Optimal Change-Point Detection with Training Sequences in the Large and Moderate Deviations Regimes

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Abstract—This paper investigates a novel offline change-point detection problem from an information-theoretic perspective. In contrast to most related works, we assume that the knowledge of the underlying pre- and post-change distributions are not known and can only be learned from the training sequences which are available. We further require the probability of the estimation error to decay either exponentially or sub-exponentially fast (corresponding respectively to the large and moderate deviations regimes in information theory parlance). Based on the training sequences as well as the test sequence consisting of a single change-point, we design a change-point estimator and further show that this estimator is optimal by establishing matching (strong) converses. This leads to a full characterization of the optimal confidence width (i.e., half the width of the confidence interval within which the true change-point is located at with high probability) as a function of the undetected error, under both the large and moderate deviations regimes.

Index Terms—Change-point detection, Training sequences, Error exponent, Moderate deviations regime, Optimal confidence width

1. INTRODUCTION AND MOTIVATION

The change-point detection (CPD) problem consists in finding changes in the underlying statistical model of data sequences that are modelled as time series. This problem has a plethora of applications in industrial systems [1], medical diagnostics [2], environmental monitoring [3], speech processing [4], finance, economics, and so on [5]. The CPD problems can be divided into two main types: offline CPD and online CPD [6]; the latter is also known as sequential CPD. This depends on whether the data sequence is fixed or obtained in a real-time setting. Offline CPD is a problem that is studied in, for example, anomaly detection problems such as detecting climate change based on existing and known statistics. Online CPD is studied in, for example, signal segmentation problems such as extracting information from streaming audio signals. In classical CPD problems, researchers either assume that the underlying distributions are known [7]–[11] or they assume that there is only access to a sequence of test data samples without knowledge of any underlying distributions [12]–[14]. However, in recent times, there has been a rising trend to adopt a modern statistical approach that makes use of training sequences to augment classical problems, including learning of the change-point(s) in the test sequence. To the best of the authors’ knowledge, existing works have not considered this problem setup.

Different from the online setting where authors of [14]–[16] are primarily concerned with the tradeoff between average detection delay and the probability of false alarm, in the offline setting, which is what this paper focuses on, authors are instead concerned with the tradeoff between the confidence width (i.e., half the width of the confidence interval within which the true change-point is located at with high probability) and the estimation (equivalently, undetected) error probability (i.e., the probability that the distance between the estimated and true change-points is larger than the confidence width). They mainly derived the order of growth of the confidence width between the estimated and true change-points. For example, the results in [17]–[21] indicated that if we require the error probability in detecting the (single) true change-point within an \( n \)-length data sequence to decay polynomially fast, there exists an algorithm or strategy such that the confidence width is \( O(\log n) \) as \( n \to \infty \). The authors [22]–[25] as well as [26, Chapter 1] showed that if we only require that the error probability vanishes (at an arbitrarily slow speed) then there exists algorithms for which the confidence width is of order \( O(1) \). These works either do not provide closed-form expressions for the implied constants in these asymptotic results (in terms of the underlying distributions) or do not prove the converse of the derived convergence rate. This is, in part, what this paper sets out to do, albeit for a different regime of undetected error probability. In the spirit of classical information-theoretic problems [27], we demand that the undetected error probability decays exponentially (or subexponentially) fast with a certain error exponent \( \lambda > 0 \). Consequently, we show that the optimal confidence width scales as \( \Theta(n) \) in the exponentially decaying case, and \( \Theta(n^{1-t}) \) (for \( t \in (0, 1/2) \)) in the subexponentially decaying case. We also characterize the exact pre-constant in the \( \Theta(\cdot) \) notation. En route to this endeavor, we show that the pre-constant factor is optimal by deriving a strong converse in the sense of Wolfowitz [28]. Our problem setting, at least from a theoretical perspective, should be of sufficient interest to the information theory community who seek to understand the fundamental performance (e.g. the error exponents) of practical problems.

In this paper, we study the offline CPD problem with a single change-point when the underlying distributions are unknown. We are instead equipped with test and training sequences to augment classical problems, including learning of the change-point(s) in the test sequence. To the best of the authors’ knowledge, existing works have not considered this problem setup.

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sequences of the pre- and post-change distributions. This setting is applicable to many real-life situations. A simple example is detecting when a light switches between on or off in a room with a sensor. The training sequences can be obtained by taking a collection of samples when the light is on, and taking another collection of samples when the light is off. Based on the training sequences, we can then design an estimator in the sensor to detect when the status of the light changes. In this paper, we assume that the test sequence is such that its first and second parts are independently and identically distributed (i.i.d.) according to two unknown pre- and post-change distributions. Two labelled training sequences that are sampled i.i.d. from the pre- and post-change distributions are also provided to the learner. Our problem setup is simple but can give fundamental and insightful results. Our objective is to deduce the fundamental performance limits of the CPD problem, i.e., the asymptotically optimal confidence width between the estimated and true change-points.

A. Main Contributions

We formulate an offline single-CPD problem of finding the optimal confidence width between the estimated and true change-points without the knowledge of the underlying distributions but given training sequences and a test sequence with a single change-point; the extension to multiple change-points is feasible. In this setup, we assume that the length of the test sequence is proportional to the length of training sequences. Our main contributions are as follows.

Firstly, inspired by Gutman [29], we derive a type-based estimator (i.e., an estimator based on empirical distributions) under two different asymptotic regimes, namely the regimes in which the worst-case undetected error probability decays exponentially fast and sub-exponentially fast, which is more stringent than the requirements in existing works [17]–[21]. These two regimes are respectively known as the large and moderate deviations [30] regimes. Moreover, we allow for the rejection option (also known as the erasure option) [31], used when one is not sufficiently confident to decide which point is the true change-point. Since erasures are much less costly compared to undetected errors, we assume that the asymptotic worst-case erasure probability is upper bounded by a constant $\epsilon \in [0, 1)$.

Secondly, we derive the asymptotically optimal confidence width under both the large and moderate deviations regimes by showing that the above-mentioned type-based estimator is asymptotically optimal. By an appropriate use of the Berry-Esseen theorem, we also prove the strong converse, i.e., the optimal confidence width in both asymptotic regimes do not depend on the upper bound on the worst-case erasure probability $\epsilon$.

Finally, we study the dependence of the optimal confidence width on various parameters, e.g., the ratio between the lengths of training and test sequences, the distance between the two underlying pre- and post-change distributions and the exponent of the undetected error probability. In the moderate deviations regime, the optimal confidence width is a function of a symmetrized version of the chi-square distance, which implies that this new divergence-like quantity finds an operational meaning based on the CPD problem. We note that the usual asymmetric version of the chi-square distance has found operational interpretations in information-theoretic contexts such as covert communications [32]–[34].

B. Related Works

The CPD problem has been studied by numerous authors for many years and it would be futile to survey all existing works. Here, we attempt to highlight the most significant ones, partitioned according to whether the underlying distributions are known or not. We also survey some works on information-theoretic limits of statistical classification.

a) Known distributions: In the early years in which the CPD problem was studied, authors commonly assumed that the distributions are known. In the offline scenario, Smith [35] adopted a Bayesian approach to infer the change-point in a sequence with known or partially-known underlying parameters. Some other probabilistic Bayesian approaches for offline CPD studied in [36], [37] were based on retrospective segmentation strategies. Shen and Srivastava [38] adopted a non-Bayesian approach to detect a change in mean of normal distributions. Lavielle [39] studied the problem of detecting multiple changes by using maximum likelihood estimation. In the online setting, the first works date back to the 1950s, done by Page [7], [8]. The author considered detecting a change of known parameters and proposed the cumulative sums (CUSUM) algorithm that can be applied to both offline and online CPD problems. Later, Page’s optimal stopping time result was further generalized in various directions by Mostakides [9]. Papers [40], [41] provided optimal results for online CPD problems in non-Bayesian settings.

b) Unknown distributions: In the 1970s, there was an increasing number of works that began to study CPD problem when the distributions were unknown. In the offline setting, Pettitt [13] introduced non-parametric techniques for the CPD problem and analyzed them based on different types of observations. Mattheson and James [25] derived a non-parametric method for detecting multiple change-points. Empirical likelihood tests were also proposed to detect single and multiple change-points in [24], [42]. Harchaoui and Cappé [43] proposed a kernel-based retrospective algorithm to detect multiple change-points in a sequence. For the online case, McGilchrist and Woodyer [12] proposed a distribution-free CUSUM algorithm. Shiryaev [44] provided an optimal method in quickest CPD where the unknown parameter varies over time according to a Markov process. Lai [45] proposed a nearly optimal window-limited generalized likelihood ratio test that can be implemented on parallel processors for distribution-free sequential CPD. Xie, Wang, and Thompson [46] studied detecting change-points in high-dimensional signal vectors using statistics based on the generalized likelihood ratio. However, they are not aware of using training sequences to augment the performance of the CPD problem. Perhaps the work that is most related to the current one is that by Gruner and Johnson [47] in which the authors considered two sets of strings and attempted to detect the time when...
their distributions started to change from being the same to being different (also see point 3 in Sec. IV). A type-based

generalized likelihood ratio test similar to that of Gutman [29]

was used. However, while interesting numerical experiments

on distributed detection problems are performed in [47], no

theoretical guarantees are provided.

c) Information-theoretic limits for classification: This

paper is mainly inspired by existing works on classification

with test and training sequences. Gutman [29] was the first
to propose asymptotically optimal type-based tests for the

binary and multiple hypothesis testing problems. Merhav

and Ziv [48] derived a Bayesian approach for classification

of Markov sources with unknown parameters. Unnikrishnan

[49] extended Gutman’s results to matching multiple sequences
to source sequences and proposed a symmetric type-based test

compared to Gutman’s test. Zhou, Tan, and Motani [50] proved

that Gutman’s results are second-order asymptotically optimal.

Recently, He, Zhou, and Tan [51] proposed an asymptotically

optimal type-based test for the distributed detection problem

with test and training sequences. Some of the proof techniques

used in this paper leverage the techniques introduced in these

papers.

II. PROBLEM FORMULATION

In this section, we start by introducing the offline single-

CPD problem with test and training sequences. Let $X$ be a

finite set. We assume that there is a sequence of observations

$X^n = (X_1, \ldots, X_n) \in \mathcal{X}^n$, in which there is a single

change-point $C \in \{1 : n\}$ (or $C \in [n]$ for short), where

$a = b$ is the length of the test sequence), and simply write

that one of $n$ points of the test sequence is the change-point

or to declare that an “erasure” has occurred when we are

not sufficiently confident in declaring which of the points

corresponds to the change-point. Given any true change-point

$C \in [n]$, let us define the set of all test and training sequences

$(x^n, y^n_1, y^n_2)$ that results in an undetected error as

$$E_C := \{(x^n, y^n_1, y^n_2) \in \mathcal{X}^{n+2N} :$$
$$\gamma(x^n, y^n_1, y^n_2) \notin [C \pm \Delta] \cup \{e\}\},$$

(1)

where $\Delta$ represents the confidence width (i.e., half the width

of the confidence interval within which the true change-point

is with high probability) and $a = b := [a - b : a + b]$. The set

of test and training sequences that leads to an erasure event

is defined as

$$E_e := \{(x^n, y^n_1, y^n_2) \in \mathcal{X}^{n+2N} : \gamma(x^n, y^n_1, y^n_2) = e\}.$$  (2)

For any true change-point $C$ and any tuple of distributions

$(P_1, P_2, P_1, P_2) \in \mathcal{P}(\mathcal{X})^4$, we define $P_C$ and $P_{\tilde{C}}$ to be probability measures satisfying

$P_C \circ (X^n, X^n_{n+1}, Y^n_1, Y^n_2)^{-1} = P_1 \times P_2^{-C} \times P_1 \times P_2^C$ and $P_{\tilde{C}} \circ (X^n, X^n_{n+1}, Y^n_1, Y^n_2)^{-1} = P_1 \times \tilde{P}_2^{-C} \times \tilde{P}_1 \times \tilde{P}_2^C$, respectively. That is, $P_1 \times P_2^{-C} \times P_1 \times P_2$ is the pushforward measure of $P_C$

corresponding to the random vector (a measurable function)

$(X^n, X^n_{n+1}, Y^n_1, Y^n_2)$. Then the undetected error probability is defined as

$$P_C\{E_C\} := \Pr \{\gamma(X^n, Y^n_1, Y^n_2) \notin [C \pm \Delta] \cup \{e\}\},$$

(3)

and the erasure probability is defined as

$$P_{\tilde{C}}\{E_e\} := \Pr \{\gamma(X^n, Y^n_1, Y^n_2) = e\},$$

(4)

where in (3) and (4), $(X^n, Y^n_1, Y^n_2)$ is distributed as $X^n \sim P_1 \times P_2^{-C}$, $Y^n_1 \sim P_1^n$, and $Y^n_2 \sim P_2^n$.

We observe that the error and erasure probabilities depend

on $n$ (the length of the test sequence), $N = rn$ (the length of the training sequences), $\Delta$ (the confidence width) as well as the generating distributions $P_1$ and $P_2$. We can further define the performance of any estimator $\gamma$ as follows.

**Definition 1** (Good Estimator). For any $\Delta \in [0, n/2)$, $r \in \mathbb{R}$, $\lambda \in \mathbb{R}_+$, $\epsilon \in [0, 1)$, and $t \in (0, 1/2)$, given any pair of distributions $(P_1, P_2) \in \mathcal{P}(\mathcal{X})^2$, an estimator $\gamma : \mathcal{X}^{n+2N} \rightarrow [n] \cup \{e\}$ is said to be $(n, \Delta, r, \lambda, \epsilon, t)$-good if

$$\max_{C \in [n]} P_C\{E_e\} \leq \epsilon,$$

(5)

and for all $(\tilde{P}_1, \tilde{P}_2) \in \mathcal{P}(\mathcal{X}^2)$,

$$\max_{E \in [n]} P_{\tilde{C}}\{E_e\} \leq \exp(-n^{1-t} \lambda).$$

(6)

When we design an estimator for the CPD problem, since we do not know the underlying distributions $(P_1, P_2)$ of the test and training sequences, we cannot design one with respect to a particular pair of distributions. Thus, we are interested in designing a universal estimator such that the worst-case undetected error probability has good performance for all pairs of possible distributions $(P_1, P_2)$ and at the same time, constrain the erasure probability with respect to a particular pair of distributions $(P_1, P_2)$. That is, we can design an
estimator based on $\lambda$ regardless of the underlying distributions. This is similar in spirit to Gutman’s work [29] as well as several other follow-up works [50], [51].

Note that the definition above corresponds to different asymptotic regimes for the undetected error probability as we vary $t$. If $t = 0$, the undetected error probability in (6) is required to decay exponentially fast; this corresponds to the large deviations regime. If instead $t \in (0, 1/2)$, the undetected error probability in (6) is required to decay subexponentially fast with rate $\exp(-\Theta(n^{1-t}))$; this corresponds to the moderate deviations regime, which has been popularized in the information theory literature by Altug and Wagner [30] among others. Although in standard moderate deviations regime, $t$ is allowed to vary between $(0, 1)$, here we restrict $t \in (0, 1/2)$ due to technical limitations in the proofs. Compared to existing works [17]–[26] in which the upper bound in (6) is replaced by a polynomially decaying sequence or that the undetected error probability is only required to vanish (at an arbitrarily slow speed), our formulation is one in which the constraint on the undetected error probability is more explicit and indeed far more stringent. However, since there exists a natural tradeoff between the optimal confidence width and the undetected error probability, consequently, it is also natural to expect that the optimal confidence width in our setting is larger. In fact, it is of order $\Theta(n)$ when $t = 0$ (or $\Theta(n^{1-t/2})$ when $t \in (0, 1/2)$) in (6); see Definition 2. Our main contribution in this paper is to characterize the optimal confidence width exactly up to the pre-constant term in $\Theta(\cdot)$.

In this CPD problem, our goal is to design an estimator that is $(n, \Delta, r, \lambda, \epsilon, t)$-good. Intuitively, increasing the confidence width $\Delta$ allows us to design estimators with smaller error probabilities, and vice versa. Thus, keeping all other parameters $(r, \lambda, \epsilon, t)$ fixed, we are primarily interested in the smallest $\Delta$ such that (5) and (6) hold.

