Scale invariance in Newton–Cartan and Hořava–Lifshitz gravity

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Abstract
We present a detailed analysis of the construction of $z = 2$ and $z \neq 2$ scale invariant Hořava–Lifshitz gravity. The construction procedure is based on the realization of Hořava–Lifshitz gravity as the dynamical Newton–Cartan geometry as well as a non-relativistic tensor calculus in the presence of the scale symmetry. An important consequence of this method is that it provides us with the necessary mechanism to distinguish the local scale invariance from the local Schrödinger invariance. Based on this result we discuss the $z = 2$ scale invariant Hořava–Lifshitz gravity and the symmetry enhancement to the full Schrödinger group.

Keywords: Newton–Cartan gravity, Schrödinger symmetries, Hořava–Lifshitz gravity

(Some figures may appear in colour only in the online journal)

1. Introduction

Non-relativistic gravity theories are of considerable interest for numerous reasons that are rooted in their applications in condensed matter and non-relativistic holography. Particularly two noteworthy models, Newton–Cartan and Hořava–Lifshitz gravity, found themselves a wide range of application in recent years. For instance, Newton–Cartan gravity, which is originally developed as the generally covariant description of the Newtonian gravity [1, 2], is known be useful in the effective field theory description of the quantum Hall effect [3–6]. Hořava–Lifshitz gravity, on the other hand, was proposed for UV completion of General
Cartan geometry is encoded in a spatial metric

The Newton–Cartan geometry and its torsional extensions in section 2.

properly, there is an enhancement to the full Schrödinger symmetry. In section 4, we make the distinction between the scale and Schrödinger symmetry explicit by constructing \( z = 2 \) scale invariant models. This is achieved by introducing a particular non-metricity condition to the metric compatibility equation.

For arbitrary values of \( z \), we cannot go beyond the scale symmetry to the full Schrödinger symmetry unless \( z \) is fixed to \( z = 2 \). For \( z = 2 \) the Jacobi identity is satisfied and the last part of this section is devoted to the Newton–Cartan geometry and its torsional extensions in the presence of Schrödinger symmetry.

To develop the framework necessary for the scale invariant tensor calculus, we start with a review of the basics of Newton–Cartan geometry and its torsional extensions in section 2.

The Newton–Cartan geometry is encoded in a spatial metric \( h^{\mu \nu} \), a temporal vielbein \( \tau_\mu \), and a U(1) gauge field \( m_\mu \). An important notion in this geometric setting is that there is no unique way to define the inverse of spatial metric or the temporal vielbein. The definition of the inverse metrics are based on orthogonality relations for the spatial and temporal metrics, but one is always free to choose a different set of inverse metrics in a way that the orthogonality conditions are satisfied. This freedom is known as the Milne boosts and it plays a crucial role in the torsional extension of the Newton–Cartan geometry. Thus, we pay a special attention to the definition of the inverse metrics, their Milne invariance, and the map between different choices of inverse metrics when the Newton–Cartan geometry is extended with torsion.

In section 3, we introduce a scale and Schrödinger symmetry to Newton–Cartan geometry. This is achieved by introducing a particular non-metricity condition to the metric compatibility equation. For arbitrary values of \( z \), we cannot go beyond the scale symmetry to the full Schrödinger extension since a particular Jacobi identity do not close in the presence of Schrödinger symmetry. For \( z = 2 \) the Jacobi identity is satisfied and the last part of this section is devoted to the Newton–Cartan geometry and its torsional extensions in the presence of Schrödinger symmetry.

In section 4, we make the distinction between the scale and Schrödinger invariance explicit by constructing \( z = 2 \) scale invariant models. This is
most easily done by gauging the relevant non-relativistic spacetime symmetries. Therefore we separate this section into two main subsections. In the first subsection we give a brief review of Bargmann algebra, which is the central extension of the Galilean algebra, as well as its $z = 2$ scale and Schrödinger extensions. Based on the gauging of the Bargmann algebra and its extensions, we then construct the relation between the non-relativistic geometric quantities and the group theoretical elements with the relevant symmetry. The second subsection introduces a $z = 2$ scale invariant tensor calculus, and includes the construction of the relevant $z = 2$ scale invariant geometric quantities. Finally, we show that although our models exhibit scale invariance in general, a particular choice of free parameters lead us to a Schrödinger gravity. In section 5, we go beyond the $z = 2$ theories. To achieve that we repeat procedure that we built in section 4 and establish the relation between the non-relativistic geometric quantities and the group theoretical elements of $z \neq 2$ scale extended Bargmann algebra. Next, we develop a scale invariant tensor calculus and construct $z \neq 2$ non-relativistic gravity models. Finally, using the dictionary between the Newton–Cartan geometry and the Hořava–Lifshitz gravity [17], we construct the $z \neq 2$ scale extension of the Hořava–Lifshitz gravity, which is one of the major results of this work. We conclude in section 6.

2. Newton–Cartan gravity

We begin with a review of the Newton–Cartan geometry and its torsional extensions. Unlike the relativistic case, where the fundamental geometric structure is the non-degenerate metric of a pseudo-Riemannian manifold, the Newton–Cartan geometry is described by a degenerate spatial metric $h^\mu\nu$ of rank-d and a temporal vielbein $\tau^\mu$ of rank-1, together with a connection $\Gamma^\rho_{\mu\nu}$ on an orientable manifold $M$. Here, degeneracy imply that $h^\mu\nu \tau^\nu = 0$. 

In order to discuss the notions of parallel transport and geodesics we need to provide a suitable connection. In the relativistic case, the torsion-free connection is uniquely fixed by the metric. As we will show in what follows, the connection in Newton–Cartan geometry is quite different: The uniqueness of the torsion-free compatible connection is lost and introducing torsion can break the invariance of the connection under Milne boosts. Moreover, the inclusion of non-metricity modifies the anti-symmetric part of the connection. As we will see, the degenerate nature of the Newton–Cartan geometry allows other geometric structures in addition to the ones we discussed above. Along the way, we will introduce necessary data to fix the connection uniquely. In the next two sections our focus will be understanding the connections with/without torsion which will be crucial obtaining the Newton–Cartan geometry from the gauging procedure.

2.1. Torsionless Newton–Cartan geometry

Let us start our discussion with the torsionless Newton–Cartan geometry. With that in mind, we first impose that the connection $\Gamma^\rho_{\mu\nu}$ is symmetric and solve the metric compatibility conditions

$$\nabla_\mu \tau_\nu = \partial_\mu \tau_\nu - \Gamma^\rho_{\mu\nu} \tau_\rho = 0,$$

$$\nabla_\mu h^\nu{}^\rho = \partial_\mu h^\nu{}^\rho + \Gamma^\nu_{\sigma\mu} h^\sigma{}^\rho + \Gamma^\rho_{\sigma\mu} h^\sigma{}^\nu = 0,$$

where the covariant derivative $\nabla$ is with respect to a connection $\Gamma^\rho_{\mu\nu}$. As the connection is symmetric, the antisymmetric part of the temporal metric compatibility condition implies
\[ \tau_\mu = \partial_\mu f, \]  
(2.3)

for a scalar function \( f(x^\mu) \), which is chosen to be the absolute time \( t \) so that the \( f = \text{const.} \) simultaneity leaves foliate the spacetime. The temporal metric compatibility condition also fixes the temporal part of the connection as

\[ \tau_\mu \Gamma^\nu_{\mu\nu} = \partial_\mu \tau_\nu. \]  
(2.4)

Having determined the temporal part of the connection let us proceed with the spatial part. For that, we need to introduce two new tensors: The spatial inverse metric \( h_{\mu\nu} \) and the temporal inverse vielbein \( \tau^\mu \) which satisfies the following relations

\[ h^{\mu\sigma} h_{\nu\sigma} = \delta^\mu_\nu - \tau^\mu \tau_\nu, \quad \tau^\mu \tau_\mu = 1, \quad h_{\mu\nu} \tau^\nu = 0. \]  
(2.5)

Using the inverse quantities, the most general symmetric connection compatible with the conditions (2.2) is given by [22]

\[ \Gamma^\rho_{\mu\nu} = \tau^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} \left( \partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu} \right) - h^{\rho\sigma} \tau_{(\mu} F_{\nu)\sigma}, \]  
(2.6)

for an arbitrary 2-form \( F_{\mu\nu} \). In order to make a contact with the covariant form of Newtonian gravity, i.e. Newton–Cartan gravity, the following conditions should be satisfied by the aptly named Newtonian connection:

1. The geodesic equation based on \( \Gamma^\nu_{\mu\nu} \) should give rise to the classical equation of motion of a massive particle

\[ \frac{d^2 x^\alpha(t)}{dt^2} + \frac{\partial \phi(x)}{dx^\alpha} = 0, \]  
(2.7)

where \( x^\alpha(t) \) are the spatial coordinates, \( t \) is the absolute time and \( \phi(x) \) is the Newtonian potential.

2. The only non-vanishing component of the Riemann tensor

\[ R^\mu_{\nu\rho\sigma}(\Gamma) = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho} \Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma} \Gamma^\alpha_{\nu\rho}. \]  
(2.8)

for the Newtonian connection (2.6) should give rise to the Poisson equation for the Newtonian potential,

\[ \nabla^2 \phi = 4\pi G \rho, \]  
(2.9)

where \( \rho \) is the mass density.

These two conditions can be satisfied given that the Riemann tensor (2.8) satisfies the so-called Trautman [23] and Ehlers [24] conditions

\[ h^{\lambda\rho} R_{\nu\rho\sigma}(\Gamma) = 0, \]  
(2.10)

\[ h^{\rho\lambda} R_{\mu\nu\rho\sigma}(\Gamma) = 0 \quad \text{or} \quad \tau^{\lambda} R_{\nu\rho\sigma}(\Gamma) = 0 \quad \text{or} \quad h^{\rho\lambda} R_{\nu\rho\sigma}(\Gamma) = 0, \]  
(2.11)

where the Trautman condition (2.10) further implies that for the connection to be Newtonian, \( F_{\mu\nu} \) must be closed, i.e.

\[ F_{\mu\nu} = 2 \partial_{(\mu} m_{\nu)}, \]  
(2.12)
where \( m_\mu \) is a \( U(1) \) connection. With these conditions in hand, it is straightforward to show that the only non-vanishing component of the connection and the Riemann tensor are given by [21]

\[
\Gamma^a_{00} = \delta^a_b \partial_b \phi, \quad R^a_{\alpha\beta0} = \nabla^2 \phi = 4\pi G \rho, \tag{2.13}
\]

which satisfies the properties of a Newtonian connection. Thus, we conclude that the Newton–Cartan gravity is given by two degenerate metrics \( h^{\mu\nu} \) and \( \tau^\mu \) and a \( U(1) \) connection \( m_\mu \) equipped with the Trautman (2.10) and Ehlers (2.11) conditions.

Before proceeding any further, it is worth mentioning the Milne boost symmetry of the Newton–Cartan geometry and the invariant quantities in the presence of the \( U(1) \) connection. First of all, while the fundamental temporal and spatial metrics \( \tau^\mu \) and \( h^{\mu\nu} \) are uniquely defined, the inverse metrics \( \tau^\mu \) and \( h^{\mu\nu} \) in (2.5) are not unique, e.g. considering a 1-form \( \psi_\mu \) we can define [20]

\[
\tau'^\mu = \tau^\mu + h^{\mu\nu} \psi_\nu, \quad h'^{\mu\nu} = h^{\mu\nu} - (\tau_\mu P^\rho_\nu + \tau_\nu P^\rho_\mu) \psi_\rho + \tau_\mu \tau_\nu h^{\rho\sigma} \psi_\rho \psi_\sigma, \tag{2.14}
\]

that still satisfies the inversion relations (2.5). These redefinitions are referred to as Milne boosts. The quantities built by using the connection \( \Gamma \) are covariant if the connection itself is invariant under the redefinition (2.14). This would require the following Milne transformation property for the \( U(1) \) connection \( m_\mu \) [20]

\[
m'_\mu = m_\mu - P^\rho_\mu \psi_\rho + \frac{1}{2} \tau_\mu h^{\rho\sigma} \psi_\rho \psi_\sigma, \tag{2.15}
\]

in which case, the invariance of the connection \( \Gamma \) is satisfied given that its temporal part is symmetric [20]. Thus, it is worth emphasizing that when introducing a temporal torsion, or torsion in general, one must be careful with the transformation of the connection under Milne boosts.

2.2. Twistless torsional Newton–Cartan geometry

In this subsection, we introduce a ‘twistless torsion’ to the Newton–Cartan geometry. As we will discuss in the detail below, the defining data of the twistless-torsional Newton–Cartan geometry (TTNC) is encoded in the following set of fields

\[
(h^{\mu\nu}, \tau^\mu, b_\mu, M_\mu), \tag{2.16}
\]

where \( b_\mu \) and \( M_\mu \) are the necessary additional vector fields. To see the role of \( b_\mu \), we first consider the temporal component of the connection, which is fixed by the temporal metric compatibility condition (2.2)

\[
\tau^\rho \Gamma^\mu_\rho = \partial_\mu \tau^\nu. \tag{2.17}
\]

As a result, the time component of the torsion is fixed as

\[
\tau^\rho \Gamma^\mu_\rho = \partial_\mu \tau^\nu. \tag{2.18}
\]

The ‘twistless torsion’ condition is given by [25]

\[
\tau^\lambda \partial_\mu \Gamma^\nu_\mu = \partial_\nu \tau^\mu = 0, \tag{2.19}
\]

which indicates that the twistless torsional Newton–Cartan structure includes an additional Milne-invariant vector \( b_\mu \) by virtue of Frobenius theorem [19].
\[ \partial_{\mu} \tau_{\nu} = z b_{\mu} \tau_{\nu}, \]  
(2.20)

where we introduced the coefficient \( z \), the dynamical critical exponent, for later convenience.

