The shift operators related to the Fourier cosine and sine transforms and visualization of their action

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Abstract. We consider the shift operators generalized by the convolutions for the Fourier cosine and sine transforms. Three out of four considered operators are not the generalized shift operator of the Levitan’s type. The basic properties of the shift operators are presented. The GeoGebra applets have been created to visualize the actions of the shift operators.

1. Introduction
This paper is a continuation of the study of shift operators given in [1–2] for the Fourier and Hankel transforms. Here we consider the shift operators generated by the convolutions for the Fourier cosine and sine transforms.

There are two main approaches to the constructing of convolutions for integral transforms. The first of them is based on the constructing of a generalized shift operator (also called generalized translation operator, or generalized displacement operator). Then the classical translation operator (ordinary translation) in the convolution is replaced by the generalized shift operator, and we get a generalized convolution. Usually, the generalized shift operators of the Delsarte–Levitan–Povzner type are used in these constructions. For example, this approach is usually served to introduce the classical convolution for the Hankel transform [3–5].

Here we use the idea of the second approach which rest on the works by Valentin Kakichev. His convolution constructing method is based on factorization equality [6]. In 1997 year, V.A. Kakichev generalized this approach and introduced the concept of polyconvolution or generalized convolution [7].

Definition 1. Let $A_i$, $i = 1, 2, 3$ be linear operators mapping linear spaces $U_i(T_i)$ to an algebra $W(X)$. By polyconvolution (generalized convolution) of function $f(t)$ and $g(t)$ generated by these operators with weighted function $\alpha(x)$ we mean the function $h(t)$ denoted by $\left( f_a^{a_1} \ast g_{a_2}^{a_2} \right)^{A_3} (t)$ for which the following factorization property is valid:

$$ (A_3 h)(x) = A_3 \left( (f_a^{a_1} \ast g_{a_2}^{a_2})^{A_3} \right)(x) = \alpha(x) \cdot (A_1 f)(x) \cdot (A_2 g)(x). $$

(1)

Here the symbol “·” denotes a multiplication in the algebra $W(X)$.

Using this definition, the polyconvolutions generated by various linear operators can be constructed. In particular, this method can be used to construct the convolutions for integral transforms [8–17]. These convolutions define shift operators for which some properties of generalized shift operators cannot be satisfied. In doing so, we use the Levitan’s definition of shift operator (see, for example, [18–19]).

Throughout the article, we will deal with weighted Lebesgue spaces $L_p(\mathbb{R}_+; \omega(t)dt), 1 \leq p < \infty$ with respect to a positive measure $\omega(t)dt$ equipped with the norm for which
\[ \|f\|_{L_p(\mathbb{R}_+; \omega(t)dt)} = \left( \int_0^\infty |f(t)|^p \omega(t)dt \right)^{1/p} < \infty \]

Let \( T^\tau \) be a family of operators depending on the parameter \( \tau \in \mathbb{R}_+ \). Thus, to every function \( f(t) \subset L_p(\mathbb{R}_+; \omega(t)dt) \) there corresponds a function \( T^\tau f(t) \) of two variables.

Denote by \( \tilde{T}^\tau \) the adjoined operator which is defined by the relation

\[ \int T^\tau f(t) g(t) \omega(t)dt = \int f(t) \tilde{T}^\tau g(t) \omega(t)dt. \]

**Definition 2.** Then let the operators \( T^\tau \) satisfy the following conditions, which we will call the conditions of generalized shift:

I. Linearity and homogeneity: \( T^\tau \{ a f(t) + b g(t) \} = a T^\tau f(t) + b T^\tau g(t) \) for all \( a, b \in \mathbb{R} \).

II. The single element \( e \in \mathbb{R}_+ \) exists such that \( T^e f(t) = f(t) \), \( T^e f(t) = f(\tau) \).

III. Associativity: \( T^r T^s f(t) = T^{r+s} f(t) \).

IV. Boundedness:

\[ \left( \int_0^\infty |T^\tau f(t)|^p \omega(t)dt \right)^{1/p} \leq A_p(\tau) \left( \int_0^\infty |f(t)|^p \omega(t)dt \right)^{1/p}, \]

\[ \left( \int_0^\infty |\tilde{T}^\tau f(t)|^p \omega(t)dt \right)^{1/p} \leq A'_p(\tau) \left( \int_0^\infty |f(t)|^p \omega(t)dt \right)^{1/p}. \]

where \( A_p(\tau), A'_p(\tau) \) are positive functions bounded on every compact set \( T \subset \mathbb{R}_+ \).

