A SPIN DECOMPOSITION OF THE VERLINDE FORMULAS FOR TYPE A MODULAR CATEGORIES

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Abstract. A modular category is a braided category with some additional algebraic features. The interest of this concept is that it provides a Topological Quantum Field Theory in dimension 3. The Verlinde formulas associated with a modular category are the dimensions of the TQFT modules. We discuss reductions and refinements of these formulas for modular categories related with $SU(N)$. Our main result is a splitting of the Verlinde formula, corresponding to a brick decomposition of the TQFT modules whose summands are indexed by spin structures modulo an even integer. We introduce here the notion of a spin modular category, and give the proof of the decomposition theorem in this general context.

0. Introduction

Given a simple, simply connected complex Lie group $G$, the Verlinde formula \cite{37} is a combinatorial function $V_G : (K, g) \mapsto V_G(K, g)$ associated with $G$ (here the integers $K$ and $g$ are respectively the level and the genus). In conformal field theory this formula gives the dimension of the so called conformal blocks. Its combinatorics was intensively studied since this formula has a deep interpretation as the rank of a space of generalized theta functions (sections of some bundle over the moduli space of $G$-bundles over a Riemann surface) \cite{6, 5, 15, 28}. See \cite{8, 9}, for a development using methods of symplectic geometry.

We will consider here a purely topological approach to Verlinde formulas related with $SU(N)$. The genus $g$ Verlinde formula associated with a modular category \cite{32} is the dimension of the TQFT-module of a genus $g$ surface; the general formula is given in \cite{32} IV,12.1.2. Various constructions of modular categories are known, either from quantum groups \cite{2, 4, 29} or from skein theory \cite{36, 11, 7}. The geometric Verlinde formula for the group $SU(N)$ at level $K$ is recovered from the so called $SU(N, K)$ modular category. This modular category can be

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obtained either from the quantum group $U_q \mathfrak{sl}(N)$ when $q = s^2$ is a primitive $(N + K)$-th root of unity or from Homfly skein theory. Its simple objects correspond to the weights in the fundamental alcove.

One may also consider a modular category with less simple objects. This was done for $\gcd(N, K) = 1$ by restricting to representations whose heighest weight is in the root lattice, and was called the projective or $PSU(N)$ theory \cite{17, 38, 22, 18, 19, 30}. Using an appropriate choice of the framing parameter in Homfly skein theory, we have obtained in \cite{11} a variant which is defined for all $N, K$. We are not aware of a quantum group approach to these reduced modular categories for $\gcd(N, K) > 1$. Nevertheless we find it convenient to call them $PU(N, K)$ modular categories. In our construction the simple object corresponding to the deformation of the determinant of the vector representation of $\mathfrak{sl}(N)$ may be non trivial; we think that a version of the quantum group $U_q(\mathfrak{gl}(N))$ could be used here.

As well known, the Verlinde formula for the $SU(N, K)$ modular category coincides with the formula in conformal fields theory for the group $SU(N)$;

$$d_{N,K}(g) = \mathcal{V}_{SU(N)}(K, g).$$

We show that for the $PU(N, K)$ modular category the Verlinde formula is

$$\tilde{d}_{N,K}(g) = \frac{d_{N,K}(g)}{N'g},$$

where $N' = \frac{N}{\gcd(N,K)}$. These integral numbers satisfy the level-rank duality relation

$$\tilde{d}_{N,K}(g) = \tilde{d}_{K,N}(g),$$

which is an integral version of a reciprocity formula in \cite{26} (see also \cite{17}).

Our main contribution here is to show that under certain condition the TQFT modules decompose in blocks indexed by spin type structures (respectively 1-dimensional cohomology classes) on the surface, and compute the corresponding refined Verlinde formulas.

An important part of this paper is devoted to the spin decomposition theorem. The proof is given in the general case of a modulo $d$ spin modular category; this notion, developed in Section \ref{section:decomposition} is new and appears in the $\mathbb{Z}/d$ graded cases which are not weakly non-degenerate in \cite{19}. As a motivation, we give below the combinatorial counterpart of this theorem for the A series, in the special case where the rank is even and divides the level (Theorem \ref{theorem:A-series}). We consider the action of $\mathbb{Z}/N$ on the set

$$\Gamma_{N,K} = \{ \lambda = (\lambda_1, \ldots, \lambda_N), \ K \geq \lambda_1 \geq \cdots \geq \lambda_{N-1} \geq \lambda_N = 0 \} ,$$
given for the generator of the cyclic group \( \mathbb{Z}/N \) by

\[
(\lambda_1, \ldots, \lambda_{N-1}, 0) \mapsto (K, \lambda_1, \ldots, \lambda_{N-1}) - (\lambda_{N-1}, \ldots, \lambda_{N-1}) .
\]

We denote by \( \# \text{orb}(\lambda) \) the cardinality of the orbit of \( \lambda \), and by \( \text{Stab}(\lambda) \) the stabilizer subgroup. For \( a, b \in \mathbb{Z}/N \), the numbers \( \epsilon_\lambda(a, b) \in \{0, 1, -1/2, 1/2\} \) are defined as follows.

If \( \# \text{orb}(\lambda) \) is even, then

\[
\epsilon_\lambda(a, b) = \begin{cases} 
1 & \text{if } a \text{ and } b \text{ are zero modulo } |\text{Stab}(\lambda)|, \\
0 & \text{else}.
\end{cases}
\]

If \( \# \text{orb}(\lambda) \) is odd, then

\[
\epsilon_\lambda(a, b) = \begin{cases} 
\frac{1}{2} \left( -1 \right)^{2a} \frac{2b}{|\text{Stab}(\lambda)|} & \text{if } a \text{ and } b \text{ are zero modulo } \frac{|\text{Stab}(\lambda)|}{2}, \\
0 & \text{else}.
\end{cases}
\]

**Theorem 0.1.** Suppose that \( N \) is even, and that \( K/N \) is an odd integer.

a) For \( (a, b) \in (\mathbb{Z}/N)^g \times (\mathbb{Z}/N)^g \), the formula

\[
d_{(a, b)}^{(a, b)}(g) = ((N + K)^{N-1}N)^{g-1} \sum_{\lambda \in \Gamma_{N,K}} \prod_{\nu=1}^g \frac{\epsilon_\lambda(a_\nu, b_\nu)}{\left( \# \text{orb}(\lambda) \right)^2} \\
\times \prod_{1 \leq i < j \leq N} \left( 2 \sin \left( \lambda_i - i - \lambda_j + j \right) \frac{\pi}{N + K} \right)^{2-2g} .
\]

defines a natural number \( d_{(a, b)}^{(a, b)}(g) \).

b) There exists a splitting of the \( SU(N) \) Verlinde formula at level \( K \)

\[
\mathcal{V}_{SU(N)}(K, g) = \sum_{(a, b) \in (\mathbb{Z}/N)^g \times (\mathbb{Z}/N)^g} d_{(a, b)}^{(a, b)}(g) .
\]

For \( N = 2 \), the spin TQFT producing the above decomposition was studied in [13], and a nice algebro-geometric interpretation was obtained by Andersen and Masbaum [1]. We quote that for \( N > 2 \), the involved spin structures are not the usual ones. These structures have coefficients modulo an even integer; they can be understood as something intermediate between usual spin structures (with modulo 2 coefficients) and complex spin structures. The convenient formalism for the TQFT involving these structures should be a slightly extended version of Homotopy Quantum Field Theory as developed by Turaev [33, 34].

The paper is organized as follows. In Section 1 we study spin structures modulo an even integer. In Section 2 we define our spin modular categories. In Section 3 we establish the spin decomposition of the
TQFT in a general context. In Section 4 we consider Verlinde formulas for modular categories of the A series. In Section 5 we establish similar decomposition theorems based on 1-dimensional cohomology classes. In Section 6 we give computer results for small values of $N$ and $K$.

1. Spin structures modulo an even integer

Let $d$ be an even integer. We recall here the topological definition for modulo $d$ spin structures that was given in [10, 11].

There exists, up to homotopy, a unique non-trivial map $g$ from the classifying space $BSO$ to the Eilenberg-MacLane space $K(\mathbb{Z}/d, 2)$. Define the fibration

$$\pi_d : BSpin(\mathbb{Z}/d) \to BSO$$

to be the pull-back, using $g$, of the path fibration over $K(\mathbb{Z}/d, 2)$. The space $BSpin(\mathbb{Z}/d)$ is a classifying space for the non-trivial central extension of the Lie group $SO$ by $\mathbb{Z}/d$, which we denote by $Spin(\mathbb{Z}/d)$. For $d = 2$, this group $Spin(\mathbb{Z}/2) = Spin$ is the universal cover of $SO$, and for general $d$, we have

$$Spin(\mathbb{Z}/d) = \frac{Spin \times \mathbb{Z}/d}{(-1, d/2)}.$$

Now we can use the fibration $\pi_d$ to define structures. Let $E_{Spin(\mathbb{Z}/d)} = \pi_d^*(E_{SO})$ be the pull-back of the canonical vector bundle over $BSO$.

**Definition 1.1.** A modulo $d$ spin structure (or $Spin(\mathbb{Z}/d)$ structure) on a manifold $M$ is an homotopy class of fiber maps from the stable tangent bundle $\tau_M$ to $E_{Spin(\mathbb{Z}/d)}$.

