Gravitational waves from a particle in circular orbits around a rotating black hole to the 11th post-Newtonian order

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We compute the energy flux of the gravitational waves radiated by a particle of mass $\mu$ in circular orbits around a rotating black hole of mass $M$ up to the 11th post-Newtonian order (11PN), i.e. $v^{22}$ beyond the leading Newtonian approximation where $v$ is the orbital velocity of the particle [L. Blanchet, Living Rev. Relativity 5, 3 (2002)]. By comparing the PN results for the energy flux with high-precision numerical results in black hole perturbation theory, we find the region of validity in the PN approximation becomes larger with increasing PN order. If one requires the relative error of the energy flux in the PN approximation to be less than $10^{-5}$, the energy flux at 11PN (4PN) can be used for $v \lesssim 0.33$ ($v \lesssim 0.13$). The region of validity can be further extended to $v \lesssim 0.4$ if one applies a resummation method to the energy flux at 11PN. We then compare the orbital phase during a two-year inspiral from the PN results with the high-precision numerical results. We find that, for late (early) inspirals when $q \leq 0.3$ ($q \leq 0.9$), where $q$ is the dimensionless spin parameter of the black hole, the difference in the phase is less than 1 ($10^{-4}$) rad and hence these inspirals may be detected in the data analysis for space detectors such as eLISA/New Gravitational wave Observatory by the PN templates. We also compute the energy flux radiated into the event horizon for a particle in circular orbits around a non-rotating black hole at 22.5PN, i.e. $v^{45}$ beyond the leading Newtonian approximation, which is comparable to the PN order derived in our previous work for the energy flux to infinity at 22PN.

Subject Index  E01, E02, E20, E31, E36

1. Introduction

Extreme mass-ratio inspirals (EMRIs) are among the main candidate sources of gravitational waves (GWs) for future space-based detectors, such as eLISA [1]. In EMRIs, a stellar-mass compact object of mass $\mu$ orbits around a super-massive black hole of mass $M$. Due to the loss of the energy and the angular momentum by the emission of gravitational waves, the compact object spirals into the super-massive black hole. A conventional method to detect gravitational waves and to extract the physical information on the sources is matched filtering, which correlates the template bank of theoretical waveforms of GWs with the noisy data stream of the detector. In order to avoid significant dephasing in the matched filtering, we need to prepare theoretical waveforms whose accuracy is at least one part in $10^5$–$10^6$, since the accumulated phase of gravitational waves from EMRIs during mission time for future space-based detectors, $\sim$ yr, is millions of radians.
Since the mass ratio is very small $\mu/M \ll 1$, EMRIs can be described by the black hole perturbation theory in which the mass ratio is used as an expansion parameter [2,3]. To the lowest order in the mass ratio, the small object moves on a geodesic of the black hole spacetime. To the first order in the mass ratio, the orbit deviates from the geodesic because of the gravitational self-force [4–6]. In the black hole perturbation theory, one may accurately compute the gravitational waves and the self-force in the strong field since there is no assumption on the velocity of the small object. However, costs for numerical calculations are so high that one cannot perform calculations for all the parameter space of EMRIs with sufficient accuracy [7]. Thus, from the point of view of computational cost, it is useful if there are analytic methods to investigate gravitational waves from EMRIs.

The post-Newtonian (PN) approximation to the Einstein equations is a standard method to compute gravitational waveforms from inspiraling compact binaries [8]. In the PN approximation to the compact binary system, one assumes that the velocities of the binary are much smaller than the speed of light, $v/c \ll 1$. In the standard PN approximation, the amplitude of gravitational waves and the orbital phase are, respectively, derived up to 3PN and 3.5PN, i.e. $v^6$ and $v^7$ beyond the leading order for the non-spinning compact binaries in quasi-circular orbits [9–15]. (Note that the 3.5PN amplitudes for $(\ell, m) = (2, 2), (3, 3)$, and $(3, 1)$ modes are derived in Refs. [16,17].) For the case of the spinning compact binaries in quasi-circular orbits, spin–orbit effects in the orbital phase are derived up to 4PN [18]. Spin–spin effects in the orbital phase are derived up to 2PN [19–22].

Using the PN approximation in the black hole perturbation theory, high PN order expressions for gravitational waves can be obtained more systematically than using the standard PN approximation [2]. The energy flux to infinity up to 5.5PN (4PN) for the case of a test particle in circular orbits around a Schwarzschild (Kerr) black hole was derived in Ref. [23] (Ref. [24]) by solving the Teukolsky equation [25], which is the fundamental equation in the black hole perturbation theory. More recently, very high PN order expressions in the energy flux to infinity and gravitational waveforms for a test particle in circular orbits around a Schwarzschild black hole were derived up to 22PN [26,27] using a more systematic method to solve the Teukolsky equation [28,29]. It was shown that dephases between 22PN waveforms and very highly accurate waveforms during two-year inspirals can be less than $10^{-2}$ rad, and hence 22PN expressions might be used to detect gravitational waves from EMRIs. In this paper, by extending our previous results in Refs. [26,27,30,31], we derive the gravitational energy flux at 11PN for a test particle in circular orbits around a Schwarzschild black hole and investigate how high PN order expressions for gravitational waves can improve the accuracy in PN results. We also obtain the gravitational energy flux into the horizon at 22.5PN for a test particle in circular orbits around a Schwarzschild black hole to fill the gap in the PN order between the energy flux at infinity, currently known at 22PN, and the horizon, previously known at 6.5PN beyond the Newtonian approximation [32].

The paper is organized as follows. In Sect. 2, we give a brief review of a formalism developed by Teukolsky and describe the necessary formulas in the paper. In Sect. 3, we show analytic expressions for the energy flux to infinity at 7.5PN and to the horizon at 7PN beyond the Newtonian approximation. (The full analytic expressions will be shown online [33].) In order to investigate the accuracy of the PN results in the paper and the applicability to eLISA data analysis, comparisons between PN results and very accurate numerical results are done in Sect. 4. Section 5 is devoted to a summary and discussions. Finally, in the appendices, we describe supplemental equations to practically compute the formulas in Sect. 2. Throughout this paper, we use geometrized units with $c = G = 1$. 

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2. Basic formulation

We solve the Teukolsky equation to calculate gravitational radiation from a particle orbiting around a Kerr black hole. In the Teukolsky formalism, the gravitational perturbation of the Kerr black hole is described by a master variable. If we consider the outgoing radiation to infinity, the master variable, the Newman–Penrose quantity $\Psi_4$, is related to the amplitude of the gravitational wave at infinity by

$$\Psi_4 \rightarrow \frac{1}{2}(\hat{h}_+ - i \hat{h}_\times) \quad \text{for} \quad r \rightarrow \infty. \quad (1)$$

The Teukolsky equation can be solved by the decomposition of $\Psi_4$ as

$$\Psi_4 = \frac{1}{(r - ia \cos \theta)^4} \sum_{\ell,m} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} e^{im\varphi} \sqrt{2\pi} -2 S^{\omega \omega}_{\ell m}(\theta) R_{\ell m\omega}(r), \quad (2)$$

where $a$ is the angular momentum of the black hole and the angular function $-2 S^{\omega \omega}_{\ell m}(\theta)$ is the spin-weighted spheroidal harmonic with spin weight $s = -2$, normalized as

$$\int_0^\pi (-2 S^{\omega \omega}_{\ell m}(\theta))^2 \sin \theta \, d\theta = 1. \quad (3)$$

The decomposition of $\Psi_4$, Eq. (2), leads to the separation of the Teukolsky equation into radial and angular parts,

$$\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) - a^2 \omega^2 \sin^2 \theta - \frac{(m + s \cos \theta)^2}{\sin^2 \theta} - 2a \omega \cos \theta + s - 2ma \omega + \lambda \right] -2 S^{\omega \omega}_{\ell m}(\theta) = 0, \quad (4)$$

$$\left[ \Delta \frac{d}{dr} \left( \frac{1}{\Delta} \frac{d}{dr} \right) - \left( -\frac{K^2 + 4i(r - M) K}{\Delta} + 8i \omega r + \lambda \right) \right] R_{\ell m\omega}(r) = T_{\ell m\omega}(r), \quad (5)$$

where $\lambda$ is the eigenvalue of $-2 S^{\omega \omega}_{\ell m}$, $\Delta = r^2 - 2Mr + a^2$, $K = (r^2 + a^2)\omega - ma$, and $T_{\ell m\omega}$ is the source term of the particle.

To solve Eq. (5), we define two independent homogeneous solutions of the radial Teukolsky equation as

$$R^\text{in}_{\ell m\omega} = \begin{cases} B^\text{trans}_{\ell m\omega} \Delta^2 e^{-ikr^*} & \text{for} \quad r^* \rightarrow -\infty, \\ r^3 B^\text{ref}_{\ell m\omega} e^{i\omega r^*} + r^{-1} B^\text{inc}_{\ell m\omega} e^{-i\omega r^*} & \text{for} \quad r^* \rightarrow +\infty, \end{cases}$$

$$R^\text{up}_{\ell m\omega} = \begin{cases} C^\text{up}_{\ell m\omega} e^{ikr^*} + \Delta^2 C^\text{ref}_{\ell m\omega} e^{-i\omega r^*} & \text{for} \quad r^* \rightarrow -\infty, \\ r^3 C^\text{trans}_{\ell m\omega} e^{i\omega r^*} & \text{for} \quad r^* \rightarrow +\infty, \end{cases} \quad (6)$$

where $k = \omega - ma/(2Mr_\pm)$ and $r^*$ is the tortoise coordinate defined as

$$r^* = r + \frac{2Mr_+}{r_+ - r_-} \ln \frac{r - r_+}{2M} - \frac{2Mr_-}{r_+ - r_-} \ln \frac{r - r_-}{2M}, \quad (7)$$

with $r_\pm = M \pm \sqrt{M^2 - a^2}$.

