Eigenvalue Ratios for vibrating string equations with single-well densities

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Abstract
In this paper, we prove the optimal upper bound \( \frac{\lambda_n}{\lambda_m} \leq (\frac{n}{m})^2 \) of vibrating string
\[-y'' = \lambda \rho(x) y,\]
with Dirichlet boundary conditions for single-well densities. The proof is based on the inequality \( \frac{\lambda_n(\rho)}{\lambda_m(\rho)} \leq \frac{\lambda_n(L)}{\lambda_m(L)}, \) with \( L \) must be a stepfunction. We also prove the same result for the Dirichlet Sturm-Liouville problems.

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1 Introduction
We consider the Sturm-Liouville equation acting on \([0, 1]\)
\[-(p(x)y')' + q(x)y = \lambda \rho(x)y, \tag{1.1}\]
with Dirichlet boundary conditions
\[y(0) = y(1) = 0, \tag{1.2}\]
where \( p > 0, \rho > 0 \) and \( q \) (may change sign) are continuous coefficients on \([0, 1]\). Here we limit ourselves to the case \( \rho > 0 \). The case \( \rho < 0 \) has been considered for related problems providing different results, we refer to pioneering works \([5, 6]\) and some refer therein.

As is well-known (see \([13]\)), there exist two countable sequences of eigenvalues
\[\lambda_1 < \lambda_2 < \cdots < \lambda_n \cdots \infty.\]

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The issues of optimal estimates for the eigenvalue ratios $\frac{\lambda_n}{\lambda_m}$ have attracted a lot of attention (cf. [1, 3, 4, 7, 8, 9, 10, 11, 13]) and references therein. Ashbaugh and Benguria proved in [3] that if $q \geq 0$ and $0 < k \leq p \rho(x) \leq K$, then the eigenvalues of (1.1)-(1.2) satisfy

$$\frac{\lambda_n}{\lambda_1} \leq \frac{Kn^2}{k}.$$ 

They also established the following ratio estimate (of two arbitrary eigenvalues)

$$\frac{\lambda_n}{\lambda_m} \leq \frac{Kn^2}{km^2}, \quad n > m \geq 1,$$

with $q \equiv 0$ and $0 < k \leq p \rho(x) \leq K$. Later, Huang and Law [9] extended the results in [3] to more general boundary conditions.

In the case where $p \equiv 1$ and $q \equiv 0$, Huang proved in [7] that the eigenvalues for the string equation

$$-y'' = \lambda \rho(x)y, \quad (1.3)$$

with Dirichlet boundary conditions (1.2) satisfy $\frac{\lambda_2}{\lambda_1} \leq 4$ for symmetric single-well density $\rho$ and $\frac{\lambda_2}{\lambda_1} \geq 4$ for symmetric single-barrier density $\rho$. The later one has been extended by Horváth [10] for single-barrier (not necessarily symmetric) density $\rho$. In 2006, Kiss [13] showed that $\frac{\lambda_n}{\lambda_1} \leq n^2$ for symmetric single-well densities and $\frac{\lambda_n}{\lambda_1} \geq n^2$ for symmetric single-barrier densities.

Recall that $f$ is a single-barrier (resp. single-well) function on $[0, 1]$ if there is a point $x_0 \in [0, 1]$ such that $f$ is increasing (resp. decreasing) on $[0, x_0]$ and decreasing (resp. increasing) on $[x_0, 1]$ (see [2]).

In this paper, we prove the optimal upper bound $\frac{\lambda_n}{\lambda_m} \leq (\frac{n}{m})^2$ of (1.3)-(1.2) for single-well density $\rho$ (not necessarily symmetric). The main step to prove this result is the inequality $\frac{\lambda_n}{\lambda_m} \leq \frac{\lambda_n(L)}{\lambda_m(L)}$, with $L$ being a stepfunction. We also prove an result for the Dirichlet Sturm-Liouville problems (1.1)-(1.2). More precisely, we show that $\frac{\lambda_n}{\lambda_m} \leq (\frac{n}{m})^2$ for $q$ is single-barrier and $pp$ is single-well with transition point $x_0 = \frac{1}{2}$ such that $0 < \min(\hat{\mu}_1, \tilde{\mu}_1)$, where $\hat{\mu}_1$ and $\tilde{\mu}_1$ are the first eigenvalues of the Neumann boundary problems defined on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively.

