The stable Derrida–Retaux system at criticality

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Dedicated to the memory of Vladas Sidoravicius

Summary. The Derrida–Retaux recursive system was investigated by Derrida and Retaux [8] as a hierarchical renormalization model in statistical physics. A prediction of [8] on the free energy has recently been rigorously proved ([1]), confirming the Berezinskii–Kosterlitz–Thouless-type phase transition in the system. Interestingly, it has been established in [1] that the prediction is valid only under a certain integrability assumption on the initial distribution, and a new type of universality result has been shown when this integrability assumption is not satisfied. We present a unified approach for systems satisfying a certain domination condition, and give an upper bound for derivatives of all orders of the moment generating function. When the integrability assumption is not satisfied, our result allows to identify the large-time order of magnitude of the product of the moment generating functions at criticality, confirming and completing a previous result in [5].

Keywords. Derrida–Retaux recursive system, moment generating function.

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1 Introduction

Fix an integer $m \geq 2$. Let $X_0$ be a random variable taking values in $\mathbb{Z}_+ := \{0, 1, 2, \ldots \}$. To avoid trivial discussion, it is assumed, throughout the paper, that $\mathbb{P}(X_0 \geq 2) > 0$. Let us
consider the Derrida–Retaux recursive system \((X_n, n \geq 0)\) defined as follows: for all \(n \geq 0\),

\[
X_{n+1} = (X_{n,1} + \cdots + X_{n,m} - 1)^+,
\]

where \(X_{n,i}, i \geq 1\), are independent copies of \(X_n\). This was investigated by Derrida and
Retaux \cite{DerridaRetaux} as a toy model to study depinning in presence of impurities \cite{Derrida1, Derrida2, Derrida3, Derrida4, Derrida5, Derrida6, Derrida7}.

We refer to \cite{Derrida8} for an overview on rigorous results and predictions about the Derrida–Retaux
system.

Assuming \(E(X_0) < \infty\), it is immediate from (1.1) that \(E(X_{n+1}) \leq m E(X_n)\), so the free
energy

\[
F_\infty := \lim_{n \to \infty} \frac{E(X_n)}{m^n} \in [0, \infty),
\]
is well-defined. A remarkable result by Collet et al. \cite{Collet} tells us that assuming \(E(X_0 m^{X_0}) < \infty\)
(which we take for granted throughout the paper) and writing \(\eta := (m - 1)E(X_0 m^{X_0}) - E(m^{X_0})\), then \(F_\infty > 0\) if \(\eta > 0\), and \(F_\infty = 0\) if \(\eta \leq 0\).

As such, it is natural to say that the system \((X_n, n \geq 0)\) is supercritical if \(\eta > 0\), is
critical if \(\eta = 0\), and is subcritical if \(\eta < 0\).

It has been conjectured by Derrida and Retaux \cite{Derrida8} that if \(\eta > 0\), then we would have

\[
F_\infty = \exp \left( - \frac{C + o(1)}{\eta^{1/2}} \right), \quad \eta \to 0^+,
\]

for some constant \(C \in (0, \infty)\) possibly depending on the law of \(X_0\). A (somehow weak)
result has been proved in \cite{Derrida9}: assuming \(E(X_0^3 m^{X_0}) < \infty\),

\[
F_\infty = \exp \left( - \frac{1}{\eta^{1/2 + o(1)}} \right), \quad \eta \to 0^+.
\]

This confirms that the Derrida–Retaux system has a Berezinskii–Kosterlitz–Thouless-type
phase transition of infinite order. The integrability assumption \(E(X_0^3 m^{X_0}) < \infty\) might look
exotic, but it is optimal. [We believe that there should be a change-of-measures argument,
and that the assumption is equivalent to saying that \(X_0\) has a finite second moment under a
new probability measure; however, we have not succeeded in making this idea into a rigorous
argument.] In fact, it has also been proved in \cite{Derrida9} that if \(P(X_0 = k) \sim c m^{-k} k^{-\alpha}, k \to \infty,\)
for some \(2 < \alpha < 4\) and \(c > 0\)\footnote{Notation: \(a_k \sim b_k, k \to \infty\), means \(\lim_{k \to \infty} \frac{a_k}{b_k} = 1\).}, then

\[
F_\infty = \exp \left( - \frac{1}{\eta^{\alpha/2 + o(1)}} \right), \quad \eta \to 0^+.
\]
\[ \nu = \nu(\alpha) := \frac{1}{\alpha - 2}. \]

In other words, (1.2) predicts only a small part of universalities, under the assumption \( E(X_0^3 m^{X_0}) < \infty \), while other universality phenomena are described by (1.3). We expect many other universality results in the latter setting (for example, corresponding to those in [3] for an analogous continuous-time model); unfortunately, they are currently only on a heuristic level.

It is well-known that sum of i.i.d. random variables, after an appropriate normalization, converges to a Gaussian limiting law under the condition of finiteness of second moment, and to a stable limiting law under a weaker integrability condition. We say that the Derrida–Retaux system has a “finite variance” if \( E(X_0^3 m^{X_0}) < \infty \), and that it is a stable system if integrability condition holds for lower orders. In this paper, we are interested in the stable system when it is critical, i.e., when \((m - 1)E(Y_0 m^{Y_0}) = E(m^{Y_0})\). [We are going to see in Section 2 quite easily, that this implies \((m - 1)E(X_n m^{X_n}) = E(m^{X_n})\) for all \(n \geq 0\).] We write \((Y_n, n \geq 0)\) instead of \((X_n, n \geq 0)\) in order to insist on criticality. From now on, we assume \((Y_n, n \geq 0)\) to be a Derrida–Retaux system satisfying \((m - 1)E(Y_0 m^{Y_0}) = E(m^{Y_0}) < \infty\), such that

\[
\begin{align*}
P(Y_0 = k) & \sim c_0 m^{-k} k^{-\alpha}, \quad k \to \infty,
\end{align*}
\]

for some \(2 < \alpha < 4\) and \(c_0 > 0\). We intend to prove the following result.

**Theorem 1.1.** Let \((Y_n, n \geq 0)\) be such that \((m - 1)E(Y_0 m^{Y_0}) = E(m^{Y_0}) < \infty\). Under assumption (1.4), there exist constants \(c_2 \geq c_1 > 0\) such that for all \(n \geq 1\),

\[
c_1 n^{\alpha - 2} \leq \prod_{i=0}^{n-1} [E(m^{Y_i})]^{m-1} \leq c_2 n^{\alpha - 2}.
\]

When the system is of “finite-variance” (i.e., \(E(Y_0^3 m^{Y_0}) < \infty\)), the analogue of Theorem 1.1 was known ([3], [4]), and has played an important role in the study of the asymptotics of \(P(Y_n > 0)\) and \(E(Y_n)\) in [2]. It would be tempting to believe that Theorem 1.1 could play an equally important role in the study of the same problems for the stable system.

Just like the usual random walk has a nice continuous-time analogue which is Brownian motion, the Derrida–Retaux system has analogues in continuous time (Derrida and Retaux [8], Hu, Mallein and Pain [13]), defined via appropriate integro-differential equations. For the continuous-time analogue of the stable Derrida–Retaux system, see [3]. These
continuous-time models have been studied in depth in [8], [13] and [3], while most of the corresponding problems remain open for the original Derrida–Retaux system.

With the exception of the case \( \alpha = 3 \), Theorem 1.1 was already stated in Collet et al. [5]: its proof in case \( 2 < \alpha < 3 \) was indicated, whereas the proof in case \( 3 < \alpha < 4 \) was only summarized in a “very succinct account”. By means of the notion of dominability (see the forthcoming Definition 2.1), we give a unified approach to the system in both situations, i.e., either it is stable (no need for discussions separately on the cases \( 2 < \alpha < 3 \) and \( 3 < \alpha < 4 \)) or is of “finite variance”. Concretely, in both situations, we use a truncating argument by considering a bounded random variable defined by

\[
Z_0 = Z_0(M) := Y_0 1_{\{Y_0 \leq a(M)\}},
\]

where \( a(M) \in [1, \infty] \) can be possibly infinite (in which case there is no need for truncation), whose value depends on an integer parameter \( M \geq 1 \). Consider the Derrida–Retaux system \((Z_n, n \geq 0)\) whose initial distribution is given by \( Z_0 \).

We prove, in Theorem 4.1, that in both situations, it is possible to choose a convenient value of \( a(M) \) such that the new system \((Z_n, n \geq 0)\) is dominable (in the sense of Definition 2.1), while it is possible to connect the moment generating functions of \( Y_n \) and \( Z_n \). In Theorem 2.3, we give an upper bound for the moment generating function of any dominable system \((Z_n, n \geq 0)\). As such, a combined application of Theorems 4.1 and 2.3 will yield information for the moment generating function of the original Derrida–Retaux system, in both situations. In the stable case, it will yield Theorem 1.1, whereas in the case of “finite variance”, under a stronger integrability assumption on the law of \( Y_0 \), it will give the following result:

Theorem 1.2. Let \((Y_n, n \geq 0)\) be such that \((m - 1)E(Y_0 m^{Y_0}) = E(m^{Y_0})\). If \( E(s^{Y_0}) < \infty \) for some \( s > m \), then there exists a constant \( c_3 > 0 \) such that for all integers \( n \geq 1 \) and \( k \geq 1 \),

\[
\left. \frac{d^k}{du^k} E(u^{Y_n}) \right|_{u=m} \leq k! e^{c_3 k n^{k-1}}.
\]

In the “finite-variance” case \( E(Y_0^3 m^{Y_0}) < \infty \) for \( k \in \{1, 2, 3\} \) was known: the case \( k = 1 \) is simple because by criticality, \( E(Y_n m^{Y_n-1}) = \frac{1}{m(m-1)} E(m^{Y_n}) \) which is bounded in \( n \) \([3, 4]\), the case \( k = 3 \) was proved in [3], and the case \( k = 2 \), stated in [1], follows

\footnote{Strictly speaking, it is a sequence of Derrida–Retaux systems, indexed by \( M \).}
immediately from the cases $k = 1$ and $k = 3$ by means of the Cauchy–Schwarz inequality. More generally, if $\mathbb{E}(Y_0^s m^{Y_0}) < \infty$ for some integer $\ell \geq 1$, then for all $k \in [1, \ell] \cap \mathbb{Z}$, it is quite easy to prove (2.2) by induction in $k$, using the recursion (1.1), that there exists a constant $c > 0$ such that for all integer $n \geq 1$,

$$\left. \frac{d^k}{du^k} \mathbb{E}(u^{Y_0}) \right|_{u=m} \leq c n^{k-1}. \tag{2.1}$$

Theorem 1.2 gives information about the dependence in $k$ of the constant $c$, under the integrability assumption $\mathbb{E}(s^{Y_0}) < \infty$ for some $s > m$.

