Implications of the recent high statistics determination of the pion electromagnetic form factor in the timelike region

B. Ananthanarayan, 1, Irinel Caprini, 2 and I. Sentitemsu Imsong 1

1 Centre for High Energy Physics, Indian Institute of Science, Bangalore 560 012, India
2 National Institute of Physics and Nuclear Engineering POB MG 6, Bucharest, R-76900, Romania

The recently evaluated two-pion contribution to the muon g − 2 and the phase of the pion electromagnetic form factor in the elastic region, known from ππ scattering by Fermi-Watson theorem, are exploited by analytic techniques for finding correlations between the coefficients of the Taylor expansion at t = 0 and the values of the form factor at several points in the spacelike region. We do not use specific parametrizations and the results are fully independent of the unknown phase in the inelastic region. Using for instance, from recent determinations, (r2) = (0.435 ± 0.005) fm2 and F(−1.6 GeV2) = 0.243 ± 0.022, we obtain the allowed ranges 3.75 GeV−4 ≤ c ≤ 3.98 GeV−4 and 9.91 GeV−6 ≤ d ≤ 10.46 GeV−6 for the curvature and the next Taylor coefficient, with a strong correlation between them. We also predict a large region in the complex plane where the form factor cannot have zeros.

PACS numbers: 11.55.Fv, 13.40.Gp, 25.80.Dj

I. INTRODUCTION

The pion electromagnetic form factor F(t), defined by the matrix element

⟨π+(p′)|Jμ|π+(p)⟩ = (p + p′)μF(t)

where q = p − p′ and t = q2, plays a central role in strong interaction dynamics. From the general principles of quantum field theory, it follows that F(t) is normalized to F(0) = 1 and is a real analytic function in the spacelike region. Using for instance, from recent determinations, (r2) = (0.435 ± 0.005) fm2 and F(−1.6 GeV2) = 0.243 ± 0.022, we obtain the allowed ranges 3.75 GeV−4 ≤ c ≤ 3.98 GeV−4 and 9.91 GeV−6 ≤ d ≤ 10.46 GeV−6 for the curvature and the next Taylor coefficient, with a strong correlation between them. We also predict a large region in the complex plane where the form factor cannot have zeros.

The pion electromagnetic form factor F(t) is very rich. This quantity was measured at spacelike values Q2 > 0 with increasing precision from electron-pion scattering and pion electroproduction from nucleons [14–25]. On the timelike cut, where the form factor is complex, the Fermi-Watson theorem implies that in the elastic region its phase is equal to the phase-shift of the P-wave of the ππ amplitude, calculated recently with precision using Roy equations and fixed-t dispersion relations [24, 26]. The modulus has been measured from the cross section of e+e− → π+π− by several groups in the past [27–36], and more recently to high accuracy by the BABAR [37] and KLOE [38, 39] collaborations. These data have been used for an accurate evaluation of the two-pion contribution to the muon anomalous magnetic moment [40, 41].

The constraints imposed on the pion form factor by analyticity and unitarity have been exploited in many works (the list [42–61] covers only partly a very rich literature). Different analytic representations, either as standard dispersion relations [17], phase ( Omn`es-type) [18, 48, 51] or modulus [52] representations, as well as expansions based on conformal mappings [18, 48, 51] or Pad`e-type approximants [61], have been constructed in order to correlate the low- and high-energy properties of the form factor. Of special interest is the issue of the zeros of the form factor, investigated by means of dispersive sum-rules [18, 48, 52] or by the more powerful techniques of analytic optimization theory [42, 47, 48]. In [61] similar functional-analytic techniques were applied for deriving bounds on the expansion coefficients at t = 0, from an weighted integral of the modulus squared along the cut, known from unitarity and dispersion relations for a related QCD correlator.

In the present paper we address the same problem, i.e. to find constraints on the coefficients appearing in the
Taylor expansion

\[ F(t) = 1 + \frac{1}{6}(t^3) + \cdot \cdot \cdot \]  

(3)

from a well-defined input on the timelike axis, and also include information coming from high precision experiments that measure the form factor in the spacelike region. We also consider the problem of the zeros, and obtain a region in the complex \( t \)-plane where zeros are excluded. The main reason of revisiting the problem is the recent high statistics measurement of the modulus \( |F(t)| \) on the unitarity cut by BABAR \[37\] and KLOE \[38, 39\] experiments. As we will show, this information leads to stringent constraints, of a remarkable level for a specific parametrization.

We apply a technique discussed in \[61, 68\], which makes use of information on both the phase and modulus, and was shown recently \[60, 70\] to place stringent bounds on the \( K \pi \) weak form factors. As first input we use the Fermi-Watson theorem, according to which one has, modulo \( \pi \),

\[ \text{Arg}[F(t + i\epsilon)] = \delta^1(t), \quad t_+ < t < t_{\text{in}}, \]  

(4)

where \( \delta^1(t) \) is the phase-shift of the \( P \)-wave of \( \pi \pi \) elastic scattering and \( t_{\text{in}} \) the first inelastic threshold. As discussed previously \[13, 14\], inelasticity in the case of the pion \( \omega \) channel, so we take \( t_{\text{in}} = (M_\pi + M_\omega)^2 \). Below this energy, the phase \( \delta^1(t) \) is known with precision from Roy equations and fixed-\( t \) dispersion relations for \( \pi \pi \) scattering \[23, 26\].