Using the two independent solutions in Eq. (6), with the Green function method one can construct a solution of the radial Teukolsky equation that is purely outgoing at infinity and purely incoming at
the horizon:

\[ R_{\ell m\omega}(r) = \frac{1}{W_{\ell m\omega}} \left( R_{\ell m\omega}^{up}(r) \int_{r_{+}}^{r} dr' \frac{R_{\ell m\omega}^{in} T_{\ell m\omega}}{\Delta^2} + R_{\ell m\omega}^{in}(r) \int_{r}^{\infty} dr' \frac{R_{\ell m\omega}^{up} T_{\ell m\omega}}{\Delta^2} \right), \]  

(8)

where the Wronskian \( W_{\ell m\omega} \) is given as

\[ W_{\ell m\omega} = 2i \omega C_{\ell m\omega}^{\text{trans}} B_{\ell m\omega}^{\text{inc}}. \]  

(9)

Then, the solution has an asymptotic form at the horizon as

\[ R_{\ell m\omega}(r \rightarrow r_{+}) = \frac{P_{\ell m\omega}^{\text{trans}}}{2i \omega C_{\ell m\omega}^{\text{trans}}} B_{\ell m\omega}^{\text{inc}} \int_{r_{+}}^{\infty} dr' \frac{R_{\ell m\omega}^{up} T_{\ell m\omega}}{\Delta^2} \equiv Z_{\ell m\omega}^{H} \Delta^2 e^{-ikr^*}, \]  

(10)

while the solution has the following asymptotic form at infinity as

\[ R_{\ell m\omega}(r \rightarrow \infty) = \frac{r^3 e^{i\omega r^*}}{2i \omega B_{\ell m\omega}^{\text{inc}}} \int_{r_{+}}^{\infty} dr' \frac{R_{\ell m\omega}^{up} T_{\ell m\omega}}{\Delta^2} \equiv Z_{\ell m\omega}^{\infty} r^3 e^{i\omega r^*}. \]  

(11)

Using the formula of the source term \( T_{\ell m\omega} \) \([2,3]\), \( Z_{\ell m\omega}^{\infty, H} \) are expressed as

\[ Z_{\ell m\omega}^{H} = \frac{\mu P_{\ell m\omega}^{\text{trans}}}{2i \omega C_{\ell m\omega}^{\text{trans}} B_{\ell m\omega}^{\text{inc}}} \int_{-\infty}^{\infty} dt e^{i\omega t - im\phi(t)} T_{\ell m\omega}^{H}[r(t), \theta(t)], \]  

\[ Z_{\ell m\omega}^{\infty} = \frac{\mu}{2i \omega B_{\ell m\omega}^{\text{inc}}} \int_{-\infty}^{\infty} dt e^{i\omega t - im\phi(t)} T_{\ell m\omega}^{\infty}[r(t), \theta(t)], \]  

(12)

where

\[ T_{\ell m\omega}^{H} = \left[ R_{\ell m\omega}^{up} (A_{n0} + A_{\bar{m}0}) - \frac{d R_{\ell m\omega}^{up}}{dr} (A_{\bar{m}n1} + A_{\bar{m}\bar{m}1}) + \frac{d^2 R_{\ell m\omega}^{up}}{dr^2} A_{\bar{m}\bar{m}2} \right]_{\tau = r(t), \theta = \tilde{\theta}(t)}, \]  

\[ T_{\ell m\omega}^{\infty} = \left[ R_{\ell m\omega}^{in} (A_{n0} + A_{\bar{m}0}) - \frac{d R_{\ell m\omega}^{in}}{dr} (A_{\bar{m}n1} + A_{\bar{m}\bar{m}1}) + \frac{d^2 R_{\ell m\omega}^{in}}{dr^2} A_{\bar{m}\bar{m}2} \right]_{\tau = r(t), \theta = \tilde{\theta}(t)} \]  

(13)

and where \( A_{n0} \) and other terms are given in Appendix A.

When the particle follows the bound geodesics of Kerr spacetime, there exist three fundamental frequencies for the orbits \([34]\) and hence the frequency spectrum of \( T_{\ell m\omega} \) becomes discrete. In the case of circular orbits, \( Z_{\ell m\omega}^{\infty, H} \) in Eq. (12) takes the form

\[ Z_{\ell m\omega}^{\infty, H} = \bar{Z}_{\ell m\omega}^{\infty, H} \delta(\omega - m \Omega), \]  

(14)

where \( \Omega = v_r / (r_0(1 + q v_r^2)) \) is the angular frequency of the particle, \( r_0 \) is the orbital radius, \( q = a / M \), and \( v_r = \sqrt{M / r_0} \) is the orbital velocity.
The time-averaged gravitational wave luminosity at infinity is then given by [35]

$$\left\langle \frac{dE}{dt} \right\rangle_\infty = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\tilde{Z}_{\ell m}^{\infty}}{4\pi \omega^2} \equiv \left( \frac{dE}{dt} \right)_N \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \eta_{\ell m}^\infty,$$

(15)

where \( \langle \cdots \rangle \) represents the time average, \( \omega = m \Omega \), and \( (dE/dt)_N \) is the Newtonian quadrupole formula defined by

$$\left( \frac{dE}{dt} \right)_N = \frac{32}{5} \left( \frac{\mu}{M} \right)^2 v^{10},$$

(16)

with \( v \equiv (M \Omega)^{1/3} \). Similarly, the time-averaged gravitational wave luminosity at the horizon becomes [35]

$$\left\langle \frac{dE}{dt} \right\rangle_H = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \alpha_{\ell m} \frac{\tilde{Z}_{\ell m}^H}{4\pi \omega^2} \equiv \left( \frac{dE}{dt} \right)_N \nu^5 \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \eta_{\ell m}^H,$$

(17)

where

$$\alpha_{\ell m} = \frac{256(2Mr_+)^5 k(k^2 + 4\tilde{e}^2)(k^2 + 16\tilde{e}^2)\omega^3}{|C|^2},$$

(18)

with \( \tilde{e} = \sqrt{M^2 - a^2}/(4Mr_+) \) and

$$|C|^2 = \left[ (\lambda + 2)^2 + 4 \lambda a \omega m - 4 a^2 \omega^2 \right] \left[ \lambda^2 + 36 \lambda a \omega m - 36 a^2 \omega^2 \right] + (2 \lambda + 3)(96 a^2 \omega^2 - 48 \lambda a \omega m) + 144 \omega^2 (M^2 - a^2).$$

Finally, the gravitational waveforms are given in terms of \( \tilde{Z}_{\ell m}^{\infty} \) as

$$h_+ - i h_\times = -\frac{2}{r} \sum_{\ell,m} \tilde{Z}_{\ell m}^{\infty} \frac{e^{im\phi}}{\sqrt{2\pi}} - 2S_{\ell m}(\theta) e^{i \omega (r^2 - t)}.$$

(19)

In this paper, using Eqs. (15), (17), and (19) we compute the gravitational energy flux and waveforms in the post-Newtonian approximation, i.e., in the expansion with respect to \( v = (M \Omega)^{1/3} \). For this purpose, it is necessary to compute the asymptotic amplitudes \( \tilde{Z}_{\ell m}^{\infty} \), which involve calculations of the angular Teukolsky function \( -2S_{\ell m}^{\omega}(\theta) \) and the radial Teukolsky functions \( R_{\ell m}\omega(r) \). To this end, in Appendices B and C, we give a short review of the calculations of the series expansions of \( -2S_{\ell m}^{\omega}(\theta) \) and \( R_{\ell m}\omega(r) \) in terms of \( \epsilon \equiv 2M\omega = O(v^3) \) and \( z = \omega r = O(v) \).

3. 11PN results for the time-averaged energy flux

In this paper, we derive the 11PN formula for the energy flux in the case of a test particle in a circular orbit around the equatorial plane of a Kerr black hole. Since the expressions are very long, we exhibit the 7.5PN expression for the energy flux at infinity in Sect. 3.1 and the next new 7PN terms in the energy flux into the horizon in Sect. 3.2. We also compute the energy flux into the event horizon for a particle in a circular orbit around a Schwarzschild black hole up to 22.5PN beyond the Newtonian approximation. The complete expressions for the energy flux will be publicly available online [33].
3.1. Infinity flux

The 7.5PN energy flux to infinity is given by

\[
\left( \frac{dE}{dt} \right)_\infty = \left( \frac{dE}{dt} \right)_N \left[ 1 + (q \text{-independent terms}) - \frac{11}{4} q v^3 + \frac{33}{16} q^2 v^4 - \frac{59}{16} q^3 v^5 
+ \left( -\frac{65}{6} \pi q + \frac{611}{504} q^2 \right) v^6
+ \left( \frac{162035}{3888} q + \frac{65}{8} \pi q^2 - \frac{71}{24} q^3 \right) v^7
+ \left( -\frac{359}{14} \pi q + \frac{22667}{4536} q^2 + \frac{17}{16} q^4 \right) v^8
+ \left( -\frac{9828207709}{52390800} + \frac{40939}{315} \ln 2 - \frac{43}{3} \pi^2 + \frac{6841}{105} \gamma + \frac{6841}{105} \ln v \right) q
+ \frac{8447}{672} \pi q^2 - \frac{112025}{4536} q^3 \right) v^9
+ \frac{23605}{144} \pi q + \left( \frac{93301799461}{628689600} - \frac{27499}{420} \ln 2 + \frac{43}{4} \pi^2 - \frac{4601}{140} \gamma - \frac{4601}{140} \ln v \right) q^2
- \frac{45}{4} \pi q^3 + \frac{731}{126} q^4 \right) v^{10}
+ \left( -\frac{244521688471}{272432160} - \frac{128079110584}{128079110584} \ln 2 - \frac{671}{12} \pi^2 + \frac{128459}{7560} \gamma + \frac{486243}{3136} \ln 3
+ \frac{18459}{7560} \ln v \right) q + \frac{3421}{1512} \pi q^2 - \frac{257407}{9072} q^3 + \frac{33}{8} \pi q^4 - \frac{1}{8} q^5 \right) v^{11}
+ \left( -\frac{270159823411}{558835200} - \frac{81878}{315} \pi \gamma + \frac{54514}{105} \pi \ln 2 + \frac{81878}{315} \pi \ln v \right) q
+ \left( -\frac{13501670684927}{28605376800} - \frac{22901}{196} \gamma + \frac{24763}{756} \pi^2 - \frac{537013}{3780} \ln 2
- \frac{142155}{1568} \ln 3 - \frac{22901}{196} \ln v \right) q^2 + \frac{67426}{567} \pi q^3 + \frac{24397}{1008} q^4 \right) v^{12}
+ \left( \frac{256}{15} \ln \kappa - \frac{51606979451}{6735960} - \gamma - \frac{16342453091}{47151720} \ln 2 - \frac{67221333}{86240} \ln 3
+ \frac{526805}{2916} \pi^2 + \frac{1290587071610633}{606842636400} \right) q
+ \left( -\frac{18297}{70} \pi \ln 2 - \frac{27499}{210} \pi \gamma + \frac{1241484285313}{2235340800} \right) q^2
+ \left( -\frac{416537225257}{1414551600} - \frac{263}{18} \pi^2 + \frac{31467}{140} \ln 2 + \frac{256}{5} \ln \kappa + \frac{174659}{1260} \gamma \right) q^3
+ \frac{60869}{2016} \pi q^4 + \frac{11311}{1008} q^5 + \left( \frac{256}{15} q + \frac{256}{5} q^3 \right) \Psi_{0}^{(0.2)}(q)
+ \left( -\frac{5045737157}{6735960} - \frac{27499}{210} \pi q^2 + \frac{239171}{1260} q^3 \right) \ln v \right) v^{13}
+ \left( -\frac{2628337}{15120} \pi \ln 2 + \frac{1458729}{1568} \pi \ln 3 + \frac{8005397}{21168} \pi \gamma - \frac{748453609847597}{15256009600} \pi \right) q
\]
among these terms, the \( q \)-dependent terms in Eq. (20) are the new terms derived by the post-Newtonian approximation in this paper. Among these terms, the \( \mathcal{O}(v^9) - \mathcal{O}(v^{11}) \) terms agree with the analytic expressions in Ref. [36], which determined the post-Newtonian coefficients of the energy flux up to 20PN by

