A Measure of Non-Markovianity for Unital Quantum Dynamical Maps

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In this paper based on the features of a back-flow information from an environment to an open quantum system in a non-Markovian process, we introduce a new witness and measure for non-Markovianity which are used from properties completely positive trace preserving unital dynamical maps. These witness and measure is constructed based on eigenvalues rate of qubit dynamics in open quantum system. The merit of this measure for these kinds of maps in compared with other measures is simplicity in calculations and the optimization procedure just is taken over the initial state of the single qubit system which it is not required complicated statistical method for optimization.

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I. INTRODUCTION

A real quantum system is constantly interacting with its surrounding environment, therefore decoherence is an inevitable phenomena. These type of quantum systems well known as open quantum systems. The study of the open quantum systems dynamics is one of the most interesting issues in the field of quantum information and computation. In term of the memory effects of the environment, and direction of information flow, dynamics of open quantum systems can be classified into two categories: (i) Markovian(memoryless): Where, the state of the system in the future just depends on the system state at the present time. In this case, we have only the flow of information from system to environment i.e. system loss its information (ii) Non-Markovian(with memory): Here, the evolution of the state of system take effect from the history of the system. This type of process compete with the back flow of information from environment to system [1]. In the theoretical study often we use Markovian approximation to simplify calculations and it does not exist in the real world. Markovian dynamics of open quantum systems can be described by a quantum dynamical semigroup with generator in Lindblad form [1], which leads us to local in time master equation. When the dynamical map has semigroup property [2], dynamics is Markovian. In fact, in most cases the dynamics of open quantum systems is non-Markovian, because of this, the studies of non-Markovian dynamics and its effects on correlations have attracted much attention in recent years [3][4]. Detecting non-Markovianity has been one of the interesting and challenging subject of studies on dynamics of open quantum systems in the last decade. There have been many attempts to introduce a general criterion for detecting non-Markovian feature of quantum evolution. For instance, Breuer et al. introduced one measure based on contractivity property of trace distance under completely positive trace preserving maps, which is well known as BLP measure [7,8], this measure requires optimization procedure over all pairs of initial state, which requires complicated numerical calculations. In Ref.[9] Rivas et al. proposed a measure based on Choi-Jamiolkowski isomorphism and divisibility of quantum dynamical (CP) maps, which is call RHP measure. Wolf et al. use semigroup property [10], Vasile et al. use fidelity [11], Lu et al. use quantum fisher information [12], and Luo et al. use quantum mutual information [13], to detect and measure the degree of non-Markovianity. In order to introduce one measure for detecting non-Markovianity, firstly we should answer to this question, what made a process to be non-Markovian? In BLP measure violation of the contractivity property of trace distance under completely positive maps use as witness for non-Markovianity, and in RHP measure violation of divisibility in quantum dynamical maps help us to detect non-Markovianity of quantum process. In this work we focus on back flow of information concept as a symptom of non-Markovian process to detect non-Markovianity. For this purpose, we use non-decreasing property of von Neumann entropy under completely positive unital maps and connect the degree of non-Markovianity to dynamics of the eigenvalues of quantum states. In other words, we concentrate on the eigenvalues dynamics of the open quantum system. Then, we link non-Markovianity measure to non-monotonicity behavior of the dynamics of the density matrix eigenvalues. If we consider the state of the system in the spectral decomposition representation, its eigenvalues of density matrix can be interpreted as probability of being in an eigenstates correspond to those distinct eigenvalues. We show that the dynamics of eigenvalues rate are monotonically increasing or decreasing, the time evolution of open quantum system is Markovian and otherwise is non-Markovian process. Remarkably, the advantage of our measure is which the calculations are simple and optimization procedure will be taken only over the initial state of the system. It is greatly increases the practical relevance of the proposed measure.

The work is organized as follows. In Sec.II the notion of unital quantum maps will be reviewed and we describe the conditions that must be hold for being a map unital.
In Sec. III we introduce our measure based on eigenvalues dynamics by starting from the non-decreasing property of von Neumann entropy. To apply our measure we investigate two example into dynamics of single qubit, in one of them single qubit interact with bosonic reservoir through certain hamiltonian, in the other example the single qubit subject to external random field. Finally in Sec.IV we present our conclusions.

