A Novel Algorithm for Linear Programming

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Abstract—The problem of optimizing a linear functional subject to a set of linear constraints (the so called Linear Programming or LP problem) has attracted many researchers; the first fundamental contributions to the LP problem was done by Kantorovich [1] and Dantzig [2], who first discovered the Simplex method, which is essentially a search method. After many years, the next break through came through Kachiyan [3] who proved that the problem can be solved, in theory, in polynomial time and then subsequently Karmarkar [4] discovered a method of search involving points inside the feasible space. Another proposed method of tackling the problem was the gravitational method [5]. However, in spite of all these many developments [6,7] the LP problem did not permit an easy resolution and even now a satisfactory solution to the problem is yet to be had. The mathematician Stephen Smale considers it as one of his 18 unsolved problems in mathematics.

In this paper we present an alternative approach to the solution of the LP problem, the method has the advantage that an LP problem of n dimensions (excluding “slack” variables) involving an objective function of n variables, is reduced to another LP problem in n-1 dimensions and so on. We show and prove by a rigorous theorem, that by invoking the convexity properties of the feasible region this stage by stage dimensional reduction of the problem is made possible.

The next section gives the details of the method.

I. INTRODUCTION

THE problem of optimizing a linear functional subject to a set of linear constraints has been a long standing problem ever since the times of Kantorovich, Dantzig and von Neuman. These developments have been followed by a different approach pioneered by Kachiyan and Karmarkar.

In this paper we present an entirely new method for solving an old optimization problem in a novel manner, a technique that reduces the dimension of the problem step by step and interestingly is recursive. A theorem which proves the correctness of the approach is given.

The method can be extended to other types of optimization problems in convex space, e.g. for solving a linear optimization problem subject to nonlinear constraints in a convex region.

Index Terms—linear programming, optimization, dimension reduction.

II. DESCRIPTION OF PROPOSED METHOD

The LP problem that will be dealt in this paper is concerned with the task of maximizing the objective function \( Z \) defined as:

\[
Z = d_1x_1 + d_2x_2 + .....d_nx_n
\]

subject to the constraints

\[
Ax \leq r
\]

where \( A \) is a \( m \times n \) matrix and \( x \) is a \( n \times 1 \) variable vector and \( r \) is a \( m \times 1 \) constant vector. Thus we are optimizing an objective function \( Z \), involving \( n \) variables (dimensions), given \( m \) constraint planes.

As is well known the feasible region will be a convex polytope whose boundaries are the constraint planes. The LP problem then consists of searching for that point \( x \), in the feasible region which has the maximum value of \( Z \). Several facts about the LP problem are well known and bears repetition: (i) The optimum point if it exists will be a vertex and lie on the boundary, (ii) the optimum vertex will be at the intersection of at least \( n \) boundary planes, (iii) the feasible region is a convex region, by which if there are any two points which are in the feasible region then the line joining P and Q will also lie in the feasible region and (iv) the last statement will be true even if P and Q are on the boundary of two planes, if this happens, then the line PQ then will either be entirely inside the feasible region or lie on the boundary.

In the following it is assumed that an optimum vertex to the chosen LP problem exists. Before, we proceed further it is necessary to define some terms:

We will call those planes whose intersections constitute the optimum vertex as “roof” planes, from (ii) above, we can see that there must obviously be at least \( n \) roof planes. We define \( n_1 \) as the unit vector which points in the optimum direction, that is \( n_d \) will have components proportional to \( \{d_1, d_2, d_3, ..., d_n\} \). Similarly we define \( n_k \) as the unit vector which is the outward normal of the \( k^{th} \) boundary plane, in this case \( n_k \) is defined as that vector whose components are proportional to the coefficients of the \( k^{th} \) row of the matrix \( A \), hence \( n_k \) is proportional to the vector \( \{a_{k1}, a_{k2}, a_{k3}, ..., a_{kn}\} \). It is assumed that all the normal vectors are made to point outwards, away from the feasible region.

