The Finiteness Result for Khovanov Homology and Localization in Monoidal Categories.

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Abstract.

In [S1] we constructed the local system of Khovanov complexes on the Vassiliev’s space of knots and extended it to the singular locus. In this paper we introduce the definition of the homology theory (local system or sheaf) of finite type and prove the first finiteness result: the Khovanov local system restricted to the subcategory of knots of the crossing number at most $n$ is the theory of type $\leq n$. This result can be further generalized to the categorification of Birman-Lin theorem [S2].

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1. Introduction.

In [S2] we outlined a program of classification of homological knot theories, such as Khovanov’s categorification of the Jones polynomial [Kh], Ozsvath-Szabo categorification of Alexander polynomial [OS] and Khovanov-Rozansky homology [KR]. This program can be further generalized to 3 and 4-manifolds.

Namely, we want to classify the following functors (embedded TQFT’s):

$\xrightarrow{\text{Spaces functor}} \xrightarrow{(e.g. \text{Khovanov})} \text{Triang.Cat}$

which behave in a prescribed way (via the wall-crossing morphisms) under cobordisms. By Spaces we understand the moduli spaces of manifolds, including the singular ones (e.g. Vassiliev’s space of knots).

We consider a knot homology theory as a local system, or a constructible sheaf on the space of all objects (knots, including singular ones), extend this local system to the singular locus and introduce the analogue of the ”Vassiliev derivative” for categorifications.

We get the correspondence between knots (possibly singular) and objects of the triangulated category (spectra), satisfying the exact triangle relations.

Our classification is given by the ”type” of the theory, the homological condition on its extension to the strata of the discriminant, similar to the Vassiliev’s classification of knot invariants.

Recall that by Vassiliev, the knot invariant is of finite type $n$, if for any selfintersection of the discriminant of the space of knots of codimension $n + 1$, the alternated sum of the invariants of knots from $2^n+1$ adjacent chambers is zero.

The main example of this paper is the Khovanov homology, however our results can be generalized to Khovanov-Rozansky theory. Recall that M. Khovanov [Kh] categorified the Jones polynomial, i.e. he found a homology theory, the Euler characteristics of which equals the Jones polynomial. From the diagram of the knot he constructs a bigraded complex, associated with this diagram, using 0 and 1- resolutions of the knot crossings.

The complex becomes the sum of the tensor products of the vector space $V$, where the homological degree is given by the number of 1’s in the complete resolution of the knot diagram. In [S1] we constructed the local system of Khovanov complexes on the space of knots, the
wall-crossing morphisms for the local system and introduced the definition of the Khovanov homology of a singular knot.

The discriminant of the space of knots corresponds to knots with transversal self-intersection, it is an algebraic hypersurface which is cooriented, so that all cobordisms between knots are directed, i.e. moving between chambers we change overcrossing to undercrossing by passing through a knot with a single double point. We extended the Khovanov local system to the discriminant by the cone of a wall-crossing morphism [S1]:

**Definition 1 [S1].** The Khovanov homology of the singular knot (with a single double point) is a bigraded complex

\[ X^\bullet \oplus Y^\bullet[1] \]  
with the matrix differential \[ d_{C_\omega} = \begin{pmatrix} dX & \omega \\ 0 & dY[1] \end{pmatrix} \]

where \( X^\bullet \) is Khovanov complex of the knot with overcrossing, \( Y^\bullet \) is the Khovanov complex of the knot with undercrossing and \( \omega \) is the wall-crossing morphism.

In this paper we give the geometric interpretation of the homology of the cone of the wall-crossing morphism:

**Definition 2.** The Khovanov homology of the singular knot \( K \) of \( n \) crossings (with \( k \)th single double point) is a bigraded complex

\[ C^\bullet \oplus C^\bullet[2] \]

with the matrix differential \[ d_{C_\omega} = \begin{pmatrix} dC & 0 \\ 0 & dc[2] \end{pmatrix} \]

where \( C^\bullet \) is the Khovanov complex of the knot of \((n-1)\) crossings where \( k \)th double point of \( K \) is given 1-resolution.

This implies the geometric definition of the local system being of finite type \( n \):

**Definition (G).** The local system of Khovanov complexes, extended to the discriminant of the space of manifolds via the cone of morphism, is a **local system of order** \( n \) if for any self-intersection of the discriminant of codimension \( n \), its \( n \)'s cone is quasiisomorphic to

\[ C^0_nX^\bullet \oplus C^1_nX^\bullet[2] \ldots \oplus C^n_nX^\bullet[2n] \]

where \( X^\bullet \) is the Khovanov complex of the disconnected sum of unknots.

In the triangulated category to every morphism between complexes there corresponds an object (up to isomorphism), the mapping cone, which fits into an exact sequence. By assigning cones of wall-crossing morphisms to singular knots we get the structure of the triangulated category for the Khovanov’s sheaf. We observe, that the constructed category has the monoidal symmetric structure: the Khovanov complex of the disconnected sum of knots is the tensor product of the corresponding complexes.

To give the definition of the local system of finite type and to prove the main result we construct the category of the Khovanov spectra. We stabilize this triangulated monoidal category by taking disconnected sums of a knot with the collection of circles, i.e. by taking tensor products of the sheaf with the complexes, corresponding to the disconnected sums of circles. (Tensoring with the Khovanov complex of a circle can be viewed as the suspension.) We use
the reduced version of Khovanov homology [Kh1]. As the result of the localization we get the category of Khovanov spectra $\tilde{D}$ which is again the symmetric monoidal category [Vo].

Next we construct the sequence of derived categories, a filtration, obtained by factorization over the ideals, generated by the Khovanov complexes of the the special form, supported on the strata of the discriminant:

$$\tilde{D} = \tilde{D}_\infty \supset \ldots \tilde{D}_n \supset \tilde{D}_{n-1} \supset \ldots \supset \tilde{D}_1$$

This filtration is an analogue of the filtration in the theory of invariants of finite type. We show that in these subcategories the Khovanov theory satisfies the finiteness condition introduced in [S1].

