Stable forms and special metrics

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March 28, 2022

Dedicated to the memory of Alfred Gray.

Abstract

We show how certain diffeomorphism-invariant functionals on differential forms in dimensions 6, 7 and 8 generate in a natural way special geometrical structures in these dimensions: metrics of holonomy $G_2$ and $Spin(7)$, metrics with weak holonomy $SU(3)$ and $G_2$, and a new and unexplored example in dimension 8. The general formalism becomes a practical tool for calculating homogeneous or cohomogeneity one examples, and we illustrate this with some newly discovered examples of $Spin(7)$ and $G_2$ metrics.

1 Introduction

One of Alfred Gray’s most original concepts was that of weak holonomy [G]. This was an idea clearly ahead of its time, as became evident in the later work on Killing spinors of Baum et al. [B], setting it in a natural context. We shall give here another natural approach to both weak holonomy
and special holonomy in low dimensions through the use of certain invariant functionals of differential forms. This provides both a canonical setting for these structures and sometimes an effective means of finding them.

Our starting point is the question: “What is a non-degenerate form?” Symplectic geometry is the geometry determined by a closed non-degenerate 2-form where non-degeneracy means $\omega^m \neq 0$. Another way to describe such genericity at each point is to note that the orbit of a nondegenerate $\omega \in \Lambda^2 V^*$ under the natural action of $GL(V)$ on $\Lambda^2 V^*$ is open. We shall say in general that $\rho \in \Lambda^p V^*$ is stable if it lies in an open orbit. (The use of this word rather than “non-degenerate” avoids possible confusion: a 2-form in odd dimensions may be stable but is always degenerate as a bilinear form. Also, $\rho$ is stable in the sense of deformation invariance: all forms in a neighbourhood of $\rho$ are $GL(V)$-equivalent to $\rho$).

When does stability occur? Clearly not very often since $\dim GL(V) = n^2$ is usually much smaller than $\dim \Lambda^p V^* = n!/p!(n - p)!$. The result (over the complex numbers) is classical [R], [S], [Gu1], [Gu2] but Robert Bryant explained to me all the real cases. Apart from the obvious case of $p = 1, 2$ there are essentially only three more, where $p = 3$ and $n = 6, 7, 8$. In these three cases the stabilizer subgroup of $\rho \in \Lambda^p V^*$ in $GL(V)$ is a real form of one of the complex groups

$$SL(3) \times SL(3), \quad G_2, \quad PSL(3)$$

respectively. We shall be concerned here only with the real forms $SL(3, C)$ and the compact groups $G_2$ and $PSU(3)$. Thus if a manifold $M^n$ admits a global stable $p$-form, it has a $G$-structure where $G$ is one of these groups.

Note that if $GL(V)$ has an open orbit in $\Lambda^p V^*$ then it also does on the dual space $\Lambda^p V \cong \Lambda^{n-p} V^* \otimes \Lambda^n V$. Since for $p \neq 0$ the scalars act non-trivially, there is then an open orbit on $\Lambda^{n-p} V^*$. Thus if we are in a dimension $n$ where stable $p$-forms exist, we can also consider stable $(n - p)$-forms.

The three stabilizers of $\rho$ above, as well as the symplectic group, each preserve a volume element $\phi(\rho) \in \Lambda^n V^*$. Thus if we have a compact oriented manifold $M^n$ and a $p$-form $\rho$ which is everywhere stable, we can integrate $\phi(\rho)$ to obtain a volume $V(\rho)$. Openness of the orbit implies that nearby forms are also stable, so that the volume functional is defined and smooth on an open set of forms.
We now set up a number of variational problems involving this functional. First we consider a critical point of $V(\rho)$ restricted to a fixed cohomology class of closed forms in $H^p(M, \mathbb{R})$, performing a non-linear version of Hodge theory. If $\rho$ is everywhere stable we find the following structures:

- for $n = 2m$ and $p = 2$ or $2m - 2$, a symplectic manifold,
- for $n = 6$ the structure of a complex 3-manifold with trivial canonical bundle (this is described in some detail in [H]),
- for $n = 7$ and $p = 3$ or 4 a Riemannian manifold with holonomy $G_2$.

The critical points for $n = 8$ and $p = 3$ or 5 form a class of geometric structures which is largely unexplored (though I have benefited from Robert Bryant’s thoughts on these). We study them briefly in Section 4, showing that they admit a solution to the Rarita-Schwinger equations, which imposes constraints on the Ricci tensor. The lack of concrete compact examples beyond the 8-manifold $SU(3)$ itself is currently a stumbling block in taking the analysis of these further, but the fact that they arise from the same variational origins as $G_2$-manifolds suggests that they ought to exist in abundance.

To continue with other variational characterizations, we note that there is a canonically defined indefinite quadratic form

$$Q(d\alpha) = \int_M \alpha \wedge d\alpha$$

on the space of exact 4-forms $d\alpha$ on a 7-manifold $M$. Thus $Q$ defines an indefinite metric on the closed 4-forms in a fixed cohomology class. We then find:

- the stable critical points of $V(\rho)$ on the trivial cohomology class, subject to the constraint $Q(d\alpha) = \text{const}$ define a 7-manifold with a weak holonomy $G_2$ structure,
- the gradient flow of $V(\rho)$ on a fixed degree 4 cohomology class yields a Riemannian metric of holonomy $Spin(7)$ on $M \times \mathbb{R}$.  

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Finally, consider a 6-manifold $M$ and a stable closed 3-form $\rho$ together with a stable closed 4-form $\sigma$. We note that the spaces of exact 3-forms $d\alpha$ and exact 4-forms $d\beta$ on a 6-manifold are formally dual to each other via the pairing
\[ \langle d\alpha, d\beta \rangle = \int_M \alpha \wedge d\beta. \]

We find

- for $\rho$ and $\sigma$ exact, the stable critical points of $3V(\rho) + 8V(\sigma)$ subject to the constraint $\langle \sigma, \rho \rangle = \text{const}$ define a manifold with weak holonomy $SU(3)$,

- the pairing defines a formal symplectic structure on the product of a degree 3 and degree 4 cohomology class and the Hamiltonian flow of the functional $V(\rho) - 2V(\sigma)$, with an appropriate initial condition, generates a Riemannian metric with holonomy $G_2$ on $M \times \mathbb{R}$.

(The particular coefficients here are a computational convenience and have no geometrical meaning.)

The formal variational setting of all these special geometries is perfectly general – any such structure appears in this way – but it also has a practical value. In the homogeneous or cohomogeneity one situation we merely set up the same variational equations using invariant forms and easily derive the corresponding equations. We demonstrate this in the case of $S^7$ by deriving the equations recently used by Čvetic et al. [C] to find a new $Spin(7)$ metric on $\mathbb{R}^8$, and in the work of Brandhuber et al. [Br] for a new example of a metric of holonomy $G_2$.

The author wishes to thank the Universidad Autónoma, Madrid and the Programa Catédras Fundación Banco de Bilbao y Vizcaya for support during the preparation of this paper.

2 The linear algebra of stable forms

Let $\rho \in \Lambda^p V^*$ be stable, in the sense described above, i.e. it lies in an open orbit $U$ of $GL(V)$. We consider the cases:
1. \( \dim V = 2m, p = 2 \) or \( 2m - 2 \): stabilizer \( Sp(2m, \mathbb{R}) \)

2. \( \dim V = 6, p = 3 \): stabilizer \( SL(3, \mathbb{C}) \)

3. \( \dim V = 7, p = 3 \) or \( 4 \): stabilizer \( G_2 \)

4. \( \dim V = 8, p = 3 \) or \( 5 \): stabilizer \( PSU(3) \)

We see that each stabilizer preserves a volume form: \( G_2 \) and \( PSU(3) \) are compact so this is the volume form of an invariant positive definite inner product on \( V \). The symplectic group fixes the Liouville volume and \( SL(3, \mathbb{C}) \) preserves a complex 3-form \( \Omega \) and hence the real 6-form \( i\Omega \wedge \bar{\Omega} \). In the appendix we give the concrete expression of this form in each case. It is algebraically determined by \( \rho \) and smooth on \( U \).

