Simulating the sensitivity of hypothetical km$^3$ hydrophone arrays to fluxes of UHE neutrinos

Modelling the effects of refraction

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Abstract. The velocity of sound in the sea is not a constant, hence sound rays are subject to refraction. The work described here, attempts to model the distortion of the acoustic “pancake”, emitted when a neutrino with over a joule of energy, scatters in the sea. The detection of such events by hypothetical arrays of hydrophones, distributed over volumes of several cubic kilometres is discussed.

1. Introduction
To date, in attempting to predict the ability of hypothetical cubic kilometre-scale acoustic sensor arrays, to detect ultra high energy (UHE) neutrinos, the presence of a non-constant sound velocity has been dealt with in several ways. What follows is a discussion of the inclusion of refraction in existing simulation work from groups active in the field of acoustic detection of UHE neutrinos. Particular attention is paid to the progress made by the ACoRNE collaboration toward the modelling of thermoacoustic radiation in the presence of refraction and attempts to reconstruct event vertices.

2. Limit on source distance
An acoustic array in the deep sea resides far away from the thermally mixed layers of water close to the surface, in a region where the sound velocity profile (SVP) is essentially linear. In a medium where the refractive index has a linear gradient, rays propagate along circular trajectories with radius $R = c_0/g$ where $c_0$ is the velocity $c = c(z)$ at $z = 0$ and $g$ is the sound velocity gradient $dc(z)/dz$ [1]. $R$ is typically around 90 km in the deep Mediterranean, as measured by the ANTARES collaboration [2].

In the sea, the velocity of sound increases with depth as a function of temperature, pressure and salinity. Because sound rays are lazy, they tend toward regions of lowest sound velocity, hence, if the SVP increases with depth, they are refracted toward the surface. If one is conservative in their approach and neglects reflections from the surface and the seabed, there exists a natural limit on the source distance from which a given acoustic array can be reached.
if the sea in which it sits is of finite depth. For example: a 2 km tall array anchored 1 km from the sea bed, has $dy = 3$ km according to Figure 1; if $R = c_0/g = 90$ km then:

$$dx = \sqrt{R^2 - (R - dY)^2} = \sqrt{90^2 - 87^2} = 23 \text{ km}$$

**Figure 1.** A natural limit on the distance from which a source can be heard exists as a result of the circular ray trajectories. Rays emitted in any direction within the shaded region and beyond will terminate at the seabed or surface before intersecting the detector.

One published simulation [3] uses this upper limit to set the radius of the volume in which event vertices are generated. Events beyond this distance are not considered. Throughout the remaining calculations a constant sound velocity $c$ is used, and event vertices, $\vec{v}$, originating at time $t_0$, are reconstructed according to the following equation:

$$|\vec{v} - \vec{r}_i|^2 = c^2(t_i - t_0)^2$$

where a receiver at location $\vec{r}_i$, hears a signal at time $t_i$.

3. **Modelling refraction by ray tracing**

Instead of making the above approximation, one can employ a ray tracing algorithm to compute circular ray trajectories. A full derivation of the ray tracing equations in the general case, where the sound velocity gradient is not linear can be found in [1]. For the geometry illustrated in the left of Figure 2 with a constant sound velocity gradient, it can be shown that:

$$x_1 - x_0 = \frac{c_0}{g} \left( \frac{\sin \theta_0 - \sin \theta_1}{\cos \theta_0} \right)$$

$$s_1 - s_0 = \frac{c_0}{g \cos \theta_0} (\theta_0 - \theta_1)$$

$$z_1 - z_0 = \frac{c_0}{g \cos \theta_0} (\cos \theta_1 - \cos \theta_0)$$

$$t_1 - t_0 = -\frac{1}{g} \ln \left( \frac{\cos \theta_0 (1 + \sin \theta_1)}{\cos \theta_1 (1 + \sin \theta_0)} \right)$$

