Thermalization of a quantum Newton’s cradle in a one-dimensional quasicondensate

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We study the nonequilibrium dynamics of the quantum Newton’s cradle in a one-dimensional (1D) Bose gas in the weakly interacting quasicondensate regime. This is the opposite regime to the original quantum Newton’s cradle experiment of Kinoshita et al. [Nature 440, 900 (2006)], which was realised in the strongly interacting 1D Bose gas. Using finite temperature c-field methods, we calculate the characteristic relaxation rates to the final equilibrium state. Hence, we identify the different dynamical regimes of the system in the parameter space that characterizes the strength of interatomic interactions, the initial temperature, and the magnitude of the Bragg momentum used to initiate the collisional oscillations of the cradle. In all parameter regimes, we find that the system relaxes to a final equilibrium state for which the momentum distribution is consistent with a thermal distribution. For sufficiently large initial Bragg momentum, the system can undergo hundreds of repeated collisional oscillations before reaching the final thermal equilibrium. The corresponding thermalization timescales can reach tens of seconds, which is only an order of magnitude smaller than in the strongly interacting regime, and at least three orders of magnitude larger than the characteristic dephasing timescale in a related experiment of Hofferberth et al. [Nature 449, 324 (2007)] on phase relaxation between coherently split quasicondensates.

I. INTRODUCTION

The question of how isolated quantum systems relax after a disturbance [1, 2] and reach a seemingly thermal equilibrium state [3] has been a topic of much interest in recent years (for reviews, see [4–8]). The experimental realisation of a quantum Newton’s cradle [9]—a strongly interacting one-dimensional (1D) Bose gas undergoing repeated collisions in a harmonic trap—was one of the first demonstrations of an isolated quantum system that did not thermalize over observable time scales, which were on the order of tens of seconds, corresponding to thousands of collisions. It was conjectured at the time that the lack of thermalization was due to the fact that the system is well approximated by an integrable Lieb-Liniger model [10]. Such integrable systems are now understood to generally relax to non-thermal states described by a generalised Gibbs ensemble [2, 5, 11–14], which puts further constraints on the system dynamics compared to those present in generic (nonintegrable) systems described by conventional ensembles of statistical mechanics.

On the other hand, an experiment performed by Hofferberth et al. [15], studying the relative phase dynamics of a coherently split 1D quasicondensate [16], i.e., a phase fluctuating 1D Bose gas in the weakly interacting regime, have initially suggested that these dynamics relax over millisecond time scales. This is orders of magnitude smaller than in the quantum Newton’s cradle, despite the fact that the strictly 1D quasicondensates can be well approximated by the same integrable Lieb-Liniger model, with the proviso that the harmonic trapping potential breaks the exact integrability (just as it does in the strongly interacting Newton’s cradle case). Hence, it was a question of interest whether the stark difference in characteristic relaxation times in these two experiments was caused by the proximity of the system to the integrable regime, or the specifics of the dynamical scenario being considered.

Since these early experiments, there has been significant progress in the theoretical understanding of these seemingly contradictory results. Nevertheless, the question of how long would it take a strictly 1D quasicondensate, i.e., a weakly- rather than strongly-interacting 1D Bose gas, to relax in the same quantum Newton’s cradle setting of Ref. [9] remained unanswered. In this paper we address this question by performing a numerical experiment of the quantum Newton’s cradle setup in a finite-temperature 1D quasicondensate in the quantum degenerate regime. We find that such a quasicondensate can take several seconds to thermalize. While this is somewhat shorter compared to the equivalent experiment in the strongly interacting regime, the approach to thermal equilibrium is at least three orders of magnitude slower than the relaxation of phase dynamics in the split quasicondensate case. This is the key finding of this paper.

To clarify the context of our work, we summarize the relevant aspects of theoretical understanding developed since the early experiments. Firstly, fast relaxation to a seemingly thermal state in the experiments with split quasicondensates can be understood as a result of purely phase dynamics and the phenomenon of prethermalization: the system dephases to a prethermalized state on a short time scale, where certain observables (in this case interference contrast [17]) of a nonequilibrium, long-lived transient state become indistinguishable from those of a thermal equilibrium state [18–20]. Relaxation to a final equilibrium state in these experiments takes much longer and depends on the evolution of the quasicondensate density [18, 20]: the splitting of the initial equi-
librium quasi-condensate here creates two initially phase-coherent samples, which are in a nonequilibrium (excited) state of the underlying trapping potential, and while the relative phase relaxes rather quickly, the density profiles of the two quasi-condensates start to undergo breathing-mode oscillations which take much longer to damp out. A single 1D quasi-condensate undergoing damping of this kind, excited by an alternative but equivalent [21] confinement quench, has been observed experimentally in Ref. [22] and was also evident in the theoretical study of Ref. [23]. However, a detailed understanding of such damping is yet to be developed. While the present work also does not analyse this situation, our numerical study of the quantum Newton’s cradle in a 1D quasi-condensate regime represents an example of relaxation scenario in which the approach to the final equilibrium involves the evolution of both the phase and density.

