A stratification of the moduli space of vector bundles on curves

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Dedicated to M. Knese

Introduction

Let $E$ be a vector bundle of rank 2 on a smooth projective curve $C$ of genus $g \geq 2$ over an algebraically closed field $K$ of arbitrary characteristic.

The invariant

$$s_1(E) := \deg E - 2 \max \deg(L),$$

where the maximum is taken over all line subbundles $L$ of $E$, is just the minimum of the self intersection numbers of all sections of the ruled surface $P(E) \to C$. Note that $E$ is stable (respectively semistable) if and only if $s_1(E) \geq 1$ (respectively $\geq 0$). According to a Theorem of C. Segre $s_1(E) \leq g$. Moreover, the function $s_1$ is lower semicontinuous. Thus $s_1$ gives a stratification of the moduli space $\mathcal{M}(2, d)$ of stable vector bundles of rank 2 and degree $d$ on $C$, into locally closed subsets $\mathcal{M}(2, d, s)$ according to the value $s$ of $s_1(E)$.

It is shown in [8] that for $s > 0$, $s \equiv d \mod 2$ the algebraic variety $\mathcal{M}(2, d, s)$ is non-empty, irreducible and of dimension

$$\dim \mathcal{M}(2, d, s) = \begin{cases} 3g + s - 2 & s \leq g - 2 \\ 4g - 3 & s \geq g - 1 \end{cases}$$

(A)

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Let $M_1(E)$ denote the set of line subbundles of $E$ of maximal degree. The set $M_1(E)$ can be considered as an algebraic scheme in a natural way. Maruyama proved in [10] (see also [8] for different proofs) the following statement:

For a general vector bundle $E$ in $M(2, d, s)$:

$$\dim M_1(E) = \begin{cases} 1 & s = g \\ 0 & s \leq g - 1 \end{cases}$$

It is the aim of the present paper, to generalize statements (A) and (B) to vector bundles of arbitrary rank $r \geq 2$.

Let now $E$ be a vector bundle of rank $r \geq 2$ over $C$. For any integer with $1 \leq k \leq r - 1$ we define

$$s_k(E) := k \deg E - r \max \deg F.$$ 

where the maximum is taken over all subbundles $F$ of rank $k$ of $E$. There is also an interpretation of the invariant $s_k(E)$ in terms of self intersection numbers in the ruled variety $P(E)$ (see [3]). According to a theorem of Hirschowitz (see [4])

$$s_k(E) \leq k(r - k)(g - 1) + (r - 1).$$

Moreover, the function $s_k$ is lower semicontinuous. Thus $s_k$ gives a stratification of the moduli space $M(r, d)$ of stable vector bundles of rank $r$ and degree on $d$ on $C$ into locally closed subsets $M(r, d, k, s)$ according to the value of $s$ and $k$. There is a component $M^0(r, d, k, s)$ of $M(r, d, k, s)$ distinguish by the fact that a general $E \in M^0(r, d, k, s)$ admits a stable subbundle $F$ such that $E/F$ is also stable.

In this paper we prove (see Theorem 4.2):

For $g \geq \frac{r+1}{2}$ and $0 < s \leq k(r - k)(g - 1) + (r + 1)$, $s \equiv kd \mod r$, $M^0(r, d, k, s)$ is non-empty, and its component $M^0(r, d, k, s)$ is of dimension

$$\dim M^0(r, d, k, s) = \begin{cases} (r^2 + k^2 - rk)(g - 1) + s - 1 & s < k(r - k)(g - 1) \\ r^2(g - 1) + 1 & s \geq k(r - k)(g - 1) \end{cases}$$

The bound $g \geq \frac{r+1}{2}$ works for all $k, 1 \leq k \leq r - 1$ simultaneously. For some special $k$ the result is better. For example, for $k = 1$ or $r - 1$ statement (C) is valid for all $g \geq 2$ (see Remark 3.3).
Let $M_k(E)$ denote the set of maximal subbundles of rank $k$ of $E$ and $\overline{M}_k(E)$ denote the closure of the set of stable subbundles $F$ of rank $k$ of $E$ of maximal degree such that $E/F$ is also stable in $M_k(E)$. Also $M_k(E)$ can be considered as an algebraic scheme in a natural way and $\overline{M}_k(E)$ is a union of components of $M_k(E)$.

Theorem 4.4 below says:

For a general vector bundle $E$ in $\mathcal{M}^0(r, d, k, s)$ we have

$$\dim M_k(E) = \max(s - k(r - k)(g - 1), 0)$$

again under the hypothesis $g \geq \frac{r+1}{2}$.

Note that (C) and (D) are direct generalizations of (A) and (B).

The main difficulty in the proofs is to show that $\mathcal{M}(r, d, k, s)$ is nonempty. (see Theorem 3.2 below). This was shown in [4] and [6] for $s \geq k(r - k)(g - 1)$ i.e. in the generic case and for $s \leq \frac{k(r-k)(g-1)}{\max(k,r-k)}$ in [4]. For a generic curve of genus $g \geq 2$ this was proven by M. Teixidor in [16]. Special cases were also considered by B. Russo in [14], (see also [2]) and M. Teixidor in [15]. The idea of proof is to start with a general vector bundle $E_0$ with $s_k(E_0) \geq k(r - k)(g - 1)$, then construct a sequence of elementary transformations to find a vector bundle in $\mathcal{M}(r, d, k, s)$. Statement (D) is an easy consequence of statement (C).