The volume form \( \phi(\rho) \) associated to \( \rho \) defines a \( GL(V) \)-invariant map

\[
\phi : U \to \Lambda^n V^*.
\]

Applying invariance to the action of the scalar matrices, we see that

\[
\phi(\lambda^p \rho) = \lambda^n \phi(\rho)
\]

so that \( \phi \) is homogeneous of degree \( n/p \).

The derivative of \( \phi \) at \( \rho \) is an invariantly defined element of \( (\Lambda^p V^*)^* \otimes \Lambda^n V^* \). Since \( (\Lambda^p V^*)^* \cong \Lambda^{n-p} V^* \otimes \Lambda^n V \), the derivative lies in \( \Lambda^{n-p} V^* \) so there is a unique element \( \hat{\rho} \in \Lambda^{n-p} V^* \) for which

\[
D\phi(\hat{\rho}) = \hat{\rho} \wedge \hat{\rho}
\]

Taking \( \hat{\rho} = \rho \), Euler’s formula for a homogeneous function gives

\[
\hat{\rho} \wedge \rho = \frac{n}{p} \phi(\rho)
\]

**Example:** If \( \dim V = 2m \) and \( \omega \in \Lambda^2 V^* \) is stable (meaning non-degenerate here), then we take the Liouville volume form

\[
\phi(\omega) = \frac{1}{m!} \omega^m \in \Lambda^{2m} V^*
\]
This is clearly homogeneous in $\omega$ of degree $m = 2m/2 = n/p$. Differentiating, we see that

$$D\phi(\dot{\omega}) = \frac{1}{(m-1)!} \dot{\omega} \wedge \omega^{m-1}$$

so that

$$\dot{\omega} = \frac{1}{(m-1)!} \omega^{m-1}.$$ 

We may also consider a stable $\rho \in \Lambda^{2m-2}V^*$, which is in the open orbit $U$ consisting of forms $\rho = \omega^m/(m-1)!$ for a non-degenerate $\omega$. In this case we have $\hat{\rho} = \omega/(m-1) \in \Lambda^2 V^*$.

The precise form of $\hat{\rho}$ is determined by seeing which elements of $\Lambda^{n-p}V^*$ are fixed by the stabilizer. The symplectic case is done in the example above. For the others we see easily that:

- for $n = 6$, $p = 3$, $\hat{\rho}$ is determined by the property that $\Omega = \rho + i\hat{\rho}$ is a complex $(3,0)$-form preserved by $SL(3, \mathbb{C})$,
- for $n = 7$, $p = 3$ or 4, $\hat{\rho} = \ast \rho$, where $\ast$ is the Hodge star operator for the inner product on $V$,
- for $n = 8$, $p = 3$ or $p = 5$, $\hat{\rho} = - \ast \rho$.

**Remark:** There is clearly a choice in what we call the volume in each case. There are conventions – the Liouville volume $\omega^m/m!$ in the symplectic case for example. For most purposes it makes no difference to the results that follow, but when we need to find a metric, as in Sections 5 and 6, we shall make a more explicit choice to aid the calculations.

### 3 Critical points

Suppose now that $M$ is a closed, oriented $n$-manifold. If $\rho \in \Omega^p(M)$ is a global $p$-form, then it is a section of $\Lambda^p T^*$. Suppose its value is stable at each point (such an assumption requires of course the reduction of the structure
group of the tangent bundle to one of the stabilizers above). Then we can define a functional by taking the total volume:

$$V(\rho) = \int_M \phi(\rho).$$

By definition of stability, nearby forms will be stable and so we can differentiate the functional. We shall set up a variational problem by considering the volume restricted to closed stable $p$-forms in a given cohomology class.

**Theorem 1** A closed stable form $\rho \in \Omega^p(M)$ is a critical point of $V(\rho)$ in its cohomology class if and only if $d\hat{\rho} = 0$.

**Proof:** Take the first variation of $V(\rho)$:

$$\delta V(\hat{\rho}) = \int_M D\phi(\hat{\rho}) = \int_M \hat{\rho} \wedge \hat{\rho}$$

from (1). But the variation is within a fixed cohomology class so $\hat{\rho} = d\alpha$. Thus

$$\delta V(\hat{\rho}) = \int_M \hat{\rho} \wedge d\alpha = \pm \int_M d\hat{\rho} \wedge \alpha$$

and the variation vanishes for all $d\alpha$ if and only if

$$d\hat{\rho} = 0.$$

$\square$

**Example:** A closed stable 2-form $\omega$ on an even-dimensional manifold is a symplectic form. The volume $V(\omega)$ is then constant on a fixed cohomology class $[\omega]$ since it is just the evaluation of the cup product:

$$V(\omega) = \frac{1}{m!} [\omega]^m [M].$$

Rather trivially a symplectic manifold appears as a critical point here. On the other hand if we take $\rho = \omega^{m-1}/(m-1)!$ to be the closed form then the condition $d\rho = 0$ is (for $m > 2$) weaker than $d\omega = 0$. The functional genuinely varies and the critical points are where

$$(m-1)d\hat{\rho} = d\omega = 0.$$

This is an alternative way of obtaining a symplectic manifold as a critical point.
Example: A 7-manifold $M$ which has either a closed stable 3-form or a closed stable 4-form which is a critical point for $V$ in its cohomology class has the structure of a Riemannian manifold with holonomy $G_2$. This follows from the theorem of M. Fernández and A. Gray [F] that the holonomy reduces to $G_2$ if and only if
\[ d\rho = d\ast\rho = 0. \]

Example: The case of $n = 6$ means that we have a complex closed locally decomposable 3-form $\rho + i\dot{\rho}$. This yields the structure of a complex 3-manifold with trivial canonical bundle as shown in [H].

The 8-dimensional case will occupy us next.

4 Eight-manifolds with $PSU(3)$ structure

Suppose that $M$ is a compact 8-manifold with a stable 3-form $\rho$ such that, with respect to the metric determined by $\rho$,
\[ d\rho = d\ast\rho = 0. \]

An example is $SU(3)$ itself, where $\rho$ is covariant constant and is a multiple of the standard bi-invariant form
\[ \rho = \text{tr}(g^{-1}dg)^3. \]

We shall never find a metric on a compact simply-connected $M^8$ whose holonomy is $PSU(3)$ other than this example, from Berger’s classification of Riemannian holonomy groups. This is not the case of weak holonomy either – there are no Killing spinors. We do however have one interesting object: a Rarita-Schwinger field. In physics terminology this is a spin $3/2$ field. We take one of the two spinor bundles (say $S^+$) and consider a spinor-valued 1-form – a section $\gamma$ of $S^+ \otimes \Lambda^1$. This satisfies the Rarita-Schwinger equation if
\[ D\gamma = 0 \]
and
\[ d^*\gamma = 0. \]
Here
\[ D : C^\infty(S^+ \otimes \Lambda^1) \to C^\infty(S^- \otimes \Lambda^1) \]
is the Dirac operator with coefficients in the bundle of 1-forms \( \Lambda^1 \) and
\[ d^* : C^\infty(S^+ \otimes \Lambda^1) \to C^\infty(S^+) \]
is the covariant \( d^* \) operator on 1-forms with coefficients in the spinor bundle \( S^+ \).

First we describe \( \gamma \) as a \( PSU(3) \)-invariant object.

**Lemma 2** Let \( S^+, S^- \) be the two spin representations and \( \Lambda^1 \) the standard vector representation of \( Spin(8) \). Then restricted to the lift \( PSU(3) \subset Spin(8) \) of the adjoint representation, these three representations are equivalent.

**Proof:** If \( \pm x_1, \ldots, \pm x_4 \) are the weights of the 8-dimensional vector representation of \( Spin(8) \), the weights of the spin representations \( S^\pm \) are
\[ \frac{1}{2}(\pm x_1 \pm x_2 \ldots \pm x_4) \quad (3) \]
where there is an even number of minus signs for \( S^+ \) and an odd number for \( S^- \).

