velocity in disc-shaped configuration

In this case the trapping potential is given by equation (14). We assume that \( W(x, y) = 0 \) and the system to be in the stationary regime: \( \psi(x, y, t) = 0 \). As before, let us presume we create a small perturbation in the system with respect to the equilibrium,
\[
n(x, y, t) = n_{eq} + \tilde{n}(x, y, t) \quad \psi(x, y, t) = \tilde{\psi}(x, y, t). \tag{38}
\]

By using these two equations in the continuity equation (18) and in the motion equation (20) and neglecting the quadratic terms in the perturbation, we get
\[
\frac{\partial \tilde{n}}{\partial t} + n_{eq} \nabla \cdot \tilde{\psi} = 0, \tag{39}
\]
and
\[
n_{eq} \frac{\partial \tilde{v}}{\partial t} + \frac{\alpha (\gamma - 1)n_{eq}^{-1}}{m \gamma^{1/2} \pi^{\gamma - 1} \eta^{(\gamma - 1)/2}} \nabla \tilde{n} - \frac{\hbar^2}{4m^2} \nabla^2 \tilde{n} = 0. \tag{40}
\]

In this case, by employing the same arguments followed in the two previous subsections, the sound velocity \( c_s \) is
\[
c_s = \left( \frac{\alpha (\gamma - 1)}{m \gamma^{1/2}} \right)^{1/2} \left( \frac{n_{eq}}{\pi ^{1/2} \eta(n_{eq})} \right)^{(\gamma - 1)/2}, \tag{41}
\]
with the axial density \( \eta \) which has to satisfy equation (23), that, for this case (with \( n_1 \) replaced by \( n_{eq} \)), has the form
\[
\eta^{(\gamma + 1)} - \frac{\lambda^2 a_\perp^2 \eta^{\gamma - 3}}{\gamma^2 \pi^{\gamma - 1} m \omega^2} = 0. \tag{42}
\]

### 6. Analysis

To analyze the behavior of the sound velocity as a function of the density, one has to find \( \sigma(n_{eq}) \) and \( \eta(n_{eq}) \)—from the constraints (37) and (42)—that when employed at the right-hand side of equations (36) and (41) will provide \( c_s = c_s(n_{eq}) \). We study the widths of the gas cloud and the sound velocity both in the CS configuration and in the DS one in two cases by assuming that the intensity of gradient correction \( \lambda \) is equal to 1.
The first case that we analyze is $\gamma = \frac{5}{3}$ ($y \ll -1$, BCS limit) when $\alpha = \frac{\hbar^2}{2m} (3\pi^2)^{\frac{1}{3}}$. In this situation, for the CS configuration, equations (36) and (37) become

$$c_s = \left( \frac{3}{5} \right)^{1/2} \frac{(9\pi^2)^{1/6}}{\hbar} \left( \frac{n_{eq}}{\sigma^2} \right)^{1/5}, \quad (43)$$

and

$$\sigma^4 = \frac{18 (9\pi^2)^{1/3}}{125} a_{eq}^2 n_{eq} \sigma^5 - a_{\perp}^4 = 0. \quad (44)$$

In the DS configuration, equations (41) and (42) read

$$c_s = \left( \frac{2}{5} \right)^{1/2} (a_3 N)^{1/2} \frac{\hbar}{m} \left( \frac{n_{eq}}{\sigma^2} \right)^{1/5}, \quad (45)$$

and

$$\eta^4 = \frac{18 \pi^3}{5^{5/2}} a_{eq}^4 n_{eq} \eta^5 - a_{\perp}^4 = 0. \quad (46)$$

The second case that we deal with is $\gamma = 2$ ($y \gg 1$, BEC limit) when $\alpha = \frac{4\pi^2 a_3 N}{m}$. Then, for the CS configuration, equations (36) and (37) will read

$$c_s = \left( \frac{2}{5} \right)^{1/2} (a_3 N)^{1/2} \frac{\hbar}{m} \left( \frac{n_{eq}}{\sigma^2} \right)^{1/5}, \quad (47)$$

$$\sigma^4 - a_{\perp}^4 (1 + 2a_3 n_{eq}) = 0. \quad (48)$$

In the DS configuration, equations (41) and (42) assume the following form:

$$c_s = 2 \left( \frac{1}{2} \right)^{1/2} \left( \frac{\hbar}{m} \right)^{1/2} \left( \frac{n_{eq}}{\eta} \right)^{1/5}, \quad (49)$$

$$\eta^4 - 2 (2\pi)^{1/2} a_{eq}^4 N n_{eq} \eta - a_{\perp}^4 = 0. \quad (50)$$

Our results are shown in figure 1 and in figure 2. The former corresponds to $\gamma = 5/3$, the latter to $\gamma = 2$. The left part of these two figures represent the CS case, and the right part the DS one. In both cases ($\gamma = 5/3$, $\gamma = 2$), the left-top panel shows the atomic cloud radial width $\sigma$ as a function of the equilibrium density $n_{eq}$, while the right-top panel shows the axial width $\eta$ studied by varying the density. From such panels, one can see that when the density is sufficiently low, $\sigma$ and $\eta$ return respectively, the radial—$a_{\perp}$—and axial—$a_c$—characteristic harmonic oscillator lengths.

