Calculation of Exclusive Cross Sections with the Lorentz Integral Transform Method

Alessandro La Piana\textsuperscript{1} and Winfried Leidemann\textsuperscript{1,2}
\textsuperscript{1}Dipartimento di Fisica, Università di Trento, I-38050 Povo, Italy;
\textsuperscript{2}Istituto Nazionale di Fisica Nucleare, Gruppo collegato di Trento.
(March 30, 2022)

Abstract

The longitudinal structure function of the $d(e,e'p)$ exclusive cross section is calculated with the Lorentz integral transform method. In this approach final state interaction is fully taken into account, but without using a final state wave function. Cross sections are obtained via the inversion of the transform. It is shown that the inversion results are very stable. The comparison to a conventional calculation with an explicit $np$ final state wave function shows that the obtained results are also very precise. Thus the method opens up the possibility to obtain exclusive cross sections for reactions with more than two particles, where it is generally very difficult to calculate the exact final state wave function.

21.45.+v, 25.30.Fj, 02.30.Qy

Keywords: few-body, final state interaction, exclusive reaction, Lorentz integral transform
I. INTRODUCTION

The study of inclusive and exclusive cross sections in inelastic reactions is an essential tool in understanding the underlying dynamics of a particle system. For systems with more than two constituents a major problem in the calculation of such reactions consists in the exact knowledge of the final state wave function in the continuum. Its calculation is far more difficult than the corresponding bound state calculation. As a matter of fact, today a computation of an intermediate energy continuum wave function is out of reach for a system with more than three particles. An exact calculation, however, can be carried out in an alternative way by using proper integral transforms. They allow one to take into account final state interactions (FSI) rigorously without using final state wave functions explicitly. In fact in recent years the Lorentz integral transform (LIT) method [1] has been successfully applied to various inclusive breakup cross sections in few-body physics. The LIT is expressed in terms of square integrable functions which are obtained from inhomogeneous differential equations. The differential equations can be solved with similar methods as a bound state problem. After having calculated the transform one obtains the cross section by inversion of the transform. A recent overview of the results obtained with the LIT method is given in [2].

The LIT method has not yet been used for the calculation of exclusive cross sections. The only exception is a calculation of the \(^4\)He spectral function [3], but in this case the method proceeds along the same lines as for an inclusive reaction. However, it is in principle possible to apply the integral transform method to general exclusive reactions as shown by Efros for the Stieltjes transform [4]. In the present paper we illustrate the details of the calculation for the \(d(e', ep)\) reaction with the LIT. It will also allow us to check the precision of the obtained results by comparing them to those of a conventional calculation with an explicit \(np\) final state wave function. Such a reliability check is necessary, since the use of integral transforms is not always unproblematic for the calculation of cross sections. In fact one may encounter problems in the inversion of the transform. For example in Ref. [5] it was shown for the case of an inclusive reaction that the Stieltjes transform is not very appropriated, since it samples contributions over a large energy range making the inversion extremely difficult. Later the LIT was proposed and it was shown for the test case of the longitudinal inelastic deuteron form factor in electron scattering that inclusive reactions can be safely calculated with this method. Whether the LIT is as appropriated also for exclusive reactions cannot be said a priori, since such a calculation is more complicated. The aim of the present work is to investigate this question.

The paper is organized as follows. In Sec. II we describe how cross sections are calculated with the LIT. Details of the calculation for the specific reaction under consideration \((d(e, e'p)n)\) are given in Sec. III. The results are illustrated in Sec. IV and a conclusion is drawn in Sec. V.

II. THE LIT METHOD

The starting point of the LIT method [1] is the calculation of an integral transform with a Lorentz kernel
\[ L(\sigma) = \int d\omega \frac{F(\omega)}{(w - \sigma_R)^2 + \sigma_I^2}. \]  

(1)

The function \( F \) depends on the internal excitation energy \( \omega = E_f - E_0 \) of a given particle system and contains information about the transition of the system from the ground state \( |\Psi_0\rangle \), with energy \( E_0 \), to the final state \( |\Psi_f\rangle \), with energy \( E_f \), induced by an external probe. In case of an inclusive reaction \( F(\omega) \) denotes the response function

\[ F(\omega) = \int d\Psi_f |\langle \Psi_f |\hat{O} |\Psi_0\rangle|^2 \delta(E_f - E_0 - \omega), \]  

(2)

where \( \hat{O} \) is a transition operator which characterizes the specific process under consideration.

The key point of the LIT method is an evaluation of \( L(\sigma) \) without explicit knowledge of \( F(\omega) \). In a second step the function \( F \) is obtained from the inversion of the transform. The great advantage of the method lies in the fact that a calculation of the generally very complicated final state wave function \( |\Psi_f\rangle \) can be avoided as will be discussed below. On the contrary a conventional calculation of \( F(\omega) \) can only be carried out with the explicit knowledge of \( |\Psi_f\rangle \).

Using completeness one can show that in order to obtain \( L(\sigma) \) one has to solve the following differential equation

\[ (H - E_0 - \sigma^*)|\tilde{\Psi}_1(\sigma)\rangle = \hat{O}|\Psi_0\rangle \]  

(3)

with

\[ \sigma = \sigma_R + i\sigma_I \quad \sigma_R, \sigma_I > 0, \]  

(4)

where \( H = T + V \) is the Hamiltonian of the system under consideration. Note that the corresponding homogeneous equation has only the trivial solution, since \( H \) has a real eigenvalue spectrum.