If \( \alpha, \beta, \alpha + \beta \) are the positive roots of \( SU(3) \) then substituting
\[ x_1 = 0, \quad x_2 = \alpha, \quad x_3 = \beta, \quad x_4 = \alpha + \beta \]
we have from (3) for \( S^+ \) and \( S^- \) the same weights \( 0, \pm \alpha, \pm \beta, \pm \alpha + \beta \) as the adjoint representation. \( \square \)

**Remark:** The lemma implies a rather interesting property of the Lie algebra of \( SU(3) \). Recall that Clifford multiplication of vectors on spinors is skew adjoint and satisfies \( x^2 = -(x, x)1 \). In eight dimensions the three representations \( S^+, S^- \) and \( \Lambda^1 \) are all real and 8-dimensional (this is triality). If \( \varphi \in S^\pm \) and \( x \in \Lambda^1 \) then
\[ (x \varphi, x \varphi) = (-x^2 \varphi, \varphi) = (x, x)(\varphi, \varphi) \]
so that Clifford multiplication

$$\Lambda^1 \otimes S^+ \to S^-$$

is an orthogonal product. From the lemma all three representations are equivalent under $SU(3)$ so we must have an $SU(3)$-invariant orthogonal multiplication on the Lie algebra of $3 \times 3$ skew-hermitian matrices with trace zero. Here it is:

$$A \times B = \omega AB - \bar{\omega} BA - \frac{i}{\sqrt{3}} \text{tr}(AB)I$$

where $\omega = (1 + i\sqrt{3})/2$.

The $SU(3)$-invariant isomorphism $S^+ \cong \Lambda^1$ defines a section of $(S^+)^* \otimes \Lambda^1 \cong S^+ \otimes \Lambda^1$ and this is what we take to be $\gamma$. There is of course an equivalent section of $S^- \otimes \Lambda^1$. We now prove:

**Theorem 3** If $M$ is an 8-manifold with a $PSU(3)$ structure defined by a 3-form $\rho$ with $d\rho = d^* \rho = 0$, then the section $\gamma \in C^\infty(S^+ \otimes \Lambda^1)$ satisfies the Rarita-Schwinger equations.

**Proof:** The covariant derivative of the 3-form $\rho$ at any point can be written

$$\nabla \rho = A(\rho)$$

where $A \in \Lambda^2 \otimes \Lambda^1$ and acts on $\rho$ by identifying $\Lambda^2$ with the Lie algebra of $SO(8)$. Since $\rho$ is fixed by $SU(3)$ we may as well assume that $A \in \Lambda^2_0 \otimes \Lambda^1$ where $\Lambda^2_0$ is the orthogonal complement in $\Lambda^2$ to the Lie algebra of $SU(3) \cong \Lambda^1$. Since $\rho$ is not covariant constant, $A$ will not vanish. However, the harmonicity condition on $\rho$ will force many of its components (as representations of $SU(3)$) to vanish.

We shall index representations by their highest weight – the adjoint representation $\Lambda^1$ of $SU(3)$ has highest weight $\alpha + \beta$. If we decompose the tensor product $\Lambda^1 \otimes \Lambda^1$ into irreducible representations as in Chapter 6 of Salamon’s book [Sal], we find that $\Lambda^2_0$ is the direct sum of two 10-dimensional irreducibles $\Lambda_+^2, \Lambda_-^2$ with highest weight $2\alpha + \beta$ and $\alpha + 2\beta$ respectively. These
are interchanged under a change of orientation, just like self-dual and anti-self-dual forms in four dimensions. Similarly $V(2\alpha + \beta) \otimes V(\alpha + \beta)$ breaks up into irreducibles with highest weights

$$3\alpha + 2\beta, \; 2\alpha + 2\beta, \; 2\alpha + \beta, \; \alpha + \beta$$

and each with multiplicity one. Robert Bryant informed me of the following result:

**Lemma 4** If $d\rho = d^*\rho = 0$, the components of $A$ with highest weight $\alpha + \beta$, $2\alpha + \beta$, $\alpha + 2\beta$ and $2\alpha + 2\beta$ all vanish.

**Proof:** Since we are considering exterior powers of the Lie algebra it is convenient to think of these as spaces of left-invariant forms on the group. Then, for example, $\Lambda^2_0$ is the space of coclosed invariant 2-forms. The 3-form $\rho$ is built out of the structure constants of the Lie algebra $\mathfrak{su}(3)$ and as a consequence of this, the action of $a \in \Lambda^2$ on $\rho$ can be rewritten as $da$ if we consider $a$ as a 2-form on the group. Thus, if

$$A = \sum_i a_i \otimes e_i \in \Lambda^2_0 \otimes \Lambda^1$$

then

$$d\rho = \sum_i e_i \wedge da_i$$

and

$$d^*\rho = \sum_i \iota(e_i) da_i.$$

Now if $x_\gamma \in \mathfrak{su}(3) \otimes \mathbb{C}$ denotes a root vector for the root $\gamma$, the vector

$$A = x_\alpha \wedge x_{\alpha+\beta} \otimes x_\beta$$

lies in $V(2\alpha + 2\beta) \subset V(2\alpha + \beta) \otimes V(\alpha + \beta)$. Let $h$ be an element of the Cartan subalgebra, then because $\iota(h)dx_\alpha = \mathcal{L}_hx_\alpha = \alpha(h)x_\alpha$ etc.

$$\iota(h)d\rho = \iota(h)(x_\beta \wedge d(x_\alpha \wedge x_{\alpha+\beta})) = \beta(h)x_\beta \wedge x_{\alpha+\beta} \wedge x_\alpha.$$ 

Since we can find $h$ for which $\beta(h) \neq 0$ this shows that the irreducible representation $V(2\alpha + 2\beta) \subset \Lambda^2_+ \otimes \Lambda^1$ maps non-trivially into $\Lambda^3$ under $A(\rho)$. 

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That representation is also contained in $\Lambda^2 \otimes \Lambda^1$, interchanging the roles of $\alpha$ and $\beta$. However, since we can choose $h$ such that $\alpha(h) = 0$ and $\beta(h) \neq 0$ and vice-versa, $V(2\alpha + 2\beta)$ appears twice in $\Lambda^3$, and so if $d\rho = 0$, then both of these components in $A$ vanish.

We now work similarly with $V(\alpha + \beta) \subset V(2\alpha + \beta) \otimes V(\alpha + \beta)$ using the vector

$$A = x_\alpha \wedge x_{\alpha+\beta} \otimes x_{-\alpha}.$$  

We deduce that if $d\rho = 0$, then both of the $V(\alpha + \beta)$ components in $A$ vanish.

Now consider $V(2\alpha + \beta) = \Lambda^2_+ \subset \Lambda^2_+ \otimes \Lambda^1$. The inclusion is unique and is given by

$$a \mapsto \sum_i \mathcal{L}_{e_i} a \otimes e_i$$

for an orthonormal basis $\{e_i\}$ of the Lie algebra $\mathfrak{su}(3) \cong \Lambda^1$. In this case

$$d^* \rho = \sum_i \iota(e_i) d(\mathcal{L}_{e_i} a) = \sum_i \mathcal{L}_{e_i}^2 a - d(\iota(e_i) \mathcal{L}_{e_i} a)$$

and $a \mapsto \iota(e_i) \mathcal{L}_{e_i} a$ is an invariant map from the irreducible $\Lambda^2_+$ to the irreducible $\Lambda^1$ and thus must vanish. Hence $d^* \rho$ is the Casimir $\sum_i \mathcal{L}_{e_i}^2 a$ which is a non-zero scalar multiple of $a$. It follows that if $d^* \rho = 0$, the component $V(2\alpha + \beta)$ (and similarly $V(\alpha + 2\beta)$) in $A$ vanishes. This completes the proof of the lemma.

