Diffusion-limited Reactions of hard-core Particles in one-dimension

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(Revised October, 1998)

We investigate three different methods to tackle the problem of diffusion-limited reactions (annihilation) of hard-core classical particles in one dimension. We first extend an approach devised by Lushnikov and calculate for a single species the asymptotic long-time and/or large distance behavior of the two-point correlation function. Based on a work by Grynberg et al., which was developed to treat stochastic adsorption-desorption models, we provide in a second step the exact two-point correlation function (both for one and two-time) of Lushnikov’s model. We then propose a new formulation of the problem in terms of path integrals for pseudo-fermions. This formalism can be used to advantage in the multi-species case, specially when applying perturbative renormalization group techniques.

PACS number(s): 05.70 Ln; 47.70-n; 82.20 Mj; 02.50-r

I. INTRODUCTION

The recent interest in modelling low dimensional diffusion-limited reactions has been stimulated in part by the experimental observation of anomalous kinetics in low-dimensional systems [1]. The traditional approach to chemical reactions is based on mean-field theory, i.e., rate equations for the densities of the various reactants. The latter describe well the reaction kinetics in three dimensions because diffusive transport of reactants allows to eliminate the spatial fluctuations of the concentrations. However, in lower dimensions, due to the lack of phase space, the reactants spatial fluctuations can grow and develop inhomogeneities in the concentrations. Furthermore, even in the spatially homogeneous case, the rate equations are not applicable in less than three dimensions; for example, in the two-species diffusion-limited annihilation the concentration of the particles decays (for identical initial concentrations) slower than the mean-field theory predicts. Thus the fluctuation-dominated dynamics is beyond the classical theories, yet can be accounted for by simple one-dimensional models of hard-core particles. The latter are solved either numerically or analytically by applying techniques from (classical or quantum) statistical mechanics [1]. In particular, exact solutions have been obtained by inter-particle distribution methods, by relating the systems to dual and solvable 1D models [3,5–7] (Kinetic Ising and Potts) and by mapping the diffusion-reaction processes to an imaginary-time dynamics of quantum spin chains with non-hermitian Hamiltonians [8]. An alternative and fruitful approach has been developed by Cardy and collaborators [9,10]: the idea is to reformulate the original problem in terms of a field theory of interacting bosons and subsequently use renormalization group techniques. This is a powerful method as it applies to arbitrary dimension and low densities of particles, a regime where universal behavior (scaling) is usually observed. Despite the progress achieved in this field, the multi-species case is still poorly understood. Furthermore, when the density of particles is high, the hard-core constraint on the dynamics of the diffusing particles becomes important. Experimentally the single-species fusion model used to describe the photoluminescence of excitons diffusing along one dimensional chains is seen to apply only as long as the initial exciton densities are small [1]. To our knowledge, there has been no systematic effort yet to investigate theoretically the regime of high densities of reactants.

This is the first in a series of technical papers where we will explore a new approach in an attempt to remedy the difficulties encountered so far. We propose to start from a quantum spin chain formulation of the master equation, fermionize and subsequently apply renormalization group techniques to deal with the interaction terms arising in the multi-species case. Note that in the single species case, the method applies at arbitrary densities as the hard-core of the classical particles is automatically accounted for by the Fermi statistics. When various species react and diffuse, it is appropriate to distinguish two cases: i) the various species have infinite on-site repulsion with themselves only; this can be treated readily following the methods outlined in the following sections; ii) the particles of different species all have a hard core constraint; this is far more difficult and will be investigated elsewhere. As with the other methods devised so far, there is a price to pay: the calculations involved are sometimes extremely tedious.

The purpose of this first paper is modest as we will focus on the single-species case for which fermion-fermion interactions do not arise as a consequence of mapping classical particles to fermions. We want to show explicitly how by elementary means we can reproduce known results (the long-time behaviour of the density), and derive some new results, i.e., the explicit and exact analytical form of the two-point correlation function of one and two-time. The paper is intended as an introduction to the fermionic functional integration approach which we will combine in...
forthcoming papers with the renormalization techniques to treat the multi-species case.

In the first section, we define the model, introduce notations and review an extension of a method developed by Lushnikov. The second section is devoted to the application of an elegant technique introduced by Grynberg et al. in a different context to evaluate the two-point (one-time) correlation function. The third section deals with a new and powerful formulation of the problem in terms of path integrals of pseudo-fermionic variables. We illustrate the technique by computing the two-time correlation function. A brief discussion concludes the paper.

II. LUSHNIKOV’S APPROACH

This section introduces notations, summarizes and extends Lushnikov’s genuine approach. We consider a lattice of $N$ (even) sites (length $L = Na$, $a = 1$ and assume $N/2$ even), with periodic boundary conditions, on which classical (spinless) particles with hard-core can diffuse (annihilate) to adjacent empty (occupied) sites with rate $D$. Whenever the arrival site is occupied, an annihilation-reaction ($1 + 1 \rightarrow 0$) takes place. A source of intensity $J$ injects pairs of particles on adjacent sites ($0 \rightarrow 1 + 1$). Lushnikov has managed to rewrite the master equation that describes the annihilation and diffusion processes described above in terms of an imaginary-time Schrödinger equation:

$$\frac{d}{dt} |\psi(t)\rangle = \mathcal{L} |\psi(t)\rangle,$$

where $\mathcal{L}$ denotes the so called “Liouvillian”, which by abuse of language we will call “non-hermitian Hamiltonian”. $|\psi(t)\rangle$ represents the state of the system at time $t$,

$$|\psi(t)\rangle = \sum_{\{n\}} \left( P(\{n\}, t) \prod_{m' \in \{\{n\}\}} \sigma^+_m \right)|0\rangle,$$

where $m'\in\{\{n\}\}$ represents the sites of configuration $\{n\}$ which are occupied. The Liouvillian is given by

$$\mathcal{L} = (J + D) \sum_m \left( \sigma^+_m \sigma^-_{m+1} + \sigma^+_m \sigma^-_{m-1} + \sigma^-_m \sigma^+_{m+1} + \sigma^-_m \sigma^+_{m-1} \right) + D \sum_m \left( \sigma^-_{m+1} \sigma^-_m - \sigma^+_{m+1} \sigma^+_m - 2 \sigma^+_m \sigma^-_m \right) - JN$$

When $J = 0$, i.e., no source, in addition to diffusive processes with rate $D$, only irreversible reactions ($1 + 1 \rightarrow 0$), with rate $2D$ take place.

For finite ($J > 0$) source, the diffusive processes ($1 + 0 \rightarrow 0 + 1$ and $0 + 1 \rightarrow 0 + 1$) take place with rate $J + D$ and we also have reversible reactions: particles are annihilated ($1 + 1 \rightarrow 0$) with rate $J + 2D$ and created ($0 \rightarrow 1 + 1$) with rate $J$. It is worth emphasizing that these rates are not independent, and are chosen such that the Liouvillian is quadratic in the spin variables for a single species (the higher order terms cancel due to the properties of Pauli matrices). We also point out that in this model the “annihilation rate” ($J + 2D$) is always bigger to the “creation rate” ($J$). In the $n$-species case, this property does no longer hold if we assume hard core repulsion between all species. Indeed, one obtains a spin Hamiltonian ($S = n/2$) that is a polynomial of higher order in the spin operators and in general cannot be solved exactly. If however, we assume infinite on-site repulsion only between particles of the same species, we can rewrite the Hamiltonian as a quadratic form of coupled spins $1/2$. The latter can be solved by the techniques presented below.

To solve the Schrödinger equation in imaginary time, Lushnikov performs a Jordan-Wigner transformation and introduces the fermionic operators $a_m = \prod_{j<m} (1 - 2n_j) \sigma^-_m$.

