Anisotropic spacetimes in \( f(T, B) \) theory IV: Noether symmetry analysis

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Abstract The Noether symmetry analysis is applied for the analysis of the field equations in an anisotropic background in \( f(T, B) \)-theory. We consider the \( f(T, B) = T + F(B) \) which describes a small deviation from TEGR introduced by the boundary scalar \( B \). For the Bianchi I, Bianchi III and Kantowski–Sachs geometries, there exists a minisuperspace description and Noether’s theorems are applied. We investigate the existence of invariant point transformations. We find that for the Bianchi I spacetime, the gravitational field equations are Liouville integrable for the \( F(B) = -\frac{B}{2} \ln B \) theory. The analytic solution is derived, and the application of Noether’s symmetries to the Wheeler–DeWitt equation of quantum cosmology is discussed.

1 Introduction

The need to explain cosmological observations [1–4] has led the scientific society to the introduction of new dynamical degrees of freedom in Einstein’s field equations [5]. The main idea is the new degrees of freedom to drive the dynamics of the cosmological evolution such that the observable phenomena can be explained. These additional degrees of freedom correspond to exotic matter source components or they describe geometrodynamical degrees of freedom which are the result of the modification of the gravitational action integral [6–11].

General relativity and its modifications are nonlinear theories. Thus, in order to extract important physical information from a specific gravitational theory methods from the study of dynamical systems in analytic mechanics should be applied. Some powerful methods of analytic mechanics that have been applied to gravitational physics are the analysis of the stationary points [12–16], the Eisenhart lift [17, 18], the singularity analysis [19, 20] and the symmetry analysis [21, 22]. The two latter methods are related to the determination of exact and analytic solutions and to the construction of invariant manifolds in the space of the dynamical variables.

The study for the existence and the determination of exact and analytic solutions for a given dynamical system is an essential approach to the investigation of any dynamical system in applied mathematics. Exact and analytic solutions when they exist for a dynamical system indicate that there are actual solutions which describe the trajectories of the dynamical system. We can study in details the behavior of the trajectories, i.e., solutions, and understand the initial value problem. Dynamical systems which admit exact and analytic solutions can be used as important toy models for the study of the real-world phenomena. For instance, the well-known oscillator is a simple integrable dynamical system which has applications in all areas of physics.

A powerful approach for the study of cosmological theories is the Noether symmetry analysis [23–32]. In 1918 [33], Emmy Noether published two theorems which connect transformations of the action integral with the invariance of the variational principle and the existence of invariant functions known as conservation laws. Noether’s work introduces a simple and systematic way for the construction of conservation laws for nonlinear differential equations. It has been widely applied for the study of various cosmological models and inspired by the work of Ovsiannikov [34]; the Noether symmetry analysis has been used to classify the unknown functions and parameters of a given gravitational model according to the admitted Noether algebra. However, from the discussion in [35] we can conclude that the Noether classification scheme has a geometric character related to that of gravity. The symmetry vectors for the field equations are generated by the minisuperspace, the geometric space, where the dynamical variables are defined.

This piece of work is the last of a series of studies on the analysis of anisotropic spacetimes in modified teleparallel \( f(T, B) \)-theory [36, 37]. In previous studies, we investigated the dynamical behaviour of the gravitational field equations in Bianchi I [38], in Kantowski–Sachs [39] and in Bianchi III geometries [40]. We focused on the \( f(T, B) = T + F(B) \) theory, and we investigate whether the specific theory can solve the flatness and isotropic problems when the initial conditions are that of anisotropic spacetimes. We found that for all the possible sets of initial conditions there always exists an attractor which describes a spatially flat accelerated FLRW Universe.

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\( f(T, B) \)-theory is a fourth-order theory, and with the use of a Lagrange multiplier, the higher-order derivatives can be attributed to a scalar field such that the field equations are of second order but with additional dynamical variables \([41]\). In addition for the anisotropic spacetimes of our consideration, the field equations have a minisuperspace description which means that the Noether symmetry analysis can be applied. There are various studies in the literature on the application of Noether’s symmetries in teleparallel theory and its modifications, see, for instance, \([42–46]\). In \([45]\), the Noether symmetries were investigated in \( f(T, B) \)-theory for the Bianchi I spacetime; however, only a special case of Noether’s theorem was applied and the Noether analysis is not complete. The plan of the paper is as follows.

