Nonlocal fractional elliptic and parabolic equations in Besov spaces and applications
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ABSTRACT

The maximal $B_{p,q}^s$-regularity properties of a nonlocal fractional elliptic equation is studied. Particularly, it is proven that the operator generated by this nonlocal elliptic equation in $B_{p,q}^s$ is sectorial and also is a generator of an analytic semigroup. Moreover, well-posedness of nonlocal fractional parabolic equation in Besov spaces are established.

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1. Introduction, notations and background

In the last years, fractional elliptic and parabolic equations have found many applications in physics (see [2, 5], [7-11], [18] and the references therein). The regularity properties of fractional differential equations (FDEs) have been studied e.g. in [3, 8, 9, 12-16]. The main objective of the present paper is to discuss the $B_{p,q}^s(\mathbb{R}^n)$-maximal regularity of the nonlocal elliptic FDE with parameter

$$\sum_{|\alpha|\leq l} a_{\alpha} \ast D^\alpha u + \lambda u = f(x), \quad x \in \mathbb{R}^n,$$

(1.1)

where $a_{\alpha}$ are complex valued functions, $\lambda$ is a complex parameter and

$$D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}...D_n^{\alpha_n}, \quad \alpha_i \in [0, \infty), \quad \alpha = (\alpha_1, \alpha_2, ..., \alpha_n).$$

Here $D_k^{\alpha_k}$ are Caputo type fractional partial derivatives of order $\alpha_k \in [m-1, m)$ with respect to $x_k \in (a, b)$ i.e.

$$D_k^{\alpha_k}u = \frac{1}{\Gamma(m-\alpha_k)} \int_a^{x_k} (x_k - \tau)^{m-\alpha_k-1} u^{(m)}(\tau) \, d\tau,$$

(1.2)
$\Gamma(\alpha_k)$ is Gamma function for $\alpha_k > 0$ (see e.g. [5, 7, 11]), $m \geq 1$ is a positive integer, $a_\alpha * u$ is a convolution of $a_\alpha$ and $u$.

Let $E$ be a Banach space. Here, $L_p(\Omega;E)$ denotes the space of $E$-valued strongly measurable complex-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm given by

$$\|f\|_{L_p(\Omega;E)} = \left( \int_\Omega \|f(x)\|_E^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$ 

$$L_p(\Omega;E)$$ is well defined and extends to a bounded linear operator on $L_p(\Omega;E)$.

Let $m_i$, $s_i$ be positive integers, $k_i$ be nonnegative integers, $m_i > s_i - k_i > 0$, $i = 1, 2, \ldots, n$ and $s = (s_1, s_2, \ldots, s_n)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $0 < h_0 < \infty$. The $E$-valued Besov space $B^{s}_{p,q}(\Omega;E)$ are defined as

$$B^{s}_{p,q}(\Omega;E) = \{ f : f \in L_p(\Omega;E), \|f\|_{B^{s}_{p,q}(\Omega;E)} = \sup_{\theta = \alpha} \frac{\|\Delta_\theta f\|_{L_p(\Omega;E)}}{\theta^{s}} < \infty \}.$$

Let $m_i$, $s_i$ be nonnegative integers, $k_i$ be nonnegative integers, $m_i > s_i - k_i > 0$, $i = 1, 2, \ldots, n$ and $s = (s_1, s_2, \ldots, s_n)$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $0 < h_0 < \infty$. The $E$-valued Besov space $B^{s}_{p,q}(\Omega;E)$ are defined as

$$B^{s}_{p,q}(\Omega;E) = \{ f : f \in L_p(\Omega;E), \|f\|_{B^{s}_{p,q}(\Omega;E)} = \sup_{\theta = \alpha} \frac{\|\Delta_\theta f\|_{L_p(\Omega;E)}}{\theta^{s}} < \infty \}.$$