\[
\left\{ - \frac{18 591 892}{661 5} \gamma \ln 2 + \frac{12 168}{785 620} \gamma - \frac{228 312 000}{2831 932 303 200} \pi q^2 - \frac{238 825}{1008} \pi q^3 \right. \\
+ \frac{238 825}{1008} \pi q^3 + \left. \frac{238 825}{1008} \pi q^3 \right\} v^{14}
\]

where \( \kappa = \sqrt{1 - q^2} \), \( \gamma \) is the Euler constant, \( \zeta(n) \) is the zeta function,

\[
\Psi_A^{(m,n)}(q) = \frac{1}{2} \left[ \Psi_A^{(m,n)} \left( 3 + \frac{i m q}{\sqrt{1 - q^2}} \right) + \Psi_B^{(m,n)} \left( 3 - \frac{i m q}{\sqrt{1 - q^2}} \right) \right],
\]

\[
\Psi_B^{(m,n)}(q) = \frac{1}{2i} \left[ \Psi_A^{(m,n)} \left( 3 + \frac{i m q}{\sqrt{1 - q^2}} \right) - \Psi_B^{(m,n)} \left( 3 - \frac{i m q}{\sqrt{1 - q^2}} \right) \right],
\]

and \( \Psi^{(n)}(z) \) is the polygamma function.

The \( \mathcal{O}(v^9) - \mathcal{O}(v^{11}) \) terms in Eq. (20) are the new terms derived by the post-Newtonian approximation in this paper. Among these terms, the \( \mathcal{O}(v^9) - \mathcal{O}(v^{11}) \) terms agree with the analytic expressions in Ref. [36], which determined the post-Newtonian coefficients of the energy flux up to 20PN by

\[1\]

We note that the "\( q \)-independent terms" in Eq. (20) coincide with those for the Schwarzschild black hole, since the \( q \)-dependent terms in Eq. (20), e.g. \( v^j q^k \ln \kappa \) and \( v^j q^k \Psi_A^{(m,n)}(q) \) where \( j \) and \( k \) are integers, vanish when \( q = 0 \). For 8PN and higher PN orders, however, there are \( q \)-dependent terms, e.g. \( v^j \kappa \) and
fitting with very accurate, one part in $10^{600}$, numerical calculation of the energy flux. For the 6PN and higher PN order energy flux at infinity, in Ref. [36], some of the post-Newtonian coefficients are not extracted as analytic values but as numerical values. This is not only because it is difficult to numerically extract analytic coefficients for combinations of transcendental numbers such as $\pi$, Euler’s constant, and logarithms of prime numbers, but also because numerical fitting of post-Newtonian coefficients is done by presenting these coefficients as a polynomial in $q$, although irrational functions in $q$ such as polygamma functions and logarithms appear from 6.5PN onward, as shown in Eq. (20). Further, by performing a small $q$ expansion of our 11PN expression, we also find that our 11PN energy flux to infinity is consistent with the one in Ref. [36] up to 11PN.

From Eq. (20), we find the coefficient in $q (\ln v)^0$ at 6PN is given by

$$-rac{270}{558} \frac{1598234411}{835200} \pi + \frac{81878}{315} \frac{\pi}{v} + \frac{54514}{105} \pi \ln 2. \quad (21)$$

The above analytic value is consistent with the numerical value of the coefficient in $q (\ln v)^0$ at 6PN energy flux to infinity in Ref. [36], 83.160 390 235 770 41 . . .

### 3.2. Horizon flux

The next new 7PN terms for the energy flux into the horizon are given by

$$\left\langle \frac{dE}{dt} \right\rangle_{H}^{(9)} = \frac{2204129}{22050} - \frac{4\pi^2}{3} - \frac{856}{105} \ln 2 - \frac{856}{105} \ln \kappa - \frac{856}{105} \ln \frac{\kappa}{\kappa} - \frac{856}{105} \frac{2}{\kappa}$$

$$- 4\pi^2 \kappa + \frac{2204129}{22050 \kappa} \left\{ - \frac{3424}{35} \ln 2 - \frac{24687352}{33075} \frac{2}{\kappa} - \frac{3424}{35} \ln \kappa - \frac{3424}{35} \frac{2}{\kappa} \right\} q^2$$

$$+ \left\{ \frac{856}{15} \ln 2 - \frac{50225669}{176400} + \frac{28}{3} \frac{\pi^2}{\kappa} - \frac{1573}{105} \ln \kappa + \frac{856}{15} \frac{2}{\kappa} + \frac{856}{15} \ln \frac{\kappa}{\kappa} \right\} q^4$$

$$+ \left\{ \frac{19195}{168} + \frac{1712}{35} \ln \kappa + \frac{8\pi^2}{\kappa} - \frac{2293757}{29400\kappa} + \frac{1712}{35} \ln \kappa + \frac{1712}{35} \ln 2 \right\} q^6 - \frac{621}{28} \frac{q^8}{\kappa}$$

$$+ \left\{ \frac{428}{315} q^2 + \frac{107}{105} q^4 \right\} \Psi_{A}^{(0,1)}(q)$$

$$+ \left\{ - \frac{856}{105} \frac{856}{105\kappa} + \left( - \frac{9976}{105} - \frac{3424}{35\kappa} \right) q^2 + \left( - 16 + \frac{856}{15\kappa} \right) q^4 + \frac{1712}{35} \frac{q^6}{\kappa} \right\} \Psi_{B}^{(0,2)}(q) \right\} \Psi_{A}^{(0,2)}$$

$\psi^{(n,m)}(q)$, which do not vanish when $q = 0$ and hence the “$q$-independent terms” do not agree with those for the Schwarzschild black hole.

$^2$ It might be noted from Eq. (20) that, if one includes $\kappa$, $\ln \kappa$, and $\Psi_{A,B}^{(n,m)}(q)$ for the numerical fitting in Ref. [36], one might be able to obtain a more accurate fitting formula.
To investigate the accuracy of the energy flux in the post-Newtonian approximation, we compare 11PN results and numerical results, based on a method in Refs. [38,39]. With this numerical method, one can investigate gravitational waves with an accuracy of about 14 significant figures in double significant figures in double
precision calculations. Hence one can use the numerical results to estimate the accuracy in the PN results by a comparison. For the comparison in this section, we need to compute the energy flux using Eqs. (15) and (17). To numerically compute the energy flux, we set the maximum value of $\ell$ to 15, which gives the relative error in the energy flux better than $10^{-5}$ for the comparisons in this section. For the energy flux at 11PN, we need to compute $\ell$ up to 13 (5) for the energy flux to infinity (the horizon).

In Sect. 4.1, comparisons for the the energy flux are done for several values of the spin of the Kerr black hole. In Sects. 4.2 and 4.3, the same comparisons are done using resummation techniques, the factorized resummation introduced in Ref. [40] and the exponential resummation in Ref. [41], for the post-Newtonian approximation to the energy flux. We will see how resummation methods improve the performance in the post-Newtonian approximation for the energy flux. Finally, in Sect. 4.4, we compare the total cycle of orbits during a two-year inspiral for representative binaries in the eLISA frequency band.

4.1. Energy flux: Taylor expanded PN approximation

Figures 1 and 2 show the relative error in the total energy flux from numerical results and PN approximations as a function of the orbital velocity up to the innermost stable circular orbit (ISCO). From these figures, one will find that the relative error becomes smaller with increasing PN order for $v \leq 0.3$, except for accidental agreements for certain values of the velocity. However, the relative error around ISCO does not necessarily become smaller with increasing PN order when $q > 0.3$. The relative error for 11PN is smaller than $10^{-5}$ when $v \lesssim 0.33$, irrespective of the values of $q$ investigated in the paper.

Figure 3 shows the relative error in the energy flux down the horizon from numerical results and PN approximations as a function of the orbital velocity up to ISCO in the case of the Schwarzschild black hole. The agreement between the numerical energy flux and post-Newtonian energy flux becomes better when the PN order is higher, even around ISCO. The relative error in the 22.5PN energy flux into the horizon around ISCO is about $10^{-5}$, which is comparable to the one for the 22PN energy flux to infinity in Ref. [27].

4.2. Energy flux: Factorized resummation to PN approximation

In this section, we compare the total energy flux from numerical results with PN results using a factorized resummation. The factorized resummation was introduced to improve the convergence in the PN energy flux to infinity for a test particle moving in Schwarzschild spacetime [40,42,43] and Kerr spacetime [31]. The factorized resummation was then extended to the PN energy flux down the horizon of the Schwarzschild black hole in Ref. [44] and the Kerr black hole in Ref. [45].

4.2.1. Factorization of the energy flux at infinity

In the factorized resummation of the energy flux at infinity, we decompose the multipolar gravitational waveforms into five factors as

$$h_{\ell m} = h^{(N,\epsilon_p)}_{\ell m} S^{(\epsilon_p)}_{\text{eff}} T_{\ell m} e^{\delta_{\ell m} (\rho_{\ell m})^\ell}, \quad (23)$$

where $\epsilon_p$ denotes the parity of the multipolar waveforms, $h^{(N,\epsilon_p)}_{\ell m}$ represents the Newtonian contribution to waveforms, $S^{(\epsilon_p)}_{\text{eff}}$ an effective source term for partial waves in the perturbation formalism, $T_{\ell m}$ resums the leading logarithms of the tail effects, $\delta_{\ell m}$ is the supplemental phase factor, and $\rho_{\ell m}$ is the $\ell$th root of the amplitude of the waveforms, which takes care of a term linear in $\ell$ at 1PN in the
more details see, e.g., Refs. [40,42,43]).

