Pathwise asymptotic behavior of random determinants in the uniform Gram and Wishart ensembles

A. Rouault *

October 29, 2018

Summary. This paper concentrates on asymptotic properties of determinants of some random symmetric matrices. If $B_{n,r}$ is an $n \times r$ rectangular matrix and $B'_{n,r}$ its transpose, we study $\det(B'_{n,r}B_{n,r})$ when $n, r$ tends to infinity with $r/n \to c \in (0,1)$. The $r$ column vectors of $B_{n,r}$ are chosen independently, with common distribution $\nu_n$. The Wishart ensemble corresponds to $\nu_n = \mathcal{N}(0, I_n)$, the standard normal distribution. We call uniform Gram ensemble the ensemble corresponding to $\nu_n = \sigma_n$, the uniform distribution on the unit sphere $S_{n-1}$. In the Wishart ensemble, a well known Bartlett’s theorem decomposes the above determinant into a product of chi-square variables. The same holds in the uniform Gram ensemble. This allows us to study the process $\{\frac{1}{n} \log \det(B'_{n,\lfloor nt\rfloor}B_{n,\lfloor nt\rfloor}), t \in [0,1]\}$ and its asymptotic behavior as $n \to \infty$: a.s. convergence, fluctuations, large deviations. We connect the results for marginals (fixed $t$) with those obtained by the spectral method.

Key words. Random matrices, Hadamard ratio, Wishart ensemble, Gram ensemble, determinant, invariance principle, large deviations.

A.M.S. Classification. 15 A 52, 15 A 15, 60F 10, 60F 17, 62 H 10

1 Introduction

For $n, r \in \mathbb{N}$ such that $r \leq n$, let $M_{n,r}(\mathbb{R})$ be the set of $n \times r$ matrices with real entries. A matrix $B \in M_{n,r}(\mathbb{R})$ consists in $r$ column vectors $b_1, \cdots, b_r$ of $\mathbb{R}^n$. We denote by $B'$ its transpose, so that $B'B \in M_{r,r}$ is symmetric. We provide $\mathbb{R}^n$ with the usual Euclidean norm.

In 1893, Hadamard [27] proved that

$$\det(B'B) \leq \|b_1\|^2 \cdots \|b_r\|^2$$

with equality if and only if $b_1, \cdots, b_r$ are orthogonal. That means that the volume (or $r$-content) of the parallelootope built from $b_1, \cdots, b_r$ is maximal when the vectors are orthogonal.

* LAMA, Bâtiment Fermat, Université de Versailles F-78035 Versailles. e-mail: rouault@math.uvsq.fr
Consequently, the quantity

\[
h(B) = \frac{\det(B'B)}{||b_1||^2 \ldots ||b_r||^2}
\]

is usually called the Hadamard ratio (cf. [17]); in the basis reduction problem ([1],[3],[5]), the quantity \(1/\sqrt{h(B)}\) is called the orthogonality defect. Some papers ([1],[17]) are concerned with the tightness of the bound \(h(B) \leq 1\) when \(B\) is random and \(n = r\). Writing \(B_{n,r}\) instead of \(B\) to stress on dimensions, it is interesting to study the asymptotic behavior of the sequence of random variables \(h(B_{n,r})\), in particular when \(n, r \rightarrow \infty\) with \(r/n \rightarrow c \in [0, 1]\).

We consider independent random vectors \(b_i, i = 1, \ldots, r\) with the same distribution \(\nu_n\) in \(\mathbb{R}^n\). It seems natural to choose \(\nu_n = \sigma_n\), the uniform distribution on the unit sphere \(S_{n-1}\). The corresponding ensemble for \(B\) is called Uniform Spherical Ensemble in [18]. The matrix ensemble for \(B'B\) is called the Gram ensemble in [18] since \(B'B\) is the Gram matrix built from vectors \(b_i\)'s. To stress on the distribution, we call it uniform Gram ensemble.

More generally, if \(\nu_n\) is isotropic, (i.e. \(\nu_n(\{0\}) = 0\) and \(\nu_n\) invariant by rotation), it is well known that \(\tilde{b}_1 := b_1/\|b_1\|\) is \(\sigma_n\) distributed and independent of \(\|b_1\|\). Denoting by \(\tilde{B}\) the matrix of unitary vectors, we see that \(\tilde{B}\tilde{B}\) is in the uniform Gram ensemble. It makes possible to study \(\det(B'B)\) in its own, since the decomposition in independent factors

\[
\det(B'B) = \det(\tilde{B}'\tilde{B}) \times \prod_{i=1}^r \|b_i\|^2
\]

reduces this case to the previous one if the distribution of \(\|b_1\|^2\) is well behaved.

The most important example is the Gaussian one with \(\nu_n = \mathcal{N}(0; I_n)\): all the entries of \(B\) are i.i.d. \(\mathcal{N}(0; 1)\) and \(B'B\) is in the Wishart ensemble. Moreover \(\|b_1\|^2\) is \(\chi^2\) distributed. Our paper is concerned essentially with these two cases.

We introduce a probability space on which all uniform Gram and Wishart matrices are defined simultaneously. It is just the infinite product space generated by a double infinite sequence of i.i.d. \(\mathcal{N}(0; 1)\) variables \(\{b_{i,j}\}_{i,j=1}^\infty\). Then we take \(B_{n,r} = \{b_{i,j}, i = 1, \ldots, r, j = 1, \ldots r\}\) and omitting the dimension index \(n\), we set \(\tilde{b}_{i,j} = b_{i,j}/(\sum_{k=1}^n b_{k,j}^2)^{1/2}\) and \(\tilde{B}_{n,r} = \{\tilde{b}_{i,j}, i = 1, \ldots r, j = 1, \ldots r\}\).

In Section 2 we recall some known results. Using the classical QR decomposition of \(B\) with \(Q \in M_{n,r}\) orthogonal and \(R \in M_{r,r}\) uppertriangular ([10]), we get

\[
\det(B_{n,r}'B_{n,r}) = \prod_{j=1}^r R_{jj}^2.
\]

In the Wishart case, the variables \(R_{jj}^2, j = 1, \ldots, r\) are independent and \(\chi^2\) distributed with respective parameters \(n - j + 1, j = 1, \ldots, r\). This result is known as the celebrated Bartlett decomposition. In the Gram case, the corresponding variables \(\tilde{R}_{jj}^2, j = 2, \ldots, r\) are independent and beta distributed with respective parameters \(\left(n-j+1, \frac{j-1}{2}\right)\). Therefore we will consider \(\{\log \det(B_{n,r}'B_{n,r}), r = 1, \ldots, n\}\) and its “tilde” version as triangular arrays and
prove pathwise\(^1\) results for the sequence of processes \(\left\{ \frac{1}{n} \log \det(B'_{n,[nt]} B_{n,[nt]}), \; t \in [0,1] \right\}\).

In Section 3 we present the spectral approach. It starts from

\[
\log \det(B'_{n,r} B_{n,r}) = \sum_{k=1}^{r} \log \lambda^{(k)}_{n,r},
\]

where \(\lambda^{(k)}_{n,r}, k = 1, \cdots, r\) are the (real) eigenvalues of \(B'_{n,r} B_{n,r}\). We may take advantage of known results (recalled in Section 3) on the convergence of the empirical spectral distribution (ESD) to the Marcenko-Pastur distribution as \(r/n \to c \in (0,1)\) (\[36\] for the Wishart ensemble and \[30\], \[13\] for the uniform Gram ensemble). This allows in the following sections to recover results for marginals (only) and in the Wishart case to discover fruitful connection between the two methods.

In Section 4 we study the Gram ensemble and set

\[
\mathcal{Y}_{n,r} = \bar{B}'_{n,r} \bar{B}_{n,r}, \quad \Upsilon_{n,r} = \log \det \mathcal{Y}_{n,r}.
\]

We state the a.s. convergence of \(\left\{ \frac{1}{n} \Upsilon_{n,[nt]}, \; t \in [0,1] \right\}\) (Theorem 4.2), the weak convergence of fluctuations (Theorem 4.3) and a large deviation principle (Theorem 4.4 and Theorem 4.5).

In Section 5 we study the Wishart ensemble. Since \(\mathbb{E}\|b_1\|^2 = n\), it is natural to normalize Wishart matrices and set

\[
\mathcal{X}_{n,r} = \frac{1}{n} B'_{n,r} B_{n,r}, \quad \Xi_{n,r} = \log \det \mathcal{X}_{n,r}. \quad (3)
\]

The asymptotic behavior of the process \(\left\{ \frac{1}{n} \Xi_{n,[nt]}, \; t \in [0,1] \right\}\) is easily deduced from (2) and the above results in the uniform Gram case. (Of course, it is also possible to use the Bartlett decomposition).

Section 6 is devoted to some remarks about extensions to matrices with entries with Gaussian entries in other fields (complex, quaternionic), and even with non Gaussian entries. An extension to the Jacobi ensemble will be considered in a forthcoming paper.

The proofs of results are located in Section 7 and 8.

All along the paper we use the function \(\ell(x) = \log \Gamma(x)\), and its derivative \(\Psi = \ell'\) which is the digamma function. Some useful properties of \(\ell\) and \(\Psi\) are given in Appendix. We use also the following functions:

\[
\mathcal{J}(u) = \begin{cases} 
  u \log u - u + 1, & \text{for } u > 0 \\
  1, & \text{for } u = 0 \\
  +\infty, & \text{for } u < 0,
\end{cases}
\]

and for \(t \geq 0\),

\[
F(t) = \int_0^t \mathcal{J}(u) \, du = \frac{t^2}{2} \log t - \frac{3t^2}{4} + t.
\]

\(^1\)We stress that this study is pathwise in the parameter \(t = r/n\) and not in the "time" parameter as in Wishart processes defined from Brownian matrices.
Recall also that for \( a > 0 \), the \( \chi^2_a \) distribution has density

\[
\frac{x^{\frac{a}{2}-1}}{2^{\frac{a}{2}}\Gamma \left( \frac{a}{2} \right)} e^{-\frac{x}{2}} \quad (x > 0)
\]

and that, for \( \alpha > 0, \beta > 0 \) the beta(\( \alpha, \beta \)) distribution has density

\[
\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad (x > 0)
\]

Let us end this introduction with some comments. Wishart matrices were first introduced in multivariate statistical analysis as sample covariance matrices: \( B \in M_{n,r} \) is a data matrix where \( r \) is the number of variates and \( n \) is the sample size. Then \( \det(B'B) \) is Wilks’ generalized variance (up to a transformation). It is used to build tests on the covariance matrix ([6, 39]). In contemporary multivariate analysis, it is common to consider large \( r \) and large \( n \) (see examples in [31]) , although it may seem non standard to increase the number of variables for a given size of the sample. Besides, in stochastic geometry it seems interesting to describe the evolution of the \( r \)-content of a random \( r \)-parallelotope as \( r \) increases. In quantum dynamics, the uniform Gram ensemble is introduced by De Conck et al. in [21] and they called \( c \) (the limit of \( r/n \)) a time-parameter, although they assumed it fixed.

2 Decompositions

This section consists in notation and recalls. The key point is a decomposition of determinants in products and its consequence for random Gaussian matrices.

2.1 Some linear algebra

Every matrix \( B \in M_{n,r} \) may be decomposed (see [10]) in a product \( B = QR \) of an uppertriangular matrix \( R \in M_{r,r}(\mathbb{R}) \) and an orthogonal matrix \( Q \in M_{n,r}(\mathbb{R}) \). If the vectors \( b_i, i = 1, \ldots, r \) are linearly independent, the decomposition is unique if we force diagonal elements of \( R \) to be positive. By the Gram-Schmidt method, we set \( c_1 = b_1 \) and for \( 2 \leq j \leq r \)

\[
c_j = b_j - \sum_{k=1}^{j-1} \frac{\langle c_k, b_j \rangle}{\|c_k\|^2} c_k,
\]

and then build the orthonormal system:

\[
f_j = \frac{c_j}{\|c_j\|}, \quad 1 \leq j \leq r.
\]

This yields

\[
b_j = \langle f_j, b_j \rangle > f_j + \sum_{k=1}^{j-1} \langle f_k, b_j \rangle f_k
\]
Now $Q$ consists in $f_1, \cdots, f_r$ and $R$ is given by

$$R_{j,j} = \|c_j\| = \langle f_j, b_j \rangle, \quad 1 \leq j \leq r,$$

and for $2 \leq j \leq r$ and $k \leq j - 1$:

$$R_{k,j} = \langle f_k, b_j \rangle = \frac{\langle c_k, b_j \rangle}{\|c_k\|}.$$  \hspace{1cm} (6)

From (4) we deduce

$$\|b_j\|^2 = R_{j,j}^2 + \sum_{k=1}^{j-1} |R_{k,j}|^2.$$  \hspace{1cm} (7)

We can write $b_i = \|b_i\|\tilde{b}_i$ with $\tilde{b}_i \in S_{n-1}$, so that $f_1, \cdots, f_r$ depend only upon $\tilde{b}_1, \cdots, \tilde{b}_r$. We have

$$R_{j,j}^2 = \|b_j\|^2\tilde{R}_{j,j}^2, \quad \tilde{R}_{j,j}^2 := \langle f_j, \tilde{b}_j \rangle^2.$$  \hspace{1cm} (8)

Since $R$ is upper triangular and $B'B = R'R$ we get easily

$$\det(B'B) = \prod_{j=1}^{r} R_{j,j}^2,$$

and from (1) and (8)

$$h(B) = \det(\tilde{B}'\tilde{B}) = \prod_{j=2}^{r} \tilde{R}_{j,j}^2.$$  \hspace{1cm} (10)

(It is clear, of course that $h(B) \leq 1$, as Hadamard noticed).