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1 The invariants $s_k(E)$

Let $C$ be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field $K$ of arbitrary characteristic. and let $E$ denote a vector bundle of rank $r \geq 2$ over $C$. For any integer $k$ with $1 \leq k \leq r - 1$ let $Sb_k(E)$ denote the set of subbundles of rank $k$ of $E$. If we denote by $\xi$ the generic point of the curve $C$, then it is easy to see that there is a canonical bijection between $Sb_k(E)$ and the set of $k$-dimensional subvector spaces of the $K(\xi)$-vector space $E(\xi)$. For any subbundle $F \in Sb_k(E)$ define
the integer $s_k(E, F)$ by

$$s_k(E, F) := k \deg E - r \deg F.$$ 

The vector bundle $E$ does not admit subbundles of arbitrarily high degree. Hence

$$s_k(E) := \min_{F \in S_{b_k}(E)} s_k(E, F)$$

is a well defined integer depending only on $E$ and $k$.

**Remark 1.1** The slope of a vector bundle $F$ on $C$ is defined as $\mu(F) = \frac{\deg F}{rk F}$. If $F$ is a subbundle of rank $k$ of $E$ then

$$s_k(E, F) = k(r - k) (\mu(E/F) - \mu(F)).$$

In particular

$$s_k(E) = k(r - k) \cdot \min_{F \in S_{b_k}(E)} (\mu(E/F) - \mu(F))$$

So instead of the invariant $s_k(E)$ one could also work with the invariant

$$\min_{F \in S_{b_k}(E)} (\mu(E/F) - \mu(F)).$$

However, for some proofs it is more convenient to work with integers. Note that there is also a geometric interpretation of the invariant $s_k(E)$ in terms of intersection numbers on the associated projective bundle $P(E)$ (see [9]).

**Remark 1.2** The following properties of the invariant $s_k(E)$ are easy to see (see [3])

(a) $s_k(E \otimes L) = s_k(E)$ for all $L \in Pic(C)$.

(b) $s_k(E) = s_{r-k}(E^*)$.

(c) $E$ is stable (respectively semistable) if and only if $s_k(E) > 0$ (respectively $s_k(E) \geq 0$) for all $1 \leq k \leq r - 1$.

(d) Let $T$ be an algebraic scheme over $K$ and $\mathcal{E}$ a vector bundle of rank $r$ on $C \times T$. For any point $t \in T$, let $\bar{t}$ denote a geometric point over $t$. The function $s_k : T \to \mathbb{Z}$ defined as $t \mapsto s_k(\mathcal{E}|_{C \times \{t\}})$ is well defined and lower semicontinuous.

Whereas the function $s_k$ may take arbitrarily negative values (for suitable direct sums of line bundles). However, it is shown in [12] and [4] that $s_k(E) \leq k(r - k)g$. Hirschowitz gives in [4] the better bound,

$$s_k(E) \leq k(r - k)(g - 1) + (r - 1).$$
We want to study the behaviour of the invariant $s_k(E)$ under an elementary transformation of the vector bundle $E$. Recall that an elementary transformation $E'$ of $E$ is defined by an exact sequence

$$0 \to E' \to E \xrightarrow{\ell} K(x) \to 0$$

where $K(x)$ denotes the skyscraper sheaf with support $x \in C$ and fibre $K$. Since $\ell$ factorizes uniquely via a $k$-linear form $E(x) \xrightarrow{\ell} K(x)$, also denoted by $\ell$, the set of elementary transformations of $E$ is parametrized by pairs $(x, \ell)$ where $x$ is a closed point of $C$ and $\ell$ is a linear form on the vector space $E(x)$.

So the set of elementary transformations of $E$, which we denote by $elm(E)$, forms a vector bundle of rank $r$ over the curve $C$. Note that for any $E' \in elm(E)$

$$\text{rk}(E') = \text{rk}(E) \quad \text{and} \quad \text{deg} E' = \text{deg} E - 1.$$

**Lemma 1.3** For any $E' \in elm(E)$ the map $\varphi : Sb_k(E) \to Sb_k(E')$ defined by $F \mapsto F \cap E'$ is a bijection.

**Proof.** For the proof only note that the inverse map is given as follows: Suppose $F' \in Sb_k(E')$. Consider $F'$ as a subsheaf of $E$ and let $F$ denote the subbundle of $E$ generated by $F'$. The map $F' \mapsto F$ is inverse to $\varphi$. \qed

Now consider a subbundle $F \in Sb_k(E)$ and denote $F' = \varphi(F) \in Sb_k(E')$. In order to compute the number $s_k(E', F')$ we have to distinguish two cases. We say that the subbundle $F$ is of type I with respect to $E'$ if $F \subseteq E'$ and $F$ is of type II with respect to $E'$ otherwise. Let $(x, \ell) \in elm(E)$ denote the pair defining the elementary transformation $E'$ of $E$. We obviously have

**Lemma 1.4** The subbundle $F \in Sb_k(E)$ is of type I with respect to $E'$ if and only if the linear form $\ell : E(x) \to K(x)$ vanishes on the subvector space $F(x)$ of $E(x)$.

