A remark on the motive of the Fano variety of lines of a cubic

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Abstract Let $X$ be a smooth cubic hypersurface, and let $F$ be the Fano variety of lines on $X$. We establish a relation between the Chow motives of $X$ and $F$. This relation implies in particular that if $X$ has finite–dimensional motive (in the sense of Kimura), then $F$ also has finite–dimensional motive. This proves finite–dimensionality for motives of Fano varieties of cubics of dimension 3 and 5, and of certain cubics in other dimensions.

Résumé Soit $X$ une hypersurface cubique lisse, et soit $F$ la variété de Fano paramétrant les droites contenues dans $X$. On établit une relation entre les motifs de Chow de $X$ et de $F$. Cette relation implique le fait que $F$ a un motif de dimension finie (au sens de Kimura) à condition que $X$ ait un motif de dimension finie. En particulier, si $X$ est une cubique lisse de dimension 3 ou 5, alors $F$ a un motif de dimension finie.

Keywords Algebraic cycles · Chow groups · motives · finite–dimensional motives · cubics · Fano variety of lines

Mathematics Subject Classification (2010) 14C15, 14C25, 14C30, 14J70, 14N25

1 Introduction

The notion of finite–dimensional motive, developed independently by Kimura and O’Sullivan [15], [1], [19], [14], [10] has given important new impetus to the study of algebraic cycles. To give but one example: thanks to this notion, we now know the Bloch conjecture is true for surfaces of geometric genus zero that are rationally dominated by a product of curves [15]. It thus seems worthwhile to find concrete examples of varieties that have finite–dimensional motive, this being (at present) one of the sole means of arriving at a satisfactory understanding of Chow groups.

The present note aims to contribute something to the list of examples of varieties with finite–dimensional motive, by considering Fano varieties of lines of smooth cubics over $\mathbb{C}$. The main result is as follows:

Theorem (main theorem) Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth cubic hypersurface, and let $F(X)$ denote the Fano variety of lines on $X$. If $X$ has finite–dimensional motive, then also $F(X)$ has finite–dimensional motive.

In particular, this implies that for smooth cubics $X$ of dimension 3 or 5, the Fano variety $F(X)$ has finite–dimensional motive. In the first case, the dimension of $F(X)$ is 2, while in the second case it is 6. The case $n = 3$ is also proven (in a different way) in [5]. Some more examples where theorem 4 applies are given in corollary [7].

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Theorem 4 follows from a more general result. This more general result relates the Chow motives of $X$ and $F = F(X)$ for any smooth cubic:

**Theorem (theorem 5)** Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth cubic hypersurface. Let $F := F(X)$ denote the Fano variety of lines on $X$, and let $X^{[2]}$ denote the second Hilbert scheme of $X$. There is an isomorphism of Chow motives

$$h(F)(2) \oplus \bigoplus_{i=0}^{n} h(X)(i) \cong h(X^{[2]})$$

in $\mathcal{M}_{\text{rat}}$.

This relation of Chow motives is inspired by (and formally similar to) a relation between $X$ and $F$ in the Grothendieck ring of varieties that was discovered by Galkin–Shinder [7] (cf. remark 16).

**Conventions** All varieties will be projective irreducible varieties over $\mathbb{C}$.

All Chow groups will be with rational coefficients: for $X$ smooth of dimension $n$, we will write $A^j(X) = A^{n-j}(X)$ for the Chow group of codimension $j$ cycles with $\mathbb{Q}$–coefficients modulo rational equivalence. We will write $A^j_{\text{hom}}(X)$ and $A^j_{\text{AJ}}(X)$ for the subgroups of homologically trivial (resp. Abel–Jacobi trivial) cycles.

The category $\mathcal{M}_{\text{rat}}$ will denote the contravariant category of pure motives with respect to rational equivalence, as in [23], [19]. For a morphism $f: X \to Y$ between smooth varieties, we will write $\Gamma_f \in A^{\dim Y}(X \times Y)$ for the graph of $f$.

### 2 Finite–dimensionality

We refer to [15], [1], [19], [10], [14] for basics on the notion of finite–dimensional motive. An essential property of varieties with finite–dimensional motive is embodied by the nilpotence theorem:

**Theorem 1 (Kimura [15])** Let $X$ be a smooth projective variety of dimension $n$ with finite–dimensional motive. Let $\Gamma \in A^n(X \times X)$ be a correspondence which is numerically trivial. Then there is $N \in \mathbb{N}$ such that

$$\Gamma^\circ N = 0 \in A^n(X \times X).$$

Actually, the nilpotence property (for all powers of $X$) could serve as an alternative definition of finite–dimensional motive, as shown by a result of Jannsen [14, Corollary 3.9]. Conjecturally, all smooth projective varieties have finite–dimensional motive [15]. We are still far from knowing this, but at least there are quite a few non–trivial examples:

**Remark 2** The following varieties have finite–dimensional motive: abelian varieties, varieties dominated by products of curves [15], $K3$ surfaces with Picard number 19 or 20 [21], surfaces not of general type with $p_g = 0$ [8, Theorem 2.11], certain surfaces of general type with $p_g = 0$ [8, 22, 31], Hilbert schemes of surfaces known to have finite–dimensional motive [1], generalized Kummer varieties [33, Remark 2.9(ii)], threefolds with nef tangent bundle [11] (an alternative proof is given in [26, Example 3.16]), fourfolds with nef tangent bundle [12], log–homogeneous varieties in the sense of [2] (this follows from [12, Theorem 4.4]), certain threefolds of general type [28, Section 8], varieties of dimension $\leq 3$ rationally dominated by products of curves [25, Example 3.15], varieties $X$ with $A^i_{\text{AJ}}(X) = 0$ for all $i$ [25, Theorem 4], products of varieties with finite–dimensional motive [15].