If non-empty the set of these structures, denoted by $Spin(M; \mathbb{Z}/d)$, is affinely isomorphic to $H^1(M; \mathbb{Z}/d)$, by obstruction theory. Moreover the obstruction for existence is a class $w_2(M; \mathbb{Z}/d) \in H^2(M; \mathbb{Z}/d)$, which is the image of the Stiefel-Whitney class $w_2(M)$ under the homomorphism induced by the inclusion of coefficients $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/d$. The Stiefel-Whitney class $w_2(M)$ is zero for every compact oriented manifold whose dimension is lower or equal to 3, hence spin structures modulo $d$ exist on these manifolds.

The various descriptions of usual spin structures [23] apply to modulo $d$ spin structures. The above definition defines, up to equivalence, a $Spin(\mathbb{Z}/d)$ principal bundle over the stable oriented framed bundle $PTM$ (with fiber $SO$) whose restriction to the fiber is equivalent to the cover map $Spin(\mathbb{Z}/d) \to SO$. The cover of $PTM$ defined by the modulo $d$ spin structure is classified by a cohomology class $\sigma \in H^1(PTM, \mathbb{Z}/d)$ whose restriction to the fiber is non-trivial. The
above correspondence is one to one; this gives an alternative definition, and we will identify $\text{Spin}(M; \mathbb{Z}/d)$ with the corresponding affine sub-space of $H^1(PTM, \mathbb{Z}/d)$.

**Definition 1.2** (Alternative definition of modulo $d$ spin structures).

A modulo $d$ spin structure on an oriented manifold $M$ is a cohomology class $\sigma \in H^1(PTM, \mathbb{Z}/d)$ whose restriction to the fiber is non-trivial.

Observe that a spin structure can be evaluated on a framed 1-cycle in the manifold.

Let us consider an oriented surface $\Sigma$. An immersed curve has a preferred framing defined by using the tangent vector. If a closed embedded curve $\gamma$ bounds a disc, then the evaluation of a modulo $d$ spin structure on the corresponding framed 1-cycle $\tilde{\gamma}$ is $\frac{d}{2}$. Following [3, 16], we get the theorem below which gives a convenient description of modulo $d$ spin structures on the oriented surface $\Sigma$.

**Theorem 1.1.**

a) Let $\gamma$ denotes an embedded closed curve with $\sharp \gamma$ components. The assignement $\gamma \mapsto \sigma(\tilde{\gamma}) + (\sharp \gamma)\frac{d}{2}$ extends to a well defined map $q_{\sigma} : H_1(\Sigma, \mathbb{Z}/d) \to \mathbb{Z}/d$.

b) The map $\sigma \mapsto q_{\sigma}$ defines a canonical bijection between $\text{Spin}(\Sigma, \mathbb{Z}/d)$ and the set of maps $q : H_1(\Sigma, \mathbb{Z}/d) \to \mathbb{Z}/d$ such that for all $x, y$ one has

\[
q(x + y) = q(x) + q(y) + \frac{d}{2}x \cdot y.
\]

Here $\cdot$ denotes the intersection form on $H_1(\Sigma, \mathbb{Z}/d)$.

**Proof.** The formula $\sigma(\tilde{\gamma}) + (\sharp \gamma)\frac{d}{2}$ is unchanged if we add or remove to the embedded curve $\gamma$ a trivial component. Let us denote by $\gamma$ (resp. $\gamma'$) the left handed (resp. right handed) curve in the band move represented in Figure 1. We have that $\sharp \gamma' - \sharp \gamma = \pm 1$. By considering the Gauss map, we see that the cycle $\tilde{\gamma}' - \tilde{\gamma}$ is homologous in $PT\Sigma$ to $\tilde{u}$ where $u$ bounds a disc on the surface. We get that the formula is also unchanged under this band move. We deduce that homologous curves in $\Sigma$ give the same result; hence we have that $q_{\sigma}$ is well defined on $H_1(\Sigma, \mathbb{Z})$. Let $\gamma$ be a generic immersed curve. Smoothing a crossing changes $\sharp \gamma$ by $\pm 1$ and does not change the 1-cycle $\tilde{\gamma}$. Hence one has $q_{\sigma}(\lceil \gamma \rceil) =
σ(γ) + (♯γ + I(γ))\frac{d}{2}, \text{ where } I(γ) \text{ is the number of double points. It follows that for any } x, y \in H_1(Σ, \mathbb{Z}), \text{ Property (1) holds. We deduce that } q_σ \text{ is well defined on } H_1(Σ, \mathbb{Z}/d). \text{ Bijectivity is established by using that the map } q_σ \text{ commutes with the action of } H^1(Σ, \mathbb{Z}/d). \square

Let \( M = S^3(L) \) be obtained by surgery on the framed link \( L \) in the 3-sphere. We want to give a combinatorial description for modulo \( d \) spin structures on \( M \). Recall that \( M \) is the boundary of a 4-manifold \( W_L \) called the trace of the surgery. To each \( σ \in \text{Spin}(M; \mathbb{Z}/d) \) is associated a relative obstruction \( w_2(σ; \mathbb{Z}/d) \) in \( H^2(W_L, M; \mathbb{Z}/d) \). The group \( H^2(W_L, M; \mathbb{Z}/d) \) is a free \( \mathbb{Z}/d \) module of rank \( m = \#L \). Taking the coordinates of the relative obstruction in the preferred basis (the basis which is Poincaré dual to the cores of the handles), we get a map \( ψ_L : \text{Spin}(M; \mathbb{Z}/d) \to (\mathbb{Z}/d)^m \).

The following theorem is proved in [11]. Here \( B_L = (b_{ij}) \) is the linking matrix.

**Theorem 1.2.** The map \( ψ_L : \text{Spin}(M; \mathbb{Z}/d) \to (\mathbb{Z}/d)^m \) is injective, and its image is the set of those \((c_1, \ldots, c_m)\) which are solutions of the following \( \mathbb{Z}/d \)-characteristic equation

\[
B_L \begin{pmatrix}
c_1 \\
\vdots \\
c_m
\end{pmatrix} = \frac{d}{2} \begin{pmatrix}
b_{11} \\
\vdots \\
b_{mm}
\end{pmatrix} \pmod{d}.
\]

2. Spin modular categories

A ribbon category is a category equipped with tensor product, braiding, twist and duality satisfying compatibility conditions [32]. If we are given a ribbon category \( C \), then we can define an invariant of links whose components are colored with objects of \( C \). This invariant extends to a representation of the \( C \)-colored tangle category and more generally to a representation of the category of \( C \)-colored ribbon graphs [32, I.2.5]. In a ribbon category there is a notion of trace of morphisms and dimension of objects. The trace of a morphism \( f \) is denoted by \( \langle f \rangle \).

\[
\langle f \rangle = \text{tr}(f)
\]

The dimension of an object \( V \) is the trace of the identity morphism \( \mathbb{1}_V \); we will use the notation \( \langle V \rangle \) as well as \( \langle \mathbb{1}_V \rangle \). We often say quantum
trace and dimension to distinguish from the usual trace and dimension in vector spaces.

Let \( k \) be a field. A ribbon category is said to be \( k \)-additive if the Hom sets are \( k \)-vector spaces, composition and tensor product are bilinear, and \( \text{End} \) (trivial object) = \( k \).

We first recall the definition of a modular category \[32, 27\]. A modular category over \( k \) is a \( k \)-additive ribbon category in which there exists a finite family \( \Gamma \) of simple objects \( \lambda \) (here simple means that \( u \mapsto u_1 \lambda \) from \( k = \text{End} \) (trivial object) to \( \text{End} \) (\( \lambda \)) is an isomorphism) satisfying the axioms below.

- (Domination axiom) For any object \( V \) in the category there exists a finite decomposition \( 1 = \sum_i f_i 1_{\lambda_i} g_i \), with \( \lambda_i \in \Gamma \) for every \( i \).
- (Non-degeneracy axiom) The following matrix is invertible.

\[
S = (S_{\lambda \mu})_{\lambda, \mu \in \Gamma}
\]

where \( S_{\lambda \mu} \in k \) is the endomorphism of the trivial object associated with the \( (\lambda, \mu) \)-colored, 0-framed Hopf link with linking +1.

It follows that \( \Gamma \) is a representative set of isomorphism classes of simple objects; note that the trivial object \( \Theta \) is simple, so that we may suppose that \( \Theta \) is in \( \Gamma \). If we replace the non-degeneracy axiom by the non-singularity condition below then we have the definition of a pre-modular category (a morphism \( f \in \text{Hom} (V, W) \) is called negligible if for any \( g \in \text{Hom} (W, V) \) we have \( \langle fg \rangle = 0 \)):

- (Non-singularity) The category has no non-trivial negligible morphism.

A general modularization procedure for pre-modular categories, and a criterion for existence are developed by Brugières \[14\], and by Müger in the context of \( \ast \)-categories \[24\]. Note that after quotienting by negligible we get the non-singularity condition. This property gives that the pairing

\[
\text{Hom} (V, W) \otimes \text{Hom} (W, V) \mapsto k \quad f \otimes g \mapsto \langle fg \rangle
\]

is non singular. We can deduce that there exists no non-trivial morphism between non-isomorphic simple objects.

One may ask further that the category has direct sums. In fact direct sums may be added in a formal way, and a pre-modular category with direct sums is abelian. This latter fact was pointed out to us by Brugières.
In a modular category $\mathcal{C}$, with representative set of simple objects $\Gamma$, the Kirby color $\Omega = \sum_{\lambda \in \Gamma} \langle \lambda \rangle \lambda$ is used to define an invariant of closed oriented manifolds with colored graph. If $M = S^3(L)$ is obtained by surgery on the framed link $L$ in the sphere and contains a colored graph $K$, then a formula for this invariant is

$$\tau_{\mathcal{C}}(M, K) = \frac{\langle L(\Omega, \ldots, \Omega), K \rangle}{\langle U_1(\Omega) \rangle^{b_+} \langle U_1(\Omega) \rangle^{b_-}}.$$ 

Here $b_+$ (resp. $b_-$) is the number of positive (resp. negative) eigenvalues of the linking matrix $B_L$, and $U_{\pm 1}$ denotes the unknot with framing $\pm 1$.

