Abstract crystals for quantum generalized Kac-Moody algebras

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ABSTRACT. In this paper, we introduce the notion of abstract crystals for quantum generalized Kac-Moody algebras and study their fundamental properties. We then prove the crystal embedding theorem and give a characterization of the crystals \( B(\infty) \) and \( B(\lambda) \).

Introduction

The purpose of this paper is to develop the theory of abstract crystals for quantum generalized Kac-Moody algebras. In [6], the third author introduced the crystal basis theory for quantum groups associated with symmetrizable Kac-Moody algebras. (In [10], Lusztig constructed canonical bases for quantum groups of ADE type.) It has become one of the most central themes in combinatorial representation theory, for it provides us with a very powerful combinatorial tool to investigate the structure of integrable modules over quantum groups and Kac-Moody algebras.

The generalized Kac-Moody algebras were introduced by Borcherds in his study of Monstrous Moonshine [1]. The Monster Lie algebra, an example of generalized Kac-Moody algebras, played a crucial role in his proof of the Moonshine conjecture [2]. In [5], the second author constructed the quantum generalized Kac-Moody algebra \( U_q(\mathfrak{g}) \) as a deformation of the universal enveloping algebra of a generalized Kac-Moody algebra \( \mathfrak{g} \). He also showed that, for a generic \( q \), the Verma modules and the unitarizable highest weight modules

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over \(\mathfrak{g}\) can be deformed to those over \(U_q(\mathfrak{g})\) in such a way that the dimensions of weight spaces are invariant under the deformation.

In [4], the first three authors developed the crystal basis theory for quantum generalized Kac-Moody algebras. More precisely, they defined the notion of crystal bases for \(U_q(\mathfrak{g})\)-modules in the category \(\mathcal{O}_{\text{int}}\) (see § 1), proved standard properties of crystal bases including the tensor product rule, and showed that there exists a crystal basis (and a global basis) of the negative part \(U_q^-\mathfrak{g}\) of a quantum generalized Kac-Moody algebra and one of the irreducible \(U_q(\mathfrak{g})\)-module \(V(\lambda)\) with a dominant integral weight \(\lambda\) as its highest weight.

In this paper, we introduce the notion of abstract crystals for quantum generalized Kac-Moody algebras and investigate their fundamental properties. We then prove the crystal embedding theorem, which yields a procedure to determine the structure of the crystal \(B(\infty)\) in terms of elementary crystals. Finally, as an application of the crystal embedding theorem, we provide a characterization of the crystals \(B(\infty)\) and \(B(\lambda)\). We also include an explicit description of the crystals \(B(\infty)\) and \(B(\lambda)\) for quantum generalized Kac-Moody algebras of rank 2 and for the quantum Monster algebra.

1. Generalized Kac-Moody algebras

Let \(I\) be a finite or countably infinite index set. A real matrix \(A = (a_{ij})_{i,j \in I}\) is called a \textit{Borcherds-Cartan matrix} if it satisfies the following conditions:

(i) \(a_{ii} = 2\) or \(a_{ii} \leq 0\) for all \(i \in I\),
(ii) \(a_{ij} \leq 0\) if \(i \neq j\),
(iii) \(a_{ij} \in \mathbb{Z}\) if \(a_{ii} = 2\),
(iv) \(a_{ij} = 0\) if and only if \(a_{ji} = 0\).

In this paper, \textit{we assume that} \(A\) \textit{is even and integral}; i.e., \(a_{ii} \in 2\mathbb{Z}_{\leq 1}\) for all \(i \in I\) and \(a_{ij} \in \mathbb{Z}\) for all \(i, j \in I\). Furthermore, we also assume that \(A\) is \textit{symmetrizable}; i.e., there exists a diagonal matrix \(D = \text{diag}(s_i \in \mathbb{Z}_{>0}; i \in I)\) such that \(DA\) is symmetric.

We say that an index \(i \in I\) is \textit{real} if \(a_{ii} = 2\) and \textit{imaginary} if \(a_{ii} \leq 0\). We denote by \(I^r = \{i \in I ; a_{ii} = 2\}\) and \(I^\text{im} = \{i \in I ; a_{ii} \leq 0\}\) the set of real indices and the set of imaginary indices, respectively.

A \textit{Borcherds-Cartan datum} \((A, P, \Pi, \Pi^\vee)\) consists of
(i) a Borcherds-Cartan matrix \( A = (a_{ij})_{i,j \in I} \),
(ii) a free abelian group \( P \), the weight lattice,
(iii) \( \Pi = \{ \alpha_i \in P \, ; \, i \in I \} \), the set of simple roots,
(iv) \( \Pi^\vee = \{ h_i \, ; \, i \in I \} \subset P^\vee := \text{Hom}(P, \mathbb{Z}) \), the set of simple coroots,
satisfying the properties:

(a) \( \langle h_i, \alpha_j \rangle = a_{ij} \) for all \( i, j \in I \),
(b) for any \( i \in I \), there exists \( \Lambda_i \in P \) such that \( \langle h_j, \Lambda_i \rangle = \delta_{ij} \) for all \( j \in I \),
(c) \( \Pi \) is linearly independent.

We denote by \( P^+ = \{ \lambda \in P \, ; \, \lambda(h_i) \geq 0 \, \text{ for all } i \in I \} \) the set of dominant integral weights. We also use the notation \( Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \) and \( Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \).

Let \( q \) be an indeterminate and set \( q_i = q^{s_i} \) (\( i \in I \)). For an integer \( n \in \mathbb{Z} \), define
\[
[n]_q = \frac{q^n - q^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \left[ \frac{m}{n} \right] = \frac{[m]_i!}{[n]_i! [m-n]_i!}.
\]

Let \( (A, P, \Pi, \Pi^\vee) \) be a Borcherds-Cartan datum. The quantum generalized Kac-Moody algebra \( U_q(\mathfrak{g}) \) associated with \( (A, P, \Pi, \Pi^\vee) \) is defined to be the associated algebra over \( \mathbb{Q}(q) \) with 1 generated by the elements \( e_i, f_i \) (\( i \in I \)), \( q^h \) (\( h \in P^\vee \)) with the following defining relations:
\[
q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee,
\]
\[
q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad \text{for } h \in P^\vee, i \in I,
\]
\[
e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad \text{for } i, j \in I, \text{ where } K_i = q^{s_i h_i},
\]
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \binom{1-a_{ij}}{k} \right] e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad \text{if } i \in I^e \text{ and } i \neq j,
\]
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \binom{1-a_{ij}}{k} \right] f_j f_i^k e_i^{1-a_{ij}-k} f_j = 0 \quad \text{if } i \in I^e \text{ and } i \neq j,
\]
\[
e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0 \quad \text{if } a_{ij} = 0.
\]

Let us denote by \( U_q^+(\mathfrak{g}) \) (resp. \( U_q^-(\mathfrak{g}) \)) the subalgebra of \( U_q(\mathfrak{g}) \) generated by the \( e_i \)'s (resp. the \( f_i \)'s). Let us denote by \( \mathcal{O}_{\text{int}} \) the abelian category of \( U_q(\mathfrak{g}) \)-modules \( M \) satisfying the following properties:
It is proved in [4] that the abelian category $O$ irreducible objects is of the form $V$ with the defining relation:

(i) $M$ has the weight decomposition: $M = \oplus_{\lambda \in P} M_\lambda$, where $M_\lambda := \{ u \in M ; q^h u = q^{\lambda(h)} u \text{ for any } h \in P^\vee \}$,

(ii) the action of $U_q^+(\mathfrak{g})$ is locally finite, i.e., $\dim U_q^+(\mathfrak{g})u < \infty$ for any $u \in M$,

(iii) $\text{wt}(M) := \{ \lambda \in P ; M_\lambda \neq 0 \} \subset \{ \lambda \in P ; \langle h_i, \lambda \rangle \geq 0 \text{ for any } i \in I^\infty \}$,

(iv) $f_i M_\lambda = 0$ for any $i \in I^\infty$ and $\lambda \in P$ such that $\langle h_i, \lambda \rangle = 0$,

(v) $e_i M_\lambda = 0$ for any $i \in I^\infty$ and $\lambda \in P$ such that $\langle h_i, \lambda \rangle \leq -a_{ii}$.

