MAXWELL FIELD ON THE POINCARÉ GROUP

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Abstract

The massless field of spin 1 is defined on the eight-dimensional configuration space; this space is a direct product of Minkowski space and of a two-dimensional complex sphere. Field equations for the spin-one field are derived from a Dirac-like Lagrangian separately for the translation group and Lorentz group parts. It is shown that a Dirac form of Maxwell equations (so-called Majorana-Oppenheimer formulation of electrodynamics) follows directly from the field equations of translation group part. The photon field is realized via Biedenharn type functions on the Poincaré group. This allows us to consider both Dirac and Maxwell fields on an equal footing, as the functions on the Poincaré group.

Keywords: Dirac-like form of Maxwell equations; quantum field theory on the Poincaré group; relativistic wavefunctions.

1 Introduction

Historically, an analysis of the Maxwell equations with respect to Lorentz transformations is one of the first objects of relativity theory. As known, any quantity which transforms linearly under Lorentz transformations is a spinor. For that reason spinor quantities are considered as fundamental in quantum field theory and basic equations for such quantities should be written in a spinor form. The spinor form of Dirac equations was first given by Van der Waerden [1], he showed that Dirac’s theory can be expressed completely in this form (see also Ref. 2). In turn, a spinor formulation of Maxwell equations was studied by Laporte and Uhlenbeck [3]. In 1936, Rumer [7] showed that spinor forms of Dirac and Maxwell equations look very similar. Further, Majorana [8] and Oppenheimer [9] proposed to consider the Maxwell theory of electromagnetism as the wave mechanics of the photon. They introduced a wave function of the form \( \psi = E - iB \) satisfying the massless Dirac-like equations. Maxwell equations in the Dirac form considered during long time by many authors [12]–[24]. The

\[^1\] Of course, there is no an equivalence between Dirac and Maxwell equations as it claimed recently by Campolattaro et al. [4, 5] (see also a discussion concerning this question [6]). A principal difference between these equations lies in the fact that Dirac and Maxwell fields have different spin tensor dimensionalities. These fields are transformed correspondingly within \((1/2, 0) \oplus (0, 1/2)\) and \((1, 0) \oplus (0, 1)\) finite dimensional representations of the Lorentz group.

\[^2\] In contrast to the Gupta-Bleuler method [10, 11], where the nonobservable four-potential \(A_\mu\) is quantized, the main advantage of the Majorana-Oppenheimer formulation of electrodynamics lies in the fact that it deals directly with observable quantities, such as the electric and magnetic fields.
interest to the Majorana-Oppenheimer formulation of electrodynamics has grown in recent years [25]–[33].

It is widely accepted in modern theoretical physics that the majority of “elementary” particles have a very complicated internal structure, that is, elementary particles seem to be spatially extended. For that reason realistic particles cannot be considered as point-like objects (of course, these more realistic fields are nonlocal fields in general). Aside string models, there is another way for description of spatially extended particles proposed by Finkelstein [34], he showed that elementary particle models with internal degrees of freedom can be described on manifolds larger then Minkowski space (homogeneous spaces of the Poincaré group $P$). All the homogeneous spaces of $P$, which contain Minkowski space, were given by Finkelstein [34] and by Bacry and Kihlberg [35]. In 1964, under the influence of Regge pole theory, Lurçat suggested to construct quantum field theory on the group manifold of $P$ [36], where one of the main motivations was to give a dynamical role to the spin. The constructions of quantum fields theories on different homogeneous spaces of $P$ were given in Refs. [37]–[46].

The main goal of the present paper is a synthesis of the two mentioned above directions (Dirac-like formulation of Maxwell equations and quantum field theory on the Poincaré group). The Maxwell field is represented by Biedenharn type functions [47] on the group manifold $M_{10}$ (this manifold is a direct product of Minkowski space and of the manifold of the Lorentz subgroup). It is shown that a general form of the wavefunction inherits its structure from the semidirect product $SL(2, \mathbb{C}) \circ T_4$ and for that reason the Maxwell field on $M_{10}$ is defined by a factorization $\psi(x)\psi(g)$, where $x \in T_4$, $g \in SL(2, \mathbb{C})$. It is obvious that the Dirac-like form of Maxwell equations should be derived from a Dirac-like Lagrangian. Using a Lagrangian formalism on the tangent bundle $T_M_{10}$ of the manifold $M_{10}$, we obtain field equations separately for the parts $\psi(x)$ and $\psi(g)$. Solutions of the field equations for $\psi(x)$ (Maxwell equations in Dirac-like form) are obtained via the plane-wave approximation. In turn, solutions of the field equations with $\psi(g)$ have been found in the form of expansions in associated hyperspherical functions.

2 Preliminaries

Let us consider some basic facts concerning the Poincaré group $P$. First of all, the group $P$ has the same number of connected components as with the Lorentz group. Later on we will consider only the component $P_+^\uparrow$ corresponding the connected component $L_+^\uparrow$ (so-called special Lorentz group [49]). As known, an universal covering $\overline{P}_{+}^\uparrow$ of the group $P_{+}^\uparrow$ is defined by a semidirect product $\overline{P}_{+}^\uparrow = SL(2, \mathbb{C}) \circ T_4 \simeq Spin_+(1, 3) \circ T_4$, where $T_4$ is a subgroup of four-dimensional translations.

The each transformation $T_\alpha \in P_+^\uparrow$ is defined by a parameter set $\alpha(\alpha_1, \ldots, \alpha_{10})$, which can be represented by a point of the space $M_{10}$. The space $M_{10}$ possesses locally euclidean properties, therefore, it is a manifold called a group manifold of the Poincare group. It is easy to see that the set $\alpha$ can be divided into two subsets, $\alpha(x_1, x_2, x_3, x_4, g_1, g_2, g_3, g_4, g_5, g_6)$, where $x_i \in T_4$ are parameters of the translation subgroup, $g_j$ are parameters of the group

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3As known, exponentials define unitary representations of the translation subgroup $T_4$. In a sense, the functions $e^{ikx}$ can be understood as “matrix elements” of $T_4$.

4Matrix elements of both spinor and principal series representations of the Lorentz group are expressed via the hyperspherical functions [48] (see Appendix).
In turn, the transformation \( T_6 \) is defined by a set \( g(\mathfrak{g}_1, \ldots, \mathfrak{g}_6) \), which can be represented by a point of a six-dimensional submanifold \( \mathcal{L}_6 \subset \mathcal{M}_{10} \) called a group manifold of the Lorentz group.

