A FUNCTIONAL-ANALYTIC TECHNIQUE FOR THE STUDY OF ANALYTIC SOLUTIONS OF PDES

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Abstract. A functional-analytic method is used to study the existence and the uniqueness of bounded, analytic and entire complex solutions of partial differential equations. As a benchmark problem, this method is applied to the nonlinear Benjamin–Bona–Mahony equation and the associated to this, linear equation. The predicted solutions are in power series form and two concrete examples are given for specific initial conditions.

1. Introduction. In this paper, a functional-analytic technique will be employed for the study of bounded, analytic and entire complex solutions of partial differential equations (PDEs). This technique was introduced in [14], where it was used for establishing a necessary and sufficient condition for the existence of polynomial solutions of a class of linear PDEs. Here, the technique will be extended so as to cover other kinds of linear PDEs as well as nonlinear PDEs.

Actually, this technique is an extension of an old functional-analytic technique for the study of analytic solutions of initial value problems of ordinary differential equations (ODEs) introduced in [8] by Ifantis and systemized in [9], [10]. Moreover, the authors of the present paper have applied this technique to various ODEs in several papers, including ODEs other than those treated in [9], [10], see e.g. [13]. Recently, this technique was utilized in the framework of finding discrete equivalents of ODEs and combined with a similar technique for ordinary difference equations, in order to be proposed as a “discretization” technique for the solution of ODEs, see [15], [17]. Moreover, although this “discretization” technique is intended for the study of initial value problems of ODEs, it can be easily combined with a standard “shooting” method and give more than satisfactory results, in the numerical approximation of solutions of boundary value problems of ODEs as well, see [16]. It would be interesting to apply such a “discretization” technique to PDEs and the present paper constitutes also a step to this direction.

The study of analytic solutions of PDEs is an old and important on its own problem. Maybe the most well-known result in this field is the Cauchy–Kowalewski theorem (see e.g. [5, p. 239–244] or [18, p. 15–16]) and the results of the present paper could be considered as of Cauchy–Kowalewski type. From the huge number of papers regarding various results

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on analytic solutions of PDEs, [6], [11], [12], [20] and [21] are indicatively mentioned, as well as the very recent [2], [3], [7].

The spaces usually involved in such kind of problems are the Sobolev spaces or the $L_2(\Delta^2)$ space of square integrable functions or subspaces of them. In the present study, the spaces involved are the Hilbert space:

$$H_2(\Delta^2) = \{ f : \Delta^2 \to \mathbb{C}, \text{ where } f(x, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{ij} x^{i-1} t^{j-1}, \text{ analytic in } \Delta^2 \}$$

with

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |f_{ij}|^2 < +\infty,$$

where $\Delta^2 = \Delta \times \Delta$, $\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$, with inner product defined by

$$(f_1(x, t), f_2(x, t))_{H_2(\Delta^2)} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \overline{f}_{ij} b_{ij},$$

where $f_1(x, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{ij} x^{i-1} t^{j-1}$ and $f_2(x, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_{ij} x^{i-1} t^{j-1}$ are elements of $H_2(\Delta^2)$, as well as the Banach space

$$H_1(\Delta^2) = \{ f : \Delta^2 \to \mathbb{C}, \text{ where } f(x, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{ij} x^{i-1} t^{j-1} \in H_2(\Delta^2),$$

for which

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |f_{ij}| < +\infty,$$

with norm $\| f(x, t) \|_{H_1(\Delta^2)} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |f_{ij}|$. (The one dimensional spaces $H_2(\Delta)$, $H_1(\Delta)$ are analogously defined with only one series involved in their definitions.)

The PDEs under consideration involve the unknown function $\hat{u}(\hat{x}, \hat{t})$, with $|\hat{x}| < X$, $|\hat{t}| < T$, $X$, $T > 0$ finite numbers. These PDEs are transformed through the simple transformations $\hat{x} = x \cdot X$ and $\hat{t} = t \cdot T$, to PDEs involving the new unknown function $u(x, t)$, for which conditions are given so as $u(x, t) \in H_2(\Delta^2)$ or $H_1(\Delta^2)$.

The reasons for studying PDEs in $H_2(\Delta^2)$ or $H_1(\Delta^2)$ come from various problems. One advantage of these spaces is that each one of their elements is represented by exactly one function and not by a class of equivalent functions, as in the case of $L_2(\Delta^2)$. Moreover, these spaces contain polynomial functions and therefore are suitable for problems involving polynomial solutions of PDEs. Also, when establishing solutions of PDEs in $H_2(\Delta^2)$ or $H_1(\Delta^2)$, they are in power series form which by definition are convergent. Even more, if the coefficients of the power series are uniquely determined, this power series solution can be considered as an “exact” solution of the PDE under consideration. Finally, in problems of quantum mechanics an arbitrary one-mode state can be represented as $f(z) = \sum_{N=1}^{\infty} f_N z^{N-1}$, where $|f| > = \sum_{N=1}^{\infty} f_N |N| >$, $\sum_{N=1}^{\infty} |f_N|^2 = 1$, and $|N|$ are the number of eigenstates. Clearly $f(z) \in H_2(\Delta)$. (For more details see e.g. [19].) This can be extended to two-mode systems considering the $|N, M >$ two-mode eigenstates $|g > = \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} g_{NM} |N, M >$, $\sum_{N=1}^{\infty} \sum_{M=1}^{\infty} |g_{NM}|^2 = 1$. Now $g(z_1, z_2) = \sum_{N=1}^{\infty} \sum_{M=1}^{\infty} g_{NM} z_1^{N-1} z_2^{M-1} \in H_2(\Delta^2)$.

