Mean field limit of point vortices with environmental noises to deterministic 2D Navier-Stokes equations

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March 9, 2022

Abstract

We consider point vortex systems on the two dimensional torus perturbed by environmental noise. It is shown that, under a suitable scaling of the noises, weak limit points of the empirical measures are solutions to the vorticity formulation of deterministic 2D Navier-Stokes equations.

Keywords: Mean field limit, point vortex, environmental noise, 2D Navier-Stokes equation, entropy

Mathematics Subject Classification: 60K35, 60K37, 35R60

1 Introduction

The mean field limit is widely used to derive macroscopic PDEs from large systems of interacting particles, as a useful method to reduce the complexity of systems. It is often natural to consider particle systems subjected to random perturbations, mainly by independent Brownian motions; the limit mean field PDEs in this case are nonlinear parabolic equations, called the McKean-Vlasov equations [19]. The coupling method is very efficient to treat Lipschitz continuous interaction kernels, giving rise to explicit convergence rate of empirical measures to solutions of the mean field equation, see the classical work of Sznitman [29] and also [20]. Particle systems with singular kernels have attracted a lot of attention, a notable example being the point vortex model for 2D Euler equations with the Biot-Savart kernel as the interaction kernel, see e.g. [18, 26] for the deterministic case and [23, 21, 11, 15] for the stochastic case. For exchangeable systems, it is well known that the mean field limit is equivalent to the phenomenon of propagation of chaos, cf. [29, p.177, Proposition 2.2] and also [22, 13] for stronger notions of chaos. The readers can find more detailed accounts of the literature in the introduction of [11] and in the nice survey [14].

The paper [11], dealing with the mean field limit of stochastic point vortices to the deterministic 2D Navier-Stokes equations, the noise being additive and independent for each particle, states at page 1425 an open problem concerning the generalization of the result to the case of environmental noise, which means that the same space-dependent noise acts on all particles – the action differing just by the position of the particle, where the noise is evaluated. This open problem, which is at the origin of the present work, has two possible faces. One is the convergence of the empirical measure to a stochastic 2D Euler equation, where stochasticity reflects the random environment, still present in the limit. This has been done in [3] under

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Lipschitz condition on the interaction kernel; see [5] for a scaling limit result on point vortices with regularized Biot-Savart kernel and suitably chosen regularizing parameter, and the recent preprint [24] for a mean field limit without smoothing the Biot-Savart kernel. Another face, the one considered here, is to rescale the space covariance of the noise, simultaneously with the increasing number of particles, in such a way that the noise becomes more and more uncorrelated, going heuristically in the direction of the independent additive noises acting on different particles. We present here a partial solution to this second case, showing that any weak limit of the empirical measures is a probability measure, time dependent, solution to the 2D Navier-Stokes equations.

The equation for the empirical measure of point vortices contains a martingale which has to converge to zero in a scaling regime leading to the deterministic 2D Navier-Stokes equation in vorticity formulation. In the classical case of independent noise on each particle, this convergence is standard. In the case of environmental noise, it may be a difficult problem, as it is here (cf. Proposition 3.3). To overcome this difficulty we have developed nontrivial estimates based on entropy, inspired by [11].

It turns out that the entropy estimate plays also an important role in proving the convergence of the nonlinear part in the equation for empirical measures. Indeed, using Young’s inequality, we are able to derive a uniform estimate on the expected value of the Hamiltonian of random point vortices, see Lemma 3.4 below. As in the deterministic theory (see e.g. [26]), such estimate implies non-concentration of point vortices, as well as the fact that weak limits of empirical measures are continuous measures containing no delta Dirac mass. These results are crucial for showing the convergence of the nonlinear part to the desired limit as the number of vortices tends to infinity.

For technical reasons, we consider point vortices on the torus $T^2 = [-1/2, 1/2]^2$, endowed with periodic boundary condition. This allows the explicit choice of a family of divergence free vector fields $\{\sigma_k\}_k$ (see below) and simplifies some computations; moreover, the compactness of torus makes it easier for integrability arguments (see e.g. Proposition 3.3). Let $K : T^2 \to \mathbb{R}^2$ be the Biot-Savart kernel on the 2D torus, whose basic properties will be recalled at the beginning of Section 2. Here we just mention that $K$ is singular near the origin since $|K(x)| \sim \frac{1}{|x|}$ as $|x| \to 0$. We consider the following system of $N$-point vortices perturbed by multiplicative noises: for $i = 1, \ldots, N$,

$$dX_{t}^{N,i} = \frac{1}{N} \sum_{j=1, j \neq i}^{N} K(X_{t}^{N,j} - X_{t}^{N,i}) \, dt + dW(t, X_t^{N,i}), \quad X_0^{N,i} = X_0^i, \quad (1.1)$$

where $\{X_t^i\}_{i \geq 1}$ is an i.i.d. sequence of random variables on $T^2$ whose law will be specified below, and $W(t, x)$ is a space-time noise, white in time and colored in space, modelling the random environment in which the vortices evolve. Unlike in the usual particle systems where different particles are perturbed by mutually independent Brownian noises (see e.g. [29, 11, 15] and the survey [14]), the noise in (1.1) is the same for each particle, that is, the random vector field $W(t, x)$. Such noise is called an environmental noise.

Under quite general conditions on the spatial covariance function of $W(t, x)$ (cf. the second paragraph on p.107 of [16] or Theorem 4.2.5 therein for an abstract result), one can represent the field $W(t, x)$ as a random series. In this work, we assume that

$$W(t, x) = \sum_{k \in \mathbb{Z}_0^2} \theta_k \sigma_k(x) W^k_t,$$

where $\mathbb{Z}_0^2 = \mathbb{Z}^2 \setminus \{0\}$ is the set of nonzero integer points, $\{\sigma_k\}_{k \in \mathbb{Z}_0^2}$ a family of divergence free vector fields on $T^2$ defined as in (2.2) below, $\{W^k\}_{k \in \mathbb{Z}_0^2}$ a family of independent standard
Brownian motions defined on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\); finally, \(\theta \in \ell^2(\mathbb{Z}_0^2)\), the latter being the usual space of square summable real sequences indexed by \(\mathbb{Z}_0^2\). It is enough to consider those \(\theta\) with only finitely many nonzero components (see for instance the example in Remark 1.2 below), and satisfying the symmetry property:

\[
\theta_k = \theta_j \quad \text{whenever } |k| = |j|.
\]

Using these notations, the point vortex system (1.1) can be written more precisely as follows: for \(i = 1, \ldots, N\),

\[
dX^N_{t,i} = \frac{1}{N} \sum_{j=1, j \neq i}^N K(X^N_{t,i} - X^N_{t,j}) \, dt + \sum_{k} \theta^N_k \sigma_k(X^N_{t,i}) \, dW^k_t, \quad X^N_{t,i} = X_0^i.
\]

Before moving forward, we remark that, under suitable nondegeneracy conditions on the noise, the stochastic point vortex system is globally well posed for Lebesgue almost every initial configuration, cf. [8, p.1456, Theorem 8] (note that this result does not require the bracket generating condition in Hypothesis 1 on p.1451); see also [17, Theorem 1.2] for a similar result on the vortex model of mSQG equations.

As mentioned above, if we fix a noise \(W(t,x)\) (i.e. fix some \(\theta \in \ell^2\)) and consider the mean field limit of empirical measures, then the limit equation will be a stochastic PDE, cf. [4, 24]. In order to get a deterministic limit equation we need to introduce a scaling parameter in the noise part. Therefore, we take a family \(\{\theta^N\}_{N \in \mathbb{N}} \subset \ell^2\) satisfying (1.2) for each \(N \in \mathbb{N}\), and consider the point vortex system

\[
dX^N_{t,i} = \frac{1}{N} \sum_{j=1, j \neq i}^N K(X^N_{t,i} - X^N_{t,j}) \, dt + \varepsilon_N \sum_{k} \theta^N_k \sigma_k(X^N_{t,i}) \, dW^k_t, \quad X^N_{t,i} = X_0^i
\]

for \(i = 1, \ldots, N\), where \((\nu > 0)\) is the noise intensity

\[
\varepsilon_N = \frac{2\sqrt{\nu}}{\|\theta^N\|_{\ell^2}}.
\]

Such scaling of noise is motivated by recent works [12, 7], where the linear transport or 2D Euler equations driven by multiplicative noise of transport type are shown to converge to deterministic parabolic equations or 2D Navier-Stokes equations (see also [10] where the limit equation is the 2D Navier-Stokes driven by space-time white noise).

