GALOIS COHOMOLOGY AND COMPONENT GROUP OF A REAL REDUCTIVE GROUP

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To the memory of Arkadi L’vovich Onishchik

Abstract. Let $G$ be a connected reductive group over the field of real numbers $\mathbb{R}$. Using results of our previous joint paper, we compute combinatorially the first Galois cohomology set $H^1(\mathbb{R}, G)$ in terms of reductive Kac labelings. Moreover, we compute the group of connected components $\pi_0G(\mathbb{R})$ of the real Lie group $G(\mathbb{R})$ and the maps in exact sequences containing $\pi_0G(\mathbb{R})$ and $H^1(\mathbb{R}, G)$.

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Introduction

In this article, by an $\mathbb{R}$-group we mean an algebraic group, not necessarily linear or connected, over the field of real numbers $\mathbb{R}$. We denote $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$, the Galois group of $\mathbb{C}$ over $\mathbb{R}$, where $\gamma$ is the complex conjugation.
0.1. For the definition of the first (nonabelian) Galois cohomology set $H^1(G) := H^1(R, G)$ of a real algebraic group $G$ see Serre’s book [Ser97, Section I.5]; see also Section 5 below. Galois cohomology can be used to answer many natural questions; see Serre [Ser97, Section III.1].

The Galois cohomology set $H^1(G)$ is finite; see Example (a) in Section III.4.2 and Theorem 4 in Section III.4.3 of Serre’s book [Ser97]. Moreover, when $G$ is nonabelian, the set $H^1(G)$ has a canonical neutral element, but no natural group structure, and one is tempted to conclude that to compute $H^1(G)$ is the same as to compute the cardinality $\#(H^1(G))$. However, for applications in classification problems over $R$, one needs *explicit cocycles* representing the cohomology classes; see, for instance, [Djo83, Section 8] or [BGL21, Section 3.3].

The first Galois cohomology sets of the classical groups $G$ are well known. The cardinalities of the Galois cohomology sets $H^1(G)$ were computed by Adams and Taïbi [AT18] for “most” of the absolutely simple $R$-groups $G$, in particular, for all simply connected absolutely simple $R$-groups and all adjoint absolutely simple $R$-groups. In [BE16], explicit cocycles were computed for all simply connected, absolutely simple $R$-groups. Thus $H^1(G)$ is known for all simply connected semisimple $R$-groups $G$; see [BT21, Introduction] for details. On the other hand, a slight modification of the method of Kac [Kac69] in the version of Onishchik and Vinberg [OV90, Section 4.4] and Gorbatsevich, Onishchik, and Vinberg [GOV94, Section 3.3] gives $H^1(G)$ (explicit cocycles) for all absolutely simple, adjoint $R$-groups $G$. Thus one obtains $H^1(G)$ for all adjoint semisimple $R$-groups $G$; see [BT21, Introduction] for details.

In [BS64, Theorem 6.8] (see also [Ser97, Section III.4.5, Theorem 6 and Example (a)]), Borel and Serre computed $H^1(G)$ for a compact group $G$. For a compact connected $R$-group $G$ (which is automatically reductive), they constructed a canonical bijection $T(R)^{(2)}/W \cong H^1(G)$, where $T \subseteq G$ is a maximal torus, $T(R)^{(2)}$ is its subgroup of real points of order dividing 2, and $W = W(G, T)$ is the Weyl group. Generalizing the result of Borel and Serre, the first-named author [Bor88] computed $H^1(G)$ for a connected reductive $R$-group, not necessarily compact (see also [Bor14, Theorem 9]). Note that, similarly to the formula of Borel and Serre, the result of [Bor88] describes $H^1(G)$ as the set of orbits of a certain Coxeter group $W_0$ of large order in a set of large cardinality. For example, if the adjoint group $G^{ad} := G/Z(G)$ is an inner form of a classical simple compact group of absolute rank $\ell$, then $W_0 = W$ has order at least $\ell!$. However, the cardinality of $H^1(G)$ is usually much smaller. Therefore, it is a challenging problem to find a transparent efficiently computable description of Galois cohomology of connected reductive $R$-groups that allows one to write down easily representatives of cohomology classes.

In [BT21], using ideas of Kac [Kac68], [Kac69], ideas and results of Gorbatsevich, Onishchik, and Vinberg [OV90], [GOV94], and the result of [Bor88], we computed the Galois cohomology set $H^1(G)$ for all semisimple $R$-groups $G$, not necessarily simply connected or adjoint, via Kac labelings. In the present article, using results of [BT21], we compute $H^1(G)$ for a connected reductive $R$-group $G$ via reductive Kac labelings. The formulas for $H^1(G)$ involve some combinatorial constructions and notation; we refer to Section 7 for the statement and proof of our result. Note that we obtain $H^1(G)$ as the set of orbits of a certain finite abelian group $F_0$ of small order acting on a relatively small set of reductive Kac labelings explicitly described in combinatorial terms. If $Z^{sc}$ is the center of the universal cover $G^{sc}$ of the commutator subgroup $[G, G]$ of $G$, then the order of $F_0$ is a divisor of the order of the group $Z^{sc}(C)$; see Remark 8.14. Our description of $H^1(G)$ allows one to enumerate the cohomology classes and to write down explicit cocycles representing them. Note that computation of Galois cohomology of *any connected* linear algebraic $R$-group reduces to the reductive case (see Subsection 0.6 below).

The set $H^1(G)$ has certain additional structures: it is a functor of $G$, there is a twisting map (see Serre [Ser97, Section I.5.3]), and there is an action of the abelian group $H^1(Z(G))$ on $H^1(G)$ (see Serre, [Ser97, Section 1.5.7]). We discuss these additional structures in Section 8.
We also compute the abelian cohomology group $H_{1}^{\text{ab}}G$ and the abelianization map
$$
\text{ab}^{1} : H^{1}G \to H_{1}^{\text{ab}}G
$$
introduced in [Bor98]; see Proposition 8.21 and Theorem 8.22. One needs the abelian group $H_{1}^{\text{ab}}G$ and the abelianization map in order to describe the Galois cohomology of a reductive group over a number field; see [Bor98, Theorem 5.11].

0.2. For a connected reductive $R$-group $G$, consider the group of connected components
$$
\pi_{0}^{R}G := \pi_{0}G(R) \text{ of the real Lie group } G(R).
$$
In the case of an absolutely simple $R$-group $G$ of adjoint type, the group $\pi_{0}^{R}G$ was tabulated in the papers [Mat64], [Ngu00], and [AT18]. For a general connected reductive group $G$, the only known to us result is that of Matsumoto [Mat64, Corollary of Theorem 1.2], see also Borel and Tits [BT65, Theorem 14.4], saying that if $T_{s}$ is a maximal split torus of $G$, then the canonical homomorphism $\pi_{0}^{R}T_{s} \to \pi_{0}^{R}G$ is surjective. From this result it follows that $\pi_{0}^{R}G \simeq (Z/2Z)^{d}$, where $d \leq \text{rank}_{R}(G) := \dim T_{s}$. In particular, the group $\pi_{0}^{R}G$ is abelian.

In this article, we compute $\pi_{0}^{R}G$ for a connected reductive $R$-group $G$ in terms of the algebraic fundamental group $\pi_{1}^{\text{alg}}G$ introduced in [Bor98].

Let $G^{\text{sc}}$ denote the universal cover of the commutator subgroup $[G, G]$ of $G$. Let $\rho : G^{\text{sc}} \to [G, G] \to G$ denote the natural homomorphism. Let $T \subseteq G$ be a maximal torus. We set $T^{\text{sc}} = \rho^{-1}(T) \subseteq G^{\text{sc}}$, which is a maximal torus in $G^{\text{sc}}$.

0.3. Definition ([Bor98]). The algebraic fundamental group of $G$ is
$$
\pi_{1}^{\text{alg}}G = \text{coker} \left[ \rho_{*} : X_{*}(T^{\text{sc}}) \to X_{*}(T) \right],
$$
where $X_{*}(T) := \text{Hom}(G_{m, \mathbb{C}}, T)$ denotes the cocharacter group of the complex torus $T := T \times_{\mathbb{R}} \mathbb{C}$, and similarly for $X_{*}(T^{\text{sc}})$.

As an abstract group, $\pi_{1}^{\text{alg}}G$ is isomorphic (non-canonically) to the topological fundamental group of the complex Lie group $G(\mathbb{C})$. The Galois group $\Gamma$ naturally acts on $\pi_{1}^{\text{alg}}G$, and the $\Gamma$-module $\pi_{1}^{\text{alg}}G$ is well defined (does not depend on the choice of $T$); see [Bor98, Lemma 1.2].

0.4. Construction. We wish to compute $\pi_{0}^{R}G$. Let $G^{\text{ad}} := G/Z(G)$ denote the corresponding semisimple group of adjoint type. Set $T^{\text{ad}} = T/Z(G) \subset G^{\text{ad}}$. Write $C = \pi_{1}^{\text{alg}}G^{\text{ad}} = X_{*}(T^{\text{ad}}) / X_{*}(T^{\text{sc}})$. The homomorphism $\text{Ad} : G \to G^{\text{ad}}$ induces a homomorphism
$$
\text{Ad}_{*} : H^{0}\pi_{1}^{\text{alg}}G \to H^{0}\pi_{1}^{\text{alg}}G^{\text{ad}} = H^{0}C,
$$
where $H^{0}\pi_{1}^{\text{alg}}G := \widehat{H}^{0}(\Gamma, \pi_{1}^{\text{alg}}G)$ (zeroth Tate cohomology), and similarly for $H^{0}C$. We note that there is a $\Gamma$-anti-equivariant isomorphism of $\Gamma$-groups $C \sim \to Z^{\text{sc}}$ that induces a canonical isomorphism $H^{0}C \sim \to H^{1}Z^{\text{sc}}$, where $Z^{\text{sc}} = Z(G^{\text{sc}})$, the center of $G^{\text{sc}}$; see Definition 1.7 and Lemma 1.8 below. The inclusion homomorphism $\iota : Z^{\text{sc}} \to G^{\text{sc}}$ induces a map $\iota_{*} : H^{1}Z^{\text{sc}} \to H^{1}G^{\text{sc}}$. Consider the composite map
$$
\phi : H^{0}\pi_{1}^{\text{alg}}G \xrightarrow{\text{Ad}_{*}} H^{0}C \xrightarrow{\sim} H^{1}Z^{\text{sc}} \xrightarrow{\iota_{*}} H^{1}G^{\text{sc}}.
$$
We write $(H^{0}\pi_{1}^{\text{alg}}G)_{1} := \ker \phi \subseteq H^{0}\pi_{1}^{\text{alg}}G$, the preimage of the neutral element $[1] \in H^{1}G^{\text{sc}}$.

0.5. Theorem.

(i) The subset $(H^{0}\pi_{1}^{\text{alg}}G)_{1} \subseteq H^{0}\pi_{1}^{\text{alg}}G$ is a subgroup.

(ii) There is a canonical group isomorphism $\psi : (H^{0}\pi_{1}^{\text{alg}}G)_{1} \sim \to \pi_{0}^{R}G$.

Moreover, we show that the subgroup $(H^{0}\pi_{1}^{\text{alg}}G)_{1}$ is the stabilizer of a Kac labeling $q \in \mathcal{K}(\check{D})$ defining the real form $G^{\text{sc}}$ of the complex semisimple group $G^{\text{sc}}$, under a certain action
of the group $H^0 \pi_1^{alg} G$ on the affine Dynkin diagram $\tilde{D}$ of $G$; see Section 9. See Definition 7.9 for the notion of a Kac labeling and the notation $K(\tilde{D})$.

0.6. We describe the structure of the article. In Sections 1–6 we gather old and new results on group cohomology and hypercohomology of abstract $\Gamma$-groups and on Galois cohomology of linear $R$-groups. In particular, for an $R$-torus $T$, in Theorem 3.6 we compute the Tate cohomology groups $H^k T$ for $k \in \mathbb{Z}$, and in Corollary 3.10 we compute the component group $\pi_0^R T$. Moreover, for an $R$-quasi-torus $A$ (R-group of multiplicative type), in Theorem 3.15 we compute the Tate cohomology groups $H^k A$. Furthermore, for a short exact sequence of $R$-groups

$$1 \to G_1 \xrightarrow{i} G_2 \xrightarrow{j} G_3 \to 1$$

(not necessarily connected or linear), in Section 5 we construct an exact sequence

$$(0.7) \quad \pi_0^R G_1 \xrightarrow{i_*} \pi_0^R G_2 \xrightarrow{j_*} \pi_0^R G_3 \xrightarrow{\delta^0} H^1 G_1 \xrightarrow{i_*} H^1 G_2 \xrightarrow{j_*} H^1 G_3.$$  

In Section 6 we show that if $G$ is a connected linear algebraic $R$-group, $G^u$ is its unipotent radical, $G^{red} := G/G^u$ is the corresponding reductive $R$-group, and $G \to G^{red}$ is the canonical homomorphism, then the induced maps $\pi_0^R G \to \pi_0^R G^{red}$ and $H^1 G \to H^1 G^{red}$ are bijective. This reduces computing the Galois cohomology and component group of a connected linear algebraic $R$-group to the case of a connected reductive $R$-group; see Remark 6.6.

In Section 7 we state and prove Theorem 7.14 that computes $H^1 G$ for a reductive $R$-group $G$ in terms of reductive Kac labelings. In Section 9 we prove Theorem 0.5. These are our main results. In Section 8 we discuss additional structures on $H^1 G$: functoriality, twisting, action of $H^1 Z(G)$, and abelianization. In Section 10 we compute the connecting map $\delta^0$ in the exact sequence (0.7) in the case when $G_2$ and $G_3$ are connected reductive $R$-groups, and $G_1$ is either a connected reductive $R$-group or an $R$-quasi-torus (an $R$-group of multiplicative type).

In Section 11 we consider examples: we compute $H^1 G$ and $\pi_0^R G$ for certain connected reductive $R$-groups $G$.

In Appendix A we give a short elementary proof of the known classification of $\Gamma$-lattices (finitely generated free abelian groups with $\Gamma$-action).

0.8. Notation and conventions.

- $\mathbb{Z}$ denotes the ring of integers.
- $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational numbers, of real numbers, and of complex numbers, respectively.
- $i \in \mathbb{C}$ is such that $i^2 = -1$. (Our results do not depend on the choice of $i$.)
- $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$, the Galois group of $\mathbb{C}$ over $\mathbb{R}$, where $\gamma$ is the complex conjugation. The action of $\gamma$ on an element $s$ of a set $S$ is denoted by $s \mapsto \gamma s$.
- We denote real algebraic groups by boldface letters $G, T, \ldots$, their complexifications by respective italic (non-bold) letters $G = G \times_{\mathbb{R}} \mathbb{C}, T = T \times_{\mathbb{R}} \mathbb{C}, \ldots$, the corresponding complex Lie algebras by respective lowercase Gothic letters $g = \text{Lie} G, \ t = \text{Lie} T, \ldots$, and the corresponding real Lie algebras by respective boldface lowercase Gothic letters $g(\mathbb{R}) = \text{Lie} G, \ t(\mathbb{R}) = \text{Lie} T, \ldots$.
- $G(\mathbb{R})$ denotes the set of real points of a real algebraic group $G$, and $G(\mathbb{C})$ denotes the set of complex points. By abuse of notation we identify $G$ with $G(\mathbb{C})$. In particular, we write $g \in G$ for $g \in G(\mathbb{C})$.
- For a homomorphism $\varphi : G \to H$ of algebraic (or Lie) groups, the differential at the unity $d\varphi : g \to \mathfrak{h}$ is a homomorphism of Lie algebras. By abuse of notation, we often write $\varphi$ instead of $d\varphi$.
- $G^0$ denotes the identity component of an algebraic (or Lie) group $G$. 

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• \( G \) is an \( R \)-group, not necessarily linear, connected, or reductive. In Section 6, \( G \) is linear. Starting from Section 7, \( G \) is connected and reductive. Moreover, in Section 7 \( G \) is compact.
• \( Z(G) \) denotes the center of \( G \).
• \( G_{m,c} \) and \( G_{m,R} \) denote the multiplicative groups over \( C \) and \( R \), respectively.
• \( X^\ast(T) = \text{Hom}(T, G_{m,c}) \), the group of complex characters of an algebraic \( R \)-group \( T \).

When \( T \) is a torus, we regard \( X^\ast(T) \) as a lattice in the dual space \( t^\ast \) of \( t \), in view of the canonical embedding \( X^\ast(T) \hookrightarrow t^\ast \), \( \chi \mapsto d\chi \).
• \( X_s(T) = \text{Hom}(G_{m,c}, T) \), the group of complex cocharacters of an \( R \)-torus \( T \). We regard \( X_s(T) \) as a lattice in \( t \), in view of the canonical embedding \( X_s(T) \hookrightarrow t, \nu \mapsto d\nu(1) \).
• \( A \cong B \) means that two groups (or algebraic groups) \( A \) and \( B \) are isomorphic.
• \( A \cong B \) means that \( A \) and \( B \) are canonically isomorphic.
• By an exact commutative diagram we mean a commutative diagram with exact rows.

1. Abelian cohomology

1.1. Let \( A \) be a \( \Gamma \)-module, that is, an abelian group written additively, endowed with an action of \( \Gamma = \{1, \gamma \} \). We consider the first cohomology group \( H^1(\Gamma, A) \). We write \( H^1A \) for \( H^1(\Gamma, A) \). Recall that
\[
H^1A = Z^1A/B^1A, \quad \text{where} \quad Z^1A = \{a \in A \mid \gamma a = -a\}, \quad B^1A = \{\gamma a' - a' \mid a' \in A\}.
\]

We define the second cohomology group \( H^2A \) by
\[
H^2A = Z^2A/B^2A, \quad \text{where} \quad Z^2A = A^\Gamma := \{a \in A \mid \gamma a = a\}, \quad B^2A = \{\gamma a' + a' \mid a' \in A\}.
\]

For \( k \in \mathbb{Z} \) we define the coboundary operator
\[
d^k : A \to A, \quad a \mapsto \gamma a + (-1)^{k+1}a.
\]
In other words, \( d^k = \gamma + (-1)^{k+1} \in \mathbb{Z}[\Gamma] \), where \( \mathbb{Z}[\Gamma] = \mathbb{Z} \oplus \mathbb{Z}\gamma \) is the group ring of \( \Gamma \). We calculate:
\[
d^k \circ d^{k-1} = (\gamma + (-1)^{k+1})(\gamma + (-1)^k) = (\gamma - (-1)^k)(\gamma + (-1)^k) = \gamma^2 - (-1)^{2k} = 0.
\]

We define the Tate cohomology groups \( \hat{H}^kA \) for all \( k \in \mathbb{Z} \) by
\[
\hat{H}^kA = Z^kA/B^kA,
\]
where
\[
Z^kA = \ker d^k = \{a \in A \mid \gamma a = (-1)^k a\}, \quad B^kA = \text{im } d^{k-1} = \{\gamma a' + (-1)^k a' \mid a' \in A\}.
\]

Then clearly
\[
\hat{H}^kA = H^1A \quad \text{when } k \text{ is odd}, \quad \text{and} \quad \hat{H}^kA = H^2A \quad \text{when } k \text{ is even}.
\]

In this article, for any \( k \in \mathbb{Z} \) we write \( H^kA \) for \( \hat{H}^kA \). In particular,
\[
H^0A = Z^0A/B^0A = A^\Gamma/\{\gamma a' + a' \mid a' \in A\}, \quad \text{and not } A^\Gamma.
\]

If \( z \in Z^kA \), we write \([z] = z + B^kA \in H^kA \) for the cohomology class of \( z \).

1.2. Remark. In the standard exposition, our definitions become theorems; see [CE56, Chapter XII, Section 7, p. 251] or [AW67, Theorem 5 in Section 8].

1.3. Lemma (See, for instance, [AW67, Section 6, Corollary 1 of Proposition 8]). For any \( k \in \mathbb{Z} \) and \( \xi \in \hat{H}^kA \), we have \( 2\xi = 0 \).
Proof. Let $\xi = [z]$, $z \in \mathbb{Z}^kA$. Then $z = (-1)^k \cdot \gamma z$. Hence
$$2z = z + (-1)^k \cdot \gamma z = (\gamma + (-1)^k)\gamma z = d^{k-1}(\gamma z) \in B^kA.$$ Thus $2\xi = [2z] = 0$. \hfill \Box

1.4. Corollary. If $A$ is a $\Gamma$-module such that the endomorphism
$$2: A \to A, \quad a \mapsto 2a$$
is invertible, then $H^kA = 0$ for all $k$.

1.5. Let
$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$
be a short exact sequence of $\Gamma$-modules. It gives rise to a cohomology exact sequence
$$(1.6) \quad \cdots \to H^{k-1}C \to H^kA \xrightarrow{i_*} H^kB \xrightarrow{j^*} H^kC \to H^{k+1}A \to \cdots$$
We recall the formula for $\delta^k$. We identify $A$ with $i(A) \subseteq B$. Let $[c] \in H^kC$, $c \in Z^kC$. We lift $c$ to some $b \in B$ and set $a = d^k b$. Then $a \in Z^{k+1}A$. We set $\delta^k[c] = [a] \in H^{k+1}A$. In particular, we have
$$\delta^0[c] = [\gamma b - b] \text{ for } c \in Z^0C, \quad \delta^1[c] = [\gamma b + b] \text{ for } c \in Z^1C.$$  

1.7. Definition. Let $A, A'$ be two $\Gamma$-modules. By a $\Gamma$-anti-equivariant homomorphism $A \to A'$ we mean a homomorphism of abelian groups
$$\varphi: A \to A'$$
such that $\gamma \varphi(a) = -\varphi(\gamma a)$ for all $a \in A$.

1.8. Lemma (obvious). Let $\varphi: A \to A'$ be a $\Gamma$-anti-equivariant homomorphism of $\Gamma$-modules. Then for any $k$ in $\mathbb{Z}$, the homomorphism $\varphi$ restricts to homomorphisms
$$Z^kA \to Z^{k+1}A' \quad \text{and} \quad B^kA \to B^{k+1}A'$$
and induces a homomorphism on cohomology
$$\varphi^k_*: H^kA \to H^{k+1}A'.$$ If, moreover, $\varphi$ is an isomorphism of abelian groups, then $\varphi^k_*$ is an isomorphism for each $k$.