II. UNITAL QUANTUM MAPS

We consider an open quantum system \( S \) with Hilbert space \( \mathcal{H}_s \), and arbitrary density matrix \( \rho^S \) belong to, all bounded linear operators acting on Hilbert space, \( B(\mathcal{H}_s) \). As we know, every completely positive map can be represented in Kraus form \([1, 2, 13]\)

\[
\Phi(\rho^S) = \sum_k E_k^\dagger \rho^S E_k, \tag{1}
\]

where \( E_k \in B(\mathcal{H}_s) \), and from trace-preserving property of quantum map we have \( \sum_k E_k E_k^\dagger = \mathbb{I} \).

**Definition:** The completely positive map \( \Phi \) is unital iff \( \sum_k E_k^\dagger E_k = \mathbb{I} \), i.e. \( \Phi \) maps identity operator to identity operator in same space, \( \Phi(\mathbb{I}) = \mathbb{I} \). For single qubit systems, the unital maps can be expressed in terms of convex combination of Pauli operators which they are also well known as Pauli maps \([13]\). From the geometrical point of view, the unital maps take the center of Bloch sphere to itself. In other word, these maps take maximally mixed state to itself. In the following, we parameterize the unital maps for the simple one-qubit case.

Let \( \rho^S \) be any state on the two-dimensional Hilbert space \( \mathcal{H}_s = \mathbb{C}^2 \) representing in term of the identity operator \( \mathbb{I} \) and Pauli matrices \( \{\sigma_x, \sigma_y, \sigma_z\} \) in the \( B(\mathcal{H}_s) \) space as

\[
\hat{\rho}^S = \frac{1}{2}(\mathbb{I} + r \cdot \vec{\sigma}), \tag{2}
\]

where \( r \in \mathbb{R}^3 \) is Bloch vector. Every quantum dynamical map \( \Phi : B(\mathcal{H}_s) \to B(\mathcal{H}_s) \) in this basis may be written as 4 \( \times \) 4 matrix \( L_\Phi \)

\[
L_\Phi = \begin{pmatrix} 1 & 0 \\ \vec{t} & M \end{pmatrix}, \tag{3}
\]

where \( \vec{t} \) is a vector in \( \mathbb{R}^3 \) and \( M \) is a \( 3 \times 3 \) matrix. Therefore, we have \( \Phi(\rho) = L_\Phi \rho L_\Phi^\dagger \) representing affine transformation on Bloch vector \( r \) as

\[
\vec{r}' = \vec{t} + \vec{M} \vec{r}. \tag{4}
\]

In this parameterization, the unital dynamical maps are characterized in a simple way: \( \Phi \) is unital iff \( t = 0 \) \([16]\).

III. MEASURE FOR NON-MARKOVIANITY

Quantum dynamical semigroup are a particular case of Markovian dynamics. Quantum dynamical semigroup is a set of quantum linear map with following properties \(2\)

1. \( \Phi_t \) is a dynamical map,
2. \( \Phi_t \Phi_s = \Phi_{t+s} \),
3. \( Tr(\Phi_t(\rho) A) \) is a continuous function of \( t, \forall A \in B(\mathcal{H}_s) \).

If \( \mathcal{L} \) be the generator of the semigroup, then the semigroup has exponential form \( \Phi_t = e^{(t \mathcal{L})} \), where the dynamics of system is described by the following master equation

\[
\frac{d}{dt} \rho^S(t) = \mathcal{L} \rho^S(t), \tag{5}
\]

where \( \mathcal{L} \) is in Lindblad form \([1]\) as

\[
\mathcal{L} \rho^S(t) = -i[H, \rho^S(t)] + 
+ \sum_k \gamma_k \{F_k \rho^S(t) F_k^\dagger - \frac{1}{2} \{F_k^\dagger F_k, \rho^S(t)\}\}, \tag{6}
\]

