Quasi-isometric rigidity for $PSL_2(\mathbb{Z}_{\frac{1}{p}})$

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Abstract

We prove that $PSL_2(\mathbb{Z}_{\frac{1}{p}})$ gives the first example of groups which are not quasi-isometric to each other but have the same quasi-isometry group. Namely, $PSL_2(\mathbb{Z}_{\frac{1}{p}})$ and $PSL_2(\mathbb{Z}_{\frac{1}{q}})$ are not quasi-isometric unless $p = q$, and, independent of $p$, the quasi-isometry group of $PSL_2(\mathbb{Z}_{\frac{1}{p}})$ is $PSL_2(\mathbb{Q})$. In addition, we characterize $PSL_2(\mathbb{Z}_{\frac{1}{p}})$ uniquely among all finitely generated groups by its quasi-isometry type.

1 Introduction

Combining the work of many people yields a complete quasi-isometry classification of irreducible lattices in semisimple Lie groups (see [F] for an overview of these results). One of the first general results in this classification is the complete description, up to quasi-isometry, of all nonuniform lattices $\Lambda$ in semisimple Lie groups of rank 1, proved by R. Schwartz [S2]. He shows that every quasi-isometry of such a lattice $\Lambda$ is equivalent to a unique commensurator of $\Lambda$. (A commensurator of $\Lambda \subset G$ is an element $g \in G$ so that $g\Lambda g^{-1} \cap \Lambda$ has finite index in $\Lambda$.) We will call this result commensurator rigidity, although it is a different notion than the commensurator rigidity of Margulis. In [FS] it was conjectured that commensurator rigidity, or at least a slightly weaker statement, “quasi-isometric iff commensurable,” should apply to nonuniform lattices in a wide class of Lie groups. Here we prove that both of these statements are true for $PSL_2(\mathbb{Z}_{\frac{1}{p}})$.

In a different direction, B. Farb and L. Mosher proved analogous quasi-isometric rigidity results for the the solvable Baumslag-Solitar groups. These groups are given by the presentation

$$BS(1, n) = \langle a, b | aba^{-1} = b^n \rangle$$

and are not lattices in any Lie group.
The group \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) is a nonuniform (i.e. non cocompact) lattice in \( PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p) \), analogous to the classical Hilbert modular group \( PSL_2(O_d) \) in \( PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R}) \). It is also a basic example of an \( S \)-arithmetic group. The proofs of Theorems A, B and C (stated below) combine techniques from the two types of quasi-isometric rigidity results mentioned above. When we construct a space \( \Omega_p \) on which \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) acts properly discontinuously and cocompactly by isometries, we see that the horospheres forming the boundary components of \( \Omega_p \) carry the geometry of the group \( BS(1,p) \). In this way the results of [FM] play a role in the quasi-isometric rigidity of \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \).

1.1 Statement of Results

In this paper we prove the following quasi-isometric rigidity results for the finitely generated groups \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \), where \( p \) is a prime. Theorem A may be viewed as a strengthening of strong (Mostow) rigidity for \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \).

**Theorem A (Main Theorem).** Every quasi-isometry of \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) is equivalent to a commensurator of \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \). Hence the natural map \( \text{Comm}(PSL_2(\mathbb{Z}[\frac{1}{p}])) \to QI(PSL_2(\mathbb{Z}[\frac{1}{p}])) \) is an isomorphism.

Since for any prime \( p \) the commensurator group of \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) in \( PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p) \) is \( PSL_2(\mathbb{Q}) \), the quasi-isometry group is also \( PSL_2(\mathbb{Q}) \). (See §2.2 for a model of \( PSL_2 \) as an algebraic group.) Thus we cannot distinguish the quasi-isometry classes of these groups via their quasi-isometry groups. However, using a result of B. Farb and L. Mosher [FM] (Theorem 2.1 below), we are able to prove the following.

**Theorem B (Quasi-isometric iff commensurable).** Let \( p \) and \( q \) be primes. Then \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) and \( PSL_2(\mathbb{Z}[\frac{1}{q}]) \) are quasi-isometric if and only if they are commensurable, which occurs only when \( p = q \).

Theorems A and B together give the first example of groups which have the same quasi-isometry group but are not quasi-isometric.

The following theorem characterizes \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) uniquely among all finitely generated groups by its quasi-isometry type.

**Theorem C (Quasi-isometry characterization).** Let \( \Gamma \) be any finitely generated group. If \( \Gamma \) is quasi-isometric to \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \), then there is a short exact sequence

\[ 1 \to N \to \Gamma \to \Lambda \to 1 \]
where $N$ is a finite group and $\Lambda$ is abstractly commensurable to $PSL_2(\mathbb{Z}[\frac{1}{p}])$.

Two groups are abstractly commensurable if they have isomorphic finite index subgroups.

1.2 An outline of the proofs of Theorems A and B

The group $PSL_2(\mathbb{Z}[\frac{1}{p}])$ is a nonuniform (i.e. non-cocompact) lattice in $G = PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p)$ under the diagonal embedding sending a matrix $M$ to the pair $(M,M)$. The group $G$ acts on $H^2 \times T_p$, where $H^2$ is the hyperbolic plane and $T_p$ is the Bruhat-Tits-Serre tree associated to $PGL_2(\mathbb{Q}_p)$. Let $f : PSL_2(\mathbb{Z}[\frac{1}{p}]) \to PSL_2(\mathbb{Z}[\frac{1}{q}])$ be a quasi-isometry. The proofs of Theorems A and B both begin as follows.

**Step 1 (The geometric model).** We construct a space $\Omega_p$ on which $PSL_2(\mathbb{Z}[\frac{1}{p}])$ acts properly discontinuously and cocompactly by isometries (hence by a result of Milnor and Svarc [Mi], $PSL_2(\mathbb{Z}[\frac{1}{p}])$ and $\Omega_p$ are quasi-isometric). The space $\Omega_p$ has a boundary consisting of horospheres of $H^2 \times T_p$, each of which is a quasi-isometrically embedded copy of the group $BS(1,p)$. The quasi-isometry $f : PSL_2(\mathbb{Z}[\frac{1}{p}]) \to PSL_2(\mathbb{Z}[\frac{1}{q}])$ then induces a quasi-isometry, also denoted $f$, from $\Omega_p$ to $\Omega_q$.

**Step 2 (The Boundary Detection Theorem).** This theorem shows that for every horosphere boundary component of $\partial \Omega_p$, there is a corresponding horosphere boundary component of $\partial \Omega_q$ so that $f$ restricts to a quasi-isometry of horospheres. The proof of this theorem uses the Coarse Separation Theorem of [FS] and [S2], and the geometry of $\Omega_p$.

**Remark.** We are now able to prove Theorem B. The initial quasi-isometry $f : PSL_2(\mathbb{Z}[\frac{1}{p}]) \to PSL_2(\mathbb{Z}[\frac{1}{q}])$ induces a quasi-isometry $f : \Omega_p \to \Omega_q$. From the Boundary Detection Theorem, we obtain a quasi-isometry $\hat{f} : \sigma \to \tau$ by restriction, where $\sigma$ and $\tau$ are horosphere boundary components of $\Omega_p$ and $\Omega_q$, respectively. By Step 1, the map $\hat{f}$ can be considered as a quasi-isometry of Baumslag-Solitar groups, namely $\hat{f} : BS(1,p) \to BS(1,q)$. From Theorem 2.1 [FM] we conclude that $p = q$.

The proof of Theorem A continues with the following steps. We are now considering a quasi-isometry $f : \Omega_p \to \Omega_p$.

**Step 3 (The geometry of $\Omega_p$).** For any two horosphere boundary components $\sigma_1$ and $\sigma_2$ of $\Omega_p$, there is a unique line $l \subset T_p$ so that $\sigma_1$ and $\sigma_2$ are a specified fixed distance apart in $H^2 \times t$, for all vertices
t of $l$. This line is called the *closeness line* of $\sigma_1$ and $\sigma_2$. The set of closeness lines of all horospheres is preserved under quasi-isometry. This geometric result replaces the usual group-equivariance assumed in Mostow-Prasad rigidity, and provides the structure necessary for Step 4.

**Step 4 (S-Arithmetic Action Rigidity).** Here we prove an 2-dimensional $S$-arithmetic version of the Action Rigidity Theorem of R. Schwartz [51]. The Action Rigidity Theorem concludes that the map induced by $f$ on the set of horospheres of $\partial \Omega_p$ (which is indexed by $\mathbb{Q} \cup \{\infty\}$) is given by an affine map of $\mathbb{Q} \cup \{\infty\}$.

**Step 5 (Conclusion of the proof).** From Step 4, we are able to choose a specific commensurator $g$ of $PSL_2(\mathbb{Z}[\frac{1}{p}])$ so that the composite map $f \circ g$ is a bounded distance from the identity map. This finishes the proof of Theorem A.

2 Preliminary material

2.1 Quasi-isometries

**Definition.** Let $K \geq 1$ and $C \geq 0$. A $(K,C)$-quasi-isometry between metric spaces $(X,d_X)$ and $(Y,d_Y)$ is a map $f : X \to Y$ satisfying:

1. $\frac{1}{K}d_X(x_1,x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq Kd_X(x_1,x_2) + C$ for all $x_1, x_2 \in X$.
2. For some constant $C'$, the $C'$ neighborhood of $f(X)$ is all of $Y$.

We often omit the constants $K$ and $C$ and simply refer to $f$ as a quasi-isometry. A quasi-isometry $f$ can always be changed by a bounded amount using the standard “connect-the-dots” procedure so that it is continuous. (See, e.g. [FS].) A quasi-isometry also has a coarse inverse, i.e. there is a quasi-isometry $g : Y \to X$ so that $f \circ g$ and $g \circ f$ are a bounded distance from the appropriate identity map in the sup norm. A map satisfying 1. but not 2. in the definition above is called a quasi-isometric embedding.

We define the *quasi-isometry group* of a space $X$, denoted $QI(X)$, to be the set of all self quasi-isometries of $X$, modulo those a bounded distance from the identity in the sup norm, under composition of quasi-isometries. Inverses exist in $QI(X)$ since every quasi-isometry has a coarse inverse. A quasi-isometry between two metric spaces $X$ and $Y$ induces an isomorphism between $QI(X)$ and $QI(Y)$.
2.2 \( PSL_2 \) as a algebraic group

We will use the following model of \( PSL_2 \) as an algebraic group. Consider the map \( Ad : SL_2(C) \to GL(sl_2) \), where we view the Lie algebra \( sl_2 \) as a vector space. Let \( G' = Ad(SL_2(C)) \). Then \( G' \) is a model for \( PSL_2 \) as an algebraic group, since the center of \( SL_2(C) \) vanishes under the map \( Ad \). By \( PSL_2(Q) \) we mean the \( Q \)-points of \( G' \), denoted \( G'_Q \).

2.3 The geometry of \( BS(1,n) \)

The Baumslag-Solitar group \( BS(1,n) \) acts properly discontinuously and cocompactly by isometries on a metric 2–complex \( X_n \) defined explicitly in [FM]. This complex \( X_n \) is topologically \( T_n \times R \), where \( T_n \) is a regular \((n + 1)\)-valent tree, directed so that each vertex has 1 incoming edge and \( n \) outgoing edges. (Figure 1.)