For this result, we modify the inverse Liouville substitution (e.g., see [16] pp. 51], [3]) in order to transform Equation (1.1) into (1.3), whose the density is single-well. Therefore, we can use the result of section 2 on the ratio of eigenvalues $\frac{\lambda_n}{\lambda_m}$ for the string equations with single-well densities.
2 Eigenvalue ratio for the vibrating string equations

Denote by \( u_n(x) \) be the \( n-th \) eigenfunction of (1.3) corresponding to \( \lambda_n \), normalized so that

\[
\int_0^1 \rho(x) u_n^2(x) \, dx = 1.
\]

It is well known that the \( u_n(x) \) has exactly \( (n-1) \) zeros in the open interval \((0, 1)\). The zeros of the \( n-th \) and \((n+1)st\) eigenfunctions interlace, i.e. between any two successive zeros of the \( n-th \) eigenfunction lies a zero of the \((n+1)st\) eigenfunction. We denote by \((y_i)\), the zeros of \( u_n \) and \((z_i)\), the zeros of \( u_{n-1} \), then in view of the comparison theorem (see [15, Chap.1]), we have \( y_i < z_i \). We may assume that \( u_n(x) > 0 \) and \( u_{n-1}(x) > 0 \) on \((0, y_i)\), then we have \( \frac{u_n(x)}{u_{n-1}(x)} \) is strictly decreasing on \((0, 1)\). Indeed,

\[
\left( \frac{u_n(x)}{u_{n-1}(x)} \right)' = \frac{u_n'(x)u_{n-1}(x) - u_n(x)u_{n-1}'(x)}{u_{n-1}^2(x)} = \frac{w(x)}{u_{n-1}^2(x)}.
\]

We find

\[
w'(x) = u''_n(x)u_{n-1}(x) - u''_{n-1}(x)u_n(x) = (\lambda_{n-1} - \lambda_n)\rho(x)u_n(x)u_{n-1}(x),
\]

this implies that \( w(x) < 0 \) on \((0, 1)\). Hence \( \frac{u_n(x)}{u_{n-1}(x)} \) is strictly decreasing on \((0, 1)\).

From this, there are points \( x_i \in (y_i, z_i) \) such that

\[
\begin{cases}
  u^2_n(x) > u^2_{n-1}(x), & x \in (x_{2i}, x_{2i+1}), \\
  u^2_n(x) < u^2_{n-1}(x), & x \in (x_{2i+1}, x_{2i+2}).
\end{cases}
\]

Let \( \rho(., \tau) \) is a one-parameter family of piecewise continuous densities such that \( \frac{\partial \rho(., \tau)}{\partial \tau} \) exists, and let \( u_n(x, \tau) \) be the \( n-th \) eigenfunction of (1.3) corresponding to \( \lambda_n(\tau) \) of the corresponding String equation (1.3) with \( \rho = \rho(., \tau) \). From Keller in [12], we get

\[
\frac{d}{d\tau} \lambda_n(\tau) = -\lambda_n(\tau) \int_0^1 \frac{\partial \rho}{\partial \tau}(x, \tau)u^2_n(x, \tau) \, d\tau.
\]

By straightforward computation that, yields

\[
\frac{d}{d\tau} \left[ \frac{\lambda_n(\tau)}{\lambda_m(\tau)} \right] = \frac{\lambda_n(\tau)}{\lambda_m(\tau)} \int_0^1 \frac{\partial \rho}{\partial \tau}(x, \tau)(u^2_m(x, \tau) - u^2_n(x, \tau)) \, d\tau.
\]

We first prove.

