A General Variational Principle of Classical Field
and Its Application to General relativity I

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Abstract
A general variational principle of classical fields with a Lagrangian
containing field quantity and its derivatives of up to the N-order is pre-
sented. Noether’s theorem is derived. The generalized Hamilton-Jacobi’s
equation for the Hamilton’s principal functional is obtained. These results
are surprisingly in great harmony with each other. They will be applied
to general relativity in the subsequent articles, especially the generalized
Noether’s theorem will be applied to the problem of conservation and
non-conservation in curved spacetime.

1 Introduction
The aim of this series of articles is to explore the conservation and non-conservation
in curved spacetime, especially to explore the difficulty of energy-momentum
conservation in general relativity and the gravitational energy-momentum. We
start with presenting a general variational principle for classical fields with a
Lagrangian containing the field quantity and its derivatives of up to the N-th
order (part I), Then the general results from part I are applied to general rela-
tivity, especially the generalized Noether’s theorem is applied to the problem of
conservation and non-conservation in general relativity (part II). The last part
(part III) is devoted to the difficulty of conservation of energy-momentum in
curved spacetime and to the problem whether the metric field carries energy-
momentum or not.

The developments of modern physics, such as the founding of statistical
mechanics and quantum mechanics, have proved that the variational prin-ciple approach to dynamics is not only an alternative and equivalent version to
the naive, intuitive approach, but also yields deeper insights into the underlying
physics. For instance, it is hard to imagine that the statistical mechanics
could have been established without using the concepts of phase space, and the
quantum mechanics could have been established without using the concept of
Hamiltonian. Therefore, we will found our argument on a general variational
principle for classical field. It might be for the same reason, soon after Einstein proposed his general theory of relativity, Hilbert made the first attempt to get Einstein’s equation by using the least action principle. The Lagrangian being used for vacuum Einstein’s equation, \((16\pi G)^{-1}R\), is the only independent scalar constructed in terms of the metric field and its derivatives of no higher than the second order. However, because the Ricci scalar curvature \(R\) contains the second order derivatives of the metric field \(g_{\mu\nu}(x)\), which is now the dynamic variable, the least action principle for Lagrangians containing only the field quantity and its first order derivatives does not lead to Einstein’s field equation. The generally accepted solution to this difficulty is adding the Gibbons-Hawking boundary term to the Hilbert action and keeping the least action principle unchanged[1]. But there is another solution to this difficulty, which is adopted in the present paper. The least action principle will be restated and the Hilbert action will still be used for the vacuum Einstein’s equation. In order to show this is proper and natural, we will consider classical fields with a Lagrangian containing the field quantity and its derivatives of up to the \(N\)-th order. In our opinion, acting at a distance is not acceptable, so we assume that the Lagrangian does not contain the integral of the field quantity. In section 2, a general Lagrangian formalism for classical fields is presented. In section 3, the Hamiltonian formalism is discussed. In section 4, the Noether’s theorem is derived and the conservation law due to the ”coordinate shift” invariance is established. The generalized Hamilton-Jacobi’s equation is obtained in section 5. All the results obtained above are in great harmony with each other and apply to various classical fields, say, those with Galilean covariance, with Lorentzian covariance, with general covariance, or without such covariance. Part I finishes with a remark. In part II, this general variational principle of classical fields developed in part I is applied to general relativity and quite a few conserved quantities corresponding to the coordinate ”shift” invariance, coordinate ”rotation” invariance etc. are found. And the properties of these conservation laws are discussed. In part III, after some general consideration, the introducing of gravitational energy-momentum is reviewed. The difficulties of conservation of energy-momentum in general relativity are explored by using Noether’s theorem and observations from geometry. It is pointed out that the metric field does not carry energy-momentum, and the law of conservation of energy-momentum no longer holds in curved spacetime.

2 Lagrangian formulation of classical fields

Suppose that our spacetime \(M\) is a smooth manifold which is differentially homeomorphic to \(\mathbb{R}^4\). Choose a chart \((M, \varphi)\), and denote by \((x^0, x^1, x^2, x^3)\) the corresponding coordinates. Suppose the action over any spacetime region \(\Omega \subset M\) of the classical field \(\{\Phi_a(x)\}\) is

\[
A = \int_{\Omega} d^4x L(x, \Phi(x), \partial \Phi(x), \partial^2 \Phi(x), \ldots, \partial^N \Phi(x)) = A[\Phi]
\]
where the Lagrangian $L$ is a function of the spacetime coordinate, the field $\{\Phi_a(x)\}$ and its derivatives of no higher than the $N$-th order. $L$ does not contain the integral of $\{\Phi_a(x)\}$, since acting at a distance is not acceptable. In order to develop a general variational principle for all locally interacting coordinates, and the Galilean invariance, Lorentzian invariance or the general invariance are not assumed for the time being. For the sake of simplicity, we have assumed that our spacetime manifold is $(3+1)$-dimensional. However, our presentation has nothing to do with the spacetime dimensionality. It still holds for an $(n+1)$-dimensional spacetime.