We denote the empirical measure by

\[
S^N_t = \frac{1}{N} \sum_{i=1}^N \delta_{X^N_{t,i}}
\]

and define the covariance function

\[
Q^N_N(x,y) = \sum_{k \in \mathbb{Z}_0^2} (\theta^N_k)^2 \sigma_k(x) \otimes \sigma_k(y), \quad x,y \in \mathbb{T}^2.
\]

It can be shown that \(Q^N_N(x,y)\) depends only on the difference \(x - y\) (cf. the proof of Lemma 2.1) and thus it will be denoted as \(Q_N(x - y)\), where \(Q_N\) is a \(2 \times 2\) matrix-valued function defined on \(\mathbb{T}^2\). Due to the choice of \(\varepsilon_N\) in (1.4) and equality (2.3) below, it holds that

\[
\varepsilon^2_N Q_N(0) = 2\nu I_2 \quad \text{for all } N \geq 1,
\]

where \(I_2\) is the \(2 \times 2\) identity matrix. Finally, let \(\mathcal{P}(\mathbb{T}^2)\) be the collection of probability measures on \(\mathbb{T}^2\) and \(H^s(\mathbb{T}^2)\) \((s \in \mathbb{R})\) the usual Sobolev spaces on \(\mathbb{T}^2\).

Our purpose is to prove
Theorem 1.1. Let $T > 0$ be given. Assume that

(a) the initial data $\{X^i_0\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. $\mathcal{F}_0$-measurable random variables with law $\mu_0 = f_0 \, dx \in \mathcal{P}(\mathbb{T}^2)$ for some density function $f_0 : \mathbb{T}^2 \to \mathbb{R}_+$ with finite entropy;

(b) the sequence $\{\theta^N\}_{N \in \mathbb{N}}$ satisfies

$$
\lim_{N \to \infty} \varepsilon^2_N Q_N(x) = 0 \quad \text{for all } x \in \mathbb{T}^2 \setminus \{0\}.
$$

Then the laws $\eta^N$ of $S^N \cdot (\cdot \mathcal{N})$ are tight on $C([0,T],H^{-s}(\mathbb{T}^2))$ for any $s > 1$, and any weak limit of $\{\eta^N\}_N$ is supported on weak solutions of the deterministic 2D Navier-Stokes equations in vorticity form:

$$
\partial_t \xi + (K \ast \xi) \cdot \nabla \xi = \nu \Delta \xi, \quad \xi|_{t=0} = f_0.
$$

Remark 1.2. (i) Unfortunately, we cannot prove that the weak limits are the unique solution to (1.7), due to the lack of good estimates on the empirical measures. Indeed, we can only prove that the weak limits $\tilde{\xi} \in L^2(0,T;H^{-1}(\mathbb{T}^2))$ almost surely, and thus the corresponding velocity $\tilde{u} = K \ast \xi \in L^2(0,T;L^2(\mathbb{T}^2))$, cf. Corollary 3.6 below. In [3, Theorem 1.5] Cheskidov and Luo proved that weak solutions in this class to the velocity form of the 2D Navier-Stokes equations are not unique. Thus the problem we leave open is an interesting one for future research.

(ii) The condition (1.6) is used to prove that the martingale part in (2.4) (the equation for empirical measures) tends to 0 in mean square. Here is a simple example for (1.6). Let

$$
\theta^N_k = \frac{1}{|k|} \mathbf{1}_{|k| \leq N}, \quad k \in \mathbb{Z}_0^2, \quad N \in \mathbb{N},
$$

then it is clear that

$$
\varepsilon^2_N = 4\nu \left( \sum_{|k| \leq N} \frac{1}{|k|^2} \right)^{-1} \sim \frac{4\nu}{\log N}
$$

and (cf. the proof of Lemma 2.1) $k^\perp = (k_2, -k_1)$

$$
Q_N(x) = \sum_{|k| \leq N} k^\perp \otimes k^\perp |k|^4 \cos(2\pi k \cdot x).
$$

We know that $\lim_{N \to \infty} Q_N(x)$ exists for all $x \in \mathbb{T}^2 \setminus \{0\}$, thus the condition (1.6) holds.

This paper is organized as follows. In Section 2 we first briefly recall the basic properties of the Biot-Savart kernel on $\mathbb{T}^2$ and define the vector fields $\{\sigma_k\}_{k \in \mathbb{Z}_0^2}$ used above; then, we turn to derive the equation for the empirical measures $S^N_t$, and establish a uniform estimate on the entropy of joint density functions of random point vortices (1.3). The proof of Theorem 1.1 is given in Section 3 where the main difficulty is to show that the martingale parts in the equations of empirical measures vanish in the scaling limit, as well as the convergence of the nonlinear parts. The proofs rely heavily on the entropy estimate in Section 2.2.
2 Preparations

First, we recall some basic properties of the Biot-Savart kernel $K$ on $\mathbb{T}^2$. We have $K = \nabla^\perp G = (\partial_2 G, -\partial_1 G)$, where $G$ is the Green function on $\mathbb{T}^2$. On the whole space $\mathbb{R}^2$, we have the simple expression $G_{\mathbb{R}^2}(x) = \frac{1}{2\pi} \log|x|$; on $\mathbb{T}^2$, it is known that

$$G(x) = \frac{1}{2\pi} \log|x| + r(x), \quad x \in \mathbb{T}^2 \setminus \{0\}, \quad (2.1)$$

where $r$ is a smooth function on $\mathbb{T}^2$. By definition $K$ is smooth and divergence free away from the origin $0 \in \mathbb{T}^2$, and $K(-x) = -K(x)$ for all $x \neq 0$; moreover, it holds that

$$|K(x)| \sim \frac{1}{2\pi|x|} \text{ as } |x| \to 0.$$  

Next we define the vector fields $\sigma_k$, $k \in \mathbb{Z}^2_0$ as follows:

$$\sigma_k(x) = \frac{k^\perp}{|k|}e_k(x), \quad x \in \mathbb{T}^2, \quad k \in \mathbb{Z}^2_0, \quad (2.2)$$

where $k^\perp = (k_2, -k_1)$ and

$$e_k(x) = \sqrt{2} \begin{cases} \cos(2\pi k \cdot x), & k \in \mathbb{Z}^2_+; \\ \sin(2\pi k \cdot x), & k \in \mathbb{Z}^2_- \end{cases},$$

with $\mathbb{Z}^2_+ = \{k \in \mathbb{Z}^2_0 : (k_1 > 0) \text{ or } (k_1 = 0, k_2 > 0)\}$ and $\mathbb{Z}^2_- = -\mathbb{Z}^2_+$. Then $\{\sigma_k\}_{k \in \mathbb{Z}^2_0}$ is a CONS of the space of square integrable and divergence free vector fields on $\mathbb{T}^2$ with zero mean.

The rest of this section consists of two parts. In Section 2.1 we derive the equation fulfilled by the empirical measure $S_t^N$, see (2.4). We introduce in Section 2.2 the rescaled entropy functional and prove a uniform estimate for the entropy of joint density functions of point vortices. This estimate will play a crucial role in the proof of the main result.

2.1 Equation for empirical measures

We want to find the equation satisfied by the empirical measure $S_t^N$, $t \geq 0$. Let $\phi \in C^2(\mathbb{T}^2)$; by (1.3) and the Itô formula,

$$d\phi(X_t^{N,i}) = \frac{1}{N} \sum_{j \neq i} K(X_t^{N,i} - X_t^{N,j}) \cdot \nabla \phi(X_t^{N,i}) dt + \varepsilon_N \sum_{k \in \mathbb{Z}^2_0} \theta_k^N(\sigma_k \cdot \nabla \phi)(X_t^{N,i}) dW_t^k$$

$$+ \frac{\varepsilon^2}{2} \sum_{k \in \mathbb{Z}^2_0} (\theta_k^N)^2 \text{Tr}[(\sigma_k \otimes \sigma_k) \nabla^2 \phi](X_t^{N,i}) dt.$$

The proof of the following key identity is similar to [10 Lemma 2.6], see also [12 Section 2].