Because of the form of the resulting non-hermitian Hamiltonian, it is appropriate to work with Fourier modes $a_q = \frac{e^{i q}}{\sqrt{N}} \sum_m a_m e^{-i q m}$. The antiperiodic boundary conditions lead to: $q = \pm (2l - 1) \pi / N$, $l = 1, 2, \ldots, N/2$. On Fourier transformation, the evolution operator reads $\mathcal{L} = \sum_{q \geq 0} \mathcal{L}_q$, where $(n_q \equiv a^+_q a_q)$

$$\mathcal{L}_q = 2(J + D) \left\{ \cos q(n_q + n_{-q}) + \sin q(a_q a_{-q} - a^+_q a^+_{-q}) \right\} + 2D \left\{ \sin q(a_q a_{-q} + a^+_q a^+_{-q}) - (n_q + n_{-q}) \right\} - JN$$

Now, by a BCS-like Ansatz,

$$|\psi(t)\rangle = \prod_{q > 0} |\psi_q(t)\rangle = \prod_{q > 0} \left( A_q(t) a^+_q a^+_{-q} + B_q(t) \right)|0\rangle,$$

Lushnikov is able to decouple the dynamical equation as
\[
\frac{d}{dt} |\psi_q(t)\rangle = \mathcal{L}_q |\psi_q(t)\rangle, \quad \forall q > 0
\]

For a lattice which is initially completely occupied, i.e., \(A_q(0) = 1, B_q(0) = 0\), one solves the above equations by

\[
A_q(t) = \frac{1}{4(J + 2D) \sin^2 \left(\frac{q}{2}\right)} (p_2 e^{pt} - p_1 e^{pt}), \quad B_q(t) = \frac{-(J + 2D) \sin q}{2(J + 2D) \sin^2 \left(\frac{q}{2}\right)} (e^{pt} - e^{pt})
\]

where

\[
p_1 = -2(J + 2D)(1 - \cos q), \quad p_2 = 2J(1 + \cos q)
\]

In the absence of source \((J = 0)\), the solution simplifies considerably to

\[
A_q(t) = \exp(-4Dt(1 - \cos q)), \quad B_q(t) = \cot \left(\frac{q}{2}\right) \left(\exp(-4Dt(1 - \cos q)) - 1\right)
\]

At this point it is worth noting that the ket \(|\psi(t)\rangle\) characterizes the state of the system at any time without however being an eigenvector of \(\mathcal{L}\) (The Liouvillian is not normal, however see below).

In ref. \([2]\), Lushnikov calculates the density of particles by the method of the generating function, which we extend in order to evaluate the two-point correlation function. The density reads

\[
\rho(t) = \sum_{\{n\}} \bar{n}_i(\{n\}) P(\{n\},t), \forall i,
\]

Similarly, the two-point correlation function is written as

\[
G_r(t) \equiv \langle \bar{n}_i \bar{n}_{i+r} \rangle(t) = \sum_{\{n\}} \bar{n}_i(\{n\}) \bar{n}_{i+r}(\{n\}) P(\{n\},t),
\]

where the translational symmetry of the system has been used. We observe that in this formalism

\[
\rho(t) = \langle 0 | \exp \left(\sum_n \sigma_n^- n_i |\psi(t)\rangle\right), \quad G_r(t) = \langle 0 | \exp \left(\sum_n \sigma_n^- n_i n_{i+r} |\psi(t)\rangle\right)
\]

as one can check using the explicit form of \(|\psi(t)\rangle\) (develop the exponential, order each term and perform a Jordan-Wigner transformation \([3]\)). It is appropriate to consider the following generating function:

\[
G(x, y, z, t) = \langle 0 | \exp \left(x \sigma_i^- + y \sigma_{i+r}^- + z \sum_{n \neq i, i+r} \sigma_n^- \right) |\psi(t)\rangle = \sum_{\{n\}} x^{n_i} y^{n_{i+r}} z \sum_{n} \bar{n}_{i-j} \bar{n}_{i+n+r} P(\{n\},t)
\]

So we have,

\[
\rho(t) = \frac{\partial}{\partial x} G(x, y, z, t) \bigg|_{x,y,z=1}, \quad G_r(t) = \frac{\partial^2}{\partial x \partial y} G(x, y, z, t) \bigg|_{x,y,z=1}
\]

To compute the generating function, we rewrite the state as

\[
|\psi(t)\rangle = \prod_{q > 0} \left( A_q(t) a_q^+ a_q^+ + B_q(t) \right) |0\rangle = \prod_{q > 0} \left( B_q(t) - \frac{2A_q(t)}{N} \sum_{n > m} a_n^+ a_n^+ \sin q(n - m) \right) |0\rangle
\]

Note the normalisation condition due to the conservation of probability:

\[
\sum_{\{n\}} P(\{n\},t) = \prod_{q > 0} \left( B_q(t) - A_q(t) \cot \left(\frac{q}{2}\right) \right) = 1
\]

Next expand the argument of the exponential as

\[
G(x, y, z, t) = \langle 0 | \left( 1 + (xa_i + ya_{i+r} + z \sum_{n \neq i, i+r} a_n) + \left(z^2 \sum_{i \neq i' \neq n \neq i'' \neq i} a_{i} a_{i'} a_{i''} + xz \sum_{n < i} a_n \sum_{n > i} a_n + \sum_{n > i, n \neq i+r} a_n a_i \right) \\
+ yz \left( \sum_{n > i+r} a_n a_{i+r} + a_{i+r} \sum_{n < i+r, n \neq i} a_n + xy a_{i+r} a_i \right) \cdots \right) \prod_{q > 0} \left( \frac{2A_q(t)}{N} \sum_{n > m} a_n^+ a_n^+ \sin q(n - m) - B_q(t) \right) |0\rangle
\]
In this expression, only the terms proportional to “xy” contribute to $G_r(t)$. Let us call the first of these terms $G_1$, 

$$G_1 = \langle 0 | xy a_{i+} a_j | 0 \rangle \sum_{q>0} \left\{ \frac{2 A_q(t)}{N} \sum_{n>m} a^+_n a^+_m \frac{\sin q(n-m)}{A_q(t) \cot \frac{q}{2} - B_q(t)} \prod_{q \neq q'>0} \left( \frac{B_{q'}(t)}{B_q(t) - A_q(t) \cot \frac{q}{2}} \right) \right\} \langle 0 | =

= x y \frac{2}{N} \left( \sum_{q>0} \frac{\sin q r}{\cot \frac{q}{2} - B_q(t)} \right) \prod_{q \neq q'>0} \left( \frac{1}{\frac{1}{A_{q'}(t)} \cot \frac{q}{2}} \right)$$

(18)

In the absence of source, we have $A_q(t) \to 0$ and $B_q(t) \to - \cot \frac{q}{2}$ exponentially fast (see [3]), so that in the thermodynamic limit ($N \to \infty$), the asymptotic behaviour of $G_r(t)$ follows as

$$G_r(t) \sim \frac{\partial^2}{\partial x \partial y} G_1 \to \frac{1}{\pi} \int_0^\pi \frac{dq \sin q r}{\cot \frac{q}{2} - B_q(t)}$$

(19)

We can do the same for the density and in the thermodynamic limit, one obtains

$$\rho(t) = \frac{\partial}{\partial x} \prod_{q>0} \langle 0 | \left( 1 + (xa_i + z \sum_{n \neq i} a_n) + \left( \frac{z^2}{2} \sum_{n \neq i, n' > m} a_n a_{n'} + x z (a_i \sum_{n < i} a_n + \sum_{n > i} a_n a_i + \ldots) \right) \right) \times

\times \left( \frac{2 A_q(t)}{N} \sum_{n>m} a^+_n a^+_m \sin q(n-m) - B_q(t) \right) \langle 0 | \right\} \bigg|_{z=1, x=0} \to \frac{1}{\pi} \int_0^\pi \frac{dq}{1 - B_q(t)}$$

(20)

In the above, we used the following identities:

$$\sum_{m>i} \sin q \langle m-i \rangle = \frac{1}{2} \left( \cot \frac{q}{2} + \frac{\cos q(i-\frac{1}{2})}{\sin \frac{q}{2}} \right), \quad \sum_{m<i} \sin q \langle m-i \rangle = \frac{1}{2} \left( \cot \frac{q}{2} - \frac{\cos q(i-\frac{1}{2})}{\sin \frac{q}{2}} \right)$$

(21)

The evaluation of the two-point correlation function requires the calculation of all the terms proportional to $xy$, which in general is a very hard task. In the following sections we will be able to solve this difficulty by reformulating the problem in a different language.