Consider the difference between actions over $\Omega$ of two possible movements close to each other. Using integration by parts and Stokes theorem, one gets

$$\delta A[\Phi] = \int_{\Omega} d^4x \delta \Phi_a(x) \left[ \frac{\partial L}{\partial \Phi_a(x)} - \frac{\partial}{\partial \lambda_1} \frac{\partial L}{\partial \lambda_1 \Phi_a(x)} - \cdots \ight.$$ \begin{align*}
+ \left. (-1)^N \partial_{\lambda_1} \cdots \partial_{\lambda_N} \frac{\partial L}{\partial \lambda_1 \cdots \partial_{\lambda_N} \Phi_a(x)} \right] 
+ \int_{\partial \Omega} ds \left[ B^a_{\lambda_1} \delta \Phi_a(x) + B^a_{\lambda_1 \cdots \lambda_N} \delta \partial_{\lambda_1} \cdots \partial_{\lambda_N} \Phi_a(x) \right],
\end{align*}$$

where the Greek indices go through 0, 1, 2, 3, and

$$B^a_{\lambda} = \frac{\partial L}{\partial \lambda \Phi_a(x)} - \partial_{\mu_1} \partial_{\lambda} \partial_{\mu_1} \Phi_a(x) + \partial_{\mu_2} \partial_{\lambda} \partial_{\mu_2} \Phi_a(x) - \cdots$$

$$B^a_{\lambda_1 \cdots \lambda_N} = \frac{\partial L}{\partial \lambda_1 \cdots \partial_{\lambda_N} \Phi_a(x)} - \partial_{\mu_1} \partial_{\lambda_1} \partial_{\mu_1} \partial_{\lambda_1} \Phi_a(x) + \partial_{\mu_2} \partial_{\lambda_1} \partial_{\mu_2} \partial_{\lambda_1} \Phi_a(x) - \cdots$$

Equation (2) suggests the least action principle reads as follows.

For any spacetime region $\Omega$, among all possible movements in $\Omega$ with the same boundary condition

$$\delta \Phi|_{\partial \Omega} = 0, \delta \partial \Phi|_{\partial \Omega} = 0, \ldots, \delta \partial^{N-1} \Phi|_{\partial \Omega} = 0,$$

the real movement corresponds to the stationary value of the action over $\Omega$.

Combining eqns. (2), (4), one obtains the field equation (Euler-Lagrange equation) satisfied by the real movement.
3 Hamiltonian formulation of classical fields

To formulate the Hamiltonian formalism of classical fields, one needs to specify a reference coordinate system \((\xi^0, \xi^1, \xi^2, \xi^3)\), such that the hyper-surfaces, \(\Sigma_{\xi^0}\), of constant \(\xi^0\) are spacelike Cauchy hyper-surfaces and the curves, \(t_{\xi}^{-}\) of constant \(\xi^{-}\) are timelike world lines of the observer at \(\xi^{-}\). We will consider the state of the field on hyper-surfaces, \(\Sigma_{\xi^0}\), and investigate the change of state (evolution) with \(\xi^0\). We will observe the state on \(\Sigma_{\xi^0}\) and the evolution with \(\xi^0\) from any reference coordinate system in the same way. As has been pointed out[2], the properly formulated Hamiltonian formalism is compatible with all dynamic systems, Galilean invariant, Lorentzian invariant, general invariant and so on. The invariance is the heritage from the Lagrangian being used. The 3 + 1 decomposition of spacetime proposed above is more general than the one generally accepted in General relativity. The latter relies on the unknown dynamical variable, the metric field. Suppose the Lagrangian functional is

\[
\Lambda = \int d^3\xi L(\xi, \Phi(\xi), \partial \Phi(\xi), \ldots, \partial^N \Phi(\xi)) = \Lambda[\xi^0, \Phi[\xi^0], \partial_0 \Phi[\xi^0], \ldots, \partial^N_0 \Phi[\xi^0]],
\]

Consider the difference between Lagrangian functionals of two states close to each other. Using the general formula (2), which is independent of the dimensionality, one gets

\[
\delta \Lambda = \int d^3\xi \{ \frac{\partial L}{\partial \Phi_a(\xi)} \delta \Phi_a(\xi) - \partial_i \frac{\partial L}{\partial \partial_i \Phi_a(\xi)} \} + \cdots + (-1)^N \partial_{i_1} \cdots \partial_{i_N} \frac{\partial L}{\partial \partial_{i_1} \cdots \partial_{i_N} \Phi_a(\xi)} \delta \Phi_a(\xi)
\]

\[
\delta \Lambda = \int d^3\xi \{ \frac{\partial L}{\partial \Phi_a(\xi)} \delta \Phi_a(\xi) - \partial_i \frac{\partial L}{\partial \partial_i \Phi_a(\xi)} \} + \cdots + (-1)^N \partial_{i_1} \cdots \partial_{i_N} \frac{\partial L}{\partial \partial_{i_1} \cdots \partial_{i_N} \Phi_a(\xi)} \delta \Phi_a(\xi)
\]

\[
\frac{\delta A}{\delta \Phi_a(x)} = \frac{\partial L}{\partial \Phi_a(x)} \partial_x \frac{\partial L}{\partial \partial_x \Phi_a(x)} + \cdots + (-1)^N \partial_{\lambda_1} \cdots \partial_{\lambda_N} \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_N} \Phi_a(x)} = 0.
\]
\[ + \cdots + \left[ \frac{\partial L}{\partial \partial_0 a^N-1 \Phi_a(\xi)} - C_N^{-1} \partial_1 \frac{\partial L}{\partial \partial_1 \partial_0 a^N-1 \Phi_a(\xi)} \right] \delta \partial_0^{N-1} \Phi_a(\xi) \]

\[ + \frac{\partial L}{\partial \partial_0 \Phi_a(\xi)} \delta \partial_0^{N} \Phi_a(\xi) \] + 

\[ + \int d^3 \xi \partial_k \left\{ K^{ai} \delta \Phi_a(\xi) + K^{aik_1} \delta \partial_{k_1} \Phi_a(\xi) + K^{aik_1 k_2} \delta \partial_{k_1} \partial_{k_2} \Phi_a(\xi) \right. \]

\[ + \cdots + K^{aik_1 \cdots k_{N-1}} \delta \partial_{k_1} \cdots \partial_{k_{N-1}} \Phi_a(\xi) \]

\[ + \left[ K_1^{a} \delta \partial_0 \Phi_a(\xi) + K_1^{aik_1} \delta \partial_{k_1} \partial_0 \Phi_a(\xi) + K_1^{aik_1 k_2} \delta \partial_{k_1} \partial_{k_2} \partial_0 \Phi_a(\xi) \right] \]

