Research Article

Existence and Stability of Solutions of Fuzzy Fractional Stochastic Differential Equations with Fractional Brownian Motions

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Received 26 May 2021; Revised 17 June 2021; Accepted 11 August 2021; Published 2 September 2021

1. Introduction

There appears to be confusion of various kinds in the modeling of several real world systems, such as trying to characterize a physical system and opinions on its parameters. To deal with this ambiguity, the fuzzy set theory will be used [1]. It is able to handle such linguistic statements mathematically using this theory, such as “large” and “less.” The capacity to investigate fuzzy differential equations (FDEs) in modeling numerous phenomena, including imprecision, is provided by a fuzzy set. In particular, the fuzzy stochastic differential equations (FSDEs), in instance, might be used to investigate a variety of economics and engineering problems that involve two types of uncertainty: randomness and fuzziness.

The fuzzy Itô stochastic integral was powered in [2, 3]. In [4, 5], the fuzzy stochastic integral is driven by the Wiener process as a fuzzy adapted stochastic process. In [6], Fei et al. studied the existence and uniqueness of solutions to the (FSDEs) under non-Lipschitzian condition. In [7], Jafari et al. study FSDEs driven by fBm. Jialu Zhu et al., in [8], prove existence of solutions to SDEs with fBm. Ding and Nieto [9] investigated analytical solutions of multitime-scale FSDEs driven by fBm. Vas’kovskii et al. [10] prove that the pth moments, \( p \geq 1 \), of strong solutions of a mixed-type SDEs are driven by a standard Brownian motion and a fBm. Despite the fact that some research exists on the problem of the uniqueness and existence of solutions to SDEs and FSDEs which are disturbed by Brownian motions or semimartingales [4, 11–15], a kind of the FFSDEs driven by an fBm has not been investigated. Agarwal et al. [16, 17] considered the concept of solution for FDEs with uncertainty and some results on FFDEs and optimal control nonlocal evolution equations. Recently, Zhou et al., in [18–20], gave some important works on the stability analysis of such SFDEs. Our results are inspired by the one in [21] where the existence and uniqueness results for the FSDEs with local martingales under the Lipschitzian conditions are studied. The rest of this paper is given as follows. Section 2 summarizes the fundamental aspects. In Section 3, existence and uniqueness of solutions to the FFSDs are proved. Moreover, the stability of solutions is studied in Section 4. Finally, in Section 5, a conclusion is given.

2. Preliminaries

This part introduces the notations, definitions, and background information that will be utilized throughout the article.

Let \( K(\mathbb{R}^n) \) be the family of nonempty convex and compact subsets of \( \mathbb{R}^n \). In \( K(\mathbb{R}^n) \), the distance \( d_H \) is defined by
We denote by $\mathcal{M}(\Omega, \mathcal{A}; K(\mathbb{R}^n))$ the family of $\mathcal{A}$-measurable multifunction, taking value in $K(\mathbb{R}^n)$.

**Definition 1** (see [21, 22]). A multifunction $G \in \mathcal{M}(\Omega, \mathcal{A}; K(\mathbb{R}^n))$ is called $\mathcal{L}^p$-integrably bounded if $\exists h \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathbb{R}^n)$ such that $\|G\| \leq h$ a.e. where

$$\mathcal{L}^p(\Omega, \mathcal{A}, P; K(\mathbb{R}^n)) = \{ G \in \mathcal{M}(\Omega, \mathcal{A}; K(\mathbb{R}^n)) : \|G\| \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathbb{R}^n) \}. \quad (3)$$

