Variability Regions for the Third Derivative of Bounded Analytic Functions

Gangqiang Chen

Abstract
Let $z_0$ and $w_0$ be given points in the open unit disk $D$ with $|w_0| < |z_0|$, and $\mathcal{H}_0$ be the class of all analytic self-maps $f$ of $D$ normalized by $f(0) = 0$. In this paper, we establish the third-order Dieudonné’s Lemma and apply it to explicitly determine the variability region $\{f'''(z_0) : f \in \mathcal{H}_0, f(z_0) = w_0, f'(z_0) = w_1\}$ for given $z_0, w_0, w_1$ and give the form of all the extremal functions.

Keywords Bounded analytic functions · Schwarz’s Lemma · Dieudonné’s Lemma · Variability region

Mathematics Subject Classification 30C80 · 30F45

1 Introduction

We denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in the complex plane $\mathbb{C}$ and by $\mathcal{H}_0$ the set of all analytic self-maps $f$ of $\mathbb{D}$ normalized by $f(0) = 0$. In 1890, Schwarz proved that $|f(z_0)| \leq |z_0|$ and $|f'(0)| \leq 1$ hold for all $f \in \mathcal{H}_0$ and $z_0 \in \mathbb{D}$, which gives sharp estimates of the values of $f(z_0)$ and $f'(0)$. Indeed, this classical result describes the variability region of $f(z_0)$ for $z_0 \in \mathbb{D}$ when $f$ ranges over $\mathcal{H}_0$. In order to illustrate this, we would like to introduce some new notation to rewrite Schwarz’s Lemma. For $c \in \mathbb{C}$ and $\rho > 0$, we define the disks $\mathbb{D}(c, \rho)$ and $\overline{\mathbb{D}}(c, \rho)$ by $\mathbb{D}(c, \rho) := \{\zeta \in \mathbb{C} : |\zeta - c| < \rho\}$, and $\overline{\mathbb{D}}(c, \rho) := \{\zeta \in \mathbb{C} : |\zeta - c| \leq \rho\}$. Let $z_0, w_0 \in \mathbb{D}$ be given points with $|w_0| < |z_0|$. Then, Schwarz’s Lemma can be restated as follows.

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Schwarz’s Lemma. Let \( z_0 \in \mathbb{D} \). Then, \( \{ f(z_0) : f \in \mathcal{H}_0 \} = \overline{\mathbb{D}}(0, |z_0|) \) and \( \{ f'(0) : f \in \mathcal{H}_0 \} = \overline{\mathbb{D}} \). Furthermore, \( f(z_0) \in \partial \mathbb{D}(0, |z_0|) \) and \( f'(0) \in \partial \mathbb{D} \) hold if and only if \( f(z) = e^{i\theta} z \) for some \( \theta \in \mathbb{R} \).

Since the discovery of the celebrated Schwarz’s Lemma, more and more famous mathematicians have devoted themselves to the extensions and generalizations of this classical result. For instance, in 1934, Rogosinski [15] established an assertion which can be considered as a sharpened version of Schwarz’s Lemma. His result describes the variability region of \( f(z) \) for \( z \in \mathbb{D}, f \in \mathcal{H}_0 \) with \( |f'(0)| < 1 \), proved by calculating the envelop of a certain union of disks (for the details of the proof, see [6,7]). Therefore, it is worth mentioning this famous refinement of Schwarz’s Lemma.

Rogosinski’s Lemma. If \( f \in \mathcal{H}_0 \) and \( f'(0) \) is fixed, then for \( z \in \mathbb{D} \setminus \{0\} \), the region of values of \( f(z) \) is the closed disk \( \overline{\mathbb{D}}(c, r) \), where

\[
c = \frac{zf'(0)(1 - z^2)}{1 - |z|^2|f'(0)|^2}, \quad r = |z|^2 \frac{1 - |f'(0)|^2}{1 - |z|^2|f'(0)|^2}.
\]

In 1996, Mercer [8] determined the variability region of \( f(z) \) for \( z \in \mathbb{D}, f \in \mathcal{H}_0 \) with \( f(z_0) = w_0(z_0 \neq 0) \), which can be reduced to Rogosinski’s Lemma as \( z_0 \to 0 \). More recently, a number of the important papers on regions of variability have been obtained [11–13,16–18]. Among others, Dieudonné [5] first obtained a description of the variability region of \( f'(z_0), f \in \mathcal{H}_0, \) at a fixed point \( z_0 \in \mathbb{D}, \) which is an improvement of the derivative part of Schwarz’s Lemma. If we define the Möbius transformation

\[
T_a(z) = \frac{z + a}{1 + \overline{a}z}, \quad z, a \in \mathbb{D},
\]

and write

\[
\Delta(z_0, w_0) = \overline{\mathbb{D}} \left( \frac{w_0}{z_0}, \frac{|z_0|^2 - |w_0|^2}{|z_0|(1 - |w_0|^2)} \right),
\]

then his observation can be restated as \( \{ f'(z_0) : f \in \mathcal{H}_0, f(z_0) = w_0 \} = \Delta(z_0, w_0) \). Furthermore, by the uniqueness part of the Schwarz’s lemma, we have \( f'(z_0) \in \partial \Delta(z_0, w_0) \) for \( \theta \in \mathbb{R} \) if and only if \( f(z) = zT_{w_0/z_0}(e^{i\theta} T_{z_0}(z)) \), which is a Blaschke product of degree 2 fixing 0 (see also [1,2,14]). Here, we remark that a Blaschke product of degree \( n \in \mathbb{N} \) takes the form

\[
B(z) = e^{i\theta} \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z_j} z}, \quad z, z_j \in \mathbb{D}, \theta \in \mathbb{R}.
\]
In 2013, Rivard [14] proved the so-called second-order Dieudonné’s Lemma which tells us that if \( f \in \mathcal{H}_0 \) is not an automorphism of \( \mathbb{D} \), then

\[
\left| \frac{1}{2} \overline{z}_0 f''(z_0) - \frac{z_0w_1 - w_0}{1 - |z_0|^2} + \frac{\overline{w}_0(z_0w_1 - w_0)^2}{|z_0|^2 - |w_0|^2} \right| + \frac{|z_0||z_0w_1 - w_0|^2}{|z_0|^2 - |w_0|^2} \leq \frac{|z_0|(|z_0|^2 - |w_0|^2)}{(1 - |z_0|^2)^2},
\]

(1.1)

where \( f(z_0) = w_0 \) and \( f'(z_0) = w_1 \in \Delta(z_0, w_0) \). Equality in (1.1) holds if and only if \( f(z) = zg(z) \) where \( g(z) \) is a Blaschke product of degree 1 or 2 (see also [4]). The original version can be appropriately modified as follows. Suppose that \( |z_0| = r \), \( |w_0| = s \) and \( \beta \in \overline{\mathbb{D}} \). Then,

\[
V(z_0, w_0, \beta) = \left\{ f''(z) : f \in \mathcal{H}_0, f(z_0) = w_0, f'(z_0) = \frac{w_0}{z_0} + \frac{r^2 - s^2}{z_0(1 - r^2)} \beta \right\}
\]

\[
= \frac{2(r^2 - s^2)}{r^2(1 - r^2)^2} \mathbb{D}(c(\beta), \rho(\beta)),
\]

where

\[
c(\beta) = \frac{\overline{z}_0}{z_0^2} \beta (1 - \overline{w}_0 \beta), \quad \rho(\beta) = r(1 - |\beta|^2).
\]

Meanwhile, for \( \beta \in \mathbb{D} \), \( f(z_0) \in \partial V(z_0, w_0, \beta) \) if and only if \( f(z) = zT_{w_0}(T_{-z_0}(z_0)T_v(e^{i\theta}T_{-z_0}(z))) \), where \( \theta \in \mathbb{R}, u_0 = w_0/z_0 \) and \( v_0 = \overline{z}_0^2 \beta / r^2 \). By using this result, the author [2] obtained the sharp upper bound for \( |f''(z_0)| \) depending only on \( |z_0| \). In addition, the author and Yanagihara [3] made use of this consequence to precisely determine the variability region \( V(z_0, w_0) = \{ f''(z_0) : f \in \mathcal{H}_0, f(z_0) = w_0 \} \).