In the present paper we restricted ourselves by a consideration of finite dimensional representations of the Poincaré group. The group \( T_4 \) of four-dimensional translations is an Abelian group, formed by a direct product of the four one-dimensional translation groups, each of which is isomorphic to additive group of real numbers. Hence it follows that all irreducible representations of \( T_4 \) are one-dimensional and expressed via the exponential. In turn, as it showed by Naimark [50], spinor representations exhaust all the finite dimensional irreducible representations of the group \( SL(2, \mathbb{C}) \). Any spinor representation of \( SL(2, \mathbb{C}) \) can be defined in the space of symmetric polynomials of the following form

\[
p(z_0, z_1, \bar{z}_0, \bar{z}_1) = \sum_{(\alpha_1, \ldots, \alpha_k)\ (\bar{\alpha}_1, \ldots, \bar{\alpha}_r)} \frac{1}{k! r!} a^{\alpha_1 \cdots \alpha_k, \bar{\alpha}_1 \cdots \bar{\alpha}_r} \alpha_1 \cdots \alpha_k z_{\alpha_1} \cdots \bar{z}_{\bar{\alpha}_1} \cdots \bar{z}_{\bar{\alpha}_r} \tag{1}
\]

where the numbers \( a^{\alpha_1 \cdots \alpha_k, \bar{\alpha}_1 \cdots \bar{\alpha}_r} \) are unaffected at the permutations of indices. The expressions (1) can be understood as functions on the Lorentz group. When the coefficients \( a^{\alpha_1 \cdots \alpha_k, \bar{\alpha}_1 \cdots \bar{\alpha}_r} \) in (1) are depend on the variables \( x_i \in T_4 \ (i = 1, 2, 3, 4) \), we come to the Biedenharn type functions [47]:

\[
p(x, z, \bar{z}) = \sum_{(\alpha_1, \ldots, \alpha_k)\ (\bar{\alpha}_1, \ldots, \bar{\alpha}_r)} \frac{1}{k! r!} a^{\alpha_1 \cdots \alpha_k, \bar{\alpha}_1 \cdots \bar{\alpha}_r}(x) z_{\alpha_1} \cdots z_{\alpha_k} \bar{z}_{\bar{\alpha}_1} \cdots \bar{z}_{\bar{\alpha}_r} \tag{2}
\]

The functions (2) should be considered as the functions on the Poincaré group. Some applications of these functions contained in Refs. 51,45. Representations of the Poincaré group \( SL(2, \mathbb{C}) \otimes T(4) \) are realized via the functions (2).

Let \( L(\alpha) \) be a Lagrangian on the group manifold \( \mathcal{M}_{10} \) of the Poincaré group (in other words, \( L(\alpha) \) is a 10-dimensional point function), where \( \alpha \) is the parameter set of this group. Then an integral for \( L(\alpha) \) on some 10-dimensional volume \( \Omega \) of the group manifold we will call an action on the Poincaré group:

\[
A = \int_{\Omega} d\alpha L(\alpha),
\]

where \( d\alpha \) is a Haar measure\(^5\) on the group \( \mathcal{P} \).

Let \( \psi(\alpha) \) be a function on the group manifold \( \mathcal{M}_{10} \) (now it is sufficient to assume that \( \psi(\alpha) \) is a square integrable function on the Poincaré group) and let

\[
\frac{\partial L}{\partial \psi} - \frac{\partial}{\partial \alpha} \frac{\partial L}{\partial \frac{\partial \psi}{\partial \alpha}} = 0 \tag{3}
\]

\(^5\)The invariant measure \( d\alpha \) on the Poincaré group can be factorized as \( d\alpha = dx dg \), where \( dg \) is a Haar measure on the Lorentz group.
be Euler-Lagrange equations on $\mathcal{M}_{10}$ (more precisely speaking, the equations (3) act on the tangent bundle $T\mathcal{M}_{10} = \bigcup_{\alpha \in \mathcal{M}_{10}} T\alpha \mathcal{M}_{10}$ of the manifold $\mathcal{M}_{10}$, see Ref. 52). Let us introduce a Lagrangian $\mathcal{L}(\alpha)$ depending on the field function $\psi(\alpha)$ as follows:

$$\mathcal{L}(\alpha) = -\frac{1}{2} \left( \psi^*(\alpha)B_\mu \frac{\partial \psi(\alpha)}{\partial \alpha_\mu} - \frac{\partial \psi^*(\alpha)}{\partial \alpha_\mu} B_\mu \psi(\alpha) \right) - \kappa \psi^*(\alpha)B_{11} \psi(\alpha),$$

where $B_\nu (\nu = 1, 2, \ldots, 10)$ are square matrices. The number of rows and columns in these matrices is equal to the number of components of $\psi(\alpha)$, $\kappa$ is a non-null real constant.

Further, if $B_{11}$ is non-singular, then we can introduce the matrices

$$\Pi_\mu = B_{11}^{-1} B_\mu, \quad \mu = 1, 2, \ldots, 10,$

and represent the Lagrangian $\mathcal{L}(\alpha)$ in the form

$$\mathcal{L}(\alpha) = -\frac{1}{2} \left( \overline{\psi}(\alpha) \Pi_\mu \frac{\partial \psi(\alpha)}{\partial \alpha_\mu} - \frac{\partial \overline{\psi}(\alpha)}{\partial \alpha_\mu} \Pi_\mu \psi(\alpha) \right) - \kappa \overline{\psi}(\alpha) \psi(\alpha),$$

where

$$\overline{\psi}(\alpha) = \psi^*(\alpha)B_{11}.$$

In case of the massless field $(j, 0) \oplus (0, j)$ we will consider on the group manifold $\mathcal{M}_{10}$ a Lagrangian of the form

$$\mathcal{L}(\alpha) = -\frac{1}{2} \left( \overline{\psi}(\alpha) \Pi_\mu \frac{\partial \psi(\alpha)}{\partial x_\mu} - \frac{\partial \overline{\psi}(\alpha)}{\partial x_\mu} \Pi_\mu \psi(\alpha) \right) - \frac{1}{2} \left( \overline{\psi}(\alpha) \Upsilon_\nu \frac{\partial \psi(\alpha)}{\partial g_\nu} - \frac{\partial \overline{\psi}(\alpha)}{\partial g_\nu} \Upsilon_\nu \psi(\alpha) \right), \quad (4)$$

As a direct consequence of (2), the relativistic wavefunction $\psi(\alpha)$ on the group manifold $\mathcal{M}_{10}$ is represented by a following factorization

$$\psi(\alpha) = \psi(x)\psi(g) = \psi(x_1, x_2, x_3, x_4)\psi(\varphi, \epsilon, \theta, \tau, \phi, \varepsilon), \quad (5)$$

where $\psi(x_i)$ is a function depending on the parameters of the subgroup $T_4$, $x_i \in T_4 (i = 1, \ldots, 4)$, and $\psi(g)$ is a function on the Lorentz group, where six parameters of this group are defined by the Euler angles $\varphi, \epsilon, \theta, \tau, \phi, \varepsilon$ which compose complex angles of the form $\varphi^c = \varphi - i\epsilon, \theta^c = \theta - i\tau, \phi^c = \phi - i\varepsilon$ (see Appendix).

