Screening properties of a two-component charge Bose gas

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The excitation spectrum of a two-component Bose-Einstein condensate has been discussed by Ref. [1] using the Bogoliubov equations. Here we make the additional contributions of the linear response function and dielectric function for a two-component CBG in the condensation phase as well as the confirmation of the plasmon excitation spectrum using the Gross-Pitaevskii (G-P) equations.

INTRODUCTION

The properties of a two-component Bose-Einstein condensate were discussed by Alexandrov and Kabanov [1]. There the Bogoliubov theory was extended for a two-component, non-converting three-dimensional charge or neutral Bose gas. Here we make the additional contributions of the linear response function and dielectric function for a two-component CBG in the condensation phase ($\mu_j = 0$) as well as the confirmation of the plasmon excitation spectrum using the Gross-Pitaevskii (G-P) equations.

Studies of the Bose condensation temperature in the cuprates [2, 3] and also recent experiments on BEC in $^{87}$Rb atoms in two different hyperfine states [4] gives the main motivation. Such a model could apply for the cuprates since the two components may describe bipolarons travelling in either the x- or y-direction.

OPTICAL AND ACOUSTIC PLASMON MODES

We give an outline of the main results from studies by Alexandrov et al. [1]. In an external vector potential $A(r, t)$ and a scalar potential $U_j(r, t)$, the Hamiltonian for a two-component ($j=1,2$) Bose gas is given as

$$H = \sum_{j=1,2} \int dr \psi_j^\dagger(r) \left[ -\left(\frac{\nabla - iq_j A(r,t)}{2m_j}\right)^2 + U_j(r,t) - \mu_j \right] \psi_j(r) + \frac{1}{2} \sum_{j,j'} \int dr \int dr' V_{jj'}(r-r') \psi_j^\dagger(r) \psi_{j'}^\dagger(r') \psi_{j'}(r') \psi_j(r),$$

where $q_j$ and $m_j$ are the charge and mass of the $j^{th}$ Bosons. Provided the two body interaction $V_{jj'}(r)$ is weak, one may assume the ground state is macroscopically occupied. Hence one can apply a Bogoliubov displacement transformation

$$\psi(r, t) = \phi(r, t) + \tilde{\psi}(r, t)$$

and consider the excitations as fluctuations from the ground state. Furthermore one then applies a Bogoliubov linear transformation for the fluctuations

$$\tilde{\psi}(r, t) = \sum_n u_{nj}(r, t)(\alpha_n + \beta_n) + v_{nj}(r, t)(\alpha_n^\dagger + \beta_n^\dagger).$$

One obtains the Bogoliubov-de Gennes (BdG) equations:

$$\frac{d}{dt} u_{j'}(r, t) - \hbar u_{j'}(r, t) = \sum_{j''} \int dr'' V_{j'j''}(r-r'') |\phi_{j''}(r', t)|^2 u_{j''}(r, t) + \phi_{j''}^\dagger(r', t) \phi_{j'}(r, t) u_{j'}(r', t) + \phi_{j'}(r, t) \phi_{j''}(r, t) v_{j''}(r', t)$$

and
\[-i \frac{d}{dt} v_j(r,t) - \hat{h} v_j(r,t) = \sum_{j'} \int d^{3}r' V_{j,j'}(r-r') [\phi_{j'}(r',t)]^2 v_j(r,t) + \phi_j(r',t) \phi_{j'}^*(r,t)v_j(r',t) + \phi_j^*(r',t) \phi_{j'}(r,t) u_{j'}(r',t), \]

where

\[\sum_{n} [u_{nj}(r,t)u_{nj}^*(r',t) + v_{nj}(r,t)v_{nj}^*(r',t)] = \delta(r-r')\]  

and

\[\hat{h} = -\frac{(\nabla - iq_{j}A(r,t))^2}{2m_j} + U_j(r,t) - \mu_j.\]

Using these BdG equations one finds the elementary excitations for a two-component Bose condensate with arbitrary interactions:

\[E_{1,2}(k) = \frac{1}{2\sqrt{2}} \left( \varepsilon_1^2(k) + \varepsilon_2^2(k) \right)^{1/2} \]

where we assume no external fields and that the excitation coherence factors are plane waves

\[u_{kj}(r,t) = u_{k,j} e^{i(k \cdot r - E_{kt})}\]

\[v_{kj}(r,t) = v_{k,j} e^{i(k \cdot r - E_{kt})}.\]