We also include information on the modulus, generically expressed by an integral relation

\[ \frac{1}{\pi} \int_{t_{\text{in}}}^{t_{\text{max}}} dt \rho(t)|F(t)|^2 \leq I, \]  

(5)

where \( \rho(t) \) is a positive definite weight in the region of integration and \( I \) is a known quantity. Actually, \( \rho(t) \) does not fully account for the present information on \( |F(t)| \); indeed, except for a small region near the threshold \( t_+ = 4M_\pi^2 \), the modulus is measured also below the inelastic threshold \( t_{\text{in}} \). This makes it possible to place stringent bounds on the \( K \pi \) weak form factors, at every \( \delta^1(t) \) is known with precision from Roy equations and fixed-\( t \) dispersion relations for \( \pi \pi \) scattering \[23, 26\].

We also consider the problem of the zeros, and was shown recently \[69, 70\] to place stringent constraints, of a remarkable level for a specific parametrization. As we will show, this information leads to stringent constraints, of a remarkable level for a specific parametrization.

The practical motivation of this particular choice is that an accurate evaluation of the two-pion contribution to the muon anomaly, taking into account the correlations between different points, is available from the refs. \[40, 41\]. As a result, this choice guarantees a very precise input. We must emphasize that, once the input \( \delta^1(t) \) is adopted, the treatment is optimal and no information is lost. A posteriori, it turns out that the results given by this choice are quite stringent.

In addition to the above input from the timelike axis, we include the values of \( F(t) \) measured experimentally at some spacelike points

\[ F(t_n) = f_n + \delta f_n, \quad t_n < 0, \quad n = 1, \ldots, N, \]  

(8)

where we use the most recent high precision experimental information from \[22, 23\]. Thus, we will be employing as input Eqs. \[4, 5\] in order to obtain correlations between the coefficients of the Taylor expansion \[3\]. We will investigate also the issue of the possible zeros of the form factor, deriving regions where zeros are forbidden.

In Sec. \[II\] we briefly review the mathematical method and in Sec. \[III\] the experimental information that goes into our computation. In Sec. \[IV\] we present our results for the parameters \( (c, d) \) and compare them with results available in the literature. In Sec. \[V\] we derive regions where zeros are excluded along the real axis and in the complex \( t \)-plane, and in Sec. \[VI\] some discussions and our conclusions are presented.

II. BASIC FORMULAE

For solving the problem we follow a mathematical method presented in \[61, 68\]. We first define the Omnic...
function

\[ O(t) = \exp \left( \frac{t}{\pi} \int_{t_n}^{\infty} dt' \frac{\delta(t')}{t'(t'-t)} \right) , \]  

(9)

where \( \delta(t) = \delta_1^1(t) \) for \( t \leq t_n \), and is an arbitrary function, sufficiently smooth (i.e. Lipschitz continuous) for \( t > t_n \). As shown in [68], the results do not depend on the choice of the function \( \delta(t) \) for \( t > t_n \). A crucial remark is that the function \( h(t) \) defined by

\[ F(t) = O(t)h(t) \]  

(10)
is analytic in the \( t \)-plane cut only for \( t > t_n \). The equality [5], written in terms of \( h(t) \) as

\[ \frac{1}{\pi} \int_{t_n}^{\infty} dt' \rho(t')|O(t')| = h(t) = \tilde{\alpha}_\mu^{\pi} , \]  

(11)
can be expressed in a canonical form, if we perform the conformal transformation

\[ \tilde{z}(t) = \frac{\sqrt{t_n} - \sqrt{t_n-t}}{\sqrt{t_n} + \sqrt{t_n-t}} , \]  

(12)
which maps the complex \( t \)-plane cut for \( t > t_n \) onto the unit disk \( |z| < 1 \) in the \( z \)-plane defined by \( z \equiv \tilde{z}(t) \), and define a function \( g(z) \) analytic in \( |z| < 1 \) by

\[ g(z) = w(z)\omega(z) F(\tilde{z}(z))[O(\tilde{z}(z))]^{-1} . \]  

(13)

In this relation \( \tilde{z}(z) \) is the inverse of \( z = \tilde{z}(t) \), for \( z \equiv \tilde{z}(t) \) as defined in [12], and the last two factors give the function \( h(\tilde{z}(z)) \) defined in [10], which is analytic in \( |z| < 1 \). Finally, \( w(z) \) and \( \omega(z) \) are outer functions, i.e. functions analytic and without zeros in \( |z| < 1 \), defined in terms of their modulus on the boundary, related to \( \sqrt{\rho(t)} \) and \( |O(t)| \), respectively. Equivalent integral representations of the outer functions in terms of their modulus can be written either in the \( z \) or \( t \) variables. In particular, we use

\[ w(z) = \exp \left[ \frac{1}{2\pi} \int_{0}^{2\pi} d\theta \frac{\zeta+z}{\zeta-z} \ln |w(\zeta)| \right] , \quad \zeta = \exp(it) , \]  