The algebraic, Vassiliev-type definition of finiteness now becomes as follows. Consider a new invariant, an additive functor from complexes to the 2-torsion in their cohomology:

$$T_2 : C \in \text{Ob}(\mathcal{D}) \rightarrow \text{Tor}_2(H^*(C))$$

If the local system of Khovanov’s complexes $CKh$ has $T_2^n = 0$, i.e. $T_2(H^*(CKh))|_{D_n} = 0$ everywhere on $D_n$ - the codimension $n$ of the discriminant, but is not quasiisomorphic to $(\mathbb{Z}^2, 0, \mathbb{Z}^2)$ (to eliminate the case of the Hopf link), form a factor-category in a sense of Verdier $D_n = \mathcal{D}/I_n$, where the category $I_n$ is supported on the codimension $n$ of the discriminant.

Now can give an algebraic (torsion) definition, similar to the original one of Vassiliev (triviality or acyclicity of complexes in codimension $n$) [S1].

**Definition (T).** The local system is of finite type $n$ if for any codimension $n$ selfintersection of the discriminant its $n$th cone is not quasiisomorphic to $(\mathbb{Z}^2, 0, \mathbb{Z}^2)$, and has torsion-free homology (i.e. the image of $T_2$ is zero).

The Vassiliev-type definition becomes:

**Definition (V).** The local system of Khovanov complexes is of finite type $n$ if for any codimension $n$ selfintersection of the discriminant the $n$th cone is zero in $\tilde{D}_n$ but not in $\tilde{D}_{n+1}$.

The main results of this paper is as follows:

**Theorem 1.** Restricted to the subcategory of knots with the crossing number at most $n$, $n \geq 3$, Khovanov local system is of finite type $\leq n$.

We will further generalize this result [S2] and get the "categorification of Birman-Lin theorem" [S2].

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2. The geometric interpretation of the cone of the wall-crossing morphism.

In this paragraph we give the geometric interpretation of the cone of the wall-crossing morphism for Khovanov homology, by raising the skein relation for the Jones polynomial to the level of complexes.

First we recall the definition of the wall-crossing morphism \( \omega_k \), corresponding to the crossing change for the kth point in the knot projection.

\[
\omega_k : A^\bullet_0(k) \xrightarrow{Id} B^\bullet_0[1](k)
\]

\[
\omega_k : A^\bullet_1(k)[1] \xrightarrow{0} B^\bullet_1[1](k)
\]

Where \( A^\bullet \) is the Khovanov complex, corresponding to the knot with the kth overcrossing and \( B^\bullet[1] \) to the knot with kth undercrossing.

When we change kth overcrossing to undercrossing, 0 and 1-resolutions are exchanged, so \( A^\bullet = A^\bullet_0(k) \oplus A^\bullet_1(k), B^\bullet[1] = B^\bullet_0(k) \oplus B^\bullet_1[1](k) \), and for every k we get morphism \( \omega_k \).

According to the Definition 2 to show that the local system is of finite type (n-1), we have to prove that for any point of selfintersection of the discriminant of codimension n the corresponding n-cones are acyclic complexes, or that the convolution of n dimensional hypercube is acyclic.

We first observe the following simple properties of the Khovanov complex:

**Proposition 1.** For any commutative n-dimensional cube, with \( 2^n \) complexes of length \( (n+1) \) at its vertices, the last complex (via coorientation) is the dual of the first one.

**Proof.** Notice, that by changing all overcrossings to undercrossings on the diagram projection we exchange 0-resolutions of the diagram to 1-resolutions and move from the knot to its mirror. One can easily see that the Khovanov complex will be dualized.

Now we want to study the restrictions of our local system. Recall that a **subcategory** of a category \( C \) is a category \( S \) whose objects are objects in \( C \) and whose arrows \( f : A \to B \) are arrows in \( C \) (with the same source and target).

For the category of knots \( \mathcal{K} \) we can define a sequence of subcategories \( \mathcal{K}_n \). Objects of \( \mathcal{K}_n \) are knots with a crossing number at most \( n \).

We refer to [GM] for the definition and properties of Postnikov towers and convolutions. Here we show that for the restriction of the Khovanov local system to the subcategory \( \mathcal{K}_n \) of knots with at most \( n \) crossings, complexes forming equators of the hypercubes will fit into a sequence.
Proposition 2. If $n$ complexes $X^\bullet, \ldots, Z^\bullet$ form the equator of the $n$-hypercube in the restriction of the Khovanov local system to $\mathcal{K}_n$, the subcategory of knots with at most $n$ crossings, then $X^\bullet, \ldots, Z^\bullet$ and the wall-crossing morphisms $\omega$ fit into a sequence:

$$
\rightarrow Z^\bullet[-n+1] \rightarrow^u X^\bullet \rightarrow^\omega \ldots \rightarrow^\omega Z^\bullet \rightarrow^u X^\bullet[n-1]
$$

where $u$ is an isomorphism between $X^n$ and $Z^0$.

Proof. If $(n+1)$ complexes $X^\bullet, \ldots, Z^\bullet$ form the equator in the $n$-dimensional commutative hypercube, there is a connecting map $u$ from the last complex into the first one, shifted by $[n-1]$, [GM]. Given Proposition 1, the connecting map $u$ is the isomorphism between $Z_0$ and $X_n$.

Proposition 3. The cone of the wall-crossing morphism between Khovanov complexes corresponding to $K$ and $K'$, where the $i$th double point of the projection of $K$ is overcrossing and the $i$th double point of $K'$ is undercrossing, is quasiisomorphic to

$$
X^\bullet \oplus X^\bullet[2]
$$

where $X^\bullet$ is the Khovanov complex of the knot of $(n-1)$ crossings where $i$th double point of $K$ is given 1-resolution.

Proof. It is clear from the definition of $\omega_k$ and the way we defined the local system, that the wall-crossing morphism corresponding to the $i$th crossing will map isomorphically the parts of the complex, which have 0-resolution of the crossing (and will be quasiisomorphic to zero in the cone of the wall-crossing morphism). The parts, that are mapped by zero, are the 1-resolutions of the $k$th crossing. In particular, we see that the last component of the complex contributes nontrivially to the homology of the cone.

So the homology of the cone of the wall crossing morphism is isomorphic to the homology of the knot with the $i$th intersection point given 1-resolution.

This allows us to give the geometric version of the Definition 1:

Let $D$ be the projection of the knot of $n$ crossings with $k$th single double point. We call $D$ the projection of the singular knot.

