A NONVANISHING CONJECTURE FOR COTANGENT BUNDLES

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Abstract. In this paper we study the positivity of the cotangent bundle of projective manifolds. We conjecture that the cotangent bundle is pseudoeffective if and only if the manifold has non-zero symmetric differentials. We confirm this conjecture for most projective surfaces that are not of general type.

1. Introduction

1.A. Main result. A central part in the minimal model program in algebraic geometry is the so-called nonvanishing conjecture: given a projective manifold or, more generally, a variety with klt singularities, \( X \), whose canonical class \( K_X \) is pseudoeffective, one has

\[
H^0(X, \mathcal{O}_X(mK_X)) \neq 0
\]

for some positive integer \( m \). This conjecture has been proven some time ago in dimension at most three, but is wide open in higher dimensions.

In analogy to the nonvanishing conjecture, one might ask for

1.1. Conjecture. Let \( X \) be a normal projective variety with klt singularities. Let \( 1 \leq q \leq \dim X \). Then \( \Omega_X^{[q]} \), the sheaf of reflexive holomorphic differentials in degree \( q \), is pseudoeffective, (see Definition 3.5), if and only for some positive integer \( m \) one has

\[
H^0(X, S^{[m]}\Omega_X^{[q]}) \neq 0.
\]

In the case \( q = \dim X \), this is of course the nonvanishing conjecture stated above. The only general result confirming Conjecture 1.1 is given in [HP19, Thm.1.6]: Suppose \( X \) is klt and smooth in codimension two with \( K_X \equiv 0 \). If \( \Omega_X^{[1]} \) is pseudoeffective, there is a quasi-étale cover \( \tilde{X} \rightarrow X \) such that \( q(\tilde{X}) > 0 \). In particular one has \( H^0(X, S^{[m]}\Omega_X^{[1]}) \neq 0 \) for some positive integer \( m \) ([Ane18, Prop.2.2], see also Lemma 4.6). While the pseudoeffectivity of \( K_X \) is equivalent to the non-uniruledness of the manifold, we do not know many examples where \( \Omega_X^{[1]} \) is pseudoeffective, but not big. We expect that this property is actually quite restrictive, our Theorem 1.2 confirms this intuition in the first non-trivial case.

In this paper we are mainly interested in the case \( q = 1 \). Already for smooth surfaces, Conjecture 1.1 is delicate. In this case we can assume without loss of generality that \( X \) is minimal, see Proposition 4.1. By surface classification, see Corollary 4.14 the problem starts with \( \kappa(X) = 1 \). We basically settle this case:
1.2. Theorem. Let $f : X \to B$ be a (minimal) smooth elliptic surface with $\kappa(X) = 1$ such that $\Omega_X^1$ is pseudoeffective. Suppose one of the following.

a) $f$ is not isotrivial

b) $f$ is isotrivial and the general fiber does not have complex multiplication

c) The tautological class on $\mathbb{P}(\Omega_X^1)$ is nef in codimension one.

Then $\tilde{q}(X) > 0$ (see Definition 2.2), so there is a positive integer $m$ such that $H^0(X, S^m \Omega_X^1) \neq 0$.

Note that each of the cases requires a different proof, Theorem 1.2 is obtained as the union of Corollary 5.5, Corollary 6.8 and Corollary 6.14.

In general, the pseudoeffectivity of $\Omega_X^1$ does not imply $\tilde{q}(X) > 0$: a smooth complete intersection surface $X \subset \mathbb{P}^N$ is simply connected, so $\tilde{q}(X) = 0$. However, if $N \geq 4$ and the multidegrees are sufficiently high, the cotangent bundle $\Omega_X^1$ is ample [Bro14].

For surfaces of general type, Conjecture 1.1 is open. If $c_2^1(X) > c_2(X)$, then by Bogomolov’s vanishing theorem

$$h^0(X, S^m \Omega_X^1) \sim \left(\frac{c_2^1(X) - c_2(X)}{6}\right)m^3,$$

but already the boundary case $c_2^1(X) = c_2(X)$ is unclear.

In higher dimension, things get worse due to the singularities of minimal models. For example, we know Conjecture 1.1 for terminal threefolds with numerically trivial canonical class, but we cannot deduce easily Conjecture 1.1 for smooth threefolds $X$ with $\kappa(X) = 0$, although $X$ has a terminal minimal model as above.

We would finally like to point out the connection to a question posed by H. Esnault, see [BKT13]: let $X$ be a projective (or compact Kähler) manifold whose fundamental group $\pi_1(X)$ is infinite. Does there exist a positive integer $m$ such that $H^0(X, S^m \Omega_X^1) \neq 0$?

An intermediate step might be to prove that $\Omega_X^1$ is pseudoeffective. Brunebarbe-Klingler-Totaro confirm Esnault’s conjecture if there is a representation $\pi_1(X) \to \text{GL}(N, \mathbb{C})$ with infinite image. The key point of their proof is to show that in many cases, the cotangent bundle $\Omega_X^1$ is even big.

Theorem 1.2 fits in the framework of Esnault’s conjecture: it is well-known to experts that if the cotangent bundle of an elliptic surface is not pseudoeffective, then $\pi_1(X)$ is finite (see Appendix A for a proof). In view of Theorem 1.2, the two properties should actually be equivalent (which is actually the case up to the exceptional isotrivial case of Theorem 1.2) and imply the existence of symmetric differentials.

1.B. Strategy of the proof. Let $X$ be a smooth projective surface such that $K_X$ is nef and $c_1(K_X)^2 = 0$. Then $K_X$ is semiample, so we have the Iitaka fibration $f : X \to B$ such that the general fibre $F$ is elliptic and $mK_X \simeq f^* A$ with $A$ an ample divisor. Assume now that $\Omega_X := \Omega_X^1$ is pseudoeffective, then one expects that there exists
a pseudoeffective subsheaf of $\Omega_X$ that is induced by a pull-back from the base $B$. Let $f^*\Omega_B \to \Omega_X$ be the cotangent map, and denote by
\[ f^*\Omega_B(D) \subset \Omega_X \]
the saturation. If $f^*\Omega_B(D)$ is pseudoeffective, Proposition 5.2 shows that $\tilde{q}(X) > 0$. Thus the main issue in Theorem 1.2 is to show that $f^*\Omega_B(D)$ is pseudoeffective. A natural approach is to show that the sheaf $\Omega_X \to \Omega_{X/B}(-D)$ is not pseudoeffective if $f$ is not almost smooth. However, by a theorem of Brunella [Bru06], the line bundle $\omega_{X/B}(-D)$ is always pseudoeffective! This leads us to considering the more refined quotient sheaf
\[ \Omega_X \to \mathcal{I}_Z \otimes \omega_{X/B}(-D) \to 0, \]
where $Z$ has support in the singular points of the reduction of the fibres. The basic idea of the proof of Theorem 1.2 is to show that the torsion-free sheaf $\mathcal{I}_Z \otimes \omega_{X/B}(-D)$ is not strongly pseudoeffective (see Definition 3.7), although its biidual is a pseudoeffective line bundle. Thanks to a result of Demailly-Peternell-Schneider [DPS94] this idea leads immediately to the result in the non-isotrivial case, see Section 5. For an isotrivial fibration this approach only yields the weaker statement appearing as part c) of the main theorem, see Subsection 6.3. On the other hand we know that $X$ is birational to a quotient $(C \times E)/G$, so we aim to compute explicitly the spaces of global sections
\[ H^0(X, S^i \Omega_X \otimes \mathcal{O}_X(jA)), \]
following a strategy introduced by Sakai [Sak79]. Apart from the technical setup, the main difficulty is to understand the local obstruction near the fixed points of the group action. For $A_1$-singularities this information is provided by [BTVA19, Prop.3.2.]. We expect that a similar description of the local obstruction for klt singularities would allow to handle the case when the elliptic curve has complex multiplication.

1.C. Structure of the paper. In Section 3 we introduce a positivity notion ("strongly pseudoeffective") that is adapted for this type of torsion-free sheaf, and present material on pseudoeffective torsion free sheaves which will be used in later sections. Section 4 is concerned with some general results on varieties with pseudoeffective cotangent sheaves. In particular, generalised Kodaira dimensions are introduced and a relation to the MRC fibration is studied. The last two sections are devoted to the proof of Theorem 1.2. Section 5 gives the general setup and settles the case that the elliptic fibration is not isotrivial. The surprisingly difficult isotrivial case finally is studied in Section 6.

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\footnote{See the introduction of Section 5 for the almost smooth case.}
2. Basic notations

We work over the complex numbers, for general definitions we refer to [Har77]. We use the terminology of [Deb01] and [KM98] for birational geometry and notions from the minimal model program and [Laz04] for notions of positivity. Manifolds and varieties will always be supposed to be irreducible and reduced.

2.1. Notation. Let $X$ be a normal complex variety. As usual, $\Omega^1_X$ denotes the sheaf of Kähler differentials, and we set

$$\Omega^{[q]}_X := \left( \bigwedge^q \Omega^1_X \right)^{**}.$$  

If $X$ is klt and $\mu : \hat{X} \to X$ a desingularization, then by [GKKP11, Thm.1.4],

$$\Omega^{[q]}_X = \mu^* (\Omega^{[q]}_{\hat{X}}).$$

If $q = 1$ and $X$ smooth, we simply set $\Omega_X := \Omega^1_X$. Finally, for any normal variety $X$, we denote by $T_X := (\Omega^1_X)^*$ its tangent sheaf.

A finite surjective map $\gamma : X' \to X$ between normal varieties is quasi-étale if its ramification divisor is empty (or equivalently, by purity of the branch locus, $\gamma$ is étale over the smooth locus of $X$).

2.2. Definition. Let $X$ be a normal projective variety with klt singularities. Then, as usual,

$$q(X) = h^1(X, \mathcal{O}_X) = h^0(X, \Omega^{[1]}_X)$$

is the irregularity of $X$. Further, we denote by $\tilde{q}(X)$ the maximal irregularity $q(\tilde{X})$, where $\tilde{X} \to X$ is any quasi-étale cover.

While the irregularity $q(X)$ is a birational invariant of projective varieties with klt singularities, this is not the case for $\tilde{q}(X)$:

2.3. Example. Let $\tau : E_1 \to \mathbb{P}^1$ be a (hyper)elliptic curve, and denote by $i_{E_1}$ the involution induced by the double cover. Let $E_2$ be an elliptic curve, and denote by $i_{E_2}$ the involution defined by $z \mapsto -z$. The surface

$$X' := (E_1 \times E_2)/\langle i_{E_1} \times i_{E_2} \rangle$$

is normal and has $A_1$-singularities in the branch points of the quasi-étale map $E_1 \times E_2 \to X'$. The projection on the first factor induces an isotrivial elliptic fibration

$$f' : X' \to \mathbb{P}^1 = E_1/\langle i_{E_1} \rangle$$

that has exactly $2g(E_1) + 2$ singular fibres, all of them are multiple fibres of multiplicity 2 such that the reduction is isomorphic to $\mathbb{P}^1 = E_2/\langle i_{E_2} \rangle$. By construction we have $\tilde{q}(X') \geq g(E_1) + g(E_2) > 0$.

Denote by $\mu : X \to X'$ the minimal resolution, then the induced elliptic fibration $f : X \to \mathbb{P}^1$ is relatively minimal, isotrivial and has exactly $2g(E_1) + 2$ singular fibres, all of them of type $I_0^*$ (in Kodaira’s terminology, see [BHPVdV04, V, Table 3]).

In the classical case where $E_1$ is an elliptic curve, the surface $X$ is a K3 surface of Kummer type. In particular we have $\tilde{q}(X) = 0$. 


3. Pseudoeffective sheaves

3.1. Notation. Let $\mathcal{G}$ be a coherent sheaf on a variety $X$, and let $\mathcal{T} \subset \mathcal{G}$ its torsion subsheaf. Then we denote by $\mathcal{G}/\mathcal{T}$ the quotient $\mathcal{G}/\mathcal{T}$. Furthermore, we set $S^{[m]}(\mathcal{G}) := (S^m \mathcal{G})^{**}$.

3.A. Projectivization of sheaves.

3.2. Definition. Let $\mathcal{F}$ be a coherent sheaf on a variety $X$. Then we denote by $\pi : \mathbb{P} (\mathcal{F}) \to X$ the projectivisation of $\mathcal{F}$ in the sense of [AT82, II, §2, Sect.2]. We denote by $\zeta_{\mathbb{P}(\mathcal{F})}$ (or $\zeta$ when no confusion is possible) the Cartier divisor class associated to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$.

Remark. The reduction of any fibre of $\pi$ is a projective space, and $\pi$ is locally trivial if and only if $\mathcal{F}$ is locally free [AT82, p.27]. For locally free sheaves, the following definition of pseudoeffectivity is now in common.