In Sect. 2, we present the basic properties and definitions of Noether’s work. In Sect. 3, we present the minisuperspace description for the field equation in \( f(T, B) \)-theory for a background geometry described by the Bianchi I, the Kantowski–Sachs or the LRS Bianchi III spacetime. The Noether symmetry analysis is presented in Sect. 4 where we show how the symmetry vectors can be applied in order to derive exact solutions. Finally, our conclusions are given in Sect. 5.

2 Noether symmetries

In this section, we present the basic results of Noether’s work for one-parameter point transformations. Because in this work we deal with second-order differential equations, we focus on Lagrangian functions of the form \( L(t, \mathbf{q}, \dot{\mathbf{q}}) \), where \( t \) is the independent variable and \( \mathbf{q} \) remarks the dependent variables, where \( \dot{\mathbf{q}} = \frac{d\mathbf{q}}{dt} \).

Assume now the action integral \([33]\)

\[
A = \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}}) dt, \tag{1}
\]

and the one-parameter infinitesimal transformation

\[
\tilde{t} = t + \varepsilon \xi (t, \mathbf{q}), \quad \tilde{\mathbf{q}} = \mathbf{q} + \varepsilon (t, \mathbf{q}) \tag{2}
\]

where \( \varepsilon \) is an infinitesimal parameter, i.e., \( \varepsilon^2 \to 0 \). The generator vector field for the latter infinitesimal transformation is \( X = \tau \partial_t + \eta \partial_{\mathbf{q}} \).

Under the application of the transformation (2), the action integral reads

\[
\tilde{A} = \int_{\tilde{t}_0}^{\tilde{t}_1} \tilde{L}(\tilde{t}, \tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) d\tilde{t}
\]

where now \( \tilde{\mathbf{q}} \) is defined as \( \dot{\tilde{\mathbf{q}}} = \frac{d\tilde{\mathbf{q}}}{d\tilde{t}} \).

By expanding around \( \varepsilon^2 \to 0 \), the action integral \( \tilde{A} \) becomes

\[
\tilde{A} = A + \varepsilon \int_{t_0}^{t_1} \left( \xi \frac{\partial L}{\partial t} + \frac{\partial L}{\partial \mathbf{q}} + \tau \frac{\partial L}{\partial \dot{\mathbf{q}}} + \dot{\xi} L \right) dt + \varepsilon F, \tag{4}
\]

where \( F = -\int_{t_0}^{t_1} \dot{f} dt \) and \( \dot{f} = \mathbf{P} - \dot{\mathbf{q}} \dot{\xi} \).

Thus, the action integral remains invariant under the application of the infinitesimal transformation (2) if and only if \( \tilde{A} = A \), that is, \([33]\)

\[
\xi \frac{\partial L}{\partial t} + \frac{\partial L}{\partial \mathbf{q}} + \tau \frac{\partial L}{\partial \dot{\mathbf{q}}} + \dot{\xi} L = \dot{f}. \tag{5}
\]

The latter expression is Noether’s first theorem, and the generator \( X \) is called the Noether symmetry for the Lagrangian function \( L(t, \mathbf{q}, \dot{\mathbf{q}}) \).

Noether’s second theorem indicates that for every Noether symmetry \( X \) there exists a function \( I(t, \mathbf{q}, \dot{\mathbf{q}}) \) defined as \([33]\)

\[
I(t, \mathbf{q}, \dot{\mathbf{q}}) = f - \left[ \xi L + \left( -\dot{\mathbf{q}} \dot{\xi} \right) \frac{\partial L}{\partial \dot{\mathbf{q}}} \right], \tag{6}
\]

such that \( I(t, \mathbf{q}, \dot{\mathbf{q}}) \) is a conservation law, that is, \( \dot{I}(t, \mathbf{q}, \dot{\mathbf{q}}) = 0 \). For more details on Noether’s theorem and extensions, we refer the reader to the recent discussion in \([47]\).

