LOWER BOUND OF MODIFIED $K$-ENERGY ON A FANO MANIFOLD
WITH DEGENERATION FOR KÄHLER-RICCI SOLITONS

LIANG ZHANG

ABSTRACT. In this paper, we extend Tosatti’s method to study the lower boundedness of modified $K$-energy on a Fano manifold and apply this result to study the relative $K$-stability of the deformation space of a Kähler Ricci soliton.

CONTENTS

0. Introduction 1
1. Preliminary 2
2. Proof of Theorem 0.1 3
3. Proof of the Theorem 0.2 5
References 6

0. Introduction

Let $(M, J)$ be a Fano manifold with soliton vector field $X$. By the virtue of Yau-Tian-Donaldson conjecture, the study of Kähler Ricci soliton is related to the notion of $K$-stability for $(M, X)$. For example, it is well known that the existence is equivalent to the $K$-polystability (see [2] or [13]). We are interested to establish the semistable version of Yau-Tian-Donaldson correspondence.

When $X = 0$, Li [11] solved this problem by showing a lot of equivalent characterization of $K$-semistability. The most important contribution of his proof is the implication from $K$-semistability to the lower boundedness of $K$-energy. This is a generalization of the result of Chen [4] and Tosatti [15] who derived the lower boundedness under the assumption that $M$ admits a smooth degeneration with Kähler Einstein metric.

However, for the nontrivial soliton case, it seems that the implication remains unknown. Fortunately, we still know that the $K$-semistability is equivalent to the existence of $K$-polystable degeneration [8]. Thus this problem can be reduced to researching whether the existence of polystable degeneration implies the lower boundedness of the modified $K$-energy. The main purpose of this paper is to derive the implication under the assumption that the polystable degeneration is smooth.

Our method is a generalization of Tosatti’s proof [15] for the Kähler Einstein case. The key technique of his proof is a slope-type inequality about the $K$-energy, which was discovered by Chen [3]. This inequality was proved by many different methods (see also [5]) and had also been used to prove the lower boundedness of K energy along Calabi flows [6].

Note that this slope-type inequality can be generated for the modified $K$-energy (and other energy in more general situations [1]). We will prove the following theorem in Section 2:

Theorem 0.1. Let $\pi : M \rightarrow \mathbb{C}$ be a smooth special degeneration associated to the soliton action induced by $X$. Suppose that there is a $T \times S^1$ invariant Kähler metric near central fiber (c.f. Section 1) and the central fiber $M_0$ admits a Kähler Ricci soliton. Then the modified $K$-energy on $M$ is bounded from below.
After establishing this theorem, we can apply it to study the deformation space of Eiji Inoue [10], which is the same as the definition of the kernel space of second order variation of Perelman’s entropy [16]. We will prove:

**Theorem 0.2.** Let \((M, J_0)\) be a Fano manifold which admits a Kähler Ricci soliton, \((M, J)\) be a sufficiently small deformation of \((M, J_0)\). Suppose that the soliton vector field on \((M, J_0)\) can be lifted to \((M, J)\). Then the modified K energy on \((M, J)\) is bounded from below.

We obtain a family of smooth manifolds on each of which the modified K energy is bounded from below. Furthermore, the lower boundedness of modified K-energy for \((M, X)\) implies that the energy level of \(M\) satisfies

\[
\sup_{\omega_g \in 2\pi c_1(M)} \lambda(g) = (2\pi)^{-n}(nV - N_X(c_1(M)))
\]

(see [7] and [17]). Thus we derive that the energy level of manifold in Theorem 0.2 is independent of the complex structure, which has been observed in [16] by the method of Kähler Ricci flow.

This paper is organized as follows:

In Section 1, we recall the notion of special degeneration and study some basic setups. In Section 2, we prove Theorem 0.1. Finally in Section 3, we prove Theorem 0.2 by showing that the manifold appearing in Theorem 0.2 admits a smooth special degeneration.

1. Preliminary

In this section, we recall the notion of special degeneration and study some basic setups. Let \(M\) be a Fano manifold with \(X\) being a soliton vector field on \(M\).

