The description of phase transition of Bardeen black hole in the Ehrenfest scheme

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Abstract

The phase transition of a Bardeen black hole is studied by considering Ehrenfest’s equations. The thermodynamic variables such as entropy, potential and heat capacity are calculated from the first law of thermodynamics for black holes. That no discontinuity in entropy and potential of the black holes means that the first order phase transition will not generate for the Bardeen black holes. However, the divergence of heat capacity at constant potential and satisfaction of the Ehrenfest’s equations indicates that the second order phase transition of Bardeen black hole will appear.

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1. Introduction

It is important that black holes can be treated as thermodynamic objects with physical temperature and entropy. Bekenstein has claimed that the black holes should have temperature which is not equal to zero and the intrinsic entropy of a black hole is proportional to the area of event horizon [1-3]. It has been shown that a black hole has a thermal radiation with the temperature due to its surface gravity through studying the quantum mechanics of scalar particles around a black hole by Hawking et.al [4-6].

Bradeen model describes a "regular" spacetime without a singularity but with a horizon [7-9]. The Bardeen black hole and several other regular black holes can be presented as the exact solutions of nonlinear electrodynamics coupled to Einstein gravity [10-13]. Subsequently, the gravitational lensing of Bardeen black holes was studied in Ref.[14]. The gravitational and electromagnetic stability of Bardeen black hole was explored [15]. The quasinormal modes of the Bardeen black hole were also discussed [16]. Sharif and Javed have noticed the thermodynamical quantities as temperature and entropy for a Bardeen black hole by means of a quantum tunneling approach over semiclassical approximations [17]. The effects of space noncommutativity on the thermodynamics of a Bardeen black hole was also examined in Ref.[18]. However, we notice that the entropy of the Bardeen black hole is a function of not only the event horizon radius of the black hole but also the charge of magnetic monopole swallowed by the black hole. During the calculation of electric potential, we set the entropy as a constant and obtain an electric potential which is different from that in Ref.[17] and Ref.[18].

The Ehrenfest scheme is a powerful tool to describe the phase transition of black holes [19]. In the context of this scheme, Banerjee et. al confirmed that the phase transitions from liquid-vapour systems to Reissner-Nordstrom-AdS or Kerr-AdS black holes belong to the second order [20, 21]. We also discuss the phase transition of a black hole with conformal anomaly with the help of Ehrenfest’s equation [22]. In this process the basic classical thermodynamic method is transplanted to study the order of phase transition in black holes. If the Gibbs potential of the system continues while the first order derivative of Gibbs potential is discontinuous, the phase transition is so-called first order. The so-called second order phase transition can be thought as that the Gibbs potential of the system and its first order derivative is continuous while the second order derivative of the potential is not continuous. The higher order transitions can be studied in the way on the analogy of this.

In this paper, we would like to study the phase transitions of Bardeen black holes in virtue of the Ehrenfest’s scheme with analytical approximation to the critical points of phase transitions. First we conform to Hawking temperature and the first law of thermodynamics, then obtain the black hole entropy with the logarithmic term and the characteristic electric potential which is distinguished from the results before [17,18]. Absence of any discontinuity in entropy and potential of the black hole as the functions with horizon radius and infinite divergences in heat capacity at constant potential indicates that the higher order phase transition of Bardeen black hole occurs.
The explicit relation of critical horizon radius and charge is given. We then show the first and second Ehrenfest’s equations are established at the critical phase transition point. The phase transition of second order will be confirmed. The paper is organized as follows. In section 2, we recapitulate the Bardeen black hole. The explicit expressions of thermodynamic quantities like temperature, entropy, electric potential and heat capacity of the black hole are given, and we discuss the critical points of phase transition. In section 3, we prove that the phase transition of a Bardeen black hole is second order with the help of Ehrenfest equation. Finally, the discussion and conclusions are emphasized.