### 2.2 Random Gaussian Matrices and Bartlett’s decomposition

In the sequel, we study models of random matrices in which all entries are independent and $\mathcal{N}(0,1)$ distributed.

We are in the situation of Section 2.1. It is clear that for every $j = 1, \cdots, r$,

$$\text{Span} \{b_1, \cdots, b_r\} = \text{Span} \{c_1, \cdots, c_r\} = \text{Span} \{f_1, \cdots, f_r\},$$

and $R_{j,j}$ is a measurable function of $(b_1, \cdots, b_j)$ thanks to (5) and (6).

**Proposition 2.1**  \hspace{1cm} 1) If $1 \leq r \leq n$, the random variables $R_{j,j}^2$, $j = 1, \cdots, r$ are independent and

$$R_{j,j}^2 \overset{D}{=} \chi_{n-(j-1)}^2,$$

where $\overset{D}{=} \text{stands for equality in distribution}.$
2) If $2 \leq r \leq n$, the random variables $\widetilde{R}_{jj}^2$, $j = 2, \cdots, r$ are independent and

$$\widetilde{R}_{jj}^2 \overset{D}{=} \text{beta}\left(\frac{n-j+1}{2}, \frac{j-1}{2}\right).$$

The first claim is the celebrated Bartlett decomposition ([3]). It is quoted in many books and articles in particular [6] pp. 170-172, [39] pp. 99 th. 3.2.14, [34]. The second claim comes from 1) and (8). For the sake of completeness, we give here the proof of 1), with the so-called "random orthogonal transformation", which may be found in [33].

**Proof:**

1) Let us fix $2 \leq j \leq r$, and condition upon $(b_1, \cdots, b_{j-1})$. From equation (6) we have $R_j := (R_{1,j}, \cdots, R_{(j-1),j})' = F b_j$ where $F = (f_1, \cdots, f_{j-1}) \in M_{n,j-1}$ is a (known) orthogonal matrix. The Cochran theorem and (7) imply that $R_j$ is $N(0; I_{j-1})$ distributed, that $\|R_j\|_2 = \chi^2_{j-1}$ distributed and that $R_{jj}^2 = \|b_j\|^2 - \|R_j\|^2$ is independent of $\|R_j\|^2$ and $\chi^2_{n-(j-1)}$ distributed. Since in all the above statements, the conditioning variables $(b_1, \cdots, b_{j-1})$ did not appear, these statements are true unconditionally. In particular, $R_{jj}^2$ is independent of $(b_1, \cdots, b_{j-1})$. This yields that all the variables $R_{jj}^2$, $j = 1, \cdots, r$ are independent.

2) From (8) and the previous remarks, we see that the variables $\widetilde{R}_{jj}^2$ are also independent.

To get the distributions, recall that if $U \overset{D}{=} \chi^2_a$ and $V \overset{D}{=} \chi^2_b$ are independent variables, then $U/(U + V) \overset{D}{=} \text{beta}(a/2, b/2)$.

To consider the asymptotic behavior of uniform Gram and Wishart determinants in a dynamic (or pathwise) way, let us give some notation.

Set $\Upsilon_{n,0} = 0$, $\Upsilon_{n,1} = 0$ and for $2 \leq r \leq n$

$$\Upsilon_{n,r} = \sum_{k=2}^r \log \widetilde{R}_{jj}^2, \quad (11)$$

which provides a first triangular array. Besides, from (6) and (9), for $r = 1, \cdots, n$

$$\Xi_{n,r} = \sum_{j=1}^r \log \frac{R_{jj}^2}{n}. \quad (12)$$

provides a second triangular array. Actually, in that case, we have also, from (8) and (10):

$$\Xi_{n,r} = \Upsilon_{n,r} + S_{n,r}, \quad r = 1, \cdots, n, \quad (13)$$

where

$$S_{n,r} := \sum_{k=1}^r \log \frac{\|b_k\|^2}{n}, \quad r = 1, \cdots, n.$$

In this auxiliary triangular array, the independent variables $(\|b_k\|^2, k = 1, \cdots, n)$ are independent of $(\Xi_{n,r}, r = 1, \cdots, n)$ and $\|b_k\|^2 \overset{D}{=} \chi^2_k(n)$. The three processes are denoted by

$$\Upsilon_n(t) := \Upsilon_{n,\lfloor nt \rfloor}, \quad \Xi_n(t) = \Xi_{n,\lfloor nt \rfloor}, \quad S_n(t) = S_{n,\lfloor nt \rfloor}, \quad t \in [0,1].$$
3 The spectral method

Beside the above "decomposition method" we will use the spectral approach which we describe now.

Let $\lambda^{(k)}_{n,r}, k = 1, \cdots, r$ be the (real) eigenvalues of $X_{n,r}$ in the regime $n, r \to \infty$ with $r/n \to c < 1$ fixed. We set

$$\mu_{n,r} = \frac{1}{r} \sum_{k=1}^{r} \delta_{\lambda^{(k)}_{n,r}}$$

the empirical spectral distribution (ESD). In particular

$$\int \log x d\mu_{n,r}(x) = \frac{1}{r} \sum_{k=1}^{r} \log \lambda^{(k)}_{n,r} = \frac{1}{r} \log \det X_{n,r}.$$ 

For $c > 0$ and $\sigma > 0$, let $\pi_{c}^{\sigma^2}$ be the probability distribution on $\mathbb{R}$ defined by

$$\pi_{c}^{\sigma^2}(dx) = (1 - c^{-1})_+ \delta_0(dx) + \frac{\left((x - \sigma^2 a(c))(\sigma^2 b(c) - x)\right)^{1/2}}{2\pi \sigma^2 c x} dx,$$

where $\delta_0$ is the Dirac mass in 0, $x_+ = \max(x, 0)$ and

$$a(c) = (1 - \sqrt{c})^2, \quad b(c) = (1 + \sqrt{c})^2.$$ (15)

It is called the Marčenko-Pastur distribution with ratio index $c$ and scale index $\sigma^2$ ([7], p.621).

It is well known ([36], [7] section 2.1.2) that as $n, r \to \infty$ with $r/n \to c \in (0, \infty)$, the family of empirical spectral distributions $(\mu_{n,r})$ converges a.s. weakly to the Marčenko-Pastur distribution $\pi_{c}^{\sigma^2}$. If we replaced the common law $\mathcal{N}(0; 1)$ by $\mathcal{N}(0, \sigma^2)$ then the limiting distribution would be $\pi_{c}^{\sigma^2}$.

Recently, De Cock et al. ([13]) and Jiang ([30]) proved that the same result holds true in the uniform Gram ensemble. If $\tilde{\lambda}^{(k)}_{n,r}, k = 1, \cdots, r$ be the (real) eigenvalues of $Y_{n,r}$ and

$$\tilde{\mu}_{n,r} = \frac{1}{r} \sum_{k=1}^{r} \delta_{\tilde{\lambda}^{(k)}_{n,r}}$$

then, as $n \to \infty$ and $r/n \to c \in (0, \infty)$, the family $(\tilde{\mu}_{n,r})$ converges a.s. to $\pi_{c}^{\sigma^2}$.

In both cases (uniform Gram and Wishart) we examine in Section 4 and 5 the connection between the "decomposition" method and the spectral method, at the level of marginals.

4 Determinants in the uniform Gram ensemble

The proofs of the results of this Section are in Section 7. The subscript or superscript $G$ (resp. $W$) for the limiting quantities refers to the uniform Gram ensemble (resp. the Wishart ensemble).

Let us notice that in the paper [16], a decomposition in product of beta variables for a completely different problem leads to similar results.
4.1 Two first moments and almost sure convergence

Proposition 4.1 For the mean, we have

\[
\lim_{n} \sup_{p \leq n} \left| \frac{1}{n} \mathbb{E} \Upsilon_{n,p} + \mathcal{J} \left( 1 - \frac{p}{n} \right) \right| = 0, \tag{16}
\]

and actually,

\[
\forall t \in [0, 1), \quad \left( \mathbb{E} \Upsilon_n(t) + n \mathcal{J} \left( 1 - \frac{|nt|}{n} \right) \right) \to d_G(t) := t + \frac{1}{2} \log(1 - t) \tag{17}
\]

\[
\left( \mathbb{E} \Upsilon_{n,n} + \frac{1}{2} \log n \right) \to \frac{-\gamma - \log 2 + 3}{2} \tag{18}
\]

where \(\gamma\) is the Euler constant.

For the variance, we have

\[
\forall t \in [0, 1), \quad \text{Var} \ \Upsilon_n(t) \to \sigma^2_G(t) := -2 \log(1 - t) - 2t \tag{19}
\]

\[
(\text{Var} \ \Upsilon_{n,n} - 2 \log n) \to \frac{8 \gamma + \pi^2}{4}. \tag{20}
\]

Theorem 4.2 Almost surely,

\[
\lim_{n} \sup_{t \in [0, 1]} \left| \frac{\Upsilon_n(t)}{n} + \mathcal{J}(1 - t) \right| = 0. \]

The formulae (18) and (20) are due to Abbott and Mulders (lemmas 4.2 and 4.4 in [1]), using a variant of the decomposition method.

If we want to use the spectral method (with fixed \(t = c < 1\)) we may start with

\[
\frac{1}{n} \Upsilon_n(c) = \frac{|nc|}{n} \int \log x \ d\tilde{\mu}_{n,|nc|}(x)
\]

use the weak convergence of \(\tilde{\mu}_{n,|nc|}\) towards \(\pi_1^c\), (see Section 3). To conclude that

\[
\lim_{n} \int \log x \ d\tilde{\mu}_{n,|nc|}(x) = \int \log x \ d\pi_1^c(x), \tag{21}
\]

an additional control is necessary, since \(x \mapsto \log x\) is not bounded. In [30], Jiang proved recently that the largest and the smallest eigenvalue of \(\mathcal{Y}_{n,r}\) converge a.s., as \(r/n \to c < 1\) to \(b(c) < \infty\) and \(a(c) > 0\) respectively (remember the definitions of \(a\) and \(b\) in (15)). So, (21) is true. Moreover, it is known ([32] p.31 and [8] p. 596-597) that :

\[
c \int \log x \ d\pi_1^c(x) = \int_{a(c)}^{b(c)} \frac{\log x}{2\pi x} \sqrt{(x - a(c))(b(c) - x)} \ dx = (c - 1) \log(1 - c) - c = -\mathcal{J}(1 - c) \tag{22}
\]

This matches with the result of Theorem 4.2.
4.2 Fluctuations

Let $D_T = \{ v \in \mathcal{D}([0,T]) : v(0) = 0 \}$ and $D = \{ v \in \mathcal{D}([0,1]) : v(0) = 0 \}$ the set of càdlàg functions on $[0,T]$ and $[0,1)$, respectively, starting from 0.

**Theorem 4.3** 1. Let for $n \geq 1$

$$\eta^n_G(t) := \Upsilon_n(t) + n\mathcal{J}\left(1 - \frac{|nt|}{n}\right), \quad t \in [0,1).$$

Then as $n \to \infty$

$$(\eta^n_G(t); t \in [0,1)) \Rightarrow (Y_t; t \in [0,1)),$$  \hspace{1cm} (23)

where $Y$ is the (Gaussian) diffusion solution of the stochastic differential equation:

$$dY_t = \frac{1-2t}{2(1-t)} \, dt + \sqrt{\frac{2t}{1-t}} \, dB_t,$$  \hspace{1cm} (24)

with $Y_0 = 0$, $B$ is a standard Brownian motion and $\Rightarrow$ stands for the weak convergence of distributions in $D$ provided with the Skorokhod topology.

2. Let

$$\tilde{\eta}^n_G(1) = \frac{\Upsilon_{n,n} + n + \frac{1}{2} \log n}{\sqrt{2 \log n}}.$$

Then as $n \to \infty$, $\tilde{\eta}^n_G(1) \Rightarrow N$ where $N$ is a standard normal variable independent of $B$, (and $\Rightarrow$ stands for the weak convergence of distribution in $\mathbb{R}$).

