CHERN-OSSERMAN TYPE EQUALITY FOR COMPLETE SURFACES
IN $\mathbb{R}^N$

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Abstract. We obtain a Chern-Osserman type equality of a complete properly immersed surface in Euclidean space, provided the $L^2$-norm of the second fundamental form is finite. Also, by using a monotonicity formula, we prove that if the $L^2$-norm of mean curvature of a noncompact surface is finite, then it has at least quadratic area growth.

1. Introduction

Let $M$ be a complete minimal surface in $\mathbb{R}^n$ with finite total curvature, Chern and Osserman [2], [7] proved that

$$-\chi(M) \leq -\frac{1}{2\pi} \int_M K - k,$$

where $K$ is the Gauss curvature of $M$, $\chi(M)$ is the Euler characteristic of $M$ and $k$ is the number of ends of $M$. Further results were obtained by Jorge and Meeks [5] that

$$-\chi(M) = -\frac{1}{2\pi} \int_M K - \lim_{t \to \infty} \frac{\text{area}(M \cap B(t))}{\pi t^2},$$

where $B(t)$ is the extrinsic ball of radius $t$.

When $M$ is a general surface properly immersed in $\mathbb{R}^n$ with $\int_M |A|^2 < \infty$, where $A$ is the second fundamental form of the immersion, White [9] proved that $\frac{1}{2\pi} \int_M K$ must be an integer. In this paper, we present a general version of (1.2), where $M$ is a general surface properly immersed in $\mathbb{R}^n$ with the $L^2$-norm of the second fundamental form is finite.

Theorem 1.1. Let $M$ be a complete properly immersed noncompact oriented surface in $\mathbb{R}^n$, $A$ the second fundamental form of the immersion, $r$ the distance of $\mathbb{R}^n$ from a fixed point and $M_t = \{x \in M : r(x) < t\}$, $\chi(M)$ the Euler characteristic of $M$. Suppose $\int_M |A|^2 < \infty$, then

1. $\lim_{t \to \infty} \frac{\text{area}(M_t)}{\pi t^2}$ exists and is a positive integer;

2. $\lim_{t \to \infty} \frac{\text{area}(M_t)}{\pi t^2} = \chi(M) - \frac{1}{2\pi} \int_M K$.

Since $\int_M |A|^2 < +\infty$, then $\int_M |K| < +\infty$ by Gauss equation. When $M$ is a complete surface with finite total Gaussian curvature, Huber [4] proved that $M$ has finite topological...
type. And Cohn-Vossen [3] obtained:
\[ 2\pi \chi(M) - \int_M K \geq 0. \]
The explicit equality was obtained by Shiohama [8]:
\[ \chi(M) - \frac{1}{2\pi} \int_M K = \lim_{t \to \infty} \frac{D(t)}{\pi t^2}, \]
where \( D(t) \) denote the area of geodesic balls of radius \( t \) at a fixed point. Our theorem shows that (1.4) also holds with extrinsic balls instead of geodesic balls if \( M \) is properly immersed in \( \mathbb{R}^n \).

The proof of Theorem 1.1 is based on two monotonicity formulas (Theorem 2.4). The monotonicity formulas also have an interesting application, namely, if the \( L^2 \)-norm of mean curvature \( H \) of the surface is finite, then it has at least quadratic area growth.

**Corollary 1.2.** (see also Corollary 2.5) Let \( M \) be a complete properly immersed noncompact surface in \( \mathbb{R}^n \) with \( \int_M |H|^2 < \infty \), then the volume of the intersection of \( M \) and the extrinsic balls has at least quadratic area growth.

