TRIVALENT EXPANDERS AND HYPERBOLIC SURFACES

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ABSTRACT. We introduce a family of trivalent expanders which tessellate compact hyperbolic surfaces with large isometry groups. We compare this family with Platonic graphs and modifications of them and prove topological and spectral properties of these families.

1. Introduction and statement of results

In this article, we consider a family of surface tessellations with interesting discrete spectral gap properties. More specifically, our family of graphs, denoted by $T_k$ ($k \geq 2$), are trivalent expander graphs tessellating hyperbolic surfaces with large isometry groups growing linear on genus. As we show below, from $k \geq 3$ onwards, there is no direct relation between our family of graphs and other graphs associated to modular congruence subgroups.

Let us first give a brief overview over the construction of $T_k$, and then go into some details. We start with a sequence of 2-groups $G_k$, following the construction in [17, Section 2]. Then we consider 6-valent Cayley graphs $X_k$ of these groups and apply $(\Delta - Y)$-transformations in all triangles of $X_k$, to finally obtain the trivalent graphs $T_k$. The $(\Delta - Y)$-transformations are standard operations to simplify electrical circuits, and were also used in [1] in connection with Colin de Verdière’s graph parameter.

The finite groups $G_k$ are constructed as follows. We start with the infinite group $\tilde{G}$ of seven generators and seven relations:

\begin{equation}
\tilde{G} = \langle x_0, \ldots, x_6 \mid x_i x_{i+1} x_{i+3} \text{ for } i = 0, \ldots, 6 \rangle,
\end{equation}

where the indices are taken modulo 7. As explained in [8], this group acts on a thick Euclidean building of type $\tilde{A}_2$. Let $S = \{x_0^{\pm 1}, x_1^{\pm 1}, x_3^{\pm 1}\}$, and consider the index two subgroup $G \leq \tilde{G}$, generated by $S$. (Note that $x_3 = x_1^{-1} x_0^{-1}$.) As explained in [17 Section 2], we use a representation of the group $G$ by infinite (finite band) upper triangular Toeplitz matrices.
matrices. The entries of these Toeplitz matrices are elements of the ring $M(3, \mathbb{F}_2)$ (i.e., $3 \times 3$-matrices over $\mathbb{F}_2$) with special periodicity properties. We denote the group of all these Toeplitz matrices by $H$, and by $H_k \leq H$ the normal subgroup of matrices whose first $k$ upper diagonals are zero. The groups $G_k$ are then the quotients $G/(G \cap H_k)$. The finite width conjecture in [17] claims that the groups $G_k$ have another purely abstract group theoretical description via the lower exponent-2 series

$$G = P_0(G) \geq P_1(G) \geq P_2(G) \geq \cdots,$$

with $P_k(G) = [P_{k-1}(G), G]P_{k-1}(G)^2$ for $k \geq 1$: namely, $G \cap H_k = P_k(G)$ for $k \geq 1$ (see [17, Conj. 1]). MAGMA computations confirm this conjecture for all indices up to $k = 100$. For simplicity, we use the same notation for the elements $x_0, x_1, x_3$ in $G$ and their images in the quotients $G_k$. Then $X_k = \text{Cay}(G, S)$, and $T_k$ are their $(\Delta - Y)$-transformations.

The graphs $T_k$ can be naturally embedded as tessellations into both compact hyperbolic surfaces $S(T_k)$ and non-compact finite area hyperbolic surfaces $S_\infty(T_k)$. The edges of the tessellation are geodesics and the vertices are their end points. Our results are given in the following theorem:

**Theorem 1.1.** Let $k \geq 2$. Then every eigenvalue $\mu \neq 3$ of $X_k$ gives rise to a pair $\pm \sqrt{\mu + 3}$ of eigenvalues of the bipartite graph $T_k$. In particular, there exists a positive constant $C < 6$ such that

1. the graphs $X_k$ are 6-valent expanders with spectrum in $[-3, C] \cup \{6\}$,
2. the bipartite graphs $T_k$ are trivalent expanders with spectrum in $[-\sqrt{C + 3}, \sqrt{C + 3}] \cup \{-3\}$.

Moreover, the isometry group of the compact hyperbolic surface $S(T_k)$ has order $\geq |V(T_k)|/2$, where $V(T_k)$ denotes the set of vertices of $T_k$. Let

$$r = \lfloor \log_2 k \rfloor + 1 \quad \text{and} \quad K = 8|k/3| + 3 \cdot (k \text{ mod } 3).$$

Then we have $|V(T_k)| \geq 2^K$, $|E(T_k)| \geq 3 \cdot 2^{K-1}$ and $|F(T_k)| \geq 3 \cdot 2^{K-r-1}$ for the vertices, edges, and faces of $T_k$, and all faces of $T_k$ are regular $2^{r+1}$-gons. The genus of $S(T_k)$ can be estimated by

$$g = |V(T_k)| - |E(T_k)| + |F(T_k)| \geq 1 + 2^{K-2} - 3 \cdot 2^{K-r-2}.$$

The above-mentioned finite width conjecture would imply that the inequalities for the vertices, edges, faces of $T_k$ and the genus of $S(T_k)$ in Theorem 1.1 hold with equality. Theorem 1.1 is proved in Section 2.2.
It is instructive to compare our tessellations $T_k$ to the well studied tessellations of hyperbolic surfaces by Platonic graphs $\Pi_N$, which are defined as follows. Let $N$ be a positive integer $\geq 2$. The vertices of $\Pi_N$ are equivalence classes $[\lambda, \mu] = \{\pm (\lambda, \mu)\}$ with
\[
\{(\lambda, \mu) \in \mathbb{Z}_N \times \mathbb{Z}_N \mid \gcd(\lambda, \mu, N) = 1\}.
\]
Two vertices $[\lambda, \mu]$ and $[\nu, \omega]$ are connected by an edge if and only if
\[
\det \begin{pmatrix} \lambda & \nu \\ \mu & \omega \end{pmatrix} = \lambda \omega - \mu \nu = \pm 1.
\]

Note that every vertex of $\Pi_N$ has degree $N$. These graphs can also be viewed as triangular tessellations of finite area hyperbolic surfaces $S_\infty(\Pi_N) = \mathbb{H}^2/\Gamma(N)$, where $\mathbb{H}^2$ denotes the hyperbolic upper half plane and $\Gamma(N)$ is a principal congruence subgroup of the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$. These and related graphs have been thoroughly investigated by several different communities. For example, in the general framework of regular maps, they were studied by D. Singerman and co-authors (see [12, 19, 11]). For odd prime numbers $N = p$, the graphs $\Pi_p$ have maximal vertex connectivity $p$, diameter 3, and are Ramanujan graphs. Analogous properties hold for the induced subgraphs $\Pi'_p$, where $\Pi'_p$ is obtained from $\Pi_p$ by removing the set of vertices $[\lambda, 0]$ with vanishing second coordinate and all their adjacent edges. Note that $\Pi'_p$ is a $(p-1)$-valent graph tessellating the same surface $S_\infty(\Pi_p)$. As in the case of our family $T_k$, the graphs $\Pi_N$ and $\Pi'_N$ can also be embedded into smooth compact hyperbolic surfaces, denoted by $S(\Pi_N)$.

It turns out that the graphs $T_2^*$ and $\Pi_8$ are isomorphic. Since the valence of the dual graph $T_k^*$ is a power of 2, any isomorphism of $T_k^*$ with a Platonic graph $\Pi_N$ would imply $N = 2^\rho$ with $\rho = \lfloor \log_2 k \rfloor + 2$. However, this leads to a contradiction for all $k \geq 3$. The next proposition summarizes the comparison between our graphs and Platonic graphs showing that, generally, these two families are of very different nature. The proof is given in Section 2.4.

**Proposition 1.2.** The graph $T_2$ is the dual of the Platonic graph $\Pi_8$ in the unique genus 5 hyperbolic surface $S(T_2) = S(\Pi_8)$ with maximal automorphism group of order 192. For $k \geq 3$, there is no graph isomorphism between $T_k^*$ and $\Pi_N$, for any $N$.

