Global Correlations for Low-Lying Collective $2^+$ States

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Abstract. By using the triaxial rotor model and the anharmonic vibrator model with phonon mixing, we derive a global correlation between the quadrupole moments of the two lowest $2^+$ states in collective nuclei that had previously been observed in experimental data across the periodic table. We then derive other electromagnetic properties for these two models of nuclear structure and compare them globally with experimental data. We find that both models are able to robustly describe the experimental data across the region of nuclei for which the models are applicable, including a large number that they have in common. We then show that there seems to exists a robust orthogonal transformation between these two models for realistic nuclear systems, suggesting that these two seemingly diverse descriptions of quadrupole collective phenomena seem to act in a similar model space and may therefore have a common origin.

1. Introduction

A large range of nuclei across the periodic table exhibit quadrupole collectivity. This includes nuclei that exhibit rotational, vibrational, triaxial and gamma-unstable features. Though these nuclei seem to have a wide variety of distinct properties, there nevertheless appear some global features common to all. Here we will focus on one such feature reported recently in an experimental survey by Allmond [1]. What Allmond found is that for a wide range of masses and for $R = E(4_1^+)/E(2_1^+)$ values ranging from 2 (the vibrational limit) to 10/3 (the rotational limit) the quadrupole moments of the lowest $2^+$ states for most of the nuclei in which both are measured satisfy the property that $Q(2_1^+) = Q(2_2^+)$. Allmond noted that this feature is reproduced theoretically for most nuclei by the IBM-1.

In this work, we will discuss two phenomenological models of nuclear structure that analytically reproduce this global $Q(2_1^+) = -Q(2_2^+)$ correlation. One is the Triaxial Rotor Model (TRM) [2] and the other is the Anharmonic Vibrator Model (AVM) [3, 4, 5] with phonon configuration mixing. As we will see, both not only incorporate the above quadrupole moment correlation but both also do an excellent job of reproducing collective properties of low-lying quadrupole states across the periodic table.
The outline of the presentation is as follows. In Section 2, we briefly review the key elements of the Triaxial Rotor Model and the Anharmonic Vibrator Model with phonon mixing, focussing on those features of the models needed for the analysis to follow. As we will see the quadrupole moment correlation noted above is analytically present in both models. Then in Section 3, we confront these two models with experimental data across several regions of the nuclear periodic table in which quadrupole collectivity is observed. As we will see both models are able to successfully reproduce the data. In Section 4, we discuss the possible relationship between these two apparently distinct collective models of nuclear structure by analyzing the orthogonal transformation that connects their bases for $2^+$ states. In Section 5, we present further numerical support for the quadrupole moment correlation underlying this work. Finally in Section 6 we summarize the key conclusions of this work and describe some possible avenues for future investigation. Further details on the work described herein can be found in Ref. [6].

2. Two phenomenological models of nuclear collective motion

2.1. The TRM

The Triaxial Rotor Model (TRM) is a phenomenological model of nuclear structure. The model contains a total of five parameters and has been used extensively to describe experimental data on E2 collectivity across several regions of the periodic table [7, 8, 9, 10].

Rather than describe the model in any detail, we refer the reader to a nice article where those details can be found [11]. We will, however, briefly summarize the key components of the formalism, as needed for our study.

The Hamiltonian matrices for $2^+$ and $4^+$ states in the model are written schematically as:

\[
H_{TRM}^{2^+} = \begin{pmatrix}
6A & 4\sqrt{3}G \\
4\sqrt{3}G & 6A + 4F
\end{pmatrix},
\]

\[
H_{TRM}^{4^+} = \begin{pmatrix}
20A & 12\sqrt{5}G & 0 \\
12\sqrt{5}G & 20A + 4F & 4\sqrt{7}G \\
0 & 4\sqrt{7}G & 20A + 16F
\end{pmatrix},
\]

(1)

where $A$, $G$ and $F$ are Hamiltonian parameters related to the three components of the inertia tensor.

The $2^+$ states are orthogonal combinations of the $K = 0$ and $K = 2$ basis configurations, according to

\[
|2^+_1\rangle = \cos \Gamma |K = 0\rangle - \sin \Gamma |K = 2\rangle,
\]

\[
|2^+_2\rangle = \sin \Gamma |K = 0\rangle + \cos \Gamma |K = 2\rangle.
\]

(2)

Note that $K$ is the projection of the angular momentum with respect to the intrinsic coordinate system, and $\tan 2\Gamma = 2\sqrt{3}G/F$ specifies the mixing of these basis configurations.

