GRIFFITHS-HARRIS RIGIDITY OF COMPACT HERMITIAN SYMMETRIC SPACES

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Abstract. I show that any complex manifold that resembles a rank two compact Hermitian symmetric space (other than a quadric hypersurface) to order two at a general point must be an open subset of such a space.

1. Introduction

Let $X \subset \mathbb{CP}^{n+a}$ be a variety and let $x \in X$ be a smooth point. The projective second fundamental form of $X$ at $x$ (see [3, 14, 4, 1]) is a basic differential invariant that measures how $X$ is moving away from its embedded tangent projective space at $x$ to first order. It determines a system of quadrics $|II_{X,x}| \subset S^2T^*_xX$. Now let $X$ be a general point, to what extent does $|II_{X,x}|$ determine $X$?

Let $X$ be such that $|II_{X,x}|$ is an isolated point in the moduli space of $a$-dimensional linear subspaces of the space of quadratic forms on $\mathbb{C}^n$ up to linear equivalence. We say $X$ is infinitesimally rigid at order two or is Griffiths-Harris rigid if whenever $Y \subset \mathbb{P}^N$ is a complex manifold, $y \in Y$ is a general point and $|II_{Y,y}| = |II_{X,x}|$, then $\overline{Y} = X$.

In [3], Griffiths and Harris posed the question as to whether the Segre variety $\text{Seg}(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$ was infinitesimally rigid to order two and in [1] I answered the question affirmatively and showed that all rank two compact Hermitian symmetric spaces (in their minimal homogeneous embeddings) except for the quadric hypersurface, the Grassmanian $G(\mathbb{C}^2, \mathbb{C}^5) \subset \mathbb{P}^9$ and the spinor variety $D_5/P_5 = S_{10} \subset \mathbb{P}^{15}$ were infinitesimally rigid at order two. The quadric is not rigid to order two and Fubini showed [3] it is rigid to order three when $n > 1$ and it is rigid to order five when $n = 1$. In this paper I resolve the two remaining cases, and explain shorter and less computational proofs for the other cases presented in [3]. I also reprove the rigidity of the three Severi varieties that are rigid to order two to illustrate the method. The new proofs use two tools, a higher order Bertini theorem, and elementary representation theory.

In [11, 12] we showed that all rational homogeneous varieties other than the rank two compact Hermitian symmetric spaces fail to be rigid to order two, so the result of this paper is the best possible in this sense. One can compare this type of rigidity to that studied by Hwang and Mok, see, [4, 5]. Some differences are: in their study they require global hypotheses where here the hypotheses are at the level of germs (this is because the systems of quadrics under study admit no local deformations); in their study the objects of interest are not a priori given an embedding (although since they assume the Picard number is one, one gets something close to an embedding); and in their study the object of interest is the cone of minimal degree rational curves through a general point, which, a priori, has nothing to do with the cone of asymptotic directions I use here (in the systems under consideration, the base locus of $II$ determines $II$).

Theorem 1.1. Let $X^n \subset \mathbb{CP}^{n+a}$ be a complex submanifold. Let $x \in X$ be a general point. If $|II_{X,x}| \simeq |II_{Z,z}|$ where $Z$ is a compact rank two Hermitian symmetric space in its natural embedding, other than a quadric hypersurface, then $\overline{X} = Z$.

Some open questions and relations with the Fulton-Hansen connectedness theorem are discussed in [3]. Another application of the techniques used here is given in [3].
2. MOVING FRAMES

For more details throughout this section, see any of [5] [6] [10] [11] [4].

Once and for all fix index ranges \(1 \leq \alpha, \beta, \gamma \leq n, n+1 \leq \mu, \nu \leq n+a\).

Let \(X^n \subset \mathbb{CP}^{n+a} = \mathbb{P}V\) be a complex submanifold and let \(x \in X\) be a general point. Let \(\pi: F^1 \to X\) denote the bundle of bases of \(V\) (frames) preserving the flag \(\hat{x} \subset \hat{T}_x X \subset V\).

