RENAULT’S $j$-MAP FOR FELL BUNDLE $C^*$-ALGEBRAS

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Abstract. If $p: B \to G$ is a Fell bundle over an étale groupoid, then we show that there is an norm reducing injective linear map $j: C^*_r(G; B) \to \Gamma_0(G; B)$ generalizing the well know map $j: C^*_r(G) \to C_0(G)$ in the case of an étale groupoid.

1. Introduction

One of the key tools in working with étale groupoids is Renault’s “$j$-map” from [Ren80, Proposition II.4.2]. Specifically, Renault shows that there is an injective norm reducing linear map $j: C^*_r(G) \to C_0(G)$ such that $j(f)(\gamma) = f(\gamma)$ when $f \in C_c(G)$. Crucially, $j$ also preserves the algebraic operations so that if $a, b \in C^*_r(G)$, then

$$j(a^*)(\gamma) = \overline{j(a)(\gamma^{-1})} \quad \text{and} \quad j(a \ast b)(\gamma) = \sum_{\eta \in G^r(\gamma)} j(a)(\eta) j(b)(\eta^{-1}\gamma).$$

In fact, Renault works with a continuous 2-cocycle, and has observed in [Ren08] that, more generally, a similar result holds for twists over étale groupoids. A proof is supplied in [BFPR21, Proposition 2.8].

The purpose of this note is to extend this result to the $C^*$-algebra of a Fell bundle $p: B \to G$ over an étale groupoid $G$. Specifically, we prove the following.

Theorem 1.1. Suppose that $p: B \to G$ is a separable, saturated Fell bundle over a second countable, locally compact Hausdorff étale groupoid $G$. Then there is an injective norm reducing linear map $j: C^*_r(G; B) \to \Gamma_0(G; B)$ such that $j(f)(\gamma) = f(\gamma)$ if $f \in \Gamma_c(G; B)$. Furthermore, for all $a, b \in C^*_r(G; B)$ and $\gamma \in G$, we have

$$j(a^*)(\gamma) = j(a)(\gamma^{-1})^*$$
and

$$j(a \ast b)(\gamma) = \sum_{\eta \in G^r(\gamma)} j(a)(\eta) j(b)(\eta^{-1}\gamma)$$

where the sum converges in the norm topology in $B_\gamma$.

Date: 2 March 2022.

This research was supported by the Edward Shapiro fund at Dartmouth College. The first-named author was partially supported by a RITA Investigator grant (IV017).
The existence of $j$ is reasonably straightforward as is the adjoint property (1.2). However, to be truly useful in applications, we clearly need to establish the convolution result (1.3). Unlike the situation of Renault’s original result, the convolution result requires considerable technology. Renault is able to show that the sum in (1.1) converges absolutely basically using Hölder’s inequality. Here we have to use the internal tensor product of Hilbert modules. As a consequence, we are only able to show convergence of the sum in (1.3) in the strong sense described in Remark 2.3. We suspect that the sum does not converge absolutely in general.

Our result subsumes earlier results for crossed products by discrete groups twisted by a unitary 2-cocycle in [ZM68].

**Assumptions.** Whenever possible we assume that our topological spaces are second countable and that our $C^*$-algebras are separable. In particular, $G$ will always denote a second countable, locally compact Hausdorff étale groupoid. We also adopt the standard convention that homomorphisms between $C^*$-algebras are $*$-preserving.

## 2. Preliminaries

Fell bundles over groupoids are a natural generalization of Fell’s $C^*$-algebraic bundles from [FD88a, Chapter VIII]. They were introduced in [Yam87]. For more details, see [Kum98, MW08]. Roughly speaking, a Fell bundle $\mathcal{B}$ over a locally compact Hausdorff groupoid $G$ is a (upper-semicontinuous) Banach bundle $p: \mathcal{B} \to G$ endowed with a continuous involution $b \mapsto b^*$ and a continuous multiplication $(a, b) \mapsto ab$ from $\mathcal{B}^{(2)} = \{(b, b') : (p(b), p(b') \in G^{(2)}\}$ to $\mathcal{B}$ such that—with respect to the operations, actions, and inner products induced by the involution and multiplication—the fibres $B_u = p^{-1}(u)$ over units $u \in G^{(0)}$ are $C^*$-algebras and such that each fibre $B_\gamma = p^{-1}(\gamma)$ is a $B_r(\gamma)-B_s(\gamma)$-imprimitivity bimodule.

Our references for Banach bundles are [MW08, Appendix A] and, for the $C^*$-bundle case, [Wil07, Appendix C]. As is now common practice, we drop the adjective “upper-semicontinuous” in front of the term “Banach bundle”. If the map $a \mapsto \|a\|$ is continuous rather than merely upper-semicontinuous, then we include the adjective continuous. An excellent reference for continuous Banach bundles is §§13–14 of [FD88a, Chap. II]. If $p: \mathcal{B} \to X$ is a Banach bundle, we will write $\Gamma_0(X; \mathcal{B})$ for the continuous sections of $\mathcal{B}$ which vanish at infinity. Furthermore, $\Gamma_0(X; \mathcal{B})$ is a Banach space with respect to the supremum norm (see [DG83, p. 10] or [Wil07, Proposition C.23]).

We write $\Gamma_c(G; \mathcal{B})$ for the $*$-algebra of continuous compactly supported sections of $\mathcal{B}$ under convolution and involution. If $Y \subset G$, then we write $\Gamma_c(Y; \mathcal{B})$ for the continuous compactly supported sections of $\mathcal{B}$ restricted to $Y$. Of course such sections take values in $\mathcal{B}_Y := p^{-1}(Y)$ which is usually called the the restriction of $\mathcal{B}$ to $Y$. If $Y$ is closed, then any $f \in \Gamma_c(Y; \mathcal{B}_Y)$ is the restriction of a section in $\Gamma_c(G; \mathcal{B})$ to $Y$ by the Tietze Extension Theorem for Banach bundles [MW08, Proposition A.5].
As in [SW13], we note that $A = \Gamma_0(G^{(0)}; \mathcal{B})$ is a $C^*$-algebra called the $C^*$-algebra of $\mathcal{B}$. If $u \in G^{(0)}$, then we often write $A_u$ for $B_u$ when we want to think of $B_u$ as a $C^*$-algebra. Then, for example, $B_\gamma$ is an $A_{r(\gamma)} - A_{s(\gamma)}$-imprimitivity bimodule. We rely on [SW13, §4] for the definition of, and standard results for, the reduced norm on $\Gamma_c(G; \mathcal{B})$. In keeping with our standing assumptions, we assume that our Banach bundles $p: \mathcal{B} \to G$, and hence our Fell bundles, are separable in the sense that $\Gamma_0(G; \mathcal{B})$ is a separable Banach space.

