On the prime factors of a quasiperfect number

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Abstract: A positive integer N is said to be quasiperfect if \( \sigma(N) = 2N + 1 \) where \( \sigma(N) \) is the sum of the positive divisors of \( N \). So far no quasiperfect number is known. If such \( N \) exists, let \( \gamma(N) \) denote the product of the distinct primes dividing \( N \). In this paper, we obtain a lower bound for \( \gamma(N) \) in terms of \( r = \omega(N) \), the number of distinct prime factors of \( N \). Also, we show that every quasiperfect number \( N \) is divisible by a prime \( p \) with: (i) \( p \equiv 1 \pmod{4} \), (ii) \( p \equiv 1 \pmod{5} \) if \( 5 \nmid N \) and (iii) \( p \equiv 1 \pmod{3} \), if \( 3 \nmid N \).

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1 Introduction

For any natural number \( N \) let \( \sigma(N) \) denote the sum of its positive divisors. W. Sierpinski [6] asked whether there is any natural number \( N \) satisfying

\[
\sigma(N) = 2N + 1,
\]

which is unanswered till date. Calling such \( N \), if it exists, a quasiperfect number, Cattaneo [2] initiated the study of such numbers. H. L. Abbott et. al. [1] continued the investigations and proved the following:
If a quasiperfect number \( N \) exists and if \( \omega(N) \) is the number of distinct prime factors of \( N \) then

\[
\omega(N) \geq 5 \text{ and } N > 10^{20} \ (\text{[1], Theorem 2 and 4}) \quad (1.2)
\]

and

\[
\omega(N) \geq 15 \text{ and } N > 10^{57} \text{ if } (N, 15) = 1 \ (\text{[3]}) \quad (1.3)
\]

In [4] M. Kishore improved (1.2) to \( \omega(N) \geq 6 \) and \( N > 10^{40} \) while a further refinement of it to \( \omega(N) \geq 7 \) and \( N > 10^{35} \) was obtained by G.L. Cohen and Peter Hagis Jr. [3].

For other details of research on quasiperfect numbers one can see the excellent book of J. Sandor and B. Crstici ([5], p. 38-39).

Recently the authors [7] have given a different proof for the first part of (1.3) for which Theorem 2.4 (given in Section 2 below) was used.

For any positive integer \( n \) let \( \gamma(n) \) denote the product of its distinct prime factors (\( \gamma(n) \) is called the radical of the integer \( n \); and it is the maximal squarefree divisor of \( n \), that is, the greatest divisor of \( n \) having no square factor > 1).

In this paper we obtain a lower bound for \( \gamma(N) \) in terms of \( r = \omega(N) \) for a quasiperfect number \( N \). Also we prove that every quasiperfect number is divisible by a prime \( p \) with

(i) \( p \equiv 1 \pmod{4} \),
(ii) \( p \equiv 1 \pmod{5} \) if \( 5 \nmid N \) and
(iii) \( p \equiv 1 \pmod{3} \) if \( 3 \nmid N \).

\section{Preliminaries}

Throughout the rest of the paper \( N \) stands for a quasiperfect number. We first state a theorem due to Cattaneo [2] needed for our purpose:

\textbf{Theorem 2.1.}

(a) If \( N \) exists, then it is of the form

\[
N = p_1^{2e_1} p_2^{2e_2} \cdots p_r^{2e_r},
\]

where \( p_1, p_2, \ldots, p_r \) are distinct odd primes and \( e_i \geq 1 \) for \( i = 1, 2, 3, \ldots, r \).

(b) If \( p_i \equiv 1 \pmod{8} \), then \( e_i \equiv 0 \) or \( 1 \pmod{4} \); if \( p_i \equiv 3 \pmod{8} \), then \( e_i \equiv 0 \pmod{2} \) and if \( p_i \equiv 5 \pmod{8} \), then \( e_i \equiv 0 \) or \( -1 \pmod{4} \).

(c) If \( M \) is a natural number such that \( \sigma(M) \geq 2M \), then no non-trivial multiple of \( M \) is quasiperfect.

\textbf{Remark 2.3.} It follows from Theorem 2.1 that every quasiperfect number is the square of an odd integer and that \( \sigma(d) < 2d \) for every divisor \( d \) of \( N \).

In [7] the authors have proved:

\textbf{Theorem 2.4.} If \( N \) exists and is of the form (2.2), then an odd number of \( p_i^{2e_i} \) are such that either \( p_i \equiv 1 \pmod{8} \) and \( e_i \equiv 1 \pmod{4} \) or \( p_i \equiv 5 \pmod{8} \) and \( e_i \equiv -1 \pmod{4} \).