3.3. Definition. Let $X$ be a projective variety, and let $\mathcal{F}$ be a locally free sheaf on $X$. Denote by $\pi : \mathbb{P}(\mathcal{F}) \to X$ the projectivisation, and by $\zeta$ the tautological class on $\mathbb{P}(\mathcal{F})$. We say that $\mathcal{F}$ is pseudoeffective if $\zeta$ is a pseudoeffective Cartier divisor class.

Remark. By [Dru18, Lemma 2.7], the locally free sheaf $\mathcal{F}$ is pseudoeffective if and only if for some ample Cartier divisor $H$ on $X$ and for all $c > 0$ there exist numbers $j \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $i > cj$ and

$$H^0(X, S^i \mathcal{F} \otimes \mathcal{O}_X(jH)) \neq 0.$$ 

We will use the following lemma, which will be generalised below.

3.4. Lemma. Let $f : X \to Y$ be a surjective morphism of projective varieties and $\mathcal{F}$ a locally free sheaf on $Y$. Then $\mathcal{F}$ is pseudoeffective if and only if $f^*(\mathcal{F})$ is pseudoeffective.

Remark. Note that the statement applies in particular to the normalisation, so for locally free sheaves pseudoeffectivity can be verified on the normalisation.

Proof. Recall the pull-back formula for the tautological classes

$$\zeta_{\mathbb{P}(f^*(\mathcal{F}))} = p^*(\zeta_{\mathbb{P}(\mathcal{F})}),$$

where $p : \mathbb{P}(f^*(\mathcal{F})) = \mathbb{P}(\mathcal{F}) \times_Y X \to \mathbb{P}(\mathcal{F})$ is the canonical projection. Thus we are reduced to the case where $\mathcal{F}$ has rank one, which is immediate by [Laz04, Thm.2.2.26, Prop.2.2.43].

3.B. Strongly pseudoeffective torsion-free sheaves. For the purpose of this paper it is not sufficient to discuss the positivity of locally free sheaves, in fact we will need the more subtle positivity properties of torsion-free sheaves. It will suffice to consider normal varieties.

We first recall the definition of pseudoeffectivity for reflexive sheaves from [Dru18 and HP19 Defn.2.1].
3.5. Definition. Let $X$ be a normal projective variety, and let $\mathcal{F}$ be a reflexive sheaf on $X$. Then $\mathcal{F}$ is pseudoeffective if for some ample Cartier divisor $H$ on $X$ and for all $c > 0$ there exist numbers $j \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $i > cj$ and

$$H^0(X, S^{[i]}\mathcal{F} \otimes \mathcal{O}_X(jH)) \neq 0.$$ 

Remark. An equivalent definition using an adapted resolution of singularities of $\mathbb{P}(\mathcal{F})$ is given in [HP19].

3.6. Example. Our definition of pseudoeffectivity is less restrictive than [BDPP13 Defn.7.1]: if $G \subset \mathcal{F}$ is a pseudoeffective reflexive subsheaf, then $\mathcal{F}$ is pseudoeffective. In particular if $\mathcal{F} = L \oplus H$ where $L$ is pseudoeffective and $H$ an antiample reflexive sheaf, then $\mathcal{F}$ is pseudoeffective in the sense of Definition 3.5 but not in the sense of [BDPP13 Defn.7.1].

Definition 3.5 makes also sense for torsion-free sheaves, but would not be very useful: by definition a torsion-free sheaf would be pseudoeffective if and only if its bidual is pseudoeffective. The following definition takes this difference into account:

3.7. Definition. Let $X$ be a normal projective variety, and let $\mathcal{F}$ be a torsion free sheaf on $X$. We say that $\mathcal{F}$ is strongly pseudoeffective if for some ample Cartier divisor $H$ on $X$ and for all $c > 0$ there exist numbers $j \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $i > cj$ and

$$H^0(X, (S^i\mathcal{F})/\text{Tor} \otimes \mathcal{O}_X(jH)) \neq 0.$$ 

3.8. Remark. For locally free sheaves, the Definitions 3.6 3.7 3.8 obviously coincide. Even for reflexive sheaves, Definition 3.7 is more restrictive than Definition 3.5 in general $(S^i\mathcal{F})/\text{Tor}$ is not reflexive and has less global sections than its bidual, so we might have

$$H^0(X, (S^i\mathcal{F})/\text{Tor} \otimes \mathcal{O}_X(jH)) = 0,$$

although $\mathcal{F}$ is pseudoeffective in the sense of Definition 3.5. We thank C. Gachet for the following example:

3.9. Example. Let

$$\mathbb{C}^2 \rightarrow X = \{(x, y, z) \in \mathbb{C}^3 \mid xy - z^2 = 0\}, \ (u, v) \mapsto (u^2, v^2, uv)$$

be the double cover of the $A_1$-singularity, we identify the polynomial ring of $X$ to its image $\mathbb{C}[u^2, v^2, uv]$ in $\mathbb{C}[u, v]$. The invariant elements under the involution

$$j : \mathbb{C}[u, v] \rightarrow \mathbb{C}[u, v], \ f(p) \mapsto -f(-p)$$

are exactly the odd polynomials, i.e. the polynomials that can be written as $uf + vg$ with $f, g \in \mathbb{C}[u^2, v^2, uv]$. Denote this set by $\mathbb{C}[u, v]^{[2]}$, then $\mathbb{C}[u, v]^{[2]}$ has a natural structure of $\mathbb{C}[u^2, v^2, uv]$-module that is reflexive, but not (locally) free. The tensor power $(\mathbb{C}[u, v]^{[2]})^\otimes 2$ is generated by $u^2, v^2, uv$, so it naturally embeds into $\mathbb{C}[u^2, v^2, uv]$. Remembering that this ring is actually the function ring of the $A_1$-singularity, we see that $(\mathbb{C}[u, v]^{[2]})^\otimes 2$ is isomorphic to the maximal ideal defining the origin.

Let now $q : A \rightarrow X$ be the quotient of an abelian surface under the involution $z \mapsto -z$, so $X$ is the singular Kummer surface. Since $S^{[2]}\Omega_X$ is globally generated, the sheaf of reflexive differentials $\Omega_X^{[1]}$ is pseudoeffective. Let us see that it is not strongly pseudoeffective: we have $\Omega_X^{[1]} \simeq \mathcal{F} \oplus \mathcal{F}$, where $\mathcal{F}$ is the sheaf of $\mathbb{Z}_2$-invariants for the natural action on $\Omega_A \simeq \mathcal{O}_Adz_1 \oplus \mathcal{O}_Adz_2$. It is immediate to see that near

$$\text{singularity }, \text{we see that}$$

$$f,g$$

$\text{coincide. Even for reflexive sheaves, Definition 3.7 is more restrictive than Definition}$

$$\text{and for all} \ \mathbb{C}[u^2, v^2, uv]$$

$$\text{is generated by} \ \mathbb{C}[u, v]^{[2]}$$

$$\text{is reflexive, but not (locally) free. The tensor power} \ \mathbb{C}[u, v]^{[2]}$$

$$\text{generates by} \ u^2, v^2, uv$$

$$\text{natural structure of} \ \mathbb{C}[u^2, v^2, uv]$$

$$\text{function ring of the} \ A_1$$

$$\text{quotient of an abelian surface under the involution}$$

$$\text{Kummer surface. Since} \ S^{[2]}\Omega_X$$

$$\text{globally generated, the sheaf of reflexive differentials} \ \Omega_X^{[1]}$$

$$\text{pseudoeffective. Let us see that it is not}$$

$$\text{strongly pseudoeffective: we have} \ \Omega_X^{[1]} \simeq \mathcal{F} \oplus \mathcal{F}$$

$$\text{sheaf of} \ \mathbb{Z}_2$$

$$\text{natural action on} \ \Omega_A \simeq \mathcal{O}_Adz_1 \oplus \mathcal{O}_Adz_2.$$ 

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$$\text{natural action on} \ \Omega_A \simeq \mathcal{O}_Adz_1 \oplus \mathcal{O}_Adz_2.$$ 

It is immediate to see that near
the fixed points, the \( \mathbb{Z}_2 \)-action on \( O_A dz_1 \) identifies to the action \( j \) in the paragraph above. Thus, using the local computation, we see that

\[
F^\otimes i \simeq \begin{cases} 
I_{X_{\text{sing}}}^i & \text{if } i \text{ even}, \\
I_{X_{\text{sing}}}^i \otimes F & \text{if } i \text{ odd}.
\end{cases}
\]

Combined with Example 3.12 this shows that \( F \) is not strongly pseudoeffective.

In general it is not clear if one can check strong pseudoeffectivity by looking at a tautological class on (a modification of) the projectivisation. However there is a natural construction in a special case:

3.10. Setup. Let \( F \) be a torsion-free sheaf on a normal projective variety \( X \) such that

\[
F \simeq I_Z \otimes \mathcal{E}
\]

where \( I_Z \) is an ideal sheaf and \( \mathcal{E} \) is a locally free sheaf.

Let \( \mu : \hat{X} \to X \) be the blow-up of the ideal sheaf \( I_Z \), then \( \hat{X} \) is a (not necessarily normal) variety [Har77, II, Prop.7.16]. We denote by \( \mathcal{O}_{\hat{X}}(1) := \mu^{-1}(I_Z) \mathcal{O}_{\hat{X}} \)

the tautological sheaf on \( \hat{X} \). Recall that by the definition of the blow-up [AT82 II, §3] one has

\[
(1) \quad \mu_*(\mathcal{O}_X(i)) = I_Z^i \quad \forall \ i \geq 0.
\]

Note also that if \( Z \) is locally generated by a regular sequence, one has \( S^i I_Z \simeq I_Z^i \) for all \( i \geq 0 \) (e.g. [BC18 Prop.2.2.8]). In particular the blowup \( \text{Bl}_{I_Z}(X) \) coincides with the projectivisation \( \mathbb{P}(I_Z) \).

3.11. Lemma. In the situation of Setup 3.10, the torsion-free sheaf \( F \) is strongly pseudoeffective if and only if the locally free sheaf \( \mathcal{O}_{\hat{X}}(1) \otimes \mu^* \mathcal{E} \) on the variety \( \hat{X} \) is pseudoeffective.

Proof. Let \( H \) be an ample Cartier divisor on \( X \). By the projection formula and (1) one has

\[
\mu_*(\mu^*(\mathcal{O}_X(jH) \otimes S^i \mathcal{E}) \otimes \mathcal{O}_X(i)) \simeq \mathcal{O}_X(jH) \otimes S^i \mathcal{E} \otimes I_Z^i
\]

for all \( i \geq 0 \). Moreover we know by [Mic64] that

\[
(S^i I_Z)/\text{Tor} \simeq I_Z^i.
\]

Thus we obtain that

\[
H^0(\hat{X}, S^i(\mathcal{O}_{\hat{X}}(1) \otimes \mu^* \mathcal{E}) \otimes \mathcal{O}_X(jH)) \simeq H^0(X, (S^i \mathcal{F})/\text{Tor} \otimes \mathcal{O}_X(jH))
\]

for all \( i \geq 0 \). Now we apply [Dru18 Lemma 2.7] (see [HLS20 Lemma 2.2.] for the case where the divisor is only big). \( \square \)

3.12. Example. Let \( X \) be a normal projective variety, and let \( I_Z \subset O_X \) an ideal sheaf. Then \( I_Z \) is not strongly pseudoeffective by Lemma 3.11.

On the other hand let \( Z \subset \mathbb{P}^2 \) be a point, then \( I_Z \otimes O_{\mathbb{P}^2}(1) \) is strongly pseudoeffective. Indeed the locally free sheaf \( \mathcal{O}_{\hat{X}}(1) \otimes \mu^* \mathcal{O}_{\mathbb{P}^2}(1) \) on the blowup \( \hat{X} \simeq \mathbb{P}_1 \) is pseudoeffective, since it is isomorphic to \( \mathcal{O}_{\mathbb{P}_1}(F) \) where \( F \) is the strict transform of a line through \( Z \).
3.13. Corollary. In the situation of Setup 3.10 suppose that the ideal sheaf $I_Z$ is locally generated by a regular sequence (e.g. if $Z$ is a locally complete intersection scheme). Then the following statements are equivalent:

a) The sheaf $F$ is strongly pseudoeffective;

b) The locally free sheaf $O_X(1) \otimes \mu^*E$ on the blow-up

$$\mu : \hat{X} = Bl_{I_Z}X \rightarrow X,$$

is pseudoeffective;

c) The tautological line bundle $O_{\mathbb{P}(F)}(1)$ on the projectivisation $\mathbb{P}(F)$ is pseudoeffective.

Proof. The equivalence between a) and b) is shown in Lemma 3.11.