2. The thermodynamic quantities of Bardeen black holes

At first we will give a brief introduction to the Bardeen black hole as a regular solution of Einstein equations coupled to a nonlinear electrodynamics which provide a monopole charge $q$ [10]. The action was proposed as,

$$S = \int dv \left( \frac{1}{16\pi} R - \frac{1}{4\pi} \mathcal{L}(F) \right),$$

where $R$ is the scalar curvature, and $\mathcal{L}$ is the Lagrangian, a function of $F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, and $F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]}$ is the electromagnetic strength. $\mathcal{L}$ as the nonlinear electrodynamics source is denoted as,

$$\mathcal{L}(F) = \frac{3M}{|q|q^2} \left( \frac{\sqrt{2q^2 F}}{1 + \sqrt{2q^2 F}} \right)^{\frac{3}{2}}.$$

Here $q$ and $M$ are magnetic charge and the mass of the magnetic monopole respectively. The Einstein-nonlinear electrodynamics field equations are

$$G^{\nu}_{\mu} = 2 \left( \frac{\partial \mathcal{L}}{\partial F^{\nu\lambda}} F^{\mu\lambda} - \delta^{\nu}_{\mu} \mathcal{L} \right),$$

$$\nabla_{\mu} \left( \frac{\partial \mathcal{L}}{\partial F^{\alpha\mu}} F^{\alpha\mu} \right) = 0.$$

The Bardeen black hole solution with a static and spherically symmetric configuration is provided by

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

where

$$f(r) = 1 - \frac{2Mr^2}{(r^2 + q^2)^{\frac{3}{2}}}.$$

Note for $q = 0$, it recovers to the well-known Schwarzschild metric. Contrary to traditional black hole spacetime which are expected to have horizons and singularities covered by the horizons, the Bardeen model has a horizon but no singularity. We find the event horizon $r_H$ of the black hole as a solution of $f(r) = 0$. The total mass of the Bardeen black hole can be presented as
\[ M = \frac{(r_H^2 + q^2)^{\frac{3}{2}}}{2r_H^2}. \]  

(7)

The relation between the mass and the event horizon radius is drawn in Figure 1. There is a minimum \( M_0 = \frac{3\sqrt{3}}{4}q \) for \( r_H = \sqrt{2}q \).

We shall calculate the thermodynamic variables of the Bardeen black hole in order to discuss the phase transition happened in this kind of black hole. We start with Hawking temperature defined as

\[ T_H(q, r_H) = -\frac{1}{4\pi} \left[ \sqrt{-g^{tt}g^{rr}} \frac{dg_{tt}}{dr} \right]_{r=r_H} = \frac{1}{4\pi} \frac{r_H^2 - 2q^2}{r_H(r_H^2 + q^2)} \]  

(8)

shown in Figure 2. From Figure 2, it is clear that the temperature of the Bardeen black hole cannot exceed the threshold. The maximum temperature \( T_0 \) of an extremal configuration with radius \( r_0 \) are obtained,

\[ T_0 = \frac{(r_0^2 - 2q^2)}{4\pi r_0(r_0^2 + q^2)}, \]  

(9)

\[ r_0 = \frac{\sqrt{14 + 2\sqrt{57}}}{2}q. \]  

(10)

The first law of thermodynamics for black holes can be written as

\[ dM = T_H dS + \phi dq, \]  

(11)

where \( S \) is the modified entropy and \( \phi \) is the electric potential. Here the semiclassical entropy can be expressed as

\[ S = \int \left( \frac{1}{T} \frac{\partial M}{\partial r_H} \right)_q r_H \frac{\pi}{r_H} (r_H^2 - 2q^2) \sqrt{r_H^2 + q^2 + 3q^2} \ln \left( r_H + \sqrt{r_H^2 + q^2} \right). \]  

(12)

The subscript \( q \) requires that the charge has to be treated as an invariant during calculating the integration. According to Eq. (7), the total mass of the black hole involves the charge \( q \). It recovers to area entropy \( \pi r_H^2 \), if charge \( q \) reduces to zero. In Figure 3, there is a continuous curve which shows the relation between the entropy and the event horizon radius of the Bardeen black.