Here \(\hat{T}_x X\) denotes the affine tangent space (the cone over the embedded tangent projective space). Let \((e_0, \ldots, e_{n+a})\) be a basis of \(V\) with dual basis \((e^0, \ldots, e^{n+a})\) adapted such that \(e_0 \in \hat{x}\) and \(\{e_0, e_\alpha\} \) span \(\hat{T}_x X\). I ignore twists and obvious quotients, writing \(e_\alpha\) for \((e_\alpha \mod e_0) \otimes e^0 \in T_x X\) and \(e_\mu\) for \((e_\mu \mod T_x X) \otimes e^0 \in N_x X = T_x \mathbb{P}V/T_x X\). Moreover, if \(x\) and \(X\) understood, I write \(T = T_x X\) and \(N = N_x X\).

The fiber of \(\pi: F^1 \to X\) over a point is isomorphic to the group

\[
G_1 = \left\{ g = \begin{pmatrix} g_{0}^{0} & g_{0}^{\beta} & g_{0}^{\nu} \\ 0 & g_{\beta}^{\beta} & g_{\beta}^{\nu} \\ 0 & 0 & g_{\nu}^{\nu} \end{pmatrix} \mid g \in GL(V) \right\}.
\]

While \(F^1\) is not in general a Lie group, since \(F^1 \subset GL(V)\), we may pullback the Maurer-Cartan from on \(GL(V)\) to \(F^1\). Write the pullback of the Maurer-Cartan form to \(F^1\) as

\[
\omega = \begin{pmatrix} \omega^0_0 & \omega^0_\beta & \omega^0_\nu \\ \omega^\beta_0 & \omega^\beta_\beta & \omega^\beta_\nu \\ \omega^\nu_0 & \omega^\nu_\beta & \omega^\nu_\nu \end{pmatrix}.
\]

The adaptation implies that \(\omega^\mu_0 = 0\) and the Maurer-Cartan equation \(d\omega = -\omega \wedge \omega\) together with the Cartan Lemma implies that for all \(\mu, \alpha\), \(\omega^\mu_\alpha = q^\mu_{\alpha \beta} \omega^\beta_0\) for some functions \(q^\mu_{\alpha \beta} : F^1 \to \mathbb{C}\). These functions determine the projective second fundamental form

\[
II = F_2 = \omega^\alpha_0 \otimes e_\mu = q^\mu_{\alpha \beta} \omega^\beta_0 \otimes e_\mu \in \Gamma(X, S^2T^*X \otimes NX).
\]

While \(II\) descends to be a section of \((S^2T^*X \otimes NX)\), higher order derivatives provide relative differential invariants \(F_k \in \Gamma(F^1, \pi^*(S^kT^* \otimes N))\). For example,

\[
F_3 = r^\mu_{\alpha \beta \gamma} \omega^\alpha_0 \omega^\beta_0 \omega^\gamma_0 \otimes e_\mu
\]

\[
F_4 = r^\mu_{\alpha \beta \gamma \delta} \omega^\alpha_0 \omega^\beta_0 \omega^\gamma_0 \omega^\delta_0 \otimes e_\mu
\]

where the functions \(r^\mu_{\alpha \beta \gamma}, r^\mu_{\alpha \beta \gamma \delta}\) are given by

\[
r^\mu_{\alpha \beta \gamma} \omega^\gamma_0 = -d q^\mu_{\alpha \beta} - q^\mu_{\alpha \beta} \omega^0_0 - q^\mu_{\alpha \beta} \omega^\mu_0 + q^\mu_{\alpha \beta} \omega^\beta_0 + q^\mu_{\alpha \beta} \omega^\nu_0
\]

\[
-3d^\mu_{\alpha \beta} \omega^\gamma_0 - q^\mu_{\alpha \beta} q^\nu_0 \omega^\gamma_0 + q^\mu_{\alpha \beta} q^\nu_0 \omega^\gamma_0 + q^\mu_{\alpha \beta} q^\nu_0 \omega^\gamma_0.
\]