Secondly, while the long relaxation times of the original Newton’s cradle experiment can indeed be understood by the proximity of the system to integrability, the said proximity here is two-fold. The first is the proximity to a uniform 1D Bose gas that can be described by the integrable Lieb-Liniger model [10] of bosons interacting via a pairwise contact interaction. Even though the harmonic trap (in which the two momentum components of the gas repeatedly collide) breaks this integrability, the breakdown can be considered weak because the longitudinal trapping is relatively weak. Accordingly, the apparent lack of relaxation to a conventional thermal state (i.e., one described by a conventional grand-canonical ensemble) can be understood as a quantum analog of the Kolmogorov-Arnold-Moser (KAM) theorem [5, 24] and the approximate applicability of the generalised Gibbs ensemble. The second aspect of the proximity to the integrability point is that the experiments of Ref. [9] were performed in the strongly interacting regime, which is closer to the Tonks-Girardeau gas of impenetrable (or hard-core) bosons [25]. The strictly Tonks-Girardeau gas corresponds to the limit of infinitely strong interactions of the Lieb-Liniger model and is integrable even in a harmonic trap [26–29] (theoretical modelling of the quantum Newton’s cradle in the Tonks-Girardeau regime can be found in Refs. [29, 30]). Therefore a harmonically trapped 1D Bose gas with large but finite interaction strength (which is the case in practice) is closer to being integrable in this sense compared to a similarly trapped but weakly interacting 1D quasi-condensate. Accordingly, such a gas might be expected to relax on longer time scales than a quasi-condensate, which is what our results demonstrate.

Finally, additional practical matters can compromise integrability and affect relaxation pathways and time scales. Such factors include deviations of interparticle interactions from point-like character, deviations of the trapping potential from being purely harmonic or truly 1D [9, 31], and virtual transverse excitations that can lead to effective three-body collisional relaxation. For example, Refs. [32, 33] show that such virtual three-body collisions result in dramatically different relaxation rates in the weakly- and strongly-interacting quasi-1D Bose gases, hence providing an alternative explanation of the comparatively shorter relaxation times in the experiments with split quasi-condensates. In light of this situation, the question of how 1D quasi-condensates in a quantum Newton’s cradle setup relax in a harmonic trap is a different scenario to those considered previously, and is yet to be addressed theoretically. It is also within reach of being addressed experimentally [34], hence the motivation for the present work.

The paper is organised as follows. In Sec. II, we describe the c-field method used in the numerical simulations of the quantum Newton’s cradle in a 1D quasi-condensate, and parametrize the system in terms of the relevant dimensionless parameters. In Secs. III A and III B, we present examples of relaxation dynamics in three typical regimes and discuss simple physical considerations required for observing persistent collisional oscillations in certain parameter regimes. In Sec. III C, we characterise the system dynamics in terms of relaxation rates and identify the three different dynamical regimes of the system in the relevant space of dimensionless parameters. In Sec. IV, we summarise our findings and discuss their implications.

II. THE C-FIELD METHOD FOR SIMULATING THE QUANTUM NEWTON’S CRADLE

The c-field (or classical field) method [35–37] is a widely used approach to studying equilibrium and nonequilibrium properties of finite temperature Bose gases, including a weakly interacting 1D Bose gas in the quasi-condensate regime [23]. The essence of the method is to treat the low-energy coherent band of the quantum Bose field as a single classical field \( \psi_c(x,t) \), thus ignoring all quantum fluctuations and the discrete nature of the particles that make up the field. In this approach, finite-temperature equilibrium configurations are sampled by integrating the stochastic projected Gross-Pitaevskii equation (SPGPE) for sufficiently long times that the memory of the initial trial state is lost. These are then used to establish the initial state of a harmonically trapped 1D quasi-condensate before setting it up into the dynamical Newton’s cradle regime. The subsequent real-time dynamical evolution, which is set by applying a Bragg pulse that initiates the Newton’s cradle collisions, is then described by evolving an ensemble of individual realisations from the SPGPE according to the projected Gross-Pitaevskii equation (GPE).

A. Parameterization of the model

To write down the c-field equations using an efficient parametrization, we recall from the works of Lieb-Liniger [10] and Yang-Yang [38] that the properties of a uniform
1D Bose gas, with linear (1D) density $\rho$ and temperature $T$, can be completely characterized by two dimensionless parameters [39, 40]: the dimensionless interaction strength $\gamma = mg/(\hbar^2 \rho)$ and dimensionless temperature $T = 2\hbar^2 k_B T/(mg^2)$, where $m$ is the mass of the particles, $k_B$ is the Boltzmann constant, and $g$ is the 1D interaction strength, which can be expressed via the 3D scattering length $a$ and the frequency of the transverse harmonic potential $\omega_\perp$ as $g \approx 2\hbar \omega_\perp a$ far away from confinement induced resonances [41].