**Lemma 2.1.** It holds that

$$\sum_{k \in \mathbb{Z}^2_0} (\theta_k^N)^2(\sigma_k \otimes \sigma_k)(x) \equiv \frac{1}{2} \|\theta^N\|^2_{l^2} I_2, \quad x \in \mathbb{T}^2.$$  

(2.3)

where $I_2$ is the $2 \times 2$ identity matrix.
Proof. We give the proof for the reader’s convenience. For any \( x, y \in \mathbb{T}^2 \), using the definition of \( \sigma_k \) and the fact that \( \theta^N \) satisfies (1.2), we have

\[
Q_N(x, y) = \sum_{k \in \mathbb{Z}_0^2} (\theta^N_k)^2 \sigma_k(x) \otimes \sigma_k(y) = \sum_{k \in \mathbb{Z}_0^2} (\theta^N_k)^2 \frac{k^\perp \otimes k^\perp}{|k|^2} \epsilon_k(x) \epsilon_k(y)
\]

\[
= 2 \sum_{k \in \mathbb{Z}_0^2} (\theta^N_k)^2 \frac{k^\perp \otimes k^\perp}{|k|^2} \left[ \cos(2\pi k \cdot x) \cos(2\pi k \cdot y) + \sin(2\pi k \cdot x) \sin(2\pi k \cdot y) \right]
\]

\[
= 2 \sum_{k \in \mathbb{Z}_0^2} (\theta^N_k)^2 \frac{k^\perp \otimes k^\perp}{|k|^2} \cos(2\pi k \cdot (x - y)) = \sum_{k \in \mathbb{Z}_0^2} (\theta^N_k)^2 \frac{k^\perp \otimes k^\perp}{|k|^2} \cos(2\pi k \cdot (x - y)).
\]

Therefore, \( Q_N(x, y) \) only depends on the displacement \( x - y \), which, for simplicity of notation, will be denoted by \( Q_N(x - y) \). In particular,

\[
Q_N(0) = \sum_{k \in \mathbb{Z}_0^2} (\theta^N_k)^2 \sigma_k(x) \otimes \sigma_k(x) = \sum_{k \in \mathbb{Z}_0^2} (\theta^N_k)^2 \frac{k^\perp \otimes k^\perp}{|k|^2} = \sum_{k \in \mathbb{Z}_0^2} (\theta^N_k)^2 \begin{pmatrix} k_2^2 & -k_1 k_2 \\ -k_1 k_2 & k_1^2 \end{pmatrix}.
\]

First, we have

\[
Q^1_N(0) = -\sum_{k \in \mathbb{Z}_0^2} (\theta^N_k)^2 \frac{k^\perp \otimes k^\perp}{|k|^2} k_1 k_2 = 0
\]

since, by (1.2), the sum of the four terms involving \((k_1, k_2), (-k_1, k_2), (k_1, -k_2), (-k_1, -k_2)\) cancel each other. Next, using again (1.2),

\[
Q^1_N(0) = \sum_{k \in \mathbb{Z}_0^2} \frac{(\theta^N_k)^2}{|k|^2} k_2^2 = \sum_{k \in \mathbb{Z}_0^2} \frac{(\theta^N_k)^2}{|k|^2} k_1^2 = Q^2_N(0)
\]

since the points \((k_1, k_2)\) and \((k_2, k_1)\) appear in pair. Therefore,

\[
Q^1_N(0) = Q^2_N(0) = \frac{1}{2} \sum_{k \in \mathbb{Z}_0^2} \frac{(\theta^N_k)^2}{|k|^2} (k_1^2 + k_2^2) = \frac{1}{2} \sum_{k \in \mathbb{Z}_0^2} (\theta^N_k)^2 = \frac{1}{2} \|\theta^N\|^2_{\ell^2}.
\]

This completes the proof. \( \square \)

Hence, by (2.3) and the definition (2.4) of \( \varepsilon_N \), we obtain

\[
\frac{\varepsilon^2}{2} \sum_{k \in \mathbb{Z}_0^2} \frac{(\theta^N_k)^2}{|k|^2} \mathrm{Tr}[(\sigma_k \otimes \sigma_k) \nabla^2 \phi](X^N_{t,i}) = \nu \Delta \phi(X^N_{t,i}).
\]

As a result, for \( 1 \leq i \leq N \),

\[
d\phi(X^N_{t,i}) = \frac{1}{N} \sum_{j \neq i} K(X^N_{t,i} - X^N_{t,j}) \cdot \nabla \phi(X^N_{t,i}) dt + \nu \Delta \phi(X^N_{t,i}) dt
\]

\[
+ \varepsilon_N \sum_{k \in \mathbb{Z}_0^2} \theta^N_k (\sigma_k \cdot \nabla \phi)(X^N_{t,i}) dW^k_t.
\]
Denoting by \( \langle S_t^N, \phi \rangle = \frac{1}{N} \sum_{i=1}^N \phi(X_t^{N,i}) \), then we have

\[
d(S_t^N, \phi) = \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} K(X_t^{N,i} - X_t^{N,j}) \cdot \nabla \phi(X_t^{N,i}) \, dt + \nu(S_t^N, \Delta \phi) \, dt + \varepsilon_N \sum_{k \in \mathbb{Z}_0^2} \theta_k^N \langle S_t^N, \sigma_k \cdot \nabla \phi \rangle \, dW_t^k.
\]

Using the fact that \( K(-x) = -K(x) \) for all \( x \in \mathbb{T}^2 \setminus \{0\} \), we can rewrite the first term on the right hand side as \( \langle S_t^N \otimes S_t^N, H_\phi \rangle \), where

\[
H_\phi(x,y) = \frac{1}{2} K(x-y) \cdot (\nabla \phi(x) - \nabla \phi(y))
\]
is a symmetric function on \( \mathbb{T}^2 \times \mathbb{T}^2 \), with the convention that \( H_\phi(x,x) = 0 \). We remark that \( H_\phi \) is smooth off the diagonal and bounded by \( C\|\nabla^2 \phi\|_\infty \) for some \( C > 0 \) independent of \( \phi \). Therefore, we get the equation for the empirical measure:

\[
d(S_t^N, \phi) = \langle S_t^N \otimes S_t^N, H_\phi \rangle \, dt + \nu(S_t^N, \Delta \phi) \, dt + \varepsilon_N \sum_{k \in \mathbb{Z}_0^2} \theta_k^N \langle S_t^N, \sigma_k \cdot \nabla \phi \rangle \, dW_t^k. \tag{2.4}
\]

### 2.2 Entropy for density functions

The relative entropy \( h_N(F) \) of probability density functions \( F \) on \( \mathbb{T}^{2N} := (\mathbb{T}^2)^N \) is defined as

\[
h_N(F) = \frac{1}{N} \int_{\mathbb{T}^{2N}} F(X) \log F(X) \, dX,
\]
where \( dX = dx_1 \ldots dx_N \) is the Lebesgue measure on \( \mathbb{T}^{2N} \). The simple inequality \( s \log s \geq s - 1 \) \( (s \geq 0) \) implies that \( h_N(F) \) is always nonnegative. As in [11], we add the factor \( 1/N \) so that if \( F(x_1, \ldots, x_N) = f(x_1) \ldots f(x_N) \) for some probability density \( f \) on \( \mathbb{T}^2 \) with finite entropy, then one has \( h_N(F) = h_1(f) \). The functional \( h_N \) enjoys the following important property: if \( F \) is exchangeable (i.e. \( F \) is invariant under permutations of its variables) and \( F^{(2)} \) is the marginal distribution of \( F \) on \( \mathbb{T}^4 \), then

\[
h_N(F) \geq \frac{N-1}{N} h_2(F^{(2)}) \quad \text{for all } N \geq 2. \tag{2.5}
\]

