THE CAUCHY-DIRICHLET PROBLEM FOR THE FENE DUMBBELL MODEL OF POLYMERIC FLUIDS

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Abstract. The FENE dumbbell model consists of the incompressible Navier-Stokes equation and the Fokker-Planck equation for the polymer distribution. In such a model, the polymer elongation cannot exceed a limit $\sqrt{b}$, yielding all interesting features near the boundary. In this paper we establish the local well-posedness for the FENE dumbbell model under a class of Dirichlet-type boundary conditions dictated by the parameter $b$. As a result, for each $b > 0$ we identify a sharp boundary requirement for the underlying density distribution, while the sharpness follows from the existence result for each specification of the boundary behavior.

It is shown that the probability density governed by the Fokker-Planck equation approaches zero near boundary, necessarily faster than the distance function $d$ for $b > 2$, faster than $d|\ln d|$ for $b = 2$, and as fast as $d^{b/2}$ for $0 < b < 2$. Moreover, the sharp boundary requirement for $b \geq 2$ is also sufficient for the distribution to remain a probability density.

1. Introduction

Let $N \geq 2$ be an integer. We consider a dimer – an idealized polymer chain – as an elastic dumbbell consisting of two beads joined by a spring that can be modeled by an elongation vector $m \in \mathbb{R}^N$ (see e.g. [3]), with $\Psi$ being the elastic spring potential defined by

\[
\Psi(m) = -\frac{Hb}{2} \log \left(1 - \frac{|m|^2}{b}\right), \quad m \in B.
\]

Here $B := B(0, \sqrt{b})$ is a ball in $\mathbb{R}^N$ with radius $\sqrt{b}$ denoting the maximum dumbbell extension. In the limiting case, this reduces to the Hookean model with $\Psi(m) = H|m|^2/2$. A general bead-spring chain model may contain more than two beads coupled with elastic springs to represent a polymer chain.

Polymers as such when put into an incompressible, viscous, isothermal Newtonian solvent are modeled by a system coupling the incompressible Navier-Stokes equation for the macroscopic velocity field $v(t,x)$ with the Fokker-Planck equation for the
probability distribution function $f(t,x,m)$:

$$
\begin{align*}
\partial_t v + (v \cdot \nabla)v + \nabla p &= \nabla \cdot \tau + \nu_k \Delta v, \\
\nabla \cdot v &= 0, \\
\partial_t f + (v \cdot \nabla)f + \nabla \cdot (\nabla \Psi(m)f) &= \frac{2}{\zeta} \nabla_m \cdot \nabla \Psi(m)f + \frac{2k_BT_a}{\zeta} \Delta mf,
\end{align*}
$$

where $x \in \mathbb{R}^N$ is the macroscopic Eulerian coordinate and $m \in B \subset \mathbb{R}^N$ is the microscopic molecular configuration variable. The model describes diluted solutions of polymeric liquids with noninteracting polymer chains (dimers). Note that the Fokker-Planck equation can be conveniently augmented to incorporate other effects such as inertial forces (see [14]).

In Navier-Stokes equation (1.2), $p$ is hydrostatic pressure, $\nu_k$ is the kinematic viscosity coefficient, and $\tau$ is a tensor representing the polymer contribution to stress,

$$
\tau = \lambda_p \int m \otimes \nabla \Psi(m) f dm,
$$

where $\lambda_p$ is the polymer density constant. In the Fokker-Planck equation (1.4), $\zeta$ is the friction coefficient of the dumbbell beads, $T_a$ is the absolute temperature, and $k_B$ is the Boltzmann constant. We refer to [6, 13, 43] for a comprehensive survey of the physical background, and [42] for the computational aspect.

Since $B$ is a bounded domain, one has to add an appropriate boundary condition for $f$ on the boundary $\partial B$. However, the singularity of the Fokker-Planck equation near $\partial B$ makes the boundary issue rather subtle, and presents various challenges. To address the boundary issue, several transformations relating to the equilibrium solution have been introduced in literature (see, e.g. [16, 24, 35, 36]). It was shown in [35] that $b = 2$ is a threshold in the sense that for $b > 2$ any preassigned boundary value of the ratio of the distribution and the equilibrium will become redundant, and for $b < 2$ that value has to be a priori given.

Our main quest in this paper is that what is the least boundary requirement for $f$ so that both existence and uniqueness of solutions to the FENE model can be established, also the solution remains a probability density.

We addressed this issue in [36] for the microscopic FENE model alone and when $b > 2$. In this article we consider the well-posedness of the coupled system (1.2)-(1.4). A general discussion of this problem and background references are given in the introduction to [36]. Here we have two objectives:

1. to identify sharp boundary conditions on $\partial B$ for all $b > 0$.
2. to prove well-posedness for the coupled FENE dumbbell model under the identified boundary condition.

The setting for our problem is the coupled system subject to the initial data

\begin{align*}
\begin{align*}
 v(0,x) &= v_0(x) \\
 f(0,x,m) &= f_0(x,m),
\end{align*}
\end{align*}

with the following boundary requirement

$$
f(t,x,m)\nu^{-1}|_{\partial B} = q(t,x,m)|_{\partial B}.
$$
Here $\nu$ depends on $b$ through the distance function, and $q$ is a given function measuring the relative ratio of $f/\nu$ near boundary. Our goal is to investigate solvability of the above system with the Cauchy-Dirichlet data. Note that our boundary condition is more or less a boundary behavior requirement for $f$, instead of the Dirichlet data in the traditional sense.

Instead of using the distance function $d = \sqrt{b - |m|}$ we shall use a regularized distance function $\rho = b - |m|^2$ when describing the solution behavior near boundary. Our main observation is the form of $\nu$

$$\nu = \begin{cases} 
\rho^{b/2}, & 0 < b < 2, \\
\rho \ln \frac{\rho}{b}, & b = 2, \\
\rho, & b > 2.
\end{cases}$$

(1.8)

With some regularity requirement on $q$ as well as on initial data we prove local well-posedness for the Cauchy-Dirichlet problem in a weighted Sobolev space for each given $q$. Our results indicate that simply putting $f = 0$ on boundary does not guarantee uniqueness of the solution.

For the Dirichlet-type boundary condition (1.7) considered in this paper, our strategy is to study the transformed problem via

$$w = \frac{f}{\nu} - q$$

with $\nu$ defined in (1.8) so as to extract useful info for $f$. Inspired from [38], for the coupled FENE system we use weak norm in $m$ and strong norm in $x$, this enables us to prove wellposedness for all cases of $b > 0$ and any given smooth $q$.

For the case $b \geq 2$ of physical interest, we prove that $f$ remains a density distribution if and only if $q|_{\partial B} = 0$. We thus identify a sharp boundary requirement for each $b > 0$ for the underlying density distribution, while the sharpness is a consequence of the existence result for each $q \neq 0$. In particular, our result asserts that near boundary the probability density governed by the Fokker-Planck equation approaches zero, necessarily faster than the distance function $d$ for $b > 2$, faster than $d|\ln d|$ for $b = 2$, and as fast as $d^{b/2}$ for $0 < b < 2$. But within our current framework we have not been able to identify a non-trivial $q$ for $0 < b < 2$ such that the corresponding solution is a density distribution.

We remark that the sharp boundary condition presented in this work provides a threshold on the boundary requirement: subject to this condition or any stronger ones incorporated through a weighted function space [17] or just zero flux [38], the Fokker-Planck dynamics will select the physically relevant solution, which is a probability density, any weaker boundary requirement can lead to many solutions, each depending on the ratio of $f/\nu$ near boundary.

This article is organized as follows. In Section 2, we state our main results and mains ideas of the proofs. In Section 3, we study the Fokker-Planck operator and well-posedness of the initial boundary value problem for the Fokker-Planck equation alone. This improves upon our previous work in [36]. The main result is summarized in Theorem 13. The Fokker-Planck problem involving spatial variable $x$ is investigated in Section 4. Well-posedness of the coupled system is proved in Section 5. In Section 6, we sketch the proof of well-posedness for the coupled system with $b \geq 6$ in a different function space than what we used in Section 5. Some concluding remarks
are drawn in Section 7.

We conclude this section by some bibliographical remarks.

Existence results for the FENE model are usually limited to small-time existence and uniqueness of strong solutions. We refer to [44] for the local existence on some related coupled systems, [22] for the FENE model (in the setting where the Fokker-Planck equation is formulated by a stochastic differential equation) with \( b > 6 \), [17] for a polynomial force. More related to this paper are the work by Zhang and Zhang [47] for the FENE model when \( b > 76 \), and Masmoudi [38] for \( b > 0 \). Global existence results are usually limited to solutions near equilibrium, see [28, 33], or to some 2D simplified models [10, 12, 27, 41]. For results concerning the existence of weak solutions to the coupled FENE system we refer to [2, 3, 4, 5, 34, 39, 45, 48].

Boundary behavior of the polymer distribution governed by the FENE model is also essential in several other aspects, including the study of large time behavior, see [1, 20, 23, 45]; and development of numerical methods, see, e.g., [8, 9, 16, 24, 37, 46]. We also refer to [21] for references on numerical aspects of polymeric fluid models.

There are also some interesting works on non-Newtonian fluid models derived through a closure of the linear Fokker-Planck equation (see, e.g., [15, 16]). We can refer to the pioneering work [18, 19], and more recently to [11, 29, 30, 31, 32]. However, none of these works is concerned with the sharpness of boundary conditions in terms of the elongation parameter.

2. MAIN RESULTS

After a suitable scaling and choice of parameters we arrive at the following Cauchy-Dirichlet problem for the coupled system

\[
\begin{align*}
\partial_t v + (v \cdot \nabla)v + \nabla p &= \nabla \cdot \tau + \Delta v, \quad x \in \mathbb{R}^N, \quad t > 0, \\
\nabla \cdot v &= 0, \\
\partial_t f + (v \cdot \nabla)f + \nabla_m \cdot (\nabla vmf) &= \frac{1}{2} \nabla_m \cdot \left( \frac{bm}{\rho} f \right) + \frac{1}{2} \Delta_m f, \quad m \in B, \\
\tau &= \int m \otimes \frac{bm}{\rho} f dm, \\
v(0, x) &= v_0(x), \\
f(0, x, m) &= f_0(x, m), \\
f(t, x, m) \nu^{-1}|_{\partial B} &= q(t, x, m)|_{\partial B}.
\end{align*}
\]

To present our main results we first fix notations to be used throughout this article. We fix an exponent \( s \), which is an integer in the range \( s > N/2 + 1 \). We use \( C \) to denote various constants depending on \( s, b \) and on some other quantities which we will indicate in the sequel. A \( b \)-dependent weight function is defined as

\[
\mu = \begin{cases} 
\rho^{b/2}, & 0 < b < 2, \\
\rho \ln^2 \frac{\rho}{\xi}, & b = 2, \\
\rho^{2-b/2}, & b > 2.
\end{cases}
\]
For $b \geq 6$, we also use
\begin{equation}
\mu_0 = \rho^\theta, \quad -1 < \theta < 1, \quad b \geq 6.
\end{equation}

Other notations are listed as below as well.

- $L^2_\mu = \{ \phi : \int_B \phi^2 \mu dm < \infty \}$
- $H^1_\mu = \{ \phi : \phi, \partial_m \phi \in L^2_\mu, j = 1 \ldots N \}$
- $H^1_\mu$ denotes the completion of $C^\infty_c$ with $H^1_\mu$ norm.
- $H^*$ is a dual space of $H$
- $H^*_x$ is the usual Sobolev space with respect to $x$

\[
|v|^2_s = \sum_{|\alpha| \leq s} \int_{\mathbb{R}^N} |\partial^\alpha v|^2 dx,
\]
\[
|w|^2_{0,s} = \sum_{|\alpha| \leq s} \int_{\mathbb{R}^N} \int_B |\partial^\alpha w|^2 \mu dm dx,
\]
\[
|w|^2_{1,s} = |w|_{0,s}^2 + |\nabla_m w|_{0,s}^2,
\]
\[
\|w\|^2_{1,1,s} = \sup_t (|w|_{1,s}^2 + |\partial_t w|_{1,s}^2),
\]
\[
\|q\|_s = \|q\|_{H^1_\mu} + \|\partial_t q\|_{H^1_\mu}.
\]

- $H^*_x L^2_\mu = \{ \phi : |\phi|_{0,s} < \infty \}$, $H^*_x H^1_\mu = \{ \phi : |\phi|_{1,s} < \infty \}$.
- $L^2_t H = L^2((0,T); H)$, $C_t H = C([0,T]; H)$ for $0 < t < T$.
- $\mathcal{H} = \{ \phi : ||\phi||_{L^2_t H^1_\mu} + ||\phi_t||_{L^2_t (H^1_\mu)^*} < \infty \}$
- $\mathcal{H}^\circ = \{ \phi(t, \cdot) \in H^1_\mu : ||\phi||_{L^2_t H^1_\mu} + ||\phi_t||_{L^2_t (H^1_\mu)^*} < \infty \}$.
- $X_\mu = [C_t H^s_x \cap L^2_t H^{s+1}_x] \times [C_t H^s_x L^2_t \cap L^2_t H^s_x H^1_\mu]$.