The rest of the paper is organized as follows. In Section 2 we introduce the notion of dominable systems. Theorem 2.3, which gives an upper bound for the moment generating function of dominable systems, is the main technical result of the paper. The brief Section 3 is devoted to the proof of Theorem 1.2, obtained as a simple consequence of Theorem 2.3. In Section 4, for both “finite-variance” and stable systems, we construct a dominable system $(Z_n, n \geq 0)$ such that $Z_0$ is obtained from an appropriate truncation of $Y_0$. Finally, Theorem 1.1 is proved in Section 5 also as a consequence of Theorem 2.3.

## 2 Dominable systems

We introduce the notion of dominable systems and prove a general upper bound for the moment generating function of such systems (Theorem 2.3). As before, we talk about the Derrida–Retaux system $(Z_n, n \geq 0)$, while it is, in fact, a sequence of Derrida–Retaux systems $(Z_n(M), n \geq 0)$ indexed by the integer-valued parameter $M$.

**Definition 2.1.** Let $\gamma > 0$. The system $(Z_n, n \geq 0)$ is said to be $\gamma$-dominable if for all sufficiently large integer $M$, say $M \geq M_0$, $Z_0 = Z_0(M)$ is bounded, and there exists a constant $\vartheta(M) \geq 1$ such that $M \mapsto \vartheta(M)$ is non-decreasing in $M \geq M_0$, and that

$$\mathbb{E}(Z_0^k m^{Z_0}) \leq M^{k-3}(\vartheta(M) + k!), \quad k \geq 3, \tag{2.1}$$

$$\vartheta(n \lor M) \prod_{i=0}^{n-1} \left[\mathbb{E}(m^{Z_i})\right]^{m-1} \leq \gamma (n \lor M)^2, \quad n \geq 1. \tag{2.2}$$

$^5$In (2.1) and (2.2), $k \geq 3$ and $n \geq 1$ are integers. Notation: $a \lor b := \max\{a, b\}$. 

5
Remark 2.2. Let \((Z_n, n \geq 0)\) be \(\gamma\)-dominable. By (2.2) and the trivial inequality \(E(mZ_i) \geq 1\) \((\forall i \geq 0)\), we have \(E(Z_n) \leq E(mZ_n) \leq [\gamma (n \vee M)^2]^{1/(m-1)}\), so the free energy \(F_\infty := \lim_{n \to \infty} E(Z_n) / m\) vanishes; by the criterion of Collet et al. [5] recalled in the introduction, the system \((Z_n, n \geq 0)\) is subcritical or critical for all \(M \geq M_0\); we have \((m-1)E(Z_0 mZ_0) \leq E(mZ_0)\). \(\square\)

**Theorem 2.3.** Let \(\gamma > 0\). Let \((Z_n, n \geq 0)\) be a \(\gamma\)-dominable system, and let \(H_n(u) := E(u Z_n)\). There exists a constant \(c_4 \geq 1\), depending only on \((m, \gamma)\), such that for all integers \(k \geq 3, n \geq 1\) and \(M \geq M_0\),

\[
H_n^{(k)}(m) \leq k! c_4^k (n \vee M)^{k-1},
\]

where \(H_n^{(k)}(\cdot)\) stands for the \(k\)-th derivative of \(H_n(\cdot)\).

**Corollary 2.4.** Let \(\gamma > 0\). Let \((Z_n, n \geq 0)\) be a \(\gamma\)-dominable system. There exists a constant \(c_5 > 0\), depending only on \((m, \gamma)\), such that for \(M \geq M_0, n \geq 1\) and \(v := m + \frac{1}{2c_4(n \vee M)}\),

\[
E(Z_n^2 v Z_n) \leq c_5 [\vartheta(n \vee M)]^{1/2} \prod_{i=0}^{n-1} [E(mZ_i)]^{(m-1)/2};
\]

in particular, we have, with \(c_6 := \gamma^{1/2}c_5\),

\[
E(Z_n^2 v Z_n) \leq c_6 (n \vee M).
\]

The rest of the section is devoted to the proof of Theorem 2.3 and Corollary 2.4. We start by mentioning a general technique going back to Collet et al. [5]. Let \((X_n, n \geq 0)\) denote a Derrida–Retaux system satisfying \(E(X_0 mX_0) < \infty\). Let

\[
G_n(u) := E(u X_n), \quad n \geq 0.
\]

The iteration formula (1.1) is equivalent to:

\[
G_{n+1}(u) = \frac{1}{u} [G_n(u)]^m + (1 - \frac{1}{u}) [G_n(0)]^m, \quad n \geq 0.
\]
A useful trick of Collet et al. \cite{5} consists in observing that this yields
\[(u - 1)u G_{n+1}'(u) - G_{n+1}(u) = [m(u - 1) G_n'(u) - G_n(u)] [G_n(u)]^{m-1}.\]
In particular, taking \(u = m\) yields that
\[G_{n+1}(m) - (m - 1)m G_n'(m) = [G_n(m) - (m - 1)m G_n'(m)] [G_n(m)]^{m-1}.\]
Iterating this formula gives that for \(n \geq 1\),
\[(2.6)\]
\[G_n(m) - (m - 1)m G_n'(m) = [G_0(m) - (m - 1)m G_0'(m)] \prod_{i=0}^{n-1} [G_i(m)]^{m-1}.\]
In particular, \((2.6)\) tells us that the sign of \(E(m X_n) - (m - 1)E(X_0 m X_0)\) remains identical for all \(n \geq 0\): it either is always positive (meaning that the system is supercritical), or is always negative (subcritical), or vanishes identically (critical).

A couple of known results which we are going to use for the subcritical or critical system: if \((X_n, n \geq 0)\) is a Derrida–Retaux satisfying \(E(m X_0) \leq (m - 1)E(X_0 m X_0) < \infty\), then for all \(n \geq 0\),
\[(2.7)\]
\[(m - 1)E(X_n m X_n) \leq E(m X_n) \leq m^{1/(m-1)}, \quad n \geq 0,\]
\[(2.8)\]
\[\prod_{i=0}^{n-1} [E(m X_i)]^{m-1} \leq c_7 n^{2}, \quad n \geq 1,\]
where \(c_7 > 0\) is a constant depending on \(m\) and on the law of \(X_0\). See \cite{1} (3.11)] for the second inequality in \((2.7)\) (proved in \cite{1} for critical systems, and the same proof valid for subcritical systems as well), and \cite{4, Proposition 1} for \((2.8)\).

The proof of Theorem 2.3 relies on the following preliminary result. Recall that \(H_n(u) := E(u^Z_n)\).

**Lemma 2.5.** Let \((Z_n, n \geq 0)\) be a \(\gamma\)-dominable system for some \(\gamma > 0\), in the sense of Definition 2.1. There exist constants \(c_8 > 0\) and \(c_9 > 0\), depending only on \(m\), such that for \(M \geq M_0\) and all integer \(n \geq 1\),
\[(2.9)\]
\[H''_n(m) \leq c_8 [\vartheta(M)]^{1/2} \prod_{i=0}^{n-1} H_i(m)^{(m-1)/2},\]
\[(2.10)\]
\[H''_n(m) \leq c_9 \vartheta(M) \prod_{i=0}^{n-1} H_i(m)^{m-1}.\]
Proof. Let

\[ D_n(u) := (m - 1)u^3H'''_n(u) + (4m - 5)u^2H''_n(u) + 2(m - 2)uH'_n(u), \quad n \geq 0. \]

Recall from Remark 2.2 that \( m(m - 1)H'_0(m) \leq H_0(m). \) By [4, Equation (19)], this yields \( D_{n+1}(m) \leq D_n(m)H_n(m)^{m-1} \) for all \( n \geq 0. \) [In [4], it was proved that if \( m(m - 1)H'_0(m) = H_0(m), \) then \( D_{n+1}(m) = D_n(m)H_n(m)^{m-1}, \) but the proof is valid for inequalities in place of equalities.] Accordingly, for all \( n \geq 1, \)

\[ D_n(m) \leq D_0(m) \prod_{i=0}^{n-1} H_i(m)^{m-1}. \]

By definition, \( D_n(m) \geq m^3(m - 1)H'''_n(m), \) so

\[ (2.11) \quad H'''_n(m) \leq \frac{D_0(m)}{m^3(m - 1)} \prod_{i=0}^{n-1} H_i(m)^{m-1}. \]

For any \( j \geq 0 \) and \( \ell \in \{1, 2, 3\}, \) writing \( Z_j^3m^{Z_j} \leq \frac{9m^3}{2} Z_j(Z_j - 1)(Z_j - 2)m^{Z_j-3} \mathbf{1}_{Z_j \geq 3} + 8m^2, \) and \( Z^\ell_j \leq Z_j^3, \) we obtain

\[ (2.12) \quad \max_{\ell \in \{1, 2, 3\}} \mathbb{E}(Z^\ell_j m^{Z_j}) \leq \frac{9m^3}{2} H'''_j(m) + 8m^2. \]

