Construction of exact solutions of Bloch-Maxwell equation based on Darboux transformation

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Abstract

A new strategy, using Darboux transformations, of finding self-switching solutions of $i\dot{\rho} = [H, f(\rho)]$ is introduced. Unlike the previous ones, working for any $f$ but for Hamiltonians whose spectrum contains at least three equally spaced eigenvalues, the strategy does not impose any restriction on the discrete part of spectrum of $H$. The strategy is applied to the Bloch-Maxwell system.

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I. INTRODUCTION

The nonlinear von Neumann equation (NvNE) \[ i\dot{\rho} = [H, f(\rho)] \] (1) was recently shown to be relevant for two apparently disconnected fields of research: Optical solitons \[5\] and chemical kinetics \[6\]. In both cases we are interested in solutions \(\rho\) that have a form of \(n \times n\) matrices. In optical applications the dimension \(n\) is related to \(n\)-level atoms. For chemical-type kinetics the kinetic variables are related to real and imaginary parts of off-diagonal matrix elements of \(\rho\) evaluated in the basis of eigenvectors of \(H\). Since any chemical kinetics involves a finite number of substrates, the dimension \(n\) must also be finite. It has to be stressed, however, that a class of infinite-dimensional solutions of (1) was also found, albeit for a rather artificial example, in \[4\]. Moreover, as shown in \[7\] the NvNE contains as special cases many known lattice equations that also lead to infinitely-dimensional solutions.

The soliton-type solutions of (1) were first found in \[2\] for \(f(\rho) = \rho^2\) and a special class of Hamiltonians \(H\). The Hamiltonians \(H\) could be characterized as those that contained at least three equally spaced eigenvalues in discrete parts of their spectra. There are, of course, many physical examples of such systems, the most notable cases including harmonic oscillators, quantum fields or spin systems. However, the problem that was not solved until now was how to construct solutions for \(H\) that has an arbitrary spectrum. In the present paper we show how to deal with any discrete spectrum and illustrate the technique on a harmonic oscillator (HO) and a hydrogen-type atom (HA).

The method we employ is based on a Darboux transformation. The main advantage of the technique is that it allows to construct solutions that are practically impossible to find by other means, especially if \(n\) is large or infinite. The disadvantage lies in the need of starting with a known solution that, in addition, guarantees nontriviality of the solution generated by the Darboux transformation.

We work out explicitly two important cases: \(n = 2\) and \(n = 3\). The cases are important for two main reasons. Firstly, the two dimensional case is easy to handle and allows for a seed solution having a sufficient number of arbitrary parameters, no matter what spectrum of \(H\) one encounters. Secondly, once we have two-dimensional seed solutions, we can use them to construct highly nontrivial “self-scattering” solutions involving any even \(n\). If \(n\) is odd we can employ a combination of two- and three-dimensional seed solutions.
In the present paper we restrict the analysis to the essential building blocks with $n = 2$ and $n = 3$. But even these cases, especially $n = 3$, are interesting in themselves. For quantum optics we can apply them to three-level atoms and the result is a dynamics analogous to the celebrated “sech” soliton $^{5}$. In application to chemical kinetics the case results in an Oregonator-type dynamics with four substrates $^{6}$.

In both cases it is sufficient to work with quadratic nonlinearities. Kinetic autocatalytic-type equations are found immediately by considering separately real and imaginary parts of matrix elements $\langle k | \rho | l \rangle$ where $|k\rangle$, $|l\rangle$ are eigenvectors of $H$ normalized to $\langle k | l \rangle = \delta_{kl}$.

In order to relate the dynamics to $n$-level atoms one makes the trick introduced in $^{5}$. Take a solution $\rho$ of $i \dot{\rho} = [H, \rho^2]$ and employ the algebraic formula $[H, \rho^2] = [H\rho + \rho H, \rho]$. For a given solution $\rho$ rewrite $H\rho + \rho H$ as $H' + V$ where $H'$ has the same form as $H$ but with modified parameters. Then $V$ is a time-dependent interaction term that has a form $-d \cdot E(t, 0)$ where $E(t, 0)$ is a field typical of an optical soliton at the origin $x = 0$. As pointed out in $^{5}$ the construction is analogous to SUSY quantum mechanics since we have in addition the Darboux transformation $\rho \mapsto \rho_1$ that allows for the transformation $V \mapsto V_1$. We leave these details for a later work but in the present paper concentrate on technicalities leading to the solutions for arbitrary $H$.

