Chromatic Number and Dichromatic Polynomial of Digraphs

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Abstract

Let $G$ be a graph of order $n$. It is well-known that $\alpha(G) \geq \sum_{i=1}^{n} \frac{1}{1+d_i}$, where $\alpha(G)$ is the independence number of $G$ and $d_1, \ldots, d_n$ is the degree sequence of $G$. We extend this result to digraphs by showing that if $D$ is a digraph with $n$ vertices, then $\alpha(D) \geq \sum_{i=1}^{n} \left( \frac{1}{1+d_i} + \frac{1}{1+d_i} - \frac{1}{1+d_i} \right)$, where $\alpha(D)$ is the maximum size of an acyclic vertex set of $D$. Golowich proved that for any digraph $D$, $\chi(D) \leq \left\lceil \frac{k}{2} \right\rceil + 2$, where $k = \max(\Delta^+(D), \Delta^-(D))$. We give a short and simple proof for this result. Next, we investigate the chromatic number of tournaments and determine the unique tournament such that for every integer $k > 1$, the number of proper $k$-colorings of that tournament is maximum among all strongly connected tournaments with the same number of vertices. Also, we find the chromatic polynomial of the strongly connected tournament with the minimum number of cycles.

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1 Introduction

Throughout this paper all graphs and digraphs are loopless. The vertex set and the edge set of a digraph $D$ are denoted by $V(D)$ and $E(D)$ respectively. This digraph is called
strict if for any two distinct vertices $u$ and $v$ of $D$, there is at most one edge from $u$ to $v$. All digraphs in this paper are assumed to be strict. A digraph is called digon-free if it has no directed cycle of length 2. An acyclic vertex set of a digraph $D$ is a subset $S$ of $V(D)$ such that the induced subdigraph of $D$ on $S$ has no directed cycle. The maximum size of an acyclic vertex set of $D$ is called the independence number of $D$ and is denoted by $\alpha(D)$. A proper $k$-coloring of $D$ is a partition of the vertex set of $D$ into at most $k$ acyclic subsets. The minimum integer $k$ for which $D$ has a proper $k$-coloring is called the chromatic number of $D$ and is denoted by $\chi(D)$. This definition for the coloring of digraphs was first appeared in [10]. The independence and chromatic number of the undirected graph $G$ is also denoted by $\alpha(G)$ and $\chi(G)$, respectively. It turns out that the number of proper $k$-colorings of $D$ is a polynomial in $k$ [6] which is called the chromatic polynomial of $D$ and is denoted by $P(D; k)$. In what follows, $N^+(v)$ and $N^-(v)$ denote the set of out-neighbors and the set of in-neighbors of $v$ and their cardinalities are called the in-degree and the out-degree of $v$, respectively. The girth of a digraph is defined as the length of its shortest directed cycle.

Section 2 is about the independence number of digraphs and some lower bounds on that number. In particular, we find a lower bound for the independence number of digraphs which is similar to a well-known bound for graphs proved in [2] and [14]. Moreover, we find another lower bound for the independence number of a digraph, in terms of its girth and number of induced cycles.

In Section 3 we investigate various upper bounds on the chromatic number of digraphs. Note that a trivial upper bound for the chromatic number of a digraph with underlying graph $G$ is $\chi(G)$. We show that the equality holds for some digraphs. In [11] Golowich proved that for any digraph $D$, $\chi(D) \leq \left\lceil \frac{k^2}{4} \right\rceil + 2$, where $k = \max(\Delta^+(D), \Delta^-(D))$. Also in [4], an upper bound for the chromatic number of digraphs with some forbidden cycle lengths is given. Here we give short and simple proofs for these two results. Moreover, we give a new upper bound for the chromatic number in terms of the girth.

The dichromatic polynomial of digraphs is the subject of Section 4. We give an interpretation for some coefficients of this polynomial, and determine the tournament which has the maximum number of proper $k$-colorings for every $k$ between all strongly connected tournaments with $n$ vertices.
2 Lower Bounds on the Independence Number

Caro [2] and Wei [14] found the following lower bound for the independence number of a graph.

