Neutron–proton spin–spin correlations in the ground states of $N = Z$ nuclei

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Abstract We present expressions for the matrix elements of the spin–spin operator $S_n \cdot S_p$ in a variety of coupling schemes. These results are then applied to calculate the expectation value $\langle S_n \cdot S_p \rangle$ in eigenstates of a schematic Hamiltonian describing neutrons and protons interacting in a single-$l$ shell through a Surface Delta Interaction. The model allows us to trace $\langle S_n \cdot S_p \rangle$ as a function of the competition between the isovector and isoscalar interaction strengths and the spin–orbit splitting of the $j = l \pm \frac{1}{2}$ shells. We find negative $\langle S_n \cdot S_p \rangle$ values in the ground state of all even–even $N = Z$ nuclei, contrary to what has been observed in hadronic inelastic scattering at medium energies. We discuss the possible origin of this discrepancy and indicate directions for future theoretical and experimental studies related to neutron–proton spin–spin correlations.

1 Introduction

The nuclear pairing mechanism [1] has been, for many years, a central subject of study in low-energy nuclear physics [2]. Although the energy gain of the nuclear system due to pairing is relatively modest, pairing correlations have a strong influence on many properties of the nucleus including the moment of inertia, deformation and excitation spectra [3]. The dominant pairing in almost all known nuclei with $N > Z$ is that in which ”superconducting” pairs of neutrons (nn) and protons (pp) couple to a state with angular momentum zero and isospin $T = 1$, known as isovector or spin-singlet pairing. However, for nuclei with $N \approx Z$, neutrons and protons occupy the same single-particle orbits at their respective Fermi surfaces and Cooper pairs, consisting of a neutron and a proton (np), may form. These types of pairs may couple in either isovector or isoscalar (spin-triplet with $J = 1$ and $T = 0$) modes, the latter being allowed by the Pauli principle. Contrary to the case of nuclei with large isospin imbalance, where the spin–orbit suppresses pairing in the triplet channel, in nuclei with $N = Z$ the isoscalar mode is expected to dominate. Since the nuclear force is charge independent, one would also expect that pairing should manifest equivalently for np pairs with $T = 1$ and $S = 0$, akin to nn and pp pairs. While there are convincing arguments for the existence of isovector np pairs, the existence of a correlated isoscalar np pair in condensate form, and the magnitude of such collective pairing, remains an intriguing and controversial topic in nuclear-structure physics [4].

Long-standing theoretical predictions of the onset of isoscalar pairing, the interplay between both pairing modes, and the presence of a condensate composed of both isoscalar and isovector pairs have remained without experimental confirmation [4–6]. This is mainly because the region of the nuclear landscape near the proton drip line, where such phenomena are expected to appear, is largely unreachable and because the experimental observables are either inconclusive and/or complicated to interpret. Two-neutron transfer reactions such as (p,t) and (t,p) have provided a key probe to understand neutron pairing correlations in nuclei [7,8]. The rapid quenching of np pairs as one moves away from $N = Z$ [9] suggests that the transfer of a np pair from even–even to odd–odd self-conjugate nuclei could also be a sensitive tool to study np correlations. Hence, reactions such as ($^3$He,p) and (p,$^3$He) are among the best choices [10,11].

A different and elegant approach has been proposed by the Osaka group [12,13] and consists of the study of neutron–proton spin–spin correlations in the ground states of $N \approx Z$ nuclei. The relevant observable is $(S_n \cdot S_p)$, the scalar product between the total spins of the neutrons and protons, which can be measured by spin $M1$ excitations produced by inelastic hadronic scattering at medium energies. In Fig. 1 we illustrate why this quantity can inform us on the nature of the pairing correlations.
condensate. It can be seen that given the distinctive values in
the two-particle system, \( \langle S_n \cdot S_p \rangle \) will also depend strongly
on the type of pairs being scattered across the Fermi surface,
as will be discussed in Sect. 3.