**Lemma 1.5** If $E'$ is an elementary transformation of $E$, $F \in Sb_k(E)$ and $F' = \varphi(F)$ then,

1. $s_k(E', F') = s_k(E, F) - k$ if $F$ is of type I with respect to $E'$
2. $s_k(E', F') = s_k(E, F) + (r - k)$ if $F$ is of type II with respect to $E'$. 

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Proof. If $F$ is of type I, we have the following diagram

$$
\begin{array}{cccc}
0 & 0 & \downarrow & \\
\downarrow & \downarrow & \downarrow & \\
0 & F' & E' & E' / F' & 0 \\
\| & \downarrow & \downarrow & \\
0 & F & E & E / F & 0 \\
\downarrow & \downarrow & K(x) = K(x) \\
\downarrow & \downarrow & \\
0 & 0 & 
\end{array}
$$

Hence $s_k(E', F') = k(\deg E - 1) - r(\deg F - 1) = s_k(E, F) - k$. If $F$ is type II we have the following diagram

$$
\begin{array}{cccc}
0 & 0 & \downarrow & \\
\downarrow & \downarrow & \downarrow & \\
0 & F' & E' & E' / F' & 0 \\
\downarrow & \downarrow & \| & \\
0 & F & E & E / F & 0 \\
\downarrow & \downarrow & K(x) = K(x) \\
\downarrow & \downarrow & \\
0 & 0 & 
\end{array}
$$

Hence $s_k(E', F') = k(\deg E - 1) - r(\deg F - 1) = s_k(E, F) + (r - k)$. \hfill \Box

A maximal subbundle $F \in Sb_k(E)$ is by definition a subbundle of rank $k$ of maximal degree of $E$. Note that $F \in Sb_k(E)$ is a maximal subbundle if and only if $s_k(E) = s_k(E, F)$.

An elementary transformation $E'$ of $E$ will be called of $k$-type I if $E$ admits a maximal subbundle of rank $k$ which is of type I with respect to $E'$. Otherwise $E'$ will be called of $k$-type II.

**Proposition 1.6** If $E'$ is an elementary transformation of $E$, then

$$
\begin{align*}
\quad \quad s_k(E') = \begin{cases} 
    s_k(E) - k & \text{if } E' \text{ is of } k\text{-type I} \\
    s_k(E) + (r - k) & \text{if } E' \text{ is of } k\text{-type II}.
\end{cases}
\end{align*}
$$

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**Proof.** Let $F \in Sb_k(E)$ and $F' = \varphi(F) \in Sb_k(E')$. Suppose first $E'$ is of $k$-type I. If $F$ is maximal and of type I with respect to $E'$, then $s_k(E', F') = s_k(E) - k$. If $F$ is maximal and of type II, then $s_k(E', F') = s_k(E) + (r - k)$. If $F$ is not maximal, then $\deg F \leq \frac{1}{k}(\deg E - s_k(E)) - 1$ and so

$$s_k(E, F) \geq k \deg E - r \deg F \geq s_k(E) + r \tag{2}$$

Hence $s_k(E', F') \geq s_k(E, F) - k \geq s_k(E) + (r - k)$. This implies the assertion if $E'$ is of $k$-type I.

Suppose now $E'$ is of $k$-type II. Any maximal subbundle $F \subset E$ is by assumption of type II with respect to $E'$. Hence according to Lemma 1.5 $s_k(E', F') = s_k(E) + (r - k)$. If $F$ is not maximal, then (2) and Lemma 1.5 imply

$$s_k(E', F') \geq s_k(E, F) - k \geq s_k(E) + (r - k).$$

\[\square\]

**Remark 1.7** One immediately deduces from the proof of Proposition 1.6:

(i) If $E'$ is of $k$-type I, then the maximal subbundles of rank $k$ of $E'$ are exactly the maximal subbundles of rank $k$ of $E$ which are of type I with respect to $E'$.

(ii) If $E'$ is of $k$-type II, then the maximal subbundles of rank $k$ of $E'$ are exactly the subbundles $F' = \varphi(F)$, where $F \in Sb_k(E)$ is either maximal or of degree one less than the degree of a maximal subbundle and of type I with respect to $E'$.

Dualizing the exact sequence (1) we obtain an exact sequence

$$0 \to E^* \to E'^* \to K(x) \to 0$$

Hence $E^*$ is an elementary transformation of $E'^*$, called the dual elementary transformation.

**Corollary 1.8** For an elementary transformation $E'$ of $E$ the following conditions are equivalent

(i) $E'$ is of $k$-type I.

(ii) The dual elementary transformation $E^*$ of $E'^*$ is of $(r - k)$-type II.
Proof: According to Proposition 1.6 and Remark 1.7, (i) holds if and only if \( s_k(E') = s_k(E) - k \). But \( s_{r-k}(E^*) = s_k(E) \) (see Remark 1.2, (b)). Hence (i) holds if and only if \( s_{r-k}(E^*) = s_{r-k}(E^*) - k \) i.e. if and only if \( s_{r-k}(E^*) = s_{r-k}(E^*) + r - (r - k) \). Applying Proposition 1.6 again gives the assertion.