(14)

where

\[ |w(\zeta)|^2 = \rho(\tilde{z}(z)) \left| \frac{d\tilde{z}(z)}{d\zeta} \right| , \]  

(15)
and

\[ \omega(z) = \exp \left( \frac{\sqrt{t_n} - \tilde{z}(z)}{\pi} \int_{t_n}^{\infty} \frac{\ln |O(t')| dt'}{\sqrt{t'-t_n(t'-\tilde{z}(z))}} \right) . \]  

(16)

Then [11] can be written as

\[ \frac{1}{2\pi} \int_{0}^{2\pi} d\theta |g(\zeta)|^2 = \tilde{\alpha}_\mu^{\pi} . \]  

(17)

\[ \text{From [12] it follows that the origin } t = 0 \text{ of the } t \text{-plane is mapped onto the origin } z = 0 \text{ of the } z \text{-plane. Therefore, from [13] it follows that each coefficient } g_k \in R \text{ of the expansion} \]

\[ g(z) = g_0 + g_1 z + g_2 z^2 + g_3 z^3 + \ldots \]  

(18)
is expressed in terms of the coefficients of order lower or equal to \( k \), of the Taylor expansion [9]. Moreover, the values \( F(t_n) \) of the form factor at a set of real points \( t_n < 0 \), \( n = 1, 2, \ldots, N \), lead to the values

\[ g(z_n) = w(z_n)\omega(z_n) F(t_n)[O(t_n)]^{-1} \quad z_n = \tilde{z}(t_n) . \]  

(19)

Then the \( L^2 \) norm condition [17] implies the determinant inequality (for a proof and older references see [68]):

\[ \begin{bmatrix} \tilde{I} & \xi_1 & \xi_2 & \cdots & \xi_N \\ \\
\xi_1 & 1 - \tilde{z}_1 K & (z_1 z_2)^K & \cdots & (z_1 z_N)^K \\
\xi_2 & 1 - z_1 z_2 & 1 - z_2^2 & \cdots & (z_2 z_N)^K \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\xi_N & (z_1 z_N)^K & (z_2 z_N)^K & \cdots & z_N^2K \end{bmatrix} \geq 0 , \]  

(20)

where \( K \geq 1 \) is an arbitrary integer and

\[ \tilde{I} = \tilde{\alpha}_\mu^{\pi} - \sum_{k=0}^{K-1} g_k^2 \quad \xi_n = g(z_n) - \sum_{k=0}^{K-1} g_k z_n^k . \]  

(21)

The same relation [20] holds if we replace \( \tilde{\alpha}_\mu^{\pi} \) by an upper bound of this quantity and the equality sign in [17] by the \( \leq \) sign. Moreover, as shown in [68], the results depend in a monotonic way on the value of the rhs of [17], becoming weaker when this value is increased.

The extension to the case of complex points \( t_n \), which enters in pairs since \( F(t^*) = F^*(t) \), is straightforward and will be discussed in Sec. V.

III. EXPERIMENTAL INPUT

We take \( \sqrt{t_m} = 0.917 \text{ GeV} \), which corresponds to the first important inelastic threshold, due to the \( \omega \pi \) pair. The choice of a lower value of \( t_m \) is legitimate in the present formalism, and we will work also with \( \sqrt{t_m} = 0.8 \text{ GeV} \), which will allow us to compare the constraining power of the input conditions [10] and [4].

Very precise parametrizations of the phase-shift \( \delta_1^1 \) are given in [24, 26]. We use as phenomenological input the phase parametrized as [26]

\[ \cot \delta_1^1(t) = \frac{\sqrt{7}}{2k_\rho^2} \left( M_\rho^2 - t \right)^{3/2} \]  

(22)

\[ \cdot \left( \frac{2 M_\rho^3}{M_\rho^2 \sqrt{t}} + B_0 + B_1 \frac{\sqrt{t} - \sqrt{t_0} - t}{\sqrt{t} + \sqrt{t_0} - t} \right) , \]  

where \( k_\rho \) is a free parameter.
TABLE I: $\pi^+\pi^-$ contribution to the muon anomaly for energies above $\sqrt{t_{\text{in}}}$.

| $\sqrt{t_{\text{in}}}$ (GeV) | $\hat{a}_{\mu}^{\pi\pi}$ | $\hat{a}_{\mu}^{\pi\pi}$ |
|-----------------------------|--------------------------|--------------------------|
| 0.800                       | $95.23 \times 10^{-10}$  | $95.23 \times 10^{-10}$  |
| 0.917                       | $22.17 \times 10^{-10}$  | $22.17 \times 10^{-10}$  |

where $k_\pi = \sqrt{t/4 - M_\pi^2}$ and

$$\sqrt{t_0} = 1.05 \text{ GeV}, \quad M_\mu = 773.6 \pm 0.9 \text{ MeV}, \quad B_0 = 1.055 \pm 0.011, \quad B_1 = 0.15 \pm 0.05.$$ (23)

The function $\delta_1^2$ obtained from $^{22}$ is practically identical with the phase-shift obtained in $^{24}$ from Roy equations for $\sqrt{t} \leq 0.8 \text{ GeV}$. The uncertainty is very small and we have checked that the results are practically insensitive to the variation of the phase-shift within the errors.

