Model identification in $\mu^- \rightarrow e^- \ X$ conversion with invisible boson emission using muonic atoms

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(Dated: May 19, 2020)

In this article, we investigate the $\mu^- \rightarrow e^- \ X$ process in a muonic atom, where $X$ is a light neutral boson. By calculating the spectrum of the emitted electron for several cases, we discuss the model-discriminating power of the process. We report the strong model dependence of the process near a high-energy endpoint. Our findings show that future experiments using muonic atoms are helpful to identify the properties of exotic bosons.

I. INTRODUCTION

Though the standard model (SM) of particle physics is consistent with almost all experimental data, it still leaves many unanswered questions: the existence of dark matter, the origin of the neutrino masses, and so on. To build physics beyond the SM, physicists have searched for direct or indirect clues for a long period of years. Since we have many candidates for the SM extension, we need to try various complementary methods to probe the effects of new physics. Interestingly, several candidates predict light particles that interact feebly with the SM particles. For the feebly-interacting light particles, it is preferable to take a different approach from heavy particle searches.

If there is such a neutral boson $X$ with a mass smaller than a muon mass $m_{\mu} = 105.658$ MeV, the boson $X$ induces an exotic muon decay $\mu \rightarrow eX$. In fact, some promising phenomenological models include a new particle whose mass is of MeV or less and which induces the lepton flavor violation: e.g., light scalars such as majorons, familons, and axion-like particles [1–8], or light extra gauge bosons [9–11]. To investigate them generally, the authors of Ref. [12] carried out a comprehensive study about $\ell \rightarrow \ell 'X$ processes where the emitted $X$ decays into lighter SM particles like an electron-positron pair or a photon pair.

Let us consider cases that the $X$ has a sufficiently long lifetime or decays into invisible particles. The general searches for the two-body muon decay $\mu^+ \rightarrow e^+X$ have been performed in some experiments. Even if we do not care about the decay property of the $X$, we can search for its trace by careful measurement of a positron energy spectrum in the muon decay. Let $m_X$ be the mass of $X$, and you find the spectrum enhanced at $E_e \simeq (m_{\mu}^2 - m_X^2) / (2m_{\mu})$. An inevitable background on this kind of search is positrons emitted from the ordinary muon decay, $\mu^+ \rightarrow e^+\nu_\mu\bar{\nu}_\mu$, which is especially serious for a small $m_X$. To suppress this background, the authors of Ref. [13] accumulated $1.8 \times 10^7$ polarized positive muons and counted emitted positrons in the opposite direction to the polarization of muons. As a result, they concluded that the constraint for the branching ratio was $Br(\mu^+ \rightarrow e^+X) < 2.6 \times 10^{-6}$, assuming that the momentum distribution of signal positrons is spherically symmetric and the $X$ is massless. Under this assumption, this constraint is still more stringent than those of any other experiments. In 2015, the TWIST experiment [14] reported the latest search for $\mu^+ \rightarrow e^+X$. They analyzed $5.8 \times 10^8$ muons and obtained the branching ratio limits of $O(10^{-5})$ for various decay asymmetries and masses of $13 \text{MeV} < m_X < 80 \text{MeV}$. In the near-future, Mu3e collaboration is going to investigate $\mu^+ \rightarrow e^+X$ with sensitivity of $Br \sim O(10^{-8})$. According to [15,16], the explorable mass region of the search is $25 \text{MeV} < m_X < 95 \text{MeV}$. This lower restriction comes from the difficulty of calibration due to the steep edge of the background spectrum, and the significant update of the constraints for $m_X \lesssim 25 \text{MeV}$ would be challenging.

A different method to investigate the $\mu \rightarrow eX$ process is to use muonic atoms instead of free muons, which was proposed in Ref. [18]. According to the literature [18], coming experiments using muonic atoms, such as COMET [19] and Mu2e [20], could explore the $\mu \rightarrow eX$ process at the same level as the past experiments using free muons.

