Relativistic linear stability equations for the nonlinear Dirac equation in Bose-Einstein condensates

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Abstract – We present relativistic linear stability equations (RLSE) for quasi-relativistic cold atoms in a honeycomb optical lattice. These equations are derived from first principles and provide a method for computing stabilities of arbitrary localized solutions of the nonlinear Dirac equation (NLDE), a relativistic generalization of the nonlinear Schrödinger equation. We present a variety of such localized solutions: skyrmions, solitons, vortices, and half-quantum vortices, and study their stabilities via the RLSE. When applied to a uniform background, our calculations reveal an experimentally observable effect in the form of Cherenkov radiation.

Progress in condensed matter and particle physics has been periodically marked by significant mutual exchanges between the two disciplines, many proposals for which are realized in model systems of ultracold quantum gases in optical lattices [1–3]. Recent active areas of research include holographic dualities such as AdS/CFT [4], theoretical constructions of superstrings in ultracold quantum gases [5], chiral confinement in quasi-relativistic Bose-Einstein condensates (BECs) [6], and our own derivation of the nonlinear Dirac equation (NLDE) describing ultracold bosons in a honeycomb optical lattice [7]. Generally, Dirac theories arising from a honeycomb lattice geometry appear in a variety of interesting settings [8–10]. Our investigation into relativistic effects in BECs is motivated by this spirit of cross fertilization with the aim of tying in theory to experiment.

In this letter, we develop the relativistic linear stability equations (RLSE) for the NLDE. Moreover, we find emergent nonlinear localized solutions [11] to the NLDE, including solitons, vortices, skyrmions, and half-quantum vortices, the latter so far unobserved in BECs. Although most of these objects have been studied in multicomponent BECs, such models lie within the usual Schrödinger many-body paradigm. In contrast to this paradigm, our investigations reside within a relativistic framework in which the elementary excitations are governed by a Dirac-like equation. This provides a fundamentally different context distinguished by the presence of a nontrivial Berry phase when circling a vortex core. The presence of a Berry phase and, indeed, the full Dirac structure of our theory was first pioneered by condensed matter theorists within the context of graphene [12–14]. In the case of a Bose-Einstein condensate confined in a trap and subjected to a periodic lattice potential, questions regarding stability may be addressed by applying the method of Bogoliubov theory directly at the lattice scale. Unfortunately, for lattices with interesting geometries, this approach stops short without shedding light on the fascinating emergent physics that is revealed by examining the long-wavelength fluctuations interacting with the lattice background. Consequently, in order to determine the quasi-particle states and energies we cannot rely on the Bogoliubov-de Gennes equations (BdGE) since these are based on nonrelativistic quantum mechanics. Instead, we derive, from first principles, the RLSE which give the correct low-energy dynamics for an arbitrary background condensate. The RLSE are reducible to the BdGE in certain limits, and so may naturally be considered relativistic generalizations of the latter. Based on the RLSE we predict Cherenkov radiation that can be measured in experiments: the combination of lattice and particle interactions results in a rich spatial distribution that is not seen in the BdGE for the uniform case [15].

The RLSE is relevant to a broad range of optical lattice constructions. For example, the RLSE also applies to bosons in a square optical lattice with a staggered gauge field similar to the arrangement described in the work by Lim et al. [16] and, generally, to any boson-lattice...
system with a bi-partite lattice structure and linear dispersion [17]. It is also important to emphasize that the optical lattice set-up that we describe here is well founded experimentally. Experiments with cold bosons in two-dimensional lattices are commonplace, and have been studied extensively [18]. The application of the RLSE to soliton and vortex solutions as well as to the uniform case gives richer physical results than one finds in either the usual single-component BECs or in the case of hyperline multi-component BECs. The unique feature of our theory is that it reveals a relativistic Fermi structure within a cold bosonic system and the RLSE are the key equations for probing this system.