It is natural for us to further study the third-order derivative \( f''' \) of \( f \in \mathcal{H}_0 \). In fact, the purpose of this present paper is to establish a third-order Dieudonné’s Lemma and then apply it to a variability region problem. Before the statement of our main result, we denote \( c \) and \( \rho \) by

\[
\begin{align*}
{c} &= c(z_0, w_0, w_1, w_2) = \frac{6(r^2 - s^2)}{z_0^3(1 - r^2)^3} \left( B + z_0\mu(1 - |\lambda|^2)(1 + r^2 - 2\overline{w}_0\lambda - z_0\overline{\lambda}\mu) \right); \\
{\rho} &= \rho(z_0, w_0, w_1, w_2) = \frac{6(r^2 - s^2)}{r(1 - r^2)^3} (1 - |\lambda|^2)(1 - |\mu|^2),
\end{align*}
\]

where

\[
B = \overline{w}_0^2\lambda^3 - \overline{w}_0^2(1 + r^2)\lambda^2 + r^2\lambda.
\]
Theorem 1.1 (The third-order Dieudonné’s Lemma) Let \( z_0, w_0 \in \mathbb{D}, \lambda, \mu \in \mathbb{D} \) with \(|w_0| = s < r = |z_0|\), 
\[ w_1 = \frac{w_0}{z_0} + \frac{r^2 - s^2}{z_0(1 - r^2)} \lambda, \]
\[ w_2 = \frac{2(r^2 - s^2)\lambda(1 - \overline{w_0}\lambda)}{z_0^2(1 - r^2)^2} + \frac{2(r^2 - s^2)(1 - |\lambda|^2)}{z_0(1 - r^2)^2} \mu. \]

Suppose that \( f \in \mathcal{H}_0, f(z_0) = w_0, f'(z_0) = w_1 \) and \( f''(z_0) = w_2 \). Set \( u_0 = w_0/z_0, v_0 = r^2\lambda/z_0^2 \).

1. If \( |\lambda| = 1 \), then \( f'''(z_0) = c \) and \( f(z) = zT_{u_0}(v_0T_{z_0}(z)) \).
2. If \( |\lambda| < 1, |\mu| = 1 \), then \( f'''(z_0) = c \) and \( f(z) = zT_{u_0}(T_{z_0}(z)T_{v_0}(T_{z_0}(z))) \), where \( \tau = \overline{z_0}\mu/z_0 \).
3. If \( |\lambda| < 1, |\mu| < 1 \), then the region of values of \( f'''(z_0) \) is the closed disk \( \overline{D}(c, \rho) \). Furthermore, \( f'''(z_0) \in \partial \overline{D}(c, \rho) \) if and only if \( f(z) = zT_{u_0}(T_{z_0}(z)T_{v_0}(T_{z_0}(z)T_{\tau}(e^{i\theta}T_{z_0}(z)))) \), where \( \theta \in \mathbb{R} \) and \( \eta = \frac{r^2\mu}{z_0^2} + \frac{\lambda^2r^2w_0 - z_0^2\overline{w_0}}{z_0^3(1 - |\lambda|^2)} \).

In Sect. 3, we will make use of the third-order Dieudonné’s Lemma to determine the region of values of \( f'''(z_0), f \in \mathcal{H}_0, \) in terms of \( z_0, f(z_0), f'(z_0) \). More precisely, we shall explicitly describe the variability region \( \{f'''(z_0) : f \in \mathcal{H}_0, f(z_0) = w_0, f'(z_0) = w_1, f''(z_0) = w_2 \} \) for given points \( z_0, w_0, w_1 \), and determine all the extremal functions. For this purpose, we restate Case (3) in Theorem 1.1 as follows. Under the same hypotheses as in Theorem 1.1 except that \( \lambda \in \mathbb{D} \), then

\( V(z_0, w_0, \lambda, \mu) = \{f'''(z_0) : f \in \mathcal{H}_0, f(z_0) = w_0, f'(z_0) = w_1, f''(z_0) = w_2 \} = \overline{D}(c, \rho) \).

The study on the third derivative of bounded analytic functions in this paper is not exhaustive but could, in our opinion, serve as a basis for further investigations such as the subordination and the extremal problems.

2 Proof of the Third-Order Dieudonné’s Lemma

We begin this section with some fundamental concepts which is convenient for understanding the proof of Theorem 1.1. First, we give an introduction to the definition of Peschl’s invariant derivatives. For \( g: \mathbb{D} \to \mathbb{D} \) holomorphic, Peschl [9] defined the so-called Peschl’s invariant derivatives \( D_ng(z) \) with respect to the hyperbolic metric by the Taylor series expansion:

\[ z \to \frac{g(z + z_0)}{1 + g(z_0) \overline{g(z + z_0)}} = \sum_{n=1}^{\infty} \frac{D_ng(z_0)}{n!} z^n, \quad z, z_0 \in \mathbb{D}. \]
For example, precise forms of $D_n g(z)$, $n = 1, 2, 3$, are given by

$$D_1 g(z) = \frac{(1 - |z|^2)g'(z)}{1 - |g(z)|^2},$$

$$D_2 g(z) = \frac{(1 - |z|^2)^2}{1 - |g(z)|^2} \left[ g''(z) - \frac{2zg'(z)}{1 - |z|^2} + \frac{2g(z)g'(z)^2}{1 - |g(z)|^2} \right],$$

$$D_3 g(z) = \frac{(1 - |z|^2)^3}{1 - |g(z)|^2} \left[ g'''(z) - \frac{6z^2g'(z)}{1 - |z|^2} + \frac{6g(z)g''(z)g'(z)}{1 - |g(z)|^2} + \frac{6z^2g'(z)}{1 - |g(z)|^2} \right] - \frac{12g(z)g'(z)^2}{(1 - |z|^2)(1 - |g(z)|^2)} + \frac{6g(z)^2g'(z)^3}{(1 - |g(z)|^2)^2}.$$

In 2012, Cho et al. [4] proved the following inequality in terms of Peschl’s invariant derivatives, from which we can derive a concrete inequality for $g'''(z)$ in terms of $z$, $g(z)$, $g'(z)$ and $g''(z)$.

**Lemma 2.1** ([4]) If $g : \mathbb{D} \to \mathbb{D}$ is holomorphic, then

$$\left| \frac{D_3 g(z)}{6} (1 - |D_1 g(z)|^2) + D_1 g(z) \left( \frac{D_2 g(z)}{2} \right)^2 \right| + \left| \frac{D_2 g(z)}{2} \right|^2 \leq (1 - |D_1 g(z)|^2)^2,$$

(2.1)

equality holds for a point $z \in \mathbb{D}$ if and only if $g$ is a Blaschke product of degree at most 3.