According to the first law of thermodynamics, the analytic expression of electric potential is written as

\[ \phi = \left( \frac{\partial M}{\partial q} \right)_S = \frac{3q}{4r_H^2(r_H^2 + q^2)(r_H + \sqrt{r_H^2 + q^2})} [3r_H^4 + 3r_H^3\sqrt{r_H^2 + q^2} + 4q^2r_H^2 - 2q^4 - 2r_H(r_H^2 - 2q^2)(r_H + \sqrt{r_H^2 + q^2})ln(r_H + \sqrt{r_H^2 + q^2})]. \]  

(13)
The expression of potential is more cumbersome due to the tangle among the entropy, event horizon radius and charge. Since the entropy is not simply proportional to the area of event horizon, it no longer works here for choosing horizon radius \( r_H \) as an invariant instead of entropy \( S \). The trick we used here is treating horizon radius as a function of charge. We plot the electric potential dependent on black hole radius for \( q = 0.9 \) in Figure 4. Neither entropy and potential have a discontinuity upon the behavior of horizon radius, so the first order phase transition of the Bardeen black hole is ruled out.

In addition, the Gibbs free energy for Bardeen black hole is defined as [20],

\[
G = M - TS - \phi q. \tag{14}
\]

With thermodynamic quantities Eq. (7), Eq. (8), Eq. (12) and Eq. (13), the free energy \( G \) can be expressed as a function of event horizon radius \( r_H \) and magnetic charge \( q \), showed as a continuous curve in Figure 5. It can be easily checked that the non-negative definiteness \( T_H \) give a strong constraint on the horizon radius \( r_H \geq \sqrt{2}q \).

The same procedure may be easily adapted to obtain heat capacity at constant potential

\[
C_\phi = \left( T \frac{\partial S}{\partial T} \right)_\phi = \frac{B(r_H, q)}{A(r_H, q)}, \tag{15}
\]

where the function \( A(r_H, q) \) is

\[
A(r_H, q) = -2\sqrt{r_H^2 + q^2(-3r_H^4 - 14r_H^4q^2 + 3r_H^2q^4 + 2q^6) - r_H^2(2q^6 - 5r_H^2q^4 - 31r_H^4q^2 - 6r_H^6)} + 2\ln(r_H + \sqrt{r_H^2 + q^2})(-r_H^4 + 7q^2r_H^2 + 2q^4)r_H(r_H + \sqrt{r_H^2 + q^2})^2.
\]

and \( B(r_H, q) \) is

\[
B(r_H, q) = \frac{-\pi}{r_H + \sqrt{r_H^2 + q^2}}[12(-r_H^4 + 7q^2r_H^2 + 2q^4)\ln(r_H + \sqrt{r_H^2 + q^2})^2r_H^2q^2 \times (4r_H^6 + 3r_H^2q^2 + (4r_H^2 + q^2)(\sqrt{r_H^2 + q^2})^2) - 4r_H\ln(r_H + \sqrt{r_H^2 + q^2})(4r_H^{10} - 55r_H^8q^2 - 56r_H^6q^4 + 70r_H^4q^6 + 59r_H^2q^8 + 10q^{10} + r_H\sqrt{r_H^2 + q^2}(4r_H^8 - 57r_H^6q^2 - 27r_H^4q^4 + 76r_H^2q^6 + 24q^8)) + \sqrt{r_H^2 + q^2}(-2r_H^8q^2 - 59r_H^6q^4 + 82r_H^4q^6 + 51r_H^2q^8 + 16q^{10} + 24r_H^{10}) + r_H(24r_H^{10} - 63r_H^6q^4 + 21r_H^4q^6 + 36q^{10} + 124r_H^2q^8 + 10r_H^2q^2)]. \tag{16}
\]

