QUANTUM GROUP DUALITY AND THE CUNTZ ALGEBRA

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Abstract. The Cuntz algebra carries in a natural way the structure of a module algebra over
the quantized universal enveloping algebra $U_q(g)$, and the structure of a co-module algebra over
the quantum group $G_q$ associated with $U_q(g)$. These two algebraic structures are dual to each
other via the duality between $G_q$ and $U_q(g)$.

1. Introduction

This paper is motivated by earlier work of one of us [12] on co-actions of (Woronowicz’ [15])
quantum groups on Cuntz algebras [2] and by studies by Doplicher and Roberts [4] of group
actions on Cuntz algebras.

The general theory of Hopf algebras [11] suggests that we should seek actions of the Drinfeld-
Jimbo [3, 10] deformations of universal enveloping algebras of semi-simple Lie algebras dual to
the co-actions (found in [12]) of the Woronowicz quantum groups. This is achieved in this paper
in a natural way. An interesting feature is the appearance of braid groups as the generators of
the fixed and co-fixed points. By analogy with [4] this paper provides interesting parallels with
the work on braid group statistics in two dimensional quantum field theory [7].

To summarize our results we collect them here in the introduction into two theorems. To
make the proofs easily accessible we include some expository material in Section 2 where we
introduce notation and basic facts. We denote by $U_q(g)$ the quantized universal enveloping
algebra of a simple Lie algebra $g$. When $0 < q < 1$ the algebraic dual of $U_q(g)$ contains the
Woronowicz quantum group $\overline{G_q^{(\pi)}}$. (The bar indicates closure in Woronowicz’ $C^*$-norm of a dense
Hopf subalgebra $G_q^{(\pi)}$ introduced in subsection 2.2). In Section 3 we describe realizations of the
braid group in the Cuntz algebra for generic $q$ (that is, not a root of unity).

When $0 < q < 1$ a co-action of $\overline{G_q^{(\pi)}}$ on the Cuntz algebra $O_d$ on $d$ generators (where $d$ is the
dimension of a representation of $g$ chosen so that its tensor powers contain all the irreducible
representations of $g$ as subrepresentations) was discovered in [12]. It was also shown that there is
a homomorphism $\alpha$ of the braid group $B_n$ on $n$ generators into $O_d$ such that the co-fixed points
under the $\overline{G_q^{(\pi)}}$ co-action were generated by $\alpha(B_n)$. In section 3 we review this embedding of the
braid groups in $O_d$.

The main results proved in this paper in Sections 4 and 5 may be stated as follows.

Theorem 1: For generic $q$ there is an action of $U_q(g)$ on the Cuntz algebra $O_d$ by non-unital non-
star densely defined endomorphisms with the following properties:
(i) The domain of the endomorphisms is the dense subalgebra $O_d^0$ of $O_d$ given by polynomials in the
generators.
(ii) There is a co-action of $G_q^{(\pi)}$ on $O_d^0$ dual to this $U_q(g)$ action. This co-action coincides for
0 < q < 1 with the co-action of $G_q^{(π)}$ discovered in [12];
(iii) The braid group elements in $O^0_d$ are both fixed and co-fixed.
(iv) The fixed and co-fixed elements of $O^0_d$ coincide.

We relate this to the results in [12] at the end of Section 5 in the following way.

**Theorem 2:** (i) There is a distinguished element of $O^0_d$ which we call the rank $d_q$-antisymmetric tensor which is fixed and co-fixed by the action and co-actions of theorem 1.
(ii) When $0 < q < 1$ the braid group elements together with the rank $d_q$-antisymmetric tensor algebraically generate the fixed points of the $U_q(g)$ action and generate, in the topological sense, the co-fixed subalgebra of $O_d$ under the $G_q^{(π)}$ action.

Section 4 contains the definition of the $U_q(g)$ action, and also discusses the braids as fixed points. Section 5 gives the dual co-action of $G_q^{(π)}$ and completes the proof of both theorems. A natural question to ask is what happens if one dualizes the co-action and section 6 is devoted to showing that we recover the given action of $U_q(g)$. There is a slight technical point in that the Hopf dual of $G_q^{(π)}$ is not obviously $U_q(g)$. There is a possibly non-trivial ideal which one needs to factor out. However in the Cuntz realization this ideal acts trivially so that we have duality working fully.

In order to understand the observations of this paper at a more fundamental level we conclude our discussion in the final section by developing the action and co-action within the framework of braided tensor categories (cf [12]). The viewpoint in this paper is algebraic in that we have not attempted to analyse the role played by the $C^*$-algebra topology on the Cuntz algebra from the viewpoint of the $U_q(g)$ action. It may be that there is an interesting connection between this topology, that on $G_q^{(π)}$ described in [15], and some, as yet undetermined, topology on $U_q(g)$.

2. Quantized universal enveloping algebra and dual quantum group

We review the definitions and some properties of the quantized universal enveloping algebras and their dual quantum groups in this section.

2.1. Quantized universal enveloping algebra $U_q(g)$. Let $g$ be any finite dimensional simple Lie algebra over the complex field $\mathbb{C}$. Denote by $Φ^+$ the set of the positive roots of $g$ relative to a base $Π = \{α_1, ..., α_r\}$, where $r$ is the rank of $g$. Define $E = \bigoplus_{i=1}^{r} \mathbb{R}_{α_i}$. Let $(\ , \ ) : E \times E \to \mathbb{R}$ be an inner product of $E$ such that the Cartan matrix $A$ of $g$ is given by

$$A = (a_{ij})_{ij=1}^{r}, \quad a_{ij} = \frac{2(α_i, α_j)}{(α_i, α_i)}.$$

The Jimbo version [11] of the quantized universal enveloping algebra $U_q(g)$ is defined to be the unital associative algebra over $\mathbb{C}$, generated by $\{h_i, e_i, f_i | i = 1, ..., r\}$ with the following relations

$$k_i k_j = k_j k_i, k_i^{-1} = 1,$$
$$k_i e_j k_i^{-1} = q_i^2 e_j k_i f_j k_i^{-1} = q_i^{-2} f_j,$$
$$[e_i, f_j] = δ_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{pmatrix} 1 & -a_{ij} \\ t & 0 \end{pmatrix}_{q_i} (e_i)^t e_j (e_i)^{1-a_{ij}-t} = 0, \quad i \neq j,$$
\[
\sum_{t=0}^{1-a_{ij}} (-1)^t \left[ \frac{1 - a_{ij}}{t} \right]_{q_i} (f_i)^t f_j (f_i)^{1-a_{ij} - t} = 0, \quad i \neq j, \tag{1}
\]

where \( q \) is a complex parameter, which is assumed to be non-zero, and is not a root of unity. Also, \( \left[ \begin{array}{c} s \\ t \end{array} \right]_q \) is the Gauss polynomial, and \( q_i = q^{(\alpha_i, \alpha_i)/2} \).