One expected advantage of muonic atoms is to evade the background problem we mentioned above. The signal energy is monochromatic in the decay of a free muon, while the electron energy spectrum in the decay of a muon in orbit has a finite width because of the nuclear recoil. This fact allows us to search for the signal in a preferable energy region where the signal-to-background ratio is large. In the special case of a small $m_X$, the maximum energy of the signal is close to the signal energy of the $\mu^- \rightarrow e^- \ X$ conversion, which is the main purpose of the COMET and Mu2e experiments. This means that the electron detector for the $\mu^- \rightarrow e^- \ X$ conversion is also optimized for the $\mu^+ \rightarrow e^+X$ search. Thus the searches for $\mu^- \rightarrow e^- X$ using muonic atoms will be complementary to searches using free muon decays.
Another merit of muonic atoms is that the shape and the nuclear dependence of the electron spectrum are available to obtain detailed information on new physics. The model identification by measuring such characteristic observables has been discussed in another lepton-flavor-violating process, $\mu^- e^- \rightarrow e^- e^-$ in a muonic atom.\textsuperscript{21, 23} Despite its importance, no one has studied the model dependence of observables in the $\mu^- \rightarrow e^- X$ process.

Our goal of this article is to understand the model-discriminating power of the $\mu^- \rightarrow e^- X$ process in a muonic atom. For a simple discussion of the model dependence, we introduce three effective models in Sec. III. Then, we formulate the rate of $\mu^- \rightarrow e^- X$ in a nuclear Coulomb potential. In Sec. IV, we show numerical results and discuss the model dependence of observables. Finally, we summarize this article in Sec. V.

II. FORMULATION

In this section, we formulate the spectrum of an emitted electron from the $\mu^- \rightarrow e^- X$ process in a muonic atom. Here, we assume a boson $X$ lighter than muons. To investigate the model dependence, we consider three simple effective models, called $S_0$, $S_1$, and $V_1$, as follows:

First, we assume that $X$ is a scalar field and the effective interaction Lagrangian to charged leptons is given as

$$L_{S_0} = \mathcal{X} \pi \left( g_{L}^{S_0} P_L + g_{R}^{S_0} P_R \right) \mu + [H.c.] ,$$  

where $P_{L/R} = (1 \mp \gamma_5)/2$ is a projection operator, and $g_{L/R}^{S_0}$ are dimensionless coupling constants. This type of Lagrangian was also analyzed in Ref. \textsuperscript{12, 18}. In this model, keeping an electron mass $m_e = 0.510999$ MeV, we find the rate of the exotic free muon decay $\mu \rightarrow e X$ to be

$$\Gamma_0 = \frac{m_\mu}{32\pi} \Delta (1, r_X^2, r_e^2) \left\{ \left( |g_{L}^{S_0}|^2 + |g_{R}^{S_0}|^2 \right)^2 \right\} (1 - r_X^2 + r_e^2) + 4 r_e r_X \text{Re} \left( g_{L}^{S_0} g_{R}^{S_0 \ast} \right) ,$$  

where $r_X = m_X/m_\mu$, $r_e = m_e/m_\mu$, and

$$\Delta (x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2xy - 2yz - 2xz} .$$

Multiplying it with the lifetime of muon $\tau_\mu = 192\pi^3/(G_F^2 m_\mu^5)$, where $G_F = 1.166 \times 10^{-5}$ GeV$^{-2}$ is the Fermi coupling constant, we obtain the branching ratio for the free muon, $Br(\mu \rightarrow e X) = \tau_\mu \Gamma_0$. For reference, suppose that $g_{L}^{S_0} = g_{R}^{S_0}(= g_{S_0})$ and $m_X = 0$. Then, using $Br < 2.6 \times 10^{-6}$,\textsuperscript{13} we obtain the constraint for the coupling constant,

$$|g_{S_0}|^2 < 3.7 \times 10^{-22} .$$

Second, we assume the following derivative coupling for the scalar $X$,

$$L_{S_1} = -i \partial^a \left( \frac{\mathcal{X} \pi}{\Lambda_{S_1}} \gamma_\alpha \left( g_{L}^{S_1} P_L + g_{R}^{S_1} P_R \right) \mu + [H.c.] \right) ,$$

where $\Lambda_{S_1}$ is an arbitrary energy scale to keep coupling constants $g_{L/R}^{S_1}$ dimensionless. The rate of the free muon decay is given as

$$\Gamma_0 = \frac{m_\mu}{32\pi} \Delta (1, r_e^2, r_X^2) \left( \frac{m_\mu}{\Lambda_{S_1}} \right)^2 \left\{ \left( |g_{L}^{S_1}|^2 + |g_{R}^{S_1}|^2 \right)^2 \right\} (1 - r_e^2 + r_X^2) + 4 r_e r_X \text{Re} \left( g_{L}^{S_1} g_{R}^{S_1 \ast} \right) .$$

Now we mention that, when both leptons are free and on mass shell, Eq. (5) is effectively equivalent to Eq. (1) due to the Dirac equation, $(i\partial - m) \psi = 0$. Here, we have the relation of coupling constants given as

$$g_{L/R}^{S_0} = \frac{1}{\Lambda_{S_1}} \left( m_\mu g_{L/R}^{S_1} - m_e g_{L/R}^{S_1} \right) .$$

Applying the relation, we easily prove the equality of Eqs. (2) and (6). However, Eq. (4) no longer holds in a Coulomb potential. For the process in a muonic atom, it is worth investigating quantitative differences of observables between the two models.