V. Continuity: if \( f(t) \subset L_p(\mathbb{R}_+; \omega(t)dt) \) then for each \( \epsilon > 0 \) there is the neighborhood \( U \) such that if \( s, r \in U \) then

\[ \int_0^\infty |T^s f(t) - T^r f(t)|^p \omega(t)dt < \epsilon^p, \int_0^\infty |\tilde{T}^s f(t) - \tilde{T}^r f(t)|^p \omega(t)dt < \epsilon^p. \]

Shift operators \( T^\tau \) play an important role in harmonic analysis, for example, they appear in the definitions of almost periodic functions, positive definite functions, convolutions, etc. The term “generalized shift operator” belongs to J. Delsarte [20–21]. Important ideas and a few original results in this field are also related to him. The systematic construction of the theory of generalized shift operators was given in the works of B.M. Levitan (see, for example, [18–19, 22]).

This research focuses on the shift operators associated with the generalized convolutions for the Fourier cosine and sine transforms. We present the GeoGebra applets [23] for visualization of the shift operator’s action. Using these applets the action of the studied operators on any function defined on \( \mathbb{R}_+ \) can be investigated.

2. **Main results**

It is widely known that the Fourier cosine and sine transforms is well-defined on the space \( L_1(\mathbb{R}_+; dt) \)

\[ V_{[c]}[f](x) = \int_0^\infty f(t) \begin{cases} \cos xt \\ \sin xt \end{cases} dt, \quad f(t) \in L_1(\mathbb{R}_+; dt). \]  \( (2) \)

Moreover, if \( F_{[c]}[x] = V_{[c]}[f](x) \in L_1(\mathbb{R}_+; dt) \) we have the reciprocal inversion formula

\[ f(t) = \frac{2}{\pi} V_{[c]} \left[ F_{[c]} \right] (t). \]

In the case of \( L_2(\mathbb{R}_+; dt) \)-space we should define the cosine Fourier transform in the mean-square convergence sense, namely

\[ V_{[c]}[f](x) = \lim_{N \to \infty} \int_{1/N}^N f(t) \begin{cases} \cos xt \\ \sin xt \end{cases} dt, \quad f(t) \in L_2(\mathbb{R}_+; dt). \]  \( (3) \)
and familiar Plancherel’s theorem says that $V_{C_{\mathbb{L}}}: L_2(\mathbb{R}_+; dt) \leftrightarrow L_2(\mathbb{R}_+; dx)$ is an isometric isomorphism and Parseval’s equality holds

$$\left\|F_{C_{\mathbb{L}}}f(x)\right\|_{L_2(\mathbb{R}_+, dx)} = \left\|V_{C_{\mathbb{L}}}f(x)\right\|_{L_2(\mathbb{R}_+, dt)} \quad (4)$$

In these function spaces we can introduce various convolutions which are generated by the Fourier cosine and sine transforms. Some of these convolutions was well studied, for example, the following constructions without weight functions $[9–10, 13, 24–25]$: two commutative convolutions

$$(f_c * g_c)_c(t) = \frac{1}{2} \int_0^\infty f(t+\tau) [g(t+\tau) + g(|t-\tau|)] d\tau,$$  

$$(f_s \ast g_s)_c(t) = \frac{1}{2} \int_0^\infty f(t+\tau) [g(t+\tau) - \text{sign}(t-\tau) g(|t-\tau|)] d\tau,$$  

and noncommutative convolution

$$(f_c \ast g_s)_s(t) = \frac{1}{2} \int_0^\infty f(t+\tau) [g(t+\tau) + \text{sign}(t-\tau) g(|t-\tau|)] d\tau,$$  

The convolutions $(5)$–$(8)$ generate four shift operators $[1]$, $t, \tau \in \mathbb{R}_+$:

$$4T^s_1f(t) = \frac{1}{2} [f(t+\tau) + f(t-\tau)],$$

$$3T^s_1f(t) = \frac{1}{2} [f(|t-\tau|) - f(t-\tau)],$$

$$3T^s_2f(t) = \frac{1}{2} [f(t+\tau) + \text{sign}(t-\tau) f(|t-\tau|)],$$

$$4T^s_2f(t) = \frac{1}{2} [f(t+\tau) - \text{sign}(t-\tau) f(|t-\tau|)].$$

The shift operator $(9)$ generated by the convolution $(5)$ is well known. This is the operator of Levitan’s type, other operators $(10)$–$(12)$ do not satisfy to some conditions of the Definition II.

