ON NON-RIEMANNIAN PARALLEL TRANSPORT
IN REGGE CALCULUS

by

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Abstract

We discuss the possibility of incorporating non-Riemannian parallel transport into Regge calculus. It is shown that every Regge lattice is locally equivalent to a space of constant curvature. Therefore well known-concepts of differential geometry imply the definition of an arbitrary linear affine connection on a Regge lattice.

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1. Introduction

The interest in Regge calculus [1] and its use as an approximation scheme for Riemannian manifolds has increased during the last fifteen years. Regge calculus was applied to a variety of problems in both classical and quantum gravity, see [2] for a recent review. Regge calculus requires the approximation of smooth manifolds by piecewise flat spaces which are built from flat simplexes. The curvature of such an \( n \)-dimensional piecewise flat space resides in its \((n-2)\)-dimensional subsimplexes, usually referred to as the “hinges” of the “Regge lattice”.

Regge calculus is a useful tool for numerical calculations in Riemannian spacetimes. It is unsatisfactory that it cannot be applied to non-Riemannian manifolds so far. Possible applications of a non-Riemannian Regge calculus would include gauge theories of gravity [3] and supergravity in ordinary spacetime [4]. These Yang-Mills type extensions of general relativity are especially important when it comes to the inclusion of fermionic matter fields. Fermionic matter fields are known to be possible sources of torsion. Their introduction requires a linear connection that a priori is independent of the metric, i.e., one has to use non-Riemannian geometries as an appropriate framework. But an enlargement of Regge calculus to non-Riemannian manifolds is also of general interest. It should lead to a better understanding of the Regge lattice, showing more clearly its relation to ordinary differential geometry.

In order to extend Regge calculus to the non-Riemannian case, it is inevitable to investigate the notion of a linear connection in a Regge lattice. The metric of the Regge lattice is given by its link lengths. In order to implement a non-Riemannian connection, it is usually proposed to let a parallelly transported frame undergo a non-trivial rotation [5–8], a linear transformation [9], or a Poincaré transformation [10] while it is passing from one \( n \)-simplex to another. It was argued in [6] that the non-trivial rotations do not contribute to the Einstein–Cartan action. This is not quite correct because the contributions from the non-trivial rotations to the curvature associated with the hinges were not properly calculated. In fact, the nontrivial rotations can be represented by a discrete version of the contortion, i.e. the non-Riemannian part of the linear connection. A non-trivial contortion will generally produce torsion and a non-Riemannian piece to the curvature.

It is the purpose of this paper to clarify the geometric content of an arbitrary Regge lattice of dimension two, three, and four. We show in Sec.2 that the geometry of such a Regge lattice is locally equivalent to the geometry of a specific space of constant curvature. This is done by an isometric embedding of the basic building block of Regge calculus, the \((\varepsilon - n)\)-cone, into \(R^{n+1}\). Parallel transport on this embedded cone is, in turn, seen to be equivalent to parallel transport in a space of constant curvature. This, together with a short analysis of Cartan’s structure equations, reveals in Sec.3 how a general parallel transport should be defined in a Regge lattice. It is shown how non-Riemannian quantities appear by parallel transport around hinges. We summarize the results in Sec.4.
2. Isometric embedding of Regge’s $(\varepsilon - n)$–cone for $n = 2, 3, 4$

2.1. Preliminaries

The basic building block of Regge calculus is Regge’s $(\varepsilon - n)$–cone [1]. The $(\varepsilon - n)$–cones are approximatively realized in an $n$–dimensional Regge lattice by the $n$–simplexes.

An $(\varepsilon - 2)$–cone can be defined as follows: Take an Euclidean plane and introduce polar coordinates $\rho, \xi$, where points with the same $\rho$ and angles $\xi$ modulo $2\pi$ are identified. We obtain the $(\varepsilon - 2)$–cone by replacing $2\pi$ with $2\pi - \varepsilon$. Note that the deficit angle $\varepsilon$ is allowed to be negative. Turning to higher dimensions, we define an $(\varepsilon - n)$–cone as the direct product $\mathbb{R}^{n-2} \times (\varepsilon - 2)$–cone, compare Fig.1. It is convenient to choose cylindrical coordinates $\rho, \xi, z_1, \ldots, z_{n-2}$, thus

$$ds^2 = d\rho^2 + \rho^2 d\xi^2 + dz_1^2 + \ldots + dz_{n-2}^2.$$  \hspace{1cm} (2.1)

**Fig.1:** Definition of Regge’s $(\varepsilon - n)$– cone. The disc with deficit angle $\varepsilon$ represents the $(\varepsilon - 2)$–cone. It is parametrized by polar coordinates $\rho, \xi$. The straight line symbolizes $\mathbb{R}^{n-2}$, which is the subset for $\rho = 0$.