It is proved in [4] that the abelian category $O_{\text{int}}$ is semisimple, and any of its irreducible objects is of the form $V(\lambda)$ for $\lambda \in P^+$, where $V(\lambda) = U_q(\mathfrak{g})u_\lambda$ with the defining relation:

(a) $u_\lambda$ has weight $\lambda$,

(b) $e_i u_\lambda = 0$ for all $i \in I$,

(c) $f_i^{(h_i, \lambda)+1} u_\lambda = 0$ for any $i \in I^w$,

(d) $f_i u_\lambda = 0$ if $i \in I^\infty$ and $\langle h_i, \lambda \rangle = 0$.

Let $A_0 = \{ f/g \in Q(q) : f, g \in Q[q], g(0) \neq 0 \}$. Let $M$ be a $U_q(\mathfrak{g})$-module in $O_{\text{int}}$. For each $i \in I$, every weight vector $u \in M_\mu$ has an $i$-string decomposition $u = \sum_{k \geq 0} f_i^{(k)} u_k$ with $u_k \in M_{\mu+n\alpha_i}$ such that $e_i u_k = 0$,

where

$$f_i^{(k)} = \begin{cases} f_i^k/[k]! & \text{if } i \in I^e, \\ f_i^k & \text{if } i \in I^\infty. \end{cases}$$

Such a decomposition is unique. We define the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ ($i \in I$) by

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.$$

A crystal basis $(L, B)$ of $M$ is a pair of a free $A_0$-submodule $L$ of $M$ and a basis $B$ of the $Q$-vector space $L/qL$ satisfying the following conditions,

(i) $L$ generates $M$ as a $Q(q)$-vector space,

(ii) $L$ has the weight decomposition $L = \oplus_{\lambda \in P} L_\lambda$ where $L_\lambda = L \cap M_\lambda$,

(iii) $B$ has the weight decomposition $B = \sqcup_{\lambda \in P} B_\lambda$ where $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,

(iv) $f_i L \subset L$ and $\tilde{e}_i L \subset L$ for any $i \in I$,

(v) $\tilde{f}_i B \subset B \cup \{ 0 \}$ and $\tilde{e}_i B \subset B \cup \{ 0 \}$ for any $i \in I$,

(vi) for $b, b' \in B$ and $i \in I$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$,
It is proved in [4] that every $M \in \mathcal{O}_{\text{int}}$ has a crystal basis unique up to an automorphism.

For $\lambda \in P^+$, let $L(\lambda)$ be the $A_0$-submodule of $V(\lambda)$ generated by 
\[
\left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda; r \geq 0, i_k \in I \right\},
\]
and set $B(\lambda) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda + qL(\lambda); r \geq 0, i_k \in I \right\} \setminus \{0\}$. Then $(L(\lambda), B(\lambda))$ is a crystal basis of $V(\lambda)$.

2. Abstract Crystals

By abstracting the properties of crystal bases of $U_q(\mathfrak{g})$-modules in $\mathcal{O}_{\text{int}}$, we shall introduce the notion of abstract crystals.

**Definition 2.1.** An abstract $U_q(\mathfrak{g})$-crystal or simply a crystal is a set $B$ together with the maps $\text{wt}: B \to P$, $\tilde{e}_i, \tilde{f}_i: B \to B \sqcup \{0\}$ and $\varepsilon_i, \varphi_i: B \to \mathbb{Z} \sqcup \{-\infty\}$ ($i \in I$) satisfying the following conditions:

(i) $\text{wt}(\tilde{e}_i b) = \text{wt} b + \alpha_i$ if $i \in I$ and $\tilde{e}_i b \neq 0$,

(ii) $\text{wt}(\tilde{f}_i b) = \text{wt} b - \alpha_i$ if $i \in I$ and $\tilde{f}_i b \neq 0$,

(iii) for any $i \in I$ and $b \in B$, $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt} b \rangle$,

(iv) for any $i \in I$ and $b, b' \in B$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$,

(v) for any $i \in I$ and $b \in B$ such that $\tilde{e}_i b \neq 0$, we have

(a) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $i \in I^\text{re}$,

(b) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + a_{ii}$ if $i \in I^\text{im}$,

(vi) for any $i \in I$ and $b \in B$ such that $\tilde{f}_i b \neq 0$, we have

(a) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $i \in I^\text{re}$,

(b) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - a_{ii}$ if $i \in I^\text{im}$,

(vii) for any $i \in I$ and $b \in B$ such that $\varphi_i(b) = -\infty$, we have $\tilde{e}_i b = \tilde{f}_i b = 0$.

We sometimes write
\[
\text{wt}_i(b) = \langle h_i, \text{wt} b \rangle \quad \text{for } i \in I \text{ and } b \in B.
\]

**Remark 2.2.** Almost all crystals appearing in this paper have the following properties:

(a) $\text{wt}_i(b) \geq 0$ for any $i \in I^\text{im}$ and $b \in B$,

(b) $\varepsilon_i(b) \in \mathbb{Z}_{\geq 0} \sqcup \{-\infty\}$ and $\varphi_i(b) \in \mathbb{Z}_{\geq 0} \sqcup \{-\infty\}$ for any $i \in I^\text{im}$ and $b \in B$.

Hence we could include these properties in the axiom of crystals.
We shall define the morphisms of crystals.

**Definition 2.3.** Let $B_1$ and $B_2$ be crystals. A morphism of crystals or a crystal morphism $\psi : B_1 \to B_2$ is a map $\psi : B_1 \to B_2$ such that

(i) for $b \in B_1$ we have
$$\text{wt}(\psi_b) = \text{wt}(b), \text{ and } \varepsilon_i(\psi_b) = \varepsilon_i(b), \varphi_i(\psi_b) = \varphi_i(b) \text{ for all } i \in I,$$

(ii) if $b \in B_1$ and $i \in I$ satisfy $\tilde{f}_i b \in B_1$, then we have $\psi(\tilde{f}_i b) = \tilde{f}_i \psi_b$.

**Remark 2.4.** If $b \in B_1$ satisfies $\tilde{e}_i b \in B_1$, one can deduce $\psi(\tilde{e}_i b) = \tilde{e}_i \psi_b$ for all $i \in I$ using Definition 2.1 (iv) and Definition 2.3 (ii).

Then the crystals form a category.

**Definition 2.5.** Let $\psi : B_1 \to B_2$ be a morphism of crystals.

(a) $\psi$ is called a strict morphism if
$$\psi(\tilde{e}_i b) = \tilde{e}_i \psi_b, \psi(\tilde{f}_i b) = \tilde{f}_i \psi_b \text{ for all } i \in I \text{ and } b \in B_1.$$ Here we understand $\psi(0) = 0$.