Let us define the the angle \( \theta_k \) by the dot product relationship \( \cos(\theta_k) = n_k \cdot n_d \). We now define the “flattest plane” as that plane \( k \) which is such that \( \theta_k \leq \theta_j \), for \((j = 1, 2, ..., m)\). Obviously, if \( k \) is the flattest plane then \( n_k \cdot n_d \geq n_j \cdot n_d \) for \((j = 1, 2, ..., m)\)

For the sake of our argument, let us assume that the optimum direction is “upwards”, (there is no loss of generality in this assumption). Now we make a crucial observation: if the problem has a single (unique) solution, then the optimum vertex will lie on the plane which is the “flattest”. This observation follows from the fact that the feasible region is convex, and because of this the flattest plane, must be one of
the “roof” planes, see figure. In fact, if the flattest plane is NOT one of the roof planes, that is if a steeper feasible plane is “above” the flattest plane, then the feasible region cannot be convex - see figure. The conclusion is true for 2-dimensions, 3-dimensions and for n-dimensions see figure. We give a formal proof in Section V.

From the above observation we can build an algorithm which is described:

1) We start with a properly defined LP problem such that all the constraints are in the form given in Eq. (2), we then calculate all the outward normals $n_j, j = 1, 2, ..., m$ of each of the $m$ constraint planes.

2) We take dot products of all the normals with the object function direction i.e. we find all $n_j.n_d, j = 1, 2, ..., m$. By examining each such dot product we identify the “flattest” plane. If $k$ is the flattest plane then it will have the property $n_k.n_d \geq n_j.n_d$ for all $(j = 1, 2, ..., m)$.

3) Now since $k$ is the flattest plane it must contain the optimum vertex. We then examine all the coefficients of the $k^{th}$ plane and choose that coefficient which is the largest (say $a_{kn}$), we then use the $k^{th}$ inequality as an equation, and get an expression for $x_n$, in terms of $x_1, x_2, ..., x_{n-1}$.

4) We substitute for $x_n$ using the above expression in all the other constraint planes and delete the $k^{th}$ plane. We will have $m-1$ constraint planes each of which is a function of only $n-1$ variables. Next we substitute for $x_n$ in the expression for the objective function $Z$, the new $Z$ will be a linear function of only $n-1$ variables $x_1, x_2, ..., x_{n-1}$. (However, we need to retain the equation of the deleted $k^{th}$ plane, for backsubstitution later on).

The idea behind elimination is simple: Since the optimum vertex will be on the flattest plane, future searches need be conducted only in this plane i.e. $n-1$ dimension space. Eliminating one variable by using the equation of this plane ensures that a search is conducted in this plane, but with a new objective function which does not have this variable.

5) The objective function and the $m-1$ constraints obtained in step 4 represents a LP problem of reduced dimensions. We now go back to step 2 and reduce the problem to $n-2$ dimensions and so on...

6) After we have recursively reduced the problem to a single variable, say $x_1$, we find out that value of $x_1$ which maximizes the objective function $Z$ which is now a function of this single variable and which satisfies all the constraints.

7) Having found $x_1$ the rest of the variables $x_2, ..., x_n$ can be found by backsubstituting in the equations representing the planes which were used for the elimination of variables, starting from the last plane and proceeding to the first in reverse order.

8) The value of $x_1, x_2, ..., x_n$, finally obtained represents the coordinates of the optimum vertex.

The number of steps in the reduction from an $n$-dimensional problem to 1 is $n-1$, however there is a word of caution: Every time we have finish task 2), we must ensure that the current flattest plane is NOT a redundant plane, by the latter we mean a plane which is entirely outside the feasible region. In case the current flattest plane is a redundant plane then it must be deleted and the next flattest plane should be chosen (after testing that it is not redundant).

III. DETAILS OF METHOD:

In this section we briefly write down the various steps involved in the method.

Denoting the objective function vector $d$ and the vector of the fundamental variables $\bar{x}$ as:

\[ d = \{d_1, d_2, d_3, .. d_n\}, \]

\[ \bar{x} = \{x_1, x_2, x_3, .. x_n\}, \]

then the LP problem involves the task of finding out the optimum value $V$ defined as the maximum value of $Z$, where

\[ Z = d^T \bar{x} \]

subject to the constraints:

\[ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + .. + a_{1n}x_n \leq r_1 \]

\[ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + .. + a_{2n}x_n \leq r_2, \]

...........................................

\[ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + .. + a_{mn}x_n \leq r_m \]

Normalization:

It will be assumed that in the above equations the objective function vector $d$ is a unit vector and that each row of the constraint equations have been normalized, so that the squares of the coefficients sum up to unity. If this has not been done (say for the $j^{th}$ row) one can evaluate $N_j$ where $N_j = \sqrt{a_{j1}^2 + a_{j2}^2 + .. + a_{jn}^2}$ and then divide the $j^{th}$ constraint by $N_j$ and redefining $a_{ji}/N_j$ by the coefficient $a_{ij}$ for each $i = 1, 2, .., n$ and the constant $r_j$ by $r_j/N_j$. This procedure is adopted for convenience because then the array $\{a_{j1}, a_{j2}, a_{j3}, .. a_{jn}\}$ become the components of the normal vector $n_j$ to the $j^{th}$ constraint plane.

