FROM HARMONIC MAPPINGS TO RICCI FLOWS DUE TO THE BOCHNER TECHNIQUE

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Abstract. The present paper is devoted to the study a global aspect of the geometry of harmonic mappings and, in particular, infinitesimal harmonic transformations, and represents the application of our results to the theory of Ricci solitons. These results will be obtained using the methods of Geometric analysis and, in particular, due to theorems of Yau, Li and Schoen on the connections between the geometry of a complete smooth manifold and the global behavior of its subharmonic functions.

Keywords and phrases: harmonic mapping, Ricci flow, Ricci soliton, Bochner technique, subharmonic function.

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1. Introduction. It is well known that there exists a connection between harmonic maps and Ricci flows due to the “de Turck trick” that modifies the Ricci flow into a nonlinear parabolic equation (see [9, pp. 113—118]). On the other hand, it is well known that self-similar solutions of the Ricci flow are Ricci solitons (see [8, p. 22]). At the same time, the vector field that turns a Riemannian metric into a Ricci soliton is an infinitesimal harmonic transformation (for the proof, see [25, 29]). In accordance with the above, the main purpose of the present paper is to make an observation of a function-theoretic nature in global differential geometry of harmonic mappings (see [11, 17, 24, 35]) and, in particular, infinitesimal harmonic transformations (see [20, 26]) and Ricci solitons (see, e.g., [8, 12]), and the Ricci flows (see, e.g., [9]). To implement this, our paper is organized as follows. In Sec. 2, we give a brief survey of basic facts of the geometry “in the large” of harmonic mappings between Riemannian manifolds. In particular, we prove that the classical theorems on harmonic mappings are consequences of well-known assertions on subharmonic functions. Results of Sec. 3 are obtained as analogs of results of Sec. 2. In turn, results of Sec. 4 are applications of results obtained in Sec. 3. In conclusion, we consider the evolution equations for the scalar curvature and the Ricci tensor from the point of view of the Bochner technique.

The Bochner technique and its generalized version will help us to connect these various research topics. We must recall here that the classic Bochner technique is an analytical method of obtaining vanishing theorems for some topological or geometrical invariants on a compact (without boundary) Riemannian manifold, under some curvature assumption (see [2, 3, 34] and [22, Chap. 9]). Proofs of such theorems are reduced to applying the Bochner maximum principle and the Green theorem (see [3, p. 30-31]). In the present paper, we will also use a generalized version of the Bochner technique (see, e.g., [23]) and, in particular, we will use the Hopf maximum principle (see [4]), results of Yau, Li, and Schoen on the connections between the geometry of a complete smooth manifold and the global behavior of its subharmonic functions (see, e.g., [19, 36]).

Assertions proved in this paper complement our results from [26, 27, 30, 31, 33] and results of other authors from [7, 12, 32, 34] and [18, p. 57].

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2. Harmonic mappings from the point of view of the Bochner technique. In this section, we give a brief survey of basic facts of the differential geometry of harmonic mappings between Riemannian manifolds (see the monographs [17, 35]). The subject of harmonic maps is vast and has found many applications, and it would require a very long reading to cover all aspects, even superficially (see [17, p. 417]). We have made a choice; in particular, highlighting the key question of nonexistence of harmonic maps between given complete Riemannian manifolds. We will survey some of the main method of geometric analysis for answering this question (see, e.g., [22, Chap. 9] and [23]). In particular, we will prove new versions of two well-known vanishing theorems of harmonic mappings from [11, 24]. We note here that the question of proving the existence of harmonic mappings was studied in [17].

Let \((M, g)\) and \((\overline{M}, \overline{g})\) denote complete Riemannian manifolds of dimensions \(n\) and \(\overline{n}\), respectively. The energy density of a smooth map \(f: (M, g) \to (\overline{M}, \overline{g})\) is the nonnegative scalar function \(e(f): M \to \mathbb{R}\) such that \(e(f) = \frac{1}{2} \|f_*\|^2\), where \(\|f_*\|^2\) denotes the squared norm of the differential \(f_*\), with respect to the induced metric \(\overline{g}\) on the vector bundle \(T^*M \otimes f^*\overline{T\overline{M}}\) by \(g\) and \(\overline{g}\) (see also [11]).

It is well known that \(f: (M, g) \to (\overline{M}, \overline{g})\) is a harmonic mapping if and only if it satisfies the Euler–Lagrange equation:

\[
\text{trace}_g(\nabla^2 f) = 0, \tag{2.1}
\]

where \(\nabla = \nabla \oplus \nabla\) is the canonical connection in the vector bundle \(T^*M \otimes f^*\overline{T\overline{M}}\) (see [9, p. 117] and [11, 17]). Moreover, if \(f\) is a harmonic mapping, then a standard calculation yields (see also [11, p. 123]):

\[
\Delta e(f) = \|\nabla f_*\|^2 + Q(f) \tag{2.2}
\]

for the Laplace–Beltrami operator \(\Delta := \text{div} \nabla\) and the scalar function:

\[
Q(f) = g(\text{Ric} f^*\overline{g}) - \text{trace}_g \bigl(\text{trace}_g(f^*\overline{R})\bigr), \tag{2.3}
\]

where \(\overline{R}\) is the Riemannian curvature tensor of \((\overline{M}, \overline{g})\) and \(\text{Ric}\) is the Ricci tensor of \((M, g)\).

