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Periodic Solution for a Complex-Valued Network Model with Discrete Delay

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1. Introduction

Recently, various complex-valued network models with or without time delays have been studied [1-4,6-20]. For example, Ji et al. have investigated the following complex-valued Wilson-Cowan neural network model:

\[ \begin{align*}
    w_1'(t) &= -w_1(t) + a_1g(w_1(t)) + a_2g(w_2(t - \tau)) + P \\
    w_2'(t) &= -w_2(t) + a_3g(w_1(t - \tau)) + a_4g(w_2(t)) + Q 
\end{align*} \]  

(1)

By using proper translations and coordinate transformations, system (1) has been decomposed the functions \( g(w_1) \), \( g(w_2) \) and \( a_1, a_2, a_3, a_4 \) into their real and imaginary parts, thus an equivalent real-valued system has been constructed. Then, the sufficient conditions for the Hopf bifurcation and its directions were provided [1]. Hang et al. have investigated a two-node network system as follows [2]:

\[ \begin{align*}
    Dz_1(t) &= -\mu z_1(t) + \alpha f\left( z_1(t - \tau) \right) + \beta f\left( z_2(t - \tau) \right) \\
    Dz_2(t) &= -\mu z_2(t) + \alpha f\left( z_1(t - \tau) \right) + \beta f\left( z_2(t - \tau) \right) 
\end{align*} \]  

(2)

About the dynamical behaviors, local asymptotical stability and the Hopf bifurcation were studied, the important conditions of emergence of bifurcation were also given. Li et al. [3] extended a real-valued network model into a complex-valued model as the following:
Regarding the discrete time delay as the bifurcating parameter, the problem of the Hopf bifurcation in the newly-proposed complex-valued neural network model was investigated under the assumption that the activation function can be separated into its real and imaginary parts. Based on the normal form theory and center manifold theorem, some sufficient conditions which determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions were established. Zhang et al. have considered a complex value delayed Hopfield neural networks model:

\[ f_{xy}(x,y) = \begin{cases} f_{xx}(x) + f_{yy}(y) & \text{if } x, y \text{ are real numbers,} \\
\end{cases} \]

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In this paper, we extend a real six-neuron network to the following complex-valued model:

\[ \begin{align*}
\dot{z}_1(t) &= (a_1 + ib_1)z_1(t) + (c_1 + id_1)f(z_1(t)) + (m_{11} + in_{11})f(w_1(t - \tau)) \\
\dot{z}_2(t) &= (a_2 + ib_2)z_2(t) + (c_2 + id_2)f(z_2(t)) + (m_{12} + in_{12})f(w_2(t - \tau)) \\
\dot{z}_3(t) &= (a_3 + ib_3)z_3(t) + (c_3 + id_3)f(z_3(t)) + (m_{13} + in_{13})f(w_3(t - \tau)) \\
\dot{z}_4(t) &= (a_4 + ib_4)z_4(t) + (c_4 + id_4)f(z_4(t)) + (m_{14} + in_{14})f(w_4(t - \tau)) \\
\dot{z}_5(t) &= (a_5 + ib_5)z_5(t) + (c_5 + id_5)f(z_5(t)) + (m_{15} + in_{15})f(w_5(t - \tau)) \\
\dot{z}_6(t) &= (a_6 + ib_6)z_6(t) + (c_6 + id_6)f(z_6(t)) + (m_{16} + in_{16})f(w_6(t - \tau)) \\
\end{align*} \]

where \( z_k = x_k + iy_k, k = 1, 2, ..., 6 \) are real numbers, \( f_{kj} \) are activation functions, \( k, j = 1, 2, ..., 6 \). We will discuss the dynamic behavior of the solutions of system (5).