For a generic constant independent of $T$ and $a \in L^2_t$ we denote
\begin{equation}
F(a) = C \left( T + \int_0^T |a(t)|^2 dt \right).
\end{equation}

Due to such a constant, any two instances of $F$ should be presumed to be with different constants.

We now state our main theorem as follows:

**Theorem 1.** Let $b > 0$ and $s$ be an integer such that $s > N/2 + 1$. Suppose that $v_0 \in H^s_x$, $f_0 \nu^{-1} \in H^s_x L^2_\mu$, and $q \in C_t^1 H^{s+1}_x H^1_\mu$. Then, for some $T > 0$ there exists a unique solution $(v, f)$ to the coupled problem (2.1) such that
\[
(v, f \nu^{-1}) \in X_\mu.
\]

It is known from (25) that $H^1_\mu = H^1_\mu$ for $b \geq 6$ with $\mu$ defined in (2.2). Thus, the boundary condition (2.1g) is nothing but the zero dirichlet boundary condition under the assumption on $q$ in Theorem 1. For non-trivial $q$ when $b \geq 6$, we show the well-posedness in a different weighted Sobolev space. The result summarized as below.
Theorem 2. Let $b \geq 6$ and $s$ be an integer such that $s > N/2 + 1$. Suppose that $v_0 \in H^s_x$, $f_0 \nu^{-1} \in H^2_x L^2_{\mu_0}$, and $q \in C^1 H^{s+1}_x H^{-1}_\mu_0$ with $\mu_0$ defined in (2.2). Then, for some $T > 0$ there exists a unique solution $(v, f)$ to the coupled problem (2.1) such that

$$(v, f \nu^{-1}) \in X_{\mu_0}.$$  

Theorem 1 and 2 tell us that for each given $q$, which denotes the rate of $f$ approaching to zero relative to $\nu$ near $\partial B$, there exists a unique solution $(v, f)$. Also, they indicate that any weaker boundary requirement may lead to more than one solutions to (2.1). For instance, the boundary condition

$$f \nu^{-1} \rho^\varepsilon|_{\partial B} = 0, \quad \varepsilon > 0$$

gives infinitely many solutions to (2.1). Precisely we state the following non-uniqueness result.

Theorem 3. Let $\tilde{\nu}$ be a smooth function of $\rho$ such that

$$\lim_{\rho \to 0} \frac{\nu}{\tilde{\nu}} = 0.$$  

Then, the coupled problem (2.1) with (2.1g) replaced by

$$f(t, x, m) \nu^{-1}|_{\partial B} = 0$$

has infinitely many solutions in $X_\mu$ and $X_{\mu_0}$ for $0 < b < 6$ and $b \geq 6$ respectively.

The natural question is for what $q$ the obtained distribution $f$ is a probability distribution. The answer when $b \geq 2$ is given in the following theorem.

Theorem 4. Suppose that $b \geq 2$ and $q|_{\partial B} \geq 0$. Under the assumption of Theorem 1 or 2, the unique solution $f$ to the Cauchy-Direchlet problem (2.1) is a probability distribution if and only if $q|_{\partial B} = 0$. That is, $f \geq 0$ if $f_0 \geq 0$, and for any $t > 0$, $x \in \mathbb{R}^N$,

$$\int_B f(t, x, m) dm = \int_B f_0(x, m) dm.$$  

Theorem 1 is proven by a fixed point argument, which is now outlined. Given $(u, g)$, we first solve the Navier-Stokes equation (NSE):

$$\partial_t v + (u \cdot \nabla) v + \nabla p = \nabla \cdot \tau + \Delta v,$$

$$\nabla \cdot v = 0,$$

$$v(0, x) = v_0(x),$$

$$\tau = \int m \otimes \frac{bm}{\rho} g dm.$$  

With the obtained $v$ we solve the Fokker-Planck equation (FPE):

$$\partial_t f + (v \cdot \nabla) f + \nabla_m \cdot (\nabla_m f) = \frac{1}{2} \nabla_m \cdot \left( \frac{bm}{\rho} f \right) + \frac{1}{2} \Delta_m f,$$

$$f(0, x, m) = f_0(x, m),$$

$$f(t, x, m) \nu^{-1}|_{\partial B} = q(t, x, m)|_{\partial B}.$$  

The above two systems define a mapping $(u, g) \rightarrow (v, f)$, the existence of problem (2.1) is equivalent to existence of a fixed point of this mapping.
The main difficulty lies in monitoring the boundary behavior of \( f \). Our strategy is to apply the transformation
\[
(2.10) \quad f = \nu(w + q),
\]
to \( (2.9) \) to obtain a \( w \)-problem
\[
(2.11a) \quad \mu(\partial_t + v \cdot \nabla)w + L[w] = \mu h,
\]
\[
(2.11b) \quad w(0, x, m) = w_0(x, m),
\]
\[
(2.11c) \quad w(t, x, m)|_{\partial B} = 0.
\]
Here the operator \( L \) is induced from the Fokker-Planck operator, \( \nu \) and \( \mu \) are weights depending on the distance functions defined in (1.8) and (2.2) respectively. The source term is obtained from
\[
(2.12) \quad h = - \partial_t q - (v \cdot \nabla)q - \mu^{-1}L[q],
\]
and the initial data is given by
\[
(2.13) \quad w_0(x, m) := f_0(x, m)\nu^{-1} - q(0, x, m).
\]
For given \((u, \varpi)\) with \( g = \nu(\varpi + q)\), we arrive at a map \( \mathcal{F} \).
\[
\mathcal{F}: \ M \rightarrow M
\]
\[
(u, \varpi) \mapsto (v, w)
\]
Here \( M \) is a subset of
\[
C_t H_x^s \times [C_t H_x^s L_\mu \cap L^2_t H_x^s H_\mu^1]
\]
such that
\[
M = \left\{ (v, w) : \sup_{0 \leq t \leq T} |v|_x^2 \leq A_1, \sup_{0 \leq t \leq T} |w|_{0, s}^2 + \frac{1}{2} \int_0^T |\nabla_m w|_{0, \alpha}^2 dt \leq A_2 \right\}.
\]
The strategy for the fixed point proof, which we implement in sections to follow, is to first prove that \( \mathcal{F} \) is well defined for some \( T, A_1 \) and \( A_2 \), then show that \( \mathcal{F} \) is actually a contraction map in a weak norm. Moreover, we will show that
\[
(2.14) \quad \mathcal{F}(M) \subset X_\mu.
\]
This proves Theorem 1 for
\[
q \in C^1_t H_x^{s+1} H_\mu^1 \subset [C_t H_x^s L_\mu^2 \cap L^2_t H_x^s H_\mu^1].
\]
Theorem 2 is proved in the same manner. A sketch of proof is presented in Section 6.

In order to prove Theorem 3 we pick \( q(t, x, \cdot) \in C^\infty(B) \cap C(\overline{B}) \) and \( q|_{\partial B} \neq 0 \) such that
\[
q \in \left\{ \begin{array}{ll}
C^1_t H_x^{s+1} H_\mu^1, & 0 < b < 6, \\
C^1_t H_x^{s+1} H_{10}^1, & b \geq 6.
\end{array} \right.
\]
Note that existence of such a \( q \) follows from the density of the weighted Sobolev space (see [23] for details). Then for each \( q \) we have a unique solution \((v, f)\) to the coupled problem (2.11) from Theorem 1 and Theorem 2. Now, we check the boundary condition (2.6).
\[
fv^{-1}|_{\partial B} = f\nu^{-1}v|_{\partial B} = q\nu^{-1}v|_{\partial B},
\]
which vanishes since \( q|_{\partial B} \) is bounded and condition (2.5) holds. This proves Theorem 3.

Theorem 4 follows from Propositions 15 and 16 via a flow map to be described in Section 4. The case for \( b \geq 6 \) can be proved by a simple modification, which is also sketched in Section 6.

3. THE FOKKER-PLANCK OPERATOR

We start with (2.9) when \( x \) is not involved. In such a case it reduces to the following problem:

\[
\begin{align*}
\frac{\partial}{\partial t} f + \mathcal{L}[f] &= 0, \quad m \in B, t > 0, \\
 f(0, m) &= f_0(m), \\
 f(t, m) \nu^{-1}|_{\partial B} &= q(t, m)|_{\partial B}.
\end{align*}
\]

Here

\[
(3.2) \quad \mathcal{L}[f] := \nabla \cdot (\kappa mf) - \frac{1}{2} \nabla \cdot \left( \frac{bm}{\rho} f \right) - \frac{1}{2} \Delta f,
\]

\( \kappa = \kappa(t) \) is a square integrable matrix function such that \( \text{Tr}(\kappa) = 0 \). We omit \( m \) from \( \nabla_m \) in (3.2) for notational convenience.

The goal of this section is two folds;

(1) to provide tools for subsequent sections.

(2) to elaborate on this model alone as an extension of our previous work \[36\].

3.1. Transformed operator. The transformation (2.10) leads to

\[
\begin{align*}
\frac{\partial}{\partial t} w_\mu + L[w] &= \mu h, \quad m \in B, t > 0, \\
 w(0, m) &= w_0, \\
 w(t, m)|_{\partial B} &= 0,
\end{align*}
\]

with the transformed operator \( L \) determined by

\[
(3.4) \quad L[w] = \nu^{-1} \mathcal{L}[\nu w].
\]

The source term \( h = -\partial_t q - \mu^{-1} L[q] \) and initial data for \( w \) is \( w_0 = f_0 \nu^{-1} - q(0, m) \).

From a direct calculation with the choice of \( \mu \) in (2.2), and \( \nu \) in (1.8), (3.4) can be expressed as

\[
(3.5) \quad L[w] = -\frac{1}{2} \nabla \cdot (\nabla w_\mu) + \nabla \cdot (\kappa mw_\mu) - Kw,
\]

where

\[
(3.6) \quad K = \begin{cases} 
0, & 0 < b < 2, \\
(N + 2\kappa m \cdot m) \ln \rho, & b = 2, \\
(N + 2\kappa m \cdot m)(b/2 - 1)\rho^{1-b/2}, & b > 2.
\end{cases}
\]

Associated with the operator \( L \), we define its time-dependent bilinear form

\[
(3.7) \quad B[w, \phi; t] := \int \left( \frac{1}{2} \nabla w \cdot \nabla \phi + w \mu m \cdot \nabla \phi - Kw \phi \right) dm
\]

for \( \phi, w \in H^1_\mu \) and fixed \( t > 0 \).
We now describe the weak solution which we are looking for.

**Definition 5.** A function \( w \in \overset{\circ}{\mathcal{H}} \) is a weak solution of \( w \)-problem (3.3), provided

1. For each \( \phi \in \overset{\circ}{H}^1_\mu \) and almost every \( 0 \leq t \leq T \),
   \[
   (\partial_t w, \phi)_{\overset{\circ}{H}^1_\mu} + B[w, \phi; t] = (h, \phi)_{\overset{\circ}{H}^1_\mu}.
   \]
2. \( w(0, m) = w_0(m) \) in \( L^2_\mu \) sense, i.e.
   \[
   \int_B |w(0, m) - w_0(m)|^2 \mu dm = 0.
   \]

**Remark 6.** In (3.8), \( (\psi, \phi)_{\overset{\circ}{H}^1_\mu} \) is a dual pair for \( \psi \in (\overset{\circ}{H}^1_\mu)^* \) and \( \phi \in \overset{\circ}{H}^1_\mu \), and can be regarded as \( L^2_\mu \) inner product. Indeed, from the Riesz representation theorem, for each \( \psi \in (\overset{\circ}{H}^1_\mu)^* \) there exists a unique \( u \in \overset{\circ}{H}^1_\mu \) such that
   \[
   (\psi, \phi)_{\overset{\circ}{H}^1_\mu} = \int_B (\nabla u \cdot \nabla \phi + u \phi) \mu dm.
   \]
   Formally, the right hand side will be
   \[
   \int_B (\nabla \cdot (\nabla u \mu)^{-1} + u) \phi \mu dm.
   \]
   We identify \( \psi \) as \( \nabla \cdot (\nabla u \mu)^{-1} + u \) and the dual pair will be the \( L^2_\mu \) inner product.