**Definition 2 (Optimal Normalized Confidence Width (NCW)).** Fix parameters $r \in \mathbb{R}^+$, $\lambda \in \mathbb{R}^+$, $\epsilon \in (0, 1)$ and $t \in [0, 1/2)$. We say that $\Delta$ is a $(r, \lambda, \epsilon, t)$-achievable normalized confidence width (NCW) if there exists a sequence of $(n, \Delta_n, r, \lambda, \epsilon, t)$-good estimators such that

$$\lim_{n \to \infty} \epsilon_n \leq \epsilon, \quad \text{and} \quad \lim_{n \to \infty} \frac{\Delta_n}{n^{1-t/2}} \leq \Delta.$$  \hspace{1cm} (7)

The $t$-optimal NCW

$$\bar{\Delta}_t^\ast(r, \lambda, \epsilon) := \inf \{ \bar{\Delta} : \bar{\Delta} \text{ is } (r, \lambda, \epsilon, t)\text{-achievable} \}.$$  \hspace{1cm} (8)

If $t = 0$ (corresponding to the large deviations regime), we simply write $\bar{\Delta}_0^\ast(\cdot)$ as $\bar{\Delta}^\ast(\cdot)$ (instead of $\bar{\Delta}_0^\ast(\cdot)$) and call the corresponding quantity the optimal NCW.

We now comment on why we restrict our attention to $\Delta < n/2$ in Definition 1. Note that if $\Delta \geq n/2$, there exists a $C \in [n]$ (e.g. $C = n/2$) such that $[C \pm \Delta C] \cap [n] = \emptyset$. Then we can easily design a trivial estimator (e.g. $\gamma(X^n, Y^n_{1}, Y^n_{2}) = n/2$) such that $\max_{C \subseteq [n]} \mathbb{P}_C[\mathcal{E}_C] = 0$ and $\max_{C \subseteq [n]} \mathbb{P}_C[\mathcal{E}_C] = 0$. Hence, if $\Delta \geq n/2$, the problem is vacuous and thus we only consider the case in which $\Delta < n/2$.

For any pair of distributions $(Q, \tilde{Q}) \in \mathcal{P}(\mathcal{X})^2$, let us define the chi-square distance between $Q$ and $\tilde{Q}$ as

$$\chi_2(Q\|\tilde{Q}) := \sum_{x \in \mathcal{X}} \frac{(Q(x) - \tilde{Q}(x))^2}{Q(x)}. \hspace{1cm} (9)$$

Note that $\chi_2(Q\|\tilde{Q}) < \infty$ for all pairs of $(Q, \tilde{Q}) \in \mathcal{P}(\mathcal{X})^2$ such that $\text{supp}(Q) = \text{supp}(\tilde{Q}) = |\mathcal{X}|$.

For any $a \in \mathbb{R}_+$, let us further define the generalized Jensen-Shannon divergence

$$\text{GJS}(Q, \tilde{Q}, a) := a D \left( \frac{aQ + \tilde{Q}}{a + 1} \right) + D \left( \frac{\tilde{Q}}{a + 1} \right). \hspace{1cm} (10)$$

The GJS$(\cdot)$ quantifies, in an $a$-weighted manner, the distance between $Q$ and $\tilde{Q}$. This quantity has featured prominently in information-theoretic decision problems in which there are training and test sequences [29], [50], [51], [54].

**Lemma 1.** For any pair of $(Q, \tilde{Q}) \in \mathcal{P}(\mathcal{X})^2$ such that $Q \neq \tilde{Q}$, GJS$(Q, \tilde{Q}, a)$ is increasing in $a$.

Lemma 1 can be obtained by calculating the first and second order derivatives of (10) and the proof is omitted for brevity.

For any set of distributions $(Q_1, Q_2, \tilde{Q}_1, \tilde{Q}_2) \in \mathcal{P}(\mathcal{X})^4$, any $\beta \in (0, 1)$ and $r \in \mathbb{R}_+$, we define the following linear combination of generalized Jensen-Shannon divergences as follows

$$L(Q_1, Q_2, \tilde{Q}_1, \tilde{Q}_2, \beta, r) := r \text{GJS}(Q_1, \tilde{Q}_1, \beta) + r \text{GJS}(Q_2, \tilde{Q}_2, 1 - \beta). \hspace{1cm} (11)$$

The function $L(\cdot)$, on the other hand, additionally quantifies, in a $(\beta, r)$-weighted manner, the sum of the distances between $Q_1$ and $\tilde{Q}_1$ as well as $Q_2$ and $\tilde{Q}_2$.

**Lemma 2.** For any $r \in \mathbb{R}_+$ and any pair of $(P_1, P_2) \in \mathcal{P}(\mathcal{X})^2$, GJS$(tP_1 + (1-t)P_2, P_2, \frac{1}{r})$ and GJS$(tP_1 + tP_2, P_1, \frac{1}{r})$ are strictly increasing and convex functions of $t \in [0, 1]$.

The proof of Lemma 2 is provided in Appendix A. For any $\alpha \in [0, 1]$ and any $\beta \in [0, 1]$, define

$$G_1(\alpha, \beta) := \text{GJS} \left( \frac{(1-\beta)P_1 + (1-\alpha)P_2}{1-\beta}, P_2, \frac{1-\beta}{r} \right), \hspace{1cm} (12)$$

and for any $\beta \in [0, \alpha]$, define

$$G_2(\alpha, \beta) := \text{GJS} \left( \frac{\alpha P_1 + (\alpha-\beta)P_2}{\beta}, P_1, \frac{1}{r} \right). \hspace{1cm} (13)$$

Given any $\Delta \in [0, 1/2)$, let $\Delta_L(\Delta) \in [0, 2\Delta]$ be the unique number such that

$$G_1(\Delta_L(\Delta), 0) = G_2(\Delta_L(\Delta), 2\Delta) \hspace{1cm} (14)$$

and $\Delta_R(\Delta) \in [0, 2\Delta]$ be the unique number such that

$$G_1(1 - \Delta_R(\Delta), 1 - 2\Delta) = G_2(1 - \Delta_R(\Delta), 1). \hspace{1cm} (15)$$

The illustrations of $\Delta_L(\Delta)$ and $\Delta_R(\Delta)$ are given in Figure 1.

**Corollary 3.** $\Delta_L(\Delta)$ and $\Delta_R(\Delta)$ are both monotonically increasing in $\Delta \in [0, 1/2)$. 

Corollary 3 can be deduced from Lemma 1 and Lemma 2. Furthermore, for any $\alpha \in [\Delta_L(\hat{\Delta}), 1 - \Delta_R(\hat{\Delta})]$, let $\iota(\alpha)$ be such that
\[ G_1(\alpha, \iota(\alpha) - \hat{\Delta}) = G_2(\alpha, \iota(\alpha) + \hat{\Delta}); \tag{16} \]
for any $\alpha \in [0, \Delta_L(\hat{\Delta})]$, let $\iota(\alpha) = \hat{\Delta}$; for any $\alpha \in (1 - \Delta_R(\hat{\Delta}), 1]$, let $\iota(\alpha) = 1 - \hat{\Delta}$. The illustration of $\iota(\alpha)$ is given in Figure 1(b).

Given any $r \in \mathbb{R}_+$ and any pair of $(P_1, P_2) \in \mathcal{P}(\mathcal{X})^2$, let us define the function $G_{\text{min}} : [0, 1/2) \to \mathbb{R}_+$ as follows
\[ G_{\text{min}}(\hat{\Delta}) := r \min_{\alpha \in [\Delta_L(\hat{\Delta}), 1 - \Delta_R(\hat{\Delta})]} G_1(\alpha, \iota(\alpha) - \hat{\Delta}). \tag{17} \]

Corollary 4. For any $r \in \mathbb{R}_+$, $G_{\text{min}}(\hat{\Delta})$ is a strictly increasing function of $\hat{\Delta} \in [0, 1/2]$.

The proof of Corollary 4 is provided in Appendix B. Note that $G_{\text{min}}(0) = 0$. From these facts, we deduce that the inverse $G_{\text{min}}^{-1}(y)$ for $y \in [0, G_{\text{min}}(1/2))$ exists and is also strictly increasing in $y$.

### III. MAIN RESULTS

The following theorem constitutes our main result and presents a single-letter expression for the optimal NCW in both asymptotic regimes.

**Theorem 1.** For any $r \in \mathbb{R}_+$, $\varepsilon \in [0, 1)$, any pair of distributions $(P_1, P_2) \in \mathcal{P}(\mathcal{X})^2$, the optimal NCW is
\[ \Delta^*(r, \lambda, \varepsilon) = \begin{cases} G_{\text{min}}^{-1}(\lambda), & \lambda \in \left(0, G_{\text{min}}\left(\frac{1}{2}\right)\right), \\ \frac{1}{2}, & \text{otherwise}; \end{cases} \tag{18} \]

In the moderate deviations regime, for any $P_1 \neq P_2$, the $t$-optimal NCW for any $t \in (0, 1/2)$ and $\lambda > 0$ is given in (19) at the top of next page.

The proof of Theorem 1 is given in Section V. Several remarks are in order.

First, we define the estimation scheme. Let us define $\rho \in (0, r)$ such that $\rho n \in \mathbb{Z}_+$. In the following, for the sake of brevity, we omit the integer constraint on $\rho n$ and simply write $\rho \in (0, r)$. For any tuple of sequences $(X^n, Y_1^N, Y_2^N) \in \mathcal{X}^{n+2N}$ and any $j \in [n]$, construct two new test vectors
\[ \tilde{X}_1^{j+\rho n} = (X_{1,i}^N)^j, \quad \tilde{X}_2^{j+\rho n} = (X_{2,i}^N)^j. \tag{20} \]

by concatenating different parts of $X^n, Y_1^N, Y_2^N$ as illustrated in Figure 2, and let
\[ T_j^0 := (T_{X_{1,i}^{j+\rho n}, T_{X_{2,i}^{j+\rho n}}, T_{Y_{1,i}^N, Y_{2,i}^N}). \tag{22} \]

denote the tuple of their sub-types. We recall that the type [27] or empirical distribution of $Z^n = (Z_1, \ldots, Z_n) \in Z^n$ is
\[ T_Z^n(z) := \frac{1}{n} \sum_{i=1}^n \mathbb{I}\{Z_i = z\} \] for all $z \in Z$. In the achievability proof (estimation scheme) of Theorem 1, given any confidence width $\Delta_n \in [0, n/2 - 1]$, we make use of the estimator $\gamma(X^n, Y_1^N, Y_2^N)$ that first computes
\[ S_i := \min_{j \in \{i \pm \Delta_n\} \cap [n]} (1 + 2\rho)L(T_j^0, j + \rho n - 2\rho, n + 2\rho, 1 + 2\rho), \forall i \in [n]. \tag{23} \]

The estimator is defined as
\[ \gamma(X^n, Y_1^N, Y_2^N) = \begin{cases} e, & \text{if } S_i \leq \lambda, \forall i \in [n], \\ I, & \text{if } \exists i \in [n] \text{ s.t. } S_i > \lambda, \end{cases} \tag{24} \]
where $I$ is any $i \in [n]$ such that $S_i > \lambda$. We show that this estimator is asymptotically optimal by taking $\rho \approx n^{-1/4}$ (we specify the exact value of $\rho$ in Section V) in large deviations regime and also in moderate deviations regime. In the latter regime, we replace the threshold $\lambda$ in (24) by $\lambda n^{-1}$. This estimator is based on the partial types and types of the test and training sequences; this is what we call a type-based estimator.

Intuitively, the estimator declares a point $k$ to be the change-point when $T_{X_{1,i}^{k+\rho n}}$ is deemed to be sufficiently close enough to $T_{Y_{1,i}^N}$ and $T_{Y_{2,i}^{k-\rho n}}$ is similarly to be sufficiently close enough to $T_{Y_{2,i}^N}$, where the proximity is measured in terms of the function $L$, defined in (11). If no such point exists, then the estimator declares an “eraser”.

Second, Theorem 1 implies that when the undetected error probability decays exponentially fast with exponent $\lambda$ (i.e., $t = 0$ in (6)), the asymptotically optimal confidence width is of order $\Theta(n)$ and the pre-constant term is specified by (18); when the undetected error decays subexponentially as...
\[
\Delta^*_n(r, \lambda, \epsilon) = \max_{\alpha \in [0,1]} \frac{\sqrt{\lambda} (\sqrt{\alpha(1+r)} \chi_2(P_1 \parallel P_2) + \sqrt{(1-\alpha)(1-r)} \chi_2(P_2 \parallel P_1))}{\sqrt{2r \chi_2(P_1 \parallel P_2) \chi_2(P_2 \parallel P_1)}}.
\]

**Fig. 2:** Illustration of concatenating test and parts of training sequences.

We can also see that \(\Delta^*_n(r, \lambda, \epsilon)\) (for \(r \in (0, 1/2)\)), the asymptotically optimal confidence width is of order \(\Theta(n^{1-t/2})\) and the pre-constant term is specified by (19). The former result is similar in spirit to those of Garreau and Arlot [18] and Carlstein [55, Theorem 2] who showed for a general class nonparametric distributions that, in the absence of training sequences, a confidence width of order \(\Theta(n)\) is achievable when the error probability decays exponentially fast; however, exact pre-constant terms were not derived in [55] and lower bounds for the NCW (the converse parts) were also not derived in both works. Although in [17]–[26], confidence widths of order \(O(\log n)\) or even \(O(1)\) were shown to be achievable, these results were derived under the less stringent constraint that the error probability decays at most polynomially fast or simply vanishes arbitrarily slow. Thus, it is reasonable that our derived optimal confidence width is larger than those in [17]–[26]. Moreover, the statements in Theorem 1 consist of accompanying strong converses (in the spirit of Wolfowitz [28]) while the results in [17]–[26] do not include their impossibility counterparts.

Third, it can be seen from Figures 3–4 that for any \(r \in [0, 1/2)\), \(\Delta^*_n(r, \lambda, \epsilon)\) increases as \(\lambda\) increases, \(r\) decreases, and the distance between \(P_1\) and \(P_2\) decreases. The observations can be intuitively explained as follows.

- As \(\lambda\) increases, the requirement in (6) concerning the worst-case undetected error probability \(\max_{C \in [n]} P(\hat{C}_C)\) becomes more stringent and thus the optimal NCW between the true and estimated change-points increases. This is illustrated in Figures 3–4.

- As \(r\) decreases, the length of the training sequences relative to the test sequence decreases, and thus less knowledge about distributions \(P_1\) and \(P_2\) can be learned from the training sequences. Therefore, to maintain the same undetected error exponent \(\lambda\) and erasure probability \(\epsilon\), the confidence width \(\Delta\) should be enlarged correspondingly. This is illustrated in Figure 3.

- As the distance between \(P_1\) and \(P_2\) decreases, it is harder to distinguish between them and thus the accuracy of detection decreases, leading to a larger confidence width. This is illustrated in Figure 4.

We can also see that \(\Delta^*_n(r, \lambda, \epsilon) = 0\) when \(\lambda = 0\), which means that if the confidence width is of order \(o(n^{1-t/2})\), the undetected error probabilities cannot decay faster than \(\exp(-n^{1-t}\lambda)\). Note that \(\Delta^*_n(r, \lambda, \epsilon)\) for any \(t \in [0, 1/2]\) is independent of \(\epsilon\), which implies that strong converses hold.