Using the explicit form of $A_q(t)$ and $B_q(t)$ ([8]), and the results of appendix A, we find the asymptotic behaviour of the density in the (irreversible) critical case as

$$\rho(t) = \frac{e^{-4Dt}}{\pi} \int_0^\pi dq e^{4Dt \cos q} = e^{-4Dt} I_0(4Dt) \sim \frac{1}{\sqrt{8 \pi Dt}}$$

(22)

Similarly the asymptotic behaviour of the two-point correlation function is

$$G_r(t) \sim \frac{e^{-4Dt}}{\pi} \int_0^\pi dq \sin q r (1 - \cos q) e^{4Dt \cos q}

= e^{-4Dt} \sum_{0 \leq n < r} \left\{ I_{2n-r+1}(4Dt) - I_{2n-r}(4Dt) \right\} \sim \frac{\pi r}{(8 \pi Dt)^{\frac{3}{2}}}$$

(23)

which implies

$$C_r(t) \sim - \frac{1}{8 \pi Dt}$$

(24)

Unfortunately, in the massive case (when the source intensity is finite), this method applies only to the computation of the density. Assuming $Dt, Jt \gg 1$, we find

$$\rho(t) = \rho_{eq} + 2(J + 2D) \int_0^\infty dt' e^{-4(J+D)t'} \left( I_0(4Dt') - I_1(4Dt') \right)

\sim \frac{\sqrt{J}}{\sqrt{J+J+2D}} + \left( 1 + \frac{J}{2D} \right) \frac{e^{-4Jt}}{8Jt \sqrt{8 \pi Dt}}$$

(25)

(26)

where $\rho_{eq} = \frac{\sqrt{J}}{\sqrt{J+J+2D}}$ represents the equilibrium value of the density, in agreement with Lushnikov’s result [3]. In the next section we provide the full two-point correlation function when $J > 0$. 

4
III. THE PSEUDO-FERMIONIC APPROACH

In this section we evaluate the full two-point correlation function in the general case by means of a powerful formalism. The central idea is to perform on the fermionic non-hermitian (and non-normal) Hamiltonian a generalized Bogoliubov transformation which allows us to work with a diagonal evolution operator (see Ref. [13,14]). Following previous works [13–15], we denote each of the $2^N$ possible configurations by a ket $|n⟩$:

$$⟨n|n′⟩ = δ_{n,n′}, \quad \sum_n |n⟩⟨n| = \mathbb{1} \quad \text{(27)}$$

In this Fock space, we can efficiently record the probabilities for the various configurations in the ket

$$|P(t)⟩ = \sum_n P(n,t)|n⟩ \quad \text{(28)}$$

The master equation governing the dynamics of the annihilation and diffusion processes described in the previous section can be rewritten as

$$\frac{∂}{∂t}|P(t)⟩ = \mathcal{U}|P(t)⟩ = \sum_n \partial_t P(n,t)|n⟩ = \sum_{n,n′} \left( A(n′ → n)P(n′,t) - A(n → n′)P(n,t) \right)|n⟩ \quad \text{(29)}$$

where $\mathcal{U}$ denotes the evolution operator, $A(n′ → n)$ and $A(n → n′)$ represent the transition rates $J$ and $D$ in Lushnikov’s formulation. The matrix elements for the operator $\mathcal{U}$ are

$$⟨n′|\mathcal{U}|n⟩ = A(n → n′), \forall n′ \neq n \quad \text{(30)}$$

$$⟨n|\mathcal{U}|n⟩ = -\sum_{n′ \neq n} A(n → n′) \quad \text{(31)}$$

This Fock-space formulation was used in Ref. [13,14] to study a stochastic adsorption-desorption problem. In what follows, we will specifically focus on the reaction-diffusion problem in one dimension. Let us now introduce the left and right steady-states, respectively

$$⟨\bar{\chi}⟩ ≡ \sum_n ⟨n⟩, \quad |\bar{\chi}⟩ ≡ \sum_n P(n,eq)|n⟩ \quad \text{(32)}$$

where $P(n,eq)$ denotes the probability for a configuration $|n⟩$ at equilibrium. It is easy to check that $e^{\mathcal{U}t}$ has no effect on $|\chi⟩$ and $⟨\bar{\chi}|$, and the conservation of probability leads to $⟨\bar{\chi}|\chi⟩ = 1$.

The transition probability from a configuration $|n⟩$ to $|n′⟩$ is simply: $W_{n,n′}(t) ≡ ⟨n′|e^{\mathcal{U}t}|n⟩$. We intend to calculate the density and two-point one-time correlation functions of a system initially in the state $|φ_0⟩ = \sum_n P(n,t = 0)|n⟩$. The occupation number operator $n_r$, being diagonal in the basis $\{|n⟩\}$, we have

$$\rho(t) = \sum_{n,n′} ⟨n′|n⟩e^{\mathcal{U}t}P(n,t = 0) = \sum_{n,n′} ⟨n′|n⟩e^{\mathcal{U}t}P(n,t = 0) = ⟨\bar{\chi}|n⟩e^{\mathcal{U}t}|φ_o⟩ \quad \text{(33)}$$

and similarly,

$$\mathcal{G}_r(t) = \sum_{n,n′} ⟨n|n⟩e^{\mathcal{U}t}P(n,t = 0) = \sum_{n,n′} ⟨n|n⟩e^{\mathcal{U}t}P(n,t = 0) = ⟨\bar{\chi}|n⟩e^{\mathcal{U}t}|φ_o⟩ \quad \text{(34)}$$

where $r = |m − l|$. At this point, we perform a generalized Bogoliubov transformation (rotation supplemented by a rescaling),

$$\begin{pmatrix} \xi^+_q \\ \xi^-_{−q} \end{pmatrix} = \begin{pmatrix} \alpha \cos θ_q & \alpha^{-1} \sin θ_q \\ -\alpha \sin θ_q & \alpha^{-1} \cos θ_q \end{pmatrix} \begin{pmatrix} a^+_q \\ a_{−q} \end{pmatrix} \quad \text{(35)}$$

in order to obtain a diagonal representation for the evolution operator. This transformation is orthogonal, i.e., invertible and canonical, in the sense that it preserves the anticommutation relations of the $a_q$’s, namely:

$$\{ξ^+_q,ξ_{q′}\} = δ_{q,q′}, \{ξ^+_q,ξ^+_{q′}\} = \{ξ_q,ξ_{q′}\} = 0 \quad \text{(36)}$$
Despite the fact that the $\xi_q$ and $\xi^+_q$ are not adjoint of each other, this representation will be very useful in the following. We set $\alpha = \left(\frac{J}{J + 2D}\right)^{\frac{1}{2}}$, so that the mode $q$ evolution operator becomes

$$
-L_q = 2\left(D(1 - \cos q) - J\right)\left(\xi^+_q\xi_q + \xi^+_q\xi^{}_{-q} + \sin \theta_q^2(\xi^{}_{-q}\xi^+_q + \xi^{}_q\xi^{}_{-q}) + 2(\xi^+_q\xi^{}_{-q} + \xi^{}_q\xi^{}_{-q})\right)
+ \sqrt{J(J + 2D)}\sin q\left(\cos 2\theta_q(\xi^+_q\xi^{}_{-q} + \xi^{}_q\xi^{}_{-q}) + \sin 2\theta_q(\xi^{}_q\xi^{}_{-q} + \xi^{}_{-q}\xi^{}_{-q})\right) + 2J
$$

(37)

To get rid of the terms that do not conserve the number of pseudo-particles, we choose $\theta_q$ as

$$
\tan 2\theta_q = \frac{\sqrt{J(J + 2D)}\sin q}{(J + D)\cos q - D}
$$

(38)

so that the Hamilton operator becomes

$$
\mathcal{L} = -\sum_{q > 0} \lambda_q \left(\xi^+_q\xi_q + \xi^+_q\xi_{-q}\right) = -\sum_q \lambda_q \xi^+_q\xi_q
$$

(39)

where

$$
\lambda_q = 2\left(D(1 - \cos q) + J\right)
$$

(40)

on account of the periodic boundary conditions ($\sum_{q > 0} \cos q = 0$). Now it is readily seen that $\langle \bar{\chi} \rangle$ and $|\chi\rangle$ act, respectively, as left and right vacua, i.e., $\xi_q |\chi\rangle = 0$ and $\langle \bar{\chi} |\xi^+_q\rangle = 0$. To simplify the calculations, we express the initial ket-state $|\phi_0\rangle$ in terms of the steady state $|\bar{\phi}_0\rangle$. We consider here two kinds of initial conditions:

i) The whole lattice is initially filled. We write $|\phi_0\rangle = |all\rangle$ and immediately conclude $a^+_q|all\rangle = 0$. Using the inverse of (35), one can check that

$$
|all\rangle = \prod_{q > 0} \left(1 - \cot \theta_q \xi^+_q\xi^{}_{-q}\right)|\chi\rangle = \exp\left(-\sum_q \frac{\cot \theta_q}{2} \xi^+_q\xi^{}_{-q}\right)|\chi\rangle
$$

