MODULI OF STABLE SHEAVES SUPPORTED ON CURVES OF GENUS THREE CONTAINED IN A QUADRIC SURFACE

MARIO MAICAN

ABSTRACT. We study the moduli space of stable sheaves of Euler characteristic 1 supported on curves of arithmetic genus 3 contained in a smooth quadric surface. We show that this moduli space is rational. We compute its Betti numbers by studying the variation of the moduli spaces of $\alpha$-semi-stable pairs. We classify the stable sheaves using locally free resolutions or extensions. We give a global description: the moduli space is obtained from a certain flag Hilbert scheme by performing two flips followed by a blow-down.

1. Introduction

Let $\mathbb{P}^1$ be the projective line over $\mathbb{C}$ and consider the surface $\mathbb{P}^1 \times \mathbb{P}^1$ with fixed polarization $\mathcal{O}(1,1) = \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. For a coherent algebraic sheaf $F$ on $\mathbb{P}^1 \times \mathbb{P}^1$, with support of dimension 1, the Euler characteristic $\chi(F(m,n))$ is a polynomial expression in $m, n$, of the form

$$P_F(m,n) = rm + sn + t,$$

where $r, s, t$ are integers depending only on $F$. This is the Hilbert polynomial of $F$. The slope of $F$ is $p(F) = t/r + s$. Let $M(P)$ be the coarse moduli space of $S$-equivalence classes of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ that are semi-stable with respect to the fixed polarization and that have Hilbert polynomial $P$. We recall that $F$ is semi-stable, respectively, stable, if it is pure and for any proper subsheaf $F' \subset F$ we have $p(F') \leq p(F)$, respectively, $p(F') < p(F)$. According to [10], $M(P)$ is projective, irreducible, and smooth at points given by stable sheaves. Its dimension is $2rs + 1$ if $r > 0$ and $s > 0$. The spaces $M(rr + n + 1), M(2m + 2n + 1)$ and $M(2m + 2n + 2)$ were studied in [1]. In fact, it is not difficult to see that $M(rr + n + 1)$ consists of the structure sheaves of curves of degree $(1, r)$, so it is isomorphic to $\mathbb{P}^{2r+1}$. The space $M(3m + 2n + 1)$ was studied in [4] and [12]. We refer to the introductory section of [12] for more background information.

This paper is concerned with the study of $M = M(4m + 2n + 1)$. The closed points of $M$ are in a bijective correspondence with the isomorphism classes $[F]$ of stable sheaves $F$ supported on curves of degree $(2, 4)$ and satisfying the condition $\chi(F) = 1$. As already mentioned, $M$ is a smooth irreducible projective variety of dimension 17. For any $t \in \mathbb{Z}$, twisting by $\mathcal{O}(t, t)$ gives an isomorphism $M \simeq M(4m + 2n + 6t + 1)$. According to [12 Corollary 1], $M \simeq M(4m + 2n - 1)$. In the following theorem we classify the sheaves in $M$.

**Theorem 1.1.** The variety $M$ can be decomposed into an open subset $M_0$, two closed irreducible subsets $M_2, M'_2$, each of codimension 2, a locally closed irreducible subset $M_3$ of codimension 3, and a locally closed irreducible subset $M_4$ of codimension 4. These subsets are defined as follows: $M_0$ is the set of sheaves $F$ having a resolution of the form

$$0 \rightarrow \mathcal{O}(-1, -3) \oplus \mathcal{O}(0, -3) \oplus \mathcal{O}(-1, -2) \overset{\varphi}{\rightarrow} \mathcal{O}(0, -2) \oplus \mathcal{O}(0, -2) \oplus \mathcal{O} \rightarrow F \rightarrow 0,$$

2010 Mathematics Subject Classification. Primary 14D20, 14D22.

Key words and phrases. Moduli spaces, Semi-stable sheaves, Wall crossing.
where the entries \( \varphi_{12} \) and \( \varphi_{22} \) are linearly independent and the maximal minors of the matrix \( (\varphi_{ij})_{i,j=1,2,3} \), describing the corestriction of \( \varphi \) to the first two summands, have no common factor; \( M_2 \) is the set of sheaves \( \mathcal{F} \) having a resolution of the form

\[
0 \to \mathcal{O}(-2, -2) \oplus \mathcal{O}(-1, -3) \xrightarrow{\varphi} \mathcal{O}(-1, -2) \oplus \mathcal{O}(0, 1) \to \mathcal{F} \to 0,
\]

with \( \varphi_{11} \neq 0, \varphi_{12} \neq 0 \); \( M_2 \) is the set of sheaves \( \mathcal{F} \) having a resolution of the form

\[
0 \to \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -4) \xrightarrow{\varphi} \mathcal{O}(-1, -1) \oplus \mathcal{O} \to \mathcal{F} \to 0,
\]

with \( \varphi_{11} \neq 0, \varphi_{12} \neq 0 \); \( M_3 \) is the set of extensions of the form

\[
0 \to \mathcal{O}_Q \to \mathcal{F} \to \mathcal{O}_L(1, 0) \to 0
\]

satisfying the condition \( \mathcal{H}^0(\mathcal{F}) \cong \mathbb{C} \), where \( Q \subset \mathbb{P}^1 \times \mathbb{P}^1 \) is a quintic curve of degree \( (2, 3) \) and \( L \subset \mathbb{P}^1 \times \mathbb{P}^1 \) is a line of degree \( (0, 1) \); \( M_4 \) is the set of extensions of the form

\[
0 \to \mathcal{O}_Q(p) \to \mathcal{F} \to \mathcal{O}_L \to 0,
\]

where \( \mathcal{O}_Q(p) \) is a non-split extension of \( \mathbb{C}_p \) by \( \mathcal{O}_Q \) for a point \( p \in Q \), and satisfying the condition \( \mathcal{H}^0(\mathcal{F}) \cong \mathbb{C} \).

Moreover, \( M_2 \) is the Brill-Noether locus of sheaves for which \( \mathcal{H}^1(\mathcal{F}) \neq \{0\} \).

The proof of Theorem 1.11 given in Section 3 relies on the Beilinson spectral sequence, which we recall in Section 4. The varieties \( X \) that appear in this paper have no odd homology, so we can define the Poincaré polynomial

\[
P(X)(\xi) = \sum_{i \geq 0} \dim_{\mathbb{C}} H^i(X, \mathbb{Q})\xi^{i/2}.
\]

**Theorem 1.2.** The Euler characteristic of \( M \) is 288. The Poincaré polynomial of \( M \) is

\[
\xi^{17} + 3\xi^{16} + 8\xi^{15} + 16\xi^{14} + 21\xi^{13} + 23\xi^{12} + 24\xi^{11} + 24\xi^{10} + 24\xi^9 \\
+ 24\xi^8 + 24\xi^7 + 24\xi^6 + 23\xi^5 + 21\xi^4 + 16\xi^3 + 8\xi^2 + 3\xi + 1.
\]

The proof of this theorem rests on the wall-crossing method of Choi and Chung [3]. In Section 4, we investigate how the moduli spaces \( M^{\alpha}(4m+2n+1) \) of \( \alpha \)-semi-stable pairs with Hilbert polynomial \( 4m+2n+1 \) change as the parameter \( \alpha \) varies. In Theorem 4.11, we find that \( M^{\alpha}(4m+2n+1) \) are related by two explicitly described flipping diagrams. Combining this with Proposition 4.12, yields a global description: \( M \) is obtained from the flag Hilbert scheme of three points on curves of degree \( (2, 4) \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \) by performing two flips followed by a blow-down centered at the Brill-Noether locus \( M_2 \).

The total space \( X \) of \( \omega_{\mathbb{P}^1 \times \mathbb{P}^1} \) is a Calabi-Yau threefold. For a homology class \( \beta = (r, s) \in H_2(\mathbb{P}^1 \times \mathbb{P}^1) \subset H_2(X) \) let \( N_{\beta}(X) \) be the genus zero Gromov-Witten invariant of \( X \) and let \( n_{\beta}(X) \) be the genus zero Gopakumar-Vafa invariant of \( X \), as introduced in [8]. It was noticed in [4] that, up to sign, the latter is the Euler characteristic of a moduli space:

\[
n_{\beta}(X) = (-1)^{\dim M(4m+2n+1)}e(M(4m+2n+1)).
\]

In [8], Katz conjectured the relation

\[
N_{\beta}(X) = \sum_{k \mid \beta} \frac{n_{\beta/k}(X)}{k^3}.
\]

For \( \beta = (4, 2) \), this conjecture reads

\[
N_{(4,2)}(X) = (-1)^{\dim M(4m+2n+1)}e(M(4m+2n+1)) + \frac{1}{8}(-1)^{\dim M(2m+n+1)}e(M(2m+n+1)) = (-1)^{\dim M}e(M) + \frac{1}{8}(-1)^{\dim \mathbb{P}^5}e(\mathbb{P}^5) = (-1)^{17}288 + \frac{1}{8}(-1)^{5}6 = -288.75.
\]
2. Preliminaries

Our main technical tool in Section 3 will be the Beilinson spectral sequence. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. According to [2, Lemma 1], we have a spectral sequence converging to $\mathcal{F}$, whose first level $E_1$ has display diagram

\[
(1) \quad H^2(\mathcal{F}(-1, -1)) \otimes \mathcal{O}(-1, -1) = E_1^{2,1} \longrightarrow E_1^{1,2} \longrightarrow E_1^{0,2} = H^2(\mathcal{F}) \otimes \mathcal{O}
\]

\[
H^1(\mathcal{F}(-1, -1)) \otimes \mathcal{O}(-1, -1) = E_1^{1,1} \theta_1 E_1^{0,1} \quad \text{and} \quad H^0(\mathcal{F}(-1, -1)) \otimes \mathcal{O}(-1, -1) = E_1^{0,0} \theta_2 E_1^{0,0} = H^0(\mathcal{F}) \otimes \mathcal{O}
\]

where $E_1^{i,j} = \{0\}$ if $i \notin \{-2, -1, 0\}$ or if $j \notin \{0, 1, 2\}$ and

\[
(2) \quad E_1^{-1-j} = H^j(\mathcal{F}(0, -1)) \otimes \mathcal{O}(0, -1) \oplus H^j(\mathcal{F}(-1, 0)) \otimes \mathcal{O}(-1, 0).
\]

If $\mathcal{F}$ has support of dimension 1, then the first row of (1) vanishes and the convergence of the spectral sequence forces $\theta_2$ to be surjective and yields the exact sequence

\[
(3) \quad 0 \longrightarrow \text{Ker}(\theta_1) \theta_2 \text{Coker}(\theta_1) \longrightarrow \mathcal{F} \longrightarrow \text{Ker}(\theta_2)/\text{Im}(\theta_1) \longrightarrow 0.
\]

An application of the Beilinson spectral sequence is the following lemma that will be used in Section 3

**Lemma 2.1.** Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a zero-dimensional subscheme of length 3 that is not contained in a line of degree $(1, 0)$ or $(0, 1)$. Then the ideal of $Z$ has resolution

\[
0 \longrightarrow 2\mathcal{O}(-2, -2) \xrightarrow{\zeta} \mathcal{O}(-1, -2) \oplus \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -1) \longrightarrow \mathcal{I}_Z \longrightarrow 0,
\]

where the maximal minors of $\zeta$ have no common factor. The dual of the structure sheaf of $Z$ has resolution

\[
0 \longrightarrow \mathcal{O}(-2, -4) \longrightarrow \mathcal{O}(-1, -3) \oplus \mathcal{O}(0, -3) \oplus \mathcal{O}(-1, -2) \xrightarrow{\zeta} 2\mathcal{O}(0, -2) \longrightarrow \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O}) \longrightarrow 0.
\]

