A Bijection Between Partially Directed Paths in the Symmetric Wedge and Matchings

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Abstract

We give a bijection between partially directed paths in the symmetric wedge $y = \pm x$ and matchings, which sends north steps to nestings. This gives a bijective proof of a result of Prellberg et al. that was first discovered through the corresponding generating functions: the number of partially directed paths starting at the origin confined to the symmetric wedge $y = \pm x$ with $k$ north steps is equal to the number of matchings on $[2n]$ with $k$ nestings.

Key Words: partially directed path, matching, nesting

AMS subject classification: 05A15, 05A18

1 Introduction

The purpose of this paper is to give a bijective proof of a fact that was discovered unexpectedly and connects two seemingly different branches of combinatorics. One of the branches is the study of matchings and set partitions and, more specifically, the statistics crossings and nestings. The other one is the study of directed paths in the plane.

Based on Touchard’s work [7], Riordan [5] derived a formula for the number of matchings with $k$ crossings. Since then, a lot of results connected to this topic have been obtained. We mention a few. M. de Sainte-Catherine in [1] bijectively shows that the number of matchings with $k$ crossings is equal to the number of matchings with $k$ nestings. This bijection also implies symmetric joint distribution of crossings and nestings. More than two decades later, Kasraoui and Zeng, in [2], extended this bijection to show that the same result holds for set partitions. Martin Klazar [3] studied the distribution of these statistics on subtrees of the generating tree of matchings, and the same questions for set partitions were studied in [4].

In another line of work, Prellberg et al. in [8] worked on founding a generating function of self-avoiding partially directed paths in the wedge $y = \pm px$ consisting of east, north and
south steps. Using the kernel method they were able to derive explicitly the generating function for the case $p = 1$. The generating function revealed that the number of such paths which end at $(n, -n)$ with $k$ north steps is the same as the number of matchings on $[2n]$ with $k$ nestings. For a nice survey of the history of the problem and how this fact was discovered see [6].

A matching on the set $[2n] = \{1, \ldots, 2n\}$ is a family of $n$ two-element disjoint subsets of $[2n]$. In particular, it is a set-partition with all the blocks of size two. It is convenient to represent a matching with its standard diagram consisting of arcs connecting $2n$ vertices on a horizontal line (see Figure 1). The vertices are numbered in increasing order from left to right. The set of all matchings of $[2n]$ is denoted by $M_n$. We say that two edges $(a, b)$ and $(c, d)$ form a crossing if $a < c < b < d$ (i.e. if the cross) and they form a nesting if $a < c < d < b$ (i.e. if one covers the other). If they are neither crossed nor nested we say they form an alignment. The number of nestings in a matching $M$ is denoted by $ne(M)$.

Figure 1: Diagram of a matching with 10 vertices and edges: $e_1 = (1, 3), e_2 = (2, 7), e_3 = (4, 6), e_4 = (5, 8),$ and $e_5 = (9, 10)$. This matching has 3 crossings formed by the pairs of edges: $(e_1, e_2), (e_2, e_4),$ and $(e_3, e_4)$, one nesting $(e_2, e_3)$, and all the other pairs of edges form alignments.

A partially directed path in the plane is a path starting at the origin and consisting of unit east, north, and south steps. We consider all such paths confined to the symmetric wedge defined by the lines $y = \pm x$. Let $\mathcal{P}_n$ be the set of all such paths ending at the line $y = -x$ with $n$ horizontal steps.

**Theorem 1.1.** There is a bijection $\Phi: \mathcal{P}_n \to M_n$ that takes the number of north steps of $P \in \mathcal{P}_n$ to the number of nestings of $\Phi(P)$.

**Remark.** While preparing the present paper, we found out about the very recent work of Martin Rubey [6] in which he presents a bijective proof of the same result. However, our bijection is different from Rubey’s, as illustrated in Example [23]. In particular, $\Phi$ may be of special interest in the study of matchings because a key part of it is a bijection on matchings which, unlike the other bijections used in the literature, does not preserve the type of the matching, i.e., the sets of minimal and maximal elements of the blocks. This may
give further insight into the interaction between matchings of different type when various statistics of matchings are studied.