We recall the main properties of the operators $(9)$–$(12)$. Assume that the functions $f(t)$ and $g(t)$ are piecewise continuous and bounded on $\mathbb{R}_+$, i.e. $\sup_{t \in \mathbb{R}_+} |f(t)| < \infty$ and $\sup_{t \in \mathbb{R}_+} |g(t)| < \infty$. In this case the functions of two variables $nT^s_nf(t)$ and $nT^s_n g(t)$ exist for any $n = 1, 4$ and the following properties are satisfied.

- Linearity and homogeneity: $nT^s_nf(t \cdot a + b \cdot g(t)) = a \cdot nT^s_nf(t) + b \cdot nT^s_ng(t)$ for all $a, b \in \mathbb{R}$, $\forall n = 1, 4$.
- If $f(t) \equiv 0$ for $t \geq a \geq 0$ then $nT^s_nf(t) \equiv 0$ for $|t-\tau| \geq a$, $\forall n = 1, 4$.
- Continuity of operators: if a sequence of continuous functions $f_k(t)$ converges uniformly to a function $f(t)$ in each finite interval, then the sequence of functions of two variables $nT^s_nf_k(t)$ converges uniformly to the function $nT^s_nf(t)$ in each finite region ($\forall n = 1, 4$).
- $\left| nT^s_nf(t) \right| \leq \sup_{t \in \mathbb{R}_+} |f(t)|$, $\forall n = 1, 4$.
- The operators $(9)$–$(10)$ is symmetric: $nT^s_nf(t) = nT^s_nf(\tau)$, $n = 1, 2$. And the operators $(11)$–(12) are not symmetric: $3T^s_1f(t) = 4T^s_2f(\tau)$ and $4T^s_2f(t) = 3T^s_1f(\tau)$.

The other properties we present for each operator separately and demonstrate their action with help graphs created by GeoGebra applets [20]. For the action visualization we take the function defined on $\mathbb{R}_+$ that is given by the formula

$$f(t) = \begin{cases} \sin \left( t - \frac{\pi}{2} \right) + 1, & 0 \leq t \leq 2\pi, \\ 0, & t > 2\pi. \end{cases}$$

The Levitan’s type shift operator $(9)$ generated by the convolution $(5)$ has the following additional properties.
Boundary conditions: \( 1T^0_t f(t) = f(t), \quad 1T^r_t f(t) = f(r). \)

1. \( 1T^t_t b = b, \) where \( b \) is a constant, \( b \in \mathbb{R}. \)

2. \( 1T^t_t \cos(t) = \cos(t) \cos(r). \)

If \( f(t) \in L_1(\mathbb{R}_+; dt) \) or \( f(t) \in L_2(\mathbb{R}_+; dt) \) then the generalized shift (9) of the Fourier cosine transform (2)–(3) exists and the equality \( 1T^y_x V_c[f](x) = V_c[f(t) \cos(yt)](x) \) holds.

**Proof.** The Fourier cosine transform (2) of the function belongs to the weighted Lebesgue space \( L_1(\mathbb{R}_+; dt) \) exists and

\[
1T^y_x V_c[f](x) = 1T^y_x \int_0^\infty f(t) \cos(xt) \, dt = \int_0^\infty f(t) \, 1T^y_x \cos(xt) \, dt
\]

\[= \int_0^\infty f(t) \cos(xt) \cos(yt) \, dt = V_c[f(t) \cos(yt)](x). \]

The property is similarly proved in the space \( L_2(\mathbb{R}_+; dt). \)

Reciprocally, using the inversion formula we obtain

\[
1T^t_t f(t) = 1T^t_t V^{-1}_c[F_c](t) = V^{-1}_c[F_c(x) \cos(t)](t), \quad f(t) \in L_2(\mathbb{R}_+; dt).
\]