Next we determine the most general connection that is compatible with (2.2) by solving the compatibility condition for \( h^{\mu \nu} \) [26]

\[ \Gamma_{\mu \nu}^{\rho} = \tau^{\rho} \partial_{\mu} \tau_{\nu} + \frac{1}{2} h^{\rho \sigma} \left( \partial_{\rho} h_{\sigma \mu} + \partial_{\mu} h_{\sigma \nu} - \partial_{\sigma} h_{\mu \nu} \right) - h^{\rho \sigma} \tau_{(\mu} F_{\nu)\sigma} - K_{\mu \nu}^{\rho}, \]
(2.21)

where \( F_{\mu \nu} = 2 \partial_{(\mu} m_{\nu)} \) and \( K_{\mu \nu}^{\rho} \) is the spatial contorsion tensor

\[ \tau_{\rho \sigma} K_{\mu \nu}^{\rho} = 0, \quad h_{\rho \sigma} (-K_{\mu \nu}^{\rho} + K_{\mu \nu}^{\sigma}) = h_{\rho \sigma} T_{\mu \nu}^{\sigma}, \]
(2.22)

where \( h_{\rho \sigma} T_{\mu \nu}^{\sigma} \) defines the spatial part of the torsion. Recall that, previously the invariance of the connection under Milne boosts rely on the condition \( \partial_{[\mu} \tau_{\nu]} = 0 \). In TTNC geometry we give up this condition, so the variation of the connection (2.21) is given by [20]

\[ \delta_{\mu} \Gamma_{\nu}^{\rho} = h^{\rho \sigma} \left( (P_{\sigma}^{\rho} \partial_{[\mu} \tau_{\nu]} + P_{\mu}^{\rho} \partial_{[\rho} \tau_{\sigma]} + P_{\sigma}^{\rho} \partial_{[\sigma} \tau_{\mu]} ) \psi_{\alpha} + \frac{1}{2} h^{\rho \sigma \alpha \beta} \psi_{\alpha} \psi_{\beta} (\tau_{\nu} \partial_{[\mu} \tau_{\sigma]} + \tau_{\mu} \partial_{[\nu} \tau_{\sigma]} ) \right) \]
(2.23)

Therefore, assuming that the contorsion tensor is U(1) invariant, \( \delta_{U(1)} K_{\mu \nu}^{\rho} = 0 \), we can split the TTNC geometry into two cases depending on the Milne transformation of the contorsion tensor

c1. If the spatial contorsion tensor is Milne-invariant \( \delta_{M} K_{\mu \nu}^{\rho} = 0 \), then we need to construct Milne-invariant inverse temporal and spatial metrics and a U(1) connection that still satisfies the inversion relations (2.5). The connection must be re-written based on the new inverse elements.

c2. If the spatial contorsion transforms under the Milne transformation, then it must satisfy

\[ \delta_{M} K_{\mu \nu}^{\rho} = h^{\rho \sigma} \left( (P_{\sigma}^{\rho} \partial_{[\mu} \tau_{\nu]} + P_{\mu}^{\rho} \partial_{[\rho} \tau_{\sigma]} + P_{\sigma}^{\rho} \partial_{[\sigma} \tau_{\mu]} ) \psi_{\alpha} + \frac{1}{2} h^{\rho \sigma \alpha \beta} \psi_{\alpha} \psi_{\beta} (\tau_{\nu} \partial_{[\mu} \tau_{\sigma]} + \tau_{\mu} \partial_{[\nu} \tau_{\sigma]} ) \right). \]
(2.24)

In the following, we will first consider these two cases separately and then show that one can transform between them by means of a linear transformation.

c1. Milne invariant spatial contorsion tensor

We first consider a Milne invariant spatial contorsion tensor, \( \delta_{M} K_{\mu \nu}^{\rho} = 0 \), which includes a vanishing spatial contorsion as a special case. The only way to make the connection Milne invariant is to construct the connection in terms of Milne invariant objects. As given in (2.14) and (2.15), \( \tau^{\rho}, h_{\mu \nu}, \) and \( m_{\mu} \) are not invariant under Milne boosts. Milne invariance can be achieved by combining these quantities. However, as \( m_{\mu} \) is the U(1) connection, such combinations would fail the U(1) invariance. Therefore, we add a scalar field \( \chi \) to the Newton–Cartan structure that transforms as shift under U(1) symmetry transformation

\[ \delta_{U(1)} \chi = \sigma \]
(2.25)

where \( \sigma \) is the transformation parameter for the U(1) symmetry. We can now define a U(1) invariant vector field

\[ M_{\mu} = m_{\mu} - \partial_{\mu} \chi, \]
(2.26)

that still transforms as (2.15) under the Milne boosts. Using this vector, we define a new, Milne invariant set of inverse metric fields [17, 27]
\[ \hat{\tau}^\mu = \tau^\mu + h^\mu_{\nu} M_\nu, \quad \hat{h}_{\mu\nu} = h_{\mu\nu} - \tau_{\mu} M_\nu - \tau_{\nu} M_\mu + 2 \tau_{\mu} \tau_{\nu} \Phi, \]

where \( \Phi \), the so-called Newton potential, is defined as

\[ \Phi = \tau^\alpha M_\sigma + \frac{1}{2} h^{\rho\sigma} M_\rho M_\sigma. \]

This new set of fields \((\hat{\tau}^\mu, \hat{h}_{\mu\nu}, M_\mu)\) satisfy the inversion relations \((2.5)\), and we give the connection that solves the metric compatibility condition \((2.2)\) as

\[ \hat{\Gamma}^\rho_{\mu\nu} = \hat{\tau}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} \left( \partial_\mu \hat{h}_{\sigma\nu} + \partial_\nu \hat{h}_{\sigma\mu} - \partial_\sigma \hat{h}_{\mu\nu} \right) + h^{\rho\sigma} \tau_\mu \tau_\nu \partial_\sigma \Phi - K_{\mu\nu}^\rho, \]

where the spatial contorsion tensor is now invariant under Milne boosts and \( U(1) \) transformations. Here, we also introduce a hatted-connection, \( \hat{\Gamma} \), in order to emphasize that this connection is constructed by use of hatted inverse metrics. Note that the penultimate term in the connection \((2.29)\) corresponds to a special choice of the arbitrary function \( F_{\mu\nu} \), which we made by demanding that when both the spatial and the temporal torsion vanishes we recover the standard Milne invariant Newton–Cartan connection \((2.6)\), i.e.

\[ \partial_\mu \tau_\nu = 0 \quad \text{and} \quad K_{\mu\nu}^\rho = 0 \quad \Rightarrow \quad \hat{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}. \]

\section*{C2. Non-invariant spatial contorsion tensor}

The second choice is to work with a non-invariant contorsion tensor, then its Milne transformation is given by \((2.24)\). In this case, we work with the original set of inverse metrics \((2.5)\) and the connection is given by \((2.21)\). Note that the connection includes \( m_\mu \) via its field strength, therefore the use of \( M_\mu \) leaves the connection unchanged.

Although we investigated the twistless torsional case in two separate cases, they are not independent from each other since the new set of inverse elements \((\hat{\tau}^\mu, \hat{h}_{\mu\nu})\) and the original set \((\tau^\mu, h_{\mu\nu})\) can be transformed to each other by means of a linear transformation \((2.27)\), see figure 1. To see that, we can consider a connection \( \Gamma^\rho_{\mu\nu} \) with a non-invariant contorsion tensor \( K^\rho_{\mu\nu} \) given by \((2.21)\) and replace \((\tau^\mu, h_{\mu\nu})\) with \((\hat{\tau}^\mu, \hat{h}_{\mu\nu})\) via

\[ \tau^\mu = \hat{\tau}^\mu - h^\mu_{\nu} M_\nu, \quad h_{\mu\nu} = \hat{h}_{\mu\nu} + \tau_{\mu} M_\nu + \tau_{\nu} M_\mu - 2 \tau_{\mu} \tau_{\nu} \Phi. \]

Upon this replacement, we obtain a connection with a Milne-invariant contorsion tensor

\[ \Gamma^\rho_{\mu\nu} = \hat{\Gamma}^\rho_{\mu\nu} - K^\rho_{\mu\nu}, \]

where the Milne-invariant contorsion, \( K^\rho_{\mu\nu} \), is related to the non-invariant contorsion tensor \( K^\rho_{\mu\nu} \) via

\[ K^\rho_{\mu\nu} = K^\rho_{\mu\nu} - h^{\rho\sigma} (M_\sigma \partial_\mu \tau_\nu + M_\nu \partial_\mu \tau_\sigma + M_\sigma \partial_\nu \tau_\sigma - 2 \Phi \tau_{\mu} \tau_{\sigma} \tau_\nu), \]

With this result in hand, it is straightforward to generalize our discussion to an arbitrary torsion. If we introduce an arbitrary torsion to the Newton–Cartan geometry, the temporal component of the torsion is not subject to any constraint, and is again fixed by the temporal metric compatibility \((2.2)\). Furthermore, the most general connection is still given by \((2.21)\) and the Milne invariance of the connection is again achieved by following the previous discussion for the TTNC.
3. Non-relativistic scale symmetry and Newton–Cartan geometry

In this section, we introduce the non-relativistic analogue of the scale symmetry to Newton–Cartan geometry. The defining property of the scale symmetry is via breaking of the compatibility condition (2.2) by a particular non-metricity tensor

\[ \nabla_\mu \tau_\nu = z b_\mu \tau_\nu, \quad \nabla_\mu h^{\nu\rho} = -2 b_\mu h^{\nu\rho}, \]

which is preserved by the following transformations

\[ \tau_\mu \rightarrow e^{z \Lambda_{\mu}(x)} \tau_\mu, \quad h^{\mu\nu} \rightarrow e^{-2 \Lambda_{\mu}(x)} h^{\mu\nu}, \quad b_\mu \rightarrow b_\mu + \partial_\mu \Lambda_D(x), \quad \Gamma_\mu^{\nu\rho} \rightarrow \Gamma_\mu^{\nu\rho}, \]

where vector field \( b_\mu \) is the gauge field for the scale transformations and is Milne invariant

\[ \delta_M b_\mu = 0. \]

Here, we purposefully represent the gauge field of the scale transformations with the same field that we used to define the twistless torsional condition (2.20) to keep the number of fields minimum. Furthermore, it is important to note that in general the spatial metric and the temporal vielbein have different scaling dimensions which only coincide for \( z = 1 \). This is reminiscent of the Schrödinger symmetries in \( d \) spatial dimensions with \( z \) critical exponent that transform the time (\( t \)) and space (\( x \)) coordinates under dilatation with a rigid dilatation parameter \( \lambda \) as follows

\[ x \rightarrow \lambda x, \quad t \rightarrow \lambda^z t. \]

In order to solve the connection in terms of the Newton–Cartan variables, we first consider the scale covariant temporal compatibility condition (3.1) which fixes the temporal part of the connection

\[ \tau_\mu \Gamma^{\nu}_{\mu\rho} = \partial_\mu \tau_\nu - z b_\mu \tau_\nu. \]

At this point, some clarifications are in order

- Unlike the relativistic scenarios, the inclusion of the non-metricity modifies the antisymmetric part of the connection

\[ \tau_\mu \Gamma^\rho_{[\mu\nu]} = \partial_{[\mu} \tau_{\nu]} - z b_{[\mu} \tau_{\nu]}. \]

Figure 1. The schematic relation between the c1 and c2 cases. The defining data of TTNC geometry is given by \((h^{\mu\nu}, \tau_\mu, b_\mu, M_\mu)\) and the inverse vielbein and the spatial metric are chosen depending on the Milne transformation of the contorsion tensor. Different choices are related to each other by means of linear maps (2.27) and (2.31).
Thus, when the twistless condition is imposed
\[ \partial_{[\mu} \tau_{\nu]} = z b_{[\mu} \tau_{\nu]}, \]  
(3.7)
the anti-symmetric part of the temporal part of the connection, thereby the temporal torsion, vanishes.

- In principle, one might think that a twistless torsion can be introduced if the twistless condition is imposed by means of another vector field \( A_{\mu} \) such that
\[ \partial_{[\mu} \tau_{\nu]} = A_{[\mu} \tau_{\nu]}, \]  
(3.8)
However, in this case, the scaling transformation of \( \tau_{\mu} \) forces us to set \( A_{\mu} = z b_{\mu} \).

- When the torsion is arbitrary we only impose
\[ \partial_{[\mu} \tau_{\nu]} \neq z b_{[\mu} \tau_{\nu]}, \]  
(3.9)
and the temporal part of the connection reads (3.5).

In the following, we construct the connection in terms of the Milne invariant set of inverse fields \((\hat{\tau}_{\mu}, \hat{h}_{\mu\nu}, M_{\mu})\). The scaling properties of the Milne invariant set are given by
\[ \hat{\tau}_{\mu} \rightarrow e^{-(2-z)\Lambda_0} \hat{\tau}_{\mu}, \quad \hat{h}_{\mu\nu} \rightarrow e^{2\Lambda_0} \hat{h}_{\mu\nu}, \quad M_{\mu} \rightarrow e^{-(z-2)\Lambda_0} M_{\mu}. \]  
(3.10)
Solving the scale covariant compatibility conditions, the connection reads
\[ \hat{\Gamma}_{\rho}^{\mu\nu} = \hat{\tau}_{\rho} D_{[\mu} \tau_{\nu]} + \frac{1}{2} \hat{h}^{\sigma\rho} \left( D_{[\nu} \hat{h}_{\sigma\mu]} + D_{\sigma} \hat{h}_{[\nu\mu]} - D_{\sigma} \hat{h}_{\nu\mu] \right) + h^{\rho\sigma} \tau_{\mu} \tau_{\nu} D_{\sigma} \Phi - K_{\mu\nu}^{\rho}, \]  
(3.11)
where \( K_{\mu\nu}^{\rho} \) is a scale and Milne invariant spatial contorsion tensor
\[ \delta_D K_{\mu\nu}^{\rho} = \delta_M K_{\mu\nu}^{\rho} = 0, \]  
(3.12)
and the scale-covariant derivatives are defined as
\[ D_{[\mu} \tau_{\nu]} = \partial_{[\mu} \tau_{\nu]} - z b_{[\mu} \tau_{\nu]}, \quad D_{\mu} \hat{h}_{\nu\rho} = \partial_{\mu} \hat{h}_{\nu\rho} - 2 b_{\mu} \hat{h}_{\nu\rho}, \quad D_{\mu} \Phi = \partial_{\mu} \Phi + (2z-2) b_{\mu} \Phi. \]  
(3.13)
Note that in the presence of the scale transformations, the definition of \( M_{\mu} \) in terms of \( m_{\mu} \) and \( \chi \) as given in (2.26) needs to be modified with a \( b_{\mu} \) dependent term as [19]
\[ M_{\mu} = m_{\mu} - \partial_{\mu} \chi - (z-2) b_{\mu} \chi, \]  
(3.14)
where the scale transformations of \( m_{\mu} \) and \( \chi \) are given by
\[ m_{\mu} \rightarrow e^{(2-z)\Lambda_0} m_{\mu}, \quad \chi \rightarrow e^{(2-z)\Lambda_0} \chi. \]  
(3.15)
The definition of \( M_{\mu} \) as given in (3.14) also implies that the U(1) transformation of \( m_{\mu} \) must be modified with a \( b_{\mu} \) dependent term as follows
\[ m_{\mu} \rightarrow m_{\mu} + \partial_{\mu} \sigma + (z-2) b_{\mu} \sigma. \]  
(3.16)
Based on the scale-invariant connection (3.11) in hand, it is now straightforward to write a scale-invariant Riemann tensor [28] as well as a \((d+1)\) Milne-invariant tensor \( g_{\mu\nu} \) [20]
\[ g_{\mu\nu} = \hat{h}_{\mu\nu} + \tau_{\mu} \tau_{\nu}, \quad g^{\mu\nu} = h^{\mu\nu} + \hat{\tau}^{\mu} \hat{\tau}^{\nu}, \]  
(3.17)
to be used to define a volume form on \( \mathcal{M} \). With these results in hand, one can also construct non-relativistic scale invariant gravity actions or scalar field equations by utilizing a real scalar field with a Weyl weight \( \omega \) that transforms as...
Finally, it is important to note that although we worked with a particular set of inverse fields, we expect that it is always possible to switch between different sets as described in figure 1. However, due to scale invariance of the connection, the relation between the contorsion tensors (2.33) is modified as

\[ K'_{\mu\nu} = K_{\mu\nu} - h_{\rho\sigma}(M_{\sigma}[\rho \tau_{\nu}] + M_{\nu}[\sigma \tau_{\rho}] - 2\Phi_{\tau_{\rho}D[\sigma \tau_{\nu}] - 2\Phi_{\tau_{\nu}D[\sigma \tau_{\rho}]}]. \]

(3.19)

This indicates that as \( D_{[\mu \tau_{\nu}] = 0 \) for the twistless torsional case, it is not possible to introduce a contorsion with a non-trivial Milne transformation—the Galilean invariance of the connection is maintained by the non-metricity property of the temporal vielbein \( \tau_{\mu} \).

Having extended the Newton–Cartan geometry with a scale symmetry, let us consider the case when \( z = 2 \), the value for which the non-relativistic scale symmetry can be enhanced to the Schrödinger symmetry, i.e. the non-relativistic analogue of the conformal symmetry, by introducing a special conformal transformation. This is done by imposing the \( z = 2 \) scale covariant compatibility conditions (3.1)

\[ \nabla_{\mu} \tau_{\nu} = 2b_{\mu} \tau_{\nu}, \quad \nabla_{\mu} h_{\rho\sigma} = -2h_{\rho\sigma}, \]

(3.20)

which is preserved by the non-relativistic special conformal transformation

\[ b_{\mu} \rightarrow b_{\mu} + \tau_{\mu} \Lambda_{K}(x). \]

(3.21)

Since the compatibility conditions are not modified, the connection is still given by (3.11), which transforms non-trivially under special conformal transformations due to appearance of \( b_{\mu} \) in its definition

\[ \Gamma_{\mu\nu}^{\rho} \rightarrow \Gamma_{\mu\nu}^{\rho} - \tau_{\rho} \delta_{\mu\nu}^{\rho} \Lambda_{K} - \tau_{\nu} \delta_{\mu\nu}^{\rho} \Lambda_{K}. \]

(3.22)

Therefore, we conclude that as the temporal component of \( b_{\mu} \) is the only field that transforms non-trivially under the special conformal transformation, see table 1, a Schrödinger invariant gravity means that a \( z = 2 \) scale invariant gravity that does not contain any \( \tau_{\mu} b_{\mu} \) term.