The algebra \( U_q(g) \) has in addition the structure of a Hopf algebra. We take the following co-multiplication

\[
\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}, \\
\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i, \\
\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i.
\]

The co-unit \( \epsilon : U_q(g) \to \mathbb{C} \) and antipode \( \gamma : U_q(g) \to U_q(g) \) are respectively given by

\[
\epsilon(e_i) = \epsilon(f_i) = 0, \quad \epsilon(k_i^{\pm 1}) = \epsilon(1) = 1, \\
\gamma(e_i) = -e_i k_i^{-1}, \\
\gamma(f_i) = -k_i f_i, \\
\gamma(k_i^{\pm 1}) = k_i^{\mp 1}.
\]

Later in the paper we will need the notion of a ‘universal \( R \)-matrix’ in the sense used in connection with the Yang-Baxter equations. This \( R \)-matrix does not live naturally in the Jimbo picture. For this we need the Drinfeld version \[5\]. If we set

\[
q = \exp(\zeta),
\]

and regard \( \zeta \) as a formal indeterminate, then the Drinfeld version \[5\] of the quantized universal enveloping algebra is an associative algebra over \( \mathbb{C}[[\zeta]] \) completed with respect to the \( \zeta \)-adic topology for \( \mathbb{C}[[\zeta]] \). It is generated by \( \{ e_i, f_i, h_i, i = 1, 2, ..., r \} \) with

\[
k_i^{\pm 1} = q^{\pm h_i},
\]

subject to the same relations (1).

We use the notation \( U_\zeta(g) \) to denote this algebra which in the terminology of Drinfeld is a quasi-triangular Hopf algebra. This means it admits an invertible \( R \in U_\zeta(g) \otimes U_\zeta(g) \) ( \( \otimes \) represents tensor product completed with respect to the \( \zeta \)-adic topology ), called the universal \( R \)-matrix, which satisfies the following defining relations

\[
R \Delta(a) = \Delta'(a) R, \quad \forall a \in U_\zeta(g), \\
(\Delta \otimes \text{id}) R = R_{13} R_{23}, \\
(\text{id} \otimes \Delta) R = R_{13} R_{12}.
\]

Further general properties of \( R \) are

\[
(\gamma \otimes \text{id}) R = (\text{id} \otimes \gamma) R = R^{-1} \\
(\epsilon \otimes \text{id}) R = (\text{id} \otimes \epsilon) R = 1, \\
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12},
\]

where the last equation is the celebrated quantum Yang-Baxter equation, which is a direct consequence of the defining relations of \( R \).
The universal $R$-matrix is of the form
\[ R = q \sum_{i,j} (B^{-1})_{ij} h_i \otimes h_j [1 \otimes 1 + \sum C(s) E(s) \otimes F(s)], \]
where
\[ B = ((\alpha_i, \alpha_j)) \]
$E(s)$ are combinations of products of $e_i$'s,
$F(s)$ are combinations of products of $f_i$'s,
$C(s)$ are scalars in $\mathbb{C}[[\zeta]]$.

Moreover,
\[ R^{-1} \otimes 1 = \sum_{i,j} (B^{-1})_{ij} h_i \otimes h_j + \sum_{\alpha \in \Phi^+} e_\alpha \otimes f_\alpha + o(\zeta), \tag{2} \]
where $e_\alpha$ and $f_\alpha$ are the quantum analogs of the Cartan-Weyl generators.

Given any nontrivial irreducible representation $\pi$ of $U_\zeta(g)$, we set
\[ T^+ = (\pi \otimes \text{id}) R^T, \quad T^- = (\pi \otimes \text{id}) R^{-1}. \tag{3} \]

**Lemma 1:** The matrix elements of $T^+$ and $T^-$ generate the entire Drinfeld algebra $U_\zeta(g)$ topologically.

**Proof.** Recall that the Drinfeld algebra is a deformation of the universal enveloping algebra $U(g)$ of the simple Lie algebra $g$ in the sense of Gerstenhaber [8]. Therefore
\[ U(g) = U_\zeta(g) / \zeta U_\zeta(g). \]

Denote by $U(g) [[\zeta]]$ the universal enveloping algebra of $g$ over $\mathbb{C}[[\zeta]]$ completed with respect to the $\zeta$-adic topology. The rigidity of $U(g)$ leads to the conclusion that $U(g) [[\zeta]]$ and $U_\zeta(g)$ are isomorphic as associative algebras and hence they have the same representation theory. Given these facts we can easily deduce that in order to prove the Lemma, it suffices to show that the matrix elements of
\[ \frac{1}{\zeta} \left( T^\pm - I \otimes 1 \right) / \zeta U_\zeta(g) \]
generate $U(g)$. Here $\zeta U_\zeta(g)$ denotes the ideal in $U_\zeta(g)$ generated by $\zeta$. Using the well known fact that the trace over any finite dimensional irreducible representation of the simple Lie algebra $g$ defines an invariant non-degenerate bilinear form, we obtain from (2)
\[ \text{tr}_{\pi} \left\{ \frac{1}{\zeta} \left( T^\pm - I \otimes 1 \right) [\pi(h_i) \otimes 1] \right\} = \pm c_\pi h_i + o(\zeta), \]
\[ \text{tr}_{\pi} \left\{ \frac{1}{\zeta} \left( T^+ - I \otimes 1 \right) [\pi(e_i) \otimes 1] \right\} = c_\pi e_i + o(\zeta), \]
\[ \text{tr}_{\pi} \left\{ \frac{1}{\zeta} \left( T^- - I \otimes 1 \right) [\pi(f_i) \otimes 1] \right\} = -c_\pi f_i + o(\zeta), \]
where $c_\pi \in \mathbb{C}$ is a nonvanishing constant, and the $\hat{h}_i$ are independent linear combinations of the $h_j$ such that
\[ [\hat{h}_i, e_j] = a_{ij} e_j. \]
Quotienting by $\zeta U_\zeta(g)$ is equivalent to taking $\zeta \to 0$ in the preceding equations whose right hand sides then clearly generate $U(g)$, thus completing the proof of the Lemma.
Observe also the following important fact: \( T^{(\pm)} \) can be expressed solely in terms of \( e_i, f_i, k_i^{\pm 1} \), and \( k_i^{\pm 1} = q^{\pm h_i^\mu} \), with \( \mu \) being the highest weight of \( \pi \), and \( h_i^\mu \) a linear combination of the \( h_i \) such that \( [h_i, e_i] = (\mu, \alpha_i) e_i \). For some Lie algebras it can happen that \( \mu \) is not in the root lattice of \( g \). In that case, \( k_i^{\pm 1} \) cannot be expressed as products of integer powers of the \( k_i^{\pm 1} \).

Let us now specialize \( q \) to a complex parameter. The matrix elements of \( T^{(\pm)} \) generate, algebraically, an associative algebra \( U_q(g) \) over the complex field \( \mathbb{C} \). If \( \mu \) belongs to the root lattice of \( g \), this algebra coincides with the Jimbo quantized universal enveloping algebra, otherwise, it contains \( U_q(g) \) as a subalgebra. The action of the Jimbo algebra we define in section 4 extends to the algebra \( \tilde{U}_q(g) \) but not uniquely in the latter case. The extensions differ however only by the fact that one needs to choose a particular (complex) root of \( q \), and are related to one another by an action of the appropriate group of roots of unity as automorphisms. At generic \( q \), this difference between the two algebras \( U_q(g) \) and \( \tilde{U}_q(g) \) is thus not at all important. Finally we remark that if we further assume that \( q \) is real, then \( U_q(g) \) has the structure of a Hopf \( * \)-algebra.