We showed in [32] the condition when \(Q(f)\) is a quasi-positive scalar function on a connected open domain \(U \subset M\), i.e., \(Q(f)\) is nonnegative everywhere in \(U\) and \(Q(f)\) is positive at least at one point of \(U\). Namely, if the sectional curvature \(\text{sec}\) of \((\overline{M}, \overline{g})\) is nonnegative at an arbitrary point of \(f(U) \subset \overline{M}\) and \(\text{Ric} \geq f^*\overline{\text{Ric}}\) at each point of \(U\) for the Ricci tensors \(\overline{\text{Ric}}\) of \((\overline{M}, \overline{g})\), then \(Q(f) \geq 0\). Furthermore, if there is at least one point of \(U\) where \(\text{Ric} > f^*\overline{\text{Ric}}\), then \(Q(f) > 0\) at this point. In this case, \(Q(f)\) is also quasi-positive scalar function defined in \(U\). As a result, the energy density function \(e(f)\) satisfies the inequality \(\Delta e(f) \geq 0\) at each point of \(U\), by (2.2). Therefore, \(e(f)\) is a subharmonic function. The assumption that the energy density function \(e(f)\) attains a local maximum at some point \(x \in U\), then \(e(f)\) is a constant \(C\) in \(U\), by the Hopf maximum principle (see [3, Theorem 2.1] and [4, Theorem 1]). If \(C > 0\), then \(\nabla f\) is nowhere zero. Now, at a point where the \(Q(f)\) is positive, the left-hand side of (2.2) is zero while the right-hand side is positive. This contradiction shows that \(C = 0\) and hence \(f\) is constant in \(U\). Thus we have proved the following lemma.

**Lemma 2.1.** Let \(f: (M, g) \to (\overline{M}, \overline{g})\) be a harmonic mapping such that its energy density \(e(f)\) has local maximum at some point \(x\) in a connected open domain \(U \subset M\). If, in addition, the sectional curvature \(\text{sec}\) of \((\overline{M}, \overline{g})\) is nonnegative at an arbitrary point of \(f(U) \subset \overline{M}\), \((M, g)\) has the Ricci tensor \(\text{Ric}\) such that \(\text{Ric} \geq f^*\overline{\text{Ric}}\) at each point of \(U\), and there is at least one point of \(U\) where \(\text{Ric} > f^*\overline{\text{Ric}}\), then \(f\) is constant in the domain \(U\).

Using Lemma 2.1 and the Bochner maximum principle (see [3, Theorem 2.2]), we obtain the following theorem proved in [32], which is a consequence of Lemma 2.1.

**Theorem 2.2.** Let \(f: (M, g) \to (\overline{M}, \overline{g})\) be a harmonic mapping between Riemannian manifolds \((M, g)\) and \((\overline{M}, \overline{g})\). Assume that the sectional curvature \(\text{sec}\) of the second manifold \((\overline{M}, \overline{g})\) is nonnegative at every point of \(f(M)\) and \((M, g)\) is a compact manifold with the Ricci tensor \(\text{Ric} \geq f^*\overline{\text{Ric}}\). Then \(f\) is a totally geodesic mapping with constant energy density \(e(f)\). Furthermore, if there is at least one point of \(M\) where \(\text{Ric} > f^*\overline{\text{Ric}}\), then \(f\) is a constant mapping.
Remark 2.3. We recall here that Eells and Sampson proved in [11] the following celebrated vanishing theorem on harmonic maps: If \( f : (M, g) \to (\overline{M}, \overline{g}) \) is a harmonic mapping between a compact Riemannian manifold \((M, g)\) with the Ricci tensor \(\text{Ric} \geq 0\) and a Riemannian manifold \((\overline{M}, \overline{g})\) with the sectional curvature \(\text{sec} \leq 0\), then \( f \) is totally geodesic and has constant energy density \( e(f) \). Furthermore, if there is at least one point of \( M \) where \( \text{Ric} > 0 \), then every harmonic map \( f : (M, g) \to (\overline{M}, \overline{g}) \) is constant.

Li and Schoen proved in [19] that there is no nonconstant, nonnegative, \( L^p \)-integrable \((0 < p < \infty)\) subharmonic function \(u\) on any complete Riemannian manifold \((M, g)\) with nonnegative Ricci tensor. In other word, if we assume that \( \text{Ric} \geq 0 \) and \( \int_M (\|u\|)^p d\text{Vol}_g < \infty \) for a complete Riemannian manifold \((M, g)\), then \( u = C \) for some constant \( C \). In this case, we have \( C^p \int_M d\text{Vol}_g < \infty \).

If \( C > 0 \), then \( u \) is nowhere zero and the volume of \((M, g)\) is finite. At the same time, we know from [36] that every complete, noncompact Riemannian manifold \((M, g)\) with nonnegative Ricci tensor has infinite volume. This contradiction shows \( C = 0 \) and hence \( u \equiv 0 \). Therefore, we can formulate the following lemma.

Lemma 2.4. Let \((M, g)\) be a complete, noncompact Riemannian manifold with nonnegative Ricci tensor. Then there is no nonzero, nonnegative, \( L^p \)-integrable \((0 < p < \infty)\) subharmonic function on \((M, g)\).

Let \((M, g)\) be a complete, noncompact Riemannian manifold. Given a smooth map \( f : (M, g) \to (\overline{M}, \overline{g}) \), we define its energy as (see [11, 24])

\[
E(f) = \int_M e(f) d\text{Vol}_g.
\]

The energy \( E(f) \) can be infinite or finite for a smooth map \( f : (M, g) \to (\overline{M}, \overline{g}) \) of a complete Riemannian manifold \((M, g)\). In particular, \( E(f) < +\infty \) for a compact Riemannian manifold \((M, g)\) (see [24]).

