HYPERSPACES WITH THE ATTTOUCH-WETS TOPOLOGY
HOMEOMORPHIC TO $\ell_2$

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Abstract. It is shown that the hyperspace of all nonempty closed subsets $\text{Cld}_{AW}(X)$ of a separable metric space $(X,d)$ endowed with the Attouch-Wets topology is homeomorphic to $\ell_2$ if and only if the completion $\overline{X}$ of $X$ is proper, locally connected, contains no bounded connected component, $X$ is topologically complete and not locally compact at infinity.

1. Introduction

For a metric space $X = (X,d)$, let $C(X)$ be the set of all continuous real valued functions on $X$ and $\text{Cld}(X)$ be the set of all nonempty closed subsets of $X$. Identifying each $A \in \text{Cld}(X)$ with the continuous function $X \ni x \mapsto d(x,A) \in \mathbb{R}$, we can embed $\text{Cld}(X)$ into the function space $C(X)$.

The function space $C(X)$ carries at least three natural topologies: of point-wise convergence, of uniform convergence and of uniform convergence on bounded subsets of $X$. Those three topologies of $C(X)$ induce three topologies on the hyperspace $\text{Cld}(X)$: the Wijsman topology, the metric Hausdorff topology and the Attouch-Wets topology. The hyperspace $\text{Cld}(X)$ endowed with one of these topologies is denoted by $\text{Cld}_W(X)$, $\text{Cld}_H(X)$, and $\text{Cld}_{AW}(X)$, respectively. The Wijsman topology coincides with the Attouch-Wets topology if and only if bounded subsets of $X$ are totally bounded [Be, Theorem 3.1.4]. On the other hand, the Attouch-Wets topology coincides with the Hausdorff metric topology if and only if $(X,d)$ is a bounded metric space [Be, Exercise 3.2.2]. The Hausdorff metric topology on $\text{Cld}_H(X)$ is generated by the Hausdorff metric $d_H(A,B) = \sup_{x \in X} |d(x,A) - d(x,B)|$, where $A,B \in \text{Cld}(X)$.

In [BKS] Theorem 5.3 it is proved that for an infinite-dimensional Banach space $X$ of weight $w(X)$, the hyperspace $\text{Cld}_{AW}(X)$ is homeomorphic to $(\cong)$ the Hilbert space of weight $2^{w(X)}$. In particular, for an infinite-dimensional separable Banach space $X$, the hyperspace $\text{Cld}_{AW}(X)$ is homeomorphic to $\ell_2(2^{w_0})$. On the other hand, for each finite-dimensional normed linear space $X$, since every bounded closed set in $X$ is compact, the Attouch-Wets topology on $\text{Cld}(X)$ agrees with the Fell topology [Be, p.144]. Then, by [SY], $\text{Cld}_{AW}(X)$ is homeomorphic to $\mathbb{Q} \setminus \{p\}.$ Thus, for a Banach space $X$ the hyperspace $\text{Cld}_{AW}(X)$ is either locally compact or non-separable. In [BKS] the authors asked: does there exist an unbounded metric space $X$ such that $\text{Cld}_{AW}(X) \cong \ell_2$? And, more generally, what are the necessary and sufficient conditions under which the hyperspace $\text{Cld}_{AW}(X)$ is homeomorphic to $\ell_2$? In this paper we answer these questions proving the following characterization theorem.

Theorem 1. The hyperspace $\text{Cld}_{AW}(X)$ of a metric space $X$ is homeomorphic to $\ell_2$ if and only if the completion $\overline{X}$ of $X$ is proper, locally connected, contains no bounded connected component, $X$ is topologically complete and not locally compact at infinity.

A metric space $X$ is defined to be

• proper if each closed bounded subset of $X$ is compact;
• not locally compact at infinity if no bounded subset of $X$ has locally compact complement.