Let \( F_t^N \) be the density function of the law on \( \mathbb{T}^{2N} \) of the particles \( (X_t^{N,1}, \ldots, X_t^{N,N}) \) associated to (1.3); then \( F_0^N(X) = f_0(x_1) \ldots f_0(x_N) \). We want to prove an estimate on \( h_N(F_t^N) \), for which we need to introduce some notations. Define the dispersion vector fields on \( \mathbb{T}^{2N} \):

\[
A_k(X) = (\sigma_k(x_1), \ldots, \sigma_k(x_N)), \quad X = (x_1, \ldots, x_N) \in \mathbb{T}^{2N}, \quad k \in \mathbb{Z}_0^2,
\]
and the drift field \( A_0 \) via

\[
A_0^i(X) = \frac{1}{N} \sum_{j=1, j \neq i}^N K(x_i - x_j), \quad X = (x_1, \ldots, x_N) \notin D_N, \quad 1 \leq i \leq N,
\]
where \( D_N = \{ X = (x_1, \ldots, x_N) \in \mathbb{T}^{2N} : \exists i \neq j \text{ such that } x_i = x_j \} \) is the generalized diagonal of \( \mathbb{T}^{2N} \). It is clear that all the vector fields \( A_k (k \in \mathbb{Z}_0^2) \) are divergence free, so is \( A_0 \) on \( D_N \).
Denoting by \( X_t^N = (X_{t,N}^1, \ldots, X_{t,N}^N) \), \( t \geq 0 \); then the system \([3]\) of SDEs can be simply written as
\[
dX_t^N = A_0(X_t^N) \, dt + \varepsilon_N \sum_{k \in \mathbb{Z}_0^2} \theta_k^N A_k(X_t^N) \, dW_t^k.
\]
We remark that the equation can also be written in the Stratonovich form since \( A_k \cdot \nabla_N A_k = 0 \) for all \( k \in \mathbb{Z}_0^2 \), where \( \nabla_N = (\nabla_{x_1}, \ldots, \nabla_{x_N}) \) is the gradient operator on \( \mathbb{T}^{2N} \). The associated infinitesimal generator has the form
\[
\mathcal{L} \Phi(X) = \frac{\varepsilon_N^2}{2} \sum_{k \in \mathbb{Z}_0^2} (\theta_k^N)^2 \langle A_k, \nabla_N (A_k, \nabla_N \Phi) \rangle_{\mathbb{R}^{2N}} + \langle A_0, \nabla_N \Phi \rangle_{\mathbb{R}^{2N}}, \quad \Phi \in C^2(\mathbb{T}^{2N}).
\]

**Lemma 2.2.** For all \( t > 0 \),
\[
h_N(F_t^N) \leq h_N(F_0^N) = h_1(f_0).
\]

**Proof.** The proof below is a little formal, but it can be made rigorous by approximating the initial density \( f_0 \) and the kernel \( K : \mathbb{T}^2 \to \mathbb{R}^2 \) with smooth objects, cf. [9, Sect. 4.2]. The density function \( F_t^N \) satisfies the Fokker–Planck equation
\[
\partial_t F_t^N = \mathcal{L}^* F_t^N = \frac{\varepsilon_N^2}{2} \sum_{k \in \mathbb{Z}_0^2} (\theta_k^N)^2 \langle A_k, \nabla_N (A_k, \nabla_N F_t^N) \rangle_{\mathbb{R}^{2N}} - \langle A_0, \nabla_N F_t^N \rangle_{\mathbb{R}^{2N}}.
\]

Therefore,
\[
\partial_t (F_t^N \log F_t^N) = (1 + \log F_t^N) \partial_t F_t^N
\]
\[
= \frac{\varepsilon_N^2}{2} \sum_{k \in \mathbb{Z}_0^2} (\theta_k^N)^2 (1 + \log F_t^N) \langle A_k, \nabla_N (A_k, \nabla_N F_t^N) \rangle_{\mathbb{R}^{2N}}
\]
\[
- (1 + \log F_t^N) \langle A_0, \nabla_N F_t^N \rangle_{\mathbb{R}^{2N}}.
\]

Note that \( (1 + \log F_t^N) \langle A_0, \nabla_N F_t^N \rangle_{\mathbb{R}^{2N}} = \langle A_0, \nabla_N (F_t^N \log F_t^N) \rangle_{\mathbb{R}^{2N}} \) and that all the vector fields \( A_k \) and \( A_0 \) are divergence free. Integrating both sides of the above equation on \( \mathbb{T}^{2N} \) and applying integration by parts yield
\[
\frac{d}{dt} h_N(F_t^N) = \frac{1}{N} \int_{\mathbb{T}^{2N}} (1 + \log F_t^N) \partial_t F_t^N \, dX
\]
\[
= -\frac{\varepsilon_N^2}{2N} \sum_{k \in \mathbb{Z}_0^2} (\theta_k^N)^2 \int_{\mathbb{T}^{2N}} \frac{\langle A_k, \nabla_N F_t^N \rangle_{\mathbb{R}^{2N}}^2}{F_t^N} \, dX,
\]
which immediately gives us the desired result.

**Remark 2.3.** Unlike in \([11]\), we are unable to derive, from the identity \([2.6]\), estimate on the Fisher information, which was used in \([11]\, Lemma 3.3] to show that particles are not too close to each other.

3 Scaling limit of random point vortices

Recall the empirical measures \( \{S_t^N : t \in [0,T]\}_{N \geq 1} \) defined in Section 1. For any \( \phi \in C^\infty(\mathbb{T}^2) \) it is obvious that \( |\langle S_t^N, \phi \rangle| \leq \|\phi\|_{\infty} \), thus, using the definition of Sobolev norm in \( H^s(\mathbb{T}^2) \), one can easily show that, for any \( s > 1 \),
\[
\sup_{0 \leq t \leq T} \| S_t^N \|_{H^{-s}} \leq C_s < \infty \quad \mathbb{P}\text{-a.s.}
\]
In particular, $S^N$ has trajectories in $L^\infty(0,T; H^{-s}(\mathbb{T}^2))$, $s > 1$.

Let $\eta_N$, $N \in \mathbb{N}$ be the laws of $S^N$; we want to show that the family $\{\eta_N\}_{N \geq 1}$ is tight on $C([0,T]; H^{-s}(\mathbb{T}^2))$ for any $s > 1$. By (3.11) and Simon’s compactness result (cf. (27) p. 90, Corollary 9)), it is sufficient to show that $\{S^N\}_{N \geq 1}$ is bounded in probability in $W^{1/3,4}(0,T; H^{-\beta}(\mathbb{T}^2))$ for some $\beta > 5$. This is an immediate consequence of the fact below:

$$
\sup_{N \geq 1} \mathbb{E} \int_0^T \int_0^T \frac{||S^N_t - S^N_s||_{H^{-\beta}}^4}{|t-s|^{7/3}} \, dt \, ds < +\infty. \quad (3.2)
$$

**Lemma 3.1.** There exists a constant $C = C(T, \nu) > 0$ such that for any $k \in \mathbb{Z}^2$, it holds

$$
\mathbb{E}\left(\left\langle S^N_t - S^N_s, e_k \right\rangle^4\right) \leq C|k|^8(t-s)^2, \quad 0 \leq s < t \leq T.
$$

**Proof.** By (2.4), we have

$$
\left\langle S^N_t - S^N_s, e_k \right\rangle = \int_s^t \left\langle S^N_r \otimes S^N_r, H_{ek} \right\rangle \, dr + \nu \int_s^t \left\langle S^N_r, \Delta e_k \right\rangle \, dr + \varepsilon_N \sum_i \theta_i^N \int_s^t \left\langle S^N_r, \sigma_i \cdot \nabla e_k \right\rangle \, dW^i_r.
$$

First, by the Burkholder-Davis-Gundy inequality,

$$
\mathbb{E}\left[\left(\varepsilon_N \sum_i \theta_i^N \int_s^t \left\langle S^N_r, \sigma_i \cdot \nabla e_k \right\rangle \, dW^i_r\right)^4\right] \leq C\varepsilon_N^4 \mathbb{E}\left[\left(\sum_i (\theta_i^N)^2 \int_s^t \left\langle S^N_r, \sigma_i \cdot \nabla e_k \right\rangle^2 \, dr\right)^2\right].
$$

Using Cauchy’s inequality, we obtain

$$
\sum_i (\theta_i^N)^2 \left\langle S^N_r, \sigma_i \cdot \nabla e_k \right\rangle^2 \leq \sum_i (\theta_i^N)^2 \frac{1}{N} \sum_{i=1}^N \left[ (\sigma_i \cdot \nabla e_k)(X^N_{t,i}) \right]^2
$$

$$
= \frac{1}{N} \sum_{i=1}^N \frac{1}{2} \left\| \theta^N \right\|_{L^2}^2 \left| \nabla e_k \left( X^N_{t,i} \right) \right|^2 \leq 4\pi^2 \left\| \theta^N \right\|_{L^2}^2 |k|^2,
$$

where in the second step we have used (2.3). By the definition of $\varepsilon_N$, we arrive at

$$
\mathbb{E}\left[\left(\varepsilon_N \sum_i \theta_i^N \int_s^t \left\langle S^N_r, \sigma_i \cdot \nabla e_k \right\rangle \, dW^i_r\right)^4\right] \leq C\nu^2 |k|^4(t-s)^2.
$$

Combining this estimate with the following facts

$$
\left| \left\langle S^N_r \otimes S^N_r, H_{ek} \right\rangle \right| \leq \| H_{ek} \|_{\infty} \leq C|k|^2, \quad \left| \left\langle S^N_r, \Delta e_k \right\rangle \right| \leq \| \Delta e_k \|_{\infty} \leq 4\pi^2 |k|^2,
$$

we can easily prove the desired estimate. 