Also we have used the definitions

\[\varepsilon_1(k) = \sqrt{\frac{k^4}{4m_1^2} + \frac{k^2V_{k1n_1}}{m_1}}\]

\[\varepsilon_2(k) = \sqrt{\frac{k^4}{4m_2^2} + \frac{k^2U_{k2n_2}}{m_2}}\]

and where \(V_{k1}, U_{k1}\) and \(W_{k}\) are the Fourier components of the interactions \(V_{11}(r), V_{22}(r)\) and \(V_{12}(r) = V_{21}(r)\) respectively. This result implies there are two branches, acoustic and optical. The acoustic branch is quadratic in \(k\) and means that the condition for superfluidity is not met (Landau criterion). However such a problem can be avoided by considering more than one interaction potential, for example a hard-core and a long-range potential. Therefore a two-component bipolaron gas may still be a superfluid.

**LINEAR RESPONSE AND THE DIELECTRIC FUNCTION**

Here we derive the linear response function for a two-component CBG. From this we obtain the dielectric function and examine its behaviour. In the process we find the same excitation spectrum found earlier using the Bogolubov equations of the previous section.

The G-P equation in the condensed phase is written:

\[-i \frac{d}{dt} \psi_{0j}(r,t) = -\left(\nabla - ie^{-}A\right)^2 \psi_{0j}(r,t) + \sum_{j'} \int d^{3}r' V_{j,j'}(r-r')|\psi_{0j'}(r',t)|^2 \psi_{0j}(r,t)\]

We allow for small fluctuations of the order parameter as

\[\psi_{0j}(r,t) = \sqrt{n_{0j}} + \phi_j(r,t)\]

and keeping terms linear in \(A\) and fluctuations \(\phi_j(r,t)\) and assuming we have a homogeneous background charge:

\[\int d^{3}r' V(r') = 0,\]
we obtain

\[
i \frac{d}{dt} \phi_j(\mathbf{r}, t) = -\frac{\nabla^2}{2m_j} \phi_j(\mathbf{r}, t) + \frac{e^* \sqrt{n_{oj}}}{2m_j} \nabla \cdot \mathbf{A}(\mathbf{r}, t) + \sum_{j'} \int d\mathbf{r}' V_{jj'}(\mathbf{r} - \mathbf{r}') \sqrt{n_{o_j} n_{o_{j'}}} \left[ \phi_{j'}(\mathbf{r}', t) + \phi_{j'}^*(\mathbf{r}', t) \right]. \tag{16}
\]

By Fourier transformation this becomes

\[
\omega \phi_j(\mathbf{q}, \omega) = q^2 \frac{1}{2m_j} \phi_j(\mathbf{q}, \omega) - \frac{e^* \sqrt{n_{o_j}}}{2m_j} \mathbf{q} \cdot \mathbf{a}(\mathbf{q}, \omega) + \sum_{j'} V_{jj'}(\mathbf{q}, \omega) \sqrt{n_{o_j} n_{o_{j'}}} \left[ \phi_{j'}(\mathbf{q}, \omega) + \phi_{j'}^*(-\mathbf{q}, -\omega) \right]. \tag{17}
\]

The complex conjugate is easily found to be

\[
-\omega \phi_j^*(-\mathbf{q}, -\omega) = q^2 \frac{1}{2m_j} \phi_j^*(-\mathbf{q}, -\omega) - \frac{e^* \sqrt{n_{o_j}}}{2m_j} \mathbf{q} \cdot \mathbf{a}(\mathbf{q}, -\omega) + \sum_{j'} V_{jj'}(\mathbf{q}, \omega) \sqrt{n_{o_j} n_{o_{j'}}} \left[ \phi_{j'}(\mathbf{q}, \omega) + \phi_{j'}^*(-\mathbf{q}, -\omega) \right] \tag{18}
\]

where

\[
\int d\mathbf{r}' V(\mathbf{r} - \mathbf{r}') e^{-i \mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} = V(\mathbf{q}) \tag{19}
\]

and \( \mathbf{a}(\mathbf{q}, \omega) = \mathbf{a}^*(-\mathbf{q}, -\omega) \) is the Fourier transform of the external vector potential. The 4 equations can be written in a matrix form as