In this paper, we restrict ourselves to the study of a weakly interacting ($\gamma \ll 1$) 1D Bose gas in a phase-fluctuating quasicondensate regime. This corresponds to a quantum degenerate gas in which the interparticle interactions play a nonperturbative role. The observable that we use to characterise the relaxation dynamics is the momentum distribution of the gas. Accordingly, as long as the (low-momentum) bulk of the momentum distribution is concerned, the quasicondensate regime is defined by the condition $g \rho e^{-2\pi/\sqrt{T}} \ll k_B T \ll \sqrt{\hbar^2 \rho^2}/m$ [42, 43]. At these temperatures, the highly-occupied, low momentum modes are dominated by thermal fluctuations rather than vacuum fluctuations, ensuring the applicability of the c-field method. Vacuum fluctuations become important at temperatures $k_B T \ll g \rho e^{-2\pi/\sqrt{T}}$, which are exponentially small in the regime of weak interactions, $\gamma \ll 1$, and are beyond the reach of current ultracold atom experiments. Thus, for all practical purposes, the quasicondensate regime corresponds to completely characterise the system. In this nonuniform ultracold atom experiments, the quasicondensate regime is defined by the condition $g \rho e^{-2\pi/\sqrt{T}} \ll k_B T \ll \sqrt{\hbar^2 \rho^2}/m$ [44]. Moreover, as discussed in Refs. [23, 35, 42], in the c-field approach the system (a uniform quasicondensate) can be completely characterised by a single dimensionless parameter, $\chi = \frac{1}{2}\gamma^{3/2}T$, which is a combination of $\gamma$ and $T$ and which satisfies $\chi \ll 1$ (according to $k_B T \ll \sqrt{\hbar^2 \rho^2}/m$).

For a harmonically trapped (nonuniform) 1D quasicondensate with longitudinal potential $V(x) = \frac{1}{2}m\omega^2 x^2$, one needs an additional parameter—the trap frequency $\omega$—to completely characterise the system. In this nonuniform case, the linear density becomes position dependent and describes the density profile of the gas $\rho(x)$. Accordingly, the dimensionless interaction strength also becomes position dependent, $\gamma(x) = mg/(\hbar^2 \rho(x))$, while the dimensionless temperature $T$ continues to serve as the global equilibrium temperature of the system. For a given chemical potential $\mu$, which fixes the total number of particles $N$ in the system, the density profile $\rho(x)$ is unique. Therefore the peak density $\rho_0 \equiv \rho(0)$ in the trap centre $x = 0$ can be used to define a dimensionless interaction strength $\gamma_0 = mg/(\hbar^2 \rho_0)$ that plays the role of a global interaction parameter for the trapped system.

The essence of the efficient parametrization of the trapped 1D quasicondensate lies in the fact that, apart from the dependence on the longitudinal trapping frequency $\omega$, it can still be completely characterized by a single dimensionless parameter, which is a combination of the dimensionless interaction strength in the trap center, $\gamma_0$, and the global dimensionless temperature $T$:

$$\chi_0 = \frac{1}{2}\gamma_0^{3/2}T.$$  \hspace{1cm} (1)

Explicitly, $\chi_0$ is given by $\chi_0 \equiv k_B T/[(\hbar \rho_0 \sqrt{g \rho_0}/m)]$, similar to $\chi$ it must satisfy $\chi_0 \ll 1$ in the quasicondensate regime; and we note that its definition is related to the dimensionless parameters $A$ and $\chi_0$ [35, 42], used, respectively, in Refs. [23, 35] (see also [42]). In particular, in Ref. [35], $\chi_0$ [see, Eq. (77)] is defined in terms of a product of ratios of characteristic energy- and length-scales of the uniform problem, $\chi_0 = \frac{1}{2}\gamma^{3/2}T$, where $\gamma = \sqrt{2\pi \hbar^2/m k_B T}$ is the thermal de Broglie wavelength, and our $\chi$ is related to $\chi_0$ via $\chi = 1/\sqrt{\chi_0}$, whereas our $\chi_0$ is defined similarly but in terms of the peak density $\rho_0$.

### B. Dimensionless c-field equations in 1D

To arrive at the dimensionless form of the SPGPE and GPE using this parametrization, we introduce the dimensionless coordinate $\xi = x/x_0$, time $\tau = t/t_0$, and dimensionless field $\varphi_C(\xi, \tau) = \psi_C(x, t)/\psi_0$, using the respective length-, time-, and field-scales introduced according to:

$$x_0 = \frac{\hbar^{4/3}}{m^{2/3}g^{1/3}(k_B T)^{1/3}}$$ \hspace{1cm} (2)

$$t_0 = \frac{m \hbar^2}{\hbar^{5/3}} = \frac{m^{1/3} g^{2/3}(k_B T)^{2/3}}{\hbar^{5/3}}$$ \hspace{1cm} (3)

$$\psi_0 = \frac{\hbar}{x_0 \sqrt{mg}} = \left(\frac{m(k_B T)^2}{\hbar^2 g}\right)^{1/6}.$$ \hspace{1cm} (4)

