General vorticity conservation

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Abstract

The motion of an incompressible fluid in Lagrangian coordinates involves infinitely many symmetries generated by the left Lie algebra of group of volume preserving diffeomorphisms of the three dimensional domain occupied by the fluid. Utilizing a 1 + 3-dimensional Hamiltonian setting an explicit realization of this symmetry algebra and the related Lagrangian and Eulerian conservation laws are constructed recursively. Their Lie algebraic structures are inherited from the same construction. The laws of general vorticity and helicity conservations are formulated globally in terms of invariant differential forms of the velocity field.
1 Introduction

The configuration space of an incompressible fluid is the group of volume preserving transformations of the three dimensional region in $\mathbb{R}^3$ containing the fluid. The motion is generated by the left action of the group by composition and hence the velocity field is right invariant. Thus, the generators of the right action which form the infinite dimensional left Lie algebra are infinitesimal symmetries of the velocity field $\mathfrak{g}$. In the fluid mechanical context, this is known as the particle relabelling symmetry $\mathfrak{g}$, also named as gauge transformations in Ref.[5] and as trivial displacements in Ref.[6]. It has been the subject of many investigations to discover the corresponding Lagrangian and Eulerian invariants in their most general form and to express for them the so-called general vorticity conservation law. (see Ref.[4] for a discussion and references).

One the best illustrative example for this general conservation principle may be the equivalences of the preservation of coadjoint orbits of vorticity, the right invariance on the cotangent bundle of the group of volume preserving diffeomorphisms, the Helmholtz’ vorticity and the Kelvin’s circulation theorems (see [3] or section(6.5)). Another formulation of the same principle results, in addition, to the Eulerian conservation laws for helicity [4]. Both examples have in common the property of being derived from the particle relabelling symmetry.

In spite of the fact that the particle relabelling symmetry is a direct consequence of the description of motion itself, many investigations on the structure of symmetries and invariants employ the techniques of analysing the defining equations for them thereby obtaining an algebraically and quantitatively incomplete picture of symmetries, invariants and their connections.

In this work we shall present a geometric framework for a systematic study of symmetries and invariants which will provide, by addressing explicitly to Lagrangian and Eulerian formulations, a better understanding of them in the description of motion.

1.1 Summary and content

We shall consider the Lagrangian description of fluid motion in the framework of a formal geometric structure to obtain a realization of the symmetry algebra, the corresponding Lagrangian invariants and the general vorticity
conservation law.

We shall start from the observation that the Euler equations themselves are the conditions for the vorticity field to be an infinitesimal symmetry of the velocity field. We shall use this symmetry to construct a symplectic structure on the time-extended domain in \( R \times R^3 \). This will define the Hamiltonian structure of the suspended velocity field. Identifying a Hamiltonian vector field as a new symmetry of the velocity field, we shall be able to generate the infinitesimal symmetries recursively.

The vorticity is, by construction, an automorphism of the Hamiltonian structure. Since any Hamiltonian vector field is also an automorphism, we shall conclude that the resulting infinite hierarchy of vector fields do form the Lie algebra of symmetries.

Associated to each infinitesimal symmetry we shall then introduce invariant differential forms of the velocity field. Among a plethora of differential invariants of various degrees we shall identify a hierarchy and use it as the basic ingredient for the global expressions of the laws of general vorticity and helicity conservations.

After a brief description of fluid motion in the next section we shall recall from Ref.\[7\] the symplectic structure of the suspended velocity field on the time-extended space \( R \times R^3 \). In section\(\[8\]\), we shall obtain the time-dependent symmetries on \( R \times R^3 \) of the velocity field and show that an equivalent representation of them are Hamiltonian vector fields on \( R^3 \). In section\(\[9\]\), we shall give explicit expressions for the laws of general vorticity and helicity conservations. All of these will arise in connection with the infinitesimal automorphisms, restriction to spatial domain in \( R^3 \), and the associated invariant forms of the symplectic structure.

## 2 Three dimensional fluid motion

Let the open set \( D \subset R^3 \) be the domain occupied initially by an incompressible fluid and \( x(t = 0) = x_0 \in D \) be the initial position, i.e., a Lagrangian label. For a fixed initial position \( x_0 \), the Eulerian coordinates \( x(t) = g_t(x_0) \) define a smooth curve in \( R^3 \) describing the evolution of fluid particles. For each time \( t \in R \), the volume preserving embedding \( g_t : D \rightarrow g_t(D) = D_t \subset R^3 \) describes a configuration of fluid. A flow is then a curve \( t \mapsto g_t \) in the space of all such transformations. The time-dependent Eulerian (spatial) velocity
field $v_t$ that generates $g_t$ is defined by

$$\frac{dx}{dt} = \frac{dg_t(x_0)}{dt} = (v_t \circ g_t)(x_0) = v(t, x)$$

(1)

where $v_t \circ g_t$ is the corresponding Lagrangian (material) velocity field \[2\],\[3\].

Since $g_t$ is volume preserving, $v_t(x)$ is a divergence-free vector field over $\mathbb{R}^3$ and Eq. (1) is a non-autonomous dynamical system associated with it. The system (1) can, equivalently, be represented as an autonomous system defined by the suspended velocity field

$$\partial_t + v(t, x), \quad v = \mathbf{v} \cdot \nabla$$

(2)

on the time-extended space $\mathbb{R} \times \mathbb{R}^3$.