\[
\begin{pmatrix}
\frac{e^* \sqrt{n_{o1}}}{2m_1} \\
\frac{e^* \sqrt{n_{o2}}}{2m_2}
\end{pmatrix}
\cdot \mathbf{a} =
\begin{pmatrix}
q^2 \frac{1}{2m_1} + V_{11} n_{o1} - \omega & V_{12} \sqrt{n_{o1} n_{o2}} & V_{11} n_{o1} & V_{12} \sqrt{n_{o1} n_{o2}} \\
V_{21} \sqrt{n_{o2} n_{o1}} & \frac{q^2}{2m_2} + V_{22} n_{o2} - \omega & V_{21} \sqrt{n_{o2} n_{o1}} & V_{22} n_{o2} \\
V_{11} n_{o1} & V_{12} \sqrt{n_{o1} n_{o2}} & \frac{q^2}{2m_1} + V_{11} n_{o1} + \omega & V_{12} \sqrt{n_{o1} n_{o2}} \\
V_{21} \sqrt{n_{o2} n_{o1}} & V_{22} n_{o2} & V_{21} \sqrt{n_{o2} n_{o1}} & \frac{q^2}{2m_2} + V_{22} n_{o2} + \omega
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_1^* \\
\phi_2^*
\end{pmatrix} \tag{20}
\]

Solving this gives the excitation spectrum of a two-component CBG in an external field with arbitrary interactions. It is useful to make sure the excitation spectrum is the same as that found by using the BdG equations \[1\]. In the presence of no external fields the determinant is zero and hence gives

\[
\omega_{1,2}^2(\mathbf{k}) = \frac{1}{2} \left( \varepsilon_1^2(\mathbf{k}) + \varepsilon_2^2(\mathbf{k}) \pm \sqrt{[\varepsilon_1^2(\mathbf{k}) - \varepsilon_2^2(\mathbf{k})]^2 + \frac{4k^4}{m_1 m_2} V^2 n_{o1} n_{o2}} \right) \tag{21}
\]
where we assume that $V = V_{11} = V_{22} = V_{12} = V_{21}$. This result is equivalent to eqn 8. Again this leads to the acoustic and plasmon modes.

\[ D_j = -e^* \left( \frac{q^2}{2m_j} \sqrt{n_{0j}/m_j} + \frac{2V(q)n_{0j}}{m_j} + \frac{2V(q)n_{0j}/\sqrt{n_{0j}}}{m_j'} \right) q \cdot a(q, \omega) \]  

(22)

and

\[ B_j = \phi_j(q, \omega) - \phi^*_j(-q, -\omega) \]  

(23)

In the presence of external fields ($A \neq 0$) we can solve for $B_j = \phi_j(q, \omega) - \phi^*_j(-q, -\omega)$:

\[ B_j = \frac{dD_j - bD_{j'}}{\text{Det} \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \]  

(25)

\[ = -e^* \left( \frac{\omega^2 - \varepsilon_j^2}{\varepsilon_j^2} \right) \sqrt{n_{0j}/m_j} + \frac{V_n q_{0j} q^2}{m_j'} - q \cdot a(q, \omega) \]  

\[ - e^* \frac{2V \sqrt{n_{0j}/m_j}}{\omega^2 - \varepsilon_j^2} \left( \varepsilon_j^2 + \frac{V_n q_{0j} q^2}{m_j'} \right) \]  

\[ - \frac{V_n}{\omega^2 - \varepsilon_j^2} \left( \varepsilon_j^2 + \frac{V_n q_{0j} q^2}{m_j'} \right) q \cdot a(q, \omega) \]  

(26)

This equation simplifies into the case for a single component charge Bose gas as found in ref \textsuperscript{5}. This can be done by taking the second component number density $n_{0j} \rightarrow 0$:

\[ B_j = \phi_j(q, \omega) - \phi^*_j(-q, -\omega) \]  

\[ = -e^* \frac{\sqrt{n_{0j}/m_j}}{\omega^2 - \varepsilon_j^2} \left( \varepsilon_j^2 + \frac{V_n q_{0j} q^2}{m_j'} \right) \]  

(27)

Back to the two-component system, the linear-response function $K'^{mn}$ is defined for each component $j$ of the charge Bose gas as

\[ J_j^m(q, \omega) = K'^{mn} a_n(q, \omega) \]  

(28)

For the purposes of the linear response function, we write the definitions

\[ J_j^m(q, \omega) = \psi^*_j(r, t) \nabla \psi_j(r, t) - (\nabla \psi^*_j(r, t)) \psi_j(r, t) \]  

\[ - \frac{e^*}{m_j} \psi^*_j(r, t) | \rho_j^m(r, t) |^2 A(r, t) \]  

(29)

where as before:

\[ \psi_j(r, t) = \sqrt{n_{0j}} + \phi_j(r, t) \]  

(30)