In the laboratory, the NLDE can be obtained by cooling bosons into the lowest Bloch band of a honeycomb optical lattice [19]; the lattice is constructed by establishing three phase-locked interfering laser beams in a plane while freezing out excitations in the vertical direction as in fig. 1. To obtain the desired Dirac structure, particles are first condensed into the lowest-energy state (zero crystal momentum) of the lattice and then adiabatically translated to the Dirac point at the band edge (see fig. 1) by adiabatically tuning the relative phases between the laser beams. We emphasize that the Dirac point, which is key to the NLDE and our predictions, is maintained even in the presence of the shallow harmonic trap endemic to atomic BECs [20].

Nonlinear phenomena in BECs have been studied extensively over the past decade [11,21], from single-component vortices in rotating, trapped BECs [22] to complex multi-component order parameters [23,24] resulting from interactions between the different components and the possibility of nontrivial topological windings of the internal symmetry space around a singular vortex core. Some form of BdGE analysis plays a central role in such constructions as a means of probing stability as well as for gaining a deeper understanding of the low-energy fluctuations.

**Derivation of the RLSE.** — Since the system we describe in this letter is a BEC confined strictly to two spatial dimensions, it is appropriate to recall the justification for such a construction before presenting the RLSE. For uniform 2D systems the Mermin-Wagner theorem forbids the formation of a true condensate defined by an infinite phase coherence length. This comes from the fact that the density of states diverges in the 2D case for finite $T$. Instead, one sees the formation of a quasi-condensate characterized by local phase coherence restricted to finite-size regions. The size of these regions greatly exceeds the healing length so that all of our solutions are realizable in this picture. However, the inclusion of a harmonic confining potential allows the formation of a true 2D condensate. The potential places a lower bound on the energy for fluctuations and, since it is these long-wavelength fluctuations that are responsible for destroying long-range order, the trap provides a means of expanding the spatial range of validity of the mean-field description.

To obtain the low-energy excitations of solutions of the NLDE, we must find the correct set of equations that describe quasi-particle states analogous to the BdGE equations for the general case. These are obtained from the Hamiltonian for a weakly interacting Bose gas, $H = \int \mathrm{d}r \tilde{\psi}^\dagger H_0 \tilde{\psi} + \frac{U}{2} \int \mathrm{d}r \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \tilde{\psi}$, $H_0 \equiv \hbar^2 \nabla^2 / 2M + V(r)$, and working through four steps [25]. 1) Take $\tilde{\psi} = \Psi_\epsilon (r) + \delta \tilde{\psi}_q (r)$ (condensate + quasi-particles), with $\delta \tilde{\psi}_q$ small. 2) Impose a constraint on $\Psi_\epsilon$ to eliminate linear terms in $\delta \tilde{\psi}_q (r)$, keep only quadratic terms in $\delta \tilde{\psi}_q (r)$, and expand as a sum of particle and hole creation operators. 3) Invoke Bloch-state expansions for $\Psi_\epsilon (r)$ and $\delta \tilde{\psi}_q (r)$ and take the lowest band. 4) Take the long-wavelength limit while taking momentum with respect to the Dirac point $\mathbf{K}$, such that $\mathbf{k} \ll \mathbf{q} \ll \mathbf{K}$, where $\mathbf{q}$ is the momentum of the condensate relative to the Dirac point $\mathbf{K}$ and $\mathbf{k}$ is the quasi-particle momentum measured relative to $\mathbf{q}$, and finally diagonalize the quasi-particle part of the Hamiltonian. One finds the RLSE:

\[
\begin{align*}
\tilde{\mathcal{D}}_{\mathbf{u} \mathbf{k}} - U \tilde{\Psi}_{\mathbf{u} \mathbf{k}} &= \tilde{E}_k \mathbf{u}_k, \\
\tilde{\mathcal{D}}^*_{\mathbf{v} \mathbf{k}} - U \tilde{\Psi}_{\mathbf{v} \mathbf{k}} &= -\tilde{E}_k \mathbf{v}_k,
\end{align*}
\]

where the matrix coefficients are defined as

\[
\tilde{\Psi} \equiv \text{diag}(|\Psi_A|^2, |\Psi_B|^2),
\]

\[
\tilde{E}_k \equiv \text{diag}(E_k, E_k),
\]