Definition 2. The Khovanov homology of the singular knot $K$ of $n$ crossings (with $k$th single double point) is a bigraded complex

$$
C^\bullet \oplus C^\bullet[2] \quad \text{with the matrix differential } \quad d_{C_\omega} = \begin{pmatrix}
  d_C & 0 \\
  0 & d_C[2]
\end{pmatrix},
$$

where $C^\bullet$ is the Khovanov complex of the knot of $(n-1)$ crossings where $k$th double point of $K$ is given 1-resolution.
On the level of the Euler characteristics one has:

\[ \chi(A^*[1]) = -\chi(A^*[1]) \]

\[ \chi(C_f) = \chi(A^*) - \chi(B^*) \]

Since the Euler characteristics of Khovanov complex is the Jones polynomial, recall its skein relation:

\[ q^{-1}J_{L_+} - qJ_{L_-} = (q^{1/2} - q^{-1/2})J_{L_0} \]

where \( L_+ \) is the knot with overcrossing, \( L_- \) knot with undercrossing and \( L_0 \) is the knot, where the crossing point is given 1-resolution.

**Proposition 4.** Let \( K \) be the knot with \( n \) crossings, denote it’s projection \( D_{i_1,\ldots,i_n} \), where each index \( i_n \) can have values + for overcrossing, − for undercrossing and 0 for 1-resolution of the crossing point. Then the \( m \)’th cone of the local system, for which \( K \) is the ”first” knot is given by formula (for simplicity we assume that the wall-crossings happen for the first \( m \) indices):

\[
C^0_m CKh(D_{0,0,\ldots,0,i_{m+1},\ldots,i_n}) \oplus C^1_m CKh(D_{0,0,\ldots,0,i_{m+1},\ldots,i_n})[2] \oplus \ldots \\
\ldots \oplus C^l_m CKh(D_{0,0,\ldots,0,i_{m+1},\ldots,i_n})[2l] \oplus \ldots \oplus CKh(D_{0,0,\ldots,0,i_{m+1},\ldots,i_n})[2m]
\]

**Proof.** We will prove this formula by induction. The first cone was described in Proposition 3. In our notation it is given by formula \( CKh(D_{0,i_2,\ldots,i_n}) \oplus CKh(D_{0,i_2,\ldots,i_n})[2] \). If we take the second cone, we get

\[ CKh(D_{0,0,i_3,\ldots,i_n}) \oplus CKh(D_{0,0,i_3,\ldots,i_n})[2] \oplus CKh(D_{0,0,i_3,\ldots,i_n})[2] \oplus CKh(D_{0,0,i_3,\ldots,i_n})[4] \]

Suppose we proved the formula for \( m-1 \), let’s show it is true for \( m \).

The \( m - 1 \)st cone is found to be

\[
C^0_{m-1} CKh(D_{0,0,\ldots,0,i_m,\ldots,i_n}) \oplus C^1_{m-1} CKh(D_{0,0,\ldots,0,i_m,\ldots,i_n})[2] \oplus \ldots \\
\ldots \oplus C^l_{m-1} CKh(D_{0,0,\ldots,0,i_m,\ldots,i_n})[2l] \oplus \ldots \oplus CKh(D_{0,0,\ldots,0,i_m,\ldots,i_n})[2m - 2]
\]

When taking the last \( m \)th cone, we will take pairwise cones of corresponding summands and can use Proposition 3 to show that we will be getting the components of the complex, described in Proposition 4.

We just have to show that the coeffitients of the formula will be given by binomial coeffitients of proposition 4. This can be proved by using the identities for the binomial coeffitients. For any \( n, m \) we use the formula:

\[ C^m_n = C^m_{n-1} \oplus C^m_{n-1} \]

The geometric version of the definition of finiteness follows from the Definition 2:
Definition (G). The local system of Khovanov complexes, extended to the discriminant of the space of manifolds via the cone of morphism, is a **local system of order** n if for any selfintersection of the discriminant of codimension n, its n’s cone is quasiisomorphic to
\[
C_0^n X^\bullet \oplus C_1^1 X^\bullet [2] \oplus \ldots \oplus C_n^n X^\bullet [2n]
\]
where \(X^\bullet\) is the Khovanov complex of the disconnected sum of circles.

In paragraphs 5,7 we will construct the sequence of derived categories, in which Definitions 2 and 4 will become equivalent.

### 3. Jones polynomial and invariants of finite type.

The notion of the invariant of finite type was introduced by V. Vassiliev in 1989 as a filtration in the spectral sequence. Later it was interpreted by Birman and Lin as a “Vassiliev derivative” and led to the following skein relation.

If \(\lambda\) be an arbitrary invariant of oriented knots in oriented space with values in some abelian group \(A\). Extend \(\lambda\) to be an invariant of 1-singular knots \(L_1\) (knots that may have a single singularity that locally looks like a double point), using the formula
\[
\lambda(L_1) = \lambda(L_+) - \lambda(L_-)
\]
where as before \(L_+\) is the knot with overcrossing, \(L_-\) knot with undercrossing.

Further extend \(\lambda\) to the set of \(n\)-singular knots \(L_n\) (knots with \(n\) double points) by repeatedly using the skein relation.

**Definition** We say that \(\lambda\) is of type \(n\) if its extension to \((n + 1)\)-singular knots vanishes identically. We say that \(\lambda\) is of finite type if it is of type \(n\) for some \(n\).

Let \(L_n\) be invariants of knots (with values in \(Q\)) of order \(\leq n\), then \(L_n/L_{n-1}\) invariants of knots of order exactly \(n\).

Let \(L\) - formal linear combinations of knots, then the singular knot is a linear combination of \(2^n\) terms.

Let \(L^n\) - subspace of \(L\), generated by knots with \(n\) double points.

**Fact 1.** \(L^{n+1} \subset L^n\).

**Fact 2.** Spaces \(L_n/L_{n-1}\) and \(L^n/L^{n+1}\) are dual to each other.

Birman and Lin (1993) showed that substituting the power series for \(e^x\) as the variable in the Jones polynomial yields a power series coefficients of which are Vassiliev invariants:

**Theorem** [BL] Let \(K\) be a knot and \(J_t(K)\) be its Jones polynomial. let \(U_k(x)\) be obtained from \(J_t(K)\) by replacing the variable \(t\) with \(e^x\). Express \(U_k(x)\) as power series in \(x\):
\[
U_k(x) = \sum u_i(K)x^i
\]
then \(u_0(K) = 1\) and each \((K) \geq 1\) is a Vassiliev invariant of order \(i\).
Their result implies in particular that the values of the Jones polynomial are not of finite type, but the values of the truncations are.

