A distance between channels: the average error of mismatched channels

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Abstract
Two channels are equivalent if their maximum likelihood (ML) decoders coincide for every code. We show that this equivalence relation partitions the space of channels into a generalized hyperplane arrangement. With this, we define a coding distance between channels in terms of their ML-decoders which is meaningful from the decoding point of view, in the sense that the closer two channels are, the larger is the probability of them sharing the same ML-decoder. We give explicit formulas for these probabilities.

Keywords Mismatched channels · Maximum likelihood decoding · Space of channels

Mathematics Subject Classification 68P30 · 51E22 · 52C35

1 Introduction
A communication channel cannot generally be chosen in application, but is rather considered to be a “fact of life”. The most that is possible, is to make measurements to characterize the type of noise and model the errors.

Some channels are simpler to handle than others. If a channel is metrizable, for example, one can use methods from classical coding theory which make use of the metric. If, furthermore, the metric is translation invariant one can use syndrome decoding which greatly reduces decoding complexity.

The question of being metrizable is one that underlines many aspects in coding theory, but is seldom stated in an explicit way: a channel with equal input and output sets of messages

1 For the use of general distances in coding theory see [1,6].

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is said to be *metrizable* if there is a metric such that, for any code, the maximum likelihood (ML) and minimum distance decoders coincide.\(^2\)

A metric is just one kind of structure that makes a channel more manageable, and sometimes it may be worth to consider an alternative channel model which is less accurate as a model for noise and errors but has other desirable properties, like more efficient decoding algorithms. In this sense, the long term goal is to develop an approximation-theory-like approach to coding theory. To do so, the first step is to determine a distance in the space of channels which translates their probabilistic structure. This is the main goal of this work.

**Previous work**

The notion of an equivalence between channels, Definition 1, was first presented in [4]. This notion was used in [2,4] to characterize under which conditions a channel is metrizable. In [4], it is shown that any metric, up to a decoding equivalence, can be isometrically embedded into the hypercube with the Hamming metric. In [3], bounds on the dimension of these embeddings are given.

**Contributions**

In this paper we show, in Theorems 1 and 2, that the space of channels, under decoding equivalence, has the structure of a special kind of hyperplane arrangement known as the braid arrangement.

In Sect. 5 we present the decoding distance, in Corollary 1, which measures the probability, after a code-word is received, that two channels have the same ML decoder, when a code is chosen uniformly at random. In Corollary 2 we show that the decoding distance is a modified version of the Kendall tau distance.

In Sect. 6 we present the radial decoding distance, in Corollary 3, which measures the probability that two channels have the same ML decoder, when a code is chosen uniformly at random.

**2 Related work**

The study of the space of channels at its own sake is related, although not equivalent, to other subjects that have been studied, namely mismatched decoding and partial ordering of channels. We give a brief overview of these topics, pointing the similarities and differences with our approach.

**2.1 Mismatched decoding**

Our approximation-theory-like approach is similar to the setting of *mismatched decoding*. In this setting instead of using the ML decoder determined by the channel \(P\) (ML\(_P\)-decoding), a different decoding criterion is used. In practice, this might occur due to inaccuracies in the measurement of a channel. In this case we are using an ML\(_Q\)-decoding, where \(Q\) is the non-accurate measured channel. Another reason for mismatched decoding arises when there are no reasonable algorithms for implementing ML\(_P\)-decoding.

Mismatched decoding has an extensive literature ([7] has many relevant references on the subject). The approach, however, is essentially information theoretical, guided by the fundamental question of determining what can, in principle, be done. This means that most of the work in the area aims to understand what is achievable asymptotically, for example,

\(^2\) For a deeper look into conditions for metrization see [2–5] and [10,11,13].
what are the achievable rates for families of channels with the input–output sets’ sizes going

to infinity. Those are very difficult questions and hence a significant part of the effort is
directed to find bounds for those rates (and other significant invariants).

Our approach is less concerned with the asymptotic aspects of achievability. Once the
input and output sets $X$ and $Y$ are given (and fixed) and supposing that the actual channel,
$P$, is known, how much are we expected to lose once we decode a code, chosen uniformly
at random, using the ML-decoding criterion determined by a different channel, $Q$. Our
measure of expected loss is the overall probability of error in the whole process of encoding,
transmitting, and decoding. In this sense, we may say that we are considering the mismatched
decoding problem in the finite block length regime. A similar approach can be found in [12],
where the authors restrict the problem to consider a memoryless channel that is mismatched by
a “decoding metric” which is assumed to be, up to a rescaling, another memoryless channel.
The same restriction emerged in Seguin’s work [13] where he classified the channels that can
be matched to an additive metric.