If one chooses local affine coordinates and writes \(X\) as a graph

\[
x^\mu = q^\mu_{\alpha \beta} x^\alpha x^\beta + r^\mu_{\alpha \beta \gamma} x^\alpha x^\beta x^\gamma + r^\mu_{\alpha \beta \gamma \delta} x^\alpha x^\beta x^\gamma x^\delta + ...
\]

then there exists a local section of \(F^1\) such that

\[
F_2 = q^\mu_{\alpha \beta} dx^\alpha dx^\beta \otimes \frac{\partial}{\partial x^\mu}
\]

\[
F_3 = r^\mu_{\alpha \beta \gamma} dx^\alpha dx^\beta dx^\gamma \otimes \frac{\partial}{\partial x^\mu}
\]

\[
F_4 = r^\mu_{\alpha \beta \gamma \delta} dx^\alpha dx^\beta dx^\gamma dx^\delta \otimes \frac{\partial}{\partial x^\mu}
\]

etc...
Since an analytic variety is uniquely determined by its Taylor series at a point, to show \( Z \) is rigid to order two, it is sufficient to show that over varieties \( X \) with \(|II_{X,x}| = |II_{Z,z}|\) there exists a subbundle of \( \mathcal{F}^1 \) such that the \( f_k \)'s of \( X \) coincide with those of \( Z \). Moreover, the minuscule varieties, that is, the compact Hermitian symmetric spaces in their natural projective embeddings, have the property that on a reduced frame bundle all the differential invariants except for their fundamental forms are zero and in our case the only nonzero fundamental form is \( II \).

The method in [1] was first to use the equations above to calculate relations among the coefficients of \( F_3 \). Enough relations were found that, combined with the coefficients that were normalizable to zero, I obtained that \( F_3 \) was zero, and the same technique was used for higher order invariants.

In this paper I decompose \( S^3T^* \otimes N \) into irreducible \( R \)-modules, where \( R \subset GL(T) \times GL(N) \) is the subgroup preserving \( II \in S^2T^* \otimes N \). I also systematize the vanishing of coefficients of the \( F_k \) somewhat using higher order Bertini theorems which I now describe. It would be nice to have a way to apply the higher order Bertini theorems directly to the irreducible modules instead of using the coefficients of \( F_3 \).

3. Vanishing tools

3.1. Higher order Bertini. Let \( T \) be a vector space. The classical Bertini theorem implies that for a linear subspace \( A \subset S^2T^* \), if \( q \in A \) is generic, then \( v \in \text{Sing}(q) := \{ v \in T \mid q(v, w) = 0 \ \forall w \in T \} \) implies \( v \in \text{Base}(A) := \{ v \in T \mid Q(v, v) = 0 \ \forall Q \in A \} \).

**Theorem 3.1** (Higher order Bertini). Let \( X^n \subset \mathbb{P}V \) be a complex manifold and let \( x \in X \) be a general point.

1. Let \( q \in |II_{X,x}| \) be a generic quadric. Then \( q_{\text{sing}} \subset \text{Base}(F_2, ..., F_k) \) for all \( k \). I.e., \( q_{\text{sing}} \) is tangent to a linear space on the completion of \( X \).

2. Let \( q \in |II_{X,x}| \) be any quadric, let \( L \subset q_{\text{sing}} \cap \text{Base}|II_{X,x}| \) be a linear subspace. Then for all \( v, w \in L \), \( F_k^q(v, w, \cdot) = 0 \).

3. With \( L \) as in 2., if \( L' \subset (\text{Base}|II_{X,x}|, F_3) \) \( L \) is a linear subspace then \( F_4^q(u, v, w, \cdot) = 0 \) for all \( u, v, w \in L' \) and so on for higher orders. Here \( F_k^q \) denotes the polynomial in \( F_k \) corresponding to the conormal direction of \( q \). This is well defined by the lower order vanishing.

