Playing with Neutrino Masses

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Most of what is known about neutrino masses and mixings results from studies of oscillation phenomena. We focus on those neutrino properties that are not amenable to such studies: $\Sigma$, the sum of the absolute values of the neutrino masses; $m_\beta$, the effective mass of the electron neutrino; and $m_{\beta\beta}$, the parameter governing neutrinoless double beta decay. Each of these is the subject of ongoing experimental or observational studies. Here we deduce constraints on these observables resulting from any one of six ad hoc hypotheses that involve the three complex mass parameters $m_i$: (1) Their product or (2) sum vanishes; (3) Their absolute values, like those of charged leptons or quarks of either charge, do not form a triangle; (4) The $e-e$ entry of the neutrino mass matrix vanishes; (5) Both the $\mu-\mu$ and $\tau-\tau$ entries vanish; (6) All three diagonal entries are equal in magnitude. The title of this note reflects the lack of any theoretical basis for any of these simple assertions.
And thus do we by indirection find directions out.  Wm. Shakespeare

Much has been learned about neutrino masses and their mixing parameters. While numerous theoretical models and symmetry schemes have been proposed, none evoke the clear ring of truth. The structure of the neutrino mass matrix remains largely unknown. Rather than launch yet another speculative model, here we explore a wide variety of potential simplicities.

We presume neutrino phenomena to involve just three left-handed states ($\nu_e$, $\nu_\mu$, $\nu_\tau$) whose Majorana masses are described in a flavor basis by the complex symmetric $3 \times 3$ matrix $\mathcal{M}$. We are not now concerned with the underlying origin of neutrino masses, only with their observable attributes. Recall that the neutrino mass matrix may be written as:

$$\mathcal{M} = U^* \mathcal{D} U^\dagger,$$

where $U$, the neutrino analog to the CKM matrix, is expressed in terms of the CP-violating parameter $\delta$ of the neutrino sector and the three mixing angles: $\theta_1 \equiv \theta_{23}$, $\theta_2 \equiv \theta_{31}$ and $\theta_3 \equiv \theta_{12}$. The diagonal matrix $\mathcal{D}$ displays three complex mass parameters $m_i$, of which $m_3$ is chosen to be real and positive with no loss of generality.

First I summarize what is known about these parameters from current neutrino oscillation experiments\[1\], whose best-fit values will suffice for our subsequent analysis:

$$\Delta_a \equiv m_3^2 - (m_1^2 + m_2^2)/2 \simeq \pm 2390 \ (\text{meV})^2$$
$$\Delta_s \equiv m_2^2 - m_1^2 \simeq 77 \ (\text{meV})^2$$
$$\cos 2\theta_1 \simeq 0.068 \quad \sin^2 \theta_2 \leq 0.030 \quad \sin^2 \theta_3 \simeq 0.312$$

where we have adopted a somewhat conservative upper bound for $\theta_2$.

We examine the phenomenological consequences of any one of several simple constraints that can be imposed upon $\mathcal{M}$. Each of these constraints yields a neutrino mass matrix that is fully compatible with all current oscillation data. Hence, we study the effects of these constraints on those observable quantities that are not directly related to such data, notably:
(1) $\Sigma = |m_1| + |m_2| + m_3$, the sum of the neutrino masses, for which several astrophysical upper bounds have been alleged, ranging from 190 meV to 1190 meV. For purposes of our subsequent analysis, we arbitrarily adopt the bound $\Sigma \leq 750$ meV.

(2) $m_\beta \equiv (c_2^2 s_3^2 m_1^2 + c_2^2 s_3^2 m_2^2 + s_2^2 m_3^2)^{1/2} \approx (c_3^2 m_1^2 + s_3^2 m_2^2)^{1/2}$, the rms mass of the electron neutrino, which can reveal itself by precision studies of beta-decay spectra, especially that of tritium. We use the notation $s_3 \equiv \sin \theta_3$, $m_1^2 \equiv |m_1^2|$, &c. Under most circumstances $m_\beta \simeq |m_1| \simeq |m_2|$. 

(3) $m_{\beta\beta} \equiv |c_2^2 c_3^2 m_1 + c_2^2 s_3^2 m_2 + s_2^2 m_3| \approx |c_3^2 m_1 + s_3^2 m_2|$, which governs the rate of neutrinoless double beta decay. Its value depends on the relative phase of $m_1$ and $m_2$, whose cosine we denote by $\mu$. Under most circumstances (when $|m_1| \simeq |m_2|$) we have $m_{\beta\beta} \simeq m_\beta \sqrt{1-\zeta}$, where $\zeta \equiv \sin^2 2\theta_3 (1-\mu)/2 \simeq 0.43 \times (1-\mu)$. The parameters $\mu$ and $\zeta$ play further roles in what follows.

