Do Majorana Fermions really ObeY Non-Abelian Statistics?

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Recently, Majorana fermions (MFs) have attracted intensive attention because of their possible non-Abelian statistics. This paper points out MFs in topological superconductors neither obey non-Abelian statistics nor Abelian statistics. Instead, MFs obey $\gamma$-statistics that is an anti-commuting representation of braiding group. “Smoking-gun” numerical evidence to identify MF’s statistics is presented. The implications of this work are given for topological quantum computation of MFs in topological superconductors.

Majorana fermions (MFs) $\gamma_i$ are their own antiparticle and constitute ‘half’ of ordinary fermions \[1–3\] that satisfies $\gamma_i = \gamma_i^\dagger$ and $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. It is still unclear whether MFs exist in nature as elementary building blocks, but in condensed matter systems, MFs may appear as Majorana bound states (MBSs)\[4,13\]. Recently, due to their potential applications in topological quantum computation (TQC)\[13,21,22,23,26\], the search for exotic states supporting MFs has attracted increasing interest in condensed matter physics. Possible example of such quantum exotic states is the $p_x + ip_y$ topological superconductor (SC). The quantized vortex ($\pi$-flux) in two dimensional (2D) $p_x + ip_y$ topological SCs trap MBSs\[24,27\]. Another creative proposal is the interface of topological superconductors. The quantized vortex ($\pi$-flux) in two dimensional (2D) $p_x + ip_y$ topological SCs trap MBSs\[24,27\].

In Ref.\[24\], it was pointed out that the MFs trapped by vortices in $p_x + ip_y$-wave topological SC obey non-Abelian statistics\[29,30\]. After exchanging two MFs in the 2D topological SC, the braiding operation can be represented by $\gamma_1 \rightarrow \gamma_2, \gamma_2 \rightarrow -\gamma_1 \gamma_2$. For the MFs at the ends of line-defect in SCs, the braiding operations were illustrated along $T$-junction paths\[11\]. Based on the arguments in Ref.\[11\], the line-defect-induced MFs were found to obey non-Abelian statistics. As a result, the MFs in different models (both vortex-induced MFs and the line-defect-induced MFs) were believed to obey the same type of non-Abelian statistics.

In this paper, we will show quite different results on the statistics of MFs: MFs neither obey non-Abelian statistics nor Abelian statistics. Instead, MFs in (topological) SCs obey a new type statistics - $\gamma$-statistics. We then propose an direct numerical approach to identify the statistics of MFs in topological SCs. The results will be helpful to learn the properties of MFs and topological states and sheds light on the correct path forward in TQC.

In quantum mechanics, there are two possibilities of the many-body wave-function to change by a $\pm$ sign under a single particle-interchange (the so-called braiding operation in 2D systems), corresponding to the cases of bosons and fermions, respectively. Then after braiding two identical particles, the initial many-body wave-function $|\psi_i\rangle = a_1^\dagger(r_1)a_2^\dagger(r_2)|0\rangle$ changes by a phase to be the final state $|\psi_f\rangle = e^{i\theta}a_2^\dagger(r_2)a_1^\dagger(r_1)|0\rangle$. $\theta = 0, \pi$ correspond to bosons and fermions, respectively.