In particular, with \( j = 0 \) and \( \ell \in \{1, 2\}, \) this gives \( \max\{H'_0(m), H''_0(m)\} \leq \frac{9m^3}{2} H'''_0(m) + 8m^2. \) Consequently, with \( c_{10} := m^3(m - 1), c_{11} := m^2(4m - 5) \) and \( c_{12} := 2m(m - 2), \)

\[ D_0(m) = c_{10} H''_0(m) + c_{11} H'_0(m) + c_{12} H'_0(m) \leq c_{13} H'''_0(m) + c_{14}, \]

where \( c_{13} := c_{10} + \frac{9m^3}{2} (c_{11} + c_{12}) \) and \( c_{14} := 8m^2(c_{11} + c_{12}). \) Since \( H'''_0(m) \leq \phi(M) + 6 \leq 7\phi(M) \) by assumption (2.11) (applied to \( k = 3; \) recalling that \( \phi(M) \geq 1 \)), we get \( D_0(m) \leq 7c_{13} \phi(M) + c_{14} \leq c_{15} \phi(M) \) with \( c_{15} := 7c_{13} + c_{14}. \) Going back to (2.11), we have

\[ H'''_n(m) \leq \frac{c_{15}}{m^3(m - 1)} \phi(M) \prod_{i=0}^{n-1} H_i(m)^{m-1}, \]

proving (2.10) with \( c_9 := \frac{c_{15}}{m^3(m - 1)}. \)

It remains to prove (2.9). By (2.12) (applied to \( j = n), \)

\[ \mathbb{E}(Z^3_n m^{Z_n}) \leq \frac{9m^3}{2} H'''_n(m) + 8m^2, \]
whereas by (2.7), $E(Z_n^2 Z_n) \leq \frac{m^{1/(m-1)}}{m-1} =: c_{16}$, it follows from the Cauchy–Schwarz inequality that

$$E(Z_n^2 Z_n) \leq c_{16}^{1/2} \left( \frac{9m^3}{2} H''_n(m) + 8m^2 \right)^{1/2}.$$

Recall from (2.10), which we have just proved, that $H''_n(m) \leq c_9 \vartheta(M) \prod_{i=0}^{n-1} H_i(m)^{m-1}$.

Writing $8m^2 \leq 8m^2 \vartheta(M) \prod_{i=0}^{n-1} H_i(m)^{m-1}$ (because $\vartheta(M) \geq 1$ and $H_i(m) \geq 1$), this yields

$$E(Z_n^2 Z_n) \leq c_{17} \left[ \vartheta(M) \right]^{1/2} \prod_{i=0}^{n-1} H_i(m)^{(m-1)/2},$$

with $c_{17} := c_{16}^{1/2} \left( \frac{9m^3}{2} c_9 + 8m^2 \right)^{1/2}$. Since $H''_n(m) \leq E(Z_n^2 Z_n)$, we obtain (2.9) with $c_8 := c_{17}$. \hfill \Box

**Remark 2.6.** We often use the following inequalities for dominable systems:

(2.13) \quad $H''_n(m) \leq c_8 \left[ \vartheta(n \lor M) \right]^{1/2} \prod_{i=0}^{n-1} H_i(m)^{(m-1)/2},$

(2.14) \quad $H'''_n(m) \leq c_9 \vartheta(n \lor M) \prod_{i=0}^{n-1} H_i(m)^{m-1},$

They are immediate consequences of Lemma 2.5 and the monotonicity of $M \mapsto \vartheta(M)$. \hfill \Box

We also need an elementary inequality.

**Lemma 2.7.** Let $\ell \geq 4$ be an integer, and let

(2.15) \quad $B_\ell := \{ u := (u_1, \ldots, u_m) \in ([0, \ell - 1] \cap \mathbb{Z})^m : u_1 + \cdots + u_m = \ell \}.$

There exists a constant $c_{18} > 0$, depending only on $m$, such that for all $y \geq 3m^6$:

(2.16) \quad $\sum_{u := (u_1, \ldots, u_m) \in B_\ell} y^{(\ell - \eta(u) - 2)^+} \prod_{i : u_i \geq 3} \frac{1}{u_i(u_i - 1)} \leq c_{18} \frac{y^{\ell-4}}{\ell^2},$

where $a^+ := \max\{a, 0\}$ as before, and

(2.17) \quad $\eta(u) := \sum_{i=1}^{m} 1_{\{u_i \geq 1\}} \geq 2.$

\textsuperscript{6}Strictly speaking, we should write $\prod_{i : 1 \leq i \leq m, u_i \geq 3} \frac{1}{u_i(u_i - 1)}$ for $\prod_{i : u_i \geq 3} \frac{1}{u_i(u_i - 1)}$. Notation: $\prod_{\emptyset} := 1.$
Proof. The sum over $u := (u_1, \ldots, u_m) \in B_\ell$ satisfying $u_{\max} := \max_{1 \leq i \leq m} u_i \leq 2$ is very simple: in this case, $\ell \leq 2m$; since $\eta(u) \geq 2$, we have $(\ell - \eta(u) - 2) \leq \ell - 4$. The number of such $u$ being smaller than $3^m$, we get

$$\sum_{u \in B_\ell : u_{\max} \leq 2} y^{(\ell - \eta(u) - 2)^+} \prod_{i : u_i \geq 3} \frac{1}{u_i(u_i - 1)} \leq 3^m y^{\ell - 4} \leq c_{19} \frac{y^{\ell - 4}}{\ell^2},$$

with $c_{19} := 3^m (2m)^2$. [Notation: $\sum_{\emptyset} := 0$.]

Let LHS$_{(2.16)}$ denote the expression on the left-hand side of (2.16). Then

$$\text{LHS}_{(2.16)} \leq c_{19} \frac{y^{\ell - 4}}{\ell^2} + \frac{2m^2}{\ell^2} \sum_{j=2}^{\ell m} m^j y^{(j - 2)^+} \sum_{(u_1, \ldots, u_j) i : 1 \leq i \leq j, u_i \geq 3} \prod_{u_i(u_i - 1)} \frac{1}{u_i(u_i - 1)},$$

where, on the right-hand side, $\sum_{(u_1, \ldots, u_j)}$ sums over all $(u_1, \ldots, u_j) \in \mathbb{Z}_+^j$ with $1 \leq u_1 \leq u_2 \leq \cdots \leq u_j$ such that $u_1 + \cdots + u_j = \ell$ and that $u_j \geq 3$. Note that $u_j \geq 3$ implies $j \leq \ell - 2$, thus $(\ell - j - 2)^+ = \ell - j - 2$. Moreover, we have $u_j \geq \frac{\ell}{2} \geq \frac{\ell}{m}$, thus $u_j(u_j - 1) \geq \frac{1}{2} u_j^2 \geq \frac{\ell^2}{2m^2}$. Consequently, $\prod_{i \leq j : u_i \geq 3} \frac{1}{u_i(u_i - 1)}$ is bounded by $\frac{2m^2}{\ell^2} \prod_{i \leq j - 1 : u_i \geq 3} \frac{1}{u_i(u_i - 1)}$. This leads to (using $\binom{m}{j} j! \leq m^j$):

$$\text{LHS}_{(2.16)} \leq c_{19} \frac{y^{\ell - 4}}{\ell^2} + \frac{2m^2}{\ell^2} \sum_{j=2}^{\ell m} m^j y^{(j - 2)^+} \sum_{(u_1, \ldots, u_j) i : 1 \leq i \leq j, u_i \geq 3} \prod_{u_i(u_i - 1)} \frac{1}{u_i(u_i - 1)}$$

$$\leq c_{19} \frac{y^{\ell - 4}}{\ell^2} + \frac{2m^2}{\ell^2} \sum_{j=2}^{\ell m} m^j y^{(j - 2)^+} \sum_{i=1}^{j-1} \left(1 + \sum_{u=3}^{\infty} \frac{1}{u(u - 1)}\right).$$

Of course, $\sum_{u=3}^{\infty} \frac{1}{u(u - 1)} = \frac{1}{2}$; also, we bound $\sum_{j=2}^{\ell m} m^j y^{(j - 2)^+}$ by $\sum_{j=2}^{\infty} \frac{3}{2} j^{-1}$. This yields that

$$\text{LHS}_{(2.16)} \leq c_{19} \frac{y^{\ell - 4}}{\ell^2} + \frac{2m^2}{\ell^2} \sum_{j=2}^{\infty} m^j y^{(j - 2)^+} \left(\frac{3}{2} j^{-1}\right).$$

On the right-hand side, write $\sum_{j=2}^{\infty} m^j y^{(j - 2)^+} \left(\frac{3}{2} j^{-1}\right) = m^2 y^{\ell - 4} \sum_{j=2}^{\infty} \left(\frac{m^2}{y}\right)^{j-2} \left(\frac{3}{2}\right)^{-1}$; in view of our choice $y \geq 3m$, this is bounded by $m^2 y^{\ell - 4} \sum_{j=2}^{\infty} \left(\frac{1}{3}\right)^{-j-2} \left(\frac{3}{2}\right)^{-1} = 3m^2 y^{\ell - 4}$. As a consequence,

$$\text{LHS}_{(2.16)} \leq c_{19} \frac{y^{\ell - 4}}{\ell^2} + \frac{6m^4 y^{\ell - 4}}{\ell^2},$$

yielding (2.16) with $c_{18} := c_{19} + 6m^4$. 

\[ \square \]
We have all the ingredients for the proof of Theorem 2.3.

**Proof of Theorem 2.3.** Let $\gamma > 0$ and let $(Z_n, n \geq 0)$ be a $\gamma$-dominable system. Write $H_n(u) := \mathbb{E}(u^{Z_n})$ as before. Recall $\vartheta(M) \geq 1$ (for all $M \geq M_0$) from (2.1) and (2.2). Write, for brevity,

\[ Q_n := \prod_{j=0}^{n-1} H_j(m)^{m-1} = \prod_{j=0}^{n-1} \left[ \mathbb{E}(m^{Z_j}) \right]^{m-1}, \]

\[ M_n := n \vee M, \]

\[ \vartheta_n := \vartheta(M_n) = \vartheta(n \vee M), \quad n \geq 1, \quad M \geq M_0. \]

By assumption (2.2),

\[ (2.18) \quad Q_n \leq \vartheta_n Q_n \leq \vartheta_n M_n^2, \quad n \geq 1. \]

We claim that for all integer $k \geq 3$,

\[ (2.19) \quad H_n^{(k)}(m) \leq c_1^{-1} (k-2)! M_n^{k-3} \vartheta_n Q_n, \quad n \geq 1, \]

where $c_1 := \max\{4 + \frac{c_{18}c_{20}}{m}, c_9^{1/2}, \gamma \}$, with $c_{20} := m^{m/(m-1)}(c_8^m \vee 1)(\gamma^{3m/2} \vee 1)$. Since $\vartheta_n Q_n \leq \vartheta_n M_n^2$ (see (2.18)), (2.19) will imply Theorem 2.3.