Fourth, we analyze the impact of the availability of training data on the optimal NCW by considering the scenarios in which \(r\) assumes its extremal values. We note that as \(r \to \infty\), in the large deviations regime, for any \(\lambda \in (0, G_{\min}(1/2))\), let

\[
\tilde{\Delta}^*(\lambda, \epsilon) = \lim_{r \to \infty} \Delta^*(r, \lambda, \epsilon),
\]

(25)

\[
\tilde{\Delta}_L(\tilde{\Delta}^*(\lambda, \epsilon)) = \lim_{r \to \infty} \Delta_L(\tilde{\Delta}^*(r, \lambda, \epsilon)), \quad \text{and}
\]

(26)

\[
\tilde{\Delta}_R(\tilde{\Delta}^*(\lambda, \epsilon)) = \lim_{r \to \infty} \Delta_R(\tilde{\Delta}^*(r, \lambda, \epsilon)).
\]

(27)

Then \(\tilde{\Delta}_L(\tilde{\Delta}^*(\lambda, \epsilon))\) and \(\tilde{\Delta}_R(\tilde{\Delta}^*(\lambda, \epsilon))\) satisfy

\[
D(\tilde{\Delta}_L(\tilde{\Delta}^*(\lambda, \epsilon)) P_1 + (1 - \tilde{\Delta}_L(\tilde{\Delta}^*(\lambda, \epsilon))) P_2) P_2) =
\]

\[
D(\tilde{\Delta}_L(\tilde{\Delta}^*(\lambda, \epsilon)) P_1 + (2 \tilde{\Delta}^*(\lambda, \epsilon) - \tilde{\Delta}_L(\tilde{\Delta}^*(\lambda, \epsilon))) P_2)
\]

\[
\leq D\left(\frac{2 \tilde{\Delta}^*(\lambda, \epsilon)}{2 \tilde{\Delta}^*(\lambda, \epsilon)} P_1 + \tilde{\Delta}_R(\tilde{\Delta}^*(\lambda, \epsilon)) P_2\right).
\]

(28)

(29)

Let \(\tilde{i}(\alpha)\) be the unique solution to

\[
D\left(\frac{(\alpha - \tilde{i}(\alpha) + \tilde{\Delta}^*(\lambda, \epsilon)) P_1 + (1 - \alpha) P_2}{\tilde{i}(\alpha) + \tilde{\Delta}^*(\lambda, \epsilon)}\right)
\]

\[
= D\left(\frac{\alpha P_1 + (\alpha - \tilde{i}(\alpha) + \tilde{\Delta}^*(\lambda, \epsilon) - \alpha) P_2}{\tilde{i}(\alpha) + \tilde{\Delta}^*(\lambda, \epsilon)}\right)
\]

(30)

and let

\[
\tilde{G}_{\min}(\tilde{\Delta}^*(\lambda, \epsilon)) := \lim_{r \to \infty} G_{\min}(\tilde{\Delta}^*(r, \lambda, \epsilon))
\]

(31)

\[
= \min_{\alpha \in [\Delta_L(\tilde{\Delta}^*(\lambda, \epsilon)), 1 - \Delta_R(\tilde{\Delta}^*(\lambda, \epsilon))]}
\]

\[
D\left(\frac{(\alpha - \tilde{i}(\alpha) + \tilde{\Delta}^*(\lambda, \epsilon)) P_1 + (1 - \alpha) P_2}{1 - \tilde{i}(\alpha) + \tilde{\Delta}^*(\lambda, \epsilon)}\right).
\]

(32)

Then we have \(\tilde{\Delta}^*(\lambda, \epsilon) = \tilde{G}_{\min}^{-1}(\lambda)\). In the moderate deviations regime,

\[
\lim_{r \to \infty} \tilde{\Delta}^*_n(r, \lambda, \epsilon) = \max_{\alpha \in [0,1]} \frac{\sqrt{\lambda} (\sqrt{\alpha(1+r)} \chi_2(P_1 \parallel P_2) + \sqrt{(1-\alpha)(1-r)} \chi_2(P_2 \parallel P_1))}{\sqrt{2r \chi_2(P_1 \parallel P_2) \chi_2(P_2 \parallel P_1)}}.
\]

(33)
As $r \to \infty$, the length of the training sequences far exceeds that of the test sequence, implying that we can estimate the underlying distributions $P_1$ and $P_2$ arbitrarily accurately (e.g., using their types). Thus, as $r \to 0$, in large deviations regime, if $\lambda \to 0$, $\Delta^*(r, \lambda, \epsilon) \to 0$; otherwise, $\Delta^*(r, \lambda, \epsilon) = 1/2$ for all $\lambda \in (0, \infty)$; in moderate deviations regime, we have $(a + r)/r \to \infty$ and $(1 - \alpha + r)/r \to \infty$, and if $\lambda \to 0$, $\lim_{r \to 0} \Delta^*(r, \lambda, \epsilon) \to \infty$; if $\lambda$ assumes a positive value, $\lim_{r \to 0} \Delta^*(r, \lambda, \epsilon) = \infty$. These limiting scenarios can be intuitively explained as follows. When the lengths of the training sequences are significantly shorter than that of the test sequence, the error probabilities cannot vanish exponentially fast if the confidence width is smaller than the maximal value of $n/2$ or sub-exponentially fast if the confidence width is of order $o(n^{1-t/2})$.

Finally, in the moderate deviations regime, we see that the optimal NCW $\Delta^*(r, \lambda, \epsilon)$ is a function of the symmetrized chi-square distance $\sqrt{\chi_2(P_1)\chi_2(P_2)\chi_2(P_1)}$. Thus, by studying the CPD problem according to our unique setup, we assign an operational interpretation of $\sqrt{\chi_2(P_1)\chi_2(P_2)\chi_2(P_1)}$. Since the chi-squared distance $\chi_2(P_1)\chi_2(P_2)$ is, in general, asymmetric, our moderate deviations result shows that the symmetrized version is operationally meaningful in problems such as CPD with training samples.

IV. Conclusion and Future Works

In this paper, we derived the optimal NCW for the CPD problem as a function of the ratio of the lengths of the training to test sequences, the exponent of the undetected error probability, and the distance between the distributions among other parameters. We proposed an asymptotically optimal...
type-based estimator in (24). We also proved strong converse statements, namely, that the optimal NCWs in both the large and moderate deviation regimes are independent of the bound on the erasure probability. Our results provide new insights on the fundamental limits of the CPD problem when side information in the form of training data is available. Our moderate deviations result demonstrates the operational significance of the symmetrized chi-square distance $\sqrt{\chi^2(P_1\|P_2)}\sqrt{\chi^2(P_2\|P_1)}$.

This work opens up a multitude of research directions, some of which are listed as follows.

1) While $\tilde{\Delta}^*_t(r, \lambda, \epsilon)$ for any $t \in (0, 1/2)$ is independent of $\epsilon \in [0, 1]$—a strong converse statement—a natural question beckons. What are the second-order terms [50], [56] of the non-normalized optimal confidence widths in both regimes? We believe a more intricate and careful analysis that is largely based on the use of various strengthenings of the central limit theorem may provide satisfactory answers. These second-order terms would, in general, depend on $\epsilon$ and shed light on the finite length performance of optimal tests.

2) Other problem settings can also be explored. For example, the techniques herein do not directly extend to the more practical setting of online CPD problem with training sequences. For this setting, we may need to leverage ideas from sequential hypothesis testing or sequential classification; see [54]. Another setting that is worth investigating is that of detecting multiple change-points. For this proposed extension, we expect the majority of techniques here to carry through.

3) Finally, the techniques contained herein may be utilized to provide theoretical guarantees for the setting considered in Gruner and Johnson [47]. In that problem, the authors, motivated by problems in distributed detection, consider two sets of sequences $\{x_1, x_2, \ldots, x_{C-1}, x_C, \ldots, x_N\} \subset X^n$ and $\{y_1, y_2, \ldots, y_{C-1}, y_C, \ldots, y_N\} \subset X^n$. At times $i \leq C - 1$, the vectors $x_i$ and $y_i$ have the same distribution. At times $i \geq C$, the vectors $x_i$ and $y_i$ have different distributions. Even though training sequences are not explicitly provided, we believe that by adding a rejection option to this setting [47], we can obtain the optimal tradeoff between confidence width for detecting $C$, the erasure probability $\epsilon$, and the exponent of the undetected error probability $\lambda$.

V. PROOF OF THEOREM 1

A. Preliminaries for the Proofs

Before presenting the proof of Theorem 1, we will find it convenient to collect the following preliminary definitions and preparatory results.

a) Notation: To simplify notation, for any $\rho \in (0, r)$, let us define two convex combinations of $P_1(x)$ and $P_2(x)$ as follows:

$$\tilde{P}^-_j(x) := \frac{(C - j)P_1(x) + (n + \rho n - C)P_2(x)}{n + \rho n - j}, \quad \text{and}$$

$$\tilde{P}^+_j(x) := \frac{(C + \rho n)P_1(x) + (j - C)P_2(x)}{j + \rho n}.$$  (34)

We will use these distributions in which $j$ is constrained to be in the interval $[n]$. In the following, we employ the sequence

$$\kappa_n := \frac{3}{4} + \frac{\log \log n}{2 \log n},$$  (35)

which clearly converges to $3/4$ as $n \to \infty$.

b) Tools for Bounding Probabilities: In anticipation of applying the central limit and Berry-Esseen theorems to bound the erasure probability, for any pair of distributions $(Q_1, Q_2) \in \mathcal{P}(X)^2$, let us define the following variance-like quantity

$$V(Q_1, Q_2, j, r, \rho) := \begin{cases} 
\frac{n + \rho n - j}{n + 2\rho n} \Var_{Q_1} \left[ \log \frac{Q_1((n+j)-N)Q_1(X)}{Q_1((n+j)-N)Q_2(X)} \right] & \text{if } j \in [1, C); \\
\frac{j + \rho n}{n + 2\rho n} \Var_{Q_1} \left[ \log \frac{(j+N)Q_1(X)}{(j+N)Q_2(X)} \right] & \text{if } j \in [C, n].
\end{cases}$$  (36)

In (36), $\Var_{Q}[\cdot]$ means that the random variable $X$ that appears in the variance operator has distribution $Q$. We can analogously define the third absolute moment as $\Var_{Q}[\cdot]$ with $\mathbb{E}[X]$. We now use the fact that $j \in [n]$ to show that the variances and third absolute moments are sufficiently well-behaved so that we can apply the Berry-Esseen theorem in the following.

Lemma 5. For any pair of distributions $(P_1, P_2) \in \mathcal{P}(X)^2$ with $\supp(P_1) = \supp(P_2) = |X|$ such that $P_1 \neq P_2$, any $\rho \in (0, r)$, any $C \in [n]$ and $j \in [C \pm (\Delta n - n^\kappa)]$ with some $\Delta > 0$, we have the following points:

- $0 < \tilde{q}_1(\rho) < V(\tilde{P}_j^-, P_2, j, \rho, \rho), V(\tilde{P}_j^+, P_1, j, \rho, \rho) < \tilde{q}_1(\rho)$, and $T(\tilde{P}_j^-, P_2, j, \rho, \rho), T(\tilde{P}_j^+, P_1, j, \rho, \rho) < \tilde{q}_2(\rho)$ where $\tilde{q}_1(\rho), \tilde{q}_1(\rho)$, and $\tilde{q}_2(\rho)$ are positive and finite for every fixed $\rho$ and $n$.
- $\limsup_{n \to \infty} \tilde{q}_1(\rho), \limsup_{n \to \infty} \tilde{q}_2(\rho)$ and $\liminf_{n \to \infty} \tilde{q}_1(\rho)$ are positive and finite for every fixed $\rho$.
- For any $t \in (0, 1/2)$ and any $j \notin [C \pm (\Delta_t n^{-1/2} - n^\kappa)]$ with some $\Delta_t > 0$, we have $\tilde{q}(\rho)$ is positive and finite for every fixed $\rho$.

The proof of Lemma 5 is provided in Appendix C.

c) Bounds on Probabilities of Atypical Events: The sequence $\kappa_n$ defined in (35) is used to define various properties of atypical events in the following. Given any distribution $P \in \mathcal{P}(X)$, define the following typical set

$$\mathcal{B}(P) := \left\{ x^n \in X^n : \max_{a \in \mathcal{X}} |T^n_{x^n}(a) - P(a)| \leq \sqrt{\frac{\log n}{n}} \right\},$$  (37)

where the lengths of the sequences contained in instantiations of the set $\mathcal{B}(\cdot)$ in the proofs below are implicit and not
necessarily \( n \). For any \( j \in [n] \) and \( i < j - n^{\alpha_n} \), let us define the atypical event
\[
\mathcal{A}_{1,j}^\rho := \{ X_{1}^{j+n^\rho} \notin B(P_1) \text{ or } X_{2}^{j+n^\rho} \notin B(P_2) \text{ or } X_{2}^{n-j+n^\rho} \notin B(P_2) \text{ or } Y_1^{n-j+n^\rho} \notin B(P_1) \text{ or } Y_2^{n-j+n^\rho} \notin B(P_2) \}.
\]
(38)

By Hoeffding’s inequality, we can show (see Appendix D) that
\[
P_j(\mathcal{A}_{1,j}^\rho) = O\left( \frac{1}{n^{2\alpha_n}} \right).
\]
(39)

d) **Monotonicity of Functions:** Fix any \( \zeta \in (0, 1) \), any \( \rho \in (0, r) \) and recall that \( C = \alpha n \). Define the functions
\[
g_1(\alpha, \zeta) := \text{GJS} \left( \frac{\tilde{P}_{\alpha-\zeta n} - P_2, n^\rho - \rho n}{\rho n - \rho n} \right), \quad \text{and} \quad \frac{\alpha + n^\rho}{\rho n - \rho n}
\]
(40)
\[
g_2(\zeta, \rho) := \text{GJS} \left( \frac{\tilde{P}_{\alpha+\zeta n} - P_1, n^\rho + \zeta n}{\rho n + \zeta n} \right)
\]
(41)

**Lemma 6.** For any \( \zeta \in (0, 1) \), the function \( g_1(\alpha, \zeta) \) is monotonically increasing in \( \alpha \in [\zeta, 1] \) and \( g_2(\zeta, \rho) \) is monotonically decreasing in \( \alpha \in [0, 1 - \zeta] \).

The proof of Lemma 6 is provided in Appendix E.

Given any \( \Delta \in [0 : n/2 - 1] \), \( C \in [n] \) and \( \rho \in (0, r) \), as shown in Figure 5(a), we define \( \Delta_{n,L}^C(\Delta) \in [0, 2\Delta] \) and \( \Delta_{n,R}^C(\Delta) \in [0, 2\Delta] \) to be the unique numbers such that \( \Delta_{n,L}^C(\Delta) + \Delta_{n,R}^C(\Delta) = 2\Delta \) and
\[
g_1 \left( \frac{C}{n}, \frac{\Delta_{n,L}^C(\Delta) + 1}{n} \right) = g_2 \left( \frac{C}{n}, \frac{\Delta_{n,R}^C(\Delta) + 1}{n} \right).
\]
(42)

Specifically, by letting \( C = \Delta_{n,L}^C(\Delta) = 1 \), we denote the unique solution to (42) as \( \Delta_{n,L}^C(\Delta) \); by letting \( C = \Delta_{n,R}^C(\Delta) + 1 \) and replacing \( \Delta_{n,L}^C(\Delta) \) with \( 2\Delta - \Delta_{n,R}^C(\Delta) \) in (42), we denote the unique solution to (42) as \( \Delta_{n,R}^C(\Delta) \). Note that \( \Delta_{n,L}^C(\Delta) \) and \( \Delta_{n,R}^C(\Delta) \) exist only when \( C \in [\Delta_{n,L}^C(\Delta) + 2 : n - \Delta_{n,R}^C(\Delta) - 1] \).

**Corollary 7.** For any \( C \in [n] \) and any \( \rho \in (0, r) \), if \( \Delta_{n,L}^C(\Delta) \) and \( \Delta_{n,R}^C(\Delta) \) exist, they both are monotonically increasing in \( \Delta \in [0 : n/2 - 1] \). Furthermore, for any \( \Delta \in [0 : n/2 - 1] \), \( \Delta_{n,L}^C(\Delta) = \Theta(\Delta) \) and \( \Delta_{n,R}^C(\Delta) = \Theta(\Delta) \).

The proof of Corollary 7 is provided in Appendix F.

**B. Achievability Proof of (18)**

In this subsection, we prove that for any \( \xi > 0 \), \( G_{\text{min}, 1}(\lambda) + \xi \) is a \((r, \lambda, \epsilon, 0)\)-achievable NCW. That is, here we consider the achievability part for the large deviations regime in which \( t = 0 \).

Recall the definitions of \( T_{n,\alpha}^\rho \) and \( L \left( T_{n,\alpha}^\rho, j^\rho \right) \) from (11) and under (21). We assume that the estimator is given by (24), with \( \rho = n^{\alpha_n-1} \) \((\rho \in (0, r) \) for \( n \) sufficiently large), \( \Delta_n = n(G_{\text{min}, 1}(\lambda) + \xi) = \Theta(n) \) and the threshold \( \lambda \) is replaced by \( \lambda = \lambda + \sigma_n \), where \( \sigma_n = \frac{1}{2} \left( |X| \log((N + n + 1)^2 + \log N) \right) \) is a vanishing sequence. Clearly \( \lim_{n \to \infty} \Delta_n/n \leq G_{\text{min}, 1}(\lambda) + \xi \).
Remark 1 (Values and properties of $\alpha$). Let $P_{\alpha}$ be the undetected error probability based on distributions $(\hat{P}_1, \hat{P}_2)$, upper bounded by $\exp(-n\lambda)$, where $\lambda$ is strictly decreasing in $\alpha$. For any pair of $(\alpha, i)$, we can obtain that $i_{\alpha}(C)$ satisfies

$$
2^{-1} \log \left[ \frac{1 - \frac{1}{\Delta_n}}{\Delta_n} \right] = g_2 \left( \frac{i_{\alpha}(C)}{n}, \frac{\Delta_n + 1}{n} \right)
$$

This is also illustrated in Figure 5(b).