Let $E_0$ and $E$ be two Banach spaces and $E_0$ be continuously and densely embedded into $E$. Moreover, let $l \in \mathbb{R}_+$. Consider the Sobolev-Besov space $B^{l,s}_{p,q}(\Omega;E)$ and $B^{s}_{p,q}(\Omega;E)$ with the norm

$$\|u\|_{B^{l,s}_{p,q}(\Omega;E)} = \|u\|_{B^{s}_{p,q}(\Omega;E)} + \sum_{\|\alpha\| \leq l} \|D^\alpha u\|_{B^{s}_{p,q}(\Omega;E)} < \infty.$$

Here, $C$ denotes the set of complex numbers. For $E_0 = E = \mathbb{C}$ the spaces $L_p(\Omega;E)$, $B^{s}_{p,q}(\Omega;E)$, $B^{s}_{p,q}(\Omega;E)$ will be denoted by $L_p(\Omega)$, $B^{s}_{p,q}(\Omega)$ and $B^{s}_{p,q}(\Omega)$, respectively.

Let $S(\mathbb{R}^n)$ denote Schwartz class, i.e., the space of all rapidly decreasing smooth functions on $\mathbb{R}^n$ equipped with its usual topology generated by seminorms. A function $\Psi \in C(\mathbb{R}^n)$ is called a Fourier multiplier from $B^{s}_{p,q}(\mathbb{R}^n)$ to $B^{s}_{p,q}(\mathbb{R}^n)$ if the map

$$u \to \Lambda u = F^{-1}\Psi(\xi) Fu, \quad u \in S(\mathbb{R}^n)$$

is well defined and extends to a bounded linear operator

$$\Lambda : B^{s}_{p,q}(\mathbb{R}^n) \to B^{s}_{p,q}(\mathbb{R}^n).$$

We prove that problem (1.1) has a unique solution $u \in B^{l,s}_{p,q}(\mathbb{R}^n)$ for $f \in B^{s}_{p,q}(\mathbb{R}^n)$ and the following uniform coercive estimate holds

$$\sum_{\|\alpha\| \leq l} |\lambda|^{1-\frac{\|\alpha\|}{l}} \|a_\alpha * u\|_{B^{l,s}_{p,q}(\mathbb{R}^n)} \leq C \|f\|_{B^{s}_{p,q}(\mathbb{R}^n)}.$$

(1.3)
The estimate (1.3) implies that the operator $O$ that generated by problem
(1.1) has a bounded inverse from $B^s_{p,q}(\mathbb{R}^n)$ into the space $B^s_{p,q}(\mathbb{R}^n)$. Particularly, from the estimate (1.3) we obtain that $O$ is uniformly sectorial operator in $B^s_{p,q}(\mathbb{R}^n)$. By using the coercive properties of elliptic operator, we prove the well posedness of the Cauchy problem for the nonlocal fractional parabolic differential equation:

$$\frac{\partial u}{\partial t} + \sum_{|\alpha| \leq \ell} a_\alpha * D^\alpha u = f(t, x), \quad u(0, x) = 0, \quad (1.4)$$

in the Besov space $Y^s = B^s_{p,q}(\mathbb{R}_+; B^s_{p,q}(\mathbb{R}^n))$.

In other words, we show that problem (1.4) for each $f \in Y^s$ has a unique solution $u \in B^{1,l,s}_{p,q}(\mathbb{R}^{n+1})$ with $p = (p, p_1)$ satisfying the coercive estimate

$$\left\| \frac{\partial u}{\partial t} \right\|_{Y^s} + \sum_{|\alpha| \leq \ell} \|a_\alpha * D^\alpha u\|_{Y^s} + \|A * u\|_{Y^s} \leq M \|f\|_{Y^s}. \quad (1.5)$$

Let $S_\varphi = \{ \lambda : \lambda \in \mathbb{C}, \ |\arg \lambda| \leq \varphi \} \cup \{0\}, \ 0 \leq \varphi < \pi$.