II. LAX PAIR AND DARBOUX TRANSFORMATION

Following $^{2,4,8}$ one starts with three overdetermined linear systems (Lax pairs)

\[ z_k |\varphi_k\rangle = (\rho - \mu_k H) |\varphi_k\rangle, \]
\[ i |\dot{\varphi}_k\rangle = \frac{1}{\mu_k} f(\rho) |\varphi_k\rangle, \]

where $\rho$, $H$ are self-adjoint operators on some Hilbert space $H$, $\mu_k$, $z_k$ are complex numbers, $f$ is real function and $|\varphi_k\rangle$ are elements of $H$ for $k = 1, 2, 3$. If we fix an operator $H$ to be time-independent then there exist an operator $\rho$ and a vector $|\varphi_k\rangle$ that fulfil the Lax pair only if $i \dot{\rho} |\varphi_k\rangle = [H, f(\rho)] |\varphi_k\rangle$. The basic theorem $^{4}$ is the following:

**Theorem.** Assume $|\varphi_k\rangle$ for $k = 1, 2, 3$ are solutions of $^{2}$ respectively and $|\psi[1]\rangle$, $\rho[1]$, are defined by

\[ |\psi[1]\rangle = \left(1 + \frac{\mu_2 - \overline{\mu}_3}{\overline{\mu}_3 - \mu_1} P^r\right) |\varphi_1\rangle, \]

3
$$\rho[1] = \left( 1 + \frac{\mu_3 - \bar{\mu}_2}{\bar{\mu}_2} \right) \rho \left( 1 + \frac{\bar{\mu}_2 - \mu_3}{\mu_3} \right),$$  \hspace{1cm} (5)$$

$$f(\rho[1]) = \left( 1 + \frac{\mu_3 - \bar{\mu}_2}{\bar{\mu}_2} \right) f(\rho) \left( 1 + \frac{\bar{\mu}_2 - \mu_3}{\mu_3} \right),$$  \hspace{1cm} (6)$$

$$P = \frac{|\varphi_1\rangle\langle\varphi_2|}{\langle\varphi_2|\varphi_1\rangle}. \hspace{1cm} (7)$$

Then

$$z_1|\psi[1]\rangle = (\rho[1]^* - \mu_k H)|\psi[1]\rangle,$$  \hspace{1cm} (8)$$

$$i|\dot{\psi}[1]\rangle = \frac{1}{\mu_1} f(\rho[1])^* |\psi[1]\rangle,$$  \hspace{1cm} (9)$$

If we put \(\mu_3 = \mu_2\) and \(|\varphi_1\rangle = |\varphi_2\rangle\) then \(\rho[1]\) is also a density matrix and spectra of \(\rho\) and \(\rho[1]\) are identical. In this case (6) is simply equivalent with definition of function \(f\) of operator \(\rho[1]\) given by the spectral theorem. We can prove that if \(\rho\) is a solution of (1) then \(\rho[1]\) is also a solution of (1).

III. TWO-DIMENSIONAL CASE

We start with a Hamiltonian having discrete spectrum: \(H = \sum_n h_n Q_n\), where \(Q_n\) and \(h_n\) are spectral projectors and eigenvalues of \(H\) respectively. Next we take a density matrix \(\rho\) with two spectral projectors \(P_1\) and \(P_2\) which satisfy the following conditions: \([P_1, Q_n] = [P_2, Q_n] = 0\) for all \(n \neq 1, 2\) Therefore \(P = P_1 + P_2 = Q_1 + Q_2\) is a projector on the two-dimensional common eigensubspace. Let us define \(\rho_1 = P \rho P\), \(H_1 = PHP\), \(\rho' = (1 - P)\rho(1 - P)\) and \(H' = (1 - P)H(1 - P)\). Then equation \(i \dot{\rho} = [H, f(\rho)]\) separates into two pieces \(i \dot{\rho}_1 = [H_1, f(\rho_1)]\) and \(i \dot{\rho}' = [H', f(\rho')]\). By using the relation \([P, H_1] = 0\) and properties of such evolution \(\dot{P} = 0\) the first equation can be described by spectral projectors:

$$i \dot{\rho}_1 = i(\lambda_1 P_1 + \lambda_2 P_2) = i(\lambda_1 - \lambda_2) \dot{P}_1$$

