Elliptic quantum groups

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This note gives an account of a construction of an “elliptic quantum group” associated with each simple classical Lie algebra. It is closely related to elliptic face models of statistical mechanics, and, in its semiclassical limit, to the Wess-Zumino-Witten model of conformal field theory on tori. More details are presented in [Fe] and complete proofs will appear in a separate publication.

Quantum groups (Drinfeld-Jimbo quantum enveloping algebras, Yangians, Sklyanin algebras, see [D], [Sk]) are the algebraic structures underlying integrable models of statistical mechanics and 2-dimensional conformal field theory, and found applications in several other contexts. However, from the point of view of statistical mechanics, the picture is not quite complete. In particular, elliptic interaction-round-a-face models of statistical mechanics have so far escaped a description in terms of quantum groups (expect in the $sl_N$ case). In this paper, we give such a description. It is hoped that the construction will shed light in other contexts, such as a description of the category of representation of quantum affine Kac–Moody algebras, or the elliptic version of Macdonald’s theory.

Our definition is motivated by the following known construction that links conformal field theory to the semiclassical version of quantum groups. Conformal blocks of WZW conformal field theory on the plane obey the consistent system of Knizhnik-Zamolodchikov (KZ) differential equations for a function $u(z_1,\ldots,z_n)$ taking values in the tensor product of $n$ finite dimensional representations of a simple Lie algebra $g$: [KZ]

$$\partial_{z_i} u = \sum_{j:j\neq i} r(z_i - z_j)^{(ij)} u$$

(1)

Here, the “classical $r$-matrix” $r(z)$ is the tensor $C/z$, where $C \in g \otimes g$ is a symmetric invariant tensor. We use the notation $X^{(i)}$, for $X \in g$ or $\text{End}(V_i)$, to

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denote $\text{Id} \otimes \cdots \otimes X \otimes \text{Id} \otimes \cdots \otimes \text{Id}$, an element of $U(\mathfrak{g})^{\otimes n}$ (or an endomorphism of the tensor product $\otimes_j V_j$). Similarly, if $X = \sum_{\alpha} x_{\alpha} \otimes y_{\alpha}$, $X^{(ij)}$ means $\sum_{\alpha} x_{\alpha}^{(i)} y_{\alpha}^{(j)}$.

The classical Yang–Baxter equation in $\mathfrak{g} \otimes \mathfrak{g} \subset U(\mathfrak{g}) \otimes U(\mathfrak{g})^\vee$ is, in this setting, the consistency condition of (1). More precisely, if $r(z) \in \mathfrak{g} \otimes \mathfrak{g}$ is a tensor satisfying $r(z) + r(21)(-z) = 0$, then the system (1) is consistent for all sets of representations if and only if $r$ obeys the classical Yang–Baxter equation. Therefore we can consider a Knizhnik–Zamolodchikov equation for each solution of the classical Yang–Baxter equation. Solutions were partially classified (with a non-degeneracy hypothesis) in [BD], and come in three families, rational, trigonometric and elliptic. It is however only in the rational case that they have a direct relation to conformal field theory.

The relation to integrable models of statistical mechanics is based on the remark that (2) in $\text{End}(V \otimes V \otimes V)$ for some vector space $V$, is the semiclassical limit of the Yang–Baxter equation $R^{(12)}(z) R^{(13)}(z + w) + R^{(23)}(w) + [R^{(13)}(z + w), R^{(23)}(w)] = 0$, (2) is, in this setting, the consistency condition of (1). More precisely, if $r(z) \in \mathfrak{g} \otimes \mathfrak{g}$ is a tensor satisfying $r(z) + r(21)(-z) = 0$, then the system (1) is consistent for all sets of representations if and only if $r$ obeys the classical Yang–Baxter equation. Therefore we can consider a Knizhnik–Zamolodchikov equation for each solution of the classical Yang–Baxter equation. Solutions were partially classified (with a non-degeneracy hypothesis) in [BD], and come in three families, rational, trigonometric and elliptic. It is however only in the rational case that they have a direct relation to conformal field theory.

The relation to integrable models of statistical mechanics is based on the remark that (2) in $\text{End}(V \otimes V \otimes V)$ for some vector space $V$, is the semiclassical limit of the Yang–Baxter equation.