\[
[\tilde{\mathcal{D}}]_{1,1} \equiv m_{\text{eff}} - \mu - 2U |\Psi_A|^2 - i \nabla \Phi_A \cdot \nabla \\
+ |\nabla \Phi_A| - i \left( \nabla^2 \Phi_A \right),
\]

\[
[\tilde{\mathcal{D}}]_{1,2} \equiv m_{\text{eff}} - \mu - 2U |\Psi_B|^2 - i \nabla \Phi_B \cdot \nabla \\
+ |\nabla \Phi_B| - i \left( \nabla^2 \Phi_B \right),
\]

\[
[\tilde{\mathcal{D}}]_{1,2} = [\tilde{\mathcal{D}}^*]_{2,1} \equiv \mathcal{D}^*.
\]

Here, $\mathcal{D} = (\partial_x + i \partial_y)$ is the single-particle Dirac operator. Also, $\Psi = (\Psi_A, \Psi_B)$ is the BEC order parameter at the $\mathbf{K}$ Dirac point, with normalization on sublattice
components \( \int dr (|\Psi_A|^2 + |\Psi_B|^2) = 1 \). Analogous equations hold for the inequivalent Dirac point at \(-K\). Cast in this highly compact form, eqs. (1), (2) are reminiscent of the BdGE and may be solved for the spinor quasi-particle amplitudes \( u_k(r) = [u_{k_A}(r), u_{k_B}(r)]^T \) and \( v_k(r) = [v_{k_A}(r), v_{k_B}(r)]^T \) and the quasi-particle energy \( E_k \). The components of these 2-spinors represent quantum fluctuations of the sublattice condensate order parameters \( \Psi_A \) and \( \Psi_B \) which in general are nonuniform \( C \)-functions on the plane. The presence of the local phase of the condensate \( \phi_{A(B)}(r) \) indicates the complex interaction between the local superfluid velocity of the condensate \( v_{s,A(B)}(r) = \nabla \phi_{A(B)}(r) \) and the spinor quasi-particles \( u_k(r) \) and \( v_k(r) \). We have taken \( \hbar = c = 1 \) for simplicity, where \( c_t \) is the effective speed of light in the NLDE. Note also that we have included an effective mass \( m_{\text{eff}} \) (anisotropic lattice) that competes with the chemical potential \( \mu \).

It is important to note that for a moving condensate, the negative-energy modes cannot be removed and are crucial indicators of Cherenkov radiation. However, in our case, the Dirac Hamiltonian is not positive-definite since our theory is defined at zero lattice energy, not the lowest-energy Bloch state, so we must respect the presence of both energy raising and lowering modes. Another important feature is that the RLSE are reducible to the BdGE when the local lattice potential energy is the main contributor to the condensate chemical potential and the condensate is slowly varying (quasi-uniform background), i.e., \( |\mu| \gg U, E_j \gg |\mu| \approx |\Sigma_0| \), where \( \Sigma_0 = \Sigma_{0A(B)} \equiv -\int dr \, w_{A(B)}^* H_0 w_{A(B)} \) is the local self-energy for an arbitrary lattice site, with \( w_{A(B)}(r) = w(r - r_{A(B)}) \) the Wannier functions.

