COORDINATE-FREE SOLUTIONS FOR COSMOLOGICAL SUPERSPACE

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Hamilton-Jacobi theory for general relativity provides an elegant covariant formulation of the gravitational field. A general 'coordinate-free' method of integrating the functional Hamilton-Jacobi equation for gravity and matter is described. This series approximation method represents a large generalization of the spatial gradient expansion that had been employed earlier. Additional solutions may be constructed using a nonlinear superposition principle. This formalism may be applied to problems in cosmology.

I. INTRODUCTION

General relativity was formulated to describe the gravitational field in a manner which was independent of any particular choice of reference frame. It is invariant under 'general coordinate transformations'. However, when one actually solves the Einstein field equations, say in a cosmological setting, one typically invokes a particular choice of coordinates. As an improvement to this situation, one would prefer to utilize a 'coordinate-free' method of computing solutions to Einstein gravity. In fact, Hamilton-Jacobi (HJ) theory for general relativity provides such a description of the gravitational field [1, 2, 3]. In solving the HJ equation, one need not specify the choice of time hypersurface nor the spatial coordinates.

Coordinate-free descriptions have proven useful in many fields of theoretical physics. Vector analysis for Euclidean space is a good example. Although one normally associates coordinates with vectors, it is possible to interpret a vector geometrically in a coordinate free way using a magnitude and a direction. In addition, by using his bra and ket notation, Dirac [4] was able to formulate quantum mechanics in a form which was independent of the choice of basis states. The bra could be interpreted as an abstract vector without referring to a particular basis.

It has been known since the early 1960’s that the HJ equation for general relativity does not refer to the lapse and shift parameters which characterize the gauge freedom of gravity [5]. Hence the HJ equation is the natural starting point for a coordinate-free analysis of gravity. However, the HJ equation is a nonlinear, functional differential equation which characterizes an ensemble of universes, superspace. It was generally believed that superspace was too complicated to solve in its entirety. Beginning with Misner [6], researchers were content to solve homogeneous minisuperspace models where one considered only a finite number of degrees of freedom for the gravitational field. These investigations have been proven to be quite fruitful in quantum cosmology (see, e.g., Louko and Ruback [7].)

For a discussion of future trends in quantum cosmology, consult Hartle [8] and Barvinsky and Kamenshchik [9].) However, there are some questions that these models cannot address since they do not include inhomogeneities. For such models, the choice of time hypersurface is degenerate: a hypersurface of uniform \( \phi \) is the same as that of uniform scalar factor \( a \). In order to obtain a deeper understanding for the nature of time in a semiclassical setting [10, 11], it is essential to include the role of inhomogeneities. (However, there are many additional obstacles in addressing the question of time in a quantum context [12, 13, 14].)

It is somewhat surprising that one can actually obtain solutions to semiclassical superspace using a series approximation method. The prototype solution for the Hamilton-Jacobi (HJ) equation for general relativity utilized a spatial gradient approximation. It was developed by Salopek and Stewart [15] and advanced further by Parry, Salopek and Stewart [16]. In effect, they demonstrated that one can decompose superspace into a discrete sum of minisuperspaces. Here, a general method will be given for constructing (HJ) solutions which utilize various other series approximations. These techniques may be applied profitably to cosmology and other areas of astrophysics.

A Hamilton-Jacobi description of general relativity is attractive for several other reasons:

(1) **Primitive formulation of quantum theory for the gravitational field.** In the semiclassical approximation, the wavefunctional is given by a phase factor, \( \Psi \sim e^{iS/\hbar} \), where Planck’s constant \( \hbar \) is assumed to be tiny. The generating functional \( S \) then satisfies the HJ equation. If \( S \) is real, then one is describing classical phenomena. If \( S \) is complex, one may describe quantum phenomena such as tunnelling or the initial wavefunction of the universe. It is generally believed that the fluctuations for structure formation as well as microwave background fluctuations are generated during an inflationary epoch. For such models, it is absolutely essential to quantize the gravitational field, including both scalar and tensor modes (see, for example, [17, 18]). The current status of inflation after the detection of the cosmic microwave background anisotropy is discussed in refs. [19, 20].

(2) **Systematic solution of constraint equations.** In superspace, the momentum constraint may be given a simple geometric interpretation and it is easy to solve. The energy constraint fully characterizes the dynamics of the gravi-
tational system, and it is difficult to integrate directly. As will be demonstrated in the present work, it will be solved using a series approximation.

(3) Nonlinear analysis of gravity. In the quasi-nonlinear regime, one may employ HJ theory to analyze the formation of cosmological pancakes [13] during the matter-dominated epoch of the Universe [18], [19].