Both for $\gamma = 5/3$ and $\gamma = 2$, in the left-bottom and right-bottom panels we show the behavior of the sound velocity $c_s$ as a function of the equilibrium density. The solid lines have been obtained by equations (43), (45), (47), and (49), with the atomic cloud widths ($\sigma$ for the cigar, and $\eta$ for the disc) as functions of the density found by numerically solving, respectively, equations (44), (46), (48), and (50). The dotted lines represent the sound velocity in the pure 3D regime. This has been obtained by equations (43), (45), (47), and (49); for this case the cloud widths in terms of density is given by the asymptotic behavior ($n_{eq} \rightarrow \infty$) of the solutions of equations (44), (46), (48), and (50) respectively. Finally, the dotted lines have been obtained by solving equations (43) and (47) with $\sigma = a_{\perp}$, that is the 1D case, and equations (45) and (49) with $\eta = a_c$, that is the 2D case. Then, from the left-bottom and right-bottom panels of figures 1 and 2, it can be seen that our numerical solutions (solid lines) approximates very well the results of the 1D (2D) regime (dashed lines) for sufficiently low densities, while the full 3D regime behavior (dotted lines) is well described by our solutions for high densities. So our
solutions capture the physics both in the extreme regimes and in the intermediate region. In determining the properties of the system, the density plays an important role. In fact, when this latter quantity is low, the particles experience few interactions and therefore the dynamics take place mainly where the harmonic confinement is absent. On the other hand, in the presence of sufficiently high densities the interactions between the particles are enhanced. This implies that the previously forbidden directions (i.e. those where there is the harmonic trap) begin to be involved in the dynamics.

As a conclusive remark, we observe that it is possible to study the sound velocity behavior by following another approach as well. We can, in fact, determine analytically $n_{\text{eq}}(\sigma)$ and $n_{\text{eq}}(\eta)$ (instead of $\sigma(n_{\text{eq}})$ and $\eta(n_{\text{eq}})$) from equations (37) and (42) respectively, that when employed in equations (36) and (41) give

$$c_s = \omega_{\perp} \left( \frac{\gamma(\sigma^4 - \lambda a_\perp^4)}{2\sigma^2} \right)^\frac{1}{2}$$

(51)

in the CS case, and

$$c_s = \omega_z \left( \frac{\gamma(\eta^4 - \lambda a_z^4)}{2\eta^2} \right)^\frac{1}{2}$$

(52)

for the DS configuration.
Figure 2. Cigar-shaped (CS) case and disc-shaped (DS) one with $\gamma = 2$ and $\lambda = 1$. Left: CS configuration. Top panel: radial width $\sigma$ versus $\nu_{n_{eq}}$ ($\nu = a_s N, a_s$ in units of $a_\perp$), equation (48). Bottom panel: sound velocity $c_s$ versus $\nu_{n_{eq}}$. Solid line: equation (47) solved with the constraint (48). Dashed line: 1D regime, equation (47) with $\sigma = 1$. Dotted line: 3D regime (see the text). Units as in figure 1. Right: DS configuration. Top panel: axial width $\eta$ versus $\nu_{n_{eq}}$ ($\nu = a_s N, a_s$ in units of $a_z$), equation (50). Bottom panel: sound velocity $c_s$ versus $\nu_{n_{eq}}$. Solid line: equation (49) solved with the constraint (50). Dashed line: 2D regime, equation (49) with $\eta = 1$. Dotted line: 3D regime (see the text). Units as in figure 1.

These formulas could have a greater utility in experiments rather than a theoretical use, because they allow one to determine the wave propagation velocity by simply measuring the average width of the superfluid confined gas.

7. Conclusions

We have considered a confined two-component Fermi gas, at zero temperature, both in the cigar- and in the disc-shaped configuration realized by a strong harmonic potential in the radial plane and in the axial direction respectively. We have studied this system via the Popov Lagrangian density by employing a Gaussian ansatz on the density. By integrating out the directions where the dynamics are frozen, a dimensional reduced Lagrangian has been achieved. We have studied the associated Euler–Lagrange equations by presuming to create a sufficiently weak perturbation with respect to the equilibrium when the system is in the stationary regime. We have numerically determined the behavior of the sound velocity as a function of the equilibrium density, both in the BCS side and in the BEC region of the crossover, which has been compared to that of the pure three-dimensional (3D) regime and to those corresponding to the pure one-dimensional (1D) (for the cigar-shaped case) and two-
dimensional (2D) (for the disc-shaped case) regimes. Thanks to this comparison, we have pointed out that our numerical solutions well approximate the 3D results for sufficiently high densities and the 1D (2D) behavior when the density is low enough, our numerical solutions being able to capture the physics in the intermediate regime as well.