For any $C \in [\Delta_n, \Delta_n + 1], i_{\alpha}(C) = \Delta_n + 1$. For any $C \in [n - \Delta_n, \Delta_n + 1], i_{\alpha}(C) = n - \Delta_n$.

From Lemma 6, we can deduce that $i_{\alpha}(C)$ is monotonically increasing on $[\Delta_n, \Delta_n + 1] + 2 : n - \Delta_n - 1$.
With Lemma 5, we can then apply the central limit theorem, more precisely a uniform version of it such as the Berry-Esseen theorem, to bound (51).

Let us define the following events

\[
\mathcal{F}_1 := \left\{ \min_{j \notin [\rho(C) \pm 2\Delta_n]} (1 + 2\rho) L\left( T_{j, n}^\rho, \frac{j + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho} \right) \leq \tilde{\lambda} \right\}, \quad \text{and} \quad (62)
\]

\[
\mathcal{F}_2 := \left\{ \arg\min_{j \notin [\rho(C) \pm 2\Delta_n]} L\left( T_{j, n}^\rho, \frac{j + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho} \right) \in \Gamma_n \left( i_p(C), \Delta_n \right) \right\}. \quad (63)
\]

It can then be verified from the definitions of these events and that of \( \Gamma_n(C, \Delta_n) \)

\[
\mathcal{F}_1 \cap \mathcal{F}_2 = \left\{ \min_{j \in \Gamma_n(i_p(C), \Delta_n)} (1 + 2\rho) L\left( T_{j, n}^\rho, \frac{j + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho} \right) \leq \tilde{\lambda} \right\}. \quad (64)
\]

With Lemma 9 and Lemma 10, we can bound the erasure probability as follows:

\[
\mathbb{P}_C(\mathcal{E}_e) = \mathbb{P}_C \left\{ \min_{j \notin [\rho(C) \pm 2\Delta_n]} (1 + 2\rho) L\left( T_{j, n}^\rho, \frac{j + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho} \right) \leq \tilde{\lambda}, \forall i \in [n] \right\}
\]

\[
= \mathbb{P}_C \left\{ \min_{j \notin [\rho(C) \pm 2\Delta_n]} (1 + 2\rho) L\left( T_{j, n}^\rho, \frac{j + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho} \right) \leq \tilde{\lambda}, \exists \tilde{i} \in [n] \right\}
\]

\[
\leq \mathbb{P}_C \left\{ \min_{j \notin [\rho(C) \pm 2\Delta_n]} (1 + 2\rho) L\left( T_{j, n}^\rho, \frac{j + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho} \right) \leq \lambda \right\}
\]

\[
\leq \mathbb{P}_C \left\{ \min_{j \notin [\rho(C) \pm 2\Delta_n]} (1 + 2\rho) L\left( T_{j, n}^\rho, \frac{j + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho} \right) \leq \lambda, \Gamma_p(\rho(C) \pm n^{\rho n}) \right\} + \mathbb{P}_C \left\{ \Gamma_p(\rho(C) \pm n^{\rho n}) \right\}
\]

\[
\leq \mathbb{P}_C \left\{ \min_{j \notin [\rho(C) \pm 2\Delta_n]} (1 + 2\rho) L\left( T_{j, n}^\rho, \frac{j + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho} \right) \leq \lambda \right\} + O\left( \frac{1}{\sqrt{n}} \right)
\]

\[
\leq \mathbb{P}_C \left\{ \min_{\Gamma_p(\rho(C), \Delta_n)} (1 + 2\rho) L\left( T_{j, n}^\rho, \frac{j + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho} \right) \leq \lambda \right\} + O\left( \frac{1}{\sqrt{n}} \right)
\]

\[
\leq \mathbb{P}_C \left\{ \min_{j \notin [\rho(C) \pm 2\Delta_n]} (1 + 2\rho) L\left( T_{j, n}^\rho, \frac{j + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho} \right) \leq \lambda \right\} + O\left( \frac{1}{\sqrt{n}} \right)
\]

\[
\text{where (66) follows from Lemma 9, (69) follows from Lemma 10 and (70) follows from the union bound. Note that in (70) the first term is equal to 0 if } i_p(C) < \Delta_n + 2 \text{ (i.e. } C < \Delta_n \text{ according to Remark 1) and the second term is equal to 0 for } i_p(C) > n - \Delta_n - 1 \text{ (i.e. } C > n - \Delta_{n, R}(\Delta_n) - 1 \text{ according to Remark 1). For simplicity, we let } \tau_n = \Delta_{n, L}(\Delta_n) + 2 \text{ and } \tau'_n = n - \Delta_{n, R}(\Delta_n) - 1 \text{ in the following analyses.}
\]

We then define

\[
C_L^* := \arg\min_{j \in \Gamma_p^L(\rho(C), \Delta_n)} \left( T_{j, n}^p, \frac{j + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho} \right), \quad \text{and} \quad (71)
\]

In the following, we provide an upper bound on \( \mathbb{P}_C \{ (1 + 2\rho) L(T_{C_L^*}^p, \frac{C_L^* + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho}) \leq \tilde{\lambda} \} \) by first conditioning on typical random vectors of sufficiently long lengths and then approximating the function \( L(\cdot) \) with sums of independent log-likelihood terms. Finally, we bound \( \mathbb{P}_C \{ (1 + 2\rho) L(T_{C_L^*}^p, \frac{C_L^* + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho}) \leq \tilde{\lambda} \} \) using the uniform version of central limit theorem.

Recall the definition of the atypical event \( A_{C_L^*} \) in (38). On the event \( \left( A_{C_L^*}^p \right)^C \), all partial test \( X \_{1}^{C_L^* + \rho m}, X \_{C_L^* + 1}^{C_L^* + \rho m} \) and full training \( Y \_{1}^{n - m}, Y \_{2}^{n - m} \) sequences are typical according to the definition in (37). Thus, by applying a Taylor expansion to the function \( L(\cdot) \) and using the bound in (39), we have (72)–(74) on the top of next page, where (74) follows from the bound on the probability of the atypical set in (39) and the fact that \( C - C_L^* = \Theta(\Delta_n) = \Theta(n) \) from Corollary 7. Now since the random variables involved in (74) are independent, by applying the uniform version of the central limit theorem, we have

\[
\lim_{n \to \infty} \max_{C \in [\tau_n, n]} \mathbb{P}_C \left\{ L\left( T_{C_L^*}^p, \frac{C_L^* + \rho m}{n + 2\rho m}, \frac{r - \rho}{1 + 2\rho} \right) \leq \tilde{\lambda} \right\}
\]

\[
\leq \lim_{n \to \infty} \max_{C \in [\tau_n, n]} \max_{C_L^* \in \Gamma_p^L(\rho(C), \Delta_n)} \Phi \left( \frac{1}{1 + 2\rho}, \frac{r - \rho}{1 + 2\rho} \right)\left\{ \frac{1}{1 + 2\rho} \right\} (\tilde{\lambda}) - r - \rho \right) GJS \left( P_{C_L^*}^r, \frac{n - C_L^* + \rho m}{(r - \rho)n} + o \left( \frac{\log n}{n} \right) \right)
\]

\[
\times \sqrt{\frac{n + 2\rho m)}{(n - C_L^* + \rho m)} \left( \frac{1}{1 + 2\rho} \right)} \left( \frac{1}{1 + 2\rho} \right)} (\tilde{\lambda}) - r - \rho \right) GJS \left( P_{C_L^*}^r, \frac{n - C_L^* + \rho m}{(r - \rho)n} + o \left( \frac{\log n}{n} \right) \right)
\]

\[
\times \sqrt{\frac{n + 2\rho m)}{(n - C_L^* + \rho m)} \left( \frac{1}{1 + 2\rho} \right)} \left( \frac{1}{1 + 2\rho} \right)} (\tilde{\lambda}) - r - \rho \right) GJS \left( P_{C_L^*}^r, \frac{n - C_L^* + \rho m}{(r - \rho)n} + o \left( \frac{\log n}{n} \right) \right)
\]
\[ \mathbb{P}_C \left\{ (1 + 2\rho) \mathbb{P}_{C_L}^\rho \left( \frac{C_L^* + \rho m}{n + 2\rho m + 1 + 2\rho} \left( r - \rho \right) \right) \leq \hat{\lambda} \right\} \leq \mathbb{P}_C \left\{ (1 + 2\rho) \mathbb{P}_{C_L}^\rho \left( \frac{C_L^* + \rho m}{n + 2\rho m + 1 + 2\rho} \left( r - \rho \right) \right) \leq \hat{\lambda}, \left( A_{C_L^*, C}^\rho \right)^C \right\} + \mathbb{P}_C \left\{ A_{C_L^*, C}^\rho \right\} \]

\[ = \mathbb{P}_C \left\{ \frac{1}{n + 2\rho m} \sum_{i \in [C_L^* + 1:n]} \log \frac{(n - C_L^* + N) \tilde{P}_{C_L^*}^{-\rho}(X_i)}{(n - C_L^* + N) \tilde{P}_{C_L^*}^{-\rho}(Y_{2,i}) + (N - \rho m) P_2(Y_{2,i})} \right. \]

\[ + \sum_{i \in [N - \rho m : N]} \log \frac{(n - C_L^* + N) \tilde{P}_{C_L^*}^{-\rho}(Y_{2,i}) + (N - \rho m) P_2(Y_{2,i})}{(n - C_L^* + N) \tilde{P}_{C_L^*}^{-\rho}(Y_{2,i}) + (N - \rho m) P_2(Y_{2,i})} + O \left( \frac{\log(C - C_L^*)}{C - C_L^*} \right) \leq \frac{\hat{\lambda}}{1 + 2\rho}, \left( A_{C_L^*, C}^\rho \right)^C \right\} \]

\[ + \mathbb{P}_C \left\{ A_{C_L^*, C}^\rho \right\} \]

\[ \leq \mathbb{P}_C \left\{ \frac{1}{n + 2\rho m} \sum_{i \in [C_L^* + 1:n]} \log \frac{(n - C_L^* + N) \tilde{P}_{C_L^*}^{-\rho}(X_i)}{(n - C_L^* + N) \tilde{P}_{C_L^*}^{-\rho}(X_i) + (N - \rho m) P_2(X_i)} \right. \]

\[ + \sum_{i \in [N - \rho m : N]} \log \frac{(n - C_L^* + N) \tilde{P}_{C_L^*}^{-\rho}(Y_{2,i}) + (N - \rho m) P_2(Y_{2,i})}{(n - C_L^* + N) \tilde{P}_{C_L^*}^{-\rho}(Y_{2,i}) + (N - \rho m) P_2(Y_{2,i})} + O \left( \frac{\log n}{n} \right) \leq \frac{\hat{\lambda}}{1 + 2\rho} + O \left( \frac{1}{n^{2\rho}} \right). \]

In a symmetric fashion, we obtain

\[ \limsup_{n \to \infty} \max_{C \in [1, \tau_n]} \mathbb{P}_C \left\{ L \left( \frac{C_L^* + \rho m}{n + 2\rho m + 1 + 2\rho} \left( r - \rho \right) \right) \leq \frac{\hat{\lambda}}{1 + 2\rho} \right\} \]

\[ \leq \Phi \left( \limsup_{n \to \infty} \max_{C \in [1, \tau_n]} C_L^* \in \Gamma^R_{\nu}((\nu(C), \Delta_n)) \right) \]

\[ - (r - \rho) \text{GJS} \left( \tilde{P}_{C_R^*}^+ P_1, \frac{C_R^* + \rho}{(r - \rho) n} \right) + O \left( \frac{\log n}{n} \right) \]

\[ \times \sqrt{\frac{n}{C_2(\rho)}} \]  \quad (78)
The following lemma shows that for any arbitrary estimator $\gamma$.
Given any arbitrary estimator $\gamma$ under (21). We denote an estimator that only makes use of $\sum \max_{n \rightarrow \infty} \frac{\lambda}{n+1} \leq 1 + 2\rho$
and any estimator $\bar{\gamma}_{\rho,C}$ given any type-based estimator $\gamma_{\rho,C}$ that also satisfies $\bar{\gamma}_{\rho,C}(E_C) \leq \exp(-\eta \lambda)$ for all $(\bar{P}_1, \bar{P}_2) \in P(\mathcal{X})^2$.
The erasure probability of it can be approximately lower bounded by that of the estimator in (24).

**Lemma 12.** For any $\lambda \in \mathbb{R}_+$, any $\rho \in (0, r)$, and any tuple of type-based estimators $\{\gamma_{\rho,C}(\cdot)\}_{i \in [n]}$ such that for all pairs of distributions $(\bar{P}_1, \bar{P}_2) \in P(\mathcal{X})^2$.
and for any $C \in [n]$,
we have that for any particular pair of distributions $(\bar{P}_1, \bar{P}_2) \in P(\mathcal{X})^2$ for and any $C \in [n]$, 
\[
\mathbb{P}_C\{\gamma_{\rho,C}(T^\rho) \notin [C \pm \Delta] \cup \{e\}\} \leq \exp(-\eta \lambda),
\]
(93)
The proof of Lemma 12 is provided in Appendix J. The following corollary shows that given any estimator $\gamma$ with undetected error probability upper bounded by $\exp(-\eta \lambda)$, the erasure probability of $\gamma$ can be approximately lower bounded in terms of the tail probability of the function $L(\cdot, \cdot)$, defined in (11). Let $\Xi_n := [n^{\eta \alpha - 1}, r - r^{\eta \alpha - 1}].$

**Corollary 13.** For any $\lambda \in \mathbb{R}_+$ and any estimator $\gamma$ such that for all pairs for $(\bar{P}_1, \bar{P}_2) \in P(\mathcal{X})^2$.
\[
\max_{C \in [n]} \mathbb{P}_C\{\gamma(X^n, Y^N_1, Y^N_2) \notin [C \pm \Delta] \cup \{e\}\} \leq \exp(-\eta \lambda),
\]
(95)
we have that for any pair of $(\bar{P}_1, \bar{P}_2) \in P(\mathcal{X})^2$ and any $C \in [n]$, 
\[
\mathbb{P}_C\{\gamma(X^n, Y^N_1, Y^N_2) = e\} \\
\geq \left(1 - \frac{1}{n}\right) \min_{\rho \in \Xi_n} \mathbb{P}_C\left\{\min_{j \in [\pm \Delta]} \left(1 + 2\rho\right) \left(L(\bar{T}^\rho, j + \rho \eta n + 2\rho) - \frac{1}{1 + 2\rho} + \delta_n\right) \leq \lambda, \forall i \in [n]\right\}.
\]
Combining Lemma 11 and 12 and setting $\eta_l = 1/n^2$ for all $l \in [n]$, we obtain Corollary 13.
Let $\hat{C}_L := \hat{i}_p(C) - \Delta - n^\kappa - 1$ and $\hat{C}_R := \hat{i}_p(C) + \Delta + n^\kappa + 1$. For any $C \in [n]$ and any estimator $\gamma$ satisfying (95), the erasure probability can be lower bounded as follows:

\[
P_C(\mathcal{E}_n) \geq \left(1 - \frac{1}{n}\right) \min_{\rho \in \mathbb{E}_n} \min_{\tau \in \mathbb{E}_n} \text{L} \left(\mathbf{T}_j^\rho, \frac{j + \rho n}{n + 2\rho}, \frac{r - \rho}{1 + 2\rho}\right) + 2 \log n \leq \frac{\lambda}{1 + 2\rho} - \delta_n, \forall i \in [n]\right)\}

(97)

\[
\geq \left(1 - \frac{1}{n}\right) \min_{\rho \in \mathbb{E}_n} \min_{\tau \in \mathbb{E}_n} \text{L} \left(\mathbf{T}_j^\rho, \frac{j + \rho n}{n + 2\rho}, \frac{r - \rho}{1 + 2\rho}\right) + 2 \log n \leq \frac{\lambda}{1 + 2\rho} - \delta_n, \forall i \in [n]\right)\}

(98)

\[
\geq \left(1 - \frac{1}{n}\right) \min_{\rho \in \mathbb{E}_n} \min_{\tau \in \mathbb{E}_n} \text{L} \left(\mathbf{T}_j^\rho, \frac{j + \rho n}{n + 2\rho}, \frac{r - \rho}{1 + 2\rho}\right) + 2 \log n \leq \frac{\lambda}{1 + 2\rho} - \delta_n, \forall i \in [n]\right)\}

(99)

\[
\geq \left(1 - \frac{1}{n}\right) \min_{\rho \in \mathbb{E}_n} \min_{\tau \in \mathbb{E}_n} \text{L} \left(\mathbf{T}_j^\rho, \frac{j + \rho n}{n + 2\rho}, \frac{r - \rho}{1 + 2\rho}\right) + 2 \log n \leq \frac{\lambda}{1 + 2\rho} - \delta_n, \forall i \in [n]\right)\}

(100)

\[
\geq \left(1 - \frac{1}{n}\right) \max_{\rho \in \mathbb{E}_n} \min_{\tau \in \mathbb{E}_n} \text{L} \left(\mathbf{T}_j^\rho, \frac{\hat{C}_L + \rho n}{n + 2\rho}, \frac{r - \rho}{1 + 2\rho}\right) + 2 \log n \leq \frac{\lambda}{1 + 2\rho} - \delta_n\}

(101)

where (99) follows since $[n] \cap [i_\rho(C) + (\Delta + n^\kappa)] \subset [n] \cap [I_\rho^* + \Delta]$. (100) follows from the fact that $\text{P}_C(D \cap A^c) \geq \text{P}_C(D) - \text{P}_C(A)$ given any two events $D$ and $A$ and (101) follows from Lemma 9.