With these scaled variables, the dimensionless time-dependent GPE reads as

$$\frac{\partial \varphi_C(\xi, \tau)}{\partial \tau} = -i\mathcal{L} \varphi_C(\xi, \tau),$$ \hspace{1cm} (5)

where the nonlinear operator $\mathcal{L}$ is defined via

$$\mathcal{L} \equiv -\frac{1}{2} \frac{\partial^2}{\partial \xi^2} + |\varphi_C|^2 + \frac{1}{2} \varphi_C^2 \omega^2 \xi^2,$$ \hspace{1cm} (6)

whereas the dimensionless SPGPE reads as

$$d \varphi_C(\xi, \tau) = [-i \mathcal{L} + \tilde{\kappa}_{th}(\mu - \mathcal{L})] \varphi_C d \tau + \sqrt{2\tilde{\kappa}_{th} dW}.$$ \hspace{1cm} (7)

Here, $\omega = \omega_0$ is the dimensionless trap frequency, $\mu = \mu_0/h$ is the dimensionless chemical potential and $\tilde{\kappa}_{th} = h \kappa_{th}$ is the rescaled growth rate with a numerical value that has no consequence for the final equilibrium configurations and can be chosen for numerical convenience [23]. Additionally, $dW$ is a complex delta-correlated noise satisfying $\langle dW(\xi, \tau) dW(\xi', \tau') \rangle = \delta(\xi - \xi') d\tau$. We point out that even though we are referring to Eq. (7) as the SPGPE, the actual projection operator, which sets up
the high-energy cutoff for the classical field region, is effectively imposed by the choice of numerical grid for simulations. Due to the strictly 1D nature of the problem at hand, the actual cutoff dependence of the dynamics is very weak (see Ref. [23] for further details).

In the above dimensionless form, the nonlinearity constant in the GPE is always equal to unity, and the normalization condition that gives the total number of particles in the system reads $N = \int |\psi(x, t)|^2 dx = \psi_0^2 x_0 N$, where $N = \int |\varphi(\xi, \tau)|^2 d\xi$.

In the Thomas-Fermi (TF) limit of an inverted parabolic density profile, the chemical potential of a harmonically trapped quasicondensate is given by $\mu = g \rho_0$, and thus the dimensionless chemical potential $\bar{\mu}$ can be expressed in terms of, and interchanged with, the earlier introduced dimensionless parameter $\chi_0$ as

$$\bar{\mu} = \chi_0^{-2/3}. \quad (8)$$

Given that $\rho_0 = (9m \omega^2 N^2/32g)^{1/3}$ in the TF limit, $\chi_0$ itself can be expressed as $\chi_0 = 4\sqrt{2} \mu k_B T/(3N \hbar \omega)$ [using $N = \psi_0^2 x_0 N$ and $N = 4\sqrt{2} \mu^{3/2} / (3\omega) = 4\sqrt{2} / (3\chi_0 \omega)$]. Beyond the TF limit, the dimensionless chemical potential $\bar{\mu}$ can still be traded off with $\chi_0$ as an input parameter, with the understanding that the simple relationship $\mu = g \rho_0$ between $\mu$ and the peak density $\rho_0$ is now only approximate, whereas the exact relationship has to be determined numerically a posteriori.

Thus, a harmonically trapped 1D quasicondensate can be completely characterized by just two dimensionless parameters, $\bar{\mu}$ and $\bar{\omega}$, or by $\chi_0 \ll 1$ and $\bar{\omega}$. Each choice of $\chi_0$ can be realized with a range of values of the dimensionless interaction $\gamma_0$ and temperature $T$ according to Eq. (1). The condition that the density profile is well approximated by the TF inverted parabola, $R_{TF} \gg l_{ho}$, is equivalent to $\bar{\omega} \ll \bar{\mu}$. This implies that for each choice of $\chi_0 \ll 1$, the dimensionless trap frequency $\bar{\omega}$ must further satisfy $\bar{\omega} \ll \chi_0^{-2/3}$. Furthermore, $\bar{\omega}$ can be expressed via the ratio of the length scale $x_0$ and the harmonic oscillator length $l_{ho} = \sqrt{\hbar/m \omega}$ as

$$\bar{\omega} = \left(\frac{x_0}{l_{ho}}\right)^2, \quad (9)$$

or explicitly $\bar{\omega} = \hbar^{3/2} m^{1/2} \omega / [g^{3/2}(k_B T)^{3/2}]$. Therefore, rewriting this as $\bar{\omega}^{-3/2} = m g^3 k_B T / (m^{3/2} \omega^{3/2} \hbar^{3/2})$, one can see that for a specific atomic species (with given $m$ and $g$) and any particular choice of the trap frequencies $\omega$ and $\omega_{\perp}$ (the latter setting the value of the coupling constant $g = 2\hbar \omega_{\perp} a$), fixing the temperature of the gas $T$ is equivalent to fixing the value of $\bar{\omega}$ and vice versa. With $\bar{\omega}$ constant, the value of the dimensionless parameter $\chi_0$ is now governed only by the choice of the total number of atoms $N$ or equivalently the peak density $\rho_0$.

