Space-time dimensionality $D$ as complex variable: calculating loop integrals using dimensional recurrence relation and analytical properties with respect to $D$.

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Abstract

We show that dimensional recurrence relation and analytical properties of the loop integrals as functions of complex variable $D$ (space-time dimensionality) provide a regular way to derive analytical representations of loop integrals. The representations derived have a form of exponentially converging sums. Several examples of the developed technique are given.

Key words: loop integrals, dimensional recurrence relation, complex analysis

1 Introduction

High-order radiative corrections to different amplitudes and cross section become nowadays more and more important both in QED and QCD. In QED this is due to the high precision of the modern spectroscopy experiments, while in QCD the radiative corrections play important role due to the strength of the interaction. The radiative corrections are expressed in terms of the loop integrals, so the possibility to calculate the latter is also very important. Several powerful techniques have been developed for the calculation of the multiloop integrals. Among them is the IBP (integration-by-part) reduction procedure [12], which allows one to reduce arbitrary loop integral to some finite set of master integrals. Owing to IBP reduction, the problem of calculation of loop integrals is reduced to the calculation of master integrals. It is important to

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understand that IBP reduction also helps in the calculation of the master integrals. Indeed, differentiating the master integral with respect to some external invariant or mass, we obtain linear combination of integrals of the same topology or its subtopologies. Applying IBP reduction procedure, we can express this combination via the same set of master integrals. Acting in this way, one can obtain the system of ordinary differential equations for all master integrals \[3,4\]. The general solution of this system depends on several constants parametrizing the solution of the corresponding homogeneous system. These constants can be fixed from the value of the integrals at some specific kinematic parameters, or from their asymptotics in which one invariant becomes big, or small, in comparison with others. This method can be applied to the integrals with several external parameters.

However, the integrals with one scale are also important for the applications. For them, the method of differential equations does not work and one has to rely on other methods. In addition to the direct calculation using, e.g., Mellin-Barnes representation, one can try the method of difference equations. One of the variants of the difference equation is described by Laporta in Ref. [5]. In this method one considers the generalized master integral which is formed by raising one of the massive denominators of the master integral of interest to arbitrary power \(x\). Using IBP reduction one can obtain the difference equation with respect to \(x\). The part, corresponding to the general solution of the homogeneous equation is then fixed from the large-\(x\) asymptotics. The resulting expression for the master integral has the form of factorial series with power-like (harmonic) convergence (the general term of the series falls down as \(k^{-\alpha}\), where \(k\) is the summation variable, \(\alpha > 1\)). This method has been successfully applied for several multiloop tadpole integrals, see, e.g., Refs. [6,7]. Due to the power-like convergence of sums, the difficulties of obtaining high-precision results rapidly grow with the number of digits. The multi-digit results, on the other hand, are important for the \texttt{ps1q} algorithm [12], which allows one to restore the analytical result from the high-precision numerical one, provided the basis of transcendental numbers is known.

Another variant of the difference equation is the dimensional recurrence relation connecting the master integrals in different dimensions, suggested by Tarasov in Ref. [8]. We remind, that, in contrast to the differential equation, where the solution of the homogeneous equation is parametrized by several arbitrary constants, in the case of difference equation the corresponding solution is parametrized by several arbitrary periodic functions. In Refs. [39] the large-\(D\) asymptotics has been used for fixing these functions. This asymptotics has been derived from the explicit parametric form of the integrals. However, for multiloop integrals the calculation of this asymptotics appears to be hardly accessible. This is why the method of the dimensional recurrence relation was mainly applied to the calculation of one-loop integrals. It is worth noting that for the case of several external invariants, one may rely on the combination of
differential and difference equations methods, see Ref. [10].

Recently, in Ref. [11] the method of the dimensional recurrence relation was successfully applied to the calculation of certain four-loop tadpole. In that paper, an arbitrary periodic function parametrizing the solution of the homogeneous equation was numerically shown to be equal to zero by using the Laporta-like difference equation with respect to the power of massive denominator. However this vanishing of the homogeneous part was rather an exception than a rule, and in many known cases the homogeneous part is not zero. Needless to say that in these cases the correct choice of the periodic functions not only can be hardly justified, but plainly difficult to guess. On the other hand, all examples of application of the dimensional recurrence relation show that results obtained with the help of this method have a form of exponentially converging series (the general term of the series falls down as $a^{-k}$, where $k$ is the summation variable, $a > 1$). The rapid convergence of these series allows one to use the \texttt{pslq} algorithm to express their $\epsilon$-expansion in terms of conventional transcendental numbers, like multiple zeta-functions.

In this paper, we describe a method of the calculation of loop integrals based on the usage of Tarasov’s dimensional recurrence relation. The key point of our approach is the use of the analytical properties of the integrals as functions of the complex variable $D$. These properties allow one to fix the periodic functions up to some constants which can be found from the calculation of the integral at some definite $D$.

The paper is organized as follows. In the next Section, we introduce notations and formulate a general path of finding the master integrals. In Section III we briefly analyse the analytical properties of the parametric representation needed for our consideration. In Section IV we give several examples of the application of the formulated approach. Section V contains discussions of the generalization of the formulated approach to the topologies with more than one master integral. The last Section gives short conclusion.

2 Dimensional recurrence relation

Assume that we are interested in the calculation of the $L$-loop integral depending on $E$ linearly independent external momenta $p_{1}, \ldots, p_{E}$. There are $N = L(L+1)/2 + LE$ scalar products depending on the loop momenta $l_{i}$:

$$s_{ik} = l_{i} \cdot q_{k}, \quad i = 1, \ldots, L, \quad k = 1, \ldots, L + E,$$

(1)

where $q_{1, \ldots, L} = l_{1, \ldots, L}, \quad q_{L+1, \ldots, L+E} = p_{1, \ldots, E}$. 
The loop integral has the form

$$J^{(D)}(\mathbf{n}) = \int \frac{d^D l_1 \ldots d^D l_L}{\pi^{L \mathcal{D}/2} D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}}$$

(2)

where the scalar functions $D_\alpha$ are linear polynomials with respect to $s_{ij}$. The functions $D_\alpha$ are assumed to be linearly independent and to form a complete basis in the sense that any non-zero linear combination of them depends on the loop momenta, and any $s_{ik}$ can be expressed in terms of $D_\alpha$. For further purposes it is convenient to introduce the operators $A_\alpha$ and $B_\alpha$, see Ref. [13], acting as follows

$$\left( A_i J^{(D)} \right)(n_1, \ldots , n_N) = n_i J^{(D)}(n_1, \ldots , n_i + 1, \ldots , n_N),$$

$$\left( B_i J^{(D)} \right)(n_1, \ldots , n_N) = J^{(D)}(n_1, \ldots , n_i - 1, \ldots , n_N).$$

(3)

Relation between the loop integrals in different dimensions has been first introduced in Ref. [8]. It was derived using the parametric representation for the loop integrals. For the integral given by some graph, the result of Ref. [8] may be represented as follows

$$J^{(D-2)}(\mathbf{n}) = \mu^L \sum_{\text{trees}} \left( A_{i_1} \ldots A_{i_L} J^{(D)} \right)(\mathbf{n}),$$