Let $E^n$ denote the set of the fuzzy $x : \mathbb{R}^n \rightarrow [0, 1]$ such that $[x]_a \in K(\mathbb{R}^n)$, for every $a \in [0, 1]$, where $[x]_a = \{ a \in \mathbb{R}^n : x(a) \geq a \}$, for $a \in (0, 1]$, and $[x]_\infty = \{ a \in \mathbb{R}^n : x(a) > 0 \}$. Let the metric be $d_{\infty}(x, y) = \sup_{a \in [0, 1]} d_H(x, y)$, $[x]_a \in K(\mathbb{R}^n)$, in $E^n$, $a \in \mathbb{R}$; we have $d_{\infty}(x + z, y + z) = d_{\infty}(x, y)$, $d_{\infty}(x + y, z + w) \leq d_{\infty}(x, z) + d_{\infty}(x, y) + d_{\infty}(y, w)$, and $d_{\infty}(ax, ay) = |a| d_{\infty}(x, y)$.

**Definition 2** (see [23]). Let $f : [c, d] \rightarrow E^n$; the fuzzy Riemann–Liouville integral of $f$ is given by

$$\left(J_c^a f\right)(u) = \frac{1}{\Gamma(a)} \int_c^u (u - v)^{a-1} f(v)dv. \quad (4)$$

If $g(u) = b$ is constant on $[0, T)$, the previous inequality is transformed into

$$f(u) \leq b E_a(KT(u)a^u), \quad u \in [0, T), \quad (8)$$

where $E_a$ is given by

$$E_a(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(ma + 1)}. \quad (9)$$

**Remark 1** (see [24]). For all $u \in [0, T)$, $\exists N^*_k > 0$ does not depend on $b$ such that $f(u) \leq N^*_k b$.

**Definition 4** (see [21, 22]).

A function $f : \Omega \rightarrow E^n$ is said fuzzy random variable if $[f]^a$ is an $\mathcal{A}$-measurable random variable $\forall a \in [0, 1]$;

A fuzzy random variable $f : \Omega \rightarrow E^n$ is said $\mathcal{L}^p$-integrably bounded, $p \geq 1$, if $[f]^a \in \mathcal{L}^p(\Omega, \mathcal{A}, P; K(\mathbb{R}^n))$, $\forall a \in [0, 1]$.

Let $\mathcal{L}^p(\Omega, \mathcal{A}, P; E^n)$ denote the set of all fuzzy random variables; they are $\mathcal{L}^p$-integrably bounded.

For the notion of an fBm, we referred to [25].

Let us define a sequence of partitions of $[a, b]$ by $\{ \psi_m, m \in \mathbb{N} \}$ such that $|\psi_m| \rightarrow 0$ as $m \rightarrow \infty$. If, in $L^2(\Omega, \mathcal{A}, P)$, $\sum_{i=0}^{m-1} \phi(t_i^m)(B^i(t_i^m) - B^i(t_i^{m+1}))$ converge to the same limit for all this sequences $\{ \psi_m, m \in \mathbb{N} \}$, then this limit is said a Stratonovich-type stochastic integral and noted by $\int_\phi \phi(s)dB^H(s)$. Let $J : [0, T]$, where $0 < T < \infty$.

**Definition 5** (see [21, 22]).

A function $f : J \times \Omega \rightarrow E^n$ is called fuzzy stochastic process; if $\forall t \in J$, $f(t, \cdot) = f(t) : \Omega \rightarrow E^n$ is a fuzzy random variable.

A fuzzy stochastic process $f$ is continuous; if $f(\cdot, v) : J \rightarrow E^n$ are continuous, and it is $\mathcal{A}^H_t$-adapted if for every $a \in [0, 1]$ and for all $t \in J$,

$$[f(t)]^a : \Omega \rightarrow K(\mathbb{R}^n) \text{ is } \mathcal{A}^H_t \text{-measurable.}$$

On the other hand, in fuzzy fractional calculus, the fractional integral of a fuzzy set $\bar{A}$ is defined as

$$J^\alpha \bar{A}(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{(x - u)^{\alpha-1}}{\Gamma(\alpha)} \bar{A}(u)du, \quad \alpha > 0.$$
Now, we investigate the FFSDEs driven by an fBm given by

\[ f(s) = \int_0^s \psi(w)dB_H(w) \]

where \( \psi \) is a \( \mathcal{F}_t \)-adapted process with \( \sup_{s \in [0, T]} \| \psi(s) \|_{L^2} < \infty \).