It remains to prove (2.19), which we do by induction in $k \geq 3$.

By (2.14), $H_n''(m) \leq c_9 \vartheta_n Q_n$ for $n \geq 1$. So (2.19) holds for $k = 3$ since $c_9 \leq c_1^3$.

Let $\ell \geq 4$ be an integer. Suppose (2.19) holds for all $k \in \{3, 4, \ldots, \ell - 1\}$. We need to prove (2.19) for $k = \ell$.

We first prove that the induction assumption yields that for $n \geq 1$ and $u := (u_1, \ldots, u_m) \in B_\ell$ (defined in (2.15)), we have, with $c_{20} := m^{m/(m-1)}(c_8^m \vee 1)(\gamma^{3m/2} \vee 1)$ as before,

\[ (2.20) \quad \prod_{i=1}^{m} H_n^{(u_i)}(m) \leq c_2 \ell^{\ell-2} M_n^{(\ell-\eta(u)-2)/2} \vartheta_n Q_n \prod_{i: u_i \geq 3} (u_i - 2)! \]

where $\eta(u) := \sum_{i=1}^{m} 1_{\{u_i \geq 1\}}$ is as in (2.17).

To check (2.20), let $n \geq 1$ and $u \in B_\ell$. Since $H_n(m) \leq m^{1/(m-1)} =: c_21$ (see (2.7)) and $H_n'(m) \leq \frac{1}{m-1} \mathbb{E}(m^{Z_n}) \leq \frac{c_{21}}{m-1} \leq c_{21}$, we have

\[ \prod_{i=1}^{m} H_n^{(u_i)}(m) \leq c_21 H_n''(m)^{\lambda_2(u)} \prod_{i: u_i \geq 3} H_n^{(u_i)}(m), \]
where $\lambda_2(u) := \sum_{i=1}^m 1_{(u_i=2)}$. By (2.13), we have $H''_n(m) \leq c_8 \frac{1}{2} Q_n^{1/2}$; thus with $c_2 := c_2^m \max\{c_8^m, 1\}$,

$$
\prod_{i=1}^m H''_n(u_i)(m) \leq c_{22} (\vartheta_n Q_n)^{\lambda_2(u)/2} \prod_{i:u_i \geq 3} H''_n(u_i)(m).
$$

By the induction assumption in (2.19), $H''_n(u_i)(m) \leq c_4^{u_i-1}(u_i - 2)! M_{n_i}^{u_i-3} \vartheta_n Q_n$ if $u_i \geq 3$. As such, we have

$$
\prod_{i=1}^m H''_n(u_i)(m) \leq c_{22} (\vartheta_n Q_n)^{\lambda_2(u)/2} \prod_{i:u_i \geq 3} (c_4^{u_i-1}(u_i - 2)! M_{n_i}^{u_i-3} \vartheta_n Q_n)
$$

$$
= c_{22} (\vartheta_n Q_n)^{\lambda_2(u)/2} (c_4 M_{n_i})^{\sum_{i:u_i \geq 3} (u_i - 1)} (M_{n_i}^{-2} \vartheta_n Q_n)^{\lambda_3(u)} \prod_{i:u_i \geq 3} (u_i - 2)!
$$

where $\lambda_3(u) := \sum_{i=1}^m 1_{(u_i \geq 3)}$. Note that $\sum_{i:u_i \geq 3} (u_i - 1) = \sum_{i=1}^m (u_i - 1)^+ - \lambda_2(u) = \ell - \eta(u) - \lambda_2(u) \leq \ell - 2$. So $c_4^{\sum_{i:u_i \geq 3} (u_i - 1)} \leq c_4^{\ell - 2}$ (using $c_4 > 1$). This leads to:

$$
\prod_{i=1}^m H''_n(u_i)(m) \leq c_{22} c_4^{\ell - 2} (\vartheta_n Q_n)^{\lambda_2(u) / 2 + \lambda_3(u)} M_{n}^{\ell - \eta(u) - \lambda_2(u) - 2\lambda_3(u)} \prod_{i:u_i \geq 3} (u_i - 2)!
$$

$$
= c_{22} c_4^{\ell - 2} (\vartheta_n Q_n)^{\lambda_2(u) / 2 + \lambda_3(u) - 1} M_{n}^{\ell - \eta(u) - \lambda_2(u) - 2\lambda_3(u)} \vartheta_n Q_n \prod_{i:u_i \geq 3} (u_i - 2)!
$$

Assume for the moment $\lambda_2(u) / 2 + \lambda_3(u) \geq 1$. By (2.18), $\vartheta_n Q_n \leq \gamma M_n^2$, so we have $(\vartheta_n Q_n)^{\lambda_2(u) / 2 + \lambda_3(u) - 1} \leq c_{23} M_{n}^{\lambda_2(u) / 2 + 2\lambda_3(u) - 2}$, with $c_{23} := \max\{\gamma^{3m/2}, 1\}$. Since $c_{20} = c_{22} c_{23}$, we get

$$
\prod_{i=1}^m H''_n(u_i)(m) \leq c_{20} c_4^{\ell - 2} M_{n}^{\ell - \eta(u) - 2} \vartheta_n Q_n \prod_{i:u_i \geq 3} (u_i - 2)!
$$

yielding (2.20). If, on the other hand, $\lambda_2(u) / 2 + \lambda_3(u) < 1$, then $\lambda_2(u) \leq 1$ and $\lambda_3(u) = 0$, i.e., $\max_{1 \leq i \leq m} u_i \leq 2$. This time, $\ell - \eta(u) = \sum_{i=1}^m (u_i - 1)^+ \leq 1$. The situation is very simple if we look at $\prod_{i=1}^m H''_n(u_i)(m)$ directly: at most one term among $H''_n(u_i)(m)$ is $H''_n(m)$ (which is bounded by $c_8 \vartheta_n^{1/2} Q_n^{1/2}$ as we have seen in (2.13)), while all the rest is either $H''_n(m)$ (which is bounded by $m^{1/(m-1)} =: c_{21}$) or $H''_n(m)$ (which is bounded by 1 because by (2.7), $H''_n(m) = \mathbb{E}(Z_n m^{z_n-1}) \leq \frac{m^{1/(m-1)}}{m(m-1)} \leq 1$). Hence

$$
\prod_{i=1}^m H''_n(u_i)(m) \leq c_{21}^m c_8 \vartheta_n^{1/2} Q_n^{1/2} \leq c_{24} \vartheta_n Q_n,
$$

12
with $c_{24} := c_{21}^m c_8$. [We have used $\vartheta_n Q_n \geq 1$.] This again gives (2.20) because $c_{24} \leq c_{20}$ and $c_4 > 1$.

Now that (2.20) is proved, it is painless to complete the proof of Theorem 2.3. Indeed, by (2.5),

$$sH_{n+1}(s) = [H_n(s)]^m + (s-1)[H_n(0)]^m.$$  

On both sides, we differentiate $\ell$ times with respect to $s$ (recalling that $\ell \geq 4$, so the last term on the right-hand side, being affine in $s$, makes no contribution to the derivatives), and apply the general Leibniz rule to the first term on the right-hand side; this leads to:

$$sH^{(\ell)}_{n+1}(s) + \ell H^{(\ell-1)}_{n+1}(s) = \sum_{(u_1, \ldots, u_m) \in \mathbb{Z}_{>0}^m: u_1 + \cdots + u_m = \ell} \frac{\ell!}{u_1! \cdots u_m!} \prod_{i=1}^m H_n^{(u_i)}(s) = mH_n^{(\ell)}(s)H_n(s)^{m-1} + \sum_{\mathbf{u} \in B_\ell} \frac{\ell!}{u_1! \cdots u_m!} \prod_{i=1}^m H_n^{(u_i)}(s).$$

Note that the expression on the left-hand side is at least $sH^{(\ell)}_{n+1}(s)$. We take $s = m$ to see that

$$H^{(\ell)}_{n+1}(m) - H^{(\ell)}_n(m)H_n(m)^{m-1} \leq \frac{1}{m} \sum_{\mathbf{u} \in B_\ell} \frac{\ell!}{u_1! \cdots u_m!} \prod_{i=1}^m H_n^{(u_i)}(m).$$

By (2.20), we have $\prod_{i=1}^m H_n^{(u_i)}(m) \leq c_{20} c_4^{\ell-2} M_n^{(\ell-\eta(u)-2)^+} \vartheta_n Q_n \prod_{i: u_i \geq 3}(u_i - 2)!$, where $\eta(u) := \sum_{i=1}^m 1_{\{u_i \geq 1\}}$ is as in (2.17). Hence

$$H^{(\ell)}_{n+1}(m) - H^{(\ell)}_n(m)H_n(m)^{m-1} \leq \frac{c_{20}}{m} c_4^{\ell-2} \ell! \vartheta_n Q_n \sum_{\mathbf{u} \in B_\ell} M_n^{(\ell-\eta(u)-2)^+} \prod_{i: u_i \geq 3} \frac{1}{u_i(u_i - 1)},$$

which, in view of Lemma 2.7 (applied to $y := M_n$), yields that, for $n \geq 1$,

$$H^{(\ell)}_n(m) \leq H^{(\ell)}_n(m)H_n(m)^{m-1} + c_{18} \frac{c_{20}}{m} c_4^{\ell-2} (\ell - 2)! M_n^{\ell-4} \vartheta_n Q_n.$$  