**Proof.** We apply the spectral sequence (1) to the sheaf $\mathcal{F} = \mathcal{I}_Z(1, 1)$. By hypothesis, $H^0(\mathcal{I}_Z(1, 0)) = \{0\}$ and $H^0(\mathcal{I}_Z(0, 1)) = \{0\}$ hence, from [2], we obtain the vanishing of $E_1^{-1,0}$. Since $H^0(\mathcal{I}_Z) = \{0\}$, also $E_1^{-2,0}$ vanishes. From the short exact sequence

\[
0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Z \longrightarrow 0
\]

we obtain the vanishing of $H^2(\mathcal{I}_Z)$. Analogously, $H^2(\mathcal{I}_Z(1, 0))$, $H^2(\mathcal{I}_Z(0, 1))$, and $H^2(\mathcal{I}_Z(1, 1))$ vanish. The first row of (1) vanishes. Denote $d = \dim_\mathbb{C} H^1(\mathcal{I}_Z(1, 1))$. Display diagram (1) now takes the simplified form

\[
\begin{array}{c c c c c}
0 & \longrightarrow & \mathcal{I}_Z & \longrightarrow & \mathcal{O} \\
& & \theta_1 & \longrightarrow & \mathcal{O}(0, -1) \oplus \mathcal{O}(-1, 0) \theta_2 \longrightarrow d\mathcal{O} \\
& & \theta_2 & \longrightarrow & (d + 1)\mathcal{O}
\end{array}
\]

From the convergence of the spectral sequence we see that $\theta_2$ is surjective. There is no surjective morphism $\theta_2: \mathcal{O}(0, -1) \oplus \mathcal{O}(-1, 0) \rightarrow d\mathcal{O}$ for $d \geq 1$, hence $d = 0$. Thus, Ker$(\theta_1)$ is a subsheaf of $\mathcal{O}$. We claim that
$\ker(\theta_1) = \{0\}$. Indeed, if $\ker(\theta_1)$ were non-zero, then $\mathcal{O}/\ker(\theta_1)$ would be a torsion subsheaf of $\mathcal{I}_Z(1,1)$. Combining the exact sequences

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{I}_Z(1,1) \rightarrow \text{Coker}(\theta_1) \rightarrow 0,$$

$$0 \rightarrow 2\mathcal{O}(-1,-1) \rightarrow \mathcal{O}(0,-1) \oplus \mathcal{O}(-1,0) \rightarrow \text{Coker}(\theta_1) \rightarrow 0$$

yields the resolution

$$0 \rightarrow 2\mathcal{O}(-1,-1) \rightarrow \mathcal{O}(0,-1) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O} \rightarrow \mathcal{I}_Z(1,1) \rightarrow 0.$$ Applying $\text{Hom}(-, \mathcal{O}(-1,-3))$, we obtain resolution \([11]\). If the maximal minors of the matrix representing $\zeta$ had a common factor $f$, then the reduced support of $\text{Coker}(\zeta)$ would contain the curve $\{f = 0\}$. But this is impossible because $\text{Ext}^2(\mathcal{O}_Z, \mathcal{O})$ has support of dimension zero.

**Lemma 2.2.** Let $S$ be a smooth projective surface and let $C \subset S$ be a locally Cohen-Macaulay curve. Let $Z$ be a coherent sheaf on $S$ with support of dimension zero. Let $F$ be an extension of $Z$ by $\mathcal{O}_C$ without zero-dimensional torsion. Then $F$ is uniquely determined up to isomorphism, meaning that if $F'$ is another extension of $Z$ by $\mathcal{O}_C$ without zero-dimensional torsion, then $F' \cong F$. Moreover, $Z \cong \text{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{O}_S)$ for a subscheme $Z \subset C$ of dimension zero, so we have the exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow F \rightarrow \text{Ext}^2_{\mathcal{O}_S}(\mathcal{O}_Z, \mathcal{O}_S) \rightarrow 0.$$ **Proof.** This lemma is a direct consequence of \([13]\) Proposition B.5). Indeed, given an exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow F \rightarrow Z \rightarrow 0$$

in which $F$ has no zero-dimensional torsion, then the pair $(\mathcal{O}_C, F)$ is a stable pair supported on $C$, in the sense of \([13]\). By \([13]\) Lemma B.2, we have $\text{Ext}^1_{\mathcal{O}_C}(F, \mathcal{O}_C) = \{0\}$. Applying $\text{Hom}_{\mathcal{O}_C}(\cdot, \mathcal{O}_C)$ to \([6]\), yields the exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{O}_C}(F, \mathcal{O}_C) \rightarrow \mathcal{O}_C \rightarrow \text{Ext}^1_{\mathcal{O}_C}(Z, \mathcal{O}_C) \rightarrow 0.$$ Thus, $\text{Ext}^1_{\mathcal{O}_C}(Z, \mathcal{O}_C)$ is the structure sheaf $\mathcal{O}_Z$ of a zero-dimensional subscheme $Z \subset C$. Under the bijection of \([13]\) Proposition B.5) between stable pairs supported on $C$ and zero-dimensional subschemes of $C$, the pair $(\mathcal{O}_C, F)$ corresponds to $Z$, so it is uniquely determined, up to isomorphism. Tensoring \([7]\) with the dualising line bundle $\omega_C$ on $C$, yields the exact sequence

$$0 \rightarrow \text{Hom}(F, \omega_C) \rightarrow \omega_C \rightarrow \mathcal{O}_Z \rightarrow 0.$$ We claim that $\text{Hom}(F, \omega_C) \cong \text{Ext}^1(F, \omega_S)$. This follows by applying $\text{Hom}(F, \cdot)$ to the exact sequence

$$0 \rightarrow \omega_S \rightarrow \omega_S \otimes \mathcal{O}(C) \rightarrow \omega_S \otimes \mathcal{O}(C)|_C \cong \omega_C \rightarrow 0.$$ We obtain the exact sequence

$$0 \rightarrow \text{Hom}(F, \omega_C) \rightarrow \text{Ext}^1(F, \omega_S) \rightarrow \text{Ext}^1(F, \omega_S \otimes \mathcal{O}(C)).$$ The last morphism is locally multiplication with an equation $f$ defining $C$. But $C = \text{supp}(F)$, hence $f$ annihilates $F$, and hence $f$ annihilates $\text{Ext}^1(F, \omega_S)$. This proves the claim. According to \([11]\) Remark 4), $\text{Ext}^1(\text{Ext}^1(F, \omega_S), \omega_S) \cong F$. Clearly,

$$\text{Ext}^1(\mathcal{O}_Z, \omega_S) = \{0\}, \quad \text{Ext}^1(\omega_C, \omega_S) \cong \mathcal{O}_C, \quad \text{Ext}^2(\omega_C, \omega_S) = \{0\}.$$ Applying $\text{Hom}(\cdot, \omega_S)$ to \([8]\) yields extension \([5]\). Comparing with \([11]\), we see that $Z \cong \text{Ext}^2(\mathcal{O}_Z, \mathcal{O}_S)$. □

Crucial for our classification of semi-stable sheaves is the following vanishing result that should be compared with \([12]\) Proposition 4). We fix vector spaces $V_1$ and $V_2$ over $\mathbb{C}$ of dimension 2 and we identify $\mathbb{P}^1 \times \mathbb{P}^1$ with $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$. Let $\{x, y\}$ be a basis of $V_1^*$ and let $\{z, w\}$ be a basis of $V_2^*$. A morphism $\mathcal{O}(i,j) \rightarrow \mathcal{O}(k,l)$ will be represented by a form in $S^{k-j}V_1^* \otimes S^{l-i}V_2^*$.

**Proposition 2.3.** Assume that the sheaf $F$ gives a point in $\mathbf{M}$. 

(i) We have \( H^0(\mathcal{F}(-1, -1)) = \{0\} \) and \( H^1(\mathcal{F}(0, -1)) = \{0\} \).

(ii) If \( \mathcal{F} \) satisfies the vanishing condition \( H^0(\mathcal{F}(0, -1)) = \{0\} \), then \( H^1(\mathcal{F}) = \{0\} \).

Proof. (i) The vanishing of \( H^0(\mathcal{F}(-1, -1)) \) follows from \cite{12} Proposition 2(ii). To prove the vanishing of \( H^0(\mathcal{F}(0, -1)) \) we can argue as in the proof of \cite{12} Proposition 3.

(ii) Assume now that \( H^0(\mathcal{F}(0, -1)) = \{0\} \). From \cite{2} and part (i) of the proposition, we deduce that \( E^{-1,1}_1 \simeq \mathcal{O}(0, -1) \oplus 3\mathcal{O}(-1, 0) \). Denote \( d = \dim_\mathbb{C} H^1(\mathcal{F}) \). There is no surjective morphism

\[
\theta_2: O(0, -1) \oplus 3O(-1, 0) \longrightarrow \mathcal{O}
\]

for \( d \geq 4 \), hence \( d \leq 3 \). Assume that \( d = 3 \). The maximal minors for a matrix representation of \( \theta_2 \) have no common factor, otherwise \( \theta_2 \) would not be surjective. Thus, \( \text{Ker}(\theta_2) \simeq \mathcal{O}(-3, -1) \), hence \( \theta_2 = 0 \), and hence, from the exact sequence \([3]\), we obtain a surjective morphism \( \mathcal{F} \rightarrow \mathcal{O}(-3, -1) \). This is absurd. Thus, the case when \( d = 3 \) is unfeasible.

Consider now the case when \( d = 2 \). If \( \theta_2 \) is represented by a matrix of the form

\[
A = \begin{bmatrix}
0 & * & * & * \\
0 & * & * & *
\end{bmatrix},
\]

then \( \text{Ker}(\theta_2) \simeq \mathcal{O}(0, -1) \oplus \mathcal{O}(-3, 0) \), hence \( \mathcal{O}(-3, 0) \) is a direct summand of \( \text{Ker}(\theta_2)/\text{Im}(\theta_1) \), and hence, from the exact sequence \([3]\), we obtain a surjective morphism \( \mathcal{F} \rightarrow \mathcal{O}(-3, 0) \). This is absurd. If \( \theta_2 \) is represented by a matrix of the form

\[
B = \begin{bmatrix}
* & * & * & 0 \\
* & * & * & 0
\end{bmatrix},
\]

then \( \text{Ker}(\theta_2) \simeq \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, 0) \), hence \( \mathcal{O}(-2, -1) \) is a direct summand of \( \text{Ker}(\theta_2)/\text{Im}(\theta_1) \), and hence we obtain a surjective morphism \( \mathcal{F} \rightarrow \mathcal{O}(-2, -1) \). This is absurd. If \( \theta_2 \) is not represented by a matrix of the form \( A, B \) or \( C \), then \( \theta_2 \) is represented by a matrix of the form

\[
C = \begin{bmatrix}
1 \otimes u & v \otimes 1 & 0 & 0 \\
0 & 0 & x \otimes 1 & y \otimes 1
\end{bmatrix},
\]

then \( \text{Ker}(\theta_2) \simeq \mathcal{O}(-1, -1) \oplus \mathcal{O}(-2, 0) \) and we obtain a surjective morphism \( \mathcal{F} \rightarrow \mathcal{O}(-2, 0) \). This is absurd.

We claim that, if \( \theta_2 \) is not of the form \( A, B \) or \( C \), then \( \theta_2 \) is represented by a matrix of the form \( D \), with \( v \neq 0 \). Indeed, since \( \theta_2 \sim A \) and \( \theta_2 \sim B \), we may write

\[
\theta_2 = \begin{bmatrix}
1 \otimes u & v_1 \otimes 1 & v_2 \otimes 1 & 0 \\
1 \otimes u_1 & 0 & x \otimes 1 & y \otimes 1
\end{bmatrix},
\]

with \( u \neq 0, v \neq 0 \). Since \( \theta_2 \sim B \), \( v_1 \) and \( v_2 \) cannot both be zero. If \( v_1 \) and \( v_2 \) are linearly independent, then \( \theta_2 \sim D \). If \( v_1 \) and \( v_2 \) span a one-dimensional vector space, then, since \( \theta_2 \sim B \), we may write

\[
\theta_2 = \begin{bmatrix}
1 \otimes u & v_1 \otimes 1 & 0 & 0 \\
1 \otimes u_1 & 0 & x \otimes 1 & y \otimes 1
\end{bmatrix}.
\]

Since \( \theta_2 \sim C \), we have \( u_1 \neq 0 \), forcing \( \theta_2 \sim D \). In the case when \( \theta_2 = D \), it is easy to see that the morphism \( \theta_1: 5\mathcal{O}(-1, -1) \longrightarrow \mathcal{O}(0, -1) \oplus 3\mathcal{O}(-1, 0) \) is represented by a matrix of the form

\[
\begin{bmatrix}
x \otimes 1 & y \otimes 1 & 0 & 0 & 0 \\
1 \otimes z & 0 & 0 & 0 & 0 \\
0 & 1 \otimes z & 0 & 0 & 0 \\
* & * & 0 & 0 & 0
\end{bmatrix}.
\]
hence \( \operatorname{Ker}(\theta_1) \simeq 3\mathcal{O}(-1, -1) \), and hence \( \operatorname{Coker}(\theta_2) \) has Hilbert polynomial \( 3m + 3n + 3 \). But then, in view of the exact sequence (3), \( \operatorname{Coker}(\theta_3) \) is a destabilizing subsheaf of \( \mathcal{F} \). Thus, the case when \( d = 2 \) is also unfeasible.