Indeed, if we have a one parameter family of solutions of the Yang–Baxter equation with $R(z) = \text{Id} - 2\eta r(z) + O(\eta^2)$, as the parameter $\eta$ goes to zero, then $r$ obeys the classical Yang–Baxter equation. If $R$ is “unitary”, i.e., if $R(z)R(21)(-z) = \text{Id}V \otimes V$, then $r(z) + r(21)(-z) = 0$.

Roughly speaking, to each solution of the Yang–Baxter equation there corresponds a bialgebra or quantum group, defined by quadratic relations [ST]. Starting from rational and trigonometric of the classical Yang-Baxter equation, we arrive in this way to Yangians and affine quantum universal enveloping algebras, which are Hopf algebras, see [F]. In the elliptic case, this construction works only in the $\mathfrak{sl}_N$ case and leads to Sklyanin algebras [Sk], [Ch]. Also, the KZ equation quantizes to a difference equation [S], [FR], which in the rational and trigonometric case is an equation for form factors of integrable quantum field theory in two dimensions.

Let us see how the above construction can be generalized to the genus one case. Our starting point is the set of genus one Knizhnik–Zamolodchikov–Bernard (KZB) equations, obtained by Bernard [BL, B2] as generalization of the KZ equations. These equations have been studied recently in [FG, EK, FW].

Let $\mathfrak{g}$ be a simple complex Lie algebra with invariant bilinear form normalized in such a way that long roots have square length 2. Fix a Cartan subalgebra $\mathfrak{h}$. The KZB equations are equations for a function $u(z_1, \ldots, z_n; \tau, \lambda)$ with values in the weight zero subspace (the subspace killed by $\mathfrak{h}$) of a tensor product of irreducible finite dimensional representations of $\mathfrak{g}$. The arguments $z_1, \ldots, z_n, \tau$ are complex numbers with $\tau$ in the upper half plane, and the $z_i$ are distinct.
modulo the lattice \( \mathbb{Z} + \tau \mathbb{Z} \), and \( \lambda \in \mathfrak{h} \). Let us introduce coordinates \( \lambda = \Sigma \lambda_\nu h_\nu \) in terms of an orthonormal basis \( \{ h_\nu \} \) of \( \mathfrak{h} \). In the formulation of [FW], the KZB equations take the form

\[
\kappa \partial_z u = - \sum \limits_\nu h_\nu^{(j)} \partial_{\lambda_\nu} u + \sum \limits_{i \neq j} \Omega^{(j,j)}(z_j - z_i, \tau, \lambda) u.
\]

Here \( \kappa \) is an integer parameter which is large enough depending on the representations in the tensor product and \( \Omega \in \mathfrak{g} \otimes \mathfrak{g} \) is a tensor preserving the weight zero subspace that we now describe. Let \( \mathfrak{g} = \mathfrak{h} + \sum _{\alpha \in \Delta} \mathfrak{g}_\alpha \) be the root decomposition of \( \mathfrak{g} \), and \( C \in S^2 \mathfrak{g} \) be the symmetric invariant tensor dual to the invariant bilinear form on \( \mathfrak{g} \). Write \( C = \sum _{\alpha \in \Delta \cup \{ 0 \}} C_\alpha \), where \( C_0 = \sum _{\nu} h_\nu \otimes h_\nu \) and \( C_\alpha \in \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \). Let \( \theta_1(t, \tau) \) be Jacobi's theta function

\[
\theta_1(t, \tau) = - \sum \limits_{j = -\infty}^{\infty} e^{\pi i (j + \frac{1}{2})^2 \tau + 2\pi i (j + \frac{1}{2})(t + \frac{1}{2})}.
\]

and introduce functions \( \rho, \sigma \):

\[
\rho(t) = \partial_t \log \theta_1(t, \tau),
\]

\[
\sigma(w, t) = \frac{\theta_1(w - t, \tau) \partial \theta_1(0, \tau)}{\theta_1(w, \tau) \partial \theta_1(t, \tau)}.
\]

The tensor \( \Omega \) is given by

\[
\Omega(z, \tau, \lambda) = \rho(z) C_0 + \sum _{\alpha \in \Delta} \sigma(\alpha(\lambda), z) C_\alpha
\]

As shown in [FW], the functions \( u \) from conformal field theory have a special dependence on the parameter \( \lambda \). For fixed \( z, \tau \), the function \( u \), as a function of \( \lambda \), belongs to a finite dimensional space of antiinvariant theta function of level \( \kappa \) (and obey certain vanishing conditions). Therefore the right way to look at these equations is to consider \( u \) as a function of \( z_1, \ldots, z_n, \tau \) taking values in a finite dimensional space of functions of \( \lambda \).