4.3 Large deviations

All along this section, as in Section 5.3 and in the proof Sections 7.4 and 8.2, we use the notation of Dembo-Zeitouni [15]. In particular we write LDP for Large Deviation Principle. The reader may have some interest in consulting [16] where a similar method is used for a different model, but here we use a slightly different topology to be able to catch the marginals in $T$.

For $T < 1$, let $\mathcal{M}_T$ be the set of signed measures on $[0,T]$ and let $\mathcal{M}_<$ be the set of measures whose support is a compact subset of $[0,1)$.

We provide $D$ with the weakened topology $\sigma(D,\mathcal{M}_<)$. So, $D$ is the projective limit of the family, indexed by $T < 1$ of topological spaces $(D_T,\sigma(D_T,\mathcal{M}_T))$.

Let $V_\ell$ (resp. $V_\ell$) be the space of left (resp. right) continuous $\mathbb{R}$-valued functions with bounded variations. We put a superscript $T$ to specify the functions on $[0,T]$. There is a one-to-one correspondence between $V_\ell^T$ and $\mathcal{M}_T$:

- for any $v \in V_\ell^T$, there exists a unique $\mu \in \mathcal{M}_T$ such that $v = \mu([0,\cdot])$; we denote it by $\dot{v}$
- for any $\mu \in \mathcal{M}_T$, $v = \mu([0,\cdot])$ stands in $V_\ell$. 
For the following statement, we need some notation. Let $H$ be the entropy function:

$$H(x|p) = x \log \frac{x}{p} + (1 - x) \log \frac{1 - x}{1 - p},$$  \hspace{1cm} (25)$$

and set

$$L^G_a(t, y) = \frac{1}{2} H(1 - t|e^y) \delta(y|(-\infty, 0)) , \quad L^G_s(t, y) = -\frac{1}{2} (1 - t)y \delta(y|(-\infty, 0)).$$  \hspace{1cm} (26)$$

\textbf{Theorem 4.4} The sequence $\{\frac{1}{n} \Upsilon_n\}$ satisfies a LDP in $(D, \sigma(D, \mathcal{M}_<))$ at scale $n^2$ with good rate function given for $v \in D$ by:

$$I^{G}_{[0,1)} (v) = \int_{[0,1)} L^G_a \left( t, \frac{d\hat{v}_a}{dt}(t) \right) dt + \int_{[0,1)} L^G_s \left( t, \frac{d\hat{v}_s}{d\mu}(t) \right) d\mu(t) \quad \text{if} \quad v \in V_r,$$

where $\hat{v} = \hat{v}_a + \hat{v}_s$ is the Lebesgue decomposition of the measure $\hat{v} \in \mathcal{M}([0,1))$ in absolutely continuous and singular parts with respect to the Lebesgue measure and $\mu$ is any bounded positive measure dominating $\hat{v}_s$. If $v \notin V_r$, then $I^{G}_{[0,1)} (v) = \infty$.

That means, roughly speaking, that

$$\mathbb{P}(\Upsilon_n \simeq v) \approx e^{-n^2 I^{G}_{[0,1)} (v)}.$$

The proof, in Section needs several steps. First we show that $\{\frac{1}{n} \hat{\Upsilon}_n\}$ satisfies a LDP in $\mathcal{M}_T$ provided with the topology $\sigma(\mathcal{M}_T, V_\ell)$. Then we carry the LDP to $(\mathcal{D}_T, \sigma(\mathcal{D}_T, \mathcal{M}_T))$ with good rate function:

$$I^{G}_{[0,T]} (v) = \int_{[0,T]} L^G_a \left( t, \frac{d\hat{v}_a}{dt}(t) \right) dt + \int_{[0,T]} L^G_s \left( t, \frac{d\hat{v}_s}{d\mu}(t) \right) d\mu(t).$$

(28)

To end the proof we apply the Dawson-Gärtner theorem on projective limits (Theorem 4.6.1, see also Proposition A2).

Let us notice that $I^{G}_{[0,T]}$ vanishes only for $\frac{d\hat{v}_a}{dt}(t) = \log(1 - t)$ and $\frac{d\hat{v}_s}{d\mu}(t) = 0$ (essentially) i.e. for $v(t) = -\mathcal{J}(1 - t)$, which is consistent with the result of Theorem 4.2.

The LDP for marginals is given in the following theorem, where a rate function with affine part appears.

\textbf{Theorem 4.5} For every $T < 1$, the sequence $\{\frac{1}{n} \Upsilon_{n, \lfloor nT \rfloor}\}$ satisfies a LDP in $\mathbb{R}$ at scale $n^2$ with good rate function denoted by

$$I^{G}_{T} (\xi) = \inf \{ I^{G}_{[0,T]} (v) : v(T) = \xi \}.$$  \hspace{1cm} (29)$$

\text{we set} \delta(y | A) = 0 \text{ if } y \in A \text{ and } = \infty \text{ if } y \notin A$
1. If $\xi \geq -T$ the equation
\[ J(1 + 2\theta) - J(1 - T + 2\theta) - T \log(1 + 2\theta) = \xi, \] (30)
has a unique solution, and we have
\[ I_T^G(\xi) = \theta \xi + \frac{T}{2} J(1 + 2\theta) + \frac{1}{2} (F(1) - F(1 - T) - F(1 + 2\theta) + F(1 - T + 2\theta)). \] (31)

2. If $\xi < -T$, we have
\[ I_T^G(\xi) = I_T^G(-T) - \frac{1 - T}{2}(\xi + T). \] (32)

5 Determinants in the Wishart ensemble

5.1 Introduction

Three ways are possible to study the asymptotic behavior of the determinant of
\[ X_{n,r} = \frac{1}{n} B_{n,r}' B_{n,r}. \]

a) The spectral approach if we are interested only in marginals ($r/n \to c$ fixed).

b) The Bartlett’s decomposition method for a dynamical study. The representation [12] leads to results similar to those of the above section. Let us remark that at the level of marginals, [10] gives the Mellin transform:
\[ \mathbb{E}(\det X_{r,n})^s = 2^{rs} \frac{\Gamma_r \left( \frac{r}{2} + s \right)}{\Gamma_r \left( \frac{r}{2} \right)}, \quad \text{for } \Re s > -\frac{n - r + 1}{2}, \]
where
\[ \Gamma_r(\alpha) = \pi^{r(r-1)/4} \Gamma(\alpha) \Gamma \left( \alpha - \frac{1}{2} \right) \cdots \Gamma \left( \alpha - \frac{r-1}{2} \right) \]
(see for instance [38] Theorem 1 p. 347). This yields the density of $\det X_{r,n}$ (cf. [37], [11] formula 2.4, when $r = n$), in which the Meijer function is involved.

c) Actually, we prefer to establish these results from the representation [13] which we recall here:
\[ \log \det X_{n,r} = \log \det Y_{n,r} + S_{n,r}, \]
where
\[ S_{n,r} = \sum_{k=1}^{r} \log \frac{\|b_k\|^2}{n}, \]
for $r = 1, \ldots, n$, where the variables $\|b_k\|^2, k = 1, \ldots, n$ are independent, $\chi_n^2$ distributed, and independent of $(\log \det Y_{n,r} = Y_{n,r}, r = 1, \ldots, n)$.

We state also connections with known results deduced from the spectral approach. The proofs are in Section [7].
5.2 Almost sure convergence and fluctuations

By extension of the study in Section 4.1 (method b) above, we get the following.

**Proposition 5.1** For the mean we have

\[
\lim_{n \to \infty} \sup_{p \leq n} \left| \frac{1}{n} \mathbb{E} \Xi_{n,p} + \mathcal{J}(1 - \frac{p}{n}) \right| = 0.
\]

Moreover, as \( n \to \infty \)

\[
\forall t \in [0, 1) \quad \mathbb{E} \Xi_{n,\lfloor nt\rfloor} + n\mathcal{J}\left(1 - \frac{|nt|}{n}\right) \to d_W(t) = \frac{1}{2} \log(1 - t)
\]

\[
\mathbb{E} \Xi_{n,n} + n + \frac{1}{2} \log n \to -\gamma + \log 2 - \frac{1}{2}.
\]

For the variance we have

\[
\forall t \in [0, 1) \quad \text{Var} \Xi_{n,\lfloor nt\rfloor} \to \sigma_W^2(t) := -2 \log(1 - t)
\]

\[
\text{Var} \Xi_{n,n} - 2 \log n \to \frac{2\gamma + \pi^2 + 8}{4}.
\]

**Theorem 5.2** Almost surely,

\[
\lim_{n \to \infty} \sup_{t \in [0,1]} \left| \frac{1}{n} \mathbb{E} \Xi_{n,\lfloor nt\rfloor} + \mathcal{J}(1 - t) \right| = 0.
\]

**Theorem 5.3** Let

\[
\eta_{n}^W(t) := \Xi_{\lfloor nt\rfloor,n} + n\mathcal{J}\left(1 - \frac{|nt|}{n}\right), \quad t \in [0,1],
\]

\[
\hat{\eta}_{n}^W(1) = \frac{\Xi_{n,n} + n + \frac{1}{2} \log n}{\sqrt{2 \log n}}.
\]

Then as \( n \to \infty \)

\[
(\eta_{n}^W(t); t \in [0,1)) \Rightarrow (X_t, \ t \in [0, 1))
\]

\[
\hat{\eta}_{n}^W(1) \Rightarrow N
\]

where \( X \) is the Gaussian diffusion solution of the stochastic differential equation:

\[
dX_t = -\frac{1}{2(1-t)} dt + \sqrt{\frac{2}{1-t}} dB_t,
\]

with \( X_0 = 0 \), where \( B \) is a standard Brownian motion and \( N \) is a standard normal variable independent of \( B \).
With the direct method b), Jonsson proved (33) for fixed $t$ (i.e. convergence in distribution of the marginal) and deduced a convergence in probability of $rac{1}{n} \Xi_{n,\lfloor nt \rfloor}$ towards $-\mathcal{J}(1-t)$ (Theorem 5.1 p.29 and Corollary 5.1 p.30 of [32]).

Let us explain now the results which may be obtained by the spectral method a).

It is well known that the empirical spectral distribution of $\mathcal{X}_{n,r}$ converges weakly a.s. when $r/n \to c$ towards $\pi_1^c$. Moreover if $c < 1$ the largest (resp. smallest) eigenvalue converges a.s. to $b(c)$ (resp. to $a(c)$). For comments on these results and references, one may consult [7] sections 2.1.2 and 2.2.2.

In our context, this implies easily that

$$\frac{1}{n} \Xi_{n,r} = \frac{1}{n} \log \det \mathcal{X}_{n,r} = \frac{r}{n} \int \log x \: d\mu_{n,r}(x) \to c \int \log x \: d\pi_1^c(x)$$

a.s. when $r/n \to c \in (0,1)$. We already see in [22] the value of this integral. Claim (33) is then consistent with Theorem 112.

The fluctuations were studied recently by Bai and Silverstein [8] (in the case of complex entries). They obtained

$$\log \det \mathcal{X}_{n,r} + n\mathcal{J}\left(1 - \frac{r}{n}\right) \Rightarrow \mathcal{N}(\log(1-c) ; -2 \log(1-c)) ,$$

which is consistent with (the marginal version of) (33).

## 5.3 Large deviations

Again, the three routes are possible to tackle the problem of large deviations for determinant of Wishart matrices. A direct method would use the cumulant generating function from Section 2.2 and would meet computations similar to those seen in the Gram case.

To avoid repetitions, we use the b) method, drawing benefit from an auxiliary study of $S_{n,r}$.

**Lemma 5.4** The sequence $\{\frac{1}{n} S_{n}(t), t \in [0,1]\}_n$ satisfies a LDP in $(D, \sigma(D, \mathcal{M}_\infty))$ at scale $n^2$ with good rate function

$$I^S_{[0,1]}(v) = \int_{[0,1]} L^S_a \left( \frac{dv_a}{dt}(t) \right) dt + \int_{[0,1]} L^S_s \left( \frac{dv_s}{d\mu}(t) \right) d\mu(t)$$

where

$$L^S_a(y) = \frac{1}{2} (e^y - y - 1) , \quad L^S_s(y) = -\frac{y}{2} \delta(y|(-\infty,0)) ,$$

and $\mu$ is any measure dominating $dv_s$.

Let us stress that the instantaneous rate functions are time homogeneous and then we may write $[0,1]$ instead of $[0,1]$. 