The method that will be presented in this paper is described in §3. According to this method, the PDE under consideration is equivalently reduced to an operator equation in an abstract separable Hilbert space $H$, which is called the abstract form of the corresponding PDE. Then, techniques and theorems of functional analysis, such as theorems for the inversion of operators or fixed point theorems are utilized, in order to guarantee the existence and uniqueness of the solution of this abstract equation in $H$. These results are then “translated” in terms of the PDE under consideration. One important advantage of this method, is that the conditions accompanying the PDE are incorporated in the equivalent operator equation.
In order to demonstrate this technique, the Benjamin–Bona–Mahony equation
\[ \ddot{u} + \dddot{u} + u \dddot{u} - \dddot{u} = 0 \]  
(1)
as well as the associated to this, linear equation
\[ \ddot{u} + \dddot{u} - \dddot{u} = 0 \]  
(2)
are considered, where \( \ddot{u} = \ddot{u}(\dddot{x}, \dddot{t}), |\dddot{x}| < X, |\dddot{t}| < T \) with \( X, T > 0 \) finite numbers. Equation (1) models long water waves in nonlinear dispersive systems and is considered as an alternative to the KdV equation [1]. It has been extensively studied both analytically and numerically for various kinds of solutions and initial data.

The main results of the present paper regarding equations (1) and (2) are presented in §2, whereas their proofs are given in §4. These results concern the existence of unique entire analytic solutions of equations (1) and (2), respectively. Although the equations considered are homogeneous with constant coefficients, the method can be successfully applied to nonhomogeneous equations. See Remarks 2 and 5. Moreover, due to the constructive character of their proofs, a bound of these solutions is given and the explicit predicted power series solution can be obtained. In order to demonstrate this, two concrete examples are given in §5.

2. Main results. In order to keep the statements of the main results simple, equations (1) and (2) will be transformed using the transformations \( \tilde{x} = x \cdot X, \tilde{t} = t \cdot T, \ddot{u}(\dddot{x}, \dddot{t}) = u(x, t) \) already mentioned in §1. Indeed, equations (2) and (1) become
\[ X^2 u_t + TXu_x - u_{xxt} = 0, \]  
(3)
and
\[ X^2 u_t + TXu_x + XTuu_x - u_{xxt} = 0, \]  
(4)
respectively, where \( x, t \in \Delta \). Now the main results of the present paper are:

Theorem 2.1. Consider the problem consisting of equation (3) and the conditions:
\[ u(x, 0) = \phi_1(x), \ u_t(0, t) = \phi_2(t), \ u_{xxt}(0, t) = \phi_3(t). \]  
(5)
If \( \phi_1(x), \phi_2(t), \phi_3(t) \in H_2(\Delta) \), then (3), (5) has a unique solution in \( H_2(\Delta^2) \).

Remark 1. Equivalently, when returning to the original variables \( \tilde{x}, \tilde{t} \), the corresponding problem for equation (2) has a unique entire solution \( \ddot{u}(\dddot{x}, \dddot{t}) \) under the assumptions of Theorem 2.1.

Remark 2. Theorem 2.1 remains true for the nonhomogeneous equation
\[ X^2 u_t + TXu_x - u_{xxt} = X^2 T g(x, t), \]
under the additional assumption that \( g(x, t) \in H_2(\Delta^2) \).

Theorem 2.2. Consider the problem consisting of equation (4) and the conditions:
\[ u(x, 0) = \phi_1(x), \ u_t(0, t) = \phi_2(t), \ u_{xxt}(0, t) = \phi_3(t). \]  
(6)
If \( \phi_1(x), \phi_2(t), \phi_3(t) \in H_1(\Delta) \), then there exists an \( L > 0 \) such that if:
\[ ||\phi_1(x)||_{H_1(\Delta)} + ||\phi_2(t)||_{H_1(\Delta)} + ||\phi_3(t)||_{H_1(\Delta)} < \frac{1}{L^2 XT} \]  
(7)
then, (4), (6) has a unique solution in \( H_1(\Delta^2) \) bounded by \( \frac{1}{2XT} \).

Remark 3. The previous theorem establishes a purely local result, in the sense that \( L \) is not explicitly determined and thus this result is more useful for theoretical considerations and not for more practical applications. However, by adding one more condition, Theorem 2.2 becomes more concrete and takes the next form.
Theorem 2.3. Consider the problem consisting of equation (4) and the conditions (6). If
\[ \phi_1(x), \phi_2(t), \phi_3(t) \in H_1(\Delta), \]
\[ X(X + T) < 2 \quad (8) \]
\[ \|\phi_1(x)\|_{H_1(\Delta)} + \|\phi_2(t)\|_{H_1(\Delta)} + \|\phi_3(t)\|_{H_1(\Delta)} < \frac{(2 - X^2 - XT)^2}{4XT} \quad (9) \]
then, (4), (6) has a unique solution in \( H_1(\Delta^2) \) bounded by \( \frac{2 - X(X + T)}{XT} \).

Remark 4. Equivalently, when returning to the original variables \( \tilde{x}, \tilde{t} \), the corresponding problem for equation (1) has a unique solution \( \tilde{u}(\tilde{x}, \tilde{t}) \) in the form of a convergent double power series in \( \tilde{x}, \tilde{t} \) under the assumptions of Theorem 2.2 or 2.3.

Remark 5. Theorems 2.2 and 2.3 remain true for the nonhomogeneous equation
\[ X^2u_t + TXu_x + XTuu_x - u_{xxt} = X^2Tg(x,t), \]
under the additional assumption that \( g(x,t) \in H_1(\Delta^2) \) and after substituting conditions (7) and (9) by conditions:
\[ \frac{X^2T}{2} \|g(x,t)\|_{H_1(\Delta^2)} + \|\phi_1(x)\|_{H_1(\Delta)} + \|\phi_2(t)\|_{H_1(\Delta)} + \|\phi_3(t)\|_{H_1(\Delta)} < 1/(L^2XT) \]
\[ \frac{X^2T}{2} \|g(x,t)\|_{H_1(\Delta^2)} + \|\phi_1(x)\|_{H_1(\Delta)} + \|\phi_2(t)\|_{H_1(\Delta)} + \|\phi_3(t)\|_{H_1(\Delta)} < \frac{(2 - X^2 - XT)^2}{4XT^2} \]
respectively.