Keeping terms linear in $A$ and fluctuations $\phi_j(r, t)$ and performing the Fourier transform we obtain

\[ J_j^m(q, \omega) = \frac{e^*}{2m_j} \left( \phi_j(q, \omega) - \phi^*_j(-q, -\omega) \right) q \]  

\[ + \frac{e^*}{m_j} a(q, \omega) \]  

(31)

The first term contains $B_j$ and so the current for one component includes interference effects from the other. Subbing in we obtain for the current
\[ J_j(q, \omega) = \frac{-e^2 n_0}{m_j} \left[ \left\{ \frac{a_j^2 (\omega^2 - \varepsilon_j^2)}{m_j} (\varepsilon_j^2 + \frac{V_{n_0} q^2}{m_j}) + \frac{V_{n_0} q}{m_j} \right\} q \cdot a_j \right] + a_j. \]  

We can therefore write the linear-response function as

\[ K_{mn}^j = -\frac{e^2 n_0}{m_j} \left[ \frac{a_m a_n}{\omega^2} \left( \varepsilon_j^2 + \frac{V_{n_0} q^2}{m_j} \right) + \frac{V_{n_0} q}{m_j} q^m q_n \left( \varepsilon_j^2 + \frac{V_{n_0} q^2}{m_j} \right) \right] + \delta^{mn} \]  

where the linear-response function has been divided into a longitudinal and transverse part. Since we ultimately want to examine the dielectric function \( \epsilon(q, \omega) \) we examine the longitudinal part which can be reduced down to

\[ K_{mm}^m = -\frac{e^2 n_0}{m_j} \frac{\omega^2 \left[ (\omega^2 - \varepsilon_j^2) (\varepsilon_j^2 + \frac{V_{n_0} q^2}{m_j}) - \omega_{\rho_0}^2 \varepsilon_j^2 \right]}{(\omega^2 - \varepsilon_j^2)(\omega^2 - \varepsilon_j^2) - \omega_{\rho_0}^2 \omega_{\rho_0}^2} \]  

where \( \omega_{\rho_0}^2 = V_{n_0} q^2 / m_j \). To obtain the dielectric func-

![Graph](image)

**FIG. 1:** The inverse of the dielectric function for a 2 component charge Bose gas, plotted with varying density ratio of the 2 components.

We use the continuity equation

\[ \nabla \cdot \mathbf{J} = -\frac{d}{dt} \rho_{\text{ind}}, \]  

one of the Maxwell equations

\[ \nabla \cdot \mathbf{D} = 4\pi \rho_{\text{ext}}, \]  

the current-external electric field relation

\[ \mathbf{J}(q, \omega) = \frac{K_{mm}^m}{i\omega} \mathbf{D}(q, \omega) \]  

and the usual definition for the dielectric function

\[ \frac{1}{\epsilon(q, \omega)} = \frac{\rho_{\text{tot}}}{\rho_{\text{ext}}}. \]  

Here \( \rho_{\text{ind}}, \rho_{\text{ext}} \) and \( \rho_{\text{tot}} \) are induced, external and total charge density. We find for the inverse dielectric function

\[ \frac{1}{\epsilon(q, \omega)} = 1 - \frac{4\pi}{\omega^2} \sum_j K_{mm}^m. \]  

Doing the summation

\[ \frac{1}{\epsilon(q, \omega)} = 1 + \frac{\omega_{p_01}^2 [\omega^2 + \omega_{p_02}^2 - \varepsilon_j^2] + \omega_{p_02}^2 [\omega^2 + \omega_{p_01}^2 - \varepsilon_j^2]}{(\omega^2 - \varepsilon_j^2)(\omega^2 - \varepsilon_j^2) - \omega_{p_01}^2 \omega_{p_02}^2} \]  

and rearranging for the static case we have

\[ \epsilon(q, 0) = 1 + \left( \frac{q_s}{q} \right)^4 \]  

where

\[ q_s^4 = 4m_1^2 \omega_{p_01}^2 + 4m_2^2 \omega_{p_02}^2. \]  

The result is exactly the same as for a single component system except here the screening wavenumber is effectively a sum of number densities. In fig 1 we plot the inverse static dielectric function. We notice that as the second components number density is increased the screening becomes stronger at intermediate \( q \) values as expected.

**SUMMARY**

A two-component CBG with arbitrary interactions has been examined. The study of its linear response
and dielectric function yield similar results to a single component CBG. The excitation spectrum in an external field was derived, whilst it was used to re-confirm previous results which were derived with no external field. This study of a two-component CBG should be useful for further studies in bipolaronic systems applied to lattices. Also it will be useful to work in atomic physics.

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