4. With \( L' \) as in 3., if \( L'' \subset L' \cap (F_3^q)_{\text{sing}} \) is a linear space, then for all \( u, v \in L'' \), \( F_4^q(u, v, \cdot, \cdot) = 0 \).

Analogous results hold for higher orders.

**Proof.** Note that 1. is classical, but we provide a proof for completeness. Assume \( v = e_1 \) and \( q = q^\mu \). Our hypotheses imply \( q_{11}^\mu = 0 \) for all \( \beta \). Formula (1) reduces to

\[
r_{11, \beta}^{\mu, \nu} \omega_0^\beta = -q_{11}^\nu \omega_0^\nu.
\]

If \( q \) is generic we are still working on \( \mathcal{F}^1 \) and so the \( \omega_0^\mu \) are independent of the semi-basic forms, thus the coefficients on both sides of the equality are zero, proving both the classical Bertini theorem and 1 in the case \( k = 3 \).

If \( q \) is not generic, in order to have \( q = q^\mu, v = e_1 \) we have reduced to a subbundle \( \mathcal{F}' \subset \mathcal{F}^1 \) and we no longer have the \( \omega_0^\nu \) independent. However hypothesis 2 states that \( q_{11}^\nu = 0 \) for all \( \nu \) and the required vanishing still holds.

For 3., note that \( r_{11}^{\nu, \mu} \omega_0^\beta = r_{11}^{\mu, \nu} \omega_0^\beta + e r_{11}^{\mu, \omega_0^\nu} 1 + q_{11}^\nu q_{11}^\mu \) and the right hand side is zero under our hypotheses. Part 4 is proven similarly.

The extension to linear spaces holds by polarizing the forms. The analogous equation at each order proves the next higher order.

\[\Box\]
Example 3.2. Let $X = G(2, m)$ and let $V = \Lambda^2 \mathbb{C}^m$ have basis $e_{st}$ with $1 \leq s < t \leq m$. At $x = [e_{12}]$ we have the adapted flag

$$\{e_{12}\} \subset \{e_{1j}, e_{2j}\} \subset V$$

where $3 \leq i < j \leq m$, and $SL_2 \times SL_{m-2}$ acts transitively on $N_x \simeq \{e_{ij}\}$. So here $\alpha = \{(1j), (2j)\}, \mu = \{(ij)\}$. In these frames $II = (\omega_0^{(1)}, \omega_0^{(2)j}) - \omega_q^{(1)}, \omega_q^{(2)}) \otimes e_{ij}$.

If $m = 5$, then $q^{45}$ is a generic quadric with $e_{13} \subset q^{45}_{sing}$. Thus we have

$$r^{(13)(13)} = 0 \forall \mu$$

$$r^{(13)(13)\beta} = 0 \forall \beta.$$ 

If $m > 5$ then $q^{45}$ is no longer generic, but since $e_{(13)} \in \text{Base}|II_{X,x}|$ we still may conclude

$$r^{(13)(13)\beta} = 0 \forall \beta.$$ 

3.2. Normalizations. $F_3$ is translated in the fiber of $F^1$ by the action of $T \otimes N^*$ and $T^*$ (the $g^\alpha_\mu$ and the $g^0_\alpha$). We may decompose $T \otimes N^*$ and $T^*$ into irreducible $R$ modules and determine which of these act nontrivially. In the case the variety is modeled on a rank two minuscule variety, we will have that all of $T \otimes N^*$ acts effectively, but the $T^*$ action duplicates a factor in $T \otimes N^*$. This is because in the homogeneous model, the forms $\omega_0^\alpha$ are independent and the forms $\omega_\mu^0$ are linear combinations of the $\omega_0^\beta$. We will let $F^n$ denote the bundle where the action of $T \otimes N^*$ has been used to kill the corresponding components of $F_3$. Similarly, on $F^n$, $F_4$ is translated by the action of $N$ and we will let $F^N$ denote the subbundle of $F^n$ where the component of $N$ in $F_4$ has been normalized to zero.