and satisfies this master equation \( \Phi_t = \mathcal{L} \Phi_t \). \( H \) is effective Hamiltonian, \{\( F_k \)\} is a set of time-independent operator, which is well known as Lindblad operator and \( \gamma_k \) is relaxation rate that must be positive. In the case of time-dependent effective Hamiltonian the generator of the quantum dynamical map satisfies \( \frac{d}{dt} \rho^S(t) = \mathcal{L}_t \rho^S(t) \), and has the following form

\[
\mathcal{L}_t \rho^S(t) = -i[H, \rho^S(t)] + 
+ \sum_k \gamma_k(t) \{F_k(t) \rho^S(t) F_k^\dagger(t) - \frac{1}{2} \{F_k^\dagger(t) F_k(t), \rho^S(t)\}\}. \tag{7}
\]

The dynamical maps correspond to this generator is a Markovian(CP and divisible) if and only if \( \gamma_k(t) \geq 0 \). If \( \gamma_k(t) < 0 \) the maps are not completely positive and so are non-Markovian. Now we are in a position to construct our measure for non-Markovianity which is based on monotonically increasing property of von Neumann entropy under CPTPPU maps. If we consider the state of the system \( \rho \) as spectral decomposition

\[
\rho^S = \sum_i \lambda_i |\phi_i\rangle \langle \phi_i|, \tag{8}
\]

where \( \lambda_i \) and \( |\phi_i\rangle \) are eigenvalues and eigenvectors of state system, respectively. Von Neumann entropy can be interpreted as lack of our knowledge about the state which is define by

\[
S(\rho^S) = -Tr(\rho^S \log \rho^S) = -\sum_i \lambda_i \log_2 \lambda_i, \tag{9}
\]

where eigenvalues of \( \lambda_i \) have probabilistic interpretation. Here, we will mention some of important properties of von Neumann entropy:
i) Pure state has minimum von Neumann entropy i.e. \( S(\rho_{\text{pure}}) = 0 \).

ii) If density operator has rank \( d \) then, \( 0 \leq S(\rho) \leq \log_2 d \).

iii) For isolated quantum system, i.e. when the evolution of quantum system is unitary, von Neumann entropy does not change during the evolution, \( S(\rho) = S(U\rho U^\dagger) \), and the entropy is always independent of time \( dS(\rho)/dt = 0 \) \[14\] \[15\].

Now we want to show the property of von Neumann entropy which is the monotonically increasing under completely positive unital maps. To achieve this property, we start from contractivity behavior of relative entropy under CP maps

\[
S(\Phi(\rho^S) \parallel \Phi(\sigma^S)) \leq S(\rho^S \parallel \sigma^S).
\]

Let us to consider \( \sigma^S = \frac{1}{d} I \), then

\[
S(\rho^S \parallel \frac{1}{d} I) = -S(\rho^S) + \log d.
\]

With the assumption that \( \Phi \) be a CPTPU map, and by substituting Eq.(11) into Eq.(10) we conclude that the von Neumann entropy is non-contracive under CPTPU \[16\] \[17\]

\[
S(\Phi(\rho^S)) \geq S(\rho^S),
\]

where indicate this fact the purity of system under above dynamical maps decreases. Regarding the above interpretation, now we are in a position to introduce witness for non-Markovianity: unital quantum dynamical map \( \Phi_t: \rho^S(0) \rightarrow \rho^S(t) = \Phi_t(\rho^S(0)) \) is non-Markovian if

\[
\frac{d}{dt} S(\rho^S(t)) < 0.
\]

According to this fact that the von Neumann entropy is invariant under unitary evolution, the non-monotonicity property of von Neumann entropy of an open quantum system can be interpreted as back flow of information from environment to the system, so dynamical maps which violate inequality Eq.(12) are not completely positive and consequently are non-Markovian. As we know for Markovian process the state of quantum system loses its purity due to the decoherence which give rise to increase the value of von Neumann entropy and can be interpreted as loss of information about the system. Similarly, increasing purity of open quantum state, or reduction of von Neumann entropy, can be considered as back flow of information from environment to system which is known as non-Markovianity process.