**Remark 3** It is an embarrassing fact that up till now, all examples of finite-dimensional motives happen to lie in the tensor subcategory generated by Chow motives of curves, i.e. they are “motives of abelian type” in the sense of [26]. On the other hand, there exist many motives that lie outside this subcategory, e.g. the motive of a very general quintic hypersurface in $\mathbb{P}^4$ [3, 7.6].
3 Main theorem

This section contains the proof of the main result of this note, as announced in the introduction:

**Theorem 4** Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth cubic hypersurface, and let $F := F(X)$ denote the Fano variety of lines on $X$. If $X$ has finite–dimensional motive (resp. motive of abelian type), then $F$ has finite–dimensional motive (resp. motive of abelian type).

Theorem 4 follows from a more general result:

**Theorem 5** Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth cubic hypersurface. Let $F := F(X)$ denote the Fano variety of lines on $X$, and let $X^{[2]}$ denote the second Hilbert scheme of $X$. There is an isomorphism of Chow motives

$$h(F)(2) \oplus \bigoplus_{i=0}^{n} h(X)(i) \cong h(X^{[2]}) \text{ in } \mathcal{M}_{\text{rat}}.$$

**Proof** The argument hinges on the following geometric relation between $X$ and $F$, which is specific to cubics:

**Proposition 6** (Galkin–Shinder [7], Voisin [30]) Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth cubic hypersurface, and let $X^{[2]}$ denote its second Hilbert scheme. There exists a birational map

$$\phi: \ X^{[2]} \dashrightarrow W,$$

where $W$ is a $\mathbb{P}^n$–bundle over $X$. The map $\phi$ admits a resolution of indeterminacy

$$\begin{array}{ccc}
\phi_1 & \leftarrow & Y \\
\phi & \rightarrow & \phi_2 \\
X^{[2]} & \leftarrow & W.
\end{array}$$

Here the morphism $\phi_1: Y \to X^{[2]}$ is the blow–up with center $\tau: Z \subset X^{[2]}$ of codimension 2, and $Z$ has the structure of a $\mathbb{P}^2$–bundle $p: Z \to F$. The morphism $\phi_2: Y \to W$ is the blow–up with center $\tau': Z' \subset W$ of codimension 3, and $Z'$ has the structure of a $\mathbb{P}^1$–bundle $p': Z' \to F$.

Moreover, the diagram

$$
\begin{array}{ccc}
E & \leftarrow & F \\
p & \leftarrow & p' \leftarrow & Z' \\
& \leftarrow & f' \leftarrow & f
\end{array}
$$

commutes, where $E \subset Y$ denotes the exceptional divisor of $\phi_1$ and $\phi_2$, and $f$ (resp. $f'$) denotes the restriction of $\phi_1$ (resp. $\phi_2$) to $E$.

**Proof** The map $\phi$ is defined in [7, Proof of Theorem 5.1]. The existence of the variety $Y$ with two different blow–up structures as indicated is [30 Proposition 2.9].

For the “Moreover” part, we inspect the proof of [30 Proposition 2.9]. This proof contains an explicit description of the exceptional divisor $E$ (denoted $Q_{P_2}$ in loc. cit.):

$$E = \left\{ \left( \ell, x+y, [\ell] \right) \mid \ell \subset X, \ x+y \in \ell^{[2]}, \ u \in \ell \right\},$$

where the pair $x+y$ is in $X^{[2]}$ and $\ell$ denotes a line. The morphism $f$ sends a triple $(u, x+y, [\ell])$ to the pair $(x+y, [\ell])$. The image $f(E)$ is the locus of length 2 subschemes $x+y \in X^{[2]}$ contained in a line $\ell$. Thus, $f(E)$ identifies with $Z$ (denoted $P_2$ in loc. cit.), and $p \circ f$ sends $(u, x+y, [\ell])$ to $[\ell] \in F$. The morphism $f'$ (which is $\phi$ restricted to $Q_{P_2}$ in the notation of loc. cit.) sends a triple $(u, x+y, [\ell])$ to the pair $(u, [\ell])$. The image $Z' = f'(E)$ (which is denoted $P$ in loc. cit.) has a $\mathbb{P}^1$–bundle structure $p: Z' \to F$ obtained by sending $(u, [\ell])$ to $[\ell] \in F$. This proves the “Moreover” assertion of proposition 6.
We now proceed with the proof of theorem\textsuperscript{[5]} As is well-known, a birational map $\phi: X^{[2]} \to W$ induces homomorphisms

$$
\phi_*: A^j(X^{[2]}) \to A^j(W), \\
\phi^*: A^j(W) \to A^j(X^{[2]}),
$$
defined by the correspondence $\Gamma_\phi$ (the closure of the graph of $\phi$) resp. its transpose. As a first step, we relate $F$ and $X^{[2]}$ on the level of Chow groups:

**Proposition 7** Let $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a smooth cubic hypersurface, and let $F = F(X)$ be its Fano variety of lines. The map

$$
A^{j-2}(F) \oplus A^j(W) \to A^j(X^{[2]}), \\
(a, b) \mapsto \tau \cdot p^*(a) + \phi^*(b)
$$
is an isomorphism for all $j$.