In a series of experiments carried out at the RCNP [13]
facility, high-energy-resolution proton inelastic scattering at
\( E_p = 295 \) MeV was studied in \(^{24}\)Mg, \(^{28}\)Si, \(^{32}\)S and \(^{36}\)Ar.
The results give positive values of \( \langle S_n \cdot S_p \rangle \approx 0.1 \) for the \( sd \)
shell suggesting a predominance of quasi-deuterons, somewhat
at variance with the discussions above and USD shell-
mmodel calculations that are unable to reproduce the experi-
mental results. However, shell-model wave functions that
take into account an enhanced spin-triplet pairing seem to
reproduce the measured spin–spin correlations [15]. Also,
the no-core shell model with realistic interactions [16] pre-
dicts positive values (lower limits due to convergence) that
could be attributed to mixing with higher-lying orbits due to
the tensor correlation.

It seems clear to us that further work is required to fully
assess the origin of the spin–spin correlation and its micro-
scopic origin. For example: Are the observed spin–spin corre-
lations between neutrons and protons connected to (a) our
beloved surface pairing BCS condensate [1], (b) aligned np
pairs [17] or (c) effects of the tensor force [18]? These are
questions that remain to be answered.

To shed light on these questions, we develop in this work
the formalism to calculate the matrix elements of the \( S_n \cdot S_p \)
operator in a variety of coupling schemes and apply it to
the solution of a schematic model consisting of nucleons in
a single-\( l \) shell. In spite of its simplicity, the model allows
us to study the behaviour of \( \langle S_n \cdot S_p \rangle \) as a function of the
competition between the isovector and isoscalar components
of the effective force between nucleons, and the spin–orbit
splitting of the \( j = l \pm \frac{1}{2} \) shells. In Sect. 2 we discuss the
structure of the \( S_n \cdot S_p \) operator and we calculate its matrix
elements in Sect. 3. In Sect. 4, following a short discussion
of the model, we present and discuss our results for several
cases involving particles occupying shells with \( l = 1–5 \) and
contrast these with the experimental observations to date.
Finally, Sect. 5 is devoted to the summary and conclusions
of our work.

2 The \( S_n \cdot S_p \) operator

The \( S_n \cdot S_p \) operator is given by

\[
S_n \cdot S_p = \sum_{k \in \{u\}} \sum_{k' \in \{p\}} s(k) \cdot s(k'),
\]

where the sums are over the neutrons and over the protons
in the nucleus. Introducing the isospin projection operator \( t_z \),
which gives \(+\frac{1}{2}\) acting on a neutron and \(-\frac{1}{2}\) acting on a
proton, we rewrite this operator as

\[
S_n \cdot S_p = \sum_{kk'} \left( \frac{1}{2} + t_z(k) \right) \left( \frac{1}{2} - t_z(k') \right) s(k) \cdot s(k')
= \frac{1}{4} \sum_{kk'} s(k) \cdot s(k') - \sum_{kk'} t_z(k) t_z(k') s(k) \cdot s(k'),
\]

where the sums are over all nucleons in the nucleus. It follows
that \( S_n \cdot S_p \) contains an isoscalar as well as an isotensor part.
Let us consider the case of nucleons occupying a single-l shell. We introduce the spin, isospin and spin–isospin operators

\[ T^{(010)}_{0\mu 0} = \sqrt{2(2l + 1)} \left( a^{\dagger}_l |l/2\rangle \times \tilde{a}_l |l/2\rangle \right)_{0\mu 0} , \]

\[ T^{(001)}_{00\nu} = \sqrt{2(2l + 1)} \left( a^{\dagger}_l |l/2\rangle \times \tilde{a}_l |l/2\rangle \right)_{00\nu} , \]

\[ T^{(011)}_{0\mu \nu} = \sqrt{2l + 1} \left( a^{\dagger}_l |l/2\rangle \times \tilde{a}_l |l/2\rangle \right)_{0\mu \nu} , \]

in terms of the nucleon creation operators \( a^{\dagger}_l |l/2\rangle \) and the modified annihilation operators \( \tilde{a}_l |l/2\rangle \). The operators (3) are scalar with respect to the orbital angular momentum and generate Wigner’s SU(4) supermultiplet algebra [19]. The representation (2) shows that \( S_n \cdot S_p \) can be written as

\[ S_n \cdot S_p = \frac{1}{4} \sum_{\mu} T^{(010)}_{0\mu 0} T^{(010)}_{0\mu 0} - \sum_{\mu} T^{(011)}_{0\mu \nu} T^{(011)}_{0\mu \nu} , \]

which proves that it is an element of the SU(4) algebra. The SU(4) tensor character of \( S_n \cdot S_p \) is derived in the Appendix.