\[ + \cdots + K_1^{aik_1 \cdots k_{N-2}} \delta \partial_{k_1} \cdots \partial_{k_{N-2}} \partial_0 \Phi_a(\xi) \]

\[ + \left[ K_2^{ai} \delta \partial_0^2 \Phi_a(\xi) + K_2^{aik_1} \delta \partial_{k_1} \partial_0^2 \Phi_a(\xi) + K_2^{aik_1 k_2} \delta \partial_{k_1} \partial_{k_2} \partial_0^2 \Phi_a(\xi) \right] \]

\[ + \cdots + K_2^{aik_1 \cdots k_{N-3}} \delta \partial_{k_1} \cdots \partial_{k_{N-3}} \partial_0^2 \Phi_a(\xi) \]

\[ + \cdots + \left[ K_{N-2}^{ai} \delta \partial_0^{N-2} \Phi_a(\xi) + K_{N-2}^{aik_1} \delta \partial_{k_1} \partial_0^{N-2} \Phi_a(\xi) \right] + K_{N-1}^{ai} \delta \partial_0^{N-1} \Phi_a(\xi) \}, \]

where the domain of integration is \( \mathbb{R}^3 \), the latin indices go through 1, 2, 3, and

\[ K^{ai} = \frac{\partial L}{\partial \partial_i \Phi_a(\xi)} - \partial_{j_1} \frac{\partial L}{\partial \partial_{j_1} \partial_{j_2} \Phi_a(\xi)} + \partial_{j_1} \partial_{j_2} \frac{\partial L}{\partial \partial_{j_1} \partial_{j_2} \partial_{j_3} \Phi_a(\xi)} - \cdots + \]

\[ + (-1)^{N-1} \partial_{j_1} \cdots \partial_{j_{N-1}} \frac{\partial L}{\partial \partial_{j_1} \partial_{j_2} \cdots \partial_{j_{N-1}} \Phi_a(\xi)}, \]

\[ K^{aik_1} = \frac{\partial L}{\partial \partial_{i_1} \Phi_a(\xi)} - \partial_{j_1} \frac{\partial L}{\partial \partial_{i_1} \partial_{j_1} \Phi_a(\xi)} + \partial_{j_1} \partial_{j_2} \frac{\partial L}{\partial \partial_{i_1} \partial_{j_1} \partial_{j_2} \Phi_a(\xi)} - \cdots + \]

\[ + (-1)^{N-2} \partial_{j_1} \cdots \partial_{j_{N-2}} \frac{\partial L}{\partial \partial_{j_1} \partial_{j_2} \cdots \partial_{j_{N-2}} \partial_{j_{N-1}} \Phi_a(\xi)} \cdots, \]

\[ K^{aik_1 \cdots k_{N-2}} = \frac{\partial L}{\partial \partial_{i_1} \partial_{i_2} \cdots \partial_{k_{N-1}} \Phi_a(\xi)} - \partial_{j_1} \frac{\partial L}{\partial \partial_{i_1} \partial_{j_1} \cdots \partial_{k_{N-1}} \Phi_a(\xi)}, \]

\[ K^{aik_1 \cdots k_{N-1}} = \frac{\partial L}{\partial \partial_{i_1} \partial_{i_2} \cdots \partial_{k_{N-1}} \Phi_a(\xi)}, \]

\[ K^{ai}_1 = c_1^1 \frac{\partial L}{\partial \partial_{i_1} \partial_0 \Phi_a(\xi)} - c_2^1 \partial_{j_1} \frac{\partial L}{\partial \partial_{j_1} \partial_0 \Phi_a(\xi)} + c_3^1 \partial_{j_1} \partial_{j_2} \frac{\partial L}{\partial \partial_{j_1} \partial_{j_2} \partial_0 \Phi_a(\xi)} \]

\[ - \cdots + (-1)^{N-2} c_N^1 \partial_{j_1} \cdots \partial_{j_{N-2}} \frac{\partial L}{\partial \partial_{j_1} \cdots \partial_{j_{N-2}} \partial_0 \Phi_a(\xi)}, \]

\[ K^{aik_1}_1 = c_1^1 \frac{\partial L}{\partial \partial_{i_1} \partial_{k_1} \partial_0 \Phi_a(\xi)} - c_2^1 \partial_{j_1} \frac{\partial L}{\partial \partial_{j_1} \partial_{k_1} \partial_0 \Phi_a(\xi)} + c_3^1 \partial_{j_1} \partial_{j_2} \frac{\partial L}{\partial \partial_{j_1} \partial_{j_2} \partial_{k_1} \partial_0 \Phi_a(\xi)} - \cdots + \]