(41)

ii) The lattice is initially empty; one can check in the same way [14] that:

$$
|\phi_0\rangle = |0\rangle = \prod_{q > 0} \left(1 + \tan \theta_q \xi^+_q\xi^{}_{-q}\right)|\chi\rangle = \exp\left(\sum_q \frac{\tan \theta_q}{2} \xi^+_q\xi^{}_{-q}\right)|\chi\rangle
$$

(42)

The time dependence of $\xi_k(t)$ and $\xi^+_k(t)$ follow as

$$
\xi_k(t) = e^{-\mathcal{L}t} \xi_k e^{\mathcal{L}t} = e^{-\lambda^+_k} \xi_k
$$

(43)

$$
\xi^+_k(t) = e^{-\mathcal{L}t} \xi^+_k e^{\mathcal{L}t} = \lambda^+_k \xi^+_k
$$

(44)

Furthermore, we have

$$
\langle \xi_{k_1}\xi_{k_2}\rangle(t = 0) \equiv \langle \bar{\chi}|\xi_{k_1}\xi_{k_2}|all\rangle = \cot \theta_{k_1}\delta_{k_1,-k_2}
$$

(46)

and

$$
\langle \xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}\rangle(t = 0) = \cot \theta_{k_1}\cot \theta_{k_3}\delta_{k_1,-k_2}\delta_{k_3,-k_4} + \cot \theta_{k_1}\cot \theta_{k_2}\left(\delta_{k_1,-k_4}\delta_{k_2,-k_3} - \delta_{k_1,-k_3}\delta_{k_2,-k_4}\right)
$$

(47)

as one can check by applying Wick’s theorem (see also section IV). Using the properties of Fourier transform and the generalized Bogoliubov transformation (35), the expression of the density and the two-point correlation function become respectively (for a lattice initially filled, $|\phi_0\rangle = |all\rangle$):

$$
\rho(t) = \frac{1}{N} \sum_{k} \sin^2 \theta_k - \sum_{k, k'} \frac{e^{i(k' - k)t}}{N} \sin \theta_k \cos \theta_{k'} \langle \bar{\chi} |\xi_{-k}\xi_{k'} e^{\mathcal{L}t}|all\rangle
$$

(48)
and one can do the same for unconnected one-time correlation function $G_r(t)$ [34].

To derive tractable formulas, we have performed tedious but straightforward calculations. Indeed, we have extracted the time-dependence of $\rho(t)$ and $G_r(t)$ using [43] and commuted all the pseudo-creation operators to the left of the pseudo-annihilation operators. In the expression $\rho(t)$ and $G_r(t)$ only terms like $\langle \tilde{\chi}\xi_{k}\xi_{q}\rangle|\text{all}\rangle = \langle \xi_{-k}\xi_{q}\xi_{q'}\rangle(t = 0)$ and $\langle \tilde{\chi}\xi_{k}\xi_{-q}\xi_{q'}\rangle|\text{all}\rangle$ survive; these were evaluated with the help of [46 [47].

In the thermodynamic limit, we arrive at

$$\rho(t) = \frac{2}{N} \sum_{k>0} \left( \sin^2 \theta_k + e^{-2\lambda_k t} \sin \theta_k \cos \theta_k \right)$$

$$\rightarrow \frac{1}{\pi} \int_0^\pi dq \left( \sin^2 \theta_q + e^{-2\lambda_q t} \cos \theta_q \right) = \frac{\sqrt{J}}{\sqrt{J + \sqrt{2}D}} + 2(J + 2D) \int_t^\infty dt' \left( I_0(4Dt') - I_1(4Dt') \right) e^{-4(J + D)t'} \quad (49)$$

which coincides with [23].

It is worth emphasizing that this result is general and works for both, the massive ($J \neq 0$) and the critical ($J = 0$) cases. The point here is that the limit $J \rightarrow 0$ is not singular, despite the divergence of $\cot \theta \rightarrow -\infty$. In fact, integration over $k$ and $k'$ of terms proportional to $\sin \theta_k \cot \theta_{k'}$ yields finite results. Therefore, we can perform the computations [43] at $J$ finite and set subsequently $J = 0$ in $\rho(t)$ and $G_r(t)$.

Similarly, the two-point correlation function (of one-time) is evaluated as $\langle r = |m - l|\rangle$:

$$G_r(eq) = \rho_c^2 + \frac{1}{\pi^2} \left( \int_0^\pi dq \sin^2 \theta_q \cos qr \right) \left( \int_0^\pi dq' \cos^2 \theta_{q'} \cos qr' \right) + \left( \frac{1}{2\pi} \int_0^\pi dq \sin 2\theta_q \sin qr \right)^2 \quad (50)$$

$$G_r(t) - G_r(eq) = \left( \rho(t)^2 - \rho_c^2 \right) + \frac{1}{\pi^2} \left( \int_0^\pi dq e^{-2\lambda_q t} \cos^2 \theta_q \cos qr \right) \left( \int_0^\pi dq' e^{-2\lambda_q t} \cos^2 \theta_{q'} \cos qr' \right) + \frac{1}{2\pi^2} \left( \int_0^\pi dq \sin 2\theta_q \sin qr \right) \left( \int_0^\pi dq' e^{-2\lambda_q t} \cos^2 \theta_{q'} \cos qr' \right) - \frac{1}{\pi^2} \left( \int_0^\pi dq \sin^2 \theta_q \cos qr \right) \left( \int_0^\pi dq' e^{-2\lambda_q t} \cos^2 \theta_{q'} \cos qr' \right) - \frac{1}{4\pi^2} \left( \int_0^\pi dq \sin 2\theta_q \sin qr \right) \left( \int_0^\pi dq' e^{-2\lambda_q t} \sin 2\theta_{q'} \sin qr' \right) + \left( \frac{1}{\pi^2} \int_0^\pi dq e^{-2\lambda_q t} \cos^2 \theta_q \cos qr \right)^2 - \left( \frac{1}{\pi^2} \int_0^\pi dq e^{-2\lambda_q t} \cos^2 \theta_q \cot \theta_q \sin qr \right) \times \left( \int_0^\pi dq e^{-2\lambda_q t} \sin^2 \theta_q \cot \theta_q \sin qr \right) \quad (51)$$

where we separated the static contribution to the correlation function from the dynamic one. Using known properties of modified Bessel functions given (see appendix) and writing $I_n'(x) = \frac{1}{2\pi} \int_0^{\pi} \sin \theta \cos n \theta \sin qr \sin \theta_q \cos qr'$, we finally infer (notice that $G_r(eq) = \rho_c^2$)

$$G_r(t) = \rho(t)^2 + 2(J + 2D)(A_0 - C_0) \int_t^\infty dt' e^{-4(J + D)t'} \left( I_r(4Dt') - I_r'(4Dt') \right)$$

$$+ \frac{\sqrt{J}(J + 2D)B_0}{2} \int_t^\infty dt' e^{-4(J + D)t'} \frac{r}{2Dt'} I_0(4Dt')$$

$$- \frac{(J + 2D)^{\frac{3}{2}}}{\sqrt{J}} \sum_{0 \leq n < r} \left( \int_t^\infty dt' e^{-4(J + D)t'} \left( 2I_{2n-r+1}(4Dt') - 2I_{2n-r}(4Dt') + I_{2n-r}(4Dt') \right) \right)$$

$$- \left( 2(J + 2D) \int_t^\infty dt' e^{-4(J + D)t'} \left( I_r(4Dt') - I_r'(4Dt') \right) \right)^2$$

$$+ 2(J + 2D)^2 \left( \sum_{0 \leq n < r} \left( 2n - r \right) \left( \int_t^\infty dt' e^{-4(J + D)t'} I_{2n-r}(4Dt') \right) \right)$$

$$\times \left( \sum_{0 \leq n < r} \left( \int_t^\infty dt' e^{-4(J + D)t'} \left( 2I_{2n-r+1}(4Dt') - 2I_{2n-r}(4Dt') + I_{2n-r}(4Dt') \right) \right) \right) \quad (52)$$

In the above formula $A_0$, $B_0$, $C_0$ have been defined by:
Here we investigate the behavior of $\rho$ with the notation critical case it has been shown [6,7] that the one-time correlation function obeys a scaling form.