**Remark 7.** With the weight function \( \mu \) so chosen as (2.2), we observe that if \( \phi \in H^1_\mu \), then \( \phi \in W^{1,1} \) since
   \[
   \int_B (|\phi| + |\nabla \phi|) dm \leq C \left( \int_B (|\phi|^2 + |\nabla \phi|^2) \mu dm \right)^{1/2} \left( \int_B \mu^{-1} dm \right)^{1/2} < \infty.
   \]
   From the standard trace theorem, the map
   \[
   \mathcal{T} : \overset{\circ}{H}^1_\mu(B) \rightarrow L^1(\partial B)
   \]
   \[
   \phi \mapsto \phi|_{\partial \Omega}
   \]
   is well defined. Thus, the element in \( \overset{\circ}{H}^1_\mu \) is characterized by the zero trace, and the Dirichlet data (3.3c) makes sense.

The well-posedness of the \( w \)-problem (3.3) is stated in the following.

**Theorem 8.** Suppose that \( w_0 \in L^2_\mu \), \( h \in L^2_\mu(\overset{\circ}{H}^1_\mu)^* \) and \( \kappa \in L^2_t \) with \( \text{Tr}(\kappa) = 0 \). Then the \( w \)-problem (3.3) has a unique weak solution in \( \overset{\circ}{\mathcal{H}} \) such that
   \[
   ||w||^2_{\overset{\circ}{\mathcal{H}}} \leq e^{F(|\kappa|)}(||w_0||^2_{L^2_\mu} + ||h||^2_{L^2_t(\overset{\circ}{H}^1_\mu)^*})
   \]
   with \( F \) defined in (2.4).

This result when \( b > 2 \) and \( q = 0 \) was proved in [36]. For general case we proceed in several steps.
An embedding theorem. We define
\begin{equation}
\mu^* = \begin{cases} 
\rho^{b/2 - 2}, & 0 < b < 2, \\
\rho^{-1}, & b = 2, \\
\rho^{-b/2}, & b > 2.
\end{cases}
\end{equation}
We call \( \mu^* \) as the conjugate of \( \mu \) due to the Sobolev inequalities in the following lemma.

**Lemma 9.** If \( \phi \in \overset{\circ}{H}^1_{\mu} \), then
\begin{equation}
\int |\phi|^2 \mu^* dm \leq C \int (|\phi|^2 + |\nabla \phi|^2) \mu dm.
\end{equation}
Also, if \( \phi \in H^1_{\rho^\theta} \) for \( \theta \leq 1 \), then for any \( \delta > 0 \)
\begin{equation}
\int |\phi|^2 \rho^{-1+\delta} dm \leq C \int (|\phi|^2 + |\nabla \phi|^2) \rho^\delta dm.
\end{equation}

**Proof.** We refer to [25] for a proof of (3.11) when \( b \neq 2 \), as well as (3.12). Here, we prove only the case \( b = 2 \).

First for \( C = \max_{1 \leq \rho \leq 2} [\rho \mu]^{-1} \) we have
\[
\int_B |\phi|^2 / \rho dm \leq C \int_{1 \leq \rho \leq 2} |\phi|^2 \mu dm + \int_{0 \leq \rho \leq 1} |\phi|^2 / \rho dm
\leq C \int_B |\phi|^2 \mu dm + \int_0^1 \frac{G^2}{\rho} d\rho,
\]
where we have used the spherical coordinate representation with \( \rho = 2 - r^2 \) and
\begin{equation}
G^2(\rho) = - \int_{|\xi| = 1} |\phi(\rho \xi)|^2 r^{N-1} dS_\xi \cdot \left( \frac{d \rho}{dr} \right)^{-1} = \frac{1}{2} \int_{|\xi| = 1} |\phi(\rho \xi)|^2 r^{N-2} dS_\xi.
\end{equation}

Note that from \( \phi \in \overset{\circ}{H}^1_{\mu} \) one can verify that \( G(0) = 0 \). It is known (see [26]) that
\[
\int_0^1 \left( \int_0^x g(t) dt \right)^2 \frac{1}{x} dx \leq C \int_0^1 g^2(x) x \ln x^2 dx.
\]
Thus,
\begin{equation}
\int_0^1 \frac{G^2}{\rho} d\rho \leq C \int_0^1 (G_\rho)^2 \rho |\ln \rho|^2 d\rho \leq C \int_0^1 \frac{G^2}{r^2} \mu d\rho \leq C \int_0^1 (G_r)^2 \mu d\rho,
\end{equation}
where we have used the fact that \( \rho |\ln \rho|^2 \leq \mu = \rho \ln^2 (e/\rho) \). Differentiation of (3.13) in term of \( r \) leads to
\[
2GG_r = \int_{|\xi| = 1} \phi \nabla \phi \cdot \xi r^{N-2} dS_\xi + \frac{N-2}{2} \int_{|\xi| = 1} |\phi(\rho \xi)|^2 r^{N-3} dS_\xi.
\]
Squaring both sides and using the Cauchy-Schwartz inequality we obtain
\[
4G^2(G_r)^2 \leq 2 \int_{|\xi| = 1} \phi^2 r^{N-2} dS_\xi \int_{|\xi| = 1} |\nabla \phi|^2 r^{N-2} dS_\xi + \frac{(N-2)^2}{2} \left( \int_{|\xi| = 1} \phi^2 r^{N-2} dS_\xi \right)^2.
\]
where we have used the fact $r \geq 1$. Hence
\[(G_r)^2 \leq \int_{|\xi|=1} |\nabla \phi(r \xi)|^2 r^{N-2} dS_\xi + \frac{(N-2)^2}{2} G^2,\]
which inserted into (3.14) ensures that the term $\int_0^1 \frac{G^2}{\rho} d\rho$ is also bounded by $C\|\phi\|_{H^1}$.

The proof is now complete. □

**Energy estimates.** We return now to the bilinear operator $B$.

**Lemma 10** (Energy estimates). For any $t$, there exists a constant $C$ which is dependent on $N, b$ such that

1. for $w(t, \cdot) \in H^1_\mu$

\[
\frac{1}{4} \int |\nabla w|^2 \mu dm \leq B[w, w; t] + C(1 + |\kappa|^2) \int w^2 \mu dm;
\]

2. for $\psi(t, \cdot) \in H^1_\mu$ and $\phi \in H^1_\mu$,

\[
|B[\psi, \phi; t]| \leq C(1 + |\kappa|) \|\psi\|_{H^1_\mu} \|\phi\|_{H^1_\mu}.
\]

**Proof.** From (3.7) it follows

\[
\frac{1}{2} \int \nabla w \cdot \nabla \phi \mu dm = B[w, \psi; t] + \int \kappa m \cdot \nabla \phi w \mu dm + \int K w \phi dm,
\]

where $K$ is given in (3.6).

1. If $0 < b < 2$, then $K = 0$; hence

\[
\frac{1}{2} \int |\nabla w|^2 \mu dm = B[w, w; t] + \int \kappa m \cdot \nabla w w \mu dm
\]

\[\leq B[w, w; t] + \frac{1}{4} \int |\nabla w|^2 \mu dm + b|\kappa|^2 \int w^2 \mu dm\]

and

\[
|B[\psi, \phi; t]| \leq \frac{1}{2} \int |\nabla \psi||\nabla \phi| \mu dm + \sqrt{b}|\kappa| \int |\psi||\nabla \phi| \mu dm
\]

\[\leq C(1 + |\kappa|) \|\psi\|_{H^1_{\mu}} \|\nabla \phi\|_{L^2_{\mu}}.
\]

2. For $b \geq 2$, it suffices to estimate the $K$-related term. If $b = 2$, we have

\[K = (N + 2\kappa m \cdot m) \ln \frac{e}{\rho} \leq (N + 2b|\kappa|) \sqrt{\mu \mu'}.
\]

If $b > 2$, we have

\[K = \left(\frac{b}{2} - 1\right) \rho^{1-b/2}(N + 2\kappa m \cdot m)
\]

\[\leq \left(\frac{b}{2} - 1\right) (N + 2b|\kappa|) \sqrt{\mu \mu'}.
\]
Hence for $b \geq 2$ we have

$$\int Kw^2dm \leq C(1 + |\kappa|) \int w^2 \nu_\mu dm$$

$$\leq \varepsilon \int w^2 \nu_\mu dm + C_\varepsilon(1 + |\kappa|^2) \int w^2 \mu dm.$$  

This when added upon right side of (3.18) using (3.11) with some small $\varepsilon$ leads to (3.15). Using (3.11) again we have

$$\left| \int K\psi \phi dm \right| \leq C(1 + |\kappa|) \int |\psi| |\phi| \nu_\mu dm \leq C(1 + |\kappa|) ||\psi||_{H^1_\mu} ||\phi||_{H^1_\mu},$$

which when combined with the above estimate for $b < 2$ gives (3.16).

**A priori estimate.**

**Lemma 11** (A priori estimates). Let $w$ be a weak solution to (3.3). Then

$$\sup_t ||w(t, \cdot)||^2_{L^2_\mu} + \frac{1}{2} ||w||^2_{L^2_\mu H^1_\mu} \leq e^{F(|\kappa|)} \left( ||w_0||^2_{L^2_\mu} + ||h||^2_{L^2_\mu(H^1_\mu)^*} \right).$$

with $F$ defined in (2.4), and furthermore

$$||w||^2_\mu \leq e^{F(|\kappa|)}(||w_0||^2_{L^2_\mu} + ||h||^2_{L^2_\mu(H^1_\mu)^*}).$$

**Proof.** From the weak solution definition in (3.8) we have for any $\phi \in H^1_\mu$

$$\left( \partial_t w, \phi \right)_{H^1_\mu} + \mathcal{B}[w, \phi; t] = (h, \phi)_{H^1_\mu}.$$  

By (3.16), $(\partial_t w, \phi)_{H^1_\mu}$ is bounded by

$$||h||_{(H^1_\mu)^*} ||\phi||_{H^1_\mu} + C(1 + |\kappa|)||w||_{H^1_\mu} ||\phi||_{H^1_\mu}.$$  

Hence

$$||\partial_t w||_{(H^1_\mu)^*} \leq ||h||_{(H^1_\mu)^*} + C(1 + |\kappa|) ||w||_{H^1_\mu}.$$  

Next we set $\phi = w$ in (3.21) and use (3.15) to have

$$\frac{1}{2} \frac{d}{dt} ||w||^2_{L^2_\mu} + \frac{1}{4} \int |\nabla w|^2 \mu dm \leq ||h||_{(H^1_\mu)^*} ||w||_{H^1_\mu} + C(1 + |\kappa|^2) ||w||^2_{L^2_\mu}$$

$$\leq 2 ||h||^2_{(H^1_\mu)^*} + \frac{1}{8} ||w||^2_{H^1_\mu} + C(1 + |\kappa|^2) ||w||^2_{L^2_\mu}.$$  

Hence

$$\frac{d}{dt} ||w||^2_{L^2_\mu} + \frac{1}{4} ||w||^2_{H^1_\mu} \leq C(1 + |\kappa|^2) ||w||^2_{L^2_\mu} + 4 ||h||^2_{(H^1_\mu)^*},$$

and therefore by Gronwall’s inequality,

$$\sup_t ||w(t, \cdot)||^2_{L^2_\mu} + \frac{1}{2} ||w||^2_{L^2_\mu H^1_\mu} \leq e^{C(T + \int_0^T |\kappa|^2 dt)} \left( ||w_0||^2_{L^2_\mu} + ||h||^2_{L^2_\mu(H^1_\mu)^*} \right),$$

which together with (3.22) yields (3.20).
Proof of Theorem 8 We construct a weak solution to (3.3) using the Galerkin approximation. Let \( \{ \phi_i \} \) be a basis of \( \hat{H}^1_{\mu} \) and \( L^2_{\mu} \). Then an approximate solution \( w_l \) in a finite dimensional space is defined as \( w_l = \sum_{i=1}^l \phi_i(t) \). Here \( \phi_i(t) \) is a unique solution to a system of linear differential equations,
\[
(\partial_t w_l, \phi_j)_{\hat{H}^1_{\mu}} + B[w_l, \phi_j; t] = (h, \phi_j)_{\hat{H}^1_{\mu}},
\]
\[
d_l(0) = ((\phi_i, \phi_j)_{L^2_{\mu}})(w_0, \phi),
\]
where \( \phi = (\phi_1, \ldots, \phi_l)^T \). Using the same argument as that in the proof of Lemma 11 we obtain estimates for \( w_l \) such that
\[
||w_l||^2_{L^2(T; \hat{H}^1_{\mu})} + ||\partial_t w_l||^2_{L^2(T; \hat{H}^1_{\mu})} \leq C F(|\kappa|) ||w_0||^2_{L^2_{\mu}} + ||h||^2_{L^2(T; \hat{H}^1_{\mu})}.
\]
Extracting a subsequence and passing to the limit give a weak solution \( w \) in \( \hat{H} \). The uniqueness follows from the a priori estimate (3.20).