$L(E_1, E_2)$ denotes the space of bounded linear operators from $E_1$ to $E_2$. For $E_1 = E_2 = E$ it denotes by $L(E)$. Let $D(A), R(A)$ denote the domain and range of the linear operator in $E$, respectively. Let Ker $A$ denote a null space of $A$. A closed linear operator $A$ is said to be $\varphi$-sectorial (or sectorial for $\varphi = 0$) in a Banach space $E$ with bound $M > 0$ if Ker $A = \{0\}$; $D(A)$ and $R(A)$ are dense on $E$, and $\left\| (A + \lambda I)^{-1} \right\|_{L(E)} \leq M |\lambda|^{-1}$ for all $\lambda \in S_\varphi$, $\varphi \in [0, \pi)$, where $I$ is an identity operator in $E$. Sometimes $A + \lambda I$ will be written as $A + \lambda$ and will be denoted by $A_\lambda$. It is known [17, §1.15.1] that the powers $A^\theta$, $\theta \in (-\infty, \infty)$ for a sectorial operator $A$ exist.

Here, $S' = S'(\mathbb{R}^n)$ denotes the space of linear continuous mappings from $S(\mathbb{R}^n)$ into $\mathbb{C}$ and it is called the Schwartz distributions. For any $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha_i \in [0, \infty)$, $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$ the function $(i\xi)^\alpha$ will be defined as:

$$(i\xi)^\alpha = \begin{cases} (i\xi_1)^{\alpha_1} \cdots (i\xi_n)^{\alpha_n}, & \xi_1 \xi_2 \cdots \xi_n \neq 0 \\ 0, & \xi_1, \xi_2, ..., \xi_n = 0, \end{cases}$$

where

$$(i\xi_k)^{\alpha_k} = \exp \left[ \alpha_k \left( \ln |\xi_k| + \frac{i\pi}{2} \text{sgn} \xi_k \right) \right], \quad k = 1, 2, ..., n.$$
**Theorem A.** Suppose $1 < p \leq p_1 < \infty$, $l$ is a positive integer and $s \in (0, \infty)$ with $\nu = \frac{1}{l} \left[ |\alpha| + n \left( \frac{1}{p} - \frac{1}{p_1} \right) \right] \leq 1$ for $0 \leq \mu \leq 1 - \nu$, then the embedding

$$D^\alpha B^{s,l}_{p,q}(\mathbb{R}^n) \subset B^{s}_{p_1,q}(\mathbb{R}^n)$$

is continuous and there exists a constant $C_\mu > 0$, depending only on $\mu$ such that

$$\|D^\alpha u\|_{B^{s}_{p_1,q}(\mathbb{R}^n)} \leq C_\mu \left[ h^\mu \|u\|_{B^{s,l}_{p,q}(\mathbb{R}^n)} + h^{-(1-\mu)} \|u\|_{B^{s}_{p,q}(\mathbb{R}^n)} \right]$$

for all $u \in B^{s,l}_{p,q}(\mathbb{R}^n)$ and $0 < h \leq h_0 < \infty$.

2. **Nonlocal fractional elliptic equation**

Consider the problem (1.1).

**Condition 2.1.** Assume $a_\alpha \in L^\infty(\mathbb{R}^n)$ such that

$$L(\xi) = \sum_{|\alpha| \leq l} \hat{a}_\alpha(\xi) (i\xi)^\alpha \in S_{\rho_1}, \quad |L(\xi)| \geq C \sum_{k=1}^n |\hat{a}_{\alpha(l,k)}| |\xi_k|^l,$$  \hspace{1cm} (2.1)

for

$$\alpha(l,k) = (0,0,...,l,0,0,...,0), \ i.e \ \alpha_i = 0, \ i \neq k.$$  

Consider operator functions

$$\sigma_1(\xi,\lambda) = \lambda \sigma_0(\xi,\lambda), \quad \sigma_2(\xi,\lambda) = \sum_{|\alpha| \leq l} |\lambda|^{-|\alpha|} \hat{a}_\alpha(\xi) (i\xi)^\alpha \sigma_0(\xi,\lambda),$$ \hspace{1cm} (2.2)

where

$$\sigma_0(\xi,\lambda) = [L(\xi) + \lambda]^{-1}.$$  

Let

$$X = B^s_{p,q}(\mathbb{R}^n), \quad Y = B^{s,l}_{l,p,q}(\mathbb{R}^n).$$

In this section we prove the following:

**Theorem 2.1.** Assume that the Condition 2.1 is satisfied and $p, q \in [1, \infty]$. Suppose that $\gamma \in (1,2]$, and $\lambda \in S_{\rho_2}$. Then for $f \in X$, $0 \leq \varphi_1 < \pi - \varphi_2$ and $\varphi_1 + \varphi_2 \leq \varphi$ there is a unique solution $u$ of the equation (1.1) belonging to $Y$ and the following coercive uniform estimate holds

$$\sum_{|\alpha| \leq l} |\lambda|^{-|\alpha|} \|a \ast D^\alpha u\|_X + \|u\|_X \leq C \|f\|_X.$$ \hspace{1cm} (2.3)

For proving of Theorem 2.1 we need the following lemmas:
Lemma 2.1. Assume Condition 2.1 holds and \( \lambda \in S_{\varphi_2} \) with \( \varphi_2 \in [0, \pi) \), where \( \varphi_1 + \varphi_2 < \pi \), then the operator functions \( \sigma_i (\xi, \lambda) \) are uniformly bounded, i.e.,

\[
|\sigma_i (\xi, \lambda)| \leq C, \ i = 0, 1, 2.
\]

Proof. By virtue of \([4, \text{Lemma 2.3}]\), for \( L(\xi) \in S_{\varphi_1}, \lambda \in S_{\varphi_2} \) and \( \varphi_1 + \varphi_2 < \pi \) there exists a positive constant \( C \) such that

\[
|\lambda + L(\xi)| \geq C (|\lambda| + |L(\xi)|).
\]

(2.4)

Since \( L(\xi) \in S_{\varphi_1} \), in view of Condition 2.1 and (2.4) the function \( \sigma_0 (\xi, \lambda) \) is uniformly bounded for all \( \xi \in \mathbb{R}^n, \lambda \in S_{\varphi_2} \), i.e.,

\[
\sigma_0 (\xi, \lambda) \leq (|\lambda| + |L(\xi)|)^{-1} \leq M_0.
\]

Moreover, we have

\[
|\sigma_1 (\xi, \lambda)| \leq M |\lambda| (|\lambda| + |L(\xi)|)^{-1} \leq M_1.
\]

Next, let us consider \( \sigma_2 \). It is clear to see that

\[
|\sigma_2 (\xi, \lambda)| \leq C \sum_{|\alpha| \leq l} |\lambda| \prod_{k=1}^{n} \left[ |\xi_k| |\lambda|^{-1} \right]^{|\alpha_k|} |\sigma_0 (\xi, \lambda)|.
\]

(2.5)

By setting \( y_k = \left( |\lambda|^{-1} |\xi_k| \right)^{\alpha_k} \) in the following well known inequality

\[
y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n} \leq C \left( 1 + \sum_{k=1}^{n} y_k \right), \ y_k \geq 0, \ |\alpha| \leq l
\]

(2.6)

we get

\[
\|\sigma_2 (\xi, \lambda)\|_{B(E)} \leq C \sum_{|\alpha| \leq l} |\lambda| \left[ 1 + \sum_{k=1}^{n} |\xi_k|^l |\lambda|^{-1} \right] |\lambda + L(\xi)|^{-1}.
\]

Taking into account the Condition 2.1 and (2.5) – (2.6) we obtain

\[
|\sigma_2 (\xi, \lambda)| \leq C \left( |\lambda| + \sum_{k=1}^{n} |\xi_k|^l \right) (|\lambda| + |L(\xi)|)^{-1} \leq C.
\]

Lemma 2.2. Assume Condition 2.1 holds. Suppose \( \hat{\alpha}_\alpha \in C^{(n)} (\mathbb{R}^n) \) and

\[
|\xi|^{|\beta|} |D^\beta \hat{\alpha}_\alpha (\xi)| \leq C_1, \ \beta_k \in \{0, 1\}, \ \xi \in \mathbb{R}^n \setminus \{0\}, \ 0 \leq |\beta| \leq n,
\]

(2.7)

Then, operators \( |\xi|^{|\beta|} D^{\beta} \sigma_1 (\xi, \lambda), i = 0, 1, 2 \) are uniformly bounded.