Note that for any $\rho \in \mathbb{E}_n$, if $i_\rho(C) < \Delta + n^\kappa + 2$ (i.e. $C < \Delta^\rho_{n,L}(\Delta + n^\kappa) + 2$ according to Remark 1), the first probability term in (101) is equal to 0; on the other hand if $i_\rho(C) > \Delta - n^\kappa - 1$ (i.e. $C > \Delta^\rho_{n,R}(\Delta + n^\kappa) - 1$ according to Remark 1), the second probability term in (101) is equal to 0. For simplicity, let $\hat{\tau}_n = \max_{\rho \in \mathbb{E}_n} \Delta^\rho_{n,L}(\Delta + n^\kappa) + 2$ and $\hat{\tau}'_n = n - \max_{\rho \in \mathbb{E}_n} \Delta^\rho_{n,R}(\Delta + n^\kappa) - 1$. Since the left and right-hand sides of (42) increases as $\rho$ decreases (a fact that can be proved similarly as that of Lemma 6 by using (55)), we can subsequently deduce that $\Delta^\rho_{n,L}(\Delta + n^\kappa)$ and $\Delta^\rho_{n,R}(\Delta + n^\kappa)$ both increase as $\rho$ decreases and thus, $\hat{\tau}_n = \Delta^\rho_{n,L}(\Delta + n^\kappa) + 2$ and $\hat{\tau}'_n = n - \Delta^\rho_{n,R}(\Delta + n^\kappa) - 1$.

Next, for any $\rho \in \mathbb{E}_n$, we have (102)–(105) on the top of next page, where (103) follows from the same steps as those leading to (73) in the achievability part and (105) again follows from the fact that $\text{P}_C(D \cap A^c) \geq \text{P}_C(D) - \text{P}_C(A)$ and from the probability of the atypical event in (39). Since the random variables involved in (105) are independent, by applying the uniform version of central limit theorem, we have

\[
\liminf_{n \to \infty} \text{min}_{\rho \in \mathbb{E}_n} \text{P}_C \left[ \text{L} \left(\mathbf{T}_j^\rho, \frac{\hat{C}_L + \rho n}{n + 2\rho}, \frac{r - \rho}{1 + 2\rho}\right) + 2 \log n \leq \frac{\lambda}{1 + 2\rho} - \delta_n \right] \geq \liminf_{n \to \infty} \text{min}_{\rho \in \mathbb{E}_n} \Phi \left(\lambda \right)

(106)

\[
\geq \liminf_{n \to \infty} \text{min}_{\rho \in \mathbb{E}_n} \Phi \left(\lambda \right)

(107)

\[
\geq \liminf_{n \to \infty} \text{min}_{\rho \in \mathbb{E}_n} \Phi \left(\lambda \right)

(108)

\[
\geq \Phi \left(\liminf_{n \to \infty} \text{min}_{\rho \in \mathbb{E}_n} \Phi \left(\lambda \right)

(109)

where (108), in which $0 < \ell_1(\rho) < \infty$ is a constant, uses the bounds on $V(\hat{P}_C^\rho, P_L, \mathcal{C}_1, r, \rho)$ as stated in Lemma 5, and the final step follows from the continuity and monotonicity of $\Phi(\cdot)$. In a symmetrical fashion, we obtain

\[
\liminf_{n \to \infty} \text{min}_{\rho \in \mathbb{E}_n} \Phi \left(\lambda \right)

(110)

\[
\geq \Phi \left(\liminf_{n \to \infty} \text{min}_{\rho \in \mathbb{E}_n} \Phi \left(\lambda \right)

(111)
for some $0 < \hat{c}_2(\rho) < \infty$.

Recall the definitions of $\Delta_L(\cdot)$, $\Delta_R(\cdot)$ in (14) and (15). Let $\Delta = n(G_{\min}^{-1}(\lambda) - \xi)$ for some arbitrarily small constant $\xi > 0$ and let $\iota(\rho) = \lim_{n \to \infty} \iota(\rho(C)/n$. Note that $\lim_{n \to \infty} \tilde{\eta}_n/\eta_n = \lim_{n \to \infty} (\Delta_{n,k}^{R,n} - 1) / \eta_n = 1 - \Delta_R(G_{\min}^{-1}(\lambda) - \xi)$. Using the continuity of GJS in the distributions and the third argument, we have $(121)-(117)$, where $(114)$ follows since the function of $(\alpha, \rho)$ in (113) decreases as $\rho$ increases for any fixed $\alpha$ (a fact that can be proved similarly as that of Lemma 6 by using (55)), (115) follows since $\iota(\rho)$ is fixed for all $\rho \in [1-\Delta_R(G_{\min}^{-1}(\lambda) - \xi)]$ and from the proof of Lemma 2, (116) follows from Corollary 4 and $\xi_1'' > 0$ is a function of $\xi$ satisfying $\xi_1'' \to 0^+$ as $\xi \to 0^+$.

Similarly, we have

\[ \lambda - \xi_2'', \quad \text{where } \xi_2'' > 0 \text{ satisfies } \xi_2'' \to 0^+ \text{ as } \xi \to 0^+ \]

Using these bounds and the continuity of $\Phi(\cdot)$, we have

\[ \lim_{n \to \infty} \min_{\rho \in [\xi_1, \xi_1'')] \max_{C \in [\xi_1', \xi_1'']} \mathbb{P}_C \left\{ L \left( \mathbf{T}^p_{C_L} \right) \left( \frac{C_L + \rho m}{n + 2\rho m} \frac{r - \rho}{1 + 2\rho} \right) + \frac{2\log n}{(1 + 2\rho)n} \leq \frac{\lambda}{1 + 2\rho} - \delta_n \right\} \]

(120)

and

\[ \lim_{n \to \infty} \max_{\rho \in [\xi_2, \xi_2'')] \min_{C \in [\xi_2', \xi_2'']} \mathbb{P}_C \left\{ L \left( \mathbf{T}^p_{C_L} \right) \left( \frac{C_L + \rho m}{n + 2\rho m} \frac{r - \rho}{1 + 2\rho} \right) + \frac{2\log n}{(1 + 2\rho)n} \leq \frac{\lambda}{1 + 2\rho} - \delta_n \right\} \]

(122)

From Lemma 5, we have that $0 < \lim_{n \to \infty} \max_{\rho \in [\xi_1, \xi_1'']} \hat{c}_1(\rho)$, $\lim_{n \to \infty} \max_{\rho \in [\xi_2, \xi_2'']} \hat{c}_2(\rho) < \infty$. Since $\xi_1'', \xi_2'' > 0$, we can see that both of the lower bounds in (120) and (122) tend to one, which means that the erasure probability of all estimators, which is lower bounded as in (97), cannot be made less than $\epsilon$ for any $\epsilon \in [0, 1]$. This constitutes a strong converse. Thus, the optimal NCW cannot be smaller than or equal to $G_{\min}^{-1}(\lambda) - \xi$, i.e., $\Delta^*(r, \lambda, c) > G_{\min}^{-1}(\lambda) - \xi$. Since $\xi > 0$ can be made arbitrarily small, it means that

$\Delta^*(r, \lambda, c) \geq G_{\min}^{-1}(\lambda)$, \hspace{1cm} (124)

completing the proof of the converse of (18).

D. Achievability Proof of (19)

Throughout the proof, we let $\rho = n^{\varepsilon_n^{-1}}$ and let $\Delta$ be equal to the right-hand side of (19). We seek to prove that for any $\xi > 0$, $\Delta + \xi$ is an $(r, \lambda, \epsilon, t)$-achievable NCW. That is, here we consider the achievability part for the moderate deviations regime in which $t \in (0, 1/2)$. Let $\Delta_n = n^{1-t/2}(\Delta + \xi) = \frac{r - \rho}{1 + 2\rho}$.
\[
\limsup_{n \to \infty} \min_{C \in [\tau_n, n]} \max_{\rho \in [0, n]} \max (r - \rho) GJS \left( \tilde{P}_{\tilde{C}_L}, P_2, \frac{n - \tilde{C}_L + \rho n}{(r - \rho)n} \right) = \min_{\alpha \in [\Delta_L(G_{\min}^{-1}(\lambda) - \xi), 1]} \max_{\rho \in (0, \rho)} GJS \left( \frac{(\alpha - \lambda' + G_{\min}^{-1}(\lambda) - \xi)P_1 + (1 - \lambda' + \rho)P_2}{1 - \lambda' + G_{\min}^{-1}(\lambda) - \xi + \rho}, \frac{P_2, 1 - \lambda' + G_{\min}^{-1}(\lambda) - \xi + \rho}{r - \rho} \right)
\]

\[
= \min_{\alpha \in [\Delta_L(G_{\min}^{-1}(\lambda) - \xi), 1]} \max_{\rho \in (0, \rho)} rGJS \left( \frac{(\alpha - \lambda' + G_{\min}^{-1}(\lambda) - \xi)P_1 + (1 - \lambda' + \rho)P_2}{1 - \lambda' + G_{\min}^{-1}(\lambda) - \xi}, \frac{P_2, 1 - \lambda' + G_{\min}^{-1}(\lambda) - \xi + \rho}{r} \right)
\]

\[
= \min_{\alpha \in [\Delta_L(G_{\min}^{-1}(\lambda) - \xi), 1 - \Delta_R(G_{\min}^{-1}(\lambda) - \xi)]} \max_{\rho \in (0, \rho)} rGJS \left( \frac{(\alpha - \lambda' + G_{\min}^{-1}(\lambda) - \xi)P_1 + (1 - \lambda' + \rho)P_2}{1 - \lambda' + G_{\min}^{-1}(\lambda) - \xi}, \frac{P_2, 1 - \lambda' + G_{\min}^{-1}(\lambda) - \xi + \rho}{r} \right)
\]

\[
= \min_{\alpha \in [\Delta_L(G_{\min}^{-1}(\lambda) - \xi), 1 - \Delta_R(G_{\min}^{-1}(\lambda) - \xi)]} \max_{\rho \in (0, \rho)} rGJS \left( (\alpha - \lambda' + G_{\min}^{-1}(\lambda))(1 - \lambda')P_1 + (1 - \lambda')P_2, 1 - \lambda' + G_{\min}^{-1}(\lambda) \right) - \xi''
\]

\[
\Theta(n^{1-t/2}) \text{ and clearly } \limsup_{n \to \infty} \Delta_n/n^{1-t/2} \leq \tilde{\Delta} + \xi.
\]

Since \( \xi \) is defined under (77). In a completely similar and symmetric fashion, we have

\[
\lambda - \xi''.
\]
and
\[
\liminf_{n \to \infty} \min_{C \in \{1, \cdots, n\}} \min_{C_R \in \mathcal{P}^n_{\mathcal{I}_n}(C, \Delta_n)} (r - \rho)n^{\epsilon} \text{GJS} \left( \hat{P}^{r \xi}_C, P_1, \frac{C_n + \rho n}{(r - \rho)n} \right)
\]
\[
= \min_{\alpha \in [0, 1]} \frac{r(\Delta_{1, R}^n(\Delta_1 + \xi))^2 \chi_2(P_2||P_1)}{2\alpha(\alpha + r)} \quad (132)
\]
\[
= \min_{\alpha \in [0, 1]} h_2(\alpha) \quad (133)
\]
Note that in (131) and (133), \( h_1(\alpha) = h_2(\alpha) \) for any \( \alpha \in [0, 1] \) from the definitions of \( \Delta_{1, R}^n(\Delta_1), \Delta_{C,R}^n(\Delta_n) \) in (42) as well as the values and property of \( \mathcal{I}_n(C) \) in Remark 1. Since \( \Delta_{1, R}^n(\Delta_1 + \xi) = 2(\Delta_1 + \xi) - \Delta_{1, R}^n(\Delta_1 + \xi) \), by letting \( h_1(\alpha) = h_2(\alpha) \), we can obtain
\[
\Delta_{1, R}^n(\Delta_1 + \xi)
\]
\[
= \frac{2(\Delta_1 + \xi) \sqrt{(1 - \alpha)(1 - \alpha + r) \chi_2(P_2||P_1)}}{\alpha(\alpha + r) \chi_2(P_2||P_1) + \sqrt{(1 - \alpha)(1 - \alpha + r) \chi_2(P_2||P_1)}} \quad (134)
\]
By plugging (134) back to (131), we have (135)-(137), where \( \xi_1 > 0 \) is a function of \( \xi \) satisfying \( \xi_1 \to 0^+ \) as \( \xi \to 0^+ \); and
\[
\liminf_{n \to \infty} \min_{C \in \{1, \cdots, n\}} \min_{C_R \in \mathcal{P}^n_{\mathcal{I}_n}(C, \Delta_n)} \left( r - \rho \right)n^{\epsilon} \text{GJS} \left( \hat{P}^{r \xi}_C, P_1, \frac{C_n + \rho n}{(r - \rho)n} \right) = \lambda + \xi_2 \quad (138)
\]
where \( \xi_2 > 0 \) is a function of \( \xi \) satisfying \( \xi_2 \to 0^+ \) as \( \xi \to 0^+ \).

Using these bounds and the continuity of \( \Phi(\cdot) \), we have
\[
\limsup_{n \to \infty} \max_{C \in \{1, \cdots, n\}} \mathbb{P}_C \left\{ L(\text{T}^{\rho}_C, C_n + \rho n, \frac{r - \rho}{n + 2\rho}) \leq \frac{\lambda \cdot \xi_1}{1 + 2\rho} \right\}
\]
\[
\leq \Phi \left( \limsup_{n \to \infty} \frac{\lambda - \lambda - \xi_1}{\sqrt{n/(2\chi_2(1)\rho)}} \right) \quad (139)
\]
\[
\leq \Phi \left( \limsup_{n \to \infty} \frac{\lambda - \lambda - \xi_2}{\sqrt{n/(2\chi_2(1)\rho)}} \right) \quad (140)
\]
Recall that \( \lambda = \lambda + o(n^{-t}) \). From Lemma 5, we have \( 0 < \liminf_{n \to \infty} \xi_1(\rho), \liminf_{n \to \infty} \xi_2(\rho) < \infty \). Since \( \xi_1, \xi_2 > 0 \) and \( t \in (0, 1/2) \), we have that for any \( \lambda \in (0, \infty) \), the limsup’s in both (139) and (140) are equal to \( -\infty \), which guarantees that both the upper bounds in (139) and (140) vanish. Then the sequence of estimators as defined in (24) yields a sequence of erasure probabilities \{\( e_n \}_{n=1}^\infty \} and a sequence of confidence widths \{\( \Delta_n \}_{n=1}^\infty \} such that \( \limsup_{n \to \infty} e_n = 0 \leq \epsilon \) (for any \( \epsilon \in (0, 1) \)) and \( \limsup_{n \to \infty} \Delta_n/n^{1-t/2} = \Delta + \xi \).