Spatial gradient expansions in general relativity have a long history [20]. Within a HJ context, a spatial gradient expansion has been applied successfully to cosmology including a detailed computation of microwave background fluctuations and galaxy-galaxy correlations function arising from inflation [21], [22]. Soda et al [23] have generalized the expansion to encompass Brans-Dicke gravity, and Chiba [24] has formulated the gradient expansion for n-dimensional Einstein gravity. The HJ equation for long-wavelength fields has often been invoked in an attempt to recover the inflation potential from cosmological observations [25], [26], [27].

In Sec. II, the HJ equation for a scalar field interacting with gravity is presented along with the analogous equation for a dust field interacting with gravity. I quickly review the spatial gradient expansion of the generating functional in a way which is easy to generalize to other situations. In Sec. III, the HJ equation including a scalar field is solved using a Taylor series in the scalar field \( \phi \). In Sec. IV, the analogous method is applied for a dust field, which describes collisionless, pressureless matter. In Sec. V, the Superposition Principle for Hamilton-Jacobi theory allows one to construct complicated solutions of the HJ from known solutions which depend on a parameter. Conclusions and a summary follow in Sec. VI.

(Units are chosen so that \( c = 8\pi G = 8\pi/m_P^2 = 1 \). The sign conventions of Misner, Thorne and Wheeler [28] will be adopted throughout.)

II. HAMILTON-JACOBI EQUATION FOR GENERAL RELATIVITY WITH MATTER

In the present work, two situations will be considered: (1) a scalar field interacting with gravity and (2) a dust field interacting with gravity.

A. Scalar field interacting with gravity

The action for a single scalar field interacting with Einstein gravity is

\[
I = \int d^4x \sqrt{-g} \left( \frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right),
\]

(2.1)

\(^{(4)}R\) is the Ricci scalar of the space-time metric \( g_{\mu\nu} \). For simplicity, the scalar field potential is assumed to be

\[
V(\phi) = \frac{1}{2} m^2 \phi^2 + V_0
\]

(2.2)

which describes a massive scalar field with a cosmological constant term. (In general the solution methods described in this paper will work for all potentials which are regular at \( \phi = 0 \).)

In the ADM formalism the line element is written as

\[
d s^2 = ( - N^2 + \gamma^{ij} N_i N_j ) dt^2 + 2 N_i dt dx^i + \gamma_{ij} dx^i dx^j ,
\]

(2.3)

where \( N \) and \( N_i \) are the lapse and shift functions respectively, and \( \gamma_{ij} \) is the 3-metric. (For a elegant review, consult D’Eath [29].) One can then rewrite the action in Hamiltonian form,

\[
I = \int d^4x \left( \pi^\phi \dot{\phi} + \pi^{ij} \dot{\gamma}_{ij} - N \mathcal{H} - N^i \mathcal{H}_i \right),
\]

(2.4)

where \( \mathcal{H} \) and \( \mathcal{H}_i \) are the energy and momentum densities, respectively. In the Hamilton-Jacobi formalism, one replaces the momenta by functional derivatives of the generating functional, \( S[\gamma_{ab}(x), \phi(x)] \),

\[
\pi^{ij}(x) = \frac{\delta S}{\delta \gamma_{ij}(x)} , \quad \pi^\phi(x) = \frac{\delta S}{\delta \phi(x) } ;
\]

(2.5)

the generating functional associates a complex number for each field configuration \( \phi(x) \) on a space-like hypersurface whose geometry is described by the 3-metric \( \gamma_{ij}(x) \). The energy and momentum constraints are now given by:
\[ 0 = \mathcal{H}(x) = \gamma^{-1/2} \left[ 2\gamma_{ik}(x)\gamma_{jl}(x) - \gamma_{ij}(x)\gamma_{kl}(x) \right] \frac{\delta S}{\delta \gamma_{ij}(x)} \frac{\delta S}{\delta \gamma_{kl}(x)} + \]
\[ \frac{1}{2} \gamma^{-1/2} \left( \frac{\delta S}{\delta \phi(x)} \right)^2 + \gamma^{1/2}V[\phi(x)] - \frac{1}{2} \gamma^{-1/2} R + \frac{1}{2} \gamma^{1/2} \gamma^{ij} \phi_i \phi_j, \] (2.6a)

and

\[ 0 = \mathcal{H}_i(x) = -2 \left( \gamma_{ik} \frac{\delta S}{\delta \gamma_{kj}(x)} \right)_{,j} + \frac{\delta S}{\delta \gamma_{kl}(x)} \gamma_{kl,i} + \frac{\delta S}{\delta \phi(x)} \phi_i. \] (2.6b)

The Hamilton-Jacobi equation (2.6a) governs the evolution of the generating functional, \( S \equiv S[\gamma_{ab}(x), \phi(x)] \), in superspace. It is quadratic in the momenta of fields. The momentum constraint is linear in momenta. Higgs showed that it legislates that the generating functional is invariant under reparametrizations of the spatial coordinates (‘spatial gauge-invariance’). It may be solved, for example, by assuming that the generating functional is an integral of some function of the curvature, say \( \int d^3x \gamma^{1/2} f(R) \), or some other combination of spatial invariants.