Note, that there is another normalization of the Jones polynomial, when it is considered not as a polynomial in \((q + q^{-1})\), but in \((q - 1)\). This renormalization implies the Jones polynomial, determined by the skein relation

\[ q^2 J_{L_+} - q^{-2} J_{L_-} = (q - q^{-1})J_{L_0} \]

and normalized by \(J(\bigcirc) = 1\).

The Birman-Lin theorem holds for this renormalized polynomial in \(q\) without substituting \(e^x\).

The Khovanov theory, categorifying this polynomial, is called the reduced Khovanov homology.

Since the Euler characteristics of the cone of the morphism is just the difference of the values of the Jones polynomials of knots from adjacent chambers., our local system is not expected to be of finite type, however, as we will see later, it’s restrictions can be of finite type.

4. The Grothendieck group and the reduced Khovanov homology.

In this paragraph we start to establish the correspondence between Vassiliev-type definition of finiteness given in [S1] and the geometric one of section 2. We introduce the Grothendieck group of Khovanov homology to be able to factorize over subgroups, generated by the Khovanov homology of the collection of circles and their shifts.

To construct the Grothendieck group of a commutative monoid \(M\), one forms the Cartesian product

\[ M \times M \]

The two coordinates represent first and second part:

\[(a, b)\]

which corresponds to

\[ a - b \]

Addition is defined as follows:

\[(a, b) + (c, d) = (a + c, b + d)\]

Next we define an equivalence relation on \(M \times M\): \((a, b)\) is equivalent to \((c, d)\) if, for some element \(k\) of \(M\) if \(a + d + k = b + c + k\). It is easy to check that the addition operation is compatible with the equivalence relation. The identity element is now any element of the form \((a, a)\), and the inverse of \((a, b)\) is \((b, a)\).
The Grothendieck group can also be constructed using generators and relations: denoting by \((\mathbb{Z}(M), +)\) the free abelian group generated by the set \(M\), the Grothendieck group is the quotient of \(\mathbb{Z}(M)\) by the subgroup generated by \(\{a + b - (a + b) \mid a, b \in M\}\).

The Grothendieck group of an abelian category \(\mathcal{M}\) is an abelian group with generators \([M]\) for all objects and relations \([M[1]] = -[M]\) and \([M_2] = [M_1] + [M_3]\) for all exact sequences:

\[
0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0
\]

The Grothendieck group of the category \(\mathcal{K}(\mathcal{M})\) of bounded complexes up to chain homotopies is an abelian group with generators \([M]\) for all objects \(M\) and relations \([M[1]] = -[M]\) and \([M_2] = [M_1] + [M_3]\) for all exact sequences of complexes as above for all components of the complexes.

The Grothendieck group of a triangulated category \(\mathcal{T}\) is an abelian group with generators \([M]\) for all objects \(M\) of \(\mathcal{T}\) and relations \([M[1]] = -[M]\) and \([M_2] = [M_1] + [M_3]\) for all distinguished triangles

\[
\cdots \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_1[1] \rightarrow \cdots
\]

It is easy to see that the Grothendieck group of the bounded derived category \(D^b(\mathcal{M})\) is isomorphic to the Grothendieck groups of \(\mathcal{K}(\mathcal{M})\) and \(\mathcal{M}\).

The Grothendieck group was originally introduced for the study of Euler characteristics.

Now we want to apply the above constructions to the derived category of Khovanov complexes.

As it was shown in the original Khovanov paper, if \(D_1, D_2\) are diagrams of oriented links \(L_1, L_2\) then for a diagram \(D_1 \sqcup D_2\) of the disjoint union \(L_1 \sqcup L_2\), there is an isomorphism of cochain complexes

\[
C(D_1 \sqcup D_2) = C(D_1) \otimes C(D_2)
\]

of free graded abelian groups.

From the Künneth formula Khovanov derives the following formulas for the cohomology of the disjoint union:

**Proposition.** There is a short split exact sequence of cohomology groups

\[
0 \rightarrow \bigoplus_{i,j \in \mathbb{Z}} (H^{i,j}((D_1) \otimes H^{k-i,m-j}(D_2))) \rightarrow H^{k,m}(D_1 \sqcup D_2) \rightarrow \bigoplus_{i,j \in \mathbb{Z}} \text{Tor}_{\mathbb{Z}}^1(H^{i,j}(D_1), H^{k-i+1,m-j}(D_2)) \rightarrow 0
\]

**Corollary.** For each \(k, m \in \mathbb{Z}\) there is an equality of isomorphism classes of abelian groups

\[
H^{k,m}(L_1 \sqcup L_2) = \bigoplus_{i,j \in \mathbb{Z}} (H^{i,j}(L_1) \otimes H^{k-i,m-j}(L_2)) \oplus \bigoplus_{i,j \in \mathbb{Z}} \text{Tor}_{\mathbb{Z}}^1(H^{i,j}(L_1), H^{k-i+1,m-j}(L_2))
\]

Over \(\mathbb{Q}\) these formulas will imply that the derived category of Khovanov complexes form a tensor category which has a monoidal structure.

Given the disconnected sum of two knots \(K_1, K_2\) one gets a tensor product of Khovanov groups:
\[ H^{a,b}(K_1) \otimes H^{c,d}(K_2) \]

We form the Grothendieck group as follows:
Khovanov homology are the bigraded groups, so we will be taking sums over all products with fixed bigradings \( a + c, b + d \):

\[ H^{a,b} \otimes H^{c,d} \rightarrow H^{a+c,b+d} \]

In the original Khovanov's paper [Kh] the homology of a loop is

\[ H(\bigotimes) = \mathbb{Z}\{-1\} \oplus \mathbb{Z}\{1\} \]

which corresponds to the normalization of the Jones polynomial, s.t.

\[ J(\bigotimes) = q + q^{-1} \]

Then the Khovanov homology of the disconnected sum of \( m \) circles is

\[ H(\bigotimes^m) = (\mathbb{Z}\{-1\} \oplus \mathbb{Z}\{1\})^{\otimes m} \]

Thus in the Grothendieck group the identity will be formed by the elements of the form \( H^{0,j} \) and after factorizing by the disconnected sum of circles and their shifts we get the new version of Khovanov homology \( 'Kh \) we will get the identity:

\[ 'H^{i,j}(D) = H^{i+j,j}(D) \]

The version of the Khovanov homology which was introduced in [Kh1] takes care of the above problem. It categorifies the Jones polynomial, determined by the skein relation

\[ q^2 J_{L_+} - q^{-2} J_{L_-} = (q - q^{-1}) J_{L_0} \]

and normalized by \( J(\bigotimes) = 1 \).

