An Explicit Construction of Self-dual 2-forms in Eight Dimensions

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Abstract

The geometry of self-dual 2-forms in 2n dimensional spaces is studied. These 2-forms determine a $n^2 - n + 1$ dimensional manifold $S_{2n}$. We prove that $S_{2n}$ has only one-dimensional linear submanifolds for n odd. In eight dimensions the self-dual forms of Corrigan et al constitute a seven dimensional linear subspace of $S_8$ among many other equally interesting linear subspaces.
1. Introduction

The concept of self-duality of a 2-form in four dimensions is generalised to any higher even dimensional space in our previous paper [1]. We recall here that self-duality can be defined as an eigenvalue criterion in the following way. (Here we adopt a different terminology, and use self-dual rather than strongly self-dual as it is used in Ref.[1]) Suppose $F$ is a real 2-form in $2n$ dimensions, and let $\Omega$ be the corresponding $2n \times 2n$ skew-symmetric matrix with respect to some local orthonormal basis. Then by a change of basis, $\Omega$ can be brought to the block-diagonal form

$$
\begin{pmatrix}
0 & \lambda_1 \\
-\lambda_1 & 0 \\
& & \ddots \\
& & & 0 & \lambda_n \\
& & & -\lambda_n & 0
\end{pmatrix}
$$

where $\lambda_1, ..., \lambda_n$ are the eigenvalues of $\Omega$. The 2-form $F$ is called self-dual or anti-self-dual provided the absolute values of the eigenvalues are all equal, that is

$$|\lambda_1| = |\lambda_2| = \ldots = |\lambda_n|.$$

To distinguish between the two cases, orientation must be taken into account. We define $F$ to be self-dual, if $\Omega$ can be brought with respect to an orientation-preserving basis change to the above block-diagonal form such that $\lambda_1 = \lambda_2 = \ldots = \lambda_n$. Similarly, we define $F$ to be anti-self-dual, if $\Omega$ can be brought to the same form by an orientation-reversing basis change. It is not difficult to check that in dimension $D=4$, the above definition coincides with the usual definition of self-duality in the Hodge sense. We studied in our previous work self-dual 2-forms in $D=8$ and have shown that:

i) Yang-Mills 2-forms satisfying a certain set of 21 linear equations, first derived by Corrigan, Devchand, Fairlie, Nuyts (CDFN) [2] by other means, are self-dual in the above sense.

ii) Each self-dual 2-form $F$, satisfying $\ast(F \wedge F) = F \wedge F$ [3] is self-dual in the above sense.

In this letter, starting from the self-duality condition on eigenvalues we obtain the CDFN self-dual 2-form. We also explain the construction of new families of self-dual 2-forms.
2. The Geometry of Self-dual 2-forms.

In this section we describe the geometrical structure of self-dual 2-forms in arbitrary even dimensions. In the following $I$ denotes an identity matrix of appropriate dimension.

**Definition 1.** Let $A_{2n}$ be the set of antisymmetric matrices in $2n$ dimensions. Then $S_{2n} = \{ A \in A_{2n} \mid A^2 + \lambda^2 I = 0, \lambda \in \mathbb{R}, \lambda \neq 0 \}$.

Note that if $A \in S_{2n}$, and $A^2 = 0$, then $A = 0$, and if $A \in S_{2n}$, then $\lambda A \in S_{2n}$ for $\lambda \neq 0$.

**Proposition 2.** The set $S_{2n}$ is diffeomorphic to $(O(2n) \cap A_{2n}) \times \mathbb{R}^+.$

**Proof.** Let $A \in S_{2n}$ with $A^2 + \lambda^2 I = 0$. Note that $\lambda^2 = -\frac{1}{2n} tr A^2$. Define $\kappa = [-\frac{1}{2n} tr A^2]^{1/2}$, $\bar{A} = \frac{1}{\kappa} A$. Then, $\bar{A}^2 + I = 0$, hence $\bar{A} \bar{A}^t = I$. Consider the map $\varphi : S_{2n} \to (O(2n) \cap A_{2n}) \times \mathbb{R}^+$ defined by $\varphi(A) = (\bar{A}, \kappa)$. The map $\varphi$ is 1-1, onto and differentiable. Its inverse is given by $(B, \alpha) \to \alpha B$ is also differentiable, hence $\varphi$ is a diffeomorphism. e.o.p.