Third, in addition to the scalar cases, we consider another case that $X$ is a vector field and the effective interaction is given as

$$L_{V_1} = \frac{\mathcal{X} \pi}{2\Lambda_{V_1}} \sigma_{\alpha\beta} \left( g_{L}^{V_1} P_L + g_{R}^{V_1} P_R \right) \mu + [H.c.] ,$$

where $\sigma_{\alpha\beta}$ is the Pauli matrix.
where $X^{\alpha\beta} = \partial^\alpha X^\beta - \partial^\beta X^\alpha$ is the field strength of the $X$. The couplings $g_{Li/R}$ are dimensionless again due to the arbitrary scale $\Lambda_{V_i}$. As with the previous models, the decay rate for free muon is given as

$$
\Gamma_0 = \frac{m_\mu}{32\pi} \Delta \left( 1, r_0^2, r_0^2 \right) \left( \frac{m_\mu}{\Lambda_{V_i}} \right)^2 \left\{ \left| g_{Li}^{\nu} \right|^2 + \left| g_{Ri}^{\nu} \right|^2 \right\} \{ 2 - r_0^2 - r_0^2 (4 + r_0^2) + 2r_0^4 \} - 12r_0^2 \text{Re} \left[ g_{Li}^{\nu} g_{Ri}^{\nu*} \right].
$$

(9)

Next, we formulate the rate of the $\mu^- \rightarrow e^- X$ process in a muonic atom. We assume the independent particle model of a muonic atom and an initial muon in a $1s$ orbit. We define the transition amplitude $\mathcal{M}$ as

$$
2\pi \delta(E_X + E_e - m_\mu^*) \mathcal{M} = \left\langle \phi_{1s}, X_{pX} \bigg| \int d^4x \mathcal{L}_M \bigg| \phi_{1s} \right\rangle,
$$

(10)

where we take only the leading order of effective interaction. For simplicity, we omit the spin indices. Here, $E_X$ and $E_e$ are the energies of the emitted $X$ and electron in the final state, respectively. $m_\mu^* = m_\mu - B_{\mu N}^{1s}$ indicates the energy of the bound muon, where $B_{\mu N}^{1s}$ is the binding energy between the nucleus and muon in a $1s$ state. The $M$ connects to the decay rate by

$$
d\Gamma = \frac{d^3p_e}{(2\pi)^3 2E_e} \frac{d^3p_X}{(2\pi)^3 2E_X} (2\pi) \delta \left( E_X + E_e - m_\mu^* \right) \frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2.
$$

(11)

The factor of $1/2$ comes from the spin average of the initial bound muon.

The transition amplitude $\mathcal{M}$ includes the overlap integrals of lepton wave functions that are solutions of the Dirac equation with the nuclear Coulomb potential $\left[ 24, 25 \right]$. In the central force system, one can represent the wave function of the bound muon as

$$
\psi_{1s}^\mu (r) = \left( \begin{array}{c} G(r) \chi_{\nu_s}^{s_{\mu}}(\hat{r}) \\ iF(r) \chi_{\nu_s+1}^{s_{\mu}}(\hat{r}) \end{array} \right),
$$

(12)

with a normalization condition

$$
\int d^3r \overline{\psi}_{\mu s}^\mu (r) \psi_{\mu s'}^\mu (r) = \delta_{s_s, s'}.
$$

(13)

The angular parts $\chi$ are two-component spinors, which is determined analytically. Furthermore, we obtain the radial part and the binding energy by solving an eigenvalue problem for the radial Dirac equations,

$$
\frac{d}{dr} \left( G(r) \right) = \left( \begin{array}{cc} 0 & E_\mu + m_{\mu N} + eV_C(r) \\ -E_\mu + m_{\mu N} - eV_C(r) & -2/r \end{array} \right) \left( \begin{array}{c} G(r) \\ F(r) \end{array} \right).
$$

(14)

The nuclear Coulomb potential $V_C$ in the equations is given as

$$
V_C(r) = \int_0^\infty dr' r'^2 \left[ \theta (r - r') \frac{1}{r} + \theta (r' - r) \frac{1}{r'} \right] \rho (r'),
$$

(15)

with a nuclear charge density $\rho(r)$. Here, we use the reduced mass $m_{\mu N} = m_\mu m_N / (m_N + m_\mu)$ with a nuclear mass $m_N$. After obtaining the solution where $E_\mu$ is minimized, we determine the binding energy of the $1s$ state by $B_{\mu N}^{1s} = m_{\mu N} - E_\mu$ $\left[ 20 \right]$.