Denote by $\hat{\pi} : \mathbb{P}(\mu^*E) \rightarrow \hat{X}$ the projectivisation, and by $O_{\mathbb{P}(\mu^*E)}(1)$ its tautological sheaf. Since $O_X(1)$ is $\mu$-ample and $O_{\mathbb{P}(\mu^*E)}(1)$ is $\hat{\pi}$-ample and $\mu \circ \hat{\pi}$-nef, we know that $\hat{\pi}^*(O_X(1)) \otimes O_{\mathbb{P}(\mu^*E)}(1)$ is $\mu \circ \hat{\pi}$-ample. Since $I_Z$ is locally generated by a regular sequence, we have $S^iI_Z \simeq I_{\hat{Z}}^i$ for all $i \geq 0$ (e.g. [BC18, Prop. 2.2.8]). Thus for all $i \geq 0$ we have

$$(\mu \circ \hat{\pi})_*((\hat{\pi})^*(O_X(i)) \otimes O_{\mathbb{P}(\mu^*E)}(i)) \simeq \mu_*((O_X(i) \otimes \mu^*S^iE) \simeq I_{\hat{Z}}^i \otimes S^iE \simeq S^i(I_Z \otimes E).$$

Therefore we have an isomorphism $\psi : \mathbb{P}(\mu^*E) \rightarrow \mathbb{P}(F)$ such that $\psi^*O_{\mathbb{P}(F)}(1) = \hat{\pi}^*(O_X(1)) \otimes O_{\mathbb{P}(\mu^*E)}(1)$. Thus b) and c) are equivalent. \qed

3.14. Definition. Let $X$ be a normal projective variety, and let $F$ be a reflexive sheaf on $X$. Let $\pi : P \rightarrow X$ be a desingularization of the normalization of the unique component $\mathbb{P}(F)$ of $\mathbb{P}(F)$ dominating $X$ such that the preimage of the singular locus of $X$ and of the singular locus of the sheaf $F$ is a divisor in $P$. Let $\zeta$ be a tautological class on $P$, [HP19, Defn. 2.2]. Then we define

$$\kappa(X, F) = \kappa(P, \zeta).$$

By construction $\kappa(P, \zeta) \geq 0$ if and only if $H^0(X, S^m(\zeta)) \neq 0$ for some $m \in \mathbb{N}$.

Remark. If $\pi : P \rightarrow X$ denotes the projection, then

$$H^0(P, O_P(m\zeta)) = H^0(X, S^m(\zeta))$$

for all positive numbers $m$, and therefore the definition is independent on the choices made.

We will use Definition 3.14 in Section 4 to introduce a generalised Kodaira dimension of $X$ (Definition 4.5).

The next result generalizes Lemma 3.3 for finite morphisms.

3.15. Lemma. Let $f : \tilde{X} \rightarrow X$ be a finite morphism of normal projective varieties. Let $F$ be a reflexive sheaf on $X$. Then the following holds:

- The reflexive pullback $f^![\zeta](F)$ is pseudoeffective if and only if $F$ is pseudoeffective.

- One has $\kappa(\tilde{X}, f^![\zeta](F)) = \kappa(X, F)$. 

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Proof. Let \( \pi : P \to X \) be the projective manifold from Definition 3.14 and \( \zeta \) a tautological class on \( P \). By the construction of \( \zeta \) (see [HP19, Defn.2.2]), we have
\[
\pi_*(\mathcal{O}_P(m\zeta)) \simeq S^{|m|}(\mathcal{F})
\]
for all positive integers \( m \). We introduce the fibre product
\[
P := P \times_X \tilde{X}.
\]
Let \( \sigma : \tilde{P} \to P \) be a desingularization, which is an isomorphism outside the singular locus of \( P \) and set \( \tilde{\zeta} := \sigma^*p_1^!(\zeta) \), where \( p_1 : \tilde{P} \to P \) denotes the projection. Let \( \tilde{\pi} : \tilde{P} \to \tilde{X} \) be the projection and \( \tilde{\sigma} := \tilde{\pi} \circ \sigma \).

Claim: There exists a \( \tilde{\sigma} \)-exceptional divisor \( \tilde{D} \) on \( \tilde{P} \), such that
\[
\tilde{\pi}_*(\mathcal{O}_{\tilde{P}}(m(\tilde{\zeta} + \tilde{D}))) \simeq S^{|m|}f^*(\mathcal{F})
\]
for all \( m \in \mathbb{N} \).

Proof of the claim: By [Nak04, III.5.10.3] there exists \( \tilde{\sigma} \)-exceptional divisor \( \tilde{D} \) on \( \tilde{P} \) such that
\[
\tilde{\pi}_*(\mathcal{O}_{\tilde{P}}(m(\tilde{\zeta} + \tilde{D})))
\]
is reflexive for all \( m \in \mathbb{N} \).

Let \( X_0 \subseteq X \) be the locus where \( X \) is smooth and \( \mathcal{F} \) is locally free. Since \( X \) is normal and \( \mathcal{F} \) is reflexive, the complement of \( X_0 \) has codimension at least two. Set now \( \tilde{X}_0 := f^{-1}(X_0) \). Since \( X_0 \) is smooth, the restriction \( f|_{\tilde{X}_0} \) is flat. Thus by flat base change [Har77, III, Prop.9.3] we have an isomorphism
\[
\tilde{\pi}_*(p_1^*(\mathcal{O}_P(m\zeta))) \simeq f^* S^{|m|}(\mathcal{F})
\]
over \( \tilde{X}_0 \). Since \( \mathcal{F} \) is locally free on \( X_0 \), we have
\[
f^* S^{|m|}(\mathcal{F}) \simeq f^{|m|}(\mathcal{F})
\]
over \( \tilde{X}_0 \). Thus \( \tilde{\pi}_*(\mathcal{O}_{\tilde{P}}(m(\tilde{\zeta} + \tilde{D}))) \) and \( f^{|m|}(\mathcal{F}) \) are isomorphic over \( \tilde{X}_0 \). Since they are both reflexive, they are isomorphic: indeed the complement of \( \tilde{X}_0 \) has codimension at least two, since \( f \) is finite. For the same reason we have
\[
f^{|m|}(\mathcal{F}) \simeq S^{|m|}f^*(\mathcal{F}),
\]
which shows the claim.

Proof of the first statement: If \( \mathcal{F} \) is pseudoeffective, fix an ample Cartier divisor \( AH \) on \( X \). Since \( f \) is finite, a section of \( S^{|i|}\mathcal{F} \otimes \mathcal{O}_X(jH) \) pulls back to a section \( S^{|i|}(f^{|i|}(\mathcal{F})) \otimes \mathcal{O}_X(jf^*H) \). Thus \( f^{|i|}(\mathcal{F}) \) is pseudoeffective, by Definition 3.35.

Assume now that \( f^{|i|}(\mathcal{F}) \) is pseudoeffective, so the divisor class \( \tilde{\zeta} + \tilde{D} \) is pseudoeffective by [HP19, Lemma 2.3]. Let now \( \mu : \tilde{P} \to P' \) and \( p'_1 : P' \to P \) be the Stein factorisation of the generically finite morphism \( p_1 \circ \sigma \), i.e. \( \mu \) is birational onto the normal variety \( P' \) and \( p'_1 \) is finite. Then \( \tilde{\zeta} = (p_1 \circ \sigma)^*(\zeta) = (p'_1 \circ \mu)^*(\zeta) \), so
\[
\mu_*((\tilde{\zeta} + \tilde{D})) = (p'_1)^*(\zeta) + \mu_*(\tilde{D})
\]
is a pseudoeffective Weil divisor class (cf. [Nak04, II, Defn.5.5]). Setting \( D = (p'_1)^*(\mu_*(\tilde{D})) \), we have an inclusion of Weil divisors \( \mu_*(\tilde{D}) \subset (p'_1)^*(\zeta + D) \). Thus we have an inclusion of Weil divisor classes
\[
(p'_1)^*(\zeta) + \mu_*(\tilde{D}) \subset (p'_1)^*(\zeta + D),
\]
which shows that \((p'_1)^*(\zeta + D)\) is pseudoeffective. Since \(\zeta + D\) is Cartier, this shows that \(\zeta + D\) is pseudoeffective. Since \(\tilde{D}\) is \(\pi\)-exceptional, the effective divisor \(D\) is \(\pi\)-exceptional. By (2) we thus have
\[
\pi_*(\mathcal{O}_F(m(\zeta + D)))^{**} = \pi_*(\mathcal{O}_F(m\zeta))^{**} \simeq S^{[m]}(\mathcal{F})
\]
for all \(m \in \mathbb{N}\). This shows that
\[
S^{[m]}(\mathcal{F}) \simeq \pi_*(\mathcal{O}_F(m\zeta)) \hookrightarrow \pi_*(\mathcal{O}_F(m(\zeta + D))) \hookrightarrow \pi_*(\mathcal{O}_F(m(\zeta + D)))^{**} \simeq S^{[m]}(\mathcal{F})
\]
is a chain of isomorphisms for all \(m \in \mathbb{N}\). Hence the reflexive sheaf \(\mathcal{F}\) is pseudoeffective by [HP19] Lemma 2.3.

**Proof of the second statement:** As for the first statement, the inequality \(\kappa(\tilde{X}, F^*[\mathcal{F}]) \geq \kappa(X, F)\) is immediate. Let us show the other inequality: by the claim we know that \(\hat{\zeta} + \tilde{D}\) is a tautological class on \(\tilde{P}\). Thus by assumption, one has
\[
\kappa(\tilde{P}, \zeta + \tilde{D}) = \kappa((\tilde{X}, f^*[\mathcal{F}]) \geq 0.
\]
Hence \(\kappa(P', (p'_1)^*(\zeta + \mu_*(\tilde{D})) = \kappa((\tilde{X}, f^*[\mathcal{F}]) \geq 0\), and,
\[
\kappa(P', (p'_1)^*(\zeta + D)) \geq \kappa(P', (p'_1)^*(\zeta + \mu_*(\tilde{D})),
\]
where we use again the inclusion \((p'_1)^*(\zeta + \mu_*(\tilde{D})) \subset (p'_1)^*(\zeta + D)\). Since
\[
\kappa(P', (p'_1)^*(\zeta + D)) = \kappa(P, \zeta + D),
\]
by [Uen75] Thm 5.13, the statement follows. \(\square\)

In Section 4 this will be applied to sheaves of reflexive differentials, Corollary prop- cover2.

3.D. **Pseudoeffective sheaves on fibered surfaces.** The results of this section will be relevant to the study of elliptic surfaces. Let us recall the Zariski decomposition on surfaces [Bau09 Thm.], [Laz04 Thm.2.3.19]:

- Let \(D\) be an effective \(\mathbb{Q}\)-divisor on a smooth surface. Then there exist uniquely determined effective \(\mathbb{Q}\)-divisor \(P\) and \(N\) with
\[
D = P + N
\]
such that \(P\) is nef, the divisor \(N = \sum a_jN_j\) is zero or has negative definite intersection matrix and \(P \cdot N_j = 0\) for all \(j\).
- The same statement holds if \(D\) is pseudoeffective, except that in this case \(P\) is not necessarily effective.

The following lemma is well-known to experts, we give the details in order to prepare the proof of its singular version in Lemma 3.14

3.16. **Lemma.** Let \(X\) be a smooth projective surface, and let \(f : X \to B\) be a fibration over a smooth curve \(B\). Let \(L\) be a pseudoeffective line bundle on \(X\) such that \(L_F \simeq \mathcal{O}_F\) for the general fiber \(F\) of \(f\). Let \(D\) be a Cartier divisor such that \(L \simeq \mathcal{O}_X(D)\), and let
\[
D = P + N
\]
its Zariski decomposition. Then the following holds:

a) Up to taking multiples, one has \(\mathcal{O}_X(P) \simeq f^*M\) with \(M\) a nef line bundle.

b) If \(P \neq 0\), one has \(\kappa(L) \geq 1\).
c) If $P \equiv 0$, there exists $m \in \mathbb{N}$ and a numerically trivial line bundle $M$ on $B$ such that $\kappa(X, L^{\otimes m} \otimes f^* M^*) = 0$.

Proof. Note that the statements are invariant under taking multiples. Up to replacing $L$ by some multiple, we can assume that $P$ and $N$ are Cartier divisors. Since $N$ is effective and

$$\mathcal{O}_F \simeq \mathcal{O}_F(D) \simeq \mathcal{O}_F(P) \otimes \mathcal{O}_F(N)$$

we see that $N$ has support in the fibres of $f$ and $\mathcal{O}_F(N) \simeq \mathcal{O}_F(P) \simeq \mathcal{O}_F$. Thus the direct image sheaf $f_* (\mathcal{O}_X(P))$ is locally free of rank one, and we have

$$\mathcal{O}_X(P) = f^*(f_* (\mathcal{O}_X(P))) \otimes \mathcal{O}_X(E)$$

where $E$ is an effective divisor, supported in fibers of $f$. Since $P^2 \geq 0$ and $E^2 \leq 0$ \cite[Lemma 8.2]{BHPvD}, it follows $E^2 = 0$. Hence by (ibid) there exists a number $k$ such that $kE = f^*(H)$ with some effective divisor $H$. Thus, again up to replacing $L$ by some multiple, we have $\mathcal{O}_X(P) = f^* M$ for some line bundle $M$ on $B$.