\([H_1, f(\rho_1)] = [(f(\lambda_1) - f(\lambda_2)) H_1, P_1]\),  \hspace{1cm} (10)$$

so

$$i \dot{P}_1 = [\alpha(\lambda) H_1, P_1]$$

and similarly:

$$i \dot{P}_2 = [\alpha(\lambda) H_1, P_2],$$
where $\alpha(\lambda) = \frac{f(\lambda_1) - f(\lambda_2)}{\lambda_1 - \lambda_2}$. Solutions of these equations are the following:

$$P_i(t) = e^{-i\alpha(\lambda)H_1 t} P_i(0) e^{i\alpha(\lambda)H_1 t}$$

for $i = 1, 2$ and we have the general solution of two-dimensional problem:

$$\rho_1(t) = e^{-i\alpha(\lambda)H_1 t} \rho_1(0) e^{i\alpha(\lambda)H_1 t}.$$

We want to use it in further constructions, therefore we have to examine its properties. Let us assume that $\rho_1$ together with $H_1$ and fixed $z_\mu$, $\mu$ and $|\varphi_1\rangle$ fulfill the Lax pair on $\mathbf{P\mathcal{H}}$:

$$z_\mu |\varphi_1\rangle = (\rho_1 - \mu H_1) |\varphi_1\rangle$$

$$i |\dot{\varphi}_1\rangle = \frac{1}{\mu} f_c(\rho_1) |\varphi_1\rangle,$$

where $f_c(\rho_1) = f(\rho_1) + c \mathbf{P}$.

Put $|\psi_1\rangle = e^{i\alpha(\lambda)H_1 t} |\varphi_1\rangle$, then

$$z_\mu |\psi_1\rangle = (\rho_1(0) - \mu H_1) |\psi_1\rangle$$

$$i |\dot{\psi}_1\rangle = \frac{1}{\mu} f_c(\rho_1(0)) - \alpha(\lambda) H_1 |\psi_1\rangle.$$ (11)

So $|\psi_1(t)\rangle = e^{-i(\frac{1}{\mu} f_c(\rho_1(0)) - \alpha(\lambda) H_1)t} |\psi_1(0)\rangle$. It is easy to check that the generator of time evolution of $|\psi_1\rangle$ commutes with $\rho_1(0) - \mu H_1$.

Taking this into account we obtain

$$z_\mu |\psi_1(0)\rangle = (\rho_1(0) - \mu H_1) |\psi_1(0)\rangle$$

$$|\varphi_1(t)\rangle = e^{-i\alpha(\lambda)H_1 t} e^{-i(\frac{1}{\mu} f_c(\rho_1(0)) - \alpha(\lambda) H_1)t} |\psi_1(0)\rangle.$$ (13)

We can conclude that if $\rho_1$ is a solution of the two-dimensional problem then it is enough to consider only the Lax pair for time $t = 0$.

IV. THREE-DIMENSIONAL CASE

We construct a density matrix $\rho$ using only three eigenprojectors: $P_1$, $P_2$ and third spectral projector is equal to $Q_3$. So $\rho' = \lambda_3 P_3 = \lambda_3 Q_3$. Denote by $|\omega_i(0)\rangle$ eigenvectors of $\rho(0)$, so $P_i(0) = |\omega_i(0)\rangle \langle \omega_i(0)|$ and denote by $P_{i,j}(0) = |\omega_i(0)\rangle \langle \omega_j(0)|$. Now we want to add to the Lax pair a component on this one-dimensional space in such a way that together
they will be the three-dimensional Lax pair with slightly modified \( f \). For this case we can also restrict to initial conditions and for given \( z_\mu, \mu \) and \( h_3 \) we can choose \( \lambda_3 \) satisfying the Lax pair:

\[
z_\mu |\omega_3(0)\rangle = (\lambda_3 - \mu h_3) |\omega_3(0)\rangle
|\omega_3(t)\rangle = |\omega_3(0)\rangle.
\]

(14)

Put \( |\varphi(t)\rangle = \gamma_1 |\varphi_1(t)\rangle + \gamma_3 |\omega_3(0)\rangle \) for arbitrary \( \gamma_1, \gamma_3 \in \mathcal{C} \). Then we have the three-dimensional Lax pair:

\[
z_\mu |\varphi\rangle = (\rho - \mu H) |\varphi\rangle
i |\dot{\varphi}\rangle = \frac{1}{\mu} (f(\rho) - f(\lambda_3)) |\varphi\rangle \equiv \frac{1}{\mu} f'(\rho) |\varphi\rangle
\]