Therefore, \( \Delta_1 + t \) is a \( (r, \rho, \epsilon, t) \)-achievable NCW. Since \( \xi > 0 \) is arbitrary, taking \( \xi \to 0^+ \), we see that
\[
\Delta_1^*(r, \rho, \epsilon) \leq \Delta_1
\]
\[
= \max_{\alpha \in [0, 1]} \frac{\sqrt{\alpha(\alpha + r) \chi_2(P_2||P_1)}}{\sqrt{2r \chi_2(P_2||P_2) \chi_2(P_2||P_1)}} (1 \pm \sqrt{(1 - \alpha)(1 - \alpha + r) \chi_2(P_2||P_1)}) \quad (141)
\]

E. Converse Proof of (19)

The converse proof of (19) follows along the same lines as that of (18) in Section V-C up to and including (111) with \( \lambda \) replaced by \( \lambda n^{-t} \). Given any \( \xi > 0 \) and recalling the definition of \( \Delta_1 \) in Section V-D, we let
\[
\Delta = n^{1-t/2}(\Delta_1 - \xi),
\]
throughout this converse proof. Then from Corollary 7, \( \tau_n = \Delta_n^{1,n^{-t}}(\Delta + n^{\rho n}) + 2 = \Theta(\Delta + n^{\rho n}) = \Theta(n^{1-t/2}) \) and \( \tau_n = n - \Delta_n^{1,n^{-t}}(\Delta + n^{\rho n}) = n - \Theta(n^{1-t/2}) = \Theta(n) \).

Since \( \frac{1}{n}(C - C_L) = \frac{1}{n}(\Delta_n^{1,n^{-t}}(\Delta + n^{\rho n}) + 1) = \Theta(n^{-t/2}) \) and \( \frac{1}{n}(C - C_L) = \frac{1}{n}(\Delta_n^{1,n^{-t}}(\Delta + n^{\rho n}) + 1) = \Theta(n^{1-t/2}) \), we can reuse the Taylor expansions of GJS as in (127) and (128).

Using the continuity of GJS in the distributions and the third argument, and following the monotonicity used in (114) and similar techniques from (125)-(138), we see that
\[
\limsup_{n \to \infty} \min_{C \in \{1, \cdots, n\}} \max_{\rho \in \mathbb{E}_n} \mathbb{P}_C \left\{ L(\text{T}^{\rho}_C, \frac{C_n + \rho n}{r - \rho}, \frac{r - \rho}{n + 2\rho}) \leq \frac{\lambda \cdot \xi_1}{1 + 2\rho} \right\}
\]
\[
\leq \Phi \left( \limsup_{n \to \infty} \frac{\lambda - \lambda - \xi_1}{\sqrt{n/(2\chi_2(1)\rho)}} \right) \quad (145)
\]
\[
\geq \Phi \left( \liminf_{n \to \infty} \frac{\lambda \cdot \xi_1}{1 + 2\rho} \right) \quad (146)
\]
and
\[
\limsup_{n \to \infty} \min_{C \in \{1, \cdots, n\}} \max_{\rho \in \mathbb{E}_n} \mathbb{P}_C \left\{ L(\text{T}^{\rho}_C, \frac{C_n + \rho n}{r - \rho}, \frac{r - \rho}{n + 2\rho}) \leq \frac{\lambda \cdot \xi_2}{1 + 2\rho} \right\}
\]
\[
\leq \Phi \left( \limsup_{n \to \infty} \frac{\lambda - \lambda - \xi_2}{\sqrt{n/(2\chi_2(1)\rho)}} \right) \quad (147)
\]
\[
\geq \Phi \left( \liminf_{n \to \infty} \frac{\lambda \cdot \xi_2}{1 + 2\rho} \right) \quad (148)
\]
From Lemma 5, we have \( 0 < \limsup_{n \to \infty} n^{t} \max_{\rho \in \mathbb{E}_n} \hat{c}_1(\rho), \limsup_{n \to \infty} n^{t} \max_{\rho \in \mathbb{E}_n} \hat{c}_2(\rho) < \infty \). Since \( \xi_1, \xi_2 > 0 \) and \( t \in (0, 1/2) \), we have that
\[
\limsup_{n \to \infty} \frac{\xi_1}{n^{1/2} \hat{c}_1(\rho)} = \limsup_{n \to \infty} \frac{\xi_1}{n^{1/2} \hat{c}_1(\rho)} = \infty \quad (149)
\]
\[
\limsup_{n \to \infty} \frac{\xi_2}{n^{1/2} \hat{c}_2(\rho)} = \limsup_{n \to \infty} \frac{\xi_2}{n^{1/2} \hat{c}_2(\rho)} = \infty \quad (150)
\]
which means that at both of the two bounds in (145) and (147) tend to one; that is, the erasure probability \( \max_{C \in \{1, \cdots, n\}} \mathbb{P}_C \{ \mathcal{E}_n^r \} \).
cannot be made less than \( \varepsilon \) for any \( \varepsilon \in [0,1) \). Thus, the \( t \)-optimal NCW cannot be smaller than or equal to \( \Delta_t - \xi \), i.e., \( \Delta^*_t(r, \lambda, \varepsilon) \geq \Delta_t - \xi \). Since \( \xi > 0 \) can be made arbitrarily small, it means that

\[
\Delta^*_t(r, \lambda, \varepsilon) \geq \Delta_t = \lambda + \xi_1. \tag{137}
\]

completing the proof of the converse of (19).

**APPENDIX**

A. Proof of Lemma 2

Let \( \beta = 1/r \). Since \( \text{GJS}(tP_1 + (1 - t)P_2, P_2, \beta) \) and \( \text{GJS}((1 - t)P_1 + tP_2, P_1, \beta) \) are symmetric, it suffices to prove that for any \( \beta > 0 \) and any pair of \( (P_1, P_2) \), \( \text{GJS}(tP_1 + (1 - t)P_2, P_2, \beta) \) is a strictly increasing function of \( t \in [0,1] \).

The first and second derivatives of \( \text{GJS}(tP_1 + (1 - t)P_2, P_2, \beta) \) with respect to \( t \) are

\[
\frac{\partial \text{GJS}(tP_1 + (1 - t)P_2, P_2, \beta)}{\partial t} = \sum_{x} (P_1(x) - P_2(x)) \log \left( \frac{1 + \beta(tP_1(x) + (1 - t)P_2(x))}{\beta tP_1(x) + (\beta - \beta t + 1)P_2(x)} \right) - \sum_{x} (tP_1(x) + (1 - t)P_2(x)) \left( \frac{\beta tP_1(x) - P_2(x)}{\beta tP_1(x) + (\beta - \beta t + 1)P_2(x)} \right) \tag{152}
\]

and

\[
\frac{\partial^2 \text{GJS}(tP_1 + (1 - t)P_2, P_2, \beta)}{\partial t^2} = \sum_{x} \frac{(P_1(x) - P_2(x))^2}{tP_1(x) + (1 - t)P_2(x)} - 2 \sum_{x} \frac{\beta (P_1(x) - P_2(x))^2}{\beta tP_1(x) + (\beta - \beta t + 1)P_2(x)} + \sum_{x} \frac{\beta^2 (P_1(x) - P_2(x))^2 (tP_1(x) + (1 - t)P_2(x))}{(\beta tP_1(x) + (\beta - \beta t + 1)P_2(x))^2} \tag{153}
\]

B. Proof of Corollary 4

We have the following deduction based on the existing lemmas and corollaries,

- From Corollary 3, the minimization interval \([\Delta_L(\Delta), 1 - \Delta_R(\Delta)]\) shrinks as \( \Delta \) increases.
- From Corollary 7, we can deduce that with fixed \( \alpha \), as \( \Delta \) increases, both \( \alpha - \ell(\alpha) + \Delta \) and \( 1 - \ell(\alpha) + \Delta \) increase and thus, the function \( \text{GJS}(\alpha - \ell(\alpha) + \Delta, P_1 + (1 - \ell(\alpha) + \Delta)P_2, P_2, 1 - \ell(\alpha) + \Delta) \) increases based on Lemma 1 and Lemma 2.

Consequently, by combining these two points, Corollary 4 is proved.

C. Proof of Lemma 5

1) Upper Bound: For any \( j \in [1, C - (\Delta n - n^\varepsilon n)] \) and any \( \rho \in (0, r) \), let \( \beta = j/n \) and we recall that \( V(\tilde{P}_j, P_2, j, r, \rho) \) is given by

\[
V(\tilde{P}_j, P_2, j, r, \rho) = \frac{n + \rho n - j}{n + 2\rho n} \times \text{Var}_{\tilde{P}_j} \left[ \log \frac{(n - j + N)\tilde{P}_j(X)}{(n + \rho n - j + N)\tilde{P}_j(X) + (N - \rho n)P_2(X)} \right] + \frac{N - \rho n}{n + 2\rho n} \times \text{Var}_{P_2} \left[ \log \frac{(n - j + N)P_2(X)}{(n + \rho n - j + N)P_2(X) + (N - \rho n)P_2(X)} \right] \tag{156}
\]


\[
\begin{align*}
\text{On the other hand, when } & V \cdot \rho \leq R \rho \text{ and only if } \\
& \lim_{n \to \infty} \frac{\log(1 - \beta + r) (\alpha - \beta) P_1(x) + (1 - \alpha + r) P_2(x)}{1 - \beta + r} P_1(x) + (1 - \alpha + r) P_2(x) \leq A_2(\rho), \\
& \text{as } n \to \infty. \quad (158)
\end{align*}
\]

for some \( A_1(\rho), A_2(\rho) \in \mathbb{R}_+. \) Then we have

\[
\begin{align*}
1 + \rho - \beta & \frac{\text{Var}_{(\alpha - \beta / \rho) + \alpha + \rho)} P_2(x)}{1 - \beta + r} P_1(x) + (1 - \alpha + r) P_2(x) \\
& \leq 1 + \rho - \beta \frac{\text{Var}_{(\alpha - \beta / \rho) + \alpha + \rho)} P_2(x)}{1 - \beta + r} P_1(x) + (1 - \alpha + r) P_2(x) \\
& \leq 1 + \rho - \beta \frac{\text{Var}_{(\alpha - \beta / \rho) + \alpha + \rho)} P_2(x)}{1 - \beta + r} P_1(x) + (1 - \alpha + r) P_2(x) \\
& \leq 1 + \rho - \beta \frac{\text{Var}_{(\alpha - \beta / \rho) + \alpha + \rho)} P_2(x)}{1 - \beta + r} P_1(x) + (1 - \alpha + r) P_2(x) \\
& < 1 + \rho - \beta (A_2(\rho) + A_2(\rho)) =: A_3(\rho),
\end{align*}
\]

for some \( A_3(\rho) \in \mathbb{R}_+. \)

Similarly, the second term in (157) is smaller than some constant \( A_4(\rho) \in \mathbb{R}_+. \) Hence, for any \( j \in [1, C - (\Delta n - n \cdot \alpha)] \) and any \( \rho \in (0, r), V(P_j^+, P_j, j, r, \rho) < A_5(\rho) + A_4(\rho). \) Similarly, for any \( j \in (C + (\Delta n - n \cdot \alpha), n] \) and any \( \rho \in (0, r), V(P_j^+, P_j, j, r, \rho) < A_5(\rho) \) for some \( A_5(\rho) \in \mathbb{R}_+. \) Let \( \pi_1(\rho) := \max\{A_3(\rho), A_4(\rho), A_5(\rho)\}. \) Since \( 0 < \pi_1(\rho) < \infty \) holds for any \( \rho \in (0, r), \) when \( n \to \infty, \) we can similarly upper bound \( \lim_{n \to \infty} V(P_j^+, P_j, j, r, \rho) \) by \( \limsup_{n \to \infty} \pi_1(\rho) \in \mathbb{R}_+ \) for any \( \rho \in (0, r). \) In a similar manner, we can also show that for any \( \rho \in (0, r), T(P_j^+, P_j, j, r, \rho), T(P_j^+, P_j, j, r, \rho) < \pi_2(\rho) \) for some \( \pi_2(\rho) \in \mathbb{R}_+ \) and \( \limsup_{n \to \infty} \pi_2(\rho) < \infty. \)

2) Lower Bound: On the other hand, when \( P_1 \neq P_2, \) for any \( \rho \in (0, r), \)

- \( V(P_j^+, P_j, j, r, \rho) = V(P_j^+, P_j, j, r, \rho) = 0 \) if and only if \( j = C, \)
- \( \lim_{n \to \infty} V(P_j^+, P_j, j, r, \rho) = \lim_{n \to \infty} V(P_j^+, P_j, j, r, \rho) = 0 \) if and only if \( \lim_{n \to \infty} |j - C|/n = 0, \)
- By taking Taylor expansion of \( V(P_j^+, P_j, j, r, \rho) \) around \( P_j^- = P_2, \) for any \( j \in [1, C - (\Delta n^1 - t^2 / 2 - n \cdot \alpha)] \) with

some \( \Delta_t \in \mathbb{R}_+, \) any \( t \in (0, 1/2) \) and any \( \rho \in (0, r), \) we have

\[
\begin{align*}
n^t V(P_j^+, P_j, j, r, \rho) & = \Theta(n^t (\alpha - \beta)^2) \\
& = \Omega((\Delta_t - n \cdot \alpha)^{-1} t^2 / 2)^2). \quad (163)
\end{align*}
\]

Thus, \( \lim_{n \to \infty} n^t V(P_j^+, P_j, j, r, \rho) = \lim_{n \to \infty} n^t V(P_j^+, P_j, j, r, \rho) = 0 \) if \( \Delta_t = 0. \)

For \( j \notin [C \pm (\Delta n - n \cdot \alpha)] \) with some constant \( \Delta > 0, \) we have \( \min\{V(P_j^+, P_j, j, r, \rho) \} \geq g_3(\rho) > 0 \) for some \( g_3(\rho) \in \mathbb{R}_+ \) and we also have \( \liminf_{n \to \infty} g_3(\rho) > 0 \) for any \( \rho \in (0, r). \) For \( j \notin [C \pm (\Delta n^1 - t^2 / 2 - n \cdot \alpha)] \), we have \( \liminf_{n \to \infty} n^t g_3(\rho) > 0 \) for any \( \rho \in (0, r). \)

D. Probability of Atypical Set

For any \( i, j \in [n] \) and \( i < j, \) when \( j \) is the true changepoint, the probability of \( \tilde{X}_i^+ \in B(P_i) \) can be bounded using Hoeffding’s inequality and the union bound as

\[
\begin{align*}
\mathbb{P}_j \{X_{i}^+ \not\in B(P_i)\} \leq 2[X]/(\iota + \rho \eta)^2. \quad (164)
\end{align*}
\]

Reusing this calculation on the other typical sequences, the probability of the atypical set \( A_{i,j} \) can be bounded as

\[
\begin{align*}
\mathbb{P}_j \{d^t_{i,j} \in B(P_i)\} & \leq \mathbb{P}_j \{X_{i}^+ \not\in B(P_i)\} + \mathbb{P}_j \{X_{i+1}^+ \not\in B(P_i)\} \\
& \leq \mathbb{P}_j \{X_{i}^+ \not\in B(P_i)\} + \mathbb{P}_j \{X_{i}^+ \not\in B(P_i)\} \\
& \leq 2[X]/(\iota + \rho \eta)^2 + 2[X]/(\iota + \rho \eta)^2 \\
& \leq 2[X]/(\iota + \rho \eta)^2 + 2[X]/(\iota + \rho \eta)^2. \quad (165)
\end{align*}
\]

E. Proof of Lemma 6

First of all, the functions \( g_1(\alpha, \zeta), g_2(\alpha, \zeta) \) can be rewritten as

\[
\begin{align*}
g_1(\alpha, \zeta) \quad & = \text{GJS} \left( \frac{C_{2} + (1 + \rho - \alpha) P_2, 1 + \rho - \alpha + \zeta}{1 + \rho - \alpha + \zeta} \right). \quad (167) \\
g_2(\alpha, \zeta) \quad & = \text{GJS} \left( \frac{C_{2} + (1 + \rho - \alpha + \zeta)}{1 + \rho - \alpha + \zeta} \right). \quad (168)
\end{align*}
\]

Fix any \( \zeta \in (0, r). \) Define the function

\[
t(\alpha) = \frac{\zeta}{1 + \rho - \alpha + \zeta} \in \left[\frac{\zeta}{1 + \rho - \alpha + \zeta}, \frac{\zeta}{1 + \rho - \alpha + \zeta} \right] \subset (0, 1). \quad (169)
\]

Note that \( t(\alpha) \) is a strictly increasing function of \( \alpha \in [\delta, 1]. \) In the following, we write \( t \) instead of \( t(\alpha) \) for brevity.