**Remark 2.1.** It was observed in [BMZ13, Lemma 3.16] that any Fell bundle over a group is necessarily a continuous Banach bundle. This is also the case for a Fell bundle $\mathcal{B}$ over a groupoid $G$ in the case where the associated $C^*$-algebra $A = \Gamma_0(G^{(0)}; \mathcal{B})$ is a continuous $C^*$-bundle over $G^{(0)}$.

Note that the topology on the total space $\mathcal{B}$ is not necessarily Hausdorff when $p: \mathcal{B} \to G$ is not a continuous Banach bundle (see [Wil07, Example C.27]). However, the following observation often allows us to finesse this difficulty. We include a proof for convenience.

**Lemma 2.2.** Let $p: \mathcal{B} \to X$ be a Banach bundle over a locally compact Hausdorff space $X$. Then the relative topology on each fibre $\mathcal{B}_x$ is the norm topology.

**Proof.** Suppose $(a_i)$ is a net in $\mathcal{B}_x$ converging to $a$ in $\mathcal{B}$. Since $X$ is Hausdorff and $p$ continuous, $a \in \mathcal{B}_x$, and $(a_i - a)$ converges to $0_x$ in $\mathcal{B}$. Since $b \mapsto \|b\|$ is upper-semicontinuous, $\{b: \|b\| < \epsilon\}$ is a neighborhood of $0_x$ in $\mathcal{B}$. It follows that $\|a_i - a\| \to 0$.

If $(a_i) \to a$ in norm in $\mathcal{B}_x$, then $\|a_i - a\| \to 0$. It now follows from the Banach bundle axioms (for example axiom B4 from [MW08, Definition A.1]), that $a_i - a \to 0_x$ in $\mathcal{B}$. Then $a_i \to a$ in $\mathcal{B}$. \qed

**Remark 2.3 (Sums).** We should explain what we mean by sums such as (1.3) when $Gr^{(\gamma)}$ is infinite. If $f: X \to V$ is a function on a set $X$ taking values in a topological vector space $V$, then

\begin{equation}
\sum_{x \in X} f(x)
\end{equation}

is defined to be the limit, if it exists, of the net $(s_F)$ where $F$ is finite subset of $X$, $s_F = \sum_{x \in F} f(x)$, and $(s_F)$ is directed by containment: $F_1 \leq F_2$ if $F_1 \subset F_2$. It is not hard to see that if $X$ is countably infinite and if $(x_n)$ is any enumeration of $X$, then

\begin{equation}
\sum_{n=1}^{\infty} f(x_n)
\end{equation}

converges if (2.1) does and then the sums coincide. In particular, the sum in (2.2) is invariant under rearrangement if (2.1) converges.

\footnote{For continuous bundles, this is [FD88a, Proposition II.13.11].}
Note that if for some enumeration of $X$, the sum in (2.2) converges, it is not necessarily the case that (2.1) converges—consider a conditional convergent series in $\mathbb{R}$. But if $V$ is a $C^*$-algebra and if $f(x)$ is always positive in $V$, then the converse holds.

Recall from [RW98, pp. 49–50] that if $X$ is an $A$–$B$-imprimitivity bimodule, then the dual module $\tilde{X}$ is the $B$–$A$-imprimitivity bimodule defined as follows. We let $\tilde{X}$ be the conjugate vector space to $X$. This means that $\tilde{X}$ is equal to $X$ as a set. If $\flat: X \rightarrow \tilde{X}$ is the identity map, then $\flat(\lambda \cdot x) = \overline{\lambda} \cdot \flat(x)$ for $\lambda \in \mathbb{C}$. The left $B$-action on $\tilde{X}$ is then given by $b \cdot \flat(x) = \flat(b^* \cdot x)$ and the left $B$-valued inner product is given by $B\langle \flat(x) , \flat(y) \rangle = \langle x , y \rangle_B$. Similar formulas hold for the right $A$-action and right $A$-valued inner product.

If $X$ is an $A$–$B$-imprimitivity bimodule, and if $Y$ is a $B$–$C$-imprimitivity bimodule, then the internal tensor product $X \otimes_B Y$ is an $A$–$C$-imprimitivity bimodule with respect to the obvious actions and the inner product given, for example, in [RW98, Proposition 3.16].

**Lemma 2.4.** Suppose that $X$ is a right Hilbert $B$-module and that $Y$ is a $B$–$A$-imprimitivity bimodule. Then if $x \otimes y \in X \otimes_B Y$, then

$$\|x \otimes y\|_{X \otimes_B Y} \leq \|x\|_X \|y\|_Y.$$  

**Proof.** We have

$$\|x \otimes y\|_{X \otimes_B Y}^2 = \|\langle x \otimes y , x \otimes y \rangle_A\| = \|\langle x , x \rangle_B \cdot y , y \rangle_A\|$$

which, by Cauchy–Schwarz (see [RW98, Lemma 2.5 and Corollary 2.7]), is

$$\leq \|\langle x , x \rangle_B \cdot y\|_Y \|y\|_Y = \|\langle x , x \rangle_B\|_Y^2 = \|x\|_X^2 \|y\|_Y^2. \quad \Box$$

### 3. The Module $X_u$

Since we are always assuming that $G$ is étale, for any $u \in G^{(0)}$, $G_u$ is a closed, discrete subset of $G$. Hence $\Gamma_c(G_u; \mathcal{B})$ is just the set of finitely supported functions $f$ on $G_u$ such that $f(\eta) \in B_\eta$ for each $\eta \in G_u$. We let $X_u$ be the full right Hilbert $A_u$-module that is the completion of $\Gamma_c(G_u; \mathcal{B})$ with respect to the pre-inner product

$$\langle h , k \rangle_{A_u} = \sum_{\eta \in G_u} h(\eta)^* k(\eta),$$

equipped with the obvious right $A_u$-action. As always, denote by $\mathcal{L}(X_u)$ the $C^*$-algebra of adjointable operators on $X_u$. 
Lemma 3.1. For each \( u \in G^{(0)} \), there is a homomorphism \( V_u : C^*_r(G; B) \to \mathcal{L}(X_u) \) such that for all \( \zeta \in G_u \)
\[
(3.1) \quad V_u(f)(h)(\zeta) = \sum_{\eta \in G^{(0)}(u)} f(\eta) h(\eta^{-1}\zeta) \quad \text{for } f \in \Gamma_c(G; B) \text{ and } h \in \Gamma_c(G_u; B).
\]
Furthermore, for all \( a \in C^*_r(G; B) \), \( \|a\| = \sup_{u \in G^{(0)}} \|V_u(a)\|_{X_u} \) where \( \| \cdot \|_{X_u} \) is the norm on \( X_u \).