**Theorem 1.** Let $G$ be a graph and $d_1, \ldots, d_n$ be the degree sequence of $G$. Then the following inequality holds:

$$\alpha(G) \geq \sum_{i=1}^{n} \frac{1}{1+d_i}$$

A probabilistic proof of this result can be found in [1]. In [3] and [13], this result has been generalized to hypergraphs.

Here we prove an analogous inequality for the independence number of a digraph.

**Theorem 2.** Let $D$ be a digraph with the vertex set $\{v_1, \ldots, v_n\}$, and let $d_i, d^+_i, d^-_i$, $i = 1, \ldots, n$, be the degree, the out-degree and the in-degree of $v_i$, respectively. Then the following inequality holds:

$$\alpha(D) \geq \sum_{i=1}^{n} \left( \frac{1}{1+d^+_i} + \frac{1}{1+d^-_i} - \frac{1}{1+d_i} \right)$$

**Proof.** Choose $\sigma \in S_n$ uniformly at random. For $i = 1, \ldots, n$, let $A_i$ be the event that for $j = 1, \ldots, n$, if $v_{\sigma(j)} \in N^+(v_i)$, then $\sigma(j) > i$. In other words, if we arrange $v_{\sigma(1)}, \ldots, v_{\sigma(n)}$ on a line from left to the right, then $A_i$ is the event that all of the out-neighbors of $v_i$ lie on the right of $v_i$. Similarly, let $B_i$ be the event that for $j = 1, \ldots, n$, if $v_{\sigma(j)} \in N^-(v_i)$, then $\sigma(j) > i$. It is not hard to see that:

$$\Pr(A_i) = \frac{1}{1+d^+_i}, \quad \Pr(B_i) = \frac{1}{1+d^-_i},$$

$$\Pr(A_i \cap B_i) = \frac{1}{1+d_i},$$

and thus, by the principle of inclusion and exclusion:

$$\Pr(A_i \cup B_i) = \frac{1}{1+d^+_i} + \frac{1}{1+d^-_i} - \frac{1}{1+d_i}.$$ 

Now, let $X_i$ be the indicator random variable of $A_i \cup B_i$ (that is, $X_i = 1$ if $A_i \cup B_i$ occurs and $X_i = 0$ otherwise). If $S_\sigma$ is the set of all $1 \leq i \leq n$ such that $A_i \cup B_i$ occurs, then $|S_\sigma| = \sum_{i=1}^{n} X_i$. So we have the following,

$$\mathbb{E}(|S_\sigma|) = \sum_{i=1}^{n} \mathbb{E}(X_i) = \sum_{i=1}^{n} \Pr(A_i \cup B_i),$$

which implies that,
\[ \mathbb{E}(|S_\sigma|) = \sum_{i=1}^{n} \left( \frac{1}{1 + d_i^+} + \frac{1}{1 + d_i^-} - \frac{1}{1 + d_i} \right). \]

Thus, there is a \( \sigma \in S_n \) for which

\[ |S_\sigma| \geq \sum_{i=1}^{n} \left( \frac{1}{1 + d_i^+} + \frac{1}{1 + d_i^-} - \frac{1}{1 + d_i} \right). \]

Now, to complete the proof, it suffices to show that for each \( \sigma \), \( \{v_i : i \in S_\sigma\} \) is an acyclic subset of the vertices of \( D \). Assume on the contrary that there is a cycle \( C \) between the vertices in \( \{v_i : i \in S_\sigma\} \). Let \( i \) be the largest integer that \( v_i \) is a vertex of \( C \) (that is, \( v_i \) is the right-most vertex of \( C \) in the random order of the vertices of \( D \)). Since \( C \) is a cycle, there are two vertices \( v_{\sigma(j)} \) and \( v_{\sigma(k)} \) in \( V(C) \) such that \( v_{\sigma(j)} \in N^+(v_i) \) and \( v_{\sigma(k)} \in N^-(v_i) \). Since \( v_{\sigma(j)} \in N^+(v_i) \) and \( \sigma(j) < i \), \( A_i \) does not occur, and similarly, since \( v_{\sigma(k)} \in N^-(v_i) \) and \( \sigma(k) < i \), \( B_i \) does not occur. But since \( i \in S_\sigma \), at least one of the \( A_i \) and \( B_i \) should occur, which is a contradiction. \( \square \)

Using this theorem, one can obtain some lower bounds for the independence number of a digraph which is completely determined by the underlying graph.