Now by Cauchy’s inequality and Lemma 3.1,

$$
\mathbb{E}(||S^N_t - S^N_s||_{H^{-\beta}}^4) = \mathbb{E}\left[\left( \sum_k \frac{(S^N_k - S^N_s, e_k)^2}{(1 + |k|^2)^{\beta}} \right)^2\right]
$$

$$
\leq \left( \sum_k \frac{1}{(1 + |k|^2)^{\beta}} \right) \left( \sum_k \mathbb{E}(\frac{(S^N_k - S^N_s, e_k)^4}{(1 + |k|^2)^{3\beta}}) \right)
$$

$$
\leq C_{\beta} \sum_k \frac{C|k|^8(t-s)^2}{(1 + |k|^2)^{3\beta}} \leq C'_{\beta}(t-s)^2,
$$

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where the last inequality is due to $\beta > 5$. From this result we immediately get (3.2).

Summarizing the above discussions, we deduce that $\{\eta_N\}_{N \geq 1}$ is tight on $C([0, T]; H^{-\beta}(\mathbb{T}^2))$ for any $s > 1$. Therefore, Prohorov’s theorem (see [2] p.59, Theorem 5.1) implies the existence of a subsequence $\{\eta_{N_i}\}_{i \geq 1}$ converging weakly to some probability measure $\eta$ supported on $C([0, T]; H^{-s}(\mathbb{T}^2))$. By Skorokhod’s representation theorem (see [2] p.70, Theorem 6.7), there exist a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, a sequence of random variables $\{\tilde{S}^{N_i}\}_{i \geq 1}$ and a random variable $\tilde{\xi}$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, such that

(i) $\tilde{\xi}$ has law $\eta$ and $\tilde{S}^{N_i}$ has law $\eta_{N_i}$ ($i \geq 1$);

(ii) $\tilde{P}$-a.s., $\tilde{S}^{N_i}$ converges to $\tilde{\xi}$ in the topology of $C([0, T]; H^{-s}(\mathbb{T}^2))$.

**Remark 3.2.** Notice that, for $\tilde{P}$-a.s. $\tilde{\omega} \in \tilde{\Omega}$, $\{\tilde{S}^{N_i}(\tilde{\omega})\}_{i \geq 1}$ is a sequence of functions with values in the space $\mathcal{P}(\mathbb{T}^2)$ of probability measures; moreover, the above arguments show that they are equivalent in time in some negative Sobolev space. Therefore, up to a further subsequence, $\tilde{S}^{N_i}(\tilde{\omega})$ converges weakly-$*$ in the space of functions with values in $\mathcal{P}(\mathbb{T}^2)$; see the third paragraph in [26, p.915] for similar remarks. As a result, for $\tilde{P}$-a.s. $\tilde{\omega} \in \tilde{\Omega}$ and a.e. $t \in (0, T)$, one has $\tilde{\xi}(\tilde{\omega}) \in \mathcal{P}(\mathbb{T}^2)$.

By assertion (i), for any $i \geq 1$, $\tilde{S}^{N_i}$ fulfills an equation as (2.4); therefore, for any $\phi \in C^\infty(\mathbb{T}^2)$, for any $t \in [0, T]$,

$$
\langle \tilde{S}_t^{N_i}, \phi \rangle = \langle \tilde{S}_0^{N_i}, \phi \rangle + \int_0^t \langle \tilde{S}_s^{N_i} \otimes \tilde{S}_s^{N_i}, H_\phi \rangle ds + \nu \int_0^t \langle \tilde{S}_s^{N_i}, \Delta \phi \rangle ds
+ \varepsilon_{N_i} \sum_{k \in \mathbb{Z}_2^d} \theta_k^{N_i} \int_0^t \langle \tilde{S}_s^{N_i}, \sigma_k \cdot \nabla \phi \rangle d\tilde{W}_s^k,
$$

(3.3)

where $\{\tilde{W}_s^k\}_{k \in \mathbb{Z}_2^d}$ is a family of independent standard Brownian motions on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. It remains to let $i \to \infty$ in the above equation and prove that the limit $\tilde{\xi}$ fulfills the weak vorticity form of the deterministic 2D Navier-Stokes equation. For this purpose, it is sufficient to show the convergence of the nonlinear part and the martingale part. In the following, for simplicity of notations, we omit the tilde over $\xi$, $\tilde{S}^{N_i}$ and $\tilde{W}_s^k$, and write $N$ instead of $N_i$.

We first deal with the martingale part:

$$
M_{t}^{\phi, N} := \varepsilon_N \sum_{k \in \mathbb{Z}_2^d} \theta_k^{N_i} \int_0^t \langle S_s^N, \sigma_k \cdot \nabla \phi \rangle dW_s^k.
$$

(3.4)

**Proposition 3.3.** Under the condition (1.6) it holds that

$$
\lim_{N \to \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} |M_{t}^{\phi, N}|^2 \right] = 0.
$$

**Proof.** We have

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} |M_{t}^{\phi, N}|^2 \right] \leq \mathbb{E} \left[ |M_{T}^{\phi, N}|^2 \right] = C \varepsilon_N^2 \sum_{k \in \mathbb{Z}_2^d} (\theta_k^{N_i})^2 \mathbb{E} \int_0^T \langle S_t^N, \sigma_k \cdot \nabla \phi \rangle^2 dt.
$$

Noticing that

$$
\langle S_t^N, \sigma_k \cdot \nabla \phi \rangle^2 = \frac{1}{N^2} \sum_{i,j=1}^N (\sigma_k \cdot \nabla \phi)(X_t^{N,i}) (\sigma_k \cdot \nabla \phi)(X_t^{N,j}),
$$
the right hand side of the above inequality can be decomposed as the sum of the following two terms:

\[ I_1 = C \varepsilon_N^2 N^2 \sum_{k \in \mathbb{Z}_0^2} \sum_{i=1}^N \left( \theta_k^N \right)^2 \mathbb{E} \int_0^T \left[ (\sigma_k \cdot \nabla \phi) (X_{t_i}^{N,i}) \right]^2 dt, \]

\[ I_2 = C \varepsilon_N^2 N^2 \sum_{1 \leq i \neq j \leq N} \sum_{k \in \mathbb{Z}_0^2} \left( \theta_k^N \right)^2 \mathbb{E} \int_0^T (\sigma_k \cdot \nabla \phi) (X_{t_i}^{N,i}) (\sigma_k \cdot \nabla \phi) (X_{t_j}^{N,j}) dt. \]

It follows from (2.3) and (1.4) that

\[ |I_1| = C \varepsilon_N^2 N^2 \sum_{i=1}^N \left\| \theta_i^N \right\|^2 \mathbb{E} \int_0^T \left| \nabla \phi (X_{t_i}^{N,i}) \right|^2 dt \leq \frac{CT \| \nabla \phi \|^2_{\infty}}{N^2} \to 0 \quad (3.5) \]

as \( N \to \infty \). Here \( \| \nabla \phi \|_{\infty} \) is the supremum norm of \( \nabla \phi(x) \) on \( \mathbb{T}^2 \).