**Corollary 1.** If \( f(t) \in L_2(\mathbb{R}_+; dt) \) then \( 1T^t_t f(t) \) belongs to \( L_2(\mathbb{R}_+; dt) \) for all \( r \in \mathbb{R}_+ \) and

\[
\|1T^t_t f\|_{L_2(\mathbb{R}_+; dt)}^2 \leq \|f\|_{L_2(\mathbb{R}_+; dt)}^2.
\]

**Corollary 2.** If \( f(t) \in L_2(\mathbb{R}_+; dt) \) then the Fourier cosine transform of the shift \( 1T^t_t f(t) \) can be presented through the Fourier cosine transform of the function \( f(t) \):

\[
V_c\left[1T^t_t f(t)\right](x) = F_c(x) \cos(xr).
\]

The action of the shift operator (9) on any function can be seen in the GeoGebra book [23]. Here, some graphs for function (13) are presented on figure 1.

The other shift operator (10) generated by the convolution (7) is not the operator of Levitan’s type because the condition II of the Definition 2 is not satisfied. In this case the boundary conditions are written in form

\[
z2T^0_t f(t) \equiv 0, \quad z2T^r_t f(t) \equiv 0.
\]

Also, the following properties hold.

1. \( z2T^t_t b = 0 \) for any constant \( b \in \mathbb{R}. \) This property also shows that the operator (10) is not Levitan’s type operator.

2. \( z2T^t_t \cos(t) = \sin(t) \sin(r). \)

If \( f(t) \in L_1(\mathbb{R}_+; dt) \) or \( f(t) \in L_2(\mathbb{R}_+; dt) \) then the generalized shift (10) of the Fourier cosine transform (2)–(3) exists and the equality \( z2T^y_x V_c[f](x) = V_c[f(t) \sin(yt)](x) \) holds.

Similarly, using the inversion formula we obtain

\[
z2T^t_t f(t) = z2T^t_t V^{-1}_c[F_c](t) = V^{-1}_s[F_c(x) \sin(xr)](t), \quad f(t) \in L_2(\mathbb{R}_+; dt).
\]

\[
= f(t - r)
\]

\[
= 1T^t_t f(t)
\]
\[ \tau = 0 \quad \tau = \pi/4 \quad \tau = \pi/2 \]
\[ \tau = 3\pi/4 \quad \tau = \pi \quad \tau = 5\pi/4 \]
\[ \tau = 3\pi/2 \quad \tau = 7\pi/4 \quad \tau = 2\pi \]
\[ \tau = 5\pi/2 \]

Figure 1. The action the shift operator (9) on the function (13).

Therefore, if \( f(t) \in L_2(\mathbb{R}_+; dt) \) then \( z\tau f(t) \) belongs to \( L_2(\mathbb{R}_+; dt) \) for all \( \tau \in \mathbb{R}_+ \) and

\[
\| z\tau f \|_{L_2(\mathbb{R}_+; dt)}^2 \leq \| f \|_{L_2(\mathbb{R}_+; dt)}^2
\]
and the Fourier sine transform of the shift $2T_\tau^f f(t)$ can be presented through the Fourier cosine transform of the function $f(t)$:

$$V_s \left[ 2T_\tau^f f(t) \right](x) = F_s(x) \sin(x\tau).$$

The action of the shift operator (10) on any function can be seen in the GeoGebra book [23]. Here, some graphs for function (13) are presented on figure 2.

The shift operator (11) generated the convolution (8) is not the operator of Levitan's type because the condition II of the Definition 2 is not satisfied. In this case the boundary conditions are written in form

$$3T_\tau^0 f(t) = f(t), \quad 3T_\tau^0 f(t) \equiv 0.$$

The other properties are presented below.

- $3T_\tau^T \{ b \} = \frac{b}{2} \left[ 1 + \text{sign}(t - \tau) \right] = \begin{cases} b, & t > \tau, \\ \frac{b}{2}, & t = \tau, \\ 0, & t < \tau, \end{cases}$ for any constant $b \in \mathbb{R}$. This property also shows that the operator (12) is not Levitan's type operator.

- $3T_\tau^T \sin(t) = \sin(t) \cos(\tau)$.

- If $f(t) \in L_2(\mathbb{R}_+;dt)$ or $f(t) \in L_2(\mathbb{R}_+;dt)$ then the generalized shift (12) of the Fourier sine transform (2)-(3) exists and the equality $3T_\tau^V V_s[f](x) = V_s[f(t) \cos(yt)](x)$ holds.