**Proposition 1** (see [5]). For \( f \in \mathcal{L}^p(J \times \Omega; E^n) \) and \( p \geq 1 \), we have \( J \times \Omega \ni (t, \omega) \mapsto \int_0^t f(s, \omega)ds \in \mathcal{L}^p(J \times \Omega; E^n) \) and \( d_{\infty} \)-continuous.

**Proposition 2** (see [5]). For \( f, g \in \mathcal{L}^p(J \times \Omega; E^n) \) and \( p \geq 1 \), we have

\[ \mathbb{E} \sup_{t \in [0, T]} \| f(t, \omega) - g(t, \omega) \|_{L^p} \leq 2^{p-1} \| f - g \|_{L^p} \]

**3. Main Result**

Now, we investigate the FFSDEs driven by an fBm given by

\[ f(s) = \int_0^s \psi(w)dB_H(w) \]

where \( \psi \) is a \( \mathcal{F}_t \)-adapted process with \( \sup_{s \in [0, T]} \| \psi(s) \|_{L^2} < \infty \).

**Definition 7** (see [21, 22]). A fuzzy process \( f: J \times \Omega \rightarrow E^n \) is said \( \mathcal{L}^p \)-integrally bounded if \( \exists h \in \mathcal{L}^p(J \times \Omega; N; R) \)

\[ d_{\infty} \left( f(s, \omega), 0 \right) \leq h(s, \omega) \]

We denote by \( \mathcal{L}^p(J \times \Omega; N; E^n) \) the set of all \( \mathcal{L}^p \)-integrally bounded and nonanticipating fuzzy stochastic processes.

**Proposition 3** (see [26]). Let \( \psi: J \rightarrow \mathbb{R}^n \); then, for \( t \in J \),

\[ \sup_{a \in [0, t]} \mathbb{E} \left( \int_0^a \psi(s)dB_H(s) \right)^2 \leq c_t \int_0^t \| \psi(s) \|^2 ds. \]

Let us define the embedding of \( \mathbb{R}^n \) to \( E^n \) as \( \langle \cdot \rangle: \mathbb{R}^n \rightarrow E^n \):

\[ \langle r \rangle(a) = \begin{cases} 1, & \text{if } a = r, \\ 0, & \text{if } a \neq r. \end{cases} \]

Now, we define \( d_{\infty}(x, y) \) as

\[ d_{\infty}(x, y) = \sup_{a \in [0, T]} \mathbb{E} \left( \int_0^a \psi(s)dB_H(s) \right)^2 \]

where \( \psi \) is a \( \mathcal{F}_t \)-adapted process with \( \sup_{s \in [0, T]} \| \psi(s) \|_{L^2} < \infty \).

**Definition 8**. A process \( x: J \times \Omega \rightarrow E^n \) is said to be a solution to equation (14) if the following holds:

(i) \( x \in \mathcal{L}^2(J \times \Omega; N; E^n) \).
(ii) \( x \) is \( d_{\infty} \)-continuous.
(iii) We have \( x(t) = x_0 + \mathbb{E} \int_0^t f(s, x(s))ds \leq x_0 + \mathbb{E} \int_0^t g(s)dB_H(s) \).