13
Theorem 5.5 The sequence \( \{ \frac{1}{n} \Xi_{n,[nt]}, t \in [0,1) \} \) satisfies a LDP in \((D, \sigma(D, \mathcal{M}_<))\), at scale \( n^2 \) with good rate function

\[
I^W_{[0,1]}(v) = \int_{[0,1]} L^W_a \left( t, \frac{dv_a}{dt}(t) \right) dt + \int_{[0,1]} L^W_s \left( t, \frac{dv_s}{d\mu}(t) \right) d\mu(t)
\]

where

\[
L^W_a(t, y) = \frac{1}{2} (e^y - 1) - \frac{1}{2}(1 - t)y + \frac{1}{2}J(1 - t)
\]

\[
L^W_s(t, y) = -\frac{1}{2} (1 - t)y \delta(y|(-\infty,0)).
\]

and \( \mu \) is any measure dominating \( dv_s \).

Let us notice that the restriction of \( I^W_{[0,1]} \) to \([0,T]\) vanishes only for \( dv_a(t) = \log(1 - t) \) and \( dv_s(t) d\mu(t) = 0 \) (essentially) i.e. for \( v(t) = -J(1 - t) \), which agrees with the result of Theorem 5.2.

For marginals, we give (without proof) the exact analogue of Theorem 4.5.

Theorem 5.6 For every \( T < 1 \), the sequence \( \{ \frac{1}{n} \Xi_{n,[nT]} \} \) satisfies a LDP in \( \mathbb{R} \) at scale \( n^2 \) with good rate function denoted by \( I^W_T \).

1. If \( \xi \geq \xi_T := J(T) - 1 \) the equation

\[
J(1 + 2\theta) - J(1 - T + 2\theta) = \xi
\]

has a unique solution, and we have

\[
I^W_T(\xi) = \theta \xi + \frac{1}{2} (F(1) - F(1 - T) - F(1 + 2\theta) + F(1 - T + 2\theta))
\]

2. If \( \xi < \xi_T \), we have

\[
I^W_T(\xi) = I^W_T(T) + \frac{1 - T}{2} (\xi_T - \xi).
\]

Let us comment the spectral approach. Hiai et Petz [28] proved that if \( n \to \infty, r/n \to c < 1 \), then \( \{ \mu_{n,r} \} \) satisfies a LDP at scale \( n^2 \) with some explicit good rate function \( I^{(sp)}_c \) given below in (41, 42, 43). If the contraction \( \mu \mapsto \int \log x d\mu(x) \) was continuous, we would claim that \( \{ \frac{1}{n} \log \det X_{n,[nT]} \} \) satisfies a LDP with good rate function

\[
I^W_T(\xi) = \inf \left\{ I^{(sp)}_T(\mu) : T \int \log x d\mu(x) = \xi \right\}.
\]

Actually

\[
I^{(sp)}_c(\mu) := -\frac{c^2}{2} \Sigma(\mu) + \frac{c}{2} \int \left( x - (1 - c) \log x \right) d\mu(x) + B(c)
\]

14
where
\[ \Sigma(\mu) := \int \int \log |x - y| \, d\mu(x) d\mu(y) \quad (42) \]
is the so-called logarithmic entropy and for \( c \in (0, 1) \)
\[ B(c) = -\frac{1}{4} \left( 3c - c^2 \log c + (1 - c)^2 \log(1 - c) \right). \quad (43) \]

We do not know if the contraction \( \mu \mapsto \int \log x \, d\mu(x) \) works, although not continuous. However we will prove the following result, where for \( u \in \mathbb{R} \) we set \( \mathcal{A}(u) = \{ \mu : \int \log x \, d\mu(x) = u \} \).

**Proposition 5.7** For \( \xi \geq \xi_T \) and \( \theta \) solution of (39), let \( \sigma^2 = 1 + 2\theta \). Then the infimum of \( I_T^{(sp)}(\mu) \) on \( \mathcal{A}(\xi/T) \) is uniquely achieved for \( \pi_{\sigma^2/T} \) and
\[ I_T^W(\xi) = I_T^{(sp)}(\pi_{\sigma^2/T}) = \inf \{ I_T^{(sp)}(\mu) ; \mu \in \mathcal{A}(\xi/T) \}. \quad (44) \]

**Remark 5.8**  
1. The endpoint is \( \xi_T = J(T) - 1 \), corresponding to \( 1 + 2\theta = T \), i.e. \( \sigma^2 = T \).
2. For \( \xi < \xi_T \) we do not know what happens. We can imagine that the infimum in (44) has a solution in some extended space.

### 6 Extensions

We examine now some possible extensions of the previous results. We focus on assumptions on the entries of the matrix \( B \). We keep the same asymptotics and notation than in above sections.

Let us mention that the methodology of the present paper will be applied to the Jacobi ensemble in a forthcoming paper.

**6.1 Independent Gaussian non real entries**

In the previous sections, entries of \( B \) were real numbers. We consider now entries in \( \mathbb{C} \) and in \( \mathbb{H} \), the set of real quaternions. Recall that an element of \( \mathbb{H} \) may be viewed as a 2 \times 2 matrix of the form
\[ q = \begin{pmatrix} z & w \\bar{w} & \bar{z} \end{pmatrix} \]
where \( z \) and \( w \) are complex numbers. Its dual (or conjugate) is
\[ \bar{q} = \begin{pmatrix} \bar{z} & -w \\bar{w} & z \end{pmatrix}. \]
denoted by $z_{jk}$ for $j = 1, \ldots, n$ and $k = 1, \ldots, r$.

Let $B$ be a $n \times r$ random matrix, and suppose the entries of $B$ are determined by a parameter $\beta = 1, 2$ or 4. These entries are i.i.d. random variables $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ valued with Gaussian densities

$$\frac{1}{\sqrt{2\pi}}e^{-b_{jk}^2/2} , \quad \frac{1}{\pi}e^{-|z_{jk}|} , \quad \frac{2}{\pi}e^{-2|z_{jk}|} , \quad \frac{2}{\pi}e^{-2|w_{jk}|} ,$$

in the three cases $\beta = 1, 2$ and 4 respectively. If $B^\dagger$ denotes the transpose of the conjugate of $B$, then the $r \times r$ matrix $X = B^\dagger B$ belongs to the Laguerre orthogonal (resp. unitary, resp. symplectic) ensemble denoted LOE (resp. LUE, resp. LSE).

Two main features of the LOE are shared by the LUE and LSE.

- The Barlett decomposition still holds. For $\beta = 2$ the references are for instance [2a, 26] (see also [22] Proposition 2.12 and 2.13). By the same argument as in section 2.2, the random variables $(R_k,j, k = 1, \ldots, j - 1)$ are complex i.i.d. normal of variance 1, hence

$$2|R_k,j|^2 \overset{D}{=} \chi_2^2.$$ Since $2\|b_j\|_2 \overset{D}{=} \chi_2^n$ we conclude that $2R_{j,j}^2 \overset{D}{=} \chi_{2n-2(j-1)}^2$. For $\beta = 4$, the reference is [22] Exercise 2.5.5. We conclude that $4R_{j,j}^2 \overset{D}{=} \chi_{2n-4(j-1)}^2$.

A pathwise study of $\log \det X$ in the cases $\beta = 2$ or 4 needs only slight modifications of arguments and would lead to results very similar to those of Section 5.

- The spectral approach is built on the probability density of the eigenvalues $\lambda_j, j = 1, \ldots, r$ of $X$ which is proportional to

$$\prod_{j=1}^r \lambda_j^{\beta(n-r+1)/2} \cdot e^{-\beta\lambda_j/2} \prod_{1 \leq j < k \leq m} |\lambda_k - \lambda_j|^{\beta}$$

Convergence of the ESD is known not only for $\beta = 2, 4$ but for every $\beta > 0$ (see for instance the Dumitriu thesis [19] Theorem 6.5.1).

Besides, the large deviations treated in [25] are stated for the (real) Wishart ensemble, but of course are available in the general case with slight modifications since everything rests on their Theorem 1.

### 6.2 Independent isotropic columns

We keep independence of vectors $b_1, \ldots, b_n$ but assume only isotropy (in $\mathbb{R}^n$) of their common distribution $\nu_n$. The polar decomposition allows to obtain similar results as in Section 5 under convenient assumptions on the radial distribution. Let $\varepsilon_n = \log \|b_1\|^2 - \log \mathbb{E}\|b_1\|^2$ (remember that we omit the dimension index $n$).

To get convergence and fluctuations it is enough to assume

$$n\mathbb{E}\varepsilon_n \rightarrow a_1 , \quad n\text{Var} \varepsilon_n \rightarrow a_2 , \quad n\mathbb{E}(\varepsilon_n - \mathbb{E}\varepsilon_n)^4 \rightarrow 0 .$$

(45)

To get large deviations, it would be sufficient to assume that $\frac{1}{n^2} \sum_{k=1}^n \log \mathbb{E}\exp(n\varphi(k/n)\varepsilon_n)$ has a limit for some convenient functions $\varphi$.

Notice that in [3] and [5], the authors use the uniform distribution in the unit ball, so that the distribution of $\|b_1\|^2$ is beta$(\frac{n}{2}, 1)$ and (45) is satisfied with $a_1 = -2$, $a_2 = 0$. Here the contribution of the radial part is roughly “deterministic” since $\mathbb{E}\|b_1\|^2$ is bounded.
6.3 Independent identically distributed (non Gaussian) entries

If we restrict ourselves to marginals only, we may leave the Gaussian world. Let us assume i.i.d. (real) entries with finite variance. In \([32], [43]\), the authors proved the convergence of the spectral distribution, using Stieljes’ transform \([36]\). In \([8]\) Bai and Silverstein assumed 
\[
E b_{11}^4 = 3 \quad \text{(real entries)} \quad \text{or} \quad E b_{11}^2 = 0 \quad \text{and} \quad E b_{11}^4 = 2 \quad \text{(complex entries)},
\]
and proved a central limit theorem for linear statistics, with the meaningful example of logarithm of the determinant.

The Bartlett’s decomposition is not possible in the general case. Nevertheless, a product formula for the determinant is well known (see for example Lemma 3.1 p.9 and formula 4.3 p.15 in \([23]\))
but nothing can be said about the distribution of the components of the product.

Moreover, using again the norming of column vectors as in previous sections, we may define \(\widetilde{B}\) with

\[
\widetilde{b}_{ij} = \frac{b_{ij}}{\sqrt{\sum_{k=1}^{n} b_{kj}^2}}
\]

A slight modification of this matrix is used in multivariate analysis to test that variates are uncorrelated. The matrix \(Y = \widetilde{B}'\widetilde{B}\) is a Gram matrix, built from independent vectors, identically distributed and living on \(S_{n-1}\). In this context, Jiang (30) recently proved the convergence of the ESD to \(\pi_1^c\) and also the convergence of the extreme eigenvalues. It is then easy to deduce the convergence of the normalized logarithm of the determinant.

This model is clearly an extension of the uniform Gram ensemble, for which De Conck et al. \([21]\) proved the convergence of the ESD to \(\pi_1^c\) with an independent method.

6.4 Independent isotropic rows

We keep independence of rows of the random matrix \(B\) and assume that they are identically distributed with an isotropic distribution on \(\mathbb{R}^r\). Actually in the data matrix \(B_{n,r}\), the index \(n\) is, as previously, the size of the sample and \(r\) is the number of variates. In \([44]\), Yin and Krishnaiah proved the convergence of the ESD of \(X_{n,r} = \frac{1}{n} B_{n,r}' B_{n,r}\) but the limiting distribution was not known. Actually, when the underlying distribution is uniform on \(S_{r-1}\) we can identify the limiting distribution from the result of Jiang \([30]\) or De Cock et al. \([13]\).

We set \(C = B'\) and then \(C'C\) is in the uniform Gram ensemble (in \(M_{nn}\)). The eigenvalues of \(C'C\) are (except 0 with multiplicity \(n-r\)) the same as those of \(CC' = B'B\). If \(\mu_{n,r}\) is the ESD of \(CC'\), the ESD of \(C'C\) is then

\[
\mu_{n,r} = \frac{r}{n} \mu_{n,r}^* + (1 - \frac{r}{n}) \delta_0
\]

If \(r/n \to c < 1\), hence \(n/r \to 1/c > 1\),

\[
\lim_{n,r \to \infty} \mu_{n,r} = \pi_1^{1/c}
\]

\([13]\) Theorem 10 or \([30]\) Theorem 2) so that,

\[
\lim_{n,r \to \infty} \mu_{n,r}^* = \frac{1}{c} \left( \pi_1^{1/c} - (1 - c) \delta_0 \right)
\]

17
and from (14) we see that

$$\lim_{n,r \to \infty} \mu_{n,r}^* = \pi_1^c$$

Besides, Yin and Krisnaiah scaling is $1/n$, so that the limit of the ESD of $X_{n,r}$ is the image of $\pi_1^c$ by the dilatation $D_c : x \mapsto cx$ i.e. $\pi_1^c$ (see Section 3). Moreover, the results on extreme eigenvalues obtained by Jiang are easily carried.