Parallel transport in an $(\varepsilon - n)$–cone is trivial except for encircling the $(n-2)$–dimensional flat submanifold $\rho = 0$. In order to be flat, the plane $z_i = \text{const}$, and hence the $(\varepsilon - n)$–cone, is lacking an angle $\varepsilon$. Therefore parallel transport around the submanifold $\rho = 0$ causes a rotation by an angle $\varepsilon$ within this plane, orthogonal to the submanifold. This is why the curvature of the $(\varepsilon - n)$–cone resides at the $(n-2)$–dimensional submanifold $\rho = 0$.

The next step is the definition of what we will call a $C^n$–cone. This $C^n$–cone will turn out to be the isometrically embedded $(\varepsilon - n)$–cone in $\mathbb{R}^{n+1}$ $(n = 2, 3, 4)$. We concentrate on a positive deficit angle first, $\varepsilon > 0$: Consider $S^n(r)$, the $n$–sphere of radius $r$ embedded in $\mathbb{R}^{n+1}$. By introducing spherical coordinates $r, \varphi, \theta_1, \ldots, \theta_{n-1}$, it is possible to characterize a parallel of latitude of the sphere by $\theta_i = const$. We define
a specific $C^n$–cone to be spanned by the tangent vectors $\frac{\partial}{\partial \theta_i}$ at a parallel of latitude of a specific $n$–sphere:

$$\vec{x} = \vec{p}(\varphi) + t_1 \frac{\partial}{\partial \theta_1} + ... + t_{n-1} \frac{\partial}{\partial \theta_{n-1}}.$$ (2.2)

The symbol $\vec{p}(\varphi)$ denotes an arbitrary point of the parallel of latitude, the parameters $t_1, ..., t_{n-1}$ are real, and $\vec{x}$ is a point of the $C^n$–cone. Accordingly a $C^n$–cone is parametrized by $n$ parameters $\varphi, t_1, ..., t_{n-1}$ and determined by the $n$ quantities $r, \theta_1, ..., \theta_{n-1}$ which characterize the parallel of latitude. The tangent spaces of the $C^n$–cone coincide at the parallel of latitude with the tangent spaces of the related $n$–sphere. Hence, $C^n$–cone and $n$–sphere are (to first order) indistinguishable there. Fig.2 shows the example of a $C^2$–cone.

**Fig.2:** The $C^2$–cone that is related to a specific parallel of latitude (this determines $\theta$) of a specific 2–sphere (this determines $r$). It is spanned by the (embedded) tangent vectors $\frac{\partial}{\partial \theta}$ of the parallel of latitude.

We focus now on four dimensions, $n = 4$. This contains the lower dimensional cases $n = 2, 3$ as special cases. The parametrization of the $C^4$–cone is given by

$$\vec{x} = \vec{p}(\varphi) + t_1 \frac{\partial}{\partial \theta_1} + t_2 \frac{\partial}{\partial \theta_2} + t_3 \frac{\partial}{\partial \theta_3},$$ (2.3)
or explicitly

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 
\end{pmatrix}
\begin{pmatrix}
  \cos \varphi \sin \theta_1 \sin \theta_2 \sin \theta_3 \\
  \sin \varphi \sin \theta_1 \sin \theta_2 \sin \theta_3 \\
  \cos \theta_1 \sin \theta_2 \sin \theta_3 \\
  \cos \theta_2 \sin \theta_3 \\
  \cos \theta_3
\end{pmatrix}
+ t_1 r
\begin{pmatrix}
  \cos \varphi \cos \theta_1 \sin \theta_2 \sin \theta_3 \\
  \sin \varphi \cos \theta_1 \sin \theta_2 \sin \theta_3 \\
  - \sin \theta_1 \sin \theta_2 \sin \theta_3 \\
  0 \\
  0
\end{pmatrix}
+ t_2 r
\begin{pmatrix}
  \cos \varphi \sin \theta_1 \cos \theta_2 \sin \theta_3 \\
  \sin \varphi \sin \theta_1 \cos \theta_2 \sin \theta_3 \\
  \cos \theta_1 \cos \theta_2 \sin \theta_3 \\
  - \sin \theta_2 \sin \theta_3 \\
  0
\end{pmatrix}
+ t_3 r
\begin{pmatrix}
  \cos \varphi \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
  \sin \varphi \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
  \cos \theta_2 \cos \theta_3 \\
  - \sin \theta_3
\end{pmatrix}
\]  