If $M \not\equiv 0$, it is ample on $B$, so

$$1 = \kappa(M) = \kappa(P) = \kappa(L).$$

If $P \equiv 0$, then $M \equiv 0$, so $\kappa(L \otimes f^* M^*) = \kappa(N) = 0$. $\square$

3.17. Lemma. Let $\hat{X}$ be an irreducible reduced projective surface, and let $\hat{f} : \hat{X} \to B$ be a fibration over a smooth curve $B$ such that the general fibre $F$ is smooth. Let $\hat{L}$ be a pseudoeffective line bundle on $\hat{X}$ such that $\hat{L}|_F \simeq \mathcal{O}_F$.

a) There exists $m \in \mathbb{N}$ and a numerically trivial line bundle $M$ on $B$ such that $h^0(\hat{X}, \hat{L}^{\otimes m} \otimes \hat{f}^* M) \geq 0$.

b) If $h^0(\hat{X}, \hat{L}^{\otimes m} \otimes \hat{f}^* M) > 1$ for some numerically trivial line bundle $M$, then $h^0(X, \hat{L}^{\otimes k}) > 1$ for some $k \in \mathbb{N}$.

3.18. Remark. We will frequently apply the lemma in the case where $B$ is a rational curve. In this case one obtains $h^0(\hat{X}, \hat{L}^{\otimes m}) > 0$ for some $m \in \mathbb{N}$.

Since the dimension of the space of global sections is not necessarily invariant under normalisation, the statement requires some work:

Proof. Note that the statement is invariant under taking multiples. Let $\mu : X \to \hat{X}$ be the composition of normalisation and desingularisation, set $f := \hat{f} \circ \mu$. Then $\mu^* \hat{L} =: \mathcal{O}_X(D)$ is pseudoeffective, and we denote by

$$D = P + N$$

the Zariski decomposition. By Lemma 3.16 a) we have, up to taking multiples, that $\mathcal{O}_X(P) \simeq f^* M$ for some line bundle on $M$. Thus we see that $\hat{L} \simeq f^* M \otimes \hat{N}$, where $\hat{N}$ is a Cartier divisor on $\hat{X}$ such that $N \equiv \mu^* \hat{N}$. By Lemma 3.16 b),c) it is sufficient to show that we can choose $\hat{N}$ to be effective.

Proof of the claim. Since

$$\hat{N} \simeq \hat{L} \otimes (\hat{f})^* M^*,$$
we have $\mathcal{O}_F(\hat{N}) \simeq \mathcal{O}_F$. Thus $(\hat{f})_*\mathcal{O}_\hat{X}(\hat{N})$ is locally free of rank one, hence for some $m \gg 0$, we have

$$H^0(\hat{X}, \mathcal{O}_\hat{X}(mF) \otimes \mathcal{O}_X(\hat{N})) \simeq H^0(B, \mathcal{O}_B(m) \otimes (\hat{f})_*\mathcal{O}_X(\hat{N})) \neq 0.$$  

Thus we can fix an effective Cartier divisor $E$ on $\hat{X}$ such that $\mathcal{O}_\hat{X}(E) \simeq \mathcal{O}_X(mF) \otimes \mathcal{O}_\hat{X}(\hat{N})$. We can decompose

$$E = f^*E_B + R$$  

where $E_B$ is an effective $\mathbb{Q}$-divisor on $B$ and $R$ is an effective divisor such that for every connected component $R' \subset R$ we have a strict set-theoretical inclusion $R' \subset \hat{f}^{-1}(\hat{f}(R'))$. Up to taking multiples and replacing $E_B$ by a linearly equivalent divisor, we can also suppose that $\text{supp}(f^*E_B) \subset X_{\text{nons}}$. Now observe that

$$\mu^*E = \mu^*f^*E_B + \mu^*R$$  

is a Zariski decomposition. Since

$$\mu^*E \equiv mF + N.$$  

is also a Zariski decomposition and the negative part of the Zariski decomposition is unique in the numerical equivalence class, we finally obtain $N = \mu^*R$. \hfill \Box

3.19. Corollary. Let $X$ be a smooth projective surface, and let $f : X \to B$ a fibration over a smooth rational curve $B$. Let $Z \subset X$ be a local complete intersection of codimension 2. Let $L$ be a line bundle on $X$ such that $L_F \simeq \mathcal{O}_F$ for the general fiber $F$ of $f$. Then $I_Z \otimes L$ is strongly pseudoeffective if and only if $\kappa(X, I_Z \otimes L) \geq 0$, i.e., if there exists a positive integer $m$ such that

$$H^0(X, I_Z \otimes L^m) \neq 0.$$  

Proof. One direction being obvious, so assume that $I_Z \otimes L$ is pseudoeffective. Let

$$\mu : \hat{X} \to X$$  

be the blow-up of $X$ along $Z$ and denote by $E$ the exceptional divisor. Since $Z$ does not surject onto $B$, the general fibre of $f \circ \mu$ is smooth. By Corollary 3.13 the the line bundle $\mathcal{O}_\hat{X}(1) \otimes \mu^*(L) \simeq \mathcal{O}_\hat{X}(-E) \otimes \mu^*(L)$ is pseudoeffective. By Remark 3.18 we see that there exists some $m \in \mathbb{N}$ such that

$$H^0(\hat{X}, \mathcal{O}_\hat{X}(-mE) \otimes \mu^*(L \otimes m)) \neq 0.$$  

Since $Z$ is a local complete intersection, we have $\mu_*(\mathcal{O}_\hat{X}(-mE)) = I_Z^m$. Thus we conclude by the projection formula. \hfill \Box

4. PSEUDEFFECTIVE COTANGENT SHEAVES AND THE NONVANISHING CONJECTURE

In this section we gather some basic facts on the behaviour of pseudoeffective cotangent sheaves under birational maps and finite covers. We also establish a relation with the MRC fibration of the variety.

4.1. Proposition. Let $\mu : \hat{X} \to X$ be a birational morphism of normal projective varieties.

a) If $\Omega^{[q]}_{\hat{X}}$ is pseudoeffective, so does $\Omega^{[q]}_X$.

b) Suppose that $X$ is smooth. Then the converse also holds.
Proof. a) We choose ample divisors $\hat{H}$ on $\hat{X}$ and $H$ on $X$ such that

$$\hat{H} + E = \mu^*(H)$$

with $E$ an effective divisor supported on the exceptional locus of $\mu$. By assumption, for all $c > 0$, there are numbers $i$ and $j$ with $i > cj$ such that

$$H^0(\hat{X}, S^{|i|}\Omega^{|q|}_X \otimes O_\hat{X}(j\hat{H})) \neq 0.$$  

In particular,

$$0 \neq H^0(\hat{X}, S^{|i|}\Omega^{|q|}_X \otimes O_\hat{X}(j\mu^*H)) = H^0(X, \mu_*(S^{|i|}\Omega^{|q|}_X \otimes O_X(jH))).$$

Since $\mu_*(S^{|i|}\Omega^{|q|}_X) \subset S^{|i|}\Omega^{|q|}_X$, we conclude.

b) Suppose that $X$ is smooth. By a), we may assume $\hat{X}$ to be smooth as well. Then all the involved sheaves are locally free, in particular

$$S^i\mu^*(\Omega^{|q|}_X) = \mu^*(S^i\Omega^{|q|}_X) \subset S^i\Omega^{|q|}_X,$$

and the claim follows.

4.2. Example. Assertion 4.1,b) fails in general if $X$ is singular, even if $X$ has only canonical singularities. In fact, the paper [GKP16] constructs a normal projective surface $X$ with the following properties.

- $X$ has only ADE singularities;
- the minimal desingularization $\hat{X}$ is rationally connected;
- $H^0(X, S^{|2|}\Omega^{|1|}_X) \neq 0$.

Thus $\Omega^{|1|}_X$ is not pseudoeffective in contrast to $\Omega^{|1|}_X$.

Another example is a K3 surface of Kummer type, see Example 2.3.

4.3. Corollary. Let $X$ be a normal projective variety with klt singularities, and let $X \rightarrow X'$ be a composition of divisorial contractions and flips. If $\Omega^{|q|}_X$ is pseudoeffective, so does $\Omega^{|q|}_X$.

Proof. By Proposition 4.1,a) it suffices to treat the case of a flip $\mu : X \rightarrow X'$. Since a flip is an isomorphism in codimension two, one has

$$H^0(X, S^{|i|}\Omega^{|q|}_X \otimes O_X(jH)) \simeq H^0(X', S^{|i|}\Omega^{|q|}_X \otimes O_{X'}(j\mu_*H))$$

for all $i, j \in \mathbb{N}$. Thus the condition in Definition 3.3 holds for a big $\mathbb{Q}$-Cartier divisor, which is sufficient (see [HLS20, Lemma 2.2]).

4.4. Proposition. Let $f : \tilde{X} \rightarrow X$ be a finite surjective morphism of normal projective varieties. If $\Omega^{|q|}_X$ is pseudoeffective, so does $\Omega^{|q|}_{\tilde{X}}$. If $f$ is quasi-étale, the converse also holds.

Proof. Over the smooth locus of $X$ we have a injective morphism

$$f^*(S^{|i|}\Omega^{|q|}_{X_{\text{nons}}}) \rightarrow S^{|i|}\Omega^{|q|}_{\tilde{X}}.$$  

Since the complement of $f^{-1}(X_{\text{nons}})$ has codimension at least two, the morphism extends to

$$f^{|i|} (S^{|i|}\Omega^{|q|}_{X}) \rightarrow S^{|i|}(\Omega^{|q|}_{\tilde{X}}),$$

which gives the first claim.
Assume now that $f$ is quasi-étale and that $\Omega^{[q]}_X$ is pseudoeffective. Then $f^{[*]}(\Omega^{[q]}_X) \simeq \Omega^{[q]}_{\tilde{X}}$ is pseudoeffective. Now we conclude by Lemma 3.15.

At this point we introduce generalised Kodaira dimension:

4.5. Definition. Let $X$ be a normal projective variety with klt singularities and $1 \leq q \leq n$. Then we define

$$\kappa_q(X) = \kappa(X, \Omega^{[q]}_X).$$

In case $q = \dim X$, we have of course $\kappa_q(X) = \kappa(X)$.

As a corollary to Lemma 3.15 we obtain a generalisation of [An18, Prop. 2.2]:

4.6. Proposition. Let $\tilde{X} \to X$ be a quasi-étale morphism of projective varieties with klt singularities. Then

$$\kappa_q(\tilde{X}) = \kappa_q(X)$$

for all $1 \leq q \leq \dim X$.

Proof. This follows from Lemma 3.15 since for a quasi-étale morphism

$$f^{[*]}(\Omega^{[q]}_X) = \Omega^{[q]}_{\tilde{X}}.$$

4.7. Remark. Let $\mu : \tilde{X} \to X$ be a birational morphism of normal projective varieties with klt singularities. Then $\kappa_q(\tilde{X}) \leq \kappa_q(X)$ with equality if $X$ is smooth. The same inequality holds if $\mu : \tilde{X} \to X$ is a composition of divisorial contractions and flips.

Although Conjecture [11] can be formulated for any $p$, we are mainly interested in the case $p = 1$. We next confirm the conjecture for $p = 1$ in case $K_X \equiv 0$.

4.8. Proposition. Let $X$ be a normal projective variety with klt singularities such that $K_X \equiv 0$. Assume that $X$ is smooth in codimension two, e.g., $X$ has terminal singularities. Then the following are equivalent:

a) $\Omega^{[1]}_X$ is pseudoeffective ;

b) we have $\overline{q}(X) > 0$, i.e., there exists a quasi-étale cover $\tilde{X} \to X$ such that $H^0(\tilde{X}, \Omega^{[1]}_{\tilde{X}}) \neq 0$ ;

c) we have $H^0(X, S^{[m]}\Omega^{[1]}_X)) \neq 0$ for some positive integer $m$, i.e., $\kappa_1(X) \geq 0$.

Proof. By [HP19 Thm.1.6] we know that 1) implies 2). By Proposition 4.6 we know that 2) implies 3) which obviously implies 1).

We will now discuss the relation between the pseudoeffectivity of $\Omega^{[q]}_X$ and the MRC fibration. First, the rational connectedness criterion given in [CDP15] can be stated as follows.