### 3 Matrix elements of the \( S_n \cdot S_p \) operator

One-body matrix elements of \( S_n \cdot S_p \) vanish,

\[ \langle ml | S_n \cdot S_p | ml \rangle = 0. \]

Two-body matrix elements can be derived in LS or in \( jj \) coupling, and in both cases in an isospin or in a neutron–proton basis. Since \( S_n \cdot S_p \) is a scalar in orbital angular momentum, spin and total angular momentum, the associated projections \( M_L, M_S \) and \( M_J \) can be suppressed. It is, however, not a scalar in isospin and therefore its matrix elements depend on the projection \( M_T \). In an \( LS \) basis the two-body matrix elements are

\[ \langle l^2 LS M_T = 0 | S_n \cdot S_p | l^2 LS M_T = 0 \rangle = \frac{1}{4} [2S(S + 1) - 3], \]

\[ \langle l^2 LS M_T = \pm 1 | S_n \cdot S_p | l^2 LS M_T = \pm 1 \rangle = 0, \]

where it is assumed that \( L + S + T \) is odd. In a \( JT \) basis the two-body matrix elements are

\[ \langle l^2 JT M_T = 0 | S_n \cdot S_p | l^2 JT M_T = 0 \rangle = \frac{1}{2} \sum_{LS} S(S + 1) \left[ \begin{array}{cc} 1 \downarrow \frac{1}{2} \downarrow j \\ \uparrow \frac{1}{2} \downarrow j \\ L \ S \ J \end{array} \right] - \frac{3}{4}, \]

\[ \langle l^2 JT M_T = \pm 1 | S_n \cdot S_p | l^2 JT M_T = \pm 1 \rangle = 0, \]

where it is assumed that \( J + T \) is odd and that the sum runs over odd \( L + S + T \).

Equation (7) can be applied if the two nucleons are in the same \( l \) shell. If the nucleons occupy an \( l \) shell, matrix elements of \( S_n \cdot S_p \) are needed with one nucleon in the \( l + \frac{1}{2} \) and the other in the \( l - \frac{1}{2} \) shell. In this case it is more convenient to consider the problem in a neutron–proton basis. The expression for the two-body matrix elements of \( S_n \cdot S_p \) is particularly simple in an \( LS \)-coupled neutron–proton basis, where the only non-zero matrix element is

\[ \langle (l_n^0)^2 (l_p^0)^2 | S_n \cdot S_p | (l_n^0)^2 (l_p^0)^2 \rangle = \left( \begin{array}{c} 3 \end{array} \right) \delta_{l_n l_p} \delta_{J_n J_p} (J_n + J_p) , \]

in terms of the reduced matrix elements

\[ \langle J||S||J\rangle = \left( \begin{array}{c} 3(2j + 1)(2J + 1) \end{array} \right)^{\frac{1}{2}} \left( \begin{array}{c} j \ rac{1}{2}j \ j' \ \frac{1}{2}j' \end{array} \right) \]

If the neutron and proton occupy the same shell, \( j_n = j_p \equiv j \), the matrix element reduces to

\[ \langle j_n j_p | S_n \cdot S_p | j_n j_p \rangle = \frac{J(J + 1) - 2j(j + 1)}{2(2l + 1)^2} . \]

For the deuteron \( l = 0 \) and \( j = \frac{1}{2} \), and one recovers the familiar values of \(-\frac{3}{4}\) for \( J = 0 \) (isovector or spin singlet) and \(+\frac{1}{4}\) for \( J = 1 \) (isoscalar or spin triplet).