(b) $\psi$ is called an embedding if the underlying map $\psi : B_1 \to B_2$ is injective. In this case, we say that $B_1$ is a subcrystal of $B_2$. If $\psi$ is a strict embedding, we say that $B_1$ is a full subcrystal of $B_2$.

**Remark 2.6.** If $B_1$ is a full subcrystal of $B_2$, then we have
$$B_2 \cong B_1 \oplus (B_2 \setminus B_1).$$

Let us give two examples of crystals.

**Example 2.7 ([4]).** Let $(L, B)$ be a crystal basis of $M \in \mathcal{O}_{\text{int}}$. Then, $B$ has a crystal structure, where the maps $\varepsilon_i, \varphi_i \ (i \in I)$ are given by

$$\varepsilon_i(b) = \begin{cases} \max \{k \geq 0 ; \tilde{e}_i^k b \neq 0\} & \text{for } i \in I^e, \\ 0 & \text{for } i \in I^m, \end{cases}$$

$$\varphi_i(b) = \begin{cases} \max \{k \geq 0 ; \tilde{f}_i^k b \neq 0\} & \text{for } i \in I^e, \\ \text{wt}_i(b) & \text{for } i \in I^m, \end{cases}$$

(This definition is different from the one given in [4].) Such a crystal $B$ has the following properties:
(a) \( \varepsilon_i(b), \varphi_i(b) \geq 0 \) for any \( b \in B \) and \( i \in I \),
(b) if \( i \in I^{im} \) and \( \mathrm{wt}_i(b) = 0 \), then \( \tilde{e}_ib = \tilde{f}_ib = 0 \),
(c) if \( i \in I \) and \( \varphi_i(b) > 0 \), then \( \tilde{f}_ib \neq 0 \).

Let \( V(\lambda) \) be the irreducible highest weight \( U_q(g) \)-module with highest weight \( \lambda \in P^+ \). Let us recall that \( V(\lambda) \) has a crystal basis \( (L(\lambda), B(\lambda)) \). Hence \( B(\lambda) \) has a crystal structure.

Example 2.8 ([4]). Fix \( i \in I \). For any \( u \in U_q^-(g) \), there exist unique \( v, w \in U_q^-(g) \) such that
\[
e_iu - u e_i = \frac{K_i v - K_i^{-1}w}{q_i - q_i^{-1}}.
\]
We define the endomorphism \( e'_i : U_q^-(g) \to U_q^-(g) \) by \( e'_i(u) = w \). Then every \( u \in U_q^-(g) \) has a unique \( i \)-string decomposition
\[
u = \sum_{k \geq 0} f_i^{(k)}u_k, \quad \text{where } e'_i u_k = 0 \text{ for all } k \geq 0,
\]
and the Kashiwara operators \( \tilde{e}_i, \tilde{f}_i \) \( (i \in I) \) are defined by
\[
\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)}u_k, \quad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)}u_k.
\]
Let \( L(\infty) \) be the \( A_0 \)-submodule of \( U_q^-(g) \) generated by
\[
\left\{ \tilde{f}_i \cdots \tilde{f}_i 1 : r \geq 0, i_k \in I \right\},
\]
and
\[
B(\infty) = \left\{ \tilde{f}_i \cdots \tilde{f}_i 1 + qL(\infty) : r \geq 0, i_k \in I \right\} \setminus \{0\} \subset L(\infty)/qL(\infty),
\]
where \( 1 \) is the multiplicative identity in \( U_q^-(g) \). Then \( B(\infty) \) becomes a crystal with the maps \( \text{wt}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i \) \( (i \in I) \), where
\[
\text{wt}(b) = -(\alpha_{i_1} + \cdots + \alpha_{i_r}) \quad \text{for} \quad b = \tilde{f}_i \cdots \tilde{f}_i 1 + qL(\infty),
\]
\[
\varepsilon_i(b) = \begin{cases} 
\max\{k \geq 0 : \tilde{e}_i^k b \neq 0\} & \text{for } i \in I^{re}, \\
0 & \text{for } i \in I^{im},
\end{cases}
\]
\[
\varphi_i(b) = \varepsilon_i(b) + \text{wt}(b) \quad (i \in I).
\]
We have \( \tilde{f}_ib \in B(\infty) \) for any \( i \in I \) and \( b \in B(\infty) \).
The crystals $B(\lambda)$ and $B(\infty)$ are closely related as seen in the following proposition.

**Proposition 2.9** ([4]). For every $\lambda \in P^+$, there exists a map $\pi_\lambda : B(\lambda) \to B(\infty)$ such that

(i) $\pi_\lambda$ is injective,
(ii) $\pi_\lambda(u_\lambda) = 1$,
(iii) $\pi_\lambda \circ \tilde{f}_i(b) = \tilde{f}_i \circ \pi_\lambda(b)$ for any $i \in I$ and $b \in B(\lambda)$ such that $\tilde{f}_ib \neq 0$,
(iv) $\pi_\lambda \circ \tilde{e}_i(b) = \tilde{e}_i \circ \pi_\lambda(b)$ for all $i \in I$ and $b \in B(\lambda)$.
(v) $wt(\pi_\lambda(b)) = wt(b) - \lambda$, $\varepsilon_i(\pi_\lambda(b)) = \varepsilon_i(b)$ for any $b \in B(\lambda)$ and $i \in I$.

*Proof.* See Propositions 7.12, 7.13, 7.23, 7.34 in [4].

We define the tensor product of a pair of crystals as follows: for two crystals $B_1$ and $B_2$, their tensor product $B_1 \otimes B_2$ is $\{b_1 \otimes b_2 ; b_1 \in B_1, b_2 \in B_2\}$ with the following crystal structure. The maps $wt, \varepsilon_i, \varphi_i$ are given by

$$
\begin{align*}
wt(b \otimes b') &= wt(b) + wt(b'), \\
\varepsilon_i(b \otimes b') &= \max(\varepsilon_i(b), \varepsilon_i(b') - wt_i(b)), \\
\varphi_i(b \otimes b') &= \max(\varphi_i(b) + wt_i(b'), \varphi_i(b')).
\end{align*}
$$

For $i \in I$, we define

$$
\tilde{f}_i(b \otimes b') =
\begin{cases}
\tilde{f}_ib \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b'), \\
b \otimes \tilde{f}_ib' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'),
\end{cases}
$$

For $i \in I^{re}$, we define

$$
\tilde{e}_i(b \otimes b') =
\begin{cases}
\tilde{e}_ib \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b'), \\
b \otimes \tilde{e}_ib' & \text{if } \varphi_i(b) < \varepsilon_i(b'),
\end{cases}
$$

and, for $i \in I^{im}$, we define

$$
\tilde{e}_i(b \otimes b') =
\begin{cases}
\tilde{e}_ib \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') - a_{ii}, \\
0 & \text{if } \varepsilon_i(b') < \varphi_i(b) \leq \varepsilon_i(b') - a_{ii}, \\
b \otimes \tilde{e}_ib' & \text{if } \varphi_i(b) \leq \varepsilon_i(b').
\end{cases}
$$
(This tensor product rule is different from the one given in [4]. But when $B_1 = B(\lambda)$ and $B_2 = B(\mu)$ for $\lambda, \mu \in P^+$, the two rules coincide.)