Proof. Observe that \( X_u \) is the right Hilbert \( A_u \)-module constructed in [SW13, §4.1] for the subgroupoid \( H = \{u\} \); that is, \( X_u \) is the module used to induce representations from \( A_u \) to \( C^*(G; B) \). Moreover, as described in [SW13, §4.2], such induced representations are regular representations and factor through \( C^*_r(G; B) \). Furthermore, there is a nondegenerate homomorphism \( V : C^*(G; B) \to \mathcal{L}(X_u) \) satisfying (3.1). Hence if \( \pi_u \) is a faithful representation of \( A_u \), then as in [SW13, Example 13], the regular representation \( \text{Ind} \pi_u \) acts on the completion of \( \Gamma_c(G_u; B) \circlearrowright \mathcal{H}_{\pi_u} \) with respect to the pre-inner product determined on elementary tensors by
\[
(f \otimes h \mid g \otimes k) = \sum_{\eta \in G_u} (\pi_u(g(\eta)^* f(\eta)))h \mid k).
\]
In particular, arguing as in the proof of [SW13, Lemma 9], we see that \( \ker V = \ker(\text{Ind} \pi_u) \). Therefore \( \ker V \supset \{ a \in C^*(G; B) : \|a\| = 0 \} \) and \( V \) factors through a homomorphism \( V_u \) as claimed. Since \( \bigoplus_{u \in G^{(0)}} \pi_u \) is a faithful representation of \( A = \Gamma_0(G^{(0)}; B) \), the \( V_u \) determine the reduced norm as claimed. \( \square \)

Lemma 3.2. We can realize \( X_u \) as the right Hilbert \( A_u \)-module
\[
(3.2) \quad \left\{ x \in \prod_{\eta \in G_u} B_\eta : \sum_{\eta \in G_u} x(\eta)^* x(\eta) \text{ converges in } A_u \right\}
\]
equipped with the inner product and right \( A_u \)-action given by
\[
(3.3) \quad \langle x, y \rangle_{A_u} = \sum_{\eta \in G_u} x(\eta)^* y(\eta) \quad \text{and} \quad (x \cdot a)(\eta) = x(\eta) \cdot a.
\]

Remark 3.3. Part of the conclusion of Lemma 3.2 is that the sum in (3.3) converges in \( A_u \) as in Remark 2.3.

Proof. Arguing as in [RW98, Proposition 2.15], or more generally, [Lan95, p. 6], (3.2) determines a right Hilbert \( A_u \)-module. Let \( \| \cdot \|_0 \) be the corresponding Hilbert \( A_u \)-module norm induced by the inner product given in (3.3). If \( x \) belongs to (3.2) and if \( F \) is a finite subset of \( G_u \), then we let \( x_F = 1_F x \). We claim that the net \( (x_F) \) converges to \( x \). In fact, \( x = x_F \| = \| x - x_F \|_{A_u} \).}

Since \( X_u \) is Hilbert \( A_u \)-module, its norm, \( \| \cdot \|_{X_u} \), is often denoted by \( \| \cdot \|_{A_u} \). However, this notation would prove confusing in the sequel.
where the last equality follows from the fact that

\( (3.4) \)

\[
\langle x, x \rangle_{A_u} - \langle x, x_F \rangle_{A_u} - \langle x_F, x \rangle_{A_u} + \langle x_F, x_F \rangle_{A_u} \]

(3.4)

\[
= \| \langle x, x \rangle_{A_u} - \langle x_F, x_F \rangle_{A_u} \|
\]

where the last equality follows from the fact that

\[
\langle x, x_F \rangle_{A_u} = \langle x_F, x \rangle_{A_u} = \langle x_F, x_F \rangle_{A_u}.
\]

Clearly, \( (3.4) \) tends to zero with \( F \). Hence, we can view \( \Gamma_c(G_u; \mathcal{B}) \) as a dense subspace. The result follows from this.

\[ \square \]

\begin{notation}
If \( \gamma \in G_u \) and \( b \in B_\gamma \), we let \( h^b_\gamma \) be the section that takes the value \( b \) at \( \gamma \) and is zero elsewhere. Since \( G_u \) is discrete, \( h^b_\gamma \) is an element of \( \Gamma_c(G_u; \mathcal{B}) \), and elements of this form span a dense subspace of \( X_u \). Note that
\end{notation}

\[
(3.5) \quad \| h^b_\gamma \|_{X_u} = \| b \|.
\]

\begin{proposition}
There is an injective norm reducing linear map \( j: \mathcal{C}^*_r(G; \mathcal{B}) \to \Gamma_0(G; \mathcal{B}) \) such that \( j(f)(\gamma) = f(\gamma) \) for all \( f \in \Gamma_c(G; \mathcal{B}) \).
\end{proposition}

\begin{proof}
Since

\[
\langle h, V_u(f)k \rangle_{A_u} = \sum_{\zeta \in G_u} h(\zeta)^* [V_u(f)k](\zeta) = \sum_{\zeta \in G_u} \sum_{\eta \in G^u(\zeta)} h(\zeta)^* f(\eta) k(\eta^{-1}\zeta),
\]

we have

\[
(3.6) \quad \langle h^b_\gamma, V_u(f)h^c_\beta \rangle_{A_u} = b^* f(\gamma/b^{-1})c \quad \text{for all } \gamma, \beta \in G_u \text{ and } f \in \Gamma_c(G; \mathcal{B}).
\]

In particular, if \((e_n)\) is an approximate identity for \( A_u \)

\[
\langle h^f(\gamma), V_u(f)h^e_n \rangle_{A_u} = f(\gamma)^* f(\gamma)e_n.
\]

Thus, by Cauchy–Schwarz,

\[
\| f(\gamma)^* f(\gamma)e_n \| = \| \langle h^f(\gamma), V_u(f)h^e_n \rangle_{A_u} \| \leq \| h^f(\gamma) \| \| V_u(f) \| \| h^e_n \|
\]

which, by \( (3.5) \) and since \( \| e_n \| \leq 1 \), is

\[
\leq \| f(\gamma) \|_{B_u} \| V_u(f) \|
\]

\[
\leq \| f(\gamma) \|_r \| f \|_r.
\]

On the other hand, \( f(\gamma)^* f(\gamma)e_n \to f(\gamma)^* f(\gamma) \) in \( A_u \). Therefore

\[
\lim_n \| f(\gamma)^* f(\gamma)e_n \|_{A_u} = \| f(\gamma)^* f(\gamma) \|_{A_u} = \| f(\gamma) \|_{B_u}^2.
\]

It follows that for any \( \gamma \in G_u \),

\[
\| f(\gamma) \|_{B_u} \leq \| f \|_r.
\]

Since \( u \in G^{(0)} \) was arbitrary, we have

\[
\| f \|_\infty \leq \| f \|_r \quad \text{for any } f \in \Gamma_c(G; \mathcal{B}).
\]
It follows that the map sending \( f \in \Gamma_c(G; \mathcal{B}) \subset C_r^*(G; \mathcal{B}) \) to \( f \in \Gamma_0(G; \mathcal{B}) \) is a bounded linear map. Hence we can extend to the completion and obtain a linear map

\[
j : C_r^*(G; \mathcal{B}) \to \Gamma_0(G; \mathcal{B})
\]
such that \( \|j(f)\|_\infty \leq \|f\|_r \) and such that \( j(f) = f \) for all \( f \in \Gamma_0(G; \mathcal{B}) \).