Reduction procedure:

We now demonstrate the dimension of the LP problem is reduced step by step.

(i) Find flattest plane:

calculate $t_r = \cos(\theta_r)$ for all $r = (1, 2, .. m)$ as:

\[ t_r = \sum_{i=1}^{n} a_{ri} d_i \]

(ii) Find that plane $k$, such that

\[ t_k \geq t_r \quad (r = 1, 2, ..., m) \]

the constraint plane $k$ then will be the flattest plane and therefore will contain the optimum vertex. Since the optimum vertex at the intersection of $n$ planes, it is necessary to find the
other n-1 planes chosen out of m, whose common intersection point is the optimum vertex. Now since we know that
the optimum point lies on this plane k we enforce this latter condition, by using the kth constraint as an equation and
then eliminating one of the variables \( \{x_1, x_2, x_3, .., x_n\} \). The variable to be eliminated will be that which has the largest
coefficient in the kth equation - to reduce round-off errors.

At this point it is assumed, for the argument, that the kth plane has been tested for nonredundancy, then the algorithm
proceeds to step (iii). (If the plane is redundant, see Appendix, then it must be deleted and the next flattest but not redundant
plane must be chosen).

(iii) The kth equation is

\[
a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + ... + a_{kn}x_n - r_k = 0
\]  

(9)

find the coefficient with the largest magnitude, say, it is the
jth that is

\[
|a_{kj}| \geq \max_{i=1,2,...,n} |a_{ki}|
\]

the above inequalities indicate that \( a_{kj} \) is the largest coef-
ficient in the kth constraint plane , so the variable \( x_j \) can be
eliminated from all the rest of the inequalities and objective
function, since

we can write the kth constraint equation as:

\[
x_j = \frac{r_k}{a_{kj}} - \frac{a_{k1}}{a_{kj}} x_1 - \frac{a_{k2}}{a_{kj}} x_2 - ... - \frac{a_{kn}}{a_{kj}} x_n
\]  

(10)

(iv) Now we will substitute (10) for variable \( x_j \) wherever it
occurs in all the other constraints, typically if we substitute it
in the u th constraint equation, the coefficients will be redefined
and it will not have the \( x_j \). Hence, we then have (for this
u th equation):

\[
a_{u1} \leftarrow (a_{u1} - a_{uj} \frac{a_{k1}}{a_{kj}})
\]

\[
a_{u2} \leftarrow (a_{u2} - a_{uj} \frac{a_{k2}}{a_{kj}})
\]

\[
... \quad \quad \quad \quad \quad \quad ...
\]

(11)

\[
ar_{u} \leftarrow (r_u - a_{uj} \frac{r_k}{a_{kj}})
\]

Note \( a_{uj} \) is zero. The above replacements are done for con-
straints u (u=1,2,....,m), except k which is (saved in memory),
but deleted and not considered further consideration in the
reduction process.

(v) The new objective function

After eliminating \( x_j \) the n coefficients of the objective
function become:

\[
d_1 \leftarrow (d_1 - d_j \frac{a_{k1}}{a_{kj}})
\]

\[
d_2 \leftarrow (d_2 - d_j \frac{a_{k2}}{a_{kj}})
\]

and we have a constant term \( d_0 \) which is initially zero,
becomes nonzero

\[
d_0 \leftarrow (d_0 + d_j \frac{r_k}{a_{kj}})
\]

\[
Z = d_0 + d_1 x_1 + d_2 x_2 + ... + d_{j-1} x_{j-1} + d_{j+1} x_{j+1} + ... + d_n x_n
\]  

(12)

It may be noticed that the objective function has one less
variable: \( x_j \) is not present.

