COMPOSITIONAL CONSTRUCTION OF APPROXIMATE ABSTRACTIONS OF INTERCONNECTED CONTROL SYSTEMS

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ABSTRACT. We consider a compositional construction of approximate abstractions of interconnected control systems. In our framework, an abstraction acts as a substitute in the controller design process and is itself a continuous control system. The abstraction is related to the concrete control system via a so-called simulation function: a Lyapunov-like function, which is used to establish a quantitative bound between the behavior of the approximate abstraction and the concrete system. In the first part of the paper, we provide a small gain type condition that facilitates the compositional construction of an abstraction of an interconnected control system together with a simulation function from the abstractions and simulation functions of the individual subsystems. In the second part of the paper, we restrict our attention to linear control system and characterize simulation functions in terms of controlled invariant, externally stabilizable subspaces. Based on those characterizations, we propose a particular scheme to construct abstractions for linear control systems. We illustrate the compositional construction of an abstraction on an interconnected system consisting of four linear subsystems. We use the abstraction as a substitute to synthesize a controller to enforce a certain linear temporal logic specification.

1. INTRODUCTION

One way to address the inherent difficulty in modeling, analyzing and controlling complex, large-scale, interconnected systems, is to apply a divide-and-conquer scheme [18]. In this approach, as a first step, the overall system is partitioned in a number of reasonably sized components, i.e., subsystems. Simultaneously, a number of appropriate interfaces to connect the individual subsystems are introduced. Subsequently, the analysis and the design of the overall system is reduced to those of the subsystems. There exist different reasoning schemes to ensure the correctness of such a component-based, compositional analysis and design procedure. One scheme, which is often invoked in the formal methods community, is called assume-guarantee reasoning, see e.g. [22, 15, 11]. Here, one establishes the correctness of the composed system by guaranteeing that each subsystem is correct, i.e., satisfies its specification, under the assumption that all other subsystems are correct. The assume-guarantee reasoning is always correct, if there is no circularity between assumptions and guarantees. In the case of circular reasoning, some additional “assume/guarantee” assumptions are imposed. Another approach, which is known from control theory, invokes a so called small gain condition, see e.g. [17, 10, 6, 7] to establish the stability of the interconnected system. For example in [6, 7], the authors assume that the gain functions that are associated with the Lyapunov functions of the individual subsystems satisfy a certain “small gain” condition. The condition certifies a small (or weak) interaction of the subsystems, which prevents an amplification of the

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signals across possible feedback interconnections. Similarly to the assume-guarantee reasoning, the small gain condition is always satisfied in the absence of any feedback interconnection [7, and references therein].

In this paper, we use the later reasoning and present a method for the compositional construction of approximate abstractions of interconnected nonlinear control systems. In our approach, an abstraction is itself a continuous control system (possibly with lower dimension), which is used as a substitute in the controller design process. The correctness reasoning from the abstraction to the concrete system is based on a notion of simulation function, which relates the concrete system with its abstraction. Simulation functions provide a quantitative bound between the behavior of the concrete systems and their abstractions. We employ a small gain type condition to construct a simulation function that relates the abstract interconnected system to the concrete interconnected system from the simulation functions of the individual subsystems. In the second part of the paper, we focus on the construction of abstractions (together with the associated simulation functions) of linear control systems. First, we characterize simulation functions in terms of controlled invariant, externally stabilizable subspaces. Subsequently, we propose a particular construction of abstractions of linear control systems. We conclude the paper with the construction of an abstraction together with a simulation function of an interconnected system consisting of four linear subsystems. We use the constructed abstraction as a substitute in the controller synthesis procedure to enforce a certain linear temporal logic property [2] on the concrete interconnected system. As we demonstrate, the controller synthesis would not have been possible without the use of the abstraction.

Related Work. Compositional reasoning schemes for verification in connection with abstractions of control systems are developed in [31, 11, 19]. The methods employ exact notions of abstractions which are based on simulation relations [11, 19] and simulation maps [31], for which constructive procedures exist only for rather restricted classes of control systems, e.g. linear control systems [9] and linear hybrid automata [11]. In contrast to the exact notions, the approximate abstractions which we study in this paper are based on simulation functions whose structures are closely related to (incremental) Lyapunov functions. Thus, advanced nonlinear control techniques developed to construct Lyapunov functions have the potential to also be used to construct simulation functions. For example the toolbox developed in [23] uses sum-of-squares techniques to construct bisimulation functions to relate nonlinear control systems.

An early approach to the compositional construction of simulation functions is given in [13], where the interconnection of two subsystems is studied. Compositional schemes for general interconnected systems for the construction of finite abstractions of linear and nonlinear control systems are presented in [33] and [24], respectively. Like in this paper, small gain type conditions are used to facilitate the compositional construction. As in our framework an abstraction is itself a continuous control system (potentially with lower dimension), the benefits of the proposed scheme are not limited to synthesis procedures based on finite abstractions, and therefore are potentially useful for a great variety of controller synthesis schemes, most notably computationally expensive schemes (in terms of the state space dimension of the system) such as [5, 4, 34, 26]. Nevertheless, as we demonstrate by an example, even for a synthesis scheme based on finite abstractions, we can apply our results as a first pre-processing step to reduce the dimensionality of
a given control system, before the construction of the finite abstraction, and therefore substantially reduce the computational complexity.

As we seek abstractions with reduced state space dimensions, our approach is closely related to the rich theory of model order reduction [1]. Specifically, the construction of abstractions of linear control systems (similar to the Krylov subspace methods and balanced order reduction schemes) can be classified as projection based methods [8]. Additionally, similar to [29], the proposed compositional construction of abstractions of interconnected control systems leads to a structure preserving reduction technique. While in [1, 8, 29] the model mismatch is established with respect to $\mathcal{H}_2/\mathcal{H}_\infty$ norms, we use simulation functions to derive $L_\infty$ error bounds, which are essential to reason about complex properties, e.g. linear temporal logic properties [2], across related systems.

To summarize, our contribution is twofold: 1) We present a small gain type condition to construct an abstraction of an interconnected system and a corresponding simulation function from the abstractions of the subsystems and their simulation functions. It is neither limited to two interconnected systems [13], nor to synthesis schemes based on finite abstractions [33, 24]. 2) We characterize simulation functions for linear subsystems in terms of controlled invariant, externally stabilizable subspaces, which leads to constructive procedures to determine abstractions of linear systems. Simulation functions for linear systems have been used in [14, 32, 12]. However, a geometric characterization of simulation functions, similar to [9], was missing. Moreover, this characterization allows to show that the conditions proposed in [14] to construct abstractions are not only sufficient, but actually also necessary.

A preliminary version of this work appeared in [27]. In this paper we present a less restrictive small gain condition and provide a novel geometric characterization of simulation functions for linear control systems.

2. Notation and Preliminaries

We denote by $\mathbb{N}$ the set of non-negative integers and by $\mathbb{R}$ the set of real numbers. We annotate those symbols with subscripts to restrict those sets in the obvious way, e.g. $\mathbb{R}_{>0}$ denotes the positive real numbers. We use $\mathbb{R}^{n \times m}$, with $n, m \in \mathbb{N}_{\geq 1}$, to denote the vector space of real matrices with $n$ rows and $m$ columns. The identity matrix in $\mathbb{R}^{n \times n}$ is denoted by $I_n$. For $a, b \in \mathbb{R}$ with $a \leq b$, we denote the closed, open and half-open intervals in $\mathbb{R}$ by $[a, b], (a, b], [a, b]$ and $]a, b]$, respectively. For $a, b \in \mathbb{N}$ and $a \leq b$, we use $[a; b], [a; b], [a; b]$ and $]a; b]$ to denote the corresponding intervals in $\mathbb{N}$. Given $N \in \mathbb{N}_{\geq 1}$, vectors $x_i \in \mathbb{R}^n$, $n_i \in \mathbb{N}_{\geq 1}$ and $i \in [1; N]$, we use $x = (x_1; \ldots; x_N)$ to denote the vector in $\mathbb{R}^N$ with $N = \sum_i n_i$ consisting of the concatenation of vectors $x_i$.

We use $|\cdot|$ to denote the Euclidean norm of vectors in $\mathbb{R}^n$ as well as the spectral norm, of matrices in $\mathbb{R}^{n \times m}$. Also for $\xi : \mathbb{R}_{\geq 1} \to \mathbb{R}^n$ we introduce $||\xi||_\infty := \sup_{t \in \mathbb{R}_{\geq 0}} |\xi(t)|$.

Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$ and $\bar{x} \in \mathbb{R}^m$, we use $f \equiv \bar{x}$ to denote that $f(x) = \bar{x}$ for all $x \in \mathbb{R}^n$. If $\bar{x}$ is the zero vector, we simply write $f \equiv 0$. The identity function in $\mathbb{R}^n$ is denoted by id, where the dimension is always clear from the context. We use $DV : \mathbb{R}^n \to \mathbb{R}_{\geq 0}^{1 \times n}$ to denote the gradient of a scalar function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $D^+V(x, v) = \lim_{t \to 0^+, t \geq 0} \frac{1}{t}(V(x + tv) - V(x))$ to denote the upper-right Dini derivative in the direction of $v$. Given two subsets $A, B \subseteq \mathbb{R}^n$, we use $A + B = \{a + b \mid a \in A, b \in B\}$ to denote the Minkowsky set addition.

We use the usual notation $\mathcal{K}, \mathcal{K}_{\infty}$ and $\mathcal{K}\mathcal{L}$ to denote the different classes of comparison functions, see e.g. [7]. Moreover, we use $\text{MAF}_n$ to denote the set of monotone aggregation...
functions \([4] \), i.e., the class of functions \( \mu : \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0} \) that satisfy: i) \( \mu(s) \geq 0 \) for all \( s \in \mathbb{R}_{\geq 0}^n \) and \( \mu(s) = 0 \) if \( s = 0 \); ii) for \( s, r \in \mathbb{R}_{\geq 0}^n \) \( s_i > r_i \) for all \( i \in [1; n] \) implies \( \mu(s) > \mu(r) \); iii) \( |s| \to \infty \) implies \( \mu(s) \to \infty \).

We recall some concepts from the geometric approach to linear systems theory \([3]\). Let \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \). We use the usual symbols \( \text{im} B \) and \( \ker B \) to denote image and kernel of \( B \). A linear subspace \( S \subseteq \mathbb{R}^n \) is called \((A, B)\)-controlled invariant if there exists a matrix \( K \) (of appropriate dimension) such that \((A + BK)S \subseteq S\), where the matrix-subspace product is given by \( AS := \{ x \in \mathbb{R}^n | \exists y \in S \ x = Ay \} \). An \((A, B)\)-controlled invariant subspace \( S \subseteq \mathbb{R}^n \) is \((A, B)\)-externally stabilizable if there exists a matrix \( K \) (of appropriate dimension) such that \((A + BK)S \subseteq S\) and \((A + BK)_{\mathbb{R}^n/S} \) is Hurwitz, i.e., the real parts of all the eigenvalues are strictly less than 0. Here, \((A + BK)_{\mathbb{R}^n/S} \) denotes the map induced by \((A + BK) \) on the quotient space \( \mathbb{R}^n/S \), see \([3] \) Def. 3.2.2].