2.2 Partial ordering of channels

Our approach to study the geometry of the space of channels has an intersection with the
concept of channel inclusion, as introduced by Shannon [15] and as presented, for example, by
Makur and Polyanskiy [9]. Using the notation of Makur and Polyanskiy, given two channels
with transition matrices $P$ and $Q$ of size $N_P \times M_P$ and $N_Q \times M_Q$ respectively, with $N_Q \leq N_P$
and $M_Q \leq M_P$, one says that $P$ includes $Q$ if there are two families $(A_k)_{k=1}^m$ and $(B_k)_{k=1}^m$
of channels (with $A_k$ being an $M_P \times M_Q$ transition matrix and $A_K$ an $N_Q \times N_P$ transition
matrix) and probability mass function $g$ over the set $\{1, 2, \ldots, m\}$ such that

$$Q = \sum_{k=1}^m g(k) B_k P A_k.$$ 

This concepts embraces many different situations, some of which can be understood with
our definition of the space of channels with the decoding equivalence. The first example
introduced in Fig. 1 of Shannon’s work corresponds to the situation where $m = 1$ and
both $A_1$ and $B_1$ is determined by a projection matrix. If we allow $A_1$ and $B_1$ to correspond
to a projection or a permutation matrix (or a combination of both), we actually have a
hyperplane, $P_\pi \subset \mathbb{R}^{N \times M}$, with a braid arrangement structure (see Sect. 4) induced from the
braid arrangement structure of $\mathbb{R}^{N \times M}$, by considering the intersection of a decoding cone
$\text{cone}(x) \cap P$.

3 Preliminaries

In this section we start with a list of definitions and notations used throughout this work.
Since these concepts are well known we present them very succinctly, citing references for
details. After that, in Sect. 3.2, we present the decoding equivalence between channels, first
presented in [4], and define the space of channels.

3.1 Notation

We consider the basic setting of information theory [14], where a transmitter sends a message
to a receiver passing through a channel. Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of input messages
which the transmitter can send and let \( Y = \{y_1, y_2, \ldots, y_m\} \) be the set of output messages which the receiver can receive. It is common for the messages to come from some alphabet in which case the sets \( X \) and \( Y \) are exponential on the block length with respect to the size of the alphabet.

A channel is a \( n \times m \) probabilistic matrix \( P \) such that \( P_{ij} = \Pr(y_j \text{ received} | x_i \text{ sent}) \), the probability of receiving \( y_j \) given that \( x_i \) was sent (the rows sum to 1).

Given a nonempty code \( C \subseteq X \), a ML decoder is such that \( y \in Y \) is decoded as some \( c \in C \) which maximizes \( \Pr(y \text{ received} | c \text{ sent}) \). The set of ML decoders of the channel \( P \) for a code \( C \) is denoted by \( \hat{D}_{\text{ec}}(P) \).

Following the definitions given in [1], a distance on a set \( X \) is a function \( d : X \times X \to \mathbb{R} \) which satisfies

1. \( d(x, y) \geq 0 \) (non-negativity),
2. \( d(x, y) = d(y, x) \) (symmetry),
3. \( d(x, x) = 0 \) (reflexivity).

If the distance also satisfies property 4 it is called a semimetric and if in addition it satisfies property 5 it is called a metric.

\[
(4) \quad d(x, y) = 0 \text{ if and only if } x = y \text{ (identity of indiscernibles)},
\]

\[
(5) \quad d(x, z) \leq d(x, y) + d(y, z) \text{ (triangle inequality)}.
\]

If the function \( d \) only satisfies properties 1, 3, and 4 it is called a quasi-semimetric and, if it also satisfies 5, it is called a quasi-metric.

A weak order over a set \( X \) is a triple \((X, \prec, \simeq)\), where \( \prec \) and \( \simeq \) are binary relations on \( X \) satisfying, for all \( x, y, z \in X \):

1. \( x \prec y \) and \( y \prec z \) implies that \( x \prec z \),
2. \( \simeq \) is an equivalence relation,
3. exactly one of \( x \prec y \), \( y \prec x \) or \( x \simeq y \) holds.

We denote the set of all weak orders over \( n \) objects by \( W_n \).

We denote the symmetric group over \( n \) objects by \( S_n \), using lowercase Greek letters for elements of this set \((\sigma, \tau, \phi \in S_n)\) and denoting the identity by \( \epsilon \in S_n \).

A set \( A \subseteq \mathbb{R}^n \) is convex if it contains the segment joining any two of its points, i.e. \( \alpha x + (1 - \alpha) y \in A \) for every \( x, y \in A \) and \( 0 \leq \alpha \leq 1 \).
A hyperplane is a set $H \subseteq \mathbb{R}^n$ of the form $H = \{ x \in \mathbb{R}^n : \alpha \cdot x = a \}$ where $0 \neq \alpha \in \mathbb{R}^n$, $a \in \mathbb{R}$ and $\alpha \cdot x := \sum_{i=1}^n \alpha_i x_i$ is the usual dot product.

A hyperplane arrangement $A$ [16] is a set of hyperplanes. A region of an arrangement is a connected component of the complement of the hyperplanes, $X = \mathbb{R}^n - \bigcup_{H \in A} H$. The set of regions is denoted by $\mathcal{R}(A)$ and $r(A) := \# \mathcal{R}(A)$.

Each hyperplane divides $\mathbb{R}^n$ into two subsets known as half-spaces. The two half spaces corresponding to $H = \{ x \in \mathbb{R}^n : \alpha \cdot x = a \}$ are $\{ x \in \mathbb{R}^n : \alpha \cdot x \leq a \}$ and $\{ x \in \mathbb{R}^n : \alpha \cdot x \geq a \}$.