For the electron in the final state, it is convenient to use the multipole expansion of the state with momentum $p_e$. The electron scattering state with the incoming boundary condition is expressed as follows:

$$
\psi_{e, p_e}^{s_{\mu}} (r) = \sum_{\kappa, j_{\kappa}, m} 4\pi i s_{\kappa} (l_{\kappa}, m, 1/2, s_{\kappa} | j_{\kappa}, \nu) Y^{m_{\mu}}_{l_{\kappa}}(\hat{p}_e) e^{-i\delta_{e}} \left( \frac{g_{E_{\mu}}^{\nu} (r) \chi_{\nu}^{s_{\kappa}}(\hat{r})}{i F_{E_{\mu}}^{\nu} (r) \chi_{s_{\kappa}}^{s_{\mu}}(r)} \right),
$$

(16)

with the Clebsch-Gordan coefficients, $(l_{\kappa}, m, 1/2, s_{\kappa} | j_{\kappa}, \nu)$, and spherical harmonics, $Y^{m_{\mu}}_{l_{\kappa}}(\hat{p}_e)$. Here, $\kappa$ is a nonzero integer to label partial waves. For the index $\kappa$, the total angular momentum $j_{\kappa}$ and the orbital angular momentum $l_{\kappa}$ are determined by

$$
j_{\kappa} = |\kappa| - \frac{1}{2},
$$

(17)

$$
l_{\kappa} = j_{\kappa} + \frac{1}{2} |\kappa|.
$$

(18)
\( \delta_\kappa \) is the phase shift of a partial wave labeled by \( \kappa \). To obtain the radial wave functions for a given \( E_e \) and \( \kappa \), we solve
\[
\frac{d}{dr} \left( \frac{g_{E_e}^E(r)}{f_{E_e}^E(r)} \right) = \left( -E_e + m_{eN} - eV_C(r) \right) \frac{d}{dr} \left( \frac{g_{E_e}^E(r)}{f_{E_e}^E(r)} \right).
\] (19)

The normalization is taken to be
\[
\int d^3r \psi_{e,p_e}^* \psi_{e,p_e} = 2E_e (2\pi)^3 \delta(3) (p_e' - p_e) \delta_{s,s'}.
\] (20)

Using the expressions of the effective interactions, we find that the electron spectra for the three models \((M = S_0, S_1, V_1)\) are universally represented as
\[
\frac{d\Gamma}{dE_e} = \sqrt{E_e^2 - m_e^2} \sqrt{E_e^2 - m_{eN}^2} \sum_\kappa (2j_e + 1) \left\{ \left( |g_L^M| + |g_R^M|^2 \right) \left( P_\kappa^M + \overline{P}_\kappa^M \right) + 2Re \left[ g_L^M g_R^M \right] \left( P_\kappa^M - \overline{P}_\kappa^M \right) \right\},
\] (21)
where \( E_X \) is a function of \( E_e \) determined by the energy conservation. To take into account nuclear recoil through \( E_X \), we apply the well-known prescription as follows \([18, 27, 28]\):
\[
E_X = m_{eN} - E_e \to m_{eN} - E_e - \frac{E_e^2}{2m_{eN}}.
\] (22)

This additional term represents the kinetic energy of the recoiled nucleus, and the term is sizable only at high \( E_e \) but negligible at low \( E_e \). Thus, even though we do not completely consider the nuclear motion, we believe that this treatment yields a good approximation for any \( E_e \).