### 3 The field $(1, 0) \oplus (0, 1)$

Before we proceed to start this section let us repeat that the Dirac-like form of Maxwell equations, considered by many authors, should be derived from a Dirac-like Lagrangian. It is one of the main assumptions which we will prove in this paper. Let us rewrite (4) in the form

$$\mathcal{L}(\alpha) = -\frac{1}{2} \left( \overline{\psi}(\alpha) \Gamma_\mu \frac{\partial \psi(\alpha)}{\partial x_\mu} - \frac{\partial \overline{\psi}(\alpha)}{\partial x_\mu} \Gamma_\mu \psi(\alpha) \right) - \frac{1}{2} \left( \overline{\psi}(\alpha) \Upsilon_\nu \frac{\partial \psi(\alpha)}{\partial g_\nu} - \frac{\partial \overline{\psi}(\alpha)}{\partial g_\nu} \Upsilon_\nu \psi(\alpha) \right), \quad (6)$$
where $\psi(\alpha) = \psi(x)\psi(x)$ ($\mu = 0, 1, 2, 3$, $\nu = 1, \ldots, 6$), and
\[
\Gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & -\alpha_1 \\ \alpha_1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & -\alpha_2 \\ \alpha_2 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & -\alpha_3 \\ \alpha_3 & 0 \end{pmatrix},
\]
\[
\psi^{\ast} = \begin{pmatrix} 0 & \Lambda_1^* \\ \Lambda_1 & 0 \end{pmatrix}, \quad \psi_{\mu}^{\ast} = \begin{pmatrix} 0 & \Lambda_2^* \\ \Lambda_2 & 0 \end{pmatrix}, \quad \psi^{\ast} = \begin{pmatrix} 0 & \Lambda_3^* \\ \Lambda_3 & 0 \end{pmatrix},
\]
\[
\psi^{\ast} = \begin{pmatrix} 0 & i\Lambda_1^* \\ i\Lambda_1 & 0 \end{pmatrix}, \quad \psi_{\mu}^{\ast} = \begin{pmatrix} 0 & i\Lambda_2^* \\ i\Lambda_2 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & i\Lambda_3^* \\ i\Lambda_3 & 0 \end{pmatrix},
\]
\[
\alpha_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \end{pmatrix},
\]
\[
\Lambda_1 = \frac{c_{11}}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda_2 = \frac{c_{11}}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \Lambda_3 = c_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Varying independently $\psi(x)$ and $\overline{\psi}(x)$ in the Lagrangian (4), we come to the following equations:
\[
\Gamma_\mu \frac{\partial \psi(x)}{\partial x_\mu} = 0, \quad \Gamma_\mu^{\ast} \frac{\partial \overline{\psi}(x)}{\partial x_\mu} = 0.
\]

The equation (11) can be written as follows:
\[
\left[ \frac{ih}{c} \frac{\partial}{\partial t} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} - \frac{ih}{\alpha_i} \frac{\partial}{\partial x} \begin{pmatrix} 0 & -\alpha_i \\ \alpha_i & 0 \end{pmatrix} \right] \begin{pmatrix} \psi(x) \\ \psi^*(x) \end{pmatrix} = 0,
\]

where
\[
\begin{pmatrix} \psi(x) \\ \psi^*(x) \end{pmatrix} = \begin{pmatrix} E - iB \\ E + iB \end{pmatrix} = \begin{pmatrix} E_1 - iB_1 \\ E_2 - iB_2 \\ E_3 - iE_3 \\ E_1 + iB_1 \\ E_2 + iB_2 \\ E_3 + iB_3 \end{pmatrix}.
\]

From the equation (13) it follows that
\[
\left( \frac{ih}{c} \frac{\partial}{\partial t} - ih\alpha_i \frac{\partial}{\partial x} \right) \psi(x) = 0,
\]
\[
\left( \frac{ih}{c} \frac{\partial}{\partial t} + ih\alpha_i \frac{\partial}{\partial x} \right) \psi^*(x) = 0.
\]

The latter equations with allowance for transversality conditions ($p \cdot \psi = 0$, $p \cdot \psi^* = 0$) coincide with the Maxwell equations. Indeed, taking into account that $(p \cdot \alpha)\psi = h\nabla \times \psi$, we obtain
\[
\frac{ih}{c} \frac{\partial \psi}{\partial t} = -h\nabla \times \psi,
\]
\[
\frac{ih}{c} \nabla \cdot \psi = 0.
\]
Whence
\[ \nabla \times (E - iB) = -\frac{i}{c} \frac{\partial(E - iB)}{\partial t}, \]
\[ \nabla \cdot (E - iB) = 0 \]
(the constant \( \hbar \) is cancelled). Separating the real and imaginary parts, we obtain Maxwell equations
\[ \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}, \]
\[ \nabla \times B = \frac{1}{c} \frac{\partial E}{\partial t}, \]
\[ \nabla \cdot E = 0, \]
\[ \nabla \cdot B = 0. \]

It is easy to verify that we come again to Maxwell equations starting from the equations
\[ \left( \frac{i\hbar}{c} \frac{\partial}{\partial t} + i\alpha \frac{\partial}{\partial x} \right) \psi^*(x) = 0, \quad (18) \]
\[ -i\hbar \nabla \cdot \psi^*(x) = 0. \quad (19) \]

In spite of the fact that equations (14) and (15) give rise to the same Maxwell equations, the physical interpretation of these equations is different (see Refs. 28,32). Namely, the equations (14) and (15) are equations with negative and positive helicity, respectively\(^6\).

As usual, the conjugated wavefunction \( \overline{\psi}(x) = \psi^0 \Gamma_0 = (\psi(x), \psi^*(x)) \) corresponds to antiparticle (it is a direct consequence of the Dirac-like Lagrangian \( \mathcal{L} \), \( \psi(x) \) is a complex function). Therefore, we come to a very controversial conclusion that the equations (12) describes the antiparticle (antiphoton) and, moreover, hence it follows that there exist the current and charge for the photon field. At first glance, we come to a drastic contradiction with the widely accepted fact that the photon is truly neutral particle. However, it is easy to verify that equations (12) give rise to the Maxwell equations also. It means that the photon coincides with its “antiparticle”. Following to the standard procedure given in many textbooks, we can define the “charge” of the photon by an expression
\[ Q \sim \int d\mathbf{x} \overline{\psi} \Gamma_0 \psi, \quad (20) \]