A convex polytope is the intersection of a finite set of half-spaces which is bounded.

A set $C \subseteq \mathbb{R}^n$ is a convex cone if $ax + \beta y \in C$ for every $x, y \in C$ and $a, \beta \geq 0$. The dimension of a convex cone is the dimension of the smallest affine space that contains it.

We are particularly interested in the braid arrangement, $B_n$, which consists of the $\binom{n}{2}$ hyperplanes: $x_i - x_j = 0$ for $1 \leq i < j \leq n$. Specifying to which side of the hyperplane a point $a \in \mathbb{R}^n$ belongs to is equivalent to determining whether $a_i < a_j$ or $a_j < a_i$. Doing so for every hyperplane is equivalent to imposing a linear order on the $a_i$. So to each permutation $\sigma \in S_n$ there corresponds a region $R_\sigma \in \mathcal{R}(B_n)$ given by $R_\sigma = \{ x \in \mathbb{R}^n : a_{\sigma(1)} < a_{\sigma(2)} < \cdots < a_{\sigma(n)} \}$. Thus, $r(B_n) = n!$.

The Iverson bracket will be used in our definitions and proofs: for a statement $S$, the bracket $[S]$ equals 1 if statement $S$ is true and equals 0 otherwise.

### 3.2 The space of channels

The results in this section appear in more detail in [2–4].

Consider the space $\mathbb{R}^n_{\geq 0}^{\times m}$ of matrices with non-negative entries. The space of all $n \times m$ channels, $Ch_{n \times m}$, is a subset of this space.

**Definition 1** Two channels $P, Q \in Ch_{n \times m}$ are decoding equivalent, $P \sim Q$, if, for any code $C \subseteq X$, they have the same ML decoders, i.e. for every $C \subseteq X$, $\hat{D}\hat{e}c_C(P) = \hat{D}\hat{e}c_C(Q)$.

Our next definition will help characterize decoding equivalence.

**Definition 2** Given a matrix $M \in \mathbb{R}^n_{\geq 0}^{\times m}$, its weak order matrix is the matrix $O^{-}\!M$ such that $(O^{-}\!M)_{ij} = k$ if $M_{ij}$ is the $k$th largest element (allowing ties) in the $j$th column of $M$.

**Example 1** If $M = \begin{pmatrix} 9 & 2 & 1 \\ 9 & 7 & 0 \\ 8 & 6 & 8 \end{pmatrix}$, then $O^{-}\!M = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 3 \\ 2 & 2 & 1 \end{pmatrix}$.

**Proposition 1** Two channels $P, Q \in Ch_{n \times m}$ are decoding equivalent if and only if $O^{-}\!P = O^{-}\!Q$.

**Proof** Corollary 3 in [4].

The decoding equivalence can be extended to the whole of $\mathbb{R}^n_{\geq 0}^{\times m}$ by defining $M \sim N$ if $O^{-}\!M = O^{-}\!N$.

The decoding equivalence partitions $\mathbb{R}^n_{\geq 0}^{\times m}$ into $|W_n|^m$ cones, $(n!)^m$ of which are full dimensional. We denote the decoding cone containing a matrix $M$ by $Cone(M)$ and note that they are the fibers of $O^{-}\!$, i.e. $Cone = (O^{-}\!)^{-1} \circ O^{-}\!$. For details see [3, Sect. 3].

As we shall see, the space of channels, with the decoding equivalence, has the structure of a hyperplane arrangement and the simplicial structure of the hyperplane arrangement reflects the structure of ML decoding.
4 Decoding equivalence and the braid arrangement

ML decoding is done comparing entries in the column corresponding to the received message. Considering a column as a vector \( x \in \mathbb{R}^n \), we show, in Theorem 1, that the decoding equivalence partitions \( \mathbb{R}^n \) into generalized regions of the braid arrangement. We then extend this result, in Theorem 2, to \( \mathbb{R}^{n \times m} \).

We first define the Order function.

**Definition 3** The Order function, \( \text{Order} : \mathbb{R}^n_{\geq 0} \rightarrow W_n \), takes a vector \( x \in \mathbb{R}^n_{\geq 0} \) to the weak ordering of its coordinates.

So, for example, \( \text{Order} \left( \sqrt{2}, \frac{-1}{2}, \sqrt{2} \right) = \text{Order}(2, 1, 2) = (2 < 1 \simeq 3) \).

**Proposition 2** Two vectors \( x, y \in \mathbb{R}^n_{\geq 0} \) are decoding equivalent if and only if \( \text{Order}(x) = \text{Order}(y) \).

**Proof** This follows because \( \text{Order}(x) = \text{Order}(y) \) if and only if \( O^{-x} = O^{-y} \).

The fibers of the Order function, i.e., the inverse images \( \text{Order}^{-1}(y) \), partition \( \mathbb{R}^n \) into the decoding equivalence class.

**Definition 4** The cone function is given by \( \text{Cone} : \mathbb{R}^n_{\geq 0} \rightarrow 2^{\mathbb{R}^n_{\geq 0}} \) such that \( \text{Cone}(x) = (\text{Order})^{-1} \circ \text{Order} \). We call \( \text{Cone}(x) \) the decoding cone of \( x \).