Heat capacity diverges if the denominator \( A(r_H, q) \) is equal to zero. Then we find that the critical radius point \( r_c \) is a root of equation \( A(r_c, q) = 0 \). For charge \( q = 0.9 \), we get a reasonable critical horizon radius \( r_c = 4.877810899 \) which is larger than \( \sqrt{2}q \). Further we substitute \( r_c \) into equation
(8) to find the critical temperature \( T_c \) where some discontinuities happen in the heat capacity at constant potential while the entropy and potential of the black hole is continuous. So the phase transition of Bardeen black hole must higher than the first order. We plot heat capacity at constant potential expressed with equations (15), (16) and (17) in Figure 6 and Figure 7 as functions of horizon radius and temperature respectively. In Figure 6, the heat capacity at constant potential dips from positive infinity to negative infinity along with increase of event horizon radius. A thermodynamically stable case requires a positive heat capacity. Noticing that horizon radius has to be larger than \( \sqrt{2}q \), we find the range of horizon radius \( r_m < r_H < r_c \) for the heat capacity at constant potential greater than zero. Here \( r_m \) is the maximum root of equation \( C_\phi = 0 \). It demonstrates that nearby the critical point the Bardeen black hole transform from an unstable phase with larger horizon radius \( r_H > r_c \) where heat capacity in this situation is negative into a stable phase with smaller black hole radius \( r_m < r_H < r_c \) where \( C_\phi \) is positive. The larger Bardeen black holes can not survive. From Figure 1 and Figure 2, a Bardeen black hole in an unstable phase has larger mass \( M > M|_{r_H=r_c} \) and lower temperature \( T_H < T_c \). It can not keep this phase for long time and eventually transmutes through critical point into a stable black hole with smaller mass \( M_0 < M < M|_{r_H=r_c} \) but higher temperature \( T_0 > T_H > T_c \). We should pay attention to the critical temperature \( T_c \) in Figure 7 less than the maximal value \( T_0 \), which means that this phase transition can happen in an appropriate temperature. Furthermore, no discontinuity for heat capacity appears if \( q = 0 \) corresponding to general Schwarzchild black hole spacetime.

3. The description of phase transition in Ehrenfest relation

In this section, we will examine whether the phase transition of a Bardeen black hole is second order by adopting Ehrenfest equation. It has been showed that the special entropy and electric potential are continuous, so through the critical point the right-hand-side of Clausius-Clapeyron equation \(- \frac{\partial \phi}{\partial T} = \frac{\Delta S}{\Delta q} \) become indeterminate form \( \frac{0}{0} \). Take advantage of L’Hospital’s rule, then two Ehrenfest’s equations for Bardeen spacetime can be easily found,

\[
- \left( \frac{\partial \phi}{\partial T_H} \right)_q = \frac{\alpha_2 - \alpha_1}{\kappa_2 - \kappa_1},
\]  

(18)

\[
- \left( \frac{\partial \phi}{\partial T_H} \right)_S = \frac{C_{\phi 2} - C_{\phi 1}}{T q (\alpha_2 - \alpha_1)},
\]  

(19)

where \( \alpha \) is the charge growth coefficient analogy to volume expansion coefficient, and \( \kappa \) is the isothermal compressibility defined as \( \alpha = \frac{1}{q} \left( \frac{\partial q}{\partial T_H} \right)_\phi \) and \( \kappa = \frac{1}{T} \left( \frac{\partial T}{\partial \phi} \right)_H \). With thermodynamic quantities of this system including magnetic charge \( q \), Hawking temperature \( T_H \), and electric potential \( \phi \), the specific expressions of charge growth coefficient and isothermal compressibility are given by

\[
\alpha = \frac{C(r_H, q)}{A(r_H, q)},
\]  

