On topological restrictions of the spacetime in cosmology

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In this paper we discuss the restrictions of the spacetime for the standard model of cosmology by using results of the differential topology of 3- and 4-manifolds. The smoothness of the cosmic evolution is the strongest restriction. The Poincare model (dodecaeder model), the Picard horn and the 3-torus are ruled out by the restrictions but a sum of two Poincare spheres is allowed.

Keywords: topological restrictions; smoothness of cosmic evolution; homology 3-spheres.

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1. Introduction

In the 80’s, there were a growing understanding of 3- and 4-manifolds. Mike Freedman proved the (topological) Poincare conjecture in dimension 4 and classifies closed, compact, simple-connected, topological 4-manifolds in 1982. Bill Thurston presented its geometrization conjecture at the same year (the geometrization conjecture was proved by Perelman). Soon afterward, Simon Donaldson found a large class of non-smoothable closed, compact, simple-connected 4-manifolds leading to the first examples of exotic $\mathbb{R}^4$. Beginning with this development, our understanding of 3- and 4-manifolds as well its relation to each other is now in a better state. In physics, 4-manifolds are models for the spacetime and 3-manifolds are the spatial part (like in global hyperbolic spacetimes $\Sigma \times \mathbb{R}$ with the 3-manifold $\Sigma$ as Cauchy surface). There are only few papers discussing the physical implications of the new 3- and 4-dimensional results. But we do not know any paper with a look for the cosmological implications.

2. Preliminaries: 3- and 4-manifolds

This section serves only as a short introduction into the theory of 3- and 4-manifolds. Further details can be found in the books.
2.1. 3-manifolds and geometric structures

A connected 3-manifold \( N \) is prime if it cannot be obtained as a connected sum of two manifolds \( N_1 \# N_2 \) (see the appendix Appendix A for the definition) neither of which is the 3-sphere \( S^3 \) (or, equivalently, neither of which is the homeomorphic to \( N \)). Examples are the 3-torus \( T^3 \) and \( S^1 \times S^2 \) but also the Poincare sphere. According to \( 10 \), any compact, oriented 3-manifold is the connected sum of an unique (up to homeomorphism) collection of prime 3-manifolds (prime decomposition). A subset of prime manifolds are the irreducible 3-manifolds. A connected 3-manifold is irreducible if every differentiable submanifold \( S \) homeomorphic to a sphere \( S^2 \) bounds a subset \( D \) (i.e. \( \partial D = S \)) which is homeomorphic to the closed ball \( D^3 \). The only prime but reducible 3-manifold is \( S^1 \times S^2 \). For the geometric properties (to meet Thurston’s geometrization theorem) we need a finer decomposition induced by incompressible tori. A properly embedded connected surface \( S \subset N \) is called 2-sided if its normal bundle is trivial, and 1-sided if its normal bundle is nontrivial. A 2-sided connected surface \( S \) other than \( S^2 \) or \( D^2 \) is called incompressible if for each disk \( D \subset N \) with \( D \cap S = \partial D \) there is a disk \( D' \subset S \) with \( \partial D' = \partial D \). The boundary of a 3-manifold is an incompressible surface. Most importantly, the 3-sphere \( S^3 \), \( S^2 \times S^1 \) and the 3-manifolds \( S^3 / \Gamma \) with \( \Gamma \subset SO(4) \) a finite subgroup do not contain incompressible surfaces. The class of 3-manifolds \( S^3 / \Gamma \) (the spherical 3-manifolds) include cases like the Poincare sphere (\( \Gamma = I^* \) the binary icosahedron group) or lens spaces (\( \Gamma = \mathbb{Z}_p \) the cyclic group). Let \( K_i \) be irreducible 3-manifolds containing incompressible surfaces then we can \( N \) split into pieces (along embedded \( S^2 \))

\[
N = K_1 \# \cdots \# K_{n_1} \# S^1 \# S^2 \# S^3 / \Gamma, \quad (1)
\]