It is clear, from the definition, that two channels in the same decoding cone determine the same ML criteria for every code.

We generalize the definition of the region of a hyperplane arrangement.

**Definition 5** A generalized region of a hyperplane arrangement \( \mathcal{A} \) is a connected component of \( \bigcap_{H \in \mathcal{A}_1} H - \bigcup_{H \in \mathcal{A}_2} H \), where \( \mathcal{A}_1, \mathcal{A}_2 \) is a disjoint partition of \( \mathcal{A} \). We denote the sets of generalized regions by \( \mathcal{GR}(\mathcal{A}) \) and \( gr(\mathcal{A}) = \#\mathcal{GR}(\mathcal{A}) \).

As stated in Sect. 3.1, the braid arrangement consists of the \( \binom{n}{2} \) hyperplanes: \( H_{ij} = \{ x \in \mathbb{R}^n : x_i = x_j \} \) for \( 1 \leq i < j \leq n \). Theorem 1 shows that the decoding equivalence partitions \( \mathbb{R}^n \) into generalized regions of the braid arrangement.

**Theorem 1** Let \( x, y \in \mathbb{R}^n_{\geq 0} \). Then, \( x \) is decoding equivalent to \( y \) if and only if \( x, y \in R \) for some \( R \in \mathcal{GR}(\mathcal{B}_n) \), where \( \mathcal{B}_n \) is the braid arrangement.

**Proof** Specifying to which generalized region, \( R_x \in \mathcal{GR}(\mathcal{B}_n) \), a point \( x \in \mathbb{R}^n \) belongs to is equivalent to determining whether \( x_i < x_j, x_i = x_j \) or \( x_i > x_j \) for every \( 1 \leq i < j \leq n \). This is equivalent to imposing a weak order on the coordinates of \( x \). But this implies that \( y \in R_x \) if and only if \( \text{Order}(y) = \text{Order}(x) \). The result then follows from Proposition 2.

In other words, if \( R \in \mathcal{GR}(\mathcal{B}_n) \) then \( x \in R \) if and only if \( R = \text{Cone}(x) \), i.e. the decoding cones are the generalized regions of the braid arrangement.

We now extend the results from \( \mathbb{R}^n \) to \( \mathbb{R}^{n \times m} \).

**Definition 6** The Order function, \( \text{Order} : \mathbb{R}^{n \times m}_{\geq 0} \rightarrow W_n^m \), is defined as

\[
\text{Order}(M) = \text{Order}(M[\cdot][1]) \times \text{Order}(M[\cdot][2]) \times \cdots \times \text{Order}(M[\cdot][m]),
\]

where \( \text{Order}(M[\cdot][j]) \) is the order function in Definition 3 applied to the \( j \)th column of \( M \). The decoding cone of \( M \) is \( \text{Cone}(M) = \text{Order}^{-1} \circ \text{Order}(M) \).
The following result is analogous to Theorem 1.

**Theorem 2** Let \( M, M' \in \mathbb{R}^{n \times m} \). Then, \( M \) is decoding equivalent to \( M' \) if and only if \( M[-][j], M'[\cdot][j] \in R_j \) for some \( R_j \in \mathcal{GR}(\mathcal{B}_n) \), where \( \mathcal{B}_n \) is the braid arrangement.

**Proof** The proof is equivalent to that of Theorem 1 by using Definition 6. \( \square \)

5 A decoding distance between permutations

Having an appropriate model for the transmission channel is not always good enough to establish all the necessities in the communication process. Many other questions, such as the complexity of the decoding algorithms, need to be taken into consideration. For this reason, for example, even when the channel is not the binary symmetric channel, the Hamming metric is commonly used.

In this sense, it may be interesting to develop an “approximation theory” for channels. The idea is that we can use an approximate simpler channel (or a distance matched to it) in place of the original one.

The most basic and mandatory tool for the development of an approximation theory is a distance in the space \( \text{Chan}_{n \times m} \) which is adequate in some sense. If \( P \) is a channel, \( P_y \) denotes the column corresponding to receiving \( y \). We will propose a relevant distance on \( \text{Chan}_{n \times m} \) which relates to the following:

Let \( P, Q \in \text{Chan}_{n \times m} \) be two different channels and suppose we know what output \( y \in Y \) is received. Choosing a code \( C \subseteq X \) from the set of all codes with uniform distribution, what is the probability that \( \hat{D}_{\text{ec}}(P_y) \cap \hat{D}_{\text{ec}}(Q_y) \neq \emptyset \)?

When we say that the distance is related to that question it means that the probability that \( \hat{D}_{\text{ec}}(P_y) \cap \hat{D}_{\text{ec}}(Q_y) \neq \emptyset \) decreases with the purposed distance: the closer the channels are, the more probable they are to determine the same decoders. Here we assume the uniform distribution on the codes.

Since we know what output \( y \) is received, only its corresponding column matters for decoding. Thus we are dealing with the decoding equivalence in \( \mathbb{R}^n \).