**Proposition 4** (see [5]). Assume that the function \( \psi: J \rightarrow \mathbb{R}^n \) satisfies \( \int_0^T \| \psi(s) \|^p ds < \infty \). Then,

(i) The fuzzy stochastic integral \( \langle \int_0^T \psi(s)dB_H(s) \rangle (\omega) \in L^2(J \times \Omega; N; E^n) \).
(ii) For \( x \in L^2(J \times \Omega; N; E^n) \), for all \( u \in J \),

\[ x(t) \leq x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} dB_H(s) \]

We will assume that all through this paper, \( f: (J \times \Omega) \times E^n \rightarrow E^n \) is \( \mathcal{L}_0 \)-measurable. Let the following assumptions be introduced:

(\( \mathcal{A}1 \)) If \( x_0 \) is \( \mathcal{L}_0 \)-measurable, we have

\[ \mathbb{E}d_{\infty}(x_0, 0) < \infty. \]

(\( \mathcal{A}2 \)) For \( f(s, \omega) \) and \( g \), we have

\[ \max \{d_{\infty}(f(s, \omega), 0), \| g \| \} \leq c, \]

for every \( s \in J \).

(\( \mathcal{A}3 \)) For all \( x, w \in E^n \),

\[ d_{\infty}(f(s, x), f(s, w)) \leq d_{\infty}(x, w). \]
where \( c \) is equal to one in (\( \mathcal{A}2 \)).

Let us now introduce the main theorem in this part.

**Theorem 1.** Under assumptions (\( \mathcal{A}1 \))-(\( \mathcal{A}3 \)) and \( x_0 \in L^2(\Omega, \mathcal{A}_0, P; E^n) \), the equation (14) has a unique solution \( x(t) \).

\[
x_n(t) = x_0 + \frac{1}{\Gamma(a)} \int_0^t f\left(s, x_{n-1}(s)\right) (t - s)^{-a} ds + \frac{1}{\Gamma(a)} \int_0^t \frac{g(s)}{(t - s)^{1-a}} dB^H(s).
\]

It is clear that \( x_n(s) \) are in \( L^2(\Gamma \times \Omega; E^n) \) and \( d_{\alpha} \)-continuous. Indeed, we have \( x_0 \in L^2(\Gamma \times \Omega; E^n) \) and \( x_0 \) is \( d_{\alpha} \)-continuous.

**Proof.** The method of successive approximations will be used to demonstrate the existence of a solution to (1). Therefore, define a sequence \( x_n; t \times \Omega \rightarrow E^n \) as follows:

\[
x_0(t) = x_0, \quad x_n(t) = x_{n-1}(t) + \frac{1}{\Gamma(a)} \int_0^t f\left(s, x_{n-1}(s)\right) (t - s)^{-a} ds + \frac{1}{\Gamma(a)} \int_0^t \frac{g(s)}{(t - s)^{1-a}} dB^H(s).
\]

and for \( n = 1, \ldots \),

\[
K_1(t) = \sup_{0 \leq s \leq t} E\tilde{d}_{\alpha}^2 \left( \frac{1}{\Gamma(a)} \int_0^u f\left(s, x_0(s)\right) (u - s)^{-a} ds + \frac{1}{\Gamma(a)} \int_0^u \frac{g(s)}{(u - s)^{1-a}} dB^H(s), \tilde{0} \right)
\]

\[
\leq 2 \sup_{0 \leq u \leq t} \left[ E\tilde{d}_{\alpha}^2 \left( \frac{1}{\Gamma(a)} \int_0^u f\left(s, x_0(s)\right) (u - s)^{-a} ds, s \right) + \frac{1}{\Gamma(a)} \int_0^u \frac{g(s)}{(u - s)^{1-a}} dB^H(s) \right]^2
\]

\[
\leq 2 \sup_{0 \leq u \leq t} \left[ 2E\tilde{d}_{\alpha}^2 \left( \frac{1}{\Gamma(a)} \int_0^u f\left(s, x_0(s)\right) (u - s)^{-a} ds, s \right) + \frac{1}{\Gamma(a)} \int_0^u \frac{g(s)}{(u - s)^{1-a}} dB^H(s) \right] + 2E\sup_{u \in [0,u]} E\tilde{d}_{\alpha}^2 \left( \frac{1}{\Gamma(a)} \int_0^u \frac{g(s)}{(u - s)^{1-a}} dB^H(s), \tilde{0} \right)
\]