where \( \#_n \) denotes the \( n \)-fold connected sum and \( \Gamma \subset SO(4) \) is a finite subgroup. The decomposition of \( N \) is unique up to the order of the factors. The irreducible 3-manifolds \( K_1, \ldots, K_{n_1} \) are able to contain incompressible tori and one can split \( K_i \) along the tori into simpler pieces \( K = H \cup T^2 \ G \) (called the JSJ decomposition). The two classes \( G \) and \( H \) are the graph manifold \( G \) and hyperbolic 3-manifold \( H \) (see Fig. 1). The hyperbolic 3-manifold \( H \) has a torus boundary \( T^2 = \partial H \), i.e. \( H \) admits a hyperbolic structure in the interior only. One property of hyperbolic 3-manifolds is central: Mostow rigidity. As shown by Mostow \( 12 \), every hyperbolic \( n \)-manifold \( n > 2 \) has this property: Every diffeomorphism (especially every conformal transformation) of a hyperbolic \( n \)-manifold is induced by an isometry. Therefore one cannot scale a hyperbolic 3-manifold and the volume is a topological invariant. Together with the prime and JSJ decomposition

\[
N = (H_1 \cup T^2 \ G_1) \# \cdots \# (H_{n_1} \cup T^2 \ G_{n_1}) \# S^1 \# S^2 \# S^3 / \Gamma, \quad (1')
\]

\( a \)The ‘sides’ of \( S \) then correspond to the components of the complement of \( S \) in a tubular neighborhood \( S \times [0, 1] \subset N \).
we can discuss the geometric properties central to Thurston's geometrization theorem: Every oriented closed prime 3-manifold can be cut along tori (JSJ decomposition), so that the interior of each of the resulting manifolds has a geometric structure with finite volume. Now, we have to clarify the term "geometric structure". A model geometry is a simply connected smooth manifold $X$ together with a transitive action of a Lie group $G$ on $X$ with compact stabilizers. A geometric structure on a manifold $N$ is a diffeomorphism from $N$ to $X/\Gamma$ for some model geometry $X$, where $\Gamma$ is a discrete subgroup of $G$ acting freely on $X$. It is a surprising fact that there are also a finite number of three-dimensional model geometries, i.e. 8 geometries with the following models: spherical ($S^3, O_4(\mathbb{R})$), Euclidean ($E^3, O_3(\mathbb{R}) \ltimes \mathbb{R}^3$), hyperbolic ($H^3, O_{1,3}(\mathbb{R})^+$), mixed spherical-Euclidean ($S^2 \times \mathbb{R}, O_3(\mathbb{R}) \times \mathbb{R} \times \mathbb{Z}_2$), mixed hyperbolic-Euclidean ($H^2 \times \mathbb{R}, O_{1,3}(\mathbb{R})^+ \times \mathbb{R} \times \mathbb{Z}_2$) and 3 exceptional cases called $\tilde{SL}_2$ (twisted version of $H^2 \times \mathbb{R}$), $\text{NIL}$ (geometry of the Heisenberg group as twisted version of $E^3$), $\text{SOL}$ (split extension of $\mathbb{R}^2$ by $\mathbb{R}$, i.e. the Lie algebra of the group of isometries of 2-dimensional Minkowski space). We refer to [13] for the details.

### 2.2. 4-manifolds and smoothness

In this subsection we will shortly discuss the relation between 3-manifolds and 4-manifolds. At first, every oriented, compact 3-manifold is the boundary of a compact, simple-connected 4-manifold (Theorem 2 in chapter VII of [5]). Therefore we have to concentrate on simple-connected 4-manifolds, which are classified by Freedman [11] topologically. The topological classification based on the intersection form $\sigma(M)$ of a simple-connected, compact, closed 4-manifold. The intersection form is a quadratic form over the second homology $H_2(M)$ (with integer coefficients). An algebraic splitting of the form $\sigma(M) = \sigma_1 \oplus \sigma_2$ is realized by a (topological) splitting of the 4-manifold (along homology 3-spheres, see [14]). At this point, there is a big difference between the smooth and the topological case: this algebraic splitting of the form is not always realized by a smooth splitting of the 4-manifold. A direct consequence is the fact that all homology 3-spheres are bounding contractable, topological 4-manifolds which are not always smoothable. An example is the Poincare sphere, i.e. there is no contractable, smooth 4-manifold with boundary the Poincare sphere.
sphere. Therefore, the assumption of a smooth spacetime is very restrictive as we will see below.