To return to the Fokker-Planck problem (3.1) we will also need the following

Lemma 12. Let \( h = -\partial_t q - \mu^{-1}L[q] \). If \( q \in C^1[H^1_{\mu}] \) and \( \kappa \in L^2_{\mu} \) with \( Tr(\kappa) = 0 \), then
\[
(3.24) \quad ||h||^2_{L^2(T; \hat{H}^1_{\mu})} \leq C \int_0^T (1 + |\kappa|^2)||q(t)||^2 dt.
\]

Proof. For \( q \in C^1[H^1_{\mu}] \), it is obvious that \( \partial_t q \in L^2(T; \hat{H}^1_{\mu})^* \) since \( H^1_{\mu} \subset (H^1_{\mu})^* \subset (\hat{H}^1_{\mu})^* \).
In order to show \( \mu^{-1}L[q] \in L^2(T; \hat{H}^1_{\mu})^* \), we use integration by parts and (3.16) to get
\[
\left| \int_{\mu^{-1}L[q] \phi_\mu dm} = |B[q, \phi; t]| \leq C(1 + |\kappa|)||q(t, \cdot)||H^1_{\mu}||\phi||H^1_{\mu}, \forall \phi \in C^\infty_c.
\]
Since \( C^\infty_c \) is a dense subset of \( \hat{H}^1_{\mu} \), for any \( \phi \in \hat{H}^1_{\mu} \) with \( ||\phi||H^1_{\mu} = 1 \), we have
\[
(3.25) \quad ||(\mu^{-1}L[q], \phi)_{\hat{H}^1_{\mu}}| \leq C(1 + |\kappa|)||q(t, \cdot)||H^1_{\mu}.
\]
Taking the \( L^2 \) norm in \( t \) leads to the desired estimate.

Theorem 8 and Lemma 12 lead to the following result for problem (3.1) with general Dirichlet boundary condition.

Theorem 13. Suppose that \( f_0 \nu^{-1} \in L^2_{\mu} \), \( q \in C^1[H^1_{\mu}] \) and \( \kappa \in L^2_{\mu} \) with \( Tr(\kappa) = 0 \). Then for any \( T > 0 \) the Fokker-Planck problem (3.7) has a unique solution \( f \) such that
\[
(3.26) \quad f = \nu(w + q) \quad \text{with} \quad w \in \mathcal{H} \quad \text{for} \quad 0 < t \leq T.
\]
Moreover, for \( F \) defined in (2.4),
\[
(3.27) \quad \sup_t ||w(t, \cdot)||^2_{L^2_{\mu}} + \frac{1}{2}||w||^2_{L^2(T; \hat{H}^1_{\mu})} \leq C F(|\kappa|) \left( ||w_0||^2_{L^2_{\mu}} + \int_0^T (1 + |\kappa(t)|^2)||q(t)||^2 dt \right).
\]
Theorem 13. Suppose $b > q$ with $w$ solves w-problem (3.3) with $w_0 \equiv h \equiv 0$. Hence $w \equiv 0$, leading to $f_1 = f_2$. □

Remark 14. As mentioned in Section 2 that $\hat{H}^1_\mu = H^1_\mu$ if $b \geq 6$, i.e., the trace of $q \in H^1_\mu$ vanishes if $b \geq 6$. Thus, the boundary condition (3.1c) is nothing but a zero Dirichlet boundary condition. In Section 6, we show the well-posedness with a nonzero Dirichlet boundary condition for $b \geq 6$ using yet a different transformation.

3.2. Probability density function. So far we have discussed well-posedness of the initial-boundary value problem (3.1) for $b > 0$ and any given $q$. We now turn to the question of which $q$ corresponds to the probability density, i.e., non-negative solution with constant mass for all time.

Proposition 15. Let $f(t,m)$ be the solution to problem (3.1) obtained in Theorem 13. If $f_0 \geq 0$ and $q(t,m)|_{\partial B} \geq 0$ almost everywhere, then $f$ remains nonnegative for $t > 0$.

Proof. We adapt an idea from [7]. Let $f^\pm$ be the positive and negative parts of the solution $f$ such that $f = f^+ - f^-$. Obviously, $w^\pm := f^\pm \nu^{-1} \in H^1_\mu$ and $q|_{\partial B} \geq 0$. This implies that the trace of $w^-$ at the boundary vanishes, so

\[ w^- \in \hat{H}^1_\mu. \]

From the equation

\[ \partial_t w + L[w] = 0, \]

which is transformed from (3.1a) it follows that

\[ (\partial_t w, w^-)_{H^1_\mu} + B[w, w^-; t] = 0. \]

Since $(\partial_t w^+, w^-)_{H^1_\mu}$ and $\int L[w^+] w^- dm$ vanish, hence

\[ \frac{1}{2} \frac{d}{dt} \left( \int |w^-|^2 \mu dm \right) + \mathcal{B}[w^-, w^-; t] = 0. \]

The coercivity of $\mathcal{B}$, (3.15), gives

\[ \frac{1}{2} \frac{d}{dt} \left( \int |w^-|^2 \mu dm \right) + \frac{1}{4} \int |\nabla w^-|^2 \mu dm \leq C(1 + |\kappa|^2) \int |w^-|^2 \mu dm. \]

Hence

\[ \sup_t ||w^-(t, \cdot)||_{L^2_\mu}^2 \leq ||w_0^-||_{L^2_\mu}^2 e^{F(|\kappa|)} \]

for $T > 0$. Since $w_0^- = 0$, $||w^-(t, \cdot)||_{L^2_\mu}^2 = 0$ for all $0 \leq t \leq T$. □

Proposition 16. Let $f$ be a solution to the Fokker-Planck problem (3.1) obtained in Theorem 13. Suppose $b \geq 2$ and $q(t,m)|_{\partial B} \geq 0$. If $q|_{\partial B} = 0$ for all $t \in [0,T]$, then

\[ \int f(t, \cdot) dm = \int f_0 dm, \quad t \in [0,T], \]

and vice versa.
Proof. It suffices to prove the claim for smooth enough \( f \) since the general case can be treated by an approximation as in [36]. We rewrite (3.1a) as

\[
\partial_t f = -\nabla \cdot (\kappa m f) + \nabla \cdot \left( \rho^{b/2} \nabla \frac{f}{\rho^{b/2}} \right).
\]

First, we take a test function \( \phi_\varepsilon(m) = \phi_\varepsilon(|m|) \in C^\infty(\mathbb{R}^N) \) converging to \( \chi_B \) as \( \varepsilon \to 0 \) such that

\[
\phi_\varepsilon(|m|) = \begin{cases} 
1, & |m| \leq \sqrt{b} - \varepsilon \\
0, & |m| \geq \sqrt{b} - \varepsilon/2 
\end{cases}, \quad |\nabla \phi_\varepsilon| \leq C \frac{1}{\varepsilon}
\]

and for any smooth \( g \)

\[
(3.28) \quad \int_{\sqrt{\sqrt{b} - \varepsilon}^{\sqrt{b} - \varepsilon/2}} g(r) \phi_\varepsilon'(r) dr \to -g(\sqrt{b}) \quad \text{as} \quad \varepsilon \to 0,
\]

where \( \phi_\varepsilon'(r) = \nabla \phi_\varepsilon \cdot \frac{m}{|m|} \).

One can construct such a \( \phi_\varepsilon \) by mollifiers, for example

\[
\phi_\varepsilon(m) = \int_{B_{\sqrt{b} - 3\varepsilon/4}} \eta_{\varepsilon/4} (m - m') dm'.
\]

where

\[
\eta_\varepsilon(m) = \frac{1}{\varepsilon^N} \eta(m/\varepsilon), \quad \eta(m) = \begin{cases} 
C e^{-\frac{1-|m^2|}{\varepsilon}}, & |m| < 1 \\
0, & |m| \geq 1 
\end{cases},
\]

and \( C \) is the normalizing constant.

Since \( \nabla \phi_\varepsilon \) is supported in \( B^\varepsilon := B_{\sqrt{b} - \varepsilon/2} \setminus B_{\sqrt{b} - \varepsilon} \), hence

\[
(3.29) \quad \frac{d}{dt} \int_B \phi_\varepsilon dm = \int_{B^\varepsilon} f \kappa m \cdot \nabla \phi_\varepsilon dm - \int_{B^\varepsilon} \rho^{b/2} \nabla \left( \frac{f}{\rho^{b/2}} \right) \cdot \nabla \phi_\varepsilon dm.
\]

By \( w = f\nu^{-1} \), the right hand side reduces to

\[
(3.30) \quad \int_{B^\varepsilon} (w \kappa m - \nabla w) \cdot \nabla \phi_\varepsilon \nu dm - \int_{B^\varepsilon} w \rho^{b/2} \nabla \phi_\varepsilon \cdot \nabla (\nu \rho^{-b/2}) dm.
\]

The first term converges to 0. Indeed,

\[
\left| \int_{B^\varepsilon} (w \kappa m - \nabla w) \cdot \nabla \phi_\varepsilon \nu dm \right| \leq \left( \int_{B^\varepsilon} |w \kappa m - \nabla w|^2 \mu dm \right)^{1/2} \left( \int_{B^\varepsilon} |\nabla \phi_\varepsilon|^2 \nu^2 \mu dm \right)^{1/2}.
\]

Since \( \nu^2/\mu = \rho^{b/2} \) for \( b \geq 2 \), by mean value theorem there exists \( r \in (\sqrt{b} - \varepsilon, \sqrt{b} - \varepsilon/2) \) such that

\[
\int_{B^\varepsilon} |\nabla \phi_\varepsilon|^2 \frac{\nu^2}{\mu} dm = \frac{\varepsilon}{2} \int_{\partial B^\varepsilon} |\nabla \phi_\varepsilon|^2 \rho^{b/2} dS \leq C \varepsilon^{b/2 - 1},
\]

which is uniformly bounded for \( b \geq 2 \). Using \( w \in H^1_{\mu} \), we obtain \( \int_{B^\varepsilon} |w \kappa m - \nabla w|^2 \mu dm \to 0 \) as \( \varepsilon \to 0 \). Hence the first term in (3.30) converges to 0.
On the other hand, for \( C_0 = \begin{cases} -2, & b = 2 \\ 2 - b, & b > 2 \end{cases} \)

\[
- \int_{B^c} \rho^{b/2} \nabla \phi \cdot \nabla (\nu \rho^{b/2}) \, dm = C_0 \int_{B^c} \rho^{b/2} \nabla \phi \cdot m \, dm \\
= \frac{C_0}{\sqrt{b - \epsilon}} \frac{b}{2} \int_{\partial B^c} \rho^{b/2} (r) \, dS \int_{\partial B^c} \rho^{b/2} (r) \, dS \\
= \frac{C_0}{\sqrt{b - \epsilon}} \frac{b}{2} \int_{\partial B^c} \rho^{b/2} (r) \, dS.
\]

Due to (3.28) this converges to

\[
-C_0 \sqrt{b} \int_{\partial B} w \, dS = -C_0 \sqrt{b} \int_{\partial B} q \, dS.
\]

Since \( C_0 \neq 0 \), this shows that \( \frac{d}{dt} \int_B f \, dm = 0 \) if and only if \( \int_{\partial B} q \, dS = 0 \), or \( q |_{\partial B} = 0 \).

\[\square\]

**Remark 17.** In Proposition 16, the assumption \( b \geq 2 \) is sharp. In the case \( b < 2 \), we need to consider nontrivial \( q \neq 0 \) since the equilibrium profile \( f_{eq} = \rho^{b/2} \) satisfies

\[
q |_{\partial B} = \rho^{b/2} \nu^{-1} |_{\partial B} = 1.
\]

This requirement is also consistent with [35], in which it was shown that when \( b < 2 \), \( f_{eq} \nu^{-1} |_{\partial B} = q |_{\partial B} \) is necessarily prescribed and each solution depends on the choice of \( q \). It would be interesting to figure out a particular \( q \) for which the corresponding solution when \( b < 2 \) is a probability density.