After straightforward calculation, we obtain the explicit formulas for \( P_\kappa^M \) and \( \overline{P}_\kappa^M \). For \( M = S_0 \), it is found that
\[
P_{\kappa}^{S_0} = \left| I_{jG}^{\kappa,(L)} - I_{jF}^{\kappa,(L)} \right|^2,
\] (23)
\[
\overline{P}_{\kappa}^{S_0} = \left| I_{jG}^{\kappa,(L+)} + I_{jF}^{\kappa,(L+)} \right|^2.
\] (24)

Here, we define the overlap integral, \( I_{jH}^{\kappa,(L)} \) \((h = g, f \text{ and } H = G, F)\), as
\[
I_{jH}^{\kappa,(L)} = \int_0^\infty drr^2 j_L \left( \sqrt{E_X^2 - m_{eN}^2} r H(r) \right) h_{E_e}^\kappa(r).
\] (25)

where \( j_l \) is the \( l \)-order spherical Bessel function. \( h \) indicates the radial wave function of the scattering electron, and \( H \) indicates that of the bound muon. This formula for \( S_0 \) is consistent with that in Ref. \([18]\). More complicated expressions for \( P_{\kappa}^{S_1}, \overline{P}_{\kappa}^{S_1}, P_{\kappa}^{V_1}, \) and \( \overline{P}_{\kappa}^{V_1} \) are given in Appendix A.

If we neglect the electron mass, we find that the components of the transition probability satisfy
\[
\overline{P}_{-\kappa}^M = P_{\kappa}^M,
\] (26)
which is valid regardless of \( M \). Due to this symmetry, the cross term between \( g_L^M \) and \( g_R^M \) disappears after summing over \( \kappa \). This observation is understandable because the interference between left- and right-handed components should vanish for the final electron if \( m_e = 0 \).

The endpoint energy \( E_{\text{end}}^{mx} \) of the electron spectrum is kinematically determined as
\[
E_{\text{end}}^{mx} = \frac{\left( m_N + m_e^* - m_X \right)^2 - m_N^2 + m_e^2}{2 \left( m_N + m_e^* - m_X \right)},
\] (27)
which is obtained by solving the relativistic relation of the energy-momentum conservation. Approximately, Eq. (27) is represented to
\[
E_{\text{end}}^{mx} \approx m_e^* - m_X - \frac{(m_e^* - m_X)^2}{2m_N},
\] (28)
where the third term is interpreted as the kinetic energy of the recoiled nucleus.
### III. NUMERICAL RESULTS

To obtain the radial wave functions of charged leptons and the binding energy of a muonic atom, we solve the differential equations, Eq. (14) for the initial muon and Eq. (19) for the final electron. In solving the differential equations, we use the fourth-order Runge-Kutta method. The correctness of our calculation code is numerically checked by comparing it with the analytic result for a point-charge density.

For reference, we focus on two kinds of nuclei as a target material. One is aluminum, $^{27}\text{Al}$, which will be used in the coming COMET and Mu2e experiments. The other is gold, $^{197}\text{Au}$, which was used in the SINDRUM II experiment [29]. For both nuclei, we assume the two-parameter-Fermi distribution as the nuclear charge density, given as

\[
\rho(r) = \frac{Z e}{4\pi} \frac{\rho_0}{1 + \exp \left(\frac{r - r_0}{a}\right)},
\]

where $Z$ is the proton number of the target nucleus and $e$ is the magnitude of the elementary charge. The parameters of the distribution, $r_0$ and $a$, are given in Table I and $\rho_0$ is a normalization factor. By solving Eq. (14), we obtain the values of the binding energy $B_{\mu N}^{1s}$ shown in Table I. Substituting the binding energy into Eq. (27), we find the endpoint energy $E_{\text{end}}^{m_X}$ for an arbitrary $m_X$. The values of $E_{\text{end}}^{m_X}$ for $m_X = 0, 25\text{MeV}, 50\text{MeV}$ are shown in Table II.

| Nuclei | $Z$ | $A$ | $m_X$ [MeV] | $r_0$ [fm] | $a$ [fm] | $E_{\mu}$ [MeV] | $B_{\mu N}^{1s}$ [MeV] |
|--------|-----|-----|-------------|-------------|----------|----------------|-----------------|
| $^{27}\text{Al}$ | 13 | 27 | 25133 | 2.845 | 0.569 | 104.75 | 10.46 |
| $^{197}\text{Au}$ | 79 | 197 | 183473 | 6.38 | 0.535 | 95.48 | 10.12 |

| Nuclei | $E_{\text{end}}^{0}$ [MeV] | $E_{\text{end}}^{25\text{MeV}}$ [MeV] | $E_{\text{end}}^{50\text{MeV}}$ [MeV] |
|--------|-----------------|-----------------|-----------------|
| $^{27}\text{Al}$ | 104.98 | 80.07 | 55.13 |
| $^{197}\text{Au}$ | 95.51 | 70.52 | 45.53 |