3 Anisotropic spacetimes in \( f(T, B) \)-theory

In \( f(T, B) \)-theory, the fundamental geometric invariants are \( T \) is the torsion scalar for the Weitzenböck connection and the boundary term \( B = 2e^{-1} \partial_{\nu}(e T_{\mu}^{\phantom{\mu}\nu}) \). The gravitational action integral is \([36]\)

\[
S_{f(T, B)} = \frac{1}{16\pi G} \int d^4 x ef(T, B), \tag{7}
\]
in which the field equations are

\[ 0 = e f, T G^i_a + \left[ \frac{1}{4} (T f, T - f) e h_a^i + e (f, T) \mu \nu \right] \]

\[ + \left[ e (f, B) \mu \nu \right] S_a^{\mu \nu} \frac{1}{2} e \left( h_a^\sigma (f, B)_{, \sigma} - h_a^\lambda (f, B)_{\mu \nu} \right) + \frac{1}{4} e B h_a^\lambda f, B, \right]. \tag{8} \]

where \( f, T = \frac{\partial f}{\partial T} \), \( f, B = \frac{\partial f}{\partial B} \), \( G^{\lambda}_{\mu \nu} \) is the Einstein tensor and \( e \) is the vierbein field with \( [e_{\mu}, e_{\nu}] = c^{\beta}_{\mu \nu} e_\beta \) where \( c^{\beta}_{\nu \mu} = 0 \). The antisymmetric connection is defined as \( \hat{T}^{\mu}_{\nu \beta} = \frac{1}{2} \eta^{\mu \rho \sigma} (c_{\nu \rho, \beta} - c_{\rho \nu, \beta} - c_{\rho \beta, \nu}) \). Moreover, \( S^{\mu \nu} = \frac{1}{2} (K^{\mu \nu} + \delta^{\mu}_{\alpha} T^{\nu \beta} - \delta^{\nu}_{\beta} T^{\mu \alpha}) \) is the superpotential tensor defined by the torsion tensor, \( T^{\mu}_{\nu \beta} = \hat{T}^{\mu}_{\nu \beta} - \hat{T}^{\nu \beta}_{\nu \lambda} \) and \( K^{\mu \nu}_{\beta} = -\frac{1}{2} (T^{\nu \mu}_{\beta} - T^{\nu}_{\beta \mu} - T^{\mu \beta}_{\nu}) \). Finally, \( e_a = h^a_\lambda (x) \partial_\lambda \) is the coordinate basis for the vierbein fields.

The three anisotropic spacetimes of our consideration, Bianchi I, Kantowski–Sachs and LRS Bianchi III are described by the line element

\[ ds^2 = -N^2 (t) dt^2 + e^{2a(t)} \left( e^{2 \beta(t)} dx^2 + e^{-\beta(t)} (dy^2 + f^2 (y) dz^2) \right) \tag{9} \]

where \( f (y) = 1 \) is for Bianchi I space, \( f (y) = \sin (y) \) for the Kantowski–Sachs geometry and \( f (y) = \sinh (y) \) for the Bianchi III spacetime. Parameters \( a(t) \) and \( \beta (t) \) are the two scale factors; specifically, \( a(t) \) remarks the radius of the three-dimensional hypersurface, while \( \beta (t) \) is the anisotropic parameter.

In teleparallelism, we should define the proper frame for each spacetime in order the limit of general relativity to be recovered. Indeed, following [37] we assume the following vierbein fields.

For the Bianchi I spacetime, we consider the diagonal frame

\[ e^1 = N dt \ , \ e^2 = e^{a+\beta} dx \ , \ e^3 = e^{a-\beta} dy \ , \ e^4 = e^{a-\beta} dz. \]

For the Kantowski–Sachs spacetime, we assume

\[ e^1 = N dt, \]

\[ e^2 = e^{a+\beta} \cos z \sin y \ dx + e^{a-\beta} \cos y \sin z \ dy - sin y \sin z \ dz, \]

\[ e^3 = e^{a+\beta} \sin y \sin z \ dx + e^{a-\beta} \cos y \sin z \ dy - sin y \sin z \ dz, \]

\[ e^4 = e^{a+\beta} \cos y \sin z \ dx - e^{a-\beta} \sin y \ dy, \]

while for the Bianchi III spacetime it follows from where we calculate the torsion scalar

\[ e^1 = N dt, \]

\[ e^2 = i e^{a+\beta} \cos z \sinh y \ dx + e^{a-\beta} \cos y \sin z \ dy - \sinh y \sin z \ dz, \]

\[ e^3 = i e^{a+\beta} \sin y \sin z \ dx + e^{a-\beta} \cos y \sin z \ dy - \sinh y \sin z \ dz, \]

\[ e^4 = -e^{a+\beta} \cosh y \ dx - i e^{a-\beta} \sinh y \ dy. \]