\[ + (-1)^{N-3} c_N^1 \partial_{j_1} \cdots \partial_{j_{N-3}} \frac{\partial L}{\partial \partial_{j_1} \cdots \partial_{j_{N-3}} \partial_{k_1} \partial_0 \Phi_a(\xi)} \cdots, \]
\[
K_{N-3}^{aik_1 \ldots k_{N-3}} = c_{N-1}^{1} \frac{\partial L}{\partial \partial_{a_{3}} \partial_{k_{1}} \ldots \partial_{k_{N-3}} \partial_{0} \Phi_{a}(\xi)} - c_{N}^{1} \partial_{j_{1}} \frac{\partial L}{\partial \partial_{a_{3}} \partial_{j_{1}} \partial_{k_{1}} \ldots \partial_{k_{N-3}} \partial_{0} \Phi_{a}(\xi)},
\]
\[
K_{N-2}^{aik_1 \ldots k_{N-2}} = c_{N}^{1} \frac{\partial L}{\partial \partial_{a_{3}} \partial_{k_{1}} \ldots \partial_{k_{N-2}} \partial_{0} \Phi_{a}(\xi)},
\]
\[
K_{a}^{2} = c_{3}^{3} \frac{\partial L}{\partial \partial_{i} \partial_{k_{1}} \partial_{0}^{2} \Phi_{a}(\xi)} - c_{4}^{2} \frac{\partial L}{\partial \partial_{i} \partial_{j_{1}} \partial_{k_{1}} \partial_{0}^{2} \Phi_{a}(\xi)} + c_{5}^{2} \frac{\partial L}{\partial \partial_{i} \partial_{j_{2}} \partial_{j_{1}} \partial_{k_{1}} \partial_{0}^{2} \Phi_{a}(\xi)} - + \cdots 
+ (-1)^{N-3} c_{N}^{3} \frac{\partial L}{\partial \partial_{i} \partial_{j_{1}} \ldots \partial_{j_{N-3}} \partial_{0}^{2} \Phi_{a}(\xi)},
\]
\[
K_{2}^{aik_1 \ldots k_{N-4}} = c_{N}^{2} \frac{\partial L}{\partial \partial_{i} \partial_{k_{1}} \ldots \partial_{k_{N-4}} \partial_{0}^{2} \Phi_{a}(\xi)} - c_{N-1}^{2} \frac{\partial L}{\partial \partial_{i} \partial_{j_{1}} \partial_{k_{1}} \ldots \partial_{k_{N-4}} \partial_{0}^{2} \Phi_{a}(\xi)},
\]
\[
K_{2}^{aik_1 \ldots k_{N-3}} = c_{N}^{2} \frac{\partial L}{\partial \partial_{i} \partial_{k_{1}} \ldots \partial_{k_{N-3}} \partial_{0}^{2} \Phi_{a}(\xi)},
\]
\[
K_{N-2}^{a} = c_{N}^{N-2} \frac{\partial L}{\partial \partial_{i} \partial_{k_{1}} \partial_{0}^{N-2} \Phi_{a}(\xi)} - c_{N-1}^{N-2} \partial_{j_{1}} \frac{\partial L}{\partial \partial_{i} \partial_{j_{1}} \partial_{k_{1}} \partial_{0}^{N-2} \Phi_{a}(\xi)},
\]
\[
K_{N-1}^{a} = c_{N}^{N-1} \frac{\partial L}{\partial \partial_{i} \partial_{0}^{N-1} \Phi_{a}(\xi)}.
\]
Hence
\[
\frac{\delta \Lambda}{\delta \Phi_{a}(\xi)} = \frac{\partial L}{\partial \partial_{a_{3}} \Phi_{a}(\xi)} - \partial_{i_{1}} \frac{\partial L}{\partial \partial_{i_{1}} \Phi_{a}(\xi)} \partial_{a_{3}} \Phi_{a}(\xi) + \cdots + (-1)^{N} \partial_{i_{1}} \cdots \partial_{i_{N-1}} \frac{\partial L}{\partial \partial_{i_{1}} \cdots \partial_{i_{N-1}} \Phi_{a}(\xi)},
\]
\[
\frac{\delta \Lambda}{\delta \partial_{0} \Phi_{a}(\xi)} = \frac{\partial L}{\partial \partial_{0} \Phi_{a}(\xi)} - c_{1}^{1} \partial_{i_{1}} \frac{\partial L}{\partial \partial_{i_{1}} \partial_{0} \Phi_{a}(\xi)} \partial_{0} \Phi_{a}(\xi) + \cdots + (-1)^{N-1} C_{N}^{1} \partial_{i_{1}} \cdots \partial_{i_{N-1}} \frac{\partial L}{\partial \partial_{i_{1}} \cdots \partial_{i_{N-1}} \partial_{0} \Phi_{a}(\xi)},
\]
\[
\frac{\delta \Lambda}{\delta \partial_{0} \Phi_{a}(\xi)} = \frac{\partial L}{\partial \partial_{0} \Phi_{a}(\xi)} - c_{3}^{2} \partial_{i_{1}} \frac{\partial L}{\partial \partial_{i_{1}} \partial_{0}^{2} \Phi_{a}(\xi)} \partial_{0} \Phi_{a}(\xi) + \cdots + (-1)^{N-2} C_{N}^{2} \partial_{i_{1}} \cdots \partial_{i_{N-2}} \frac{\partial L}{\partial \partial_{i_{1}} \cdots \partial_{i_{N-2}} \partial_{0}^{2} \Phi_{a}(\xi)},
\]
\[\ldots\]
\[
\frac{\delta \Lambda}{\delta \phi^{N-1}_{a}(\xi)} = \frac{\partial L}{\partial \phi^{N-1}_{a}(\xi)} - C_{N}^{N-1}\frac{\partial L}{\partial \phi^{N-1}_{a}(\xi)}, \\
\frac{\delta \Lambda}{\delta \phi_{a}(\xi)} = \frac{\partial L}{\partial \phi_{a}(\xi)}.
\] (9)

Consider the difference between actions of two possible movements close to each other.

\[
\delta A = \int_{t_{0}}^{t} d\xi^{0} \Delta \Lambda(\xi, \partial_{0}\Phi_{e^{0}}, \ldots, \partial_{0}^{N}\Phi_{e^{0}}) = \int_{t_{0}}^{t} d\xi^{0} \int d^{3}\xi L(\Phi(\xi), \partial\Phi(\xi), \ldots, \partial^{N}\Phi(\xi))
\]

\[
= \int_{t_{0}}^{t} d\xi^{0} \int d^{3}\xi \left[ \frac{\delta \Lambda}{\delta \phi_{a}(\xi)} \delta \Phi_{a}(\xi) + \frac{\delta \Lambda}{\delta \phi_{a}(\xi)} \delta \Phi_{a}(\xi) + \frac{\delta \Lambda}{\delta \phi_{a}(\xi)} \delta \Phi_{a}(\xi) + \cdots \right] + \int_{t_{0}}^{t} d\xi^{0} \int d^{3}\xi \partial_{1}[K^{a_{1}} \delta \Phi_{a}(\xi) + K^{a_{1}k_{2}} \delta \partial_{k_{2}} \Phi_{a}(\xi) + \cdots] + \cdots
\]

