Hall-type algebras for categorical Donaldson–Thomas theories on local surfaces

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Abstract
We show that the categorified cohomological Hall algebra structures on surfaces constructed by Porta–Sala descend to those on Donaldson–Thomas categories on local surfaces introduced in the author’s previous paper. A similar argument also shows that Pandharipande–Thomas categories on local surfaces admit actions of categorified COHA for zero dimensional sheaves on surfaces. We also construct annihilator actions of its simple operators, and show that their commutator in the K-theory satisfies the relation similar to the one of Weyl algebras. This result may be regarded as a categorification of Weyl algebra action on homologies of Hilbert schemes of points on locally planar curves due to Rennemo, which is relevant for Gopakumar–Vafa formula of generating series of PT invariants.

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1 Introduction

1.1 Motivation and background

The Donaldson–Thomas (DT for short) invariants virtually count stable coherent sheaves on Calabi–Yau (CY for short) threefolds [26], and play important roles in the recent study of curve counting theories and mathematical physics. The original DT invariants are integer valued invariants, which coincide with weighted Euler characteristics of Behrend functions [4] on moduli spaces of stable sheaves on CY threefolds. The Behrend functions are point-wise Euler characteristics of vanishing cycles, so locally admit natural refinements such as motivic vanishing cycles, perverse sheaves of vanishing cycles. Based on this fact, Kontsevich-Soibelman [15,16] proposed several refinements of DT invariants, e.g. motivic DT invariants, cohomological DT invariants, whose foundations are now available in [3,6]. Among them, cohomological DT invariants are expected to carry an algebra structure, which globalize critical cohomological Hall algebras for quivers with super-potentials constructed in [8,16].

In the author’s previous paper [32], we proposed further refinement of DT invariants to triangulated (or dg) categories, called categorical DT theories (or DT categories for short). The DT categories should be constructed as gluing of locally defined categories of matrix factorizations, but their general construction is still beyond our scope (see also
see [13, (J)], [31, Section 6.1]). In [32] we constructed $\mathbb{C}^*$-equivariant DT categories in the special case of CY threefolds, called local surfaces, i.e. the total spaces of canonical line bundles on surfaces. In this case, they are defined to be the Verdier quotients of derived categories of coherent sheaves on derived moduli stacks of coherent sheaves on surfaces, by the subcategory of objects whose singular supports (in the sense of Arinkin-Gaitsgory [1]) are contained in the unstable loci. Via Koszul duality, the DT category is locally equivalent to the category of $\mathbb{C}^*$-equivariant matrix factorizations. The DT categories are expected to recover the cohomological DT invariants on local surfaces by taking the periodic cyclic homologies, so their Euler characteristics should recover the original DT invariants.

The purpose of this paper is to construct Hall-type algebra structures on DT categories for local surfaces. Our construction is induced by the categorification of cohomological Hall algebra (COHA for short) structures on surfaces constructed by Porta–Sala [21]. A new point in this paper is that their product structure is compatible with singular supports in a certain sense, so that it descends to the product structure on DT categories. By taking the associated product on K-theory, we obtain a globalization of K-theoretic Hall algebras for quivers with super-potentials constructed by Păduaruiu [20]. Moreover by taking periodic cyclic homologies, we expect that our construction gives a critical COHA restricted to the semistable locus, giving a globalization of critical COHA defined for quivers with super-potentials [8,16].

One of our motivations of this study is to construct some algebra actions on Pandharipande–Thomas (PT for short) categories for local surfaces, which are relevant for the Gopakumar–Vafa (GV for short) formula of the generating series of PT invariants. By a similar argument as above, we also show that PT categories for local surfaces admit actions of DT categories of zero dimensional sheaves. We also construct annihilator actions of its simple operators, and show that their commutator in the K-theory satisfies the relation similar to the one of Weyl algebras. This result may be regarded as a categorification of Weyl algebra action on homologies of Hilbert schemes of points on locally planar curves due to Rennemo, where he derived GV formula of the generating series of Euler characteristics of these Hilbert schemes. Therefore we expect that this work may be relevant for categorical understanding of conjectural PT/GV formula, which will be pursued in a future work.

1.2 Categorical DT theory for local surfaces

Let $S$ be a smooth projective surface over $\mathbb{C}$. We consider the total space of its canonical line bundle, which is a non-compact CY threefold

$$\pi : X := \text{Tot}_S(\omega_S) \to S.$$  \hspace{1cm} (1.1)

Let $N(S)$ be the numerical Grothendieck group of $S$ and take $v \in N(S)$. For a choice of a stability condition $\sigma$ on the abelian category of compactly supported coherent sheaves on $X$ (e.g. Gieseker stability condition), we have the moduli stack $\mathcal{M}^{ss}_X(v)$ of $\sigma$-semistable sheaves $E$ on $X$ with $[\pi_\ast E] = v$, together with the commutative
Here $\mathcal{M}_S(v)$ is the derived moduli stack of coherent sheaves $F$ on $S$ with $[F] = v$, $t_0(-)$ means the classical truncation with $\mathcal{M}_S(v) = t_0(\mathcal{M}_S(v))$, and $\Omega\mathcal{M}_S(v)[-1]$ is the $(-1)$-shifted cotangent stack over $\mathcal{M}_S(v)$. The top left horizontal arrow is an open immersion and the right horizontal arrows are closed immersions.

In [32], the $\mathbb{C}^*$-equivariant categorical DT theory (or simply DT category) associated with the moduli stack $\mathcal{M}_X^{\sigma\text{-ss}}(v)$ is defined to be the Verdier quotient (see Definition 3.3)

$$\text{DT}^{\mathbb{C}^*}(\mathcal{M}_X^{\sigma\text{-ss}}(v)) := \mathcal{D}^b_{\text{coh}}(\mathcal{M}_S(v)^{\text{fin}})/\mathcal{C}_{Z_{\sigma\text{-us}}(v)^{\text{fin}}}.$$  \hspace{1cm} (1.3)

Here $\mathcal{M}_S(v)^{\text{fin}}$ is a derived open substack of $\mathcal{M}_S(v)$ which is of finite type and contains the image of the map $\pi^*$ in (1.1). Furthermore the subcategory

$$\mathcal{C}_{Z_{\sigma\text{-us}}(v)^{\text{fin}}} \subset \mathcal{D}^b_{\text{coh}}(\mathcal{M}_S(v)^{\text{fin}})$$

consists of objects $E \in \mathcal{D}^b_{\text{coh}}(\mathcal{M}_S(v)^{\text{fin}})$ satisfying that

$$\text{Supp}^S(E) \subset Z_{\sigma\text{-us}}(v)^{\text{fin}} := t_0(\Omega\mathcal{M}_S(v)^{\text{fin}}[-1])\setminus\mathcal{M}_X^{\sigma\text{-ss}}(v).$$

Here $\text{Supp}^S(E)$ is the singular support of $E$ introduced by Arinkin-Gaitsgory [1], following an earlier work by Benson-Iyengar-Krause [5]. Via Koszul duality, the category (1.3) is locally equivalent to the category of $\mathbb{C}^*$-equivariant matrix factorizations of functions, whose critical loci locally describe $\mathcal{M}_X^{\sigma\text{-ss}}(v)$. If there is no strictly $\sigma$-semistable sheaves, the $\mathbb{C}^*$-rigidified version of the triangulated category (1.3) is expected to recover the cohomological Donaldson–Thomas invariants associated with (1.2) by taking its periodic cyclic homology. A key point of the construction (1.3) is that, we capture (un)stable loci on moduli stacks of sheaves on threefolds via derived structures of those on surfaces using singular supports. In other words, we define the category (1.3) as if we have the ‘dimension reduction’ for DT categories, similarly to the dimension reduction for critical COHA for some quivers with super-potentials coming from preprojective algebras proved in [25,36].

1.3 Categorified COHA

Our first result is to show the existence of Hall-type algebra structure on the DT categories (1.3). Let us recall Porta–Sala’s construction [21] of categorified COHA
for surfaces. We take $v_\bullet = (v_1, v_2, v_3) \in N(S)^{\times 3}$ with $v_2 = v_1 + v_3$. Then the functor

$$D^b_{\text{coh}}(\mathcal{M}_S(v_1)) \times D^b_{\text{coh}}(\mathcal{M}_S(v_3)) \to D^b_{\text{coh}}(\mathcal{M}_S(v_2)) \quad (1.4)$$

is constructed in [21] by using the following Hall-type diagram

$$\begin{diagram}
  \mathcal{M}_S^{\text{ext}}(v_\bullet) & \rTo{ev_2} & \mathcal{M}_S(v_2) \\
\downTo{(ev_1, ev_3)} & & \downTo{} \\
\mathcal{M}_S(v_1) \times \mathcal{M}_S(v_3).
\end{diagram} \quad (1.5)$$

Here $\mathcal{M}_S^{\text{ext}}(v_\bullet)$ is the derived moduli stack of short exact sequences of coherent sheaves on $S$

$$0 \to F_1 \to F_2 \to F_3 \to 0, \quad [F_i] = v_i, \quad \text{ev}_i(F_\bullet) = F_i.$$

The functor (1.4) is defined by pull-back/push-forward of the diagram (1.5), and regarded as a categorification of two dimensional cohomological Hall algebra by Kapranov-Vasserot [17].

We show that the product functor (1.4) descends to the functor on DT categories. In order to state the statement, we fix a polynomial $\chi \in \mathbb{Q}[m]$ and denote by $N(S)_\chi \subset N(S)$ the subgroup of $v \in N(S)$ whose reduced Hilbert polynomial is equal to $\chi$, or $v = 0$. Our first result is as follows.

**Theorem 1.1** (Theorem 3.5) For $(v_1, v_2, v_3) \in N(S)_\chi$ with $v_2 = v_1 + v_3$, the functor (1.4) descends to the functor

$$\text{DT}^{C^*} \left( \mathcal{M}_{\chi}^{\sigma-\text{ss}}(v_1) \right) \times \text{DT}^{C^*} \left( \mathcal{M}_{\chi}^{\sigma-\text{ss}}(v_3) \right) \to \text{DT}^{C^*} \left( \mathcal{M}_{\chi}^{\sigma-\text{ss}}(v_2) \right).$$

In particular, the direct sum of the $K$-theory

$$\bigoplus_{v \in N(S)_\chi} K \left( \text{DT}^{C^*} \left( \mathcal{M}_{\chi}^{\sigma-\text{ss}}(v) \right) \right) \quad (1.6)$$

has a structure of an associative algebra.

We note that if any $\sigma$-semistable sheaf on $X$ is push-forward to a $\sigma$-semistable sheaf on $S$, then we have

$$\bigoplus_{v \in N(S)_\chi} K \left( \text{DT}^{C^*} \left( \mathcal{M}_{\chi}^{\sigma-\text{ss}}(v) \right) \right) = \bigoplus_{v \in N(S)_\chi} K \left( \mathcal{M}_S^{\sigma-\text{ss}}(v) \right)$$

1 More precisely Porta–Sala’s construction work in a dg-categorical setting, where not only the stacks of exact sequences but also higher parts of the Waldhausen construction are required in order to control the higher associativity. Porta and Sala pointed out that the same would apply to our situation, see Remark 3.9.
and the algebra structure on it is essentially the same one in [21]. For example as we will see in (1.9), this happens when \( \chi \equiv 1 \). However this is not the case in general, and our construction of the algebra (1.6) is new when the push-forward does not preserve the \( \sigma \)-semistability.

### 1.4 Categorical PT theories on local surfaces

By definition, a PT stable pair on \( X \) consists of a pair

\[
(F, s), \quad s: O_X \rightarrow F
\]

where \( F \) is a compactly supported pure one dimensional coherent sheaf on \( X \) and \( s \) is surjective in dimension one. For \( \beta \in \text{NS}(S) \) and \( n \in \mathbb{Z} \), we have the moduli space of Pandharipande–Thomas stable pairs

\[
P_n(X, \beta)
\]

which parametrizes pairs (1.7) satisfying \( \pi_*[F] = \beta \) and \( \chi(F) = n \). The DT type invariants defined from \( P_n(X, \beta) \), called PT invariants, are defined in [22] and play an important role in the recent study of curve counting invariants on CY threefolds (see [7,23,33,35] for example).

The moduli space \( P_n(X, \beta) \) fits into the commutative diagram

\[
P_n(X, \beta) \xrightarrow{\pi_*} t_0(\Omega_{\mathcal{M}^\dagger_S(\beta, n)}[1]) \xrightarrow{\chi} \Omega_{\mathcal{M}^\dagger_S(\beta, n)}[1]
\]

Here \( \mathcal{M}^\dagger_S(\beta, n) \) is the derived moduli stack of pairs \( (F, \xi) \), where \( F \) is a one dimensional coherent sheaf on \( S \) satisfying \( [F] = \beta \), \( \chi(F) = n \), and \( \xi: O_S \rightarrow F \) is a morphism. Similarly to (1.3), we also introduced the notion of \( \mathbb{C}^* \)-equivariant categorical PT theory in [32] by the following (see Definition 4.9 for details)

\[
\mathcal{DT}^{\mathbb{C}^*}(P_n(X, \beta)) := D^b_{\text{coh}}(\mathcal{M}_S^\dagger(\beta, n)_{\text{fin}})/\mathcal{C}_{\mathcal{Z}_{F\text{-us}}(\beta, n)_{\text{fin}}}.
\]

On the other hand if we take \( \chi \) to be the constant polynomial 1, then \( N(S)_{\chi} \) consists of \( m[pt] \) where \( [pt] \) is the numerical class of \( O_x \) for \( x \in S \). Then (1.6) is

\[
\bigoplus_{m \geq 0} K(\mathcal{DT}^{\mathbb{C}^*}(\mathcal{M}_S^{\sigma-ss}(m[pt]))) = \bigoplus_{m \geq 0} K(\mathcal{M}_S(m[pt]))
\]

where \( \mathcal{M}_S(m[pt]) \) is the moduli stack of zero dimensional sheaves on \( S \) with length \( m \). The above algebra is nothing but the K-theoretic Hall algebra for zero dimensional sheaves on the surface \( S \) constructed by Zhao [37], and is related to the shuffle algebra as proved in loc. cit. We show that the above algebra acts on PT categories.
Theorem 1.2 (Theorem 4.10) There exist functors

\[ \mathcal{DT}_C^* (P_n(X, \beta)) \times \mathcal{DT}_C^* (\mathcal{M}_X^{ss}(m[pt])) \to \mathcal{DT}_C^* (P_{n+m}(X, \beta)) \quad (1.10) \]

which induce the right action of the K-theoretic Hall algebra of zero dimensional sheaves (1.9) to the following direct sum

\[ \bigoplus_{n \in \mathbb{Z}} K(\mathcal{DT}_C^* (P_n(X, \beta))). \quad (1.11) \]

In [32], we also defined MNOP categories associated with moduli spaces of one or zero dimensional subschemes in \( X \). In Sect. 7, we also show that the algebra (1.9) acts on MNOP categories from the left (not right). Since the PT moduli spaces and MNOP moduli spaces are related by wall-crossing, the fact that the direction of the action changes may be an incarnation of wall-crossing phenomena in terms of actions of Hall-type algebras.

We will also consider the algebra (1.6) for one dimensional semistable sheaves on \( X \), and show that it acts on the left/right on DT type categories associated with stable D0–D2–D6 bound states. Similarly to above, the direction of the action of DT type categories of one dimensional semistable sheaves also changes when we crosses the wall of weak stability conditions on the category of D0–D2–D6 bound states. Since the wall-crossing in D0–D2–D6 bound states is relevant for GV formula of generating series of PT invariants (see [35]), the above observation may be relevant for categorical understanding of GV formula. This direction of research will be pursued in future.

1.5 Commutator relations of Hecke actions

Since we have \( \mathcal{M}_S([pt]) = S \times B\mathbb{C}^* \), we have the decomposition

\[ K(\mathcal{M}_S([pt])) = \bigoplus_{k \in \mathbb{Z}} K(S)_k \]

where \( K(S)_k \) is the \( \mathbb{C}^* \)-weight \( k \)-part, which is isomorphic to \( K(S) \). Therefore the action (1.10) on the K-theory for \( m = 1 \) and weight \( k \)-part induces the creation operators

\[ \mu^+_{E,k} : K(\mathcal{DT}_C^* (P_n(X, \beta))) \to K(\mathcal{DT}_C^* (P_{n+1}(X, \beta))) \]

for each \( E \in K(S) \). We also construct annihilator operators

\[ \mu^-_{E,k} : K(\mathcal{DT}_C^* (P_{n+1}(X, \beta))) \to K(\mathcal{DT}_C^* (P_n(X, \beta))) \]
and form the following maps of degree ±1

\[ \mu_{E}^{\pm}(z) := \sum_{k \in \mathbb{Z}} \frac{\mathcal{H}_{E,k}^{\pm}}{z} : \bigoplus_{n \in \mathbb{Z}} K(DT^{C^*}(P_{n}(X, \beta))) \to \bigoplus_{n \in \mathbb{Z}} K(DT^{C^*}(P_{n}(X, \beta)))\{z\}. \]

We then compute the commutator relation of the above operators. For a fixed \( \beta \), we set

\[ \mathcal{M}_{S}^{\dagger}(\beta) := \coprod_{n \in \mathbb{Z}} \mathcal{M}_{S}^{\dagger}(\beta, n) \]

and we denote by

\[ (\mathcal{O}_{S} \times \mathcal{M}_{S}^{\dagger}(\beta) \to \mathcal{F}(\beta)) \in \text{Perf}(S \times \mathcal{M}_{S}^{\dagger}(\beta)) \]

the universal pair. Let \( p_{\mathfrak{M}} : S \times \mathcal{M}_{S}^{\dagger}(\beta) \to \mathcal{M}_{S}^{\dagger}(\beta) \) be the projection. We have the following result.

**Theorem 1.3** (Theorem 6.8) *For a K-group element (−) represented by perfect complexes, we have the following commutator relation*

\[ [\mu_{E_{1}}^{+}(z), \mu_{E_{2}}^{-}(w)](−) = (−) \otimes p_{\mathfrak{M}*}((\mathcal{E}_{1} \otimes \mathcal{E}_{2} \otimes \omega_{S}) \boxtimes \frac{h^{+}(z) - h^{-}(w)}{q_{S} - 1} \delta \left(\frac{w}{z}\right)). \]

(1.12)

Here \( \delta(x) = \sum_{k \in \mathbb{Z}} x^{k}, h^{\pm}(z) \) is the expansion of the following rational function at \( z = \infty \) and \( z = 0 \)

\[ h(z) = \left(1 - \frac{1}{z}\right)^{\mathcal{F}(\beta)} \delta \left(\frac{q_{S}^{-1} - 1}{z}\right). \]

Also \( q_{S} \) is the pull-back of the class \([\omega_{S}] \in K(S)\).

The proof for Theorem 1.3 much relies on arguments of Negut [19], where he studied Hecke operators for K-groups of moduli spaces of stable sheaves on surfaces, and computed several commutator relations. Contrary to the case of [19], our moduli stacks may be singular so we can prove the identity (1.12) only for elements represented by a perfect complex. Also it seems likely that, following the arguments of [19], we can compute more commutator relations such as \([\mu_{E_{1}}^{+}(z), \mu_{E_{2}}^{+}(w)], [\mu_{E_{1}}^{-}(z), \mu_{E_{2}}^{-}(−)]\). We will not pursue these computations in this paper.

As a corollary of Theorem 1.3, we have the following relation for \( k = 0 \) (see Corollary 6.9)

\[ [\mu_{E_{1},0}^{+}, \mu_{E_{2},0}^{-}](−) = (−) \otimes p_{\mathfrak{M}*}((\mathcal{E}_{1} \otimes \mathcal{E}_{2} \otimes \omega_{S}) \boxtimes \mathcal{F}(\beta)^{\vee}). \]

(1.13)
The existence of operators $\mu_{E,0}^\pm$ satisfying the relation (1.13) may be regarded as a categorification of Weyl algebra action on homologies of Hilbert schemes of points $C[[n]]$ on locally planar curves $C$ constructed by Rennemo [24] (see Remark 6.11), which is an analogy of Grojnowski and Nakajima’s Heisenberg action for homologies of Hilbert schemes of points on surfaces [11,18]. The moduli space of stable pairs $P_n(X, \beta)$ is much more generalized version of $C[[n]]$, so we expect a similar Weyl algebra action on the vanishing cycle cohomology of $P_n(X, \beta)$. The operators $\mu_{E}^\pm$ together with the relation (1.13) gives a further categorification of such expected Weyl algebra action. Using the Weyl-algebra action, the GV form of the generating series of $\chi(C[[n]])$ is derived in [24], see [24, Section 1.4] for details. We expect that the result of Theorem 1.3 is relevant for the categorification of GV form of PT categories, which will be pursued in a future work.

1.6 Notation and convention

In this paper, all the schemes or derived stacks are defined over $\mathbb{C}$. For a scheme or a derived stack $A$ and a quasi-coherent sheaf $F$ on it, we denote by $S(F) = \text{Sym}_A^*(F)$ its symmetric product. For a derived stack $\mathcal{M}$, we always denote by $t_0(\mathcal{M})$ its classical truncation. For a triangulated category $D$ and a set of objects $S \subset D$, we denote by $\langle S \rangle_{\text{ext}}$ the extension closure, i.e. the smallest extension closed subcategory which contains $S$.

2 Singular supports of coherent sheaves and Fourier–Mukai transforms

The notion of singular supports for (ind) coherent sheaves on quasi-smooth derived stacks was developed by Arinkin-Gaitsgory [1], in order to formulate a categorical geometric Langlands conjecture. In the author’s previous paper [32], we used singular supports to capture (un)stable sheaves on threefolds from the derived geometry on moduli stacks of sheaves the surface. In this section, we review the theory of singular supports and see how they interact with Fourier-Mukai transforms.

2.1 Singular supports of coherent sheaves

Let $A$ be an affine $\mathbb{C}$-scheme and $V \to A$ a vector bundle on it. For a section $s : A \to V$, we consider the affine derived scheme $\mathfrak{U}$ given by the derived zero locus of $s$

$$\mathfrak{U} = \text{Spec} \mathcal{R}(V \to A, s)$$

(2.1)

where $\mathcal{R}(V \to A, s)$ is the Koszul complex

$$\mathcal{R}(V \to A, s) := \left( \cdots \to V^\vee \to V^\vee \to \mathcal{O}_A \right).$$
The classical truncation of $\mathfrak{U}$ is the closed subscheme of $A$ given by

$$\mathcal{U} := \mathfrak{t}_0(\mathfrak{U}) = (s = 0) \subset A.$$ 

On the other hand, let $w : V^\vee \to \mathbb{C}$ be the function defined by

$$w(x, v) = \langle s(x), v \rangle, \ x \in A, \ v \in V^\vee|_x.$$ 

Then its critical locus is the classical truncation of $(-1)$-shifted cotangent scheme over $\mathfrak{U}$ (or called dual obstruction cone, see [14])

$$\text{Crit}(w) = \mathfrak{t}_0(\Omega_\mathfrak{U}[-1]) = \text{Spec} \mathcal{S}(\mathcal{H}^1(\mathbb{T}_\mathfrak{U})).$$

Below we take the fiberwise weight two $\mathbb{C}^*$-action on the total space of the bundle $V^\vee \to A$, so that $w$ is of weight two.

Let $\text{HH}^*\mathfrak{U}$ be the Hochschild cohomology

$$\text{HH}^*\mathfrak{U} := \text{Hom}^*_{\mathfrak{U} \times \mathfrak{U}}(\Delta_* \mathcal{O}_\mathfrak{U}, \Delta_* \mathcal{O}_\mathfrak{U}).$$

Here $\Delta : \mathfrak{U} \to \mathfrak{U} \times \mathfrak{U}$ is the diagonal. Then it is shown in [1, Section 4] that there exists a canonical map $\mathcal{H}^1(\mathbb{T}_\mathfrak{U}) \to \text{HH}^2\mathfrak{U}$. So for $F \in D^b_{\text{coh}}\mathfrak{U}$, we have the map of graded rings

$$\mathcal{O}_{\text{Crit}(w)} = S(\mathcal{H}^1(\mathbb{T}_\mathfrak{U})) \to \text{HH}^2\mathfrak{U} \to \text{Hom}^2(F, F). \quad (2.2)$$

Here the last arrow is defined by taking Fourier-Mukai transforms associated with morphisms $\Delta_* \mathcal{O}_\mathfrak{U} \to \Delta_* \mathcal{O}_\mathfrak{U}[2*]$. The above map defines the $\mathbb{C}^*$-equivariant $\mathcal{O}_{\text{Crit}(w)}$-module structure on $\text{Hom}^2(F, F)$, which is finitely generated by [1, Theorem 4.1.8].

The singular support of $F$

$$\text{Supp}^g(F) \subset \text{Crit}(w) \quad (2.3)$$

is defined to be the support of $\text{Hom}^2(F, F)$ as a graded $\mathcal{O}_{\text{Crit}(w)}$-module. Note that the singular support is a conical (i.e. $\mathbb{C}^*$-invariant) closed subscheme of $\text{Crit}(w)$. For a conical closed subset $Z \subset \text{Crit}(w)$, we denote by

$$\mathcal{C}_Z \subset D^b_{\text{coh}}\mathfrak{U}$$

the triangulated subcategory of objects $F \in D^b_{\text{coh}}\mathfrak{U}$ whose singular supports are contained in $Z$. Via Koszul duality, we have the following relation with the category of matrix factorizations (see [32, Corollary 2.11])

$$D^b_{\text{coh}}\mathfrak{U}/\mathcal{C}_Z \sim \mathcal{M}C^*(V^\vee \setminus Z, w). \quad (2.4)$$

Here the right hand side is the triangulated category of $\mathbb{C}^*$-equivariant matrix factorizations for $w : V^\vee \setminus Z \to \mathbb{C}$.
2.2 Quasi-smooth derived stacks

Below, we denote by $\mathcal{M}$ a derived Artin stack over $\mathbb{C}$. This means that $\mathcal{M}$ is a contravariant $\infty$-functor from the $\infty$-category of affine derived schemes over $\mathbb{C}$ to the $\infty$-category of simplicial sets

$$\mathcal{M} : d\text{Aff}^{\text{op}} \to \text{SSets}$$

satisfying some conditions (see [30, Section 3.2] for details). Here $d\text{Aff}^{\text{op}}$ is defined to be the $\infty$-category of commutative simplicial $\mathbb{C}$-algebras, which is equivalent to the $\infty$-category of commutative differential graded $\mathbb{C}$-algebras with non-positive degrees. All derived stacks considered in this paper are locally of finite presentation. The classical truncation of $\mathcal{M}$ is denoted by

$$\mathcal{M} := t_0(\mathcal{M}) : \text{Aff}^{\text{op}} \hookrightarrow d\text{Aff}^{\text{op}} \to \text{SSets}$$

where the first arrow is a natural functor from the category of affine schemes to affine derived schemes.