**Proposition 1** Let \( \rho > 0 \) be monotone decreasing in \([0, 1]\) and let \( L(x) = \rho(x_{2i+1}) \) (where \( x_i \) the points such that \( u^2_n(x_i) = u^2_{n-1}(x_i) \)), then

\[
\frac{\lambda_n(\rho)}{\lambda_m(\rho)} \leq \frac{\lambda_n(L)}{\lambda_m(L)}.
\]

with equality if and only if \( \rho \equiv L \).
Proof Define $\hat{\rho}(x, \tau) = \tau \rho(x) + (1 - \tau)L(x)$. Using (2.1), one gets
\[
\frac{d}{d\tau} \left[ \frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \right] = \frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \int_0^1 \frac{\partial}{\partial \tau}(x, \tau)(u_{n-1}^2(x, \tau) - u_n^2(x, \tau))d\tau
\]
\[= \frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \sum_{i=0}^{x_{2i+2}} (\rho(x) - L(x))(u_{n-1}^2(x, \tau) - u_n^2(x, \tau))d\tau. \tag{2.3}\]

We notice that
\[
\int_{x_{2i}}^{x_{2i+2}} (\rho(x) - L(x))(u_{n-1}^2(x, \tau) - u_n^2(x, \tau))d\tau \leq 0.
\]
It then follows that $\frac{d}{d\tau} \left[ \frac{\lambda_n(\tau)}{\lambda_{n-1}(\tau)} \right] \leq 0$. Thus, by the continuity of eigenvalues, we obtain
\[
\frac{\lambda_n(\rho)}{\lambda_{n-1}(\rho)} = \frac{\lambda_n(1)}{\lambda_{n-1}(1)} \leq \frac{\lambda_n(0)}{\lambda_{n-1}(0)} = \frac{\lambda_n(L)}{\lambda_{n-1}(L)}.
\]
And hence
\[
\frac{\lambda_n(\rho)}{\lambda_m(\rho)} \leq \frac{\lambda_n(L)}{\lambda_m(L)}.
\]
Equality holds iff $\rho = L$. \qed

We are now in position to state our main result.

**Theorem 1** Let $\rho$ be a single-well density on $[0, 1]$. Then the eigenvalues of the Dirichlet problem (1.3) - (1.2) satisfy
\[
\frac{\lambda_n}{\lambda_m} \leq \left(\frac{n}{m}\right)^2, \tag{2.4}
\]
with equality if and only if $\rho$ is constant.

The proof of Theorem 1 will be given in section 3.

3 Proof of Theorems 1

**Corollary 1** Consider equation (1.3) with the Dirichlet boundary conditions (1.2). If the density $\rho$ is decreasing in $[0, 1]$, then the $m$ - th and $n$ - th eigenvalues with $m < n$ satisfy
\[
\frac{\lambda_n}{\lambda_m} \leq \left(\frac{n}{m}\right)^2.
\]
Equality holds iff $\rho$ is constant.
In order to prove Corollary 1 we need some preliminary results. Let \( y(x, z) \) be the unique solution of the initial value problem

\[
\begin{aligned}
-\ddot{y} &= z^2 \rho(x) y, \quad x \in [0, 1], \quad z > 0, \\
y(0) &= 0, \quad \dot{y}(0) = \rho_{\frac{1}{4}}(0).
\end{aligned}
\]  

(3.1)

We shall apply to System (3.1), the modified Prüfer substitution as introduced in [13].

\[
\begin{align*}
y(x, z) &= r(x, z) \frac{1}{z} \rho_{\frac{1}{4}} \sin \varphi(x, z), \\
\dot{y}(x, z) &= r(x, z) \rho_{\frac{1}{4}} \cos \varphi(x, z), \\
\varphi(0, z) &= 0,
\end{align*}
\]

(3.2)

where \( r(x, z) > 0 \), and then let \( \theta(x, z) = \frac{\varphi(x, z)}{z} \). We denote by prime (resp. dot) the derivative with respect to \( x \) (resp. \( z \)).