We only consider the cases for which \( \text{Cone}(P_y) \) and \( \text{Cone}(Q_y) \) are \( n \)-dimensional and leave the general case for future work. We say that a channel \( P \) such that \( \text{Cone}(P) \) is full dimensional is a stable channel, since small perturbations of the channel probabilities do not affect the decoding decisions, i.e. \( \text{Cone}(P) \) is an open set. In this case \( \hat{D}_{\text{ec}}(P_y) \cap \hat{D}_{\text{ec}}(Q_y) \neq \emptyset \) is equivalent to \( \hat{D}_{\text{ec}}(P_y) = \hat{D}_{\text{ec}}(Q_y) \).

By Theorem 1, each \( n \)-dimensional decoding cone corresponds to a region of the braid arrangement \( \mathcal{B}_n \). As noted in Sect. 3.1 to each \( \sigma \in S_n \) there corresponds a region \( R_{\sigma} \in \mathcal{R}(\mathcal{B}_n) \). We can therefore identify every \( n \)-dimensional decoding cone with a permutation in \( S_n \).

**Example 2** Consider \( \mathbb{R}_+^3 \). The identity element, \( \epsilon \in S_3 \), corresponds to the cone with ordering \((1 < 2 < 3)\). The transposition \((13) \in S_3 \), corresponds to the cone with ordering \((3 < 2 < 1)\).

Since decoding depends exclusively on the decoding cone, we can extend the definition of \( \hat{D}_{\text{ec}} \) to permutations in the following way.

**Definition 7** Let \( \sigma \in S_n \), \( R_{\sigma} \in \mathcal{R}(\mathcal{B}_n) \) be its corresponding decoding cone and \( P \in \text{Chan}_{n \times m} \) be such that \( P \in R_{\sigma} \). We define \( \hat{D}_{\text{ec}}(\sigma) = \hat{D}_{\text{ec}}(P) \) for every \( C \subseteq X \).

The leading question we posed in the beginning of this section can now be restated in terms of permutation groups as follows:
Given two permutations \( \sigma, \phi \in S_n \), what is the probability that \( D\hat{e}_C(\sigma) = D\hat{e}_C(\phi) \) if \( C \subseteq X \) is chosen uniformly at random?

More precisely, we are interested in computing the following distance.

**Definition 8** The **decoding distance** between two permutations, \( \sigma, \phi \in S_n \), is

\[
d_{\text{dec}}(\sigma, \phi) = 1 - \Pr(D\hat{e}_C(\sigma) = D\hat{e}_C(\phi)),
\]

where a code \( C \subseteq X \) is chosen uniformly at random.

We will determine the decoding distance by elementary counting.

**Definition 9** Let \( \sigma, \phi \in S_n \). We denote by \( S(\sigma, \phi) \) the number of codes \( C \) for which \( D\hat{e}_C(\sigma) = D\hat{e}_C(\phi) \).

We aim to relate \( S(\sigma, \phi) \) to \( d_{\text{dec}}(\sigma, \phi) \). We first remark that the function \( S \) is permutation invariant.

**Proposition 3** Let \( \sigma, \phi, \tau \in S_n \). Then, \( S(\tau \circ \sigma, \tau \circ \phi) = S(\sigma, \phi) \).

**Proof** This follows from the fact that if you permute the rows of a channel, the same permutation on a ML decoder of it will yield a ML decoder of the permuted channel.

Thus, we can define \( S(\sigma) = S(\epsilon, \sigma) \) and then \( S(\sigma, \phi) = S(\phi^{-1} \circ \sigma) \).

We now show how to compute this function.

**Theorem 3** Let \( \sigma \in S_n \) and let us define \( f_i(\sigma) = \sum_{j=i+1}^{n} [\sigma^{-1}(i) \leq \sigma^{-1}(j)] \). Then, \( S(\sigma) = \sum_{i=1}^{n} 2 f_i(\sigma) \).

**Proof** We want to count how many codes, \( C \subseteq X \), satisfy \( D\hat{e}_C(\epsilon) = D\hat{e}_C(\sigma) \). The identity element represents the order \( (1 < 2 < \cdots < n) \) and \( \sigma \) represents the order \( (\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n)) \).

Recall that \( \sigma \) corresponds to a channel \( P \) with input messages \( X = \{x_1, \ldots, x_n\} \). Consider the codes, \( C \subseteq X \), such that \( x_1 \in C \). The identity element, \( \epsilon \), decodes any message as \( x_1 \), since \( x_1 \) corresponds to the entry 1 in \( (1 < 2 < \cdots < n) \). Thus, \( D\hat{e}_C(\epsilon) = D\hat{e}_C(\sigma) \) if and only if \( \sigma \) also decodes as \( x_1 \). For this to happen, the code, \( C \), can only contain elements, \( x_i \), such that \( \sigma^{-1}(i) \leq \sigma^{-1}(j) \). But \( f_i(\sigma) \) counts precisely how many of these exist. So the total number of codes satisfying \( x_1 \in C \) and \( D\hat{e}_C(\epsilon) = D\hat{e}_C(\sigma) \) is \( 2 f_i(\sigma) \).

Now, consider the codes \( C \subseteq X \) such that \( x_1 \notin C \) and \( x_2 \in C \). The same reasoning yields the total number of codes satisfying \( x_1 \notin C \), \( x_2 \in C \), and \( D\hat{e}_C(\epsilon) = D\hat{e}_C(\sigma) \) as \( 2^{f_2(\sigma)} \).