Finally, it is of use to find the reduced matrix elements in \( LST \) coupling of the separate isoscalar and isotensor parts of \( S_n \cdot S_p \). We write

\[ S_n \cdot S_p = T^{(000)}_{0000} + T^{(002)}_{0000} , \]

where the upper indices refer to the tensor character in \( LST \) and the lower indices to the projections \( M_L M_S M_T \). The following relations are valid

\[ \langle l^2 LST = M_T = 0 | S_n \cdot S_p | l^2 LST = M_T = 0 \rangle = \langle l^2 LST = 0 | T^{(000)}_{0000} | l^2 LST = 0 \rangle, \]

\[ \langle l^2 LST = 1, M_T = 0 | S_n \cdot S_p | l^2 LST = 1, M_T = 0 \rangle = \langle l^2 LST = 1 | T^{(000)}_{0000} | l^2 LST = 1 \rangle. \]
where the double-barred matrix elements are reduced in $L$, $S$ and $T$. With the help of the expressions (6) one deduces

\[
\langle l^2\text{LST} \parallel T^{(000)} \parallel l^2\text{LST} \rangle = \left[ \frac{2L + 1}{16(2S + 1)} \right]^{1/2} [2S(S + 1) - 3].
\]

\[
\langle l^2\text{LST} \parallel T^{(002)} \parallel l^2\text{LST} \rangle = -\delta_{T1} \left[ \frac{5(2L + 1)(2S + 1)}{96} \right]^{1/2} [2S(S + 1) - 3].
\]  

(14)

4 Schematic model

We consider a single-$l$ shell, corresponding to two $j$ shells, $j = l \pm 1/2$, together with the schematic Hamiltonian

\[
H = \epsilon_- n_- + \epsilon_+ n_+ - 4\pi \sum_{T=0,1} a'_T \sum_{i<j} \delta(r_i - r_j) \delta(r_i - R_0).
\]

(15)

where $n_\pm$ are the number operators for the $j = l \pm 1/2$ shells and the last term represents a surface delta interaction (SDI). Following Brussaard and Glaudemans [22] we introduce the isoscalar and isovector strengths, $a_T = a'_T C(R_0)$, where $C(R_0)$ is a radial integral, and we adopt the notation $a \equiv a_0$ and $a(1-x) \equiv a_1$, so that $x$ indicates the relative importance of both strengths. We note that, as long as one considers a single-$l$ or single-$j$ shell, as is done in the following, results obtained with SDI are identical to those with a delta interaction, except for an overall scaling of the strengths. We also note that the additional terms introduced in the modified SDI, although important to reproduce nuclear binding energies [22], do not alter wave functions and therefore do not influence expectation values of $L_n \cdot S_p$. For any combination of its parameters the eigenstates of the Hamiltonian (15) carry good angular momentum $J$ and isospin $T$. For such eigenstates we calculate the expectation value of the operator $S_n \cdot S_p$, which, as shown above, contains an isoscalar and an isovector component.

The spectrum of the Hamiltonian (15) depends on four parameters whereas relative energies are determined by the three parameters $\Delta \epsilon \equiv \epsilon_- - \epsilon_+, a$ and $x$. Eigenfunctions depend on only two dimensionless parameters $\Delta \epsilon/a$ and $x$, which are varied in order to study their influence on the expectation value of $S_n \cdot S_p$. A bounded parameter can be defined as

\[
y \equiv \frac{\Delta \epsilon}{a} \leq |\Delta \epsilon/a|.
\]

(16)

In most cases rapid changes in the expectation value $\langle S_n \cdot S_p \rangle$ occur for $|\Delta \epsilon/a| \approx 5$ (see Sect. 4.3). With the choice of 5 in the denominator this corresponds to $|y| \approx 0.5$. In the convention of a positive strength $a$ for an attractive force and with a spin–orbit interaction that favours the alignment of spin and orbital angular momentum ($\epsilon_- \geq \epsilon_+$), its domain is $0 \leq y \leq 1$.

Calculations can be restricted to the lower half of the $l$ shell because the results for the upper half can be obtained through the application of a particle–hole transformation. The Hamiltonian (15) is not invariant under particle–hole conjugation since this transformation induces the change $\epsilon_+ \to -\epsilon_-$. In the $(x, y)$ parametrisation introduced above the particle–hole transformation leaves $x$ invariant and induces a sign change in $y$. We may therefore restrict calculations to the lower half of the $l$ shell provided we extend the parameter domain to $-1 \leq y \leq +1$. This covers all possible parameter values for all possible nucleon numbers.

A number of limiting cases of interest occur, which are illustrated in the next subsections.