A height function on \( T_n \) is a continuous function \( h : T_n \to R \) which maps each oriented edge of \( T_n \) homeomorphically onto an oriented interval of a given length \( d \). A vertex of \( T_n \) whose height under \( h \) is \( kd \) is said to have combinatorial height \( k \). Fix a basepoint for \( T_n \) with height 0. This determines a height function on \( T_n \). Let \( \pi : T_n \times R \to T_n \) denote projection. Then \( h \circ \pi \) is a height function on \( X_n \).

A proper line in \( T_n \) is the image of a proper embedding \( R \to T_n \). A coherently oriented proper line is one on which the height function is strictly monotone. We will use the term “line” to mean a proper line in \( T_n \). The metric on \( X_n \) is defined so that for each infinite, coherently
oriented line \( l \subset T \), the plane \( l \times \mathbb{R} \) is isometric to a hyperbolic plane.

When studying the geometry of \( X_n \), there are two “boundaries” of the complex which play an important role. (Figure 1.) The lower boundary, denoted \( \partial_l X_n \), is homeomorphic to \( \mathbb{R} \) and is the common lower boundary of all hyperbolic planes in \( X_n \).

The upper boundary, denoted \( \partial^u X_n \), is defined to be the space of hyperbolic planes in \( X_n \) with the following metric. If \( Q_1 \) and \( Q_2 \) are hyperbolic planes in \( X_n \) which agree below combinatorial height \( k \), define the distance between them to be \( n^{-k} \). With this metric, \( \partial^u X_n \) is isometric to the set of \( n \)-adic rational numbers, \( \mathbb{Q}_n \), with the metric defined by the \( n \)-adic absolute value.

B. Farb and L. Mosher obtain the following quasi-isometric rigidity results for \( BS(1,n) \). [FM]

**Theorem 2.1 ([FM]).** For integers \( m,n \geq 2 \), the groups \( BS(1,m) \) and \( BS(1,n) \) are quasi-isometric if and only if they are commensurable. This happens if and only if there exist integers \( r,j,k > 0 \) such that \( m = rj \) and \( n = rk \).

**Theorem 2.2 ([FM]).** The quasi-isometry group of \( BS(1,n) \) is given by the following isomorphism:

\[
QI(BS(1,n)) \cong Bilip(\mathbb{R}) \times Bilip(\mathbb{Q}_n).
\]

A quasi-isometry \( f \in QI(BS(1,n)) \) induces bilipschitz maps \( f^u \) and \( f^l \) on the upper and lower boundaries of \( X_n \), respectively [FM]. From Theorem 2.2 we see that the map \( QI(BS(1,n)) \to Bilip(\mathbb{R}) \times Bilip(\mathbb{Q}_n) \) given by \( f \to (f^l,f^u) \) is an isomorphism. It is perhaps surprising that \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) should have such a small quasi-isometry group while \( BS(1,p) \) has such a large quasi-isometry group.

### 3 The geometry of \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \)

#### 3.1 The action of \( PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p) \) on \( \mathbb{H}^2 \times T_p \)

We will consider \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \subset PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p) \) as the image of the diagonal map \( \eta : PSL_2(\mathbb{Z}[\frac{1}{p}]) \to PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p) \) given by \( \eta(M) = (M,M) \). Viewed in this way, \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) is a lattice in the group \( PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p) \), for any prime \( p \).

We now define the Bruhat-Tits tree \( T_p \) associated to \( PGL_2(\mathbb{Q}_p) \). We consider a tree \( T \) to be a set of vertices \( Vert(T) \) together with a set of adjacency relations among the vertices. Let \( Vert(T_p) \) be the set of equivalence classes of \( \mathbb{Z}_p \)-lattices in \( \mathbb{Q}_p \times \mathbb{Q}_p \). Two lattices \( L_1 \) and \( L_2 \) are equivalent if \( L_2 = \alpha L_1 \), where \( \alpha \in \mathbb{Q}_p - \{0\} \). Two vertices \([L_1] \) and
are adjacent if there exist representatives \( L_1 \in [L_1] \) and \( L_2 \in [L_2] \) with \( L_1 \subset L_2 \) and \([L_2 : L_1] = p\). An example of two adjacent vertices is \([\mathbb{Z}_p \times \mathbb{Z}_p] \) and \([p\mathbb{Z}_p \times \mathbb{Z}_p] \). The tree \( T_p \) is a regular \((p + 1)\)-valent tree. We will fix \([L_0] = [\mathbb{Z}_p \times \mathbb{Z}_p] \) as the basepoint of \( T_p \) as well as a height function \( h \) giving \([L_0] \) height 0. An element \( g \in PSL_2(\mathbb{Q}_p) \) acts on \([L] \in Vert(T_p)\) by matrix multiplication on the basis vectors of a representative lattice in the equivalence class \([L] \). Note that \( PSL_2(\mathbb{Q}_p) \) also acts on \( T_p \) in this way.

Let \( \mathbb{H}^2 \) denote 2–dimensional hyperbolic space in the upper half plane model, i.e. \( \mathbb{H}^2 = \{(x, y)| x \in \mathbb{R}, y > 0\} \) with the metric \( \frac{dx^2 + dy^2}{y^2} \). We define the action of an element \((g_1, g_2) \in PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p)\) on a point \((x, [L]) \in \mathbb{H}^2 \times T_p\). The element \( g_1 \in PSL_2(\mathbb{R}) \) acts on \( x \in \mathbb{H}^2 \) by fractional linear transformations. The element \( g_2 \in PSL_2(\mathbb{Q}_p) \) acts on \([L] \in Vert(T_p)\) by matrix multiplication on the basis vectors of a representative lattice in the equivalence class \([L] \). When we are considering the action of \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) on \( \mathbb{H}^2 \times T_p \), we have \( g_1 = g_2 \). We will use only one coordinate to represent the elements of \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \subset PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p) \). So \( g \in PSL_2(\mathbb{Z}[\frac{1}{p}]) \) would correspond to \((g, g) \in PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p) \). Hence we can refer to \( g \in PSL_2(\mathbb{Z}[\frac{1}{p}]) \) acting on either \( \mathbb{H}^2 \) or \( T_p \) or \( \mathbb{H}^2 \times T_p \) in the appropriate manner.

3.2 Constructing the space \( \Omega_p \)

We want to construct a space \( \Omega_p \subset \mathbb{H}^2 \times T_p \) on which \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) acts properly discontinuously and cocompactly by isometries. The Milnor-Svarc criterion states that if a finitely generated group \( \Gamma \) acts properly discontinuously and cocompactly by isometries on a space \( X \), then \( \Gamma \) is quasi-isometric to \( X \). We then refer to the geometry of \( X \) as the large scale geometry of \( \Gamma \). This additional geometric information associated to \( \Gamma \) is often useful in determining rigidity properties of \( \Gamma \).

Although \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) acts by isometries on \( \mathbb{H}^2 \times T_p \), it does not act cocompactly, because the fundamental domain for the action of \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) on a fixed \( \mathbb{H}^2 \) is the same as the fundamental domain for the action of \( PSL_2(\mathbb{Z}) \) on \( \mathbb{H}^2 \), which is unbounded in one direction.

Let \( w \) be the segment of the horocircle based at \( \infty \) of height \( h_0 \) in this fundamental domain (in the upper half space model of hyperbolic space), for \( h_0 \) sufficiently large. Fix \( H > 1 \). Lift the segment \( w \) to \( \mathbb{H}^2 \times [L_0] \) to obtain a horocyclic segment at height \( H \) whose orbit under \( PSL_2(\mathbb{Z}) = \text{Stab}_{PSL_2(\mathbb{Z}[\frac{1}{p}])}([L_0]) \) is a disjoint collection of horocircles of \( \mathbb{H}^2 \), centered at \( Q \cup \{\infty\} \subset \partial_{\infty} \mathbb{H}^2 \). The orbit of the lift of \( w \) under the entire group \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) gives a \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \)-equivariant collection of horocircles in \( \mathbb{H}^2 \times T_p \), based at \( Q \cup \{\infty\} \) in each copy of
Figure 2: This is a piece of the horosphere $\sigma_\infty$ in $H^2 \times T_3$, the space associated to $PSL_2(\mathbb{Z}[\frac{1}{3}])$. Notice how the height of the horosphere increases in each successive copy of $H^2$. Viewing the bold black lines as part of the tree $T_3$ helps one to see that this horosphere is topologically $T_3 \times \mathbb{R}$.

$H^2 \subset H^2 \times T_p$.

We now define a horosphere of $H^2 \times T_p$ based at $\alpha \in \mathbb{R} \cup \{\infty\}$ to be the collection of horocircles, one in each $H^2 \times [L]$ for every $[L] \in Vert(T_p)$, all based at $\alpha$. According to the above construction, there is exactly one such horocircle in each $H^2 \times [L]$. We will denote this horosphere of $H^2 \times T_p$ by $\sigma_\alpha$.

In order to have a “connected” picture of a horosphere, we can put an edge $e$ between any two adjacent vertices of $T_p$ and extend the horospheres linearly in $H^2 \times e$. So we can think of a horosphere as a hollow tube, or in the case of $\sigma_\infty$ as a flat sheet, whose image under the projection $\pi : H^2 \times T_p \to T_p$ is all of $T_p$. (Figures 2 and 3.)

We define the space $\Omega_p$, where $PSL_2(\mathbb{Z}[\frac{1}{3}])$ acts properly discontinuously and cocompactly by isometries, to be $H^2 \times T_p$ with the interiors of all the horospheres removed. The interior of a horosphere is the union of the interiors of the component horocircles.

### 3.3 The metric

The following theorem allows us to use the product metric $d_{H^2} \times d_T$ on $\Omega_p$. Although it is stated for semisimple Lie groups, it is proven in the more general context of $S$-arithmetic groups.
**Theorem 3.1.** [LMR] If $G$ is a semisimple Lie group of rank at least 2 and $\Gamma$ is an irreducible lattice in $G$ then, $d_R$ restricted to $\Gamma$ is Lipschitz equivalent to $d_W$, where $d_W$ is the word metric on $\Gamma$ and $d_R$ is the left invariant Riemannian metric on $\Gamma$.

By construction, $PSL_2(\mathbb{Z}_p)$ acts properly discontinuously and cocompactly by isometries on $\Omega_p$.

### 3.4 The closeness line

We are now interested in the packing of the horospheres of $H^2 \times T_p$, i.e. how the horosphere “tubes” fit together. We consider two horospheres, without loss of generality $\sigma_0$ and $\sigma_1$, and the distance between them when restricted to $H^2 \times [L]$, for any $[L] \in Vert(T_p)$. We use the notation $\sigma|_{[L]}$ for the horosphere $\sigma \subset \partial \Omega_p$ restricted to $H^2 \times [L]$. We will show that there is a unique line $l \subset T_p$ so that $\sigma_0$ and $\sigma_1$ remain a constant distance apart when restricted to $H^2 \times [L]$, for all vertices $[L]$ lying on $l$. In addition, we will show that all other lines $l' \subset T_p$ have the property that $d_H^{2}(\sigma_0|_{[L]}, \sigma_1|_{[L]})$ increases without bound as the vertices $[L]$ lying on $l'$ increase in height. We call $l$ the **closeness line** of $\sigma_0$ and $\sigma_1$. By symmetry, any two horospheres $\sigma_\alpha$ and $\sigma_\beta$ have a unique closeness line.