(4)

where $i_1, \ldots , i_L$ numerate the chords of the tree, and $\mu = \pm 1$ for the euclidean/pseudoeuclidean case, respectively. It is natural to label this relation as a raising one since it expresses any integral in $D - 2$ dimensions via the integrals in $D$ dimensions. It may be also convenient to use the lowering dimensional relation which we will obtain now. Our derivation is based on the Baikov’s approach [14] in which one passes from the integration over loop momenta to the integration over scalar products. The master formula can be represented as

$$\int \frac{d^D l_1 \ldots d^D l_L}{\pi^{L \mathcal{D}/2}} f = \frac{\mu^L \pi^{-L \mathcal{E}/2 - L(L-1)/4}}{\Gamma[(D - \mathcal{E} - L + 1)/2, \ldots , (D - \mathcal{E})/2]} \times \int \left( \prod_{i=1}^{L} \prod_{j=i}^{L+E} d s_{ij} \right) \frac{\left[ V(l_2, \ldots l_L, p_1, \ldots , p_E) \right]^{(D-E-L-1)/2}}{\left[ V(p_1, \ldots , p_E) \right]^{(D-E-1)/2}} f,$$

(5)

where

$$V(q_1, \ldots , q_M) = \begin{vmatrix} q_1^2 & \cdots & q_1 \cdot q_M \\ \vdots & \ddots & \vdots \\ q_1 \cdot q_M & \cdots & q_M^2 \end{vmatrix}$$

(6)

is a Gram determinant constructed on the vectors $q_1, \ldots , q_M$ and $f$ is arbitrary function of scalar products $s_{ij}$. For our purposes, of course, we choose $f =
Due to the aforementioned properties of basis \( \{D_1, \ldots, D_N\} \), the function \( V(l_2, \ldots, l_L, p_1, \ldots, p_E) \) has the form of some polynomial of degree \( L + E \) of \( D_1, \ldots, D_N \) (and of external invariants, which is not important for our purposes):

\[
V(l_2, \ldots, l_L, p_1, \ldots, p_E) = P(D_1, D_2, \ldots, D_N) \quad (7)
\]

Replacing \( D \to D+2 \) in Eq. (5), we obtain the lowering dimensional recurrence relation

\[
J^{(D+2)}(n) = \frac{(2\mu)^L [V(p_1, \ldots, p_E)]^{-1}}{(D - E - L + 1)_L} \left( P(B_1, B_2, \ldots, B_N) J^{(D)} \right)(n). \quad (8)
\]

Comparing Eqs. (4) and (8) we see that in order to obtain the former, one has to analyse the graph corresponding to the loop integral, while the latter can be obtained without any reference to the graph. So, for computer implementation the lowering dimensional relation can be more convenient. On the other hand, Eq. (4) is more "economic" since the right-hand side of it contains integrals with indices shifted by at most \( L \), while the right-hand side of Eq. (8) may contain integrals with indices shifted by \( L + E \). Of course, after IBP reduction, both raising and lowering recurrence relations are equivalent.

In case when the integral in the left-hand side of Eq. (4) or Eq. (8) is master integral, IBP reduction of the right-hand side gives a difference equation for this master integral. If the topology contains only one master integral, the right-hand side, after IBP reduction, contains this master integral and master integrals of the subtopologies, which we assume to be found by the same method, or some other. In case of several master integrals in the topology, we can obtain in the same way a system of difference equations for them. We will concentrate below on the case of one master integral in the topology of interest. The discussion of the case of several-master topology is given in Section V. So, after the IBP reduction we obtain from, e.g., raising relation the following dimensional recurrence relation

\[
J^{(D-2)} = C(D) J^{(D)} + R(D), \quad (9)
\]

where \( R(D) \) is non-homogeneous part constructed of the master integrals of subtopologies in \( D \) dimensions. The coefficient \( C(D) \) is a rational function of \( D \) which can always be represented as

\[
C(D) = \frac{c \prod_i (a_i - D/2)}{\prod_j (b_j - D/2)} \quad (10)
\]

where \( c \) is some constant, and \( a_i \) (\( b_j \)) are the zeros of the numerator (denominator). We can construct the solution of the homogeneous part of the equation
as follows:

$$J_0^{(D)} = \frac{\omega(D)}{\Sigma(D)},$$

(11)

where \(\omega(D) = \omega(D + 2)\) is arbitrary periodic function of \(D\) and \(\Sigma^{-1}(D)\) is any specific non-zero solution of the homogeneous equation. One of possible choices of \(\Sigma(D)\) is

$$\Sigma^{-1}(D) = c^{-D/2} \prod_{i=1}^{n} \Gamma (a_i - D/2) \prod_{j=1}^{m} \Gamma (b_j - D/2).$$

(12)

By analogy with the notion of integrating factor in the theory of differential equations, we will call \(\Sigma(D)\) the summing factor. The choice of the summing factor is by no means unique. Given one summing factor, we may multiply it by any periodic function of \(D\) to obtain another. In particular, another natural choice is

$$\Sigma^{-1}(D) = \left[(-1)^{n+m} c\right]^{-D/2} \prod_{i=1}^{n} \Gamma (D/2 + 1 - b_j) \prod_{j=1}^{m} \Gamma(D/2 + 1 - a_i).$$

(13)

Let us now substitute \(J^{(D)} = \Sigma^{-1}(D) g(D)\) in Eq. (9). After multiplying both parts by \(\Sigma(D - 2)\), we obtain

$$g(D - 2) = g(D) + r(D),$$

(14)

where \(r(D) = R(D) \Sigma(D - 2)\). Suppose now that we can decompose \(r(D)\) into two, \(r_+(D)\) and \(r_-(D)\), decreasing fast enough at \(\pm \infty\), respectively:

$$r(D) = r_+(D - 2) + r_(D), \quad r_\pm(D \pm 2k) \xrightarrow{k \to \infty} a^k, \quad |a| < 1 \quad (15)$$

Note the shift of the argument in the definition of \(r_+(D)\). This decomposition also determines the corresponding decomposition

$$R(D) = R_+(D - 2) + R_-(D),$$

(16)

such that

$$\left| \lim_{D \to +\infty} \frac{C(D + 2) R_+(D + 2)}{R_+(D)} \right| < 1,$$

$$\left| \lim_{D \to -\infty} \frac{C^{-1}(D - 2) R_-(D - 2)}{R_-(D)} \right| < 1. \quad (17)$$
The general solution reads

\[ J^{(D)} = \Sigma^{-1} (D) \omega (D) + \sum_{k=0}^{\infty} s_+ (D, k) - \sum_{k=0}^{\infty} s_- (D, k), \]  

(18)

\[ s_+ (D, k) = \frac{r_+ (D + 2k)}{\Sigma (D)} = \left[ (-1)^{n+m} c \right]^k \prod_{i=1}^{n} \left( \frac{D}{2} + 1 - a_i \right)_k R_+ (D + 2k), \]

\[ s_- (D, k) = \frac{r_- (D - 2k)}{\Sigma (D)} = c^{-1-k} \prod_{j=1}^{m} \left( b_j - \frac{D}{2} \right)_{k+1} R_- (D - 2k). \]

