APPLICATIONS OF CHIRAL SYMMETRY

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ABSTRACT

I discuss several topics in the applications of chiral symmetry at nonzero temperature. First, where does the rho go? The answer: up. The restoration of chiral symmetry at a temperature $T_X$ implies that the $\rho$ and $a_1$ vector mesons are degenerate in mass. In a gauged linear sigma model the $\rho$ mass increases with temperature, $m_{\rho}(T_X) > m_{\rho}(0)$. I conjecture that at $T_X$ the thermal $\rho - a_1$ peak is relatively high, at about $\sim 1 \text{ GeV}$, with a width approximately that at zero temperature (up to standard kinematic factors). The $\omega$ meson also increases in mass, nearly degenerate with the $\rho$, but its width grows dramatically with temperature, increasing to at least $\sim 100 \text{ MeV}$ by $T_X$. I also stress how utterly remarkable the principle of vector meson dominance is, when viewed from the modern perspective of the renormalization group. Secondly, I discuss the possible appearance of disoriented chiral condensates from “quenched” heavy ion collisions. It appears difficult to obtain large domains of disoriented chiral condensates in the standard two flavor model. This leads to the last topic, which is the phase diagram for $QCD$ with three flavors, and its proximity to the chiral critical point. $QCD$ may be very near this chiral critical point, and one might thereby generated large domains of disoriented chiral condensates.

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1. Introduction

In this paper I review several topics in which effective models are used to study the dynamics of chiral symmetry at nonzero temperature. The order is somewhat jumbled, and approximately in reverse chronological order from that in which the work was done. Whatever else, this has the virtue of presenting what I am most excited about (since it is most recent) first.

2. Vector mesons and chiral symmetry

The spectrum of dileptons in the collisions of heavy ions at ultrarelativistic energies provides a window into “hot” regions of the collision, whereby the formation of a quark-gluon plasma might be observed. For any effects of a quark-gluon plasma to be distinguishable from the background of ordinary hadronic processes, the system
must last for a long period of time at a temperature $T_\chi$, burning off the entropy of a quark-gluon phase. This is true for a first order transition with a large latent heat, but applies even if there is no true phase transition, as long as there is a large jump in the entropy in a narrow region of temperature. Numerical simulations appear to find such a jump in the entropy.

There are two distinct ways in which vector mesons at a temperature $T_\chi$ can be affected by the quark-gluon plasma, or more generally, by a hot hadronic plasma. We should speak of either type of plasma, since if there is no true phase transition the two cannot be distinguished, even in principle. (I speak only of nonzero temperature, and not of a system with nonzero baryon density. Cold, dense systems can be treated by an extension of the present methods, but may well involve new phenomenon.)

The first is if the vector meson has a large decay width so that its lifetime is short, less than $1\, fm/c$, and it decays inside the plasma. Then if its effective mass at $T_\chi$ is different from that at zero temperature, as it surely must be, in principle the shift in its mass, and so the corresponding peak in the dilepton spectra, is observable. At zero temperature the only example which appears in the dilepton spectra is the peak for the $\rho$ meson, with a width of $\sim 150\, MeV$. I suggested such a shift in the "thermal" $\rho$ peak some time ago.

The second way in which a vector meson can be affected is if it has a long lifetime, much greater than $1\, fm/c$, so that it decays outside of the plasma. Then even if its effective mass at $T_\chi$ is different from that at zero temperature, the overwhelming bulk of decays occurs outside of the plasma, and the peak for this vector meson in dileptons doesn’t move. Instead, the peak shrinks, as the hot medium shakes the bound state apart. The $J/\Psi$ peak is suppressed in this way.

At first thought, one might expect that the $\omega$ meson is like the $J/\Psi$ meson, since at zero temperature its width is small relative to that of the $\rho$ meson, of order $\sim 10\, MeV$. In this work I argue that this expectation is wrong: the width of the $\omega$ meson quickly grows with temperature, and is large at $T_\chi$, at least $\sim 100\, MeV$. Thus thermal $\omega$ mesons decay inside, and not outside, the plasma, and the shift of their masses is in principle observable. The width of the $\rho$ meson does not significantly increase with temperature.

The first question to settle is: where do the $\rho$ and $\omega$ mesons go? Do their effective masses go up, or down, with increasing temperature?

My first guess was “down” — that the effective mass decreased with temperature. If the phase transition to a quark-gluon phase is primarily one of deconfinement, then this may be modeled by an effective bag constant which decreases with temperature. The effective mass of the $\rho$ meson, like all hadronic bound states, should then decrease with temperature.

Numerical simulations of lattice gauge theory, however, demonstrate that the phase diagram of $QCD$ with three flavors of quarks is rather involved, and depends crucially on the values of the quark masses. This material will be discussed at greater
Figure 1: The phase diagram of a gauge theory with three colors and 2 + 1 flavors, following the results of the Columbia group. The quark masses are held in fixed ratio, so up = down quark mass, \( M \), and the strange quark mass = 32\( M \). The y-axis is \( M \), running from 0 to \( \infty \), while the x-axis is the temperature \( T \). There are two critical points, the deconfining critical point, \( M_D \), and the chiral critical point, \( M_C \). As drawn, QCD appears to lie above but near the chiral critical point.

length in sec. 3, so I just summarize the results here. For simplicity I work with “2+1” flavors, holding the up, down, and strange quarks in fixed ratio, \( M \equiv M_{up} = M_{down} = rM_{strange} \), with \( r \sim 1/32 \). At infinite \( M \) there is a deconfining phase transition of first order, which might be — but at least to my eyes, today, need not be — modeled by a decreasing bag constant.

(Perhaps it is worth elaborating on my concerns. At infinite \( M \) we are dealing with the phase transition in the pure glue theory. This is first order, with masses which shift with temperature on either side of the phase transition. But why should the masses decrease as the transition is approached from below? In particular, I can certainly rule out any transition in which the effective bag constant vanishes identically. If this were to happen, then by construction an infinite number of bound states would become massless. But there is no known critical point where an infinite number of states become massless: for all known critical points in 3 + 1 dimensions, only a finite number of fields become massless.)

Going down from infinite \( M \) there is a line of first order deconfining phase transitions; for three colors and 2 + 1 flavors, this line of first order deconfining phase
transitions does not extend indefinitely, but terminates in a deconfining critical point, \( \mathcal{M}_D \).

The opposite limit of \( \mathcal{M} = 0 \) is the point of chiral symmetry for three flavors; here the chiral phase transition is expected to be of first order. Again, as \( \mathcal{M} \) increases from zero there is a line of first order chiral transitions, which for three colors and 2+1 flavors terminates in a chiral critical point, \( \mathcal{M}_C \). Thus the crucial feature of the phase diagram is a gap, for \( \mathcal{M}_D > \mathcal{M} > \mathcal{M}_C \), in which there is no true phase transition. Current lattice data indicates that while QCD lies in this gap, \( \mathcal{M}_D > \mathcal{M}_{QCD} > \mathcal{M}_C \), \( \mathcal{M}_{QCD} \) lies much closer to the chiral critical point than to the deconfining critical point. Thus QCD appears to be described by a smooth transition which is dominated by the (approximate) restoration of chiral symmetry. This phase diagram is illustrated in fig. (1), and will be considered at length later.

2.1. Chiral symmetries

Consequently we require a model which treats the restoration of chiral symmetry for both spin zero and spin one mesons. Before constructing the model, it is worth reviewing how the underlying quark fields, and so the composite meson fields constructed from them, transform under the global chiral symmetries.

For simplicity I only consider the case of two flavors; since the \( \phi \) meson is known experimentally to be predominantly \( \bar{s}s \), this is probably adequate for treating \( \rho \) and \( \omega \) mesons. It is surely inadequate for mesons such as the \( \eta \) and \( \eta' \), which are not flavor, but \( SU(3) \), eigenstates.