To return to the theorem, consider $D\gamma \in C^\infty(S^- \otimes \Lambda^1)$. The right hand side, since $S^- \cong \Lambda^1$, is a vector bundle associated to the representation $V(\alpha + \beta) \otimes V(\alpha + \beta)$. The skew part we calculated to have highest weights $2\alpha+\beta, \alpha+2\beta$ and $\alpha+\beta$. The calculation gives at the same time the symmetric part to have weights $0, \alpha + \beta$ and $2\alpha + 2\beta$. But $D\gamma$ is the image under an invariant map of $A$ which from the lemma only has components with highest weights $3\alpha + 2\beta$ and $2\alpha + 3\beta$. Since these do not occur in $S^- \otimes \Lambda^1$ we deduce that $D\gamma = 0$. Similarly, since $S^+ = V(\alpha + \beta)$, we see that $d^*\gamma = 0$.

The existence of a Rarita-Schwinger field implies the vanishing of certain components of the Ricci tensor. In a sense the Einstein equations are the integrability condition for a linear system given by the Rarita-Schwinger
operator $[\mathcal{D}]$. More concretely, given $\gamma \in C^\infty(S^+ \otimes \Lambda^1)$ we form the second covariant derivative

$$\nabla^2 \gamma \in C^\infty(S^+ \otimes \Lambda^1 \otimes \Lambda^1 \otimes \Lambda^1)$$

with components $\gamma_{ijk}$ in an orthonormal basis. Covariantly differentiating the equation $D\gamma = 0$ gives

$$\sum_j e_j \gamma_{ijk} = 0$$

and by contraction

$$\sum_{i,j} e_j \gamma_{iji} = 0. \quad (4)$$

But differentiating the equation $d^* \gamma = 0$ gives

$$\sum_{i,j} \gamma_{ij} = 0$$

and hence also

$$\sum_{i,j} e_j \gamma_{iji} = 0. \quad (5)$$

The difference of (4) and (5) is skew-symmetric in the last two indices and can therefore be rewritten in terms of the action of the curvature tensor on $S^+ \otimes \Lambda^1$. This only involves the Ricci tensor:

$$\sum_{i,j} R_{ij} e_i \gamma_j = 0 \quad (6)$$

The expression $\sum_{i,j} R_{ij} e_i \gamma_j$ in (6) defines an $SU(3)$-invariant map from the space of symmetric tensors $R_{ij}$ to $S^-$. We have seen that $S^- \cong V(\alpha + \beta)$ and this representation occurs with multiplicity one in $Sym^2 \Lambda^1$. Thus equation (6) implies the vanishing of 8 of the 36 components of the Ricci tensor.

**Remark:** One consequence of the theorem is that the operator

$$\mathcal{D} : \Omega^1(M) \to \Omega^1(M)$$

defined by the orthogonal multiplication

$$\mathcal{D} \alpha = \sum_i e_i \times \nabla_i \alpha$$
is equivalent (under the isomorphisms $\gamma$ for $S^+$ and $S^-$) to the Dirac operator. The multiplication $\alpha \times \beta$ is of course Clifford multiplication under these isomorphisms, but the Levi-Civita connections on $\Lambda^1$ on $S^\pm$ are different. Nevertheless, $D\gamma = 0$ implies that the Dirac operators correspond.

5 Constrained critical points

On a compact oriented manifold $M^n$ there is a non-degenerate pairing between the spaces of forms $\Omega^p(M)$ and $\Omega^{n-p}(M)$ defined by

$$\int_M \alpha \wedge \beta.$$ 

If $\alpha = d\gamma \in \Omega^p(M)$ is exact, then by Stokes’ theorem

$$\int_M d\gamma \wedge \beta = (-1)^p \int_M \gamma \wedge d\beta$$

which vanishes for all $\gamma$ if and only if $\beta$ is closed. We thus have a non-degenerate pairing between $\Omega^p_{exact}(M)$ and

$$\Omega^{n-p}(M)/\Omega^{n-p}_{closed}(M)$$

Since the exterior derivative $d$ maps this latter space isomorphically onto $\Omega^{n-p+1}_{exact}(M)$, then formally (in the sense of non-degenerate pairings) we can say that

$$\Omega^p_{exact}(M)^* \cong \Omega^{n-p+1}_{exact}(M)$$ (7)

Now consider $n = 7$ and $p = 4$. We have seen that a metric of holonomy $G_2$ arises from a stable critical point for the functional $V(\rho)$ on a fixed cohomology class $[\rho]$ of closed 4-forms. On the other hand since $*\rho \wedge \rho = 7\phi(\rho)/4$ and $\rho$ is both closed and coclosed, we must have $[*\rho] \cup [\rho][M] = V(\rho) \neq 0$ and in particular $[\rho] \neq 0$. For the trivial cohomology class it follows that there are no stable critical points for this functional. However, we have from (4)

$$\Omega^4_{exact}(M)^* \cong \Omega^4_{exact}(M)$$

and hence a non-degenerate quadratic form on $\Omega^4_{exact}(M)$ given by

$$Q(d\gamma) = \int_M \gamma \wedge d\gamma.$$
We now have two natural functionals on the space of exact 4-forms: $V$ and $Q$.

**Theorem 5** An exact stable 4-form $\rho$ on a compact 7-manifold is a critical point of $V(\rho)$ subject to the constraint $Q(\rho) = \text{const.}$ if and only if $\rho$ defines a metric with weak holonomy $G_2$.

**Proof:** From Theorem (1), the first variation of $V$ at $\rho = d\gamma$ is

$$\delta V(d\gamma) = \int_M \ast \rho \wedge d\gamma$$

and the first variation of the quadratic form $Q$ is

$$\delta Q(d\gamma) = 2 \int_M \dot{\gamma} \wedge \rho.$$

Thus, introducing a Lagrangian multiplier, the constrained critical point is given by

$$d(\ast \rho) = \lambda \rho$$

and from [Fr], this is equivalent to the structure of a manifold with weak holonomy $G_2$ (sometimes called a nearly parallel $G_2$ structure). $\square$

**Remark:** A 7-manifold with weak holonomy $G_2$ is an Einstein manifold with positive scalar curvature. There are many examples, such as 3-Sasakian manifolds (see [Bo]) and their squashed versions (see [Fr]).

Next consider the case of $n = 6$ and $p = 3$. Then (7) tells us

$$\Omega_{\text{exact}}^3(M) \ast \cong \Omega_{\text{exact}}^4(M)$$

so that the spaces of exact 3-forms and exact 4-forms are formally dual to each other. The pairing for an exact 3-form $\rho = d\alpha$ and an exact 4-form $\sigma = d\beta$ is

$$\langle \rho, \sigma \rangle = \int_M \alpha \wedge \sigma = -\int_M \rho \wedge \beta$$

(8)

We shall consider in 6 dimensions a variational problem involving the three functionals $V(\rho), V(\sigma)$ and $\langle \rho, \sigma \rangle$, but before proving a theorem, let us see
how $\rho$ and $\sigma$ define a reduction of the structure group of the tangent bundle of $M^6$ to $SU(3)$. The 3-form $\rho$, being stable, provides a reduction to $SL(3,\mathbb{C})$ and $\sigma = \omega^2/2$ to $Sp(6,\mathbb{R})$. The group $SU(3)$ is an intersection of these two but there are two compatibility conditions for $\rho$ and $\sigma$ to achieve this. The first is

$$\omega \wedge \rho = 0 \quad (9)$$

From the point of view of $\rho$ and the complex structure it describes, this says that $\omega$ is of type (1,1). Since $\rho + i\hat{\rho}$ is a (3,0) form the property $\omega \wedge \hat{\rho} = 0$ follows from (8). From the point of view of the symplectic form $\omega$, the equation says that $\rho$ is primitive. For a stable 3-form $\rho$ and a stable 4-form $\sigma = \omega^2/2$, we shall say that the pair $(\rho, \sigma)$ is of positive type if the almost complex structure $I$ determined by $\rho$ (see Appendix (8.2)) makes $\omega(X, IX)$ a positive definite form. This is clearly an open condition. If $\omega \wedge \rho = 0$, the condition of positive type means that the Hermitian form defined by $\omega$ is positive definite.

The second condition is

$$\phi(\rho) = c\phi(\sigma) \quad (10)$$

For a constant $c$. This says that the complex 3-form $\rho + i\hat{\rho}$ has constant length relative to the Hermitian metric.