\[
= \int_{t_{0}}^{t} d\xi^{0} \int d^{3}\xi \frac{\delta \Lambda}{\delta \phi_{a}(\xi)} - \partial_{0} \frac{\delta \Lambda}{\delta \phi_{a}(\xi)} + \partial_{0}^{2} \frac{\delta \Lambda}{\delta \phi_{a}(\xi)} + \cdots \]

\[
+ (-1)^{N} \partial_{0}^{N} \frac{\delta \Lambda}{\delta \phi_{a}(\xi)} \delta \phi_{a}(\xi) + \int_{t_{0}}^{t} d\xi^{0} \int d^{3}\xi \partial_{1} \left[ B^{a\lambda_{1}} \delta \Phi_{a}(\xi) + B^{a\lambda_{1}} \delta \partial_{\lambda_{1}} \Phi_{a}(\xi) + \cdots \right]
\]

Using the least action principle, one re-obtains the **Euler-Lagrange equation**

\[
\frac{\delta \Lambda}{\delta \phi_{a}(\xi)} - \partial_{0} \frac{\delta \Lambda}{\delta \phi_{a}(\xi)} + \partial_{0}^{2} \frac{\delta \Lambda}{\delta \phi_{a}(\xi)} + \cdots + (-1)^{N} \partial_{0}^{N} \frac{\delta \Lambda}{\delta \phi_{a}(\xi)} = 0 \tag{11}
\]

Noting eqn.(9), one easily sees that eqns.(11) and (5) are exactly the same.

Let

\[
H = \int d^{3}\xi \left[ \pi_{1}^{a}(\xi) \partial_{0} \Phi_{a}(\xi) + \pi_{2}^{a}(\xi) \partial_{0}^{2} \Phi_{a}(\xi) + \cdots + \pi_{N}^{a}(\xi) \partial_{0}^{N} \Phi_{a}(\xi) \right] - \Delta \tag{12}
\]
One easily gets

$$
\delta H = \int d^3\xi [\delta_0 \Phi_a(\xi) \delta \pi_1^a(\xi) + \partial_0^2 \Phi_a(\xi) \delta \pi_2^a(\xi) + \cdots + \partial_0^N \Phi_a(\xi) \delta \pi_N^a(\xi) - \frac{\delta \Lambda}{\delta \Phi_a(\xi)} \delta \Phi_a(\xi)]
$$

$$
- \int d\sigma \{ [K^a_{\xi} \delta \Phi_a(\xi) + K^a_{\xi \xi} \delta \partial_0 \Phi_a(\xi) + K^a_{\xi \xi \xi} \delta \partial_0^2 \Phi_a(\xi) + \cdots + K^a_{\xi \cdots \xi N} \delta \partial_0^N \Phi_a(\xi)]
+ \{ K^a_{\xi \xi} \partial_0^2 \Phi_a(\xi) + K^a_{\xi \xi \xi} \partial_0^3 \Phi_a(\xi) + K^a_{\xi \cdots \xi N} \partial_0^N \Phi_a(\xi) \}
+ \cdots + [K^a_{\xi \cdots \xi N} \delta \partial_0^N \Phi_a(\xi) + K^a_{\xi \cdots \xi N} \delta \partial_0^N \Phi_a(\xi)]
+ K^a_{\xi \cdots \xi N} \delta \partial_0^N \Phi_a(\xi) \}
\}.
$$

(13)

This suggests that the Hamiltonian \( H \) is a functional of \( \{ \Phi_a, \pi_1^a, \ldots, \pi_N^a \} \)

$$
H = H[\xi^0, \Phi, \pi_1, \pi_2, \ldots, \pi_N],
$$

(14)

$$
\frac{\delta H}{\delta \Phi_a(\xi)} = - \frac{\delta \Lambda}{\delta \Phi_a(\xi)}, \quad
\frac{\delta H}{\delta \pi_1^a(\xi)} = \partial_0^0 \Phi_a(\xi), \quad
\frac{\delta H}{\delta \pi_2^a(\xi)} = \partial_0^2 \Phi_a(\xi), \quad \ldots, \quad
\frac{\delta H}{\delta \pi_N^a(\xi)} = \partial_0^N \Phi_a(\xi),
$$

(15)

and

$$
\Lambda = \int d^3\xi \frac{\delta H}{\delta \pi_1^a(\xi)} \pi_1^a(\xi) + \frac{\delta H}{\delta \pi_2^a(\xi)} \pi_2^a(\xi) + \cdots + \frac{\delta H}{\delta \pi_N^a(\xi)} \pi_N^a(\xi) - H
$$

(16)

From the Euler-Lagrange equation (11), one gets

$$
\partial_0 \pi_1^a(\xi) - \partial_0^2 \pi_2^a(\xi) - \cdots - (-1)^N \partial_0^N \pi_N^a(\xi) = - \frac{\delta H}{\delta \Phi_a(\xi)}.
$$

(17)

Eqn.(17) and eqn.(18)

$$
\partial_0 \Phi_a(\xi) = \frac{\delta H}{\delta \pi_1^a(\xi)} \partial_0^0 \Phi_a(\xi) = \frac{\delta H}{\delta \pi_2^a(\xi)} \partial_0^2 \Phi_a(\xi) \ldots \partial_0^N \Phi_a(\xi) = \frac{\delta H}{\delta \pi_N^a(\xi)},
$$

(18)

constitute the canonical equations. Note that when \( N = 1 \) (all pre-G.R. field theories belong to this case), canonical equations (17), (18) read

$$
\partial_0 \pi_1^a(\xi) = - \frac{\delta H}{\delta \Phi_a(\xi)},
$$

$$
\partial_0 \Phi_a(\xi) = \frac{\delta H}{\delta \pi_1^a(\xi)}.
$$

(19)
And one has

\[
\frac{d}{d\xi^0}H = \partial_0 H. \tag{20}
\]