We begin by exhibiting the bounds on these observables resulting from our assumed upper limit to $\Sigma$ and the known values of $\Delta_a$ and $\Delta_s$, with no further hypotheses. Hereafter, neutrino masses are expressed in milli-electron-volts:

- Normal Hierarchy: $750 \geq \Sigma \geq 58$, $250 \geq m_\beta \geq 5$, $250 \geq m_{\beta\beta} \geq 0$
- Inverted Hierarchy: $750 \geq \Sigma \geq 98$, $375 \geq m_\beta \geq 49$, $375 \geq m_{\beta\beta} \geq 31$

Current and anticipated measurements of $m_\beta$ and $m_{\beta\beta}$ are summarized elsewhere, e.g., [1]. We proceed to examine the consequences of any one of an astonishing variety of additional hypotheses. The results are essentially unaffected by our neglect of terms involving $s_2^2$ and, when appropriate, of those involving $\Delta_s$.

I. Suppose that $\det \mathcal{M} = 0$. There are two relevant cases with one vanishing neutrino mass. (a) If $m_1 = 0$, the hierarchy is normal and:

$$\Sigma \simeq 57, \quad m_\beta = 5, \quad m_{\beta\beta} \simeq 3.$$ 

(b) If $m_3 = 0$, the hierarchy is inverted and:

$$\Sigma \simeq 98, \quad m_\beta \simeq 49, \quad 49 \geq m_{\beta\beta} \geq 31.$$
II. \( M_{ee} = 0 \), for which \( m_{\beta\beta} \) is set equal to zero \textit{ab initio} and we obtain the relation:
\[ c_2^2 m_1 + s_2^2 m_2 = 0. \]
The neutrino mass hierarchy is normal and all three observables are tiny:
\[ \Sigma \simeq 64, \quad m_\beta \simeq 7, \quad m_{\beta\beta} = 0. \]

III. Neither the masses of the three charged leptons, nor those of the three \( Q = \frac{2}{3} \) quarks, nor those of the three \( Q = -\frac{1}{3} \) quarks satisfy triangle equalities. Perhaps that property is shared by the three neutrino masses. There are two cases.

(a) If the hierarchy is normal, we require \( |m_3| > |m_1| + |m_2| \). Neglecting \( \Delta_s \), we set \( |m_3| > 2|m_1| \simeq 2|m_2| \). From \( \Delta_a = |m_3|^2 - |m_1|^2 \) we find: \( |m_1| < \sqrt{\Delta_a/3} \simeq 29 \) and \( |m_3| < 2\sqrt{\Delta_a/3} \), which imply:
\[ 58 < \Sigma < 116, \quad m_{\beta\beta} \leq m_\beta < 29. \]

(b) If the hierarchy is inverted, we require \( |m_3| < |m_2| - |m_1| \). This may only be achieved for \( |m_3| < 1 \), which effectively yields case Ib.

The preceding scenarios would not please those experimenters seeking definitive measurements of \( \Sigma, m_\beta \) or \( m_{\beta\beta} \). Let them read on!

IV. Suppose \( M \) to be traceless so that \( m_1 + m_2 + m_3 = 0 \). This possibility was first considered by He and Zee\[3\]. We must stress that the phases of the flavor eigen-fields may be adjusted arbitrarily in the standard model or its simplest extensions. Thus the condition \( \text{Tr} M = 0 \) is ordinarily senseless because it is not not preserved by such phase redefinitions. Nevertheless one may conceive of new physics that would fix these phases, thus rendering the condition meaningful and making its consequences worth considering. Furthermore, we note (as did He and Zee) that in general \( \text{Tr} M \neq \Sigma m_i \). However, their difference is \( \sim \sin^2 \theta_2 \), which we may safely ignore. Our results depend sensitively on the relative phase of \( m_1 \) and \( m_2 \), which (due to the preceding argument) may not coincide with its value in the Kobayashi-Maskawa ansatz. We obtain:
\[ |m_1| \simeq |m_2| \simeq \sqrt{\frac{\Delta_a}{1 + 2\mu}} \quad m_3 \simeq \sqrt{\frac{2(1 + \mu) \Delta_a}{1 + 2\mu}}. \]
Notice that neutrino masses approach degeneracy and diverge as $\mu \to -\frac{1}{2}$. The hierarchy is normal for $\mu > -\frac{1}{2}$, inverted for $\mu < -\frac{1}{2}$, and nearly degenerate for $\mu \sim -\frac{1}{2}$. Several examples may be worth consideration:

(a) $\mu = 1$ \quad $\Sigma = 116$, \quad $m_\beta = 29$, \quad $m_{\beta\beta} = 29$

(b) $\mu \simeq -0.5$ \quad $\Sigma = 750$, \quad $m_\beta = 250$, \quad $m_{\beta\beta} = 149$

(c) $\mu = -1$ \quad $\Sigma = 98$, \quad $m_\beta = 49$, \quad $m_{\beta\beta} = 31$

In (a), we have a normal hierarchy with $|m_3| \simeq 2|m_{1,2}|$. In (b), the neutrino masses are nearly degenerate and $\mu$ is chosen to be close enough to $-\frac{1}{2}$ for $\Sigma$ to attain its upper bound of 750 meV. Case (c) yields a special case of Ib: an inverted hierarchy with $|m_3| = 0$.