3. The restrictions to smooth cosmological spacetimes

According to the cosmological principle, our expanding universe, although it is so complex, can be considered at very large scale homogeneous and isotropic. The exact solution to Einstein’s equation, describing a homogeneous, isotropic universe, is in general called the Friedmann-Lemaitre- Robertson-Walker (FLRW) metric. The FLRW model shows that the universe should be, at a given moment of time, either in expansion, or in contraction. From Hubble’s observations, we know that the universe is currently expanding. The FLRW-model shows that, long time ago, there was a very high concentration of matter, which exploded in what we call the Big Bang. The singularity theorems of Hawking and Penrose\textsuperscript{15} showed the necessity for the appearance of singularities including the Big Bang under general circumstances. In this paper we will consider the main hypothesis:

**Main Hypothesis:** Our universe is a compact 3-manifold $\Sigma$ expanding smoothly so that the spacetime is a smooth 4-manifold $M$. The topology of the 3-manifold is allowed to change. The spacetime is a compact, smooth 4-manifold, i.e. we consider only the finite time period from the Big Bang to the current universe so that $\partial M = \Sigma$.

The compactness of the 3-manifold $\Sigma$ is a hypothesis motivated by the WMAP data\textsuperscript{16,17,18}. One of the enigmas of the cosmic microwave background (CMB) is the low power in the temperature correlations at large angles. Especially the gaps in the spectrum of the background radiation have a simple explanation by assuming a compact universe. Current further investigations\textsuperscript{19} gave the conclusion that the low power at large angles is real with high probability. One explanation of this suppression of power could be that the universe possesses a non-trivial topology (multi-connected spatial space i.e. compact 3-manifolds with non-trivial fundamental group). Currently three models are discussed, the Poincare (or dodecaeder) model\textsuperscript{20} (positive curvature), the Picard horn\textsuperscript{21} (negative curvature) and the flat universe with the 3-torus (homogeneous model\textsuperscript{16,18}) or the half-turn space (inhomogeneous model\textsuperscript{22}). But the current data are unable to decide between these cases\textsuperscript{19}.

Now we study the implications of this hypothesis by using some results of the differential topology of 3- and 4-manifolds.

3.1. Classical case

In the classical case we assume a singularity, i.e. the existence of a point in the past attracting all geodesics (pointed backward). A simple example is given by a cone over the circle. All normals to the circle (the geodesics of the cone) converge to the apex. At the same time one can interpret the cone as a process to contract the circle to a point. Then one uses these geodesics to construct this (continuous)
contraction map. To generalize this case, we remark that the cone is homeomorphic (actually also diffeomorphic after smoothing the apex) to the disk $D^2$. Therefore we take the 4-disk $D^4$ homeomorphic to the cone $\text{Cone}(S^3) \simeq D^4$ over $S^3$ (\(\simeq\) denotes "homeomorphic"). This 4-manifold (compact with boundary) serves as a model for the "classical" Big Bang, starting with a singularity (the apex) and evolving to the 3-sphere after a finite time. Topologically, the model can be characterized by the property: the cone $\text{Cone}(S^3)$ is contractable to a point, i.e. there is a continuous (actually also a smooth) homotopy $\text{Cone}(S^3) \to \{\star\}$. But we may ask: Is this model the most general case? We will partly answer this question in the following.

To fix the problem, we have to consider the class of smooth, compact, contractable 4-manifolds. Contractability is needed (but not necessary) to obtain a Lorentz metric outside of the Big Bang singularity. Furthermore contractability guarantees that there is no topology change after the Big Bang. We will later discuss the case of an explicit topology change. We remark that in topology there exists also a general procedure to construct the cone $\text{cone}(\Sigma)$ over a topological space $\Sigma$ by

$$\text{cone}(\Sigma) = \frac{\Sigma \times [0,1]}{\Sigma \times \{1\} \sim \{\star\}}$$