4.1 SU(4) symmetry

If $a_0 = a_1$ and $\epsilon_- = \epsilon_+$ (or $x = 1/2$ and $y = 0$), the Hamiltonian (15) conserves orbital angular momentum $L$, spin $S$, isospin $T$ and in addition has an SU(4) symmetry. Since $S_n \cdot S_p$ can be written in terms of SU(4) generators, its expectation value in the ground state depends solely on the supermultiplet labels $(\lambda, \mu, \nu)$ and on $(\text{LST})$ in the ground state. For example, even–even $N = Z$ nuclei have the ground-state labels $(\lambda, \mu, \nu) = (000)$ and $(\text{LST}) = (000)$. Odd–odd $N = Z$ nuclei have a ground-state configuration with $(\lambda, \mu, \nu) = (010)$, which contains two degenerate states with $(\text{LST}) = (010)$ (isoscalar) or $(001)$ (isovector). These labels completely determine the expectation value of $S_n \cdot S_p$, which therefore is independent of the nucleon number. For a SDI all $N = Z$ nuclei have $L = 0$ in the ground state. Denoting the ground state of an even–even $N = Z$ nucleus as $|J^kL = 0ST\rangle$ and of an odd–odd $N = Z$ nucleus as $|J^{k+2}L = 0ST\rangle$, we conclude that the following expectation values are valid:
The expectation value \( \langle S_n \cdot S_p \rangle \) as a function of \( x = a_0/(a_0 + a_1) \) in the yrast \( L = 0 \) eigenstate of the Hamiltonian (15) for even–even \( N = Z \) systems. a Ground state for two neutrons and two protons in a \( p \) (black), \( d \) (blue), \( f \) (red) or \( g \) (purple) shell. b Ground state for \( k \) neutrons and \( k \) protons in the \( d \) shell with \( k = 0 \) (black), \( k = 2 \) (blue) and \( k = 4 \) (red).

\[
\langle l^{4k} 000 | S_n \cdot S_p | l^{4k} 000 \rangle = 0, \quad N = Z \text{ even},
\]
\[
\langle l^{4k+2} 010 | S_n \cdot S_p | l^{4k+2} 010 \rangle = + \frac{1}{4}, \quad N = Z \text{ odd},
\]
\[
\langle l^{4k+2} 001 | S_n \cdot S_p | l^{4k+2} 001 \rangle = - \frac{3}{4}, \quad N = Z \text{ odd}. \quad (17)
\]

As far as the expectation value of \( S_n \cdot S_p \) is concerned, the ground state of an odd–odd \( N = Z \) nucleus therefore behaves as a deuteron by virtue of the SU(4) symmetry.

### 4.2 LS coupling

If \( a_0 \neq a_1 \) and \( \epsilon_- = \epsilon_+ \) (or \( x \neq \frac{1}{2} \) and \( y = 0 \)), the Hamiltonian (15) breaks SU(4) symmetry but conserves orbital angular momentum \( L \), spin \( S \) and isospin \( T \). The energy matrix associated with the Hamiltonian (15) can therefore be constructed in an \( LST \) basis. The \( S_n \cdot S_p \) operator is not an \( LST \) scalar, however, since it has an isoscalar as well as an isovector piece. Its matrix elements can be calculated from the application of the Wigner–Eckart theorem [20,21]

\[
\langle l^n LST M_T | S_n \cdot S_p | l^n LST M_T \rangle = \frac{1}{\sqrt{(2L+1)(2S+1)(2T+1)}} \langle l^n LST | T^{(000)} | l^n LST \rangle
\]
\[
+ \frac{(-)^{T-M_T}}{\sqrt{(2L+1)(2S+1)}} \begin{pmatrix} T & 2 & T \\ T & 0 & T \\ -M_T & 0 & M_T \end{pmatrix} \times \langle l^n LST | T^{(002)} | l^n LST \rangle. \quad (18)
\]

The \( n \)-particle \( LST \)-reduced matrix elements of \( T^{(000)} \) and \( T^{(002)} \) can be related recursively to the two-particle matrix elements (14) by means of coefficients of fractional parentage (CFPs) in \( LST \) coupling.