Using Equation (1.3) together with (3.2), one finds the following differential equations for \( r(x, z) \) and \( \varphi(x, z) \):

\[
\begin{align*}
\varphi' &= z \rho_{\frac{1}{4}} + \frac{1}{4} \frac{\rho'}{\rho} \sin(2\varphi), \\
\frac{r'}{r} &= -\frac{1}{4} \frac{\rho'}{\rho} \cos(2\varphi).
\end{align*}
\]  

(3.3)  

(3.4)

Lemma 1

\[
\dot{\varphi} = \int_0^x \frac{1}{\rho(t)} \frac{r^2(t, z)}{r^2(x, z)} dt.
\]  

(3.5)

Proof Differentiate equation (3.3) with respect to \( z \):

\[
\dot{\varphi}' = \rho_{\frac{1}{4}} + \frac{1}{4} \frac{2 \varphi'}{\rho} \frac{\rho'}{\rho} \cos(2\varphi).
\]  

(3.6)

Multiplying both sides by \( e^{\int_0^x \frac{r'(t)}{r(t)} dt} \), yields

\[
\dot{\varphi} = \int_0^x \rho_{\frac{1}{4}}(t) \frac{r^2(t, z)}{r^2(x, z)} dt.
\]

Corollary 2

\[
\dot{\theta}(x, z) = \frac{1}{z^2 r^2(x)} \int_0^x r^2(t) \left[ 2z \rho_{\frac{1}{4}}(t) + \frac{1}{4} \frac{\rho'(t)}{\rho(t)} \left( \sin(2\varphi(t)) + 2\varphi(t) \cos(2\varphi) \right) \right] dt.
\]  

(3.7)
Proof

\[ \dot{\theta}(x, z) = \frac{\dot{\varphi}(x, z)}{\varphi(x, z)} = \frac{1}{z} \int_0^x \rho^2(t) \frac{r^2(t, z)}{r^2(x, z)} dt - \frac{\varphi(x, z)}{z^2} \]

\[ = \frac{1}{z^2 r^2(x, z)} \left[ \int_0^x z \rho^2(t) r^2(t, z) dt - r^2(x, z) \varphi(x, z) \right] \]

\[ = \frac{1}{z^2 r^2(x, z)} \left[ \int_0^x r^2(t, z) z \rho^2(t) dt - 2 \int_0^x r(t) r'(t) \varphi(t, z) dt + \int_0^x r^2(t) \varphi'(t, z) dt \right] \]

\[ = \frac{1}{z^2 r^2(x, z)} \left[ \int_0^x r^2(t, z) z \rho^2(t) + \varphi'(t, z) dt - 2 \int_0^x r(t) r'(t) \varphi(t, z) dt \right] \]

\[ = \frac{1}{z^2 r^2(x, z)} \left[ \int_0^x r^2(t, z) z [2 z \rho^2(t) + \varphi'(t, z)] dt + 2 \int_0^x r^2(t, z) [r(t, z) \varphi(t, z) + \varphi(t, z) \cos(2 \varphi)] dt \right] \]

\[ = \frac{1}{z^2 r^2(x)} \int_0^x r^2(t) \left[ 2 z \rho^2(t) + \frac{1}{4} \rho(t) \left( \sin(2 \varphi(t)) + 2 \varphi(t) \cos(2 \varphi) \right) \right] dt. \]

\[ \square \]

We can now prove Corollary \[1\[Int]

Proof Let \( L(x) = \rho(x_{2i+1}) \), for all \( x \in (x_{2i}, x_{2i+2}) \), then \( L' \equiv 0 \) for all \( x \in (x_{2i}, x_{2i+2}) \). Using Corollary \[2\[Int]\ we obtain

\[ \dot{\theta}(x, z) = \frac{2}{z r^2(x)} \int_0^x r^2(t) L'(t) dt \geq 0. \]

Therefore, \( \dot{\theta}(x, z) \geq 0 \). Let \( m \) be less than \( n \). Then \( \frac{m}{z_m} = \theta(z_m) \leq \frac{n}{z_n} = \theta(z_n) \), and thus \( \frac{m}{z_m} \leq \frac{n}{m} \) and \( \frac{\lambda_n(t)}{\lambda_m(t)} \leq \left( \frac{m}{n} \right)^2 \). Then from Proposition \[1\[Int]\ we get

\[ \frac{\lambda_n}{\lambda_m} \leq \left( \frac{n}{m} \right)^2. \]