The Lagrangian description of fluid motion is the description by trajectories, that is, by solutions of non-autonomous ordinary differential equations (1). Equivalence of (1) to the autonomous system associated with (2) means that the trajectories can be obtained by describing streamlines at each time. In Ref. \[7\], using the Eulerian dynamical equations, we constructed a formal symplectic structure for (2) on a time-extended domain in $\mathbb{R} \times \mathbb{R}^3$. We shall now summarize this construction.

### 2.1 Symplectic structure

A symplectic structure \[2\],\[3\],\[8\] on a manifold $N$ of even dimension $2n$ is defined by a closed, non-degenerate two-form $\Omega$. It is exact if there exists a one-form $\theta$ such that $\Omega = -d\theta$. Darboux’s theorem guarantees the existence of local coordinates $(q^i, p_i) i = 1, \ldots, n$ in which $\Omega$ has the canonical form $dq^i \wedge dp_i$. The $2n$-form $(-1)^n \Omega^n/n!$ is called the Liouville volume. A vector field $V$ on $N$ is called Hamiltonian if there exists a function $h$ on $N$ such that

$$i(V)(\Omega) = dh$$

(3)

where $i(V)(\cdot)$ denotes the inner product with $X$. The identity $i(V)(dh) = 0$ which follows from (3) is the expression for conservation of $h$ under the flow of $V$. With the correspondence (3) between functions and vector fields, the Poisson bracket of functions on $N$ defined by

$$\{f, g\} = \Omega(V_f, V_g) = \Omega^{-1}(df, dg)$$

(4)
satisfies the conditions of bilinearity, skew-symmetry, the Jacobi identity and
the Leibniz rule. This enables us to write the dynamical system associated
with the vector field $V_h$ in the form of Hamilton’s equations
\[
\frac{dx}{dt} = \{x, h\}.
\]

**Proposition 1** Let the dynamics of the velocity field $\mathbf{v}$ be governed by
\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{F}
\]
and assume that the divergence-free field $\mathbf{B}$ and the function $\varphi$ satisfy
\[
\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \quad \frac{\partial \varphi}{\partial t} + \mathbf{v} \cdot \nabla \varphi = 0
\]
which are the frozen field equations. Then

1. $\partial_t + \mathbf{v}$ is a Hamiltonian vector field with the symplectic two-form
   \[
   \Omega = -(\mathbf{v} \cdot \mathbf{B}) \cdot \mathbf{dx} \wedge dt + \mathbf{B} \cdot (\mathbf{dx} \wedge \mathbf{dx})
   \]
   and the Hamiltonian function $\varphi$.
2. $\rho_\varphi \equiv -\mathbf{B} \cdot \nabla \varphi$ is the invariant Liouville volume density.
3. If moreover $\mathbf{B} = \nabla \times \mathbf{A}$ for some vector potential $\mathbf{A}$, then $\Omega$ is exact
   \[
   \Omega = -d\psi, \quad -\psi = \varphi + P/\rho + \frac{v^2}{2}
   \]
   where $\psi$ is determined by the equation
   \[
   \frac{\partial \mathbf{A}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla (\varphi + \psi).
   \]
4. In Darboux coordinates, $\Omega = dq \wedge dp + dt \wedge dh_{\text{can}}$ and the potential field has the Clebsch representation
   \[
   \mathbf{A} = \nabla s + p\nabla q
   \]
   where $s$ is the generating function of the canonical transformation.

The Euler flow of an ideal isentropic fluid is characterized by $\mathbf{F} = -\nabla (P/\rho)$, where $P$ and $\rho$ are pressure and density, respectively. In this case, $\mathbf{A}$ is replaced by the velocity field itself, $\mathbf{B}$ becomes the vorticity $\mathbf{w} = \nabla \times \mathbf{v}$ and Eqs.(10) reduces to the expression
\[
-\psi = \varphi + P/\rho + \frac{v^2}{2}
\]
for the scalar potential. The invariant volume density $\rho_\varphi = -\mathbf{w} \cdot \nabla \varphi$ is also known as potential vorticity.
3 Symmetries

A time-dependent vector field \( U = \xi \partial_t + u \) is an infinitesimal geometric symmetry of dynamical system described by \( v \) if the criterion

\[
[\partial_t + v, \xi \partial_t + u] = (\xi_t + v(\xi))(\partial_t + v)
\]

is satisfied. These are the most general symmetries of the system (1) of first order ordinary differential equations [8].

The frozen field equation (7) for \( B \) is an expression for it to be a symmetry of \( v \). Since \( \rho_\varphi \) is a conserved function, \( \rho_\varphi^{-1} B \) is also a symmetry. We observe that this initial symmetry leaves the symplectic two-form (8) invariant. More precisely, one checks that the Lie derivative of \( \Omega \) with respect to the normalized field \( \rho_\varphi^{-1} B \cdot \nabla \) vanishes.