For left and right handed quark fields \( q_l \) and \( q_r \), \( q_{l,r} = (1 \pm \gamma_5)q/2 \), under a global chiral symmetry transformation in the group \( SU_l(2) \times SU_r(2) \times U_a(1) \),

\[
q_l \rightarrow e^{i\theta_a} U_l q_l , \quad q_r \rightarrow e^{-i\theta_a} U_r q_r \, .
\]  

(1)

Here \( U_{l,r} \) are transformations under \( SU_{l,r}(2) \), and \( e^{i\theta_a} \) that for axial \( U_a(1) \).

The spin zero mesons constructed from the quark fields include two singlet fields,

\[
\sigma_\eta = \bar{q} q , \quad \eta = \bar{q} \gamma_5 q ,
\]  

(2)

and two isotriplet fields,

\[
\sigma_\pi = \bar{q} \bar{\ell} q , \quad \pi = \bar{q} \gamma_5 \bar{\ell} q ,
\]  

(3)

where the \( \bar{\ell} \) are proportional to the Pauli matrices for \( SU(2) \). The \( \eta \) and \( \pi \)'s are \( J^P = 0^- \) states, while the \( \sigma_\eta \) and the \( \sigma_\pi \) are \( J^P = 0^+ \). The identification of the \( 0^+ \) states is highly controversial; the candidates in the Particle Data Tables are the \( f_0(975) \) and the \( a_0(980) \). Both of these states have narrow widths, each less than 60 MeV, and so are very possibly some other type of state, such as a \( \bar{K}K \) molecule.
The spin one mesons are constructed analogously by adding the Dirac matrix $\gamma_\mu$ everywhere. In terms of the plausible candidates in the Particle Data tables these fields are the isosinglet states,

$$\omega_\mu = \bar{q} \gamma^\mu q \; , \; f_1^\mu = \bar{q} \gamma_5 \gamma^\mu q ,$$

and the isotriplet states,

$$\vec{\rho}_\mu = \bar{q} \gamma^\mu \vec{t} q \; , \; \vec{a}_{1}^\mu = \bar{q} \gamma_5 \gamma^\mu \vec{t} q .$$

The $\omega(783)$, the $\rho(770)$, and the $a_1(1260)$ are all familiar; the only unfamiliar state is the $f_1$, which I identify with the lightest $f_1$ in the Particle Data Table, the $f_1(1285)$. The $\omega$ and $\rho$ are $J^P = 1^- $; the $f_1$ and the $a_1$ are $J^P = 1^+ $.

I would like to make a side remark. From a study of the $QCD$ phase diagram as a function of the current quark mass $M$, it is known that how far $QCD$ is from the chiral critical point depends crucially on the value of the mass for the isosinglet $0^+$, the $\sigma_\eta$, at zero temperature. Of course in $QCD$ this is inevitably complicated by the decay channel into two pions, by mixing with $KK$ molecules, and a myriad of other details.

But in the quenched approximation, all of these details are absent, since by construction all mesons have zero width. Thus one can ask:

**In the quenched approximation, is the mass of the $\sigma_\eta$ less than, or greater than, the mass of the $\rho$?**

I am not suggesting that the answer in the quenched approximation is the same as for $QCD$; it could well differ. I am suggesting, however, that it is a well defined question, with an answer amenable by present day techniques.

To return to the question of chiral symmetry, under $SU_l(2) \times SU_r(2)$ transformations the scalars mix with each other as:

$$\sigma_\eta \leftrightarrow \vec{\pi} \; , \; \eta \leftrightarrow \sigma_\pi ,$$

while the vectors mix as:

$$\omega \& f_1, \; \text{invariant} \; ; \; \vec{\rho} \leftrightarrow \vec{a}_1 .$$

For the scalars, the $0^+$ and the $0^-$ states mix, while for the vectors, while the $1^-$ $\rho$ and the $1^+ a_1$ mix, the singlet states, the $\omega$ and the $f_1$, are left unchanged. The singlets are invariant because they correspond to currents for fermion number and axial fermion number, which are conserved classically in the massless theory. From Eqs. (3) and (4), in a chirally symmetric phase the masses of the $\sigma_\eta$ and the $\pi$, the $\eta$ and the $\sigma_\pi$, and the $\rho$ and the $a_1$, are equal to each other; there is no prediction for the masses of the $\omega$ and the $f_1$, since they don’t mix with any other states under the chiral symmetry.
The axial $U_a(1)$ symmetry is broken quantum mechanically. Nevertheless, for completeness I list how the states transform under $U_a(1)$:

$$\sigma_\eta \leftrightarrow \eta \ , \ \sigma_{\vec{\pi}} \leftrightarrow \vec{\pi}.$$  \hfill (8)

In contrast to the scalars, all vectors — the $\omega$, $f_1$, $\rho$, and the $\sigma_1$, are invariant under $U_a(1)$ transformations. This follows because of the conservation of $U_a(1)$ at the classical level.

At zero temperature we know that the effects of the anomaly, and so the quantum mechanical breaking of the axial $U_a(1)$ symmetry, are large, because the $\eta'$ (which for two flavors is the $\eta$ meson) is heavy. It is reasonable to suspect that at very high temperature the axial $U_a(1)$ symmetry is at least partially restored. One way of seeing this is to see if the masses of the states which mix with each other under Eq. (8) are approximately equal. While this is possible for the singlet states, the $\eta$ and the $\sigma_\eta$, it is probably much easier for the isotriplet states, the $\vec{\pi}$ and the $\sigma_{\vec{\pi}}$.

**Thus on the lattice, one can indirectly look for the restoration of axial $U_a(1)$ by measuring the mass splitting between the $\pi$’s and the $\sigma_\pi$’s.** This provides another motivation for measuring the properties of the scalar particles.

In this paper I will ignore the possible effects of the restoration of axial $U_a(1)$, and concentrate on the restoration of $SU_l(2) \times SU_r(2)$. For the scalars, this implies that I forget the $\eta$ and $\sigma_\pi$ fields, keeping only the $\pi$ and $\sigma_\eta$ mesons. Since the there is just one sigma field, the isosinglet $\sigma_\eta$, I will refer to that simply as the $\sigma$ field. I stress, however, that even for two flavors there is another isotriplet of scalar fields which in principle should be included, the $\sigma_{\vec{\pi}}$’s. As will be discussed in sec. 3, for three flavors there is a full nonet of sigma mesons which go along with the usual nonet of pseudoscalar Goldstone bosons.

If we only wanted to describe the lightest fields at zero temperature, which are the pions, then it would suffice to use a nonlinear sigma model. While this is currently a very popular approach, I stress that at best it is most awkward to describe the phase transition to a chirally symmetric phase. The reason is rather trivial: above the temperature for the restoration of chiral symmetry, all particles must fall into degenerate multiplets of $SU_l(2) \times SU_r(2)$, and possibly $U_a(1)$ as well. It is simply easier if we start with a theory in which all the fields which we need to form the complete multiplets are there to begin with, rather than having to generate them dynamically. Of course if we don’t put the full multiplet in by hand, they will be generated dynamically. For example, in the nonlinear sigma model in $2+\epsilon$ dimensions, by using the large $N$ expansion it can be shown that in the chirally symmetric phase the $\sigma$ field is generated dynamically, as a bound state of two pions.

Part of the prejudice against the linear sigma model, as opposed to the nonlinear model, is due to the fact that the sigma meson is very broad at zero temperature, from its decay mode into two pions. Thus the feeling is, why bother? But even if the sigma is broad at zero temperature, inevitably it will become narrow at high temperature. This is trivial in a chirally symmetric phase, since then the sigma and the pions are
degenerate. When the pion is massive at zero temperature, a narrow sigma can even emerge before the temperature of (approximate) chiral symmetry restoration. This happens because as the temperature is raised, the sigma mass goes down, while the pion mass goes up; thus the (thermal) sigma mass falls below twice the (thermal) pion mass before the temperature of chiral symmetry restoration.

Thus the sigma meson(s), which is a broad state, lost in the hadronic mud at zero temperature, eventually emerges as a clean, narrow state at nonzero temperature.