\[
\leq 2E\sup_{0 \leq u \leq t} E\tilde{d}_{\alpha}^2 \left( f\left(s, x_0(s)\right), f\left(s, x_0(s)\right)\right) + 4T\sup_{0 \leq u \leq t} E\tilde{d}_{\alpha}^2 \left( f\left(s, x_0(s)\right), f\left(s, x_0(s)\right)\right) + 4T \frac{c_{T,H}}{\Gamma(a)} \left( \int_0^u \frac{g(s)}{(u - s)^{1-a}} ds \right)
\]

\[
\leq 4T \cdot \frac{ct^{\frac{a}{2}}}{\Gamma(a)} \left( x_0(s), \tilde{0} \right) + 4T \frac{ct^{\frac{a}{2}}}{\Gamma(a)} \left( x_0(s), \tilde{0} \right) + 2c_{T,H} \left( \int_0^u \frac{g(s)}{(u - s)^{1-a}} ds \right)
\]

\[
= \frac{l_1 t^a}{\Gamma(a + 1)}
\]

where \( l_1 = 4cTd^2_{\alpha}(x_0, \tilde{0}) + 4Tc + 2c^2c_{T,H} \). Moreover, similarly, we have

\[
K_{n+1}(t) = \sup_{0 \leq s \leq t} E\tilde{d}_{\alpha}^2 \left( \frac{1}{\Gamma(a)} \int_0^u f\left(s, x_n(s)\right) (u - s)^{-a} ds + \frac{1}{\Gamma(a)} \int_0^u \frac{g(s)}{(u - s)^{1-a}} dB^H(s), \frac{1}{\Gamma(a)} \int_0^u f\left(s, x_{n-1}(s)\right) (u - s)^{-a} ds + \frac{1}{\Gamma(a)} \int_0^u \frac{g(s)}{(u - s)^{1-a}} dB^H(s) \right)
\]

\[
\leq 2 \sup_{0 \leq u \leq t} E\tilde{d}_{\alpha}^2 \left( \frac{1}{\Gamma(a)} \int_0^u f\left(s, x_n(s)\right) (u - s)^{-a} ds, \tilde{0} \right) + \frac{1}{\Gamma(a)} \int_0^u \frac{g(s)}{(u - s)^{1-a}} dB^H(s)
\]

\[
\leq 2 \frac{t}{\Gamma(a)} \left( \int_0^u (t - s)^{a-1} E\tilde{d}_{\alpha}^2 \left( f\left(s, x_n(s)\right), f\left(s, x_{n-1}(s)\right)\right) ds \right)
\]

\[
\leq 2 \frac{tc}{\Gamma(a)} \left( \int_0^u (t - s)^{a-1} \sup_{|u| \leq a} E\tilde{d}_{\alpha}^2 \left( x_n(s), x_{n-1}(s)\right) ds \right)
\]

\[
\leq 2 \frac{tc}{\Gamma(a)} \left( \int_0^u (t - s)^{a-1} K_n(s) ds \right).
\]
Thus, we obtain
\[ K_n(t) = \frac{l_1}{l_2} \frac{(l_2 t^n)^n}{n!}, \quad \forall t \in J, \ n \in \mathbb{N}, \] (24)
where \( l_2 = 2Tc \).

Hence, from Chebyshev’s inequality and (24), we obtain
\[ \mathbb{P} \left( \sup_{u \in J} d^2_{\infty} (x_n(u), x_{n-1}(u)) > \frac{1}{4^n} \right) \leq \frac{l_1}{l_2} \frac{(4l_2 T^n)^n}{n!}. \] (25)

Since the series \( \sum_{n=1}^{\infty} (4l_1 T^n)^n/n! \) converges, according to Borel–Cantelli lemma, we obtain
\[ \mathbb{P} \left( \sup_{u \in J} d_{\infty} (x_n(u), x_{n-1}(u)) > \frac{1}{2^n} \right) = 0. \] (26)

Thus, the sequence \( \{x_n(\cdot, v)\} \) is uniformly convergent to \( \bar{x}(\cdot, v) \): \( J \rightarrow \mathbb{R}^n \) for \( v \in \Omega_e \), where \( \Omega_e \in \mathcal{A} \) and \( \mathbb{P}(\Omega_e) = 1 \).