2.2. The quantum group \( G_q^{(\pi)} \). We now move on to set up the notation and properties of the dual to \( U_q(g) \), the Woronowicz quantum group. The finite dual \( (U_q(g))^0 \) of \( U_q(g) \) has a natural Hopf algebra structure, with the multiplication \( m_0 \), co-multiplication \( \Delta_0 \), unit \( 1_\pi \), co-unit \( \epsilon_0 \), and antipode \( \gamma_0 \) defined in the standard fashion [11]. We consider a subalgebra \( G_q^{(\pi)} \) of \( (U_q(g))^0 \) defined in the following way. Let \( \pi \) be a finite dimensional non-trivial representation of \( U_q(g) \), which can be assumed to be irreducible without losing generality. Set \( d = \dim_{\mathbb{C}} \pi \). Consider the matrix

\[
U = (u_{ij})_{i,j=1}^d, \quad u_{ij} \in (U_q(g))^0,
\]

defined by

\[
\langle u_{ij}, a \rangle = \pi(a)_{ij}, \quad \forall a \in U_q(g).
\]

**Definition:** We define \( G_q^{(\pi)} \) as the associative subalgebra of \( (U_q(g))^0 \) generated by the matrix elements of \( U \), with the multiplication defined by

\[
\langle u_{ij} u_{kl}, a \rangle = \sum_{(a)} \pi_{ij}(a_{(1)}) \pi_{kl}(a_{(2)}), \quad \forall a \in U_q(g).
\]

For \( 0 < q < 1 \) in [13] this algebra is completed in an appropriate \( C^* \)-algebra norm. For the most part we will not need this topology here and will work with the uncompleted algebra. When necessary we denote the \( C^* \) completion by a bar. Clearly the unit of \( G_q^{(\pi)} \) coincides with the co-unit \( \epsilon \) of \( U_q(g) \). Set

\[
R^{(\pi)} = (\pi \otimes \pi) R,
\]

where \( R \) is the universal \( R \)-matrix of \( U_q(g) \).

**Lemma 2:** \( U \) satisfies the following quadratic relation

\[
R_{12}^{(\pi)} U_1 U_2 = U_2 U_1 R_{12}^{(\pi)}.
\]

**Proof:** We first note that the left-hand side contracted with any \( a \in U_q(g) \) gives

\[
R_{12}^{(\pi)} U_1 U_2(a) = R^{(\pi)}(\pi \otimes \pi) \Delta(a);
\]

while the right hand side yields

\[
U_2 U_1 R_{12}^{(\pi)}(a) = (\pi \otimes \pi) \Delta'(a) R^{(\pi)}.
\]
Then equation (4) immediately follows from the fact that
\[ R^{(\pi)}(\pi \otimes \pi)\Delta(a) = (\pi \otimes \pi)\Delta'(a)R^{(\pi)}, \quad \forall a \in U_q(g), \]
completing the proof.

It is also easy to see that \( G_q^{(\pi)} \) has the structure of a bi-algebra. The co-multiplication is given by
\[ \Delta_0(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad (5) \]
which follows from the equation
\[ (\Delta_0(u_{ij}), a \otimes b) = \pi_{ij}(ab), \quad \forall a, b \in U_q(g), \]
while the co-unit is \( 1_{U_q(g)} \)
\[ u_{ij}(1_{U_q(g)}) = \delta_{ij}. \]
Furthermore, \( G_q^{(\pi)} \) admits an antipode \( \gamma_0 \). Let \( \pi^\dagger \) be the irreducible representation of \( U_q(g) \) dual to \( \pi \). If \( V \) is the module furnishing the representation \( \pi \), we take \( V^* \) to be the dual vector space on which the dual representation is defined by \( (\pi^\dagger(a), v^*)(v') = v^*(\pi(\gamma(a)), v') \), where \( \gamma \) is the antipode for \( U_q(g) \). Then the antipode for \( G_q^{(\pi)} \) is given by
\[ \gamma_0(U) = U^{-1}, \quad (6) \]
where \( U^{-1} \) is another \( d \otimes d \) matrix, the entries of which belong to \( (U_q(g))^0 \) and satisfy
\[ (U^{-1})_{ij}(a) = \pi^\dagger(a)_{ij}, \quad \forall a \in U_q(g). \]
For this to make sense, we need to show that

**Lemma 3:** The elements of \( U^{-1} \) belong to \( G_q^{(\pi)} \).

**Proof.** The Lemma is equivalent to the statement that some repeated tensor product of \( \pi \) (with respect to the co-multiplication \( \Delta \)) contains the dual representation \( \pi^\dagger \) as an irreducible component. For this to be true, it suffices to show that a one dimensional representation can arise from nontrivial tensor products of \( \pi \). Let \( V \) be the module which furnishes the representation \( \pi \). We claim that there exists a nonvanishing \( \Lambda \in V^{\otimes d} \) which generates a one dimensional representation of \( U_q(g) \). By calling upon the Lusztig-Rosso theorem, \[ 3, 13 \] we conclude that there exists a nonvanishing \( \Lambda \in V^{\otimes d} \), which reduces to \( \Lambda^{(0)} \) in the \( q \to 1 \) limit, such that
\[ x \cdot \Lambda^{(0)} = 0, \quad \forall x \in g. \]

By calling upon the Lusztig-Rosso theorem, \[ 3, 13 \] we conclude that there exists a nonvanishing \( \Lambda \in V^{\otimes d} \) which reduces to \( \Lambda^{(0)} \) in the \( q \to 1 \) limit, such that
\[ a \cdot \Lambda = \epsilon(a), \quad \forall a \in U_q(g). \]
This completes the proof of the Lemma.

**Definition.** We will call \( \Lambda \) the rank \( d(= \dim \pi) \) \( q \)-antisymmetric tensor of \( V \).
2.3. Modules and co-modules. We begin with some generalities on co-actions for Hopf algebras. Let $A$ be a Hopf algebra with co-multiplication $\Delta$, co-unit $\epsilon$ and antipode $\gamma$. Let $V$ be a left $A$-module, which is assumed to be locally finite, i.e., it satisfies the following properties: corresponding to each $v \in V$, we can find a finite set of elements $v_i \in V$, $i = 1, 2, \ldots, N$, such that

$$v = \sum_{i=1}^{N} c_i v_i, \quad c_i \in \mathbb{C},$$

$$a \circ v_i = \sum_{j=1}^{N} \psi_{ji}(a) v_j, \quad \forall a \in A,$$

where $\psi_{ji}(a) \in \mathbb{C}$.

Let $A^0$ be the finite dual Hopf algebra with multiplication $m_0$, unit $\epsilon$, co-unit $1_A$, co-multiplication $\Delta_0$ and antipode $\gamma_0$. The left $A$-module $V$ automatically carries a right $A^0$ co-module structure

$$\omega : V \rightarrow V \otimes A^0$$

defined in the following way. For any element $v \in V$, if we write

$$\omega(v) = \sum_{(v)} v_{(1)} \otimes v_{(2)}, \quad v_{(2)} \in A^0,$$

then for all $a \in A$,

$$\omega(v)(a) = \sum_{(v)} v_{(1)} v_{(2)}(a) = a \circ v$$

To be more explicit, we consider

$$\omega(v_i) = \sum_{j} v_j \otimes u_{ji}, \quad (7)$$

where $u_{ij}$ are elements of $A^0$ which are uniquely determined by the requirement that

$$u_{ij}(a) = \psi(a)_{ij}, \quad \forall a \in A.$$ 

It follows from

$$\{(id \otimes \Delta_0)\omega(v_i)\}(a \otimes b) = \omega(v_i)(ab)$$

that

$$\Delta_0(u_{ij}) = \sum_{k} u_{ik} \otimes u_{kj}.$$ 

Now

$$(id \otimes \Delta_0)\omega(v_i) = \sum_{j,k} v_{j} \otimes u_{jk} \otimes u_{ki}.$$ 

On the other hand,

$$(\omega \otimes id)\omega(v_i) = \sum_{j} \omega(v_i) \otimes u_{ji} = \sum_{j,k} v_{k} \otimes u_{kj} \otimes u_{ji},$$

hence

$$(id \otimes \Delta_0)\omega = (\omega \otimes id)\omega.$$ 

Denote by $\epsilon^*$ the co-unit of $A^0$, i.e., for any $a^* \in A^0$, $\epsilon^*(a^*) = \langle a^*, 1_A \rangle$. Then

$$(id_V \otimes \epsilon^*)\omega(v_i) = \sum_{j} v_{j} \otimes \delta_{ji} = v_i,$$

i.e.

$$(id_V \otimes \epsilon^*)\omega = id_V.$$
The above equations show that $\omega$ is indeed a right-co-module action of $A^0$ on $V$.