Our intention now is to find another infinitesimal invariance of \( \Omega \) which is a symmetry of the velocity field as well. One of the best candidates for this is a Hamiltonian vector field because the symplectic two-form is invariant under the flows of Hamiltonian vector fields. This can easily be seen from the identity \( \mathcal{L}_U = i(U)d + d\mathcal{L}(U) \) for the Lie derivative together with Eq.(3) and the closure of \( \Omega \). If \( U \) is such a field, called an automorphism of \( \Omega \), then it follows from the identity

\[
\mathcal{L}_{[\rho_\varphi^{-1} B, U]} = \mathcal{L}_{\rho_\varphi^{-1} B} \mathcal{L}_U - \mathcal{L}_U \mathcal{L}_{\rho_\varphi^{-1} B}
\]

that \( [\rho_\varphi^{-1} B, U] \) also leaves \( \Omega \) invariant. Replacing \( \rho_\varphi^{-1} B \) with \( [\rho_\varphi^{-1} B, U] \) in the identity (14) we see that one can generate an infinite dimensional algebra of Hamiltonian vector fields (over simply connected domains of fluid) as invariants of the symplectic two-form.

Thus, if we can find \( U \) which is also a symmetry of the velocity field, this so-called algebra of symplectic automorphisms of \( \Omega \) will be carried over to the symmetry algebra of the velocity field. The identity (14) will then become the Jacobi identity of the algebra of vector fields and will enable us to obtain the generators recursively.

**Proposition 2** The Hamiltonian vector field

\[
U = \rho_\varphi^{-1}[-B(h)(\partial_t + v) + \frac{dh}{dt} B + \nabla \varphi \times \nabla h \cdot \nabla]
\]
associated with the symplectic two-form $\Omega$ and the arbitrary smooth function $h$ is an infinitesimal symmetry of $v$ if $h$ is invariant under the flow of $v$, that is, $\frac{dh}{dt} = h_t + v(h) = 0$. In this case, the brackets

$$ [...[[\rho^{-1}_\varphi B, U], U], ... ] $$

(16)

generate an infinite hierarchy of time-dependent infinitesimal Hamiltonian symmetries of the velocity field $v$.

The infinitesimal symmetries (16) are of the form of

$$ U_k = \xi_k(\partial_t + v) + \hat{u}_k, \quad k = 0, 1, 2,... $$

(17)

where $\xi_k$’s are conserved functions of $v$ and $\hat{u}_k$’s are vector fields on the spatial domain in $R^3$. We compute $\xi_0 = 0$, $\xi_1 = -B(h)$, $\xi_2 = -B^2(h)$, $\xi_3 = \hat{u}_2(\xi_1) - \hat{u}_1(\xi_2)$, $\xi_4 = \hat{u}_3(\xi_1) - \hat{u}_1(\hat{u}_2(\xi_1)) + \hat{u}_1^2(\xi_2)$, ... to list a few of these functions. The vector fields $\hat{u}_k$ have a well-known interpretation in symmetry analysis of differential equations [8]. They are the unique characteristic (or evolutionary) forms of $U_k$’s along the velocity field $v$. It follows from Eq.(13) by direct computation that the symmetry condition for $\hat{u}_k$ reduces to

$$ [\partial_t + v, \hat{u}_k] = 0. $$

(18)

We shall now switch to this equivalent representation of symmetries on the flow space $R^3$ because this form is more suitable to present their geometric features. Since the bracket of characteristic vector fields is the same as the characteristic form of the bracket, the hierarchy (16) can be written as

$$ [...[[\rho^{-1}_\varphi B, \hat{u}], \hat{u}], ... ] $$

and generates time-dependent vector fields on $R^3$ satisfying Eqs.(18). The Hamiltonian vector fields (16) are divergence-free with respect to the Liouville volume. To this end, it would be appropriate to set $\rho_{\varphi} = 1$. This greatly simplifies the general expressions for symmetries and allows a better manifestation of their generic properties.

Using notations of three dimensional vector calculus and the divergence-free property of $B, \hat{u}_1$, the hierarchy of symmetries (16) in their characteristic form can be obtained recursively from

$$ \hat{u}_k = (\nabla \times \hat{u}_1 \times)^k B = \nabla \times (\hat{u}_1 \times \hat{u}_{k-1}), \quad k = 2, 3, 4,... $$

(19)

where the operator $\nabla \times \hat{u}_1 \times$ is the “curl of cross-product with $\hat{u}_1$”. We find

$$ \hat{u}_1 \equiv \hat{u} = \nabla \varphi \times \nabla h, \quad \hat{u}_2 = \nabla \varphi \times \nabla B(h), $$
$$ \hat{u}_3 = \nabla \varphi \times \nabla [\nabla B(h) \cdot (\nabla \varphi \times \nabla h)], \ldots, \hat{u}_k = \nabla \varphi \times \nabla \hat{u}_{k-1}(h), $$

(20)
for the first few and the generic members of the hierarchy. They are made up of gradients of conserved functions and this property is preserved under the action of symmetries. Hence, their flow lines are described by intersections of surfaces defined by conserved functions $\varphi$, $h$, $B \cdot \nabla \varphi$, $B \cdot \nabla h$, $(\hat{u}_1 \cdot \nabla)(B \cdot \nabla h)$, ... of the velocity field.

The vector fields $\hat{u}_k$, $k = 1, 2, 3, ...$ are manifestly divergence-free (cf. Eqs. (19)) and are in Clebsch form (cf. Eqs. (20)). Then, as pointed out in (Exercise 1.4-1 of) Ref. [3], we can introduce vector potentials $A_k$ and, moreover, write $\hat{u}_k$’s as Hamiltonian vector fields on $\mathbb{R}^3$. From Eqs. (19) we can read off the potentials as

$$A_1 = \varphi \nabla h, \quad A_k = \hat{u}_1 \times \hat{u}_{k-1} = (\hat{u}_1 \times \nabla)(\hat{u}_k - 1)A, \quad k = 2, 3, 4, ... \quad (21)$$

where the operator generating $A_k$ from the potential $A$ of $B$ is to "take curl then multiply with $\hat{u}_1$".