Then,
\[ \lim_{n \rightarrow \infty} \sup_{t \in J} \mathbb{E} d^2_{\infty} (x_n(t), \bar{x}(t)) = 0. \] (27)

Let us define \( x: J \times \Omega \rightarrow \mathbb{E}^n \) as follows:
\[ x(\cdot, v) = \begin{cases} \bar{x}(\cdot, v), & \text{if } v \in \Omega_e, \\ \text{freely chosen}, & \text{if } v \in \Omega \setminus \Omega_e. \end{cases} \] (28)

By the triangle inequality, \((\mathcal{H}1)-(\mathcal{H}3)\), and Propositions 2 and 3, we have
\[ \psi_n(t) \leq 3 \mathbb{E} d^2_{\infty} (x_0, \bar{0}) + 3 \sup_{0 \leq s \leq t} \mathbb{E} d^2_{\infty} \left( \frac{1}{\Gamma(\alpha)} \int_0^u (u-s)^{\alpha-1} f(s, x_{n-1}(s)) ds, 0 \right) + 3 \mathbb{E} \sup_{0 \leq s \leq t} \left\{ \frac{1}{\Gamma(\alpha)} \int_0^u (u-s)^{\alpha-1} g(s) dB^H(s) \right\}^2. \] (33)

We obtain
\[ \psi_n(t) \leq A_1 + A_2 T^\alpha \int_0^t (t-s)^{\alpha-1} \psi_{n-1}(s) ds, \] (35)
where \( A_1 = 3 \mathbb{E} d^2_{\infty} (x_0, \bar{0}) + (6ct \Gamma(\alpha + 1) \Gamma(\alpha + 1)) \) and \( A_2 = 6ct/l_\mathcal{L} \Gamma(\alpha) \).

According to Lemma 1 and Remark 1, there exist a constant \( M_A > 0 \) independent of \( A_1 \) such that
\[ \psi_n(t) \leq M_A A_1. \] (36)

Due to \((\mathcal{H}1),(31)\), and (36), we obtain
\[ \sup_{0 \leq s \leq t} \mathbb{E} d^2_{\infty} (x(s), \bar{0}) \leq 2 \sup_{0 \leq s \leq t} \mathbb{E} d^2_{\infty} (x(s), x_n(s)) + 2 \sup_{0 \leq s \leq t} \mathbb{E} d^2_{\infty} (x_n(s), \bar{0}) \leq 2M_A A_1 + 2M_A A_1 < \infty. \] (37)
which implies
\[
\int_0^T \mathbb{E}_{\infty}^2(x(s),\tilde{0})\,ds \leq T \sup_{t \in J} \mathbb{E}_{\infty}^2(x(t),\tilde{0}) < \infty. \tag{38}
\]

Thus, we get \( x \in L^2(J \times \Omega, \mathbb{N}; \mathbb{E}^n) \).

On the contrary, we have
\[
\sup_{t \in J} \mathbb{E}_{\infty}^2\left(x(t),x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t f(s,x(s))\,ds + \frac{1}{\Gamma(\alpha)} \int_0^t g(s)\,d\mathbb{B}^H(s)\right) = 0. \tag{39}
\]