Note that the specific nonhomogeneous solution does not depend on the particular choice of the summing factor. Eq. (18) clearly demonstrates that, in order to find \( J^{(D)} \), we have to fix the periodic function \( \omega (D) \). This function can be conveniently understood as a function of the complex variable

\[ z = e^{\pi D} \]  

(19)

and below we switch the notations \( \omega (D) \rightarrow \omega (z) \). Understanding \( \omega \) as a function of \( z \), we automatically account for the periodicity due to identity \( e^{\pi(D+2)} = e^{\pi D} \). Our idea of fixing \( \omega (z) \) is simple and yet quite efficient as we will demonstrate in Section IV. The functions \( r_{\pm} \) in the right-hand side of Eq. (18) are assumed to be known and, in particular, their analytical properties are known. The integral \( J^{(D)} \) in the left-hand side is not known (this is the purpose of our investigation), but we can discover some of its analytical properties from, e.g., its parametric representation. Note that we do not have to analyse the integral in the whole complex plane of \( D \), but only in an arbitrary stripe of width 2, i.e., in \( S = \{ D \mid \ d < \text{Re} D \leq d + 2 \} \), which we will call the basic stripe. The factor \( \Sigma (D) \) is also a known analytical function of \( D \). This allows us to fix \( \omega (z) \) up to some constant which can be found from the value of the integral \( J^{(D)} \) at one given value of \( D \). There is a number of technical conveniences in this approach. First, as we have said above, the summing factor \( \Sigma (D) \) can be chosen in different forms and we are free to choose the one with the most convenient analytical properties. Second, we can pass from using one integral of certain topology as a master to using another integral of the same topology as a master. Finally, we are free to choose the convenient basic stripe in which we perform the analysis.

So, summarizing, the suggested path of finding a master integral is the following (we assume we can do IBP reduction):

1. Make sure all master integrals in subtopologies are known. If it is not so, start from calculating them.
2. Pass to a suitable master integral. It is convenient to choose a master
integral which is finite in some interval \( \mathcal{D} \in (d, d+2) \). For this purpose, e.g., increase powers of some massive propagators.

(3) Construct dimensional recurrence relation for this master integral. Due to step 1, the nonhomogeneous part of the recurrence relation is known.

(4) Find a general solution of this recurrence relation depending on function \( \omega(z) \).

(5) Fix the singularities of this function by analysing the analytical properties of the master integrals and summing factor. The convenient choices of summing factor and of basic stripe are at your disposal.

(6) If needed, fix the remaining constants from the value of the integral at some space-time dimension \( \mathcal{D} \).

The step 5 is the key point of our approach. We will present the applications of the formulated approach on several examples, but first we would like to demonstrate how some of the analytical properties of the integral can be determined from its parametric representation.

3 Parametric representation

Let us consider the parametric representation of some \( L \)-loop integral in Euclidean space with \( I \) internal lines (see, e.g., Ref. [15]):

\[
J^{(D)} = \Gamma (I - LD/2) \int dx_1 \cdots dx_I \delta (1 - \sum x_i) \left( \frac{[Q(x)]^{D_L/2-I}}{[P(x)]^{D(L+1)/2-I}} \right) .
\]  

The polynomials \( Q(x) \) and \( P(x) \) are determined by the graph. For our consideration it is important only that both these functions are nonnegative in the whole integration region,

\[
Q(x) \geq 0, \quad P(x) \geq 0.
\]

Suppose now that the parametric representation converges for all \( D \) in some interval \( (d_1, d_2) \). Then it is easy to see that Eq. (20) determines \( J^{(D)} \) as a holomorphic function on the whole stripe \( S = \{ D | \quad \text{Re} \ D \in (d_1, d_2) \} \). Moreover,
as $\text{Im } D$ tends to $\pm \infty$ while $\text{Re } D = d \in (d_1, d_2)$ is fixed, we can estimate

$$|J^{(D)}| = |\Gamma (I - LD/2)| \int dx_1 \ldots dx_I \delta (1 - \sum x_i) \frac{[Q (x)]^{dL/2-I}}{[P (x)]^{d(L+1)/2-I}}$$

$$\leq |\Gamma (I - LD/2)| \int dx_1 \ldots dx_I \delta (1 - \sum x_i) \frac{[Q (x, P, m)]^{dL/2-I}}{[P (x)]^{d(L+1)/2-I}}$$

$$= |\Gamma (I - LD/2)| \int dx_1 \ldots dx_I \delta (1 - \sum x_i) \frac{[Q (x, P, m)]^{dL/2-I}}{[P (x)]^{d(L+1)/2-I}}$$

$$= \frac{|\Gamma (I - LD/2)|}{\Gamma (I - Ld/2)} J^{(d)} \lesssim \text{const} \times e^{-\frac{\pi |\text{Im } D|}{4}} |\text{Im } D|^{-1/2-L \text{Re } D/2} \quad (21)$$

These properties of the parametric representation will be used in the examples in the next Section.

4 Examples

In this Section we give several examples of the application of our method to different loop integrals. Except for the last example, our consideration is restricted by the one-scale integrals, since several-scale integrals can be treated by the differential equation method. In all cases, the IBP reduction has been performed using the Mathematica program, based on Ref. [13]. All integrals considered in this paper are known due to some other techniques. Our representations for one-scale cases, however, differ from the known ones, being exponentially convergent sums.

4.1 Three-loop sunrise tadpole: basics of the technique

Let us consider the three-loop integral

$$J_1^{(D)} = \int = \int \frac{d^D k \, d^D l \, d^D r}{\pi^{3D/2} [k^2 + 1] [l^2 + 1] [r^2 + 1] [(k + l + r)^2 + 1]} \quad (22)$$

This integral has been investigated in Refs. [16,17] and the result is expressed in terms of $3F_2 \ldots 1$. We will follow the path formulated in the end of Section II.

(1) There is one master integral in subtopologies:

$$J_{1a}^{(D)} = \Gamma^3 (1 - D/2) \quad (23)$$
The integral \( J_1^D \) is a holomorphic function in the stripe \( S = \{ D \mid \text{Re } D \in [-2, 0) \} \) which can be easily understood from the parametric representation:

\[
J_1^D = \Gamma \left( 4 - 3D/2 \right) \int \frac{dx_1 dx_2 dx_3 dx_4 \delta \left( 1 - \sum x_i \right)}{[x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4]^{D/2}}
\] (24)

It is easy to realise that any Euclidean integral with all internal propagators massive is holomorphic in the whole half-plane \( \text{Re } D < 0 \).

The dimensional recurrence reads

\[
J_1^{(D-2)} = -\frac{(3D - 10)_3 (D - 2)}{128(D - 4)} J_1^D - \frac{(11D - 38)(D - 2)^3}{64(D - 4)} J_1^D
\] (25)

where \( \alpha_n = \alpha (\alpha + 1) \ldots (\alpha + n - 1) \) denotes the Pochhammer symbol.

We choose the summing factor as

\[
\Sigma^{-1} (D) = 4^D \frac{\Gamma \left( 3/2 - D/2 \right) \Gamma \left( 3 - 3D/2 \right)}{\Gamma \left( 2 - D/2 \right)}
\] (26)

Note that the chosen summing factor has neither poles nor zeros in \( S \).