2 Maximal subbundles

Let \( E \) denote a vector bundle of rank \( r \) and degree \( d \) on the curve \( C \). In this section we study the set \( M_k(E) \) of maximal subbundles of rank \( k \) of \( E \). Let \( d_k \) denote the common degree of the maximal subbundles of rank \( k \) of \( E \). The following lemma shows that \( M_k(E) \) admits a natural structure of a projective scheme over \( K \). Denote by \( Q := \text{Quot}_{E}^{r-k,d-d_k} \) the Quot scheme of coherent quotients of rank \( r - k \) and degree \( d - d_k \) of \( E \).

Lemma 2.1 There is a canonical identification of \( M_k(E) \) with the set of closed points of \( Q \).

Proof. If \( F \in M_k(E) \), then \( E \to E/F \) gives a closed point of \( Q \). On the other hand if \( E \to G \to 0 \) corresponds to a closed point of \( Q \), then \( F = \ker p \in M_k(E) \).

Let \( G \) denote the universal quotient sheaf on \( C \times Q \). The maximality condition implies that \( G \) is locally free. Hence if \( \text{Grass}_{r-k}(E) \to C \) denotes the Grassmanian scheme of \((r-k)\)-dimensional quotient vector spaces of the fibres \( E(x) \) and \( p^*E \to U \to 0 \) the universal quotient on \( \text{Grass}_{r-k}(E) \), then any \( F \in M_k(E) \) corresponds on the one hand to a closed point \( t \) of \( Q \) and on the other hand to a section \( \sigma_t : C \to \text{Grass}_{r-k}(E) \). This leads to a morphism

\[
\phi : \begin{cases} 
C \times Q \to \text{Grass}_{r-k}(E) \\
(x, t) \mapsto \sigma_t(x)
\end{cases}
\]

with the property that \( G = \phi^*U \).

Lemma 2.2 The morphism \( \phi : C \times Q \to \text{Grass}_{r-k}(E) \) is finite.

For a proof we refers to [12] or [9], Lemma 3.9.

The geometric interpretation of Lemma 2.2 is
Corollary 2.3 Let $x \in C$ and $V \subset E(x)$ and $k$-dimensional subvector space. There are most finitely many maximal subbundles $F$ of rank $k$ of $E$ such that $F(x) = V$.

Corollary 2.4 $\dim M_k(E) \leq k(r-k)$.

Proof: From Lemma 2.1 there is a canonical identification of $M_k(E)$ with the set of closed points of $Q$. Since $\phi$ is finite, we have that $\dim Q \leq \dim \text{Grass}_{r-k}(E) - \dim C = k(r-k) + 1 - 1$. $\square$

Assume now that $\dim M_k(E) = n$, where $n \leq k(r-k)$ according to Corollary 2.4, and let $E'$ be an elementary transformation of $E$. We want to estimate $\dim M_k(E')$. Suppose that $E'$ corresponds to the pair $(x, \ell)$ with exact sequence (1) of Section 1. For any $(r-1)$-dimensional subvector space $V$ of the vector space $E(x)$ consider the Schubert cycle

$$\sigma_k(V) := \{F \in \text{Grass}_{r-k}(E(x)) | F \subset V\}.$$

Proposition 2.5 If $E'$ is of $k$-type I, then $\dim M_k(E') \geq \dim M_k(E) - k$.

Proof. Denote $V = \ker(\ell : E(x) \rightarrow K)$. According to Remark 1.7, there is a canonical identification

$$M_k(E') = \{F \in M_k(E) | F(x) \subset V\}.$$ 

Defining

$$M_k(E)(x) := \{F(x) | F \in M_k(E)\},$$

we have from Corollary 2.3

$$\dim M_k(E') = \dim M_k(E')(x)$$

$$= \dim (M_k(E)(x) \cap \sigma_k(V))$$

$$\geq \dim M_k(E)(x) + \dim \sigma_k(V) - \dim \text{Grass}_{r-k}(E(x))$$

$$= n + k(r - 1 - k) - k(r - k) = n - k.$$ $\square$

This completes the proof of the proposition. $\square$

Proposition 2.6 If $E' \in \text{elm}(E)$ is of $k$-type II, then $\dim M_k(E') \leq \dim M_k(E) + r - k$. 

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Proof. According to Corollary 1.8, the dual elementary transformation $E^*$ of $E'^*$ is of $(r - k)$-type I. Hence by Proposition 2.5

$$\dim M_{r-k}(E^*) \geq \dim M_{r-k}(E'^*) - (r - k).$$

But dualizing induces a canonical isomorphism $M_k(E) \sim \to M_{r-k}(E^*)$ and similarly for $E'$. This completes the assertion. \hfill \Box

3 Stable extensions

Suppose now that the genus $g$ of the curve $C$ is $\geq r + 1$. Let $r, d, k$ and $s$ be integers with $r \geq 2$, $1 \leq k \leq r - 1$, $0 < s \leq k(r - k)(g - 1) + (r + 1)$ and $s \equiv kd \mod r$. The aim of this section is the proof of the following theorem.

**Theorem 3.1** There exists an extension $0 \to F \to E \to G \to 0$ of vector bundles on the curve $C$ with the following properties

(i) $rk E = r$, deg $E = d$.

(ii) $F$ is a maximal subbundle of rank $k$ of $E$ with $s_k(E, F) = s$.