C. Choice of simulation parameters

For the numerical simulations we have chosen $\bar{\omega} = 0.0696$ and varied $\chi_0 \in [0.01, 0.1]$, a range corresponding from deep within the thermal quasicondensate to a near degenerate ideal Bose gas regime (see Refs. [40, 45, 46] for further details on the regimes of an interacting 1D Bose gas). The lower (upper) bound in $\chi_0$ corresponds to higher (lower) atom number $N$ and hence to conditions that are effectively deeper in the low-temperature quasicondensate regime (or are further away from the degenerate ideal Bose gas regime). Considering $^{87}$Rb atoms ($m \simeq 1.44 \times 10^{-25}$ kg, $a \approx 5.31$ nm) in a harmonic trap of frequency $\omega/2\pi = 3$ Hz as an example, these choices of dimensionless parameters can be realised with $\omega_{\perp}/2\pi = 6$ kHz, $T = 13$ nK, and by varying the atom number $N \in [8.5 \times 10^2, 8.5 \times 10^3]$, which is typical of current experiments on 1D quasicondensates. Note that while this is just one choice of dimensionless parameters, the same $\omega$ and $\chi_0$ can be realised with other combinations of trap frequencies, temperatures, atom numbers, and atomic species.

To initiate the Newton’s cradle dynamics, we model the application of a sequence of Bragg pulses [47] which split the initial equilibrium quasicondensate into two counter-propagating wavepackets with momenta $\pm 2\hbar k_0$ corresponding to the lowest diffraction orders of Bragg scattering. Ideally, this initializes the $c$-field configuration denoted $\psi_C(x, t = 0^+)$ into a superposition of the form

$$\psi_C(x, t = 0^+) = \frac{\psi_C(x, t = 0^-)}{\sqrt{2}} (e^{i2k_0 x} + e^{-i2k_0 x}), \quad (10)$$

where $\psi_C(x, t = 0^-)$ is the finite temperature equilibrium configuration. This state is then evolved in real time according to the GPE. This idealized form of the Bragg pulse is a good approximation to a realistic experimental implementation of a sequence of Bragg pulses tuned to operate in the Bragg regime [47], which corresponds to the condition that the Bragg momentum $2\hbar k_0$ is much greater than the width of the momentum distribution of the quasicondensate. A comparison of the numerical implementations of the above idealised Bragg pulse and the more realistic pulse presented in Ref. [47] showed small differences in our results during the initial collisional oscillation cycles (see also the discussion of Figs. 5 and 6 below), however, no appreciable differences were found in the approach to the final relaxed state for all parameter regions.

In our simulations, we considered momentum kicks of $k_0 \in [10^6, 10^7]$ m$^{-1}$. For $^{87}$Rb atoms and other relevant parameter choices as above (namely, $\omega_{\perp}/2\pi = 6$ kHz and $T = 13$ nK), this corresponds to a dimensionless momentum $q_0 = k_0 x_0$ in the range $q_0 \in [1.16, 11.6]$. In harmonic oscillator units, $q_0$ can be converted to a dimensionless momentum $q_0^{(ho)} \equiv k_0 l_{ho} = q_0 / \sqrt{\bar{\omega}}$, and for $\bar{\omega} = 0.0696$ this range of $q_0$ would correspond to $q_0^{(ho)} \in [4.4, 44]$. 
FIG. 1. Evolution of the position-space (left) and momentum-space (right) density distributions in the quantum Newton’s cradle setup of a harmonically trapped 1D quasicondensate, generated from $c$-field simulations. The three examples displayed all have parameter values of $\bar{\omega} = 0.0696$ and $\chi_0 = 0.0562$ for the initial thermal equilibrium state, whereas the wavenumber for the Bragg pulse varies as follows: (a) $q_0 = 1.16$; (b) $q_0 = 2.06$; and (c) $q_0 = 3.67$.

From the practical point of view, considering momentum kicks significantly beyond the considered range of $q_0 \in [1.16, 11.6]$ would be either of no physical interest in terms of producing (for smaller $q_0$) the collisional dynamics that we are interested in, or would be computationally too demanding (for larger $q_0$) in terms of the numerical grids required to capture the relevant physics at high momenta with sufficient resolution in both the momentum and position spaces.

III. RESULTS AND DISCUSSION

A. Examples of relaxation dynamics

Typical examples of $c$-field simulations of the quantum Newton’s cradle for a 1D quasicondensate are illustrated in Fig. 1, where we show the relaxation dynamics of the real-space density profile and the momentum distribution of the gas following the Bragg pulse. Figure 1(a) represents an example of fast relaxation, occurring in the regime where the clouds are extremely wide compared to the maximum spatial separation due to the weak momentum kick, and hence are always overlapped to some extent. This causes the system to relax without developing any appreciable collisional oscillations. Figure 1(b) is an example in the intermediate quasi-periodic regime with a stronger momentum kick; the clouds separate completely, but the momentum kick is still not large enough to lead to persisting collisions and periodic oscillations. Finally, Fig. 1(c) illustrates an example in the periodic regime, with the momentum kick larger than in (b); the collisional oscillations persist in this system with no noticeable damping for many (tens to a hundred) periods, and take over a hundred oscillations to fully dampen.