For single qubit dynamics of open system the density matrix at time \( t \)

\[
\rho^S(t) = \begin{pmatrix} \rho_{11}^S(t) & \rho_{12}^S(t) \\ \rho_{21}^S(t) & \rho_{22}^S(t) \end{pmatrix},
\]

where the von Neumann entropy is \( S(\rho^S(t)) = -\lambda_+^*(t) \log_2 \lambda_+(t) - \lambda_-^*(t) \log_2 \lambda_-(t) \), \( \lambda_\pm(t) \) in which the eigenvalues of density matrix are

\[
\lambda_\pm(t) = \frac{1 \pm \sqrt{1 - 4(\rho_{11}^S(t) - |\rho_{12}^S(t)|^2)}}{2}.
\]

By considering non-contarctivity of von Neumann entropy with respect to CPTPU map Eq.(12), we obtain the following relation

\[
\frac{d\lambda_+(t)}{dt} \log_2 \frac{\lambda_+(t)}{\lambda_-(t)} \geq 0.
\]

Now, we enable to introduce a non-Markovianity witness and measure for CPTPU maps in terms of eigenvalues of density matrix for the qubit dynamics open quantum systems:

Non-Markovianity Witness: The dynamical unital map \( \Phi(t,0) \) for a qubit system is non-Markovian if \( \eta_- = \frac{d\lambda_-(t)}{dt} < 0 \) or \( \eta_+ = \frac{d\lambda_+(t)}{dt} > 0 \). In other words violation of Eq.(16) is sufficient condition for non-Markovianity of unital dynamical maps.

The question is: how can we define the measure for non-Markovianity by using this criteria? For this purpose, we have to measure the total amount of information which is fluow back to the qubit system which it arise to increase the purity of the systems.

Measure of Non-Markovianity: According to the above notions, we can define the measure of non-Markovianity as

\[
N_\epsilon(\Phi) = \max_{\{\rho^S(0)\}} \left\{ \int_{\eta_+ > 0} \eta_+ dt + \int_{\eta_- < 0} \eta_- dt \right\}.
\]

Integral is over all time intervals \( t \in (a_i, b_i) \) in which for the first relation in Eq.(17), \( \eta_+ > 0 \) and for the latter \( \eta_- < 0 \), and the maximization is taken over all input state of the system i.e. \( \rho^S(0) \). Our non-Markovian measurement depends on choosing the initial state, this is the common feature in other measures which were proposed by others. Since non-Markovianity should be a property of dynamics, a logical method to decline this dependence is the optimization procedure. Fortunately, from Eq.(17) we observe that, the optimization procedure just is taken over the initial state of the single qubit system, which will simplify calculations. Also the measure can be rewritten as

\[
N_\epsilon(\Phi) = \max_{\{\rho^S(0)\}} \sum_i (\lambda_+(b_i) - \lambda_+(a_i)) = \max_{\{\rho^S(0)\}} \sum_i (\lambda_-(a_i) - \lambda_-(b_i)).
\]
**Example 1:** We consider the dynamics of a single qubit which is described by following Hamiltonian \[ H = \omega_0 \sigma_z + \sum_i \omega_i a_i^\dagger a_i + \sum_i \sigma_z (g_i a_i^\dagger + g_i^* a_i). \] \[ (19) \]

In order to study the dynamics of the system we work in interaction picture

\[ H_I(t) = \sum_i \sigma_z (g_i a_i^\dagger e^{i\omega_i t} + g_i^* a_i e^{-i\omega_i t}). \] \[ (20) \]