The tensor \( \Omega \) has the skew-symmetry property \( \Omega(z) + \Omega(-z) = 0 \), and commutes with \( X^{(1)} + X^{(2)} \) for all \( X \in \mathfrak{h} \). The compatibility condition of (3) is then the modified classical Yang–Baxter equation [FW]

\[
\sum \limits_\nu \partial_{\lambda_\nu} \Omega^{(1,2)} h_\nu^{(3)} + \sum \limits_\nu \partial_{\lambda_\nu} \Omega^{(2,3)} h_\nu^{(1)} + \sum \limits_\nu \partial_{\lambda_\nu} \Omega^{(3,1)} h_\nu^{(2)}
\]

\[
-\left[ \Omega^{(1,2)}, \Omega^{(1,3)} \right] - \left[ \Omega^{(1,2)}, \Omega^{(2,3)} \right] - \left[ \Omega^{(1,3)}, \Omega^{(2,3)} \right] = 0
\]

in \( \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \). In this equation, \( \Omega^{(ij)} \) is taken at \( (z_i - z_j, \tau, \lambda) \).

Let us turn to the question of finding the quantum version of the modified classical Yang–Baxter equation, which is the elliptic version of the Yang–Baxter...
equation. To do this, let us take some distance from Lie algebras and consider the following setting.

Let \( h \) be the complexification of a Euclidean space \( h_r \) and extend the scalar product to a bilinear form on \( h \). View \( h \) as an Abelian Lie algebra. We consider finite dimensional diagonalizable \( h \)-modules \( V \). This means that we have a weight decomposition \( V = \bigoplus_{\mu \in h} V[\mu] \) such that \( \lambda \in h \) acts as \((\mu, \lambda)\) on \( V[\mu] \). Let \( P_\mu \in \text{End}(V) \) be the projection onto \( V[\mu] \).

It is convenient to introduce the following notation. Suppose \( V_1, \ldots, V_n \) are finite dimensional diagonalizable \( h \)-modules. If \( f(\lambda) \) is a meromorphic function on \( h \) with values in \( \bigotimes_i V_i = V_1 \otimes \cdots \otimes V_n \) or \( \text{End}(\bigotimes_i V_i) \), and \( \eta_i \) are complex numbers, we define a function on \( h \)

\[
\sum_{\mu_1, \ldots, \mu_n} \prod_{i=1}^n P_{\mu_i} f(\lambda + \Sigma \eta_i \mu_i),
\]

taking values in the same space as \( f \).

Given \( h \) and \( V \) as above, the quantization of (4) is an equation for a meromorphic function \( R \) of the spectral parameter \( z \in \mathbb{C} \) and an additional variable \( \lambda \in h \), taking values in \( \text{End}(V \otimes V) \)

\[
\begin{align*}
R^{(12)}(z_{12}, \lambda + \eta h^{(i)}) & R^{(13)}(z_{13}, \lambda - \eta h^{(2)}) R^{(23)}(z_{23}, \lambda + \eta h^{(1)}) = \\
& = R^{(23)}(z_{23}, \lambda - \eta h^{(1)}) R^{(13)}(z_{13}, \lambda + \eta h^{(2)}) R^{(12)}(z_{12}, \lambda - \eta h^{(3)}). \tag{5}
\end{align*}
\]

Here \( \eta \) is a parameter and \( z_{ij} \) stands for \( z_i - z_j \). This equation forms the basis for the subsequent analysis. Let us call it modified Yang–Baxter equation (MYBE). Note that a similar equation, without spectral parameter, has appeared for the monodromy matrices in Liouville theory, see [GN], [Ba], [AF]. We supplement it by the “unitarity” condition

\[
R^{(12)}(z_{12}, \lambda) R^{(21)}(z_{21}, \lambda) = \text{Id}_{V \otimes V}, \tag{6}
\]

and the “weight zero” condition

\[
[X^{(1)} + X^{(2)}, R(z, \lambda)] = 0, \quad \forall X \in h. \tag{7}
\]

We say that \( R \in \text{End}(V \otimes V) \) is a generalized quantum \( R \)-matrix if it obeys (5), (6), (7).

If we have a family of solutions parametrized by \( \eta \) (the same parameter entering the MYBE) in some neighborhood of the origin, and \( R(z, \lambda) = \text{Id}_{V \otimes V} - 2\eta \Omega(z, \lambda) + O(\eta^2) \) has a “semiclassical asymptotic expansion”, then (5) reduces to the modified classical Yang–Baxter equation (4).