When \(N = 2\) (G.R. is this case), the canonical equations read

\[
\begin{align*}
\partial_0 \pi_1^a(\xi) - \partial_0^2 \pi_2^a(\xi) &= -\frac{\delta H}{\delta \Phi_a(\xi)}, \\
\partial_0 \Phi_a(\xi) &= \frac{\delta H}{\delta \pi_1^a(\xi)}, \\
\partial_0^2 \Phi_a(\xi) &= \frac{\delta H}{\delta \pi_2^a(\xi)}. \tag{21}
\end{align*}
\]

And one has

\[
\frac{d}{d\xi^0}\{H - \int d^3\xi [\partial_0 \Phi_a(\xi) \partial_0 \pi_2^a(\xi)]\} = \partial_0 H. \tag{22}
\]

4 Noether’s theorem

4.1 Proof of Noether’s theorem for Lagrangians containing up to \(N\)-th derivatives of field

Now we have to deal with two kinds of derivatives of \(L(x, \Phi(x), \partial \Phi(x), \partial^2 \Phi(x), \ldots, \partial^N \Phi(x))\) with respect to coordinates, \(\partial_{\sigma} L = \partial L/\partial x^\sigma\) and \(\partial_{\sigma} L = \partial L/\partial x^\sigma\), relating to each other through the following equation,

\[
\begin{align*}
\frac{\partial L}{\partial x^\sigma} &= \frac{\partial L}{\partial x^\sigma} + \frac{\partial L}{\partial \Phi_a(x)} \partial_{\sigma} \Phi_a(x) + \frac{\partial L}{\partial \partial_{\lambda_1} \Phi_a(x)} \partial_{\sigma} \partial_{\lambda_1} \Phi_a(x) \\
&\quad + \cdots + \frac{\partial L}{\partial \partial_{\lambda_1} \cdots \partial_{\lambda_N} \Phi_a(x)} \partial_{\sigma} \partial_{\lambda_1} \cdots \partial_{\lambda_N} \Phi_a(x) \tag{23}
\end{align*}
\]

**Theorem 1** If the action of classical fields over every spacetime region \(\Omega\) remains unchanged under the following \(r\)-parameter family of infinitesimal transformation of coordinates and fields

\[
x^\lambda \mapsto \tilde{x}^\lambda = x^\lambda + \delta x^\lambda, \\
\Phi_a(x) \mapsto \tilde{\Phi}_a(x) = \Phi_a(x) + \delta \Phi_a(x), \tag{24}
\]

then there exist \(r\) conserved quantities.

**Proof.** From eqn. (24) one has
\[
\delta d^4 x = (\partial_\sigma \delta x^\sigma) d^4 x, \\
\delta \partial_\lambda = - (\partial_\lambda \delta x^\sigma) \partial_\sigma,
\]
\[
\delta[\partial_{\lambda_1} \Phi_a(x)] = (\delta \partial_{\lambda_1}) \Phi_a(x) + \partial_{\lambda_1} \delta \Phi_a(x) = \partial_{\lambda_1} \delta \Phi_a(x) - \partial_\sigma \Phi_a(x) \partial_{\lambda_1} \delta x^\sigma,
\]
\[
\delta[\partial_{\lambda_1} \partial_{\lambda_2} \Phi_a(x)] = \partial_{\lambda_1} \partial_{\lambda_2} \delta \Phi_a(x) - \partial_{\lambda_1} \partial_\sigma \Phi_a(x) \partial_{\lambda_2} \delta x^\sigma - \partial_{\lambda_1} \partial_\sigma \Phi_a(x) \partial_{\lambda_2} \delta x^\sigma
\]
\[
- \partial_\sigma \Phi_a(x) \partial_{\lambda_1} \partial_{\lambda_2} \delta x^\sigma,
\]
\[
\ldots
\]
\[
\delta[\partial_{\lambda_1} \partial_{\lambda_2} \cdots \partial_{\lambda_N} \Phi_a(x)] = \partial_{\lambda_1} \partial_{\lambda_2} \cdots \partial_{\lambda_N} \delta \Phi_a(x) - \sum_{1 \leq i \leq N} \partial_\sigma \partial_{\lambda_1} \cdots \partial_{\lambda_i} \cdots \partial_{\lambda_N} \Phi_a(x) \partial_{\lambda_i} \delta x^\sigma
\]
\[
- \sum_{1 \leq i < j \leq N} \partial_\sigma \partial_{\lambda_1} \cdots \partial_{\lambda_j} \cdots \partial_{\lambda_N} \Phi_a(x) \partial_{\lambda_i} \partial_{\lambda_j} \delta x^\sigma
\]
\[
- \sum_{k \leq j \leq N} \partial_\sigma \partial_{\lambda_1} \cdots \partial_{\lambda_k} \partial_{\lambda_i} \cdots \partial_{\lambda_N} \Phi_a(x) \partial_{\lambda_i} \partial_{\lambda_k} \delta x^\sigma
\]
\[
- \delta_\sigma \Phi_a(x) \partial_{\lambda_1} \cdots \partial_{\lambda_N} \delta x^\sigma
\]