4. The Fokker-Planck equation

In this section, we show the well-posedness of the FPE (2.9) including \( x \) variable. The result is stated as follows.

**Theorem 18.** Suppose that for \( b > 0 \) and any integer \( s > N/2 + 1 \), \( \nabla \cdot v = 0 \) and

\[
v \in C^1_l H^s_x \cap L^2_t H^{s+1}_x, \quad f_0 \nu^{-1} \in H^s_x L^2_t \mu, \quad q \in C^1_l H^{s+1}_x H^1_\mu, \quad 0 < t < T
\]

for any \( T > 0 \). Then (2.9) has a unique solution \( f = \nu (w + q) \) satisfying

\[
\sup_t |w|_{0,s}^2 + \frac{1}{2} \int_0^T |\nabla m w|_{0,s}^2 \, dt \leq e^{F(|v|_{s+1})} \left( |w_0|_{0,s}^2 + \|q\|_{1,1,s+1}^2 \right),
\]

where \( F \) was defined in (2.4).

The proof of Theorem 18 consists of two parts: first we show the existence of the solution \( f \) to problem (2.9) by using the flow map, followed by proving regularity in \( x \) inductively such that \( w \in C^1_l H^s_x L^2_t \mu \cap L^2_t H^2_x H^1_\mu \) with \( v, f_0 \) and \( q \) given in (4.1). In the second step, we derive estimate (4.2) directly from (2.9) to control \( f \) in terms of the given data. The uniqueness can be obtained from the estimation (4.2) as performed in the proof of Theorem 13.

First, we state a technical lemma.
Lemma 19. Suppose that $\psi \in H^1_\mu$ and $\phi \in \overset{\circ}{H}^1_\mu$. Then for the trace map $T : W^{1,1}(B) \rightarrow L^1(\partial B)$

(4.3) $T(\psi\phi\mu) = 0$.

Proof. Since $C^\infty_c$ is a dense subset of $\overset{\circ}{H}^1_\mu$, it suffices to show that for a fixed $\psi \in H^1_\mu$ and any $\phi \in C^\infty_c$ (4.4) $||\psi\phi\mu||_{W^{1,1}} \leq C||\phi||_{H^1_\mu}$.

Then, the standard trace theorem asserts that $T(\psi\phi\mu)$ is well-defined in $L^1(\partial B)$ and it vanishes, also $T$ is a continuous map with respect to $\phi$, we can thus conclude (4.3) for any $\phi \in \overset{\circ}{H}^1_\mu$ by passing to the limit of sequence $\phi_n \in C^\infty_c$ such that $\phi_n \rightarrow \phi$.

(4.4) is indeed the case. It is obvious that $\psi\phi\mu, \nabla_m\psi\phi\mu$ and $\psi\nabla_m\phi\mu$ are integrable.

For $b \neq 2, |\nabla_m\mu| \leq C(\ln \frac{e}{\rho} + \ln \frac{\rho}{\mu}) \leq C(\ln \frac{e}{\rho} + \sqrt{\mu\mu^*})$.

Using (3.12) and $\psi \in H^1_\mu$, we obtain $\psi \in L^{2-1+\delta}$ for any $\delta > 0$. Hence

$$\left| \int \psi\phi \ln \frac{e}{\rho} dm \right| \leq C \left( \sqrt{\int \psi^2 \rho^{-1} dm} \sqrt{\int \phi^2 \rho^{1-\delta} \ln \frac{e}{\rho} dm} \right).$$

It follows that for any $b > 0$

$$\int |\psi\phi\mu| + |\nabla_m(\psi\phi\mu)| dm < C||\psi||_{H^1_\mu}||\phi||_{H^1_\mu}$$
as we desired. \qed

The main ingredient for the proof of Theorem 18 is to use the calculus inequalities in the Sobolev spaces, see Appendix 3.5 of [40]: for any positive integer $r > 0$ and $u, v \in L^\infty_\delta \cap H^r_x$,

(4.5) $\sum_{|\alpha| \leq r} ||\partial^\alpha(uv) - u\partial^\alpha v||_{L^2} \leq C(||\nabla u||_{L^\infty}||v||_{H^{r-1}} + ||u||_{H^r}||v||_{L^\infty})$,

(4.6) $||uv||_{H^r} \leq C(||u||_{L^\infty}||v||_{H^r} + ||u||_{H^r}||v||_{L^\infty})$.

Note that (4.5) remains valid when $\partial^\alpha$ on the left hand is replaced by the corresponding difference operator.

Proof of Theorem 18

Step1 (well-posedness) Let a particle path be defined by

$$\partial_t x(t,y) = v(t,x(t,y)), \quad x(0,y) = y,$$
along which the distribution function \( \tilde{f}(t, y, m) := f(t, x(t, y), m) \) solves

\[
\begin{align*}
(4.7a) \quad & \partial_t \tilde{f} + \mathcal{L}[\tilde{f}] = 0, \\
(4.7b) \quad & \tilde{f}(0, y, m) = f_0(y, m), \\
(4.7c) \quad & \tilde{f}(t, y, m)\nu^{-1}|_{\partial \Omega} = \tilde{q}(t, y, m)|_{\partial \Omega}.
\end{align*}
\]

Here \( \mathcal{L} \) is defined in (3.22) with \( \kappa \) replaced by \( \tilde{\kappa}(t, y) = \nabla v(t, x(t, y)) \), and \( \tilde{q}(t, y, m) := q(t, x(t, y), m) \).

In order to show existence of the solution to (2.9) under the conditions \( v \in C^s H_s^x \cap L^2 H_{s+1}^x \) and \( \nabla \cdot v = 0 \), it suffices to show that (4.7) has a solution \( \tilde{f} = \nu(\tilde{w} + \tilde{q}) \) such that

\[
\tilde{w} := w(t, x(t, y), m) \in C^s H_s^y L^2_{\mu} \cap L^2_{\mu} H^1_y H_{\mu}^1,
\]
assuming that

\[
(4.8) \quad \tilde{\kappa} \in L^2 H_{\mu}^s, \quad w_0 \in H^s_{\mu} L^2_{\mu}, \quad \tilde{q} \in C^1_{s} H^s_y H_{\mu}^1.
\]

These follow from (4.11) since \( |\tilde{\kappa}(t)|_s \leq C |v(t)|_{s+1} \) for \( t > 0, w_0(x, m) = f_0 \nu^{-1} - \tilde{q}(t = 0) = w_0(y, m) \), and \( \|\tilde{q}\|_{1, s} \leq C \|q\|_{1, 1, s+1} \), for which we have used \( \partial \tilde{q} = \partial q + v \cdot \nabla q \).

Using Theorem 13 for each \( y \), there exists a unique solution \( \tilde{f} \) such that

\[
\tilde{f} = \nu(\tilde{w} + \tilde{q})
\]
with \( \tilde{w} \) satisfying (3.27), i.e.,

\[
\begin{align*}
\sup_t ||\tilde{w}(t, y, \cdot)||_{L^2_{\mu}}^2 + \frac{1}{2}||\tilde{w}(\cdot, y, \cdot)||_{L^2_{\mu} H_{\mu}^s}^2 & \leq e^{F(|\tilde{\kappa}|_s)} \left( ||w_0(y, \cdot)||_{L^2_{\mu}}^2 \\
& \quad + \int_0^T (1 + |\tilde{\kappa}(\cdot, y)|^2)||\tilde{q}(t, \cdot, \cdot)||^2 dt \right).
\end{align*}
\]

Integration of (4.9) with respect to \( y \), upon exchanging the order of integration in \( y \) and \( m \), and using the Sobolev inequality, \( \sup_y ||\tilde{\kappa}| \leq C |\tilde{\kappa}|_{s-1} \), gives

\[
(4.10) \quad \sup_t ||\tilde{w}||^2_{0, 0} + \frac{1}{2} \int_0^T ||\tilde{w}||^2_{1, 0} dt \leq e^{F(|\tilde{\kappa}|_{s-1})} \left( ||w_0||^2_{0, 0} + ||\tilde{q}||^2_{1, 1, 0} \right).
\]

Hence \( \tilde{w} \in C^s H^s_{\mu} L^2_{\mu} \cap L^2_{\mu} H^s_{\mu} \). On the other hand, the right hand side of (4.9) is uniformly bounded in \( y \), taking \( \sup_y \) of (4.9) gives

\[
(4.11) \quad \sup_{t,y} ||\tilde{w}(t, y, \cdot)||^2_{L^2_{\mu}} \leq e^{F(|\tilde{\kappa}|_{s-1})} (||w_0||^2_{0, s-1} + ||\tilde{q}||^2_{1, 1, s-1}).
\]

We now use an induction argument to prove that \( \tilde{w} \in C^s H^s_{\mu} L^2_{\mu} \cap L^2_{\mu} H^s_{\mu} \) for \( 0 \leq r \leq s \), and

\[
(4.12) \quad \sup_t ||\tilde{w}||^2_{0, r} + \frac{1}{2} \int_0^T ||\tilde{w}||^2_{1, r} dt \leq e^{F(|\tilde{\kappa}|_{r})} (||w_0||^2_{0, s} + ||\tilde{q}||^2_{1, 1, s}).
\]

The case \( r = 0 \) has been proved as shown in (4.10). Suppose (4.12) holds for \( r = k \), we only need to show (4.12) for \( r = k + 1 \leq s \).

To prove regularity of \( \tilde{f} \) in the \( y \) variable, we use difference quotients. Define the difference operator in the \( y \) variable as

\[
\delta^\gamma := \delta_1^{\gamma_1} \cdots \delta_N^{\gamma_N}, \quad \delta_i u(y) := \frac{1}{\eta}[u(y + \eta e_i) - u(y)].
\]
Apply $\delta^\gamma$ to (4.7) with $|\gamma| \leq s$, then

\begin{align}
(4.13a) \quad & \partial_t \delta^\gamma \tilde{f} + \mathcal{L} [\delta^\gamma \tilde{f}] = \nabla_m \cdot J, \\
(4.13b) \quad & \delta^\gamma \tilde{f}(0, y, m) = \delta^\gamma f_0(y, m), \\
(4.13c) \quad & \delta^\gamma \tilde{f}(t, y, m) \nu^{-1}|_{\partial B} = \delta^\gamma \tilde{q}(t, y, m)|_{\partial B},
\end{align}

where

\begin{equation}
(4.14) \quad J = \tilde{k} m \delta^\gamma \tilde{f} - \delta^\gamma (\tilde{k} m \tilde{f}).
\end{equation}

This when transformed into the w-problem of form (3.3) involves the following non-homogeneous term

\begin{equation}
(4.15) \quad h = -\partial_t \delta^\gamma \tilde{q} - \mu^{-1} L [\delta^\gamma \tilde{q}] + \nabla_m \cdot J \nu^{-1}.
\end{equation}

Using Theorem 13 again for each $y$, $\delta^\gamma \tilde{f}$ is the unique solution to (4.13) as long as $h \in L^2_t(\tilde{H}^1_\mu)^*$. Moreover,

\[ \delta^\gamma \tilde{f} = \nu (\delta^\gamma \tilde{w} + \delta^\gamma \tilde{q}), \]

where $\delta^\gamma \tilde{w}$, using (3.19), satisfies

\[ \sup_t \|\delta^\gamma \tilde{w}(t, y, \cdot)\|_{L^2_{\tilde{h}_\mu}}^2 + \frac{1}{2} \|\delta^\gamma \tilde{w}(\cdot, y, \cdot)\|_{L^2_{\tilde{h}_\mu}}^2 \leq e^{F(|\tilde{k}(\cdot, y)|)} \left( \|\delta^\gamma w_0\|_{L^2_{\tilde{h}_\mu}}^2 + \|h\|^2_{L^2_t(\tilde{H}^1_\mu)^*} \right). \]