For the above three different vierbein fields, we derive the torsion scalar

\[ T = \frac{1}{N^2} \left( 6 \dot{\alpha}^2 - \frac{3}{2} \dot{\beta}^2 \right) = 2 K e^{-2a+\beta} \tag{10} \]

where \( K = 0 \) corresponds to the Bianchi I space, \( K = 1 \) is for Kantowski–Sachs spacetime and \( K = -1 \) corresponds to the Bianchi III geometry. Moreover, \( K \) is the value of the spatially curvature in the limit, the anisotropic line element (9) becomes isotropic, and the Friedmann–Lemaître–Robertson–Walker (FLRW) geometry is recovered. Indeed, Bianchi I spacetime is related to the spatially flat FLRW space, while the Kantowski–Sachs and the Bianchi III spacetimes provide in the limit of isotropization the closed and open FLRW spacetimes, respectively.

Furthermore, the boundary term \( B \) is derived to be

\[ B = \frac{6}{N^2} \left( \dot{\alpha} - \frac{\dot{N}}{N} + 3 \dot{\alpha}^2 \right), \tag{11} \]

from where we observe that \( B \) is the same for the spacetimes of our consideration.
3.1 Minisuperspace description

An important characteristic of the spacetimes of our consideration is that they admit a minisuperspace description [48]. Not all cosmological theories have a minisuperspace description. For a full-scale factor matrix and a non-vanishing shift, the minisuperspace description admits the Bianchi class A models and the Bianchi V spacetime [48]. Recall that Bianchi I, Bianchi III and Bianchi IX belong to the family of Class A spaces.

By the minisuperspace description, we mean that there exists a Lagrangian function of the form

\[ S = \int \sqrt{-g} \mathcal{L} dx^4 \rightarrow S = \int \mathcal{L}(N, \mathbf{q}, \dot{\mathbf{q}}) dt, \]

whose variation produces the gravitational field equations.

For a specific family of gravitational models, Lagrangian \( \mathcal{L}(N, \mathbf{q}, \dot{\mathbf{q}}) \) is of the form of a point-like Lagrangian

\[
\mathcal{L}(N, \mathbf{q}, \dot{\mathbf{q}}) = \left[ \frac{1}{2N} \gamma_{\alpha \beta}(\mathbf{q}) \dot{q}^\alpha \dot{q}^\beta - N \mathbf{U}(\mathbf{q}) \right].
\]

where now \( \gamma_{\alpha \beta}(\mathbf{q}) \) is a second-rank tensor called the minisuperspace metric. Function \( \mathbf{U}(\mathbf{q}) \) is called the effective potential for the given theory which drives the dynamics.

The existence of a point-like Lagrangian for a cosmological model is an important theory. Indeed, the point-like Lagrangian can be used to write the field equations in Hamiltonian-formalism and perform quantization, or to apply important techniques for the study of the field equations such as the Noether symmetry analysis which we perform in this work.

In order to derive the minisuperspace Lagrangian, we follow the procedure applied in [38]. We introduce the Lagrange multipliers \( \lambda_1 \) and \( \lambda_2 \) such that the gravitational action integral becomes

\[
S_{f(T, B)} = \frac{1}{16\pi G} \int d^4 x N e^{3\alpha} (f(T, B)).
\]

\[
+ \frac{1}{16\pi G} \int d^4 x N e^{3\alpha} \lambda_1 \left( T - \frac{1}{3N} \left( 6\dot{\alpha}^2 + \frac{3}{2} \dot{\beta}^2 \right) + 2K e^{-2\alpha+\beta} \right)
\]

\[
+ \frac{1}{16\pi G} \int d^4 x N e^{3\alpha} \lambda_2 \left( B - \frac{6}{N^2} \left( \ddot{\alpha} - \frac{N}{N^2} + 3\dot{\alpha}^2 \right) \right).
\]

Thus, by replacing in (14) with respect to the scalars \( T \) and \( B \) we find \( \lambda_1 = -f, \dot{T} \) and \( \lambda_2 = -f, \dot{B} \).