\( ^6 \)It is interesting to note a correspondence between photon helicity states and a complexification of the group \( SU(2) \) presented by a local isomorphism \( SU(2) \otimes SU(2) \cong SL(2, \mathbb{C}) \). As known, a root subgroup of a semisimple Lie group \( O_4 \) (a rotation group of the 4-dimensional space) is a normal divisor of \( O_4 \). For that reason the 6-parameter group \( O_4 \) is semisimple, and is represented by a direct product of the two 3-parameter unimodular groups. By analogy with the group \( O_4 \), a double covering \( SL(2, \mathbb{C}) \) of the proper orthochronous Lorentz group (a rotation group of the 4-dimensional spacetime continuum) is semisimple, and is represented by a direct product of the two 3-parameter special unimodular groups, \( SL(2, \mathbb{C}) \cong SU(2) \otimes SU(2) \). An explicit form of this isomorphism can be obtained by means of a complexification of the group \( SU(2) \), that is, \( SL(2, \mathbb{C}) \cong \text{complex}(SU(2)) \cong SU(2) \otimes SU(2) \) \([53]\). Moreover, in the works \([51][55]\), inspired by Ryder book \([56]\), the Lorentz group is represented by a product \( SU_R(2) \otimes SU_L(2) \), and spinors \( \psi(p^\mu) = (\phi_R(p^\mu), \phi_L(p^\mu))^T \) are transformed within \( (j_1, j_2) \oplus (j_2, j_1) \) representation space. The components \( \phi_R(p^\mu) \) and \( \phi_L(p^\mu) \) correspond to different helicity states (right- and left-handed spinors). Hence it follows an analogy with the photon spin states. Namely, the operators \( X = J + iK \) and \( Y = J - iK \) correspond to the right and left polarization states of the photon, where \( J \) and \( K \) are generators of rotation and Lorentz boosts, respectively.
where $\psi \Gamma_0 \psi = 2(E^2 + B^2)$. However, Newton and Wigner [57] showed that for the photon there exist no localized states. Therefore, the integral in the right side of (20) presents an indeterminable expression. Since the integral (20) does not exist in general, then the “charge” of the photon cannot be considered as a constant magnitude (as it takes place for the electron field which has localized states [77] and a well-defined constant charge). In a sense, one can say that the “charge” of the photon is equal to the energy $E^2 + B^2$ of $\gamma$-quantum.

We see that the equation (13) gives rise to the two Dirac-like equations (14) and (15) which in combination with the transversality conditions (17) and (19) are equivalent to the Maxwell equations. Let us represent solutions of (14) in a plane-wave form

$$\psi(x) = \varepsilon(k) \exp[i\hbar^{-1}(k \cdot x - \omega t)].$$

(21)

After substitution of (21) into (14) we come to the following matrix eigenvalue problem

$$-c \begin{pmatrix} 0 & ik_3 & -ik_2 \\ -ik_3 & 0 & ik_1 \\ ik_2 & -ik_1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix} = \omega \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}.$$

It is easy to verify that we come to the same eigenvalue problem starting from (15). The secular equation has the solutions

$$\omega = \pm c|k|.$$

Therefore, solutions of (13) in the plane-wave approximation are expressed via the functions

$$\psi_\pm(k; x, t) = \left\{2(2\pi)^{3/2}\right\}^{-1/2} \begin{pmatrix} \varepsilon_\pm(k) \\ \varepsilon_\pm(k) \end{pmatrix} \exp[i(k \cdot x - \omega t)],$$

$$\psi_0(k; x) = \left\{2(2\pi)^{3/2}\right\}^{-1/2} \begin{pmatrix} \varepsilon_0(k) \\ \varepsilon_0(k) \end{pmatrix} \exp[ik \cdot x]$$

and the complex conjugate functions $\psi_\pm^*(k; x, t)$ and $\psi_0^*(k; x)$ ($E + iB$) corresponding to positive helicity, here $\omega = c|k|$ and $\varepsilon_\lambda(k)$ ($\lambda = \pm, 0$) are the polarization vectors of a photon:

$$\varepsilon_\pm(k) = \left\{2|k|^2(k_1^2 + k_2^2)\right\}^{-1/2} \begin{bmatrix} -k_1k_3 \pm ik_2|k| \\ -k_2k_3 \mp ik_1|k| \\ k_1^2 + k_2^2 \end{bmatrix},$$

$$\varepsilon_0(k) = |k|^{-1} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}.$$

Varying now $\psi(g)$ and $\bar{\psi}(g)$ in the Lagrangian (6), we come to equations

$$\Gamma_\nu \frac{\partial \psi(g)}{\partial g_\nu} = 0,$$

$$\Gamma^T_\nu \frac{\partial \bar{\psi}(g)}{\partial g_\nu} = 0.$$ (22)

The latter equations are written in the parameters of $SL(2, \mathbb{C})$. Since an universal covering $SL(2, \mathbb{C})$ of the proper orthochronous Lorentz group is a complexification of the group $SU(2)$ (see, for example, Ref. 58), then it is more convenient to express six parameters of the group.
\( SL(2, \mathbb{C}) \) via three parameters \( a_1, a_2, a_3 \) of \( SU(2) \). It is obvious that \( g_1 = a_1, g_2 = a_2, g_3 = a_3, g_4 = ia_1, g_5 = ia_2, g_6 = ia_3 \). Taking into account the structure of \( \Upsilon_* \) given by (7)–(8), we can rewrite the first equation from (22) as follows

\[
\sum_{k=1}^{3} \Lambda_k^* \frac{\partial \psi}{\partial a_k^*} - i \sum_{k=1}^{3} \Lambda_k \frac{\partial \psi}{\partial a_k^*} = 0,
\]

\[
\sum_{k=1}^{3} \Lambda_k \frac{\partial \psi}{\partial \bar{a}_k} + i \sum_{k=1}^{3} \Lambda_k^* \frac{\partial \psi}{\partial \bar{a}_k} = 0.
\]  

(23)

where \( a_1^* = -ig_4, a_2^* = -ig_5, a_3^* = -ig_6, \) and \( \bar{a}_k, \bar{a}_k^* \) are parameters corresponding to dual basis. In essence, equations (23) are defined in a three-dimensional complex space \( \mathbb{C}^3 \). In turn, the space \( \mathbb{C}^3 \) is isometric to a six-dimensional bivector space \( \mathbb{R}^6 \) (a parameter space of the Lorentz group [50]). The bivector space \( \mathbb{R}^6 \) is a tangent space of the group manifold \( \mathcal{L}_6 \) of the Lorentz group, that is, the group manifold \( \mathcal{L}_6 \) in each its point is equivalent to the space \( \mathbb{R}^6 \). Thus, for all \( \mathbf{g} \in \mathcal{L}_6 \), we have \( T_\mathbf{g} \mathcal{L}_6 \cong \mathbb{R}^6 \). Taking into account the explicit form of the matrices \( \Lambda_i \) given by (10), we can rewrite the system (23) in the following form

\[
\frac{\sqrt{2}}{2} \frac{\partial \psi_2}{\partial x_1} - \frac{\sqrt{2}}{2} \frac{\partial \psi_2}{\partial x_2} + \frac{\partial \psi_1}{\partial x_3} = 0,
\]

\[
\frac{\sqrt{2}}{2} \frac{\partial \psi_1}{\partial x_1} + \frac{\sqrt{2}}{2} \frac{\partial \psi_2}{\partial x_2} = 0,
\]

\[
\frac{\sqrt{2}}{2} \frac{\partial \psi_1}{\partial x_1} + \frac{\sqrt{2}}{2} \frac{\partial \psi_3}{\partial x_2} + \frac{\partial \psi_1}{\partial x_3} = 0,
\]

\[
\frac{\sqrt{2}}{2} \frac{\partial \psi_3}{\partial x_1} + \frac{\sqrt{2}}{2} \frac{\partial \psi_3}{\partial x_2} + \frac{\partial \psi_3}{\partial x_3} = 0.
\]  