### B. Dust field interacting with gravity

The case of dust interacting with gravity is of high interest in cosmology. It is generally believed that most of the Universe consists of dark-matter whose dynamics may be described by a dust field which is pressureless and collisionless. Secondly, hypersurfaces of uniform \( \chi \) in general relativity come closest to describing Lorentz frames that proved so useful in flat spacetime. In fact, a Lorentz frame may be considered to be a properly synchronized collection of dust particles. It may be necessary to introduce dust to interpret a quantum theory of the gravitational field [81], [13].

The action

\[ \mathcal{I} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \gamma^{ij} \gamma^{kl} \partial_{\mu} \chi \partial_{\nu} \chi + 1 \right], \] (2.7)

for a dust field, \( \chi \), interacting with gravity, is similar to that of a scalar field. The new ingredient is the rest number density \( n \equiv n(t, x) \) which is a Lagrange multiplier that ensures that the 4-velocity

\[ U^\mu = -g^{\mu\nu} \chi_{,\nu} \] (2.8)

satisfies \( U^\mu U_\mu = -1 \). Hence \( \chi \) may be interpreted as a velocity potential. In Hamiltonian form, the action is

\[ \mathcal{I} = \int d^4x \left( \pi^\chi \dot{\chi} + \pi^{ij} \dot{\gamma}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i \right). \] (2.9)

where \( \pi^\chi \) is the canonical momentum of the dust field. Replacing the momenta by functional derivatives of \( S \),

\[ \pi^{ij}(x) = \frac{\delta S}{\delta \gamma_{ij}(x)}, \quad \pi^\chi(x) = \frac{\delta S}{\delta \chi(x)}, \] (2.10)

the energy and momentum constraint equations become:

\[ 0 = \mathcal{H}(x) = \gamma^{-1/2} \left[ 2\gamma_{ik}(x)\gamma_{jl}(x) - \gamma_{ij}(x)\gamma_{kl}(x) \right] \frac{\delta S}{\delta \gamma_{ij}(x)} \frac{\delta S}{\delta \gamma_{kl}(x)} \right] + \sqrt{1 + \gamma^{ij} \chi_{,i} \chi_{,j}} \frac{\delta S}{\delta \chi(x)} \gamma_{kl,i} - \frac{1}{2} \gamma^{1/2} R, \] (2.11a)

\[ 0 = \mathcal{H}_i(x) = -2 \left( \gamma_{ik} \frac{\delta S}{\delta \gamma_{kj}(x)} \right)_{,j} + \frac{\delta S}{\delta \gamma_{kl}(x)} \gamma_{kl,i} + \frac{\delta S}{\delta \chi(x)} \chi_{,i}. \] (2.11b)

In contrast to the Hamiltonian for a scalar field (2.6a), the energy constraint (2.11a) is linear in the canonical momentum of the dust field. In a quantum context, the Hamiltonian constraint for dust and gravity is thus very similar to the Schrodinger equation (or its covariant generalization, the Tomonoga-Schwinger equation [30]) that has been so successful in flat space-time.
C. Review of Spatial Gradient Expansion

The spatial gradient approximation for the HJ equation (2.6a) with a scalar field will be quickly reviewed. The essential aspects will highlighted in order to illustrate what must be done in other situations.

One expands the generating functional

\[ S[\gamma_{ab}(x), \phi(x)] = \sum_{n=0}^{\infty} S^{(2n)} \]  

(spatial gradient expansion) (2.12)

in a series of terms according to the number of spatial gradient terms that they contain. It is quite important that each term in the series expansion satisfy the momentum constraint,

\[ 0 = H^{(2n)}(x) \equiv -2 \left( \frac{\delta S^{(2n)}}{\delta \gamma_{kj}(x)} \right)_{ij} + \frac{\delta S^{(2n)}}{\delta \gamma_{kl}(x)} \gamma_{ki,l} + \frac{\delta S^{(2n)}}{\delta \phi(x)} \phi_{,i} . \]  

(2.13)

The zeroth order term

\[ S^{(0)} = -2 \int d^3 x \gamma^{1/2} H(\phi) , \]  

(2.14)

is the simplest such term that one can imagine; it contains no spatial gradients where the function \( H(\phi) \) satisfies

\[ H^2 = \frac{2}{3} \left( \frac{\partial H}{\partial \phi} \right)^2 + \frac{1}{3} V(\phi) . \]  

(2.15)

The volume element \( d^3 x \gamma^{1/2} \) appearing in eq. (2.14) is obviously invariant under spatial coordinate transformations. The second order term contains two spatial gradients and is an integral over the 3-curvature and a term quadratic in spatial derivatives of \( \phi \):

\[ S^{(2)} = \int \gamma^{1/2} d^3 x \left( J(\phi) R + K(\phi) \gamma^{ab} \phi_{,a} \phi_{,b} \right) ; \]  

(2.16)

\( J \) and \( K \) are arbitrary functions of \( \phi \) which are chosen to satisfy the second order HJ equation. The higher order terms proceed along similar lines: e.g., \( S^{(4)} \) consists of all invariant terms with four spatial derivatives.