3. The functional-analytic method. The method employed in the present paper is an extension of the method introduced in [14] and various results will be given without proofs. Denote by \( H \) an abstract separable Hilbert space over the complex field \( \mathbb{C} \) with orthonormal basis \( \{e_{i,j}\}_{i,j=1}^\infty \). The inner product and the induced norm will be denoted by \( \langle \cdot, \cdot \rangle, \| \cdot \| \). Define also the shift operators \( V_1, V_2 \) on \( H \) as follows:
\[ V_1e_{i,j} = e_{i+1,j}, \; i, j = 1, 2, ..., \quad V_2e_{i,j} = e_{i,j+1}, \; i, j = 1, 2, ... \]
and their adjoint operators \( V_1^*, V_2^* \) as:
\[ V_1^*e_{i,j} = e_{i-1,j}, \; i = 2, 3, ..., j = 1, 2, ... \quad V_1^*e_{1,j} = 0, \; j = 1, 2, ... \]
\[ V_2^*e_{i,j} = e_{i,j-1}, \; i = 1, 2, ..., j = 2, 3, ... \quad V_2^*e_{i,1} = 0, \; i = 1, 2, ... \]
The operators \( V_i, V_i^* \), \( i, j = 1, 2 \) commute as long as the indices are different. For example, it is true that \( V_1V_2 = V_2V_1 \) or \( V_1V_2^* = V_2^*V_1 \). Moreover:
\[ V_1^*V_1 = I, \quad V_2^*V_2 = I, \quad \|V_1\| = \|V_2\| = \|V_1^*\| = \|V_2^*\| = 1 \quad (10) \]
where \( I \) is the identity operator. Then the following are true:

Proposition 1 (See [14, Proposition 1]). Every point \( xt \), with \( x, t \in \Delta = \{x \in \mathbb{C} : |x| < 1\} \), belongs to the point spectrum of \( V_1^*V_2^* \) and the set of the eigenvalues:
\[ f_{x,t} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x^{i-1}t^{j-1}e_{i,j}, \quad f_{0,t} = \sum_{j=1}^{\infty} t^{j-1}e_{1,j}, \quad f_{x,0} = \sum_{i=1}^{\infty} x^{i-1}e_{i,1}, \quad f_{0,0} = e_{1,1} \quad (11) \]
forms a complete system in \( H \) i.e., if \( f \) is orthogonal to \( f_{x,t} \) \( \forall x, t \in \Delta \), then \( f = 0 \).

Proposition 2 (See [14, §3.2]). The mapping \( \phi : H \to H_2(\Delta^2) \) with
\[ \phi(f) = (f_{x,t}, f) = f(x,t), \]
is a one-to-one mapping from \( H \) onto \( H_2(\Delta^2) \), which preserves the norm.
Actually, for every \( f(x,t) = \sum_{i=1}^{\infty} \frac{x^i}{i!} t^{i-1} f_i \in H_2(\Delta^2) \), there exists the element \( f = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{ij} e_{i,j} \in H \) such that \( \phi(f) = f(x,t) \), which will be called the abstract form of \( f(x,t) \). Conversely, if \( f = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (f, e_{i,j}) e_{i,j} \), then due to (12), \( f(x,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (f, e_{i,j}) x^{i-1} t^{j-1} \).

For the implementation of the method, the abstract forms of all the appearing terms in the PDEs, are needed. For all the linear terms, the corresponding abstract forms have been found in [14, Proposition 2]. More precisely,

\[
\frac{\partial^n f(x,t)}{\partial x^n} = (f_{xt}, (C_1^{(0)}) \hat{V}_1^n f), \quad n = 1, 2, \ldots
\]

\[
\frac{\partial^n f(x,t)}{\partial t^n} = (f_{xt}, (C_2^{(0)}) \hat{V}_2^n f), \quad n = 1, 2, \ldots
\]

\[
\frac{\partial^n f(x,t)}{\partial x^n \partial t^l} = (f_{xt}, (C_1^{(0)}) \hat{V}_1^n (C_2^{(0)}) \hat{V}_2^l f), \quad n, m, l = 1, 2, \ldots, \quad m + l = n,
\]

where \( C_1^{(0)}, C_2^{(0)} \) are the diagonal operators defined on \( H \) as follows:

\[
C_1^{(0)} e_{i,j} = i e_{i,j}, \quad C_2^{(0)} e_{i,j} = j e_{i,j}, \quad i, j = 1, 2, \ldots.
\]

These operators have the following properties [14, Remark 3]:

(i) They have a self-adjoint extension with discrete spectrum, i.e., the definition domain of \( C_1^{(0)}, C_2^{(0)} \) can be extended to the range of the bounded operators \( B_1^{(0)}, B_2^{(0)} \), respectively, defined by: \( B_1^{(0)} e_{i,j} = \frac{1}{i} e_{i,j}, B_2^{(0)} e_{i,j} = \frac{1}{j} e_{i,j}, \quad i, j = 1, 2, \ldots \).

(ii) The definition domains of the operators \( (C_1^{(0)})^p, (C_2^{(0)})^p \) are extended to the range of the bounded operators \( (B_1^{(0)})^p, (B_2^{(0)})^p, p = 2, 3, \ldots, k \), respectively.

(iii) The range of \( (B_1^{(0)})^p (B_2^{(0)})^p \) in \( H , \quad p = 1, 2, \ldots, k \), i.e., the definition domain of \( (C_1^{(0)})^p (C_2^{(0)})^p \) is isomorphic to the linear manifold in \( H_2(\Delta^2) \) which consists of functions with derivatives with respect to \( x (t) \) up to order \( p \) in \( H_2(\Delta^2) \).