At the same time, we can formulate an alternative theorem for harmonic maps from complete Riemannian manifolds to Riemannian manifolds with nonnegative sectional curvature (see [24] and [35, p. 25]). In this case, we can also use Lemma 2.1 on nonzero, nonnegative, \( L^p \)-integrable \((0 < p < \infty)\) subharmonic functions on a complete Riemannian manifold with nonnegative Ricci curvature for the proof of this theorem. Namely, the following theorem holds.

Theorem 2.5. Let \( f : (M, g) \to (\overline{M}, \overline{g}) \) be a harmonic mapping with finite energy. If \((M, g)\) is a complete, noncompact Riemannian manifold with the Ricci tensor \(\text{Ric} \geq f^*\overline{\text{Ric}}\), where \(\overline{\text{Ric}}\) denotes the Ricci tensor of \((\overline{M}, \overline{g})\), and \((\overline{M}, \overline{g})\) is a Riemannian manifold with nonnegative sectional curvature \(\text{sec} \leq 0\) at every point of \(f(M)\), then \( f \) is a constant map.

Remark 2.6. We recall here that Yau and Schoen proved the following well-known vanishing theorem (see [24]): A harmonic map of finite energy \( E(f) \) from a complete, noncompact manifold \((M, g)\) with the Ricci tensor \(\text{Ric} \geq 0\) to a compact manifold \((\overline{M}, \overline{g})\) with the sectional curvature \(\text{sec} \leq 0\) is homotopic to a constant map.
3. Infinitesimal harmonic transformations from the function-theoretic point of view.

The main results of this section are obtained as analogs of results of Sec. 2.

A vector field $\xi$ on a complete Riemannian manifold $(M, g)$ is called an infinitesimal harmonic transformation (see [20]) if $\xi$ generates a flow, which is a local one-parameter group of harmonic transformations $(t, x) \in \mathbb{R} \times M \to \phi_t(x) \in M$ (in other words, local harmonic diffeomorphisms). Analytic characteristic of such vector field has the form

$$\text{trace}_g (L_\xi \nabla) = 0,$$

where $L_\xi$ is the Lie derivative along $\xi$ (see [20, 26]). This formula is an analog of the formula (2.1).

In addition, we proved in [26] that a vector field $\xi$ is an infinitesimal harmonic transformation if and only if

$$\tilde{\Delta} \theta = 2 \text{Ric}(\xi, \cdot)$$

for the 1-form $\theta$ corresponding to $\xi$ under the duality defined by the metric $g$ and the Hodge–de Rham Laplacian $\tilde{\Delta}$ (see [2, p. 158]).

In accordance with the theory of harmonic maps (see [11]), we define the energy density of the flow on $(M, g)$ generated by an infinitesimal harmonic transformation $\xi$ as the scalar function $e(\xi) = \frac{1}{2} \|\xi\|^2$, where $\|\xi\|^2 = g(\xi, \xi)$. Then the Laplace–Beltrami operator $\Delta e(\xi)$ for the energy density $e(\xi)$ of an infinitesimal harmonic transformation $\xi$ has the form

$$\Delta e(\xi) = \|\nabla \xi\|^2 - \text{Ric}(\xi, \xi) \quad (3.1)$$

(see [25, 27, 30]). The formula (3.1) is an analog of the formula (2.2) for the energy density $e(f)$ of a harmonic map $f$. In this case, the following theorem is valid.

**Theorem 3.1.** Let $(M, g)$ be a Riemannian manifold and $U \subset M$ be a connected, open domain. If the energy density of the flow $e(\xi)$ generated by an infinitesimal harmonic transformation $\xi$ has a local maximum at some point of $U$ and the Ricci curvature of $(M, g)$ is quasi-negative in $U$, then $\xi \equiv 0$ everywhere in $U$.

**Proof.** Let the Ricci curvature of $(M, g)$ be quasi-negative everywhere in a connected, open domain $U \subset M$; then due (3.8) the energy density function $e(\xi)$ satisfies the inequality $e(\xi) \geq 0$. This means that $e(\xi)$ is a subharmonic function. Now assume that the energy density function $e(\xi)$ attains a local maximum at a point $x \in U$; then $e(\xi)$ is a constant $C$ in $U$ by the Hopf maximum principle (see [3, Theorem 2.1] and [4, Theorem 1]). If $C > 0$, then $\xi$ is nowhere zero. Now, at a point where the Ric is negative, the left-hand side of (3.1) is zero while the right-hand side is positive. This contradiction shows that $C = 0$ and hence $\xi \equiv 0$ everywhere in the domain $U$. $\square$

**Remark 3.2.** Theorem 3.1 is a direct generalization of Theorem 4.3 presented in Kobayashi’s monograph on transformation groups (see [18, p. 57]) and Wu’s proposition on a Killing vector field whose length achieves a local maximum (see [34]).

As an analog of Theorem 2.2, we can formulate the following theorem, which can be proved by using the Bochner maximum principle (see [3, Theorem 2.2]).

**Theorem 3.3.** A compact Riemannian manifold $(M, g)$ with quasi-negative Ricci curvature does not possess nonzero infinitesimal harmonic transformations.