What precisely is the expansion parameter in the gradient expansion? To what does the index \( 2n \) refer? The index \( 2n \) is related to the conformal weight of the functional \( S^{(2n)} \). To clarify this point, it is useful to introduce a scaling factor, \( s \). If one rescales the 3-metric using the homogeneous conformal factor, \( s \),

\[ \gamma_{ab}(x) \rightarrow s^2 \gamma_{ab}(x) \]  

(2.17)

one finds that

\[ S^{(2n)}[s^2 \gamma_{ab}, \phi] = s^{(3-2n)} S^{(2n)}[\gamma_{ab}, \phi] . \]  

(2.18)

Hence the gradient expansion is an expansion of the generating functional in powers of the scaling factor \( s \):

\[ S[s^2 \gamma_{ab}, \phi] = \sum_{n=0}^{\infty} s^{(3-2n)} S^{(2n)}[\gamma_{ab}, \phi] , \]  

(2.19)

where \( S^{(2n)}[\gamma_{ab}, \phi] \) are simply the coefficients of \( s^{(3-2n)} \). At the end of the calculation, one sets the scaling factor to unity because it is just a counting parameter.

The interpretation of the spatial gradient expansion as an expansion in powers of the conformal weight may be trivially extended to other situations. For example, instead of rescaling the metric, one may choose to rescale the matter field, \( \phi(x) \rightarrow s\phi(x) \) (or \( \chi(x) \rightarrow s\chi(x) \)) and then expand in powers of \( s \). Such a simple adjustment leads to radically different forms of the generating functional as will be demonstrated in Sec. III and Sec. IV. (In order to reduce the introduction of extraneous notation, one in practice expands in powers of the desired field, and typically foregoes any mention of the scaling factor \( s \).)
III. TAYLOR SERIES EXPANSION IN THE SCALAR FIELD

For the case of a scalar field interacting with gravity, we will now consider a solution for the generating functional in the form of a power series of the scalar field:

$$ S[\gamma_{ab}(x), \phi(x)] = \sum_{m=0}^{\infty} S^{(m)}(x) \quad \text{(series in } \phi). \quad (3.1) $$

The zeroth order term will be assumed to be independent of $\phi$, but otherwise, it is an arbitrary functional of the 3-metric,

$$ S^{(0)} \equiv S^{(0)}[\gamma_{ab}(x)], \quad (3.2a) $$

which is invariant under reparametrizations of the spatial coordinates:

$$ 0 = -2 \left( \frac{\delta S^{(0)}}{\delta \gamma_{ij}(x)} \right)_{ij} + \frac{\delta S^{(0)}}{\delta \gamma_{kl}(x)} \gamma_{kl}, \quad (3.2b) $$

A. Equations for Scalar Field

One substitutes the series into the HJ equation [2.6a], and collects terms of like order, to find:

$$ \left( \frac{\delta S^{(1)}}{\delta \phi(x)} \right)^2 = \gamma R - 2 \gamma V_0 - 2 \left[ 2 \gamma_{il}(x) \gamma_{jk}(x) - \gamma_{ij}(x) \gamma_{kl}(x) \right] \frac{\delta S^{(0)}}{\delta \gamma_{il}(x)} \frac{\delta S^{(0)}}{\delta \gamma_{jk}(x)}, \quad \text{(zeroth order terms),} \quad (3.3a) $$

$$ \frac{\delta S^{(1)}}{\delta \phi(x)} \frac{\delta S^{(2)}}{\delta \phi(x)} = -2 \left[ 2 \gamma_{il}(x) \gamma_{jk}(x) - \gamma_{ij}(x) \gamma_{kl}(x) \right] \frac{\delta S^{(0)}}{\delta \gamma_{il}(x)} \frac{\delta S^{(0)}}{\delta \gamma_{jk}(x)}, \quad \text{(first order terms)} \quad (3.3b) $$

$$ \frac{\delta S^{(1)}}{\delta \phi(x)} \frac{\delta S^{(3)}}{\delta \phi(x)} = -1 \left( \frac{\delta S^{(2)}}{\delta \phi(x)} \right)^2 - \frac{\gamma}{2} m^2 \phi^2 - \frac{\gamma}{2} \gamma^{ab} \phi, a \phi, b \quad \text{(second order terms)} \quad (3.3c) $$

$$ - \left[ 2 \gamma_{il}(x) \gamma_{jk}(x) - \gamma_{ij}(x) \gamma_{kl}(x) \right] \left( \frac{\delta S^{(1)}}{\delta \gamma_{il}(x)} \frac{\delta S^{(1)}}{\delta \gamma_{jk}(x)} + 2 \frac{\delta S^{(0)}}{\delta \gamma_{ij}(x)} \frac{\delta S^{(2)}}{\delta \gamma_{kl}(x)} \right), \quad (3.3d) $$

It is straightforward to derive a general expression for higher order terms in analogy with ref. [2].