The norm of the solution \( \tilde{\Psi}_1(\sigma) \) determines the LIT directly:

\[ L(\sigma) = \langle \tilde{\Psi}_1(\sigma)|\tilde{\Psi}_1(\sigma)\rangle. \]  

(5)

Different from a Schrödinger equation at positive energies, one has for the solution of Eq. (3) a very simple boundary condition. Due to the localized source at the right hand side (rhs) of Eq. (3) \( \tilde{\Psi}_1(\sigma) \) vanishes at large distances similar to a bound state wave function. Therefore one can apply the same techniques as for the calculation of a bound state wave function.

The response function \( F(\omega) \) serves only for the determination of inclusive cross sections. For an exclusive process one needs a more detailed information about the transition of the system. In fact one has to be able to calculate transition matrix elements of the form

\[ T_{fi}(E_f) = \langle \Psi_f |\hat{O} |\Psi_0\rangle. \]  

(6)

How such a calculation can be carried out with an integral transform was shown for the case of the Stieltjes transform \( [4] \). For the LIT the calculation proceeds analogously as outlined in the following (see also \([3,4]\)).
For simplicity we will consider an exclusive reaction leading to a final state with two fragments, but the method can be applied also for channels with more than two particles. Besides the correct final state wave function $|\Psi_f\rangle$ we also introduce the corresponding plane wave

$$\Phi_{PW}(\vec{r}) = \mathcal{A}\Psi_1\Psi_2\frac{\exp(i\vec{k}\cdot\vec{r})}{(2\pi)^{3/2}},$$  \hspace{1cm} (7)

where $\mathcal{A}$ is a proper antisymmetrizer and $\Psi_1$ and $\Psi_2$ are the internal wave functions of the two fragments, while $\vec{k}$ and $\vec{r}$ are the usual relative coordinates for momentum and position of the two-body system formed by the two fragments.

In order to calculate the $T$ matrix element one starts from the Lippmann-Schwinger equation for the final state

$$\langle \Psi_f | = \langle \Phi_{PW} | + \langle \Phi_{PW} | \hat{V} \frac{1}{E_f + i\epsilon - H}$$  \hspace{1cm} (8)

with

$$\hat{V} = \sum_{i,j} \hat{V}_{ij}, \quad i \in \mathcal{F}_1, j \in \mathcal{F}_2,$$  \hspace{1cm} (9)

where $\mathcal{F}_1$ and $\mathcal{F}_2$ contain all particles of the first and second fragment, respectively. Inserting the above expression in Eq. (3) one obtains a sum of two pieces, a Born term

$$T_{fi}^{\text{Born}}(E_f) = \langle \Phi_{PW} | \hat{O} \Psi_0 \rangle,$$  \hspace{1cm} (10)

and a term depending on FSI

$$T_{fi}^{\text{FSI}}(E_f) = \langle \Phi_{PW} | \hat{V} \frac{1}{E_f + i\epsilon - H} \hat{O} | \Psi_0 \rangle.$$  \hspace{1cm} (11)

The calculation of the Born term is rather simple. The main difficulty of the calculation is the determination of the matrix element depending on FSI.

Using the completeness of the eigenstates $|\Psi(E)\rangle$ of $H$ one can rewrite $T_{fi}^{\text{FSI}}$ as follows

$$T_{fi}^{\text{FSI}}(E_f) = \int dE \langle \Phi_{PW} | \hat{V} | \Psi(E) \rangle \langle \Psi(E) | \frac{1}{E_f + i\epsilon - H} \hat{O} | \Psi_0 \rangle$$

$$= \int dE F_{fi}(E) \frac{1}{E_f + i\epsilon - E},$$  \hspace{1cm} (12)

with

$$F_{fi}(E) = \langle \Phi_{PW} | \hat{V} | \Psi(E) \rangle \langle \Psi(E) | \hat{O} | \Psi_0 \rangle.$$  \hspace{1cm} (13)

One obtains the following formal solution for the FSI term

$$T_{fi}^{\text{FSI}}(E_f) = -i\pi F_{fi}(E_f) + \mathcal{P} \int_{E_0}^{\infty} dE \frac{F_{fi}(E)}{E_f - E}.$$  \hspace{1cm} (14)
For a calculation of $T_{fi}$ one needs to know $F_{fi}$ for any given energy. A direct calculation of $F_{fi}$ is of course in general far too difficult, since one has to determine final state wave functions $\Psi$ for the whole eigenvalue spectrum of $H$. On the other hand an indirect calculation via the LIT is possible. To this end one performs a Lorentz integral transform of $F_{fi}$, i.e.

$$L(\sigma) = \int_{E_0}^{\infty} dE \frac{F_{fi}(E)}{(E - \sigma)(E - \sigma^*)}, \quad \sigma_R > E_0.$$  \hspace{1cm} (15)

Inserting the definition of $F_{fi}$ one finds

$$L(\sigma) = \int_{E_0}^{\infty} dE \langle \Phi^{PW} | \hat{V} \frac{1}{H - \sigma} | \Psi(E) \rangle \langle \Psi(E) | \frac{1}{H - \sigma^*} \hat{O} | \Psi_0 \rangle$$

$$= \langle \Phi^{PW} | \hat{V} \frac{1}{H - \sigma} \frac{1}{H - \sigma^*} \hat{O} | \Psi_0 \rangle$$

$$= \langle \tilde{\Psi}_2(\sigma) | \tilde{\Psi}_1(\sigma) \rangle , \quad (16)$$

with

$$| \tilde{\Psi}_1(\sigma) \rangle = \frac{1}{H - \sigma_R + i\sigma_I} \hat{O} | \Psi_0 \rangle , \quad (17)$$
$$| \tilde{\Psi}_2(\sigma) \rangle = \frac{1}{H - \sigma_R + i\sigma_I} \hat{V} | \Phi^{PW} \rangle . \quad (18)$$