Then we can further rewrite \( g_1(\alpha, \zeta) \) as \( \text{GJS}(t(P_1 + (1 - t)P_2, P_2, t/\zeta)) \). Let \( r' = \zeta/(r - \rho) \). Next, it suffices to prove that \( \text{GJS}(t(P_1 + (1 - t)P_2, P_2, r'/t)) \) is an increasing function of \( t \in (0, 1). \)

\[
\text{GJS} \left( t(P_1 + (1 - t)P_2, P_2, r'/t) \right) = D \left( P_2, t \right) + \frac{r'}{t} \left( t(P_1 + (1 - t)P_2, P_2, r'/t) \right)
\]
Thus, we have
\[ t P_1 + (1 - t) P_2 \frac{\partial (t P_1 + (1 - t) P_2 + P_2)}{1 + \frac{r'}{r}}. \]

Let \( \beta = \frac{r'}{r} \), which increases as \( t \) increases. Then
\[ \frac{\partial D(P_2 \parallel \beta P_1 + (1 - \beta) P_2)}{\partial \beta} = \sum_x P_2(x) \frac{(P_1(x) - P_2(x))}{\beta P_1(x) + (1 - \beta) P_2(x)}. \]

The derivatives of \( D(P_2 \parallel \beta P_1 + (1 - \beta) P_2) \) with respect to \( \beta \) are given by
\[ \frac{\partial^2 D(P_2 \parallel \beta P_1 + (1 - \beta) P_2)}{\partial \beta^2} = \sum_x P_2(x) \frac{(P_1(x) - P_2(x))^2}{(\beta P_1(x) + (1 - \beta) P_2(x))^2} > 0. \]

Thus, we have
\[ \frac{\partial D(P_2 \parallel \beta P_1 + (1 - \beta) P_2)}{\partial \beta} \bigg|_{\beta = 0} = 0, \]
which implies that \( D(P_2 \parallel \beta P_1 + (1 - \beta) P_2) \) is an increasing function of \( \beta \in (0, 1) \). Thus, the KL divergence on the left of the equality in (171) is an increasing function of \( t \in (0, 1) \).

The derivatives of the second term in (170) with respect to \( t \) are given by
\[ \frac{\partial^\prime}{\partial t} D \left( t P_1 + (1 - t) P_2 \right) \frac{\partial (t P_1 + (1 - t) P_2 + P_2)}{1 + \frac{r'}{r}}. \]

Let us compute the derivatives of the function on the right-hand side of (179) as follows:
\[ \frac{\partial}{\partial t} \sum_x \frac{t P_1(x) + (1 - t) P_2(x) - P_2(x)}{t P_1(x) + (r' - r't + t) P_2(x)} \]
\[ = \sum_x \frac{2P_2^3(x)(P_2(x) - P_1(x))r'^2}{(r't P_1(x) + (r' - r't + t) P_2(x))^3}. \]

When \( P_2(x) > P_1(x) \),
\[ \sum_{x : P_2(x) > P_1(x)} \frac{-2P_2^3(x)(P_2(x) - P_1(x))r'^2}{(r't P_1(x) + (r' - r't + t) P_2(x))^3} < \sum_{x : P_2(x) > P_1(x)} \frac{2(P_1(x) - P_2(x))r'^3}{(t + r')^3}, \]
and when \( P_2(x) \leq P_1(x) \),
\[ \sum_{x : P_2(x) \leq P_1(x)} \frac{2P_2^3(x)(P_1(x) - P_2(x))r'^2}{(r't P_1(x) + (r' - r't + t) P_2(x))^3} \leq \sum_{x : P_2(x) \leq P_1(x)} \frac{2(P_1(x) - P_2(x))r'^3}{(t + r')^3}, \]
and we have
\[ \sum_x \frac{-2P_2^3(x)(P_2(x) - P_1(x))r'^2}{(r't P_1(x) + (r' - r't + t) P_2(x))^3} \leq 0. \]

Thus,
\[ \frac{\partial}{\partial t} \left| \sum_x \frac{(t + r')(P_1(x) + (1 - t) P_2(x))P_2(x)}{r't P_1(x) + (r' - r't + t) P_2(x)} \right|_{t=0} = 0, \]
and
\[\sum_{x} (t + r')(t P_1(x) + (1 - t) P_2(x)) P_2(x) \leq \left. \sum_{x} \left( (t + r')(t P_1(x) + (1 - t) P_2(x)) P_2(x) \right) \right|_{t=0} = 1, \]
which implies that the second term in (170) is an increasing function of \( t \in (0, 1) \).

Finally, we can show that \( GJS(t P_1 + (1 - t) P_2, P_2', r'/t) \) is an increasing function of \( t \in (0, 1) \), and thus \( g_1(\alpha, \zeta) \), defined in (40), is an increasing function of \( \alpha \in [\zeta, 1] \).

Following similar steps, we can also prove that \( g_2(\alpha, \zeta) \), defined in (41), is a decreasing function of \( \alpha \in [0, 1 - \zeta] \). 

\section*{F. Proof of Corollary 7}

The first statement can be easily deduced from Lemma 1 and Lemma 2.

For the second statement, by applying the Taylor expansion of \( GJS(\cdot, P_2, P_1, n - \Delta_\kappa, -\Delta_{C,\rho} + (1 + \rho n) \) around \( P_2 \) and that of \( GJS(\cdot, P_1, n - \Delta_\kappa, -\Delta_{C,\rho} + (1 + \rho n) \) around \( P_1 \), we have
\[
GJS \left( \frac{\tilde{P}_j - \Delta_{C,\rho}(\Delta_n)}{n} \right) = O \left( \frac{\Delta_{n,\rho}(\Delta_n)}{n} \right),
\]
\[
GJS \left( \frac{\tilde{P}_j + \Delta_{C,\rho}(\Delta_n)}{n} \right) = O \left( \frac{2 \Delta_n - \Delta_{n,\rho}(\Delta_n)}{n} \right).
\]

Since the left-hand sides of (187) and (188) are equal according to (42), to make the right-hand sides to be equal, we must have \( \Delta_{n,\rho}(\Delta) = \Delta(\Delta) \). Similarly, we also have \( \Delta_{n,\rho}(\Delta) = \Delta(\Delta) \).

\section*{G. Proof of Lemma 8}

Recall the definitions of \( \kappa_\alpha \) in (35), \( B(P) \) in (37) and \( \rho \in [n^{\kappa_\alpha - 1}, r - n^{\kappa_\alpha - 1}] \). According to (39), for any \( i, j \in [n] \) such that \( i < j - n^{\kappa_\alpha} \),
\[
\mathbb{P}_j \{ X_{i + 1}^{+\rho} \notin B(P) \text{ or } X_{i + 1}^{+\rho} \notin B(P) \} \]
\[
= \mathbb{P}_j \{ X_{i + 1}^{+\rho} \notin B(P) \} \text{ or } Y_{i + 1}^{N - \rho} = B(P) \}
\]
\[
= \mathbb{P}_j \{ X_{i + 1}^{+\rho} \notin B(P) \} \text{ or } Y_{i + 1}^{N - \rho} = B(P) \}
\]
\[
= O(n^{-2\kappa_\alpha}).
\]

Recall the definitions of \( \tilde{P}_j^{-\rho} \) and \( \tilde{P}_j^{+\rho} \) in (34). Fix any \( j \in [1, C - n^{\kappa_\alpha}] \) and let \( \mathbb{P}_j := (P_i, \tilde{P}_j^{-\rho}, P_1, P_2) \) and \( \mathbb{P}_j := (P_j, \tilde{P}_j^{+\rho}, P_1, P_2) \). Notice that
\[
\mathbb{L} \left( \mathbb{P}_j, \frac{\tilde{P}_j^{-\rho} + \rho}{n + 2\rho, 1 + 2\rho} \right) = \Omega(n^{2\kappa_\alpha - 2})
\]
because \( \mathbb{L}(\mathbb{P}_j^{-\rho}, \frac{\tilde{P}_j^{-\rho} + \rho}{n + 2\rho, 1 + 2\rho}) \) is strictly decreasing in \( j \) (a fact that can be proved similarly as that of Lemma 2) and from the Taylor expansion of the function \( \mathbb{L}(\cdot, \frac{\tilde{P}_j^{-\rho} + \rho}{n + 2\rho, 1 + 2\rho}) \) around \( P_2 \); see (127) in which \( C_+ \) is replaced by \( C - n^{\kappa_\alpha} \). Similarly for any \( j \in [C + n^{\kappa_\alpha}, n] \) we have
\[
\mathbb{L} \left( \mathbb{P}_j, \frac{\tilde{P}_j^{-\rho} + \rho}{n + 2\rho, 1 + 2\rho} \right) = \Omega(n^{2\kappa_\alpha - 2}).
\]

For any \( j \in [1, C - n^{\kappa_\alpha}] \), the Taylor expansion of \( \mathbb{L}(\mathbb{P}_j, \frac{\tilde{P}_j^{-\rho} + \rho}{n + 2\rho, 1 + 2\rho}) \) around \( \mathbb{P}_2 \) follows from Hoeffding’s inequality and \( M_1(\rho) \neq M_2(\rho) \) are two constants depending only on \( \rho \) such that for any \( j \in [1, C - n^{\kappa_\alpha}] \),
\[
M_1(\rho) \leq \log \frac{(n - j + N) \tilde{P}_j(X)}{(n + (j + n) P_2(X) + (N - \rho) P_2(X)}, \]
\[
\log \frac{(n - j + N) \tilde{P}_j(X)}{(n + (j + n) P_2(X) + (N - \rho) P_2(X) \leq M_2(\rho), \ a.s.,}
\]
\[
(197) \text{ follows since } \mathbb{L}(\mathbb{P}_j, \frac{\tilde{P}_j^{-\rho} + \rho}{n + 2\rho, 1 + 2\rho}) \text{ is strictly decreasing in } j \text{ and its Taylor expansion of the first argument around } \mathbb{P}_j^{-\rho} \text{ is}
\]
\[
\mathbb{L} \left( \mathbb{P}_j^{-\rho}, \frac{i - n^{\kappa_\alpha} + \rho}{n + 2\rho, 1 + 2\rho} \right)
\]
\[
= \frac{\nu}{r - \rho} \mathbb{GJS} \left( \frac{\tilde{P}_j^{-\rho}, P_2, n - i + n^{\kappa_\alpha} + \rho}{n + 2\rho, 1 + 2\rho} \right)
\]
\[
= \frac{\nu}{r - \rho} \mathbb{GJS} \left( \frac{\tilde{P}_j^{-\rho}, P_2, n - i + n^{\kappa_\alpha} + \rho}{n + 2\rho, 1 + 2\rho} \right) + \Theta(n^{\kappa_\alpha - 1})
\]
\[
= \mathbb{L} \left( \mathbb{P}_j^{-\rho}, \frac{i + n^{\kappa_\alpha} + \rho}{n + 2\rho, 1 + 2\rho} \right) + \Theta(n^{\kappa_\alpha - 1})
\]
\[
(198) \text{ follows since } \exp \left\{ -c' n^{2\kappa_\alpha - 1} \right\} = \exp \left\{ -c' n^{2\kappa_\alpha - 1} \right\} < \frac{1}{n^{\kappa_\alpha}}, \text{ for } n \text{ sufficiently large.}
\]

Similarly, using (194), for any \( i > C \) and \( j \in [i + n^{\kappa_\alpha}, n] \), we also have
\[
\mathbb{P}_C \{ \mathbb{L}(\mathbb{P}_j, \frac{\tilde{P}_j^{-\rho} + \rho}{n + 2\rho, 1 + 2\rho}) \leq \mathbb{L}(\mathbb{P}_j, \frac{i + n^{\kappa_\alpha} + \rho}{n + 2\rho, 1 + 2\rho}) \}
\]
\[
= \Omega \left( \frac{1}{n^{\kappa_\alpha}} \right).
\]

This completes the proof of Lemma 8.
\[ \begin{align*}
L\left( T_{j,i}^{\rho}, \frac{j + \rho n}{n + 2\rho n}, r - \rho \right) &= \frac{n - j + \rho n}{n + 2\rho n} \sum_x T_{x}^{n-j+n}(x) \log \frac{(n - j + N) \tilde{P}_j^-(x)}{(C - j) P_1(x) + (n - C + N) P_2(x)} \\
&+ \frac{N - \rho n}{n + 2\rho n} \sum_x T_{x}^{N-n}(x) \log \frac{(n - j + N) P_2(x)}{(C - j) P_1(x) + (n - C + N) P_2(x)} \\
&+ O\left( \frac{\log(j + \rho n)}{j + \rho n} \right) - O\left( \frac{\log(C - j)}{C - j} \right) + O\left( \frac{\log(n - C + \rho n)}{n - C + \rho n} \right) + O\left( \frac{\log(N - \rho n)}{N - \rho n} \right) \\
&= \frac{1}{n + 2\rho n} \left( \sum_{i \in [j+1:n]} \log \frac{(n - j + N) \tilde{P}_j^+(y_{i,1})}{(j + \rho n) P_j^+(y_{i,1}) + (N - \rho n) P_1(y_{i,1})} + \sum_{i \in [j]} \log \frac{(n - j + N) \tilde{P}_j^+(y_{i,1})}{(j + \rho n) P_j^+(y_{i,1}) + (N - \rho n) P_1(y_{i,1})} \right) \\
&+ \frac{1}{n + 2\rho n} \left( \sum_{i \in [N-n+1:N]} \log \frac{(n - j + N) \tilde{P}_j^+(y_{i,1})}{(j + \rho n) P_j^+(y_{i,1}) + (N - \rho n) P_1(y_{i,1})} + \sum_{i \in [j]} \log \frac{(n - j + N) \tilde{P}_j^+(y_{i,1})}{(j + \rho n) P_j^+(y_{i,1}) + (N - \rho n) P_1(y_{i,1})} \right) \\
&+ \frac{1}{n + 2\rho n} \left( \sum_{i \in [N-n]} \log \frac{(n - j + N) \tilde{P}_j^+(y_{i,1})}{(j + \rho n) P_j^+(y_{i,1}) + (N - \rho n) P_1(y_{i,1})} + \sum_{i \in [j]} \log \frac{(n - j + N) \tilde{P}_j^+(y_{i,1})}{(j + \rho n) P_j^+(y_{i,1}) + (N - \rho n) P_1(y_{i,1})} \right)
\end{align*} \]
H. Proof of Lemma 9

Recall $\Gamma_n(i, \Delta_n)$ defined in (60) for any $i \in [n]$. Given any true change-point $C \in [n]$, we have that for any $i \in [C \pm \Delta_n]$, using the union bound over all $O(n)$ values of $j \notin [i \pm (\Delta_n + n^{s_n})]$ and Lemma 8,

$$P_C \left\{ \min_{j \notin [i \pm \Delta_n]} L \left( T_{ij}, j + \rho n \frac{r - \rho}{n + 2\rho \pm 1 + 2\rho} \right) \right\} = O \left( \frac{1}{\sqrt{n}} \right).$$

Similarly, for any $i < C - \Delta_n$, using the union bound over all $O(n)$ values of $j \in [i - \Delta_n]$, we have

$$P_C \left\{ \min_{j \in [i - \Delta_n]} L \left( T_{ij}, j + \rho n \frac{r - \rho}{n + 2\rho \pm 1 + 2\rho} \right) \right\} = O \left( \frac{1}{\sqrt{n}} \right).$$

Thus, combining (206) and (207), we have for any $i < C - \Delta_n$

$$P_C \left\{ \arg \min_{j \notin [i \pm \Delta_n]} L \left( T_{ij}, j + \rho n \frac{r - \rho}{n + 2\rho \pm 1 + 2\rho} \right) \notin [C \pm n^{s_n}] \right\} \leq O \left( \frac{1}{\sqrt{n}} \right).$$

where (209) follows from (206). In a completely symmetric manner, (210) also holds for $i > C + \Delta_n$.