Using the second order time-convolutionless master equation and work in zero temperature \( T = 0 \), we derive the master equation in form

\[ \mathcal{L}_e(\rho^S(t)) = \frac{\gamma(t)}{2} (\sigma_z \rho^S(t) \sigma_z - \rho^S(t)), \] \[ (21) \]

where its solution of this master equation represents the pure dephasing map

\[ \Phi_t(\rho^S(0)) = \begin{pmatrix} \rho_{11}^S(0) & \rho_{12}^S(0)e^{-i\Gamma(t)} \\ \rho_{21}^S(0)e^{i\Gamma(t)} & \rho_{22}^S(0) \end{pmatrix}, \] \[ (22) \]

\[ \Gamma(t) = 4 \int d\omega J(\omega) \frac{1 - \cos \omega t}{\omega^2}, \quad \gamma(t) = \frac{d\Gamma(t)}{dt}. \] \[ (23) \]

The eigenvalues rate of the evolved density matrix is straightforwardly obtained as

\[ \frac{d\lambda_+}{dt} = \frac{-8\gamma(t) e^{-2\Gamma(t)} |\rho_{12}^S(0)|^2}{4\sqrt{1 - 4(\rho_{11}^S(0)\rho_{22}^S(0) - e^{-2\Gamma(t)}|\rho_{12}^S(0)|^2)}}, \] \[ (24) \]

where due to trace preserving of the dynamical maps we have \( \frac{d\lambda_+}{dt} = -\frac{d\lambda_-}{dt} \). By referring to our non-Markovianity witness mentioned one can obtain

\[ \frac{d\lambda_+}{dt} > 0 \implies \gamma(t) < 0, \] \[ (25) \]

where agrees with the result obtained by RHP measure in Ref.[13, 18]. For more details of behavior of the our non-Markovianity measure with respect to environment in this example, we consider the following spectral density for reservoir modes [13, 18]

\[ J(\omega) = \omega_c 1 - s \omega \exp(-\omega_c/\omega). \] \[ (26) \]

where \( \omega_c \) and \( s \) are the cutoff frequency and ohmicity, respectively. The \( s \) parameter can take the values, \( s < 1 \), \( s = 1 \), \( s > 1 \), correspond to sub-Ohmic, ohmic and super-Ohmic spectral densities, respectively. Here, the maximum in Eqs.(17) and (18) is achieved for the initial states \( \rho^S(0) = |\pm\rangle\langle\pm| \) where, \( |\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle) \). In Fig.(1) we plot the rate of the eigenvalue, \( \eta_+ \), for this state as a function of time \( t \) and parameter \( s \). As can be seen in the Fig.(1), for some intervals \( s \in [2.5, 5.5] \) the process is non-Markovian due to the positivity of \( \eta_+ \). In other words, the non-Markovian behavior appears when the qubit interact with super-Ohmic reservoirs. Positivity of \( \eta_+ \) or negativity of \( \eta_- \) means that the probability of finding the state of the system in the initial state increase in some time intervals during the process, which we interpret this process as the back-flow of information from environment to the system (purity increases). By straightforward calculation based on the introduced measure in Eq.(17), one can get the degree of non-Markovianity which the behavior of \( N_e(\Phi) \) has been plotted as a function of Ohmicity parameter \( s \) in Fig.(2). This results, which is obtained based on the introduced measure in Eq.(17), agree with the results obtained by BLP measure [20].

**Example 2:** Dynamics of qubit in the presence of random telegraph field [20], which is described by the time dependent Hamiltonian \[ H(t) = \Gamma(t) \sigma_z, \] \[ (27) \]

where \( \Gamma(t) \) is a random variable obeying the statistics of a random telegraph signal. \( \Gamma(t) \) is related to variable \( n(t) \) by equation \( \Gamma(t) = an(t) \) which \( n(t) \) has a poisson distribution with a mean equal to \( t/2\tau \), and \( a \) is a coin-flip random variable possessing the values \( \pm a \). We start from von Neumann equation, given by