Continuing with the same argument yields our result.

**Remark 1** In Theorem 3 we give an expression for \( S(\sigma) \) which consider the probability of a mismatched decoding considering the family of all possible codes, each one being picked uniformly at random. To restrict the family of codes, to say, codes of a fixed rate, or to change the probability distribution on the codes being picked, privileging, for example, codes with larger minimal distance, remains as an interesting open question that will demand significant changes to Theorem 3.

The next theorem answers the question posed in the beginning of this section.

**Theorem 4** Let \( \sigma, \phi \in S_n \). If a code, \( C \subseteq X \), is picked uniformly at random from the space of all codes, then \( \Pr(D\hat{e}_C(\sigma) = D\hat{e}_C(\phi)) = S(\text{fix}_{\phi} \circ \sigma) / 2^n \).

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As we shall see on the following theorem, differently from the Kendall tau distance where \(d = \phi\) and is defined as the minimum number of adjacent permutations to describe the function \(f\), the decoding distance of the graph.

**Proof** By definition, the function \(S(\phi^{-1} \circ \sigma)\) counts the number of codes such that \(D \hat{e}c_C(\sigma) = D \hat{e}c_C(\phi)\). Elementary probability says we must divide this by the total number of codes. \(\square\)

As a direct corollary we compute the decoding distance.

**Corollary 1** The decoding distance between two permutations \(\sigma, \phi \in S_n\) is

\[
d_{dec}(\sigma, \phi) = 1 - \frac{S(\phi^{-1} \circ \sigma)}{2^n - 1}.
\]

**Remark 2** It is clear that the decoding distance is a semimetric. One may be interested in knowing if it is also a metric, i.e. if it satisfies the triangle inequality. However, every semimetric, \(d\), on a finite space \(X\) can be transformed into a metric,

\[
d'(x, y) = 1 + \frac{d(x, y)}{\max_{u, v \in X} d(u, v)}
\]

[and \(d(x, x) = 0\)], which is equivalent to it, in the sense that \(d(x, y) < d(x, z)\) implies in \(d'(x, y) < d'(x, z)\). Thus, if the triangle inequality is desired, the decoding distance can be transformed to a metric.

In the context of the braid arrangement there exists already a natural distance between permutations. It is known as the Kendall tau distance [8], which we denote by \(d_t(\sigma, \phi)\), and is defined as the minimum number of adjacent permutations \(\tau_1, \tau_2, \ldots, \tau_{d_t(\sigma, \phi)}\) so that \(\phi = \sigma \circ \tau_1 \circ \tau_2 \circ \cdots \circ \tau_{d_t(\sigma, \phi)}\).

Consider the graph whose vertices are the regions of the braid arrangement and such that two vertices share an edge if their corresponding regions are adjacent to each other (so that each edge corresponds to a hyperplane). Then, the Kendall tau distance is the shortest path distance of the graph.

In technical terms: if \(\sigma, \tau \in S_n\) where \(\tau = (r, r + 1)\), then,

\[
d_t(\epsilon, \tau \circ \sigma) - d_t(\epsilon, \sigma) = \begin{cases} 1 & \text{if } \sigma^{-1}(r) < \sigma^{-1}(r + 1), \\ -1 & \text{if } \sigma^{-1}(r) > \sigma^{-1}(r + 1). \end{cases}
\]

In the next theorem we show that the decoding distance is a weighted version of the Kendall tau distance. The function, \(f_r(\sigma) = \sum_{j=i+1}^{\sigma^{-1}(i) \leq \sigma^{-1}(j)}\), used in Theorem 3 to describe the function \(S(\sigma)\) differs from the Kendall tau distance, since it considers not only the number of transpositions \((r, r + 1)\) but also the value of \(r\): \(f_r(\tau_r) = n - r - 1\). As we shall see on the following theorem, differently from the Kendall tau distance where \(|d_t(\epsilon, \tau \circ \sigma_r) - d_t(\epsilon, \sigma)| = 1\) independently of \(r\), the difference \(|S(\tau_r \circ \sigma) - S(\sigma)|\) decreases with \(r\).

**Theorem 5** Let \(\sigma, \tau \in S_n\) where \(\tau := \tau_r = (r, r + 1)\). Then,

\[
S(\tau \circ \sigma) - S(\sigma) = \begin{cases} -2f_r(\sigma) & \text{if } \sigma^{-1}(r) < \sigma^{-1}(r + 1), \\ 2f_{r+1}(\sigma) & \text{if } \sigma^{-1}(r) > \sigma^{-1}(r + 1). \end{cases}
\]

**Proof** Since \((\tau \circ \sigma)^{-1}(r) = \sigma^{-1}(r + 1)\) and \((\tau \circ \sigma)^{-1}(j) = \begin{cases} \sigma^{-1}(r) & \text{if } j = r + 1 \\ \sigma^{-1}(j) & \text{if } j > r + 1 \end{cases}\) for \(j \geq r + 1\) it follows that

\[
f_r(\tau \circ \sigma) = \sum_{j=r+1}^{n} [(\tau \circ \sigma)^{-1}(r) \leq (\tau \circ \sigma)^{-1}(j)]
\]

\[
= [\sigma^{-1}(r + 1) \leq \sigma^{-1}(r)] + f_{r+1}(\sigma).
\]