### 2.2. Sigma models and vector dominance

Having fixed upon a linear sigma model, we then have to include the vector mesons. Here we can rely upon one of the unjustly forgotten triumphs of the 1960’s, which is the model of vector dominance; indeed, it was Sakurai’s gauge theory of the $\rho$ which lead to the standard model, and thence to QCD. I hope to emphasize, however, that whatever its historical antecedents, that vector dominance is an amazing thing indeed.

From the $SU(2)$ Pauli matrices $\sigma^a$, I introduce $t^a = \sigma^a / 2$ and $t^0 = 1 / 2$; these matrices are normalized so that $tr(t^a t^b) = \delta^{ab} / 2$, etc. Since we are forgetting about the $\eta$ and the $\sigma_\pi$ fields, for the scalar fields we only need an $SU(2) \times SU(2) \sim O(4)$ vector,

$$\Phi = \sigma t^0 + i \vec{\pi} \cdot \vec{t}. \quad (9)$$

Under a $SU_l(2) \times SU_r(2)$ chiral rotation, $\Phi$ transforms as

$$\Phi \rightarrow \Omega_l^\dagger \Phi \Omega_r. \quad (10)$$

For the vector fields, I introduce left and right handed fields as

$$A_\mu^l = (\omega^\mu + f_\mu^l) + (\vec{\rho}_\mu + \vec{a}_\mu^l) \cdot \vec{t},$$

$$A_\mu^r = (\omega^\mu - f_\mu^l) + (\vec{\rho}_\mu - \vec{a}_\mu^l) \cdot \vec{t}. \quad (11)$$

The central question is how the vector fields transform under chiral rotations. The obvious, and certainly the most natural guess, is that they transform as one would expect under a global chiral rotation, which is homogeneously:

$$A_{l,r}^\mu \rightarrow \Omega_{l,r}^\dagger \ A_{l,r}^\mu \Omega_{l,r}. \quad (12)$$

I now show that while the transformation of Eq. (12) is standard, it does not lead to a model with vector dominance; the following repeats the discussion of Gasiorowicz and Geffen. I introduce the abelian field strength tensor for the left and right handed fields,

$$F_{\mu\nu}^{l,r}_{not VDM} = \partial^\mu A_{l,r}^\nu - \partial^\nu A_{l,r}^\mu, \quad (13)$$
and introduce the effective lagrangian for the gauge fields,

\[ \mathcal{L}_{\text{not VDM}} = \frac{1}{2} tr \left( (F_{\mu\nu}^l)_{\text{not VDM}}^2 + (F_{\mu\nu}^r)_{\text{not VDM}}^2 \right) + m^2 \left( (A_{\mu}^l)^2 + (A_{\mu}^r)^2 \right) \]  

(14)

This is a reasonable effective lagrangian for the vector fields. It respects the global chiral symmetry of \( SU_l(2) \times SU_r(2) \), while the mass term for the gauge fields will ensure that the gauge fields are massive. Of course we should also add the coupling to the scalar field \( \Phi \), but let us neglect that for the time being.

The current for \( \mathcal{L}_{\text{not VDM}} \) is computed by the standard Noether construction. For the left handed fields, say, take \( \Omega_l = \exp(i\vec{\omega}_l \cdot \vec{t}) \), and compute the infinitesimal variation of the lagrangian for a spatially dependent \( \omega_l \); then the current is the coefficient of \( \partial_{\mu} \omega_l \). For infinitesimal \( \omega_l \), in group space the change in the vector potential is \( \delta A_{\mu}^l = [A_{\mu}^l, \omega_l] \). Immediately we see that the mass term in \( \mathcal{L}_{\text{not VDM}} \) doesn’t contribute to the current, since \( \delta tr((A_{\mu}^l)^2) = tr([A_{\mu}^l, A_{\mu}^l], \omega_l]) = 0 \) by the cyclic property of the trace. That doesn’t mean that there isn’t a current; the abelian field strength tensor gives rise to a perfectly fine (left handed) current,

\[ \vec{j}_{\text{not VDM}} = [A_{\mu}^l, \partial_{\mu} A_{\nu}^l - \partial_{\nu} A_{\mu}^l] \]  

(15)

From the standpoint of a current, this is a very reasonable expression, bilinear in the fields. This is analogous to the current we know for a scalar field, \( \phi^* \partial_{\mu} \phi - \partial_{\mu} \phi^* \phi \), and that for a fermion field, \( \bar{\psi} \gamma^\mu \psi \).

But it’s not the current required for vector dominance. Vector dominance is the statement that the largest coupling of hadrons to photons is through a current linear in the appropriate vector fields, with a coupling proportional to the mass (squared) of the vector fields. We are not used to dealing with currents linear in the fields!

Such a current can be constructed, but at the expense of quite a leap. After all, we are dealing with a theory which has only a global chiral symmetry. Vector dominance instructs us to promote this symmetry to one which is local. At least for myself this was a remarkable step which I strenuously resisted, to no avail. The basic point can be understand by changing the transformation of Eq. (12) to that for a local chiral rotation,

\[ A_{\mu}^l \rightarrow \frac{1}{-i\tilde{g}} \Omega_{l,r}^\dagger D_{l,r}^\mu \Omega_{l,r} \]  

(16)

I introduce the covariant derivatives \( D_{l,r}^\mu = \partial^\mu - i\tilde{g} A_{l,r}^\mu \) and the coupling constant \( \tilde{g} \). Notice that the coupling constant \( \tilde{g} \) is that for vector dominance — it has nothing (directly) to do with the coupling constant of \( QCD \), and is generically a large coupling, of order one.

The amazing point is that by assuming that the vector fields transform inhomogeneously under (local) chiral rotations, we are automatically guaranteed that the mass term
gives the proper current. This is simply because for small \( \omega_l \), \( \delta A^\mu_l = i \partial^\mu \omega_l/\tilde{g} + [A^\mu_l, \omega_l] \).

Now for the mass term, the commutator term doesn’t contribute to the current, as before. But the inhomogeneous part of the transformation of the gauge field then gives, trivially, a contribution to the current as

\[
j_l^\mu = \frac{m^2}{\tilde{g}} A_l^\mu. \tag{17}\]

This is the type of expression required for vector dominance: the current is linear in the gauge fields, with a coefficient proportional to the mass (squared) divided by the vector coupling constant.

Since this is the current we want, we then have to ensure that no other terms in the lagrangian contribute to the current. But having taken the giant step of introducing a local chiral symmetry, the rest is easy. To ensure that no other terms in the effective lagrangian contribute to the current, we require that all other terms couple to the vector fields in a gauge invariant manner. As long as the couplings are gauge invariant, then, by definition the rest of the lagrangian will be invariant under the gauge transformation of Eq. (16)!

Hence we introduce the nonabelian field strengths,

\[
F_{l,r}^{\mu\nu} = \partial^\mu A_{l,r}^\nu - \partial^\nu A_{l,r}^\mu - i\tilde{g} [A_{l,r}^\mu, A_{l,r}^\nu]. \tag{18}\]

For the scalar field \( \Phi \) the appropriate covariant derivative acts as

\[
D^\mu \Phi = \partial^\mu \Phi - i\tilde{g} (A_l^\mu \Phi - \Phi A_l^\mu). \tag{19}\]

There is a relative minus sign between the coupling constant for the left and right handed fields because from Eq. (10), \( \Phi \) transforms under \( \Omega_r \) and \( \Omega_l^\dagger \). Given these quantities, the effective lagrangian for the gauged linear sigma model which respects vector dominance is then

\[
\mathcal{L} = \text{tr} |D_\mu \Phi|^2 - h \text{tr} (\Phi) + \mu^2 \text{tr} |\Phi|^2 + \frac{\lambda}{2} (\text{tr} |\Phi|^2)^2 \\
+ \frac{1}{2} \text{tr} \left( (F^\mu_{l})^2 + (F^\mu_{r})^2 \right) + m^2 \text{tr} \left( (A_l^\mu)^2 + (A_r^\mu)^2 \right). \tag{20}\]

The scalar potential is standard, with a term linear in \( \Phi \), \( \sim h \text{tr} (\Phi) \), added to ensure that the pions are massive.