**Proof** It will be convenient to prove proposition\textsuperscript{[7]} in a more abstract set-up. That is, we forget for the time being that we are dealing with cubics and Fano varieties and we only keep the geometric structure provided by proposition\textsuperscript{[8]} In this abstract set-up, we will prove the isomorphism of proposition\textsuperscript{[7]}

**Proposition 8** Let $V$ and $V'$ be smooth projective varieties of dimension $m$. Assume there is a birational map

$$
\phi: V \dashrightarrow V',
$$
and a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\phi_1} & Y' \\
V & \xrightarrow{\phi} & V'
\end{array}
$$

where $\phi_1$ is the blow-up with smooth codimension 2 center $\tau: Z \subset V$, and $\phi_2$ is the blow-up with smooth codimension 3 center $\tau': Z' \subset V'$. Assume moreover there is a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{j} & F \\
Z & \xrightarrow{p} & Z'
\end{array}
$$

where $E$ denotes the exceptional divisor of $\phi_1$ and $\phi_2$, and $j$ (resp. $j'$) denotes the restriction of $\phi_1$ (resp. $\phi_2$) to $E$, and $p$ (resp. $p'$) is a $\mathbb{P}^2$-bundle (resp. $\mathbb{P}^3$-bundle) over a smooth projective variety $F$. Then the map

$$
A^{j-2}(F) \oplus A^j(V') \to A^j(V), \\
(a, b) \mapsto \tau \cdot p^*(a) + \phi^*(b)
$$
is an isomorphism for all $j$.

Proposition\textsuperscript{[7]} is then the conjunction of propositions\textsuperscript{[6] and 8]. We now prove proposition\textsuperscript{[8]}

For any $j$, there is a diagram with split-exact rows

\begin{equation}
\begin{aligned}
0 & \to A^{j-2}(Z) \xrightarrow{\alpha} A^{j-1}(E) \oplus A^j(V) \xrightarrow{\beta} A^j(Y) \to 0 \\
& \uparrow (\phi_* \phi^*) \quad \quad \quad \quad \uparrow \tau \\
0 & \to A^{j-3}(Z') \xrightarrow{\alpha'} A^{j-1}(E) \oplus A^j(V') \xrightarrow{\beta'} A^j(Y) \to 0
\end{aligned}
\end{equation}

Here, the arrow labelled $\alpha$ is defined as $(\epsilon_1(G) \cdot f^*(\cdot), \tau_*)$ where $G$ is the excess normal bundle of the embedding $Z \subset V$ (as defined in \textsuperscript{[6]} Section 6.7). A left-inverse to $\alpha$ is given by $(a, b) \mapsto f_*(a)$. The arrow labelled $\beta$ is defined as $i_* - (\phi_1)^*$, where $i: E \to Y$ denotes the inclusion morphism. The arrow labelled $\alpha'$ is defined as
is commutative (here $G'$ is the excess normal bundle of the embedding $Z' \subset V'$. A left-inverse to $\alpha'$ is given by $(c, d) \mapsto (f')^*(c)$. The arrow labelled $\beta'$ is defined as $i_* - (\phi_2)^*$, and for the upper square we have used $[6, \text{Proposition } 6.7\text{(a)})].$

The commutativity of (2) and (3) proves commutativity of diagram (1). Thus, we have equality $\phi^* = (\phi_1)_*(\phi_2)^*$. The square

\[
\begin{array}{ccc}
A^j(Y) & \xrightarrow{(\phi_1)^*} & A^j(Y) \\
\uparrow \phi^* & & \uparrow \tau \\
A^j(Y') & \xrightarrow{(\phi_2)^*} & A^j(Y)
\end{array}
\]

is commutative. The arrow labelled $\psi$ is defined as $f^* f_* (\cdot) \cdot c_1(G)$. The diagram

\[
\begin{array}{ccc}
A^{j-1}(E) & \overset{i_*}{\rightarrow} & A^j(Y) \\
\uparrow f^* c_1(G) & & \uparrow (\phi_1)^* \\
A^{j-2}(Z) & \overset{(\tau_2)_*}{\rightarrow} & A^j(V) \\
\uparrow f_* & & \uparrow (\phi_1)_* \\
A^{j-1}(E) & \overset{i_*}{\rightarrow} & A^j(Y)
\end{array}
\]

is commutative (here $i_Z$ is the inclusion $Z \rightarrow V$, and for the upper square we have used $[6, \text{Proposition } 6.7\text{(a)})]$. The commutativity of (2) and (3) proves commutativity of diagram (1).

Since the diagram (1) is commutative with exact rows, there exists a map $\gamma$ making the diagram

\[
0 \rightarrow A^{j-2}(Z) \rightarrow A^{j-1}(E) \oplus A^j(V) \rightarrow A^j(Y) \rightarrow 0
\]

commute. Applying the snake lemma to diagram (4), we find an exact sequence

\[
\ker \psi \oplus \ker (\phi^*) \xrightarrow{g} \ker \tau \rightarrow \text{Coker } \gamma \xrightarrow{(h_1, h_2)} \text{Coker } \psi \oplus \text{Coker } (\phi^*) \xrightarrow{k} \text{Coker } \tau
\]

We now state some lemmas about the arrows in (3):

**Lemma 9** The arrow labelled $g$ in (3) is surjective.

**Proof** Let $c$ be an element in $\ker \tau$, i.e. $c \in A^j(Y)$ with $(\phi_1)^*(\phi_1)_*(c) = 0$. As $(\phi_1)^*$ is injective, we must have $(\phi_1)_*(c) = 0$, and so as $c$ restricts to 0 in $A^j(Y \setminus E)$ the element $c$ comes from an element $d \in A^{j-1}(E)$. The element $d$ can be written in a unique way as

\[
d = d_1 + d_2 \in A^{j-1}(E),
\]

where $d_1 = f^*(b_1)$ for $b_1 \in A^{j-1}(Z)$, and $d_2 = f^*(b_2) \cdot c_1(G)$ for $b_2 \in A^{j-2}(Z)$. Using the commutativity of diagram (3), we find that

\[
i_*(f^* f_* (d) \cdot c_1(G)) = (\phi_1)^*(\phi_1)_*(c) = 0 \quad \text{in } A^j(Y).
\]