In certain topological SCs, MFs with zero energy (the Majorana zero modes) may emerge around topological defects (for example, the quantized vortex or the 1D nanowire). In general, people can obtain the function $|\gamma\rangle$ of an emergent Majorana zero mode by solving the Bogolubov-de Gennes (BdG) equations numerically. To describe the Majorana zero mode, a real fermion field called Majorana fermion $\gamma_1 = \int dr |u_0 \psi^* + v_0 \psi\rangle (\gamma_1^\dagger = \gamma)$ is introduced. To describe the subspace of the system with two degenerate modes ($|\gamma_1\rangle, |\gamma_2\rangle$) (or two MFs ($\gamma_1, \gamma_2$)), we introduce the Fermion-parity operator $\hat{P} = (-i\gamma_1\gamma_2)$. Since $\hat{P}^2 = 1$, $\hat{P}$ has two eigenvalues $\pm 1$, called even and odd Fermion-parities, respectively. In a 2D gapped SC, the quantum eigen-states including Majorana modes must have a determinant parity. Then, we label a pair of MFs ($\gamma_1, \gamma_2$) by a complex fermion as $\gamma_1 = c + c^\dagger, \gamma_2 = -i(c - c^\dagger)$ where $c^\dagger/c$ is the creation/annihilation operator of spinless fermion. With the help of $c$ and $c^\dagger$, two Majorana (zero) modes ($|\gamma_1\rangle, |\gamma_2\rangle$) form the physical Fermion-parity qubit: $|0\rangle = (|\gamma_1\rangle + i|\gamma_2\rangle)/\sqrt{2}$ is a quantum state with even Fermion-parity and $c^\dagger|0\rangle = |1\rangle = (|\gamma_1\rangle - i|\gamma_2\rangle)/\sqrt{2}$ is a quantum state with odd Fermion-parity. Generally, there exists the coupling between two MFs and the effective Hamiltonian is given by $i\mathcal{J}\gamma_1\gamma_2 = 2\mathcal{J}(c^\dagger c - \frac{1}{2})$ where $\mathcal{J}$ is the coupling constant.

To distinguish the statistics of MFs, we study their braiding operations. MFs could be braided by adiabatically deforming the system, for example, changing the length of a 1D nano-wire or moving the vortices. In particular, during braiding operations, the adiabatic evolutions of the degenerate energy levels (Majorana zero modes, $|\gamma_1\rangle, |\gamma_2\rangle$) lead to a nontrivial change. Owning

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to the conservation rule of Fermion-parities, the transition between Majorana states \(|0\rangle\) and \(|1\rangle\) is forbidden. Then the final functions of the Majorana modes to which the system returns after the braiding is identical to the initial one, up to a phase,

\[
|\psi_i\rangle = \begin{cases} |1\rangle & \text{if } i = 1 \\ |0\rangle & \text{if } i = 0 \end{cases} \rightarrow |\psi'_i\rangle = \begin{cases} e^{i(\varphi_{1D} + \psi_{in})}|1\rangle & \text{if } i = 1 \\ e^{-i(\varphi_{0D} + \psi_{on})}|0\rangle & \text{if } i = 0 \end{cases}
\]

where the Berry phases \(\varphi_{1B}, -\varphi_{0B}\) don’t depend on how long the process takes and the dynamical phase \(\varphi_{1D} = -\varphi_{0D} = \int_{t_f}^{t_0} \mathcal{J}(t)dt\) depends on the energy of the MBSS \(\mathcal{J}(t)\) and the length of time for the process \(t_f - t_0\).

Let us derive their braiding operators for MFs. When we consider the initial states \(|\psi_i\rangle\) to be \(|\gamma_{1i}\rangle\), after the braiding operation, \(2 \leftrightarrow 1\), the final states \(|\psi'_i\rangle\) turn into \(e^{i\phi_1}|\gamma_{2i}\rangle\), where \(e^{i\phi_1}\) and \(e^{i\phi_2}\) are phase factors to be solved. On the other hand, when we consider the initial states \(|\psi_i\rangle\) to be \(|\frac{1}{\sqrt{2}}\rangle\), after the braiding operation, \(2 \leftrightarrow 1\), due to PH symmetry the final states \(|\psi'_i\rangle\) turn into \(e^{i\varphi}|\frac{1}{\sqrt{2}}\rangle\). Here \(\phi_1, \phi_2\) and \(\varphi\) are real numbers to be determined. From the definition of \(|\frac{1}{\sqrt{2}}\rangle = \frac{|\gamma_{1i}\rangle - e^{-i\pi/4}|\gamma_{2i}\rangle}{\sqrt{2}}\), we can obtain two solutions: one is \(\phi_1 = \pi, \phi_2 = 0, \varphi = -\frac{\pi}{2}\), the other is \(\phi_1 = 0, \phi_2 = \pi, \varphi = \frac{\pi}{2}\). In this paper, due to equivalence property, we focus on the first solution. As a result, on the basis of \(|\frac{1}{\sqrt{2}}\rangle\), the braiding operator becomes \(R = e^{-i\sigma_z\pi/2}\). If we consider the initial states to be \(|\gamma_{1i}\rangle\), after the braiding operation, the final states are \(|\gamma_{2i}\rangle\).