Following [30], we define the dg-category of quasi-coherent sheaves on $\mathcal{M}$ as

$$L_{\text{qcoh}}(\mathcal{M}) := \lim_{\triangleleft \mathcal{M}} L_{\text{qcoh}}(\mathfrak{U}). \quad (2.5)$$

Here $\mathfrak{U} = \text{Spec} A$ is an affine derived scheme for a cdga $A$. The category $L_{\text{qcoh}}(\mathfrak{U})$ is defined to be the dg-category of dg-modules over $A$ localized by quasi-isomorphisms (see [28, Section 2.4]), so that its homotopy category is equivalent to the derived category $D_{\text{qcoh}}(\mathfrak{U})$ of dg-modules over $A$. The limit in (2.5) is taken for all the diagrams

$$\begin{array}{ccc}
\mathfrak{U} & \xrightarrow{f} & \mathfrak{U}' \\
\downarrow \alpha & & \downarrow \alpha' \\
\mathcal{M},
\end{array} \quad (2.6)
$$

where $f$ is a 0-representable smooth morphism, and $\alpha' \circ f$ is equivalent to $\alpha$. The homotopy category of $L_{\text{qcoh}}(\mathcal{M})$ is denoted by $D_{\text{qcoh}}(\mathcal{M})$. We have the triangulated subcategory

$$D^b_{\text{coh}}(\mathcal{M}) \subset D_{\text{qcoh}}(\mathcal{M})$$

consisting of objects which have bounded coherent cohomologies. We note that there is a bounded t-structure on $D^b_{\text{coh}}(\mathcal{M})$ whose heart coincides with $\text{Coh}(\mathcal{M})$.

A morphism of derived stacks $f : \mathcal{M} \to \mathcal{N}$ is called quasi-smooth if $L_f$ is perfect such that for any point $x \to \mathcal{M}$ the restriction $L_f|_x$ is of cohomological amplitude $[-1, 1]$. Here $L_f$ is the $f$-relative cotangent complex. A derived stack $\mathcal{M}$ over $\mathbb{C}$ is
called quasi-smooth if $M \to \text{Spec } \mathbb{C}$ is quasi-smooth. By [2, Theorem 2.8], the quasi-smoothness of $M$ is equivalent to that $M$ is a 1-stack, and any point of $M$ lies in the image of a 0-representable smooth morphism

$$\alpha : \mathcal{U} \to M$$

(2.7)

where $\mathcal{U}$ is an affine derived scheme of the form (2.1). Furthermore following [9, Definition 1.1.8], a derived stack $M$ is called QCA (quasi-compact and with affine automorphism groups) if the following conditions hold:

(i) $M$ is quasi-compact;
(ii) The automorphism groups of its geometric points are affine;
(iii) The classical inertia stack $I_M := \Delta \times_M M \times_M \Delta$ is of finite presentation over $M$.

Below our derived stack $M$ always satisfies (ii), (iii), but we often encounter derived stacks which are not of finite type and in this case (i) may not be satisfied. In such a case we may take a derived open substack $M^{\text{fin}} \subset M$ of finite type, and then $M^{\text{fin}}$ is QCA.

For a quasi-smooth derived stack $M$ and an object $E \in \text{Perf}(M)$, we set

$$\rho : \mathbb{V}(E) := \text{Spec}_{\mathbb{R}} S(E) \to M.$$ 

If $E|_x$ is of cohomological amplitude $[-1, 1]$ for any point $x \to M$, then $\mathbb{V}(E)$ is quasi-smooth and $\rho$ is a quasi-smooth morphism. We also set

$$\rho' : \mathbb{P}(E) := (\mathbb{V}(E) \setminus 0_{\mathbb{R}})/\mathbb{C}^* \to M.$$ 

Here $\mathbb{C}^*$ acts by weight $m$ on $S^m(E)$, and $0_{\mathbb{R}}$ is the zero section of $\rho$. Under the above situation, $\mathbb{P}(E)$ is quasi-smooth and $\rho'$ is a quasi-smooth morphism. If furthermore $E|_x$ is of cohomological amplitude $[-1, 0]$ for any point $x \to M$, then $\rho'$ is a proper morphism, i.e. $\rho'$ is a representable morphism such that any pull-back by $U \to M$ for a scheme $U$ is a proper morphism of schemes.

### 2.3 Singular supports for quasi-smooth derived stacks

Let $\Omega_{\mathbb{R}}[-1]$ be the $(-1)$-shifted cotangent stack over $M$, i.e.

$$\Omega_{\mathbb{R}}[-1] = \mathbb{V}(\mathcal{T}_{\mathbb{R}}[1]).$$

Let $M_1, M_2$ be quasi-smooth derived stacks with truncations $M_i = t_0(M_i)$. Let $f : M_1 \to M_2$ be a morphism. Then the morphism $f^* L_{\mathbb{R}2} \to L_{\mathbb{R}1}$ induces the diagram
\[ t_0(\Omega_{\mathfrak{M}_1}[-1]) \xleftarrow{f^\diamond} f^*t_0(\Omega_{\mathfrak{M}_2}[-1]) \xrightarrow{f^\bullet} t_0(\Omega_{\mathfrak{M}_2}[-1]) \] (2.8)

It is easy to see that \( f \) is quasi-smooth if and only if \( f^\diamond \) is a closed immersion. \( f \) is smooth if and only if \( f^\circ \) is an isomorphism. Let \( \Omega_f[-2] \) is the \((2)-\)shifted conormal stack

\[ \Omega_f[-2] := \mathbb{V}(T_f[2]). \]

Here \( T_f \) is the dual of \( \mathbb{L}_f \). We have the following lemma.

**Lemma 2.1** We have the isomorphism over \( \mathcal{M}_1 \)

\[ t_0(\Omega_f[-2]) \cong (f^\circ)^{-1}(0_{\mathcal{M}_1}). \] (2.9)

Here \( 0_{\mathcal{M}_1} \) is the zero section of the left horizontal arrow in (2.8).

**Proof** By the definition of \( \Omega_f[-2] \), we have

\[ t_0(\Omega_f[-2]) = \text{Spec}_{\mathcal{M}_1} S(H^2(T_f)). \]

Therefore for each \( p \in \mathcal{M}_1 \), the fiber of \( t_0(\Omega_f[-2]) \to \mathcal{M}_1 \) at \( p \) is \( H^{-2}(\mathbb{L}_f|_p) \). On the other hand, we have the distinguished triangle

\[ f^*\mathbb{L}_{\mathfrak{M}_2} \to \mathbb{L}_{\mathfrak{M}_1} \to \mathbb{L}_f. \] (2.10)

By restricting it to \( p \) and the associated long exact sequence of cohomologies, we have the exact sequence

\[ 0 \to H^{-2}(\mathbb{L}_f|_p) \to H^{-1}(\mathbb{L}_{\mathfrak{M}_2}|f(p)) \to H^{-1}(\mathbb{L}_{\mathfrak{M}_1}|p). \]

Therefore the fibers of both sides of (2.9) are naturally identified. \( \square \)

The smooth morphism (2.7) induces the diagram

\[ t_0(\Omega_{\mathfrak{M}}[-1]) \xleftarrow{\alpha^\circ} t_0(\alpha^*\Omega_{\mathfrak{M}}[-1]) \xrightarrow{\alpha^\bullet} t_0(\Omega_{\mathfrak{M}}[-1]). \] (2.11)

A closed substack of \( t_0(\Omega_{\mathfrak{M}}[-1]) \) is called conical if it is closed under the fiberwise \( \mathbb{C}^* \)-action on \( t_0(\Omega_{\mathfrak{M}}[-1]) \). For a conical closed substack \( \mathcal{Z} \subset t_0(\Omega_{\mathfrak{M}}[-1]) \), we have the conical closed subscheme

\[ \alpha^* \mathcal{Z} := \alpha^\circ(\alpha^\bullet)^{-1}(\mathcal{Z}) \subset t_0(\Omega_{\mathfrak{M}}[-1]) = \text{Crit}(w). \]
We define
\[ C_Z \subset D^b_{\text{coh}}(\mathcal{M}) \]
to be the triangulated subcategory consisting of objects whose singular supports are contained in \( Z \), i.e. those of objects \( E \in D^b_{\text{coh}}(\mathcal{M}) \) such that for any map \( \alpha \) as in (2.7), we have
\[ \text{Supp}^{\text{sg}}(\alpha^*E) \subset \alpha^*Z. \]

Below we sometimes take a derived open substack \( \mathcal{M}^{\text{fin}} \subset \mathcal{M} \) and work on \( \mathcal{M}^{\text{fin}} \). In this case, for a conical closed substack \( Z \subset t_0(\Omega_{\mathcal{M}}[-1]) \), we set
\[ Z^{\text{fin}} := Z \times_{\mathcal{M}} \mathcal{M}^{\text{fin}} \subset t_0(\Omega_{\mathcal{M}^{\text{fin}}[-1]}) \]
where \( \mathcal{M}^{\text{fin}} \) is the classical truncation of \( \mathcal{M}^{\text{fin}} \). Then we have the subcategory \( C_{Z^{\text{fin}}} \subset D^b_{\text{coh}}(\mathcal{M}^{\text{fin}}) \), and the quotient category
\[ D^b_{\text{coh}}(\mathcal{M}^{\text{fin}})/C_{Z^{\text{fin}}} \]
is our model for the definition of DT category in [32]. By the equivalence (2.4), the above quotient category may be regarded as a gluing of matrix factorizations.

2.4 Functoriality of singular supports

Let \( \mathcal{M}_1, \mathcal{M}_2 \) be quasi-smooth derived stacks, and take a morphism
\[ f : \mathcal{M}_1 \to \mathcal{M}_2 \]
First suppose that \( f \) is a quasi-smooth morphism. Then we have the pull-back functor (see [21, Section 4.2])
\[ f^* : D^b_{\text{coh}}(\mathcal{M}_2) \to D^b_{\text{coh}}(\mathcal{M}_1). \]  
(2.12)
In this case, the morphism \( f^{\diamond} \) in the diagram (2.8) is a closed immersion. Therefore for any conical closed substack \( Z_2 \subset t_0(\Omega_{\mathcal{M}_2}[-1]) \), we have the conical closed substack
\[ f^{\diamond}(f^\bullet)^{-1}(Z_2) \subset t_0(\Omega_{\mathcal{M}_1}[-1]). \]  
(2.13)

**Lemma 2.2** Suppose that a conical closed substack \( Z_1 \subset t_0(\Omega_{\mathcal{M}_1}[-1]) \) contains (2.13). Then the functor (2.12) sends \( C_{Z_2} \) to \( C_{Z_1} \).

**Proof** The \( f^! \)-version is proved in [1, Lemma 8.4.2], i.e. the functor \( f^! \) sends \( C_{Z_2} \) to \( C_{Z_1} \). As \( f \) is quasi-smooth, we have \( f^!(-) = f^*(-) \otimes \omega_f[\text{vdim} f] \) where \( \omega_f \) is a \( f \)-relative canonical line bundle of \( f \) and \( \text{vdim} f \) is the virtual dimension of \( f \) (see [9, (3.12)]). Since \( \otimes \omega_f[\text{vdim}] \) sends \( C_{Z_1} \) to \( C_{Z_1} \), we obtain the lemma. \( \square \)
Next suppose that $f$ is a proper morphism. Then we have the push-forward functor (see [21, Section 4.2])

$$f_*: D^b_{c\text{oh}}(\mathcal{M}_1) \to D^b_{c\text{oh}}(\mathcal{M}_2). \quad (2.14)$$

In this case, the morphism $f^\bullet$ in the diagram (2.8) is a proper morphism of schemes. Therefore for any conical closed substack $Z_1 \subset t_0(\Omega \mathfrak{M}_1[-1])$, we have the conical closed substack

$$f^\bullet(f^\diamond)^{-1}(Z_1) \subset t_0(\Omega \mathfrak{M}_2[-1]). \quad (2.15)$$

The following lemma is proved in [1, Lemma 8.4.5].

**Lemma 2.3** Suppose that a conical closed substack $Z_2 \subset t_0(\Omega \mathfrak{M}_2[-1])$ contains (2.15). Then the functor (2.14) sends $C_{Z_1}$ to $C_{Z_2}$.

Let $\mathfrak{N}$ be another quasi-smooth derived stack with a diagram

$$\mathfrak{M}_1 \xleftarrow{f_1} \mathfrak{N} \xrightarrow{f_2} \mathfrak{M}_2. \quad (2.16)$$

Suppose that $f_1$ is quasi-smooth and $f_2$ is a proper morphism. Then for any $\mathcal{P} \in \text{Perf}(\mathfrak{N})$, we have the functor

$$\text{FM}_{\mathcal{P}}(-) = f_2^*(f_1^*(-) \otimes \mathcal{P}): D^b_{c\text{oh}}(\mathfrak{M}_1) \to D^b_{c\text{oh}}(\mathfrak{M}_2). \quad (2.17)$$

Let $f$ be the morphism

$$f = (f_1, f_2): \mathfrak{N} \to \mathfrak{M}_1 \times \mathfrak{M}_2.$$

Then we have the diagram

Here the middle square is Cartesian by the isomorphism (2.9). Note that $g_2$ is a closed immersion as $f_1^\diamond$ is, and $f_2^\bullet$ is proper as $f_2$ is. Therefore $h_2$ is also proper. We obtain
the following commutative diagram

\[
\begin{array}{c}
t_0(\Omega_{\mathcal{M}_1}[-1]) & \xleftarrow{h_1} & t_0(\Omega_f [-2]) & \xrightarrow{h_2} & t_0(\Omega_{\mathcal{M}_2}[-1]) \\
\mathcal{M}_1 & \xleftarrow{f_1} & \mathcal{N} & \xrightarrow{f_2} & \mathcal{M}_2.
\end{array}
\]  

(2.19)

Here \( \mathcal{M}_i = t_0(\mathcal{M}_i) \) and \( \mathcal{N} = t_0(\Omega) \). For a conical closed substack \( \mathcal{Z}_1 \subset t_0(\Omega_{\mathcal{M}_1}[-1]) \), we have the following conical closed substack

\[
(h_2(h_1)^{-1}(\mathcal{Z}_1)) \subset t_0(\Omega_{\mathcal{M}_2}[-1]).
\]  

(2.20)

**Proposition 2.4** Suppose that a conical closed substack \( \mathcal{Z}_2 \subset t_0(\Omega_{\mathcal{M}_2}[-1]) \) contains (2.20). Then the functor \((2.17)\) sends \( \mathcal{C}_{\mathcal{Z}_1} \) to \( \mathcal{C}_{\mathcal{Z}_2} \).

**Proof** Note that since \( \mathcal{P} \) is perfect, the functor \( \otimes \mathcal{P} \) takes \( \mathcal{C}_{\mathcal{W}} \) to \( \mathcal{C}_{\mathcal{W}} \) for any conical closed substack \( \mathcal{W} \subset t_0(\Omega_{\mathcal{M}_1}[-1]) \). Therefore by Lemma 2.2 and Lemma 2.3, the functor \((2.17)\) sends \( \mathcal{C}_{\mathcal{Z}_1} \) to \( \mathcal{C}_{\mathcal{Z}_2} \) where \( \mathcal{Z}_2' \) is

\[
\mathcal{Z}_2' = f_2^\ast(f_2^\diamond)^{-1}f_1^\diamond(f_1^\ast)^{-1}(\mathcal{Z}_1) = f_2^\ast g_2^\ast g_1^\ast(f_1^\ast)^{-1}(\mathcal{Z}_1) = h_2(h_1)^{-1}(\mathcal{Z}_1).
\]

By the assumption we have \( \mathcal{Z}_2' \subset \mathcal{Z}_2 \), hence \( \mathcal{C}_{\mathcal{Z}_2} \subset \mathcal{C}_{\mathcal{Z}_2} \). Therefore the proposition holds. \( \square \)

In the situation above, let \( \mathcal{M}_i^0 \subset \mathcal{M}_i \) for \( i = 1, 2 \) be derived open substacks satisfying

\[
\mathcal{N}^0 := f_2^{-1}(\mathcal{M}_2^0) \subset f_1^{-1}(\mathcal{M}_1^0).
\]

Then a diagram (2.16) restricts to the diagram

\[
\begin{array}{c}
\mathcal{M}_1^0 & \xrightarrow{f_1^0} & \mathcal{N} & \xrightarrow{f_2^0} & \mathcal{M}_2^0.
\end{array}
\]  

(2.21)

Note that \( f_1^0 \) is quasi-smooth and \( f_2^0 \) is proper. Therefore for \( \mathcal{P}^0 := \mathcal{P}|_{\Omega^0} \), we have the functor

\[
\text{FM}_{\mathcal{P}^0}(-) = f_2^0(f_1^{\circ^0}(-) \otimes \mathcal{P}^0) : D^b_{\text{coh}}(\mathcal{M}_1^0) \to D^b_{\text{coh}}(\mathcal{M}_2^0).
\]

For conical closed substacks \( \mathcal{Z}_i \subset t_0(\Omega_{\mathcal{M}_i}[-1]) \), we set \( \mathcal{Z}_i^0 := \mathcal{Z}_i \times_{\mathcal{M}_i} \mathcal{M}_i^0 \) where \( \mathcal{M}_i^0 = t_0(\mathcal{M}_i^0) \).

**Lemma 2.5** Under the same assumption of Proposition 2.4, the functor \( \text{FM}_{\mathcal{P}^0} \) sends \( \mathcal{C}_{\mathcal{Z}_i} \) to \( \mathcal{C}_{\mathcal{Z}_i^0} \).
Proof Similarly to (2.19), the diagram (2.21) induces the diagram

\[
\begin{array}{ccc}
t_0(\Omega \mathcal{M}_{1}[-1]) & \xleftarrow{h_1^0} & t_0(\Omega f^*[2]) & \xrightarrow{h_2^0} & t_0(\Omega \mathcal{M}_{2}[-1]) \\
\mathcal{M}_1^o & \xleftarrow{f_1^0} & \mathcal{N}^o & \xrightarrow{f_2^0} & \mathcal{M}_2^o.
\end{array}
\]

(2.22)

Here \(\mathcal{N}^o = t_0(\mathcal{N}^o)\) and \(f^o = (f_1^o, f_2^o): \mathcal{N}^o \to \mathcal{M}_1^o \times \mathcal{M}_2^o\). As we have the Cartesian square

\[
\begin{array}{ccc}
\mathcal{M}_1^o \times \mathcal{M}_2^o & \xrightarrow{\square} & \mathcal{M}_1^o \times \mathcal{M}_2^o \\
\mathcal{N}^o & \xrightarrow{f^o} & \mathcal{N}^o \\
\mathcal{N} & \xrightarrow{f} & \mathcal{N}^o.
\end{array}
\]

we have \(t_0(\Omega f^*[2]) = t_0(\Omega f[-2]) \times_{\mathcal{N}} \mathcal{N}^o\). It follows that the right square of (2.22) is obtained from the right square of (2.19) via \((-) \times_{\mathcal{M}_2} \mathcal{M}_2^o\). Therefore we have

\[
\begin{align*}
h_2^0(h_1^0)^{-1}(Z_1^o) &= h_2^0(h_1^0)^{-1}(Z_1) \times_{\mathcal{N}} \mathcal{N}^o \\
&= h_2(h_1^0)^{-1}(Z_1) \times_{\mathcal{M}_2} \mathcal{M}_2^o \subset Z_2 \times_{\mathcal{M}_2} \mathcal{M}_2^o = Z_2^o.
\end{align*}
\]

The lemma now follows from Proposition 2.4. \(\square\)

3 Hall-type algebras for DT categories

In this section, we prove Theorem 1.1. We first recall the relevant constructions of derived moduli stacks of sheaves and their extensions, and then recall the definition of DT categories introduced in [32]. They are defined in terms of derived categories of coherent sheaves on derived moduli spaces of sheaves on surfaces, together with the notion of singular supports. We then show that the categorified COHA structure by Porta–Sala [21] for derived moduli spaces of sheaves on surfaces is compatible with singular supports, so that it descends to the algebra structure on DT categories.

3.1 Derived moduli stacks of coherent sheaves on surfaces

Let \(S\) be a smooth projective surface over \(\mathbb{C}\). We consider the derived Artin stack

\[
\mathcal{M}_S: dAff^{op} \to \text{SSets}
\]

(3.1)

which sends an affine derived scheme \(T\) to the \(\infty\)-groupoid of perfect complexes on \(T \times S\), whose restriction to \(t_0(T) \times S\) is a flat family of coherent sheaves on \(S\) over \(t_0(T)\). Note that we have an open immersion
\[ \mathcal{M}_S \subset \mathcal{P}erf_S \]

where \( \mathcal{P}erf_S \) is the derived moduli stack of perfect complexes on \( S \) constructed in [27]. Since any object in \( \text{Coh}(S) \) is perfect as \( S \) is smooth, the derived Artin stack \( \mathcal{M}_S \) is the derived moduli stack of objects in \( \text{Coh}(S) \). The classical truncation of \( \mathcal{M}_S \) is denoted by \( \mathcal{M}_S = t_0(\mathcal{M}_S) \).

Let \( N(S) \) be the numerical Grothendieck group of \( S \)

\[ N(S) := K(S) / \equiv \quad (3.2) \]

where \( F_1, F_2 \in K(S) \) satisfies \( F_1 \equiv F_2 \) if \( \text{ch}(F_1) = \text{ch}(F_2) \). Then \( N(S) \) is a finitely generated free abelian group. We have the decompositions into open and closed substacks

\[ \mathcal{M}_S = \bigsqcup_{v \in N(S)} \mathcal{M}_S(v), \quad \mathcal{M}_S = \bigsqcup_{v \in N(S)} \mathcal{M}_S(v) \]

where each component corresponds to sheaves \( F \) on \( S \) with \( [F] = v \). We denote by

\[ \mathcal{F}(v) \in \text{Perf}(S \times \mathcal{M}_S(v)) \]

the universal object on the component \( \mathcal{M}_S(v) \).

We also define the derived moduli stack of exact sequences of coherent sheaves on \( S \), following [21, Section 3]. It is given by the derived Artin stack

\[ \mathcal{M}_S^{\text{ext}} : dAff^{\text{op}} \to SSets \]

which sends an affine derived scheme \( T \) to the \( \infty \)-groupoid of fiber sequences of perfect complexes on \( T \times S \),

\[ \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \quad (3.4) \]

whose restrictions to \( t_0(T) \times S \) are flat families of exact sequences of coherent sheaves on \( S \) over \( t_0(T) \). The classical truncation of \( \mathcal{M}_S^{\text{ext}} \) is denoted by \( \mathcal{M}_S^{\text{ext}} := t_0(\mathcal{M}_S^{\text{ext}}) \).

We have the decompositions into open and closed substacks

\[ \mathcal{M}_S^{\text{ext}} = \bigsqcup_{v_\bullet = (v_1, v_2, v_3)} \mathcal{M}_S^{\text{ext}}(v_\bullet), \quad \mathcal{M}_S^{\text{ext}} = \bigsqcup_{v_\bullet = (v_1, v_2, v_3)} \mathcal{M}_S^{\text{ext}}(v_\bullet) \]

where each component corresponds to exact sequences \( 0 \to F_1 \to F_2 \to F_3 \to 0 \) with \( [F_i] = v_i \).

By sending a sequence \( (3.4) \) to \( \mathcal{F}_i \), we have the evaluation morphisms

\[ \text{ev}_i : \mathcal{M}_S^{\text{ext}}(v_\bullet) \to \mathcal{M}_S(v_i), \quad 1 \leq i \leq 3. \]
Below we use the following diagram

\[
\begin{array}{c}
\mathcal{M}^\text{ext}_S(v) \\
\downarrow \text{(ev}_1, \text{ev}_3) \\
\mathcal{M}_S(v_1) \times \mathcal{M}_S(v_3).
\end{array}
\]

(3.5)

Then the vertical map in (3.5) is described as the relative spectrum (see [21, Proposition 3.8])

\[
\mathcal{M}^\text{ext}_S(v) = \mathbb{V}(\mathcal{H}\text{om}_{p\mathcal{M}_S \times \mathcal{M}_S}(\mathcal{F}(v_3), \mathcal{F}(v_1)[1])^\vee).
\]

(3.6)

Here we have denoted by $\mathcal{F}(v_i)$ the pull-back of the universal object (3.3) by the projection

\[
S \times \mathcal{M}_S(v_1) \times \mathcal{M}_S(v_3) \to S \times \mathcal{M}_S(v_i)
\]

and $p_{\mathcal{M}_S \times \mathcal{M}_S}$ is the projection to $\mathcal{M}_S(v_1) \times \mathcal{M}_S(v_3)$. Since for $(F_1, F_3) \in \mathcal{M}_S(v_1) \times \mathcal{M}_S(v_3)$ the cohomological amplitude of $\mathcal{R}\text{Hom}(F_3, F_1)[1]$ is $[-1, 1]$, the morphism $(\text{ev}_1, \text{ev}_3)$ is quasi-smooth. In particular, we have the well-defined functor

\[
(\text{ev}_1, \text{ev}_3)^* : D^b_{\text{coh}}(\mathcal{M}_S(v_1) \times \mathcal{M}_S(v_3)) \to D^b_{\text{coh}}(\mathcal{M}^\text{ext}_S(v))
\]

On the other hand, the horizontal morphism in (3.5) is a proper morphism. Indeed for a point $x \to \mathcal{M}_S(v_2)$ corresponding to coherent sheaf $F_2$ on $S$, the fiber product

\[
\mathcal{M}^\text{ext}_S(v) \times_{\text{ev}_2, \mathcal{M}_S(v_2)} x
\]

is the derived Quot scheme parameterizing quotients $F_2 \twoheadrightarrow F_3$ with $[F_3] = v_3$. Therefore we have the well-defined functor

\[
\text{ev}_{2*} : D^b_{\text{coh}}(\mathcal{M}^\text{ext}_S(v)) \to D^b_{\text{coh}}(\mathcal{M}_S(v_2)).
\]

As we mentioned in the introduction, the following composition

\[
D^b_{\text{coh}}(\mathcal{M}_S(v_1)) \times D^b_{\text{coh}}(\mathcal{M}_S(v_3)) \overset{\Box}{\longrightarrow} D^b_{\text{coh}}(\mathcal{M}_S(v_1)) \\
\times \mathcal{M}_S(v_3) \overset{\text{ev}_2(\text{ev}_1, \text{ev}_3)^*}{\longrightarrow} D^b_{\text{coh}}(\mathcal{M}_S(v_2))
\]

(3.7)

was considered in [21], as a categorification of cohomological Hall algebra on surfaces [17].
3.2 Moduli stacks of coherent sheaves on local surfaces

We next consider similar (but underived) moduli stacks on the threefold $X$, defined by

$$X := \text{Tot}_S(\omega_S) \xrightarrow{\pi} S.$$ 

Here $\pi$ is the projection. Note that $X$ is a non-compact CY threefold, called *local surface*. We denote by $\text{Coh}_{\text{cpt}}(X) \subset \text{Coh}(X)$ the subcategory of compactly supported coherent sheaves on $X$. We consider the classical Artin stack

$$\mathcal{M}_X : \text{Aff}^{\text{op}} \to \text{Groupoid}$$

whose $T$-valued points for $T \in \text{Aff}$ form the groupoid of $T$-flat families of objects in $\text{Coh}_{\text{cpt}}(X)$. We have the decomposition into open and closed substacks

$$\mathcal{M}_X = \bigsqcup_{v \in N(S)} \mathcal{M}_X(v)$$

where each component corresponds to objects $E \in \text{Coh}_{\text{cpt}}(X)$ with $[\pi_* E] = v$.