(2.4)

From (2.4) we find the metric of the \(C^4\)-cone which is induced by the Euclidean metric of \(R^5\):

\[
ds^2 = dE^2 d\varphi^2 + r^2 \sin^2 \theta_2 \sin^2 \theta_3 \, dt_1^2 + r^2 \sin^2 \theta_3 \, dt_2^2 + r^2 \, dt_3^2,
\]

(2.5)

where

\[
d_E = r(\sin \theta_1 \sin \theta_2 \sin \theta_3 + t_1 \cos \theta_1 \sin \theta_2 \sin \theta_3 + t_2 \sin \theta_1 \cos \theta_2 \sin \theta_3
+ t_3 \sin \theta_1 \sin \theta_2 \cos \theta_3).
\]

(2.6)

The label \(d_E\) denotes the Euclidean distance between a point of the \(C^4\)-cone and the subset \(x_1 = x_2 = 0\) in \(R^5\). The “tip” of the \(C^4\)-cone is defined by the equation \(d_E = 0\). This, together with (2.4), yields

\[
x_1 = x_2 = 0, \ x_3 \cos \theta_1 \sin \theta_2 \sin \theta_3 + x_4 \cos \theta_2 \sin \theta_3 + x_5 \cos \theta_3 = r.
\]

(2.7)

Equation (2.7) represents a parametrization of the two-dimensional tip in \(R^5\). From (2.5) one may derive that the Riemannian curvature of the \(C^4\)-cone vanishes, except for the tip where the metric is ill defined.

### 2.2. Explicit construction of the isometry

According to (2.1), the metric of the \((\varepsilon - 4)\)-cone reads

\[
ds^2 = \rho^2 d\xi^2 + d\rho^2 + dz_1^2 + dz_2^2.
\]

(2.8)

We relate the orthonormal coframes in (2.5) and (2.8) by a rigid rotation, i.e. by a constant orthogonal matrix, according to

\[
\begin{pmatrix}
  dE d\varphi \\
  r \sin \theta_2 \sin \theta_3 \, dt_1 \\
  r \sin \theta_3 \, dt_2 \\
  r dt_1
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & e_1 & -e_2 & -e_3 \\
  0 & e_2 & e_1 + \frac{e_2}{e_2^2 + e_3^2}(1 - e_1) & -\frac{e_2 e_3}{e_2^2 + e_3^2}(1 - e_1) \\
  0 & e_3 & -\frac{e_2 e_3}{e_2^2 + e_3^2}(1 - e_1) & e_1 + \frac{e_2^2}{e_2^2 + e_3^2}(1 - e_1)
\end{pmatrix}
\begin{pmatrix}
  \rho d\xi \\
  dp \\
  dz_1 \\
  dz_2
\end{pmatrix}.
\]

(2.9)
The constants $e_i$ are given by

$$ e_1 := \frac{\cos \theta_1}{\Omega}, \quad e_2 := \frac{\sin \theta_1 \cos \theta_2}{\Omega}, \quad e_3 := \frac{\sin \theta_1 \sin \theta_2 \cos \theta_3}{\Omega}, \quad (2.10) $$

$$ \Omega := \sqrt{1 - \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3}. $$

Equation (2.9) defines a coordinate transformation

$$(\varphi, t_1, t_2, t_3) \mapsto (\xi, \rho, z_1, z_2)$$

which allows to replace in the parametrization (2.4) of the $C^4$—cone the coordinates $(\varphi, t_1, t_2, t_3)$ by $(\xi, \rho, z_1, z_2)$. The replacement yields a mapping $f$ from the $(\varepsilon - 4)$—cone onto the $C^4$—cone:

$$ f : \quad (\varepsilon - 4) - \text{cone} \quad \mapsto \quad C^4 - \text{cone} $$