Here are examples of solutions. Take \( h \) to be the Abelian Lie algebra of diagonal \( N \) by \( N \) complex matrices, with bilinear form \( \text{Trace}(AB) \), acting on \( V = \mathbb{C}^N \). Denote by \( E_{ij} \) the \( N \) by \( N \) matrix with a one in the \( i \)th row and \( j \)th column and zeroes everywhere else. Then we have
Proposition 1 The function
\[ R(z, \lambda) = \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} \frac{\sigma(\gamma, \lambda_{ij})}{\sigma(\gamma, z)} E_{ii} \otimes E_{jj} + \sum_{i \neq j} \frac{\sigma(\lambda_{ij}, z)}{\sigma(\gamma, z)} E_{ij} \otimes E_{ji}, \]
is a “unitary” weight zero solution of the modified Yang–Baxter equation, i.e.,
it is a generalized quantum R-matrix, with \( \eta = \gamma/2 \).

Following the Leningrad school (see [STF], [F]), one associates a bialgebra with
quadratic relations to each solution of the Yang–Baxter equation. In our case we
have modify slightly the construction. Let us consider an “algebra” \( A(R) \)
associated to a generalized quantum R-matrix \( R \), generated by meromorphic functions
on \( h \) and the matrix elements (in some basis of \( V \)) of a matrix \( L(u, \lambda) \in \text{End}(V) \)
with non commutative entries, subject to the relations
\[ R^{(12)}(z_{12}, \lambda + \eta h^{(1)})L^{(1)}(z_1, \lambda - \eta h^{(2)})L^{(2)}(z_2, \lambda + \eta h^{(1)}) = \]
\[ = L^{(2)}(z_2, \lambda - \eta h^{(1)})L^{(1)}(z_1, \lambda + \eta h^{(2)})R^{(12)}(z_{12}, \lambda - \eta h). \]

Instead of giving a more precise definition of this algebra, let us define the more
important notion (for our purposes) of representation of \( A(R) \). We define a
tensor category of “representations of \( A(R) \)”.

Definition: Let \( R \in \text{End}(V \otimes V) \) be a meromorphic unitary weight zero solution
of the MYBE (a generalized quantum R-matrix). A representation of \( A(R) \) is
a diagonalizable \( h \)-module \( W \) together with a meromorphic function \( L(u, \lambda) \)
(called L-operator) on \( C \times h \) with values in \( \text{End}(V \otimes W) \) such that the identity
\[ R^{(12)}(z_{12}, \lambda + \eta h^{(3)})L^{(13)}(z_1, \lambda - \eta h^{(2)})L^{(23)}(z_2, \lambda + \eta h^{(1)}) = \]
\[ = L^{(23)}(z_2, \lambda - \eta h^{(1)})L^{(13)}(z_1, \lambda + \eta h^{(2)})R^{(12)}(z_{12}, \lambda - \eta h^{(3)}) \]
holds in \( \text{End}(V \otimes V \otimes W) \), and so that \( L \) is of weight zero:
\[ [X^{(1)} + X^{(2)}, L(u, \lambda)] = 0, \quad \forall X \in h. \]

We have natural notions of homomorphisms of representations: A homomorphism
\( \phi : (W, L) \rightarrow (W', L') \) is a linear map \( \phi(u, \lambda) \in \text{Hom}(W, W') \) depending
meromorphically on \( u, \lambda \), such that \( L'(u, \lambda) \text{Id} \otimes \phi(u, \lambda) = \text{Id} \otimes \phi(u, \lambda)L(u, \lambda) \).

Theorem 2 (Existence and coassociativity of the coproduct) Let \( (W, L) \) and
\((W', L') \) be representations of \( A(R) \). Then \( W \otimes W' \) with \( h \)-module structure
\( X(w \otimes w') = Xw \otimes w' + w \otimes Xw' \) and L-operator
\[ L^{(12)}(z, \lambda + \eta h^{(3)})L^{(13)}(z, \lambda - \eta h^{(2)}) \]
is a representation of \( A(R) \). Moreover, if we have three representation \( W, W', W'' \),
then the representations \((W \otimes W') \otimes W'' \) and \( W \otimes (W' \otimes W'') \) are isomorphic
(with the obvious isomorphism).
Note also that if $L(z, \lambda)$ is an $L$-operator then also $L(z - w, \lambda)$ for any complex number $w$. Since the MYBE and the weight zero condition mean that $(V, R)$ is a representation, we may construct representations on $V^\otimes n = V \otimes \cdots \otimes V$ by iterating the construction of Theorem 2. The corresponding $L$ operator is the “monodromy matrix” with parameters $z_1, \ldots, z_n$:

$$\prod_{j=2}^{n+1} R^{(1j)}(z - z_j, \lambda - \eta \Sigma_{1<i<j} h^{(i)} + \eta \Sigma_{j<i\leq n+1} h^{(i)}).$$