Next, we recall that the kinetic energy $E(\xi)$ of the flow on $(M, g)$ generated by a vector field $\xi$ is determined in accordance with [1, p. 2] by the following equation:

$$E(\xi) = \int_M e(\xi) d\text{Vol}_g.$$
Remark 3.4. The definition is consistent with the theory of harmonic mappings in the case of an infinitesimal harmonic transformation. Moreover, the energy $E(\xi)$ can be infinite and finite. For example, $E(\xi) < +\infty$ for a smooth complete vector field $\xi$ on a compact Riemannian manifold $(M, g)$.

As an analog of Theorem 2.5, we formulate the following theorem.

**Theorem 3.5.** Let $(M, g)$ be a complete Riemannian manifold $(M, g)$ with nonpositive Ricci curvature. Then every infinitesimal harmonic transformation with finite kinetic energy is parallel. If, in addition, the volume of $(M, g)$ is infinite or the Ricci curvature is negative at some point of $M$, then each infinitesimal harmonic transformation is identically zero everywhere on $(M, g)$.

**Proof.** For the proof, we use the well-known second Kato inequality (see [2, p. 380]):

$$-\|\xi\| \Delta \|\xi\| \leq g(\Delta \theta, \theta),$$

where $\Delta := -\text{trace}_g \nabla \circ \nabla$ is the rough Laplacian and $\theta$ is the 1-form corresponding to $\xi$ under the duality defined by the metric $g$. It is well known that the rough Laplacian satisfies the Weitzenböck formula (see [18, p. 44] and [2, p. 378]):

$$\Delta \theta = \tilde{\Delta} \theta - S \xi,$$

where $S$ is the Ricci operator defined by $g(SX, Y) = \text{Ric}(X, Y)$ for any tangent vector fields $X$ and $Y$. Therefore, the second Kato inequality can be rewritten in the form

$$2\sqrt{e(\xi)} \Delta \sqrt{e(\xi)} \geq -g(\Delta \theta, \theta) + \text{Ric}(\xi, \xi),$$

(3.2)

where $\|\xi\| = \sqrt{2e(\xi)}$. On the other hand, it was proved in [26, 27] that $\xi$ is an infinitesimal harmonic transformation on $(M, g)$ if and only if $\Delta \xi = 2S \xi$. Therefore, we obtain from (3.2) the following equation:

$$\sqrt{e(\xi)} \Delta \sqrt{e(\xi)} = -\frac{1}{2} \text{Ric}(\xi, \xi).$$

(3.3)

Assume that the Ricci tensor $\text{Ric}$ is nonpositive. It was proved in [36, p. 664] and [37] that every nonnegative smooth function $u$ defined on a complete Riemannian manifold $(M, g)$ and satisfying the conditions

$$u\Delta u \geq 0, \quad \int_M u^p \text{Vol}_g < +\infty \quad \forall p \neq 1,$$

is constant. In particular, if the volume of $(M, g)$ is infinite and $u = \text{const}$, then $u = 0$. Therefore, if a smooth manifold $(M, g)$ is complete and

$$E(\xi) = \int_M e(\xi) \text{Vol}_g < +\infty,$$

(3.4)

then the function $\sqrt{e(\xi)}$ is constant. At the same time, we obtain from (3.4) that the volume of $(M, g)$ is finite. Thus, (3.2) implies that $\nabla \xi = 0$. On the other hand, if we assume that $\text{Ric}_x < 0$ at a point $x \in M$, then this inequality contradicts Eq. (3.1). The proof is complete. □

**Remark 3.6.** If $\xi$ is a nonzero vector field such that $\text{Ric}(\xi, \xi) \leq 0$, then (3.3) implies the inequality $\Delta \sqrt{e(\xi)} \geq 0$. This means that $\sqrt{e(\xi)}$ is a subharmonic function. Therefore, the result given above is an analog of Theorem 2.5.

The following corollary is valid.

**Corollary 3.7.** Let $(M, g)$ be a connected, complete, noncompact Riemannian manifold of dimension $n \geq 2$ with irreducible holonomy group $\text{Hol}(g)$ and nonpositive Ricci curvature. Then each infinitesimal harmonic transformation on $(M, g)$ with finite kinetic energy is identically zero everywhere on $(M, g)$. 427
Proof. By Theorem 3.5, an infinitesimal harmonic transformation $\xi$ with $E(\xi) < +\infty$ on a connected, complete, noncompact Riemannian manifold $(M, g)$ with nonpositive Ricci curvature is parallel. Under the assumption that the holonomy group $\text{Hol}(g)$ is irreducible, this relation means that $\xi \equiv 0$. □

An example of a complete smooth manifold $(M, g)$ with nonpositive Ricci curvature is the well known Cartan–Hadamard manifold, i.e., a simply connected, complete Riemannian manifold of nonpositive sectional curvature (see [23, p. 90]).

**Corollary 3.8.** Let $(M, g)$ be a Cartan–Hadamard manifold of dimension $n \geq 2$ with infinite volume. Then each infinitesimal harmonic transformation on $(M, g)$ with finite kinetic energy is identically zero everywhere on $(M, g)$.

**Remark 3.9.** Other properties of infinitesimal harmonic transformations can be found in [7, 18, 25–27, 30, 31]. In particular, it was proved in [26] that the set of all infinitesimal harmonic transformations on a compact Riemannian manifold $(M, g)$ is a finite-dimensional vector space over $\mathbb{R}$. Moreover, the Lie algebra $i(M)$ of infinitesimal isometric transformation is a subspace of this vector space (see [26]). We recall here that an infinitesimal isometric transformation (infinitesimal isometry) or a Killing vector field $X$ on $(M, g)$ is defined by the well-known equation $L_X g = 0$. It is well known that $\dim i(M) = n(n + 1)/2$ on an $n$-dimensional Riemannian manifold $(M, g)$ of constant curvature (see, e.g., [18, pp. 46-47]). Therefore, the dimension of the vector space of infinitesimal harmonic transformations on an $n$-dimensional Riemannian manifold $(M, g)$ of constant curvature is at least $n(n + 1)/2$. Note that this proposition is a local result.