or more explicitly, in the massive case (with help of (A12)),

$$A_0 = \frac{1}{\pi} \int_0^\pi dq \cos^2 \theta_q \cos qr, \quad B_0 = \frac{1}{\pi} \int_0^\pi dq \sin 2\theta_q \sin qr, \quad C_0 = \frac{1}{\pi} \int_0^\pi dq \sin^2 \theta_q \cos qr \quad (53)$$

Note that the above result, though obtained by elementary means, is new. The formula can be evaluated numerically. The connected two-point correlation function $C_r(t) = C_r(t) - \rho(t)^2$ follows immediately from the above. To establish a connection with the results derived previously we explicitly evaluate $C_r(t)$ when $J = 0$. Taking due attention to the apparent singularities occurring in this limit, we find

$$C_r(t) = \frac{1}{2\pi^2} \int_0^\pi dq \sin 2\theta_q \sin qr \int_0^\pi dq' e^{-2\lambda_q t} \cot \theta_q \sin qr' - \left( \frac{1}{\pi} \int_0^\pi dq e^{-2\lambda_q t} \cot \theta_q \sin qr \right) \left( \frac{1}{\pi} \int_0^\pi dq e^{-2\lambda_q t} \sin^2 \theta_q \cot \theta_q \sin qr \right) \quad (58)$$

or in terms of elementary functions,

$$C_r(t) = \sum_{0 \leq n < r} e^{-4Dt} \left( I_{2n-1-r}(4Dt) - I_{2n-r}(4Dt) \right) - \left( \sum_{0 \leq n < r} e^{-4Dt} I_{2n-r} \right)^2 - \left( \sum_{0 \leq n < r} e^{-4Dt} I_{2n-r+1} \right)^2 \quad (59)$$

This expression is equivalent to the result derived by a well-known mapping of Lushnikov’s model to Glauber’s 1D Ising model [1].

Finally, we compute (using formula (A11)), the asymptotic behaviour of the two-point correlation function $C_r(t)$ when $Dt \gg 1, Jt \gg 1$, and $r < \infty$

$$C_r(t) \sim -\frac{\zeta^{-1} e^{-4Jt}}{8Jt^2} \left( (J + 2D)(1 - \zeta)^2 + J(1 + \zeta)^2 + \frac{r J (J + 2D)}{D} (1 - \zeta^2) \right) \frac{1}{4\sqrt{J(J + 2D)}} \quad (60)$$

As expected for a finite source intensity $J > 0$, the density and one-time correlation function decay exponentially with time. Moreover, the one-time correlation function also decays exponentially with distance. A similar calculation can be performed for the case of an initial empty lattice ($|\phi_0| = |0\rangle$). In this case the density is $\rho(t) = 2J \int_0^\infty dt ' \left( I_0(4Dt') + I_1(4Dt') \right) e^{-4(J+D)t'} \sim \rho_{eq} - e^{-4Jt}/\sqrt{8\piDt} \quad [2]$. It has been shown [13], via the mapping to the Glauber dynamics, that the nearest neighbor connected function decays $(Dt, Jt \gg 1)$ as $C(r = 1, t) \sim \rho_{eq} e^{-4Jt}/\sqrt{2\piDt}$. The large-time and large-distance behavior of the one-time correlation function can also be studied. In fact, in the critical case it has been shown [14] that the one-time correlation function obeys a scaling form. Here we investigate the behavior of $C_r(t)$ in the limit where $r$ and $Dt \to \infty$ with $u \equiv \frac{2}{\pi \sqrt{Dt}}$ finite. In the massive case, we do not expect a scaling form:

$$C_r(t) \sim -\sqrt{1 + \frac{2D}{J} \zeta^{-1} (1 - \zeta^2)} g(r^2, u) \frac{9}{4r^2} \quad (61)$$

where
\[ g(r^2, u) = \sqrt{\frac{\mu^3}{\pi}} e^{-\frac{2u}{\mu}} \] (62)

The effect of the source is to disrupt the spatial correlations, i.e. to make them short-range. In this sense the finite source prohibits the formation of arbitrarily large vacancy domains.

In the next section we will introduce a field-theoretical approach to deal with two-time correlation function. Our approach and the results obtained thereby complement the exact treatments of the critical case [16]. It is worth pointing out, that Glauber calculated in his pioneering work the two-time two-points of spin correlations functions [34].

IV. FIELD THEORETICAL REFORMULATION

The purpose of this section is to reformulate the results of the previous section in a field theoretical language, i.e., in terms of path integrals of fermionic variables (see for example [17]). We define the Grassmann numbers \( \eta_q, \eta_q^\ast \) which anticommute with each other and with the fermionic operators introduced in the previous section, i.e., \( \{ \eta_q, \eta_{q'} \} = \{ \eta_q^\ast, \eta_{q'}^\ast \} = \{ \eta_q, \eta_{q'}^\ast \} = \{ \eta_q^\ast, \eta_{q'} \} = 0 \) and \( \{ \eta_q, \xi_{q'} \} = \{ \eta_q^\ast, \xi_{q'}^\ast \} = \{ \eta_q^\ast, \xi_{q'} \} = 0 \). We follow standard practice and consider the coherent states associated to the fermionic variables. We recall that pseudo-fermionic operators required the introduction of a left vacuum \( \langle \tilde{\chi} \rangle \) and a right vacuum \( |\chi\rangle \), ergo we define the right \( |\eta\rangle \) and the left coherent states \( |\bar{\eta}\rangle \), respectively, as \( |\eta\rangle = e^{-\sum_q \eta_q \xi_q^\ast |\chi\rangle} \), \( |\bar{\eta}\rangle = e^{-\sum_q \xi_q^\ast \eta_q |\chi\rangle} \).

Despite the fact that \( \xi \) and \( \xi^\ast \) are not adjoint of each other, we find the familiar results:
\[
\langle \bar{\eta}|\chi \rangle = \langle \tilde{\chi}|\eta \rangle = 1 , \quad \langle \bar{\eta}|\eta \rangle = e^{\sum_q \eta_q^\ast \eta_q} \text{ and } \xi_q |\eta \rangle = |\eta_q \rangle , \quad \langle \bar{\eta}|\xi_q^\ast = \langle \tilde{\eta}|\eta_q^\ast , \quad \xi_q^\ast |\eta \rangle = -\frac{\partial}{\partial \eta_q}|\eta \rangle .
\]

Most importantly, the closure relation holds true, i.e.,
\[
\int \prod_q d\eta_q d\eta_q^\ast e^{-\sum_q \eta_q^\ast \eta_q} |\eta\rangle \langle \eta| = 1
\] (63)

At this point, we know from field theory how to calculate \( \langle \tilde{\chi}|n_m e^{L_{t_1}}|all \rangle \) and \( \langle \tilde{\chi}|n_m n_{m+\epsilon} e^{L_{t_1}}|all \rangle \) using the path-integral formalism. We discretize time in \( M \) infinitesimal intervals of width \( \epsilon = \lim_{M\to\infty} \frac{1}{M} \). \( L \) is normal ordered, and the closure relation is inserted into the above formulas. We have for the density
\[
\rho(t) = \int \prod_{q,\alpha=1,...,M} d\eta_q^\ast d\eta_q e^{-\sum_{\alpha,\alpha'} \eta_{\alpha,\alpha'}^\ast \eta_{\alpha,\alpha}' \langle \tilde{\chi}|n_m |\eta_M \rangle \langle \eta_M | e^{L_{t_1}} |\eta_{M-1} \rangle \cdots \langle \eta_2 | e^{L_{t_2}} |\eta_1 \rangle \langle \eta_1 | e^{L_{t_1}} |all \rangle} ,
\] (64)

where
\[
\lim_{M\to\infty} \langle \eta_1 | e^{L_{t_1}} | all \rangle = e^{-\frac{1}{2} \sum_q \cot \theta_q \eta_q^\ast \eta_q}
\] (65)
\[
\langle \tilde{\chi}|n_m |\eta_M \rangle = \frac{1}{N} \left( \sum_k \sin^2 \theta_k - \sum_{k,k'} \sin \theta_k \cos \theta_{k'} \langle \tilde{\chi}|\xi_{-k}^\ast \xi_{k'} |\eta_m \rangle \right)
\]
\[
= \frac{1}{N} \left( \sum_k \sin^2 \theta_k - \sum_{k,k'} \sin \theta_k \cos \theta_{k'} \eta_{M,k} \eta_{M,k'} \right)
\] (66)