We will use the notation $\sigma|_{[L]}$ for the horosphere $\sigma$ of $\partial \Omega_p$ restricted to $H^2 \times [L]$, for $[L] \in Vert(T_p)$. Note that a matrix $g$ which translates $\sigma|_{[L]}$ to $\sigma|_{[L]}$ will lie in $Stab_{PSL_2(\mathbb{Z}_p)}(\alpha)$ and satisfy $g[L_1] = [L_2]$. However, to move between the horocircles $\sigma|_{[L]}$ and $\sigma|_{[L]}$, for $\beta \neq \alpha$, we will need to use a different element of $PSL_2(\mathbb{Z}_p)$, i.e. $h \in Stab_{PSL_2(\mathbb{Z}_p)}(1)$ such that $[hL_1] = [L_2]$.

Products of the matrices $A = \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ give the $PSL_2(\mathbb{Q}_p)$ action on $T_p$. (See, e.g. [La].) Since

$$A \in Stab_{PSL_2(\mathbb{Z}_p)}(0) \cap Stab_{PSL_2(\mathbb{Z}_p)}(\infty),$$

letting $A^t$ act on $H^2 \times [L_0]$ moves the horocircle $\sigma_0|_{[L_0]}$ (resp. $\sigma_1|_{[L_0]}$) to the horocircle $\sigma_0|_{[A^tL_0]}$ (resp. $\sigma_1|_{[A^tL_0]}$). Moreover, $A \in PSL_2(\mathbb{R}) = Isom^+(H^2)$, so we have

$$d_H^{2}(\sigma_0|_{[L_0]}, \sigma_1|_{[L_0]}) = d_H^{2}(\sigma_0|_{[A^tL_0]}, \sigma_1|_{[A^tL_0]}),$$

where $H$ was chosen in §3.2. Let $l$ be the line in $T_p$ which is the orbit of $[L_0]$ under the cyclic group generated by the matrix $A$. We call $l$ the **diagonal line**.

Recall that we usually need two different matrices to move two different horocircles in $H^2 \times [L_0]$ to their corresponding horocircles in $H^2 \times [L_0]$. We will use the notation $\sigma_\alpha|_{[L_0]}$. By symmetry, any two horospheres $\sigma_\alpha$ and $\sigma_\beta$ have a unique closeness line.
Another computation shows that \( \sigma_{\text{matrix}}A \) or \( C_i = A \) where either \( C_i = A \) or \( C_i = B^jA \). The diagonal ray (i.e. the part of the diagonal line beginning at \([L_0]\) and moving upwards in height) is described by \( \{\Pi_{i=1}^N A_i\}_{N \in \mathbb{N}} \), and we know from the previous paragraph that \( d_H^2(\sigma_0[\Pi_{i=1}^N A], \sigma_\infty[\Pi_{i=1}^N A]) = 2\log H \) for all \( N \in \mathbb{N} \).

Now suppose that \( r \subset T_p \) is the ray based at \([L_0]\) given by the sequence \( \{\Pi_{i=1}^N C_i\}_{N \in \mathbb{N}} \) where \( C_i = B^jA \) for \( j \in \{0, 1, \cdots p - 1\} \). For any \( N_0 \in N \), the product \( \Pi_{i=1}^N C_i \) has the form \( (\Pi_{i=1}^N A)(\Pi_{i=1}^N C_i) \) where \( C_i \) is as above. Such a product is a matrix of the form \( M = \begin{pmatrix} p^S & \frac{s}{p^S} \\ 0 & \frac{1}{p^S} \end{pmatrix} \), where \( S \) or \( T \) may be 0. We may assume that \( S > T \), meaning that in the tree factor, we are considering vertices sufficiently far away from the diagonal line \( l \).

Let \( \beta = \frac{-s}{p^S} \), where \( s \) and \( S \) are as in the matrix \( M \). We can find a matrix of the form \( N = \begin{pmatrix} a & -s \\ b & p^S \end{pmatrix} \in PSL_2(\mathbb{Z}) \) where \( s, S \) are as in \( M \). Since \( N \in PSL_2(\mathbb{Z}) = \text{Stab}_{PSL_2(\mathbb{Z})}([L_0]) \) and \( N \cdot 0 = \beta \), the matrix \( N \) maps \( \sigma_0[\Pi_{i=1}^N A] \) to \( \sigma_\beta[\Pi_{i=1}^N A] \). A computation shows that the height of \( \sigma_\beta[\Pi_{i=1}^N A] \) is \( \frac{1}{p^S + H} \). Moreover, \( M \cdot \beta = 0 \), so \( M \cdot (\sigma_\beta[\Pi_{i=1}^N A]) = \sigma_0[\Pi_{i=1}^N A] \).

Another computation shows that \( \sigma_0[\Pi_{i=1}^N A] \) has height \( \frac{1}{p^S + H} \) in \( H^2 \). The height of \( \sigma_\infty[\Pi_{i=1}^N A] \) is \( p^2(S + T)H \). Hence

\[
\begin{align*}
\sigma_0 \in H^2(\sigma_0[\Pi_{i=1}^N A], \sigma_\infty[\Pi_{i=1}^N A]) = \log(p^{2S}H^2).
\end{align*}
\]

When \( S = 0 \), the vertex \( \Pi_{i=1}^N A \) lies on the diagonal line, and the above formula gives a constant distance of \( 2\log H \) between \( \sigma_\infty \) and \( \sigma_0 \) in \( A \times l \). When \( S \neq 0 \), we see that \( S \) increases by a factor of 2 for each unit of height in \( T_p \); hence the distance between \( \sigma_\infty \) and \( \sigma_0 \) increases by a factor of \( p^4 \).

So we see that for any line \( l' \subset T \) which is not the diagonal line, \( d_H^2(\sigma_0[\Pi_{i=1}^N A], \sigma_\infty[\Pi_{i=1}^N A]) \) increases without bound as we consider vertices \([L]\) of \( l' \) of increasing height. So the diagonal line \( l \) is the closeness line of \( \sigma_0 \) and \( \sigma_\infty \).

It is clear by construction that the closeness line of \( \sigma_\alpha \) and \( \sigma_\beta \) and the closeness line of \( \sigma_\alpha \) and \( \sigma_\gamma \) are distinct.
3.5 The geometry of the horospheres

Consider the matrices
\[ A = \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]
Then
\[ A, B \in PSL_2(\mathbb{Z}[\frac{1}{p}]) \subset PSL_2(\mathbb{Q}_p) \]
and
\[ ABA^{-1} = \begin{pmatrix} 1 & p^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = Bp^2. \]
The map \( \Phi \) sending the generator \( a \) of \( BS(1,p^2) \) to the matrix \( A \) and the generator \( b \) to \( B \) is a homomorphism of \( BS(1,p^2) \) into the group \( PSL_2(\mathbb{Q}_p) \). Products of the matrices \( A \) and \( B \) form the elements of \( PGL_2(\mathbb{Q}_p) \) needed to move between vertices of \( T_p \).

If we take the orbit of the segment \( w \) from \( Hi \) to \( 1 + Hi \) in \( H^2 \times [L_0] \) under the group \( \Phi(BS(1,p^2)) \subset PSL_2(\mathbb{Q}_p) \), we obtain the horosphere \( \sigma_\infty \). The width of a horostrip in \( \sigma_\infty \) between vertices whose combinatorial heights differ by 1, in the metric on \( H^2 \times T_p \), is \( 1 + 2 \log p \), while in \( X_{p^2} \) it is \( 2 \log p \). Since these distances are comparable, the horosphere \( \sigma_\infty \) (and hence any horosphere \( \sigma_\alpha \)) is quasi-isometric to the complex \( X_{p^2} \) associated to \( BS(1,p^2) \). Since \( BS(1,p^2) \) and \( BS(1,p) \) are commensurable, hence quasi-isometric, we may assume that the horospheres are quasi-isometrically embedded copies of the complex \( X_p \) associated to \( BS(1,p) \).

3.6 Another view of the closeness lines

The following discussion of closeness lines provides some geometric intuition for understanding these lines but is not necessary for the proofs of Theorems A, B and C.

We can consider the collection of closeness lines of all horospheres of \( \partial \Omega_p \) with the horosphere \( \sigma_\infty \). This is a collection of distinct lines in the tree \( T_p \). These closeness lines can also be viewed as lying in the horosphere \( \sigma_\infty \), which is thought of as the complex \( X_p \) associated to \( BS(1,p) \). In this case, the closeness line of \( \sigma_\infty \) and \( \sigma_\alpha \) can be viewed as the set of points in \( \sigma_\infty \) closest to \( \sigma_\alpha \) which project under \( \pi : H^2 \times T_p \to T_p \) to the closeness line of \( \sigma_\infty \) and \( \sigma_\alpha \) as described in §3.4. (Figure 4.) Then each hyperbolic plane \( H^2 \subset X_p = \sigma_\infty \) intersects these closeness lines, with at most one line lying completely in that plane. (Figure 5.)

4 The Boundary Detection Theorem

The first goal of this section is to prove Theorem 4.3 (Boundary Detection Theorem), which states that a quasi-isometry \( f : \Omega_p \to \Omega_q \) maps a horosphere boundary component of \( \Omega_p \) to within a bounded Hausdorff
Figure 3: The two horospheres shown are $\sigma_\alpha$ and $\sigma_\infty$, for $\alpha \neq \infty$. The dark line in $\sigma_\infty$ represents the closest points in $\sigma_\infty$ to $\sigma_\alpha$. We can view this line as the closeness line of $\sigma_\infty$ and $\sigma_\alpha$ drawn in $\sigma_\infty$.

Figure 4: A subset of the collection of closeness lines of all horospheres with $\sigma_\infty$, drawn in the complex $X_p$ of $\sigma_\infty$. 

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distance of a horosphere boundary component of $\Omega_q$. Two subsets $X$ and $Y$ of a metric space $W$ have bounded Hausdorff distance if there exists an $\epsilon > 0$ so that $X \subset \text{Nbhd}_\epsilon(Y)$ and $Y \subset \text{Nbhd}_\epsilon(X)$. The infimum of such constants $\epsilon$ is called the Hausdorff distance between $X$ and $Y$.

The second goal is to use Theorem 4.5, combined with Theorem 2.1, to prove Theorem B, i.e. that $\text{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ and $\text{PSL}_2(\mathbb{Z}[\frac{1}{q}])$ are quasi-isometric if and only if $p = q$.

We use the notation $\text{Nbhd}_r(S)$ to be the $r$-neighborhood in $\Omega$ of a subset $S$ of $\Omega$. We say that a subset $S$ of a metric space $X$ has the strong separation property in $X$ if there is a fixed $r > 0$ with the following property. For every $k > 0$, there are at least two connected components of $X - \text{Nbhd}_r(S)$ which contain metric balls of radius $k$. We will say that $S$ separates $X$ if $S$ has the strong separation property in $X$. The constant $r$ is called the separation constant.