For this type of statistics, the braiding operation is then represented by

\[
\gamma_i \rightarrow -\gamma_i, \quad \gamma_j \rightarrow -\gamma_j
\]

Such braiding operation can also be described by an "unitary" transformation \(U_{ij} = e^{\pi i \gamma_{ij}} = \gamma_i \gamma_j\) that transforms the operators \(\gamma_k\) of MFs according to \(\gamma_k \rightarrow U_{ij} \gamma_k U_{ij}^\dagger\). For three MFs \(\gamma_1, \gamma_2, \gamma_3\), we found an anti-commutating relation as

\[
\{U_{ij}, U_{jk}\} = 0.
\]

As a result, we call this type of statistics with anti-commuting relation as \(\gamma\)-statistics. In particular, \(\gamma\)-statistics is new type of quantum statistics that is different from Abelian statistics with a commutating relation \([U_{ij}, U_{jk}] = 0\) and non-Abelian statistics with \([U_{ij}, U_{jk}] \neq 0\) and \(\{U_{ij}, U_{jk}\} \neq 0\). We may regard \(U_{ij} = \gamma_i \gamma_j\) as an anti-commutating representation of braiding group.

We consider \(2N\) MFs that composites \(2^N\) qubits, \((\gamma_{1a}, \gamma_{1b}), (\gamma_{2a}, \gamma_{2b}), \ldots (\gamma_{N,a}, \gamma_{N,b})\). Because the braiding operation can be represented by \(U_{ij} = \gamma_i \gamma_j\), an arbitrary braiding operation \(U\) on the Hilbert space of \(2^N\) qubits \(|\Gamma\rangle = (|1\rangle_1 \otimes |1\rangle_2 \otimes \ldots |1\rangle_N\) can be

\[
U |\Gamma\rangle = U_{ij} \cdot U_{ik} \cdot \ldots U_{mn} |\Gamma\rangle = \gamma_i \gamma_j \gamma_k \ldots \gamma_m \gamma_n |\Gamma\rangle
\]

(4)

where \(U_i = (\gamma_{ia} \gamma_{ib} \gamma_{jb})\) is an arbitrary braiding operator on qubit \(|\frac{1}{\sqrt{2}}\rangle\). Since the braiding operations \(U_i = (\gamma_{ia} \gamma_{ib} \gamma_{ib}) \) obey Heisenberg algebra (except for a phase factor \(e^{i\pi/2}\), \(\gamma_{ia}, \gamma_{ib} = 2\gamma_{ia} \gamma_{ib}, \{\gamma_i, \gamma_j\} = (\gamma_i \gamma_j + \gamma_j \gamma_i) = 0,\) \(\gamma_{ia} \gamma_{ib}, \gamma_{ib} \gamma_{ia}, \gamma_{ia} \gamma_{ia} = -1\), we could map the braiding operator \((\gamma_{ia} \gamma_{ib} \gamma_{ia} \gamma_{ib})\) to \((\tau_i^x, \tau_j^y, \tau_k^z)\) where \(\tau_i^{x,y,z}\) is the Pauli matrix.

To do universal quantum computation, people need to do the following four elementary gates: the Hadamard gate, the phase gate, the \(\pi/8\) gate and the Controlled NOT (CNOT) gate. However, based on braiding operations on MFs, people can only do X gate \((\gamma_{ia} \gamma_{ib})\) or \(Y\)-gate \((\gamma_{ib} \gamma_{ib})\) or \(Z\)-gate \((\gamma_{ia} \gamma_{ib} \gamma_{ia} \gamma_{ib})\). As a result, none of the four elementary gates can be done and it is no-go to do TQC based on MFs.

However, it had been pointed out that if we exchange two MFs, the resulting braiding operation is given by \(\gamma_i \rightarrow -\gamma_j, \gamma_j \rightarrow \gamma_i\) that can be described by an "unitary" transformation \(U_{ij} = e^{-i\pi \gamma_{ij}}[22]\). For three MFs \(\gamma_i, \gamma_j, \gamma_k\), due to \([U_{ij}, U_{jk}] \neq 0\) and \([U_{ij}, U_{jk}] \neq 0\), an adiabatic braiding operation obviously shows a non-Abelian character of the MFs. On the basis of \(|\frac{1}{\sqrt{2}}\rangle\), the resulting braiding operator is \(e^{-i\pi \gamma_{ij} / 4}\); on the basis of \(|\frac{1}{\sqrt{2}}\rangle\), the resulting braiding operator is \(\frac{1}{\sqrt{2}}(1, -1, -1)\).