3. BACKGROUND AND MOTIVATION

In this work, we study nonlinear control systems of the following form.

**Definition 1.** A control system \( \Sigma \) is a tuple

\[
\Sigma = (X, U, W, \mathcal{U}, W, f, Y, h),
\]

where \( X \subseteq \mathbb{R}^n \), \( U \subseteq \mathbb{R}^m \), \( W \subseteq \mathbb{R}^p \), and \( Y \subseteq \mathbb{R}^q \) are the state space, external input space, internal input space, and output space, respectively. We use the symbols \( \mathcal{U} \) and \( \mathcal{W} \) to, respectively, denote the set of piecewise continuous functions \( \nu : \mathbb{R}_{\geq 0} \to U \) and \( \omega : \mathbb{R}_{\geq 0} \to W \). The function \( f : X \times U \times W \to \mathbb{R}^n \) is the vector field and \( h : X \to Y \) is the output function.

In our definition of a control system, we distinguish between external inputs \( u \in U \) and internal inputs \( w \in W \). The purpose of this distinction will become apparent in Section \([4]\) where we introduce the interconnection of systems. Basically, we use the internal inputs to define the interconnection. For now, without referring to the interconnection, we can interpret the internal inputs as disturbances over which we have no control and the external inputs as control inputs which we are allowed to modify.

A control system \( \Sigma \) induces a set of trajectories by the differential equation

\[
\begin{align*}
\dot{\xi}(t) &= f(\xi(t), \nu(t), \omega(t)), \\
\zeta(t) &= h(\xi(t)).
\end{align*}
\]

A trajectory of \( \Sigma \) is a tuple \( (\xi, \zeta, \nu, \omega) \), consisting of a state trajectory \( \xi : \mathbb{R}_{\geq 0} \to X \), an output trajectory \( \zeta : \mathbb{R}_{\geq 0} \to Y \), and input trajectories \( \nu \in \mathcal{U} \) and \( \omega \in \mathcal{W} \), that satisfies \([2]\) for almost all times \( t \in \mathbb{R}_{\geq 0} \). We often use \( \xi_{x, \nu, \omega} \) and \( \zeta_{x, \nu, \omega} \) to denote the state trajectory and output trajectory associated with input trajectories \( \nu \in \mathcal{U} \), \( \omega \in \mathcal{W} \) and initial state \( x = \xi(0) \), without explicitly referring to the tuple \( (\xi, \zeta, \nu, \omega) \).

Throughout the paper, we impose the usual regularity assumptions \([20]\) on \( f \) and assume that \( X \) is strongly invariant and \( \Sigma \) is forward complete, so that for every initial state and input trajectories, there exists a unique state trajectory which is defined on the whole semi-axis.

We recall the notion of simulation function, introduced in \([14]\), which we adapt here to match our notion of control system with internal and external inputs. As we show in Section \([5]\) for the case of linear control systems, our notion of simulation function is related to the notion of simulation relation used in \([9]\).
**Definition 2.** Let $\Sigma = (X, U, W, U, W, f, Y, h)$ and $\hat{\Sigma} = (\hat{X}, \hat{U}, \hat{W}, \hat{U}, \hat{W}, \hat{f}, \hat{Y}, \hat{h})$ be two control systems with $p = \hat{p}$ and $q = \hat{q}$. A continuous function $V : \hat{X} \times X \to \mathbb{R}_{\geq 0}$, locally Lipschitz on $(\hat{X} \times X) \setminus V_0$ with $V_0 = \{(\hat{x}, x) \mid V(\hat{x}, x) = 0\}$, is called a simulation function from $\hat{\Sigma}$ to $\Sigma$ if for every $x \in X$, $\hat{x} \in \hat{X}$, $\hat{u} \in \hat{U}$, $\hat{w} \in \hat{W}$, there exists $u \in U$ so that for all $w \in W$ we have the following inequalities

$$\alpha(||\hat{h}(\hat{x}) - h(x)||) \leq V(\hat{x}, x),$$

$$D^+V\left(\begin{bmatrix} \hat{f}(\hat{x}, \hat{u}, \hat{w}) \\ f(x, u, w) \end{bmatrix}\right) \leq -\lambda(V(\hat{x}, x))$$

$$+ \rho(||\hat{u}||) + \mu(||w_1 - \hat{w}_1||, \ldots, ||w_p - \hat{w}_p||).$$

for some fixed $\alpha, \lambda \in \mathcal{K}_{\infty}$, $\rho \in \mathcal{K} \cup \{0\}$ and $\text{MAF}_p \mu$.

Let us point out some differences between our definition of simulation function and Definition 1 in [14]. Here, for the sake of a simpler presentation, we simply assume that for every $x, \hat{x}, \hat{u}, \hat{w}$ there exists a $u$ so that (4) holds for all $w$. While in [14] the authors use an interface function $k$ to provide the input $u = k(x, \hat{x}, \hat{u}, \hat{w})$ that enforces (4). Moreover, in Definition 1 in [14] there is no distinction between internal and external inputs and, therefore, $\mu(||w_1 - \hat{w}_1||, \ldots, ||w_p - \hat{w}_p||)$ does not appear on the right-hand-side of (4). Furthermore, we formulate the decay condition (4) in “dissipative” form [6], while in [14] Def. 1 the decay condition is formulated in “implication” form [6].

The following theorem shows the existence of a simulation function according to Definition 2.

**Theorem 1.** Consider $\Sigma = (X, U, W, U, W, f, Y, h)$ and $\hat{\Sigma} = (\hat{X}, \hat{U}, \hat{W}, \hat{U}, \hat{W}, \hat{f}, \hat{Y}, \hat{h})$ with $q = \hat{q}$ and $\hat{p} = p$. Suppose $V$ is a simulation function from $\hat{\Sigma}$ to $\Sigma$. Then, there exist a KL function $\beta$ and $\mathcal{K} \cup \{0\}$ functions $\gamma_{\text{ext}}$, $\gamma_{\text{int}}$ such that for any $x \in X$, $\hat{x} \in \hat{X}$, $\hat{w} \in \hat{W}$, there exists $\nu \in U$ so that for all $w \in W$ and $t \in \mathbb{R}_{\geq 0}$ we have

$$\left|\xi_{x,\nu,\hat{w}}(t) - \xi_{x,\nu,\hat{w}}(t)\right| \leq \beta(V(\hat{x}, x), t)$$

$$+ \gamma_{\text{ext}}(||\hat{w}||_\infty) + \gamma_{\text{int}}(||w - \hat{w}||_\infty).$$

The proof, which is given in the appendix, follows the usual arguments that are known from similar results in the context of input-to-state Lyapunov functions, e.g. see [30].

We need the following technical corollary later in the proof of Theorem 3.

**Corollary 1.** Given the assumptions of Theorem 1 there exist a KL function $\beta$ and $\mathcal{K} \cup \{0\}$ functions $\gamma_{\text{ext}}$, $\gamma_{\text{int}}$ such that for any $\hat{w} \in \hat{W}$, $\hat{\omega} \in \hat{\mathcal{W}}$, $x \in X$, and $\hat{x} \in \hat{X}$ there exists $\nu \in U$ so that for every $w \in W$ and $t \in \mathbb{R}_{\geq 0}$ we have

$$\left|\xi_{x,\nu,\hat{w}}(t) - \xi_{x,\nu,\hat{w}}(t)\right| \leq \beta(V(\hat{x}, x), t)$$

$$+ \gamma_{\text{ext}}(||\hat{w}||_\infty) + \gamma_{\text{int}}(||w - \hat{w}||_\infty),$$

where the KL function $\beta$ satisfies $\beta(\nu, 0) = r$ for all $r \in \mathbb{R}_{\geq 0}$.

The proof is provided in the appendix.

**Remark 1.** If we are given an interface function $k$ that maps every $x, \hat{x}, \hat{u}$ and $\hat{w}$ to an input $u = k(x, \hat{x}, \hat{u}, \hat{w})$ so that (4) is satisfied, then, the input $\nu \in U$ that realizes (5) is readily given by $\nu(t) = k(\xi(t), \hat{\xi}(t), \hat{\nu}(t), \hat{\omega}(t))$, see [27] Thm. 1.

Given an interface function, we might exploit the usefulness of simulation functions as follows. For various reasons (e.g. lower dimension) it might be easier to synthesize a
controller for the system \( \hat{\Sigma} \) enforcing some complex specifications, e.g. given as formulae in linear temporal logic \[2\], rather than for the original system \( \Sigma \). Then we can use the interface function \( k \) to transfer or refine the controller that we computed for \( \hat{\Sigma} \) to a controller for the system \( \Sigma \) (cf. example in Section 6). In this context, we refer to \( \hat{\Sigma} \) as an approximate abstraction and to \( \Sigma \) as the concrete system. A quantification of the error that is introduced in the design process by taking the detour through the abstraction is given by (5). A uniform error bound can be obtained by bounding the difference of the initial states (measured in terms of \( V(\hat{x}, x) \)) together with bounds on the infinity norms of \( \hat{\nu} \) and \( \omega - \hat{\omega} \).

Remark 2. In case that a control system does not have internal inputs, the definition (1) reduces to \((X,U,U,F,Y,h)\) and the vector field becomes \( f : X \times U \rightarrow \mathbb{R}^n \). Correspondingly, the definition of simulation functions simplifies, i.e., in (4) we do not quantify the inequality over \( w, \hat{w} \) and the term \( \mu(|w_1 - \hat{w}_1|, \ldots, |w_p - \hat{w}_p|) \) is omitted. Similarly, the results in Theorem 1 and Corollary 1 are modified, i.e., inequalities (5) and (6) are not quantified over \( \omega, \hat{\omega} \in \mathcal{W} \) and the term \( \gamma_{\text{int}}(||\omega - \hat{\omega}||_\infty) \) is omitted.

4. Compositionality Result

In this section, we analyze interconnected control systems and show how to construct an approximate abstraction of an interconnected system and the corresponding simulation function from the abstractions of the subsystems and their corresponding simulation functions, respectively. The definition of the interconnected control system is based on the notion of interconnected systems introduced in [33].

4.1. Interconnected Control Systems. We consider \( N \in \mathbb{N}_{\geq 1} \) control systems

\[
\Sigma_i = (X_i, U_i, W_i, U_i, W_i, f_i, Y_i, h_i), \quad i \in [1; N]
\]

with partitioned internal inputs and outputs

\[
w_i = (w_{i1}; \ldots; w_{i(i-1)}; w_{i(i+1)}; \ldots; w_{iN}), \quad y_i = (y_{i1}; \ldots; y_{iN}), \tag{7}
\]

with \( w_{ij} \in W_{ij} \subseteq \mathbb{R}^{p_{ij}}, y_{ij} \in Y_{ij} \subseteq \mathbb{R}^{q_{ij}} \), and output function

\[
h_i(x_i) = (h_{i1}(x_i); \ldots; h_{iN}(x_i)), \tag{8}
\]

as depicted schematically in Figure 1.