We follow [38]; thus, from the variation of (14) with respect to the scalars \( T \) and \( B \) we find \( \lambda_1 = -f, \dot{T} \) and \( \lambda_2 = -f, \dot{B} \).

4 Noether symmetry analysis

Consider now the infinitesimal transformation with generator the vector field

\[
X = \xi(t, \alpha, \beta, \phi) \partial_t + \eta^1(t, \alpha, \beta, \phi) \partial_\alpha + \eta^2(t, \alpha, \beta, \phi) \partial_\beta + \eta^3(t, \alpha, \beta, \phi) \partial_\phi.
\]
Hence, for the Lagrangian function (18) and the symmetry condition (5) we end with a system of partial differential equations which constrain the generator $X$, the scalar field potential $V(\phi)$ and the curvature constant $K$. In order to solve the symmetry conditions, we follow the approach presented in [52]. The results are summarized in the following proposition.

**Proposition 1** For a nonlinear scalar field potential $V(\phi)$, the Lagrangian function (18) admits for arbitrary potential the symmetry vector $X^1 = \partial_t$. Furthermore, when $V(\phi) = V_0 e^{-\lambda \phi}$, the Noether symmetry vector exists $X^2 = 2t \partial_t + \frac{2}{3} \partial_\alpha + \frac{4}{7} \partial_\delta$. In the special case of Bianchi I spacetime, for arbitrary potential function $V(\phi)$ there exists the additional Noether symmetry $X^3 = \partial_\beta$.

Therefore, with the use of Noether’s second theorem, that is, from expression (6) for each symmetry vector we can construct a conservation law. The conservation laws are presented in the following.

**Proposition 2** The gravitational field equations described by the point-like Lagrangian (18) admit for arbitrary potential the conservation law

$$\mathcal{H} = e^{3\alpha} \left( 6\dot{\alpha}^2 - \frac{3}{2} \dot{\beta}^2 \right) - 6e^{3\alpha} \dot{\phi} - e^{3\alpha} V(\phi) + 2K e^{2\alpha + \beta},$$

related to the vector field $X^1 = \partial_t$. However, $\mathcal{H}$ is nothing else than the constraint equation, that is $\mathcal{H} = 0$. When $K = 0$, there exists the additional conservation law

$$I(X^3) = e^{3\alpha} \dot{\beta}.$$ (20)

Finally, for $V(\phi) = V_0 e^{-\lambda \phi}$, the conservation law related to the vector field $X^2$ is

$$I(X^2) = 2t \mathcal{H} - \frac{2e^{3\alpha}}{\lambda} \left( 4(\lambda - 3) \dot{\alpha} - 2\lambda \dot{\phi} \right).$$ (21)

At this point, it is important to mention that for the exponential potential $V(\phi) = V_0 e^{-\lambda \phi}$ with $\lambda = 3$, from [38–40] we know that the future attractor of the field equations is the de Sitter Universe. We observe that the functional form of the conservation law $I(X^2)$ depends on the parameter $\lambda$, and when $\lambda = 3$, with the use of the constraint equation in the limit $I(X^2)$ we find $\phi(t) = \frac{3}{2} \beta (t) + \phi_0$.

The Noether symmetry vectors can be used to construct invariant functions. Indeed, from $X^2$ we can define the invariant functions $u_1 = \alpha - \frac{1}{3} \ln t$, $u_2 = \beta + \frac{1}{3} \ln t$ and $u_3 = \phi - \frac{2}{3} \ln t$. However, these invariant functions are related to the exact solutions found previously in [39, 40]. What it is of special interest is the case of Bianchi I spacetime for the exponential potential. For this cosmological model, the field equations admit three conservation laws which are independent and involution, which means that the field equations are Liouville integrable.