The left panel (a) of Fig. 1 shows the electron spectra for the aluminum nucleus. The spectra are normalized by the rate for a free muon, whose expression for each model is given in Sec. II. Here, we plot only the spectrum of $S_0$ model, because the differences between the models are too small to recognize in this energy scale. Each curve in Fig. 1 corresponds to $m_X=0, 25\text{MeV}, 50\text{MeV}$ are shown in Table II. In this figure, one can recognize the endpoint energy as a reference because the high-energy endpoint of $\mu^-\rightarrow e^-X$ conversion in an experiment which is optimized to detect high-energy electrons. Then, it is useful to focus on the spectrum near the high-energy endpoint. Hereafter we set $m_X=0$ as a reference because the high-energy endpoint of $\mu^-\rightarrow e^-X$ is close to the signal energy of the $\mu^-\rightarrow e^-\text{conversion}$. We plot the spectra for $^{27}\text{Al}$ in the range of $0.99 \leq E_e/E_{\text{end}}^{0} \leq 1$ in Fig. 2. In this figure, one can recognize the difference between the models of the boson X. In particular, the high-energy tail of the $V_1$ model, indicated by the dotted (green) curve, is larger than the others. This observation suggests that the analysis of the endpoint spectrum is more sensitive to the $V_1$ model than the others.

We should comment on the spectrum for the $S_0$ model, shown by the solid (red) curve in Fig. 2. One may find that the spectrum for the $S_0$ model is unnaturally suppressed near $E_e/E_{\text{end}}^{0} \approx 0.998$, which is clearly seen in (b) of Fig. 2. This happens due to the following three facts: First, the spectrum is dominated by the contribution of $\kappa = -1$ in Eq. (23). Second, $P_{X}^{S_0}$ vanishes when $E_e = E_{\mu}$ [27]. Third, $E_{\mu}$ is slightly smaller than $E_{\text{end}}^{0}$ due to the finite

TABLE I. Parameters for each nucleus and calculated energies. The forth and fifth columns are the parameters in Eq. (29), given by Ref. [31]. The sixth and seventh columns are $E_{\mu}$ and $B_{\mu N}^{1s}$ obtained by our calculation.

TABLE II. Endpoint energies $E_{\text{end}}^{m_X}$ for $m_X = 0, 25\text{MeV}, 50\text{MeV}$. The left panel (a) of Fig. 1 shows the electron spectra for the aluminum nucleus. The spectra are normalized by the rate for a free muon, whose expression for each model is given in Sec. II. Here, we plot only the spectrum of $S_0$ model because the differences between the models are too small to recognize in this energy scale. Each curve in Fig. 1 corresponds to $m_X=0, 25\text{MeV}, 50\text{MeV}$, which is clearly seen in (b) of Fig. 1. This is because the momentum uncertainty of the initial muon is large as the nucleus has a stronger Coulomb field.

We suppose that simultaneous searches for the $\mu^-\rightarrow e^-X$ process with the $\mu^-\rightarrow e^-\text{conversion}$ in an experiment which is optimized to detect high-energy electrons. Then, it is useful to focus on the spectrum near the high-energy endpoint. Hereafter we set $m_X=0$ as a reference because the high-energy endpoint of $\mu^-\rightarrow e^-X$ is close to the signal energy of the $\mu^-\rightarrow e^-\text{conversion}$. We plot the spectra for $^{27}\text{Al}$ in the range of $0.99 \leq E_e/E_{\text{end}}^{0} \leq 1$ in Fig. 2. In this figure, one can recognize the difference between the models of the boson X. In particular, the high-energy tail of the $V_1$ model, indicated by the dotted (green) curve, is larger than the others. This observation suggests that the analysis of the endpoint spectrum is more sensitive to the $V_1$ model than the others.

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nuclear mass. Organizing them, we notice that the main contribution of the spectrum vanishes at $E_e = E_\mu \lesssim E_{\text{end}}^0$, which is close to but smaller than $E_{\text{end}}^0$. This interesting property characterizes the $S_0$ model. In practice, after the confirmation of $X$, we need much more careful measurement to identify the spectrum shape.

Also, Fig. 2 shows the spectrum for $^{197}$Au in the range of $0.99 \leq E_e/E_{\text{end}}^0 \leq 1$. We find that the high-energy tail is much larger than $^{27}$Al. As with $^{27}$Al, the tail of the $V_1$ model is the largest of the three models. We cannot recognize the suppression of the spectrum near the endpoint for the $S_0$ model in $^{27}$Al case, because the nuclear mass $m_N$ is so heavy that $E_\mu$ is sufficiently close to $E_{\text{end}}^0$.