Modular $G$-categories, with $G$ a group have been introduced by Turaev in [34]; details in the case of an abelian group $G$, and examples derived from quantum groups are given in [19].

Let $G$ be an abelian group. A $G$ grading of a $k$-additive monoidal category $\mathcal{C}$ is a family of full sub-categories $\mathcal{C}_j$, $j \in G$, such that

(i) for any pair of objects $V \in \text{Obj}(\mathcal{C}_j)$, $V' \in \text{Obj}(\mathcal{C}_{j'})$, one has $V \otimes V' \in \text{Obj}(\mathcal{C}_{j+j'})$;

(ii) if for some pair of objects $V \in \text{Obj}(\mathcal{C}_j)$, $V' \in \text{Obj}(\mathcal{C}_{j'})$ one has $\text{Hom}_{\mathcal{C}}(V, V') \neq \{0\}$, then $j = j'$;

(iii) each object of $\mathcal{C}$ is either in $\cup_j \text{Obj}(\mathcal{C}_j)$, or a direct sum of objects in $\cup_j \text{Obj}(\mathcal{C}_j)$.

Axiom (iii) asks that every object splits as a direct sum of homogeneous objects. Axiom (i) asks that tensor product is homogeneous, and axiom (ii) that any non-zero morphism with source or target an homogeneous object is homogeneous; this implies that the dual of an homogeneous object has opposite grading.

Let $\mathcal{C}$ be a modular category. We denote by $U(\mathcal{C})$ the abelian group of isomorphism classes of invertible objects in $\mathcal{C}$ (the law is tensor product). If $U$ is a subgroup of $U(\mathcal{C})$ and $G = \hat{U}$ is the group of characters $\chi : U \to k^*$, then the category $\mathcal{C}$ is $G$ graded. A simple object $\lambda$ is an object in $\mathcal{C}_\chi$ if and only if for every $J \in U$ equality in Figure 2 holds.

A modular $G$-category [19] over $k$ is a $G$ graded $k$-additive ribbon category $(\mathcal{C}; \mathcal{C}_j, j \in G)$ in which there exists finite families $\Gamma_j \subset \text{Obj}(\mathcal{C}_j)$, $j \in G$, of simple objects $\lambda$ satisfying the axioms below.

- (Domination axiom) For any object $V$ in $\mathcal{C}_j$ there exists a finite decomposition $1_V = \sum_i f_i \lambda_i g_i$, with $\lambda_i \in \Gamma_j$ for every $i$.
- (Non-degeneracy axiom) The following matrix is invertible.

$$S = (S_{\lambda \mu})_{\lambda, \mu \in \Gamma_0}$$
where $S_{\lambda\mu} \in \mathbf{k}$ is the endomorphism of the trivial object associated with the $(\lambda, \mu)$-colored, 0-framed Hopf link with linking +1.

It is shown in [34] that a modular $G$-category with $G$ an abelian group gives invariants of 3-manifolds equipped with a 1-dimensional cohomology class. A modular $G$-category may not be a modular category, even in the case where $G$ is a finite abelian group (see [19, Section 1.6]). We point out that a modular category with a $G$ grading is not necessarily a modular $G$-category. The reason is that the $S$-matrix restricted to zero graded objects may be non-invertible. In addition, the zero graded subcategory may be non-modularizable, so that there is no hope to get a modular $G$-category by using some modularization procedure. The latter fact implies that the modular $G$-category is not weakly non-degenerate [19]; it is verified for the class of modular categories we consider below. These categories have a $\mathbb{Z}/d$ grading with $d$ even and give invariants of 3-manifolds equipped with modulo $d$ spin structures; the relevant version of Homotopy Quantum Field Theories as considered by Turaev, should be understood in relation with [34, Remark 7.4.6].

For a simple object $\lambda$ the twist coefficient $\theta_\lambda$ is defined by the figure 3. In the quantum group context, this coefficient is given by the action of the so called quantum Casimir.
Definition 2.1. Let $d$ be an even integer (resp. an integer). A modular category is modulo $d$ spin\(^1\) (resp. modulo $d$ cohomological) if it is equipped with an invertible object $\rho$ whose order is $d$ and whose twist coefficient is $\theta_\rho = -1$ (resp. $\theta_\rho = 1$).

In the following we will mainly discuss the spin case; the cohomological case will be considered in section 5.

Let $(\mathcal{C}, \rho)$ be a modulo $d$ spin modular category, with $\Gamma$ as a representative set of simple objects. The object $\rho^d$ is isomorphic to the trivial, hence we have $\langle \rho^d \rangle = 1$. The dual objects $\rho$ and $\rho^{d-1}$ have the same quantum dimension. We deduce that $d = \langle \rho \rangle = \pm 1$. Note that invertible objects are simple, hence the braiding for $\rho^2$ is identity up to a scalar. By closing we get this scalar and establish the following identity.

\[ \rho \rho = -d \rho \]

The next identity is obtained in a similar way.

\[ \rho \rho = d \rho \]

It is convenient to fix a primitive $d$-th root of unity $\zeta$, and to identify the group of characters $\chi : \{ \rho^j, j \in \mathbb{Z}/d \} \to \mathbb{k}^*$ with $\mathbb{Z}/d$. Then the category $\mathcal{C}$ is $\mathbb{Z}/d$ graded. A simple object $V$ has degree equal to $j$ if and only if the equality in figure 4 holds.

\[^1\text{Invariants associated with a (mod. 2) spin modular category are considered in [31] Theorem 2b]. Our definition here is more general.}\]
The Kirby color decomposes according to this grading.

\[ \Omega = \sum_{\lambda \in \Gamma} <\lambda> \lambda = \sum_{j \in \mathbb{Z}/d} \Omega_j \]

Here the notation \(<\lambda>\) is the quantum dimension of \(\lambda\).

The proof of the theorem below is the same as in the ungraded case (see e.g. [7]). The statement holds for any \(G\) graded pre-modular category with \(G\) an abelian group [19, Prop. 1.4].

**Theorem 2.1.** (Graded sliding property) Suppose that \(V_j\) is an object in \(C_j\), then the equality in Figure 5 holds for any \(j' \in \mathbb{Z}/d\). Here the framed knot labeled with \(\Omega_{j'}\) may be knotted or linked with the other component labeled \(V_j\); this fact is represented by the dashed part in the figure.

The following theorem is proved from the graded sliding property as was done in [11, Theorem 4.2]. We suppose that \(C\) is a modulo \(d\) spin modular category, and that

\[ \Omega = \sum_{\lambda \in \Gamma} <\lambda> \lambda = \sum_{j \in \mathbb{Z}/d} \Omega_j \]

is the graded decomposition of the Kirby color; note [11, lemma 4.5] that \(<U_{\pm 1}(\Omega)> = <U_{\pm 1}(\Omega_{d/2})>\).

**Theorem 2.2.** Let \(C\) be a modulo \(d\) spin modular category, and \(\Omega = \sum_{j \in \mathbb{Z}/d} \Omega_j\) be the graded decomposition of the Kirby element. Provided \(c = (c_1, \ldots, c_m)\) satisfies the modulo \(d\) characteristic condition, the formula

\[ \tau_c^{\text{spin}}(M, \sigma) = \frac{\langle L(\Omega_{c_1}, \ldots, \Omega_{c_m}) \rangle}{\langle U_1(\Omega) \rangle^{b_+} \langle U_{-1}(\Omega) \rangle^{b_-}} \]
defines an invariant of the surgered manifold $M = S^3(L)$ equipped with the modulo $d$ spin structure $\sigma = \psi^{-1}(c_1, \ldots, c_m)$. Moreover,

$$\forall M \quad \tau_C(M) = \sum_{\sigma \in \text{Spin}(M; \mathbb{Z}/d)} \tau^\text{spin}_C(M, \sigma)$$

3. The spin decomposition of the Verlinde formula

If we are given a modular category $\mathcal{C}$ then we get a TQFT. In brief we have a functor $V_C$ from a cobordism category in dimension 3 to vector spaces. If $\mathcal{C}$ is a modulo $d$ spin modular category, then we will construct here a decomposition of the TQFT modules $V_C(\Sigma_g)$ of a genus $g$ surface and compute the ranks of the summands.

The TQFT gives a normalized invariant for a closed 3-manifold $M$ equipped with $p_1$-structure or 2-framing $\alpha$ and colored graph $K$. We extend the scalar field $k$ if necessary, and fix $\kappa$ such that $\kappa^6 = \langle U_1(\Omega) \rangle \langle U_{-1}(\Omega) \rangle$. Let $D = \kappa^{-3}\langle U_1(\Omega) \rangle = \kappa^3\langle U_{-1}(\Omega) \rangle$; note that $D^2 = \langle \Omega \rangle$.

The normalized invariant of a connected closed 3-manifold $M = (M, \alpha, K)$ is then

$$Z_C(M, \alpha, K) = D^{-1-b_1(M)} \kappa^{\sigma(\alpha)} \tau_C(M, K).$$

Here $b_1(M)$ is the first Betti number, and $\sigma(\alpha)$ is the sigma invariant: $\sigma(\alpha) = 3 \text{signature}(W_L) - \langle p_1(W_L, \alpha), [W_L] \rangle$, where $W_L$ is the trace of the surgery and $p_1(W_L, \alpha) \in H^4(W_L, S^3(L))$ is the relative obstruction to extending $\alpha$.