Recall that a special degeneration of a Fano manifold \(M\) is a normal variety \(\mathcal{M}\) with a \(\mathbb{C}^*\)-action satisfying the follow conditions [13]:

1. There exists a flat \(\mathbb{C}^*\)-equivariant map \(\pi : \mathcal{M} \to \mathbb{C}\) such that \(\pi^{-1}(t)\) is biholomorphic to \(M\) for any \(t \neq 0\);
2. There exists an holomorphic line bundle \(L\) on \(\mathcal{M}\) such that for any \(t \neq 0\), \(L|_{\pi^{-1}(t)}\) is isomorphic to \(K_{\mathcal{M}}^{-r}\) for some integer \(r > 0\);
3. The center \(M_0 = \pi^{-1}(0)\) is a \(Q\)-Fano variety.

The following definition can be seen in [18].

**Definition 1.1.** \(\mathcal{M}\) is called a special degeneration associated to the soliton action induced by \(X\) if \(\sigma^\pi_t\) communicates to \(\sigma^X_t\), where \(\sigma^X_t\) and \(\sigma^\pi_t\) are two lifting one-parameter subgroups on \(\mathcal{M}\) induced by \(X\) and the holomorphic vector field \(v\) associated to the \(\mathbb{C}^*\) action, respectively.

If \(M_0\) is smooth and there exists an neighborhood \(\Delta = \{|z| < \epsilon\}\) such that \(\pi^{-1}(\Delta)\) admits a \(T \times S^1\) invariant Kähler metric \(\Omega\). We call \(\mathcal{M}\) a smooth special degeneration with a \(T \times S^1\) invariant Kähler metric near central fiber. Here \(T\) and \(S\) are one-parameter subgroups on \(\mathcal{M}\) induced by \(\xi = \text{Im}(X)\) and \(\text{Im}(v)\), respectively.

Since \(M_0\) is smooth, we know that \(\mathcal{M}\) is smooth and \(\pi\) is holomorphic proper submersion. By Ehremsmann’s theorem, we can find a neighborhood \(\Delta = \{|z| < \epsilon\}\) of 0 and a diffeomorphism

\[
F : M \times \Delta \mapsto \pi^{-1}(\Delta)
\]

such that \(\pi(F(m, z)) = z\). Here we use \(M\) to denote the underlying differential manifold of \((M, J)\).

By the definition of \(\mathcal{M}\), there is a \(T \times \mathbb{C}^*\) action on \(\mathcal{M}\) such that \(\pi\) is \(T \times \mathbb{C}^*\) equivalent. We may induce a local action of \(T \times \mathbb{C}^*\) on \(M \times \Delta\) by \(F\), which satisfying:

\[
(w, s) \cdot (m, z) = F^{-1}((w, s) \cdot F(m, z)),
\]

if \(sz \in \Delta\). Note that \(T \times S^1\) maps \(M \times \Delta\) to itself. Hence this local action forces \(M \times \Delta\) to admit a \(T \times S^1\) action.
We can also induce a Kähler metric on $M \times \Delta$ through $F$. Since $F$ is $T \times S^1$ equivalent, this metric is also $T \times S^1$ invariant. We still denote by $\Omega$. Let $V$ be the real vector field on $M \times \Delta$ which generates the action of $S^1$ on $M \times \Delta$. Thus we have

$$L_V \Omega = d\iota_V \Omega = 0.$$  

Since $H^1(M \times \Delta, \mathbb{R}) = 0$, we may find a smooth function $H_V$ on $M \times \Delta$ such that

$$\iota_V \Omega = dH_V.$$  

Similarly, let $W$ be the real vector field on $M \times \Delta$ which generates the action of $T$ on $M \times \Delta$, and we may find a smooth function $H_W$ such that

$$\iota_W \Omega = dH_W.$$  

Let $\mathcal{J} = F^* J_M$ be the complex structure induced by $F$. Here $J_M$ is the complex structure of $M$. It is easy to see that $\sqrt{-1} W + J_W$ tangents to each fiber $M_z$ and it’s restriction $X_z = \sqrt{-1} W|_{M_z} + J|_{M_z} W|_{M_z}$ is the soliton vector field on $M_z$. By restricting (1.5) we see that the soliton potential of $X_z$ on $M_z$ respect to $\Omega|_{M_z}$ is $H_W|_{M_z}$.