Next, we would like to show some fundamental argument which simplifies the proof of Theorem 1.1. For brevity, we assume that $z_0 = re^{i\varphi}$, $w_0 = se^{i\xi} \in \mathbb{D}$ with $s < r$. We also define the ‘rotation function’ $\tilde{f}(z) = e^{-i\xi} f(e^{i\varphi}z)$, then we have $\tilde{f}'(r) = e^{i(\varphi - \xi)} f'(z_0) \in \Delta(r, s)$, $\tilde{f}''(r) = e^{i(2\varphi - \xi)} f''(z_0)$ and $\tilde{f}'''(r) = e^{i(3\varphi - \xi)} f'''(z_0)$. Indeed, we can relabel $\tilde{f}$ as $f$, and assume that

$$z_0 = r, \quad w_0 = s, \quad w_1 = \frac{s}{r} + \frac{r^2 - s^2}{r(1 - r^2)} \lambda, \quad \lambda \in \overline{\mathbb{D}},$$

$$w_2 = \frac{2(r^2 - s^2)}{r^2(1 - r^2)^2} (\lambda(1 - s\lambda) + r(1 - |\lambda|^2)\mu), \quad \mu \in \overline{\mathbb{D}}.$$

Correspondingly, we define $c_0$ and $\rho_0$ by

$$\begin{cases}
    c_0 = c_0(r, s, \lambda, \mu) = A \left( B + r \mu(1 - |\lambda|^2)(1 + r^2 - 2s\lambda - r\lambda \mu) \right); \\
    \rho_0 = \rho_0(r, s, \lambda, \mu) = Ar^2(1 - |\lambda|^2)(1 - |\mu|^2),
\end{cases}$$

\[ Springer \]
where
\[
A = \frac{6(r^2 - s^2)}{r^3(1 - r^2)^3}, \quad B = s^2\lambda^3 - s(1 + r^2)\lambda^2 + r^2\lambda.
\] (2.2)

Assume that \( g(z) = f(z)/z \), then \( g \) is an analytic self-map of \( \mathbb{D} \). A straight computation shows that \( D_1g(r) = \lambda, D_2g(r) = 2\mu(1 - |\lambda|^2) \) and
\[
D_3g(r) = \frac{r(1 - r^2)^3}{r^3(r^2 - s^2)} f''''(r) + \frac{6b}{r^2},
\]
where
\[
b = -s^2\lambda^3 + s(1 + r^2)\lambda^2 - r^2\lambda + r\mu(-1 - r^2 + 2s\lambda)(1 - |\lambda|^2).
\]

From Lemma 2.1, we have
\[
\left| \frac{D_3g(r)}{6} + \lambda\mu^2(1 - |\lambda|^2) \right| \leq (1 - |\lambda|^2)(1 - |\mu|^2),
\]

Then, we obtain
\[
\left| f''''(r) + \frac{6(r^2 - s^2)}{r^3(1 - r^2)^3}(b + r^2\lambda\mu^2(1 - |\lambda|^2)) \right| \leq \frac{6(r^2 - s^2)}{r(1 - r^2)^3}(1 - |\lambda|^2)(1 - |\mu|^2),
\]
which is
\[
|f''''(r) - c_0| \leq \rho_0. \tag{2.3}
\]

Equality in (2.3) holds if and only if \( f(z) = zg(z) \), where \( g \) is a Blaschke product of degree 1, 2 or 3 and satisfies
\[
\begin{align*}
g(r) &= \frac{s}{r}; \\
g'(r) &= \frac{r^2 - s^2}{r^2(1 - r^2)}\lambda; \\
g''(r) &= \frac{2(r^2 - s^2)}{r^3(1 - r^2)^2} (-s\lambda^2 + r^2\lambda + r\mu(1 - |\lambda|^2)).
\end{align*}
\] (2.4)

Therefore, Theorem 1.1 is reduced to the following corollary.

**Corollary 2.1** Let \( 0 \leq s < r < 1, \lambda, \mu \in \mathbb{D} \) with \( w_1 = \frac{s}{r} + \frac{r^2 - s^2}{r(1 - r^2)}\lambda, \)
\[
w_2 = \frac{2(r^2 - s^2)}{r^2(1 - r^2)^2} (\lambda(1 - s\lambda) + r(1 - |\lambda|^2)\mu).
\]
Suppose that $f \in \mathcal{H}_0$, $f(r) = s$, $f'(r) = w_1$ and $f''(r) = w_2$. Set $u_0 = s/r$ and $v_0 = \lambda$.

1. If $|\lambda| = 1$, then $f'''(r) = c_0$ and $f(z) = zT_{u_0}(\lambda T_{-r}(z))$.
2. If $|\lambda| < 1$, $|\mu| = 1$, then $f'''(r) = c_0$ and $f(z) = zT_{u_0}(T_{-r}(z)T_{\lambda}(\mu T_{-r}(z)))$.
3. If $|\lambda| < 1$, $|\mu| < 1$, then the region of values of $f'''(z_0)$ is the closed disk $\overline{D}(c_0, \rho_0)$. Furthermore, $f'''(z_0) \in \partial D(c_0, \rho_0)$ if and only if $f(z) = zT_{u_0}(T_{-r}(z)T_{\lambda}(T_{-r}(z)T_{\mu}(e^{i\theta}T_{-r}(z))))$, where $\theta \in \mathbb{R}$.

**Proof** We can easily prove Case (1) and (2) by using the same method in the proof of [2, Lemma 2.2].

For Case (3), the inequality (2.3) means that $f'''(r)$ lies in $\overline{D}(c_0, \rho_0)$. To show that $\overline{D}(c_0, \rho_0)$ is covered, let $\alpha \in \overline{D}$, $u_0 = s/r$ and set $f(z) = zg(z)$, where

$$g(z) = T_{u_0}(T_{-r}(z)T_{\lambda}(T_{-r}(z)T_{\mu}(\alpha T_{-r}(z)))).$$ 

Then, $f(0) = 0$ and $f(r) = s$. Next, we need to show that $f'(r) = w_1$. Note that

$$T_{-u_0} \circ g(z) = T_{-r}(z)T_{\lambda}(T_{-r}(z)T_{\mu}(\alpha T_{-r}(z))). \quad (2.5)$$

Differentiating both sides, we get

$$(T_{-u_0})'(g(z))g'(z) = T'_{-r}(z)T_{\lambda}(T_{-r}(z)T_{\mu}(\alpha T_{-r}(z))) + T_{-r}(z)T'_{\lambda}(T_{-r}(z)T_{\mu}(\alpha T_{-r}(z)))$$

$$+ T_{-r}(z)T_{\lambda}(T_{-r}(z)T_{\mu}(\alpha T_{-r}(z)))$$

$$+ (T'_{-r}(z)T_{\mu}(\alpha T_{-r}(z)) + T_{-r}(z)T'_{\mu}(\alpha T_{-r}(z))\alpha T'_{-r}(z))). \quad (2.6)$$

for all $z \in \mathbb{D}$. Substituting $z = r$ into this equation, we have

$$(T_{-u_0})'(g(r))g'(r) = T'_{-r}(z_0)T_{\mu}(0),$$

which implies

$$g'(r) = \frac{(r^2 - s^2)\lambda}{r^2(1 - r^2)}.$$

Thus, we obtain that $f$ satisfies

$$f'(r) = g(r) + rg'(r) = w_1.$$
Similarly, differentiating both sides of (2.6), we obtain

\[
(T_{-u_0})''(g(z))(g'(z))^2 + (T_{-u_0})'(g(z))g''(z)
= T_{-r}(z)T_\lambda(T_{-r}(z)T_\mu(\alpha T_{-r}(z)))
+ 2T'_{-r}(z)T'_\lambda(T_{-r}(z)T_\mu(\alpha T_{-r}(z)))(T'_{-r}(z)T_\mu(\alpha T_{-r}(z))
+ T_{-r}(z)T''_\mu(\alpha T_{-r}(z))T_\mu(\alpha T_{-r}(z)))
+ T_{-r}(z)T''_\mu(T_{-r}(z)T_\mu(\alpha T_{-r}(z)))(T'_{-r}(z)T_\mu(\alpha T_{-r}(z))
+ T_{-r}(z)T''_\mu(\alpha T_{-r}(z))\alpha T'_{-r}(z))
+ T_{-r}(z)T''_\mu(T_{-r}(z)T_\mu(\alpha T_{-r}(z)))
+ 2T'_{-r}(z)T'_\mu(\alpha T_{-r}(z))\alpha T'_{-r}(z))
+ T_{-r}(z)T''_\mu(T_{-r}(z))(\alpha T_{-r}(z))2
+ T_{-r}(z)T''_\mu(T_{-r}(z))(\alpha T_{-r}(z))\alpha T''_{-r}(z)), \quad z \in \mathbb{D}.
\]