It is called the reduced homology and is defined as follows:

Let \( \mathcal{A} = Q[X]/(X^2) \) is the base ring and \( H^{i,j}(D) \) is the complex of finite-dimensional \( Q \)-vector spaces. Khovanov constructs a map of complexes \( \mathcal{A} \otimes C(D) \rightarrow C(D) \) via a geometric construction: choose a segment of the knot diagram \( D \) that doesn’t contain crossing, place an unknotted circle next to it and consider cobordism, which merges circle into \( D \). Ridemeister move, which happens away from this cobordism, induces a chain homotopy equivalence between complexes of \( \mathcal{A} \)-modules. It also establishes a bijection between \((1,1)\)-tangles and oriented links with marked component.

If \( Q = \mathcal{A}/\mathcal{A} \mathcal{A} \) is onedimensional representation of \( \mathcal{A} \), then the reduced complex is defined as

\[ \tilde{C}(D) = C(D) \otimes_{\mathcal{A}} Q \]

and its homology is the reduced homology of \( D \). The analogous construction can be carried out over integers.
In our case we mark any arc of the ”first” knot, which will become a circle after making 1-resolutions of all crossing points. By the cobordism construction all other circles of the resolved link can be merged to this component.

After we made a derived category of Khovanov complexes into a ”group”, the first guess of how to match the geometric and the Vassiliev-type definitions would be: factorize the category by the complexes corresponding to the disconnected sums of circles. However, disconnected sums with unknots will give acyclic complexes and the category will become trivial.

**Note.** Quotients in the DG categories were studied by V. Drinfeld [D]. However, these are not the quotients that we would like to consider, since we don’t want homology of the unknot to be zero.

We will do the factorization in two steps: first we construct a stable category, in which the disconnected sum with a circle is viewed as a suspension, so that the homology of the disconnected sum of a knot and a circle would be quasiisomorphic to the homology of the knot. (In that case we won’t have to remember how many circles our theory decomposed into).

Next we form the filtration, on the factors of which the geometric definition will become equivalent to the Vassiliev’s one.

We will be using the reduced version of Khovanov homology.

5. $S^1$- stable Homotopy. Category of Khovanov Spectra.

We introduce a new derived category $\mathcal{D}$ in which the reduced Khovanov homology of the disconnected sums of circles, discussed in the previous paragraph, is quasiisomorphic to the Khovanov homology of a circle.

This can be understood by considering a linear equivalence relation, generated by an object - unknot $u$:

$$X \sim Y \iff X \otimes u_1 = Y \otimes u_2$$

where $u_1$ and $u_2$ are disconnected sums of unknots.

What we will construct in this section can be viewed as a version of **Spanier-Whitehead category**, objects of which are called **spectra**.

This construction can be carried out in any **symmetric monoidal** category and it was proved that the result of this localization is again a symmetric monoidal category, e.g. [Vo].

Recall the definition of the monoidal category:

**Definition.** A **monoidal category** (or tensor category) is a category $\mathcal{M}$ equipped with

1) a binary functor $\otimes: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ called the tensor product,
2) an object $I$ called the unit object,
3) three natural isomorphisms subject to certain coherence conditions expressing the fact that the tensor operation $\otimes$ is associative, i.e. there is a natural isomorphism, called associativity, with components $\alpha_{A,B,C}: (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $\otimes$ has $I$ as left and right identity: there are two natural isomorphisms, with components $\lambda_A: I \otimes A \to A$ and $\rho_A: A \otimes I \to A.$
The coherence conditions for these natural transformations, given by pentagon diagrams which commute for all objects $A, B, C, D \in \mathcal{M}$ (see Fig. 1 on the next page).

The tensor category, which we are considering is also braided, i.e. it is equipped with the braiding isomorphism

$$\gamma_{A,B} : A \otimes B \rightarrow B \otimes A$$

A symmetric monoidal category is a braided monoidal category whose braiding satisfies

$$\gamma_{B,A} \gamma_{A,B} = 1_{A \otimes B}.$$

It is obvious that the derived category of Khovanov complexes satisfies all these conditions and is a symmetric monoidal category [Kh2].

**Definition.** Suppose $\tilde{D}$ satisfies the Definition 1 for countable coproducts. Let

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \ldots$$

be a sequence of objects and morphisms in $\tilde{D}$. The homotopy colimit of the sequence denoted $\mathcal{Hocolim}_{i \to +\infty}(X_i)$ is by definition given, up to non-canonical isomorphism by the triangle:

$$
\prod X_i \xrightarrow{[1]-shift} \prod X_i \xrightarrow{\mathcal{Hocolim}(X_i)} \Sigma \{ \prod X_i \}
$$

here the shift map $[1] - shift$ is the infinite matrix:

$$
\begin{pmatrix}
1_{X_0} & j_1 & 0 & 1 & \ldots \\
0 & 1_{X_1} & j_2 & 0 & \ldots \\
0 & 0 & 1_{X_2} & j_3 & \ldots \\
0 & 0 & 0 & 1_{X_3} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
$$
The coherence conditions:

\[
\begin{align*}
(A \otimes B) \otimes (C \otimes D) \\
\left[(A \otimes B) \otimes C\right] \otimes D \\
\left[(A \otimes B) \otimes (C \otimes D)\right] \\
A \otimes (B \otimes (C \otimes D)) \\
\end{align*}
\]

One can think of taking a disconnected sum with the circle as a suspension on Khovanov complex. Indeed, connect the circle to the knot by two strands, forming the overcrossing point in projection:

then the resolutions are
The resulting complex will be of length \((n + 1)\) if the original was of length \(n\) and will have shifted grading, like in the case of usual suspensions. Desuspending will correspond to eliminating fake loops as above.