In addition, we may construct a family of metric on $M$ by using the action of $\mathbb{C}^*$. Let

$$F_t : M \times \Delta \mapsto M \times \Delta, F_t(m,z) = e^{-t} \cdot (m,z), t > 0$$

and $f_t = F_t \circ i$, where $i : M \mapsto M \times \Delta, i(m) = (m, 1)$. We can define

$$\omega_t = f_t^* \Omega$$

as a family of Kähler metric on $M$. We will show that this family decay fast in some sense.

Let $\rho_t : M_{e^{-t}} \mapsto M$ be the inverse of $f_t : M \mapsto f_t(M)$. Note that $\rho_t^* \omega_t = \Omega|_{M_{e^{-t}}}$. We conclude that

$$\|\rho_t^* \omega_t - \Omega|_{M_0}\|_g \leq C e^{-t}. $$

Here $g$ is a fixed Riemannian metric on $M$.

In addition, we may write $\omega_t$ as $\omega_t = \omega_0 + dd^c \varphi_t$. Since

$$\frac{d}{dt} \omega_t = dd^c f_t^* H_V.$$  

We may assume that $\varphi_t = f_t^* H_V$. As a result, we have that

$$\|\rho_t^* \varphi_t - H_V|_{M_0}\|_g \leq C e^{-t}. $$

Finally, since the isomorphism $f_t$ pulls back the soliton vector field $X_{e^{-t}}$ on $M_{e^{-t}}$ to $X$, we conclude that the soliton potential $\theta_t = \theta_X(\omega_t)$ of $X$ respect to $\omega_t$ is $f_t^* H_W$. Consequently, we have

$$\|\rho_t^* \theta_t - H_W|_{M_0}\|_g \leq C e^{-t}. $$

2. Proof of Theorem 0.1

In this section we prove the Theorem 0.1.

Proof of Theorem 0.1. Let $\Omega$ be a $T \times S^1$ invariant Kähler metric on $M \times \Delta$.

Claim 2.1. We may assume that $\Omega|_{M_0}$ is the soliton metric of $M_0$ respect to soliton vector field $X_0$.

Let $\omega_t$ be the family of metric on $M$ defined in Section 1 and $\omega = \omega_0$. We will prove that $\mu_\omega$ is bounded from below.

Let $\varphi \in \mathcal{M}_\omega$, where

$$\mathcal{M}_\omega = \{ \varphi \in C^\infty(M) | \omega + dd^c \varphi > 0, \text{Im}(X)(\varphi) = 0 \}. $$
We may choose a path \( \varphi_t, t \in [-1, 0] \) such that \( \varphi_{-1} = \varphi_1 \) and \( \varphi_0 = 0 \). Connecting it with \( \varphi_t, t \geq 0 \) we get a ray \( \{ \varphi_t : t \geq -1 \} \). Then for \( t > 0 \), the derivative of \( \mu_\omega(\varphi_t) \) is
\[
\frac{d}{dt}\mu_\omega(\varphi_t) = -\int_M \hat{\varphi}^*(t)(\Delta \hat{g}_t + X)(h_{\omega_t} - \theta_t)\omega_t^n
\]
(2.2) 
\[
= -\int_M \rho_t^* \hat{\varphi}(t)(\Delta \rho_t^* g_t + X_{e^{-t}})(h_{\rho_t^* \omega_t} - \rho_t^* \theta_t)\rho_t^* \omega_t^n.
\]

Since \( \Delta \rho_t^* g_t h_{\rho_t^* \omega_t} = R(\rho_t^* g_t) - n \), by (1.8) we see that
\[
(2.3) \quad \| h_{\rho_t^* \omega_t} - h_{\Omega|M_0} \| \leq Ce^{-t}.
\]