3.3. Remarks on decompositions of the $F_k$ and vanishing. Let $II \in S^2 T^* \otimes N$ arise from a trivial representation of a reductive group $R \subset GL(T) \times GL(N)$. Let $X^\eta \subset \mathbb{P}^{n+\eta}$ be a complex submanifold, let $x \in X$ be a general point and suppose $II_{X,x} = II$.

1. Since the orbit of a highest weight vector in any module spans the module, the component of $F_k$ in an irreducible module $V$ is zero if its highest weight vector is zero.

2. An irreducible module in $S^3 T^* \otimes N$ can occur in $F_3$ only if it also occurs in $(T \otimes T^*)^c \otimes T^* + (N \otimes N^*)^c \otimes T^*$. Here, the Lie algebra of $R$, occurs as a submodule of $T \otimes T^*$ and $N \otimes N^*$ and $(T \otimes T^*)^c$ denotes the complement of $T$ in $T \otimes T^*$. This is because the tangential and normal connection forms, $\omega_\beta^0, \omega_\mu^0$ may be decomposed into $\rho_T(\tau)$-valued (resp. $\rho_N(\tau)$-valued) forms and semi-basic forms. (Here $\rho_T, \rho_N$ denote the representations of $\tau$ on $T$ and $N$.) The coefficients for the semi-basic components are linear combinations of the $\tau^0_{\alpha\beta\gamma}$, all of the same weight. On the other hand the coefficients give rise to $R$-modules respectively in $(T \otimes T^*)^c \otimes T^*$ and $(N \otimes N^*)^c \otimes T^*$. 

3. Similarly, if $F_3 = 0$, and the normalizations of $F_3$ are exactly by $T \otimes N^*$, then an irreducible module in $S^4 T^* \otimes N$ can occur in $F_4$ only if it also occurs in $(T \otimes N^*)^c \otimes T^*$, where $(T \otimes N^*)^c$ is the complement of $T^*$ in $T \otimes N^*$. Thus in our normalizations, the forms $\omega^0_\beta$ will remain independent and independent of the semi-basic forms. On the other hand the forms $\omega^0_0$ will become dependent on the semi-basic forms and the $\omega^0_0$. Again, the components that will depend on the semi-basic forms will have coefficients consisting of linear combinations of monomials in $F_4$ of the same weight.

4. If $F_3, F_4 = 0$ (after normalizations), then an irreducible module in $S^5 T^* \otimes N$ can occur in $F_4$ only if it also occurs in $N$.

5. If $F_3, F_4, F_5 = 0$ (after normalizations), then all higher $F_k$ are zero as well, see \[.\]
4. Case of $G(2,5)$ and $S_{10}$

4.1. Model for $G(2,5)$. Write $T = A^* \otimes B$. We index bases of $T$ and $N$ as above. $R = \mathfrak{sl}(A) + \mathfrak{sl}(B) + \mathbb{C} = \mathfrak{sl}_2 + \mathfrak{sl}_3 + \mathbb{C}$. We write $A_j$ for the representation of $\mathfrak{sl}(A)$ with highest weight $j$ and $B_{ij}$ for the representation of $\mathfrak{sl}(B)$ of highest weight $i\omega_1 + j\omega_2$. Here and throughout we use the notations and ordering of the weights of $[1]$. The relevant modules are summarized in the table below.

4.2. Model for $S_{10}$. Write $\mathbb{C}^{16} = \text{Clifford}(\mathbb{C}^5) \simeq \Lambda^\text{even} \mathbb{C}^5$ with $\lambda \simeq \Lambda^0 W$, $T \simeq \Lambda^2 W, N \simeq \Lambda^4 W \simeq W^*$. We let $e_{st} = e_s \wedge e_t$, $1 \leq s < t \leq 5$ index a basis of $T$ and $e^s$ index a basis of $N$. Note that, as with $G(2,5)$, $R$ acts transitively on $N$ so all quadrics in $|II|$ are generic.

