Robustness of Adiabatic Quantum Computing

Seth Lloyd, MIT

Abstract: Adiabatic quantum computation for performing quantum computations such as Shor’s algorithm is protected against thermal errors by an energy gap of size $O(1/n)$, where $n$ is the length of the computation to be performed.

Adiabatic quantum computing is a novel form of quantum information processing that allows one to find solutions to NP-hard problems with at least a square root speed-up [1]. In some cases, adiabatic quantum computing may afford an exponential speed-up over classical computation. It is known that adiabatic quantum computing is no stronger than conventional quantum computing, since a quantum computer can be used to simulate an adiabatic quantum computer. Aharonov et al. showed that adiabatic quantum computing is no weaker than conventional quantum computation [2]. This paper presents novel models for adiabatic quantum computation and shows that adiabatic quantum computation is intrinsically protected against thermal noise from the environment. Indeed, thermal noise can actually be used to ‘help’ an adiabatic quantum computation along.

A simple way to do adiabatic versions of ‘conventional’ quantum computing is to use the Feynman pointer consisting of a line of qubits [3]. The Feynman Hamiltonian is

$$H = - \sum_{\ell=0}^{n-1} U_{\ell} \otimes |\ell+1\rangle \langle \ell| + U_{\ell}^\dagger \otimes |\ell\rangle \langle \ell+1|, \quad (1)$$

where $U_{\ell}$ is the unitary operator for the $\ell$’th gate and $|\ell\rangle$ is a state of the pointer where the $\ell$’th qubit is 1 and all the remaining qubits are 0. Clearly, $H$ is local and each of its
terms acts on four qubits at once for two-qubit gates. If we consider the pointer to be a ‘unary’ variable, then the each of the terms of $H$ acts on two qubits and the unary pointer variable.

Assume that the computation has been set up so that all qubits start in the state $0$. The computation takes place and the answer is placed in an answer register. Now a long set of steps, say $n/2$ takes place in which nothing happens. Then the computation is undone, returning the system to its initial state at the $n - 1$’th step. The computational circuit then wraps around and begins again. The eigenstates of $H$ then have the form

$$|b, k\rangle = \frac{1}{\sqrt{n}}\sum_{\ell=0}^{n-1} e^{i2\pi k\ell/n} U_\ell \ldots U_0 |b\rangle \otimes |\ell\rangle,$$  \hspace{1cm} (2)

with eigenvalue $-2\cos 2\pi k/n$, since $H|b, k\rangle = -(e^{-i2\pi k/n} + e^{i2\pi k/n})|b, k\rangle$. These can be thought of as momentum eigenstates for the propagation of the pointer qubit down the chain. The $|0, k\rangle$ momentum eigenstates have the nice feature that if you measure the answer register, the probability of obtaining the answer is $1/2$.

Feynman used this Hamiltonian to set up a traveling state of the pointer (a Gaussian superposition of the momentum eigenstates) so that the computation could take place sequentially. This propagating state is mathematically equivalent to a coherent quantum walk down the chain [4-5]. Landauer pointed out that for any realistic implementation of such a system, imperfections in the Hamiltonian would result in Anderson localization and the computation would get ‘stuck’ and fail to propagate to the end [6]. Localization is a significant problem for quantum walks in general and for the Feynman quantum computing model in particular.

One can also use Feynman’s Hamiltonian to implement the computation adiabatically. In this case, as will be seen, localization does not affect the computation. Let

$$H_0 = \sum_j (1 - |0\rangle_j\langle 0|) \otimes |\ell = 0\rangle\langle \ell = 0|$$  \hspace{1cm} (3)

be the Hamiltonian one of whose ground states is the state with all the computer qubits equal to zero and the pointer at the zero spot of the line. $H_0$ is degenerate: states where the pointer is at other places also have zero energy, regardless of the values of the computational qubits. First, prepare the system in the state $|00\ldots 0\rangle|\ell = 0\rangle$ by applying the Hamiltonian $\eta H_0 + H_1$, where $H_1 = -|\ell = 0\rangle\langle \ell = 0|$ and $\eta$ is a positive term that sets the overall energy scale.
Now gradually turn on an \( H \) term while turning off the \( H_1 \) term: the total Hamiltonian is \( \eta H_0 + (1 - \lambda)H_1 + \lambda H \). As \( \lambda \) is turned on, the system goes to its new ground state \( |b = 0, k = 0\rangle \). It can be verified numerically and analytically [7] that the minimum energy gap in this system occurs at \( \lambda = 1 \): consequently, the minimum gap goes as \( 1/n^2 \). In fact, the energy gap due to the interaction between the \( H_1 \) and the \( H \) terms is just the energy gap of the simpler system consisting just of the chain qubits on their own, confined to the subspace in which exactly one qubit is 1: that is, it is the energy gap of a qubit chain with Hamiltonian \( (1 - \lambda)H_1 + \lambda H' \), where \( H' = -\sum_{\ell} |\ell + 1\rangle \langle \ell| + |\ell\rangle \langle \ell + 1| \). This gap goes as \( 1/n^2 \). Accordingly, the amount of time required to perform the adiabatic passage is polynomial in \( n \).