The Hamiltonian structure of $\hat{u}_k$’s is defined by the Poisson bracket

$$\{f, g\}_\varphi = \nabla f \cdot \nabla g$$

which is characterized by the Hamiltonian function $\varphi$ of the suspension (3). The Hamiltonian functions for the vector fields (20) are

$$h_1 = h, \quad h_2 = \hat{u}_0(h), \quad h_3 = -\hat{u}_1(\hat{u}_0(h)), \quad \cdots \quad (23)$$

from which it follows by induction that for the symmetry $\hat{u}_k$

$$h_k = (-1)^k \hat{u}_1^{k-2}(\hat{u}_0(h)), \quad k = 3, 4, 5, ... \quad (24)$$

is the Hamiltonian function. These are all conserved functions of the velocity field which are in the form of potential vorticity. Thus, the bracket (22) may be interpreted as to define the Poisson bracket algebra of generalized potential vorticities on the flow space. We note finally that the bracket (22) is induced from and is compatible with the Poisson bracket (1) on $\mathbb{R} \times \mathbb{R}^3$ defined by the symplectic two-form $\Omega$. 

8
4 Invariant differential forms

We shall construct invariant differential forms of the velocity field associated with the infinitesimal symmetries. The techniques of constructing such invariants together with the infinite dimensionality of the symmetry algebra will result in proliferation of invariant forms. However, a detailed analysis of them will lead us to identify a basic hierarchy to formulate the laws of general vorticity and helicity conservations.

Since the time components $\xi_k$ of symmetries (17) are conserved functions, the right hand side of the symmetry criterion (13) vanishes and this makes the three-form

$$\alpha_k = i(U_k)(\mu) = \xi_k dx \wedge dy \wedge dz - (\xi_k v + \hat{u}_k) \cdot d\mathbf{x} \wedge dt$$

which is independent of our choice for $\rho_\varphi$, an absolute invariant of $\partial_t + v$. This means,

$$\mathcal{L}_{\partial_t + v}(\alpha_k) = i([\partial_t + v, U_k])(\mu) = 0$$

where we used the identity

$$\mathcal{L}_V i(U) - i(U) \mathcal{L}_V = i([V, U])$$

and that $v$ is divergence-free. $\alpha_k$’s are closed via conservation of $\xi_k$’s. The commutativity of the Lie derivative with the interior product for $\partial_t + v$ implies that the two-forms

$$\Theta_k = i(\partial_t + v)(\alpha_k) = -\hat{u}_k \cdot d\mathbf{x} \wedge dt - (\hat{u}_k \times v) \cdot d\mathbf{x} \wedge dt$$

are also absolutely invariant. They are closed by Eq.(28) and the closure of $\alpha_k$’s or, by direct computation using Eqs.(18) and divergence-free property of $\hat{u}_k$’s. Employing the Poincaré lemma, we introduce one-forms $\theta_k$

$$\Theta_k = -d\theta_k, \quad \theta_k = \psi_k dt + A_k \cdot d\mathbf{x}, \quad \hat{u}_k = \nabla \times A_k$$

and $\psi_k$ are determined from the equations

$$\frac{\partial A_k}{\partial t} + \hat{u}_k \times \mathbf{v} = \nabla \psi_k.$$

Commutativity of the Lie derivative with the exterior derivative and the Poincaré lemma imply that $\theta_k$’s are relative invariants. That means, their Lie derivatives with respect to $\partial_t + v$ are exact differentials

$$\mathcal{L}_{\partial_t + v}(\theta_k) = di(\partial_t + v)(\theta_k) = d\chi_k, \quad \chi_k = \psi_k + A_k \cdot \mathbf{v}$$

which follow from the definitions (28) and (29).
### 4.1 A plethora of invariants

It can be seen using the identities (14) and (27) that for an invariant form $\alpha$ and an infinitesimal symmetry $U_l$ of $v$, the Lie derivative $\mathcal{L}_{U_l}(\alpha)$ and the interior product $i(U_l)(\alpha)$ are also invariants. In fact, $\Theta_k$ is obtained from $\alpha_k$ by taking interior product with $\partial_t + v$ which is trivially a symmetry.

Similarly, we have the absolutely invariant two-forms $\alpha_{lk} = i(U_l)(\alpha_k)$ obtained from $\alpha_k$'s. Their Lie derivatives give $d\alpha_{lk} = i([U_l,U_k])(\mu)$ which may be regarded as redefinitions of $\alpha_k$’s in the algebra of vector fields (16).

From $\Theta_k = \alpha_{0k}$ we get absolute invariants $\Theta_{lk} = i(U_l)(\Theta_k)$ and $d\Theta_{lk} = i(\partial_t + v)(d\alpha_{lk})$, the latter of which have similar interpretations of being redefinitions of $\Theta_k$’s.

The relatively invariant one-forms $\theta_k$ result in conserved functions $\theta_{lk} = \xi_k \psi_k + \hat{u}_k \cdot A_k$ of the velocity field whenever $\chi_k$ in Eqs.(31) are conserved under the flow of the symmetry $U_l$. Lie derivatives of $\theta_k$’s add the differential $-d\theta_{lk}$ to the absolute invariants $\Theta_{lk}$ to produce new invariant one-forms which are also absolute invariants if $\theta_{lk}$ are conserved functions of $v$.