To see that \( j \) is injective, it suffices to see that if \( a \in C_r^*(G; \mathcal{B}) \) and \( a \neq 0 \), then \( j(a) \neq 0 \). But if \( a \neq 0 \), then by Lemma [3.1] there is a \( u \in G^{(0)} \) such that \( V_u(a) \neq 0 \).

Since elements of the form \( h^b_\gamma \) with \( \gamma \in G_u \) and \( b \in B_\gamma \) span a dense subspace of \( X_u \), there must be vectors \( h^b_\gamma \) and \( h^c_\beta \) in \( \Gamma_c(G_u; \mathcal{B}) \subset X_u \) such that

\[
\langle h^b_\gamma, V_u(a)h^c_\beta \rangle_{A_u} \neq 0.
\]

Let \((f_n)\) be a sequence in \( \Gamma_c(G; \mathcal{B}) \) converging to \( a \) in \( C_r^*(G; \mathcal{B}) \). Since \( j \) is bounded, \( j(f_n) \to j(a) \) in \( \Gamma_0(G; \mathcal{B}) \). In particular,

\[
j(a)(\gamma^{-1}) = \lim_n j(f_n)(\gamma^{-1}).
\]

Since multiplication in \( \mathcal{B} \) is continuous and associative, this implies

\[
b^*j(a)(\gamma^{-1})c = \lim_n b^*j(f_n)(\gamma^{-1})c.
\]

On the other hand, by [3.6],

\[
b^*j(f_n)(\gamma^{-1})c = b^*f_n(\gamma^{-1})c = \langle h^b_\gamma, V_u(f_n)h^c_\beta \rangle_{A_u}.
\]

Since \( V_u \) is bounded,

\[
\lim_n \langle h^b_\gamma, V_u(f_n)h^c_\beta \rangle_{A_u} = \langle h^b_\gamma, V_u(a)h^c_\beta \rangle_{A_u}.
\]

We have shown that \( b^*j(f_n)(\gamma^{-1})c \) converges to both \( b^*j(a)(\gamma^{-1})c \) and \( \langle h^b_\gamma, V_u(a)h^c_\beta \rangle_{A_u} \) in a fibre of \( \mathcal{B} \). Since those fibres are Hausdorff by Lemma [22], we conclude

\[
\langle h^b_\gamma, V_u(a)h^c_\beta \rangle_{A_u} = b^*j(a)(\gamma^{-1})c.
\]

Since the left-hand side of [3.7] is nonzero by assumption, it follows that \( j(a)(\gamma^{-1}) \neq 0 \). Thus, \( j \) is injective.

**Corollary 3.6.** If \( a \in C_r^*(G; \mathcal{B}) \), then \( j(a^*)(\eta) = j(a)(\eta^{-1})^* \).

**Proof.** Let \((f_n)\) be a sequence in \( \Gamma_c(G; \mathcal{B}) \) such that \( f_n \to a \) in \( C_r^*(G; \mathcal{B}) \). Then for all \( \eta \in G \), \( f_n(\eta) = j(f_n(\eta)) \to j(a(\eta)) \) and \( f^*_n(\eta) = j(f^*_n(\eta)) \to j(a^*)(\eta) \). But by definition, \( f^*_n(\eta) = f_n(\eta^{-1})^* \). Hence \((f_n(\eta^{-1})^*)\) converges to both \( j(a)(\eta^{-1})^* \) and \( j(a^*)(\eta) \) in \( B_\eta \). Since the relative topology on \( B_\eta \) is the norm topology, it is Hausdorff, and \( j(a^*)(\eta) = j(a)(\eta^{-1})^* \) as required.

If \( a \in C_r^*(G; \mathcal{B}) \), then for any \( u \in G^{(0)} \), the restriction of \( j(a) \) to \( G_u \) defines an element \( j_u(a) \) of \( \prod_{\eta \in G_u} B_\eta \). In particular, we have the following.
Proposition 3.7. If $a \in C_r^*(G; \mathcal{B})$ and $u \in G^{(0)}$, then $j_u(a) \in X_u$ and

(3.8) \[ \|j_u(a)\|_{X_u} \leq \|a\|_r. \]

Proof. Suppose that $a = f \in \Gamma_c(G; \mathcal{B})$. Then if $(e_n)$ is an approximate identity for $A_u$, we have

\[ [V_u(f)h_u^e_n](\gamma) = f(\gamma)e_n. \]

Thus, on the one hand,

\[ \|V_u(f)h_u^e_n\|_{X_u}^2 \leq \|f\|_r^2\|h_u^e_n\|_{X_u}^2 \leq \|f\|_r^2. \]

On the other hand,

\begin{align*}
\|V_u(f)h_u^e_n\|_{X_u}^2 &= \|(V_u(f)h_u^e_n, V_u(f)h_u^e_n)_{A_u}\|_{A_u} \\
&= \left\| \sum_{\eta \in G_u} e_n f(\eta)^* f(\eta) e_n \right\|_{A_u} = \left\| e_n \left( \sum_{\eta \in G_u} f(\eta)^* f(\eta) \right) e_n \right\|_{A_u}.
\end{align*}

(3.9)

Note that as $f$ is compactly supported, only finitely many of the summands in each sum in (3.9) are nonzero, so each sum converges to an element of $A_u$. Thus, for all $n$,

\[ \left\| e_n \left( \sum_{\eta \in G_u} f(\eta)^* f(\eta) \right) e_n \right\|_{A_u} \leq \|f\|_r^2. \]

We conclude that for all $f \in \Gamma_c(G; \mathcal{B})$,

(3.10) \[ \|f\|_{X_u}^2 = \left\| \sum_{\eta \in G_u} f(\eta)^* f(\eta) \right\|_{A_u} \leq \|f\|_r^2. \]

Thus, (3.8) holds provided $a = f \in \Gamma_c(G; \mathcal{B})$.

Suppose that $(f_n)$ is a sequence in $\Gamma_c(G; \mathcal{B})$ such that $f_n \to a$ in $C_r^*(G; \mathcal{B})$. Since

\[ \|j_u(f_n) - j_u(f_m)\|_{X_u} = \|j_u(f_n - f_m)\|_{X_u} \leq \|f_n - f_m\|_r \]

by (3.10), it follows that $(j_u(f_n))$ is Cauchy in $X_u$. Since $X_u$ is complete, $(j_u(f_n))$ converges to some $y \in X_u$. Since the map $x \mapsto x(\eta)$ is clearly a bounded linear map from $X_u$ to $B_{\eta}$, “evaluation at $\eta$” is continuous on $X_u$. Hence $j_u(f_n)(\eta) \to y(\eta)$ in $B_\eta$ for all $\eta \in G_u$. Further, as $j$ is continuous, we have $j(f_n) \to j(a)$ uniformly. Since $B_\eta$ is Hausdorff, we conclude that $y = j_u(a)$. Hence, $j_u(a) \in X_u$ and $\|j_u(a)\|_{X_u} = \lim_u \|j_u(f_n)\|_{X_u} \leq \lim_n \|f_n\|_r = \|a\|_r$. \qed

As a corollary of the proof, we have the following.