**Remark 3.** $O(2n) \cap A_{2n}$ is a fibre bundle over the sphere $S^{2n-2}$ with fibre $O(2n - 2) \cap A_{2n-2}$. (See Steenrod, Ref.[4])

For our purposes the following description of $S_{2n}$ is more useful.

**Proposition 4.** $S_{2n}$ is diffeomorphic to the homogeneous manifold $(O(2n) \times R^+)/U(n) \times \{1\}$, and $\text{dim} S_{2n} = n^2 - n + 1$.

**Proof.** Let $G$ be the product group $O(2n) \times \mathbb{R}^+$, where $\mathbb{R}^+$ is considered as a multiplicative group. $G$ acts on $S_{2n}$ by $(P, \alpha) \bar{A} = \alpha (P^t A P)$, where $P \in O(2n)$, $\alpha \in \mathbb{R}^+$, $A \in S_{2n}$, and $t$ indicates the transpose. Since all matrices in $S_{2n}$ are conjugate to each other up to a multiplicative constant, this action is transitive, and actually any $A \in S_{2n}$ can be written as $A = \lambda P^t J P$, where $\lambda = [-\frac{1}{2n} tr A^2]^{1/2}$, with $P \in O(2n)$ and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. It can be seen that the isotropy subgroup of $G$ at $J$ is $U(n)$ [5] and $G/U(n)$ is diffeomorphic to $S_{2n}$ (Ref.[6] p.132, Thm.3.62). Then $\text{dim} S_{2n} = \text{dim} (O(2n) \times \mathbb{R}^+/U(n))$ can be easily computed as $\text{dim} S = \text{dim} O(2n) + 1 - \text{dim} U(n) = (2n^2 - n + 1) - n^2 = n^2 - n + 1$. e.o.p.

In particular, in eight dimensions, $S_8$ is a 13 dimensional manifold.

As $O(2n)$ has two connected components ($SO(2n)$ and $O(2n) \setminus SO(2n)$), $U(n)$ is connected and $U(n) \subset SO(2n)$, $S_{2n}$ has two connected components. One of them (that contains $J$) consists of the self-dual forms and the other of the anti-self-dual forms.

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3. Linear submanifolds of $S_8$

The defining equations of the set $S_8$ are homogeneous quadratic polynomial equations for the components of the curvature 2-form. Thus they correspond to differential equations quadratic in the first derivative for the connection. Thus the study of their solutions, hence the study of the moduli space of self-dual connections is rather difficult. Therefore one might hope to restrict the notion of self-duality, to the linear submanifolds of $S_8 \cup \{0\}$. But there are plenty of them (at least in $S_8$) and there is no plausible reason to single out a specific one of them. In Ref. [1] we have shown that the 2-forms satisfying a set of 21 equations proposed by Corrigan et al belong to $S_8$ and we shall give below a natural way of arriving at them, but it will depend on a reference form. Changing the reference form one obtains translates of this 7-dimensional plane, which in some cases look more pregnant than the original one.

Note that we excluded the zero matrix from $S_{2n}$ in our definition in order to obtain its manifold structure. We denote $S_{2n} = S_{2n} \cup \{0\}$. By linearity of the action of $O(2n)$ on $S_{2n}$ we obtain the following

**Lemma 4.** Let $\mathcal{L}$ be a linear submanifold of $S_{2n}$. Then $\mathcal{L}_P = P^t \mathcal{L} P$, $P \in O(2n)$ is also a linear submanifold of $S_{2n}$.

Let $J_0 = \text{diag}(\epsilon, \epsilon, \epsilon, \epsilon)$, where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that any $A \in S_8$ is conjugate to $J_0$, hence any linear subset of $S_8$ can be realized as the translate of a linear submanifold containing $J_0$ under conjugation. Thus without loss of generality we can concentrate on linear subsets containing $J_0$.

**Proposition 5.** If $A \in S_8$ and $(A + J_0) \in S_8$, where $J_0 = \text{diag}(\epsilon, \epsilon, \epsilon, \epsilon)$, with $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then

$$A = \begin{pmatrix} k\epsilon & r_1 S(\alpha) & r_2 S(\beta) & r_3 S(\gamma) \\ -r_1 S(\alpha) & k\epsilon & r_3 S(\gamma') & -r_2 S(\beta') \\ -r_2 S(\beta) & -r_3 S(\gamma') & k\epsilon & r_1 S(\alpha') \\ -r_3 S(\gamma) & r_2 S(\beta') & -r_1 S(\alpha') & k\epsilon \end{pmatrix}$$

where $k \in R$, $r_1$, $r_2$, $r_3$ are in $R^+$, and $S(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$, and $\alpha$, $\alpha'$, $\beta$, $\beta'$, $\gamma$, $\gamma'$ satisfy

$$\alpha + \alpha' = \beta + \beta' = \gamma + \gamma'.$$
Proof. If $A$ and $A + J_o$ are both in $S_8$, then the matrix $AJ_o + J_oA$ is proportional to identity. This gives a set of linear equations whose solutions can be obtained without difficulty to yield