In addition to taking Lie derivative and interior product with infinitesimal symmetries, forming wedge products of invariants also produces new invariants. For example, $\theta_k \wedge \Theta_l$, $\theta_k \wedge \alpha_l$ and $\theta_k \wedge \Theta_l \wedge \theta_l$ are relative invariants while $\Theta_k \wedge \Theta_l = 0 \forall k,l = 1,2,3,...$

Thus, the above procedures seem to generate a plethora of invariant forms for the velocity field. However, we find some of them to be identical

$$\theta_{lk} = \theta_{ik} \mu = [\chi_k \xi_l + A_k \cdot \hat{u}_l] \mu$$  \hspace{1cm} (32)

$$\theta_k \wedge \Theta_l \wedge \theta_l = \chi_l (A_k \cdot \hat{u}_l) \mu$$  \hspace{1cm} (33)

and that they consist of functions available from more simple invariants. Moreover, we can get rid of the proliferation of invariants by imposing physical restrictions and utilizing the Lie algebraic structure of the infinitesimal symmetries.

We first observe that the invariants $\alpha_k$ consist of two parts which decouple the velocity field from its symmetries and hence are physically not much relevant. To see this we write $\alpha_k$ as the sum of two absolute invariants

$$\alpha_k = \xi_k \sigma - \hat{u}_k \cdot d\mathbf{x} \wedge d\mathbf{x} \wedge dt$$  \hspace{1cm} (34)

where $\sigma$ is the well-known invariant three-form that appears in the formulation of continuity equation and has no effect in the definition of $\Theta_k$’s.
The invariants $\alpha_{lk}$, on the other hand, have a similar structure as those of $\Theta_l$ involving appropriate differences of the symmetries $\hat{u}_l$ and $\hat{u}_k$ and thus can be taken to be equivalent. Likewise, due to the Lie bracket relations between symmetries we can regard $d\alpha_{lk}$ and $d\Theta_{lk}$ to be equivalent to (24) and (28), respectively.

5 General conservation laws

This analysis of invariants leads us to conclude that appropriate expressions of the laws of general vorticity and helicity conservations may be best given in terms of the basic hierarchy consisting of $\Theta_k$ and $\theta_k$. We shall formulate the conservation laws as a restatement of infinitesimal conditions on these invariants in terms of integration over submanifolds of the flow domain.

**Proposition 3** Let $S_t$ and $D_t$ be two and three dimensional regions in $R \times R^3$ advected by the graph $(t, g_t)$ of the flow of the velocity field $v$. Denote their smooth boundaries by $\partial S_t$ and $\partial D_t$. Then,

1. The exact two-forms $\Theta_k = -d\theta_k$ are the basic ingredients of general vorticity conservation law which can be expressed as

$$\frac{d}{dt} \int_{S_t} \Theta_k = -\frac{d}{dt} \oint_{\partial S_t} \theta_k = -\oint_{\partial S_t} d\chi_k \equiv 0, \quad k = 0, 1, 2, ...$$

where $\Theta_0$ is the symplectic two-form $\Omega$.

2. The relatively invariant three-forms $\theta_k \wedge d\theta_l$ define the general helicity conservation laws

$$\frac{d}{dt} \int_{D_t} \theta_k \wedge d\theta_l = \oint_{\partial D_t} \chi_k d\theta_l = -\oint_{\partial D_t} d\chi_k \wedge \theta_l = 0$$

if either $\hat{u}_l$ is tangent to the projection onto spatial domain of $\partial D_t$ or, $\chi_k \equiv i(\partial_t + v)(\theta_k) = constant$.

3. If, on the other hand, $\hat{u}_l$ is tangent to the surfaces $\chi_k = constant$ then the generalized helicity densities in $\theta_k \wedge d\theta_l$ are Lagrangian invariants, that is, conserved functions of the velocity field. Moreover, they belong to the Poisson bracket algebra (22) on $R^3$ of generalized potential vorticities.

For an equivalent formulation, using the one-forms $\theta_k$, of the first part of proposition(3) one replaces exactness and absolute invariance with Eqs.(30)
and relative invariance, respectively. In obtaining the second part, we used
the properties of the velocity field that characterize it as the generator of
volume preserving diffeomorphisms. Namely, it is divergence-free and is tan-
gential to the boundary of $D_t$. Eqs. (35) and the condition that the helicity
is conserved for $\chi_k = \text{constant}$ follow from the fact that the integral of exact
forms over boundaries is identically $(\partial \circ \partial \cdot \equiv)$ zero via Stokes’ theorem.
Thus, in the latter case, there will be no boundary terms contributed by $\hat{\mathbf{u}}_l$,
meaning that the three-forms are absolutely invariant and they result in the
last conclusion of the proposition. To complete the proof of proposition(3) we
shall supply, in the next section, coordinate expressions for these geometric
arguments as well as for the global formalism of conservation laws.

We end this section with a remark on the boundaries of the integration
domains $S_t$ and $D_t$ which are two and three dimensional subsets of the time-
extended space $R \times R^3$. For such decomposition of space-time, the one and
two dimensional boundaries $\partial S_t$ and $\partial D_t$ do not close up, but rather have
extent along a closed, finite interval of the time axis. Their projection
onto spatial domain are a closed curve and a closed surface in the usual
sense. For example, a circular helix whose finite-length axis is directed along
the time axis is the boundary of a two-surface and thus is a closed curve in
$R \times R^2$.