The E2 properties of the lowest $2^+$ states can be expressed fairly simply as

\[
B(E2; 2^+_1 \to 0^+_1) = \frac{Q_0^2}{16\pi} \cos^2(\gamma + \Gamma),
\]

\[
B(E2; 2^+_2 \to 0^+_1) = \frac{Q_0^2}{16\pi} \sin^2(\gamma + \Gamma),
\]

\[
B(E2; 2^+_2 \to 2^+_1) = \frac{5Q_0^2}{56\pi} \sin^2(\gamma - 2\Gamma),
\]

\[
Q(2^+_1) = -\frac{2}{7} Q_0 \cos(\gamma - 2\Gamma) = -Q(2^+_2).
\]

(3)
Here $Q_0$ is the static quadrupole moment, and $\gamma$ is a parameter related to the nuclear quadrupole deformation. Adding these to the three independent inertia parameters gives a total of five parameters in the model.

Note from the last line of eq.(3) that a $Q(2^+_1) = -Q(2^+_2)$ correlation is unconditionally conserved in the TRM.

2.2. The AHV

In the AHV description of nuclei [4], the model space of $2^+$ states is constructed from one- and two-phonon excitations of the phonon vacuum $|0\rangle$ (the $0^+$ ground state of the system). We will denote them as $|1\rangle$ and $|2\rangle$, respectively.

In this model space, the Hamiltonian matrix for $2^+$ states is given by

$$H^{2^+}_{AHV} = \begin{pmatrix} \hbar \omega & \lambda \\ \lambda & 2\hbar \omega \end{pmatrix},$$

where $\hbar \omega$ is the phonon energy and $\lambda$ defines the mixing between one- and two-phonon configurations. The eigenstates of such a Hamiltonian are given by

$$|2^+_1\rangle = a_1|1\rangle + a_2|2\rangle, \quad |2^+_2\rangle = -a_2|1\rangle + a_1|2\rangle,$$

where $a_1^2 + a_2^2 = 1$.

The E2 operator in the model is $\hat{Q} = \chi (\hat{b}^\dagger + \hat{b})$, in terms of phonon creation and annihilation operators and a free parameter $\chi$. Quadrupole properties in the model can be summarized as

$$B(E2; 2^+_1 \rightarrow 0^+_1) = \frac{\chi^2 a_1^2}{5} \langle 0|\hat{b}^\dagger\hat{b}|1\rangle^2,$$

$$B(E2; 2^+_2 \rightarrow 0^+_1) = \frac{\chi^2 a_2^2}{5} \langle 0|\hat{b}^\dagger\hat{b}|1\rangle^2,$$

$$B(E2; 2^+_2 \rightarrow 2^+_1) = \frac{\chi^2 (a_1^2 - a_2^2)^2}{5} \langle 1|\hat{b}^\dagger\hat{b}|2\rangle^2,$$

$$Q(2^+_1) = \frac{8\chi a_1 a_2}{5} \sqrt{\frac{2\pi}{7}} \langle 1|\hat{b}^\dagger\hat{b}|2\rangle = -Q(2^+_2).$$

Here too we find the desired quadrupole moment correlation, independent of the spectral behavior. However, we should emphasize that this only arises when we ignore multi-phonon states. The fact that data suggests that the quadrupole moment correlation exists is an indication that multi-phonon mixing should not be too important.

3. Systematic Comparison with Data

3.1. Spectral properties

Diagonalization of Eq. (1) for given values of $A$, $F$ and $G$ provides excitation energies of $2^+_1$, $2^+_2$ and $4^+_1$ states in the TRM approach. We refer to those nuclei for which the approach leads to a fit to the data [12] as TRM-solvable nuclei and present in Fig. 1 the $R = E(4^+_1)/E(2^+_1)$ distribution of these 203 nuclei. As is evident from the figure, the distribution spreads over the whole $R > 2$ region, indicating that the TRM can indeed provide non-rotor-like spectra, while still maintaining the rotational quadrupole moment correlation. We also note that there are two peaks in the distribution. The peak near $R = 10/3$ peak corresponds to rotational nuclei with nearly perfect axially-symmetric deformation. The peak near $R = 5/2$ corresponds to $\gamma$-unstable nuclei, or nuclei near the $SO(6)$ limit of the IBM [13].
Figure 1. \( R = E(4^+_1)/E(2^+_1) \) distribution of the 203 TRM-solvable nuclei in the ENSDF. Peaks for \( \gamma \)-instability (\( R \approx 5/2 \)) and an axially-symmetric rotor (\( R \approx 10/3 \)) are highlighted.