## 2. Preliminaries

Let \( x : M \to \mathbb{R}^n \) be a complete properly immersed surface in \( \mathbb{R}^n \), \( r \) the distance function of \( \mathbb{R}^n \) from a fixed point. For simplicity, we always assume the fixed point to be 0, unless otherwise specified. Denote the covariant derivative of \( \mathbb{R}^n \) and \( M \) by \( \nabla \) and \( \nabla \) respectively. Let \( X, Y \) be two tangent vector fields of \( M \), then
\[
(\nabla^2 r)(X,Y) = XY(r) - \nabla_X Y(r) = (\nabla^2 r)(X,Y) - \langle A(X,Y), \nabla r \rangle.
\]
The equality (2.1), together with the fact that \( \nabla^2 r = \frac{1}{r}(g_{st} - dr \otimes dr) \), where \( g_{st} \) denotes the standard metric of \( \mathbb{R}^n \), implies

**Proposition 2.1.** For any unit tangent vector \( e \) of \( M \),
\[
(\nabla^2 r)(e,e) = \frac{1}{r}(1 - \langle e, \nabla r \rangle^2) + \langle A(e,e), \nabla^\perp r \rangle,
\]
where \( \nabla^\perp r \) is the projection of \( \nabla \) onto the normal of \( M \).

By Sard’s theorem, for a.e. \( t > 0 \), \( M_t = \{ x \in M : r(x) < t \} \) is a related compact open subset of \( M \) with the boundary \( \partial M_t \) being a closed immersed curve of \( M \). Let \( v(t) = area M_t \), \( A \) the second fundamental form of \( M \), and \( H = tr A \) the mean curvature vector.

**Proposition 2.2.** Suppose \( M \) is a complete properly immersed surface in \( \mathbb{R}^n \). Then for a.e. \( t > 0 \),
\[
2\pi \chi(M_t) - \int_{M_t} K = \frac{1}{t} \left( v'(t) + \int_{\partial M_t} \langle \nabla^\perp r, H \rangle \right) - \int_{\partial M_t} \langle A(\nabla r, \nabla r), \nabla^\perp r \rangle,
where \( x^\perp \) is the projection of position vector \( x \) onto the normal of \( M \).

**Proof.** By the Gauss-Bonnet formula, it’s sufficient to verify

\[
(2.2) \quad \int_{\partial M_t} k_g = \frac{1}{t} v'(t) + \int_{\partial M_t} \left\langle \frac{x^\perp}{|\nabla r|}, H \right\rangle - \int_{\partial M_t} \left\langle A(\frac{\nabla r}{|\nabla r|}, \frac{\nabla^2 r}{|\nabla r|^2}), \frac{\nabla^\perp r}{|\nabla r|} \right\rangle,
\]

where \( k_g \) denote the geodesic curvature of \( \partial M_t \) in \( M \).

Suppose \( e \) is the unit tangent vector of \( \partial M_t \). Since the normal of \( \partial M_t \) is \( \nabla r / |\nabla r| \),

\[
k_g = -\left\langle \nabla e, \frac{\nabla r}{|\nabla r|} \right\rangle
= \frac{1}{|\nabla r|} (\nabla^2 r)(e, e)
= \frac{1}{|\nabla r|} \left( \frac{1}{r} + \left\langle A(e, e), \frac{\nabla^\perp r}{|\nabla r|} \right\rangle \right)
= \frac{1}{|\nabla r|} \left( \frac{1}{r} + \left\langle H - A(\frac{\nabla r}{|\nabla r|}, \frac{\nabla^2 r}{|\nabla r|^2}), \frac{\nabla^\perp r}{|\nabla r|} \right\rangle \right),
\]

where the third equality follows by Proposition 2.1. Then by using co-area formula,

\[
v'(t) = \int_{\partial M_t} \frac{1}{|\nabla r|} \text{ and the fact that } \nabla^\perp r = \frac{x^\perp}{r}, \text{ we obtain (2.2).} \quad \square
\]

**Proposition 2.3.** Let \( M \) be a complete properly immersed surface in \( \mathbb{R}^n \), then

\[
tv'(t) = t \int_{\partial M_t} \frac{1}{|\nabla r|} |\nabla^\perp r|^2 + 2v(t) + \int_{M_t} \langle x^\perp, H \rangle.
\]