The first factor $h_{\ell m}^{(N,\epsilon_p)}$ takes the form

$$h_{\ell m}^{(N,\epsilon_p)} = \frac{GMv}{c^2 r} n_{\ell m}^{(\epsilon_p)} c_{\ell+\epsilon_p}(v) v^{\ell+\epsilon_p} Y^{\ell-\epsilon_p,-m} \left(\frac{\pi}{2}, \phi\right), \quad (24)$$

where $\phi$ is the orbital phase and $n_{\ell m}^{(\epsilon_p)}$ are

$$n_{\ell m}^{(0)} = (im)^\ell \frac{8\pi}{(2\ell + 1)!!} \sqrt{\frac{(\ell + 1)(\ell + 2)}{\ell(\ell - 1)}}, \quad (25a)$$

$$n_{\ell m}^{(1)} = -(im)^\ell \frac{16\pi i}{(2\ell + 1)!!} \sqrt{\frac{(2\ell + 1)(\ell + 2)(\ell^2 - m^2)}{(2\ell - 1)(\ell + 1)(\ell - 1)}}, \quad (25b)$$

and $c_{\ell+\epsilon_p}(v)$ are functions of the symmetric mass ratio $\nu \equiv \mu M/(M + \mu)^2$, defined by

$$c_{\ell+\epsilon_p}(v) = \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4v}\right)^{\ell+\epsilon_p-1} + (-)^{\ell+\epsilon_p} \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4v}\right)^{\ell+\epsilon_p-1}. \quad (26)$$

**Fig. 1.** Absolute values of the relative error in the total energy flux from numerical results and PN approximations as a function of the orbital velocity, $v = (M/r_0)^{1/2}[1 + q (M/r_0)^{3/2}]^{-1/3}$, up to ISCO for $q = 0.1, 0.3, 0.5, 0.7, 0.9, \text{and} 0.998$. The relative error for 11PN is smaller than $10^{-5}$ when $v \lesssim 0.33$.
|1-\frac{dE}{dt}|_{PN}/|1-\frac{dE}{dt}|_{numerical}|
|1-\frac{dE}{dt}|_{PN}/|1-\frac{dE}{dt}|_{numerical}|
|1-\frac{dE}{dt}|_{PN}/|1-\frac{dE}{dt}|_{numerical}|
|1-\frac{dE}{dt}|_{PN}/|1-\frac{dE}{dt}|_{numerical}|
|1-\frac{dE}{dt}|_{PN}/|1-\frac{dE}{dt}|_{numerical}|

Fig. 2. Same as Fig. 1 but for \( q = -0.01, -0.05, -0.1, -0.3, -0.5, \) and -0.9. The relative error for 11PN is smaller than \( 10^{-5} \) when \( v \lesssim 0.33 \).

Fig. 3. (Left) Absolute values of the difference in the energy flux down the horizon from numerical results and PN approximation as a function of the orbital velocity up to ISCO for \( q = 0 \). (Right) Same as the left figure but for the total energy flux, which includes fluxes to infinity and the horizon.

The second factor \( \hat{S}_{\text{eff}}^{(\epsilon_p)} \) in Eq. (23) is defined by

\[
\hat{S}_{\text{eff}}^{(\epsilon_p)} = \begin{cases} 
\tilde{E} & \text{for } \epsilon_p = 0 (\ell + m = \text{even}), \\
v\tilde{L}/M & \text{for } \epsilon_p = 1 (\ell + m = \text{odd}),
\end{cases}
\]
where $\bar{E}$ and $\bar{L}_z$ are the specific energy and the angular momentum of the particle, given by

$$
\bar{E} = \frac{1 - 2v_r^2 + q v_r^3}{\sqrt{1 - 3v_r^2 + 2q v_r^3}}, \quad \bar{L}_z = \frac{r_0 v_r (1 - 2q v_r^3 + q^2 v_r^4)}{\sqrt{1 - 3v_r^2 + 2q v_r^3}}.
$$

(28)

The third factor $T_{\ell m}$ in Eq. (23) is defined by

$$
T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i m \Omega)}{\Gamma(\ell + 1)} e^{i m \Omega} e^{2i m \Omega \ln(2m \Omega r_0)},
$$

(29)

where $r_0 = 2M/\sqrt{\bar{e}}$ is introduced to reproduce the test-particle limit waveforms [31].

The fourth and fifth factors in Eq. (23), $\delta_{\ell m}$ and $\rho_{\ell m}$, can be derived by comparing the multi-polar waveforms (23) with those obtained from Eq. (19). For the comparison, it is useful to express waveforms (19) in terms of the $-2$ spin-weighted spherical harmonics $-2Y_{\ell m}(\theta, \phi) \equiv -2P_{\ell m}(\theta) e^{i m \phi}/\sqrt{2\pi}$ [31]:

$$
\tilde{h}_+ - i \tilde{h}_\times = -2 \sum_{\ell, m} \frac{\tilde{Z}_{\ell m}^{i\omega} e^{i m \phi}}{\omega^2 \sqrt{2\pi}} -2 \tilde{S}_{\ell m}^{i\omega}(\theta) e^{i \omega(r^+ - t)},
$$

$$
\tilde{h}_+ - i \tilde{h}_\times = -2 \sum_{\ell, m} \frac{\tilde{C}_{\ell m}^{i\omega} e^{i m \phi}}{\omega^2 \sqrt{2\pi}} -2 P_{\ell m}(\theta) e^{i \omega(r^+ - t)},
$$

(30)

where $-2P_{\ell m}(\theta)$ is defined as

$$
-2P_{\ell m}(\theta) = (-1)^m \sqrt{\frac{(l + m)!(l - m)!(2l + 1)}{2(l + 2)!(l - 2)!}} \sin^2 \left(\frac{\theta}{2}\right)
$$

$$
\times \sum_{r = 0}^{l+2} \left(\begin{array}{c} l + 2 \\ r \end{array}\right) \left(\begin{array}{c} l - 2 \\ r - 2 - m \end{array}\right) (-1)^{l-r+2} \cot^{2r-2-m} \left(\frac{\theta}{2}\right).
$$

(31)

From Eq. (30), one can compute $\tilde{C}_{\ell m}^{i\omega}$ from $\tilde{Z}_{\ell m}^{i\omega}$:

$$
\tilde{C}_{\ell m}^{i\omega} = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \sum_{\ell'} \sum_{m'=-\ell'}^{\ell'} \tilde{Z}_{\ell m'}^{i\omega}(\theta) -2 \tilde{S}_{\ell m'}^{i\omega}(\theta) -2P_{\ell m}(\theta) e^{i (m' - m) \phi},
$$

$$
\tilde{C}_{\ell m}^{i\omega} = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \sum_{\ell'} \tilde{Z}_{\ell m}^{i\omega} -2 \tilde{S}_{\ell m}^{i\omega}(\theta) -2P_{\ell m}(\theta),
$$

(32)

where we used the orthogonality condition of the $-2$ spin-weighted spherical harmonics,

$$
\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta -2Y_{\ell m}(\theta, \phi) -2\tilde{Y}_{\ell m}(\theta, \phi) = \delta_{\ell \ell'} \delta_{mm'},
$$

(33)

and $\tilde{X}$ is the complex conjugate of $X$. Note that in Eq. (32) the mixing of $\tilde{Z}_{\ell m}^{i\omega}$ happens among the same $m$ and different $\ell$ modes [31]. Note that the infinite summation over $\ell'$ in Eq. (32) can be truncated at a certain $\ell'$ for a given post-Newtonian order since $\tilde{Z}_{\ell m}^{i\omega} = O(\nu^{\ell'+2+\epsilon\nu})$ (see, e.g., Ref. [30]).
Once we obtain $\tilde{C}_{\ell m o}^\infty$, from Eq. (32), it is straightforward to compute $\hat{s}_{\ell m}$ and $\rho_{\ell m}$ from the following relation between $h_{\ell m}$ and $\tilde{C}_{\ell m o}^\infty$ [14,30]:

$$h_{\ell m} = \int \sin \Theta \, d\Theta \, d\Phi \, (h_+ - i \, h_\times) \, -2 \tilde{Y}_{\ell m}(\Theta, \Phi),$$

$$\begin{array}{l}
= -2 \sum_{\ell', m'} \frac{\tilde{C}_{\ell m o}^\infty e^{im'\Omega r^*}}{(m'^2 \Omega^2)^2} \int \sin \Theta \, d\Theta \, d\Phi \, e^{-im'\Omega r^*} \tilde{Y}_{\ell' m'}(\Theta, \varphi) \, -2 \tilde{Y}_{\ell m}(\Theta, \Phi), \\
= -2 \frac{\tilde{C}_{\ell m o}^\infty e^{im\Omega r^*}}{(m^2 \Omega^2)^2} e^{im\varphi},
\end{array}$$

(34)

Using the factorized waveforms $h_{\ell m}$, Eq. (23), computed from Eqs. (32) and (34), the time-averaged energy flux to infinity is computed as

$$\left\langle \frac{dE}{dt} \right\rangle_\infty = \frac{1}{16\pi} \sum_{\ell=2}^{\infty} \left( \sum_{m=-\ell}^{\ell} (mM\Omega)^2 \right) \frac{r}{M} \left| h_{\ell m} \right|^2.$$

(35)

4.2.2. Factorization of the energy flux down the horizon

For the factorized resummation of the energy flux down the horizon, the modal energy flux is decomposed as [44,45]

$$\eta_{\ell m}^H = \left( 1 - \frac{2v^3r_+}{a} \right) \eta_{\ell m}^{N,H} \left( \hat{S}_{\ell m}^{(\epsilon_p)} \right)^2 \left( \rho_{\ell m}^H \right)^{2\ell},$$

(36)

where the factor $\left( 1 - \frac{2v^3r_+}{a} \right)$ is motivated by the factor $k = \omega - ma/(2Mr_+) = m(\Omega - a/(2Mr_+))$ in Eq. (18), which is responsible for the sign of the modal energy flux to the horizon.

The second factor $\eta_{\ell m}^{N,H}$ represents the leading term in the modal energy flux into the horizon and takes the form

$$\eta_{\ell m}^{N,H} = v^{4(\ell - 2) + 2\epsilon_p} \eta_{\ell m}^{(H,\epsilon_p)} \epsilon_{\ell m}^{H}(q),$$

(37)

where

$$\eta_{\ell m}^{(H,0)} = -\frac{5}{32} \frac{(\ell + 1)(\ell + 2)}{\ell(\ell - 1)} \frac{2\ell + 1}{[(2\ell + 1)!]^2} \frac{(\ell - m)!}{[(\ell - m)!]^2} \frac{(\ell + m)!}{[(\ell + m)!]^2},$$

$$\eta_{\ell m}^{(H,1)} = -\frac{5}{8\ell^2} \frac{(\ell + 1)(\ell + 2)}{\ell(\ell - 1)} \frac{2\ell + 1}{[(2\ell + 1)!]^2} \frac{[(\ell - m)!]^2}{(\ell - m)!} \frac{[(\ell + m)!]^2}{(\ell + m)!},$$

(38a)

and $\epsilon_{\ell + \epsilon_p}(v)$

$$\epsilon_{\ell m}^H(q) = \frac{1}{q} \prod_{k=0}^{\ell} \left[ k^2 + (m^2 - k^2) \right] q^2,$$

$$= q \, m^2 \left( 1 - q^2 \right)^\ell \left( 1 - \frac{imq}{\sqrt{1 - q^2}} \right)^\ell \left( 1 + \frac{imq}{\sqrt{1 - q^2}} \right)^\ell,$$

(39)

where $(z)_n = \Gamma(z + n)/\Gamma(z)$.