It remains to examine the case when \( d = 1 \). Recall that \( \theta_2 \) is surjective, hence it can have two possible forms. Firstly, if

\[
\theta_2 = \begin{bmatrix} 0 & x \otimes 1 & y \otimes 1 & 0 \end{bmatrix},
\]

then \( \operatorname{Ker}(\theta_2) \simeq \mathcal{O}(0, -1) \oplus \mathcal{O}(-2, 0) \oplus \mathcal{O}(-1, 0) \) and we obtain a surjective morphism \( \mathcal{F} \to \mathcal{O}(-2, 0) \), which is absurd. The second form is

\[
\theta_2 = \begin{bmatrix} -1 \otimes z & x \otimes 1 & y \otimes 1 & 0 \end{bmatrix}.
\]

If \( \theta_1 \) is represented by a matrix having two zero columns, then \( \operatorname{Ker}(\theta_1) \simeq 2\mathcal{O}(-1, -1) \), hence \( \operatorname{Coker}(\theta_5) \) has Hilbert polynomial \( 2m + 2n + 2 \), and hence \( \operatorname{Coker}(\theta_5) \) is a destabilizing subsheaf of \( \mathcal{F} \). Thus, we may write

\[
\theta_1 = \begin{bmatrix} x \otimes 1 & y \otimes 1 & 0 & 0 & 0 \\ 1 \otimes z & 0 & 0 & 0 & 0 \\ 0 & 1 \otimes z & 0 & 0 & 0 \\ 0 & 0 & 1 \otimes z & 1 \otimes w & 0 \end{bmatrix},
\]

hence \( \operatorname{Ker}(\theta_1) \simeq \mathcal{O}(-1, -2) \oplus \mathcal{O}(-1, -1) \), and hence \( \operatorname{Coker}(\theta_5) \) has Hilbert polynomial \( 3m + 2n + 2 \). But then \( \operatorname{Coker}(\theta_5) \) is a destabilizing subsheaf of \( \mathcal{F} \). We deduce that the case when \( d = 1 \) is also unfeasible. □

3. Classification of sheaves

We begin our classification of semi-stable sheaves by examining the Brill-Noether locus of sheaves that do not satisfy the first vanishing condition in Proposition 2.3(ii).

**Proposition 3.1.** The sheaves \( \mathcal{F} \) in \( \mathbf{M} \) satisfying the condition \( H^0(\mathcal{F}(0, -1)) \neq \{0\} \) are precisely the non-split extension sheaves of the form

\[
0 \to \mathcal{O}_C(0, 1) \to \mathcal{F} \to C_p \to 0,
\]

where \( C \subset \mathbb{P}^1 \times \mathbb{P}^1 \) is a curve of degree \((2, 4)\) and \( p \) is a point on \( C \). Moreover, the sheaves from (9) are precisely the sheaves \( \mathcal{F} \) having a resolution of the form

\[
0 \to \mathcal{O}(-2, -2) \oplus \mathcal{O}(-1, -3) \to \mathcal{O}(-1, -2) \oplus \mathcal{O}(0, 1) \to \mathcal{F} \to 0,
\]

with \( \varphi_{11} \neq 0, \varphi_{12} \neq 0 \). Let \( \mathbf{M}_2 \subset \mathbf{M} \) be the subset of sheaves \( \mathcal{F} \) from (7). Then \( \mathbf{M}_2 \) is closed, irreducible, of codimension 2, and is isomorphic to the universal curve of degree \((2, 4)\) in \( \mathbb{P}^1 \times \mathbb{P}^1 \). Thus, \( \mathbf{M}_2 \) is a fiber bundle with fiber \( \mathbb{P}^3 \) and base \( \mathbb{P}^1 \times \mathbb{P}^1 \).

**Proof.** Let \( \mathcal{F} \) give a point in \( \mathbf{M} \) and satisfy \( H^0(\mathcal{F}(0, -1)) \neq \{0\} \). As in the proof of [12, Proposition 2], there is an injective morphism \( \mathcal{O}_C \to \mathcal{F}(0, -1) \) for a curve \( C \) of degree \((s, r)\), \( 0 \leq s \leq 2, 0 \leq r \leq 4, 1 \leq r + s \leq 6 \). From the stability of \( \mathcal{F} \) we have the inequality

\[
p(\mathcal{O}_C(0, 1)) = \frac{r + 2s - rs}{r + s} \leq \frac{1}{6} = p(\mathcal{F}),
\]

which has the unique solution \((s, r) = (2, 4)\). We obtain extension (9). Conversely, let \( \mathcal{F} \) be given by the non-split extension (9). As in the proof of [12, Proposition 3], we can show that \( \mathcal{O}_C(0, 1) \) is stable, from which it immediately follows that \( \mathcal{F} \) gives a point in \( \mathbf{M} \) and that \( H^0(\mathcal{F}(0, -1)) \neq \{0\} \). Choose \( \varphi_{11} \in V_1^* \otimes \mathbb{C} \) and \( \varphi_{12} \in \mathbb{C} \otimes V_2^* \) defining \( p \). Since \( p \in C \), we can find \( \varphi_{21} \in S^2 V_1^* \otimes S^3 V_2^* \) and \( \varphi_{22} \in V_1^* \otimes S^4 V_2^* \) such that the polynomial \( \varphi_{11} \varphi_{22} - \varphi_{12} \varphi_{21} \) defines \( C \). Consider the morphism

\[
\varphi: \mathcal{O}(-2, -2) \oplus \mathcal{O}(-1, -3) \to \mathcal{O}(-1, -2) \oplus \mathcal{O}(0, 1),
\]

\[
\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}.
\]
From the snake lemma we see that $\text{Coker}(\varphi)$ is an extension of $\mathbb{C}_p$ by $\mathcal{O}_C(0,1)$. Since $\text{Coker}(\varphi)$ has no zero-dimensional torsion, we can apply Lemma 2.2 to deduce that $\mathcal{F} \simeq \text{Coker}(\varphi)$. Thus, $[\mathcal{F}] \in \mathcal{M}_2$ if and only if $\mathcal{F}$ has resolution (10).

In the remaining part of this section we will assume that $\mathcal{F}$ satisfies both vanishing conditions from Proposition 2.3(ii). The exact sequence (3) takes the form

\begin{equation}
\begin{aligned}
0 &\rightarrow \text{Ker}(\theta_1) \xrightarrow{\varrho} \mathcal{O} \rightarrow \mathcal{F} \rightarrow \text{Coker}(\theta_1) \rightarrow 0,
\end{aligned}
\end{equation}

where

\begin{equation}
\begin{aligned}
\theta_1 : 5\mathcal{O}(-1,-1) &\rightarrow \mathcal{O}(0,-1) \oplus 3\mathcal{O}(-1,0).
\end{aligned}
\end{equation}

**Proposition 3.2.** Assume that $[\mathcal{F}] \in \mathcal{M}$ and that $H^0(\mathcal{F}(0,-1)) = \{0\}$. Assume that the maximal minors of $\theta_1$ have a common factor. Then $\mathcal{F}$ is an extension of the form

\begin{equation}
\begin{aligned}
0 &\rightarrow \mathcal{O}_Q \rightarrow \mathcal{F} \rightarrow \mathcal{O}_L(1,0) \rightarrow 0
\end{aligned}
\end{equation}

for a quintic curve $Q \subset \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2,3)$ and a line $L \subset \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(0,1)$, or is an extension of the form

\begin{equation}
\begin{aligned}
0 &\rightarrow \mathcal{O}_Q(p) \rightarrow \mathcal{F} \rightarrow \mathcal{O}_L \rightarrow 0,
\end{aligned}
\end{equation}

where $\mathcal{O}_Q(p)$ is a non-split extension of $\mathbb{C}_p$ by $\mathcal{O}_Q$ for a point $p \in Q$.

Conversely, any extension $\mathcal{F}$ as in (12) or (13) satisfying the condition $H^0(\mathcal{F}) \simeq \mathbb{C}$ is semi-stable. Let $\mathcal{M}_1 \subset \mathcal{M}$ be the subset of sheaves $\mathcal{F}$ as in (12) satisfying the condition $H^0(\mathcal{F}) \simeq \mathbb{C}$. Let $\mathcal{M}_3 \subset \mathcal{M}$ be the subset of sheaves $\mathcal{F}$ as in (13) satisfying the condition $H^0(\mathcal{F}) \simeq \mathbb{C}$. Then $\mathcal{M}_3$ and $\mathcal{M}_4$ are locally closed, irreducible subsets, of codimension 3, respectively, 4.

**Proof.** Let $\eta_i$ be the maximal minor of a matrix representing $\theta_1$ obtained by deleting column $i$. Denote $g = \gcd(\eta_1, \ldots, \eta_5)$. Let $(s, r) = (2, 4) - \deg(g)$. It is easy to check that the sequence

\begin{equation}
\begin{aligned}
0 &\rightarrow \mathcal{O}(-s, -r) \xrightarrow{\eta} 5\mathcal{O}(-1,-1) \xrightarrow{\theta_1} \mathcal{O}(0,-1) \oplus 3\mathcal{O}(-1,0),
\end{aligned}
\end{equation}

is exact. From (11) we see that $\text{Coker}(\theta_5)$ is a subsheaf of $\mathcal{F}$, hence we have the inequality

\begin{equation}
\begin{aligned}
1 - \frac{rs}{r+s} = p(\text{Coker}(\theta_5)) \leq p(\mathcal{F}) = \frac{1}{6},
\end{aligned}
\end{equation}

forcing $(s, r) = (2, 3)$ or $(s, r) = (2, 2)$. If $(s, r) = (2, 2)$, then $P_{\text{Coker}(\theta_1)} = 2m + 1$ and $\text{Coker}(\theta_1)$ is semi-stable, which follows from the semi-stability of $\mathcal{F}$. But, according to [10 Proposition 10], $M(2m + 1) = 0$.

This contradiction shows that $(s, r) \neq (2, 2)$, hence $(s, r) = (2, 3)$. From (11) we obtain the extension

\begin{equation}
\begin{aligned}
0 &\rightarrow \mathcal{O}_Q \rightarrow \mathcal{F} \rightarrow \text{Coker}(\theta_1) \rightarrow 0.
\end{aligned}
\end{equation}

If $\text{Coker}(\theta_1)$ has no zero-dimensional torsion, we obtain extension (12). Otherwise, the zero-dimensional torsion has length 1, its pull-back in $\mathcal{F}$ is a semi-stable sheaf $\mathcal{O}_Q(p)$, and we obtain extension (13).

Conversely, let $\mathcal{F}$ be an extension as in (12) satisfying $H^0(\mathcal{F}) \simeq \mathbb{C}$. Assume that $\mathcal{F}$ had a destabilizing subsheaf $\mathcal{F}'$. Let $G$ be the image of $\mathcal{F}'$ in $\mathcal{O}_L(1,0)$. According to (12) Proposition 1, $\mathcal{O}_Q$ is stable, hence $\chi(\mathcal{F}) \leq -1$. Since $\chi(\mathcal{F}) \geq 1$, we see that $\chi(G) \geq 2$, hence $G \simeq \mathcal{O}_L(1,0)$ and $\mathcal{O}_Q \nsubseteq \mathcal{F}'$. Thus $H^0(\mathcal{F}) = \{0\}$, hence the map $H^0(\mathcal{F}') \rightarrow H^0(\mathcal{O}_L(1,0))$ is injective. But this map factors through $H^0(\mathcal{F}') \rightarrow H^0(\mathcal{O}_L(1,0))$, which, by hypothesis, is the zero map. We deduce that $H^0(\mathcal{F}') = \{0\}$, which yields a contradiction. Thus, there is no destabilizing subsheaf. The same argument applies for extensions (13) satisfying $H^0(\mathcal{F}) \simeq \mathbb{C}$.