We now turn to the second term \( I_2 \). By the exchangeability,

\[ I_2 = C \varepsilon_N^2 \frac{N-1}{N} \sum_{k \in \mathbb{Z}_0^2} \left( \theta_k^N \right)^2 \mathbb{E} \int_0^T (\sigma_k \cdot \nabla \phi) (X_{t_i}^{N,1}) (\sigma_k \cdot \nabla \phi) (X_{t_i}^{N,2}) dt \]

\[ = C \varepsilon_N^2 \frac{N-1}{N} \sum_{k \in \mathbb{Z}_0^2} \left( \theta_k^N \right)^2 \int_0^T \int_0^T (\sigma_k \cdot \nabla \phi) (x_1) (\sigma_k \cdot \nabla \phi) (x_2) F_{t_1}^{N,2} (x_1, x_2) dx_1 dx_2 dt, \]

where \( F_{t_i}^{N,2} \) is the joint density function of \( (X_{t_i}^{N,1}, X_{t_i}^{N,2}) \). We have, by the definition \( (3.5) \) of the covariance function \( Q_N \),

\[ I_2 = C \varepsilon_N^2 \frac{N-1}{N} \int_0^T \int_{\mathbb{T}^4} (\nabla \phi (x_1))^* Q_N (x_1 - x_2) \nabla \phi (x_2) F_{t_1}^{N,2} (x_1, x_2) dx_1 dx_2 dt. \]

Therefore, for any \( M > 1 \),

\[ |I_2| \leq C \| \nabla \phi \|^2_{\infty} \varepsilon_N^2 \int_0^T \int_{\mathbb{T}^4} \left| Q_N (x_1 - x_2) \right| F_{t_1}^{N,2} (x_1, x_2) dx_1 dx_2 dt \]

\[ = C \| \nabla \phi \|^2_{\infty} \varepsilon_N^2 \int_0^T \int_{\mathbb{T}^4} \left| Q_N (x_1 - x_2) \right| F_{t_1}^{N,2} (x_1, x_2) dx_1 dx_2 dt \]

\[ + C \| \nabla \phi \|^2_{\infty} \varepsilon_N^2 \int_0^T \int_{\mathbb{T}^4} \left| Q_N (x_1 - x_2) \right| F_{t_1}^{N,2} (x_1, x_2) dx_1 dx_2 dt \]

\[ =: J_N^{(1)} + J_N^{(2)}. \]

For the first term, one has

\[ J_N^{(1)} \leq CM \| \nabla \phi \|^2_{\infty} \varepsilon_N^2 \int_0^T \int_{\{ F_{t_1}^{N,2} \leq M \}} \left| Q_N (x_1 - x_2) \right| dx_1 dx_2 dt \]

\[ \leq CM \| \nabla \phi \|^2_{\infty} T \varepsilon_N^2 \int_{\mathbb{T}^4} \left| Q_N (x_1 - x_2) \right| dx_1 dx_2. \]

By the proof of Lemma (2.4), it is easy to see that

\[ \varepsilon_N^2 \left| Q_N (x_1 - x_2) \right| \leq 2\nu \quad \text{for all} \ x_1, x_2 \in \mathbb{T}^2. \quad (3.7) \]
Hence by (1.6) and the dominated convergence theorem,
\[
\lim_{N \to \infty} J^{(1)}_N = 0. \tag{3.8}
\]

Next, using again the inequality (3.7),
\[
J^{(2)}_N \leq 2\nu C\|\nabla \phi\|_\infty \int_0^T \int_{\{F^{N,2}_{t} \to M\}} F^{N,2}_{t}(x_1, x_2) \, dx_1 \, dx_2 \, dt \\
\leq \frac{C'}{\log M} \int_0^T \int_{\{F^{N,2}_{t} \to 1\}} \left( F^{N,2}_{t} \log F^{N,2}_{t} \right)(x_1, x_2) \, dx_1 \, dx_2 \, dt.
\]

Recalling the simple fact that \( s \log s \in [-e^{-1}, 0] \) for all \( s \in [0, 1] \), one can easily prove
\[
\int_{\{F^{N,2}_{t} \to 1\}} \left( F^{N,2}_{t} \log F^{N,2}_{t} \right)(x_1, x_2) \, dx_1 \, dx_2 \leq 2h_2(F^{N,2}_{t}) + e^{-1}.
\]

Therefore, by (2.5) and Lemma 2.2 above, we obtain
\[
J^{(2)}_N \leq \frac{C'T}{\log M} [h_1(f_0) + e^{-1}].
\]

Combining this estimate with (3.6) and (3.8), we conclude that \( I_2 \) tends to 0 as \( N \to \infty \). In view of (3.5), this completes the proof. \( \square \)

Next we turn to prove the convergence of the nonlinear part in (3.3). For this purpose, we make some preparations by introducing the Hamiltonian of point vortices: for \( X = (x_1, \ldots, x_N) \in \mathbb{T}^2N \),
\[
\mathcal{H}_N(X) = \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \left[ c_0 - G(x_i - x_j) \right],
\]

where \( c_0 \) is a constant such that \( G(x) \leq c_0, x \in \mathbb{T}^2 \). We introduce the constant so that \( \mathcal{H}_N \) is nonnegative. Recall that \( X^N_t = (X^N_{1,t}, \ldots, X^N_{N,t}) \), \( t \geq 0 \) is the solution to the particle system (1.3). We will show

Lemma 3.4. It holds that
\[
\sup_{N \geq 2} \sup_{t \geq 0} \mathbb{E}\mathcal{H}_N(X^N_t) < +\infty.
\]

Proof. By the definition of \( \mathcal{H}_N(X^N_t) \) and the exchangeability,
\[
\mathbb{E}\mathcal{H}_N(X^N_t) = \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \mathbb{E}[c_0 - G(X^N_{i,t} - X^N_{j,t})] = \frac{N-1}{N} \mathbb{E}[c_0 - G(X^N_{1,t} - X^N_{2,t})].
\]

Using the joint density function \( F^{N,2}_{t} \) of \( \left(X^{N,1}_{t}, X^{N,2}_{t}\right) \), we have
\[
\mathbb{E}\mathcal{H}_N(X^N_t) \leq \int_{\mathbb{T}^4} [c_0 - G(x - y)] F^{N,2}_{t}(x, y) \, dx \, dy.
\]

Thanks to the formula (2.1) of the Green function \( G \) on \( \mathbb{T}^2 \), we can find some big \( c_1 \) such that
\[
c_0 - G(x - y) \leq \log \frac{c_1}{|x - y|^{1/2}}, \quad x, y \in \mathbb{T}^2, \quad x \neq y.
\]
Therefore,

\[
\mathbb{E} H_N(X_t^N) \leq \int_{\mathbb{T}^d} \left( \log \frac{c_1}{|x-y|^{1/2}} \right) f_t^{N,2}(x,y) \, dx \, dy \\
\leq \log \left( \int_{\mathbb{T}^d} \exp \left( \log \frac{c_1}{|x-y|^{1/2}} \right) \, dx \, dy \right) + \int_{\mathbb{T}^d} \left( f_t^{N,2} \log f_t^{N,2} \right)(x,y) \, dx \, dy,
\]

where the second step follows from Young’s inequality (cf. [28 Lemma 6.45] or [11 Lemma 2.4]) and the fact that \( f_t^{N,2} \) is a probability density on \( \mathbb{T}^d \). Note that the last integral is nothing but \( 2h_2(f_t^{N,2}) \); by (2.5) and Lemma 2.2 we deduce that

\[
\mathbb{E} H_N(X_t^N) \leq \log \left( \int_{\mathbb{T}^d} \frac{c_1}{|x-y|^{1/2}} \, dx \, dy \right) + 2h_2(f_t^{N,2}) \leq c_2 + 4h_1(f_0)
\]

for some constant \( c_2 > 0 \). The above bound is independent of \( t \geq 0 \) and \( N \geq 1 \). \( \Box \)

We can deduce the following non-concentration result for point vortices. Let \( B_r(x) \) be a ball with center \( x \in \mathbb{T}^2 \) and radius \( r > 0 \).

**Corollary 3.5.** It holds that

\[
\lim_{n \to \infty} \lim_{r \to 0} \sup_{N \geq n} \mathbb{E} \left[ \sup_{x \in \mathbb{T}^2} S_t^N(B_r(x)) \right] = 0.
\]

In particular, \( \mathbb{P} \)-a.s., for all \( t \in [0,T] \), the limit \( \xi_t \) is a continuous measure on \( \mathbb{T}^2 \), i.e. it does not contain delta Dirac mass.