(24)

A separation of variables in (24) is realized via the following factorizations:

\[
\psi_1 = f^{l_1}_{1,1}(r) \mathcal{M}_1^l(\varphi, \epsilon, \theta, \tau, 0, 0),
\]

\[
\psi_2 = f^{l_0}_{1,0}(r) \mathcal{M}_0^0(0, 0, \theta, \tau, 0, 0),
\]

\[
\psi_3 = f^{l_1}_{1,-1}(r) \mathcal{M}_1^{-1}(\varphi, \epsilon, \theta, \tau, 0, 0),
\]

\[
\psi_4 = f^{l_1}_{1,1}(r^*) \mathcal{M}_1^l(\varphi, \epsilon, \theta, \tau, 0, 0),
\]

\[
\psi_5 = f^{l_0}_{1,0}(r^*) \mathcal{M}_0^0(0, 0, \theta, \tau, 0, 0),
\]

\[
\psi_6 = f^{l_1}_{1,-1}(r^*) \mathcal{M}_1^{-1}(\varphi, \epsilon, \theta, \tau, 0, 0),
\]

where \( \mathcal{M}_m^l(\varphi, \epsilon, \theta, \tau, 0, 0) \) (\( \mathcal{M}_m^l(\varphi, \epsilon, \theta, \tau, 0, 0) \)) are associated hyperspherical functions defined on the surface of the two-dimensional complex sphere of the radius \( r \), \( f^{l_0}_{l_0}(r) \) and \( f^{l_0}_{l_0}(r^*) \)

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\(^7\)The matrices [10] can be derived at \( l = 1 \) from the more general expressions (62)–(67) given in the Ref. 62.
are radial functions (for more details about two-dimensional complex sphere see Refs. 60–62).\footnote{Choosing the two-dimensional complex sphere as an internal spin space, we see that our configuration space \( \mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathcal{L}_6 \) reduces to \( \mathcal{M}_8 = \mathbb{R}^{1,3} \times \mathbb{C}^2 \). Bary and Kihlberg\cite{16} claimed that the 8-dimensional configuration space \( \mathcal{M}_8 \) is the most suitable for a description of both half-integer and integer spins. \( \mathcal{M}_8 \) is a homogeneous space of the Poincaré group. Indeed, the space \( \mathbb{C}^2 \) is homeomorphic to an extended complex plane \( \mathbb{C} \cup \infty \) which presents an absolute (a set of infinitely distant points) of a Lobachevskii space \( S^{1,2} \). At this point, the group of fractional linear transformations of the plane \( \mathbb{C} \cup \infty \) is isomorphic to a motion group of \( S^{1,2} \). In turn, the Lobachevskii space \( S^{1,2} \) is an absolute of the Minkowski world \( \mathbb{R}^{1,3} \) and, therefore, the group of fractional linear transformations of \( \mathbb{C} \cup \infty \) double covers a rotation group of \( \mathbb{R}^{1,3} \), that is, the Lorentz group. It is not hard to see that the two-dimensional complex sphere coincides with a well-known Penrose’s celestial sphere \cite{17}. Further, using a canonical projection \( \pi : \mathbb{C}^2 \to S^2 \), where \( \mathbb{C}^2 = \mathbb{C}^2 / \{0,0\} \) and \( S^2 \) is a two-dimensional real sphere, we see that the configuration space \( \mathcal{M}_8 = \mathbb{R}^{1,3} \times \mathbb{C}^2 \) reduces to \( \mathcal{M}_6 = \mathbb{R}^{1,3} \times S^2 \). The real two-sphere \( S^2 \) has a minimal possible dimension among the homogeneous spaces of the Lorentz group. For that reason \( \mathcal{M}_6 \) is a minimal homogeneous space of the Poincaré group (the spacetime translations act trivially on \( S^2 \)). Field models on the configuration space \( \mathcal{M}_6 \) have been considered in recent works\cite{12,13,14}. In the Ref. 43 Drechsler considered the two-sphere as a “spin shell” \( S_r^2 \) of radius \( r = 2s \), where \( s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \).}

Repeating for the case \( l = 1 \) all the transformations presented for the general relativistically invariant system\cite{62}, we come to the following system (the system (126) in the Ref. 62 at \( l = 1 \)):

\[
\begin{align*}
&2 \frac{d f_{1,1}^l(r)}{dr} - \frac{1}{r} f_{1,1}^l(r) - \frac{\sqrt{2l(l + 1)}}{r} f_{1,0}^l(r) = 0, \\
&\quad - \frac{\sqrt{2l(l + 1)}}{r} f_{1,1}^{l-1}(r) + \frac{\sqrt{2l(l + 1)}}{r} f_{1,1}^l(r) = 0, \\
&-2 \frac{d f_{1,1}^{l-1}(r)}{dr} + \frac{1}{r} f_{1,1}^{l-1}(r) + \frac{\sqrt{2l(l + 1)}}{r} f_{1,0}^{l-1}(r) = 0, \\
&\quad 2 \frac{d f_{1,1}^l(r^*)}{dr^*} - \frac{1}{r^*} f_{1,1}^l(r^*) - \frac{\sqrt{2l(l + 1)}}{r^*} f_{1,0}^l(r^*) = 0, \\
&\quad - \frac{\sqrt{2l(l + 1)}}{r^*} f_{1,1}^{l-1}(r^*) + \frac{\sqrt{2l(l + 1)}}{r^*} f_{1,1}^l(r^*) = 0, \\
&-2 \frac{d f_{1,1}^{l-1}(r^*)}{dr^*} + \frac{1}{r^*} f_{1,1}^{l-1}(r^*) + \frac{\sqrt{2l(l + 1)}}{r^*} f_{1,0}^{l-1}(r^*) = 0,
\end{align*}
\]  

(25)

From the second and fifth equations it follows that \( f_{1,1}^{l-1}(r) = f_{1,1}^l(r) \) and \( f_{1,1}^{l-1}(r^*) = f_{1,1}^l(r^*) \). Taking into account these relations we can rewrite the system (25) as follows

\[
\begin{align*}
&2 \frac{d f_{1,1}^l(r)}{dr} - \frac{1}{r} f_{1,1}^l(r) - \frac{\sqrt{2l(l + 1)}}{r} f_{1,0}^l(r) = 0, \\
&\quad -2 \frac{d f_{1,1}^{l-1}(r)}{dr} + \frac{1}{r} f_{1,1}^{l-1}(r) + \frac{\sqrt{2l(l + 1)}}{r} f_{1,0}^{l-1}(r) = 0, \\
&\quad 2 \frac{d f_{1,1}^l(r^*)}{dr^*} - \frac{1}{r^*} f_{1,1}^l(r^*) - \frac{\sqrt{2l(l + 1)}}{r^*} f_{1,0}^l(r^*) = 0, \\
&\quad -2 \frac{d f_{1,1}^{l-1}(r^*)}{dr^*} + \frac{1}{r^*} f_{1,1}^{l-1}(r^*) + \frac{\sqrt{2l(l + 1)}}{r^*} f_{1,0}^{l-1}(r^*) = 0,
\end{align*}
\]  