Observe that under the conditions of Theorem 1 the Attouch-Wets topology coincides with the Wijsman topology (cf. [Be, Theorem 3.1.4]). So, for free, we obtain the following

Corollary 1. For a metric space $(X,d)$ the hyperspace $\text{Cld}_W(X)$ is homeomorphic to $\ell_2$ if the completion $\overline{X}$ of $X$ is proper, locally connected, contains no bounded component, $X$ is topologically complete and not locally compact at infinity.
Applying Theorem 1 and Corollary 1 to the space $\mathbb{P}$ of irrational numbers of the real line we obtain:

**Corollary 2.** $\text{Cld}_{AW}(\mathbb{P}) = \text{Cld}_{W}(\mathbb{P}) \cong \ell_2$.

As a by-product of the proof of Theorem 1 we obtain the following characterization of metric spaces whose hyperspaces with the Attouch-Wets topology are separable absolute retracts.

**Theorem 2.** The hyperspace $\text{Cld}_{AW}(X)$ of a metric space $X$ is a separable absolute retract if and only if the completion $\bar{X}$ of $X$ is proper, locally connected and contains no bounded connected component.

Our Theorems 1 and 2 are “Attouch-Wets” counterparts of the following two results from [BV2]:

**Theorem 3.** The hyperspace $\text{Cld}_H(X)$ of a metric space $(X, d)$ is homeomorphic to $\ell_2$ if and only if $X$ is a topologically complete nowhere locally compact space and the completion $\bar{X}$ of $X$ is compact, connected, and locally connected.

**Theorem 4.** The hyperspace $\text{Cld}_H(X)$ of a metric space $X$ is a separable absolute retract if and only if the completion $\bar{X}$ of $X$ is compact, connected and locally connected.

2. Topology of Lawson semilattices and some auxiliary facts

Theorem 2 will be derived from a more general result concerning Lawson semilattices. By a topological semilattice we understand a pair $(L, \lor)$ consisting of a topological space $L$ and a continuous associative commutative idempotent operation $\lor : L \times L \to L$. A topological semilattice $(L, \lor)$ is a Lawson semilattice if open subsemilattices form a base of the topology of $L$. A typical example of a Lawson semilattice is the hyperspace $\text{Cld}(X)$ endowed with the operation of union $\cup$.

Each semilattice $(L, \lor)$ carries a natural partial order: $x \leq y$ iff $x \lor y = y$. A semilattice $(L, \lor)$ is called complete if each subset $A \subseteq L$ has the smallest upper bound $\sup A \in L$. It is well-known (and can be easily proved) that each compact topological semilattice is complete.

**Lemma 1.** If $L$ is a locally compact Lawson semilattice, then each compact subset $K \subseteq L$ has the smallest upper bound $\sup K \in L$. Moreover, the map $\sup : \text{Comp}(L) \to L$, $\sup : K \mapsto \sup K$, is a continuous semilattice homomorphism. Also for every subset $A \subseteq L$ with compact closure $\overline{A}$ we have $\sup A = \sup \overline{A}$.

This lemma easily follows from its compact version proved by J. Lawson in [Law].

In Lawson semilattices many geometric questions reduce to the one-dimensional level. The following fact illustrating this phenomenon is proved in [KSY].

**Lemma 2.** Let $X$ be a dense subsemilattice of a metrizable Lawson semilattice $L$. If $X$ is relatively $LC^0$ in $L$ (and $X$ is path-connected), then $X$ and $L$ are ANRs (ARs) and $X$ is homotopy dense in $L$.

A subset $Y \subseteq X$ is defined to be relatively $LC^0$ in $X$ if for every $x \in X$ each neighborhood $U$ of $x$ in $X$ contains a smaller neighborhood $V$ of $x$ such that every two points of $V \cap Y$ can be joined by a path in $U \cap Y$.

Under a suitable completeness condition, the density of a subsemilattice is equivalent to the homotopical density. A subset $Y$ of a topological space $X$ is homotopy dense in $X$ if there is a homotopy $(h_t)_{t \in [0, 1]} : X \to X$ such that $h_0 = \text{id}$ and $h_t(X) \subseteq Y$ for every $t > 0$.