2 Definition and properties of the bijection Φ

Below we define a bijection \( \Phi: \mathcal{P}_n \rightarrow \mathcal{M}_n \) that takes the number of north steps of \( P \in \mathcal{P}_n \) to the number of nestings of \( \Phi(P) \). The map \( \Phi \) is defined as the composition of two maps: \( \Phi = \phi \circ \psi \), where \( \psi: \mathcal{P}_n \rightarrow \mathcal{M}_n \) and \( \phi: \mathcal{M}_n \rightarrow \mathcal{M}_n \).

2.1 Bijection \( \psi \) from \( \mathcal{P}_n \) to \( \mathcal{M}_n \)

Every path \( P \in \mathcal{P}_n \) is determined by the \( y \)-coordinates of its east steps, i.e., a sequence \( a_1, \ldots, a_n \) of integers such that \(- (i - 1) \leq a_i \leq i - 1 \). Set \( b_i = a_{n+1-i} + n + 1 - i \). Note that \( 1 \leq b_i \leq 2(n + 1 - i) - 1 \). Define a matching \( M \) on \([2n]\) by connecting the first available vertex from the left to the \( b_i \)-th available vertex to its right, one by one for each \( i = 1, \ldots, n \) in that order. Note that before the \( i \)-th step there are \( 2(n + 1 - i) \) vertices that are not connected yet, so each step is possible. We define \( \psi(P) = M \). It is not hard to see that knowing \( M \), one can reverse the steps one by one and find the \( b_i \)'s, which determine a path \( P \). So \( \psi \) is a bijection. Figure 2 shows a path \( P \in \mathcal{P}_7 \) and \( \psi(P) \).

Definition 2.1. Let \( M \in \mathcal{M}_n \). Suppose the edges \( e_1, \ldots, e_n \) of \( M \) are ordered according to their left endpoints in ascending order. Suppose \( e_i = (a, b) \) and \( e_{i+1} = (c, d) \). Define

\[
st_i(M) := \begin{cases} 
\{v : d \leq v \leq b, v \text{ is a vertex of } e_k, k > i\}, & \text{if } e_i \text{ and } e_{i+1} \text{ are nested} \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
st(M) = \sum_{i=1}^{n-1} st_i(M).
\]

Lemma 2.2. The number of north steps of \( P \) is equal to \( st(\psi(P)) \).

Proof. Let \( M = \psi(P) \). The number of north steps of \( P \) is

\[
s(n+1) \sum_{a_{i+1} > a_i} (a_{i+1} - a_i) \leq \sum_{b_{n-i} \geq b_{n-i+1} + 2} (b_{n-i} - b_{n-i+1} - 1)
\]

(2.1)
Figure 2: Path $P \in \mathcal{P}_7$ and the corresponding matching $\psi(P)$.

So, it suffices to show that

$$st_i(M) = \begin{cases} b_i - b_{i+1} - 1, & \text{if } b_i \geq b_{i+1} + 2 \\ 0, & \text{otherwise} \end{cases} \quad (2.2)$$

After the $i$-th edge $e_i$ is drawn in the construction of $M$, there are $b_i - 1$ unconnected vertices below it. In the case $b_i \geq b_{i+1} + 2$, we have $b_i - 1 \geq b_{i+1} + 1$ which implies $e_{i+1}$ is nested below $e_i$ and $st_i(M) = b_i - b_{i+1} - 1$. In the other case, when $b_i < b_{i+1} + 2$, we have $b_i - 1 < b_{i+1} + 1$ and hence the edge $e_{i+1}$ and $e_i$ are crossed (if $b_i > 1$) or aligned (if $b_i = 1$). In either case, $st_i(M) = 0$.