**Proof.** Since \( \frac{1}{2} \Delta r^2 = 2 + \langle x, H \rangle \), integrating over \( M_t \) and using the Green’s formula,

\[
(2.4) \quad t \int_{\partial M_t} |\nabla r| = 2v(t) + \int_{M_t} \langle x, H \rangle.
\]

By the co-area formula,

\[
v'(t) = \int_{\partial M_t} \frac{1}{|\nabla r|}.
\]

So we have,

\[
tv'(t) = t \left( \int_{\partial M_t} \frac{1}{|\nabla r|} - \int_{\partial M_t} |\nabla r| \right) + t \int_{\partial M_t} |\nabla r|
= t \int_{\partial M_t} |\nabla^\perp r|^2 + 2v(t) + \int_{M_t} \langle x^\perp, H \rangle. \quad \square
\]

**Theorem 2.4.** Let \( M \) be a complete properly immersed surface in \( \mathbb{R}^n \), \( r \) the distance of \( \mathbb{R}^n \) from a fixed point \( x_0 \), \( H \) the mean curvature of \( M \), \( M_t = \{ x \in M : r(x) < t \} \), \( v(t) = \text{area}M_t \), then both

\[
u_1(t) \triangleq \frac{v(t)}{t^2} = \frac{1}{24} \int_{M_t} |(x - x_0)^\perp| |H| + \frac{1}{16} \int_{M_t} |H|^2
\]
and
\[ u_2(t) = v(t) - \frac{1}{t^2} \int_{M_t} |(x - x_0)^\perp| H | + \frac{1}{4} \int_{M_t} |H|^2 \]
are monotone nondecreasing in \( t \).

**Proof.** For simplicity, we assume \( x_0 = 0 \). By Proposition 2.3, we have
\[ tv'(t) \geq t \int_{\partial M_t} \frac{|\nabla_r^\perp|^2}{|\nabla_r|} + 2v(t) - \int_{M_t} |x^\perp| H |. \quad (2.5) \]
By co-area formula and the weighted mean value inequalities,
\[ \frac{d}{dt} \left( \int_{M_t} |x^\perp| H | \right) = \int_{\partial M_t} \frac{|\nabla_r^\perp| H |}{|\nabla_r|} \]
\[ = t \int_{\partial M_t} \frac{|\nabla_r^\perp| H |}{|\nabla_r|} \]
\[ \leq 2 \int_{\partial M_t} \frac{|\nabla_r^\perp|^2}{|\nabla_r|} + \frac{t^2}{8} \int_{\partial M_t} |H|^2. \quad (2.6) \]
Combining (2.5) and (2.6), we have
\[ tv'(t) \geq \frac{t}{2} \left( t \int_{M_t} |x^\perp| H | \right)' - \frac{t^2}{8} \int_{\partial M_t} \frac{|H|^2}{|\nabla_r|} \]
\[ + 2v(t) - \int_{M_t} |x^\perp| H |, \quad (2.7) \]
or equivalently,
\[ tv'(t) - 2v(t) - \frac{1}{2} \left( t \int_{M_t} |x^\perp| H | \right)' - 2 \int_{M_t} |x^\perp| H | \]
\[ - \frac{t^3}{16} \int_{\partial M_t} \frac{|H|^2}{|\nabla_r|} \geq 0. \quad (2.8) \]
Dividing both sides of (2.8) by \( t^3 \) yields
\[ \frac{d}{dt} \left( \frac{v(t)}{t^2} - \frac{1}{2} \int_{M_t} \frac{|x^\perp| H |}{t^2} + \frac{1}{16} \int_{M_t} |H|^2 \right) \geq 0, \quad (2.9) \]
this proves that \( u_1(t) \) is monotone nondecreasing in \( t \).