The definition for the third factor $\hat{S}_{\ell m}^{(\epsilon_p)}$ is the same as in Eq. (27), which is used for the resummed multipolar waveforms (23). The fourth factor $\rho_{\ell m}^H$ is the $2\ell$th root of the residual amplitude of the
Fig. 4. Same as Fig. 1 but using factorized resummation to the energy flux in the post-Newtonian approximation. The relative error for 11PN is less than $10^{-5}$ when $v \lesssim 0.4$, whose region is larger than $v \lesssim 0.33$ for the Taylor expanded energy flux in Fig. 1.

modal energy flux and can be derived by comparing the Taylor expanded modal energy flux $\eta_{\ell m}^H$ with the factorized modal energy flux, Eq. (36).

4.2.3. Comparisons with numerical results
Figures 4 and 5 show the relative error in the total energy flux from numerical results and PN approximations as a function of the orbital velocity up to ISCO using the factorized resummation to PN approximations [40]. From these figures, one will find that the relative error becomes smaller as PN order becomes higher for $v \leq 0.3$, except for accidental agreements for certain values of the velocity. However, the relative error around ISCO does not necessarily become smaller for higher PN orders when $q > 0.3$. The relative error for 11PN is smaller than $10^{-5}$ when $v \lesssim 0.4$, irrespective of the values of $q$ investigated in the paper. We note that the region of the velocity, $v \lesssim 0.4$, is larger than the one using the Taylor expanded PN energy flux, $v \lesssim 0.33$.

4.3. Energy flux: Exponential resummation to PN approximation
In this section, we compare the total energy flux from numerical results with PN results using the exponential resummation [41,46].
Fig. 5. Same as Fig. 2 but using factorized resummation to the energy flux in the post-Newtonian approximation. The relative error for 11PN is less than $10^{-10}$ when $v < 0.4$, whose region is larger than $v < 0.33$ for the Taylor expanded energy flux in Fig. 2.

In the exponential resummation, the modal energy fluxes to infinity, $\eta^\infty_{\ell m}$, and the horizon, $\eta^H_{\ell m}$, are decomposed as

$$
\eta^\infty_{\ell m} = \frac{1}{1 - 3v_r^2 + 2q v^3_r} \eta^N_{\ell m} \exp \left[ \ln (\eta^\infty_{\ell m}) \right], \quad \eta^H_{\ell m} = \frac{1 - 2v^3 r_+ / a}{1 - 3v_r^2 + 2q v^3_r} \eta^N_{\ell m} \exp \left[ \ln (\eta^H_{\ell m}) \right].
$$

(40)

where $\eta^N_{\ell m,\infty}$ and $\eta^N_{\ell m, H}$ are the leading terms for $\eta^\infty_{\ell m}$ and $\eta^H_{\ell m}$, respectively, the denominator $(1 - 3v_r^2 + 2q v^3_r)$ is the square of the denominator of $\hat{S}^{(\ell p)}_{\text{eff}}$ in Eq. (23), and the factor $(1 - 2v^3 r_+ / a)$ is motivated by the factor $k = \omega - ma / (2Mr_+) = m / M (v^3 - a / (2r_+))$ in Eq. (18), which is again responsible for the sign of the modal energy flux to the horizon.

Similarly to $\eta^N_{\ell m, H}$ defined in Eq. (37), the explicit expression for $\eta^N_{\ell m, \infty}$ can be derived from factors $n_{\ell m}^{(\ell p)}$ and $c_{\ell + \epsilon_p} (v)$ in the Newtonian contribution to waveforms $h^{(N, \epsilon_p)}_{\ell m}$, Eq. (24), as

$$
\eta^N_{\ell m, \infty} = \frac{5}{256 \pi^2} \frac{m^2}{\ell !} \left| n_{\ell m}^{(\ell p)} \right|^2 (c_{\ell + \epsilon_p} (v))^2 v^{2(\ell - 2) + 2\epsilon_p} \left( P_{\ell - \epsilon_p, -m} \left( \frac{\pi}{2} \right) \right)^2.
$$

(41)
The factors $\hat{\eta}_\infty^L$ and $\hat{\eta}_H^L$ in the exponential resummation, Eq. (40), can be derived by comparing with the Taylor expanded modal energy fluxes $\hat{\eta}_\infty^L$ and $\hat{\eta}_H^L$. 

Figures 6 and 7 show the relative error in the total energy flux from numerical results and PN approximations as a function of the orbital velocity up to ISCO using exponential resummation to PN approximations [41]. From these figures, one will find that the relative error becomes smaller as the PN order becomes higher for $v \leq 0.3$, except for accidental agreements for certain values of the velocity. However, the relative error around ISCO does not necessarily become smaller at higher PN orders when $q > 0.3$. The relative error for 11PN is smaller than $10^{-5}$ when $v \lesssim 0.4$, irrespective of the values of $q$ investigated in the paper. Again, we note that the region of the velocity, $v \lesssim 0.4$, is larger than the one using the Taylor expanded PN energy flux, $v \lesssim 0.33$.

4.4. Phase difference during the two-year inspiral

We compare the orbital phase from PN results with numerical results during two-year inspirals to estimate the applicability of the PN results in the data analysis. For the comparison, we choose two representative systems of EMRIs in the eLISA frequency band, System-I and System-II, following

Fig. 6. Same as Fig. 1 but using exponential resummation to the energy flux in the post-Newtonian approximation. The relative error for 11PN is less than $10^{-5}$ when $v \lesssim 0.4$, whose region is larger than $v \lesssim 0.33$ for the Taylor expanded energy flux in Fig. 1.

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Fig. 7. Same as Fig. 2 but using exponential resummation to the energy flux in the post-Newtonian approximation. The relative error for 11PN is less than $10^{-5}$ when $v \lesssim 0.4$, whose region is larger than $v \lesssim 0.33$ for the Taylor expanded energy flux in Fig. 2.

Refs. [26,27,47,48]. System-I is an early inspiral of an EMRI with masses $(M, \mu) = (10^5, 10)M_\odot$, i.e. $\mu/M = 10^{-4}$, which reaches $r_0 \simeq 16M$ after the two-year inspiral. System-II is a late inspiral of an EMRI with masses $(M, \mu) = (10^6, 10)M_\odot$, i.e. $\mu/M = 10^{-5}$, which reaches ISCO after the two-year inspiral. Although the initial values for orbital radius, velocity, and GW frequency depend on the spin of the Kerr black hole, System-I inspirals from $r_0 \simeq 29M$ to $r_0 \simeq 16M$ with associated velocities $v \in [0.2, 0.25]$ and frequencies $f_{GW} \in [4 \times 10^{-3}, 10^{-2}]$ Hz, while System-II explores orbital separation in the range $r_0/M \in [r_{ISCO}, 11M]$, velocities $v \in [0.3, v_{ISCO}]$, and frequencies $f_{GW} \in [10^{-3}, f_{ISCO}]$ Hz. The orbital phase for System-I (System-II) after the two-year inspiral is about $10^9 (5 \times 10^5)$ rad. Moreover, System-I (System-II) sweeps the high- (low-) frequency region of the eLISA frequency band.

For the calculation of the orbital phase, we define the phase as $\Psi_{\ell m}(t) = m \int_0^t \Omega(t') dt'$, where $\Omega(t) = M^{1/2}/r(t)^{3/2}/(1 + qM^{3/2}/r(t)^{3/2})$ is the angular frequency of the particle and $r(t)$ is the orbital radius as a function of time. The orbital radius $r(t)$ is derived as $r(t) = \int (dr/dt') dt' = \int (\partial r/\partial E) (dE/dt') dt'$, where $(dE/dt)$ is computed by the energy balance equation, $(dE/dt) = -\langle dE/dt \rangle_{\infty} - \langle dE/dt \rangle_{H}$. To save computation time, we apply cubic spline interpolation [49] to
Fig. 8. Absolute values of the dephasing during the two-year inspiral between the factorized PN and the numerical results for the dominant $\ell = m = 2$ mode as a function of time in months when $q = 0.1, 0.3, 0.5$, and 0.9. These panels show the dephases for System-I with masses $(M, \mu) = (10^5, 10)M_\odot$, which inspirals from $r_0 \simeq 29M$ to $r_0 \simeq 16M$ with associated frequencies $f_{\text{GW}} \in [4 \times 10^{-3}, 10^{-2}]$ Hz. These inspirals represent the early inspiral phase in the eLISA band. The dephases between the 11PN results and numerical results after the two-year inspiral are less than $10^{-4}$ rad.

perform the integration using $10^3$ data points for $(v, (d\tilde{E}/dt))$, i.e. $(v, dr/dt)$, in the range from $v = 0.01$ to $v = r_{\text{ISCO}}$ [26,27,47,50]. The computation time to perform the numerical integration is less than a second if we use the cubic spline interpolation.

Figures 8 and 9 show absolute values of the difference in the orbital phase for the dominant $\ell = m = 2$ mode between the PN and the numerical results during two-year inspirals for several values of the spin of the black hole. As for the PN approximations, we show results using the factorized resummation in Sect. 4.2, which are better than those using the Taylor expanded PN energy flux and comparable to those using the exponential resummation. The dephases between the 11PN results and numerical results after the two-year inspiral are less than $10^{-4}$ rad for System-I. However, the dephases after the two-year inspiral become larger than a radian for System-II when $q > 0.3$. Thus, one has to derive higher PN order results for the energy flux to achieve a dephase of less than a radian for System-II when $q > 0.3$, which represents a stronger-field situation than the one for System-I.

5. Summary

We have investigated gravitational waves from a particle moving in circular orbits in Kerr spacetime using the post-Newtonian approximation and computed the energy flux up to 11PN. We have also computed the energy flux down the event horizon for a particle in circular orbits around a Schwarzschild black hole at 22.5PN beyond the Newtonian approximation to fill the gap in the PN
order between the energy flux at infinity, currently known at 22PN, and the event horizon, previously known at 6.5PN beyond the Newtonian approximation.

To investigate how higher PN order expressions improve the applicability to data analysis of eLISA/New Gravitational wave Observatory (NGO), comparisons between PN results and high-precision numerical results in black hole perturbation theory have been done. We first compared PN energy flux to numerical energy flux and found that the region of validity in the PN energy flux becomes larger as the PN order becomes higher. If the relative error of the energy flux in the PN approximation should be less than $10^{-5}$, the energy flux at 11PN satisfies this requirement for $v \lesssim 0.33$, which clearly shows an improvement from $v \lesssim 0.13$ in an earlier work at 4PN [24]. The region of validity in the 11PN energy flux can become larger, $v \lesssim 0.4$, if one uses resummation techniques such as factorized resummation [40] and exponential resummation [41]. The region of validity might become even larger if one takes more careful account of the structure of homogeneous solutions of the Teukolsky equation [46].