We can then show that $I^*_\rho \in [C \pm \Delta_n]$ with high probability for $n$ large enough:

$$P_C \left\{ I^*_\rho \notin [C \pm \Delta_n] \right\} \leq P_C \left\{ \min_{j \notin [i \pm \Delta_n]} L \left( T_{ij}, j + \rho n \frac{r - \rho}{n + 2\rho \pm 1 + 2\rho} \right) \leq \min_{j \notin [i \pm \Delta_n]} L \left( T_{ij}, j + \rho n \frac{r - \rho}{n + 2\rho \pm 1 + 2\rho} \right) \right\} \leq O \left( \frac{1}{\sqrt{n}} \right).$$

where (216) follows from (214), (217) follows from the union bound over all values of $i \in [C \pm \Delta_n] \setminus [i_{\rho}(C) \pm n^{s_n}]$. (218) follows by letting $i' = i_{\rho}(C)$ and (219) follows by choosing one of the values of $j \notin [i \pm \Delta_n]$ and from (205).
Recall $P_j = P_j^+ = (P_j, P_j^-, P_j^+, P_j)$ for $j \leq C$, $P_j = P_j^+ = (P_j, P_j^-, P_j^+, P_j)$ for $j > C$ and the definition of $J_i^*(i^*)$ for any $i \in [n]$ in (53). Let $c$ be some positive constant. According to the Taylor expansions of $L(T_j^+ \in \frac{n+\rho_{n}^n}{n+\rho_{n}^n}, \frac{r-\rho}{1+2p})$ around $T_j^+ = P_j$ (cf. (193) and (194)) and using the Hoeffding's inequality similar to (196), for any $i \in \{\nu_i(C) + n^{\alpha n} + 1 : C + \Delta_n\}$, we have (220)–(224) on the top of next page, where (222) follows from (55) and from the Taylor expansions of $L(P_j^+, \frac{n+\rho_{n}^n}{n+\rho_{n}^n}, \frac{r-\rho}{1+2p})$ around $J^*(i^*_P(C)) = \nu_i(C)$ and $\Delta_n = 1$ for $J^*(i^*_P(C)) \in \Gamma_n^\nu_i(C, \Delta_n + n^{\alpha n})$ and $J^*(i^*_P(C)) = \nu_i(C) + \Delta_n + 1$ for $J^*(i^*_P(C)) \in \Gamma_n^\nu_i(C, \Delta_n + n^{\alpha n})$:

$$\mathbb{E}_{i^*_P(C)} \left[ L \left( P_j^+, \frac{n+\rho_{n}^n}{n+\rho_{n}^n}, \frac{r-\rho}{1+2p} \right) \right] = \mathbb{E} \left( P_j^+, \frac{n+\rho_{n}^n}{n+\rho_{n}^n}, \frac{r-\rho}{1+2p} \right) + O(n^{\alpha n-1}),(225)$$

and (232) follows since $L(P_j^+, \frac{n+\rho_{n}^n}{n+\rho_{n}^n}, \frac{r-\rho}{1+2p})$ is strictly decreasing in $j$ (a fact that can be proved similarly as that of Lemma 2) and from the Taylor expansions of $L(P_j^+, \frac{n+\rho_{n}^n}{n+\rho_{n}^n}, \frac{r-\rho}{1+2p})$ around $i = \nu_i(C)$.

In a completely symmetric manner, we have for any $i \in \{C - \Delta_n : \nu_i(C) - n^{\alpha n} - 1\},$

$$\mathbb{P}_C \left\{ \mathbb{E}_{i^*_P(C)} \left[ L \left( P_j^+, \frac{n+\rho_{n}^n}{n+\rho_{n}^n}, \frac{r-\rho}{1+2p} \right) \right] = \mathbb{E} \left( P_j^+, \frac{n+\rho_{n}^n}{n+\rho_{n}^n}, \frac{r-\rho}{1+2p} \right) \right\} = O \left( \frac{1}{n^{3/2}} \right). \quad (226)$$

Finally, combining (224) and (226), we can upper bound (219) as follows:

$$\mathbb{P}_C \left\{ I^*_P \notin \{\nu_i(C) \pm n^{\alpha n}\} \right\} = O \left( \frac{1}{n^{3/2}} \right). \quad (227)$$

1. Proof of Lemma 11

For any $\rho \in (0, r)$ and any $i \in [n]$, recall $\hat{X}_i^{\rho+mn}, \hat{X}_i^{\rho-n^{\alpha n}+mn}$ defined in (20) and (21). Let $P_{\rho+mn,n-i,\rho-2n-2\rho+2m}(X) := P_{\rho+mn}(X) \times P_{\rho-n}(\rho \times \rho) \times P_{n-2\rho+2m}(\rho \times \rho)$. For any $\hat{Q}_{\rho}^c$, we use $\hat{T}_{\rho+2n}^{\rho+2n}$ to denote the tuple of sequences for $(X_1, X_2) \in \hat{T}_{\rho+2n}$ such that $X_1^{\rho+mn} \in \hat{T}_{\rho+2n}^{\rho+2n}, X_2^{\rho-n^{\alpha n}+mn} \in \hat{T}_{\rho+2n}^{\rho-n^{\alpha n}+2m}, Y_1^{\rho+n^{\alpha n}} \in \hat{T}_{\rho-2n}^{\rho+2n}$ and $Y_2^{\rho+n^{\alpha n}} \in \hat{T}_{\rho-2n}^{\rho-n^{\alpha n}}$.

Given any estimator $\gamma$, we define the following sets of test and training sequences:

$$A_k(\gamma) = \{x^n, y^n \in \hat{Q}_{\rho}^c : \gamma(x^n, y^n) = k\}, \forall k \in [n], \quad (228)$$

$$A_k(\gamma) = \{x^n, y^n, y^n \in \hat{Q}_{\rho}^c : \gamma(x^n, y^n, y^n) = e\} = \bigcup_{k \in [n]} A_k. \quad (229)$$

Fix any $\eta \in [0, 1]^n$. Given any $j \in [n]$ and any tuple of types $\hat{Q}_{\rho,j}^c$, we can construct the following type-based estimator $\gamma^{type}_{\rho,j}$:

- if $|A_k(\gamma) \cap \hat{T}_{\rho+2n}^{\rho+2n} \geq \eta_k |T_{\rho+2n}^{\rho+2n}|$ and $|A_l(\gamma) \cap \hat{T}_{\rho+2n}^{\rho+2n} < \eta_l |T_{\rho+2n}^{\rho+2n}|$ for all $l < k$, we define $\gamma^{type}_{\rho,j}(Q_{\rho,j}^c) := k$;
- if $|A_k(\gamma) \cap \hat{T}_{\rho+2n}^{\rho+2n} < \eta_k |T_{\rho+2n}^{\rho+2n}|$ for all $k \in [n]$, we define $\gamma^{type}_{\rho,j}(Q_{\rho,j}^c) := e$.

Then for any $j \in [n]$, we have

$$\mathbb{P}_j \{ \gamma(X^n, Y_1^n, Y_2^n) \notin [j \pm \Delta] \cup \{e\} \} = \sum_{k \notin [j \pm \Delta] \cup \{e\}} \mathbb{P}_j \{ A_k \} \quad (230)$$

$$\geq \sum_{k \notin [j \pm \Delta] \cup \{e\}} \mathbb{P}_j \{ A_k \cap \hat{T}_{\rho+2n}^{\rho+2n} \} \quad (231)$$

and

$$\mathbb{P}_j \{ \gamma(X^n, Y_1^n, Y_2^n) = e \} \geq \sum_{k \in \mathbb{P}_j \{ A_k \} \cap \hat{T}_{\rho+2n}^{\rho+2n} \} \quad (232)$$

$$\geq \sum_{k \notin [j \pm \Delta] \cup \{e\}} \eta_k \mathbb{P}_j \{ \gamma^{type}_{\rho,j}(Q_{\rho,j}^c) = k \} \quad (233)$$

$$\geq \eta_{min} \mathbb{P}_j \{ \gamma^{type}_{\rho,j}(Q_{\rho,j}^c) \notin [j \pm \Delta] \cup \{e\} \}, \quad (234)$$

Finally, combining (224) and (226), we can upper bound (219) as follows:

$$\mathbb{P}_C \{ I^*_P \notin \{\nu_i(C) \pm n^{\alpha n}\} \} = O \left( \frac{1}{n^{3/2}} \right). \quad (227)$$

J. Proof of Lemma 12

For any $\rho \in (0, r)$ and any $i \in [n]$, recall the tuple of types $T_{\rho,j}^{\rho}$ defined under (21). Given any $C \in [n]$ and any type-based estimator $\gamma^{type}_{\rho,j}$, let

$$\Omega_{\rho,j}^{\rho} := \{x^n, y^n, y^n \in \hat{Q}_{\rho,j}^c : \gamma^{type}_{\rho,j}(T_{\rho,j}^{\rho}) = i\}. \quad (240)$$
\[ \mathbb{P}_C \left\{ L \left( T^\rho_{i,\Delta_n-1}, \frac{i - \Delta_n - 1 + \rho n}{n + 2\rho n}, \frac{r - \rho}{1 + 2\rho} \right) > \min_{j \in \Gamma_n(i_\rho(C), \Delta_n + \alpha n \rho)} L \left( T^\rho_{j, \frac{j + \rho n}{n + 2\rho}}, \frac{r - \rho}{1 + 2\rho} \right) \right\} \] 

\[ \leq \exp \left\{ -cn \left( \mathbb{E}_{\mathbb{P}^*} \left[ L \left( T^\rho_{i_\rho(C)}, n + 2\rho n \right), 1 + 2\rho \right] \right) \right\} \] 

\[ + O \left( \frac{1}{n^{3/2}} \right) \] 

\[ = \exp \left\{ -cn \left( \frac{L \left( \frac{i - \Delta_n - 1 + \rho n}{n + 2\rho n}, \frac{r - \rho}{1 + 2\rho} \right)}{\left( M(\rho) - M(2\rho) \right)^2} \right) \right\} \] 

\[ + O \left( \frac{1}{n^{3/2}} \right) \] 

\[ = O \left( \frac{1}{n^{3/2}} \right), \] 

Suppose \( \rho \exp C = \exp + \exp \left( P\tilde{\Lambda} \right) \). 

\[ \rho \exp C = \exp \left( P\tilde{\Lambda} \right) \] 

\[ = \exp \left\{ -cn \left( \mathbb{E}_{\mathbb{P}^*} \left[ L \left( T^\rho_{i_\rho(C)}, n + 2\rho n \right), 1 + 2\rho \right] \right) \right\} \] 

\[ + O \left( \frac{1}{n^{3/2}} \right) \] 

\[ = \exp \left\{ -cn \left( \frac{L \left( \frac{i - \Delta_n - 1 + \rho n}{n + 2\rho n}, \frac{r - \rho}{1 + 2\rho} \right)}{\left( M(\rho) - M(2\rho) \right)^2} \right) \right\} \] 

\[ + O \left( \frac{1}{n^{3/2}} \right) \] 

\[ = O \left( \frac{1}{n^{3/2}} \right) \] 

\[ \Omega^C_i := \left( \bigcup_{i \in [n]} \Omega^C_i \right)^C = \left\{ \left( x^n, y^N, y^N_2 \right) : \gamma_{\rho,i}^C \left( T^C_\rho \right) = e \right\}, \] 

\[ \Lambda^C_i := \left\{ \left( x^n, y^N, y^N_2 \right) : \right\} \] 

\[ (1 + 2\rho) \left( L \left( T^C_\rho, \frac{C + \rho n}{n + 2\rho n}, \frac{r - \rho}{1 + 2\rho} \right) + \delta_n \right) > \lambda, \] 

\[ \text{which means } \bigcup_{i \notin [C \Delta]} \Omega^C_i \subset \tilde{\Lambda}^C_i. \]
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REFERENCES

[1] L. Marti, N. Sanchez-Pi, J. M. Molina, and A. C. B. Garcia, “Anomaly detection based on sensor data in petroleum industry applications,” Sensors, vol. 15, no. 2, pp. 2774–2797, 2015.

[2] N. R. Zhang, D. O. Siegmund, H. Ji, and J. Z. Li, “Detecting simultaneous change-points in multiple sequences,” Biometrika, vol. 97, no. 3, pp. 631–645, 2010.

[3] J. Reeves, J. Chen, X. L. Wang, R. Lund, and Q. Q. Lu, “A review and comparison of change-point detection techniques for climate data,” Journal of Applied Meteorology and Climatology, vol. 46, no. 6, pp. 900–915, 2007.

[4] Z. Harchaoui, F. Vallet, A. Lung-Yut-Fong, and O. Cappe, “A regularized kernel-based approach to unsupervised audio segmentation,” in IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP). IEEE, 2009, pp. 1665–1668.

[5] A. Tartakovsky, N. Nikiforov, and M. Basseville, Sequential analysis: Hypothesis testing and change-point detection. CRC Press, 2014.

[6] E. S. Page, “Continuous inspection schemes,” Biometrika, vol. 41, no. 1/2, pp. 100–115, 1954.

[7] A. Pettitt, “A test for a change in a parameter occurring at an unknown point,” Biometrika, vol. 42, no. 3/4, pp. 523–527, 1955.

[8] G. V. Moustakides, “Optimal stopping times for detecting changes in distributions,” The Annals of Statistics, vol. 14, no. 4, pp. 1379–1387, 1986.

[9] D. Siegmund, Sequential analysis: Tests and confidence intervals. Springer Science & Business Media, 2013.

[10] M. Basseville and I. V. Nikiforov, Detection of abrupt changes: Theory and application. Prentice Hall Englewood Cliffs, 1993, vol. 104.

[11] C. McGilchrist and K. Woodyer, “Note on a distribution-free cusum technique,” Technometrics, vol. 17, no. 3, pp. 321–325, 1975.

[12] G. Lorden, “Procedures for reacting to a change in distribution,” The Annals of Mathematical Statistics, vol. 46, no. 6, pp. 1897–1908, 1971.

[13] J. H. Einmahl and I. W. McKeague, “Empirical likelihood based hypothesis testing,” Bernoulli, vol. 9, no. 2, pp. 267–290, 2003.

[14] A. Pettitt, “A non-parametric approach to the change-point problem,” Journal of the Royal Statistical Society: Series B (Methodological), vol. 51, no. 2, pp. 273–284, 1989.

[15] V. V. Veeravalli and T. Banerjee, “Quickest change detection,” in IEEE Transactions on Information Theory, vol. 57, no. 3, pp. 1604–1614, 2011.

[16] J. Reeves, J. Chen, X. L. Wang, R. Lund, and Q. Q. Lu, “A review and comparison of change-point detection techniques for climate data,” Journal of Applied Meteorology and Climatology, vol. 46, no. 6, pp. 900–915, 2007.

[17] A. Tartakovsky, N. Nikiforov, and M. Basseville, Sequential analysis: Hypothesis testing and change-point detection. CRC Press, 2014.

[18] E. S. Page, “Continuous inspection schemes,” Biometrika, vol. 41, no. 1/2, pp. 100–115, 1954.

[19] A. Pettitt, “A test for a change in a parameter occurring at an unknown point,” Biometrika, vol. 42, no. 3/4, pp. 523–527, 1955.

[20] G. V. Moustakides, “Optimal stopping times for detecting changes in distributions,” The Annals of Statistics, vol. 14, no. 4, pp. 1379–1387, 1986.

[21] E. Brodsky and B. S. Darkhovsky, Nonparametric methods in change point problems. Springer Science & Business Media, 2013, vol. 243.

[22] V. V. Veeravalli and T. Banerjee, “Quickest change detection,” in Academic Press Library in Signal Processing. Elsevier, 2013, vol. 3, pp. 209–255.

[23] J. Unnikrishnan, V. V. Veeravalli, and S. P. Meyn, “Minimax robust quickest change detection,” IEEE Transactions on Information Theory, vol. 57, no. 3, pp. 1604–1614, 2011.

[24] B. S. Darkhovsky and E. Brodsky, Asymptotically optimal change-point detection: Penalization, cusum and optimality. Springer, 1979.

[25] R. Baranowski, Y. Chen, and P. Fryzlewicz, “Range-corrected thresholding for multiple change-points and change-point-like features,” arXiv preprint arXiv:1609.00293, 2016.

[26] D. Garreau and S. Aloat, “Consistent change-point detection with kernels,” Electronic Journal of Statistics, vol. 12, no. 2, pp. 4440–4466, 2018.

[27] D. Wang, Y. Yu, and A. Rinaldo, “Univariate mean change point detection,” in IEEE Global Conference on Signal and Information Processing (GlobalSIP). IEEE, 2015, pp. 78–82.

[28] C. M. Gruner and D. H. Johnson, “Detection of change in periodic, nonstationary data,” in IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP). IEEE, 1996, pp. 2471–2474.

[29] N. Matthay and J. Ziv, “A Bayesian approach for classification of Markov sources,” IEEE Transactions on Information Theory, vol. 37, no. 4, pp. 1067–1071, 1991.

[30] J. Unnikrishnan, “Asymptotically optimal matching of multiple sequences to source distributions and training sequences,” IEEE Transactions on Information Theory, vol. 61, no. 1, pp. 452–468, 2014.

[31] L. Zhou, Y. Y. F. Tan, and M. Motani, “Second-order asymptotically optimal statistical classification,” Information and Inference: A Journal of the IMA, vol. 9, no. 1, pp. 81–111, 2020.

[32] H. He, L. Zhou, and Y. Y. F. Tan, “Distributed detection with empirically observed statistics,” IEEE Transactions on Information Theory, vol. 66, pp. 4349–4367, Jul 2020.
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