Let $\Sigma$ be an oriented surface with structure (a marking or a $p_1$-structure). We use the object $\rho$ to define a group action on $V_C(\Sigma)$ as follows. To an embedded oriented curve $\gamma$ in $\Sigma$ we associate the TQFT operator $\phi_\gamma : V_C(\Sigma) \to V_C(\Sigma)$ corresponding to a trivial cobordism $[0, 1] \times \Sigma$ equipped with a colored link $\gamma_2(\rho)$. Here $\gamma_2$ is the link $\frac{1}{2} \times \gamma$ equipped with the framing given by the orientation and the normal vector parallel to $\Sigma$. The components of this link are colored with $\rho$.

The spectral projector of $\phi_\gamma$ corresponding to the eigenvalue $\zeta^\nu$ is equal to $\frac{1}{d} \sum_{j=0}^{d-1} \zeta^{-\nu j} \phi_j^\gamma$. This projector is represented by a trivial cobordism with colored link $\gamma(\pi_\nu)$ where the color $\pi_\nu$ is defined by

$$\pi_\nu = \frac{1}{d} \sum_{j=0}^{d-1} \zeta^{-\nu j} \rho^j.$$

Using the definition of the grading, we get the following lemma.
Lemma 3.1. Let $V$ be an object in $\mathcal{C}_j$; denote by $\delta_{\nu j}$ the Kronecker symbol. One has the equality in Figure 6.

We denote by $PT\Sigma$ the principal $SO$-bundle of oriented orthonormal frames in the stabilized tangent bundle to $\Sigma$ (we could stabilize only once).

We denote by $\tilde{\gamma}$ the lift in $PT\Sigma$, using the unit tangent vector, of the embedded curve $\gamma$.

Proposition 3.2. There exists a well defined action of the group $H_1(PT\Sigma, \mathbb{Z}/d)$ on $\mathcal{V}_C(\Sigma)$, which maps $x = [\tilde{\gamma}]$ to the operator $\psi_x = (-d)^{\tilde{\gamma}} \phi_\gamma$.

Proof. The $\mathbb{Z}/d$ module $H_1(PT\Sigma, \mathbb{Z}/d)$ is generated by the 1-cycles $\tilde{\gamma}$ associated with embedded curves $\gamma$. A trivial circle represents the generator on the fiber; this generator has order 2. A disjoint union represents the sum. All the other relations are given by the modified band move in Figure 7. By relation $\text{3}$ the modified band move doesn’t change the number of components modulo 2, hence by relation $\text{2}$, $\psi_x$ is well defined by the formula $\psi_x = (-d)^{\tilde{\gamma}} \phi_\gamma$. Here $\gamma$ is an embedded curve such that the lift $\tilde{\gamma}$ represents $x$. A crossing resolution changes by $\pm 1$ the number of components, hence in the above formula we can use an immersed curve as well. If $\gamma, \gamma'$ represent $x$ and $x'$, then we can isotope $\gamma'$ so that $\gamma \cup \gamma'$ is an immersed curve. This shows that for all $x, x'$, one has $\psi_x \psi_{x'} = \psi_{x+x'}$. 

As a consequence, we have a decomposition of $\mathcal{V}_C(\Sigma)$ indexed by the group $H^1(PT\Sigma, \mathbb{Z}/d)$ identified with the characters on $H_1(PT\Sigma, \mathbb{Z}/d)$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{Figure 6. Modified band move}
\end{figure}
Recall that we have chosen a primitive $d$-th root of unity denoted by $\zeta$. A vector $v$ belongs to the component indexed by $\sigma$ if and only if for every $x \in H_1(PT\Sigma, \mathbb{Z}/d)$ one has $\psi_x v = \zeta^{\sigma(x)} v$. Since the generator of the fiber acts by $-1$, only the classes whose restriction to the fiber is non-trivial, i.e. $Spin(\mathbb{Z}/d)$ structures, will correspond to non-trivial summands.

If $\sigma$ is a modulo $d$ spin structure on the genus $g$ oriented surface $\Sigma_g$, we denote by $V_C(\Sigma_g, \sigma)$ the corresponding summand and by $d_C(\Sigma_g, \sigma)$ its dimension.

$$V_C(\Sigma_g, \sigma) = \{ v \in V_C(\Sigma_g), \forall x \in H_1(PT\Sigma, \mathbb{Z}/d) \psi_x v = \zeta^{\sigma(x)} v \}$$

**Theorem 3.3.** a) There exists a splitting of the TQFT module $V_C(\Sigma_g) = \bigoplus_{\sigma \in Spin(\Sigma_g, \mathbb{Z}/d)} V_C(\Sigma_g, \sigma)$.

b) Suppose that the scalar field $k$ has characteristic zero, then the refined Verlinde formula is the following

$$d_C(\Sigma_g, \sigma) = \langle \Omega \rangle^{g-1} \sum_{\lambda \in \Gamma} \langle \lambda \rangle^{2-2g} \times \prod_{\nu=1}^{g} \frac{\epsilon_\lambda(a_\nu(\sigma), b_\nu(\sigma))}{(\# orb(\lambda))^2}.$$ 

Here $(a(\sigma), b(\sigma)) \in (\mathbb{Z}/d)^g \times (\mathbb{Z}/d)^g$ is given by the values of $q_\sigma$ on a symplectic basis.

If $\# orb(\lambda)$ is even, then

$$\epsilon_\lambda(a, b) = \begin{cases} 1 & \text{if } a \text{ and } b \text{ are zero modulo } |Stab(\lambda)|, \\ 0 & \text{else}. \end{cases}$$

If $\# orb(\lambda)$ is odd, then

$$\epsilon_\lambda(a, b) = \begin{cases} \frac{1}{2}(-1)^{\frac{2a}{|Stab(\lambda)|} - \frac{2b}{|Stab(\lambda)|}} & \text{if } a \text{ and } b \text{ are zero modulo } \frac{|Stab(\lambda)|}{2}, \\ 0 & \text{else}. \end{cases}$$

**Remark.** Any element in $Stab(\lambda)$ has quantum dimension equal to one. In the case where $d = -1$, the group $Stab(\lambda)$ is generated by an even power of $d$, and $\# orb(\lambda)$ is even. We do not know examples with $d = -1$.

**Remark.** If the scalar field $k$ has characteristic $p > 0$, then statement b) computes the dimension mod. $p$.

**Proof.** The formula in a) follows from the decomposition of the vector space $V_C(\Sigma_g)$ described above. Moreover the dimension $d_C(\Sigma_g, \sigma)$ of a summand is the trace of the corresponding projector. This projector can be represented by a cobordism $[0, 1] \times \Sigma_g$ in which we have inserted a convenient skein element. By a standard TQFT argument we get

$$d_C(\Sigma_g, \sigma) = Z_C(S^1 \times \Sigma_g, \text{skein element}).$$
The 3-manifold $S^1 \times \Sigma_g$ is obtained by surgery on the borromean link with $2g + 1$ components represented in Figure 8 [20, Th. 14.12].

In this presentation, a meridian around the bigger component corresponds to $S^1 \times pt$, and the $2g$ meridians around the other components correspond to a system of $2g$ fundamental curves in $\Sigma_g$; these curves are framed by using the meridian disc. The skein element which arises here is represented by these $2g$ curves, decorated with some $\pi$.

If $d = 1$, then $\nu$ is the value of the quadratic form $q_\sigma$ on the curve, and if $d = -1$, then $\nu$ is the value of $\sigma$ on the 1-cycle represented by the curve.

By using (4) and lemma (3.1) we get

$$d_C(\Sigma_g, \sigma) = D^{-(2g+2)} \sum_{\lambda} \langle \lambda \rangle B_\lambda = \langle \Omega \rangle^{-1-g} \sum_{\lambda} \langle \lambda \rangle B_\lambda,$$

where $B_\lambda$ is the invariant of the colored borromean link in figure 8. Here $(a_1, b_1), \ldots, (a_g, b_g)$ are given by the values of the quadratic form $q_\sigma$ on the corresponding curves if $d = 1$, and are equal to the value of $\sigma$ on the 1-cycle represented by the curve if $d = -1$.

Recall that the cyclic group generated by the object $\varrho$, identified with $\mathbb{Z}/d$, acts on the set $\Gamma$ of (representatives of) isomorphism classes of simple objects. If $j$ is in the stabilizer subgroup of $\lambda$, then we choose a basis for the 1-dimensional vector spaces $\text{Hom}_C(\varrho^j, \lambda^* \otimes \lambda)$ and the dual basis for $\text{Hom}_C(\lambda^* \otimes \lambda, \varrho^j)$. We denote these bases by the trivalent vertices in Figure 9.

We then have the relations in Figure 10; recall that in the case $d = -1$, $j$ must be even.
Figure 9. Trivalent vertices.

Figure 10. Relations for trivalent vertices.