4.9. Theorem. Let $X$ be a projective manifold of dimension $n$. Then $X$ is rationally connected if and only if for all $1 \leq q \leq n$ the vector bundle $\Omega^{[q]}_X$ is not pseudoeffective.
Proof. If \( X \) is rationally connected, it is dominated by very free rational curves \[\text{[Kol96, IV.3.9].} \] It is then straightforward to verify the vanishing condition in Definition \ref{vanishing-condition}.

Assume that \( \Omega^q_X \) is not pseudoeffective for all \( 1 \leq q \leq n \). Let \( \mathcal{F} \subset \Omega^q_X \) be an invertible subsheaf, then \( \mathcal{F} \) is not pseudoeffective (see Example \ref{example}). Hence by \[\text{[CDP15, Thm.1.1].} \] \( X \) is rationally connected. \( \Box \)

Theorem \ref{main-theorem} can be generalised as follows.

4.10. Theorem. Let \( X \) be a projective variety of dimension \( n \). Fix some \( r \in \{1, \ldots, n\} \) and assume that \( \Omega^q_X \) is not pseudoeffective for all \( r \leq q \leq n \).

Then \( X \) is uniruled, and the base \( Z \) of the MRC fibration satisfies \( \dim Z \leq r - 1 \).

Proof. By Proposition \ref{smoothness} we can assume without loss of generality that \( X \) is smooth. Since \( K_X = \Omega^n_X \) is not pseudoeffective, the manifold \( X \) is uniruled by \[\text{[BDPP13].} \] Hence we consider the MRC fibration \( f : X \rightarrow Z \).

Up to replacing \( Z \) by a resolution and \( X \) by a blow-up, we may assume, by Proposition \ref{smoothness}, that \( Z \) is smooth and \( f \) is a morphism.

Arguing by contradiction we suppose that \( d := \dim Z \geq r \). By \[\text{[GHS03],} \] the variety \( Z \) is not uniruled, hence \( K_Z \) is pseudoeffective. Choose \( \mathcal{H}_Z \) ample on \( Z \) and set \( \mathcal{H}_X = f^*(\mathcal{H}_Z) \). Since \( K_Z \) is pseudoeffective, for all \( c > 0 \) there exist integers \( i \) and \( j \) with \( i > cj \) such that \( H^0(Z, \mathcal{O}_Z(iK_Z) \otimes \mathcal{O}_Z(j\mathcal{H}_Z)) \neq 0 \).

Since \( 0 \neq f^*(\Omega^d_Z) \subset \Omega^d_X \), it follows that \( H^0(X, S^i \Omega^d_X \otimes \mathcal{O}_X(j\mathcal{H}_X)) \neq 0 \).

Thus \( \Omega^{d}_X \) is pseudoeffective and \( d \geq r \), a contradiction to our assumption. \( \Box \)

If \( X \) is smooth, the converse to Theorem \ref{main-theorem} is also true:

4.11. Proposition. Let \( X \) be a uniruled projective manifold of dimension \( n \), and let \( f : X \rightarrow Z \) be the MRC fibration. If \( d = \dim Z \), then for every \( \Omega^q_X \) is not pseudoeffective for all \( d + 1 \leq q \leq n \).

Proof. Let \( F \) be a general fiber of \( f \). Then \( F \) is rationally connected of dimension \( \dim F = n - d \). Let \( C \subset F \) be a general very free rational curve \[\text{[Kol96, IV.3.9],} \] so \( T_F|_C \) is ample. Then \( T_X|_C \simeq \mathcal{O}^{\oplus d}_C \oplus T_F|_C \).

Thus for every \( q \geq d + 1 \), the exterior power \( \wedge^q T_X|_C \) is ample. Hence \( \Omega^{q}_X \) is not pseudoeffective. \( \Box \)

For smooth varieties, the MRC-fibration should allow to reduce Conjecture \ref{main-conjecture} to non-uniruled varieties:

4.12. Conjecture. Let \( X \) be a uniruled projective manifold, and let \( f : X \rightarrow Z \) be the MRC fibration to the projective manifold \( Z \). Let \( 1 \leq q \leq n \). Then \( \Omega^q_X \) is pseudoeffective if and only if \( \Omega^q_Z \) is pseudoeffective.
Note that, by Proposition 4.11, we may assume $f$ holomorphic. Then one direction is clear: if $\Omega^2_X$ is pseudoeffective, then so does $f^*(\Omega^2_Y)$ by Lemma 3.4. Hence $\Omega^2_X$ is pseudoeffective, see Example 3.9. Vice versa assume that $\Omega^2_X$ is pseudoeffective. Applying [BCHM10, Cor.1.3.2] $f$ factors into a sequence of divisorial contractions and flips, ending with a Mori fiber space $f' : X' \to Z$ of relative Picard number one. By Corollary 4.13 we may therefore assume that $f$ is a Mori contraction, but now $X$ may have terminal singularities instead of being smooth.

4.13. Proposition. Conjecture 4.12 is true in dimension three.

Proof. As just noticed it suffices to treat Mori contractions $f : X \to Z$ where $X$ has terminal singularities. Since $Z$ is not uniruled, we may assume that $\dim Z = 2$; otherwise $Z$ is a curve of genus $g \geq 1$ and there is nothing to prove. Further, since $K_Z$ is pseudoeffective, only the case $g = 1$ needs to be treated. Now $f$ is a generically a conic bundle [AW97, 4.1]. More precisely, the singular locus of $X$ being finite, there is a finite set $B = f(Sing(X))$ in $Z$ such that, setting $Z_0 = Z \setminus B$ and $X_0 = f^{-1}(Z_0)$, the map $f_0 : X_0 \to Z_0$ is a conic bundle with only finitely many non-reduced fibers. Furthermore, $f^{-1}(B)$ is one-dimensional.

Since $-K_X$ is relatively ample and relatively globally generated on $X_0$, we can choose a very ample Cartier divisor $H$ on $Z$ such that $-K_{X/Z} + f^*H =: A$ is ample and satisfies $H^0(X, \mathcal{O}_X(A)) \neq 0$. In particular there is an injection

$$\omega_{X/Z} \to \mathcal{O}_X(f^*H).$$

We claim that for every $c > 1$ there exist positive integers $k, j$ such that $k \geq cj$ such that

$$H^0(X_0, f^*\omega^i X \otimes \mathcal{O}_X(j(A + f^*H))) \neq 0.$$

Since $f^{-1}(B)$ has codimension at least two, this shows that $f^*\Omega_Z$ is pseudoeffective. Thus $\Omega_Z$ is pseudoeffective by Lemma 3.4.

Proof of the claim. Since $\Omega_X$ is pseudoeffective, there exist positive integers $i, j$ such that $i \geq 2cj$ such that

$$H^0(X_0, S^i \Omega_X \otimes \mathcal{O}_X(jA)) \neq 0.$$

We consider the canonical exact sequence

$$0 \to f^*(\Omega_{Z_0}) \xrightarrow{df} \Omega_{X_0} \to \Omega_{X_0/Z_0} \to 0.\tag{4}$$

Since $X_0 \to Z_0$ is a conic bundle, we know that $df$ cannot vanish along a divisor $D$. Thus $\Omega_{X/Z}$ is torsion free and the singular locus of $\Omega_{X/Z}$ is at most one-dimensional. Thus we get

$$H^0(X_0, f^*\omega^k X \otimes \mathcal{O}_X(jA)) \neq 0.$$

for some $k \in \{0, \ldots, i\}$. Since $\omega^k X \otimes \mathcal{O}_X(jA)$ has negative degree on the fibres of $f$ if $i-k > j$ we see that $k \in \{i-j, \ldots\}$. Note that since $i \geq 2cj$ and $c > 1$ this implies that $k \geq j$. Using the morphism (3) obtain

$$H^0(X_0, f^*\omega^k Z_0 \otimes \mathcal{O}_X(jA + (i-k)f^*H)) \neq 0.$$

Since $i-k \leq j$ we finally obtain the claim. \qed
Remark. The key point of the proof above is that the morphism \( df \) does not vanish along a divisor \( D_0 \). In higher dimension, since the total space of the Mori fibre space is not necessarily smooth, this might very well happen. Then these type of arguments only show that \( f^*(\Omega^q(D)) \) is pseudoeffective where \( D \) has support inside the support of \( D_0 \). At least in dimension three we also see that \( H^0(X, S^m\Omega_X) = H^0(Z, S^m\Omega_Z) \)

where \( X \to Z \) is the MRC fibration of a uniruled smooth threefold \( X \) and \( Z \) is smooth.

4.14. Corollary. Let \( X \) be a smooth projective surface such that \( \Omega_X \) is pseudoeffective. If \( \kappa(X) \leq 0 \), then Conjecture 1.1 holds.

Proof. If \( \kappa(X) = -\infty \), the surface \( X \) is uniruled. Since \( \Omega_X \) is pseudoeffective, by Proposition 4.11 the base of the MRC fibration is a curve of genus at least one. Thus we have \( q(X) > 0 \).

If \( \kappa(X) = 0 \), let \( X_{\text{min}} \) be the minimal model of \( X \). Then \( \Omega_{X_{\text{min}}} \) is pseudoeffective by Proposition 4.11. Thus by Proposition 4.8 one has \( H^0(X_{\text{min}}, S^m\Omega^1_{X_{\text{min}}}) \neq 0 \) for some positive integer \( m \). Since \( X_{\text{min}} \) is smooth, we have an isomorphism

\[
H^0(X, S^m\Omega^1_X) \cong H^0(X_{\text{min}}, S^m\Omega^1_{X_{\text{min}}}).
\]

\( \square \)

In the next two sections we will deal with surfaces \( X \) of Kodaira dimension \( \kappa(X) = 1 \).

5. Elliptic surfaces: general set-up and the non-isotrivial case

We are starting here to study elliptic fibrations \( f : X \to B \) with \( \kappa(X) = 1 \) towards Conjecture 1.1. If \( f \) is almost smooth, i.e., the only singular fibers are multiples of elliptic curves, then \( c_2(X) = e(X) = 0 \) \cite[III, Prop.11.4]{BHPV}. Thus by Noether’s formula \( \chi(X, \mathcal{O}_X) \leq 0 \), and therefore \( q(X) > 0 \), so there is nothing to prove. We first fix notations.

5.A. Elliptic fibrations: the setup.

5.1. Setup. Let \( X \) be a smooth projective surface, and let \( f : X \to B \) be an elliptic fibration onto a smooth curve \( B \). We set

\[
D = \sum_{b \in B} f^*b - (f^*b)_\text{red},
\]

so \( D \) is an effective divisor having support exactly on the irreducible components of a fibre that are not reduced. The exact sequence

\[
0 \to f^*\Omega_B \to \Omega_X \to \Omega_{X/B} \to 0
\]

induces an exact sequence

\[
0 \to f^*\Omega_B(D) \to \Omega_X \to \mathcal{I}_Z \otimes \omega_{X/B}(-D) \to 0,
\]

where \( Z \) is a local complete intersection scheme of codimension two whose support coincides with the singular points of the reduction \( (f^*b)_\text{red} \) of the fibres \cite[Prop.3.1(iii)]{Ser96}.
We denote by $\pi : \mathbb{P}(\Omega_X) \to X$ the projectivisation, and by $\zeta : \mathbb{P}(\Omega_X)$ the tautological class.

We set

\[ Y := \mathbb{P}(I_Z \otimes \omega_{X/B}(-D)) \subset \mathbb{P}(\Omega_X). \]

Since $I_Z$ is a local complete intersection of codimension two, the projectivisation coincides with the blow-up of the ideal sheaf $I_Z$ (see Setup 3.10). In particular $Y$ is a prime divisor in $\mathbb{P}(\Omega_X)$ and

\[ [Y] = \zeta - \pi^*c_1(f^*\Omega_B(D)). \]

Denote by $K \subset B$ the finite set of points such that the fibre $f^*b$ is not multiple and not reduced. The divisor $D$ can be decomposed as

\[ D = \sum_{i=1}^s (m_i - 1)F_i + \sum_{b \in K} D_{0,b}, \]

where the $F_i$ are the reductions of multiple $f$-fibres and

\[ D_0 := \sum_{b \in K} D_{0,b} \]

is simply the remainder, i.e. the part of $D$ coming from non-multiple, non-reduced fibres. It follows from Kodaira’s classification [BHPVdV04, V, Sect. 7, Table 3] that the support of $D_0$ does not contain any fibre, so the intersection matrix of $D_0$ is negative definite by Zariski’s lemma [BHPVdV04, III, Lemma 8.2]. It is now straightforward to check that (8) is the Zariski decomposition of $D$ (see Subsection 3.1) with $P = \sum_{i=1}^s (m_i - 1)F_i$.