where $t_0$ is the time at the point of origin and $t_1$ is the time at point $(x_1, z_1)$. Hence, for a known interaction vertex $(x_0, y_0, z_0)$ and a known hydrophone position $(x_h, y_h, z_h)$ one can compute the effect of refraction on the fictitious linear ray that intersects the two coordinates, this is illustrated in the right of Figure 2. The Equations 2 to 5 thus allow for the computation
**Figure 2.** (Left): Geometry of a refracted sound ray. (Right): The effect of refraction on the fictitious linear ray $i$ that intersects the vertex at $(x_0, y_0, z_0)$ and the hydrophone position $(x_1, y_1, z_{hyd})$ is to deflect the real ray $d$ to some coordinate at $(x_h, y_h, z_{def})$. The deflection has occurred along the $z$-axis. The angles $\theta_i$ and $\theta_d$ represent the angle between the undeflected ray and the origin, and the deflected ray and the origin respectively. The difference between the angle of deflection and the start angle of the undeflected ray is $\Delta \theta = \theta_d - \theta_i$. The start angle for the real refracted ray $R$ is therefore given by $\theta_0 = \theta_i - \Delta \theta$.

**Figure 3.** Schematic of the energy deposition (Left). At $E_\nu = 10^{20}$ eV, in water, a hadronic shower deposits 99% of its energy in a cylinder of length $L = 20m$ and radius $R = 20cm$. Refraction occurs only in the vertical plane; there are two extreme cases: that when the shower is horizontal (Centre) and that when the shower is vertically inclined (Right). In the former case, there is no deformation of the pancake such that the radiation field is bent outside the plane orthogonal to the shower axis.

of the circular ray path length and the time at which the signal is received for a given set of source and receiver locations. If the speed of sound were constant, the thermoacoustic emission resulting from the interaction of a UHE neutrino in water would be contained in a narrow “pancake”, emanating from the centre of the induced particle cascade, perpendicular to the cascade axis. This is illustrated in Figure 3.

In the presence of a linear SVP the pancake is no longer confined to a plane. The circular
ray trajectories result in a parabolic radiation pattern. Equation 1 is no longer valid. There is no analytical solution to the vertex location from the times of the signals received. A method of vertex reconstruction is in use, that uses precomputed tables of signal times [4, 5]. First, one populates a lattice of points, in a volume surrounding the detector and computes the arrival times on each receiver corresponding to a source at that point. Secondly a minimisation of the following metric is performed:

\[ M = \frac{1}{N-1} \sum_{i<j} |t_{ij} - t_{ij}(\vec{r})|^2 \]  

(6)

where \( t_{ij}^{meas} = t_i^{meas} - t_j^{meas} \) and \( t_{ij}(\vec{r}) = t_i(\vec{r}) - t_j(\vec{r}) \). An off-lattice solution is interpolated from the point that gave the minimum value of \( M \). This solution to the vertex location problem is suitable for relatively small fiducial volumes, less than say, 10 km\(^3\) but becomes impractical for volumes of the order a hundred km\(^3\), which is potentially the requirement for detection of GZK neutrinos [3, 6]. An alternative solution is sought, that keeps the number of phase-space iterations to a minimum.

4. Reconstruction of event vertices in the presence of refraction

A “best guess” of the vertex location can be computed. We start with Equation 1, expanding the bracketed terms gives:

\[ \frac{v^2 + r_i^2 - 2\vec{v} \cdot \vec{r}_i}{(r_i^2 - r_j^2) - c^2[(t_i - t_0)^2 - (t_j - t_0)^2]} = \frac{c^2(t_i - t_0)^2}{2\vec{v} \cdot (\vec{r}_i - \vec{r}_j)} \]

which can be expressed in the form of a matrix equation:

\[ \vec{R} + t_0 \vec{T} = \mathbf{M} \times \vec{v} \]

(7)

where, in the limiting case of four receivers:

\[ \vec{R} = r_i^2 - r_j^2 - c^2(t_i^2 - t_j^2) \quad (i, j = 1, 2; 1, 3; 1, 4) \]

\[ \vec{T} = 2c^2(t_i - t_j) \quad (i, j = 1, 2; 1, 3; 1, 4) \]

\[ \mathbf{M} = 2 \begin{pmatrix} dx_{12} & dy_{12} & dz_{12} \\ dx_{13} & dy_{13} & dz_{13} \\ dx_{14} & dy_{14} & dz_{14} \end{pmatrix} \]