The following statement on infinitesimal isometric transformations is a well-known oldest result (see, e.g., [3, p. 57] and [18, p. 44]). If $\xi$ is an infinitesimal isometry, it satisfies the following differential equations:

$$
\Delta \theta = 2\text{Ric}(\xi, \cdot); \tag{3.5}
$$
$$
div \xi = 0, \tag{3.6}
$$

for the 1-form $\theta$ corresponding to $\xi$ under the duality defined by the metric $g$. Conversely, if $(M, g)$ is compact and $\xi$ satisfies (3.5) and (3.6), then $\xi$ is an infinitesimal isometry.

Equation (3.6) is more rigid than is required for the statement above. In turn, we can formulate and prove an alternative version of this statement.

**Theorem 3.10.** Let $(M, g)$ be a Riemannian manifold and $\xi$ be a vector field on $(M, g)$. If $\xi$ is an infinitesimal isometry, then it satisfies the following conditions: $\xi$ is an infinitesimal harmonic transformation and

$$
L_\xi \text{div} \xi \geq 0. \tag{3.7}
$$

Conversely, if $(M, g)$ is compact and $\xi$ satisfies the above two conditions, then $\xi$ is an infinitesimal isometry.

**Proof.** We have already proved that if $\xi$ is an infinitesimal harmonic transformation, then it satisfies (3.5). If, in addition, $\xi$ is an infinitesimal isometry, then the equation $\text{div} \xi = 0$ holds. This implies (3.7). To prove the converse, we may assume that $M$ is compact and orientable. (If $M$ is not orientable, consider its orientable double covering.) Let us consider the vector field $X = (\text{div} \xi)\xi$ for an arbitrary infinitesimal harmonic transformation $\xi$ on $(M, g)$. The divergence $\text{div} X$ has the form

$$
\text{div} X = L_\xi (\text{div} \xi) + (\text{div} \xi)^2. \tag{3.8}
$$

If we apply the classic Green theorem

$$
\int_M \text{div} X \, d\text{Vol}_g = 0
$$

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to the vector field \( X = (\text{div} \xi) \xi \), then we obtain the integral formula

\[
\int_M (L_\xi (\text{div} \xi) + (\text{div} \xi)^2) \, d\text{Vol}_g = 0,
\]

where \( d\text{Vol}_g \) is the canonical measure (the volume element) associated to the metric \( g \). If the inequality \( L_\xi (\text{div} \xi) \geq 0 \) holds anywhere on \((M, g)\), then from (3.9) we conclude that \( \text{div} \xi = 0 \). Next, for complete the proof we can refer to Theorem 3.10.

\( \square \)

**Remark 3.11.** We can define the divergence of a vector field using the hydrodynamical approach. Namely, the divergence of a smooth vector field \( \xi \) on \((M, g)\) is the scalar function defined as follows (see, e.g., [18, p. 6] and [21, p. 195]):

\[
(\text{div} \xi) \, d\text{Vol}_g = L_\xi (d\text{Vol}_g).
\]

Specifically, the formula (3.10) shows how the volume form changes along the flow of the vector field. Due to (3.10), the function \( \text{div} \xi \) was called in [21, p. 195] the *acceleration of volumetric expansion*, i.e., the acceleration of change of the volume element \( d\text{Vol}_g \) along trajectories of the flow with the velocity vector \( \xi \). In particular, the condition \( L_\xi \text{div} \xi \geq 0 \) means that \( d\text{Vol}_g \) is a nondecreasing function along trajectories of this flow.

**Remark 3.12.** The equality \( L_\xi g = 0 \) also implies the invariance condition for the Ricci tensor \( L_\xi \text{Ric} = 0 \). In General Relativity, there are investigations (see, e.g., [10, 14]), where the “weakened condition” of the form \( \text{trace}_g (L_\xi \text{Ric}) = 0 \) is studied instead of the condition \( L_\xi \text{Ric} = 0 \).

It is of interest to note, that, in the case of a compact manifold \((M, g)\), the addition of the conditions \( L_\xi \text{Ric} \geq 0 (\leq 0) \) to Eq. (3.1) implies that infinitesimal harmonic transformation \( \xi \) will actually be an infinitesimal isometry. Namely, the following theorem holds.

**Theorem 3.13.** Let \((M, g)\) be a Riemannian manifold and \( \xi \) be a vector field on \((M, g)\). If \( \xi \) is an infinitesimal isometry, it satisfies the following conditions: \( \xi \) is an infinitesimal harmonic transformation and

\[
\text{trace}_g (L_\xi \text{Ric}) \geq 0 (\leq 0).
\]

Conversely, if \((M, g)\) is compact and \( \xi \) satisfies the above two conditions, then \( \xi \) is an infinitesimal isometry.

**Proof.** We have already proved that if \( \xi \) is an infinitesimal harmonic transformation, then it satisfies (3.5), which coincides with the first condition of the theorem. If, in addition, \( \xi \) is an infinitesimal isometry, then the equation \( L_\xi \text{Ric} = 0 \) holds. To prove the converse, we may assume that \( M \) is compact. If \( \xi \) is an infinitesimal harmonic transformation, then we have the differential equation

\[
\Delta \text{div} \xi = \text{trace}_g (L_\xi \text{Ric})
\]

(see [28]). Using the *Bochner maximum principle* (see [3, Theorem 2.2]), we conclude that \( \text{div} \xi = \text{const.} \).