Proof. Consider the term \( |\xi|^{|\beta|} D^{\beta} \sigma_0 (\xi, \lambda) \). By using the Conditin 2.1 and the estimates (2.4) – (2.6), we get

\[
|\xi_k| |D_{\xi_k} \sigma_0 (\xi, \lambda)| \leq
\]
\[
\left| \xi_k \right| \frac{\partial}{\partial \xi_k} \hat{a}_\alpha (\xi) + \alpha_k |\hat{a}_\alpha (\xi)| \right| \prod_{k=1}^n (i\xi_k)^{\alpha_k} \left| |L (\xi) + \lambda|^{-2} \right| < \infty.
\]

It easy to see that operators \( |\xi|^{\beta} D_{\xi}^{\beta} \sigma_0 (\xi, \lambda) \) contain the similar terms as in \( |\xi|^{\beta} \left| D_{\xi}^{\beta} \sigma_0 (\xi, \lambda) \right| \) for all \( \beta_k \in \{0, 1\} \). Hence we get
\[
|\xi|^{\beta} \left| D_{\xi}^{\beta} \sigma_0 (\xi, \lambda) \right| < \infty.
\]

In a similar way, by using the Conditin 2.1 and the estimates (2.4) – (2.7) we obtain
\[
|\xi|^{\beta} \left| D_{\xi}^{\beta} \sigma_i (\xi, \lambda) \right| < \infty, \ i = 1, 2. \tag{2.8}
\]

**Proof of Theorem 2.1.** By applying the Fourier transform to equation (1.1) we get
\[
\hat{u} (\xi) = \sigma_0 (\xi, \lambda) \hat{f} (\xi), \ \sigma_0 (\xi, \lambda) = [L (\xi) + \lambda]^{-1} . \tag{2.9}
\]

Hence, the solution of (1.1) can be represented as \( u (x) = F^{-1} D_{\xi} (\xi, \lambda) \hat{f} \) and by Lemma 2.1 there are positive constants \( C_1 \) and \( C_2 \) such that
\[
C_1 |\lambda| \|u\|_X \leq \left\| F^{-1} \left[ \lambda \sigma_0 (\xi, \lambda) \hat{f} \right] \right\|_X \leq C_2 |\lambda| \|u\|_X ,
\]
\[
C_1 \sum_{|\alpha| \leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_X \leq \left\| F^{-1} \left[ \sigma_2 (\xi, \lambda) \hat{f} \right] \right\|_X \leq C_2 \sum_{|\alpha| \leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_X . \tag{2.10}
\]

Therefore, it is sufficient to show that the operators \( \sigma_i (\xi, \lambda) \) are multipliers in \( X \). But, by Lemma 2.2 and by virtue of Fourier multiplier theorem in Besov spaces \( B^{p,q}_{\nu} (\mathbb{R}^n) \) (see e.g [6, Corollary 4.11]) we get that \( \sigma_i (\xi, \lambda) \) are multipliers in \( X \). So, we obtain the assertion.