(15)

and we have to consider only

\[
z_\mu |\psi(0)\rangle = (\rho(0) - \mu H) |\psi(0)\rangle
|\varphi(t)\rangle = e^{-i\alpha(\lambda) H t} e^{-i(\frac{1}{\mu} f'(\rho(0)) - \alpha(\lambda) H) t} |\psi(0)\rangle \equiv U |\psi(0)\rangle
\]

(16)

with \( |\psi\rangle = e^{i\alpha(\lambda) H t} |\varphi\rangle \).

Now we have the Lax pair expressed only by initial conditions and can construct the operator \( P \) which defines the Darboux transformation (see 7):

\[
\begin{align*}
P &= \frac{|\varphi(t)\rangle \langle \varphi(t)|}{|\varphi(t)\rangle \langle \varphi(t)|} \frac{U |\psi(0)\rangle \langle \psi(0)| e^{i(\frac{1}{\mu} f'(\rho(0)) - \alpha(\lambda) H) t} e^{i\alpha(\lambda) H t}}{|\varphi(t)\rangle \langle \varphi(t)|} \\
&= e^{-i\alpha(\lambda) H t} P_{\text{int}} e^{i\alpha(\lambda) H t} \equiv V P_{\text{int}} V^*,
\end{align*}
\]

(17)

where

\[
P_{\text{int}} = \frac{|\psi(t)\rangle \langle \psi(t)|}{|\varphi(t)\rangle \langle \varphi(t)|} = \frac{|\psi(t)\rangle \langle \psi(t)|}{|\psi(t)\rangle \langle \psi(t)|}.
\]

Then

\[
\rho [1] (t) = (1 + \frac{\mu - \mu P}{\mu} P)(\rho(0)(1 + \frac{\mu - \mu P}{\mu} P)
= V(1 + \frac{\mu - \mu P}{\mu} P_{\text{int}})\rho(0)(1 + \frac{\mu - \mu P_{\text{int}}}{\mu} P_{\text{int}}) V^* = V \rho_{\text{int}} [1] (t) V^*.
\]

(18)

From the first equation of Lax pair \([2]\) we have

\[
H |\psi\rangle = \frac{1}{\mu} (\rho(0) - z_\mu) |\psi\rangle
\]
It can be written as

\[
\langle \psi(t) \rangle = e^{\frac{1}{\mu} \int (f'(\rho(0)) - \alpha(\lambda) [\rho(0) - z_\mu]) t} \langle \psi(0) \rangle = e^{\frac{1}{\mu} \int \sum_{i,j=1}^{3} \alpha_{\lambda} (\lambda - \mu_{ij}) P_{ij}(0) t} \langle \psi(0) \rangle \sum_{i,j=1}^{3} \beta_{\lambda} P_{ij}(0),
\]

where \( \beta_{\lambda} = f'(\lambda) - \alpha(\lambda)[\lambda - \mu_{ij}] \). Inserting this into the definition of \( P_{\text{int}} \) we obtain

\[
P_{\text{int}} = \frac{\langle \psi(t) \rangle \langle \psi(t) \rangle}{\langle \psi(t)|\psi(t) \rangle} = \frac{1}{F(t)} \times \left\{ \sum_{i,j=1}^{3} e^{-i \frac{\beta(\lambda) t}{\mu} (\lambda - \mu_{ij}) P_{ij}(0) t} \langle \psi(0) \rangle \langle \psi(0) \rangle P_{j}(0) \right\}
\]

where \( a_{ij} = \langle \omega_i(0) | \psi(0) \rangle \langle \psi(0) | \omega_j(0) \rangle \), \( b_{ij} = \beta(\lambda) - \beta(\lambda_{ij}) \). Since \( b_{11} = b_{22} = b_{12} = b_{21} = b \) and \( \beta(\lambda_3) = -\alpha(\lambda) \mu \) then \( b_{33} = 0 \). So \( F(t) = \sum_{i,j=1}^{3} a_{ij} e^{-ib_{ij} t} = e^{-ib_{11}(a_{11} + a_{22}) + a_{33}} \). If \( \gamma_3 = 0 \) then the projector \( P_{\text{int}} \) does not depend on time, so we need at least a three dimensional space to have a nontrivial Darboux transformation. We are in position to give the form of \( \rho_{\text{int}}[1](t) \) in terms of spectral projectors of \( \rho(0) \):