**Figure 1.** Input/output configuration of subsystem \( \Sigma_i \).

We interpret the outputs \( y_{ii} \) as external outputs, whereas the outputs \( y_{ij} \) with \( i \neq j \) are internal outputs which are used to define the interconnected systems. In particular, we assume that the dimension of \( w_{ij} \) is equal to the dimension of \( y_{ij} \), i.e., the following interconnection constraints hold:

\[
\forall i, j \in [1; N], \ i \neq j : \ q_{ij} = p_{ji}, \quad Y_{ij} \subseteq W_{ji}. \tag{9}
\]

If there is no connection from subsystem \( \Sigma_i \) to \( \Sigma_j \), we simply set \( h_{ij} \equiv 0 \).
Consider $N \in \mathbb{N}_{\geq 1}$ control systems $\Sigma_i = (X_i, U_i, W_i, U_i, W_i, f_i, Y_i, h_i)$, $i \in [1; N]$, with the input-output structure given by $[7]$. The interconnected control system $\Sigma = (X, U, f, Y, h)$, denoted by $I(\Sigma_1, \ldots, \Sigma_N)$, is given by $X := X_1 \times \cdots X_N$, $U := U_1 \times \cdots \times U_N$, $Y := Y_{11} \times \cdots \times Y_{NN}$ and functions

$$f(x, u) := (f_1(x_1, u_1, w_1); \ldots; f_N(x_N, u_N, w_N)),$$

$$h(x) := (h_{11}(x_1); \ldots; h_{NN}(x_N)),$$

where $u = (u_1; \ldots; u_N)$ and $x = (x_1; \ldots; x_N)$ and with the interconnection variables constrained by $w_{ij} = y_{ji}$ for all $i, j \in [1; N], i \neq j$.

An example of an interconnection of two control subsystems $\Sigma_1$ and $\Sigma_2$ is illustrated in Figure 2.

**Figure 2.** Interconnection of two control subsystems $\Sigma_1$ and $\Sigma_2$.

### 4.2. Compositional Construction of Approximate Abstractions and Simulation Functions

In this subsection, we assume that we are given $N$ subsystems $\Sigma_i = (X_i, U_i, W_i, U_i, W_i, f_i, Y_i, h_i)$, together with their abstractions $\hat{\Sigma}_i = (\hat{X}_i, \hat{U}_i, \hat{W}_i, \hat{U}_i, \hat{W}_i, \hat{f}_i, \hat{Y}_i, \hat{h}_i)$ and the simulation functions $V_i$ from $\hat{\Sigma}_i$ to $\Sigma_i$, with the associated comparison functions denoted by $\alpha_i$, $\lambda_i$, $\rho_i$, and $\mu_i$. We assume that the arguments of $\mu_i$ are partitioned according to the interconnection scheme, i.e., $\mu_i \in \text{MAF}_{N-1}$ and the internal inputs appear in $[4]$ for $\Delta w_{ij} := |w_{ij} - \hat{w}_{ij}|$ according to

$$\mu_i(\Delta w_{i1}, \ldots, \Delta w_{i(i-1)}, \Delta w_{i(i+1)}, \ldots, \Delta w_{iN}).$$

We follow $[6]$ and use an operator $\Gamma : \mathbb{R}^N_{\geq 0} \rightarrow \mathbb{R}^N_{\geq 0}$ to formulate a small gain condition. Each component $\Gamma_i$ with $r_j := \alpha_j^{-1}(s_j)$ is given by

$$\Gamma_i(s) := \begin{cases} 
\mu_i(r_2, \ldots, r_N) & i = 1 \\
\mu_i(r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_N) & i \in ]1; N[ \\
\mu_N(r_1, \ldots, r_{N-1}) & i = N.
\end{cases}$$

For $\varepsilon_i \in \mathcal{K}_\infty$, $i \in [1; N]$ we introduce $D : \mathbb{R}^N_{\geq 0} \rightarrow \mathbb{R}^N_{\geq 0}$ with $D(s) = (s_1 + \varepsilon_1(s_1); \ldots; s_N + \varepsilon_N(s_N))$ as well as $\Lambda^{-1} : \mathbb{R}^N_{\geq 0} \rightarrow \mathbb{R}^N_{\geq 0}$ by $\Lambda^{-1}(s) = (\lambda_1^{-1}(s_1); \ldots; \lambda_N^{-1}(s_N))$. The nonlinear small gain condition is given by $D \circ \Gamma \circ \Lambda^{-1} \not\geq$ id, i.e., for any $s \in \mathbb{R}^N_{\geq 0}$, at least one component of $D \circ \Gamma \circ \Lambda^{-1}(s)$ is strictly less than the corresponding component of $s$. One of the main results in $[7]$ shows that if $D \circ \Gamma \circ \Lambda^{-1}$ is irreducible and satisfies the small gain condition, then there exist $\mathcal{K}_\infty$ functions $\sigma_i$, $i \in [1; N]$ so that $\sigma(r) = (\sigma_1(r); \ldots; \sigma_N(r))$ satisfies\footnote{We interpret the inequality $[12]$ component-wise, i.e., for $x \in \mathbb{R}^N$ we have $x < 0$ if and only if every entry $x_i < 0$, $i \in [1; N]$.}

$$D \circ \Gamma \circ \Lambda^{-1}(\sigma(r)) < \sigma(r) \quad \text{for all} \quad r \in \mathbb{R}_{>0}.$$
Subsequently, we term \( N \) \( K_\infty \) functions \( \sigma_i \) that satisfy (12) for some \( D \) as \( \Omega \)-path [7, Def. 5.1].

Suppose that \( N = 2 \) and \( \alpha_i = \text{id} \) for \( i \in [1; 2] \). The small gain condition requires that there exist \( \varepsilon_i \in K_\infty \) so that either \((\text{id} + \varepsilon_1) \circ \mu_1 \circ \lambda_2^{-1}(s_2) < s_1 \) or \((\text{id} + \varepsilon_2) \circ \mu_2 \circ \lambda_1^{-1}(s_1) < s_2 \) holds for all \( s \in \mathbb{R}^2_0 \). This follows, e.g. by the small gain condition used in [8].

\[
\exists \varepsilon \in \mathbb{R} \forall r \in \mathbb{R}^2_0 (1 + \varepsilon) \mu_2 \circ \lambda_1^{-1} \circ (1 + \varepsilon) \mu_1 \circ \lambda_2^{-1}(r) < r. \tag{13}
\]

The main technical result in [16], which enables the small gain theorem, shows that (13) implies the existence of \( \sigma_2 \in K_\infty \) so that \((1 + \varepsilon) \mu_2 \circ \lambda_1^{-1}(r) < \sigma_2(r) < \lambda_2 \circ \mu_1^{-1}(r^\varepsilon) \) holds for all \( r > 0 \). It is easy to check that \( \sigma(s) = (s_1, \sigma_2(s_2)) \) satisfies (12). In the context of simulation functions, condition (13) ensures that the output mismatch propagated through the interconnected systems is not amplified. For general interconnected systems, the small gain condition can also be interpreted as the requirement that the “loop-gains” associated with the cycles of the interconnection graph are strictly less than one, see [7, Sec. 8.4].

If the functions \( \alpha_i, \mu_i \) and \( \lambda_i \) are linear, the existence of an \( \Omega \)-path follows from \( \Gamma \Lambda^{-1} \) having spectral radius strictly less than one [7, Thm. 5.1]. In this case, the right eigenvector \( \eta \in \mathbb{R}^N_\infty \) associated with the spectral radius has positive entries, and it follows that \( D \Gamma \Lambda^{-1} \eta < \eta \) for some appropriately picked \( \varepsilon_i > 0 \). Hence, \( \sigma(r) = (\eta_1 r; \ldots; \eta_N r) \), is an \( \Omega \)-path.

In the following theorem, similar to [6, Thm 4.5], we use the technical assumption on the derivative \((\sigma_i^{-1} \circ \lambda_i)'\) of the functions \( \sigma_i^{-1} \circ \lambda_i \) which reads

\[
\forall i \in [1; N] \forall \kappa \in K_\infty \exists \varepsilon \in \mathbb{R} \forall r \in \mathbb{R}^N_0 : \kappa(r) \leq \kappa(r)(\sigma_i^{-1} \circ \lambda_i)'(r) \leq \kappa(r)(\sigma_i^{-1} \circ \lambda_i)'(r) \leq \kappa(r). \tag{14}
\]

**Theorem 2.** Consider the interconnected control system \( \Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N) \) induced by \( N \in \mathbb{N}_{\geq 1} \) control subsystems \( \Sigma_i \). Suppose that for each subsystem \( \Sigma_i \), we are given \( \Sigma_i \) together with a simulation function \( V_i \) from \( \Sigma_i \) to \( \Sigma_i \) with comparison functions \( \alpha_i, \lambda_i, \rho_i \), and \( \mu_i \). Suppose that there exists an \( \Omega \)-path \( \sigma \) and for every \( i \in [1; N] \) \( \sigma_i^{-1} \circ \lambda_i \) is differentiable on \( \mathbb{R}^N_0 \) and (14) holds. Then

\[
V(\hat{x}, x) = \max_{i \in [1; N]} \{\sigma_i^{-1} \circ \lambda_i \circ V_i(\hat{x}_i, x_i)\} \tag{15}
\]

is a simulation function from \( \hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \ldots, \hat{\Sigma}_N) \) to \( \Sigma \).