2.4. Fixed points and co-fixed points. We continue the notation of the previous subsection. As $V$ is a module over $A$, and at the same time, a co-module over $A^0$, we introduce the notation $(V)^A$ for the fixed point set of $V$ under the action of $A$, and $(V)_{A^0}$ for the co-fixed point set of $V$ under the co-action of $A^0$. Thus

$$(V)^A = \{ v \in V \mid a \circ v = \epsilon(a)v, \forall a \in A \},$$

$$(V)_{A^0} = \{ u \in V \mid w(u) = u \otimes 1_{A^0} \}.$$ 

Consider $v \in V$. If $v$ is co-fixed by $A^0$, i.e.,

$$\omega(v) = v \otimes \epsilon,$$

then

$$a \circ v = \omega(v)(a) = \epsilon(a)v,$$

hence $v$ is also fixed by $A$. On the other hand, if $v \in V$ is fixed by $A$, then

$$a \circ v = \epsilon(a)v = \sum_{(v)} v_{(1)}v_{(2)}(a), \forall a \in A,$$

i.e.

$$\omega(v) = v \otimes \epsilon,$$

thus $v$ is also co-fixed. Therefore, $(V)^A = (V)_{A^0}$.

2.5. The $q$-determinant. Returning now to the discussion begun in subsection 2.2 we will need the notion of $q$-determinant of the matrix $U$ used to define $G_q^{(\pi)}$. Let $V$ denote the representation space of $\pi$. By the discussion of the preceding two subsections we may consider the co-action of $G_q^{(\pi)}$, denoted by $\omega$, on $V$. Now this co-action when applied to the $q$-antisymmetric tensor $\Lambda$ yields

$$\omega(\Lambda) = \Lambda \otimes \det_q U,$$

where $\det_q U$ is some element of $G_q^{(\pi)}$.

**Definition.** We call $\det_q U$ the $q$-determinant of $U$.

However, invoking the argument of the preceding subsection, $\Lambda$ must be a co-fixed point of $G_q^{(\pi)}$ as $\Lambda$ generates a trivial module of $U_q(g)$. Hence

$$\det_q U = \epsilon,$$

i.e., $U$ has $q$-determinant $1_{G_q}(= \epsilon)$.

Note that in [13] the quantum determinant is set equal to the identity as an additional relation, while this relation is built into the definition of $G_q^{(\pi)}$ used here.

**Remarks:** We need to comment on the representation $\pi$ and the dependence on it of $G_q^{(\pi)}$. If $\pi$ can generate all the finite dimensional representations of $U_q$ by repeated tensor products, then $G_q^{(\pi)}$ will contain all $G_q^{(\pi')}$ as Hopf subalgebras, where $\pi'$ is any representation of $U_q(g)$. The vector representation of $U_q(sl(n))$ has this property, as does the spinor representation of $U_q(so(n))$. A second point is that for a generic $q$, all finite dimensional representations of $U_q(g)$ are completely reducible, so we can simply take $\pi$ to be an irreducible representation. In general, repeated tensor products of $\pi$ can be expressed as direct sums of a subclass of irreps of $U_q(g)$. Components of these irreps form a basis of $G_q^{(\pi)}$, and this statement may be regarded as an algebraic analogue.
of the Peter-Weyl theorem in the quantum setting. The multiplication rule of \( G_h^{(\pi)} \) in terms of such a basis is the Clebsch-Gordon decomposition of representations of \( U_q(g) \).

2.6. The dual Hopf algebra of \( G_h^{(\pi)} \). In this subsection we regard \( G_h^{(\pi)} \) as a Hopf algebra in its own right. It is defined to be the Hopf algebra generated by the matrix elements of \( U \), subject to the relations (4) together with the \( q \)-determinant condition

\[
\det_q U = 1_{G_h^{(\pi)}}.
\]

The co-multiplication, co-unit and antipode are as given before. We want to investigate the finite dual of \( G_h^{(\pi)} \).

Consider the set of elements \( \{ l_{ij}^{(\pm)} \in (G_h^{(\pi)})^* | i, j = 1, \ldots, d \} \), which satisfy the following properties: write

\[
L^{(\pm)} = \sum e_{ij} \otimes l_{ij}^{(\pm)}.
\]

Then

\[
\langle 1, U^\otimes r \rangle = I^\otimes r
\]

\[
\langle L^{(\pm)}, U_1 U_2 \ldots U_r \rangle = R^{(\pm)}_1 R^{(\pm)}_2 \ldots R^{(\pm)}_r,
\]

\[
\langle L^{(-)}, U_1 U_2 \ldots U_r \rangle = R^{(-)}_1 R^{(-)}_2 \ldots R^{(-)}_r,
\]

(8)

where

\[
R^{(\pm)} = PR^{(\pi)} P, \quad R^{(-)} = (R^{(\pi)})^{-1},
\]

and our notation is largely the same as that of \([4]\). By considering the co-multiplication of \( G_h^{(\pi)} \), one can show \([4]\) that \( L^{(\pm)} \) satisfy the following relations

\[
R^{(\pm)}_1 R^{(\pm)}_2 L^{(\pm)}_2 L^{(\pm)}_1 = L^{(\pm)}_1 L^{(\pm)}_2 R^{(\pm)}_1 R^{(\pm)}_2,
\]

\[
R^{(\pm)}_1 R^{(-)}_2 L^{(-)}_2 L^{(\pm)}_1 = L^{(-)}_1 L^{(\pm)}_2 R^{(-)}_1 R^{(\pm)}_2.
\]

(9)

Let us denote by \( U'_q \) the algebra generated by the matrix elements of \( L^{(\pm)} \). A co-multiplication for \( U'_q \) is given by

\[
\Delta(L^{(\pm)}) = L^{(\pm)} \otimes L^{(\pm)} \quad \text{(i.e. } \Delta(l_{ij}^{(\pm)}) = \sum_k l_{ik}^{(\pm)} \otimes l_{kj}^{(\pm)})
\]

and the corresponding co-unit and antipode are respectively given by

\[
\epsilon(L^{(\pm)}) = I, \quad \gamma(L^{(\pm)}) = (L^{(\pm)})^{-1}.
\]

Define a linear map

\[
L^{(\pm)} \mapsto T^{(\pm)}, \quad L^{(-)} \mapsto T^{(-)}.
\]

This map extends in a unique way to a Hopf algebra homomorphism \( \phi : U'_q \to \tilde{U}_q(g) \), which is clearly surjective. The kernel of \( \phi \) is a Hopf ideal of \( U'_q \), and \( U'_q / \ker \phi \cong \tilde{U}_q(g) \) as Hopf algebras.
3. Cuntz algebra realizations of the braid group

3.1. Representations of the braid group. Consider $R^{(\pi)}$, where $\pi$ is a $d$-dimensional representation of $U_q(g)$ and let $P : V \otimes V \to V \otimes V$ be the flip: $P(a \otimes b) = b \otimes a$. Define

$$\sigma = PR^{(\pi)}.$$ 

It follows from the defining relations of the universal $R$ matrix that

$$[\sigma, (\pi \otimes \pi)\Delta(a)] = 0, \quad \forall \ a \in U_q(g),$$

$$(\sigma \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1) = (1 \otimes \sigma)(\sigma \otimes 1)(1 \otimes \sigma)$$

Note that $\sigma$ acts on $V \otimes V$, while the above equation holds as endomorphisms of $V \otimes V \otimes V$.