A metric space $X$ is called uniformly contractible if there is a function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ with the following property. If $f : \Delta \to X$ is a map of a finite simplicial complex, and $f(\Delta) \subset B_r$, where $B_r \subset X$ is a metric $r$-ball, then $f(\Delta)$ is contractible in $B_{\alpha(r)}$, where $\alpha$ is independent of $\dim(\Delta)$. Any contractible space admitting a cocompact group of isometries is uniformly contractible.

We will need the following coarse topology results which we state as special cases of Theorem 5.2 and Corollary 5.3 of [FS]. We are using the bounded geometry metric $d$ on $\mathbb{R}^3$ described below which comes from choosing a proper embedding of the tree $T_p \to \mathbb{R}^2$. This allows us to consider $T_p \times \mathbb{R}$ as a subset of $(\mathbb{R}^3, d)$. In applying the results of [FS] we use the fact that $\mathbb{R}^3$ in this bounded geometry metric is uniformly contractible and contains spheres of arbitrarily large radius (the “expanding spheres” condition of [FS]).

**Theorem 4.1 (Coarse Separation [FS]).** Suppose $\phi : (\mathbb{R}^3, d) \to Y$ is a $(K, C)$—quasi-isometric embedding of $(\mathbb{R}^3, d)$ into a uniformly contractible Riemannian manifold $Y$ diffeomorphic to $\mathbb{R}^4$. Then $\phi(\mathbb{R}^3)$ separates $Y$, where the separation constant depends on $(K, C)$.

**Corollary 4.2 (Packing Theorem [FS]).** Suppose $\phi : (\mathbb{R}^3, d) \to (\mathbb{R}^3, d)$ is a $(K, C)$—quasi-isometric embedding. Then $\phi$ is a $(K', C')$—quasi-isometry, for some constants $(K', C')$ depending on $(K, C)$.

### 4.1 Separation

Let $\sigma$ be any horosphere boundary component of $\Omega_p$ and $f : \Omega_p \to \Omega_q$ a quasi-isometry. In this section we show that the image $f(\sigma)$ separates $H^2 \times T_q$. To prove this, we extend the quasi-isometric embedding $f|_\sigma : \sigma \to H^2 \times T_q$ to a quasi-isometric embedding $\tilde{f} : (\mathbb{R}^3, d) \to H^2 \times \mathbb{R}^2$. 


to which we can apply Theorem 4.1 (Coarse Separation). As above, $d$ is the bounded geometry metric on $\mathbb{R}^3$ which comes from choosing a proper embedding of the tree $T_p \to \mathbb{R}^2$.

Consider $\sigma$ as the complex $X_p$ associated to $BS(1,p)$ and choose a homeomorphism $\beta : X_p \to T_p \times \mathbb{R}$. Let $\alpha_p : T_p \to \mathbb{R}^2$ be any proper embedding. Then we can consider $X_p$ as a subset of $\mathbb{R}^3$ via the map $(\alpha \times Id) \circ \beta$. It is shown in [FM] that $\mathbb{R}^3$ can be given a bounded geometry metric $d$ for which this map is an isometric embedding, as follows. The boundary of each connected component $C$ of $\mathbb{R}^3 - X_p$ is topologically a plane. We use two coordinates $(t, r)$ on this plane $\partial C$, where $t \in T_p$ and $r \in \mathbb{R}$. Choose a homeomorphism which identifies $\partial C \cup C$ with $\partial C \times [0, \infty)$. Then a point in $C$ has three coordinates: $(t, r, s)$ where $t$ and $r$ as above give a point in $\partial C$ and $s \in [0, \infty)$. We use the product metric on each component.

The quasi-isometric embedding $f|_{\sigma}$ has image in $\mathbb{H}^2 \times T_q$. Analogous to the above situation, we choose a proper embedding $\alpha_q : T_q \to \mathbb{R}^2$ and view $\mathbb{H}^2 \times T_q$ as a subset of $\mathbb{H}^2 \times \mathbb{R}^2$ via the map $Id \times \alpha$. As above, we obtain a metric $d'$ on $\mathbb{H}^2 \times \mathbb{R}^2$ for which this map is an isometric embedding. To apply Theorem 4.1 we use the fact that $(\mathbb{H}^2 \times \mathbb{R}^2, d')$ is diffeomorphic to $\mathbb{R}^4$, uniformly contractible and contains arbitrary large metric balls.

We can now extend the map $f|_{\sigma}$ to a quasi-isometric embedding

$$\tilde{f} : (\mathbb{R}^3, d) \to (\mathbb{H}^2 \times \mathbb{R}^2, d')$$

by

$$\tilde{f}(t, r, s) = (f(t, r), s).$$

Note that $f(t, r)$ is a point in $\mathbb{H}^2 \times T_q$, and so provides two coordinates. Also, $\tilde{f}|_{\sigma} = f$.

Proposition 4.3 (Separation). Let $f : \Omega_p \to \Omega_q \subset \mathbb{H}^2 \times T_q$ be a $(K, C)$–quasi-isometry, and let $\sigma \subset \partial \Omega_p$ be any horosphere boundary component. Then $f(\sigma)$ separates $\mathbb{H}^2 \times T_q$, where the separation constant depends on $(K, C)$.

Proof. Extend $f|_{\sigma}$ as above to a quasi-isometric embedding

$$\tilde{f} : (\mathbb{R}^3, d) \to (\mathbb{H}^2 \times \mathbb{R}^2, d').$$

It is understood that a path “avoiding” $f(\sigma)$ or $\tilde{f}(\mathbb{R}^3)$ stays outside the neighborhood $\text{Nbhd}_r(f(\sigma))$ or $\text{Nbhd}_r(\tilde{f}(\mathbb{R}^3))$ for a constant $r$ or $r'$. Also note that $f(\sigma) = \tilde{f}(\sigma)$.

The map $\tilde{f} : (\mathbb{R}^3, d) \to (\mathbb{H}^2 \times \mathbb{R}^2, d')$ satisfies the conditions of Theorem 4.1, so $\tilde{f}(\mathbb{R}^3)$ separates $\mathbb{H}^2 \times \mathbb{R}^2$.

Suppose that $\tilde{f}(\mathbb{R}^3)$ separates $\mathbb{H}^2 \times \mathbb{R}^2$ and $\tilde{f}(\sigma)$ does not separate $\mathbb{H}^2 \times T_q$. This means that any two points in $(\mathbb{H}^2 \times T_q) - \tilde{f}(\sigma)$ can be
joined by a path avoiding \( \hat{f}(\sigma) \). We will show that if \( \hat{f}(\sigma) \) does not separate \( H^2 \times T_q \), any two points in \( (H^2 \times \mathbb{R}^2) - \hat{f}(\mathbb{R}^3) \) can be connected by a path avoiding \( \hat{f}(\mathbb{R}^3) \), contradicting the fact that \( \hat{f}(\mathbb{R}^3) \) separates \( H^2 \times \mathbb{R}^2 \). Let \( x_1 \) and \( x_2 \) be any two points in \( (H^2 \times \mathbb{R}^2) - \hat{f}(\mathbb{R}^3) \). Each \( x_i, i = 1, 2 \), has coordinates \( (\alpha_i, t_i, s_i) \) as above. Since \( x_i \notin \hat{f}(\mathbb{R}^3) \), each line \( \{(\alpha_i, t_i, s)|s \in \mathbb{R}\} \) is not contained in \( \hat{f}(\mathbb{R}^3) \) by construction. When \( s = 0 \), each line gives a point in \( H^2 \times T_q \) not contained in \( \hat{f}(\sigma) \). Call these points \( \beta_i \). Since \( \hat{f}(\sigma) \) does not separate \( H^2 \times T_q \), we can connect \( \beta_1 \) to \( \beta_2 \) by a path \( \gamma \) lying in \( H^2 \times T_q \) which avoids \( \hat{f}(\sigma) \) and hence \( \hat{f}(\mathbb{R}^3) \). So \( x_1 \) and \( x_2 \) are connected by the path

\[
(\alpha_1, t_1, [0, s_2]) \ast \gamma \ast (\alpha_2, t_2, [0, s_1])^{-1}
\]

which avoids \( \hat{f}(\mathbb{R}^3) \). Thus, if \( \hat{f}(\mathbb{R}^3) \) separates \( \mathbb{R}^2 \times H^2 \) then \( f(\sigma) = \hat{f}(\sigma) \) separates \( H^2 \times T_q \). \( \square \)

We now prove a lemma which shows that the space \( \Omega_p \) with the neighborhood of any horosphere removed is path connected. This lemma is needed in the proof of Theorem 1.5.

**Lemma 4.4.** Let \( r \) be any positive real number. For any horosphere boundary component \( \sigma \) of \( \Omega_p \), the space \( \Omega_p - \text{Nbhd}_r(\sigma) \) is path connected.

**Proof.** We will show that \( \Omega_p - \text{Nbhd}_r(\sigma) \) is path connected for any \( \sigma \subset \partial \Omega \) and for any \( r \). For convenience we may assume that \( \sigma = \sigma_\infty \).

For any horosphere \( \tau \subset \partial \Omega_p \), consider the set of vertices of \( T_p \) where \( \tau \) intersects \( \text{Nbhd}_r(\sigma_\infty) \):

\[
S_{\tau, \sigma} = \{[L] \in \text{Vert}(T_p) \ | \ [L] \cap \text{Nbhd}_r(\sigma_\infty) \neq \emptyset\}.
\]

Since each \( \tau \) has a closeness line with \( \sigma_\infty \), for large enough \( r \) the set \( S_{\tau, \sigma} \) is the neighborhood in \( T_p \) of a line \( l_\tau \subset T_p \). There is one line \( l_\tau \) for each horosphere \( \tau \neq \sigma_\infty \). Since the horosphere boundary components of \( \Omega_p \) are indexed by rationals and the set of lines of \( T_p \) is indexed by elements of \( \mathbb{Q}_p \), there are lines \( l \subset T_p \) such that no horosphere intersects \( \text{Nbhd}_r(\sigma_\infty) \) over every point of \( l \). Let \( l' \) be such a line.

Choose two points \( x \) and \( y \) in \( \Omega_p - \text{Nbhd}_r(\sigma) \). If \( x \) and \( y \) lie in \( H^2 \times l' \), then there is a path connecting them. If this is not the case, let \( \alpha \) be any path from \( x \) to a point \( x' \in H^2 \times l' \), and \( \beta \) any path from \( y \) to a point \( y' \in H^2 \times l' \). Let \( \gamma \) be a path in \( H^2 \times l' \) connecting \( x' \) and \( y' \). The composite path \( \beta^{-1} \circ \gamma \circ \alpha \) connects \( x \) to \( y \) and lies in \( \Omega_p - \text{Nbhd}_r(\sigma) \), proving the theorem. \( \square \)
4.2 Proof of the Boundary Detection Theorem

We now state and prove Theorem 4.5, which plays a role in the proofs of Theorems A and B.