When the adiabatic passage is complete, the energy gap of the \( H \) term on its own goes as \( 1/n^2 \) from the cosine dependence of the eigenvalues of \( H \): it is also just the energy gap of the simplified system in the previous paragraph. This implies that the adiabatic passage can accurately be performed in a time polynomial in \( n \). Measuring the answer register now gives the answer to the computation with probability 1/2. This is an alternative (and considerably simpler) demonstration to that of [2] that ‘conventional’ quantum computation can be performed efficiently in an adiabatic setting.

An interesting feature of this procedure is that the adiabatic passage can be faulty and still work just fine: all of the energy eigenstates in the \( |b = 0, k\rangle \) sector give the correct answer to the computation with probability 1/2, for any \( k \). The real issue is making sure we do not transition to the \( |b \neq 0, k\rangle \) sector. But the Hamiltonians \( H_1 \) and \( H \) do not couple to this sector: so in fact, we can perform the passage non-adiabatically and still get the answer to the computation. For example, if we turn off the \( H_1 \) Hamiltonian very rapidly and turn on the \( H \) Hamiltonian at the same time, the system is now in an equal superposition of all the \( |b = 0, k\rangle \) eigenstates. If we wait for a time \( \propto n^2 \) (corresponding to the inverse of the minimum separation between the eigenvalues of \( H \)), then the state of the system will be spread throughout the \( |b = 0, k\rangle \) sector, and we can read out the answer with probability 1/2. This method effectively relies on dispersion of the wavepacket to find the answer.

Since coherence of the pointer doesn’t matter, we can also apply a Hamiltonian to the pointer that tilts the energy landscape so that higher pointer values have lower energy. (E.g., \( H_{\text{pointer}} = -\sum_{\ell} \ell E|\ell\rangle \langle \ell| \).) Starting the pointer off in the initial state above and letting it interact with a thermal environment will obtain the answer to the computation.
in time of $O(n)$. Similarly, in the absence of an environment, starting the pointer off in a wavepacket with zero momentum at time 0 and letting it accelerate will get the answer to the computation in even shorter time.

Clearly, this method is quite a robust way of performing quantum computation. Let us look more closely at the sources of this robustness. If $\eta$ is big, then there is a separation of energy of $O(\eta/n)$ between the $|b = 0, k\rangle$ sector — states which give the correct answer to the computation — and the $|b \neq 0, k\rangle$ sector — states which give the incorrect answer to the computation. This is because $\langle b \neq 0, k | \eta H_0 | b \neq 0, k \rangle = \eta/n$. This energy gap goes down only linearly in the length of the computation and can be made much larger than the gap between the ground and first excited state by increasing $\eta >> 1$.

This second energy gap is very useful: it means that thermal excitations with an energy below the gap will not interfere with obtaining the proper answer. That is, this method is intrinsically error-tolerant in the face of thermal noise. Indeed, it is this $O(\eta/n)$ gap that determines how rapidly the computation can take place rather than the $O(1/n^2)$ gap between the ground and excited states.

Of course, the actual errors in a system that realizes the above scheme are likely to arise from variability in the applied Hamiltonians. The energy gap arguments for robustness only apply to the translational dynamics of the system (this is what makes the analysis of the system tractable in the first place). That is, errors that affect each $U_\ell$ on its own are not protected against: but these are the errors that cause the computation to come out wrong. Of course, one can always program the circuit to perform ‘conventional’ robust quantum computation to protect against such errors. One must be careful, however, that errors that entangle the pointer with the logical qubits do not contaminate other qubits: conventional robust quantum computation protocols will have to be modified to address this issue. Farhi et al. have recently exhibited error correcting codes for ‘conventional’ adiabatic quantum computation [1] that can protect against such computational errors [8].

The use of error correcting codes to correct the variation in the $U_\ell$ may well be overkill. In any system manufactured to implement adiabatic quantum computing, these errors in the $U_\ell$ are essentially deterministic: the $U_\ell$ could in principle be measured and their variation from their nominal values compensated for by tuning the local Hamiltonians. Because it involves no added redundancy, such an approach is potentially more efficient than the use of quantum error correcting codes. Exactly how to detect and correct variations in the $U_\ell$ will depend on the techniques (e.g., quantum dots or superconducting systems) used to
construct adiabatic quantum circuits.