Let us say a few more words about the Platonic graphs $\Pi_p$ and their modifications $\Pi'_p$. The modified graphs $\Pi'_p$ have an alternative description as Cayley graphs of the quotients $\Gamma_0(p)/\Gamma(p)$ of congruence subgroups (see end of Section 3.3). However, we did not find these modified graphs explicitly in the literature (for example, they do not
Cheeger constant estimates for $\Pi_p$ have been obtained, e.g., in \cite{7, 13}, but we do not know of any reference for the maximal vertex connectivity, and present a proof of this fact for both graph families $\Pi_p$ and $\Pi'_p$ in Section 3.3. To our knowledge, all proofs for the Ramanujan property of the graphs $\Pi_p$ in the literature (see, e.g., \cite{10, 14, 9}) are based on some amount of number theory (characters of representations). We think it is remarkable that there is also an easy proof for the Ramanujan properties of the graphs $\Pi_p$ and $\Pi'_p$ with no reference to number theory other than the irrationality of $\sqrt{p}$ (see Section 3.4). These facts are summarized in the following theorem.

**Theorem 1.3.** Let $p$ be an odd prime. Then the graphs $\Pi_p$ and $\Pi'_p$ have diameter 3 and maximal vertex connectivity $p$ and $p - 1$, respectively. Moreover, the spectrum of the graph $\Pi'_p$ consists of

(i) $p - 1$ with multiplicity one,

(ii) $-1$ with multiplicity $p - 1$,

(iii) 0 with multiplicity $(p - 3)/2$, and

(iv) $\pm \sqrt{p}$ with multiplicity $(p - 1)(p - 3)/4$, each.

In particular, the graphs $\Pi'_p$ are Ramanujan.

As mentioned earlier, our family $T_k$ is based on powers of the prime number 2. The question whether the Ramanujan property of $\Pi_N$ for primes $N = p$ still holds for composite numbers $N$, or, at least, for prime powers $N = p^r$, was answered in the negative in \cite{10, Prop. 4.7}. But the Ramanujan property for prime powers is preserved if one considers Platonic graphs over finite fields $\mathbb{F}_{p^r}$ instead of the rings $\mathbb{Z}_{p^r}$ (see \cite{9}). However, we do not know how these Ramanujan graphs associated to prime powers could be naturally embedded into appropriate surfaces.

It is easily checked that any triangular tessellation $X$ of a compact oriented surface $S$ satisfies $|E(X)| = 3(|V(X)| - 2) + 6g(S)$, i.e., the number of edges of every triangulation with at least two vertices is $\geq 6g(S)$. Therefore, the ratio

$$\frac{6g(S)}{|E(X)|} \leq 1,$$

measures the non-flatness of such a triangulation, i.e., how effectively the edges of $X$ are chosen to generate a surface of high genus. The following asymptotic results hold for the triangulations $\Pi_N$ and $T^*_k$.

**Proposition 1.4.** We have

$$\lim_{N \to \infty} \frac{6g(S(\Pi_N))}{|E(\Pi_N)|} = 1,$$

(3)
and
\[
\lim_{k \to \infty} \frac{6g(\mathcal{S}(T_k))}{|E(T_k)|} = 1.
\]

In the second case, note that the dual graph \(T_k^*\) is a triangulation of \(\mathcal{S}(T_k)\) and that the number of edges of \(T_k\) and \(T_k^*\) coincide.

The two formulas in this proposition are proved in Sections 2.3 and 3.2.

Our trivalent expander graphs \(T_k\) can also be used to construct another family of compact hyperbolic surfaces \(\hat{\mathcal{S}}(T_k)\) by glueing together regular \(Y\)-pieces, as explained in Buser [6]. The surfaces \(\hat{\mathcal{S}}(T_k)\) can be viewed as tubes around the graphs \(T_k\) with a hyperbolic metric. Using the results in [6], the expander properties of \(T_k\) translate directly into a uniform lower bound of the first non-trivial eigenvalue \(\lambda_1\) of the Laplacian on these surfaces.

**Corollary 1.5.** The compact hyperbolic surfaces \(\hat{\mathcal{S}}(T_k)\) \((k \geq 2)\) have genus \(1 + |V(T_k)|/2\) and isometry groups of order \(\geq |V(T_k)|/2\). They form a tower of coverings
\[
\cdots \to \hat{\mathcal{S}}(T_{k+1}) \to \hat{\mathcal{S}}(T_k) \to \hat{\mathcal{S}}(T_{k-1}) \to \cdots
\]
where all the covering indices are powers of 2. There is a positive constant \(\epsilon > 0\) such that we have, for all \(k\),
\[
\lambda_1(\hat{\mathcal{S}}(T_k)) \geq \epsilon.
\]

Corollary 1.5 is proved in Section 2.5. There is a well-known classical result by Randol [18] which is in some sense complementary to this corollary. Namely, there exist finite coverings \(\hat{\mathcal{S}}\) of every compact hyperbolic surface \(\mathcal{S}\) with arbitrarily small first eigenvalues. It would be interesting to find out whether there are also uniform positive lower bounds for \(\lambda_1\) of our other compact hyperbolic surfaces \(\mathcal{S}(T_k)\). It seems that the methods in [4, 5] are not applicable in this case, since the shapes of the hyperbolic triangles tessellating these surfaces are changing with \(k\).

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2. Properties of the tessellations \((T_k, \mathcal{S}(T_k))\)

2.1. The surfaces \(S_\infty(T_k)\) and \(\mathcal{S}(T_k)\). The explicit construction of the Cayley graphs \(X_k = \text{Cay}(G_k, S)\) and of the trivalent graphs \(T_k\) was explained in the introduction. For the underlying 2-groups \(G_k\), we refer the reader to [17] Section 2]. Let us now construct the hyperbolic
surface $S_\infty(T_k)$: We start with a 3-punctured sphere $S_0$, by glueing together two ideal hyperbolic triangles along their corresponding edges. Note that $S_0$ carries a hyperbolic metric. It is useful to think of the two ideal triangles of $S_0$ to be coloured black and white. Let $P_0 \in S_0$ be the center of the black triangle. Choose a geometric basis $\gamma_0, \gamma_1, \gamma_2 \in \pi(S_0, P_0)$ such that $\gamma_i$ is a simple counterclockwise look around the $i$-th cusp of $S_0$ and $\gamma_0 \gamma_1 \gamma_2 = e$. The surjective homomorphism

$$
\Psi : \pi(S_0, P_0) \to G_k,
$$
given by $\Psi(\gamma_0) = x_0, \Psi(\gamma_1) = x_1$ and $\Psi(\gamma_2) = x_3$, induces a Riemannian covering map $\pi : S_\infty \to S_0$. The surface $S_\infty$ is a hyperbolic surface, tessellated by $2|G_k|$ ideal hyperbolic triangles, half of them black and the others white. Hurwitz’s formula yields

$$
g(S_\infty) = 1 + \frac{1 - \mu_k}{2}|G_k|,
$$
where

$$
\mu_k = \frac{1}{\text{ord}(x_0)} + \frac{1}{\text{ord}(x_1)} + \frac{1}{\text{ord}(x_3)}.
$$

In the case $k = 2$ we have $|G_2| = 32$ and $\text{ord}(x_0) = \text{ord}(x_1) = \text{ord}(x_3) = 4$, which leads to

$$
g(S_\infty) = 1 + \frac{1}{8} \cdot 32 = 5.
$$

$G_k$ acts simply transitive on the black triangles of $S_\infty$. Let $V = \pi^{-1}(P_0)$ and $V_{\text{black}}, V_{\text{white}} \subset V$ be the sets of centers of black and white triangles, respectively. Choose a reference point $P \in V_{\text{black}}$, and identify the vertices of the Cayley graph $X_k$ with the points in $V_{\text{black}}$ by $G_k \ni h \mapsto hP \in V_{\text{black}}$. Then two adjacent vertices in $X_k$ are the centers of two black triangles which share a white triangle as their common neighbour. The corresponding edge is then the minimal geodesic passing through these three ideal triangles and connecting these two vertices.