If we make slight modifications to (2.5) and (2.6), we have
\[ tv'(t) \geq t \int_{\partial M_t} \frac{|\nabla_r^\perp|^2}{|\nabla_r|} + 2v(t) - 2 \int_{M_t} |x^\perp| H |, \quad (2.5)' \]
and
\[ \frac{d}{dt} \left( \int_{M_t} |x^\perp| H | \right) \leq \int_{\partial M_t} \frac{|\nabla_r^\perp|^2}{|\nabla_r|} + \frac{t^2}{4} \int_{\partial M_t} \frac{|H|^2}{|\nabla_r|}. \quad (2.6)' \]
Combining (2.5)' and (2.6)', we obtain
\[ \frac{d}{dt} \left( \frac{v(t)}{t^2} - \frac{1}{2} \int_{M_t} \frac{|x^\perp| H |}{t^2} + \frac{1}{4} \int_{M_t} |H|^2 \right) \geq 0, \quad (2.9)' \]
i.e. \( u_2(t) \) is monotone nondecreasing in \( t \).  
\( \square \)
Remark 2.4. From the proof, we can see that the theorem is also valid for noncomplete surface, for \( t \) with \( \partial M \cap B_{x_0}(t) = \emptyset \), where \( B_{x_0}(t) \) is the ball in \( \mathbb{R}^n \) of radius \( t \) and centered at \( x_0 \).

By Theorem 2.4, we can get various volume estimates under suitable restrictions on mean curvature \( H \).

**Corollary 2.5.** Let \( M \) be a complete properly immersed noncompact surface in \( \mathbb{R}^n \) with \( \int_M |H|^2 < \infty \), then the volume of the intersection of \( M \) and the extrinsic balls has at least quadratic area growth.

**Proof.** Without loss of generality, we assume the center of the extrinsic balls to be 0. Since \( \int_M |H|^2 < \infty \), for a given \( \varepsilon > 0 \), there exists \( R > 0 \), such that

\[
\int_{M \setminus B_0(R)} |H|^2 < \varepsilon.
\]

Now for \( t > R \) large enough, choosing a point \( p \in M \cap \partial B_0(\frac{t+R}{2}) \), then \( B_p(\frac{t-R}{2}) \subset B_0(t) \setminus B_0(R) \), so we have

\[
\int_{M \cap B_p(\frac{t-R}{2})} |H|^2 < \varepsilon, \quad \text{Vol}(M \cap B_0(t)) \geq \text{Vol}(M \cap B_p(\frac{t-R}{2})).
\]

Taking \( x_0 = p \) in Theorem 2.4, then we have

\[
u_1(\frac{t-R}{2}) \geq \lim_{r \to 0} u_1(r) = \pi.
\]

Combining (2.10) and (2.11), we obtain

\[
\text{Vol}(M \cap B_0(t)) \geq \text{Vol}(M \cap B_p(\frac{t-R}{2})) \\
\geq \frac{(t-R)^2}{4} \left( u_1(\frac{t-R}{2}) - \frac{1}{16} \int_{M \setminus \frac{t-R}{2}} |H|^2 \right) \\
\geq \frac{\pi - \varepsilon}{4} (t-R)^2.
\]

The conclusion follows by choosing \( \varepsilon \) small. \( \Box \)

3. **Proof of Theorem 1.1**

**Lemma 3.1.** Let \( M \) be as in Theorem 1.1, then both \( \lim_{t \to \infty} \frac{v(t)}{t^2} \) and \( \lim_{t \to \infty} \frac{\int_{M_t} x^+ \cdot |H|}{t^2} \) exist.

**Proof.** First we prove:

**Claim:** \( \liminf_{t \to \infty} \frac{b}{t^2} < +\infty \).
Proof of the claim: Since by the weighted mean value inequality,

\begin{equation}
\frac{1}{t} \int_{\partial M_t} \frac{\langle x^+, H \rangle}{|\nabla r|} \leq \int_{\partial M_t} \frac{|H|}{|\nabla r|} \leq \frac{1}{2} \left( t \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} + \frac{v'}{t} \right),
\end{equation}

by Proposition 2.2, we have

\begin{equation}
2\pi \chi(M_t) - \int K \geq \frac{1}{t} \left( v'(t) - \int_{\partial M_t} \frac{\langle x^+, H \rangle}{|\nabla r|} \right) - \int_{\partial M_t} |A| \frac{|\nabla^T r|}{|\nabla r|}
\end{equation}