Substitute eqns. (24) and (25) into the following equation:
\[
\delta A = \int_\Omega (\delta d^4 x) L + \int_\Omega d^4 x \left[ \frac{\partial L}{\partial \Phi_a(x)} \delta \Phi_a(x) + \frac{\partial L}{\partial \partial_{\lambda_1} \Phi_a(x)} \partial_{\lambda_1} \delta \Phi_a(x) \right] + \ldots + \left[ \frac{\partial L}{\partial \partial_{\lambda_1} \partial_{\lambda_2} \cdots \partial_{\lambda_N} \Phi_a(x)} \partial_{\lambda_1} \partial_{\lambda_2} \cdots \partial_{\lambda_N} \delta \Phi_a(x) \right]
\]

one gets
\[
\delta A = \int_\Omega d^4 x \left[ \frac{\partial L}{\partial \Phi_a(x)} - \partial_\lambda \frac{\partial L}{\partial \partial_{\lambda_1} \Phi_a(x)} \right] \delta \Phi_a(x)
\]
\[
\cdot (\delta \Phi_a(x) - \partial_\sigma \Phi_a(x) \delta x^\sigma) + \int_\Omega d^4 x \partial_\lambda \left[ L \delta \Phi_a(x) - B^{\lambda \sigma} \delta x^\sigma + B^{\lambda \mu_1 \mu_2} \partial_{\mu_1} \partial_{\mu_2} \delta \Phi_a(x) - \partial_\sigma \Phi_a(x) \delta x^\sigma \right]
\]
\[
+ \sum_{\mu_1 \mu_2 \mu_N} B^{\lambda \mu_1 \cdots \mu_N} \partial_{\mu_1} \cdots \partial_{\mu_N} (\delta \Phi_a(x) - \partial_\sigma \Phi_a(x) \delta x^\sigma)
\]

The first integral at rhs vanishes for real movement, hence the second integral does too. One gets the following equation due to the arbitrariness of \(\Omega\).
\[
0 = \delta_\lambda \left[ L \delta \Phi_a(x) + B^{\lambda \sigma} \delta \Phi_a(x) - \partial_\sigma \Phi_a(x) \delta x^\sigma \right] + B^{\lambda \mu_1 \mu_2} \partial_{\mu_1} \partial_{\mu_2} \delta \Phi_a(x) - \partial_\sigma \Phi_a(x) \delta x^\sigma
\]
\[
+ \sum_{\mu_1 \mu_2 \mu_N} B^{\lambda \mu_1 \cdots \mu_N} \partial_{\mu_1} \cdots \partial_{\mu_N} \delta \Phi_a(x) - \partial_\sigma \Phi_a(x) \delta x^\sigma
\]

Noting that both \(\delta x^\sigma\) and \(\delta \Phi_a(x)\) depend on \(r\) real parameters, one can consider eqn. (27) as a conservation laws. ■
4.2 "Conservation law due to "coordinate shift" invariance

In this subsection, we restrict our discussion to Lagrangians which do not manifestly contain coordinates and is invariant under "coordinate shift". In this case, the action (1) remains unchanged under the following "coordinate shift".

$$\delta x^\sigma = \epsilon^\sigma, \delta \Phi_a(x) = 0.$$ (28)

In this case, eqn.(27) reads

$$\partial_\lambda \tau^\lambda_\sigma = 0,$$ (29)

where

$$\tau^\lambda_\sigma = B^{a\lambda} \partial_\sigma \Phi_a(x) + B^{a\lambda \nu_1} \partial_{\nu_1} \partial_\sigma \Phi_a(x) + B^{a\lambda \nu_1 \nu_2} \partial_{\nu_1} \partial_{\nu_2} \partial_\sigma \Phi_a(x)$$

$$+ \cdots + B^{a\lambda \nu_1 \cdots \nu_{N-1}} \partial_{\nu_1} \cdots \partial_{\nu_{N-1}} \partial_\sigma \Phi_a(x) - L \delta^\lambda_\sigma (30)$$

is usually called energy-momentum tensor. Notice that "coordinate shift" eqn.(28) is not an invariant concept under general coordinate transformation. This is easily seen from the active viewpoint of transformation. This explains why $\tau^\lambda_\sigma$ in not a tensor under general coordinate transformation. We will get back to this problem later.

5 Hamilton’s principal functional and Hamilton-Jacobi’s equation

Let us consider the difference between actions over spacetime region $\Omega$ of two real movements close to each other. Using eqns.(2) and (5), one gets, for real movements

$$\delta A[\Phi] = \int_{\partial \Omega} ds \left[ B^{a\lambda} \delta \Phi_a(x) + B^{a\lambda \nu_1} \delta \partial_\nu_1 \Phi_a(x) + B^{a\lambda \nu_1 \nu_2} \delta \partial_\nu_1 \partial_\nu_2 \Phi_a(x)$$

$$+ \cdots + B^{a\lambda \nu_1 \cdots \nu_{N-1}} \delta \partial_{\nu_1} \cdots \partial_{\nu_{N-1}} \Phi_a(x) \right] (31)$$

From eqn.(31), one sees that the action over a spacetime region $\Omega$ of a real movement is determined by the closed hyper-surface $\partial \Omega$, and $\Phi|_{\partial \Omega}, \partial \Phi|_{\partial \Omega}, \ldots, \partial^{N-1} \Phi|_{\partial \Omega}$. It will be called the generalized Hamilton’s principal functional and denoted by

$$S = S[\partial \Omega, \Phi|_{\partial \Omega}, \partial \Phi|_{\partial \Omega}, \ldots, \partial^{N-1} \Phi|_{\partial \Omega}] (32)$$

Re-write eqn.(31) as
\[ \delta S = \int_{\partial \Omega} ds_\lambda \left[ B^{a\lambda \nu_1} \delta \Phi_\alpha(x) + B^{a\lambda \nu_1 \nu_2} \delta \partial_{\nu_1} \Phi_\alpha(x) + \cdots + B^{a\lambda \nu_1 \cdots \nu_{N-1}} \delta \partial_{\nu_1} \cdots \partial_{\nu_{N-1}} \Phi_\alpha(x) \right] \]

Note that when \( \Phi_\alpha|_{\partial \Omega} \) is given, only one of the four derivatives \( \partial_\lambda \Phi_\alpha|_{\partial \Omega} (\lambda = 0, 1, 2, 3) \) is independent; when \( \partial_\lambda \Phi_\alpha|_{\partial \Omega} \) is given, only one of the four derivatives \( \partial_\mu \partial_\lambda \Phi_\alpha|_{\partial \Omega} (\mu = 0, 1, 2, 3) \) is independent; and so on. Thus for a given suffix \( a \), only \( N \) items from \( \Phi_\alpha|_{\partial \Omega}, \partial_\lambda \Phi_\alpha|_{\partial \Omega}, \ldots, \partial_\lambda, \ldots, \partial_{\lambda_{N-1}} \Phi_\alpha|_{\partial \Omega} \) \( (\lambda_j = 0, 1, 2, 3) \) are independent.