In Fig. 2 we show the momentum distributions of the initial ($t = 0^-$) and the final relaxed states for the examples of Fig. 1, and compare the final distributions with those of an equilibrium thermal state at a certain (higher) temperature. The best-fit thermal state, in which the temperature serves as a fitting parameter, is generated using the same SPGPE equation as the one used to ini-
The initial and relaxed distributions displayed in (a), (b), and (c) are, respectively, for the same parameters as in Fig. 1, i.e., for $(\chi_0, \bar{\omega}) = (0.0562, 0.0696)$ initially and three different values of $q_0$ as shown, whereas the thermal fitted distributions, plotted for comparison with the relaxed ones, are for the following fitting parameters: (a) $(\chi_0, \bar{\omega}) = (0.250, 0.045)$; (b) $(\chi_0, \bar{\omega}) = (0.238, 0.0462)$, and (c) $(\chi_0, \bar{\omega}) = (0.500, 0.150)$.

The initial equilibrium quasicondensate, $(\chi_0, \bar{\omega})$, and the Bragg momentum $2\hbar k_0$ imparted onto each half of the split cloud. The interplay between these three parameters can, in the first instance, be analyzed using simple geometric considerations. This will allow us to construct a qualitative overview of the expected different types of dynamical behaviour, which can be broadly classified as (I) aperiodic—displaying fast thermalization; (II) quasiperiodic—displaying intermediate to slow thermalization timescales, and (III) periodic—in which case the Newton’s cradle collisional oscillations persist for many oscillation periods and the thermalization is the slowest.

The first of the simple geometric considerations is the requirement that in order to set the initial quasicondensate into a well-defined Newton’s cradle collisional regime and observe persisting oscillations over many periods, the two momentum components of the cloud must be well separated in momentum space. This means that the difference between the Bragg momenta $\pm 2\hbar k_0$ must be much larger than the characteristic momentum width of the cloud, which we denote via $\sigma_k$:

\[
4\hbar k_0 \gg \sigma_k.
\]

The second consideration is that, even if the initial momentum components are well separated in momentum...
space, the two clouds will relax quickly, within the a few oscillation cycles. This occurs if the respective position-space density distributions remain largely overlapping during the collisional period and thus do not separate well in position-space. Thus, the second requirement for setting the initial quasicondensate into a a well-defined Newton’s cradle collisional regime is that the maximum separation of the clouds $x_{\text{max}}$ is much larger than the characteristic width of the cloud in position space, $\sigma_x$: $x_{\text{max}} \gg \sigma_x$. Given that for a simple harmonic motion with a maximum momentum $2k_0$, the maximum displacement is given by $x_{\text{max}} = 2k_0/m\omega$, this condition can be approximated via

$$2k_0 \gg \sigma_x/l_{\text{ho}}^2,$$ \hspace{1cm} (12)

Equations (11) and (12) can be further simplified and rewritten in terms of our dimensionless parameters if we approximate the characteristic size of the quasicondensate $\sigma_x$ by the TF radius, $R_{\text{TF}} = \sqrt{2\mu/m\omega^2}$ and the momentum width of the cloud by the inverse of the temperature-dependent phase coherence length in the trap centre [23], $\sigma_k \approx 1/l_\phi$, with $l_\phi = \hbar^2\rho_0/mkB_T$. Introducing the dimensionless Bragg momentum $q_0 \equiv k_0x_0$, Eqs. (11) and (12) can now be rewritten, respectively, as:

$$q_0 \gg \frac{1}{4\mu} = \frac{\chi_0^{2/3}}{4},$$ \hspace{1cm} (13)

$$q_0 \gg \sqrt{\frac{\mu}{2}} = \frac{1}{\sqrt{2\chi_0^{1/3}}},$$ \hspace{1cm} (14)

where we note that the quasicondensate regime requires $\chi_0 \ll 1$, and the condition that the density profile is well approximated by the TF inverted parabola, $R_{\text{TF}} \gg l_{\text{ho}}$, is equivalent to $\bar{\omega} \ll \mu = \chi_0^{-2/3}$.

Using the right-hand-sides of Eqs. (13) and (14) as crossover boundaries of different types of behaviour we can now construct a candidate diagram of the dynamical regimes (see Fig. 4) for observing different relaxation scenarios of the quasicondensate Newton’s cradle. This will be verified numerically in the next section. In the regions I and II we expect fast thermalisation due to the fact that the two momentum components of the split quasi-condensate do not separate well in momentum (region I) or in position spaces (region II); the example in Fig. 1(a) corresponds to conditions from region II. Region III corresponds to quasiperiodic behaviour and intermediate thermalization timescales, and this is when, despite the fact that the two components are well separated in both momentum and position spaces, the dephasing during the first few collisions due to the strong nonlinearity acts as a strong perturbation to persistent periodic behaviour; the examples (b) and (c) in Fig. 1 corresponds to conditions from region III.