6 Demonstrations

We shall give explicit coordinate expressions for the general conservation laws
of proposition(3). In the particular case of the Euler flow we shall show that
the dynamical equations themselves, Bernoulli’s equation, the Kelvin’s cir-
culation theorem, the Helmholtz’ vorticity theorem, conservations of helicity
and potential vorticity can be obtained from the invariance conditions of the
basic hierarchy $\Theta_k, \theta_k$ and from the global formulations (35) and (36).

6.1 Bernoulli’s equation

For ideal fluids and for $k = 0$, we consider the invariance condition (31) of
the canonical one-form in Eq. (3) with $\psi$ as given in Eqs. (12). Using the
rotational form
\[
\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \mathbf{w} = -\nabla \left( \frac{P}{\rho} + \frac{1}{2} \mathbf{v}^2 \right)
\] (37)
of the Euler equations we obtain from Eq.(31) the Bernoulli’s equation
\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{v}^2 \right) + \mathbf{v} \cdot \nabla \left( \frac{P}{\rho} + \frac{1}{2} \mathbf{v}^2 \right) = 0
\] (38)
where \( P + \rho v^2/2 \) is the total (or stagnation) pressure. This says that the non-invariance of the kinetic energy under the flow of velocity field is due to the existence of pressure.

Although, Eq.(38) is a direct consequence of (37), our purpose to include this and the next example is to demonstrate the interplay between the present geometric framework for the Lagrangian description and the Eulerian dynamical equations.

### 6.2 The Euler equations

The hierarchy of Eqs.(35) contains, in particular, the dynamical equations for the Euler flow. This follows from the relative invariance of the one-forms \( \theta_k \). Eq.(31) for \( k = 0 \) can be solved for the Lie derivative
\[
\mathcal{L}_{\partial_t + \mathbf{v}} (\mathbf{v} \cdot d\mathbf{x}) = \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{v}^2 \right) dt + \nabla \left( \frac{1}{2} \mathbf{v}^2 - \frac{P}{\rho} \right) \cdot d\mathbf{x}
\] (39)
where we used Bernoulli’s equation to simplify the right hand side. Note that the Lie derivative is the one on \( \mathbb{R} \times \mathbb{R}^3 \). In terms of the differential operators on \( \mathbb{R}^3 \) this expression reduces to
\[
\frac{\partial}{\partial t} (\mathbf{v} \cdot d\mathbf{x}) + \mathcal{L}_\mathbf{v} (\mathbf{v} \cdot d\mathbf{x}) = d \left( \frac{1}{2} \mathbf{v}^2 - \frac{P}{\rho} \right)
\] (40)
which is the invariant form of the Euler equations of ideal fluid. It is this form that can be generalized for flows on general Riemannian manifolds of arbitrary dimensions [3].

### 6.3 Helmholtz’ vorticity theorems

We first remark that in the integration of the two-forms \( \Theta_k \) over two-surfaces advected by the fluid motion, the term \( (\hat{u}_k \times \mathbf{v}) \cdot d\mathbf{x} \wedge dt \) has no contribution
because its pull-back to Lagrangian coordinates by the solution $g_t$ of (1) vanishes. Then, (35) becomes an expression for the general form of Helmholtz’ vorticity theorem

$$\frac{d}{dt} \int_{S_t} \Theta_k = \frac{d}{dt} \int_{S_t} \hat{u}_k \cdot d\mathbf{x} \wedge d\mathbf{x} = 0 \quad (41)$$

which reduces for $k = 0$, $\hat{u}_0 = \mathbf{w} = \nabla \times \mathbf{u}$ to the conservation of vorticity flux and, for $\hat{u}_0 = \mathbf{B}$ to the Alfvén’s theorem of dynamo theory.

### 6.4 Kelvin’s circulation theorems

For two-surfaces $S_t$ with smooth boundary $\partial S_t$ the invariance of the two-forms $\Theta_k$ gives

$$0 = \frac{d}{dt} \oint_{\partial S_t} \theta_k = \frac{d}{dt} \oint_{\partial S_t} A_k \cdot d\mathbf{x} \quad (42)$$

via Stokes’ theorem and the generic form of symmetry generators as curl vectors. This is a generalization of the Kelvin’s circulation theorem for which the usual form is obtained when $k = 0$. It is the conservation of circulation of the velocity field $A_0 = \mathbf{v}$ for the Euler flow and of the vector potential $A_0 = \mathbf{A}$ for magnetization problems.

### 6.5 General vorticity conservations

An immediate generalization to the infinite dimensional left Lie algebra of symmetries of the well-known equivalence of the Helmholtz’ and the Kelvin’s theorems follows directly from Eqs. (35), (41) and (42). For two-surfaces $S_t = g_t(S)$ with smooth boundary $\partial S_t$ we obtain

$$\int_S g_t^* (\hat{u}_k \cdot d\mathbf{x} \wedge d\mathbf{x}) = \int_{S_t} \hat{u}_k \cdot d\mathbf{x} \wedge d\mathbf{x} = \int_{\partial S_t} A_k \cdot d\mathbf{x} = \text{constant} \quad (43)$$

where $g_t$ is a flow of the velocity field. The constancy of the first integral is the preservation of coadjoint orbits of $\Theta_k$’s, of the second is the Helmholtz’ theorem on conservation of flux of flow lines of the infinitesimal symmetry $\hat{u}_k$, and of the third is the Kelvin’s theorem on conservation of circulation of vector potential $A_k$ for $\hat{u}_k$. The fact that, for fixed $\hat{u}_k$, these expressions hold for all solutions $g_t$ of the velocity field is a manifestation of the right invariance on the cotangent bundle of the group of volume preserving diffeomorphisms.
6.6 General helicity conservations