We have the natural push-forward morphism

$$\pi_* : \mathcal{M}_X(v) \to \mathcal{M}_S(v), \ E \mapsto \pi_* E$$

which realizes $\mathcal{M}_X(v)$ as the dual obstruction cone over $\mathcal{M}_S(v)$, i.e. $\mathcal{M}_X(v)$ is the classical truncation of the $(-1)$-shifted cotangent stack of $\mathcal{M}_S(v)$ (see [32, Lemma 5.1])

$$\mathcal{M}_X(v) = t_0(\Omega_{\mathcal{M}_S(v)}[-1]). \tag{3.8}$$

Similarly we consider the classical Artin stack of short exact sequences of compactly supported coherent sheaves on $X$. It is given by the 2-functor

$$\mathcal{M}^\text{ext}_X : \text{Aff}^{\text{op}} \to \text{Groupoid}$$

whose $T$-valued points for $T \in \text{Aff}$ form the groupoid of exact sequences of coherent sheaves on $X \times T$,

$$0 \to \mathcal{E}_1 \to \mathcal{E}_3 \to \mathcal{E}_3 \to 0, \ \mathcal{E}_i \in \mathcal{M}_X(T).$$

We have the decomposition into open and closed substacks

$$\mathcal{M}^\text{ext}_X = \bigsqcup_{v_* = (v_1, v_2, v_3)} \mathcal{M}^\text{ext}_X(v_*)$$
where each component corresponds to exact sequences \( 0 \to E_1 \to E_2 \to E_3 \to 0 \) with \([\pi_* E_i] = v_i\). We have the evaluation morphisms

\[
ev^X_i : \mathcal{M}^\text{ext}_X(v_\bullet) \to \mathcal{M}_X(v_i), \ E_\bullet \mapsto E_i
\]

and obtain the diagram

\[
\begin{array}{c}
\mathcal{M}^\text{ext}_X(v_\bullet) \xrightarrow{\ev^X_2} \mathcal{M}_X(v_2) \\
\mathcal{M}_X(v_1) \times \mathcal{M}_X(v_3).
\end{array}
\] (3.9)

We also have the morphism given by the push-forward along \( \pi \)

\[
\pi_* : \mathcal{M}^\text{ext}_X(v_\bullet) \to \mathcal{M}^\text{ext}_S(v_\bullet), \ E_\bullet \mapsto \pi_* E_\bullet
\] (3.10)

Here note that \( \pi_* \) preserves the exact sequences of coherent sheaves as \( \pi \) is affine.

### 3.3 The relation of \( \mathcal{M}^\text{ext}_X \) and \( \mathcal{M}^\text{ext}_S \)

Here we relate \( \mathcal{M}^\text{ext}_X \) and \( \mathcal{M}^\text{ext}_S \) as a \((-2)\)-shifted conormal stack. Let \( \ev \) be the morphism

\[
\ev = (\ev_1, \ev_2, \ev_3) : \mathcal{M}^\text{ext}_S(v_\bullet) \to \mathcal{M}_S(v_1) \times \mathcal{M}_S(v_2) \times \mathcal{M}_S(v_3).
\]

We have the \((-2)\)-shifted conormal stack and its truncation

\[
\Omega_{\ev}[-2] \to \mathcal{M}^\text{ext}_S(v_\bullet), \ t_0(\Omega_{\ev}[-2]) \to \mathcal{M}^\text{ext}_S(v_\bullet).
\]

**Proposition 3.1** There is an isomorphism over \( \mathcal{M}^\text{ext}_S(v_\bullet) \)

\[
\mathcal{M}^\text{ext}_X(v_\bullet) \xrightarrow{\cong} t_0(\Omega_{\ev}[-2]).
\] (3.11)

**Proof** Let us take a point \( p \in \mathcal{M}^\text{ext}_S(v_\bullet) \) represented by an exact sequence \( 0 \to F_1 \to F_2 \to F_3 \to 0 \) on \( S \). The fiber of the morphism (3.10) at this point is identified with the stack of morphisms \( \phi_i : F_i \to F_i \otimes \omega_S \) for \( i = 1, 2, 3 \) which fit into the commutative diagram

\[
\begin{array}{c}
0 \to F_1 \xrightarrow{i} F_2 \xrightarrow{j} F_3 \to 0 \\
\phi_1 \downarrow \quad \phi_2 \downarrow \quad \phi_3 \\
0 \to F_1 \otimes \omega_S \xrightarrow{i} F_2 \otimes \omega_S \xrightarrow{j} F_3 \otimes \omega_S \to 0
\end{array}
\] (3.12)
Indeed such $\phi_i$ determines an $\mathcal{O}_X$-module structure on $F_i$, and the diagram (3.12) is an exact sequence of $\mathcal{O}_X$-modules. From the diagram (3.12), $\phi_1, \phi_3$ are uniquely determined by $\phi_2$. Conversely given $\phi_2$, it induces $\phi_1, \phi_3$ in the diagram (3.12) if and only if the composition

$$F_1 \xrightarrow{i F_2 \xrightarrow{\phi_2} F_2 \otimes \omega_S \xrightarrow{j F_3 \otimes \omega_S}$$

is zero. Therefore the fiber of the morphism (3.10) is identified with

$$\pi^{-1}_*(p) = \text{Ker}(\text{Hom}(F_2, F_2 \otimes \omega_S) \xrightarrow{j_0(-)oj} \text{Hom}(F_1, F_3 \otimes \omega_S)). \quad (3.13)$$

On the other hand, we have the commutative diagram

$$\begin{array}{cccc}
(ev_1, ev_3)^*\mathcal{L}M_S(v_1) \times \mathcal{M}_S(v_3) & \xrightarrow{} & (ev_1, ev_3)^*\mathcal{L}M_S(v_1) \times \mathcal{M}_S(v_3) \\
\downarrow & & \downarrow \\
\mathcal{L}M^{\text{ext}}(v) & \xrightarrow{} & \mathcal{L}ev \\
\downarrow & & \downarrow \\
ev_2^*\mathcal{L}M_S(v_2) & & (ev_1, ev_3)^*(\mathcal{H}om_{pM \times M}(\mathcal{F}_3, \mathcal{F}_1)[1])^\vee
\end{array}$$

Here each horizontal and vertical sequences are distinguished triangle. The middle horizontal sequence is obtained by the description (3.6). By taking the cone, we have the distinguished triangle

$$\ev_2^*\mathcal{L}M_S(v_2) \rightarrow (ev_1, ev_3)^*(\mathcal{H}om_{pM \times M}(\mathcal{F}_3, \mathcal{F}_1)[1])^\vee \rightarrow \mathcal{L}ev.$$

By restricting it to $p$ and the associated exact sequence of cohomologies, we see that

$$\mathcal{H}^{-2}(\mathcal{L}ev|_p) = \text{Ker}(\text{Ext}^2_{\mathcal{S}}(F_2, F_2)^\vee \rightarrow \text{Ext}^2_{\mathcal{S}}(F_3, F_1)^\vee)$$

$$= \text{Ker}(\text{Hom}(F_2, F_2 \otimes \omega_S) \xrightarrow{\eta} \text{Hom}(F_1, F_3 \otimes \omega_S)).$$

Here the second identity is due to Serre duality, and one can check that the map $\eta$ is identified with the map in (3.13). Since the left hand side is the fiber of $t_0(\Omega_{ev}[-2]) \rightarrow \mathcal{M}^{\text{ext}}_S$ at $p$, we have the equivalence of $\mathbb{C}$-valued points

$$\mathcal{M}^{\text{ext}}_X(v_*)(\mathbb{C}) \sim t_0(\Omega_{ev}[-2])(\mathbb{C}).$$

It is straightforward to generalize the above arguments for $T$-valued point of $\mathcal{M}^{\text{ext}}_S(v_*)$ for a $\mathbb{C}$-scheme $T$, and therefore we obtain the isomorphism (3.11).

Let us take derived open substacks $\mathcal{M}_S(v_i)^{\text{fin}} \subset \mathcal{M}_S(v_i)$ of finite type satisfying

$$\mathcal{M}^{\text{ext}}_S(v_*)^{\text{fin}} := \ev_2^{-1}(\mathcal{M}_S(v_2)^{\text{fin}}) \subset (ev_1, ev_3)^{-1}(\mathcal{M}_S(v_1)^{\text{fin}} \times \mathcal{M}_S(v_3)^{\text{fin}}). \quad (3.14)$$
Then the diagram (3.5) restricts to the diagram

\[
\begin{array}{c}
\mathcal{M}_S^{\text{ex}}(v)_{\text{fin}} \xrightarrow{\text{ev}_2} \mathcal{M}_S(v)_{\text{fin}} \\
\downarrow \text{ev}_1 \cdot \text{ev}_3 \\
\mathcal{M}_S(v_1)_{\text{fin}} \times \mathcal{M}_S(v_3)_{\text{fin}}.
\end{array}
\]

Note that the vertical arrow is quasi-smooth and the horizontal arrow is proper. Therefore we have the induced functor

\[
ev^*_2(\text{ev}_1, \text{ev}_3) : D^b_{\text{coh}}(\mathcal{M}_S(v_1)_{\text{fin}}) \times D^b_{\text{coh}}(\mathcal{M}_S(v_3)_{\text{fin}}) \to D^b_{\text{coh}}(\mathcal{M}_S(v_2)_{\text{fin}}).
\]  

We have the following corollary of Proposition 3.1.

**Corollary 3.2** For conical closed substacks \( Z_i \subset \mathcal{M}_X(v_i) \), suppose that

\[
(\text{ev}_1^X, \text{ev}_3^X)^{-1}((Z_1 \times \mathcal{M}_X) \cup (\mathcal{M}_X \times Z_3)) \subset (\text{ev}_2^X)^{-1}(Z_2).
\]  

Then the functor (3.15) descends to the functor

\[
D^b_{\text{coh}}(\mathcal{M}_S(v_1)_{\text{fin}})/C_{Z_1_{\text{fin}}} \times D^b_{\text{coh}}(\mathcal{M}_S(v_3)_{\text{fin}})/C_{Z_3_{\text{fin}}} \to D^b_{\text{coh}}(\mathcal{M}_S(v_2)_{\text{fin}})/C_{Z_2_{\text{fin}}}.
\]  

**Proof** By (2.18), the diagram (3.5) induces the diagram

\[
\begin{array}{c}
t_0(\Omega_{\text{ev}}[-1]) \\
\downarrow \theta_0(\mathcal{M}_S(v_2)[-1]) \\
t_0(\Theta_{\mathcal{M}_S(v_1)}[-1]) \times t_0(\Theta_{\mathcal{M}_S(v_3)}[-1]).
\end{array}
\]

Under the isomorphisms (3.8), (3.11), one can check that the above diagram is identified with the diagram (3.9). Therefore by Proposition 2.4 and Lemma 2.5, the assumption (3.16) implies that the functor (3.15) restricts to the functors

\[
C_{Z_1_{\text{fin}}} \times D^b_{\text{coh}}(\mathcal{M}_S(v_3)_{\text{fin}}) \to C_{Z_2_{\text{fin}}}, \quad D^b_{\text{coh}}(\mathcal{M}_S(v_1)_{\text{fin}}) \times C_{Z_3_{\text{fin}}} \to C_{Z_2_{\text{fin}}}.
\]

Therefore the functor (3.15) descends to the functor (3.17). \(\square\)

### 3.4 Categorical DT theory of stable sheaves on local surfaces

Let us take an element

\[
\sigma = B + iH \in \text{NS}(S)_{\mathbb{C}}
\]  

(3.18)
such that $H$ is an ample class. For an object $E \in \text{Cohcpt}(X)$, its $B$-twisted reduced Hilbert polynomial is defined by

$$
\overline{\chi}_\sigma(E, m) := \frac{\chi(\pi_* E \otimes \mathcal{O}_S(-B + mH))}{c} \in \mathbb{Q}[m].
$$

(3.19)

Here $c$ is the coefficient of the highest degree term of the polynomial $\chi(\pi_* E \otimes \mathcal{O}_S(-B + mH))$ in $m$. Note that the polynomial $\overline{\chi}_\sigma(E, m)$ is determined by the numerical class of $E$, so it extends to the map

$$
\overline{\chi}_\sigma(-, m) : N(S) \to \mathbb{Q}[m]
$$

such that $\overline{\chi}_\sigma(E, m) = \overline{\chi}_\sigma([E], m)$.

By definition, an object $E \in \text{Cohcpt}(X)$ is $\sigma$-(semi)stable if it is a pure dimensional sheaf, and for any subsheaf $0 \neq E' \subseteq E$ we have

$$
\overline{\chi}_\sigma(E', m) < (\leq) \overline{\chi}_\sigma(E, m), \ m \gg 0.
$$

We have the open substack and the conical closed substack

$$
\mathcal{M}_{\sigma-ss}^X(v) \subset \mathcal{M}_X(v), \ Z_{\sigma-us}(v) \subset \mathcal{M}_X(v)
$$

(3.20)

where $\mathcal{M}_{\sigma-ss}^X(v)$ corresponds to $\sigma$-semistable sheaves and $Z_{\sigma-us}(v)$ is its complement. The moduli stack $\mathcal{M}_{\sigma-ss}^X(v)$ is of finite type, while $\mathcal{M}_S(v)$ is not of finite type in general, so not necessary QCA. Therefore we take a derived open substack $\mathcal{M}_S(v)_{\text{fin}} \subset \mathcal{M}_S(v)$ of finite type satisfying the condition

$$
\mathcal{M}_{\sigma-ss}^X(v) \subset \pi_*^{-1}(\mathcal{M}_S(v)_{\text{fin}}) = t_0(\Omega_{\mathcal{M}_S(v)_{\text{fin}}}[−1]).
$$

(3.21)

The $\mathbb{C}^*$-equivariant categorical DT theory for $\mathcal{M}_{\sigma-ss}^X(v)$ is defined as follows:

**Definition 3.3** The $\mathbb{C}^*$-equivariant categorical DT theory for the moduli stack $\mathcal{M}_{\sigma-ss}^X(v)$ is defined by

$$
\text{DT}^{\mathbb{C}^*}(\mathcal{M}_{\sigma-ss}^X(v)) := D^b_{\text{coh}}(\mathcal{M}_S(v)_{\text{fin}})/\mathcal{C}_{Z_{\sigma-us}(v)_{\text{fin}}}.
$$

**Remark 3.4** By [32, Lemma 3.10], the categorical DT theories in Definition 3.3 are independent of a choice of a finite type open substack $\mathcal{M}_S(v)_{\text{fin}}$ of $\mathcal{M}_S(v)$ satisfying (3.21), up to equivalence. We use the QCA condition for the above independence.

### 3.5 Categorical COHA for categorical DT theories

For each polynomial $\overline{\chi} \in \mathbb{Q}[m]$, we set

$$
N(S)_{\overline{\chi}} := \{v \in N(S) : \overline{\chi}_\sigma(v, m) = \overline{\chi}\} \cup \{0\}.
$$

The following is the main result in this section.
Theorem 3.5 For \(v_\bullet = (v_1, v_2, v_3) \in N(S)^3\) with \(v_2 = v_1 + v_3\), the functor (3.7) descends to the functor

\[
\mathcal{D}T^C^* (\mathcal{M}_X^{\sigma, \text{ss}}(v_1)) \times \mathcal{D}T^C^* (\mathcal{M}_X^{\sigma, \text{ss}}(v_2)) \to \mathcal{D}T^C^* (\mathcal{M}_X^{\sigma, \text{ss}}(v_3)).
\]

(3.22)

Proof By Corollary 3.2, it is enough to show that

\[
(\text{ev}_1^X, \text{ev}_3^X)^{-1}((Z_{\sigma, \text{us}}(v_1) \times \mathcal{M}_X(v_3)) \cup (\mathcal{M}_X(v_1) \times Z_{\sigma, \text{us}}(v_3))) \subset (\text{ev}_2^X)^{-1}(Z_{\sigma, \text{us}}(v_2)).
\]

The above inclusion follows from Lemma 3.6 below. Therefore we obtain the induced functor (3.22). \(\square\)

We have used the following lemma, whose proof is obvious from the definition of stability condition.

Lemma 3.6 For \(v_\bullet \in N(S)^3\) and a point of \(\mathcal{M}_X^{\text{ext}}(v_\bullet)\) corresponding to an exact sequence on \(X\)

\[
0 \to E_1 \to E_2 \to E_3 \to 0
\]

the object \(E_2\) is \(\sigma\)-semistable if and only if both of \(E_1, E_3\) are \(\sigma\)-semistable.

By taking the associated morphisms on K-groups, we obtain the following corollary.

Corollary 3.7 The functors (3.22) determine the associative algebra structure on

\[
\bigoplus_{v \in N(S)_{T}} K(\mathcal{D}T^C^* (\mathcal{M}_X^{\sigma, \text{ss}}(v))).
\]

(3.23)

Proof By the construction, the functor (3.22) fits into the commutative diagram

Here the top horizontal arrow is given by (3.7) and the vertical arrows are compositions of restrictions to \(\mathcal{M}_S(v_i)^{\text{fin}}\) and quotient functors. The vertical arrows are surjective on K-theory, so we have the surjective map of algebras

\[
\bigoplus_{v \in N(S)_{T}} K(\mathcal{M}_S(v)) \to \bigoplus_{v \in N(S)_{T}} K(\mathcal{D}T^C^* (\mathcal{M}_X^{\sigma, \text{ss}}(v))).
\]

The left hand side is associative by [21], hence (3.23) is also an associative algebra. \(\square\)
Remark 3.8 As we mentioned in Remark 3.4, the DT categories $\mathcal{DT}(\mathcal{M}_X^{\sigma,ss}(v))$ are independent of choices of $\mathcal{M}_S(v)^{\text{fin}}$. It is straightforward to check that, under the above equivalences, the functor (3.22) is also independent of choices of $\mathcal{M}_S(v_i)^{\text{fin}}$ satisfying (3.14). The same also applies for later constructions in Theorem 4.10, Theorem 7.3, Theorem 7.8 and Theorem 7.10.

Remark 3.9 Porta and Sala pointed out to the author that the result of Theorem 3.5 would apply to a dg-categorical setting, so that the dg-enhancements $\mathcal{DT}_{\text{dg}}(\mathcal{M}_X^{\sigma,ss}(v))$ form an $E_1$-algebra. Similarly to [21, Section 4], we need to involve higher parts of the Waldhausen constructions for the dg-enhancements of $\text{Coh}(S)$ in order to control the higher associativity.

As we mentioned in the introduction, if we have

$$\mathcal{M}_X^{\sigma,ss}(v) = t_0(\Omega^1_{\mathcal{M}_S^{\sigma,ss}(v)}[1])$$

then we have (see [32, Lemma 5.7])

$$\bigoplus_{v \in N(S)_\mathcal{X}} K(\mathcal{DT}(\mathcal{M}_X^{\sigma,ss}(v))) = \bigoplus_{v \in N(S)_\mathcal{X}} K(\mathcal{M}_S^{\sigma,ss}(v))$$

and the algebra structure on it is essentially the same one in [21]. Of course in general the condition (3.24) does not hold, and in this case the algebra structure of (3.23) is more difficult to describe (for example, see [32, Example 5.8]). Here we give some examples of the algebra (3.25) when (3.24) holds.

Example 3.10 (i) If we take $\mathcal{X} \equiv 1$, then we have

$$N(S)_\mathcal{X} = \mathbb{Z} \cdot [\text{pt}], \ [\text{pt}] := [\mathcal{O}_x]$$

where $x \in S$. Since the stack $\mathcal{M}_X^{\sigma,ss}(m[\text{pt}])$ coincides with the stack of zero dimensional sheaves on $X$ with length $m$, the condition (3.24) is satisfied. Then the algebra (3.25)

$$\bigoplus_{v \in N(S)_\mathcal{X}} K(\mathcal{DT}(\mathcal{M}_X^{\sigma,ss}(v))) = \bigoplus_{m \geq 0} K(\mathcal{M}_S(m[\text{pt}]))$$

is nothing but the K-theoretic Hall algebra of zero dimensional sheaves constructed by Zhao [37], which admits a morphism to the shuffle algebra.

(ii) Let $C = \mathbb{P}^1 \subset S$ be a $(-1)$-curve. Suppose that we have

$$N(S)_\mathcal{X} = \mathbb{Z} \cdot [\mathcal{O}_C(k)], \ k \in \mathbb{Z}$$

for a fixed $k$. Since $\mathcal{M}_X^{\sigma,ss}(m[\mathcal{O}_C(k)])$ consists of the direct sum $\mathcal{O}_C(k)^{\oplus m}$, the condition (3.24) is satisfied. The algebra (3.25) is

$$\bigoplus_{v \in N(S)_\mathcal{X}} K(\mathcal{DT}(\mathcal{M}_X^{\sigma,ss}(v))) = \bigoplus_{m \geq 0} K(BGL_m(\mathbb{C})) = \text{Sym}^\bullet_{\mathbb{Z}}(\mathbb{Z}[z^{\pm 1}]).$$

(3.26)
For \( f \in S^n(\mathbb{Z}[z^{\pm 1}]) \) and \( g \in S^m(\mathbb{Z}[z^{\pm 1}]) \), its product is given by (see [20, Proposition 9.1])

\[
 f \cdot g(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m}) = \text{Sym} \left( \frac{f(z_1, \ldots, z_n)g(z_{n+1}, \ldots, z_{n+m})}{\prod_{i=1}^{n} \prod_{j=n+1}^{n+m} (1 - z_i z_j^{-1})} \right).
\]

Here Sym means the symmetrization.

### 4 An action of zero dimensional DT categories to PT categories

In this section, we prove Theorem 1.2. We introduce the moduli stacks of pairs and their extensions, and recall the definition of PT categories on local surfaces defined in [32]. We then construct an action of DT categories of zero dimensional sheaves on them. By taking the associated action on the K-theory, we obtain a representation of K-theoretic Hall algebra of zero dimensional sheaves in Example 3.10 (i).

#### 4.1 Moduli stacks of pairs

For a smooth projective surface \( S \), let \( \mathcal{M}_S \) be the derived moduli stack of coherent sheaves on \( S \) considered in (3.1), and \( \mathfrak{g} \) the universal object (3.3). We define the derived stack \( \mathcal{M}_S^\dagger \) by

\[
 \rho^\dagger : \mathcal{M}_S^\dagger := \mathcal{V}((p_{\mathcal{M}_S}^\star \mathfrak{g})^\vee) \to \mathcal{M}_S.
\]

Here \( p_{\mathcal{M}_S} : S \times \mathcal{M}_S \to \mathcal{M}_S \) is the projection. For \( T \in \text{Aff} \), the \( T \)-valued points of \( \mathcal{M}_S^\dagger \) form the \( \infty \)-groupoid of pairs

\[
 (\mathfrak{g}, \xi) : \mathcal{O}_{S \times T} \to \mathfrak{g}
\]

where \( \mathfrak{g} \) is a \( T \)-valued point of \( \mathcal{M}_S \).

The classical truncation of \( \mathcal{M}_S^\dagger \) is a 1-stack

\[
 \mathcal{M}_S^\dagger := t_0(\mathcal{M}_S^\dagger) = \text{Spec}_{\mathcal{M}_S}(S(\mathcal{H}^0((p_{\mathcal{M}_S}^\star \mathcal{F})^\vee))). \quad (4.1)
\]

We have the universal pair on \( \mathcal{M}_S^\dagger \)

\[
 \mathcal{I}^\star = (\mathcal{O}_{S \times \mathcal{M}_S^\dagger} \to \mathcal{F}).
\]

Then we have the following description of the cotangent complex of \( \mathcal{M}_S^\dagger \)

\[
 \mathbb{L}_{\mathfrak{g}} |_{\mathcal{M}_S^\dagger} = \left( \mathcal{H}om_{\mathcal{M}_S^\dagger}(\mathcal{I}^\star, \mathcal{F}) \right)^\vee. \quad (4.2)
\]
Here $p_{\mathcal{M}^\dagger}: S \times \mathcal{M}^\dagger_S \to \mathcal{M}^\dagger_S$ is the projection. Also we have the decompositions into open and closed substacks

$$\mathcal{M}^\dagger_S = \bigsqcup_{v \in N(S)} \mathcal{M}^\dagger_S(v), \quad \mathcal{M}^\dagger_S = \bigsqcup_{v \in N(S)} \mathcal{M}^\dagger_S(v),$$

where each component corresponds to pairs $(F, \xi)$ such that $[F] = v$.

Let $\text{Coh}_{\leq 1}(S) \subset \text{Coh}(S)$ be the subcategory of sheaves $F$ with $\dim \text{Supp}(F) \leq 1$. We define the subgroup $N_{\leq 1}(S) \subset N(S)$ to be

$$N_{\leq 1}(S) := \text{Im}(K(\text{Coh}_{\leq 1}(S)) \to N(S)).$$

Note that we have an isomorphism

$$N_{\leq 1}(S) \cong N_S(S) \oplus \mathbb{Z}, \quad F \mapsto (l(F), \chi(F)) \quad (4.3)$$

where $l(F)$ is the fundamental one cycle of $F$. Below we identify an element $v \in N_{\leq 1}(S)$ with $(\beta, n) \in N_S(S) \oplus \mathbb{Z}$ by the above isomorphism.

For $v \in N_{\leq 1}(S)$, the derived stack $\mathcal{M}^\dagger_S(v)$ is quasi-smooth (see [32, Lemma 6.1]). We have the $(-1)$-shifted cotangent stack, and its classical truncation

$$\Omega_{\mathcal{M}^\dagger_S(v)}[-1] \to \mathcal{M}^\dagger_S(v), \quad t_0(\Omega_{\mathcal{M}^\dagger_S(v)}[-1]) \xrightarrow{\eta} \mathcal{M}^\dagger_S(v). \quad (4.4)$$

From (4.2), the fiber of the morphism $\eta$ at the pair $(F, \xi)$ is

$$\eta^{-1}((F, \xi)) = \text{Hom}_S(I^*, F[1])^\vee = \text{Hom}_S(F \otimes \omega^{-1}_S, I^*[1]). \quad (4.5)$$

Here $I^*$ is the two term complex $(O_S \xrightarrow{\xi} F)$ such that $O_S$ is located in degree zero.