The assumption that the spin one mesons couple to the scalar field only through coupling which are locally gauge invariant greatly restricts the possible couplings. If only a global chiral symmetry is imposed, many more terms are possible. For example, the terms \( A_l^\mu \Phi \) and \( \Phi A_l^\mu \) both transform under global chiral rotations like \( \Phi \) itself, Eq. (10). Since QCD preserves parity, we can require that all possible terms be invariant under the interchange of left and right handed gauge fields. For example, for the quartic terms, besides \( |A_l^\mu \Phi - \Phi A_l^\mu|^2 \), the terms \( |A_l^\mu \Phi + \Phi A_l^\mu|^2 \) and \( |A_l^\mu \Phi|^2 + |\Phi A_l^\mu|^2 \) are also allowed under the global chiral symmetry.
From the viewpoint of the renormalization group, this restriction in possible terms is most striking. The standard prescription of the renormalization group is the following. Start with the terms with the lowest mass dimension, which are usually the mass terms. Write down all mass terms consonant the relevant symmetries. For terms with higher mass dimension, which are typically cubic or quartic in the fields, and whose coupling constants are less relevant, all possible couplings which respect the symmetries are allowed. For example, for a theory with an $N$ component vector $\vec{\phi}$, if the symmetry is $O(N)$, only terms as $\vec{\phi} \cdot \vec{\phi}$ arise. But if the mass term is only invariant under $O(N - 2)$, many more terms are possible, as long as they are $O(N - 2)$ invariant.

What is so peculiar about the model of vector dominance is that the mass term for the gauge fields is the most relevant operator and manifestly breaks the gauge symmetry. Yet for all other terms — the coupling of the gauge fields to themselves, and the coupling of the gauge fields to the scalar fields — one uses this local gauge symmetry, most crucially, to greatly restrict the possible couplings of the gauge field. Of course what is unusual about the model of vector dominance is that it is a local, and not simply a global, symmetry which arises. Somehow this must be crucial to its use and eventual consistency. But it seems to me as if something more interesting than mere dusty phenomenology is afoot.

I only use the effective lagrangian of Eq. (20) at the tree level. One simplification follows immediately: the equations of motion for the gauge field are $D^\mu F_\mu^\nu = m^2 A_\nu^\mu + j_\nu^\mu$, where $j_\nu^\mu$ is the current from the scalar field. To lowest order this reduces to $\partial^\mu F_\mu^\nu = m^2 A_\nu^\mu$, which is only consistent is the gauge field is in Landau gauge, $\partial_\mu A_\mu^\nu = 0$. Thus in perturbation theory the propagator for the gauge field is $(\delta^{\mu\nu} - p^\mu p^\nu/(p^2 + m^2))$, which falls off like $\sim 1/p^2$ at large momenta. Of course in gauges other than Landau, the propagator behaves like $(\delta^{\mu\nu} - p^\mu p^\nu/m^2)/(p^2 + m^2)$, and is of order $\sim 1$ at large momenta. So at least the ultraviolet behavior of the gauge propagator isn’t as horrible as it might be.

Of course there is still the question of how to consistently quantize the theory. The usual linear sigma model at least has the virtue of perturbative renormalizability, although one may well be pushing it into a highly nonperturbative regime. The effective lagrangian of Eq. (20) does not have such a virtue, however, and it is far from clear how to treat the standard problems of such theories: the coupling of (unphysical) massless modes which are present in the gauge propagator to physical modes, unitarity, lack of renormalizability at two loop order, and so on. It may well be possible to find some extension of Eq. (20) which is (at least) renormalizable, but I haven’t thought seriously about it.

Much of the physics of vector dominance can be read off from the coupling of the spin one fields to the scalars through the covariant derivative. After working out the matrix algebra, one finds

$$tr|D_\mu \Phi|^2 = \frac{1}{2} (\partial^\mu \sigma + \vec{\bar{g}} \vec{a}_1^\mu \cdot \vec{\pi})^2$$
\[
\frac{1}{2} (\partial^\mu \vec{\pi} + \tilde{g}(\vec{\rho}^\mu \times \vec{\pi}) - \tilde{g} \sigma \vec{a}_1^\mu)^2 + \frac{\tilde{g}^2}{2} (\sigma^2 + \vec{\pi}^2) f_1^2.
\]  
(21)

Notice that the \(\omega\) meson drops out completely: from Eq. (11), in the vector fields \(A_{l,r}\), the \(\omega\) meson is proportional to the unity matrix, and so the \(\omega\) cancels in the covariant derivative in Eq. (19). There is a coupling of the \(\omega\) meson, due to the anomaly, but I will only discuss that in passing here.

The principal couplings of interest in Eq. (21) are the coupling of the \(\rho\) to two pions, as \(\vec{\rho}^\mu \cdot (\vec{\pi} \times \partial^\mu \vec{\pi})\), and the coupling of the \(a_1\) to the \(\sigma\) and \(\vec{\pi}\), \(\sigma \vec{a}_1^\mu \cdot \vec{\pi}\). Both of these couplings are proportional to the coupling of vector dominance, \(\tilde{g}\), and are presumably responsible for the principal decays of these particles.

To obtain more realistic expressions, I assume that the potential of the scalars is such that a vacuum expectation value for the sigma field develops, \(\mu^2 = \mu_0^2 < 0\). Because of the background magnetic field \(\sim h\), the vacuum expectation value of \(\phi\) must be along the \(\sigma\) direction. After shifting \(\sigma \to \sigma_0 + \sigma\), however, a complication arises which is special to the gauged sigma model: there is a cross term between the \(a_1\) and the pion, as \(\tilde{g} \vec{a}_1^\mu \cdot \partial^\mu \vec{\pi}\). This cross term must be eliminated by a shift in the \(a_1\) field,

\[
\vec{a}_1^\mu \to \vec{a}_1^\mu - \frac{g\sigma_0}{m^2 + (g\sigma_0)^2} \partial^\mu \vec{\pi}.
\]  
(22)

One can show that the equations of motion for the shifted \(a_1\) field are such that the shifted (and not the unshifted) \(a_1\) field should be in Landau gauge.

After this shift, for the vector fields the mass term written in Eq. (20), \(tr(A_{l,r}^2)\), gives

\[
m^2_\rho = m^2_\omega = m^2, \quad m^2_{a_1} = m^2_{f_1} = m^2 + (\tilde{g}\sigma_0)^2.
\]  
(23)

This is in good accord with experiment, where the \(\rho\) and the \(\omega\), as well as the \(a_1\) and the \(f_1\), are very nearly degenerate. Yet I find it mystifying. Besides the mass term \(tr(A_{l,r}^2)\), there is a second set of mass terms which are invariant under global \(SU_l(2) \times SU_r(2)\) rotations, \((tr(A_{l,r}))^2\). These new mass terms only contribute to the masses of the \(\omega\) and the \(f_1\), but experimentally they do not seem to be there: the masses of the \(\omega\) and the \(\rho\), and the masses of the \(a_1\) and the \(f_1\), are each within 2\% of each other! Perhaps my understanding of vector dominance is faulty, but there appears to be a further principle at work, something that tells us that such flavor singlet mass terms are very small. Of course such singlet mesons can only be treated realistically within the context of a full, three flavor model, but I believe that the paradox will remain.