By Serre duality

\begin{equation}
\begin{aligned}
\text{Ext}^1(\mathcal{O}_L(1,0), \mathcal{O}_Q) &\simeq \text{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L(-1,-2))^*.
\end{aligned}
\end{equation}
From the short exact sequence
\[ 0 \to \mathcal{O}(-2, -3) \to \mathcal{O} \to \mathcal{O}_Q \to 0, \]
we obtain the long exact sequence
\[ \{0\} = H^0(\mathcal{O}_L(-1, -2)) \to H^0(\mathcal{O}_L(1, 1)) \cong \mathbb{C}^2 \to \text{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L(-1, -2)) \to H^1(\mathcal{O}_L(-1, -2)) = \{0\}. \]
Thus \(\text{Ext}^1(\mathcal{O}_L(1, 0), \mathcal{O}_Q) \cong \mathbb{C}^2\), hence \(M_4\) is isomorphic to an open subset of a \(\mathbb{P}^1\)-bundle over \(\mathbb{P}^{11} \times \mathbb{P}^1\).
By Serre duality we have
\[ \text{Ext}^1(\mathcal{O}_L, \mathcal{O}_Q(p)) \cong \text{Ext}^1(\mathcal{O}_Q(p), \mathcal{O}_L(-2, -2))^*. \]
Using Lemma 2.2 it is easy to see that the sheaves \(\mathcal{O}_Q(p)\) are precisely the sheaves having a resolution of the form
\[ (14) \quad 0 \to \mathcal{O}(-2, -2) \oplus \mathcal{O}(-1, -3) \xrightarrow{\psi} \mathcal{O}(-1, -2) \oplus \mathcal{O} \to \mathcal{O}_Q(p) \to 0, \]
where \(\psi_{11} \neq 0, \psi_{12} \neq 0\) (cf. Proposition 3.3). From resolution (14) we obtain the long exact sequence
\[ \text{Ext}^1(\mathcal{O}_Q(p), \mathcal{O}_L(-2, -2)) \to H^0(\mathcal{O}_L \oplus \mathcal{O}_L(-1, 1)) \cong \mathbb{C} \to \text{Ext}^1(\mathcal{O}_Q(p), \mathcal{O}_L(-1, -2)) \to H^1(\mathcal{O}_L \oplus \mathcal{O}_L(-1, 1)) = \{0\}. \]
Thus, \(\text{Ext}^1(\mathcal{O}_L, \mathcal{O}_Q(p)) \cong \mathbb{C}^2\), hence \(M_3\) is obviously. □

**Lemma 3.3.** Assume that \([\mathcal{F}] \in M\) and \(H^0(\mathcal{F}(0, -1)) = \{0\}\). Assume that the maximal minors of \(\theta_1\) have no common factor. Then \(\text{Ker}(\theta_1) \cong \mathcal{O}(-2, -4)\) and \(\text{Coker}(\theta_1) \cong \text{Ext}^2(\mathcal{O}_Z, \mathcal{O})\) with \(Z\) described below. We have an extension
\[ (15) \quad 0 \to \mathcal{O}_C \to \mathcal{F} \to \text{Ext}^2(\mathcal{O}_Z, \mathcal{O}) \to 0, \]
where \(C\) is a curve of degree \((2, 4)\) and \(Z \subset C\) is a subscheme of dimension zero and length 3. Moreover, \(Z\) is not contained in a line of degree \((0, 1)\).

**Proof.** The fact that \(\text{Ker}(\theta_1) \cong \mathcal{O}(-2, -4)\) is well-known. The Hilbert polynomial of \(\text{Coker}(\theta_1)\) is 3, hence \(\text{Coker}(\theta_1)\) has dimension zero and length 3. From (14), we obtain the exact sequence
\[ 0 \to \mathcal{O}_C \to \mathcal{F} \to \text{Coker}(\theta_1) \to 0. \]
We can now apply Lemma 2.2 to obtain the extension (15) and the isomorphism \(\text{Coker}(\theta_1) \cong \text{Ext}^2(\mathcal{O}_Z, \mathcal{O})\).

Assume that \(Z\) is contained in a line \(L\) of degree \((0, 1)\). Then \(\mathcal{O}_Z \cong \text{Ext}^2(\mathcal{O}_Z, \mathcal{O})\). Choose \(\varphi_{11} \in \mathbb{C} \otimes V_2^*\) defining \(L\). Choose \(\varphi_{12} \in S^3V_1^* \otimes \mathbb{C}\) such that \(\varphi_{11}\) and \(\varphi_{12}\) define \(Z\). If \(L \not\subset C\), then \(L.C = 2\), which contradicts the fact that \(Z \subset L \cap C\). Thus \(L \subset C\), so there is \(\varphi_{22} \in S^2V_1^* \otimes S^3V_2^*\) such that \(\varphi_{11} \varphi_{22}\) is a defining polynomial of \(C\). Consider the exact sequence
\[ 0 \to \mathcal{O}(1, -4) \oplus \mathcal{O}(-2, -3) \xrightarrow{\varphi} \mathcal{O}(1, -3) \oplus \mathcal{O} \to \mathcal{F}' \to 0, \]
where
\[ \varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ 0 & \varphi_{22} \end{bmatrix}. \]
Then \(\mathcal{F}'\) is an extension of \(\mathcal{O}_Z\) by \(\mathcal{O}_C\) without zero-dimensional torsion. Since, from the exact sequence (15), \(\mathcal{F}\) is an extension of \(\mathcal{O}_Z\) by \(\mathcal{O}_C\) without zero-dimensional torsion, we can apply Lemma 2.2 to deduce that \(\mathcal{F} \cong \mathcal{F}'\). We obtain a contradiction from the isomorphisms \(\mathbb{C} \cong H^0(\mathcal{F}) \cong H^0(\mathcal{F}') \cong \mathbb{C}^3\). □

**Proposition 3.4.** Let \(M_0 \subset M\) be the subset of sheaves \(\mathcal{F}\) for which \(H^0(\mathcal{F}(0, -1)) = \{0\}\), \(\text{Ker}(\theta_1) \cong \mathcal{O}(-2, -4)\) and \(\text{supp}(\text{Coker}(\theta_1))\) is not contained in a line of degree \((1, 0)\) or \((0, 1)\). Then \(M_0\) is open and can be described as the subset of sheaves \(\mathcal{F}\) having a resolution of the form
\[ (16) \quad 0 \to \mathcal{O}(-1, -3) \oplus \mathcal{O}(0, -3) \oplus \mathcal{O}(-1, -2) \xrightarrow{\varphi} \mathcal{O}(0, -2) \oplus \mathcal{O}(0, -2) \oplus \mathcal{O} \to \mathcal{F} \to 0, \]
where \( \varphi_{12} \) and \( \varphi_{22} \) are linearly independent and the maximal minors of the matrix \((\varphi_{ij})_{i=1,2; j=1,2,3}\) have no common factor.

**Proof.** Let \( \mathcal{F} \) give a point in \( M_0 \). Let \( Z \) and \( C \) be as in Lemma 3.5. By hypothesis \( Z \) is not contained in a line of degree \((1,0)\) or \((0,1)\), hence \( \text{Ext}^2(\mathcal{O}_Z, \mathcal{O}) \simeq \text{Coker}(\zeta) \) as in (1). Let \( \zeta_1, \zeta_2, \zeta_3 \) be the maximal minors of \( \zeta \). They are the defining polynomials of \( Z \), hence we can find \( \varphi_{31} \in V_1^* \otimes S^3 V_2^* \), \( \varphi_{32} \in C \otimes S^4 V_2^* \), \( \varphi_{33} \in V_1^* \otimes S^2 V_2^* \) such that \( \zeta_1 \varphi_{31} - \zeta_2 \varphi_{32} + \zeta_3 \varphi_{33} \) is the polynomial defining \( C \). Let

\[
\varphi = \begin{bmatrix} \varphi_{31} & \varphi_{32} & \varphi_{33} \end{bmatrix}.
\]

Then \( \text{Coker}(\varphi) \) is an extension of \( \text{Ext}^2(\mathcal{O}_Z, \mathcal{O}) \) by \( \mathcal{O}_C \) without zero-dimensional torsion and, by Lemma 3.5, the same is true of \( \mathcal{F} \). From Lemma 2.2 we deduce that \( \mathcal{F} \simeq \text{Coker}(\varphi) \). By Proposition 2.3 \( H^0(\mathcal{F}) \simeq \mathbb{C} \), hence the map \( H^1(\mathcal{O}(0, -3)) \rightarrow H^1(2\mathcal{O}(0, -2)) \) is injective, which is equivalent to saying that \( \varphi_{12} \) and \( \varphi_{22} \) are linearly independent. We have shown that \( \mathcal{F} \) has resolution (10).

Conversely, assume that \( \mathcal{F} \) has resolution (10). Then \( H^0(\mathcal{F}) \simeq \mathbb{C} \) because \( \varphi_{12} \) and \( \varphi_{22} \) are linearly independent. From the snake lemma we see that \( \mathcal{F} \) is an extension of \( \text{Ext}^2(\mathcal{O}_Z, \mathcal{O}) \) by \( \mathcal{O}_C \), where \( Z \) is the zero-dimensional scheme of length 3 given by the maximal minors of the matrix obtained by deleting the third row of \( \varphi \), and \( C \) is the curve of degree \((2,4)\) defined by \( \det(\varphi) \). Thus, \( H^0(\mathcal{F}) \) generates \( \mathcal{O}_C \). We will show that \( \mathcal{F} \) is semi-stable. Assume that \( \mathcal{F} \) had a destabilizing subsheaf \( \mathcal{F}' \). Then \( \chi(\mathcal{F}') < 0 \) and \( \chi(\mathcal{F}') \leq \dim_{\mathbb{C}} H^0(\mathcal{F}) = 1 \), hence \( \chi(\mathcal{F}') = 1 \), forcing \( H^0(\mathcal{F}') \simeq \mathbb{C} \). Hence \( H^0(\mathcal{F}) \) gives a point in \( M \). Since \( \varphi_{12} \) and \( \varphi_{22} \) are linearly independent, we have \( H^0(\mathcal{F}(0,1)) = \{0\} \). Since \( H^0(\mathcal{F}) \) generates \( \mathcal{O}_C \), \( \ker(\theta_1) \simeq \mathcal{O}(-2, -4) \) and \( \text{Coker}(\theta_1) \simeq \text{Ext}^2(\mathcal{O}_Z, \mathcal{O}) \). Note that \( Z \) is not contained in a line of degree \((1,0)\) or \((0,1)\). In conclusion, \( \mathcal{F} \) gives a point in \( M_0 \). \( \square \)

**Proposition 3.5.** The variety \( M \) is rational.

**Proof.** By Lemma 2.4, Lemma 2.2, Lemma 3.5 and Proposition 3.4, the open subset of \( M_0 \), given by the condition that \( Z \) consist of three distinct points, is a \( \mathbb{P}^1 \)-bundle over an open subset of \( \text{Hilb}_{\mathbb{P}^1 \times \mathbb{P}^1}(3) \), so it is rational. \( \square \)

**Proposition 3.6.** Let \( \mathcal{F} \) be an extension as in (10) without zero-dimensional torsion, for a curve \( C \) of degree \((2,4)\) and a subscheme \( Z \subset C \) that is the intersection of two curves of degree \((1,0)\), respectively, \((0,3)\). Then \( \mathcal{F} \) gives a point in \( M \). Let \( M'_2 \subset M \) be the subset of such sheaves \( \mathcal{F} \). Then \( M'_2 \) is closed, irreducible, of codimension 2, and can be described as the set of sheaves \( \mathcal{F} \) having a resolution of the form

\[
0 \rightarrow \mathcal{O}(-2, -1) \oplus \mathcal{O}(-1, -4) \xrightarrow{\varphi} \mathcal{O}(-1, -1) \oplus \mathcal{O} \rightarrow \mathcal{F} \rightarrow 0,
\]

with \( \varphi_{11} \neq 0 \), \( \varphi_{12} \neq 0 \).