4. \( z = 2 \) scale and Schrödinger symmetry

In the previous section, we approach to the Newton–Cartan gravity as well as its torsional, scale and Schrödinger extensions from a geometric perspective. In particular, we put a distinction between the realization of local scale and Schrödinger symmetry in non-relativistic gravity, and stated that a Schrödinger invariant gravity is a \( z = 2 \) scale invariant gravity that does not contain any \( \tau_{\mu} b_{\mu} \) term. We now want to make this distinction explicit by constructing scale invariant Newton–Cartan models and show that the scale symmetry is enhanced to the non-relativistic conformal symmetry when particular models are combined to annihilate all \( \tau_{\mu} b_{\mu} \) terms. Such constructions are most simply done by gauging the relevant spacetime symmetries. Thus, we dedicate this section to introduce the Bargmann algebra and its scale and Schrödinger extensions.

4.1. Bargmann algebra

To set the stage, we first review the basics of the Bargmann algebra, the central extension of the Galilean algebra that is generated by the time translations \( H \), space translations \( P_{\mu} \), Galilean boosts \( G_{\mu} \), and the spatial rotations \( J_{\alpha\beta} \). Here, Galilean boosts represent
the infinitesimal realization of the Milne boosts that we discussed in the previous section. Considering Newton–Cartan gravity as a gauge theory is based on the Bargmann algebra with the following generators

\[
\{H, Pa, J_{ab}, Ga, N\},
\]

(4.1)

where \(N\) for central charge transformations. The commutation relations between the generators of the Bargmann algebra is given by [21]

\[
\begin{align*}
[H, G_a] &= P_a, \\
[P_a, G_b] &= \delta_{ab}N, \\
[J_{ab}, P_c] &= 2\delta_{[a}P_{b]c}, \\
[J_{ab}, G_c] &= 2\delta_{[a}G_{b]c}, \\
[J_{ab}, J_{cd}] &= 4\delta_{[a}J_{b]c+d].
\end{align*}
\]

(4.2)

The gauge fields corresponding to these generators are

\[
h^{A}_\mu = \{\tau^{A}_\mu, \ e^{a}_\mu, \ \omega^{ab}_\mu, \ \omega^{a}_\mu, \ m^{a}_\mu\},
\]

(4.3)

where the Latin indices \(a, b, c, \ldots\) refer to the spatial local Galilean frame while the Greek indices \(\mu, \nu, \ldots\) refer to the coordinate frame and labels all spacetime coordinates, \(x \equiv (t, x^i)\).

The transformations are generated by operators according to

\[
\delta = \xi H + \zeta^{a} P_{a} + \frac{1}{2} \lambda^{a b} J_{a b} + \lambda^{a} G_{a} + \sigma N,
\]

(4.4)

where \(\xi, \zeta^{a}, \lambda^{a}, \lambda^{a b}\) and \(\sigma\) are the parameters for the time translations, space translations, Galilean boosts, spatial rotations and central charge transformations in the respective order. Using the structure constants of the Bargmann algebra [21] and the standard rules

\[
\begin{align*}
\delta h^{A}_\mu &= \partial_{\mu} e^{A} + f_{BC}^{A} h^{B}_\mu \ e^{C}, \\
R_{\mu\nu}^{A} &= 2\partial_{[\mu}h^{A}_{\nu]} + f_{BC}^{A} h^{B}_{\mu} h^{C}_{\nu}.
\end{align*}
\]

(4.5)

we give the transformation rules for the gauge fields as [21]
\[ \delta \tau_\mu = \partial_\mu \xi, \]
\[ \delta e_\mu^a = \partial_\mu e^a - \omega_\mu^{ab} \xi_b + \lambda^a_b e_\mu^b + \lambda^a_\mu - \omega_\mu^a e_\mu, \]
\[ \delta \omega_\mu^{ab} = \partial_\mu \lambda^{ab} + 2\lambda^{[a} e_\mu^{b]}, \]
\[ \delta \omega_\mu^a = \partial_\mu \lambda^a - \omega_\mu^{ab} \lambda_b + \lambda^a b \omega_\mu, \]
\[ \delta m_\mu = \partial_\mu \sigma - \xi^a \omega_\mu a + \lambda e_\mu^a. \]

and the corresponding curvatures are given by
\[ R_{\mu \nu} (H) = 2 \partial_{[\mu} \tau_{\nu]}, \]
\[ R_{\mu \nu}^a (P) = 2 \partial_{[\mu} e^a_{\nu]} - 2 \omega_{\mu [a}^b e_{\nu]\b] - 2 \omega_{[\mu}^a \tau_{\nu]}, \]
\[ R_{\mu \nu} (J) = 2 \partial_{[\mu} \omega_{\nu]} - 2 \omega_{[a} e_{\mu]}^b, \]
\[ R_{\mu \nu}^a (G) = 2 \partial_{[\mu} \omega_{\nu]}^a + 2 \omega_{[a}^b \omega_{\nu]}^b, \]
\[ R_{\mu \nu} (N) = 2 \partial_{[\mu} m_{\nu]} - 2 \omega_{[a}^a e_{\nu]}^a. \]

In order to leave the \( \tau_\mu, e_\mu^a \) and \( m_\mu \) as the only independent fields, we impose the following constraints [21]
\[ R_{\mu \nu}^a (P) = 0, \quad R_{\mu \nu} (N) = 0, \]
and the Bianchi identity on \( R_{\mu \nu}^a (P) \) and \( R_{\mu \nu} (N) \) gives rise to the following relations between curvatures
\[ e_{[\mu} R_{\nu]a}^a (J) + \tau_{[\mu} R_{\nu]a}^a (G) = 0, \quad e_{[\mu} R_{\nu]a} (G) = 0. \]

Here, first two equations in (4.8) correspond to the non-relativistic version of the torsionless structure equations (2.2). Furthermore, as the first equation reads \( \partial_{[\mu} \tau_{\nu]} = 0 \), we may take
\[ \tau_\mu = \partial_\mu \tau. \]

In order to solve the last two equations in (4.8) to obtain composite expressions for \( \omega_\mu^{ab} \) and \( \omega_\mu^a \), we introduce two new fields: The inverse temporal vielbein \( e_\mu^a \) and the inverse spatial vielbein \( e_a^\mu \) with the following properties
\[ \tau_\mu e_\mu^a = 0, \quad \tau^\mu e_\mu^a = 0, \quad \tau_\mu e_\mu^a = 1, \quad \delta e_\mu^a = \delta a b, \quad \delta e_\mu^a = \delta a^b - \tau_\mu \tau^a. \]

Note that these properties are invariant under the Galilean boost transformations given that \( \tau^\mu \) and \( e_\mu^a \) transforms under Galilean boost transformations as
\[ \delta \tau^\mu = - \lambda^a e_\mu^a, \quad \delta e_\mu^a = 0. \]

Furthermore, any tensor \( T_\mu \) can be decomposed into its spatial and temporal part using the inverse temporal and spatial vielbein
\[ T_\mu = \tau_\mu T_0 + e_\mu^a T_a, \quad \text{such that} \quad T_0 = \tau^\mu T_\mu, \quad \text{and} \quad T_a = e^a_\mu T_\mu. \]

Using these inverse fields, we give the composite expressions for \( \omega_\mu^{ab} \) and \( \omega_\mu^a \) as
\[ \omega_\mu^{ab} = -2 e^{[a} \partial_{[\mu} e_{\nu]} b] + e^{[a} \xi^{b]} \partial_{[\mu} e_{\nu]} c - \tau_\mu e^{a} \xi^{b} \partial_{[\mu} m_{\nu]}, \]
\[ \omega_\mu^a = \tau^a \partial_{[\mu} e_{\nu]} + e^{a} \tau^\nu \partial_{[\mu} e_{\nu]} + \tau_\mu \tau^a \partial_{[\mu} m_{\nu]}. \]

In order to make contact with Newton–Cartan geometry, we first determine \( \tau_\mu \) as the temporal metric and define the spatial metric \( h^{\mu \nu} \) in terms of the inverse spatial vielbein \( e_\mu^a \) as
\[ h^{\mu\nu} = e^a_\mu e^\nu_b \delta^{ab}. \] (4.15)

Then, the connection can be determined by the spatial and temporal vielbein postulates

\[ \partial_\mu e^a_\nu - \omega^a_{\mu b} e^b_\nu = 0, \quad \partial_\nu e^a_\mu - \Gamma^a_{\mu \nu} = 0. \] (4.16)

These two equations fix the connection as

\[ \Gamma^a_{\mu \nu} = \tau^a_\rho \partial_\mu \tau_\nu + e^a_\rho \left( \partial_\mu e^\nu_b - \omega^b_{\mu a} e^a_\nu - \omega^a_{\mu} \right), \] (4.17)

which is the same symmetric connection that we obtained by imposing the torsionless metric compatibility condition (2.6) upon using the definition of the inverse spatial metric (4.15) and the composite expressions (4.14). The Riemann tensor corresponding to this connection is given in terms of \( R_{\mu \nu \sigma}^{ab}(J) \) and \( R_{\mu \nu \sigma}^{ab}(G) \) as [21]

\[ R^{\rho}_{\mu \nu \sigma}(\Gamma) = -e^a_\rho \left( \tau_{\mu \nu}^{ab} R^{ab}(G) + e^{ab} R_{\rho \sigma}^{ab}(J) \right). \] (4.18)

In order for this connection to be the Newtonian connection, we must satisfy the Trautman (2.10) and Ehlers conditions (2.11), which can be achieved by the following curvature constraint [18]

\[ e^\nu_a R^{ab}_{\mu \nu}(J) = 0. \] (4.19)

This constraint implies that the only non-vanishing component of \( R_{\mu \nu}^{ab}(G) \) is related to the only non-zero component of the Riemann tensor [21]

\[ \tau^\mu e^{\nu(a} R^{b)}_{\mu \nu}(G) = \delta^{(a} R^{b)}_{\nu \sigma} \delta_{00}(\Gamma). \] (4.20)

Therefore we obtain the desired geodesic equation (2.7) and the Poisson equation (2.9) by means of the gauging of the Bargmann algebra.

### 4.2. \( z = 2 \) non-relativistic scale invariance

When an additional local scale invariance is demanded, we can improve the Bargmann group with the scale symmetry generator \( D \)

\[ \{ H, \ P_a, \ J_{ab}, \ G_a, \ N, \ D \} \] (4.21)

which introduces an additional gauge field for scale transformations to (4.3), which we represent by \( b_\mu \). The commutation relations between the generators of the \( z = 2 \) scale extended Bargmann algebra is given by

\[
[D, P_a] = -P_a, \\
[D, H] = -2H, \\
[H, G_a] = P_a, \\
[P_a, G_b] = \delta^{ab}N, \\
[D, G_a] = G_a, \\
[J_{ab}, P_c] = 2\delta_{[a}^{[c} P_{b]}], \\
[J_{ab}, G_c] = 2\delta_{[a}^{[c} G_{b]}, \\
[J_{ab}, J_{cd}] = 4\delta_{[a}^{[c} J_{b]}_{d]}. 
\] (4.22)

As mentioned in the previous section, the addition of scale invariance can be made while keeping the dynamical critical exponent \( z \) arbitrary. Here, we are interested in comparing the scale invariance with the Schrödinger invariance, thus we set \( z = 2 \) for the rest of this section.

Using the structure constants for the scale-extended Bargmann group [19], we give the transformation rules for the gauge fields as
\[ \delta \tau_{\mu} = \partial_{\mu} \xi - 2 \zeta \partial_{\mu} \tau^0 + 2 \Lambda \partial_{\mu} \tau_0, \]
\[ \delta \epsilon_{\mu} = \partial_{\mu} \epsilon_{0}^a - \omega_{\mu}^{ab} \epsilon_{b} - b_{\mu} \epsilon_{0}^a + \lambda_{a}^{ab} \epsilon_{a} + \lambda_{a}^{a} \tau_{\mu} - \omega_{\mu}^{a} \epsilon_{a} + \Lambda D \epsilon_{a}. \]
\[ \delta \omega_{\mu}^{ab} = \partial_{\mu} \omega_{ab} + 2 \lambda_{c}^{[a} \omega_{b]} \epsilon_{c}. \]
\[ \delta \omega_{\mu}^{a} = \partial_{\mu} \omega_{a} - \omega_{\mu}^{ab} \omega_{b} + \lambda_{a}^{b} \omega_{b} + \lambda_{a}^{a} \mu_{b} - \Lambda D \omega_{a}. \]
\[ \delta m_{\mu} = \partial_{\mu} \sigma - \xi^{a} \omega_{\mu a} + \lambda \epsilon_{\mu a}. \]
\[ \delta b_{\mu} = \partial_{\mu} \Lambda D. \] (4.23)

and the corresponding curvatures are given by
\[ R_{\mu\nu}(H) = 2 b_{\mu} \tau_{\nu} - 4 b_{\mu} \partial_{\nu} \tau_{0}, \]
\[ R_{\mu\nu}^{a}(P) = 2 b_{\mu} e_{\nu}^{a} - 2 \partial_{\mu} \epsilon_{b} e_{\nu}^{b} - 2 \omega_{\mu}^{ab} \tau_{\nu} - 2 b_{\mu} e_{\nu}^{a}, \]
\[ R_{\mu\nu}^{ab}(J) = 2 b_{\mu} \omega_{\nu}^{ab} - 2 \partial_{\mu} \epsilon_{b} \omega_{\nu}^{b}, \]
\[ R_{\mu\nu}^{a}(G) = 2 b_{\mu} \omega_{\nu}^{a} + 2 \omega_{\mu}^{ab} \omega_{b}^{a} - 2 \omega_{\mu}^{a} b_{\nu}, \]
\[ R_{\mu\nu}(D) = 2 b_{\mu} b_{\nu}, \]
\[ R_{\mu\nu}(N) = 2 b_{\mu} m_{\nu} - 2 \omega_{\mu}^{a} e_{\nu}^{a}. \] (4.24)

Note that in the presence of the dilatation, the \( G \) and \( D \) transformation rules for the inverse vielbein and the inverse temporal vielbein are given by
\[ \delta \tau^{\mu} = -2 \Lambda D \tau^{\mu} - \lambda^{a} \epsilon^{a}, \quad \delta \epsilon^{\mu} = -\Lambda D \epsilon^{a}. \] (4.25)

In order to make contact to the scale-invariant generalization of the Newton–Cartan geometry, which we established in section 3, we want to solve \( \omega^{ab} \) and \( \omega^{a} \) in terms of the other fields to leave \( \tau_{\mu}^{e}, e^{a} \), \( m_{\mu} \) and \( b_{\mu} \) as the set of fields that characterizes the scale-invariant Newton–Cartan geometry. This can be achieved by the following set of constraints
\[ R_{\mu\nu}(H) = 0, \quad R_{\mu\nu}^{a}(P) = 0, \quad R_{\mu\nu}(N) = 0, \] (4.26)
which results to the following further constraints by Bianchi identities
\[ e_{\mu} b_{\nu}^{a} e_{\nu}^{b} (J) + e_{\mu}^{a} R_{\nu\rho}^{a} (D) + \tau_{\mu}^{a} R_{\nu\rho}^{a} = 0, \quad \tau_{\mu}^{a} R_{\nu\rho}^{a} = 0, \quad e_{\mu}^{a} R_{\nu\rho}^{a} = 0. \] (4.27)

The first constraint implies that the twistless condition is satisfied, thereby the torsion vanishes
\[ R_{\mu\nu}(H) = 0 \quad \Rightarrow \quad \partial_{\mu} \tau_{\nu} = 2 b_{\mu} \partial_{\nu} \tau_{0}. \] (4.28)