1.9. Definition. Let $A$ be a $\Gamma$-module. We denote by $(i)_! A$ the abelian group consisting of formal expressions $\{(i)_! a \mid a \in A\}$ with the addition law
$$(i)_! a + (i)_! a' = (i)_! (a + a').$$
Then $(i)(-a) = -(i)_! a$. We define a $\Gamma$-action on $(i)_! A$ by
$$\gamma ((i)_! a) = -(i)_! \gamma a.$$ We have a canonical isomorphism of $\Gamma$-modules
$$(i)_!(i)_! A \xrightarrow{\sim} A, \quad (i)_! (i)_! a \mapsto -a.$$  

1.10. Corollary (from Lemma 1.8). For any $\Gamma$-module $A$, the canonical $\Gamma$-anti-equivariant isomorphism
$$A \to (i)_! A, \quad a \mapsto (i)_! a \quad \text{for } a \in A,$$induces canonical isomorphisms
$$H^kA \xrightarrow{\sim} H^{k+1}(i)_! A$$for all $k \in \mathbb{Z}$.

2. Hypercohomology

2.1. Definition. A short complex of $\Gamma$-modules is a morphism of $\Gamma$-modules $A_1 \xrightarrow{\partial} A_0.$
2.2. For a short complex of \( \Gamma \)-modules \( A_1 \overset{\partial}{\to} A_0 \) and for \( k \in \mathbb{Z} \), we define a differential
\[
D^k : A_1 \oplus A_0 \to A_1 \oplus A_0, \quad D^k(a_1, a_0) = (-d^{k+1}a_1, d^k a_0 - \partial a_1) \text{ for } (a_1, a_0) \in A_1 \oplus A_0.
\]
We calculate:
\[
D^k(D^{k-1}(a_1, a_0)) = D^k(-d^k a_1, d^{k-1} a_0 - \partial a_1)
\]
\[
= (d^{k+1}d^k a_1, d^k d^{k-1} a_0 - d^k \partial a_1 + \partial d^k a_1) = (0, 0).
\]
Thus \( D^k \circ D^{k-1} = 0 \).

We define the \( k \)-th Tate hypercohomology group \( H^k(A_1 \overset{\partial}{\to} A_0) \) by
\[
H^k(A_1 \overset{\partial}{\to} A_0) = Z^k(A_1 \overset{\partial}{\to} A_0) / B^k(A_1 \overset{\partial}{\to} A_0),
\]
where
\[
Z^k(A_1 \overset{\partial}{\to} A_0) = \ker D^k = \{(a_1, a_0) \in A_1 \oplus A_0, \mid d^{k+1} a_1 = 0, d^k a_0 = \partial a_1 \},
\]
\[
B^k(A_1 \overset{\partial}{\to} A_0) = \text{im } D^{k-1} = \{(-d^k a'_1, d^{k-1} a'_0 - \partial a'_1) \mid (a'_1, a'_0) \in A_1 \oplus A_0 \}.
\]
For simplicity we write \( H^k(A_1 \to A_0) \) instead of \( H^k(A_1 \overset{\partial}{\to} A_0) \). If \((a_1, a_0) \in Z^k(A_1 \to A_0)\), we write \([a_1, a_0] \in H^k(A_1 \to A_0)\) for the cohomology class of \((a_1, a_0)\).

2.3. Examples.

(1) We have an isomorphism
\[
H^k A_0 \xrightarrow{\sim} \mathbb{H}^k(0 \to A_0), \quad [a_0] \mapsto [0, a_0].
\]

(2) We have an isomorphism
\[
\mathbb{H}^k(A_1 \to 0) \xrightarrow{\sim} H^{k+1}A_1, \quad [a_1, 0] \mapsto [a_1].
\]

The correspondence \((A_1 \to A_0) \mapsto H^k(A_1 \to A_0)\) is a functor from the category of short complexes of \( \Gamma \)-modules to the category of abelian groups. Moreover, a short exact sequence of short complexes of \( \Gamma \)-modules
\[
0 \to (A_1 \to A_0) \overset{\alpha}{\to} (B_1 \to B_0) \overset{\beta}{\to} (C_1 \to C_0) \to 0
\]
gives rise to a hypercohomology exact sequence
\[
\cdots \quad \mathbb{H}^k(A_1 \to A_0) \overset{\delta^k}{\to} \mathbb{H}^k(B_1 \to B_0) \overset{\beta}{\to} \mathbb{H}^k(C_1 \to C_0) \overset{\delta^k}{\to} \mathbb{H}^{k+1}(A_1 \to A_0) \cdots
\]

We specify the maps \( \delta^k \) in the sequence (2.4). Let \((c_1, c_0) \in Z^k(C_0 \to C_1) \subseteq C_1 \oplus C_0\). We lift \((c_1, c_0)\) to some \((b_1, b_0)\) in \( B_1 \oplus B_0 \) and set \((a_1, a_0) = D^k(b_1, b_0)\). Then \((a_1, a_0) \in Z^k(A_1 \to A_0)\), and we set \( \delta^k[c_1, c_0] = [a_1, a_0] \in H^{k+1}(A_1 \to A_0)\).

2.5. Example. Applying (2.4) to the short exact sequence of complexes
\[
0 \to (0 \to A_0) \overset{\lambda}{\to} (A_1 \to A_0) \overset{\mu}{\to} (A_1 \to 0) \to 0
\]
gives rise to a hypercohomology exact sequence
\[
\cdots \quad H^k A_1 \overset{\delta^k}{\to} H^k A_0 \overset{\lambda^k}{\to} H^k(A_1 \to A_0) \overset{\mu^k}{\to} H^{k+1} A_1 \overset{\delta^{k+1}}{\to} H^{k+1} A_0 \cdots
\]

2.7. Lemma. The maps \( \lambda^k, \mu^k, \) and \( \delta^{k+1} \), in (2.6) are the following:
\[
\lambda^k : H^k A_0 \to H^k(A_1 \to A_0), \quad [a_0] \mapsto [0, a_0],
\]
\[
\mu^k : H^k(A_1 \to A_0) \to H^{k+1} A_1, \quad [a_1, a_0] \mapsto [a_1],
\]
\[
\delta^{k+1} : H^{k+1} A_1 \to H^{k+1} A_0, \quad [a_1] \mapsto [\partial a_1].
\]
Identifying where \(-\) to \((j\) Then the canonical homomorphism 2.8. Lemma (well-known) \(\partial\) Since \(\text{im} \implies \text{ker}\). Then \(\partial\) \(A\) \(\oplus\) \(A\), and apply \(D^k\) to the lift. We obtain \(D^k(a_1,0) = (-d^{k+1}a_1, -\partial a_1) = (0, -\partial a_1) \in Z^{k+1}(0 \to A_0)\). Identifying \(H^{k+1}(0 \to A_0) \cong H^{k+1}A_0\), we obtain \(\begin{bmatrix} 0, -\partial a_1 \end{bmatrix} = -[\partial a_1] = [\partial a_1] \in H^{k+1}A_0\), where \(-[\partial a_1] = [\partial a_1]\) by Lemma 1.3. 

2.8. Lemma (well-known). Let \(\partial: A_1 \to A_0\) be an injective homomorphism of \(\Gamma\)-modules. Then the canonical homomorphism \(j^k: H^k(A_1 \xrightarrow{\partial} A_0) \to H^k(0 \to \text{coker} \, \partial) = H^k \text{coker} \, \partial, \quad [a_1, a_0] \mapsto [0, a_0 + \text{im} \, \partial] \mapsto [a_0 + \text{im} \, \partial]\) induced by the canonical morphism of complexes \(j: (A_1 \to A_0) \to (0 \to \text{coker} \, \partial), \quad (a_1, a_0) \mapsto (0, a_0 + \text{im} \, \partial)\) is an isomorphism.

Proof. We prove the surjectivity. Let \(a_0 + \text{im} \, \partial \in Z^k \text{coker} \, \partial\). Then \(d^k(a_0 + \text{im} \, \partial) = 0\), that is, \(d^k a_0 = \partial a_1\) for some \(a_1 \in A_1\). We have \(\partial d^{k+1}a_1 = d^{k+1} \partial a_1 = d^{k+1}d^k a_0 = 0\).

Since \(\partial\) is injective, we conclude that \(d^{k+1}a_1 = 0\). We see that \((a_1, a_0) \in Z^k(A_1 \to A_0)\) and \(j^k[a_1, a_0] = [a_0 + \text{im} \, \partial]\), which proves the surjectivity.

We prove the injectivity. Let \((a_1, a_0) \in Z^k(A_1 \to A_0), \quad [a_1, a_0] \in \ker j^k\). Then \(a_0 + \text{im} \, \partial = d^{k-1}(a'_0 + \text{im} \, \partial)\) for some \(a'_0 \in A_0\), that is, \(a_0 = d^{k-1}a'_0 - \partial a'_1\) for some \(a'_0 \in A_0, \ a'_1 \in A_1\), and, moreover, \((0, 0) = D^k(a_1, a_0) = (-d^{k+1}a_1, d^k a_0 - \partial a_1)\).

We see that \(\partial a_1 = d^k a_0 = d^k(d^{k-1}a'_0 - \partial a'_1) = -d^k \partial a'_1 = -\partial d^k a'_1\).

Since \(\partial\) is injective, we conclude that \(a_1 = -d^k a'_1\). It follows that \((a_1, a_0) = (-d^k a'_1, d^{k-1}a'_0 - \partial a'_1) = D^{k-1}(a'_1, a'_0)\).

Thus \([a_1, a_0] = 0\) and the homomorphism \(j^k\) is injective, which completes the proof of the lemma.

2.9. Example. Applying (2.4) to the short exact sequence of complexes \(0 \to (\ker \partial \to 0) \to (A_1 \xrightarrow{\partial} A_0) \to (\text{im} \, \partial \leftarrow A_0) \to 0\), where \(H^k(\text{im} \, \partial \leftarrow A_0) \cong H^k \text{coker} \, \partial\) by Lemma 2.8, we obtain an exact sequence \(\begin{align*}
0 & \to (\ker \partial \to 0) \to (A_1 \xrightarrow{\partial} A_0) \to (\text{im} \, \partial \leftarrow A_0) \to 0,
\end{align*}\)

where \(H^k(\text{im} \, \partial \leftarrow A_0) \cong H^k \text{coker} \, \partial\) by Lemma 2.8, we obtain an exact sequence \(\begin{align*}
0 & \to (\ker \partial \to 0) \to (A_1 \xrightarrow{\partial} A_0) \to (\text{im} \, \partial \leftarrow A_0) \to 0,
\end{align*}\)
2.11. Definition. A morphism of short complexes

\[(2.12) \varphi: (A_1 \xrightarrow{\partial} A_0) \rightarrow (A'_1 \xrightarrow{\partial'} A'_0)\]

is called a quasi-isomorphism if the induced homomorphisms

\[\ker \partial \rightarrow \ker \partial' \quad \text{and} \quad \coker \partial \rightarrow \coker \partial'\]

are isomorphisms.

2.13. Examples.

(1) If \(A_1 \hookrightarrow A_0\) is injective, then \((A_1 \hookrightarrow A_0) \rightarrow (0, \coker [A_1 \hookrightarrow A_0])\) is a quasi-isomorphism.

(2) If \(A_1 \twoheadrightarrow A_0\) is surjective, then \((\ker [A_1 \twoheadrightarrow A_0] \rightarrow 0) \rightarrow (A_1 \twoheadrightarrow A_0)\) is a quasi-isomorphism.

2.14. Proposition (well-known). A quasi-isomorphism of complexes of \(\Gamma\)-modules \((2.12)\) induces isomorphisms on the hypercohomology

\[\varphi^k: \mathbb{H}^k(A_1 \rightarrow A_0) \xrightarrow{\sim} \mathbb{H}^k(A'_1 \rightarrow A'_0).\]

Idea of an elementary proof. Using \((2.10)\), we obtain from \((2.12)\) an exact commutative diagram

\[
\begin{array}{ccccccc}
H^{k-1}\ker \partial & \rightarrow & H^k\ker \partial & \rightarrow & H^k(A_1 \xrightarrow{\partial} A_0) & \rightarrow & H^{k+1}\ker \partial \\
\Rightarrow & & \Rightarrow & & \varphi^k & & \Rightarrow \\
H^{k-1}\ker \partial' & \rightarrow & H^k\ker \partial' & \rightarrow & H^k(A'_1 \xrightarrow{\partial'} A'_0) & \rightarrow & H^{k+1}\ker \partial'
\end{array}
\]

in which the middle vertical arrow \(\varphi^k\) is an isomorphism by the five lemma. \(\square\)

2.15. Example. We have \(\mathbb{H}^k(A_1 \rightarrow 0) \cong \mathbb{H}^{k+1}A_1\). Hence, if a homomorphism \(\partial: A_1 \rightarrow A_0\) is surjective, then by Example 2.13(2) and Proposition 2.14 we have

\[\mathbb{H}^k(A_1 \xrightarrow{\partial} A_0) \cong \mathbb{H}^k(\ker \partial \rightarrow 0) \cong \mathbb{H}^{k+1}\ker \partial.\]

2.16. Let

\[0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0\]

be a short exact sequence of \(\Gamma\)-modules, where we identify \(A\) with \(i(A) \subseteq B\). The quasi-isomorphism

\[i^\#: (A \rightarrow 0) \rightarrow (B \rightarrow C)\]

induces isomorphisms

\[i^\#: H^{k+1}A \rightarrow \mathbb{H}^k(B \rightarrow C), \ [a] \mapsto [a, 0].\]

Combining the cohomology exact sequences \((1.6)\) and \((2.6)\), we obtain a diagram with exact rows

\[(2.17)\]

\[\begin{array}{ccccccc}
H^kC \xrightarrow{\delta^k} H^{k+1}A & \xrightarrow{i^k} & H^{k+1}B \\
\Rightarrow & & \Rightarrow \\
H^kC \xrightarrow{\lambda^k} \mathbb{H}^k(B \rightarrow C) & \xrightarrow{\mu^k} & H^{k+1}B
\end{array}\]

2.18. Lemma. The diagram \((2.17)\) is commutative.
Proof. The right-hand rectangle clearly commutes, and therefore it sufficed to show that the left-hand rectangle commutes. We perform a calculation. Let \( c \in \mathbb{Z}^k \). We lift \( c \) to some \( b \in B \). Then \( i_b^k(\delta^k[c]) = [d^k(b), 0] \) and \( \lambda^k_b[c] = [0, c] = [0, j(b)] \), whence
\[
i_b^k(\delta^k[c]) = -\lambda^k_c[c] + [d^k b, j(b)].
\]
Since \( (d^kB, j(b)) = D^{k-1}(-b, 0) \in B^k(B \rightarrow C) \), we see that \( i_b^k(\delta^k[c]) = -\lambda^k_c[c] \). By Lemma 1.3 we have \(-[c] = [c]\), and hence \( i_b^k(\delta^k[c]) = \lambda^k_c[c] \), as required. \( \square \)

3. Galois cohomology of tori and quasi-tori

3.1. Let \( T \) be an \( \mathbb{R} \)-torus. Consider the cocharacter group
\[
X_\ast(T) = \text{Hom}(G_m, C, T).
\]
The group \( \Gamma \) acts on \( X_\ast(T) \) by
\[
(\gamma \nu)(z) = \gamma(\nu(\gamma^{-1} z)) \quad \text{for} \quad \nu \in X_\ast(T), \ z \in C^x
\]
(where in our case \( \gamma^{-1} = \gamma \)). We see that \( X_\ast(T) \) is a \( \Gamma \)-lattice, that is, a finitely generated free abelian group with \( \Gamma \)-action.

Let \( L \) be a nonzero \( \Gamma \)-lattice. We say that \( L \) is indecomposable if it is not a direct sum of its two nonzero \( \Gamma \)-sublattices. Clearly, any \( \Gamma \)-lattice is a direct sum of indecomposable lattices.

3.2. Proposition (well-known). Up to isomorphism, there are exactly three indecomposable \( \Gamma \)-lattices:

\begin{enumerate}
\item \( \mathbb{Z} \) with trivial action of \( \gamma \);
\item \( \mathbb{Z} \) with the action of \( \gamma \) by \(-1\);
\item \( \mathbb{Z} \oplus \mathbb{Z} \) with the action of \( \gamma \) by the matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).
\end{enumerate}

Proof. See Casselman [Cas08, Theorem 2] or Appendix A below. \( \square \)

3.3. Let \( \varphi : T \rightarrow S \) be a homomorphism of \( \mathbb{R} \)-tori; then \( \varphi_\ast : X_\ast(T) \rightarrow X_\ast(S) \) is a homomorphism of \( \Gamma \)-lattices. In this way we obtain an equivalence between the category of \( \mathbb{R} \)-tori and the category of \( \Gamma \)-lattices.

We say that an \( \mathbb{R} \)-torus is indecomposable if it is not a direct product of its two nontrivial subtori. Clearly, every \( \mathbb{R} \)-torus is a direct product of indecomposable \( \mathbb{R} \)-tori. It is also clear that a torus \( T \) is indecomposable if and only if its cocharacter lattice \( X_\ast(T) \) is indecomposable.

3.4. Corollary (of Proposition 3.2; see, for instance, Voskresenskiî [Vos98, Section 10.1]). Up to isomorphism, there are exactly three indecomposable \( \mathbb{R} \)-tori:

\begin{enumerate}
\item \( T^1 = G_{m, \mathbb{R}} = (C^x, z \mapsto z) \), with group of \( \mathbb{R} \)-points \( R^x \), a one-dimensional split torus;
\item \( T^1_c = (C^x, z \mapsto z^{-1}) \), with group of \( \mathbb{R} \)-points \( U(1) = \{ z \in C^x \mid z \bar{z} = 1 \} \), a one-dimensional compact torus;
\item \( R_{C/R} G_{m, C} = (C^x \times C^x, (z_1, z_2) \mapsto (\bar{z}_2, \bar{z}_1)) \), with group of \( \mathbb{R} \)-points \( C^x \).
\end{enumerate}

3.5. Notation. Let \( A \) be a commutative algebraic \( \mathbb{R} \)-group, written multiplicatively. For \( k \in \mathbb{Z} \), write
\[
H^k A = H^k(R, A) := H^k(\Gamma, A(C)).
\]

3.6. Theorem. Let \( T \) be a \( \mathbb{R} \)-torus. Consider the canonical \( \Gamma \)-equivariant evaluation homomorphism
\[
ev : X_\ast(T) \rightarrow T(C), \quad \nu \mapsto \nu(-1) \quad \text{for} \quad \nu \in X_\ast(T).
\]
Then for any $k \in \mathbb{Z}$, the induced homomorphism
\begin{equation}
\text{ev}_s^k : H^k \chi_s(T) \to H^k T, \quad [\nu] \mapsto [\nu(-1)]
\end{equation}
is an isomorphism.

We give three different proofs of this theorem.

A proof using reduction to the split and compact cases. One checks immediately that the theorem holds for $T^1_s$ and $T^1_c$. It follows that the theorem holds for the split tori and for the compact tori.

Now let $T$ be an arbitrary $\mathbb{R}$-torus, and let $T_0 \subseteq T$ denote its maximal compact subtorus. Then $T/T_0$ is a split torus. We have an exact commutative diagram
\[
\begin{array}{c}
\begin{array}{c}
0 \longrightarrow X(T_0) \longrightarrow X_s(T) \longrightarrow X_s(T/T_0) \longrightarrow 0 \\
1 \longrightarrow T_0(C) \longrightarrow T(C) \longrightarrow (T/T_0)(C) \longrightarrow 1
\end{array}
\end{array}
\]
in which the vertical arrows are the evaluation homomorphisms. We obtain a commutative diagram in which the rows are the corresponding cohomology exact sequences. Since the theorem holds for the compact torus $T_0$ and for the split torus $T/T_0$, by the five lemma it holds for $T$ as well.

A proof using the exponential map. We write $t = \text{Lie } T$ and identify $X_s(T)$ with an additive subgroup of $t$ via the embedding $\nu \mapsto a\nu(1)$ for $\nu \in X_s(T)$. The short exact sequence of $\Gamma$-modules
\[
0 \longrightarrow iX_s(T) \longrightarrow t \longrightarrow E \longrightarrow 1,
\]
where $E(x) = \exp 2\pi x$ for $x \in t$, gives rise to a cohomology long exact sequence
\[
\cdots \longrightarrow H^k t \xrightarrow{\delta^k} H^k T \xrightarrow{\delta^k} H^{k+1} iX_s(T) \xrightarrow{\iota_{k+1}} H^{k+1} t \longrightarrow \cdots,
\]
in which $H^k t = 0$ for all $k \in \mathbb{Z}$ by Corollary 1.4. Hence the connecting homomorphism $\delta^k$ is an isomorphism.

Let $i\nu \in Z^{k+1} iX_s(T)$; then $\gamma \nu = (-1)^k \nu$. It follows that $\nu(-1) \in Z^k T$. We show that the isomorphism $\delta^k$ sends $[\nu(-1)]$ to $[i\nu]$. Indeed, take
\[
x = i\nu/2 \in t, \quad t = \nu(-1) = \exp \pi i\nu = E(x).
\]
Then $\delta^k$ sends $[t]$ to the class of
\[
d^k x = \gamma(i\nu/2) + (-1)^{k+1}i\nu/2 = -\frac{i}{2} (\gamma \nu + (-1)^k \nu) = (-1)^{k+1} i\nu.
\]
By Lemma 1.3 we have $[(1)^{k+1} i\nu] = [i\nu]$. Thus $\delta^k$ indeed sends $[\nu(-1)]$ to $[i\nu]$.