V. Here we consider a previously studied\cite{4} model wherein two diagonal entries of the neutrino mass matrix vanish: $M_{\mu\mu} = M_{\tau\tau} = 0$. Two useful relations result from these constraints:

\[ \cot 2\theta_1 \cot 2\theta_3 = s_2 \cos \delta \]

\[ \sin^2 \theta_3 m_1 + \cos^2 \theta_3 m_2 + m_3 = 0. \]

The first relation correlates the small observed value of $\theta_2$ with the observation of maximal or nearly maximal atmospheric oscillations. The second relation is relevant to the issues at hand, from which we obtain:

\[ |m_1| = \sqrt{\Delta_a / \zeta}, \quad |m_3| = \sqrt{\Delta_a (1 - \zeta) / \zeta}, \]

where $\zeta$ was defined earlier. We examine three examples:

1 - $\mu = 2$ \quad $\Sigma = 126$, \quad $m_\beta = 53$, \quad $m_{\beta\beta} = 32$

1 - $\mu = 1$ \quad $\Sigma = 206$, \quad $m_\beta = 75$, \quad $m_{\beta\beta} = 56$

1 - $\mu \simeq 0.09$ \quad $\Sigma \simeq 750$, \quad $m_\beta \simeq 251$, \quad $m_{\beta\beta} = 248$

For the third example we chose $\mu \simeq 0.91$, small enough to saturate the bound on $\Sigma$. The hierarchy is necessarily inverted and the neutrino masses both diverge and approach degeneracy as $\mu \to -1$. The relative (Majorana) phase of $m_1$ and $m_2$ is related to the CP-violating phase $\delta$:

\[ 1 - \mu = \frac{2 \cos \delta}{\cos^2 \theta_3 + \cos \delta \sin^2 \theta_3} \]
This relation shows that $1 \geq \cos \delta > 0$, with a lower bound of about 0.03 corresponding to our constraint on $\mu$. Relatively large values of $m_\beta$ and $m_\beta \beta$ are permitted, but only for $\delta$ close to $\pi/2$, i.e., for nearly maximal CP violation in oscillation phenomena.

VI. Suppose the three diagonal entries of the neutrino mass matrix to be equal in magnitude: $|M_{ee}| = |M_{\mu \mu}| = |M_{\tau \tau}|$. With appropriate phase redefinitions, this condition is seen to be equivalent to the strictly off-diagonal Zee matrix, augmented by a multiple of the unit matrix. While the Zee model is experimentally disfavored, this model is compatible with all available oscillation data. We examine its consequences neglecting relatively small terms involving $\Delta_s$, $s_2$ and $\cos 2 \theta_1$. With these approximations, which cannot substantially alter our result, the condition reduces to a single real quadratic equation:

$$R^2 + 2R(\mu_1 c_3^2 + \mu_2 s_3^2) - 3(1 - \xi)$$

where $R = |m_3/m_1| \simeq |m_3/m_2|$ and $\mu_i = \cos \phi_i$ with $\phi_i$ being the Majorana phases of $m_1$ and $m_2$. Recall that we have chosen $m_3$ to be real and positive and $\xi = \sin^2 2\theta_3(1 - \mu)/2$, where $\mu = \cos (\phi_2 - \phi_1)$. Again I offer three examples:

(a) $R$ attains its maximum value of 3 for $\phi_1 = \phi_2 = \pi$. This yields a normal hierarchy with:

$$\Sigma = 190, \quad m_\beta = m_\beta \beta = 17.$$

(b) Degenerate neutrino masses result from the choice $\phi_1 = \phi_2 = 0$, we obtain $R = 1$. Saturating our bound on $\Sigma$ near this point yields:

$$\Sigma = 750, \quad m_\beta = m_\beta \beta = 250.$$

(c) The smallest possible value of $R$ is obtained for $\phi_1 = -\phi_2 = \pi/2$, whence $R \simeq 0.65$. This yields an inverted hierarchy with:

$$\Sigma = 199, \quad m_\beta = 75, \quad m_\beta \beta = 57.$$

Perhaps our discussion will prove stimulating to both experimenters and model builders. That so many disparate hypotheses remain compatible with what we already know about neutrinos reflects the depth of our ignorance about the mechanism responsible for neutrino masses and mixings. As experimenters and astrophysicists become ever more clever in constraining $\Sigma$, $m_\beta$ and $m_\beta \beta$, perhaps we shall learn which, if any, nature has chosen of our six scenarios.

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