A subsemilattice $X$ of semilattice $L$ is defined to be relatively complete in $L$ if for any subset $A \subseteq X$ having the smallest upper bound $\sup A$ in $L$ this bound belongs to $X$.

**Proposition 1.** Let $L$ be a locally compact locally connected Lawson semilattice. Each dense relatively complete subsemilattice $X \subseteq L$ is homotopy dense in $L$.

**Proof.** According to Lemma 2 it suffices to check that $X$ is relatively $LC^0$ in $L$. Given a point $x_0 \in L$ and a neighborhood $U \subseteq L$ of $x_0$, consider the canonical retraction $\sup : \text{Comp}(L) \to L$. Using the ANR-property of $L$ and continuity of $\sup$, find a path-connected neighborhood $V \subseteq L$ of $x_0$ such that $\sup(\overline{V}) \subseteq U$. We claim that any two points $x, y \in X \cap V$ can be connected by a path in $X \cap U$.

First we construct a path $\gamma : [0, 1] \to \overline{V}$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma^{-1}(X)$ is dense in $[0, 1]$. Let $\{q_n : n \in \omega \}$ be a countable dense subset in $[0, 1]$ with $q_0 = 0$ and $q_1 = 1$. The space $L$, being locally compact, admits a complete metric $d$. The path-connectedness of $V$ implies the existence of a continuous map $\gamma_n : [0, 1] \to V$ such that $\gamma_n(0) = x$ and $\gamma_n(1) = y$. Using the local path-connectedness of $L$ we can construct inductively a sequence of functions $\gamma_n : [0, 1] \to V$ such that
• $\gamma_n(q_k) = \gamma_{n-1}(q_k)$ for all $k \leq n$;
• $\gamma_n(q_{n+1}) \in X$;
• $\sup_{t \in [0,1]} \rho(\gamma_n(t), \gamma_{n-1}(t)) < 2^{-n}$.

Then the map $\gamma = \lim_{n \to \infty} \gamma_n : [0, 1] \to V$ is continuous and has the desired properties: $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma(q_n) \in X$ for all $n \in \omega$.

For every $t \in [0,1]$ consider the set $\Gamma(t) = \{ \gamma(s) : |t - s| \leq \text{dist}(t, \{0,1\}) \}$. It is clear that the map $\Gamma : [0,1] \to \text{Comp}(L)$ is continuous and so is the composition $\sup \circ \Gamma : [0,1] \to L$. Observe that $\sup \circ \Gamma(0) = \sup \{ \gamma(0) \} = \gamma(0) = x$, $\sup \circ \Gamma(1) = y$, and $\sup \circ \Gamma([0,1]) \subseteq \sup(\text{Comp}(V)) \subseteq U$. Since for every $t \in (0,1)$ the set $\Gamma(t) = \overline{\Gamma(t) \cap X}$, we get $\sup \Gamma(t) = \sup(\Gamma(t) \cap X) \in X$ by the relative completeness of $X$ in $L$. Thus $\sup \circ \Gamma : [0,1] \to U \cap X$ is a path connecting $x$ and $y$ in $U$. \hfill $\Box$

For a metric space $X$ by $\text{Fin}(X)$ we denote the subspace of $\text{Comp}(X)$ consisting of non-empty finite subspaces of $X$.

**Lemma 3.** If $Y$ is a subset of a locally path-connected space $X$, then the subset $L = \text{Fin}(X) \setminus \text{Fin}(Y)$ is relatively $LC^0$ in $\text{Comp}(X)$.