The proposition 3.8 below is the key point in the computation. By using this proposition, we get

\[ B_\lambda = \langle \lambda \rangle \prod_{\nu=1}^{g} \sum_{j,j' \in \text{Stab}(\lambda)} \zeta^{j_{\nu}+j'_{\nu}} (-1)^{j_{\nu}j'_{\nu}} \frac{\langle \Omega \rangle^2}{\langle \lambda \rangle^2 d^2} \]

Let \( l = \sharp \text{orb}(\lambda) \) and \( l' = \frac{d}{4} = |\text{Stab}(\lambda)| \). The stabilizer subgroup is then \( \text{Stab}(\lambda) = \{ l_s, \ 0 \leq s < l' \} \).
If \( l \) is even then \( \sum_{j,j' \in \text{Stab}(\lambda)} \zeta^{ja_\nu+j'b_\nu}(-1)^{jj'} \) is zero unless \( \zeta^{la_\nu} = \zeta^{lb_\nu} = 1 \), and we get

\[
\sum_{j,j' \in \text{Stab}(\lambda)} \zeta^{ja_\nu+j'b_\nu}(-1)^{jj'} = \begin{cases} |\text{Stab}(\lambda)|^2 & \text{if } a_\nu \equiv b_\nu \equiv 0 \mod. |\text{Stab}(\lambda)|, \\ 0 & \text{else}. \end{cases}
\]

If \( l \) is odd, then we decompose the sum \( \sum_{j,j' \in \text{Stab}(\lambda)} \zeta^{ja_\nu+j'b_\nu}(-1)^{jj'} \) according to the parity of the indices. The sum is zero if \( \zeta^{2a_\nu} \neq 1 \) or \( \zeta^{2b_\nu} \neq 1 \). It remains four cases to consider according to \( \zeta^{la_\nu} = \pm 1, \zeta^{lb_\nu} = \pm 1 \).

Case \( a_\nu \equiv b_\nu \equiv 0 \mod. |\text{Stab}(\lambda)| \).

\[
\sum_{j,j' \in \text{Stab}(\lambda)} \zeta^{ja_\nu+j'b_\nu}(-1)^{jj'} = \sum_{j,j' \text{ even}} + \sum_{j \text{ even}, j' \text{ odd}} + \sum_{j' \text{ even}, j \text{ odd}} - \sum_{j,j' \text{ odd}} = \frac{4}{4} + \frac{\nu^2}{4} - \frac{\nu^2}{4} = \frac{\nu^2}{2}.
\]

Case \( a_\nu \equiv b_\nu \equiv \frac{|\text{Stab}(\lambda)|}{2} \mod. |\text{Stab}(\lambda)| \).

\[
\sum_{j,j' \in \text{Stab}(\lambda)} \zeta^{ja_\nu+j'b_\nu}(-1)^{jj'} = \sum_{j,j' \text{ even}} + \sum_{j \text{ even}, j' \text{ odd}} + \sum_{j' \text{ even}, j \text{ odd}} - \sum_{j,j' \text{ odd}} = \frac{4}{4} - \frac{\nu^2}{4} - \frac{\nu^2}{4} = -\frac{\nu^2}{2}.
\]

Case \( a_\nu \equiv 0 , b_\nu \equiv \frac{|\text{Stab}(\lambda)|}{2} \mod. |\text{Stab}(\lambda)| \).

\[
\sum_{j,j' \in \text{Stab}(\lambda)} \zeta^{ja_\nu+j'b_\nu}(-1)^{jj'} = \sum_{j,j' \text{ even}} + \sum_{j \text{ even}, j' \text{ odd}} + \sum_{j' \text{ even}, j \text{ odd}} - \sum_{j,j' \text{ odd}} = \frac{4}{4} + \frac{\nu^2}{4} + \frac{\nu^2}{4} = \frac{\nu^2}{2}.
\]

Case \( b_\nu \equiv 0 , a_\nu \equiv \frac{|\text{Stab}(\lambda)|}{2} \mod. |\text{Stab}(\lambda)| \).

\[
\sum_{j,j' \in \text{Stab}(\lambda)} \zeta^{ja_\nu+j'b_\nu}(-1)^{jj'} = \sum_{j,j' \text{ even}} + \sum_{j \text{ even}, j' \text{ odd}} + \sum_{j' \text{ even}, j \text{ odd}} - \sum_{j,j' \text{ odd}} = \frac{\nu^2}{4} - \frac{\nu^2}{4} + \frac{\nu^2}{4} = \frac{\nu^2}{2}.
\]
In all cases we get the formula below.

\[(7) \sum \zeta^{j a_\nu + j' b_\nu} (-1)^j j' = \epsilon_\lambda(a_\nu, b_\nu) |\text{Stab}(\lambda)|^2.\]

In the case where \(d = -1\), then \(l\) is even, \(l' = |\text{Stab}(\lambda)|\) divides \(\frac{d}{2}\) and we have \(\epsilon_\lambda(a_\nu, b_\nu) = \epsilon_\lambda(a_\nu + \frac{d}{2}, b_\nu + \frac{d}{2})\). So that we may define \((a_1, b_1), \ldots, (a_g, b_g)\) by the values of the quadratic form \(q_\sigma\) as well.

We will now establish statement b).

\[d_C(\Sigma g, \sigma) = \langle \Omega \rangle^{-1-g} \sum_\lambda \langle \lambda \rangle^{2-2g} \prod_{\nu=1}^g \epsilon_\lambda(a_\nu, b_\nu) |\text{Stab}(\lambda)|^2 \frac{\langle \Omega \rangle^2}{d^2}\]

\[= \langle \Omega \rangle^{g-1} \sum_\lambda \langle \lambda \rangle^{2-2g} \prod_{\nu=1}^g \epsilon_\lambda(a_\nu, b_\nu) \#_{\text{orb}(\lambda)}^2.\]  

\[\square\]

**Lemma 3.4.** For any \(i \in \mathbb{Z}/d\) the subcategory \(C_i\) contains at least one simple object, and for any simple object \(\lambda_i\) in \(C_i\), one has

\[\langle \lambda_i \rangle \Omega_{i+j} = \lambda_i \otimes \Omega_j.\]

In a modular category the dimension of a simple object is non-zero, hence we have that for any \(i\)

\[(8) \langle \Omega_i \rangle = \langle \Omega_0 \rangle = \frac{1}{d} \langle \Omega \rangle\]

**Proof.** Let \(\nu\) be a generator for the subgroup of \(\mathbb{Z}/d\) formed with all \(i\) such that \(C_i\) contains at least one non-trivial object. Suppose that \(\nu\) has order \(d'\), then \(\varrho^{d'}\) is a simple object whose contribution in the \(S\) matrix is the same as that of the trivial. This object is isomorphic to the trivial, and we deduce that \(d' = d\). This proves the first part of the lemma. The second part follows from the graded sliding property (see [19, Section 1.3]).  

\[\square\]

**Lemma 3.5.** Let \(\lambda\) be a simple object in \(C\), for any \(i\) in \(\mathbb{Z}/d\) the following morphism is non-zero if and only if \(\lambda\) is isomorphic to \(\varrho^j\) for some \(j\).

\[\begin{array}{c}
\Omega_i \\
\downarrow \\
\lambda
\end{array}\]

\[\square\]

\[Proof.\] If \(\lambda\) is equal to \(\varrho^j\) then the morphism is equal to \(\frac{1}{d} \langle \Omega \rangle \zeta^{ij} \mathbf{1}_\lambda\), and so is not zero.
Suppose now that for some simple object $\lambda$ the above morphism is not zero. By using Lemma 3.4, we obtain a scalar $t_\lambda$ such that

$$\Omega_i \lambda = t_\lambda \Omega_\lambda \lambda.$$

Note that $t_\lambda^d = 1$, hence there exists $j$ such that $t_\lambda = \zeta^j$. By the graded sliding property we deduce that the contribution of $\lambda$ in the $S$ matrix is the same as that of $\varrho^j$, and we get the required isomorphism.

**Lemma 3.6.** Let $\lambda$ be a simple object in $\mathcal{C}$, then one has the relation in Figure 11.

**Proof.** We first use the domination axiom. The decomposition of the identity of $\lambda^* \otimes \lambda$ is given by a so called fusion formula (see e.g. [7, Section 1.2]). Note that in this formula the multiplicity of an invertible objet is one if it belongs to the stabilizer subgroup of $\lambda$ and zero else. We then apply Lemma 3.5. The result follows.

**Lemma 3.7.** For $i, j \in \text{Stab}(\lambda)$, one has the relation in Figure 12.

**Proof.** The first equality uses the defining property of a modulo $d$ spin modular category. The second one comes from the definition of the trivalent vertices.

**Proposition 3.8.** The formula in Figure 13 holds.

**Proof.** By using Lemma 3.6 twice (firstly for the component colored by $\Omega_b$), we get the formula in Figure 14. After an isotopy, we apply
lemma 3.7. The result follows.
4. VERLINDE FORMULAS FOR TYPE A MODULAR CATEGORIES

4.1. The $SU(N, K)$ modular category. We first consider the so-called $SU(N, K)$ modular category. The construction can be done either from the representation theory of the quantum group $U_q\mathfrak{sl}(N)$ at a convenient root of unity [2, 35, 4] or from skein theory [38, 11]. In the following we will use Young diagrams to denote the corresponding simple object. Here a Young diagram (or partition) $\lambda$ is a finite non-increasing sequence of non-negative integers. A cell for this partition is a pair $c = (i, j)$ with $1 \leq j \leq \lambda_i$. We denote by $\lambda^\vee$ the transpose of $\lambda$; $(i, j)$ is a cell in $\lambda^\vee$ if and only if $(j, i)$ is a cell in $\lambda$. The content and hook-length for a cell $c = (i, j)$ are defined respectively by

$$cn(c) = j - i, \quad hl(c) = \lambda_i + \lambda_j^\vee - i - j + 1.$$ 

The size of $\lambda$ is $|\lambda| = \sum_1^N \lambda_i$.