Since $\rho$ and $\sigma$ satisfying these conditions define a metric, it is natural to normalize the constant $c$ above so that the volume forms $V(\rho)$ and $V(\sigma)$ are fixed multiples of the metric volume form. The most convenient way to do this is to see the normal form of the $G_2$ 3-form $\varphi$ and its dual $*\varphi$ in terms of an orthonormal basis as in [Sa1]:

$$\varphi = e_7(e_5e_6 + e_1e_4 + e_3e_2) + (e_1e_2e_3 - e_3e_4e_5 + e_1e_3e_6 - e_4e_2e_6)$$

$$*\varphi = e_7(e_3e_4e_6 - e_1e_2e_6 + e_1e_3e_5 - e_4e_2e_5) + (e_1e_3e_4e_2 + e_5e_6e_2e_3 + e_5e_6e_4e_1).$$

The 6-dimensional geometry is defined by $\omega = e_5e_6 + e_1e_4 + e_3e_2$ and $\rho = e_1e_2e_3 - e_3e_4e_5 + e_1e_3e_6 - e_4e_2e_6$, and then $\hat{\rho} = e_3e_4e_6 - e_1e_2e_6 + e_1e_3e_5 - e_4e_2e_5$. We have from the above expressions

$$\phi(\sigma) = \frac{1}{6}\omega^3 = \frac{1}{4}\hat{\rho} \wedge \rho = \frac{1}{2}\phi(\rho) \quad (11)$$

and

$$\varphi = e_7 \wedge \omega + \rho, \quad *\varphi = e_7 \wedge \hat{\rho} - \sigma \quad (12)$$

We now prove the theorem.
Theorem 6 A pair \((\rho, \sigma)\) of exact, stable forms of positive type on a compact 6-manifold forms a critical point of \(3V(\rho) + 8V(\sigma)\) subject to the constraint \(\langle \rho, \sigma \rangle = \text{const.}\) if and only if \(\rho\) and \(\sigma\) define a metric with weak holonomy \(SU(3)\).

Proof: From Theorem (1), the first variation of \(3V(\rho) + 8V(\sigma)\) is

\[
3 \int_M \dot{\rho} \wedge \dot{\rho} + 4 \int_M \omega \wedge \dot{\sigma}.
\]

The first variation of \(P = \langle \rho, \sigma \rangle\) is

\[
\delta P(\dot{\rho}, \dot{\sigma}) = \int_M \dot{\rho} \wedge \beta + \int_M \dot{\sigma} \wedge \alpha
\]

where \(\rho = d\alpha\) and \(\sigma = d\beta\). Using Stokes' theorem and a Lagrange multiplier 12\(\lambda\) we find the constrained critical point to be given by the equations 3\(d\dot{\rho} = -12\lambda \sigma = -6\lambda \omega^2\) and 4\(d\omega = 12\lambda \rho\), i.e.

\[
d\dot{\rho} = -2\lambda \omega^2 \quad (13)
\]

\[
d\omega = 3\lambda \rho \quad (14)
\]

The compatibility conditions (9), (10) actually follow from these equations. From (14) we have

\[
\lambda \omega \wedge \rho = \frac{1}{3} \omega \wedge d\omega = \frac{1}{6} d(\omega^2) = 0
\]

since \(\sigma = \omega^2/2\) is closed. Moreover, from (13)

\[
2\lambda \omega^3 = -\omega \wedge d\dot{\rho} = -d(\omega \wedge \dot{\rho}) + d\omega \wedge \dot{\rho} = 3\lambda \rho \wedge \dot{\rho}
\]

using (14) and the fact that \(\omega \wedge \dot{\rho} = 0\), which, as we have seen, follows from \(\omega \wedge \rho = 0\). Thus \(\omega^3 = 3\dot{\rho} \wedge \rho/2\) as in (11).

The equations (13), (14) give a metric of weak holonomy \(SU(3)\), sometimes called a nearly Kähler metric. To see this, consider the 3-form on \(M \times \mathbb{R}\)

\[
\varphi = \frac{r^2}{\lambda} dr \wedge \omega + r^3 \rho.
\]
This is stable, and from (14) closed. Moreover, by comparing with the normal forms above, we see that the $G_2$ metric it defines is
\[(dr/\lambda)^2 + r^2g\]
where $g$ is the $SU(3)$ metric. From (12) we have
\[\ast \varphi = \frac{r^3}{\lambda} dr \wedge \hat{\rho} - \frac{r^4}{2} \omega^2\]
and then from (13), $d \ast \varphi = 0$. It follows that the cone metric above is a Riemannian metric of holonomy $G_2$. However, from [Ba], this implies that $M^6$ is nearly Kähler. \hfill \Box

**Remark:** The particular coefficients of $V(\rho)$ and $V(\sigma)$ in the theorem are not crucial. Any two positive numbers will, after a rescaling of metrics, give the same result.

**Remark:** Manifolds with weak holonomy $SU(3)$ structure are (currently) far less plentiful than their 7-dimensional counterparts with weak holonomy $G_2$. They are again Einstein manifolds with positive scalar curvature, but the only known examples are $S^6$, the twistor spaces of $S^4$ and $\mathbb{C}P^2$ and $S^3 \times S^3$.

### 6 Evolution equations

We return to a 7-manifold and consider a fixed cohomology class $\mathcal{A}$ of closed 4-forms. This is an infinite dimensional affine space, whose tangent space at each point is naturally isomorphic to $\Omega^4_{\text{exact}}(M)$. The quadratic form $Q$ defined above provides an indefinite metric on $\mathcal{A}$:
\[(\hat{\rho}, \hat{\rho}) = \int_M \gamma \wedge d\gamma\]
where $\hat{\rho} = d\gamma$.

In this setting the functional $V(\rho)$ defines a gradient vector field $X$ on $\mathcal{A}$, and the critical points of $V(\rho)$ are the zeros of $X$. More generally, we can consider the gradient flow, and we find:
Theorem 7  Let \( M \) be a closed 7-manifold and suppose \( \rho(t) \) is a closed stable 4-form which evolves via the gradient flow of the functional \( V(\rho) \) restricted to a cohomology class in \( H^4(M, \mathbb{R}) \). Then the 4-form

\[
\varphi = dt \wedge * \rho + \rho
\]
defines a metric with holonomy \( \text{Spin}(7) \) on the 8-manifold \( N = M \times (a, b) \) for an interval \((a, b)\). Conversely, if \( N \) is an 8-manifold with holonomy \( \text{Spin}(7) \), foliated by equidistant compact hypersurfaces diffeomorphic to \( M \), the restriction of the defining 4-form to each hypersurface evolves as the gradient flow of \( V(\rho) \).

Proof:  The gradient flow is the solution of the equation

\[
\frac{dx}{dt} = X
\]
where \((X, Y) = D\phi(Y)\) for any vector field \( Y \). In our case we have \( x \) described by a 4-form \( \rho(t) \) so using the inner product above with \( Y = d\gamma \),

\[
\left( \frac{dx}{dt}, Y \right) = \int_M \frac{\partial \rho}{\partial t} \wedge \gamma = D\phi(Y) = \int_M *\rho \wedge d\gamma
\]
since \( D\phi = *\rho \). But this equation holds for all 3-forms \( \gamma \) and thus yields the gradient flow equation

\[
\frac{\partial \rho}{\partial t} = d(*\rho) \tag{15}
\]
But \( d\rho = 0 \), so we obtain

\[
d(dt \wedge *\rho + \rho) = 0
\]
and from [Sal] we see that the 4-form \( dt \wedge *\rho + \rho \) defines a metric with holonomy \( \text{Spin}(7) \). In this metric \( ||dt||^2 = 1 \).