If $f$ is relatively minimal, the canonical bundle formula [BHPVdV04, V, Thm.12.1, Prop.12.2] holds:

\[ \omega_X \simeq f^*(\omega_B \otimes (R^1f_*\mathcal{O}_X)^*) \otimes \mathcal{O}_X(\sum_{i=1}^s (m_i - 1)F_i), \]

and $\deg(R^1f_*\mathcal{O}_X)^* = \chi(X, \mathcal{O}_X)$.

Finally assume that $f$ is relatively minimal and $B = \mathbb{P}^1$. Then (9) implies that

\[ K_{X/B} - D \sim aF - D_0 \]

where $a = \chi(X, \mathcal{O}_X)$ and $F$ is a general fiber.

5.2. Proposition. In the situation of Setup 5.1, suppose that $f^*\Omega_B(D)$ is pseudoeffective. Then we have $\hat{q}(X) > 0$.

Proof. The statement is trivial if $g(B) \geq 1$, so assume $B \simeq \mathbb{P}^1$. We follow the philosophy of [Cam04, Sect.3.5]. By Remark 3.1 there exists $\kappa(f^*\Omega_B(D)) \geq 0$. Choose a positive integer $m$ and a non-zero section $s \in H^0(X, (f^*\Omega_B(D))^\otimes m)$. Let $E$ be the divisor defined by $s$. Note that $E$ is supported on fibers of $f$ and that

\[ E \sim m(D - 2F), \]

where $F$ is a general fiber of $f$. By the discussion in Setup 5.1 we know that the nef part of $D$ is represented by $\sum_{i=1}^s (m_i - 1)F_i \equiv \lambda F$ where the $F_i$ are the reductions of multiple $f$-fibres. Since

\[ D \sim \frac{1}{m} E + 2F, \]

we have $\hat{q}(X) > 0$.
is a decomposition in effective divisors and $2F$ is nef, we obtain $\lambda \geq 2$.

Introducing the $\mathbb{Q}$-divisor $\Delta = \sum_{i=1}^{s}(1 - \frac{1}{m_i})p_i$ with $p_i = f(F_i)$, we have $f^*\Delta = \sum_{i=1}^{s}(m_i - 1)F_i$, so

$$\sum_{i=1}^{s}(1 - \frac{1}{m_i}) = \lambda \geq 2.$$  

Thus $f$ has at least three multiple fibers. Then there is a ramified base change $\tilde{B} \to B$ inducing an étale map

$$\tilde{X} \to X$$

such that $\tilde{f} : \tilde{X} \to \tilde{B}$ has no multiple fibers and $g(\tilde{B}) \geq 1$, see e.g. [FK80] IV.9.12.

Thus $\tilde{q}(X) \neq 0$.

5.B. The non-isotrivial case. In the situation of Setup 5.1 assume furthermore that the elliptic fibration $f$ is not isotrivial. Let $F$ be a general fibre, then its Kodaira-Spencer class is not zero. Thus we have a non-split extension defined by the restriction of $\mathcal{D}$ to $F$

$$0 \to \mathcal{O}_F \to \Omega^1_X|_F \to \Omega^1_F \cong \mathcal{O}_F \to 0.$$ 

Denote by $\zeta_F \to \mathbb{P}(\Omega^1_X|_F)$ the tautological class. Then the $(1,1)$-class $\zeta_F$ is nef with $\zeta_F^2 = 0$, moreover it is represented by the current of integration over the curve $C$ defined by the quotient $\Omega^1_X|_F \to \mathcal{O}_F$. We recall the following result:

5.3. Lemma. [DPS94] Ex. 1.7 Let $h_F$ be any singular metric on $\zeta_F$ such that the curvature current $\Theta_{h_F}(\zeta_F)$ is positive. Then

$$\Theta_{h_F}(\zeta_F) = [C]$$

where $[C]$ is the current of integration over $C$.

5.4. Proposition. In the situation of Setup 5.1 assume that $f$ is not isotrivial. If $\zeta$ is pseudoeffective, the line bundle $f^*\Omega_B(D)$ is pseudoeffective.

Proof. By assumption there exists a singular metric $h$ on $\zeta$ such that $\Theta_{h}(\zeta) \geq 0$. Then we can consider the Siu decomposition

$$\Theta_{h}(\zeta) = \sum_k \nu(\Theta_{h}(\zeta), Y_k)[Y_k] + P$$

where $Y_k \subset \mathbb{P}(\Omega_X)$ are prime divisors and $P$ is a positive closed current such that the (countably many) irreducible components of $E_+(P)$ [Kou04] Sect.2.2.1 have codimension at least two. Let now $F$ be a very general $f$-fibre, so that $\mathbb{P}(\Omega^1_X|_F)$ does not contain any positive dimensional irreducible component of $E_+(P)$ then the restriction $h_F$ of the metric $h$ to $\mathbb{P}(\Omega^1_X|_F)$ is well-defined and yields a singular metric on $\zeta_F$. Moreover we have a decomposition

$$\Theta_{h_F}(\zeta_F) = (\Theta_{h}(\zeta))|_F = \sum_k \nu(\Theta_{h}(\zeta), Y_k)[Y_k \cap F] + P|_F.$$ 

By Lemma 5.3 we see that $P|_F = 0$, and there exists a unique $Y_k$ (say $Y_1$) such that $Y_1 \cap F = C$ and $\nu(\Theta_{h}(\zeta), Y_1) = 1$.

Since the intersection $Y_1 \cap F$ coincides with $C$ on all the general fibres and the quotient $\Omega^1_X|_F \to \Omega^1_F$ is the restriction of the quotient $\Omega^1_X \to I_Z \otimes \omega_X/B(-D)$ we obtain that $Y_1 = Y$. Thus we have

$$\Theta_{h}(\zeta) - [Y] = \sum_{k \geq 2} \nu(\Theta_{h}(\zeta), Y_k)[Y_k] + P$$
which is a positive current. By (7) this implies the statement. □

By Proposition 5.2 and 4.6, we conclude

5.5. Corollary. Let $X$ be a smooth projective surface, and let $f : X \to B$ be a non-isotrivial elliptic fibration. Then the following are equivalent:

a) $\Omega^1_X$ is pseudoeffective;

b) we have $\bar{q}(X) > 0$;

c) we have $H^0(X, S^m\Omega^1_X)) \neq 0$ for some positive integer $m$

In summary, Conjecture 1.1 holds for non-isotrivial elliptic fibrations.

6. Elliptic surfaces: the isotrivial case

In this section we treat isotrivial elliptic fibrations and prove parts (b) and (c) of Theorem 1.2.

6.A. Notation. In the situation of Setup 5.1 assume that $f : X \to B$

is relatively minimal and isotrivial. Denote by $E$ the elliptic curve such that a general $f$-fibre is isomorphic to $E$. By [Ser96, Sect.2] there exists a smooth curve $C$ and a finite group $G$ acting diagonally on the product $C \times E$ such that $X$ is birational to the quotient $(C \times E)/G$ and the fibration $f$ corresponds to the fibration $(C \times E)/G \to C/G \simeq B$ induced by projection on the first factor.

More precisely, denote by $q : C \times E \to (C \times E)/G$ the quotient map, and by $\bar{p}_C : (C \times E)/G \to C/G$ the map induced by the projection $p_C$. Denote by $\lambda : R \to (C \times E)/G$ the minimal resolution of singularities, then the exceptional divisors are Hirzebruch-Jung strings [Ser96, 2.0.2] and the singular fibres of $f_R := \bar{p}_C \circ \lambda : R \to B$

are described in [Ser96, Thm.2.1]. Following Serrano, we call $f_R$ the standard model of the isotrivial fibration $f$.

The standard isotrivial fibration $f_R$ factors through its relative minimal model. Since we assumed that $f$ is relatively minimal and the relative minimal model of an elliptic surface is unique, we have a birational morphism $\mu : R \to X$ such that $f_R = f \circ \mu$. We summarise the construction in a commutative diagram:

\begin{equation}
\begin{array}{ccc}
R & \downarrow \mu & C \times E \\
\| & \quad \lambda \quad & q \\
X & \downarrow \bar{p}_C & (C \times E)/G \\
\| & f & \downarrow \quad \\
B & \quad & C/G
\end{array}
\end{equation}

In general the birational map $\mu$ is not an isomorphism [Ser96 (2.4)], but as shown in Lemma 6.3, it is an isomorphism unless $E$ has complex multiplication.
6.B. Relatively minimal standard isotrivial fibrations.

6.1. Assumption. In the whole subsection we assume that

\[ f : X \to B \]

is a relatively minimal, isotrivial elliptic fibration that is standard, i.e. the birational map \( \mu \) (cf. Notation 6.A) is an isomorphism. For simplicity of the exposition we thus identify \( X = R \).

The following lemmas are well-known, we include them for lack of reference:

6.2. Lemma. Using the Notation 6.A, let \( x \in C \) be a point with non-trivial stabiliser \( G_x \) on \( C \). then \( G_x \) acts either by translation or as the involution \( z \mapsto -z \) on the elliptic curve \( x \times E \).

In particular all the singular fibers of \( f \) are either multiple elliptic curves or curves of type \( I^*_0 \) in Kodaira’s classification, see e.g., [BHPV dV04, V, Sect. 7, Table 3].

Proof. We follow the description of the singularities of \( (C \times E)/G \) in [Ser96, 2.0.2]: the group \( G \) acts on \( C \times E \), let \( b \in B \) a point with non-trivial stabiliser \( G_b \subset G \). The group \( G_b \) is cyclic [FK80, III.7.7, Cor.]. If it acts freely on \( E \), the quotient \( (B' \times E)/G \) is smooth near the fibre which is multiple elliptic. Assume now that there exists a point \( e \in E \) with non-trivial stabiliser \( G_{b,e} \subset G_b \). Then \( G_{b,e} \subset Aut(E,e) \). In view of [Har77, IV, Cor.4.7] this strongly limits the possibilities:

If \( G_{b,e} = \mathbb{Z}_2 \), the group acts as \( z \mapsto -z \) on \( E \) and we are done.

If \( G_{b,e} = \mathbb{Z}_4 \), the group acts as \( z \mapsto iz \) on the elliptic curve \( C/\mathbb{Z} \oplus i\mathbb{Z} \). This action fixes the origin and \( \frac{1}{2} + \frac{1}{2}i \), so we obtain two singularities of type \( A_{1,4} \) in the quotient. The points \( \frac{1}{2} \) and \( \frac{1}{2}i \) have non-trivial stabiliser, but are in the same orbit, so we obtain one singular point of type \( A_{1,2} \) in the quotient. The minimal resolution of \( R \to (C \times E)/G \) thus yields a fibre with a central component of multiplicity 4, two components of multiplicity 1 and self-intersection \(-4\) and one component of multiplicity 2 and self-intersection \(-2\). This fibre is the log-resolution of a fibre of Kodaira’s type \( III \), but it is not relatively minimal.

If finally \( G_{b,e} = \mathbb{Z}_6 \), then we see in a similar fashion that the action of \( \zeta \) on the elliptic curve \( C/\mathbb{Z} \oplus \zeta \mathbb{Z} \) (with \( \zeta = e^{i\pi/3} \)) leads to a log resolution of the configuration obtained for type \( IV \) and the action of \(-\zeta \) leads to a log resolution of the configuration obtained for type \( II \). The contradiction is then as before.

□

6.3. Lemma. Let \( f : X \to B \) be a minimal isotrivial elliptic fibration as in Notation 6.A. Assume that if \( x \in C \) is a point with non-trivial stabiliser \( G_x \) on \( C \), then \( G_x \) acts either by translation or as the involution \( z \mapsto -z \) on the elliptic curve \( x \times E \). Then \( f : X \to B \) is standard. In particular, if \( E \) does not have complex multiplication, then \( f : X \to B \) is standard.

Proof. Under our assumptions, the singular fibers of \( \overline{\pi}_C \circ \lambda = f \circ \mu \) are either multiple elliptic curves or of type \( I^*_0 \). But then \( \mu \) must be an isomorphism, due to Kodaira’s classification, applied to \( f \).

□

We have the following crucial
6.4. Lemma. Using the Notation \[6.3\] let \( Z \) denote the set of points in \( x \in C \) such that \( G_x \) acts as the involution \( z \mapsto -z \) on the elliptic curve \( C \times E \).

If \( f^*\Omega_B(D) \) is not pseudoeffective, then \( Z \) has at least \( 2g(C) - 1 \) elements.

Proof. Since \( f^*\Omega_B(D) \) is not pseudoeffective, we have \( B \cong \mathbb{P}^1 \).