\[
\rho_{\text{int}}[1](t) = (1 + \frac{\mu - \bar{\mu}}{\mu} P_{\text{int}} \rho(0) (1 + \frac{\bar{\mu} - \mu}{\mu} P_{\text{int}})
\]

\[
= \rho(0) + \frac{(\mu - \bar{\mu})}{F(t)^2 |\mu|^2} \sum_{i,j=1}^{3} c_{ij} P_{ij}(0),
\]

where \( c_i = (\mu - \bar{\mu}) a_{ii} e^{-ib_{ij} t}[(\lambda_i - \lambda_j) a_{jj} e^{-ib_{ij} t} + (\lambda_i - \mu_k) a_{kk} e^{-ib_{kj} t}] \) and \( c_{ij} = a_{ij} e^{-ib_{ij} t}[(\lambda_j - \lambda_i) (\mu a_{ij} e^{-ib_{ij} t} + \bar{\mu} a_{jj} e^{-ib_{ij} t}) + (\bar{\mu} - \mu) \lambda_k + \mu \lambda_j - \mu \lambda_i) a_{kk} e^{-ib_{kj} t}] \) and here \( j \), \( k \) in the first term and \( j \) in the second term mean the remaining indexes.

V. MODELS

We want to see whether we can find appropriate \( \mu \) and a density matrix \( \rho \) for a construction of “self-scattering” solution for any Hamiltonian \( H \) with a discrete part of spectrum
and nonlinearity $f$. We use the spectral representation of $H$ and take any three of its eigenvectors:

$$H = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}$$  \hspace{1cm} (22)

We can decompose $H$ in two pieces in two different ways:

$$H = \begin{pmatrix} \frac{1}{2}(h_1 - h_2) & 0 & 0 \\ 0 & -\frac{1}{2}(h_1 - h_2) & 0 \\ 0 & 0 & h_3 - \frac{1}{2}(h_1 + h_2) \end{pmatrix} + [h_1 - \frac{1}{2}(h_1 - h_2)]I$$  \hspace{1cm} (23)

for $h_3 - \frac{1}{2}(h_1 + h_2) \neq 0$ and

$$H = \begin{pmatrix} B & 0 & 0 \\ 0 & 3B & 0 \\ 0 & 0 & 2B \end{pmatrix} + AI$$  \hspace{1cm} (24)

for equally-spaced spectrum. Clearly, if the density matrix $\rho$ is a solution of the equation $i\dot{\rho} = [H, f(\rho)]$, then it also satisfies the equation with $H' = H + CI$, so we can consider Hamiltonians with special forms. In the first case, for the operator

$$\rho - \mu H = \begin{pmatrix} \varrho_1 - \frac{\mu}{2}(h_1 - h_2) & \varrho & 0 \\ \overline{\varrho} & \varrho_2 + \frac{\mu}{2}(h_1 - h_2) & 0 \\ 0 & 0 & \varrho_3 - \mu[h_3 - \frac{1}{2}(h_1 + h_2)] \end{pmatrix}$$  \hspace{1cm} (25)

we can choose $\varrho_1 - \varrho_2 = \alpha(h_1 - h_2), \beta^2(h_1 - h_2)^2)4|\varrho|^2, |\varrho|^2 = \beta^2[h_3(h_1 + h_2 - h_3) - h_1h_2]$ and $\varrho_1 + \varrho_2 = 2\varrho_3 - 2\alpha h_3 + \alpha(h_1 + h_2)$, where $\mu = \alpha + i\beta$ to have an appropriate eigenvector. The compatibility of the third assumption is satisfied if we arrange eigenvalues of $H$ in such a way that $h_3$ is a number between $h_1$ and $h_2$. Under these assumptions we can take $\varrho_1 = \alpha(h_1 - h_3) + \varrho_3, \varrho_2 = \alpha(h_2 - h_3) + \varrho_3$ and $|\varrho|^2 = \beta^2[h_3(h_1 + h_2 - h_3) - h_1h_2]$. To get $\rho$ positive we need $\varrho_1|0$ and $\varrho_1|\varrho_2||\varrho|^2$. We can easy control these conditions by choosing an appropriate $\varrho_3, \alpha$ and $\beta$. In order to have $\rho$ as a density matrix we have to put $\alpha = \frac{1 - 3\varrho_3}{h_1 + h_2 - 2h_3}$.