The receiver with index 1 is taken as the reference receiver, \( dx, dy, dz \) are the difference in \( x, y \) and \( z \) coordinates respectively, between this receiver and the other receivers that detect a signal. Additionally \( c = c(z) \), where \( z \) is the depth of the receiver with index 1 and \( \mathbf{M} \) is always a \( 3 \times 3 \) square matrix. Solving for the vertex \( \vec{v} \) gives:

\[ \vec{v} = \mathbf{M}^{-1} \vec{R} + t_0 \mathbf{M}^{-1} \vec{T} \]

(8)

which is solved via the propagation time equation:

\[ |\vec{v} - \vec{r}_i|^2 = c^2(t_i - t_0)^2 = |\mathbf{M}^{-1} \vec{R} + t_0 \mathbf{M}^{-1} \vec{T} - \vec{r}_i|^2 \]

(9)
The mean vertex $\vec{v}$ from $N$ receiver signals is thus calculated as:

$$\vec{v} = \frac{1}{N-3} \sum_{i=0}^{i=(N-3)} \vec{v}_i$$  \hspace{1cm} (10)$$

and constitutes an estimate of the real vertex location. Now, one can interpolate toward the true vertex, by minimising the metric $M$, having circumvented the necessity for iterations over a large number of test coordinates.

5. Neutrino pointing

In addition to reconstruction of the event vertex, one also desires some knowledge of the neutrino trajectory. In the case of an unrefracted pancake, those receivers which register a signal are confined to a plane orthogonal to the direction of the particle cascade, which evolves collinearly to the neutrino trajectory. It is easy therefore, to project a normal to this plane, through the reconstructed vertex and thus resolve the direction of neutrino.

If the pancake is warped by refraction no such plane exists. A naive solution is to take each triplet of receivers that register a signal located at $\vec{r}_a$, $\vec{r}_b$ and $\vec{r}_c$ respectively. The normal to the plane in which these three receivers lie is given simply by:

$$\vec{B}\vec{A} \times \vec{B}\vec{C}$$  \hspace{1cm} (11)$$

where $\vec{B}\vec{A} = \vec{r}_a - \vec{r}_b$ and $\vec{B}\vec{C} = \vec{r}_c - \vec{r}_b$. A mean pointing vector, weighted by the signal amplitudes measured by each receiver, is then computed:

$$\vec{P} = \frac{1}{W} \sum_{i=0}^{i=(N-2)} (\vec{B}\vec{A} \times \vec{B}\vec{C})w_i$$  \hspace{1cm} (12)$$

where $w_i$ is the mean pressure of the triplet and $W$ is the sum of the triplet pressures.

Clearly the ability of a given acoustic array to reconstruct interaction vertices, along with the orientation of the induced cascades, is dependent on the geometry and performance of the instrumentation. It has been suggested that an acoustic module (an omnidirectional detector that has a probability of detecting a signal above threshold equal to unity) density of 200 acoustic modules per cubic kilometre is optimal [3]. This result is derived from a simulation in which the velocity of sound is assumed to be constant. It therefore remains to be seen whether this result is still valid if the refraction of found rays is included. An example of the effect of refraction on pointing accuracy is illustrated in Figure 4.

6. Outlook

A sensitivity calculation requires that a vertex is reconstructed in order to pass as a bona fide detection. This condition, therefore, has an inherent requirement for some limit on resolution. Without such an imposition, any event that returns a non-null solution to Equation 1 will pass the detection criterion. This subsequently poses the following question: to what accuracy does one require event vertices and neutrino trajectories to be reconstructed? Only in answering this can we optimise the geometry and density of our detector elements.

Acknowledgments

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Figure 4. The refracted (solid) mean pointing resolution $\sim 19^\circ$ and unrefracted (dashed) mean pointing resolution $\sim 8^\circ$ for an array of one thousand hydrophones, distributed randomly in a volume of one kilometre cubed. Event vertices are generated up to a distance of 10 km from the instrumented volume. The location of the receivers is know to within 0.1 m in accordance with the performance of the ANTARES neutrino telescope [2].

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