$$ f(\xi, \rho, z_1, z_2) = \begin{pmatrix}
\rho \Omega \cos(\xi), \\
\rho \Omega \sin(\xi), \\
\rho \sin \theta_1 \sin \theta_2 \sin \theta_3 k_{11} + z_1 k_{12} + z_2 k_{13}, \\
\rho \sin \theta_1 \sin \theta_2 \sin \theta_3 k_{21} + z_1 k_{22} + z_2 k_{23}, \\
\rho \sin \theta_1 \sin \theta_2 \sin \theta_3 k_{31} + z_1 k_{32} + z_2 k_{33}
\end{pmatrix}, \quad (2.11) $$

where

$$ k_{11} := -\frac{\cos \theta_1 \sin \theta_2 \sin \theta_3}{\Omega}, \quad k_{12} := \frac{\cos \theta_2}{\Omega}, \quad k_{13} := \frac{\sin \theta_2 \cos \theta_3}{\Omega}, $$

$$ k_{21} := -\frac{\cos \theta_2 \sin \theta_3}{\Omega}, \quad k_{22} := -\frac{\sin \theta_2 \cos \theta_1}{\Omega} - \frac{\cos \theta_3 \sin \theta_2 (1 - \frac{\cos \theta_1}{\Omega})}{\sin^2 \theta_2 + \cos^2 \theta_2 \cos^2 \theta_3}, $$

$$ k_{23} := \frac{\cos \theta_1 \cos \theta_2 \cos \theta_3}{\Omega} + \frac{\cos \theta_2 \cos \theta_3 (1 - \frac{\cos \theta_1}{\Omega})}{\cos^2 \theta_2 + \sin^2 \theta_2 \cos^2 \theta_3}, \quad k_{31} := -\frac{\cos \theta_3}{\Omega}, $$

$$ k_{32} := \frac{\sin \theta_2 \sin \theta_3 \cos \theta_2 \cos \theta_3 (1 - \frac{\cos \theta_1}{\Omega})}{\cos^2 \theta_2 + \sin^2 \theta_2 \cos^2 \theta_3}, $$

$$ k_{33} := -\frac{\sin \theta_3 \cos \theta_1}{\Omega} - \frac{\cos^2 \theta_2 \sin \theta_3 (1 - \frac{\cos \theta_1}{\Omega})}{\cos^2 \theta_2 + \sin^2 \theta_2 \cos^2 \theta_3}. $$

Both cones are equivalent since the rotation in (2.9) just corresponds to a different choice of an orthonormal coframe. Therefore the mapping $f$ should be an isometry. We prove this by choosing

$$ v := \rho \frac{\partial}{\partial \rho} + \xi \frac{\partial}{\partial \xi} + z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}, \quad (2.13) $$
and calculating

\[< \vec{v}, \vec{v}>_{R^4} = \rho^2 + \rho^2 \xi^2 + z_1^2 + z_2^2, \quad (2.14)\]

\[< f_*^*v, f_*^*v >_{R^5} = < \frac{\partial f^i}{\partial v^j} v^j, \frac{\partial f^i}{\partial v^j} v^j >_{R^5} = \rho^2 + \rho^2 \xi^2 + z_1^2 + z_2^2. \quad (2.15)\]

Hence

\[< \vec{v}, \vec{v}>_{R^4} = < f_*^*v, f_*^*v >_{R^5}, \quad (2.16)\]

i.e. \( f \) is an isometry, indeed.

### 2.3. Negative deficit angles

So far we dealt exclusively with positive deficit angles. This can be seen by calculating \( \varepsilon \) explicitly: The distance \( \rho \) between the tip of a \( C^4 \)-cone, given by (2.7), and a parallel of latitude turns out to be

\[\rho = r \frac{\sin \theta_1 \sin \theta_2 \sin \theta_3}{\sqrt{1 - \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3}}. \quad (2.17)\]

Thus the length of a parallel of latitude is given by \((2\pi - \varepsilon)\rho\) or, via the embedding, by \(2\pi r \sin \theta_1 \sin \theta_2 \sin \theta_3\). Equating both expressions yields

\[\varepsilon = 2\pi (1 - \sqrt{1 - \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3}) > 0. \quad (2.18)\]

The embedding of an \((\varepsilon - n)\)-cone with a negative deficit angle is analogous to the case with positive angle. We replace \( S^n(r) \) by \( H^n(r) \), which is one sheet of a two-sheeted hyperboloid:

\[H^n(r) = \{ \vec{x} \in R^{n+1} | < \vec{x}, \vec{x}>_{L^{n+1}} = -x_1^2 - x_2^2 ... - x_n^2 + x_{n+1}^2 = r^2, \ x_{n+1} > 0\} \quad (2.19)\]

The Lorentzian metric \(<, >_{L^{n+1}}\) induces a Riemannian metric on \( H^n(r) \). The sheet \( H^n(r) \) possesses a constant Riemannian curvature of amount \(-\frac{1}{r^2}\) with respect to this metric.