The equality iff \( \frac{\lambda_n(t)}{\lambda_m(t)} \) is a constant. From \( (3.3) \), \( \dot{\theta}(x, z) = 0 \), which implies that \( \rho \equiv \dot{\rho} \equiv cte \). This completes the proof of the theorem. \( \square \)

Proof of Theorem \[1\[Int]\ We define \( \ddot{\rho}(x) = \rho(1-x) \), then \( \ddot{\rho}(x) \) is monotone decreasing in \([0, 1-x_0]\) and monotone increasing in \([1-x_0, 1]\). According to Proposition \[1\[Int]\ together with Corollary \[1\[Int]\ yields

\[ \frac{\lambda_n}{\lambda_m} = \frac{\lambda_n(\rho)}{\lambda_m(\rho)} \leq \frac{\lambda_n(L)}{\lambda_m(L)} \leq \left( \frac{n}{m} \right)^2. \]

The equality holds, if \( \rho \) is a constant. \( \square \)
4 Eigenvalue ratios for Sturm-Liouville problems

In this section, we modify the inverse Liouville substitution (e.g., see \cite{16, pp. 51, 3}) in order to transform Equation (1.1) into (1.3), whose the density is single-well. Therefore, we can use the result of section 2 on the ratio of eigenvalues $\frac{\lambda_n}{\lambda_m}$ for the string equations with single-well densities.

**Theorem 2** Let $q$ be a single-barrier potential and $pp$ be a single-well function with transition point $x_0 = \frac{1}{2}$ such that $0 < \min(\hat{\mu_1}, \tilde{\mu_1})$ where $\hat{\mu_1}$ and $\tilde{\mu_1}$ are the first eigenvalues of the Neumann boundary problems defined on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively. Then the eigenvalues of Problem (1.1)-(1.2) satisfy

$$\frac{\lambda_n}{\lambda_m} \leq \left(\frac{n}{m}\right)^2. \quad (4.1)$$

Equality holds iff $q \equiv 0$ and $pp$ is constant in $[0, 1]$.

The following result is stated without assumptions on the monotonicity on the potential $q$.

**Corollary 3** If $q$ is nonnegative and $pp$ is single-well with transition point $x_0 = \frac{1}{2}$, then

$$\frac{\lambda_n}{\lambda_m} \leq \left(\frac{n}{m}\right)^2. \quad (4.1)$$

Equality holds iff $q \equiv 0$ and $pp$ is constant on $[0, 1]$.

For the proof of Theorem 2, we need some preliminary results.

Let $h(x, \lambda)$ be the unique solution of Equation (1.1) satisfying the initial conditions

$$h(1/2) = 1, \quad h'(1/2) = 0. \quad (4.2)$$

We introduce the meromorphic function

$$F(x, \lambda) = \frac{ph'(x, \lambda)}{h(x, \lambda)}. \quad (4.3)$$

Let $\hat{\eta}_1$ and $\tilde{\eta}_1$ be the first eigenvalues of the problems determined by Equation (1.1) and the boundary conditions

$$y(0) = y'(1/2) = 0, \quad (4.4)$$

$$y'(1/2) = y(1) = 0, \quad (4.5)$$

respectively.
Lemma 2

- The function \( F(0, \lambda) \) is increasing along the interval \((-\infty, \hat{\eta}_1)\).
- The function \( F(1, \lambda) \) is decreasing along the interval \((-\infty, \tilde{\eta}_1)\).

**Proof** The proof is similar to that of Lemma 3 in [4]. □

We are now ready to prove Theorem 2.

**Proof** Firstly, if \( q \equiv 0 \) then by use the Legendre substitution [14, pp. 227-228]

\[
t(x) = \frac{1}{\sigma} \int_0^x \frac{1}{p(z)} dz, \quad \sigma = \int_0^1 \frac{1}{p(z)} dz,
\]