Furthermore, the last two constraints in (4.26) gives rise to the solution of \( \omega^{ab} \) and \( \omega^{a} \)
\[ \omega_{\mu}^{ab} = -2 e_{\nu}^{a} e_{\nu}^{b} \partial_{\mu} b_{\nu}, \]
\[ \omega_{\mu}^{a} = \tau^{a} b_{\mu} e_{\nu}^{a} + e_{\nu}^{a} \tau^{e} b_{\mu} e_{\nu}^{c} - \tau^{c} e_{\nu}^{a} b_{\mu} e_{\nu}^{c} + 2 e_{\nu}^{b} e_{\nu}^{c} b_{\mu}^{c}, \]
\[ \omega_{\mu}^{a} = \tau^{a} b_{\mu} e_{\nu}^{a} + e_{\nu}^{a} \tau^{e} b_{\mu} e_{\nu}^{c} - \tau^{c} e_{\nu}^{a} b_{\mu} e_{\nu}^{c} + 2 e_{\nu}^{b} e_{\nu}^{c} b_{\mu}^{c}. \] (4.29)

We can now define the connection for the scale invariant gravity in terms of the elements of the scale-extended Bargmann algebra by means of scale covariant metric compatibility conditions
\[ 0 = \partial_{\mu} \tau_{\nu} - \Gamma_{\mu\nu}^{e} \tau_{\rho} + 2 b_{\mu} \partial_{\nu} \tau_{0}, \]
\[ 0 = \partial_{\mu} e_{\nu}^{a} - \Gamma_{\mu\nu}^{e} e_{\rho}^{a} - \omega_{\mu}^{ab} e_{\rho}^{b} - \omega_{\mu}^{a} \tau_{\nu} - b_{\mu} e_{\nu}^{a}. \] (4.30)

These conditions uniquely determine \( \Gamma \) as
\[ \Gamma_{\mu\nu}^{e} = \tau^{a} \partial_{\mu} b_{\nu} + \frac{1}{2} h_{\rho}^{e} \left( D_{\rho} b_{\tau_{\mu}} + D_{\rho} b_{\tau_{\nu}} - D_{\rho} b_{\tau_{\mu\nu}} \right) - h_{\rho}^{e} \tau_{\rho}(F_{\nu})_{\tau} \sigma. \] (4.31)
where the scale-covariant objects are as defined as
\[ D_\mu \tau_\nu = \partial_\mu \tau_\nu - 2b_\mu \tau_\nu, \quad D_\mu h_\nu^\rho = \partial_\mu h_\nu^\rho - 2b_\mu h_\nu^\rho. \]  
(4.32)

If desired, we can simply go to the “hatted frame” by introducing a compensating scalar \( \chi \), defining the vector \( M_\mu \) via
\[ m_\mu = M_\mu + \partial_\mu \chi \]  
(4.33)
and finally using the map (2.31) in which case we obtain the connection given in (3.11) for \( z = 2 \) and \( K_\mu \rho^\sigma = 0 \). Finally, we give the corresponding scale invariant Riemann tensor in terms of \( R_{\mu \nu}^a \), \( R_{\mu \nu}^a (G) \) and \( R_{\mu \nu}^a (D) \) as
\[ R^a_{\mu \nu \sigma} (\Gamma) = -e^a_{\mu} \left( \tau_\mu R^a_{\nu \sigma} (G) + e_{\mu \nu} R^a_{\sigma \nu} (J) + e_{\mu \nu} R^a_{\sigma \nu} (D) \right) - 2\tau_\mu R^a_{\nu \sigma} (D), \]  
(4.34)

As described in section 3, we can use this scale invariant Riemann tensor to construct scale-invariant non-relativistic gravity actions or field equations by introducing a compensating scalar field \( \phi \) and a rank-\((d + 1)\) Milne-invariant tensor \( g_{\mu \nu} \), see (3.17).

### 4.3. Schrödinger algebra

When the symmetries are extended to the Schrödinger extension of the Bargmann algebra, we include the non-relativistic analogue the special conformal symmetry generator \( K \)
\[ \{ H, \ P_\mu, \ J_{ab}, \ G_\mu, \ N, \ D, \ K \} \]  
(4.35)

which introduces an additional gauge field \( f_\mu \). The commutation relations between the generators of the Schrödinger algebra is given by [31]
\[ [D, P_\mu] = -P_\mu, \quad [D, H] = -2H, \quad [H, G_\mu] = P_\mu, \quad [P_\mu, G_\mu] = \delta_{ab} N, \]
\[ [D, G_\mu] = G_\mu, \quad [D, K] = 2K, \quad [K, P_\mu] = -G_\mu, \quad [H, K] = D, \]
\[ [J_{ab}, P_\mu] = 2\delta_{a[\mu} P_{\nu]}; \quad [J_{ab}, G_\mu] = 2\delta_{[\mu} G_{\nu]}; \quad [J_{ab}, J_{cd}] = 4\delta_{[\mu} J_{\nu]d}. \]  
(4.36)

Using the structure constants for the Schrödinger group [19], we give the transformation rules for the gauge fields as
\[
\begin{align*}
\delta \tau_\mu &= \partial_\mu \xi + 2\Lambda_D \tau_\mu - 2b_\mu \xi, \\
\delta e^{\mu}_{a} &= \partial_\mu \xi^{a} - \omega^{ab}_{b} \xi_{b} - b_\mu \xi^{a} + \lambda^{a}_{b} e^{b}_{\mu} + \lambda^{a}_{b} \tau_\mu - \omega^{a}_{b} \xi + \Lambda_D e^{a}_{\mu}, \\
\delta \omega^{ab}_{\mu} &= \partial_\mu \lambda^{ab} + 2\lambda^{[a \omega_{b}\mu]}, \\
\delta \omega^{a}_{\mu} &= \partial_\mu \lambda^{a} + 2b_\mu \lambda^{a} - \omega^{ab}_{b} \lambda_{b} - \omega^{a}_{b} \Lambda_D - f_{\mu} \xi^{a} + e^{a}_{\mu} \Lambda_k + \lambda^{a}_{b} \omega^{b}_{\mu}, \\
\delta b_\mu &= \partial_\mu \Lambda_D + \tau_\mu \Lambda_K - \xi f_\mu, \\
\delta f_\mu &= \partial_\mu \Lambda_K + 2\Lambda_K b_\mu - 2f_\mu \Lambda_D, \\
\delta m_\mu &= \partial_\mu \sigma + \lambda^{a} e^{a}_{\mu} + \omega^{a}_{b} \xi_{a}.
\end{align*}
\]  
(4.37)
These constraints imply further constraints due to Bianchi identities \[19\]

Using these constraints, we find that the composite expressions for \(\omega_{\mu}^{ab}\) and \(\omega_{\mu}^{\tau\nu}\) are still given by (4.29) while the solutions for \(e_{\mu}^{a}b_{\nu}\) and \(f_{\mu}\) read [19]

\[\tau^\mu f_{\mu} = \frac{1}{d} \tau^\nu e_{\nu} R_{\mu \nu}^{a} (G) + 2 \tau^\mu M^{\nu} R_{\mu \nu}^{a} (J) + M^{\nu} e^{\nu} e_{\mu} R_{\mu a} (J) = 0.\]

\[e_{\mu}^{a} = \frac{1}{d} \tau^\nu e_{\nu} \partial_{\mu} b_{\nu}, \quad e^{a} b_{\mu} = e^{a} \tau^{\nu} \partial_{\mu} b_{\nu}.\]

In order to make contact to the conformal generalization of the Newton–Cartan geometry that we mentioned in the previous section, we want to solve \(\omega_{\mu}^{ab}\), \(\omega_{\mu}^{\tau\nu}\), \(e_{\mu}^{a}\), \(b_{\mu}\) and \(f_{\mu}\) in terms of the other fields to leave \(\tau_{\mu}, e_{\mu}^{a}, m_{\mu}\) and \(\tau^{a} b_{\mu}\) as the set of fields that characterizes the conformal extension of the Newton–Cartan geometry. This can be achieved in two distinct way.

- In the first case, we impose the following set of constraints

\[
\begin{align*}
R_{\mu \nu} (H) &= 0, \quad R_{\mu \nu}^{a} (P) = 0, \quad R_{\mu \nu} (N) = 0, \\
R_{\mu \nu} (D) &= 0, \quad e^{\mu} e^{\tau} R_{\mu \nu}^{a} (G) = 0, \quad R_{\mu \nu}^{ab} (J) = 0.
\end{align*}
\]

These constraints imply further constraints due to Bianchi identities

\[
\begin{align*}
R_{\mu a} (G) &= 0, \quad R_{\mu a} (P) = 0, \quad R_{\mu a} (N) = 0, \\
R_{\mu a} (D) &= 0, \quad R_{\mu a} (K) = 0.
\end{align*}
\]

From these constraints, we find the solutions of \(\omega_{\mu}^{ab}\), \(\omega_{\mu}^{\tau\nu}\), \(e_{\mu}^{a}\), \(b_{\mu}\) and \(f_{\mu}\) as

\[
\begin{align*}
\omega_{\mu}^{ab} &= 0, \quad \tau^\mu f_{\mu} = \frac{1}{d} R_{\mu \nu}^{a} (G), \quad e^{\mu} e^{\nu} = 2 e^{\mu} \tau^{\nu} \partial_{\nu} b_{\mu}, \\
\omega_{\mu}^{\tau\nu} &= \tau^\nu \partial_{\mu} e_{\nu} + e^{\mu} \tau^{\nu} \partial_{\nu} e_{\mu} + e^{\mu} \tau^{\nu} \partial_{\nu} m_{\mu} + e^{\mu} \tau^{\nu} \partial_{\nu} b_{\mu}, \quad e^{a} b_{\mu} = e^{a} \tau^{\nu} \partial_{\mu} b_{\mu}.
\end{align*}
\]

where

\[
R_{\mu \nu}^{a} (G) = 2 \partial_{\mu} \omega_{\nu}^{a} + 2 \omega_{\mu}^{b} \omega_{\nu}^{a} b - 2 \omega_{\mu}^{a} b_{\nu}
\]

- In the second case, we first introduce a scalar field \(\chi\) and define the U(1) invariant vector \(M_{\mu}\) as in (2.26). Next, we impose the following set of constraints [19]

\[
\begin{align*}
R_{\mu \nu} (H) &= 0, \quad R_{\mu \nu}^{a} (P) = 0, \quad R_{\mu \nu} (N) = 0, \\
R_{\mu \nu} (D) &= 0, \quad \tau^\mu e_{\mu} R_{\mu \nu}^{a} (G) + M^{\nu} e^{\nu} e_{\mu} R_{\mu a} (J) + M^{\nu} e^{\nu} e_{\mu} R_{\mu a} (J) = 0.
\end{align*}
\]

These constraints imply further constraints due to Bianchi identities [19]

\[
\begin{align*}
R_{\mu a} (G) &= 0, \quad R_{\mu a} (P) = 0, \quad R_{\mu a} (N) = 0, \\
2 R_{\mu a} (J) &= 0, \quad R_{\mu a} (K) = 0.
\end{align*}
\]

Using these constraints, we find that the composite expressions for \(\omega_{\mu}^{ab}\) and \(\omega_{\mu}^{\tau\nu}\) are still given by (4.29) while the solutions for \(e_{\mu}^{a}b_{\nu}\) and \(f_{\mu}\) read [19]

\[
\begin{align*}
\tau^\mu f_{\mu} &= \frac{1}{d} \tau^\nu e_{\nu} R_{\mu \nu}^{a} (G) + 2 \tau^\mu M^{\nu} R_{\mu \nu}^{a} (J) + M^{\nu} M^{\nu} R_{\mu a} (J), \\
e^{a} b_{\mu} &= e^{a} \tau^{\nu} \partial_{\mu} b_{\nu}, \quad e^{a} b_{\mu} = e^{a} \tau^{\nu} \partial_{\mu} b_{\nu}.
\end{align*}
\]
As shown in [18], the second case is the most convenient to construct Schrödinger invariant gravity models as the first case sets \( \omega_{\mu}^{ab} = 0 \) by imposing \( R_{\mu \nu}^{ab} (J) = 0 \). Thus, we will utilize the second set of constraints. With \( e_\mu^a, \tau_\mu, \tau^\mu b_\mu \) and \( M_\mu \) being the independent fields of the Schrödinger invariant non-relativistic gravity, we observe that \( \tau^\mu b_\mu \) is the only independent field that transforms non-trivially under \( K^- \)transformations as in the geometric construction (3.21). Furthermore, the scale-covariant compatibility conditions (4.30) remains unchanged in the presence of the special conformal symmetry, thereby the connection is again given by (4.31) which has a non-trivial \( K^- \)transformation due to the appearance of \( \tau^\mu b_\mu \) terms in its definition as shown in (3.22).

4.4. Scale invariant non-relativistic gravity

The purpose of this section is to classify the scale invariant scalar field theories that are relevant to the Hořava–Lifshitz gravity after gauge fixing of the scale symmetry. In order to do so, we introduce two scalar fields \( \phi \) and \( \chi \), which has the following transformations under scale and \( U(1) \) transformations

\[
\delta \phi = \omega \Lambda D \phi, \quad \delta \chi = \sigma, \quad (4.46)
\]

where \( \omega \) refers to the scaling dimension of the scalar field \( \phi \). Note that we purposefully used \( \phi \) and \( \chi \) to refer to the scalars that we used in the geometric construction of the scale and Schrödinger invariant models, see (3.18) and (2.25). Furthermore, we will also use a complex scalar field \( \Psi \) that is defined in terms of \( \phi \) and \( \chi \) as

\[
\Psi = \phi e^{i M \chi}, \quad (4.47)
\]

which transforms homogeneously under dilatations and \( U(1) \) transformations

\[
\delta \Psi = \omega \Lambda D \Psi + i M \sigma \Psi. \quad (4.48)
\]

The models that we will introduce here fall in three classes. First two class are separated depending on the number of time derivatives acting on the scalar fields. The third class is necessary to introduce scale-invariant models that do not arise as a scalar field theory as they include group theoretical curvatures given in (4.24).

- **Potential terms**: Lagrangians that are zeroth, second and fourth order in spatial derivatives \( (n_s = 0, 2, 4) \), and no time derivative \( (n_t = 0) \).
- **Kinetetic terms**: Lagrangians that are first and second order in time derivative \( (n_t = 1, 2) \).
- **Curvature terms**: Lagrangians that are constructed by using the group theoretical curvatures which we defined in (4.24).

Before we proceed to the construction of the scale invariant models, we would like to point out that any non-relativistic Lagrangian can be made scale invariant by using the above mentioned compensating fields. Here, the challenge is to find the class of ‘any non-relativistic Lagrangians’ which is not as trivial as the relativistic case due to Galilean transformations.

4.4.1. Potential terms. In this section, we construct the Lagrangians that are zeroth, second and fourth order in spatial derivatives, while keeping the time derivative at zeroth order.

- \( n_s = 0 \): The only possibility for a scalar field theory that has no space or time derivatives is given by
\[ S^{(0)} = \int dt \, d^4x \, e \, \Lambda_0 \, \phi^2 \]  
\hspace{1cm} (4.49)

where \( \Lambda_0 \) is an arbitrary constant and \( e = \det(\tau_{\mu}, e_\mu^a) \) has the following scaling transformation
\[ \delta_D e = (d + 2) \Lambda_0 \, e. \]  
\hspace{1cm} (4.50)

Here, we also fixed the scaling dimension of the scalar field \( \phi \) as \( \omega = -\frac{d+2}{2} \) for the scale invariance of the action.