Define

$$b_i = 1 \otimes \cdots \otimes 1 \otimes \sigma \otimes 1 \otimes \cdots \otimes 1, \quad i = 1, 2, \ldots, n$$

Then we have the braid relations

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$$

$$b_i b_j = b_j b_i, \quad |i - j| \geq 2$$

Also,

$$[b_i, \pi^{\otimes (n+1)}\Delta^{(n)}(a)] = 0, \quad \forall \ a \in U_q(g)$$

3.2. Cuntz algebra realization of the braid group. Let $O_d$ be the Cuntz algebra on $d$ generators. This is the universal $C^*$-algebra generated by $\{s_j|j = 1, 2, \ldots, d\}$ satisfying $s_i^* s_j = \delta_{ij}$ and $\sum_j s_j s_j^* = 1$ where 1 denotes the identity of $O_d$.

Let $\{e_{ij}|i, j = 1, 2, \ldots, d\}$ be the matrix units for $\text{End} V$, which obey

$$e_{ij} e_{kl} = \delta_{jk} e_{il}.$$ 

There exists a well known algebra homomorphism

$$\eta : \text{End} V \to O_d,$$

defined by

$$\eta(e_{ij}) = s_i s_j^*.$$ 

This extends to an algebra homomorphism $\eta : \text{End} V^{\otimes m} \to O_d$ for each $m$, defined by

$$\eta(e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes \cdots \otimes e_{i_m j_m}) = s_{i_1} s_{i_2} \cdots s_{i_m-1} s_{i_m} s_{j_m}^* s_{j_{m-1}}^* \cdots s_{j_2}^* s_{j_1}^*.$$ 

We notice that $\sigma \in \text{End} V$ can be expressed as

$$\sigma = \sum_{i,j,k,l=1}^d \sigma_{ijkl} e_{ij} \otimes e_{kl},$$

which, under $\eta$, maps to

$$\theta = \eta(\sigma) = \sum \sigma_{ijkl} s_i s_k s_j^* s_l^*.$$ (11)

More generally, we have

$$\theta_i = \eta(b_i) = \sum_{\{l\}} s_{i_1} s_{i_2} \cdots s_{i_{l-1}} \theta s_{i_{l-1}}^* \cdots s_{i_2} s_{i_1}^*, \quad i = 1, 2, \ldots, n.$$ (12)
As elements of the Cuntz algebra, the $\theta_i$'s satisfy the braid group relations.

4. The Cuntz algebra as a module algebra over $U_q(g)$

Introduce the notation:

$$H = \bigoplus_{i=1}^d \mathbb{C}s_i, \quad H^0 = \mathbb{C}, \quad H^r = H^\otimes r, \quad \mathcal{H} = \bigoplus_{r=0}^\infty H^r,$$

where $\mathcal{H}$ is the algebraic direct sum. Let $H^*$ be the dual vector space of $H$,

$$H^* = \bigoplus_{1}^d \mathbb{C}s_i^*,$$

with the pairing $H^* \otimes H \rightarrow \mathbb{C}$ given by

$$\langle s_i^*, s_j \rangle = \delta_{ij}.$$

Set

$$H^{*r} = H^{*\otimes r}, \quad \mathcal{H}^* = \bigoplus_{r=0}^\infty H^{*r}.$$

The pairing between $H^{*r}$, $H^t$ is given by

$$\langle s_j^*, s_{j_{r-1}}^* \ldots s_{j_1}^*, s_{i_1} s_{i_2} \ldots s_{i_t} \rangle = \delta_{rt} \delta_{i_1 j_1} \delta_{i_2 j_2} \ldots \delta_{i_t j_t}.$$

This definition is compatible with Cuntz multiplication when $r$ and $t$ are equal. We now build up our action of $U_q(g)$ on the dense subalgebra of $O_d$ consisting of polynomials in the generators in three steps outlined in the following subsections. For convenience we introduce a notation for this subalgebra.

**Definition:** Let $O_d^0$ denote the dense subalgebra of $O_d$ consisting of polynomials in the generators.

4.1. $U_q(g)$ action on $\mathcal{H}$ and $\mathcal{H}^*$. We begin by defining the action of $U_q(g)$ on $\mathbb{C}$ by

$$a \circ c = \epsilon(a)c.$$

That is, we regard $\mathbb{C}$ as a trivial module over $U_q(g)$. Now let $\pi$ be the $d$-dimensional nontrivial irreducible representation of $U_q(g)$ introduced earlier. A $U_q(g)$-module action

$$U_q(g) \otimes H \xrightarrow{\circ} H$$

can be defined by setting

$$a \circ s_i = \sum_{j=1}^d \pi(a)_{ji}s_j.$$  \hspace{1cm} (14)

Since $\pi$ is irreducible, $H$ contains a unique (up to scalar multiples) highest weight vector $s^\pi$.

Now each $H^r$ furnishes a $U_q(g)$-module with the module action defined by the co-multiplication:

$$a \circ (s_{i_1}s_{i_2} \ldots s_{i_r}) = \sum_{\{j\}} \pi(a_{(1)})_{j{i_1}}\pi(a_{(2)})_{j{i_2}} \ldots \pi(a_{(r)})_{j{i_r}} \times s_{j_1}s_{j_2} \ldots s_{j_r}.$$

Now $\mathcal{H}$ becomes a $U_q(g)$-module, provided that we consider only finite linear combinations of vectors in the direct sum.
The dual vector spaces $H^{sr}$ have a natural $U_q(g)$-module structure. On $\mathcal{H}^*$, the $U_q(g)$ action is defined by requiring

$$\langle a \circ s_i^*, s_j \rangle = \langle s_i^*, \gamma(a) \circ s_j \rangle = \pi(\gamma(a))_{ij}.$$ 

Then each $H^{sr}$ becomes a tensor product module, and $\mathcal{H}^*$ is the module obtained as the algebraic direct sum of the $H^{sr}$'s. Explicitly the action of $U_q(g)$ on $H^{sr}$ is given by

$$\langle a \circ (s_j^s, s_{j-1}^s \cdots s_{j_1}^s), s_{i_1} s_{i_2} \cdots s_{i_v} \rangle = (\pi_{j_1 i_1} \otimes \cdots \otimes \pi_{j_v i_v}) \Delta^{(r-1)}(\gamma(a)).$$

Note that if we write

$$\Delta^{(k-1)}(a) = \sum (a(1) \otimes \cdots \otimes a(k),$$

then

$$\Delta^{(k-1)}(\gamma(a)) = \sum \gamma(a(k)) \otimes \gamma(a(k-1)) \otimes \cdots \otimes \gamma(a(1))$$