Finally, we discuss which nucleus is preferable for the $\mu^- \rightarrow e^- X$ search. Suppose that the new physics search using muonic atoms is performed by measuring the number of electrons with an energy close to the signal energy of $\mu^- \rightarrow e^- X$ conversion, which is equal to $E_{\text{end}}^0$. We define a net branching ratio as

$$Br_x(Z) = \tilde{\tau}_\mu \int_x^1 \frac{dE_e}{E_{\text{end}}^0} \frac{E_{\text{end}}^0}{\Gamma_{\text{end}}} d\Gamma_{\text{e}} dE_e,$$

(30)

where $\tilde{\tau}$ is the lifetime of a muonic atom, listed in Ref. [31]. This value corresponds to the number of electrons with $E_e \geq xE_{\text{end}}^0$ ($x < 1$) coming from $\mu^- \rightarrow e^- X$, normalized by the created number of muonic atoms. For further convenience, we define

$$R_x(Z) = \frac{\tilde{\tau}_\mu}{\tau_\mu} \int_x^1 \frac{dE_e}{E_{\text{end}}^0} \frac{E_{\text{end}}^0}{\Gamma_0} d\Gamma_{\text{e}} dE_e,$$

(31)

so that

$$Br_x(Z) = R_x(Z) Br(\mu^+ \rightarrow e^+ X).$$

(32)
FIG. 3. Spectra of the emitted electron for $^{197}\text{Au}$. See the caption of Fig. 2 for an explanation of the axes and curves.

FIG. 4. The $Z$ dependence of $R_{0.9}(Z)$ defined in Eq. (31). Sampled points are shown by crosses. For the simplicity of calculation, we use the uniform distribution with the nuclear radius of $1.2A^{1/3}\text{fm}$ as the nuclear charge density. We take the mass number $A$ of the most abundant isotope for each $Z$.

Setting $x = 0.9$, we find that $Z$ dependence of $R_{0.9}(Z)$ is shown in Fig. 4. One can see that the typical value of $R_{0.9}(Z)$ is $O(10^{-9} - 10^{-8})$. As larger nuclei, the lifetime of muonic atoms is shorter, but the high-energy tail of the electron spectrum gets larger. Due to the cancellation of the two effects [18], the $Z$ dependence of $R$ is not so strong above $Z \approx 30$. Considering the current experimental constraint of $\text{Br}(\mu^+ \rightarrow e^+ X)$, we find that the current upper limit of the net branching ratio is $\text{Br}_x(Z) < O(10^{-15} - 10^{-14})$. Since the goal of the created number of muons in the planned $\mu^- \rightarrow e^- X$ conversion searches [19, 20] is $O(10^{18})$, it would be possible to reach the constraint by the near-future muon sources.

IV. SUMMARY

We have investigated the $\mu^- \rightarrow e^- X$ process in muonic atoms as an interesting candidate to constrain the property of light neutral bosons. Assuming three simple effective models of the unknown boson, we have discussed the model dependence of the electron spectrum. As a result, we found that the spectrum near the endpoint strongly depends on the property of the boson $X$. We also showed that the nuclear dependence of the net branching ratio is moderate.

A remaining theoretical problem is to include radiative corrections in the calculation for the spectrum near the high-energy endpoint, which is shown to be important for ordinary decay of muon in orbit [32]. Although we need further studies for the realistic sensitivity of experiments, we believe that careful measurements for the electron spectrum in a muon decay are useful to find unknown invisible bosons and to identify their property.
ACKNOWLEDGMENTS

We thank Y. Kuno, C. Wu, T. Xing, J. Sato, and T. Sato for fruitful comments. This work was supported by the JSPS KAKENHI Grants No. 18H01210 and the Sasakawa Scientific Research Grant from the Japan Science Society.