- \( n_s = 2 \): For this case, the only scale-invariant model is

\[ S^{(1)} = \int dt \, d^4x \, e \, D_\mu \phi \, D^\mu \phi, \]  
\hspace{1.5cm} (4.51)

where \( D\phi \) is the spatial part of the gauge-covariant derivative \( D_\mu \phi \)
\[ D_\mu \phi = e_\mu^a D^a \phi \quad \text{where} \quad D_\mu \phi = \partial_\mu \phi - \omega b_\mu \phi. \]  
\hspace{1cm} (4.52)

In this case the scaling dimension of the scalar field is fixed to \( \omega = -\frac{d}{2} \). In principle, one can also have an action that includes \( \phi D^a D_\mu \phi \) term. However, as noted in [18], it is related to \( S^{(1)} \) up to a boundary term.

- For \( n_s = 4 \), we have three distinct actions that contributes as potential terms

\[ S^{(2)} = \int dt \, d^4x \, e \, (D_\mu \phi \, D^\mu \phi)^2, \]  
\[ S^{(3)} = \int dt \, d^4x \, e \, (D_\mu \phi \, D^\mu \phi) \Delta \phi, \]  
\[ S^{(4)} = \int dt \, d^4x \, e \, (\Delta \phi)^2, \]  
\hspace{1.5cm} (4.53)

where
\[ \Delta \phi = D^a D_\mu \phi = e^{ transmitted } \left[ (\partial_\mu - (\omega - 1)b_\mu) D_\mu \phi - \omega b_\mu D_\mu \phi \right]. \]  
\hspace{1cm} (4.54)

Note that in this case, the scaling dimension of the scalar field \( \phi \) as \( \omega = -\frac{d+2}{2} \). In principle, one can also have two other \( n_s = 4 \) actions given by \( \phi \Delta^2 \phi \) and \( D_\mu \phi \, D^\mu \phi \, D^b \phi \). However, as noted in [18], these terms can be written in terms of a combination of \( n_s = 4 \) and curvature terms up to boundary terms. Thus, we will not include them in our list of \( n_s = 4 \) actions.

In principle, we can also produce potential terms with the scalar field \( \chi \). However, as shown in [18], such potential terms arises in the kinetic terms of a complex scalar field, which we will consider in the next section. Thus, these actions complete our list of potential terms.

4.4.2. Kinetic terms. In this section, we construct the actions that are first and second order in time derivative by utilizing the complex scalar field \( \Psi \) that we introduced in (4.47).

- At first order in time derivative, \( n_t = 1 \), we have

\[ S^{(5)} = \int dt \, d^4x \, e \, \psi^* \Box \psi. \]  
\hspace{1.5cm} (4.55)
Here, we defined the scale-covariant d’Alambertian operator that is given by
\[
\Box \Psi = \left( iD_0 - \frac{1}{2M} \triangle \right) \Psi
\]  
\tag{4.56}
where
\[
D_\mu \Psi = \partial_\mu \Psi - \omega b_\mu \Psi - iMm_\mu \Psi.
\]
\[
\triangle \Psi = e^{\mu a} \left( (\partial_\mu - (\omega - 1)b_\mu - iMm_\mu)D_\mu \Psi - \omega_{\mu a}D_a \Psi + iM\omega_{\mu a} \Psi \right).
\]  
\tag{4.57}
Note that the scale invariance fixes the scaling dimension of $\Psi$ as $\omega = -\frac{d}{2}$ while U(1) charge of $\Psi$ remains arbitrary.

- $n_t = 2$: For this case, we have the following set of invariant modes
\[
S^{(6)} = \int dt d^4x e^{\square \Psi},
\]
\[
S^{(7)} = \int dt d^4x e |\Box \Psi|^2,
\]
\[
S^{(8)} = \int dt d^4x e \left| \Box \frac{\Box \Psi D_a \Psi}{\Psi} \right|^2,
\]
\[
S^{(9)} = \int dt d^4x e (\Psi^* \Psi - 1) \left( (i\Psi^* D_0 \Psi - i\Psi D_0 \Psi^*) \frac{\Box D_a \Psi D^a \Psi}{\Box D_a \Psi D^a \Psi} \right)^2,
\]
\[
S^{(10)} = \int dt d^4x e \phi^{-2} D_a \phi^a \phi \left( (\Psi^* D_0 \Psi - i\Psi D_0 \Psi^*) \frac{\Box D_a \Psi D^a \Psi}{\Box D_a \Psi D^a \Psi} \right),
\]
\[
S^{(11)} = \int dt d^4x e \phi^{-1} \Delta \phi \left( (\Psi^* D_0 \Psi - i\Psi D_0 \Psi^*) \frac{\Box D_a \Psi D^a \Psi}{\Box D_a \Psi D^a \Psi} \right) \tag{4.58}
\]
where we have defined
\[
D_\mu \Psi = \tau^\mu \left( (\partial_\mu - (\omega - 2)b_\mu - iMm_\mu)D_\mu \Psi + \omega_{\mu a}D_a \Psi \right),
\]
\[
\triangle D_\mu \Psi = \tau^\mu \left( (\partial_\mu - (\omega - 2)b_\mu - iMm_\mu)\triangle \Psi + 2iM\omega_{\mu a}D_a \Psi \right),
\]
\[
\triangle D_\mu \Psi = e^{\mu a} \left( (\partial_\mu - (\omega - 3)b_\mu - iMm_\mu)D_a D_\mu \Psi - \omega_{\mu a}b_\mu D_a \Psi \right)
\]
\[
+ \omega_{\mu a}b_\mu D_a D_\mu \Psi + iM\omega_{\mu a}D_a \Psi \right),
\]
\[
\triangle^2 \Psi = e^{\mu a} \left( (\partial_\mu - (\omega - 3)b_\mu - iMm_\mu)D_a \triangle \Psi - \omega_{\mu a}b_\mu D_a \triangle \Psi \right)
\]
\[
+ 2iM\omega_{\mu a}b_\mu D_a \triangle \Psi + iM\omega_{\mu a} \triangle \Psi \right),
\]
\[
\Box^2 \Psi = \left( -D_0^2 + \frac{1}{4M^2} \triangle^2 - \frac{i}{2M} D_0 \triangle - \frac{i}{2M} \triangle D_0 \right) \Psi \tag{4.59}
\]
Note that the action $S^{(8)}$ is actually a potential term for the complex scalar field $\Psi$, however we include it in the kinetic actions for future purposes. The order of derivatives matter in the Lagrangians due to the following non-vanishing commutation relations
\[
[D_0, \triangle] \Psi = -\omega D^a R_{a b}(D) \Psi - (2\omega - 1)R_{a b}(D)D^a \Psi + R_{a b}(J)D^a \Psi + iM R_{a b}(G) \Psi,
\]
\[
[D_0, D_a] \Psi = -\omega R_{a b}(D) \Psi \tag{4.60}
\]
Here, the scale invariance of the action fixes the scaling dimension of both the real scalar \( \phi \) and the complex scalar field \( \Psi \) as \( \omega = -\frac{d-2}{2} \).

### 4.4.3. Curvature terms.

In this section, we consider the curvature invariants, i.e. Lagrangians that are constructed by using the group theoretical curvatures. Such actions cannot be obtained by gauging a globally scale-invariant scalar field theory, thus deserved a special attention \([18]\). In particular, the following two actions play an important rôle in comparing scale and Schrödinger invariant models

\[
S^{(12)} = \int dt dx e \left( \tau^{\mu} e^{\nu} a R_{\mu\nu}^{\ a} (G) + M^b [2 \tau^{\mu} e^{\nu} a R_{\mu\nu}^{\ ab} (J) \right. \\
\left. - d \tau^{\mu} e^{b} R_{\mu\nu}^{\ b} (D) + M^e e^{\mu} e^{\nu} a R_{\mu\nu}^{\ ab} (J)] \right) (\Psi \Psi^*) ,
\]

\[
S^{(13)} = \int dt dx e \left( - \frac{i}{2M} \left[ \tau^{\mu} e^{\nu} a R_{\mu\nu}^{\ b} (D) \cal{D}^{\rho} \Psi + \cal{D}^{\rho} (\tau^{\mu} e^{\nu} a R_{\mu\nu}^{\ b} (D) \Psi) \right] \right. \\
\left. + M^a e^{\nu} a \tau^{\mu} R_{\mu\nu}^{\ a} (D) \Psi \right) \Psi^* \tag{4.61}
\]

where the scaling dimension of the complex scalar \( \Psi \) is given by \( \omega = -\frac{d-2}{2} \). We can construct further independent invariants that cannot be obtained from a scalar field theory by using the spatial rotation curvature \( R_{abcd} (J) \) \([18]\)

\[
S^{(14)} = \int dt dx e R(J) \phi^2 \\
S^{(15)} = \int dt dx e R(J)^2 \phi^2 \\
S^{(16)} = \int dt dx e \cal{D}_a \phi \cal{D}^a \phi R(J) \\
S^{(17)} = \int dt dx e \phi \Box \phi R(J) \\
S^{(18)} = \int dt dx e \cal{D}_a \phi \cal{D}_b \phi R^{ab} (J) \\
S^{(19)} = \int dt dx e \phi^2 R_{ab} (J) R^{ab} (J) \\
S^{(20)} = \int dt dx e \phi^2 R_{abcd} (J) R^{abcd} (J), \\
S^{(21)} = \int dt dx e \phi^2 \left( i \Psi^* \cal{D}_0 \Psi - i \Psi \cal{D}_0 \Psi^* + \frac{1}{M} \cal{D}_a \Psi^* \cal{D}^a \Psi \right) R(J) \tag{4.62}
\]

where

\[
R(J) \equiv R^{ab} (J), \quad \text{and} \quad R_{ab} (J) \equiv R^{abc} (J). \tag{4.63}
\]

Finally, we give the scale invariant action for \( R_{\mu\nu}^{\ \ ab} (D) \), which is, as we will discuss below, a crucial action to distinguish scale invariance from Schrödinger invariance

\[
S_D = \int dt dx e \phi^2 \tau^{\mu} \tau^{\rho} e^{\nu} a e^{\sigma} a R_{\mu\nu}^{\ ab} (D) R_{\rho\sigma} (D) \tag{4.64}
\]

where the scaling dimension of the real scalar \( \phi \) is given by \( \omega = \frac{4-d}{2} \).
4.5. Schrödinger gravity

We now have all the desired actions to proceed to the Schrödinger gravity models and compare them with the scale-invariant non-relativistic gravity theories. Since the difference between the scale and the Schrödinger symmetry is the existence of $\tau^\mu b_\mu$ terms, it is worth mentioning such terms in the scale-invariant models that we constructed above.

- **Potential terms:** None of the potential terms constructed here includes a $\tau^\mu b_\mu$ terms. Thus, they exhibit a Schrödinger invariance.

- **Kinetic terms:** $S^{(6)}, S^{(7)}$ and $S^{(8)}$ fail the Schrödinger invariance as they include explicit $\tau^\mu b_\mu$ terms. Other kinetic actions exhibit a Schrödinger invariance.

- **Curvature terms:** Only $S^{(12)}$ and $S^{(13)}$ fail the Schrödinger invariance as they include explicit $\tau^\mu b_\mu$ terms. Other curvature actions exhibit a Schrödinger invariance.

As the scale invariant potential terms are already Schrödinger invariant, we start our investigation with the kinetic terms. For an arbitrary scaling dimension of the complex scalar $\Psi$, the $b_0 \equiv \tau^\mu b_\mu$ terms in the terms relevant to $S^{(6)}, S^{(7)}$ and $S^{(8)}$ are given by

$$
\Psi^* \Box^2 \Psi |_{\tau^\mu b_\mu} = -i(2\omega - 2 + d) b_0 (\Psi^* \Box \Psi) - \frac{i}{2M} (\omega + \frac{d}{2}) \Psi^* \partial_j (\tau^\mu e^\nu a R_{\mu\nu}(D)) \\
- \frac{i}{2M} (\omega + \frac{d}{2}) \Psi^* \partial_j (\tau^\mu e^\nu a R_{\mu\nu}(D) \Psi) + (\omega + \frac{d}{2}) (\Psi^* \Psi) \partial_0 b_0 \\
+ (\omega + \frac{d}{2})^2 (\Psi^* \Psi) b_0^2.
$$

(4.65)

$$
\Box \Psi |_{\tau^\mu b_\mu} = -i(\omega + \frac{d}{2}) b_0 \Psi,
$$

(4.66)

$$
\triangle \Psi |_{\tau^\mu b_\mu} = iMdB_0 \Psi.
$$

(4.67)

From these expressions, we first observe that given the scaling dimension of $\Psi$ is $\omega = -\frac{d-2}{2}$, the following combination does not have an explicit $b_0$ term and is invariant under the full scale-extended Bargmann group, thus exhibits a Schrödinger invariance

$$
S = \int d^4 x e \left( \Box \Psi + \frac{1}{Md} \left( \triangle \Psi - \frac{D^a \Psi D_a \Psi}{\Psi} \right) \right)^2.
$$

(4.68)

This action is precisely the one of the Schrödinger invariant actions found in [18]. Thus, in order to distinguish between the scale and Schrödinger invariance, we can introduce a parameter $\alpha$ that measures the deviation from Schrödinger symmetry while preserving the scale invariance

$$
S^{(22)} = \int d^4 x e \left( \Box \Psi + \frac{\alpha}{Md} \left( \triangle \Psi - \frac{D^a \Psi D_a \Psi}{\Psi} \right) \right)^2.
$$

(4.69)

Here, when $\alpha = 1$ we have a full Schrödinger invariance, otherwise the model exhibits only scale invariance.

For the $b_0$ terms in $\Psi^* \Box^2 \Psi$, it is not possible to find a vanishing combination by use of only kinetic terms and choosing a weight, thus we turn our attention to the curvature actions $S^{(12)}$ and $S^{(13)}$. Given the scaling dimension of the complex scalar field $\Psi$ is $\omega = -\frac{d-2}{2}$ we first note that the first expression in (4.65) drops out, and the second and the third expressions can be compensated by $S^{(13)}$. Furthermore, the $b_0$ terms in $S^{(12)}$ is given by
\[ S^{(12)}_{\tau_n b_0} = \int dt \, d^4x \, e \left( \Psi^* \Psi \right) \left( d(\partial_0 b_0 + b_0^2) - dM^\mu \tau^\mu e'_{\rho} R_{\mu\nu}(D) \right) \]  

(4.70)

which has the correct structure to cancel out the last two \( b_0 \) structure in (4.65). Thus, we find that the following scale-invariant combination also exhibits a Schrödinger invariance

\[ S = S^{(6)} - \frac{1}{\alpha} \left( S^{(12)}_{\tau_n b_0} + dS^{(13)}_{\tau_n b_0} \right). \]  

(4.71)

Once again, we make the distinction between the scale and Schrödinger invariance explicit by introducing a free parameter \( \alpha \) such that

\[ S^{(23)} = S^{(6)} - \frac{\alpha}{d} \left( S^{(12)}_{\tau_n b_0} + dS^{(13)}_{\tau_n b_0} \right). \]  

(4.72)

If \( \alpha \neq 1 \), the action (4.72) preserves the scale symmetry but no longer invariant under the special conformal transformations. When \( \alpha = 1 \), the model reduces to the following Schrödinger invariant action [18]

\[ S|_{\alpha=1} = \int dt \, d^4x \, e \Psi^* \Box_{\text{Sch}}^2 \Psi, \]  

(4.73)

where the \( \Box_{\text{Sch}}^2 \) is the square of the Schrödinger invariant d’Alambertian operator

\[ \Box_{\text{Sch}}^2 \Psi = \left( -D_0^2 + \frac{1}{4M^2} \Lambda^2_{\text{Sch}} - \frac{i}{2M} D_0 \Lambda_{\text{Sch}} - \frac{i}{2M} \Lambda_{\text{Sch}} D_0 \right) \Psi, \]  