We consider the $\Gamma$-anti-equivariant isomorphism $X_s(T) \to iX_s(T), \quad \nu \mapsto i\nu,$
which by Lemma 1.8 induces an isomorphism
\[
H^k X_s(T) \xrightarrow{\sim} H^{k+1} iX_s(T).
\]
The composite isomorphism
\[
H^k X_s(T) \xrightarrow{\sim} H^{k+1} iX_s(T) \xrightarrow{(\delta^k)^{-1}} H^k T
\]
sends $[\nu] \in H^k X_s(T)$ to $[i\nu] \in H^{k+1} iX_s(T)$ and then to $[\nu(-1)] \in H^k T$, which completes the proof of the theorem.
A proof using the Tate–Nakayama theorem. Write \( A = \mathbb{C}^\times \); then
\[
\text{H}^1(\Gamma, A) = \{1\} \quad \text{and} \quad \#\text{H}^2(\Gamma, A) = 2.
\]
Let \( a \in \text{H}^2(\Gamma, A) \) denote the only nontrivial element of this group. The triple \((\Gamma, A, a)\) trivially satisfies the assumptions of the Tate–Nakayama theorem; see [Ser79, Section IX.8, Theorem 14]. By this theorem, for any torsion free \( \Gamma \)-module \( M \) and for any integer \( k \), the cup product with \( a \) induces an isomorphism
\[
\text{H}^{k-2}(\Gamma, M) \xrightarrow{\sim} \text{H}^k(\Gamma, M \otimes A).
\]
We take \( M = X_*(T) \); then \( T(\mathbb{C}) \cong X_*(T) \otimes \mathbb{C}^\times = M \otimes A \), and we obtain an isomorphism
\[
\text{H}^k(\Gamma, X_*(T)) \cong \text{H}^{k-2}(\Gamma, X_*(T)) \xrightarrow{\sim} \text{H}^k(\Gamma, X_*(T) \otimes \mathbb{C}^\times) = \text{H}^kT.
\]
One can check that this isomorphism is induced by the evaluation homomorphism (3.7). \( \square \)

3.9. Proposition. Let \( T \) be an \( \mathbb{R} \)-torus. The canonical surjective homomorphism
\[
\text{Z}^0T = T(\mathbb{R}), \quad \text{induces an isomorphism}
\]
\[
\pi_0T(\mathbb{R}) \xrightarrow{\sim} \text{H}^0T.
\]
Proof. The homomorphism \( t \to t(\mathbb{R}), \quad x \mapsto x + \gamma x \), is surjective, and therefore the image \( B^0T \) of the homomorphism of real Lie groups
\[
d^{-1}: T(\mathbb{C}) \to T(\mathbb{R}), \quad t \mapsto t \cdot \gamma t
\]
is open. Since the complex Lie group \( T(\mathbb{C}) \) is connected, its image \( d^{-1}(T(\mathbb{C})) = B^0T \) is connected. It follows that \( B^0T = T(\mathbb{R})^0 \), and the proposition follows. \( \square \)

3.10. Corollary ([Cas08, Section 5]). The homomorphism (3.7) induces isomorphisms
\[
\text{H}^0X_*(T) \xrightarrow{\sim} \text{H}^0T \xrightarrow{\sim} \pi_0T(\mathbb{R}).
\]

3.11. Let \( T_1 \xrightarrow{\partial} T_0 \) be a short complex of \( \mathbb{R} \)-tori. Consider the short complex of \( \Gamma \)-modules
\[
X_*(T_1) \xrightarrow{\partial} X_*(T_0).
\]
Formula (3.7) permits us to define the evaluation morphism of short complexes of \( \Gamma \)-modules
\[
ev: (X_*(T_1) \to X_*(T_0)) \to (T_1(\mathbb{C}) \to T_0(\mathbb{C})), \quad (\nu_1, \nu_0) \mapsto (\nu_1(-1), \nu_0(-1))
\]
which in general is not a quasi-isomorphism.

3.13. Proposition. The morphism of short complexes (3.12) induces isomorphisms on hypercohomology
\[
ev^k_x: \text{H}^k(X_*(T_1) \to X_*(T_0)) \xrightarrow{\sim} \text{H}^k(T_1 \to T_0), \quad [\nu_1, \nu_0] \mapsto [\nu_1(-1), \nu_0(-1)].
\]
Proof. Using the sequence (2.6) for the short complexes \((X_*(T_1) \to X_*(T_0))\) and \((T_1 \to T_0)\), we obtain an exact commutative diagram
\[
\begin{array}{ccccccc}
\text{H}^kX_*(T_1) & \longrightarrow & \text{H}^kX_*(T_0) & \longrightarrow & \text{H}^k(X_*(T_1) \to X_*(T_0)) & \longrightarrow & \text{H}^{k+1}X_*(T_1) \\
\cong & & & & & & \cong \\
\text{H}^kT_1 & \longrightarrow & \text{H}^kT_0 & \longrightarrow & \text{H}^k(T_1 \to T_0) & \longrightarrow & \text{H}^{k+1}T_1
\end{array}
\]
in which by Theorem 3.6 the four vertical arrows labeled with \( \cong \) are isomorphisms. By the five lemma, the fifth vertical arrow in the diagram (labelled \( \ev^k_x \)) is an isomorphism as well. \( \square \)
3.14. Following Gorbatsevich, Onishchik, and Vinberg [GOV94, Section 3.3.2], we say that a quasi-torus over $R$ is a commutative algebraic $R$-group $A$ such that all elements of $A(C)$ are semisimple. In other words, $A$ is an $R$-group of multiplicative type; see Milne [Mil17, Corollary 12.21]. In other words, $A$ is an $R$-subgroup of some $R$-torus $T$; see, for instance, [BGR22, Section 2.2]. Set $T' = T/A$; then $A$ is the kernel of a surjective homomorphism of tori $T \to T'$. The following theorem computes the Galois cohomology of a quasi-torus $A$ in terms of the lattices $X_*(T)$ and $X_*(T')$.

3.15. **Theorem.** Let $A$ be an $R$-quasi-torus, the kernel of a surjective homomorphism of $R$-tori $j: T \to T'$. Then there are canonical isomorphisms

$$\vartheta^k: H^k(X_*(T) \to X_*(T')) \cong H^{k+1}A.$$

**Proof.** We have a short exact sequence

$$1 \to A \xrightarrow{i} T \xrightarrow{j} T' \to 1,$$

whence we obtain a quasi-isomorphism

$$i_*: (A \to 1) \to (T \to T'),$$

an induced isomorphism

$$i^k_\#: H^{k+1}A \cong H^k(T \to T'),$$

and a composite isomorphism

$$\vartheta^k: H^k(X_*(T) \to X_*(T')) \xrightarrow{ev_k} H^k(T \to T') \xrightarrow{(i^k_\#)^{-1}} H^{k+1}A,$$

as required. \qed

3.17. **Lemma.** The following exact diagram, in which the arrows in the top and middle rows are from (2.6), and the arrows in the bottom row are from (1.6), is commutative:

$$\begin{array}{cccccc}
H^kX_*(T') & \overset{\cong}{\longrightarrow} & H^k(X_*(T) \to X_*(T')) & \overset{\cong}{\longrightarrow} & H^{k+1}X_*(T) \\
\cong & \cong & \cong & \cong & \\
H^kT' & \xrightarrow{\cong} & H^k(T \to T') & \xrightarrow{\cong} & H^{k+1}T \\
\cong & \cong & \cong & \cong & \\
H^kT' & \xrightarrow{i^k_\#} & H^{k+1}A & \xrightarrow{\cong} & H^{k+1}T \\
\end{array}$$

**Proof.** The top half of the diagram is clearly commutative, and the bottom half is commutative by Lemma 2.18. \qed
4. Nonabelian cohomology for abstract $\Gamma$-groups

4.1. Let $A$ be a $\Gamma$-group (written multiplicatively), that is, a group (not necessarily abelian) endowed with an action of $\Gamma$. We consider the first cohomology $H^1(\Gamma, A)$. We write $H^1 A$ for $H^1(\Gamma, A)$. Recall that

$$H^1 A = Z^1 A/\sim,$$

where $Z^1 A = \{ a \in A \mid a \cdot \gamma a = 1 \}$, and two 1-cocycles (elements of $Z^1 A$) $a_1$, $a_2$ are equivalent or cohomologous (we write $a_1 \sim a_2$) if there exists $a' \in A$ such that

$$a_2 = a' \cdot a \cdot (\gamma a')^{-1}.$$

If $a \in Z^1 A$, we write $[a] \in H^1 A$ for the cohomology class of $a$. The set $H^1 A$ has a canonical neutral element $[1]$, the class of the cocycle $1 \in Z^1 A$. The correspondence $A \sim H^1 A$ is a functor from the category of $\Gamma$-groups to the category of pointed sets.

If the group $A$ is abelian, then

$$H^1 A = Z^1 A/B^1 A,$$

where the abelian subgroups $Z^1 A$ and $B^1 A$ were defined as in Subsection 1.1. Thus $H^1 A$ is naturally an abelian group in this case.

4.2. Construction. Let

$$(4.3) \quad 1 \to A \to B \to C \to 1$$

be a short exact sequence of $\Gamma$-groups. Then we have a cohomology exact sequence

$$1 \to A^\Gamma \to B^\Gamma \to C^\Gamma \to H^1 A \to H^1 B \to H^1 C;$$

see Serre [Ser97, I.5.5, Proposition 38]. We recall the definition of the map $\delta$ in our case of the group $\Gamma$ of order 2.

Let $c \in C^\Gamma$. We lift $c$ to an element $b \in B$ and set $a = b^{-1} \cdot \gamma b \in B$. It is easy to check that in fact $a \in Z^1 A \subseteq A$. We set $\delta(c) = [a] \in H^1 A$.

4.4. Construction. Assume that in (4.3) the subgroup $A$ is central in $B$. Then we have a cohomology exact sequence

$$1 \to A^\Gamma \to B^\Gamma \to C^\Gamma \to H^1 A \to H^1 B \to H^1 C \to H^2 A;$$

see Serre [Ser97, I.5.7, Proposition 43]. We recall the definition of the map $\delta^1$ in our case of the group $\Gamma$ of order 2.

Let $c \in Z^1 C \subseteq C$; then $c \cdot \gamma c = 1$. We lift $c$ to some element $b \in B$; then $b \cdot \gamma b \in A$. We set $a = b \cdot \gamma b$. We have $\gamma a = \gamma b \cdot b$. Since $\gamma a \in A$ and $A$ is central in $B$, we have

$$\gamma a = \gamma b^{-1} \cdot \gamma a \cdot \gamma b = \gamma b^{-1} \cdot \gamma b \cdot b \cdot \gamma b = b \cdot \gamma b = a.$$

Thus $a \in Z^2 A$, and we set $\delta^1[c] = [a] = [b \cdot \gamma b] \in H^2 A$.

Note that when the groups $A$, $B$, and $C$ are abelian, we have $\delta(c) = \delta^0[c]$ for $c \in C^\Gamma = Z^0 C$, where $\delta^0 : H^0 C \to H^1 A$ is the homomorphism of Section 1.5. Moreover, then our map $\delta^1$ coincides with the homomorphism $\delta^1$ of Section 1.5.

5. Nonabelian Galois cohomology of real algebraic groups

5.1. Notation. Let $G$ be an algebraic $R$-group, not necessarily abelian. We write

$$H^1 G = H^1(\Gamma, G) := H^1(\Gamma, G(\mathbb{C})).$$

The group $G(\mathbb{C})$ is a complex Lie group and $G(\mathbb{R})$ is a real Lie group. If $G$ is connected in the Zariski topology, then $G(\mathbb{C})$ is connected in the usual Hausdorff topology, but $G(\mathbb{R})$ is
not necessarily connected even if $G$ were connected. Let $G(R)^0 \subseteq G(R)$ denote the identity component, which is clearly normal in $G(R)$ and open for the Hausdorff topology. We write

$$\pi_b^R G = \pi_0 G(R) := G(R)/G(R)^0.$$  

If $g \in G(R)$, we write $[g] = gG(R)^0 \in \pi_b^R G$ for the class of $g$.

5.2. Let

$$1 \to A \xrightarrow{i} B \xrightarrow{j} C \to 1$$

be a short exact sequence of real algebraic groups (not necessarily linear or connected). Then we have a short exact sequence of $\Gamma$-groups

$$1 \to A(C) \xrightarrow{i} B(C) \xrightarrow{j} C(C) \to 1,$$

whence a cohomology exact sequence

$$1 \to A(R) \xrightarrow{i} B(R) \xrightarrow{j} C(R) \xrightarrow{\delta} H^1 A \xrightarrow{i_1} H^1 B \xrightarrow{j_1} H^1 C.$$

5.4. Proposition.

(i) The map $\delta : C(R) \to H^1 A$ induces a map

$$\delta^0 : \pi_b^R C \to H^1 A.$$  

(ii) The following sequence is exact:

$$\pi_b^R A \xrightarrow{i_0} \pi_b^R B \xrightarrow{j_0} \pi_b^R C \xrightarrow{\delta^0} H^1 A \xrightarrow{i_1} H^1 B \xrightarrow{j_1} H^1 C.$$  

Proof. We show that $j(B(R)^0) = C(R)^0$. Indeed, since $B(R)^0$ is connected, we have $j(B(R)^0) \subseteq C(R)^0$. On the other hand, since the homomorphism $j : B(C) \to C(C)$ is surjective, we see that the differential

$$dj : \text{Lie } B \to \text{Lie } C$$

is surjective (over $C$, and hence over $R$), and therefore, the image $j(B(R)^0) \subseteq C(R)^0$ contains an open neighborhood of $1$. It follows that $j(B(R)^0) = C(R)^0$.

We prove (i). We define the map $\delta^0$ by $\delta^0[c] = \delta(c)$ for $c \in C(R)$. We show that the map $\delta^0$ is well defined. Indeed, let $c_1, c_2 \in C(R)$, $c_2 = c_0c_1$ for some $c_0 \in C(R)^0$. Then $c_0 = j(b_0)$ for some $b_0 \in B(R)^0 \subseteq B(R)$, and hence $\delta(c_1) = \delta(c_2)$, as required.

We prove (ii). We show that the sequence (5.5) is exact at $\pi_b^R C$. Indeed, let $[c_1], [c_2] \in \pi_b^R C, c_1, c_2 \in C(R)$. Assume that $\delta^0[c_1] = \delta^0[c_2]$. Then $\delta(c_1) = \delta(c_2)$, and hence, $c_2 = j(b)c_1$ for some $b \in B(R)$. Thus $[c_2] = j^0[b] \cdot [c_1]$, which shows that the sequence (5.5) is exact at $\pi_b^R C$. Moreover, the map $\delta^0$ induces a map

$$\pi_b^R C / j_0^0(\pi_b^R B) \to H^1 A.$$  

We show that the sequence (5.5) is exact at $\pi_b^R B$. Let $[b] \in \pi_b^R B$, where $b \in B(R)$. Assume that $j^0[b] = 1$. Then $j(b) \in C(R)^0$. It follows that $j(b) = j(b_0)$ for some $b_0 \in B(R)^0$. Then $b = i(a) \cdot b_0$ for some $a \in A(R)$, and hence $[b] = i_1^0[a] \cdot [b_0] = i_1^0[a]$, as required.

The exactness of (5.5) at $H^1 A$ and at $H^1 B$ follows from the exactness of (5.3). \qed
6. Taking Quotient by the Unipotent Radical

6.1. Let $U$ be a unipotent $R$-group. We may assume that $U$ is embedded into $GL_{n,R}$ for some natural number $n$. We consider the exponential and logarithm maps

$$\exp: u \to U, \quad x \mapsto 1 + x + x^2/2! + x^3/3! + \cdots \quad \text{for } x \in u;$$

$$\log: U \to u, \quad u \mapsto y - y^2/2 + y^3/3 - \cdots \quad \text{for } u = 1 + y \in U.$$ 

Since $U$ is unipotent, these two maps are regular (polynomial). They are mutually inverse and defined over $R$.

6.2. Lemma (well-known). Let $U$ be a unipotent $R$-group. Then:

(i) $\pi^R_0 U = \{1\}$;

(ii) $H^1 U = \{1\}$.

Proof. (i) The exponential map $\exp: u \to U$ is an isomorphism of real algebraic varieties, and hence it induces an isomorphism of real analytic varieties $u(R) \cong U(R)$ and an isomorphism of their component groups $\{1\} = \pi_0 u(R) \cong \pi^R_0 U$.

(ii) See Serre [Ser97, Section III.2.1] for a proof using induction on $\dim U$ (over an arbitrary perfect field). Here we give a constructive proof over $R$ in one step. Let $z \in Z U$. We have $z \cdot \gamma z = 1$, whence $\gamma z = z^{-1}$. Set $u = \exp(-\frac{1}{2} \log z)$; then $\gamma u = u^{-1}$ and $u^2 = z^{-1}$. We have $u^{-1} \cdot \gamma u = u^{-2} = z$, whence $z \sim 1$ and $[z] = [1]$, as required. □

6.3. Let $G$ be a linear $R$-group, not necessarily connected. Let $G^u$ denote the unipotent radical of $G$, that is, the largest normal unipotent subgroup of $G$. We say that $G$ is reductive if $G^u = \{1\}$. For any linear algebraic $R$-group $G$, we set $G^{\text{red}} = G/G^u$, which is a reductive $R$-group (not necessarily connected).

6.4. Theorem (Mostow). Let $G$ be a linear algebraic $R$-group, not necessarily connected. Then the short exact sequence

$$1 \to G^u \longrightarrow G \longrightarrow G^{\text{red}} \to 1$$

splits, that is, there exists a homomorphism of $R$-groups (a splitting) $s: G^{\text{red}} \to G$ such that $r \circ s = \text{id}_{G^{\text{red}}}$. 

Proof. See Mostow [Mos56, Theorem 7.1], or Hochschild [Hoc81, Section VIII.4, Theorem 4.3], or [grp], for proofs over any field of characteristic 0. □

6.5. Theorem (well-known). Let $G$ be a linear $R$-group, not necessarily connected. Consider the canonical surjective homomorphism $r: G \to G^{\text{red}}$. Then:

(i) The induced map $r_1^*: H^1 G \to H^1 G^{\text{red}}$ is bijective.

(ii) The induced homomorphism $r_0^*: \pi^R_1 G \to \pi^R_1 G^{\text{red}}$ is an isomorphism.

Proof. We prove assertion (i). It follows from Sansuc’s lemma; see [San81, Lemma 1.13] or [BDR21, Proposition 3.1]. Here we deduce it from Theorem 6.4. Since $r \circ s = \text{id}$, we obtain

$$r_1^* \circ s_1^* = \text{id}: H^1 G^{\text{red}} \to H^1 G \to H^1 G^{\text{red}},$$

whence we see that the map $r_1^*$ is surjective.

We prove the injectivity. Write $U = G^u$. Let $g \in Z G$. By Serre [Ser97, I.5.5, Corollary 2 of Proposition 39], the fiber of the map $r_1^*$ over $r_1^*[g] \in H^1 G^{\text{red}}$ is in a canonical bijection with the quotient of the set $H^1_{\partial g} U$ by the group $(\partial g G^{\text{red}})(R)$. Here the left subscript $g$ denotes the twisting of an $R$-group structure by the cocycle $g$; see [Ser97, Section I.5.3]. Since the $R$-group $\partial U$ is unipotent, by Lemma 6.2(ii) the cohomology set $H^1_{\partial g} U$ is a singleton. We
conclude that the fiber of the map \( r_1 \) over \( r_1^*[g] \) is a singleton. Thus the map \( r_1 \) is indeed injective.

We prove assertion (ii). We use Theorem 6.4. We have an isomorphism of \( \mathbb{R} \)-varieties
\[
\mathbf{U} \times G^{\text{red}} \rightarrow G, \quad (u, h) \mapsto u \cdot s(h) \quad \text{for} \quad u \in U, \ h \in G^{\text{red}}.
\]
Passing to \( \mathbb{R} \)-points, we obtain an isomorphism of real analytic manifolds
\[
\mathbf{U}(\mathbb{R}) \times G^{\text{red}}(\mathbb{R}) \overset{\sim}{\rightarrow} G(\mathbb{R}),
\]
where \( \mathbf{U}(\mathbb{R}) \) is connected by Lemma 6.2(i). We obtain an isomorphism
\[
\pi_0^R G^{\text{red}} = \pi_0^R \mathbf{U} \times \pi_0^R G^{\text{red}} \overset{\sim}{\rightarrow} \pi_0^R G,
\]
which is clearly inverse to the homomorphism \( r_1^* \).

\[\Box\]

6.6. Remark. Let \( G \) be a connected linear algebraic \( \mathbb{R} \)-group; then \( G^{\text{red}} \) is a connected reductive group. In Sections 7 and 9 we shall compute \( H^1 G^{\text{red}} \) and \( \pi_0^R G^{\text{red}} \). By Theorem 6.5 this will give us \( H^1 G \) and \( \pi_0^R G \).

7. Galois cohomology of a reductive group

In this section we compute the Galois cohomology of an arbitrary connected reductive \( \mathbb{R} \)-group in transparent combinatorial terms. We freely use the notation of [BT21]. From now on, by a semisimple or reductive algebraic group we always mean a connected semisimple or reductive algebraic group, respectively. We use the following notation:

7.1. Notation.

- \( G \) is a connected reductive \( \mathbb{R} \)-group. In this section \( G \) is compact (anisotropic).
- \( G^{ss} = [G, G] \), the commutator subgroup of \( G \), which is semisimple.
- \( G^{ss} \) is the universal cover of \( G^{ss} \), which is simply connected.
- \( \rho: G^{ss} \rightarrow G^{ss} \leftarrow G \) is the canonical homomorphism.
- \( Z = Z(G) \), the center of \( G \).
- \( Z^{ss} = Z(G^{ss}) = \rho^{-1}(Z) \).
- \( G^{\text{ad}} = G/Z \cong G^{ss}/Z^{ss} \), which is a semisimple group of adjoint type.
- \( T \subset G \) is a maximal torus.
- \( T^{ss} = T \cap G^{ss} \subset G^{ss} \).
- \( T^{ss} = \rho^{-1}(T) \subset G^{ss} \).
- \( T^{\text{ad}} = T/Z \subset G^{\text{ad}} \).
- \( S = Z(G)^0 \), the identity component of \( Z = Z(G) \).
- \( \mathcal{E}: t \rightarrow T, \mathcal{E}^{ss}: t^{ss} \rightarrow T^{ss}, \mathcal{E}^{ss}: t^{ss} \rightarrow T^{ss}, \mathcal{E}^{\text{ad}}: t^{\text{ad}} \rightarrow T^{\text{ad}} \) and \( \mathcal{E}_S: s \rightarrow S \) are the scaled exponential maps given by the formula \( x \mapsto \exp 2\pi x \) (note that we identify \( t^{ss} = t^{\text{ad}} \)).
- \( A^{(2)} \) denotes the set of elements of order dividing 2 in a subset \( A \) of some group.