Representing \( J_1^D = \Sigma^{-1} (D) g (D) \), we obtain

\[
g (D - 2) = g (D) + r (D)
\]

\[
r (D) = -\frac{(11D - 38) \Gamma^4 (2 - D/2)}{4^D \Gamma \left( 5/2 - D/2 \right) \Gamma \left( 6 - 3D/2 \right)}
\] (27)

The nonhomogeneous part \( r (D - 2k) \) falls down as \((16/27)^k\) when \( k \) goes to \( \infty \), so we can write the solution in the form

\[
J_1^D = \omega (z) / \Sigma (D) + t (D) = 4^D \frac{\Gamma \left( 3/2 - D/2, 3 - 3D/2 \right)}{\Gamma \left( 2 - D/2 \right)} \omega (z)
\]

\[+ \frac{1}{16 \Gamma \left( 2 - D/2 \right)} \sum_{k=1}^{\infty} \frac{(11D - 16 - 22k) \Gamma^4 (1 + k - D/2)}{(3/2 - D/2)_k \left( 3 - 3D/2 \right)_k} 16^k
\] (28)

Let us consider analytical properties of functions entering Eq. (28) in \( S \). The integral \( J_1^D \) is a holomorphic function in \( S \). When \( \text{Im } D \rightarrow \pm \infty \), it falls down at least as

\[
\left| J_1^D \right| \lesssim \left| \Gamma \left( 4 - 3D/2 \right) \right| \sim e^{-\frac{3|\text{Im } D|}{4}} |\text{Im } D|^{7/2 - 3 \text{Re } D/2}
\] (29)

which can be clearly seen from the parametric representation (24). The summing factor is also holomorphic function in \( S \) and

\[
|\Sigma (D)| \lesssim e^{\frac{3|\text{Im } D|}{4}} |\text{Im } D|^{-2 + 3 \text{Re } D/2}
\] (30)
in the limit \( \text{Im} \mathcal{D} \to \pm \infty \). Finally, the specific solution \( t(\mathcal{D}) \) of the non-homogeneous equation is also a holomorphic function in \( S \), and

\[
|t(\mathcal{D})| \lesssim e^{-\frac{3\epsilon \text{Im} \mathcal{D}}{4}} |\text{Im} \mathcal{D}|^\sigma
\]  

(31)

Note that the limits \( \text{Im} \mathcal{D} \to \pm \infty \) correspond to the limits \( z \to 0, \infty \). So, from Eqs. (28), (29), (30), and (31) we conclude that

\[
\omega(z) \sim |\text{Im} \mathcal{D}|^\nu,
\]  

(32)

where \( \nu \) is some real number not essential for our consideration. Taking into account that

\[
\lim_{z \to \infty} |\text{Im} \mathcal{D}|^\nu |z|^{-\alpha} = \lim_{z \to 0} |\text{Im} \mathcal{D}|^\nu |z|^{-\alpha} = 0
\]  

(33)

for any \( \nu \) and any \( \alpha > 0 \), we conclude that \( \omega(z) \) is holomorphic function in the extended complex plane of \( z \), except, may be \( z = 0 \) and \( z = \infty \) and growing slower than any positive(negative) power of \( |z| \) when \( z \) tends to infinity (zero). This is sufficient to claim that \( \omega(z) \) is holomorphic function in the extended complex plane of \( z \), being therefore a constant.

(6) We can fix this constant by the condition \( J_1^{(D=0)} = 1 \), which, e.g., follows from the parametric representation. We finally obtain

\[
J_1^{(D)} = 4^D \frac{\Gamma(3/2 - D/2, 3 - 3D/2)}{\Gamma(2 - D/2)} \frac{1}{\sqrt{\pi}} \sum_{k=0}^\infty \frac{(1 + 11k/8)}{(3/2)_k (3/2)_k} \frac{1}{16^k} + \frac{1}{16^D} \frac{\Gamma(2 - D/2)}{\Gamma(2 - D/2)} \sum_{k=1}^\infty \frac{(11D - 16 - 22k)}{(3/2 - D/2)_k} \frac{1}{(3 - 3D/2)_k} \frac{1}{16^k}
\]  

(34)

It is easy to understand that general terms of both sums in Eq. (34) fall down as \( \left(\frac{16}{27}\right)^k \) thus providing fast convergence of sums which can be conveniently determined as \( 1/\log_{10}(27/16) \approx 4.4 \) terms per decimal digit. We may have expressed our result (34) in terms of \( {}_pF_q \ldots \left[\frac{16}{27}\right] \), but we consider the sums to be more treatable than these functions. In particular, in order to obtain the \( \epsilon \)-expansion of \( J_1^{(4-2\epsilon)} \), we can expand under the sum sign. It is easy to check that, up to a \( O(\epsilon^0) \), this expansion is given by the first term of the second sum:

\[
J_1^{(4-2\epsilon)} = -\frac{4}{3} \frac{11\epsilon - 3}{(2\epsilon - 1)(3\epsilon - 1)(3\epsilon - 2)(\epsilon - 1)} \Gamma^3[\epsilon] + O(\epsilon)
\]  

(35)

In fact, the next 5 terms of \( \epsilon \)-expansion of \( J_1^{(4-2\epsilon)} \) can be found in Refs. [16][17]. In order to demonstrate the fast convergence of the obtained representation we present here the 40-digit values of coefficients of \( J_1^{(4-2\epsilon)} \) expansion.

11
up to $O(\epsilon^6)$ obtained in less than 2 minutes on Mathematica:

\[
J_1^{(4-2\epsilon)} = +2.000000000000000000000000000000000 \times \epsilon^{-3} \\
+ 4.203372677257469503027594126172252080413 \times \epsilon^{-2} \\
+ 12.1574432207889158433285265524001769362 \times \epsilon^{-1} \\
+ 10.34395517124061695632484451407869433 \times \epsilon^0 \\
+ 22.40040337934740516273278836060402645950 \times \epsilon \\
- 192.9905185401601934396793596955193083 \times \epsilon^2 \\
- 298.298921053537223686516742365651739123 \times \epsilon^3 \\
- 3327.9312325386248395591754768685145911 \times \epsilon^4 \\
- 5027.397388051400645438297589076204952450 \times \epsilon^5 \\
- 37321.16865675250679268222655965209414217 \times \epsilon^6 \\
+ O(\epsilon^7) \quad (36)
\]

4.2 Four-loop tadpole: Dealing with massless internal lines.

Let us now consider the following four-loop tadpole integral:

\[
J_2^{(D)} = \int \frac{d^D k \, d^D l \, d^D r \, d^D p}{\pi^{2D} k^2 l^2 r^2 \left[ (k+p)^2 + 1 \right] \left[ (l+p)^2 + 1 \right] \left[ (r+p)^2 + 1 \right]} \quad (37)
\]

The explicit expression for this integral and expansions near $D = 4$ and $D = 3$ can be found in Ref. [7]. In this example we will demonstrate how to deal with the massless propagators.