**Corollary 1.** Let \( D \) be a digraph with \( n \) vertices and \( m \) edges, and let \( d_1, \ldots, d_n \) be the degrees of vertices of the underlying graph of \( D \). Then

1. \( \alpha(D) \geq \sum_{i=1}^{n} \frac{3d_i + 2}{(d_i + 1)(d_i + 2)} \),
2. If \( D \) has no isolated vertices and \( k = \frac{m}{n} \), then \( \alpha(D) \geq \frac{n}{4k + 1} \),

**Proof.** (i) If \( d_i^+ \) and \( d_i^- \) denote the out-degree and the in-degree of the \( i \)-th vertex, then

\[ \frac{1}{1 + d_i^+} + \frac{1}{1 + d_i^-} \geq \frac{2}{1 + \frac{d_i^+ + d_i^-}{2}} = \frac{4}{2 + d_i} \]

and applying Theorem \( \square \) yields the first part. (ii) For any real number \( x \geq 1 \) define:

\[ f(x) = \frac{4}{x + 2} - \frac{1}{x + 1} = \frac{3x + 2}{(x + 1)(x + 2)}. \]

Since for \( x \geq 1 \)

\[ f''(x) = \frac{8}{(x + 2)^3} - \frac{2}{(x + 1)^3} \geq 0, \]

\( f \) is convex and thus,

\[ \sum_{i=1}^{n} \frac{3d_i + 2}{(d_i + 1)(d_i + 2)} = \sum_{i=1}^{n} f(d_i) \geq nf \left( \frac{\sum_{i=1}^{n} d_i}{n} \right) = nf \left( \frac{2m}{n} \right). \]
This inequality together with the previous part, gives the following, \[
\alpha(D) \geq n - \frac{3k + 1}{(k + 1)(2k + 1)} \geq \frac{n}{\frac{2k}{3} + 1}
\]
which completes the proof. \[\square\]

In [6], it is conjectured that for a digon-free digraph \(D\), \(\chi(D) \leq \lfloor \frac{\Delta}{2} \rfloor + 1\), where \(\Delta = \max\{\sqrt{d^+(v)d^-(v)}| v \in V(D)\}\). If this conjecture is true, then \(\alpha(D) \geq \frac{n}{\lfloor \frac{\Delta}{2} \rfloor + 1}\). In the next corollary, a weaker result about the independence number is proved.

**Corollary 2.** Let \(D\) be a digon-free digraph. Then \(\alpha(D) \geq \frac{n}{\frac{2\Delta}{3} + 1}\).

**Proof.** Let \(V(D) = \{v_1, \ldots, v_n\}\) and \(d_i^+ = d^+(v_i), d_i^- = d^-(v_i)\) and \(d_i = d_i^+ + d_i^-\), \(p_i = d_i^+ d_i^-\). We have the following,

\[
\frac{1}{1 + d_i^+} + \frac{1}{1 + d_i^-} - \frac{1}{1 + d_i} = \frac{2 + d_i}{1 + d_i + p_i} - \frac{1}{1 + d_i}.
\]

It can be seen that the above expression is an increasing function of \(d_i\) (for a fixed \(p_i\)), thus, since \(d_i \geq 2\sqrt{p_i}\),

\[
\frac{1}{1 + d_i^+} + \frac{1}{1 + d_i^-} - \frac{1}{1 + d_i} \geq \frac{2 + 2\sqrt{p_i}}{1 + 2\sqrt{p_i} + p_i} - \frac{1}{1 + 2\sqrt{p_i}} = \frac{1 + 3\sqrt{p_i}}{(1 + \sqrt{p_i})(1 + 2\sqrt{p_i})}.
\]