Hence

$$a \circ (s_j^s, s_{j-1}^s \cdots s_{j_1}^s) = \sum (a(1) \circ s_j^s)(a(2) \circ s_{j-1}^s) \cdots (a(v) \circ s_{j_1}^s)$$

4.2. $U_q(g)$ actions on $\mathcal{H} \otimes \mathcal{H}^*$ and $\mathcal{H}^* \otimes \mathcal{H}$. The actions are defined by the co-multiplication in the obvious way, namely, for $u \in \mathcal{H}$, $v^* \in \mathcal{H}^*$,

$$a \circ (u \otimes v^*) = \sum (a(1) \circ u) \otimes a(2) \circ v^*,$$

$$a \circ (v^* \otimes u) = \sum a(1) \circ v^* \otimes a(2) \circ u.$$ 

They have the following useful properties

$$\sum_i a \circ (u s_i^s \otimes v^*) = \sum_j (a(1) \circ u) s_j \otimes s_j^s (a(2) \circ v^*),$$

$$a \circ (v^* s_i^s \otimes s_j u) = \sum_{kl} (a(1) \circ v^*) \pi(\gamma(a(2)))_{ik} s_k^s \otimes \pi(a(3))_{lj} s_l (a(4) \circ u).$$

We wish to examine the properties of the module actions under Cuntz multiplication. Consider

$$a \circ (s_{j_1}^s, s_{j-1}^s \cdots s_{j_1}^s \otimes s_{i_1} s_{i_2} \cdots s_{i_t})$$

$$= \sum (a(1) \circ s_{j_1}^s)(a(2) \circ s_{j-1}^s) \cdots (a(v) \circ s_{j_1}^s) \otimes (a(v+1) \circ s_{i_1}) \cdots (a(r+t) \circ s_{i_t}).$$

Direct calculations can establish that

$$a \circ (s_{j_1}^s, s_{j-1}^s \cdots s_{j_1}^s \otimes s_{i_1} s_{i_2} \cdots s_{i_t})$$

$$= \sum (a(1) \circ s_{j_1}^s)(a(2) \circ s_{j-1}^s) \cdots (a(v) \circ s_{j_1}^s)(a(v+1) \circ s_{i_1}) \cdots (a(r+t) \circ s_{i_t}),$$

where both sides of the above equation are regarded as elements of the Cuntz algebra. It is also clearly true that

$$a \circ (s_{i_1} s_{i_2} \cdots s_{i_t} s_{j_1}^s, s_{j_2}^s)$$

$$= \sum (a(1) \circ s_{i_1})(a(2) \circ s_{i_2}) \cdots (a(t) \circ s_{i_t})(a(t+1) \circ s_{j_1}^s)(a(t+2) \circ s_{j_2}^s) \cdots (a(r+t) \circ s_{j_1}^s).$$

Therefore, the $U_q(g)$ action preserves the Cuntz multiplication.
4.3. $O^0_d$ as a module algebra over $U_q(g)$. The results of the last subsection suggest that a $U_q(g)$-module action on $O^0_d$:

\[ U_q(g) \otimes O^0_d \rightarrow O^0_d, \]

can be introduced directly in which each element of $U_q(g)$ acts by a non-unital endomorphism (by which we mean it preserves the multiplication but not the $*$-operation or identity) of $O^0_d$. This is achieved by defining

\[
\begin{align*}
a \circ 1 &= \epsilon(a) \\
a \circ s_i &= \sum_j \pi(a)_{ji}s_j, \\
a \circ s_i^* &= \sum_j \pi(\gamma(a))_{ij}s_j^*, \\
a \circ (xy) &= \sum_{(a)} (a(1) \circ x)(a(2) \circ y), \quad x, y \in O^0_d. \\
\end{align*}
\]

The defining relations of the Cuntz algebra are clearly preserved,

\[
a \circ (s_i^*s_j) = \sum_{(a)} (a(1) \circ s_i^*)(a(2) \circ s_j) = \delta_{ij}a \circ 1,
\]

\[
a \circ (\sum_{1 \leq i \leq d} s_is_i^*) = \sum_{1 \leq i \leq d} \sum_{(a)} (a(1) \circ s_i)(a(2) \circ s_i^*) = a \circ 1.
\]

Therefore, $O^0_d$ defines a module algebra over $U_q(g)$ under the action (15).

This module algebra structure of $O^0_d$ over $U_q(g)$ can be straightforwardly extended to $\tilde{U}_q(g)$ by specifying the action of $k_\pi$ on the highest weight vector $s^\pi$ of $H$,

\[ k_\pi \cdot s^\pi = q^{(\mu, \mu)} s^\pi. \]  

Note that when the highest weight $\mu$ of $\pi$ does not belong to the root lattice, $(\mu, \mu)$ is in general a rational number, and in that case we make a choice of the complex value of $q^{(\mu, \mu)}$.

4.4. Braids as fixed points. In the previous subsection we established the first claim of Theorem 1. To verify the remaining claims we now consider the fixed points. The fixed point set of $O^0_d$ under the $U_q(g)$ action (15) is

\[
\{ u \in O^0_d \mid a \circ u = \epsilon(a)u, \quad \forall a \in U_q(g) \}.
\]

Since $O^0_d$ is a $U_q(g)$ module algebra, the fixed point set defines a subalgebra of $O^0_d$. Consider the $U_q(g)$ action on the braid generator $\theta = \sum \sigma_{ij,kl}s_is_ks_i^*s_j^*$

\[
\begin{align*}
a \circ \theta &= \sum \left[ (\pi \otimes \pi)\Delta(a(1)) \cdot \sigma \cdot (\pi \otimes \pi)\Delta(\gamma(a(2))) \right]_{ij,kl} s_is_ks_i^*s_j^* \\
&= \sum \left[ (\pi \otimes \pi)\Delta(a(1)\gamma(a(2))) \cdot \sigma \right]_{ij,kl} s_is_ks_i^*s_j^* \\
&= \epsilon(a)\theta,
\end{align*}
\]

where we have used the fact that the braid generator $\sigma$ commutes with $(\pi \otimes \pi)\Delta(a)$, $\forall a \in U_q(g)$. More generally,

\[
\begin{align*}
a \circ \theta_{r+1} &= \sum a(1) \cdot (s_i \ldots s_{ir}) \cdot [a(2) \circ \eta(\sigma)]a(3) \circ (s_{ir}^* \ldots s_i^*) \\
&= \epsilon(a)\theta_{r+1}.
\end{align*}
\]

Thus we have shown that the braids $\theta_r \in O^0_d$, $\forall r = 0, 1, 2, \ldots$, are fixed points of the $U_q(g)$-action.
5. $O_d^0$ as a co-module algebra over $G_q^{(\pi)}$

5.1. The algebra $O_d^0$ as a co-module algebra over $G_q^{(\pi)}$. Following the general method discussed in Section 2 we can define a co-module action of $G_q^{(\pi)}$ on $O_d^0$ by

$$\omega(1) = 1 \otimes \epsilon,$$

$$\omega(s_i) = \sum_{j=1}^d s_j \otimes u_{ij},$$

$$\omega(s_i^*) = \sum_{j=1}^d s_j^* \otimes \gamma_0(u_{ij}),$$

$$\omega(xy) = \omega(x)\omega(y), \quad \forall \ x, y \in O_d^0,$$

where the multiplication on the right hand side of the last equation is the natural one for the algebra $O_d^0 \otimes G_q^{(\pi)}$ induced by the multiplication of $O_d^0$ and that of $G_q^{(\pi)}$. The consistency of this definition is confirmed by the simple calculations below:

$$\omega(x)\omega(y)(a) = \sum_{(x),(y)} x_1(y_1) \langle x_2y_2, a \rangle$$

$$= \sum_{(qa)} \sum_{(x),(y)} x_1(a_1) y_1 \langle y_2, a_2 \rangle$$

$$= \sum_{(a)} (a_1 \circ x)(a_2 \circ y) = a \circ (xy)$$

$$= \omega(xy)(a), \quad \forall a \in A.$$  \hspace{1cm} (17)

Hence $O_d^0$ is a co-module algebra over $G_q^{(\pi)}$.