**Lemma 2.10.** By the definition above, $B_1 \otimes B_2$ is a crystal.

**Proof.** The properties (i), (ii), (iii), (vii) are clear.

Suppose $\tilde{f}_i(b \otimes b') = b_1 \otimes b'_1 \neq 0$ for $i \in I^{im}$. If $\varphi_i(b) > \varepsilon_i(b')$, then $b_1 = \tilde{f}_i b \neq 0$ and $b'_1 = b'$. In this case, we have $\varphi_i(b_1) = \varphi_i(\tilde{f}_i b) = \varphi_i(b) - a_{ii} > \varepsilon_i(b') - a_{ii} = \varepsilon_i(b'_1) - a_{ii}$. Hence we obtain $\tilde{e}_i(b_1 \otimes b'_1) = \tilde{e}_i b_1 \otimes b'_1 = b \otimes b'$. Also we have $\varepsilon_i(b_1) = \varepsilon_i(b) > \varepsilon_i(b') - \text{wt}_i(b) \geq \varepsilon_i(b') - \text{wt}_i(\tilde{f}_i b)$. Hence $\varepsilon_i(\tilde{f}_i(b \otimes b')) = \varepsilon_i(b_1) = \varepsilon_i(b \otimes b')$.

If $\varphi_i(b) \leq \varepsilon_i(b')$, then $b_1 = b$ and $b'_1 = \tilde{f}_i b' \neq 0$. In this case, we have $\varphi_i(b_1) = \varphi_i(b) \leq \varepsilon_i(b') = \varepsilon_i(b'_1)$. Hence we obtain $\tilde{e}_i(b_1 \otimes b'_1) = b_1 \otimes \tilde{e}_i b'_1 = b \otimes b'$. Also we have $\varepsilon_i(b_1) \leq \varepsilon_i(b') - \text{wt}_i(b) = \varepsilon_i(\tilde{f}_i b') - \text{wt}_i(b)$. Hence $\varepsilon_i(\tilde{f}_i(b \otimes b')) = \varepsilon_i(b') - \text{wt}_i(b) = \varepsilon_i(b \otimes b')$.

Therefore (vi)(b) and the half of (iv) are proved. Now (v)(b) follows easily from the property (iii). The rest may be proved similarly. \qed

Remark that, for a crystal basis $(L_i, B_i)$ of $M_i \in \mathcal{O}_{\text{int}}$ ($i = 1, 2$), $(L_1, B_1) \otimes (L_2, B_2) := (L_1 \otimes_{A_0} L_2, B_1 \otimes B_2)$ is a crystal basis of $M_1 \otimes M_2$, and the crystal structure on $B_1 \otimes B_2$ coincides with the tensor product of the crystals $B_1$ and $B_2$.

It is also easy to check the following associativity law for the tensor product, and the category of crystals has a structure of a tensor category (see e.g. [9]).

**Lemma 2.11.** For three crystals, $B_\nu$ ($\nu = 1, 2, 3$), the map $(b_1 \otimes b_2) \otimes b_3 \mapsto b_1 \otimes (b_2 \otimes b_3)$ gives an isomorphism of crystals:

$$\Phi : (B_1 \otimes B_2) \otimes B_3 \sim \rightarrow B_1 \otimes (B_2 \otimes B_3).$$

**Proof.** We shall only show here that $\Phi(\tilde{e}_i((b_1 \otimes b_2) \otimes b_3)) = \tilde{e}_i(b_1 \otimes (b_2 \otimes b_3))$ for $i \in I^{im}$, leaving the proof of the rest to the reader.

**Case 1:** $\varphi_i(b_1 \otimes b_2) > \varepsilon_i(b_3) - a_{ii}$.

In this case, we have $\tilde{e}_i((b_1 \otimes b_2) \otimes b_3) = (\tilde{e}_i(b_1 \otimes b_2)) \otimes b_3$.

If $\varphi_i(b_1) > \varepsilon_i(b_2) - a_{ii}$, then $\tilde{e}_i((b_1 \otimes b_2) \otimes b_3) = (\tilde{e}_i b_1 \otimes b_2) \otimes b_3$. Since $\varphi_i(b_1) > \varepsilon_i(b_2)$, we have $\varphi_i(b_1 \otimes b_2) = \varphi_i(b_1) + \text{wt}_i(b_2)$. Hence we obtain $\varphi_i(b_1) > \varepsilon_i(b_3) - \text{wt}_i(b_2) - a_{ii}$. Since $\varphi_i(b_1) > \varepsilon_i(b_2) - a_{ii}$, we obtain $\varphi_i(b_1) > \varepsilon_i(b_2 \otimes b_3) - a_{ii}$, which implies $\tilde{e}_i(b_1 \otimes (b_2 \otimes b_3)) = \tilde{e}_i b_1 \otimes (b_2 \otimes b_3)$. 

If \( \varepsilon_i(b_2) < \varphi_i(b_1) \leq \varepsilon_i(b_2) - a_{ii} \), we will show \( \tilde{e}_i(b_1 \otimes (b_2 \otimes b_3)) = 0 \). Since \( \varepsilon_i(b_2) < \varphi_i(b_1) \), we have \( \varphi_i(b_1) + \text{wt}_i(b_2) = \varphi_i(b_1 \otimes b_2) > \varepsilon_i(b_3) - a_{ii} \). Since

\[ \varepsilon_i(b_3) - a_{ii} - \text{wt}_i(b_2) < \varphi_i(b_1) \leq \varepsilon_i(b_3) - a_{ii} \],

we have \( \varepsilon_i(b_3) - \text{wt}_i(b_2) < \varepsilon_i(b_2) \), which implies \( \varepsilon_i(b_2 \otimes b_3) = \varepsilon_i(b_2) \). Now we have \( \varepsilon_i(b_2 \otimes b_3) < \varphi_i(b_1) \leq \varepsilon_i(b_2 \otimes b_3) - a_{ii} \). Therefore we get the desired result.

If \( \varphi_i(b_1) \leq \varepsilon_i(b_2) \), it suffices to show that \( \tilde{e}_i(b_1 \otimes (b_2 \otimes b_3)) = b_1 \otimes (\tilde{e}_i b_2 \otimes b_3) \).

In this case, we have \( \varphi_i(b_1 \otimes b_2) = \varphi_i(b_2) \). Hence by our assumption, we obtain

\[ (2.1) \quad \varphi_i(b_2) > \varepsilon_i(b_3) - a_{ii}. \]

In particular, \( \varepsilon_i(b_2 \otimes b_3) = \varepsilon_i(b_2) \geq \varphi_i(b_1) \), which yields \( \tilde{e}_i(b_1 \otimes (b_2 \otimes b_3)) = b_1 \otimes \tilde{e}_i(b_2 \otimes b_3) \). By (2.1), we get what we wanted.