(vi) The equation (12) now needs to be maximized with the
m-1 constraints given by equations (11) for all planes u except k.
Hence (12) and (11) now represent an LP problem but with
one less dimension i.e. n-1.

Hence, we normalize the set of equations (11) and (12) as
stated above. Calculate the outward normals \( n_j \) of the new
constraint planes, for which we need one feasible point and
then we put all the constraints in the standard form (viz. eqs (4)
to (6)) and proceed to paragraph (i) and find the new flattest
plane in n-1 dimensions wrt the new objective function.

In this manner, the problem is reduced to a single variable
involving an objective function of a single variable say \( x_1 \)
which can be maximized within the constraints.

Having, found \( x_1 \), then the previous plane which was
just eliminated, is read from storage, this is an equation
in two variables, it will have \( x_1 \) and another variable, say
\( x_2 \), using this \( x_2 \) is found. After this the plane which was
eliminated second-last is retrieved from storage, this contains
an additional variable say \( x_3 \) which is then found. This back
substitution process which is nothing but a version of Gauss-
Seidel elimination process, is used to find the coordinates of the
optimum vertex namely \( (x_1, x_2, x_3, .., x_n) \).
IV. Figures

We see that with respect to the objective function direction, the "flattest" plane contains the optimum point P. PQ and PM are roof planes.

In all the figures the optimum direction is assumed upwards, see arrow.

In Fig 1b we consider a case when the flattest plane is not a roof plane, the plane AB is the flattest and is not a roof plane, we see that in such a case the feasible space is NOT convex.

To briefly describe the working of the algorithm, it is perhaps worthwhile to say that as we reduce the dimensions step by step we will be choosing the "flattest" plane with respect to the present optimum direction. For instance, suppose we are now in the r+1 th step, that is we are now searching in n-r dimensional space, we will find that in this space some of the planes which were having a feasible region in the previous n-r+1 dimension space, may now become redundant, i.e. they will lie outside the present feasible region. Of course, it is not necessary for us to know which one unless it happens to be the “flattest”, in the latter case it has to be eliminated. In other words in the algorithm we need to test only those planes for redundancy, which have currently qualified as “flattest”. Though it is acknowledged that the test for redundancy, is a bit of a downside, the dimension reduction along with the discovery of a roof plane at each step registers a plus score for the algorithm. After all there are only n roof planes to be discovered and one is being discovered in each cycle (step).

It is quite possible that the method may be extendable to other types of optimization problems, for instance if one is to deal with the case of optimizing a linear objective function, but under nonlinear (but convex) constraints which still maintain a convex feasible region, then this method is useful. Because such constraint surfaces can be approximated by a piece-wise patchwork of planes.

V. Theorem

Theorem: The optimum vertex, if it exists lies on the flattest plane.

Proof:

We assume that the flattest plane, in the collection of m planes given in Eq. (2) is A, and given by the equation:

\[ a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + \ldots + a_{kn}x_n = r_k \]

We now show that there cannot be another plane, say B, which is “steeper” than A and is a roof plane and contains a point, Q, whose optimum value Z(Q), is higher than Z(P) of a point P, on A. Let us assume, for the sake of argument such a B actually exists and is given by the equation:

\[ a_{j1}x_1 + a_{j2}x_2 + a_{j3}x_3 + \ldots + a_{jn}x_n = r_j \]

as the proof proceeds it will be shown that such a B cannot exist.

Now since A and B are non-parallel they will “intersect” in a region R. (For the case when n=2, R is a single point, and if n=3, R is a line and if n=4, R is a 2-d plane and so on).

We will now perform the following coordinate transformations \( x \rightarrow x' \):

(i) We first shift the origin to some point in the region R, the constant terms in the r.h.s. of the above equations are zero; as the origin is assumed, by our choice, to lie on both R, the constant terms in the r.h.s. of the above equations are zero.

(ii) We then rotate the coordinate system such that \( x_2' \) is along the optimum direction, and the coordinate \( x_1' \) is perpendicular to it as shown, the other n-2 coordinate axis will be \( x_3', x_4', \ldots, x'_n \). We will be able to write down the equations for A and B in terms of a new set of coefficients such as:

Plane A:

\[ \sin(\alpha)x_1' - \cos(\alpha)x_2' + a_3'x_3' + \ldots + a_n'x_n' = 0 \]

Plane B:

\[ \sin(\beta)x_1' - \cos(\beta)x_2' + b_3'x_3' + \ldots + b_n'x_n' = 0 \]

Note since we have chosen the new origin to be in the region R, the constant terms in the r.h.s. of the above equations are zero; as the origin is assumed, by our choice, to lie on both planes.