Eulerian invariants are conservation laws of dynamical equations (1) which are divergence expressions of the form

\[ \frac{\partial T}{\partial t} + \nabla \cdot P = 0 \]  

where the conserved density \( T \) and the flux \( P \) are functions of \( t, x \), the Eulerian fields and their derivatives. Observing that Eq. (44) can be expressed as the closure of the three-form

\[ -T dx \wedge dy \wedge dz + P \cdot dx \wedge dx \wedge dt \]  

we shall derive the Eulerian conservation laws as secondary invariants obtained from the basic hierarchy \((23)\) and \((29)\). These are the relative invariants \( \theta_k \wedge d\theta_l \) which are closed via Eqs.\((18)\) and \((30)\). The corresponding conservation laws are expressed by Eqs.\((36)\) of proposition\((3)\).

The case \( k = l = 0 \) is exceptional because \( \Omega \) is non-degenerate. In this case, the identity

\[ d(\theta \wedge \Omega) + \Omega \wedge \Omega = 0 \]  

gives the conservation of usual (magnetic) helicity

\[ \frac{\partial}{\partial t} (A \cdot B) + \nabla \cdot ((A \cdot B)v - (\psi + \varphi + v \cdot A)B) = 0 \]  

at which a hierarchy of others is anchored. For \( k, l \neq 0 \) we find

\[ \frac{\partial}{\partial t} (A_k \cdot \nabla \times A_l) + \nabla \cdot (A_k \times (v \times (\nabla \times A_l)) - \psi_k \nabla \times A_l) = 0 \]  

associated with a pair of symmetries. Using the identity

\[ A_k \times (v \times \hat{u}_l) = (\hat{u}_l \cdot A_k)v - (v \cdot A_k)\hat{u}_l \]  

and the divergence-free properties of \( v, \hat{u}_l \) we can write Eq. \((18)\) as

\[ \frac{\partial}{\partial t} (A_k \cdot \hat{u}_l) + v \cdot \nabla (A_k \cdot \hat{u}_l) - \hat{u}_l \cdot \nabla \chi_k = 0 \]
from which one can immediately conclude the results in the second and the third items of proposition (3). Thus, the functions

\[ A \cdot \nabla \times A, \quad A \cdot \nabla \varphi \times \nabla h, \quad \varphi B \cdot \nabla h, \]

\[(\hat{u}_1 \times \nabla \times)^{k-1} A \cdot \nabla \times (\hat{u}_1 \times \nabla \times)^{l-1} A, \quad k, l = 2, 3, ... \quad (51)\]

are the first few and the generic members of the conserved densities associated to the general helicity conservation \([30]\). The usual helicity and the potential vorticity are included as the first and the third elements of this hierarchy.

The conservation of potential vorticity can also be obtained from the absolute invariance of the four-form \(\Omega \wedge \Omega\) by integration over a region in \(R \times R^3\). This, in turn, implies the constancy of the integral of the three-form \(\theta \wedge \Omega\) over closed three dimensional domains in \(R \times R^3\). For \(k = l = 1, 2, 3, ...\) the densities \((51)\) vanish because the symmetries \(\hat{u}_k\) have Clebsch representations.

### 6.7 Topological interpretations

In the particular case of \(l = 0\) and \(k = 0, 1, 2, ...\), the hierarchy \(\hat{u}_0 \cdot A_k\) of invariants is related to the linking number of trajectories of vorticity \(\hat{u}_0 = w\) or magnetic \(\hat{u}_0 = B\) fields with those of symmetries \([9]-[12]\). For \(k = 0\), the integral of this function coincides either with the self-linking number of vorticity or magnetic field, which is also called helicity, or, with the linking number of \(w\) and \(B\) expressed by the integrals of \(w \cdot A\) or \(v \cdot B\). This latter result, when first proven with the second integrand in \([12]\) for ideal magnetohydrodynamic equations, was interpreted to be unexpected because \(w\) is not frozen. Here, we see that it is sufficient to have \(B\), and in general \(\hat{u}_k\), to be frozen.

The closed two-forms \(\Theta_k\) can be associated with two-dimensional foliations of four-space by the integral curves of \(U_k\) and \(V\). The conditions \(\Theta_k \wedge \Theta_l = 0\) means that intersection of leaves of any two such foliations is a one-dimensional foliation. The leaves of the latter are trajectories of the suspended velocity field. Moreover, for any \(k, l\), \((V, U_k, U_l)\) form a three-dimensional foliation. In this case, the integrals of the relative invariants \(\theta_l \wedge \theta_k \wedge \Theta_k\) are shown to be related to the average linking number of the foliation of \(\Theta_k\) with the vector field \(U_{kl}\) defined by \(i(U_{kl})(\mu) = d(\theta_k \wedge \theta_l)\) \([13],[14]\). The characteristic form of this is the vector field \(\hat{u}_{kl} = \chi_k \hat{u}_l - \chi_l \hat{u}_k\) on \(R^3\).
We conclude from these examples that it is the hierarchy of secondary and higher order invariants of particle relabelling symmetry which contains topological informations about the flow domain.