The spectral structure of nuclei within the AHV approach is simpler than for the TRM. Since the \( 2^+_1 \) and \( 2^+_2 \) states arise from mixing one- and two-phonon configurations, the energy of the lowest \( 2^+_1 \) state must be below the one-phonon excitation energy, \( \hbar \omega \), whereas the energy of the second \( 2^+_1 \) state must be above \( 2\hbar \omega \). Fig. 2 shows all available experimental data on nuclei with \( E(2^+_2) \) versus \( E(2^+_1) \) in the range \( R = 2.05 \sim 3.15 \), nuclei that were assigned to the AHV by Casten et al. This ensemble includes 177 nuclei in the ENSDF. Clearly most such nuclei sit above the \( E(2^+_2) = 2E(2^+_1) \) line. This suggests that the ensemble of \( R = 2.05 \sim 3.15 \) nuclei is indeed a reasonable sample of AHV-applicable nuclei, in agreement with the classification of Casten et al. [14].

3.2. \( E2 \) collectivity

We now focus on correlations between \( E2 \) transition rates and the quadrupole moments of the two lowest \( 2^+_1 \) states in the two models. From earlier formulae presented for TRM we can obtain the following approximate relation between the \( \text{BE}2 \) values and quadrupole moments of \( 2^+_1 \) states:

\[
B(E2; 2^+_1 \rightarrow 0^+ + 1^+) + B(E2; 2^+_2 \rightarrow 0^+ + 1^+) \approx 0.7 B(E2; 2^+_1 \rightarrow 2^+_1) + 0.244 Q^2(2^+_1). \tag{7}
\]

For the AHV, we can derive an analogous formula:

\[
B(E2; 2^+_1 \rightarrow 0^+ + 1^+) + B(E2; 2^+_2 \rightarrow 0^+ + 1^+) \approx 0.5 B(E2; 2^+_2 \rightarrow 2^+_1) + 0.174 Q^2(2^+_1). \tag{8}
\]

Both are in the general form of Kumar-Cline sum rules and can be more generally written as

\[
B(E2; 2^+_1 \rightarrow 0^+ + 1^+) + B(E2; 2^+_2 \rightarrow 0^+ + 1^+) = c_1 B(E2; 2^+_1 \rightarrow 2^+_1) + c_2 Q^2(2^+_1), \tag{9}
\]

where \( c_1 \) and \( c_2 \) are free variables.

We perform a multiple linear fitting of this formula (9) to all available experimental data from the ENSDF with \( c_1 \) and \( c_2 \) as fitting parameters. A total of 78 nuclei are considered in the fit of the left side to the right side, with the correlation coefficient \( r = 0.980 \) that emerges
being very close to 1. This confirms that B(E2) values between ground states and low-lying 2\(^+\) states are highly correlated with \(Q(2_1^+)\) in the ENSDF, as expected for the two models under consideration. The best-fit results are \(c_1 = 0.479\) and \(c_2 = 0.188\), substantially closer to the AHV values.

Using Eqs. (7) for the TRM and (8) for the AHV, we estimate the magnitudes of the quadrupole moments \(Q(2_1^+)\) from the experimentally available B(E2) values. In Fig. 3, we plot the \(|Q(2_1^+)|\) values that emerge in comparison with the associated experimental values. For both models, the data points of both models scatter fairly closely around the diagonal line, confirming that both the TRM and AHV approaches provide meaningful global descriptions of low-lying E2 collectivity.

3.3. Other collective observables
We have also analyzed magnetic moments of the two lowest 2\(^+\) states within the TRM and AHV frameworks. We summarize here the key results, with further details available in Ref. [6].

- In the AHV approach with mixing between one- and two-phonon configuration only, another exact correlation emerges, namely that \(\mu(2_1^+) = \mu(2_2^+)\). Experimentally, this correlation is borne out extremely well for most of the AHV-applicable nuclei.
- In the TRM approach, the \(\mu(2_1^+) = \mu(2_2^+)\) correlation that is present in experimental data only emerges if we assume that \(g_R = g_K\). When the accepted g-factors and the parameters emerging from the fits to energies described earlier are used, one typically finds \(\mu(2_1^+) \approx 0.8\mu(2_2^+)\) in the TRM.
- Thus, as for quadrupole moments, magnetic moments seem to be somewhat better described by the AHV approach than the TRM approach.