On the other hand, we have

\[
f^* f_* (d) \cdot c_1(G) = f^* f_* (d_2) \cdot c_1(G) = f^*(b_2) \cdot c_1(G) = d_2 \quad \text{in } A^{j-1}(E)
\]

(here, we have used the splitting property $f_* (f^*(b_2) \cdot c_1(G)) = b_2$ in $A^{j-2}(Z)$ mentioned above), and so

\[
i_*(d_2) = 0 \quad \text{in } A^j(Y).
\]

Thus, we have equality $c = i_*(d_1)$ in $A^j(Y)$ and $d_1 \in \ker \psi$, proving the arrow $g$ is surjective.

**Lemma 10** The arrow labelled $h_1$ in (3) is 0.
Proof The map
\[ A^{j-2}(Z) \to A^{j-1}(E) \]
in diagram (4) is defined as \( f^* \cdot c_2(G) \). By definition of \( \psi := f^*f_* : A^{j-2}(Z) \to A^{j-1}(E) \), the image of \( A^{j-2}(Z) \) in \( A^{j-1}(E) \) is in the image of \( \psi \), and so the arrow \( h_1 \) is 0.

**Lemma 11** The arrow labelled \( k \) in (5) is 0 when restricted to \( \text{Coker}(\phi^*) \).

Proof The map
\[ A^j(V) \to A^j(Y) \]
in diagram (4) is defined as \((\phi_1)^*\). But \((\phi_1)^* = (\phi_1)^*(\phi_1) : A^j(V) \to A^j(Y)\), and so
\[ \text{Im}(A^j(V) \xrightarrow{(\phi_1)^*} A^j(Y)) \subseteq \text{Im}(A^j(Y) \xrightarrow{(\phi_1)^*(\phi_1)} A^j(Y)) \]
which shows the arrow \( k \) is 0.

Applying lemmas 9, 10, 11 to the exact sequence (5), we find that the sequence (5) contains an isomorphism
\[ \text{Coker}(A^{j-3}(Z) \xrightarrow{\gamma} A^{j-2}(Z)) \cong \text{Coker}(A^j(V') \xrightarrow{\phi^*} A^j(V)) \, (6) \]

Let us now determine the cokernel of the map \( \gamma \):

**Lemma 12** There exist isomorphisms
\[ A^{j-2}(Z) \cong A^{j-2}(F) \oplus A^{j-3}(F) \oplus A^{j-4}(F) \]
\[ A^{j-3}(Z') \cong A^{j-3}(F) \oplus A^{j-4}(F) \]
such that the map
\[ \gamma : A^{j-3}(Z') \to A^{j-2}(Z) \]
(defined by diagram (4)) sends \( A^{j-3}(F) \) isomorphically to \( A^{j-3}(F) \), and \( A^{j-4}(F) \) isomorphically to \( A^{j-4}(F) \).

Proof Since \( p' : Z' \to F \) is a \( \mathbb{P}^1 \)-bundle, we can write any \( a' \in A^{j-3}(Z') \) uniquely as
\[ a' = (f')^*(f_3) + (f')^*(f_4) \cdot h' \, \text{in} \, A^{j-3}(Z') \]
where \( f_k \in A^k(F) \) and \( h' \in A^1(Z') \) denotes the tautological class. This furnishes the second isomorphism required in lemma 12.

We now consider the image of \( a' \) under the induced map
\[ \gamma : A^{j-3}(Z') \to A^{j-2}(Z) \, . \]
By the above description of the maps in the diagram (4) defining \( \gamma \), we have that
\[ \gamma(a') = f_* \left( f_* f_* (c_2(G') \cdot (f')^*(a')) \cdot c_1(G) \right) \]
\[ = f_* (c_2(G') \cdot (f')^*(a')) \, \text{in} \, A^{j-2}(Z) \, . \]
(Here we have used the splitting property \( f_* (f^*(b) \cdot c_1(G)) = b \) mentioned above.)

In particular, a cycle of the form \( (f')^*(f_3) \) in \( A^{j-3}(Z') \) is mapped to
\[ \gamma((f')^*(f_3)) = f_* \left( c_2(G') \cdot (f')^*(f_3) \right) \]
\[ = f_* c_2(G') \, \text{in} \, A^{j-2}(Z) \, . \]
(Here, we have used the “Moreover” part of proposition 6 plus the projection formula.)
Likewise, a cycle of the form \((p')^*(f_{j-4}) \cdot h'\) in \(A^{j-3}(Z')\) is mapped to
\[
\gamma((p')^*(f_{j-4}) \cdot h') = f_*\left(\left(c_2(G') \cdot (f')^*(h') \cdot (f')^*(p')^*(f_{j-4})\right)\right)
\]
\[
= f_*\left(\left(c_2(G') \cdot (f')^*(h') \cdot f^*p^*(f_{j-4})\right)\right)
\]
\[
= f_*\left(c_2(G') \cdot (f')^*(h')\right) \cdot p^*(f_{j-4}) \text{ in } A^{j-2}(Z).
\]

Let us now define
\[
h_1 := f_*c_2(G') \in A^1(Z),
\]
\[
h_2 := f_*\left(c_2(G') \cdot (f')^*(h')\right) \in A^2(Z).
\]