Recall that $Q_n := \prod_{j=0}^{n-1} H_j(m)^{m-1}$. Iterating this inequality, and by means of the monotonicity of $n \mapsto M_n^{\ell-4} \vartheta_n Q_n$, we get

$$H^{(\ell)}_n(m) \leq H^{(\ell)}_0(m)Q_n + \sum_{j=0}^{n-1} \frac{c_{18} c_{20}}{m} c_4^{\ell-2}(\ell - 2)! M_n^{\ell-4} \vartheta_n Q_n = \left( \frac{H^{(\ell)}_0(m)}{\vartheta_n} + n \frac{c_{18} c_{20}}{m} c_4^{\ell-2}(\ell - 2)! M_n^{\ell-4} \right) \vartheta_n Q_n.$$  

13
We use $n \leq M_n$ so that $nM_n^{\ell-4} \leq M_n^{\ell-3}$. On the other hand, $H_0^{(\ell)}(m) \leq M_n^{\ell-3}(\vartheta(M) + \ell!)$ (by assumption (2.1)), which is bounded by $M_n^{\ell-3} \vartheta_n(1 + \ell!)$. Thus

$$H_n^{(\ell)}(m) \leq (1 + \ell! + \frac{c_{18}c_{20}}{m}c_4^{\ell-2}(\ell - 2)! )M_n^{\ell-3} \vartheta_nQ_n,$$

$$\leq (1 + \ell(\ell - 1) + \frac{c_{18}c_{20}}{m}c_4^{\ell-2}) (\ell - 2)! M_n^{\ell-3} \vartheta_nQ_n.$$ 

Since $\ell \geq 4$, we have $1 + \ell(\ell - 1) \leq \ell^2 \leq 2\ell \leq 4c_4^{\ell-2}$ (because $c_4 \geq 2$), so $1 + \ell(\ell - 1) + \frac{c_{18}c_{20}}{m}c_4^{\ell-2} \leq 4c_4^{\ell-2} + \frac{c_{18}c_{20}}{m}c_4^{\ell-2} \leq c_4^{\ell-1}$ by means of the fact that $c_4 := \max\{4 + \frac{c_{18}c_{20}}{m}, c_9^{1/2}, \gamma\}$. Consequently,

$$H_n^{(\ell)}(m) \leq c_4^{\ell-1}(\ell - 2)! M_n^{\ell-3} \vartheta_nQ_n,$$

implying (2.19) for $k = \ell$, and completing the proof of Theorem 2.3.

Proof of Corollary 2.4. Only (2.3) needs proving because (2.4) will follow immediately from (2.3) and assumption (2.2).

Let $n \geq 1$ and $s \in [m, m + \frac{1}{2c_4M_n}]$, where we keep using the notation

$$M_n := n \lor M.$$

Write $H_n(u) := E(uZ_n)$ as before. Then

$$E(Z_n^2sZ_n) = sH_n'(s) + s^2H_n''(s).$$

Since $u \mapsto H_n''(u)$ is non-decreasing, we have $H_n'(s) \leq H_n'(m) + (s - m)H_n''(s)$; hence

$$E(Z_n^2sZ_n) \leq sH_n'(m) + s(s - m)H_n''(s) + s^2H_n''(s) = sH_n'(m) + s(2s - m)H_n''(s).$$

On the right-hand side, we use $H_n'(m) \leq 1$ (which has already been observed as a consequence of (2.7)), and $m \leq s \leq m + 1$ (so $2(2s - m) \leq (m + 1)(m + 2)$), to see that

$$E(Z_n^2sZ_n) \leq m + 1 + (m + 1)(m + 2)H_n''(m + \frac{1}{2c_4M_n}).$$

To bound $H_n''(m + \frac{1}{2c_4M_n})$, we recall that $Z_n$ is bounded for each $n$ (which is a consequence of the boundedness of $Z_0$), so by Taylor expansion,

$$H_n''(m + \frac{1}{2c_4M_n}) - H_n''(m) = \sum_{k=3}^{\infty} \frac{(2c_4M_n)^{-(k-2)}}{(k - 2)!}H_n^{(k)}(m).$$
By (2.19), we have $H_n^{(k)}(m) \leq c_4^{k-1}(k-2)!M_n^{k-3}\vartheta_nQ_n$ (for $k \geq 3$). Hence

$$H''_n(m + \frac{1}{2c_4M_n}) - H''_n(m) \leq \sum_{k=3}^{\infty} \frac{(2c_4M_n)^{(k-2)}c_4^{k-1}(k-2)!}{(k-2)!}M_n^{k-3}\vartheta_nQ_n = c_4\frac{\vartheta_nQ_n}{M_n}.$$ 

As such, we arrive at:

$$E(Z_n^2s^{Z_n}) \leq m + 1 + (m + 1)(m + 2)\left(H''_n(m) + c_4\frac{\vartheta_nQ_n}{M_n}\right).$$

By (2.13), $H''_n(m) \leq c_8\vartheta_n^{1/2}Q_n^{1/2}$, whereas according to assumption (2.2), $M_n \geq \frac{1}{\gamma^{1/2}}\vartheta_n^{1/2}Q_n^{1/2}$, this readily yields (2.3) (recalling that $\vartheta_nQ_n \geq 1$). Corollary 2.4 is proved. 

3 Proof of Theorem 1.2

Let $(Y_n, n \geq 0)$ be a critical system such that $E(s^{Y_0}) < \infty$ for some $s > m$. We claim that with $c_{25} := \sup_{x>0} x(\frac{s}{m})^{-x/e} \vee 1 \in [1, \infty)$, we have, for all integer $k \geq 1$,

$$(3.1) \quad E(Y_0^km^{Y_0}) \leq c_{25}^kE(s^{Y_0})k!.$$ 

Indeed, by definition,

$$x \leq c_{25}(\frac{s}{m})^{x/e},$$

for all $x > 0$. Taking to the power $k$ on both sides and with $x := \frac{e\ell}{k}$, we see that for all integers $k \geq 1$ and $\ell \geq 1$,

$$\left(\frac{e\ell}{k}\right)^k \leq c_{25}^k(\frac{s}{m})^\ell.$$ 

Since $k! \geq (\frac{e}{k})^k$ by Stirling’s formula, this yields $\ell^km^\ell \leq c_{25}^k s^\ell k!$, from which (3.1) follows.

Let $L \geq 1$ be an integer, and let $Z_0 = Z_0(M, L) := Y_01_{Y_0 \leq L}$. Then $Z_0$ is bounded, and does not depend on $M$, though we still treat it as indexed by $M$. Let us check that assumptions (2.1) and (2.2) in Definition 2.1 are satisfied.

Since $Z_0 \leq Y_0$, it follows from (3.1) that $E(Z_0^km^{Z_0}) \leq c_{26}c_{25}^k E(s^{Y_0})k!$ for $k \geq 1$, where $c_{26} = c_{26}(s) := E(s^{Y_0}) \geq 1$. Let $M \geq c_{26}c_{25}^4$. Then $M^{k-3} \geq (c_{26}c_{25}^3)^{k-3}c_{25}^{k-3} \geq c_{26}c_{25}^3c_{25}^{k-3} = c_{26}c_{25}^{k}$ for all integer $k \geq 4$, so $c_{26}c_{25}^k k! \leq M^{k-3}k!$ for $k \geq 4$. For $k = 3$, we have $c_{26}c_{25}^33! \leq c_{27} + 3!$ with $c_{27} := 6(c_{26}c_{25}^3 - 1) \vee 1 \in [1, \infty)$. As such, we see that for all integers $k \geq 3$ and $M \geq c_{26}c_{25}^4$,

$$E(Z_0^km^{Z_0}) \leq M^{k-3}(c_{27} + k!);$$

and
in words, assumption (2.1) is satisfied with \( \vartheta(M) := c_{27} \) for all \( M \geq M_0 := \lceil c_{26}c_{25} \rceil. \)

Assumption (2.2) is easily seen to be satisfied: by (2.3), \( \prod_{i=0}^{n-1} [\mathbb{E}(m_i)]^{m_i-1} \leq c_7 n^2 \) for all \( n \geq 1 \), so (2.2) holds with \( \gamma := c_7c_{27}. \)

So we are entitled to apply Theorem 2.3 to see that for \( k \geq 3 \), \( n \geq 1 \) and \( M \geq M_0 := \lceil c_{26}c_{25} \rceil, \)

\[
\mathbb{E}[Z_n(Z_n - 1) \cdots (Z_n - k + 1) m_{Z_n}^{k-1}] \leq k! c_4^k (n \vee M)^{k-1}.
\]

We take \( M := M_0. \) Recall that \( Z_0 = Z_0(M_0, L) := Y_0 1_{\{Y_0 \leq L\}}. \) Letting \( L \to \infty \), and applying the monotone convergence theorem, we get, for \( k \geq 3 \) and \( n \geq 1, \)

\[
\mathbb{E}[Y_n(Y_n - 1) \cdots (Y_n - k + 1) m_{Y_n}^{k-1}] \leq k! c_4^k (n \vee M_0)^{k-1}.
\]

This implies the desired inequality for \( k \geq 3. \) The case \( k = 1 \) has already been implicitly treated in the proof of Corollary 2.4 in Section 2. \( \mathbb{E}(Y_n m_{Y_n}^{n-1}) = \frac{\mathbb{E}(m_{Y_0})}{m(m-1)} \leq \frac{m^{1/m(m-1)}}{m(m-1)} \) (see (2.7)). The case \( k = 2 \) follows from the cases \( k = 1 \) and \( k = 3 \) by the Cauchy–Schwarz inequality.

4 Truncating the critical system

Let \((Y_n, n \geq 0)\) be a Derrida–Retaux system satisfying \( \mathbb{E}(m_{Y_0}) = (m - 1)\mathbb{E}(Y_0 m_{Y_0}) < \infty \) (so the system is critical). Recall that the system \((Y_n, n \geq 0)\) is of “finite variance” if \( \mathbb{E}(Y_0^3 m_{Y_0}) < \infty, \) and is stable if \( \mathbb{P}(Y_0 = j) \sim c_0 m^{-j} j^{-\alpha}, j \to \infty, \) for some \( c_0 > 0 \) and \( 2 < \alpha < 4 \) as in (1.4). The following theorem tells that in either case, we can define

\[
Z_0 := Y_0 1_{\{Y_0 \leq a(M)\}},
\]

for some appropriate \( a(M) \in [1, \infty) \) such that \((Z_n, n \geq 0)\) is dominable in the sense of Definition 2.1 the values of \( a(M) \) and \( \vartheta(M) \) (as defined in Definition 2.1) are also given as they are often useful in the applications.