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ -a_{12} & 0 & a_{14} & -a_{13} & a_{16} & -a_{15} & a_{18} & -a_{17} \\ -a_{13} & -a_{14} & 0 & a_{12} & a_{35} & a_{36} & a_{37} & a_{38} \\ -a_{14} & a_{13} & -a_{12} & 0 & a_{36} & -a_{35} & a_{38} & -a_{37} \\ -a_{15} & -a_{16} & -a_{35} & -a_{36} & 0 & a_{12} & a_{57} & a_{58} \\ -a_{16} & a_{15} & -a_{36} & a_{35} & -a_{12} & 0 & a_{58} & -a_{57} \\ -a_{17} & -a_{18} & -a_{37} & -a_{38} & -a_{57} & -a_{58} & 0 & a_{12} \\ -a_{18} & a_{17} & -a_{38} & a_{37} & -a_{58} & a_{57} & -a_{12} & 0 \end{pmatrix}$$

Then the requirement that the diagonal entries in $A^2$ be equal to each other gives the following equations after some algebraic manipulations.

$$a_{13}^2 + a_{14}^2 = a_{57}^2 + a_{58}^2$$
$$a_{15}^2 + a_{16}^2 = a_{37}^2 + a_{38}^2$$
$$a_{17}^2 + a_{18}^2 = a_{35}^2 + a_{36}^2$$

Thus we can parametrize $A$ by

$$a_{13} = r_1 \cos \alpha, \quad a_{15} = r_2 \cos \beta, \quad a_{17} = r_3 \cos \gamma$$
$$a_{14} = r_1 \sin \alpha, \quad a_{16} = r_2 \sin \beta, \quad a_{18} = r_3 \sin \gamma$$
$$a_{57} = r_1 \cos \alpha', \quad a_{37} = r_2 \cos \beta', \quad a_{45} = r_3 \cos \gamma'$$
$$a_{58} = r_1 \sin \alpha', \quad a_{38} = r_2 \sin \beta', \quad a_{36} = r_3 \sin \gamma'$$

Finally the requirement that the off diagonal terms in $A^2$ be equal to zero gives quadratic equations, which can be rearranged and using trigonometric identities they give $\alpha + \alpha' = \beta + \beta' = \gamma + \gamma'$. e.o.p.

Thus the set of matrices $A \in S_8$ such that $(A + J_o) \in S_8$ constitutes an eight parameter family and the equations of CDFN correspond to the case $\alpha' + \alpha = \beta' + \beta = \gamma' + \gamma = 0$. Thus we have an invariant description of these equations, that we repeat here for convenience.

$$F_{12} - F_{34} = 0 \quad F_{12} - F_{56} = 0 \quad F_{12} - F_{78} = 0$$
$$F_{13} + F_{24} = 0 \quad F_{13} - F_{57} = 0 \quad F_{13} + F_{68} = 0$$
$$F_{14} - F_{23} = 0 \quad F_{14} + F_{67} = 0 \quad F_{14} + F_{58} = 0$$
\[\begin{align*}
F_{15} + F_{26} &= 0 \\
F_{16} - F_{25} &= 0 \\
F_{17} + F_{28} &= 0 \\
F_{18} - F_{27} &= 0
\end{align*}\]
\[\begin{align*}
F_{15} + F_{37} &= 0 \\
F_{16} - F_{38} &= 0 \\
F_{17} - F_{35} &= 0 \\
F_{18} + F_{36} &= 0
\end{align*}\]
\[\begin{align*}
F_{15} - F_{48} &= 0 \\
F_{16} - F_{47} &= 0 \\
F_{17} + F_{46} &= 0 \\
F_{18} + F_{45} &= 0
\end{align*}\]