**FIG. 1:** The variations of \( \eta_+ \) as a function of time \( t \) and \( s \).

**FIG. 2:** The degree of non-Markovianity for different type of reservoir, \( s < 1 \) sub-Ohmic, \( s = 1 \) Ohmic, \( s > 1 \) super-Ohmic.
Kraus operators for Single qubit system are given by
\[
\rho^S(t) = -(i/\hbar)[H, \rho^S(t)] = -(i/\hbar)\Gamma(t)[\sigma_z, \rho^S(t)],
\]
which has the solution
\[
\rho^S(t) = \rho^S(0) - i \int_0^t \Gamma(t_1)[\sigma_z, \rho^S(t_1)]dt_1. \tag{28}
\]
By substituting this equation into the von Neumann equation and performing the stochastic averages, we have
\[
\rho^S(t) = -\int_0^t e^{-\tau(t-\tau)/\tau} a^2[\sigma_z, \rho^S(\tau)]d\tau,
\]
where kernel function comes from the correlation function of random variable \( \langle \Gamma(t)\Gamma(\tau) \rangle = a^2 \exp(-|t-\tau|/\tau) \). Next by following the procedure which proposed by Daffer et al. we can solve master equation. In this model Kraus operators for Single qubit system are given by
\[
A_1 = \sqrt{\frac{1+\Lambda(\nu)}{2}} I_{2\times 2}, \quad A_2 = \sqrt{\frac{1-\Lambda(\nu)}{2}} \sigma_z, \tag{30}
\]
where \( \nu = t/2\tau \) is dimensionless time, \( \Lambda(\nu) = e^{-\nu \cos(\mu \nu) + \sin(\mu \nu)}/\mu \), and \( \mu = \sqrt{(4\sigma^2 - 1)} \). We can obtain the time evolution of single qubit system by using Kraus representation as
\[
\rho^S(t) = \sum_i A_i \rho^S(0) A_i^\dagger. \tag{31}
\]
Again in this kind of dynamics, the maximum in Eqs.(17) and (18) is achieved for the initial states \( \rho^S(0) = |\pm\rangle\langle\pm| \) where, \( |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \).

\[
\frac{d\lambda_+}{dt} = \frac{d\Lambda(\nu)}{d\nu} \sqrt{1 - 4(\rho^S_{11}(0)\rho^S_{22}(0) - \Lambda(\nu)^2|\rho^S_{12}(0)|^2)^\nu} \tag{32}
\]
From Eq.(32) and by making use of our witness, we find that in this case the quantum dynamical map is non-Markovian if \( \frac{d\Lambda(\nu)}{d\nu} > 0 \). In Fig.(3), we plot the rate of the eigenvalue \( \eta_+(t) \) for this state as a function of time and \( \sigma \tau \). As we have been seen in some time interval and for fluctuation rate \( \sigma \tau \geq 1/2 \) the rate of eigenvalue \( \eta_+(t) \) is positive, thus dynamics is non-Markovian. Also, one can obtain the degree of non-Markovianity based on the introduced measure in Eq.(17) which the behavior of \( N_e(\Phi) \) has been plotted in terms of fluctuation rate of environment \( \sigma \tau \) in Fig.(4).

What clear is that from these two examples is that, the structure of reservoir affects on non-Markovianity character of dynamics. In example 1 non-Markovianity is revealed in the super Ohmic environment regime and in example 2, the degree of non-Markovianity increases by increasing the fluctuation rate of external field.

IV. CONCLUSIONS

We have proposed a non-Markovianity measure \( N_e \), which is expressly connected to the rate of the density matrix eigenvalues. In our measure, we have used the back flow of information phenomena from an environment to an open quantum system as a feature of non-Markovian process. We have shown that the rate of the eigenvalues, \( \eta_+ \) and \( \eta_- \), have positive and negative values, respectively, then the purity of the state system increases during the time evolution with respect to its initial state, which means that the information flow from the environment to the system and the dynamics is non-Markovian. In addition, as we have seen in Fig.(2) and Fig(4) the structure of reservoir affects on the Markovianity and non-Markovianity character of the open quantum system dynamics \( \Phi \). The merit of this measure compared with other measures is simplicity in calculations and its optimization procedure is only taken over initial input state of the single qubit system. Although in this letter we just only focus on CPTPU maps, however we should be point out that for these kinds of maps making use of this measure is simpler than other measures, this is because, in this measure we do not require complicated statistical methods, such as Monte Carlo sampling of pairs of initial states in order to do optimization procedure.
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