Now, the above expression is a decreasing function of \(p_i\), and since \(\tilde{\Delta} = \max\{\sqrt{p_i}| i = 1, \ldots, n\}\), we find the following,

\[
\frac{1}{1 + d_i^+} + \frac{1}{1 + d_i^-} - \frac{1}{1 + d_i} \geq \frac{1 + 3\tilde{\Delta}}{(1 + \Delta)(1 + 2\Delta)} \geq \frac{1}{\frac{2\Delta}{3} + 1}.
\]

This inequality and the Corollary 1 gives the result. \[\square\]

In the next theorem we give a lower bound on the independence number of a digraph in terms of its girth and the number of induced cycles. Note that a subset of vertices of a digraph is acyclic if and only if it contains no induce cycles.

**Theorem 3.** Let \(D\) be a digraph with \(n\) vertices and girth \(g\), and let \(t\) be the number of induced directed cycles of \(D\). If \(tg \geq n\) then

\[
\alpha(D) \geq \frac{g - 1}{g} \left(\frac{n^g}{tg}\right)^{\frac{1}{tg}}
\]
Proof. Let \( C_1, \ldots, C_t \) be all of the induced cycles in \( D \). Let \( p \) be any number in \([0,1]\). Choose a random subset \( S \) of \( V(D) \), such that for each \( v \in V(D) \), \( Pr(v \in S) = p \). Let \( Y \) be the number of induced cycles in \( D[S] \). For each \( i = 1, \ldots, t \), the probability that \( V(C_i) \subseteq S \) is \( p^{|V(C_i)|} \), so
\[
E(Y) = \sum_{i=1}^{t} p^{|V(C_i)|} \leq tp^g.
\]
Thus \( E(|S| - Y|) \geq np - tp^g \). So there is a subset \( S \) of vertices such that after removing one vertex of each induced cycle in \( D[S] \), at least \( np - tp^g \) vertices remain. But the remaining vertices constitute an acyclic set and so we have \( \alpha(D) \geq np - tp^g \), for each \( p \in [0,1] \). Since \( tg \geq n \) we can set \( p = \left( \frac{n}{tg} \right)^{\frac{1}{g-1}} \) which gives the desired result. \( \square \)

**Corollary 3.** If \( T \) is a tournament with \( n \) vertices and \( t \) directed triangles such that \( n \geq 3t \), then \( \alpha(T) \geq \frac{2}{3 \sqrt{3t} n} \).

**Proof.** In a tournament, induced directed cycles are directed triangle and the result follows from the previous theorem. \( \square \)

### 3 Bounds on the Chromatic Number

There are some known bounds for the chromatic number of digraphs, see \([4,9,11]\). Here we show new bounds for the chromatic number and give simpler proofs for some known results.

It is obvious that if \( D \) is any orientation of the graph \( G \) with chromatic number \( k \), then \( \chi(D) \leq k \). We show that this bound is tight for every \( k \).

**Lemma 1.** Let \( n \geq 10 \) and \( t \geq 3 \log_2 n \) be two positive integers. Then there exists an orientation of \( K_{n,n} \) such that each of its \( K_{t,t} \)-subdigraph has a directed cycle.