**Proof.** By the argument of [CN] we can show that $\text{Fin}(X)$ is relatively $LC^0$ in $\text{Comp}(X)$. Consequently, for every compact set $K \subseteq \text{Comp}(X)$ and a neighborhood $U \subseteq \text{Comp}(X)$ of $K$ there is a neighborhood $V \subseteq \text{Comp}(X)$ of $K$ such that any two points $A, B \in \text{Fin}(X) \cap V$ can be linked by a path in $\text{Fin}(X) \cap U$. Since $\text{Comp}(X)$ is a Lawson semilattice, we may assume that $U$ and $V$ are subsemilattices of $\text{Comp}(X)$.

We claim that any two points $A, B \in L \cap V$ can be connected by a path in $L \cap U$. Since $L \subseteq \text{Fin}(X)$, there is a path $\gamma : [0,1] \to L \cap \text{Fin}(X)$ such that $\gamma(0) = A$ and $\gamma(1) = B$. Define a new path $\gamma' : [0,1] \to U \cap \text{Fin}(X)$ letting $\gamma'(t) = \gamma(max\{0,2t-1\}) \cup \gamma(min\{2t,1\})$. Observe that $A \subseteq \gamma'(t)$ if $t \leq 1/2$ and $B \subseteq \gamma'(t)$ if $t \geq 1/2$. Since $A, B \not\in \text{Fin}(Y)$, we conclude that $\gamma'([0,1]) \subseteq L \cap U$. \hfill $\Box$

We also need the following nontrivial fact from [BV1, Corollary 2].

**Lemma 4.** Let $X$ be a dense subset of a metric space $M$. Then the hyperspace $\text{Cl}_{H}(X)$ is an $A(N)R$ if and only if so is the hyperspace $\text{Cl}_{H}(M)$.

The proof of Theorem 4 and Theorem 5 relies on the next lemma due to D. Curtis [CN].

**Lemma 5.** A homotopy dense $G_{\delta}$-subset $X \subseteq Q$ with homotopy dense complement in the Hilbert cube $Q$ is homeomorphic to $\ell_2$.

3. **The metrics $d_{AW}$ and $d_{H}$ on $\text{Cl}_{d}(X)$**

Let $X = (X, d)$ be a metric space. The $\varepsilon$-neighborhood of $x \in X$ (i.e., the open ball centered at $x$ with radius $\varepsilon$) is denoted by $B(x, \varepsilon)$. Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. The excess of $A$ over $B$ with respect to $d$ is defined by the formula

$$e_d(A, B) = \sup \{d(a, B) : a \in A\}.$$ 

Here we assume that $e_d(A, \emptyset) = +\infty$. For the Hausdorff metric we have the following:

$$d_H(A, B) = \max\{e_d(A, B), e_d(B, A)\} = \sup_{x \in X} |d(x, A) - d(x, B)|.$$ 

Now we define the metric $d_{AW}$.

Fix $x_0 \in X$ and let $X_i = \{x \in X : d(x, x_0) \leq i\}$. The following metric $d_{AW}$ on $\text{Cl}_{d}(X)$ generates the Attouch-Wets topology: \footnote{In [Be], the following metric is adopted}$^1$

$$d_{AW}(A, B) = \sup_{i \in \mathbb{N}} \min \left\{ \frac{1}{i}, \sup_{x \in X_i} |d(x, A) - d(x, B)| \right\}.$$ 

It should be noticed that

$$d_{AW}(A, B) \leq d_H(A, B) \quad \text{for every } A, B \in \text{Cl}_{d}(X).$$ 

\footnote{In [Be], the following metric is adopted}
We need the following fact for the Attouch-Wets convergence in terms of excess, see [3] Theorem 3.1.7.

Proposition 2. Let \((X, d)\) be a metric space, and \(A, A_1, A_2, \ldots\) be nonempty closed subsets of \(X\), \(x_0 \in X\) be fixed. The following are equivalent:

1. \(\lim_{n \to \infty} d_{AW}(A_n, A) = 0\);
2. For each \(i \in \mathbb{N}\), we have both \(\lim_{n \to \infty} e_d(A \cap X_i, A_n) = 0\) and \(\lim_{n \to \infty} e_d(A_n \cap X_i, A) = 0\).