**Proof.** Note that \( \mathcal{O}_Z \simeq \text{Ext}^2(\mathcal{O}_Z, \mathcal{O}) \). Let \( \mathcal{F} \) be an extension of \( \mathcal{O}_Z \) by \( \mathcal{O}_C \) without zero-dimensional torsion. Let \( \varphi_{11} \in V_1^* \otimes C \) and \( \varphi_{12} \in C \otimes S^4 V_2^* \) be the defining polynomials of \( Z \). We can find \( \varphi_{21} \in S^2 V_1^* \otimes V_2^* \) and \( \varphi_{22} \in V_1^* \otimes S^4 V_2^* \) such that \( \varphi_{11} \varphi_{22} - \varphi_{12} \varphi_{21} \) is the defining polynomial of \( C \). Then the cokernel of \( \varphi = (\varphi_{ij})_{1 \leq i,j \leq 2} \) is an extension of \( \mathcal{O}_Z \) by \( \mathcal{O}_C \) without zero-dimensional torsion, hence, by Lemma 2.2, \( \mathcal{F} \simeq \text{Coker}(\varphi) \). Conversely, arguing as in Proposition 3.4, we can show that any sheaf of the form \( \text{Coker}(\varphi) \), with \( \varphi \) as in (17), is semi-stable. \( \square \)

**Proof of Theorem 7.7.** By Propositions 3.1, 3.2, 3.4 and 3.6, \( M \) is the union of the subvarieties \( M_0, M_2, M'_2, M_3, M_4 \). For \( [\mathcal{F}] \in M_2 \), we have \( H^0(\mathcal{F}) \simeq \mathbb{C}^2 \), whereas, for \( [\mathcal{F}] \) in any of the other subvarieties, we have \( H^0(\mathcal{F}) \simeq \mathbb{C} \). Thus, \( M_2 \) is disjoint from the other subvarieties. For \( [\mathcal{F}] \in M_2 \cup M'_2 \), \( H^0(\mathcal{F}) \) generates the structure sheaf of a curve \( C \) of degree \((2,4)\), whereas, for \( [\mathcal{F}] \in M_3 \cup M_4 \), \( H^0(\mathcal{F}) \) generates the structure sheaf of a curve \( Q \) of degree \((2,3)\). Thus, \( M_2 \cup M'_2 \) is disjoint from \( M_3 \cup M_4 \). For \( [\mathcal{F}] \in M_0 \), the support of \( \mathcal{F}/\mathcal{O}_C \) is not contained in a line of degree \((1,0)\), whereas, for \( [\mathcal{F}] \in M'_2 \), the support of \( \mathcal{F}/\mathcal{O}_C \) is contained
in a line of degree $(1, 0)$. Thus, $M_0$ is disjoint from $M'_0$. For $[F] \in M_3$, $F/O_Q$ has zero-dimensional torsion, whereas, for $[F] \in M_4$, $F/O_Q$ is pure. Thus, $M_3$ is disjoint from $M_4$. In conclusion, the subvarieties in question form a decomposition of $M$. 

4. Variation of the moduli spaces of $\alpha$-semi-stable pairs

A coherent system $\Lambda = (\Gamma, F)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ consists of a coherent algebraic sheaf $F$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and a vector subspace $\Gamma \subset H^0(F)$. Let $\alpha$ be a positive real number and let $P_F(m, n) = rm + sn + t$ be the Hilbert polynomial of $F$. We define the $\alpha$-slope of $\Lambda$ as the ratio

$$p_\alpha(\Lambda) = \frac{\alpha \dim \Gamma + t}{r + s}.$$ 

We say that $\Lambda$ is $\alpha$-semi-stable, respectively, $\alpha$-stable, if $F$ is pure and for any proper coherent subsystem $\Lambda' \subset \Lambda$ we have $p_\alpha(\Lambda') \leq p_\alpha(\Lambda)$, respectively, $p_\alpha(\Lambda') < p_\alpha(\Lambda)$. According to [9] and [7], for fixed positive real number $\alpha$, non-negative integer $k$ and linear polynomial $P(m, n)$, there is a coarse moduli space, denoted $Syst(\mathbb{P}^1 \times \mathbb{P}^1, \alpha, k, P)$, which is a projective scheme whose closed points are in a bijective correspondence with the set of $S$-equivalence classes of $\alpha$-semi-stable coherent systems $(\Gamma, F)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ for which $\dim \Gamma = k$ and $P_\infty = P$. When $k = 0$ this space is $M(P)$. A coherent system for which $\dim \Gamma = 1$ will be called a pair. Our main concern is with the moduli space of $\alpha$-semi-stable pairs $M^\alpha(P) = Syst(\mathbb{P}^1 \times \mathbb{P}^1, \alpha, 1, P)$. It is known that there are finitely many positive rational numbers $\alpha_1 < \ldots < \alpha_n$, called walls, such that the set of $\alpha$-semi-stable pairs with Hilbert polynomial $P$ remains unchanged as $\alpha$ varies in one of the intervals $(0, \alpha_1)$, or $(\alpha_i, \alpha_{i+1})$, or $(\alpha_n, \infty)$. In fact, from the definition of $\alpha$-semi-stability, we can see that, if $\alpha$ is a wall, then there is a strictly $\alpha$-semi-stable pair, i.e. a pair $\Lambda$ for which there exists a subpair or quotient pair $\Lambda'$, such that $p_\alpha(\Lambda) = p_\alpha(\Lambda')$. This equation has only rational solutions in $\alpha$. For $\alpha \in (0, \alpha_1)$ we write $M^\infty(P) = M^\alpha(P)$. For $\alpha \in (0, 1)$ we write $M^{++}(P) = M^\alpha(P)$. If $\gcd(r + s, t) = 1$, then, from the definition of $\alpha$-semi-stability, we see that $(\Gamma, F) \in M^{++}(P)$ if and only if $F$ is semi-stable. At the other extreme we have the following proposition due to Pandharpande and Thomas.

**Proposition 4.1.** For $\alpha \gg 0$, a pair $\Lambda = (\Gamma, F)$ is $\alpha$-semi-stable if and only if $F$ is pure and $F/O_C$ has dimension zero or is zero, where $O_C$ is the subsheaf of $F$ generated by $\Gamma$. In particular, $t \geq r + s - rs$.

The scheme $M^\infty(rm + sn + t)$ is isomorphic to the relative Hilbert scheme of zero-dimensional schemes of length $t - r - s + rs$ contained in curves of degree $(s, r)$.

**Proof.** Assume that $(\Gamma, F)$ is $\alpha$-semi-stable for $\alpha \gg 0$. If $P_{O_C}(m, n) = r'm + s'n + t'$ with $r' + s' < r + s$, then

$$p_\alpha(\Gamma, O_C) = \frac{\alpha + t'}{r' + s'} > \frac{\alpha + t}{r + s} = p_\alpha(\Lambda) \quad \text{for } \alpha \gg 0,$$

which contradicts our hypothesis. Thus, $P_{O_C}(m, n) = rm + sn + r + s - rs$. Conversely, assume that $O_C$ has this Hilbert polynomial and that $F$ is pure. Let $\Lambda' = (\Gamma', F') \subset \Lambda$ be a proper coherent subsystem with $P_{F'}(m, n) = r'm + s'n + t'$. If $\Gamma' = \{0\}$, then

$$p_\alpha(\Lambda') = \frac{t'}{r' + s'} < \frac{\alpha + t}{r + s} = p_\alpha(\Lambda) \quad \text{for } \alpha \gg 0.$$ 

If $\Gamma' = \Gamma$, then $O_C \subset F'$, hence $r' = r$, $s' = s$, $t' < t$, and we have

$$p_\alpha(\Lambda') = \frac{\alpha + t'}{r' + s'} < \frac{\alpha + t}{r + s} = p_\alpha(\Lambda).$$ 

The isomorphism between $M^\infty(P)$ and the relative Hilbert scheme is a particular case of [13, Proposition B.8]. As a map, it is given by $(\Gamma, F) \mapsto (Z, C)$, where $Z \subset C$ is the subscheme introduced at Lemma 2.2.2.

**Corollary 4.2.** The scheme $M^\infty(4m + 2n + 1)$ is isomorphic to a fiber bundle with fiber $\mathbb{P}^1$ and base the Hilbert scheme of three points in $\mathbb{P}^1 \times \mathbb{P}^1$, so it is smooth.
Proof. The relative Hilbert scheme of pairs \((Z, C)\), where \(C \subseteq \mathbb{P}^1 \times \mathbb{P}^1\) is a curve of degree \((2, 4)\) and \(Z \subseteq C\) is a subscheme of dimension zero and length 3, has fiber \(\mathbb{P}(\mathbb{H}^0(I_Z(2,4)))\) over \(Z\). If \(Z\) is not contained in a line of degree \((0,1)\) or \((1,0)\), then, from Lemma 2.1, we deduce that \(\mathbb{H}^0(I_Z(2,4)) \simeq \mathbb{C}^{12}\). If \(Z\) is contained in such a line, then it is straightforward to check that \(\mathbb{H}^0(I_Z(2,4)) \simeq \mathbb{C}^{12}\).

Lemma 4.3. Assume that \(M^\alpha(rm+sn+t) \neq \emptyset\). Then \(t \geq r+s-rs\). For \(r, s\) non-negative integers, not both zero, and \(\alpha \in (0, \infty)\), we have

\[
M^\alpha(rm+sn+r+s-rs) \simeq M^\infty(rm+sn+r+s-rs).
\]

Proof. We use induction on \(r+s\). If \(r+s = 1\), or if there is no wall in \([\alpha, \infty)\), then \(M^\alpha(rm+sn+t) = M^\infty(rm+sn+t)\) and the conclusion follows from Proposition 1.4. Assume that \(r+s > 1\) and that there is a wall \(\alpha' \in [\alpha, \infty)\). There is a pair \(\Lambda \in M^\alpha(rm+sn+t)\) and a subpair or quotient pair \(\Lambda' \in M^\gamma(r'm+s'n+t')\), such that \(p_{\alpha'}(\Lambda) = p_{\alpha'}(\Lambda')\). We have \(0 \leq r' \leq r, 0 \leq s' \leq s, 1 \leq r' + s' < r + s\),

\[
\frac{\alpha' + t'}{r' + s'} = \frac{\alpha + t}{r + s}.
\]

hence

\[
t = \frac{(r + s - r' - s')\alpha' + (r + s)t'}{r' + s'} > \frac{r + s}{r' + s'}t'
\]

\[
\geq \frac{r + s}{r' + s'}(r' + s' - r's') \quad \text{(by the induction hypothesis)}
\]

\[
= r + s - \frac{r + s}{r' + s'}r's' \geq r + s - rs.
\]

If \(t = r + s - rs\), then there is no wall in \([\alpha, \infty)\), hence we have an isomorphism as in the lemma.

Proposition 4.4. With respect to \(P(m,n) = 4m + 2n + 1\) there are only two walls at \(\alpha_1 = 5\) and \(\alpha_2 = 11\).

Proof. Assume that \(\alpha\) is a wall. Then there are pairs \(\Lambda \in M^\alpha(4m+2n+1)\) and \(\Lambda' \in M^\alpha(rm+sn+t)\) such that \(\Lambda'\) is a subpair or a quotient pair of \(\Lambda\) and

\[
(18) \quad \frac{\alpha + t}{r + s} = \frac{\alpha + 1}{6}.
\]

Here \(0 \leq r \leq 4, 0 \leq s \leq 2, 1 \leq r + s \leq 5\). By Lemma 1.3, we also have \(t \geq r + s - rs\). Assume that \(r = 3, s = 2, t \geq -1\). Equation (18) has solutions \(\alpha_1 = 5\) for \(t = 0\) and \(\alpha_2 = 11\) for \(t = -1\). Assume that \(r = 2, s = 2, t \geq 0\). Equation (18) has solution \(\alpha = 2\) for \(t = 0\). In this case either \(\Lambda \in \text{Ext}^1(\Lambda', \Lambda'')\) or \(\Lambda \in \text{Ext}^1(\Lambda'', \Lambda')\) for some \(\Lambda'' \in M(2m + 1)\). However, according to Proposition 10, \(M(2m + 1) = \emptyset\). Thus, there is no wall at \(\alpha = 2\). For all other choices of \(r\) and \(s\) equation (18) has no positive solution in \(\alpha\).