(26)
It is easy to see that the first equation is equivalent to the second, and third equation is equivalent to the fourth. Thus, we come to the following inhomogeneous differential equations of the first order:

\[ 2r \frac{df_{1,1}^l(r)}{dr} - f_{1,1}^l(r) - \sqrt{2l(l+1)}f_{1,0}^l(r) = 0, \]
\[ 2r^2 \frac{df_{1,1}^l(r^*)}{dr^*} - f_{1,1}^l(r^*) - \sqrt{2l(l+1)}f_{1,0}^l(r^*) = 0, \]

where the functions \( f_{1,0}^l(r) \) and \( f_{1,0}^l(r^*) \) are understood as inhomogeneous parts. Solutions of these equations are expressed via the elementary functions:

\[ f_{1,1}^l(r) = C\sqrt{r} + \sqrt{2l(l+1)}r; \]
\[ f_{1,1}^l(r^*) = \dot{C}\sqrt{r^*} + \sqrt{2l(l+1)}r^*. \]

Therefore, solutions of the radial part have the form:

\[ f_{1,1}^l(r) = f_{1,1}^l(r) = C\sqrt{r} + \sqrt{2l(l+1)}r; \]
\[ f_{1,0}^l(r) = \sqrt{2l(l+1)}r; \]
\[ f_{1,1}^l(r^*) = f_{1,1}^l(r^*) = \dot{C}\sqrt{r^*} + \sqrt{2l(l+1)}r^*; \]
\[ f_{1,0}^l(r^*) = \sqrt{2l(l+1)}r^*. \]

In such a way, solutions of the \( SL(2, \mathbb{C}) \)-field equations [24] are

\[ \psi_1(r, \varphi^c, \theta^c) = f_{1,1}^l(r)m_{l}^1(\varphi, \epsilon, \theta, \tau, 0, 0), \]
\[ \psi_2(r, \varphi^c, \theta^c) = f_{1,0}^l(r)m_{l}^0(0, 0, 0, \theta, \tau, 0, 0), \]
\[ \psi_3(r, \varphi^c, \theta^c) = f_{1,1}^l(r)m_{l}^{-1}(\varphi, \epsilon, \theta, \tau, 0, 0), \]
\[ \dot{\psi}_1(r^*, \varphi^c, \theta^c) = f_{1,1}^l(r^*)m_{l}^1(\varphi, \epsilon, \theta, \tau, 0, 0), \]
\[ \dot{\psi}_2(r^*, \varphi^c, \theta^c) = f_{1,0}^l(r^*)m_{l}^0(0, 0, \theta, \tau, 0, 0), \]
\[ \dot{\psi}_3(r^*, \varphi^c, \theta^c) = f_{1,1}^l(r^*)m_{l}^{-1}(\varphi, \epsilon, \theta, \tau, 0, 0), \]

where

\[ l = 1, 2, 3, \ldots \]
\[ \hat{l} = 1, 2, 3, \ldots \]

\[ m_{l}^\pm(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{\mp(\epsilon+i\varphi)}Z_{l}^{\pm}(\theta, \tau), \]

\[ Z_{l}^{\pm}(\theta, \tau) = \cos^{2l} \frac{\theta}{2} \cosh^{2l} \frac{r}{2} \sum_{k=-l}^{l} \tan^{\pm l-k} \frac{\theta}{2} \tanh^{k} \frac{\tau}{2} \times \]
\[ _2F_1 \left( \begin{array}{c} \pm l + 1, -l, -k \\ \pm 1, -k \end{array} \right) \frac{i^2 \tan^2 \frac{\theta}{2}}{l-k+1} \tanh^2 \frac{\tau}{2} \right), \]
\[ \mathcal{M}_l^0(0, 0, \theta, \tau, 0, 0) = Z_l^0(\theta, \tau), \]

\[ Z_l^0(\theta, \tau) = \cos^{2l} \frac{\theta}{2} \cosh^{2l} \frac{\tau}{2} \sum_{k=-l}^{l} i^{-k} \tan^{-k} \frac{\theta}{2} \tanh^{-k} \frac{\tau}{2} \times \]

\[ {}_2F_1\left( -l + 1, 1 - l - k \atop -k + 1 \right) \equiv \frac{\theta}{2} \right| \tanh^2 \frac{\tau}{2}, \]

\[ \mathcal{M}_l^{\pm 1}(\varphi, \epsilon, \theta, \tau, 0, 0) = e^{\mp i \epsilon} Z_l^{\pm 1}(\theta, \tau), \]

\[ Z_l^{\pm 1}(\theta, \tau) = \cos^{2l} \frac{\theta}{2} \cosh^{2l} \frac{\tau}{2} \sum_{k=-l}^{l} i^{\pm 1-k} \tan^{\pm 1-k} \frac{\theta}{2} \tanh^{-k} \frac{\tau}{2} \times \]

\[ {}_2F_1\left( \pm 1 - \frac{i}{2} + 1, 1 - \frac{i}{2} - k \atop \pm 1 - \frac{i}{2} - k \right) \equiv \frac{\theta}{2} \right| \tanh^2 \frac{\tau}{2}, \]

\[ \mathcal{M}_l^0(0, 0, \theta, \tau, 0, 0) = Z_l^0(\theta, \tau), \]

\[ Z_l^0(\theta, \tau) = \cos^{2l} \frac{\theta}{2} \cosh^{2l} \frac{\tau}{2} \sum_{k=-l}^{l} i^{-k} \tan^{-k} \frac{\theta}{2} \tanh^{-k} \frac{\tau}{2} \times \]

\[ {}_2F_1\left( -i + 1, 1 - i - k \atop -k + 1 \right) \equiv \frac{\theta}{2} \right| \tanh^2 \frac{\tau}{2}, \]

where \( \mathcal{M}_l^0(0, 0, \theta, \tau, 0, 0) \) (\( Z_l^0(\theta, \tau) \)) are zonal hyperspherical functions (see Ref. 48).