Recall that the Attouch-Wets topology depends on the metric for \(X\), that is, the space \(\text{Cld}_{AW}(X)\) is not a topological invariant for \(X\). Concerning conditions that two metrics for \(X\) induce the same topology, see [3] Theorem 3.3.3.

4. Embedding \(\text{Cld}_{AW}(X)\) in \(\text{Cld}_H(\alpha X)\)

The main idea in proving Theorems 1 and 2 is the following: we reduce the Attouch-Wets topology on \(\text{Cld}(X)\) to the Hausdorff metric topology on \(\text{Cld}(\alpha X)\) for a suitable one-point extension \(\alpha X\) of \(X\). The metric space \((\alpha X, \rho)\) is obtained by adding the infinity point \(\infty\) to the space \(X\). More precisely, we endow the space \(\alpha X\) with the metric

\[
\rho(x, y) = \begin{cases} 
\min\left\{d(x, y), \frac{1}{1+d(x, x_0)}, \frac{1}{1+d(y, x_0)}\right\}, & \text{if } x, y \in X \\
\frac{1}{1+d(x, y)} & \text{if } x \in X, \ y = \infty \\
\frac{1}{1+d(y, x)} & \text{if } y \in X, \ x = \infty \\
0, & \text{if } x = y = \infty.
\end{cases}
\]

Here, \(x_0 \in X\) is a fixed point. Note that \((X, d)\) is homeomorphic to \((\alpha X \setminus \{\infty\}, \rho)\) and \(\text{diam}(\alpha X) < 2\).

Remark 1. We can obtain the space \((\alpha X, \rho)\) in the following way: embed \(X\) in \(X \times [0,1]\) by the formula \(x \mapsto (x, \frac{d(x, x_0)}{1+d(x, x_0)})\) and consider the cone metric on this space (induced by the suitable metrization of the quotient space \(X \times [0,1]/X \times \{1\}\)).

Proposition 3. The function \(e : \text{Cld}_{AW}(X) \to \text{Cld}_H(\alpha X)\) defined by the formula \(e(A) = A \cup \{\infty\}\) is an embedding.

Proof. Let \(\lim_{n \to \infty} d_{AW}(A_n, A) = 0\). Assume to the contrary that \(\lim_{n \to \infty} \rho_H(e(A_n), e(A)) \neq 0\). This means that there exists some \(\varepsilon_0 > 0\) such that we can find either a sequence \(x_{n_k} \in A_{n_k}, k \in \mathbb{N}\), with \(\rho_H(x_{n_k}, A \cup \{\infty\}) \geq \varepsilon_0\), or there exists a sequence \((y_k) \subset A\) with \(\rho_H(A_{n_k} \cup \{\infty\}, y_k) \geq \varepsilon_0\) for all \(k \in \mathbb{N}\).

In the former case, since \(\infty \in e(A)\), we have \(\rho(x_{n_k}, \infty) = \frac{1}{1+d(x_{n_k}, x_0)} \geq \varepsilon_0\) for each \(k \in \mathbb{N}\). Hence, there exists some \(i_0 \in \mathbb{N}\) with \((x_{n_k}) \subset X_{i_0}, k \in \mathbb{N}\). For every \(y \in A\) and \(k \in \mathbb{N}\) \(\rho(x_{n_k}, y) \geq \varepsilon_0\), and so \(d(x_{n_k}, y) \geq \varepsilon_0\). This implies that for each \(k \in \mathbb{N}\) \(\sup_{x \in X_{i_0}} |d(x, A_{n_k}) - d(x, A)| \geq d(x_{n_k}, A) \geq \varepsilon_0\). Combining this with the definition of the Attouch-Wets metric we get \(d_{AW}(A_{n_k}, A) \geq \min\{\frac{1}{1+i_0}, \varepsilon_0\}\) for all \(k \in \mathbb{N}\). This is a contradiction. In the latter case, similar to the above, we have for each \(k \in \mathbb{N}\) \(\sup_{x \in X_{i_0}} |d(x, A_{n_k}) - d(x, A)| \geq d(y_k, A_{n_k}) \geq \varepsilon_0\). Hence, \(d_{AW}(A_{n_k}, A) \geq \min\{\frac{1}{1+i_0}, \varepsilon_0\}\) for all \(k \in \mathbb{N}\).