Finally, we compared the orbital phase during the two-year inspiral using the factorized resummed PN flux and the high-precision numerical flux. We found that the dephase is less than 1 ($10^{-4}$) rad for late (early) inspirals when $q \leq 0.3$ ($q \leq 0.9$). This implies that the 11PN factorized resummed flux

\[ r < \frac{3}{2} \]

\[ v \leq \frac{1}{\Omega_1} \]

\[ f_{GW} \in [10^{-3}, f_{ISCO}] \]

\[ q = 0.5 \]

\[ \Omega \lesssim \Omega_{ISCO} \]

\[ q = 0.9 \]

\[ \Omega_{ISCO} \]

\[ \Omega \lesssim \Omega_{ISCO} \]

\[ q = 0.9 \]
may be used to detect early inspirals in the data analysis of eLISA/NGO. To detect gravitational waves from late inspirals when \( q > 0.3 \), however, it is necessary to obtain higher PN order expressions than 11PN. From numerical calculations in black hole perturbation theory, it is estimated that we may need to compute at least up to \( \ell = 30 \), i.e. 28PN, to obtain the relative error of \( 10^{-5} \) in the energy flux at ISCO for \( q = 0.9 \) [27]. If it is not possible to perform such a high PN order calculation, it may be necessary to use other approaches that compute unknown PN coefficients by numerical fitting [36,47,48].

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Appendix A. Source term of the Teukolsky equation

\( A_{nn0} \) etc. in Eq. (13) are defined as

\[
A_{n n 0} = -\frac{2}{\sqrt{2 \pi} \Delta^2} C_{n n} \rho^{-2} \frac{d}{d \tau} \left\{ \rho^{-4} C_2^2 \left( \rho^{-3} - 2 S_{1 m}^0(\omega) \right) \right\},
\]

\[
A_{\bar{m} \bar{n} 0} = 2 \frac{2}{\sqrt{2 \pi} \Delta} C_{\bar{m} \bar{n}} \rho^{-3} \left[ \left( C_2^2 - 2 S_{1 m}^0(\omega) \right) \left( \frac{i K}{\Delta} + \rho + \bar{\rho} \right) - a \sin \theta \right] \left( - 2 S_{1 m}^0(\omega) \frac{K}{\Delta} (\bar{\rho} - \rho) \right),
\]

\[
A_{m m 0} = - \frac{1}{\sqrt{2 \pi} \Delta} \rho^{-3} \bar{\rho} C_{m m} - 2 S_{1 m}^0(\omega) \left[ -i \left( \frac{K}{\Delta} \right) - \frac{K^2}{\Delta^2} + 2i \rho \frac{K}{\Delta} \right],
\]

\[
A_{m \bar{m} 1} = \frac{2}{\sqrt{2 \pi} \Delta} \rho^{-3} \bar{\rho} C_{m \bar{m}} - 2 S_{1 m}^0(\omega) \left( i \frac{K}{\Delta} - \rho \right),
\]

\[
A_{\bar{m} \bar{m} 2} = - \frac{1}{\sqrt{2 \pi} \Delta} \rho^{-3} \bar{\rho} C_{\bar{m} \bar{m}} - 2 S_{1 m}^0(\omega), \tag{A1}
\]

where \( C_{\sigma}^{\ell} = 0 \) - \( m/\sin \theta + a \omega \sin \theta + \sigma \cot \theta, \rho = 1/(r - ia \cos \theta) \), and

\[
C_{n n} = - \frac{1}{4 \Sigma^2 \ell} \left[ \tilde{E} (r^2 + a^2) - a \tilde{L}_z + \Sigma \frac{d r}{d \tau} \right]^2,
\]

\[
C_{m m} = - \frac{1}{2 \sqrt{2 \Sigma^2 \ell}} \left[ \tilde{E} (r^2 + a^2) - a \tilde{L}_z + \Sigma \frac{d r}{d \tau} \right] \left[ i \sin \theta \left( a \tilde{E} - \frac{\tilde{L}_z}{\sin^2 \theta} \right) \right],
\]

\[
C_{m \bar{m}} = \frac{\rho^2}{2 \Sigma \ell} \left[ i \sin \theta \left( a \tilde{E} - \frac{\tilde{L}_z}{\sin^2 \theta} \right) \right]^2, \tag{A2}
\]

with \( \ell = d t / d \tau \) and \( \Sigma = r^2 + a^2 \cos^2 \theta \).
Appendix B. Spin-weighted spheroidal harmonics

Using \( x = \cos \theta \), the angular Teukolsky equation (4) takes the form
\[
\left[ (1 - x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} + \xi^2 x^2 - \frac{m^2 + s^2 + 2msx}{1 - x^2} - 2s \xi x + s E_{\ell m}(\xi) \right] s S^{a \omega}_{\ell m}(x) = 0, \tag{B1}
\]
where \( \xi = a \omega \) and \( s E_{\ell m}(\xi) = \lambda + s(s + 1) - a^2 \omega^2 + 2a m \omega \).

When \( \xi = 0 \), the solutions \( s S^{a \omega}_{\ell m}(x) \) in Eq. (B1) reduce to the spin-weighted spherical harmonics and the eigenvalue \( s E_{\ell m}(\xi) \) becomes \( \ell (\ell + 1) \) [35]. Thus, it might be useful to express the spin-weighted spheroidal harmonics in a series of the spin-weighted spherical harmonics [3,24,51,52].

Taking account of singularities at \( x = \pm 1 \) and \( \infty \) in the differential equation (B1), it is also possible to expand the spin-weighted spheroidal harmonics in a series of Jacobi polynomials [38,53]. For this purpose, we introduce new functions \( s U_{\ell m}(x) \) and \( s V_{\ell m}(x) \) through
\[
s S^{a \omega}_{\ell m}(x) = e^{\xi x} \left( \frac{1 - x}{2} \right)^{\frac{\alpha}{2}} \left( \frac{1 + x}{2} \right)^{\frac{\beta}{2}} s U_{\ell m}(x), \tag{B2}
\]
and
\[
s S^{a \omega}_{\ell m}(x) = e^{-\xi x} \left( \frac{1 - x}{2} \right)^{\frac{\alpha}{2}} \left( \frac{1 + x}{2} \right)^{\frac{\beta}{2}} s V_{\ell m}(x), \tag{B3}
\]
where \( \alpha = |m + s| \) and \( \beta = |m - s| \). Note that Eqs. (B2) and (B3) imply
\[
s V_{\ell m}(x) = \exp(2\xi x) s U_{\ell m}(x). \tag{B4}
\]

Substituting Eqs. (B2) and (B3) into Eq. (B1), \( s U_{\ell m}(x) \) and \( s V_{\ell m}(x) \), respectively, satisfy the differential equations as
\[
(1 - x^2) s U''_{\ell m}(x) + \left[ \beta - \alpha - (2 + \alpha + \beta)x \right] s U'_{\ell m}(x)
+ \left[ s E_{\ell m}(\xi) - \frac{\alpha + \beta}{2} \left( \frac{\alpha + \beta}{2} + 1 \right) \right] s U_{\ell m}(x)
= \xi \left[ -2(1 - x^2) s U'_{\ell m}(x) + (\alpha + \beta + 2s + 2)x \right] s U_{\ell m}(x)
- (\xi - \beta + \alpha) s U_{\ell m}(x), \tag{B5}
\]
and
\[
(1 - x^2) s V''_{\ell m}(x) + \left[ \beta - \alpha - (2 + \alpha + \beta)x \right] s V'_{\ell m}(x)
+ \left[ s E_{\ell m}(\xi) - \frac{\alpha + \beta}{2} \left( \frac{\alpha + \beta}{2} + 1 \right) \right] s V_{\ell m}(x)
= \xi \left[ 2(1 - x^2) s V'_{\ell m}(x) - (\alpha + \beta - 2s + 2)x \right] s V_{\ell m}(x)
- (\xi - \beta + \alpha) s V_{\ell m}(x). \tag{B6}
\]

When \( \xi = 0 \), Eqs. (B5) and (B6) reduce to the differential equation for Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \):
\[
(1 - x^2) P_n^{(\alpha, \beta)''}(x) + \left[ \beta - \alpha - (\alpha + \beta + 2)x \right] P_n^{(\alpha, \beta)'}(x)
+ n(n + \alpha + \beta + 1) P_n^{(\alpha, \beta)}(x) = 0. \tag{B7}
\]
provided the eigenvalue $s^2E_{\ell m}(\xi)$ in Eqs. (B5) and (B6) becomes $\ell(\ell+1)$, where $\ell = n + (\alpha + \beta)/2 = n + \max(|m|, |s|)$. Here the Jacobi polynomials are defined by Rodrigue's formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\left(\frac{d}{dx}\right)^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}].$$  \hfill (B8)

If we expand $sU_{\ell m}(x)$ and $sV_{\ell m}(x)$ as infinite series of Jacobi polynomials,

$$sU_{\ell m}(x) = \sum_{n=0}^{\infty} sA_{\ell m}^{(n)}(\xi) P_n^{(\alpha,\beta)}(x),$$ \hfill (B9)

and

$$sV_{\ell m}(x) = \sum_{n=0}^{\infty} sB_{\ell m}^{(n)}(\xi) P_n^{(\alpha,\beta)}(x),$$ \hfill (B10)

we obtain three-term recurrence relations for the expansion coefficients $sA_{\ell m}^{(n)}(\xi)$ and $sB_{\ell m}^{(n)}(\xi)$, respectively, as

$$\alpha^{(0)} sA_{\ell m}^{(1)}(\xi) + \beta^{(0)} sA_{\ell m}^{(0)}(\xi) = 0, \quad \alpha^{(n)} sA_{\ell m}^{(n+1)}(\xi) + \beta^{(n)} sA_{\ell m}^{(n)}(\xi) + \gamma^{(n)} sA_{\ell m}^{(n-1)}(\xi) = 0, \quad (n \geq 1),$$ \hfill (B11)

with

$$\alpha^{(n)} = \frac{4\xi(n+\alpha+1)(n+\beta+1)(n+(\alpha+\beta)/2+1-s)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)},$$

$$\beta^{(n)} = sE_{\ell m}(\xi) + \xi^2 - \left(n + \frac{\alpha + \beta}{2}\right)\left(n + \frac{\alpha + \beta}{2} + 1\right) + \frac{2\xi s(\alpha - \beta)(\alpha + \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)},$$

$$\gamma^{(n)} = -\frac{4\xi n(n+\alpha+\beta)(n+(\alpha+\beta)/2+s)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)},$$ \hfill (B12)

and

$$\tilde{\alpha}^{(0)} sB_{\ell m}^{(1)}(\xi) + \tilde{\beta}^{(0)} sB_{\ell m}^{(0)}(\xi) = 0, \quad \tilde{\alpha}^{(n)} sB_{\ell m}^{(n+1)}(\xi) + \tilde{\beta}^{(n)} sB_{\ell m}^{(n)}(\xi) + \tilde{\gamma}^{(n)} sB_{\ell m}^{(n-1)}(\xi) = 0, \quad (n \geq 1)$$ \hfill (B13)

with

$$\tilde{\alpha}^{(n)} = -\frac{4\xi(n+\alpha+1)(n+\beta+1)(n+(\alpha+\beta)/2+1+s)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)},$$

$$\tilde{\beta}^{(n)} = sE_{\ell m}(\xi) + \xi^2 - \left(n + \frac{\alpha + \beta}{2}\right)\left(n + \frac{\alpha + \beta}{2} + 1\right) + \frac{2\xi s(\alpha - \beta)(\alpha + \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)},$$

$$\tilde{\gamma}^{(n)} = \frac{4\xi n(n+\alpha+\beta)(n+(\alpha+\beta)/2-s)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)}.$$ \hfill (B14)

Note that, for deriving Eqs. (B11) and (B13), we use recurrence relations for Jacobi polynomials [53].