**Physical parameters and regimes.** – We list first the fundamental dimensionful parameters that we use. They are as follows: the average particle density \( n_0 \), the chemical potential \( \mu \), the lattice spacing \( a \), the \( s \)-wave scattering length \( \alpha_s \), the mass of the constituent bosons \( M \), and the lattice well depth \( V_0 \). Several relevant composite quantities may be constructed from these. These are the effective speed of light \( c_t = t_h a \sqrt{3}/2h \), the sound speed \( c_s = \sqrt{U n_0/M} \), the interaction strength \( U = 4\pi \hbar^2 a_s / M \), the healing length \( \xi = t_h a \sqrt{3}/2h n_0 U \), and the hopping energy \( t_h = \int d^2 r \, w^* H_0 w \), where \( t_h \) depends on \( a \) and \( V_0 \), respectively, over the overlap of Wannier functions and the lattice potential inside \( H_0 \). Two fundamentally important constraints regarding these quantities should be stated. First, in order to avoid reaching the Landau velocity at the band edge and creating unwanted excitations we require \( c_t < c_s \), where \( c_s \) is the sound speed. Thus we require \( t_h a \sqrt{3}/2h < \sqrt{U n_0/M} \) or \( (t_h a \sqrt{3}/2h)(4\pi \hbar^2 a_s n_0 M)^{-1/2} < 1 \). For \(^{87}\text{Rb}\) with \( t_h = \hbar \times 10^3 \) Hz, \( a = 0.5 \times 10^{-7} \) m, \( a_s = 5 \times 10^{-9} \) m, \( n_0 = 2 \times 10^{12} \) cm\(^{-3} \), we get \( c_t/c_s \lesssim 0.17 \). Second, in order for our long-wavelength approximation to be correct, we require the NLDE healing length \( \xi \equiv t_h a \sqrt{3}/2h n_0 U \gg a \); using the same values for the physical parameters, we find \( \xi \approx 9.43 \) a.

Next, we discuss the physical regimes for our theory. In this discussion we take \( n_0 U/t_h \ll 1 \). At length scales much larger than the healing length, \( \xi \ll 1 \) where \( k \) is a characteristic quasi-particle momentum, excitations are comprised of correlated particle-hole pairs that propagate with a dispersion given by \( E \propto k^{1/2} \). This is a Bose gas of composite particles in the sense that excitations of opposite spin are paired up (albeit nonlocally) to form bosons. In contrast, for \( \xi \gg 1 \), excitations are particle-like which corresponds to the case where spin eigenstates are excited independently. These states reflect the bipartite structure of the lattice, multi-component with a Dirac-like dispersion \( \propto k \), but are local objects and so also reflect the bosonic nature of the fundamental constituent particles. In this sense, they form a hybrid Dirac-Bose gas.

### Uniformly moving condensate.

Now we return to the RLSE and solve them for the simplest case of a uniform background \( \Psi(r) \equiv \mathbf{\sqrt{C_0}} e^{i\mathbf{q} \cdot r}(1, C_0)^T \), where \( C_0 \in \mathbb{C} \) contains a relative phase, \( n_0 \) is the average particle density, and \( \mathbf{q} \) is the condensate momentum measured with respect to the Dirac point. In order to obtain the coherence factors and quasi-particle dispersion, we must then solve a 4 × 4 eigenvalue problem; the RLSE yield: \( E_k = c_t^2 q^2 \pm \sqrt{c_t^4 q^4 + 4n_0 C_0 \hbar c_t} \).

In keeping with the usual Bogoliubov notation found in the literature, we may write \( E_k = (c_t^2 q^2 + \epsilon) \pm \sqrt{c_t^4 q^4 + 4n_0 C_0 \hbar^2 c_t} \) where \( \epsilon \equiv \mathbf{c}_t \mathbf{h} \) is the single-quasi-particle energy for zero interaction and \( E_0^2 = \sqrt{c_t^4 q^4 + 4n_0 C_0 \hbar^2 c_t} \) is the quasi-particle energy for a static background. The associated coherence factors can then be written as \( |u_{k_{A(B)}}| = (E_0^2 + c_t \hbar) / \sqrt{4E_0^2 c_t \hbar} \), \( |v_{k_{A(B)}}| = |u_{k_{A(B)}}| (\pm \rightarrow) \).

The full interacting Hamiltonian is given by \( H_{\text{RLSE}} = \frac{1}{2} \mathbf{T} n_0 \mathbf{A} + c_t \mathbf{h} - \mathbf{E}_k \mathbf{r}^A - c_t \mathbf{H}_k \mathbf{c}_t^A \), where \( A \) is the area of the plane. The first three terms are the mean-field and quantum corrections to the condensate energy and the last accounts for the number of quasi-particles present in the system. The constant \( c_t^A \) is determined in terms of the overlap integral between Wannier states at neighboring lattice sites by \( c_t^A = \sqrt{3} a \pi / 2h \), where \( r_{A,B} \equiv -\int dr \, w^*_A \nabla w_B \) and \( \tau = |r_{A,B}| \) (see footnote \(^1\)).