- Reciprocally, for the function $f(t) \in L_2(\mathbb{R}_+;dt)$. we obtain

$$3T_\tau^V f(t) = 3T_\tau^V V_s^{-1}[F_3](t) = V_s^{-1}[F_3(x) \cos(x\tau)](t)$$

Hence if $f(t) \in L_2(\mathbb{R}_+;dt)$ then $3T_\tau^V f(t)$ belongs to $L_2(\mathbb{R}_+;dt)$ for all $\tau \in \mathbb{R}_+$ and

$$\| 3T_\tau^V f \|^2_{L_2(\mathbb{R}_+,dt)} \leq \| f \|^2_{L_2(\mathbb{R}_+,dt)}.$$

In this case the Fourier sine transform of the shift $3T_\tau^V f(t)$ can be presented through the Fourier sine transform of the function $f(t)$:

$$V_s \left[ 3T_\tau^V f(t) \right](x) = F_s(x) \cos(x\tau).$$

The action of the shift operator (11) on any function can be seen in the GeoGebra book [23]. Here, some graphs for function (13) are presented on figure 3.

The shift operator (12) generated the convolution (6) is not the operator of Levitan’s type because the condition II of the Definition 2 is not satisfied. In this case the boundary conditions are written in form

$$4T_\tau^0 f(t) = 3T_\tau^0 f(\tau) \equiv 0, \quad 4T_\tau^0 f(t) = 3T_\tau^0 f(\tau) = f(\tau).$$

The following properties hold.

- $4T_\tau^T \{ b \} = 3T_\tau^T \{ b \} = \frac{b}{2} \left[ 1 - \text{sign}(t - \tau) \right] = \begin{cases} 0, & t > \tau, \\ \frac{b}{2}, & t = \tau, \\ b, & t < \tau, \end{cases}$ for any constant $b \in \mathbb{R}$. This property also shows that the operator (12) is not Levitan’s type operator.

- $4T_\tau^T \sin(t) = 3T_\tau^T \sin(t) = \sin(t) \sin(\tau)$.

- If $f(t) \in L_1(\mathbb{R}_+;dt)$ or $f(t) \in L_2(\mathbb{R}_+;dt)$ then the generalized shift (12) of the Fourier sine transform (2)–(3) exists and the equality $4T_\tau^V V_s[f](x) = V_s[f(t) \sin(yt)](x)$ holds.
\[ \tau = 0 \]
\[ \tau = \pi /4 \]
\[ \tau = \pi /2 \]
\[ \tau = 3\pi /4 \]
\[ \tau = \pi \]
\[ \tau = 5\pi /4 \]
\[ \tau = 3\pi /2 \]
\[ \tau = 7\pi /4 \]
\[ \tau = 2\pi \]
\[ \tau = 5\pi /2 \]

**Figure 2.** The action the shift operator (10) on the function (13).
\[ \tau = 0 \quad \tau = \pi /4 \quad \tau = \pi /2 \]
\[ \tau = 3\pi /4 \quad \tau = \pi \quad \tau = 5\pi /4 \]
\[ \tau = 3\pi /2 \quad \tau = 7\pi /4 \quad \tau = 2\pi \]
\[ \tau = 5\pi /2 \]

Figure 3. The action the shift operator (11) on the function (13).
\[ \tau = 0 \]
\[ \tau = \pi /4 \]
\[ \tau = \pi /2 \]
\[ \tau = 3\pi /4 \]
\[ \tau = \pi \]
\[ \tau = 5\pi /4 \]
\[ \tau = 3\pi /2 \]
\[ \tau = 7\pi /4 \]
\[ \tau = 2\pi \]
\[ \tau = 5\pi /2 \]

Figure 4. The action the shift operator (12) on the function (13).
Similarly, we obtain
\[ 4T_t^2 f(t) = 4T_t^2 V_\tau^{-1}[F_\tau(t)](t) = V_\tau^{-1}[F_\tau(x) \sin(\tau t)](t) \]
for the function \( f(t) \in L_2(\mathbb{R}_+; dt) \). Therefore,
\[ \| 4T_t^2 f \|_{L_2(\mathbb{R}_+; dt)}^2 \leq \| f \|_{L_2(\mathbb{R}_+; dt)}^2, \]
the Fourier cosine transform of the shift \( 4T_t^2 f(t) \) can be presented through the Fourier sine transform of the function \( f(t) \):
\[ V_\tau[4T_t^2 f(t)](x) = F_\tau(x) \sin(\tau t). \]

The action of the shift operator (12) on any function can be seen in the GeoGebra book [23]. Here, some graphs for function (13) are presented on figure 4.