The substitutions \(\sin \theta_{n-1} \rightarrow \sinh \tilde{\theta}_{n-1}, \cos \theta_{n-1} \rightarrow \cosh \tilde{\theta}_{n-1}\) lead from spherical coordinates on the sphere to hyperbolical coordinates on \( H^n(r) \). The isometrically embedded \((\varepsilon - n)\)-cone with negative deficit angle is given by (2.3), where in this case \( \vec{p}(\varphi) \) denotes a point of a parallel of latitude \( \theta_i = \text{const.} \) on \( H^n(r) \). This can be proven as in the case with positive deficit angle: We rewrite \( \theta_{n-1} \) as \(i(-i\theta_{n-1})\) and use \(-i\tilde{\theta}_{n-1}\) as a new complex variable \( \tilde{\theta}_{n-1} \). Then \( \sinh \tilde{\theta}_{n-1} = i \sin \tilde{\theta}_{n-1} \) and \( \cosh \tilde{\theta}_{n-1} = \cos \tilde{\theta}_{n-1} \). This leads exactly to the former starting point (2.5). The factor \( i \) in front of \( \sin \tilde{\theta}_{n-1} \) takes into account the Lorentzian metric of \( R^{n+1} \). Computation
of the distance \( \rho \) between the tip of a cone with negative deficit angle and a related parallel of latitude yields

\[
\rho = r \frac{\sin \theta_1 \sin \theta_2 \sinh \theta_3}{\sqrt{1 + \sin^2 \theta_1 \sin^2 \theta_2 \sinh^2 \theta_3}},
\]

(2.20)

and we obtain

\[
\varepsilon = 2\pi \left(1 - \sqrt{1 + \sin^2 \theta_1 \sin^2 \theta_2 \sinh^2 \theta_3}\right) < 0.
\]

(2.21)

Calculation of \( \varepsilon \) by means of the parallel transport equation related to the Riemannian connection of \( S^n(r) \) or \( H^n(r) \) yields, of course, the same result.

3. General parallel transport on a Regge lattice

The notion of parallel transport on a Regge lattice becomes transparent now:

Riemannian parallel transport on a Regge lattice is trivial except for surrounding hinges with nonzero deficit angle. The result of parallel transport around such a hinge is independent of the path chosen. Thus we may choose a parallel of latitude of the related \( (\varepsilon - n) \)-cone and identify this, via the isometrical embedding (2.11), with a parallel of latitude of a specific space of constant curvature.

In order to relate a specific hinge of the Regge lattice (with adjacent \( n \)-simplexes) to a certain \( (\varepsilon - n) \)-cone and, in turn, to a parallel of latitude of a space of constant curvature, it is required to read off the values of \( \varepsilon \) and \( \rho \) from the lattice. Then we obtain for a positive deficit angle by (2.17) and (2.18) values for \( r \) and \( \sin \theta_1 \sin \theta_2 \sin \theta_3 \), while for a negative deficit angle we use (2.20) and (2.21) to obtain values for \( r \) and \( \sin \theta_1 \sin \theta_2 \sinh \theta_3 \). The deficit angle \( \varepsilon \) can be read off exactly, but how do we read off \( \rho \)? The distance \( \rho \) between the tip of a \( (\varepsilon - n) \)-cone and a parallel of latitude corresponds in a Regge lattice to the distance between a hinge and a path that encircles the hinge. If we restrict ourselves to define paths in a Regge lattice exclusively on a corresponding dual lattice, we may assign to \( \rho \) an average distance between a hinge and a path that encircles it. Adopting, for example, a barycentric dual lattice, the distance \( \rho \) can be defined by the average of the Euclidean distances between the hinge and the barycenters of adjacent \( n \)-simplexes. This procedure is the only approximation that enters the transition from the path surrounding a hinge on a Regge lattice to a parallel of latitude of \( S^n(r) \) or \( H^n(r) \) and expresses the approximative character of the Regge lattice. The transition requires the metric of the Regge lattice since \( \varepsilon \) and \( \rho \) are functions of the link lengths.