Denote \(M^\alpha = M^\alpha(4m+2n+1)\). For \(\alpha \in (11, \infty)\), write \(M^\alpha = M^\infty\). For \(\alpha \in (5,11)\), write \(M^\alpha = M^{11-}\). For \(\alpha \in (0,5)\), write \(M^\alpha = M^{0+}\). The inclusions of sets of \(\alpha\)-semi-stable pairs induce the birational morphisms

\[
\begin{align*}
M^\infty &\to M^{11-} \\
M^{11-} &\to M^{5+} \\
M^{5+} &\to M^{0+}
\end{align*}
\]

In view of Theorem 4.11, the above are flipping diagrams (consult [12, Remark 5] for details).
Remark 4.5. From the proof of Proposition 4.4, we see that an S-equivalence class of strictly α-semistable elements in \( M^{11} \) consists of (split or non-split) extensions of \((\Gamma_1, E_1)\) by \((0, O_L(1,0))\), together with the extensions of \((0, O_L(1,0))\) by \((\Gamma_1, E_1)\). Here \((\Gamma_1, E_1)\) lies in \( M^{11}(3m + 2n -1) \) and \( L \subset \mathbb{P}^1 \times \mathbb{P}^1 \) is a line of degree \((0,1)\). We say, for short, that the strictly \( \alpha \)-semi-stable elements of \( M^{11} \) are of the form \((\Gamma, E) \oplus (0, O_L(1,0))\). According to Lemma 4.3 and Proposition 4.1, \( E \) is semi-stable, in fact \( [\alpha] \approx O_Q \) for a quintic curve \( Q \subset \mathbb{P}^1 \times \mathbb{P}^1 \) of degree \((2,3)\). Thus, \( M^{11}(3m + 2n -1) \approx \mathbb{P}^{11} \).

Again from the proof of Proposition 4.4, we see that the strictly \( \alpha \)-semi-stable elements in \( M^5 \) are of the form \((\Gamma, E) \oplus (0, O_L)\), where \((\Gamma, E) \in M^5(3m + 2n)\). We claim that \( M^5(3m + 2n) \approx M^5(3m + 2n) \). To see this, we will show that there are no walls relative to the Polynomial \( P(m,n) = 3m + 2n \). As in the proof of Proposition 4.4, we attempt to solve the equation

\[
\frac{\alpha + t}{r + s} = \frac{\alpha}{5}
\]

with \(0 \leq r \leq 3, 0 \leq s \leq 2, 1 \leq r + s \leq 4, t \geq r + s - rs\). For all choices of \( r \) and \( s \) we have \( t \geq 0 \), hence the above equation has no positive solutions in \( \alpha \). From Proposition 4.4 we see that \( M^5(3m + 2n) \) isomorphic to the universal quintic of degree \((2,3)\), so it is a \( \mathbb{P}^{10} \)-bundle over \( \mathbb{P}^1 \times \mathbb{P}^1 \). More precisely, the elements in \( M^5(3m + 2n) \) are of the form \((\mathbb{H}^0(O_Q(p)), O_Q(p))\), where \( O_Q(p) \) is a non-split extension of \( C_p \) by \( O_Q \).

Proposition 4.6. Let \( Q \subset \mathbb{P}^1 \times \mathbb{P}^1 \) be a quintic curve of degree \((2,3)\), let \( p \in Q \) be a point, let \( O_Q(p) \) be a non-split extension of \( C_p \) by \( O_Q \), and let \( L \subset \mathbb{P}^1 \times \mathbb{P}^1 \) be a line of degree \((0,1)\). Then any non-split extension sheaf \( F \) as in (1.2) is semi-stable. The set of such sheaves is the closure of \( \mathcal{M}_3 \) in \( \mathcal{M} \). The boundary \( \mathcal{M}_3 \setminus \mathcal{M}_3 \) is contained in \( \mathcal{M}_2 \), more precisely, it consists of extensions as in \([7]\) in which \( C = Q \cup L \) and \( p \in Q \).

Proof. The case when \( H^0(F) \simeq C \) was examined at Proposition 3.2, so we need only consider the case when \( H^0(F) \simeq C^2 \). In this case the canonical morphism \( \mathcal{O} \to O_L \) lifts to a morphism \( \mathcal{O} \to F \), hence we can combine resolution (1.4) with the standard resolution of \( O_L \) to obtain the resolution

\[
0 \to O(-2, -2) \oplus O(-1, -3) \oplus O(0, -1) \xrightarrow{\varphi} O(-1, -2) \oplus O \oplus O \to F \to 0,
\]

where \( \varphi \) is given by the equations \( \varphi_{11} \neq 0, \varphi_{12} \neq 0, \) and \( \varphi_{23} \) and \( \varphi_{33} \) are linearly independent. Note that \( p \) is given by the equations \( \varphi_{11} = 0, \varphi_{12} = 0 \). From the snake lemma, we obtain an extension

\[
0 \to F' \to F \to C_p \to 0,
\]

where \( F' \) is given by the resolution

\[
0 \to O(-2, -3) \oplus O(0, -1) \xrightarrow{\varphi'} 2O \to F' \to 0,
\]

where \( \varphi' = \begin{bmatrix} \varphi_{11} & \varphi_{23} \\ \varphi_{12} & \varphi_{33} \end{bmatrix} \), \( \varphi'_{11} = \varphi_{11} \varphi_{22} - \varphi_{12} \varphi_{21} \).

We claim that \( F' \simeq O_C(0,1) \), where \( C = Q \cup L \). In view of Proposition 3.2, the claim implies that \( F \) is semi-stable, in fact \([F] \in \mathcal{M}_2 \). It remains to prove the claim. Let \( K \) be the kernel of the canonical morphism \( O_C \to O_Q \). Since \( K \) has no zero-dimensional torsion and \( P_K = m - 1, K \simeq O_L(-2,0) \). Applying \( \mathcal{H}om(-, \omega) \) to the exact sequence

\[
0 \to O_L(-2,0) \to O_C(0,1) \to O_Q(0,1) \to 0,
\]

yields the exact sequence

\[
0 \to \mathcal{E}xt^1(O_Q(0,1), \omega) \to \mathcal{E}xt^1(O_C(0,1), \omega) \to \mathcal{E}xt^1(O_L(-2,0), \omega) \to 0,
\]
Proposition 4.7. The preimages of the sets of strictly semi-stable elements are the flipping loci: 

Assume now that \( \Gamma \in \Lambda \), then \( \Lambda \) is semi-stable. We have in this case

\[
\begin{align*}
F^\infty &= \rho_\infty^{-1}(M^{11}(3m+2n-1) \times M(m+2)) \subset M^{\infty}, \\
F^{11} &= \rho_{11}^{-1}(M^{11}(3m+2n-1) \times M(m+2)) \subset M^{11^-}, \\
F^{5} &= \rho_5^{-1}(M^5(3m+2n) \times M(m+1)) \subset M^{5^+}, \\
F^{0} &= \rho_0^{-1}(M^5(3m+2n) \times M(m+1)) \subset M^{0^+}.
\end{align*}
\]

\textbf{Proposition 4.7.} Consider \( \Lambda_1 \in M^{11}(3n+2n-1) \), \( \Lambda_2 \in M(m+2), \Lambda_3 \in M^{5}(3m+2n), \text{ and } \Lambda_4 \in M(m+1) \).

(i) Over a point \((\Lambda_1, \Lambda_2)\), \( F^{\infty} \) has fiber \( \mathbb{P}(\text{Ext}^1(\Lambda_1, \Lambda_2)) \).

(ii) Over a point \((\Lambda_1, \Lambda_2)\), \( F^{11} \) has fiber \( \mathbb{P}(\text{Ext}^1(\Lambda_2, \Lambda_1)) \).

(iii) Over a point \((\Lambda_3, \Lambda_4)\), \( F^{5} \) has fiber \( \mathbb{P}(\text{Ext}^1(\Lambda_3, \Lambda_4)) \).

(iv) Over a point \((\Lambda_3, \Lambda_4)\), \( F^{0} \) has fiber \( \mathbb{P}(\text{Ext}^1(\Lambda_4, \Lambda_3)) \).

\textbf{Proof.} (i) We refer to the argument at [12] Remark 2.

(ii) Assume that \( \Lambda = (\Gamma, F) \in F^{11} \) lies over \((\Lambda_1, \Lambda_2)\). Then \( \Lambda \) is a non-split extension of \( \Lambda_1 \) by \( \Lambda_2 \), or, vice versa, of \( \Lambda_2 \) by \( \Lambda_1 \). If \( \Lambda_2 \subset \Lambda \), then

\[
p_\alpha(\Lambda_2) = 2 > \frac{\alpha + 1}{6} = p_\alpha(\Lambda) \quad \text{for } \alpha \in (5, 11),
\]

which violates the semi-stability of \( \Lambda \). Thus \( \Lambda \in \mathbb{P}(\text{Ext}^1(\Lambda_2, \Lambda_1)) \). Conversely, given such \( \Lambda \), we need to show that \( \Lambda \in M^a \) for \( \alpha \in (5, 11) \). Write \( \Lambda_1 = (\Gamma_1, O_Q), \Lambda_2 = (0, O_L(1, 0)) \). We have a non-split extension of sheaves

\[
0 \longrightarrow O_Q \longrightarrow \mathcal{F} \longrightarrow O_L(1, 0) \longrightarrow 0.
\]

Let \( \Lambda' = (\Gamma', F') \) be a proper coherent subsystem of \( \Lambda \). Let \( \mathcal{G} \) be the image of \( \mathcal{F} \) in \( O_L(1, 0) \). If \( \mathcal{F}' \cap O_Q = \{0\} \), then \( \mathcal{G} \neq O_L(1, 0) \), forcing \( \chi(F') = \chi(G) \leq 1 \). If \( \mathcal{F}' \cap O_Q \neq \{0\} \), then \( \chi(F' \cap O_Q) \leq -1 \) because, by virtue of [11] Lemma 9], \( O_Q \) is semi-stable. We have in this case \( \chi(F') = \chi(F' \cap O_Q) + \chi(G) \leq -1 + 2 = 1 \). If \( \Gamma' = \{0\} \), then

\[
p_\alpha(\Lambda') = p_\alpha(\mathcal{F}') \leq 1 < \frac{\alpha + 1}{6} = p_\alpha(\Lambda) \quad \text{for } \alpha \in (5, 11).
\]

Assume now that \( \Gamma' \neq \{0\} \). Then \( \Gamma' = \Gamma = H^0(O_Q) \), hence \( O_Q \subset F' \). If \( O_Q = F' \), then

\[
p_\alpha(\Lambda') = \frac{\alpha - 1}{5} < \frac{\alpha + 1}{6} = p_\alpha(\Lambda) \quad \text{for } \alpha \in (5, 11).
\]

If \( O_Q \not\subset F' \), then \( r(F') + s(F') = 6 \), hence \( \chi(F') \leq 0 \), and hence

\[
p_\alpha(\Lambda') = \frac{\alpha + \chi(F')}{6} \leq \frac{\alpha}{6} < \frac{\alpha + 1}{6} = p_\alpha(\Lambda).
\]
In all cases we have the inequality $p_\alpha(\Lambda') < p_\alpha(\Lambda)$, hence $\Lambda \in \mathbf{M}^\alpha$, for $\alpha \in (5,11)$.