### 2.2 Bijection $\phi$ from $\mathcal{M}_n$ to $\mathcal{M}_n$

We describe $\phi$ by a series of transformations on the diagrams of the matchings. This map preserves the first edge. For $M \in \mathcal{M}_n$, $N = \phi(M)$ is constructed inductively as follows. If $n = 1$ set $\phi(M) = M$. If $n > 1$, let $M_1$ be the matching obtained from $M$ by deleting its first edge $e_1 = (1, r)$ and let $N_1 = \phi(M_1)$. Let $N_2$ be the matching obtained by adding back the edge $e_1$ in the same position as it was in $M$. Denote by $e_2$ the second edge of $N_2$ (which was also the second edge of $M$). There are three cases:

**case 1:** $e_1$ and $e_2$ were aligned

In this case set $N = \phi(M) = N_2$.

**case 2:** $e_1$ and $e_2$ were crossed

Let $f_2 = e_2 = (l_2, r_2), f_3 = (l_3, r_3), \ldots, f_k = (l_k, r_k)$ be the edges in $N_2$ crossing $e_1$ ordered by their left endpoints $2 = l_2 < l_3 < \cdots < l_k$. Rearrange them in the following way: connect $r_2$ to $l_3$, $r_3$ to $l_4$, $\ldots$, $r_{k-1}$ to $l_k$. Finally, insert one additional vertex
right before \( r \) and connect it to \( r_k \). Delete the vertex \( l_2 \) and renumber the remaining vertices (see Figure 3). Note that the position of the first edge in the matching \( N \) obtained this way is the same as in \( M \). Set \( \phi(M) = N \).

**Figure 3:** Definition of \( \phi \) when \( e_1 \) and \( e_2 \) are crossed. Dashed lines are used to represent edges whose left endpoints have been changed.

**case 3:** \( e_1 \) and \( e_2 \) were nested
In \( N_2 \), let \( f_1 = (l_1, r_1), \ldots, f_p = (l_p, r_p) \) be the edges crossing both \( e_1 = (1, r) \) and \( e_2 = (2, q) \), and let \( f_{p+1} = (l_{p+1}, r_{p+1}), \ldots, f_{p+s} = (l_{p+s}, r_{p+s}) \) be the edges crossing \( e_1 \) but not \( e_2 \), such that \( l_1 < \cdots < l_p < q < l_{p+1} < \cdots < l_{p+s} \). For easier notation let \( \{l_1 < \cdots < l_p < q < l_{p+1} < \cdots < l_{p+s}\} = \{v_1 < \cdots < v_p < v_{p+1} < v_{p+2} < \cdots < v_{p+s+1}\} \). Add one vertex right before \( r \) and connect it to \( v_{s+1} \). "Rearrange" the edges \( f_1, \ldots, f_{p+s} \) so that \( r_1, \ldots, r_{p+s} \) are connected to \( v_1, \ldots, v_s, v_{s+2}, \ldots, v_{p+s+1} \) in that order. Finally, delete the vertex 2 and renumber the remaining vertices. See Figure 4 for an illustration when \( p = 3 \) and \( s = 2 \). Call the matching obtained this way \( N \). The first edge of \( N \) is the same as in \( M \). Set \( \phi(M) = N \).

**Figure 4:** Example of case 3 for \( p = 3 \) and \( s = 2 \).

**Example 2.3.** Figure 5 shows step-by-step construction of \( \phi(M) \) for the matching \( M \) from Figure 2. So, for the path \( P \) given in Figure 2, the corresponding matching is \( \Phi(P) = \{(1, 4), (2, 14), (3, 12), (5, 8), (6, 9), (7, 11), (10, 13)\} \). Note that the image of \( P \) under Rubey's bijection defined in [6] is \( \{(1, 4), (2, 14), (3, 11), (5, 8), (6, 9), (7, 13), (10, 12)\} \). Hence the two bijections are different.
Theorem 2.4. The map $\phi$ is a bijection and $ne(\phi(M)) = st(M)$.

Proof. To show that $\phi$ is bijective, we explain how to define the inverse map. Note that the matching resulting from case 1 above has the property that its first edge is $e_{1}$. In the matching resulting from case 2 (case 3 respectively), the vertex preceding the right endpoint of the first edge $e_{1}$ is a left endpoint (right endpoint respectively) of an edge different than $e_{1}$. Since all the steps in the definition of $\phi$ are invertible, we simply perform the inverse steps of the corresponding case.