(by Proposition 2.3)

\begin{align*}
&= \frac{v'}{2t} - \frac{t}{2} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} - \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} - \frac{tv' - (2v(t) + \int_{M_t} \langle x^+, H \rangle)}{2t^2} \\
&= \frac{v}{t^2} - \frac{t}{2} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} - \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} + \int_{M_t} \langle x^+, H \rangle \frac{2t^2}{|\nabla r|} \\
&\geq \frac{v}{t^2} - \frac{t}{2} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} - \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} - \frac{\sqrt{v} \int_{M_t} |H|^2}{2t} \\
&\geq \frac{v}{t^2} - \frac{t}{2} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} - \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} - \frac{1}{8} \int_{M_t} |H|^2
\end{align*}

where we use the weighted mean value inequalities in the second and the last equality, while the second equality count backwards follows from Cauchy’s inequality.

Since \( \int_M |H|^2 + \int_M |A|^2 < +\infty \), there exists a sequence \( \{\tau_i\} \) diverging to infinity such that

\begin{equation}
t(\int_{\partial M_t} \frac{|H|^2}{|\nabla r|} + \int_{\partial M_t} \frac{|A|^2}{|\nabla r|}) \bigg|_{t=\tau_i} \to 0 \quad \text{as} \quad i \to \infty.
\end{equation}

Otherwise, we must have \( \liminf_{t \to \infty} t(\int_{\partial M_t} \frac{|H|^2}{|\nabla r|} + \int_{\partial M_t} \frac{|A|^2}{|\nabla r|}) = \delta > 0 \). So for sufficient large \( t \), we have

\( t(\int_{\partial M_t} \frac{|H|^2}{|\nabla r|} + \int_{\partial M_t} \frac{|A|^2}{|\nabla r|}) > \frac{\delta}{2} \),

i.e.

\( \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} + \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} > \frac{\delta}{2t} \).

When you integrate \( t \), by the co-area formula, it is as bounded on the left as it is diverging on the right, a contradiction.

Then taking \( t = \tau_i \) in (3.2), together with the fact that

\( \chi(M_t) \leq 1, \quad \left| \int_{M_t} K \right| \leq \frac{1}{2} \int_M |A|^2 < +\infty \) and \( \int_{M_t} |H|^2 \leq 2 \int_M |A|^2 < +\infty \),
we have \( \limsup_{i \to \infty} \frac{v(t_i)}{t_i^2} < +\infty \), which implies \( \liminf_{t \to \infty} \frac{v(t)}{t^2} \leq \limsup_{i \to \infty} \frac{v(t_i)}{t_i^2} < +\infty \). This proves the claim.

Let \( u_1(t) \) and \( u_2(t) \) be as in Theorem 2.4 with \( x_0 = 0 \). By the claim, we have

\[
\liminf_{t \to \infty} u_1(t) \leq \liminf_{t \to \infty} \frac{v(t)}{t^2} + \frac{1}{16} \int_M |H|^2 < +\infty,
\]

\[
\liminf_{t \to \infty} u_2(t) \leq \liminf_{t \to \infty} \frac{v(t)}{t^2} + \frac{1}{4} \int_M |H|^2 < +\infty.
\]

Combining (3.4) and Theorem 2.4, we know that both \( u_1(t) \) and \( u_2(t) \) have finite limit as \( t \to \infty \).

Since derivative of each function in left side of (3.6) is non-negative, we have

\[
\int_M x^\perp \|H\| = 2u_1(t) - 2u_2(t) + \frac{3}{8} \int_M |H|^2,
\]

we conclude that both \( \lim_{t \to \infty} \frac{v(t)}{t^2} \) and \( \lim_{t \to \infty} \int_M \frac{x^\perp \|H\|}{t^2} \) exist. \( \square \)