(Such \( p_i^{2e_i} \) are called special factors of \( N \) in [7])
3 Lower bound for $\gamma(N)$

Suppose $A = \{a_1, a_2, \ldots, a_r\}$ is a set of positive real numbers and for any $k (1 \leq k \leq r)$ suppose $S_k(A)$ is the sum of the products of the elements in the $k$-element subsets of $A$. That is,

$$S_k(A) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq r} a_{i_1}a_{i_2}\ldots a_{i_k} \tag{3.1}$$

For example, $S_1(A) = \sum_{i=1} r a_i$ and $S_2(A) = \sum_{1 \leq i_1 < i_2 \leq r} a_{i_1}a_{i_2}$.

Note that

$$\prod_{i=1}^r (1 + a_i) = 1 + \sum_{k=1}^r S_k(A) \tag{3.2}$$

Observe that $S_k(A)$ has $\binom{r}{k}$ terms and that each $a_j \in A$ occurs exactly in $\binom{r-1}{k-1}$ terms of it. Therefore the product $P_k(A)$ of the terms in $S_k(A)$ is given by

$$P_k(A) = (a_1a_2\ldots a_r)^{\binom{r-1}{k-1}} \tag{3.3}$$

Therefore, the inequality between the arithmetic mean and the geometric mean gives

$$\frac{S_k(A)}{\binom{r}{k}} > (P_k(A))^{\frac{1}{\binom{r}{k}}}$$

(the strict inequality is due to the fact that $a_j$ are distinct)

which, in view of (3.3), shows that

$$S_k(A) > \binom{r}{k} (a_1a_2\ldots a_r)^{\frac{1}{r}} \tag{3.4}$$

**Theorem 3.5.** If $N$ exists and is of the form (2.2), then

$$\gamma(N) > A_r,$$

where $A_r = \frac{1}{(2^r - 1)^r}$

**Proof.** Here $\gamma(N) = p_1p_2\ldots p_r$ is a divisor of $N$ so that by Remark 2.3 and (3.2) we have

$$2 > \frac{\sigma(\gamma(N))}{\gamma(N)} = \prod_{i=1}^r \frac{\sigma(p_i)}{p_i} = \prod_{i=1}^r \left(1 + \frac{1}{p_i}\right) = 1 + \sum_{k=1}^r S_k(B),$$

where $B = \left\{ \frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_r} \right\}$. Therefore, by (3.4), it follows that

$$2 > 1 + \sum_{k=1}^r \binom{r}{k} \left( \frac{1}{p_1p_2\ldots p_r} \right)^{\frac{1}{k}}$$

$$= 1 + \sum_{k=1}^r \binom{r}{k} \left\{ \gamma(N)^{-\frac{1}{r}} \right\}^k$$

$$= \left\{ 1 + \gamma(N)^{-\frac{1}{r}} \right\}^r,$$

which proves the theorem. \(\square\)
Remark 3.6. One of the reviewers has pointed out that a better lower bound for $\gamma(N)$ than $A_r$ can be obtained by using known estimates for some functions over primes and this will be investigated later. Another reviewer has observed that the proof of Theorem 3.5 bears a close resemblance to the proof of a result of Anirudh Prabhu’s paper available online via arXiv at https://arxiv.org/pdf/1008.1114.pdf and the authors were not aware of the paper earlier.

4 On prime factors of $N$

Theorem 4.1. If $N$ is of the form (2.2), then $p_i \equiv 1 \pmod{4}$ for some $i$.

Proof. If not, $p_i \equiv 3$ or $7 \pmod{8}$ for each $i$, contradicting Theorem 2.4. \hfill \Box

Theorem 4.2. If $N$ is of the form (2.2) and $(N, 5) = 1$, then $p_i \equiv 1 \pmod{5}$ for some $i$.

Proof. If $(N, 5) = 1$ then $p_i \equiv \pm 1$ or $\pm 2 \pmod{5}$

First suppose $p_i \equiv \pm 1 \pmod{5}$ so that $p_i^2 \equiv 1 \pmod{5}$ and therefore

$$\sigma(p_i^{2e_i}) = (1 + p_i)(1 + p_i^2 + \ldots + p_i^{2e_i - 2}) + p_i^{2e_i} \equiv (1 + p_i)e_i + 1 \pmod{5}$$

$$\equiv \begin{cases} 2e_i + 1 \pmod{5} & \text{if } p_i \equiv 1 \pmod{5} \\ 1 \pmod{5} & \text{if } p_i \equiv -1 \pmod{5} \end{cases} \tag{4.3}$$