After shifting the \(a_1\) field as in Eq. (22), the kinetic term for the \(\pi\) becomes

\[
\frac{1}{2} \left( \frac{m^2}{m^2 + (\tilde{g}\sigma_0)^2} \right) (\partial^\mu \vec{\pi})^2.
\]  
(24)

This demonstrates what is known as the “partial” Higgs effect. In the limit that the mass of the vector meson becomes very heavy, \(m \to \infty\), all effects of the vector
mesons should decouple, and the pions (sigmas, etc.) are unaffected. The opposite limit, when the mass of the vector meson vanishes, \( m \to 0 \), is also familiar. From Eq. (24), the kinetic term for the pions vanishes as \( m \to 0 \), but this is simply because in this limit there is a true gauge symmetry and a true Higgs effect, with the pions turning into the longitudinal components of the \( a_1 \)’s. For finite but nonzero \( m \), what happens is that ratio \( \sqrt{m^2 + (\tilde{g}\sigma_0)^2}/m = m_{a_1}/m_\rho \) enters. Experimentally this is a significant number, \( \sim 1.6 \).

In this vein, I should also mention what is known as the KSRF relation. This relation predicts that the ratio of the \( a_1 \) to \( \rho \) mass is fixed, \( \sigma = \sqrt{2} \). Within the general context of gauged linear sigma models, there is no reason why the KSRF relation should be satisfied; see, for example, the comments of Lee and Nieh. One could argue that the actual value of \( m_{a_1}/m_\rho \sim 1.6 \) is not too far from \( \sqrt{2} \sim 1.414 \ldots \), but I see no particular virtue in adhering to the KSRF relation. Instead, I use the \( m_{a_1}/m_\rho \) ratio to fix the value of the vector meson coupling \( \tilde{g} \).

After rescaling \( \pi \to (m_{a_1}/m_\rho)\pi \), the physical quantities of interest are:

\[
m_{\pi}^2 = \left(\frac{m_{a_1}}{m_\rho}\right)^2 \frac{h}{\sigma_0} , \quad m_\sigma^2 = \frac{h}{\sigma_0} + 2\lambda \sigma_0^2 , \quad f_\pi = \frac{m_\rho}{m_{a_1}} \sigma_0 . \tag{25}
\]

Fitting the above expressions with \( m_\pi = 137 \text{ MeV} \), \( m_\sigma = 600 \text{ MeV} \), \( m_\rho = 770 \text{ MeV} \), and \( m_{a_1} = 1260 \text{ MeV} \), the results are

\[
\sigma_0 = 152 \text{ MeV} , \quad h = (102 \text{ MeV})^3 , \quad \mu_0^2 = -(588 \text{ MeV})^2 , \quad \lambda = 7.6 . \tag{26}
\]

For comparison, if we send the mass of the vector mesons to infinity, and denote the analogous quantities by a superscript \( \text{nv} \) for “no vectors”, with the same masses for the \( \pi \) and the \( \sigma \) I find

\[
\sigma_0^{\text{nv}} = 93 \text{ MeV} , \quad h^{\text{nv}} = (120 \text{ MeV})^3 , \quad (\mu_0^{\text{nv}})^2 = -(567 \text{ MeV})^2 , \quad \lambda^{\text{nv}} = 19.7 . \tag{27}
\]

From Eq. (25), the “partial” Higgs effect tends to increase the pion mass and to decrease the pion decay constant \( f_\pi \). This description is somewhat misleading, because it is phrased in terms of the quantities \( h, \sigma_0, \) and \( \mu_0^2 \), which are not directly physical; the physical quantities are the masses and \( f_\pi \). One quantity which is physical is the value of the scalar self coupling, \( \lambda \); that is a dimensionless number, which tells one how close the linear model is to the nonlinear model, the nonlinear model being recovered as \( \lambda \to \infty \). I find it notable that adding the vectors changes this coupling from \( \lambda^{\text{nv}} = 19.7 \) to a much smaller value, \( \lambda = 7.6 \). Presumably with the \( \rho \) and \( a_1 \) vector mesons, some of the scalar interactions, which otherwise arise solely from their self interaction, \( \lambda(|\Phi|^2)^2 \), can be made up by exchange of vector mesons, giving a smaller value of \( \lambda \).
In conclude this section by discussing how vector meson dominance works in the limit of a large number of colors, \( N_c \to \infty \); if the nonabelian coupling of the \( SU(N_c) \) gauge theory is \( g^2 \), \( g^2 N_c \equiv \alpha_s \) is held fixed and of order one as \( N_c \to \infty \). Consider, in particular, the spectrum of dileptons, which is given by the imaginary part of the two point function of hadronic currents, \( \langle j^\mu(x) j^\nu(0) \rangle \). In perturbation theory this graph starts with a free quark loop, which is of order \( N_c \) for quarks in the fundamental representation of \( SU(N_c) \), plus an infinite series of corrections in \( \alpha_s \). Now at large \( N_c \) it is known that all color singlet currents are saturated by hadronic states, where mesonic masses are of order one. This works fine for the vector channel considered here if the coupling of \( j^\mu \) to the \( \rho \) is of order \( \sqrt{N_c} \), which is standard at large \( N_c \).

This can also be seen from vector dominance, assuming that vector dominance holds in the large \( N_c \) limit. From Eq. (17), \( j^\mu = m^2 A^\mu / \tilde{g} \); \( m = m_\rho \) is of order one, as is the vector field \( A^\mu \), since it is just another mesonic field. The coupling of vector meson dominance, however, is a trilinear coupling between three mesons. This is typically vanishes at large \( N_c \) as \( \tilde{g} \sim 1/\sqrt{N_c} \), so Eq. (17) becomes \( j^\mu \sim \sqrt{N_c} A^\mu \), and agrees with the counting of quark loops.

All of this is totally standard. Let us then push on and consider corrections to the large \( N_c \) limit. The coupling of the \( \rho \) to two pions is again a trilinear meson coupling, of order \( \tilde{g} \sim 1/\sqrt{N_c} \), so the width of the \( \rho \) meson is of order \( \tilde{g}^2 \sim 1/N_c \). Thus while the height of the \( \rho \) peak is very large, of order \( N_c \), it is very narrow, of order \( 1/N_c \), so the total area under the \( \rho \) peak is of order one. At order one there are other contributions to dilepton production, such as from pion pairs. Since the coupling of hadrons to photons is electromagnetic, and pions don’t interact at large \( N_c \), the pion contribution to dileptons is just given by the imaginary part of the one loop diagram of pions. Since both the electromagnetic coupling of the pions and their number are of order one, so is the total pion contribution. Thus the large \( N_c \) limit is consistent with vector meson dominance as follows: vector mesons, such as the \( \rho, \omega, \) etc. give tall but narrow peaks, under which is a smooth continuum for \( \pi^+\pi^-, K^+K^- \) pairs, and so on. The total area under the vector meson peaks, and the continuum from scalar meson pairs, is the same, of order one.

The above counting just shows that the large \( N_c \) limit is compatible with vector meson dominance; it does not show that in fact vector meson dominance emerges at large \( N_c \). Nevertheless, it is important to see that there is nothing inconsistent between vector meson domainance and the limit of large \( N_c \).

2.3. Vector mesons at nonzero temperature

With all of this introduction, the extension to nonzero temperature is relatively straightforward. I deal exclusively with mean field theory, since at least that gives a well defined approximation. I am not claiming that it is a good approximation, simply that it is well defined.
Consider the mass terms in the effective lagrangian of Eq. (20),

$$\mathcal{L}_{\text{mass}} = \mu^2 tr|\Phi|^2 + m^2 tr\left((A_\mu^a)^2 + (A_\mu^b)^2\right)$$

Diagrammatically, both masses increase with increasing temperature. For example, from the scalar self interactions, a term as $\sim \lambda T^2 tr|\Phi|^2$ is generated from the interactions with a pion loop; similarly, the interactions of vector mesons with a pion loop also produce a term $\sim \tilde{g}^2 T^2 tr(A_1^2 + A_2^2)$. \textit{A priori}, I do not see any reason why the contribution of a pion loop to one term should be significantly larger, or smaller, than the other.

This allows me to make what is mathematically trivial, but physically profound, observation: that the effects of thermal fluctuations are always to cause both $\mu^2$ and $m^2$ to increase with increasing temperature. The details of their relative increase surely depends on details of the model — the number of flavors, colors, the values of the self couplings, \textit{etc.} But in any gauged sigma model, both mass terms go one way and only one way with increasing temperature, and that is \textit{up}.