We could instead start the process by glueing together two compact hyperbolic triangles with angles $\pi/\text{ord}(x_0), \pi/\text{ord}(x_1)$ and $\pi/\text{ord}(x_3)$, and obtain an orbifold $S_0$. The same arguments then lead to an embedding of $X_k$ into a smooth compact hyperbolic surface $S$ with the same genus as $S_\infty$. The surface $S$ is triangulated by compact black and white triangles, and every black triangle contains a vertex of $X_k$. In fact, $G_k$ acts on the surface $S$ by isometries.

The abstract $(\Delta - Y)$-transformation of a graph adds a new vertex $v$ for every triangle, removes the three edges of this triangle and replaces them by three edges connecting $v$ with the vertices of this triangle. We
apply this rule to our graph $X_k$ and obtain a graph $T_k$, which we can view as an embedding in $S$ with the following properties: The vertex set of $T_k$ coincides with $V$, and there is an edge (geodesic segment) connecting every black/white vertex in $V$ with the vertices in the three neighbouring white/black triangles. The best way to illustrate this transformation is to present it in the universal covering of the surface $S$, i.e., the Poincaré unit disc $\mathbb{D}$ (see Figure 1, the new vertices replacing every triangle are green). Note that $T_k$ has twice as many vertices as $X_k$, which shows that the isometry group of the above compact surface $S = S(T_k)$ has order $\geq |G_k| = |V(T_k)|/2$. Moreover, $T_k$ is indeed the dual of the triangulation of $S$ by the abovementioned compact black and white triangles.

![Figure 1. The lifts of the Cayley graph $X_2$ (left) and of the $(\Delta - Y')$-transformation $T_2$ (right) to the Poincaré unit disc $\mathbb{D}$](image)

### 2.2. Proof of Theorem 1.1

We first establish the expander properties of $X_k$ and $T_k$ and the relations between their eigenvalues, stated in Theorem 1.1. It was proved in [17, Section 2] that the group $\tilde{G}$ generated by $x_0, x_1 \in \tilde{G}$ is an index two subgroup of the group $G$ in (1). $G$ is explicitly given by $G = \langle x_0, x_1 \mid r_1, r_2, r_3 \rangle$ with

\begin{align*}
    r_1(x_0, x_1) &= (x_1x_0)^3x_1^{-3}x_0^{-3}, \\
    r_2(x_0, x_1) &= x_1x_0^{-1}x_1^{-1}x_0^{-3}x_1^2x_0^{-1}x_1x_0x_1, \\
    r_3(x_0, x_1) &= x_1^3x_0^{-1}x_1x_0x_1^2x_0^2x_1x_0x_1x_0.
\end{align*}

(Note that our group $\tilde{G}$ is denoted in [17] by $\Gamma$, which is reserved for $\text{PSL}(2, \mathbb{Z})$ in this paper.) Moreover, both groups $\tilde{G}$ and $G$ have Kazhdan property (T) (see [17, Section 3]). Using [15, Prop. 3.3], we conclude that the Cayley graphs $X_k$ are expanders.
The adjacency operator $A$, acting on functions on the vertices of a graph, is defined as

$$Af(v) = \sum_{w \sim v} f(w).$$

Note that $V(X_k)$ is a subset of $V(T_k)$. We have the following relations between the eigenfunctions of the adjacency operators on $X_k$ and $T_k$.

**Theorem 2.1.**

(a) Every eigenfunction $F$ on $T_k$ to an eigenvalue $\lambda \in [-3, 3]$ gives rise to an eigenfunction $f$ to the eigenvalue $\mu = \lambda^2 - 3 \in [-3, 6]$ on $X_k$ (with $f(v) = F(v)$ for all $v \in V(X_k)$).

(b) Every eigenfunction $f$ on $X_k$ to an eigenvalue $\mu \in [-6, 6] - \{3\}$ gives rise to two eigenfunctions $F_{\pm}$ to the eigenvalues $\pm \sqrt{\mu + 3}$ on $T_k$ with

$$F_{\pm}(v) = \begin{cases} 
  f(v) & \text{if } v \in V(X_k), \\
  \pm \frac{1}{\sqrt{\mu + 3}} \sum_{w \sim v} f(w) & \text{if } v \in V(T_k) - V(X_k).
\end{cases}$$

(c) An eigenfunction $f$ on $X_k$ to the eigenvalue $-3$ gives rise to an eigenfunction $F$ to the eigenvalue 0 of $T_k$ with

$$F(v) = \begin{cases} 
  f(v) & \text{if } v \in V(X_k), \\
  0 & \text{if } v \in V(T_k) - V(X_k),
\end{cases}$$

if and only if we have, for all triangles $\Delta$ in $X_k$, $\sum_{v \in V(\Delta)} f(v) = 0$.

**Proof.** (a) Let $f$ and $F$ be two functions on $X_k$ and $T_k$, related by $f(v) = F(v)$ for all $v \in V(X_k)$. Then

$$A_{X_k} f(v) = \sum_{w \sim v} f(w) = \sum_{d_{T_k}(w,v) = 2} F(w) = (A_{T_k})^2 F(v) - 3F(v),$$

which can also be written as $A_{X_k} = (A_{T_k})^2 - 3$. (Note that $\sim_{X_k}$ denotes adjacency in $X_k$, and $d_{T_k}$ is the combinatorial distance in $T_k$.) This implies immediately the connection between the eigenfunctions and eigenvalues.

(b) Let $A_{X_k} f = \mu f$ and $F_{\pm}$ be defined as in the theorem. Let $\lambda = \pm \sqrt{\mu + 3}$. Then we have for $v \in V(X_k)$:

$$A_{T_k} F_{\pm}(v) = \sum_{w \sim v} F_{\pm}(w) = \frac{1}{\lambda} \sum_{w \sim v} \sum_{x \sim w} F_{\pm}(x)$$

$$= \frac{1}{\lambda} \left( \sum_{w \sim X_k v} f(w) + 3f(v) \right) = \frac{\mu + 3}{\lambda} f(v) = \lambda F_{\pm}(v),$$
and for $v \in V(T_k) - V(X_k)$:

$$A_{T_k} F_\pm(v) = \sum_{w \sim v} F_\pm(w) = \lambda \left( \frac{1}{\lambda} \sum_{w \sim v} f(w) \right) = \lambda F_\pm(v).$$

Note that $1/\lambda$ is well defined since $\lambda = \pm \sqrt{3} \neq 0$.

(c) In the case of $\mu = -3$ we have $\lambda = 0$, and the above calculation for $v \in V(X_k)$ goes through without changes. For $v \in V(T_k) - V(X_k)$, the condition

$$0 = A_{T_k} F_\pm(v) = \sum_{w \sim T_k v} f(v)$$

translates into the condition that the summation of $f$ over the vertices of all the triangles must vanish.

Theorem 2.1 implies that the expander property of the family $X_k$ carries over to the graphs $T_k$. Moreover, the spectrum of $X_k$ cannot contain eigenvalues in the interval $[-6, -3)$, since this would lead to non-real eigenvalues of $T_k$. This finishes the proof of the spectral statements in Theorem 1.1.

Since $X_k$ are Cayley graphs of quotients of the group $G$ with property (T), not all of these graphs can be Ramanujan (see [15, Prop. 4.5.7]). But what can we say about their $(\Delta - Y)$-transformations $T_k$? Theorem 2.1 implies that $T_k$ is Ramanujan if and only if the largest non-trivial eigenvalue of $X_k$ is $< 5$. MAGMA computations provide the following numerical results:

| graph | number of vertices | largest non-trivial eigenvalue |
|-------|-------------------|-------------------------------|
| $X_2$ | 32                | 2.828427124746190...          |
| $X_3$ | 128               | 4.340172973252067...          |
| $X_4$ | 1024              | 4.475244292138809...          |
| $X_5$ | 8192              | 5.160252515773351...          |

This implies that only $X_2$ and $X_3$ are Ramanujan; their largest non-trivial eigenvalue needs to be $< 2\sqrt{3} = 4.472135...$, which is no longer true for $k = 4$. Moreover, we have $\sigma(X_k) \subset \sigma(X_{k+1})$, since the graphs $X_k$ are a tower of coverings. Similarly, only $T_2, T_3, T_4$ are Ramanujan, since the covering properties of $X_k$ carry over to their $(\Delta - Y)$-transformations $T_k$.