Integration in $y$ gives

\begin{equation}
(4.16) \quad \sup_t \|\delta^\gamma \tilde{w}\|_{0,0}^2 + \frac{1}{2} \int_0^T \|\delta^\gamma \tilde{w}\|_{1,0}^2 dt \leq e^{F(\sup_y |\tilde{k}(\cdot, y)|)} \left( \|\delta^\gamma w_0\|_{0,0}^2 + \|h\|^2_{L^2_t(\tilde{H}^1_\mu)^*} \right)
\end{equation}

\[ \leq e^{F(|\tilde{k}|_{s-1})} \left( \|w_0\|_{0,s}^2 + \|h\|^2_{L^2_t(\tilde{H}^1_\mu)^*} \right). \]

We now turn to bound the last term in the above inequality. For any $\phi \in \tilde{H}^1_\mu$ and $J$ defined in (4.14), Lemma 19 allows the use of integration by parts. Hence,

\[ \left| \int \nabla_m \cdot J \nu^{-1} \phi \mu dm \right| \leq \left( \int |J \nu^{-1}||\nu \nabla_m \frac{\mu}{\nu}||\phi| dm + \int |J \nu^{-1}||\nabla_m \phi| \mu dm \right) \]

\[ \leq C \|J \nu^{-1}\|_{L^2_{\tilde{h}_\mu}} (\|\phi\|_{L^2_{\tilde{h}_\mu}} + \|\nabla_m \phi\|_{L^2_{\tilde{h}_\mu}}) \]

\[ \leq C \|J \nu^{-1}\|_{L^2_{\tilde{h}_\mu}} \|\phi\|_{H^1_{\tilde{h}_\mu}}. \]

Here we have used $|\nu \nabla_m \frac{\mu}{\nu}| \leq C \sqrt{\mu^* \mu}$ and the embedding theorem (3.11). This together with Lemma 12 and (4.15) yields

\begin{equation}
(4.17) \quad \|h\|^2_{L^2_t(\tilde{H}^1_\mu)^*} \leq C \int_0^T (1 + \sup_y |\tilde{k}(t, y)|^2) \int \|\delta^\gamma \tilde{q}(t, y, \cdot)\|^2 dy dt + C \int_0^T |J \nu^{-1}|_{0,0}^2 dt.
\end{equation}

For $|\gamma| \leq s$, the first term on the right side is bounded by

\begin{equation}
(4.18) \quad F(|\tilde{k}|_{s-1})\|\delta^\gamma \tilde{q}\|_{1,0,0}^2 \leq F(|\tilde{k}|_{s-1})\|\tilde{q}\|_{1,1,s}^2.
\end{equation}
To obtain (4.12) for \( r = k + 1 \leq s \), it remains to estimate the last term in (4.17) with \( |\gamma| = k + 1 \). In fact,
\[
|J^\nu|^{-1}_{0,0}^2 = |(\delta^\gamma(\tilde{k}m\tilde{f}) - \tilde{k}m\delta^\gamma\tilde{f})\nu^{-1}|^2_{0,0} \\
\leq C(\sup_y |\nabla_y \tilde{k}|^2|\tilde{f}\nu^{-1}|^2_{0,k} + |\tilde{k}|^2_{k+1} \sup_y \|\tilde{f}\nu^{-1}\|_{L^2_y}^2) \\
\leq C|\tilde{k}|^2_s(\tilde{w})^2_{0,k} + \sup_y \|\tilde{w}\|_{L^2_y}^2 + \|	ilde{q}\|_{1,1,s}^2,
\]
where we have used (4.5) with \( \partial^\gamma \) replaced by \( \delta^\gamma \).

Using (4.12) for \( r = k \) and (4.11) we have
\[
\int_0^T |J^\nu|^{-1}_{0,0}^2 dt \leq e^{F(\tilde{k}|\cdot|)}(\|w_0\|_{0,s}^2 + \|\tilde{q}\|_{1,1,s}^2).
\]

This and (4.18) when inserted into (4.17) gives a bound for \( \|h\|_{L^2_tL^2_y(H^s_x)} \). That bound combined with (4.10) yields
\[
\sup_t |\delta^\gamma \tilde{w}|^2_{0,0} + \frac{1}{2} \int_0^T |\delta^\gamma \tilde{w}|^2_{1,0} dt \leq e^{F(\tilde{k}|\cdot|)}(\|w_0\|_{0,s}^2 + \|\tilde{q}\|_{1,1,s}^2) < \infty, \quad |\gamma| = k + 1.
\]

Sending \( \eta \to 0 \) we obtain (4.12) with \( r = k + 1 \). Hence, (4.12) holds for any \( r \leq s \), and thus the solution \( f \) to (2.9) exists, and
\[
\sup_t |w|^2_{0,s} + \frac{1}{2} \int_0^T |w|^2_{1,s} dt < \infty.
\]

One may obtain an upper bound from (4.12) with \( r = s \) using the inverse map of \( x = x(t, y) \). Nevertheless, the next step gives the claimed bound in (4.2).

**Step 2 (a priori estimate)** For a priori estimate, we consider the w-problem (2.11)
\[
(4.19) \quad \mu(\partial_t + v \cdot \nabla)w + L[w] = -\mu(\partial_t + v \cdot \nabla)q - L[q].
\]

Recall that
\[
L[w] = -\frac{1}{2} \nabla_m \cdot (\nabla_m w\mu) + \nabla_m \cdot (\kappa m w\mu) - Kw.
\]

Take \( \gamma \) derivative in \( x \)-variable. Then, the left and right hand side of (4.19) will be
\[
\begin{align*}
I &= \mu(\partial_t + v \cdot \nabla)\partial^\gamma w - \frac{1}{2} \nabla_m \cdot (\nabla_m \partial^\gamma w\mu) \\
+ \mu[\partial^\gamma((v \cdot \nabla)w) - (v \cdot \nabla)\partial^\gamma w] \\
+ \nabla_m \cdot (\partial^\gamma(\kappa m w\mu)) \\
- \partial^\gamma(Kw),
\end{align*}
\]
\[
II = -\mu\partial_\nu\partial^\gamma q + \frac{1}{2} \nabla_m \cdot (\nabla_m \partial^\gamma q\mu) \\
- \mu\partial^\gamma((v \cdot \nabla)q) \\
- \nabla_m \cdot (\partial^\gamma(\kappa m q\mu)) \\
+ \partial^\gamma(Kq).
\]
We now estimate term by term of
\begin{equation}
\sum_{|\gamma| \leq s} \int \partial^\gamma w \mu dm = \sum_{|\gamma| \leq s} \int \partial^\gamma w \mu dm.
\end{equation}

Since \( v \) is divergence free, the first two terms on the left hand side will be

\[
\frac{1}{2} \frac{d}{dt} |w|_{0,s}^2 + \frac{1}{2} |\nabla_m w|_{0,s}^2.
\]

Indeed, Cauchy inequality shows that the term related to \( (4.21) \) is bounded by

\[
\varepsilon |w|_{0,s}^2 + C_\varepsilon \sum_{|\gamma| \leq s} \int \int |\partial^\gamma ((v \cdot \nabla) w - (v \cdot \nabla) \partial^\gamma w)|^2 \mu dm.
\]

Now, we exchange the order of integration in \( x \) and \( m \), and apply \( (4.5) \) to obtain

\[
\varepsilon |w|_{0,s}^2 + C_\varepsilon \int \left( \|\nabla v\|_{L^\infty}^2 \|\nabla w(\cdot, m)\|_{H_{x-1}^s}^2 + \|v\|_{L^\infty} \|\nabla w(\cdot, m)\|_{L^\infty}^2 \right) \mu dm
\]

\[
\leq \varepsilon |w|_{0,s}^2 + C_\varepsilon |v|_{s+1}^2 |w|_{0,s}^2,
\]

where the Sobolev inequality, \( |u|_0 \leq C |u|_{s-1} \) for any \( u \in H_{x}^{s-1} \), is invoked in the last inequality. Similarly, the term with \( (4.22) \) will be estimated as follows due to \( (4.6) \):

\[
\varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon \sum_{|\gamma| \leq s} \int \int |\partial^\gamma (\kappa m w)|^2 \mu dm
\]

\[
\leq \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon |\kappa|_{s+1}^2 |w|_{0,s}^2
\]

Recall that

\[
K = \begin{cases} 
0, & 0 < b < 2, \\
(N + 2 \kappa m \cdot m) \ln \frac{\rho}{\rho}, & b = 2, \\
(N + 2 \kappa m \cdot m)(b/2 - 1) \rho^{1-b/2}, & b > 2.
\end{cases}
\]

Thus, we can express \( K \) as

\begin{equation}
K = c_1 \sqrt{\mu \mu^*} + c_2 \kappa m \cdot m \sqrt{\mu \mu^*}
\end{equation}

for some positive constant \( c_i \) depending on \( N \) and \( b \). We now estimate the last term on the left hand side, by using

\[
\partial^\gamma (K w \partial^\gamma w = c_1 |\partial^\gamma w|^2 \sqrt{\mu \mu^*} + c_2 \partial^\gamma (\kappa m \cdot m \mu^* \partial^\gamma w \sqrt{\mu \mu^*}.
\]

The Cauchy inequality and the embedding theorem \( (3.11) \) give

\[
c_1 \sum_{|\gamma| \leq s} \int \int |\partial^\gamma w|^2 \sqrt{\mu \mu^*} dm = c_1 \int |w(t, \cdot, m)|_{s}^2 \sqrt{\mu \mu^*} dm
\]

\[
\leq \varepsilon \int |w(t, \cdot, m)|_{s}^2 \mu^* dm + C_\varepsilon \int |w(t, \cdot, m)|_{s}^2 \mu dm
\]

\[
\leq \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon |w|_{0,s}^2.
\]
Similarly,

\[ c_2 \sum_{|\gamma| \leq s} \int \int |\partial^\gamma (\kappa \cdot m \cdot w) \partial^\gamma w| \sqrt{\mu} \mu \, dm \, dx \leq \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon \int \int |\partial^\gamma (\kappa \cdot m \cdot w)|^2 \mu \, dm \, dx. \]

The last term, using (4.6) and the Sobolev inequality for \( \kappa = \nabla v \), is then bounded by

\[ C_\varepsilon |v|_{s+1}^2 |w|_{0,s}^2. \]

Hence,

\[
\left| \sum_{|\gamma| \leq s} \int \int \partial^\gamma (Kw) \partial^\gamma w \, dmdx \right| \leq \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon (|v|_{s+1}^2 + 1) |w|_{0,s}^2.
\]

Now we turn to the right hand side, related to (4.24)-(4.27). The estimation is similar to that for the left hand side. Except that here we have to assume higher regularity of \( q \) in \( x \) than that of \( w \) since \( \int v \cdot \int \nabla \partial^\gamma q \partial^\gamma w \, dmdx \) does not vanish as \( \int v \cdot \int \nabla \partial^\gamma w \partial^\gamma w \, dmdx \). Indeed, the first two terms, related to (4.24) are bounded by

\[ \varepsilon |w|_{0,s}^2 + C_\varepsilon |\partial_t q|_{0,s}^2 + \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon |\nabla m q|_{0,s}^2, \]

and the other terms are estimated as follows;

\[
\begin{align*}
&\sum_{|\gamma| \leq s} \left| \int \int \partial^\gamma (v \cdot \nabla q) \partial^\gamma w \, dmdx \right| \leq \varepsilon |w|_{0,s}^2 + C_\varepsilon |v|_{s}^2 |q|_{0,s+1}^2, \\
&\sum_{|\gamma| \leq s} \left| \int \int \partial^\gamma (\kappa m q) \nabla_m \partial^\gamma w \, dmdx \right| \leq \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon |v|_{s+1}^2 |q|_{0,s}^2, \\
&\sum_{|\gamma| \leq s} \left| \int \int \partial^\gamma (Kq) \partial^\gamma w \, dmdx \right| \leq \varepsilon |\nabla_m w|_{0,s}^2 + C_\varepsilon |q|_{0,s}^2 + C_\varepsilon |v|_{s+1}^2 |q|_{0,s}^2.
\end{align*}
\]

We combine all estimates for sufficiently small \( \varepsilon \) to obtain

\[ \partial_t |w|_{0,s}^2 + \frac{1}{2} |\nabla_m w|_{0,s}^2 \leq C (|v|_{s+1}^2 + 1) \left( |w|_{0,s}^2 + (|q|_{1,s+1}^2 + |\partial q|_{1,s+1}^2) \right). \]

We deduce that

\[ |w|_{0,s}^2 + \frac{1}{2} \int_0^t |\nabla_m w|_{0,s}^2 \, dt \leq e^{F(|v|_{s+1})} \left( |w_0|_{0,s}^2 + F(|v|_{s+1}) |q|_{1,s+1}^2 \right). \]

Replacing \( Fe^F \) by \( e^F \) leads to (4.2).

\[ \]
Lemma 20. Suppose that $\phi \in \overset{\circ}{H}^1_\mu$. For any $\varepsilon > 0$ there exists $C_\varepsilon$ such that

\[
(5.1) \quad \left| \int \phi \nu \rho^{-1} dm \right|^2 \leq C_\varepsilon \int |\phi|^2 \mu dm + \varepsilon \int |\nabla_m \phi|^2 \mu dm.
\]

Proof. For $b > 2$, the Cauchy-Schwartz inequality yields

\[
\left| \int \phi dm \right|^2 \leq \int |\phi|^2 \mu dm \int \mu^{-1} dm.
\]

For any $\varepsilon > 0$, taking $C_\varepsilon = \int \mu^{-1} dm < \infty$, we obtain (5.1) for $b > 2$.