Corollary 3.8. Suppose that $(f_n) \subset \Gamma_c(G; \mathcal{B})$ converges to $a \in C_r^*(G; \mathcal{B})$. Then $(j_u(f_n))$ converges to $j_u(a)$ in $X_u$. 
4. The Module $\mathcal{W}_\gamma$

Fix $\gamma \in G$ and let $r(\gamma) = u$ and $s(\gamma) = v$. Let $\mathcal{W}_0^0$ be the vector space of finitely supported functions $\xi : G_u \to \mathcal{B}$ such that $\xi(\eta) \in B_{\eta \gamma}$. Then $\mathcal{W}_0^0$ carries an obvious right $A_u$-action: $\xi \cdot a(\eta) = \xi(\eta)a$. If $\xi, \zeta \in \mathcal{W}_0^0$, then for each $\eta \in G_u$, $\xi(\eta)^* \zeta(\eta)$ is just the $A_u$-valued inner product in $B_{\eta \gamma}$. Hence

$$\langle \xi, \zeta \rangle_{A_u} = \sum_{\eta \in G_u} \xi(\eta)^* \zeta(\eta)$$

is easily seen to be an $A_u$-valued inner product on $\mathcal{W}_0^0$ as in [RW98, Lemma 2.16]. We denote the right Hilbert module completion by $\mathcal{W}_\gamma$.

Remark 4.1. Since both $\xi$ and $\zeta$ have finite support, the norm of (4.1) is bounded by a multiple of $\|\xi\|_\infty \|\zeta\|_\infty$. Thus, if $(\xi_i)$ converges uniformly to $\xi$ with supports all contained in a fixed finite set, then $\xi_i \to \xi$ in the norm induced by the inner product.

In particular, since $p : \mathcal{B} \to G$ is saturated, sections of the form $\eta \mapsto f(\eta)b$ with $f \in \Gamma_c(G_u; \mathcal{B})$ and $b \in B_\gamma$ span a dense subspace of $\mathcal{W}_0^0$.

Since $X_u$ is a full right Hilbert $A_u$-module, it is also a $\mathcal{K}(X_u) - A_u$-imprimitivity bimodule where $\mathcal{K}(X_u)$ is the ideal in $\mathcal{L}(X_u)$ generated by the “rank-one” operators $\kappa_{(x,y)}(x, y)$, defined for all $x, y \in X_u$ by $\kappa_{(x,y)}(x, y)(z) := x \cdot (y, z)_{A_u}$. We let

$$\mathcal{K}_0(X_u) = \text{span}\{ \kappa_{(x,y)}(f, g) : f, g \in \Gamma_c(G_u; \mathcal{B}) \},$$

a dense subalgebra of $\mathcal{K}(X_u)$.

Note that, if $f, g, h \in \Gamma_c(G_u; \mathcal{B})$, then $\kappa_{(x,y)}(f, g)(h) \in \Gamma_c(G_u; \mathcal{B})$, and

$$\kappa_{(x,y)}(f, g)(h)(\eta) = f(\eta) \cdot (g, h)_{A_u}$$

for all $\eta \in G_u$, where the product on the right-hand side is the $A_u$-action on the fibre $B_\eta$. It follows that, for $\gamma \in G_u^\times$, the interior tensor product $X_u \otimes_{A_u} B_\gamma$ is a $\mathcal{K}(X_u) - A_u$-imprimitivity bimodule when equipped with the inner products from [RW98, Proposition 3.16].

Lemma 4.2. The map $\Psi$ determined by $f \otimes b \mapsto (\eta \mapsto f(\eta)b)$ from $\Gamma_c(G_u; \mathcal{B}) \otimes B_\gamma$ to $\mathcal{W}_0^0$ extends to a right Hilbert $A_u$-module isomorphism of $X_u \otimes_{A_u} B_\gamma$ onto $\mathcal{W}_\gamma$.

Proof. The map $\Psi$ is clearly $A_u$-balanced and bilinear. Hence $\Psi$ indeed defines a map from $\Gamma_c(G_u; \mathcal{B}) \otimes B_\gamma$ to $\mathcal{W}_0^0$. Since

$$\langle f \otimes b, g \otimes c \rangle_{A_u} = \langle \langle g, f \rangle_{A_u} \cdot b, c \rangle_{A_u}$$

$$= b^* \langle f, g \rangle_{A_u} c$$

$$= \sum_{\eta \in G_u} b^* f(\eta)^* g(\eta) c$$

$$= \langle \Psi(f \otimes b), \Psi(g \otimes c) \rangle_{A_u},$$

it follows that $\Psi$ preserves the inner product. It follows from Remark 4.1 that $\Psi$ has dense range and the result follows. \hfill $\Box$
Using Lemma 4.2 we obtain the following.

**Proposition 4.3.** We can view $W_\gamma$ as a $\mathcal{K}(X_u) - A_u$-imprimitivity bimodule with respect to the left $\mathcal{K}(X_u)$-action determined by

\[
\langle \mathcal{K}(X_u), (f, g) \cdot \xi \rangle(\eta) = f(\eta) \sum_{\tau \in G_u} g(\tau)^* \xi(\tau) \quad \text{for } f, g \in \Gamma_c(G_u; \mathcal{B}) \text{ and } \xi \in W_0^\gamma,
\]

and the left $\mathcal{K}(X_u)$-valued inner product determined by

\[
\langle \mathcal{K}(X_u), \xi \cdot \zeta \rangle(\eta) = \xi(\eta) \sum_{\tau \in G_u} \zeta(\tau)^* \omega(\tau) \quad \text{for } \xi, \zeta, \omega \in W_0^\gamma.
\]

**Sketch of the Proof.** Since we can identify $\mathcal{K}(X_u \otimes A_u, B_\gamma)$ with $\mathcal{K}(X_u)$, and since by [RW98, Lemma 4.55], $T \mapsto \Psi T \Psi^{-1}$ is an isomorphism of $\mathcal{K}(X_u)$ with $\mathcal{K}(W_\gamma)$, it suffices to see that the induced action of $\mathcal{K}(X_u)$ and the left inner product are as specified on $\mathcal{K}_0(X_u)$.

Suppose that $\xi = \Psi(h \otimes b)$. Then

\[
\Psi(\mathcal{K}(X_u), (f, g)(h) \otimes b)(\eta) = \mathcal{K}(X_u)(f, g)(h)(\eta)b
\]

\[
= f(\eta) \cdot \langle g, h \rangle_{A_u} b
\]

\[
= f(\eta) \sum_{\tau \in G_u} g(\tau)^* \xi(\tau).
\]

Hence for $\xi$ of the form $\eta \mapsto h(\eta)b$, the formula for $\mathcal{K}(X_u), (f, g) \cdot \xi$ in [4.2] indeed coincides with $\Psi(\mathcal{K}(X_u), (f, g) \cdot \Psi^{-1}(\xi))$. Since such elements $\xi$ span a dense subspace, we have established (4.2).