(iii) $E, F$ and $G$ are stable.

Let $d_1$ be the unique integer with $s = kd - rd_1$ and $d_2 = d - d_1$. According to [13] Proposition 2.4 there are finite étale coverings

$$\pi_1 : \tilde{M}_1 \to \mathcal{M}(k, d_1) \quad \text{and} \quad \pi_2 : \tilde{M}_2 \to \mathcal{M}(r - k, d_2)$$

such that there are vector bundles $\mathcal{F}_i$ on $C \times \tilde{M}_i$ whose classifying map is just $id \times \pi_i$ for $i = 1, 2$. Let $p_{ij}$ denote the canonical projections of $C \times \tilde{M}_1 \times \tilde{M}_2$ for $i, j = 0, 1, 2$. According to [4], Lemma 4.1 the sheaf $R^1p_{12*}(p_{02}^*\mathcal{F}_2^* \otimes p_{01}^*\mathcal{F}_1)$ is locally free of rank $k(r - k)(g - 1) + s$ on $\tilde{M}_1 \times \tilde{M}_2$. Let

$$\pi : P := P(R^1p_{12*}(p_{02}^*\mathcal{F}_2^* \otimes p_{01}^*\mathcal{F}_1)) \to \tilde{M}_1 \times \tilde{M}_2$$

denote the corresponding projective bundle. According to [4], Corollary 4.5 there is an exact sequence

$$0 \to \pi^*p_{01}^*\mathcal{F}_1 \otimes \mathcal{O}_P(1) \to \mathcal{E} \to \pi^*p_{02}^*\mathcal{F}_2 \to 0 \quad (3)$$
on $C \times P$, universal in a sense which is outlined in that paper. In particular this means that for every closed point $q \in P$ the restriction of the exact sequence (3) to $C \times \{q\}$ is just the extension of $\mathcal{F}_2|_{C \times \{p_2(q)\}}$ by $\mathcal{F}_1|_{C \times \{p_1(q)\}}$ modulo $K^*$, which is represented by the point $q$. Here $p_i : P \to \tilde{M}_i$ denotes the canonical map.

With $r, k, d$ and $s$ as above consider the set

$$U(r, d, k, s) := \{q \in P : \mathcal{E}|_{C \times \{q\}} \text{ is stable with } s_k(\mathcal{E}|_{C \times \{q\}}) = s\}$$

From the lower semicontinuity of the function $s_k$ and stability being an open condition we deduce that the set $U(r, d, k, s)$ is an open subset of $P$. Hence, Theorem 3.1 is equivalent to the following theorem.

**Theorem 3.2** For any $r, k, d$ and $s$ as above the set $U(r, d, k, s)$ is nonempty.

**Proof.** It suffices to show that the set

$$U(r, d, k, s, i) := \{q \in P : s_i(\mathcal{E}|_{C \times \{q\}}) > 0 \text{ and } s_k(\mathcal{E}|_{C \times \{q\}}) = s\}$$

is nonempty for any $i = 1, ..., r - 1, i \neq k$, since

$$U(r, d, k, s) = \bigcap_{i=1}^{r-1} U(r, d, k, s, i)$$

and the function $s_i$ is lower semicontinuous. According to Remark 1.2 (b) dualization gives a canonical bijection

$$U(r, d, k, s, i) \sim U(r, -d, r - k, s, r - i)$$

Hence it suffices to show that

$$U(r, d, k, s, i) \neq \emptyset,$$

for all $r, d, k, s$ as above and all $1 \leq i \leq k - 1$. Choose a positive integer $N_k$ such that

$$k(r - k)(g - 1) \leq s + N_k k \leq k(r - k)(g - 1) + r - 1$$

and denote $\tilde{d} := d + N_k$.

We call a vector bundle $E$ out of the moduli space $\mathcal{M}(r, \tilde{d})$ general, if for all $0 < j < r$ the number $s_j(E)$ takes a maximal value, say $s_{j,\max}$. By the semicontinuity of the
function $s_j$ the set of general vector bundles is open and dense in $M(r, \bar{d})$. According to a theorem of Hirschowitz (see [4], Théorème p. 153):

$$s_{j, \text{max}} = j(r-j)(g-1) + \epsilon_j$$

where $\epsilon_j$ is the unique integer with $0 \leq \epsilon_j \leq r - 1$ such that $j(r-j)(g-1) + \epsilon_j \equiv j\bar{d} \mod r$. Moreover, it is shown in [6] (p. 458), that $U(r, \bar{d}, k, s_{k, \text{max}})$ is non-empty and its image is open and dense in $M(r, \bar{d})$ for all $r, \bar{d}$ and $k$.

Let $0 \to F_0 \to E_0 \to G_0 \to 0$ be an exact sequence corresponding to a general point in $U(r, \bar{d}, k, s_{k, \text{max}})$. Then $E_0, F_0$ and $G_0$ respectively are general vector bundles in $\mathcal{M}(r, \bar{d}), \mathcal{M}(k, d_k)$ and $\mathcal{M}(r-k, \bar{d}-d_k)$ respectively, with $d_k = \frac{1}{r}(k\bar{d} - s_{k, \text{max}})$ and $s_k(E_0, F_0) = s_{k, \text{max}}$. Choose inductively for any $\nu = 1, \ldots, N_k$ an elementary transformation $E_\nu$ of $k$-type $I$ of $E_{\nu-1}$.