The above method has the advantage of requiring the diagonalisation of matrices of only modest dimension but it has the drawback that CFPs have to be calculated recursively in \( LST \) coupling for the total number of nucleons. It is therefore more efficient to consider the problem in a neutron–proton \( LS \) basis. In this basis matrices are still of reasonable dimension and CFPs can be evaluated for the neutrons and the protons separately. For example, for \( 5 \) (7) neutrons and 5 (7) protons in the \( d \) (\( f \) shell with \( L = 0 \), the dimensions are 26 (731) for \( S = 0 \) and 42 (1407) for \( S = 1 \).

Figure 2 shows the expectation value \( \langle S_n \cdot S_p \rangle \) as a function of \( x = a_0/(a_0 + a_1) \) in the ground state of \( N = Z \) nuclei, for two neutrons and two protons in a \( p \), \( d \), \( f \) or \( g \) shell and for even numbers of neutrons and protons in the \( d \) shell. The ground state has \( (LST) = (000) \) for the entire parameter range. The expectation value \( \langle S_n \cdot S_p \rangle \) is 0 at \( x = \frac{1}{2} \), its value in the SU(4) limit, and becomes more negative as the \( l \) of the shell and/or the number of nucleons increases. Note that for \( l = 0 \) two neutrons and two protons fill the \( s \) shell (not shown in Fig. 2) and \( \langle S_n \cdot S_p \rangle = 0 \), independent of the Hamiltonian. It should also be noted that \( \langle S_n \cdot S_p \rangle \) is invariant under the exchange of \( a_0 \) and \( a_1 \).

Although no data are available at present for odd–odd \( N = Z \) nuclei, for completeness we show in Fig. 3 \( \langle S_n \cdot S_p \rangle \) as a function of \( x \) in the yrast eigenstates with \( (LST) = (001) \) and \( (010) \), for three neutrons and three protons in a \( p \), \( d \), \( f \) or \( g \) shell and for odd numbers of neutrons and protons in the \( d \) shell. For \( x > \frac{1}{2} \), the isoscalar interaction is dominant and the ground state has \( (LST) = (010) \); for \( x < \frac{1}{2} \) the isovector interaction is dominant and the ground state has \( (LST) = (001) \). Figure 3 shows \( \langle S_n \cdot S_p \rangle \) for both states over the entire range of values \( 0 \leq x \leq 1 \). For all \( x \), \( \langle S_n \cdot S_p \rangle \) is below its value at \( x = \frac{1}{2} \), where one recovers the SU(4) values \( -\frac{3}{4} \) and \( \frac{1}{4} \) for \( S = 0 \) and \( S = 1 \), respectively. As the \( l \) of the shell and/or the number of nucleons increases, \( \langle S_n \cdot S_p \rangle \) further decreases.

### 4.3 Spin–orbit interaction

The single-particle energies of the \( l + \frac{1}{2} \) and \( l - \frac{1}{2} \) shells are not expected to be degenerate and, because of the spin–orbit component of the nuclear interaction, the former is the lowest,
The expectation value $\langle S_n \cdot S_p \rangle$ as a function of $x = a_0 / (a_0 + a_1)$ in yrast $L = 0$ eigenstates of the Hamiltonian (15) for odd–odd $N = Z$ systems. a, c Eigenstates with $a S = 0$ and $c S = 1$ for three neutrons and three protons in a $p$ (black), $d$ (blue), $f$ (red) or $g$ (purple) shell. b, d Eigenstates with $b S = 0$ and $d S = 1$ for $k$ neutrons and $k$ protons in the $d$ shell with $k = 1$ (black), $k = 3$ (blue) and $k = 5$ (red).

Fig. 4 The expectation value $\langle S_n \cdot S_p \rangle$ as a function of $y$ defined in Eq. (16) in the $J = 0$ ground state of the Hamiltonian (15) for even–even $N = Z$ systems. a Ground state for two neutrons and two protons in a $p$ (black), $d$ (blue), $f$ (red) or $g$ (purple) shell. b Ground state for $k$ neutrons and $k$ protons in the $d$ shell with $k = 0$ (black), $k = 2$ (blue) and $k = 4$ (red). The isoscalar and isovector strengths are assumed equal, $a_0 = a_1$.

$\epsilon_+ < \epsilon_-$. We assume for simplicity in this subsection that the isoscalar and isovector strengths are the same, $a_0 = a_1$.