### 4.1 Bianchi I analytic solution

We proceed with the derivation of exact solutions and the analytic solution for the Bianchi I model. For the Bianchi I spacetime and the exponential potential $V(\phi) = V_0 e^{-\lambda \phi}$, $V_0$ can be a positive or negative real parameter. The exponential potential for this specific gravitational theory has appeared before and in the case of isotropic universe. For more details on the symmetry analysis in the isotropic limit, we refer the reader to [42], where more potential functions in which the field equations have the integrability property exist.

For the derivation of the solutions, we define the new scalar field $\psi = \phi - \frac{6}{\lambda} \alpha$, such that the point-like Lagrangian of the field equations is

$$\mathcal{L}(\alpha, \dot{\alpha}, \dot{\beta}, \psi, \dot{\psi}) = \frac{1}{N} e^{3\alpha} \left( \frac{6}{\lambda} (\lambda - 6) \dot{\alpha}^2 - \frac{3}{2} \dot{\beta}^2 - 6\dot{\alpha} \dot{\psi} \right) + V_0 Ne^{-3\alpha - \lambda \psi}. $$ (22)

#### 4.1.1 Exact solutions

For $N = 1$, the field equations read

$$e^{3\alpha} \left( \frac{6}{\lambda} (\lambda - 6) \dot{\alpha}^2 - \frac{3}{2} \dot{\beta}^2 - 6\dot{\alpha} \dot{\psi} \right) - V_0 e^{-3\alpha - \lambda \psi} = 0, $$ (23)

$$6e^{3\alpha} \left( \dot{\alpha}^2 + 3\dot{\beta}^2 \right) - V_0 \lambda e^{-3\alpha - \lambda \psi} = 0, $$ (24)

and

$$\ddot{\beta} + 3\dot{\beta} \dot{\beta} = 0 $$ (25)

and

$$6(\lambda - 6) \dot{\alpha}^2 - 3\lambda \dot{\alpha} \dot{\psi} + 2(\lambda - 6) \dot{\alpha} - \lambda \dot{\psi} = 0. $$ (26)
We assume the linear function \( \alpha(t) = H_0 t \), and we replace in the field equations from where it follows

\[
\begin{align*}
\beta(t) &= \beta_0 - \frac{\beta_1}{3H_0} t^3 H_0, \\
\psi(t) &= -\frac{6H_0}{\lambda} t + \ln \left( \frac{\lambda V_0}{18H_0^2} \right),
\end{align*}
\]

with constraint

\[
\frac{4}{\lambda} H_0^2 (\lambda - 3) - e^{-6H_0 t} \beta_1^2 = 0,
\]

from where we observe that for \( H_0 t >> 1 \), it follows that the de Sitter universe is an asymptotic solution when \( \lambda = 3 \). This is in agreement with the result found before with the method of dynamical analysis presented in [38].

On the other hand, for the function \( \alpha(t) = p \ln t \), it follows

\[
\begin{align*}
\beta(t) &= \frac{\beta_1}{1 - 3p} t^{1-3p} + \beta_0, \\
\psi(t) &= \frac{1}{\lambda} \ln \left( \frac{t^{2(1-3p)}}{6p(3p-1)} V_0 \lambda \right),
\end{align*}
\]

with constraint

\[
\frac{4p}{\lambda} (p(\lambda - 3) - 1) - \beta_1^2 t^{2(1-3p)} = 0.
\]

Therefore, for \( 1 - 3p < 0 \) there exists the asymptotic isotropic solution for \( p(\lambda - 3) - 1 = 0 \), that is, \( 3 < \lambda < 6 \) which was found before in [38].