Finally, we obtain from [17, Cor. 2.3] that

$$|V(T_k)| = 2|G_k| = 2[G : (G \cap H_k)] \geq 2^K.$$  

Since Conjecture 1 in [17] (i.e., $G \cap H_k = P_k(G)$) holds for all $3 \leq k \leq 100$, (8) holds for these indices with equality. The faces of $T_k$ are
determined by the orders of the generators \( x_0, x_1, x_3 \) in the group \( G_k \), which we determine next. Let

\[
\alpha_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1
\end{pmatrix},
\beta_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
\alpha_1 = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix},
\beta_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{pmatrix},
\]

\[
\alpha_3 = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix},
\beta_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}.
\]

Lemma 2.2. We have, in the notation of [17], for \( i \in \{0, 1, 3\} \):

\[
x_i^{2^l} = \begin{cases} 
M_{2^l-1}(\alpha_i, \ldots) & \text{if } l \text{ is even}, \\
M_{2^l-1}(\beta_i, \ldots) & \text{if } l \text{ is odd}.
\end{cases}
\]

This implies, in particular, that \( \text{ord}_{\text{G}_k}(x_i) = 2^r \) with \( r \) given in (2).

Proof. Since \( G_k \) is a 2-group, \( \text{ord}_{\text{G}_k}(x_i) \) has to be a power of 2. The formulas (9) follow from a straightforward calculation from \( x_i = M_0(\alpha_i, \ldots) \) and Proposition 2.5 in [17]. This implies that \( \text{ord}_{\text{G}_k}(x_i) = 2^r \) if and only if \( 2^{r-1} - 1 \leq k - 1 < 2^r - 1 \), i.e., \( r = \lfloor \log_2 k \rfloor + 1 \).

Lemma 2.2 implies that the faces of \( T_k \) are regular \( 2^{r+1} \)-gons. The estimates for \( |E(T_k)| \) and \( |F(T_k)| \) in Theorem 1.1 follow immediately from this fact, the inequality (8), and the trivalence of the graphs \( T_k \). The genus estimate for \( S(T_k) \) can then be deduced from (5) and (6). This finishes the proof of Theorem 1.1.

2.3. Proof of (4) in Proposition 1.4. We conclude from the trivalence of \( T_k \) and (5) that

\[
\frac{6g(S(T_k))}{|E(T_k^*)|} = 6 \frac{1 + (1 - \mu_k)|V(T_k)|/4}{3|V(T_k)|/2}.
\]

Note that \( |V(T_k)| = 2|H_k| \geq 2^K \to \infty \), which implies that

\[
\lim_{k \to \infty} \frac{6g(S(T_k))}{|E(T_k^*)|} = 1 - \lim_{k \to \infty} \mu_k.
\]

Recall from (6) and Lemma 2.2 that \( \mu_k = 3/\text{ord}_{\text{G}_k}(x_0) \to 0 \) as \( k \to \infty \), finishing the proof of (4).
2.4. **Proof of Proposition 1.2.** We first recall a few important facts about the Platonic graphs $\Pi_N$ and the surfaces $S_\infty(\Pi_N)$ and $S(\Pi_N)$. For more details, see [11]. Let $\mathcal{F}$ be the Farey tessellation of the hyperbolic upper half plane $\mathbb{H}^2$, and let $\Omega(\mathcal{F})$ be the set of oriented geodesics in $\mathcal{F}$. Recall that the Farey tessellation is a triangulation of $\mathbb{H}^2$ with vertices on the line at infinity $\mathbb{R} \cup \{\infty\}$, namely, the subset of extended rationals $\mathbb{Q} \cup \{\infty\}$. Two rational vertices with reduced forms $a/c$ and $b/d$ are joined by an edge, a geodesic of $\mathbb{H}^2$, if and only if $ad - bc = \pm 1$ (see [11, Fig. 1] for an illustration of the Farey tessellation). The group of conformal transformations of $\mathbb{H}^2$ that leave $\mathcal{F}$ invariant is the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$, which acts transitively on $\Omega(\mathcal{F})$. The principal congruence subgroups of $\Gamma$ are normal subgroups defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ mod } N \right\}.$$

It is well known (see, e.g., [11]) that $\mathcal{F}/\Gamma(N)$ and $\Pi_N$ are isomorphic, and $\mathcal{F}/\Gamma(N)$ is a triangulation of the surface $S_\infty(\Pi_N) = \mathbb{H}^2/\Gamma(N)$ by ideal triangles (the vertices are, in fact, the cusps of $S_\infty(\Pi_N)$). The tessellation $(\Pi_N, S_\infty(\Pi_N))$ can be interpreted as a map $\mathcal{M}_N$ in the sense of Jones/Singerman [12]. The group $\text{Aut}(\mathcal{M}_N)$ of automorphisms of $\mathcal{M}_N$ is the group of orientation preserving isometries of $S_\infty(\Pi_N)$ preserving the triangulation. As $\Gamma(N)$ is normal in $\Gamma$, we have that the map $\mathcal{M}_N$ is regular, meaning that $\text{Aut}(\mathcal{M}_N)$ acts transitively on the set of directed edges of $\Pi_N$ (see [12, Thm 6.3]). Moreover, by [12, Thm 3.8],

$$\text{Aut}(\mathcal{M}_N) \cong \Gamma/\Gamma(N) \cong \text{PSL}(2, \mathbb{Z}_N).$$

(Note that in the case of a prime power $N = p^r$, $\text{PSL}(2, \mathbb{Z}_N)$ is the group defined over the ring $\mathbb{Z}_N$ and not over the field with $p^r$ elements.)

Let $N \geq 7$. Noticing that all vertices of $\Pi_N$ have degree $N$, we obtain a smooth compact surface $S(\Pi_N)$ by substituting every ideal triangle in $(\Pi_N, S_\infty(\Pi_N))$ by a compact hyperbolic $(2\pi/N, 2\pi/N, 2\pi/N)$-triangle, and glueing them along their edges in the same way as the ideal triangles of $S_\infty(\Pi_N)$. The group of orientation preserving isometries of $S(\Pi_N)$ preserving this triangulation is, again, isomorphic to $\text{PSL}(2, \mathbb{Z}_N)$. Hence, the automorphism group of the triangulation $(\Pi_N, S(\Pi_N))$ is $\text{PSL}(2, \mathbb{Z}_N)$ of order 192. This implies that $S(\Pi_8)$ is the unique compact hyperbolic surface of genus 5 with maximal automorphism group (see [2]).

The $\Pi_8$-triangulation of $S(\Pi_8)$ is shown in Figure 2; the black-white pattern on the triangles is a first test whether this triangulation can be isomorphic to the $T_3^5$-triangulation of $S(T_2)$. (The $\Pi_N$-triangulations for $3 \leq N \leq 7$ can be found in Figs. 3 and 4 of [11].) $\text{PSL}(2, \mathbb{Z}_8)$
Figure 2. The Platonic graph $\Pi_8$: Each triangle corresponds to a hyperbolic $(\pi/4, \pi/4, \pi/4)$-triangle of the tessellation of $S(S_8)$. The edges along the boundary path are pairwise glued to obtain $S(\Pi_8)$.

acts simply transitively on the directed edges of this triangulation. Consider now a refinement of this triangulation by subdividing each $(\pi/4, \pi/4, \pi/4)$-triangle into six $(\pi/2, \pi/3, \pi/8)$-triangles. It is easily checked that the smaller $(\pi/2, \pi/3, \pi/8)$-triangles admit also a black-white colouring such that the neighbours of all smaller black triangles are white triangles and vice versa. Each black $(\pi/2, \pi/3, \pi/8)$-triangle is in 1-1 correspondence to a half-edge of $\Pi_8$ which, in turn, can be identified with a directed edge of $\Pi_8$. Consequently, the orientation preserving isometries of the surface $S(\Pi_8)$ corresponding to the elements in $PSL(2, \mathbb{Z}_8)$ act simply transitively on the black $(\pi/2, \pi/3, \pi/8)$-triangles. In fact, $PSL(2, \mathbb{Z}_8)$ can be interpreted as a quotient of the
triangle group $\Delta^+(2, 3, 8)$, namely,

$$\text{PSL}(2, \mathbb{Z}_8) \cong \langle x^2, y^3, z^8, xyz, (xz^2x^5)^2 \rangle,$$

where $x, y, z$ correspond to rotations by $\pi, 2\pi/3, \pi/4$ about the three vertices of a given $(\pi/2, \pi/3, \pi/8)$-triangle.