(1) There is one master integral in subtopologies:

\[
J_{2a}^{(D)} = \infty = - \frac{\Gamma (4-3D/2) \Gamma^2 (3-D) \Gamma^2 (1-D/2) \Gamma (D/2-1)}{\Gamma (6-2D)} \quad (38)
\]

(2) Due to the presence of massless propagators, there is an infrared divergence at $D = 2$ in $J_2^{(D)}$. It is also easy to see that there is an ultraviolet divergence at $D = 3$. Thus, the integral is a holomorphic function only in the stripe $S = \{ D \mid \text{Re} \, D \in (2, 3) \}$. This is not sufficient for our purposes and we instead consider another integral, with massive propagators squared:

\[
\tilde{J}_2^{(D)} = \int \frac{d^D k \, d^D l \, d^D r \, d^D p}{\pi^{2D} k^2 l^2 r^2 \left[ (k+p)^2 + 1 \right]^2 \left[ (l+p)^2 + 1 \right]^2 \left[ (r+p)^2 + 1 \right]^2} \quad (39)
\]
Due to IBP identities, the original integral $J_2^{(D)}$ can be expressed via $\tilde{J}_2^{(D)}$ and $J_{2a}^{(D)}$ as follows:

$$J_2^{(D)} = -\frac{3(3D - 11)(3D - 10)}{4(D - 4)(D - 3)^3(2D - 7)}\tilde{J}_2^{(D)} - \frac{3(D - 2)(3D - 8)(13D^2 - 88D + 148)}{128(D - 3)^2(2D - 7)^2}J_{2a}^{(D)}$$ \hspace{1cm} (40)

Thus, we can restrict ourselves to the calculation of the integral $\tilde{J}_2^{(D)}$. This integral is obviously a holomorphic function of $D$ when $\text{Re} D \in \left(2, 4\right)$. This is sufficient for our purposes. So we choose the basic stripe as $S = \{D | \text{Re} D \in (2, 4]\}$. Let us make one observation which essentially simplifies the following consideration. The two terms in the right-hand side of Eq. (40) have poles of the third order at $D = 3$, while the integral $J_2^{(D)}$ clearly has only first-order pole at $D = 3$. Thus, from Eq. (40), we can extract two terms of expansion of $\tilde{J}_2^{(D)}$ near $D = 3$:

$$\tilde{J}_2^{(3-2\epsilon)} = \frac{\pi^2}{4} + \frac{\pi^2}{4} (11 - 4\gamma - 8 \ln 2) + O(\epsilon)$$ \hspace{1cm} (41)

(3) The dimensional recurrence for $\tilde{J}_2^{(D)}$ reads

$$\tilde{J}_2^{(D)} = -\frac{4(2D - 8)4(D - 3)^3(D - 1)_2}{3(3D - 11)_5}J_{2(D+2)}^{(D)} + R(D)$$ \hspace{1cm} (42)

(4) It is easy to establish from the explicit form of known integral $J_{2a}^{(D)}$ that in this case $R_-(D) = 0$. The summing factor obeys the equation

$$\sum(D + 2) = \frac{4(2D - 8)4(D - 3)^3(D - 1)_2}{3(3D - 11)_5}$$ \hspace{1cm} (43)

As we noticed above, we can use different forms of the summing factor and we choose

$$\sum(D) = \frac{\sin\left(\frac{\pi}{2}D - \frac{2\pi}{3}\right)\sin\left(\frac{\pi}{2}D - \frac{\pi}{3}\right)\sin^2\left(\frac{\pi D}{2}\right)}{2^D \sin\left(\frac{\pi}{2}D - \frac{5\pi}{6}\right)\sin\left(\frac{\pi}{2}D - \frac{\pi}{6}\right)} \times \frac{\Gamma\left(7 - \frac{3D}{2}\right)\Gamma\left(\frac{D}{2} - \frac{3}{2}\right)\Gamma(D - 1)}{\Gamma(9 - 2D)\Gamma(5 - D)\Gamma\left(\frac{3D}{2} - \frac{11}{2}\right)}$$ \hspace{1cm} (44)

Our choice of the summing factor needs some explanations. When choosing the summing factor, we used the following "rules of thumb".
(a) "No undetermined singularities in basic stripe": The function \( \Sigma (D) \) is chosen in such a way that \( \Sigma (D) J_2^{(D)} \) has only known singularities in \( S \). We did not eliminate the simple pole at \( D = 3 \) because the singular part of \( \Sigma (D) J_2^{(D)} \) near \( D = 3 \) is known due to Eq. (41).

(b) "Good behaviour at infinity": We require that the function \( \Sigma (D) J_2^{(D)} \) obey the following constraint

\[
\frac{\Sigma (D) J_2^{(D)}}{\exp \left[ \pi |D| \right]} \xrightarrow{\text{Im } D \to \pm \infty} 0
\]

when \( \text{Re } D \in (2, 4) \). It is very easy to establish from the Feynman parametrization, that

\[
J_3^{(D)} \xrightarrow{\text{Im } D \to \pm \infty} \Gamma \left[ 9 - 2D \right] |\text{Im } D|^\nu,
\]

so we can easily obtain the desired estimate for \( \Sigma (D) J_2^{(D)} \). We also require \( r_+ (D) = \Sigma (D) R (D) \) to obey the same constraint

\[
\frac{r_+ (D)}{\exp \left[ \pi |D| \right]} \xrightarrow{\text{Im } D \to \pm \infty} 0
\]

for \( \text{Re } D > 2 \).

(c) "Pole minimization": The function \( r_+ (D) = \Sigma (D) R (D) \) has some poles. We try to minimize their orders and/or number in the region \( \text{Re } D > 2 \) by adding some factors of the form \( \sin \left( (D - d) \pi / 2 \right) \). This requirement explains appearance of \( \sin \) functions in the numerator in Eq. (44). At some point we were not able to add \( \sin \) factors due to the condition b.

The pole structure of

\[
r_+ (D) = -\frac{3\pi^{5/2} \tilde{\omega}^{2D - 15} D \cos \left( \frac{3\pi D}{2} \right) \Gamma (7 - 3D) \Gamma \left( 4 - \frac{3D}{2} \right) \Gamma \left( \frac{D}{2} - \frac{1}{2} \right)^2}{\sin (\pi D) \tan \left( \frac{\pi D}{2} - \frac{\pi}{6} \right) \tan \left( \frac{\pi D}{2} + \frac{\pi}{6} \right) \Gamma (8 - 2D)^2 \Gamma \left( \frac{D}{2} + 1 \right)}
\]

\[
\times \left( 3299D^6 - 59493D^5 + 444098D^4 - 1756164D^3 + 3879800D^2 - 4540224D + 2198784 \right)
\]

(47)

is the following. There are 5 series of simple poles in

\[
D = D_1 (k), D_2 (k), D_3 (k), D_4 (k), D_5 (k), \quad k = 0, 1, \ldots
\]

\[
D_1 (k) = \frac{4}{3} + 2k, \quad D_2 (k) = \frac{2}{3} + 2k,
\]

\[
D_3 (k) = \frac{3}{2} + 2k, \quad D_4 (k) = \frac{3}{2} + 2k,
\]

\[
D_5 (k) = 4 + 2k,
\]

and also a single pole at \( D = 3 \). This pole structure is depicted in Fig. 1.
The general solution of the recurrence relation reads

\[ \tilde{J}_2^\mathcal{D} = \Sigma^{-1} (\mathcal{D}) \left[ \omega (z) + \sum_{k=0}^{\infty} r_+ (\mathcal{D} + 2k) \right], \]  

(49)

where \( \omega (z) = \omega (e^{i\pi \mathcal{D}}) \) is the arbitrary periodic function of \( \mathcal{D} \) we need to fix.