Also since we have assumed that Plane B is steeper than plane A w.r.t. the optimum direction \( x_2' \), we must have:

\[ \beta > \alpha \]

Now we show that the above inequality is untenable, with the condition that the optimum value lies on B rather than A. Now convexity implies that for every feasible point P which lies on A and another feasible point Q which lies on B, the line segment PQ must be in the feasible region. We will now show that this cannot happen for the planes A and B chosen as above. To prove the latter sentence we need is to choose two feasible points P, Q lying on A and B respectively, and show that the line PQ cannot be in the feasible region.

We choose P as follows:

Let P be the point whose coordinate is \((x_P, y_P, 0, 0, \ldots, 0)\), that is we have chosen, \( x_1' = x_P, x_2' = y_P, x_3' = 0, x_4' = 0, \ldots, x_n' = 0 \). Similarly we choose Q as the point with coordinate \((x_Q, y_Q, 0, 0, \ldots, 0)\), that is we have chosen \( x_1' = x_Q, x_2' = y_Q, x_3' = 0, x_4' = 0, \ldots, x_n' = 0 \). Substituting these two coordinates in the equation for their corresponding planes we have the conditions, for P and Q to lie on A and B respectively as:
\[ \sin(\alpha)x_P - \cos(\alpha)y_P = 0 \]  
\[ \sin(\beta)x_Q - \cos(\beta)y_Q = 0 \]  
(13)  
(14)

In order to draw a 2-dimension figure for an n-dimensional situation, we do as follows: consider a planar section containing the origin O and the \((x'_1, x'_2)\) axis, in this figure, P and Q will appear as points lying on lines OA and OB, which represent the respective planes, A and B. Now the value of the objective function \(Z\) at points P and Q are \(Z(P) = y_P\) and \(Z(Q) = y_Q\), since \(\beta > \alpha\), we can see from the figure that \(y_Q > y_P\), i.e. \(Z(Q) > Z(P)\), but the line segment PQ is outside the feasible region. Hence, plane B, having the property as above, cannot exist, thus the theorem is proved QED.

VI. A BRIEF ON REDUNDANT PLANES

In the description of the present method, we had said that the number of steps in reducing the problem from n dimensions to 1 would be n-1, however much depends on the actual geometry of the polytopes. It must be ensured that the current flattest plane is not a redundant plane, that is, it should not be a plane completely “above” the feasible region, if such a thing happens then obviously the optimum vertex point cannot lie on it and then there is no sense using the equation to the flattest plane for elimination of a variable/reduction - the redundant plane must simply be deleted and the next “flattest” plane must be found. The Linear Programming literature contains a number of techniques of detecting redundant planes and these techniques may be used.

Since we need to only test one plane at a time, the current flattest plane to ensure that it is not redundant, perhaps, the simplest way to begin is to use Monte Carlo techniques (e.g. see [8]). That is randomly generate coordinates of many feasible points inside the polytope and then from each of these points “draw” straight lines in the optimum direction, all of them will intersect some plane or the other and out of these a few will hit the “flattest” plane (see Figure 2). The points where the flattest plane is hit will become feasible points in the n-1 dimension space upon dimension reduction (one of the feasible points can also be used to calculate the outward normals of the new system of planes).

If it so happens that the flattest plane is redundant then the plane will not be hit before another “lower” plane, proving that the plane is redundant and can be deleted. The diagram in Fig 3, shows LM as the “flattest” but redundant plane, hence this will be deleted by the algorithm and the next “flattest” plane PQ will be retained. It is not necessary to remove all redundant planes, but only those which happen to qualify as “flattest” at any stage.

This method has the advantage that it can be easily implemented by an algorithm which is parallelizable, thus one can very efficiently use multiple processors.

VII. CONCLUSIONS

A new method of solving the LP problem which is done by dimensional reduction and is aesthetically pleasing has been found. This paper also shows that there exists a recursive method of solving the LP problem, a fact which was not known and therefore novel. The possible utility of the strategy described in this paper for dealing with other types of optimization problems involving nonlinear constraints have also been briefly indicated.

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