MAGMA computations show that $\text{PSL}(2, \mathbb{Z}_8)$ has a unique normal subgroup $N$ of index 6, generated by the elements $X = x^{-1}z^2x$, $Y = y^{-1}z^2y$ and $Z = z^2$, which is isomorphic to the triangle group quotient $\Delta^+(4, 4, 4)/P_2(\Delta^+(4, 4, 4))$ via the explicit isomorphism

$$(10) \quad X \mapsto \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad Z \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$\hfill

Note that the matrices in (10), viewed as elements in $\text{PSL}(2, \mathbb{Z})$, generate a group acting simply transitively on the black triangles of the Farey tessellation in $\mathbb{H}^2$, as illustrated in Figure 3. The images of a black triangle $\mathcal{T}$ with vertices $0, 1, \infty$ under $\{X^\pm 1, Y^\pm 1, Z^\pm 1\}$ are the six black triangles each sharing a common white triangle with $\mathcal{T}$.

MAGMA computations also show that we have the explicit isomorphism

$$N = \langle X, Y, Z \rangle \cong G_2 = \langle x_0, x_1, x_3 \rangle,$$

given by $X \mapsto x_0, Y \mapsto x_1, Z \mapsto x_3$. The normal group $N \triangleleft \text{PSL}(2, \mathbb{Z}_8)$ is of order 32 and the quotient $\mathcal{S}_0 = (S_{\mathbb{Z}_8}, S(S_{\mathbb{Z}_8}))/N$ is an orbifold consisting of two hyperbolic $(\pi/4, \pi/4, \pi/4)$-triangles (one of them black and the other white). We conclude from the explicit isomorphism
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N ≡ G_2 that the covering procedure discussed in Section 2.1 leads to isometric surfaces S(Π_8) and S(T_2), and that Π_8 ⊂ S(Π_8) is dual to the tessellation (T_2, S(T_2)).

On the spectral side, the adjacency operators on the graphs X_2 and Π_8 compare as follows:

| eigenvalue | 8 | 6 | √8 | 2 | 0 | -2 | -√8 | -4 | Total |
|------------|---|---|-----|---|---|-----|------|----|-------|
| multiplicity in X_2 | 0 | 1 | 6 | 6 | 4 | 9 | 6 | 0 | 32 |
| multiplicity in Π_8 | 1 | 0 | 6 | 0 | 9 | 0 | 6 | 2 | 24 |

We also like to mention that, for k = 2, the (∆ − Y)-transformation X_2 → T_2 has a group theoretical interpretation. There exists a group extension ̃G_2 of G_2 by Z_2, generated by involutions A, B, C satisfying X = AB, Y = BC and Z = CA, and T_2 is the Cayley graph of ̃G_2 with respect to the generators A, B, C. This group theoretic interpretation of the (∆ − Y)-transformation fails for k ≥ 5. In fact, the group T = ⟨A, B, C | A^2, B^2, C^2, r_1(AB, BC), r_2(AB, BC), r_3(AB, BC)⟩ with r_1, r_2, r_3 given in (7) is finite and of order 6144. If the introduction of the above involutions A, B, C would lead to a group extension ̃G_k, then ̃G_k would have to be of order 2|G_k| and a quotient of T and, therefore, of order ≤ 6144. However, we have 2|G_5| = 16384 in contradiction to the second condition. Thus we do not obtain a Cayley graph representation of the graphs T_k for k ≥ 5 via this procedure.

Let us finally explain why we can no longer have an isomorphism T_k^* ≡ Π_2^ρ for k ≥ 3, with ρ appropriately chosen. Let us assume that Π_2^ρ = T_k^*. Comparison of the vertex degrees of Π_2^ρ and T_k^* leads to ρ = r + 1, with r given in (2). Moreover, we conclude from (12) below that |V(Π_2^ρ)| = 3 ⋅ 2^{2ρ−3}. We know from Theorem 1.1 that

\[ |V(T_k^*)| = |F(T_k)| \geq 3 \cdot 2^{K−r−1}.\]

The condition |V(Π_2^ρ)| = |V(T_k^*)| together with ρ = r + 1 implies that 3r ≥ K with K in (2), i.e.,

\[ 3[\log_2 k] + 3 \geq 8[k/3] + 3 \cdot (k \mod 3).\]

But one easily checks that this inequality holds only for k = 1, 2. (In the case k = 1, we have Π_4 = T_1^*, since T_1 is combinatorially the cube and Π_4 is the octagon.) This shows that the graph family Π_N cannot contain any of the dual graphs T_k^*, for indices k ≥ 3.
2.5. **Proof of Corollary 1.5.** The identity $2g - 2 = |V(T_k)|$ between the genus of the surface $\hat{S}(T_k)$ and the number of vertices of the trivalent graph $T_k$ is easily checked. Moreover, every automorphism of the graph $T_k$ induces an isometry on $\hat{S}(T_k)$. Since the graphs $T_k$ form a power of coverings with powers of 2 as covering indices, the same holds true for the associated surfaces $\hat{S}(T_k)$. Now, [6] shows that $\lambda_1(\hat{S}(T_k))$ can be estimated from below by a fixed multiple of the isoperimetric Cheeger constant of $T_k$. The expander property implies that the Cheeger constants of $T_k$ have a uniform lower positive bound. This finishes the proof of the Corollary 1.5.

3. **The Platonic graphs**

3.1. **Algebraic description of vertices and axes.** Let us briefly recall some algebraic facts from [11]. Both groups $\Gamma = \text{PSL}(2, \mathbb{Z})$ and $\text{PSL}(2, \mathbb{Z}_N)$ act on $V(\Pi_N)$ via

$$(a \ b \ c \ d) [\lambda, \mu] = [a\lambda + b\mu, c\lambda + d\mu],$$

and there is a 1-1 correspondence between the vertex set $V(\Pi_N)$ and the cosets $\Gamma/\Gamma_1(N)$. Here, $\Gamma_1(N)$ is the congruence subgroup given by

$$\Gamma_1(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \mid \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \left( \begin{array}{cc} 1 & * \\ 0 & 1 \end{array} \right) \mod N \right\}.$$ 

In [11], the set of vertices was partitioned into *axes*. Two vertices belong to the same axis if they have the same stabilizer in $\text{PSL}(2, \mathbb{Z}_N)$. Since $\text{PSL}(2, \mathbb{Z}_N)$ acts transitively on $V(\Pi_N)$, all axes have the same number of vertices. An interesting observation is that if an element of $\text{PSL}(2, \mathbb{Z}_N)$ leaves a vertex $[\lambda, \mu]$ invariant, then any vertex $[\nu, \omega]$ with $\lambda \omega - \mu \nu = 0$ is also invariant under the same element. Thus the axis containing $[1, 0]$ is given by

$$(11) \quad A_{\text{princ}} = \{ [\lambda, 0] \mid \gcd(\lambda, N) = 1 \},$$

and we call this axis the *principal axis* of $\Pi_N$. The set of all axes of $\Pi_N$ is denoted by $A(\Pi_N)$. There is a 1-1 correspondence between the axes of $\Pi_N$ and the cosets $\Gamma/\Gamma_0(N)$, where $\Gamma_0(N)$ is the congruence subgroup

$$\Gamma_0(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \mid \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \left( \begin{array}{cc} * & * \\ 0 & * \end{array} \right) \mod N \right\}.$$ 

From the 1-1 correspondences with the cosets of $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$, we can immediately obtain the numbers of directed edges, vertices and
axes of \( \Pi_N \) as the indices of these subgroups in \( \Gamma \):

\[
|A(\Pi_N)| = |\Gamma : \Gamma_0(N)| = N \prod_{p|N} \left(1 + \frac{1}{p}\right),
\]

(12)

\[
|V(\Pi_N)| = |\Gamma : \Gamma_1(N)| = \frac{N^2}{2} \prod_{p|N} \left(1 - \frac{1}{p^2}\right),
\]

\[
|E(\Pi_N)| = \frac{1}{2} |\Gamma : \Gamma(N)| = \frac{N^3}{4} \prod_{p|N} \left(1 - \frac{1}{p^2}\right),
\]

where the products run over the distinct prime divisors of \( N \), see for example [16]. In particular, for a prime \( p \), \( \Pi_p \) has \( p^2 + 1 \) axes, \( (p^2 - 1)/2 \) vertices and \( p(p^2 - 1)/4 \) undirected edges.