5.2. The braids as co-fixed points. We have already shown that the braids are fixed by the $U_q(g)$ action, thus they must be co-fixed by $G_q^{(\pi)}$. Nevertheless, we look at an example to illustrate the general result.

Let

$$\theta = \sum \sigma_{ijkl}s_is_ks_l^*s_j^*$$

be a braid embedded in $O_d^0$. The coefficient matrix $\sigma$ commutes with $(\pi \otimes \pi)\Delta(a) \forall a \in U_q(g)$. This translates, for the dual Hopf algebra $G_q^{(\pi)}$, to the following relations

$$\sigma_{12}U_1U_2 = U_1U_2\sigma_{12}.$$  

Now

$$\omega(\theta) = \sum s_is_ks_l^*s_j^* \otimes (U_1U_2\sigma_{12}\gamma_0(U_1U_2))_{ijkl}$$

$$= \theta \otimes \epsilon.$$

The discussion of this section completes the proof of Theorem 1.

5.3. Proof of Theorem 2. As shown in 4.1, the linear span $H$ of the generators $s_i$ of $O_d$ furnishes an irreducible $U_q(g)$ module. It follows the results of 2.2 that there exists a nonvanishing rank $d$ $q$-antisymmetric tensor $S_q \in H^d$, which generates a trivial $U_q(g)$ module,

$$a \circ S_q = \epsilon(a)S_q.$$
Clearly \( S_q \) is also co-fixed by \( G_q^{(\pi)} \). This \( S_q \) is the distinguished element of Theorem 2.

To complete the proof of Theorem 2 we note that any element \( x \) of \( O_0^{d q} \) which is fixed under the \( U_q(g) \) action is also co-fixed under the \( G_q^{(\pi)} \) action. For \( 0 < q < 1 \) this means \( x \) is co-fixed under the co-action of \( G_q^{(\pi)} \). Hence by [12] \( x \) is in the subalgebra generated topologically by the rank \( d q \)-antisymmetric tensor together with the braid group elements. On the other hand \( x \in O_0^d \) and so lies in the algebraic subalgebra generated by the rank \( d q \)-antisymmetric tensor together with the braid group elements.

6. The \( G_q^{(\pi)} \) co-module \( O_0^d \) as a \( U'_q \) module

In the above calculations we started with \( O_0^d \) as a \( U_q(g) \) module and showed how to obtain the dual co-action of \( G_q^{(\pi)} \). It is of some interest to understand whether, when the reverse procedure is adopted, namely regarding \( O_0^d \) as a \( G_q^{(\pi)} \) co-module as in [12] and constructing the dual action, one recovers the given \( U_q(g) \) action. This is indeed the case but it requires us to understand \( U_q(g) \) in a different way namely in terms of \( U'_q \).

6.1. Co-module of a Hopf algebra as a module of the dual Hopf algebra. As in Section 2 we adopt a general viewpoint first and specialise later. In the notation of Section 2, \( A \) is a Hopf algebra with the finite dual \( A^0 \), which is also a Hopf algebra. Let \( V \) be a co-module of \( A \),

\[ V \xrightarrow{\omega} V \otimes A. \]

The defining property of \( \omega \) is that

\[ (\text{id}_V \otimes \Delta) \omega = (\omega \otimes \text{id}_A) w : V \rightarrow V \otimes A \otimes A, \]

\[ (\text{id}_V \otimes \epsilon) \omega = \text{id}_V : V \rightarrow V. \]

Using Sweedler’s sigma notation [14], we write, for \( v \in V \),

\[ w(v) = \sum_{(v)} v_{(1)} \otimes v_{(2)}, \quad v_{(1)} \in V, \quad v_{(2)} \in A, \]

where the right hand side is assumed to be a finite sum.

We observe that \( V \) has a natural \( A^0 \)-module structure,

\[ A^0 \otimes V \xrightarrow{0} V \]

defined, for any \( v \in V \) by,

\[ a \circ v = (\text{id}_V \otimes a) w(v), \quad a \in A^0 \]

or more explicitly,

\[ a \circ v = \sum_{(v)} v_{(1)} \langle a, v_{(2)} \rangle. \]

To check that this indeed defines a module over \( A^0 \), consider

\[ b \circ (a \circ v) = \sum_{(v)} v_{(1)} \langle b, v_{(2)} \rangle \langle a, c_{(3)} \rangle \]

\[ = \sum_{(v)} v_{(1)} \langle ba, v_{(2)} \rangle = (ba) \circ v. \]
The unit of \( A^0 \) is \( \epsilon \),
\[
\epsilon \circ v = \sum_{(v)} v(1) \langle \epsilon, v(2) \rangle = v.
\]

6.2. \( O^0_d \) as a module over \( U'_q \subset (O_q^d)^\ast \). The co-action of \( G^d_q \) on \( O^0_d \) is defined by (17). It follows from the discussion of the last subsection that this co-action dualizes an action of \( U'_q \) on the Cuntz algebra:
\[
a \circ 1 = \epsilon(a),
\]
\[
a \circ s_i = \sum_j s_j(a, u_{ji}), \quad a \circ s_i^* = \sum_j s_j^*(a, \gamma_0(u_{ij})),
\]
\[
a \circ (xy) = \sum a(w(xy)) = \sum x(1)y(1) \langle a, x(2)y(2) \rangle = \sum (a_1(1) \circ x)(a_2(2) \circ y).
\]
By recalling the defining relations (8) of \( U'_q \), we can see that \( \text{Ker}\phi \) annihilates \( O^0_d \). Hence the actions of \( U'_q \) and \( \tilde{U}_q(g) \) on \( O^0_d \) coincide,
\[
a \circ x = \phi(a) \circ x, \forall a \in U'_q, x \in O^0_d,
\]
where the action of \( \tilde{U}_q(g) \) on \( O_d \) appearing on the right hand side is that defined by (15) and (16). Therefore, we have recovered \( \tilde{U}_q(g) \) action and hence \textit{a fortiori} the \( U'_q(g) \) action on the Cuntz algebra from the \( G^d_q \) co-action.

7. Categorical interpretation

There is a way to interpret our earlier constructions more abstractly. This may be useful in order to relate this paper to other examples. Let \( A \) be a Hopf \( * \)-algebra generated by the elements of \( (u_{ij})_{i,j=1}^d \). We assume that \( u \) is unitary, that is, \( uu^* = I = uu^\ast \). Let \( H \) be a Hilbert space of dimension \( d < \infty \), which furnishes a co-representation of \( A \), \( \omega : H \rightarrow H \otimes A, \omega(e_i) = \sum_j e_j \otimes u_{ji}. \) Tensor products of \( H \) with itself yield co-representations of \( A \) in a natural way.