The low-energy behavior of the uniform condensate has a rich structure. The \( \mathbf{q} = \mathbf{0} \) case corresponds to a condensate with zero crystal momentum measured from the Dirac point but with momentum \( \mathbf{K} \) relative to the lowest Bloch state of the crystal. The idea of a condensate in motion relative to its background has been treated both in free space and in the case of a moving background lattice \([18,26]\). Physically, the lattice potential is moving relative to the stationary condensate (laboratory frame). Two-body collisions reduce the momentum of some particles relative to the lattice (slowing down)

\(^1\)Note that whereas for the effective speed of light we have \( |c_t^A| = m^* \cdot s^{-1} \), in contrast \( |c_t^A| = m^* \cdot s^{-1} \) since \( r_{A,B} \) is an integral over the gradient operator rather than the Laplacian.
and increase the momentum of others (speeding-up) corresponding to a finite depletition of the condensate. In the laboratory frame, a two-particle collision appears as one particle gaining a component of momentum to the left and the other a component to the right. This is consistent with the well-known particle-hole symmetry of the Dirac Hamiltonian: negative-energy states can be interpreted as positive-energy states that propagate in the opposite direction. In our theory these are quasi-particles with momentum $K - k$ (for the K-Dirac point) relative to the lowest Bloch state.

For $q = 0$ then, we get $E^{\pm}_k = \pm E_k^0 = \pm \sqrt{c_l^2 + n_0 U c_l^2}$. The two energy regimes evident here are separated by the condition $c_l \hbar k / n_0 U \equiv \xi k \approx 1$. At short wavelength, $k \xi > 1$ so that $E^{\pm}_k \approx \pm (c_l \hbar k + n_0 U / 2)$, where the dominant first term reflects only the presence of the honeycomb lattice, while the second term is a small mean-field Hartree shift due to the interaction with the background. When $k \xi < 1$, we find $E^{\pm}_k \approx \pm \sqrt{\xi k}$. These are collective excitations induced by the particle interactions just above the condensate energy. The presence of negative-energy modes means that the condensate may lower its energy through spontaneous emission of radiation. This process can be suppressed by introducing an anisotropy in the lattice by breaking the A-B sublattice degeneracy with a deeper optical lattice in one direction [27]. This results in an additional term in the dispersion opening up a mass gap $2m_{\text{eff}}$ at the Dirac point. For the negative-energy modes we then have $E^{(-)}_{k}(m_{\text{eff}}) = 2m_{\text{eff}} - \sqrt{(c_l^0)^2 + n_0 U c_l^0}$ so that excitations require a minimum momentum determined by $c_l \hbar k_{\text{spin}} = \sqrt{4m_{\text{eff}}^2 + n_0^2 U^2}$. Alternatively, we can consider the effect of the confining potential: this sets a lower bound for quasi-particle energy given by $|E_{k_{\text{min}}}^{(-)}| \approx \sqrt{(c_l \hbar 2\pi / R_{\perp})^2 + n_0 U c_l \hbar 2\pi / R_{\perp}}$, where $R_{\perp}$ is the characteristic trap radius in the 2D plane.

**Cherenkov radiation.** – This usually refers to the anisotropic emission of electromagnetic radiation from a source whose speed exceeds the local speed of light in some medium [28]. This concept generalizes to any source moving through a medium at a speed that exceeds the phase velocity of the elementary excitations of the medium. For example, a BEC moving in the laboratory frame, or with respect to a background, will “radiate” (emit particles) when its speed exceeds the sound speed. Moreover, the radiation will be emitted in a cone subtended by a specific angle in the direction opposite the motion of the BEC. The RLSE can be used to demonstrate this effect in the present context of a BEC in a honeycomb optical lattice.