**Lemma 3.2.** There exists a sequence \( \{t_k\} \) diverging to infinity such that

(i) \( \lim_{k \to \infty} \frac{v'(t_k)}{t_k} = \lim_{k \to \infty} \frac{2v(t_k)}{t_k} \lim_{k \to \infty} \frac{2}{t_k} \int_{\partial M_{t_k}} x^\perp \|H\| = \lim_{k \to \infty} \frac{1}{t_k} \int_{\partial M_{t_k}} x^\perp \|H\| = 0, \)

(ii) \( \lim_{k \to \infty} t_k \int_{\partial M_{t_k}} \frac{|H|^2}{\|\nabla r\|} = 0, \lim_{k \to \infty} t_k \int_{\partial M_{t_k}} \frac{|A|^2}{\|\nabla r\|} = 0. \)

**Proof.** Let \( u_1(t), u_2(t) \) be as in Lemma 3.1. Since \( u_1(t) + u_2(t) + \int_M |H|^2 + \int_M |A|^2 \) is bounded, arguing as in the proof of the claim in Lemma 3.1, we know that there is a sequence \( \{t_k\} \) diverging to infinity such that

\[
\left. \frac{d}{dt} \left( u_1(t) + u_2(t) + \int_M |H|^2 + \int_M |A|^2 \right) \right|_{t=t_k} \to 0 \text{ as } k \to \infty.
\]

Since derivative of each function in left side of (3.6) is nonnegative, we have

\[
t_k u_1'(t_k) \to 0, \quad t_k u_2'(t_k) \to 0 \quad \text{and}
\]

\[
t \left( \int_M |H|^2 \right)' \bigg|_{t=t_k} \to 0, \quad t \left( \int_M |A|^2 \right)' \bigg|_{t=t_k} \to 0
\]

as \( k \to \infty \). Combining (3.5) and (3.7), we get

\[
\left. t \left( \frac{v(t)}{t^2} \right)' \right|_{t=t_k} \to 0, \quad \left. t \left( \int_M \frac{x^\perp \|H\|}{t^2} \right)' \right|_{t=t_k} \to 0 \quad \text{and}
\]

\[
\left. t \left( \int_M |H|^2 \right)' \right|_{t=t_k} \to 0, \quad \left. t \left( \int_M |A|^2 \right)' \right|_{t=t_k} \to 0
\]
as \( k \to \infty \). So we obtain

\[
\lim_{k \to \infty} v'(t_k) = \lim_{k \to \infty} \frac{2v(t_k)}{t_k^2}, \quad \lim_{k \to \infty} \frac{1}{t_k} \left( \int_{M_{t_k}} |x^\perp||H| \right)' = \lim_{k \to \infty} \frac{2}{t_k^2} \int_{M_{t_k}} |x^\perp||H| \quad \text{and}
\]

\[
\lim_{k \to \infty} t_k \int_{\partial M_{t_k}} |H|^2 \frac{1}{\nabla r} = 0, \quad \lim_{k \to \infty} t_k \int_{\partial M_{t_k}} |A|^2 \frac{1}{\nabla r} = 0,
\]

where we use the fact that \( \lim_{t \to \infty} v(t) \) and \( \lim_{t \to \infty} \int_M |x^\perp||H| \) exist by Lemma 3.1, this proves (ii).

By co-area formula, when \( k \to \infty \),

\[
\frac{1}{t_k} \int_{D_{t_k}} \left( \int_{M_t} |x^\perp||H| \right)_{t=t_k} = \frac{1}{t_k} \int_{\partial M_{t_k}} |x^\perp||H| \frac{1}{\nabla r} \leq \int_{\partial M_{t_k}} \frac{|H|}{\nabla r} \leq \sqrt{v'(t_k)} \int_{\partial M_{t_k}} \frac{|H|^2}{\nabla r} = \sqrt{v'(t_k)} t_k \int_{\partial M_{t_k}} \frac{|H|^2}{\nabla r} \to 0.
\]

Combining (3.9) and (3.10), we have

\[
\lim_{k \to \infty} \frac{2}{t_k} \int_{M_{t_k}} |x^\perp||H| = \lim_{k \to \infty} \frac{1}{t_k} \int_{\partial M_{t_k}} |x^\perp||H| \frac{1}{\nabla r} = 0.
\]

Then (i) follows from (3.9) and (3.11).