**Theorem 4.5 (Boundary Detection Theorem).** Let \( f : \Omega_p \to \Omega_q \) be a \((K,C)\)-quasi-isometry. There exist constants \((K',C')\) depending on \((K,C)\) and the spaces \(\Omega_p\) and \(\Omega_q\), with the following property. For every horosphere boundary component \(\sigma\) of \(\Omega_p\), there is a horosphere boundary component \(\tau\) of \(\Omega_q\) so that \(f|_{\sigma} : \sigma \to \tau\) is a \((K',C')\)-quasi-isometry.

The following lemmas form the two major components of the proof of Theorem 4.5. In these lemmas, \(f : \Omega_p \to \Omega_q\) will be a \((K,C)\)-quasi-isometry.

**Lemma 4.6.** There exists a constant \(\epsilon\) depending on \((K,C)\) and the spaces \(\Omega_p\) and \(\Omega_q\), with the following property. For every horosphere boundary component \(\sigma\) of \(\Omega_p\), there is a horosphere boundary component \(\tau\) of \(\Omega_q\) so that \(\tau \subset \text{Nbhd}_\epsilon(f(\sigma))\).

**Proof.** Consider a horosphere boundary component \(\sigma\) of \(\Omega_p\). We will show that any two points \(x_1\) and \(x_2\) of \(H^2 \times T_q\) which are outside \(\text{Nbhd}_\delta(f(\sigma))\) (for some constant \(\delta\)) can be connected by a path avoiding \(\text{Nbhd}_\delta(f(\sigma))\). If the lemma is false, then any horosphere \(\tau \subset \partial \Omega_q\) contains points not within \(\epsilon\) of \(f(\sigma)\). For the proper choice of \(\epsilon\), this will contradict the fact that \(f(\sigma)\) separates \(H^2 \times T_q\).

Choose points \(x_1, x_2 \notin \text{Nbhd}_\delta(f(\sigma))\). If \(x_i \notin \Omega_q\), \(i = 1, 2\), we can find a path connecting \(x_i\) to some \(x'_i \in \Omega_q\) which avoids \(\text{Nbhd}_\delta(f(\sigma))\). Hence we will assume that \(x_i \in \Omega_q\).

Let \(f^{-1}\) be a coarse inverse to \(f\) and consider the points \(y_i = f^{-1}(x_i) \subset \Omega_p\) for \(i = 1, 2\). If \(\delta\) is sufficiently large, these points lie outside \(\text{Nbhd}_{\delta_1}(\sigma)\), for some \(\delta_1\). From Lemma 4.4 we know that there is a path \(\gamma\) between \(y_1\) and \(y_2\) which avoids \(\text{Nbhd}_{\delta_1}(\sigma)\). Then for some \(\epsilon\), \(f(\gamma)\) is a path in \(\Omega_q\) connecting \(x_1\) and \(x_2\) but avoiding \(\text{Nbhd}_\delta(f(\sigma))\). (Figure 7.) If our initial \(\delta\) was large enough, then \(\epsilon > \delta\) and we can make this argument with \(x_1 \in \tau\), \(x_1 \notin \text{Nbhd}_\delta(f(\sigma))\), contradicting the fact that \(f(\sigma)\) separates \(H^2 \times T_q\). \(\square\)

**Lemma 4.7.** There exists a constant \(\epsilon'\) depending on \((K,C)\) and the spaces \(\Omega_p\) and \(\Omega_q\), with the following property. For every horosphere boundary component \(\sigma\) of \(\Omega_p\), there is a horosphere boundary component \(\tau\) of \(\Omega_q\) so that \(f(\sigma) \subset \text{Nbhd}_{\epsilon'}(\tau)\).

**Proof.** Consider a horosphere boundary component \(\sigma\) of \(\Omega_p\). From Lemma 4.6, there is a horosphere boundary component \(\tau\) of \(\Omega_q\) and a constant \(\epsilon\) so that \(\tau \subset \text{Nbhd}_\epsilon(f(\sigma))\). Define a map \(\psi : \tau \to \sigma\) by
\[ \psi(y) = x \in \sigma, \] where \( x \) is any point so that \( f(x) \) is metrically closest to \( y \). If there is more than one such point, choose randomly. From Lemma 4.6, we see that \( \psi(\tau) \) separates \( \mathbb{H}^2 \times T_q \). So for some constant \( \delta' \), the horosphere \( \sigma \) is contained in \( \text{Nbhd}_{\delta'}(\psi(\tau)) \), i.e. every point of \( \sigma \) is within a constant \( \delta' \) of some point \( x \in \psi(\tau) \) which maps to a point \( f(x) \) within \( \epsilon \) of a point of \( \tau \). By enlarging \( \delta' \) to a constant \( \epsilon' \), we see that \( f(\sigma) \subset \text{Nbhd}_{\epsilon'}(\tau) \). Since all the horospheres are isometric, the constant \( \epsilon' \) is independent of the choice of \( \sigma \).

**Proof of the Boundary Detection Theorem.** Apply Lemma 4.7 to both \( f \) and \( f^{-1} \). Compose \( f \) with a nearest point projection to obtain a quasi-isometric embedding \( \tilde{f} : \sigma \to \tau \). As in §4.1, embed \( \sigma \) and \( \tau \) isometrically in \( (\mathbb{R}^3, d) \) and extend \( f \) to a quasi-isometric embedding \( f : (\mathbb{R}^3, d) \to (\mathbb{R}^3, d) \). (Recall that \( d \) was the bounded geometry metric on \( \mathbb{R}^3 \) described in §4.1.) From the Packing Theorem, this map is a \((K', C')\)-quasi-isometry, where the pair \((K', C')\) depends on \((K, C)\). Then by the construction of \( \tilde{f} \), the map \( f \) must also be a \((K', C')\)-quasi-isometry. It is now clear that closeness lines are preserved under quasi-isometry.

### 4.3 Proof of Theorem B

Theorem 4.3 allows us to prove Theorem B.

**Proof of Theorem B.** If \( PSL_2(\mathbb{Z}\left[\frac{1}{p}\right]) \) and \( PSL_2(\mathbb{Z}\left[\frac{1}{q}\right]) \) are commensurable, then they are automatically quasi-isometric.

Let \( f : PSL_2(\mathbb{Z}\left[\frac{1}{p}\right]) \to PSL_2(\mathbb{Z}\left[\frac{1}{q}\right]) \) be a \((K, C)\)-quasi-isometry. Construct the spaces \( \Omega_p \) and \( \Omega_q \) corresponding to \( PSL_2(\mathbb{Z}\left[\frac{1}{p}\right]) \) and \( PSL_2(\mathbb{Z}\left[\frac{1}{q}\right]) \), respectively. Then \( f \) induces a quasi-isometry, also denoted \( f \), from \( \Omega_p \) to \( \Omega_q \). Theorem 4.3 allows us to restrict \( f \) to a quasi-isometry \( \hat{f} \) on horospheres. In §2.3 we showed that a horosphere of \( \mathbb{H}^2 \times T_p \) has the geometry of the group \( BS(1, p) \). Hence \( f \) is a quasi-isometry from \( BS(1, p) \) to \( BS(1, q) \). According to Theorem 2.1, we must have \( p = q \) for such a quasi-isometry to exist.

### 5 Theorems A and C

Every commensurator of \( PSL_2(\mathbb{Z}\left[\frac{1}{p}\right]) \) acts as a quasi-isometry of \( \Omega_p \). To prove Theorem A, we must show that by composing an element \( f \in QI(PSL_2(\mathbb{Z}\left[\frac{1}{p}\right])) \) with a specific commensurator, we obtain a map which is a bounded distance from the identity map. It is Theorem 4.3 (stated below) that tells us which commensurator to choose for this.
purpose. In the Appendix, we show that the commensurator group of \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) in \( PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p) \) is \( PSL_2(\mathbb{Q}) \), where \( PSL_2 \) is viewed as an algebraic group as in \( \text{§2.2} \).

In all that follows, we assume that \( f : \Omega \to \Omega_p \) is a \((K,C)\)-quasi-isometry which has been changed by a bounded amount using the “connect the dots” procedure so that it is continuous. (See, e.g. [FS].) Since \( \text{Comm}(PSL_2(\mathbb{Z}[\frac{1}{p}])) = PSL_2(\mathbb{Q}) \) acts transitively on pairs of distinct points of \( R \cup \{\infty\} \), we can assume that \( f \) has been composed with a commensurator so that \( f(\sigma_\infty) = \sigma_\infty \) and \( f(\sigma_0) = \sigma_0 \). (Note that these horospheres are not necessarily fixed pointwise.)

### 5.1 Action Rigidity

We will now define a boundary of the space \( \mathbb{H}^2 \times T_p \). Let \( t \) be the common endpoint of any two lines in \( T_p \) and consider \( \partial_\infty(\mathbb{H}^2 \times t) = \{\infty\} \cong \mathbb{R} \). Recall from \( \text{§2.2} \) that we can consider \( \sigma_\infty \subset \partial_\infty \) as a quasi-isometrically embedded copy of the 2-complex \( X_p \) associated to \( BS(1,p) \). So we can refer to the lower boundary \( \partial_l(\sigma_\infty) \) of \( \sigma_\infty \) (resp. the upper boundary \( \partial_u(\sigma_\infty) \)) as the lower (resp. upper) boundary of \( X_p \). The inclusion \( i : \sigma \to \mathbb{H}^2 \times T_p \) induces an identification between \( \partial_l(\sigma_\infty) \) and the copy of \( \mathbb{R} \) described above. Since \( f(\sigma_\infty) = \sigma_\infty \), the quasi-isometry \( f \) induces bilipschitz maps \( f_l \) and \( f_u \) on \( \partial_l(\sigma_\infty) \) and \( \partial_u(\sigma_\infty) \) of \( \sigma_\infty \), respectively. The restriction of \( f \) to \( R = \partial_\infty(\mathbb{H}^2 \times t) = \{\infty\} \) determines the permutation of the horospheres of \( \mathbb{H}^2 \times T_p \) under the map \( f \). Hence, from the identification induced by the inclusion map \( i \), the map \( f_l \) is exactly the map which determines the permutation of the horospheres under \( f \).