**Proof.** Let \( X, Y \) be two parts of \( K_{n,n} \). Let \( D \) be a random orientation of \( G \). For each \( I \subseteq X \) and \( J \subseteq Y \) with \( |I| = |J| = t \), let \( A_{I,J} \) be the event that the induced subdigraph of \( D \) on \( I \cup J \) is acyclic. Since an acyclic orientation of a digraph gives an ordering of its vertices, we find that
\[
Pr(A_{I,J}) \leq \frac{(2t)!}{2^{t^2}}.
\]
So
\[
Pr(\bigcup_{I,J} A_{I,J}) \leq \left( \frac{n}{t} \right)^2 \frac{(2t)!}{2^{t^2}} < \frac{n^{2t}(2t)!}{2^{t^2} (t!)^2} \leq \frac{n^{2t}t^t}{2^{t^2}}.
\]
On the other hand, by the assumption
\[
\frac{n^{2t^t}}{2^{t^t}} < 2^{2t^t/3t \log t} < 1.
\]
Thus there is an orientation of \(G\) such that no \(A_{I,J}\) occurs, and the proof is complete. \(\square\)

**Theorem 4.** For any positive integer \(k\), there exists a graph \(G\) with an orientation \(D\) such that \(\chi(D) = \chi(G) = k\).

*Proof.* Let \(n, t\) be two positive integers that satisfy \(n > (k - 1)t\) and \(t \geq 3 \log_2 n\). Let \(G\) be the complete \(k\)-partite graph with parts \(X_1, \ldots, X_k\) such that the size of each part is \(n\). For each \(i \neq j\), by Lemma 1, we can orient the edges between \(X_i\) and \(X_j\), in such a way that each \(K_{t,t}\)-subdigraph of \(G[X_i \cup X_j]\) has a directed cycle. Denote the resulting orientation of \(G\) by \(D\). We claim that \(\chi(D) = k\). Assume on the contrary that \(D\) has a proper \((k - 1)\)-coloring. Since \(n > (k - 1)t\), in each part there are at least \(t\) vertices that receive the same color. For each \(i = 1, \ldots, k\), let \(c_i\) be the color that is appeared on maximum number of vertices of \(X_i\). Since there are \(k\) colors, there are \(i \neq j\) such that \(c_i = c_j\). But there are \(A \subseteq X_i\) and \(B \subseteq X_j\) such that \(|A|, |B| \geq t\) and all vertices of \(A\) have color \(c_i\) and all vertices of \(B\) have color \(c_j\). By our choice of orientation, \(D[A \cup B]\) has a directed cycle, a contradiction. \(\square\)

Now, we present another upper bound for the chromatic number of a digraph. In [6], it is proved that for any digraph \(D\), \(\chi(D) \leq \min\{\Delta^+(D), \Delta^-(D)\} + 1\). Also they have proposed the following conjecture.

**Conjecture 1.** For every \(k\)-regular digraph \(D\), we have \(\chi(D) \leq \lfloor \frac{k}{2} \rfloor + 1\).

In [11] they proved that if \(D\) is a digraph and \(k = \max\{\Delta^+(D), \Delta^-(D)\}\), then \(\chi(D) \leq \lfloor \frac{4k}{5} \rfloor + 2\). We provide a shorter and easier proof for this theorem.

**Theorem 5.** If \(D\) is a digraph and \(k = \max\{\Delta^+(D), \Delta^-(D)\}\), then \(\chi(D) \leq \lfloor \frac{4k}{5} \rfloor + 2\).

*Proof.* Let \(t = \lfloor \frac{2k+1}{3} \rfloor\). Partition the vertices of \(D\) into \(t\) parts such that the number of edges between these parts is maximum possible. Now, we claim that every vertex at most \(4\) neighbors in its part. To see this, note that if the number of neighbors of a vertex in its part is at least \(5\), then its degree in one of the other parts should be at most \(4\). So one may change the part of this vertex and make the number of edges between parts larger, which is impossible. Thus, by Theorem 2.3 of [9], we can color the vertices of each part with just \(2\) colors. So we can color all vertices of \(D\) with \(2t = 2|\lfloor \frac{2k+1}{3} \rfloor| \leq \lfloor \frac{4k}{5} \rfloor + 2\) colors such that the subdigraph of each color is acyclic. \(\square\)
Theorem 6. Let $D$ be a digraph with girth $g$ and $n$ vertices. Then $\chi(D) \leq \lceil \frac{n-1}{g-1} \rceil + 1$.