It follows from (1.11) and (2.3) that
\[
(2.4) \quad \| h_{\rho_t^* \omega_t} - \rho_t^* \theta_t - (h_{\Omega|M_0} - H_W|\Omega|M_0) \| \leq Ce^{-t}.
\]

Note that \( \Omega|M_0 \) is a soliton metric and \( H_W|\Omega|M_0 \) is soliton potential. We have
\[
(2.5) \quad h_{\Omega|M_0} = H_W|\Omega|M_0.
\]

It follows that
\[
(2.6) \quad \| h_{\rho_t^* \omega_t} - \rho_t^* \theta_t \| \leq Ce^{-t}.
\]

Meanwhile, by (1.10) and (1.8) and the fact that
\[
(2.7) \quad \| X_{e^{-t}} - X_0 \| \leq Ce^{-t},
\]
we derive that \( \frac{d}{dt}\mu_\omega(\varphi_t) \) converges exponentially to
\[
(2.8) \quad -\int_M H_V(\Delta g|\Omega|M_0 + X_0)(h_{\Omega|M_0} - H_W|\Omega|M_0)\Omega|M_0^n = 0.
\]

As a result, we have
\[
(2.9) \quad \mu_\omega(\varphi_t) \geq -C.
\]

Furthermore, we have the Chen inequality (see Corollary 1 in [1]) for modified K-energy
\[
(2.10) \quad \mu_\omega(\varphi_{-1}) \geq \mu_\omega(\varphi_t) - d(\varphi_{-1}, \varphi_t)\sqrt{Ca(\omega_t)}.
\]

Here
\[
(2.11) \quad d(\varphi_{-1}, \varphi_t) = \int_{-1}^t \int_M (\hat{\varphi}(s))^2 \omega_s^n ds
\]
and
\[
(2.12) \quad Ca(\omega_t) = \int_M [(\Delta g_t + X)(h_{\omega_t} - \theta_t)]^2 e^{2\theta_t} \omega_t^n.
\]

By (1.8) and (2.6), we conclude that
\[
(2.13) \quad |Ca(\omega_t)| \leq \int_M [(\Delta g_t + X_0)(h_{\Omega|M_0} - H_W|\Omega|M_0)]^2 e^{2H_W|\Omega|M_0} \Omega|M_0^n | \leq Ce^{-2t}.
\]

Hence by (2.5), it follows that
\[
(2.14) \quad Ca(\omega_t) \leq Ce^{-2t}.
\]

Finally, we see that for \( s > 0 \),
\[
(2.15) \quad \int_M (\hat{\varphi}(s))^2 \omega_s^n = \int_M (\rho_t^* \hat{\varphi}(s))^2 \rho_t^* \omega_s^n.
\]

It follows from (1.8) and (1.10) that \( \int_M (\hat{\varphi}(s))^2 \omega_s^n \) is uniformly bounded for \( s > 0 \). Thus we have
\[
(2.16) \quad d(\varphi_{-1}, \varphi_t) = \int_{-1}^0 \int_M (\hat{\varphi}(s))^2 \omega_s^n ds + \int_0^t \int_M (\hat{\varphi}(s))^2 \omega_s^n ds \leq Ct + D.
\]
Combining (2.10), (2.11), (2.14) and (2.16), we conclude that
\[(2.17) \quad \mu_{\omega}(\varphi) = \mu_{\omega}(\varphi_{-1}) \geq -C.\]
Thus \(\mu_{\omega}\) is bounded from below. We finish the proof. \(\square\)

To complete the proof, we prove Claim (2.1) as following:

**Proof of Claim (2.1).** Since we assume that \(M_0\) admits a Kähler Ricci soliton, and the action of \(S^1\) commutes with the action of \(T\) on \(M_0\), we may find a \(T \times S^1\) invariant function \(\hat{\psi}\) on \(M_0\) such that \(\Omega|_{M_0} + d\bar{\partial}\hat{\psi}\) is the soliton metric of \(M_0\) with respect to soliton vector field \(X_0\). As \(T \times S^1\) is compact, we can extend \(\hat{\psi}\) to be a \(T \times S^1\) invariant smooth function on \(\hat{M} \times \Delta\). We denote it by \(\psi\). Shrinking \(\Delta\) if it is necessary, we may assume that \(\Omega + d\bar{\partial}\psi + add^c|z|^2\) is a \(T \times S^1\) invariant Kähler metric on \(\hat{M} \times \Delta\) such that \((\Omega + d\bar{\partial}\psi + add^c|z|^2)|_{M_0}\) is a soliton metric of \(M_0\) respect to soliton vector field \(X_0\). Here \(a > 0\) is a big positive number. As a result, replacing \(\Omega\) by \(\Omega + d\bar{\partial}\psi + add^c|z|^2\), we conclude that Claim (2.1) is true. \(\square\)

3. PROOF OF THE THEOREM 0.2

In this section we prove the Theorem 0.2

**Proof of the Theorem 0.2.** First at all, we may construct a smooth special degeneration associated to the soliton action on \((M, J)\). We refer the readers to the proof of Theorem 0.2 in [16] for the details. Since the soliton vector field of \((M, J)\) can be lifted to \((M, J)\), we know that the Kähler Ricci flow \((M, g(t))\) on \((M, J)\) converges smoothly to a Kähler Ricci soliton \((M_\infty, J_\infty, g_\infty)\) by the Theorem 0.1 in that paper. Then we can embed \((M, g(t))\) to a projective space \(\mathbb{P}^N\) by partial \(C^0\)-estimate for \(t \geq t_0\) with \(\sigma_\infty\) being regarded as a subgroup of \(\text{SL}(N + 1, \mathbb{C})\). By GIT, we will find a fixed number \(t_1 \geq t_0\) and a one parameter subgroup \(\sigma_t \subseteq \text{SL}(N + 1, \mathbb{C})\) which commutes with \(\sigma_\infty\) such that \(\sigma_t(M_{t_1})\) converges to a limit cycle \(\hat{M}_\infty\) which is isomorphic to \((M_\infty, J_\infty)\). Hence we can construct a special degeneration \(\mathcal{M} \subset \mathbb{P}^N \times \mathbb{C}\) as the compactification of
\[(3.1) \quad S = \{(x, t) \in \mathbb{P}^N \times \mathbb{C} | x \in \sigma_t(M_{t_1})\},\]
whose central fiber is \(\tilde{M}_\infty\) [12]. There is a nature way to introduce the action of \(\mathbb{C}^* \times \mathbb{C}^*\) on \(\mathbb{P}^N \times \mathbb{C}\) as
\[(3.2) \quad (t, s)(x, z) = (\sigma_t \sigma_\infty(x), tz), \quad (t, s) \in \mathbb{C}^* \times \mathbb{C}^*, \quad (x, z) \in \mathbb{P}^N \times \mathbb{C}.
\]
Note that \(S\) is invariant under the action of \(\mathbb{C}^* \times \mathbb{C}^*\). We know that \(\mathcal{M}\) is also invariant. Thus \(\mathcal{M}\) is a special degeneration of \((M, J)\) associated to the soliton action. Since the central fiber is \(\tilde{M}_\infty\) and this family is flat, we conclude that it is also a smooth special degeneration and \(\mathcal{M}\) is a smooth submanifold of \(\mathbb{P}^N \times \mathbb{C}\) (see proposition 10.2 in [9]).

Secondly, as the compact subgroup of \(\sigma_t\) commutes with the compact subgroup of \(\sigma_\infty\), we may find a Kähler metric \(\omega\) of \(\mathbb{P}^N\) such that \(\omega\) is invariant under the action of these two compact subgroups. Therefore, we can construct a \(T \times S^1\) invariant metric \(\mathbb{P}^N \times \mathbb{C}\) as
\[(3.3) \quad \Omega = \omega + \sqrt{-1}dz \wedge d\bar{z}.\]
Restricting \(\Omega\) to the \(\mathcal{M}\), we derive a \(T \times S^1\) invariant metric of \(\mathcal{M}\).