It is evident that (17) leads essentially to the same differential equation as for the inclusive process:

$$(H - \sigma_R + i\sigma_I) | \tilde{\Psi}_1(\sigma) \rangle = \hat{O} | \Psi_0 \rangle . \quad (19)$$

From Eq. (18) one obtains

$$\langle H - \sigma_R + i\sigma_I | \tilde{\Psi}_2(\sigma) \rangle = \hat{V} | \Phi^{PW} \rangle , \quad (20)$$

which is similar to Eq. (19), but for a different source term on the rhs. For a finite range potential one has also in this case a vanishing source term for large distances. This guarantees also for $\tilde{\Psi}_2$ an asymptotic boundary condition similar to a ground state problem. If $\hat{V}$ contains also the Coulomb potential one cannot proceed exactly in the same way as shown here. In this case one has to start from a modified Lippmann-Schwinger equation, where Coulomb wave function are taken into account \[8,7\].

As shown by Efros \[4,7\] the integral transform method can be extended to exclusive processes with more than two fragments. In principle one obtains equations similar to Eqs. (16,19,20). However, one cannot guarantee, as in the two fragment case, that the potential $\hat{V}$ vanishes asymptotically, since two of the fragments could remain close to each other. Therefore it is necessary to choose a different solution to the problem. In fact one can rewrite $L(\sigma)$ as follows

$$L(\sigma) = \langle \Phi^{PW} | \hat{V} | \tilde{\Psi}_1 \rangle , \quad (21)$$

where $| \tilde{\Psi}_1 \rangle$ is obtained from the solution of the following differential equation
\[(H - \sigma_R - i\sigma_I)|\tilde{\Psi}_1(\sigma)\rangle = |\tilde{\Psi}_1(\sigma)\rangle. \tag{22}\]

Since \(|\tilde{\Psi}_1\rangle\) vanishes at large distances, one can again use bound state methods for the solution of this differential equation.

There is another possibility to calculate the LIT \[\text{[7]}\]. Starting from the following identity
\[\frac{1}{(H - \sigma)(H - \sigma^*)} = \frac{1}{2i\sigma_I}\left(\frac{1}{H - \sigma} - \frac{1}{H - \sigma^*}\right), \tag{23}\]
and defining
\[|\tilde{\Psi}'_1(\sigma)\rangle = \frac{1}{H - \sigma_R - i\sigma_I} \hat{O} |\Psi_0\rangle, \tag{24}\]
one gets for the transform
\[L(\sigma) = \frac{1}{2i\sigma_I}\left(\langle \Phi^{PW} | \hat{V} | \tilde{\Psi}'_1(\sigma)\rangle - \langle \Phi^{PW} | \hat{V} | \tilde{\Psi}_1(\sigma)\rangle\right). \tag{25}\]

It is seen that one has to solve only one type of differential equation. In fact for the solution of \(|\tilde{\Psi}'_1(\sigma)\rangle\) it is sufficient to solve Eq. (19) a second time replacing \(\sigma^*\) by \(\sigma\).

Which of the two approaches for a nonvanishing source term is more appropriate depends also on the possibility to obtain a precise numerical solution. Small errors in the solution of Eq. (19) could lead in both cases to larger errors for the calculation of the LIT. Therefore we will also consider in Sec. [IV] these additional possibilities to calculate the LIT for our realistic test case of the electromagnetic deuteron breakup.

**III. EXCLUSIVE DEUTERON BREAKUP**

The exclusive deuteron breakup \(d(e,e'p)n\) is governed by the four structure functions \(f_L, f_T, f_{LT}, \text{and } f_{TT}\). In the following we will only consider the longitudinal \(f_L\). We use the same notation as in Ref. \[\text{[8]}\], where the structure functions are calculated in the final np c.m. system with \(f_L = f_L(E_{np}, |\vec{q}_{c.m.}|, \theta)\). In this system \(\vec{q}_{c.m.}\) denotes the momentum transfer, \(E_{np}\) the relative np energy, and the relative np momentum \(\vec{k}\) has the angle \(\theta\) with respect to \(\vec{q}_{c.m.}\). The structure functions are expressed in terms of the transition matrix \(T_{S\mu\mu_\text{np}m_d}\) for the process \(e + d \rightarrow e' + np\). The quantum numbers \(S\) and \(m\) denote the spin and spin projection of the outgoing np pair with respect to \(\vec{k}\), \(\mu\) characterizes the transition operator \(\hat{O}\), and \(m_d\) is the projection of the deuteron spin with respect to \(\vec{q}_{c.m.}\). For the longitudinal structure function, i.e. \(\mu = 0\), one has the following transition operator
\[\hat{O} = \sum_j G_{E,j}(q_{j\mu}^2) \exp(i\vec{q}_{c.m.} \cdot \vec{r}_j), \tag{26}\]
where \(G_{E,j}\) denotes the electric form factor of the j-th nucleon with \(q_{j\mu}^2\) being the four-momentum transfer squared. Here we use the electric dipole form factor for the proton, while the neutron electric form factor is set to zero. It is evident that the above operator does not affect the spin, i.e. \(S = 1\), and one gets \[\text{[8]}\].
\[ f_L(E_{np}, q_{c.m.}, \theta) = \sum_{mm_d} T_{1m0m_d} T_{1m0m_d}^* , \]

where

\[ T_{1m0m_d} = C \langle 1m | \hat{O} | m_d \rangle \]

with

\[ C = -(2\pi)^{3/2} \sqrt{\frac{kM\alpha}{4\pi}}, \]

where M denotes the nucleon mass and \( \alpha \) is the fine structure constant. Like \( f_L \) the transition matrix depends on \( E_{np}, q_{c.m.}, \) and \( \theta \). In the following we suppress the dependences on \( q_{c.m.} \) and \( \theta \), while the dependence on \( E_{np} \) is made explicit in most cases.