(iii) We will show that every $\Lambda = (\Gamma, \mathcal{F}) \in \mathbb{P}(\text{Ext}^1(\Lambda_3, \Lambda_4))$ gives a point in $\mathbf{M}^\alpha$ for $\alpha \in (5,11)$. Write $\Lambda_3 = (\Gamma_3, \mathcal{O}_Q(p))$, $\Lambda_4 = (0, \mathcal{O}_L)$. We have a, possibly split, extension of sheaves

$$0 \to \mathcal{O}_L \to \mathcal{F} \to \mathcal{O}_Q(p) \to 0.$$ 

Let $\Lambda' = (\Gamma', \mathcal{F}')$ be a proper coherent subsystem of $\Lambda$. Let $\mathcal{G}$ be the image of $\mathcal{F}'$ in $\mathcal{O}_Q(p)$. Using the fact that $\mathcal{O}_Q$ is semi-stable, it is easy to see that $\mathcal{O}_Q(p)$ is semi-stable, as well. Thus, $\chi(\mathcal{G}) \leq 0$, hence $\chi(\mathcal{F}') = \chi(\mathcal{F}' \cap \mathcal{O}_L) + \chi(\mathcal{G}) \leq 1 + 0 = 1$. If $\Gamma' = \{0\}$, then

$$p_\alpha(\Lambda') = p(\mathcal{F'}) \leq 1 < \frac{\alpha + 1}{6} = p_\alpha(\Lambda) \quad \text{for } \alpha \in (5,11).$$

Assume now that $\Gamma' \neq \{0\}$, i.e. $\Gamma' = \Gamma$. Then $\mathcal{O}_Q \subset \mathcal{G}$. If $\mathcal{F}' \cap \mathcal{O}_L = \{0\}$, then $\mathcal{F}' \not\cong \mathcal{O}_Q(p)$, otherwise $\Lambda \simeq \Lambda_3 \oplus \Lambda_4$. In this case $\mathcal{F}' \simeq \mathcal{O}_Q$, hence

$$p_\alpha(\Lambda') = \frac{\alpha - 1}{5} < \frac{\alpha + 1}{6} = p_\alpha(\Lambda) \quad \text{for } \alpha \in (5,11).$$

Assume now that $\mathcal{F}' \cap \mathcal{O}_L \neq \{0\}$. Then $r(\mathcal{F}') + s(\mathcal{F}') = 6$, hence $\chi(\mathcal{F}') \leq 0$, and hence $p_\alpha(\Lambda') < p_\alpha(\Lambda)$.

(iv) If $(\Gamma, \mathcal{F}) \in \mathbb{P}(\text{Ext}^1(\Lambda_1, \Lambda_4))$, then we have the non-split extension (13), hence, by Proposition 4.6, $\mathcal{F}$ is semi-stable. Thus $(\Gamma, \mathcal{F}) \in \mathbf{M}^{p+}$, i.e. $(\Gamma, \mathcal{F}) \in \mathbb{F}^0$.

**Proposition 4.8.** ([7 Corollaire 1.6]) Let $\Lambda = (\Gamma, \mathcal{F})$ and $\Lambda' = (\Gamma', \mathcal{F}')$ be two coherent systems on a separated scheme of finite type over $\mathbb{C}$. Then there is a long exact sequence

$$0 \to \text{Hom}(\Lambda, \Lambda') \to \text{Hom}(\mathcal{F}, \mathcal{F}') \to \text{Hom}(\Gamma, \text{H}^0(\mathcal{F}')/\mathcal{I}') \to \text{Ext}^1(\Lambda, \Lambda') \to \text{Ext}^1(\mathcal{F}, \mathcal{F}') \to \text{Hom}(\Gamma, \text{H}^1(\mathcal{F}')) \to \text{Ext}^2(\Lambda, \Lambda') \to \text{Ext}^2(\mathcal{F}, \mathcal{F}') \to \text{Hom}(\Gamma, \text{H}^2(\mathcal{F}')).$$

**Proposition 4.9.** The flipping loci $F^\infty$, $F^{11}$, $F^5$, $F^0$ are smooth bundles with fibers $\mathbb{P}^3$, $\mathbb{P}^1$, $\mathbb{P}^2$, respectively, $\mathbb{P}^1$.

**Proof.** We need to determine the extension spaces of pairs occurring at Proposition 4.7.

(i) Choose $\Lambda_1 = (\Gamma_1, \mathcal{O}_Q)$ and $\Lambda_2 = (0, \mathcal{O}_L(1,0))$. From Proposition 4.8 we have the long exact sequence

$$0 \to \text{Hom}(\Lambda_1, \Lambda_2) \to \text{Hom}(\mathcal{O}_Q, \mathcal{O}_L(1,0)) \to \text{Hom}(\Gamma_1, \text{H}^0(\mathcal{O}_L(1,0))) \simeq \mathbb{C}^2$$

$$\to \text{Ext}^1(\Lambda_1, \Lambda_2) \to \text{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L(1,0)) \to \text{Hom}(\Gamma_1, \text{H}^1(\mathcal{O}_L(1,0))) = \{0\}.$$ 

For $\alpha \gg 0$, $\Lambda_1$ and $\Lambda_2$ are $\alpha$-stable coherent systems of different slopes, hence $\text{Hom}(\Lambda_1, \Lambda_2) = \{0\}$. From the short exact sequence

$$0 \to \mathcal{O}(-2,-3) \to \mathcal{O} \to \mathcal{O}_Q \to 0,$$

we obtain the long exact sequence

$$0 \to \text{Hom}(\mathcal{O}_Q, \mathcal{O}_L(1,0)) \to \text{H}^0(\mathcal{O}_L(1,0)) \simeq \mathbb{C}^2 \to \text{H}^0(\mathcal{O}_L(3,3)) \simeq \mathbb{C}^4$$

$$\to \text{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L(1,0)) \to \text{H}^1(\mathcal{O}_L(1,0)) = \{0\}.$$ 

Combining the last two long exact sequences, we obtain the isomorphism $\text{Ext}^1(\Lambda_1, \Lambda_2) \simeq \mathbb{C}^4$.

(ii) From Proposition 4.8 we have the exact sequence

$$\{0\} = \text{Hom}(0, \text{H}^0(\mathcal{O}_Q)/\Gamma_1) \to \text{Ext}^1(\Lambda_2, \Lambda_1) \to \text{Ext}^1(\mathcal{O}_L(1,0), \mathcal{O}_Q) \simeq \text{Ext}^1(\mathcal{O}_Q, \mathcal{O}_L(-1,-2))^* \to \text{Hom}(0, \text{H}^1(\mathcal{O}_Q)) = \{0\}.$$
From resolution \([12]\), we obtain the exact sequence
\[
\{0\} = H^0(O_L(-1, -2)) \to H^0(O_L(1, 1)) \cong \mathbb{C}^2 \to \text{Ext}^1(O_Q, O_L(-1, -2)) \to H^1(O_L(-1, -2)) = \{0\}.
\]
Combining the last two exact sequences, we obtain the isomorphism \(\text{Ext}^1(\Lambda_2, \Lambda_1) \cong \mathbb{C}^2\).

(iii) Choose \(\Lambda_3 = (\Gamma, O_Q(p))\) and \(\Lambda_4 = (0, O_L)\). From Proposition \([13]\) we have the long exact sequence
\[
\{0\} = \text{Hom}(\Lambda_3, \Lambda_4) \to \text{Hom}(O_Q(p), O_L) \to \text{Hom}(\Gamma, H^0(O_Q)) \cong \mathbb{C} \to \text{Ext}^1(\Lambda_3, \Lambda_4) \to \text{Ext}^1(O_Q(p), O_L) \to \text{Hom}(\Gamma, H^1(O_Q)) = \{0\}.
\]
From resolution \([14]\) we obtain the exact sequence
\[
0 \to \text{Hom}(O_Q(p), O_L) \to H^0(O_L(1, 2) \oplus O_L) \cong \mathbb{C}^3 \to H^0(O_L(2, 2) \oplus O_L(1, 3)) \cong \mathbb{C}^5
\]
\[
\to \text{Ext}^1(O_Q(p), O_L) \to H^1(O_L(1, 2) \oplus O_L) = \{0\}.
\]
Combining the last two exact sequences, it follows that \(\text{Ext}^1(\Lambda_3, \Lambda_4) \cong \mathbb{C}^3\).

(iv) From Proposition \([15]\) we obtain the exact sequence
\[
\{0\} = \text{Hom}(0, H^0(O_Q(p))/\Gamma) \to \text{Ext}^1(\Lambda_4, \Lambda_3) \to \text{Ext}^1(O_L, O_Q(p)) \cong \text{Ext}^1(O_Q(p), O_L(-2, -2))^* \to \text{Hom}(0, H^1(O_Q(p))) = \{0\}.
\]
From resolution \([14]\) we obtain the exact sequence
\[
\{0\} = H^0(O_L(-1, 0) \oplus O_L(-2, -2)) \to H^0(O_L \oplus O_L(-1, 1)) \cong \mathbb{C} \to \text{Ext}^1(O_Q(p), O_L(-2, -2))
\]
\[
\to H^1(O_L(-1, 0) \oplus O_L(-2, -2)) \cong \mathbb{C} \to H^1(O_L \oplus O_L(-1, 1)) = \{0\}.
\]
Combining the last two exact sequences, it follows that \(\text{Ext}^1(\Lambda_4, \Lambda_3) \cong \mathbb{C}^2\).

\[\square\]

**Lemma 4.10.** (i) For \(\Lambda \in F^{11}\) we have \(\text{Ext}^2(\Lambda, \Lambda) = \{0\}\).

(ii) For \(\Lambda \in F^0\) we have \(\text{Ext}^2(\Lambda, \Lambda) = \{0\}\).

**Proof.** (i) In view of the exact sequence
\[
0 \to \Lambda_1 \to \Lambda \to \Lambda_2 \to 0
\]
it is enough to show that \(\text{Ext}^2(\Lambda_1, \Lambda_j) = \{0\}\) for \(i, j = 1, 2\). From Proposition \([4.8]\) we have the exact sequence
\[
\{0\} = \text{Hom}(\Gamma_1, H^1(O_L(1, 0))) \to \text{Ext}^2(\Lambda_1, \Lambda_2) \to \text{Ext}^2(O_Q, O_L(1, 0)) \cong \text{Hom}(O_L(1, 0), O_Q(-2, -2))^*.
\]
The group on the right vanishes because \(H^0(O_Q(-3, -2)) = \{0\}\). Thus, \(\text{Ext}^2(\Lambda_1, \Lambda_2) = \{0\}\). From the exact sequence
\[
\{0\} = \text{Hom}(0, H^1(O_Q)) \to \text{Ext}^2(A_2, A_1) \to \text{Ext}^2(O_L(1, 0), O_Q) \cong \text{Hom}(O_Q, O_L(-1, -2))^* = \{0\}
\]
we obtain the vanishing of \(\text{Ext}^2(\Lambda_2, \Lambda_1)\). From the exact sequence
\[
\{0\} = \text{Hom}(0, H^1(O_L(1, 0))) \to \text{Ext}^2(A_2, A_2)
\]
\[
\to \text{Ext}^2(O_L(1, 0), O_L(1, 0)) \cong \text{Hom}(O_L(1, 0), O_L(-1, -2))^* = \{0\}
\]
we obtain the vanishing of \(\text{Ext}^2(\Lambda_2, \Lambda_2)\). From Proposition \([15]\) we have the exact sequence
\[
\{0\} = \text{Hom}(\Gamma_1, H^0(O_Q))/\Gamma_1) \to \text{Ext}^1(\Lambda_1, \Lambda_1) \to \text{Ext}^1(O_Q, O_Q) \to \text{Hom}(\Gamma_1, H^1(O_Q)) \cong \mathbb{C}^2
\]
\[
\to \text{Ext}^2(\Lambda_1, \Lambda_1) \to \text{Ext}^2(O_Q, O_Q) \cong \text{Hom}(O_Q, O_Q(-2, -2))^* = \{0\}.
\]
According to [7, Théorème 3.12], \( \text{Ext}^1(\Lambda_1, \Lambda_1) \) is isomorphic to the tangent space of \( M^1(3m + 2n - 1) \cong \mathbb{P}^{11} \) (see Remark 4.5) at \( \Lambda_1 \), so it is isomorphic to \( \mathbb{C}^{11} \). From resolution (19), we obtain the exact sequence

\[
0 \to \text{Hom}(O_Q, O_Q) \cong H^0(O_Q) \to H^0(O_Q(2, 3)) \cong \mathbb{C}^{11} \\
\to \text{Ext}^1(O_Q, O_Q) \to H^1(O_Q) \cong \mathbb{C}^2 \to H^1(O_Q(2, 3)) = \{0\}.
\]

Combining the last two exact sequences we obtain the vanishing of \( \text{Ext}^2(\Lambda_1, \Lambda_1) \).