The expressions for the masses of the vector mesons given in Eq. (23) remain valid as $m^2$ and $\mu^2$ (and so $\sigma_0^2$) change. Thus a gauged sigma model predicts that the mass of the $\rho$ increases monotonically with temperature; similarly, that the masses of the $\omega$ and the $\phi$ also increase. Because of the constraints of chiral symmetry, the masses of the other vector mesons, the $a_1$ and the $f_1$ (and their strange partners) behave in a more complicated fashion: while $m^2$ increases, $\mu^2$ becomes less negative, so $\sigma_0$ decreases. Thus the $m_{a_1}$ and $m_{f_1}$ first decrease with temperature, until they become (approximately) degenerate with the $\rho - \omega$; then the masses of all states continue to increase with temperature.

The above assumes, implicitly, that there are no singlet masses for the vector fields, $(tr(A_{l,r}))^2$. Such singlet mass will split the $\omega$ and $f_1$ apart from the $\rho$ and $a_1$. Even if such a term is not present at zero temperature — because of experimental constraints on the known masses of the $\rho$ and $\omega$ — unless there is a symmetry reason why such a term cannot arise, it will be generated at nonzero temperature. If such a term is generated, the degeneracy between the $\omega$ and the $\rho$ will be lifted, but both masses will still increase with temperature. In any case, as we shall see, all states becomes broad at nonzero temperature, so a relatively small shift in the real part of the masses may not be significant.

The prediction that the mass of the $\rho$ (and those of the $\omega$ and $\phi$) increase with temperature is not standard lore. It is a prediction of a gauged sigma model, although the generality of the remark does not seem to have been stressed in previous studies. Different models, such as those of Brown and Rho, predict the opposite: that the mass of the $\rho$ decreases as the temperature goes up. In principle this could be studied by means of other phenomenological approaches, such as sum rules. Here one runs into a problem: extending sum rules to nonzero temperature is not free of ambiguity. Thus while some papers find that the mass of the $\rho$ decreases with
increasing temperature, others find that it increases. Besides my obvious prejudice in the matter, it is worth noting that the studies which claim to find that the \( \rho \) mass increase with temperature find that it does so in a manner consistent with the restoration of chiral symmetry. Of course chiral symmetry alone is not enough to claim that the \( \rho \) mass goes up with temperature; all chiral symmetry tells you is that the \( \rho \) and \( a_1 \) should be approximately degenerate.

One approach which differs from a gauged linear sigma model, and yet incorporates chiral symmetry in a crucial way, is the study of Nambu-Jona-Lasinio (NJL) models. If the predictions of the gauged linear sigma model have any generality whatsoever, they should hold for a NJL. There is a technical caveat: what is usually studied are (chirally) symmetric NJL models with the simplest possible interaction, \((\bar{\psi}i\tau\gamma_5\psi)^2\). This type of interaction is certainly adequate for treating the scalar and pseudoscalar particles, since undoing the quartic interactions by means of auxiliary fields with automatically generate effective fields for these particles. To study vector and axial vector particles, however, I suggest that it is crucial to include interactions such as \((\bar{\psi}i\tau\gamma_\mu\psi)^2 + (\bar{\psi}i\tau\gamma_5\gamma_\mu\psi)^2\), in what is termed the extended NJL model.

With this cautionary aside, there is a unique prediction: in any extended NJL model, the mass of the \( \rho \) should increase, monotonically, with temperature. If not, there is something seriously wrong with a gauged linear sigma model, or with vector meson dominance.

In principle, the masses of the vector mesons could also be studied by numerical simulations of lattice gauge theory. This is not elementary to date all studies have concentrated on the (manageable) task of computing static correlation lengths. While the \( \pi \) and \( \sigma \) channels are special, in all other channels one finds that two quark states have a static correlation length \( \sim 1/(2\pi T) \), three quark states one of \( \sim 1/(3\pi T) \), and so on. The usual interpretation is that this represents the propagation of free quarks, dressed by interactions. I suggest that if the deconfining transition is far afield, then what one is seeing really is the propagation of mesons (and baryons) in a hot hadronic gas. To definitively answer the problem, however, it is necessary to look not at static correlation lengths, but at the true poles in the appropriate (effective) propagators. These are defined in real time, by the analytic continuation of the spectral densities from euclidean momenta. Thus it is necessary to measure not just the static correlation lengths, at \( p^0 = 0 \), but the correlation lengths for \( p^0 = \pm 2\pi T \), \( \pm 4\pi T \), etc., and then fourier transform. This is a very difficult problem.

At this point I note an alternate scenario by Georgi. He works in a nonlinear model, with the term for the \( \rho \) mass squared proportional to a dimensionless coupling constant times \( f_\pi^2 \). He then argues that as \( f_\pi \to 0 \), \( m_\rho \to 0 \). In terms of the linear sigma model, it is not apparent to me why the mass term for the \( \rho \) should involve \( f_\pi \) at all; why is it simply not some other dimensional parameter? One can rephrase the argument in terms of the linear sigma model: Georgi’s term corresponds to assuming that there is no bare mass term for the \( \rho \), such as \( m^2 tr(A_i^2 + A_k^2) \), but only a term
as $tr(\Phi^\dagger \Phi)tr(A_i^2 + A_j^2)$, which would have a dimensionless coupling constant. In the linear sigma model this is odd: why should there be no bare mass term for the vector mesons, yet one induced by spontaneous symmetry breaking? After all, the bare mass term is more relevant than Georgi’s term. Leaving prejudice aside, I appeal to the lattice data. Admittedly, while static screening lengths are not automatically the (inverse) pole masses, it is true that if the $\rho$ were massless at $T_\chi$, then its static screening length would diverge, like that for the $\sigma$ and the $\pi$. This is not seen in present day lattice simulations: only the $\sigma$ and the $\pi$ have divergent static screening lengths (at least for two flavors, where the transition appears to be of second order).

Returning to the matter at hand, one of the drawbacks of the gauged sigma model is that it does not predict how much $\mu^2$ and $m^2$ increase with temperature. I thus adopt what is, most honestly, a wild guess. As the temperature increases, the mass of the $\rho$ goes up, while that of the $a_1$ goes down, until they meet (approximately), after which they both increase. I define the temperature of chiral symmetry restoration $T_\chi$ as that for which the two masses are approximately equal, and assume that they obey the rule that their mass at $T_\chi$ is the arithmetic mean of their masses at zero temperature,

$$m_\rho(T_\chi) \approx m_{a_1}(T_\chi) \sim \frac{1}{2} (m_\rho(0) + m_{a_1}(0)) = 1 \text{ GeV}. \quad (29)$$

I emphasize that this is only a guess; at least it is easy to remember. It probably is an upper bound on how much the $\rho$ mass can increase with temperature. For example, Song[14] has done calculations in a gauged nonlinear sigma model, and finds that with different parametrizations, the mass of the $\rho$ can even stay (almost) constant with temperature, or increase as much as in Eq. (29). The former seems extremely unlikely to me; again, the pion loops that drive $\mu^2$ up also contribute to driving $m^2$ up. Nevertheless, how much the $\rho$ mass increases with temperature is probably only something which can be settled by lattice simulations.

The virtue of Eq. (29) is that it gives a unique parametrization of how masses change with temperature; I find that if $\mu_0^2 = -(588 \text{ MeV})^2$ and $m_0^2 = (770 \text{ MeV})^2$ are the values at zero temperature, then Eq. (29) is satisfied for

$$m^2 = m_0^2 + .6 \frac{(\mu^2 - \mu_0^2)}{\mu_0^2}. \quad (30)$$

One can then compute the properties of the model as a function of a single parameter, $\mu^2$; all effects of increasing temperature are then modeled by increasing $\mu^2$. Again, I stress that Eqs. (29) and (30) are only guesses.