For a BEC with momentum $q > 0$ measured from the Dirac point, examination of the angular dependence of $E_k$ reveals an intriguing structure for the emission of Cherenkov radiation. We observe the following properties for $E_k$: 1) When $v < c_l$, where $v = c_q q$ is the condensate speed, all excitations have positive energy regardless of the angle of emission. 2) When $v > c_l$, quasi-particle energies are positive only for emission angles (measured relative to $q$) for which $\theta < \theta_c \equiv \cos^{-1}(-c_l / v)$ while all other modes have negative energy corresponding to the emission of radiation in a backwards cone bounded by $\theta_c$. When $v = c_l$, $\theta_c = \pi$ marks the onset of radiation, in which case radiation is only emitted in the direction opposite $q$. This unique directional property of the radiation suggests an obvious detectable signature in the laboratory: a time-of-flight analysis of a BEC prepared with precise values of the parameters should show a predictable shift in the momentum distributions between the forward and backward directions.

**Nonlinear localized modes.** – In situations where the nonlinearity of the NLDE is Lorentz invariant, solutions may be obtained directly by exploiting the associated conservation equations [29]. This is not the case for our NLDE so we must employ other means. To obtain solutions of the NLDE that are localized in $x, y$ for $U > 0$, we substitute the plane-polar ansatz $\Psi_A(r) = c_A \exp[ip_A(\theta)]F_A(\rho)$, $\Psi_B(r) = c_B \exp[ip_B(\theta)]F_B(\rho)$ into the NLDE. Then $c_A = i$, $c_B = 1$, and there are two possible combinations for the angular functions: i) $p_A(\theta) = (l - 1)\theta$, $p_B(\theta) = \theta$; ii) $p_A(\theta) = (l - 1)\theta$, $p_B(\theta) = (l + 1)\theta$, with $l \in \mathbb{Z}$. In particular, $l = 0$ in i) corresponds to a vortex configuration in $\Psi_A$ filled in at the core with a nonzero soliton for $\Psi_B$. Solutions of this type exist for different relative values of $\mu$ and $U$ and for several asymptotic values of the components: $l \to \infty (\Psi_A, \Psi_B) \in \{-i \sqrt{F}, 0\}, \{-i \sqrt{F}, \sqrt{F}\}, \{0, 0\}$, for $l = 1$, we obtain the same types of solutions but with $\Psi_A$ and $\Psi_B$ exchanged. For $l > 1$, centripetal terms are present for both $F_A(\rho)$ and $F_B(\rho)$ so that we must have $\Psi_A(0) = \Psi_B(0) = 0$ and both components are vortices with zero core densities. For the $l = 1$ case, we also obtain a skyrmion solution for which the pseudospin $S = (\Psi^* \sigma \Psi)$ (with Pauli vector $\sigma$) exhibits an integral number of flips near the core and approaches a constant value far from the core. This feature is encoded in a topologically conserved charge $(1/8\pi) \int \Omega d\rho\cdot e^\parallel S \cdot \partial S \cdot \partial S$ which one recognizes as the Pontryagin index that classifies the mapping $S^{\parallel}_{\text{spin}} \to S^{\parallel}_{\text{Dir}}$ where the two circles $S^{\parallel}_{\text{spin}}$ and $S^{\parallel}_{\text{Dir}}$ parameterize the rotations between the densities $\rho_A(\rho)$ and the polar angle rotation on the 2D boundary $\partial S$ at spatial infinity. In general, similar types of solutions exist for ii) above. Analytical and numerical solutions are plotted in fig. 2 for which different values of $\mu / U$ and $l$ allow us to obtain the different asymptotic forms.

Besides vortices with integer phase winding, we also find solutions with fractional phase winding, called half-quantum vortices (HQVs). Ordinarily, analyticity (single-valuedness) of the order parameter forbids the rotation of the phase of $\Psi$ around the core to take on fractional values. In the NLDE, $\Psi$ can acquire a coherent internal Berry phase in addition to an external phase whose angles are identified with the polar angle $\theta$ [30,31]. Such states
may have half-integer winding in both the internal and external phase angles while remaining single-valued overall. We obtain HQVs with asymptotic form \( \lim_{r \to \infty} \Psi_{\text{HQV}}(r) = 2i\sqrt{n_0/2} e^{-i\theta/2} | \cos(\theta/2), i \sin(\theta/2) \rangle \)T; the complete solution is shown in fig. 2(c).