The action of \( G \) on the curve \( C \) defines a Galois cover \( \psi : C \to C/G \cong \mathbb{P}^1 \) of degree \( d = |G| \). By the Hurwitz formula we have

\[
2g(C) - 2 = \deg K_C = -2d + \deg R,
\]

where \( R \) is the ramification divisor of \( \psi \). If \( x \in Z \), then it is a point with stabiliser \( G_x \cong \mathbb{Z}_2 \), hence \( \psi \) ramifies with order 2 in \( x \). Thus \( \deg R_x = 1 \) and

\[
\#(Z) = 2g(C) - 2 + 2d - \deg R_t,
\]

where \( R_t \subset R \) is the ramification corresponding to points \( y \in C \) such that \( G_y \cong \mathbb{Z}_{m_y} \) acts by translation on \( y \times E \) (see Lemma \[6.2\]). If \( y \in C \) is a point with this property, every point in its orbit \( G_y \) has the same property. Since the orbit has length \( \frac{d}{m_y} \), this induces a ramification divisor of order \( \frac{m_y - 1}{m_y} \).

Now note that the corresponding \( f \)- fibre over \( \bar{y} = \psi(y) \) is multiple elliptic with multiplicity \( m_y \), so it defines a component of \( D \) that is numerically equivalent to \( \frac{m_y - 1}{m_y} F \) where \( F \) is a general fibre of \( f \). Thus we see that \( f^*\Omega_{\mathbb{P}^1}(D) \) is pseudoeffective if

\[
\sum_{\bar{y} \in \mathbb{P}^1, G_y \text{acts by translation}} \frac{m_y - 1}{m_y} \geq 2.
\]

Suppose now that this is not the case: then we have

\[
\deg R_t = \sum_{y \in C, G_y \text{acts by translation}} d \frac{m_y - 1}{m_y} < 2d,
\]

hence \( \#(Z) = 2g(C) - 2 + 2d - \deg R_t > 2g(C) - 2. \) \( \square \)

6.5. Construction.

We again use the Notation \[6.3\]. Let \( \bar{A} \) be an ample Cartier divisor on \( (C \times E)/G \), and set \( A_X = \lambda^* \bar{A} \). Then by definition (and \[HLS20, Lemma 2.2\]) the vector bundle \( \Omega_X \) is not pseudoeffective if and only if there exists a \( c > 0 \) such that for all \( i, j \in \mathbb{N} \) such that \( i > cj \) one has

\[
H^0(X, S^i \Omega_X \otimes \mathcal{O}_X(jA_X)) = 0.
\]

Denote by \( D \subset X \) the exceptional locus of \( \lambda \), then we have morphisms

\[
\begin{align*}
H^0(X, S^i \Omega_X \otimes \mathcal{O}_X(jA_X)) &\quad \to \quad H^0((C \times E)/G \setminus \lambda(D), S^i \Omega_{(C \times E)/G} \otimes \mathcal{O}_{(C \times E)/G}(jq^* \bar{A})) \\
\to \quad &\quad H^0(C \times E \setminus q^{-1}(\lambda(D)), S^i \Omega_{C \times E} \otimes \mathcal{O}_{C \times E}(jq^* \bar{A})) \\
&\quad \to \quad H^0(C \times E, S^i \Omega_{C \times E} \otimes \mathcal{O}_{C \times E}(jq^* \bar{A}))
\end{align*}
\]

\[22\]
where the last isomorphism is due to the fact that $S^i\Omega_{C \times E}$ is reflexive, since $C \times E$ is smooth.

Finally let $A_C$ be an ample divisor on $C$ and $A_E$ an ample divisor of degree one on $E$. Set

$$A := p_C^*(A_C) \otimes p_E^*(A_E)$$

with the canonical projections $p_C : C \times E \to C$ and $p_E : C \times E \to E$. Then for some $l \in \mathbb{N}$ sufficiently high we have an inclusion

$$\mathcal{O}_{C \times E}(q^* \tilde{A}) \hookrightarrow \mathcal{O}_{C \times E}(lA),$$

so for all $i, j \in \mathbb{N}$ we obtain an inclusion

$$\Phi = \Phi_{i,j,l} : H^0(X, S^i \Omega_X \otimes \mathcal{O}_X(jA_X)) \hookrightarrow H^0(C \times E, S^i \Omega_{C \times E} \otimes \mathcal{O}_{C \times E}(jlA)).$$

6.6. Definition. We say that a $\eta \in H^0(C \times E, S^i \Omega_{C \times E} \otimes \mathcal{O}_{C \times E}(jA))$ induces a holomorphic symmetric differential with values in $A_X$ on $X$ if it is in the image of an inclusion $\Phi$ in (12).

Remarks. The definition is a slight abuse of terminology, since the inclusion $\Phi$ in (12) is only defined for $j$ divisible by $l$. Since in the definition of pseudoeffectivity we can always replace $i$ and $j$ by $il$ and $jl$, we will, for the simplicity of notation, ignore this point.

Note also that the chain of inclusions does not use that $C$ is proper, so the terminology also applies to an analytic open subset $\Delta \times E \subset C \times E$ with some subgroup $G_x \subset G$ acting on $\Delta \times E$.

6.C. The nonvanishing conjecture for standard isotrivial fibrations. The goal of this subsection is to prove the following

6.7. Theorem. Let $X$ be a smooth projective surface that admits relatively minimal, isotrivial elliptic fibration

$$f : X \to B$$

that is standard (see Subsection 6.A). If $f^*\Omega_B(D)$ is not pseudoeffective, then $\Omega_X$ is not pseudoeffective.

By Proposition 5.2 and 4.6 we thus obtain:

6.8. Corollary. Let $f : X \to B$ be a standard isotrivial fibration; e.g., $f$ is an isotrivial elliptic fibration such that the general fiber does not have complex multiplication. Then the following are equivalent:

a) $\Omega^1_X$ is pseudoeffective;

b) we have $\bar{q}(X) > 0$;

c) we have $H^0(X, S^m \Omega_X^1) \neq 0$ for some positive integer $m$.

The proof of Theorem 6.7 is done by showing that if $i \gg j$, a non-zero

$$\eta \in H^0(C \times E, S^i \Omega_{C \times E} \otimes \mathcal{O}_{C \times E}(jA))$$

does not induce a holomorphic symmetric differential on $X$. In other words, $\Phi = 0$. Since the details are somewhat technical, let us first recall and reprove a result of Sakai.
6.9. Example. [Sak79, §4, (D)] Let $C$ be a hyperelliptic curve, and let $\tau = (i_C, i_E)$ be the involution on $C \times E$ defined by the hyperelliptic involution $i_C$ on $C$ and the map $i_E : E \to E, z \mapsto -z$ on an elliptic curve $E$. The minimal resolution $X \to (C \times E)/(\tau)$, has an isotrivial fibration $f : X \to \mathbb{P}^1$ such that all the singular fibres are of type $I_\delta^0$, in particular it is relatively minimal and standard. Then one has

$$H^0(X, S^i \Omega_X) = 0 \quad \forall i \in \mathbb{N}.$$  

Proof: Let

$$p^*_C \alpha \otimes p^*_E \beta \in H^0(C \times E, S^i \Omega_{C \times E}) = \bigoplus_{i=0}^{\infty} H^0(C \times E, p^*_C \omega_C^{\otimes i} \otimes p^*_E \omega_E^{\otimes i - l})$$

be a rank one tensor, i.e. $\alpha \in H^0(C, \omega_C^{\otimes l})$ and $\beta \in H^0(E, \omega_E^{\otimes i - l})$ for some $l \in 0, \ldots, i$.

Assume that $\alpha \otimes \beta$ induces a holomorphic symmetric differential on $X$. Arguing by contradiction we assume that $\alpha \neq 0$ and $\beta \neq 0$. If $(x, 0)$ is a fixed point of $\tau$, choose local coordinates $z_1$ on $C$ and $z_2$ on $E$ such that locally near $(x, 0)$, the involution is given by $(z_1, z_2) \mapsto (-z_1, -z_2)$. In these local coordinates we write $\alpha = f_\alpha(z_1)dz_1^i$ and $\beta = f_\beta(z_2)dz_2^{i-l}$. For a general point in the exceptional divisor over the point $(x, 0)$, we can choose local coordinates $(u, v)$ on $X$ such that $z_1 = u^2, z_2 = uv$. In these coordinates the exceptional divisor is given by $u = 0$. Substituting $(z_1, z_2)$ by these coordinates we see that $\alpha \otimes \beta$ induces the meromorphic symmetric differential

$$f_\alpha(\sqrt{u}) \cdot f_\beta(\sqrt{u}) \cdot du \frac{(u\sqrt{v} + vdu)^{i-l}}{(2\sqrt{u})^i}.$$  

on $X$. Looking at the term of $du^i$ we see that the differential is holomorphic along the exceptional divisor if and only if

$$f_\alpha(\sqrt{u}) \cdot f_\beta(\sqrt{u})$$

is holomorphic, i.e. if and only if $\alpha \otimes \beta$ vanishes with order at least $i$ in the fixed point. Since $0 \neq \beta \in H^0(E, \omega_E^{\otimes i-l}) \simeq H^0(E, \mathcal{O}_E)$ does not vanish, this shows that $\alpha \in H^0(C, \omega_C^{\otimes i})$ vanishes with order $i$ in $x$. Since the involution $i_C$ has $2g(C) + 2$ fixed points, we obtain that $\alpha$ vanishes along a divisor of degree at least $i \cdot (2g(C) + 2)$. Since $\deg \omega_C^{\otimes i} = l \cdot (2g(C) - 2) < i \cdot (2g(C) + 2)$ we obtain that $\alpha = 0$. Since $H^0(E, \omega_E^{\otimes i-l})$ has dimension one for every $i - l$ one can reduce the general case to rank one tensors, so the statement follows. This settles the proof of the example.

In the proof of Theorem 6.7 we have $h^0(E, \omega_E^{\otimes i-l} \otimes \mathcal{O}_E(jA_E)) > 1$, so the symmetric differentials are not global rank one tensors. Somewhat surprisingly, this leads to a much weaker local obstruction (cf. [BTV19, Prop.3.2]), i.e., the vanishing order of the symmetric differential in a fixed point can be strictly smaller than $i$. We will improve this local estimate by taking into account that the vanishing order along $E$ is bounded by $j \deg A_E$.

6.10. The local obstruction - setup. Let us describe the local obstruction for a holomorphic symmetric differential on $C \times E$ to induce a holomorphic symmetric differential on $X$: using the notation of Lemma 6.4, fix a point $x \in Z \subset C$, and let
\(\tau_x\) be the generator of \(G_x \simeq \mathbb{Z}_2\). Let \(0 \in E\) be a fixed point of \(\tau_x\), for this local computation we choose \(A_E := 0\) to be the corresponding ample divisor of degree one\(^{\text{2}}\).

Let \(x \in \Delta \subset C\) be a small disc and choose a local coordinate \(z_1\) such that the action of \(G_x\) is given by \(z_1 \mapsto -z_1\) (in particular \(x = 0\)). We have

\[
H^0(\Delta \times E, \mathcal{O}_{\Delta \times E}(jA)) \simeq \mathbb{C}\{z_1\} \otimes H^0(E, \mathcal{O}_E(jA_E)),
\]

where \(z_1\) is a local coordinate on \(\Delta\) and \(\mathbb{C}\{z_1\}\) denotes the algebra of convergent power series in \(z_1\).

Since \(E\) is an elliptic curve, we have \(h^0(E, \mathcal{O}_E(jA_E)) = j\). and we may choose a basis of sections \(s_{j,0}, \ldots, s_{j,j-2}, s_{j,j}\) such that \(s_{j,k}\) vanishes with order exactly \(k\) in the neutral element 0 of the elliptic curve. In particular we have a \(\mathbb{C}\)-basis of \(H^0(\Delta \times E, \mathcal{O}_{\Delta \times E}(jA))\)

\[
z_1^{n-k} s_{j,k} \quad n \in \mathbb{N}, \quad k = 0, \ldots, j - 2, j.
\]

Note that by construction \(z_1^{n-k} s_{j,k}\) vanishes with order exactly \(n\) in \((0,0)\).

Let \(z \in \Delta \subset C\) be the holomorphic \(1\)-form with values in \(A\) giving in local coordinates the form \(z\).

Let \(\omega \in H^0(\Delta \times E, S^i \mathcal{O}_{\Delta \times E} \otimes \mathcal{O}_{\Delta \times E}(jA))\), and let

\[
\omega = \sum_{n \in \mathbb{N}} \omega_n
\]

be the decomposition such that \(\omega_n \in V_n\). Finally let

\[
M := z_1 dz_2 - s_{1,1} dz_1
\]

be the holomorphic \(1\)-form with values in \(A\) giving in local coordinates the form \(z_1 dz_2 - z_2 dz_1\).