Therefore, in accordance with the factorization \( \psi(\alpha) \) an explicit form of the relativistic wavefunction \( \psi(\alpha) = \psi(x)\psi(g) \) on the Poincaré group in the case of \( (1, 0) \oplus (0, 1) \) representation (Maxwell field) is defined by the following expressions (complete set)

\[ \psi_1(\alpha) = \psi_+(k; x, t)\psi_1(g) = \]

\[ \{ 2(2\pi)^3 \}^{-\frac{1}{2}} \left( \frac{\epsilon_+(k)}{\epsilon_+(k)} \right) \exp[i(k \cdot x - \omega t)] f_{1,1}^l(r) \mathcal{M}_l^0(\varphi, \epsilon, \theta, \tau, 0, 0), \]

\[ \psi_0(\alpha) = \psi_0(k; x)\psi_0(g) = \{ 2(2\pi)^3 \}^{-\frac{1}{2}} \left( \frac{\epsilon_0(k)}{\epsilon_0(k)} \right) \exp[i k \cdot x] f_{1,1}^l(r) \mathcal{M}_l^0(0, 0, \theta, \tau, 0, 0), \]

\[ \psi_{-1}(\alpha) = \psi_-(k; x, t)\psi_{-1}(g) = \]

\[ \{ 2(2\pi)^3 \}^{-\frac{1}{2}} \left( \frac{\epsilon_-(k)}{\epsilon_-(k)} \right) \exp[i(k \cdot x - \omega t)] f_{1,1}^l(r) \mathcal{M}_{l-1}^0(\varphi, \epsilon, \theta, \tau, 0, 0), \]

\[ \psi_1(\alpha) = \psi_+(k; x, t)\psi_1(g) = \]

\[ \{ 2(2\pi)^3 \}^{-\frac{1}{2}} \left( \frac{\epsilon_+(k)}{\epsilon_+(k)} \right) \exp[-i(k \cdot x - \omega t)] f_{1,1}^l(r^*) \mathcal{M}_l^0(\varphi, \epsilon, \theta, \tau, 0, 0), \]
\[
\dot{\psi}_0(\alpha) = \psi_0^*(k;x)\dot{\psi}_0(g) = \left\{2(2\pi)^3\right\}^{-\frac{1}{2}} \begin{pmatrix} \varepsilon_0^+(k) \\ \varepsilon_0^-(k) \end{pmatrix} \exp[-i k \cdot x] f_{1,0}^I(r^*) \mathcal{M}_{i}(0,0,\theta,\tau,0,0), \\
\dot{\psi}_{-1}(\alpha) = \psi_{-1}^*(k;x,t)\dot{\psi}_{-1}(g) = \\
\left\{2(2\pi)^3\right\}^{-\frac{1}{2}} \begin{pmatrix} \varepsilon_1^+(k) \\ \varepsilon_1^-(k) \end{pmatrix} \exp[-i (k \cdot x - \omega t)] f_{1,-1}^I(r^*) \mathcal{M}_{-1}(\varphi,\epsilon,\theta,\tau,0,0),
\]

The set (27) consists of the transverse solutions \(\psi_{\pm 1}(\alpha)\) (positive energy), \(\dot{\psi}_{\pm 1}(\alpha)\) (negative energy) and the zero-eigenvalue (longitudinal) solutions \(\psi_0(\alpha)\) and \(\dot{\psi}_0(\alpha)\). The negative energy solutions \(\dot{\psi}_{\pm 1}(\alpha)\) should be omitted, since photons have no antiparticles. The longitudinal solutions \(\psi_0(\alpha)\) and \(\dot{\psi}_0(\alpha)\) do not contribute to a real photon due to their transversality conditions (17) and (19). Thus, any real photon should be described by only \(\psi_{\pm 1}(\alpha)\):

\[
\psi_{\pm 1}(\alpha) = \psi_{\pm}(k;x,t)\psi_{\pm}(g) = \\
\left\{2(2\pi)^3\right\}^{-\frac{1}{2}} \begin{pmatrix} \varepsilon_{\pm}(k) \\ \varepsilon_{\pm}(k) \end{pmatrix} \exp[i (k \cdot x - \omega t)] f_{1,\pm 1}^I(r^*) \mathcal{M}_{\pm 1}(\varphi,\epsilon,\theta,\tau,0,0),
\]

In such a way, we obtain a solution set defining the Maxwell field \((1,0) \oplus (0,1)\) on the Poincaré group (or, equally, on the group manifold \(M_4\)). It should be noted that obtained previously solutions for the field \((1/2,0) \oplus (0,1/2)\) (Dirac field) \([64]\) have the analogous mathematical structure, that is, they are the functions on the Poincaré group. This circumstance allows us to consider the fields \((1/2,0) \oplus (0,1/2)\) and \((1,0) \oplus (0,1)\) on an equal footing, from the one group theoretical viewpoint. The following step is a definition of field operators in the obtained solutions for the Dirac and Maxwell fields via harmonic analysis on the Poincaré group. A construction of quantum electrodynamics in terms of so-defined field operators will be studied in a separate work.

### Appendix. Complex two-sphere and hyperspherical functions

Let us construct in \(\mathbb{C}^3\) a two–dimensional complex sphere from the quantities \(z_k = x_k + iy_k\), \(z_k = x_k - iy_k\) as follows

\[
z^2 = z_1^2 + z_2^2 + z_3^2 = x^2 - y^2 + 2i xy = r^2 
\]  

(A.1)

and its complex conjugate (dual) sphere

\[
\bar{z}^2 = z_1^2 + z_2^2 + z_3^2 = x^2 - y^2 - 2i xy = \bar{r}^2.
\]  

(A.2)

It is well-known that both quantities \(x^2 - y^2\), \(xy\) are invariant with respect to the Lorentz transformations, since a surface of the complex sphere is invariant (Casimir operators of the Lorentz group are constructed from such quantities, see also (A.3)). Moreover, since the real and imaginary parts of the complex two-sphere transform like the electric and magnetic fields, respectively, the invariance of \(z^2 \sim (E + iB)^2\) under proper Lorentz transformations is evident.
The group $SL(2, \mathbb{C})$ of all complex matrices

$$
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
$$

of 2-nd order with the determinant $\alpha \delta - \gamma \beta = 1$, is a complexification of the group $SU(2)$. The group $SU(2)$ is one of the real forms of $SL(2, \mathbb{C})$. The transition from $SU(2)$ to $SL(2, \mathbb{C})$ is realized via the complexification of three real parameters $\varphi, \theta, \psi$ (Euler angles). Let $\theta^c = \theta - i\pi, \varphi^c = \varphi - i\epsilon, \psi^c = \psi - i\epsilon$ be complex Euler angles, where

$$
0 \leq \text{Re}\theta^c = \theta \leq \pi, \quad -\infty < \text{Im}\theta^c = \tau < +\infty,\quad 0 \leq \text{Re}\varphi^c = \varphi < 2\pi, \quad -\infty < \text{Im}\varphi^c = \epsilon < +\infty,
$$

(A.3)

As known, for the Lorentz group there are two independent Casimir operators

$$
X^2 = X_1^2 + X_2^2 + X_3^2 = \frac{1}{4}(A^2 - B^2 + 2iAB),
$$

$$
Y^2 = Y_1^2 + Y_2^2 + Y_3^2 = \frac{1}{4}(\tilde{A}^2 - \tilde{B}^2 - 2i\tilde{A}\tilde{B}).
$$

(A.4)