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References

[1] Eagles D M 1969 Phys. Rev. 186 456
[2] Leggett A J 1980 Modern Trends in the Theory of Condensed Matter ed A Pekalski and J Przystawa (Berlin: Springer) p 13
[3] Nozieres P and Schmitt-Rink S 1985 J. Low Temp. Phys. 59 195
[4] Greiner M, Regal C A and Jin D S 2003 Nature 426 537
[5] Regal C A, Greiner M and Jin D S 2004 Phys. Rev. Lett. 92 040403
[6] Kinast J, Hemmer S L, Gehm M E, Turlapov A and Thomas J E 2004 Phys. Rev. Lett. 92 150402
[7] Zwierlein M W et al 2004 Phys. Rev. Lett. 92 120403
[8] Chin C et al 2004 Science 305 1128
[9] Bartenstein M et al 2004 Phys. Rev. Lett. 92 203201
[10] Moerdijk A J, Verhaar B J and Axelsson A 1995 Phys. Rev. A 51 4852
[11] Görlitz A et al 2001 Phys. Rev. Lett. 87 130402
[12] Schreck F, Khaykovich L, Corwin K L, Ferrari G, Bourdel T, Cubizolles J and Salomon C 2001 Phys. Rev. Lett. 87 080403
[13] Kinoshita T, Wenger T and Weiss D S 2004 Science 305 1125
[14] Oshianii M 1998 Phys. Rev. Lett. 81 938
[15] Jackson A D, Kavoulakis G M and Pethick C J 1998 Phys. Rev. A 58 2417
[16] Chiofalo M L and Tosi M P 2000 Phys. Lett. A 268 406
[17] Shlyapnikov D S, Petrov G V and Walraven J T M 2000 Phys. Rev. Lett. 84 2551
[18] Salasnich L, Parola A and Reatto L 2002 Phys. Rev. A 65 043614
[19] Salasnich L, Parola A and Reatto L 2004 Phys. Rev. A 69 045601
[20] Schneider J and Wallis H 1998 Phys. Rev. A 57 1253
[21] Salasnich L 2000 J. Math. Phys. 41 8016
[22] Salasnich L, Pozzi B, Parola A and Reatto L 2000 J. Phys. B: At. Mol. Opt. Phys. 33 3943
[23] Salasnich L, Reatto L and Parola A 2001 Perspectives in Theoretical Nuclear Physics VIII ed G Pisent et al (Singapore: World Scientific) pp 391–46
[24] Vignolo P and Minguzzi A 2003 Phys. Rev. A 67 053601
[25] Bruun G M and Clark C W 1999 Phys. Rev. Lett. 83 5415
[26] Minguzzi A, Vignolo P, Chiofalo M L and Tosi M P 2001 Phys. Rev. A 64 033605
[27] Das K K 2003 Phys. Rev. Lett. 90 170403
[28] Salasnich L, Adhikari S K and Toigo F 2007 Phys. Rev. A 75 023616
[29] Capuzzi P, Vignolo P, Federici F and Tosi M P 2004 J. Phys. B: At. Mol. Opt. Phys. 37 S91
[30] Ghosh T K and Machida K 2006 Phys. Rev. A 73 013613
[31] Capuzzi P, Vignolo P, Federici F and Tosi M P 2006 Phys. Rev. A 73 021603
[32] Capuzzi P, Vignolo P, Federici F and Tosi M P 2006 Phys. Rev. A 74 057601
[33] Shanenko A A, Croitoru M D, Vagov A V, Axt V M, Perali A and Peeters F M 2012 Phys. Rev. A 86 033612
[34] Popov V N 1983 Functional Integrals in Quantum Field Theory and Statistical Physics (Dordrecht: Reidel)
[35] Stritto M D and De Silva T N 2012 Phys. Lett. A 376 2298–305
[36] von Weizsacher C F 1935 Z. Phys. 96 431
[37] Kirzhnits D A 1957 Sov. Phys.—JEPT 5 64
Kirzhnits D A 1967 Field Theoretical in Many-Body Systems (London: Pergamon)
[38] Holas A, Kozlowski P M and March N H 1991 J. Phys. A: Math. Gen. 24 4249
[39] Salasnich L 2007 J. Phys. A: Math. Theor. 40 9987
[40] Salasnich L and Toigo F 2008 Phys. Rev. A 78 053626
[41] Salasnich L 2009 Laser Phys. 19 642–6