**Proof.** We follow the arguments in [6, Thm. 4.5]. Let us first point out, that \( \sigma_i^{-1} \circ \lambda_i \) and \( V_i \) being differentiable and locally Lipschitz, respectively, implies that \( V \) is locally Lipschitz. Let us show inequality (3) for \( x = (x_1; \ldots; x_N) \in X \) and \( \hat{x} = (\hat{x}_1; \ldots; \hat{x}_N) \in \hat{X} \). We derive

\[
|\hat{h}(\hat{x}) - h(x)| \leq \sqrt{N} \max_{i \in [1; N]} |\hat{h}_{ii}(\hat{x}_i) - h_{ii}(x_i)|
\]

\[
\leq \sqrt{N} \max_{i \in [1; N]} \alpha_i^{-1} \circ \lambda_i \circ V_i(\hat{x}_i, x_i)
\]

\[
\leq \tilde{\alpha}(\max_{i \in [1; N]} \sigma_i^{-1} \circ \lambda_i \circ V_i(\hat{x}_i, x_i)) = \tilde{\alpha}(V(\hat{x}, x))
\]

where \( \tilde{\alpha}(r) = \sqrt{N} \max_{i \in [1; N]} \alpha_i^{-1} \circ \lambda_i^{-1} \circ \sigma_i(r) \) which is a \( K_\infty \) function and (3) holds with \( \alpha = \tilde{\alpha}^{-1} \).
We continue with showing (4). Let $z,v \in \hat{X} \times X$. Using a straightforward extension of [21] Thm. 1, we obtain
\[
D^+ V(z,v) \leq \max \{ D^+ (\sigma_i^{-1} \circ \lambda_i \circ V_i)(z,v) \mid i \in I(z) \}
\]
where $I(z) = \{ i \in [1; N] \mid V(z) = \sigma_i^{-1} \circ \lambda_i \circ V_i(z) \}$. Moreover, by Lemma \[1\] in the appendix, we have
\[
D^+ (\sigma_i^{-1} \circ \lambda_i \circ V_i)(z,v) \leq (\sigma_i^{-1} \circ \lambda_i)'(V(z)) D^+ V_i(z,v).
\]
We fix $x = (x_1; \ldots; x_N), \hat{x} = (\hat{x}_1; \ldots; \hat{x}_N)$, in $\hat{X} \times X \setminus V_0, \hat{u} = (\hat{u}_1; \ldots; \hat{u}_N) \in \hat{U}$ and $u = (u_1; \ldots; u_N) \in U$, where we pick $u_i$ to satisfy (4) with the internal inputs given by $w_{ij} = h_{ji}(x_j)$ and $\hat{w}_{ij} = \hat{h}_{ji}(\hat{x}_j)$. We define $\Delta w_{ij} := |w_{ij} - \hat{w}_{ij}|, \Delta y_{ji} := |y_{ji} - \hat{y}_{ji}|$ and $V_{vec} = (V_1; \ldots; V_N)$, then we get
\[
\begin{align*}
\mu_i(\Delta w_{i1}, \ldots, \Delta w_{i(i-1)}, \Delta w_{i(i+1)}, \ldots, \Delta w_{i1}) &= \mu_i(\Delta y_{i1}, \ldots, \Delta y_{i(i-1)}, \Delta y_{i(i+1)}, \ldots, \Delta y_{i1}) \\
&\leq \mu_i(\alpha_{i-1}^{-1}(V_1), \ldots, \alpha_{i-1}^{-1}(V_{i-1}), \alpha_{i+1}^{-1}(V_{i+1}), \ldots, \alpha_N^{-1}(V_N)) \\
&\leq \Gamma_i(V_{vec}).
\end{align*}
\]
Moreover, we see that $\Gamma_i(V_{vec})$ equals
\[
\Gamma_i \circ \Lambda^{-1}(\sigma_1 \circ \sigma_{i-1} \circ \lambda_i(V_1); \ldots; \sigma_N \circ \sigma_N^{-1} \circ \lambda_N(V_N)).
\]
Using (12), i.e., $\Gamma \circ \Lambda^{-1}(\sigma(r)) < D^{-1} \circ \sigma(r)$, and (15) we obtain a bound of (10) by $(id + \varepsilon_i)^{-1} \circ \sigma_i(V)$. Let us slightly abuse notation and use $V_i$ and $D^+ V_i$ for $V_i(\hat{x}, x_i)$ and $D^+ V_i((\hat{x}_i, x_i), (f(\hat{x}_i, \hat{u}_i, \hat{w}_i), f(x_i, u_i, w_i)))$. Similarly, we simplify the notation for $V$ and $D^+ V$. Let $i \in I((\hat{x}, x))$, then we compute
\[
\begin{align*}
D^+ V_i &\leq -\lambda_i(V_i) + (id + \varepsilon_i)^{-1} \circ \sigma_i(V) + \rho_i(\|\hat{u}_i\|) \\
&\leq -\sigma_i(V) + (id + \varepsilon_i)^{-1} \circ \sigma_i(V) + \rho_i(\|\hat{u}_i\|) \\
&\leq -\varepsilon_i \circ (id + \varepsilon_i)^{-1} \circ \sigma_i(V) + \rho_i(\|\hat{u}_i\|).
\end{align*}
\]
Using (14), it follows that there exist $\kappa_i \in \mathcal{K}_\infty$ and $\overline{\kappa}_i \in \mathcal{K} \cup \{0\}$ so that $\kappa_i \circ \frac{\varepsilon_i}{\|\hat{u}_i\| + \varepsilon_i} \circ \sigma_i(\|\hat{u}_i\|) \leq (\sigma_i^{-1} \circ \lambda_i)'(r) \frac{\varepsilon_i}{\|\hat{u}_i\| + \varepsilon_i} \circ \sigma_i(\|\hat{u}_i\|)$ and $(\sigma_i^{-1} \circ \lambda_i)'(r) \rho_i(\|\hat{u}_i\|) \leq \kappa_i \circ \rho_i(\|\hat{u}_i\|)$ holds for all $r \in \mathbb{R}_{>0}$. We define $\lambda \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K} \cup \{0\}$ by
\[
\lambda(r) = \min_{i \in [1; N]} \{ \kappa_i \circ \frac{\varepsilon_i}{\|\hat{u}_i\| + \varepsilon_i} \circ \sigma_i(\|\hat{u}_i\|), \rho(r) = \max_{i \in [1; N]} \{ \overline{\kappa}_i \circ \rho_i(\|\hat{u}_i\|) \} \}.
\]
Using (16) and (17) we get $D^+ V \leq -\lambda(V) + \rho(\|\hat{u}\|)$ which completes the proof. $\square$

**Remark 3.** In the linear case, with $\sigma(r) = (\eta_1 r; \ldots; \eta_N r)$ we get $V(\hat{x}, x) = \max_{i \in [1; N]} V_i(\hat{x}_i, x_i)$ with $\lambda = \min_{i \in [1; N]} \{ \lambda_i \frac{\varepsilon_i}{\|\hat{u}_i\| + \varepsilon_i} \}$ and $\rho = \max_{i \in [1; N]} \{ \rho_i \frac{\lambda_i}{\|\hat{u}_i\|} \}$, where we abuse notation and identify linear functions $\alpha(r) = ar$ with their coefficients, i.e., $a = a$.

## 5. Approximate Abstractions and Simulation Functions for Linear Systems

In this section, we focus on linear control systems $\Sigma$ and square-root-of-quadratic simulation functions $V$.

In the first part, we follow the geometric approach to linear control systems, and characterize simulation functions for linear control systems $\Sigma$ in terms of controlled invariant externally stabilizable subspaces [3]. The results are closely connected to the characterization of simulation relations developed in [9]. In the second part, we use the characterization of simulation functions to actually construct
abstractions of linear control subsystems whose existence was assumed in the first part of the paper.

5.1. Characterization of Simulation Functions. A linear control system is defined as a control system with the vector field and output function given by the following linear maps

\[
\begin{align*}
\dot{\xi}(t) &= A\xi(t) + B\nu(t) + D\omega(t), \\
\zeta(t) &= C\xi(t),
\end{align*}
\]  

(18)

with the state space, external input space, internal input space and output space given by \(\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p\) and \(\mathbb{R}^q\), respectively. The dimensions of the matrices follow by

\[
A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, D \in \mathbb{R}^{n \times p}, \text{ and } C \in \mathbb{R}^{q \times n}.
\]  

(19)

Henceforth, we simply use the tuple \(\Sigma = (A, B, C, D)\) to refer to a control system with vector field and output function of the form of (18) with the dimension of the corresponding matrices specified by (19). As the co-domain of the internal and external inputs are implicitly determined by the dimension of \(B\) and \(D\), we do not include the sets \(U\) and \(W\) in the system tuple.

In the following we characterize simulation functions from \(\Sigma_1 = (A_1, B_1, C_1, D_1)\) to \(\Sigma_2 = (A_2, B_2, C_2, D_2)\) in terms of the auxiliary matrices given by

\[
A_{12} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},
\]

(20)

Theorem 3 (Necessity). Consider two linear control systems \(\Sigma_i = (A_i, B_i, C_i, D_i), i \in \{1, 2\}\) with the same internal input space dimension and the same output space dimension. Let the matrices \(A_{12}, B_{12}, B_{21}, C_{12}, D_{12}\) be given by (20). Suppose there exists a simulation function \(V\) from \(\Sigma_1\) to \(\Sigma_2\), then there exists a relation \(R \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\) which is a subspace that satisfies

\[
\begin{align*}
R \text{ is } (A_{12}, B_{12})\text{-externally stabilizable} & \quad (21a) \\
A_{12}R \subseteq R + \text{im } B_{12} & \quad (21b) \\
\text{im } D_{12} \subseteq R + \text{im } B_{12} & \quad (21c) \\
R \subseteq \ker C_{12}. & \quad (21d)
\end{align*}
\]

If the function \(\rho\) associated with \(V\) equals to zero, then

\[
\text{im } B_{21} \subseteq R + \text{im } B_{12}.
\]  

(22a)

Proof of Theorem

Let \(R \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\) be the smallest subspace in \(\mathbb{R}^{n_1 \times n_2}\) that contains the set \(S = \{ (x_1; x_2) \mid V(x_1, x_2) = 0 \}\). By definition, any element of \(R\) follows by applying scalar multiplication and addition to elements in \(S\), and therefore, we obtain \(R \subseteq \ker C_{12}\). Now let \(\nu_1 \equiv 0\) and \(\omega_1 = 0\). Choose \(x_1, x_2\) such that \(V(x_1, x_2) = 0\), then it follows from Corollary 1 that there exists \(\nu_2\) such that \(V(\xi_1, x_1, \nu_1, \omega_1(t), \xi_2, x_2, \nu_2, \omega_2(t)) = 0\) holds for all \(t \in \mathbb{R}_{\geq 0}\). By the linearity of solutions of linear systems, we have \((x_1, x_2) \in R\) implies that there exists \(\nu_2\) such that \((\xi_1, x_1, \nu_1, \omega_1(t), \xi_2, x_2, \nu_2, \omega_2(t)) \in R\) holds for all \(t \in \mathbb{R}_{\geq 0}\), which shows that \(R\) is \((A_{12}, B_{12})\)-controlled invariant, see [3 Thm. 4.1.1]
and (21b) follows. As the choice of \( x_1, x_2 \) and \( \omega_1 \) is arbitrary, we invoke the fundamental lemma of the geometric approach \([3]\) Lem. 3.2.1] and obtain that for every \( x_1, x_2, w_1 \) there is \( u_2 \) so that \((A_1 x_1 + D_1 w_1, A_2 x_2 + D_2 w_1 + B_3 u_2) \in R\). By setting \( x_1 = x_2 = 0 \), we obtain (21c). We continue to show (21a). For \( \nu_1 \equiv 0 \), \( \omega_1 \equiv \omega_2 \), Corollary \([\text{3}]\) implies that for every \((x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\), there exists \( u_2 \in U_2 \) such that \( \lim_{t \to \infty} V(\xi_{1,x_1,x_1,\omega_1}(t), \xi_{2,x_2,x_2,\omega_2}(t)) = 0 \), which implies that \((\xi_{1,x_1,x_1,\omega_1}(t), \xi_{2,x_2,x_2,\omega_2}(t))\) converges to \( R \). Then using \([\text{3}]\) Def. 4.1.6 and Prp. 4.1.14 we conclude that \( R \) is \((A_{12}, B_{12})\)-externally stabilizable.