Using the parameters (A.3), we obtain for the Casimir operators the following expressions

$$
X^2 = \frac{\partial^2}{\partial \theta^c^2} + \cot \theta^c \frac{\partial}{\partial \theta^c} + \frac{1}{\sin^2 \theta^c} \left[ \frac{\partial^2}{\partial \varphi^c^2} - 2 \cos \theta^c \frac{\partial}{\partial \varphi^c} \frac{\partial}{\partial \psi^c} + \frac{\partial^2}{\partial \psi^c^2} \right],
$$

$$
Y^2 = \frac{\partial^2}{\partial \theta^c^2} + \cot \theta^c \frac{\partial}{\partial \theta^c} + \frac{1}{\sin^2 \theta^c} \left[ \frac{\partial^2}{\partial \varphi^c^2} - 2 \cos \theta^c \frac{\partial}{\partial \varphi^c} \frac{\partial}{\partial \psi^c} + \frac{\partial^2}{\partial \psi^c^2} \right].
$$

(A.5)

Matrix elements of unitary irreducible representations of the Lorentz group are eigenfunctions of the operators (A.5):

$$
[X^2 + l(l + 1)] M^i_{mn}(\varphi^c, \theta^c, \psi^c) = 0,
$$

$$
[Y^2 + l(l + 1)] M^i_{mn}(\varphi^c, \theta^c, \psi^c) = 0,
$$

(A.6)

where

$$
M^i_{mn}(\varphi^c, \theta^c, \psi^c) = e^{-i(m\varphi^c + n\psi^c)} Z^i_{mn}(\theta^c),
$$

$$
M^i_{mn}(\varphi^c, \theta^c, \psi^c) = e^{-i(m\varphi^c + \tilde{n}\psi^c)} Z^i_{m\tilde{n}}(\tilde{\theta}^c).
$$

(A.7)

Substituting the functions (A.7) into (A.6) and taking into account the operators (A.5), we obtain a complex analog of the Legendre equations:

$$
(1 - z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} - \frac{m^2 + n^2 - 2mnz}{1 - z^2} + l(l + 1) Z^i_{mn} = 0,
$$

(A.8)

$$
(1 - \tilde{z}^2) \frac{d^2}{d\tilde{z}^2} - 2\tilde{z} \frac{d}{d\tilde{z}} - \frac{\tilde{m}^2 + \tilde{n}^2 - 2\tilde{m}\tilde{z}}{1 - \tilde{z}^2} + \tilde{l}(\tilde{l} + 1) Z^i_{m\tilde{n}} = 0,
$$

(A.9)
where \( z = \cos \theta^c \) and \( \bar{z} = \cos \bar{\theta}^c \). The latter equations have three singular points \(-1, +1, \infty\). Solutions of (A.8) have the form

\[
Z_{mn}^l = \sum_{k=-l}^{l} \, i^{m-k} \, \sqrt{\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(l-k+1)\Gamma(l+k+1)} \times \\
\cos^{2l} \theta \tan^{m-k} \frac{\theta}{2} \times \\
\frac{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+k-j+1)\Gamma(m-k+j+1)}{\Gamma(l-n+1)\Gamma(l+n+1)\Gamma(l-k+1)\Gamma(l+k+1)} \cosh^{2l} \frac{\tau}{2} \tanh^{n-k} \frac{\tau}{2} \times \\
\left( \sum_{j=\max(0,k-m)}^{\min(l-m,l+k)} \frac{i^{2j} \tan^{2j} \frac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+k-j+1)\Gamma(m-k+j+1)} \\
\times \right) \\
\left( \sum_{s=\max(0,k-n)}^{\min(l-n,l+k)} \frac{\tanh^{2s} \frac{\tau}{2}}{\Gamma(s+1)\Gamma(l-n-s+1)\Gamma(l+k-s+1)\Gamma(n-k+s+1)} \right).
\]  
(A.10)

We will call the functions \( Z_{mn}^l \) in (A.10) as hyperspherical functions\(^9\). The functions \( Z_{mn}^l \) can be written via the hypergeometric series as follows:

\[
Z_{mn}^l = \cos^{2l} \frac{\theta}{2} \cosh^{2l} \frac{\tau}{2} \sum_{k=-l}^{l} \, i^{m-k} \, \tan^{m-k} \frac{\theta}{2} \tanh^{n-k} \frac{\tau}{2} \times \\
\left( \sum_{k=\max(0,k-m)}^{\min(l-m,l+k)} \frac{i^{2k} \tan^{2k} \frac{\theta}{2}}{\Gamma(k+1)\Gamma(l-m-k+1)\Gamma(l+k-m+1)\Gamma(m-k+1)} \right) \times \\
\left( \sum_{s=\max(0,k-n)}^{\min(l-n,l+k)} \frac{\tanh^{2s} \frac{\tau}{2}}{\Gamma(s+1)\Gamma(l-n-s+1)\Gamma(l+k-s+1)\Gamma(n-k+s+1)} \right) \\
\times \left( \sum_{j=\max(0,k-m)}^{\min(l-m,l+k)} \frac{i^{2j} \tan^{2j} \frac{\theta}{2}}{\Gamma(j+1)\Gamma(l-m-j+1)\Gamma(l+k-j+1)\Gamma(m-k+j+1)} \right) \times \\
\left( \sum_{s=\max(0,k-n)}^{\min(l-n,l+k)} \frac{\tanh^{2s} \frac{\tau}{2}}{\Gamma(s+1)\Gamma(l-n-s+1)\Gamma(l+k-s+1)\Gamma(n-k+s+1)} \right). \\
\]  
(A.11)

Therefore, matrix elements are expressed by means of the function (a generalized hyperspherical function)

\[
\mathcal{M}_{mn}^l (g) = e^{-m(\epsilon+i\phi)} Z_{mn}^l (\cos \theta^c) e^{-n(\epsilon+i\psi)},
\]  
(A.12)

where

\[
Z_{mn}^l (\cos \theta^c) = \sum_{k=-l}^{l} P_{mk}^l (\cos \theta) \mathcal{P}_{kn}^l (\cosh \tau),
\]  
(A.13)

here \( P_{mn}^l (\cos \theta) \) is a generalized spherical function on the group SU(2) (see Ref. 66), and \( \mathcal{P}_{mn}^l \) is an analog of the generalized spherical function for the group QU(2) (so-called Jacobi function ). QU(2) is a group of quasiunitary unimodular matrices of second order. As well as the group SU(2), the group QU(2) is one of the real forms of SL(2,\( \mathbb{C} \)) (QU(2) is noncompact). Other designation of this group is SU(1,1) known also as three-dimensional Lorentz group (this group is isomorphic to SL(2,\( \mathbb{R} \)). Associated hyperspherical functions are derived from (A.12) at \( n = 0 \). They have the form

\[
\mathcal{M}_{mn}^l (g) = e^{-m(\epsilon+i\phi)} Z_{mn}^l (\cos \theta^c).
\]
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