It is left to prove $ne(\phi(M)) = st(M)$. For shortness, for any matching $M$, let $ne(e, M)$ denote the number of edges in $M$ below the edge $e$. Let $M, M_{1}, N_{1}, N_{2}$, and $N$ be the same as in the definition of $\phi$. By inductive hypothesis, $ne(N_{1}) = st(M_{1}) = st(M) - st_{1}(M)$. So we just need to prove

$$ne(N) = ne(N_{1}) + st_{1}(M) \quad (2.3)$$

It is clear that

$$ne(N_{2}) = ne(N_{1}) + ne(e_{1}, N_{2}) \quad (2.4)$$
In the first case of the definition of $\phi$, (2.3) clearly follows since $st_1(M) = 0$ and we do not add nestings to $N_1$ by adding back $e_1$.

In the second case, $st_1(M) = 0$, so we need to show that $ne(N) = ne(N_1)$. To this end, if $e$ is an edge in $N_2$ different from $f_2, \ldots, f_k$ (notation from the definition of $\phi$), let $r(e)$ be the edge in $N$ that corresponds to $e$ in the obvious way, and let $r(f_i)$ be the edge with right endpoint $r_i$, for $i = 2, \ldots, k$. It is clear that $ne(e, N_2) = ne(r(e), N)$ for any edge $e \notin \{e_1, f_2, \ldots, f_k\}$. Note that the left endpoint of $r(f_i)$ in $N$ is $l_i - 1$ because the vertex 2 from $N_2$ was deleted (see Figure 3). So, for $2 \leq i < k$

$$ne(f_i, N_2) - ne(r(f_i), N) = |\{\text{edges in } N \text{ below } e_1 \text{ with left endpoint between } l_i - 1 \text{ and } l_{i+1} - 1\}| \quad (2.5)$$

$$ne(f_k, N_2) - ne(r(f_k), N) = |\{\text{edges in } N \text{ below } e_1 \text{ with left endpoint between } l_k - 1 \text{ and } r\}| \quad (2.6)$$

By subtracting the following equalities

$$ne(N_2) = \sum_{i=2}^{k} ne(f_i, N_2) + \sum_{e \notin \{f_2, \ldots, f_k\}} ne(e, N_2) \quad (2.7)$$

$$ne(N) = \sum_{i=2}^{k} ne(r(f_i), N) + \sum_{e \notin \{f_2, \ldots, f_k\}} ne(r(e), N) \quad (2.8)$$

and using (2.5) and (2.6) we get

$$ne(N_2) - ne(N) = ne(e_1, N) = ne(e_1, N_2) \quad (2.9)$$

This together with (2.4) gives $ne(N) = ne(N_1)$.

In the third case, similarly, denote by $r(f_i)$ the edge in $N$ that ends with vertex $r_i$, $i = 1, \ldots, p + s$, by $r(e_2)$ the edge that ends with the vertex $r - 1$, and for every other edge $e$ in $N_2$, denote by $r(e)$ the edge in $N$ that corresponds to $e$ in the natural way. In $N_2$, define $a$ to be the number of edges below $e_1$ and crossing $e_2 = (2, q)$ and $b$ to be the number of those edges below $e_1$ with a left endpoint right of $q$. In what follows, $v_i$ are the vertices
defined in case 3 of the definition of $\phi$. Then

\begin{align}
st_1(M) &= 1 + a + 2b + s \\
n e(r(e_2), N) &= |\{\text{edges in } N_2 \text{ below } e_1 \text{ with left endpoint between } v_{s+1} \text{ and } r\}| \quad (2.11) \\
n e(N_2) &= n e(N_1) + n e(e_2, N_2) + 1 + a + b \\
n e(N_2) &= n e(e_1, N_2) + n e(e_2, N_2) + \sum_{i=1}^{p+s} n e(f_i, N_2) + \sum_{e \notin \{e_1, e_2, f_1, \ldots, f_{p+s}\}} n e(e, N_2) \\
n e(N) &= n e(e_1, N) + n e(r(e_2), N) + \sum_{i=1}^{p+s} n e(r(f_i), N) + \sum_{e \notin \{e_1, e_2, f_1, \ldots, f_{p+s}\}} n e(r(e), N_2) \quad (2.13)
\end{align}