(5) Let us now analyse the singularities of \( \omega (z) \). Rewriting Eq.(49) as

\[ \omega (z) = \Sigma (\mathcal{D}) \tilde{J}_2^\mathcal{D} - \sum_{k=0}^{\infty} r_+ (\mathcal{D} + 2k), \]  

(50)

we see that singularities of this function are determined by the singularities of the terms in the right-hand side. Due to our choice of the summing factor (see 4a), the first term has only known singularities in \( \mathcal{S} \). Namely, it has simple pole at \( \mathcal{D} = 3 \) with the residue, determined from Eq. (41):

\[ \Sigma (3 - 2\epsilon) \tilde{J}_2^{(3-2\epsilon)} = \frac{3\pi^{5/2}}{256\epsilon} + O \left( \epsilon^0 \right) \]  

(51)

From the pole structure of \( r_+ (\mathcal{D}) \) we conclude that the first term of the sum in the right-hand side of Eq.(50) also has a pole at \( \mathcal{D} = 3 \), while the other terms of the sum have no singularities in this point. Moreover,

\[ r_+ (3 - 2\epsilon) = \frac{3\pi^{5/2}}{256\epsilon} + O \left( \epsilon^0 \right), \]  

(52)

so the singularities of the right-hand side of Eq.(50) at \( \mathcal{D} = 3 \) cancel. Thus, \( \omega (z) \) has no singularities in \( z = \exp [i\pi \times 3] = -1 \). The five series of poles (48) give rise to the poles of \( \omega (z) \) in

\[ z = e^{i\pi/3}, e^{i2\pi/3}, e^{-i2\pi/3}, e^{-i\pi/3}, 1. \]  

(53)
From the constraints (45) and (46) we conclude that \( \omega(z) \) is regular at \( z = 0, \infty \). Thus, the general form of \( \omega(z) \), mimicking the structure of the poles in the right-hand side of Eq. (50), is the following

\[
\omega(z) = a_0 + \frac{a_1}{z - e^{i\pi/3}} + \frac{a_2}{z - e^{i2\pi/3}} + \frac{a_3}{z - e^{-i\pi/3}} + \frac{a_4}{z - e^{-i2\pi/3}} + \frac{a_5}{z - 1} \tag{54}
\]

It is convenient to rewrite this form in terms of cotangents. Using the identity

\[
\cot\left(\frac{\pi}{2}(D - D_0)\right) = i\frac{z + e^{i\pi D_0}}{z - e^{i\pi D_0}} \tag{55}
\]

we represent

\[
\omega(z) = b + b_1 \cot\left(\frac{\pi}{2} (D - 2\frac{1}{3})\right) + b_2 \cot\left(\frac{\pi}{2} (D - 2\frac{2}{3})\right) + b_3 \cot\left(\frac{\pi}{2} (D - 3\frac{1}{3})\right) + b_4 \cot\left(\frac{\pi}{2} (D - 3\frac{2}{3})\right) + b_5 \cot\left(\frac{\pi}{2} (D - 4)\right) \tag{56}
\]

The constants \( b_{-5} \) are fixed by the singularities of the sum in the right-hand side of Eq. (50):

\[
b_i = -\frac{\pi}{2} \sum_{k=0}^{\infty} \text{Res}_{D_0 = D_i(k)} r_+(D), \tag{57}
\]

where \( D_i(k) \) are determined in Eq. (48). They can be expressed as linear combinations of \( {}_pF_q \). Using a guess \( b_i = f_i \pi^{7/2} \) with \( f_i \) being rationals, we were able to establish the simple form of these coefficients:

\[
b_1 \overset{N}{=} -b_2 \overset{N}{=} -b_3 \overset{N}{=} b_4 \overset{N}{=} -\frac{1}{4} b_5 \overset{N}{=} -\frac{3\pi^{7/2}}{32}, \tag{58}
\]

where \( \overset{N}{=} \) denotes the equality checked numerically with at least 10\(^3\) digits.

(6) The last coefficient \( b \) in \( \omega(z) \) can be shown to be equal to zero by considering the limit of the right-hand side of Eq. (50) at \( D \to 3 \) and using Eq. (41). Note that in this limit only the first term in the sum contribute. Thus, we obtain finally

\[
\tilde{J}^{(D)}_2 = \Sigma^{-1}(D) \left[ -\frac{3\pi^{7/2}}{32} \left( \cot\left(\frac{\pi}{2} (D - 2\frac{1}{3})\right) - \cot\left(\frac{\pi}{2} (D - 2\frac{2}{3})\right) \right. \\
- \cot\left(\frac{\pi}{2} (D - 3\frac{1}{3})\right) + \cot\left(\frac{\pi}{2} (D - 3\frac{2}{3})\right) - 4 \cot\left(\frac{\pi}{2} (D - 4)\right) \\
\left. + \sum_{k=0}^{\infty} r_+(D + 2k) \right] , \tag{59}
\]

where \( \Sigma^{-1}(D) \) and \( r_+(D) \) are determined in Eqs. (44) and (47), respectively. Again, we can express the sum via \( {}_pF_q \). Thus, the rate of convergence in representation (59) is \( 1/\log_{10}(3^9/2^{14}) \approx 12.5 \) terms per decimal digit. We have checked numerically the agreement of our representation with the known expansions around \( D = 4 \) and \( D = 3 \).
This example demonstrates how the method works in the case when both $R_+$ and $R_-$ are not zero. Let us consider the integral

$$J_3^{(D)} = \int \frac{d^D l d^D r}{\pi^2 l^2 r^2 (l-p)^2 (r-q)^2 (l-r-p)^2 (r-l-q)^2}$$

with

$$p^2 = q^2 = 0, \quad (p+q)^2 = -1.$$  

In contrast to the previous examples, here all scalar products are pseudoeuclidean, e.g., $l^2 = l_0^2 - l^2$. This integral in arbitrary dimension has been calculated in Ref. [18] and the result is expressed via $q+1F_q[\ldots|1]$ with $q = 3, 4$.

Let us again apply our technique.

1. There are two master integrals in the subtopologies:

$$J_{3a}^{(D)} = \int \frac{d^D l d^D r}{\pi^2 l^2 r^2 (l-p)^2 (r-q)^2 (l-r-p)^2 (r-l-q)^2} = \frac{\Gamma(3-D) \Gamma \left( \frac{D}{2} - 1 \right)^3}{\Gamma \left( \frac{3D}{2} - 3 \right)}$$

$$J_{3b}^{(D)} = \int \frac{d^D l d^D r}{\pi^2 l^2 r^2 (l-p)^2 (r-q)^2 (l-r-p)^2 (r-l-q)^2} = -\frac{\Gamma(4-D) \Gamma \left( 2 - \frac{D}{2} \right) \Gamma \left( \frac{D}{2} - 1 \right)^2 \Gamma(D-3)^2}{\Gamma(D-2) \Gamma \left( \frac{3D}{2} - 4 \right)}$$

2. The integral has collinear divergences at $D = 4$ and ultraviolet divergence at $D = 6$. So, we choose the basic stripe as $S = \{D| \text{Re}D \in (4, 6)\}$. Thorough analysis of Feynman parametrization justifies that $J_3^{(D)}$ is finite when $\text{Re}D \in (4, 6)$. We also note that at $D = 6$ the integral has simple pole.