B. Solutions for a Scalar Field

The generating functional $S^{(n)}$ for any order may be written in terms of a recursion relation. The first few terms are:

$$ S^{(0)} \equiv S^{(0)}[\gamma_{ab}(x)], \quad (3.4a) $$

$$ S^{(1)} = \int d^3x \gamma^{1/2} \phi(x) \left[ R - 2 \gamma V_0 - 2 \gamma^{-1} \left[ 2 \gamma_{il}(x) \gamma_{jk}(x) - \gamma_{ij}(x) \gamma_{kl}(x) \right] \frac{\delta S^{(0)}}{\delta \gamma_{il}(x)} \frac{\delta S^{(0)}}{\delta \gamma_{jk}(x)} \right]^{1/2}, \quad (3.4b) $$

$$ S^{(2)} = - \int d^3x \phi(x) \left[ 2 \gamma_{il}(x) \gamma_{jk}(x) - \gamma_{ij}(x) \gamma_{kl}(x) \right] \frac{\delta S^{(0)}}{\delta \gamma_{il}(x)} \frac{\delta S^{(0)}}{\delta \gamma_{jk}(x)} \left/ \frac{\delta S^{(1)}}{\delta \phi(x)} \right., \quad (3.4c) $$

$$ S^{(3)} = \frac{1}{3} \int d^3x \phi(x) \left\{ - \frac{1}{2} \left( \frac{\delta S^{(2)}}{\delta \phi(x)} \right)^2 - \frac{\gamma}{2} m^2 \phi^2 - \frac{\gamma}{2} \gamma^{ab} \phi, a \phi, b \right\} \left/ \frac{\delta S^{(1)}}{\delta \phi(x)} \right., \quad (3.4d) $$

The validity of these these formulae will be demonstrated by deriving $S^{(2)}$. The functional equation (3.3b) defining $S^{(2)}$ may be rewritten in the form
\[
\frac{\delta S^{(2)}}{\delta \phi(x)} = -2 \left[ 2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x) \right] \frac{\delta S^{(0)}}{\delta \gamma_{ij}(x)} \frac{\delta S^{(1)}}{\delta \gamma_{kl}(x)} \frac{\delta S^{(1)}}{\delta \phi(x)}.
\]

Given \( S^{(1)} \), the right hand side is known. Hence eq. (3.5) has the form of an infinite dimensional gradient which may be integrated using a contour integral \( \int_\gamma \phi \) in \( \phi \) field-space (the 3-metric is held fixed during such an integration). For the same reasons given in earlier work \([10]\), one may choose an arbitrary contour of integration, and I will use a straight line parameterized by

\[
\gamma(x) = t\phi(x), \quad 0 \leq t \leq 1,
\]

to connect the origin, \( \phi_0(x) = 0 \) to the ‘final point’ \( \phi_f(x) = \phi(x) \):

\[
S^{(2)} = \int d^3x \int_0^1 dt \phi(x) \frac{\delta S^{(2)}}{\delta \phi(x)}
\]

After counting powers of \( t \), the integral over \( t \) gives \( 1/2 \), and hence \( S^{(2)} \) is given by eq. (3.4c). Integrability of \( S^{(2)} \) in eq. (3.5) is guaranteed if the previous order terms, \( S^{(0)} \) and \( S^{(1)} \), are gauge-invariant \([10]\).  

1. **First Example for a Scalar Field**

We will now give some explicit examples. Since \( S^{(0)}[\gamma_{ab}(x)] \) is arbitrary, the possibilities are limitless, but a sampling is instructive.

\[
S^{(0)} = -2H_0 \int d^3x \gamma^{1/2}, \quad H_0 = \sqrt{V_0/3},
\]

\[
S^{(1)} = \int d^3x \gamma^{1/2} \phi(x) R^{1/2},
\]

\[
S^{(2)} = -H_0 \int d^3x \gamma^{1/2} \left[ \phi^2 + \left( \frac{\phi}{\sqrt{R}} \right) |k \left( \frac{\phi}{\sqrt{R}} \right) |k \right].
\]

No spatial derivatives of \( \phi \) appear in \( S^{(1)} \). However, they do appear in \( S^{(2)} \), and they cannot be removed by integration by parts.