**Case 2:** \( \varepsilon_i(b_1) < \varphi_i(b_1 \otimes b_2) \leq \varepsilon_i(b_3) - a_{ii}. \)

We will show \( \tilde{e}_i(b_1 \otimes (b_2 \otimes b_3)) = 0 \). By our assumption, we have

\[ (2.2) \quad \varphi_i(b_1) + \text{wt}_i(b_2) \leq \varepsilon_i(b_3) - a_{ii} \quad \text{and} \quad \varphi_i(b_2) \leq \varepsilon_i(b_3) - a_{ii}. \]

If \( \varphi_i(b_2) \leq \varepsilon_i(b_3) \), we will show \( \varphi_i(b_1) + a_{ii} \leq \varepsilon_i(b_2 \otimes b_3) < \varphi_i(b_1) \). Since \( \varepsilon_i(b_2 \otimes b_3) = \varepsilon_i(b_3) - \text{wt}_i(b_2) \), we must show \( \varphi_i(b_1) + a_{ii} \leq \varepsilon_i(b_3) - \text{wt}_i(b_2) < \varphi_i(b_1) \). By (2.2), it suffices to show the second inequality. Since \( \varphi_i(b_2) \leq \varepsilon_i(b_3) < \varphi_i(b_1 \otimes b_2) \), we have \( \varphi_i(b_1 \otimes b_2) = \varphi_i(b_1) + \text{wt}_i(b_2) \). Hence we obtain

\[ \varphi_i(b_1) > \varepsilon_i(b_3) - \text{wt}_i(b_2). \]

If \( \varepsilon_i(b_3) < \varphi_i(b_2) \), by the first inequality of (2.2), we know \( \varphi_i(b_1) \leq \varepsilon_i(b_2 \otimes b_3) - a_{ii} \). Hence \( \tilde{e}_i(b_1 \otimes (b_2 \otimes b_3)) = 0 \) or \( \tilde{e}_i(b_1 \otimes (b_2 \otimes b_3)) = b_1 \otimes \tilde{e}_i(b_2 \otimes b_3) \).

By our assumption and the second inequality of (2.2), the latter is also 0.

**Case 3:** \( \varphi_i(b_1 \otimes b_2) \leq \varepsilon_i(b_3) \).

In this case, we have \( \tilde{e}_i((b_1 \otimes b_2) \otimes b_3) = (b_1 \otimes b_2) \otimes \tilde{e}_i b_3 \). Hence it is enough to show \( \varphi_i(b_1) \leq \varepsilon_i(b_2 \otimes b_3) \) and \( \varphi_i(b_2) \leq \varepsilon_i(b_3) \). By the definition of \( \varphi_i \), we have (a) \( \varphi_i(b_1) + \text{wt}_i(b_2) \leq \varepsilon_i(b_3) \) and (b) \( \varphi_i(b_2) \leq \varepsilon_i(b_3) \). Hence we have \( \varphi_i(b_1) \leq \varepsilon_i(b_3) - \text{wt}_i(b_2) \leq \varepsilon_i(b_2 \otimes b_3) \), in which the first inequality follows from (a).

\[ \square \]

**Remark 2.12.** The category of crystals \( B \) such that

\[ \varepsilon_i(b) = 0, \quad \varphi_i(b) = \text{wt}_i(b) \geq 0 \]

for any \( i \in I^{\text{im}} \) and \( b \in B \)

is closed under tensor product. Note that \( B(\infty) \) and \( B(\lambda) \) (\( \lambda \in P^+ \)) belong to this category.
Example 2.13. For $\lambda \in P$, let $T_\lambda = \{t_\lambda\}$ and define

$$\text{wt}(t_\lambda) = \lambda, \quad \tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0 \quad \text{for all } i \in I,$$

$$\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty \quad \text{for all } i \in I.$$

Then $T_\lambda$ is a crystal. We have $T_\lambda \otimes T_\mu \simeq T_{\lambda + \mu}$. Note that $T_0$ is a unit object of the tensor category of crystals (see e.g. [9]). Using this crystal, Proposition 2.9 can be translated into following the statement:

(2.3) for every $\lambda \in P^+$, there exists an embedding $\pi_\lambda : B(\lambda) \to B(\infty) \otimes T_\lambda$.

Example 2.14. For each $i \in I$, let $B_i = \{b_i(-n); n \geq 0\}$. Then $B_i$ is a crystal with the maps defined by

$$\text{wt } b_i(-n) = -n \alpha_i,$$

$$\tilde{e}_i b_i(-n) = b_i(-n + 1), \quad \tilde{f}_i b_i(-n) = b_i(-n - 1),$$

$$\varepsilon_i(b_i(-n)) = n, \quad \varphi_i(b_i(-n)) = -n \quad \text{if } i \in I^\text{re},$$

$$\varepsilon_i(b_i(-n)) = 0, \quad \varphi_i(b_i(-n)) = \text{wt}_i(b_i(-n)) = -n \varepsilon_i; \quad \text{if } i \in I^\text{im},$$

$$\varepsilon_j(b_i(-n)) = \varphi_j(b_i(-n)) = -\infty \quad \text{if } j \neq i.$$

Here, we understand $b_i(-n) = 0$ for $n < 0$. The crystal $B_i$ is called an elementary crystal.

Example 2.15. For $\lambda, \mu \in P^+$, the tensor product $B(\lambda) \otimes B(\mu)$ is a crystal associated with $V(\lambda) \otimes V(\mu)$. There exists a unique strict embedding:

$$\Phi_{\lambda, \mu} : B(\lambda + \mu) \longrightarrow B(\lambda) \otimes B(\mu),$$

which sends $u_{\lambda + \mu}$ to $u_\lambda \otimes u_\mu$.

Example 2.16. Let $C = \{c\}$ be the crystal with $\text{wt}(c) = 0$ and $\varepsilon_i(c) = \varphi_i(c) = 0, \tilde{f}_i c = \tilde{e}_i c = 0$ for any $i \in I$. Then $C$ is isomorphic to $B(0)$. For a crystal $B,$
$b \in B$ and $i \in I$, we have

\[ \begin{align*}
\text{wt}(b \otimes c) &= \text{wt}(b), \\
\varepsilon_i(b \otimes c) &= \max(\varepsilon_i(b), -\text{wt}(b)), \\
\varphi_i(b \otimes c) &= \max(\varphi_i(b), 0), \\
\tilde{\varepsilon}_i(b \otimes c) &= \begin{cases} 
\tilde{\varepsilon}_i b \otimes c & \text{if } \varphi_i(b) \geq 0 \text{ and } i \in I^e, \\
\tilde{\varepsilon}_i b \otimes c & \text{if } \varphi_i(b) + a_{ii} > 0 \text{ and } i \in I^m, \\
0 & \text{otherwise},
\end{cases} \\
\tilde{f}_i(b \otimes c) &= \begin{cases} 
\tilde{f}_i b \otimes c & \text{if } \varphi_i(b) > 0, \\
0 & \text{otherwise}.
\end{cases}
\end{align*} \]

In general, $B \otimes C$ is not isomorphic to $B$.