3. The dimensional recurrence reads

$$J_3^{(D)} = -\frac{(2D-7)^4}{(D-4)^2} J_3^{(D+2)} + R_+ (D) + R_- (D+2)$$

$$R_+ (D) = -\frac{2 (43D^4 - 478D^3 + 1963D^2 - 3530D + 2352)}{(D-3) (D-4)^3} J_3^{(D)},$$

$$R_- (D+2) = -\frac{2 (37D^3 - 313D^2 + 858D - 752)}{(3D-8) (D-4)^2} J_3^{(D)}.$$  

Here we used the lowering recurrence relation, thus, appearance of $J_{3a, b}^{(D)}$ in the right-hand side. It is easy to check, that $R_\pm (D)$ satisfy the conditions (17).

4. We choose the summing factor as

$$\Sigma (D) = 4^D (4-D) \sin \left( \frac{\pi}{2} (D-5) \right) \sin^2 \left( \frac{\pi}{2} (D-4) \right) \Gamma \left( D - \frac{7}{2} \right).$$
Note the quadratic zero at \( D = 6 \), which provides that
\[
\lim_{D \to 6} \sum (D) J_3^{(D)} = 0, \tag{66}
\]
and, in particular, that the function \( \sum (D) J_4^{(D)} \) is regular at \( D = 6 \). The general solution reads
\[
J_3^{(D)} = \Sigma^{-1} (D) \left[ \omega (D) + \sum_{k=0}^{\infty} r_+ (D + 2k) - \sum_{k=0}^{\infty} r_- (D - 2k) \right], \tag{67}
\]
\[
r_+ (D) = \frac{\sqrt{2} \pi \sin(\pi D) \Gamma \left( \frac{3}{2} - \frac{D}{2} \right) \Gamma \left( \frac{D}{2} - 2 \right) \Gamma \left( D - \frac{7}{2} \right)}{(D-3) \Gamma \left( \frac{3D}{2} - 3 \right)} \times \left( 43D^4 - 478D^3 + 1963D^2 - 3530D + 2352 \right),
\]
\[
r_- (D) = -\frac{\pi 2^{2D-6} \Gamma \left( \frac{D}{2} - 3 \right) \Gamma \left( D - \frac{11}{2} \right)}{(5-D) \Gamma \left( \frac{3D}{2} - 6 \right)} (37D^3 - 535D^2 + 2554D - 4016).
\]

(5) Due to our choice of the summing factor, the function \( r_+ (D) \) does not have any singularities in the region \( \text{Re} \, D > 4 \). The function \( r_- (D) \) has simple poles at
\[
D = D_1 (k), D_2 (k), 5, 6 \quad \text{with } D_1 (k) = 5 \frac{1}{2} - 2k, \quad D_2 (k) = 4 \frac{1}{2} - 2k, \quad k = 0, 1, \ldots \tag{68}
\]
Taking into account that \( \sum (D) J_4^{(D)} \) is holomorphic in \( S \), we obtain
\[
\omega (D) = b + b_1 \cot \left( \frac{\pi}{2} (D - 5 \frac{1}{2}) \right) + b_2 \cot \left( \frac{\pi}{2} (D - 4 \frac{1}{2}) \right) + b_3 \cot \left( \frac{\pi}{2} (D - 6) \right),
\]
\[
b_1 = \frac{\pi}{2} \sum_{k=0}^{\infty} \text{Res}_{D=D_1(k)} r_- (D - 2k), \quad b_2 = \frac{\pi}{2} \sum_{k=0}^{\infty} \text{Res}_{D=D_2(k)} r_- (D - 2k),
\]
\[
b_3 = \frac{\pi}{2} \text{Res}_{D=5} r_- (D) = 256 \pi^{7/2}, \quad b_4 = \frac{\pi}{2} \text{Res}_{D=6} r_- (D) = 1280 \pi^{7/2} \tag{69}
\]
Again, making a guess \( b_{1,2} = f_{1,2} \pi^{7/2} \), supported by the values of \( b_{3,4} \), we check with the accuracy as high as \( 10^3 \) digits that
\[
b_1 = b_2 = -256 \pi^{7/2} \tag{70}
\]
(6) The constant \( b \) can be fixed using the condition (66) and the properties of \( r_\pm \):
\[
r_+ (6 + 2k) = 0, \quad r_- (6 - 2\epsilon) - \frac{2560 \pi^{5/2}}{-2\epsilon} = O (\epsilon)
\]
We obtain
\[
b = \sum_{k=0}^{\infty} r_- (4 - 2k) \overset{N}{=} 0. \tag{71}
\]
Therefore, Eqs. (67), (65), (69), (70), and (71) give us the result for \( J_3^{(D)} \). Note that the sums of \( r_\pm \) in Eq. (67) can be expressed via \(_5F_4 \left[ \ldots | -\frac{16}{27} \right]\) and \(_4F_3 \left[ \ldots | \frac{27}{64} \right]\), respectively. The arguments of the hypergeometric functions determine the rate of convergency of the sums: the first converges with the rate 4.5 terms per decimal digit, while the second converges with the rate 2.7 terms per decimal digit. We have checked numerically the agreement of our result for the \( \epsilon \)-expansion of \( J_3^{(4-2\epsilon)} \) with the corresponding result in Ref. [18].

4.4 Two-loop two-masses tadpole: Two-scale case

Let us now examine how the method works for the integrals with several scales. Consider the integral

\[
J_4^{(D)} = \int \frac{d^D l \, d^D r}{\pi^D [l^2 + 1] [r^2 + 1] [(l - r)^2 + m^2]} \tag{72}
\]

Of course, we can obtain the differential equation with respect to \( m \) and put the initial condition in the point \( m = 1 \). However, we can also proceed as before.

1. Master integrals in subtopologies are

\[
J_{4a}^{(D)} = \infty = m^{D-4} \infty = \Gamma^2 [1 - D/2] \tag{73}
\]

2. The basic stripe can be chosen, e.g., as \( S = \{D| \quad \text{Re} \, D \in [0, 2)\} \).

3. The dimensional recurrence reads

\[
J_4^{(D)} = \frac{m^2 (4 - m^2)}{(D - 2)(D - 3)} J_4^{(D-2)} + \frac{m^2 (4 - 2m^{D-2})}{(D - 2)(D - 3)} \Gamma^2 (2 - D/2) \tag{74}
\]

4. Let us assume that \( m < 2 \) for the moment. The result for \( m > 2 \) can be obtained by the analytical continuation. We choose the summing factor as

\[
\Sigma (D) = \frac{m^{-D} (4 - m^2)^{-D/2}}{\Gamma(2 - D)} \tag{75}
\]

After the replacement \( J_4^{(D)} = g(D)/\Sigma (D) \) we obtain

\[
g(D) = g(D - 2) + r(D) \\
r(D) = \frac{(m^2 - 2 - 2m^{D-2}) \Gamma^2 (2 - D/2)}{m^D (4 - m^2)^{D/2} \Gamma(4 - D)} \tag{76}
\]