For $b \leq 2$, we define for fixed $M$,

\[
G = \{ \phi \in \overset{\circ}{H}^1_\mu : \int \phi \nu \rho^{-1} dm = 1, ||\phi||_{H^1_\mu} \leq M \}.
\]

It suffices to prove

\[
l := \inf_{\phi \in G} \int |\phi|^2 \mu dm > 0.
\]

Let $\{ \phi_n \} \subset G$ be a sequence such that

\[
\lim_{n \to \infty} \int |\phi_n|^2 \mu dm = \inf_{\phi \in G} \int |\phi|^2 \mu dm.
\]

Since $\{ \phi_n \}$ is bounded in $H^1_\mu$, by embedding theorem (3.11), there exists a subsequence $\{ \phi_{n_k} \}$ such that

\[
\phi_{n_k} \rightharpoonup \phi^* \quad \text{in } H^1_\mu,
\]

\[
\phi_{n_k} \rightharpoonup \phi^* \quad \text{in } L^2_\mu,
\]

\[
\phi_{n_k} \rightharpoonup \phi^* \quad \text{in } L^2_{\mu^*}.
\]

Furthermore, since $\sqrt{\frac{\mu}{\mu^*}} \in L^2_\mu$, for $b \leq 2$

\[
\int \phi^* \nu \rho^{-1} dm = \int \phi^* \sqrt{\frac{\mu}{\mu^*}} \mu^* dm
\]

\[
= \lim_{n_k \to \infty} \int \phi_{n_k} \sqrt{\frac{\mu}{\mu^*}} \mu^* dm = 1.
\]

This shows that $\phi^* \in G$. On the other hand,

\[
\int |\phi^*|^2 \mu dm \leq \lim_{n_k \to \infty} \int |\phi_{n_k}|^2 \mu dm = l.
\]

If $l = 0$, then $\phi^* = 0$ which is a contradiction to $\phi^* \in G$. \qed

The zero trace of $\phi$ is essential for the estimate (5.1). For the general case, i.e., for $\phi \in H^1_\mu$, one can only have a weaker estimate.

Lemma 21. If $\phi \in H^1_\mu$, then there exists $C$ such that

\[
(5.2) \quad \left| \int \phi \nu \rho^{-1} dm \right|^2 \leq C ||\phi||^2_{H^1_\mu}.
\]
Proof. For \( b > 2 \), we have
\[
\left| \int \phi \nu \rho^{-1} dm \right|^2 \leq C \int |\phi|^2 \mu dm, \quad C := \int \mu^{-1} dm < \infty.
\]
For \( b \leq 2 \),
\[
\left| \int \phi \nu \rho^{-1} dm \right|^2 \leq C_\delta \int |\phi|^2 \rho^{-\delta} dm, \quad C_\delta := \left( \int \nu^2 \rho^{-1-\delta} dm \right).
\]
We choose \( \delta > 0 \) small enough so that \( C_\delta \) is bounded. On the other hand, by (3.12) in Lemma 9 we have
\[
\int |\phi|^2 \rho^{-1+\delta} dm \leq C \int (|\phi|^2 + |\nabla_m \phi|^2) \rho^{1/2} dm = C \int (|\phi|^2 + |\nabla_m \phi|^2) \mu dm, \quad b < 2
\]
\[
\int |\phi|^2 \rho^{-1+\delta} dm \leq C \int (|\phi|^2 + |\nabla_m \phi|^2) \rho dm \leq C \int (|\phi|^2 + |\nabla_m \phi|^2) \mu dm, \quad b = 2.
\]
This completes the proof. \( \square \)

We now turn to the map
\[
\mathcal{F} : \quad \mathbf{M} \to \mathbf{M}
\]
\[
(u, \varpi) \mapsto (v, w),
\]
and
\[
\mathbf{M} = \left\{ (v, w) : \sup_{0 \leq t \leq T} |v|^2_s \leq A_1, \sup_{0 \leq t \leq T} |w|^2_{0,s} + \frac{1}{2} \int_0^T |\nabla_m w|^2_{0,s} dt \leq A_2 \right\}.
\]
We first prove that, given \( v_0 \in H^s_x, f_0 \nu^{-1} \in H^s_x L^2 \) and \( q \in C_t^1 H^{s+1}_x H^1_t \), the map \( \mathcal{F} \) is well defined, i.e., \( \mathcal{F}(\mathbf{M}) \subset \mathbf{M} \) for some \( A_1, A_2, T \).

Let \( (u, \varpi) \in \mathbf{M} \). It is now well known that (2.8) has a unique solution \( v \) such that
\[
\sup_t |v|^2_s + \int_0^T |v|^2_{s+1} dt \leq |v_0|^2_s + C \int_0^T |u_s| |v|^2_s dt + \frac{1}{2} \int_0^T |\nabla m v|^2_s dt, \quad s > N/2 + 1.
\]
By Gronwall’s inequality and \( \sup_{0 \leq t \leq T} |u|^2_s \leq A_1 \), we have
\[
\sup_t |v|^2_s + \int_0^T |v|^2_{s+1} dt \leq \left( |v_0|^2_s + \int_0^T |\nabla^s v|^2 dt \right) e^{C \sqrt{\lambda} T}.
\]
We proceed to estimate the stress term
\[
\int_0^T |\nabla^s v|^2 dt = \int_0^T \sum_{|\alpha| \leq s} \int |\nabla^\alpha \nabla| v|^2 dx dt,
\]
where using Lemma 20,
\[
|\nabla^\alpha \nabla|^2 = b^2 \left| \int_B m \otimes m \nabla (\varpi + q) \nu \rho^{-1} dm \right|^2 \leq C \left( \int |\nabla \varpi|^2 \mu dm + \frac{\varepsilon}{2} \int |\nabla \nabla \varpi|^2 \mu dm + 2b^4 \left| \int \nabla q \nu \rho^{-1} dm \right|^2. \right.
\]
Using (5.2) the last term is uniformly bounded by
\[ C\|\partial^\gamma q(t,x,\cdot)\|_{H^1_{\mu}}^2 \leq C\|q\|_{1,1,s+1}^2. \]

Hence for \((u,\varpi) \in \mathbb{M}\) we obtain
\[ (5.5) \int_0^T |\tau_s|^2 dt \leq C\epsilon T A_2 + \epsilon A_2 + CT|q|_{1,1,s+1}^2 \leq CT(A_2 + \|q\|_{1,1,s+1}^2) + \epsilon A_2, \]

where we have used the assumption \(q \in C^1_t H_x^{s+1} H_{\mu}^1\).

We choose \(A_1\) as
\[ (5.6) A_1 = 2|v_0|^2 s e, \]

\(A_2\) as
\[ (5.7) A_2 = (|w_0|^2_{0,s} + \|q\|_{1,1,s+1}^2)e^{C(T+A_1)} \]

for \(T \leq 1/(C \sqrt{A_1})\).

Hence, if \(T\) and \(\epsilon\) are chosen small enough so that
\[ CT(A_2 + \|q\|_{1,1,s+1}^2) + \epsilon A_2 \leq \frac{1}{2e} A_1, \]

we get
\[ (5.8) e^{C\sqrt{T}T}(|v_0|^2_s + CT(A_2 + |q|_{1,1,s+1}^2) + \epsilon A_2) \leq e(|v_0|^2_s + \frac{1}{2e} A_1) \leq A_1. \]

This together with (5.4), (5.5) gives
\[ (5.9) \sup_t |v|^2_s + \int_0^{T_1} |v|^2_{s+1} dt \leq A_1. \]

Estimate (4.2) in Theorem 18 (5.7) and (5.9) yield
\[ (5.10) \sup_t |w|^2_{0,s} + \frac{1}{2} \int_0^T |\nabla w|^2_{0,s} dt \leq A_2. \]

So the map \(\mathcal{F}\) is well defined in \(\mathbb{M}\).

Next, we show that \(\mathcal{F}\) is a contraction mapping for small enough \(T\) using a weak norm on \(\mathbb{M}\), i.e.
\[ (5.11) ||(v,w)||_\mathbb{M} := \sup_t |v|^2_0 + \sup_t |w|^2_{0,0} + \frac{1}{2} \int_0^T |\nabla w|^2_{0,0} dt. \]

Suppose that \(v_i(i = 1, 2)\) are solutions of the NSE (2.8) with \(u_i(i = 1, 2)\) and \(\tau_i(i = 1, 2)\) computed from \(\varpi_i(i = 1, 2)\) respectively. Then we obtain
\[ (5.12) \partial_t v + (u_2 \cdot \nabla)v + (u \cdot \nabla)v_1 + \nabla p = \nabla \cdot \tau + \Delta v, \quad v(0, \cdot) = 0, \]

where \(v = v_2 - v_1, u = u_2 - u_1, p = p_2 - p_1, \tau = \tau_2 - \tau_1\) and \(\varpi = \varpi_2 - \varpi_1\). Multiplication by \(v\) to (5.12) and integration with respect to \(x\) yield
\[ \frac{1}{2} \frac{d}{dt} |v|^2_0 + \int (u \cdot \nabla v_1)v dx = -\int \tau \nabla v dx - \int |\nabla v|^2 dx. \]
Hence
\[ \frac{d}{dt}|v|^2_0 + |\nabla v|^2_0 \leq |u|^2_0 + |\tau|^2 + \sup_x |\nabla v_1|^2 |v|^2_0 \]
\[ \leq |u|^2_0 + |\tau|^2 + A_1 |v|^2_0. \]  
(5.13)

Let \( f_i \) be the solutions to (5.9) associated with \( v_i (i = 1, 2) \). Then
\[ w = (f_2 - f_1)\nu^{-1} =: w_2 - w_1 \]
solves
\[ \begin{align*}
\partial_t w + v_2 \cdot \nabla w + L_2[w] &= -v \cdot \nabla w_1 - \nabla_m \cdot (\nabla vmw_1) \frac{\mu}{\nu}, \\
(5.14a) \quad w(0, x, m) &= 0, \\
(5.14b) \quad w(t, x, m)|_{\partial B} &= 0,
\end{align*} \]
where \( L_2[w] = L[w] \) defined in (3.3) with \( \kappa = \nabla v_2 \). Note that \( w_i|_{\partial B} = q|_{\partial B} \), i.e. \( w_i(t, x, \cdot) \in H^1_\mu \), so \( w(t, x, \cdot) \in \dot{H}^1_\mu \).