Unlike the spatial gradient expansion, the above series is not analytic since \( S^{(1)} \) contains a square root of \( R \). If \( R < 0 \), \( S \) is imaginary, which is classically forbidden. If the sign of the square root is chosen accordingly, these solutions would be exponentially suppressed in a quantum analysis:

\[
\Psi \sim e^{iS}, \quad |\Psi|^2 = \exp \left[ -2 \int d^3x \gamma^{1/2} \phi \sqrt{-R} \right], \quad \text{(for } \phi \geq 0). \]

2. **Second Example for a Scalar Field**

Another simple example arises if we take \( S^{(0)} = 0 \):

\[
S^{(0)} = 0,
\]

\[
S^{(1)} = \int d^3x \gamma^{1/2} \phi (R - 2V_0)^{1/2},
\]

\[
S^{(2)} = 0,
\]

\[
S^{(3)} = -\frac{1}{3} \int d^3x \gamma^{1/2} \frac{\phi}{\sqrt{R - 2V_0}} \left[ \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \gamma^{ab} \phi_{,a} \phi_{,b} + \gamma^{-1} \left[ 2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x) \right] \frac{\delta S^{(1)}}{\delta \gamma_{ij}(x)} \frac{\delta S^{(1)}}{\delta \gamma_{kl}(x)} \right],
\]

where

\[
\frac{\delta S^{(1)}}{\delta \gamma_{ab}(x)} = \frac{\gamma^{1/2}}{2} \left[ (R - 2V_0)\gamma^{ab} - R^{ab} + D^{ab} - \gamma^{ab}D^2 \right] \frac{\phi (R - 2V_0)^{-1/2}}{2}.
\]

If \( R > 2V_0 \), then \( S^{(1)} \) describes the classically forbidden regime.
IV. TAYLOR SERIES EXPANSION IN DUST FIELD

A. Equations for Dust Field

I will now consider an expansion of $S$ in powers of $\chi$:

$$S[\gamma_{ab}(x), \chi(x)] = \sum_{m=0}^{\infty} S^{(m)} \quad \text{(series in } \chi).$$

(4.1)

This expansion is quite general, and it will be applicable in most instances except when one encounters a singular point (which will require a separate treatment). The zeroth order term will be assumed to be independent of $\chi$:

$$S^{(0)} \equiv S^{(0)}[\gamma_{ab}(x)].$$

(4.2)

Expanding all terms in powers of $\chi$, one obtains the following equations:

$$\frac{\delta S^{(1)}}{\delta \chi(x)} = \frac{1}{2} \gamma^{1/2} R - \gamma^{-1/2} [2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x)] \frac{\delta S^{(0)}}{\delta \gamma_{ij}(x)} \frac{\delta S^{(0)}}{\delta \gamma_{kl}(x)},$$

(4.3a)

$$\frac{\delta S^{(2)}}{\delta \chi(x)} = -2\gamma^{-1/2} [2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x)] \frac{\delta S^{(0)}}{\delta \gamma_{ij}(x)} \frac{\delta S^{(1)}}{\delta \gamma_{kl}(x)},$$

(4.3b)

$$\frac{\delta S^{(3)}}{\delta \chi(x)} = -\frac{1}{2} \left( \chi_{[a} \chi^{a]} \right) \frac{\delta S^{(1)}}{\delta \chi(x)} - \gamma^{-1/2} [2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x)] \left[ \frac{\delta S^{(1)}}{\delta \gamma_{ij}(x)} \frac{\delta S^{(1)}}{\delta \gamma_{kl}(x)} + 2 \frac{\delta S^{(1)}}{\delta \gamma_{ij}(x)} \frac{\delta S^{(2)}}{\delta \gamma_{kl}(x)} \right].$$

(4.3c)

B. Solutions for Dust Field

The above equations may be integrated immediately:

$$S^{(0)} \equiv S^{(0)}[\gamma_{ab}(x)],$$

(4.4a)

$$S^{(1)} = \int d^3 x \chi(x) \left[ \frac{1}{2} \gamma^{1/2} R - \gamma^{-1/2} [2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x)] \frac{\delta S^{(0)}}{\delta \gamma_{ij}(x)} \frac{\delta S^{(0)}}{\delta \gamma_{kl}(x)} \right],$$

(4.4b)

$$S^{(2)} = -\int d^3 x \chi(x) \gamma^{-1/2} [2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x)] \frac{\delta S^{(0)}}{\delta \gamma_{ij}(x)} \frac{\delta S^{(1)}}{\delta \gamma_{kl}(x)},$$

(4.4c)

$$S^{(3)} = \frac{1}{3} \int d^3 x \chi(x) \left\{ -\frac{1}{2} \left( \chi_{[a} \chi^{a]} \right) \frac{\delta S^{(1)}}{\delta \chi(x)} - \gamma^{-1/2} [2\gamma_{il}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{kl}(x)] \left[ \frac{\delta S^{(1)}}{\delta \gamma_{ij}(x)} \frac{\delta S^{(1)}}{\delta \gamma_{kl}(x)} + 2 \frac{\delta S^{(1)}}{\delta \gamma_{ij}(x)} \frac{\delta S^{(2)}}{\delta \gamma_{kl}(x)} \right] \right\}. $$

(4.4d)

One may interpret the higher order terms, $S^{(1)}, S^{(2)}, \ldots$, as describing the evolution in time $\chi(x)$ of the the initial state $S^{(0)}[\gamma_{ab}(x)]$.