In order to formulate the generalized Hamilton-Jacobi’s equation, one needs a new type of functional derivative.

**Definition 2** Let \( \Sigma \) be a hypersurface in spacetime \( M \), \( \Psi \) a function defined on \( M \), and \( F = F[\Sigma, \Psi|_\Sigma] \) a functional of \( \Sigma \) and \( \Psi|_\Sigma \). The functional derivatives are defined as follows. If the variation of \( F \) can be written as

\[ \delta F[\Sigma, \Psi|_\Sigma] = \int_\Sigma ds_\lambda \left\{ Y[\Sigma, \Psi|_\Sigma, x]^{\lambda}_\mu \delta \Sigma^\mu(x) + Z[\Sigma, \Psi|_\Sigma, x]^{\lambda} \delta \Psi(x) \right\} \]  

then \( Y[\Sigma, \Psi|_\Sigma, x]^{\lambda}_\mu \) and \( Z[\Sigma, \Psi|_\Sigma, x]^{\lambda} \) are called the functional derivative of \( F \) with respect to \( \Sigma^\mu(x) \) and \( \Psi(x) \), and denoted by

\[ Y[\Sigma, \Psi|_\Sigma, x]^{\lambda}_\mu = \left( \frac{\delta F}{\delta \Sigma^\mu(x)} \right)^\lambda, \quad Z[\Sigma, \Psi|_\Sigma, x]^{\lambda} = \left( \frac{\delta F}{\delta \Psi(x)} \right)^\lambda \]

respectively.

Hence we have

\[ \delta F[\Sigma, \Psi|_\Sigma] = \int_\Sigma ds_\lambda \left[ \left( \frac{\delta F}{\delta \Sigma^\mu(x)} \right)^\lambda \delta \Sigma^\mu(x) + \left( \frac{\delta F}{\delta \Psi(x)} \right)^\lambda \delta \Psi(x) \right] \]

The hypersurface \( \Sigma \) is given by the parameter equation

\[ x^\mu = \Sigma^\mu(\theta^1, \theta^2, \theta^3) \]

The \( \delta \Sigma^\mu(x) \) in eqn.(33) is

\[ \delta \Sigma^\mu(x) = \bar{\Sigma}^\mu(\theta^1, \theta^2, \theta^3) - \Sigma^\mu(\theta^1, \theta^2, \theta^3). \]

Now, from eqn.(33) we have

\[ \left( \frac{\delta S}{\delta \Phi_\alpha(x)} \right)^\lambda = B^{a\lambda \nu_1}, \left( \frac{\delta S}{\delta \partial_{\nu_1} \Phi_\alpha(x)} \right)^\lambda = B^{a\lambda \nu_1 \nu_2}, \ldots, \left( \frac{\delta S}{\delta \partial_{\nu_1} \cdots \partial_{\nu_{N-1}} \Phi_\alpha(x)} \right)^\lambda = B^{a\lambda \nu_1 \cdots \nu_{N-1}}. \]

Follow the evolution of one real movement and observe the change of its action.
\[ \delta S = \int_{\partial \Omega} ds \lambda \delta \Sigma^\sigma \delta \Sigma (x) \]
\[ = \int_{\partial \Omega} ds \lambda \left( \frac{\delta S}{\delta \Sigma^\sigma (x)} \right) \delta \Sigma^\sigma (x) + \left( \frac{\delta S}{\delta \Phi_a (x)} \right) \partial_\sigma \Phi_a(x) \delta \Sigma^\sigma (x) \]
\[ + \left( \frac{\delta S}{\delta \partial_{v_1} \Phi_a (x)} \right) \partial_\sigma \partial_{v_1} \Phi_a(x) \delta \Sigma^\sigma (x) + \cdots \]
\[ + \left( \frac{\delta S}{\delta \partial_{v_1} \cdots \partial_{v_{N-1}} \Phi_a (x)} \right) \partial_\sigma \partial_{v_1} \cdots \partial_{v_{N-1}} \Phi_a(x) \delta \Sigma^\sigma (x) \]  
(40)

From eqns. (30), (39) and (40), we get the generalized Hamilton-Jacobi’s equation.

\[ \left( \frac{\delta S}{\delta \Sigma^\sigma (x)} \right) \lambda + \tau_\sigma^\lambda = 0. \]  
(41)

**Remark 3** So far we have presented a general variational principle for classical fields. The only postulate made in this formalism is the least action principle. This formalism applies to all the classical fields \( \{ \Phi_a (x) \} \) with a Lagrangian \( L(x, \Phi(x), \partial \Phi(x), \partial^2 \Phi(x), \ldots, \partial^N \Phi(x)) \), say, Newtonian fluid mechanics, Maxwell’s electromagnetic field, general relativity, etc. The specific symmetries and covariance of a classical field are the heritage from the Lagrangian, not from this formalism. This formalism yields manifestly Galilean (Lorentzian, general) covariant field theory when the inputted Lagrangian is Galilean (Lorentzian, general) covariant. It is worth noting that all the results obtained above, are in great harmony with each other. We will apply this general variational principle to general relativity, especially apply the generalized Noether’s theorem to the long standing problem, conservation and non-conservation in curved spacetime in part II and part III.

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**References**

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