In fig. (2) I give the values of the masses of the $\pi$, $\sigma$, $\rho$, and $a_1$ as a function of increasing $\mu^2$ ($\sim$ temperature squared). There is no prediction whatsoever for $T_\chi$, since one can only see that there is first approximate chiral symmetry restoration for $\mu_\chi^2 \sim +(150 \text{ MeV})^2$. At $T_\chi$, $m_\sigma \sim m_\pi \sim 200 \text{ MeV}$; by assumption, $m_\rho \sim m_{a_1} \sim 1 \text{ GeV}$. In fig. (3) I show $f_\pi$ versus $\mu^2$; the relevant observation is that at $\mu_\chi^2$, $f_\pi$
Figure 2: Masses of the vector and scalar particles versus the mass parameter $\mu^2$ in a two flavor, gauged linear sigma model. The masses are plotted against the $y$-axis in MeV; from top to bottom, these are the masses of the $a_1$, $\rho$, $\sigma$, and $\pi$, respectively. $\mu^2$ is along the $x$-axis: the values at zero temperature are at the extreme left, with increasing $\mu^2$ equivalent to increasing temperature squared.

has decreased to about a third of its value at zero temperature. This decrease in $f_\pi$ is elementary: for vanishing pion mass $f_\pi \sim \sigma$, and vanishes at the temperature for chiral restoration (if the transition is second order).

We can use the effective model to make more detailed predictions about the widths of the states at $T_\chi$. At present I give only a very crude estimate, with careful estimates given later. There is certainly new kinematics which opens up: at zero temperature I assume that $m_\sigma = 600 \text{ MeV}$, so the $\rho$ can’t decay into $\sigma$’s plus $\pi$’s. That changes at $T_\chi$, since then $m_\sigma \sim m_\pi$, and furthermore the $\rho$ is heavier.

The decay modes of the thermal $\rho$ can be read off from the effective lagrangian of Eq. (21), but the allowed channels follow from general principles. Even if it is possible kinematics, the $\rho$ doesn’t decay into two $\sigma$’s because of isospin symmetry; also, $\rho$ doesn’t decay into $\sigma\pi$ because of $G$-parity. Of course at nonzero temperature $\rho \rightarrow \pi\pi$, as at zero temperature. By similar arguments one can see that the only new three body decay mode is $\rho \rightarrow \pi\pi\sigma$; this respects both isospin and $G$-parity,
Figure 3: The value of the pion decay constant, $f_\pi$, versus the mass parameter $\mu^2$ in a two flavor, gauged linear sigma model. The y-axis is the value of $f_\pi$ in MeV, while $\mu^2$ is along the x-axis: the values at zero temperature are at the extreme left, with increasing $\mu^2$ equivalent to increasing temperature squared.

and with the assumed masses has nonzero phase space. However, I do not believe that this three body decay gives a significant contribution; the coupling constant for $\rho \to \pi\pi$ is $\sim \tilde{g}$; from Eq. (21), that for $\rho \to \pi\pi\sigma$ is $\tilde{g}^2 f_\pi m_\sigma/m_\rho^2$, or $\sim \tilde{g}/3$. Thus the amplitude for $\rho \to \pi\pi\sigma$ is about 10% that for $\rho \to \pi\pi$, even without the further restrictions which arise kinematically from a three body, versus a two body, decay. For this reasons I neglect the three body decays of the thermal $\rho$.

For the remaining mode of $\rho \to \pi\pi$, I make life even easier for myself by neglecting thermal effects entirely, computing the decay as if it were at zero temperature. This is valid because the pions are so energetic; if $n(E) = 1/(\exp(E/T) - 1)$ is the Bose-Einstein distribution function for the pions, at $E \sim 500$ MeV and $T \sim 150$ MeV, $2n(E/T) \sim .07$. Consequently,

$$ \Gamma_\rho(T_\chi) \sim \frac{(m_\rho(T_\chi)^2 - 4m_\pi^2(T_\chi))^{3/2}}{m_\rho^2(T_\chi)} \Gamma_\rho(0) \sim 175$ MeV. \hspace{1cm} (31) $$

As the mass of the $\pi$ goes up, the decay width of the $\rho$ goes down, to about 100 MeV for $m_\rho \sim 1 GeV$ and $m_\pi \sim 300$ MeV. Given the crudeness of the model, I conclude
Figure 4: The width of the $\rho$, under the extreme assumptions discussed in the text. Essentially, the formula for the width is assumed to be the same as at zero temperature, with the only difference from the mass shifts of the $\rho$ and $\pi$. The $y$-axis is $\Gamma_\rho(T_\chi)$ in $MeV$, the $x$-axis the value of $m_\pi(T_\chi)$ in $MeV$.

that the width of the thermal $\rho$ at 1$GeV$ is approximately equal to its width at zero temperature. In fig. (4) I present a graph of $\Gamma_\rho(T_\chi)$ for $m_\rho = 1\, GeV$, and different values of $m_\pi$, under the above assumptions. Due to phase space, the decay width decreases from that in Eq. (31).

Implicitly I am assuming that the coupling constant for vector meson dominance, $\tilde{g}$, does not change with temperature. This is probably wrong, but consideration of the one loop $\beta$-functions suggests that $\tilde{g}$ should decrease with temperature near a critical point, making the width even smaller.

Computing the width of the $\omega$ meson is more difficult. A proper analysis requires an understanding of the Wess-Zumino-Witten term\cite{12} which I defer for another day.\footnotemark[18] For now I make a very simple assumption, that the decay of the $\omega$ is dominated by what is known as the Gell-Mann Sharp Wagner (GSW) mechanism. This is the statement that the decay $\omega \rightarrow 3\, \pi$ proceeds by $\omega \rightarrow \rho\, \pi$, with the virtual $\rho$ decaying as $\rho \rightarrow \pi\pi$. The amplitude for the first process, $\omega \rightarrow \rho\, \pi$, is dominated by the anomaly, and proportional to $1/f_\pi$; the amplitude for the second process is proportional to $\tilde{g}$ times kinematic factors. At present\cite{13} I ignore all of these details to concentrate simply
Figure 5: Masses of the scalar particles versus the mass parameter $\mu^2$ in a three flavor linear sigma model which is not gauged. The masses are plotted against the $y$-axis in MeV; $\mu^2$ is along the $x$-axis: the values at zero temperature are at the extreme left, with increasing $\mu^2$ equivalent to increasing temperature squared. Notice that unlike the two flavor model, for three flavors I take $m_{\sigma'^{\prime}}(0) = 975$ MeV. The assumptions which produce this plot are described in section (3.2.1). The point of note here is to see that the kaon increases in mass, to approximately 650 MeV.

on the change in $f_\pi$ with temperature; since $f_\pi(T_\chi) \sim f_\pi(0)/3$ and $\Gamma_\omega(0) \sim 10$ MeV,

$$\Gamma_\omega(T_\chi) \sim \left( \frac{f_\pi(0)}{f_\pi(T_\chi)} \right)^2 \Gamma_\omega(0) \sim 100$ MeV. \hspace{1cm} (32)$$

This estimate is very crude, and clearly requires a more careful analysis. The basic point, however, is bound to be correct: as chiral symmetry is (approximately) restored, $f_\pi$ decreases, leading to a much broader $\omega$ meson than at zero temperature. Kinematic factors will only help, as the $\omega$ moves up in mass, so the allowed phase space increases as well.

Lastly, although the model discussed does not incorporate it, it is reasonable to ask about the $\phi$ meson. The chiral partner of the $\phi(1020)$ is one of the two $f_1$ states, either the $f_1(1420)$ or the $f_1(1510)$. There is some controversy concerning the $f_1(1420)$, so I take the $f_1(1420)$ as some sort of special quark state, and assume that
the chiral partner of the $\phi$ is the $f_1(1510)$. Using the same kind of simple minded prediction for the mass at $T_\chi$ as in Eq. (29), I suggest that

$$m_\phi(T_\chi) \approx m_{f_1}(T_\chi) \sim \frac{1}{2} \left( m_\phi(0) + m_{f_1(1510)}(0) \right) = 1250 \text{ MeV}.$$  \hspace{1cm} (33)

Even with this generous increase in the mass of the thermal $\phi$, the thermal $K$ mass is like that of the $\pi$, and also increases, to at least 650 MeV; see, for example, fig. (5). With these numbers, the $\phi$ does not have phase space to decay as $\phi \to K K$ (except for thermal processes, which should be small). Even if the masses were to change so that $\phi \to K K$ were allowed, it is very unlikely for the width of the $\phi$ to change so much that it decays inside the plasma.