By what we have just seen, the map \(\gamma: A^{j-3}(Z') \to A^{j-2}(Z)\) verifies
\[
\gamma((p')^*(f_{j-3})) = h_1 \cdot p^*(f_{j-3}),
\]
\[
\gamma((p')^*(f_{j-4}) \cdot h') = h_2 \cdot p^*(f_{j-4}),
\]
and this completely determines the map \(\gamma\). The isomorphism
\[
A^{j-2}(Z) \cong A^{j-2}(F) \oplus A^{j-3}(F) \oplus A^{j-4}(F)
\]
required in lemma 12 is now furnished by the following sublemma:

**Sublemma 13** Any \(a \in A^{j-2}(Z)\) can be written uniquely as
\[
a = p^*(f_{j-2}) + h_1 \cdot p^*(f_{j-3}) + h_2 \cdot p^*(f_{j-4}) \text{ in } A^{j-2}(Z),
\]
where \(f_k \in A^k(F)\).

**Proof** First, we claim that \(h_1 \in A^1(Z), h_2 \in A^2(Z)\) have the following property:
\[
p_*h_2 = [F] \text{ in } A^0(F),
\]
\[
p_*h_1 = [F] \text{ in } A^0(F),
\]
for some \(g \in A^1(Z)\).

To see this, note that
\[
p_*h_2 = p_*f_*\left(c_2(G') \cdot (f')^*(h')\right)
\]
\[
= (p')_*\left(c_2(G') \cdot (f')^*(h')\right)
\]
\[
= (p')_*\left((f')^*c_2(G') \cdot h'\right)
\]
\[
= (p')_*\left([Z] \cdot h'\right) = (p')_*h' = [F] \text{ in } A^0(F).
\]

(Here, the first equality is just the definition of \(h_2\); the second equality is the “Moreover” part of proposition 5; the third equality is the projection formula; the fourth equality is the fact that \(-\) as noted above- \((p')_*\) is a left-inverse to the arrow \(\alpha'\).) This proves the first part of the claim.

For the second part of the claim, let \(h\) be the tautological class of the \(\mathbb{P}^2\)-bundle \(p: Z \to F\). We can write
\[
h_1 = c_1h + p^*(d) \text{ in } A^1(Z),
\]
where \(c_1 \in \mathbb{Q}\) and \(d \in A^3(F)\). Let us suppose for a moment that \(c_1 = 0\), so \(h_1 = p^*(d)\). Then we would have for any \(f_{j-3} \in A^{j-3}(F)\) that
\[
\gamma((p')^*(f_{j-3})) = h_1 \cdot p^*(f_{j-3}) = p^*(f_{j-3} \cdot d) \text{ in } A^{j-2}(Z).
\]
In particular, taking \(j = m - 1\) we have \(f_{m-4} \cdot d = 0\) (since \(\dim F = m - 4\)), and so this would imply that
\[
\gamma((p')^*A^{m-4}(F)) = 0.
\]
Since we know that
\[ A^{m-4}(F) \oplus A^{m-5}(F) \rightarrow A^{m-4}(Z'), \]
\[(f_{m-4}, f_{m-5}) \mapsto (p')^*(f_{m-4}) + (p')^*(f_{m-5}) \cdot h' \]
is an isomorphism, this would imply that
\[ \text{Im}(A^{m-4}(Z') \rightarrow A^{m-3}(Z)) = \text{Im}(A^{m-5}(F) \rightarrow A^{m-4}(Z') \rightarrow A^{m-3}(Z)). \]

In view of the description of $\gamma$ given in (9), this would imply that
\[ \text{Im}(A^{m-4}(Z') \rightarrow A^{m-3}(Z)) \subset \text{Im}(A^{m-5}(F) \rightarrow A^{m-3}(Z)). \]

But then we would have
\[ \text{Coker}(A^{m-4}(Z') \rightarrow A^{m-3}(Z)) \neq 0 \]

indeed, the map
\[ A^{m-4}(F) \oplus A^{m-5}(F) \rightarrow A^{m-3}(Z), 
(f_{m-4}, f_{m-5}) \mapsto p^*(f_{m-4}) \cdot h + p^*(f_{m-5}) \cdot h_2 \]
is an isomorphism, and so any cycle of the form $p^*(f_{m-4}) \cdot h$ in $A^{m-3}(Z)$ will be in the cokernel of $\gamma$. In view of the isomorphism (6), this would mean that also
\[ \text{Coker}(A^{m-1}(V') \rightarrow A^{m-1}(V)) \neq 0. \]

But this is a contradiction: any curve class on $V$ is represented by a cycle supported on the open $V \setminus Z$ (and likewise on $V'$), and so there is an isomorphism $\phi^*: A^{m-1}(V') \cong A^{m-1}(V)$. It follows that $c_1 \neq 0$ and so
\[ p_*(h_1 \cdot h) = p_*(c_1 h^2) = c_1[F] \text{ in } A^0(F). \]

Setting $g := \frac{1}{c_1} h$, this proves the second part of the claim.

Sublemma[13] is now readily proven: it follows from the equalities (8) there are relations
\[ h = \frac{1}{c_1} h_1 + p^*(d) \text{ in } A^1(Z), \]
\[ h^2 = h_2 + p^*(d_{21}) \cdot h + p^*(d_{22}) \text{ in } A^2(Z), \]
for some $d, d_{21} \in A^1(F), d_{22} \in A^2(F)$.