In order to complete the proof of Theorem 3.2 it suffices to show that

$$E_{N_k} \in U(r, d, k, s, i)$$

But

$$s_k(E_{N_k}) = s_k(E_0) - N_k i = s_{k, \text{max}} - N_k i = s$$

and

$$s_i(E_{N_k}) \geq s_i(E_0) - N_k i \quad \text{(by Proposition 1.6)}$$

$$\geq i(r-i)(g-1) - \frac{i}{k}(k(r-k)(g-1) - s + r - 1)$$

(since $E_0$ is general and using (4).)

$$\geq i(k-i)(g-1) - \frac{i}{k}(r-2) \quad \text{(since } s \geq 1)$$

$$> 0 \quad \text{(since } g \geq \frac{r+1}{2} \text{ by assumption)}$$

Remark 3.3  
(a) The assumption on the genus $g$ in Theorems 3.1 and 3.2 is imposed by the last line in the proof of Theorem 3.2. The bound $g \geq \frac{r+1}{2}$ works for any $r$, for all $k, 1 \leq k \leq r - 1$ simultaneously. If one fixes also $k$, the bound is slightly better. In fact, if $k = 1$ or $r - 1$, the proof shows that Theorems 3.1 and 3.2 are valid for any $g \geq 2$. (For the proof note that in both cases using duality one only has to check that $i(r - 1 - i)(g - 1) - \frac{i}{r-1}(r-2) > 0$ for all $1 \leq i \leq r - 2$. But this is valid for all $g \geq 2$.) For $2 \leq k \leq r - 2$ denote $k = \frac{r+1}{2}$ with $0 \leq n \leq r - 4$. Then the Theorems are valid for any $g \geq 3 + 2\frac{n-1}{r-n}$.

(b) There is a modification of the proof, for which the bound for $g$ is also slightly better. The duality can also be used to reduce the proof to the case $k \geq \frac{r}{2}$. Then
one has also to show that $s_i(E_{N_k}) > 0$ for $k < i < r$. For this one has to choose the sequence of bundles $E_0, E_1, ..., E_{N_k}$ more carefully: Whenever possible one should use an elementary transformation of $k$-type I which is of $i$-type II.

4 Stratification of $\mathcal{M}(r, d)$ according to the invariant $s_k$

The function $s_k : \mathcal{M}(r, d) \to \mathbb{Z}$ defined by $E \mapsto s_k(E)$ is lower semicontinuous and this induces a stratification of the moduli space $\mathcal{M}(r, d)$ into locally closed subvarieties

$$\mathcal{M}(r, d, k, s) := \{ E \in \mathcal{M}(r, d) : s_k(E) = s \}$$

according to the value $s$ of $s_k$. It is not clear (to us) whether $\mathcal{M}(r, d, k, s)$ is irreducible or consists of several components. Consider the natural map

$$\phi : U(r, d, k, s) \to \mathcal{M}(r, d, k, s) \subset \mathcal{M}(r, d).$$

As an image of an irreducible variety $\text{Im} \phi$ is irreducible. Let $\mathcal{M}^0(r, d, k, s)$ denote the Zariski closure of $\text{Im} \phi$ in $\mathcal{M}(r, d, k, s)$.

**Lemma 4.1** $\mathcal{M}^0(r, d, k, s)$ is an irreducible component of $\mathcal{M}(r, d, k, s)$, if is nonempty.

**Proof:** According to [13], Proposition 2.4 there is a finite étale covering $\pi : \tilde{M} \to \mathcal{M}(r, d, k, s) \subset \mathcal{M}(r, d)$ and a vector bundle $\mathcal{E}$ on $C \times \tilde{M}$ such that $id \times \pi$ is just the classifying map. Let $Q_{\mathcal{E}}$ denote the Quot scheme of $\mathcal{E}$ and

$$0 \to \mathcal{F} \to p^* \mathcal{E} \to \mathcal{G} \to 0$$

the universal exact sequence on $C \times \tilde{M} \times Q_{\mathcal{E}}$. Here $p : C \times \tilde{M} \times Q_{\mathcal{E}} \to C \times \tilde{M}$ denotes the projection map. Certainly there are finitely many components of $Q_{\mathcal{E}}$, the union of which we denote by $Q_{\mathcal{E}}^{r-k,d-d_h}$, such that for all closed points $(e, x) \in \tilde{M} \times Q_{\mathcal{E}}^{r-k,d-d_h}$ the restriction $\mathcal{F}_{|C \times \{(e, x)\}}$ is a maximal subbundle of rank $k$ and degree $d_k = \frac{1}{r}(s - kd)$ of $E = \mathcal{E}_{|C \times \{(e, x)\}}$ and moreover every maximal subbundle occurs as a restriction of the exact sequence (5) to $C \times \{(e, x)\}$ for some $(e, x) \in \tilde{M} \times Q_{\mathcal{E}}^{r-k,d-d_h}$. As in section 2, the maximality condition implies that $\mathcal{F}$ and $\mathcal{G}$ are vector bundles. Since stableness is an open condition, it follows that the set of points $(e, x) \in \tilde{M} \times Q_{\mathcal{E}}^{r-k,d-d_h}$ such that $\mathcal{F}_{|C \times \{(e, x)\}}$ and $\mathcal{G}_{|C \times \{(e, x)\}}$ are stable consists of whole components of $\tilde{M} \times Q_{\mathcal{E}}^{r-k,d-d_h}$.