We also obtain one-dimensional kink-soliton, skyrmion, and line-soliton solutions. The kink and skyrmion solutions are obtained by a straightforward substitution of the ansatz \( \Psi(x) = \eta | \cos(\varphi) \rangle, | \sin(\varphi) \rangle \)T into the NLDE and then considering the distinct cases where \( \varphi = \text{constant} \) (kink) or \( \eta = \text{constant} \) (skyrmion). The line-soliton solution is obtained when both \( \eta \) and \( \varphi \) are functions of \( x \) with the additional condition that, at the origin, \( \eta \) remains below a certain value. This ensures that \( \mu^2 < \sqrt{U/8} < (U+1)/2U \Rightarrow U < 3.365 \) and \( \mu^2 < 0.649 \), which allows the wave function to collapse away from the \( y \)-axis while the nonzero wave function near and along the \( y \)-axis has a Lorentzian form in the \( x \)-direction due to the attractive effect from the kinetic terms.

**Localized mode stability.** – We can now apply the RLSE to our localized solutions. In particular, for the case where the condensate wave function is in the vortex/soliton configuration, we have obtained the exact solution: \( \Psi_{\text{vs}}(r,t) = e^{-i\mu t/h} \sqrt{\frac{n_0}{1+(r/\xi)^2}} \left( ie^{i\theta/r/\xi}, 1 \right) \). The upper component is a vortex with rotation speed \( v = c_0 \xi/r \), and the lower component is a soliton centered at the core of the vortex. We solved the RLSE numerically for the vortex/soliton and found that the lowest excitation, with angular momentum \( m = -1 \) (relative to the condensate), has the eigenenergy \( E_{-1} = -3.9274 + 0.0020 i \), in units of \( n_0U \). This excitation, having angular momentum equal in magnitude but opposite to the vortex rotation, perturbs the vortex by adding a component of the state that has zero rotation, effectively driving the vortex energy towards the Dirac point. The corresponding coherence factors are shown in fig. 3. They peak in the region \( \xi < r < 2\xi \) with \( |u_{A(B),1}|^2 \sim 10^{-2} \) and \( |v_{A(B),1}|^2 \sim 10^{-5} \) so that \( u_{A(B),1} \gg v_{A(B),1} \), which results in a positive normalizaiton integral \( \int d^2r (u_{1}^2 - v_{-1}^2) \). This combination of positive norm and negative energy signals the presence of the anomalous mode which also occurs for vortices in single-component trapped BECs [32]. To the left of the peaks, near the core where \( 0 < r < \xi \), excitations are particle-like. There, the tangential rotation speed of the vortex exceeds the critical velocity for emissions of Cherenkov radiation so that, in the presence of a mechanism for dissipation, particles are freely radiated out of the condensate. In contrast, for \( r \approx 5\xi \), we find that \( |u_{-1}| \approx |v_{-1}| \) so that excitations are roughly equal admixtures of particles and holes and no radiation is expected.

The anomalous mode has a direct physical interpretation in terms of the precession of the vortex around the central core. To see this, first compute the density fluctuation in the anomalous mode:

\[
\delta n_{\text{vs},-1} = |\langle \Psi_{\text{vs}}^{*} | \Psi_{\text{vs}} \rangle^2 - |\Psi_{\text{vs}}^{*} | \Psi_{\text{vs}} \rangle^2 |u_{-1}^{T} | \Psi_{\text{vs}}^{*} - \Psi_{\text{vs}} |v_{-1}^{T} | e^{-iE_{-1}t/h} + |\Psi_{\text{vs}}^{*} | u_{-1}^{T} | \Psi_{\text{vs}}^{*} - \Psi_{\text{vs}} |v_{-1}^{T} | e^{-iE_{-1}t/h}. \tag{8}
\]