We will latter discuss this condition. For such a specification $\rho - \mu H$ has the eigenvalue

$z_\mu = \varrho_3 + \frac{1}{2}(1 - 3\varrho_3) - i\beta[h_3 - \frac{1}{2}(h_1 + h_2)]$.

In the second case:

$$\rho' - \mu H = \begin{pmatrix} \varrho_1' - \mu B & \varrho' & 0 \\ \overline{\varrho'} & \varrho_2' - 3\mu B & 0 \\ 0 & 0 & \varrho_3' - 2\mu B \end{pmatrix}$$  \hspace{1cm} (26)
has an appropriate two-dimensional eigenspace if, for example, we take $\phi_1 - \phi_2 = -2\alpha B$ and $|\phi'|^2 = \beta^2 B^2$. To obtain density matrix we can put $\phi_3 = \frac{1}{3}$, $\phi_1' = \frac{1}{3} - \alpha B$, $\phi_2' = \frac{1}{3} + \alpha B$, $|\phi'|^2 = \beta^2 B^2$ and $|\mu|^2 B^2(\frac{1}{3})$. Then $z_\mu = \frac{1}{3} - 2\mu B$.

Let us choose a hydrogen-like atom Hamitonian (HA) as an example of an inhomogeneous spectrum and a harmonic oscillator (HO) for the homogeneous one. We choose neighbouring levels for simplicity: $h_1^{(HA)} = -\frac{B^{(HA)}}{n^2}$, $h_2^{(HA)} = -\frac{B^{(HA)}}{(n+2)^2}$, $h_3^{(HA)} = -\frac{B^{(HA)}}{(n+1)^2}$. To have solutions not so complicated in form and more legible we take $\alpha = 0$. Otherwise the calculations become very complicated. We find $\theta_3^{(HA)} = \frac{1}{3}$, $\theta_1^{(HA)} = (n+2)^2(2n+1) - \frac{1}{2}(6n^3+21n^2+24n+8)$, $\theta_2^{(HA)} = \frac{1}{2}(6n^3+15n^2+12n+4) - 2n^3 - 3n^2$, and $|\theta^{(HA)}|^2 = \frac{\beta^2 (B^{(HA)})^2 (4n^2 + 8n + 3)}{n^2(n+2)^2(n+1)^4}$.

Then for both Hamiltonians we obtain the same type of solution

$$\rho_{int}[1](t) = \begin{pmatrix} 1/3 + \zeta(t) & 0 & \xi(t) \\ 0 & 1/3 - \zeta(t) & \xi(t) \\ \xi(t) & \xi(t) & 1/3 \end{pmatrix}. \quad (27)$$

For HO we have

$$\zeta(t) = \frac{|\phi'|^2 t}{2 + |D|^2 e^{2|\phi'|^2 t}}$$

$$= |\phi'| | \tanh \left( \frac{|\phi'|^2 t}{\beta} + \ln \left( \frac{|D|}{\sqrt{2}} \right) \right) \equiv |\phi'| | \tanh (\theta' t + \phi')$$

$$\xi(t) = -\frac{2|\phi'| D (1 + i) e^{2|\phi'|^2 t}}{2 + |D|^2 e^{2|\phi'|^2 t}}$$

$$= -\frac{D}{\sqrt{2}|D|} |\phi'| \sech \left( \frac{|\phi'|^2 t}{\beta} + \ln \left( \frac{|D|}{\sqrt{2}} \right) \right) \equiv Z \sech (\theta' t + \phi') \quad (28)$$

and for HA

$$\zeta(t) = \frac{|\phi^{(HA)}|^2 t}{2d(n) + |D|^2 e^{2|\phi^{(HA)}|^2 t}}$$

$$= |\phi^{(HA)}| | \tanh \left( \frac{|\phi^{(HA)}|^2 t}{\beta} + \ln \left( \frac{|D|}{\sqrt{2d(n)}} \right) \right) \equiv |\phi^{(HA)}| | \tanh (\theta^{(HA)} t + \phi^{(HA)})$$

$$\xi(t) = -\frac{2|\phi^{(HA)}| |e(n) D e^{2|\phi^{(HA)}|^2 t}}{2d(n) + |D|^2 e^{2|\phi^{(HA)}|^2 t}} = -\frac{D e(n)}{\sqrt{2d(n)} |D|} |\phi^{(HA)}| | \sech \left( \frac{|\phi^{(HA)}|^2 t}{\beta} + \ln \left( \frac{|D|}{\sqrt{2d(n)}} \right) \right)$$