On the other hand, let $B_S$ be the category of diagrams

$$\xymatrix{ 0 \ar[r] & \mathcal{V} \ar[r] & \mathcal{U} \ar[r] & F \otimes \omega^{-1}_S \ar[r] & 0 \ar[d]^-{\phi} \\ & & & F \ar[lu]^-{\xi} } \quad (4.6)$$

for $\mathcal{V} \in \langle O_S \rangle_{\text{ex}}$ and $F \in \text{Coh}_{\leq 1}(S)$. Here the top sequence is an exact sequence of coherent sheaves on $S$. Then $B_S$ is an abelian category, and we denote by $B_{\leq 1}^\circ \subset B_S$ the subcategory of diagrams (4.6) with rank$(\mathcal{V}) \leq 1$. The following result was proved in [32].

**Theorem 4.1** ([32, Theorem 6.3]) *The stack $t_0(\Omega_{\mathcal{M}^\dagger_S(v)}[-1])$ is isomorphic to the stack of diagrams (4.6) with $\mathcal{V} = O_S$ and $[F] = v$. Under the above isomorphism, the map $\eta$ in (4.4) sends the diagram (4.6) to the pair $(F, \xi)$.***
The correspondence in Theorem 4.1 is explained in the following way. Over the pair $(F, \xi)$, the diagram (4.6) with $V = \mathcal{O}_S$ and $[F] = \nu$ determines a point in the fiber (4.5) by

$$
F \otimes \omega_S^{-1}[-1] \xleftarrow{\sim} (\mathcal{O}_S \to \mathcal{U}) \xrightarrow{(\text{id}, \phi)} (\mathcal{O}_S \to F) = I^*.
$$

Conversely given a morphism $\vartheta : F \otimes \omega_S^{-1} \to I^*[1]$, then we associate the commutative diagram

\[
\begin{array}{ccccccccc}
\mathcal{O}_S & \rightarrow & \mathcal{U} & \rightarrow & F \otimes \omega_S^{-1} & \rightarrow & \mathcal{O}_S[1] \\
\uparrow & & & & \uparrow & & \\
I^* & \rightarrow & \mathcal{O}_S & \rightarrow & F & \rightarrow & I^*[1] & \rightarrow & \mathcal{O}_S[1].
\end{array}
\]

Here horizontal sequences are distinguished triangles. Therefore there exists a morphism $\phi : \mathcal{U} \rightarrow F$ which makes the above diagram commutative, and can be shown to be unique (see [32, Lemma 9.2]).

**Remark 4.2** Here we note that the above morphism $\vartheta$ is regarded as a morphism in some abelian category. Indeed let $\mathcal{T}, \mathcal{F} \subset \text{Coh}(S)$ be subcategories such that $\mathcal{T}$ consists of torsion sheaves and $\mathcal{F}$ consists of torsion free sheaves. Then $(\mathcal{T}, \mathcal{F})$ forms a torsion pair, and we have the associated tilting [12]

$$
\text{Coh}^\varpi(S) := \langle \mathcal{F}[1], \mathcal{T} \rangle \subset D^b_{\text{coh}}(S).
$$

As a general result of tilting, $\text{Coh}^\varpi(S)$ is the heart of a t-structure on $D^b_{\text{coh}}(S)$, hence an abelian category. Then both of $F \otimes \omega_S^{-1}$ and $I^*[1]$ are objects in $\text{Coh}^\varpi(S)$ and $\vartheta$ is a morphism in $\text{Coh}^\varpi(S)$.

### 4.2 Derived moduli stacks of extensions of pairs

In this subsection, we introduce the derived moduli stack of extensions of pairs, and extend the categorified COHA structure to the module structure over it on the derived category of coherent sheaves on derived moduli stacks of pairs. We define the derived stack $\mathcal{M}^\text{ext,†}_S$ by the Cartesian square

\[
\begin{array}{ccc}
\mathcal{M}^\text{ext,†}_S & \rightarrow & \mathcal{M}^\text{ext}_S \\
(\text{ev}_1^\text{†}, \text{ev}_3^\text{†}) \downarrow & \square & \downarrow (\text{ev}_1, \text{ev}_3) \\
\mathcal{M}_S^\text{†} \times \mathcal{M}_S & \rightarrow & \mathcal{M}_S \times \mathcal{M}_S.
\end{array}
\]
For \( T \in \mathcal{d} \mathcal{A} f\), the \( T \)-valued points of \( \mathcal{M}_S^\dagger \) form the \( \infty \)-groupoid of diagrams

\[
\begin{array}{ccc}
\mathcal{O}_{S \times T} & \xrightarrow{\xi} & \mathcal{F}_1 \\
\downarrow & & \downarrow \\
\mathcal{F}_2 & \rightarrow & \mathcal{F}_3.
\end{array}
\] (4.10)

Here the bottom sequence is a \( T \)-valued point of \( \mathcal{M}_S^\text{ext} \). Let us take \( v_\bullet = (v_1, v_2, v_3) \in N_{\leq 1}(S)^{\times 3} \). We have the open and closed derived substack

\[
\mathcal{M}_S^\text{ext,}^\dagger(v_\bullet) \subset \mathcal{M}_S^\text{ext}.
\]

Lemma 4.3 The derived stack \( \mathcal{M}_S^\text{ext,}^\dagger(v_\bullet) \) is quasi-smooth.

Proof The lemma follows since \((\text{ev}_1^\dagger, \text{ev}_3^\dagger)\) is quasi-smooth and both of \( \mathcal{M}_S^\dagger(v_1)\), \( \mathcal{M}_S(v_3) \) are quasi-smooth. \( \square \)

We have the diagram

\[
\begin{array}{ccc}
\mathcal{M}_S^\text{ext,}^\dagger(v_\bullet) & \xrightarrow{\text{ev}_2^\dagger} & \mathcal{M}_S^\dagger(v_2) \\
\downarrow & \downarrow & \\
\mathcal{M}_S^\dagger(v_1) \times \mathcal{M}_S(v_3).
\end{array}
\] (4.11)

Here \( \text{ev}_2^\dagger \) is obtained by sending a diagram (4.10) to the composition \( \mathcal{O}_{S \times T} \xrightarrow{\xi} \mathcal{F}_1 \rightarrow \mathcal{F}_2 \). Note that the vertical arrow is quasi-smooth. As for the horizontal arrow, we have the following.

Lemma 4.4 The morphism \( \text{ev}_2^\dagger \) in the diagram (4.11) is proper.

Proof For a point of \( \mathcal{M}_S^\dagger(v_2) \) corresponding to a pair \((F_2, \xi_2)\), the fiber of \( \text{ev}_2^\dagger \) at this point corresponds to diagrams

\[
\mathcal{O}_S \xrightarrow{\xi_2} F_2 \rightarrow F_3, \quad [F_3] = v_3,
\]

whose composition is zero. The classical moduli space of such diagrams is a closed subscheme of the Quot scheme parameterizing quotients \( F_2 \twoheadrightarrow F_3 \) with \([F_3] = v_3\), hence it is a proper scheme. Therefore \( \text{ev}_2^\dagger \) is proper. \( \square \)

By Lemma 4.4, the diagram (4.11) induces the functor

\[
\text{ev}_{2*}^\dagger(\text{ev}_1^\dagger, \text{ev}_3^\dagger)^\circ : \mathcal{D}_\text{coh}(\mathcal{M}_S^\dagger(v_1)) \times \mathcal{D}_\text{coh}(\mathcal{M}_S(v_3)) \rightarrow \mathcal{D}_\text{coh}(\mathcal{M}_S^\dagger(v_2)).
\] (4.12)
4.3 Moduli stacks of D0–D2–D6 bound states

In this subsection, we recall the notion of D0–D2–D6 bound states and their moduli stacks. Let $\overline{X}$ be the projective compactification of $X$

$$X \subset \overline{X} := \mathbb{P}_S(\omega_S \oplus \mathcal{O}_S) = X \cup S_\infty.$$ 

Here $S_\infty$ is the divisor at the infinity. The category of D0–D2–D6 bound states on the non-compact CY threefold $X = \text{Tot}_S(\omega_S)$ is defined by the extension closure in $D^b_{\text{coh}}(\overline{X})$

$$\mathcal{A}_X := \langle \mathcal{O}_{\overline{X}}, \text{Coh}_{\leq 1}(X)[−1]\rangle_{\text{ex}} \subset D^b_{\text{coh}}(\overline{X}).$$

Here $\text{Coh}_{\leq 1}(X)$ is the subcategory of objects in $\text{Coh}_{\text{cpt}}(X)$ whose supports have dimensions less than or equal to one. We regard $\text{Coh}_{\leq 1}(X)$ as a subcategory of $\text{Coh}(\overline{X})$ by the push-forward of the open immersion $X \subset \overline{X}$. The arguments in [33, Lemma 3.5, Proposition 3.6] show that $\mathcal{A}_X$ is an abelian subcategory of $D^b_{\text{coh}}(\overline{X})$. We denote by $\mathcal{A}^{\leq 1}_X \subset \mathcal{A}_X$ the subcategory of objects $E \in \mathcal{A}_X$ with $\text{rank}(E) \leq 1$.

There is a group homomorphism

$$\text{cl}: K(\mathcal{A}_X) \rightarrow \mathbb{Z} \oplus N\leq 1(S)$$

characterized by the condition that $\text{cl}(\mathcal{O}_X) = (1, 0)$ and $\text{cl}(F) = (0, [\pi_* F])$ for $F \in \text{Coh}_{\leq 1}(X)$. We define the (classical) moduli stack of rank one objects in $\mathcal{A}_X$ to be the 2-functor

$$\mathcal{M}_X^\dagger: \text{Aff}^{\text{op}} \rightarrow \text{Groupoid}$$

whose $T$-valued points for $T \in \text{Aff}$ form the groupoid of data

$$E \in D^b_{\text{coh}}(\overline{X} \times T), \quad \lambda: E \otimes \mathcal{O}_{S_\infty \times T} \xrightarrow{\cong} \mathcal{O}_{S_\infty \times T}$$

such that for any closed point $x \in T$, we have

$$E_x := L\lambda^* E \in \mathcal{A}_X, \quad i_x: \overline{X} \times \{x\} \hookrightarrow \overline{X} \times T.$$ 

The isomorphisms of the groupoid $\mathcal{M}_X^\dagger(T)$ are given by isomorphisms of objects $E_T$ which commute with the trivializations at the infinity. We have the decomposition of $\mathcal{M}_X^\dagger$ into open and closed substacks

$$\mathcal{M}_X^\dagger = \coprod_{v \in N\leq 1(S)} \mathcal{M}_X^\dagger(v)$$

where $\mathcal{M}_X^\dagger(v)$ corresponds to $E \in \mathcal{A}_X$ with $\text{cl}(E) = (1, −v)$. The following result is proved in [32]:
Theorem 4.5 ([32, Theorem 6.3]) There is an equivalence of categories

\[ \mathcal{A}_X^{\leq 1} \sim \mathcal{B}_S^{\leq 1}. \]  

Moreover the above equivalence together with the isomorphism in Theorem 4.5 induce the isomorphism of stacks over \( \mathcal{M}_S^\dagger(v) \)

\[ \mathcal{M}_X(v) \xrightarrow{\cong} \iota_0(\Omega_{2R_S^\dagger(v)}[-1]). \]  

4.4 Moduli stacks of extensions in \( \mathcal{A}_X \)

In this subsection, we consider the classical moduli stack of exact sequences in \( \mathcal{A}_X \) of the form \( 0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3[-1] \rightarrow 0 \) where \( E_1, E_2 \) are rank one objects in \( \mathcal{A}_X \) and \( E_3 \in \text{Coh}_{\leq 1}(X) \). More precisely, we take \( \upsilon_\bullet = (\upsilon_1, \upsilon_2, \upsilon_3) \in N_{\leq 1}(S)^3 \) and define the classical stack

\[ \mathcal{M}_X^{\text{ext.} \dagger}(\upsilon_\bullet): \text{Aff}^{\text{op}} \rightarrow \text{Groupoid} \]

by sending \( T \in \text{Aff} \) to the groupoid of distinguished triangles \( \mathcal{E}_1 \xrightarrow{i} \mathcal{E}_2 \xrightarrow{} \mathcal{E}_3[-1] \) together with commutative diagrams

\[
\begin{array}{ccc}
\mathcal{E}_1 \otimes \mathcal{O}_{S_\infty \times T} & \xrightarrow{i} & \mathcal{E}_2 \otimes \mathcal{O}_{S_\infty \times T} \\
\lambda_1 \cong & & \lambda_2 \cong \\
\mathcal{O}_{S_\infty \times T} & = & \mathcal{O}_{S_\infty \times T}.
\end{array}
\]

Here \( (\mathcal{E}_i, \lambda_i) \) for \( i = 1, 2 \) are \( T \)-valued points of \( \mathcal{M}_X^\dagger(\upsilon_i) \) and \( \mathcal{E}_3 \) is a \( T \)-valued point of \( \mathcal{M}_X(\upsilon_3) \). We also have the evaluation morphisms

\[ \mathcal{M}_X^{\text{ext.} \dagger}(\upsilon_\bullet) \xrightarrow{(\text{ev}_1^{X, \dagger}, \text{ev}_3^{X, \dagger})} \mathcal{M}_X^\dagger(\upsilon_1) \times \mathcal{M}_X(\upsilon_3) \]  

where \( \text{ev}_i^{X, \dagger} \) sends \( \mathcal{E}_\bullet \) to \( \mathcal{E}_i \). We have the following description of each point of \( \mathcal{M}_X^{\text{ext.} \dagger}(\upsilon_\bullet) \).
Lemma 4.6  Giving a point of $\mathcal{M}_X^{\text{ext.}^\dagger}(v_\bullet)$ is equivalent to giving a point of $\mathcal{M}_S^{\text{ext.}^\dagger}(v_\bullet)$

\[ \mathcal{O}_S \rightarrow \mathcal{O}_S \]

\[ 0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0. \]

(4.17)

together with a morphism of distinguished triangles

\[ F_1 \otimes \omega_S^{-1} \rightarrow F_2 \otimes \omega_S^{-1} \rightarrow F_3 \otimes \omega_S^{-1} \]

\[ \varphi_1 \downarrow \quad \varphi_2 \downarrow \quad \varphi_3 \downarrow \]

\[ I_1^\bullet[1] \rightarrow I_2^\bullet[1] \rightarrow I_3^\bullet \rightarrow 0. \]

(4.18)

Here $I_i^\bullet = (\mathcal{O}_S \rightarrow F_i)$ and the bottom sequence of (4.18) is given by the diagram (4.17).

Proof  By the equivalence (4.14), the stack $\mathcal{M}_X^{\text{ext.}^\dagger}(v_\bullet)$ is isomorphic to the stack of exact sequences in $\mathcal{B}_S^{\leq 1}$ of the form

\[
\begin{array}{c}
0 \rightarrow \\
0 \rightarrow \mathcal{O}_S \rightarrow U_1 \rightarrow F_1 \otimes \omega_S^{-1} \rightarrow 0 \\
\downarrow \varphi \\
F
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow \\
0 \rightarrow \mathcal{O}_S \rightarrow U_2 \rightarrow F_2 \otimes \omega_S^{-1} \rightarrow 0 \\
\downarrow \varphi_2 \\
F_2
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow \\
0 \rightarrow \mathcal{O}_S \rightarrow U_3 \rightarrow F_3 \otimes \omega_S^{-1} \rightarrow 0 \\
\downarrow \varphi_3 \\
F_3
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \\
\rightarrow 0
\end{array}
\]
such that \([F_i] = v_i\). By taking the associated exact sequence for dotted arrows, we obtain the diagram (4.17). From the correspondence of objects in \(B^<\leq_{1} S\) explained after Theorem 4.1, the above diagram immediately gives the diagram (4.18).

Conversely suppose that diagrams (4.17), (4.18) are given. Then by composing the diagram (4.18) with \(I^\bullet[1] \to \mathcal{O}_S[1]\) and taking cones, we obtain the commutative diagram

\[
\begin{array}{cccc}
U_1 & \longrightarrow & U_2 & \longrightarrow & U_3 \\
\downarrow & & \downarrow & & \downarrow \\
F_1 \otimes \omega_S^{-1} & \longrightarrow & F_2 \otimes \omega_S^{-1} & \longrightarrow & F_3 \otimes \omega_S^{-1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_S[1] & \longrightarrow & \mathcal{O}_S[1].
\end{array}
\]

Here each horizontal and vertical sequences are distinguished triangles. The above diagram together with (4.17) give an exact sequence in \(B^<\leq_{1} S\), hence an exact sequence in \(\mathcal{M}^\leq_{X}\) by the equivalence (4.14).

For \(v_\bullet = (v_1, v_2, v_3) \in N_{\leq 1}(S)^{\times 3}\), we have the morphism

\[
\pi^\dagger: \mathcal{M}_{X}^{\text{ext},+}(v_\bullet) \to \mathcal{M}_{S}^{\text{ext},+}(v_\bullet)
\]

by sending a point in \(\mathcal{M}_{X}^{\text{ext},+}(v_\bullet)\) to the diagram (4.17). On the other hand, let \(\text{ev}^\dagger\) be the morphism

\[
\text{ev}^\dagger = (\text{ev}_1^\dagger, \text{ev}_2^\dagger, \text{ev}_3^\dagger): \mathcal{M}_{S}^{\text{ext},+}(v_\bullet) \to \mathcal{M}_S^+(v_1) \times \mathcal{M}_S^+(v_2) \times \mathcal{M}_S(v_3).
\]

We have the following proposition.

**Proposition 4.7** We have an isomorphism over \(\mathcal{M}_{S}^{\text{ext},+}(v_\bullet)\)

\[
\mathcal{M}_{X}^{\text{ext},+}(v_\bullet) \xrightarrow{\cong} t_0(\Omega_{\text{ev}^\dagger}[-2]).
\]

**Proof** Let \(p \in \mathcal{M}_S^{\text{ext}}(v_\bullet)\) be a point corresponding to the diagram (4.17). By Lemma 4.6, the fiber of the morphism (4.19) at \(p\) is given by the diagram (4.18). By Remark 4.2, the diagram (4.18) is regarded as a morphism of exact sequences in the abelian category \(\text{Coh}^+(S)\). Therefore similarly to the proof of Proposition 3.1, we have

\[
(p^\dagger)^{-1}(p) = \text{Ker}(\text{Hom}(F_2 \otimes \omega_S^{-1}, I^\bullet_2[1]) \xrightarrow{k_0(-)_{\text{oi}}} \text{Hom}(F_1 \otimes \omega_S^{-1}, F_3)).
\]
On the other hand, we have the commutative diagram

\[
\begin{array}{cccccc}
(e_{v_1}, e_{v_3})^* \mathbb{L}_{\mathcal{M}_S(v_1) \times \mathcal{M}_S(v_3)} & \cong & (e_{v_1}, e_{v_3})^* \mathbb{L}_{\mathcal{M}_S(v_1) \times \mathcal{M}_S(v_3)} \\
\downarrow & & \downarrow \\
\mathbb{L}_{\mathcal{M}_S(v_1) \times \mathcal{M}_S(v_2) \times \mathcal{M}_S(v_3)} & \rightarrow & \mathbb{L}_{\mathcal{M}_S^\text{ext}(v_\bullet)} \\
\downarrow & & \downarrow \\
\mathbb{L}_{\mathcal{M}_S(v_2)} & \cong & (e_{v_1}, e_{v_3})^* (\mathcal{H}om_{p}\mathfrak{M} \times \mathfrak{M} (\mathfrak{F}_3, \mathfrak{F}_1)[1])^\vee
\end{array}
\]

Here each horizontal and vertical sequences are distinguished triangle. By taking the cone, we obtain the distinguished triangle

\[
(e_{v_1}, e_{v_3})^* \mathbb{L}_{\mathcal{M}_S^\dagger(v_2)} \rightarrow (ev_{v_1}^\dagger, ev_{v_3}^\dagger)^* (\mathcal{H}om_{p}\mathfrak{M} \times \mathfrak{M} (\mathfrak{F}_3, \mathfrak{F}_1)[1])^\vee \rightarrow \mathbb{L}_{ev^\dagger}.
\]

By restricting it to \(p\) and the associated exact sequence of cohomologies, we see that

\[\mathcal{H}^{-2}(\mathbb{L}_{ev^\dagger}|_p) = \text{Ker}(\text{Hom}(F_2 \otimes \omega_S^{-1}, I_2^\bullet[1]) \rightarrow \text{Hom}(F_1, F_3 \otimes \omega_S)),\]

which is identified with (4.21). Therefore similarly to Proposition 3.1, we have the isomorphism (4.20).

Let us take finite type derived open substacks \(\mathfrak{M}_S^\dagger(v_i)^\text{fin} \subset \mathfrak{M}_S^\dagger(v_i)\) for \(i = 1, 2\) and \(\mathfrak{M}_S(v_3)^\text{fin} \subset \mathfrak{M}_S(v_3)\) satisfying

\[
\mathfrak{M}_S^\text{ext,\dagger}(v_\bullet)^\text{fin} := (ev_{v_2}^\dagger)^{-1}(\mathfrak{M}_S^\dagger(v_2)^\text{fin}) \subset (ev_{v_1}^\dagger, ev_{v_3}^\dagger)^{-1}(\mathfrak{M}_S^\dagger(v_1)^\text{fin} \times \mathfrak{M}_S(v_3)^\text{fin}).
\]

Then the diagram (4.11) restricts to the diagram

\[
\begin{array}{ccc}
\mathfrak{M}_S^\text{ext,\dagger}(v_\bullet)^\text{fin} & \xrightarrow{(ev_{v_1}^\dagger, ev_{v_3}^\dagger)} & \mathfrak{M}_S^\dagger(v_2)^\text{fin} \\
\downarrow & & \downarrow \\
\mathfrak{M}_S^\dagger(v_1)^\text{fin} \times \mathfrak{M}_S(v_3)^\text{fin}.
\end{array}
\]

The vertical arrow is quasi-smooth and the horizontal arrow is proper. Therefore we have the induced functor

\[
ev_{v_2}^\dagger (ev_{v_1}^\dagger, ev_{v_3}^\dagger)^* : D^b_{\text{coh}}(\mathfrak{M}_S^\dagger(v_1)^\text{fin}) \times D^b_{\text{coh}}(\mathfrak{M}_S(v_3)^\text{fin}) \rightarrow D^b_{\text{coh}}(\mathfrak{M}_S^\dagger(v_2)^\text{fin}).
\]

Similarly to Corollary 3.2, we also have the following corollary of Proposition 4.7.
Corollary 4.8 For conical closed substacks $Z_i \subset \mathcal{M}_X(v_i)$ for $i = 1, 2$ and $Z_3 \subset \mathcal{M}_X(v_3)$ suppose that the following condition holds

$$(\text{ev}_1^{X,\dagger}, \text{ev}_3^{X,\dagger})^{-1}((Z_1 \times \mathcal{M}_X(v_3)) \cup (\mathcal{M}_X(v_1) \times Z_3)) \subset (\text{ev}_2^{X,\dagger})^{-1}(Z_2).$$

Then the functor (4.24) descends to the functor

$$D^b_{\text{coh}}(\mathcal{M}_S^\dagger(v_1)^{\text{fin}})/C_{Z_1^\dagger} \times D^b_{\text{coh}}(\mathcal{M}_S(v_3)^{\text{fin}})/C_{Z_3^\dagger} \to D^b_{\text{coh}}(\mathcal{M}_S^\dagger(v_2)^{\text{fin}})/C_{Z_2^\dagger}.$$ 

4.5 Categorical PT theory

Here we recall the definition of PT categories and give a proof of Theorem 1.2. By definition, a PT stable pair consists of a pair [22]

$$(F, s), \ F \in \text{Coh}_{\leq 1}(X), \ s : \mathcal{O}_X \to F$$

such that $F$ is pure one dimensional and $s$ is surjective in dimension one. For $(\beta, n) \in N_{\leq 1}(S)$, we denote by

$$P_n(X, \beta)$$

the moduli space of PT stable pairs (4.25) satisfying $[\pi_* F] = (\beta, n)$. The moduli space of stable pairs $P_n(X, \beta)$ is known to be a quasi-projective scheme.

We have the open immersion

$$P_n(X, \beta) \subset \mathcal{M}_X^\dagger(\beta, n)$$

sending a pair $(F, s)$ to the two term complex $(\mathcal{O}_X \overset{s}{\to} F)$. We define the following conical closed substack

$$Z_{P,\text{us}}(\beta, n) := \mathcal{M}_X^\dagger(\beta, n) \setminus P_n(X, \beta) \subset \mathcal{M}_X^\dagger(\beta, n).$$

Since $P_n(X, \beta)$ is a quasi-projective scheme, there is a derived open substack $\mathcal{M}_S^\dagger(\beta, n)^{\text{fin}} \subset \mathcal{M}_S^\dagger(\beta, n)$ of finite type such that

$$P_n(X, \beta) \subset i_0(\Omega_{\mathcal{M}_S^\dagger(\beta, n)^{\text{fin}}})[-1]) \subset \mathcal{M}_X^\dagger(\beta, n).$$ (4.26)

Here we have used the isomorphism (4.15).

Definition 4.9 ([32, Definition 6.6]) The $\mathbb{C}^*$-equivariant categorical PT theory is defined by

$$\mathcal{D}\mathcal{T}^{\mathbb{C}^*}(P_n(X, \beta)) := D^b_{\text{coh}}(\mathcal{M}_S^\dagger(\beta, n)^{\text{fin}})/C_{Z_{P,\text{us}}(\beta, n)^{\text{fin}}}.$$
Similarly to Remark 3.4, the above definition is independent of a choice of $\mathcal{M}_S^\dagger(\beta, n)^{\text{fin}}$ satisfying (4.26). Let us take $v_\bullet \in N_{\leq 1}(S)^{\times 3}$ to be

$$v_1 = (\beta, n), \quad v_2 = (\beta, n + m), \quad v_3 = (0, m) = m[pt].$$

We also take derived open substacks of finite type

$$\mathcal{M}_S^\dagger(v_1)^{\text{fin}} \subset \mathcal{M}_S^\dagger(v_1), \quad \mathcal{M}_S^\dagger(v_2)^{\text{fin}} \subset \mathcal{M}_S^\dagger(v_2), \quad \mathcal{M}_S^{\text{fin}}(v_3) = \mathcal{M}_S(m[pt])$$

satisfying (4.26) for $v_1, v_2$ and the condition (4.22). Here we note that $\mathcal{M}_S(m[pt])$ is the derived moduli stack of zero dimensional sheaves of length $m$, so it is of finite type. Then the diagram (4.23) induces the functor

$$\text{ev}_2^\dagger(\text{ev}_1^\dagger, \text{ev}_3^\dagger)^*: D^b_{\text{coh}}(\mathcal{M}_S^\dagger(v_1)^{\text{fin}}) \times D^b_{\text{coh}}(\mathcal{M}_S(m[pt])) \to D^b_{\text{coh}}(\mathcal{M}_S^\dagger(v_2)^{\text{fin}}).$$

(4.27)

**Theorem 4.10** The functor (4.27) descends to the functor

$$\mathcal{DT}_C^a(P_n(X, \beta)) \times \mathcal{DT}_C^a(M_X^{\sigma-ss}(m[pt])) \to \mathcal{DT}_C^a(P_{n+m}(X, \beta)).$$

(4.28)

**Proof** By Lemma 4.11 below, we have the following inclusion

$$(\text{ev}_1^X, \text{ev}_3^X)^{-1}(Z_{P, \text{us}}(v_1) \times M_X(v_3)) \subset (\text{ev}_2^X)^{-1}(Z_{P, \text{us}}(v_2)).$$

Therefore the result follows from Corollary 4.8. \qed

Here we have used the following lemma.