Above $t_{\text{in}}$ we use in $^{19}$ a smooth phase $\delta(t)$, which approaches asymptotically $\pi$. As shown in $^{63}$, the dependence on $\delta(t)$ of the functions $\mathcal{O}$ and $\omega$, defined in $^{19}$ and $^{16}$, respectively, exactly compensate each other, leading to results fully independent of the unknown phase in the inelastic region.

The two-pion contribution to muon anomaly was evaluated to great precision in $^{10}$ $^{11}$. The most recent evaluation $^{11}$, based on all the available experimental data, gives for the total $\pi^+\pi^-$ contribution to muon anomaly the value $a_{\mu}^{\pi\pi} = (507.80 \pm 1.22 \pm 2.50 \pm 0.56) \times 10^{-10}$.

In our method we need the specific contribution $a_{\mu}^{\pi\pi}$ of the energies from $\sqrt{t_{\text{in}}}$ to infinity. The values given below$^1$ are based on the BABAR data $^{33}$, whose spectrum extends up to 3 GeV.

For the interval 0.917 - 3 GeV the two-pion contribution is $(21.73 \pm 0.24) \times 10^{-10}$. Increasing the central value by the error, and adding an estimate of about $0.2 \times 10^{-10}$ for the interval from 3 GeV to $\infty$, gives the close upper bound $a_{\mu}^{\pi\pi} \leq 22.17 \times 10^{-10}$ for the two-pion contribution from 0.917 GeV to $\infty$. As mentioned above, if we use in $^{17}$, instead of the exact value of $a_{\mu}^{\pi\pi}$ an upper bound on this quantity, the results are still valid, but are weaker. In order to obtain results which are in the same time unbiased and stringent, we need a conservative and accurate estimate of $a_{\mu}^{\pi\pi}$.

For the interval 0.8 - 3 GeV the two-pion contribution in $^{11}$ is $(94.25 \pm 0.77) \times 10^{-10}$. Increasing as before the central value by the error, and adding $0.2 \times 10^{-10}$ for the interval from 3 GeV to $\infty$, we obtain $a_{\mu}^{\pi\pi} \leq 95.23 \times 10^{-10}$ for the two-pion contribution from 0.8 GeV to $\infty$. The final numbers for the two choices of $t_{\text{in}}$ are compiled in Table II.

TABLE II: Spacelike data from $^{22}$ $^{23}$.

| $t$ | Value (GeV$^2$) | $F(t)$ |
|-----|----------------|--------|
| $t_1$ | $-1.60$ | $0.243 \pm 0.012^{+0.019}_{-0.008}$ |
| $t_2$ | $-2.45$ | $0.167 \pm 0.010^{+0.007}_{-0.008}$ |

Finally, we use additional spacelike data coming from $^{22}$ $^{23}$, which are collected for completeness in Table III where the first error is statistical and the second is systematical.

IV. ALLOWED DOMAIN IN THE $c - d$ PLANE

In this section, we present the constraints on the coefficients $c$ and $d$ entering the Taylor expansion $^{25}$ using the formalism developed in Sec. III. We list in Table III the various quantities required in the basic inequality $^{20}$, for two choices of $t_{\text{in}}$. We implemented the normalization $F(0) = 1$, but kept arbitrary the charge radius $\langle r_\pi^2 \rangle$ and the spacelike values $F_1$ and $F_2$. Using the input from Tables II and III one obtains easily from $^{20}$ a convex quadratic condition for the coefficients $c$ and $d$, represented as the interior of an ellipse in the $c - d$ plane.

We consider first the constraints obtained without any information at spacelike points, when the determinant $^{20}$ has only one element, $I$, and the condition $^{20}$ becomes

$$g_0^2 + g_1^2 + g_2^2 + g_3^2 + \ldots \leq \hat{a}_{\mu}^{\pi\pi}.$$ (24)

The quantities $g_i$ are calculated for $t_{\text{in}} = (0.8 \text{ GeV})^2$ using the first line of Table II and the first column of Table III and for $t_{\text{in}} = (0.917 \text{ GeV})^2$ using the quantities written in the second line of Table II and the second column of Table III.

In order to investigate the influence of the choice of the threshold $t_{\text{in}}$, we show in Fig. II the domains obtained with the two values of $t_{\text{in}}$ considered in Tables II and III. For convenience, we take $\langle r_\pi^2 \rangle = 0.43 \text{ fm}^2$ $^{22}$ $^{23}$ $^{25}$ $^{54}$ $^{55}$. The figure shows that the ellipse corresponding to $t_{\text{in}} = (0.917 \text{ GeV})^2$ is smaller and lies fully inside that of the ellipse with $t_{\text{in}} = (0.8 \text{ GeV})^2$, proving that the best results are obtained by exploiting the known phase along the whole elasticity region. Therefore, in what follows we shall adopt the choice $t_{\text{in}} = (0.917 \text{ GeV})^2$.