**Theorem 4.1.** Let \((Y_n, n \geq 0)\) be a Derrida–Retaux system satisfying \( \mathbb{E}(m_{Y_0}) = (m - 1)\mathbb{E}(Y_0 m_{Y_0}) < \infty. \) If it is either of “finite variance” or stable, then there is a dominable system \((Z_n, n \geq 0)\) such that \( Z_0 \leq Y_0 \) a.s. More precisely,

(i) if the system is of “finite variance”, we can choose \( Z_0 := Y_0 1_{\{Y_0 \leq M_0(M)\}} \) where \( \zeta(M) := -\log \mathbb{E}(Y_0^3 m_{Y_0} 1_{\{Y_0 > M_0\}}) \leq \infty \) \( \# \) with \( \vartheta(M) := \max\{\mathbb{E}(Y_0^3 m_{Y_0}), 1\}; \)

\[\text{So } \lim_{M \to \infty} \zeta(M) = \infty \text{ by the “finite variance” assumption.}\]
(ii) if the system is stable, we can choose \( Z_0 := Y_0 \mathbf{1}_{\{Y_0 \leq M\}} \), with \( \vartheta(M) := c_{30} M^{1-\alpha} \), where \( c_{30} \) is the constant in (4.5) below.

For the sake of clarity, the two situations (“finite variance”, stable) are discussed in distinct parts.

### 4.1 Proof of Theorem 4.1: the “finite variance” case

Assume \( \mathbb{E}(m^{Y_0}) = (m - 1) \mathbb{E}(Y_0 m^{Y_0}) < \infty \) and \( \mathbb{E}(Y^3_0 m^{Y_0}) < \infty \). Let

\[
\zeta(M) := -\log \mathbb{E}(Y_0 m^{Y_0} \mathbf{1}_{\{Y_0 > M\}}) \leq \infty.
\]

Since \( \lim_{M \to \infty} \zeta(M) \to \infty \), we can choose \( M \) sufficiently large so that \( M \zeta(M) \geq 2 \). Let \( Z_0 := Y_0 \mathbf{1}_{\{Y_0 \leq M\zeta(M)\}} \).

We claim that assumption (2.1) is satisfied with \( \vartheta(M) := \max\{\mathbb{E}(Y^3_0 m^{Y_0}), 1\} \) and that \( Z_0 \) is bounded. For any integer \( k \geq 3 \), we write

\[
\mathbb{E}(Z_0^k m^{Z_0}) = \mathbb{E}(Y_0^k m^{Y_0} \mathbf{1}_{\{Y_0 \leq M\}}) + \mathbb{E}(Y_0^k m^{Y_0} \mathbf{1}_{\{M < Y_0 \leq M\zeta(M)\}}) 
\leq M^{k-3} \mathbb{E}(Y_0^3 m^{Y_0} \mathbf{1}_{\{Y_0 \leq M\}}) + \mathbb{E}(Y_0^k m^{Y_0} \mathbf{1}_{\{M < Y_0 \leq M\zeta(M)\}}).
\]

The first term on the right-hand side is easy to handle: we have \( \mathbb{E}(Y_0^k m^{Y_0} \mathbf{1}_{\{Y_0 \leq M\}}) \leq M^{k-3} \mathbb{E}(Y_0^3 m^{Y_0} \mathbf{1}_{\{Y_0 \leq M\}}) \leq \mathbb{E}(Y_0^3 m^{Y_0}) M^{k-3} \). In case \( \zeta(M) = \infty \), we have \( Y_0 \leq M \) a.s., so \( Z_0 \) is bounded and the second term on the right-hand side vanishes, which yields (2.1) with \( \vartheta(M) := \max\{\mathbb{E}(Y^3_0 m^{Y_0}), 1\} \).

To treat the case \( \zeta(M) < \infty \) (in which case \( Z_0 \) is obviously bounded), let us look at the second term on the right-hand side: since \( \mathbb{E}(Y_0^3 m^{Y_0} \mathbf{1}_{\{Y_0 > M\}}) = e^{-\zeta(M)} \), we have

\[
\mathbb{E}(Y_0^k m^{Y_0} \mathbf{1}_{\{M < Y_0 \leq M\zeta(M)\}}) \leq M^{k-3} \zeta(M)^k \mathbb{E}(Y_0^3 m^{Y_0} \mathbf{1}_{\{Y_0 > M\}}) = M^{k-3} \zeta(M)^k e^{-\zeta(M)}.
\]

Applying the inequality \( e^x \geq \frac{x^{k-3}}{(k-3)!} \) (for \( x \geq 0 \) and \( k \geq 3 \)) to \( x := \zeta(M) \) yields (2.1) again with \( \vartheta(M) := \max\{\mathbb{E}(Y^3_0 m^{Y_0}), 1\} \).

Consequently, regardless of whether \( \zeta(M) \) is finite or infinite, \( Z_0 \) is bounded, and assumption (2.1) is satisfied with \( \vartheta(M) := \max\{\mathbb{E}(Y^3_0 m^{Y_0}), 1\} \). Note that \( \vartheta(M) \) does not depend on \( M \).

Assumption (2.2) is also satisfied: by (2.8), \( \prod_{i=0}^{n-1} \mathbb{E}(m^{Y_i}) \leq c_7 n^2 \) for all \( n \geq 1 \); since \( \mathbb{E}(m^{Y_i}) \leq \mathbb{E}(m^{|Y_i|}) \) for all \( i \geq 0 \), (2.2) is satisfied with \( \gamma := c_7 \max\{\mathbb{E}(Y^3_0 m^{Y_0}), 1\} \). □
4.2 Proof of Theorem 4.1: the stable case

We start with a simple inequality.

**Lemma 4.2.** Let \((X_n, n \geq 0)\) be a Derrida–Retaux system satisfying \((m - 1) \mathbb{E}(X_0 m^{X_0}) < \infty\). Then

\[
\prod_{i=0}^{\infty} \left[ \mathbb{E}(m^{X_i}) \right]^{m-1} \leq \frac{1}{\mathbb{E}(m^{X_0}) - (m - 1) \mathbb{E}(X_0 m^{X_0})}.
\]

**Proof.** For the moment, let \((X_n, n \geq 0)\) be an arbitrary Derrida–Retaux system satisfying \(\mathbb{E}(X_0 m^{X_0}) < \infty\), and such that \(\mathbb{E}(m^{X_0}) \neq (m - 1) \mathbb{E}(X_0 m^{X_0})\). By (2.6),

\[
\prod_{i=0}^{n-1} \left[ \mathbb{E}(m^{X_i}) \right]^{m-1} = \frac{\mathbb{E}(m^{X_n}) - (m - 1) \mathbb{E}(X_n m^{X_n})}{\mathbb{E}(m^{X_0}) - (m - 1) \mathbb{E}(X_0 m^{X_0})}.
\]

For the nominator in (4.1), we observe that \(\mathbb{E}(m^{X_n}) - (m - 1) \mathbb{E}(X_n m^{X_n}) = \mathbb{E}[(1 - (m - 1)X_n)m^{X_n}] \leq \mathbb{P}(X_n = 0) \leq 1\). If we assume \((m - 1) \mathbb{E}(X_0 m^{X_0}) < \mathbb{E}(m^{X_0})\), then the denominator in (4.1) is positive; the lemma follows immediately from the monotone convergence theorem by letting \(n \to \infty\). \(\square\)

We now proceed to the proof of Theorem 4.1 for stable systems. Assume \(\mathbb{E}(Y_0) = \mathbb{E}(Y_0 m^{Y_0}) < \infty\) and \(\mathbb{P}(Y_0 = j) \sim c_0 m^{-j} j^{-\alpha}, j \to \infty\), for some \(c_0 > 0\) and \(2 < \alpha < 4\) as in (1.4). This yields the existence of constants \(c_{28} \geq c_{29} > 0\) and an integer \(j_0 \geq 1\), all depending on \(m\) and on the law of \(Y_0\), such that

\[
\mathbb{P}(Y_0 = j) \leq c_{28} m^{-j} j^{-\alpha}, \quad j \geq 1,
\]

\[
\mathbb{P}(Y_0 = j) \geq c_{29} m^{-j} j^{-\alpha}, \quad j \geq j_0,
\]

Let \(M \geq j_0\) be an integer and let

\[
Z_0 := Y_0 \mathbf{1}_{\{Y_0 \leq M\}};
\]

which is a bounded random variable. For integer \(k \geq 3\), we write

\[
\mathbb{E}(Z_0^k m^{Z_0}) = \mathbb{E}(Y_0^k m^{Y_0} \mathbf{1}_{\{Y_0 \leq M\}}) \leq M^{k-3} \mathbb{E}(Y_0^3 m^{Y_0} \mathbf{1}_{\{Y_0 \leq M\}}),
\]
so by (4.2), we have
\[
E(Z_0^km^{Z_0}) \leq c_{28}M^{k-3} \sum_{j=1}^{M} j^{3-\alpha} \leq c_{28}M^{k-3} \int_{0}^{M+1} x^{3-\alpha} \, dx.
\]
\[
= c_{28}M^{k-3} \frac{(M+1)^{4-\alpha}}{4-\alpha} \leq c_{28}M^{k-3} \frac{2^{4-\alpha} M^{4-\alpha}}{4-\alpha}.
\]
As such, assumption (2.1) is satisfied with
\[
(4.5) \quad \vartheta(M) := c_{30} M^{4-\alpha},
\]
where \(c_{30} := \max\{c_{28} \frac{2^{4-\alpha}}{4-\alpha}, 1\}\). In particular, \(M \mapsto \vartheta(M)\) is non-decreasing.