In the setup just introduced, the following holds:

**6.11. Lemma.** Let \(S\) be the minimal resolution of \((\Delta \times E)/G_x\). If \(\omega\) induces a holomorphic symmetric differential with values in \(A_S\) on \(S\) (see Definition 6.6), then for all \(n \in \mathbb{N}\) the form \(\omega_n\) induces a holomorphic symmetric differential with values in \(A_S\). Moreover, if \(n < i\), there exists a differential

\[
\eta_n \in H^0(\Delta \times E, S^{i+n} \mathcal{O}_{\Delta \times E} \otimes \mathcal{O}_{\Delta \times E}(l - \frac{n}{2} A))
\]

such that \(\omega_n = \eta_n \times M \frac{i+n}{l-2n}\).

**Remark.** Since \(G_x \simeq \mathbb{Z}_2\) acts locally as \((z_1, z_2) \mapsto (-z_1, -z_2)\), one has \(n = i \mod 2\) [BTVA19 Sect.3], so \(\frac{i+n}{l-2n}\) and \(\frac{i-n}{2}\) are positive integers.

\(^{\text{2}}\)We make this choice in order to simplify the notation. Since the obstruction depends only on a small neighbourhood of \(0 \in E\), the statement in Corollary 6.12 is independent of this choice.
Proof. The property of being holomorphic is local, so we can just apply the proof of [BTVA19 Prop.3.2]. □

Since $H^0(E, \mathcal{O}_E((j - \frac{i}{2})A_E)) = 0$ for $j - \frac{i}{2} < 0$ this implies:

**6.12. Corollary.** If $\omega \in H^0(\Delta \times E, S^i\Omega_{\Delta \times E} \otimes \mathcal{O}_{\Delta \times E}(jA))$ induces a holomorphic symmetric differential with values in $A_S$ on $S$, then

$$\omega \in H^0(\Delta \times E, I_{0,0}^n \otimes S^i\Omega_{\Delta \times E} \otimes \mathcal{O}_{\Delta \times E}(jA))$$

where $I_{0,0}$ is the ideal sheaf of the origin and $n \geq i - 2j$.

**Proof of Theorem 6.7.** We use the notation of Subsection 6.A and Construction 6.3. Denote by $Z \subset C$ the set of points in $x \in C$ such that the stabiliser $G_x = \langle \tau_x \rangle$ acts as the involution $z \mapsto -z$ on the elliptic curve $x \times E$. Since $f$ is standard and $f^*\Omega_B(D)$ is not pseudoeffective, we know by Lemma 6.4 that $Z$ has at least $2g - 1$ elements where $g = g(C)$.

For every $x \in Z$ we fix a point $x' \in (x \times E)$ such that $(x, x')$ is a fixed point of $\tau_x$.

Fix some rational $\epsilon > 0$ such that $\frac{2g-2}{2g-1} + \epsilon < 1$ and fix a positive integer $N$ such that

$$\left(\frac{2g-2}{2g-1} + \epsilon\right)N \in \mathbb{N}.$$ 

Assume now that

$$\eta \in H^0(C \times E, S^i\Omega_{C \times E} \otimes \mathcal{O}_{C \times E}(jA))$$

induces a holomorphic symmetric differential with values in $A_X$ on $X$ (see Definition 6.6). Then for every $x \in Z$, the restriction to some neighbourhood $\Delta \times E$ induces a holomorphic symmetric differential on the minimal resolution $S$ of $(\Delta \times E)/G_x$.

Thus by Corollary 6.12 we have for every $i \in \mathbb{N}$ that

$$\eta \in H^0(C \times E, (\otimes_{x \in Z} I_{(x,x')}^n) \otimes S^i\Omega_{C \times E} \otimes \mathcal{O}_{\Delta \times E}(jA)).$$

with $n \geq i - 2j$.

Observe that if $i \geq \frac{2}{\frac{2g}{\frac{2g}{2g-1} - \epsilon} - 2} j$ then has

$$n \geq \left(\frac{2g-2}{2g-1} + \epsilon\right)i.$$ 

Thus we get an inclusion

$$\eta^N \in H^0(C \times E, (\otimes_{x \in Z} I_{(x,x')}^{\left(\frac{2g}{2g-1} + \epsilon\right)n_i}) \otimes S^N\Omega_{C \times E} \otimes \mathcal{O}_{C \times E}(NjA)).$$

We claim that this last vector space is zero for $i \geq \frac{2}{\frac{2g}{2g-1} - \epsilon} j$. As explained after Theorem 6.7 this shows the statement.

**Proof of the claim.** Recall that

$$S^i\Omega_{C \times E} = \bigoplus_{l=0}^i p^*_C \omega_{C}^l \otimes p^*_E \omega_{E}^{i-l} \simeq \bigoplus_{l=0}^i p^*_C \omega_{C}^l$$

since $\omega_C \rightarrow \omega_{C}^l$ for $l < i$ we are reduced to showing that

$$H^0(C \times E, (\otimes_{x \in Z} I_{(x,x')}^{\left(\frac{2g}{2g-1} + \epsilon\right)n_i}) \otimes p^*_C \omega_{C}^{N_i} \otimes \mathcal{O}_{C \times E}(NjA)) = 0$$

for $i \gg j$. By Definition 6.1 this is the same as proving that

$$\otimes_{x \in Z} I_{(x,x')}^{\left(\frac{2g}{2g-1} + \epsilon\right)n_i} \otimes p^*_C \omega_{C}^{N_i}$$

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is not strongly pseudoeffective. Arguing by contradiction assume that the sheaf is strongly pseudoeffective.

Let $\mu : S \to C \times E$ be the blowup of the (reduced) set $\cup_{x \in Z} (x, x') \subset C \times E$ and denote by $\mathcal{O}_S(1)$ the tautological sheaf, i.e. the ideal sheaf of the exceptional divisor of $\mu$. Recall that $S$ is isomorphic to the blowup of the ideal sheaf $\otimes_{x \in Z} I_{(x, x')}^{\left(\frac{2g-2}{2g-1}\right)2}$ (e.g. [Har77, II, Ex.7.11]), so by Corollary 3.13 the line bundle

$$\mathcal{O}_S((2g-2)(2g-1) + \epsilon)N \otimes \mu^* p_C^* \omega_C^N$$

is pseudoeffective. By Lemma 3.10 applied to the elliptic fibration $p_C \circ \mu : S \to C$, there exists $m \in \mathbb{N}$ and a numerically trivial line bundle $M$ on $C$ such that

$$H^0(S, \mathcal{O}_S((2g-2)(2g-1) + \epsilon)N \otimes \mu^* p_C^* (\omega_C^m N \otimes M^*)) \neq 0.$$

Thus we have

$$(13) \quad H^0(S, \mathcal{O}_S((2g-2)(2g-1) + \epsilon)N \otimes \mu^* p_C^* (\omega_C^m N \otimes M^*)) \neq 0.$$

Yet by Lemma 6.4 we know that $Z$ has at least $2g - 1$ elements, hence

$$\deg(\otimes_{x \in Z} I_x^{(\frac{2g-2}{2g-1}+\epsilon)mN}) \geq mN\left(\frac{2g-2}{2g-1} + \epsilon\right) \cdot 2g - 1 > mN(2g - 2).$$

Since $\deg(\omega_C^m N \otimes M^*) = mN(2g - 2)$, we obtain a contradiction to (13). $\square$

6.D. Isotrivial fibrations and the Zariski decomposition. Let $f : X \to \mathbb{P}^1$ be an isotrivial elliptic fibration over a rational curve, and assume that $\Omega_X$ is pseudoeffective. Since the proof of Theorem 6.7 is a bit tedious, we present here a more conceptual approach based on the ideas of Subsection 6.B. The considerations of this section are independent of whether $f$ is standard or not.

We use the notations of the setup 5.1. Let $\zeta$ be the tautological class of $\mathbb{P}(\Omega_X)$. If the elliptic fibration $f$ is not almost smooth, we would like to show that the subvariety

$$Y := \mathbb{P}(I_Z \otimes \omega_{X/B}(-D)) \subset \mathbb{P}(\Omega_X)$$

defined by $f$ is in the negative part of the Zariski decomposition of $\zeta$. If $f$ is isotrivial, the restriction $\mathbb{P}(\Omega_X|_F)$ over a general fibre $F$ is isomorphic to $\mathbb{P}^1 \times F$, so the proof of Proposition 5.4 does not apply. We therefore have to use some global information to explicitly compute the restriction $\zeta|_Y$.

6.13. Proposition. In the situation of Setup 5.1 assume that $f$ is isotrivial, relatively minimal and not almost smooth. Then $\zeta|_Y$ is not pseudoeffective.

Proof. By Corollary 5.13 the statement is equivalent to showing that $I_Z \otimes \omega_{X/B}(-D)$ is not strongly pseudoeffective. By Corollary 3.19 this is equivalent to showing that

$$H^0(X, I_Z^k \otimes (\omega_{X/B}(-D))^{\otimes k}) = 0$$

for all $k \in \mathbb{N}$. Recall the canonical bundle formula [9], formula [10] and [BHPVdV04, III,Prop.11.4, Rem.11.5]:

$$\chi(X, \mathcal{O}_X) = \frac{e(X)}{12} = \frac{1}{12} \sum_{b \in K} e(X_b).$$
Here the set \( K \subset B \) consists of all the points such that the reduction of the scheme-theoretic fibre \( X_b \) is not an elliptic curve. From these facts, we obtain,

\[
K_{X/B} - D \sim_\mathbb{Q} \sum_{b \in K} \left( \frac{e(X_b)}{12}F - D_{0,b} \right).
\]

Now the proof is finished by observing that

\[
\kappa(X, I_{Z,b} \otimes \frac{e(X_b)}{12}F) = -\infty
\]

for all \( b \in K \). This is done by a tedious case by case calculation, using that, by the proof of Lemma 6.2 the non-multiple fibres are of Kodaira's type II, III, IV or \( I_0^* \). The details are left to the interested reader. \( \square \)

As an application, we obtain

**6.14. Corollary.** Let \( X \) be a smooth projective surface such that \( K_X \) is nef and \( \kappa(X) = 1 \). Denote by \( \zeta \) the tautological divisor on \( \pi : \mathbb{P}(\Omega_X) \to X \). If \( \zeta \) is pseudoeffective and nef in codimension one, then we have \( \bar{g}(X) > 0 \). In particular, by [Aue18, Prop.2.2], we have \( H^0(X, S^m \Omega_X^1) \neq 0 \) for some positive integer \( m \).

**Proof.** If the Iitaka fibration is isotrivial and not almost smooth, we see by Proposition 6.13 that the restriction \( \zeta|_Y \) is not pseudoeffective. Thus \( Y \) is in the negative part of the divisorial Zariski decomposition, hence \( \zeta \) is not nef in codimension one. Thus we know that \( f \) is almost smooth or not isotrivial. If the Iitaka fibration is almost smooth, then \( c_2(X) = 0 \) [BHPVdV04, III,Prop.11.4] and thus \( q(X) > 0 \) as already noticed at the beginning of Section 5.

If the Iitaka fibration \( \varphi : X \to B \) is not isotrivial, we conclude by Corollary 5.5. \( \square \)

**Remark.** By Proposition 6.13 the restriction \( \zeta|_Y \) is not pseudoeffective, so by divisorial Zariski decomposition there exists a \( c > 0 \) such that \( \zeta - cY \) is pseudoeffective. If we prove that \( \zeta - cY \) is pseudoeffective for some \( c \geq 1 \), we obtain the nonvanishing conjecture as in Corollary 5.5. However this is not obvious, even in simple situations.

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**Appendix A. Fundamental group of elliptic surfaces**

The following statement is essentially a consequence of [Cam04, Sect. 3.5]

**A.1. Lemma.** Let \( f : X \to B \) be a smooth elliptic surface such that the cotangent bundle \( \Omega_X \) is not pseudoeffective. Then the fundamental group of \( X \) is finite.

**Proof.** By Proposition 4.1 we can assume without loss of generality that \( X \) is minimal. Since \( \Omega_X \) is not pseudoeffective, we have \( B \simeq \mathbb{P}^1 \). If \( f \) is almost smooth, there exists an étale cover \( B' \times E \to X \) with \( E \) an elliptic curve (see introduction to Section 5). In particular \( \Omega_{B' \times E} \) is pseudoeffective, a contradiction to Proposition 4.1. Thus \( f \) is not almost smooth, hence by [CZ79] Lemma 1.39 and [Cam11, Cor.12.10] one has \( \pi_1(X) \simeq \pi_1(B, \Delta) \), where \( (B, \Delta) \) is the orbifold structure defined by the multiple fibres. Since \( B \) is a curve, by [Cam98, App.C], we can find a finite étale cover \( X' \to X \) such that the orbifold divisor of \( X' \to B' \) is empty. Since \( \Omega_{X'} \) is not pseudoeffective by Proposition 4.1, we have \( B' \simeq \mathbb{P}^1 \). Thus \( \pi_1(X') \simeq \pi_1(B') \simeq \{1\} \). \( \square \)
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