The aim of the present work is a test of the LIT method for the exclusive deuteron breakup. Since there is no need to obtain results with a realistic potential we choose the semi-realistic TN potential [2] for this test. It is a central potential model, which is different for the various spin-isospin (ST) channels, i.e. \( V(r) = \sum_{ST} V^{ST}(r) \). Since we have \( S = 1 \), only \( V^{10} \) and \( V^{11} \) have to be considered here. In addition \( V^{11}(r) = 0 \) for the TN potential, hence FSI effects appear only in the channel \( ST = 10 \).

Due to the absence of the tensor force in the TN potential one obtains a reduced complexity of the equations to be solved. Nevertheless, as calculations with this potential model show (see e.g. [2]) results are sufficiently realistic in order to serve as test case.

As discussed in Sec. II there are two contributions for any \( T \)-matrix element, a Born and an FSI term. Considering an \( s \)-wave deuteron with radial wave function \( u(r) \) one obtains for the Born term

\[ T_{1m0m_d} = (-)^{md} \delta_{mm_d} G_{E,p} \sqrt{3kMY_{00}(\hat{k}_-)} \int_0^\infty dr u(r) j_0(k_-r), \]

where \( j_0 \) denotes the spherical Bessel function of order 0 and

\[ \hat{k}_- = \hat{k} - \hat{q}_{c.m.} \]

For the second piece, \( T^{\text{FSI}} \), it is necessary to perform a multipole decomposition. It is convenient to introduce the following expansions with projections \( M \) with respect to \( \hat{q}_{c.m.} \).

\[ \hat{\Psi}_{1,M} = \sum_{jL} i^L [Y^{[L]}(\hat{r}) \times \chi^{[1]}(\vec{\sigma}_1, \vec{\sigma}_2)]^{[j]}_M \sqrt{2L+1} C^{L1j}_{0M} r^{-1} \tilde{\psi}^{(1)}_{L_1j}(r), \]

\[ \hat{\Psi}_{2,M} = \sum_{jL,m_l} i^L [Y^{[L]}(\hat{r}) \times \chi^{[1]}(\vec{\sigma}_1, \vec{\sigma}_2)]^{[j]}_M C^{L1j}_{m_lm_l} Y^{*}_{lm_1}(\hat{k}) r^{-1} \tilde{\psi}^{(2)}_{L_1j}(r), \]

where \( \chi^{[1]}(\vec{\sigma}_1, \vec{\sigma}_2) \) denotes the spin wave function for a two-nucleon system with \( S = 1 \). For the rhs of the differential equations (19) and (20) we perform similar expansions as for \( \hat{\Psi}_1 \) and \( \hat{\Psi}_2 \) leading to

\[ \hat{O}|m_d\rangle = \sum_{jL} i^L [Y^{[L]}(\hat{r}) \times \chi^{[1]}(\vec{\sigma}_1, \vec{\sigma}_2)]^{[j]}_m \sqrt{2L+1} C^{L1j}_{0m_d} r^{-1} f_L(r), \]

\[ f_L(r) = j_L\left(\frac{q_{c.m.} r}{2}\right) u(r) \]
\[
\hat{V}|\Phi_{M}^{\text{PW}}\rangle = \sum_{j,l,m_l} \hat{i}^l [Y_{1l}^{[1]}(\hat{r}) \times \chi^{[1]}(\sigma_1, \sigma_2)]_{jM} C_{m_l m_M}^{jM} Y_{l m_l}^* (\hat{k}) r^{-1} g_{lj}(r),
\]

where \( V_{jl} \) is the potential for the NN partial wave \(^3l_j\).

For \( \tilde{\Psi}_2 \) the above multipole decompositions lead to the following coupled differential equations in real and imaginary parts

\[
\left\{ \begin{array}{l}
-\frac{\hbar^2}{M} \left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) + V_{ij}(r) - \sigma_R \Re[\tilde{\psi}_{lj}^{(1)}(\sigma, r)] - \sigma_I \Im[\tilde{\psi}_{lj}^{(1)}(\sigma, r)] = g_{lj}(r) \\
-\frac{\hbar^2}{M} \left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) + V_{ij}(r) - \sigma_R \Im[\tilde{\psi}_{lj}^{(1)}(\sigma, r)] + \sigma_I \Re[\tilde{\psi}_{lj}^{(1)}(\sigma, r)] = 0
\end{array} \right. \quad (38)
\]

As mentioned above, because of our potential model we have to consider only the channel with \( ST=10 \), thus \( V_{ij}(r) \) can be replaced by \( V^{10}(r) \). In addition the Pauli principle has to be fulfilled, i.e. \( S+T+l \) has to be odd. Therefore the differential equation has only to be solved for \( l \) even. For channels with \( l \) odd one had to consider \( V^{11}(r) \), but as already mentioned \( V^{11} \) is zero in our potential model. Note that there is no explicit dependence on \( j \) in the coupled differential equation, thus one has \( \tilde{\psi}_{lj}^{(1)} = \tilde{\psi}_{lj}^{(1)} \).

For \( \tilde{\Psi}_1 \) one finds very similar equations with the only difference that one has to replace \( g_{lj}(r) \) by \( f_{Lj}(r) \) on the rhs. Also here we have \( \tilde{\psi}_{Lj}^{(1)} = \tilde{\psi}_{Lj}^{(1)} \).