We continue with (22a). Let \( \omega_1 = \omega_2 \equiv 0 \). Since \( \gamma_{\text{ext}} \equiv 0 \), we use the same arguments as above, and obtain that for every \((x_1, x_2) \in R\) and \( \nu_1 \in U_2 \) there is \( \nu_2 \in U_2 \) so that \((\xi_{1,x_1,x_1,\omega_1}(t), \xi_{2,x_2,x_2,\omega_2}(t)) \in R\) holds for all \( t \in \mathbb{R}_{\geq 0} \). By the fundamental lemma of the geometric approach \([3]\) Lem. 3.2.1], we have that \((\xi_{1,x_1,x_1,\omega_1}(t), \xi_{2,x_2,x_2,\omega_2}(t)) \in R\) for almost all \( t \in \mathbb{R}_{\geq 0} \). This implies, for every \((x_1, x_2) \in R\) and \( u_1 \in \mathbb{R}^{m_1}\), there exists \( u_2 \in \mathbb{R}^{m_2}\) so that \((A_1 x_1 + B_1 u_1, A_2 x_2 + B_2 u_2) \in R\), which concludes the proof. \(\square\)

**Theorem 4** (Sufficiency). Consider two linear control systems \(\Sigma_i = (A_i, B_i, C_i, D_i)\), \(i \in \{1, 2\}\) with the same internal input space dimension and the same output space dimension. Let the matrices \(A_{12}, B_{12}, B_{21}, C_{12}, D_{12}\) be given by (20). Suppose there exists a linear subspace \(R \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\) that satisfies (21a), (21d), then there exists a symmetric positive semi-definite matrix \(M \in \mathbb{R}^{(n_1 + n_2) \times (n_1 + n_2)}\) so that

\[
V(x_1, x_2) = ((x_1; x_2)\top M(x_1; x_2))^{1/2}
\]

is a simulation function from \(\Sigma_1\) to \(\Sigma_2\). If additionally (22a) holds, then the function \(\rho\) associated with \(V\) equals to zero, i.e., \(\rho \equiv 0\).

**Proof of Theorem 4** We pick \(K_{12}\) so that \((A_{12} + B_{12}K_{12})R \subseteq R\) and \((A_{12} + B_{12}K_{12})|_{(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})/R}\) is Hurwitz. Let \(A = (A_{12} + B_{12}K_{12})\), from \([3]\) Proof of Thm 3.2.1 and Def. 3.2.4 it follows that for any invertible matrix \(T = [T_1 \ T_2]\) with \(\text{im} T_1 = R\) and \(T^{-1} = [\bar{T}_1 \bar{T}_2]\) we obtain

\[
T^{-1}AT = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}
\]

where \(F_{22}\) is Hurwitz, \(F_{22}\) is Hurwitz, and \(F_{22}\) is Hurwitz, so that \(\bar{T}_2 = \bar{T}_2\).

For the remainder, we use \(\bar{A} = F_{22}\) and \(\bar{\Pi} = \bar{T}_2\). Let \(x \in \text{ker} \bar{T}_2\), then we compute \(x = TT^{-1}x = T_1 y\) for \(y = \bar{T}_1 x\) and it follows that \(\text{ker} \bar{\Pi} \subseteq R\). Since \(R \subseteq \text{ker} C_{12}\), we obtain \(\text{ker} \bar{\Pi} \subseteq \text{ker} C_{12}\) and there exists \(\bar{C}\) so that \(\bar{C}\Pi = C_{12}\). As \(\bar{A}\) is Hurwitz, there exist a constant \(\lambda \in \mathbb{R}_{>0}\) and a symmetric positive definite matrix \(\bar{M}\), so that

\[
\bar{C}\top \bar{C} \leq \bar{M}
\]

\[
\bar{A}\top \bar{M} + \bar{M}\bar{A} \leq -2\lambda \bar{M}.
\]

We define \(V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}\) by

\[
V(x_1, x_2) = ((x_1; x_2)\top \Pi\top \bar{M}\Pi(x_1; x_2))^{1/2}.
\]

Clearly \(\Pi\top \bar{M}\Pi\) is symmetric positive semi-definite and it remains to show that \(V\) is indeed a simulation function from \(\Sigma_1\) to \(\Sigma_2\). First, we verify that \((3)\) holds for \(\alpha = \text{id}\) by

\[
|C_{1}x_{1} - C_{2}x_{2}|^2 = |C_{12}(x_1; x_2)|^2 = |C\Pi(x_1; x_2)|^2 \leq (x_1; x_2)\top \Pi\top \bar{M}\Pi(x_1; x_2) = V(x_1, x_2)^2.
\]
We continue to show that (4) holds as well. Let \( x_1, x_2, u_1 \) and \( w_1 \) be given. Then we pick \( w_2 = K_{12}(x_1; x_2) + K_2u_1 + u_4 \) where we pick \( u_3 \) so that \( D_{12}w_1 + B_{12}u_3 \in R \) holds, which is possible by (21c). The purpose of \( K_{12} \) will become apparent later. Let \( x = (x_1; x_2) \), then for any \( w_2 \) the left-hand-side of (4) evaluates to

\[
\frac{x^\top \Pi^\top \Pi}{V(x_1, x_2)} \left[ Ax + D_{12}w_1 + B_{12}u_3 + \begin{bmatrix} B_1 \\ B_2K_4 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ D_2 \end{bmatrix} \Delta w \right]
\]

with \( \Delta w = (w_1 - w_2) \). We use \( \overline{A} = \Pi \lambda \) and (24) to bound the first term by

\[
\frac{x^\top \Pi^\top \Pi}{V(x_1, x_2)} Ax \leq -\lambda \frac{x^\top \Pi^\top \Pi x}{V(x_1, x_2)} = -\lambda V(x_1, x_2).
\]

Moreover, \( \Pi(D_{12}w_1 + B_{12}u_3) = 0 \) as \( D_{12}w_1 + B_{12}u_3 \in R \). Then we use the Cauchy-Schwarz inequality to bound

\[
\frac{x^\top \Pi^\top \Pi}{V(x_1, x_2)} \left( \begin{bmatrix} B_1 \\ B_2K_4 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ D_2 \end{bmatrix} \Delta w \right) \leq |\sqrt{\Pi} \begin{bmatrix} B_1 \\ B_2K_4 \end{bmatrix} ||u_1|| + |\sqrt{\Pi} \begin{bmatrix} 0 \\ D_2 \end{bmatrix} ||w_2 - w_1||
\]

and we see that \( V \) is a simulation function with the associated comparison functions given by \( \alpha = \text{id} \) and for all \( r \in \mathbb{R}_{\geq 0} \) and \( s \in \mathbb{R}_{\geq 0} \) by \( \lambda(r) = \lambda r \),

\[
\rho(r) = |\sqrt{\Pi} \begin{bmatrix} B_1 \\ B_2K_4 \end{bmatrix} |r \text{ and } \mu(s) = \sum_{i=1}^{p} |\sqrt{\Pi} \begin{bmatrix} 0 \\ D_2 \end{bmatrix} |s_i.
\]

If \( \text{im} B_{21} \subseteq R + \text{im} B_{12} \), for every \( u_1 \) we choose \( w_2 \) differently by \( w_2 = K_{12}(x_1; x_2) + u_3 + u_4 \) with \( u_4 \) so that \( B_{21}u_1 + B_{12}u_4 \in R \) which implies \( \Pi(B_{21}u_1 + B_{21}u_4) = 0 \) and the term of the left-hand-side of (4) associated with \( u_1 \) vanishes.

Theorem 4 gives rise to the following definition.

**Definition 4.** Let \( \Sigma_i = (A_i, B_i, C_i, D_i), i \in \{1, 2\} \) be two linear control systems with the same internal input space dimension and the same output space dimension. Let the matrices \( A_{12}, B_{12}, C_{12}, D_{12} \) be given by (20). We say that a relation \( R \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) induces a simulation function from \( \Sigma_1 \) to \( \Sigma_2 \) if it satisfies (21a)-(21d).

Theorems 3 and 4 facilitate a direct comparison of simulation functions with the notion of a simulation relation \( R \) from \( \Sigma_1 \) to \( \Sigma_2 \). A relation \( R \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) is a simulation relation from \( \Sigma_1 \) to \( \Sigma_2 \) if for every \( (x_1, x_2) \in R \), \( \nu_1 \) and \( \omega_1 \equiv \omega_2 \), there exists \( \nu_2 \) so that

\[
\forall t \in \mathbb{R}_{\geq 0} : \left[ (\xi_{1,x_1,\nu_1,\omega_1}(t), \xi_{2,x_2,\nu_2,\omega_2}(t)) \in R \right]
\]

This notion of simulation relation was introduced in [9] in the context of verification for linear systems with two types of inputs. Due to the verification context, in [9] the internal input is interpreted as control input and the external inputs as disturbances. While in our approach, we use the external input as control input that we refine from \( \Sigma_1 \) to \( \Sigma_2 \) and the internal input is used for the interconnection of the subsystems. Nevertheless, mathematically, both notions are closely related and the authors in [9] characterized simulation relations from \( \Sigma_1 \) to \( \Sigma_2 \) in terms of conditions (21b)-(22a).

On one hand, two systems \( \Sigma_1 \) and \( \Sigma_2 \) that are related via a simulation function (or
equivalently a relation that induces a simulation function) needs to satisfy (22a) only if \( \rho \equiv 0 \) should hold. As a result, given an output trajectory \( \zeta_{1, x_1, \nu_1, \omega_1}(t) \) of \( \Sigma_1 \), there does not necessarily exist an output trajectory \( \zeta_{2, x_2, \nu_2, \omega_2}(t) \) of \( \Sigma_2 \) so that both trajectories are identical. On the other hand, a simulation relation \( R \) is not required to be externally stabilizable (21a). The external stabilizability in the context of simulation functions allows \( \zeta_{1, x_1, \nu_1, \omega_1}(t) \) and \( \zeta_{2, x_2, \nu_2, \omega_2}(t) \) to be driven by the different internal inputs \( \omega_1 \not\equiv \omega_2 \) and the initial states are not restricted to satisfy \( (x_1, x_2) \in R \). In view of (5) the effect of the different internal inputs on the output difference is bounded and the effect of the freely chosen initial states vanishes over time.

We conclude this subsection with the characterization of a relation inducing a simulation function from \( \Sigma_1 \) to \( \Sigma_2 \) (or from \( \Sigma_2 \) to \( \Sigma_1 \)) that is defined in terms of a matrix \( P \in \mathbb{R}^{n_2 \times n_1} \) by

\[
R = \{(x_1; x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid P \cdot x_1 = x_2 \}.
\] (26)

We use this result in the next subsection to construct an approximate abstraction \( \hat{\Sigma} \) of a given linear control system \( \Sigma \).