\(\square\)
Since \((\tau \circ \sigma)^{-1}(r + 1) = \sigma^{-1}(r)\) and \(r + 1 < j \Rightarrow (\tau \circ \sigma)^{-1}(j) = \sigma^{-1}(j)\) it follows that

\[
f_{r+1}(\tau \circ \sigma) = \sum_{j=r+2}^{n} [(\tau \circ \sigma)^{-1}(r + 1) \leq (\tau \circ \sigma)^{-1}(j)] + [\sigma^{-1}(r) \leq \sigma^{-1}(r + 1)] - [\sigma^{-1}(r) \leq \sigma^{-1}(r + 1)]
= f_r(\sigma) - [\sigma^{-1}(r) \leq \sigma^{-1}(r + 1)].
\]

Thus, we have

\[
S(\tau \circ \sigma) = \sum_{i=1}^{r-1} 2f_i(\sigma) + 2f_r(\tau \circ \sigma) + 2f_{r+1}(\tau \circ \sigma) + \sum_{i=r+2}^{n} 2f_i(\sigma)
= S(\sigma) + 2f_r(\sigma) \left( 2^{[\sigma^{-1}(r) \leq \sigma^{-1}(r + 1)]} - 1 \right)
+ 2f_{r+1}(\sigma) \left( 2^{[\sigma^{-1}(r + 1) \leq \sigma^{-1}(r)]} - 1 \right).
\]

As a direct corollary, using Corollary 1, we have the following result.

**Corollary 2** Let \(\sigma, \tau \in S_n\) where \(\tau = (r, r + 1)\). Then,

\[
d_{dec}(\varepsilon, \tau \circ \sigma) - d_{dec}(\varepsilon, \sigma) = \begin{cases} 2f_r(\sigma) - 1 & \text{if } \sigma^{-1}(r) < \sigma^{-1}(r + 1), \\ - \frac{2f_{r+1}(\sigma)}{2^n - 1} & \text{if } \sigma^{-1}(r) > \sigma^{-1}(r + 1). \end{cases}
\]

### 6 A distance between stable channels

In this section we extend the results of the previous one to define a distance (in some sense) between channels. As in the last section we will only consider the case where the decoding cones are full dimensional, i.e. the channel is stable.

We could define a distance by setting \(d(P, Q) = 1 - \Pr(D\hat{e}C_C(P) = D\hat{e}C_C(Q))\), but we will see that a more refined distance can be defined.

Consider three channels \(P, Q, R \in Cha_3\) such that

\[
O^{-}P = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \quad O^{-}Q = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad O^{-}R = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}.
\]

One can check, by exhaustion, that \(d(P, Q) = d(P, R) = d(Q, R) = \frac{4}{7}\). But \(Q\) differs from \(P\) in only one position of a single column, while \(R\) differs from \(P\) in one position in two different columns. If \(y_1\) or \(y_2\) (the output messages corresponding, respectively, to the first and second columns) is received, \(P\) and \(Q\) are essentially the same channel. Intuitively, however, we expect \(Q\) to be closer to \(P\) than \(R\).

This distance does not use the fact that the received message will be known at the time of decoding. We will use this fact to define a more refined distance.

If we assume that the transmission is made through the channel \(P\), and denote by \(Q_y\) the column corresponding to the received message \(y\) in \(Q\), we can calculate \(\Pr(D\hat{e}C_C(P_y) =
A distance between channels

$D\widehat{cc}_C(Q_y)$, the probability that both decoders will be equal when a message $y$ is received.\footnote{In this case both the code $C$ and the message $y$ are random variables.}

With this we can define the following distance:

**Definition 10** Let $P, Q \in Ch_{n \times m}$ and assume that $P$ is the channel being used. The radial decoding distance to $Q$ centered in $P$ is given by

$$d^P_{\text{dec}}(Q) = 1 - \Pr(D\widehat{cc}_C(P_y) = D\widehat{cc}_C(Q_y)).$$

The next theorem shows how to compute this distance.

**Theorem 6** Let $P, Q \in Ch_{n \times m}$ and $\sigma, \phi \in S_n^n$ be such that $\sigma_i$ and $\phi_i$ correspond to the ordering in the $i$th column of $O^{-1}P$ and $O^{-1}Q$, respectively. Suppose that the channel being used is $P$. If a code $C \subseteq X$ is picked uniformly at random from the space of all codes, then

$$\Pr(D\widehat{cc}_C(P_y) = D\widehat{cc}_C(Q_y)) = \frac{1}{n(2^n - 1)} \sum_{i=1}^m S(\sigma_i, \phi_i) \|P_i\|_1,$$

where $\|P_i\|_1 := \sum_{j=1}^n P_{ji}$ is the $1$-norm of the $i$th column of $P$.