Let \( \Delta_Q \) denote the lattice in \( \mathbb{R} \times \mathbb{Q}_p \) given by the diagonal \( \{(a,a)\mid a \in \mathbb{Q}\} \subset \mathbb{R} \times \mathbb{Q}_p \), and \( \Delta \) the sublattice \( \{(b,b)\mid b \in \mathbb{Z}[\frac{1}{p}]\} \). Clearly \( \Delta \subset \Delta_Q \). We view the first coordinate of the pair \((b,b)\in \Delta_Q \) as \( b \in \mathbb{Q} \subset \mathbb{R} \) denoting the basepoint of a horosphere of \( \mathbb{H}^2 \times T_p \), and the second coordinate \( b \in \mathbb{Q} \subset \mathbb{Q}_p \) as the point in \( \mathbb{Q}_p \), determined by the closeness line of \( \sigma_0 \) and \( \sigma_\infty \). Using the identification described above, we see that the map induced by \( f \) on \( \mathbb{Q} \subset \mathbb{R} \) is exactly \( f_l|\mathbb{Q} \). We can view \( \mathbb{Q} \subset \mathbb{Q}_p \) as a subset of \( \partial(\sigma_\infty) \). Since \( f(\sigma_\infty) = \sigma_\infty \), the map induced by \( f \) on \( \mathbb{Q} \subset \mathbb{Q}_p \) is exactly \( f_u|\mathbb{Q} \). The maps \( f_u|\mathbb{Q} \) and \( f_l|\mathbb{Q} \) are identical because closeness lines are preserved under quasi-isometry and \( f(\sigma_\infty) = \sigma_\infty \). Hence \( f \) induces a map of \( \Delta_Q \) given by \((f_l|\mathbb{Q}, f_u|\mathbb{Q})\), and we will use a single coordinate for points of \( \Delta_Q \). Let \( \phi \) denote the common restriction of \( f_l \) and \( f_u \) to \( \mathbb{Q} \). Then \( \phi \) is \( K_0 \)-bilipschitz, for some constant \( K_0 \) depending on the pair \((K,C)\).

Let \( H \) be the cyclic group generated by the matrix \( \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix} \). We make the following definitions relating to a group-invariant diameter
function which will allow us to state Theorem 5.3 (Action Rigidity).

Definitions.

1. For any subset $S$ of $\mathbb{R} \times \mathbb{Q}_p$ and $H$ as above, define the $H$-invariant diameter of $S$ by
   $$\delta_H(S) = \inf_{T \in H} \text{diam}(T(S)).$$
   For the remainder of this paper we will write $\delta$ for $\delta_H$.

2. For subsets $S_1, S_2$ of $\mathbb{R} \times \mathbb{Q}_p$, we say that the map $\phi : S_1 \to S_2$ is quasi-adapted to $\delta$ if there exists a map $\alpha : \mathbb{N} \to \mathbb{N}$ such that for any compact set $V$,
   $$\delta(V) \leq k \Rightarrow \delta(\phi(V)) \leq \alpha(k)$$
   and
   $$\delta(\phi(V)) \leq k \Rightarrow \delta(V) \leq \alpha(k).$$

3. Let $S$ be a subset of $\Delta_Q$. We say that $S$ has bounded height if $S \subset \frac{1}{M}\Delta$ for some $M \in \mathbb{Z}^+$ with $(M, p) = 1$.

4. A bijection $\phi : \Delta_Q \to \Delta_Q$ is said to be quasi-integral if both $\phi$ and $\phi^{-1}$ take sets of bounded height to sets of bounded height.

5. A bijection $\phi : \Delta_Q \to \Delta_Q$ is said to be quasicompatible with $H$ if $\phi$ is quasi-integral and, when restricted to sets of bounded height, both $\phi$ and $\phi^{-1}$ are quasi-adapted to $\delta$.

In a sequence of lemmas we show that the bilipschitz map $\phi : \Delta_Q \to \Delta_Q$ obtained from the original quasi-isometry $f$ is quasi-compatible with $H$.

Associate to each point of $\Delta_Q$ the horosphere boundary component of $\Omega_p$ based at that point. Then the action of $\Gamma_\infty$ on $\partial \Omega_p$ induces an action on $\Delta_Q$. In particular, this action preserves denominators, i.e. $\Gamma_\infty \cdot \frac{1}{M}\Delta \subset \frac{1}{M}\Delta$. Consider the diameter function on subsets $S$ of $\Delta_Q$ defined by
   $$\delta_\Gamma(S) = \inf_{T \in \Gamma_\infty} \text{diam} T(S).$$

Also consider the following diameter function, with $S$ as above. Let $\delta_B(S)$ be the diameter in $\Omega_p$ of the smallest metric ball which intersects all horosphere boundary components based at points of $S$. We now show that these two diameter functions are quasi-identical when restricted to $\frac{1}{M}\Delta$, i.e. if $\delta_\Gamma(S)$ is small for a subset $S$ of $\Delta_Q$, then $\delta_B(S)$ is bounded and vice versa.

**Lemma 5.1.** The restrictions of $\delta_\Gamma$ and $\delta_B$ to $\frac{1}{M}\Delta$ are quasi-identical.
Proof. Let $S$ be a subset of $\Delta_Q$ so that $\delta_B(S)$ is small, say $\delta_B(S) = \epsilon$. Then there exists a point $x \in \Omega_p$ which is within $\epsilon$ of all horosphere boundary components based at points of $S$. Since $\Omega_p/\text{PSL}_2(\mathbb{Z}[1/p])$ is compact by construction, there are only a finite number of choices for the set $S$ (modulo $\text{Aut}(\Delta)$). Thus $\delta_B(S)$ must be bounded.

Now suppose $\delta_B(S)$ is small. By compactness there are only finitely many choices for $S$ (modulo $\text{Aut}(\Delta)$). Hence $\delta_B(S)$ is bounded.

Lemma 5.2. For every $M \in \mathbb{Z}^+$, there exists $M' \in \mathbb{Z}^+$ depending on the bilipschitz constant $K_0$ of $\phi$ and the space $\Omega_p$, so that $\phi(\frac{1}{M} \Delta) \subset \frac{1}{M'} \Delta$.

Proof. Consider the action of $\Gamma_\infty = \text{Stab}_{\text{PSL}_2(\mathbb{Z}[\frac{1}{p}])}(\infty)$ on $\frac{1}{M/\Delta}$. This action preserves denominators, i.e. $\Gamma_\infty \cdot \frac{1}{M/\Delta} \subset \frac{1}{M'} \Delta$. In particular, under this action $\frac{1}{M/\Delta}/\Gamma_\infty$ consists of a finite set of points, which we view as a finite collection of horospheres $\sigma_1, \ldots, \sigma_n \subset \partial \Omega_p$. Choose a point $x \in \sigma_\infty$ and consider the smallest metric ball in $\Omega_p$ based at $x$ which intersects all of the $\sigma_i$. Let $\epsilon$ be the radius of this ball. There is an $\epsilon'$ (depending on $\epsilon$) so that the $\epsilon'$ ball around $f(x)$ must intersect all of the $f(\sigma_i)$. Thus there are only a finite number of choices of horospheres in $\partial \Omega_p$ for the images $f(\sigma_i)$. It follows that there is a number $M' \in \mathbb{N}$ so that the collection of horospheres $\{f(\sigma_i)\}$ is based at points in $\frac{1}{M'} \Delta$. Since $\Gamma_\infty$ preserves denominators, the image of $\sigma_\alpha$ (for any $\alpha \in \frac{1}{M'} \Delta$) must be a horosphere based at a point of $\frac{1}{M'} \Delta$. This is equivalent to saying that $\phi(\frac{1}{M} \Delta) \subset \frac{1}{M'} \Delta$. \[ \square \]

Lemma 5.2 shows that $\phi$ is quasi-integral, and since $\phi$ is a bilipschitz map of both $\mathbb{R}$ and $\mathbb{Q}_p$, it is quasi-adapted to $\delta$. Hence $\phi$ is quasicompatible with $H$.

We can now state Theorem 5.3. We say that a map $\psi : \mathbb{R} \times \mathbb{Q}_p \rightarrow \mathbb{R} \times \mathbb{Q}_p$ is affine if its restriction to each factor is affine. This theorem is an 2−dimensional $S$-arithmetic version of the Action Rigidity Theorem of R. Schwartz [S1].

Theorem 5.3 (Action Rigidity). Let $\Delta_Q \subset \mathbb{R} \times \mathbb{Q}_p$ be the lattice $\{(a,a)|a \in \mathbb{Q}\}$ and let $H$ be the group generated by the matrix $\begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix}$. Then any bilipschitz map $\phi : \Delta_Q \rightarrow \Delta_Q$ which is quasi-compatible with $H$ is the restriction of an affine map of $\mathbb{R} \times \mathbb{Q}_p$.

5.2 The Parallelogram Lemma

The key lemma in the proof of Theorem 5.3 is Lemma 5.4 (Parallelogram Lemma). Let $\Delta_Q$ be as in §5.3 and let $\phi : \Delta_Q \rightarrow \Delta_Q$ be quasi-compatible with $H$ and $K_0$−bilipschitz. We now make some preliminary definitions.
Definition. Let $M \in \mathbb{Z}^+$. A parallelogram $P$ in $\frac{1}{M}\Delta$ is a quadruple of points $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ with $a, b, c, d \in \frac{1}{M}\Delta$ satisfying $a - c = b - d$.

We say that $\phi(P)$ is the quadruple $\left[\begin{array}{cc} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{array}\right]$. The goal of Lemma 5.4 is to determine when $\phi(P)$ is itself a parallelogram. We now define two quantities associated to a parallelogram $P$ which will be invariant under the group action and translation.

Definition. Let $P = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ be a parallelogram. The $H$-invariant perimeter of $P$ is given by

\[ \text{per}(P) = \delta(a \cup b) + \delta(a \cup c). \]

The shape of $P$ is given by

\[ \text{shape}(P) = |\nu(b - a) - \nu(c - a)| \]

where $\nu(p^n x) = n$ is the $p$-adic valuation on $\mathbb{Q}_p$. For any $T \in H$ and $x \in \Delta_Q$ we have $\text{per}(P) = \text{per}(T(P) + x)$ and $\text{shape}(P) = \text{shape}(T(P) + x)$.

Let $P = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ be a parallelogram such that $\phi(P) = \left[\begin{array}{cc} a' & b' \\ c' & d' \end{array}\right]$ is again a parallelogram. Since $\phi$ is quasi-compatible with $H$, there exists a constant $D$ (depending on $\text{per}(P)$ and $K$) such that

\[ \delta(a' \cup b') + \delta(a' \cup c') \leq D. \]

By symmetry we also have $\delta(c' \cup d') + \delta(b' \cup d') \leq D$. From lemma 5.2 we obtain a constant $k$ such that $\phi(\frac{1}{k}\Delta) \subset \frac{1}{k}\Delta$.

We now describe the points $x \in \frac{1}{k}\Delta$, $(k, p) = 1$, satisfying $\delta(0 \cup x) \leq D$.

\[ S_{k, D} = \{ x \in \frac{1}{k}\Delta | x \neq 0 \text{ and } \delta(0 \cup x) \leq D \}. \]

The set $S_{k, D}$ is the orbit under $G$ of a finite set of points of $\frac{1}{k}\Delta$, denoted $\{ \frac{1}{k}a_i \}_{i \in I}$, where $(a_i, p) = 1$. Since $\delta$ is invariant under translation, the points of $\frac{1}{k}\Delta$ within $D$ of some point $y \in \frac{1}{k}\Delta$ are given by $S_{k, D} + y$.