## 7 Proofs of Section 4

To stress the dependence on $n$ we set for $2 \leq k \leq n$

$$h_{k,n} = \tilde{R}_{k,k}^2.$$ (and $h_{1,n} = 1$). The key tool is the cumulant generating function :

$$\Lambda_{n,k}^G(\theta) := \log \mathbb{E} \exp (\theta \log h_{k,n}) = \log \mathbb{E}(h_{k,n}^\theta).$$ (46)

From Proposition 2.1 2), we have for $2 \leq k \leq n$

$$\Lambda_{n,k}^G(\theta) = \ell \left( \frac{n-k+1}{2} + \theta \right) - \ell \left( \frac{n-k+1}{2} \right) - \ell \left( \frac{n}{2} + \theta \right) + \ell \left( \frac{n}{2} \right)$$ (47)

where $\ell = \log \Gamma$. By derivation

$$\mathbb{E}(\log h_{k,n}) = \Psi \left( \frac{n-k+1}{2} \right) - \Psi \left( \frac{n}{2} \right)$$

$$\text{Var}(\log h_{k,n}) = \Psi' \left( \frac{n-k+1}{2} \right) - \Psi' \left( \frac{n}{2} \right),$$

where $\Psi = \ell' = \Gamma'/\Gamma$ is the digamma function.

### 7.1 Proof of Proposition 4.1

We need the following lemma.

**Lemma 7.1** For every $p \leq n$, we have

$$\mathbb{E} \Upsilon_{n,p} = \frac{n-1}{2} \Psi \left( \frac{n+1}{2} \right) + \frac{n-2p}{2} \Psi \left( \frac{n}{2} \right) + 1 - p$$

$$- \frac{n-p-1}{2} \Psi \left( \frac{n-p+1}{2} \right) - \frac{n-p}{2} \Psi \left( \frac{n-p+2}{2} \right)$$ (48)

and

$$\left| \text{Var} \Upsilon_{n,p} - 2(H_n - H_{n-p} - \frac{p}{n}) \right| \leq 4 \sum_{k=n-p+1}^{n} \frac{1}{k^2}.$$ (49)
Proof of Lemma 7.1: From (11) and (47) we get
\[ \mathbb{E} \Upsilon_{n,p} = \sum_{k=2}^{p} (\Lambda_{n,k}^G)'(0) = \sum_{k=2}^{p} \left[ \psi \left( \frac{n-k+1}{2} \right) - \psi \left( \frac{n}{2} \right) \right] = \sum_{k=n-p+1}^{n} \psi \left( \frac{k}{2} \right) - p \psi \left( \frac{n}{2} \right). \]  
(50)

From the classical identity
\[ \psi \left( \frac{k+2}{2} \right) - \psi \left( \frac{k}{2} \right) = \frac{2}{k} \]
and since \( \psi(1/2) = -\gamma - 2 \log 2 \), Abbott and Mulders \( \Box \) deduced
\[ \sum_{i=1}^{k-1} \psi \left( \frac{i}{2} \right) = \frac{k-2}{2} \psi \left( \frac{k}{2} \right) - k \left( \frac{k-1}{2} \psi \left( \frac{k+1}{2} \right) + \frac{2-\gamma-2 \log 2}{2} \right). \]  
(51)

It remains to take successively \( k = n+1 \) and \( k = n-p+1 \) and use (50). Besides
\[ \text{Var} \Upsilon_{n,p} = \sum_{k=2}^{p} (\Lambda_{k,n}^G)^{''}(0) = \sum_{k=2}^{p} \left[ \psi' \left( \frac{n-k+1}{2} \right) - \psi' \left( \frac{n}{2} \right) \right], \]
so that (49) comes from (87) and from (88) with \( q = 2 \). \( \blacksquare \)

Proof of Proposition 4.1: We have only to prove (16), (17) and (19).
1) We have
\[ n \mathcal{J} \left( 1 - \frac{p}{n} \right) = p + (n-p) \log(n-p) - (n-p) \log n. \]
for \( p < n \) and \( = n \) for \( p = n \). Using (48) and (85) we get (16).
2) Now, for \( p = \lfloor nt \rfloor \), \( 0 < t < 1 \) and \( n \to \infty \) we use the more precise estimate (86) in (48). We leave the details to the reader.
3) For the variance, we start from (49) and we get easily (19). \( \blacksquare \)

7.2 Proof of Theorem 4.2
Since \( \mathcal{J} \) is uniformly continuous on \([0,1]\) we have
\[ \lim_{n \to \infty} \sup_{t \in [0,1]} \left| \mathcal{J} \left( 1 - \frac{\lfloor nt \rfloor}{n} \right) - \mathcal{J}(1-t) \right| = 0, \]
so that, owing to (13), it is enough to prove that a.s. \( \sup_{1 \leq p \leq n} |\Upsilon_{p,n} - \mathbb{E} \Upsilon_{p,n}| = o(n) \). Actually this convergence is a consequence of Borel-Cantelli’s lemma, Doob’s inequality and of the variance estimate \( \text{Var} \left( \frac{1}{n} \Upsilon_{n,n} \right) = O(n^{-2} \log n) \) coming from (20). \( \blacksquare \)
7.3 Proof of Theorem 4.3

Let us first notice that, thanks to the estimations of expectations in (17) and (18), we can reduce the problem to the centered process \( \Delta_n(t) := \Upsilon_n(t) - \mathbb{E} \Upsilon_n(t) \) and to the centered variable \( \hat{\Delta}_n = \Delta_n(1)/\sqrt{\log n} \).

1) We use the notation of (17) and (19). We have \( \Delta_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \eta_{n,k} \) where

\[
\eta_{n,k} := (\log h_{k,n}) - \mathbb{E}(\log h_{k,n}), \quad k \leq n
\]

is a row-wise independent arrow. To prove (23) it is enough to prove the convergence in distribution in \( \mathcal{D}([0, T]) \), for every \( T < 1 \), of \( \Delta_n \) to a centered Gaussian process with independent increments, and variance \( \sigma^2_G \). To this purpose we apply a version of the Lindeberg-Lévy-Lyapunov criteria (see [12] Theorem 7.4.28 of the french edition, or [29] §3c). For \( t < 1 \), from (19) it is enough to prove that

\[
\lim_{n} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}(\eta^4_{n,k}) = 0. \tag{52}
\]

We have from (16)

\[
\mathbb{E}(\eta^4_{n,k}) = (\Lambda^G_{k,n})^{(4)}(0) + 3[(\Lambda^G_{k,n})''(0)]^2. \tag{53}
\]

On the one hand, from (17), (87) and (88) for \( q = 4 \)

\[
\left| \sum_{k=1}^{p} (\Lambda^G_{k,n})^{(4)}(0) - 48 \sum_{k=1}^{p} \left[ \frac{1}{(n-k+1)^2} - \frac{1}{n^3} \right] \right| \leq 96 \sum_{k=1}^{p} \frac{1}{(n-k+1)^4}. \tag{54}
\]

which, for \( 0 < t < 1 \) and \( p = \lfloor nt \rfloor \) yields \( \lim_n \sum_{k=1}^{\lfloor nt \rfloor} (\Lambda^G_{k,n})^{(4)}(0) = 0 \). On the other hand,

\[
\sum_{k=1}^{p} ((\Lambda^G_{k,n})''(0))^2 \leq (\sup_{j \leq p} (\Lambda^G_{j,n})''(0)) \sum_{k=1}^{p} (\Lambda^G_{k,n})''(0) \tag{55}
\]

and from (19) we get \( \lim_n \sum_{k=1}^{\lfloor nt \rfloor} (\Lambda^G_{k,n})''(0) = \sigma^2_G(t) \). Besides, \( \Psi' \) is non-increasing (see (87)) so that

\[
\sup_{j \leq p} (\Lambda^G_{j,n})''(0) \leq \Psi' \left( \frac{n-p+1}{2} \right)
\]

and from (88) (again), this term tends to 0. We just checked (52), which proves that the sequence of processes \( (\Delta_n(t), t \in [0, 1])_n \) converges to a Gaussian centered process \( \mathcal{W} \) with independent increments and variance \( \sigma^2_G \). So, by (17), \( \eta^G_n \) converges to the Gaussian process \( \mathcal{W} + d_G \) with independent increments, drift \( d_G \) and variance \( \sigma^2_G \).

Finally, equation (24) comes from

\[
d_G(t) = \int_0^T \frac{1-2s}{2(1-s)} \, ds, \quad \sigma^2_G(t) = \int_0^t \frac{2s}{1-s} \, ds.
\]
2) When \( t = 1 \), most of the sums studied above explode and we need a renormalisation. In fact, for every \( n \), the process \((\Delta_n(t), t \in [0, 1])\) has independent increments. The conditional distribution of \( \Delta_n(1) \), knowing \( \Delta_n(t_1) = \varepsilon_1, \ldots, \Delta_n(t_r) = \varepsilon_r \) for \( t_1 < \cdots < t_r \) is the same as
\[
\varepsilon_r + \sum_{[nt_r] + 1}^{n} \eta_{k,n}.
\]
Formulae (49) and (20) yield
\[
\sum_n \mathbb{E}(\eta_{k,n}^2) = 2 \log n + O(1).
\]
Actually we can apply the Lindeberg’s theorem (with the criterion of Lyapunov) to the triangular array of random variables \( \tilde{\eta}_{k,n} = \eta_{k,n}/\sqrt{2 \log n} \) with \( k = [nt_r] + 1, \ldots, n \). It is enough to prove
\[
\lim_n \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}(\tilde{\eta}_{k,n}^4) = 0.
\] (56)
The route is the same as before, starting from (53), but now (54) says that the sum \( \sum_{k=1}^{n} \Lambda_{k,n}^4(0) \) is bounded. In (55), the sum on the right (with \( p = n \)) is now equivalent to \( 2 \log n \) and the supremum (with \( p = n \)) is bounded. This yields
\[
\sum_{k=1}^{n} \mathbb{E}(\tilde{\eta}_{k,n}^4) = (\log n)^{-2} \sum_{k=1}^{n} \mathbb{E}(\eta_{k,n}^4) = O((\log n)^{-1})
\]
which proves (56).

Then \( \sum_{[nt_r] + 1}^{n} \eta_{k,n}/\sqrt{2 \log n} \) converges in distribution to \( \mathcal{N}(0, 1) \), and the same is true for the conditional distribution of \( \hat{\Delta}_n \) knowing \( \Delta_n(t_1) = \varepsilon_1, \ldots, \Delta_n(t_r) = \varepsilon_r \). Since the limiting distribution does not depend on \( \varepsilon_1, \ldots, \varepsilon_r \), we have proved that \( \hat{\Delta}_n \) converges in distribution to a random variable which is \( \mathcal{N}(0, 1) \) and independent of \( \mathcal{W} \).

7.4 Proof of Theorem 4.4

As mentioned after the statement of the theorem, we have to prove the LDP for the restriction of \( \frac{1}{n} \hat{\Upsilon}_n \) to \([0, T]\), viewed as an element of \( \mathcal{M}_T \), at scale \( n^2 \) with rate function
\[
\bar{I}_[0,T]^G(m) := \int_0^T L_a^G \left( t, \frac{dm}{dt}(t) \right) dt + \int_0^T L_s^G \left( t, \frac{dm}{d\mu}(t) \right) d\mu(t).
\] (57)
Let \( V_\ell \) be the set of functions from \([0, T]\) to \( \mathbb{R} \) which are left continuous and have bounded variation, and let \( V_\ell^* \) be its topological dual when \( V_\ell \) is provided with the uniform convergence topology.

Actually \( \frac{1}{n} \hat{\Upsilon}_n \in \mathcal{M}_T \) may be identified with an element of \( V_\ell^* \) (see Appendix B): its action on \( \varphi \in V_\ell \) is given by
\[
< \frac{1}{n} \hat{\Upsilon}_n, \varphi > := \frac{1}{n} \sum_{k=1}^{[nT]} \varphi \left( \frac{k}{n} \right) \log h_{n,k}.
\]
The proof is based on the ideas of Baldi’s theorem ([15] p.?). The main tool is the normalized cumulant generated function (n.c.g.f.) which here takes the form

\[ \mathcal{L}^G_{n,\lfloor nt \rfloor}(\varphi) := \frac{1}{n^2} \log \mathbb{E} \exp n < \hat{T}^n, \varphi > \]

Owing to (46) we have

\[ \mathcal{L}^G_{n,\lfloor nt \rfloor}(\varphi) = \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \Lambda^G_{n,k} \left( n \varphi \left( \frac{k}{n} \right) \right) \]

and from (47) it is finite if \( \varphi \left( \frac{k}{n} \right) > -\frac{n-k+1}{2n} \) for \( 1 \leq k \leq \lfloor nT \rfloor \) and \( +\infty \) otherwise.