(20)
\[ \kappa = \frac{D(r_H, q)}{A(r_H, q)}, \]  
\hspace{1cm} (21)

where the function \( C(r_H, q) \) is given by,
\[ C(r_H, q) = \frac{4\pi r_H (r_H^2 + q^2)}{r_H^2 - 2q^2} \left[ 2(r_H^4 - 7q^2 r_H^2 - 2q^4) \ln(r_H + \sqrt{r_H^2 + q^2}) r_H (r_H + \sqrt{r_H^2 + q^2})^2 ight. \\
\left. - r_H (5r_H^4 q^2 - 17r_H^2 q^4 + 10r_H^6 - 6q^6) + 2(2q^4 + 5r_H^2 q^2 - 5r_H^4)(r_H^2 + q^2)^2 \right], \]  
\hspace{1cm} (22)

and the function \( D(r_H, q) \) is,
\[ D(r_H, q) = \frac{4(r_H + \sqrt{r_H^2 + q^2})^2 (r_H^2 + q^2) (r_H^2 - 7q^2 r_H^2 - 2q^4) r_H^2}{3q (r_H^2 - 2q^2)}. \]  
\hspace{1cm} (23)

They diverge while the black hole as a thermodynamic object approaches the critical point. The Figure 7 and Figure 8 describe the behavior of charge growth coefficient and isothermal compressibility depending on the event horizon radius of Bardeen black hole. Some discontinuities also emerge to exhibit the higher order phase transition in a Bardeen black hole. One can see from equations (15), (20) and (21), all of the heat capacity at constant potential, charge growth coefficient and volume expansion coefficient have the same denominator. These three thermodynamical quantities diverge at the same radius \( r_c \) although some discontinuities at other points also appear. Therefore, the critical horizon radius is still our phase transition position.

Firstly we check the left-hand-side of Ehrenfest’s equations (18), (19) at critical point in order to ensure the second order phase transition is able to occur. Combine the electric potential (13), and Hawking temperature (8) and take magnetic charge or entropy (12) as constants, the left-hand-side of equation (18) and (19) then be found
\[ - \left( \frac{\partial \phi}{\partial T_H} \right)_q |_{r_H=r_c} = - \left( \frac{\partial \phi}{\partial T_H} \right)_S |_{r_H=r_c} = -6\pi q. \]  
\hspace{1cm} (24)

Secondly we study the right-hand-side of equations (18), (19) at critical position. By regarding \( q \) invariable, \( B(r_H) \) from heat capacity at a constant potential, \( C(r_H) \) from the charge growth coefficient and \( D(r_H) \) from the isothermal compressibility are well behavior, and all of \( C_\phi, \alpha \) and \( \kappa \) have the same denominator \( A(r_H) \). We choose two arbitrary points \( r_1 \) and \( r_2 \) close to the phase transition point \( r_c \), one get \( B(r_1) = B(r_2) = B(r_c) \), \( C(r_1) = C(r_2) = C(r_c) \), and \( D(r_1) = D(r_2) = D(r_c) \), and they are all not equal to zero. However, this approximation method to critical point is not suitable for the denominator term because \( A(r_c) = 0 \). The right-hand-side of Ehrenfest’s equations become
\[ \frac{\alpha_2 - \alpha_1}{\kappa_2 - \kappa_1} |_{r_c} = \frac{C(r_c)}{Dr_c} = -6\pi q, \]  
\hspace{1cm} (25)

and
\[ \frac{C_\phi_2 - C_\phi_1}{Tq(\alpha_2 - \alpha_1)} |_{r_c} = \frac{B(r_c)}{T_H q C_{r_c}} |_{r_c} = -6\pi q. \]  
\hspace{1cm} (26)
Divergences in heat capacity, charge growth coefficient and isothermal compressibility are canceled in both of Eq.(16) and Eq.(17). From Eq.(24), Eq.(25) and Eq.(26), it is evident that the two Ehrenfest’s equations are established at the critical point. Furthermore the first and second equations are identical to each other at the critical point \( r_c \). So far we analytically prove that the phase transition of a Bardeen black hole belongs to a second order transition by means of Ehrenfest scheme.