(4.74)

where the Schrödinger invariant derivative operators read [18]

\[ D_\mu \Psi = (\partial_\mu - \omega b_\mu - iMm_\mu) \Psi, \]
\[ D_\mu D_\nu \Psi = \left( (\partial_\mu - (\omega - 2)b_\mu - iMm_\mu) D_\nu + \omega_\mu^{\ a} D_\mu + \omega_\mu^{\ b} \right) \Psi, \]
\[ D_0^2 \Psi = \tau^\mu \left( (\partial_\mu - (\omega - 2)b_\mu - iMm_\mu) D_\nu + \omega_\mu^{\ a} D_\mu + \omega_\mu^{\ b} \right) \Psi, \]
\[ D_\mu D_\nu \Psi = e_\mu^a \left( (\partial_\mu - (\omega - 1)b_\mu - iMm_\mu) D_\nu - \omega_\mu^{\ b} D_\nu + iM\omega_\mu^{\ b} \right) \Psi, \]
\[ \Delta_{\text{Sch}} \Psi = e_\mu^a \left( (\partial_\mu - (\omega - 1)b_\mu - iMm_\mu) D^a - \omega_\mu^{\ c} D_\nu + iM\omega_\mu^{\ c} \right) \Psi, \]
\[ D_\mu \Delta_{\text{Sch}} \Psi = \left( (\partial_\mu - (\omega - 2)b_\mu - iMm_\mu) \Delta_{\text{Sch}} + 2iM\omega_\mu^{\ a} D_\mu - iMf_\mu \right) \Psi, \]
\[ \Delta_{\text{Sch}} D_\mu \Psi = e_\mu^a \left( (\partial_\mu - (\omega - 3)b_\mu - iMm_\mu) D_\nu D_\mu + \omega_\mu^{\ b} D_\nu D_\mu + \omega_\mu^{\ b} D_\nu D_\mu + \right. \]
\[ \left. + iM\omega_\mu^{\ b} D_\nu D_\mu + (\omega - 1)f_\mu D_\mu \right) \Psi, \]
\[ \Delta_{\text{Sch}}^2 \Psi = e^{\mu a} \left( (\partial_\mu - (\omega - 3)b_\mu - iMm_\mu) \Delta_{\text{Sch}} + \omega_\mu^{\ b} D_\mu \right) \Delta_{\text{Sch}} \]
\[ + 2iM\omega_\mu^{\ b} D_\mu D_\nu + iM\omega_\mu^{\ a b} \Delta_{\text{Sch}} - iM(d + 2) f_\mu D_\mu \right) \Psi. \]  

(4.75)

Note that we utilized the gauge fields of the Schrödinger algebra (4.37) to define the Schrödinger covariant derivatives (4.75). Here, the major difference between the scale and Schrödinger covariant objects is the existence of the composite \( f_\mu \) field. In the case of Schrödinger invariance, the composite \( f_\mu \) comes with a fixed coefficient to cancel out the \( b_0 \) terms to preserve the special conformal symmetry. In the case of scale invariance, the combination of \( S^{(12)} \) and \( S^{(13)} \) as given in (4.72) plays the role of \( f_\mu \). Hence, when \( \alpha = 1 \) that combination completes the scale covariant d’Alambertian-squared action \( S^{(6)} \) to the Schrödinger covariant
d’Alambertian-squared action (4.73), otherwise the model only exhibits scale invariance but not special conformal invariance.

We finish this section with a comment on the necessity of the action (4.64). If this action is not present, then $b_0$ becomes an auxiliary field and can simply be eliminated by its field equation. Thus, due to this elimination, any $z = 2$ scale invariant model becomes Schrödinger invariant. Hence, when a model that aims to distinguish local scale invariant models from Schrödinger invariant models is to develop a scale extended Bargmann algebra are given by [19]

\[
\begin{align*}
[D, P_a] &= -P_a, & [D, H] &= -zH, & [H, G_a] &= P_a, \\
[P_{a\mu}, G_b] &= \delta_{ab}N, & [D, G_a] &= (z - 1)G_a, & [D, N] &= (z - 2)N, \\
[J_{ab}, P_c] &= 2\delta_{[a}{\rho}P_{b]}, & [J_{ab}, G_c] &= 2\delta_{[a}{\rho}G_{b]}, & [J_{ab}, J_{cd}] &= 4\delta_{[a}{\rho}J_{b]}{\delta}. \\
\end{align*}
\]

(5.1)

The transformation rules are given by [19]

\[
\begin{align*}
\delta \tau_\mu &= \partial_\mu \xi - z\xi \delta b_\mu + z\Lambda_D \tau_\mu, \\
\delta \omega^{a}_\mu &= \partial_\mu \omega^{a}_\mu - \omega^{ab}_\mu \xi_b + b_\mu \omega^{a}_\mu + \lambda^a \tau_\mu - \omega^{\sigma}_\mu \xi + \Lambda_D \omega^{a}_\mu, \\
\delta \omega^{ab}_\mu &= \partial_\mu \omega^{ab}_\mu - 2\lambda^b \omega^{a}_\mu b_\mu + \omega^{a}_\mu \omega^{b}_\mu + (z - 1)\lambda^a \delta b_\mu - (z - 1)\Lambda_D \omega^{ab}_\mu, \\
\delta m_\mu &= \partial_\mu \sigma - \xi^a \omega^{a}_\mu b_\mu + \lambda^a \omega^{a}_\mu b_\mu + (z - 2)\sigma \delta b_\mu - (z - 2)\Lambda_D m_\mu, \\
\delta b_\mu &= \partial_\mu \Lambda_D, \\
\end{align*}
\]

and the corresponding curvatures are given by [19]

\[
\begin{align*}
R_{\mu\nu}(H) &= 2\partial_{[\mu} \tau_{\nu]} - 2z b_{[\mu} \tau_{\nu]}, \\
R^{a}_{\mu\nu}(P) &= 2\partial_{[\mu} e_{\nu]}{\sigma} - 2\omega^{ab}_\mu e_{\nu} b_\mu - 2\omega^{a}_\mu \tau_{\nu} - 2b_{[\mu} e_{\nu]}{\sigma}, \\
R^{ab}_{\mu\nu}(J) &= 2\partial_{[\mu} \omega^{ab}_\nu{\sigma} - 2\omega^{ab}_\mu \omega^{c}_{\nu} b_{[\mu} e_{\nu]}{\sigma}, \\
R^{a}_{\mu\nu}(G) &= 2\partial_{[\mu} \omega^{a}_\nu{\sigma} + 2\omega^{b}_\mu \omega^{a}_\nu b_{[\mu} - 2(z - 1)\omega^{a}_\mu \delta b_{[\mu} e_{\nu]}, \\
R_{\mu\nu}(D) &= 2\partial_{[\mu} b_{\nu]}, \\
R_{\mu\nu}(N) &= 2\partial_{[\mu} m_{\nu]} - 2\omega^{a}_\mu e_{\nu} b_{[\mu} + 2(z - 2)b_{[\mu} m_{\nu]}, \\
\end{align*}
\]

(5.3)

5. $z \neq 2$ scale invariant Hořava–Lifshitz gravity

In the previous section, we constructed all the $z = 2$ scale invariant gravity models that are relevant to the Hořava–Lifshitz gravity and put an explicit distinction between the scale and Schrödinger invariant extension of the Hořava–Lifshitz gravity. In this section, our purpose is to develop a $z \neq 2$ scale invariant tensor calculus and construct the potential, kinetic and curvature terms that are relevant to the $z \neq 2$ scale extension of the Hořava–Lifshitz gravity. Finally, following [17], we identify the $z \neq 2$ scale extended Hořava–Lifshitz gravity.

As mentioned in section 3, when $z \neq 2$, the scale symmetry can no longer be extended to the Schrödinger symmetry by including a non-relativistic special conformal transformation. Thus, the $z \neq 2$ scale extended Bargmann algebra has the same generators and the gauge fields as in the $z = 2$ scale extended case. The commutations relations between the generators of $z \neq 2$ scale extended Bargmann algebra are given by [19]
Note that when the dynamical critical exponent $z$ is left arbitrary, the $D$ transformation rules for the inverse vielbein and the inverse temporal vielbein are given by

$$\delta \tau^\mu = -z \Lambda D \tau^\mu, \quad \delta e^\mu{}_a = -\Lambda D e^\mu{}_a.$$ (5.4)

We are now at a position to make contact to the $z \neq 2$ scale-invariant generalization of the Newton–Cartan geometry that we established in section 3. As before, this is achieved imposing a set of curvature constraints. In the case $z \neq 2$ scale symmetry, we have the following set of constraints

$$R_{\mu \nu}(H) = 0, \quad R_{\mu \nu}{}^a(P) = 0, \quad R_{\mu \nu}(N) = 0, \quad R_{\mu \nu}(D) = 0,$$ (5.5)

which results to the following further constraints by Bianchi identities

$$e[^b R_{\nu \rho}]^a_J(J) + \tau[^a R_{\nu \rho}]^b_j(G) = 0, \quad e[^a R_{\nu \rho}]^b_j(G) = 0.$$ (5.6)

The first constraint implies that the twistless condition is satisfied, thereby the torsion vanishes

$$R_{\mu \nu}(H) = 0 \quad \Rightarrow \quad \partial_{[\mu} \tau_{\nu]} = z b_{[\mu} \tau_{\nu]},$$ (5.7)

and determines the spatial part of $b_\mu$ as

$$e^\mu_a b_\mu = \frac{2}{z} e^\mu a \tau^\nu \partial_{[\mu} \tau_{\nu]}.$$ (5.8)

Furthermore, the last two constraints in (5.5) gives rise to the solution of $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$

$$\omega_\mu{}^{ab} = -2 e^\nu[^a [\partial_{\mu} e_\nu]^b] + e^\nu[^a e_\nu]^b \partial_{[\mu} e_\nu + 2 e_\nu[^a e_\nu]^b \partial_{[\mu} e_\nu,$$

$$\omega_\mu{}^a = \tau^\nu \partial_{[\mu} e_\nu a + e_\nu[a \tau^\nu \partial_{[\mu} e_\nu + e_\nu[a \tau^\nu b_\nu + e_\mu e_\nu \partial_{[\mu} m_{\nu]} + (z - 2) b_{[\mu} m_{\nu]}).$$ (5.9)

To make contact with geometry, we turn to the $z \neq 2$ scale covariant metric compatibility conditions

$$0 = \partial_{\mu} \tau_{\nu} - \Gamma^\rho_{\mu \nu} \tau_\rho - z b_{\mu} \tau_{\nu},$$

$$0 = \partial_{\mu} e_\nu a - \Gamma^\rho_{\mu \nu} e_\rho a - \omega_\mu{}^{ab} e_\rho b - \omega_\mu{}^a_{\tau \nu} - b_\mu e_\nu a.$$ (5.10)

These conditions uniquely determine $\Gamma$ as a symmetric connection

$$\Gamma^\rho_{\mu \nu} = \tau^\rho \mathcal{D}_{\mu} \tau_{\nu} + \frac{1}{2} h^\rho_{\sigma \tau} \left( \mathcal{D}_{\nu} h_{\sigma \mu} + \mathcal{D}_{\nu} h_{\tau \mu} - \mathcal{D}_\sigma h_{\mu \nu} \right) - h^\rho_{\sigma \tau} \tau_{(\mu} F_{\nu)} \sigma.$$ (5.11)

where the $z \neq 2$ scale-covariant objects are as defined as

$$\mathcal{D}_{\mu} \tau_{\nu} = \partial_{\mu} \tau_{\nu} - z b_{\mu} \tau_{\nu}, \quad \mathcal{D}_{\nu} h_{\rho \mu} = \partial_{\nu} h_{\rho \mu} - 2 b_{\mu} h_{\rho \nu}. $$ (5.12)

Finally, we give the corresponding $z \neq 2$ scale invariant Riemann tensor in terms of $R_{\mu \nu}{}^{ab}(J)$ and $R_{\mu \nu}{}^a(G)$ as

$$R^\rho_{\mu \nu \sigma}(\Gamma) = -e^\rho_{\alpha \beta} \left( \tau_\mu R_{\nu \sigma}{}^\alpha(G) + e_{\mu \nu} R_{\sigma \alpha}{}^{ab}(J) \right).$$ (5.13)
Once again, as described in section 3, we can use this Riemann tensor to construct \( z \neq 2 \) scale-invariant non-relativistic gravity actions or field equations by introducing a compensating scalar field \( \phi \) and a rank- \((d+1)\) Milne-invariant tensor \( g_{\mu\nu} \).

### 5.1. \( z \neq 2 \) scale invariant tensor calculus

Our aim is to develop a tensor calculus to construct the \( z \neq 2 \) generalization of the non-relativistic scale invariant gravity. As before, we are only interested in the set of models that are relevant to the Hořava–Lifshitz theory, thus we limit ourselves to a certain class of potential, kinetic and curvature terms. In principle, the construction procedure might seem like a straightforward generalization of what was done for the \( z = 2 \) case. Furthermore, it seems natural to expect that the \( z = 2 \) limit of the \( z \neq 2 \) construction must recover the models that we give in section 4.4. Thus, before we start with the construction procedure, it is useful to enumerate the subtleties and technical differences of \( z \neq 2 \) models.

1. First of all, when \( z \neq 2 \), the curvature of the dilatation gauge field, \( R_{\mu\nu}(D) \) is set to zero due to Bianchi identity for \( R_{\mu\nu}(N) \). Thus, \( b_\mu \) can be given as \( b_\mu = \partial_\mu \phi \) where \( \phi \) is a scalar field that transforms as shift under dilatations \( \delta \phi = \Lambda \phi \).

2. As \( R_{\mu\nu}(D) = 0 \) for \( z \neq 2 \) models, the Riemann tensor for \( z \neq 2 \) differs from the \( z = 2 \) Riemann tensor by \( R_{\mu\nu}(D) \) terms, see (4.34) and (5.13). Thus, there is no smooth \( z = 2 \) limit of \( z \neq 2 \) gravity theories that are constructed by use of the \( z \neq 2 \) Riemann tensor.

3. \( z \neq 2 \) scale-extended Bargmann algebra does not allow the existence of a scalar field with a homogeneous dilatation and U(1) transformation due to the following commutation relation

\[
[D, N] = (z - 2)N.
\]

(5.14)

Thus, as opposed to the \( z = 2 \) case, we cannot introduce a complex scalar field as given in (4.47). For \( z \neq 2 \) setting we are only allowed to work with two type of scalar fields with the following transformation rules

\[
\delta \phi = \omega \Lambda \phi, \quad \delta \chi = \sigma + (2 - z)\Lambda \partial \chi.
\]

(5.15)

4. The scalar field \( \chi \) has a non-vanishing U(1) transformation. This implies that we cannot form potential terms with \( \chi \) and it can only appear in an action by its covariant derivative, which reads

\[
D_\mu \chi = \partial_\mu \chi - (2 - z)b_\mu \chi - m_\mu.
\]

(5.16)

This is the very definition of the U(1) invariant vector field \( M_\mu \) up to an overall sign difference, see (3.14). Thus, for the \( z \neq 2 \) setting, the main elements of the scale invariant tensor calculus are the scalar field \( \phi \) and the U(1) invariant vector field \( M_\mu \).

With these points in mind, we now proceed to the construction of the relevant potential, kinetic and curvature actions of \( z \neq 2 \) scale invariant gravity.

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\*If \( R_{\mu\nu}(D) = 0 \) is chosen as the constraint as in [19] instead of \( R_{\mu\nu}(D) = 0 \), then \( b_\mu \) can no longer be set to \( b_\mu = \partial_\mu \phi \). In that case one can introduce a special conformal symmetry to the \( z \neq 2 \) scale extended Bargmann algebra by only taking the internal part of the algebra into account, which would lead one to a construction procedure similar to the construction of \( z = 2 \) models given in section 4.
5.1.1 Potential terms. The transformation rules for $\phi$ does not change in the $z \neq 2$ setting. Thus, the potential terms for $\phi$ in $z \neq 2$ scale-invariance is the same as the $z = 2$ theory. On the other hand, unlike the $z = 2$ case, we cannot define a complex scalar $\Psi$ field to include the potential terms of $\chi$ into the kinetic terms of the $\Psi$. Thus, here we give the potential terms of $\chi$ in zeroth, second and fourth order spatial derivatives.