Also, for the proofs of the main results, the following relation will be needed (see, [14, Proposition 3]):

\[
(C_1^{(0)} \hat{V}_1^k) (C_1^{(0)} + I) \cdots (C_1^{(0)} + (k - 1) I) (\hat{V}_1^k) k, \quad k = 2, 3, \ldots
\]

In order to extend the method of [14] to nonlinear equations, the approach of [9], [10] for ODEs will be followed, but adequately adapted for PDEs. First of all it should be mentioned, that various nonlinear operators are not defined on all elements of \( H_2(\Delta^2) \). For example, even the simple operator \( f(x,t) \rightarrow [f(x,t)]^2 \) does not belong to \( H_2(\Delta^2) \) for all elements of \( H_2(\Delta^2) \). One way to surpass this difficulty is to consider the linear manifold of all elements \( f(x,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{ij} x^{i-1} t^{j-1} \) of \( H_2(\Delta^2) \) which satisfy the condition \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |f_{ij}| < +\infty \). This linear manifold equipped with the norm \( \|f_1(x,t)\|_{H_1(\Delta^2)} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |f_{ij}| \) becomes the well known Banach space \( H_1(\Delta^2) \) and it “carries” the inner product of \( H_2(\Delta^2) \). The corresponding to \( H_1(\Delta^2) \) by the mapping (12), Banach abstract space will be denoted by \( H_1 \) and its norm by \( \| \cdot \|_1 \). As in [9], the following statements are true:

- \( H_1 \) is invariant under the shift operators \( V_i, V_i^* \), \( i = 1, 2 \) and their powers. Moreover, \( \|V_1\| = \|V_2\| = \|V_1^*\| = \|V_2^*\| = 1 \).
- \( H_1 \) is invariant under every bounded diagonal operator \( D e_{i,j} = d(i, j) e_{i,j} \), \( i, j = 1, 2, \ldots \) on \( H \). Moreover, \( \|D\| = \|D\| = \sup |d(i, j)| \).
- The null spaces of \( (V_1^*)^k \) and \( (V_2^*)^k \) in \( H \) belong to \( H_1 \).

In order to determine the abstract form of the nonlinear term \( uu \) appearing in the PDEs under consideration the following will be needed:
Proposition 3. The abstract form of the term \([f(x,t)]^2\) is the nonlinear operator
\[
f(V_1, V_2) f = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (f, e_{i,j}) V_1^{i-1} V_2^{j-1} f
\] (15)
and is defined on all \(H_1\) for \(f \in H_1\).

Proof. First of all, \(f(V_1, V_2) f\) is defined on all \(H_1\) if \(f \in H_1\), since
\[
\|f(V_1, V_2) f\|_1 = \left\| \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (f, e_{i,j}) V_1^{i-1} V_2^{j-1} f \right\|_1 \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |(f, e_{i,j})| \cdot \| V_1^{i-1} V_2^{j-1} f \|_1 \leq \|f\|_1 \cdot \| V_1 \|^{i-1} \cdot \| V_2 \|^{j-1} \cdot \|f\|_1 \leq \|f\|_1^2.
\]

Moreover, \((f_{xt}, f(V_1, V_2) f) = \lim_{I \to \infty} \lim_{J \to \infty} \left( f_{xt}, \sum_{i=1}^{J} \sum_{j=1}^{J} (f, e_{i,j}) V_1^{i-1} V_2^{j-1} f \right) =
\]
\[
= \lim_{I \to \infty} \lim_{J \to \infty} \sum_{i=1}^{J} \sum_{j=1}^{J} (f, e_{i,j}) f_{xt} (V_1^{i-1} V_2^{j-1} f) =
\]
\[
= \lim_{I \to \infty} \lim_{J \to \infty} \sum_{i=1}^{J} \sum_{j=1}^{J} (f, e_{i,j}) ((V_*^{i-1}) V_*^{j-1} f_{xt}, f)
\]
or due to the definition (11) of \(f_{xt}\) and the action of the shift operators
\[
(f_{xt}, f(V_1, V_2) f) = \lim_{I \to \infty} \lim_{J \to \infty} \sum_{i=1}^{J} \sum_{j=1}^{J} (f, e_{i,j}) (t^{i-1} x^{j-1} f_{xt}, f) =
\]
\[
= (f_{xt}, f) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (f, e_{i,j}) x^{i-1} t^{j-1} = f(x, t) \cdot f(x, t),
\]
due to (12). \(\square\)

Corollary 1. The abstract form of the term \(f(x,t) \frac{\partial f(x,t)}{\partial x}\) is the operator
\[
G(f) = \frac{1}{2} C_1^{(0)} V_*^t f(V_1, V_2) f.
\]

Proof. The operator \(G(f)\) is obviously defined on all \(H_1\). Moreover,
\[
(f_{xt}, G(f)) = (f_{xt}, \frac{1}{2} C_1^{(0)} V_*^t f(V_1, V_2) f) = \frac{1}{2} (f_{xt}, C_1^{(0)} V_*^t g),
\]
where \(g = f(V_1, V_2) f\). Then according to (13)
\[
(f_{xt}, G(f)) = \frac{1}{2} g(x,t) \frac{\partial g(x,t)}{\partial x},
\]
where \(g(x,t)\) the function corresponding to \(g\), i.e. \(g(x,t) = [f(x,t)]^2\). Thus,
\[
(f_{xt}, G(f)) = f(x,t) \frac{\partial f(x,t)}{\partial x},
\]
which completes the proof. \(\square\)

The next proposition establishes a very important property of \(f(V_1, V_2) f\).
Proposition 4. The operator \( f(V_1, V_2) \) defined by (15) is Frechét differentiable at \( f_0 \in H_1 \) with Frechét derivative given by:

\[
L(h) = f_0(V_1, V_2)h + h(V_1, V_2) f_0,
\]

where

\[
f_0(V_1, V_2)h = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (f_0, e_{i,j}) V_i^{j-1} V_j^{i-1} h, \quad h(V_1, V_2) f_0 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (h, e_{i,j}) V_i^{j-1} V_j^{i-1} f_0
\]