Note that operators (9)–(12) are related to each other by the following properties that are directly proved by differentiation.

Let \( D_t \) be the differentiation operator with respect to a variable \( t \), \( D_t = \frac{\partial}{\partial t} \), the function \( f(t) \) be differentiable, and the function \( f'(t) \) is piecewise continuous and bounded on \( \mathbb{R}_+ \).

\[ \begin{align*}
D_t \; 1T_t^2 f(t) &= 3T_t^2 f'(t) \quad \text{and} \quad D_t \; 1T_t^2 f(t) \big|_{t=0} = 3T_t^0 f'(t) \equiv 0. \\
D_t \; 2T_t^2 f(t) &= 4T_t^2 f'(t) \quad \text{and} \quad D_t \; 2T_t^2 f(t) \big|_{t=0} = 4T_t^0 f'(t) \equiv 0. \\
D_t \; 3T_t^2 f(t) &= -4T_t^2 f'(t) \quad \text{and} \quad D_t \; 3T_t^2 f(t) \big|_{t=0} = -4T_t^0 f'(t) = -f'(t). \\
D_t \; 4T_t^2 f(t) &= -3T_t^2 f'(t) \quad \text{and} \quad D_t \; 4T_t^2 f(t) \big|_{t=0} = -3T_t^0 f'(t) = f'(t). \\
D_t \; 1T_t^n f(t) &= 2T_t^n f'(t) \quad \text{and} \quad D_t \; 1T_t^n f(t) \big|_{t=0} = 2T_t^0 f'(t) = f'(t). \\
D_t \; 3T_t^n f(t) &= -3T_t^n f'(t) \quad \text{and} \quad D_t \; 3T_t^n f(t) \big|_{t=0} = -3T_t^0 f'(t) \equiv 0. \\
D_t \; 4T_t^n f(t) &= -4T_t^n f'(t) \quad \text{and} \quad D_t \; 4T_t^n f(t) \big|_{t=0} = -4T_t^0 f'(t) \equiv 0. \\
D_t \; nT_t^n f(t) &= -nT_t^n f'(t) \quad \text{and} \quad D_t \; nT_t^n f(t) \big|_{t=0} = -nT_t^0 f'(t) = f'(t). 
\end{align*} \]

Consequently, if the function \( f(t) \) be double differentiable, and the functions \( f'(t) \) and \( f''(t) \) are piecewise continuous and bounded on \( \mathbb{R}_+ \), \( D_t^2 = \frac{\partial^2}{\partial t^2} \), then \( D_t^2 \; nT_t^n f(t) = nT_t^n f''(t) \) and \( D_t^2 \; nT_t^n f(t) = nT_t^n f''(t) \) for any \( n = 1, 2 \) with corresponding boundary properties:

\[ \begin{align*}
D_t^2 \; 1T_t^n f(t) \big|_{t=0} = f''(t) \quad \text{and} \quad D_t^2 \; 1T_t^n f(t) \big|_{t=0} = f''(t); \\
D_t^2 \; 2T_t^n f(t) \big|_{t=0} \equiv 0 \quad \text{and} \quad D_t^2 \; 2T_t^n f(t) \big|_{t=0} \equiv 0; \\
D_t^2 \; 3T_t^n f(t) \big|_{t=0} \equiv 0 \quad \text{and} \quad D_t^2 \; 3T_t^n f(t) \big|_{t=0} = f''(t); \\
D_t^2 \; 4T_t^n f(t) \big|_{t=0} = f''(t) \quad \text{and} \quad D_t^2 \; 4T_t^n f(t) \big|_{t=0} \equiv 0. 
\end{align*} \]

We considered the basic properties of operators, except for commutativity and associativity, which hold only for the operator \( 1T_t^n f(t) \), in other cases, the operators are mixed.

3. Conclusion
The studied shift operators \( nT_t^n \) allows one to solve various equations containing the differential operator \( D_t^2 \) and (or) the shift operator itself. They can also be used in the construction of generalized wavelet transforms. The generalized convolutions (5)–(8) related to the operator (9)–(12) and the convolution applications are of particular interest.
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