7.2. As in [BT21], let \( G = (G, \sigma_c) \) be a compact connected reductive \( \mathbb{R} \)-group, with the action of \( \gamma \) on \( G \) given by an anti-regular involutive automorphism \( \sigma_c \). Let \( T \subset G \) be a maximal torus, and \( B \subset G \) be a Borel subgroup containing \( T \). Let \( \mathcal{B} = \text{BRD}(G, T, B) \) denote the based root datum of \( (G, T, B) \); see Springer [Spr79, Sections 1 and 2]. Recall that
\[
\text{BRD}(G, T, B) = (X, \chi^\vee, \mathcal{R}, \mathcal{R}^\vee, S, S^\vee)
\]
where

- \( X = X^\vee(T) \) is the character group of \( T \), and \( X^\vee = \chi^\vee(T) \) is the cocharacter group;
- \( \mathcal{R} = \mathcal{R}(G, T) \subset X \) is the root system, and \( \mathcal{R}^\vee = \mathcal{R}^\vee(G, T) \subset X^\vee \) is the coroot system;
• $S = S(G, T, B) \subset R$ is the system of simple roots, and $S^\vee = S^\vee(G, T, B) \subset R^\vee$ is the system of simple coroots with respect to $B$.

Let $\tau$ be an involutive automorphism (maybe identity) of $(G, T, B)$ coming from an automorphism of $B$. Let $T_0$ denote the identity component of the fixed point subgroup $T^\tau$, and $\theta \in \text{Aut}(G, T, B)$ be an involutive automorphism of the form $\theta = \text{inn}(t_\theta) \circ \tau$ with $t_\theta \in T_0$ and $t_\theta^2 \in Z$. Our aim is to compute $H^1(R, \theta G)$, where $\theta G$ is the corresponding twisted $R$-group with real structure (the action of $\gamma$ on $G$) given by $\sigma = \theta \circ \sigma_c$.

7.3. Recall that $S$ is the connected center and $G^{ss}$ is the derived subgroup of $G$. We have a decomposition into an almost direct product $G = G^{ss} \cdot S$. There is a chain of isogenies

$$G^{sc} \times S \rightarrow G \rightarrow G^{\text{ad}} \times \overline{S},$$

where $G^{sc}$ is the simply connected cover of $G^{ss}$, $G^{\text{ad}}$ is the adjoint group of $G$, and $\overline{S} = S/(S \cap G^{ss})$. Note that $T^{ss} = T \cap G^{ss}$ is a maximal torus in $G^{ss}$, its preimage $T^{sc}$ in $G^{sc}$ is a maximal torus in $G^{sc}$, and its image $T^{\text{ad}}$ in $G^{\text{ad}}$ is a maximal torus in $G^{\text{ad}}$. There is a chain of isogenies

$$T^{sc} \times S \rightarrow T = T^{ss} \cdot S \rightarrow T^{\text{ad}} \times \overline{S}.$$  

The respective inclusion of character lattices reads as

$$P \oplus \Lambda \supseteq X \supseteq Q \oplus M,$$

where $X = X^*(T)$, $\Lambda = X^*(S)$, and $M = X^*(\overline{S})$. As usual, we denote by $P = X^*(T^{sc})$ the weight lattice and by $Q = X^*(T^{\text{ad}})$ the root lattice of the root system $R$. Note that $P \supset Q$ and $\Lambda \supseteq M$.

7.4. Lemma. $M = X \cap \Lambda$.

Proof. We have $\overline{S} = T/T^{ss}$, and therefore $M = X^*(\overline{S})$ consists of the characters of $T$ that become trivial (identically 1) when restricted to $T^{ss}$. If we regard the characters of $T$ as characters of $T^{sc} \times S$, then $M$ consists of the characters of $T$ that are trivial on $T^{sc}$, that is, are contained in the direct summand $\Lambda$ of $P \oplus \Lambda$, as required.

7.5. The respective inclusion of cocharacter lattices reads as

$$Q^\vee \oplus \Lambda^\vee \subseteq X^\vee \subseteq P^\vee \oplus M^\vee,$$

where $X^\vee = X_\vee(T)$, $\Lambda^\vee = X_\vee(S)$, and $M^\vee = X_\vee(\overline{S})$. Then $X^\vee$ is dual to $X$, $\Lambda^\vee$ is dual to $\Lambda$, and $M^\vee$ is dual to $M$. As usual, we denote by $Q^\vee = X_\vee(T^{sc})$ and $P^\vee = X_\vee(T^{\text{ad}})$ the coroot and coweight lattice, respectively, so that the lattice $P^\vee$ is dual to $Q$, and the lattice $Q^\vee$ is dual to $P$. Note $Q^\vee \subseteq P^\vee$ and $\Lambda^\vee \subseteq M^\vee$.

7.6. Lemma. Let $A, B, L, M, Y$ be lattices (finitely generated free abelian groups) such that

$$A \supseteq B, \quad L \supseteq M, \quad A \oplus L \supseteq Y \supseteq B \oplus M.$$

Assume that $[L : M] < \infty$. Then $Y \cap L = M$ if and only if the natural map of the dual lattices $Y^\vee \rightarrow M^\vee$ induced by the inclusion $M \hookrightarrow Y$ is surjective.

Proof. We have $Y \supseteq M$ and $L \supseteq M$; hence $Y \cap L \supseteq M$. Moreover,

$$(7.7) \quad [(Y \cap L) : M] \leq [L : M] < \infty.$$

If $Y \cap L = M$, then $Y/M = Y/(Y \cap L)$ embeds into $(A \oplus L)/L = A$, and hence $Y/M$ is torsion free. It follows that $M$ is a direct summand of $Y$, and therefore the natural map $Y^\vee \rightarrow M^\vee$ is surjective, as required.

Conversely, if the map $Y^\vee \rightarrow M^\vee$ is surjective, then $Y/M$ is torsion free, and hence $(Y \cap L)/M$ is torsion free. Now it follows from (7.7) that $Y \cap L = M$, as required. \hfill \Box
Since by Lemma 7.4 we have $X \cap \Lambda = M$, by Lemma 7.6 the lattice $X^\vee$ projects onto $M^\vee$. Since $S$ embeds into $T$, the lattice $X$ projects onto $\Lambda$ in the direct sum $P \oplus \Lambda$, and by Lemma 7.6 we have $X^\vee \cap M^\vee = \Lambda^\vee$.

7.8. Consider the almost direct product decomposition $T = T_0 \cdot T_1$, where $\theta$ and $\tau$ act on $T_0$ trivially and on $T_1$ as inversion. The subtori $T_0$ and $T_1$ of $T = T \times \mathbb{R}C$ are defined over $\mathbb{R}$, and we denote the corresponding $\mathbb{R}$-subtori of $T$ by $T_0$ and $T_1$. We have similar decompositions for $S$, $\tilde{S}$, $T^{ss}$, $T^{sc}$, and $T^{ad}$.

Note that $\theta T$ is a fundamental torus in $\theta G$, that is, $\theta T_0 = T_0$ is a maximal compact torus and $\theta T$ is the centralizer of $\theta T_0$ in $\theta G$. Note also that $\theta T_1$ is a split $\mathbb{R}$-torus.

We set
\[ X_0 = X^*(T_0), \quad \Lambda_0 = X^*(S_0), \quad M_0 = X^*(\tilde{S}_0). \]
Then $X_0$ is the restriction of $X$ to $T_0$, $\Lambda_0$ is the restriction of $\Lambda$ to $S_0$, and $M_0$ is the restriction of $M$ to $\tilde{S}_0$. Note that $X_0$, $\Lambda_0$, and $M_0$ are identified with the images of $X$, $\Lambda$, and $M$, respectively, under the canonical projection $t^* \to \nu^*_0$; see [BT21, 7.3].

We also consider the dual lattices $X^\vee_0 = X_*(T_0)$, $\Lambda^\vee_0 = X_*(S_0)$, and $M^\vee_0 = X_*(\tilde{S}_0)$, which we regard as subgroups of $t$. We have
\[ X^\vee_0 = (X^\vee)^\tau = \{ \nu \in X^\vee \mid \tau(\nu) = \nu \}, \]
and similarly for $\Lambda^\vee_0$ and $M^\vee_0$; they are the intersections with $t_0$ of $X^\vee$, $\Lambda^\vee$, and $M^\vee$, respectively. Finally, let $\tilde{X}_0^\vee$, $\tilde{\Lambda}_0^\vee$, and $\tilde{M}_0^\vee$ denote the images of $X^\vee$, $\Lambda^\vee$, and $M^\vee$, respectively, under the canonical projection $t \to t_0$, $\nu \mapsto (\nu + \tau(\nu))/2$; see [BT21, 7.4]. Note that
\[ X^\vee_0 \subseteq \tilde{X}_0^\vee \subseteq \frac{1}{2} X^\vee, \]
and similarly for $\tilde{\Lambda}_0^\vee$, $\tilde{M}_0^\vee$. There are chains of inclusions:
\[ P_0^\vee \oplus \Lambda_0 \supseteq X_0 \supseteq Q_0 \oplus M_0, \]
\[ Q_0^\vee \oplus \Lambda_0^\vee \subseteq X^\vee_0 \subseteq P_0^\vee \oplus M^\vee_0, \]
\[ Q_0^\vee \oplus \tilde{\Lambda}_0^\vee \subseteq \tilde{X}_0^\vee \subseteq \tilde{P}_0^\vee \oplus \tilde{M}_0^\vee. \]

Write
\[ \theta G^{ss} = \theta G^{(1)} \times \cdots \times \theta G^{(s)}, \]
where each $\theta G^{(k)}$ is $\mathbb{R}$-simple. Let
\[ \tilde{D} = \tilde{D}^{(1)} \sqcup \cdots \sqcup \tilde{D}^{(s)} \]
denote the affine Dynkin diagram of $(\theta G, \theta T, B)$; see [BT21, Section 12]. By abuse of notation we write $\beta \in \tilde{D}$ if $\beta$ is a vertex of $\tilde{D}$. The affine Dynkin diagram $\tilde{D}$ comes with a family of positive integers $m_\beta$ for $\beta \in \tilde{D}$; see [BT21, Sections 9, 10, and 11].

7.9. Definition ([BT21]). A Kac labeling of $\tilde{D}$ is a family of nonnegative integer numerical labels $p = (p_\beta)_{\beta \in \tilde{D}}$ at the vertices $\beta$ of $\tilde{D}$ satisfying
\[ \sum_{\beta \in \tilde{D}^{(k)}} m_\beta p_\beta = 2 \quad \text{for each } k = 1, \ldots, s. \]
We denote the set of Kac labelings of $\tilde{D}$ by $K(\tilde{D})$.

To any $x \in t_0^{ss}(\mathbb{R})$, we assign a family $p = p(x) = (p_\beta)_{\beta \in \tilde{D}}$ of real numbers $p_\beta = 2x_\beta$, where the real numbers $x_\beta$ are the barycentric coordinates of $x$ defined in [BT21, Sections 9.3, 10.3, 11.3]. This correspondence identifies $t_0^{ss}(\mathbb{R})$ with the subspace of $\mathbb{R}^{\tilde{D}}$ defined by the equations (7.10). We denote the inverse correspondence by $p \mapsto x = x(p)$. Let $\Delta = \Delta^{(1)} \times \cdots \times \Delta^{(s)}$.
denote the fundamental domain for the reflection group \( \widetilde{W}_0^{sc} = \widetilde{Q}_0^0 \rtimes W_0 \) acting in \( t_0(R) \), where \( W_0 = N(T_0)/T \) and \( N(T_0) \) is the normalizer of \( T_0 \) in \( G \); see [BT21, Sections 9.3, 10.3, 11.3, and 12.2] for the description of \( \Delta \). In particular, all \( \Delta^{(k)} \) are simplices, and \( x \in \Delta \) if and only if \( \rho_\beta \geq 0 \) for all \( \beta \in \tilde{D} \).

7.11. Consider the set \( K(\tilde{D}) \) of Kac labelings of the affine Dynkin diagram \( \tilde{D} \). For any \( p \in K(\tilde{D}) \) consider the associated point \( x(p) = \frac{1}{2} P^\vee \subset t_0(R) \) in the fundamental polyhedron \( \Delta \subset t_0(R) \) of the reflection group \( \widetilde{W}_0^{sc} \); see [BT21, 12.7]. Consider the scaled exponential maps

\[
\mathcal{E}: t \rightarrow T \quad \text{and} \quad \mathcal{E}^{ad}: t^{ad} \rightarrow T^{ad}, \quad x \mapsto \exp 2\pi x.
\]

Then \( \mathcal{E}^{ad}(x(p)) \in (T_0^{ad})^{(2)} \). In particular, we may and shall assume that \( \text{inn}(t_\theta) = \mathcal{E}^{ad}(x(q)) \) and \( t_\theta = \mathcal{E}(x(q)) \) for some \( q \in K(\tilde{D}) \); see [BT21, 12.12]. We write

\[
\mathbf{G}(B, \tau, q) = \theta \mathbf{G}
\]

(the real reductive group corresponding to \( B, \tau, \text{and} q \). Recall that \( B \) is the based root datum of \( (G, T, B) \).

For any \( m \in M_0^\vee \), we write \( y(m) = \frac{1}{2} m \in M_0^\vee \subset t_0(R) \). Set

\[
\nu_{p,q,m} = \frac{1}{t}(x(p) - x(q) + y(m)) \in P_0^{ad} \oplus M_0^\vee.
\]

Recall that in [BT21, Section 12.7] we defined a pairing

\[
\langle \cdot, \cdot \rangle_P: P_0 \times K(\tilde{D}) \rightarrow Q, \quad (\lambda, p) \mapsto (\lambda, p)_P := \sum c_{\beta p \beta} \text{ for } \lambda = \sum c_\beta \beta \text{ with } c_\beta \in Q,
\]

where \( \beta \) runs over the set of restricted simple roots \( \tilde{\mathcal{S}} \subset Q_0 \). This pairing induces a well-defined pairing

\[
P_0/Q_0 \times K(\tilde{D}) \rightarrow Q/\mathbb{Z}.
\]

Furthermore, we have a canonical pairing

\[
\langle \cdot, \cdot \rangle_\Lambda: \Lambda_0 \times M_0^\vee \rightarrow C, \quad (\lambda, m) \mapsto (\lambda, m)_\Lambda \text{ for } \lambda \in \Lambda_0, \ m \in M_0^\vee,
\]

the restriction of the canonical pairing \( s_0^\vee \times s_0 \rightarrow C \). Since \( \Lambda_0 \subset M_0^{ss} \subset Q_0 \), the pairing \( \langle \cdot, \cdot \rangle_\Lambda \) takes values in \( Q \). If \( \lambda \in \Lambda_0 \) or \( m \in 2\Lambda_0^\vee \subset \Lambda_0^\vee \), then \( (\lambda, m)_\Lambda \in \mathbb{Z} \). We see that the pairing \( \langle \cdot, \cdot \rangle_\Lambda \) induces a well-defined pairing

\[
\Lambda_0/M_0 \times M_0^\vee /2\tilde{\Lambda}_0^\vee \rightarrow Q/\mathbb{Z}.
\]

Now if \( \lambda \in X_0 \subset P_0 \oplus \Lambda_0 \), we write \( \lambda = \lambda_P + \lambda_\Lambda \) with \( \lambda_P \in P_0, \ \lambda_\Lambda \in \Lambda_0 \).

7.12. Notation. We define the set of reductive Kac labelings \( K(\tilde{D}, \Lambda, X, \tau, q) \) to be the subset of \( K(\tilde{D}) \times M_0^\vee /2\tilde{\Lambda}_0^\vee \) consisting of all pairs \( (p, [m]) \) (with \( m \in M_0^\vee \)) satisfying

\[
(7.13) \quad (\lambda_P, p)_P + (\lambda_\Lambda, m)_\Lambda \equiv (\lambda_P, q)_P \pmod{\mathbb{Z}} \quad \text{for all } [\lambda] \in X_0/(Q_0 \oplus M_0).
\]

If \( \lambda \in Q_0 \oplus M_0 \), then the congruence (7.13) is satisfied for any \( p, q, m \), because \( (\lambda_P, p)_P, (\lambda_P, q)_P, \) and \( (\lambda_\Lambda, m)_\Lambda \) are integers in this case.

The finite abelian group

\[
F_0 = \bar{X}_0^\vee / (\bar{Q}_0^0 \oplus \bar{\Lambda}_0^0) \subset \bar{P}_0^0 / \bar{Q}_0^0 \oplus \bar{M}_0^0 / \bar{\Lambda}_0^0
\]

acts diagonally on \( K(\tilde{D}) \times M_0^\vee /2\tilde{\Lambda}_0^\vee \), where the action on \( K(\tilde{D}) \) is induced by the action of \( \bar{P}_0^0 / \bar{Q}_0^0 \) via automorphisms of the diagram \( \tilde{D} \) described in [BT21, §12] and the action on \( M_0^\vee /2\tilde{\Lambda}_0^\vee \) is induced by the translation action via the homomorphism

\[
\bar{M}_0^0 / \bar{\Lambda}_0^0 \rightarrow M_0^\vee /2\tilde{\Lambda}_0^\vee, \quad m + \bar{\Lambda}_0^0 \mapsto 2m + 2\tilde{\Lambda}_0^0 \in 2\bar{M}_0^0 /2\tilde{\Lambda}_0^0 \subset M_0^\vee /2\tilde{\Lambda}_0^\vee.
\]
7.14. Theorem. The group $F_0$, when acting on $\mathcal{K}(\tilde{D}) \times M_0^/ / 2\Lambda_0^\vee$, preserves the set of reductive Kac labelings $\mathcal{K}(\tilde{D}, \Lambda, X, \tau, q)$. For $p \in \mathcal{K}(\tilde{D})$, $m \in M_0^/$, we have $(p, [m]) \in \mathcal{K}(\tilde{D}, \Lambda, X, \tau, q)$ if and only if $\nu_{p,q,m} \in X_0^\vee$. The map

$$\kappa: \mathcal{K}(\tilde{D}, \Lambda, X, \tau, q) \rightarrow T_0^{(2)} \subset Z^1(\mathfrak{g}, \mathfrak{g}G),$$

(7.15) $$(p, [m]) \mapsto \exp 2\pi(x(p) - x(q) + y(m)) = \nu_{p,q,m}(-1)$$

is well defined and induces a bijection

$$\kappa_*: \mathcal{K}(\tilde{D}, \Lambda, X, \tau, q)/F_0 \simeq \mathcal{H}^1(\mathfrak{g}, \mathfrak{g}G)$$

between the set of $F_0$-orbits in $\mathcal{K}(\tilde{D}, \Lambda, X, \tau, q)$ and the first Galois cohomology set $\mathcal{H}^1(\mathfrak{g}, \mathfrak{g}G)$.

Proof. By [BT21, Prop. 5.6], the inclusion $T_0^{(2)} \subset Z^1(\mathfrak{g}, \mathfrak{g}G)$ induces a bijection between $H^1(\mathfrak{g}, \mathfrak{g}G)$ and the orbit set $T_0^{(2)}/N_\tau$ for the group $N_\tau \subset N(T_0)$ acting on $T_0^{(2)}$ by twisted conjugation; see [BT21, Sections 5.1, 5.2, and (4.4)]. The twisted conjugation action of $N_\tau$ preserves the set $T_0(\mathbb{R})$ containing $T_0^{(2)}$; see [BT21, Lemma 5.2(iii)].

We consider the semidirect product $G \rtimes \langle \hat{\tau} \rangle$, where $\langle \hat{\tau} \rangle$ is the group of order 1 or 2 acting faithfully on $G$ by conjugation so that $\hat{\tau}$ acts via $\tau$. The map

$$T_0(\mathbb{R}) \rightarrow T_0(\mathbb{R}) \cdot \hat{\tau} \subseteq G \cdot \hat{\tau}, \quad t \mapsto t\tau \cdot \hat{\tau}$$

is an $N_\tau$-equivariant bijection, where $N_\tau$ acts on $T_0(\mathbb{R}) \cdot \hat{\tau}$ by usual conjugation, and the subset $T_0^{(2)}$ maps bijectively onto

$$(T_0 \cdot \hat{\tau})^{(2)} := \{ g \in T_0 \cdot \hat{\tau} \mid g^2 = z \},$$

where $z = \tau^2_0$, see [BT21, Lemma 6.4]. The conjugation action of $N_\tau$ on $T_0(\mathbb{R}) \cdot \hat{\tau}$ factors through an effective action of $\tilde{W}_0 = N_\tau/T_0 \cong (T_0 \cap T_1) \times W_0$, where $(T_0 \cap T_1)$ acts by translations; see [BT21, Sections 8.3, 8.4, and 8.9].

Consider the orbit set $t_0(\mathbb{R})/\tilde{W}_0$, where $\tilde{W}_0 = \tilde{X}_0^\vee \times W_0$ acts on $t_0(\mathbb{R})$ by affine isometries in a natural way (see [BT21, Section 7.14]). By [BT21, Lemma 8.13], the shifted exponential map

$$\mathcal{E}: t_0(\mathbb{R}) \rightarrow T_0(\mathbb{R}) \cdot \hat{\tau}, \quad x \mapsto \exp 2\pi x \cdot \hat{\tau}$$

induces a bijection between the orbit sets $t_0(\mathbb{R})/\tilde{W}_0$ and $(T_0(\mathbb{R}) \cdot \hat{\tau})/\tilde{W}_0$.

We conclude that the map

$$t_0(\mathbb{R}) \rightarrow T_0(\mathbb{R}), \quad x \mapsto t\tau^{-1}, \quad \text{where} \quad t = \mathcal{E}(x),$$

induces a bijection between the orbit sets $t_0(\mathbb{R})/\tilde{W}_0$ and $T_0(\mathbb{R})/N_\tau$. Our aim is to identify the subset of $t_0(\mathbb{R})/\tilde{W}_0$ corresponding to $T_0^{(2)}/N_\tau \cong H^1(\mathfrak{g}, \mathfrak{g}G)$ and provide explicit orbit representatives.