Conversely, if a \( \text{Spin}(7) \) manifold is foliated by equidistant hypersurfaces, defining the function \( t \) to be the distance to a fixed hypersurface \( M \), we can write the defining 4-form in the form \( dt \wedge *\rho + \rho \) and the statement that this is closed is equivalent to the gradient flow equation \((15)\). \( \square \)

We can do something similar with the 6-dimensional case. Here we take \( A \) to be a cohomology class in \( H^3(M, \mathbb{R}) \) and \( B \) to be a class in \( H^4(M, \mathbb{R}) \).
Then the tangent space at each point of the product of affine spaces $\mathcal{A} \times \mathcal{B}$ is naturally isomorphic to

$$\Omega^3_{\text{exact}}(M) \times \Omega^4_{\text{exact}}(M)$$

and (7) shows that the pairing $\langle \rho, \sigma \rangle$ defines formally a symplectic structure on $\mathcal{A} \times \mathcal{B}$:

$$\omega((\rho_1, \sigma_1), (\rho_2, \sigma_2)) = \langle \rho_1, \sigma_2 \rangle - \langle \rho_2, \sigma_1 \rangle.$$

**Remark:** The pairing between $V$ and $V^*$ to define a symplectic structure on $V \times V^*$ can also be used to define an indefinite metric:

$$((\rho_1, \sigma_1), (\rho_2, \sigma_2)) = \langle \rho_1, \sigma_2 \rangle + \langle \rho_2, \sigma_1 \rangle.$$

The gradient flow of a function of the form $f(\rho) + g(\sigma)$ is then equivalent to the Hamiltonian flow of $f(\rho) - g(\sigma)$. With $V = \Omega^3_{\text{exact}}(M)$, the gradient flow shows, as a particular consequence, the relationship between weak holonomy $SU(3)$ and conical $G_2$ metrics as in [Ba]. Nevertheless, the Hamiltonian interpretation of the equations has certain advantages over the gradient viewpoint, which is why we adopt it in the next theorem.

**Theorem 8** Let $\mathcal{A} \in H^3(M, \mathbb{R})$ and $\mathcal{B} \in H^4(M, \mathbb{R})$ be cohomology classes and $(\rho, \sigma) \in \mathcal{A} \times \mathcal{B}$ be stable forms of positive type which evolve via the Hamiltonian flow of the functional $H = V(\rho) - 2V(\sigma)$. If for time $t = t_0$, $\rho$ and $\sigma$ satisfy the compatibility conditions $\omega \wedge \rho = 0$ and $\phi(\rho) = 2\phi(\sigma)$ then the 3-form

$$\varphi = dt \wedge \omega + \rho$$

(where $\sigma = \omega^2/2$) defines a metric with holonomy $G_2$ on the 7-manifold $M \times (a, b)$. Conversely, if $N$ is a 7-manifold with holonomy $G_2$, foliated by equidistant compact hypersurfaces diffeomorphic to $M$, the restriction of the defining closed forms $\rho, \sigma$ to each hypersurface evolves as the Hamiltonian flow of $H$.

**Proof:** As before, we have the derivative of the Hamiltonian $H = V(\rho) - 2V(\sigma)$ given by

$$DH(d\alpha, d\beta) = \int_M \hat{\rho} \wedge d\alpha - \int_M \omega \wedge d\beta$$
and the Hamiltonian vector field $X$ is defined by $\iota(X)\omega = dH$ so in our case we have
$$\int_M \dot{\rho} \wedge \beta - \int_M \dot{\sigma} \wedge \alpha = \int_M \dot{\rho} \wedge d\alpha - \int_M \omega \wedge d\beta$$
and this gives the equations:
$$\frac{\partial \rho}{\partial t} = d\omega$$  \hspace{1cm} (16)
$$\frac{\partial \sigma}{\partial t} = \omega \wedge \frac{\partial \omega}{\partial t} = -d\dot{\rho}$$  \hspace{1cm} (17)

We shall see that if the compatibility conditions (9), (10) between $\omega$ and $\rho$ hold for $t = t_0$, then they hold for all subsequent time.

First consider the condition $\omega \wedge \rho = 0$. This can be viewed as the vanishing of the moment map for the natural action of Diff($M$) on the symplectic manifold $A \times B$. To see this, for a vector field $X$ on $M$, consider the function
$$\mu_X(\rho, \sigma) = \int_M \iota(X)\sigma \wedge \rho = \int_M \iota(X)\omega \wedge \omega \wedge \rho$$  \hspace{1cm} (18)

We have
$$d\mu_X(\dot{\rho}, \dot{\sigma}) = \int_M \iota(X)\dot{\sigma} \wedge \rho + \int_M \iota(X)\sigma \wedge \dot{\rho}$$

But the first term is
$$-\int_M \dot{\sigma} \wedge \iota(X)\rho$$

since $\dot{\sigma} \wedge \rho = 0$. Thus
$$d\mu_X(\dot{\rho}, \dot{\sigma}) = \int_M \iota(X)\rho \wedge \dot{\sigma} + \int_M \iota(X)\sigma \wedge \dot{\rho}$$  \hspace{1cm} (19)

Now since $\rho$ and $\sigma$ are closed,
$$\mathcal{L}_X(\rho, \sigma) = (d\iota(X)\rho, d\iota(X)\sigma)$$
so (19) can be written
$$d\mu_X(\dot{\rho}, \dot{\sigma}) = \langle \mathcal{L}_X(\rho, \sigma), (\dot{\rho}, \dot{\sigma}) \rangle$$
using the definition of the symplectic form. We deduce that $\mu_X$ is the moment map for Diff($M$), evaluated on $X$. Since $\omega$ is non-degenerate, from (18) $\mu_X$ vanishes for all $X$ if and only if $\omega \wedge \rho = 0$. 

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Now since the functional $H$ is diffeomorphism invariant, $H$ Poisson commutes with all the functions $\mu_X$. Hence the Hamiltonian flow of $H$ is tangential to the zero set of all the functions $\mu_X$, i.e. the space of pairs $(\rho, \sigma) \in A \times B$ such that $\omega \wedge \rho = 0$. Thus if $\omega \wedge \rho = 0$ holds for $t = t_0$, it holds for all time. Note that it follows then that $\omega \wedge \hat{\rho} = 0$ for all time too.

Next consider the second compatibility condition (10). The form $\hat{\rho}$ is defined by the derivative of $\phi$, $D\phi(\hat{\rho}) = \hat{\rho} \wedge \dot{\rho}$, so the derivative of $\hat{\rho}$ can be expressed via $D^2\phi$. The volume $\phi(\rho)$ is homogeneous of degree 2, so its derivative is homogeneous of degree one and hence

$$D^2\phi(\rho, \hat{\rho}) = D\phi(\dot{\rho}) = \dot{\rho} \wedge \rho \quad (20)$$

We have

$$\frac{\partial}{\partial t} \wedge \rho = D^2\phi(\rho, \dot{\rho}) = D\phi(\frac{\partial \rho}{\partial t}) = \dot{\rho} \wedge \frac{\partial \rho}{\partial t}$$

So from (20)

$$\frac{\partial}{\partial t} (\hat{\rho} \wedge \rho) = 2\dot{\rho} \wedge \frac{\partial \rho}{\partial t} = 2\dot{\rho} \wedge d\omega$$

from (10). But $\omega \wedge \dot{\rho} = 0$, and hence $\hat{\rho} \wedge d\omega = -d\dot{\rho} \wedge \omega$ Thus

$$\frac{\partial}{\partial t} (\hat{\rho} \wedge \rho) = -2d\dot{\rho} \wedge \omega = 2\omega^2 \frac{\partial \omega}{\partial t} = \frac{2}{3} \frac{\partial}{\partial t} \omega^3$$

and so if $\phi(\rho) = 2\phi(\sigma)$ at $t = t_0$ then it holds for all $t$.

The evolution equation thus preserves the $SU(3)$ geometry on $M$, and we define the 3-form

$$\varphi = dt \wedge \omega + \rho$$

From (16), $d\varphi = 0$. From (12), using the $G_2$ metric defined by $\varphi$, we have

$$\ast \varphi = dt \wedge \hat{\rho} - \sigma$$

and from (17), $d \ast \varphi = 0$. Thus $\varphi$ defines a metric of holonomy $G_2$ on $M \times (a, b)$ for some interval.

\begin{proof}

7 Examples

It is a well-known principle that if we are looking for a $G$-invariant critical point of an invariant function, we need only consider critical points of the

\end{proof}
same function restricted to the fixed point set of $G$. Thus if $G$ acts on $M$, a $G$-invariant special metric of any of the types we are considering can be found by restricting the volume functionals to $G$-invariant forms. This applies in particular when $M$ is homogeneous under $G$, in which case we are reduced to a finite-dimensional variational problem.