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of $D$ and $k = \lceil \frac{n-1}{g-1} \rceil + 1$. Color the vertices with $k$ colors from $v_1$ to $v_n$. For each vertex $v_i$, we have at most $\lceil \frac{n-1}{g-1} \rceil$ vertex-disjoint cycles containing $v_i$, so we can use the other color and we are sure that $v_i$ is not in a directed monochromatic cycle.

In [4] the following result was proved. Here we give a short and simple proof of this result. If $T$ is a directed DFS tree of a digraph $D$, then a back edge of $D$ is an edge from a vertex to one of its ancestors.

Theorem 7. Let $k \geq 2$ be an integer. If a digraph $D$ contains no directed cycle of length 1 modulo $k$, then $\chi(D) \leq k$.

Proof. Without loss of generality, we can assume that $D$ is strongly connected. Let $T$ be a directed DFS tree of $D$ with root $v_0$. For each $v \in V(D)$, let $c(v)$ be the distance of $v$ from $v_0$ in $T$ modulo $k$. We claim that, $c$ is a proper $k$-coloring of $D$. Let $C$ be an arbitrary directed cycle of $D$. By [4] Lemma 22.11, $C$ contains a back-edge $xy$. Let $P$ be the unique directed path from $y$ to $x$ in $T$. If $c(x) = c(y)$, then the length of $P$ is divisible by $k$. So, $D$ contains a cycle of length 1 modulo $k$, a contradiction. Thus, $\chi(D) \leq k$.

4 Dichromatic Polynomial

In [5] the dichromatic polynomial of digraphs was defined. Now, we want to discuss about some properties of dichromatic polynomial for digraphs and tournaments.

Theorem 8. Let $g$ be the girth of digraph $D$. Then in $P(D; x)$ the coefficients of $x^{n-1}, ..., x^{n-g+2}$ are zero and the coefficient of $x^{n-g+1}$ is the number of cycles of length $g$ with negative sign.

Proof. Let $c_1, c_2, ..., c_k$ be the minimal cycles of $D$. By the Inclusion-exclusion principle, the number of ways for coloring the vertices of $D$ with $x$ colors without a monochromatic cycle, is equal to $x^n$ minus the number of way with at least one monochromatic cycle, plus the number of way with at least two monochromatic cycles, etc. and except $x^n$ each other term is of the form $x^k$ with $k \leq n - g + 1$. Furthermore there is exactly $g$ number of $x^{n-g+1}$ with negative sign. So the proof is complete.

Corollary 4. Let $T$ be a tournament with $n$ vertices. Then in $P(T; x)$ the coefficient of $x^{n-1}$ is zero, and the coefficient of $x^{n-2}$ is the number of directed triangles of $T$ with negative sign.
In tournaments, we know the maximum number of directed triangles and some other theorems about the directed cycles, so maybe we can compute and compare the dichromatic polynomials for some tournaments. For example let \( D \) be an acyclic tournament on the vertex set \( \{v_1, \ldots, v_n\} \), such that the out-neighbors of \( v_i \) are \( v_1, \ldots, v_{i-1} \). Now, reverse the edges of the Hamiltonian path \( v_n, v_{n-1}, \ldots, v_1 \). The resulting tournament is denoted by \( S_n \).

**Theorem 9.** For every positive integer \( n \),

\[
P(S_n; x) = \sum_{i=1}^{n} \binom{i}{n-i} x(x-1)^{i-1}
\]

*Proof.* Let \( f_n(x) = P(S_n; x) \). It can be easily checked that \( f_1(x) = x \) and \( f_2(x) = x^2 \). Let \( n \geq 3 \). The proper \( k \)-colorings of \( S_n \) can be divided into two types: the colorings in which the colors of \( v_n \) and \( v_{n-1} \) are different, and ones in which \( v_n \) and \( v_{n-1} \) have the same color.