In Subsection 7.4.1, we prove the convergence of this sequence of n.c.g.f. for a large class of functions \( \varphi \). It will be sufficient, jointly to the variational formula given in Subsection 7.4.2 to get the upperbound for probability of compact sets. Then Subsection 7.4.3 is devoted to exponential tightness, which allow to get the upperbound for closed sets. However, since the limiting n.c.g.f. is not defined everywhere, the lowerbound (for open sets) is more delicate than in Baldi’s theorem. Actually a careful study of exposed points as in [24] is managed in Subsection 7.4.4. We end the proof in 7.4.5.

7.4.1 Convergence of the n.c.g.f.

Lemma 7.2 If \( \varphi \in V_\ell \) satisfies \( \varphi(t) > -\frac{1-t}{2} \) for every \( t \in (0, T] \), then

\[ \lim_n \mathcal{L}^G_{n,\lfloor nT \rfloor}(\varphi) = \Lambda^G_{[0,T]}(\varphi) := \int_0^T g(t, \varphi(t)) \, dt , \]

where, for \( \theta > -\frac{(1-t)}{2} \)

\[ g(t, \theta) := \frac{1}{2} \left( \mathcal{J}(1-t+2\theta) - \mathcal{J}(1-t) - \mathcal{J}(1+2\theta) \right) . \]

**Proof:** The key point is a convergence of Riemann sums. From (47) and (50) we have, for any \( \theta > -\frac{n-k+1}{2n} \),

\[ \Lambda^G_{n,k}(n\theta) = \frac{n-k+2n\theta}{2} \log \left( 1 - \frac{k}{n} + 2\theta + \frac{1}{n} \right) - \frac{n-k}{2} \log \left( 1 - \frac{k}{n} + \frac{1}{n} \right) - \frac{n-1+2n\theta}{2} \log(1+2\theta) + R_{n,k}(\theta) \]

where

\[ R_{n,k}(\theta) = \int_0^\infty f(s) e^{-\frac{s}{2}} \left[ e^{-\frac{n-k+2n\theta}{2}s} - e^{-\frac{n-k}{2}s} - e^{-\frac{n-1+2n\theta}{2}s} + e^{-\frac{n-1}{2}s} \right] ds , \]

is bounded:

\[ |R_{n,k}(\theta)| \leq 2 \int_0^\infty e^{-\frac{s}{2}} f(s) \, ds . \]
If we set
\[
2\Phi_n(t) := (1 - t + 2\varphi(t)) \log(1 - t + 2\varphi(t) + \frac{1}{n}) - (1 - t) \log(1 - t + \frac{1}{n}) - \left(1 - \frac{1}{n} + 2\varphi(t)\right) \log(1 + 2\varphi(t))
\]
then, making \( \theta = \varphi(k/n) \) in (61), and adding in \( k \), we get from (58)
\[
\frac{1}{n^2} \left( \mathcal{L}^G_{n,[nt]}(\varphi) - \sum_{i=2}^{[nt]} R_{n,k}(\varphi) \right) = \frac{1}{n} \sum_{i=1}^{[nt]} \Phi_n\left(\frac{k}{n}\right) = \int_{1/n}^{[nt]/n} \Phi_n\left(\frac{nt}{n}\right) dt + \frac{1}{n} \Phi_n\left(\frac{nt}{n}\right).
\]
On the one hand, since \( \varphi \) is left continuous, \( \lim_n \Phi_n\left(\frac{nt}{n}\right) = 2g(t, \varphi(t)) \) for every \( t \in [0, T] \).
On the other hand the following double inequality holds true:
\[
2\Phi_n(t) \geq (1 - t + 2\varphi(t)) \log(1 - t + 2\varphi(t)) - (1 - t) \log(2 - t) - (1 + 2\varphi(t)) \log(1 + 2\varphi(t)) - (1 - t) |\log(1 - t + 2\varphi(t))|,
\]
and with our assumptions on \( \varphi \), these bounds are both integrable. This allows to apply the dominated convergence theorem which ends the proof of Lemma 7.2. \( \square \)

If there exists \( s > T \) such that \( 2\varphi(s) < -(1-s) \) then for \( n \) large enough, \( \mathcal{L}_{n,[nt]}(\varphi) = +\infty \) and we set \( \Lambda^G_{[0,T]}(\varphi) = \infty \). In the other cases we do not know what happens, but as in [24], we will study the exposed points. Before, we need another expression of the dual of \( \Lambda^G_{[0,T]} \).

7.4.2 Variational formula

Let us define \( \Lambda^G_{[0,T]}(\varphi) = +\infty \) if \( \varphi \) does not satisfy the assumption of Lemma 7.2. The dual of \( \Lambda^G_{[0,T]} \) is then
\[
(\Lambda^G_{[0,T]})^* (\nu) = \sup_{\varphi \in V_t^*} \left\{ \langle \nu, \varphi \rangle - \Lambda^G_{[0,T]}(\varphi) \right\}
\]
for \( \nu \in V_t^* \). Mimicking the method of Léonard (35) p. 112-113), we get
\[
(\Lambda^G_{[0,T]})^* (\nu) = \sup_{\varphi \in \mathcal{C}} \left\{ \langle \nu, \varphi \rangle - \Lambda^G_{[0,T]}(\varphi) \right\}
\]
where \( \mathcal{C} \) is the set of continuous functions from \( [0, T] \) into \( \mathbb{R} \) vanishing at 0. Then we apply Theorem 5 of Rockafellar [41]. We get
\[
(\Lambda^G_{[0,T]})^* (\nu) = \int_0^T g^* \left( t, \frac{d\nu}{dt} \right) dt + \int_0^T r \left( t, \frac{d\nu}{dt} (t) \right) d\mu(t)
\]

where
\[ g^*(t, y) = \sup_\lambda \{ \lambda y - g(t, \lambda) \delta(|\lambda|(-1/2, \infty)) \} \tag{64} \]
and \( r \) is the recession function:
\[ r(t, y) = \lim_{\kappa \to \infty} \frac{g^*(t, \kappa y)}{\kappa} . \]
Actually, if \( y < 0 \), the supremum is achieved for
\[ \lambda(t, y) := -\frac{1}{2} \left( 1 - \frac{t}{1 - e^y} \right) \tag{65} \]
and we have
\[ g^*(t, y) = \lambda(t, y) y - g(t, \lambda(t, y)) \]
\[ = \frac{1}{2} \left[ -y(1 - t) + (1 - t) \log(1 - t) + t \log t - t \log(1 - e^y) \right] \]
\[ = \frac{1}{2} \mathcal{H}(1 - t|e^y). \tag{66} \]
If \( y \geq 0 \), \( g^*(t, y) = \infty \). The recession is now \( r(t, y) = -\frac{1}{2}(1-t) \) if \( y \leq 0 \), and \( = \infty \) if \( y > 0 \). As a result
\[ g^*(t, y) = L_a^G(t, y), \quad r(t, y) = L_a^G(t, y). \tag{67} \]
So we proved the identification \( \left( \Lambda_{[0,T]}^G \right)^* = \tilde{I}_{[0,T]}^G \) (recall (67)).

7.4.3 Exponential tightness

If \( V^*_\ell \) is provided with the topology \( \sigma(V^*_\ell, V_\ell) \), the set \( B_a := \{ \mu \in V^*_\ell : |\mu|_{[0,T]} \leq a \} \) is compact according to the Banach-Alaoglu theorem. But \( \frac{1}{n} \hat{\gamma}_n \) is a positive measure and \( \frac{1}{n} \hat{\gamma}_n([0, T]) = \frac{1}{n} \gamma_n(T) \) has a n.c.g.f. given for \( \theta > 0 \) by
\[ \hat{\mathcal{L}}_{n,T}(\theta) := \frac{1}{n^2} \log \mathbb{E} \exp \{ n\theta \gamma_n(T) \} = \mathcal{L}_{n,[nT]}(\theta \mathbb{I}_{[0,T]}) \]
For \( \theta > -\frac{1-T}{2} \) let
\[ L_T(\theta) := \int_0^T g(t, \theta) dt . \tag{68} \]
Lemma [7.2] says that for fixed \( \theta > -\frac{1-T}{2} \)
\[ \lim \mathcal{L}_{n,[nT]}(\theta \mathbb{I}_{[0,T]}) = L_T(\theta) \tag{69} \]
so that
\[ \lim \sup_n \frac{1}{n^2} \log \mathbb{P} \left( \frac{1}{n} \hat{\gamma}_n \notin B_a \right) \leq \lim \sup_n \frac{1}{n^2} \log \mathbb{P} \left( \gamma_n(T) > na \right) \leq -a \theta + L_T(\theta), \]
which proves the exponential tightness, letting \( a \to \infty \).

Let us notice that it was not possible to take \( T = 1 \).
7.4.4 Exposed points

Let $\mathcal{R}$ be the set of functions from $[0,T]$ into $\mathbb{R}$ which are positive, continuous and with bounded variation. Let $\mathcal{F}$ be the set of those $m \in V^*_t$ (identified with $\mathcal{M}_T$ as in [35]) which are absolutely continuous and whose density $\rho$ is such that $-\rho \in \mathcal{R}$. Let us prove that such a $m$ is exposed, with exposing hyperplane $f_m(t) = \lambda(t,\rho(t))$ (recall (65)). Actually we follow the method of [24]. For fixed $t$, $g^*(t,.)$ is strictly convex on $(-\infty,0)$ so that, if $z \neq \rho(t)$, we have

$$g^*(t,\rho(t)) - g^*(t,z) < \lambda(t,\rho(t))(\rho(t) - z).$$

Let $d\xi = \tilde{l}(t)dt + \xi$ the Lebesgue decomposition of some element $\xi \in \mathcal{M}_T$ such that $\tilde{I}^G_{[0,T]}(\xi) < \infty$. Taking $z = \tilde{l}(t)$ and integrating, we get

$$\int_0^T g^*(t,\rho(t))dt - \int_0^T g^*(t,\tilde{l}(t))dt < \int_0^T \lambda(t,\rho(t))\rho(t)dt - \int_0^T \lambda(t,\rho(t))\tilde{l}(t)dt$$

and since $\int_0^T g^*(t,\tilde{l}(t))dt = \int_0^T L^G_a(t,\tilde{l}(t))dt \leq \tilde{I}^G_{[0,T]}(\xi)$ this yields

$$\tilde{I}^G_{[0,T]}(m) - \tilde{I}^G_{[0,T]}(\xi) < \int_0^T f_m dm - \int_0^T f_m d\xi.$$ 

Now let us prove that this set of exposed points is rich enough. We have the following lemma.

**Lemma 7.3** Let $m \in V_t$ such that $\tilde{I}^G_{[0,T]}(m) < \infty$. There exists a sequence of functions $l_n \in \mathcal{R}$ such that

1. $\lim_n l_n(t)dt = -m$ in $V^*_t$ with the $\sigma(V^*_t,V_t)$ topology
2. $\lim_n \tilde{I}^G_{[0,T]}(-l_n(t)dt) = \tilde{I}^G_{[0,T]}(m)$.

**Proof:** The method may be found in [24] and in [16]. The only difference is in the topology because we want to recover marginals. We will use the basic inequality which holds for every $\epsilon \leq 0$:

$$L^G_a(t,v + \epsilon) \leq L^G_a(t,v) - \frac{\epsilon}{2}(1-t) \quad (70)$$

Let $m = m_a + m_s$ such that $\tilde{I}^G_{[0,T]}(m) < \infty$. From (28) and (26) it is clear that $-m_a$ and $-m_s$ must be positive measures.