Conversely, let \(\lim_{n \to \infty} \rho_H(e(A_n), e(A)) = 0\). Assume to the contrary that \(\lim_{n \to \infty} d_{AW}(A_n, A) \neq 0\). This means that there exists a subsequence \((A_{n_k}) \subset (A_n)\) with \(d_{AW}(A_{n_k}, A) \geq \varepsilon_0\) for some \(\varepsilon_0 > 0\). Then, by Proposition 2 there exists \(i_0 \in \mathbb{N}\) such that either \(e(A_{n_k} \cap X_{i_0}, A) \geq \varepsilon_0\) or \(e_d(A \cap X_{i_0}, A_{n_k}) \geq \varepsilon_0\). Remark, that we can take \(i_0\) so large that \(A \cap X_{i_0} \neq \emptyset\) for all \(i \in \mathbb{N}\). Consequently, in the former case we can find a sequence \(x_k \in A_{n_k} \cap X_{i_0}\) with \(d(x_k, y) \geq \varepsilon_0\) for each \(y \in A\) and \(k \in \mathbb{N}\). Hence, \(\rho(x_k, y) = \min\{d(x_k, y), \frac{1}{1+d(x_k, x_0)}, \frac{1}{1+d(y, x_0)}\} \geq \min\{\varepsilon_0, \frac{1}{1+i_0}\}\) for all \(y \in A\) and \(k \in \mathbb{N}\). Since \(\rho(x_k, \infty) = \frac{1}{1+d(x_k, x_0)} \geq \frac{1}{1+i_0}\), it follows that \(\lim_{n \to \infty} \rho_H(A_{n_k} \cup \{\infty\}, A \cup \{\infty\}) \geq \min\{\varepsilon_0, \frac{1}{1+i_0}\} > 0\), and we have a contradiction. In the latter case, there exists a sequence \(y_k \in A \cap X_{i_0}\) with \(d(y_k, x) \geq \varepsilon_0\) for all \(x \in A_{n_k}\) and \(k \in \mathbb{N}\). Hence, for every \(k \in \mathbb{N}\) we have \(\rho(y_k, x) \geq \min\{\varepsilon_0, \frac{1}{1+d(y_k, x_0)}\} \geq \min\{\varepsilon_0, \frac{1}{1+i_0}\}\).

For a metric space \(X\) and a point \(x_0 \in X\) let

\[
\text{Cld}_H(X \setminus \{x_0\}) = \{F \in \text{Cld}_H(X) : x_0 \in F\}.
\]
It follows from Proposition 3 that
\[ e(Cld_{AW}(X)) = Cld_H(\alpha X \{\infty\}) \setminus \{\infty\} \]
and thus \(Cld_{AW}(X)\) is homeomorphic to \(Cld_H(\alpha X \{\infty\}) \setminus \{\infty\}\).

The ANR-property of the space \(Cld_H(X\{x_0\})\) was characterized in [Voy]:

**Proposition 4.** For a metric space \(X\) with a distinguished point \(x_0\) the hyperspace \(Cld_H(X\{x_0\})\) is an absolute (neighborhood) retract if and only if so is the hyperspace \(Cld_H(X)\).

Proposition 4 implies the following fact about Attouch-Wets hyperspace topology having an independent interest.

**Corollary 3.** Let \(X\) be a dense subset of a metric space \(M\). Then, the hyperspace \(Cld_{AW}(X)\) is an absolute neighborhood retract (an absolute retract) if and only if so is the hyperspace \(Cld_{AW}(M)\).