Taking the continuum limit [17], we arrive at
\[
\rho(t) = \frac{1}{N} \int \prod_q d\eta_q^\ast(t) d\eta_q(t) \left( \sum_k \sin^2 \theta_k - \sum_{k,k'} \sin \theta_k \cos \theta_{k'} \eta_q(t) \eta_q^\ast(t) \right) e^{-\sum_q \cot \theta_q \eta_q^\ast(0) \eta_q(0)} e^{-S_0(\eta^\ast(t),\eta(t))}
\] (67)

where, the Liouvillian operator \( L \) acts as the Hamiltonian of Field theory. By abuse of language we call \( S_0(\eta^\ast(t),\eta(t)) \) the Euclidian action of our problem, which is defined by
\[
S_0(\eta^\ast(t),\eta(t)) \equiv \int_0^t dt' \sum_q \left( \eta_q^\ast(t') \partial_{t'} \eta_q(t') - L(\eta_q^\ast(t'),\eta_q(t')) \right) + \sum_q \eta_q^\ast(0) \eta_q(0)
\] (68)

Taking averages \( \langle \ldots \rangle_{S_0} \) on the gaussian distribution, the density can be rewritten as
\[ \rho(t) = \frac{1}{N} \langle \left( \sum_k \sin^2 \theta_k - \sum_{k,k'} \sin \theta_k \cos \theta_{k'} \eta_q(t) \eta_{q'}(t) \right) e^{-\sum_q \cot \theta_g \eta_q(0) \eta_{-q}(0)} \rangle_S \]  
(69)

We proceed to discretize the free Euclidian action \( S_0 \) which is bilinear, as

\[ S(q) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a & 1 & 0 & \cdots & 0 \\ 0 & -a & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -a & 1 \end{pmatrix} \]  
(70)

where, \( a \equiv 1 - \frac{4}{M} \lambda_q \).

Following Ref. [17], we find \( \det(S(q)) = 1 \) and

\[ \lim_{M \to \infty} S_{\alpha,\beta}^{-1}(q) = e^{-\lambda_q(t_\alpha - t_\beta)}, \quad \forall \alpha = 1 \ldots M \geq \beta, \quad t_\alpha = \frac{\alpha t}{M} \]  
(71)

Furthermore we have, in the continuum limit

\[ \left\langle \eta_p(t_2) \eta_{p'}(t_1) \right\rangle_{S_0} = \int \prod_q d\eta_q(t_2) d\eta_{q'}(t_1) e^{-S_0} = \delta_{p,p'} e^{-\lambda_{p}(t_2 - t_1)} \]  
(72)

Applying Wick’s theorem, leads to

\[ \left\langle \xi_p \xi_{p'} \right\rangle(t) = \left\langle \eta_p(t) \eta_{p'}(t) e^{-\frac{1}{2} \sum_q \cot \theta_g \eta_q(0) \eta_{-q}(0)} \right\rangle_{S_0} = \delta_{p,p'} \cot \theta_g e^{-2 \lambda_{p} t} \]  
(73)

With this result, it is straightforward to calculate the density using (73) and (69).

The computation of correlation function requires evaluation of terms like \( \left\langle \xi_k \xi_{k'}(t) \right\rangle \). We generate in the standard fashion the multi-point correlation functions. The generating functional in discretized time reads:

\[ Z[g_{q,\alpha}; q'_{q',\alpha'}] = \left\langle e^{\sum_q \left( -\frac{1}{2} \cot \theta_g \eta_q(0) \eta_{-q}(0) + \sum_{\alpha \geq 2} (g_{q,\alpha,\eta_q,\alpha} + \eta_{q,\alpha} g_{q,\alpha}) \right)} \right\rangle_{S_0} \]  
(74)

here \( g_{q,\alpha}, g_{q',\alpha'} \) denote (Grassman-numbers) the coefficient of the source terms. Note that this functional differs from the usual field theoretical one \([17]\) by a term that codes the initial condition, i.e., \( \prod_{q>0} e^{\cot \theta_g \eta_q(0) \eta_{-q}(0)} \). Taking this term into account however is no trouble, i.e.,

\[ Z[g_{q,\alpha}; q'_{q',\alpha'}] = \prod_{q>0} \left( e^{g_{-q,1} g_{q,1}^* \cot \theta_q + g_{-q,1} g_{q,1}^* e^{\lambda_{q}(t_j)}} \right) \prod_{q,q',i,j=2}^{M} e^{g_{q,\alpha}^* S_{\alpha,\beta}(q) g_{q',\alpha'}^*} \]  
(75)

Considering that the source term will be set to zero in the end of the calculation and noting that for \( \alpha = 1 \), only pseudo-creation operators contribute, we find it convenient to work with

\[ \tilde{Z}[g_{q,\alpha}; g_{q',\alpha'}] = \prod_{q} \left( e^{\frac{1}{2} g_{-q,1} g_{q,1}^* \cot \theta_q + \sum_{q',i,j=1}^{M} g_{q,\alpha}^* e^{\lambda_{q}(t_j - 1)} g_{q',\alpha'}^*} \right) \]  
(76)

From the definition, it follows then that

\[ \left\langle \xi_k \xi_{k'} \right\rangle(t) = \frac{\delta^6 \tilde{Z}[g_{q,\alpha}; g_{q',\alpha'}]}{\delta g_{-q,1} \delta g_{k,1}^* \delta g_{k,1}^* M^2 \delta g_{k,2} \delta g_{-q,2}} \bigg|_{g=g^*=0} \]  
(77)

or more generally,
\begin{align}
&\langle \xi_k \xi_{k'} \cdots \xi_k \xi_{k''} \rangle (t) = \langle \chi | \xi_k \xi_{k'} \cdots \xi_k \xi_{k''} e^{L_t} | \text{all} \rangle = \langle \chi | \xi_k \xi_{k'} \cdots \xi_k \xi_{k''} e^{L_t} e^{-\frac{1}{2} \sum \nu \cot \theta_\nu \xi_\nu^* \xi_\nu} | \chi \rangle \\
&= \frac{\delta^{6n} Z[g_{q,\alpha}^*; g_{q',\alpha'}^*]}{\delta g_{-q,1}^* \delta g_{q,1}^* \delta g_{-q_1,1}^* \delta g_{q_1,1}^* \delta g_{k_1,M}^* \delta g_{-k_1,M}^* \delta g_{k,M}^* \delta g_{-k,M}^* \delta g_{q_{1,2}}^* \delta g_{-q_{1,2}}^*} \bigg|_{g=g^*} = 0 (78)
\end{align}

As an application of this field-theoretical formulation we will now extend our approach to the computation of the two-time correlation function, for an initial state \( |\phi_0\rangle \),

\[ G_r(t, t') \equiv \langle n_{t+r} (t + t') n_t (t) \rangle = \langle \bar{\chi} | n_t \bar{e} e^{L_t} n_{t+r} e^{L_t} | \phi_0 \rangle = \sum_{n,n'} \langle n' | n_{t+r} e^{L_t} n_t \bar{e} \rangle n_{n'} P(n, 0) \]

\[ = \sum_{n,n',n''} \langle n' | n_{t+r} | n' \rangle \langle n'' | n_t | n'' \rangle W_{n',n''} (t') W_{n''n} (t) P(n, 0) \]

(79)

As above, we consider an initially filled lattice (\( |\phi_0\rangle = |\text{all}\rangle \)).