**Proof.** The second assertion follows from the first limit, as mentioned at the bottom of [28 p.1086]. To show the limit, we follow the idea in [26 p.928, (3.5)] which deals with the deterministic setting. Given small \( r > 0 \), we have, for any \( x \in \mathbb{T}^2 \),

\[
\mathcal{H}_N(X_t^N) \log \frac{1}{2r} = \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \frac{1}{\log \frac{1}{2r}} \left[ c_0 - G(X_t^{N,i} - X_t^{N,j}) \right] \\
\geq \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \frac{1}{\log \frac{1}{2r}} \left[ c_0 - G(X_t^{N,i} - X_t^{N,j}) \right],
\]

where \( a \vee b = \max\{a,b\} \). Choosing a bigger \( c_0 \) if necessary, we can assume that

\[
c_0 - G(x-y) \geq \frac{1}{2\pi} \log \frac{1}{|x-y|} \quad \text{for all } x,y \in \mathbb{T}^2, \ 0 < |x-y| \leq \frac{1}{2}.
\] (3.9)

As a result,

\[
\mathcal{H}_N(X_t^N) \log \frac{1}{2r} \geq \frac{1}{2\pi} \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \frac{1}{\log \frac{1}{2r}} \frac{1}{|X_t^{N,i} - X_t^{N,j}|} \\
\geq \frac{1}{2\pi} \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \frac{1}{|X_t^{N,i} - X_t^{N,j}|} \frac{1}{|X_t^{N,i} - x|} \frac{1}{|X_t^{N,j} - x|} < r.
\]
where the second step follows from $|X_t^{N,i} - X_t^{N,j}| \leq |X_t^{N,i} - x| + |X_t^{N,j} - x| < 2r$. Therefore,

$$
\mathcal{H}_N(X_t^N) \geq \frac{\mathcal{H}_N(X_t^N)}{\log \frac{1}{2r}} \geq \frac{1}{2\pi} \left[ \left( \sum_{|X_t^{N,i} - x| < r} \frac{1}{N} \right)^2 - \sum_{|X_t^{N,j} - x| < r} \frac{1}{N^2} \right] \geq \frac{1}{2\pi} \left[ (S_t^N(B_r(x)))^2 - \frac{1}{N} \right].
$$

This holds for any $x \in \mathbb{T}^2$, and thus we obtain

$$
\sup_{x \in \mathbb{T}^2} S_t^N(B_r(x)) \leq \frac{1}{\sqrt{N}} + \left( \frac{2\pi \mathcal{H}_N(X_t^N)}{\log \frac{1}{2r}} \right)^{1/2}.
$$

Combining this inequality with Lemma 3.4 we immediately get the desired limit. \hfill \Box

We also have the following energy estimate on the weak limits.

**Corollary 3.6.** It holds that

$$
\sup_{t \in [0,T]} \mathbb{E}\langle -G, \xi_t \otimes \xi_t \rangle < +\infty
$$

and hence, $\mathbb{P}$-a.s., $\xi \in L^2(0,T; H^{-1}(\mathbb{T}^2))$.

**Proof.** By the definition of the Hamiltonian,

$$
\mathcal{H}_N(X_t^N) = \frac{N}{N-1} c_0 + \langle -G, S_t^N \otimes S_t^N \rangle,
$$

where we have made the convention that $G(x) \equiv 0$ for all $x \in \mathbb{T}^2$. By Lemma 3.4 there is $L > 0$ such that

$$
\sup_{N \geq 2} \sup_{t \in [0,T]} \mathbb{E}\langle -G, S_t^N \otimes S_t^N \rangle \leq L.
$$

For any $\varepsilon > 0$, take $G_\varepsilon \in C^\infty(\mathbb{T}^2)$ such that $G_\varepsilon \geq G$ and $G_\varepsilon(x) = G(x)$ for all $|x| \geq \varepsilon$. Then,

$$
\mathbb{E}\langle -G_\varepsilon, S_t^N \otimes S_t^N \rangle \leq \mathbb{E}\langle -G, S_t^N \otimes S_t^N \rangle \leq L
$$

for any $N \geq 2$ and $t \in [0,T]$. As $G_\varepsilon$ is smooth on $\mathbb{T}^2$ we can apply the dominated convergence theorem to get $\mathbb{E}\langle -G_\varepsilon, \xi_t \otimes \xi_t \rangle \leq L$. By Fatou’s lemma, letting $\varepsilon \to 0$ yields

$$
\mathbb{E}\langle -G, \xi_t \otimes \xi_t \rangle \leq L \text{ for all } t \in [0,T]. \tag{3.10}
$$

Next, denoting by $\psi_t = -G \ast \xi_t$ the stream function, then

$$
\langle -G, \xi_t \otimes \xi_t \rangle = \langle \psi_t, \xi_t \rangle \tag{3.11}
$$

where $\langle \cdot, \cdot \rangle$ is now the duality between $\psi_t$ and $\xi_t$. If we use the Fourier expansion

$$
\xi_t = \sum_{k \in \mathbb{Z}_0^2} \langle \xi_t, e_k \rangle e_k,
$$

then we have

$$
\psi_t = \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}_0^2} \frac{1}{|k|^2} \langle \xi_t, e_k \rangle e_k,
$$

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and thus
\[ \langle \psi_t, \xi_t \rangle = \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}^2} \frac{1}{|k|^2} \langle \xi_t, e_k \rangle^2 = \frac{1}{4\pi^2} \| \xi_t \|^2_{H^{-1}}. \]

We obtain from (3.10) and (3.11) that
\[ \mathbb{E}(\| \xi_t \|^2_{H^{-1}}) \leq 4\pi^2 L \]
for all \( t \in [0, T] \). This concludes the proof.

Now we proceed to the proof of convergence of the nonlinear term in (3.3); recall that we omit the tilde over \( \tilde{S}^N \) and write \( N \) instead of \( N_i \). Take \( \rho \in C^\infty(\mathbb{R}_+, [0,1]) \) with support in \( [0,1] \) and \( \rho|_{[0, 1/2]} \equiv 1 \); for any \( \delta \in (0,1) \), let \( \rho_\delta(t) = \rho(t/\delta) \), \( t \in \mathbb{R} \). Then, we have the decomposition below:
\[ \langle S^N_s \otimes S^N_s, H_\phi \rangle = \int_{T^4} H_\phi(x,y) S^N_s(dx) S^N_s(dy) = I^N_1(s) + I^N_2(s), \]
where
\[ I^N_1(s) = \int_{T^4} H_\phi(x,y)(1 - \rho_\delta(|x-y|)) S^N_s(dx) S^N_s(dy), \]
\[ I^N_2(s) = \int_{T^4} H_\phi(x,y) \rho_\delta(|x-y|) S^N_s(dx) S^N_s(dy). \]

Note that \( |H_\phi(x,y)| \leq C \| \nabla^2 \phi \|_{\infty} \) for all \( x \neq y \), and \( H_\phi(x,x) \equiv 0 \) by convention; thus, \( \mathbb{P}\)-a.s., \( I^N_1(s) \) and \( I^N_2(s) \) are uniformly bounded by \( C \| \nabla^2 \phi \|_{\infty} \) for all \( s \in (0,T) \). Moreover, \( H_\phi(x,y)(1 - \rho_\delta(|x-y|)) \) is smooth on \( T^4 \); by the \( \mathbb{P}\)-a.s. convergence of \( S^N \) to \( \xi \) in the topology of \( C([0,T]; H^{-r}(T^2)) \) for all \( r > 1 \), we have, \( \mathbb{P}\)-a.s., for all \( s \in (0,T) \),
\[ \lim_{N \to \infty} I^N_1(s) = \int_{T^4} H_\phi(x,y)(1 - \rho_\delta(|x-y|)) \xi_s(dx) \xi_s(dy). \]