Then the Freudenthal theorem implies that

\[
[X, Y] \to [\Sigma X, \Sigma Y] \to [\Sigma^2 X, \Sigma^2 Y] \to ...
\]

eventually stabilizes. The stable homotopy classes of maps from \(X\) to \(Y\) is above colimit.

The resulting category is the Spanier-Whitehead category, or the category of Khovanov spectra.

From the category viewpoint the above construction implies, that we added another axiom to the the standard axioms of triangulated category:

**Definition.** Let \(\alpha\) be an infinite cardinal. The triangulated category \(\tilde{D}\) is said to satisfy **homotopy colimits** axiom if the following holds:

For any set \(N\) of cardinality less than \(\alpha\) and any collection \(\{X_n, n \in N\}\) of objects, the coproduct \(\coprod X_n\) exists in \(\tilde{D}\).

Now we define the new category \(\tilde{D}\), by modifying morphisms:

**Definition.** The category \(\tilde{D}\) of Khovanov **spectra** is as follows

1) Objects of \(\tilde{D}\) = Objects of \(D\)
2) Morphisms of \(\tilde{D}\):

\[
\tilde{\text{Hom}}(X \to Y) = \text{colim}(\text{Hom}(X \to (Y + n(\text{Unknots}))))
\]

One can define composition of morphisms. If \(\mathcal{H}_n = \text{Hom}(X \to (Y + n(\text{Unknots})))\), then one has to take the limit

\[
\mathcal{H}_1 \to \mathcal{H}_2 \to \ldots \mathcal{H}_i \to ...
\]

where \(i\) denotes the number of unknots.

The fact, that the resulting category is triangulated and has nice properties was proved by several authors [A], [Vo], in algebraic setting for example:

**Theorem [Vo].** Let \((D, \otimes, 1)\) be a symmetric monoidal category and \(U\) be an object, s.t. the cyclic permutation on \(U \otimes U \otimes U\) equals identity in \(D[U^{-1}]\). Then there exists a symmetric monoidal structure \(\otimes\) on \(D[U^{-1}]\).

The above theorem implies that in the category \(\tilde{D} = D[U^{-1}]\) the disconnected sums of unknots are considered an identity and the resulting category is again symmetric monoidal.
6. The Poincare Polynomial.

As we observed in paragraph 2, Proposition 4, the nth cone of the Khovanov local system, restricted to the subcategory of knots of crossing number n is a complex of length $2n + 1$, with generators (unknots) in the odd homological dimensions.

**Definition.** The **Poincare polynomial** of the complex $X^\bullet = (X^0, X^1, ....X^n)$ is the polynomial $P(X, t) = \Sigma \beta_k t^k$, where $\beta_k = dim(X^k)$ is the kth Betti number.

We consider the ideals $I_n$ generated by the complexes of the form:

$$C^\bullet(n) = \begin{pmatrix} C^0_n Z \\ 0 \\ C^1_n Z \\ 0 \\ \vdots \\ 0 \\ C^n_n Z \end{pmatrix}$$

In the next paragraph we will factorize the category of complexes by these complexes. Notice that $C^\bullet(n) = (C^\bullet(1))^\otimes n$.

**Lemma.** The Poincare polynomial of the complex $C^\bullet(n)$ equals $(1 + t^2)^n$ and its Euler characteristics is $2^n$.

**Proof.** Obvious:

$$(1 + t^2)^n = \Sigma C^i_n t^{2i}$$

$$\Sigma C^i_n = 2^n.$$ 

**Note.** After grading change, we can assume that the Poincare polynomial is equal to $(t^{-1} + t)^n$.

For each n by the polynomial $(t + t^{-1})^n$ we generate an ideal $I_n$ in the ring of Laurent polynomials $Z[t, t^{-1}]$:

$$I_n = Z[t, t^{-1}]/(t^{-1} + t)^n$$

In the next chapter we are taking this factorization to the level of the derived category.
7. On the Algebraic Definition of Finiteness.

The geometric definition of finiteness, which we gave in section 2 is sufficient to prove the main result of this paper, however, we would like to give an algebraic definition, similar to the original one of Vassiliev (that complexes become zero or acyclic after the extension to the codimension \( n \) of the discriminant).

However, we would like to understand if there are computational ways to determine that the local system decomposed into a collection of circles on the strata of the discriminant, i.e. how the TQFT "knows" about it?

The idea is to factor out the free part of the homology.

Recall the Theorem of A.Shumakovich [Su] and Asaeda-Przytycki [AP] regarding the torsion in Khovanov homology:

**Theorem [Su]** The only alternating links that do not have torsion are the trivial knot, the Hopf link, their connected sums and disjoint unions. The nontrivial torsion always contains the \( Z_2 \) subgroup.

They further conjecture (and prove in many cases) that the above theorem will also hold for any, not necessarily alternating links.

The above statements allow us to reformulate the geometric definition in terms of torsion. But first we have to eliminate the case of the Hopf link.

**Calculation.** The Khovanov homology of the Hopf link is quasiisomorphic to \((Z_2,0,Z_2)\).

Assigning 0 and 1-resolutions to 2 crossing points of the Hopf link one gets 4 complete resolutions and the Khovanov complex becomes:

\[
0 \to V \otimes V \to V \oplus V \to V \otimes V \to 0
\]

recall that that the space \( V \) is generated by \( v_+ \) and \( v_- \). One can see that the 1-cycles are \((v_- \otimes v_-), (v_+ \otimes v_- - v_- \otimes v_+)\) and 1-boundaries are 0. The 2-cycles are \((v^1_+ \oplus v^2_+), (v^1_- \oplus v^2_-)\), 2-boundaries are \((v^1_+ \oplus v^2_+)\) and \((v^1_- \oplus v^2_-)\), 3-cycles are \((v_+ \otimes v_+), (v_+ \otimes v_-), (v_- \otimes v_+), (v_- \otimes v_-)\), 3-boundaries are \((v_- \otimes v_-), (v_+ \otimes v_- + v_- \otimes v_+)\). Thus the Khovanov homology of the Hopf link is \((Z_2,0,Z_2)\).