On the other hand, we have

\[
\int_M \text{div} \xi \, d\text{Vol}_g = 0.
\]

Hence, \( \text{div} \xi = 0 \), showing that \( \xi \) is an infinitesimal transformation (see Theorem 3.10). Theorem 3.13 is proved.

\( \square \)

**Remark 3.14.** Theorems 3.10 and 3.13 are alternative versions of the classic theorem on infinitesimal isometric transformations (see, e.g., [3, p. 57] and [18, p. 44]).
4. Ricci solitons from the point of view of infinitesimal harmonic transformations. The main results of this section are applications of results of Sec. 3 to the theory of Ricci solitons.

Let $g$ be a fixed Riemannian metric on a smooth manifold $M$. Consider a one-parameter family of diffeomorphisms $(t, x) \in \mathbb{R} \times M \rightarrow \phi_t(x) \in M$ generated by a smooth vector field $\xi$ on $M$. The evolutive metric $g(t) = \sigma(t)\phi_t^* g(0)$ for a positive scalar $\sigma(t)$ such that $\sigma(0) = 1$ and $g(0) = g$ is a Ricci soliton if the metric $g$ is a solution of the nonlinear stationary PDF:

$$-2\text{Ric} = L_\xi g + 2\lambda g,$$

where $\text{Ric}$ is the Ricci tensor of $g$, $L_\xi g$ is the Lie derivative of $g$ with respect to $\xi$, and $\lambda$ is a constant (see, e.g., [8, p. 22]). To simplify the notation, we denote the Ricci soliton by $(g, \xi, \lambda)$. A Ricci soliton is said to be steady, shrinking, or expanding if $\lambda = 0$, $\lambda < 0$, or $\lambda > 0$, respectively. In addition, a Ricci soliton is called Einsteinian if $L_\xi g = 0$, and it is called trivial if $\xi \equiv 0$.

In [29], the following theorem was proved.

**Theorem 4.1.** The vector field $\xi$ of an arbitrary Ricci soliton $(g, \xi, \lambda)$ on a smooth manifold $M$ is an infinitesimal harmonic transformation on the Riemannian manifold $(M, g)$.

Theorems 3.1 and 4.1 imply the following assertion.

**Corollary 4.2.** Let $(g, \xi, \lambda)$ be a Ricci soliton on a smooth manifold $M$. If the Ricci tensor $\text{Ric}$ of $g$ is quasi-negative in a connected, open domain $U \subset M$ and the energy density of the flow generated by the vector field $\xi$ has a local maximum at some point of $U$, then $(g, \xi, \lambda)$ is an expanding Ricci soliton.

**Proof.** Let $(g, \xi, \lambda)$ be a Ricci soliton on a smooth manifold $M$; then its vector field $\xi$ is an infinitesimal harmonic transformation on the Riemannian manifold $(M, g)$. If, in addition, the Ricci tensor $\text{Ric}$ of $g$ is quasi-negative in a connected, open domain $U \subset M$ and the energy density of the flow $e(\xi)$ generated by $\xi$ has a local maximum at some point $x \in U$; then $\xi \equiv 0$ at any point $y \in U$ by Theorem 3.1. In this case, we obtain from (4.1) the equality $\text{Ric}_g = -\lambda g_y$ and hence $\lambda > 0$.

**Remark 4.3.** Corollary 3.7 implies that a Ricci soliton $(g, \xi, \lambda)$ with the quasi-negative Ricci curvature of $g$ is trivial on a compact smooth manifold $M$.

In turn, from Theorem 3.3 we obtain the following consequence.

**Corollary 4.4.** Let $(g, \xi, \lambda)$ be a Ricci soliton on a compact smooth manifold $M$. If the volume element $d\text{Vol}_g$ (respectively, the scalar curvature $s$ of $g$) is a nondecreasing (respectively, nonincreasing) function along trajectories of the flow with the velocity vector $\xi$, then $(g, \xi, \lambda)$ is trivial.

**Proof.** Consider a Ricci soliton $(g, \xi, \lambda)$ on a compact smooth manifold $M$. From (4.1) we obtain $\text{div} \xi = -(s+n\lambda)$ for the scalar curvature $s = \text{trace}_g \text{Ric}$; then the acceleration of volumetric expansion of the flow generated by the vector field $\xi$ of the Ricci soliton $(g, \xi, \lambda)$ has the form

$$L_\xi(\text{div} \xi) = -L_\xi s.$$

Therefore, the acceleration of change of the volume element $d\text{Vol}_g$ along trajectories of the flow with the velocity vector $\xi$ is equal to $-L_\xi s$. In particular, the condition $L_\xi s \leq 0$ means that $d\text{Vol}_g$ is a nondecreasing function along trajectories of this flow. On the other hand, if $L_\xi s \leq 0$, then from Theorem 3.13 we obtain that $\xi$ is an infinitesimal isometry, i.e., $L_\xi g = 0$. On the other hand, an arbitrary Ricci soliton $(g, \xi, \lambda)$ on a compact smooth manifold $M$ is a gradient soliton, i.e., $\theta = \text{grad} u$ for some $u \in C^\infty M$ (see [12]). Therefore, the condition $L_\xi g = 0$ becomes $\nabla \nabla u = 0$, which implies the equation $\Delta u = 0$, i.e., $u$ is a harmonic function. In this case, $u = \text{const}$ by the Bochner maximum principle. As a result, we obtain $\theta = \text{grad} u = 0$ and hence the Ricci soliton $(g, \xi, \lambda)$ is trivial.