Since metrics of holonomy $G_2$ and $Spin(7)$ have zero Ricci tensor, we have no interesting compact homogeneous examples. We can nevertheless apply the principle to compact spaces for weak holonomy $G_2$ and $SU(3)$, and also to the evolution equation to find non-compact examples with holonomy $Spin(7)$ and $G_2$.

### 7.1 A 7-dimensional example

We consider $M^7 = S^7$ as a principal $SU(2)$ bundle over $S^4$, and $G = SO(5)$ acting transitively on it with stabilizer $SU(2)$. All the relevant cohomology classes are trivial here, so we are dealing with exact forms, which must be built out of the three invariant 1-forms $\alpha_1, \alpha_2, \alpha_3$ which are the components of the $SU(2)$-connection form, and the components $\omega_1, \omega_2, \omega_3$ of the curvature. These are defined by:

$$d\alpha_1 + 2\alpha_2 \alpha_3 = \omega_1$$

and similar expressions for $\alpha_2, \alpha_3$. The curvature forms themselves satisfy the Bianchi identity

$$d\omega_1 = 2(\omega_2 \alpha_3 - \omega_3 \alpha_2) \text{ etc.}$$

A basis for the exact invariant 4-forms is provided by

$$u_1 = d(\alpha_1 \omega_1), u_2 = d(\alpha_2 \omega_2), u_3 = d(\alpha_3 \omega_3), u_4 = d(\alpha_1 \alpha_2 \alpha_3)$$

and the quadratic form

$$Q(d\gamma) = (d\gamma, d\gamma) = \int_M \gamma d\gamma$$

defines a metric on this space. We have, for example,

$$(u_1, u_1) = \int_M \alpha_1 \omega_1 d(\alpha_1 \omega_1) = \int_M \alpha_1 \omega_1 (-2 \alpha_2 \alpha_3) \omega_1 = -2 \int_M \alpha_1 \alpha_2 \alpha_3 \omega_1^2$$
using (21) and (22). Here, because the connection is anti-self-dual, we have
\[ \omega_1^2 = \omega_2^2 = \omega_3^2 = \nu, \quad \omega_1 \omega_2 = \omega_2 \omega_3 = \omega_3 \omega_1 = 0. \]
Continuing this way, and normalizing the integral of \( \alpha_1 \alpha_2 \alpha_3 \nu \), we find the metric with respect to this basis to be
\[
(u_1, u_1) = (u_2, u_2) = (u_3, u_3) = 2, \quad (u_1, u_2) = (u_2, u_3) = (u_3, u_1) = -2
\]
\[
(u_1, u_4) = (u_2, u_4) = (u_3, u_4) = 1, \quad (u_4, u_4) = 0.
\]
We now need the volume \( V(\rho) \) for a general exact invariant 4-form
\[
\rho = \sum_{i=1}^{3} x_i d(\alpha_i \omega_i) + 2x_4 d(\alpha_1 \alpha_2 \alpha_3)
\]
which from (21) and (22) can be written
\[
\rho = (x_1 + x_2 + x_3) \nu + k_1 \alpha_1 \alpha_2 \alpha_3 \omega_1 + k_2 \alpha_3 \alpha_1 \omega_2 + k_3 \alpha_1 \alpha_2 \omega_3
\]
where \( k_1 = 2(x_4 - x_1 + x_2 + x_3) \) etc. Instead of working out the volume as in (8.4), it is easier to transform \( \rho \) to the standard form (12). In fact we can rewrite that formula as
\[
* \varphi = e_1 e_2 e_3 e_4 + e_5 e_6 (e_2 e_3 + e_4 e_1) + e_6 e_7 (e_4 e_3 + e_1 e_2) + e_7 e_5 (e_1 e_3 + e_2 e_4)
\]
Then putting \( e_5 = y_1 \alpha_1, e_6 = y_2 \alpha_2, e_7 = y_3 \alpha_3 \) and rescaling \( e_i = y_4 v_i \) for \( 1 \leq i \leq 4 \), where \( v_i \) is a local orthonormal basis such that \( \omega_1 = (v_4 v_3 + v_1 v_2) \) etc. we can transform \( \rho \) into \( * \varphi \) if
\[
k_1 = y_2 y_3 y_4^2, \quad k_2 = y_3 y_1 y_4^2, \quad k_3 = y_1 y_2 y_4^2, \quad y_4^4 = -2(x_1 + x_2 + x_3)
\]
The Riemannian volume is then
\[
V(\rho) = y_1 y_2 y_3 y_4^4
\]
Using the expression for the inner products above, we find the gradient flow equation
\[
2(-\dot{x}_1 + \dot{x}_2 + \dot{x}_3 + \dot{x}_4) = \frac{\partial V}{\partial x_1}, \text{ etc.}
\]
\[ 2(\dot{x}_1 + \dot{x}_2 + \dot{x}_3) = \frac{\partial V}{\partial x_4} \]

Converting to coordinates \( y_i \) defined by
\[ y_1 y_2 y_4^2 = 2(x_1 + x_2 - x_3 + x_4) \text{ etc., } y_4^4 = -2(x_1 + x_2 + x_3), \]

we find the equations
\[
\begin{align*}
\dot{y}_1 &= -1 + \frac{1}{2y_2y_3} (y_2^2 + y_3^2 - y_1^2) + \frac{y_1^2}{2y_4^2} \\
\dot{y}_2 &= -1 + \frac{1}{2y_3y_1} (y_3^2 + y_1^2 - y_2^2) + \frac{y_2^2}{2y_4^2} \\
\dot{y}_3 &= -1 + \frac{1}{2y_1y_2} (y_1^2 + y_2^2 - y_3^2) + \frac{y_3^2}{2y_4^2} \\
4y_4\dot{y}_4 &= -(y_1 + y_2 + y_3)
\end{align*}
\]

Putting \(-\alpha = y_2 = y_3, \beta = -y_1, \gamma = y_4\) gives the equations solved by Čvetic et al in [C]. Their solution represents a \( \text{Spin}(7) \) manifold with an extra \( S^1 \) symmetry.

The constrained variational problem to get a weak holonomy \( G_2 \) structure consists of finding critical points of \( V \) subject to the condition that the quadratic form is constant. This is precisely when the gradient field is parallel to the position vector. As an example, if \( y_1 = y_2 = y_3 = y \) in the equations above, then we obtain
\[ \lambda y = -\frac{1}{2} + \frac{y^2}{2y_4^2}, \quad 4\lambda y_4^2 = -3y \]

which yields \( y = -3/10\lambda \), and this is the squashed 7-sphere.

In this same symmetric situation the gradient flow equations become
\[ \dot{y} = -\frac{1}{2} + \frac{y^2}{y_4^2}, \quad 2y_4\dot{y}_4 = -\frac{3}{2}y \]

Putting \( s = y_4^2 \) the equation becomes
\[ y \frac{dy}{ds} + \frac{y^2}{3s} = \frac{1}{3} \]

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which is solved easily:
\[ y^2 = \frac{2}{3} s + cs^{-2/3}. \]

When \( c = 0 \) we have the cone on the squashed \( S^7 \) and for \( c \neq 0 \), the complete Bryant-Salamon metric on the spin bundle over \( S^4 \) [Sal].