The projective bundle formula implies that any $a \in A^{j-2}(Z)$ can be written as
\[ a = p^*(f_{j-2}) + h \cdot p^*(f_{j-3}) + h^2 \cdot p^*(f_{j-4}) \text{ in } A^{j-2}(Z), \]
where $f_k \in A^k(F)$. Plugging in the relations (9), we find
\[ a = p^*(f_{j-2}) + h \cdot p^*(f_{j-3}) + h^2 \cdot p^*(f_{j-4}) \]
\[ = p^*(f_{j-2}) + \left( \frac{1}{c_1} h_1 + p^*(d) \right) \cdot p^*(f_{j-3}) + \left( h_2 + p^*(d_{21}) \cdot \left( \frac{1}{c_1} h_1 + p^*(d) \right) + p^*(d_{22}) \right) \cdot p^*(f_{j-4}) \]
\[ = p^*(f_{j-2} + d \cdot f_{j-3} + d \cdot d_{21} \cdot f_{j-4} + d_{22} \cdot f_{j-4} + h_1 \cdot p^*(\frac{1}{c_1} (f_{j-3} + d_{21} \cdot f_{j-4})) + h_2 \cdot p^*(f_{j-4}) \]
\[ = p^*(f'_{j-2}) + h_1 \cdot p^*(f'_{j-3}) + h_2 \cdot p^*(f'_{j-4}) \text{ in } A^{j-2}(Z), \]
for some $f'_{j-2} \in A^k(F)$.

It remains to prove unicity in sublemma[13] suppose $f_k \in A^k(F)$ is such that
\[ p^*(f_{j-2}) + h_1 \cdot p^*(f_{j-3}) + h_2 \cdot p^*(f_{j-4}) = 0 \text{ in } A^{j-2}(Z). \]
Then in particular
\[ p_*(p^*(f_{j-2}) + h_1 \cdot p^*(f_{j-3}) + h_2 \cdot p^*(f_{j-4})) = 0 \quad \text{in} \quad A^{j-4}(F). \]
But the left-hand side equals \( p_*(h_2 \cdot p^*(f_{j-4})) = f_{j-4} \) and so \( f_{j-4} = 0 \). Similarly, the assumption implies
\[ p_*(g \cdot (p^*(f_{j-2}) + h_1 \cdot p^*(f_{j-3}))) = p_*(g \cdot h_1 \cdot p^*(f_{j-3})) = f_{j-3} = 0 \quad \text{in} \quad A^{j-3}(F) \]
(where we have used the equality \( (8) \)). Finally, the assumption implies that
\[ p_*(h_2 \cdot p^*(f_{j-2})) = p_*(h_2) \cdot f_{j-2} = f_{j-2} = 0 \quad \text{in} \quad A^{j-2}(F) \]
(where we have used again the equality \( (8) \)), and so we are done. This proves sublemma \( [13] \) and hence lemma \( [12] \).

We are now in position to wrap up the proof of proposition \( [8] \). Combining the isomorphism \( (6) \) and lemma \( [12] \) we obtain an isomorphism
\[ \text{Coker} \left( A^j(V') \xrightarrow{\phi^*} A^j(V) \right) \cong \text{Coker} \gamma \cong A^{j-2}(F). \]
This proves proposition \( [8] \). Indeed, it follows from this isomorphism of cokernels there is a commutative diagram with exact rows
\[
\begin{array}{cccccc}
0 & \rightarrow & A^{j-3}(Z') & \xrightarrow{\gamma} & A^{j-2}(Z) & \xrightarrow{\delta} & A^{j-2}(F) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cong & & \\
0 & \rightarrow & A^j(V') & \xrightarrow{\phi^*} & A^j(V) & \rightarrow & \text{Coker} (\phi^*) & \rightarrow & 0 \\
\end{array}
\]
As we have seen, the upper row is split exact (lemma \( [12] \)), and a right-inverse to \( \delta \) is given by the pull-back \( p^* \) (sublemma \( [13] \)). It follows the lower row is also split and proposition \( [8] \) is proven.

The second step of the proof of theorem \( [5] \) consists in extending proposition \( [7] \) to a “universal isomorphism” of Chow groups:

**Proposition 14** Let \( X \subset \mathbb{P}^{n+1}(\mathbb{C}) \) be a smooth cubic hypersurface, and let \( F = F(X) \) be its Fano variety of lines. Let \( M \) be any smooth projective variety. The natural map
\[
A^{j-2}(F \times M) \oplus A^j(W \times M) \rightarrow A^j(X_{[2]} \times M)
\]
\[ (a, b) \mapsto (\tau \times id_M)_*(p \times id_M)^*(a) + (\phi \times id_M)^*(b) \]
is an isomorphism for all \( j \).

**Proof** For any variety \( V \), let \( V_M \) denote the product \( V \times M \). For a morphism \( f : X \rightarrow Y \), let \( f_M : X_M \rightarrow Y_M \) denote the morphism \( f \times id_M \). Proposition \( [6] \) induces a birational map
\[
\phi_M := \phi \times id_M : \quad (X_{[2]})_M \dashrightarrow W_M.
\]
Again using proposition \( [6] \) we find that the map \( \phi_M \) admits a resolution of indeterminacy
\[
(\phi_1)_M \leftarrow Y_M \xrightarrow{\phi_M} (\phi_2)_M \rightarrow W_M.
\]
Here the morphism \( (\phi_1)_M \) is the blow-up with codimension 2 center \( Z_M \subset (X_{[2]})_M \), and the morphism \( (\phi_2)_M \) is the blow-up with codimension 3 center \((Z')_M \subset W_M \). Clearly, the exceptional divisor \( E_M \subset Y_M \) fits in a commutative diagram
\[
\begin{array}{cccccc}
E_M & \xrightarrow{f_M} & F_M \\
Z_M & \xrightarrow{p_M} & (Z')_M \\
\end{array}
\]
That is, we are in a set-up where we may apply proposition \( [8] \) (with \( V = (X_{[2]})_M \) and \( V' = W_M \)), and so proposition \( [14] \) is proven.
In the third and final step of the proof of theorem 5, we relate $F$ and $X^{[2]}$ on the level of Chow motives.