Based on braiding operations on Ising anyons that obey non-Abelian statistics, people can do the Hadamard gate, the phase gate, and the CNOT gate topology except for the \(\pi/8\) gate[22, 31, 32].

A question arises, "whether emergent MFs in topological SCs obey \(\gamma\)-statistics or non-Abelian statistics?" To answer this question, based on two typical topological SCs, 1D \(p\)-wave topological SC and 2D \(p_x + ip_y\)-wave topological SC, we study statistics of emergent MFs in SCs by using numerical approach to simulate the braiding processes. To characterize the braiding process, we introduce two parameters, the amplitude of the function-overlap \(O = |\langle \psi_i | \psi_j \rangle|\) and the relative Berry phase \(\Phi = |\varphi_{1B} - \varphi_{0B}|\). If we get \(O = 1\) and \(\Phi = \pi\), MFs obey \(\gamma\)-statistics. If we get \(O = 1\) and \(\Phi = \pi/2\), MFs obey non-Abelian statistics. Therefore, we can distinguish the statistics for the MFs by calculating \(O\) and \(\Phi\).
The first model is 1D p-wave SC on T-junction, which consists of two parts, a “−”-line (A→C, of which the length is $L_1$) and a ”|”-line (D→B of which the length is $L_2$). See the illustration in Fig.1.(a). The pair order parameter on ”−”-line is $\Delta$ and the pair order parameter on ”|”-line is $\mu$. The Hamiltonian of the system can be written as $H = H_− + H_+ + H_D$ where

$$
H_− = -J \sum_{j=1}^{L_−-1} c_{j+1}^\dagger c_j + \Delta \sum_{j=1}^{L_−-1} c_{j+1}^\dagger c_j + h.c. - \sum_{j=1}^{L_−} \mu_j c_j^\dagger c_j,
$$

$$
H_+ = -J \sum_{j=1}^{L_+} a_{j+1}^\dagger a_j + \Delta \sum_{j=1}^{L_+} a_{j+1}^\dagger a_j + h.c. - \sum_{j=1}^{L_+} \mu_j a_j^\dagger a_j,
$$

$$
H_D = J a_{j=1}^\dagger c_{jD} + i \Delta a_{j=1}^\dagger c_{jD}^\dagger + h.c.
$$

(5)

where $c_j$ ($a_j$) is the annihilation operator of spinless fermions on ”−”-line and $jD$ denotes the touching point of ”|”-line on ”−”-line (D point in Fig.1.(a)). $J$ is the hopping strength, $\Delta$ is the SC pairing order parameter and $\mu$ is the on-site chemical potential, respectively. In the followings, we choose $\Delta = J$ and the lattice constant is set to be unit.

For the case of $|\mu| < 2t$, the ground state is a topological SC; for the case of $|\mu| > 2t$, the ground state is a non-topological SC. In Fig.1.(a), we denote the topological p-wave SC by blue line and non-topological p-wave SC by red line. As a result, there always exist a pair of MFs ($\gamma_1$, $\gamma_2$) at two ends of blue lines; (b) The energy gap of the system, $\Delta_E_j$. The inset is the energy $E_0$ of Majorana state $|0\rangle$; (c) The function-overlap $\mathcal{O} = |\langle \psi_1 | \psi_1 \rangle|$ of the Majorana modes during the adiabatic evolutions (the dotted line denotes 1); (d) The relative Berry phase $\Phi$ (the dotted lines denotes $\pi$). In all these figures, we choosing the parameters as $\Delta = J$.

For the case of $|\mu| < 2t$, the ground state is a topological SC; for the case of $|\mu| > 2t$, the ground state is a non-topological SC. In Fig.1.(a), we denote the topological p-wave SC by blue line and non-topological p-wave SC by red line. As a result, there always exist a pair of MFs ($\gamma_1$, $\gamma_2$) at two ends of topological p-wave SC (a blue line). For the initial state ($t = t_0$), we set $\mu = -0.7J$ on ”−”-line and $\mu = -10J$ on ”|”-line. Thus, the p-wave SC on ”−”-line is topological; while the p-wave SC on ”|”-line is non-topological. By numerical calculations on T-junction with $L_1 = 20$, $L_2 = 10$, we find that there are two Majorana zero modes ($|\gamma_1\rangle$, $|\gamma_2\rangle$) near A and C, respectively.