**Example 2.17.** Let $i = (i_1, i_2, \ldots)$ be an infinite sequence in $I$ such that every $i \in I$ appears infinitely many times in $i$. For $k \in \mathbb{Z}_{\geq 0}$, set $B(k) = B_{i_k} \otimes \cdots \otimes B_{i_1}$. For $k_1 \leq k_2$, let $\psi_{k_2,k_1} : B(k_1) \to B(k_2)$ be the map $b \mapsto b_{i_{k_2}}(0) \otimes \cdots \otimes b_{i_{k_1+1}}(0) \otimes b$. Then $\{B(k)\}_{k \geq 1}$ is an inductive system. We shall consider the (set-theoretical) inductive limit of $B(k)$:

\[ B(i) = \{ \cdots \otimes b_{i_k}(-x_k) \otimes \cdots \otimes b_{i_1}(-x_1) \in \cdots \otimes B_{i_k} \otimes \cdots \otimes B_{i_1} \mid x_k \in \mathbb{Z}_{\geq 0}, \text{ and } x_k = 0 \text{ for } k \gg 0 \}. \]

Let $\psi_k : B(k) \to B(i)$ be the canonical injective map. The maps $\psi_{k_2,k_1}$ are not crystal morphisms, but they have the following properties:

(i) $\text{wt}$ is preserved by $\psi_{k_2,k_1}$,

(ii) for $i \in I$, $k \in \mathbb{Z}_{>0}$ and $b \in B(k)$, the sequences $\{\psi_{k'}(\tilde{\varepsilon}_i(\psi_{k',k}(b)))\}_{k' \geq k}$, $\{\tilde{f}_i(\psi_{k',k}(b))\}_{k' \geq k}$ and $\{\varepsilon_i(\psi_{k',k}(b))\}_{k' \geq k}$, $\{\varphi_i(\psi_{k',k}(b))\}_{k' \geq k}$ are stationary,

Therefore, the inductive limit $B(i)$ has a crystal structure. This is explicitly given as follows. Let $b = \cdots \otimes b_{i_k}(-x_k) \otimes \cdots \otimes b_{i_1}(-x_1) \in B(i)$. Then we have

\[ \text{wt}(b) = -\sum_k x_k \alpha_{i_k}. \]
For $i \in I^{re}$, we have
\[ \varepsilon_i(b) = \max \left\{ x_k + \sum_{l>k} \langle h_i, \alpha_i \rangle x_l ; 1 \leq k, i = i_k \right\}, \]
\[ \varphi_i(b) = \max \left\{ -x_k - \sum_{1 \leq l < k} \langle h_i, \alpha_i \rangle x_l ; 1 \leq k, i = i_k \right\}, \]
and, for $i \in I^{im}$, we have
\[ \varepsilon_i(b) = 0 \quad \text{and} \quad \varphi_i(b) = \text{wt}_i(b). \]

For $i \in I^{re}$, we have
\[ \tilde{e}_i b = \begin{cases} \cdots \otimes b_{i_n e} (-x_{n_e} + 1) \otimes \cdots \otimes b_i (-x_1) & \text{if } \varepsilon_i(b) > 0, \\ 0 & \text{if } \varepsilon_i(b) \leq 0, \end{cases} \]
\[ \tilde{f}_i b = \cdots \otimes b_{i_{n_f}} (-x_{n_f} - 1) \otimes \cdots \otimes b_i (-x_1), \]
where $n_e$ (resp. $n_f$) is the largest (resp. smallest) $k \geq 1$ such that $i_k = i$ and $x_k + \sum_{l>k} \langle h_i, \alpha_i \rangle x_l = \varepsilon_i(b)$. Note that such an $n_e$ exists if $\varepsilon_i(b) > 0$.

When $i \in I^{im}$, let $n_f$ be the smallest $k$ such that
\[ i_k = i \quad \text{and} \quad \sum_{l>k} \langle h_i, \alpha_i \rangle x_l = 0. \]

Then we have
\[ \tilde{f}_i b = \cdots \otimes b_{i_{n_f}} (-x_{n_f} - 1) \otimes \cdots \otimes b_i (-x_1) \]
and
\[ \tilde{e}_i b = \begin{cases} \cdots \otimes b_{i_{n_f}} (-x_{n_f} + 1) \otimes \cdots \otimes b_i (-x_1) & \text{if } x_{n_f} > 0 \text{ and } \sum_{k < l \leq n_f} \langle h_i, \alpha_i \rangle x_l < a_{ii} \text{ for any} \\ k \text{ such that } 1 \leq k < n_f \text{ and } i_k = i, \\ 0 & \text{otherwise.} \end{cases} \]

3. Crystal Embedding Theorem

In this section, we will prove one of the main results of this paper, the crystal embedding theorem for quantum generalized Kac-Moody algebras.
Theorem 3.1. For all \( i \in I \), there exists a unique strict embedding
\[
\Psi_i : B(\infty) \rightarrow B(\infty) \otimes B_i,
\]
called the crystal embedding.

The map \( \Psi_i \) sends \( 1 \) to \( 1 \otimes b_i(0) \), because there is a unique vector of weight 0 in \( B(\infty) \otimes B_i \).

Proof. Let \( b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} 1 \in B(\infty) \). Take \( \mu \gg 0 \) in \( P^+ \) such that \( b \in \text{Im}(\pi_\mu) \), where \( \pi_\mu : B(\mu) \rightarrow B(\infty) \) is the map given in Proposition 2.9. Hence \( b_\mu := \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\mu \in B(\mu) \) satisfies \( \pi_\mu(b_\mu) = b \).

Set \( l = \mu(h_i) \) and set \( \lambda = \mu - l\Lambda_i \in P^+ \). Then \( \lambda(h_i) = 0 \) and there is a unique strict embedding \( \Phi_{\lambda,\Lambda_i} : B(\mu) \rightarrow B(\lambda) \otimes B(l\Lambda_i) \), which sends \( u_\mu \) to \( u_\lambda \otimes u_{l\Lambda_i} \) (see Example 2.15). We claim that
\[
\Phi_{\lambda,\Lambda_i}(b_\mu) = b' \otimes \tilde{f}_{i_r}^m u_{l\Lambda_i} \quad \text{for some } b' \text{ and } n \in \mathbb{Z}_{\geq 0},
\]
moreover \( \pi_\lambda(b') \otimes b_i(-n) \) does not depend on the choice of \( \mu \gg 0 \).

Then we define \( \Psi_i(b) = \pi_\lambda(b') \otimes b_i(-n) \).

We show (3.1) by the induction on \( r \). If \( r = 0 \), our assertion is obvious.

Assume that our assertion is true for \( r - 1 \) and let \( b_1 = \tilde{f}_{i_2} \cdots \tilde{f}_{i_r} u_\mu \). By the induction hypothesis, we have
\[
\Phi_{\lambda,\Lambda_i}(b_1) = b'_1 \otimes \tilde{f}_{i_r}^m u_{l\Lambda_i},
\]
for some \( b'_1 \in B(\lambda) \) and \( m \in \mathbb{Z}_{\geq 0} \). Hence it suffices to show that
\[
\tilde{f}_{i_1}(b'_1 \otimes \tilde{f}_{i_r}^m u_{l\Lambda_i}) = b' \otimes \tilde{f}_{i_r}^m u_{l\Lambda_i} \quad \text{for some } b' \in B(\lambda) \text{ and } n \in \mathbb{Z}_{\geq 0},
\]
and \( \tilde{f}_{i_1}(\pi_\lambda(b'_1) \otimes b_i(-m)) = \pi_\lambda(b') \otimes b_i(-n) \).