It remains to check assumption (2.2), which in this case states that for some constant \(c_{31} > 0\),
\[
(4.6) \quad \prod_{i=0}^{n-1} [E(m^{Z_i})]^{m-1} \leq c_{31} (n \vee M)^{\alpha-2}, \quad M \geq j_0, \; n \geq 1.
\]

Since \(E(m^{Z_i}) \leq E(m^{Y_i})\), there is nothing to prove if \(n \leq j_0\): it suffices to take \(c_{31}\) such that \(c_{31} \geq \prod_{i=0}^{j_0-1} [E(m^{Y_i})]^{m-1}\). Let us assume \(n > j_0\). We write
\[
\prod_{i=0}^{n-1} [E(m^{Z_i})]^{m-1} \leq \prod_{i=0}^{\infty} [E(m^{Z_i})]^{m-1},
\]
so by Lemma 4.2
\[
\prod_{i=0}^{n-1} [E(m^{Z_i})]^{m-1} \leq \frac{1}{E(m^{Z_0}) - (m-1)E(Z_0^km^{Z_0})}.
\]
By definition, \(Z_0 = Y_0 1_{(Y_0 \leq M)}\), and by assumption, \(E(m^{Y_0}) = (m-1)E(Y_0 m^{Y_0})\). So
\[
\begin{align*}
E(m^{Z_0}) - (m-1)E(Z_0^km^{Z_0}) &= E[((m-1)Y_0 - 1) m^{Y_0} 1_{(Y_0 > M)}] + P(Y_0 > M) \\
&\geq E[((m-1)Y_0 - 1) m^{Y_0} 1_{(Y_0 > M)}],
\end{align*}
\]
which, by (4.3), is \(\geq c_{29} \sum_{j=M+1}^{\infty} ((m-1)j - 1)j^{-\alpha} \geq \frac{c_{29}}{M^{\alpha-2}}\) for some constant \(c_{32} > 0\) and all \(M \geq j_0\). Consequently,
\[
\prod_{i=0}^{n-1} [E(m^{Z_i})]^{m-1} \leq \frac{M^{\alpha-2}}{c_{32}},
\]
which is bounded by \(\frac{(n \vee M)^{\alpha-2}}{c_{32}}\) as \(\alpha > 2\). This yields (4.6). \(\square\)
Remark 4.3. For further use, let us note that in the stable case, \( \vartheta(M) := c_30 M^{4-\alpha} \) for all \( M \geq M_0 \) (by (4.5)), whereas \( \prod_{n=0}^{n-1} [E(m^2_n)]^{m-1} \leq \frac{M^{n-2}}{c_{32}} \) (by (4.8)), so inequality (2.3) in Corollary 2.4 implies that for \( M \geq M_0, n \geq 1 \) and \( v := m + \frac{1}{2\gamma_4(n \vee M)} \) (\( c_4 \geq 1 \) being as before the constant in Theorem 2.3),

\[
E(Z_n^2 v^2) \leq c_{33} (n \vee M)^{(4-\alpha)/2} M^{(\alpha-2)/2},
\]

where \( c_{33} := \frac{c_5 c_{30}^{1/2}}{c_{32}^{1/2}}. \)

5 Proof of Theorem 1.1

Let \( (Y_n, n \geq 0) \) be a Derrida–Retaux system such that \( E(m Y_0) = (m - 1) E(m Y_0) \leq \infty. \) We assume that \( P(Y_0 = j) \sim c_0 m^{-j} j^{-\alpha}, j \to \infty, \) for some \( c_0 > 0 \) and \( 2 < \alpha < 4 \) as in 1.4.

5.1 Upper bound

We start with a lemma.

Lemma 5.1. Let \( (X_n, n \geq 0) \) be a Derrida–Retaux system satisfying \( E(m X_0) < \infty. \) There exists a constant \( c_{34} > 0, \) depending only on \( m, \) such that \( (\frac{n^2}{2} := \infty) \)

\[
\prod_{i=0}^{n-1} [E(m^2 X_i)]^{m-1} \leq c_{34} \frac{n^2}{E(X_0^3 m X_0 1_{\{2 \leq X_0 \leq 3n\}})}, \quad n \geq 1.
\]

Proof. The lemma was known in various forms. Recall from [4] that for \( s \in (\frac{m}{2}, m), \)

\[
\prod_{i=0}^{n-1} [E(m^2 X_i)]^{m-1} \leq \left( \frac{m}{2s - m} \right)^n \frac{1}{\Delta_0(s)},
\]

where

\[
\Delta_0(s) := \sum_{k=1}^{\infty} m^k ((m - 1)k - 1)(1 - (k + 1)x^k + kx^{k+1})P(X_0 = k),
\]

with \( x := \frac{s}{m} \in (\frac{1}{2}, 1). \) We now reproduce some elementary computations from [1]. For \( k \geq 1, \) we have \( x^k \leq e^{-(1-x^k)}, \) so \( 1 - (1+k)x^k + kx^{1+k} \geq 1 - (1+u) e^{-u}, \) where \( u := (1-x)k > 0. \) Since \( 1 - (1+v) e^{-v} \geq 1 - \frac{2}{e} \) for \( v \geq 1 \)(because \( v \mapsto 1 - (1+v) e^{-v} \) is increasing on \( (0, \infty) \))
and \(1-(1+v)e^{-v} \geq \frac{v^2}{2e}\) for \(v \in (0, 1]\) (because \(v \mapsto 1-(1+v)e^{-v} - \frac{v^2}{2e}\) is increasing on \((0, 1)\)], we get, for \(k \geq 1,\)

\[
1-(k+1)x^k + kx^{k+1} \geq c_{35} \min\{(1-x)^2k^2, 1\},
\]

where \(c_{35} := \min\{1 - \frac{2}{e}, \frac{1}{2e}\} > 0\). Let \(n \geq 1\). We take \(s = s_n := (1 - \frac{1}{3n})m\) (so \(x = 1 - \frac{1}{3n}\)), to see that

\[
\Delta_0(s_n) \geq c_{35} \sum_{k=1}^{\infty} m^k((m-1)k-1) \min\{\frac{k^2}{(3n)^2}, 1\} \mathbb{P}(X_0 = k) \geq \frac{c_{35}}{(3n)^2} \sum_{k=1}^{3n} k^2 m^k((m-1)k-1) \mathbb{P}(X_0 = k).
\]

We use \((m-1)k-1 \geq \frac{m-1}{2} k\) for \(k \geq 2\), so that with \(c_{36} := c_{35} \frac{m-1}{18} > 0\)

\[
\Delta_0(s_n) \geq c_{36} \frac{3n}{n^2} \sum_{k=2}^{3n} k^3 m^k \mathbb{P}(X_0 = k) = \frac{c_{36}}{n^2} \mathbb{E}(X_0^3 m X_0 1_{\{2 \leq X_0 \leq 3n\}}).
\]

This, in view of (5.1) (applied to \(s = s_n\)), yields the lemma. \(\square\)

**Proof of the upper bound in Theorem 1.1.** By Lemma 5.1, for all \(n \geq 1,\)

\[
\prod_{i=0}^{n-1}[\mathbb{E}(mY_i)]^{m-1} \leq c_{34} \frac{n^2}{\mathbb{E}(Y_0^3 m X_0 1_{\{2 \leq Y_0 \leq 3n\}})}.
\]

Since \(\mathbb{P}(Y_0 = j) \sim c_0 m^{-j} j^{-\alpha}, j \to \infty,\) and \(2 < \alpha < 4,\) we have \(\mathbb{E}(Y_0^3 m Y_0 1_{\{2 \leq Y_0 \leq 3n\}}) \geq c_{37} n^{4-\alpha}\) for some constant \(c_{37} > 0\) and all sufficiently large \(n\); this implies the upper bound in Theorem 1.1. \(\square\)

### 5.2 Lower bound in Theorem 1.1

The proof of the lower bound in Theorem 1.1 needs some preparation.

**Lemma 5.2.** There exists a constant \(c_{38} > 0\) such that for all sufficiently large integer \(n,\) say \(n \geq n_0,\) we have

\[
either \prod_{i \in (\frac{2}{3}, n] \cap \mathbb{Z}} [\mathbb{E}(mY_i)]^{m-1} \geq 8, \quad or \quad \prod_{i=0}^{n}[\mathbb{E}(mY_i)]^{m-1} \geq c_{38} n^{\alpha-2}.
\]
Proof. Let $M_0 \geq 1$ and $c_{33} \geq 1$ be the constants in (4.9) in Remark 4.3. Let $c_{39} := [120(m - 1)c_{33}]^{2/(\alpha - 2)} \geq 120$. Let $n \geq \lceil 3c_{39} M_0 \rceil := n_0$ be an integer, and let $M = M(n) := \lceil \frac{n}{c_{39}} \rceil$. So $\frac{n}{2c_{39}} \leq M \leq \frac{n}{120}$. Let $u_n := m - \frac{cn}{n}$, where $c_{40} := 30m$. We can enlarge the value of $M_0$ if necessary to ensure that $u_n > \frac{m}{2}$.

We discuss on two possible situations, each leading to one of the inequalities stated in the lemma.