To get a coloring of the first type, it suffices to properly color the induced subgraph on \( \{v_1, \ldots, v_{n-1}\} \), which is \( S_{n-1} \), by \( k \) colors and then color \( v_n \) by a color different from the color of \( v_{n-1} \). Thus the number of colorings of this type is \( p_k \cdot x \cdot f_{n-1}(x) \).

In a similar way, for obtaining a coloring of the second type, one can properly color the induced subgraph on \( \{v_1, \ldots, v_{n-2}\} \), which is \( S_{n-2} \), by \( k \) colors and then color \( v_{n-1} \) and \( v_n \) by a color that is different from that of \( v_{n-2} \) (note that \( v_{n-2}, v_{n-1}, v_n \) form a cycle). Thus the number of the colorings of the second type is \( p_k \cdot x \cdot f_{n-2}(x) \).

Therefore the number of proper \( k \)-colorings of \( S_n \) is \( (k-1)(f_{n-1}(k) + f_{n-2}(k)) \) and we have the following recurrence relation for \( n \geq 3 \),

\[
f_n(x) = (x - 1)(f_{n-1}(x) + f_{n-2}(x)).
\]

To solve this recurrence, we use the method of the generating functions. Define

\[
T(x, y) = \sum_{n=1}^{\infty} f_n(x)y^n
\]

Using the recurrence relation and initial conditions, we have

\[
T(x, y) = xy + x^2y^2 + (x-1)\sum_{n=3}^{\infty} (f_{n-1}(x) + f_{n-2}(x))y^n
\]

\[
= xy + x^2y^2 + (x-1)y(T(x, y) - xy) + (x-1)y^2T(x, y)
\]

which gives

\[
T(x, y) = \frac{xy(y+1)}{1-(x-1)y(y+1)}.
\]
Now, Since
\[
\frac{1}{1 - (x - 1)y(y + 1)} = \sum_{i=0}^{\infty} (x - 1)^i y^i (y + 1)^i,
\]
we get
\[
T(x, y) = x \sum_{i=1}^{\infty} (x - 1)^{i-1} y^i (y + 1)^i
\]
and the coefficient of $y^n$ in $T(x, y)$, which is $f_n(x)$, equals
\[
\sum_{i=1}^{n} \binom{i}{n-i} x(x - 1)^{i-1},
\]
as desired.

Next, we introduce the unique strongly connected tournament which has the maximum number of proper $k$-colorings for each $k > 1$, among all tournament of order $n$. Let $D_n$ be the tournament with the vertex set $\{v_1, \ldots, v_n\}$ such that for $1 \leq i < j \leq n$, where $i \neq 1$ or $j \neq n$, the edge between $v_i$ and $v_j$ is oriented from $v_i$ to $v_j$, and the edge between $v_1$ and $v_n$ is oriented from $v_n$ to $v_1$. Clearly, $D_n$ is strongly connected.

For a digraph $D$, $u, v \in V(D)$ and a positive integer $k$, let $P_{u\rightarrow v}(D; k)$ be the number of proper $k$-colorings of $D$ such that $u$ and $v$ have the same colors, and $P_{u\not\rightarrow v}(D; k)$ be the number of proper $k$-colorings of $D$ such that $u$ and $v$ have different colors.

The next lemma is an obvious observation about the chromatic polynomial of $D_n$.

**Lemma 2.** For two positive integers $n, k$, $P_{v_1\rightarrow v_n}(D_n; k) = k(k - 1)^{n-2}$, $P_{v_1\not\rightarrow v_n}(D_n; k) = k^{n-1}(k - 1)$. Moreover $P(D_n; k) = k(k - 1)^{n-2} + k^{n-1}(k - 1)$.

**Lemma 3.** Let $T$ be a strongly connected tournament which has an edge $e = uv$ such that every directed cycle of $T$ contains $e$. Then $T = D_n$, $u = v_n$ and $v = v_1$.

**Proof.** Let $T' = D - e$. By assumption, $T'$ is an acyclic digraph, so there is a topological ordering of the vertices of $T'$ like $v_1, \ldots, v_n$. Since $T$ is a strongly connected tournament, $e$ should be an edge from $v_n$ to $v_1$ which implies that $T = D_n$. 