**Theorem 5.** Consider two linear control systems \( \Sigma_i = (A_i, B_i, C_i, D_i), i \in \{1, 2\} \) with the same internal input space dimension and the same output space dimension. Let \( R \) be given by (26) with the matrix \( P \in \mathbb{R}^{n_2 \times n_1} \). The relation \( R \) induces a simulation function from \( \Sigma_1 \) to \( \Sigma_2 \) iff there exists matrices \( K_1, K_2, K_3 \) of appropriate dimensions so that the following holds

\[
A_2 + B_2 K_1 \text{ is Hurwitz} \quad (27a)
\]
\[
A_2 P = PA_1 + B_2 K_2 \quad (27b)
\]
\[
D_2 = PD_1 + B_2 K_3 \quad (27c)
\]
\[
C_1 = C_2 P. \quad (27d)
\]

Moreover, (22a) holds iff there exists \( K_4 \) so that

\[
P B_1 = B_2 K_4. \quad (28a)
\]

**Proof.** First, we show that \( R \) satisfies (21b)-(21d) (21b)-(22a)) holds. By the definition of \( R \) it is straightforward to establish the equivalences (21b) \( \iff \) (27b), (21c) \( \iff \) (27c), (21d) \( \iff \) (27d) and (22a) \( \iff \) (28a). Now we assume that \( R \) is \((A_{12}, B_{12})\)-controlled invariant. Let \( K_{12} = [K'_1 \ K_1] \) so that \((A_{12} + B_{12} K_1) R \subseteq R \). Then we pick \( T \) in (23) by

\[
T = \begin{bmatrix} I_{n_1} & 0 \\ P & I_{n_2} \end{bmatrix}, \text{ where } \begin{bmatrix} I_{n_1} \\ P \end{bmatrix} = R
\]

and observe that \(-PA_1 + B_2 K'_1 + (A_2 + B_2 K_1)P = 0 \) and \( F_{22} = A_2 + B_2 K_1 \), which shows that (27b) holds and, consequently, (21a) holds iff (27a) holds. \( \square \)

The following corollary readily follows from the proofs of Theorem 4 and 5.

**Corollary 2.** Suppose that (27a)-(27d) hold. Let \( M \in \mathbb{R}^{n_2 \times n_2} \) be a symmetric positive definite matrix that satisfies

\[
C_2^T C_2 \leq M \quad (27a)
\]
\[
(A_2 + B_2 K_1)^T M + M (A_2 + B_2 K_1) \leq -2 \lambda M
\]
for some $\lambda \in \mathbb{R}_{>0}$. Then a simulation function from $\Sigma_1$ to $\Sigma_2$ is given by
\[ V(x_1, x_2) = \left((x_2 - Px_1)\top M(x_2 - Px_1)\right)^{\frac{1}{2}} \]
and the interface function that maps $x_1$, $x_2$, $u_1$, $w_1$ to $u_2$ so that (4) holds is given by
\[ u_2 = K_1(x_2 - Px_1) - K_2x_1 - K_3w_1 + K_4u_1, \]
where $K_4$ is given implicitly as the matrix that minimizes $|\sqrt{M}(PB_1(B_2K_4))|$. Let $d_i \in \mathbb{R}^n$, $i \in [1;p]$ denote the columns of $D_2$. The comparison functions associated with $V$ follow for all $r \in \mathbb{R}_{\geq 0}$ and $s \in \mathbb{R}_{\geq 0}^p$ by
\[ \alpha(r) = r, \quad \lambda(r) = r, \quad \rho(r) = |\sqrt{M}(PB_1(B_2K_4))|r, \quad \mu(s_1, \ldots, s_p) = |\sqrt{M}d_1|s_1 + \ldots + |\sqrt{M}d_p|s_p. \]

5.2. Construction of Approximate Abstractions. In this subsection, we are interested in the construction of an approximate abstraction $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, \hat{D})$ for a given linear control system $\Sigma = (A, B, C, D)$ together with a square-root-of-quadratic simulation function from $\hat{\Sigma}$ to $\Sigma$. Given the fact that any two asymptotically stable linear systems $\Sigma$ and $\hat{\Sigma}$ (with suitable internal input and output space dimensions) can be related via a simulation function, we follow the approach in [14] to construct abstractions of linear control systems, and ask not only for a simulation function from $\hat{\Sigma}$ to $\Sigma$, but additionally require that there exists a simulation relation $\hat{R}$ from $\Sigma$ to $\hat{\Sigma}$, which ensures that nice properties like controllability of $\Sigma$ are preserved on the abstraction $\hat{\Sigma}$. The construction is based on the assumption that $(A, B)$ is stabilizable,
\[ (A, B) \text{ is stabilizable}, \quad (29a) \]
and on the existence of a matrix $P \in \mathbb{R}^{n \times n}$ with a trivial kernel that satisfies
\[ \text{im} P \subseteq \text{im} P + \text{im} B \quad (30a) \]
\[ \text{im} D \subseteq \text{im} P + \text{im} B \quad (30b) \]
\[ \text{im} P + \ker C = \mathbb{R}^n. \quad (30c) \]

In [14] conditions (29a), (30a) and (30c) were used to construct an abstraction $\hat{\Sigma}$ and a square-root-of-quadratic simulation function $V$ from $\Sigma$ to $\hat{\Sigma}$ together with a simulation relation $\hat{R} = \{(x; \hat{x}) \mid \hat{P}x = \hat{x}\}$ (for some $\hat{P} \in \mathbb{R}^{n \times n}$) from $\Sigma$ to $\hat{\Sigma}$. In this paper, we extend the scheme in [14] in the following directions. First, we add condition (30b) in order to be able to account for systems with internal and external inputs. Second, we show that the simulation relation $\hat{R}$ actually induces a simulation function from $\Sigma$ to $\hat{\Sigma}$. Third, and most importantly, using the novel geometric characterization of simulation functions, we show that the conditions (29a)-(30c) are not only sufficient but actually necessary for the existence of an abstraction $\hat{\Sigma}$ so that the relation $R = \{(\hat{x}, x) \mid \hat{P}x = x\}$ induces a simulation function from $\hat{\Sigma}$ to $\Sigma$ and $\hat{R}$ induces a simulation function from $\Sigma$ to $\hat{\Sigma}$.

**Theorem 6.** Consider $\Sigma = (A, B, C, D)$ and
\[ R = \{(\hat{x}, x) \in \mathbb{R}^n \times \mathbb{R}^n \mid \hat{P}x = x\} \]

Actually, the authors of [14] show that $\hat{\Sigma}$ is $\hat{P}$-related to $\Sigma$ (see [14] Def. 3)), which, when we omit the internal inputs, is equivalent to $\hat{R}$ being a simulation relation from $\Sigma$ to $\hat{\Sigma}$. [2]
with \( P \in \mathbb{R}^{n \times \hat{n}} \), \( \ker P = 0 \). There exist \( \hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, \hat{D}) \) with the same internal input space dim. and the same output space dim. as \( \Sigma \) and \( \hat{R} = \{(x; \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^n | \hat{P}x = \hat{x}\} \) with \( \hat{P} \in \mathbb{R}^{\hat{n} \times n} \), so that \( R \) induces a simulation function from \( \hat{\Sigma} \) to \( \Sigma \) and \( \hat{R} \) induces a simulation function from \( \Sigma \) to \( \hat{\Sigma} \) iff \((29a) - (30c)\) hold.

**Proof.** Let \( R(\hat{R}) \) induces a simulation function from \( \hat{\Sigma} \) to \( \Sigma \) (\( \Sigma \) to \( \hat{\Sigma} \)). From Theorem 5 it follows that \((27a)\) implies \((29a)\), \((27b)\) implies \((30a)\) and \((27c)\) implies \((30c)\). From \((27d)\) it follows that \( C = CP \) and \( CP = C \), which implies that \( CP\hat{P} = C \). Since \( \ker P = 0 \), Lemma 3 in [1] is applicable and we obtain \((30c)\). Now suppose that \((29a) - (30c)\) hold. Let \( \hat{C} = CP \) and pick \( \hat{A} \) and \( \hat{C} \) together with \( K_1, K_2, K_3 \) so that \((27a) - (27d)\) hold for \( A, B, \hat{C}, \hat{D} \) in place of \( A_2, B_2, C_2, D_2, A_1, B_1, C_1, D_1 \), respectively. Theorem 5 shows that \( R \) induces a simulation function from \( \hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, \hat{D}) \) to \( \Sigma \) for any \( \hat{B} \) of appropriate dimension. We continue to show that \( \hat{R} \) induces a simulation function from \( \Sigma \) to \( \hat{\Sigma} \). Again we use Lemma 3 in [1] to pick \( \hat{P} \) with \( \hat{P} = \mathbb{R}^n \) so that \( \hat{C}\hat{P} = C, \hat{P}P = I_\hat{n} \) and \( \hat{P}P + E\hat{F} = I_n \) for some matrices \( E \) and \( F \) of appropriate dimension with \( \ker E = \ker C \). Let \( \hat{B} = [\hat{P}\hat{B}\hat{PA}] \). We derive \( \hat{A}\hat{P} = \hat{P}\hat{PA}\hat{P} = \hat{P}\hat{AP} - \hat{P}B(K_1 + K_1\hat{P}) = \hat{P}A - \hat{P}\hat{PA}E - \hat{P}B(K_1 + K_1\hat{P}) = \hat{P}A + \hat{B}[-(K_1 + K_1\hat{P})^T - \hat{F}]^T \) and \( \hat{D} = \hat{P}\hat{PD} = \hat{P}(D - BK_3) = \hat{P}D + \hat{B}[-K_3^T 0]^T \). Additionally, we have \( \hat{P}\hat{B} = \hat{B}[I_\hat{n} 0]^T \) and it follows that \( \hat{R} \) satisfies \((27b) - (28a)\) for \( \hat{P}, \hat{A}, \hat{B}, \hat{C}, \hat{D}, A, B, C, D \) in place of \( P, A_2, B_2, C_2, D_2, A_1, B_1, C_1, D_1 \), respectively, which shows that \( \hat{R} \) is a simulation relation from \( \Sigma \) to \( \hat{\Sigma} \) [9, Prp. 5.2]. Moreover, \( \ker \hat{P} = \mathbb{R}^n \). As \( (A, B) \) is stabilizable we use (25) to verify that \( (\hat{A}, \hat{B}) \) is stabilizable as well. Hence, there exists a matrix \( \hat{K} \) so that \((27a)\) holds. It follows that \( \hat{R} \) induces a simulation function from \( \Sigma \) to \( \hat{\Sigma} \).

We summarize the construction of an approximate abstraction of a stabilizable control system \( \Sigma = (A, B, C, D) \) in Table 1. The associated simulation function from \( \hat{\Sigma} \) to \( \Sigma \) follows from Corollary 2 to \( V(\hat{x}, x) = \sqrt{(x - \hat{P}\hat{x})^TM(x - \hat{P}\hat{x})} \) and the interface function that maps \( \hat{x}, x, \hat{u}, \hat{w} \) to \( u \) so that \((1)\) holds is given by \( u = K_1(x - \hat{P}\hat{x}) - K_2\hat{x} - K_3\hat{w} + K_4\hat{u} \). The matrix \( K_4 \) is given as the one that minimizes \( |\sqrt{M(\hat{P}\hat{B} - BK_4)}| \) which can be computed according to [14, Prp. 1].