- $n_s = 0$: $\chi$ has a non-vanishing $U(1)$ transformation, thus cannot form an action with no derivatives.
- $n_s = 2$: For the construction of potential terms with two spatial derivatives, we turn to the spatial part of the covariant derivative of $M_a$

$$\mathcal{D}_\mu M_a = \partial_\mu M_a + (z-1) b_\mu M_a - \omega_\mu \alpha M_b - \omega_\mu \beta a$$

which is invariant under Galilean transformations due to the fact that $M_a$ transforms as shift under Galilean transformations

$$\delta M_a = \lambda a.$$  \hspace{1cm} (5.18)

As the inverse spatial vielbein $e^\mu_a$ is also Galilean invariant, we can form a spatial covariant derivative of $M_a$ that is invariant under Galilean transformations

$$\delta \mathcal{D}_a M_b = - z \Lambda \mathcal{D}_a M_b.$$ \hspace{1cm} (5.19)

Thus, the only possible $n_s = 2$ potential terms for $\chi$ are

$$S^{(2)}_{z \neq 2} = \int dt d^d x \, e \, \phi \, [\mathcal{D}_a M^a]^2,$$

Here, the scaling dimension of $\phi$ is given by $\omega = - d$. Furthermore $e = \text{det}(\tau_\mu, e^\mu_a)$ has the following scaling transformation

$$\delta e = (d + z) \Lambda \mathcal{D}_a e.$$ \hspace{1cm} (5.21)

- $n_s = 4$: The construction of potential terms with four spatial derivatives can be divided into following three subclasses

a. We first consider the models such that the spatial derivatives only act on $\chi$ terms ($n_s = 4, n_\phi = 0$).

$$S^{(2)}_{z \neq 2} = \int dt d^d x \, e \, \phi \, (\mathcal{D}_a M^a)^2,$$

$$S^{(3)}_{z \neq 2} = \int dt d^d x \, e \, \phi \, (\mathcal{D}_a M_b)^2,$$

$$S^{(4)}_{z \neq 2} = \int dt d^d x \, e \, \phi \, \mathcal{D}_a M^a.$$ \hspace{1cm} (5.22)

Here, for $S^{(2)}_{z \neq 2}$ and $S^{(3)}_{z \neq 2}$, the scaling dimension of $\phi$ is given by $\omega = z - d$ while for the $S^{(4)}_{z \neq 2}$, the scaling dimension of $\phi$ is $\omega = 2 - d$. Note that it is also possible to consider the models of kind $\phi \mathcal{D}_a \triangle M^a$ or $\phi \mathcal{D}_a \mathcal{D}_b \mathcal{D}^a M^b$, however such actions are related to $S^{(4)}_{z \neq 2}$ up to curvature invariants

$$\mathcal{D}_a \triangle M^a = \triangle \mathcal{D}_a M^a + [\mathcal{D}_a, \triangle] M^a$$

$$= \triangle \mathcal{D}_a M^a - \mathcal{D}^b \left( R_{bc}(J) M^c + R_{ab}^c (G) \right).$$ \hspace{1cm} (5.23)
We will construct that curvature invariant in the next section.

b. Next, we consider \( n_{s, \chi} = 3, n_{s, \phi} = 1 \) models. In this case, the candidate models are given as

\[
\mathcal{D}_a \phi \, \triangle M^a, \quad \mathcal{D}_b \phi \mathcal{D}_a \mathcal{D}_b M^a, \quad \mathcal{D}_b \phi \mathcal{D}_a \mathcal{D}_b M^a.
\] (5.24)

however, all these models are equivalent to \( S_c^{(4)} \) up to boundary terms and the curvature term given in (5.23).

c. Finally, we consider the models with \( n_{s, \chi} = 2, n_{s, \phi} = 2 \). In this case, the candidate models are given as

\[
\triangle \mathcal{D}_a \phi, \quad \phi^{-1} \mathcal{D}_a \mathcal{M}^a \phi \mathcal{D}_b \phi, \quad \mathcal{D}_a \mathcal{M}_b \mathcal{D}_c \mathcal{D}_a \phi, \quad \phi^{-1} \mathcal{D}_a \mathcal{M}_b \mathcal{D}_a \phi \mathcal{D}_b \phi.
\] (5.25)

However, all these models are equivalent to the previously constructed ones up to boundary terms. Thus, there is no independent \( n_{s, \chi} = 2, n_{s, \phi} = 2 \) potential terms.

Note that as \( \mathcal{M}_a \) is not Galilean invariant, there is no \( n_{s, \chi} = 1, n_{s, \phi} = 3 \) class of potential terms with. Furthermore, as mentioned before, \( n_{s, \chi} = 0, n_{s, \phi} = 4 \) potential terms are the same as \( z = 2 \) theory.

5.1.2. Kinetic terms. In this section, we construct the actions that are first and second order in time derivative. In order to do so, we first construct Galilean invariant quantities that include time derivatives on \( \phi \) or \( \chi \). For the real scalar field \( \phi \), when no spatial derivative act on it, the only possible Galilean invariant quantity is

\[
\mathcal{D}_0 \phi + \mathcal{M}^a \mathcal{D}_a \phi.
\] (5.26)

If we allow a single spatial derivative to act on \( \mathcal{D}_0 \phi \) we also have a single Galilean invariant quantity

\[
\mathcal{D}_a \mathcal{D}_0 \phi + \mathcal{M}^b \mathcal{D}_b \phi.
\] (5.27)

Note that in principle we could also have \( \mathcal{D}_0 \mathcal{D}_a \phi \), but it is the same \( \mathcal{D}_a \mathcal{D}_0 \phi \) since the commutator of \( \mathcal{D}_0 \) and \( \mathcal{D}_a \) on the real scalar field vanishes. Next, we allow two spatial derivative to act on \( \mathcal{D}_0 \phi \). In this case, there are two possible independent Galilean invariant quantities

\[
\triangle \mathcal{D}_0 \phi + \mathcal{M}^a \triangle \mathcal{D}_a \phi, \quad \triangle \mathcal{D}_a \phi + \mathcal{M}^a \triangle \mathcal{D}_0 \phi.
\] (5.28)

We could also have \( \mathcal{D}_0 \triangle \phi \) and \( \mathcal{D}_a \mathcal{D}_0 \phi \) but they are related to \( \triangle \mathcal{D}_0 \phi \) by the curvature term \( R_{ab}(J) \mathcal{D}^b \phi \). Another two possible actions at this level, \( \mathcal{D}_0 \mathcal{D}_a \phi \) and \( \mathcal{D}_b \mathcal{D}_0 \mathcal{D}_a \phi \), are related to \( \mathcal{D}_0 \mathcal{D}_a \phi \) by the curvature term \( R_{ab}(J) \mathcal{D}_0 \phi \).

When it comes to \( \chi \) term, we work with the temporal component of the vector field \( \mathcal{M}_\mu \). First, we do not allow a spatial derivative to act on \( \mathcal{M}_0 \). In this case the only possible Galilean invariant quantity is given by

\[
\mathcal{M}_0 + \frac{1}{2} \mathcal{M}_a \mathcal{M}^a.
\] (5.29)

In the next step, we only allow a single spatial derivative to act on \( \mathcal{M}_0 \), in which case the only Galilean invariant quantity is

\[
\mathcal{D}_a \mathcal{M}_0 + \mathcal{M}^b \mathcal{D}_a \mathcal{M}_b.
\] (5.30)
Here, we could also have $\mathcal{D}_0M_a$ but it is equivalent to $\mathcal{D}_aM_0$ as the commutator of $\mathcal{D}_0$ and $\mathcal{D}_a$ vanishes on $\chi$. Finally, we allow two spatial derivative to act on $M_0$, in which case we have two possible independent Galilean invariant quantities

$$\triangle M_0 + M^a \triangle M_a, \quad \mathcal{D}_a \mathcal{D}_a M_0 + M^a \mathcal{D}_a \mathcal{D}_a M_a.$$  \hspace{1cm} (5.31)

Note that we could also have $\mathcal{D}_0 \mathcal{D}_a M^a$ and $\mathcal{D}_a \mathcal{D}_0 M^a$ but they are related to $\triangle M_0$ up to the curvature term $R_{a\alpha}(G) + R_{ab}^{\alpha \beta}(J)M_b$. Furthermore, $\mathcal{D}_0 \mathcal{D}_a M_b$ and $\mathcal{D}_b \mathcal{D}_0 M_a$ are related to $\mathcal{D}_a \mathcal{D}_b M_0$ by the curvature term $R_{ab}^{\alpha \beta}(G) + R_{abc}^{\alpha \beta}(J)M_c$.

When we have time derivatives at second order acting on $\phi$ and $\chi$, we do not allow any spatial derivatives to act on such terms. In this case, the Galilean invariant quantities are

$$\mathcal{D}_0^2 \phi + 2M^a \mathcal{D}_a \phi + M^a M^b \mathcal{D}_a \mathcal{D}_b \phi, \quad \mathcal{D}_0 M_0 + 2M^a \mathcal{D}_a M_a + M^a M^b \mathcal{D}_a M_b.$$  \hspace{1cm} (5.32)

With these results in hand, we have the following classification of Galilean invariant actions.

- $n_t = 1$: When we have a single time derivative acting on $\phi$ or $\chi$, we first consider the models with no spatial derivatives

$$S_{c \neq 2}^{(5)} = \int dt d^4x \phi (\mathcal{D}_0 \phi + M^a \mathcal{D}_a \phi),$$

$$S_{c \neq 2}^{(6)} = \int dt d^4x \phi^2 (M_0 + \frac{1}{2} M_a M^a).$$  \hspace{1cm} (5.33)

Here, for $S_{c \neq 2}^{(5)}$ we have $\omega = \frac{2 - d}{2}$, while for $S_{c \neq 2}^{(6)}$ we have $\omega = \frac{2 - d - 2}{2}$. The remaining $z \neq 2$ scale-invariant actions, which consists one temporal and two spatial derivatives, can be classified with respect to the scaling dimension of the scalar field $\phi$ as follows

- a. For the following two models, the scaling dimension of the scalar field $\phi$ is $\omega = \frac{2 - d}{2}$

$$S_{c \neq 2}^{(7)} = \int dt d^4x \phi \triangle \phi (\mathcal{D}_0 \phi + M^a \mathcal{D}_a \phi),$$

$$S_{c \neq 2}^{(8)} = \int dt d^4x \phi \triangle \phi (\mathcal{D}_0 \phi + M^a \mathcal{D}_a \phi).$$  \hspace{1cm} (5.34)

- b. For the next three models, the scaling dimension of the scalar field $\phi$ is $\omega = \frac{d - 2}{2}$

$$S_{c \neq 2}^{(9)} = \int dt d^4x \phi \mathcal{D}^a \mathcal{D}_a (\mathcal{D}_0 \phi + M^a \mathcal{D}_a \phi),$$

$$S_{c \neq 2}^{(10)} = \int dt d^4x \phi \triangle \phi (M_0 + \frac{1}{2} M_a M^a),$$

$$S_{c \neq 2}^{(11)} = \int dt d^4x \mathcal{D}_a \phi \mathcal{D}^a \phi (M_0 + \frac{1}{2} M_a M^a).$$  \hspace{1cm} (5.35)

- c. For the final kinetic term with a single derivative, the scaling dimension of the scalar field $\phi$ is $\omega = \frac{2 - d - 2}{2}$

$$S_{c \neq 2}^{(12)} = \int dt d^4x \phi^2 \mathcal{D}^a \mathcal{D}_a (M_0 + \frac{1}{2} M_a M^a).$$  \hspace{1cm} (5.36)

- $n_t = 2$: When we have two time derivative acting on $\phi$ or $\chi$, the possible $z \neq 2$ scale-invariant actions are
\[ S_{\phi}^{(1)} = \int dt \, d^4 x \, e \, \phi \left( \frac{1}{2} \left( D_0^2 \phi + 2 M^2 D_0 D_\mu \phi + M^4 M^6 D_\mu D_\nu \phi \right) \right), \]
\[ S_{\phi}^{(2)} = \int dt \, d^4 x \, e \, \phi^2 \left( D_0^2 M_0 + 2 M^2 D_0 M_\mu + M^4 M^6 D_\mu M_\nu \right), \]
\[ S_{\phi}^{(3)} = \int dt \, d^4 x \, e \, \phi^2 \left( M_0 + \frac{1}{2} M_2 M^8 \right). \]

Unlike the potentials and \( n_t = 1 \) models, we need to choose a different scaling dimension for each of the \( n_t = 2 \) actions due to the fact that the scaling dimension of \( M_0 \) is \( z \)-dependent, \( \delta M_0 = 2(1 - z) \Lambda_0 M_0 \). For \( S_{\phi}^{(1)} \) we have \( \omega = \frac{2 - d}{2} \), while for \( S_{\phi}^{(2)} \) we have \( \omega = \frac{4 - d}{2} \) and for \( S_{\phi}^{(3)} \) we have \( \omega = \frac{6 - d}{2} \).

In our construction above, we avoid models that are equivalent to each other by means of partial integration or combination of other invariant actions, e.g. it is possible can also produce an invariant action using (5.27) and multiplying it with the Galilean-invariant covariant derivative \( D^\mu \phi \). However, such a model can be obtained by a partial integration of \( S_{\phi}^{(1)} \).

5.1.3 Curvature terms. Following the \( z = 2 \) discussion, we will now consider the curvature invariants. First, we enumerate the curvature invariants that are required for the commutation relations as mentioned above. The models that include the non-zero curvature \( R_{\mu \nu \rho \sigma} \) are given by
\[ S_{\phi}^{(16)} = \int dt \, d^4 x \, e \, \phi^2 \left( \tau^\mu e_{\nu \rho} R_{\mu \nu \rho \sigma}(G) + M^4 [2 \tau^\mu e_{\nu \rho} R_{\mu \nu \rho \sigma}(J) + M^2 e_{\nu \rho} e_{\sigma \tau} R_{\mu \nu \rho \sigma}(J)] \right), \]
\[ S_{\phi}^{(17)} = \int dt \, d^4 x \, e \, \phi \left( R_{\mu \nu \rho \sigma}(G) + R_{\mu \nu \rho \sigma}(G) D^\mu \phi \right). \quad (5.38) \]

For \( S_{\phi}^{(16)} \), the scaling dimension of \( \phi \) is given by \( \omega = \frac{2 - d}{2} \), while for \( S_{\phi}^{(17)} \) we have \( \omega = \frac{4 - d}{2} \). We also have the following two invariants that replaces the \( z = 2 \) scale invariant action \( S^{(21)} \) given in (4.62), in the case of \( z \neq 2 \) scale-extended non-relativistic gravity
\[ S_{\phi}^{(18)} = \int dt \, d^4 x \, e \, R(J)(D_0 \phi + M^2 D_\mu \phi), \]
\[ S_{\phi}^{(19)} = \int dt \, d^4 x \, e \, \phi^2 R(J)(M_0 + \frac{1}{2} M_2 M^8). \quad (5.39) \]

For \( S_{\phi}^{(18)} \), the scaling dimension of \( \phi \) is given by \( \omega = \frac{2 - d}{2} \), while for \( S_{\phi}^{(19)} \) we have \( \omega = \frac{4 - d}{2} \). Other curvature invariants that include the contraction of the rotation curvature \( R_{\alpha \beta \gamma \delta} \) are the same as \( z = 2 \) as given in (4.62) from \( S^{(14)} \) to \( S^{(20)} \), thus we will not give them here.