Consider the normal subgroup

$$\tilde{W}_0^{sc} \times \tilde{\Lambda}_0^\vee = (\tilde{Q}_0^\vee \oplus \tilde{\Lambda}_0^\vee) \times W_0 \subseteq \tilde{X}_0^\vee \times W_0 = \tilde{W}_0$$

and the action of $\tilde{W}_0/(\tilde{W}_0^{sc} \times \tilde{\Lambda}_0^\vee)$ on $\Delta \times s_0(\mathbb{R})/\tilde{\Lambda}_0^\vee$, where the group $F_0 \subseteq (\tilde{P}_0^\vee / \tilde{Q}_0^\vee) \oplus (M_0^/ / \tilde{\Lambda}_0^\vee)$ acts on $\Delta$ via the action of $\tilde{P}_0^\vee / \tilde{Q}_0^\vee$ described in [BT21, §12], and on $s_0(\mathbb{R})/\tilde{\Lambda}_0^\vee$ via the translation action of $\tilde{M}_0^/ / \tilde{\Lambda}_0^\vee$. The set of orbits of $\tilde{W}_0^{sc} \times \tilde{\Lambda}_0^\vee$ acting on $t_0(\mathbb{R})$ is identified with $\Delta \times s_0(\mathbb{R})/\tilde{\Lambda}_0^\vee$, and the inclusion map

$$\Delta \times s_0(\mathbb{R}) \hookrightarrow t_0(\mathbb{R})$$
induces a bijection between the set of orbits of the group $F_0$ in $\Delta \times s_0(\mathbb{R})/i \tilde{\Lambda}^\vee_0$ and the set of orbits of $\tilde{W}_0$ in $t_0(\mathbb{R})$. We obtain a composite bijection

$$(\Delta \times s_0(\mathbb{R})/i \tilde{\Lambda}^\vee_0)/F_0 \cong t_0(\mathbb{R})/\tilde{W}_0 \cong T_0(\mathbb{R})/N_r.$$  

We see that every $\tilde{W}_0$-orbit in $t_0(\mathbb{R})$ is represented by a vector $x = x' + y$, where $x' \in \Delta$ and $y \in s_0(\mathbb{R})$. The orbit of $x$ corresponds to a cohomology class in $H^1(\mathbb{R}, \rho \mathcal{G})$ if and only if $\eta^{-1} = \mathcal{E}(x - x(q)) \in T_0^{(2)}$. This condition reads as $x - x(q) \in \frac{1}{2} X_0^\vee$ or, equivalently, as

$$\lambda(x) \equiv \lambda(x(q)) \pmod{\frac{1}{2} \mathbb{Z}}$$

for all $\lambda \in X_0$.

Assume that (7.16) is satisfied. Since $\gamma^2 \in Z(G)$, we have $\lambda(x(q)) \in \frac{1}{2} \mathbb{Z}$ for all $\lambda \in Q_0$. We see that for all $\lambda \in Q_0 \subseteq X_0$ we have

$$\lambda(x') = \lambda(x) \in \frac{1}{2} \mathbb{Z}.$$  

Let $(x, \beta)$ for $\beta \in \tilde{D}$ denote the barycentric coordinates of $x'$, and write $p_\beta = 2x, \beta$. Then from (7.17) and the definitions of the barycentric coordinates in [BT21, Sections 9.3, 10.3, and 11.3] it follows that all $p_\beta$ are integers. Since $x \in \Delta$, the numbers $p_\beta$ are nonnegative. It follows from [BT21, Section 12.7] that the $p_\beta$ satisfy (7.10). Thus $p = (p_\beta) \in K(\tilde{D})$ and $x = x(p)$. Similarly, we see that for all $\lambda \in M_0 \subseteq X_0$ we have $\lambda(y) = \lambda(x) \in \frac{1}{2} \mathbb{Z}$, whence $y \in \frac{1}{2} M_0'$ and therefore $y = y(m)$ for some $m \in M_0'$.

Conversely, if $p \in K(\tilde{D})$, $m \in M_0'$, $x = x(p) + y(m)$, and $t = \mathcal{E}(x)$, then $x(p) \in \Delta$ and

$$\lambda(x) \equiv \lambda(x(q)) \pmod{\frac{1}{2} \mathbb{Z}}$$

for all $\lambda \in Q_0 \oplus M_0$.

Now we see that (7.16) is equivalent to (7.13), that is, $tt^{-1} = T_0^{(2)}$ if and only if $(p, \gamma) \in K(\tilde{D}, \Lambda, X, \tau, q)$.

The congruences (7.16) can also be written as

$$\lambda(x(p) - x(q) + y(m)) \in \frac{1}{2} \mathbb{Z}, \text{ that is, } \lambda(\nu_{p,q,m}) \in \mathbb{Z} \text{ for all } \lambda \in X_0.$$  

Hence $(p, \gamma) \in K(\tilde{D}, \Lambda, X, \tau, q)$ if and only if $\nu_{p,q,m} \in X_0^\vee$, and the cocycle in $T_0^{(2)} \subseteq Z^1_{\rho \mathcal{G}}$ corresponding to $(p, \gamma)$ is

$$tt^{-1} = \mathcal{E}(x(p) - x(q) + y(m)) = \exp \pi i \nu_{p,q,m} = \nu_{p,q,m}(-1),$$

given by formula (7.15), where the last equality follows from [BT21, (7.2)].

We conclude that the Galois cohomology classes in $H^1(\mathbb{R}, \rho \mathcal{G})$ are represented by the elements $\nu_{p,q,m}(-1) \in T_0^{(2)}$ with $(p, \gamma) \in K(\tilde{D}, \Lambda, X, \tau, q)$ defined up to the action of $F_0$ and this correspondence is bijective. This completes the proof of Theorem 7.14.

8. Additional structures on Galois cohomology of a reductive group

Starting from this section, $\mathcal{G}$ is a connected reductive $\mathbb{R}$-group, not necessarily compact.

8.1. Let

$$\varphi : \mathcal{G} \to \mathcal{G}''$$

be a normal homomorphism of connected reductive $\mathbb{R}$-groups. Here “normal” means that the image $\text{im} \varphi$ is normal in $\mathcal{G}''$. We write $\mathcal{H} = \text{im} \varphi$. We wish to describe the induced map on Galois cohomology.

The normal homomorphism $\varphi$ induces a homomorphism

$$\varphi^{\text{ad}} : \mathcal{G}^{\text{ad}} \to \mathcal{G}''^{\text{ad}}$$

with normal image $\mathcal{H}^{\text{ad}}$. Clearly, $\mathcal{H}^{\text{ad}}$ is the direct factor of both $\mathcal{G}^{\text{ad}}$ and $\mathcal{G}''^{\text{ad}}$.  

Consider the affine Dynkin diagrams \( \tilde{D}' = \tilde{D}(G') \), \( \tilde{D}'' = \tilde{D}(G'') \) and \( \tilde{D} = \tilde{D}(H) \). Then \( \tilde{D} \) naturally embeds into \( \tilde{D}' \) and into \( \tilde{D}'' \). Let \( q \in \mathcal{K}(\tilde{D}) \) denote a Kac labeling defining the R-structure of \( H \) (note that \( q \) is defined not uniquely). Then we may choose Kac labelings \( q' \in \mathcal{K}(\tilde{D}') \) defining the R-structure of \( G' \) and \( q'' \in \mathcal{K}(\tilde{D}'') \) defining the R-structure of \( G'' \) such that \( q'|_{\tilde{D}} = q = q''|_{\tilde{D}} \). We may write \( G' = G(B', \tau', q') \), \( G'' = G(B'', \tau'', q'') \), \( H = G(B, \tau, q) \) with the notation of Subsection 7.11.

**8.2. Proposition.** Let \( \varphi : G' \to G'' \) be a normal homomorphism of reductive R-groups, not necessarily compact, with image \( H = \im \varphi \). Let \( q \in \mathcal{K}(\tilde{D}) \), \( q' \in \mathcal{K}(\tilde{D}') \), and \( q'' \in \mathcal{K}(\tilde{D}'') \) be as above. Let \((p', [m']) \in \mathcal{K}(\tilde{D}', \Lambda', X', \tau', q')\), \( \nu_{p',q',m'}(-1) \in (T_0^{(2)}) \subset Z^1 G' \), and \([\nu_{p',q',m'}(-1)] \in H^1 G'\) be as in Theorem 7.14. Then

\[
\varphi(\nu_{p',q',m'}(-1)) = \nu_{p'',q'',m''}(-1),
\]

and hence

\[
\varphi_*(\nu_{p',q',m'}(-1)) = [\nu_{p'',q'',m''}(-1)] \in H^1 G'',
\]

where \( m'' = \varphi_*(m') \) and where \( p'' \in \mathcal{K}(\tilde{D}'') \) is such that

\[
p''|_{\tilde{D}} = p'|_{\tilde{D}} \quad \text{and} \quad p''|_{\tilde{D}'' \setminus \tilde{D}} = q''|_{\tilde{D}'' \setminus \tilde{D}}.
\]

**Proof.** A straightforward check. \( \square \)

**8.3. Proposition.** Consider a reductive R-group \( G(B, \tau, q) \) with the notation of Subsection 7.11. Having fixed \( B \) and \( \tau \), we shall write \( G_q = G(B, \tau, q) \) for brevity. We use the notation of Theorem 7.14. Let \( q \in \mathcal{K}(\tilde{D}) \), and consider the 1-cocycle

\[
a = \mathcal{E}^{\text{ad}}(x(q') - x(q)) \in (T_0^{(2)}) \subset Z^1 G_q^{\text{ad}}.
\]

Consider the twisted group \( a G_q \); cf. [BT21, Section 14.1]. Then there is a canonical isomorphism \( a G_q \cong G_q' \).

**Proof.** Similar to that of [BT21, Proposition 14.2]. \( \square \)

**8.4.** Now let \((q', [m']) \in \mathcal{K}(\tilde{D}', \Lambda, X, \tau, q')\), that is, \( q' \in \mathcal{K}(\tilde{D}) \), \( m' \in M'_0 \), \( [m'] \in M'_0 / 2\Lambda'_0 \), and (7.13) is satisfied. With the notation of Theorem 7.14, consider the 1-cocycle

\[
a = \nu_{q',q,m'}(-1) = \mathcal{E}(x(q') - x(q) + y(m')) \in T_0^{(2)} \subset Z^1 G_q
\]

and the twisting bijection \( \mathcal{T}_a : H^1 a G_q \to H^1 G_q \) of Serre [Ser97, I.5.3, Proposition 35 bis]. By Proposition 8.3 we may identify \( a G_q \) with \( G_q' \). Thus we obtain a map

\[
\mathcal{T}_a : H^1 G_q' \to H^1 G_q
\]

sending the neutral cohomology class \([1] \in H^1 G_q'\) to \([a] \in H^1 G_q\).

**8.5. Proposition.**

\[
\mathcal{T}_a[\nu_{p'',q',m''}(-1)] = [\nu_{p'',q,m''+m'}(-1)] \quad \text{for all} \quad (p'', [m'']) \in \mathcal{K}(\tilde{D}, \Lambda, X, \tau, q').
\]

**Proof.** Note that \( \nu_{p'',q,m''+m'} = \nu_{p'',q',m''} + \nu_{q',q,m'} \). The map \( \mathcal{T}_a \) is induced by the map on cocycles

\[
Z^1 G_q' \to Z^1 G_q, \quad a'' \mapsto a'' a,
\]

sending \( \nu_{p'',q',m''}(-1) \) to

\[
\nu_{p'',q',m''}(-1) \cdot a = \nu_{p'',q',m''}(-1) \cdot \nu_{q',q,m'}(-1) = \nu_{p'',q,m''+m'}(-1).
\]

\( \square \)
Let $G = G(\mathcal{B}, \tau, q)$ with the notation of Subsection 7.11. We consider the center $Z = Z(G)$ of $G$. The group $H^1 Z$ naturally acts on $H^1 G$ by

$$ (8.7) \quad [z] \cdot [g] = [zg] \quad for \quad z \in Z^1 Z, \; g \in Z^1 G; $$

see Serre [Ser97, Section I.5.7]. We wish to compute this action in our language.

8.8. Lemma. Let $\zeta \in H^1 Z$. Then $\zeta$ can be represented by a cocycle of the form

$$ (8.9) \quad \zeta = \mathcal{E}^{ss}(i\nu_P) \cdot \mathcal{E}_S(i\nu_M/2), $$

where $\nu_P \in P^\vee$, $\nu_M \in M^\vee_0$, and the maps $\mathcal{E}^{ss} : t^\vee \to T^\vee$, $\mathcal{E}_S : s \to S$ are the restrictions of $\mathcal{E} : t \to T$, $x \mapsto \exp 2\pi x$, to $t^\vee$ and $s$, respectively.

Proof. The class $\zeta$ is represented by a cocycle $z = z^{ss} \cdot s$, where $z^{ss} \in Z(G^{ss})$, $s \in S$. Then $z^{ss} = \mathcal{E}^{ss}(i\nu_P)$, $s = \mathcal{E}_S(y)$, where $\nu_P \in P^\vee$, $y \in s$. The cocycle condition reads as

$$ z \cdot \gamma z = \mathcal{E}^{ss}(i\nu_P + i\tau(\nu_P)) \cdot \mathcal{E}_S(y + \gamma y) = 1, $$

which is equivalent to

$$ i\nu_P + i\tau(\nu_P) + y + \gamma y \in iX^\vee_0. $$

This implies $y + \gamma y = i\nu_M \in iM^\vee_0$. Put $y = y_+ + y_-$, where $\gamma y_+ = y_+$ and $\gamma y_- = -y_-$. Then

$$ s_- := \mathcal{E}_S(y_-) = \mathcal{E}_S(y_-/2) \cdot \gamma \mathcal{E}_S(y_-/2)^{-1} $$

is a coboundary in $S$. Replacing $z$ with $zs^2$ yields $y = y_+ = i\nu_M/2$. \hfill \Box

8.10. Since $X_*(T) = X^\vee$, $X_*(T^{ad}) = P^\vee$, and $T/Z = T^{ad}$, by Theorem 3.15 there is a canonical isomorphism

$$ \vartheta^0 : H^0(X^\vee \to P^\vee) \xrightarrow{\sim} H^1 Z. $$

8.11. Lemma. Let $(\nu, \nu') \in Z^0(X^\vee \to P^\vee)$, and let $\zeta = \vartheta^0[\nu, \nu'] \in H^1 Z$. Then $\zeta$ can be represented by the cocycle $z$ of the form (8.9), where $\nu_P = -\nu' \in P^\vee$ and $\nu_M$ is the image of $\nu \in X^\vee_0 \subseteq P^\vee_0 \oplus M^\vee_0$ under the projection to $M^\vee_0$.

Proof. Since $(\nu, \nu')$ is a 0-hypercocycle, we have

$$ D^0(\nu, \nu') = (-d^1 \nu, d^0 \nu' - \partial \nu) = (\tau(\nu) - \nu, -\tau(\nu') - \nu' - \nu + \nu_M) = (0, 0), $$

where $\nu_M$ is the image of $\nu \in X^\vee \subseteq P^\vee \oplus M^\vee$ under the projection to $M^\vee$. Hence $\nu \in X^\vee_0$, $\nu_M \in M^\vee_0$, and $\nu - \nu_M = -\nu' - \tau(\nu') \in P^\vee_0$.

As noted in Subsection 3.17, $\vartheta^0[\nu, \nu']$ is represented by a cocycle $z = \nu(-1) \cdot t \cdot \tau^{-1}$, where $t \in T$ and the image of $t$ in $T^{ad}$ is $\nu'(-1) = \mathcal{E}^{ad}(i\nu'/2) = \mathcal{E}^{ad}(-i\nu'/2)$. We may take $t = \mathcal{E}^{ss}(-i\nu'/2)$; then

$$ z = \mathcal{E}(i\nu/2 - i\nu'/2 + i\tau(\nu'/2)) = \mathcal{E}(-i\nu'/2 + i\nu_M/2) = \mathcal{E}^{ss}(-i\nu') \cdot \mathcal{E}_S(i\nu_M/2), $$

as required. \hfill \Box

8.12. Let $G^{sc}$ be a simply connected semisimple $R$-group. We consider the center $Z^{sc} = Z(G^{sc})$ of $G^{sc}$. Set $C = P^\vee/Q^\vee$. We embed $P^\vee$ and $Q^\vee$ into $t^{sc}$. The scaled exponential map

$$ \mathcal{E}^{sc} : t^{sc} \to T^{sc}, \quad x \mapsto \exp 2\pi x $$

has kernel $iQ^\vee$ and induces a $\Gamma$-equivariant isomorphism of abelian groups

$$ (i)C = iP^\vee/iQ^\vee \xrightarrow{\sim} Z^{sc}. $$

Using Lemma 1.8, we obtain an isomorphism on cohomology

$$ H^0 C = H^1 (i)C \xrightarrow{\sim} H^1 Z^{sc}. $$
In [BT21, Section 7.14] we defined the group \( C_0 = \tilde{P}_0^\vee / \tilde{Q}_0^\vee \). We have a canonical surjective homomorphism

\[
(8.13) \quad C \to C_0: \quad \nu + Q^\vee \mapsto \frac{1}{2} (\nu + \tau(\nu)) + \tilde{Q}_0^\vee.
\]

**8.14. Remark.** There is a similar surjective homomorphism \( F = X^\vee / (Q^\vee \oplus \Lambda^\vee) \to F_0 \). On the other hand, \( F \subseteq C \oplus M^\vee / \Lambda^\vee \) embeds into \( C \) under the natural projection. Indeed, a coset \([\nu] \in F\) represented by \( \nu \in X^\vee \) projects to \([0] \in C\) if and only if \( \nu \in Q^\vee \oplus M^\vee \); but then \( \nu = \nu_Q + \nu_M \), where \( \nu_Q \in Q^\vee \) and \( \nu_M \in X^\vee \cap M^\vee = \Lambda^\vee \), whence \([\nu] = [0] \in F\). We deduce that \( \#F_0 \) divides \( \#F \) and \( \#F \) divides \( \#C \).

**8.15.** In [BT21, Sections 9–11] we described a canonical action of the group \( C_0 \) on the twisted affine Dynkin diagram \( \tilde{D} \). Thus we obtain a canonical homomorphism

\[
(8.16) \quad \zeta: P \to C \to C_0 \to Aut \tilde{D}.
\]

Since \( \tau = -\gamma \) on \( P \), we have

\[
B^0 C = \{ c + \gamma c \mid c \in C \} = \{ c - \tau(c) \mid c \in C \} = \{ \nu - \tau(\nu) + Q^\vee \mid \nu \in P^\vee \}.
\]

We see that the canonical surjective map \((8.13)\) sends \( B^0 C \subseteq C \) to 0 and thus induces a homomorphism \( H^0 C \to C_0 \). Thus we obtain a canonical homomorphism

\[
(8.17) \quad \zeta^0: H^0 C \to C_0 \to Aut \tilde{D}.
\]

**8.18. Proposition.** For \( G \) as in 8.6, let \( \xi \in H^1 G \), \( \xi = \kappa_s[p,[m]] \) with the notation of Theorem 7.14. Let \( \zeta = [z] \in H^1 Z(G) \), where \( z = E^{sc}(iv_P \cdot E_S(iv_M/2), \) with \( \nu_P \in P^\vee \), \( \nu_M \in M^0_\bullet \) as in Lemma 8.8. Then

\[
\zeta \cdot \xi = \kappa_s[p',[m']],
\]

where \( m' = \nu_M + m \), \( p' = \zeta(\nu_P)(p) \), and \( \zeta \) is the homomorphism of \((8.16)\).

Proposition 8.18 computes the action of \( H^1 Z(G) \) on \( H^1 G \).

**Proof.** The class \( \zeta \cdot \xi \) is represented by the cocycle

\[
z \cdot \nu_{p,q,m}(-1) = E^{sc}(iv_P + x(p) - x(q)) \cdot E_S(iv_M/2 + y(m)).
\]

Put \( \nu_P = \nu_0 + \nu_1 \), where \( \nu_0 = \frac{1}{2}(\nu_P + \tau(\nu_P)) \in \tilde{P}_0^\vee \) and \( \nu_1 = \frac{1}{2}(\nu_P - \tau(\nu_P)) \). Note that

\[
\gamma E(iv_1/2) \cdot E(iv_P) \cdot E(iv_1/2)^{-1} = E(iv_0),
\]

whence

\[
\zeta \cdot \xi = [E^{sc}(iv_0 + x(p) - x(q)) \cdot E_S(y(\nu_M + m))] = [E^{sc}(x(p') - x(q)) \cdot E_S(y(m'))] = [\nu_{p',q,m'}(-1)]
\]

by the definition of the action of \( C_0 \) on the fundamental domain \( \Delta \), as desired.

**8.19. Corollary.** Let \( G^{sc} \) be a simply connected semisimple \( R \)-group with center \( Z^{sc} \). Consider the map

\[
\iota_s: H^1 Z^{sc} \to H^1 G^{sc}
\]

induced by the inclusion map \( \iota: Z^{sc} \to G^{sc} \). Let

\[
\iota^0_s: H^0 C \to H^1 Z^{sc} \to H^1 G^{sc}
\]

denote the composite map. Then

\[
ker \iota^0_s = (H^0 C)_q.
\]

the stabilizer of \( q \in K(\tilde{D}) \) under the action of \( H^0 C \) on \( K(\tilde{D}) \) induced by the action \((8.17)\) of \( H^0 C \) on \( \tilde{D} \).
Proof. It follows from the definition of the action (8.7) of $H^1 Z_{sc}$ on $H^1 G_{sc}$ that

$$\iota_* [z] = [z] \cdot [1] \in H^1 G_{sc}.$$

Therefore, $\ker \iota_* = (H^1 Z_{sc})[1]$ and $\ker \iota_0^0 = (H^0 C)[1]$, the stabilizers of the neutral element $[1] \in H^1 G_{sc}$, where $H^0 C$ acts on $H^1 G_{sc}$ via the isomorphism $H^0 C = H^1(i) C \sim H^1 Z_{sc}$. Now the corollary follows from Proposition 8.18.