![FIG. 4. Diagram of the dynamical regimes for the quasicondensate Newton’s cradle in the $\chi_0$ versus $q_0$ parameter space, with $\bar{\omega} = 0.0696$. The crossover boundaries (solid and dashed lines) between the different regimes, corresponding to fast (regions I and II) and slow (region III) relaxation, equate to the right-hand-sides of Eqs. (13) and (14). The circles labelled (a), (b) and (c) represent, respectively, the parameter combinations used in the examples displayed in Figs. 1, 2, and 7 (see text). The coloured rectangle corresponds to the scanned parameter space of $\chi_0$ and $q_0$, for which we extracted the dimensionless decay rate $\Gamma$ from the numerical simulations (see Sec. III C). As we see, the crossover boundary between the regions II and III predicted from simple arguments is in good agreement with the quantitative picture emerging from the simulations of the decay rate.]

C. Characteristic relaxation rate

The observable we use to characterize the relaxation rate of the system is the rms momentum width, for the momentum distribution averaged over each $(m\text{th}, m = 1, 2, 3, \ldots)$ period of Newton’s cradle collisional oscillations:

$$W_m = \sqrt{\frac{\int dk k^2 n_m(k)}{\int dk n_m(k)}},$$ \hspace{1cm} (15)

where

$$n_m(k) = \frac{1}{T_\omega} \int_{T_\omega \times (m-1)} dt n(k,t),$$ \hspace{1cm} (16)

is the momentum distribution averaged over the $m$th oscillation and $T_\omega = 2\pi/\omega$ is the oscillation period. Here, averaging over an oscillation period separates the oscillatory dynamics and the internal structure of instantaneous momentum distributions from the gradual net relaxation and the approach of $W_m$ to a stationary value.

Examples of the momentum distributions $n_m(k)$ averaged over the $m$th oscillation period, for $m = 1, 10, 100, 200, 300$ and the parameter values of Fig. 1(c), are shown in Fig. 5. It is interesting to see that the
three peaked structure present in the equivalent averages found experimentally in Ref. [9] and theoretically in Ref. [29] are not present in Fig. 5. In order to explain this discrepancy we performed a simulation with equivalent parameters to those in Fig. 5, however rather than applying an idealized Bragg pulse of Eq. (10) we initialize the system with ten percent of particles remaining in the zero momentum state, i.e., unperturbed. This more closely resembles the physical Bragg pulse implemented in Refs. [9, 29]. As can be seen in Fig. 6, using this initialization the three peaked distribution does occur initially ($m = 1$), in contrast to Fig. 5, however, the three peaked structure eventually dampens out as can be seen from the example for $m = 10$.

To extract the relaxation rate $\Gamma$ from the c-field simulation results, we fit an exponential decay of the form

$$W_m = A \exp(-\Gamma m) + B$$

(17)

to the numerically computed values of $W_m$. Here, $\Gamma = \Gamma T_\omega$ is the dimensionless relaxation rate, $m$ is the oscillation number, and the numerical constants $A$ and $B$ fix the initial and final (relaxed) values of $W_m$.

In Fig. 7, we plot examples of relaxation of $W_m$ for parameter values corresponding to Fig. 1 (a-c). As we can see after the initial fast pre-thermalization stage [4, 12, 18–20, 48] during which the phase dynamics damps out, the subsequent decay of the rms width $W_m$ towards the final thermal distribution, characterized in Fig. 2, is well approximated by an exponential.

![FIG. 5. Momentum distributions averaged over one ($m$th) oscillation period, $n_m(k)$, for parameter values corresponding to $(\chi_0, q_0, \bar{\omega}) = (0.0562, 3.67, 0.0696)$, i.e., example (c) in Fig. 1. Distributions for $m = 1, 10, 100, 200, \text{ and } 300$ are shown. The average distributions for the earlier oscillations, $m = 1$ and $10$, still have two separate peaks at the initial momentum kicks $\pm 2q_0$ and a smaller third peak at $q = 0$, indicating there is still some oscillatory dynamics present in the system, while the large-$m$ averages approach stationary thermal distributions as those in Fig. 2.](image1)

![FIG. 6. Momentum distributions averaged over one ($m$th) oscillation period, $n_m(k)$, for parameter values corresponding to $(\chi_0, q_0, \bar{\omega}) = (0.0562, 3.67, 0.0696)$ as in Fig. 5. However, here we initialize the system with ten percent of particles remaining unperturbed by the Bragg pulse (see text). We see that in this case the distinct three peaked distribution, observed in Refs. [9, 29], does occur initially ($m = 1$), in contrast to Fig. 5, however, the three peaked structure eventually dampens out as can be seen from the example for $m = 10$.](image2)