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Hence the closure of the set of points $E \in \mathcal{M}(r, d, k, s)$ which admits a stable maximal subbundle $F$ of rank $k$ such that $E/F$ is also stable, consists also of whole components of $\mathcal{M}(r, d, k, s)$. But since the set of such points $E$ of $\mathcal{M}(r, d, k, s)$ is just the irreducible set $\text{Im} \Phi$, this implies the assertion. \hfill \square

It would be interesting to give an example for which $\mathcal{M}^0(r, d, k, s) \neq \mathcal{M}(r, d, k, s)$. For an example where $\text{Im} \phi \neq \mathcal{M}(r, d, k, s)$ see Remark 4.5 below. The following theorem gives us the dimension of $\mathcal{M}^0(r, d, k, s)$.

**Theorem 4.2** Let $r, d, k,$ and $s$ be integers with $r \geq 2$, $1 \leq k \leq r-1$, $1 \geq s \geq k(r-k)(g-1 + (r-2)$ and $s \equiv kd \text{ mod } r$. Suppose the genus of $C$ is $g \geq \frac{r+1}{2}$. Then $\mathcal{M}^0(r, d, k, s)$ is a non-empty algebraic variety with

$$\dim \mathcal{M}^0(r, d, k, s) = \begin{cases} (r^2 + k^2 - rk)(g-1) + s + 1 & \text{if } s < k(r-k)(g-1) \\ r^2(g-1) + 1 & \text{if } s \geq k(r-k)(g-1) \end{cases}$$

**Proof:** If $s \geq k(r-k)(g-1)$, then $s = k(r-k)(g-1) + \epsilon_k$ where $\epsilon_k$ is the unique integer with $0 \leq \epsilon_k \leq r-1$ and $s \equiv kd \text{ mod } r$. Then $\text{Im} \phi$ is open and dense in $\mathcal{M}(r, d)$, which gives the assertion in this case (see [6]).

So we may assume that $s < k(r-k)(g-1)$. Consider the open set $U(r, d, k, s)$ in the variety $P = P(R_{p_2}^1, (p_{02}^* F_2 \otimes p_{01}^* F_1)^*)$ of Section 3. According to Theorem 3.2 $U(r, d, k, s)$ is non-empty, open and dense in $P$. According to the definitions of $U(r, d, k, s)$ and $s_k$ the natural map

$$\phi : U(r, d, k, s) \longrightarrow \mathcal{M}^0(r, d, k, s) \subseteq \mathcal{M}(r, d)$$

is dominant. We have to compute the dimension of $\mathcal{M}(r, d, k, s)$.

Let $q \in U(r, d, k, s)$ be a general closed point. If $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ denotes the corresponding exact sequence, then $\phi(q) = E$ and [3], Lemma 4.2 says that

$$\dim \phi^{-1}(E) \leq h^0(F^* \otimes G).$$

On the other hand, $F$ and $G$ are general vector bundles in their corresponding moduli spaces. Hence according to [3] Théorème 4.6, the vector bundle $F^* \otimes G$ is non-special, implying

$$h^0(F^* \otimes G) = 0$$
since \( \deg (F^* \otimes G) = s < k(r-k)(g-1) = \text{rk}(F^* \otimes G)(g-1) \). Hence the generic fibre of \( \phi \) is finite and thus

\[
\dim \mathcal{M}^0(r, d, k, s) = \dim U(r, d, k, s) = \dim \mathbf{P}(R^1p_{12*}(p_{02}^*\mathcal{F}_2^* \otimes p_{01}^*\mathcal{F}_1)^*)) = \dim \mathcal{M}(k, d_1) + \dim \mathcal{M}(r-k, d-d_1) + k(r-k)(g-1) + s - 1
\]

with \( d_1 = \deg F = \frac{1}{r}(kd - s) \). Note that \( \mathbf{P}(R^1p_{12*}(p_{02}^*\mathcal{F}_2^* \otimes p_{01}^*\mathcal{F}_1)^*)) \) is a projective bundle of rank \( k(r-k)(g-1) + s - 1 \) over a finite covering of \( \mathcal{M}(k, d_1) \times \mathcal{M}(r-k, d-d_1) \) (see [2], p. 455). Hence

\[
\dim \mathcal{M}^0(r, d, k, s) = k^2(g-1) + 1 + (r-k)^2(g-1) + 1 + k(r-k)(g-1) + s - 1 = (r^2 + k^2 - rk)(g-1) + s + 1. \quad \square
\]

The fibres \( \phi^{-1}(E) \) of the map \( \phi : U(r, d, k, s) \to \mathcal{M}^0(r, d, k, s) \) of the proof of Theorem 4.2 are exactly the sets of stable maximal subbundles of \( E \), whose quotient is also stable. However, maximal subbundles are not necessarily stable. Noting that a maximal subbundle of a maximal subbundle of \( E \) is also a subbundle of \( E \) (and similarly for quotient bundles) one easily shows

**Proposition 4.3** Suppose \( E \in \mathcal{M}(r, d) \) with \( s_k(E) = s \) for some \( 1 \leq k \leq r - 1 \). Let \( F \) be a maximal subbundle of rank \( k \) of \( E \). Then

(i) \( s_\nu(F) \geq \frac{1}{r}(k - \nu s) \) for all \( 1 \leq \nu \leq k - 1 \)

(ii) \( s_\nu(E/F) \geq \frac{1}{r}((r-k)-(r-k-\nu)s) \) for all \( 1 \leq \nu \leq r-k-1 \).