**Lemma 4.11** For an exact sequence $0 \to E_1 \to E_2 \to E_3 \to 0$ in $\mathcal{A}_X$, suppose that $E_2 = (\mathcal{O}_X \xrightarrow{s_2} F_2)$ for a PT stable pair $(F_2, s_2)$ and $E_3 = Q[-1]$ for a zero dimensional sheaf $Q$. Then $E_1 = (\mathcal{O}_X \xrightarrow{s_1} F_1)$ for a PT stable pair $(F_1, s_1)$.

**Proof** By taking the long exact sequence of cohomologies, we have the surjection

$$\mathcal{H}^1(E_2) = \text{Cok}(s_2) \twoheadrightarrow \mathcal{H}^1(E_3) = Q.$$  

(4.29)

On the other hand, we have the following commutative diagram

$$\begin{array}{cccc}
F_1[-1] & \to & E_1 \\
\downarrow & & \downarrow \\
F_2[-1] & \to & E_2 & \to & \mathcal{O}_X \\
\alpha[-1] & & \downarrow & & \\
Q[-1] & = & Q[-1].
\end{array}$$
Here vertical sequences are distinguished triangles. The map $\alpha : F_2 \to Q$ is the composition of $F_2 \to \text{Cok}(s_2)$ with the morphism (4.29), so it is surjective. Therefore $F_1$ is a subsheaf of $F_2$, which is a pure one dimensional sheaf. By taking the cones, we obtain the distinguished triangle $F_1[-1] \to E_1 \to \mathcal{O}_X$, hence $E_1 = (\mathcal{O}_X \overset{s_1}{\to} F_1)$ for a pair $(F_1, s_1)$. Since $(F_1, s_1)$ is isomorphic to $(F_2, s_2)$ outside the support of $Q$ and $s_2$ is surjective in dimension one, $s_1$ is also surjective in dimension one. Therefore $(F_1, s_1)$ is PT stable. \hfill $\square$

Similarly to Corollary 3.7, we have the following corollary of Theorem 4.10.

**Corollary 4.12** The functors (4.28) induce the right action of the K-theoretic Hall-algebra of zero dimensional sheaves to the direct sum of PT categories for a fixed $\beta$

$$\bigoplus_{n \in \mathbb{Z}} K(DT^C_\beta (P_n(X, \beta))) \times \bigoplus_{m \geq 0} K(DT^C_\beta (\mathcal{M}_{\sigma-ss}^X([pt]))) \to \bigoplus_{n \in \mathbb{Z}} K(DT^C_\beta (P_n(X, \beta))).$$

### 5 Hecke correspondences of categorical PT theories

In this section, we consider the simple operators in Corollary 4.12. They are induced by the stacks of Hecke correspondences, and we describe them as two kinds of projectivizations over some quasi-smooth derived stacks. Using these descriptions, we construct the annihilator operators of the above simple operators.

#### 5.1 Simple operators

Here we consider the operator (4.28) for $m = 1$

$$\phi_p : DT^C_\beta (P_n(X, \beta)) \times DT^C_\beta (\mathcal{M}_{\sigma-ss}^X([pt])) \to DT^C_\beta (P_{n+1}(X, \beta)).$$

Note that $\mathcal{M}_S([pt])$ is the derived moduli stack of skyscraper sheaves of points of $S$, which is written as

$$\mathcal{M}_S([pt]) = [S/\mathbb{C}^*].$$

Here $\mathcal{S}$ is a quasi-smooth derived scheme whose classical truncation is $S$, and $\mathbb{C}^*$ acts on it trivially. Therefore we have the decomposition

$$DT^C_\beta (\mathcal{M}_{\sigma-ss}^X([pt])) = D^b_{\text{coh}}(\mathcal{M}_S([pt])) = \bigoplus_{k \in \mathbb{Z}} D^b_{\text{coh}}(\mathcal{S})_k \quad (5.1)$$

where $D^b_{\text{coh}}(\mathcal{S})_k$ consists of objects with $\mathbb{C}^*$-weights $k$. We have the following description of the derived scheme $\mathcal{S}$. 
Lemma 5.1 We have $\mathcal{G} = \text{Spec}_S S(\omega_S[1])$.

Proof Since $\text{Ext}^2_S(\mathcal{O}_x, \mathcal{O}_x) = \omega_S \vert_x$ for each point $x \in S$, the closed immersion $S \hookrightarrow \mathcal{G}$ is a square zero extension such that we have the distinguished triangle

$$\omega_S[1] \rightarrow \mathcal{O}_{\mathcal{G}} \rightarrow \mathcal{O}_S.$$ 

Therefore the square zero extension $S \hookrightarrow \mathcal{G}$ is classified by the map $\delta$ in the following distinguished triangle (see [10, Proposition 5.4.2])

$$\omega_S[1] \rightarrow \mathbb{L}_S \rightarrow \Omega_S \xrightarrow{\delta} \omega_S[2].$$

Since the structure sheaf of the diagonal $\Delta \subset S \times S$ is the universal family on $\mathcal{G} \times S$ restricted to $S \times S$, we have the isomorphism

$$\mathbb{L}_S \vert_S \cong (\tau_{\geq 0} p_{1*} \mathcal{H}om_{S \times S}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)[1])^\vee.$$

(5.2)

Here $p_1 : S \times S \rightarrow S$ is the first projection. By the Hochschild-Kostant-Rosenberg isomorphism, we have

$$p_{1*} \mathcal{H}om_{S \times S}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)[1] \cong \mathcal{O}_S[1] \oplus T_S \oplus \wedge^2 T_S[-1].$$

It follows that (5.2) is isomorphic to $\Omega_S \oplus \omega_S[1]$, and $\delta = 0$ follows. Therefore $S \hookrightarrow \mathcal{G}$ is a trivial square zero extension, and the lemma follows. \qed

Note that have the diagram

$$S \hookrightarrow \mathcal{G} \xrightarrow{\psi} [\mathcal{G}/\mathbb{C}^*]$$

(5.3)

where the left arrow is induced by the classical truncation and the right arrow is induced by the natural morphism $B\mathbb{C}^* \rightarrow \text{Spec } \mathbb{C}$. We define

$$\lambda_k := \psi^* t_* (-) \otimes \mathcal{O}(k) : D^b_{\text{coh}}(S) \rightarrow D^b_{\text{coh}}(\mathcal{G})_k$$

where $\mathcal{O}(k)$ is a line bundle on $[\mathcal{G}/\mathbb{C}^*]$ determined by the one dimensional representation of $\mathbb{C}^*$ with weight $k$.

Definition 5.2 For each $k \in \mathbb{Z}$ and $\mathcal{E} \in D^b_{\text{coh}}(S)$, we define the functor $\mu^+_{\mathcal{E}, k}$ by

$$\mu^+_{\mathcal{E}, k}(-) := \phi_P(((-), \lambda_k(\mathcal{E}))) : \mathcal{D}T^{\mathbb{C}^*}(P_n(X, \beta)) \rightarrow \mathcal{D}T^{\mathbb{C}^*}(P_{n+1}(X, \beta)).$$

Note that the functor $\lambda_k$ induces the isomorphism on the K-theory

$$\lambda_k : K(S) \xrightarrow{\cong} K(\mathcal{G})_k.$$
Therefore under the decomposition (5.1) and the above isomorphism, the induced map of \( \phi_p \) on K-theory

\[
\phi_p : K(\mathcal{D}T^{C^*}(P_n(X, \beta))) \times K(\mathcal{D}T^{C^*}(\mathcal{M}^\text{ss}_{X}([pt]))) \to K(\mathcal{D}T^{C^*}(P_{n+1}(X, \beta)))
\]

is written as, for \( \mathcal{E}_k \in K(S) \),

\[
\phi_p \left( \left\{ (-), \sum_{k \in \mathbb{Z}} \mathcal{E}_k \right\} \right) = \sum_{k \in \mathbb{Z}} \mu_\mathcal{E}_k(k)(-).
\]

In the rest of this section, we construct functors in the opposite direction

\[
\mu_{\mathcal{E}, k}(-) : \mathcal{D}T^{C^*}(P_{n+1}(X, \beta)) \to \mathcal{D}T^{C^*}(P_n(X, \beta)) \quad (5.4)
\]

as an analogy of annihilation operators for Heinsenberg algebra action on homologies of Hilbert schemes of points \([11,18]\).

### 5.2 The stack of Hecke correspondences

We describe the operator \( \mu_{\mathcal{E}, k}^+ \) in terms of the stack of Hecke correspondences, which we will define below. For \( v \in \mathbb{N} \leq 1 (S) \), we denote by

\[
\mathcal{M}_S(v)_{\text{pure}} \subset \mathcal{M}_S(v), \quad \mathcal{M}_S(v_1)_{\text{pure}} \subset \mathcal{M}_S(v)
\]

the derived open substacks consisting of pure one dimensional sheaves \( F \), pairs \((F, \xi)\) such that \( F \) is pure one dimensional, respectively. In the notation of the diagram (4.11), we also define

\[
\mathcal{M}_S(v_2)_{\text{pure}} := (\text{ev}_2^\dagger)^{-1}(\mathcal{M}_S(v_2)_{\text{pure}}) \subset \mathcal{M}^{\text{ext.}^\dagger}_S(v_2).
\]

Since any non-zero subsheaf of a pure one dimensional sheaf is also pure one dimensional, for \( v_\bullet \in \mathbb{N}_{\leq 1}(S)^3 \) the diagram (4.11) restricts to the diagram

\[
\begin{array}{ccc}
\mathcal{M}_S^{\text{ext.}^\dagger}(v_\bullet)_{\text{fin}} & \xrightarrow{\text{ev}_2^\dagger} & \mathcal{M}_S^\dagger(v_2)_{\text{fin}} \\
\downarrow & & \downarrow \\
(\text{ev}_1^\dagger, \text{ev}_3^\dagger) & & \\
\mathcal{M}_S^\dagger(v_1)_{\text{fin}} \times \mathcal{M}_S(v_3)_{\text{fin}} & & 
\end{array}
\]

We take \( v_\bullet \in \mathbb{N}_{\leq 1}(S)^3 \) of the form

\[
v_1 = (\beta, n), \quad v_2 = (\beta, n + 1), \quad v_3 = (0, 1) = [\text{pt}].
\]
Here $\beta \in \text{NS}(S)$ is a non-zero effective curve class on $S$ and $n \in \mathbb{Z}$. Then we have the following diagram

$$
\begin{array}{c}
\text{Hecke}(v \bullet) & \xrightarrow{\pi_2} & \mathcal{M}_S^\text{ext,\dagger}(v \bullet)_{\text{pure}} & \xrightarrow{\text{ev}_2^\dagger} & \mathcal{M}_S^\dagger(v_2)_{\text{pure}} \\
& & \downarrow & \downarrow & \\
(\pi_1, \pi_3) & \xrightarrow{\tau} & \mathcal{M}_S^\dagger(v_1)_{\text{pure}} \times [S/\mathbb{C}^*] & \xrightarrow{\text{id} \times \vartheta} & \mathcal{M}_S^\dagger(v_1)_{\text{pure}} \times \mathbb{S} \\
& & \downarrow & \downarrow & \\
& & \mathcal{M}_S^\dagger(v_1)_{\text{pure}} \times S & \xrightarrow{\text{id} \times \iota} & \mathcal{M}_S^\dagger(v_1)_{\text{pure}} \times \mathbb{S} \\
\end{array}
$$

(5.5)

Here $\text{Hecke}(v \bullet)$ is defined by the top left Cartesian square. Let us take derived open substacks $\mathcal{M}_S^\dagger(v_i)_{\text{fin}} \subset \mathcal{M}_S^\dagger(v_i)_{\text{pure}}$ for $i = 1, 2$ satisfying the condition (4.26) and

$$
\text{Hecke}(v \bullet)_{\text{fin}} := \pi_2^{-1}(\mathcal{M}_S^\dagger(v_2)_{\text{fin}}) \subset \pi_1^{-1}(\mathcal{M}_S^\dagger(v_1)_{\text{fin}}). \quad (5.6)
$$

Then we have the diagram

$$
\begin{array}{c}
\text{Hecke}(v \bullet)_{\text{fin}} & \xrightarrow{\pi_2} & \mathcal{M}_S^\dagger(v_2)_{\text{fin}} \\
(\pi_1, \pi_3) & \downarrow & \downarrow \\
& \mathcal{M}_S^\dagger(v_1)_{\text{fin}} \times S. \\
\end{array}
$$

(5.7)

Here the vertical arrow is quasi-smooth and the horizontal arrow is proper. Also let $L$ be the line bundle on $\text{Hecke}(v \bullet)$ defined by

$$
L := \text{ev}_3^\ast \mathcal{O}(1)|_{\text{Hecke}(v \bullet)} \in \text{Pic}(\text{Hecke}(v \bullet)). \quad (5.8)
$$

Then for each $E \in D^b_{\text{coh}}(S)$, we have the functor

$$
\pi_2_*(\pi_1^*(-) \otimes \pi_3^* E \otimes L^k) : D^b_{\text{coh}}(\mathcal{M}_S^\dagger(\beta, n)_{\text{fin}}) \rightarrow D^b_{\text{coh}}(\mathcal{M}_S^\dagger(\eta, n + 1)_{\text{fin}}). \quad (5.9)
$$
Lemma 5.3  The functor (5.9) descends to the functor $\mu_{E,k}^+$ in Definition 5.2, i.e. the following diagram commutes

$$
\begin{array}{ccc}
D^b_{\text{coh}}(\mathcal{M}_S^+(\beta, n)^{\text{fin}}) & \longrightarrow & D^b_{\text{coh}}(\mathcal{M}_S^+(\beta, n+1)^{\text{fin}}) \\
\downarrow & & \downarrow \\
\mathcal{D}T^{C^*}(P_n(X, \beta)) & \underset{\mu_{E,k}^+}{\longrightarrow} & \mathcal{D}T^{C^*}(P_{n+1}(X, \beta)).
\end{array}
$$

(5.10)

Here the top horizontal arrow is given by (5.9) and the vertical arrows are quotient functors.

Proof  The lemma follows by using base change with respect to the diagram (5.5). Namely in the notation of the diagram (5.5), we have

$$
ev_{2*}^\dagger (ev_1^\dagger, ev_3^\dagger)^* (- \times \vartheta_{*}\mathcal{E} \otimes \mathcal{O}(k)) \cong \pi_{2*}((\pi_1, \pi_3)^* (- \otimes \mathcal{E}) \otimes (ev_3^\dagger)^* \mathcal{O}(k)) \cong \pi_{2*}((\pi_1, \pi_3)^* (- \otimes \mathcal{L}) \otimes \mathcal{L}^k).
$$

The above isomorphisms immediately implies the commutativity of the diagram (5.10).

5.3 The descriptions of the derived stack $\mathcal{H}ecke(v_{\bullet})$

In this subsection, we give two descriptions of the derived stack $\mathcal{H}ecke(v_{\bullet})$ as projectivizations of perfect objects over quasi-smooth derived stacks. These descriptions immediately show that both of the morphisms $\pi_1, \pi_2$ are quasi-smooth and proper, and these facts will be required to construct the functor (5.4). The descriptions here will be also used to compute the commutators in the next section. We first prove some lemmas.

Lemma 5.4  The stack $\mathcal{H}ecke(v_{\bullet})$ parametrizes diagrams

$$
\begin{array}{ccc}
\mathcal{O}_S & \longrightarrow & \mathcal{O}_S \\
\xi_1 \downarrow & & \xi_2 \downarrow \\
0 & \longrightarrow & F_1 \longrightarrow F_2 \rightarrow j \mathcal{O}_X \longrightarrow 0.
\end{array}
$$

(5.11)

Here the bottom sequence is a non-split exact sequence such that $F_1, F_2$ are pure one dimensional coherent sheaves on $S$ with $[F_i] = v_i$, and $x \in S$. The isomorphisms of the diagrams (5.11) are termwise isomorphisms which are identities on $\mathcal{O}_S$ and $\mathcal{O}_X$.

Proof  From the definition of $\mathcal{H}ecke(v_{\bullet})$, it is enough to show that for a diagram (5.11), the sheaf $F_2$ is pure if and only if $F_1$ is pure and the bottom sequence is non-split. The
only if direction is obvious. Suppose that $F_1$ is pure and the bottom sequence is non-split. If $F_2$ is not pure, then there is a point $y \in S$ and a non-zero map $i : \mathcal{O}_y \to F_2$. As $F_1$ is pure, we must have $y = x$. Moreover the composition

$$\mathcal{O}_x \xrightarrow{i} F_2 \xrightarrow{j} \mathcal{O}_x$$

must be non-zero, as otherwise there is a non-zero map $\mathcal{O}_x \to F_1$ which contradicts to the purity of $F_1$. Therefore the bottom sequence of (5.11) splits, so a contradiction.

Let $\mathbb{D}_S$ be the dualizing functor

$$\mathbb{D}_S = \mathcal{H}om_S(\cdot, \mathcal{O}_S) : D^b_{\text{coh}}(S)^{\text{op}} \to D^b_{\text{coh}}(S).$$

Then $\mathbb{D}_S(\text{Coh}(S))$ is another heart of a t-structure on $D^b_{\text{coh}}(S)$. By the lemma below, we can also regard $\text{Hecke}(v^\bullet)$ as the stack of short exact sequences in $\mathbb{D}_S(\text{Coh}(S))[1]$. 

**Lemma 5.5** Giving a diagram (5.11) is equivalent to giving a non-split exact sequence in $\mathbb{D}_S(\text{Coh}(S))[1]$ of the form

$$0 \to \mathcal{O}_x[-1] \to I_1^\bullet[1] \to I_2^\bullet[1] \to 0.$$  \hfill (5.12)

Here $I_i^\bullet = (\mathcal{O}_S \xrightarrow{\xi_i} F_i) \in D^b_{\text{coh}}(S)$.

**Proof** By taking the cone and shift of the diagram (5.11), we obtain the distinguished triangle $\mathcal{O}_x[-1] \to I_1^\bullet[1] \to I_2^\bullet[1]$. Since we have

$$\mathbb{D}_S(\mathcal{O}_x) = \mathcal{O}_x[-2], \quad \mathbb{D}_S(F) = \mathcal{E}xt^1_S(F, \mathcal{O}_S)[-1], \quad \mathbb{D}_S(\mathcal{O}_S) = \mathcal{O}_S$$

where $F$ is a pure one dimensional sheaf, each term of the sequence (5.12) is an object in $\mathbb{D}_S(\text{Coh}(S))[1]$. Therefore (5.12) is an exact sequence in $\mathbb{D}_S(\text{Coh}(S))[1]$. Since $F_1$ is pure, the sequence (5.12) is non-split.

Conversely suppose that a non-split exact sequence (5.12) is given. We have the distinguished triangle

$$\mathbf{R}\mathcal{H}om_S(I_2^\bullet[1], \mathcal{O}_x) \to \mathbf{R}\mathcal{H}om_S(F_2, \mathcal{O}_x) \to \mathbf{R}\mathcal{H}om_S(\mathcal{O}_S, \mathcal{O}_x) = \mathbb{C}.$$ \hfill (5.13)

By taking the associated long exact sequence of cohomologies, we see that the extension class $I_2^\bullet[1] \to \mathcal{O}_x$ of the sequence (5.12) is represented by the commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_S & \to & 0 \\
\xi_2 \downarrow & & \downarrow \\
F_2 & \to & \mathcal{O}_x.
\end{array}$$
As the map $\mathcal{J}_x^*[1] \to \mathcal{O}_x$ is non-zero, the bottom arrow is surjection. By taking the termwise kernels, we obtain the diagram (5.11).

For $i = 1, 2$, we denote by

$$
\mathcal{J}^*(v_i) = (\mathcal{O}_{\mathcal{M}_S^+(v_i)^{\text{pure}} \times S} \to \mathfrak{F}(v_i)) \in \text{Perf}(\mathcal{M}_S^+(v_i)^{\text{pure}} \times S)
$$

the object associated with the universal pair. We first describe the morphism $(\pi_1, \pi_3)$ in the diagram (5.5).

**Lemma 5.6** We have an equivalence of derived stacks

$$
\mathcal{H}ecke(v_\bullet) \sim \mathbb{P}_{\mathcal{M}_S^+(v_1)^{\text{pure}} \times S}(\mathfrak{F}(v_1)^\vee \boxtimes \omega_S[1])
$$

(5.14)

such that the projection of the right hand side to $\mathcal{M}_S^+(v_1)^{\text{pure}} \times S$ is identified with $(\pi_1, \pi_3)$. In particular $(\pi_1, \pi_3)$ and $\pi_1$ are quasi-smooth and proper.

**Proof** We consider the following diagram

$$
\begin{array}{ccc}
\mathcal{M}_S^+(v_1)^{\text{pure}} \times S & \xrightarrow{\mathbb{P}_{\mathcal{M}_S^+(v_1)^{\text{pure}} \times S}(\mathfrak{F}(v_1)^\vee \boxtimes \omega_S[1])} & \mathcal{M}_S^+(v_1)^{\text{pure}} \times S \\
\downarrow & & \downarrow \\
S & \xrightarrow{(\text{id}, \Delta)} & S \times S
\end{array}
$$

Here $\Delta$ is the diagonal and $p_{ij}$ are projections onto the corresponding factors. By the description (3.6) and Lemma 5.4, we see that the derived stack $\mathcal{H}ecke(v_\bullet)$ together with $(\pi_1, \pi_3)$ are identified with the following

$$
\left[\left(\mathbb{V}((\mathcal{H}om_{p_{13}}(p_{23}^*\mathcal{O}_\Delta, p_{12}^*\mathfrak{F}(v_1)[1])^\vee) \big\vert_{\mathcal{M}_S^+(v_1)^{\text{pure}} \times S}) / C^*\right) \to \mathcal{M}_S^+(v_1)^{\text{pure}} \times S.\right]
$$

(5.15)

Here $C^*$ acts on $\mathcal{O}_\Delta$ with weight one. We have the isomorphisms

$$
\mathcal{H}om_{p_{13}}(p_{23}^*\mathcal{O}_\Delta, p_{12}^*\mathfrak{F}(v_1)[1])^\vee \cong \mathcal{H}om_{p_{13}}(p_{12}^*\mathfrak{F}(v_1), (\text{id}, \Delta)_*(\mathcal{O} \boxtimes \omega_S[1]))
$$

$$
\cong p_{13*}(\text{id}, \Delta)_*((\text{id}, \Delta)_*p_{12}^*\mathfrak{F}(v_1)^\vee \boxtimes \omega_S[1])
$$

$$
\cong \mathfrak{F}(v_1)^\vee \boxtimes \omega_S[1].
$$

Here the first isomorphism follows from the Grothendieck duality. Therefore we have the equivalence (5.14).

Let us take a point $(F_1, \xi_1)$ of $\mathcal{M}_S^+(v_1)^{\text{pure}}$ and a point $x \in S$. By the equivalence (5.14), the fiber of $(\pi_1, \pi_3)$ at the point $((F_1, \xi_1), x)$ is given by

$$(\pi_1, \pi_3)^{-1}((F_1, \xi_1), x)) = \mathbb{P}(F_1^\vee|_x \otimes \omega_S[1]).$$
Since $F_1$ is pure one dimensional, $F_1^\vee|_S \otimes \omega_S[1]$ has cohomological amplitude $[-1, 0]$. Therefore the morphism $(\pi_1, \pi_3)$ is quasi-smooth and proper. Then $\pi_1$ is also quasi-smooth and proper, as $S$ is smooth projective. \hfill \Box

By replacing $\pi_1$ with $\pi_2$ in the diagram (5.7), we also obtain the diagram

\begin{align*}
\Hecke(v_\bullet) \xrightarrow{\tau} M_S^{\text{ext,} \dagger}(v_\bullet)^\text{pure} \xrightarrow{\text{ev}_1^\dagger} M_S^\dagger(v_1)^\text{pure}
\end{align*}

By Lemma 5.6, $\pi_1$ is quasi-smooth and proper. We also investigate the morphism $(\pi_2, \pi_3)$.

**Lemma 5.7** We have an equivalence of derived stacks

\begin{align*}
\Hecke(v_\bullet) \xrightarrow{\sim} [\mathbb{P}^{M^\dagger_S(v_2)^\text{pure} \times S} \mathcal{O}^*(v_2)[1])]
\end{align*}

such that the projection of the right hand side to $M^\dagger_S(v_2)^\text{pure} \times S$ is identified with $(\pi_2, \pi_3)$. In particular $(\pi_2, \pi_3)$ and $\pi_3$ are quasi-smooth and proper.

**Proof** We consider the following diagram

\begin{align*}
M^\dagger_S(v_1)^\text{pure} \times S \xrightarrow{(\text{id, } \Delta)} M^\dagger_S(v_1)^\text{pure} \times S \times S \xrightarrow{p_{12}} M^\dagger_S(v_1)^\text{pure} \times S \xrightarrow{p_{13}} M^\dagger_S(v_2)^\text{pure} \times S.
\end{align*}

Here $q_{ij}$ are projections onto the corresponding factors. By Lemma 5.5, a similar argument of (5.15) shows that the derived stack $\Hecke(v_\bullet)$ together with $(\pi_2, \pi_3)$ are identified with the following

\begin{align*}
\left[ \left( \mathcal{V} \mathcal{H}om_{q_{13}}(q^*_{12} \mathcal{O}(v_2)[1], q^*_{23} \mathcal{O}_\Delta)^\vee \right) \setminus 0_{M^\dagger_S(v_2)^\text{pure} \times S} \right] / \mathbb{C}^* \rightarrow M^\dagger_S(v_2)^\text{pure} \times S.
\end{align*}
Here $\mathbb{C}^*$ acts on $\mathcal{O}_\Delta$ with weight one. We have the isomorphisms

$$
\mathcal{H}om_{q_{13}}(q_{12}^* \mathcal{I}^*(v_2)[1], q_{23}^* \mathcal{O}_\Delta)^\vee \cong \mathcal{H}om_{q_{13}}(q_{12}^* \mathcal{I}^*(v_2)[1], (\text{id}, \Delta)_* \mathcal{O}_{\mathfrak{M}^+_S(v_2)\text{pure} \times S})^\vee \\
\cong \mathcal{H}om(\mathcal{I}^*(v_2)[1], \mathcal{O}_{\mathfrak{M}^+_S(v_2)\text{pure} \times S})^\vee \\
\cong \mathcal{I}^*(v_2)[1].
$$

Therefore we have the equivalence (5.17).