A precise estimate $\langle r_\pi^2 \rangle = (0.435 \pm 0.005) \text{ fm}^2$ is given in $^{23}$. In Fig. III we present the domains described by $^{20}$ for $t_{\text{in}} = (0.917 \text{ GeV})^2$ and two values of the charge radius $\langle r_\pi^2 \rangle = 0.43 \text{ fm}^2$ and $\langle r_\pi^2 \rangle = 0.44 \text{ fm}^2$ resulting from this estimate. The allowed domain is quite sensitive to the variation of $\langle r_\pi^2 \rangle$, being shifted towards the upper right end if $\langle r_\pi^2 \rangle$ is increased. To account for the uncertainty of the charge radius, we take as allowed domain

---

$^1$ We are grateful to Bogdan Malaescu for providing us these numbers.
the union of the two ellipses in Fig. 2 which leads to the ranges

\[ 3.48 \text{ GeV}^{-4} \lesssim c \lesssim 3.98 \text{ GeV}^{-4}, \]
\[ 9.36 \text{ GeV}^{-6} \lesssim d \lesssim 10.46 \text{ GeV}^{-6}, \]

with a strong correlation between the values of \( c \) and \( d \).

We implement now the value at a point on the spacelike axis, using the input given in Table III. In this case the determinant in (20) has two rows and two columns. We choose the input at the spacelike point \( t_1 \) given in Table III and account for the errors by varying \( F_1 \) inside the error bars. In Fig. 1 we present the allowed domain in the \( c - d \) plane obtained for \( \langle r^2_c \rangle = 0.43 \text{ fm}^2 \) and three values of \( F_1 \); the central value 0.243 given in Table III and the extreme values 0.265 (0.228) obtained by adding (subtracting) the corresponding errors added in quadrature. The additional information on the spacelike axis improves in a dramatic way the constraints on the \( c \) and \( d \) coefficients. The small ellipses are entirely included in the larger ellipse obtained without information on the spacelike axis, which confirms the consistency of the various pieces of the input information. Varying \( F_1 \) inside the error bars, we obtain the allowed domain of the \( c \) and \( d \) parameters at the present level of knowledge as the union of the three small ellipses in Fig. 3. This gives, for \( \langle r^2_c \rangle = (0.435 \pm 0.005) \text{ fm}^2 \), the allowed ranges

\[ 3.75 \text{ GeV}^{-4} \lesssim c \lesssim 3.98 \text{ GeV}^{-4}, \]
\[ 9.91 \text{ GeV}^{-6} \lesssim d \lesssim 10.45 \text{ GeV}^{-6}, \]

with a strong correlation between the two coefficients. The comparison with (25) shows that the information at the spacelike point improves the lower bounds on both \( c \) and \( d \), a feature seen actually from Fig. 3.

Similar results are obtained using as input the second Huber datum \( t_2 \) in Table III. Note that the formalism allows the simultaneous inclusion of several spacelike points in the determinant (20). In practice, as discussed in [62, 63], when more points are included the results are extremely sensitive to the values used as input, which requires adequate numerical methods for treating the problem.

### Table III: Tabulation of the quantities entering as input in (20) for obtaining the constraints on the \( c, d \) coefficients, for two choices of \( t_1 \).

| Quantity | \( t_1 = (0.8 \text{ GeV})^2 \) | \( t_1 = (0.917 \text{ GeV})^2 \) |
|----------|-------------------------------|-------------------------------|
| \( g_0 \) | \( 0.2284 \times 10^{-4} \) | \( 0.1238 \times 10^{-4} \) |
| \( g_1 \) | \( (0.2503\langle r^2_c \rangle - 0.0414) \times 10^{-3} \) | \( (0.1783\langle r^2_c \rangle - 0.0431) \times 10^{-3} \) |
| \( g_2 \) | \( (1.497c - 0.9547\langle r^2_c \rangle - 0.1160) \times 10^{-3} \) | \( (1.401c - 0.9773\langle r^2_c \rangle - 0.0985) \times 10^{-3} \) |
| \( g_3 \) | \( (-0.8704c + 0.3833d + 0.3879\langle r^2_c \rangle - 0.7260) \times 10^{-3} \) | \( (-1.0481c + 0.4712d + 0.3589\langle r^2_c \rangle - 0.9154) \times 10^{-3} \) |
| \( z_1 \) | -0.3033 | -0.2603 |
| \( z_2 \) | -0.3745 | -0.3285 |
| \( g(z_1) \) | \( F_1 \times 0.3051 \times 10^{-4} \) | \( F_1 \times 0.2066 \times 10^{-4} \) |
| \( g(z_2) \) | \( F_2 \times 0.3984 \times 10^{-4} \) | \( F_2 \times 0.2210 \times 10^{-4} \) |
The digression above shows also that, by imposing simultaneously the experimental values at \( t_1 \) and \( t_2 \), we can only obtain a slight improvement of the allowed domain of the parameters \( c \) and \( d \). The reason is the fact that, as already noted above, the information on \( F(t_2) \) forces \( F(t_1) \) to lie within a slightly smaller range around the central value. Since the gain is expected to be small, we keep for simplicity as input only one spacelike point, which is sufficient to produce the narrow ranges reported in (26).