where $\Sigma \times \{1\}$ is contracted to a point $\{\star\}$ (the apex). The cone $\text{cone}(\Sigma)$ is contractable but it is in most cases not a manifold but an orbifold (it fails to be locally modeled on open subsets of $\mathbb{R}^4$ but instead on quotients of open subsets of $\mathbb{R}^4$ by finite group actions). Therefore a whole neighborhood of the singularity fails to be a manifold. To illustrate this singularity we consider the cone over the 3-sphere again. The homeomorphism between the cone $\text{Cone}(S^3)$ and the 4-disk is known as Alexanders trick, i.e. a homeomorphism of the two boundaries $\partial\text{Cone}(S^3) = S^3$ and $\partial D^4 = S^3$ can be extended to a homeomorphism $\text{Cone}(S^3) \simeq D^4$. This fails for the 3-torus. The cone $\text{Cone}(T^3)$ contains a 3-torus for every value $t \in [0,1)$ but not at $t = 1$ (where it is a point). Therefore the neighborhood of this point do not look like $T^3 \times [a,b]$ (with the finite interval $[a,b]$), $t = 1$ is a so-called non-flat point in topology. Especially the neighborhood is not an open subset of $\mathbb{R}^4$ (but rather $\mathbb{R}^4/\mathbb{Z}_3$). All tangent vectors at $t = 1$ vanishes, i.e. the point is non-smooth and cannot be smoothly approximated (like in the case $\text{Cone}(S^3)$). The whole discussion remains true for every 4-space $\text{Cone}(\Sigma)$ with $\Sigma$ a 3-manifold having a different homology then the 3-sphere (see below). Our current understanding of the singularity gives a manifold structure in the neighborhood of the singularity, i.e. Einsteins equation gives a smooth solution except for the singular point. Therefore the model of the 3-torus, the Picard horn or half-turn space is singled out (or one

\[\text{\textsuperscript{b}}\text{Remember, at the singularity all geodesics (with backward time orientability) converges. To see it, consider a non-vanishing vector field which is the obstruction to introduce a Lorentz metric.}\text{\textsuperscript{23}}\text{For contractable manifold $A$ has Euler characteristics $\chi(A) = 1$ but the excision of one point gives the desired result $\chi(A \setminus \{\star\}) = 0$.}\]
has to assume a non-smooth transition). But then we are left with a contractable, smooth 4-manifold as a model of the spacetime (for finite times).

Freedman [25] solves the problem to find all compact, contractable, topological 4-manifolds. He proved that any homology 3-sphere (i.e. a compact 3-manifold with the same homology groups as the 3-sphere) bounds a compact, contractable, topological 4-manifold. But as a corollary to Donaldson’s theory [26, 27], not all homology 3-spheres are the boundary of a smooth compact contractable 4-manifold (see also [28]). One example is the Poincare homology 3-sphere bounding only a topological (but non-smooth) compact contractable 4-manifold. A further division for all possible cases is given by the inclusion of the geometric structure, i.e. the existence of homogeneous metrics on Σ. Especially we concentrated our discussion to the case of prime 3-manifolds. The following cases for prime homology 3-spheres are possible:

(1) Spherical geometry (positive scalar curvature): There are only two homology 3-spheres with this geometry: the 3-sphere $S^3$ and the Poincare sphere (or sums of these cases). The Poincare sphere can be excluded by Donaldson theory (see above). The geometry $S^2 \times \mathbb{R}$ can be also excluded, there is no homology 3-sphere with this geometry.

(2) Flat geometry (Euclidean geometry or NIL geometry): There are no homology 3-sphere carrying a homogeneous, flat metric.

(3) Negative curvatures (hyperbolic and $\tilde{SL}_2$ geometry): The class of homogeneous metrics with a negative curvature (at least along one direction) forms the largest class of possible geometric structures on homology 3-spheres. All members of the special class of Seifert fibred homology 3-spheres have the $\tilde{SL}_2$ geometry. An example is the Brieskorn sphere $\Sigma(2, 5, 7)$ defined by the set

$$\Sigma(p, q, r) = \{(u, v, w) \in \mathbb{C}^3 \mid u^p + v^q + w^r = 0\} \cup S^5$$

for $p = 2, q = 5, r = 7$ (which is the boundary of the famous Mazur manifold [29], a contractable smooth 4-manifold). Examples of hyperbolic homology 3-spheres can be constructed by using ±1 Dehn surgery [30] along hyperbolic knot [4]. Examples of hyperbolic 3-manifolds bounding contractable smooth 4-manifolds are generated by the knots $0_1$ or $8_8$ (in Rolfsen notation, see Fig. 2) whereas hyperbolic knots like the figure 8-knot $4_1$ or the knot $5_2$ are excluded (see Fig. 3).