### 4.1.2 Analytic solution

However, in order to derive the analytic solution we assume the lapse function \( N(t) = e^{\lambda t} \) in which the point-like Lagrangian reads

\[
\mathcal{L}(\alpha, \dot{\alpha}, \beta, \dot{\beta}, \psi, \dot{\psi}) = \frac{6}{\lambda} (\dot{\alpha}^2 - \frac{3}{2} \beta^2 - 6\dot{\alpha} \dot{\psi} + V_0 e^{-\lambda \psi}).
\]

Consequently, the field equations in the new variables are

\[
\begin{align*}
\frac{6}{\lambda} (\dot{\alpha}^2 - \frac{3}{2} \beta^2 - 6\dot{\alpha} \dot{\psi} - V_0 e^{-\lambda \psi}) &= 0, \\
\dot{\alpha} - \frac{\lambda V_0}{6} e^{-\lambda \psi} &= 0, \\
\dot{\psi} - \frac{V_0 (\dot{\alpha} - 6)}{3} e^{-\lambda \psi} &= 0,
\end{align*}
\]

and

\[
\dot{\beta} = 0.
\]

It is easy to see now that the additional conservation laws in the new variables are \( \dot{\beta} = const \) and \( \frac{2(\dot{\alpha} - 6)}{\lambda} \dot{\psi} = const \). Therefore, the analytic solution is

\[
\begin{align*}
\beta(t) &= \beta_0 t + \beta_1, \\
\psi(t) &= \frac{1}{\lambda} \ln \left( \frac{\lambda (\dot{\alpha} - 6) V_0}{6 \dot{\psi}^2_0} \cosh^2(\psi_0 (t - t_0)) \right),
\end{align*}
\]

and

\[
\alpha(t) = \alpha_0 + \alpha_1 t + \frac{1}{\lambda - 6} \ln(\cosh(\psi_0 (t - t_0))).
\]

Moreover, the constraint equation provides the algebraic condition for the integration constants

\[
4\alpha_1^2 (\lambda - 6)^2 - \beta_0^2 (\lambda - 6) \lambda - 4\psi_0^2 = 0.
\]

However, for \( \lambda = 6 \), the conservation laws are \( \dot{\beta} = const. \) and \( \dot{\psi} = const. \), where now the analytic solution is

\[
\begin{align*}
\beta(t) &= \beta_0 t + \beta_1, \\
\psi(t) &= \psi_0 t + \psi_1,
\end{align*}
\]
and
\[ \alpha(t) = \frac{V_0}{36\psi_0^2} e^{-6(\psi_0^2+\psi_1)} + \alpha_1 t + \alpha_0. \] (42)

Finally, the algebraic constraint for the integration constants is
\[ \beta_0^2 + 4\alpha_1 \psi_0 = 0. \] (43)

Recall the analytic solutions in this section derived for the line element
\[ ds^2 = -e^{6\alpha(t)} dt^2 + e^{2\alpha(t)} \left( e^{2\beta(t)} dx^2 + e^{-\beta(t)} (dy^2 + dz^2) \right). \] (44)

For the latter exact solution for \( \lambda = 6 \), when the linear term dominates in the scale factor, that is, \( \alpha(t) \simeq t \), it is easy to see that the asymptotic spacetime (44) describes a Kasner-like solution. On the other hand, when the exponential term dominates in the scale factor, \( \alpha(t) \simeq e^{-6\psi_0 t} \) the Hubble function \( H(t) = e^{-3\alpha} \dot{\alpha} \) is asymptotically described as \( H(t) \simeq e^{-3\alpha} \), where it is clear that \( H(t) \rightarrow 0 \) is the asymptotic limit. Similarly, the anisotropic parameter \( \sigma = e^{-3\alpha} \beta \) is calculated \( \sigma = e^{-3\alpha} \), that is \( \sigma = \alpha^{-1} \), where for large values of \( \alpha \), \( \sigma \rightarrow 0 \), that is, the isotropic limit. On the other hand, for \( \psi_0 = 0 \), \( \alpha_1 = 0 \), from the constraint equation it follows that \( \beta_0 = 0 \), which means that for this set of initial conditions, the exact solution describes an isotropic spacetime. Recall that the case \( \lambda = 6 \) is a bifurcation for the cosmological dynamics as discussed in [38].

5 Conclusions

In this work, we applied the Noether symmetry analysis to \( f(T, B) \)-theory for the field equations in anisotropic cosmological background geometries. Specifically, we focused on the case where \( f(T, B) = T + F(B) \) and we extended previous analysis on the study of the dynamics. The Noether symmetry analysis is a powerful method for the constraint of the unknown functions and parameters of the dynamical system and the construction of conservation laws.