We solve the differential equation by adding an additional homogeneous equation determining the source terms on the rhs. In this way we obtain a coupled homogeneous differential equation system. The numerical solution leads to very precise results as shown in Ref. [6].

With the solutions for \( \tilde{\psi}_{Lj}^{(1)} \) and \( \tilde{\psi}_{lj}^{(1)} \) one obtains for the scalar product (16)

\[
\langle \tilde{\Psi}_2 | \tilde{\Psi}_1 \rangle = \sum_{m_m d_m l_m} \sqrt{2l+1} Y_{l m_l}^* (\hat{k}) \sum_j C_{m_m m_M}^{[1]j} C_{d_m d_M}^{[1]l} \tilde{R}_{lj}(\sigma)
\]

with

\[
\tilde{R}_{lj}(\sigma) = \int_0^\infty dr (\tilde{\psi}_{lj}^{(2)}(\sigma, r))^* \tilde{\psi}_{Lj}^{(1)}(\sigma, r) \delta_{Ll}.
\]

To calculate the FSI contribution to the \( T \)-matrix elements one has to invert the LIT

\[
\tilde{R}_{lj}(\sigma) = \int_{E_0}^\infty dE \frac{R_{lj}(E)}{(E - \sigma_R)^2 + \sigma_I^2}
\]

in order to obtain the function \( R_{lj}(E) \). The transform can be inverted using the following ansatz

\[
R_{lj}(E) = \sum_{n=1}^N c_{n,lj} \chi_{n,lj}(E, \beta),
\]

where \( \chi_{n,lj} \) are given functions with nonlinear parameters \( \beta \). Substituting this expansion into the rhs of Eq. (42) one obtains
\[ \tilde{R}_{ij}(\sigma) = \sum_{n=1}^{N} c_{n,ij} \tilde{\chi}_{n,ij}(\sigma, \beta), \quad (44) \]

where the \( \tilde{\chi}_{n,ij} \) are the Lorentz integral transforms of the basis functions. The parameters \( c_{n,ij} \) and \( \beta \) are determined by fitting the calculated transform \( \tilde{R}_{ij}(\sigma) \) to the above expansion at many \( \sigma_R \) points for a fixed \( \sigma_I \). The number of functions \( N \) plays the role of a regularization parameter and is chosen within a stability region, i.e. where the obtained results are stable for a certain range of \( N \) (see also [10,2]). Here we use the following set of basis functions

\[ \tilde{\chi}_{n,ij}(E, \beta) = E^{l+\frac{1}{2}} e^{\frac{\beta E}{n}}. \quad (45) \]

For the parametrization of the elastic monopole transition for \( l = 0 \) we include an additional function in the set \( \chi \):

\[ \chi_{0,0j}(E, \beta) = \delta(E - E_0). \quad (46) \]

Thus the sum in Eqs. (43, 44) starts in this case with \( n = 0 \) instead of \( n = 1 \).

Once the inversion is carried out one can make use of Eq. (14). For a specific set of \( m \) and \( m_d \) one finds the following FSI contribution to the \( T \)-matrix

\[ T_{1m0m_d}^{\text{FSI}}(E_{np}) = \sum_{lm} Y_{lm}(\hat{k}) \sqrt{\frac{2l + 1}{4\pi}} t_{1m0m_d}^{\text{FSI},l}(E_{np}) \quad (47) \]

with

\[ t_{1m0m_d}^{\text{FSI},l}(E_{np}) = C \sqrt{4\pi G_{E,p}} (q_{\mu}^2) \sum_j C_{m,mm_d}^{11j} C_{0m,md}^{11j} (-i\pi R_{ij}(E_{np}) + \mathcal{P} \int_{E_0}^{\infty} dE \frac{R_{ij}(E)}{E_{np} - E}). \quad (48) \]

For our case without tensor force the following simple relations hold

\[ t_{1m0m_d}^{\text{FSI},l} = \delta_{m,m_d} t_{1m0m_d}^{\text{FSI},l}, \quad (49) \]

\[ t_{1101}^{\text{FSI},l} = t_{1000}^{\text{FSI},l} = t_{1101}^{\text{FSI},l}. \quad (50) \]

It is worth mentioning that as a byproduct of the calculation one obtains also the NN phase shifts from our calculation, i.e. without having solved the Schrödinger equation for the scattering state. The ratio of imaginary and real parts of a given transition matrix element is equal to \( \tan(\delta) \) (see e.g., Ref. [3]). For our simple potential model one obtains

\[ \delta_l = \text{atan} \left( \frac{\Im(t_{\text{FSI},l})}{\Re(t_{\text{FSI},l}) + t_{\text{Born},l}} \right), \quad (51) \]

where \( t_{\text{Born},l} \) is analogously defined as \( t_{\text{FSI},l} \), and easily evaluated from a multipole decomposition of the Born term in Eq. (34). If one is only interested in the phase shifts themselves, one can perform a simpler calculation neglecting the excitation operator \( \hat{O} \) (see Ref. [3]).
IV. DISCUSSION OF RESULTS

We test the LIT method for exclusive reactions choosing for the electromagnetic deuteron breakup three different kinematics with rather strong FSI effects: (i) in the tail region beyond the quasi-elastic peak at moderate momentum transfer \((E_{np} = 120 \text{ MeV}, q_{c.m.}^2 = 5 \text{ fm}^{-2})\), (ii) on the photon line \((E_{\gamma} = 70 \text{ MeV})\), and (iii) close to the deuteron breakup threshold \((E_{np} = 1 \text{ MeV}, q_{c.m.}^2 = 2 \text{ fm}^{-2})\). Note for kinematics (ii) that there is no longitudinal contribution to the exclusive \((e,e'p)\) cross section, but that nevertheless the structure function \(f_L\) does not vanish. In fact in applying Siegert’s theorem the longitudinal matrix elements are commonly used in photodisintegration and lead to the dominant contribution for the electric transitions.