(the factors are ordered from left to right). Although the construction is very reminiscent of the Quantum Inverse Scattering Method [F], we cannot at this point construct commuting transfer matrices by taking the trace of the monodromy matrices. Instead, as we will see now, one has to pass to (interaction-round-a-) face models.

In our setting, the relation between the generalized quantum $R$-matrix and the Boltzmann weights $W$ of the corresponding interaction-round-a-face (IRF) model [B] is very simple. Let $R \in \text{End}(V \otimes V)$ be a generalized quantum $R$-matrix, and let $V[\mu]$ be the component of weight $\mu \in h^*$ of $V$, with projection $E[\mu]: V \to V[\mu]$. Then for $a, b, c, d \in h^*$, such that $b - a, c - b, d - a$ and $c - d$ occur in the weight decomposition of $V$, define a linear map

$$W(a, b, c, d, z): V[d - a] \otimes V[c - d] \to V[c - b] \otimes V[b - a],$$

by the formula

$$W(a,b,c,d,z) = E[c - b] \otimes E[b - a]R(z, \lambda + \eta a + \eta c)[V[d - a] \otimes V[c - d]].$$

Note that $W(a + x, b + x, c + x, d + x, z, \lambda - 2\eta x)$ is independent of $x \in h \simeq h^*$. Set $W(a, b, c, d, z) = W(a, b, c, d, z, 0)$.

**Theorem 3** If $R$ is a solution of the MYBE, then $W(a, b, c, d, z)$ obeys the Star-Triangle relation

$$\sum_g W(b, c, d, g, z_{12})^{(12)} W(a, b, g, f, z_{13})^{(13)} W(f, g, d, e, z_{23})^{(23)} = \sum_g W(a, b, c, g, z_{23})^{(23)} W(g, c, d, e, z_{13})^{(13)} W(a, g, e, f, z_{12})^{(12)},$$

on $V[f - a] \otimes V[e - f] \otimes V[d - e]$.

The familiar form of the Star-Triangle relation [B, JMO] is recovered when the spaces $V[\mu]$ are 1-dimensional. Upon choice of a basis, the Boltzmann weights $W(a, b, c, d, z)$ are then numbers.

For example, if $R$ is the solution of Prop. 1, we obtain the well-known $A_n^{(1)}$ solution (see [IMO], [JKMO] and references therein).

Moreover we also have a converse of this theorem, which gives us many examples of solutions of the modified Yang–Baxter equation.
Theorem 4 Let $V$ be a finite dimensional diagonalizable $\mathfrak{h}$-module, with weight decomposition $\bigoplus_{\mu \in \mathfrak{h}} V[\mu]$. Set $A = \{ \mu \in \mathfrak{h} | V[\mu] \neq 0 \}$. Suppose that for each $(a, b, c, d) \in \mathfrak{h}^4$ such that $d - a, c - d, c - b, b - a \in A$, $W(a, b, c, d, z) \in \text{Hom}(V[d - a] \otimes V[c - d], V[c - b] \otimes V[b - a])$ is a meromorphic function of $z$, and that these functions obey the Star-Triangle relation \[ (9) \]. Assume also that $W$ obeys the relation
\[
\sum_g W(a, g, c, d, z) W(a, b, c, g, -z) = \delta_{bd}.
\]
Then
\[
R(z, \lambda) = \sum_{a,b,c,d:a+c=\lambda/\eta} W(a, b, c, d, z) E(d - a) \otimes E(c - d),
\]
is a generalized quantum $R$-matrix.

In particular, if we take the solutions of \[ \text{\cite{IMO}} \], which have $V$ as the vector representation of simple Lie algebras of type $A, B, C, D$, we obtain generalized $R$-matrices and thus elliptic quantum groups associated to all classical simple Lie algebras.

We have not discussed here the difference equations arising as quantization of the Knizhnik–Zamolodchikov-Bernard equations. See \[ \text{\cite{Fe}} \] for some detail on this point.

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