We point out that the bifurcating method is not easy to deal with system (5) if all \( a_j, b_j, m_{kj}, n_{kj} \) are different real numbers. In this paper, by means of the mathematical analysis method, we discuss the periodic oscillation for system (5). For convenience, let \( f_{kj}(t - \tau) = \int_0^{\tau} f_{kj}(t - \tau) \), then the complex-valued system (5) can be expressed by separating it into real and imaginary parts as the following:

\[ \begin{align*}
\dot{x}_1(t) &= a_1x_1(t) + b_1y_1(t) + m_{11}x_1(t - \tau) + m_{12}x_2(t - \tau) + m_{13}x_3(t - \tau) + m_{14}x_4(t - \tau) + m_{15}x_5(t - \tau) + m_{16}x_6(t - \tau) \\
\dot{y}_1(t) &= c_1x_1(t) + d_1y_1(t) + n_{11}x_1(t - \tau) + n_{12}x_2(t - \tau) + n_{13}x_3(t - \tau) + n_{14}x_4(t - \tau) + n_{15}x_5(t - \tau) + n_{16}x_6(t - \tau) \\
\end{align*} \]

Therefore, in order to discuss the periodic solution of model (5), we only consider the periodic solution of system (6). Suppose that the derivative of \( f_{kj}(t) \) with respect to \( x_j \) and \( y_j \) exist, continuous, and \( f_{kj}(0) = 0, f_{kj}'(0) = 0 \). Then the linearized system of (6) is the following:

\[ \begin{align*}
\dot{x}_j(t) &= a_jx_j(t) + b_jy_j(t) + m_{1j}x_j(t - \tau) + m_{2j}y_j(t - \tau) + m_{3j}x_j(t - \tau) + m_{4j}y_j(t - \tau) + m_{5j}x_j(t - \tau) + m_{6j}y_j(t - \tau) \\
\dot{y}_j(t) &= c_jx_j(t) + d_jy_j(t) + n_{1j}x_j(t - \tau) + n_{2j}y_j(t - \tau) + n_{3j}x_j(t - \tau) + n_{4j}y_j(t - \tau) + n_{5j}x_j(t - \tau) + n_{6j}y_j(t - \tau) \\
\end{align*} \]

The matrix form of system (7) is the following:

\[ U'(t) = AU(t) + BU(t - \tau) \]

where \( U(t) = [x_1(t), y_1(t), ..., x_6(t), y_6(t)]^T, U(t - \tau) = [x_1(t - \tau), y_1(t - \tau), ..., x_6(t - \tau), y_6(t - \tau)]^T \). Both A and B are 12 by 12 matrices as follows:

\[ A = \begin{pmatrix} \end{pmatrix}, B = \begin{pmatrix} \end{pmatrix}. \]
2. Preliminaries

Lemma 1 Assume that \( a_j > 0, b_j > 0, f^{(k)}(0, 0) = 0, f^{(j)}(0, 0) = 0 \), \( f^{(k)}(x_j, y_j) > 0, f^{(j)}(x_j, y_j) > 0 \) when \( x_j > 0, y_j > 0 \), while \( f^{(k)}(x_j, y_j) < 0, f^{(j)}(x_j, y_j) < 0 \) when \( x_j < 0, y_j < 0 \) \((k, j = 1, 2, \ldots, 6)\). C\( = A + B \) is a nonsingular matrix, then system (6) has a unique equilibrium.

Proof An equilibrium \( U^* = [x_1^*, y_1^*, \ldots, x_6^*, y_6^*]^T \) of system (8) is a constant solution of the following algebraic equation:

\[
AU^* + BU^* = CU^* = 0
\]

If \( C = A + B \) is a nonsingular matrix, then system (9) only has zero solution according to the linear algebra basic theorem. Noting that \( f^{(k)}(0, 0) = 0, f^{(j)}(0, 0) = 0 \) \((k, j = 1, 2, \ldots, 6)\). Therefore, zero is a solution of system (6). Obviously, zero is the unique equilibrium of system (6) since \( f^{(k)}(x_j, y_j) > 0, f^{(j)}(x_j, y_j) > 0 \) when \( x_j > 0, y_j > 0 \), while \( f^{(k)}(x_j, y_j) < 0, f^{(j)}(x_j, y_j) < 0 \) when \( x_j < 0, y_j < 0 \) \((k, j = 1, 2, \ldots, 6)\). Then all solutions of system (6) are uniformly bounded.