![FIG. 7. Relaxation of the rms width $W_m$ of the average momentum distribution over the $m$th oscillation period as a function of oscillation number $m$. The symbols correspond to the numerically evaluated values of the width $W_m$, for the same parameters as in Fig. 1, whereas the solid, dashed, and dash-dotted lines are based on exponential fits of Eq. (17); the widths are rescaled by their respective values of $W_1$ in order to be easily visible on the same graph. Note that the exponential fits begin after the respective prethermalization periods which are not well approximated by a single exponential. The long-time relaxation rates $\bar{\Gamma}$ extracted from these fits are as follows: (a) $\bar{\Gamma} = 0.06$; (b) $\bar{\Gamma} = 0.025$; and (c) $\bar{\Gamma} = 0.0065$.](image3)
By repeating the fitting procedure for a range of parameter values of \((\chi_0, q_0, \omega)\) and extracting the respective relaxation rates \(\Gamma\), we can verify the proposed nonequilibrium dynamical regimes of Fig. 4. This is illustrated through the colour plot embedded in Fig. 4, where we show a density plot of \(\Gamma\) over \((\chi_0, q_0)\) parameter space for a fixed value of \(\omega = 0.0696\). As we see from these results, for an experimentally feasible range of parameters, the relaxation rate \(\Gamma\) typically ranges within \(\Gamma \in [0.002, 0.2]\), and the crossover boundary between the regions II and III, evident at \(\Gamma \sim 0.1\), is in good agreement with the boundary predicted in Sec. IIIB.

Taking the trap frequency of \(\omega/2\pi = 150\) Hz as an example (corresponding to the values used in the original quantum Newton’s cradle experiment in a strongly interacting 1D Bose gas [9]), this range of the dimensionless \(\Gamma\) corresponds to \(\Gamma \in [0.3, 30]\) s\(^{-1}\), and hence the characteristic relaxation time for the quasicondensate Newton’s cradle is expected to be in the range of 0.033 to 3.3 seconds, depending on the actual interaction strength and temperature of the gas. For comparison, for \(\omega/2\pi = 5\) Hz, which is the typical value used in the coherently split quasicondensate experiments [15], our results convert approximately to typical relaxation times of 1 to 100 seconds. This is at least three orders of magnitude larger than the millisecond time scale of relaxation of phase dynamics in a 1D quasicondensate [15].

At the same time, this is comparable to the relaxation times scales observed in the Newton’s cradle setup of a strongly interacting 1D Bose gas. We thus conclude that for an equivalent quantum Newton’s cradle experiment in a weakly interacting 1D Bose gas the relaxation rate is of similar order of magnitude as in the strongly interacting case.

**IV. CONCLUSION**

In conclusion, we studied the collisional dynamics of a harmonically trapped weakly interacting 1D quasicondensate in the same quantum Newton’s cradle setting as the experiment performed by Kinoshita et al. [9] in the strongly interacting (near-Tonks-Girardeau) regime. By parametrizing the system in terms of just two (for any given trap frequency) dimensionless parameters, \(\chi_0\) and \(q_0\), which encompass the interaction strength, the temperature of the gas, and the magnitude of the Bragg momentum, we identify the different dynamical regimes of the system in terms of its relaxation rate to the final equilibrium state.

We find that the final relaxed state of the system can be well characterised by a thermal distribution, where the kinetic energy imparted to the system by the initial Bragg pulse is distributed amongst the internal degrees of the system and results in a higher equilibrium temperature of the final state. Relaxation to this thermal state, rather than to a state which would require characterisation via a generalised Gibbs ensemble, implies that harmonic confinement here cannot be regarded as a weak perturbation from integrability of the respective uniform 1D Bose gas. Accordingly, a 1D quasicondensate in this nonequilibrium scenario does not offer itself as a system to which a quantum analog of the KAM theorem could be applied.

The characteristic relaxation timescales are predicted to be of the order of tens of seconds for typical experimental parameters, which in turn correspond to hundreds of repeated collisions taking place before equilibrium is reached. This is similar to, and only an order of magnitude shorter than, the typical relaxation timescales observed in the strongly interacting quantum Newton’s cradle. The difference can be explained by the effect of repulsive interactions in the system, which strongly reduce the two-body correlations and hence suppress the two-body collisional rates in the Tonks-Girardeau regime. At the same time, such relaxation timescales are significantly longer than the typical millisecond dephasing timescales observed in the experiments of Hofferberth et al. [15] on coherently split quasicondensates.

Our system can be realized using currently available experimental techniques of creating equilibrium 1D Bose gases, except that one has to maintain an additional, dynamical 1D condition of \(\hbar^2(2k_0)^2/2m \ll \hbar \omega_L\), which is required for eliminating transverse excitations of the gas due to the (large) longitudinal kinetic energy imparted onto the atoms by the Bragg pulse and hence could be challenging. The nonequilibrium scenario that we studied represents a directly comparable counterpart of the original quantum Newton’s cradle setup, except that the collisional dynamics now takes place in a weakly interacting gas. A similarly comparable scenario, which could provide further insights into thermalisation in strongly and weakly interacting 1D Bose gases, would be to perform Hofferberth et al. experiment [15] in the strongly interacting regime.

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