In the following parts we will show how to braid the two MFs ($\gamma_1$, $\gamma_2$) at two ends of ”−”-line. As shown in Fig.1.(a), we can move $\gamma_1$ away from A by tuning the on-site chemical potential on T-shape lattice. Thus, braiding two MFs ($\gamma_1$, $\gamma_2$) is a three-step process: step 1 is to move MF $\gamma_1$ from A to B through D as A→D→B during the time period $t_1 - t_0$; step 2 is to move $\gamma_2$ from C to A through D during the time period $t_2 - t_1$; step 3 is to move $\gamma_1$ from B to C through D during the time period $t_f - t_2$.

Then, we adiabatically braid the MFs step-by-step by choosing the T-shape path. During the braiding process, the minimum value of energy gap $\Delta E_j$ is about 0.9J that protected the topological properties of the system and the stability of the MFs and the maximum value of the energy $E_0 = J/2$ ($E_1 = -J/2$) of Majorana state $|0\rangle$ ($|1\rangle$) is about $10^{-5}J$ that may lead to a small dynamical phase.

After calculating $\mathcal{T}\{\exp[i\int_0^t H(t)dt] |\psi_3\rangle\}$, we derive the dynamical phase and Berry phase of the Majorana zero modes during the braiding process. Here, $\mathcal{T}$ is the time-ordered product operator. To guarantee the adiabatic condition, $\Delta t \gg \hbar/\Delta E_j$, the time period $\Delta t$ for moving an MF one lattice constant is very large, $\Delta t = 10000t/J$. The total time period for the braiding operation is $t_f - t_0 = 58\Delta t$ and $t_1 - t_0 = t_2 - t_1 = 19\Delta t$, $t_f - t_2 = 20\Delta t$.

The final results are given in Fig.1.(c) and Fig.1.(d). For the initial state $|\psi_1\rangle$ to be $|0\rangle$ (or $|1\rangle$), after the braiding operation, we get $\Phi = \pi$ that means the Berry phases $\varphi_{AB}$ ($\varphi_{0B}$) is $-\pi/2$ (or $\pi/2$). From Fig.1.(d), one can see that the relative Berry phase $\Phi$ changes abruptly during MF $\gamma_2$ crossing D point. The fidelity $|\langle \psi_1 | \psi_1 \rangle|$ is very close to 100%. As a result, we identify the $\gamma$-statistics of MFs in 1D p-wave topological SC system and exclude the possibility of non-Abelian statistics.

The second model is $p_x + ip_y$-wave SC, of which the Hamiltonian can be written as $\mathcal{H}[24]$

$$
H = -\sum_j \sum_{\hat{\mu} = \pm \hat{x}, \hat{y}} J_j (c_{j+\hat{\mu}}^\dagger c_j + c_{j-\hat{\mu}}^\dagger c_j)/2 - \mu \sum_j c_j^\dagger c_j
$$

$$
+ \sum_j [\Delta_j (c_{j+\hat{x}}^\dagger c_{j}^\dagger + ic_{j+\hat{y}} c_{j}^\dagger)/2 + h.c.],
$$

(6)

where $c_j^\dagger / c_j$ is the creation/annihilation operator of spinless fermion. $J_j$ is the hopping strength, $\Delta_j$ is the SC pairing order parameter and $\mu$ is the chemical potential, respectively. In the followings, we choosing the parameters as $|\Delta_j| = |J_j| = J$, $\mu = J$. Now the ground state is a topological SC with non-zero Chern number. The lattice constant is also set to be unit.