$$\equiv Z^{(HA)} \sech (\theta^{(HA)} t + \phi^{(HA)}) \quad (29)$$

where $d(n) = 4(2n+1)(n+1)^3$, $e(n) = (2n+1)(n+2) + in\sqrt{(2n+3)(2n+1)}$ and $D = \frac{\text{Im}(\phi^{(HA)})}{|\phi^{(HA)}|^2}$. Rewriting $\rho_{int}[1](t)$ in spectral basis of the Hamiltonian and renumerating in the order of
increasing eigenvalues we find:

\[
\rho_{\text{int}}[1](t) = \begin{pmatrix}
\frac{1}{3} & \sqrt{2} \Re(\xi(t)) & \zeta(t) \\
\sqrt{2} \Re(\xi(t)) & \frac{1}{3} & -i \sqrt{2} \Im(\xi(t)) \\
\zeta(t) & i \sqrt{2} \Im(\xi(t)) & \frac{1}{3}
\end{pmatrix}.
\] (30)

If we compare this expression with formula (16) in [5] we can see that we have two differences. First, the diagonal elements are different but in both cases constant. The second difference is more important. (16) describes \(\rho[1](t)\) but not \(\rho_{\text{int}}[1](t)\) and we have to sandwich the latter between \(e^{i\alpha(\lambda)Ht}\) and its inverse in sense of (18) to obtain the former. After this operation the symmetry of \(\rho[1](t)\) for HO and HA will be slightly different. For HO we have exactly the same symmetry as in [5] and for HA \(\xi\) in the first row is multiplied by \(e^{i\alpha(\lambda)(h_1-h_2)t}\) and in second row by \(e^{i\alpha(\lambda)(h_2-h_3)t}\) (for HO both are the same).

Obviously these differences do not disturb the idea of [5] to construct a time dependent Hamiltonian describing a three-level perturbation of not only HO but for any system, e.g. HA interacting with an optical soliton. The only modification comes from non-equal spacings between levels of the system and produces appropriate frequencies of oscillation. We can easily observe that the only modification of \(H\) is in multiplication by the factor \(\frac{3}{2}\) that, for the quadratic case, is equal exactly to \(\alpha(\lambda)\). Therefore we can start with Hamiltonian times \(\frac{3}{2}\) and get an appropriate \(H_o\).

The relation \(i\dot{\rho} = [H, \rho^2] = [H\rho + \rho H, \rho] = [h, \rho]\) is given up to two parameters \(\varepsilon_1\) and \(\varepsilon_2\):

\[h = (H + \varepsilon_1)\rho + (H + \varepsilon_1)\rho + \varepsilon_2 1.\]

Putting \(\varepsilon_1 = -\frac{h_1 + h_2}{2}\) and \(\varepsilon_2 = \frac{3(h_1 + h_2)}{4}\) we get:

\[
h = \begin{pmatrix}
2/3h_1 & 0 & 0 \\
0 & 2/3h_2 & 0 \\
0 & 0 & 2/3h_3
\end{pmatrix} + \begin{pmatrix}
0 & a(t) & 0 \\
0 & a(t) & b(t) \\
0 & b(t) & 0
\end{pmatrix} = H_o + V = H_o - \vec{d} \cdot \vec{E},
\] (31)

where \(a(t) = (h_2 - h_3)\sqrt{2}\Re(\xi(t))e^{i2/3(h_1-h_2)t}\) and \(b(t) = -i(h_2 - h_1)\sqrt{2}\Im(\xi(t))e^{i2/3(h_3-h_2)t}\).