We have not specified the relations satisfied by the matrix elements of \( u \) at this stage. In fact, it is our purpose here to show that the defining relations can be recovered from the co-algebraic structure and the co-fixed points of the \( A \) co-action on the braided tensor category generated by \( H \).

Consider the tensor category \( F_d \), whose objects are \( H^\otimes 0 \), and arrows are the linear mappings \( T : H^\otimes 0 \rightarrow H^\otimes s \). As in Doplicher [3], we associate to \( F_d \) the Cuntz algebra \( O_d \). Then there is a natural \( A \) coaction on \( O_d \) defined by
\[
\Gamma : O_d \rightarrow O_d \otimes A
\]
\[
\Gamma(s_i) = \sum_{j=1}^d s_j \otimes u_{ji}.
\]
This co-action, when restricted to the UHF-algebra \( M^\infty_d \) of \( O_d \) through the embedding \( \eta(e_{i_1j_1} \otimes \cdots \otimes e_{i_kj_k}) = s_{i_1} \ldots s_{i_k} s^*_{j_1} \ldots s^*_{j_k}, \) gives
\[
\Gamma_{UHF}(e_{i_1j_1} \otimes \cdots \otimes e_{i_kj_k}) = \sum_{a_1 \ldots a_k \ b_1 \ldots b_k} e_{a_1b_1} \otimes \cdots \otimes e_{a_kb_k} \otimes u_{a_1i_1} \ldots u_{a_kb_k} s^*_{b_1j_1} \ldots s^*_{b_kj_k},
\]
for all positive integers \( k \).
Set \( u^k = u \otimes \cdots \otimes u \). Then for any \( x \in M_d^k \), \( \Phi(x) = \Gamma_{UHF}(x) = u^k(x \otimes 1)(u^k)^* \). The fixed points of \( \Phi \) are those \( x \) such that \( u^k(x \otimes 1)(u^k)^* = (x \otimes 1) \Rightarrow u^k(x \otimes 1) = (x \otimes 1)u^k \).

Let \( R \) be the solution of quantum Yang-Baxter equation such that \( R : H \otimes H \to H \otimes H \). Let \( P \) be the flip. Then \( \sigma = PR \) satisfies the braid group relations. As elements of \( O_d \), \( \theta = \eta(\sigma) = \sum \sigma_{ijkl} \eta(e_{ij} \otimes e_{kl}) = \sum \sigma_{ijkl}s_is_js_k^*s_j^* \) and

\[
\theta_i = \eta(b_i) = \sum_{\{t\}} s_{t_1} s_{t_2} \cdots s_{t_{i-1}} \otimes \sigma \otimes s_{t_{i-1}}^* \cdots s_{t_2}^* s_{t_1}^* 
\]
satisfy the braid group relations.

Form then a braided tensor category \((F_d, R)\) by replacing the commutativity of \( F_d \) with the braiding \( b_i \). Denote by \((O_d)^\Gamma\) the fixed point algebra of \((F_d, R)\) under the coaction \( \Gamma : (O_d)^\Gamma = \{x : \Gamma(x) = x \otimes 1\} \). Now we require that the fixed point algebra \((O_d)^\Gamma\) is generated by the braids \( b_i \). Then it follows from \( u^2(\sigma \otimes 1) = (\sigma \otimes 1)u^2 \) that

\[
R(u \otimes 1)(1 \otimes u) = (1 \otimes u)(u \otimes 1)R, 
\]

which is precisely the defining relations of a quantum group in the formulation of \([\text{I}]\). Therefore, the algebraic structure of \( A \) is recovered from the co-fixed point algebra \((O_d)^\Gamma\) associated to the braided category \((F_d, R)\).

Conversely, starting with our Hilbert space \( H \) of finite dimension \( d \) as an irreducible \( U \) module, where \( U \) is a Hopf \(*\)-algebra, we can consider the tensor category whose objects are \( H^\otimes n \) and the arrows are linear mappings \( H^\otimes n \to H^\otimes n \). Assume there exists a nondegenerate pairing between \( A \) and \( U \) which relates \( \Delta, \gamma, \epsilon \) of \( A \) with the multiplication, unit of \( U \), and relates the algebra structure of \( A \) with the comultiplication and co-unit of \( U \). As for the antipode, we have

\[
\langle \gamma(x), a \rangle = \langle x, \gamma(a) \rangle, x \in A, a \in U. 
\]

As before \( O_d \) is associated to the tensor category \( F_d \). Then there exists a natural \( U \)-module action on \( O_d \) given by \( a \circ 1 = \epsilon(a)1, a \circ s_i = \sum_j \pi(a)_{ij}s_j, a \circ s_i^* = \sum_j \pi(\gamma(a))_{ij}s_j^* \), \( a \circ (xy) = \sum_{\{l\}}(a_{(1)}x) \circ (a_{(2)}y) \forall x, y \in O_d \). This module action gives rise to a co-module action of \( A \) on \( O_d \) through equation (7) and the pairing

\[
\langle u_{ij}, a \rangle = u_{ij}(a) = \pi(a)_{ij}. 
\]

We can continue in this vein, reinterpreting the earlier explicit constructions from the more abstract viewpoint. However we desist after making one final comment. If we also assume that there exists a braid operator \( \sigma \) acting on \( H \otimes H \), such that \([\sigma, (\pi \otimes \pi)\Delta(a)] = 0, \forall a \in U \) then \( \theta = \eta(\sigma) \in O_d \) can be easily seen to be co-fixed by \( A \). Furthermore,

\[
Ru_1u_2 = u_2u_1R, 
\]

where \( R = P\sigma \) and \( u = (u_{ij})^d_{i,j=1} \).

References

[1] A. L. Carey and D. E. Evans, *On an automorphism action of U(n, 1) on O_n*, J. Funct. Analys., 70 (1987) 90.
[2] J. Cuntz, *Simple C*-algebras generated by isometries*, Commun. Math. Phys. 57 (1977) 173.
[3] S. Doplicher, *Abstract compact group duals, operator algebras and quantum field theory*, Proc. Internl. Cong. Math. Kyoto (1990).
[4] S. Doplicher, J. E. Roberts, *Duals of compact Lie groups realized in the Cuntz algebras*, J. Funct. Analys., 74 (1987) 90.
[5] V. G. Drinfeld, *Quantum groups*, Proc. Internl. Cong. Math., Berkeley, 1 (1986) 789.
[6] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J., 1 (1990) 193.

[7] K. Fredenhagen, K. H. Rehren and B. Schroer, *Superselection sectors with braid group statistics and exchange algebras I*, Commun. Math. Phys., 125 (1989) 201.

[8] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. Math. (2) 79 (1964) 59.

[9] G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras*, Adv. Math. 70 (1988) 237 - 249.

[10] M. Jimbo, *A q-difference analog of U(g) and the Yang - Baxter equation*, Lett. math. Phys., 10 (1985) 63.

[11] S. Montgomery, *Hopf algebras and their actions on rings*, Regional Conference Series in Math., 82 (1993).

[12] A. Paolucci, *Co-actions of Hopf algebras on the Cuntz algebras and their fixed point algebras*, Proc. American Math. Soc. to appear (1997).

[13] M. Rosso, *Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra*, Comm. Math. Phys. 117 (1988) 581 - 593.

[14] M. Sweedler, *Hopf algebras*, Benjamin, New York (1969).

[15] S. L. Woronowicz, *Compact matrix pseudogroups*, Commun. Math. Phys. 111 (1987) 613; *Tannha - Krein duality for compact matrix pseudogroups*, Inv. Math., 93 (1988) 35.

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