Substituting \( z = r \) into the above equation,

\[
(T_{-u_0})''(g(r))(g'(r))^2 + (T_{-u_0})'(g(r))g''(r)
= T''_{-r}(r)T_\mu(0) + 2T'_{-r}(r)T'_\lambda(0)(T'_{-r}(z)T_\mu(\alpha T_{-r}(z))).
\]

We get that

\[
g''(r) = \frac{2(r^2 - s^2)}{r^3(1 - r^2)^2}(-s\lambda^2 + r^2\lambda + r\mu(1 - |\lambda|^2)).
\]

The above, in conjunction with \( f''(z) = 2g'(z) + zg''(z) \), immediately yields \( f''(r) = w_2 \).

Next, we determine the form of \( f'''(r) \). Differentiating both sides of (2.7),

\[
(T_{-u_0})'''(g(z))(g'(z))^3 + 3(T_{-u_0})''(g(z))g''(z) + T'''_{-u_0}(g(z))g'''(z)
= T'''_{-r}(z)T_\lambda(T_{-r}(z)T_\mu(\alpha T_{-r}(z)))
+ 3T'_{-r}(z)T'_\lambda(T_{-r}(z)T_\mu(\alpha T_{-r}(z)))(T'_{-r}(z)T_\mu(\alpha T_{-r}(z))
+ T_{-r}(z)T''_\mu(\alpha T_{-r}(z))T_\mu(\alpha T_{-r}(z)))
+ 3T'_{-r}(z)T''_\mu(T_{-r}(z)T_\mu(\alpha T_{-r}(z)))(T'_{-r}(z)T_\mu(\alpha T_{-r}(z))
+ T_{-r}(z)T''_\mu(\alpha T_{-r}(z))\alpha T'_{-r}(z))
+ 3T'_{-r}(z)T''_\mu(T_{-r}(z)T_\mu(\alpha T_{-r}(z)))
+ 2T'_{-r}(z)T'_\mu(\alpha T_{-r}(z))\alpha T'_{-r}(z))
+ T_{-r}(z)T''_\mu(T_{-r}(z)T_\mu(\alpha T_{-r}(z)))(T''_{-r}(z)T_\mu(\alpha T_{-r}(z))
+ T_{-r}(z)T''_\mu(\alpha T_{-r}(z))\alpha T''_{-r}(z)))3
3T_{-r}(z)T''_\mu(T_{-r}(z)T_\mu(\alpha T_{-r}(z)))(T'_{-r}(z)T_\mu(\alpha T_{-r}(z))
+ T_{-r}(z)T''_\mu(\alpha T_{-r}(z))\alpha T'_{-r}(z)))
\]
\[
\left( T''_r(z) T'_{\mu}(\alpha T'_r(z)) \right) + 2T'_{-r}(z) T''_{\mu}(\alpha T'_{-r}(z) \alpha T'_r(z)) \\
+ T_{-r}(z) T''_{\mu}(\alpha T_{-r}(z)) (\alpha T'_{-r}(z))^2 + T_{-r}(z) T''_{\mu}(\alpha T_{-r}(z)) \alpha T''_{-r}(z) \\
T_{-r}(z) T'_{\lambda}(T_{-r}(z) T_{\mu}(\alpha T_{-r}(z))) \\
\times \left( T''_{-r}(z) T_{\mu}(\alpha T_{-r}(z)) \right) \\
+ 3T''_{-r}(z) T'_{\mu}(\alpha T_{-r}(z)) \alpha T'_{-r}(z) + 3T'_{-r}(z) T''_{\mu}(\alpha T_{-r}(z)) (\alpha T'_{-r}(z))^2 \\
+ 3T'_{-r}(z) T''_{\mu}(\alpha T_{-r}(z)) \alpha T''_{-r}(z) + T_{-r}(z) T''_{\mu}(\alpha T_{-r}(z)) (\alpha T'_{-r}(z))^3 \\
+ 3T_{-r}(z) T''_{\mu}(\alpha T_{-r}(z)) \alpha^2 T'_{-r}(z) T''_{-r}(z) + T_{-r}(z) T''_{\mu}(\alpha T_{-r}(z)) \alpha T''_{-r}(z) \right).
\]
(2.8)

and then substituting \( z = r \) into (2.8), we have

\[
(T_{-u_0})'''(g(r)) (g'(r))^3 + 3(T_{-u_0})''(g(r)) g''(r) + T'''_{-u_0}(g(r)) g'''(r) \\
= T'''_{-r}(r) T_{\lambda}(0) + 3T'''_{-r}(r) T'_{\mu}(0) (T'_{-r}(r) T_{\mu}(0)) \\
+ 3T'_{-r}(r) T''_{\mu}(0) (T'_{-r}(r) T_{\mu}(0))^2 + 3T'_{-r}(r) T''_{\mu}(0) (T''_{-r}(r) T_{\mu}(0)) \\
+ 2(T'_{-r}(r))^2 T_{\mu}(0) \alpha.
\]

Together with \( f'''(z) = 3g''(z) + zg'''(z) \), we obtain

\[
f'''(r) = c_0 + \rho_0 \alpha.
\]

Now \( \alpha \in \overline{D} \) is arbitrary, so the closed disk \( \overline{D}(c_0, \rho_0) \) is covered.

We know that \( f'''(r) \in \partial D(c_0, \rho_0) \) if and only if \( f(z) = zg(z) \), where \( g \) is a Blaschke product of degree 3 satisfying (2.4), and then we apply this fact to determine the precise form of \( g \). Set

\[
h(z) = \frac{T_{-u_0} \circ g \circ T_{\epsilon}(z)}{z}, \quad z \in \mathbb{D}.
\]

Clearly, \( h \) is a Blaschke product of degree 2 depending on \( g \) and satisfying

\[
h(0) = (T_{-u_0} \circ g \circ T_{\epsilon_0})'(0) = v_0 = \lambda.
\]

Then, \( H(z) = T_{-v_0} \circ h(z) \) is a Blaschke product of degree 2 fixing 0. Set

\[
G(z) = \frac{H(z)}{z}.
\]

Obviously, \( G \) is an automorphism of \( \mathbb{D} \) depending on \( g \) and satisfying

\[
G(0) = H'(0) = T'_{-v_0}(v_0) h'(0) = \mu.
\]
Thus, $T_{-\mu} \circ G$ is an automorphism of $D$ fixing 0, which means that $T_{-\mu} \circ G(z) = e^{i\theta} z$ for $z \in D$ and $\theta \in \mathbb{R}$. Now it is easy to check that

$$g(z) = T_{h_0} \left( T_{-r}(z) T_\lambda (T_{-r}(z) T_\mu (e^{i\theta} T_{-r}(z))) \right), \quad z \in D.$$  

Conversely, if $f(z) = z T_{h_0} \left( T_{-r}(z) T_\lambda (T_{-r}(z) T_\mu (e^{i\theta} T_{-r}(z))) \right)$, where $\theta \in \mathbb{R}$, then direct calculations gives

$$f'''(r) = c_0 + \rho_0 \rho e^{i\theta} \in \partial D(c_0, \rho_0).$$

Hence, we complete the proof. \hfill \Box

In addition, we obtain a sharp upper bound of $|f'''(r)|$ for Case (1).