In order to have a more detailed comparison between the LIT results and the results of a conventional calculation we do not simply discuss the final result for \(f_L\), but rather study directly the FSI effect on the various multipole transitions. This allows us to make a much more precise comparison between the two calculations. Because of Eqs. (49,50) it is sufficient to consider \(t_{FSI}^{ll}\) in the following.

Before turning to the above mentioned three kinematical cases we first illustrate results for the transform in a more general way choosing a constant \(q_{c.m.}^2\) of 5 fm\(^{-2}\) and various energies \(E_{np}\). In Fig. 1 we show \(\tilde{R}_{ll}(\sigma_R, \sigma_I = 20 \text{ MeV})\) for \(l = 0, 2\). Its inversion, \(\tilde{R}_{ll}\), gives a contributes to \(t_{FSI}^{l}\) (see Eq. (48)). For the \(l = 0\) transforms one has an interesting structure at small \(\sigma_R\). It originates from the rather strong monopole transition strength close to the deuteron breakup threshold. It is interesting to see that the peak becomes more and more pronounced for the case that also \(E_{np}\) moves closer to the threshold region. In addition there is a second rather sizable contribution in the low \(\sigma_R\) range. It arises from the elastic monopole contribution. Therefore one has to pay attention in the inversion of \(\tilde{R}_{00}\).

One has to check whether \(\sigma_I\) is small enough to resolve with sufficient precision the elastic contribution from the threshold contribution. For \(l = 2\) one has a rather different picture. One finds a peak in the quasi-elastic region. Note that for the considered momentum of \(q_{c.m.}^2\) of 5 fm\(^{-2}\) the quasi-elastic peak is situated at about \(E_{np} = 50 \text{ MeV}\). To find such a quasi-elastic peak for \(\tilde{R}_{22}\) is a bit surprising, since one does not expect there strong FSI effects. On the other hand FSI should be small not on an absolute scale but compared with the corresponding Born term. Furthermore, the real part of the FSI contribution is difficult to estimate from Fig. 1 because a principle value integral has to be calculated in this case (see Eq. (48)).

In Fig. 2 we show \(R_{ll}(E)\) of kinematics (i) for \(l = 0, 2, 4\) and \(\sigma_I = 5\) and 20 MeV. It is obtained from the inversion of the corresponding \(\tilde{R}_{ll}(\sigma_R, \sigma_I)\) (see Eq. (42)). One sees that six basis functions are not sufficient for the inversion, but for a higher \(N\) one obtains a very nice stability of the inversion. Comparing the results with different \(\sigma_I\), one finds a small difference for \(l = 0\) in the threshold region. The differences arise because the monopole contribution has a peak at the very threshold which has to be separated from the elastic contribution at \(E_0 = -2.225 \text{ MeV}\). From the inversion we obtain an elastic contribution of about 1.1 fm\(^3\) which is rather sizable compared to the inelastic part with a peak height of 0.045 fm\(^3\). Therefore is not surprising that the higher resolution with \(\sigma_I = 5 \text{ MeV}\) leads to a somewhat different result. However, because of the rather high \(E_{np}\) of 120 MeV, the difference at the threshold is rather unimportant. This is confirmed by the results for \(t_{FSI}^{l}\), which are shown in Fig. 3 as function of the number of basis functions used for the inversion.
In fact the agreement among the results with $\sigma_I = 5$ and 20 MeV is very good. It is seen that one obtains for $N \geq 10$ for all considered multipolarities $l$ and for both $\sigma_I$ values very similar and stable results.

Also shown in Fig. 3 is the $t^{\text{FSI}}$ of a conventional calculation. These results are very similar to the LIT results with relative differences of less than 1%. Only for the real part of the $l = 4$ transition the difference is a little bit larger. On the other hand one has also to consider that this matrix element is very small. In fact its size is only $-2.5\%$ of the corresponding $t^{\text{Born}}$ matrix element. Thus the relative difference for the total matrix element is of the order of $10^{-4}$. For such a small FSI effect a part of the differences could also be due to a not completely exact result of the conventional calculation. Different from the LIT method $t^{\text{FSI}}$ is not calculated directly, but taken indirectly from the difference of $t^{\text{total}}$ and $t^{\text{Born}}$; here $t^{\text{total}}$ corresponds to the transition with the correct $np$ final state wave function in presence of the potential. The FSI effect is much more sizable for the two other transitions. Taking also here the ratio of $t^{\text{FSI}}/t^{\text{Born}}$ for the real parts, one finds for $l = 0$ about $-40\%$ and for $l = 2$ about $-30\%$.

Results for kinematics (ii) are shown in Figs. 4 and 5. For the $R_{ll}(E)$ of Fig. 4 one finds again nice stabilities of the inversion for a larger number of basis functions. Comparing the $R_{ll}$ with the two different $\sigma_I$ one has also here differences for the monopole transition and in addition for $l = 4$. The monopole is of course not relevant for this kinematics on the photon line, since there is no corresponding electric monopole. On the other hand it is interesting to see whether one is able to separate the strongly peaked threshold strength from the dominant elastic contribution. Due to the lower momentum transfer one obtains an even larger elastic $R_{ll}$ than for kinematics (i), namely a value of about $14 \text{ fm}^2$. The $t^{\text{FSI}}$ results are shown in Fig. 5. The real parts turn out to be very stable as function of number of inversion basis functions. They are also very similar for both $\sigma_I$. Here we have the following relative FSI effects comparing with the Born term: $-100\%$ ($l = 0$), $-25\%$ ($l = 2$), $-1.5\%$ ($l = 4$).