8.20. Let $G$ be a connected reductive $R$-group (not necessarily compact) and let $T \subseteq G$ be a maximal torus. Let $T_{sc} = \rho^{-1}(T) \subseteq G_{sc}$. By [Bor98, Definition 2.2], the $k$-th abelian Galois cohomology group is defined by

$$H^k_{ab} G := H_k(T_{sc} \to T).$$

It is an abelian group, and it does not depend on the choice of $T$ (up to a canonical isomorphism). In [Bor98, Section 3], the first-named author defined the abelianization map

$$ab^1: H^1 G \to H^1_{ab} G.$$

The first abelian Galois cohomology group $H^1_{ab} G$ and the abelianization map play an important role in the description of Galois cohomology of reductive groups over number fields; see [Bor98, Theorem 5.11]. Here we compute $H^k_{ab} G$ and the abelianization map.

8.21. Proposition. Let $G$ be a connected reductive $R$-group. Then for any $k \in \mathbb{Z}$, there is a canonical isomorphism

$$H^k \pi_1^{alg} G \sim H^k_{ab} G.$$

Proof. Let $T \subseteq G$ be a maximal torus. Since the homomorphism

$$\rho_* : Q^\vee = X_*(T_{sc}) \longrightarrow X_*(T) = X^\vee$$

is injective and $\pi_1^{alg} G = \ker \rho_*$, by Lemma 2.8 we have a canonical isomorphism

$$H^k \pi_1^{alg} G \sim H^k(X_*(T_{sc}) \to X_*(T)).$$

By Proposition 3.13 we have a canonical isomorphism

$$H^k(X_*(T_{sc}) \to X_*(T)) \sim H^k(T_{sc} \to T) = H^1_{ab} G,$$

as required.

8.22. Theorem. Let $G = G(B, \tau, q)$ with the notation of Subsection 7.11 (it comes with a fundamental torus $T \subseteq G$). With the notation of Theorem 7.14, consider the map

$$\lambda : K(\tilde{D}, \Lambda, X, \tau, q) \longrightarrow H^1 \pi_1^{alg} G$$

$$(p, [m]) \mapsto [\nu_{p,q,m} + Q^\vee].$$

Then the map $\lambda$ induces a map

$$\lambda_* : K(\tilde{D}, \Lambda, X, \tau, q)/F_0 \longrightarrow H^1 \pi_1^{alg} G,$$

and the following diagram commuting:

$$K(\tilde{D}, \Lambda, X, \tau, q)/F_0 \xrightarrow{\lambda_*} H^1 \pi_1^{alg} G \xrightarrow{\kappa_*} H^1 G \xrightarrow{ab^1} H^1_{ab} G$$

where $\kappa_*$ is the bijection of Theorem 7.14 and the right-hand vertical arrow is the isomorphism of Proposition 8.21.

Theorem 8.22 computes the abelianization map $ab^1$. 
Proof. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{K}(\tilde{D}, \Lambda, X, \tau, q) & \xrightarrow{\lambda} & H^1_{\pi_1^{\text{alg}}} G \\
\downarrow \kappa & & \downarrow \nu \\
H^1 T & \xrightarrow{|t|\mapsto[1,t]} & H^1(T_{\text{sc}} \rightarrow T) \\
\downarrow i_* & & \downarrow \\
H^1 G & \xrightarrow{\text{ab}^l} & H^1_{\text{ab}} G
\end{array}
\]

in which the map \(\kappa\) is given by \((p, [m]) \mapsto \nu_{p,q,m}(-1)\) and the map \(i_*\) is induced by the inclusion map \(i: T \hookrightarrow G\). In this diagram, the top rectangle clearly commutes, and we know from the definition of the morphism of functors \(\text{ab}^l\) in terms of hypercohomology of crossed modules in [Bor98, Section 3.10] that the bottom rectangle commutes as well. Thus the diagram commutes. Since the composite left-hand vertical arrow is constant on the orbits of \(F_0\), and the right-hand vertical arrows are isomorphisms, we conclude that the map \(\lambda\) is constant on the orbits of \(F_0\), which completes the proof of the theorem. 

\[\square\]

9. The group of real connected components of a reductive group

In this section we compute the group of real connected components \(\pi_0^R G := \pi_0 G(R)\) for a connected reductive \(R\)-group \(G\) in terms of the algebraic fundamental group \(\pi_1^{\text{alg}} G\).

9.1. Let \(T \subseteq G\) be a maximal torus, not necessarily fundamental. Recall that \(\pi_1^{\text{alg}} G = X^\vee/Q^\vee\), where \(X^\vee = X_*(T)\), \(Q^\vee = X_*(T_{\text{sc}})\).

Recall that

\[
G^{\text{ad}} = G/Z(G), \quad T^{\text{ad}} = T/Z(G), \quad P^\vee = X_*(T^{\text{ad}}), \quad C = P^\vee/Q^\vee.
\]

The canonical homomorphism \(\text{Ad}: G \rightarrow G^{\text{ad}}\) induces a homomorphism of \(\Gamma\)-modules

\[
\pi_1^{\text{alg}} G \rightarrow \pi_1^{\text{alg}} G^{\text{ad}} = C,
\]

which in turn induces a homomorphism

\[
\text{Ad}_*: H^0_{\pi_1^{\text{alg}}} G \rightarrow H^0 C.
\]

Consider the composite map

\[
\phi: H^0_{\pi_1^{\text{alg}}} G \xrightarrow{\text{Ad}_*} H^0 C \xrightarrow{\kappa} H^1 Z^\text{sc} \rightarrow H^1 T^\text{sc} \rightarrow H^1 G^\text{sc},
\]

where the third and fourth arrows are induced by the inclusion homomorphisms \(Z^\text{sc} \hookrightarrow T^\text{sc} \hookrightarrow G^\text{sc}\). Explicitly:

\[
\phi[\nu + Q^\vee] = [C^{\text{sc}}(i\nu^{\text{ad}})],
\]

where \(\nu^{\text{ad}} \in P^\vee \subset T^{\text{ad}} = T^\text{sc}\) is the image of \(\nu \in X^\vee\) under the canonical homomorphism \(X^\vee \rightarrow P^\vee\). Let \((H^0_{\pi_1^{\text{alg}}} G)_1\) denote the kernel of \(\phi\), that is, the preimage in \(H^0_{\pi_1^{\text{alg}}} G\) of \([1] \in H^1 G^{\text{sc}}\).

Now write \(G = G(B, \tau, q)\) with the notation of Subsection 7.11. The group \(H^0 C\) acts on the affine Dynkin diagram \(\tilde{D}\) of \(G^{\text{sc}}\) via the homomorphism \(\varsigma^0\) of (8.17), and composing with \(\text{Ad}_*\) we obtain an action of \(H^0_{\pi_1^{\text{alg}}} G\) on \(\tilde{D}\), and hence on \(\mathcal{K}(\tilde{D})\).
9.2. Theorem.

(i) Let $G$ be a connected reductive $\mathbb{R}$-group. Then the subset $(H^0 \pi^\text{alg}_1 G)_1$ of the abelian group $H^0 \pi^\text{alg}_1 G$ is a subgroup, and there exists a canonical isomorphism of abelian groups

$$
\psi: (H^0 \pi^\text{alg}_1 G)_1 \xrightarrow{\sim} \pi^\text{R}_0 G.
$$

(ii) Write $G = G(\mathcal{B}, \tau, q)$ with the notation of Subsection 7.11. Then $(H^0 \pi^\text{alg}_1 G)_1$ is the stabilizer of $q \in K(\hat{D})$ under the action of the abelian group $H^0 \pi^\text{alg}_1 G$ on $K(\hat{D})$ described in Subsection 9.1.

The theorem will be proved in Subsections 9.18 and 9.21.

9.3. Example. Let $G = T$ be an $\mathbb{R}$-torus. Then Theorem 9.2 says that $\pi^\text{R}_0 T \cong H^0 X_4(T)$, which is Corollary 3.10.

9.4. We specify the map $\psi$ in Theorem 9.2(i). Let $\nu \in X^\vee$ be such that

\begin{align}
\nu + Q^\vee &\in Z^0 \pi^\text{alg}_1 G, \\
[\nu + Q^\vee] &\in \ker \phi.
\end{align}

Here (9.5) means that

$$\gamma \nu - \nu = \nu^\text{sc} \quad \text{for some } \nu^\text{sc} \in Q^\vee.$$

Then $\nu^\text{sc} \in Z^1 Q^\vee$. Set

$$t = \nu(-1) \in T \subseteq G \quad \text{and} \quad t^\text{sc} = \nu^\text{sc}(-1) \in Z^1 T^\text{sc} \subseteq Z^1 G^\text{sc}.$$

The following diagram commutes:

\[
\begin{array}{ccc}
X^\vee & \xrightarrow{\text{Ad}_s} & P^\vee \\
\downarrow & & \downarrow \\
Q^\vee & & \\
\end{array}
\]

Now it follows from (9.7) that

$$\gamma \nu^\text{ad} - \nu^\text{ad} = \nu^\text{sc},$$

where $\nu^\text{ad} = \text{Ad}_s(\nu) \in P^\vee$. Write $\nu^\text{ad} = \nu_0 + \nu_1$, where

$$\nu_0 = \frac{1}{2}(\nu^\text{ad} - \gamma \nu^\text{ad}) \quad \text{and} \quad \nu_1 = \frac{1}{2}(\nu^\text{ad} + \gamma \nu^\text{ad}).$$

Then $\gamma \nu_0 = -\nu_0$, $\gamma \nu_1 = \nu_1$, and $2\nu_0 = -\nu^\text{sc}$.

By (9.6) we have $[E^\text{sc}(i\nu^\text{ad})] = [1] \in H^1 G^\text{sc}$. Since

$$E^\text{sc}(i\nu_1/2)^{-1} \cdot E^\text{sc}(i\nu^\text{ad}) \cdot E^\text{sc}(i\nu_1/2) = E^\text{sc}(i\nu_0) = \exp(-\pi i \nu^\text{sc}) = t^\text{sc},$$

we see that $[t^\text{sc}] = [1] \in H^1 G^\text{sc}$ and therefore $t^\text{sc} = (g^\text{sc})^{-1} \cdot \gamma g^\text{sc}$ for some $g^\text{sc} \in G^\text{sc}$. We set

$$g = \rho(g^\text{sc}) \cdot t^{-1} \in G.$$

We compute $\gamma g$. By (9.7) we have $\gamma t = t \cdot \rho(t^\text{sc})$. By construction we have $\gamma g^\text{sc} = g^\text{sc} t^\text{sc}$.

Thus

$$\gamma g = \rho(g^\text{sc}) \cdot \gamma t^{-1} = \rho(g^\text{sc} t^\text{sc}) \cdot \rho(t^\text{sc})^{-1} \cdot t^{-1} = \rho(g^\text{sc}) \cdot t^{-1} = g.$$

We conclude that $g \in G(\mathbb{R})$. We set $\psi[\nu + Q^\vee] = [g] \in \pi^\text{R}_0 G$. 
9.9. Let \( \pi_1^{\text{top}} G \) denote the topological fundamental group of \( G = G(\mathbb{C}) \). Recall that \( \pi_1^{\text{top}} G \) is the group of equivalence classes of loops, that is, continuous maps \( l : [0, 1] \to G \) from the segment \([0, 1]\) to \( G \) with \( l(0) = l(1) = 1_G \). It is well known that \( \pi_1^{\text{top}} G \) is a finitely generated abelian group. Since the Galois group \( \Gamma \) acts on \( G(\mathbb{C}) \) continuously in the usual Hausdorff topology, it naturally acts on \( \pi_1^{\text{top}} G \). We usually write \( \pi_1^{\text{top}} G \) for \( \pi_1^{\text{top}} G \) to stress that it is a \( \Gamma \)-module with the \( \Gamma \)-action induced by the real form \( G \) of \( G \).

9.10. Example. Take \( G = G_{m,R} \). We have a canonical isomorphism of \( \Gamma \)-modules

\[
i \mathbb{Z} \xrightarrow{\sim} \pi_1^{\text{top}} G_{m,R} = \pi_1^{\text{top}} \mathbb{C}^\times, \quad i \cdot n \mapsto [t \mapsto \exp 2\pi i nt], \ n \in \mathbb{Z}, \ t \in [0, 1].
\]

9.11. Proposition ([Bor98, Proposition 1.11]). Let \( G \) be a connected reductive \( \mathbb{R} \)-group. With the above notation, there is a canonical isomorphism

\[
(i)\pi_1^{\text{alg}} G \cong iX^\vee / iQ^\vee \xrightarrow{\sim} \pi_1^{\text{top}} G.
\]

Proof. We give an alternative proof of Proposition 9.11. In this proof we introduce notions that are necessary for the proof of Theorem 9.2, and we give the explicit formula (9.14) for the isomorphism (9.12).

Recall that \( \mathfrak{s} = \text{Lie}(S) \), where \( S = Z(G)^0 \). Consider the \( \Gamma \)-group \( \tilde{G} := G^{\text{sc}} \times \mathfrak{s} \), the product of the simply connected \( \mathbb{R} \)-group \( G^{\text{sc}} \) and the commutative unipotent \( \mathbb{R} \)-group \( \mathfrak{s} \). Observe that the complex Lie group \( \tilde{G} = G(\mathbb{C}) = G^{\text{sc}} \times \mathfrak{s} \) is simply connected. Consider the surjective \( \Gamma \)-equivariant homomorphism of complex Lie groups

\[
\tilde{\rho} : \tilde{G} = G^{\text{sc}} \times \mathfrak{s} \to G^{\text{ss}} \times S \to G, \quad \tilde{g} = (g^{\text{sc}}, y) \mapsto \rho(g^{\text{sc}}) : E_S(y) \quad \text{for} \ g^{\text{sc}} \in G^{\text{sc}}, \ y \in \mathfrak{s}.
\]

(Note that \( \tilde{\rho} \) is not a homomorphism of algebraic groups!) Since each of the homomorphisms

\[
G^{\text{sc}} \to G^{\text{ss}}, \quad \mathfrak{s} \to S, \quad G^{\text{ss}} \times S \to G
\]

has discrete kernel, the homomorphism \( \tilde{\rho} \) has discrete kernel as well. We see that \( \tilde{\rho} \) is a universal covering of \( G \).

By the exact homotopy sequence for \( \tilde{\rho} \), the group \( \tilde{\mathbb{Z}} := \text{ker} \tilde{\rho} \) can be identified with \( \pi_1^{\text{top}} G \); see, for instance, [OV90, Section 1.3, Theorem 4 and Corollary 2]. This identification goes as follows. A loop \( l : [0, 1] \to G \) with \( l(0) = l(1) = 1_G \) can be uniquely lifted to a continuous path \( \tilde{l} : [0, 1] \to \tilde{G} \) with \( \tilde{l}(0) = 1_{\tilde{G}} \). Then \( \tilde{\rho}(\tilde{l}(1)) = l(1) = 1_G \), whence \( \tilde{l}(1) \in \tilde{\mathbb{Z}} \). To \( [l] \in \pi_1^{\text{top}} G \) we associate \( \tilde{l}(1) \in \tilde{\mathbb{Z}} \).

On the other hand, \( \tilde{\mathbb{Z}} \) is contained in \( T^{\text{sc}} \times \mathfrak{s} \). Let \( \tilde{t} = (t^{\text{sc}}, y) \in T^{\text{sc}} \times \mathfrak{s}, \ t^{\text{sc}} = E^{\text{sc}}(x), \ x \in t^{\text{sc}}, \ y \in \mathfrak{s} \). Then we have

\[
\tilde{\rho}(\tilde{t}) = \rho(t^{\text{sc}}) \cdot E_S(y) = E^{\text{ss}}(x) \cdot E_S(y) = E(x + y).
\]

Hence \( \tilde{l} \in \tilde{\mathbb{Z}} \) if and only \( x + y \in iX^\vee \). The homomorphism

\[
\tilde{E} : t = t^{\text{sc}} \oplus \mathfrak{s} \to T^{\text{sc}} \times \mathfrak{s}, \quad \tilde{E}(x + y) = (E^{\text{sc}}(x), y)
\]

is a universal covering and \( \text{ker} \tilde{E} = iQ^\vee \). Hence \( \tilde{\mathbb{Z}} = \tilde{E}(iX^\vee) \cong iX^\vee / iQ^\vee \cong (i)\pi_1^{\text{alg}} G \).

We obtain a \( \Gamma \)-equivariant isomorphism

\[
(i)\pi_1^{\text{alg}} G \xrightarrow{\sim} \tilde{\mathbb{Z}} \xrightarrow{\sim} \pi_1^{\text{top}} G
\]

\[
i \nu + iQ^\vee \mapsto \tilde{E}(i\nu) \mapsto [t \mapsto E(t\nu) = \nu(\exp 2\pi i t)] \quad \text{for} \ \nu \in X^\vee, \ t \in [0, 1],
\]

as required. The explicit formula (9.14) for the homomorphism (9.13) stems from an observation that the loop \( l : t \mapsto E(t\nu) \) in \( G \) lifts to the path \( \tilde{l} : t \mapsto \tilde{E}(t\nu) \) in \( \tilde{G} \) starting at \( 1_{\tilde{G}} \) and ending at \( \tilde{E}(i\nu) \). \( \square \)
9.15. Corollary. There is a canonical isomorphism
\[
H^0\pi_1^{\text{alg}} G \to H^1\pi_1^{\text{top}} G, \quad [\nu + Q^\vee] \mapsto [t \mapsto \nu(\exp 2\pi i t)] \quad \text{for } \nu \in X^\vee.
\]

Proof. The corollary follows from (9.13), (9.14), and Corollary 1.10. \qed

9.17. Proposition. Let \( G^{\text{sc}} \) be a simply connected semisimple \( \mathbb{R} \)-group. Then the real Lie group \( G^{\text{sc}}(\mathbb{R}) \) is connected.

Proof. See Borel and Tits [BT72, Corollary 4.7], or Gorbatsevich, Onishchik and Vinberg [GOV94, 4.2.2, Theorem 2.2], or Platonov and Rapinchuk [PR94, Proposition 7.6 on page 407]. \qed

9.18. Proof of Theorem 9.2(ii). The identification \( \pi_1^{\text{top}} G \cong \tilde{Z} \) yields a short exact sequence of \( \Gamma \)-groups
\[
1 \to \pi_1^{\text{top}} G \xrightarrow{i} \tilde{G} \xrightarrow{\tilde{\rho}} G \to 1,
\]
which gives rise to a cohomology exact sequence
\[
\tilde{G}(\mathbb{R}) = G^{\text{sc}}(\mathbb{R}) \times \mathfrak{s}(\mathbb{R}) \to G(\mathbb{R}) \to H^1\pi_1^{\text{top}} G \xrightarrow{i_*} H^1 \tilde{G} = H^1 G^{\text{sc}} \times H^1 \mathfrak{s} = H^1 G^{\text{sc}}.
\]

By Proposition 9.17 the Lie group \( \tilde{G}(\mathbb{R}) \) is connected, and its image in \( G(\mathbb{R}) \) is connected and open, and hence it is the identity component. Thus we obtain an injective homomorphism
\[
\pi_0^{\text{R}} G = \coker \left( \tilde{G}(\mathbb{R}) \to G(\mathbb{R}) \right) \hookrightarrow H^1\pi_1^{\text{top}} G
\]
whose image is
\[
(9.19) \quad \ker \left( H^1\pi_1^{\text{top}} G \xrightarrow{i_*} H^1 G^{\text{sc}} \right).
\]
It follows that the kernel (9.19) is a subgroup. Using the isomorphism (9.16), we obtain a group isomorphism
\[
(9.20) \quad \pi_0^{\text{R}} G \xrightarrow{\sim} \ker \left( H^0\pi_1^{\text{alg}} G \xrightarrow{\phi} H^1 G^{\text{sc}} \right) = (H^0\pi_1^{\text{alg}} G)_1.
\]
Since \( H^0\pi_1^{\text{alg}} G \) is an abelian group killed by multiplication by 2, we see that so is \( \pi_0^{\text{R}} G \).

For any \( g \in G(\mathbb{R}) \), the isomorphism (9.20) sends \([g] \in \pi_0^{\text{R}} G \) to the cohomology class in \( H^0\pi_1^{\text{alg}} G \cong H^1\pi_1^{\text{top}} G \) corresponding to the 1-cocycle \( \tilde{\zeta} = \tilde{g}^{-1} : \gamma \tilde{g} \in \tilde{Z} \), where \( \tilde{g} \in \tilde{G} \) is such that \( \tilde{\rho}(\tilde{g}) = g \). If \( g \) is given by (9.8), then, with the notation of Subsection 9.4, we may take \( \tilde{g} = g^{\text{sc}} \cdot \tilde{t}^{-1} \), where \( \tilde{t} = \tilde{E}(i\nu/2) \in T^{\text{sc}} \times \mathfrak{s} \). We have
\[
\tilde{\zeta} = \tilde{t} \cdot (g^{\text{sc}})^{-1} : \gamma g^{\text{sc}} \cdot \gamma \tilde{t}^{-1} = \tilde{t} \cdot (t^{\text{sc}})^{-1} : \gamma \tilde{t}^{-1} = \tilde{E}(i\nu/2 - i\nu^{\text{sc}}/2 + i\nu/2) = \tilde{E}(i\nu),
\]
where the last equality follows from (9.7). Then \( \tilde{\zeta} \) represents the cohomology class \([\nu + Q^\vee] \in H^0\pi_1^{\text{alg}} G \cong H^1\pi_1^{\text{top}} G \). Thus (9.20) is the inverse of the map \( \psi \) of Subsection 9.4, which completes the proof of Theorem 9.2(ii). \qed

9.21. Proof of Theorem 9.2(iii). Now we assume that the maximal torus \( T \) is fundamental. Consider the canonical isomorphism \( \text{Ad} : G \to G^{\text{ad}} \) and the commutative diagram
\[
\begin{array}{ccc}
\pi_1^{\text{top}} G & \xrightarrow{i} & \tilde{G} \\
\downarrow & & \\
\pi_1^{\text{top}} G^{\text{ad}} & \xrightarrow{i^{\text{ad}}} & G^{\text{sc}}
\end{array}
\]
which induces a commutative diagram

\[
\begin{array}{cccccc}
H^0_\pi^{-1} \mathbb{G} & \sim & H^1_\pi \mathbb{G} & \sim & H^1 \tilde{\mathbb{G}} \\
\text{Ad}^0 & & \text{Ad}^1 & & \cong \\
H^0_\pi \mathbb{G}^{\text{ad}} & \sim & H^1_\pi \mathbb{G}^{\text{ad}} & \sim & H^1 \mathbb{G}^{\text{sc}}
\end{array}
\]

The map $H^0 C = H^0_\pi \mathbb{G}^{\text{ad}} \rightarrow H^1 \mathbb{G}^{\text{sc}}$ in the bottom row of the latter diagram comes from the action of the group $H^0 C$ on the cohomology set $H^1 \mathbb{G}^{\text{sc}}$, and by Corollary 8.19 the kernel of this map is $(H^0 C)_q$. We see from the diagram (9.22) that the kernel of the map $H^0_\pi \mathbb{G} \rightarrow H^1 \mathbb{G}^{\text{sc}}$ of (9.20) is the preimage in $H^0_\pi \mathbb{G}$ of $(H^0 C)_q$ under the homomorphism $\text{Ad}^0: H^0_\pi \mathbb{G} \rightarrow H^0 C$, that is, the stabilizer of $q$ under the action of $H^0_\pi \mathbb{G}$ on $\mathcal{K}(\tilde{D})$ described in Subsection 9.1, as required.