**First step** We assume that $m = -l(t)dt - \eta$ with $l \in L^1([0,T];dt)$ and $\eta$ a singular positive measure. One can find a sequence of non negative continuous functions $h_n$ such that $h_n(t)dt \rightarrow \eta$ for the topology $\sigma(V^*_t,V_t)$. Indeed every function $\psi \in V_t$ may be written as a difference $\psi_1 - \psi_2$ of two increasing functions. There exists a unique (positive) measure $\alpha_1$ such that $\psi_1(t) = \alpha_1([t,T])$ for every $t \in [0,T]$. Moreover, the function $g = \eta([0,\cdot]) \in V_t$ is
non decreasing and may be approached by a sequence of continuously derivable and non decreasing functions \((g_n)\) such that \(g_n \leq g\). Setting \(h_n := g_n'\) and \(\nu_n = h_n(t)dt\), the dominated convergence theorem gives

\[
< \psi_1, \nu_n > = \int_0^T \nu_n([0,t])[\alpha_1(dt)] \to \int_0^T \eta([0,t])[\alpha_1(dt)].
\]

With the same result for \(\psi_2\) we get

\[
< \psi, \nu_n > = \int_0^T \nu_n([0,t])[\alpha_1(dt)] - \int_0^T \nu_n([0,t])[\alpha_2(dt)] \\
\to \int_0^T \eta([0,t])[\alpha_1(dt)] - \int_0^T \eta([0,t])[\alpha_2(dt)].
\]

or \(\lim_n < \psi, \nu_n > = < \psi, \eta >\). On the one hand, the lower semicontinuity of \(\tilde{I}_G^{[0,T]}\) yields

\[
\lim inf_n \tilde{I}_G^{[0,T]}(-(l(t) + h_n(t))dt) \geq \tilde{I}_G^{[0,T]}(m).
\]

On the other hand, integrating \((70)\) yields

\[
\tilde{I}_G^{[0,T]}(-(l(t) + h_n(t))dt) \leq \int_0^T L_a^G(t, -l(t))dt + \frac{1}{2} \int_0^T (1 - t)h_n(t)dt \\
\to \int_0^T L_a^G(t, -l(t))dt + \frac{1}{2} \int_0^T (1 - t)\eta(dt) = \tilde{I}_G^{[0,T]}(m).
\]

**Second step** Let us assume that \(m = -l(t)dt\) with \(l \in L^1([0,T]; dt)\) and for every \(n\), let us set \(l_n = \max(l, 1/n)\). It is clear that as \(n \to \infty\), then \(l_n \downarrow l\). On the one hand the lower semicontinuity gives \(\lim inf_A \tilde{I}_G^{[0,T]}(-l_n(t)dt) \geq \tilde{I}_G^{[0,T]}(-l(t)dt)\). On the other hand, by integration of inequality \((70)\), since \(l_n - l \leq 1/n\)

\[
\tilde{I}_G^{[0,T]}(-l_n(t)dt) \leq \tilde{I}_G^{[0,T]}(-l(t)dt) + \frac{1}{2n}.
\]

It is then possible to reduce the problem to the case of functions bounded below.

**Third step** Let us assume that \(m = -l(t)dt\) with \(l \in L^1([0,T]; dt)\) and bounded below by \(A > 0\). One can find a sequence of continuous functions \((h_n)\) with bounded variation such that \(h_n \geq A/2\) for every \(n\) and such that \(h_n \to l\) a.e. and in \(L^1([0,T], dt)\). We have \(h_n(t)dt \to l(t)dt\) in \(\sigma(V^*_t, V_t)\) and since \(L^G_a(t, \cdot)\) is uniformly Lipschitz on \([-\infty, -A/2]\), say with constant \(\kappa\), we get

\[
|\tilde{I}_G^{[0,T]}(-h_n(t)dt) - \tilde{I}_G^{[0,T]}(-l(t)dt)| \leq \kappa \int_0^T |h_n(t) - l(t)|dt \to 0.
\]

Actually, \(h_n \in \mathcal{R}\) and \(\varphi_n(t) := \lambda(t, -h_n(t))\) satisfies the assumption of Lemma \(7.2\) since

\[
1 + 2\varphi_n(t) - t \geq \frac{t}{1 - e^{-A/2}}.
\]
7.4.5 End of the proof of Theorem 4.4

• First step: upperbound for compact sets. We use th. 4.5.3 b) in [15] and the following lemma.

**Lemma 7.4** For every $\delta > 0$ and $m \in V_{\ell}$, there exists $\varphi_{\delta}$ fulfilling conditions of 7.2 and such that

$$
\int_0^T \varphi_{\delta} dm - \Lambda^G_T(\varphi_{\delta}) \geq \min \left[ I_{[0,T]}^G(m) - \delta, \delta^{-1} \right].
$$

(71)

• Second step: upperbound for closed sets. We use the exponential tightness.

• Third step: lowerbound for open sets. The method is classical (see [15] th. 4.5.20 c)), owing to Lemma 7.3.

To prove Lemma 7.4 we start from the definition (62) or (63). One can find $\bar{\varphi}_{\delta} \in V_{\ell}$ satisfying (71). If $\bar{\varphi}_{\delta}$ does not check assumptions of the lemma we add $\varepsilon > 0$ to $\bar{\varphi}_{\delta}$ which allows to check them and satisfy (71) up to a change of $\delta$.

7.5 Proof of Theorem 4.5

We use the contraction from the LDP for paths. Since the mapping $m \mapsto m([0,T])$ is continuous from $D$ to $\mathbb{R}$, the family $\{\Upsilon_n, nT\}$ satisfies the LDP with good rate function given by (29):

$$
I^G_T(\xi) = \inf \{ I^G_{[0,T]}(v) ; v(T) = \xi \}.
$$

Fixing $\xi$, we can look for optimal $v$. Let $\theta \in (-1-T)/2, \infty]$ (playing the role of a Lagrange multiplier).

By the duality property (64)

$$
g^* \left( t, \frac{d\hat{v}_a}{dt}(t) \right) \geq \theta \frac{d\hat{v}_a}{dt}(t) - g(t, \theta).
$$

Integrating and using (57), (67) and (68) we get

$$
I^G_{[0,T]}(v) \geq \theta \hat{v}_a([0,T]) - L_T(\theta) - \frac{1}{2} \int_0^T (1-t) \, d\hat{v}_s(t),
$$

(72)

For every $v$ such that $v(T) = \xi$ it turns out that

$$
I^G_{[0,T]}(v) \geq \theta \xi - L_T(\theta) - \frac{1}{2} \int_0^T (1-t+2\theta) \, d\hat{v}_s(t) \geq \theta \xi - L_T(\theta).
$$

(73)

Besides, from (65) the ordinary differential equation

$$
\lambda(t, \phi'(t)) = \theta \\
\phi(0) = 0,
$$

27
admits for unique solution in $C^1([0, T])$

$$t \mapsto \phi(\theta; t) := \mathcal{J}(1 + 2\theta) - \mathcal{J}(1 - t + 2\theta) - t \log(1 + 2\theta).$$

The mapping $\theta \mapsto \phi(\theta; T)$ has a positive derivative and its limit as $\theta \downarrow -\left(1 - \frac{T}{2}\right)$ is $-T$. Moreover, by duality

$$g^\star \left( t, \partial_t \phi(\theta, t) \right) = \theta \frac{\partial}{\partial t} \phi(\theta, t) - g(t, \theta).$$

There are two cases.

- If $\xi > -T$, there exists a unique $\theta_\xi$ such that $\phi(\theta_\xi, T) = \xi$ (i.e. the relation (36) is satisfied). For $v_\xi := \phi(\theta_\xi, \cdot)$, we get from (57), (67) and (68) again

$$I_{[0, T]}^G(v_\xi) = \theta_\xi \xi - L_T(\theta_\xi)$$

so that $v_\xi$ realizes the infimum in (29). A simple computation ends the proof of the first statement of Theorem 4.5.

Let us notice that at the end point $\xi = -T$, we have

$$\theta_\xi = -\frac{1 - T}{2}, \quad v_\xi(t) = \mathcal{J}(T) - \mathcal{J}(T - t) - t \log T, \quad (v_\xi)'(t) = \log(1 - t/T).$$

Finally

$$I_T^G(-T) = T(1 - T) + \frac{1}{2} \left( F(1) - F(1 - T) - F(T) + T^2 \log T \right)$$

$$= \frac{T(1 - T)}{4} + \frac{T^2 \log T}{4} - \frac{(1 - T)^2 \log(1 - T)}{4} + \frac{3}{8}.$$

- Let us assume $\xi = -T - \varepsilon$ with $\varepsilon > 0$. Plugging $\theta = -\frac{1 - T}{2}$ in (73) yields, for every $v$ such that $v(T) = \xi$

$$I_{[0, T]}^G(v) \geq -\frac{1 - T}{2} \xi - L_T \left( -\frac{1 - T}{2} \right) = \frac{1 - T}{2} \varepsilon + I_T^G(-T)$$

Moreover this lower bound is achieved by the measure $\tilde{v} = (v^{-T})'(t) dt - \varepsilon \delta_T(t)$, since

$$\int_0^T L^G_a(t, (v^{-T})'(t)) dt = I_T^G(T), \quad \int_0^T \frac{1 - t}{2} \varepsilon \delta_T(t) = \frac{(1 - T)}{2} \varepsilon.$$

That ends the proof of the second statement of Theorem 4.5.

Remark 7.5 It is possible to try a direct method to get (31), (32) using Gärtner-Ellis’ theorem ([15], Theorem 2.3.6). From (69) the limiting n.c.g.f. of $\frac{X_n(T)}{n}$ is $L_T$ which is analytic for $\theta > -\frac{1 - T}{2}$. When $\theta \downarrow -\frac{1 - T}{2}$, we have $L_T'(\theta) \downarrow -T$. We meet a case of so called non steepness. To proceed in that direction we could use the method of time dependent
change of probability (see [14]). We will not give details here. Nevertheless, this approach
allows to get one-sided large deviations in the critical case $T = 1$. Actually we get
\[
\lim_{n \to \infty} \frac{1}{n^2} \log \mathbb{P}(\Upsilon_{n,n} \geq nx) = -I^G_1(x)
\]
for $x \geq -1$. The value $x = -1$ corresponds to the limit of $\Upsilon_{n,n}/n$. Notice that the second
(right) derivative of $I^G_1$ at this point is zero (or equivalently $\lim L''_1(\theta) = \infty$ as $\theta \downarrow 0$), which
is consistent with previous results on variance. I do not know the rate of convergence to 0 of
$\mathbb{P}(\Upsilon_{n,n} \leq nx)$ for $x < -1$.

8 Proofs of Section 5

8.1 Proofs of Subsection 5.2

We use the decomposition (13). We need only to notice that
\[
\mathbb{E}(\chi_n^2)^s = 2^s \frac{\Gamma(s + \frac{n}{2})}{\Gamma(s)}
\]
hence
\[
\log \mathbb{E} \exp \theta \log \frac{\|b_1\|^2}{n} = \ell \left( \theta + \frac{n}{2} \right) - \ell \left( \frac{n}{2} \right) - \theta \log \left( \frac{n}{2} \right),
\] (74)
which provides estimates for the expectation and the variance. Differentiating once and
taking $\theta = 0$, we see that
\[
\mathbb{E} \log \frac{\|b_1\|^2}{n} = \left[ \Psi \left( \frac{n}{2} \right) - \log \left( \frac{n}{2} \right) \right] = -\frac{1}{n} - \int_0^\infty e^{-s^2/2} s f(s) \, ds = -\frac{1}{n} + O\left( \frac{1}{n^2} \right)
\]
(see (84), (83)), which gives
\[
\sup_{p \leq n} \left| \mathbb{E} S_{n,p} + \frac{p}{n} \right| = O\left( \frac{1}{n} \right). \tag{75}
\]
Besides, differentiating (74) twice and taking $\theta = 0$ again, we have
\[
\text{Var} \left( \log \frac{\|b_1\|^2}{n} \right) = \Psi' \left( \frac{n}{2} \right) = 2 \frac{n}{2} + O\left( \frac{1}{n^2} \right)
\]
(see (88)), which yields
\[
\sup_{p \leq n} \left| \text{Var} S_{n,p} - \frac{2p}{n} \right| = O\left( \frac{1}{n} \right). \tag{76}
\]
From (75) and (76) it is easy to check (via a fourth moment estimate) that $S_n$ converges in
distribution in $\mathcal{D}((0,1])$ to $\left( -t + \sqrt{2} \tilde{B}_t, \ t \in [0,1] \right)$, where $\tilde{B}$ is a Brownian motion indepen-
dent of the $\sigma$-field generated by $(\Upsilon_n, n \in \mathbb{N})$. Finally the family of processes $\eta_n^W = \eta_n^G + S_n$
converges in distribution towards \((Y_t - t + \sqrt{2} \tilde{B}_t, \ t \in [0, 1])\). It is a Gaussian process, whose drift and variance are
\[
d_W(t) = d_G(t) - t = \frac{1}{2} \log(1 - t), \quad \sigma_W^2(t) = \sigma_G^2(t) + 2t = -2 \log(1 - t).
\]
which identify the process \(X\).

Besides, we have
\[
\hat{\eta}^W_n(1) = \hat{\eta}^G_n(1) + \frac{S_n(1)}{\sqrt{2 \log n}},
\]
so that the convergence of \(\hat{\eta}^W_n(1)\) is clear. Moreover the independence properties seen in Section 4.2 remain true.

### 8.2 Proofs of Subsection 5.3

#### 8.2.1 Proof of Lemma 5.4

It is a route similar to the proof of Theorem 4.4 in Section 7.4 (see also [40]). We start from (74):
\[
\log \mathbb{E} \exp \left< n \dot{\hat{S}}_n, \gamma \right> = \sum_{k=1}^{n} \left[ n\gamma \left( \frac{k}{n} \right) \log 2 + \ell \left( n\gamma \left( \frac{k}{n} \right) + \frac{n}{2} \right) - \ell \left( \frac{n}{2} \right) \right]
\]
if \(\gamma(s) + \frac{1}{2} > 0\) for every \(s \in [0, 1]\). The limiting n.c.g.f. is
\[
\mathcal{L}^S(\gamma) = \frac{1}{2} \int_0^1 J(1 + 2\gamma(t)) dt,
\]
which yields (36) by duality (see [41] again).