We use the following notation in the statement of Lemma 5.4. Let

\[ B_1 = \max_{i,j \in I} \{ a_i - a_j \}, \quad B_2 = \max_{i \in I} \{ a_i \}, \]

\[ B_3 = \max_{i,j \in I} \{ |\nu(a_i - a_j)|, |\nu(a_i + a_j)| \} \]

and

\[ B_4 = \max_{i,j,k \in I} \{ \nu(a_i + a_j - a_k) \}. \]
Lemma 5.4 (Parallelogram Lemma). Let $\phi : \Delta_Q \rightarrow \Delta_Q$ be quasi-compatible with $H$ and $K_0$-bilipschitz. Let $\log_p(K_0) = R$ and $L \in \mathbb{N}$, $(L, p) = 1$. If $P$ is a parallelogram in $\frac{1}{p} \Delta$ with
\[
\text{per}(P) \leq L \text{ and } \text{shape}(P) > s_0
\]
then $\phi(P)$ is also a parallelogram.

Proof. Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\phi(P) = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$. We are assuming that $\phi(0) = 0$, so without loss of generality translate $P$ so that $a = 0$ and hence $a' = 0$. As stated above, there is a constant $D$ so that $\delta(0 \cup b') + \delta(0 \cup c') \leq D$ and $\delta(b' \cup d') + \delta(c' \cup d') \leq D$.

We can write $b' = p^m x$ and $c' = p^m y$ where $x = a_i$ and $y = a_j$ for some $a_i, a_j$ as above. Then $d'$ can be expressed in one of two ways. It is $\delta$-close to both $b'$ and $c'$, hence of the form $p^m x + p^N w$ and also $p^m x + p^M w$ where $z = a_i, w = a_j$ for some $i, j \in I$. Setting these expressions equal and clearing denominators yields
\[
p^n x + p^N z = p^m y + p^M w. \quad (\ast)
\]
Note that $x, y, z, w$ are all relatively prime to $p$.

We first obtain a lower bound on $n - m$, with $n, m$ as above. We can write $b = p^{n_0} \frac{x_0}{L}$ and $c = p^{m_0} \frac{x_0}{L}$, with $(x_0, p) = 1$ and $(y_0, p) = 1$, chosen so that $\text{shape}(P) = n_0 - m_0$. We know by assumption that $n_0 - m_0 > s_0$. Since $\phi$ is a $K_0$-bilipschitz map on $\mathbb{Q} \subset \mathbb{Q}_p$ and $\phi(0) = 0$, letting $R = \log_p(K_0)$ we obtain
\[
|p^{-n_0 - R} \phi(p^{n_0} \frac{x_0}{L})|_p = |p^n \frac{x}{K}|_p \leq p^{-n_0 + R}.
\]
Hence $n_0 - R \leq n \leq n_0 + R$; similarly, $m_0 - R \leq m \leq m_0 + R$. Thus,
\[
(n_0 - m_0) - 2R \leq n - m \leq (n_0 - m_0) + 2R.
\]
Using the fact that $n_0 - m_0 > s_0$, we see that
\[
n - m \geq s = \max\{2 \log_p(B_1 + B_2), 3B_3^2, 2B_4\}.
\]

Consider again the expression $(\ast)$ for $d'$. First we assume that not all exponents are equal. If all the exponents were equal, then $n$ would equal $m$, which is impossible since $n - m > s > 0$. Suppose first that $n > N$. Since the $p$-norms of both sides must be equal, we know that $N = \min\{m, M\}$.
Case 1. Suppose \( N = m \) and \( M \neq m \). Then (\( \ast \)) simplifies to
\[
p^{n-m}x = p^{M-m}w + (y - z).
\]
We know that \( n - m > s \). Suppose \( y - z \neq 0 \). The highest power of \( p \) dividing the right hand side of the equation is bounded by \( B_3 \). But by the choice of \( s \), we have \( n - m > B_3 \); hence this situation cannot occur.

If by chance \( M - m = B_3 \), and \( y - z = p^{B_3t} \) for some integer \( t \), then note that \( \nu(w + t) \) is also bounded. If this is the case, change the initial value of \( s_0 \) by the quantity \( \nu(w + t) \), and we may assume that we are not in this case.

So we must have \( y - z = 0 \), i.e. \( z = y \) and \( p^{n-m}x = p^{M-m}w \). Thus \( n = M \) and \( x = w \), and \( \phi(P) \) is a parallelogram.

Case 2. Suppose \( N = M \) and \( M \neq m \). Then (\( \ast \)) simplifies to
\[
p^{m-M}(p^{n-m}x - y) = w - z.
\]
Since \( m - M > 0 \) and \( w - z \leq B_1 \), our choice of \( s_0 \) insures that \( p^{m-M}(p^{n-m}x - y) > B_1 \), a contradiction.

Case 3. Suppose that \( N = M = m \). Then (\( \ast \)) simplifies to
\[
p^{n-m}x = y + z - w.
\]
By our choice of \( s_0 \), the exponent \( m - n > B_4 \), a contradiction.

Now suppose that \( n < N \). Then we have \( n = \min\{m, M\} \). Since \( n - m > s > 0 \), we cannot have \( n = m \).

Case 4. Suppose that \( n = M \). But then \( M < m < n = M \), a contradiction.

Lastly, we consider the case \( n = N \).

Case 5. Suppose \( n = N \). Then (\( \ast \)) simplifies to
\[
p^n(x + z) = p^my + p^MW.
\]
The quantity \( x + z \) can take one of a finite number of values, so we can write \( x + z = p^bf \) where \( (f, p) = 1 \) and \( h \) is bounded by \( B_3 \). Then we must have \( n + h = \min(m, M) \). If \( n + h = m \), then \( -h = n - m > s \) which cannot happen, since \( |h| \leq B_3 \) but \( n - m > B_3 \). If \( n + h = M \), then (\( \ast \)) becomes \( p^{n-m+b}(f - w) = y \). Again by the choice of \( s_0 \), the left hand side of the equation is greater than the right hand side.

5.3 Proof of the Action Rigidity Theorem
We now prove Theorem 5.3.
Proof of Theorem 5.3. Fix \( q \in \mathbb{N} \), \((q,p) = 1\) and let \( S \) be a generating set for \( \frac{1}{q} \Delta \) containing \( \frac{1}{q} \). Let \( s_0 \) be as in the Lemma 5.4, and let \( H(S) \) denote the orbit of \( S \) under \( H \). Given \( x, y \in \frac{1}{q} \Delta \) we say that \((x,y)\) is a distinguished pair if \( x - y \in H(S) \). Let \( C = \max_{a \in H(S)}(q, 2\delta(0 \cup a)) \).

We write \( P((x,y),(z,w)) \) if \( P = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \) is a parallelogram with \( P \subset \frac{1}{q} \Delta \), \( \text{per}(P) \leq C \) and \( \text{shape}(P) \geq s_0 \). Then by the Parallelogram lemma, \( \phi(P) \) will also be a parallelogram, i.e.

\[
\phi(x) - \phi(z) = \phi(y) - \phi(w).
\]

We write \( \overline{P}((x,y),(z,w)) \) if there is a finite sequence of pairs \((a_i,b_i)\) for \( i = 0, \cdots, n \) with \((a_0,b_0) = (x,y)\) and \((a_n,b_n) = (z,w)\) such that \( P((a_i,b_i),(a_{i+1},b_{i+1})) \). This means that we have a sequence of parallelograms between the two given pairs of points, each satisfying the conclusions of the parallelogram lemma, and with any two consecutive parallelograms between the two given pairs of points, each satisfying the conclusions of the parallelogram lemma, and with any two consecutive parallelograms in this sequence sharing a common side. Concatenating these intermediate parallelograms allows us to conclude that the parallelogram given by the original pairs also satisfies the conclusions of the Lemma 5.4.

We first need the following lemma.

**Lemma 5.5.** Let \( a \in \frac{1}{q} \Delta \) be arbitrary, and let \((u,v)\) be a distinguished pair. Then \( \overline{P}((u,v),(u+a,v+a)) \).

*Proof.* Consider \( P_y = \begin{bmatrix} u & v \\ u+y & v+y \end{bmatrix} \) for any \( y \in H(S) \). By construction, we have \( \text{per}(P_y) \leq C \). Let \( Y = Y(u,v) \subset H(S) \) denote those \( \delta \) \( \eta \) \( Y \) \( \phi \) \( \phi \) \( \text{shape}(P_y) \geq s_0 \). Let \( \Sigma_0 Y \) be the sublattice in \( \Delta \) generated by \( Y \) and \( \Sigma Y = \Sigma_0 Y \cap \frac{1}{q} \Delta \). We will first show that \( \overline{P}((u,v),(u+x,v+x)) \) for any \( x \in \Sigma Y \) and then show that \( \Sigma Y = \frac{1}{q} \Delta \). We prove this first assertion by induction. If \( x \in Y \), it is certainly true that \( P((u,v),(u+x,v+x)) \). If \( x \in \Sigma Y \), write \( x = x' + \nu \) for \( \nu \in Y \), and we have \( \overline{P}((u,v),(u+x',v+x')) \) by the induction hypothesis. Consider the parallelogram

\[
P = \begin{bmatrix} u + x' & v + x' \\ u + x' + y & v + x' + y \end{bmatrix} = \begin{bmatrix} u + x & v + x \\ u + x + v + x \end{bmatrix}.
\]

We see that \( \text{shape}(P) = \text{shape}(P_y) \geq s_0 \). Also, \( \delta(u + x' \cup u) = \delta(0 \cup y) \leq \frac{1}{C} \) and \( \delta(u + x' \cup v + x') = \delta(u \cup v) \leq \frac{1}{C} \). So we have \( \text{per}(P) \leq C \). Therefore we know that \( P((u+x',v+x'),(u+x,v+x)) \).

Now we show that \( \Sigma Y = \frac{1}{q} \Delta \). We know that \( \text{shape}(P_y) = \nu(u-v) - \nu(y) \). Let \( \omega = \nu(u-v) \). Then \( Y = \{ y \in H(S) | \omega - \nu(y) \geq s_0 \} \). Hence for large enough \( n \), we have \( \frac{1}{p^n} \in Y \), so \( \Sigma Y = \frac{1}{q} \Delta \), proving the lemma. \( \square \)

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Given $x \in \frac{1}{q}\Delta$, we can create a distinguished pair by taking $(x, x + a)$, for any $a \in S$. So for any $x, y \in \frac{1}{q}\Delta$ we know that $P((x, x + a), (y, y + a))$, or $\phi(x + a) - \phi(x) = \phi(y + a) - \phi(y)$. Since $S$ generates $\frac{1}{q}\Delta$ we know that for any $x, y, z \in \frac{1}{q}\Delta$ we have $P((x, x + z), (y, y + z))$ or $\phi(x + z) - \phi(x) = \phi(y + z) - \phi(y)$.

In particular, since $\phi(0) = 0$, we know that $\phi|_{\frac{1}{q}\Delta}$ is multiplication by a constant $C_q$. Let $q_1, q_2 \in \mathbb{N}$, with $(q_1, p) = (q_2, p) = 1$. Then we must have $C_{q_1q_2} = C_{q_1} = C_{q_2}$. Hence there is a constant $\alpha$ such that the map $\phi|_{\Delta\Omega}$ is multiplication by $\alpha$. It follows that $\phi$ is the restriction of an affine map of $\mathbb{R} \times \mathbb{Q}_p$.

5.4 Proof of Theorem A

We now use Theorem 5.3 to prove Theorem A.