It is of interest to compare our predictions with previous determinations available in the literature. First, the range of \( c \) given in (25) considerably improves the bounds obtained with similar techniques in [51, 63, 64].

The improvement is due mainly to the very accurate information available now on the modulus, expressed in the values in Table I. On the theoretical side, a fit based on ChPT to two-loop accuracy for \( \tau \) decays gives \( c = (3.2 \pm 0.5_{\exp} \pm 0.9_{\text{theor}}) \) GeV\(^{-4} \) [4]. Subsequent calculations of the electromagnetic form factors in two-loop ChPT lead to the values \( c = (3.85 \pm 0.60) \) GeV\(^{-4} \) [8], in agreement with the range given in (26), and \( c = (4.49 \pm 0.28) \) GeV\(^{-4} \) [8], slightly above that range. Finally, both the prediction \( c = (4.00 \pm 0.50) \) GeV\(^{-4} \), based on the quark-mass dependence of the form factor [62], and the range \( c = (4 \pm 2) \) GeV\(^{-4} \) quoted as a conservative next-to-next-to-leading ChPT result in the same reference, are consistent with (26). On the other hand, a recent lattice calculation with chiral extrapolation based on two-loop ChPT gives a slightly lower value, \( c = 3.22(17)(36) \) GeV\(^{-4} \) [7]. It must be noted however, that the lattice data are generated at rather high spacelike momenta, \( t \in (-0.3, -1.7) \) GeV\(^2 \). Therefore the extraction of the radius and the curvature can not be very precise and the corresponding uncertainties might be larger than estimated.

Other determinations of the curvature are based on fits of experimental data with specific analytic parametrizations of the form factor. The value \( c = (3.90 \pm 0.10) \) GeV\(^{-4} \) was obtained in [59] by a usual dispersion relation, while a fit of the ALEPH data [57] on the hadronic \( \tau \) decay rate with a Gounaris-Sakurai formula for the form factor gives \( c = (3.2 \pm 1.0) \) GeV\(^{-4} \). Several analyses are based on phase (Omnès-type) representations, with various parametrizations of the phase along the whole unitarity cut. Their predictions, like \( c = (3.79 \pm 0.04) \) GeV\(^{-4} \) [58], \( c = (3.84 \pm 0.02) \) GeV\(^{-4} \) [58] and more recently \( c = (3.75 \pm 0.33) \) GeV\(^{-4} \) [65], are in overall agreement with (26). We note also that the value \( c = (3.30 \pm 0.03_{\exp} \pm 0.33_{\text{theor}}) \) GeV\(^{-4} \), obtained recently from a fit of spacelike data with Padé approximants [64], is below our prediction (26). It may be worth investigating whether the fact that the unitarity cut and the precise data available along it are not included in this analysis is responsible for the mismatch.

As in the case of \( c \), the range of \( d \) given in (26) considerably improves the bounds obtained with similar techniques in [61, 63, 64]. The information available in the

![FIG. 3: Allowed domain in the \( c - d \) plane calculated with \( t_{\text{in}} = (0.917 \) GeV\(^2 \) and \( \langle \tau^2 \rangle = 0.43 \) fm\(^2 \), for three values of \( F(t_1) \) at the spacelike point \( t_1 = -1.6 \) GeV\(^2 \) (central value in Table III and the extreme values obtained from the error intervals). Also shown is the large ellipse when no spacelike datum is included.](image)
literature on the cubic term in the Taylor expansion (29) is not rich. Theoretical results from ChPT and lattice calculations are not yet available. From fits of the data, the value $d = (9.70 \pm 0.40) \text{GeV}^{-6}$ was obtained by means of usual dispersion relations in [53], while the Taylor expansion of the Gounaris-Sakurai parametrization [57], mentioned above, leads to $d = 9.80 \text{GeV}^{-6}$. Both values are consistent with the range (26).

V. DOMAIN WHERE ZEROS ARE EXCLUDED

As we discussed in the Introduction (see also [71]), the formalism developed in Sec. II allows one to find rigorously the domain where the form factor cannot have zeros. The method amounts to testing the consistency of the assumption that a zero is present with the other pieces of the input. Let us assume first that the formalism presented in Sec. II can be easily adapted to the domain where the form factor cannot have simple zeros in the range $-1.93 \text{ GeV}^2 \leq t_0 \leq 0.83 \text{ GeV}^2$ of the real axis. If we impose the additional constraint at a spacelike point $t_1 = -1.6 \text{ GeV}^2$, the interval for the excluded zeros is much bigger. The left end of the range is actually quite sensitive to the input value $F_1 = F(t_1)$. Using the central value $F_1 = 0.243$ given in Table II, we find from (29) that the form factor cannot have simple zeros in the range $-5.56 \text{ GeV}^2 \leq t_0 \leq 0.84 \text{ GeV}^2$. By varying $F_1$ inside the error interval given in Table II (with errors added in quadrature), we find that zeros are excluded from the range $-12.67 \text{ GeV}^2 \leq t_0 \leq 0.84 \text{ GeV}^2$ for $F_1 = 0.265$ at the upper limit of the error interval, while for the lower limit $F_1 = 0.228$ the range is $-4.46 \text{ GeV}^2 \leq t_0 \leq 0.84 \text{ GeV}^2$.