From the behavior of the three-term recurrence relation Eq. (B11) for sufficiently large $n$, there may be two independent solutions in Eq. (B11) as

$$A_{(1)}^{(n)} \sim \frac{\text{const.}(-\xi)^n}{\Gamma(n+(\alpha + \beta + 3)/2-s)},$$ \hfill (B15)

$$A_{(2)}^{(n)} \sim \text{const.}\xi^n\Gamma(n+(\alpha + \beta + 1)/2+s).$$ \hfill (B16)
According to the theory of three-term recurrence relations [54], $A^{(n)}_{(1)}$ is a minimal solution and $A^{(n)}_{(2)}$ is a dominant solution since $\lim_{n \to \infty} A^{(n)}_{(1)}/A^{(n)}_{(2)} = 0$. The series Eq. (B9) computed from the dominant solution $A^{(n)}_{(2)}$ diverges for all values of $x$ since $A^{(n)}_{(2)}$ increases with $n$, while the series Eq. (B9) computed from the minimal solution $A^{(n)}_{(1)}$ converges uniformly. Thus, we have to choose $A^{(n)}_{(1)}$ for the series expansion Eq. (B9) to obtain a solution that converges uniformly. This choice of $A^{(n)}_{(1)}$ requires that the eigenvalue $E_{\ell m}(\xi)$ satisfies a certain transcendental equation, which is expressed in terms of continued fractions.

In order to obtain the equation that determines the eigenvalue $E_{\ell m}(\xi)$, it is convenient to introduce the following quantities:

$$ R_n \equiv \frac{A^{(n)}_{(1)}}{A^{(n)}_{(1)}/A^{(n-1)}_{(1)}}, \quad L_n \equiv \frac{A^{(n)}_{(1)}}{A^{(n+1)}_{(1)}}. \quad (B17) $$

Using the three-term recurrence relation Eq. (B11) we can express $R_n$ as an infinite continued fraction,

$$ R_n = - \frac{\gamma^{(n)}}{\beta^{(n)} + \alpha^{(n)} R_{n+1}} = - \frac{\gamma^{(n)}}{\beta^{(n)} - \alpha^{(n+1)} \gamma^{(n+2)} \beta^{(n+2)} - \beta^{(n+2)}} \cdots, \quad (B18) $$

and $L_n$ as a finite continued fraction,

$$ L_n = - \frac{\alpha^{(n)}}{\beta^{(n)} + \gamma^{(n)} L_{n-1}} = - \frac{\alpha^{(n)}}{\beta^{(n)} - \alpha^{(n-1)} \gamma^{(n)} \beta^{(n-2)} - \beta^{(n-2)}} \cdots \frac{\alpha^{(1)} \gamma^{(2)} \beta^{(0)} - \alpha^{(0)} \gamma^{(1)}}{\beta^{(0)}}. \quad (B19) $$

The expression for $R_n$ is valid if this infinite continued fraction converges. Noting the properties of the three-term recurrence relations (see p. 35 in Ref. [54]), it can be proved that the continued fraction Eq. (B18) converges if the eigenvalue $E_{\ell m}(\xi)$ is finite.

We obtain the equation to determine the eigenvalue $E_{\ell m}(\xi)$ dividing Eq. (B11) by the expansion coefficients $A^{(n)}_{\ell m}(\xi)$:

$$ \beta^{(n)} + \alpha^{(n)} R_{n+1} + \gamma^{(n)} L_{n-1} = 0, \quad (B20) $$

where $R_{n+1}$ and $L_{n-1}$ are defined by the continued fractions Eqs. (B18) and (B19), which are convergent for finite values of $E_{\ell m}(\xi)$. There are many roots in Eq. (B20) for given $n, m, s$, and $\xi$. These roots are associated with the same $m, s$, and $\xi$ but with different $\ell$. In order to find the root for a given $\ell$, it is useful to choose $n = n_\ell \equiv \ell - (\alpha + \beta)/2$ in Eq. (B20), since in the limit $\xi \to 0$ all the terms in Eq. (B20) become $O(\xi^2)$. This means that the choice naturally gives the leading term of a series expansion of the eigenvalue $E_{\ell m}(\xi)$ in terms of $\xi$ as $\ell(\ell + 1)$.

When $|\xi|$ is not large, we can obtain the analytic expression of $E_{\ell m}(\xi)$ in a series of $\xi$ as

$$ E_{\ell m}(\xi) = \ell(\ell + 1) - \frac{2s^2m}{\ell(\ell + 1)} \xi + [H(\ell + 1) - H(\ell - 1)] \xi^2 + O(\xi^3), \quad (B21) $$

where

$$ H(\ell) = \frac{2(\ell^2 - m^2)(\ell^2 - s^2)^2}{(2 \ell - 1)^3(2 \ell + 1)}. \quad (B22) $$

For the numerical calculation to determine $E_{\ell m}(\xi)$, one can use the analytic expression of $E_{\ell m}(\xi)$ above as an initial value to find the root in Eq. (B20).

Once we obtain the eigenvalue, using Eqs. (B18) and (B19) we can compute all the coefficients $A^{(n)}_{\ell m}(\xi)$ from $A^{(n)}_{\ell m}(\xi)$ for a given $n$. The coefficient for $n = n_\ell = \ell - (\alpha + \beta)/2$ is usually the largest term. The ratio of other terms to the dominant term, i.e. $A^{(n)}_{\ell m}(\xi)/A^{(n)}_{\ell m}(\xi)$, can be determined using Eqs. (B18) and (B19) for $0 < n < n_\ell$ and $n > n_\ell$, respectively.
homogeneous solutions of the radial Teukolsky equation \[28,29\]. In this paper, we use a formalism developed by Mano, Suzuki, and Takasugi (MST) to compute the

From Eqs. (B25) and (B26), we can obtain the squares of \( sA_{ltm}(\xi) \) and \( sB_{ltm}(\xi) \). The final determination of the signs of \( sA_{ltm}(\xi) \) and \( sB_{ltm}(\xi) \) is made by the requirement that \( sA_{ltm}(x) \) reduces to the spin-weighted spherical harmonics in the limit \( \xi \to 0 \).

Appendix C. Homogeneous solutions of the radial Teukolsky equation

In this paper, we use a formalism developed by Mano, Suzuki, and Takasugi (MST) to compute the homogeneous solutions of the radial Teukolsky equation \[28,29\]. In the formalism, analytic expressions of homogeneous solutions are given using two kinds of series expansions in terms of hypergeometric functions and Coulomb wave functions, which are, respectively, convergent at the horizon.
and infinity. One can obtain analytic expressions of the asymptotic amplitudes of the homogeneous solutions by analytic matching of the two kinds of solutions in the overlapping region of convergence. Compared to numerical integration methods to solve the Teukolsky equation, the formalism is quite powerful for very accurate numerical calculations of gravitational waves [36,38,39,55]. The formalism is also very powerful for the performance of post-Newtonian expansions of gravitational waves at higher orders, since the series expansion of homogeneous solutions is closely related to the low-frequency expansion. Applying the formalism to the post-Newtonian approximation in black hole perturbation theory, the energy flux going down the horizon was calculated up to 6.5PN for a particle in a circular and equatorial orbit around a Kerr black hole [32] and the 2.5PN energy flux to infinity was computed for a particle in a slightly eccentric and inclined orbit around the Kerr black hole [56,57]. More recently, we applied the formalism to obtain the 5.5PN waveforms for a particle in a circular orbit around a Schwarzschild black hole [30] and the 4PN waveforms for a particle in a circular and equatorial orbit around the Kerr black hole [31], which, respectively, confirmed the 5.5PN energy flux in Ref. [23] and the 4PN energy flux in Ref. [24]. In Refs. [26,27], we extended the formalism to obtain very high PN expressions for the energy flux to infinity for the particle in a circular orbit around the Schwarzschild black hole. For more details of the formalism, we refer the reader to a recent review, Ref. [2].

In the MST formalism, one can expand a homogeneous solution of the radial Teukolsky equation in a series of Coulomb wave functions as

$$R^v_C = \hat{z}^{-1-s} \left( 1 - \frac{\epsilon \kappa}{\hat{z}} \right)^{-s-i(\epsilon+\tau)/2} \sum_{n=-\infty}^{\infty} (-i)^N \frac{(v+1+s-i\epsilon)^n}{(v+1-s+i\epsilon)^n} a^v_n F_{n+v}(-is-\epsilon, \hat{z}), \quad (C1)$$

where $\hat{z} = \omega(r-r_-)$, $\tau = (\epsilon - m q)/\kappa$, $(a)_n = \Gamma(a+n)/\Gamma(a)$, and $F_N(\eta, z)$ is a Coulomb wave function defined by

$$F_N(\eta, z) = e^{-i\hat{z}} 2^N \hat{z}^{N+1} \frac{\Gamma(N+1-i\eta)}{\Gamma(2N+2)} \Phi(N+1-i\eta, 2N+2; 2i\hat{z}), \quad (C2)$$

where $\Phi(a, \beta; z)$ is the confluent hypergeometric function, regular at $z = 0$ (see Sect. 13 in Ref. [58]). Note that the so-called renormalized angular momentum $v$ is introduced in the homogeneous solution in a series of Coulomb wave functions, Eq. (C1). $v$ is a generalization of $\ell$, which has a property such that $v \to \ell$ as $\epsilon \to 0$, and is determined through conditions that the series of Coulomb wave functions, Eq. (C1), converges and actually represents a homogeneous solution of the radial Teukolsky equation.