Let us take a point $(F_2, \xi_2)$ of $M^+_S(v_2)$pure and a point $x \in S$. By the equivalence (5.17), the fiber of $(\pi_2, \pi_3)$ at the point $((F_2, \xi_2), x)$ is given by

$$(\pi_2, \pi_3)^{-1}((F_2, \xi_2), x)) = \mathbb{P}(I^+_2[1]|_x).$$

We have the distinguished triangle

$$F_2|_x \rightarrow I^+_2[1]|_x \rightarrow \mathcal{O}_x[1].$$

Since $F_2$ is pure one dimensional, $F_2|_x$ has cohomological amplitude $[-1, 0]$. Therefore $I^+_2[1]|_x$ has also cohomological amplitude $[-1, 0]$, hence $(\pi_2, \pi_3)$ is quasi-smooth and proper. Then $\pi_2$ is also quasi-smooth and proper, as $S$ is smooth projective.

\[\square\]

5.4 The stack of pure objects in $\mathcal{A}_X$

Here we give a refinement of Lemma 4.11 which is required to construct the functors (5.4). For $v \in N_{\leq 1}(S)$, we define the following open substack of $\mathcal{M}^+_X(v)$

$$\mathcal{M}^+_X(v)_{\text{pure}} := t_0(\Omega_{\mathfrak{M}^+_S(v)\text{pure}}(v)[-1]) \subset \mathcal{M}^+_X(v).$$

Here we have used the isomorphism (4.15). Note that through the equivalence $\mathcal{A}^{\leq 1}_X \sim \mathfrak{B}^{\leq 1}_S$ in Theorem 4.5, the stack $\mathcal{M}^+_X(v)_{\text{pure}}$ parametrizes diagrams (4.6) such that $\mathcal{V} = \mathcal{O}_S$ and $F$ is a pure one dimensional sheaf with $[F] = v$. We have the following lemma.

**Lemma 5.8** For an exact sequence in $\mathcal{A}_X$ of the form

$$0 \rightarrow E_1 \rightarrow E_2 \rightarrow \mathcal{O}_x[-1] \rightarrow 0$$

suppose that $E_i$ corresponds to points of $\mathcal{M}^+_X(v_i)_{\text{pure}}$. Then $E_1$ is a PT stable pair if and only if $E_2$ is a PT stable pair.

**Proof** The if direction is proved in Lemma 4.11, so we prove the only if direction. Suppose that $E_1 = (\mathcal{O}_X \xrightarrow{s_1} F_1)$ for a PT stable pair $(F_1, s_1)$ on $X$. Then we have the following diagram
Since $\text{Hom}(\mathcal{O}_x[-2], \mathcal{O}_X) = 0$, the map $E_1 \to \mathcal{O}_X$ factors through $E_1 \to E_2 \to \mathcal{O}_X$. By taking the cones, we obtain distinguished triangles

$$F_2[-1] \to E_2 \to \mathcal{O}_X, \quad \mathcal{O}_x[-2] \to F_1[-1] \to F_2[-1]$$

for some $F_2$. By the second sequence, the object $F_2$ is a one dimensional sheaf. Then by the first sequence, $E_2 = (\mathcal{O}_X \xrightarrow{s_2} F_2)$ for a pair $(F_2, s_2)$. As $E_2$ is a point of $\mathcal{M}_X^\dagger(v_2)_{\text{pure}}$, the sheaf $F_2$ is a pure one dimensional sheaf. Moreover $(F_2, s_2)$ is isomorphic to $(F_1, s_1)$ outside $x$, so $s_2$ is also surjective in dimension one. Therefore $(F_2, s_2)$ is a PT stable pair.

The above lemma is rephrased in terms of stack of exact sequences in $\mathcal{A}_X$ in the following way. We take $v_\bullet \in N_{\leq 1}(S)^{\times 3}$ and, using the notation of the diagram (4.16), we set

$$\mathcal{M}_{X}^{\text{ext}, \dagger}(v_\bullet)_{\text{pure}} := (\text{ev}_2^X, \text{ev}_3^X)^{-1}(\mathcal{M}_X^\dagger(v_2)_{\text{pure}}) \subset \mathcal{M}_{X}^{\text{ext}, \dagger}(v_\bullet).$$

Since any subsheaf of a pure one dimensional sheaf is pure one dimensional, the diagram (4.16) restricts to the diagram

$$\begin{array}{ccc}
\mathcal{M}_{X}^{\text{ext}, \dagger}(v_\bullet)_{\text{pure}} & \xrightarrow{\text{ev}_2^X, \text{ev}_3^X} & \mathcal{M}_X^\dagger(v_2)_{\text{pure}} \\
(\text{ev}_1^X, \text{ev}_3^X) & & \\
\mathcal{M}_X^\dagger(v_1)_{\text{pure}} \times \mathcal{M}_X(v_3). & & 
\end{array}$$

We also define the following conical closed substack of $\mathcal{M}_X^\dagger(v)_{\text{pure}}$

$$Z_{P-\text{us}}(v)_{\text{pure}} := Z_{P-\text{us}}(v) \cap \mathcal{M}_X^\dagger(v)_{\text{pure}} \subset \mathcal{M}_X^\dagger(v)_{\text{pure}}.$$

Then for $v_3 = [\text{pt}]$ in the diagram (5.18), the result of Lemma 5.8 implies the following identity

$$(\text{ev}_1^X, \text{ev}_3^X)^{-1}(Z_{P-\text{us}}(v_1)_{\text{pure}} \times \mathcal{M}_X(v_3)) = (\text{ev}_2^X)^{-1}(Z_{P-\text{us}}(v_2)_{\text{pure}}).$$
5.5 The annihilator functors

Finally in this section, we construct the annihilator functors (5.4). Recall that in (5.6), we took derived open substacks \( \mathcal{M}_S(v_i)^\text{fin} \subset \mathcal{M}_S(v_i)^\text{pure} \) of finite type. Here we take another derived open substack \( \mathcal{M}_S(v_2)^\text{fin}' \subset \mathcal{M}_S(v_2)^\text{pure} \) of finite type, which contains \( \mathcal{M}_S(v_2)^\text{fin} \) and satisfies

\[
\mathcal{H}ecke(v_\bullet)^\text{fin}':=\pi_1^{-1}(\mathcal{M}_S(v_1)^\text{fin}) \subset \pi_2^{-1}(\mathcal{M}_S(v_2)^\text{fin}').
\]

Then the diagram (5.16) restricts to the diagram

\[
\begin{align*}
\mathcal{H}ecke(v_\bullet)^\text{fin}' & \quad \xrightarrow{\pi_1} \quad \mathcal{M}_S(v_1)^\text{fin} \quad \xrightarrow{(\pi_2,\pi_3)} \quad \mathcal{M}_S(v_2)^\text{fin}' \times S.
\end{align*}
\]

By Lemma 5.6 and Lemma 5.7, the vertical arrow is quasi-smooth and the horizontal arrow is proper. Therefore for each \( E \in \text{Db}_{\text{coh}}(S) \) and \( k \in \mathbb{Z} \), we have the functor

\[
\pi_1^!(\pi_2^*(-) \otimes \pi_3^* E \otimes L^k): \text{Db}_{\text{coh}}(\mathcal{M}_S(v_1)^\text{fin}') \to \text{Db}_{\text{coh}}(\mathcal{M}_S(v_2)^\text{fin}'). \tag{5.20}
\]

**Lemma 5.9** The functor (5.20) sends \( \mathcal{C}_{Z_{P-us}(\beta,n+1)^\text{fin}'} \) to \( \mathcal{C}_{Z_{P-us}(\beta,n)^\text{fin}} \).

**Proof** By Lemma 5.6, we can compute \( \omega_{\pi_1}[\text{vdim } \pi_1] \) as

\[
\omega_{\pi_1}[\text{vdim } \pi_1] = (\pi_1, \pi_3)^* \text{det } \mathcal{F}(v_1) \otimes \pi_3^* \omega_{S}[1].
\]

Since \( \pi_1! = \pi_1^*(- \otimes \omega_{\pi_1}[\text{vdim } \pi_1]) \), the functor (5.20) is written as

\[
\pi_1^*[\pi_2^*(-) \otimes \pi_3^* (E \otimes \omega_{S}) \otimes (\pi_1, \pi_3)^* \text{det } \mathcal{F}(v_1)[1] \otimes L^k].
\]

By Lemma 5.1, the morphism \( \tau \) in the diagram (5.5) is a trivial square zero extension. Therefore there is \( \mathcal{P} \in \text{Perf}(\mathcal{M}_S^\text{ext,vpure}) \) whose restriction to \( \mathcal{H}ecke(v_\bullet) \) is equivalent to \( (\pi_1, \pi_3)^* \text{det } \mathcal{F}(v_1)[1] \). Then by the similar argument of Lemma 5.3 using base-change with respect to the diagram (5.16), the functor (5.20) is written as

\[
ev_{1\#}\{\text{ev}_{2\#}^\dagger, \text{ev}_{3\#}^\dagger\}^*((-) \boxtimes \lambda_k (E \otimes \omega_{S})) \otimes \mathcal{P}\}.
\]

By Lemma 5.8 and the argument of Corollary 4.8, the above functor sends \( \mathcal{C}_{Z_{P-us}(\beta,n+1)^\text{fin}'} \) to \( \mathcal{C}_{Z_{P-us}(\beta,n)^\text{fin}} \). \qed
Definition 5.10 We define the functor
\[ \mu_{E,k}^- : DT_{C^*}^n (P_{n+1}(X, \beta)) \to DT_{C^*}^n (P_n(X, \beta)) \] (5.21)
as a descendent of the functor (5.20), which exists uniquely by Lemma 5.9.

5.6 Restrictions to perfect PT subcategories

Here we introduce the notion of perfect PT categories, and show that the operators \( \mu_{E,k}^\pm \) restrict to perfect PT categories. Let us take a derived open substack \( \mathcal{M}^+_{S}(\beta, n)^{\mathrm{fin}} \subset \mathcal{M}^+_{S}(\beta, n)^{\mathrm{pure}} \) as before. We give the following definition.

Definition 5.11 We define the perfect PT category to be
\[ DT^*_{\mathrm{perf}} (P_n(X, \beta)) := \text{Perf}(\mathcal{M}^+_{S}(\beta, n)^{\mathrm{fin}})/\text{Perf}(\mathcal{M}^+_{S}(\beta, n)^{\mathrm{fin}}) \cap C_{Z_{\mathcal{P}, \mathrm{us}}(\beta, n)^{\mathrm{fin}}} \] (5.22)

We have the composition of functors
\[ \text{Perf}(\mathcal{M}^+_{S}(\beta, n)^{\mathrm{fin}}) \hookrightarrow D^b_{\text{coh}}(\mathcal{M}^+_{S}(\beta, n)^{\mathrm{fin}}) \twoheadrightarrow DT^*_{\mathrm{perf}} (P_n(X, \beta)) \] (5.23)
whose kernel is exactly \( \text{Perf}(\mathcal{M}^+_{S}(\beta, n)^{\mathrm{fin}}) \cap C_{Z_{\mathcal{P}, \mathrm{us}}(\beta, n)^{\mathrm{fin}}} \). Therefore we have the canonical functor
\[ DT^*_{\mathrm{perf}} (P_n(X, \beta)) \to DT^*_{\mathrm{perf}} (P_n(X, \beta)). \] (5.24)

The perfect PT categories are closely related to the moduli spaces of stable pairs on \( S \), rather than those on threefolds \( X \) in the following way. Let
\[ \mathcal{P}^+_n(S, \beta) \subset \mathcal{M}^+_{S}(\beta, n)^{\mathrm{fin}} \]
be the derived open substack of PT stable pairs on \( S \), i.e. this is the substack of pairs \((F, \xi)\) such that \( \xi \) is surjective in dimension one.

Lemma 5.12 We have an equivalence
\[ DT^*_{\mathrm{perf}} (P_n(X, \beta)) \sim \text{Perf}(\mathcal{P}^+_n(S, \beta)). \] (5.25)

Proof Note that we have
\[ \text{Perf}(\mathcal{M}^+_{S}(\beta, n)^{\mathrm{fin}}) = C_{0,\mathcal{M}^+_{S}(\beta, n)^{\mathrm{fin}}} \subset D^b_{\text{coh}}(\mathcal{M}^+_{S}(\beta, n)^{\mathrm{fin}}) \]
where $0_{\mathcal{M}^\dag_{S}(\beta, n)^{\text{fin}}}$ is the zero section of $\pi_* : \mathcal{M}_X^\dag(\beta, n) \to \mathcal{M}^\dag_{S}(\beta, n)$ over $\mathcal{M}^\dag_{S}(\beta, n)^{\text{fin}}$ (see [1, Theorem 4.2.6]). It follows that we have

$$\text{Perf}(\mathcal{M}^\dag_{S}(\beta, n)^{\text{fin}}) \cap C_{Z_{P_{\text{us}}(\beta, n)^{\text{fin}}}} = \{ \mathcal{E} \in \text{Perf}(\mathcal{M}^\dag_{S}(\beta, n)^{\text{fin}}) : \text{Supp}(\mathcal{E}) \subset \mathcal{M}^\dag_{S}(\beta, n)^{\text{fin}} \setminus P_n(S, \beta) \}.$$ 

Therefore the restriction functor with respect to the open immersion (5.24) gives an equivalence (5.25).

As we see below, the operators $\mu_{E, k}^\pm$ restrict to the perfect PT categories.

**Lemma 5.13** The functors (5.9), (5.20) restrict to functors

$$\nu_{E, k}^\pm : DT_{\text{perf}}^C(P_n(X, \beta)) \to DT_{\text{perf}}^C(P_{n\pm 1}(X, \beta))$$

so that we have the commutative diagram

$$\begin{array}{ccc}
DT_{\text{perf}}^C(P_n(X, \beta)) & \xrightarrow{\nu_{E, k}^\pm} & DT_{\text{perf}}^C(P_{n\pm 1}(X, \beta)) \\
& \downarrow{\mu_{E, k}^\pm} & \\
DT_{\text{perf}}^C(P_{n\pm 1}(X, \beta)) & \xrightarrow{\nu_{E, k}^\pm} & DT_{\text{perf}}^C(P_{n\pm 1}(X, \beta)).
\end{array}$$ (5.26)

Here the horizontal arrows are given by (5.23).

**Proof** Since $\pi_1, \pi_2$ are quasi-smooth and proper, both of $\pi_i^*, \pi_i!$ preserve perfect objects (see [29, Lemma 2.2]). Therefore the lemma follows by the definitions of the functors (5.9), (5.20). The commutative diagram (5.26) is obvious by the construction.

\[\square\]

## 6 Commutator relations in K-theory

In this section, we compute the commutator relation of the operators $\mu_{E, k}^+, \mu_{E, k}^-$ for elements of K-groups of PT categories, coming from perfect PT categories. Our strategy is to use residue arguments developed by Negut [19]. We will see that the commutator relation for $k = 0$ gives an analogue of Weyl algebra action for homologies of Hilbert schemes of points on locally planar curves [24].

### 6.1 Some notation in K-theory

Here we prepare some notation in K-theory following [19]. For a derived stack $\mathcal{M}$, we set

$$K(\mathcal{M}) = K(D_{\text{coh}}^b(\mathcal{M})), \quad K_{\text{perf}}(\mathcal{M}) = K(\text{Perf}(\mathcal{M})).$$
Note that we have maps given by tensor products
\[
\otimes : \text{perf}(\mathcal{M}) \times \text{perf}(\mathcal{M}) \to \text{perf}(\mathcal{M}), \quad \otimes : \text{perf}(\mathcal{M}) \times K(\mathcal{M}) \to K(\mathcal{M})
\]
which make \( \text{perf}(\mathcal{M}) \) a commutative ring and \( K(\mathcal{M}) \) a module over it. Suppose that \( \mathcal{M} \) is quasi-smooth and let \( i : \mathcal{M} \hookrightarrow \mathcal{M} \) be the natural closed immersion for \( \mathcal{M} = t_0(\mathcal{M}) \). Then the following induced map is an isomorphism
\[
i_* : K(\mathcal{M}) \cong K(\mathcal{M}). \tag{6.1}
\]

For a vector bundle \( \mathcal{P} \to \mathcal{M} \), we set
\[
\wedge^\bullet(\mathcal{P} x) := \sum_{i \geq 0} \wedge^i \mathcal{P} \cdot (-x)^i \in \text{perf}(\mathcal{M})[x].
\]
Also for an element \( \mathcal{P} = [\mathcal{P}_0] - [\mathcal{P}_1] \in \text{perf}(\mathcal{M}) \) for vector bundles \( \mathcal{P}_0, \mathcal{P}_1 \), we define
\[
\wedge^\bullet(\mathcal{P} x) := \wedge^\bullet(\mathcal{P}_0 x) \wedge^\bullet(\mathcal{P}_1 x) \in \text{perf}(\mathcal{M})(x), \quad \wedge^\bullet\left(\frac{x}{\mathcal{P}}\right) := \wedge^\bullet(\mathcal{P}^\vee x) \in \text{perf}(\mathcal{M})(x).
\]
These are rational functions in \( x \). For a rational function \( f(x) \in \text{perf}(\mathcal{M})(x) \), we denote by
\[
f(x)|_{x=\infty} \in \text{perf}(\mathcal{M})(1/x), \quad f(x)|_{x=0} \in \text{perf}(\mathcal{M})((x))
\]
the expansions of the rational function \( f(x) \) at \( x=\infty \), \( x=0 \), respectively. For example if \( \mathcal{P} \) is a rank \( r \) vector bundle, we have
\[
\frac{1}{\wedge^\bullet(\mathcal{P} x)} \bigg|_{x=\infty} = (-1)^r \det \mathcal{P}^\vee x^{-r} \text{Sym}^\bullet(\mathcal{P}^\vee x^{-1}), \quad \frac{1}{\wedge^\bullet(\mathcal{P} x)} \bigg|_{x=0} = \text{Sym}^\bullet(\mathcal{P} x). \tag{6.2}
\]
We define
\[
f(x)|_{x=\infty-0} := f(x)|_{x=\infty} - f(x)|_{x=0} \in K(\mathcal{M})[x].
\]
We will use the following calculation.

**Lemma 6.1** For \( \mathcal{P} = [\mathcal{P}_0] - [\mathcal{P}_1] \) as above, suppose that rank(\( \mathcal{P} \)) = 0. Then for any line bundle \( \mathcal{L} \) on \( \mathcal{M} \), we have
\[
\wedge^\bullet((\mathcal{L} - 1)\mathcal{P} x))|_{x=\infty-0} = O(x^{-2}) + (1 - \mathcal{L}^\vee)\mathcal{P}^\vee x^{-1} + (\mathcal{L} - 1)\mathcal{P} x + O(x^2). \tag{6.3}
\]
Proof  By definition, we have the identity

\[ \wedge^\bullet((\mathcal{L} - 1)\mathcal{P}x)) = \frac{\wedge^\bullet\mathcal{L}\mathcal{P}_0 x}{\wedge^\bullet\mathcal{L}\mathcal{P}_1 x} \cdot \wedge^\bullet\mathcal{P}_0 x. \]

Using the identities (6.2) and the assumption that \( \text{rank}(\mathcal{P}) = 0 \), we have

\[ \wedge^\bullet((\mathcal{L} - 1)\mathcal{P}x))|_{x=\infty-0} = \wedge^\bullet(\mathcal{L}^\vee\mathcal{P}_0^\vee x^{-1}) \]
\[ \wedge^\bullet(\mathcal{P}_1^\vee x^{-1})\text{Sym}^\bullet(\mathcal{L}^\vee\mathcal{P}_1^\vee x^{-1})\text{Sym}^\bullet(\mathcal{P}_0^\vee x^{-1}) \]
\[ - \wedge^\bullet(\mathcal{L}\mathcal{P}_0 x) \wedge^\bullet(\mathcal{P}_1 x)\text{Sym}^\bullet(\mathcal{L}\mathcal{P}_1 x)\text{Sym}^\bullet(\mathcal{P}_0 x). \]

Therefore we obtain (6.3).

We also set

\[ \delta(x) := \left( \frac{1}{x-1} \right)|_{x=\infty-0} = \sum_{k \in \mathbb{Z}} x^k. \]

We have the following relations

\[ \delta \left( \frac{x}{y} \right) f(x)|_{x=\infty} = \delta \left( \frac{x}{y} \right) f(y)|_{y=\infty}, \quad \delta \left( \frac{x}{y} \right) f(x)|_{x=0} = \delta \left( \frac{x}{y} \right) f(y)|_{y=0}. \]

Also for two rational functions \( f(x), g(y) \), we define

\[ f(x)g(y)|_{(x,y)=\infty-0} := f(x)|_{x=\infty}g(y)|_{y=\infty} - f(x)|_{x=0}g(y)|_{y=0}. \]

For a two variable rational function \( h(x, y) \in K_{\text{perf}}(\mathcal{M})(x, y) \), we denote by

\[ h(x, y)|_{x=\infty-0 \atop y=\infty-0} \in K_{\text{perf}}(\mathcal{M})\{x, y\} \]

first apply \( |_{x=\infty-0} \) and then apply \( |_{y=\infty-0} \). The following lemma can be checked by a direct calculation.

Lemma 6.2 For two rational functions \( f(x), g(y) \) and non-zero \( \alpha \), we have

\[ \frac{f(x)g(y)}{y/x - \alpha}|_{x=\infty-0 \atop y=\infty-0} - \frac{f(x)g(y)}{y/x - \alpha}|_{x=\infty-0 \atop y=\infty-0} = -\frac{1}{\alpha} \delta \left( \frac{y}{\alpha x} \right) \{ f(x)g(y) \}_{(x,y)=\infty-0}. \]

(6.4)

For an object \( \mathcal{E} \in \text{Perf}(\mathcal{M}) \), suppose that \( \mathcal{E}|_x \) is of cohomological amplitude \([-1, 0]\) for any point \( x \rightarrow \mathcal{M} \). We have projectivizations

\[ \pi : \mathbb{P}_{\mathcal{M}}(\mathcal{E}) \rightarrow \mathcal{M}, \quad \pi' : \mathbb{P}_{\mathcal{M}}(\mathcal{E}^\vee[1]) \rightarrow \mathcal{M} \]

which are quasi-smooth and proper. We will use the following lemma.
Lemma 6.3 Suppose that $\mathcal{M} = [Q/G]$ where $Q$ is a quasi-projective scheme and $G$ is a reductive algebraic group which acts on $Q$. Then for any $\alpha \in K(\mathcal{M})$, we have the following relations in $K(\mathcal{M})[z]$

$$\pi_* \left[ \delta \left( \frac{O_{\pi}(1)}{z} \right) \right] \otimes \alpha = \wedge \left( - \frac{E}{z} \right) \big|_{z=\infty-0} \otimes \alpha,$$

$$\pi'_* \left[ \delta \left( \frac{O_{\pi'}(-1)}{z} \right) \right] \otimes \alpha = \wedge \left( \frac{z}{E} \right) \big|_{z=\infty-0} \otimes \alpha.$$

Here elements before taking $\otimes \alpha$ are defined in $K_{\text{perf}}(\mathcal{M})[z]$.

Proof We set $\mathcal{E} = \mathcal{E}|_{\mathcal{M}} \in \text{Perf}(\mathcal{M})$, and take the similar projectivizations on classical truncations

$$\overline{\pi} : \mathbb{P}\mathcal{M}(\mathcal{E}) \to \mathcal{M}, \quad \overline{\pi}' : \mathbb{P}\mathcal{M}(\mathcal{E}'[1]) \to \mathcal{M}.$$

By the isomorphism (6.1), we can write $\alpha = i_* \alpha'$ for some $\alpha' \in K(\mathcal{M})$. Note that for any $\beta \in K_{\text{perf}}(\mathcal{M})$ we have

$$\beta \otimes i_* \alpha' = i_*(i^* \beta \otimes \alpha').$$

Therefore it is enough to show the following identities in $K_{\text{perf}}(\mathcal{M})[z]$

$$\pi_* \left[ \delta \left( \frac{O_{\pi}(1)}{z} \right) \right] = \wedge \left( - \frac{E}{z} \right) \big|_{z=\infty-0}, \quad \pi'_* \left[ \delta \left( \frac{O_{\pi'}(-1)}{z} \right) \right] = \wedge \left( \frac{z}{E} \right) \big|_{z=\infty-0}. \quad (6.5)$$

By the assumption on $\mathcal{M}$, we can represent $\mathcal{E}$ as a two term complex

$$\mathcal{E} = (\mathcal{E}^{-1} \xrightarrow{d} \mathcal{E}^0)$$

where $\mathcal{E}^{-1}$ and $\mathcal{E}^0$ are $G$-equivariant vector bundles on $Q$ and $d$ is $G$-equivariant. Then the identities (6.5) follow from [19, Proposition 5.19].

Remark 6.4 It is well-known that any finite type derived open substack of $\mathcal{M}_S(v)$, $\mathcal{M}^+_S(v)$ satisfies the assumption of Lemma 6.3, i.e. its classical truncation is of the form $[Q/G]$ as in Lemma 6.3. For example, see [17, Proposition 4.1.1].