We now turn to the study of complex zeros. The formalism presented in Sec. II can be easily adapted to include complex values of the form factor outside the real axis. Since the form factor is real analytic, its zeros occur in complex conjugate pairs, i.e. if $F(t_0) = 0$, then also $F(t_0^*) = 0$ (a double zero occurs as $t_0$ approaches the real axis). One can show that the determinant condition (28)
We first apply the inequality (30) to illustrate the dependence of the domain without zeros on the value of $t_1$ used in the calculations. As seen from Fig. 4, the larger value $t_1 = (0.917 \text{ GeV})^2$ leads to a domain that extends to high values of $|t|$ in all the directions of the complex plane, which shows that, like in the case of the $c-d$ domain, the best results are obtained if the phase condition (4) is used along the whole range of validity.

The dependence of the domain on the variation of $\langle r_0^2 \rangle$ is shown in Fig. 5. As expected, for a larger charge radius, $\langle r_0^2 \rangle = 0.44 \text{ fm}^2$, the zeros are excluded from a bigger complex domain around the timelike axis, while the left end of the domain, around the spacelike axis, is almost insensitive to the slope at $t = 0$.

The effect of an additional input at a spacelike point is illustrated in Fig. 6, where we show the domain in the complex plane where zeros are excluded, using $t_1 = -1.6 \text{ GeV}^2$ and the value $F(t_1) = 0.243$ (the central experimental value given in [22, 23]). By comparing Fig. 6 with the large domain in Fig. 4, one can see that the knowledge of the form factor at a spacelike point excludes zeros in a larger domain near the spacelike axis, while it has a smaller influence on the right part of the domain. This feature is present also in Fig. 7 which shows the sensitivity of the domain to the input value of $F(t_1)$. As is seen in the figure, the larger value $F(t_1) = 0.265$ obtained from the upper limit of the error bar, excludes the zeros in a domain extending to considerably larger values along the spacelike axis.

The results on the zeros reported in the literature [18, 42–44, 47, 48] are rather controversial. The best results for the regions free of zeros were obtained in [42, 44, 48], by a method related to ours. However, since the experimental information at that time was poor, the authors were forced to make some ad-hoc assumptions, especially on the modulus on the timelike axis. At present the precise measurement of the modulus gives a solid basis to our results.

The issue of zeros is of relevance for the analytic representation of the form factor using phase (Omnès)- or modulus-type representations, which require the knowledge of the zeros. Such representations were extensively studied in the past [18, 46, 50, 52, 53, 65], and often are based on the assumption that zeros are absent. Our results, which show that the zeros are excluded from a rather large region at low energies, give support to such representations, and confirm also theoretical expectations based on ChPT or more general physical arguments [18].

VI. DISCUSSIONS AND CONCLUSION

The experimental information available at present on the pion electromagnetic form factor is very rich. The recent high statistics measurements of the modulus by BABAR and KLOE collaborations [37, 34], supplemented by the phase in the elastic region known with accuracy from the $P$-wave of $\pi\pi$ scattering [24, 20], are expected to considerably constrain the behavior on the timelike axis. The values of the form factor on the spacelike axis are also measured with increasing precision. Theoretically, predictions on the pion form factor at low energies are available from ChPT and lattice QCD, while perturbative QCD predicts the behavior at high energies along the spacelike axis. The transition to the perturbative regime is known to be an open problem that deserves further study in the case of the pion form factor.
Analyticity is the ideal tool for connecting the low- and high-energy regimes for physical quantities like the pion form factor. The full treatment of the present rich experimental and theoretical input, which might overconstrain the system, is a challenge for the future investigations based on analyticity. In the present work we do not perform such a complete analysis, but exploit only in part the present information on the modulus on the unitarity cut. However, even in this limited frame we obtain quite stringent conclusions on the low energy properties of the form factor.

The conditions used as input in our approach are expressed by the phase condition (4) and the integral of the modulus squared (5), which we further restricted by choosing the weight \( p(t) \) as the kernel relevant for the two-pion contribution to the muon anomaly, cf. (6) and (7). A more general class of suitable weights will be investigated in a future work. Once the input is chosen, it is exploited in an optimal way by a mathematical formalism presented in Sec. II, leading to strong correlations between the coefficients of the Taylor expansion (3) at \( t = 0 \) and the values of the form factor on the spacelike axis.

Our basic results are contained in Eqs. (20) and (21), where the input quantities are defined in Tables II and III. The numerical coefficients in Table III depend on the normalization \( F(0) = 1 \) and the phase of the form factor below the inelastic threshold \( t_{in} \), being very stable with respect to small variations of the phase. Moreover, as emphasized in Sec. II the results are independent of the unknown phase of the form factor above the inelastic threshold \( t_{in} \). In Table III the charge radius \( \langle r_{\pi}^2 \rangle \), the higher-order Taylor coefficients \( c \) and \( d \), and the values of the form factor at several spacelike points are kept free, so the formalism can be easily applied for finding model independent correlations between the values of the form factor at different points and for testing the consistency of input values known from different sources.