To perform the computation of \( G_r(t, t') \) we need to evaluate terms like

\[ \langle \bar{\chi} | \xi_p \bar{e} e^{L_{t'}} \xi_q e^{L_t} | \text{all} \rangle = \langle \eta_p (t + t') \eta_q (t + t') \eta \rangle (t) \eta_{q'} (t) e^{-\frac{1}{2} \sum \nu \cot \theta_\nu \eta^{(0)}_\nu \eta^{(0)}_\nu} \]

\[ \bigg|_{S_0} \]

(80)

This expression can be calculated by using the generating functional, where as before \( t \ (t') \) is discretized into \( M \ (N) \) infinitesimal time steps (respectively), with \( M, N \to \infty \):

\[ \langle \bar{\chi} | \xi_p \bar{e} e^{L_{t'}} \xi_q e^{L_t} | \text{all} \rangle = \frac{\delta^{12} Z[g_{q,\alpha}^*; g_{q',\alpha'}^*]}{\delta g_{-q_1,1}^* \delta g_{q_1,1}^* \delta g_{-q_{1,1}}^* \delta g_{q_{1,1}}^* \delta g_{k_1,M}^* \delta g_{-k_1,M}^* \delta g_{k,M}^* \delta g_{-k,M}^* \delta g_{q_{1,2}}^* \delta g_{-q_{1,2}}^*} \bigg|_{g=g^*} = 0 = e^{-\left( \lambda_p + \lambda_{q'} \right) \left( t + t' \right) - \left( \lambda_q + \lambda_q \right) t} \left( \cot \theta_p \cot \theta_q \delta_{p,-q} \delta_{q,-q'} + \cot \theta_p \cot \theta_{q'} \left( \delta_{-p,q} \delta_{-q,q'} - \delta_{-p,q} \delta_{-q,q'} \right) \right) \]

(81)

where the continuum limit for the time has been taken.

Applying the same technique to each term of \( G_r(t, t') \), yields:

\[ G_r(t, t') = \rho_{eq}^2 + 2(J + 2D) \rho_{eq} \int_0^\infty dt e^{-4J+8D} \]

\[ \times \left( I_0 (4DT^2) - I_1 (4DT^2) \right) \]

\[ + J(2J + 4D) \left( \int_0^\infty dt e^{-2J+8D} \left( I_r (2DT) + I_r' (2DT) \right) \right) \]

\[ \times \left( \int_0^\infty dt e^{-2J+8D} \left( I_r (2DT) - I_r' (2DT) \right) \right) \]

\[ + \frac{J(2J + 4D)}{4} \left( \int_0^\infty dt e^{-2J+8D} \right) \left( \frac{r}{2DT} I_r (2DT) \right) \]

\[ - 2(J + 2D)^2 \left( \int_0^\infty dt e^{-2J+8D} \left( I_r (2DT) - I_r' (2DT) \right) \right) \]

\[ \times \left( \int_0^\infty dt e^{-2J+8D} \right) \left( \frac{r}{2DT} I_r (2DT) \right) \]

\[ + (J + 2D)^2 \left( \int_0^\infty dt e^{-2J+8D} \right) \left( \frac{r}{2DT} I_r (2DT) \right) \]

\[ \times \left( \sum_{0 \leq n < r} \int_0^\infty dt e^{-2J+8D} \left( 2I_{2n-r+1} (2DT + t') - 2I_{2n-r} (2DT + t') + I_{2n-r} (2DT + t') \right) \right) \]

\[ \text{11} \]
To study the behavior at finite distance of the two-time correlation function distinguish the massive and the critical regime. We begin with the massive case and consider
\[ r < \infty \]

With help of (A8-A11), we have:
\[ G_r(t, t') = e^{-4D(2t + t')} \left( I_0(2D(2t + t')) - I_r(2D(2t + t')) \right)^2 + D e^{-2D(2t + t')} \left( \int_{t'}^\infty dt e^{-2DT} \frac{r}{D} I_r(2D) \right) \times \left( \sum_{0 \leq n < r} \left( I_{2n-r+1}(2D(2t + t')) - I_{2n-r}(2D(2t + t')) \right) \right) \]
\[ + e^{-4D(2t + t')} \left( \sum_{0 \leq n < r} I_{2n-r}(2D(2t + t')) \right)^2 - \left( \sum_{0 \leq n < r} I_{2n-r+1}(2D(2t + t')) \right)^2 \]  \tag{82}

In the critical case some care is required when taking \( J \to 0 \),
\[ G_r(t, t') = e^{-4D(2t + t')} \left( I_0(2D(2t + t')) \right)^2 - \left( I_r(2D(2t + t')) \right)^2 \]
\[ + D e^{-2D(2t + t')} \left( \int_{t'}^\infty dt e^{-2DT} \frac{r}{D} I_r(2D) \right) \times \left( \sum_{0 \leq n < r} \left( I_{2n-r+1}(2D(2t + t')) - I_{2n-r}(2D(2t + t')) \right) \right) \]
\[ + e^{-4D(2t + t')} \left( \sum_{0 \leq n < r} I_{2n-r}(2D(2t + t')) \right)^2 - \left( \sum_{0 \leq n < r} I_{2n-r+1}(2D(2t + t')) \right)^2 \]  \tag{83}

To study the behavior at finite distance of the two-time correlation function \( C_r(t, t') \equiv G_r(t, t') - \rho(t) \rho(t') \), we distinguish the massive and the critical regime. We begin with the massive case and consider \( Dt, Dt' \gg 1, Jt, Jt' \gg 1 \) and \( r < \infty \). With help of (A8-A11), we have:
\[ G_r(t, t') \sim \rho_{eq}^2 + \left( 1 + \frac{J}{2D} \right) \rho_{eq} \frac{e^{-4Jt}}{8Jt \sqrt{8\pi Dt}} - \frac{e^{-4Jt'}}{8Jt' \sqrt{8\pi Dt'}} \]  \tag{84}

When \( r < \infty, Dt \) and \( Jt \) are finite and \( Dt', Jt' \gg 1 \) we obtain:
\[ G_r(t, t') \sim \rho_{eq}^2 + \frac{J + 2D}{128J(2D')}^2 \frac{e^{-4Jt'}}{8Jt' \sqrt{8\pi Dt'}} + \frac{J + 2D}{16\pi D \sqrt{7}(2t + t')^2} \frac{e^{-4J(t+t')}}{16Jt' \sqrt{8\pi Dt'}} \]  \tag{85}
In the critical case \((r < \infty)\), both asymptotic behavior \(Dt, Dt' \gg 1\) and \(Dt \) finite with \(Dt' \gg 1\) of the disconnected and connected correlation function are given by:

\[
G_r(t, t') \sim \frac{r^2}{4\pi D^2(2t + t')^2} \left(1 + \frac{1}{8} \sqrt{1 + \frac{2t}{t'}}\right)
\]

\[
C_r(t, t') \sim -\frac{1}{8\pi D \sqrt{Dt}}
\]

Near the initial state (the density of particles is high), when \(Dt, Dt' \ll 1\) the decay is linear and independ of \(r\), i.e.,

\[
G_r(t, t') \sim 1 - 4D(2t + t')
\]

\[
C_r(t, t') \sim -4Dt
\]

We now provide a scaling form for the two-time correlation function \(C_r(t, t')\). It is known, from the duality with Glauber’s model, that the single-time correlation function obeys a scaling form for large-time and long-distances \((Dt\) and \(r \to \infty\) with the ratio \(r^2/Dt\) held finite \([56]\)). The scaling form is found as \(C_r(t) \sim r^{-2} f(r^2/4Dt)\), where the exponent \(-2\) is believed to be universal \([56]\). We further assume the long-time and large-distance scaling form \(C_r(t, t') \sim r^{-2} h(u, v)\), where \(r, Dt, Dt' \to \infty\) with \(u \equiv r^2/4Dt\) and \(v \equiv r^2/4Dt'\) held finite and arrive at

\[
C_r(t, t') \sim \frac{1}{\pi r^2} \left\{ K^2 e^{-K} \left( \sqrt{\pi} \operatorname{erf}\sqrt{v} - 2\sqrt{K} \right) + K \left( 1 - e^{-2K} \right) - \frac{\sqrt{uv}}{2} \right\},
\]

where \(0 < K \equiv \frac{uv}{u + 2v} < \infty\).

At this point, it is appropriate to review what we have achieved so far: we have been able to reformulate the problem of the evaluation of the multi-point correlation functions in a language that parallels the field theoretical one. This allows us to compute in an efficient and systematic way physical quantities of interest despite some technical differences to the standard approach. While this paper deals with a free “field theory” of pseudo-fermions, it is tempting to apply the same formalism to the multi-species case where two- or multi-body interactions arise. The latter however will be investigated in a future work by perturbative renormalization group techniques, as no exact solutions are available.