6.2 Actions on K-theory

Now we return to the situation in Sect. 5. For a fixed $\beta \in \text{NS}(S)$, let $\mathcal{M}^+_S(\beta)^{\text{pure}}$ be defined by

$$\mathcal{M}^+_S(\beta)^{\text{pure}} := \bigsqcup_{n \in \mathbb{Z}} \mathcal{M}^+_S(\beta, n)^{\text{pure}}.$$
Using the notation in the diagrams (5.5), (5.16), we define the following maps on the K-theory

\[ \mu^+(z) := (\pi_2, \pi_3)_* \left( \pi_1^*(-) \otimes \delta \left( \frac{L}{z} \right) \right) : K(\mathcal{M}_{S}^\dagger(\beta)^{\text{pure}}) \to K(\mathcal{M}_{S}^\dagger(\beta)^{\text{pure}}) \times S)\{z\}, \]

\[ \mu^-(z) := (\pi_1, \pi_3)_* \left( \pi_2^*(-) \otimes \delta \left( \frac{L}{z} \right) \right) : K(\mathcal{M}_{S}^\dagger(\beta)^{\text{pure}}) \to K(\mathcal{M}_{S}^\dagger(\beta)^{\text{pure}}) \times S)\{z\}. \]

(6.6)

We denote by

\[ \mathcal{J}^* (\beta) = (O_{S \times \mathcal{M}_{S}^\dagger(\beta)^{\text{pure}}} \to \mathcal{F}(\beta)) \in \text{Perf} (S \times \mathcal{M}_{S}^\dagger(\beta)^{\text{pure}}) \]

the object associated with the universal pair. By Lemma 5.7, the map \( \mu^-(z) \) is written as

\[ \mu^-(z) = (\pi_1, \pi_3)_* \left( \pi_2^*(-) \otimes \delta \left( \frac{L}{z} \right) \right) \cdot (- \det \mathcal{F}(\beta)). \]

We write \( \mu^\pm(z) \) as

\[ \mu^\pm(z) = \sum_{k \in \mathbb{Z}} \mu^\pm_k z^{-k}, \quad \mu^\pm_k : K(\mathcal{M}_{S}^\dagger(\beta)^{\text{pure}}) \to K(\mathcal{M}_{S}^\dagger(\beta)^{\text{pure}}) \times S). \]

Then by Lemma 5.3 and Definition 5.10, the functors \( \mu^\pm_{E,k} \) satisfy the following relations

\[ p_{\mathcal{M}}(\mu^+_k(-) \otimes p_{S}^*\mathcal{E}) = \mu^+_k(-), \quad p_{\mathcal{M}}(\mu^-_k(-) \otimes p_{S}^*\mathcal{E}) = \mu^-_k(-). \]

(6.7)

Here \( p_{\mathcal{M}} \) and \( p_{S} \) are the projections from \( \mathcal{M}_{S}^\dagger(\beta)^{\text{pure}} \times S \) onto corresponding factors. We then set

\[ [\mu^+(z), \mu^-(w)] : K(\mathcal{M}_{S}^\dagger(\beta)^{\text{pure}}) \to K(\mathcal{M}_{S}^\dagger(\beta)^{\text{pure}}) \times S \times S)\{z, w\} \]

by the following

\[ [\mu^+(z), \mu^-(w)] := (\mu^+(z) \otimes \text{id}_S) \circ \mu^-(w) - (\mu^-(w) \otimes \text{id}_S) \circ \mu^+(z). \]

The rest of this section is devoted to the computation of \( [\mu^+(z), \mu^-(w)] \) following the argument of [19].
6.3 Compositions of Hecke actions

In order to compute the composition $\mu^- \circ \mu^+$, we consider the following diagram

\[
\begin{array}{ccc}
\text{Hecke}(v \bullet) & \xrightarrow{q_2} & \text{Hecke}(v \bullet) \times S \xrightarrow{(\pi_1, \pi_3, \text{id}_S)} M_S^+(v_1)_{\text{pure}} \times S \times S \\
\downarrow q_1 & \text{Hecke}(v \bullet) & \downarrow (\pi_2, \text{id}_S) \\
\pi_1 & \xrightarrow{(\pi_2, \pi_3)} & M_S^+(v_2)_{\text{pure}} \times S \\
\end{array}
\]

Here $\text{Hecke}(v \bullet)$ is defined by the top left Cartesian square. From the construction, it parametrizes diagrams

\[
\begin{array}{ccc}
F_1 & \xrightarrow{x} & F_2 \\
\text{O}_S & \xrightarrow{y} & \text{O}_S \\
F'_1 & \xrightarrow{F'_2} & \\
\end{array}
\]

Here $F_1, F'_1$ are points of $M_S(v_1)_{\text{pure}}$, $F_2$ is a point of $M_S(v_2)_{\text{pure}}$, and $F_1 \xleftarrow{x} F_2$ means that $F_2/F_1 = \text{O}_x$ for $x \in S$.

Similarly we take

\[v' = (v'_1, v'_2, v'_3), \quad v'_1 = (\beta, n - 1), \quad v'_2 = v_1 = (\beta, n), \quad v'_3 = (0, 1)\]

and consider the diagram
Here $\mathfrak{Hecke}^{\Diamond}(v'_\bullet)$ is defined by the top left Cartesian square. It parametrizes diagrams

\[
\begin{array}{ccc}
F_1 & \rightarrow & y \\
\downarrow & & \downarrow \\
O_S & \rightarrow & F_0 \\
\uparrow & & \uparrow \\
x & \rightarrow & F'_1.
\end{array}
\]

Here $F_1, F'_1$ are points of $\mathcal{M}_S(v_1)^{\text{pure}}$, $F_0$ is a point of $\mathcal{M}_S(v'_1)^{\text{pure}}$. The following is an analogue of [19, Claim 3.8] for our situation.

**Lemma 6.5** Let $(F_1, F'_1, x, y)$ be given, where $F_1, F'_1$ are points of $\mathcal{M}_S(v_1)^{\text{pure}}$ and $x, y \in S$. Suppose that either $x \neq y$ or $F_1$ is not isomorphic to $F'_1$. Then giving a diagram (6.9) is equivalent to giving a diagram (6.11).

**Proof** Suppose that a diagram (6.9) is given. The assumption implies that $F_1 \neq F'_1$ inside $F_2$. Therefore by setting $F_0 = F_1 \cap F'_1$ inside $F_2$, we obtain the diagram (6.11). Conversely suppose that a diagram (6.11) is given. We set $F_2$ by the exact sequence

\[
0 \rightarrow F_0 \rightarrow F_1 \oplus F'_1 \rightarrow F_2 \rightarrow 0.
\]

We claim that, under the assumption of the lemma, the sheaf $F_2$ is a pure one dimensional sheaf. The claim is obvious if $x \neq y$, so we assume that $x = y$. If $F_2$ is not pure, there is an injection $O_x \hookrightarrow F_2$. By setting $F'_2 = F_2/O_x$, we have the morphisms

\[
F_1 \xrightarrow{(\text{id}, 0)} F_1 \oplus F'_1 \twoheadrightarrow F_2 \twoheadrightarrow F'_2.
\]
The composition of the above morphisms is generically injective, hence injective as
\( F_1 \) is pure. Since \( \chi(F_1) = \chi(F'_2) \), the morphism (6.12) is an isomorphism, \( F_1 \cong F'_2 \).
Similarly \( F'_1 \cong F'_2 \), which contradicts to the assumption that \( F_1 \) is not isomorphic to \( F'_1 \). Therefore \( F_2 \) is pure one dimensional, and we obtain the diagram (6.9).

Using diagrams (6.8), (6.10) and Lemma 6.5, we have the following lemma, which is an analogue of [19, (3.32)].

**Lemma 6.6** There exists \( \gamma \in K(\mathcal{M}_S^{\dagger}(\beta)_{\text{pure}} \times z, w) \) such that we have the commutative diagram

\[
\begin{array}{ccc}
K_{\text{perf}}(\mathcal{M}_S^{\dagger}(\beta)_{\text{pure}}) & \xrightarrow{\otimes(id, \Delta) \gamma} & K(\mathcal{M}_S^{\dagger}(\beta)_{\text{pure}}) \\
\downarrow & & \downarrow [\mu^+(z), \mu^-(w)] \\
K(\mathcal{M}_S^{\dagger}(\beta)_{\text{pure}} \times z, w) & & K(\mathcal{M}_S^{\dagger}(\beta)_{\text{pure}} \times S \times z, w).
\end{array}
\]

Here the horizontal arrow is the natural map, and \((id, \Delta): \mathcal{M}_S^{\dagger}(\beta)_{\text{pure}} \times S \to \mathcal{M}_S^{\dagger}(\beta)_{\text{pure}} \times S \times S\) is the product with diagonal \( \Delta: S \hookrightarrow S \times S \).

**Proof** From the diagrams (6.8), (6.10), we have

\[
\begin{align*}
(\mu^-(w) \otimes id_S) \circ \mu^+(z) &= (\pi_1^\bullet, p_1, p_2)_* \\
\left( (\pi_1^\bullet)^*(-) \otimes (q_1^* L) \otimes (q_2^* L) \otimes (- (\pi_1^\bullet, p_2)_* \det \mathcal{G}(\beta)) \right), \\
(\mu^+(z) \otimes id_S) \circ \mu^-(w) &= (\pi_1^\diamondsuit, p_1', p_2')_* \\
\left( (\pi_1^\diamondsuit)^*(-) \otimes (q_1'^* L) \otimes (q_2'^* L) \otimes (- (\pi_1^\diamondsuit, p_1')_* \det \mathcal{G}(\beta)) \right).
\end{align*}
\]

By Lemma 6.5, two derived stacks over \( \mathcal{M}_S^{\dagger}(v_1)_{\text{pure}} \times \mathcal{M}_S^{\dagger}(v_1)_{\text{pure}} \times S \times S \)

\[
\begin{array}{ccc}
\mathfrak{Hecke}(v'_1) & \xrightarrow{\mathfrak{Hecke}(v'_1)} & \mathfrak{Hecke}(v'_1) \\
(\pi_1^\bullet, \pi_1^\bullet, p_1, p_2) & & (\pi_1^\diamondsuit, \pi_1^\diamondsuit, p_1', p_2') \\
\mathcal{M}_S^{\dagger}(v_1)_{\text{pure}} \times \mathcal{M}_S^{\dagger}(v_1)_{\text{pure}} \times S \times S.
\end{array}
\]

are equivalent outside the diagonal

\[
(\Delta_{\mathcal{M}} \times \Delta): \mathcal{M}_S^{\dagger}(v_1)_{\text{pure}} \times S \leftrightarrow \mathcal{M}_S^{\dagger}(v_1)_{\text{pure}} \times \mathcal{M}_S^{\dagger}(v_1)_{\text{pure}} \times S \times S.
\]
such that \( q_1^* \mathcal{L}, q_2^* \mathcal{L} \) are identified and \((\pi_1^\bullet, p_2)^* \det \mathfrak{F}(\beta), (\pi_1^\circ, p_1^\prime)^* \det \mathfrak{F}(\beta) \) are identified. Therefore the difference

\[
(\pi_1^\bullet, \pi_1^\circ, p_1, p_2)^* \left\{ \delta \left( \frac{q_1^* \mathcal{L}}{z} \right) \otimes \delta \left( \frac{q_2^* \mathcal{L}}{w} \right) \otimes \left( - (\pi_1^\bullet, p_2)^* \det \mathfrak{F}(\beta) \right) \right\}
- (\pi_1^\circ, \pi_1^\circ, p_1^\prime, p_2^\prime)^* \left\{ \delta \left( \frac{q_1^* \mathcal{L}}{z} \right) \otimes \delta \left( \frac{q_2^* \mathcal{L}}{w} \right) \otimes \left( - (\pi_1^\circ, p_1^\prime)^* \det \mathfrak{F}(\beta) \right) \right\}
\]

in \( K(\mathcal{M}_S^\dagger(\beta)^\text{pure} \times \mathcal{M}_S^\dagger(\beta)^\text{pure} \times S \times S)[z, w] \) is written as \( (\Delta_M \times \Delta)_\gamma \) for some \( \gamma \in K(\mathcal{M}_S^\dagger(\beta)^\text{pure} \times S)[z, w] \). Then the commutator \( [\mu^+(z), \mu^-(w)](-) \) applied for perfect complexes coincides with \( (-) \otimes (\text{id}, \Delta)_\gamma \), therefore the lemma holds. \( \square \)

Below we take derived open substacks \( \mathcal{M}_S^\dagger(\beta, n)^\text{fin} \subset \mathcal{M}_S^\dagger(\beta, n)^\text{pure} \) for each \( n \), satisfying the condition (4.26). Then we set

\[
\mathcal{M}_S^\dagger(\beta)^\text{fin} := \bigsqcup_{n \in \mathbb{Z}} \mathcal{M}_S^\dagger(\beta, n)^\text{fin} \subset \mathcal{M}_S^\dagger(\beta)^\text{pure}.
\]

Let \( h^\pm(z) \in K_{\text{perf}}(\mathcal{M}_S^\dagger(\beta)^\text{pure} \times S)[z] \) be defined by

\[
h^+(z) = \left( 1 - \frac{1}{z} \right) \wedge^\bullet \left( (q_S^{-1} - 1) \mathfrak{F}(\beta) \right) \big|_{z=\infty},
\]

\[
h^-(z) = \left( 1 - \frac{1}{z} \right) \wedge^\bullet \left( (q_S^{-1} - 1) \mathfrak{F}(\beta) \right) \big|_{z=0}.
\]

Here \( q_S \) is the class \([\omega_S] \in K(S)\), pulled back to \( \mathcal{M}_S^\dagger(\beta)^\text{pure} \times S \). The restrictions of \( h^\pm(z) \) to \( \mathcal{M}_S^\dagger(\beta)^\text{fin} \times S \) are also denoted by \( h^\pm(z) \). We have the following proposition, which is an analogue of [19, Proposition 3.6].

**Proposition 6.7** We have the following diagram

\[
\begin{array}{ccc}
K_{\text{perf}}(\mathcal{M}_S^\dagger(\beta)^\text{pure}) & \xrightarrow{\boxtimes(\text{id}, \Delta)_\gamma} & K_{\text{perf}}(\mathcal{M}_S^\dagger(\beta)^\text{pure} \times S \times S)[z, w] \\
\otimes \rho(z, w) & \downarrow & \downarrow \\
K_{\text{perf}}(\mathcal{M}_S^\dagger(\beta)^\text{pure} \times S \times S)[z, w] & \rightarrow & K(\mathcal{M}_S^\dagger(\beta)^\text{pure} \times S \times S)[z, w]
\end{array}
\]

which commutes after restricting the both compositions to \( \mathcal{M}_S^\dagger(\beta)^\text{fin} \times S \times S \). Here \( \rho(z, w) \) is given by

\[
\rho(z, w) := (\text{id}, \Delta)_* \frac{h^+(z) - h^-(w)}{q_S - 1} \delta \left( \frac{w}{z} \right) \in K_{\text{perf}}(\mathcal{M}_S^\dagger(\beta)^\text{pure} \times S \times S)[z, w].
\]

(6.14)
**Proof** It is enough to show the following identity in $K(\mathcal{M}_S^+(\beta)^{\text{fin}} \times S)\{z, w\}$

\[
\gamma = \frac{\delta (w/z)}{q_S - 1} \left[ \left( 1 - \frac{1}{z} \right) \wedge \left( \frac{(q_S^{-1} - 1)\mathfrak{F}(\beta)}{z} \right) \right]_{z=\infty} - \left( 1 - \frac{1}{w} \right) \wedge \left( \frac{(q_S^{-1} - 1)\mathfrak{F}(\beta)}{w} \right) \bigg|_{w=0}.
\]

(6.15)

We compute $\gamma$ by the identity from Lemma 6.6

\[
[\mu^+(z), \mu^-(w)] \cdot 1 = (\text{id}, \Delta)_* \gamma.
\]

Here 1 is the class of the structure sheaf of $\mathcal{M}_S^+(\beta)^{\text{pure}}$. Let $\mathcal{L}$ be the line bundle defined in (5.8). By Lemma 5.6 and Lemma 5.7, we have

\[
\mathcal{L} = \mathcal{O}(\pi_1, \pi_3)(-1) = \mathcal{O}(\pi_2, \pi_3)(1).
\]

Therefore by Lemma 5.7, Lemma 6.3 and Remark 6.4, after restricting to $\mathcal{M}_S^+(\beta)^{\text{fin}} \times S$, we have

\[
\mu^+(z) \cdot 1 = \wedge^\bullet \left( \frac{-\mathfrak{F}(\beta)[1]}{z} \right) \bigg|_{z=\infty-0}.
\]

Below when we write some identity such as above, it always means the identity after restricting to $\mathcal{M}_S^+(\beta)^{\text{fin}}$. Similarly by Lemma 5.6 and Lemma 5.7, we have

\[
\mu^-(w) \cdot 1 = \wedge^\bullet \left( \frac{wq_S}{\mathfrak{F}(\beta)} \right) \bigg|_{w=\infty-0} = \wedge^\bullet \left( \frac{\mathfrak{F}(\beta)}{wq_S} \right) \bigg|_{w=\infty-0}.
\]

On the other hand using the diagram

\[
\text{Hecke}(v_\bullet) \xrightarrow{\text{id}_S, \pi_3} \text{Hecke}(v_\bullet) \times S \xrightarrow{(\pi_2, \text{id}_S)} \mathcal{M}_S^+(v_2)^{\text{pure}} \times S
\]

we have the distinguished triangle

\[
(\pi_1, \text{id}_S)^* \mathfrak{F}(v_1)[1] \to (\pi_2, \text{id}_S)^* \mathfrak{F}(v_2)[1] \to (\text{id}_S, \pi_3)_* \mathcal{L}.
\]

(6.16)
Then using (6.16), we have

\[
\begin{align*}
\mu^-(w)\mu^+(z) \cdot 1 &= \mu^-(w) \wedge \left( -\frac{\mathcal{F}(\beta)[1]}{z} \right) \bigg|_{z=\infty-0} \\
&= \wedge \left( -\frac{\mathcal{F}(\beta)[1]}{z} \right) \wedge \left( -\frac{w}{z} \partial \right) \mu^-(w) \cdot 1 \bigg|_{z=\infty-0} \\
&= \wedge \left( \frac{\mathcal{F}(\beta)}{wq_S(1)} \right) \wedge \left( -\frac{\mathcal{F}(\beta)[1]}{z} \right) \wedge \left( -\frac{w}{z} \partial \right) \bigg|_{w=\infty-0, z=\infty-0}.
\end{align*}
\]

(6.17)

Here the subscript (1), (2) means pulling back to \( \mathcal{M}_S^+ (\beta) \text{pure} \times S \times S \) by the first and second projection, the first and third projection, respectively. A similar computation shows that

\[
\begin{align*}
\mu^+(z)\mu^-(w) \cdot 1 &= \wedge \left( \frac{\mathcal{F}(\beta)}{wq_S(1)} \right) \wedge \left( -\frac{\mathcal{F}(\beta)[1]}{z} \right) \wedge \left( -\frac{w}{z} \partial \right) \bigg|_{w=\infty-0, z=\infty-0}.
\end{align*}
\]

(6.18)

The difference between (6.17) and (6.18) is that, in the former we first expand by \( z \) and then expand by \( w \), while in the latter we do in the opposite order. Therefore by (6.4), the difference comes from the poles in \( w/z \),

\[
\begin{align*}
[\mu^+(z), \mu^-(w)] \cdot 1 &= \sum_{\alpha \in \text{Pol}(\xi^S)} \text{Res}_{x=\alpha} \xi^S(x) \cdot \frac{1}{\alpha} \delta \left( \frac{w}{\alpha z} \right) \\
&= \left\{ \wedge \left( \frac{\mathcal{F}(\beta)}{wq_S(1)} \right) \wedge \left( -\frac{\mathcal{F}(\beta)[1]}{z} \right) \right\} \bigg|_{(z, w)=\infty-0}.
\end{align*}
\]

Here \( \xi^S(x) \in K(S \times S)(x) \) is defined by

\[
\xi^S(x) = \wedge (-x \cdot \partial) = 1 + \frac{[\partial \Delta] \cdot x}{(1 - x)(1 - xq_S)}
\]

where the second identity is due to [19, Equation (3.5), (3.6)]. Also \( \text{Pol}(\xi^S(x)) \) is the set of poles of \( \xi^S(x) \), i.e. \( \{1, q_S^{-1}\} \). Therefore we have

\[
[\mu^+(z), \mu^-(w)] \cdot 1 = \delta \left( \frac{w}{z} \right) \wedge \left( \frac{\mathcal{F}(\beta)}{zq_S} \right) \wedge \left( -\frac{\mathcal{F}(\beta)[1]}{z} \right) \wedge \left( [\partial \Delta] \cdot \frac{x}{q_S - 1} \right) \bigg|_{z=\infty-0} \\
+ \delta \left( \frac{wq_S}{z} \right) \wedge \left( \frac{\mathcal{F}(\beta)}{z} \right) \wedge \left( -\frac{\mathcal{F}(\beta)[1]}{z} \right) q_S \wedge \left( [\partial \Delta] q_S \right) \bigg|_{z=\infty-0}.
\]

(6.19)
Here we have identified $\mathfrak{H}^{(1)}(\beta) = \mathfrak{H}(\beta), \mathfrak{H}^{(2)}(\beta) = \mathfrak{H}^*(\beta)$ because of the factor $[O_\Delta]$. By the distinguished triangle

$$\mathfrak{H}(\beta) \to \mathfrak{H}^*(\beta)[1] \to O_{\mathbb{M}^\delta(\beta)_{\text{pure}} \times S}[1]$$

(6.20)

the second term of (6.19) is

$$\delta \left( \frac{wqS}{z} \right) \wedge \left( \frac{[O_{\mathbb{M}^\delta(\beta)_{\text{pure}} \times S}]}{qS-1} \right) \bigg|_{z=\infty-0} = 0.$$  

By computing the first term of (6.19) using (6.20), we have

$$[\mu^+(z), \mu^-(w)] \cdot 1 = \delta \left( \frac{w}{z} \right) \frac{[O_\Delta]}{qS-1} \left( 1 - \frac{1}{z} \right) \wedge \left( \frac{(qS-1)\mathfrak{H}(\beta)}{z} \right) \bigg|_{z=\infty-0}.$$  

Therefore we obtain (6.15).

For $E \in D^b_{\text{coh}}(S)$, let $\mu^\pm_{E, k}$ be the functors defined in Definition 5.2, Definition 5.10. We use the same notation for the induced maps on K-theories

$$\mu^\pm_{E, k} : K(DT^C_{\text{perf}}(P_n(X, \beta))) \to K(DT^C_{\text{perf}}(P_{n\pm 1}(X, \beta))).$$

Then we set

$$\mu^\pm_{E}(z) := \sum_{k \in \mathbb{Z}} \frac{h^\pm_{E, k}}{z^k} : \bigoplus_{n \in \mathbb{Z}} K(DT^C_{\text{perf}}(P_n(X, \beta))) \to \bigoplus_{n \in \mathbb{Z}} K(DT^C_{\text{perf}}(P_n(X, \beta))[z].$$

We have the following result.

**Theorem 6.8** We have the following commutative diagram

$$\begin{align*}
\bigoplus_{n \in \mathbb{Z}} K(DT^C_{\text{perf}}(P_n(X, \beta))) &\xrightarrow{\otimes \rho_{E_1, E_2}(z, w)} \bigoplus_{n \in \mathbb{Z}} K(DT^C_{\text{perf}}(P_n(X, \beta))) \\
\oplus \rho_{E_1, E_2}(z, w) &\downarrow \quad \downarrow [\mu^+_{E_1}(z), \mu^-_{E_2}(w)] \\
\bigoplus_{n \in \mathbb{Z}} K(DT^C_{\text{perf}}(P_n(X, \beta))[z, w]) &\xrightarrow{\otimes \rho_{E_1, E_2}(z, w)} \bigoplus_{n \in \mathbb{Z}} K(DT^C_{\text{perf}}(P_n(X, \beta))[z, w]).
\end{align*}$$

(6.21)

Here the horizontal arrows are natural maps, and $\rho_{E_1, E_2}(z, w)$ is given by

$$\rho_{E_1, E_2}(z, w) = p q_* \left( (E_1 \otimes E_2 \otimes \omega_S) \boxtimes \frac{h^+(z) - h^-(w)}{qS-1} \delta \left( \frac{w}{z} \right) \right)$$

$$\in \bigoplus_{n \in \mathbb{Z}} K(DT^C_{\text{perf}}(P_n(X, \beta))[z, w].$$
where \( p_\mathcal{P} : \mathcal{P}_n(S, \beta) \times S \to \mathcal{P}_n(S, \beta) \) is the projection and we have used the equivalence (5.25).

**Proof** Let \( q_{\mathfrak{M}}, r_1, r_2 \) be the projections from \( \mathcal{M}_S^+(\beta)_{\text{pure}} \times S \times S \) onto the corresponding factors. By (6.7) and noting that \( p_{\mathfrak{M}} = p_{\mathfrak{M}}(\omega_S[2]) \) for the projection \( p_\mathfrak{M} : \mathcal{M}_S^+(\beta)_{\text{pure}} \times S \to \mathcal{M}_S^+(\beta)_{\text{pure}} \), we have

\[
\mu_1^+(z) \circ \mu_2^-(w)(-) = q_{\mathfrak{M}}\{((\mu_1^+(z) \circ \mu_2^-(w)(-)) \otimes r_1^*(E_2 \otimes \omega_S) \otimes r_2^*(E_1)\},
\]

\[
\mu_2^-(w) \circ \mu_1^+(z)(-) = q_{\mathfrak{M}}\{((\mu_1^-(w) \circ \mu_2^+(z)(-)) \otimes r_1^*(E_1) \otimes r_2^*(E_2 \otimes \omega_S)\}.
\]

(6.22)

Then the commutative diagram (6.21) follows from Lemma 6.6 and Proposition 6.7. □

By taking the commutator for \( k = 0 \), we obtain the following corollary.