From this we can see that:

\[
\vec{E} = (E_x, E_y, E_z) = (E_x(0)\text{sech}(\theta t + \vartheta)\cos(\omega t), E_y(0)\text{sech}(\theta t + \vartheta)\sin(\omega t), 0)
\] (32)

and
\[d_x = \begin{pmatrix}
0 & \tilde{a}(t) \cos(\omega t) & 0 \\
\tilde{a}(t) \cos(\omega t) & 0 & \tilde{b}(t) \cos(\omega t) \\
0 & \tilde{b}(t) \cos(\omega t) & 0 
\end{pmatrix},
d_y = \begin{pmatrix}
0 & \tilde{a}(t) \sin(\omega t) & 0 \\
\tilde{a}(t) \sin(\omega t) & 0 & \tilde{b}(t) \sin(\omega t) \\
0 & \tilde{b}(t) \sin(\omega t) & 0 
\end{pmatrix}\]

\[\tilde{a}(t) = (h_2 - h_3)\sqrt{2}\Re(Z)e^{i2/3(h_1 - h_2)t}, \quad \tilde{b}(t) = -i(h_2 - h_1)\sqrt{2}\Im(Z)e^{i2/3(h_3 - h_2)t}.\]

Hence we get potential \(V\) in shape of well-known McCall-Hahn “sech” soliton [9]. Another choice of parameters \(\varepsilon_i\) leads us to a very wide class of electromagnetic impulses [10]. We can use now this solution to construct another one for the same potential in accordance with the SUSY scheme. The possibility follows from the richness of the set of solutions we obtain by the Darboux transformation.

Now we are in position to check that we have not only the solution of the Bloch equation but also of the Maxwell one. First we calculate the components of atomic polarization:

\[P_x = \text{Tr}(\rho d_x) = 4 \text{sech}(\theta t + \vartheta) \cos(\omega t)[(h_2 - h_3)\Re(Z)^2 + (h_2 - h_1)\Im(Z)^2]\]
\[P_y = \text{Tr}(\rho d_y) = 4 \text{sech}(\theta t + \vartheta) \sin(\omega t)[(h_2 - h_3)\Re(Z)^2 + (h_2 - h_1)\Im(Z)^2]\]

Let us remember that \((h_2 - h_3)\) and \((h_2 - h_1)\) have the same sign.

Because McCall-Hahn is a steady-state pulse we can use coordinate \(\varsigma = t - \frac{z}{v}\), where \(v\) is the constant pulse velocity. Hence we have \(\frac{\partial}{\partial t} \mapsto \frac{d}{d\varsigma}\) and \(\frac{\partial}{\partial z} \mapsto -\frac{1}{v} \frac{d}{d\varsigma}\) then Maxwell equation has the following shape:

\[\frac{d^2}{d\varsigma^2} \vec{E} = \left( -\frac{4\pi}{(\nu^2 - 1)} \right) \frac{d^2}{d\varsigma^2} \vec{P}.\]

By comparing (32) and (34) we can see that Maxwell equation is fulfilled if we choose parameters in such a way:

\[16\pi [(h_2 - h_3)\Re(Z)^2 + (h_2 - h_1)\Im(Z)^2] = (\frac{\varsigma}{v})^2 - 1\] \(E_o(0)\),

where \(o\) means \(x\) or \(y\).

VI. COMMENTS

We have described a construction of solutions for the two main “building blocks”: \(2 \times 2\) and \(3 \times 3\). How to get from \(2 \times 2\) blocks to infinite-dimensional and irreducible solutions
was discussed in [4]. However, in infinite dimensional examples the issues of normalization to $\text{Tr} \rho = 1$ involve certain subtleties that require a separate treatment. These difficulties do not occur if one restricts the discussion to arbitrarily large but finite matrix dimensions. The method we have described in the present paper allows for immediate generalizations to arbitrary finite dimensions if one follows the strategy employed in [4].

The technique gives a method of constructing three-dimensional solutions appropriate for different situations. It has to be stressed that not only can we use any Hamiltonian and any type of nonlinearity, but it seems that we can get arbitrary values for the eigenvalues of $\rho$ for a homogeneous nonlinearity. To see this it is enough to observe that we can generalize the scaling and the shifting properties which were observed in [2] the for quadratic nonlinearity. If $\varrho$ is solution of NvNE for $f(x) = x^k$ then $\frac{\rho}{\text{Tr} \rho} \left( \frac{1}{\text{Tr} \rho} \right)^{k-1}$ also satisfies it and $\rho + \sigma$ when $\dot{\sigma} = 0$ is solution if we put $g(x) = f(x + \sigma)$ instead of $f(x)$. Because $\varrho_1$ and $\varrho_2$ are very simple functions of the parameter $\alpha$ and $\varrho_3$, and $\varrho$ depends only on $\beta$, we can produce positive “predensity” matrices with wide spectrum of eigenvectors. How “big” is this set of solutions in the set of all the solutions of NvNE will be considered elsewhere.

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