### 7.2 A 6-dimensional example

We take now \( M^6 = S^3 \times S^3 \) and the forms to be invariant under \( G \) given by the left action of \( S^3 \times S^3 \) and anti-invariant under the \( \mathbb{Z}/2 \) action which interchanges the two factors. We have left-invariant 1-forms
\[ \sigma_1, \sigma_2, \sigma_3, \quad \Sigma_1, \Sigma_2, \Sigma_3 \]
on the two factors, satisfying
\[ d\sigma_1 = -\sigma_2 \sigma_3, \text{ etc.} \]

We shall take \( A \in H^3(S^3 \times S^3) \) to be \( \nu_1 - \nu_2 \) where \( \nu_1 \) is the pull-back of a generator of \( H^3(S^3) \) on the first factor and \( \nu_2 \) the pull-back from the second. This is clearly anti-invariant under the \( \mathbb{Z}/2 \) action. Since \( H^4(S^3 \times S^3) = 0 \), \( B \) must be the trivial class.

We find the following two three-dimensional spaces of closed invariant 3-forms \( \rho \) and 4-forms \( \sigma \):
\[ \rho = \sigma_1 \sigma_2 \sigma_3 - \Sigma_1 \Sigma_2 \Sigma_3 + x_1(\sigma_1 \Sigma_2 \Sigma_3 - \sigma_2 \sigma_3 \Sigma_1) + \cdots \tag{24} \]
\[ \sigma = y_1 \sigma_2 \Sigma_2 \Sigma_3 + y_2 \sigma_3 \Sigma_3 \Sigma_1 + y_3 \Sigma_1 \Sigma_2 \Sigma_2 \tag{25} \]

Since \( d(\sigma_1 \Sigma_1) = \sigma_1 \Sigma_2 \Sigma_3 - \sigma_2 \sigma_3 \Sigma_1 \) we see that \( \rho \) lies in the fixed cohomology class \( A \). The 4-form \( \sigma \) is clearly closed. Moreover \( \omega \wedge \rho = 0 \) for all of these forms. We can work out the pairing between invariant exact 3-forms and invariant exact 4-forms in a straightforward manner, e.g.
\[ \langle \sigma_2 \Sigma_2 \Sigma_3, d(\sigma_1 \Sigma_1) \rangle = \int_{S^3 \times S^3} \sigma_2 \Sigma_2 \sigma_3 \Sigma_3 \sigma_1 \Sigma_1 = \nu_1 [S^3] \nu_2 [S^3] \neq 0. \]

It turns out that the symplectic form is a multiple of
\[ dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3 \]
We can use (8.2) to find \( V(\rho) \) and then we find

\[
V(\rho)^2 = (1 + x_1 + x_2 + x_3)(x_2 + x_3 - x_1 - 1)(x_3 + x_1 - x_2 - 1)(x_1 + x_2 - x_3 - 1)
\]

\[
V(\sigma)^2 = y_1y_2y_3
\]

Since we are only interested in solutions where \( V(\rho) = 2V(\sigma) \), after a change of parameter \( t \), it is equivalent to consider the Hamiltonian flow of \( 4V^2(\sigma) - V^2(\rho) \) i.e. for the Hamiltonian

\[
H = 4y_1y_2y_3 - (1 + x_1 + x_2 + x_3)(x_2 + x_3 - x_1 - 1)(x_3 + x_1 - x_2 - 1)(x_1 + x_2 - x_3 - 1)
\]

The six first order equations arising from this are equivalent to those in [5], where the authors produce an explicit complete solution with \( x_2 = x_3 \) and \( y_2 = y_3 \). They express their solution using an orthonormal basis \( A_j(\sigma_j - \Sigma_j), B_j(\sigma_j + \Sigma_j) \) for the metric and our coordinates relate to theirs by \( x_1 = A_1A_2A_3 + A_1A_2B_3 + A_3B_1B_2 - A_1B_2B_3 \) etc. and \( y_1 = 4A_2B_2A_3B_3 \) etc.

When \( x_1 = x_2 = x_3 = x \) and \( y_1 = y_2 = y_3 = y \), we don’t need to solve the equations since the solutions are given by the vanishing of the Hamiltonian:

\[
4y^3 - (1 + 3x)(x - 1)^3 = 0
\]

This is the metric originally found by Bryant and Salamon [Sal] on the spin bundle over \( S^3 \).

For a weak holonomy \( SU(3) \) metric we need the cohomology classes \( \mathcal{A}, \mathcal{B} \) to be trivial, and then the “1” terms disappear in the functional. In the fully symmetric situation, we need then to find critical points of \( 8y^{3/2} + 3\sqrt{3}x^2 \) subject to the condition \( xy = c \). The solution \( y^{7/2} = \sqrt{3}c^2/2 \) gives the weak holonomy \( SU(3) \) metric on \( S^3 \times S^3 \).

8 Appendix: Definition of the volumes

We assume \( V \) to be oriented and then make all volume forms positive.
8.1 \( p = 2m - 2, n = 2m \)

The vector space \( V \) is \( 2m \)-dimensional and \( \rho \in \Lambda^{2m-2}V^* \). Use the isomorphism

\[
\Lambda^{2m-2}V^* \cong \Lambda^2 V \otimes \Lambda^{2m}V^*
\]

to write \( \rho \) as \( \sigma \in \Lambda^2 V \otimes \Lambda^{2m}V^* \). Then we have using the exterior product

\[
\sigma^{2m} \in \Lambda^{2m} V \otimes (\Lambda^{2m}V^*)^m \cong (\Lambda^{2m}V^*)^{m-1}
\]

We define

\[
\phi(\rho) = |\sigma^{2m}|^{1/(m-1)}.
\]

8.2 \( n = 6, p = 3 \)

We have a 6-dimensional space \( V \) and \( \rho \in \Lambda^3 V^* \). For \( v \in V \) define

\[
K(v) = \iota(v)\rho \wedge \rho \in \Lambda^5 V^* \cong V \otimes \Lambda^6 V^*.
\]

Then

\[
\text{tr}(K^2) \in (\Lambda^6 V^*)^2.
\]

The stable forms with stabilizer \( SL(3, \mathbb{C}) \) are characterized by \( \text{tr}(K)^2 < 0 \) and there

\[
\phi(\rho) = |\sqrt{-\text{tr}K^2}| \in \Lambda^6 V^*.
\]

8.3 \( n = 7, p = 3 \)

The space \( V \) is 7-dimensional and \( \rho \in \Lambda^3 V^* \). Given \( v, w \in V \) form

\[
\iota(v)\rho \wedge \iota(w)\rho \wedge \rho \in \Lambda^7 V^*.
\]

This is a symmetric bilinear form on \( V \) with values in \( \Lambda^7 V^* \), and so a linear map

\[
G : V \to V^* \otimes \Lambda^7 V^*
\]

We have

\[
\det G \in (\Lambda^7 V^*)^9
\]

and we define

\[
\phi(\rho) = |\det G|^{1/9}.
\]
8.4 \( n = 7, p = 4 \)

Take \( \rho \in \Lambda^4 V^* \) but identify with an element \( \sigma \in \Lambda^3 V \otimes \Lambda^7 V^* \). Now apply the construction of the previous case and we find a linear map

\[
H : V^* \rightarrow V \otimes (\Lambda^7 V^*)^2
\]

This gives

\[
\det H \in (\Lambda^7 V^*)^{12}
\]

and we define

\[
\phi(\rho) = |\det H|^{1/12}.
\]

8.5 \( n = 8, p = 3 \) and \( p = 5 \)

The space \( V \) is 8-dimensional and \( \rho \in \Lambda^3 V^* \). For \( v, w \in V \) form

\[
\iota(v)\rho \wedge \iota(w)\rho \wedge \rho \in \Lambda^7 V^* \cong V \otimes \Lambda^8 V^*
\]

This gives an element

\[
d \in V^* \otimes V^* \otimes V \otimes \Lambda^8 V^*
\]

Take \( d \otimes d \) and contract to get

\[
G \in V^* \otimes V^* \otimes (\Lambda^8 V^*)^2
\]

or equivalently

\[
G : V \rightarrow V^* \otimes (\Lambda^8 V^*)^2.
\]

Then we have

\[
\det G \in (\Lambda^8 V^*)^{18}
\]

and define

\[
\phi(\rho) = |\det G|^{1/18}.
\]

For \( p = 5 \) we use \( \Lambda^5 V^* \cong \Lambda^3 V \otimes \Lambda^8 V^* \) and proceed similarly.
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