4. What next?
We have seen that both the TRM and the AHV provide a fairly robust description of excitation energies, E2 collective features and magnetic moments, especially for collective 2\(^+\) states.
Figure 3. Plot of the estimated $|Q(2^+_1)|$ values based on the use of experimental B(E2) values according to Eqs. (7) and (8), against the experimental data, for the 78 nuclei with available data in the ENSDF. The diagonal line is a measure of the quality of the estimates.

Furthermore, in both cases, the $2^+$ states are described in terms of a two-state basis. Since both models do a reasonable job of describing the same set of properties in terms of 2x2 matrices, there should be an orthogonal transformation for each such nucleus,

$$U = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.$$  \hfill (10)

such that

$$H_{2+}^{TRM} = U H_{2+}^{AHV} U^T.$$  \hfill (11)

Is there an intrinsic relation between the TRM and the AHV models for realistic nuclei? If so, it would seem that there should be a unique orthogonal transformation that relates the two bases, namely the K basis of the TRM and the phonon basis of the AHV, \textit{viz.}

$$\begin{bmatrix} K = 0 \\ K = 2 \end{bmatrix} = U \begin{bmatrix} |1\rangle \\ |2\rangle \end{bmatrix}.$$  \hfill (12)

To see whether the U matrix, i.e. the $\theta$ angle, is globally unique or robust for realistic nuclei, we calculate the distribution of $\theta$ values. We do this in such a way as to remove any bias due to the fact that the $\theta$ value for a given nucleus should most likely be correlated with the $R$ value for that same nucleus.

We will not discuss here the details of how we do this, referring the reader to [6]. We should note, however, that we only get info on $|\theta|$, since $\theta$ and $-\theta$ define the same transformation.

The resulting distribution of $|\theta|$ values, denoted $P(|\theta|)$, is shown in Fig. 4 for the 203 nuclei in the ENSDF that are TRM- and AHV-solvable. What we see is that there is a single dominant
peak, centered near $|\theta| = 34.9^\circ$, suggesting that the bases used in the two sets of analyses (for $2^+$ states) seem to a good approximation be the same.

This raises the question of whether the underlying physics of quadrupole collective motion may have a deeper origin, with the TRM and AHV descriptions contained within a richer model of quadrupole collectivity.

As an illustration of what we have in mind, we have studied the behavior of the IBM-1 [13], but with just two bosons. As a reminder, the IBM-1 is a model of $s$ ($L = 0$) and $d$ ($L = 2$) bosons, whereby it is governed by a U(6) symmetry. The model has three dynamical symmetry limits, each describing a distinct types of quadrupole collectivity. For a perfect harmonic vibrator, the U(5) symmetry limit applies, with one-phonon and two-phonon states involving one and two $d$ bosons, respectively. If we mix these states, we obtain the AHV model described earlier. If the mixing is driven by an SU(3) Hamiltonian, however, we find that a unique linear transformation of such phonon states provides two rotational $2^+$ states, as defined in Eq. (10). We have calculated the transformation angle between the states of this basis and the lowest two $2^+$ states of the SU(3) basis numerically, and find a rotation angle of $\theta = 28.1^\circ$, not too far from the $35^\circ$ we found that relates the TRM and AHV bases fairly robustly. Of course, the TRM does not derive from the SU(3) limit of the IBM-1, which is an axially symmetric rotor, so that we cannot meaningfully compare the two angles that emerged. Nevertheless, this illustrates how two distinct models of nuclear collectivity can be a reflection of a richer model that contains both (and perhaps others).

5. The global quadrupole moment correlation in the IBM

In his article in which the global correlation $Q(2^+1) = -Q(2^+2)$ was first introduced, Allmond [1] pointed out that the correlation seems to be reproduced fairly generally in the IBM-1 in the context of a consistent-Q treatment. Numerical results were, however, not presented. Here

![Figure 4. $R$-normalized $P(|\theta|)$ distribution (square points). The error bar represents the statistical error. The peak fit (solid line) has its center at $|\theta| = 34.9(2)^\circ$, as highlighted. This figure uses the same statistical ensemble as in Fig. 1.](image-url)
we briefly expand on those considerations by presenting numerical results under a variety of different scenarios, now using the IBM-2.