In the topological phase of $p_x + ip_y$-wave SC, we study two MFs ($\gamma_1$, $\gamma_2$) around two quantized vortices ($\pi$-
fluxes) by numerical calculations on a 20 × 20 lattice. The dotted line denotes the phase branch-cut of the two π-fluxes (we call it A-B-C π-phase string), along which the hopping parameters and the pairing order parameters change sign, \( J_j \to -J_j, \Delta_j \to -\Delta_j \). Thus, there exists a Majorana zero mode around each end of the π-phase string. See the particle density distribution of Majorana modes (|\( \gamma_1 \rangle, |\gamma_2 \rangle \}) around π-fluxes in Fig.2.(a).

To braid the two MFs (\( \gamma_1, \gamma_2 \)), we choose a □-shape path rather than traditional T-shape path. As shown in Fig.2.(b), we anticlockwise move two MFs (\( \gamma_1, \gamma_2 \)) through a two-step process: step 1 is to move MF \( \gamma_1 \) from A to B together with moving MF \( \gamma_2 \) from C to D during the time period \( t_1 - t_0 \); step 2 is to move MF \( \gamma_1 \) from B to C together with moving MF \( \gamma_2 \) from D to A during the time period \( t_f - t_1 \). However, after the two-step braiding process, the A-B-C π-phase string changes into an C-D-A π-phase string. Thus, to return the initial configuration, we have to deform the C-D-A π-phase string to A-B-C π-phase string by doing local Z2 transformation on the fermion fields inside the □-shape closed loop A-B-C-D, \( e_j \to -e_j \). Eventually, we adiabatically do the braiding operation.

Then we derive the Berry phases of the Majorana states |1⟩ and |0⟩ by calculating \( \mathcal{T}\{\exp\left[i\int_{t_0}^{t_f} H(t) dt \right]|1 \rangle \langle 0|\} \). In numerical calculations, the time period for moving an MF one lattice constant is \( \Delta t = 400\hbar/J \) (or the total time period for the braiding operation is \( t_f - t_0 = 20\Delta t \) and \( t_1 - t_0 = t_f - t_1 = 10\Delta t \)). During the braiding operation, the energy gap of the system is always \( \Delta E_f = 2J \) and the energy splitting between two energy levels of Majorana modes is extremely tiny, \( \max |\Delta \rangle < 10^{-2}J \). As a result, the dynamical phases of the Majorana states |1⟩ and |0⟩ are about \( \pm 10^{-5}\pi \) that are too small to cause errors to Berry phase calculations.

The results of the amplitude of the function-overlap \( \mathcal{O} = |\langle \psi_1 | \psi_0 \rangle| \) and those of the relative Berry phase \( \Phi = |\varphi_{1B} - \varphi_{0B}| \) are given in Fig.2.(c) and Fig.2.(d), respectively. From Fig.2.(d), one can see that the relative Berry phase is about \( \Phi = 0.9998\pi \) which really closes to \( \pi \). The fidelity \( \mathcal{O} = |\langle \psi_1 | \psi_0 \rangle| \) is up to 99.98%. As a result, we claim that MFs induced by the vortices in \( p_x + ip_y \) topological SC obey γ-statistics rather than non-Abelian statistics.

Finally, we draw the conclusion. In this paper we found that the emergent MFs in topological SCs obey γ-statistics, rather than the well-known non-Abelian statistics. The numerical results for MFs in 1D and 2D topological SCs confirm our prediction. As a result, this work pointed out that the braiding operations for MFs will never lead to nontrivial physical consequences and cannot be used to do TQC directly. Thus, to do TQC, the possible candidate may be non-Abelian anyons in topological orders.

\[ \text{FIG. 2: (color online) (a) The particle density distribution of Majorana modes (|\( \gamma_1 \rangle, |\gamma_2 \rangle \}) around two π-fluxes in } p_x + ip_y \text{ topological superconductor. The dotted line denotes A-B-C π-phase string that connects the two MFs; (b) The □-shape path of two-step braiding process: I→II (t_0 < t < t_1) and II→III (t_1 < t < t_f). III→IV denotes the string deformation; (c) The function-overlap } \mathcal{O} = |\langle \psi_1 | \psi_0 \rangle| \text{ of the Majorana modes during the adiabatic evolutions (the dotted line denotes 1); (d) The relative Berry phase } \Phi \text{ (the dotted lines denote π). In all these figures, we choosing the parameters as } \Delta = J, \mu = J \text{. The last spots of the data in (c) and (d) are obtained from string deformation rather than braiding operations.} \]

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