Equation (1.1) can be rewritten in the string equation

\[-\ddot{y} = \lambda \sigma^2 \tilde{p}(t) \tilde{\rho}(t) y,\]

where \( \tilde{p}(t) = p(x) \) and \( \tilde{\rho}(t) = \rho(x) \). Thus the estimate (1.1) is direct consequence of Theorem 1. In the sequel we suppose that \( q \not\equiv 0 \). Assume that \( \hat{\mu}_1 = \min(\hat{\mu}_1, \tilde{\mu}_1) \) and let \( h \) be the unique solution of the second-order equation

\[
(p(x)y')' = q(x)y,
\]

satisfying the initial conditions (4.2). Hence, by the hypothesis and the variational principle,

\[0 < \hat{\mu}_1 < \hat{\eta}_1.\]

It is known that \( h(x, \hat{\eta}_1) > 0 \) on \((0, \frac{1}{2}]\), by Sturm comparison theorem (see [15, Chap. 1]), we have \( h(x) > 0 \) on \([0, \frac{1}{2}]\).

On the other hand, since \( F(0, \hat{\mu}_1) = 0 \), then in view of Lemma 2 \( F(0, \lambda) \leq 0 \) on \((-\infty, \hat{\mu}_1)\). Hence, from the condition \( 0 < \hat{\mu}_1 \), together with \( h(0) > 0 \), we get \( h'(0) \leq 0 \).

Taking into account that \( q \) is increasing on \([0, \frac{1}{2}]\), then it may vanish at most once, say at \( a_0 \in [0, \frac{1}{2}] \). From this and (4.7), we have \((ph')' \leq 0\) on \([0, a_0]\) and \((ph')' \geq 0\) on \([a_0, \frac{1}{2}]\), and consequently \( ph' \) is increasing on \([0, a_0]\) and increasing on \([a_0, \frac{1}{2}]\). Since \( h'(\frac{1}{2}) = 0 \) and \( h'(0) \leq 0 \), then \( h'(x) \leq 0 \) on \([0, \frac{1}{2}]\]. Using similar arguments, it can be shown that \( h'(x) \geq 0 \) on \([\frac{1}{2}, 1]\). Therefore, \( h \) is a single-well function on \([0, 1]\).

We introduce the modified inverse Liouville substitution (e.g., see [16, pp. 51], [3])

\[
z(x) = \frac{1}{c} \int_0^x \frac{1}{h^2(s)} ds, \quad \text{where} \quad c = \int_0^1 \frac{1}{h^2(s)} ds,
\]

which transforms Problem (1.1)-(1.2) into the system

\[
\begin{cases}
-(p(z)u')' = \tilde{\lambda} \tilde{h}^4(z) \rho(z) u, & z \in (0, 1), \\
u(0) = u(1) = 0,
\end{cases}
\]

\[\text{(4.9)}\]
where \( y = uh, \tilde{h}(z) = h(x) \) and \( \tilde{\lambda} = c^2 \lambda \). Using the Legendre substitution (4.6), the System (4.9), becomes a string equation

\[
-\ddot{u} = \sigma^2 \tilde{\lambda} \tilde{h}^4(t) \dot{p}(t) \dot{\rho}(t) u, \ t \in (0, 1),
\]

with Dirichlet boundary conditions

\[
u(0) = u(1) = 0,
\]

where \( \dot{p}(t) = p(x), \dot{\rho}(t) = \rho(x) \) and \( \tilde{h}(t) = h(x) \). Taking into account that \( \tilde{h} \) is a single-well on \([0, 1]\), then in view of Theorem 1, we have

\[
\tilde{\lambda}_n \tilde{\lambda}_m \leq (\frac{n}{m})^2.
\]

Since \( \tilde{\lambda}_n = c^2 \sigma^2 (\lambda_n - \mu_1) \), then \( \frac{\tilde{\lambda}_n}{\tilde{\lambda}_m} \leq (\frac{n}{m})^2 \).

Assume that there exist \( q(x), p(x) \) and \( \rho(x) \) such that \( \frac{\tilde{\lambda}_n}{\tilde{\lambda}_m} = (\frac{n}{m})^2 \), where \( q(x) \neq 0 \) or \( pp\rho(x) \) is not constant on \([0, 1]\). It is clear that the density in equation (4.10) is constant iff \( \tilde{h}^4 = \frac{\alpha}{pp} \) on \([0, 1]\) for some \( \alpha > 0 \), which is not possible from the monotonicity of \( \tilde{h}^4pp \). This is in contradiction with Theorem 1. The proof of the theorem is complete.

we can now prove Corollary 3.