If \( i_1 = i \), then \( \varphi_i(b'_1) = \varphi_i(\pi_\lambda(b'_1)) \) since \( \lambda(h_i) = 0 \), and
\[
\varepsilon_i(\tilde{f}_{i_r}^m u_{l\Lambda_i}) = \varepsilon_i(b_i(-m)) = \begin{cases} m & \text{if } i \in I^e, \\ 0 & \text{if } i \in I^m, \end{cases}
\]
and hence (3.2) is obvious. Suppose that \( i_1 \neq i \). Then
\[
\varphi_{i_1}(b'_1) = \varphi_{i_1}(\pi_\lambda(b'_1)) + \langle h_{i_1}, \lambda \rangle \gg 0 = \varepsilon_{i_1}(\tilde{f}_{i_r}^m u_{l\Lambda_i}),
\]
we have
\[
\tilde{f}_{i_1}(b'_1 \otimes \tilde{f}_{i_r}^m u_{l\Lambda_i}) = \tilde{f}_{i_1} b'_1 \otimes \tilde{f}_{i_r}^m u_{l\Lambda_i}.
\]
On the other hand we have \( \tilde{f}_i (\pi_\lambda (b'_i) \otimes b_i (-m)) = \tilde{f}_i \pi_\lambda (b'_i) \otimes b_i (-m) \), which proves our claim (3.2).

It is straightforward to verify that \( \Psi_i : B(\infty) \to B(\infty) \otimes B_i \) is a strict crystal embedding. The uniqueness of \( \Psi_i \) is obvious since \( B(\infty) \otimes B_i \) has a unique vector with weight 0. \( \square \)

The crystal embedding theorem yields a procedure to determine the structure of the crystal \( B(\infty) \) in terms of elementary crystals. Take an infinite sequence \( i = (i_1, i_2, \ldots) \) in \( I \) such that every \( i \in I \) appears infinitely many times. Such a sequence always exists since \( I \) is countable. For each \( N \geq 1 \), taking the composition of crystal embeddings repeatedly, we obtain a strict crystal embedding

\[
\Psi^{(N)} := (\Psi_{i_N} \otimes \text{id} \otimes \cdots \otimes \text{id}) \circ \cdots \circ (\Psi_{i_2} \otimes \text{id}) \circ \Psi_{i_1} : \]

\( B(\infty) \hookrightarrow B(\infty) \otimes B_{i_1} \hookrightarrow B(\infty) \otimes B_{i_2} \otimes B_{i_1} \hookrightarrow \cdots \hookrightarrow B(\infty) \otimes B_{i_N} \otimes \cdots \otimes B_{i_1}. \)

It is easily seen that, for any \( b \in B \), there exists \( N > 0 \) such that

\[
\Psi^{(N)}(b) = 1 \otimes b_{i_N} (-x_N) \otimes \cdots \otimes b_{i_1} (-x_1)
\]

for some \( x_1, \ldots, x_N \in \mathbb{Z}_{\geq 0} \). Set \( x_k = 0 \) for \( k > N \). Then for any \( k \geq N \), we have \( \Psi^{(k)}(b) = 1 \otimes b_{i_k} (-x_k) \otimes \cdots \otimes b_{i_1} (-x_1) \). Hence, associating \( \cdots \otimes b_{i_{N+1}} (0) \otimes b_{i_N} (-x_N) \otimes \cdots \otimes b_{i_1} (-x_1) \) to \( b \), we obtain a map \( B(\infty) \to B(i) \) (see Example 2.17). We can easily see that it is a crystal morphism, and we obtain the following result.

**Proposition 3.2.** \( B(\infty) \) is strictly embedded in the crystal \( B(i) \) introduced in Example 2.17.

Hence \( B(\infty) \) is isomorphic to the connected component of \( B(i) \) containing \( b(i, 0) := \cdots \otimes b_{i_2} (0) \otimes b_{i_1} (0) \).

**Example 3.3.** Let \( I = \{1, 2\} \) and consider the quantum generalized Kac-Moody algebra \( U_q(\mathfrak{g}) \) associated with a rank 2 Borcherds-Cartan matrix

\[
A = \begin{pmatrix} 2 & -a \\ -b & -c \end{pmatrix}
\]

for some \( a, b \in \mathbb{Z}_{>0} \) and \( c \in 2\mathbb{Z}_{\geq 0} \).
Take an infinite sequence \( i = (1, 2, 1, 2, \ldots) \) and let \( B \) be the set of elements of the form

\[
b(i, x) := \cdots \otimes b_2(-x_{2k}) \otimes b_1(-x_{2k-1}) \otimes \cdots \otimes b_2(-x_2) \otimes b_1(-x_1) \in B(i)
\]
satisfying the following conditions:

(i) \( ax_{2k} - x_{2k+1} \geq 0 \) for all \( k \geq 1 \),
(ii) for all \( k \geq 2 \) such that \( x_{2k} > 0 \), we have \( x_{2k-1} > 0 \) and \( ax_{2k} - x_{2k+1} > 0 \).

It is shown in [11] that \( B \) is the connected component of \( B(i) \) containing \( b(i, 0) := \cdots \otimes b_2(0) \otimes b_1(0) \otimes b_2(0) \otimes b_1(0) \). Therefore, \( B \) is isomorphic to the crystal \( B(\infty) \).

**Example 3.4.** Let \( I = \{ (i, t) : i \in \mathbb{Z}_{\geq -1}, 1 \leq t \leq c(i) \} \), where \( c(i) \) is the \( i \)-th coefficient of the elliptic modular function

\[
j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots = \sum_{i=-1}^{\infty} c(i)q^i.
\]

Consider the Borcherds-Cartan matrix

\[
A = (a_{(i,t),(j,s)})_{(i,t),(j,s) \in I}
\]
whose entries are given by \( a_{(i,t),(j,s)} = -(i + j) \). The associated generalized Kac-Moody algebra \( g \) is called the *Monster Lie algebra*, and it played a crucial role in Borcherds’ proof of the Moonshine conjecture [2]. More precisely, Borcherds derived the *twisted denominator identity* for the Monster Lie algebra with the action of the Monster, from which the replication formulae for the Thompson series follow.

In this paper, we deal with the corresponding quantum group \( U_q(g) \) which we call the *quantum Monster algebra*. Take the infinite sequence

\[
i = ([i(k)])_{k=1}^{\infty} = ((-1, 1), (1, 1), \ldots, (1, c(1)); (-1, 1), (1, 1), \ldots, (1, c(1)),
\]

\[
(2, 1), \ldots, (2, c(2)); (-1, 1), (1, 1), \ldots, (1, c(1)), (2, 1), \ldots,
\]

\[
(2, c(2)), (3, 1), \ldots, (3, c(3)); (-1, 1), \ldots).
\]

Note that \((−1, 1)\) appears at the \( b(n) \)-th position for \( n \geq 0 \), where \( b(n) = nc(1) + (n - 1)c(2) + \cdots + c(n) + n + 1 \).

For \( k \in \mathbb{Z}_{>0} \), we denote by \( k^{(-)} \) the largest integer \( l < k \) such that \( i(l) = i(k) \). If such an \( l \) does not exist, then set \( k^{(-)} = 0 \). Let \( B \) be the set of elements

\[
b(i, x) := \cdots \otimes b_{i(k)}(-x_k) \otimes \cdots \otimes b_{i(1)}(-x_1) \in B(i)
\]
satisfying the following conditions:

(i) $x_{b(1)} = 0$,
(ii) for all $n \geq 1$, we have

$$-\sum_{b(n)<l<b(n+1)} \langle h_{(-1,1)}, \alpha_{l(l)} \rangle x_l \geq x_{b(n+1)},$$

(iii) if $i(k) \neq (-1,1)$, $x_k > 0$ and $k^{(-)} > 0$, then

$$\sum_{k^{(-)}<l<k} \langle h_{i(k)}, \alpha_{i(l)} \rangle x_l < 0.$$

In addition, if $x_l = 0$ for all $k^{(-)} < l < k$ such that $i(l) \neq (-1,1)$, then we have

$$-\sum_{b(n)<l<b(n+1)} \langle h_{(-1,1)}, \alpha_{l(l)} \rangle x_l \geq x_{b(n+1)},$$

where $n$ is a unique integer such that $k^{(-)} < b(n) < k$.