It turned out in Sec.2 that the \( (\varepsilon - n) \)-cone is build from the tangent spaces of a parallel of latitude of a space of constant curvature. The \( n \)-simplexes related to a hinge locally approximate some \( (\varepsilon - n) \)-cone in a sense that each of them constitutes (a part of) the tangent space at one point of the “underlying” parallel of latitude. Therefore
each $n$–simplex represents a tangent space of the manifold that is approximated by the Regge lattice. We infer from this that a general parallel transport in a Regge lattice is provided by a prescription which connects adjacent $n$–simplexes, i.e. neighbouring tangent spaces. Such a prescription can be derived immediately by regarding the intuitive and illuminating description of parallel transport in a manifold which is due to É. Cartan [11]: Let $p$ and $p'$ be two infinitesimally close points of a manifold $M$, their affine tangent spaces are denoted by $T_p M$ and $T_{p'} M$, both are equipped with a vector basis $e_\alpha$ and $e'_\alpha$, respectively. The affine tangent spaces $T_p M$ and $T_{p'} M$ are compared by means of an affine (Cartan-)connection $(\vartheta^\alpha, \Gamma^\alpha_{\beta\gamma})$ as follows:

$$p' = p + \vartheta^\alpha(p') e_\alpha ,$$  
(3.1)

$$e'_\alpha = (\Gamma^\alpha_{\beta\gamma}(p') + \delta^\beta_\alpha) e_\beta .$$  
(3.2)

Equations (3.1) and (3.2) define an affine transformation from $T_p M$ to $T_{p'} M$ and specify parallel transport from $p$ to $p'$. Integration of the one forms $\vartheta^\alpha$ and $\Gamma^\alpha_{\beta\gamma}$ around infinitesimally closed loops yields the Cartan structure equations

$$T^\alpha := d\vartheta^\alpha + \vartheta^\beta \wedge \Gamma^\alpha_{\beta\gamma} ,$$  
(3.3)

$$R^\alpha_{\beta\gamma} := d\Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\delta} \wedge \Gamma^\delta_{\beta\gamma} .$$  
(3.4)

Therefore successive affine transformations around an infinitesimally closed loop can be replaced by one affine transformation, which is determined by the two–forms torsion $T^\alpha$ and curvature $R^\alpha_{\beta\gamma}$. The torsion determines the translational part, whereas the curvature determines the linear part of the resulting affine transformation. We may detect whether the linear part is orthogonal or not if we equip $M$ with a metric $g$. Then it is possible to express shears and/or dilations in terms of the nonmetricity one-form [12]:

$$Q^\alpha_{\beta} := -Dg_{\alpha\beta} = -dg_{\alpha\beta} + \Gamma^\gamma_{\alpha\delta}g_{\gamma\delta} + \Gamma^\gamma_{\beta\delta}g_{\alpha\gamma} .$$  
(3.5)

This implies that a general parallel transport in a Regge lattice is defined by affine transformations between adjacent $n$–simplexes. A set of affine transformations, one affine transformation for each pair of adjacent $n$–simplexes, may be called a discrete affine connection. The approximative character of the Regge lattice is expressed by the fact that the points $p$ and $p'$ are no longer infinitesimally close. Parallel transport around a hinge is performed by successive affine transformation while passing from one simplex to another. Curvature, torsion and nonmetricity associated with a hinge are defined by the affine transformation which results from this discrete parallel transport around the hinge.

To become more explicit, we consider some closed loop in a Regge lattice that surrounds one hinge by passing through $k$ different $n$–simplexes $S_1, \ldots, S_k$. Parallel transport is specified by a discrete affine connection $(\vartheta^\alpha_{\{lm\}}, \Gamma_{\{lm\}\alpha\beta})$, where the affine
transformation connecting the adjacent $n$–simplexes $l$ and $m$ is given by $\vartheta_{\{lm\}}^\alpha$ and $\Gamma_{\{lm\}}^{\alpha\beta}$. Each simplex represents an affine tangent space so that we choose an origin $P$ and a frame $e_\alpha$ in each of them. Parallel transport from simplex $S_1$ to $S_2$ is specified by (cf. (3.1), (3.2))

$$p\{2\} = p\{1\} + \vartheta_{\{12\}}^\alpha e_{\{1\}\alpha},$$  \hspace{1cm} (3.6)$$