The following theorem is proved in [11]. The result can also be obtained from [4, Th. 3.3.20] ($A_{N-1}$ case).

**Theorem 4.1.** Let $N, K \geq 2$. Suppose that $\alpha$ is a $2N(N + K)$-th root of unity in the scalar field and $s = \alpha^{-N}$.

There exists a modular category $C^{SU(N,K)}$ whose set of distinguished simple objects is

$$\Gamma_{N,K} = \{\lambda = (\lambda_1, \ldots, \lambda_N), \ K \geq \lambda_1 \geq \cdots \geq \lambda_{N-1} \geq \lambda_N = 0\}.$$ 

The quantum dimension and framing coefficient of a simple object $\lambda \in \Gamma_{N,K}$ are given by the following formulas (here $[n] = \frac{s^n - s^{-n}}{s - s^{-1}}$ denotes the quantum integer).

$$\langle \lambda \rangle = \prod_{\text{cells}} \frac{[N + cn(c)]}{[hl(c)]}$$ 

$$\lambda \bigcirc = \alpha^{\lambda |^2 \frac{s^{|\lambda| + 2 \sum_{\text{cells}} cn(c)}}{s - s^{-1}}}$$

**Remarks.** 1. In the quantum group approach, a Young diagram in $\Gamma_{N,K}$ gives a highest weight module, which is irreducible and has non-zero quantum dimension. The quantum dimension follows from Weyl’s character formula and computation with symmetric functions in [21, section I.3]. The value of the twist is obtained by the action of Drinfeld quantum Casimir.
2. In the skein theoretic approach the Young diagram gives a minimal idempotent in Hecke algebra (the deformation of the Young symmetrizer in the symmetric group algebra); this idempotent becomes a simple object in the so called Karoubi completion of the Hecke category.

We denote by $V_{N,K}(\Sigma_g)$ the TQFT vector space, associated with a genus $g$ surface $\Sigma_g$, for the modular category $\mathcal{C}^{SU(N,K)}$, and by $d_{N,K}(g)$ its rank. We give below the well known computation for this formula.

**Theorem 4.2.** The rank $d_{N,K}(g)$ is equal to the Verlinde number for the group $SU(N)$ at level $K$.

$$d_{N,K}(g) = V_{SU(N)}(K,g) = \left(\sum_{\lambda \in \Gamma_{N,K}} \langle \lambda \rangle^2\right)^{g-1} \sum_{\lambda \in \Gamma_{N,K}} \prod_{1 \leq i < j \leq N} \left(2 \sin \left(\lambda_i - i - \lambda_j + j \frac{\pi}{N+K}\right)\right)^{2-2g}$$

**Proof.** We can use Turaev’s formula [32, Corollary 12.1.2]. Note that this formula computes the TQFT-invariant of the manifold $S^1 \times \Sigma_g$.

$$d_{N,K}(g) = \left(\sum_{\lambda \in \Gamma_{N,K}} \langle \lambda \rangle^2\right)^{g-1} \sum_{\lambda \in \Gamma_{N,K}} \langle \lambda \rangle^{2-2g}$$

The computation is achieved with the lemma below. Statement a) is a standard fact on symmetric functions; statement b) is contained e.g. in the proof of lemma 2.8 in [11]. Note that the result does not depend on the choice of the root of unity with required order; it is also unchanged if $s$ is replaced by $\bar{s} = -s$. The formula agrees with the Verlinde number $V_{SU(N)}(K,g)$ [5, 28].

**Lemma 4.3.** a)

$$\langle \lambda \rangle^2 = \frac{a_{\rho+\lambda}a_{\rho+\lambda}}{a_{\rho}a_{\rho}}$$

with $\rho = (N-1, N-2, \ldots, 0)$ and, for $l = (l_1, \ldots, l_N)$

$$a_l = \det \left(s^{2(i-1)l_j}\right)_{1 \leq i, j \leq N}$$

b)

$$\sum_{\lambda \in \Gamma_{N,K}} \langle \lambda \rangle^2 = \frac{N(N + K)^{N-1}}{a_{\rho}a_{\rho}}.$$

4.2. **Spin decomposition of the Verlinde formula for $SU(N, K)$ modular category.** In the category $\mathcal{C}^{SU(N,K)}$, the object $(K)$ (a $K$ cells Young diagram with only one row) is an invertible object whose
order is \( N \). It is a generator of the group of invertible objects, and has quantum dimension 1. Its framing coefficient is equal to
\[
\theta_K = a^{K^2} s^{NK + K(K-1)} = (-a^{N+K})^K.
\]
If \( N = jl \), and \( (-a^{N+K})^j = -1 \), then the category \( C^{SU(N,K)} \) equipped with the invertible object \( \rho = (K)^{\otimes j} \) is a modulo \( l \) spin modular category.

Recall that \( a \) is a \( 2N(N + K) \)-th root of unity. Let \( d = \gcd(N, K) \), \( N = dN', K = dK' \). A convenient integer \( j \) exists if and only if either \( d \) is even, \( N' \) is odd and the exponent of 2 in \( K' \) is even \( (K' = 2^{2n}(2m + 1)) \), or \( d \) is odd and the exponent of 2 in \( N \) is an even positive number.

We emphasize the simplest case in the theorem below.

**Theorem 4.4.** If \( N \) is even and \( K' = \frac{K}{N} \) is an odd integer, then the category \( C^{SU(N,K)} \) equipped with the invertible object \( \rho = (K) \) is a modulo \( N \) spin modular category.

The following theorem is an application of 3.3.

**Theorem 4.5.** Suppose that \( N \) is even and \( K' = \frac{K}{N} \) is an odd integer.

a) There exists a splitting of the Verlinde formula
\[
d_{N,K}(g) = \sum_{\sigma \in \text{Spin}((\Sigma_g, \mathbb{Z}/d))} d_{N,K}(g, \sigma).
\]

b) The refined Verlinde formula is the following
\[
d_{N,K}(g, \sigma) = \left( (N + K)^{N-1} N \right)^{g-1} \sum_{\lambda \in \Gamma_{N,K}} \prod_{\nu=1}^{g} e_{\lambda}(a_{\nu}(\sigma), b_{\nu}(\sigma))
\[
\times \prod_{1 \leq i < j \leq N} \left( 2 \sin (\lambda_i - i - \lambda_j + j) \frac{\pi}{N + K} \right)^{2-2g}.
\]

Here \( (a(\sigma), b(\sigma)) \in \mathbb{Z}/N^g \times (\mathbb{Z}/N)^g \) are given by the values of \( q_\sigma \) on a symplectic basis. We consider the action of \( \mathbb{Z}/N \) on the set
\[
\Gamma_{N,K} = \{ \lambda = (\lambda_1, \ldots, \lambda_N), \ K \geq \lambda_1 \geq \cdots \geq \lambda_{N-1} \geq \lambda_N = 0 \},
\]
given for the generator of the cyclic group \( \mathbb{Z}/N \) by
\[
(\lambda_1, \ldots, \lambda_{N-1}, 0) \mapsto (K, \lambda_1, \ldots, \lambda_{N-1}) - (\lambda_{N-1}, \ldots, \lambda_{N-1}).
\]
We denote by \( \sharp \text{orb}(\lambda) \) the cardinality of the orbit of \( \lambda \), and by \( \text{Stab}(\lambda) \) the stabilizer subgroup. The numbers \( e_{\lambda}(a, b) \in \{0, 1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, 0\} \) are defined as follows.
If \(\#\text{orb}(\lambda)\) is even, then

\[
\epsilon_\lambda(a, b) = \begin{cases} 
1 & \text{if } a \text{ and } b \text{ are zero modulo } |\text{Stab}(\lambda)|, \\
0 & \text{else.}
\end{cases}
\]

If \(\#\text{orb}(\lambda)\) is odd, then

\[
\epsilon_\lambda(a, b) = \begin{cases} 
\frac{1}{2}(-1)^{\frac{a^2}{2|\text{Stab}(\lambda)|}} & \text{if } a \text{ and } b \text{ are zero modulo } \frac{|\text{Stab}(\lambda)|}{2}, \\
0 & \text{else.}
\end{cases}
\]

Remark. In the general case, one can use the reduction formula [11, Theorem 3.6] in order to establish a tensor product decomposition of the \(SU(N, K)\) TQFT functor \(V_{N,K}\)

\[
V_{N,K} = V_{N'}^{U(1)} \otimes \tilde{V}_{N,K},
\]

where \(\tilde{V}_{N,K}\) is the TQFT functor associated with the modular category \(C^{PU(N,K)}\) discussed below, and \(V_{N'}^{U(1)}\) (known as a \(U(1)\) theory) is associated with a modular category based on linking numbers. The latter involves a root of unity \(\eta\) whose order is \(2N'\) (resp. \(N'\)) if \(N'\) is even (resp. odd); when \(N'\) is even with even exponent of 2, then one can find \(j\) such that \(j^2 \equiv N' \pmod{2N'}\), and the category is modulo \(j\) spin modular.

4.3. The \(PU(N,K)\) modular category. The so called projective \(PSU(N,K)\) modular category was obtained for \(N\) and \(K\) coprime by restricting to simple objects in the root lattice [22, 18, 19]. The modular category \(C^{PU(N,K)}\) (denoted by \(\tilde{H}^{N,K}\) in [11]) is a generalization to the case where \(N\) and \(K\) are not required to be coprime.