**Lemma 4.** Let $T$ be a strongly connected digraph of order $n$ and $u, v$ be two distinct vertices of $T$. Then for every positive integer $k$, $P_{u\not\rightarrow v}(T; k) \leq k^{n-1}(k - 1)$, and the equality holds if and only if $T = D_n$, $u = v_n$ and $v = v_1$. 

10
Proof. Clearly $P_{u \not\approx v}(T; k) \leq k^{n-1}(k - 1)$. Now, suppose that there exist $k > 1$ such that $P_{u \not\approx v}(T; k) \leq k^{n-1}(k - 1)$. If every cycle of $T$ contains $uv$, then by Lemma 8 $T = D_n$, $u = v_n$ and $v = v_1$. Assume on the contrary that there is a cycle in $T$ not containing $uv$. It can be seen that there is a cycle $C$ in $T$ that does not contain either $u$ or $v$. Let $v \notin V(C)$ and $|V(C)| = r$. The number of proper $k$-colorings of $C$ are $k^r - k$ and there are at most $(k - 1)k^{n-r-1}$ to extend a $k$-proper coloring of $C$ to $T$. Therefore,

$$k^{n-1}(k - 1) = P_{u \not\approx v}(T; k) \leq (k^r - r)(k - 1)k^{n-r-1},$$

which implies that $k^r \leq k^r - k$, a contradiction. \hfill \Box

**Theorem 10.** Let $T \neq D_n$ be a strongly connected tournament and $k > 1$ be an integer. Then $P(T; k) < P(D_n; k)$.

Proof. The proof is by induction on $n$. For $n = 1, 2, 3$ there is no strongly connected tournament of order $n$ other than $D_n$. So let $n \geq 4$. Since $T$ is strongly connected, there is a cycle $C$ of length $n - 1$ in $T$. Let $V(T) \setminus V(C) = \{v\}$. Since $T$ is strongly connected, there is a triangle $vuw$ containing $v$. Now,

$$P_{u \approx w}(T; k) \leq (k - 1)P_{u \approx w}(T - v; k),$$

$$P_{u \not\approx w}(T; k) \leq kP_{u \not\approx w}(T - v; k).$$

Hence we find that

$$P(T; k) \leq (k - 1)P(T - v; k) + P_{u \not\approx w}(T - v; k).$$

By the induction hypothesis and Lemma 2 we have,

$$P(T - v, k) \leq P(D_{n-1}, k) = k(k - 1)^{n-3} + k^{n-2}(k - 1),$$

and by Lemma 4 one can see that,

$$P_{u \not\approx v}(T - v; k) \leq k^{n-2}(k - 1).$$

So the following holds:

$$P(T; k) \leq k(k - 1)^{n-2} + k^{n-1}(k - 1) = P(D_n; k).$$

Now, let $P(T; k) = P(D_n; k)$. Thus

$$P_{u \approx w}(T; k) = (k - 1)P_{u \approx w}(T - v; k),$$

$$P_{u \not\approx w}(T; k) = kP_{u \not\approx w}(T - v; k).$$

Then we have

$$P(T; k) < P(D_n; k).$$
Lemma 4 and the last equality imply that $T - v$ is $D_{n-1}$, $w = v_1$ and $u = v_{n-1}$. By the first equality, each proper $k$-coloring of $T - v$ in which $u$ and $w$ have the same color $c$ can be extended to a coloring of $T$ by assigning an arbitrary color different from $c$ to $v$. Suppose that there is a triangle $vu'w'$ such that $u' \neq u$ and $w' \neq w$. Choose a proper $k$-coloring of $T - v$ in which $u$ and $w$ have color 1 and $u'$ and $w'$ have color 2. To extend this coloring to a proper $k$-coloring for $T$, we have at most $k - 2$ choices for the color of $v$, contrary to the assumption. Thus there is no such triangle and it can be easily seen that $T = D_n$. 

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