Consider a new invariant, an additive functor:

\[
\mathcal{T}_2 : C \in \text{Ob}(\mathcal{D}) \to \text{Tor}_2(H^*(C))
\]

**Definition (T).** The local system is of finite type \( n \) if for any codimension \( n \) selfintersection of the discriminant the corresponding nth cone is not quasiisomorphic to \((Z_2,0,Z_2)\) and has torsion-free homology, i.e. the image of \( \mathcal{T}_2 \) is zero.

The disadvantage of this definition is that after taking torsion the theory becomes trivial on the level of the Euler characteristics. To fix this problem, we consider the ideals \( I_n \) generated by the complexes \( C(n)^* \) introduced in section 6.


8. The Filtration.

In this paragraph we take the factorization, which we discussed in section 5, to the level of the derived category.

According to [D], it is possible to divide in the tensor monoidal categories if the tensor product preserves quasiisomorphisms, i.e. an exact functor. If we consider the class of flat objects (those, on which the tensor product is exact), then one can divide by a subcategory.

Recall our geometric definition of finiteness:

**Definition (G).** The local system is of finite type n, if there exists such minimal n, s.t. for any selfintersection of the discriminant of codimension n the corresponding complex is quasiisomorphic to $C^\bullet(n)(U)$, where $U$ is the Khovanov complex of the disjoint union of unknots.

We will follow the Verdier approach, who constructed the quotient $T/S$, where $S$ is a subcategory of $T$. He showed that the factorization is well-defined if $S$ is thick. Recall the definition of a **thick** subcategory:

**Definition.** The subcategory of a triangulated category is called **thick** if it is triangulated and contains all direct summands of it's objects.

The subcategories, over which we will be factorizing are generated by $I_n$ and supported on $D_n$ - the union of strata of codimension n of the discriminant:

First we define the notion of generated subcategory:

**Definition.** Let $T$ be a triangulated category satisfying the homotopy colimits axiom. Let $\alpha$ be an infinite cardinal. Let $S$ be a class of objects of $T$. Then $< S >^\alpha$ will denote the smallest $S$, $S$ a triangulated subcategory of $T$, the **generated subcategory** satisfying:

1). The objects of $S$ lie in $S$.
2). Any coproduct of fewer that $\alpha$ objects of $S$ lies in $S$.
3). The subcategory $S \subset T$ is thick.

We refer to [N] for the proofs that $S$ is well-defined and that it is localizing [Bo]. By Verdier theorem [V] one can factorize by such subcategories.

**Proposition 5.** Let $I_n$ be the triangulated category generated by $C^\bullet(n)$. Then for any complex $X^\bullet \in I_n$ we have $\chi(X^\bullet) = 2^n \cdot k$.

**Proof.** This is a check for all operations in the triangulated category:

1) quasiisomorphic complexes have the same Euler characteristics
2) By taking the direct sum of i copies of $C^\bullet(n)$ we get a complex with Euler characteristics $2^n \cdot i$:

$$\chi(X^\bullet \oplus X^\bullet \oplus ...X^\bullet) = i\chi(X^\bullet)$$

3) By taking the cone of the map between complexes $C^\bullet(n)$ we can arrange any combination of morphisms between the components, but it is an easy check that what we get will have a trivial Euler characteristics.
In all cases the Euler characteristics of the resulting complex will be a multiple of $2^n$. Thus the divisability of the Euler characteristics by $2^n$ becomes an invariant of $I_n$.

**Proposition 6.** One gets a sequence of categories:

$$
\ldots I_n \subset I_{n-1} \subset \ldots \subset I_1
$$

**Proof.** This statement follows from the previous one: if one takes $k$ equal 2, and recall that $I_n$ is supported on $D_n$, then

$$
I_n \subset I_{n-1}
$$

**Definition.** We define the derived category $\tilde{D}_n$ as a factor category (in a sense of Verdier [V], [D]):

$$
\tilde{D}_n = \tilde{D}/I_n
$$

Where $I_n$ is a thick subcategory.

It follows from the above definition that the subcategory is thick when it is closed under cofibrations, retractions, direct sums and suspensions. When we form a thick subcategory, generated by $I_n$, we may loose the property of the Euler characteristics being divisible by $2^n$, but since $I_n$ is supported on $D_n$, this is the thick subcategory of $\tilde{D}$. The resulting category $\tilde{D}_n$ will lack the exactness property only on codimension $n$ strata.

**Example.** In the triangulated category of complexes over $\mathbb{Z}$ consider those, which have $k$-torsion in the homology. This subcategory is thick.

**Proposition 7.** The Verdier quotients of the category $\tilde{D}$ over thick subcategories $\tilde{I}_n$ defined above, form an increasing filtration of the category $\tilde{D}$:

$$
\tilde{D} = \tilde{D}_\infty \supset \ldots \tilde{D}_n \supset \tilde{D}_n^{-1} \supset \ldots \supset \tilde{D}_1
$$

**Proof.** The statement follows from the previous propositions.

**Remark 1.** Notice that in $\tilde{D}$ the Euler characteristics of complexes is well-defined modulo $2^n$.

**Remark 2.** Notice that this final definition is very similar to the one of invariants of finite type given in paragraph 3.

**Remark 3.** Yet another approach... Complexes with torsion-free homology don’t form a thick subcategory, but they are factor- functors of Tor. We show that one still can form a factor-category $\tilde{D}_{tor}$ with the same objects, morphisms (arrows) of which can factor through complexes, homology of which don’t have torsion [Ho].

**Theorem.** In the factor-category $\tilde{D}_{tor}$ objects, corresponding to complexes, homology of which have no torsion are isomorphic to zero, while complexes, homology of which have torsion are never isomorphic to zero in $\tilde{D}_{tor}$.

**Definition (V').** The local system of Khovanov complexes is of finite type $n$ if for any codimension $n$ selfintersection of the discriminant the nth cone is zero in $\tilde{D}_{tor}$.
9. The Finiteness result.

In this paragraph we prove the first simple finiteness property of the Khovanov local system.

First let’s define our categories.