19
The limit

$$\lim_{D \to -\infty} \frac{r(D + 2)}{r(D)} = \left(1 - \frac{m^2}{4}\right) \max\left(1, m^2\right) \leq 1 \quad (77)$$

shows that $r(D) = r_-(D)$ and allows us to represent the general solution in the form

$$J_4^{(D)} = \Sigma^{-1}(D) \omega(z) + t(D)$$

$$t(D) = \frac{\Gamma(1-D/2) \Gamma(2-D/2)}{2(3-D)}$$

$$\times \sum_{k=0}^{\infty} \left(1 - \frac{m^2}{4}\right)^k \left((m^2 - 2)m^{2k} - 2m^{2-2} \right) \frac{(2-D/2)_k}{(5/2 - D/2)_k}$$

$$= \frac{\Gamma(1-D/2) \Gamma(2-D/2)}{2(3-D)} \left((m^2 - 2) \frac{\theta\left(\frac{1}{2}, \frac{3-D/2}{5/2 - D/2} \right) \frac{m^2}{4}}{m^2 - \frac{m^4}{4}}\right)$$

$$-2m^{2-2} \frac{\theta\left(\frac{1}{2}, \frac{3-D/2}{5/2 - D/2} \right) \frac{m^2}{4}}{m^2 - \frac{m^4}{4}}$$

(5) The functions $\Sigma(D), J_4^{(D)}$ and $\Sigma(D), t(D)$ are holomorphic in $S$, They also behave well when $\text{Im}D \to \pm\infty$, so $\omega(z) = \text{const}.$

(6) Fixing the constant by the condition $J_4^{(0)} = m^{-2}$, we obtain

$$\omega(z) = m^{-2} - t(0) = \frac{4\pi\theta \left(2 - \frac{m^2}{2}\right)}{m^3 (4 - m^2)^{3/2}} \quad (78)$$

One can check that unusual $\theta \left(2 - \frac{m^2}{2}\right)$ in this formula cancels the discontinuity of $(m^2 - 2) \frac{\theta\left(\frac{1}{2}, \frac{3-D/2}{5/2 - D/2} \right) \frac{m^2}{4}}{m^2 - \frac{m^4}{4}}$ in $t(D)$. Using the properties of the hypergeometric function, we rewrite the final result as

$$J_4^{(D)} = \frac{\Gamma\left(1 - \frac{D}{2}\right) \Gamma\left(2 - \frac{D}{2}\right)}{D - 3} \left(2F_1\left[\frac{1,3-D/2}{5/2 - D/2} \left| \frac{m^2}{4}\right]\right] + m^{D-2} \frac{\theta\left(\frac{1}{2}, \frac{3-D/2}{5/2 - D/2} \right) \frac{m^2}{4}}{m^2 - \frac{m^4}{4}}\right),$$

which coincides with the one obtained in Ref. [19].

5 Topologies with several master integrals

We would like to discuss now the applicability of the described method for the topologies containing more then one master integral. In this case the dimensional recurrence relation (9) should be understood as the vector equation with $J^{(D)}, R(D)$ being the column vectors and $C(D)$ the matrix. As it follows from the theory of difference equations (see, e.g., Ref. [5]), the solution of the homogeneous part of (9) can be sought for in the form of factorial series and
the general form is

\[ J_0^{(D)} = F(D) \omega(z), \]  

(79)

where \( F(D) \) is the matrix of fundamental solutions and \( \omega(z) = \omega(e^{i\pi D}) \) is a column vector of arbitrary periodic functions of \( D \) (with period 2). So, to proceed in the same way as before, we have to define the summing factor as \( \Sigma(D) = F^{-1}(D) \). While the elements of the matrix \( F(D) \) have the form of factorial series and their analytical properties can be analysed in the same way as those of the nonhomogeneous terms in our consideration, this is not so for the inverse matrix \( F^{-1}(D) \). To overcome this difficulty, we can use the following transformation. The matrix \( F(D) \) satisfies the following matrix equation

\[ F(D - 2) = C(D) F(D) \]  

(80)

Taking the determinant of both sides, we have the equation

\[ f(D - 2) = c(D) f(D), \]  

(81)

where \( f(D) = \det F(D) \), \( c(D) = \det C(D) \). The coefficient \( c(D) \) is a rational function of \( D \) and can be represented as

\[ c(D) = \frac{\prod_i (a_i - D/2)}{\prod_j (b_j - D/2)} \]  

(82)

Therefore, the general form of \( f(D) \) is (cf. with Eq. (12))

\[ f(D) = c^{-D/2} \prod_{i=1}^{n} \frac{\Gamma(a_i - D/2)}{\prod_{j=1}^{m} \Gamma(b_j - D/2)} \omega_0(z) \]  

(83)

Now we may use the formula

\[ F^{-1}(D) = f^{-1}(D) \tilde{F}(D), \]  

(84)

where \( \tilde{F}(D) \) is the adjugate of \( F(D) \). This allows us to choose the summing factor in the form

\[ \Sigma(D) = c^{D/2} \prod_{j=1}^{m} \frac{\Gamma(b_j - D/2)}{\prod_{i=1}^{n} \Gamma(a_i - D/2)} \tilde{F}(D) \]  

(85)

This formula does not contain the sums in the denominator which allows one to perform the analysis of its analytical properties in the same manner as we did before. Again, the choice of the summing factor is not unique due to the possibility of its multiplication from the left by any periodic matrix \( \Omega(z) \). The examples of application of this technique to the topologies with several master integrals will be given elsewhere.
6 Conclusions

We have presented here the approach to the calculation of the master integrals based on Tarasov’s dimensional recurrence relation and on the analytical properties of the loop integrals as functions of the complex variable $D$. The results obtained within this approach have the form of exponentially converging series, in contrast to those obtained with many other methods having only power-like convergence. Fast convergence of the sums allow one to apply the \texttt{pslq} algorithm in order to obtain the representation of the $\epsilon$-expansion of the integral in terms of the conventional transcendental numbers, like multiple $\zeta$-functions. We have limited the presentation to the cases in which master integrals of subtopologies are expressed in terms of $\Gamma$-functions. In a more general case, when the master integrals of subtopologies are expressed via hypergeometric sums and/or the main topology contains more than one master integral, the analysis of the analytic properties, though being more complicated, can be performed in the same way. Presumably, this approach can be applied to the calculation of various master integrals with one scale, including, in particular, the three-loop onshell vertex integrals not yet calculated.

Acknowledgements

I am grateful to my colleagues G.G. Kirilin, A.V. Pomeransky, and I.S. Terekhov for useful discussions and for the interest to this work. Special thanks go to G.G. Kirilin for providing some of his considerations concerning second example in this paper. I appreciate the help of A.G. Grozin with finding the correct references for first example. I would like to thank also K.G. Chetyrkin and V.A. Smirnov for informative and stimulating discussions and also for the interest to the dimensional recurrence method. The third example of this paper stems from the communications with V.A. Smirnov. This work is supported by RFBR (grant No. 07-02-00953) and DFG (grant No. GZ436RUS113/769/0-2).

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