**Result 2.1.** Theorem 2.1 implies that the operator \( O \) is separable in \( X \), i.e.

for all \( f \in X \) there is a unique solution \( u \in Y \) of the problem (1.1), all terms of equation (1.1) are also from \( X \) and there are positive constants \( C_1 \) and \( C_2 \) so that
\[
C_1 \|Ou\|_X \leq \sum_{|\alpha| \leq l} \|a_\alpha * D^\alpha u\|_X + \|u\|_X \leq C_2 \|Ou\|_X .
\]

Indeed, if we put \( \lambda = 1 \) in (2.3), by Theorem 2.1 we get the second inequality. So it is remain to prove the first estimate. The first inequality is equivalent to the following estimate
\[
\sum_{|\alpha| \leq l} \left\| F^{-1} a_\alpha (i\xi)^\alpha \hat{u} \right\|_X \leq \sum_{|\alpha| \leq l} \left\| F^{-1} a_\alpha (i\xi)^\alpha \sigma_0 (\xi, \lambda) \hat{f} (\xi) \right\|_X .
\]
So, it suffices to show that the operator functions
\[ \sigma_0(\xi, \lambda), \sum_{|\alpha| \leq l} \hat{a}_\alpha (i\xi)^\alpha \sigma_0(\xi, \lambda) \]
are uniform Fourier multipliers in \(X\). This fact is proved in a similar way as in the proof of Theorem 2.1.

From Theorem 2.1, we have:

**Result 2.2.** Assume all conditions of Theorem 2.1 are satisfied. Then, for all \(\lambda \in S_\varphi\) the resolvent of operator \(O\) exists and the following sharp coercive uniform estimate holds
\[
\sum_{|\alpha| \leq l} |\lambda|^{-\frac{|\alpha|}{\theta}} \| a * D^\alpha (O + \lambda)^{-1} \|_{L(X)} + \|(O + \lambda)^{-1}\|_{L(X)} \leq C. \tag{2.11}
\]

Indeed, we infer from Theorem 2.1 that the operator \(O + \lambda\) has a bounded inverse from \(X\) to \(Y\). So, the solution \(u\) of the equation (1.1) can be expressed as \(u(x) = (O + \lambda)^{-1} f\) for all \(f \in X\). Then estimate (2.4) implies the estimate (2.11).

**Theorem 2.2.** Assume that the Condition 2.1 is satisfied, \(p, q \in [1, \infty]\) and \(\varphi \leq \varphi_2\). Then for \(f \in X, 0 \leq \varphi_1 < \varphi - \varphi_2\) and \(\varphi_1 + \varphi_2 \leq \varphi\) there is a unique solution \(u\) of (1.1) belonging to \(Y\) and the following coercive uniform estimate holds
\[
\sum_{|\alpha| \leq l} |\lambda|^{-\frac{|\alpha|}{\theta}} \| D^\alpha u \|_X + \| u \|_X \leq C \| f \|_X. \tag{2.12}
\]

**Proof.** The estimate (2.12) is derived by reasoning as in Theorem 2.2. From Theorem 2.2, we have the following results:

**Result 2.3.** There are positive constants \(C_1\) and \(C_2\) so that
\[
C_1 \| Ou \|_X \leq \sum_{|\alpha| \leq l} \| D^\alpha u \|_X + \| A u \|_X \leq C_2 \| Ou \|_X. \tag{2.13}
\]

From theorem 2.2, we obtain

**Result 2.4.** Assume all conditions of Theorem 2.2 hold. Then, for all \(\lambda \in S_\varphi\) the resolvent of operator \(O\) exists and the following sharp uniform estimate holds
\[
\sum_{|\alpha| \leq l} |\lambda|^{-\frac{|\alpha|}{\theta}} \| D^\alpha (O + \lambda)^{-1} \|_{L(X)} + \|(O + \lambda)^{-1}\|_{L(X)} \leq C. \tag{2.15}
\]

**Result 2.5.** Theorem 2.2 particularly implies that the operator \(O\) is sectorial in \(X\). Then the operators \(O^s\) are generators of analytic semigroups in \(X\) for \(s \leq \frac{1}{2}\) (see e.g. [17, §1.14.5]).