Note that Theorem 3 provides only structural conditions for the construction of approximate abstractions of linear control systems and it is an interesting open question on how to pick the different matrices outlined in Table 1 (within the allowed domains) so as to obtain approximate abstractions with optimal approximation accuracies.

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**Table 1.** Construction of an approximate abstraction \( \Sigma \).
Let us consider the compositional construction of an approximate abstraction together with a simulation function for an interconnected linear control system illustrated in Figure 3. We consider two triple integrators (\(\Sigma_1\) and \(\Sigma_3\)) which are organized in a feedback connection, where the output of \(\Sigma_3\) is directly connected to the input of \(\Sigma_1\) and the output of \(\Sigma_1\) is connected to the input of \(\Sigma_3\) via two two-dimensional systems \(\Sigma_2\) and \(\Sigma_4\). The system matrices are accordingly set to

\[
\begin{align*}
A_1 &= A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, & B_{1}^T &= B_{3}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \\
C_{11} &= C_{33} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},
\end{align*}
\]

and

\[
\begin{align*}
A_2 &= A_4 = \begin{bmatrix} 0 & 1 & 0 \\ -6 & 1 & 0 \\ -5 & 0 & 0 \end{bmatrix}, & B_{2}^T &= B_{4}^T = C_{22} = C_{44} = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\end{align*}
\]

Whereas the interconnection matrices \(C_{ij}\), \(D_{ij}\) are given by

\[
\begin{align*}
C_{14} &= C_{12} = C_{31} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, & C_{23} &= C_{43} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
D_{13} &= \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix}, & D_{21} &= D_{41} = \begin{bmatrix} -d_2 \\ 2d_2 \end{bmatrix}, & D_{34} &= D_{32} = \begin{bmatrix} 0 \\ 0 \\ d_3 \end{bmatrix},
\end{align*}
\]

for some \(d_i \in \mathbb{R}\). The remaining \(C_{ij}\) and \(D_{ij}\) are given by zero matrices. We summarize the input and output matrices by

\[
\begin{align*}
C_1 &= C_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, & C_2 &= C_4 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
D_1 &= \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix}, & D_2 &= D_4 = \begin{bmatrix} -d_2 \\ 2d_2 \end{bmatrix}, & D_3 &= \begin{bmatrix} 0 & 0 \\ 0 & d_3 \end{bmatrix}.
\end{align*}
\]

The Abstract System. We continue the example by applying the procedure outlined in Table 1 to construct an abstraction \(\hat{\Sigma}_i\) of each subsystem \(\Sigma_i\).

We start by computing \(M_i\), \(K_{i,1}\) and \(\lambda_i\), for \(i \in \{1, 3\}\), such that the matrix inequalities in 1) of Table 1 hold. To this end, we solve the linear matrix inequality given by equations (6) and (7) in [14]. We obtain

\[
M_i = \begin{bmatrix} 4.59 & 4.07 & 0.90 \\ 4.07 & 4.72 & 1.24 \\ 0.90 & 1.24 & 0.61 \end{bmatrix}, & K_{i,1} = -\begin{bmatrix} 5.13 & 7.12 & 3.03 \end{bmatrix},
\]

with \(\lambda_i = 1\). Next we determine \(P_i\) for \(\Sigma_i\) so that (30a)-(30c) hold by \(P_i = [1 \ 0 \ 0]^T\). Following 2) through 5) in Table 1 we obtain \(\hat{\Sigma}_i\) by

\[
\hat{A}_i = 0, \quad \hat{B}_i = 1, \quad \hat{D}_i = 0, \quad \hat{C}_i = 1,
\]
together with the matrices for the interface $K_{1,2} = 0$, $K_{1,3} = d_1$, $K_{3,3} = [d_3, d_3]$ and $K_{3,4} = 1.47$. The simulation functions follow by $V_i(\hat{x}_i, x_i) = (x_i - P_i\hat{x}_i)^\top M_i(x_i - P_i\hat{x}_i)$ and the associated comparison functions by $\alpha_1 = \alpha_3 = \text{id}$ and

$$
\lambda_1(r) = r, \rho_1(r) = 1.81r, \mu_1(r, r_3, r_4) = 0.78d_3r_3
$$

$$
\lambda_3(r) = r, \rho_3(r) = 1.81r, \mu_3(r_1, r_2, r_4) = 0.78d_3(r_2 + r_4).
$$

We continue with subsystems $\Sigma_2$ and $\Sigma_4$. Since the subsystems $\Sigma_2$ and $\Sigma_4$ have no external inputs, it is necessary that the matrices $A_2$ and $A_4$ are Hurwitz in order to be able to find matrices $M_2$ and $M_4$ that satisfy the matrix inequalities in 1) of Table 1. This holds for our example and we compute

$$
M_2 = M_4 = \begin{bmatrix} 26 & 10 \\ 10 & 4 \end{bmatrix}, \quad K_{2,1} = K_{4,1} = 0
$$

with $\lambda_2 = \lambda_4 = 2$. Also the conditions (30) and (30b) simplify in the absence of any external inputs. It follows that im $P_i$ needs to be an $A_i$-invariant subspace that contains im $D_i$. In this case, we can use the Algorithm 3.2.1 in [3] to compute the minimal $A_i$-invariant subspace that contains im $D_i$. We obtain $P_i = [1 - 2]^\top, i \in \{2, 4\}$, and the abstractions $\Sigma_i$ by

$$
\hat{A}_i = -2, \quad \hat{B}_i = 1, \quad \hat{D}_i = -d_2, \quad \hat{C}_i = 1.
$$

As before we obtain the square-root-of-quadratic simulation function, defined by $P_i$ and $M_i$. The associated interface follows by $k_i \equiv 0$. The comparison functions associated with the simulation function $V_i$ are given by $\alpha_i = \text{id}$, $\lambda_i(r) = 2r$, $\rho_i(r) = 1.41r$ and $\mu_i(r_1, r_2, r_3) = 1.41d_2r_1$.

The Composition. We apply Theorem 2 to obtain a simulation function from $\mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ to $\mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$. The functions $\Lambda$ and $\Gamma$ are linear and identified with

$$
\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 & 0 & 0.78d_1 & 0 \\ 1.41d_2 & 0 & 0 & 0 \\ 0 & 0.78d_3 & 0 & 0.78d_3 \\ 1.41d_2 & 0 & 0 & 0 \end{bmatrix}.
$$

In order to be able to apply Theorem 2 we need to assure that the spectral radius of $\Gamma \Lambda^{-1}$ is strictly less than one so that there exists a vector $\eta \in \mathbb{R}^4_{\geq 0}$ such that $(1 + \varepsilon)\Gamma \Lambda^{-1} \eta < \eta$ holds for some $\varepsilon > 0$. We pick $d_1 = d_2 = d_3 = 0.5$ and obtain $\lambda_{\max}(\Gamma \Lambda^{-1}) = 0.19$. We pick $\eta = [0.4 \ 0.6 \ 0.5 \ 0.6]^\top$ and verify that $(1 + \varepsilon)\Gamma \Lambda^{-1} \eta < \eta$ holds for $\varepsilon = 4$. Certainly, $\lambda_i/\eta_i r$ is differentiable and satisfies (14). We apply Theorem 2 and obtain $V(\hat{x}, x) = \max_i \frac{\lambda_i}{\eta_i} V_i(\hat{x}_i, x_i)$ as simulation function from $\mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ to $\mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$, with the associated comparison functions given by $\alpha(r) = r$, $\lambda(r) = 4/5r$ and $\rho(r) = 4.8r$, see Remark 3. Hence, we obtain the bound

$$
|\zeta(t) - \zeta(t)| \leq V(\hat{\zeta}(t), \xi(t)) \leq e^{-4/5d} V(\hat{x}, x) + 5.9\|\hat{\nu}\|_{\infty}.
$$

Let $V_{vec}(t) = (V_1(\hat{\xi}_1(t), \xi_1(t)); \ldots; V_4(\hat{\xi}_4(t), \xi_4(t)))$ and $\hat{Z} = (\rho_1(\|\hat{\nu}_1\|_{\infty}); \ldots; \rho_4(\|\hat{\nu}_4\|_{\infty}))$, then similarly to (31), we get

$$
V_{vec}(t) \leq e^{-\Lambda} V_{vec}(0) + \Gamma \Lambda^{-1} V_{vec}(t) + \Lambda^{-1} \hat{Z}
$$

which provides the bound $|\zeta(t) - \zeta(t)| \leq |V_{vec}(t)|$. 
Controller Synthesis. Let us now synthesize a controller for $\Sigma$ via the abstraction $\hat{\Sigma}$ to enforce the specification, defined by the LTL formula \[ 2 \]

$$\Box S \land \Box \Diamond T_i, \quad S, T_i \subseteq \mathbb{R}^4,$$ (33)

which requires that any output trajectory $\zeta$ of the closed loop system evolves inside the set $S$ and visits each $T_i$, $i \in [1;3]$ infinitely often, i.e., for all $t \in \mathbb{R}_{>0}$ $\zeta(t) \in S$ and for each $i \in [1;3]$ there exists $t' \geq t$ so that $\zeta(t') \in T_i$, see \[ ] [2]. The specification is illustrated in Figure 4. We use SCOTS \([28]\) to synthesize a controller for $\hat{\Sigma}$ to enforce (33). In the synthesis process we restricted the abstract inputs to $\hat{u}_1, \hat{u}_3 \in [-0.1, 0.1]$ and $\hat{u}_2 = \hat{u}_4 = 0$ for all times. Given that we can set the initial states of $\Sigma$ to $x_i = P_3 \hat{x}_i$, so that $V(\hat{x}, x) = 0$, we obtain a bound from (31) on the output difference by $|\zeta(t) - \zeta(t)| \leq V(\hat{x}(t), \xi(t)) \leq \bar{V} := 0.85$ for all $t \geq 0$. An improved bound is obtained from (32) by noting that $V_{\text{vec}}^{k+1} = \Gamma V_{\text{vec}}^k + \Lambda^{-1} \hat{Z}$ with $V_{\text{vec}}^0 = (\eta_1/\lambda_1 \hat{V}; \ldots ; \eta_4/\lambda_4 \hat{V})$ provides an upper bound $|\zeta(t) - \zeta(t)| \leq V_{\text{vec}}(t) \leq V_{\text{vec}}^{k+1}$ for any $k \geq 0$.

A closed loop trajectory of $\Sigma$ and $\hat{\Sigma}$ as well as the output difference and the theoretical bound $V_{\text{vec}}^\infty = \lim_{k \to \infty} V_{\text{vec}}^k$ are illustrated in Figure 4. A bound for $||\nu_1||_\infty$ follows by $|K_{1,1}(x_1 - P_1 \hat{x}_1)| + |K_{1,3} \hat{w}_1| + K_{1,4} \hat{u}_1| \leq 5.7$ where we used $|x_1 - P_1 \hat{x}_1| \leq V_1(x_1, \hat{x}_1)/\sqrt{\lambda_{\min}(M_1)} \leq 0.47$ and $|\hat{w}_1| = |\hat{y}_3| \leq 6$. Similarly we obtain $||\nu_3||_\infty \leq 4.4$. For the example trajectory in Figure 4 the inputs $\nu_1$ and $\nu_3$ never exceeded 1.2 and 0.31, respectively.