#### 8.2.2 Proof of Theorem 5.5

We deduce from Lemma 5.4 and Theorem 4.4 that the sum \(\frac{1}{n} \hat{\Upsilon}_n + \frac{1}{n} \dot{\hat{S}}_n\) satisfies a LDP at the same scale with good rate function obtained by inf-convolution of \(I_G^{[0,T]}\) and \(\tilde{I}_S^{[0,T]}\). To obtain (57) and (58), it is possible to compute explicitly this inf-convolution:
\[
L^W_a = \inf_v \{ L^G_a(v) + L^S_a(u - v) \}
\]
\[
L^W_s = \inf_v \{ L^G_s(v) + L^S_s(u - v) \}.
\]
Alternatively, it is possible to sum the two n.c.g.f. (39 and 77) and get the rate function by duality.

\[\square\]

**Remark 8.1** We can make the same comments as in Remark 7.3. In particular, we get
\[
\lim_{n} \frac{1}{n^2} \log \mathbb{P}(\Xi_{n,n} \geq nx) = -I^W_1(x)
\]
for \(x \geq -1\). This boundary point corresponds to the limit of \(\Xi_{n,n}/n\). The second derivative is vanishing at this point, which is consistent with the results on variance.

30
8.2.3 Proof of Proposition 5.7

Let \( \theta \) be a Lagrangian factor. We begin by minimizing

\[
I^\text{sp}\left(\mu \right) - \theta T \int \log x \, d\mu(x) = \frac{T^2}{2} \left[ -\Sigma(\mu) + 2 \int q_{\lambda,s}(x) \, d\mu(x) \right] + B(T)
\]

where

\[
q_{\lambda,s}(x) = \lambda x - s \log x, \quad \lambda = \frac{1}{2T}, \quad s = \frac{1 - T + 2\theta}{2T}.
\]

(79)

In [42] p.43 example 5.4, it is stated that for \( \lambda > 0 \) and \( 2s + 1 > 0 \) fixed, the infimum

\[
\inf_{\mu} -\Sigma(\mu) + 2 \int q_{\lambda,s}(x) \, d\mu(x)
\]

is achieved on the unique extremal measure \( \pi^c_{\sigma^2} \) with

\[
\sigma^2 = \frac{2s + 1}{2\lambda}, \quad c = \frac{1}{2s + 1},
\]

which yields, from (79):

\[
\sigma^2 = 1 + 2\theta, \quad c = \frac{T}{\sigma^2}.
\]

Now it remains to look for \( \theta \) such that the constraint \( \mu \in \mathcal{A}(\xi/T) \) is saturated. Since

\[
\int \log x \, d\pi^c_{\sigma^2}(x) = \log \sigma^2 + \int \log x \, d\pi^c_1(x) \, dx,
\]

and thanks to [22] we see that \( \theta \) must satisfy

\[
\xi = T \log \sigma^2 - T \frac{\mathcal{J}(1 - c)}{c} = \mathcal{J}(1 + 2\theta) - \mathcal{J}(1 - T + 2\theta),
\]

which is exactly exactly (39).

To compute \( I_T^{(sp)}(\pi^c_{\sigma^2}) \), we start from the definition (11):

\[
I_T^{(sp)}(\pi^c_{\sigma^2}) = -\frac{T^2}{2} \Sigma(\pi^c_{\sigma^2}) + \frac{T}{2} \int(x - (1 - T) \log x) \, d\pi^c_{\sigma^2}(x) + B(T),
\]

and transform \( \pi^c_{\sigma^2} \) to \( \pi^1_{\sigma} \) using the dilatation. In particular, [12] yields \( \Sigma(\pi^c_{\sigma^2}) = \log \sigma^2 + \Sigma(\pi^1_{\sigma}) \) and \( \Sigma(\pi^1_{\sigma}) \) may be picked from formula (13) p.10 in [28]:

\[
\Sigma(\pi^1_{\sigma}) = -1 + \frac{1}{2} \left( c^{-1} + \log c + (c^{-1} - 1)^2 \log(1 - c) \right).
\]

Besides we have easily \( \int x \, d\pi^1_1(x) = 1 \). After some tedious but elementary computations we get exactly the expression (40).
9 Appendix : Some properties of $\ell = \log \Gamma$ and $\Psi$

From the Binet’s formula (\cite{2} \cite{20} p.21), we have

$$
\ell(x) = (x - \frac{1}{2}) \log x - x + 1 + \int_{0}^{\infty} f(s)[e^{-sx} - e^{-s}] ds 
$$

(80)

$$
= (x - \frac{1}{2}) \log x - x + \frac{1}{2} \log(2\pi) + \int_{0}^{\infty} f(s)e^{-sx} ds.
$$

(81)

where the function $f$ is defined by

$$
f(s) = \left[1 - \frac{1}{s} + \frac{1}{e^s - 1}\right] \frac{1}{s} = 2 \sum_{k=1}^{\infty} \frac{1}{s^2 + 4\pi^2 k^2},
$$

(82)

and satisfies for every $s \geq 0$

$$
0 < f(s) \leq f(0) = 1/12 , \ 0 < (sf(s) + 1) < 1.
$$

(83)

By differentiation

$$
\Psi(x) = \log x - \frac{1}{2x} - \int_{0}^{\infty} sf(s)s^{-s} ds = \log x - \int_{0}^{\infty} e^{-sx}(sf(s) + \frac{1}{2}) ds
$$

(84)

As easy consequences, we have, for every $x > 0$

$$
0 < x \left(\log x - \Psi(x)\right) \leq 1,
$$

(85)

and

$$
0 < x^2 \left(\log x - \Psi(x) - \frac{1}{2x}\right) \leq \frac{1}{12}.
$$

(86)

Differentiating again we see that for $q \geq 1$

$$
\Psi^{(q)}(z) = (-1)^{q-1}q!z^{-q} + (-1)^{q-1} \int_{0}^{\infty} e^{-sz}s^q(sf(s) + \frac{1}{2}) ds
$$

(87)

and then

$$
|\Psi^{(q)}(z) - (-1)^{q-1}q!z^{-q}| \leq z^{-q-1}q!.
$$

(88)

References

[1] J. Abbott and T. Mulders. How tight is Hadamard bound? Experiment. Math., 10(3):331–336, 2001.

[2] M. Abramowitz and I.A. Stegun. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover, New York, 1972. pp. 258-259, 9th edition.
[3] A. Akhavi. Analyse comparative d'algorithmes de réduction sur les réseaux aléatoires. PhD thesis, Université de Caen, 1999.

[4] A. Akhavi. Threshold phenomena in random lattices and efficient reduction algorithms. *Theoretical Computer Science*, 257:359–385, 2002.

[5] A. Akhavi, J.F. Marckert, and A. Rouault. On the Lovasz reduction of a random basis. http://fermat.math.uvsq.fr/~rouault/publications.html.

[6] T.W. Anderson. *An introduction to multivariate statistical analysis*. Wiley Series in Probability and Statistics. John Wiley, 2003. Third edition.

[7] Z.D. Bai. Methodologies in spectral analysis of large dimensional random matrices, a review. *Statistica Sinica*, pages 611–677, 1999.

[8] Z.D. Bai and J.W. Silverstein. CLT for linear spectral statistics of large-dimensional sample covariance matrices. *The Annals of Probab.*, 32(1A):553–605, 2004.

[9] M.S. Bartlett. On the theory of statistical regression. *Proc. Royal. Soc. Edinb.*, (53):260–283, 1933.

[10] R. Bhatia. *Matrix Analysis*. Springer, 1997. Graduate text in Mathematics.

[11] G.M. Cicuta and M.L. Mehta. Probability density of determinants of random matrices. *J. Phys. A: Math. Gen.*, 33:8029–8035, 2000.

[12] D. Dacunha-Castelle and M. Duflo. *Probability and Statistics*. Springer-Verlag, 1986.

[13] M. De Cock, M. Fannes, and P. Spincemaille. On quantum dynamics and statistics of vectors. *J. Phys. A: Math. Gen.*, 32:6547–6571, 1999.

[14] A. Dembo and O. Zeitouni. Large deviations via parameter dependent change of measure, and an application to the lower tail of Gaussian processes. In *Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1993)*, volume 36 of *Progr. Probab.*, pages 111–121. Birkhauser, 1995.

[15] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*. Springer, 2nd edition, 1998.

[16] H. Dette and F. Gamboa. Asymptotic properties of the algebraic moment range process. http://www.lsp.ups-tlse.fr/Fp/Gamboa/range6.pdf, January 2005.

[17] J.D. Dixon. How good is Hadamard’s inequality for determinants? *Can. Math. Bull.*, 27(3):260–264, 1984.

[18] D. Donoho and Y. Tsaig. Breakdown of equivalence between the minimal l1-norm solution and the sparsest solution. EURASIP Signal Processing Journal, to appear, May 2005.
[19] I. Dumitriu. *Eigenvalue Statistics for Beta Ensembles*. PhD thesis, M.I.T., http://math.berkeley.edu/~dumitriu/main.pdf, 2003.

[20] A. Erdelyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi. *Higher transcendental functions*, volume I. Krieger, New-York, 1981.

[21] M. Fannes and Spincemaille P. The mutual affinity of random measures. *Periodica Mathematica Hungarica*, 47:51–71, 2003.

[22] P.J. Forrester. Log-gases and random matrices. Book available at http://www.ms.unimelb.edu.au/~matpjf/matpjf.html.

[23] S. Friedland, B. Rider, and O. Zeitouni. Concentration of permanent estimators for certain large matrices. *The Annals of Applied Probab.*, 14(3):1559–1576, 2004.

[24] F. Gamboa, A. Rouault, and M. Zani. A functional large deviation principle for quadratic forms of gaussian stationary processes. *Stat. and Probab. Letters*, 43:299–308, 1999.

[25] N.R. Goodman. Statistical analysis based on a certain multivariate complex gaussian distribution. (An introduction.). *Ann. Math. Stat.*, (34):152–177, 1963.

[26] P. Graczyk, G. Letac, and H. Massam. The complex Wishart distribution and the symmetric group. *Ann. Stat. 31*, (1):287–309, 2003.

[27] J. Hadamard. Résolution d’une question relative aux déterminants. *Bull. Sci. Math.*, 17:240–246, 1893.

[28] F. Hiai and D. Petz. Eigenvalue density of the Wishart matrix and large deviations. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 1(4):633–646, 1998.

[29] J. Jacod and A.N. Shiryaev. *Limit theorems for stochastic processes*. Springer, 1987.

[30] T. Jiang. The limiting distributions of eigenvalues of sample correlation matrices. *Sankhya*, 66(1):35–48, 2004.

[31] I. Johnstone. On the distribution of the largest eigenvalue in principal component analysis. *The Annals of Statistics*, 29(2):295–327, 2001.

[32] D. Jonsson. Some limit theorems for the eigenvalues of a sample covariance matrix. *J. Multivariate Anal.*, 12:1–38, 1982.

[33] A.M. Kshirsagar. Bartlett decomposition and Wishart distribution. *Ann. Math. Stat.*, (30):239–241, 1959.

[34] A.M. Kshirsagar. *Multivariate Analysis*. Marcel Dekker, 1972.

[35] C. Léonard. Large deviations for Poisson random measures and processes with independent increments. *Stoch. Proc. and their Appl.*, 85:93–121, 2000.
[36] V.A. Marchenko and L.A. Pastur. Distribution of eigenvalues of some sets of random matrices. Math. USSR Sb., 1:457–483, 1967.

[37] A.M. Mathai. A handbook of generalized special functions for statistical and physical sciences. OUP, Oxford, 1993.

[38] A.M. Mathai. Random $p$-content of a $p$-parallelootope in Euclidean $n$-space. Adv. Appl. Prob., 31:343–354, 1999.

[39] R. J. Muirhead. Aspects of multivariate statistical theory. John Wiley, 1982.

[40] J. Najim. A Cramer type theorem for weighted random variables. Electronic Journal of Probability, 7(4):1–32, 2002.

[41] R.T. Rockafellar. Integrals which are convex functionals, II. Pacific J. Math., 39(2):439–469, 1971.

[42] E.B. Saff and V. Totik. Logarithmic potentials with external fields. Springer, 1997.

[43] J.W. Silverstein and Z.D. Bai. On the empirical distribution of eigenvalues of a class of large dimensional random matrices. J. Multivariate Anal., (54):175–192, 1995.

[44] Y.Q. Yin and P.R. Krishnaiah. Limit theorem for the eigenvalues of the sample covariance matrix when the underlying distribution is isotropic. Theory Probab. Appl., (30):861–867, 1986.