**Remark 2.2** For $|\lambda| = 1$,

$$|f'''(r)| = \frac{6(r^2 - s^2)}{r^3(1 - r^2)^3}|(1 + r^2)s\lambda^2 - s^2\lambda^3 - r^2\lambda|$$

$$\leq \frac{6(r^2 - s^2)}{r^3(1 - r^2)^3}[(1 + r^2)s + s^2 + r^2],$$

and equality holds if and only if $\lambda = -1$, or if and only if

$$f(z) = -\frac{z - a}{1 - az},$$

where $a = \frac{r^2 + s}{r(1 + s)}$.

We end this section by asking the meaningful question: is it possible to obtain a sharp upper bound for $|f'''(z)|$ depending only on $z$?

### 3 Variability Regions for the Third Derivative

Let $\beta \in \mathbb{D}$, we begin with analyzing the structure of the variability region

$$V(z_0, w_0, \beta) = \{ f'''(z_0) : f \in \mathcal{H}_0(z_0, w_0, \beta) \},$$

where

$$\mathcal{H}_0(z_0, w_0, \beta) = \left\{ f \in \mathcal{H}_0 : f(z_0) = w_0, f'(z_0) = \frac{w_0}{z_0} + \frac{r^2 - s^2}{z_0(1 - r^2)} \beta \right\}.$$

Since the relation $V(r, s, \lambda) = e^{i(3\varphi - \xi)} V(z_0, w_0, \beta)$ holds for $\lambda = e^{-i\xi} \beta$, where $z_0 = r e^{i\varphi}, w_0 = s e^{i\xi} \in \mathbb{D}$ with $s < r$, it is sufficient to determine the variability region $V(r, s, \lambda), \lambda \in \mathbb{D}$. Now we present certain general properties of it.
**Proposition 3.1** \( V(r, s, \lambda) \) is a compact convex domain enclosed by the Jordan curve \( \partial V(r, s, \lambda) \).

**Proof** It is easy to see that \( \mathcal{H}_0(r, s, \lambda) \) is a compact subset of the linear space \( A \) of all analytic functions in \( \mathbb{D} \) endowed with the topology of uniform convergence on compact subsets of \( \mathbb{D} \). Since \( V(r, s, \lambda) \) is the image of \( \mathcal{H}_0(r, s, \lambda) \) with respect to the continuous linear functional \( \ell : A \ni f \mapsto f'''(r) \in \mathbb{C}, V(r, s, \lambda) = \ell(\mathcal{H}_0(r, s, \lambda)) \) is also compact.

Next, we take \( f_1, f_2 \in \mathcal{H}_0(z_0, w_0, \beta) \) and for \( 0 \leq t \leq 1 \), define \( f(z) = (1-t)f_1(z)+tf_2(z) \). It is easy to see \( f \in \mathcal{H}_0(z_0, w_0, \beta) \). Since \( f'''(z_0) = (1-t)f_1'''(z_0)+tf'''(z_0) \in V(z_0, w_0) \), then the convexity of \( V(z_0, w_0) \) is evident.

Furthermore, we claim that \( V(r, s, \lambda) \) has nonempty interior, because \( D(AB, Ar^2(1-|\lambda|^2)) \subset V(r, s, \lambda) \), where \( A, B \) are defined in (2.2). Therefore, \( V(r, s, \lambda) \) is a Jordan closed domain and \( \partial V(r, s, \lambda) \) is a Jordan curve. \( \square \)

We define \( c(\zeta), \rho(\zeta) \) and \( V \) by

\[
c(\zeta) = \zeta(1-\eta \zeta), \quad \rho(\zeta) = t(1-|\zeta|^2), \quad V = \bigcup_{\zeta \in \mathbb{D}} D(c(\zeta), \rho(\zeta)) \tag{3.1}
\]

where

\[
\eta = \frac{r\lambda}{1+r^2-2s\lambda}, \quad t = \frac{r}{|1+r^2-2s\lambda|}.
\]

We remark that \( \eta \in \mathbb{C} \), which is different from the case in [3]. Then, by the third-order Dieudonné’s Lemma, we have

\[
V(r, s, \lambda) = A(B + CV),
\]

where \( C\overline{D}(c, \rho) \) means \( \overline{D}(Cc, |C|\rho) \) and

\[
C = r(1-|\lambda|^2)(1+r^2-2s\lambda) \in \mathbb{C}. \tag{3.2}
\]

We claim that the set \( V \) has the same properties as \( V(r, s, \lambda) \). Firstly, it is not difficult to see that \( V \) contains \( D(0, t) \). Secondly, the compactness and convexity of \( V \) follow from the fact \( V \) corresponds to the variability region \( V(r, s, \lambda) \). Therefore, we reduce the determination of \( \partial V(r, s, \lambda) \) to that of \( V \).

We can obtain the most crucial result below, analogous to [3, Proposition 2.1 and 2.3], which gives the parameter representation of \( \partial V \). However, we will use a different method to prove it and the proof will be given in Sect. 4.

**Proposition 3.2** For \( \theta \in \mathbb{R} \), let \( t_0 \) be the unique solution to the equation

\[
|x e^{i\theta} - \eta| = 2(x^2 - |\eta|^2), \quad x > |\eta| , \tag{3.3}
\]

\( \square \) Springer
if \(|xe^{i\theta} - \eta| \geq 2(x^2 - |\eta|^2)|; otherwise let \(t_\theta = t\). Set
\[
\zeta_\theta = \frac{t_\theta e^{i\theta} - \eta}{2(t_\theta^2 - |\eta|^2)} \in \mathbb{D}.
\] (3.4)

Then, there is a unique point \(v_\theta \in \partial V\) such that
\[
\text{Re}(v_\theta e^{-i\theta}) = \max_{v \in V} \text{Re}(ve^{-i\theta})
\] (3.5)
and \(v_\theta\) can be expressed as
\[
v_\theta = \begin{cases} 
  c(\zeta_\theta) + \rho(\zeta_\theta)e^{i\theta}, & |te^{i\theta} - \eta| < 2(t^2 - |\eta|^2), \\
  c(\zeta_\theta), & |te^{i\theta} - \eta| \geq 2(t^2 - |\eta|^2).
\end{cases}
\] (3.6)

Furthermore, the mapping
\[(-\pi, \pi] \ni \theta \mapsto v_\theta \in \partial V\]
is a continuous bijection and gives a parametric representation of \(\partial V\).

With the results above, we are ready to state the following theorem, an analogue of [3, Theorem 1.2], which gives the unified parametric representation of \(\partial V(r, s, \lambda)\) and all the extremal functions. It is worth mentioning that this result directly follows from Proposition 3.2 and we shall omit its proof. Recall that \(A, B, C\) are given in (2.2), (3.2) and \(\zeta_\theta\) is defined in Proposition 3.2.

**Theorem 2.1** Let \(0 \leq s < r < 1\) and \(|\lambda| < 1\). Then, \(V(r, s, \lambda)\) is a convex closed domain enclosed by the Jordan curve \(\partial V(r, s, \lambda)\) and a parametric representation \((-\pi, \pi] \ni \theta \mapsto \gamma(\theta)\) of \(\partial V(r, s, \lambda)\) is given as follows.