The category of knots $\mathcal{K}$ is the topological category, objects of which are knots and morphism are knot cobordisms. (If knots $K_1$ and $K_2$ have the same isotopy type, i.e. lie within the same chamber of the Vassiliev space, then morphisms are just the product cobordisms, if we pass between adjacent chambers, changing one crossing, then these are genus one cobordisms).

The construction of the local system of Khovanov complexes on the Vassiliev space of knots [S1] provided us with a functor from the category of knots into the derived category of complexes, denote it by $\mathcal{K}h$.

Recall that the crossing number of the knot is the minimum of the crossing numbers over all its projections.

In the category of knots $\mathcal{K}$ we define a sequence of subcategories $\mathcal{K}_n$. Objects of $\mathcal{K}_n$ are knots with at most $n$ crossings, morphisms in $\mathcal{K}_n$ are cobordisms between knots with the crossing number at most $n$, etc. The corresponding derived category is denoted $\mathcal{K}h_n$.

**Theorem 1.** Restricted to the subcategory of knots with at most $n$ crossings, $n \geq 3$, Khovanov local system is of finite type $\leq n$.

**Proof.** We give the proof according to the geometric definitions.

If we restrict Khovanov theory to the subcategory $\mathcal{K}_n$ of knots with at most $n$ crossing, the complexes we get from knot projections are all quasiisomorphic to the ones of length at most $(n + 1)$.

Consider an $n$-dimensional commutative hypercube, corresponding to the selfintersection of the discriminant of codimension $n$.

As we have seen in the previous paragraph, complexes, corresponding to the codimension one walls of the discriminant are

$$X^\bullet \oplus X^\bullet[2] \text{ with the matrix differential } d_{C_\omega} = \begin{pmatrix} dX & 0 \\ 0 & dX[2] \end{pmatrix},$$

where $X^\bullet$ is the Khovanov complex of the knot of $(n-1)$ crossings where kth double point of $K$ is given 1-resolution.

When we pass to codimension 2 selfintersections of the discriminant, we get 4-graded complexes associated to it:

$$Y^\bullet \oplus Y^\bullet[2] \oplus Y^\bullet[2] \oplus Y^\bullet[4]$$

where $Y^\bullet$ is the Khovanov complex of the knot of $(n-2)$ crossings where kth and lth double points of $K$ are given 1-resolution.

After establishing these identities for all indices up to $n$, it is easy to see from our geometrical interpretation of the finite type condition what is the nth generalized cone of the restricted Khovanov local system.
The geometric definition implies that we calculate the nth cone by taking the "first" knot in the hypercube (via the coorientation) and giving all the crossings of its projection 1-resolutions. We end up with a collection of circles in $R^2$, corresponding to the last component of the complex, associated with the projection of the knot.

The nth cone, assigned to the selfintersection of the discriminant of codimension $n$ will be quasiisomorphic to the complex of the disconnected sum of $2^n$ copies of the collection of circles, described above, shifted in homological grading according to the coorientation. In the stable category this complex is isomorphic to the one of the circle.

Passing to the stable category means the eliminating of the "inessential" crossings of the knot projection. The discriminant of the space of knots, consists to singular knots with "essential" crossings (i.e. this singular knot can be realized without extra under/over crossings). Let $K_n$ be the subcategory of knots with the crossings number (minimum over all projections) at most $n$, then we can reformulate the main theorem as follows:

**Theorem 1'.** If the singular knot $K$ with $n$ double points can be realized without extra under/over crossings, then the extension of the Khovanov local system to $K$ is zero in $\tilde{D}_n$ (even if in the diagram of $K$ there are other under/over crossings).

Next we consider the localized local system: since the nth cone is given by the formula $C^\bullet(n) \otimes K_{1,1,\ldots,1}$, where $K_{1,1,\ldots,1}$ Khovanov complex of the knot projection, where $n$ crossing points are given 1-resolutions and $C^\bullet(n) = \begin{pmatrix}
C^n_0 Z & \\
0 & \\
\vdots & \ddots \\
0 & \\
C^n_0 Z & \\
0 & \\
\vdots & \\
0 & \\
C^n_0 Z
\end{pmatrix}$

in $\tilde{D}_n$ it will satisfy the finiteness condition, since all crossing points will be given 1-resolution and this is a local system of finite type $n$. And since we are not considering walls with knots of crossing number $n+1$, this is the only cone we can form.

**Remark.** One may not need to take the nth cone to get the finiteness condition, i.e. the knot projection may decompose into a disconnected sum of unknots earlier. (It would be interesting to get an estimate). But since $\tilde{D}_n \supset \tilde{D}_k$ for $k \leq n$, we get that the local system will be of type at most $n$.

**Example.** Consider the right-handed trefoil with three double points:

There are 8 resolutions of this singular knot, corresponding to 8 chambers (say, complexes $X,Y,A,B,C,D,W,Z$), adjacent to the selfintersection of the discriminant of codimension 3. Two of them correspond to the righthanded trefoil and its mirror image and six - to the twisted
unknots, which are obtained after changing any overcrossing in the projection of the trefiol to the undercrossing.

The third cone $C^\bullet(3)$ will be of the form:

$$C^\bullet(3) \otimes K_{1,1,1} = \begin{pmatrix} Z \\ 0 \\ 3Z \\ 0 \\ 3Z \\ 0 \\ Z \end{pmatrix} \otimes K_{1,1,1}$$

Thus the third cone is acyclic in $\tilde{D}_3$ and this is a condition for the local system to be of type 3.

In the upcoming paper [S2] we prove the generalization of Theorem 1, or the categorification of Birman-Lin theorem.
10. Further directions.

1. We would like to generalize the result of this paper to the Khovanov-Rozhansky homology [KR], for which we defined the wall-crossing morphisms [SW].

2. It would be very interesting to see, if the recent extensions to the singular locus of Ozsvath-Szabo knot invariants, done by Benjamin Audoux [A], cf. [OSS], satisfy finiteness conditions.

3. The geometric definition implies that the knot homology theory is of finite type, if after taking sufficiently high cones all objects decompose into a collection of disconnected circles. It would be very interesting to understand what are the ”building blocks” for the homology theories of higher dimensions.

4. Our theorem can be proved using the properties of the homological width of the knot. Such proof could provide another point of view on finiteness result (via the ranks of homologies).

5. We believe that our constructions will provide a better estimates on the crossing number of the knot.

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