Now we may expect that the understanding of the simplest pieces is enough to get an overview for the general case. But as usual, 4-dimensional topology is an exception. The connected sum of prime homology 3-spheres can bound a contractable

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A knot $K$ as smooth embedding $S^1 \to S^3$ is hyperbolic if its complement $S^3 \setminus (K \times D^2)$ is a hyperbolic 3-manifold (with boundary a torus), i.e. a 3-manifold with a homogeneous metric in the interior.

dThese knots are all members of the class of slice knots, i.e. knots bounding a disk.
Fig. 2. ±1 Dehn surgery along these knots (6_1, 8_8, 8_9) generates 3-manifolds bounding smoothly contractable 4-manifolds.

Fig. 3. ±1 Dehn surgery along these knots generates (4_1, 5_2, 8_1) 3-manifolds bounding only non-smoothly contractable 4-manifolds.

compact, smooth 4-manifold. As we saw above the Poincare sphere $P$ cannot be a boundary of this kind but the following sum

$$P\#P\#K_1\#K_2\#K_3$$

where $K_i$ are prime hyperbolic homology 3-spheres \(^\dagger\) (for instance generated by the knots $6_1$, $8_8$ or $8_9$ via ±1 Dehn surgery, see Fig. 2). Therefore, if the Poincare model \(^\dagger\dagger\) of the cosmos is correct then one has to use the sum of two Poincare spheres. Otherwise one has to assume a non-smooth topology change in the primordial phase of the cosmos.

3.2. Big bounce case

Our discussion has excluded (like the 3-torus) or restricted (like the Poincare sphere) many interesting models of the cosmos. Here we want to consider the possible restrictions for a model without singularity inspired by Loop quantum gravity. In \(^\dagger\dagger\) Ashtekar et.al. described a cosmological model showing the effect of a Big bounce (see also \(^\dagger\dagger\dagger\)). Then this model does not show a singularity, i.e. there is no Big Crunch and the contraction is followed by an expansion again. The crucial point is the determination of the initial state. The following assumptions are physically plausible:

\(^{\dagger}\)There is only one condition: $K_i$ has to smoothly embed into $S^2 \times S^2$. 

\(^{\dagger\dagger}\)In this context: $K_i$.

\(^{\dagger\dagger\dagger}\)In this context: $K_i$.
• The (spatial) cosmos at the "Big Bounce point" is closed (no boundary).
• It consists of a minimal number of elementary cells.

The first assumption favors a compact, closed 3-manifold. Using homology theory, we have to use at least two cells (glued together along the common boundary) to fulfill the second assumption. It singles out the 3-sphere (uniquely using the solution to the Poincare conjecture). Therefore the Big Bounce model starts with a (Planck-sized) 3-sphere $S^3$ and evolved smoothly to a closed, compact 3-manifold $\Sigma$ (see the Main hypothesis above). Although this starting point is different from the singularity in the previous subsection, the results are quite similar.

At first we remark that the evolution from the 3-sphere $S^3$ to $\Sigma$ is a smooth 4-manifold, a cobordism $W(S^3, \Sigma)$ between $S^3$ and $\Sigma$. We can close partly this cobordism along the 3-sphere by using a 4-disk, $W(S^3, \Sigma) \cup S^3 D^4$, to get a contractable 4-manifold with boundary $\Sigma$. Therefore we obtain the same restrictions as above: $\Sigma$ must be a homology 3-sphere which bounds a smooth, contractable, compact 4-manifold. Again the case of a single Poincare sphere is excluded (by Donaldson theory). The list of prime homology 3-spheres agrees with the list above.