(i) If \(t + |\eta| \leq \frac{1}{2}\), then \(|te^{i\theta} - \eta| \geq 2(t^2 - |\eta|^2)| for all \(\theta \in \mathbb{R}\) and
\[
\gamma(\theta) = A(B + Cc(\zeta_\theta)) \in \partial V(r, s, \lambda).
\]

(ii) If \(t - |\eta| \leq \frac{1}{2}\), then \(|te^{i\theta} - \eta| \leq 2(t^2 - |\eta|^2)| for all \(\theta \in \mathbb{R}\) and
\[
\gamma(\theta) = A\left(B + C(c(\zeta_\theta) + \rho(\zeta_\theta)e^{i\theta})\right) \in \partial V(r, s, \lambda).
\]

(iii) If \(t + |\eta| > \frac{1}{2}\) and \(t - |\eta| < \frac{1}{2}\), then
\[
\gamma(\theta) = \begin{cases} 
  A\left(B + C(c(\zeta_\theta) + \rho(\zeta_\theta)e^{i\theta})\right), & |te^{i\theta} - \eta| < 2(t^2 - |\eta|^2), \\
  A(B + Cc(\zeta_\theta)), & |te^{i\theta} - \eta| \geq 2(t^2 - |\eta|^2).
\end{cases}
\]
Furthermore,
\[
f'''(r) = A \left( B + C(c(\xi_\theta) + \rho(\xi_\theta)e^{i\theta}) \right) \in \partial V(r, s, \lambda),
\]
for some \( \theta \in \mathbb{R} \) with \( \xi_\theta \in \mathbb{D} \) if and only if
\[
f(z) = zT_{\frac{r}{2}} \left( T_{-r}(z)T_{\lambda}(T_{-r}(z))T_{\xi_\theta}(e^{i(\theta + \arg C)}T_{-r}(z)) \right), \quad z \in \mathbb{D}.
\]
Similarly,
\[
f'''(r) = A \left( B + Cc(\xi_\theta) \right) \in \partial V(r, s, \lambda),
\]
for some \( \theta \in \mathbb{R} \) with \( \xi_\theta \in \partial \mathbb{D} \) if and only if
\[
f(z) = zT_{\frac{r}{2}} \left( T_{-r}(z)T_{\lambda}(\xi_\theta T_{-r}(z)) \right), \quad z \in \mathbb{D}.
\]

The above theorem has a direct consequence corresponding to [3, Theorem 1.1], which shows three cases of \( \partial V(r, s, \lambda) \).

**Theorem 2.2** Let \( 0 \leq s < r < 1 \) and \( |\lambda| < 1 \). Then,

(i) If \( t + |\eta| \leq \frac{1}{2} \), then \( \partial V(r, s, \lambda) \) coincides with the Jordan curve given by

\[
\partial \mathbb{D} \ni \zeta \mapsto A \left( B + Cc(\zeta) \right).
\]

(ii) If \( t - |\eta| \geq \frac{1}{2} \), then \( \partial V(r, s, \lambda) \) coincides with the circle given by

\[
\partial \mathbb{D} \ni \zeta \mapsto A \left\{ B + \frac{1 + 4(t^2 - |\eta|^2)t e^{i\theta} - \eta}{4(t^2 - |\eta|^2)} C \right\}.
\]

(iii) If \( t + |\eta| > \frac{1}{2} \) and \( t - |\eta| < \frac{1}{2} \), then \( \partial V(r, s, \lambda) \) consists of the circular arc given by

\[
\Theta \ni \theta \mapsto A \left\{ B + \frac{1 + 4(t^2 - |\eta|^2)t e^{i\theta} - \eta}{4(t^2 - |\eta|^2)} C \right\},
\]

and the simple arc given by

\[
J \ni \zeta \mapsto A \left( B + Cc(\zeta) \right),
\]

where

\[
\Theta = \left\{ \theta \in (-\pi, \pi) : \cos (\theta + \arg(\eta)) > \frac{t^2 + |\eta|^2 - 4(t^2 - |\eta|^2)^2}{2t|\eta|} \right\}.
\]
and

\[ J = \left\{ \xi_\theta : \cos (\theta + \arg(\eta)) \leq \frac{t^2 + |\eta|^2 - 4(t^2 - |\eta|^2)^2}{2t|\eta|} \right\} \]

is the closed subarc of \( \partial \mathbb{D} \).

**Remark 3.3** \( J \) is the closed subarc of \( \partial \mathbb{D} \) which has end points \( \xi_{\theta_1} = \frac{te^{i\theta_1} - \eta}{2(t^2 - |\eta|^2)} \) and \( \xi_{\theta_2} = \frac{te^{i\theta_2} - \eta}{2(t^2 - |\eta|^2)} \), where \(-\pi < \theta_1 < \theta_2 \leq \pi \) are the two solutions of \( |te^{i\theta} - \bar{\eta}| = 2(t^2 - |\eta|^2) \).

We show these three cases of \( \partial V(r, s, \lambda) \) in Figs. 1a, b and 2.

### 4 Proof of Proposition 3.2

In preparation for the proof of Proposition 3.2, we state and prove some auxiliary results related to a compact set \( V \). First, we can easily check the following result is valid by considering the logarithmic derivative of \( h_\theta(x) \).

**Lemma 4.1** For \( \theta \in \mathbb{R} \), define a positive and continuous function \( h_\theta \) by

\[ h_\theta(x) = \frac{|xe^{i\theta} - \bar{\eta}|}{2(x^2 - |\eta|^2)}, \quad x > |\eta|. \]

Then, \( h_\theta \) is strictly decreasing in \( x > |\eta| \) for each fixed \( \theta \) and \( \lim_{x \to \infty} h_\theta(x) = 0. \)
Fig. 2 If $r = 2/3, s = 1/3, \lambda = 1/2$, then
$\eta = 3/10, t = 3/5$ and
$\partial V(r, s, \lambda)$ consists of a circular arc (solid) and a simple arc (dashed)

**Lemma 4.2** For a compact set $V \subset \mathbb{C}$, the function

$$g(\theta) = \max_{v \in V} \text{Re}(ve^{-i\theta}),$$

is continuous in $\theta \in \mathbb{R}$.

**Proof** Since $V$ is compact, then there exists a $v_\theta \in V$ such that

$$g(\theta) = \max_{v \in V} \text{Re}(ve^{-i\theta}) = \text{Re}(v_\theta e^{-i\theta}).$$

For $\theta_0 \in \mathbb{R}$, take a sequence $\theta_n$ which satisfies $\theta_n \to \theta_0$, then there are a $v^* \in V$ and a sequence $v_{\theta_n}$, such that $v_{\theta_n} \to v^*$, and we also have

$$\lim_{\theta \to \theta_0} g(\theta) = \lim_{n \to \infty} g(\theta_n) = \lim_{n \to \infty} \text{Re}(v_{\theta_n} e^{-i\theta_n})$$

$$= \text{Re}(v^* e^{-i\theta_0}) \leq \max_{v \in V} \text{Re}(ve^{-i\theta_0}) = g(\theta_0).$$

Since

$$g(\theta) = \max_{v \in V} \text{Re}(ve^{-i\theta}) \geq \text{Re}(ve^{-i\theta})$$

for any $v \in V$, we obtain

$$\lim_{\theta \to \theta_0} g(\theta) \geq \lim_{\theta \to \theta_0} \text{Re}(ve^{-i\theta}) = \text{Re}(ve^{-i\theta_0}).$$
Noting that $v$ is arbitrary, we have
\[
\lim_{\theta \to \theta_0} g(\theta) \geq \max_{v \in V} \text{Re}(ve^{-i\theta_0}) = g(\theta_0),
\]

it follows that
\[
\lim_{\theta \to \theta_0} g(\theta) = \lim_{\theta \to \theta_0} g(\theta) = g(\theta_0),
\]

thus we prove the continuity of $g(\theta)$. \hfill \Box

We recall a basic notion, the corner point, used in conformal geometry, referring to [10, Section 3.4] by Ch. Pommerenke for details. Notice that a half-plane $H$ is a supporting half-plane of $V$ if it intersects $V$ on its border and such that $V \subset H$, and $\partial H$ is called the supporting line (see Fig. 3). For a convex domain $W \subset \mathbb{C}$, the boundary point is a corner point if and only if there are at least two supporting lines of $W$ at $w$.