5.2. \( z \neq 2 \) scale invariance and the Hořava–Lifshitz gravity
In this section, our purpose is to combine the \( z \neq 2 \) scale invariant gravity models that we constructed to identify the \( z \neq 2 \) scale extension of the Hořava–Lifshitz gravity. Thus, we start this section with a brief review of the dictionary between the dynamical Newton–Cartan geometry and the Hořava–Lifshitz gravity that was put forward in [17]. We refer to [17] for readers interested in the details of the dictionary we review here.
1. **Coordinates:** In order to define the Hořava–Lifshitz variables in terms of the fields in the scale-extended Newton–Cartan geometry, we first assume the hypersurface orthogonality condition

\[ \tau_{[\mu} \partial_{\nu} \tau_{\rho]} = 0, \]  

(5.40)

which is satisfied by the \( z \neq 2 \) scale invariant theory due to the constraint \( R_{\mu\nu}(H) = 0 \), see (5.7). Next, we consider the \( (d + 1) \)-dimensional ADM decomposition of the metric tensor where metric tensor \( g_{\mu\nu} \) that we defined in (3.17). This leads to the following relations between the components of \( h_{\mu\nu}, \hat{h}_{\mu\nu}, \tau_{\mu}, \hat{\tau}_{\mu} \) and the lapse function \( N = N(t,x) \), and the \( d \)-metric \( \gamma_{ab} \) \[17\]

\[ \tau_t = N, \quad \tau_i = 0, \]
\[ h^t = h^\mu = h^i = 0, \quad h^i = \gamma^i, \]
\[ \hat{\tau}^t = N^{-1}, \quad \hat{\tau}^i = -N^{-1} N^i, \]
\[ \hat{h}_a = \gamma_a N^i, \quad \hat{h}_t = \gamma_0 N^i, \quad \hat{h}_y = \gamma_y, \]  

(5.41)

which implies the following expressions for \( h_{\mu\nu} \) and \( \tau^a \)

\[ \tau^t = N^{-1}, \quad \tau^i = 0, \quad h_a = h_i = h_y = 0, \quad h_y = \gamma_y. \]  

(5.42)

Here, it is important to note that we split the \( \mu \)-index into coordinates \( t \) and \( x^i \). Using these relations, we also identify the \( U(1) \)-invariant vector field \( M_\mu \) as

\[ M_t = -\frac{1}{2N} \gamma_0 N^i + \Phi N, \quad M_i = -\frac{1}{N} \gamma_0 N^i, \]  

(5.43)

where \( \Phi \) is the Newtonian potential that we defined in (2.28). Finally, based on the twistless condition (2.20), we observe that this condition is fixed by \( b_i = e_i^a b_a \) since the twistless condition imply

\[ \partial_{[\mu} \tau_{\nu]} = zb_{[\mu} \tau_{\nu]} = zb_a e_{[\mu}^a \tau_{\nu]}. \]  

(5.44)

Thus, in order to define the twistless condition in terms of the ADM variables, we define a vector, \( a_\mu \), as follows \[17\]

\[ a_\mu = L_\tau \tau_\mu = \hat{\tau}^a (\partial_a \tau_\mu - \partial_\mu \tau_a) = -ze_\mu^a b_a, \]  

(5.45)

where the last part of the equation is fixed by the twistless condition (2.20). Using the condition (2.20), we determine the vector \( a_\mu \) as \[17\]

\[ a_t = N^i a_i, \quad a_i = -N^{-1} \partial_i N. \]  

(5.46)

Note that temporal component \( b_0 \) does not play a role in the connection between the Hořava–Lifshitz gravity and the Newton–Cartan geometry.

2. **Geometry:** When we keep the lapse function only as a function of time only \( N = N(t) \), we are dealing with the projectable Hořava–Lifshitz gravity. On the geometry side, \( N = N(t) \) corresponds to the torsionless Newton–Cartan geometry since it gives rise to \( \partial_{[\mu} \tau_{\nu]} = 0 \). When the lapse function \( N \) is left arbitrary, we are dealing with the non-projectable Hořava–Lifshitz gravity, which corresponds to the twistless-torsional Newton Cartan geometry.
### Curvatures

The second fundamental form, or the extrinsic curvature is defined as

\[
K_{ij} = \frac{1}{2N} \left( \partial_t \gamma_{ij} - \bar{\nabla}_i N_j - \bar{\nabla}_j N_i \right),
\]

(5.47)

where \( \bar{\nabla}_i \) denotes the d-dimensional covariant derivative with respect to the d-metric \( \gamma_{ij} \)

\[
\bar{\nabla}_i N_j = \partial_i N_j - \bar{\Gamma}^k_{ij} N_k, \tag{5.48}
\]

where \( \bar{\Gamma}^k_{ij} \) denotes the components of the Christoffel connection for the d-metric \( \gamma_{ij} \)

\[
\bar{\Gamma}^k_{ij} = \frac{1}{2} \gamma^{lm} \left( \partial_l \gamma_{jm} + \partial_j \gamma_{im} - \partial_{m} \gamma_{ij} \right). \tag{5.49}
\]

In order to relate the extrinsic curvature (5.47) to the Newton–Cartan variables, we first make the following definition for the scale-covariant derivative of the \( U(1) \)-invariant vector \( M_a \)

\[
K'_{ab} = \mathcal{D}_a M_b - \bar{\nabla}_a (\phi M_b) + z b_0 (\partial_0 \phi) M_a - \delta_{ab} b_0 \tag{5.50}
\]

where \( \bar{\nabla}_a \) refers to the Galilean gauge-covariant piece of the scale covariant derivative

\[
\bar{\nabla}_a M_b = \epsilon_{ab}^\mu \left( \partial_\mu M_b - \Omega_{\mu bc} M_c - \Omega_{\mu bc} \right). \tag{5.51}
\]

Here, we also decomposed the rotation and boost gauge connections of the \( z \neq 2 \) scale extended Bargmann algebra to the of the Bargmann algebra as

\[
\omega_{\mu}^{ab} = \Omega_{\mu}^{ab} + 2 \epsilon_{\mu}^{[ab]} \phi \left( z - 2 \right) \epsilon^{\rho a} \epsilon^{\nu b} \tau_{\rho \nu \mu}, \quad \omega_{\mu}^{a} = \Omega_{\mu}^{a} + \epsilon_{\mu}^{[ab]} \phi \left( z - 2 \right) \epsilon^{\rho a} b_{[\mu} m_{\nu]} + \left( z - 2 \right) \tau_{\mu \nu}^{\rho} \epsilon^{\rho a} \phi \left( z - 2 \right) \phi \left( z - 2 \right). \tag{5.52}
\]

Here, we represent the rotation and boost gauge fields of the Bargmann algebra with \( \Omega_{\mu}^{ab} \) and \( \Omega_{\mu}^{a} \) in the respective order to distinguish these quantities with the relevant \( z \neq 2 \) scale invariant ones. Based on our previous conclusion that \( b_0 \) does not play a role in the Hořava–Lifshitz gravity, it is best to consider models where \( b_0 \) drops out. This can be achieved in two ways

i. We can decompose \( K'_{ab} \) as

\[
K'_{ab} = K_{ab} - \delta_{ab} (b^c M_c + b_0) \quad \text{such that} \quad K_{ab} = \bar{\nabla}_a (\phi M_b) + z b_0 (\phi M_a), \tag{5.53}
\]

in which case the following combination has no \( b_0 \) term

\[
K'_{ab} K^{ab} = \frac{1}{d} K'^2 = K_{ab} K^{ab} - \frac{1}{d} K^2, \tag{5.54}
\]

where \( K \equiv K_{ab} \) and \( K' \equiv K'_{ab} \). This is precisely what was found as the kinetic term of conformal Hořava–Lifshitz gravity in [17].

ii. We can consider the combination of \( K'_{ab} \) with the \( z \neq 2 \) scale covariant combination (5.26)

\[
K'_{ab} - \delta_{ab} (\phi^{-1} \mathcal{D}_a \phi + M^a \phi^{-1} \mathcal{D}_a \phi), \tag{5.55}
\]

in which case, as in the previous scenario, only the \( K_{ab} \) part of \( K'_{ab} \) survives.

Therefore, we only need to worry about the relation between the \( K_{ab} \) and \( K_{ij} \) since the remaining terms can either be absorbed into the \( z \neq 2 \) scale covariant combination (5.26).
or can be canceled out by choosing a proper combination of $K'_{ab}K^{ab}$ and $K'^2$. As noted in [18], $K_{ab}$ can be written as

$$K_{ab} = e^{\mu e^\nu} \left( \frac{1}{2} \nabla_\mu (P^\alpha_\mu M_\rho) + \frac{1}{2} \nabla_\nu (P^\alpha_\mu M_\rho) + zM_{(\mu \nu)} \right),$$

(5.56)

where

$$\nabla_\mu M_\nu = \partial_\mu M_\nu - \Gamma^\alpha_{\mu\nu} M_\alpha.$$  

(5.57)

From this expression, we observe that one can write down the kinetic terms $K_{ab}K^{ab} = K_{ij}K_{ij}$ and $K_{aa} = \gamma_{ij}K_{ij}$ upon using the map between the ADM variables and the Newton–Cartan fields (5.41).

With these results in hand, we give the $z \neq 2$ scale-extended Hořava–Lifshitz gravity as

$$S_{z \neq 2}^{HL} = S_{z \neq 2}^{(3)} - \lambda S_{z \neq 2}^{(2)} + S_V,$$

(5.58)

where $\lambda$ is an arbitrary parameter and $S_V$ represents any remaining combination of actions that we constructed for $\chi$, $\phi$ and group theoretical curvatures in the previous sections.

### 6. Conclusions

In this paper we present a detailed study on the construction of $z = 2$ and $z \neq 2$ scale invariant extension of the Hořava–Lifshitz gravity. To achieve these result, we developed a non-relativistic scale invariant tensor calculus and constructed scale invariant actions. Our results also enabled us to put an explicit distinction between the scale and Schrödinger invariance for $z = 2$ non-relativistic gravity.

There are a number of directions to pursue following our work. First of all, the formulation we present here is not torsional since the gauging procedure we applied for the $z = 2$ as well as the $z \neq 2$ theories gave rise to a symmetric connection. In order to introduce a torsion, one can follow the idea presented in a recent work [29] for the gauging of the Schrödinger algebra with torsion and repeat the construction procedure that we applied here. Furthermore, the scale or Schrödinger symmetry corresponds to a special choice of non-metricity in the compatibility equation, and it is possible to have a more general classification of non-relativistic geometries by imposing a more general non-metricity. This classification has been done for the relativistic scenarios in [30], and it would be interesting to see the full classification of non-relativistic geometries with an arbitrary vector distortion. Another interesting direction concerns the supersymmetric completion of the Hořava–Lifshitz gravity. In [31] three-dimensional $\mathcal{N} = 2$ Schrödinger supergravity was achieved by gauging the Schrödinger superalgebra. As the non-relativistic scalar multiplet includes a complex scalar, it is possible to extend the Schrödinger tensor calculus of [18] to a super-Schrödinger tensor calculus, which would than give rise to the three-dimensional $\mathcal{N} = 2$ Hořava–Lifshitz supergravity upon using the dictionary developed in [17] and gauge fixing the redundant superconformal symmetries. Finally, the Schrödinger transformations are not true analogue of the relativistic conformal symmetry as they leave the action of a massive non-relativistic particle invariant. The true non-relativistic analogue of the relativistic conformal symmetry, which leaves the action of a massless non-relativistic particle is invariant is called the Galilean conformal algebra [32]. It would interesting to see whether the non-relativistic tensor calculus can be extended to Galilean conformal algebra, and whether it is possible to distinguish a Galilean conformal gravity from a Galilean scale invariant gravity.
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References

[1] Cartan E 1923 Sur les variétés à connexion affine et la théorie de la relativité généralisée. (première partie) Ann. Sci. Ecole Norm. Sup. 40 325–412
[2] Cartan E 1924 Sur les variétés à connexion affine et la théorie de la relativité généralisée. (première partie) (Suite) Ann. Sci. Ecole Norm. Sup. 41 1–25
[3] Son D T 2013 Newton–Cartan geometry and the quantum Hall effect (arXiv:1306.0638)
[4] Gromov A and Abanov A G 2015 Thermal Hall effect and geometry with torsion Phys. Rev. Lett. 114 016802
[5] Geracie M, Son D T, Wu C and Wu S-F 2015 Spacetime symmetries of the quantum Hall effect Phys. Rev. D 91 045030
[6] Moroz S and Hoyos C 2015 Effective theory of two-dimensional chiral superfluids: gauge duality and Newton–Cartan formulation Phys. Rev. B 91 064508
[7] Horava P 2009 Quantum gravity at a Lifshitz point Phys. Rev. D 79 084008
[8] Horava P 2009 Membranes at quantum criticality J. High Energy Phys. JHEP03(2009)020
[9] Leiva C and Plyushchay M S 2003 Conformal symmetry of relativistic and nonrelativistic systems and Ads / CFT correspondence Ann. Phys. 307 372–91
[10] Balasubramanian K and McGreevy J 2008 Gravity duals for non-relativistic CFTs Phys. Rev. Lett. 101 061601
[11] Son D T 2008 Toward an AdS/cold atoms correspondence: a geometric realization of the Schrodinger symmetry Phys. Rev. D 78 046003
[12] Herzog C P, Rangamani M and Ross S F 2008 Heating up Galilean holography J. High Energy Phys. JHEP11(2008)080
[13] Duval C, Hassaine M and Horvathy P A 2009 The geometry of Schrodinger symmetry in gravity background/non-relativistic CFT Ann. Phys. 324 1158–67
[14] Kachru S, Liu X and Mulligan M 2008 Gravity duals of Lifshitz-like fixed points Phys. Rev. D 78 106005
[15] Taylor M 2008 Non-relativistic holography (arXiv:0812.0530)
[16] Wang A 2017 Horava gravity at a Lifshitz point: a progress report Int. J. Mod. Phys. D 26 1730014
[17] Hartong J and Obers N A 2015 Hořava–Lifshitz gravity from dynamical Newton–Cartan geometry J. High Energy Phys. JHEP07(2015)155
[18] Afshar H R, Bergshoeff E A, Mehra A, Parchek P and Rollier B 2016 A Schrödinger approach to Newton–Cartan and Hořava–Lifshitz gravities J. High Energy Phys. JHEP04(2016)145
[19] Bergshoeff E A, Hartong J and Rosseel J 2015 Torsional Newton–Cartan geometry and the Schrödinger algebra Class. Quantum Grav. 32 135017
[20] Jensen K 2014 On the coupling of Galilean-invariant field theories to curved spacetime (arXiv:1408.6855)
[21] Andringa R, Bergshoeff E, Panda S and de Roo M 2011 Newtonian gravity and the Bargmann algebra Class. Quantum Grav. 28 105011
[22] Dautcourt G 1990 On the Newtonian limit of general relativity Acta. Phys. Pol. B 21 766
[23] Trautman A 1963 Sur la theorie newtonienne de la gravitation C. R. Acad. Sci. Paris 247 617
[24] Ehlers J 1981 Über den Newtonschen Grenzwert Grundlagen-probleme der modernen Physik ed J Nitsch et al (Zürich: Bibliographisches Institut Mannheim)
[25] Christensen M H, Hartong J, Obers N A and Rollier B 2014 Torsional Newton–Cartan geometry and Lifshitz holography Phys. Rev. D 89 061901
[26] Banerjee R and Mukherjee P 2016 Torsional Newton–Cartan geometry from Galilean gauge theory Class. Quantum Grav. 33 225013
[27] Hartong J, Kiritsis E and Obers N A 2015 Field theory on Newton–Cartan backgrounds and symmetries of the Lifshitz vacuum J. High Energy Phys. JHEP08(2015)006
[28] Mitra A 2017 Nonrelativistic fluids on scale covariant Newton–Cartan backgrounds Int. J. Mod. Phys. A 32 1750206
[29] Bergshoeff E, Chatzistavrakidis A, Romano L and Rosseel J 2017 Newton–Cartan gravity and torsion J. High Energy Phys. JHEP10(2017)194
[30] Beltran Jimenez J and Koivisto T S 2016 Spacetimes with vector distortion: inflation from generalised Weyl geometry Phys. Lett. B 756 400
[31] Bergshoeff E, Rosseel J and Zojer T 2015 Newton–Cartan supergravity with torsion and Schrödinger supergravity J. High Energy Phys. JHEP11(2015)180
[32] Bagchi A and Gopakumar R 2009 Galilean conformal algebras and AdS/CFT J. High Energy Phys. JHEP07(2009)037