Let \(N, K \geq 2\). We suppose that in the scalar field

\[
\begin{cases} 
\text{s has order } 2(N + K) & \text{if } N + K \text{ is even,} \\
\text{s has order } N + K & \text{if } N + K \text{ is odd.}
\end{cases}
\]

Then the Hecke category completed with idempotents and quotiented with negligible, which we denote by \(H^{N,K}\) is semisimple. In addition to simple objects \(\lambda \in \Gamma_{N,K}\) there is an invertible simple object \(1^N\) and its tensor powers. The group of invertible objects is generated by \(1^N\) and \((K)^\otimes\) with the relation \((1^N)^\otimes \approx (K)^\otimes\). In order to apply the modularization procedure, we have to know which are the transparent simple objects [14]. This depends on the order \(\alpha\) of \((a^N s)^2\) and the order \(\beta\) of \((a^K s^{-1})^2\). The set of isomorphism classes of transparent simple objects is then the group generated by \((1^N)^\otimes\) and \((K)^\otimes\). We choose the framing parameter \(a\) in such a way that this group of transparent objects is as big as possible, and that the modularization criterion is satisfied.
Theorem 4.6. Set \( d = \gcd(N, K) \), \( N = dN' \), \( K = dK' \), \( d = \alpha \beta \) with \( \gcd(\alpha, K') = \gcd(\beta, N') = \gcd(\alpha, \beta) = 1 \).

Suppose that \( a \) satisfies the relations

\[
(a^N s)^\alpha = (-1)^{N+K+1}(a^K s^{-1})^\beta = (-1)^{(N+K+1)\beta}
\]

(such an \( a \) exists up to extension of the scalar field).

There exists a modular category \( \mathcal{C}^{PU(N,K)} \) in which isomorphism classes of simple objects correspond bijectively with cosets in the quotient of

\[
\hat{\Gamma}_{N,K} = \{(1^{N})^\otimes \otimes \lambda, \ 0 \leq j < \alpha, \lambda \in \Gamma_{N,K}\}
\]

under a free action of the cyclic group of order \( N/\alpha \).

The action of the generator is given by tensor product with \( (K)^{\otimes \beta} \) in the completed Homfly category \( H \). One has to iterate \( \beta \) times the rule

\[
(1^N)^{\otimes j} \otimes \lambda \mapsto (1^N)^{\otimes j'} + (K - \lambda_{N-1}, \lambda_1 - \lambda_{N-1}, \ldots, \lambda_{N-2} - \lambda_{N-1}, 0)
\]

where \( j' \equiv j + \lambda_{N-1} \mod \alpha \).

The quantum dimension and framing coefficient of a simple object \( V = (j^{\otimes N}) \otimes \lambda \) are given by the following formulas.

\[
\langle V \rangle = \langle \lambda \rangle = \prod_{\text{cells}} \frac{[N + cn(c)]}{[hl(c)]}
\]

\[
\begin{array}{c}
\left\uparrow V \bigcirc \right. \\
\end{array} = (a^N s)^{N^2+2|\lambda|a^{|\lambda|^2} s^N|\lambda|^2+2\sum_{\text{cells}} cn(c)} \bigg| V
\]

We denote by \( \tilde{V}_{N,K}(\Sigma_g) \) the TQFT vector space, associated with a genus \( g \) surface \( \Sigma_g \), for the modular category \( \mathcal{C}^{PU(N,K)} \) and by \( \tilde{d}_{N,K}(g) \) its rank.

Theorem 4.7. The rank \( \tilde{d}_{N,K}(g) \) is

\[
\tilde{d}_{N,K}(g) = \frac{d_{N,K}(g)}{N^{ng}}
\]

Proof. By Turaev’s formula we have the following,

\[
\tilde{d}_{N,K}(g) = \left( \sum_{V \in \hat{\Gamma}_{N,K}} \langle V \rangle^2 \right)^{g-1} \sum_{V \in \hat{\Gamma}_{N,K}} \langle V \rangle^{2-2g}
\]

Here \( \tilde{\Gamma}_{N,K} \subset \hat{\Gamma}_{N,K} \) is a representative set of the orbits in \( \hat{\Gamma}_{N,K} \) under the order \( N/\alpha \) free cyclic action. Note that this action preserves the
We get
\[ \tilde{d}_{N,K}(g) = \left( \frac{1}{\alpha' N} \sum_{V \in \Gamma_{N,K}} \langle V \rangle^2 \right)^{g-1} \frac{1}{\alpha' N} \sum_{V \in \Gamma_{N,K}} \langle V \rangle^{2-2g} \]

Write \( V = (1^N)^{\otimes i} \otimes \lambda, 0 \leq i < \alpha \) and \( \lambda \in \Gamma_{N,K} \).
\[ \tilde{d}_{N,K}(g) = \left( \frac{1}{N'} \sum_{V \in \Gamma_{N,K}} \langle V \rangle^2 \right)^{g-1} \frac{1}{N'} \sum_{V \in \Gamma_{N,K}} \langle V \rangle^{2-2g} = \frac{d_{N,K}(g)}{N'^g} \].

The following is an integral version of a reciprocity formula in \[26\].

**Theorem 4.8 (Level-rank duality).** One has \( \tilde{d}_{N,K}(g) = \tilde{d}_{K,N}(g) \).

**Proof.** In the construction arising from Homfly skein theory, the parameters \( N \) and \( K \) play the same role, so that we can interchange rows and columns in the description of isomorphisms classes in the modular category \( C_{PU(N,K)} \). We will get the same combinatorics as for the modular category \( C_{PU(K,N)} \). The result follows. \( \square \)

### 4.4. Spin decomposition of the Verlinde formula for \( C_{PU(N,K)} \).

Here we consider the modular category \( C_{PU(N,K)} \) in the spin case. This means that \( d = \gcd(N, K) \) is even, and that \( N' = \frac{N}{d} \) and \( K' = \frac{K}{d} \) are both odd. We fix the framing parameter \( a \) as we did above.

**Theorem 4.9.** Under the above hypothesis, the category \( C_{PU(N,K)} \) equipped with \( \varrho = (K) \otimes (1^N) \) is a modulo \( d \) spin modular category.

**Proof.** In the modular category \( C_{PU(N,K)} \) the object \( 1^N \) and \( (K) \) are invertible with respective orders the coprime integers \( \alpha \) and \( \beta \). It follows that \( \varrho \) is invertible with order \( \alpha \beta = d \). The figure below shows that the twist coefficient for \( \varrho \) is the product of the two twist coefficients for \( 1^N \) and \( (K) \) and a braiding coefficient between \( 1^N \) and \( (K) \). Using [11, Prop. 1.11] we see that the 3 coefficients are respectively \( (a^N s)^N = (-1)^\beta \), \( (A^K s^{-1})^K = (-1)^\alpha \), \( (a^N s)^{2NK} = 1 \). The product is \(-1\).

\( \square \)

If \( \sigma \) is a modulo \( d \) spin structure on the genus \( g \) oriented surface \( \Sigma_g \), we denote by \( \tilde{V}(\Sigma_g, \sigma) \) the corresponding summand and \( \tilde{d}(g, \sigma) \) its dimension. By applying 3.3 we get.
Theorem 4.10. \( a \) There exists a splitting of the Verlinde formula
\[
\tilde{d}_{N,K}(g) = \sum_{\sigma \in \text{Spin}(\Sigma_g, \mathbb{Z}/d)} \tilde{d}_{N,K}(g, \sigma).
\]

\( b \) The refined Verlinde formula is the following
\[
\tilde{d}_{N,K}(g, \sigma) = ((N + K)^{N-1}d)^{g-1} \sum_{V = (1^N) \otimes \lambda \in \Gamma_{N,K}} \prod_{\nu=1}^{g} \epsilon_{V}(a_{\nu}(\sigma), b_{\nu}(\sigma)) \prod_{1 \leq i < j \leq N} \left( 2 \sin \left( \lambda_i - i - \lambda_j + j \right) \frac{\pi}{N + K} \right)^{2-2g}.
\]

Here \( \epsilon_{V} \) and \( \# \text{Orb}(V) \) are defined as before, \( \Gamma_{N,K} \) is a representative set of the quotient of
\[
\hat{\Gamma}_{N,K} = \{(j^{\otimes N}) \otimes \lambda, \; j \in \mathbb{Z}/\alpha, \; \lambda \in \Gamma_{N,K}\}
\]
under a free action of \( \mathbb{Z}/\alpha N' \). The formula in \( b \) can be expressed as follows,
\[
\tilde{d}_{N,K}(g, \sigma) = ((N + K)^{N-1}d)^{g-1} \frac{1}{\alpha N'} \times \sum_{V = (1^N) \otimes \lambda \in \Gamma_{N,K}} \prod_{\nu=1}^{g} \epsilon_{V}(a_{\nu}(\sigma), b_{\nu}(\sigma)) \left( \frac{\alpha N'}{\# \text{Orb}(V)} \right)^2 \prod_{1 \leq i < j \leq N} \left( 2 \sin \left( \lambda_i - i - \lambda_j + j \right) \frac{\pi}{N + K} \right)^{2-2g}.
\]
We consider now the orbit \( \text{Orb}(V) \) under the action of the group \( \mathbb{Z}/\alpha \times \mathbb{Z}/N \) on \( \hat{\Gamma}_{N,K} \), where \((1,0)\) acts by
\[
(1^N)_{\otimes \nu} \otimes \lambda \mapsto (1^N)_{\otimes (\nu+1)} \otimes \lambda,
\]
and \((0,1)\) acts by
\[
(1^N)_{\otimes \nu} \otimes \lambda \mapsto (1^N)_{\otimes (\nu+\lambda_n-1)} \otimes ((K,\lambda) - \lambda_n^{N-1}).
\]

If \#\( \text{Orb}(V)/\alpha N' \) is even, then
\[
\epsilon_V(a,b) = \begin{cases} 
1 & \text{if } a \text{ and } b \text{ are zero modulo } |\text{Stab}(V)|, \\
0 & \text{else.}
\end{cases}
\]

If \#\( \text{Orb}(V)/\alpha N' \) is odd, then
\[
\epsilon_V(a,b) = \begin{cases} 
\frac{1}{2}(-1)^{\frac{2a}{|\text{Stab}(V)|}} & \text{if } a \text{ and } b \text{ are zero modulo } \frac{|\text{Stab}(V)|}{2}, \\
0 & \text{else.}
\end{cases}
\]

5. Cohomological decomposition

In this section we will establish the decomposition in the cohomological case.