By [RW98, Lemma 4.55], the $\mathcal{K}(W_\gamma)$-valued inner product on $W_\gamma$ is given by

\[
\langle \Psi(f \otimes b), \Psi(g \otimes c) \rangle_{W_\gamma} = \Psi \circ \mathcal{K}(X_u)(f \otimes b, g \otimes c) \circ \Psi^{-1}.
\]

On the other hand,

\[
\mathcal{K}(X_u)(f \otimes b, g \otimes c) = \mathcal{K}(X_u)(f, g \cdot A_u(c, b)) = \mathcal{K}(X_u)f, g \cdot cb^*).
\]

It follows that

\[
\mathcal{K}(X_u)(f \otimes b, g \otimes c)(h \otimes d) = \mathcal{K}(X_u)f, g \cdot cb^*)(h \otimes d).
\]

If we let $\xi = \Psi(f \otimes b)$, $\zeta = \Psi(g \otimes c)$, and $\omega = \Psi(h \otimes d)$, then

\[
\Psi(\mathcal{K}(X_u)(f \otimes b, g \otimes c) \cdot (h \otimes d))(\eta) = \mathcal{K}(X_u)f, g \cdot cb^*)(h)(\eta)d
\]

\[
= f(\eta) \sum_{\tau \in G_u} bc^* g(\tau)^* h(\tau)d
\]

\[
= \xi(\eta) \sum_{\tau \in G_u} \zeta(\tau)^* \omega(\tau).
\]

Hence (4.3) holds by continuity. \qed
Just as in Lemma 3.2, we can realize \( W_{\gamma} \) as follows.

**Lemma 4.4.** If \( \gamma \in G_u^v \), then we can realize \( W_{\gamma} \) with the right Hilbert \( A_v \)-module

\[
\left\{ x \in \prod_{\eta \in G_u} B_{\eta \gamma} : \sum_{\eta \in G_u} x(\eta)^* x(\eta) \text{ converges in } A_v \right\}
\]
equipped with the inner product and right \( A_v \)-action given by

\[
\langle x, y \rangle_{A_v} = \sum_{\eta \in G_u} x(\eta)^* y(\eta) \quad \text{and} \quad (x \cdot a)(\eta) = x(\eta) \cdot a.
\]

5. The Convolution Formula

**Proposition 5.1.** Suppose that \( \gamma \in G_u^v \) and that \( \tilde{X}_u \) is the \( A_u - \mathcal{K}(X_u) \)-imprimitivity bimodule dual to \( X_u \). Then there is a right Hilbert \( A_v \)-module isomorphism \( \Phi : \tilde{X}_u \otimes_{\mathcal{K}(X_u)} W_{\gamma} \rightarrow B_{\gamma} \) given on elementary tensors \( \langle f \otimes \xi \in \mathcal{B}(\Gamma_c(G_u; B)) \otimes W_0^\gamma \) by

\[
(5.1) \quad \Phi(\langle f \otimes \xi \rangle) = \sum_{\eta \in G_u} f(\eta)^* \xi(\eta).
\]

**Proof.** It is straightforward to see that \( \Phi \) is \( \mathcal{K}_0(X_u) \)-balanced, bilinear, and \( A_v \)-linear. On the other hand,

\[
\langle \langle \langle \xi \otimes \xi \rangle , \langle g \otimes g \rangle \rangle_{A_v} = \langle \mathcal{K}(X_u) \langle g , f \rangle \cdot \xi , \xi \rangle_{A_v} = \sum_{\eta \in G_u} \mathcal{K}(X_u) \langle g , f \rangle(\eta)^* \xi(\eta)
\]

\[
= \sum_{\eta \in G_u} \sum_{\tau \in G_u} (g(\eta) f(\tau)^* \xi(\tau))^* \xi(\eta)
\]

\[
= \sum_{\eta \in G_u} \sum_{\tau \in G_u} (f(\tau)^* \xi(\tau))^* g(\eta)^* \xi(\eta)
\]

\[
= \Phi(\langle f \otimes \xi \rangle) \cdot \Phi(\langle g \otimes \xi \rangle)
\]

\[
= \langle \Phi(\langle f \otimes \xi \rangle) , \Phi(\langle g \otimes \xi \rangle) \rangle_{A_v}.
\]

It follows that \( \Phi \) preserves the right \( A_v \)-pre-inner products and hence extends to the completion \( \tilde{X}_u \otimes_{\mathcal{K}(X_u)} W_{\gamma} \). Since it clearly has dense range, this completes the proof. \( \square \)

**Remark 5.2.** We could also obtain Proposition 5.1 by observing that

\[
B_{\gamma} \cong A_u \otimes_{A_v} B_{\gamma} \cong (\tilde{X}_u \otimes_{\mathcal{K}(X_u)} X_u) \otimes_{A_u} B_{\gamma} \cong \tilde{X}_u \otimes_{\mathcal{K}(X_u)} (X_u \otimes_{A_u} B_{\gamma})
\]

\[
\cong \tilde{X}_u \otimes_{\mathcal{K}(X_u)} W_{\gamma},
\]

but a direct proof will be useful in the sequel.
Corollary 5.3. After realizing $X_u$ as in Lemma 3.2 and $W_\gamma$ as in Lemma 4.4, the formula in (5.1) extends to elementary tensors in $\tilde{X}_u \otimes W_\gamma$. That is, if $x \in X_u$ and $y \in W_\gamma$, then

$$\Phi(\flat(x) \otimes y) = \sum_{\eta \in G_u} x(\eta)^*(y(\eta)).$$

In particular, the sum in (5.2) converges in $B_\gamma$, and

$$\| \sum_{\eta \in G_u} x(\eta)^* y(\eta) \| \leq \|x\|_{X_u} \|y\|_{W_\gamma}.$$

Proof. If $F$ is a finite subset of $G_u$, then let $x_F = 1_F x$ and $y_F = 1_F y$. Then, just as in the proof of Lemma 3.2, the nets $(x_F)$ and $(y_F)$ converge to $x$ in $X_u$ and $y$ in $W_\gamma$, respectively. By Lemma 2.4, the net $(\flat(x_F) \otimes y_F)$ thus converges to $\flat(x) \otimes y$ in $\tilde{X}_u \otimes_{K(X_u)} W_\gamma$. Therefore $(\Phi(\flat(x_F) \otimes y_F))$ converges to $\Phi(\flat(x) \otimes y)$ in $B_\gamma$. Since

$$\Phi(\flat(x_F) \otimes y_F) = \sum_{\eta \in F} x(\eta)^* y(\eta),$$

and since $B_\gamma$ is Hausdorff, this establishes (5.2).