Figure 4 shows the expectation value $\langle S_n \cdot S_p \rangle$ in the $J = 0$ ground state of the Hamiltonian (15) for two neutrons and two protons in a $p, d, f$ or $g$ shell, and for even numbers of neutrons and protons in the $d$ shell. Two neutrons and two protons fill the $s$ shell (not shown in Fig. 2) and $\langle S_n \cdot S_p \rangle = 0$, independent of the Hamiltonian. For $\epsilon_+ \neq \epsilon_-$, the quantum numbers $L$ and $S$ are not conserved and one has to revert to labelling states with their total angular momentum $J$. Given the definition (16), results for $y \to \pm 1$ approach those for a single shell with $j = l \pm 1/2$. This explains some of the values observed in Fig. 4 at the limits $y = \pm 1$. For example, two neutrons and two protons fill the $p_{1/2}$ shell and therefore the $p$ (black) curve in Fig. 4a necessarily must converge to 0 at $y = -1$. Likewise, four neutrons and four protons fill the $d_{3/2}$ shell and the $k = 4$ (red) curve in Fig. 4b converges to 0 at $y = -1$. Furthermore, particle–hole symmetry explains some of the results of Fig. 4. In a $d_{5/2}$ shell the ground state of a 2n–2p system is the particle–hole conjugate of that of a 4n–4p system Therefore the $k = 2$ (blue) and $k = 4$ (red) curves in Fig. 4b converge at $y = +1$.

The spin–orbit term in the nuclear mean field is $A$ dependent and, with use of its estimate given in Ref. [23], one finds a splitting of the spin–orbit partner levels of the order $\Delta \epsilon \approx 10(2l + 1)A^{-2/3}$ MeV. The strengths of the SDI are also $A$ dependent and a rough estimate is given in Ref. [22], $a_0 \approx a_1 \approx 25A^{-1}$ MeV. We arrive therefore at the follow-
For y in the limit |

Application to the ground state of the Hamiltonian (15) for various even–even \( N = Z \) systems. a, c Eigenstates with \( a J = 0 \) and \( c J = 1 \) for three neutrons and three protons in a \( p \) (black), \( d \) (blue) or \( f \) (red) shell. b, d Eigenstates with \( b J = 0 \) and \( d J = 1 \) for \( k \) neutrons and \( k \) protons in the \( d \) shell with \( k = 1 \) (black), \( k = 3 \) (blue) and \( k = 5 \) (red). The isoscalar and isovector strengths are assumed equal, \( a_0 = a_1 \) and \( d = +| \rightarrow + \rightarrow \infty \) the problem is reduced to a single- \( j \) calculation. Figure 6 shows \( \langle S_n \cdot S_p \rangle \) in the \( J = 0 \) ground state of the Hamiltonian (15) for various even–even \( N = Z \) systems confined to a single- \( j \) shell. Whether the orbital angular momentum and the spin are aligned, \( j = l + \frac{1}{2} \), or anti-aligned, \( j = l - \frac{1}{2} \), has little influence on the results. The expectation value is slightly less negative in the latter case except for extreme (and unphysical) values of \( x \).

5 Summary and outlook

This study shows that there is no ‘simple’ explanation for the positive values of \( \langle S_n \cdot S_p \rangle \) as observed in the experiments reported in Refs. [12–14]. For all possible parameter values
in the Hamiltonian (15) the expectation value \( \langle S_n \cdot S_p \rangle \) is found to be negative in the ground state of all even–even \( N = Z \) nuclei. Admittedly, the Hamiltonian (15) is of a schematic character and the analysis is carried out in a single-\( l \) shell. But our results show that the naive expectation that an increase of the isoscalar (spin-triplet) interaction strength leads to positive values of \( \langle S_n \cdot S_p \rangle \) is unfounded. Also the role of the spin–orbit term in the nuclear mean field is clearly established as it inevitably leads to more negative \( \langle S_n \cdot S_p \rangle \) values in even–even \( N = Z \) nuclei. The interpretation of the results for odd–odd \( N = Z \) nuclei is more intricate. While no yrast \( J = 0 \) state is found with positive \( \langle S_n \cdot S_p \rangle \), this might occur for yrast \( J = 1 \) eigenstates.