Substituting the homogeneous solution, Eq. (C1), into the radial Teukolsky equation (5) with $T_{lm\omega} = 0$, one obtains the following three-term recurrence relation for the expansion coefficients $a^v_n$:

$$\alpha^v_n a^v_{n+1} + \beta^v_n a^v_n + \gamma^v_n a^v_{n-1} = 0, \quad (C3)$$

where

$$\alpha^v_n = \frac{\epsilon \kappa (n+v+1+s+i\epsilon)(n+v+1+s-i\epsilon)(n+v+1+i\tau)}{(n+v+1)(2n+2v+3)}, \quad (C4a)$$

$$\beta^v_n = -\lambda - s(s+1) + (n+v)(n+v+1) + \epsilon^2 + \epsilon(e - mq) + \epsilon(e - m q)(s^2 + \epsilon^2) + \frac{\epsilon(e - mq)(s^2 + \epsilon^2)}{(n+v)(n+v+1)}, \quad (C4b)$$

$$\gamma^v_n = -\frac{i \epsilon \kappa (n+v-s+i\epsilon)(n+v-s-i\epsilon)(n+v-i\tau)}{(n+v)(2n+2v-1)}. \quad (C4c)$$
The series of Coulomb wave functions, Eq. (C1), converges and represents a homogeneous solution of the radial Teukolsky equation if \( v \) satisfies the following equation:

\[
R^v_n L^v_{n-1} = 1, \tag{C5}
\]

where \( R^v_n \) and \( L^v_n \) are defined in terms of infinite continued fractions,

\[
R^v_n \equiv \frac{a^v_n}{a^v_{n-1}} = -\frac{\gamma^v_n}{\beta^v_n + a^v_n R^v_{n+1}} = -\frac{\gamma^v_n}{\beta^v_n - \alpha^v_n} - \frac{\alpha^v_n \gamma^v_{n+1} \gamma^v_{n+2}}{\beta^v_n - \beta^v_{n+2}} - \cdots, \tag{C6}
\]

\[
L^v_n \equiv \frac{a^v_n}{a^v_{n+1}} = -\frac{\delta^v_n}{\beta^v_n + \gamma^v_n L^v_{n-1}} = -\frac{\delta^v_n}{\beta^v_n - \alpha^v_n} - \frac{\alpha^v_n \delta^v_{n-1} \delta^v_{n-2}}{\beta^v_n - \beta^v_{n-2}} - \cdots, \tag{C7}
\]

which can be derived from the three-term recurrence relation, Eq. (C3). Observe that one can obtain two kinds of expansion coefficients, \( a^v_n \), from two kinds of the continued fractions, \( R^v_n \) and \( L^v_n \). If \( v \) is chosen to satisfy Eq. (C5) for a certain \( n \), the two kinds of the expansion coefficients coincide and the series of Coulomb wave functions, Eq. (C1), converges for \( r > r_+ \).

Since \( a^{-v-1}_n = \gamma^v_n \) and \( \beta^{-v-1}_n = \beta^v_n \) in Eq. (C4), one finds that \( a^{-v-1}_n \) satisfies the same recurrence relation, Eq. (C3), as \( a^v_n \). Then it can be shown that \( R^{-v-1}_C \) is also a homogeneous solution of the Teukolsky equation, which converges for \( r > r_+ \).

Matching the solution in a series of Coulomb wave functions, which converges for \( r > r_+ \), with the one in a series of hypergeometric functions, which converges for \( r < \infty \), one can obtain the incoming solution of the radial Teukolsky equation, \( R_{in_{C\omega}}^v \), which converges in the entire region as

\[
R_{in_{C\omega}}^v = K_v R_C^v + K_{-v-1} R^{-v-1}_C, \tag{C8}
\]

where

\[
K_v = \frac{e^{i\epsilon(2\epsilon\kappa)^{1-v-N} - N \frac{2-\epsilon}{1}} N \Gamma(1 - s - \epsilon - i\tau) \Gamma(N + 2 + 2v) \Gamma(N + v + 1 + s + \epsilon) \Gamma(N + v + 1 + i\tau)}{(N + 2v + 2) \Gamma(N + v + 1 + s + \epsilon) \Gamma(N + v + 1 + i\tau)} \times \left( \sum_{n=N}^{\infty} \frac{(-1)^n \Gamma(n + N + 2 + 1)}{(n - N)!} \right) \frac{\Gamma(1 - s - \epsilon - i\tau) \Gamma(n + v + 1 + s + \epsilon) \Gamma(n + v + 1 + i\tau)}{(N - n)!} d^v_n \times \left( \frac{(-1)^n}{(N - n)!} \right) \frac{(v + 1 + s - \epsilon) n}{(N + v + 2 + 2) n} A^v_n e^{-i\epsilon(\ln \epsilon - \frac{1-\epsilon}{\epsilon})}, \tag{C9}
\]

and \( N \) is an arbitrary integer. The factor \( K_v \) is a constant to match the solutions in the overlap region of convergence, and is independent of the choice of \( N \).

By comparing \( R_{in_{C\omega}}^v \) in Eq. (6) to Eq. (C8) in the limit \( r^* \rightarrow \pm \infty \), one derives analytic expressions for the asymptotic amplitudes \( B_{trans_{C\omega}}^v, B_{inc_{C\omega}}^v, \) and \( B_{ref_{C\omega}}^v \) in Eq. (6) as

\[
B_{trans_{C\omega}}^v = \left( \frac{\epsilon \kappa}{\omega} \right)^{2v} e^{i\epsilon(\epsilon + \tau) \left( \frac{1}{2} + \frac{\ln \epsilon}{\epsilon} \right)} \sum_{n=-\infty}^{\infty} A^v_n, \tag{C10a}
\]

\[
B_{inc_{C\omega}}^v = \omega^{-1} \left[ K_v - i e^{-i\pi \epsilon} \sin \pi (v + s + i\epsilon) K_{-v-1} \right] A^v_n e^{-i\epsilon(\ln \epsilon - \frac{1-\epsilon}{\epsilon})}, \tag{C10b}
\]

\[
B_{ref_{C\omega}}^v = \omega^{-1-2s} \left[ K_v + i e^{i\pi \epsilon} K_{-v-1} \right] A^v_n e^{i\epsilon(\ln \epsilon - \frac{1-\epsilon}{\epsilon})}, \tag{C10c}
\]
where

\[ A^v_+ = 2^{-1+s-i\epsilon} e^{-\pi i (v+1-s)} e^{\frac{\pi}{2} i (v+1-s)} \frac{\Gamma(v+1-s+i\epsilon)}{\Gamma(v+1+s-i\epsilon)} \sum_{n=-\infty}^{+\infty} a_n^v, \tag{C11a}\]

\[ A^v_- = 2^{-1-s+i\epsilon} e^{\pi i (v+1+s)} \sum_{n=-\infty}^{+\infty} (-1)^n (v+1+s-i\epsilon)_n a_n^v. \tag{C11b}\]

For obtaining Eq. (C10), it is useful to note that the asymptotic form of \( r^* \) in the limit \( r^* \to \pm \infty \) takes

\[ \omega r^* \to \hat{z} + \epsilon \ln \hat{z} - \epsilon \ln \epsilon \quad \text{for} \quad r \to \infty, \tag{C12a} \]

\[ kr^* \to \frac{\epsilon + \tau}{2} \ln \left( \frac{r_+ - r}{2M\kappa} \right) + \kappa \frac{\epsilon + \tau}{2} + \frac{\kappa (\epsilon + \tau)}{1 + \kappa} \ln \kappa \quad \text{for} \quad r \to r_. \tag{C12b} \]

As for the other homogeneous solution \( R_{\pm \omega}^{\uparrow} \) in Eq. (6), we decompose the homogeneous solution in a series of Coulomb wave functions \( R_C^v \) as

\[ R_C^v = R_+^v + R_-^v, \tag{C13} \]

where

\[ R_+^v = 2 v e^{-\pi \epsilon} e^{i\pi (v+1-s)} \frac{\Gamma(v+1-s+i\epsilon)}{\Gamma(v+1+s-i\epsilon)} e^{-i\epsilon \tau} e^{(\epsilon+i\tau)/2 (z - \epsilon \kappa)^{-s-i(\epsilon+\tau)/2}} \]

\[ \times \sum_{n=-\infty}^{+\infty} i^n a_n^v (2z)^n (n + v + 1 - s + i\epsilon, 2n + 2v + 2; 2iz), \tag{C14} \]

\[ R_-^v = 2 v e^{-\pi \epsilon} e^{-i\pi (v+1+s)} e^{i\epsilon \tau} e^{(\epsilon+i\tau)/2 (z - \epsilon \kappa)^{-s-i(\epsilon+\tau)/2}} \sum_{n=-\infty}^{+\infty} i^n \]

\[ \times \frac{(v+1+s-i\epsilon)_n}{(v+1-s+i\epsilon)_n} a_n^v (2z)^n (n + v + 1 - s - i\epsilon, 2n + 2v + 2; -2iz). \tag{C15} \]

For the decomposition, we used an analytic property of the confluent hypergeometric function (see p. 259 in Ref. [59]):

\[ \Phi(\alpha, \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \alpha)} e^{i\pi \sigma} \Psi(\alpha, \gamma; x) + \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^{i\pi(\alpha - \gamma)\sigma} e^x \Psi(\gamma - \alpha, \gamma; -x), \tag{C16} \]

where \( \Psi \) is the irregular confluent hypergeometric function and \( \sigma = \text{sgn}[\text{Im}(x)] \) is assumed.

Since \( \Psi(\alpha, \beta, x) \to x^{-\alpha} \) in the limit \( |x| \to \infty \) (see Sect. 13 in Ref. [58]), one finds

\[ R_+^v = A_+^v z^{-1} e^{-i(z+\epsilon \ln z)}, \quad R_-^v = A_-^v z^{-1} e^{i(z+\epsilon \ln z)} \quad \text{for} \quad r \to \infty. \tag{C17} \]

Noting that the functions \( R_+^v \) and \( R_-^v \) have factors \( e^{-iz} \) and \( e^{iz} \), respectively, one finds that \( R_+^v \) (\( R_-^v \)) is an incoming (outgoing) wave solution at infinity. Then the upgoing solution \( R_{\pm \omega}^{\uparrow} \) is given by

\[ R_{\pm \omega}^{\uparrow} = R_+^v. \tag{C18} \]

Again noting the asymptotic form of \( r^* \) in the limit \( r^* \to \pm \infty \) in Eq. (C12) and comparing \( R_{\pm \omega}^{\uparrow} \) in Eq. (6) to Eq. (C18) in the limit \( r^* \to +\infty \), one finds the asymptotic amplitude \( C_{\pm \omega}^{\uparrow} \) as

\[ C_{\pm \omega}^{\uparrow} = \omega^{-1-2s} A_+^v e^{i(\epsilon \ln \epsilon - \frac{1-\kappa}{2} s\epsilon)}. \tag{C19} \]
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