Since $\Phi$ preserves the inner product,

$$\| \sum_{\eta \in G_u} x(\eta)^* y(\eta) \| = \|\Phi(\flat(x_F) \otimes y_F)\| = \|\flat(x_F) \otimes y_F\|.$$

Using Lemma 2.4, we conclude that

$$\| \sum_{\eta \in G_u} x(\eta)^* y(\eta) \| \leq \|x_F\|_{X_u} \|y_F\|_{W_\gamma} \leq \|x\|_{X_u} \|y\|_{W_\gamma},$$

and we thus obtain (5.3) as well. □

Corollary 5.4. Suppose that $a, b \in C_r^*(G; B)$ and that $\gamma \in G_u^\omega$. Then

$$\sum_{\eta \in G_u} j(a)(\eta) j(b)(\eta^{-1}\gamma)$$

converges in $B_\gamma$. Moreover,

$$j(a * b)(\gamma) = \sum_{\eta \in G_u} j(a)(\eta) j(b)(\eta^{-1}\gamma).$$

Proof. To see that (5.4) converges, it suffices to see that

$$\sum_{\eta \in G_u} j(a)(\eta^{-1}) j(b)(\eta \gamma)$$

converges. Using Corollary 3.6, the sum in (5.5) is the same as

$$\sum_{\eta \in G_u} j(a)(\eta)^* j(b)(\eta \gamma).$$
Now let \( x_a(\eta) = j(a^*)(\eta) \) for \( \eta \in G_u \). Then by Proposition \( 3.7 \) \( x_a \in X_u \) with \( \|x_a\|_{X_u} \leq \|a\|_r \). Note that the same proposition states that
\[
\left\| \sum_{\eta \in G_u} j(b) (\eta \gamma)^* j(b)(\eta \gamma) \right\| \leq \left\| \sum_{\eta \in G_u} j(b)(\eta)^* j(b)(\eta) \right\| \leq \|b\|_r^2.
\]
In particular, if we let \( y_b(\eta) = j(b)(\eta \gamma) \) for \( \eta \in G_u \), then \( y_b \) is an element of \( W_\gamma \) with norm bounded by \( \|b\|_r \). Now Corollary \( 5.3 \) applied to \( b(x_a) \otimes y_b \), immediately implies that \( (5.4) \) converges. Let \( (f_i) \) and \( (g_i) \) be sequences in \( \Gamma_c(G; B) \) converging in \( C^*_\gamma(G; B) \) to \( a \) and \( b \), respectively. Then by the above
\[
\|x_{f_i} - x_a\|_{X_u} = \|x_{(f_i - a)}\|_{X_u} \leq \|f_i - a\|_r.
\]
Hence \( (x_{f_i}) \) converges to \( x_a \) in \( X_u \). Similarly, \( (y_{g_i}) \) converges to \( y_b \) in \( W_\gamma \). Thus \( \Phi(b(x_{f_i}) \otimes y_{g_i}) \to \Phi(b(x_a) \otimes y_b) \). But
\[
\Phi(b(x_{f_i}) \otimes y_{g_i}) = (f_i * g_i)(\gamma) = j(f_i * g_i)(\gamma).
\]
This suffices since the continuity of \( j \) and multiplication implies that \( j(f_i * g_i)(\gamma) \) converges to \( j(a * b)(\gamma) \). \( \square \)

Our main result, Theorem \( 1.1 \) now follows from Proposition \( 3.3 \) Corollary \( 3.6 \) and Corollary \( 5.4 \).

6. Examples

As illustrated in [MW08, §2], Fell bundles and their \( C^* \)-algebras subsume most examples of groupoid dynamical systems. Hence our main theorem applies to many such examples when the groupoid involved is étale. We consider some such examples here, and the first recovers [BFPR21, Proposition 2.8].

Example 6.1 (Twists). Consider a twist \( E \) over an étale groupoid \( G \) as defined by Kumjian in [Kum86]. To be precise, we have a central groupoid extension
\[
G(0) \times T \xrightarrow{\iota} E \xrightarrow{q} G
\]
where \( \iota \) and \( q \) are continuous groupoid homomorphisms such that \( \iota \) is a homeomorphism onto its range, \( q \) is an open surjection with kernel equal to the range of \( \iota \), and
\[
\iota(r(e), z) e = e \iota(s(e), z) \quad \text{for all } e \in E \text{ and } z \in T.
\]
The associated \( C^* \)-algebra \( C^*(G; E) \) can be realized as the \( C^* \)-algebra of the Fell bundle \( p : B \to G \) where \( B \) is the quotient of \( E \times C \) by the \( T \)-action \( z \cdot (e, \lambda) = (ze, z\lambda) \) and \( p \) is given by \( p([e, z]) = q(e) \). Then \([e, \lambda][f, \tau] = [ef, \lambda\tau] \) whenever \((e, f) \in E(2)\), and \([e, \lambda]^* = [e^{-1}, \lambda] \).
As in [IKR+21, Proposition 1.4], it is not hard to see that if $\tilde{f}: G \to \mathcal{B}$ is a continuous section, then $\tilde{f}(q(e)) = [e, f(e)]$ where $f: E \to C$ is a continuous function such that $e$
ol'',
ol'\begin{equation}
ol{6.1} f(z \cdot e) = zf(e).
\end{equation}
Therefore, if we let $C_0(G; E)$ be the set of continuous functions $f: E \to C$ satisfying 
\n\n\n\[ \text{and } j(a \ast b)(e') = \sum_{q(e) \in G^{(e')}} j(a)(e) j(b)(e^{-1}e'). \]

Remark 6.2. It should be kept in mind that the sum in (6.2) is taken over the set $G^{(e')}$.
The quantity $j(a)(e) j(b)(e^{-1}e')$ depends only on $q(e)$. Hence if we define $F(q(e)) = j(a)(e) j(b)(e^{-1}e')$, then the sum is equal to
\n\begin{equation}
\sum_{\eta \in G^{(e')}} F(\eta).
\end{equation}
Furthermore, since the sum in (6.2) or (6.3) is invariant under rearrangement, the convergence is absolute.

Example 6.3 (Groupoid Crossed Products). Let $(\mathcal{E}, G, \alpha)$ be a groupoid dynamical system with $G$ étale. To be precise, $k: \mathcal{E} \to G^{(0)}$ is a $C^*$-bundle and $\alpha = \{ \alpha_\gamma \}_{\gamma \in G}$ is a family of isomorphisms $\alpha_\gamma: E_{s(\gamma)} \to E_{r(\gamma)}$ such that $\gamma \cdot a = \alpha_\gamma(a)$ is a continuous action of $G$ on the bundle $\mathcal{E}$. Then we can realize the reduced crossed product $\mathcal{E} \rtimes_{\alpha, r} G$ as the reduced $C^*$-algebra of a Fell bundle $p: \mathcal{B} \to G$ where $\mathcal{B} = r^*\mathcal{E} = \{ (a, \gamma) \in \mathcal{E} \times G : q(a) = r(\gamma) \}$ with $p(a, \gamma) = \gamma$. Then $(a, \gamma)(b, \eta) = (\alpha_\gamma(b), \gamma\eta)$ for $(\gamma, \eta) \in G^{(2)}$, and $(a, \gamma)^* = (\alpha_\gamma^{-1}(a^*), \gamma^{-1})$.