Let $V_{ijkl}$ denote the $\mathfrak{sl}_5$ module with highest weight $i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4$. $|II|$ is given by the Pfaffians of the $4 \times 4$ minors centered about the diagonal with $e^j$ corresponding to the Pfaffian obtained by removing the $j$-th row and column. The relevant modules are summarized in the following table.

$$
T = V_{0100} = A_1 \otimes B_{10} \\
T^* = V_{0010} = A_1 \otimes B_{01} \\
N = V_{0001} = A_0 \otimes B_{10} \\
S^2T^* = T^{*2} \oplus N^* \\
S^3T^* = T^{*3} \oplus N^*T^* \\
S^3T^* \otimes N = (T^{*3}N \oplus TT^{*2}) \oplus ((N^*T^*)N \oplus N^*T \oplus T^*) \\
T \otimes N^* = TN^* \oplus T^* \\
(T \otimes N^*)T^{*c} \otimes T^* = N^*TT^* \oplus N^{*2}N \oplus TN \oplus N^* \\
S^4T^* = T^{*4} \oplus N^*T^{*2} \oplus N^{*2} \\
S^4T^* \otimes N = (T^{*5} \oplus TT^{*3}N \oplus N^*T^{*3}) \oplus (N^*T^{*3} \oplus N^*TT^*N \oplus T^{*2}N \oplus N^{*2}T^* \oplus TT^*) \\
\oplus (N^{*2}N \oplus N^*)
$$

The notation is such that if $V_\lambda, W_\mu$ are the irreducible representations with highest weights $\lambda, \mu$, then $V^{k}, V W$ are respectively the representations with highest weights $k\lambda$ and $\lambda + \mu$. $V W \subset V \otimes W$ is called the Cartan component.

To obtain the vanishing of $F_3$ we need to eliminate five modules. We first eliminate two by reducing to $F^2$ as described above, so the last two factors are zero. Let $F^3 \subset F^1$ denote our new frame bundle.

On our new bundle there remains three modules to eliminate.

The first module in $S^3T^* \otimes N$ remains. In the $G(2,5)$ case $(N^*T^*)$ has highest weight a linear combination of $r_{(13)(13)(13)}$ in the $G(2,5)$ case and $r_{(12)(12)(12)}$ in the $S_{10}$ case. We already saw the $G(2,5)$ case is covered by Bertini, and the $S_{10}$ case is as well because $e_{(12)} \in q^1_{\text{sing}}$, and all quadrics in the system are generic. Thus the first two modules in $S^3T^* \otimes N$ don’t appear in $F_3$.

At this point just $(N^*T^*)N$ remains. In the $G(2,5)$ case $(N^*T^*)$ has highest weight a linear combination of $r_{(13)(13)(24)}$ and $r_{(13)(14)(23)}$. In the $S_{10}$ case it has highest weight a linear combination of $r_{(12)(12)(34)}, r_{(12)(13)(24)}$ and $r_{(12)(14)(23)}$ and thus the Cartan components respectively have highest weights linear combinations of $r_{(13)(13)(24)}^5, r_{(13)(14)(23)}^5$ and $r_{(12)(12)(34)}^5, r_{(12)(13)(24)}^5$ and $r_{(12)(14)(23)}^5$, all of which are zero by Bertini. Thus $F_3$ is zero.
We normalize away the $N$ factor in $S^4T^* \otimes N$ and study the remaining modules. Comparing $S^4T^* \otimes N$ and $(T \otimes N^*)^{T^c} \otimes T^*$ modulo $N^*$, their intersection is empty and thus $F_4 = 0$ on $F^N$.

One can check that $S^5T^* \otimes N$ does not contain a copy of $N$, so we are done. \hfill \square

Remark 4.1. If one compares modules, $N^*T^*N$ is not eliminated from $F_3$. On the other hand, if one just uses Bertini to study $F_4$, everything coming from the first factor in $S^4T^*$ and all Cartan products of $N$ with factors in $S^4T^*$ are easily seen to be zero, but a few of the other modules are more complicated to eliminate.