**Lemma 4.3** Let $V$ be a compact convex set without corner points in $\mathbb{C}$, and suppose that for each $\theta \in \mathbb{R}$, there is a unique point $v_\theta \in \partial V$ such that
\[
\text{Re}(v_\theta e^{-i\theta}) = \max_{v \in V} \text{Re}(ve^{-i\theta}). \tag{4.1}
\]

Then, the mapping
\[
(-\pi, \pi] \ni \theta \mapsto v_\theta \tag{4.2}
\]
gives a continuous bijection of $(-\pi, \pi]$ onto $\partial V$.

**Proof** First, we show the mapping $(-\pi, \pi] \ni \theta \mapsto v_\theta$ is continuous. For $\theta_0 \in \mathbb{R}$, we take a sequence $\theta_n$ which satisfies $\theta_n \to \theta_0$. Since $V$ is compact, there exists a $v^* \in V$ and a subsequence $v_{\theta_{n_k}}$, such that $v_{\theta_{n_k}} \to v^*$. As $g(\theta_n) = \text{Re}(v_{\theta_n} e^{-i\theta_n})$, we have
\[
\text{Re}(v_{\theta_0} e^{-i\theta_0}) = g(\theta_0) = \lim_{k \to \infty} g(\theta_{n_k}) = \lim_{k \to \infty} \text{Re}(v_{\theta_{n_k}} e^{-i\theta_{n_k}}) = \text{Re}(v^* e^{-i\theta_0}).
\]

From the uniqueness of $v_\theta$, we have $v_{\theta_0} = v^*$.
Since $V$ is a compact convex subset of $\mathbb{C}$ and has nonempty interior, the boundary $\partial V$ is a simple closed curve. Note that $v_\theta$ is injective continuous from $\partial \mathbb{D}$ to $\partial V$, and recall that a simple closed curve cannot contain any simple closed curve other than itself. Thus, $\partial V$ is given by

$$(-\pi, \pi] \ni \theta \mapsto v_\theta \in \partial V.$$ 

Now we begin our investigation of the properties of the set $V = \bigcup_{\zeta \in \mathbb{D}} (c(\zeta), \rho(\zeta))$.

**Lemma 4.4** For $\theta \in \mathbb{R}$, take $v_\theta \in \partial V$ such that $\text{Re}(v_\theta e^{-i\theta}) = \max_{v \in V} \text{Re}(ve^{-i\theta})$, then there is only one $\zeta_\theta \in \mathbb{D}$ such that $v_\theta = c(\zeta_\theta) + \rho(\zeta_\theta)e^{i\theta}$.

**Proof** For $\theta \in \mathbb{R}$, take $v_\theta \in \partial V$ such that $\text{Re}(v_\theta e^{-i\theta}) = \max_{v \in V} \text{Re}(ve^{-i\theta})$. Then, $\exists \zeta_\theta, \varepsilon_\theta \in \mathbb{D}$, such that $v_\theta = c(\zeta_\theta) + \rho(\zeta_\theta) \varepsilon_\theta$. From the hypotheses of the lemma, we have

$$\text{Re}\{(c(\zeta_\theta) + \rho(\zeta_\theta) \varepsilon_\theta)e^{-i\theta}\} \geq \text{Re}\{(c(\zeta) + \rho(\zeta) \varepsilon)e^{-i\theta}\}, \quad \forall \zeta, \varepsilon \in \mathbb{D}.$$ 

Substitute $\zeta = \zeta_\theta$ into this equation, we have $\text{Re}\{\rho(\zeta_\theta) \varepsilon_\theta e^{-i\theta}\} \geq \text{Re}\{\rho(\zeta_\theta) \varepsilon e^{-i\theta}\}$. Let $\varepsilon = e^{i\theta}$, we obtain $\text{Re}\{\rho(\zeta_\theta) \varepsilon_\theta e^{-i\theta}\} \geq \rho(\zeta_\theta)$. Thus, $\varepsilon_\theta = e^{i\theta}$ and $v_\theta = c(\zeta_\theta) + \rho(\zeta_\theta)e^{i\theta}$.

Therefore, we have

$$\text{Re}\{c(\zeta_\theta) e^{-i\theta}\} + \rho(\zeta) \leq \text{Re}\{c(\zeta_\theta) e^{-i\theta}\} + \rho(\zeta_\theta),$$

and

$$g_\theta(\zeta) = \text{Re}\{(c(\zeta) \cdot e^{-i\theta}) + \rho(\zeta), \quad \zeta \in \mathbb{D}$$

takes maximum at $\zeta_\theta$.

(1) For $\theta$ satisfies $|te^{i\theta} - \eta| < 2(r^2 - |\eta|^2)$, prove by contradiction, we can get that $g_\theta(\zeta)$ attains its maximum at $\zeta_\theta \in \mathbb{D}$ and satisfies

$$\frac{\partial g_\theta(\zeta)}{\partial \zeta}|_{\zeta = \zeta_\theta} = 0, \quad (4.3)$$

then we have

$$\zeta_\theta = \frac{te^{i\theta} - \eta}{2(r^2 - |\eta|^2)} \in \mathbb{D},$$

which shows that $\zeta_\theta$ is unique and depends only on $\theta$. 

\[\square\] Springer
(2) For $\theta$ satisfies $|te^{i\theta} - \eta| \geq 2(t^2 - |\eta|^2)$, by contradiction, we can conclude that $g_{\theta}(\zeta)$ takes maximum at $\zeta_{\theta} \in \partial \mathbb{D}$, $v_{\theta} = c(\zeta_{\theta})$ and for $\zeta = \rho e^{i\varphi}$, $g_{\theta}(\zeta)$ satisfies

$$\frac{\partial g_{\theta}}{\partial \varphi}(\zeta)|_{\zeta = \zeta_{\theta}} = - \text{Im}(\zeta_{\theta}(1 - 2\eta\zeta_{\theta})e^{-i\theta}) = 0,$$  \hspace{1cm} (4.4)

$$\frac{\partial g_{\theta}}{\partial \rho}(\zeta)|_{\zeta = \zeta_{\theta}} = \text{Re}(\zeta_{\theta}(1 - 2\eta\zeta_{\theta})e^{-i\theta}) - 2t \geq 0.$$  \hspace{1cm} (4.5)

Therefore, there is a $\tilde{t} \geq t$, such that $\zeta_{\theta}(1 - 2\eta\zeta_{\theta})e^{-i\theta} = 2\tilde{t}$, we have

$$\zeta_{\theta} = \frac{\tilde{t}e^{i\theta} - \eta}{2(\tilde{t}^2 - |\eta|^2)}.$$  

Since $h_{\theta}(x)$ is strictly decreasing for $x > |\eta|$, $\tilde{t}$ is the unique solution of

$$\frac{|xe^{i\theta} - \eta|}{2(x^2 - |\eta|^2)} = 1,$$

which implies the uniqueness of $\zeta_{\theta}$. This also implies the uniqueness of $v_{\theta}$.  \hspace{1cm} \square

**Proof** (Proof of Proposition 3.2) From Lemma 4.4, we get that $v_{\theta}$ is also unique and can be expressed as (3.6). We already know that $V$ is a compact convex subset and has nonempty interior, then $V$ is a convex closed domain enclosed by the Jordan curve $\partial V$.