The following consequence of Theorem 3.13 is proved similarly.
Corollary 4.5. Let \((g, \xi, \lambda)\) be a Ricci soliton on a compact smooth manifold \(M\). If \(\text{trace}_g L_\xi \text{Ric} \geq 0\) \((\leq 0)\) for the Ricci tensor \(\text{Ric}\) of \(g\), then \((g, \xi, \lambda)\) is trivial.

Remark 4.6. It is well known that every steady and expanding Ricci soliton on a compact smooth manifold \(M\) is trivial (see, e.g., [12]). On the other hand, the well-known problem presented in [12]: Are there special conditions in dimension \(n \geq 4\) under which a shrinking compact Ricci soliton is trivial? Our two corollaries give answers to this question.

The following consequence of Theorem 3.5 is obvious.

Corollary 4.7. Let \(M\) be a connected smooth manifold and \((g, \xi, \lambda)\) be a Ricci soliton with complete Riemannian metric \(g\) and nonpositive Ricci curvature on \(M\). If the kinetic energy of the flow generated by the vector field \(\xi\) is infinite, then \((g, \xi, \lambda)\) is an Einsteinian Ricci soliton. If, in addition, the volume of \((M, g)\) is infinite or the Ricci curvature is negative at some point, then \((g, \xi, \lambda)\) is a trivial Ricci soliton.

From (3.8) and (4.2) we conclude that the divergence of the acceleration vector of volumetric expansion \(X = (\text{div} \xi) \xi\) has the form
\[\text{div} X = -L_\xi s + (s + n\lambda)^2.\]  
If, in addition, \((M, g)\) is a complete and oriented Riemannian manifold such that \(||(\text{div} \xi)\xi|| \in L^1(M, g)\) and \(L_\xi s \leq 0\), then from (4.3) we obtain that \(\text{div} X \geq 0\). Then, by the generalized Green theorem (see [5, 6]), we have \(\text{div} X = 0\). This means that \(L_\xi s = 0\) and
\[s = -n\lambda.\]  
In this case, from the well-known Schur identity \(\delta \text{Ric} = -2^{-1}{\nabla} s\), we obtain the equation \(\delta \text{Ric} = 0\). We recall the well-known equality \(\delta \tilde{\Delta} = \tilde{\Delta} \delta\). Therefore, if we apply the divergence operator \(\delta\) to both sides of the equation
\[\tilde{\Delta} \theta = 2 \text{Ric}(\xi, \cdot),\]
we obtain
\[\text{trace}_g (S : \nabla \xi) = 0.\]  
In this case, from Eqs. (4.1) and (4.4) we obtain \(||\text{Ric}||^2 = n^{-1}s^2\). Then the square of the traceless part of the Ricci tensor is equal to zero, i.e.,
\[||\text{Ric} - n^{-1}sg||^2 = ||\text{Ric}||^2 - n^{-1}s^2 = 0.\]  
Therefore, we conclude that \(\text{Ric} = n^{-1}s \cdot g\) and \(L_\xi g = 0\). This means that \(\xi\) is an infinitesimal isometry and \((g, \xi, \lambda)\) is an Einsteinian soliton. Thus, we proved the following statement.

Theorem 4.8. Let \((g, \xi, \lambda)\) be a Ricci soliton with complete Riemannian metric \(g\) on a connected and oriented smooth manifold \(M\) satisfying the following conditions:

(i) the volume element \(d\text{Vol}_g\) (respectively, the scalar curvature \(s\) of \(g\)) is a nondecreasing (respectively, nonincreasing) function along trajectories of the flow with the velocity vector \(\xi\);

(ii) the logarithmic rate of volumetric expansion \(X = (\text{div} \xi) \xi\) is such that \(||X|| \in L^1(M, g)\).

Then the flow generated by \(\xi\) consists of isometric transformations and \((g, \xi, \lambda)\) is an Einsteinian Ricci soliton.

Remark 4.9. Theorem 4.8 is a generalization of [7, Theorem 5.4], which is one of the main results of this paper.

In particular, in the case \(s = \text{const}\), we have the following assertion.

Corollary 4.10. Let \((g, \xi, \lambda)\) be a Ricci soliton with complete Riemannian metric \(g\) of a constant scalar curvature on a connected and oriented smooth manifold \(M\) such that the logarithmic rate of volumetric expansion \(X = (\text{div} \xi) \xi\) satisfies the condition \(||X|| \in L^1(M, g)\). Then the flow generated by \(\xi\) consists of isometric transformations and \((g, \xi, \lambda)\) is an Einsteinian Ricci soliton.
Remark 4.11. Corollary 4.10 complements results of [13], where complete gradient Ricci solitons with constant scalar curvature were examined.

Let \((g, \xi, \lambda)\) be a shrinking Ricci soliton with complete metric \(g\) on a connected and oriented smooth manifold \(M\) such that \(\xi\) satisfies Theorem 4.8. Then from Theorem 4.1 we obtain the following inequality: Ric = \(n^{-1}s \cdot g > 0\), where \(s\) is a positive constant. Then the Myers diameter bound implies that \((M, g)\) is compact (see, e.g., [22, pp. 251 and 386]). Assume that \(\dim M = 3\); then \((M, g)\) has a constant sectional curvature and, in particular, if \(M\) is simply connected, it is isometric to an Euclidian sphere \(S^3\). In this case, we obtain the following assertion.