**Proposition 15** Let $X \subset \mathbb{P}^{n+1}(C)$ be a smooth cubic hypersurface, and let $F = F(X)$ be its Fano variety of lines. The map

$$\Gamma_\tau \circ \Gamma_p \oplus \Gamma_\phi : h(F)(2) \oplus h(W) \to h(X^{[2]}) \quad \text{in} \quad M_{\text{rat}}$$

is an isomorphism.

**Proof** This follows from proposition 14 by virtue of Manin’s identity principle [23, 2.3].

Proposition 15 proves theorem 5, since $h(W) \cong \bigoplus_{i=0}^{n} h(X)(i)$ in $M_{\text{rat}}$ (this is the projective bundle formula for the $\mathbb{P}^n$–bundle $W \to X$). Theorem 5 immediately implies theorem 4 if $X$ has finite–dimensional motive (resp. motive of abelian type), then also $X^{[2]}$ has finite–dimensional motive (resp. motive of abelian type); moreover, the property of having finite–dimensional motive (resp. motive of abelian type) is preserved under taking direct summands.

**Remark 16** In [7, Theorem 5.1], proposition 6 is used to establish a relation between a (not necessarily smooth) cubic $X \subset \mathbb{P}^{n+1}(k)$ and its Fano variety $F := F(X)$ in the Grothendieck ring of varieties:

$$[X^{[2]}] = [\mathbb{P}^n][X] + L^2[F] \quad \text{in} \quad K_0(\text{Var/k}).$$

Theorem 5 shows that for smooth cubics over $C$, a similar relation holds on the level of Chow motives.

### 4 Examples

**Corollary 17** Let $F(X)$ be the Fano variety of lines of a smooth cubic $X \subset \mathbb{P}^{n+1}(C)$. In the following cases, $F(X)$ has finite–dimensional motive (of abelian type):

(i) $n = 3$ or $n = 5$;
(ii) $X$ is a Fermat cubic

$$x_0^3 + x_1^3 + \cdots + x_{n+1}^3 = 0;$$

(iii) $n = 4$ and $X$ is defined by an equation

$$f(x_0, \ldots, x_3) + x_4^3 + x_5^3 = 0,$$

where $f(x_0, \ldots, x_3)$ defines a smooth cubic surface;

(iv) $n = 6$ and $X$ is defined by an equation

$$f_1(x_0, \ldots, x_3) + f_2(x_4, \ldots, x_7) = 0,$$

where $f_1, f_2$ define smooth cubic surfaces.

**Proof** Appealing to theorem 4 it suffices to check $X$ has motive of abelian type. In case (ii), this is well–known (it follows from the inductive structure of Fermat varieties [24]). In case (i), we have

$$A_{A,j}(X) = 0 \quad \text{for all} \quad j$$

(this is proven in [17], and alternatively in [20] and [9]). This implies the motive of $X$ is generated by curves [25, Theorem 4].

In case (iii), the argument is a combination of (i) and (ii): Let $X$ be a cubic fourfold as in (iii). There is a (Shioda–style) rational map

$$\phi : Y \times C \to X;$$
A remark on the motive of the Fano variety of lines of a cubic

where \( C \) is a cubic Fermat curve and \( Y \) the cubic threefold defined by an equation

\[
f(x_0, \ldots, x_3) + x_4^3 = 0.
\]

The indeterminacy locus \( S \) of \( \phi \) is a union of smooth cubic surfaces, and \( X \) is dominated by the blow–up of \( Y \times C \) with center \( S \) (these assertions are proven just as [24, Theorem 2]). This blow–up has motive of abelian type.

The argument for case (iv) is similar: there is a (Shioda–style) rational map

\[
\phi : X_1 \times X_2 \to X,
\]

where \( X_1, X_2 \) are the cubic threefolds defined by the equation

\[
f_1(x_0, \ldots, x_3) + x_4^3 = 0,
\]

resp.

\[
f_2(x_4, \ldots, x_7) + x_8^2 = 0.
\]

The indeterminacy locus \( S \) of \( \phi \) is a product of two smooth cubic surfaces, and \( X \) is dominated by the blow–up of \( X_1 \times X_2 \) with center \( S \). Since smooth cubic surfaces and threefolds have motive of abelian type, \( X \) has motive of abelian type.

Remark 18 Cubic fourfolds as in corollary [17](iii) appear in [29] Example 4.2. As shown in loc. cit., to such a fourfold \( X \) one can associate a \( K3 \) surface \( S_X \) with the property that there is a correspondence inducing an isomorphism

\[
A_{homo}^0(S_X) \cong A_{alg}^1(X).
\]

These \( K3 \) surfaces \( S_X \) form a 4–dimensional family of double covers of \( \mathbb{P}^2 \) ramified along a sextic.

The example of corollary [17](iii) is generalized in [16], where it is shown that smooth cubic fourfolds of type

\[
f(x_0, \ldots, x_4) + x_5^3 = 0
\]

have finite–dimensional motive. The argument is more involved.

Acknowledgements This note is a protracted after–effect of the Strasbourg 2014—2015 groupe de travail based on the monograph [32]; thanks to all the participants for the pleasant and stimulating atmosphere. Many thanks to Yasuyo, Kai and Len for not being there when I work, and for being there when I don’t.