V. CONCLUDING REMARKS

We have studied three different approaches to the problem of diffusion-annihilation of classical hard-core particles moving on a one-dimensional ring. Though Lushnikov’s contributions to the problem are genuine and undisputable, we have shown how an extension of his generating function method to evaluate the two-point correlation function can be cumbersome in practice, even in the simplest case available of a single-species. We have seen that it is advantageous to apply a generalized Bogoliubov transformation first used by Grynberg et al. \([13,14]\) in a different context. The evolution operator can be diagonalized, i.e., expressed as a quadratic form of two operators that are not adjoint of each other. Despite this fact the formalism resembles the standard one and appears as a powerful tool. Indeed, we were able to compute for the first time the full one-time and two-time correlation functions for an initially fully occupied lattice (other initial conditions can also be studied) in the presence of a finite source. We derived a scaling form for the two-time correlation function. We used the results of the first section (algebraic decay in the critical case and exponential in the presence of source) to check the asymptotic behavior of the density and two-point correlation function. We discovered that while in the absence of source, the modes at long wave lengths fully control the long-time asymptotics of the density and correlation functions, in the presence of a finite source all modes contribute. This means that in the general case, the long wave-length approximations that were so successful in strongly correlated systems (such as bosonization or conformal field theory techniques) do not work for the problem at hand. Moreover, the idea of exploiting the integrability of some spin Hamiltonians on which the multi-species Liouvillian maps, might turn out more elusive than expected. In view of the above remark, it would seem extremely difficult if not impossible to extract from the exact Bethe-ansatz solution of the non-hermitian spin Hamiltonian the relevant matrix elements that in turn allow the evaluation of correlation functions \([5,6]\). We propose to tackle the multi-species problem in terms of fermion functionals, the main difficulties arising from the two- and/or multi-body interactions occurring in the process of mapping classical particles to fermions. We will illustrate the power of the formalism in a forthcoming paper, where we will apply the renormalization group scheme. Besides the obvious advantage of formulating the problem in a field theoretical language (perturbation theory,...), the method is applicable to arbitrary densities of particles, and thus complements the approach developed by Cardy and collaborators \([5,6]\). In higher dimensions, however, Fermi statistics requires the introduction of a gauge field that is strongly coupled to the fermions. We also intend to explore this line of research in the future.
ACKNOWLEDGMENTS

We thank M. Droz, L. Frachebourg, Ch. Gruber, S. Gyger, Ph. Nozières and T.M. Rice for stimulating discussions. We are grateful to the referee for drawing our attention to references [16,4]. The support of the Swiss National Fonds is gratefully acknowledged.

APPENDIX A: USEFUL RESULTS

In this appendix we provide some useful properties of the Bessel functions of imaginary argument. We recall the definition of the modified Bessel \( I_n(z) \) function (\( n \) integer) \[18\]

\[
I_n(z) = \frac{(-i)^n}{2\pi} \int_{-\pi}^{\pi} e^{-z\sin(p)+inp} dp = \frac{1}{\pi} \int_{0}^{\pi} e^{z\cos\theta} \cos(n\theta) d\theta
\]

(A1)

with \( I_n(z) = I_{-n}(z) \). We also use in \( \text{(52,52,53)} \) the well-known properties \( I_{n-1}(z) - I_{n+1}(z) = 2\pi i I_n(z) \) and \( I_{n-1}(z) + I_{n+1}(z) = 2\pi i z I_n(z) \). The following integrals occur in the evaluation of the two-point correlation function.

\[
\tilde{I}_1(r,t) = \int_{0}^{\pi} \frac{\sin qr}{\sin q} \sin q e^{4Dt\cos q z} dq , \quad \tilde{I}_2(r,t) = \int_{0}^{\pi} \frac{\sin qr}{\sin q} \cos q e^{4Dt\cos q z} dq , \quad \tilde{I}_3(r,t) = \int_{0}^{\pi} \frac{\sin qr}{\sin q} \cos 2q e^{4Dt\cos q z} dq
\]

(A2)

Setting \( \tilde{q} = q - i\epsilon \), \( \epsilon \) real and \( \epsilon > 0 \), we have

\[
\frac{\sin qr}{\sin q} = \lim_{\epsilon \downarrow 0} \frac{\sin \tilde{q}r}{\sin \tilde{q}}
\]

(A3)

\[
\frac{\sin qr}{\sin q} = \lim_{\epsilon \downarrow 0,q \to q} \frac{\sin \tilde{q}r}{\sin \tilde{q}} = \lim_{\epsilon \downarrow 0,q \to q} \sum_{n \geq 0} \left( e^{-2i\tilde{q}(n-(\frac{1}{2}))} - e^{-2i\tilde{q}(n+(\frac{1}{2}))} \right)
\]

(A4)

Therefore,

\[
\tilde{I}_1(r,t) = \int_{0}^{\pi} \left( \lim_{\epsilon \downarrow 0,q \to q} \sum_{n \geq 0} \left( e^{-2i\tilde{q}(n-(\frac{1}{2}))} - e^{-2i\tilde{q}(n+(\frac{1}{2}))} \right) \right) e^{4Dt\cos q z} dq = 
\]

\[
= \pi \sum_{0 \leq n < r} I_{2n-r+1}(4Dt)
\]

(A5)

Similarly, we find

\[
\tilde{I}_2(r,t) = \frac{\pi}{2} \sum_{0 \leq n < r} \left\{ I_{2n-r}(4Dt) + I_{2n-r+2}(4Dt) \right\} = \pi \sum_{0 \leq n < r} I_{2n-r}(4Dt)
\]

(A6)

\[
\tilde{I}_3(r,t) = \frac{\pi}{2} \sum_{0 \leq n < r} \left\{ I_{2n-r-1}(4Dt) + I_{2n-r}(4Dt) + I_{2n-r+1}(4Dt) + I_{2n-r+3}(4Dt) \right\} = 
\]

\[
= \pi \sum_{0 \leq n < r} I_{2n-r-1}(4Dt)
\]

(A7)

With help of asymptotic behaviour of Bessel functions and using the properties of the Incomplete Gamma functions \[18\], the asymptotic behaviour \( (Dt \gg 1, Jt \gg 1) \) of the following integrals is readily found

\[
\int_{t}^{\infty} dt' \frac{e^{-4Jt'}}{\sqrt{8\pi Dt'}} \sim \frac{e^{-4Jt}}{4J\sqrt{8\pi Dt}} \left( 1 - \frac{1}{8Jt} + \mathcal{O}(Jt^{-2}) \right)
\]

(A8)

\[
\int_{t}^{\infty} dt' \frac{e^{-4Jt'}}{\sqrt{8\pi Dt'}8Dt'} \sim \frac{e^{-4Jt}}{4J\sqrt{8\pi Dt}} \left( \frac{1}{8Dt} - \frac{3}{64Jt} + \mathcal{O}(Jt^{-3}) \right)
\]

(A9)
\[ \int_{t}^{\infty} dt' \frac{e^{-4Jt'}}{\sqrt{8\pi Dt'(Dt')^2}} \sim \frac{e^{-4Jt}}{4J\sqrt{8\pi Dt}} \left( \frac{1}{(Dt)^3} + \mathcal{O}((Jt)^{-3}) \right) \]  

(A10)

A further result \((Dt \gg 1, Jt \gg 1)\) used in the evaluation of the asymptotic behaviour of the density and the two-point correlation function \(\{8\} \{8\} \{8\} \{8\} \{8\} \{8\} \{8\} \{8\} \{8\} \{8\})

\[ \int_{t}^{\infty} e^{-4(J+D)t'} I_n(4Dt') \sim \frac{e^{-4Jt}}{4J\sqrt{8\pi Dt}} \left\{ \left(1 - \frac{1}{8Jt} + \mathcal{O}((Jt)^{-2})\right) - 
\left(-\frac{n^2 - \frac{1}{4}}{8Dt} - \frac{3}{64JD^2} + \mathcal{O}((Jt)^{-3})\right) + 
\left(+\frac{n^2 - \frac{1}{4}}{4}(n^2 - \frac{9}{4})\left(\frac{1}{128D^2} + \mathcal{O}((Jt)^{-3})\right) + \ldots \right) \right\} \]  

(A11)

Finally, the calculation of the constants \(A_0, B_0, C_0\), when \(J > 0\) is performed with help of the formula \[18\]

\[ \int_{0}^{\infty} d\nu e^{-\alpha\nu} I_\nu(\beta) = \frac{\beta^\nu}{\sqrt{\alpha^2 - \beta^2}} \left(\alpha + \sqrt{\alpha^2 - \beta^2}\right)^\nu, \quad \forall \text{Re}(\nu) > -1, \ \text{Re}(\alpha) > \text{Re}|\beta| \]  

(A12)

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