**Proof.** It follows from the Propositions 3, 4 and Lemma 4.

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5. The completion \(\overline{\alpha X}\) is a Peano continuum

We need the following lemma proved in [SY, Lemma 2].

**Lemma 6.** If \(X\) is a locally connected, locally compact separable metrizable space with no compact components, then its Alexandroff one-point compactification \(\alpha X\) is a Peano continuum.

Using the previous lemma we can easily obtain

**Lemma 7.** Suppose that the completion \(\overline{X}\) of a metric space \(X\) is a proper locally connected space with no bounded connected components. Then, \(\overline{\alpha X}\) is a Peano continuum.

**Proof.** Note, that the completion \(\overline{X}\) of \(X\) satisfies the conditions of Lemma 6 and \(\overline{\alpha X}\) (the completion of \(\alpha X\)) coincides with the Alexandroff one-point compactification of \(\overline{X}\).

Then, by the Curtis-Schori Hyperspace Theorem [CS], \(Cld_H(\overline{\alpha X}) = \text{Comp}(\overline{\alpha X})\) is homeomorphic to the Hilbert cube \(Q\).

**Lemma 8.** If a metric space \(X\) has proper, locally connected completion \(\overline{X}\) having no bounded connected component, then the hyperspace \(Cld_H(\overline{\alpha X}\{\infty\})\) is homeomorphic to the Hilbert cube \(Q\).

**Proof.** Observe that \(Cld_H(\overline{\alpha X}\{\infty\})\) is a retract of \(Cld_H(\overline{\alpha X})\), and thus is a compact absolute retract. Then, we use the Characterization Theorem for the Hilbert cube, see [BRZ, Theorem 1.1.23]. By this theorem we have to check that for each \(\varepsilon > 0\), every \(n \in \mathbb{N}\), and each maps \(f_1, f_2 : I^n \to Cld_H(\overline{\alpha X}\{\infty\})\) there are maps \(f'_1, f'_2 : I^n \to Cld_H(\overline{\alpha X}\{\infty\})\) such that \(d(f_i, f'_i) < \varepsilon, i = 1, 2\), and \(f'_1(I^n) \cap f'_2(I^n) = \emptyset\).

Fix \(\varepsilon > 0\), \(n \in \mathbb{N}\), and maps \(f_1, f_2 : I^n \to Cld_H(\overline{\alpha X}\{\infty\})\). By the argument of [CN] we can show that \(\text{Fin}_H(\overline{\alpha X})\) is homotopy dense in \(Cld_H(\overline{\alpha X})\). Therefore, we can find an \(\varepsilon/2\)-close to \(f_i\) map \(g_i : I^n \to \text{Fin}_H(\overline{\alpha X})\), \(i = 1, 2\), respectively, see [BRZ, Ex. 1.2.10]. Observe, that \(d(f_i, g_i \cup \{\infty\}) < \varepsilon/2, i = 1, 2\). Then, it is easily seen that maps \(f'_1 = g_1 \cup \{\infty\}\) and \(f'_2 = g_2 \cup B(\infty, \varepsilon/2)\) are as required.

6. Proof of Theorem 2

To prove the “only if” part, assume that \(Cld_{AW}(X)\) is a separable absolute retract. The separability of \(Cld_{AW}(X)\) implies that each bounded subset of \(X\) is totally bounded [BKS, Theorem 5.2], which is equivalent to the properness of the completion \(\overline{X}\) of \(X\). By Corollary 3 the hyperspace \(Cld_{AW}(\overline{X})\) is a separable absolute retract too. In this case \(Cld_{AW}(\overline{X}) = Cld_{F}(\overline{X})\) (by \(Cld_{F}(X)\) we denote the hyperspace \(Cld(X)\) endowed with the Fell topology, see [Hec, Theorem 5.1.10]) is an absolute retract, and we can apply [SY, Propositions 1, 2] to conclude that the locally compact space \(\overline{X}\) is locally connected and contains no bounded (=compact) connected component.