Lemma 2 Assume that \( f^{(k)}(x_j, y_j), f^{(j)}(x_j, y_j) \; (k, j = 1, 2, \ldots, 6) \) are continuous bounded functions, \( a_j > 0, b_j > 0 \) \((k, j = 1, 2, \ldots, 6)\). Then all solutions of system (6) are uniformly bounded.

Proof Since \( f^{(k)}(x_j, y_j), f^{(j)}(x_j, y_j) \; (k, j = 1, 2, \ldots, 6) \) are continuous bounded functions, then from system (6) we have

\[
\begin{align*}
-x_t^j(t) & \leq -a_j x_t^j(t) + b_j x_t^j(t) + N_{ij} \\
y_t^j(t) & \leq -b_j y_t^j(t) - a_j y_t^j(t) + N_{ij}
\end{align*}
\]

where \( N_{ij}, N_{ij} \) are some positive constants. It is easily to see that all solutions of system (10) are uniformly bounded with \( a_j > 0, b_j > 0 \) \((k, j = 1, 2, \ldots, 6)\), implying that all solutions of system (6) are uniformly bounded.

3. Main Results

It is known that the instability of the trivial solution of system (7) guarantees the instability of the trivial solution of system (6). Thus, we have the following theorems.

Theorem 1 Assume that Lemma 1 and Lemma 2 hold for selecting parameter values of \( a_j, b_j, m_{ij}, n_{ij} \; (k, j = 1, 2, \ldots, 6) \). Let the eigenvalues of matrices \( A, B \) be \( \alpha_j, \beta_j \; (j = 1, 2, \ldots, 6) \) respectively. If there exists at least one eigenvalue \( \beta_k, k \in \{1, 2, \ldots, 6\} \) such that

\[
\beta_k > 0 \quad \text{or} \quad \text{Re} (\beta_j) > a_k, \quad \text{where} \quad a_k = \min_{1 \leq j \leq 6} \{ -a_j \}. \quad \text{Then system (6) generates a periodic oscillatory solution.}
\]

Proof Obviously, we only need to consider the instability of the trivial solution of system (7). Suppose that the eigenvalues of matrix \( \Lambda \) are \( \alpha_j \) then \( \alpha_1 = a_1 + ib_1, \alpha_2 = a_2 + ib_2, \ldots, \alpha_6 = a_6 + ib_6 \). Therefore, the characteristic equation corresponding to system (8) is the following

\[
\prod_{j=1}^{12} (\lambda - a_j - \beta e^{-j\lambda}) = 0 \quad (11)
\]

Noting that \( \text{Re} (\alpha_j) = -a_j < 0 \), and there exists some \( \beta_k > 0 \) or \( \text{Re} (\beta_j) > a_k \) thus system (11) has a positive real eigenvalue or an eigenvalue which has a positive real part. Therefore, the trivial solution of system (8) (or (7)) is unstable according to the basic result of delayed differential equation, implying that the trivial solution of system (6) is unstable. Since system (6) has a unique equilibrium point and all solutions are bounded, based on the extended Chafee’s criterion \([21, 22]\), this instability of the trivial solution will force system (6) to generate a limit cycle, namely, a periodic oscillatory solution.

Now set \( \sigma = \max_{1 \leq j \leq 6} \{ -a_j - b_j \}, \quad b_j = \max_{1 \leq j \leq 6, 1 \leq i \leq 12} \sum_{1 \leq j \leq 6} lb_{ij} \).

Then we have

Theorem 2 Assume that Lemma 1 and Lemma 2 hold for selecting parameter values of \( a_j, b_j, m_{ij}, n_{ij} \; (k, j = 1, 2, \ldots, 6) \). If

\[
\sigma + b > 0 \quad (12)
\]

Then the unique equilibrium point of system (6) is unstable, implying that system (6) generates a periodic oscillatory solution.