Indeed, we observe
\[
\begin{align*}
\sup_{t \in J} \mathbb{E}_{\infty}^2\left(x(t),x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t f(s,x(s))\,ds + \frac{1}{\Gamma(\alpha)} \int_0^t g(s)\,d\mathbb{B}^H(s)\right) \\
\leq 3 \left[ \sup_{t \in J} \mathbb{E}_{\infty}^2(x(t),x_n(t)) + \sup_{t \in J} \mathbb{E}_{\infty}^2\left(x_n(t),x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t f(s,x_{n-1}(s))\,ds + \frac{1}{\Gamma(\alpha)} \int_0^t g(s)\,d\mathbb{B}^H(s)\right) \right] \\
+ \sup_{t \in J} \mathbb{E}_{\infty}^2\left(x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t f(s,x(s))\,ds + \frac{1}{\Gamma(\alpha)} \int_0^t g(s)\,d\mathbb{B}^H(s)\right) \\
x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t f(s,x(s))\,ds + \frac{1}{\Gamma(\alpha)} \int_0^t g(s)\,d\mathbb{B}^H(s)) \right] = I_1 + I_2 + I_3,
\end{align*}
\]

where \( \lim_{n \to \infty} I_1 = 0 \) and \( I_2 = 0 \). For \( I_3 \), by using Propositions 2 and 3, \((\mathcal{H}3)\), and \((30)\), we have
\[
\lim_{n \to \infty} I_3 \leq \lim_{n \to \infty} \left( \frac{T^{\alpha+1}}{\Gamma(\alpha+1)} \sup_{t \in J} \mathbb{E}_{\infty}^2(x(u),x_{n-1}(u))\,du \right) = 0. \tag{41}
\]

Hence, we get \((39)\), which implies \((16)\) holds. Hence, from definition \((8)\), \( x(t) \) is a solution to equation \((14)\).

For the uniqueness of a solution \( x \), suppose that \( x, z : J \times \Omega \rightarrow \mathbb{E}^n \) are solutions to equation \((14)\). We denote by
\[
K(t) := \sup_{t \geq s \in J} \mathbb{E}_{\infty}^2(x(v),z(v)).
\]

So, for each \( t \in J \), we obtain
\[
K(t) \leq \frac{tc}{\Gamma(\alpha)} \int_0^t \mathbb{E}_{\infty}^2(x(s),z(s))\,(t-s)^{1-\alpha}\,ds \leq \frac{Tc}{\Gamma(\alpha)} \int_0^t \frac{K(s)}{(t-s)^{1-\alpha}}\,ds.
\]

Thus, by Lemma 1, we have, for \( t \in J \), \( K(t) \equiv 0 \), which implies
\[
\sup_{t \in J} \mathbb{E}_{\infty}^2(x(t),z(t)) = 0. \tag{43}
\]

4. Stability Result

In this part, we examine the stability of the solution with respect to initial values by using Henry–Gronwall inequality. Indeed, let \( x \) and \( z \) denote the solutions of the following FFSDEs:

\[
\text{Proposition 5. Suppose that } x_0, z_0 \in L^2(\Omega, \mathcal{A} \mathcal{F}_0, \mathbb{P}; \mathbb{E}^n) \text{ and } f : J \times \Omega \times \mathbb{E}^n \rightarrow \mathbb{E}^n \text{ and } g : J \rightarrow \mathbb{R}^n \text{ satisfy } (\mathcal{H}1)-(\mathcal{H}3). \text{ Then,}
\]
\[
\sup_{t \in J} \mathbb{E}_{\infty}^2(x(u),z(u)) \leq \lambda_0 M_t,
\]

where \( \lambda_0 = 2 \mathbb{E}(\mathbb{E}_{\infty}^2(x_0,z_0)) \) and \( \lambda_1 = 2Tc/\Gamma(\alpha) \). Especially, \( x(t) = z(t) \) if \( x_0 = z_0 \).