4. Discussion

In this paper, with the help of Ehrenfest schemes, we show analytically that the second order phase transition will appear for the Bardeen black holes. According to the first law of thermodynamics of black holes, we obtain the electric potential, heat capacity at constant potential, charge growth coefficient and isothermal compressibility for a regular Bardeen black hole. The entropy and potential of black hole as functions of event horizon radius have no discontinuity while the relation between heat capacity and horizon radius approaches to the infinity at the critical radius regarded as the signature of phase transition. Close to the critical point, an unstable black hole with larger event horizon radius than critical radius, larger mass and lower temperature changes to a stable phase with smaller radius, lower mass and higher temperature because the branch with large radius has negative heat capacity at constant potential but the other branch with smaller radius has positive one. Then we analysis the Ehrenfest’s equations to prove that a second order phase transition can happen for Bardeen black holes. The explicit expressions of charge growth coefficient and isothermal compressibility also reveal discontinuities at the same critical point during the phase transition.

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References

[1] J. D. Bekenstein, Lett. Nuovo Cim. 4(1972)737
[2] J. D. Bekenstein, Phys. Rev. D7(1973)2333
[3] J. D. Bekenstein, Phys. Rev. D9(1974)3292
[4] S. W. Hawking, Commun. Math. Phys. 43(1975)199
[5] J. B. Hartle, S. W. Hawking, Phys. Rev. D13(1976)2188
[6] G. W. Gibbons, S. W. Hawking, Phys. Rev. D15(1977)2752
[7] J. Bardeen, in Proceedings of GR5, Tiflis, U.S.S.R.(Unpublished)
[8] A. Borde, Phys. Rev. D 50, 3692 (1994)
[9] A. Borde, Phys. Rev. D 55,7615(1997)
[10] E. Ayón-Beato and A. García, Phys. Lett. B 493, 149(2000)
[11] E. Ayón-Beato and A. García, Phys. Lett. B 464, 25(1999)
[12] E. Ayón-Beato and A. García, Gen. Relativ. Gravit. 31, 629(1999)
[13] E. Ayón-Beato and A. García, Gen. Relativ. Gravit. 37, 635(2005)
[14] E. F. Eiroa and C. M. Sendra, Classical Quantum Gravity 28, 085008(2011)
[15] C. Moreno, O. Sarbach, Phys. Rev. D 67, 024028 (2003)
[16] S. Fernando, J. Correa, Phys. Rev. D 86.064039(2012)
[17] M. Sharif, W. Javed, J. Korean. Phys. Soc. 57(2010) 217-222
[18] M. Sharif, W. Javed, Can. J. Phys. 89 (2011) 1027-1033
[19] R. Banerjee, S. K. Modak, S. Samanta, Eur. Phys. J. C70(2010)317
R. Banerjee, S. Ghosh, D. Roychowdhury, Phys. Lett. B696(2011)156
[20] R. Banerjee, S. K. Modak and D. Roychowdhury, JHEP 10 (2012) 125
R. Banerjee and D. Roychowdhury, JHEP 11 (2011) 004
[21] R. Banerjee, S. K. Modak, S. Samanta, Phys. Rev. D84(2011)064024
[22] J. Man, H. Cheng, arXiv:1304.5685
Figure 1: The figure shows the mass of the magnetic monopole as a function of event horizon radius for $q = 0.9$. 
Figure 2: The figure shows the temperature $T_H$ as a function of the event horizon radius $r_H$ for $q = 0.9$. 
Figure 3: The relation between the entropy $S$ and event horizon radius $r_H$ for $q = 0.9$. 
Figure 4: The relation between the electric potential $\phi$ and event horizon radius $r_H$ for $q = 0.9$. 
Figure 5: The behavior of the heat capacity at constant potential with the event horizon radius for the Bardeen black hole with charge $q = 0.9$. 
Figure 6: The behavior of the heat capacity at constant potential with the temperature for the Bardeen black hole with charge $q = 0.9$. 
Figure 7: The behavior of charge growth coefficient with event horizon radius of the Bardeen black hole for $q = 0.9$. 
Figure 8: The behavior of the isothermal compressibility with event horizon radius of the Bardeen black hole for $q = 0.9$. 

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