In Sec. IV we derived stringent constraints on the allowed values of the higher-order coefficients \( c \) and \( d \) of the Taylor expansion (4). The best results are obtained with \( t_{in} = (0.917 \text{ GeV})^2 \), which corresponds to the physical inelastic threshold produced by the \( \pi \pi \) channel. The charge radius and an additional information at a spacelike point were used as input. In (25) and (26) and in Figs. 1-4 we illustrated the results for \( \langle r_{\pi}^2 \rangle \) in the range \((0.43 - 0.44) \text{ fm}^2 \) and \( F(-1.6 \text{ GeV}^2) = 0.243 \pm 0.012^{+0.019}_{-0.008} \) \( \langle r_{\pi}^2 \rangle \). It is remarkable that the allowed ranges are already comparable in precision with other determinations in the literature based on specific parametrizations.

The present method can be used also to obtain bounds on the values of the form factor along the spacelike axis, using as input the information on the timelike axis, together with some values inside the analyticity domain. As discussed in Sec. IV using as input the value \( F(t_1) \) at the first Huber point, we obtain stringent limits on the value \( F(t_2) \) at the second point, with a strong correlation between the two. Of course, the method can be applied in principle also to higher spacelike energies. However, with our choice of the weight \( p(t) \), we expect that the predictions will become gradually weaker when the energy is increased. Indeed, since \( p \) decreases rapidly at large momenta, the condition (5) provides stringent constraints on the low energy parameters like \( c \) and \( d \), but in the same time it imposes weak constraints on the behavior of the form factor at large energies. A different choice of \( p \) could lead to interesting results also for the behavior at higher energies, but this is beyond the scope of the present analysis and will be investigated in a future work.

In Sec. V we showed that the same formalism leads to an analytic description for the regions of the complex plane where the zeros of the form factor are forbidden. Our results are contained in Eqs. (25) - (29) and are illustrated in Figs. 2-7, for the same input \( \langle r_{\pi}^2 \rangle \) and \( F(t_1) \). We obtain a rather large domain where zeros are excluded, which gives support to Omnès-type representations, which often assume the absence of the zeros. Our results also confirm theoretical expectations on the absence of zeros at low energy, based on ChPT or general physical arguments. We note that by our method we can find rigorously the domains free of zeros, but we can say nothing about the remaining domains, where zeros may be present or absent. Alternative methods, based on modulus representations can rule out in principle the zeros from the whole complex plane provided they are absent. However, these methods are very sensitive to the input and led to controversial results in the past. An update of such analyses using the recent precise determination of the modulus would be of much interest.

We finally note that the mathematical formalism applied in this paper may be useful also for finding an analytic parametrization of the form factor suitable for fitting the rich amount of experimental data. Namely, the representation of \( F(t) \) that results from (13) involves the known functions \( w(z) \), which accounts for the weight \( p(t) \), \( \omega(z) \) and \( O(t) \), which implement the phase below \( t_{in} \), and the arbitrary function \( g(z) \), analytic in the \( t \)-plane cut for \( t > t_{in} \), or equivalently in the unit disc \(|z| < 1 \) of the \( z \)-plane defined by the conformal mapping (12). The expansion (13) is convergent in \(|z| < 1 \), and moreover the coefficients satisfy the inequality (23), which is very useful in order to control the higher orders of the expansion and the truncation error.

Acknowledgement: BA thanks the Department of Science and Technology, the Government of India, and the Homi Bhabha Fellowships Council for support. IC acknowledges support from CNCSIS in the Program Idei, Contract No. 464/2009. We thank B. Malaescu, G. Colangelo, M. Passera and S. Ramanan for useful correspondence.
[63] B. Ananthanarayan and S. Ramanan, Eur. Phys. J. C 60, 73 (2009) [arXiv:0811.0482 [hep-ph]].

[64] P. Masjuan, S. Peris and J. J. Sanz-Cillero, Phys. Rev. D 78, 074028 (2008) [arXiv:0807.4893 [hep-ph]].

[65] F. K. Guo, C. Hanhart, F. J. Llanes-Estrada and U. G. Meissner, Phys. Lett. B 678, 90 (2009) [arXiv:0812.3270 [hep-ph]].

[66] G. Abbas, B. Ananthanarayan and S. Ramanan, Eur. Phys. J. A 41, 93 (2009), [arXiv:0903.4297 [hep-ph]].

[67] M. Belicka, S. Dubnicka, A. Z. Dubnickova and A. Liptaj, Phys. Rev. C 83, 028201 (2011), [arXiv:1102.3122 [hep-ph]].

[68] G. Abbas, B. Ananthanarayan, I. Caprini, I.S. Imsong and S. Ramanan, Eur. Phys. J. A 45, 389 (2010), [arXiv:1004.4257 [hep-ph]].

[69] G. Abbas, B. Ananthanarayan, I. Caprini, I. Sentitemsu Imsong and S. Ramanan, Eur. Phys. J. A 44, 175 (2010) [arXiv:0912.2831 [hep-ph]].

[70] G. Abbas, B. Ananthanarayan, I. Caprini and I. Sentitemsu Imsong, Phys. Rev. D 82, 094018 (2010) [arXiv:1008.0925 [hep-ph]].