3. The Cauchy problem for fractional parabolic equation
In this section, we shall consider the following Cauchy problem for the parabolic FDE

\[
\frac{\partial u}{\partial t} + \sum_{|\alpha| \leq l} a_\alpha * D^\alpha u = f(t, x), \quad u(0, x) = 0, \ t \in \mathbb{R}_+, \ x \in \mathbb{R}^n, \tag{3.1}
\]

where \(a\) is a complex number, \(D^\alpha_x\) is the fractional derivative in \(x\) defined by (1.2).

By applying Theorem 2.1 we establish the maximal regularity of the problem (3.1) in mixed Besov spaces. Let \(O\) denote the operator generated by problem (1.1) for \(\lambda = 0\). Let

\[
X_0 = B^s_p(\mathbb{R}^n), \ Y^s = B^s_{p_1, q}(\mathbb{R}_+; X_0), \ Y_0 = B^s_{p_2, q}(\mathbb{R}^{n+1}_+),
\]

\[Y^{1,l,s} = B^{1,l,s}_{p_1, q}(\mathbb{R}^{n+1}_+), \ p = (p_1, p)\]

Let \(Y^{1,l,s}\) denotes the space of all functions \(u \in X\) possessing the generalized derivative \(D_t u = \frac{\partial u}{\partial t} \in Y_0\) and fractional derivatives \(D^\alpha_x u \in Y_0\) for \(|\alpha| \leq l\) with the norm

\[
\|u\|_{Y^{1,l,s}} = \|u\|_{Y_0} + \|\partial_t u\|_{Y_0} + \sum_{|\alpha| \leq l} \|D^\alpha_x u\|_{Y_0},
\]

where \(u = u(t, x)\).

Now, we are ready to state the main result of this section.

**Theorem 3.1.** Assume the conditions of Theorem 2.1 hold for \(\varphi \in \left(\frac{\pi}{2}, \pi\right)\) and \(p_1, p, q \in [1, \infty]\). Then for \(f \in Y\) problem (3.1) has a unique solution \(u \in Y^{1,l,s}\) satisfying the following uniform coercive estimate

\[
\left\| \frac{\partial u}{\partial t} \right\|_{Y^s} + \sum_{|\alpha| \leq l} \|a_\alpha * D^\alpha u\|_{Y^s} + \|u\|_{Y^s} \leq C \|f\|_{Y^s}.
\]

**Proof.** By definition of \(X_0\) and by definition of the mixed space \(B^s_{p,q}(\mathbb{R}^{n+1}_+)\) for \(p = (p_1, p)\), we have

\[
\|u\|_{Y^s} = \|u\|_{B^s_{p_1,q}(\mathbb{R}_+; X_0)} = \tag{3.2}
\]

\[
\|u\|_{L_{p_1}(\mathbb{R}_+; X_0)} + \int_0^{y_0} \left( \int_0^y \left( y^{-(s-k)q+1} \|\Delta^{m_k} (y, \Omega) D^k f \|_{L_{p_1}(\mathbb{R}_+; X_0)} \right) \right) dh \geq \|u\|_{B^s_{p,q}(\mathbb{R}^{n+1}_+)}.
\]

Hence, the problem (3.1) can be expressed as the following Cauchy problem for the parabolic equation

\[
\frac{du}{dt} + Ou(t) = f(t), \ u(0) = 0, \ t \in \mathbb{R}_+. \tag{3.3}
\]
Then, by virtue of [1, Proposition 8.10], we obtain that for $f \in Y^*$ problem (3.3) has a unique solution $u \in B^{s,q}_{1,p}(\mathbb{R}^+;D(O),Y^*)$ satisfying the following estimate

$$\left\| \frac{du}{dt} \right\|_{B^{s,q}_{1,p}(\mathbb{R}^+;X_0)} + \left\| Ou \right\|_{B^{s,q}_{1,p}(\mathbb{R}^+;X_0)} \leq C \| f \|_{B^{s,q}_{1,p}(\mathbb{R}^+;X_0)}.$$ 

From the Theorem 2.1, relation (3.2) and from the above estimate we get the assertion.

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