**Figure 4.** Left: The specification with closed loop trajectories of $\Sigma$ (red) and $\hat{\Sigma}$ (blue). The green dot marks the initial state. The sets $S$ and $T_i$ are given by $S = \hat{S} \setminus \bar{S}$ with $\hat{S} = [-6,6] \times [-1,1] \times [-6,6] \times [-1,1]$ and $\bar{S} = [-5,5] \times [-1,1] \times [-5,5] \times [-1,1]$, $\hat{T}_1 = \frac{1}{3}[-1,1] \times [-1,1] \times [5,6] \times [-1,1]$, $\hat{T}_2 = [-6,-5] \times [-1,1] \times [-5,-4] \times [-1,1]$, and $\hat{T}_3 = [5,6] \times [-1,1] \times [-5,-4] \times [-1,1]$. Right: The output difference (blue) and the upper bound obtained from (31) (red).

**Remark 4.** As the controller synthesis algorithms implemented in SCOTS operate on a finite abstraction of the concrete system, which is obtained by a uniform discretization of the state space, it would not have been possible to synthesize a controller for the original system $\Sigma$, without the lower dimensional intermediate approximation $\hat{\Sigma}$. 
7. Summary

In this paper we presented a compositional reasoning approach based on a small gain type argument in connection with approximate abstractions of nonlinear control systems. Given that the small gain type condition is satisfied, we showed how to construct an approximate abstraction together with a simulation function for an interconnected nonlinear control system from the abstractions and simulation functions of its subsystems. Moreover, for the special case of linear control systems, we characterized simulation functions in terms of a controlled invariant, externally stabilizable subspace. Based on this characterization, we proposed a particular scheme to construct approximate abstractions together with the associate simulation functions.

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Lemma 1. Let $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a monotonically increasing function, differentiable on $\mathbb{R}_{>0}$, and consider a function $f : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$. Then we have for all $x, v \in \mathbb{R}^n$ with $f(x) > 0$

$$D^+(\alpha \circ f)(x, v) \leq \alpha'(f(x))D^+f(x, v). \quad (34)$$

Proof. As $\alpha$ is monotonically increasing and differentiable on $\mathbb{R}_{>0}$ we have for all $y_0 \in \mathbb{R}_{>0}$

$$0 \leq \alpha'(y_0) = \lim_{y \rightarrow y_0, y < y_0} \frac{\alpha(y) - \alpha(y_0)}{y - y_0} = \limsup_{y \rightarrow y_0, y > y_0} \frac{\alpha(y) - \alpha(y_0)}{y - y_0}.$$

Let $x \in \mathbb{R}^n$ with $y_0 = f(x) > 0$ and $v \in \mathbb{R}^n$. There exists a sequence $(t_i)_{i \in \mathbb{N}}$ in $\mathbb{R}_{>0}$ with limit 0 so that

$$D^+(\alpha \circ f)(x, v) = \lim_{i \rightarrow \infty} \frac{1}{t_i} \left( \alpha(f(x + t_i v)) - \alpha(f(x)) \right).$$

If $f(x + t_i v) = f(x)$ for all $i \geq j$ for some $j \in \mathbb{N}$, we have $D^+(\alpha \circ f)(x, v) = 0$ and $D^+f(x, v) \geq 0$, which shows (34). If for every $j \in \mathbb{N}$ there exists $i \geq j$ so that $f(x + t_i v) - f(x) > 0$ holds, we set $y_i = f(x + t_i v)$, $y = f(x)$ and pick a subsequence $(t_{i_j})$ of $(t_i)$ so that $y_{i_j} > y$ for all $i_j$. Since $(\alpha(y_{i_j}) - \alpha(y))(y_{i_j} - y) \geq 0$ and $f(x + t_{i_j} v) -
\( f(x) > 0 \), for all \( j \in \mathbb{N} \) we get

\[
\lim_{j \to \infty} \frac{\alpha(y_{i_j}) - \alpha(y)}{y_{i_j} - y} = \frac{f(x + t_{i_j}v) - f(x)}{t_{i_j}} \\
\leq \limsup_{j \to \infty} \frac{\alpha(y_{i_j}) - \alpha(y)}{y_{i_j} - y} \limsup_{j \to \infty} \frac{f(x + t_{i_j}v) - f(x)}{t_{i_j}} \\
\leq \alpha'(f(x))D^+f(x,v).
\]

If \((y_i - y)_i \in \mathbb{N}\) contains infinitely negative entries, we pick a subsequence \((t_{i_j})\) of \((t_i)\) so that we have \(y_{i_j} < y\) for all \(i_j\) and use a similar reasoning as in the previous case to arrive at \((34)\). \(\square\)

**Proof of Theorem 7.** Let us define the \(K_\infty\) function \(\tilde{\mu}(s) := \mu(s, \ldots, s)\). We consider the trajectories \((\xi, \zeta, \nu, \omega)\) and \((\hat{\xi}, \hat{\zeta}, \hat{\nu}, \hat{\omega})\) of the control systems \(\Sigma\) and \(\hat{\Sigma}\), respectively. We assume that \(\nu\) is given such that \([1]\) holds with \(x = \xi(t), \dot{x} = \dot{\xi}(t), u = \nu(t), \hat{u} = \hat{\nu}(t), w = \omega(t), \hat{w} = \hat{\omega}(t)\) for all \(t \in \mathbb{R}_\geq 0\). We define \(c = \lambda^{-1}(2\mu(||\dot{\nu}\|_\infty) + 2\hat{\mu}(||\omega - \hat{\omega}\|_\infty))\) and the set \(S = \{(x; \dot{x}) \in \mathbb{R}^n \times \mathbb{R}^n \mid V(\dot{x}, x) \leq c\}\). From \([1]\), we see that \(y(t) := V(\xi(t), \xi(t))\) satisfies, whenever \((\hat{\xi}(t), \hat{\xi}(t))\) is outside the set \(S\), i.e. \(y(t) > c\), the inequality

\[
D^+y(t, 1) = D^+V \left( (\hat{\xi}(t), \xi(t)), \left[ \hat{f}(\hat{\xi}(t), \hat{\nu}(t), \hat{\omega}(t)) \right] \right) \\
\leq -\frac{1}{2} \lambda(V(\hat{\xi}(t), \xi(t)))
\]

where the equality in \((35)\) follows from \([25]\ Thm 4.3, Rmk 4.4, pp. 353\). Hence, \(y\) is decreasing for \(y(t) > c\). Suppose for all \(t \in [a, b] \subseteq \mathbb{R}_\geq 0\) we have \(y(t) > c\), then \(t', t \in [a, b]\) with \(t' < t\) implies \(y(t') < y(t) - \frac{1}{2} \int_{t'}^t y(s)ds\) \([25]\ Thm 2.3, Rmk 2.5\). We show that \(S\) is forward invariant, i.e., if there exists \(t_0 \geq 0\) with \((\xi(t_0), \hat{\xi}(t_0)) \in S\) then we have \((\xi(t), \hat{\xi}(t)) \in S\) for all \(t \geq t_0\). Let \((\xi(t_0), \hat{\xi}(t_0)) \in S\) and suppose to the contrary that the trajectories leave \(S\). Since \(S\) is closed, there exists \(t_1 > t_0\) and \(\varepsilon \in \mathbb{R}_\geq 0\) such that \(y(t_1) \geq c + \varepsilon\). Let \(t_1\) be minimal for this choice of \(\varepsilon\). Since \(y(t)\) is continuous in \(t\), there exists \(\delta > 0\) such that \(y(t) > c\) holds for all \(t \in [t_1 + \delta, t_1 + \delta]\). However, \(y\) is decreasing on \([t_1 + \delta, t_1]\) which contradicts the minimality of \(t_1\). It follows that \(S\) is forward invariant and the output trajectories satisfy for all \(t \geq t_0\) the inequality

\[
|\hat{\zeta}(t) - \hat{\zeta}(t)| \leq \alpha^{-1}(V(\hat{\xi}(t), \xi(t))) \\
\leq \alpha^{-1}(\lambda^{-1}(2\rho(||\dot{\nu}\|_\infty) + 2\hat{\mu}(||\omega - \hat{\omega}\|_\infty))) \\
\leq \gamma_{\text{ext}}(||\dot{\nu}\|_\infty) + \gamma_{\text{int}}(||\omega - \hat{\omega}\|_\infty)
\]

with \(K \cup \{0\}\) functions \(\gamma_{\text{ext}}(s) := \alpha^{-1}(\lambda^{-1}(4\rho(s)))\) and \(\gamma_{\text{int}}(s) := \alpha^{-1}(\lambda^{-1}(4\hat{\mu}(s)))\). Note that here we used the fact that for any \(K \cup \{0\}\) function \(\gamma\) the inequality \(\gamma(a + b) \leq \gamma(2a) + \gamma(2b)\) holds for all \(a, b \in \mathbb{R}_\geq 0\).

We proceed with the analysis of the trajectories outside of \(S\). We define \(t_0 = \inf\{t \mid (\xi(t), \hat{\xi}(t)) \not\in S\}\) (possibly infinite) and observe that the function \(y(t) := V(\hat{\xi}(t), \xi(t))\) is absolutely continuous, since \(V\) is locally Lipschitz and the state trajectories are absolutely continuous. Hence, \(y(t)\) is differentiable almost everywhere and \(y\) satisfies \(\dot{y}(t) \leq -\frac{1}{2}\lambda(y(t))\) for almost all \(t \in [0, t_0]\).

Then we apply Lemma 4.4 in \([20]\) and obtain a \(KL\) function \(\tilde{\beta}\) with \(\tilde{\beta}(r, 0) = r\), depending only on \(\lambda\), so that \(y(t) \leq \tilde{\beta}(y(0), t)\) holds for all \(t \in [0, t_0]\). It follows that the
output trajectories satisfy for all $t \in [0, t_0]$ the inequality

$$|\zeta(t) - \hat{\zeta}(t)| \leq \beta(V(\hat{\zeta}(0), \xi(0)), t)$$

(37)

with $\beta(r, t) = \alpha^{-1}(\bar{\beta}(r, t))$. By combining the bounds (36) and (37) we obtain the desired estimate (5).

Proof of Corollary 1. It follows immediately by the previous derivations that $V$ satisfies (6) with the $KL$ function given by $\bar{\beta}$ (as determined in the previous proof) and the $\mathcal{K} \cup \{0\}$ functions are given by $\gamma_{\text{ext}}(s) := \lambda^{-1}(4\rho(s))$ and $\gamma_{\text{int}}(s) := \lambda^{-1}(4\bar{\mu}(s))$. 

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