9.23. Proposition. Let $\varphi: \mathbb{G}' \rightarrow \mathbb{G}''$ be a homomorphism of connected reductive $\mathbb{R}$-groups. Then the induced homomorphism $\varphi^0: H^0_\pi \mathbb{G}' \rightarrow H^0_\pi \mathbb{G}''$ sends the subgroup $(H^0_\pi \mathbb{G}')_1$ into $(H^0_\pi \mathbb{G}'')_1$, and the following diagram commutes:

\[
\begin{array}{ccc}
(H^0_\pi \mathbb{G}')_1 & \rightarrow & (H^0_\pi \mathbb{G}'')_1 \\
\varphi^0 & & \varphi^0 \\
\pi_0 \mathbb{G}' & \rightarrow & \pi_0 \mathbb{G}''
\end{array}
\]

where $\psi'$ and $\psi''$ are the isomorphisms of Theorem 9.2(i).

Proof. A straightforward check.

10. Connecting maps in exact sequences

10.1. Let

\[
1 \rightarrow \mathbb{G}_1 \xrightarrow{i} \mathbb{G}_2 \xrightarrow{j} \mathbb{G}_3 \rightarrow 1
\]

be a short exact sequence of connected reductive $\mathbb{R}$-groups. By Proposition 5.4 this sequence gives rise to an exact sequence

\[
\begin{array}{cccccc}
\pi_0 \mathbb{G}_1 & \rightarrow & \pi_0 \mathbb{G}_2 & \rightarrow & \pi_0 \mathbb{G}_3 & \rightarrow & H^1 \mathbb{G}_1 & \rightarrow & H^1 \mathbb{G}_2 & \rightarrow & H^1 \mathbb{G}_3.
\end{array}
\]

In Theorems 9.2 and 7.14 we computed all groups and sets in this exact sequence. We computed the homomorphisms $i^0_*$ and $j^0_*$ in Proposition 9.23, and we computed the maps $i^1_*$ and $j^1_*$ in Proposition 8.2. Here we compute the connecting map $\delta^0: \pi_0 \mathbb{G}_3 \rightarrow H^1 \mathbb{G}_1$. It turns out that $\delta^0$ factorizes via a homomorphism into $H^1 Z(\mathbb{G}_1)$.

10.3. We have an exact commutative diagram
in which the top and the bottom rows have canonical splittings
\[ s^{sc}: G_3^{sc} \to G_2^{sc}, \quad s^{ad}: G_3^{ad} \to G_2^{ad}, \]
and these splittings are compatible with the composite vertical arrows, that is, the following diagram commutes:

\[
\begin{array}{ccc}
G_2^{sc} & \xrightarrow{s^{sc}} & G_3^{sc} \\
\rho_2 \downarrow & & \downarrow \rho_3 \\
G_2^{ad} & \xrightarrow{s^{ad}} & G_3^{ad} \\
\end{array}
\]

(10.4)

We choose a maximal torus \( T_2 \subseteq G_2 \) (not necessarily fundamental) and we set \( T_1 = i^{-1}(T_2) \subseteq G_1, T_3 = j(T_2) \subseteq G_3. \) We obtain commutative diagrams

\[
\begin{array}{ccc}
1 & \xrightarrow{1} & T_1^{sc} & \xrightarrow{\iota} & T_2^{sc} & \xrightarrow{\iota} & T_3^{sc} & \xrightarrow{1} & 1 \\
\downarrow \rho_1 & & \downarrow \rho_2 & & \downarrow \rho_3 & & \downarrow \rho_3 & & \downarrow \rho_3 \\
1 & \xrightarrow{1} & T_1^{ad} & \xrightarrow{1} & T_2^{ad} & \xrightarrow{1} & T_3^{ad} & \xrightarrow{1} & 1 \\
\end{array}
\]

and

\[
\begin{array}{ccc}
0 & \xrightarrow{0} & Q_1^{\vee} & \xrightarrow{0} & Q_2^{\vee} & \xrightarrow{0} & Q_3^{\vee} & \xrightarrow{0} & 0 \\
\downarrow \rho_1 & & \downarrow \rho_2 & & \downarrow \rho_3 & & \downarrow \rho_3 & & \downarrow \rho_3 \\
0 & \xrightarrow{0} & X_1^{\vee} & \xrightarrow{0} & X_2^{\vee} & \xrightarrow{0} & X_3^{\vee} & \xrightarrow{0} & 0 \\
\downarrow \rho_1 & & \downarrow \rho_2 & & \downarrow \rho_3 & & \downarrow \rho_3 & & \downarrow \rho_3 \\
0 & \xrightarrow{0} & P_1^{\vee} & \xrightarrow{0} & P_2^{\vee} & \xrightarrow{0} & P_3^{\vee} & \xrightarrow{0} & 0 \\
\end{array}
\]

where \( T_k^{sc} = \rho_k^{-1}(T_k), T_k^{ad} = \text{Ad}_k(T_k) \) for \( k = 1, 2, 3, \) and where \( X_k^{\vee} = X_{\iota}(T_k) \) and similarly for \( Q_k^{\vee} \) and \( P_k^{\vee}. \) In these diagrams, the rows are exact, but not the columns. Note that the top and the bottom rows of these diagrams split canonically, which can be written for the latter diagram as follows:

\[ Q_3^{\vee} = Q_1^{\vee} \oplus Q_3^{\vee} \quad \text{and} \quad P_3^{\vee} = P_1^{\vee} \oplus P_3^{\vee}. \]

These splittings are compatible with the composite vertical arrows, that is, a diagram similar to (10.4) commutes, which can be written as follows:

\[ (\text{Ad}_2 \circ \rho_2)(0, \nu_3^{sc}) = (0, (\text{Ad}_3 \circ \rho_3)(\nu_3^{sc})) \quad \text{for} \quad \nu_3^{sc} \in Q_3^{\vee}. \]

10.5. Construction. We construct a canonical coboundary homomorphism

\[ \delta_Z: H^0(Q_3^{\vee} \to X_3^{\vee}) \to H^0(X_1^{\vee} \to P_1^{\vee}). \]

We start with \([\nu_3^{sc}, \nu_3] \in H^0(Q_3^{\vee} \to X_3^{\vee}). \) Here \((\nu_3^{sc}, \nu_3) \in Z^0(Q_3^{\vee} \to X_3^{\vee}), \) that is,

\[ (10.6) \quad \nu_3^{sc} \in Q_3^{\vee}, \quad \nu_3 \in X_3^{\vee}, \quad \gamma \nu_3^{sc} + \nu_3^{sc} = 0, \quad \gamma \nu_3 - \nu_3 = \rho_3(\nu_3^{sc}). \]
We lift canonically \( \nu^c_3 \) to
\[
\nu^c_2 = (0, \nu^c_3) \in Q^\vee_1 \oplus Q^\vee_3 = Q^\vee_2,
\]
and we lift \( \nu_3 \) to some \( \nu_2 \in X^\vee_2 \). We write
\[
\Ad_2(\nu_2) = (\nu^\text{ad}_1, \nu^\text{ad}_3) \in P^\vee_1 \oplus P^\vee_3 = P^\vee_2,
\]
where \( \nu^\text{ad}_3 = \Ad_3(\nu_3) \in P^\vee_3 \) and \( \nu^\text{ad}_1 \in P^\vee_1 \). We set
\[
\nu_1 = \gamma \nu_2 - \nu_2 - \rho_2(\nu^c_2).
\]
Since by (10.6) we have
\[
\gamma \nu_3 - \nu_3 = \rho_3(\nu^c_3),
\]
we see that \( \nu_1 \in X^\vee_1 \). Straightforward calculations show that
\[
\gamma \nu_1 + \nu_1 = 0, \quad \Ad_1(\nu_1) = \gamma \nu^\text{ad}_1 - \nu^\text{ad}_1.
\]
We see that \( (\nu_1, \nu^\text{ad}_1) \in Z^0(X^\vee_1 \to P^\vee_1) \). We set
\[
\delta_2[\nu^c_2, \nu_3] = [\nu_1, \nu^\text{ad}_1] \in H^0(X^\vee_1 \to P^\vee_1).
\]
A straightforward check shows that the map \( \delta_2 \) is a well-defined homomorphism.

We have \( \pi^\text{alg}_1 \text{G}_3 = X^\vee_3 / \rho_3(Q^\vee_3) \) with injective \( \rho_3 \), whence \( H^0(Q^\vee_3 \to X^\vee_3) \cong H^0(\pi^\text{alg}_1 \text{G}_3) \).
Moreover, by Theorem 3.15 we have a canonical isomorphism \( H^0(X^\vee_1 \to P^\vee_1) \cong H^1Z(\text{G}_1) \); see Subsection 3.17 for an explicit formula for this isomorphism. Thus we obtain a canonical homomorphism
\[
\delta_2: H^0(\pi^\text{alg}_1 \text{G}_3) \to H^1Z(\text{G}_1)
\]

We show that the connecting map \( \delta^0: \pi^\text{alg}_0 \text{G}_3 \to H^1 \text{G}_1 \) in the exact sequence (10.2) factors via \( \delta_2 \).

**10.7. Theorem.** With the above assumptions and notation, the following diagram commutes:

\[
\begin{array}{ccc}
(H^0(\pi^\text{alg}_1 \text{G}_3))_1 & \xrightarrow{\delta_2} & H^1Z(\text{G}_1) \\
& \searrow & \downarrow \cong \\
\pi^R_0 \text{G}_3 & \xrightarrow{\delta^0} & H^1 \text{G}_1 \\
\end{array}
\]

where the map \( \iota_* \) is induced by the inclusion map \( \iota: Z(\text{G}_1) \hookrightarrow \text{G}_1 \).

This theorem computes the connecting map \( \delta^0 \).

**Proof.** Consider \( [\nu_3] \in (H^0(\pi^\text{alg}_1 \text{G}_3))_1 \) and \( \psi_3[\nu_3] \in \pi^R_0 \text{G}_3 \). We write:
\[
\nu_3 \in X^\vee_3, \quad \gamma \nu_3 - \nu_3 = \rho_3(\nu^c_3), \quad \nu^c_3 \in Q^\vee_3, \quad \nu^c_3(-1) = (g^c_3)^{-1} \cdot \gamma g^c_3,
\]
\[
g_3 = \rho_3(g^c_3) \cdot \nu_3(-1) \in \text{G}_3(\text{R}), \quad \psi_3[\nu_3] = [g_3] \in \pi^R_0 \text{G}_3.
\]
We lift canonically \( \nu^c_3 \) to \( \nu^c_2 \in Q^\vee_2 \) and lift canonically \( g^c_3 \) to \( g^c_2 \in \text{G}^c_2 \), that is, \( \nu^c_2 = (0, \nu^c_3) \) and \( g^c_2 = (1, g^c_3) \). We lift \( \nu_3 \) to some \( \nu_2 \in X^\vee_2 \). We set
\[
g_2 = \rho_2(g^c_2) \cdot \nu_2(-1) \in \text{G}_2.
\]
Then \( g_2 \) is a lift of \( g_3 \in \text{G}_3(\text{R}) \). We set
\[
z_1 = g_2^{-1} \cdot \gamma g_2.
\]
Then \( z_1 \in Z^1 \text{G}_1 \), and
\[
\delta^0(\psi_3[\nu_3]) = \delta^0[g_3] = [z_1] \in H^1 \text{G}_1.
\]
We calculate:
\[
z_1 = g_2^{-1} \cdot \gamma g_2 = \nu_2(-1) \cdot \rho_2(g^c_2)^{-1} \cdot \gamma \rho_2(g^c_2) \cdot \gamma \nu_2(-1)
\]
\[
= \nu_2(-1) \cdot \rho_2(\nu^c_2)^{-1}(-1) \cdot \gamma \nu_2(-1) = \nu_1(-1),
\]
where $\nu_1 = \gamma \nu_2 - \nu_2 - \rho_2(\nu_2^{sc}) \in Z^1 X_1^{\gamma}$. Thus

\begin{equation}
\delta^0(\nu_3[z_3]) = [z_1] = [\nu_1(-1)] \in H^1 G_1.
\end{equation}

Recall that $\delta Z[\nu_3^{ad}, \nu_3] = [\nu_1, \pi_1^{ad}] \in H^0(X_1^{\gamma} \to P_1^{\gamma})$. By Lemma 3.18 the image of $[\nu_1, \nu_1^{ad}]$ under the composite map

$$H^0(X_1^{\gamma} \to P_1^{\gamma}) \to H^1 Z(G_1) \to H^1 T_1$$

is $[\nu_1(-1)] \in H^1 T_1$. Since the right-hand rectangle in the diagram (10.8) clearly commutes, we see that

\begin{equation}
\nu_* (\delta Z[\nu_3]) = [\nu_1(-1)] \in H^1 G_1.
\end{equation}

By (10.9) and (10.10), the left-hand rectangle in the diagram (10.8) commutes, which completes the proof of the theorem.

**10.11.** Let

$$1 \to A_1 \overset{i}{\to} G_2 \overset{j}{\to} G_3 \to 1$$

be a short exact sequence, where $G_2$ and $G_3$ are connected reductive $R$-groups, and $A_1 = \ker [G_2 \to G_3]$ is a central subgroup. Then $A_1$ is an $R$-quasi-torus. By Proposition 5.4, the above short exact sequence gives rise to an exact sequence

$$\pi^R_0 G_2 \overset{j^0}{\to} \pi^R_0 G_3 \overset{\delta^0}{\to} H^1 A_1 \overset{i^0}{\to} H^1 G_2 \overset{j^0}{\to} H^1 G_3.$$

In Theorems 9.2, 3.15, and 7.14, we computed all groups and sets in this exact sequence. We computed the homomorphism $j^0$ in Proposition 9.23, and we computed the map $j^0$ in Proposition 8.2. The map $i^0$ factors via $H^1 Z(G_2)$ and therefore can be computed using Proposition 8.18. In the rest of this section we compute the connecting homomorphism $\delta^0$.

**10.12.** Let $T_2 \subset G_2$ be a maximal torus (not necessarily fundamental). We set $T_3 = j(T_2) \subset G_3$. For $k = 2, 3$, we define $T_k^{sc}$, $X_k^{\gamma}$, and $Q_k^{\gamma}$ as in Subsection 10.3. Then we have a commutative diagram

\[\begin{array}{ccc}
Q_2^{\gamma} & \xrightarrow{j^{sc}} & Q_3^{\gamma} \\
\rho_2 \downarrow & & \downarrow \rho_3 \\
X_2^{\gamma} & \xrightarrow{j} & X_3^{\gamma}
\end{array}\]

in which $j^{sc}$ is an isomorphism and where $\nu = \rho_2 \circ (j^{sc})^{-1}$. We obtain a morphism of short complexes

$$\nu_* : (Q_3^{\gamma} \to X_3^{\gamma}) \to (X_2^{\gamma} \to X_3^{\gamma})$$

and the induced homomorphism on hypercohomology

$$\nu_* : H^0(Q_3^{\gamma} \to X_3^{\gamma}) \to H^0(X_2^{\gamma} \to X_3^{\gamma}).$$

We identify $H^0(Q_3^{\gamma} \to X_3^{\gamma})$ with $H^0_{\gamma_1^{alg}} G_3$ by Lemma 2.8.

**10.13. Theorem.** The following diagram is commutative:

\begin{equation}
\begin{array}{ccc}
H^0_{\gamma_1^{alg}} G_3 & \xrightarrow{\nu_*} & H^0(X_2^{\gamma} \to X_3^{\gamma}) \\
\downarrow \cong \uparrow \cong & & \downarrow \cong \uparrow \cong \\
(H^0_{\gamma_1^{alg}} G_3)_1 & \xrightarrow{\psi_3 \cong \psi_1 \cong \psi_2} & H^0(T_2 \to T_3) \\
\end{array}
\end{equation}

where $\nu_3 \cong \psi_3 \cong \psi_2$.
in which the left-hand vertical arrow \( \psi_3 \) is from Theorem 9.2, and the right-hand vertical arrows are from the diagram (3.19).

This theorem computes the connecting homomorphism \( \delta^0 \). See Subsection 3.17 for an explicit formula for the isomorphism \( H^0(X'_3 \to X'_3) \cong H^1A_1 \).

**Proof.** Consider \( [\nu_3] \in (H^0_{\pi_1^{alg}}G_3)_1 \) and \( \psi_3[\nu_3] \in \pi^0BG_3 \). As in the proof of Theorem 10.7, we write:

\[
\nu_3 \in X'_3, \quad g_3 = \nu_3 - \nu_3 = \nu_3^{sc}, \quad \nu_3^{sc} \in Q'_3, \quad \nu_3^{sc}(-1) = (g_3^{sc})^{-1}. \gamma g_3^{sc},
\]

We lift canonically \( \nu_3^{sc} \) to \( \nu_2^{sc} \in Q'_2 \) and lift canonically \( g_3^{sc} \) to \( g_2^{sc} \in G_2^{sc} \). We lift \( \nu_2 \) to \( \nu_2 \) and \( \rho_2(\nu_2^{sc}) = v(\nu_2^{sc}) \). We set

\[
g_2 = \rho_2(g_2^{sc}) \cdot \nu_2(-1) \in G_2(C).
\]

Then \( g_2 \) is a lift of \( g_3 \in G_3(R) \). We set

\[
a_1 = g_2^{-1} \cdot \gamma g_2.
\]

Then \( a_1 \in Z^1A_1 \), and

\[
\delta^0(\psi_3[\nu_3]) = \delta^0[g_3] = [a_1] \in H^1G_1.
\]

We have

\[
a_1 = g_2^{-1} \cdot \gamma g_2 = \nu_2(-1) \cdot \rho_2(g_2^{sc})^{-1} \cdot \gamma \rho_2(g_2^{sc}) \cdot \nu_2(-1) = \nu_2(-1) \cdot \rho_2(g_2^{sc})\nu_2(-1).
\]

Thus

\[
\delta^0(\psi_3[\nu_3]) = [a_1] = [\nu_2(-1) \cdot \gamma \nu_2(-1) \cdot \rho_2(\nu_2^{sc})(-1)] \in H^1A_1
\]

and

\[
(i \circ \delta^0 \circ \psi_3)[\nu_3] = [\nu_2(-1) \cdot \gamma \nu_2(-1) \cdot \rho_2(\nu_2^{sc})(-1), 1] = [\rho_2(\nu_2^{sc})(-1), j_*(\nu_2)(-1)] \in H^0(T_2 \to T_3).
\]

On the other hand, we have

\[
u_3[\nu_3] = [v(\nu_3^{sc}), \nu_3] \in H^0(X'_3 \to X'_3),
\]

\[
(ev_3^* \circ \psi_3)[\nu_3] = [v(\nu_3^{sc})(-1), \nu_3(-1)] \in H^0(T_2 \to T_3).
\]

Since \( \rho_2(\nu_2^{sc}) = v(\nu_2^{sc}) \) and \( j(\nu_2) = \nu_3 \), we conclude that the diagram (10.14) indeed commutes, as required. \( \square \)

### 11. Examples

11.1. For even \( l > 4 \), consider the real spin group \( G_q^{sc} = G(D_\ell, 0, id, q) \) of type \( D_\ell \) with the notation of [BT21, Section 12.12], where \( q \in K(D) \) is a Kac labeling of the affine Dynkin diagram \( \tilde{D} = D^{(1)} \) with the notation of [OV90, Table 6]. This group comes with a compact maximal torus \( T^{sc} \) and a system of simple roots \( S = \{\alpha_1, \ldots, \alpha_\ell\} \) with the notation of [OV90, Table 1]. We consider the fundamental coweight

\[
\omega_\ell^{-1} = (\varepsilon_1 - \varepsilon_\ell + \varepsilon_\ell^-)/2 \in P_\ell \subset t,
\]

where \( \varepsilon_1, \ldots, \varepsilon_\ell \) are the cocharacters dual to the weights \( \varepsilon_1, \ldots, \varepsilon_\ell \) of the vector representation, cf. [OV90, Table 1]. Set

\[
a = \exp 2\pi i \omega_\ell^{-1} \in T^{sc}(R) \subset G_q^{sc}(R).
\]

Since \( \ell \) is even, we have \( 2\omega_\ell^{-1} \in Q_\ell = X_\ell(T^{sc}) \), and hence \( a^2 = 1 \).
We consider the one-dimensional split \( R \)-torus \( T^1_s \) and the one-dimensional compact \( R \)-torus \( T^1_c \), see Corollary 3.4. Consider the elements of order 2

\[
-1 \in T^1_s(R) \quad \text{and} \quad -1 \in T^1_c(R).
\]

We consider the reductive \( R \)-groups

\[
G_{s,q} = (G_q^s \times T^1_s) / \{1, (a, -1)\} \quad \text{and} \quad G_{c,q} = (G_q^c \times T^1_c) / \{1, (a, -1)\}.
\]

11.2. Let \( G \) be either \( G_{s,q} \) or \( G_{c,q} \). We wish to compute \( H^1G \) and \( \pi_0^R G \). We use Notation 7.1. Recall that \( S = Z(G)^0 \); then \( S \) is either \( T^1_s \) or \( T^1_c \). Recall that \( G^{ss} = [G, G] \); then \( G^{ss} = G_q^{sc} \).

We freely use the notation of [BT21, Section 16].