**Proof of Theorem 1.1** By Proposition 2.3, we have

\[
\left| \int_{\partial M_t} \left( A(\frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|}, \frac{\nabla^\perp r}{|\nabla r|}) \right) \right| \leq \int_{\partial M_t} |A| |\frac{\nabla^\perp r}{|\nabla r|}|
\]

\[
\leq \int_{\partial M_t} \left( \frac{|A|^2}{2 |\nabla r|^2} + \frac{|\nabla^\perp r|^2}{2t |\nabla r|^2} \right)
\]

\[
= \frac{t}{2} \int_{\partial M_t} |A|^2 |\nabla r| + \frac{tv'(t) + \int_M (x^\perp, H)}{2t^2},
\]

then Lemma 3.2 implies

\[
\lim_{k \to \infty} \int_{\partial M_{t_k}} \langle A(\frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|}, \frac{\nabla^\perp r}{|\nabla r|}) \rangle = 0.
\]
Taking \( t = t_k \) in Proposition 2.2 and letting \( k \to \infty \), together with (3.13) and Lemma 3.2, we get

\[
(3.14) \quad \lim_{k \to \infty} 2\pi \chi(M_{t_k}) - \int_M K = \lim_{k \to \infty} \frac{2v(t_k)}{t_k^2},
\]

which implies

\[
(3.15) \quad \lim_{t \to \infty} \frac{2v(t)}{t^2} \leq 2\pi \chi(M) - \int_M K.
\]

Since the extrinsic distance is smaller than intrinsic distance, we clearly have

\[
(3.16) \quad \lim_{t \to \infty} \frac{v(t)}{t} \geq \lim_{t \to \infty} \frac{D(t)}{t^2},
\]

where \( D(t) \) is the area of geodesic balls of radius \( t \) at a fixed point.

Combining (1.4), (3.15) and (3.16), we conclude that

\[
(3.17) \quad \lim_{t \to \infty} \frac{2v(t)}{t^2} = 2\pi \chi(M) - \int_M K.
\]

Furthermore, by the main theorem of White [9], we know that \( \frac{1}{4\pi} \int_M K \) is an integer, so is \( \lim_{t \to \infty} \frac{v(t)}{\pi t^2} \), and this limit must be positive by Corollary 2.5. This completes the proof of Theorem 1.1.

**Corollary 3.3** Let \( M \) be a complete properly immersed noncompact oriented surface in \( \mathbb{R}^n \) with \( \int_M |A|^2 < 4\pi \), then \( \chi(M) = 1 \).

**References**

[1] Q. Chen and Y. Cheng, Chern-Osserman inequality for minimal surfaces in \( H^n \), Proc. AMS. 128 (1999), 2445-2450.
[2] S. S. Chern and R. Osserman, Complete minimal surface in \( \mathbb{R}^n \), J. d'Analyse Math. 19 (1967), 15-34.
[3] S. Cohn-Vossen, Kürzeste Wege und Totalkrümmung auf Flächen, G'omposiho Math. 2 (1935), 69-133.
[4] A. Huber, On subharmonic functions and differential geometry in the large, Comment Math. Helv. 32 (1957) 13-72.
[5] L. P. Jorge and W. H. Meeks, The topology of minimal surfaces of finite total Gaussian curvature, Topology 22 (1983), 203-221.
[6] J. H. Michael and L. M. Simon, Sobolev and Mean-Value Inequalities on Generalized submanifolds of \( \mathbb{R}^n \), Comm. Pure and Appl. Math., 26 (1973), 361-379.
[7] R. Osserman, A survey of minimal surfaces, Van Norstrand Rienhold, New York, 1969.
[8] K. Shiohama, Total curvature and minimal area of complete open surfaces, Proc. AMS. 94 (1985), 310-316.
[9] B. White, Complete surfaces of finite total curvature, J.Diff.Geom., 26 (1987), 315-326.