Proof Similar to Theorem 1, we show that the trivial solution of system (7) is unstable, then the trivial solution of system (6) also is unstable. In system (7), let \( v(t) = \sum_{j=1}^{6} (x_j(t) + y_j(t)) \), then we have
\[
\frac{dv(t)}{dt} \leq \sigma v(t) + bv(t - \tau) \tag{13}
\]

Corresponding to equation (13), we consider the following equation
\[
\frac{dw(t)}{dt} = \sigma w(t) + bw(t - \tau) \tag{14}
\]

The characteristic equation associated with equation (14) is
\[
\lambda = \sigma + be^{-\lambda t} \tag{15}
\]

We claim that there exists a positive root of (15). Let \( f(\lambda) = \lambda - \sigma - be^{-\lambda t} \). Obviously, \( f(\lambda) \) is a continuous function of \( \lambda \). When \( \lambda = 0 \), we get \( f(0) = -\sigma - b = -((\sigma + b) < 0 \) since \( \sigma + b > 0 \). On the other hand, there exists a suitably large \( \lambda \), say \( \lambda_1 > 0 \) such that \( f(\lambda_1) = \lambda_1 - \sigma - be^{-\lambda_1 t} > 0 \) since \( \lim_{\lambda \to \infty} e^{-\lambda t} = 0 \). Based on the Intermediate Value Theorem, there exists a \( \lambda_0 \in (0, \lambda_1) \) such that \( f(\lambda_0) = \lambda_0 - \sigma - be^{-\lambda_0 t} = 0 \). In other words, \( \lambda_0 \) is a positive characteristic root of equation (15). Therefore, the trivial solution of equation (14) is unstable. Noting that \( v(t) \leq w(t) \). So the instability of the trivial solution of (14) implies that the trivial solution of system (7) (thus system (6)) is unstable. This instability of the trivial solution such that system (6) has a limit cycle, namely, a periodic oscillatory solution.

4. Simulation Result

This simulation is based on system (6). We first select the parameters as \( a_1=0.45, a_2=0.65, a_3=0.48, a_4=0.35, a_5=0.25, a_6=0.18, b_1=0.24, b_2=0.56, b_3=0.24, b_4=0.32, b_5=0.45, b_6=0.32; m_{14}=0.56, m_{15}=0.42, n_{14}=0.12, n_{15}=0.32, m_{24}=-0.65, m_{25}=0.36, m_{26}=0.55, n_{24}=0.32, n_{25}=0.48, n_{26}=0.25 m_{34}=0.65, m_{35}=0.45, n_{34}=0.36, n_{35}=0.36, m_{41}=0.35, m_{42}=0.58, n_{41}=0.35, n_{42}=0.65, m_{51}=0.68, m_{52}=0.56, m_{53}=0.32, n_{51}=0.45, n_{52}=0.25, n_{53}=0.25, m_{62}=0.48, m_{63}=0.32, n_{62}=0.12, n_{63}=0.18 \). The activation functions are as in Figure 3, time delay is 0.5. We see that \( \sigma = 0.5, b = 7.44 \). Therefore, \( \sigma + b > 0 \) holds. Based on Theorem 2, there exists a periodic oscillatory solution (see Figure 4).

![Figure 1. Oscillation of the solutions, activation function: tanh (x), time delay: 0.5.](image-url)
5. Conclusions

The paper has discussed the oscillatory behavior of the solutions for a complex-valued neural network model with discrete delay. By means of the mathematical analysis method, two criteria to guarantee the existence of periodic oscillatory solution are provided which are easy to be checked. In this network, we decomposed the activation functions and connection weights into their real and imaginary parts, so as to discuss an equivalent real-valued system. The activation

Figure 2. Oscillation of the solutions, activation function: tanh (z), time delay: 1.5.

Figure 3. Oscillation of the solutions, activation function: arctan (z), time delay: 0.5.

Figure 4. Oscillation of the solutions, activation function: arctan (z), time delay: 0.6.
functions affect the oscillatory behavior slightly.

**Conflict of Interest**

The author declares no conflict of interest.

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