**Proof** Following the proof of Theorem 2, let \( h \) be the solution of Problem (4.7)-(4.2). As before \( \min(\tilde{\eta}_1, \tilde{\eta}_1) > 0 \), we have \( h(x) > 0 \) on \([0, 1]\). Since \( q(x) \geq 0 \) on \([0, 1]\), then by (4.7) and (4.2), \( h'(x) \leq 0 \) on \([0, \frac{1}{2}]\) and \( h'(x) \geq 0 \) on \([\frac{1}{2}, 1]\). Therefore \( h \) is a single-well function on \([0, 1]\). The rest of the proof is similar to that of Theorem 2.

**Remark 1** The method used in the proof of Theorem 1 cannot be applied in the case of single-barrier densities. More precisely, we have

\[
\frac{\lambda_n(\rho)}{\lambda_m(\rho)} \geq \frac{\lambda_n(\tilde{\rho})}{\lambda_m(\tilde{\rho})}
\]

on the other hand,

\[
\frac{\lambda_n(L)}{\lambda_m(L)} \leq (\frac{n}{m})^2.
\]

I believe that different techniques are needed to deal with the case of single barrier densities.

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References

[1] M. S. Ashbaugh and R. D. Benguria, Optimal bounds for ratios of eigenvalues of one dimensional Schrödinger operators with Dirichlet boundary conditions and positive potentials, Comm. Math. Phys., 124, (1989), 403 – 415.

[2] M. Ashbaugh and R. Benguria, Optimal lower bound for the gap between the first two eigenvalues of one-dimensional Schrödinger operators with symmetric single-well potentials, Proc. Amer. Math. Soc., 105, (1989), 419 – 424.

[3] M. S. Ashbaugh and R. D. Benguria, Eigenvalue ratios for Sturm-Liouville operators, J. Differential Equations, 103, (1993), 205 – 219.

[4] J. Ben Amara and Jihed Hedhly, Eigenvalue ratios for Schrödinger operators with indefinite potentials, Applied Mathematics Letters, 76, (2018), 96 – 102.

[5] A. Constantin, A general-weighted Sturm-Liouville problem, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, 24, (1997), 767 – 782.

[6] A. Constantin, On the Inverse Spectral Problem for the Camassa-Holm Equation, journal of functional analysis, 155, (1998), 352 – 363.

[7] M-J. Huang, On The Eigenvalue Ratio For Vibrating Strings, Proc. Amer. Math. Soc., 127, (2006), 1805 – 1813.

[8] M. J. Huang, The eigenvalue ratio for a class of densities, J. Math. Anal. Appl., 435, (2016), 944 – 954.

[9] Y. L. Huang and C. K. Law, Eigenvalue ratios for the regular Sturm-Liouville system, Proc. Amer. Math. Soc., 124, (1996), 1427 – 1436.

[10] M. Horváth, on the first two eigenvalues of Sturm-Liouville operators, Proc. Amer. Math. Soc., 131, (2002), 1215 – 1224.

[11] M. Horváth and M. Kiss, A bound for ratios of eigenvalues of Schrödinger operators with single-well potentials, Proc. Amer. Math. Soc., 134, (2005), 1425 – 1434.

[12] J. B. Keller, The minimum ratio of two eigenvalues, SIAM J. Appl. Math., 31, (1976), 485 – 491.

[13] M. Kiss, Eigenvalue ratios of vibrating strings, Acta Math. Hungar., 110, 2006, 253 – 259.

[14] W. Leighton, Ordinary Differential Equations. 3rd ed. Wadsworth, Belmont. CA., 1970.
[15] B.M. Levitan and I. S. Sargsyan, Introduction to spectral theory: Selfadjoint Ordinary Differential Operators, American Mathematical Society, Translation of Mathematical Monographs, 39, (1975).

[16] W. Magnus and S. Winkler, Hill’s Equation. Wiley. New York. 1966, reprinted by by Dover. New York 1979. , AMS, 1975.