3.2. **Proof of (3) in Proposition 1.4.** From

\[
\chi(S(\Pi_N)) = 2 - 2g(S(\Pi_N)) = V - E/3
\]

we conclude

\[
g(S(\Pi_N)) = 1 + \frac{N^2(N - 6)}{24} \prod_{p|N} \left(1 - \frac{1}{p^2}\right),
\]

which immediately implies that

\[
\lim_{N \to \infty} \frac{6g(S(\Pi_N))}{|E(\Pi_N)|} = 1.
\]

Note also, that the genus of the surface \( S(\Pi_p) \) for a prime \( p \) is given by \( (p + 2)(p - 3)(p - 5)/24 \).

3.3. **Vertex connectivity of \( \Pi_p \) and \( \Pi'_p \).** Let \( p \) be a fixed odd prime and \( n = (p - 1)/2 \). The wheel structure of \( \Pi_p \) was already discussed in [13, Thm 2.1]. Let us present this and other geometric facts in our terminology. The principal axis of \( \Pi_p \) is given by

\[
A_{\text{princ}} = \{[i, 0] \in V(\Pi_p) \mid 1 \leq i \leq n\}.
\]

The vertices of \( A_{\text{princ}} \) and their 1-ring neighbours form a partition of \( V(\Pi_p) \) into \( n \) components with \( p + 1 \) vertices each. We call these components the wheels of \( \Pi_p \), see Figure 4. The wheel with center \([i, 0]\) (\( 1 \leq i \leq n \)) is denoted by \( W_i \) and is a subgraph of \( \Pi_p \) with \( p + 1 \) vertices and \( 2p \) edges. We also use the notation \( \partial W_i \) for the induced subgraph with vertex set \( V(\partial W_i) = V(W_i) - \{[i, 0]\} \). We call \( \partial W_i \) the boundary of the \( i \)-th wheel. Note that \( \partial W_i \) is isomorphic to the cyclic graph of \( p \) vertices.

Every vertex that is not in \( A_{\text{princ}} \) is adjacent to exactly two vertices of the boundary of any given wheel \( W_i, 1 \leq i \leq n \). Indeed, because
$\text{PSL}(2, \mathbb{Z}_p)$ acts transitively on $V(\Pi_p)$, we may consider, w.l.o.g., the vertex $[0, 1] \in \partial W_1$. The $p-1$ vertices adjacent to $[0, 1]$ that are not in $A_{\text{princ}}$ are $[1, x]$ with $x \in \{1, 2, \ldots, p-1\}$. To find the vertices $[1, x]$ in $\partial W_i$, we need to solve
$$\det \begin{pmatrix} 1 & i \\ x & 0 \end{pmatrix} = \pm 1,$$
which has exactly two solution $x = \pm i^{-1}$ (where we think of $i \in \mathbb{Z}_p$) which correspond to two distinct vertices of $\Pi_p$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{\Pi_p consists of $n = (p-1)/2$ wheels. Each vertex at the boundary of a wheel is connected with the center of its wheel and exactly two points on the boundary of any wheel (including itself).}
\end{figure}

\textbf{Lemma 3.1.} Let $i, j \in \{1, 2, \ldots, n\}$. Then we have the following facts.
\begin{enumerate}[(a)]
\item Let $x_1, x_2$ be two different vertices in $\partial W_i$ and also $y_1, y_2$ be two different vertices in the same set $\partial W_i$ ($\{x_1, x_2\} \cap \{y_1, y_2\} \neq \emptyset$ is allowed). Then there exists a permutation $\sigma \in \text{Sym}(2)$ and two vertex distinct paths $p_1, p_2$ in $\partial W_i$, such that $p_1$ connects $x_1$ with $y_\sigma(1)$ and $p_2$ connects $x_2$ with $y_\sigma(2)$.
\item Every $x \in \partial W_i$ has precisely two neighbours in $\partial W_j$.
\item Assume additionally that $i \neq j$. Then there exists a bijective map $\Phi : V(\partial W_i) \to V(\partial W_j)$ such that $v \sim \Phi(v)$ for all vertices $v \in \partial W_i$.
\end{enumerate}

\textit{Proof.} Note that $\partial W_i$ is isomorphic to the cyclic graph of $p$ vertices. (a) is then a straightforward inspection of all possible cases. (b) is already proved by our previous arguments. It remains to prove (c): Think of $i, j \in \mathbb{Z}_p - \{0\}$. Then the vertices in $\partial W_i$ are of the form $[\mu, i^{-1}]$ and the vertices in $\partial W_j$ of the form $[\nu, j^{-1}]$ with $\mu, \nu \in \mathbb{Z}_p$. The map $\phi : \mathbb{Z}_p \to \mathbb{Z}_p$, defined by $\phi(\mu) = i + ij^{-1}\mu$, is obviously a bijection, and we have $[\mu, i^{-1}] \sim [\phi(\mu), j^{-1}]$, finishing the proof. \hfill $\square$
Note that the wheel structure is not confined to the choice of the principal axis. Since the group $\text{PSL}(2, \mathbb{Z}_p)$ maps axes to axes and acts transitively on them, we can choose any axis $\mathcal{A}$ as the centers of the $n$ wheels, and Lemma 3.1 is still valid in this setting.

Now we prove that $\Pi_p$ is $p$-vertex-connected. Notice that the arguments in this proof also give $\text{diam}(\Pi_p) \leq 3$ as a by-product.

**Proof.** We will show that for any two vertices of $\Pi_p$, we can find $p$ vertex disjoint paths connecting them. Then the result will follow from Menger’s Theorem.

Since $\text{PSL}(2, \mathbb{Z}_N)$ acts transitively on $V(\Pi_p)$, we can assume that the start vertex is $[1,0] \in W_1$. Separating three cases, we will find $p$ vertex distinct paths to

(i) the vertices in $\partial W_1$,
(ii) the vertices in any $\partial W_j$ with $2 \leq j \leq n$,
(iii) the other vertices in $\mathcal{A}_p$.

Ad (i): Assume that the end vertex is $[\nu, 1]$. Then we already have three vertex disjoint paths given by $[1, 0] \rightarrow [\nu, 0], [1, 0] \rightarrow [\nu \pm 1, 1] \rightarrow [\nu, 1]$.

We need to find vertex disjoint paths starting with $[1, 0] \rightarrow [\nu \pm i, 1]$ and ending at $[\nu, 1]$, for $2 \leq i \leq n$. By Lemma 3.1(c), we can find two different vertices $x_1, x_2 \in \partial W_i$ such that $[\nu - i, 1] \sim x_1$ and $[\nu + i, 1] \sim x_2$. By Lemma 3.1(b), $[\nu, 1]$ has two different neighbours $\{y_1, y_2\}$ in $\partial W_i$. We now use Lemma 3.1(a) to complete the paths.