Corollary 4.12. Let \((g, \xi, \lambda)\) be a shrinking Ricci soliton with complete Riemannian metric \(g\) on a connected, oriented, simply connected, three-dimensional smooth manifold \(M\) such that the volume element \(d\text{Vol}_g\) is a nondecreasing function along trajectories of the flow with the velocity vector \(\xi\) and the logarithmic rate of volumetric expansion \(X = (\text{div } \xi)\xi\) satisfies the condition \(||X|| \in L^1(M, g)\). Then \((M, g)\) is isometric to a Euclidian sphere \(S^3\).

Corollary 3.7 implies the following assertion.

Corollary 4.13. Let \((g, \xi, \lambda)\) be a Ricci soliton on a connected noncompact smooth manifold \(M\) with complete metric \(g\), nonpositive Ricci curvature, and irreducible holonomy group \(\text{Hol}(g)\). If the kinetic energy of the flow generated by the vector field \(\xi\) is infinite, then \((g, \xi, \lambda)\) is a trivial soliton.

5. Evolution equations for the scalar curvature and the Ricci tensor. Let \(g(t)\) be a one-parametric family of metrics on a manifold \(M\) defined on a time interval \(J \subset \mathbb{R}\). Then the Hamiltonian Ricci flow equation has the form

\[
\frac{\partial}{\partial t} g(t) = -2 \text{Ric}(t),
\]

(5.1)

where \text{Ric}(t) is the Ricci tensor of the metric \(g = g(t)\). It is well known that for any \(C^\infty\)-metric \(g\) on a compact manifold \(M\), there exists a unique solution \(g(t), t \in [0, \varepsilon)\), to the Ricci flow equation for some \(\varepsilon > 0\) with \(g(0) = g\) (see [16]).

Remark 5.1. Compact Ricci solitons are fixed point of the Ricci flow (5.1) projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds.

Under the Ricci flow, we have an evolution equation for the scalar curvature \(s = s(t)\) (see [9, p. 99]):

\[
\frac{\partial}{\partial t} s(t) = \Delta_g(t)s(t) + 2 \|\text{Ric}(t)\|^2.
\]

(5.2)

If we assume that \(\partial s(t)/\partial t \leq 0\) for any \(t \in J\), then from (5.2) we obtain

\[
\Delta_g(t)s(t) \leq -2 \|\text{Ric}(t)\|^2 \leq 0.
\]

(5.3)

For the case of a compact manifold \(M\), we obtain \(\Delta_g(t)s(t) = 0\) and \(s = \text{const}\) by the Bochner maximum principle. Then from (5.3) we conclude that \(\text{Ric}(t) \equiv 0\). In this case, Eq. (5.2) can be rewritten in the form \(\partial g(t)/\partial t \equiv 0\) and hence the Ricci flow (5.1) is trivial. Then we can formulate the following obvious proposition.

Proposition 5.2. Let \(M\) be a compact smooth manifold with the Hamiltonian Ricci flow \(\partial g(t)/\partial t = -2 \text{Ric}(t)\) for a one-parametric family of metrics \(g(t)\) on \(M\) defined on a time interval \(J \subset \mathbb{R}\). If \(\partial s(t)/\partial t \leq 0\) for any \(t \in J\), then the Ricci flow is trivial.

In particular, we can conclude from Proposition 5.2 that the scalar curvature \(s = s(t)\) cannot be a monotonic decreasing function under a nontrivial Ricci flow. In addition, the minimum of the scalar curvature grows with time as follows (see [15]):

\[
s_{\text{min}}(t) \geq \frac{s_{\text{min}}(0)}{1 - \frac{4}{3}ts_{\text{min}}(0)}.
\]

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Remark 5.3. Corollary 4.4 automatically follows from this proposition.

On the other hand, the evolution equation of the Ricci tensor under the Ricci flow has the form

$$\frac{\partial}{\partial t} \text{Ric}(t) = \Delta_L \text{Ric}(t)$$

(5.4)

(see [9, p. 112]), where $\Delta_L$ denotes the Lichnerowicz Laplacian determined by the metric $g(t)$ (see [9, p. 109]). From (5.4) we obtain

$$\text{trace}_g(\frac{\partial}{\partial t} \text{Ric}(t)) = \Delta_g s(t).$$

(5.5)

If we assume that

$$\text{trace}_g(\frac{\partial}{\partial t} \text{Ric}(t)) \geq 0 \quad \text{or} \quad \text{trace}_g(\frac{\partial}{\partial t} \text{Ric}(t)) \leq 0 \quad \forall t \in J,$$

(5.6)

then from (5.5) we obtain that $\Delta_g s(t) \leq 0$ or $\Delta_g s(t) \geq 0$, respectively. For the case of a compact manifold $M$ we obtain $s = \text{const}$ by the Bochner maximum principle. In this case, the Ricci flow (5.1) is trivial. Then we can formulate the following obvious proposition.

Proposition 5.4. Let $M$ be a compact smooth manifold with the Hamiltonian Ricci flow $\partial g(t)/\partial t = -2 \text{Ric}(t)$ for a one-parametric family of metrics $g(t)$ on $M$ defined on a time interval $J \subset \mathbb{R}$. If the condition (5.6) holds, then the Ricci flow is trivial.

Remark 5.5. Corollary 4.5 automatically follows from this proposition.

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