We consider the one-dimensional split \( R \)-torus \( T^1 \). The automorphism \( \tau \) acts on the weights and coweights as follows:

\[
\varepsilon \mapsto -\varepsilon, \quad \varepsilon^\vee \mapsto -\varepsilon^\vee, \quad \varepsilon_i \mapsto \varepsilon_i, \quad \varepsilon_i^\vee \mapsto \varepsilon_i^\vee.
\]

Therefore, \( T_0 \) is a maximal torus in \( G^{ss} \) and

\[
\Lambda_0 = M_0 = 0, \quad \Lambda_0^\vee = M_0^\vee = 0, \quad X_0 = P,
\]

\[
\tilde{X}_0^\vee = \left\langle \frac{\pm \varepsilon_1^\vee \pm \cdots \pm \varepsilon_\ell^\vee}{2} \right\rangle \quad \text{with odd number of minuses among } \pm \varepsilon_i^\vee.
\]

Hence in Theorem 7.14 we have \( m = 0 \) and the cohomology classes correspond to Kac labelings \( p \in \mathcal{K}(\bar{D}) \).

The lattice \( X_0 \) is generated by the root lattice \( Q \) and the weights

\[
\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \cdots + \varepsilon_{\ell-3} - \varepsilon_{\ell-2} + \varepsilon_{\ell-1} + \varepsilon_\ell = \frac{\alpha_1 + \alpha_3 + \cdots + \alpha_{\ell-3} + \alpha_\ell}{2}
\]

\[
\text{and} \quad \varepsilon_{\ell-1} = \frac{\alpha_{\ell-1} + \alpha_\ell}{2}.
\]

For \( p \in \mathcal{K}(\bar{D}) \), we set:

\[
r(p) = p_1 + p_3 + \cdots + p_{\ell-3} + p_\ell \pmod{2},
\]

\[
r'(p) = p_{\ell-1} + p_\ell.
\]

The congruences (7.13) are equivalent to \( r(p) \equiv r(q), \quad r'(p) \equiv r'(q) \pmod{2} \).
The group $F_0$ is generated by the class $[\omega_{\ell-1}]$. It acts on $K(\tilde{D})$ by the reflection with respect to the vertical symmetry axis of $\tilde{D}$:

![Diagram](image_url)

We denote by $[p]$ the $F_0$-orbit of $p \in K(\tilde{D})$. Then $r(p)$ and $r'(p)$ depend only on $[p]$.

By Theorem 7.14, the set $H^1G_{s,q}$ is in a canonical bijection with the set of orbits

\[ \text{Orb}(r(q), r'(q)) = \{ [p] \mid p \in K(\tilde{D}), r(p) \equiv r(q), r'(p) \equiv r'(q) \pmod{2} \}. \]

These four sets $\text{Orb}(r, r')$ (described by representatives of orbits) are:

- $\text{Orb}(0,0)$:
  \[
  \begin{array}{cccc}
  2 & 0 \cdots & 0 & 0 \\
  0 & 2 \cdots & 0 & 0 \\
  0 & 0 \cdots & 1 & 0 \\
  0 & 0 \cdots & 0 & 0 \\
  \end{array}
  \quad \text{with 1 at } i = 2j,
  \]

- $\text{Orb}(0,1)$:
  \[
  \begin{array}{cccc}
  1 & 0 \cdots & 1 & 0 \\
  0 & 0 \cdots & 0 & 0 \\
  0 & 0 \cdots & 1 & 0 \\
  0 & 0 \cdots & 0 & 1 \\
  \end{array}
  \]

- $\text{Orb}(1,0)$:
  \[
  \begin{array}{cccc}
  1 & 0 \cdots & 0 & 0 \\
  0 & 0 \cdots & 1 & 0 \\
  0 & 0 \cdots & 0 & 0 \\
  0 & 0 \cdots & 0 & 1 \\
  \end{array}
  \quad \text{with 1 at } i = 2j + 1,
  \]

- $\text{Orb}(1,1)$:
  \[
  \begin{array}{cccc}
  1 & 0 \cdots & 0 & 0 \\
  0 & 0 \cdots & 0 & 0 \\
  0 & 0 \cdots & 0 & 1 \\
  0 & 0 \cdots & 0 & 1 \\
  \end{array}
  \]

for each integer $i$ (even or odd, respectively) with $1 < i \leq \ell/2$. We have

\[
\#\text{Orb}(0,0) = [\ell/4] + 2, \quad \#\text{Orb}(0,1) = 2, \quad \#\text{Orb}(1,0) = [\ell/4], \quad \#\text{Orb}(1,1) = 1.
\]

We see that

\[
\#H^1G_{s,q} = \#\text{Orb}(r(q), r'(q)).
\]

Thus we know the cardinalities of $H^1G_{s,q}$ for all $q \in K(\tilde{D})$:

If $r'(q) = 0$, then

\[
\#H^1G_{s,q} = \begin{cases} [\ell/4] + 2, & r(q) = 0, \\ [\ell/4], & r(q) = 1. \end{cases}
\]

If $r'(q) = 1$, then

\[
\#H^1G_{s,q} = \begin{cases} 2, & r(q) = 0, \\ 1, & r(q) = 1, \end{cases} = 2 - r(q).
\]

11.4. We compute $H^1G_{s,q}$. In this case $S = T^1_0$ is a compact torus. The difference with the previous example is that now $\tau = \text{id}$ and $T_0 = T$. It follows that

\[
\Lambda_0 = \Lambda, \quad M_0 = M, \quad X_0 = X, \quad \text{and}
\]

\[
\Lambda^\vee_0 = \tilde{\Lambda}^\vee_0 = \Lambda^\vee, \quad M^\vee_0 = \tilde{M}^\vee_0 = M^\vee, \quad X^\vee_0 = \tilde{X}^\vee_0 = X^\vee.
\]

In particular, $M^\vee_0/2\Lambda^\vee_0 = (\sqrt{2}\varepsilon^\vee)/\langle 2\varepsilon^\vee \rangle \cong \mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$. For $m \in M^\vee$, we denote its class in $\mathbb{Z}/4\mathbb{Z}$ by $[m]$, that is, $[m] = k \pmod{4}$ if $m = \frac{k}{2}\varepsilon^\vee$.

The lattice $X$ is generated by the root lattice $Q$ and the weights

\[
\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \cdots + \varepsilon_{\ell-3} - \varepsilon_{\ell-2} + \varepsilon_{\ell-1} + \varepsilon_{\ell} = \frac{\alpha_1 + \alpha_3 + \cdots + \alpha_{\ell-3} + \alpha_{\ell}}{2}
\]

and $\varepsilon_{\ell-1} + \varepsilon = \frac{\alpha_{\ell-1} + \alpha_{\ell}}{2} + \varepsilon$. 
For $m \in M^\vee$, we set
\[ r''(m) = 2\langle \varepsilon, m \rangle \mod 2 = \begin{cases} 0, & [m] = 0 \text{ or } 2, \\ 1, & [m] = 1 \text{ or } 3. \end{cases} \]

The congruences (7.13) are equivalent to $r(p) \equiv r(q)$, $r'(p) + r''(m) \equiv r'(q) \mod 2$.

The group $F_0$ is generated by the class $[\omega_{\varepsilon-1} + \frac{1}{2}\varepsilon']$. It acts on $\mathcal{K}(\tilde{D})$ by the reflection with respect to the vertical symmetry axis of $\tilde{D}$ and on $M_0^\vee/2\Lambda_0^\vee \cong \mathbb{Z}/4\mathbb{Z}$ as $0 \leftrightarrow 2, 1 \leftrightarrow 3$. Note that $r(p)$, $r'(p)$, and $r''(m)$ depend only on the $F_0$-orbit of $(p, [m]) \in \mathcal{K}(\tilde{D}) \times M_0^\vee/2\Lambda_0^\vee$.

Let $\text{Orb}(r, r', r'')$ denote the set of $F_0$-orbits of $(p, [m])$ such that
\[ r(p) \equiv r, \quad r'(p) \equiv r', \quad r''(m) \equiv r'' \mod 2. \]

The representatives of the orbits in $\text{Orb}(r, r', r'')$ are $(p, [m])$, where $p$ are the representatives of the orbits in $\text{Orb}(r, r')$ and
\[ [m] = \begin{cases} r'', & \text{if } p \text{ is fixed by } F_0, \\ r'' \text{ or } r'' + 2, & \text{otherwise}. \end{cases} \]

The cardinalities of these eight orbit sets are:
\[ \#\text{Orb}(0, 0, 0) = \#\text{Orb}(0, 0, 1) = \begin{cases} 2\lfloor \ell/4 \rfloor + 3, & \ell/2 \text{ even}, \\ 2\lfloor \ell/4 \rfloor + 4, & \ell/2 \text{ odd}, \end{cases} \quad \#\text{Orb}(0, 1, 0) = \#\text{Orb}(0, 1, 1) = 2, \]
\[ \#\text{Orb}(1, 0, 0) = \#\text{Orb}(1, 0, 1) = \begin{cases} 2\lfloor \ell/4 \rfloor, & \ell/2 \text{ even}, \\ 2\lfloor \ell/4 \rfloor - 1, & \ell/2 \text{ odd}, \end{cases} \quad \#\text{Orb}(1, 1, 0) = \#\text{Orb}(1, 1, 1) = 2. \]

By Theorem 7.14, the set $H^1 G_{c,q}$ is in a canonical bijection with the union of two orbit sets $\text{Orb}(r(q), r'(q), 0) \cup \text{Orb}(r(q), r'(q) - 1, 1)$. We obtain
\[ \#H^1 G_{c,q} = \#\text{Orb}(r(q), r'(q), 0) + \#\text{Orb}(r(q), r'(q) - 1, 1). \]

Thus if $r(q) = 0$, then
\[ \#H^1 G_{c,q} = \begin{cases} 2\lfloor \ell/4 \rfloor + 5, & \ell/2 \text{ even} \\ 2\lfloor \ell/4 \rfloor + 6, & \ell/2 \text{ odd} \end{cases} = \ell/2 + 5. \]

If $r(q) = 1$, then
\[ \#H^1 G_{c,q} = \begin{cases} 2\lfloor \ell/4 \rfloor + 2, & \ell/2 \text{ even} \\ 2\lfloor \ell/4 \rfloor + 1, & \ell/2 \text{ odd} \end{cases} = \ell/2 + 2. \]

11.5. In [Bor88] (see also [Bor14]), for any connected reductive $\mathbb{R}$-group $G$, the first-named author constructed a bijection between $H^1 G$ and the set of orbits of a certain action of $W_0$ on $H^1 T$, where $T$ is a fundamental torus of $G$, and $W_0$ is a certain subgroup of the Weyl group $W = W(G, T)$; see [BT21, Section 4]. We compare our present formula for $H^1 G$ with the old formula of [Bor88] in the cases $G = G_{s,q}$ and $G = G_{c,q}$ as defined in Subsection 11.1. In both cases we have $W_0 = W$, and so $W_0$ is a Coxeter group of type $D_\ell$ and of order $\ell!2^{\ell-1}$. Moreover, in both cases the group $F_0$ is generated by an element acting on $\tilde{D}$ by the reflection with respect to the vertical symmetry axis.

For $G = G_{s,q}$ we obtain the set $H^1 G$ of cardinality $\leq \ell/4 + 2$ by computing orbits of a group of order 2. On the other hand, with the old formula we obtain $\#H^1 G$ as the set of orbits of the group $W_0$ of order $\ell!2^{\ell-1}$ in the set $H^1 T$ of cardinality $2^{\ell-1}$.

Similarly, for $G = G_{c,q}$ we obtain the set $H^1 G$ of cardinality $\ell/2+2$ or $\ell/2+5$ by computing orbits of a group of order 2. On the other hand, with the old formula we obtain $\#H^1 G$ as the set of orbits of the group $W_0$ of order $\ell!2^{\ell-1}$ in the set $H^1 T$ of cardinality $2^{\ell+1}$. 
We conclude that our present method requires less calculations than the old one.

11.6. We compute $\pi^R_0 G$, where $G = G_{c,q}$. We have $\pi^1_{\text{alg}} G \simeq \mathbb{Z}$, where $\gamma$ acts on $\pi^1_{\text{alg}} G$ trivially, and $\tau$ acts as $-1$. We see that $H^0 \pi^1_{\text{alg}} G = 0$, and by Theorem 9.2 we have $\pi^R_0 G_{c,q} \cong (H^0 \pi^1_{\text{alg}} G)_1 = 0$.

11.7. We compute $\pi^R_0 G$, where $G = G_{s,q}$. We have $\pi^1_{\text{alg}} G \simeq \mathbb{Z}$, where $\tau$ acts on $\pi^1_{\text{alg}} G$ as $-1$, and $\gamma$ acts trivially. We see that $H^0 \pi^1_{\text{alg}} G \simeq \mathbb{Z}/2\mathbb{Z}$. We wish to compute $(H^0 \pi^1_{\text{alg}} G)_1$.

The group $H^0 \pi^1_{\text{alg}} G$ acts on $\mathcal{K}(\tilde{D})$ via the homomorphism $\text{Ad}_*: H^0 \pi^1_{\text{alg}} G \to H^0 C = C = C_0 \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, which sends the generator $[\omega_{\ell-1}^\vee + \frac{1}{2} \xi^\vee]$ of $H^0 \pi^1_{\text{alg}} G$ to $[\omega_{\ell-1}^\vee]$. As already noted in Subsection 11.3, $[\omega_{\ell-1}^\vee]$ acts on $\mathcal{K}(\tilde{D})$ by reflecting the Dynkin diagram $\tilde{D}$ with respect to the vertical symmetry axis.

By Theorem 9.2, $\pi^R_0 G_{s,q} = (H^0 \pi^1_{\text{alg}} G)_1$ is the stabilizer of $q$ under the action of $H^0 \pi^1_{\text{alg}} G$ on $\mathcal{K}(\tilde{D})$. It follows that $\pi^R_0 G_{s,q}$ is nontrivial if and only if $q$ is fixed by the aforementioned reflection. This condition is satisfied if and only if

$$q = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \end{pmatrix} \text{ with 1 at } \ell/2 \text{ or } 0 \text{ at } \ell/2$$

up to the action of $C$, that is, $G^s = G^c$ is isomorphic to either $\text{Spin}_{\ell,\ell}$ or $\text{Spin}_{2\ell}$; see [OV90, Table 7]. In this case $\pi^R_0 G_{s,q} \simeq \mathbb{Z}/2\mathbb{Z}$.

**Appendix A. Classification of $\Gamma$-lattices**

For the reader’s convenience, we provide a short elementary proof of the following known result.

**A.1. Theorem** (Curtis and Reiner [CR62, Theorem (74.3)]). Let $L$ be a lattice (a finitely generated free abelian group), and $\tau: L \to L$ be an involutive automorphism, that is, $\tau^2 = \text{id}$. Then there exists a basis of $L$ consisting of vectors $e_i$, $f_j$, $g_k$, $h_k$, on which $\tau$ acts as follows: $\tau(e_i) = e_i$, $\tau(f_j) = -f_j$, $\tau(g_k) = h_k$, $\tau(h_k) = g_k$.

**Proof.** We prove the theorem by induction on the rank $r$ of $L$. The case $r = 0$ is trivial.

Induction step: assume that $r \geq 1$. If $\tau = -\text{id}$, there is nothing to prove. Otherwise there is a nonzero $\tau$-fixed vector $e$, which we can choose to be primitive (divisible). Consider the quotient lattice $L/e$. Since $e$ is primitive, $L/e$ is a lattice (free abelian group) of rank $r-1$. Clearly, $\tau$ acts on $L/e$ as an involution. By the induction hypothesis, the quotient lattice $L/e$ has a basis $[e_i]$, $[f_j]$, $[g_k]$, $[h_k]$ with required properties, where $[v]$ denotes the coset of a vector $v$ in $L$. We consider the action of $\tau$ on the basis $e_i$, $f_j$, $g_k$, $h_k$ of $L$.

Firstly, $\tau(e_i) = e_i$. Otherwise it would be $\tau(e_i) = e_i + me$ with some nonzero $m$, but then $\tau^2(e_i) = e_i + 2me \neq e_i$, which contradicts the assumption $\tau^2 = \text{id}$.

Secondly, $\tau(g_k) = h_k + me$ for some integer $m$ (depending on $k$). After replacing $h_k$ with $h_k + me$, we have $\tau(g_k) = h_k$, and hence $\tau(h_k) = g_k$ (because $\tau$ is involutive).

Finally, $\tau(f_j) = -f_j + me$ for some integer $m$ (depending on $j$). If $m = 2n$ is even, then $\tau(f_j - ne) = -f_j + ne$, and after replacing $f_j$ with $f_j - ne$ we obtain $\tau(f_j) = -f_j$. If $m = 2n + 1$ is odd, the same replacement gives $\tau(f_j) = -f_j + e$.

If this latter case $\tau(f_j) = -f_j + e$ does not appear, the proof is complete. Otherwise let us fix some $j$, say, $j = 0$, such that $\tau(f_0) = -f_0 + e$, and consider all other $j$ for which
\[ \tau(f_j) = -f_j + e. \] After replacing \( f_j \) with \( f_j - f_0 \) for all these other \( j \), we obtain \( \tau(f_j) = -f_j \). In other words, we may assume that \( \tau(f_j) = -f_j + e \) holds for only one \( j \).

Now we replace \( e \) with \( -f_j + e \) and obtain two basis vectors \( g = f_j \) and \( h = -f_j + e \), for which \( \tau(g) = h \) and \( \tau(h) = g \). Thus we obtain a basis of \( L \) with required properties, which completes the proof of the theorem. \( \square \)

See Casselman [Cas08] for another elementary proof of Theorem A.1.

References

[AT18] Jeffrey Adams and Olivier Taïbi. Galois and Cartan cohomology of real groups. *Duke Math. J.*, 167(6):1057–1097, 2018.

[AW67] Michael F. Atiyah and Charles T. C. Wall. Cohomology of groups. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 94–115. Thompson, Washington, D.C., 1967.

[BDR21] Mikhail Borovoi, Christopher Daw, and Jinbo Ren. Conjugation of semisimple subgroups over real number fields of bounded degree. *Proc. Amer. Math. Soc.*, 149(12):4973–4984, 2021.

[BE16] Mikhail Borovoi and Zachi Evenor. Real homogeneous spaces, Galois cohomology, and Reeder puzzles. *J. Algebra*, 467:307–365, 2016.

[BGL21] Mikhail Borovoi, Willem A. de Graaf, and Hông Vân Lê. Classification of real trivectors in dimension nine. https://doi.org/10.48550/arXiv.2108.00790.

[BGR22] Mikhail Borovoi, Andrei A. Gornitskii, and Zev Rosengarten. Galois cohomology of real quasi-connected reductive groups. *Arch. Math. (Basel)*, 118(1):27–38, 2022.

[Bor88] Mikhail V. Borovoi. Galois cohomology of real reductive groups and real forms of simple Lie algebras. *Functional. Anal. Appl.*, 22(2):135–136, 1988.

[Bor98] Mikhail Borovoi. Abelian Galois cohomology of reductive groups. *Mem. Amer. Math. Soc.*, 132(626):viii+50, 1998.

[Bor14] Mikhail Borovoi. Galois cohomology of real semisimple groups via Kac labelings. *Transform. Groups*, 26(2):433–477, 2021.

[Cas08] Bill Casselman. Computations in real tori. In *Representation theory of real reductive Lie groups*, volume 472 of *Contemp. Math.*, pages 137–151. Amer. Math. Soc., Providence, RI, 2008.

[CE56] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.

[CR62] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*. Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[Djo83] Dragomir Ž. Djoković. Classification of trivectors of an eight-dimensional real vector space. *Linear and Multilinear Algebra*, 13(1):3–39, 1983.

[GOV94] Vladimir V. Gorbatsevich, Arkady L. Onishchik, and Èrnest B. Vinberg. *Lie groups and Lie algebras III. Structure of Lie groups and Lie algebras*. Transl. from the Russian by V. Minachin, volume 41. Berlin: Springer-Verlag, 1994.

[grp] grp (https://mathoverflow.net/users/26145/grp). Mostow’s theorem on algebraic groups. MathOverflow. URL: https://mathoverflow.net/q/107342 (version: 2012-09-16).

[Hoc81] Gerhard P. Hochschild. *Basic theory of algebraic groups and Lie algebras*, volume 75 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1981.

[Kac68] Victor G. Kac. Graded Lie algebras and symmetric spaces. *Functional. Anal. i Priložen.*, 2(2):93–94, 1968. English transl.: *Funct. Anal. Appl.*, 2(2):182–183, 1968.

[Kac69] Victor G. Kac. Automorphisms of finite order of semisimple Lie algebras. *Funktional. Anal. i Priložen.*, 3(3):94–96, 1969. English transl.: *Funct. Anal. Appl.*, 3(3):252–254, 1969.

[Mat64] Hideya Matsumoto. Quelques remarques sur les groupes de Lie algébriques réels. *J. Math. Soc. Japan*, 16:419–446, 1964.
[Mil17] James S. Milne. *Algebraic groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. The theory of group schemes of finite type over a field.

[Mos56] George D. Mostow. Fully reducible subgroups of algebraic groups. *Amer. J. Math.*, 78:200–221, 1956.

[Ngu00] Nguyêñ Quôç Thâńg. Number of connected components of groups of real points of adjoint groups. *Commun. Algebra*, 28(3):1097–1110, 2000.

[OV90] Arkady L. Onishchik and Èrnest B. Vinberg. *Lie groups and algebraic groups*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. Translated from the Russian and with a preface by D. A. Leites.

[PR94] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[San81] Jean-Jacques Sansuc. Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres. *J. Reine Angew. Math.*, 327:12–80, 1981.

[Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.

[Ser97] Jean-Pierre Serre. *Galois cohomology*. Springer-Verlag, Berlin, 1997. Translated from the French by Patrick Ion and revised by the author.

[Spr79] Tonny A. Springer. Reductive groups. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, Part 1, Proc. Sympos. Pure Math., XXXIII, pages 3–27. Amer. Math. Soc., Providence, R.I., 1979.

[Vos98] Valentin È. Voskresenski˘ı. *Algebraic groups and their birational invariants*, volume 179 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1998. Translated from the Russian manuscript by Boris È. Kunyavski˘ı.

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