In order to be able to apply the symmetry analysis, the field equations should admit a minisuperspace description. That is valid for the cosmological model of our consideration. Indeed, the Bianchi I, the Bianchi III and the Kantowski–Sachs geometries admit a minisuperspace. For the \( f(T, B) = T + F(B) \) theory, the Lagrangian of the field equations is point-like Lagrangian and previous studies on the analysis of the symmetries can be applied. Thus, the Noether symmetries for the field equations are determined by the collinearizations of the minisuperspace [52], and that gives a geometric character for the Noether symmetry analysis.

In terms of the point symmetries, we found that there exists a unique function \( F(B) \) that is, \( F(B) \simeq -\frac{1}{3} B \ln(B) \) where the field equations admit non-trivial Noether symmetries. Additional, in the case of the Bianchi I model the field equations admit a three-dimensional Lie algebra of the Noether symmetries, where the corresponding conservation laws are independent and involution, which means that the field equations are Liouville integrable and the generic solution to this specific model was presented in closed-form solutions.

However, because for the Bianchi I spacetime the field equations are integrable, we can investigate the existence of exact solutions, and for the Wheeler–DeWitt (WdW) equation of quantum cosmology [53]. When a minisuperspace description exists, the WdW equation is represented as a single differential equation [54–57]. Indeed, the WdW equation in this case follows from the canonical quantization of the constraint equation. Indeed, we define the momentum \( p = \frac{\partial}{\partial \phi} \) and we quantize as \( p = i \frac{\partial}{\partial \phi} \).

Hence, the WdW equation for our model is

\[ \left( \frac{1}{3} \left( \frac{\partial^2}{\partial \alpha \partial \phi} + \frac{\partial^2}{\partial \beta^2} + \frac{\partial^2}{\partial \phi^2} \right) - V_0 e^{6\alpha-\lambda \phi} \right) \Psi(\alpha, \beta, \phi) = 0. \] (45)

From the Noether symmetry vectors, we can define the quantum operators \( \left( \frac{\partial}{\partial \phi} - \beta_1 \right) \Psi = 0 \), and \( \left( \frac{\lambda}{6} \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \phi} - \beta_2 \right) \Psi = 0 \).

Finally, with the use of the latter operators we determine the closed-form solution

\[ \Psi(\alpha, \beta, \phi) = U(\psi) \exp \left( \frac{6\beta_2 \alpha + \beta_1 \beta \lambda}{\lambda} \right), \quad \psi = \frac{6}{\lambda} \alpha - \phi \] (46)

where

\[ (\lambda - 6)U, \psi + 6\beta_2 U, \psi + \beta_1^2 \lambda U - 3 V_0 \lambda e^{-\lambda \psi} U = 0. \] (47)

The latter solution is expressed in terms of the Bessel functions of the first kind, that is,

\[ U(\psi) = e^{-\frac{V_0}{\lambda} \psi} \left( U_1 J_\lambda(\sigma(\psi)) + U_2 Y_\lambda(\sigma(\psi)) \right), \] (48)
where \( \mu = - \sqrt[\lambda]{\frac{\sqrt{\lambda}}{\lambda - 6}} \), \( \sigma (\psi) = 2 \sqrt{\frac{2V_0}{\lambda (\lambda - 6)}} \exp \left( - \frac{1}{2} \psi \right) \), or

\[
U (\psi) = U_1 \exp \left( - \frac{2\beta_1^2 \psi + V_0 e^{-6\psi}}{2\beta_2} \right), \quad \lambda = 6. \tag{49}
\]

From the symmetry analysis for the anisotropic Bianchi III and Kantowski–Sachs geometries, we found that a non-trivial symmetry vector exists for the \( F (B) \simeq - \frac{1}{2} B \ln (B) \) theory. However, the existence of this symmetry vector is sufficient not to infer the integrability properties of the field equations, neither to construct any exact or analytic solution. Other techniques and methods from analytic mechanics may be applied, but such analysis is out of the scope of this work.

We stop our discussion at this point. In a future work, we plan to further investigate the WdW equation and the quantum potentiality for the \( f (T, B) \)-theory. This work concludes a series of studies on the analysis of the dynamics for the \( f (T, B) \)-theory in anisotropic cosmologies.

Data Availability Statements Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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