1. First Example for a Dust Field

If $S^{(0)}$ is proportional to the volume of a given 3-geometry, the first few terms are:

$$S^{(0)} = C \int d^3 x \gamma^{1/2},$$

(4.5a)

$$S^{(1)} = \int d^3 x \gamma^{1/2} \left[ \frac{R}{2} + \frac{3C^2}{4} \right],$$

(4.5b)

$$S^{(2)} = C \int d^3 x \gamma^{1/2} \left[ \lambda^2 \left( \frac{R}{8} + \frac{9C^2}{16} \right) + \frac{1}{2} \lambda \chi_{[a} \chi_{k]} \right].$$

(4.5c)
2. Second Example for a Dust Field

As a second example, consider a series whose first term is proportional to an integral of the spatial curvature:

$$S(0) = E \int d^3x \gamma^{1/2} R,$$

$$S(1) = \int d^3x \gamma^{1/2} \chi \left[ \frac{R}{2} - 2E^2 \left( R^{ab} R_{ab} - \frac{3}{8} R^2 \right) \right].$$

V. SUPERPOSITION OF HAMILTON-JACOBI SOLUTIONS

One of the very attractive features of quantum mechanics is the principle of linear superposition: if \( \psi_1 \) and \( \psi_2 \) are solutions of the quantum theory, then \( \psi(y) = \psi_1 + \psi_2 \) is a solution as well. This principle has no direct classical interpretation. Nonetheless, there are situations where one can indeed construct additional solutions to the HJ equation from other known solutions. One can enunciate a principle of superposition for Hamilton-Jacobi theory which is motivated by a semiclassical treatment of the quantum theory.

For a quantum system with configuration variable, \( y \), suppose that one is fortunate enough to find a solution to the Schrodinger equation, \( \psi(y|a) \) which depends on a continuous parameter \( a \). Then any linear superposition of these solutions is also a solution:

$$\psi(y) = \int da \, w(a) \psi(y|a), \quad (5.1)$$

where the weighting function \( w(a) \) is arbitrary. If we work in the semiclassical limit, \( \hbar \to 0 \), then \( \psi(y|a) \) and \( w(a) \) may be approximated by phase factors,

$$\psi(y|a) \sim e^{iS(y|a)/\hbar}, \quad w(a) = e^{ig(a)/\hbar}; \quad (5.2)$$

\( S(y|a) \) is then a solution of the HJ equation which depends on a parameter \( a \). The resulting integral \( (5.1) \) may approximated using the stationary phase approximation,

$$\psi(y) = \exp[i(S(y|a) + g(a))/\hbar] \quad (5.3a)$$

where \( a \equiv a(y) \) is now chosen so that the phase of the integrand has a maximum or minimum,

$$0 = \frac{\partial}{\partial a} [S(y|a) + g(a)]. \quad (5.3b)$$

Hence the principle of linear superposition in quantum mechanics and the stationary phase approximation lead to the principle of superposition for Hamilton-Jacobi theory:

If \( S(y|a) \) is a solution of the HJ equation which depends on a continuous parameter \( a \), then

$$T(y) = S(y|a) + g(a) \quad (5.4a)$$

is also a solution provided that \( a \equiv a(y) \) is determined by the stationary phase condition,

$$0 = \frac{\partial S(y|a)}{\partial a} + \frac{\partial g(a)}{\partial a}. \quad (5.4b)$$

The proof given above was motivated by the quantum theory. In a gravitational context, it is not clear that a consistent quantum theory exists (at present). However, one can verify the principle totally within a classical context by noting that

$$\frac{\partial T(y)}{\partial y} = \left. \frac{\partial S(y|a)}{\partial y} \right|_a + \left. \frac{\partial S(y|a)}{\partial a} \frac{\partial a}{\partial y} + \frac{\partial g(a)}{\partial a} \frac{\partial a}{\partial y} \right|_a \quad (5.5)$$

The last two terms vanish by virtue of the stationary phase condition \( (5.4b) \), and thus a derivative of \( S(y) \) with respect to \( y \) coincides with a derivative of \( S(y|a) \) with respect to \( y \) (holding \( a \) fixed):

$$\frac{\partial T(y)}{\partial y} = \left. \frac{\partial S(y|a)}{\partial y} \right|_a \quad (5.6)$$

Since \( S \) appears in the HJ equation only in terms of its derivatives with respect to the configuration variables, the principle is justified.
A. Applying the Superposition to the HJ Equation for Gravity