So in particular for highs \( s_\nu(F) \) or \( s_\nu(E/F) \) might be very negative. According to Lemma 2.1 there is a canonical identification of the set \( M_k(E) \) of maximal subbundles of \( E \) with the Quot-scheme \( Q = \text{Quot}_{E}^{r-k, d+(r-k)d+s} \). Hence there is a universal subbundle \( \mathcal{F} \) of \( p^*E \) on the scheme \( C \times M_k(E) \), where \( p : C \times M_k(E) \to C \) denotes the projection map. Denote

\[
\overline{M}_k(E) := \{ F \in M_k(E) | F \text{ and } E/F \text{ stable} \}.
\]

and by \( \overline{M}_k(E) \) the Zariski closure of \( \overline{M}_k(E) \) in \( M_k(E) \). Applying the openness of stability to the universal subbundle \( \mathcal{F} \) and the universal quotient bundle \( p^*E/F \) of \( p^*E \) one deduces that \( \overline{M}_k(E) \) consists of whole irreducibility components of \( M_k(E) \),
namely exactly of those components of $M_k(E)$ which contain a stable subbundle $F$ of $E$ such that $E/F$ is also stable. Moreover $\tilde{M}_k(E)$ is open in $\hat{M}_k(E)$.

By definition we may canonically identify

$$\tilde{M}_k(E) \to \phi^{-1}(E).$$

This implies that

$$\dim \hat{M}_k(E) = \dim \phi^{-1}(E). \quad (6)$$

and we may use the map $\phi$ to compute the dimension of $\hat{M}_k(E)$. Let $r, d, k$ and $s$ be integers as above and $g(C) \geq \frac{r+1}{2}$. According to Theorem 4.2 the variety $M^0(r, d, k, s)$ is non empty.

**Theorem 4.4** For a general vector bundle $E$ in $M^0(r, d, k, s)$ we have $\dim \hat{M}_k(E) = \max(s - k(r - k)(g - 1), 0)$.

In particular, if $E$ is general in $M(r, d)$ there is a unique integer $\epsilon_k$ with $0 \leq \epsilon_k \leq r - 1$ and $k(r - k)(g - 1) + \epsilon_k \equiv kd \mod r$ and we have

$$\dim \hat{M}_k(E) = \epsilon_k$$

If $s$ is not maximal value, i.e. $s < k(r - k)(g - 1)$, then a general vector bundle in $M^0(r, d, k, s)$ admits only finitely many stable maximal subbundles such that $E/F$ is also stable.

**Proof of Theorem 4.4:** The natural map $\phi : U(r, d, k, s) \to M^0(r, d, k, s)$ is a dominant morphism of algebraic varieties by the definition of $U(r, d, k, s)$ and $M^0(r, d, k, s)$. Let $q \in U(r, d, k, s)$ be a general point and $0 \to F \to E \to G \to 0$ be the corresponding exact sequence. According to [6] Lemma 4.2 and Hirschowitz’ Theorem (see [5] Théorème 4.6) we have

$$\dim \phi^{-1}(E) \leq h^0(F^* \otimes G) \leq \max(s - k(r - k)(g - 1), 0)$$

So equation (6) implies the assertion for $s \leq k(r - k)(g - 1)$. For $s > k(r - k)(g - 1)$ it suffices to show that the local dimension of $\phi^{-1}(E)$ at $q$ is equal to $s - k(r - k)(g - 1)$. But Mori showed in [11] that

$$h^0(F^* \otimes G) - h^1(F^* \otimes G) \leq \dim_q Q \mathcal{E}$$
Again by [4] Théorème 4.6 the vector bundle $F^* \otimes G$ is non-special implying $h^1(F^* \otimes G) = 0$ and thus

$$\dim \tilde{M}_k(E) = \dim \phi^{-1}(E) = h^0(F^* \otimes G) = s - k(r - k)(g - 1).$$

\[\square\]

**Remark 4.5** Take $r = 3$, $d = 1$, $k = 2$, $s = 2$ and $g \geq 2$. In [3] it was proved that there are extensions $0 \to \mathcal{O}^2 \to E \to L \to 0$ of a line bundle $L$ of degree 1 by the trivial bundle $\mathcal{O}^2$ such that $E$ is stable. Actually, such bundle $E$ is in $\mathcal{M}(3,1,2,2)$, since $\mu(E) < 1$ and hence $s_2(E) = 2$. However, for such bundles there is no stable subbundles of degree 0 and hence $E \notin \text{Im}\phi$. Such bundles $E$ are in the Brill-Noether locus $W_{r,d}^{k}$. An interesting problem is to study the relation between the Brill-Noether loci $W_{r,d}^k$ and the $\mathcal{M}_0^0(r, d, k, s)$ varieties.

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