Next we estimate \( I^N_2(s) \):
\[ |I^N_2(s)| \leq C \| \nabla^2 \phi \|_{\infty} \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \rho_\delta(\|X_s^{N,i} - X_s^{N,j}\|) \]
\[ \leq C \| \nabla^2 \phi \|_{\infty} \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \rho_\delta(\|X_s^{N,i} - X_s^{N,j}\|) \frac{1}{\log \frac{1}{\delta}} \frac{1}{\log \frac{1}{\delta} \|X_s^{N,i} - X_s^{N,j}\|}, \]

where the second step is due to \( \rho_\delta(\|X_s^{N,i} - X_s^{N,j}\|) = 0 \) for \( \|X_s^{N,i} - X_s^{N,j}\| > \delta \). Using (3.9) and the fact that \( 0 \leq \rho_\delta \leq 1 \) we obtain
\[ |I^N_2(s)| \leq C \| \nabla^2 \phi \|_{\infty} \frac{2\pi}{\log \frac{1}{\delta}} \frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} [c_0 - G(X_s^{N,i} - X_s^{N,j})] \]
\[ = C \| \nabla^2 \phi \|_{\infty} \frac{2\pi}{\log \frac{1}{\delta}} \mathcal{H}_N(X^N_s). \]

Combining this estimate with Lemma 3.4 gives us
\[ \mathbb{E}|I^N_2(s)| \leq C' \| \nabla^2 \phi \|_{\infty} \frac{2\pi}{\log \frac{1}{\delta}} \text{ uniformly in } s \in (0,T), \ N \geq 1. \]
Summarizing the above arguments and applying Lebesgue’s dominated convergence theorem, we conclude that

\[
\lim_{N \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \langle S^N_s \otimes S^N_s, H_\phi \rangle \, ds - \int_0^t \langle \xi_s \otimes \xi_s, H_\phi \rangle \, ds \right| \right] \\
\leq C' T \frac{\|\nabla^2 \phi\|_\infty}{\log \frac{1}{\delta}} + \mathbb{E} \int_0^T \left| \int_0^t H_\phi(x,y) \rho_\delta(|x-y|) \xi_s(dx) \xi_s(dy) \right| \, ds.
\]

By the second assertion of Corollary 3.5, the right hand side vanishes as \( \delta \to 0 \), and thus

\[
\lim_{N \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \langle S^N_s \otimes S^N_s, H_\phi \rangle \, ds - \int_0^t \langle \xi_s \otimes \xi_s, H_\phi \rangle \, ds \right| \right] = 0.
\]

Combining the above limit with Proposition 3.3, we can finally let \( i \to \infty \) in (3.3) to yield, for all \( t \in [0,T] \),

\[
\langle \xi_t, \phi \rangle = \langle f_0, \phi \rangle + \int_0^t \langle \xi_s \otimes \xi_s, H_\phi \rangle \, ds + \nu \int_0^t \langle \xi_s, \Delta \phi \rangle \, ds.
\]

Thus \( \xi \) satisfies the weak vorticity formulation of the deterministic 2D Navier-Stokes equation with initial data \( f_0 \), and thus we complete the proof of Theorem 1.1.

Acknowledgements. The second author is grateful to the financial supports of the National Key R&D Program of China (No. 2020YFA0712700), the National Natural Science Foundation of China (Nos. 11931004, 12090014) and the Youth Innovation Promotion Association, CAS (Y2021002).

References

[1] M. Arnaudon, A. Thalmaier, F.-Y. Wang, Gradient estimates and Harnack inequalities on non-compact Riemannian manifolds. *Stoch. Proc. Appl.* 119 (2009), 3653–3670.

[2] P. Billingsley. Convergence of Probability Measures. Second edition. Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. *John Wiley & Sons, Inc., New York*, 1999.

[3] A. Cheskidov, X. Luo. Sharp nonuniqueness for the Navier-Stokes equations. [arXiv:2009.06596v1](https://arxiv.org/abs/2009.06596).

[4] M. Coghi, F. Flandoli, Propagation of chaos for interacting particles subject to environmental noise. *Ann. Appl. Probab.* 26 (2016), no. 3, 1407–1442.

[5] M. Coghi, M. Maurelli, Regularized vortex approximation for 2D Euler equations with transport noise. *Stoch. Dyn.* 20 (2020), no. 6, 2040002, 27 pp.

[6] F. Delarue, F. Flandoli, D. Vincenzi, Noise prevents collapse of Vlasov–Poisson point charges. *Comm. Pure Appl. Math.* 67 (2014), no. 10, 1700–1736.

[7] F. Flandoli, L. Galeati, D. Luo, Scaling limit of stochastic 2D Euler equations with transport noises to the deterministic Navier-Stokes equations. *J. Evol. Equ.* 21 (2021), no. 1, 567–600.

[8] F. Flandoli, M. Gubinelli, E. Priola. Full well-posedness of point vortex dynamics corresponding to stochastic 2D Euler equations. *Stoch. Proc. Appl.* 121 (2011), no. 7, 1445–1463.
[9] F. Flandoli, D. Luo, \( \rho \)-white noise solution to 2D stochastic Euler equations. *Probab. Theory Relat. Fields* **175** (2019), 783–832.

[10] F. Flandoli, D. Luo, Convergence of transport noise to Ornstein-Uhlenbeck for 2D Euler equations under the enstrophy measure. *Ann. Probab.* **48** (2020), no. 1, 264–295.

[11] N. Fournier, M. Hauray, S. Mischler, Propagation of chaos for the 2D viscous vortex model. *J. Eur. Math. Soc.* **16** (2014), no. 7, 1423–1466.

[12] L. Galeati, On the convergence of stochastic transport equations to a deterministic parabolic one. *Stoch. Partial Differ. Equ. Anal. Comput.* **8** (2020), no. 4, 833–868.

[13] M. Hauray, S. Mischler, On Kac’s chaos and related problems. *J. Funct. Anal.* **266** (2014), no. 10, 6055–6157.

[14] P.-E. Jabin, Zhenfu Wang, Mean field limit for stochastic particle systems. *Active particles. Vol. 1. Advances in theory, models, and applications*, 379–402, Model. Simul. Sci. Eng. Technol., Birkhäuser/Springer, Cham, 2017.

[15] P.-E. Jabin, Zhenfu Wang, Quantitative estimates of propagation of chaos for stochastic systems with \( W^{-1,\infty} \) kernels. *Invent. Math.* **214** (2018), no. 1, 523–591.

[16] H. Kunita, Stochastic flows and stochastic differential equations. Cambridge Studies in Advanced Mathematics, 24. *Cambridge University Press, Cambridge*, 1990.

[17] D. Luo, M. Saal, Regularization by noise for the point vortex model of mSQG equations. *Acta Math. Sin. (Engl. Ser.)* **37** (2021), no. 3, 408–422.

[18] C. Marchioro, M. Pulvirenti, Hydrodynamics in two dimensions and vortex theory. *Comm. Math. Phys.* **84** (1982), no. 4, 483–503.

[19] H. P. McKean, Propagation of chaos for a class of non-linear parabolic equations. 1967 Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967) pp. 41–57 Air Force Office Sci. Res., Arlington, Va.

[20] S. Méléard, Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models. Probabilistic models for nonlinear partial differential equations, 42–95, Lecture Notes in Math., 1627, Springer, Berlin, 1996.

[21] S. Méléard, Monte-Carlo approximations for 2d Navier-Stokes equations with measure initial data. *Probab. Theory Related Fields* **121** (2001), no. 3, 367–388.

[22] S. Mischler, C. Mouhot, Kac’s program in kinetic theory. *Invent. Math.* **193** (2013), no. 1, 1–147.

[23] H. Osada, Propagation of chaos for the two-dimensional Navier-Stokes equation. Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), 303–334, Academic Press, Boston, MA, 1987.

[24] M. Rosenzweig, The Mean-Field Limit of Stochastic Point Vortex Systems with Multiplicative Noise, [arXiv:2011.12180v1](https://arxiv.org/abs/2011.12180v1).

[25] S. Schochet, The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation. *Comm. Partial Differential Equations* **20** (1995), no. 5–6, 1077–1104.

[26] S. Schochet, The point-vortex method for periodic weak solutions of the 2-D Euler equations. *Comm. Pure Appl. Math.* **49** (1996), no. 9, 911–965.

[27] J. Simon. Compact sets in the space \( L^p(0,T;B) \). *Ann. Mat. Pura Appl.* **146** (1987), 65–96.

[28] D. W. Stroock, An Introduction to the Analysis of Paths on a Riemannian Manifold, Mathematical Surveys and Monographs, 74. *American Mathematical Society, Providence, RI*, 2000.
[29] A.-S. Sznitman, Topics in propagation of chaos. *École d’Été de Probabilités de Saint-Flour XIX–1989*, 165–251, Lecture Notes in Math., 1464, *Springer, Berlin*, 1991.