Ad (ii): We assume $p \geq 5$, for otherwise there is nothing to prove. Let us assume that the end vertex is in $\partial W_i$ with $2 \leq i \leq n$, and let us denote this vertex by $w \in \partial W_i$. Let $v_-, v_+ \in \partial W_1$ be the two neighbours of $w$ in the first wheel. Choose three different vertices $v_1, v_2, v_3 \in \partial W_1 - \{v_-, v_+\}$, and use Lemma 3.1(c) to find three different vertices $w_1, w_2, w_3 \in \partial W_i - \{w\}$ such that $v_j \sim w_j$ for $1 \leq j \leq 3$. W.l.o.g., we can assume that the pair $\{w_1, w_3\}$ separates $w_2$ and $w$ within $\partial W_i$. Let $q_1, q_3$ be the two vertex disjoint paths in $\partial W_i - \{w_2\}$ connecting $w$ with $w_1$ and $w_3$, respectively. Then we already have five vertex disjoint paths given by

$[1, 0] \rightarrow v_\pm \rightarrow v, \quad [1, 0] \rightarrow v_2 \rightarrow w_2 \rightarrow [i, 0] \rightarrow w,$

and

$[1, 0] \rightarrow v_1 \rightarrow w_1 \xrightarrow{q_1} w, \quad [1, 0] \rightarrow v_3 \rightarrow w_3 \xrightarrow{q_3} w.$

Notice that for any wheel $W_j$ with $j \not\in \{1, i\}$, we have not yet used any edges with one vertex in $\partial W_j$. We will see that every such wheel allows us to create two more vertex disjoint paths from $[1, 0]$ to $w$, finishing.
this case. Let \( y_1, y_2 \) be the two different vertices in wheel \( \partial W_j \) adjacent to \( w \). Choose two different vertices \( v', v'' \in \partial W_1 \) which have not been used yet and associate to them two different vertices \( x_1, x_2 \in \partial W_j \) such that \( v' \sim x_1 \) and \( v'' \sim x_2 \), using Lemma 3.1(c). Then we can use Lemma 3.1(a) to complete the paths within \( \partial W_j \).

Ad (iii): This is the easiest case. Assume that the end vertex is \([i, 0] \in W_i \) with \( 2 \leq i \leq n \). We use the bijection \( \Phi : V(\partial W_1) \to V(\partial W_i) \) in Lemma 3.1(c) to create the \( p \) vertex disjoint paths

\[
[1, 0] \to [0, \mu] \to \Phi([0, \mu]) \to [i, 0]
\]

with \( 1 \leq \mu \leq p \).

Next, we present the proof that \( \Pi'_p \) is \((p - 1)\)-vertex-connected. In contrast to the previous proof, the arguments given here do not imply that \( \text{diam}(\Pi'_p) \leq 3 \).

**Proof.** Let \( v, w \in \Pi'_p \) be two different vertices with \( v \in \partial W_i \) and \( w \in \partial W_j \). We consider the two cases \( i = j \) and \( i \neq j \) separately:

Case \( i = j \): Obviously, we can choose two vertex disjoint paths within \( \partial W_i \) to connect \( v \) and \( w \). Next, we show that every wheel \( \partial W_j \) with \( j \neq i \) gives rise to two additional vertex disjoint paths. Let \( x_1, x_2 \in \partial W_j \) be the two distinct neighbours of \( v \), and \( y_1, y_2 \in \partial W_j \) be the two distinct neighbours of \( w \). Then we can use Lemma 3.1(a) to complete the paths within \( \partial W_j \).

Case \( i \neq j \): Let \( w_1, w_2 \in \partial W_j \) be the neighbours of \( v \) and \( v_1, v_2 \in \partial W_i \) be the neighbours of \( w \). Then, using only additional edges in \( \partial W_i \cup \partial W_j \), we can find four vertex disjoint paths \( v \to \cdots \to v_k \to w \), \( v \to w_k \to \cdots \to w \) (for \( k = 1, 2 \)). Again, every wheel \( W_i \) with \( l \notin \{i, j\} \) will give rise to two more vertex disjoint paths. Let \( x_1, x_2 \in \partial W_i \) be the neighbours of \( v \), and \( y_1, y_2 \in \partial W_i \) be the neighbours of \( w \). Use Lemma 3.1(c) to complete the paths within \( \partial W_i \).

Finally, we prove \( \text{diam}(\Pi'_p) = 3 \).

**Proof.** Let us first confirm that any two different vertices in the same wheel can be connected by a path of length 2: Let \( 1 \leq i \leq n \) and \([\mu, i^{-1}], [\nu, i^{-1}] \in \partial W_i \) (thinking of \( i \in \mathbb{Z}_p \)) be the two vertices. The required path is then given by

\[
[\mu, i^{-1}] \to [2(\mu - \nu)^{-1} \mu i^{-1} - i, 2(\mu - \nu)^{-1}] \to [\nu, i^{-1}].
\]

Now choose two vertices \( v \in \partial W_i \) and \( w \in \partial W_j \) on different wheels. Let \( v' \in \partial W_i \) be one of the two neighbours of \( w \) in the \( i \)-th wheel. Connecting \( v \) and \( v' \) by a path of length 2 (as shown before) implies that \( d(v, w) \leq d(v, v') + 1 \leq 3 \).
Note that the graph $\Pi_p'$ was initially defined as the induced subgraph of $\Pi_p$ with vertex sets $V(\Pi_p) - \mathcal{A}_{\text{princ}}$. Alternatively, $\Pi_p'$ can also be described as the Cayley graph $\text{Cay}(U_p, S)$ with $U_p = \left\{ \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}_p) \right\} \cong \Gamma_0(p)/\Gamma(p)$ and $S = \left\{ \begin{pmatrix} \ast & 1 \\ 0 & \ast \end{pmatrix} \in U_p \right\}$.

The vertices $[\lambda, \mu] \in V(\Pi_p')$ (with non-vanishing second coordinate $\mu$) are then identified with the matrices $\begin{pmatrix} \mu^{-1} & \lambda \\ 0 & \mu \end{pmatrix} \in U_p$.

3.4. Ramanujan properties "without number theory". As before, we assume that $p$ is a fixed odd prime and $n = (p - 1)/2$. The considerations of the previous section show also that $\Pi_p$ is an $n$-fold covering $\pi : \Pi_p \to K_{p+1}$ of the complete graph $K_{p+1}$, where the preimages $\pi^{-1}(v)$ correspond to the axes of $\Pi_p$. It is useful to think of the vertices in $K_{p+1}$ as the points in the finite projective line over the field $\mathbb{Z}_p$, i.e., $V(K_{p+1}) = \{0, 1, \ldots, p - 1, \infty\}$ and the covering map is then given, algebraically, by
\[
\pi([\lambda, \mu]) = \lambda \mu^{-1},
\]
with the usual convention $\infty^{-1} = 0$ and $0^{-1} = \infty$. In particular, we have $\mathcal{A}_{\text{princ}} = \pi^{-1}(\infty)$. Note that $\text{PSL}(2, \mathbb{Z}_N)$ acts also on the vertices of $K_{p+1}$ via
\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} z = (\alpha z + \beta)(\gamma z + \delta)^{-1}.
\]
One easily checks that $\pi(gv) = g\pi(v)$ for all $g \in \text{PSL}(2, \mathbb{Z}_N)$ and $v \in V(\Pi_p)$.

Let us now explicitly derive the spectra of the graphs $\Pi_p$ and $\Pi_p'$. We will use the following notation: For a linear operator $T$ on a finite dimensional vector space, we denote the eigenspace of $T$ to the eigenvalue $\lambda$ by $\mathcal{E}(T, \lambda)$.

We start with a "number theory free" proof of Theorem 4.2 in [10], using the covering $\pi : \Pi_p \to K_{p+1}$.

Proof. Every eigenfunction $f$ of $K_{p+1}$ gives rise to an eigenfunction $F : V(\Pi_p) \to \mathbb{C}$ of the same eigenvalue via $F(v) = f(\pi(v))$. The spectrum of the adjacency operator on $K_{p+1}$ is given by (see, e.g., [3, p. 17])
\[
\sigma(K_{p+1}) = \{p, -1, \ldots, -1\}^{p \text{ times}}.
\]
This implies that $\sigma(\Pi_p)$ contains the eigenvalue $p$ with multiplicity one and the eigenvalue $-1$ with multiplicity $\geq p$. 