6.8 Lagrangian invariants

Lagrangian invariants are functions which are conserved under the flow of the velocity field \( \dot{u}_k \cdot \dot{A}_l \), \( k, l \neq 0, 1 \) in the Lagrangian description when the condition \( \dot{u}_k \cdot \nabla \chi_l = 0 \) is satisfied. For the special case of \( l = 0 \) for which \( \chi_0 = \psi + \varphi + v \cdot A \), this can be verified using Eqs.(10) and the identity \( v \times (\nabla \times A) + (v \cdot \nabla)A = v^i \nabla A_i \). Similarly, one obtains from Eq.(6) and the symmetry condition for \( \dot{u}_k \) that \( \dot{u}_k \cdot v \) is a Lagrangian invariant if

\[
\dot{u}_k \cdot (F + \frac{1}{2} \nabla v^2) = 0, \quad k = 0, 1, 2, ...
\]

is satisfied by a general force field \( F \). It can be checked that for the Euler equations of ideal fluid this coincides with the general criterion.

The hierarchy of Lagrangian invariants includes the helicities \( v \cdot w, A \cdot B \), the potential vorticities \( w \cdot \nabla h, B \cdot \nabla h \) as well as the invariants \( w \cdot A \) and \( v \cdot B \) of magnetization problems and, furthermore \( v \cdot \nabla \varphi \times \nabla h, v \cdot \nabla \varphi \times \nabla h \) which are obtained for the first two values of \( k, l \). It can be seen using the Hamiltonian form with (22) of \( \dot{u}_k \)'s that the Lagrangian invariants \( \dot{u}_k \cdot A_l \) belong to the Poisson bracket algebra (22) of generalized potential vorticities \( h_k \) and can be obtained from the Poisson brackets of functions \( h_k \) and \( h_l \).

An immediate application of these conserved functions of the velocity field may be the construction of volume preserving diffeomorphisms which are also automorphisms of the symplectic two-form (5). This can be achieved by choosing a set \( y^i(t, x), i = 1, 2, 3 \) of functionally independent Lagrangian invariants which further satisfy the condition

\[
\det \left( \frac{\partial y^i}{\partial x^j} \right) = \nabla y^1 \cdot \nabla y^2 \times \nabla y^3 = 1
\]

for volume preservation. In particular, if we let \( y^1 = \varphi \) then \( \nabla y^2 \times \nabla y^3 \) must be \( B \) (recall that we set \( \rho \varphi = -B(\varphi) = 1 \)). Thus, the other two Lagrangian invariants \( y^2 \) and \( y^3 \) are the Clebsch variables \( p, q \) for \( A \) (cf. Eqs.(10)). From
proposition (1) we conclude that this particular diffeomorphism is the one which brings the symplectic two-form $\Omega$ into canonical form. It follows from Eq. (53) and (22) that
\[ \{q, p\}_\varphi = 1. \]

## 7 Discussions and conclusions

We presented a geometric approach to the explicit construction of symmetries starting from an initial one and utilizing a symplectic structure both of which are implicit in the Eulerian equations constraining the velocity field. We may conclude that, as the simplicity and the Lie-Poisson structure of Eulerian description derives from the particle relabelling symmetry [2]-[4], the Eulerian description, in its turn, enables us to construct the Lie algebra of these symmetries.

Therefore, depending on the structure of the Eulerian dynamical equations, the present construction may be generalized to other hydrodynamic systems. We demonstrated that the Euler flow in arbitrary Riemannian manifolds allows this sort of generalizations. A treatment for compressible flows may be based on the recognition that the so-called exponential vector field $\exp(\int \nabla \cdot \mathbf{v} \, dt) \mathbf{w}$ is a time-dependent infinitesimal symmetry of the velocity field. We refer to Refs. [12] and [14] for other examples of hydrodynamic systems which admit a vectorial frozen-in field and hence can be analysed with the present geometric framework.

It would be reasonable to compare the overall picture for invariants of three dimensional flow with that of two dimensional motion because, although, there is no essential difference in their group theoretical descriptions, they have been observed to have some different quantitative aspects [14]-[16], [12], [2], [3]. We showed that some properties of two dimensional flows such as the representation of area preserving diffeomorphisms as Hamiltonian vector fields and the existence of infinitely many enstrophy type integrals, can be realized, with appropriate modifications of geometric tools, for the three dimensional flows as well.

Moreover, due to dimensional reasons we obtained a much richer structure of invariants for the present case. Namely, the geometric framework at our disposal led us to discriminate the invariants into three classes each having a certain degree of relation to the particle relabelling symmetry. Thus, for three
dimensional flows, in addition to general vorticity conservation, we can speak of a general helicity conservation and moreover, under certain conditions, of general Lagrangian invariants.

The geometric character and the algebraic consequences of our construction and classification of symmetries and invariants distinguish the present work from the approaches that employ the analysis of defining equations as the main tool [5],[17],[18]. Such constructions require that the algebraic structure of the solution set consisting of symmetries and invariants be separately treated. Instead, we gave a recursive construction of symmetries via the Jacobi identity and, moreover, realized them as the Hamiltonian automorphisms of the symplectic two-form. This ensures that the infinitesimal symmetries thus obtained constitute a Lie algebra because, although not every infinite set of vector fields form a Lie algebra, Hamiltonian vector fields do so.

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