It is also interesting to note that performing quantum computation adiabatically is intrinsically more energy efficient than performing a sequence of quantum logic gates via the application of a series of external pulses. The external pulses must be accurately clocked and shaped, which requires large amounts of energy. In the schemes investigated here, the internal dynamics of the computer insure that quantum logic operations are performed in the proper order, so no clocking or external pulses need be applied. The adiabatic technique also avoids the Anderson localization problem raised by Landauer.

The above construction requires an external pointer and four qubit interactions. One can also set up a pointerless model that requires only pairwise interactions between spin 1/2 particles (compare the following method with the method proposed in reference [9]). Let each qubit in the computational circuit correspond to a particle with two internal states. Let each wire in the circuit correspond to a mode that can be occupied by a particle. The \(\ell\)'th quantum logic gate then corresponds to an operator \(\tilde{H}_\ell = A_\ell + A_\ell^\dagger\), where \(A_\ell\) is an operator that takes two particles from the two input modes and moves them to the output modes while performing a quantum logic operation on their two qubits. That is,

\[
A_\ell = a_{out1}^\dagger a_{in1} a_{out2}^\dagger a_{in2} \otimes U_\ell.
\] (5)

Note that \(A_\ell\) acts only when both input modes are occupied by a qubit-carrying particle.

If we use the Hamiltonian \(\tilde{H} = \sum_\ell \tilde{H}_\ell\) in place of \(H\) in the construction above, the ground state of this Hamiltonian is a superposition of states in which the computation is at various stages of completion. Just as above, measurement on the ground state will reveal the answer to the computation with probability \(1/2\).

Note that even though the Hamiltonian in equation (5) involves a product of operators on four degrees of freedom (the internal degrees of freedom of the particles together with their positions), it is nonetheless a physically reasonable local Hamiltonian involving pairwise interactions between spin-1/2 particles. To simulate its operation using an array of qubits as in [2] would require four qubit interactions, as in the pointer model discussed above. This point is raised here because of the emphasis in the quantum computing literature on reducing computation to pairwise interactions between qubits. Pairwise interactions between particles or fields – i.e., the sort of interactions found in nature – may correspond to interactions between more than two qubits.

Without further restrictions on the form of the quantum logic circuit, evaluating the energy gap in this particle model is difficult, even for the final Hamiltonian \(1 - \tilde{H}\). But we
can always set up the computational circuit in a way that allows the adiabatic passage to be mapped directly on the Feynman pointer model above. The method is straightforward: order the quantum logic gates as above. Now insert additional quantum logic gates between each consecutive pair of gates in the original circuit. The additional gate inserted between the $\ell$’th and $\ell+1$’th quantum logic gates couples one output qubit of the $\ell$’th quantum logic gate with one input qubit of the $\ell + 1$’th gate, and performs a trivial operation $U = 1$ on the internal qubits of these gates. The purpose of these gates is to ensure that the quantum logic operations are performed in the proper sequence. Effectively, one of the qubits from the $\ell$’th gate must ‘tag’ one of the qubits from the $\ell + 1$’th gate before the $\ell + 1$’th gate can be implemented. Accordingly, we call this trick, a ‘tag-team’ quantum circuit.

Tag-team quantum circuits are unitarily equivalent to the Feynman pointer model with an extra, identity gate inserted between each of the original quantum logic gates. Accordingly, the spectral gap for tag-team quantum circuits goes as $1/n^2$ and the quantum computation can be performed in time $O(poly(n))$. Just as for the pointer version of adiabatic quantum computing, the important spectral gap for tag-team adiabatic quantum computation is not the minimum gap, but rather the gap of size $O(\eta/n)$ between the ground-state manifold of ‘correct’ states and the next higher manifold of ‘incorrect’ states. Once again, the existence of this gap is a powerful protection against errors in adiabatic quantum computation.

The methods described above represent an alternative derivation of the fact that adiabatic quantum information processing can efficiently perform conventional quantum computation. The relative simplicity of the derivation from the original Feynman Hamiltonian [3] allows an analysis of the robustness of the scheme against thermal excitations. Adiabatic implementations of quantum computation are robust against thermal noise at temperatures below the appropriate energy gap. The appropriate energy gap is not the minimum gap, which scales as $n^2$, but the gap between the lowest sector of eigenstates, which give the correct answer, and the next sector. This gap scales as $\eta/n$, where $\eta$ is an energy parameter that is within the control of the experimentalist.

Acknowledgements: This work was supported by ARDA, ARO, DARPA, CMI, and the W.M. Keck foundation. The author would like to thank D. Gottesman and D. Nagaj for helpful conversations.
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