Our next aim is to prove that the eigenspace $\mathcal{E}(A^2, p)$ of the square of the adjacency operator on $\Pi_p$ has dimension $(p + 1)(p - 3)/2$. Let $f : V(\Pi_p) \to \mathbb{C}$ be a function satisfying

\begin{equation}
A^2 f(v) = pf(v) \quad \text{for all } v \in V(\Pi_p).
\end{equation}

Note that (13) can be viewed as a homogenous system of $(p^2 - 1)/2$ linear equations. The key observation is that all linear equations corresponding to vertices of the same axis coincide, i.e., we end up with only $p + 1$ linear independent homogeneous equations (since $p + 1$ equals the number of axes), showing that the eigenspace has dimension at least

\begin{equation}
|V(\Pi_p)| - (p + 1) = \frac{p^2 - 1}{2} - (p + 1) = \frac{(p + 1)(p - 3)}{2}.
\end{equation}

Indeed, since $PSL(2, \mathbb{Z}_p)$ acts transitively on the vertices, we only need to show that the linear equations of (13) corresponding to the vertices in the principal axis $A_p$ coincide. Recall that $S_p$ has the wheel-structure given in Figure 4. Let $v \in A_p$. Then we have

\begin{equation}
A^2 f(v) = pf(v) + 2 \sum_{i=1}^{n} \sum_{w \in \partial W_i} f(w),
\end{equation}

since there are exactly $p$ paths of length 2 from $v$ to itself, no paths of length 2 from the centers of all the other wheels to $v$, and for every $w \in \cup_i \partial W_i$ there are exactly 2 paths from $w$ to $v$ of length 2, because of Lemma 3.1(b). Note that the combination of (13) and (14) simplifies to

\begin{equation}
\sum_{i=1}^{n} \sum_{w \in \partial W_i} f(w) = 0,
\end{equation}

independently of the choice of $v \in A_p$. This shows that $\dim \mathcal{E}(A^2, p) \geq (p + 1)(p - 3)/2$. Adding up the multiplicities of all eigenvalues, we see that $\dim \mathcal{E}(A^2, p) = (p + 1)(p - 3)/2$.

If $f_1, \ldots, f_K$ span the space $\mathcal{E}(A^2, p)$, then the $2K$ functions

$$\sqrt{p} f_1 \pm Af_1, \ldots, \sqrt{p} f_K \pm Af_K$$

are eigenfunctions of $A$ to the eigenvalues $\pm \sqrt{p}$, and they also span $\mathcal{E}(A^2, p)$. This shows that we have

$$\mathcal{E}(A^2, p) = \mathcal{E}(A, \sqrt{p}) \oplus \mathcal{E}(A, -\sqrt{p}).$$

Finally, the equality

$$\dim \mathcal{E}(A, \sqrt{p}) = \dim \mathcal{E}(A, -\sqrt{p}) = \frac{(p + 1)(p - 3)}{4}$$

follows from Lemma 3.2 below. \qed
Lemma 3.2. Let $T$ be a square matrix with rational entries and $K$ be a positive integer which is not a square. Then we have

$$\dim \mathcal{E}(T, \sqrt{K}) = \dim \mathcal{E}(T, -\sqrt{K}).$$

Proof. The proof is based on the fact that $\sqrt{K}$ is irrational. Let $p(z) \in \mathbb{Q}[z]$ be the characteristic polynomial of $T$. We split $p(z)$ into its even and odd part, i.e.,

$$p(z) = p_{\text{even}}(z) + p_{\text{odd}}(z),$$

with even polynomials $p_{\text{even}}(z), p_{\text{odd}}(z)$. Note that we have

$$p(\sqrt{K}) = p_{\text{even}}(\sqrt{K}) + p_{\text{odd}}(\sqrt{K}) \sqrt{K}$$

and $p_{\text{even}}(\sqrt{K}), p_{\text{odd}}(\sqrt{K}) \in \mathbb{Q}$. Therefore, if $\sqrt{K}$ is a root of $p(z)$, then $\sqrt{K}$ is also a root of both polynomials $p_{\text{even}}(z)$ and $p_{\text{odd}}(z)$, separately. This implies that $-\sqrt{K}$ is also a root of $p(z)$. We can then split off the factor $z^2 - K$ from $p(z)$, and repeat the procedure with the remaining polynomial. \qed

Next we derive the spectrum of the modified graph $\Pi'_p$, using the $n$-fold covering map $\pi : \Pi'_p \to K_p$ and the wheel-structure, which partitions the vertex set $V(\Pi'_p)$ into $n$ disjoint sets $\partial W_i$ of $p$ vertices, each. This will finish the proof of Theorem 1.3.

Proof. The proof of the spectral statements in Theorem 1.3 proceeds in steps.

(i) Let $W$ be the vector space of all functions which are constant on the wheels. We first introduce a basis of eigenfunctions of this vector space. Let $\zeta_n = e^{2\pi i/n}$ and, for $0 \leq j \leq n - 1$, define

$$f_j(v) = \zeta_n^j \text{ if } v \in \partial W_i.$$

Note that $f_0$ is the constant function to the eigenvalue $p - 1$. It is easily checked that $Af_j = 0$ for $j \geq 1$. Since these functions are linearly independent, they form a basis of $W$. Moreover, we have $\dim \mathcal{E}(A, 0) \geq n - 1 = (p - 3)/2$.

(ii) Let $V$ be the vector space of all functions which are constant along all axes. Every such function is a lift $F(v) = f(\pi(v))$ of a function $f$ on $K_p$. Note that eigenfunctions of $K_p$ are lifted to eigenfunctions to the same eigenvalue, so $V$ can be viewed as the span of a constant function and $p - 1$ linear independent eigenfunctions to the eigenvalue $-1$. In particular, we have $\dim \mathcal{E}(A, -1) \geq p - 1$.

(iii) Note that $W \cap V = \text{span}(f_0)$. By the orthogonality of eigenfunctions, it only remains to study the eigenfunctions in the orthogonal
complement \((\mathcal{W} + \mathcal{V})^\perp\) of dimension
\[
|V(\Pi_p)| - (\dim \mathcal{W} + \dim \mathcal{V}) + 1 = \frac{(p - 1)(p - 3)}{2} = K.
\]

Let \(g_1, \ldots, g_K\) be a basis of this orthogonal complement by eigenfunctions with \(A g_i = \lambda_i g_i\). We now extend each \(g_i\) trivially to a function \(\tilde{g}_i\) on \(S_p\) by setting \(\tilde{g}_i(v) = 0\) for all \(v \in \mathcal{A}_p\). Note that these extensions are eigenfunctions of the Platonic graph \(\Pi_p\) to the same eigenvalue, i.e., \(\tilde{A} \tilde{g}_i = \lambda_i \tilde{g}_i\). Therefore, we must have \(\lambda_i \in \{p + 1, -1, \pm \sqrt{p}\}\). As discussed in the previous proof, the span of the eigenfunctions of \(\Pi_p\) to the eigenvalues \(-1\) and \(p + 1\) is obtained via lifting the eigenfunctions of \(K_{p+1}\), and the restriction of these functions to \(\Pi'_p\) must therefore lie in \(\mathcal{V}\). This shows that we must have \(\lambda_i = \pm \sqrt{p}\).

(iv) Adding up the multiplicities of all eigenvalues, we conclude that
\[
\dim \mathcal{E}(A, \sqrt{p}) \oplus \mathcal{E}(A, -\sqrt{p}) = \frac{(p - 1)(p - 3)}{2},
\]
\[
\dim \mathcal{E}(A, 0) = \frac{(p - 3)}{2},
\]
\[
\dim \mathcal{E}(A, -1) = p - 1.
\]

We finally obtain
\[
\dim \mathcal{E}(A, \sqrt{p}) = \dim \mathcal{E}(A, -\sqrt{p}) = \frac{(p - 1)(p - 3)}{4},
\]
by applying, again, Lemma 3.2.

\[\square\]

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