The superposition principle leads to some rather exotic solutions of the HJ equation for general relativity. To illustrate the basic principles, we will consider a massless scalar field, \( m = 0 \) with vanishing cosmological constant, \( V_0 = 0 \), which has the following spatial gradient expansion solution for the HJ eq. (2.6a),

\[
S[\gamma_{ab}(x), \phi(x)] = S^{(0)} + S^{(2)} + \ldots,
\]

(5.7a)
of which the first two terms in a spatial gradient expansion are

\[
S^{(0)} = -2C \int d^3x \gamma^{1/2} e^{3/2\phi},
\]

(5.7b)

\[
S^{(2)} = \int d^3x f^{1/2} \left[ \frac{1}{C} e^{-4\phi/\sqrt{6}} \left( \frac{1}{8} R^f + \frac{1}{6} f_{ab} \phi_a \phi_b \right) + E R^f \right].
\]

(5.7c)

\( C \) and \( E \) are homogeneous but arbitrary parameters. The new 3-metric \( f_{ab} \) is related to \( \gamma_{ab} \) by a conformal transformation:

\[
f_{ab} = e^{2\phi/\sqrt{6}} \gamma_{ab}.
\]

(5.8)

\( R^f \) is the Ricci Scalar associated with \( f_{ab} \).

Consider a new solution \( T \) of the HJ equation given by the sum

\[
T = S - \frac{1}{2} C^2,
\]

(5.9a)

where \( T \) satisfies the stationary phase condition with respect to \( C \):

\[
0 = \frac{\partial T}{\partial C} = \frac{\partial S}{\partial C} - C.
\]

(5.9b)

Using a spatial gradient expansion, one can solve for \( C \):

\[
C = C_0 - \frac{1}{C_0^2} \int d^3x f^{1/2} e^{-4\phi/\sqrt{6}} \left( \frac{1}{8} R^f + \frac{1}{6} f_{ab} \phi_a \phi_b \right) + \ldots
\]

(5.10a)

where the functional \( C_0 \) is given by

\[
C_0 = -2 \int d^3x \gamma^{1/2} e^{3/2\phi}.
\]

(5.10b)

\( T \) admits the following spatial gradient expansion:

\[
T = T^{(0)} + T^{(2)} + \ldots
\]

(5.11a)

with

\[
T^{(0)} = 2 \left( \int d^3x \gamma^{1/2} e^{3/2\phi} \right)^2,
\]

(5.11b)

\[
T^{(2)} = E \int d^3x f^{1/2} R^f + \left[ \int d^3x f^{1/2} e^{-4\phi/\sqrt{6}} \left( -\frac{1}{16} R^f - \frac{1}{12} f_{ab} \phi_a \phi_b \right) \right] \left/ \left( \int d^3x \gamma^{1/2} e^{3/2\phi} \right) \right.
\]

(5.11c)

One may verify directly that eq. (5.11a) is solution of the HJ eq. (2.6a).

Unlike the original gradient expansion solution (5.7a) for \( S \), the new solution \( T \) is no longer an integral over local terms, but contains products and quotients of local integrals. Hence very complicated solutions of the HJ equations may be generated using the nonlinear superposition of local solutions.
VI. CONCLUSIONS

A coordinate-free analysis of general relativity possesses distinct advantages over traditional methods of solving Einstein’s equations. When solving the field equations, one typically makes arbitrary gauge choices. In many situations, the optimal choice of gauge is not very clear, and a poor choice of gauge can complicate the analysis significantly. In the coordinate-free method expounded here, one may solve the functional HJ equation without making any gauge choices. However, applications of the HJ equation to physical problems have previously been hampered by the lack of mathematical tools. Some general and useful techniques have been developed which will be applied later in cosmology. However, the functional approach is a radically different way of describing gravity. More work is required in order to develop a more intuitive understanding of the method.

Although one can obtain exact general solutions for gravitational superspace in two spacetime dimensions (see, for example, [32]), it is doubtful that one may construct such solutions in four spacetime dimensions. One must resort to some approximation method. The spatial gradient expansion was the prototype solution of the HJ equation for gravity and matter. It is basically a Taylor series expansion in the conformal weight factor of the 3-metric. By expanding the generating functional in terms of other fields, one may construct numerous other solutions to the HJ equation for general relativity. Explicit expansions in either a scalar field $\phi$ or a dust field $\chi$ were demonstrated explicitly. These solutions describe the evolution of some gauge-invariant initial state $\mathcal{S}^{(0)}[\gamma_{ab}(x)]$ as a functional of the matter field. Integrability of these solutions is guaranteed by spatial gauge-invariance.

Many of these solutions depend on continuous parameters. One can construct additional solutions by using the Superposition Principle for Hamilton-Jacobi Theory which is motivated by the superposition principle in quantum mechanics in conjunction with the stationary phase approximation. One can in effect construct complicated solutions by integrating over the continuous parameters.

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