As above, if $\tilde{f} \in \Gamma_c(G; \mathcal{B})$, then there is a continuous function $f: G \to \mathcal{E}$ such that $f(\gamma) \in E_{r(\gamma)}$ and $\gamma \mapsto \| f(\gamma) \|$ vanishes at infinity on $G$. If we denote the collection of such functions by $C_0(r)(G, \mathcal{E})$, then Theorem [14] implies that there is a norm reducing injective linear map $j: \mathcal{E} \rtimes_{\alpha, r} G \to C_0(r)(G, \mathcal{E})$ such that $j(\tilde{f}) = f$ for all $f \in \Gamma_c(G; \mathcal{B})$. Moreover, if $a, b \in \mathcal{E} \rtimes_{\alpha, r} G$, then
\begin{equation}
\begin{aligned}
j(a^*)(\gamma) &= j(a)(\gamma^{-1})^* \\
j(a \ast b)(\gamma) &= \sum_{\eta \in G^{(\gamma)}} j(a)(\eta) \alpha_\eta(j(b)(\eta^{-1}\gamma)).
\end{aligned}
\end{equation}

\footnote{Unfortunately, the literature is inconsistent as to whether there should be a complex conjugate on the section $z$ in (6.1) when constructing the $C^*$-algebra $C^*(G; E)$. As explained in [vEW13], Example 2.3], the choice depends on whether one takes the Fell bundle $\mathcal{B}$ to be the complex line bundle defined above or its conjugate bundle $\mathcal{E}$. We have opted to stay consistent with Kunjian's choice in [Kum86]. Note that $C^*(G; E)$ and $C^*(G; \overline{E})$ each others opposite algebras—see [BS21].}
Remark 6.4. Of course, Example 6.3 applies to crossed products by discrete groups. In that case, the result has been known for some time and appears in Zeller-Meier [ZM68, Theorem 4.2] with a weaker notion of convergence for the convolution product. Zeller-Meier also allows for a unitary valued 2-cocycle. However, using the observations in [EL97], we can use a Fell bundle model to include cocycles. We omit the details.

Example 6.5 (Green-Renault Twisted Crossed Products). We can combine the idea of a twist and a groupoid crossed product to arrive at Renault’s generalization of a Green twisted crossed product from [Ren91]. As in [MW08, Example 2.5] or the slightly more general set-up in [IKR+21 §1.4], we can realize these twisted crossed products via a Fell bundle. We start with a groupoid extension that fixes the unit space,

\[ \mathcal{A} \leftarrow \iota \rightarrow \Sigma \twoheadrightarrow G \]

where \( \mathcal{A} \) is a subgroupoid group bundle of \( \Sigma \) with unit space \( G^{(0)} \), \( \iota \) is the inclusion map, and \( q \) is a continuous open surjection restricting to a homeomorphism of \( \Sigma^{(0)} \) with \( G^{(0)} \). (Hereafter, we identify \( \Sigma^{(0)} \) with \( G^{(0)} \).) We also require a groupoid dynamical system \( (\mathcal{E}, \Sigma, \alpha) \) for a \( C^* \)-bundle \( k \): \( \mathcal{E} \rightarrow G^{(0)} \) as in Example 6.3.

To define the twist, we let \( U(E_u) \) for \( u \in G^{(0)} \) be the unitary group of the \( C^* \)-algebra \( E_u \) and let \( \coprod_{u \in G^{(0)}} U(E_u) \) be the corresponding (algebraic) group bundle over \( G^{(0)} \). Then a twisting map is a unit-space fixing homomorphism \( \theta: \mathcal{A} \rightarrow \coprod_{u \in G^{(0)}} U(E_u) \) that induces an action by isometric Banach space isomorphisms of \( \mathcal{A} \) on \( \mathcal{E} \) such that \( (a, e) \mapsto a \cdot e := \theta(a)e \) is continuous from \( \mathcal{A} \times \mathcal{E} = \{ (a, e) \in \mathcal{A} \times \mathcal{E} : s(a) = k(e) \} \) to \( \mathcal{E} \), and which satisfies

\[ \alpha_a(e) = \theta(a)e\theta(a)^* \quad \text{for all} \quad (a, e) \in \mathcal{A} \times \mathcal{E}, \quad \text{and} \]

\[ \theta(\sigma a \sigma^{-1}) = \overline{\alpha}_\sigma(\theta(a)) \quad \text{for all} \quad (\sigma, a) \in \Sigma \times \mathcal{A}. \]

To get a Fell bundle, we observe that \( \mathcal{A} \) acts on \( r^*\mathcal{E} \) by \( a \cdot (\sigma, e) = (a\sigma, e\theta(a)^*) \). Then the left quotient \( \mathcal{B} = \mathcal{A}\backslash r^*\mathcal{E} \) is a Banach bundle over \( G \) with \( p: \mathcal{B} \rightarrow G \) given by \( p([\sigma, e]) = q(\sigma) \). Then \( \mathcal{B} \) is a Fell bundle with

\[ [\sigma, e][\tau, f] = [\sigma \tau, e\alpha_\sigma(f)] \quad \text{and} \quad [\sigma, e]^* = [\sigma^{-1}, \alpha_\sigma^{-1}(e^*)]. \]

Using [IKR+21 Proposition 1.4], we see that sections \( \tilde{f} \in \Gamma(G; \mathcal{B}) \) correspond to continuous functions \( f: \Sigma \rightarrow \mathcal{E} \) such that \( f(a\sigma) = f(\sigma)\theta(a)^* \) for all \( (a, \sigma) \in \mathcal{A} \times \mathcal{E} \). If we let \( C_0(G; \Sigma; \mathcal{E}) \) be the continuous functions \( f: \Sigma \rightarrow \mathcal{E} \) transforming as above and such that \( \gamma \mapsto \|\tilde{f}(\gamma)\| \) vanishes at infinity on \( G \), then Theorem [IJK+21] implies that there is an injective norm reducing linear map \( j: C^*_r(G; \mathcal{B}) \rightarrow C_0(G; \Sigma; \mathcal{E}) \) such that for all \( a, b \in C^*_r(G; \mathcal{B}) \) we have

\[ j(a^*)(\sigma) = j(a)(\sigma^{-1})^* \quad \text{and} \quad j(a \ast b)(\sigma) = \sum_{q(\tau) \in G^{(0)}} j(a)(\tau) \alpha_\tau(j(b)(\tau^{-1}\sigma)). \]
where the comments in Remark 6.2 apply to the sum since $j(a)(\tau) \alpha_r(j(b)(\tau^{-1}\sigma))$ depends only on $q(\tau)$.

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