References

1. Y. André, Motifs de dimension finie (d’après S.-I. Kimura, P. O’Sullivan,...), Séminaire Bourbaki 2003/2004, Astérisque 299 Exp. No. 929, viii, 115—145,
2. M. Brion, Log homogeneous varieties, in: Actas del XVI Coloquio Latinoamericano de Algebra, Revista Matemática Iberoamericana, Madrid 2007, arXiv: math/0609669,
3. M. de Cataldo and L. Migliorini, The Chow groups and the motive of the Hilbert scheme of points on a surface, Journal of Algebra 251 no. 2 (2002), 824—848,
4. P. Deligne, La conjecture de Weil pour les surfaces K3, Invent. Math. 15 (1972), 206—226,
5. H. Diaz, The motive of the Fano surface of lines, arXiv:1602.06403v1,
6. W. Fulton, Intersection theory, Springer–Verlag Ergebnisse der Mathematik, Berlin Heidelberg New York, Tokyo, 1984,
7. S. Galkin and E. Shinder, The Fano variety of lines and rationality problem for a cubic hypersurface, arXiv:1408.5754,
8. V. Guletskiǐ and C. Pedrini, The Chow motive of the Godeaux surface, in: Algebraic Geometry, a volume in memory of Paolo Francia (M.C. Beltrametti et alii, editors), Walter de Gruyter, Berlin New York, 2002,
9. A. Hirschowitz and J. Iyer, Hilbert schemes of fat r–planes and the triviality of Chow groups of complete intersections. In: Vector bundles and complex geometry, Contemp. Math. 522, Amer. Math. Soc., Providence 2010,
10. F. Ivorra, Finite dimensional motives and applications (following S.-I. Kimura, P. O’Sullivan and others), in: Autour des motifs, Asian-French summer school on algebraic geometry and number theory, Volume III, Panoramas et synthèses, Société mathématique de France 2011,
11. J. Iyer, Murre’s conjectures and explicit Chow–Künneth projectors for varieties with a nef tangent bundle, Transactions of the Amer. Math. Soc. 361 (2008), 1667—1681,
12. J. Iyer, Absolute Chow–Künneth decomposition for rational homogeneous bundles and for log homogeneous varieties, Michigan Math. Journal Vol. 60, 1 (2011), 79—91,
13. U. Jannsen, Motivic sheaves and filtrations on Chow groups, in: Motives (U. Jannsen et alii, editors), Proceedings of Symposia in Pure Mathematics Vol. 55 (1994), Part 1.
14. U. Jannsen, On finite–dimensional motives and Murre’s conjecture, in: Algebraic cycles and motives (J. Nagel and C. Peters, editors), Cambridge University Press, Cambridge 2007.
15. S. Kimura, Chow groups are finite dimensional, in some sense, Math. Ann. 331 (2005), 173—201.
16. R. Laterveer, A family of cubic fourfolds with finite–dimensional motive, preprint.
17. J. Lewis, Cylinder homomorphisms and Chow groups, Math. Nachr. 160 (1993), 205—221.
18. J. Murre, On a conjectural filtration on the Chow groups of an algebraic variety, parts I and II, Indag. Math. 4 (1993), 177—201.
19. J. Murre, J. Nagel and C. Peters, Lectures on the theory of pure motives, Amer. Math. Soc. University Lecture Series 61, Providence 2013.
20. A. Otwinowska, Remarques sur les groupes de Chow des hypersurfaces de petit degré, C. R. Acad. Sci. Paris Série I Math. 329 (1999), no. 1, 51—56.
21. C. Pedrini, On the finite dimensionality of a $K3$ surface, Manuscripta Mathematica 138 (2012), 59—72.
22. C. Pedrini and C. Weibel, Some surfaces of general type for which Bloch’s conjecture holds, to appear in: Period Domains, Algebraic Cycles, and Arithmetic, Cambridge Univ. Press, 2015.
23. T. Scholl, Classical motives, in: Motives (U. Jannsen et alii, editors), Proceedings of Symposia in Pure Mathematics Vol. 55 (1994), Part 1.
24. T. Shioda, The Hodge conjecture for Fermat varieties, Math. Ann. 245 (1979), 175—184.
25. C. Vial, Projectors on the intermediate algebraic Jacobians, New York J. Math. 19 (2013), 793—822.
26. C. Vial, Remarks on motives of abelian type, to appear in Tohoku Math. J.
27. C. Vial, Niveau and coniveau filtrations on cohomology groups and Chow groups, Proceedings of the LMS 106(2) (2013), 410—444.
28. C. Vial, Chow–Künneth decomposition for 3– and 4–folds fibred by varieties with trivial Chow group of zero–cycles, J. Alg. Geom. 24 (2015), 51—80.
29. C. Voisin, Remarks on zero–cycles of self–products of varieties, in: Moduli of vector bundles, Proceedings of the Taniguchi Congress (M. Maruyama, editor), Marcel Dekker New York Basel Hong Kong 1994.
30. C. Voisin, On the universal $CH_0$ group of cubic hypersurfaces, to appear in Journal Eur. Math. Soc.
31. C. Voisin, Bloch’s conjecture for Catanese and Barlow surfaces, J. Differential Geometry 97 (2014), 149—175.
32. C. Voisin, Chow Rings, Decomposition of the Diagonal, and the Topology of Families, Princeton University Press, Princeton and Oxford, 2014.
33. Z. Xu, Algebraic cycles on a generalized Kummer variety, arXiv:1506.04297v1.