Proof. The operator \( L \) is obviously linear and bounded. Moreover,

\[
(f_0(V_1, V_2) + h(V_1, V_2))(f_0 + h) - f_0(V_1, V_2) f_0 - L(h) = h(V_1, V_2)h,
\]

where \( h(V_1, V_2)h = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (h, e_{i,j}) V_i^{j-1} V_j^{i-1} h \) and as in the proof of Proposition 3 it is obtained that

\[
\| (f_0(V_1, V_2) + h(V_1, V_2))(f_0 + h) - f_0(V_1, V_2) f_0 - L(h) \|_1 \leq \| h \|_1^2 \Rightarrow
\]

\[
\frac{\| (f_0(V_1, V_2) + h(V_1, V_2))(f_0 + h) - f_0(V_1, V_2) f_0 - L(h) \|_1}{\| h \|_1} \leq \| h \|_1 \to 0,
\]

for \( \| h \|_1 \to 0. \)

\[\square\]

4. Proof of main results.

Proof of Theorem 2.1. According to §3, equation (3) is written as

\[-X^2 \left( f_{xt}, C_2^{(0)} V_2 * u \right) - XT \left( f_{xt}, C_1^{(0)} V_1 * u \right) + \left( f_{xt}, C_1^{(0)} V_2 * (C_2^{(0)} V_2 * u) \right) = 0\]

or since \( f_{xt} \) form a complete system of \( H \)

\[
(C_1^{(0)} V_1 * u)^2 (C_2^{(0)} V_2 * u) - X^2 C_2^{(0)} V_2 * u - X T C_2^{(0)} V_2 * u = 0,
\]

which is the equivalent to (3) abstract operator equation in \( H \).

Using the commutation relation (14), equation (17) becomes:

\[
\frac{C_2^{(0)} (C_1^{(0)} + I) (V_1 * V_1) C_2^{(0)} V_2 * u - X^2 C_2^{(0)} V_2 * u - X T C_1^{(0)} V_1 * u = 0}{\Rightarrow (V_1 * C_2^{(0)} V_2 * u - X^2 B_1^{(1)} (C_2^{(0)} V_2 * u) - X T B_1^{(1)} V_1 * u = 0}\]

\[
\Rightarrow V_1 C_2^{(0)} V_2 * u - X^2 V_1 B_1^{(1)} (C_2^{(0)} V_2 * u) - XTV_1 B_1^{(1)} V_1 * u = \sum_{j=1}^{\infty} A_j e_{1,j},
\]

due to the properties of \( V_1 * \), where \( B_1^{(1)} \) is the diagonal operator \( B_1^{(1)} e_{i,j} = \frac{1}{i+j} e_{i,j}, i,j = 1,2, \ldots \) and the coefficients \( A_j \) are uniquely determined by the coefficients of \( \phi_3(t) \), by taking the inner product of (18) with \( e_{1,j} \) as follows:

\[
\left( V_1 C_2^{(0)} V_2 * u, e_{1,j} \right) = A_j \Rightarrow j(u, e_{2,j+1}) = A_j.
\]

But since \( u(x,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (u, e_{i,j}) x^{i-1} t^{j-1} \), it is \( \phi_3(t) = \sum_{j=1}^{\infty} j(u, e_{2,j+1}) t^{j-1} \). Thus, the coefficients \( A_j \) are the complex conjugates of the coefficients of the power series in \( t \) of \( \phi_3(t) \). Proceeding in the same way, equation (18) is rewritten as

\[
C_2^{(0)} V_2 * u - X^2 V_1 B_1^{(1)} (C_2^{(0)} V_2 * u) - X T V_1 B_1^{(1)} V_1 * u = \sum_{j=1}^{\infty} A_j e_{2,j} + \sum_{j=1}^{\infty} B_j e_{1,j} \Rightarrow
\]

\[
\Rightarrow V_2 * u - X^2 V_1 B_1^{(1)} (C_2^{(0)} V_2 * u) - X T B_2^{(0)} V_2 B_1^{(1)} V_1 * u = \sum_{j=1}^{\infty} A_j e_{2,j} + \sum_{j=1}^{\infty} B_j e_{1,j} \Rightarrow
\]
\[
\begin{align*}
&\left( I - X^2V_2V_2^2B_1(0)V_2^* - XTV_2B_2(0)V_1^2B_1(1)V_* \right) u = \\
&\Rightarrow \sum_{j=1}^{\infty} \frac{A_j}{j} e_{2,j+1} + \sum_{j=1}^{\infty} \frac{B_j}{j} e_{1,j+1} + \sum_{i=1}^{\infty} \Gamma_i e_{i,1} \Rightarrow \\
&\Rightarrow (I-K)u = h,
\end{align*}
\]
where \(B_j\) and \(\Gamma_i\) are the complex conjugates of the coefficients of the power series in \(t\) of \(\phi_2(t)\) and \(x\) of \(\phi_1(x)\), respectively.

Since \(B_2(0)\), \(B_1(0)\) and \(B_1(1)\) are compact operators and \(V_1, V_2, V_1^*, V_2^*\) are bounded operators, \(K\) is a compact operator. Thus, according to the Fredholm alternative, either (19) has a unique solution in \(\mathcal{H}\), or the corresponding homogeneous equation

\[
(I-K)u = 0 \Rightarrow \left( I - X^2V_2V_2^2B_1(0)V_2^* - XTV_2B_2(0)V_1^2B_1(1)V_* \right) u = 0
\]

has a nontrivial solution in \(\mathcal{H}\).