Proof of Theorem A. It is clear that every commensurator of $PSL_2(\mathbb{Z}[\frac{1}{p}])$ gives rise to a unique quasi-isometry of $\Omega_p$. Given $f \in QI(PSL_2(\mathbb{Z}[\frac{1}{p}]))$, we will now choose a commensurator $g \in Comm(PSL_2(\mathbb{Z}[\frac{1}{p}]))$ so that the composition $g \circ f$ is a bounded distance from the identity map.

Let $f \in QI(PSL_2(\mathbb{Z}[\frac{1}{p}]))$. Then $f$ induces a quasi-isometry from $\Omega_p$ to itself. Compose $f$ with a commensurator so that $f(\sigma_\infty) = \sigma_\infty$ and $f(\sigma_0) = \sigma_0$. Combining Theorems 2.2 and 5.3 we know that $f$ induces a map $f_1$ on the lower boundary of $\sigma_\infty$ which is multiplication by a constant $\alpha$ and determines the permutation of the horospheres under $f$.

Recall from 2.2 that we view $PSL_2(\mathbb{Q})$ as the $\mathbb{Q}$-points $G'$ of $G' = Ad(SL_2(\mathbb{C}))$. Compose $f$ with the commensurator $g = Ad\left(\begin{array}{cc} 1/\alpha & 0 \\ 0 & 1/\alpha \end{array}\right)$. Then the permutation of the horospheres under $g \circ f$ is the identity permutation. We must now show that for any $m \in \Omega_p$, the image $f(m)$ lies in an $\epsilon$-ball around $m$, where $\epsilon$ is independent of the choice of $m$.

The set of points within $n$ units of any three distinct horospheres has bounded diameter, independent of the choice of horospheres. Also, any quasi-isometry $f$ has the property that

$$hd(f(Nbhd(A) \cap Nbhd(B)), Nbhd(f(A) \cap Nbhd(B))) < \infty$$

where $hd$ denotes Hausdorff distance. So if $x \in \Omega_p$ lies in the intersection of the $n$-neighborhoods of three horospheres, then there is a constant $n'$ so that $f(x) \in B_{n'}(x)$. Thus $f \circ g$ is a bounded distance from the identity map, so the natural map $\Psi : Comm(PSL_2(\mathbb{Z}[\frac{1}{p}])) \rightarrow QI(PSL_2(\mathbb{Z}[\frac{1}{p}]))$ is an isomorphism. □
5.5 Proof of Theorem C

We now prove Theorem C. The proof uses some standard techniques from the study of quasi-isometric rigidity for lattices in semisimple Lie groups. In addition to these techniques, Theorem A is applied, as well as the $S$-arithmetic superrigidity theorem of Margulis [M].

**Proof of Theorem C.** Since $\text{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ is quasi-isometric to the space $\Omega_p$, we get a quasi-isometry $f: \Gamma \to \Omega_p$. To obtain the exact sequence of the theorem, we will find a representation $\rho: \Gamma \to \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{Q}_p)$ with finite kernel so that $\rho(\Gamma)$ is a nonuniform lattice in $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{Q}_p)$, hence commensurable to $\text{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ by [M].

Since $\Gamma$ acts on itself by isometries via left multiplication $L_\gamma$, for all $\gamma \in \Gamma$, we obtain a uniform family of quasi-isometries $f_\gamma = f \circ L_\gamma \circ f^{-1}: \Omega_p \to \Omega_p$.

From Theorem A, we can think of $f_\gamma$ as a commensurator, hence a bounded distance from an isometry. For each $f_\gamma$, compose with a bounded alteration $B_\gamma$ to obtain a map $\rho: \Gamma \to \text{Isom}(\mathbb{H}^2 \times T_p)$.

First we show that $\rho$ is a homomorphism. Suppose $\rho(\gamma)$ is a bounded distance from the identity isometry, i.e. $d_{\mathbb{H}^2 \times T_p}(x, \rho(\gamma) \cdot x) < \epsilon$ for all $x \in \mathbb{H}^2 \times T_p$. But then it follows that $\rho(\gamma)$ is the identity, hence $\rho$ is a homomorphism. We must show that $\rho(\gamma)$ is a lattice and that $\rho$ has finite kernel.

Since $f(\Gamma)$ is a net in $\Omega_p$, $\rho(\Gamma) = \{f_\gamma\} = \{B_\gamma \circ f \circ L_\gamma \circ f^{-1}| \gamma \in \Gamma\}$ acts cocompactly on $\Omega_p$. It follows that $\rho(\Gamma)$ acts on $\mathbb{H}^2 \times T_p$ with cofinite volume.

Choose a basepoint $x \in \Omega_p$ and consider $d_{\mathbb{H}^2 \times T_p}(x, \rho(\gamma) \cdot x)$. For finitely many $\gamma \in \Gamma$, $\rho(\gamma)$ moves $x$ a bounded amount, i.e. there exists a constant $C' > 0$ so that $d_{\mathbb{H}^2 \times T_p}(x, \rho(\gamma) \cdot x) \leq C'$ for some finite set $\{\gamma_i\} \subset \Gamma$. (To find $C'$ we are using the fact that $\{f_\gamma\}$ is a uniform family of quasi-isometries.) In particular, $d_{\mathbb{H}^2 \times T_p}(x, \rho(\gamma) \cdot x) = 0$ for only finitely many $\gamma \in \Gamma$. But if $d_{\mathbb{H}^2 \times T_p}(x, \rho(\gamma) \cdot x) > 0$, then $\rho(\gamma)$ is not the identity isometry. Hence $\rho$ has finite kernel. The previous paragraph showed that $\rho(\Gamma)$ is discrete, so we can now construct the short exact sequence

$$1 \to N \to \Gamma \to \Lambda \to 1$$

where $N = \ker(\rho)$ and $\Lambda = \rho(\Gamma)$ is a lattice in $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{Q}_p)$.

To complete the proof, we apply the following theorem of Margulis [M] (Theorem 5.6), stated in the case $n = 2$, from which we conclude that the lattice $\Lambda$ obtained above is commensurable to $\text{PSL}_2(\mathbb{Z}[\frac{1}{p}])$. 26
Theorem 5.6. Let \( p \) be a prime and \( n \in \mathbb{N}^+ \), \( n \geq 2 \). If a subgroup \( \Gamma \) of \( SL_2(\mathbb{Q}) \) under the diagonal embedding in \( SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_p) \) is a lattice in \( SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_p) \), then the subgroups \( \Gamma \) and \( SL_2(\mathbb{Z}[\frac{1}{p}]) \) are commensurable.

The proof of Theorem 5.6 uses Margulis’ superrigidity theorem for \( S \)-arithmetic groups. Theorem 5.6 completes the proof of Theorem C.

APPENDIX

We now compute the commensurator group \( \text{Comm}(PSL_2(\mathbb{Z}[\frac{1}{p}])) \) of \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) in \( PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p) \).

Proposition. The commensurator subgroup for \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \) is given by
\[
\text{Comm}(PSL_2(\mathbb{Z}[\frac{1}{p}])) = PSL_2(\mathbb{Q})
\]
where
\[
\{ (a, a) | a \in PSL_2(\mathbb{Q}) \} \subset PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p).
\]

Proof. Since any commensurator \( g \) must preserve \( \Omega_p \), hence preserve the horospheres, i.e. map \( \mathbb{Q} \cup \{ \infty \} \) to \( \mathbb{Q} \cup \{ \infty \} \), we must have \( g \in PSL_2(\mathbb{Q}) \). We now show that the commensurator subgroup, which is contained in \( PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p) \), is exactly the set \( \{ (g, g) | g \in PSL_2(\mathbb{Q}) \} \).

First we show that for \( b \in PSL_2(\mathbb{Q}) \), the element \( h = (Id, b) \) is not a commensurator of \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \). We know that an element \( (a, b) \in \text{Comm}(PSL_2(\mathbb{Z}[\frac{1}{p}])) \) must induce a quasi-isometry on the space \( \Omega_p \). Since \( a \in PSL_2(\mathbb{Q}) \) \( \subset PSL_2(\mathbb{R}) \) and \( b \in PSL_2(\mathbb{Q}) \subset PSL_2(\mathbb{Q}_p) \), we can let \( a \) act by conjugation on the \( \mathbb{H}^2 \) factor and \( b \) by conjugation on \( \mathbb{T}_p \). So \( h \) acts by the identity on \( \mathbb{H}^2 \), hence \( h(\sigma_\alpha) = \sigma_\alpha \), although \( \sigma_\alpha \) is not fixed pointwise. In addition, \( h(\sigma_\infty) = \sigma_\infty \), so the closeness line between \( \sigma_\alpha \) and \( \sigma_\infty \) is also preserved. Since this is true for all \( \alpha \in \mathbb{Q} \), if we view \( h \) as a map on \( \mathbb{Q}_p \), then \( h \) fixes a copy of \( \mathbb{Q} \subset \mathbb{Q}_p \). Hence when we extend \( h \) continuously to \( \mathbb{Q}_p \), we see that it must be the identity.

We now show that if \( a \in PSL_2(\mathbb{Q}) \), then \( (a, a) \) is a commensurator of \( PSL_2(\mathbb{Z}[\frac{1}{p}]) \). A calculation shows that for any element \( (g, g) \in PSL_2(\mathbb{Z}[\frac{1}{p}]) \times PSL_2(\mathbb{Z}[\frac{1}{p}]) \subset PSL_2(\mathbb{R}) \times PSL_2(\mathbb{Q}_p) \) the expression
\[
(a, a)(g, g)(a^{-1}, a^{-1})
\]
can be written as a matrix all of whose entries have bounded denominator, say bounded by $d$, meaning that each denominator is of the form $dp^r$ for some $r \in \mathbb{Z}^+$. Thus the subgroup

$$(a, a) (\text{PSL}_2(\mathbb{Z}[\frac{1}{p}]), \text{PSL}_2(\mathbb{Z}[\frac{1}{p}])) (a^{-1}, a^{-1})$$

is of finite index in $\text{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \times \text{PSL}_2(\mathbb{Z}[\frac{1}{p}])$. Hence $(a, a)$ is a commensurator.

Now we will show that $\text{Comm}(\text{PSL}_2(\mathbb{Z}[\frac{1}{p}]))$ consists entirely of elements of the form $(a, a)$, with $a \in \text{PSL}_2(\mathbb{Q})$. Suppose $(a, b) \in \text{Comm}(\text{PSL}_2(\mathbb{Z}[\frac{1}{p}]))$, with $a, b \in \text{PSL}_2(\mathbb{Q})$. We know that $(a^{-1}, a^{-1})$ is an element of $\text{Comm}(\text{PSL}_2(\mathbb{Z}[\frac{1}{p}]))$. Compose these two elements to form a new commensurator $(a, b) \circ (a^{-1}, a^{-1}) = (\text{Id}, ba^{-1})$. By the reasoning above, if this is a commensurator, then we must have $ba^{-1} = \text{Id}$, which implies that $b = a$, so our original commensurator must have been of the form $(a, a)$ with $a \in \text{PSL}_2(\mathbb{Q})$.

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