We deduce from (5.14a) that
\[ \frac{1}{2} \frac{d}{dt} |w|^2_0 + \frac{1}{2} |\nabla w|^2_0 \leq \int \int |\nabla v_2 m \cdot \nabla_m w \nu w| dm dx + \int \int |K w|^2 dm dx \]
\[ + \int |v \cdot \nabla w_1 |dm dx + \int \int |\nabla_m \cdot (\nabla vmw_1) \frac{\mu}{\nu} w dm| dx. \]
Similar to that led to (4.2), first two terms on the right hand side are bounded by
\[ C_\varepsilon (|v_2|^2 + 1)|w|^2_0 + \varepsilon |\nabla w|^2_0. \]
and the third term
\[ \int |v \cdot \nabla w_1| dm dx \leq C \int |v|^2 \int |\nabla w_1|^2 \mu dm dx + \int |w|^2 \mu dm dx \]
\[ \leq C |v|^2_0 |w_1|^2_0 + |w|^2_0. \]
The last term, using integration by parts with vanished boundary term due to Lemma 19 is bounded by
\[ \int \left| \int \nabla_m \cdot (\nabla vmw_1) \frac{\mu}{\nu} w dm \right| dx \leq C \varepsilon |\nabla v|^2_0 |w_1|^2_0 + \varepsilon |\nabla w|^2_0. \]
Putting all together we have
\[ \frac{d}{dt} |w|^2_0 + \frac{1}{2} |\nabla w|^2_0 \leq C (|v_2|^2 + 1)|w|^2_0 + C |w_1|^2_0 (|v|^2_0 + |\nabla v|^2_0) \]
\[ \leq C (A_1 + 1)|w|^2_0 + CA_2 (|v|^2_0 + |\nabla v|^2_0). \]

Substitution of the estimates of \( |\nabla v|^2_0 \) and \( \frac{d}{dt} |v|^2_0 \) in (5.13) gives
\[ \frac{d}{dt} (|v|^2_0 + |w|^2_0) + \frac{1}{2} |\nabla w|^2_0 \leq D (|v|^2_0 + |w|^2_0) + D |w|^2_0 + D |\tau|^2_0, \]
where \( D \) is a large constant depending on \( C, A_1, A_2 \), for example we may choose
\[ D = C (A_1 + 1)(A_2 + 1). \]
The Gronwall inequality gives
\[
\sup_t (|v|^2_t + |w|^2_t) + \frac{1}{2} \int_0^T |\nabla_m w|^2_{0,0} dt \leq De^{DT} \int_0^T |u|^2_t + |\tau|^2_t dt
\]
for any \(0 < T^* \leq T\). Due to the similar estimate for \( \tau \) as (5.5), the right hand side is bounded by
\[
De^{DT} \left( T^* \sup_t |u|^2_t + C_\varepsilon T^* \sup_t |\varpi|^2_{0,0} + \varepsilon \int_0^{T^*} |\nabla_m \varpi|^2_{0,0} dt \right).
\]
We choose \( \varepsilon = \frac{1}{4De^{DT}} \), \( T^* = \frac{1}{2} \min \left\{ T, \frac{1}{(C_\varepsilon + 1) De^{DT}} \right\} \) and redefine \( T = T^* \) to obtain
\[
\|(v_2, w_2) - (v_1, w_1)\|^2_M = \|(v, w)\|^2_M \leq \frac{1}{2} \|(u_2, \varpi_2) - (u_1, \varpi_1)\|^2_M.
\]
This shows that \( F \) has a fixed point \((v, w)\) in \( M \), which is a solution to the coupled problem \((2.1)\). Since \( F(v, w) = (v, w) \), (5.3) and Theorem 13 imply that \((v, w) \in X_\mu\).

The uniqueness follows from the same computation of estimates for the contraction mapping. Let \((v_i, f_i \nu^{-1})(i = 1, 2)\) be solutions of the coupled problem \((2.1)\). Then \( v = v_2 - v_1 \) solves (5.12) with \( u_i = v_i, u = v, \) and \( \tau = \tau_2 - \tau_1 \) computed from \( f_i \). \( w = (f_2 - f_1) \nu^{-1} \) also solves (5.14) with \( w_1 = f_1 \nu^{-1} \). Similar to (5.15), we obtain
\[
\frac{d}{dt} (|v|^2_t + |w|^2_t) + \frac{1}{2} |\nabla_m w|^2_{0,0} \leq D(|v|^2_t + |w|^2_t + |\tau|^2_t).
\]
It follows from the estimate for \( \tau \) and Gronwall inequality that \((v, w) \equiv (0, 0)\), which gives the uniqueness of problem \((2.1)\).

6. A further look at \( b \geq 6 \)

In this section, we sketch proofs of Theorem 2 and Theorem 4 for the case of \( \mu = \mu_0 \).

Consider \((2.9)\) when \( x \) is not involved, i.e., (3.11). The corresponding w-problem for \( w = f \nu^{-1} - q \) with \( \mu = \mu_0 \) solves (5.3) with the operator \( L \) replaced by
\[
L_0[w] = -\frac{1}{2} \nabla \cdot (\nabla w \mu_0) + \left( 2 - \frac{1}{2} b - \theta \right) m \cdot \nabla w \rho^{\theta - 1} + \nabla \cdot (kmw \mu_0) - K_0 w,
\]
where
\[
K_0 = [N(b/2 - 1) + 2km \cdot m(1 - \theta)] \rho^{\theta - 1}.
\]
Define the conjugate of \( \mu_0 \) as \((3.10)\), \( \mu_0^* = \rho^{\theta - 2} \), then \( K_0 \) can be rewritten as
\[
K_0 = [N(b/2 - 1) + 2km \cdot m(1 - \theta)] \sqrt{\mu_0 \mu_0^*}.
\]
To ensure well-posedness of \((5.3)\), we need to check the coercivity of \( B_0[w, w; t] \), which is defined as
\[
\frac{1}{2} \int |\nabla w|^2 \mu_0 dm = B_0[w, w; t] - \left( 2 - \frac{1}{2} b - \theta \right) \int m \cdot \nabla w \rho^{\theta - 1} dm
\]
\[
- \int \nabla \cdot (kmw \mu_0) w dm + \int K_0 w^2 dm.
\]
From the proof of Lemma 10 the last two terms are bounded by

$$C_c \int w^2 \mu_0 dm + \varepsilon \int |\nabla w|^2 \mu_0 dm,$$

where the embedding theorem (3.11) has been used. For small enough \(\varepsilon\), this estimate yields

$$\frac{1}{4} \int |\nabla w|^2 \mu_0 dm \leq B_0[w, w; t] + C \int w^2 \mu_0 dm,$$

as long as

$$\int \left(2 - \frac{1}{2} b - \theta\right) m \cdot \nabla w w \rho^{\theta - 1} dm \geq 0,$$

for \(w \in \dot{H}^1_{\mu_0}\). This is indeed the case, as shown below.

**Lemma 22.** Let \(w \in \dot{H}^1_{\mu_0}\). Then

$$(6.4) \quad \int (2 - \frac{1}{2} b - \theta)m \cdot \nabla w w \rho^{\theta - 1} dm \geq 0.$$  

**Proof.** From \(-1 < \theta < 1 \) and \(b \geq 6\), we see that \((2 - b/2 - \theta) < 0\). It suffices to show

$$\int m \cdot \nabla w w \rho^{\theta - 1} dm = \frac{1}{2} \int m \cdot \nabla w^2 \rho^{\theta - 1} dm \leq 0.$$  

Integration by parts gives

$$\int m \cdot \nabla w^2 \rho^{\theta - 1} dm = - \int w^2 (N \rho^{\theta - 1} + 2(1 - \theta)|m|^{2 \rho^{\theta - 2}) dm + \int_{\partial B} w^2 \rho^{\theta - 1} m \cdot \frac{m}{|m|} dS$$

$$\leq \sqrt{b} \int_{\partial B} w^2 \rho^{\theta - 1} dS = 0.$$  

Here we use the fact that \(w^2 \rho^{\theta - 1} \in W^{1,1}\) and \(w^2 \rho^{\theta - 1}|_{\partial B} = 0\). To see this, for any \(w \in \dot{H}^1_{\mu_0}\), we estimate

$$\int w^2 \rho^{\theta - 1} + |\nabla (w^2 \rho^{\theta - 1})| dm \leq \int w^2 \rho^{\theta - 1} + 2|w\nabla w| \rho^{\theta - 1} + 2(1 - \theta)|mw^2|\rho^{\theta - 2} dm$$

$$\leq C \int w^2 \sqrt{\mu_0 \mu_0^*} + |w||\nabla w| \sqrt{\mu_0 \mu_0^*} + w^2 \mu_0^* dm$$

$$\leq C \|w\|_{\dot{H}^1_{\mu_0}}^2,$$

due to the embedding theorem (3.11). Thus \(w^2 \rho^{\theta - 1}|_{\partial B} \in L^1(\partial B)\) from the trace theorem and it is zero from the fact that \(C^\infty_c\) is a dense subset of \(\dot{H}^1_{\mu_0}\). Thus (6.4) follows.

We now turn to the FPE problem including \(x\)-variable. The first step in the proof of Theorem 18 remains valid for \(\mu = \mu_0\). To check the second part of the proof, we need only look at two extra terms beyond those in (4.28).

$$- \left(2 - \frac{1}{2} b - \theta\right) \int m \cdot \nabla_m \partial^\gamma w \rho^{\theta - 1} \partial^\gamma w dm, \quad - \left(2 - \frac{1}{2} b - \theta\right) \int m \cdot \nabla_m \partial^\gamma q \rho^{\theta - 1} \partial^\gamma w dm.$$
The first term is non-positive from Lemma 22 and the second term is bounded by
\[ C \left| \int \int m \cdot \nabla m \partial^\gamma q q^{\theta-1} \partial^\gamma w \right| \leq C_\varepsilon \int \left| \nabla m \partial^\gamma q \right|^2 \mu_0 dm + \varepsilon \int \left| \partial^\gamma w \right|^2 \mu_0^* dm. \]

These ensure the same estimate (4.30) and thus (4.2).

For the well-posedness for the coupled problem, we utilize \( \theta < 1 \) and Lemma 22. For example, for the proof of Lemma 20 with \( \mu_0 \)
\[ | \int \phi \nu \rho^{-1} dm |^2 = | \int \phi dm |^2 \leq \int \phi^2 \mu_0 dm \int \mu_0^{-1} dm. \]

Since \( \theta < 1 \) we have \( \int \mu_0^{-1} dm < \infty \), hence (5.1). Verification of other terms is omitted.

The remaining is to show Theorem 4, the solution \( f \) is a probability distribution if and only if \( q |_{\partial B} = 0 \) for \( \mu = \mu_0 \). Positivity of \( f \) follows as in Proposition 15. For the conservation of mass, as in Proposition 16, we only have to check (3.30).
\[ \int \left( w k m - \nabla w \cdot \nabla \phi \varepsilon \nu \right) dm - \int \frac{w \rho^{b/2} \nabla \phi \varepsilon \cdot \nabla (\nu \rho^{-b/2})}{2 \varepsilon} dm. \]

Since \( \nu^2 / \mu_0 = \rho^{2-\theta} \) and \( 2 - \theta > 1 \)
\[ \frac{\varepsilon}{2} \int \left| \nabla \phi \varepsilon \right|^2 \rho^{2-\theta} dS \]
converges to 0 as \( \varepsilon \to 0 \). Thus the first term converges to 0 as well. On the other hand, the same argument shows that the second term converges to \( C \int_{\partial B} q dS \) for some nonzero constant \( C \). Hence, we conclude Theorem 4 under the assumption of Theorem 2.

7. Conclusion

In this paper, we have analyzed the FENE Dumbbell model which is of bead-spring type Navier-Stokes-Fokker-Planck models for dilute polymeric fluids, with our focus on developing a local well-posedness theory subject to a class of Dirichlet-type boundary conditions
\[ f \nu^{-1} = q \quad \text{on} \quad \partial B \]
for the polymer distribution \( f \), where \( \nu \) depends on \( b > 0 \) through the distance function, and \( q \) is a given smooth function measuring the relative ratio of \( f / \nu \) near boundary. We have thus identified a sharp Dirichlet-type boundary requirement for each \( b > 0 \), while the sharpness of the boundary requirement is a consequence of the existence result for each specification of the boundary behavior. It has been shown that the probability density governed by the Fokker-Planck equation approaches zero near boundary, necessarily faster than the distance function \( d \) for \( b > 2 \), faster than \( d | \ln d | \) for \( b = 2 \), and as fast as \( d^{b/2} \) for \( 0 < b < 2 \). Moreover, the sharp boundary requirement for \( b \geq 2 \) is also sufficient for the distribution to remain a probability density.
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