By taking the inner product of (20) with \(e_{i,1}\), one obtains:

\[
(u, e_{i,1}) = 0, \quad \forall \quad i = 1, 2, \ldots.
\]

Next, by taking the inner product of (20) with \(e_{i,2}\), one obtains:

\[
(u, e_{i,2}) = 0, \quad (u, e_{2,2}) = 0 \quad (21)
\]

\[
(u, e_{i,2}) - \frac{X^2}{(i-1)(i-2)} (u, e_{i-2,2}) - \frac{XT}{i-1} (u, e_{i-1,1}) = 0 \Rightarrow \\
(u, e_{i,2}) = 0 \quad \forall \quad i = 3, 4, \ldots
\]

which is a difference equation with respect to \((u, e_{i,2})\), the solution of which under conditions (21) is \((u, e_{i,2}) = 0\). Thus, \((u, e_{i,2}) = 0 \forall i = 1, 2, \ldots\). In the same way, it can be proved by induction that \((u, e_{i,j}) = 0 \forall i, j = 1, 2, \ldots\) and as a consequence \(u = 0\) is the only solution of (20) in \(\mathcal{H}\). Thus, (19) has a unique solution in \(\mathcal{H}\), which equivalently means that the problem (3), (5) has a unique solution in \(L^2(\Delta^2)\). \(\square\)

**Proof of Theorem 2.2.** As in the proof of Theorem 2.1, the equivalent to (4) abstract operator equation in \(H_1\) is

\[
(C_1(0)V_1^*)^2(C_2(0)V_2^*)u - X^2C_2(0)V_2^2u - XTC_1(0)V_1^*u = \frac{XT}{2}C_1(0)V_1^*N(V_1, V_2)u.
\]

Using the commutation relation (14) and manipulating (22) as in the proof of Theorem 2.1, one ends up with:

\[
(I-K)u = h + \frac{XT}{2}V_2B_2(0)V_1^2B_1(1)V_*N(V_1, V_2)u,
\]

where \(h\) and \(K\) are defined as in the previous proof. By considering \((I-K)u = 0\), it can be proved as before that, \((I-K)^{-1}\) exists on all \(H_1\) and is bounded, i.e. there exists an \(L > 0\) such that \(\|(I-K)^{-1}\| \leq L\). Then, (23) becomes

\[
u = (I-K)^{-1} \left[ h + \frac{XT}{2}V_2B_2(0)V_1^2B_1(1)V_*N(V_1, V_2)u \right] = g(u).
\]

At this point the following fixed point theorem of Earle and Hamilton [4] will be applied: “If \(f : X \rightarrow X\) is holomorphic, i.e. its Fréchet derivative exists, and \(f(X)\) lies strictly inside \(X\), then \(f\) has a unique fixed point in \(X\), where \(X\) is a bounded, connected and open subset of a Banach space \(E\). (By saying that a subset \(X'\) of \(X\) lies strictly inside \(X\) it is meant that there exists an \(\epsilon_1 > 0\) such that \(\|x' - y\| > \epsilon_1\) for all \(x' \in X'\) and \(y \in E - X\).)”
Returning to (24), suppose that \( u \in B(0, R) \). Then, \( \|u\|_1 < R \) and
\[
\|g(u)\|_1 \leq L \left( \|h\|_1 + \frac{XT}{4}\|N(V_1, V_2)u\|_1 \right) \leq L\|h\|_1 + \frac{LXT}{4}\|u\|_1^2 \Rightarrow
\]
\[
\|g(u)\|_1 \leq L \left[ \|\phi_1(x)\|_{H_1(\Delta)} + \|\phi_2(t)\|_{H_1(\Delta)} + \|\phi_3(t)\|_{H_1(\Delta)} \right] + \frac{LXT}{4} R^2.
\]
(25)
Consider the function \( P(R) = R - \frac{LXT}{4} R^2 \) which attains its maximum \( R_0 = \frac{1}{LXT} \) at \( R_0 = \frac{R}{LXT} \). Then, for \( \|u\|_1 \leq R_0 - \epsilon < R_0 \), it follows that
\[
L \left[ \|\phi_1(x)\|_{H_1(\Delta)} + \|\phi_2(t)\|_{H_1(\Delta)} + \|\phi_3(t)\|_{H_1(\Delta)} \right] \leq P_0 - \epsilon < P_0,
\]
then (25) gives \( \|g(u)\|_1 \leq P_0 - \epsilon + \frac{LXT}{4} R_0^2 = R_0 - \epsilon < R_0 \). Moreover, \( g(u) \) is Fréchet differentiable and according to the theorem of Earle and Hamilton, equation (24) has a unique solution in \( H_1 \), bounded by \( R_0 \). Equivalently, if (7) holds, the problem (4), (6) has a unique solution in \( H_1(\Delta^2) \), bounded by \( R_0 \).

Proof of Theorem 2.3. According to a classical inversion theorem: “If \( T \) is a linear bounded operator of a Hilbert space \( H \), with \( \|T\| < 1 \), then \( I - T \) is invertible, defined on all \( H \) and \( \| (I - T)^{-1} \| \leq \frac{1}{1 - \|T\|} \).” Thus, the proof follows from Theorem 2.2 for \( L = \frac{L}{T - X(\Delta^2 + T)} \), provided that (8) holds.

Remark 6. The proofs of Remarks 2 and 5 are the same with the proofs of Theorems 2.1, 2.2 and 2.3, by taking into consideration the nonhomogeneous term.

5. Examples. The method described in the previous sections can also provide the unique solution established in the theorems of §2. This solution, according to (12), has the form \( u(x, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (u, e_{i,j}) x^{i-1} t^{j-1} \), where the coefficients \( (u, e_{i,j}) \) can be uniquely determined by taking the inner product of the equivalent to the PDE, abstract operator equation with the elements \( e_{i,j} \) of the base of \( H \).

In this section, two concrete examples will be given both for the linear equation (2) and the nonlinear equation (4), demonstrating this procedure. For reasons of simplicity the examples concern real solutions.