The present results call for a theoretical study in similar vein but with a more sophisticated schematic Hamiltonian. While realistic shell-model calculations are able to reproduce the observed spin–spin correlations [15, 16], it would still be worthwhile to pinpoint the exact origin of the positive \( \langle S_n \cdot S_p \rangle \) values. The positive values of \( \langle S_n \cdot S_p \rangle \), found experimentally in \( sd \)-shell nuclei [12–14], might be a consequence of mixing between configurations in the \( s \) and \( d \) shells, not considered in the present work. Alternatively, they might be due to a non-central, in particular a tensor, component of the nuclear interaction. As the tensor interaction to some extent acts as a negative spin–orbit term, it is yet not clear whether its effect on \( \langle S_n \cdot S_p \rangle \) is adequately represented in the schematic Hamiltonian considered in this work, although it could be partially captured in our dimensionless parameter \( y \). Finally, the positive \( \langle S_n \cdot S_p \rangle \) values are perhaps the result of a combination of both effects, that is, of configuration mixing and the tensor component of the nuclear interaction. Note that the positive values seen in \( ^4 \)He and \( ^{12} \)C, where the \( LS \) coupling scheme could be considered a good approximation, may indeed favour the important role of the tensor force.

This study also shows the value of extending spin–spin-correlation experiments in two directions. One is towards odd–odd \( N = Z \) nuclei where the occurrence of \( J = 0, T = 1 \) and \( J = 1, T = 0 \) states at similar energies might give complementary information. In this regard, measurements on \( ^6 \)Li and \( ^{14} \)N will be of much interest. Above \( ^{40} \)Ca, a program to study \( (p, p') \) scattering with radioactive beams in inverse kinematics, at facilities such as RIKEN [25], FRIB [26] and FAIR [27], is compelling. A second direction is to go slightly off the \( N = Z \) line. Since the \( S_n \cdot S_p \) operator is a combination of isoscalar and isotensor parts, the measurement of its expectation value in the \( J = 0 \) ground state of an even–even \( N = Z + 2 \) nucleus as well as in its isobaric analogue state in the neighbouring \( N = Z \) odd–odd nucleus determines the separate pieces. Along this line, an approved experiment at iThemba [28] will extend the studies of Ref. [13] measuring the spin–spin correlations in the ground states of \( ^{46},^{48} \)Ti. For \( N > Z \) targets, a combination of \( (p, p') \) and \( (d, d') \) scattering is required to disentangle the IS and IV components of the \( M1 \) operator.
Fig. 7 The expectation value \( \langle \mathbf{S}_n \cdot \mathbf{S}_p \rangle \) as a function of 
\( x = a_0/(a_0 + a_1) \) in yrast \( J = 0 \) and \( J = 1 \) eigenstates of the 
Hamiltonian (15) for odd-odd 
\( N = Z \) systems. a, e Eigenstates with \( a J = 0 \) and \( e J = 1 \) for 
three neutrons and three protons in a 
\( d_{5/2} \) (black), \( f_{5/2} \) (blue), 
\( g_{9/2} \) (red) or \( h_{11/2} \) (purple) 
shell. e, g Eigenstates with \( e J = 0 \) and \( g J = 1 \) for three 
novels and three protons in a 
\( f_{3/2} \) (black), \( g_{7/2} \) (blue), \( h_{9/2} \) 
(red) or \( i_{11/2} \) (purple) shell. b, f 
Eigenstates with \( b J = 0 \) and \( f J = 1 \) for \( k \) neutrons and \( k \) 
protons in the \( g_{9/2} \) shell with 
\( k = 0 \) (black), \( k = 2 \) (red) and 
\( k = 4 \) (purple). d, h Eigenstates 
with \( d J = 0 \) and \( h J = 1 \) for \( k \) 
novels and \( k \) protons in the 
\( h_{9/2} \) shell with \( k = 0 \) (black), 
\( k = 2 \) (red) and \( k = 4 \) (purple)
\[ [31] = (210) \rightarrow (01) + (10) + (11) + (12) + (21), \]
\[ [332] = (012) \rightarrow (01) + (10) + (11) + (12) + (21), \]
\[ [422] = (202) \rightarrow (00) + (11)^2 + (02) + (20) + (12) + (21) + (22), \]

it follows that \( S_n \cdot S_p \) must be a combination of \((000)\), \((020)\) and \((202)\) SU(4) tensors.

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