(ii) As above, we need to prove that \( \text{Ext}^2(\Lambda_i, \Lambda_j) = \{0\} \) for \( i, j = 3, 4 \). From Proposition 4.8, we have the exact sequence

\[
\{0\} = \text{Hom}(\Gamma, H^1(O_L)) \to \text{Ext}^2(\Lambda_3, \Lambda_4) \to \text{Ext}^2(O_Q(p), O_L) \cong \text{Hom}(O_L, O_Q(p)(-2, -2))^* = \{0\}.
\]

Thus, \( \text{Ext}^2(\Lambda_3, \Lambda_4) = \{0\} \). From the exact sequence

\[
\{0\} = \text{Hom}(0, H^1(O_Q(p))) \to \text{Ext}^2(\Lambda_4, \Lambda_3) \to \text{Ext}^2(O_L, O_Q(p)) \cong \text{Hom}(O_Q(p), O_L(-2, -2))^* = \{0\}
\]

we obtain the vanishing of \( \text{Ext}^2(\Lambda_4, \Lambda_3) \). From Proposition 4.8, we have the exact sequence

\[
\{0\} = \text{Hom}(\Gamma, H^0(O_Q(p))/\Gamma) \\
\to \text{Ext}^1(\Lambda_3, \Lambda_3) \to \text{Ext}^1(O_Q(p), O_Q(p)) \to \text{Hom}(\Gamma, H^1(O_Q(p))) \cong \mathbb{C} \\
\to \text{Ext}^2(\Lambda_3, \Lambda_3) \to \text{Ext}^2(O_Q(p), O_Q(p)) \cong \text{Hom}(O_Q(p), O_Q(p)(-2, -2))^* = \{0\}.
\]

From resolution (14), we obtain the exact sequence

\[
0 \to \text{Hom}(O_Q(p), O_Q(p)) \to H^0(O_Q(p)(1, 2)) \oplus H^0(O_Q(p)) \to H^0(O_Q(p)(2, 2)) \oplus H^0(O_Q(p)(1, 3)) \\
\to \text{Ext}^1(O_Q(p), O_Q(p)) \to H^1(O_Q(p)(1, 2)) \oplus H^1(O_Q(p)) \to H^1(O_Q(p)(2, 2)) \oplus H^1(O_Q(p)(1, 3)) \to 0.
\]

Since \( \text{Hom}(O_Q(p), O_Q(p)) \cong \mathbb{C} \), it follows that

\[
\dim_{\mathbb{C}} \text{Ext}^1(O_Q(p), O_Q(p)) = 1 - \chi(O_Q(p)(1, 2)) - \chi(O_Q(p)) + \chi(O_Q(p)(2, 2)) + \chi(O_Q(p)(1, 3)) = 13.
\]

According to [7, Théorème 3.12], \( \text{Ext}^1(\Lambda_3, \Lambda_3) \) is isomorphic to the tangent space at \( \Lambda_3 \) of \( M^5(3m + 2n) \), which, according to Remark 4.5, is smooth of dimension 12. We obtain the vanishing of \( \text{Ext}^2(\Lambda_3, \Lambda_3) \).

**Theorem 4.11.** Let \( M^\alpha \) be the moduli space of \( \alpha \)-semi-stable pairs on \( \mathbb{P}^1 \times \mathbb{P}^1 \) with Hilbert polynomial \( P(m, n) = 4m + 2n + 1 \). We have the following blowing up diagrams

\[
\begin{array}{c}
\beta_{11} \downarrow \\
\beta_{11} \downarrow \\
M^\infty & M^{11} & M^{11} & M^{5+} & M^0+ \\
\beta_{11} \downarrow & \beta_{11} \downarrow & \beta_5 \downarrow & \beta_5 \downarrow & \beta_0 \downarrow \\
M^{11} & M^{5+} & M^{5+} & M^{5+} & M^0+
\end{array}
\]

Here \( \beta_{11} \) is the blow-up along \( F^\infty \) and \( \beta_{11} \) is the contraction of the exceptional divisor \( F^\infty \) in the direction of \( \mathbb{P}^3 \), where we view \( F^\infty \) as a \( \mathbb{P}^3 \times \mathbb{P}^1 \)-bundle with base \( M^{11}(3m + 2n - 1) \times M(m + 2) \). Likewise, \( \beta_5 \) is the blow-up along \( F^5 \) and \( \beta_0 \) is the contraction of the exceptional divisor \( F^5 \) in the direction of \( \mathbb{P}^2 \), where we view \( F^5 \) as a \( \mathbb{P}^2 \times \mathbb{P}^1 \)-bundle over \( M^5(3m + 2n) \times M(m + 1) \).
Proof. A birational morphism $\beta_{11}: \tilde{M}^{\infty} \to M^{11-}$ can be constructed as at [3, Theorem 3.3] such that $\beta_{11}$ contracts $\tilde{F}^{\infty}$ in the direction of $P^3$, $\beta_{11}$ is an isomorphism outside $F^{11}$, and $\beta_{11}^{-1}(x) \simeq \mathbb{P}^3$ for any $x \in F^{11}$. We now apply the Universal Property of the blow-up [3, p. 604] to deduce that $\beta_{11}$ is a blow-up with center $F^{11}$. For this we need to know that $M^{11-}$ and $F^{11}$ are smooth. By Corollary [4.2], $M^{11-}$ is smooth, so by Proposition [3.1], the blowing up center $F^{11}$ is smooth, hence $M^{11-}$ is smooth, too. Since $\beta_{11}$ is an isomorphism outside $F^{11}$, $M^{11-} \setminus F^{11}$ is smooth. Since all points of $M^{11-}$ are $\alpha$-stable, we can apply the Smoothness Criterion [7, Théorème 3.12], which states that $\Lambda \in M^{11-}$ is a smooth point if $\text{Ext}^2(\Lambda, \Lambda) = \{0\}$. Thus, in view of Lemma [4.10(i)], $M^{11-}$ is smooth at every point of $F^{11}$. The smoothness of $F^{11}$ was proved at Proposition [4.9].

For the second blow-up diagram we reason analogously, using the facts that $F^5$ and $F^0$ are smooth, and using Lemma [4.10(ii)]. □

According to [7, Théorème 4.3], there is a universal family $(\tilde{\Gamma}, \tilde{F})$ of coherent systems on $M^{0+} \times \mathbb{P}^1 \times \mathbb{P}^1$. In particular, $\tilde{F}$ is a family of semi-stable sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert polynomial $4m + 2n + 1$, which is flat over $M^{0+}$. It induces the so called forgetful morphism $\phi: M^{0+} \to M$. We have $\phi(\Gamma, F) = [F]$. Proposition 4.12. The forgetful morphism $\phi: M^{0+} \to M$ is a blow-up with center the Brill-Noether locus $M_2$.

Proof. According to Proposition [2.3 ii), for $[F] \in M \setminus M_2$ we have $H^0(F) \simeq \mathbb{C}$, hence $\phi^{-1}([F]) = (H^0(F), F)$ is a single point. Thus, $\phi$ is an isomorphism away from $M_2$. According to Proposition [4.11] for $[F] \in M_2$ we have $H^0(F) \simeq \mathbb{C}^2$, hence $\phi^{-1}([F]) \simeq \mathbb{P}^1$. Taking into account that $M$ and $M_2$ are smooth, we can apply the Universal Property of the blow-up [6, p. 604] to conclude that $\phi$ is a blow-up with center $M_2$. □

Proof of Theorem 4.11. By virtue of Proposition 4.12 we have the relation

$$P(M) = P(M^{0+}) - \xi P(M_2).$$

According to Proposition 5.1 we have the relation

$$P(M_2) = P(\mathbb{P}^1) P(\mathbb{P}^1 \times \mathbb{P}^1).$$

By virtue of Theorem 4.11 we have the relation

$$P(M^{0+}) = P(M^{\infty}) \left( P(\mathbb{P}^1) - P(\mathbb{P}^3) \right) P(M^{11}(3m + 2n - 1) \times M(m + 2))$$

$$+ \left( P(\mathbb{P}^1) - P(\mathbb{P}^2) \right) P(M^5(3m + 2n) \times M(m + 1)).$$

In view of Corollary 4.2 and Remark 4.5 we have the relation

$$P(M^{0+}) = P(\mathbb{P}^{11}) P(\text{Hilb}_{\mathbb{P}^1 \times \mathbb{P}^1}(3)) + (P(\mathbb{P}^1) - P(\mathbb{P}^3)) P(\mathbb{P}^{11}) P(\mathbb{P}^1) + (P(\mathbb{P}^1) - P(\mathbb{P}^2)) P(\mathbb{P}^{10}) P(\mathbb{P}^1 \times \mathbb{P}^1) P(\mathbb{P}^1).$$

According to [5, Theorem 0.1], we have the equation

$$P(\text{Hilb}_{\mathbb{P}^1 \times \mathbb{P}^1}(3)) = \xi^6 + 3\xi^5 + 9\xi^4 + 14\xi^3 + 9\xi^2 + 3\xi + 1.$$

The final result reads

$$P(M) = \frac{\xi^{12} - 1}{\xi - 1}(\xi^6 + 3\xi^5 + 9\xi^4 + 14\xi^3 + 9\xi^2 + 3\xi + 1) - (\xi^3 + \xi^2) \frac{\xi^{12} - 1}{\xi - 1}(\xi + 1)$$

$$- \xi^2 \frac{\xi^{11} - 1}{\xi - 1}(\xi + 1)^3 - \xi \frac{\xi^{14} - 1}{\xi - 1}(\xi + 1)^2. \quad \square$$

Acknowledgement. The author would like to thank Jean-Marc Drézet for several helpful discussions.
References

[1] E. Ballico, S. Huh. Stable sheaves on a smooth quadric surface with linear Hilbert bipolynomials. Sci. World J. (2014), article ID 346126.
[2] N. P. Buchdahl. Stable 2-bundles on Hirzebruch surfaces. Math. Z. 194 (1987), 143–152.
[3] J. Choi, K. Chung. Moduli spaces of α-stable pairs and wall-crossing on $\mathbb{P}^2$. J. Math. Soc. Japan 68 (2016), 685–709.
[4] J. Choi, S. Katz, A. Klemm. The refined BPS index from stable pair invariants. Commun. Math. Phys. 328 (2014), 903–954.
[5] L. Göttsche. The Betti numbers of the Hilbert scheme of points on a smooth projective surface. Math. Ann. 286 (1990), 193–207.
[6] P. Griffiths, J. Harris. Principles of Algebraic Geometry. John Wiley & Sons, New York, 1994.
[7] M. He. Espaces de modules de systèmes cohérents. Int. J. Math. 9 (1998), 545–598.
[8] S. Katz. Genus zero Gopakumar-Vafa invariants of contractible curves. J. Differential Geometry 79 (2008), 185–195.
[9] J. Le Potier. Systèmes cohérents et structures de niveau. Astérisque 214, 1993.
[10] J. Le Potier. Faisceaux semi-stables de dimension 1 sur le plan projectif. Rev. Roumaine Math. Pures Appl. 38 (1993), 635–678.
[11] M. Maican. A duality result for moduli spaces of semistable sheaves supported on projective curves. Rend. Sem. Mat. Univ. Padova 123 (2010), 55–68.
[12] M. Maican. Moduli of sheaves supported on curves of genus two in a quadric surface. Geom. Dedicata 199 (2019), 307–334.
[13] R. Pandharipande, R. P. Thomas. Stable pairs and BPS invariants. J. Amer. Math. Soc. 23 (2010), 267–297.

Institute of Mathematics of the Romanian Academy, Calea Grivitei 21, Bucharest 010702, Romania
E-mail address: maican@imar.ro