Then $B$ is the connected component of $B(i)$ containing $b(i,0) = \cdots \otimes b_{i(2)}(0) \otimes b_{i(1)}(0)$ (see [11]). Therefore, $B$ is isomorphic to the crystal $B(\infty)$.

4. Characterization of $B(\infty)$ and $B(\lambda)$

As an application of the crystal embedding theorem, we will give a characterization of the crystals $B(\infty)$ and $B(\lambda)$.

**Theorem 4.1.** Let $B$ be a crystal. Suppose that $B$ satisfies the following conditions:

(i) $\text{wt}(B) \subset -Q_+$,
(ii) there exists an element $b_0 \in B$ such that $\text{wt}(b_0) = 0$,
(iii) for any $b \in B$ such that $b \neq b_0$, there exists some $i \in I$ such that $\tilde{e}_i b \neq 0$,
(iv) for all $i$, there exists a strict embedding $\Psi_i : B \rightarrow B \otimes B_i$.

Then there is a crystal isomorphism

$$B \cong B(\infty),$$

which sends $b_0$ to $1$. 
Proof. Note that \( b_0 \) is a unique element of weight 0 in \( B \). Indeed, if \( b \neq b_0 \) has weight 0, then taking \( i \in I \) such that \( \tilde{e}_i b \in B \), the weight of \( \tilde{e}_i b \) does not belong to \( -Q_+ \), which is a contradiction.

Note also that \( \tilde{e}_i b_0 = 0 \) for any \( i \in I \). Indeed, otherwise, \( \text{wt}(\tilde{e}_i b_0) \not\in -Q_+ \).

Since \( B \otimes B_i \) has a unique vector of weight 0, we have \( \Psi_i(b_0) = b_0 \otimes b_i(0) \).

Take an infinite sequence \( i = (i_1, i_2, \ldots) \) in \( I \) such that every \( i \in I \) appears infinitely many times. Similarly to the case of \( B(\infty) \), we obtain the strict crystal embeddings

\[
\Psi^{(N)} := (\Psi_{i_N} \otimes \text{id} \otimes \cdots \otimes \text{id}) \circ \cdots \circ (\Psi_{i_2} \otimes \text{id}) \circ \Psi_{i_1}:
\]

\[
B(\infty) \hookrightarrow B(\infty) \otimes B_{i_1} \hookrightarrow B(\infty) \otimes B_{i_2} \otimes B_{i_1} \hookrightarrow \cdots \hookrightarrow B(\infty) \otimes B_{i_N} \otimes \cdots \otimes B_{i_1}.
\]

We can easily see that for any \( b \in B \), there exists \( N \gg 0 \) such that

\[
\Psi^{(N)}(b) = b_0 \otimes b_{i_N}(-x_N) \otimes \cdots \otimes b_{i_1}(-x_1)
\]

for some \( x_1, \ldots, x_N \in \mathbb{Z}_{\geq 0} \). Hence we get a strict crystal embedding \( B \hookrightarrow B(i) \) given by

\[
b \mapsto \cdots \otimes b_{i_{N+1}}(0) \otimes b_{i_N}(-x_N) \otimes \cdots \otimes b_{i_1}(-x_1).
\]

Since this embedding sends \( b_0 \) to

\[
b(i,0) := \cdots \otimes b_{i_{N+1}}(0) \otimes b_{i_N}(0) \otimes \cdots \otimes b_{i_1}(0),
\]

\( B \) is isomorphic to the the connected component of \( B(i) \) containing \( b(i,0) \).

The theorem then follows from the fact that \( B(\infty) \) is also isomorphic to the smallest full subcrystal of \( B(i) \) containing \( b(i,0) \).

For \( \lambda \in P^+ \), the crystal \( B(\lambda) \) is embedded in \( B(\infty) \otimes T_\lambda \).

**Theorem 4.2.** Let \( \lambda \in P^+ \) be a dominant integral weight. Then \( B(\lambda) \) is isomorphic to the connected component of \( B(\infty) \otimes T_\lambda \otimes C \) containing \( 1 \otimes t_\lambda \otimes c \). Here \( C \) is the crystal introduced in Example 2.16.

**Proof.** Note that the embedding \( \iota : B(\lambda) \rightarrow B(\infty) \otimes T_\lambda \) commutes with \( \tilde{e}_i \).

Let us remark that for \( b \in B(\lambda) \), \( \tilde{f}_i(b) = 0 \) if and only if \( \varphi_i(b) \leq 0 \). Hence, \( \iota \) induces a strict embedding \( B(\lambda) \rightarrow B(\infty) \otimes T_\lambda \otimes C \). Hence the assertion follows. \( \square \)
Example 4.3. Let $U_q(g)$ be the quantum generalized Kac-Moody algebra associated with the rank 2 Borcherds-Cartan matrix given in Example 3.3. We shall use the notations there. Let $\lambda \in P^+$. Then $B(\lambda)$ is isomorphic to the connected component $B^\lambda$ of $B(i) \otimes T_\lambda \otimes C$ containing $b(i, 0) \otimes t_\lambda \otimes c$. It was shown in [12] that $B^\lambda$ is the set of elements of the form

$$b(i, x) \otimes t_\lambda \otimes c := \cdots \otimes b_2(-x_{2k}) \otimes b_1(-x_{2k-1}) \otimes \cdots \otimes b_2(-x_2) \otimes b_1(-x_1) \otimes t_\lambda \otimes c$$

satisfying the conditions (i) and (ii) in Example 3.3 and two additional conditions:

(a) $0 \leq x_1 \leq \langle h_1, \lambda \rangle$,
(b) if $x_2 > 0$ and $\langle h_2, \lambda \rangle = 0$, then $x_1 > 0$.

Example 4.4. Let $U_q(g)$ be the quantum Monster algebra in Example 3.4, and let $B(\lambda)$ be the irreducible highest weight crystal with $\lambda \in P^+$. Using the notations in Example 3.4, it was shown in [12] that the connected component of $B(i) \otimes T_\lambda \otimes C$ containing $b(i, 0) \otimes t_\lambda \otimes c = \cdots \otimes b_{i(k)}(0) \otimes \cdots \otimes b_{i(1)}(0) \otimes t_\lambda \otimes c$ is the set $B^\lambda$ consisting of elements of the form

$$b(i, x) \otimes t_\lambda \otimes c = \cdots \otimes b_{i(k)}(-x_k) \otimes \cdots \otimes b_{i(1)}(-x_1) \otimes t_\lambda \otimes c$$

satisfying the conditions (i)–(iii) in Example 3.4 and two additional conditions:

(a) $0 \leq x_1 \leq \langle h_{(-1,1)}, \lambda \rangle$,
(b) if $i(k) \neq (-1, 1)$, $\langle h_{i(k)}, \lambda \rangle = 0$, $x_k > 0$ and $k^{(-)} = 0$, then there exists $l$ such that $1 \leq l < k$, $\langle h_{i(k)}, \alpha_{i(l)} \rangle < 0$ and $x_l > 0$.

Hence by Theorem 4.2, we conclude that $B^\lambda$ is isomorphic to the crystal $B(\lambda)$.

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