The (skew-symmetric) matrix of such a 2-form is

\[
\begin{pmatrix}
0 & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} & F_{17} & F_{18} \\
F_{12} & 0 & -F_{13} & F_{14} & F_{15} & F_{16} & F_{17} & F_{18} \\
F_{13} & -F_{12} & 0 & F_{14} & F_{15} & F_{16} & F_{17} & F_{18} \\
F_{14} & F_{13} & F_{12} & 0 & F_{15} & F_{16} & F_{17} & F_{18} \\
F_{15} & F_{14} & F_{13} & F_{12} & 0 & F_{16} & F_{17} & F_{18} \\
F_{16} & F_{15} & F_{14} & F_{13} & F_{12} & 0 & F_{17} & F_{18} \\
F_{17} & F_{16} & F_{15} & F_{14} & F_{13} & F_{12} & 0 & F_{18} \\
F_{18} & F_{17} & F_{16} & F_{15} & F_{14} & F_{13} & F_{12} & 0
\end{pmatrix}
\]

We will refer to the plane consisting of these forms as the CDFN-plane. Let us now consider as the reference form \(J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\) instead of \(J_o\). \(J\) can be obtained from \(J_o\) by conjugation \(J = P^t J_o P\) with

\[
P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Then the conjugation of the CDFN-plane by \(P\) is given by the following remarkable (D=8 self-dual) 2-form

\[F_{12}J + \begin{pmatrix} \Omega' & \Omega'' \\ -\Omega''t & -\Omega' \end{pmatrix}\]

where \(\Omega'\) is a D=4 self-dual 2-form and \(\Omega''\) is a D=4 anti-self-dual 2-form.
4. The Geometry of $S_{4k+2}$

We prove that for odd $n$ there are no linear subspaces other than the one dimensional one.

**Proposition 6.** Let $\mathcal{M} = \{A \in S \mid (A + J_o) \in S\}$. Then $\mathcal{M} = \{kJ | k \in \mathbb{R}\}$ for odd $n$.

**Proof.** Let $A = \begin{pmatrix} A_{11} & A_{12} \\ -A_{12}^t & A_{22} \end{pmatrix}$, where $A_{11} + A_{11}^t = 0$, $A_{22} + A_{22}^t = 0$. As before if $(A + J_o) \in S$ then $AJ_o + J_oA$ is proportional to the identity. This gives $A_{11} + A_{22} = 0$ and the symmetric part of $A_{12}$ is proportional to identity. Therefore $A = kJ_o + \begin{pmatrix} A_{11} & A_{12o} \\ A_{12o} & -A_{11} \end{pmatrix}$, where $A_{12o}$ denotes the antisymmetric part of $A_{12}$ and $k$ is a constant. Then the requirement that $A \in S$ gives

$$[A_{11}, A_{12o}] = 0, \quad A_{11}^2 + A_{12o}^2 + kI = 0, \quad k \in \mathbb{R}. $$

As $A_{11}$ and $A_{12o}$ commute, they can be simultaneously diagonalizable, hence for odd $n$ they can be brought to the form

$$A_{11} = \text{diag}(\lambda_1 \epsilon, \ldots, \lambda_{(n-1)/2} \epsilon, 0)$$
$$A_{12o} = \text{diag}(\mu_1 \epsilon, \ldots, \mu_{(n-1)/2} \epsilon, 0)$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and 0 denotes a $1 \times 1$ block, up the the permutation of the blocks. If the blocks occur as shown, clearly $A_{11}^2 + A_{12o}^2$ cannot be proportional to identity. It can also be seen that except for $\lambda_i = \mu_i = 0$ the same result holds for any permutation of the blocks.

5. Conclusion

To conclude we would like to emphasise that the choice of a linear subspace of $S_{2n}$ is incidental. Instead, one should try to understand the totality of the non-linear space of self-dual 2-forms. In that respect the approach to self-duality given above might be a good starting point.
References

[1] A.H.Bilge, T.Dereli, Ş.Koçak, “Self-dual Yang-Mills fields in eight dimensions” Lett.Math.Phys. (to appear)

[2] E.Corrigan, C.Devchand, D.B.Fairlie and J.Nuyts, “First-order equations for gauge fields in spaces of dimension greater than four”, Nuclear Physics B214, 452-464, (1983).

[3] B.Grossman, T.W.Kephart, J.D.Stasheff, “Solutions to Yang-Mills field equations in eight dimensions and the last Hopf map”, Commun. Math. Phys., 96, 431-437, (1984).

[4] N.Steenrod, The Topology of Fibre Bundles (Princeton U.P., 1951)

[5] S.Kobayashi, K.Nomizu, Foundations of Differential Geometry Vol.II (Interscience, 1969)

[6] F.W.Warner, Foundations of Differentiable Manifolds and Lie Groups (Scott and Foresman, 1971)