5.1. Example 1. Linear Case. Consider equation (2) subject to:
\[
u(\hat{x}, 0) = e^{2\hat{x}}, \quad u_\hat{t}(0, \hat{t}) = \frac{2}{3} e^{2\hat{t}/3}, \quad u_{\hat{t}\hat{t}}(0, \hat{t}) = \frac{4}{3} e^{2\hat{t}/3}.
\]
(26)
By applying the transformations \( \hat{x} = x \cdot T, \hat{t} = t \cdot T \), \( \hat{u}(\hat{x}, \hat{t}) = u(x, t) \), the problem is transformed to the one consisting of equation (3) and conditions
\[
u(x, 0) = e^{2Xx} = \phi_1(x), \quad u_t(0, t) = \frac{2T}{3} e^{2Tt/3} = \phi_2(t),
\]
\[
u_{\hat{t}\hat{t}}(0, t) = \frac{4XT}{3} e^{2Tt/3} = \phi_3(t).
\]
(27)
It is obvious that all functions \( \phi_1(x), \phi_2(t) \) and \( \phi_3(t) \) of (27) belong to \( H_2(\Delta) \) and can be rewritten in power series form as:
\[
\phi_1(x) = \sum_{i=1}^{\infty} \frac{(2X)^{i-1}}{(i-1)!} x^{i-1}, \quad \phi_2(t) = \sum_{j=1}^{\infty} \left( \frac{2T}{3} \right)^j t^{j-1}, \quad \phi_3(t) = \sum_{j=1}^{\infty} 2X \left( \frac{2T}{3} \right)^j t^{j-1}.
\]
(28)
Then the equivalent to this problem abstract equation according to (19) and taking into consideration (28) is:

\[
(\mathbf{I} - X^2 V_2 V_1^2 B_1^{(1)} B_2^{(0)} V_2^* - XTV_2 B_2^{(0)} V_1^2 B_1^{(1)} V_1^*) u = \sum_{j=1}^{\infty} 2X \left( \frac{2T}{3} \right)^j \frac{1}{j!} e_{j+1} + \sum_{j=1}^{\infty} \left( \frac{2T}{3} \right)^j \frac{1}{j!} e_{j+1} + \sum_{i=1}^{(2X)^j-1} (2X)^{j-i} e_{i,j}. \tag{29}
\]

By taking the inner product of (29) with \( e_{1,j} \) and using the orthonormality of \( \{e_{i,j}\} \) one obtains:

\[
(u, e_{1,1}) = 1, \quad \text{since} \quad V_2^* e_{1,1} = 0 \Rightarrow (u, e_{1,j}) = \left( \frac{2T}{3} \right)^{j-1} \frac{1}{(j-1)!} \quad \forall \ j.
\]

By taking the inner product of (29) with \( e_{2,j} \) one obtains in the same way:

\[
(u, e_{2,1}) = 2X, \quad (u, e_{2,j}) = 2X \left( \frac{2T}{3} \right)^{j-1} \frac{1}{(j-1)!} \quad \forall \ j \neq 1 \Rightarrow (u, e_{2,j}) = 2X \left( \frac{2T}{3} \right)^{j-1} \frac{1}{(j-1)!} \quad \forall j.
\]

Continuing with the coefficients \( u, e_{3,j} \) one similarly obtains:

\[
(u, e_{3,j}) = \left( \frac{2Xe}{2} \right) \left( \frac{2T}{3} \right)^{j-1} \frac{1}{(j-1)!} \quad \forall \ j.
\]

For the rest of the coefficients the technique of mathematical induction can be applied.

Thus, it is assumed that \( (u, e_{i,j}) = \left( \frac{2X^{i-1}}{(i-1)!} \right) \left( \frac{2T}{3} \right)^{j-1} \frac{1}{(j-1)!} \) \( \forall \ i = 1, \ldots k, \ j = 1, 2, \ldots \) and it should be proved that

\[
(u, e_{k+1,j}) = \left( \frac{2X^k}{k!} \right) \left( \frac{2T}{3} \right)^{j-1} \frac{1}{(j-1)!} \quad j = 1, 2, \ldots.
\]

Indeed, by taking the inner product of (29) with \( e_{k+1,1} \), \( k \geq 4 \) it is obtained:

\[
(u, e_{k+1,1}) = \left( \sum_{i=1}^{\infty} \frac{2X^{i-1}}{(i-1)!} e_{i,1} e_{k+1,1} \right) \Rightarrow (u, e_{k+1,1}) = \frac{(2X)^k}{k!}.
\]

due to the orthonormality of \( \{e_{i,j}\} \) and the fact that \( V_2^* e_{k+1,1} = 0 \).

Next, by taking the inner product of (29) with \( e_{k+1,j} \), \( k \geq 4 \), \( j \neq 1 \) it is obtained:

\[
(u, e_{k+1,j}) = X^2 (V_1^2 B_1^{(1)} B_2^{(0)} V_2^* u, e_{k+1,j-1}) - X T (B_2^{(0)} V_1^2 B_1^{(1)} V_1^* u, e_{k+1,j-1}) = 0 \Rightarrow
\]

\[
(u, e_{k+1,j}) = X^2 \frac{k}{(k-1)} (V_2^* u, e_{k-1,j-1}) - X T \frac{1}{(j-1)} (B_1^{(1)} V_1^* u, e_{k-1,j-1}) = 0 \Rightarrow
\]

\[
(u, e_{k+1,j}) = \frac{X^2}{k(k-1)} (u, e_{k-1,j}) + X T \frac{1}{(j-1)k} (u, e_{k-1,j}) =
\]

\[
= \frac{X^2}{k(k-1)} (2X)^{k-2} \frac{2T}{(3)^{j-1}} \frac{1}{(j-1)!} + X T \frac{(2X)^{k-1}}{(j-1)k} \frac{(2T)^{j-2}}{(3)^{j-2}} \frac{1}{(j-2)!} \Rightarrow
\]

\[
(u, e_{k+1,j}) = \frac{(2X)^k}{k!} \left( \frac{2T}{3} \right)^{j-1} \frac{1}{(j-1)!}.
\]

Thus, the solution of (3) satisfying conditions (27) is

\[
u(x, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (u, e_{i,j}) x^{i-1} t^{j-1} = \sum_{j=1}^{\infty} (u, e_{1,j}) t^{j-1} + x \sum_{j=1}^{\infty} (u, e_{2,j}) t^{j-1} + \ldots =
\]

\[
= \sum_{j=1}^{\infty} \left( \frac{2T}{3} \right)^{j-1} \frac{1}{(j-1)!} t^{j-1} + x \sum_{j=1}^{\infty} 2X \left( \frac{2T}{3} \right)^{j-1} \frac{1}{(j-1)!} t^{j-1} + \ldots +
\]
Thus, the solution of (2) satisfying conditions (26) is \( \tilde{u}(x, t) = e^{2x} \), which is the unique entire solution of the problem, according to Remark 1.