**Proof**

$$\Pr(D\widehat{cc}_C(P_y) = D\widehat{cc}_C(Q_y)) = \sum_{i=1}^m \Pr(D\widehat{cc}_C(P_y) = D\widehat{cc}_C(Q_y) | y_i \text{ received}) \Pr(y_i \text{ received})$$

$$= \sum_{i=1}^m S(\sigma_i, \phi_i) \sum_{j=1}^n \Pr(y_i \text{ received} | x_j \text{ sent}) \Pr(x_j \text{ sent})$$

$$= \sum_{i=1}^m S(\sigma_i, \phi_i) \frac{1}{2^n - 1} \|P_i\|_1 \frac{1}{n}.$$

\[\square\]

In the hypothesis of Theorem 6 we assume that one of the channels is the correct one. This occurs because the expression depends on the probability of receiving $y$ which may not coincide for different channels.

**Corollary 3** Let $P, Q \in Ch_{n \times m}$ and assume that $P$ is the channel being used. The radial decoding distance to $Q$ centered in $P$ is given by

$$d^P_{\text{dec}}(Q) = 1 - \frac{1}{n(2^n - 1)} \sum_{i=1}^m S(\sigma_i, \phi_i) \|P_i\|_1.$$

**Remark 3** Since it is not symmetric, the radial decoding distance is a quasi-semimetric. The radial decoding distance can be transformed into an equivalent quasi-metric using the transformation in Remark 2.

We go back to the example discussed in the beginning of this section.
Example 3 Suppose a channel \( P = \begin{pmatrix} 5 & 1 & 2 \\ 2 & 5 & 1 \\ 1 & 2 & 5 \end{pmatrix} \) is used for transmission and \( Q, R \in \text{Cha}_3 \) are such that
\[
O - Q = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad O - R = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}.
\]

Then, by Theorem 6,
\[
\Pr(\widehat{D}\text{ec}_C(P_y) = \widehat{D}\text{ec}_C(Q_y)) = \frac{1}{21}(7 + 7 + 4) = \frac{6}{7}
\]
and
\[
\Pr(\widehat{D}\text{ec}_C(P_y) = \widehat{D}\text{ec}_C(R_y)) = \frac{1}{21}(5 + 7 + 4) = \frac{16}{21}.
\]

Thus,
\[
d_{dec}^P(Q) = \frac{1}{7} < \frac{5}{21} = d_{dec}^P(R).
\]

We note that this difference is, intuitively, compatible with the simple observation that \( Q \) differs from \( P \) in only one position of a single column, while \( R \) differs from \( P \) in one position in two different columns.

The decoding distance presented in Definition 8 of the previous section was symmetric and only depended on the equivalence classes of the permutations. In contrast, the radial decoding distance to \( Q \) centered in \( P \) is not symmetrical, and although it only depends on the equivalence class of \( Q \), it depends on the internal structure of \( P \), i.e., if \( Q \sim Q' \), then \( d_{dec}^P(Q) = d_{dec}^P(Q') \), but \( P \sim P' \) does not necessarily imply that \( d_{dec}^P(Q) = d_{dec}^P(Q') \).

7 Discussion

In this work, we gave an explicit expression for a meaningful distance in the space of all channels over given input and output sets. This establishes the ground to study the details of what can be a kind of finite approximation approach to channels and decodification problems.

A family of questions that arise in this context are the following: Let \( \overline{Ch_{n \times m}} = Ch_{n \times m}/ \sim \) be the set of decoding cones and let \( A \subset \overline{Ch_{n \times m}} \) be a subset of channels with some interesting property (for example, the set of channels that admits syndrome decoding). If we want to approximate a channel \( P \in Ch_{n \times m} \) by a decoding cone in \( A \), how much (in terms of decoding errors) should we expect to lose? From Corollary 3, we are actually interested in determining \( \max\{d_{dec}^P(Q) ; Q \in \overline{Ch_{n \times m}}\} \). Asymptotic versions arise naturally as we consider a family of increasing (in terms of \( n = |X| \) and \( m = |Y| \)) input and output sets.

This approach is similar to the one adopted in the study of mismatched channels as, for example, in [7]. The approach used in this (and other works studying mismatched channels) rests on the determination of achievable rates, that is, in proving that, for \( n \) sufficiently large there are codes that can be decoded with the approximating channel with no significant loss, that is, with probability of mis-decoding approaching 0. In our approach we are not looking at this family of codes (asymptotically the best choice of code for the mismatched channel), but on the average loss while choosing sequences of codes with a given rate.
We also stress that any prescribed deterministic decision rule can be seen as a ML decoding rule of some channel (actually an equivalence class of channels), as can be seen, for example, in [5].

Besides that, we remark that we considered the case of stable channels, i.e. the case of a decoding cone \( \text{Cone}(P) \) that is determined by a set of strict inequalities. An unstable (non-full dimensional) cone \( \text{Cone}(P) \) is determined by a set of inequalities and a non-empty set of equalities, or, in other words, \( \text{Order}(P) \) contains equivalences. It inherits its decoders from its full dimensional neighbours, that is, the cones corresponding to stable channels in which every inequality of \( \text{Cone}(P) \) also holds.

Finding explicit expressions for a distance on the set of all decoding cones, both stable and unstable, is, technically, more challenging.

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