In comparison to the conventional calculation one finds in Fig. 5 for all the real parts only very small differences of less than 1%. For the imaginary part of $t^{\text{FSI}}$ the picture is a bit different. The $l = 2$ results are very stable and agree with extremely high precision to the results of the conventional calculation. Also the $l = 4$ results are stable, but they are a few percent larger than found in the conventional calculation. However, one should note that the matrix element is very small and hence the difference of a few percent is not relevant. In fact comparing with the above mentioned size of the real part of the total matrix element the difference between both calculations is of the order of $10^{-4}$. The imaginary part of the $l = 0$ transition shows a bit less stability with the number of inversion basis function reflecting also the above mentioned problems for the separation of the elastic contribution. On the other hand one obtains reliable results for the highest $N$'s.

For the third kinematics we illustrate the results in Figs. 6 and 7. Here we consider only $l = 0$ and $l = 2$ transitions, since FSI effects do not play any role for higher transitions at threshold. In fact the FSI contribution is already very small for $l = 2$. The inversion results in Fig. 6 are again very stable, except for $l = 0$ with $\sigma_I = 20 \text{ MeV}$. Of course, again it is the problem associated with the elastic contribution ($R_{00}(E_0)$ is about $15 \text{ fm}^2$). Figure 7 shows that one obtains very good results for the real part of $t^{\text{FSI}}$ with $\sigma_I = 5 \text{ MeV}$, while there is somewhat less stability for the inversion results with $\sigma_I = 20 \text{ MeV}$. The comparison with
the conventional calculation is also here satisfactory. There are only deviations of about 1%. Again we list the relative FSI effect comparing with the Born term: $-230\% (l = 0)$, $+2.5\% (l = 2)$. The imaginary parts are somewhat more problematic. For $l = 0$ one has the already mentioned problem with the elastic contribution combined with the fact that one needs $R_{00}(E)$ close to threshold ($E_{np} = 1$ MeV), but with $\sigma_I = 5$ MeV one obtains a sufficiently good result as seen from the comparison to the result of the conventional calculation. Though the relative differences to the conventional calculation are rather large for the imaginary part of the $l = 2$ transition, its value is in principle correct, since it is more or less identical to 0. Note that it is about 200 times smaller than the already very small real FSI part of the $l = 2$ transition.

We do not show results for the angular distribution of $f_L$. However, from the discussion above it should be clear that the two different calculations lead for $f_L(\theta)$ to relative differences of considerably less than 1% for kinematics (i) and (ii) and of about 1% for kinematics (iii).

As mentioned in Sec. II one has to use a somewhat different method for the calculation of the LIT for an exclusive reaction with more than two fragments in the final state. Two other possibilities are discussed at the end of Sec. II. In both cases one has to solve different differential equations, e.g., (22) instead of Eq. (20). However, these new methods appear to be numerically more problematic. Small errors in $\Psi_1$, the solution of the differential equation (19), might lead to a much larger error for the solution of Eq. (22), where $\Psi_1$ serves as source term on the rhs. Also for the determination of the LIT via Eq. (25) it is important how precise $\Psi_1$ and $\Psi_1'$ are calculated, since one has to determine the difference $\langle \Phi_{PW} | \hat{V} | \Psi_1'(\sigma) \rangle - \langle \Phi_{PW} | \hat{V} | \Psi_1(\sigma) \rangle$. We are able to study this question for the $d(e,e'p)n$ reaction, since we can also use these alternative ways of evaluating the LIT. As a matter of fact both alternative methods lead in our case essentially to the same results with relative differences smaller than 0.01%. In Fig. 8 we show for a few selected cases these new LIT results compared to those obtained with Eq. (14). One finds relative deviations of the order of 1%. There are larger differences for the kinematics with $E_{np} = 1$ and 120 MeV beyond a $\sigma_R$ of 150 MeV, but they are rather unimportant, since both $\tilde{R}_{00}$ are very small there. In fact $\tilde{R}_{00}$ crosses zero at about 205 and 190 MeV for $E_{np} = 1$ and 120 MeV, respectively. Altogether one can say that one does not encounter greater numerical problems in evaluating the LIT with these alternative ways. Therefore also a calculation of an exclusive reaction to a three-body channel should lead to rather reliable results with the LIT method.

V. CONCLUSION

We have calculated the longitudinal response of the exclusive $d(e,e'p)$ reaction with the method of the Lorentz integral transform. This method allows one to include the complete FSI, however, without explicit use of final state wave functions. It is the first time that the LIT method is applied to an exclusive reaction. In the past only inclusive processes have been studied with the LIT. The great success of the method raised the question whether it can also be successfully used in exclusive reactions. The results in this work show that one obtains a very precise determination of the various transition matrix elements. Differences to the conventional calculation are generally below 1%. Only in the case of a very small FSI effect on the transition strength, i.e. a $10^{-4}$ effect compared to the corresponding Born
term, one can also obtain somewhat higher differences of a few percent. However, in this case differences could, as discussed in Sec. [IV], at least partly be due to a small inexactness in the conventional calculation. There is only one exception, where one can expect a somewhat larger size of the error of the LIT result. This is the case for a transition matrix element in a region with transition strength from two (or more) rather narrow lying peaks. We had chosen such a situation with our kinematics (iii), where we have a strong elastic contribution at about $E = -2.2$ MeV and another strong peak right above breakup threshold. In such a situation one should try to improve the resolution of the transform $L(\sigma_R, \sigma_I)$ by choosing a smaller $\sigma_I$. In fact our results improve significantly from $\sigma_I = 20$ MeV to $\sigma_I = 5$ MeV.