**Corollary 6.9** We have the following commutative diagram

\[
\bigoplus_{n \in \mathbb{Z}} K\left( DT_{\text{perf}}^C(P_n(X, \beta)) \right) \xrightarrow{\otimes \rho_{E_1, E_2}} \bigoplus_{n \in \mathbb{Z}} K\left( DT_{\text{perf}}^C(P_n(X, \beta)) \right) \xrightarrow{[\mu_{E_1, k=0}, \mu_{E_2, k=0}]} \bigoplus_{n \in \mathbb{Z}} K\left( DT_{\text{perf}}^C(P_n(X, \beta)) \right).
\]

Here \( \rho_{E_1, E_2} \) is given by

\[
\rho_{E_1, E_2} = p_{\mathfrak{M}}(E_1 \otimes E_2 \otimes \omega_S) \otimes \mathcal{F}(\beta)^{\vee} \in \bigoplus_{n \in \mathbb{Z}} K\left( DT_{\text{perf}}^C(P_n(X, \beta)) \right).
\]

(6.23)

**Proof** Using the expansion (6.3), we see that

\[
\wedge^\bullet \left( \frac{(q_S^{-1} - 1)\mathcal{F}(\beta)}{z} \right) \bigg|_{z=\infty} = O(z^{-1}) + (1 - q_S)[\mathcal{F}(\beta)^{\vee}]z + O(z^2).
\]

Therefore the constant term of \( \rho(z, w) \) in (6.14) is given by \((\text{id}, \Delta)_*\mathcal{F}(\beta)^{\vee}\). By taking the constant term of the diagram (6.21), we obtain the desired commutative diagram (6.23). □

**Example 6.10** In Corollary 6.8, let us take \( E_1 = \mathcal{O}_S \) and \( E_2 = \mathcal{O}_H \) for an ample divisor \( H \subset S \). Suppose that \( H \) intersects with any curve \( C \subset S \) with \([C] = \beta \) transversely. Then by setting \( \mu_S^+ = \mu_{\mathcal{O}_S, 0}^+, \mu_H^+ = \mu_{\mathcal{O}_H, 0}^+ \), the diagram (6.23) implies that

\[
[\mu_H^-, \mu_S^+](-) = [\mu_S^-, \mu_H^+](-) = (-) \otimes \mathcal{V}(\beta)
\]

(6.24)

for a vector bundle \( \mathcal{V}(\beta) \) on \( \mathcal{M}_S^+(\beta)_{\text{pure}} \) of rank \( \beta \cdot H \) and \(-\) is a K-group element coming from the perfect PT categories. We also have obvious vanishings \([\mu_S^+, \mu_S^-] = \)
\[ [\mu_H^+, \mu_H^-] = 0. \text{ However } [\mu_S^+, \mu_S^-], [\mu_H^+, \mu_H^-] \text{ do not necessary vanish (though the latter is a nilpotent operator).} \]

**Remark 6.11** In [24], Rennemo proved the following. Let \( C \) be an irreducible curve with at worst planar singularities, and \( C^{[n]} \) the Hilbert scheme of \( n \)-points on \( C \). Then there exist linear maps

\[
\mu_C^\pm: H_*(C^{[n]}) \to H_{*\pm 1}(C^{[n+1]}), \quad \mu_{pt}^\pm: H_*(C^{[n]}) \to H_{*-1}(C^{[n+1]})
\]

using the moduli space parameterizing \((Z, Z')\), \( Z \in C^{[n]}, Z' \in C^{[n+1]} \) with \( Z \subset Z' \). They satisfy the following relations of Weyl algebras

\[
[\mu_{pt}^-, \mu_C^+] = [\mu_C^-, \mu_{pt}^+] = \text{id} \quad (6.25)
\]

and all other pairs of operators commute.

Since the moduli spaces of stable pairs much generalize \( C^{[n]} \), the result of Corollary 6.8 may be regarded as a categorification of the above result by Rennemo. Indeed the relation (6.24) is regarded as a categorification of (6.25). However contrary to the vanishing of commutators of \([\mu_C^+, \mu_C^-], [\mu_{pt}^+, \mu_{pt}^-]\), the commutators \([\mu_S^+ \mu_S^-], [\mu_H^+, \mu_H^-] \) in Example 6.10 do not necessary vanish.

### 7 Some other actions of DT categories

In this section, we construct some other actions of DT categories in the following cases: the left action of DT categories of zero dimensional sheaves to MNOP categories, the right/left actions of DT categories of one dimensional semistable sheaves to DT categories for stable D0–D2–D6 bound states. Almost all the arguments are similar to those in Sect. 4, so we omit details in several places.

#### 7.1 Another stacks of extensions

For \( v_\bullet = (v_1, v_2, v_3) \in N_{\leq 1}(S)^{\times 3} \), we define the derived stack \( \mathcal{M}^{\text{ext}, +}_S(v_\bullet) \) by the Cartesian square

\[
\begin{array}{ccc}
\mathcal{M}^{\text{ext}, +}_S(v_\bullet) & \xrightarrow{\text{ev}_2^+} & \mathcal{M}^+_S(v_2) \\
\downarrow & & \downarrow \\
\mathcal{M}^{\text{ext}}_S(v_\bullet) & \xrightarrow{\text{ev}_2} & \mathcal{M}_S(v_2).
\end{array}
\]
For $T \in d\text{Aff}$, the $T$-valued points of $\mathbb{M}_S^{\dagger}$ form the $\infty$-groupoid of diagrams

$$\mathcal{O}_{S \times T} \xrightarrow{\xi} \mathcal{F}_1 \xrightarrow{} \mathcal{F}_2 \xrightarrow{} \mathcal{F}_3.$$ (7.1)

Here the bottom sequence is a $T$-valued point of $\mathbb{M}_S^{\text{ext}}(v_\bullet)$. We have the following diagram

$$\mathbb{M}_S^{\text{ext}, \dagger}(v_\bullet) \xrightarrow{(\text{ev}_1^\dagger, \text{ev}_3^\dagger)} \mathbb{M}_S^{\dagger}(v_2) \xrightarrow{} \mathbb{M}_S^{\dagger}(v_1) \times \mathbb{M}_S^{\dagger}(v_3).$$ (7.2)

Here $\text{ev}_1^\dagger$ sends a diagram (7.1) to $\mathcal{F}_1$, and $\text{ev}_3^\dagger$ sends a diagram (7.1) to the composition $\mathcal{O}_{S \times T} \xrightarrow{\xi} \mathcal{F}_2 \xrightarrow{} \mathcal{F}_3$. Note that $\text{ev}_2^\dagger$ is proper by definition.

**Lemma 7.1** The morphism $(\text{ev}_1^\dagger, \text{ev}_3^\dagger)$ is quasi-smooth. In particular, the derived stack $\mathbb{M}_S^{\text{ext}, \dagger}(v_\bullet)$ is quasi-smooth.

**Proof** Similarly to (3.6), we have

$$\mathbb{M}_S^{\text{ext}, \dagger}(v_\bullet) = \mathbb{V}(\mathcal{H}om_{p_{\mathfrak{M} \times \mathfrak{M}^{\dagger}}(\mathcal{G}^\bullet(v_3), \mathcal{F}(v_1))^\vee) \rightarrow \mathbb{M}(v_1) \times \mathbb{M}^{\dagger}(v_3).$$

Here $p_{\mathfrak{M} \times \mathfrak{M}^{\dagger}}$ is the projection from $S \times \mathbb{M}_S(v_1) \times \mathbb{M}_S^{\dagger}(v_3)$ to $\mathbb{M}_S(v_1) \times \mathbb{M}_S^{\dagger}(v_3)$. From the distinguished triangle

$$p_{\mathfrak{M} \times \mathfrak{M}^{\dagger}}\mathcal{F}(v_1) \boxtimes \mathcal{O}_{\mathfrak{M}^{\dagger}(v_3)} \rightarrow \mathcal{H}om_{p_{\mathfrak{M} \times \mathfrak{M}^{\dagger}}(\mathcal{G}^\bullet(v_3), \mathcal{F}(v_1)) \rightarrow \mathcal{H}om_{p_{\mathfrak{M} \times \mathfrak{M}^{\dagger}}(\mathcal{G}(v_3), \mathcal{F}(v_1))[1]$$

the middle term is perfect whose restriction to any point $x \rightarrow \mathbb{M}_S(v_1) \times \mathbb{M}_S^{\dagger}(v_3)$ is of cohomological amplitude $[-1, 1]$. Therefore $(\text{ev}_1^\dagger, \text{ev}_3^\dagger)$ is quasi-smooth. The last statement holds as $\mathbb{M}_S(v_1) \times \mathbb{M}_S^{\dagger}(v_3)$ is also quasi-smooth. \qed

We take $v_\bullet = (v_1, v_2, v_3) \in N_{\leq 1}(S)^{\times 3}$ and define the classical stack

$$\mathcal{M}_X^{\text{ext}, \dagger}(v_\bullet): \text{Aff}^{\text{op}} \rightarrow \text{Groupoid}$$
by sending $T \in \text{Aff}$ to the groupoid of distinguished triangles $E_1[-1] \to E_2 \xrightarrow{j} E_3$ together with commutative diagrams

\[
\begin{array}{c}
E_2 \otimes \mathcal{O}_{S_\infty \times T} \xrightarrow{j} E_3 \otimes \mathcal{O}_{S_\infty \times T} \\
\lambda_2 \cong \quad \cong \lambda_3 \\
\mathcal{O}_{S_\infty \times T} \quad \mathcal{O}_{S_\infty \times T}.
\end{array}
\]

Here $(E_i, \lambda_i)$ for $i = 2, 3$ are $T$-valued points of $\mathcal{M}_X^+(v_i)$ and $E_1$ is a $T$-valued point of $\mathcal{M}_X(v_1)$. We also have the evaluation morphisms

\[
\begin{array}{c}
\mathcal{M}_X^{\text{ext}^+}(v_\bullet) \xrightarrow{\text{ev}_2^{X,+}} \mathcal{M}_X^+(v_2) \\
(\text{ev}_1^{X,+}, \text{ev}_3^{X,+}) \downarrow \\
\mathcal{M}_X(v_1) \times \mathcal{M}_X^+(v_3)
\end{array}
\]

where $\text{ev}_i^{X,+}$ sends $E_\bullet$ to $E_i$.

Let $\text{ev}^+$ be the morphism

\[
\text{ev}^+ = (\text{ev}_1^+, \text{ev}_2^+, \text{ev}_3^+): \mathcal{M}_S^+(v_\bullet) \to \mathcal{M}_S(v_1) \times \mathcal{M}_S^+(v_2) \times \mathcal{M}_S^+(v_3).
\]

Similarly to Proposition 4.7, we can show the isomorphism over $\mathcal{M}_{S}^{\text{ext}^+}(v_\bullet)$

\[
\mathcal{M}_X^{\text{ext}^+}(v_\bullet) \cong \iota_0(\Omega_{\text{ev}^+}[-2]).
\]

### 7.2 Actions on MNOP categories

For $(\beta, n) \in N_{\leq 1}(S)$, we denote by

\[
I_n(X, \beta)
\]

the moduli space compactly supported closed subschemes $Z \subset X$ satisfying $[\pi_* \mathcal{O}_Z] = (\beta, n)$. The moduli space $I_n(X, \beta)$ is a Hilbert scheme of one or zero dimensional subschemes of $X$, so it is a quasi-projective scheme.

We have the open immersion

\[
I_n(X, \beta) \subset \mathcal{M}_X^+(\beta, n)
\]
sending a closed subscheme $Z \subset X$ to the ideal sheaf $I_Z \subset \mathcal{O}_X$. We define the following conical closed substack

$$Z_{I\text{-us}}(\beta, n) := \mathcal{M}_X^\dagger(\beta, n) \setminus I_n(X, \beta) \subset \mathcal{M}_X^\dagger(\beta, n).$$

Since $I_n(X, \beta)$ is a quasi-projective scheme, there is a derived open substack $\mathcal{M}_S^\dagger(\beta, n)_{\text{fin}} \subset \mathcal{M}_S^\dagger(\beta, n)$ of finite type such that

$$I_n(X, \beta) \subset t_0(\Omega_{\mathcal{M}_S^\dagger(\beta, n)_{\text{fin}}}[-1]) \subset \mathcal{M}_X^\dagger(\beta, n). \quad (7.5)$$

**Definition 7.2** ([32, Definition 6.6]) The $\mathbb{C}^*$-equivariant categorical MNOP theory is defined by

$$\mathcal{D}T_{\mathbb{C}^*}(I_n(X, \beta)) := D^b_{\text{coh}}(\mathcal{M}_S^\dagger(\beta, n)_{\text{fin}})/\mathcal{C}_{Z_{I\text{-us}}(\beta, n)_{\text{fin}}}.$$

Let us take $v_\bullet \in N_{\leq 1}(S)^3$ to be

$$v_1 = (0, m) = m[\text{pt}], \quad v_2 = (\beta, n + m), \quad v_3 = (\beta, n).$$

We also take derived open substacks of finite type

$$\mathcal{M}_S(v_1)_{\text{fin}} = \mathcal{M}_S(m[\text{pt}]), \quad \mathcal{M}_S(v_2)_{\text{fin}} \subset \mathcal{M}_S^\dagger(v_2), \quad \mathcal{M}_S(v_3)_{\text{fin}} \subset \mathcal{M}_S^\dagger(v_3)$$

satisfying (7.5) for $v_2, v_3$ and the condition

$$\mathcal{M}_S^{\dagger, \text{ext}}(v_\bullet)_{\text{fin}} := (\text{ev}_2^\dagger)^{-1}(\mathcal{M}_S^\dagger(v_2)_{\text{fin}}) \subset (\text{ev}_1^\dagger, \text{ev}_3^\dagger)^{-1}(\mathcal{M}_S(v_1)_{\text{fin}} \times \mathcal{M}_S^\dagger(v_3)_{\text{fin}}). \quad (7.6)$$

Then the diagram (7.2) restricts to the diagram

$$\begin{array}{ccc}
\mathcal{M}_S^{\dagger, \text{ext}}(v_\bullet)_{\text{fin}} & \xrightarrow{\text{ev}_2^\dagger} & \mathcal{M}_S^\dagger(v_2)_{\text{fin}} \\
(\text{ev}_1^\dagger, \text{ev}_3^\dagger) \downarrow & & \downarrow \\
\mathcal{M}_S(v_1)_{\text{fin}} \times \mathcal{M}_S^\dagger(v_3)_{\text{fin}}. & & \\
\end{array}$$

The vertical arrow is quasi-smooth and the horizontal arrow is proper. Therefore we have the induced functor

$$\text{ev}_2^\dagger(\text{ev}_1^\dagger, \text{ev}_3^\dagger)^*: D^b_{\text{coh}}(\mathcal{M}_S(v_1)_{\text{fin}}) \times D^b_{\text{coh}}(\mathcal{M}_S^\dagger(v_3)_{\text{fin}}) \to D^b_{\text{coh}}(\mathcal{M}_S^\dagger(v_2)_{\text{fin}}). \quad (7.7)$$

Similarly to Theorem 4.10, we have the following.
Theorem 7.3 The functor (7.7) descends to the functor
\[ DT^C(X) (M^\sigma_{ss}(m[pt])) \times DT^C(X) (I_n(X, \beta)) \rightarrow DT^C(X) (I_{n+m}(X, \beta)). \] (7.8)

Proof By Lemma 4.11 below, we have the following inclusion
\[ (ev_1^{X, \sigma}, ev_3^{X, \sigma})^{-1} (M_X(v_1) \times Z_{I-as}(v_3)) \subset (ev_2^{X, \sigma})^{-1} (Z_{I-as}(v_2)). \]

Therefore the result follows from Corollary 4.8. \( \square \)

Here we have used the following lemma.

Lemma 7.4 For an exact sequence \( 0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0 \) in \( A_X \), suppose that \( E_1 = Q[-1] \) for a zero dimensional sheaf \( Q \) and \( E_2 = I_{Z_2} \) for a compactly supported one or zero dimensional subscheme \( Z_2 \subset X \). Then \( E_3 = I_{Z_3} \) for a compactly supported one or zero dimensional subscheme \( Z_3 \subset X \).

Proof We have the distinguished triangle \( E_2 \rightarrow E_3 \rightarrow E_1[1] \), so we have \( E_3 \in A_X \cap Coh(X) \). By the definition of \( A_X \), any object in \( A_X \cap Coh(X) \) is torsion free, so \( E_3 \) is of the form \( E_3 = I_{Z_3} \) for a closed subscheme \( Z_3 \subset X \). Since \( E_3 \in A_X \), it must be trivial along the divisor at the infinity, so \( Z_3 \subset X \). \( \square \)

Similarly to Corollary 4.12, we have the following corollary.

Corollary 7.5 The functors (7.8) induce the left action of the K-theoretic Hall-algebra of zero dimensional sheaves to the direct sum of MNOP categories for a fixed \( \beta \)
\[ \bigoplus_{m \geq 0} K(DT^C(X)(M^\sigma_{ss}(m[pt]))) \times \bigoplus_{n \in \mathbb{Z}} K(DT^C(X)(I_n(X, \beta))) \rightarrow \bigoplus_{n \in \mathbb{Z}} K(DT^C(X)(I_{n+m}(X, \beta))). \]

7.3 Categorical DT theories for stable D0–D2–D6 bound states

In this subsection, we recall categorical DT theories for stable D0–D2–D6 bound states. They depend on a choice of a stability parameter \( t \in \mathbb{R} \), and the wall-crossing phenomena with respect to the above stability parameter is relevant for the rationality of generating series of PT invariants and GV formula. See [35] for details.

Below, we fix an element \( \sigma = iH \) in (3.18) for \( B = 0 \) and an ample divisor \( H \). We define the map \( \mu \) by
\[ \mu : N_{\leq 1}(S) \rightarrow \mathbb{Q} \cup \{ \infty \}, \ (\beta, n) \mapsto \frac{n}{\beta \cdot H}. \]

Here \( \mu(\beta, n) = \infty \) if the denominator is zero. For a non-zero \( F \in Coh_{\leq 1}(X) \), it is \( \sigma \)-(semi)stable if and only if we have the inequality \( \mu(F') \leq (\leq) \mu(F) \) for any non-zero subsheaf \( F' \subsetneq F \).

For each \( t \in \mathbb{R} \), we also define the following map
\[ \mu_t^+: \mathbb{Z} \oplus N_{\leq 1}(S) \rightarrow \mathbb{R} \cup \{ \infty \}, \ (r, \beta, n) \mapsto \begin{cases} t, & r \neq 0, \\ \mu(\beta, n), & r = 0. \end{cases} \]
For $E \in \mathcal{A}_X$, we set $\mu_t^\dagger(E) = \mu_t^\dagger(\text{cl}(E))$.

**Definition 7.6** An object $E \in \mathcal{A}_X$ is $\mu_t^\dagger$-(semi) stable if for any exact sequence $0 \to E' \to E \to E'' \to 0$ in $\mathcal{A}_X$ we have the inequality $\mu_t^\dagger(E') \leq (\leq) \mu_t^\dagger(E'')$.

We have the substacks

$$P_n(X, \beta)_t \subset \mathcal{M}_X^\dagger(\beta, n)$$

corresponding to $\mu_t^\dagger$-stable objects. The result of [34, Proposition 3.17] shows that the above substack is an algebraic space of finite type. Moreover there is a finite set of walls $W \subset \mathbb{Q}$ such that $P_n(X, \beta)_t$ is constant if $t$ lies on on a connected component of $\mathbb{R} \setminus W$. We say that $t \in \mathbb{R}$ lies in a chamber if $t \not\in W$.

We define the following conical closed substack

$$Z_{\mu_t^\dagger-\text{us}}(\beta, n) := \mathcal{M}_X^\dagger(\beta, n) \setminus P_n(X, \beta)_t \subset \mathcal{M}_X^\dagger(\beta, n).$$

Since $P_n(X, \beta)_t$ is of finite type, there is a derived open substack $\mathcal{M}_S^\dagger(\beta, n)_{\text{fin}} \subset \mathcal{M}_S^\dagger(\beta, n)$ of finite type such that

$$P_n(X, \beta)_t \subset t_0(\Omega_{\mathcal{M}_S^\dagger(\beta, n)_{\text{fin}}}[−1]) \subset \mathcal{M}_X^\dagger(\beta, n). \quad (7.9)$$

**Definition 7.7** ([32, Definition 6.6]) The $\mathbb{C}^*$-equivariant DT theory for stable D0–D2–D6 bound states is defined by

$$DT_{\mathbb{C}^*}^*(P_n(X, \beta)_t) := D^b_{\text{coh}}(\mathcal{M}_S^\dagger(\beta, n)_{\text{fin}})/\mathcal{Z}_{\mu_t^\dagger-\text{us}}(\beta, n)_{\text{fin}}.$$

### 7.4 Actions on categorical DT theories for stable D0–D2–D6 bound states

Here we construct actions of DT categories of one dimensional semistable sheaves to the DT categories in Definition 7.7. We fix $t = t_0 \in \mathbb{R}$, and set $t_{\pm} = t_0 \pm \epsilon$ for $0 < \epsilon \ll 1$. Then for $\overline{\mathbb{X}} = m + t_0 \in \mathbb{Q}[m]$, we have

$$N(S)_{\overline{\mathbb{X}}} = \{ (\beta, n) \in N_{\leq 1}(S) : \mu(\beta, n) = t_0 \} \cup \{0\}.$$

We take $v_\bullet = (v_1, v_2, v_3) \in N_{\leq 1}(S)$ by

$$v_1 = (\beta, n), \ v_2 = (\beta + \beta', n + n'), \ v_3 = (\beta', n') \in N(S)_{\overline{\mathbb{X}}}.$$

We then take derived open substacks

$$\mathcal{M}_S^\dagger(v_1)_{\text{fin}} \subset \mathcal{M}_S^\dagger(v_1), \ \mathcal{M}_S^\dagger(v_2)_{\text{fin}} \subset \mathcal{M}_S^\dagger(v_2), \ \mathcal{M}_S^\dagger(v_3)_{\text{fin}} \subset \mathcal{M}_S^\dagger(v_3)$$

satisfying the conditions (7.9) for $v_1, v_2$, the condition (3.21) for $v_3$ and (4.22). Then we have the following.
Theorem 7.8 The functor (4.24) descends to the functor
\[ \mathcal{DT}^c \left( P_n(X, \beta), \mathcal{M}_n^\infty \right) \times \mathcal{DT}^c \left( \mathcal{M}_n^\alpha_{n^+} \right) \rightarrow \mathcal{DT}^c \left( P_{n+n'}(X, \beta + \beta'), \mathcal{M}_{n+n'}^\infty \right). \]

Proof Similarly to Theorem 4.10, the result follows from Lemma 7.9 (i).

Here we have used the following lemma, which is an analogue of Lemmas 4.11 and 7.4.

Lemma 7.9 Let \( 0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0 \) be an exact sequence in \( \mathcal{A}_X \).

(i) Suppose that \( E_3 = F_3[-1] \) for \( F_3 \in \text{Coh}_{\leq 1}(X) \) with \( \mu(F_3) = t_0 \). Then if \( E_2 \) is \( \mu_{t_+} \)-stable, then \( F_3 \) is \( \mu \)-semistable and \( E_1 \) is \( \mu_{t_-} \)-stable.

(ii) Suppose that \( E_1 = F_1[-1] \) for \( F_1 \in \text{Coh}_{\leq 1}(X) \) with \( \mu(F_1) = t_0 \). Then if \( E_2 \) is \( \mu_{t_+} \)-stable, then \( F_1 \) is \( \mu \)-semistable and \( E_2 \) is \( \mu_{t_-} \)-stable.

Proof We only prove (i), as the proof of (ii) is similar. First we show that \( F_3 \) is \( \mu \)-semistable. Let \( F_3 \twoheadrightarrow F \) be a non-trivial surjection in \( \text{Coh}_{\leq 1}(X) \). Then we have the surjection \( E_2 \twoheadrightarrow F \twoheadrightarrow [-1] \) in \( \mathcal{A}_X \). By the \( \mu_{t_-} \)-stability of \( E_2 \), we have \( \mu(F) > t_- \), hence \( \mu(F) \geq t_0 = \mu(F_3) \). Therefore \( F_3 \) is \( \mu \)-semistable.

Next we show that \( E_1 \) is \( \mu_{t_+} \)-stable. Let \( F[-1] \hookrightarrow E_1 \) be an injection in \( \mathcal{A}_X \) for \( F \in \text{Coh}_{\leq 1}(X) \). Then we have an injection \( F[-1] \hookrightarrow E_2 \) hence \( \mu(F) < t_- \) by the \( \mu_{t_-} \)-stability of \( E_2 \). Let \( E_1 \hookrightarrow F[-1] \) be a surjection in \( \mathcal{A}_X \). Then we have exact sequences in \( \mathcal{A}_X \) and \( \text{Coh}_{\leq 1}(X) \)

\[ 0 \rightarrow E_1' \rightarrow E_2 \rightarrow F'[\cdot \cdot \cdot ] \rightarrow 0, \quad 0 \rightarrow F \rightarrow F' \rightarrow F_3 \rightarrow 0. \]

By the \( \mu_{t_-} \)-stability of \( E_2 \), we have \( \mu(F') \leq t_0 \). Then as \( \mu(F_3) = t_0 \), we have \( \mu(F) \geq t_0 > t_- \). Therefore \( E_1 \) is \( \mu_{t_-} \)-stable.

We see that after crossing the wall at \( t = t_0 \), we obtain the left action of DT categories of one dimensional semistable sheaves. We take \( v_\bullet = (v_1, v_2, v_3) \in N_{\leq 1}(S) \) as

\[ v_1 = (\beta', n') \in N(S)_X, \quad v_2 = (\beta + \beta', n + n'), \quad v_3 = (\beta, n). \]

We then take derived open substacks

\[ \mathcal{M}_S(v_1)_{\text{fin}} \subset \mathcal{M}_S(v_1), \quad \mathcal{M}_S(v_2)_{\text{fin}} \subset \mathcal{M}_S(v_2), \quad \mathcal{M}_S(v_3)_{\text{fin}} \subset \mathcal{M}_S(v_3) \]

satisfying the conditions (7.9) for \( v_2, v_3 \), the condition (3.21) for \( v_1 \) and (7.6). Then we have the following.

Theorem 7.10 The functor (7.7) descends to the functor
\[ \mathcal{DT}^c \left( \mathcal{M}_n^\alpha_{n^+} \right) \times \mathcal{DT}^c \left( \mathcal{M}_n^\alpha_{n^+} \right) \rightarrow \mathcal{DT}^c \left( P_{n+n'}(X, \beta + \beta'), \mathcal{M}_{n+n'}^\infty \right). \]

Proof Similarly to Theorem 7.3, the result follows from Lemma 7.9 (ii).

As a corollary of Theorem 7.8 and Theorem 7.10, we have the following:
Corollary 7.11 For each $t_0 \in \mathbb{R}$, the $K$-theoretic Hall-algebra of one dimensional semistable sheaves with slope $t_0$
\[
\bigoplus_{\mu(\beta, n) = t_0} K(\mathcal{D}T_{C^*}^n(\mathcal{M}_X^{\text{ss}}(\beta, n)))
\]
acts on
\[
\bigoplus_{(\beta, n) \in N_{\leq 1}(S)} K(\mathcal{D}T_{C^*}^n(P_n(X, \beta)_{t_-})), \quad \bigoplus_{(\beta, n) \in N_{\leq 1}(S)} K(\mathcal{D}T_{C^*}^n(P_n(X, \beta)_{t_+}))
\]
(7.10)
from right, left, respectively.

Example 7.12 Let $S \to \mathbb{C}^2$ be the blow-up at the origin, and $C \subset S$ is the exceptional curve. Then
\[
X = \text{Tot}_S(\omega_S) = \text{Tot}_{\mathbb{P}^1}(O_{\mathbb{P}^1}(-1)^{\oplus 2})
\]
is the resolved conifold. In this case, the set of walls is given by $W = \mathbb{Z}_{> 0} \subset \mathbb{Q}$ since any one dimensional stable sheaf on $X$ is of the form $O_C(m)$. Then we set
\[
P_n(X, d)_m := P_n(X, d[C])_{t_+}, \ t \in (m, m + 1)
\]
which makes sense by the above description of walls. By Corollary 7.11 for $t_0 = m, m + 1$, the direct sum for each $m \in \mathbb{Z}_{> 0}$
\[
\bigoplus_{(n, d) \in \mathbb{Z}^2} K(\mathcal{D}T_{C^*}^n(P_n(X, d)_m))
\]
ads right/left actions of the algebra (3.26).

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