Next, we prove the “if” part of Theorem 2. Assume that the completion \(\overline{X}\) of \(X\) is proper, locally connected with no bounded connected components. By Lemma 6, the space \(Cld_H(\overline{\alpha X}\{\infty\})\) is homeomorphic to the Hilbert cube \(Q\). Proposition 4 implies that \(Cld_H(\overline{\alpha X}\{\infty\})\) is homotopy dense in \(Cld_H(\overline{\alpha X}\{\infty\})\). Taking into account that the Hilbert cube with deleted point is an absolute retract and so is any homotopy dense subset of \(Q \setminus \{pt\}\), we conclude that \(Cld_H(\overline{\alpha X}\{\infty\}) \setminus \{\infty\}\) and its topological copy \(Cld_{AW}(X)\) are absolute retracts.
7. Proof of Theorem 1

The “only if” part. If $Cld_{AW}(X)$ is homeomorphic to $\ell_2$, then $X$ is topologically complete by [Co]. The total boundedness of each bounded subset of $X$ follows from [BKS, Theorem 5.2]. Since $\ell_2$ is a separable absolute retract, we may apply Theorem 2 to conclude that the completion $X$ of $X$ is locally connected and contains no bounded connected component. It remains to show that $X$ is not locally compact at infinity. Assume the contrary, i.e., there exists a bounded subset $B \subset X$ with locally compact complement in $X$. Then it is easily seen that the point $\infty \in \alpha X$ has an open neighborhood with compact closure. Whence, we can find a compact neighborhood of $\{\infty\}$ in $Cld_H(\alpha X \setminus \{\infty\})$. But this is impossible because of the nowhere locally compactness of the Hilbert space $\ell_2$. This proves the “only if” part of Theorem 1.

To prove the “if” part, assume that $X$ is topologically complete, not locally compact at infinity and the completion $\overline{X}$ of $X$ is proper, locally connected with no bounded connected components. By Proposition 3 we identify $Cld_{AW}(X)$ with the subspace $Cld_H(\alpha X \setminus \{\infty\}) \setminus \{\infty\}$ of $Cld_H(\alpha X \setminus \{\infty\})$. By Lemma 8 the hyperspace $Cld_H(\alpha X \setminus \{\infty\})$ is homeomorphic to $Q$. Now consider the map $e : Cld_H(\alpha X \setminus \{\infty\}) \to Cld_H(\alpha X \setminus \{\infty\})$ assigning to each closed subset $F \subset \alpha X$ its closure $\overline{F}$ in $\alpha X$ and note that this map is an isometric embedding, which allows us to identify the hyperspace $Cld_{AW}(X)$ with the subspace $\{F \subset Cld_H(\alpha X \setminus \{\infty\}) : F = \overline{F \cap \alpha X}\}$ of $Cld_H(\alpha X \setminus \{\infty\})$. It is easy to check that this subspace is dense and relatively complete in the Lawson semilattice $Cld_H(\alpha X \setminus \{\infty\})$. Then it is homotopically dense in $Cld_H(\alpha X \setminus \{\infty\})$ by Proposition 1 and Lemma 2. The subset $Cld_{AW}(X)$, being topologically complete, is a $G_\delta$-set in $Cld_H(\alpha X \setminus \{\infty\})$. The dense subsemilattice $L = \mathrm{Fin}_H(\alpha X \setminus \{\infty\}) \setminus \mathrm{Fin}_H(\alpha X \setminus \{\infty\})$ is homotopy dense in $Cld_H(\alpha X \setminus \{\infty\})$, since $X$ is not locally compact at infinity. Since $L \cap Cld_{AW}(X) = \emptyset$, we get that $Cld_{AW}(X)$ is a homotopy dense $G_\delta$-subset in $Cld_H(\alpha X \setminus \{\infty\})$ with homotopy dense complement. Applying Lemma 5 we conclude that the space $Cld_{AW}(X)$ is homeomorphic to $\ell_2$.

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