$$e_{\{2\}\alpha} = (\Gamma_{\{12\}}^{\alpha\beta} + \delta_{\alpha}^\beta)e_{\{1\}\beta}$$

$$=: \Lambda_{\{12\}}^{\alpha\beta} e_{\{1\}\beta},$$ \hspace{1cm} (3.7)$$
or, in a shorter notation, by $(\vartheta_{\{12\}}^\alpha, \Lambda_{\{12\}}^{\alpha\beta})_{S_1 \to S_2}$. Similarly, we have for parallel transport from $S_2$ to $S_3$ the expression $(\vartheta_{\{23\}}^\alpha, \Lambda_{\{23\}}^{\alpha\beta})_{S_2 \to S_3}$ such that parallel transport from $S_1$ to $S_3$ is described by

$$(\vartheta_{\{23\}}^\alpha + \vartheta_{\{12\}}^\alpha \Lambda_{\{23\}}^{\alpha\beta}, \Lambda_{\{23\}}^{\alpha\gamma} \Lambda_{\{12\}}^{\alpha\gamma})_{S_1 \to S_3} =: (\vartheta_{\{13\}}^\alpha, \Lambda_{\{13\}}^{\alpha\beta})_{S_1 \to S_3}. \hspace{1cm} (3.8)$$

Proceeding with parallel transport to $S_k$ yields

$$(\vartheta_{\{(k-1)k\}}^\alpha + \vartheta_{\{1(k-1)\}}^\alpha \Lambda_{\{(k-1)k\}}^{\alpha\beta}, \Lambda_{\{(k-1)k\}}^{\alpha\gamma} \Lambda_{\{1(k-1)\}}^{\alpha\gamma})_{S_1 \to S_k}$$

$$=: (\vartheta_{\{1k\}}^\alpha, \Lambda_{\{1k\}}^{\alpha\beta})_{S_1 \to S_k},$$ \hspace{1cm} (3.9)$$

and we obtain the affine transformation that results from parallel transport around this loop,

$$(\vartheta_{\{k1\}}^\alpha + \vartheta_{\{1k\}}^\alpha \Lambda_{\{k1\}}^{\alpha\beta}, \Lambda_{\{k1\}}^{\alpha\gamma} \Lambda_{\{1k\}}^{\alpha\gamma})_{S_1 \to S_1}$$

$$=: (T^\alpha, R^\alpha_{\alpha\beta} + \delta_{\alpha}^\beta)_{S_1 \to S_1}, \hspace{1cm} (3.10)$$

where $T^\alpha$ and $R^\alpha_{\alpha\beta}$ denote torsion and curvature associated to this loop. We note the recursive definition of $\vartheta_{\{1l\}}^\alpha$ and $\Lambda_{\{1l\}}^{\alpha\beta}$ for $l \in \{3, ..., k\}$. Equation (3.10) constitutes the discrete equivalent to (3.3) and (3.4). The nonmetricity can be calculated by using a metric $g$ in $S_1$:

$$Q_{\alpha\beta} = -g(R_{\alpha\gamma} e_\gamma, R_{\beta\gamma} e_\gamma). \hspace{1cm} (3.11)$$

The definition of torsion and curvature in both the Cartan structure equations (3.3), (3.4) and their discrete analogue (3.10) is completely independent of metric and link lengths, respectively. The link lengths determine the manifold approximated by the Regge lattice, while the discrete affine connection approximates a general parallel transport on this manifold. It is just the special case of Riemannian parallel transport that can be deduced from the link lengths via the deficit angle.
4. Summary

The isometry (2.11) proves that every Regge lattice locally, or “hingewise”, approximates a space of constant curvature. The explicit identification between a hinge of the Regge lattice and a specific space of constant curvature is due to the metric, i.e. the link lengths, of the Regge lattice. The $n-$simplexes represent affine tangent spaces of the manifold approximated. Parallel transport in a Regge lattice is a prescription that maps neighbouring $n-$simplexes onto each other. A priori this prescription is not determined but may be defined by a discrete affine connection. Such a discrete affine connection is independent of the metric of the Regge lattice. This is, of course, analogous to ordinary differential geometry: The metric alone cannot determine an arbitrary affine connection. Therefore it seems that attempts to build a non-Riemannian Regge calculus by relying exclusively on the metric are not promising.

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