5. CASE OF $\mathbb{A}P^2$

Let $A_R$ respectively denote $\mathbb{C}, \mathbb{H}, \mathbb{O}$ and let $A = A_R \otimes \mathbb{R}$.

I use the following model: $T = A \oplus A$, where $A = \mathbb{H}$, the complexified quaternions, for $G(2,6)$, and $A = \mathbb{O}$, the complexified octonions for $\mathbb{O}P^2$. I use $(a, b)$ as $A$-valued coordinates. Then $|II| = \{a\overline{a}, bb, ab\}$ where $ab$ represents $\dim A$ quads. Let $p = 3, 7$. Write $a = a_0 + a_1\epsilon_1 + \ldots + a_p\epsilon_p$. We will need to work with null vectors so let $e_1 = 1 + i\epsilon_1, \overline{e}_1 = 1 - i\epsilon_1, e_2 = 1 + i\epsilon_2$ denote elements of the first copy of $A$ (with coordinate $a$). We let $e_a$ denote the normal vector such that $q^a = a\overline{a}$ and similarly for $e_b$. Let $e_0$ denote the normal vector such that $q^0 = Re(ab)$ and $e_{ij}$ such that $q^j$ is the $e_j$ coefficient of $ab$.

Let $V_{ijk\ell m}$ denote the $\mathfrak{g}_2$-module with highest weight $i\omega_1 + j\omega_3 + k\omega_4 + l\omega_5 + m\omega_6$, and the $\mathfrak{sl}(A) + \mathfrak{sl}(B)$ modules are indexed in the obvious way. For the Segre case write $T = U_{10} \oplus W_{10}$ and $N = U_{01} \oplus W_{01}$. The remaining relevant modules are as follows:

- $T = V_{00001}$
- $T^* = V_{00010}$
- $N = N^* = V_{10000}$
- $S^2T^* = T^2 \oplus N$
- $S^3T^* = T^3 \oplus NT^*$
- $S^3T^* \otimes N = (T^2T^* \oplus NT^*) \oplus (N^2T^* \oplus NT \oplus T^*)$
- $(N \otimes N^*)^\epsilon \otimes T^* = N^2T^* \oplus NT$
- $(T \otimes T^*)^\epsilon \otimes T^* = T^* \oplus T^* \oplus NT \oplus T^*$

See [12, 13] for an explanation of $T_2$.

The decompositions above show that there are 6 components of $F_3$ on $F^1$ and four when we restrict to $F^n$.

We may choose our model such that $e_1$ is a highest weight vector (since it is in Base$|II_{X,x}|$). We may also have $e^b$ be a highest weight vector for $N^*$.

Bertini easily kills the first component of $F_3$ as it has highest weight vector $r_{111}^b$. In fact, as in the cases above, the second Cartan component is also killed by Bertini. To see this note that the two irreducible components of $S^3T^*$ respectively have highest weight vectors $r_{111}$ and a linear combination of $r_{111}$ and $r_{112}$. The two Cartan components in $S^3T^* \otimes N$ thus have highest weight vectors $r_{111}^b$ and a linear combination of $r_{111}^b$ and $r_{112}^b$. To see the second is zero, note that any linear combination of $e_1, e_2$ is in Base$|II_{X,x}|$ and Sing$q^b$.

It remains to eliminate the second and fourth modules in $S^3T^* \otimes N$, but neither of these occurs in $(T \otimes T^*)^\epsilon \otimes T^* \oplus (N \otimes N^*)^\epsilon \otimes T^*$ and we are done with $F_3$.\hfill \square
The higher order invariants are safely left to the reader. 

To compare with the $G(2,6)$ case in the standard model, we have $e_1 = e_{(13)}, e_1 = e_{(24)}, q^a = q^{34}, q^b = q^{56}$ etc....

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