Finally, we can apply the Theorem 0.1 to finish the proof of Theorem 0.2. \(\square\)

**Remark 3.1.** We have shown that for Kähler Ricci soliton \((M, J, \omega_{FS})\), the soliton metric \(\omega_{FS}\) can be viewed as a Kähler metric on each manifold appearing in the deformation family of it [10]. So we can construct a \(K\) invariant Kähler metric on the deformation space. Here \(K\) is a maximal compact subgroup of \(\text{Aut}_J(M, J)\) respect to \(\omega_{FS}\). Hence, by GIT and Eiji Inoue’s deformation Theorem [10] we may construct a smooth degeneration with \(T \times S^1\) invariant Kähler metric near central fiber for each manifold appearing in this family. As a result, we can also prove Theorem 0.2 by Theorem 0.1 and this construction.
References

[1] Abdellah, L., Convexity of the weighted Mabuchi functional and the uniqueness of weighted extremal metrics, arXiv:2007.01345.

[2] Berman, R. and Nystrom, D., Complex optimal transport and the pluripotential theory of Kähler-Ricci solitons, arXiv: 1401.8264.

[3] Chen, X., Space of Kähler metrics. III. On the lower bound of the Calabi energy and geodesic distance, Invent. Math. 175 (2009), no. 3, 453-503.

[4] Chen, X., Space of Kähler metrics (IV)–On the lower bound of the K-energy, arXiv:0809.4081.

[5] Chen, X., Space of Kähler metrics (V)—Kähler quantization, Progr. Math., 297, Birkhäuser/Springer, Basel, 2012.

[6] Chen, X. and Sun, S., Calabi flow, geodesic rays, and uniqueness of constant scalar curvature Kähler metrics, Ann. of Math. (2) 180 (2014), no. 2, 407-454.

[7] Dervan, R. and Székelyhidi G., Kähler-Ricci flow and optimal degenerations, J. Differential Geom. 116 (2020), no. 1, 187-203.

[8] Han, J.Y., Li, C., Algebraic uniqueness of Kähler-Ricci flow limits and optimal degenerations of Fano varieties, arXiv:2009.01010.

[9] Hartshorne, R., Algebraic geometry. Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp. ISBN: 0-387-90244-9 14-01

[10] Inoue, E., The moduli space of Fano manifolds with Kähler-Ricci solitons, Adv. Math. 357 (2019), 106841, 65 pp.

[11] Li, C., Yau-Tian-Donaldson correspondence for K-semistable Fano manifolds, J. Reine Angew. Math, 733 (2017), 55-85.

[12] Mumford, D., Stability of projective varieties, Enseign. Math. (2) 23 (1977), no. 1-2, 39-110.

[13] Székelyhidi, G., The Kähler-Ricci flow and K-polystability, Amer. J. Math. 132 (2010), no. 4, 1077-1090.

[14] Tian, G., Kähler-Einstein metrics with positive scalar curvature, Invent. Math. 130 (1997), no. 1, 1-37.

[15] Tosatti, V., The K-energy on small deformations of constant scalar curvature Kähler manifolds, Advances in geometric analysis, 139-147, Adv. Lect. Math. (ALM), 21, Int. Press, Somerville, MA, 2012.

[16] Tian, G. and Zhang, L. and Zhu, X. H., Kähler Ricci flow for deformed complex structure, arXiv: 2107.12680.

[17] Wang, F. and Zhu, X. H., Uniformly strong convergence of Kähler-Ricci flows on a Fano manifold, arXiv: 2009.10354

[18] Wang, F., Zhou, B. and Zhu, X. H., Modified Futaki invariant and equivariant Riemann-Roch formula Adv. Math. 289 (2016), 1205-1235.

School of Mathematical Sciences, Peking University, Beijing 100871, China.
Email address: tensor@pku.edu.cn