In Table 1, we present results at or near a variety of symmetry limits. The results denoted SU(3), SU(5) and SU(3)* refer to symmetry limits. Both SU(3) and SU(5) refer to symmetric scenarios and could be obtained alternatively using an IBM-1 code. The SU(3)* results, which refers to a pure triaxial rotor [15], required a full IBM-2 treatment. In the exact O(6) limit, for which $\chi_\pi = \chi_\nu = 0$, the quadrupole moments are identically zero. Thus we consider a slight breaking of the symmetry (denoted by $\delta$) by including a small amount of either SU(3) mixing or SU(3)* mixing. We do this by setting $\chi_\pi = 0.10$, $\chi_\nu = -0.10$ (for SU(3)* mixing).

|               | $Q(2^+_1)/Q(2^+_2)$ |
|---------------|---------------------|
| SU(3)        | -1.20               |
| SU*(3)       | -1.00               |
| U(5)         | -2.28               |
| SO(6) $+$ $\delta$(SU(3)) | -1.04               |
| SO(6) $+$ $\delta$(SU*(3)) | -1.04               |

Table 1. Selected results for the ratio of quadrupole moments of the lowest two $2^+$ states in the IBM-2 under a variety of conditions. All results are for $N_\pi = 3$ and $N_\nu = 5$ bosons.

We should note that we have also carried out calculations for the slightly larger boson numbers $N_\pi = 5$ and $N_\nu = 7$ to see whether the boson number dramatically impacted the results. Though the precise numerical results changed slightly, the same general features emerged.

In Table 2, we explore the transition between vibrational and deformed nuclei, through the use of a parametrized IBM-2 Hamiltonian

$$H = (1 - \xi)(n_{d_\pi} + n_{d_\nu}) - \xi \frac{N_\pi + N_\nu}{N_\pi N_\nu} Q_\pi \cdot Q_\nu ,$$

with $\xi$ varying from 0 to 1. Note that when $\xi = 1$, the Hamiltonian produces rotational motion, which smoothly evolves to vibrational motion as $\xi$ is decreased to 0. At $\xi \approx 0.2$, a shape phase transition takes place and the spectrum involves significant level crossings. Thus, at this point we show the ratio of $Q(2^+_1)/Q(2^+_2)$ to reflect the $2^+$ states with the same structure as prior to the transition. As is evident from the table, prior to the shape transition the ratio of quadrupole moments is roughly $-7/3$, as is typical of vibrational nuclei. After the shape transition, the ratio quickly falls to a value typical of rotational nuclei.

All results are compatible with $Q(2^+_1)$ and $Q(2^+_2)$ having opposite signs and roughly the same magnitudes (typically within a factor of roughly 2). However, as was also noted by Allmond, in the rotational scenario it is critical that the second $2^+$ state not be part of a $\beta$ band for this to emerge.
Table 2. Results for the ratio of quadrupole moments of the lowest two $2^+$ states in the IBM-2 across the transition from vibrational to rotational nuclei, as produced by the Hamiltonian in Eq. 13. All results are for $N_\pi = 3$ and $N_\nu = 5$ bosons. At $\xi = 0.2$, the results denoted by a * refer to $Q(2^+_1)/Q(2^+_4)$.

| $\xi$ | $Q(2^+_1)/Q(2^+_4)$ |
|-------|----------------------|
| 0.0 | -2.33 |
| 0.1 | -2.30 |
| 0.2* | -2.31 |
| 0.3 | -1.46 |
| 0.4 | -1.27 |
| 0.5 | -1.24 |
| 0.6 | -1.23 |
| 0.7 | -1.25 |
| 0.8 | -1.20 |
| 0.9 | -1.21 |
| 1.0 | -1.21 |

6. Summary
In this article, we discussed two phenomenological models of nuclear structure, the TRM and the AHV, both of which have the feature that they analytically satisfy the property $Q(2^+_1) = Q(2^+_2)$ that has been found globally for nuclei exhibiting quadrupole collectivity. We have shown that both of these models can describe systematic correlations between excitation energies and E2 collectivity, especially for the lowest two $2^+$ states, across a wide range of the periodic table, and that they provide in general good overall agreement with the associated experimental data. We also find that the TRM and AHV Hamiltonian matrices can be connected by an orthogonal transformation that seems to be roughly the same for most nuclei. This suggests that the TRM and AHV, though seemingly very different models of nuclear collective behavior, seem to share the same model space for realistic nuclear systems, and perhaps are just two aspects of a more general model of quadrupole collective behavior. The Neutron-Proton Interacting Boson Model (IBM-2) seems a natural framework in which to explore this further.

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