Appendix A: Full expressions of the transition provability

We show the expressions for $P^M_\kappa$ and $\overline{P}^M_\kappa$ ($M = S_1, V_1$). For the $S_1$ model,

\[ P^S_\kappa = \frac{E^2_X}{\Lambda^2} \left| \sqrt{\frac{l_\kappa + 1}{l_\kappa}} \left( \frac{l_\kappa - 1 - \kappa}{2l_\kappa + 1} I^\kappa_{gG} + \frac{l_\kappa + 1 + \kappa}{2l_\kappa + 1} I^\kappa_{ff} \right) \right|^2 + \frac{\sqrt{E^2_X - m^2_X}}{E_X} \left( \frac{2 + l_\kappa + \kappa I^\kappa_{gG}(l_\kappa + 1)}{2l_\kappa + 1} - \frac{l_\kappa - \kappa}{2l_\kappa + 1} I^\kappa_{gG}(l_\kappa + 1) \right) \]

\[ \overline{P}^S_\kappa = \frac{E^2_X}{\Lambda^2} \left| \sqrt{\frac{l_\kappa + 1}{l_\kappa}} \left( \frac{l_\kappa - 1 + \kappa}{2l_\kappa + 1} I^\kappa_{gG} + \frac{l_\kappa + 1 - \kappa}{2l_\kappa + 1} I^\kappa_{ff} \right) \right|^2 + \frac{\sqrt{E^2_X - m^2_X}}{E_X} \left( \frac{2 + l_\kappa - \kappa I^\kappa_{gG}(l_\kappa + 1)}{2l_\kappa + 1} + \frac{l_\kappa + \kappa}{2l_\kappa + 1} I^\kappa_{gG} \right) \]

For the $V_1$ model,

\[ P^V_\kappa = \frac{E^2_X}{\Lambda^2} \left| \sqrt{\frac{l_\kappa + 1}{l_\kappa}} \left( \frac{l_\kappa - 1 - \kappa}{2l_\kappa + 1} I^\kappa_{gG} + \frac{l_\kappa + 1 + \kappa}{2l_\kappa + 1} I^\kappa_{ff} \right) \right|^2 + \frac{\sqrt{E^2_X - m^2_X}}{E_X} \left( \frac{1 - \kappa}{l_\kappa (l_\kappa + 1)} \left( I^\kappa_{gG}(l_\kappa) + I^\kappa_{ff} \right) \right)^2 \]

\[ + \frac{E^2_X}{\Lambda^2} \left| \sqrt{\frac{l_\kappa + 1}{l_\kappa}} \left( \frac{l_\kappa + 1 - \kappa}{2l_\kappa + 1} I^\kappa_{gG} + \frac{l_\kappa - 1 + \kappa}{2l_\kappa + 1} I^\kappa_{ff} \right) \right|^2 + \frac{\sqrt{E^2_X - m^2_X}}{E_X} \left( \frac{1 - \kappa}{l_\kappa (l_\kappa + 1)} \left( I^\kappa_{gG}(l_\kappa + 1) - I^\kappa_{ff}(l_\kappa) \right) \right)^2 \]

\[ + \frac{E^2_X}{\Lambda^2} \left| \sqrt{\frac{l_\kappa + 1}{l_\kappa}} \left( \frac{l_\kappa + 1 + \kappa}{2l_\kappa + 1} I^\kappa_{gG} + \frac{l_\kappa - 1 - \kappa}{2l_\kappa + 1} I^\kappa_{ff} \right) \right|^2 + \frac{\sqrt{E^2_X - m^2_X}}{E_X} \left( \frac{l_\kappa + 1 - \kappa}{2l_\kappa + 1} I^\kappa_{gG} - \frac{l_\kappa - 1 + \kappa}{2l_\kappa + 1} I^\kappa_{ff} \right) \]

\[ \left. + \frac{\sqrt{E^2_X - m^2_X}}{E_X} \left( \frac{l_\kappa + 1 - \kappa}{2l_\kappa + 1} I^\kappa_{gG} - \frac{l_\kappa - 1 + \kappa}{2l_\kappa + 1} I^\kappa_{ff} \right) \right|^2 \], (A3)
\[ T_\kappa = \frac{E^2}{\Lambda^2} \left[ \sqrt{\frac{\Lambda}{\kappa^2} \left( \frac{\kappa - 1}{\kappa} \right)^2 - \frac{1}{2l - \kappa + 1} \left( \frac{\kappa - 1 + \kappa f_G^{(l+1)}}{2l - \kappa + 1} \right)^2} \right. \]

\[ + \sqrt{\frac{\Lambda}{\kappa^2} \left( \frac{\kappa + 2 - \kappa f_G^{(l+1)}}{2l - \kappa + 1} \right)^2 - \frac{1}{2l - \kappa + 1} \left( \frac{\kappa + 2 - \kappa f_G^{(l+1)}}{2l - \kappa + 1} \right)^2 \sqrt{\frac{\Lambda}{\kappa^2} \left( \frac{\kappa - 1}{\kappa} \right)^2} \right] \]