If $p_i \equiv \pm 2 \pmod{5}$, then $p_i^2 \equiv -1 \pmod{5}$ and therefore

$$\sigma(p_i^{2e_i}) = (1 + p_i)(1 + p_i^2 + \ldots + p_i^{2e_i - 2}) + p_i^{2e_i}$$

$$\equiv (1 + p_i)\{1 + (-1) + (-1)^2 + \ldots + (-1)^{e_i - 1}\} + (-1)^{e_i} \pmod{5}$$

$$\equiv \begin{cases} 1 \pmod{5} & \text{if } e_i \text{ is even} \\ 2 \pmod{5} & \text{if } e_i \text{ is odd, } p_i \equiv 2 \pmod{5} \\ -2 \pmod{5} & \text{if } e_i \text{ is odd, } p_i \equiv -2 \pmod{5} \end{cases} \tag{4.4}$$

If possible, suppose no $p_i \equiv 1 \pmod{5}$, then either $p_i \equiv -1 \pmod{5}$ or $p_i \equiv \pm 2 \pmod{5}$. Therefore, by (4.3) and (4.4), we get

$$\sigma(N) \equiv \prod_{\substack{p_i \equiv 2 \pmod{5} \\ e_i \text{ is odd}}} (2) \times \prod_{\substack{p_i \equiv -2 \pmod{5} \\ e_i \text{ is odd}}} (-2) \pmod{5}$$

$$\equiv 2^{k+k'} \cdot (-1)^{k'} \pmod{5}, \tag{4.5}$$

where $k = \#\{p_i^{2e_i} : p_i \equiv 2 \pmod{5}, e_i \text{ odd}\}$ and $k' = \#\{p_i^{2e_i} : p_i \equiv -2 \pmod{5}, e_i \text{ odd}\}$.

Also

$$2N + 1 \equiv 2 \prod_{i=1}^{r} (p_i^2)^{e_i} + 1 \equiv 2(-1)^{k+k'} + 1 \pmod{5}. \tag{4.6}$$
Now (4.5) and (4.6) imply that
\[ 2.(-1)^{k+k'} + 1 \equiv 2^{k+k'}.(-1)^{k'} \pmod{5}, \]  \hspace{1cm} (4.7)
which reduces to
\[ 2.(-1)^k + (-1)^{k'} \equiv 2^{k+k'} \pmod{5}, \]  \hspace{1cm} (4.8)
and this congruence is impossible for all choices of integers \( k \) and \( k' \), a contradiction, proving the theorem.

**Theorem 4.9.** If \( N \) is of the form (2.2) and \( (N, 3) = 1 \), then \( p_i \equiv 1 \pmod{3} \) for some \( i \).

**Proof.** If \( (N, 3) = 1 \) then \( p_i \equiv \pm 1 \pmod{3} \) for each \( i \) and since each \( p_i \) is odd it follows \( p_i \equiv \pm 1 \pmod{6} \) for each \( i \) so that \( p_i^2 \equiv 1 \pmod{6} \). Therefore,
\[ 2N + 1 \equiv 2 \prod_{i=1}^{r} (p_i^{2e_i} + 1) \equiv 3 \pmod{6} \]  \hspace{1cm} (4.10)
and for each \( i \),
\[ \sigma(p_i^{2e_i}) = (1 + p_i)(1 + p_i^2 + \ldots + p_i^{2e_i-2}) + p_i^{2e_i} \equiv (1 + p_i)e_i + 1 \pmod{6} \]
\[ \equiv \begin{cases} 
2e_i + 1 & \text{if } p_i \equiv 1 \pmod{6} \\
1 & \text{if } p_i \equiv -1 \pmod{6} 
\end{cases} \]

If possible, suppose no \( p_i \equiv 1 \pmod{6} \). Then
\[ \sigma(N) = \prod_{i=1}^{r} \sigma(p_i^{2e_i}) \equiv 1 \pmod{6} \]  \hspace{1cm} (4.11)
Now, by (4.10) and (4.11), we have
\[ 1 \equiv 3 \pmod{6} \]  \hspace{1cm} (4.12)
a contradiction. This proves the theorem.

Under certain stronger conditions we have a more general result given below:

**Theorem 4.13.** If \( N \) is of the form (2.2) and \( (N, m) = 1 \) for some odd \( m > 2 \) and if \( p_i \equiv \pm 1 \pmod{m} \) for all \( i \), then \( p_j \equiv 1 \pmod{m} \) for some \( j \) \((1 \leq j \leq r)\). Also if there is exactly one \( j \) with this property then \( e_j \equiv 1 \pmod{m} \).

**Proof.** Similar to the proof of Theorem 4.9 for the first part. If there is exactly one \( j \) with \( p_j \equiv 1 \pmod{m} \) then \( 2e_j + 1 \equiv 3 \pmod{m} \) giving \( e_j \equiv 1 \pmod{m} \) since \( m \) is odd.

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