To complete the proof, we need to distinguish two cases: $s \geq p$ and $p > s$. When $s \geq p$, close inspection of the "rearrangement" of the edges reveals:

\begin{align}n e(r(f_i), N) - n e(f_i, N_2) &= \\
&= \begin{cases} 1, & 1 \leq i \leq p \\
1 + |\{\text{edges in } N_2 \text{ below } e_1 \text{ with left vertex between } v_i \text{ and } v_{i+1}\}|, & p < i \leq s \\
0, & s < i \leq p + s \end{cases} \quad (2.15)
\end{align}

while when $p > s$, similar equalities hold:

\begin{align}n e(r(f_i)) - n e(f_i) &= \\
&= \begin{cases} 1, & 1 \leq i \leq s \\
-|\{\text{edges in } N_2 \text{ below } e_1 \text{ with left vertex between } v_i \text{ and } v_{i+1}\}|, & s < i \leq p \\
0, & p < i \leq p + s \end{cases} \quad (2.16)
\end{align}

Now, we add the equations (2.12) and (2.14) and subtract (2.13) from them. Using (2.10), (2.11), and (2.15), i.e., (2.16), we get (2.3). \hfill \square

### 2.3 Some properties of $\Phi$

First we need few definitions. We say that $\{l, l+1, \ldots, k\}$ is a component of a matching $M \in \mathcal{M}_n$ if the restrictions of $M$ on each of the sets $\{1, \ldots, l-1\}$, $\{l, l+1, \ldots, k\}$, and
\{k + 1, \ldots, n\} are matchings themselves. A matching is called irreducible if it has only one component. In terms of diagrams, a matching is irreducible if it cannot be split by vertical bars into disjoint matchings.

A component of a path \( P \in \mathcal{P}_n \) is a subsequence of consecutive steps beginning at \((l, -l)\) and ending at \((k, -k)\) such that both parts of \( P \) between \((l, -l)\) and \((k, -k)\), and between \((k, -k)\) and \((n, -n)\) when translated by the appropriate vector to the origin represent paths in \( \mathcal{P}_{k-l} \) and \( \mathcal{P}_{n-k} \) respectively. A component which does not have nontrivial subcomponents is called irreducible.

**Proposition 2.5.** For \( P \in \mathcal{P}_n \) the following are true:

(a) \( P \) has \( k \) south steps on the line \( x = n \) if and only in \( \Phi(P), 1 \) is connected to \( k + 1 \).

(b) The irreducible components of \( P \) read backwards are in one-to-one correspondence with the irreducible components of \( \Phi(P) \) from left to right.

**Proof.** (a) From the definition of \( \psi \), it is clear that \( P \) has \( k \) south steps on the line \( x = n \) if and only in \( \psi(P), 1 \) is connected to \( k + 1 \). Thus, the claim follows from the fact that \( \phi \) preserves the first edge.

(b) This statement is clearly true if we replace \( \Phi \) by \( \psi \). Hence, it suffices to observe that if the irreducible components of \( \psi(P) \) are \( C_1, \ldots, C_k \), then \( \phi(C_1), \ldots, \phi(C_k) \) are the irreducible components of \( \Phi(P) \).

**Proposition 2.6.** If \( P \) is a path with no north steps (Dyck path) then \( M = \Phi(P) \) is the unique matching with no nestings such that \( i \) is a left endpoint in \( M \) exactly when the \((2n + 1 - i)\)-th step of \( P \) is a south step.

In other words, the set of left and right endpoints of \( M \) is determined by \( P \) traced backwards.

**Proof.** It follows from the definition of \( \psi \) that the statement is true for \( \psi(P) \). Moreover, since \( \psi(P) \) has no nestings, \( \phi \) leaves \( \psi(P) \) unchanged.

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