A difference to the results above is the flat case (i.e. with Euclidean or NIL geometry). The simplest example is the 3-torus $T^3$. Now we have to consider a (smooth) cobordism $W(S^3, T^3)$ between the 3-sphere and the 3-torus. But then we obtain a topology change from the simple-connected 3-sphere to the multiple-connected 3-torus. Especially the cobordism $W(S^3, T^3)$ itself is not a simple-connected, compact 4-manifold (in contrast to a cobordism $W(S^3, \Sigma)$ between $S^3$ and the homology 3-sphere). As discussed in [33], the cobordism $W(S^3, T^3)$ (a Morse cobordism in the notation of [33]) gives rise to causal discontinuities (in the sense of the Borde-Sorkin conjecture [33], [34]) and allows the singular propagation of a quantum field (like in the trousers spacetime in 1 + 1 dimensions [35]). But there is also a second argument against the appearance of the cobordism $W(S^3, T^3)$: Clearly $W(S^3, T^3)$ is a Lorentz cobordism but the mod-2 Kervaire semi-characteristics $U(\partial W(S^3, T^3))$ gives

$$U(\partial W(S^3, T^3)) = (\dim H^0(\partial W(S^3, T^3), \mathbb{Z}_2) + \dim H^1(\partial W(S^3, T^3), \mathbb{Z}_2)) \mod 2$$

$$= 5 \mod 2 = 1$$

i.e. the cobordism $W(S^3, T^3)$ is not a Spin-Lorentz cobordism (see [36], [37]) or it do not admit a $SL(2, \mathbb{C})$–spin structure (defining an unique parallel transport of a spinor). The case of a NIL geometry is similar. The corresponding 3-manifold $\tilde{T}^3$ is a twisted 3-torus but leading to the same mod-2 Kervaire semi-characteristics $U(\partial W(S^3, \tilde{T}^3)) = 1$ with the same result: it is not a Spin-Lorentz cobordism.

\footnote{This twisted 3-torus is the mapping torus $M_f(T^2)$: Consider $T^2 \times [0, 1]$ and identify $T^2 \times \{0\}$ and $T^2 \times \{1\}$ by a diffeomorphism $f : T^2 \to T^2$. If $f$ is of finite order then $M_f(T^2)$ has Euclidean geometry and if $f$ is a Dehn twist then $M_f(T^2)$ has NIL geometry. The remaining case (Anosov map) leads to the SOL geometry (as twisted $\mathbb{H}^2 \times \mathbb{R}$ geometry with negative curvature).}
4. Conclusion

In this paper we discussed the differential-topological restrictions on the spacetime for the evolution of our universe with an explicit Big Bang singularity and for models with the Big Bounce effect. Surprisingly, the results are the same for both cases. Furthermore, we showed that the main restriction comes from the assumption of a smooth spacetime. The relaxation of this smoothness assumption causes in bad singularities like non-manifold points or complicated topology changes with causal discontinuities. Especially the Poincare sphere, the Picard horn or the 3-torus are not favored.

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Appendix A. Connected and boundary-connected sum of manifolds

Now we will define the connected sum # and the boundary connected sum ♭ of manifolds. Let $M, N$ be two $n$-manifolds with boundaries $\partial M, \partial N$. The connected sum $M \# N$ is the procedure of cutting out a disk $D^n$ from the interior $\text{int}(M) \setminus D^n$ and $\text{int}(N) \setminus D^n$ with the boundaries $S^{n-1} \sqcup \partial M$ and $S^{n-1} \sqcup \partial N$, respectively, and gluing them together along the common boundary component $S^{n-1}$. The boundary $\partial(M \# N) = \partial M \sqcup \partial N$ is the disjoint sum of the boundaries $\partial M, \partial N$. The boundary connected sum $M \# N$ is the procedure of cutting out a disk $D^{n-1}$ from the boundary $\partial M \setminus D^{n-1}$ and $\partial N \setminus D^{n-1}$ and gluing them together along $S^{n-2}$ of the boundary. Then the boundary of this sum $M \# N$ is the connected sum $\partial(M \# N) = \partial M \# \partial N$ of the boundaries $\partial M, \partial N$.

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