In case of an exclusive reaction with more than two fragments in the final state one cannot proceed exactly in the same way as for the breakup in two fragments. In this case one has to use other ways for the determination of the LIT. We could show that also these alternative methods lead to rather precise results. Therefore, in general, we may conclude that the LIT method leads to reliable results not only for inclusive, but also for exclusive reactions.

ACKNOWLEDGMENT

We thank V. D. Efros and G. Orlandini for helpful discussions.
REFERENCES

[1] V.D. Efros, W. Leidemann, and G. Orlandini, Phys. Lett B 338, 130 (1994).
[2] V.D. Efros, W. Leidemann, and G. Orlandini, Few-Body Syst. 26, 251 (1999).
[3] V.D. Efros, W. Leidemann, and G. Orlandini, Phys. Rev. C 58, 582 (1998).
[4] V.D. Efros, Sov. J. Nucl. Phys. 41, 949 (1985).
[5] V.D. Efros, W. Leidemann, and G. Orlandini, Few-Body Syst. 14, 151 (1993).
[6] A. La Piana, thesis, University of Trento (1999).
[7] V.D. Efros, Phys. of Atom. Nucl. 62, 1833 (1999).
[8] M.L. Goldberger and K.W. Watson, Collision Theory (Wiley, New York, 1964).
[9] W. Fabian and H. Arenhövel, Nucl. Phys. A314, 253 (1979).
[10] V.D. Efros, W. Leidemann, and G. Orlandini, Nucl. Phys. A631, 658c (1998).
FIGURES

FIG. 1. Lorentz integral transforms $\tilde{R}_H(\sigma_R, \sigma_I = 20 \text{ MeV})$ for $l = 0$ (top) and $l = 2$ (bottom) at $q_{c.m.}^2 = 5 \text{ fm}^{-2}$ for various $E_{np}$ as shown in the figure.

FIG. 2. Inversion result $R_H(E)$ with $\sigma_I = 5 \text{ MeV}$ (left) and $\sigma_I = 20 \text{ MeV}$ (right) for $l = 0$ (top), $l = 2$ (middle), $l = 4$ (bottom); as indicated in the figure curves are shown for a number of inversion basis functions $N = 6, 16, 18, 20$.

FIG. 3. Results for real (left) and imaginary (right) parts of $t_{1101}^{\text{FSI}}$ for kinematics (i) with $l = 0$ (top), $l = 2$ (middle), $l = 4$ (bottom) as function of the number of inversion basis functions: diamonds ($\sigma_I = 5 \text{ MeV}$), squares ($\sigma_I = 20 \text{ MeV}$); also shown are the results of a conventional calculation with explicit final state wave function (full curves) and deviations of $\pm 1\%$ from these results (dashed curves).

FIG. 4. As Fig. 2 but for kinematics (ii).

FIG. 5. As Fig. 3 but for kinematics (ii).

FIG. 6. As Fig. 2 but for kinematics (iii) and $l = 0, 2$.

FIG. 7. As Fig. 3 but for kinematics (iii) and $l = 0, 2$.

FIG. 8. Ratio of Lorentz integral transforms $\tilde{R}_{00}(\sigma_R, \sigma_I = 5 \text{ MeV})$ calculated with Eq. (21) (dots) relative to the results of Eq. (16) at $q_{c.m.}^2 = 5 \text{ fm}^{-2}$ for $E_{np} = 1 \text{ MeV}$ (top), 120 MeV (middle), 200 MeV (bottom).
\[ \sigma_{l} = 5 \text{ MeV} \]
\[ \sigma_{l} = 20 \text{ MeV} \]

\[ l=0 \]
\[ l=2 \]
\[ l=4 \]
\[ |f_{m}^{3/2}| \]

\[ E \text{ [MeV]} \]

\[ l = 0 \]
\[ l = 2 \]
\[ l = 4 \]

\( \sigma_{I} = 5 \text{ MeV} \)
\( \sigma_{I} = 20 \text{ MeV} \)

- 6 functions
- 16 functions
- 18 functions
- 20 functions
with explicit final state
5 MeV
20 MeV

\[ |f_m|^{1/2} \]

Real part

Imaginary part

l=0

l=2

N

\[ 5 \quad 10 \quad 15 \quad 20 \]

\[ -2.8 \times 10^{-04} \quad -2.6 \times 10^{-04} \quad -2.4 \times 10^{-04} \quad -2.3 \times 10^{-04} \]

\[ -2.1 \times 10^{-01} \quad -2.0 \times 10^{-01} \quad -1.8 \times 10^{-01} \quad -1.6 \times 10^{-01} \quad -1.5 \times 10^{-01} \]

\[ 0.0 \times 10^{00} \quad 1.0 \times 10^{-06} \quad 2.0 \times 10^{-06} \quad 3.0 \times 10^{-06} \quad 6.0 \times 10^{-02} \]

\[ 7.0 \times 10^{-02} \quad 8.0 \times 10^{-02} \quad 9.0 \times 10^{-02} \]

\[ -2.3 \times 10^{-04} \quad -2.4 \times 10^{-04} \quad -2.6 \times 10^{-04} \quad -2.8 \times 10^{-04} \]

\[ -1.6 \times 10^{-01} \quad -1.8 \times 10^{-01} \]

\[ 1.0 \times 10^{-01} \]
$E_{np} = 1\, \text{MeV}$

$E_{np} = 120\, \text{MeV}$

$E_{np} = 200\, \text{MeV}$