\text{Proof. Suppose that } x, z : J \times \Omega \rightarrow \mathbb{E}^n \text{ are solutions to equations } (44) \text{ and } (45), \text{ respectively. So, let } K(t) := \mathbb{E}\sup_{t \in J} \mathbb{E}_{\infty}^2(x(u),z(u)). \text{ Due to Propositions 2 and 3 and } (\mathcal{H}3), \text{ we obtain}
Then, according to Lemma 1 and Remark 1, there exist a constant $M_{A_1} > 0$ independent of $\lambda_0$ such that

$$K(t) \leq \lambda_0 M_{A_1}, \quad \forall t \in J. \quad (48)$$

Then, $\lambda_0 = 0$ if $x_{n,0} = z_0$. Therefore, we know that $x(t)\big|_{t=0} = z(t)$.

Finally, we examine the exponential stability of solutions to the FFSDEs which disturbed an fBm with respect to $f$ and $g$. Thus, let $x$ and $x_n$ denote solutions to the following FFSDEs:

$$\begin{align*}
K(t) & \leq 2E^{\alpha_0}d_2(x_0, z_0) + \frac{2}{\Gamma(\alpha)} \sup_{u \in [0,t]} E^{\alpha_0} \left( \int_0^u f(s, x(s)) \frac{ds}{(t-s)^{1-\alpha}} + \int_0^u f(s, z(s)) \frac{ds}{(t-s)^{1-\alpha}} \right) \\
& \leq 2E^{\alpha_0}d_2(x_0, z_0) + \frac{2Tc}{\Gamma(\alpha)} \int_0^t E^{\alpha_0} d_2(x(s), z(s)) \frac{ds}{(t-s)^{1-\alpha}} \\
& \leq 2E^{\alpha_0}d_2(x_0, z_0) + \frac{2Tc}{\Gamma(\alpha)} \int_0^t \sup_{u \in [0,t]} E^{\alpha_0} d_2(x(u), z(u)) \frac{ds}{(t-s)^{1-\alpha}} \\
& = 2E^{\alpha_0}d_2(x_0, z_0) + \frac{2Tc}{\Gamma(\alpha)} \int_0^t \frac{K(s)}{(t-s)^{1-\alpha}} ds \\
& = \lambda_0 + \lambda_1 \int_0^t \frac{K(s)}{(t-s)^{1-\alpha}} ds.
\end{align*}$$

respectively.

**Proposition 6.** Suppose that $x_0 \in L^2((\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{E}^n))$ and $f, f_n : J \times \mathbb{E}^n \rightarrow \mathbb{E}^n$ are solutions of equations (49) and (50), respectively.

**Proof.** According to Theorem 1, the solutions $x$ and $x_n$ are unique and exist. From Propositions 3 and 4, we deduce that, for every $t \in J$,

$$\sup_{0 \leq u \leq t} E^{\alpha_0} d_2(x(u), x_n(u)) \leq 2 \left( \frac{1}{\Gamma(\alpha)} \int_0^u f(s, x(s)) \frac{ds}{(u-s)^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_0^u f_n(s, x_n(s)) \frac{ds}{(u-s)^{1-\alpha}} \right) + 2 \sup_{0 \leq u \leq t} E^{\alpha_0} \left( \frac{1}{\Gamma(\alpha)} \int_0^u g(s) \frac{ds}{(u-s)^{1-\alpha}} dB_{H}(s) \right).$$
where

\[ \beta_1 = \frac{4T}{\Gamma(\alpha)} \int_{0}^{t} \mathbb{E} \left[ d_{\infty}^{2}(f(s, x(s)), f_n(s, x(s))) \right] ds + \frac{2c_{T,H}}{\Gamma(\alpha)} \int_{0}^{t} \frac{\|g(s) - g_n(s)\|^2}{(t-s)^{1-\alpha}} ds, \]

\[ \beta_2 = 4\frac{T}{\Gamma(\alpha)} (\alpha - 1) M \beta_1. \]

Hence, from (51) and (52), we get \( \lim_{n \to \infty} \beta_1^n = 0. \) \( \square \)

5. Conclusions

In this study, we have proved the existence and uniqueness of solutions to FFSDEs under the Lipschitzian coefficient. On the contrary, the stability of solutions to the FFSDEs is analyzed.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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