Let \( d \) be an integer, and \((\mathcal{C},\nu)\) be a modulo \( d \) cohomological modular category. This means that the object \( \nu \) has order \( d \) and twist coefficient \( \theta_\nu = 1 \). We deduce that the quantum dimension of \( \nu \) is \( \nu = \pm 1 \), and
\[
(9) \quad \nu \nu = \nu \nu
\]
\[
(10) \quad \nu \nu = \nu \nu
\]

After fixing a \( d \)-th root of unity \( \zeta \), the category is \( \mathbb{Z}/d \) graded. The Kirby color decomposes according to this grading.
\[
\Omega = \sum_{\lambda \in \Gamma} \langle \lambda \rangle \lambda = \sum_{j \in \mathbb{Z}/d} \Omega_j
\]

Using this grading we obtain the theorem below [11,19].
Theorem 5.1. Let $\mathcal{C}$ be a modulo $d$ cohomological modular category, and $\Omega = \sum_{j \in \mathbb{Z}/d} \Omega_j$ be the graded decomposition of the Kirby element. Provided $c = (c_1, \ldots, c_m) \in \mathbb{Z}^m$ is in the kernel of $B_L \otimes \mathbb{Z}/d$ the formula

$$
\tau_{\mathcal{C}}(M, \sigma) = \frac{\langle L(\Omega_{c_1}, \ldots, \Omega_{c_m}) \rangle}{\langle U_1(\Omega) \rangle^{b_+}(U_{-1}(\Omega) \rangle^{b_-} \langle U_1(\Omega) \rangle^{b_+}(U_{-1}(\Omega) \rangle^{b_-}}
$$

is an invariant of the surgered manifold $M = S^3(L)$ equipped with the modulo $d$ cohomology class $\sigma$ corresponding to $c$.

Moreover,

$$
\forall M \quad \tau(M) = \sum_{\sigma \in \text{Spin}(M; \mathbb{Z}/d)} \tau_{\mathcal{C}}(M, \sigma)
$$

Following section 2 we get the proposition below. Note that here the action given by a trivial curve $\gamma$ colored with $d\phi_\gamma$ is trivial.

Proposition 5.2. There exists a well defined action of the group $H_1(\Sigma, \mathbb{Z}/d)$ on $V_{\mathcal{C}}(\Sigma)$, which maps $x = [\gamma]$ to the operator $\psi_x = (d)^{\sharp \gamma} \phi_\gamma$.

Using this action we get the decomposition theorem below.

Theorem 5.3. Let $(\mathcal{C}, \varnothing)$ be a modulo $d$ cohomological modular category.

a) There exists a splitting of the Verlinde formula

$$
\dim(V_{\mathcal{C}}(\Sigma_g)) = \sum_{\sigma \in H_1(\Sigma_g, \mathbb{Z}/d)} \dim(V_{\mathcal{C}}(\Sigma_g, \sigma))
$$

b) The refined Verlinde formula is the following

$$
\dim(V_{\mathcal{C}}(\Sigma_g, \sigma)) = \langle \Omega \rangle^{g-1} \sum_{\lambda \in \Gamma} \langle \lambda \rangle^{2g-2} \times \prod_{\nu=1}^g \epsilon_\lambda(a_{\nu}(\sigma), b_{\nu}(\sigma)) \frac{1}{(|\text{orb}(\lambda)|)^2}.
$$

Here $(a_{\sigma}, b_{\sigma}) \in (\mathbb{Z}/d)^g \times (\mathbb{Z}/d)^g$ is given by the values of $\sigma$ on a symplectic basis, and

$$
\epsilon_\lambda(a, b) = \begin{cases} 
1 & \text{if } a \text{ and } b \text{ are zero modulo } |\text{Stab}(\lambda)|, \\
0 & \text{else}.
\end{cases}
$$

Proof. The decomposition a) follows from the action given in Proposition 5.2.

For $\sigma \in H_1(\Sigma_g, \mathbb{Z}/d)$, we have

$$
\dim(V_{\mathcal{C}}(\Sigma_g, \sigma)) = \langle \Omega \rangle^{-1-g} \sum_{\lambda} \langle \lambda \rangle B_\lambda,
$$

where $B_\lambda$ is the invariant of the colored link in figure 8. Here $(a_1, b_1), \ldots, (a_g, b_g)$ are given by the values of $\sigma$ on the corresponding curves if $d = 1$, and this values plus $\frac{d}{2}$ if $d = -1$ (in this case $d$ has to be even).
The computation is done as section 3. We have

\[ B_\lambda = \langle \lambda \rangle \prod_{\nu=1}^{g} \sum_{j, j' \in \text{Stab}(\lambda)} \zeta^{j a_\nu + j' b_\nu} \frac{\langle \Omega \rangle^2}{\langle \lambda \rangle^2} \frac{d^2}{d^2} \]

The formula follows. \[\Box\]

Let \( d = \gcd(N, K) \). If \( d \) is odd, or if \( d \) is even but \( \frac{NK}{d^2} \) is even, then the category \( C^{PU(N,K)} \) is a modulo \( d \) cohomological modular category.

If \( N = jl \), and \( (-a^{N+K})Kj^2 = 1 \), then the category \( C^{SU(N,K)} \) equipped with the invertible object \( g = (K)^{\otimes j} \) is a modulo \( l \) cohomological modular category. In particular, if \( N \) divides \( K \), and \( N \) is odd or \( \frac{K}{N} \) is even, then the category \( C^{SU(N,K)} \) is a modulo \( N \) cohomological modular category.

6. Some computations

We give below some computations obtained with MuPAD [25]. Our program implements the Verlinde formulas for the categories \( C^{SU(N,K)} \). The cardinality of the alcove increases rapidly, and we obtain results only for small values of \( N, K \). The function Verlinde\((N, K, g)\) gives \( d_{N, K}(g) \), and Spin_Verl\((N, K, [\ldots])\) computes \( d_{N, K}(g, \sigma) \), where the value of \( q_\sigma \) on the standard basis is the list \([\ldots]\). We know [13] that \( d_{2,2}(g, \sigma) \) is 0 or 1 according to the Arf invariant of the spin structure. It would be interesting to understand the combinatorics of the formula \( d_{N, K}(g, \sigma) \) in the general case.

\[ \text{Verlinde}(2, 2, 1); \]
\[ 3 \]
\[ \text{Verlinde}(2, 2, 2); \]
\[ 10 \]
\[ \text{Spin_Verl}(2, 2, [[0, 0]]); \]
\[ 1 \]
\[ \text{Spin_Verl}(2, 2, [[1, 1]]); \]
\[ 0 \]
\[ \text{Spin_Verl}(2, 2, [[0, 0], [1, 1]]); \]
\[ 0 \]
\[ \text{Spin_Verl}(2, 2, [[1, 1], [1, 1]]); \]
\[ 1 \]
\[ \text{Verlinde}(2, 6, 1); \]
\[ 7 \]
\[ \text{Verlinde}(2, 6, 2); \]
\[ 84 \]
\[ \text{Spin_Verl}(2, 6, [[0, 0]]); \]
2
Spin_Verl(2,6,[[1,1]]);
1
Spin_Verl(2,6,[[0,0],[1,1]]);
4
Spin_Verl(2,6,[[0,0],[0,0]]);
6
Verlinde(4,4,1);
35
Verlinde(4,4,2);
4680
Spin_Verl(4,4,[[0,0]]);
3
Spin_Verl(4,4,[[1,0]]);
2
Spin_Verl(4,4,[[1,1]]);
2
Spin_Verl(4,4,[[2,2]]);
2
Spin_Verl(4,4,[[0,0],[0,0]]);
24
Spin_Verl(4,4,[[1,0],[0,0]]);
18
Spin_Verl(4,4,[[1,0],[1,0]]);
18
Spin_Verl(4,4,[[2,2],[0,0]]);
20
Verlinde(6,6,1);
462
Verlinde(6,6,2);
30660988
Spin_Verl(6,6,[[0,0]]);
14
Spin_Verl(6,6,[[1,0]]);
13
Spin_Verl(6,6,[[2,0]]);
13
Spin_Verl(6,6,[[1,1]]);
12
Spin_Verl(6,6,[[2,2]]);
13
Spin_Verl(6,6,[[3,0]]);
Spin Verl(6,6,[[3,3]]);
Spin Verl(6,6,[[0,0],[0,0]]);
Spin Verl(6,6,[[1,0],[0,0]]);
Spin Verl(6,6,[[1,0],[0,0]]);
Spin Verl(6,6,[[2,0],[0,0]]);
Spin Verl(6,6,[[2,2],[0,0]]);
Spin Verl(6,6,[[3,0],[0,0]]);
Spin Verl(6,6,[[3,3],[0,0]]);

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