We just need to show that $\partial V$ has no corner points, which is to prove $(-\pi, \pi] \ni \theta \mapsto v(\theta)$ is injective. Suppose, on the contrary, $v^* \in \partial V$ is a corner point, then there are two supporting half planes $H_1, H_2$ such that $v^* \in \partial V \subset H_1 \cap H_2$, $v^* \in \partial H_1 \cap \partial H_2$ and the opening angle $\alpha$ of $H_1 \cap H_2$ is less than $\pi$ (see Fig. 4). Without generality, we suppose that $v_{\theta_1} = v_{\theta_2} = v^*$ for some $-\pi < \theta_1 < \theta_2 \leq \pi$. For $j = 1, 2$ define a half plane $H_j$ by

$$H_j = \{w \in \mathbb{C} : \text{Re}(we^{-i\theta_j}) \leq \text{Re}(v^*e^{-i\theta_j})\}.$$  

Then, $v^* \in V \subset H_1 \cap H_2$ and $v^* \in \partial H_1 \cap \partial H_2$. By a geometric consideration, $v_{\theta} \equiv v^*$ for $\theta_1 \leq \theta \leq \theta_2$.

By taking a subinterval, if necessary, we may assume that $|te^{i\theta} - \eta| > 2(t^2 - |\eta|^2)$ on $[\theta_1, \theta_2]$ at first. In this case, $\zeta_{\theta} \in \partial \mathbb{D}$ and $c(\zeta_{\theta}) = v_{\theta} \equiv v^*$ on $[\theta_1, \theta_2]$. Since $c(\zeta)$ is a nonconstant analytic function and $v_{\theta}$ is continuous in $\theta$, this implies that there exists $\zeta^* \in \partial \mathbb{D}$ with $\zeta_{\theta} \equiv \zeta^*$ on $[\theta_1, \theta_2]$. Let

$$\Phi(z) = \frac{z - \eta}{2(|z|^2 - |\eta|^2)}, \hspace{1cm} |z| > |\eta|.$$  

Then, $\zeta^* \equiv \zeta_{\theta} = \Phi(r_{\theta}e^{i\theta})$ on $[\theta_1, \theta_2]$. Since the Jacobian $J_{\Phi}$ of $\Phi$ is

$$J_{\Phi}(\zeta) := \left| \frac{\partial \Phi}{\partial \zeta}(\zeta) \right|^2 - \left| \frac{\partial \Phi}{\partial \zeta}(\zeta) \right|^2,$$

\hspace{1cm} Springer
Fig. 4 The supporting lines

\[
\Phi \text{ is locally injective and hence there exists } z^* (= \Phi^{-1}(\zeta^*)) \text{ with } t_\theta e^{i\theta} \equiv z^* \text{ on } [\theta_1, \theta_2], \text{ which is apparently a contradiction.}
\]

Next, we assume \(|te^{i\theta} - \eta| < 2(t^2 - |\eta|^2)\) on \([\theta_1, \theta_2]\). In this case, we have on \([\theta_1, \theta_2]\)

\[
\zeta_\theta = \frac{te^{i\theta} - \eta}{2(t^2 - |\eta|^2)},
\]

\[
v_\theta = c(\zeta_\theta) + \rho(\zeta_\theta)e^{i\theta} \equiv v^*.
\]

Then, by an elementary calculation we have

\[
\frac{d}{d\theta} \{v_\theta\} = c'(\zeta_\theta) \frac{d\zeta_\theta}{d\theta} - t \left( \frac{d\zeta_\theta}{d\theta} \overline{\zeta_\theta} + \overline{\frac{d\zeta_\theta}{d\theta}} \zeta_\theta \right) e^{i\theta} + it(1 - |\zeta_\theta|^2)e^{i\theta}
\]

\[
= \frac{it e^{i\theta}}{4(t^2 - |\eta|^2)} \{1 + 4(t^2 - |\eta|^2)\} \neq 0,
\]

which is a contradiction.

We remain to consider the case \(\theta_1 < \theta_2, |te^{i\theta_1} - \eta| < 2(t^2 - |\eta|^2)\) and \(|te^{i\theta_2} - \eta| > 2(t^2 - |\eta|^2)\), such that \(v_{\theta_1} = v_{\theta_2}\), then for \(\theta_1 \leq \theta \leq \theta_2\), we have \(v_\theta \equiv v_{\theta_1}\). Thus, there exists \(\theta_1 < \theta_1' < \theta_2\) such that \(|te^{i\theta_1'} - \eta| < 2(t^2 - |\eta|^2)\), \(v_{\theta_1} = v_{\theta_1'}\), which is a contradiction. Above all, we prove that \(\theta \mapsto v(\theta)\) is injective and then \(\partial V\) has no corner points.

By Lemma 4.3, the mapping \((-\pi, \pi] \ni \theta \mapsto v_\theta \in \partial V\) gives a parametric representation of \(\partial V\).

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

References

1. Beardon, A.F., Minda, D.: A multi-point Schwarz–Pick lemma. J. Anal. Math. 92, 81–104 (2004)
2. Chen, G.Q.: Estimates of the second derivative of bounded analytic functions. Bull. Aust. Math. Soc. 100(3), 458–469 (2019)
3. Chen, G.Q., Yanagihara, H.: Variability regions for the second derivative of bounded analytic functions. arXiv:2004.02405
4. Cho, K.H., Kim, S.-A., Sugawa, T.: On a multi-point Schwarz–Pick lemma. Comput. Methods Funct. Theory 12(2), 483–499 (2012)
5. Dieudonné, J.: Recherches sur quelques problèmes relatifs aux polynômes et aux fonctions bornées d’une variable complexe. Ann. Sci. Normale Supérieure 48, 247–358 (1931)
6. Duren, P.L.: Univalent Functions, vol. 259. Springer, New York (1983)
7. Goluzin, G.M.: Geometric Theory of Functions of a Complex Variable, vol. 26. Amer. Math. Soc, Providence (1969)
8. Mercer, P.R.: Sharpened versions of the Schwarz lemma. J. Math. Anal. Appl. 205(2), 508–511 (1997)
9. Peschl, E.: Les invariants différentiels non holomorphes et leur rôle dans la théorie des fonctions. Rend. Sem. Mat. Messina 1, 100–108 (1955)
10. Pommerenke, C.: Boundary Behaviour of Conformal Maps, vol. 299. Springer, Berlin (2013)
11. Ponnusamy, S., Vasudevarao, A.: Region of variability of two subclasses of univalent functions. J. Math. Anal. Appl. 332, 1323–1334 (2007)
12. Ponnusamy, S., Vasudevarao, A., Yanagihara, H.: Region of variability for close-to-convex functions. Complex Var. Elliptic Equ. 53(8), 709–716 (2008)
13. Ponnusamy, S., Vasudevarao, A., Vuorinen, M.: Region of variability for certain classes of univalent functions satisfying differential inequalities. Complex Var. Elliptic Equ. 54(10), 899–922 (2009)
14. Rivard, P.: Some applications of higher-order hyperbolic derivatives. Complex Anal. Oper. Theory 7(4), 1127–1156 (2013)
15. Rogosinski, W.: Zum Schwarzschen Lemma. Jahresbericht der Deutschen Mathematiker-Vereinigung 44, 258–261 (1934)
16. Yanagihara, H.: Regions of variability for functions of bounded derivatives. Kodai Math. J. 28(2), 452–462 (2005)
17. Yanagihara, H.: Regions of variability for convex functions. Math. Nachr. 279(15), 1723–1730 (2006)
18. Yanagihara, H.: Variability regions for families of convex functions. Comput. Methods Funct. Theory 10(1), 291–302 (2010)

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