First situation: $E[(1 - (m - 1)Y)u_{n}] < \frac{1}{2}$ for all $i \in (\frac{n}{2}, n] \cap \mathbb{Z}$. In this situation, we have, for all $i \in (\frac{n}{2}, n] \cap \mathbb{Z}$, $(m - 1)E(Y)u_{n} \geq E(u_{n}^{Y_{i}}) - \frac{1}{2} \geq \frac{1}{2}$, thus

$$E(Y_i u_{n}^{Y_{i}}) \geq \frac{1}{2(m - 1)}.$$ 

Consider the function $s \mapsto f_i(s) := E(s^{Y_{i}})$, $s \in [0, m]$. We have

$$E(m^{Y_{i}}) = f_i(m) \geq f_i(u_{n}) + (m - u_{n})f'_i(u_{n}) \geq 1 + (m - u_{n})f'_i(u_{n}).$$

Since $m - u_{n} = \frac{30m}{n}$ and $f'_i(u_{n}) = \frac{1}{u_{n}} E(Y_i u_{n}^{Y_{i}}) \geq \frac{1}{m} \frac{1}{2(m - 1)} = \frac{1}{2m(m - 1)}$, we get, for all $i \in (\frac{n}{2}, n] \cap \mathbb{Z},$

$$E(m^{Y_{i}}) \geq 1 + \frac{15}{(m - 1)n}.$$ 

Consequently,

$$\prod_{i \in (\frac{n}{2}, n] \cap \mathbb{Z}} [E(m^{Y_{i}})]^{m-1} \geq \left( 1 + \frac{15}{(m - 1)n} \right)^{(m-1)n/2} \geq 8.$$ 

Second (and last) situation: $E[(1 - (m - 1)Y_{\ell})u_{n}^{Y_{\ell}}] \geq \frac{1}{2}$ for some $\ell = \ell(n) \in (\frac{n}{2}, n] \cap \mathbb{Z}$. We will be working with this particular $\ell$ in the rest of the proof. Let $Z_0 := Y_0 1_{\{Y_0 \leq M\}}$ as in (4.4). Let $(Z_n, n \geq 0)$ be a Derrida–Retaux system whose initial distribution is given by $Z_0$.

Since $u_{n} \in [1, m]$, the function $x \mapsto (1 - (m - 1)x)u_{n}^{x}$ is decreasing on $[0, \infty)$, so

$$E[(1 - (m - 1)Z_{\ell})u_{n}^{Z_{\ell}}] \geq E[(1 - (m - 1)Y_{\ell})u_{n}^{Y_{\ell}}] \geq \frac{1}{2}.$$ 

Consider the function

$$\varphi(s) := E[(1 - (m - 1)Z_{\ell})s^{Z_{\ell}}], \quad s \geq 0,$$
which is well-defined because $Z_\ell$ is bounded. We have just proved that $\varphi(u_n) \geq \frac{1}{2}$. Let $v_n := m + \frac{1}{2c_4n}$, where $c_4 \geq 1$ is the constant in Theorem 2.3. Since $v_n > m$, we have, by concavity of $\varphi(\cdot)$,

$$\varphi(v_n) \geq \varphi(m) + (v_n - m)\varphi'(v_n).$$

By assumption, the system $(Y_n, n \geq 0)$ is critical, so $(Z_n, n \geq 0)$ is subcritical (or critical in case $Y_0 \leq M$ a.s.), which implies that $(m - 1)\mathbb{E}(Z_\ell m^{Z_\ell}) \leq \mathbb{E}(m^{Z_\ell})$. This means that $\varphi(m) \geq 0$. On the other hand, $v_n - m = \frac{1}{2c_4n}$, whereas

$$\varphi'(v_n) = \mathbb{E}[(1 - (m - 1)Z_\ell)Z_\ell v_n^{Z_\ell - 1}] \geq -(m - 1)\mathbb{E}(Z_\ell^2 v_n^{Z_\ell - 1}),$$

which is $= -\frac{m-1}{v_n} \mathbb{E}(Z_\ell^2 v_n^2) \geq -\frac{m-1}{m} \mathbb{E}(Z_\ell^2 v_n^2)$. Assembling these pieces together yields that

$$\varphi(v_n) \geq -\frac{1}{2c_4n} \frac{m-1}{m} \mathbb{E}(Z_\ell^2 v_n^2).$$

Since $v_n = m + \frac{1}{2c_4n} \leq m + \frac{1}{2c_4} = m + \frac{1}{2c_4(\ell M)}$ (to obtain the last equality, we have used the fact that $\ell \geq \frac{n}{2} \geq M$), we are entitled to apply inequality (4.19) in Remark 4.3 to see that

$$\mathbb{E}(Z_\ell^2 v_n^Z) \leq c_{33}(\ell \vee M)^{(4-\alpha)/2} M^{(\alpha-2)/2}. $$

Since $(\ell \vee M) = \ell \leq n$ and $M \leq \frac{n}{c_{39}}$, this yields

$$\mathbb{E}(Z_\ell^2 v_n^Z) \leq c_{33}n^{(4-\alpha)/2} \left(\frac{n}{c_{39}}\right)^{(\alpha-2)/2} = \frac{c_{33}}{c_{39}^{(\alpha-2)/2}} n.$$

Accordingly,

$$\varphi(v_n) \geq -\frac{1}{2c_4n} \frac{m-1}{m} \frac{c_{33}}{c_{39}^{(\alpha-2)/2}} n = -\frac{c_{41}}{c_{39}^{(\alpha-2)/2}},$$

where $c_{41} := \frac{c_{33} m-1}{2c_4 m}$. On the other hand, we have $\varphi(u_n) \geq \frac{1}{2}$ (see (5.2)). Since $m = \beta u_n + (1 - \beta) v_n$ with $\beta := \frac{1}{1+2c_4 c_{40} \in (0, 1)$, it follows from concavity of $\varphi(\cdot)$ that

$$\varphi(m) \geq \beta \varphi(u_n) + (1 - \beta) \varphi(v_n) \geq \frac{\beta}{2} - (1 - \beta) \frac{c_{41}}{c_{39}^{(\alpha-2)/2}} = \frac{1 - \frac{4c_4 c_{40} c_{41}}{c_{39}^{(\alpha-2)/2}}}{2(1+2c_4 c_{40})}.$$

Our choice of the constant $c_{39}$ ensures $\frac{4c_4 c_{40} c_{41}}{c_{39}^{(\alpha-2)/2}} = \frac{1}{2}$; hence $\varphi(m) \geq \frac{1}{4(1+2c_4 c_{40})} =: c_{42}$, i.e.,

$$\mathbb{E}(m^{Z_\ell}) - (m - 1)\mathbb{E}(Z_\ell m^{Z_\ell}) \geq c_{42}. $$

23
Recall from (4.1) that
\[
\prod_{i=0}^{\ell-1} \left[ E(mZ_i) \right]^{m-1} = \frac{E(mZ_i) - (m - 1)E(Z_i mZ_i)}{E(Z_0) - (m - 1)E(Z_0 mZ_0)}.
\]

On the right-hand side, the numerator is at least \(c_{42}\), whereas the denominator has already appeared in (4.7):
\[
E(mZ_0) - (m - 1)E(Z_0 mZ_0) = E[(m - 1)Y_0 - 1] mY_0 1_{(Y_0 > M)} + P(Y_0 > M) \leq (m - 1)E(Y_0 mY_0 1_{(Y_0 > M)}).
\]

Our assumption \(P(Y_0 = j) \sim c_0 m^{-j} j^{-\alpha}\) (for \(j \to \infty\)) yields that
\[
E(Y_0 mY_0 1_{(Y_0 > M)}) \leq c_{43} M^{-\alpha + 2},
\]
for some constant \(c_{43} > 0\) depending on the law of \(Y_0\). As a consequence,
\[
\prod_{i=0}^{\ell-1} \left[ E(mZ_i) \right]^{m-1} \geq \frac{c_{42}}{c_{43} M^{-\alpha + 2}} = c_{44} M^{\alpha - 2},
\]
where \(c_{44} := \frac{c_{42}}{c_{43}}\). This implies that
\[
\prod_{i=0}^{n-1} \left[ E(mY_i) \right]^{m-1} \geq \prod_{i=0}^{\ell-1} \left[ E(mZ_i) \right]^{m-1} \geq \prod_{i=0}^{\ell-1} \left[ E(mZ_i) \right]^{m-1} \geq c_{44} M^{\alpha - 2}.
\]

Since \(M \geq \frac{n}{2c_{39}}\), this completes the proof of the lemma.

We have all the ingredients for the proof of the lower bound in Theorem 1.1.

**Proof of the lower bound in Theorem 1.1** It follows quite easily from Lemma 5.2. Indeed, let \(n_0\) be the integer in Lemma 5.2 and let \(n \geq 2n_0\). According to Lemma 5.2 there are two possibilities:
- either we have \(\prod_{i \in [n/2i+1, n/2j] \cap \mathbb{Z} [E(mY_i)]^{m-1} \geq 8\) for all \(0 \leq j \leq \left\lfloor \frac{\log(n/n_0)}{\log 2}\right\rfloor\), in which case we have
  \[
  \prod_{i=0}^{n} [E(mY_i)]^{m-1} \geq 8^{\left\lfloor \frac{\log(n/n_0)}{\log 2}\right\rfloor} \geq 8^{\frac{\log(n/n_0)}{\log 2} - 1} = \frac{n^3}{8n_0^3},
  \]
  which yields the lower bound in Theorem 1.1 because \(3 > \alpha - 2\).
- or there exists an integer \(0 \leq j^* = j^*(n) \leq \left\lfloor \frac{\log(n/n_0)}{\log 2}\right\rfloor\) such that \(\prod_{i=0}^{n/2j^*} [E(mY_i)]^{m-1} \geq c_{38} \left(\frac{n}{2j^*}\right)^{\alpha - 2}\) and that \(\prod_{i \in [n/2j+1, n/2j] \cap \mathbb{Z} [E(mY_i)]^{m-1} \geq 8\) for all non-negative integer \(j < j^*\) (if
there is any); in this case, we have (recalling notation: $\prod_{\emptyset} := 1$)

\[
\prod_{i=0}^{n}[E(m_{Y_i})]^{m-1} \geq \prod_{i=0}^{n/2^j} [E(m_{Y_i})]^{m-1} \times \prod_{j=0}^{j^*-1} \prod_{i \in (n/2^j+1, n/2^j] \cap \mathbb{Z}} [E(m_{Y_i})]^{m-1}
\]

\[
\geq c_{38} \left( \frac{n}{2^j} \right)^{a-2} \times 8^j,
\]

which implies that

\[
\prod_{i=0}^{n}[E(m_{Y_i})]^{m-1} \geq c_{38} n^{a-2} 2^{(5-a)j^*} \geq c_{38} n^{a-2}.
\]

In both situations, the lower bound in Theorem 1.1 is valid.

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