5.2. Example 2. Nonlinear Case. Consider equation (4) subject to:

\[ u(x, 0) = x = \phi_1(x), \quad u_t(0, t) = 1 = \phi_2(t), \quad u_{xx}(0, t) = 0 = \phi_3(t). \]  \( (30) \)

It is obvious that all functions \( \phi_1(x), \phi_2(t) \) and \( \phi_3(t) \) of (30) belong to \( H_1(\Delta) \). For these data, conditions (8), (9) become \( X(X + T) < 2 \) and \( 8XT < (2 - X^2 - XT)^2 \). For reasons of simplicity, \( X, T \) are taken equal to 1/2. Then the equivalent to this problem abstract equation according to (23) and taking into consideration (30) is:

\[
\left( I - \frac{1}{4} V_2 V_1^2 B_1^{(1)} B_1^{(0)} V_2^* - \frac{1}{4} V_2 B_2^{(0)} V_1^2 B_1^{(1)} V_1^* \right) u = e_{1,2} + e_{2,1} + \frac{1}{8} V_2 B_2^{(0)} V_1^2 B_1^{(1)} V_1^* N(V_1, V_2) u.
\]  \( (31) \)

By taking the inner product of (31) with \( e_{1,j} \) one obtains:

\( (u, e_{1,1}) = 0 \), \( (u, e_{1,2}) = 1 \) and \( (u, e_{1,j}) = 0 \) for \( j \neq 1, \ j \neq 2 \).

By taking the inner product of (31) with \( e_{2,j} \) one finds:

\( (u, e_{2,1}) = 1 \) and \( (u, e_{2,j}) = 0 \) for \( j \neq 1 \).

Continuing with the coefficients \( (u, e_{3,j}) \) one finds \( (u, e_{3,1}) = 0 \) and for \( j \neq 1 \)

\[
(u, e_{3,j}) - \frac{1}{4} (V_2 B_1^{(1)} B_1^{(0)} V_2^* u, e_{3,j-1}) - \frac{1}{4} (B_2^{(0)} V_1^2 B_1^{(1)} V_1^* u, e_{3,j-1}) = \frac{1}{8} (B_2^{(0)} V_1^2 B_1^{(1)} V_1^* N(V_1, V_2) u, e_{3,j-1}) \Rightarrow
\]

\[
(u, e_{3,j}) - \frac{1}{4} (B_1^{(1)} B_1^{(0)} V_2^* u, e_{1,j-1}) - \frac{1}{4(j-1)} (B_1^{(1)} V_1^* u, e_{1,j-1}) = \frac{1}{8(j-1)} (B_1^{(1)} V_1^* N(V_1, V_2) u, e_{1,j-1}) \Rightarrow
\]

\[
(u, e_{3,j}) - \frac{1}{8} (u, e_{1,j}) - \frac{1}{8(j-1)} (u, e_{2,j-1}) = \frac{1}{16(j-1)} (N(V_1, V_2) u, e_{2,j-1}) \Rightarrow
\]

\[
(u, e_{3,j}) = \frac{1}{8} (u, e_{1,j}) + \frac{1}{8(j-1)} (u, e_{2,j-1}) + \frac{1}{16(j-1)} \left( \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (u, e_{k,l}) V_1^{k-1} V_2^{l-1} u, e_{2,j-1} \right) \Rightarrow
\]

\[
(u, e_{3,j}) = \frac{1}{8} (u, e_{1,j}) + \frac{1}{8(j-1)} (u, e_{2,j-1}) + \frac{1}{16(j-1)} \left( \sum_{k=1}^{2} \sum_{l=1}^{j-1} (u, e_{k,l}) (u, e_{3-k,j-l}) \right), \ j \neq 1.
\]

From the previous relation it can be easily found that

\( (u, e_{3,2}) = 1/4, \ (u, e_{3,3}) = 1/16, \ \text{and} \ (u, e_{3,j}) = 0, \ \text{for} \ j \geq 4. \)
The rest of the coefficients are uniquely determined in the same way. Indeed, by taking the inner product of (31) with $e_{i,j}$, $i \geq 3$ one finds:

$$
(u, e_{i,1}) = 0
$$

$$
(u, e_{i,j}) = \frac{1}{4(i-1)(i-2)}(u, e_{i-2,j}) + \frac{1}{4(j-1)(j-2)}(u, e_{i-1,j-1}) + \frac{1}{8(j-1)(j-2)}(\sum_{k=1}^{i-1}\sum_{l=1}^{j-1}(u, e_{k,l})(u, e_{k-i,j-l}))
$$

$$
, j \neq 1.
$$

Thus, the solution of (4) satisfying conditions (30) is

$$
u(x, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (u, e_{i,j})x^{i-1}t^{j-1} = \sum_{j=1}^{\infty} (u, e_{1,j})t^{j-1} + x \sum_{j=1}^{\infty} (u, e_{2,j})t^{j-1} + \ldots =
$$

$$
t + x + \frac{x^2}{4}t + \frac{x^2}{16}t^2 + \ldots,
$$

where the rest of the coefficients are uniquely determined by (32).

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