Substituting the expressions for \( \Psi_{\text{vs}}, u_{-1} \) and \( v_{-1} \) into eq. (8) allows us to obtain the density fluctuations for the individual components of the condensate. For the upper component (vortex), we obtain

\[
\delta n_{\text{vs},A,-1} = f_{A}(r) e^{-i(\theta - Re E_{-1}t/h)} e^{\text{Im} E_{-1}t/h} + g_{A}(r) e^{i(\theta - Re E_{-1}t/h)} e^{-\text{Im} E_{-1}t/h}, \tag{9}
\]

with the radial functions given by \( f_{A}(r) = \frac{n_0}{\sqrt{2}} u_{A,-1}(r) \times (r/\xi)^2/\sqrt{1+(r/\xi)^2} \) and \( g_{A}(r) = \frac{n_0}{\sqrt{2}} v_{A,-1}(r) (r/\xi)^2/\sqrt{1+(r/\xi)^2} \). Fluctuations are strongest near \( r \approx 1.5 \xi \) with the main contribution coming from the first term in eq. (9), which describes particle excitations with only a minimal hole component. For the fluctuations of the lower component of the condensate (soliton), we obtain an expression similar to eq. (9) but with the radial
functions given by $f_B(r) = n_0^{1/2} u_B,_{-1}(r)/\sqrt{1 + (r/\xi)^2}$ and $g_B(r) = n_0^{1/2} v_B,_{-1}(r)/\sqrt{1 + (r/\xi)^2}$. The fluctuations of the soliton contain nonzero angular terms which is a consequence of the coupling between the vortex and soliton through the Dirac kinetic terms; in time, through quantum fluctuations, the soliton will develop a finite rotation. The first term in eq. (9) grows exponentially and is proportional to $\exp[i(\Re E_{-1}/t/\hbar - \theta)] \times \exp(\Im E_{-1} t/\hbar)$. The complex factor describes the precession of the density fluctuation in the anticlockwise direction. The additional factor that grows in time does not appear in the analog case for a vortex in the Gross-Pitaevskii formalism and arises here from the coupling of the spinor components through the derivative terms in the Dirac Hamiltonian.

We can use eq. (8) to estimate the finite lifetime of the vortex due to this effect. By requiring that the total number of particles be conserved, we compute the value: $\tau = \ln 2/(2\Im E_{-1}) \approx 1.88 s$. We note that this value does not take into account interactions between the vortex and thermal cloud at finite temperature which would further reduce the value of $\tau$. Thus, we expect the lifetime for Dirac vortices to be shorter than vortices in condensates which are stationary with respect to the lattice. This is expected since the lattice provides a source of friction (dissipation) and bosons are free to drop to lower-energy states. This is of course not the case for fermions in graphene, for example, due to the presence of the Fermi level which coincides with the Dirac point. For several of our solutions, we find that the characteristic time associated with this instability is experimentally reasonable. The lowest quasiparticle energies for the other localized solutions are: $-3.9276 + 0.0019 i; \ 2.634 \times 10^{-2} + 9.96 \times 10^3 i; \ -3.9274 + 0.0019 i; \ 7.8409 \times 10^{-3} - 9.9993 \times 10^2 i; \ 7.9349 \times 10^{-3} - 9.9993 \times 10^2 i$; for the ring-vortex/soliton, half-quantum vortex, planar skyrmion, line-skyrmion, and line-soliton respectively, where all quantities are given in units of $\hbar U$.

In conclusion, we have shown that an effective quasi-relativistic system with a Dirac-like structure may be designed using ordinary cold bosonic atoms as the underlying degrees of freedom. We solved the resulting NLDE for different classes of nonlinear modes including half-quantum vortices. We derived and solved relativistic linear stability equations and gave explicit criteria for experimental observation of Cherenkov radiation, as well as predicting an anomalous mode for the vortex/soliton solution. Density profiles may be observed by time-of-flight techniques to detect both massive and massless Dirac fermions in the laboratory [27, 33]; nonlinear modes involving phase winding can be created by techniques analogous to those used at JILA [34]; and we anticipate that Bragg scattering can be used to populate the Dirac cones at both $\mathbf{K}$ and $\mathbf{K}'$ points, leading to arbitrary superpositions over our localized solution types between the two cones, and thereby populating all four components of the Dirac spinor.

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