In summary, for the thermal vector mesons at the temperature of chiral symmetry restoration, $T_\chi$, the width of the $\rho$ remains about that at zero temperature, and decays inside the plasma; the $\omega$ becomes much broader, so much so that most thermal $\omega$’s decay inside the plasma; the $\phi$ remains narrow, decaying primarily outside of the plasma. I do not expect a more careful analysis to significantly alter these conclusions.

3. Disoriented Chiral Condensates

If an extraordinarily large number of pions are produced in a hadronic collision, it is natural to think that these unusual events might proceed by the coherent decay of a (semi-) classical pion field. There might be a very distinctive signature of such a coherent decay: a given classical pion field points in some direction (or directions) in isospin space, so coherent production is necessarily one in which isospin symmetry would appear to be, at least locally, badly violated. That is, most of the pions from a given region would be primarily charged, or conversely, primarily neutral. (Of course in the total collision, isospin will always, in the end, average out.) Such behavior may have been observed in Centauro events in cosmic ray collisions. Experimentally the situation is unclear: some, but not all, cosmic ray experiments see Centauro behavior, while collider experiments do not see Centauro type events. Hopefully the “Mini Max” experiment at FNAL may help settle the issue. In the interim, it behooves theorist to consider seriously the possibility.

I introduce the quantity $R_3$, which is the ratio of neutral to the total number of pions. In the average, hadronic collisions conserve isospin. This will be true if one averages over different events (say in $pp$ collisions), or over all regions in rapidity of $AA$ collisions. Thus in the average there will be a binomial distribution in $R_3$, strongly peaked about the isospin symmetric value of 1/3.

But the behavior of the average might not be representative of all events. Suppose that in individual events single domains are produced, in which the pion field has a nonzero vacuum expectation value which points in a given, fixed direction in isospin space. If all directions in isospin space are equally likely, while the average value
of $\mathcal{R}_3$ in that domain is $1/3$, the distribution is far from binomial, $= 1/(2\sqrt{\mathcal{R}_3})^2$.

At this point I haven’t spoken of how, in detail, such domains could be produced dynamically; a possible mechanism is the subject of the following.

Note that while I have chosen the 3 direction in isospin to be that for neutral pions, there is nothing special in this choice. For example, if $\mathcal{R}_1$ is the fraction of pions in the isospin 1 direction, then the distribution in $\mathcal{R}_1$ is $1/(2\sqrt{\mathcal{R}_1})$. This is obvious, since by an isospin rotation the 3 direction can be relabeled as the 1 direction. The detailed form of the distribution in $\mathcal{R}_3$ or $\mathcal{R}_1$ is a consequence of the symmetry group of isospin being $O(3)$, and differs for other symmetry groups. The important point is that the distribution is far from binomial.

Bjorken, Kowalski, and Taylor have proposed a “Baked Alaska” scenario wherein such a distribution is produced in $pp$ collisions; the idea here is to trigger specifically on $pp$ collisions in which the multiplicity is very large. $^2$ They refer to a domain with a fixed direction in isospin space as a “Disoriented Chiral Condensate” (DCC); it is disoriented because in the true vacuum the pion field doesn’t point in this direction (instead the sigma field acquires a vacuum expectation value). In some abuse of terminology, I refer to any system which generates an isospin distribution $= 1/(2\sqrt{\mathcal{R}_3})$ as that of a DCC. It is worth remembering, however, that the way in which DCC’s can arise in hadronic collisions can, and in general will be, very different from that in $AA$ collisions.

The behavior of central $AA$ collisions at high energies is manifestly an example of a system with a large number of pions; for example, at RHIC and LHC, there will be on the order of thousands of pions per unit rapidity. The hope is then that while the total isospin in conserved for the total collision, that for given slices in rapidity, the distribution in $\mathcal{R}_3$ is not binomial, but that of a DCC. As emphasized by Blaizot and Krzywicki, $^2$ the difficulty is understanding why domains should be large in the transverse dimension: many transverse domains, oriented in different directions, washes out the DCC effect to give the usual binomial distribution.

Rajagopal and Wilczek $^2$ (R&W) proposed a dynamical mechanism for obtaining DCC distributions in heavy ion collisions. They use a linear sigma model to describe the chiral behavior over large distances, and make a dramatic assumption about the dynamical evolution in time. They assume that the dynamics is that of a “quench”: the initial state is chirally symmetric, as appropriate at high temperature, but its evolution forward in time is by means of of the (classical) equations of motions at zero temperature. R&W assert that this dynamics, which is very far from equilibrium, produces amplification and coherent oscillation of pions at low momentum; that is, large DCC domains.

In this talk I review two attempts by S. Gavin, A. Gocksch and I to understand how disoriented chiral condensates might arise in heavy ion collisions. $^2$ Our work was directly motivated by the work of Bjorken, Kowalski, and Taylor, and also by the work of Rajagopal and Wilczek. At the outset I must confess that while we all speak of DCC’s, in detail the mechanisms of Bjorken and of Rajagopal and Wilczek
are really very different. In the first section we consider the work of Rajagopal and Wilczek in detail, extending their results. Unfortunately, our results are negative: we find that the only way to get large DCC’s is if there is at least one light field about. In the third section I turn to what seems a long digression on a disconnected topic: the phase diagram of QCD with 2 + 1 flavors. This leads, however, to the speculation that the equilibrium phase diagram for 2 + 1 flavors could itself generate a large distance scale. This then might generate large domains of DCC’s.

3.1. DCC’s in a two flavor model

In this subsection I review results which extend previous work by Rajagopal and Wilczek, R&W. Our major purpose was to concentrate on the question of domain size and on the experimentally relevant question of the nature of the distribution in \( R_3 \). While our results, like those of R&W, are the product of numerical simulations, much of our understanding follows from the analysis of Boyanovsky et al., who consider the question of domain growth following a quench. While the work of (30) was done primarily in weak coupling (as applies in inflationary cosmology), it is direct to extend it to strong coupling, at least qualitatively.

For the purposes of discussion we take the assumptions of R&W for granted. This means that we consider only the effects of two quark flavors by means of a linear sigma model, ignoring vector mesons. This involves an \( O(4) \) vector \( \Phi = (\sigma, \vec{\pi}) \) where \( \sigma \) is an isosinglet \( J^P = 0^+ \) field. The action for the \( \Phi \) field is

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi)^2 - H \cdot \Phi + \frac{\lambda}{2} v^2 \Phi^2 + \frac{\lambda}{4} (\Phi^2)^2.
\]  

At zero temperature the vacuum state is \( \langle \Phi \rangle = (v, \vec{0}) \). The parameters of the model are the quartic coupling \( \lambda \), the vacuum expectation value of the sigma field, \( v \), and a background magnetic field, \( H = h(1, \vec{0}) \), which makes the pions massive. To treat the theory in strong coupling we discretize it by putting it on a spatial lattice with a lattice spacing of \( a = 1 \) fermi (\( fm \)). R&W studied the physically relevant case of strong coupling: \( \lambda = 20 \), \( v = 87.4 MeV \), and \( h = (119 MeV)^3 \). With these values the pion decay constant \( f_\pi = 92.5 MeV \), the mass of the pion is \( m_\pi = 135 MeV \), while that of the sigma is \( m_\sigma = 600 MeV \).

(The value of \( m_\sigma = 600 MeV \) is traditional is the two flavor model. For the realistic case of three flavors in the next subsection, we shall instead identify the sigma field with a heavier particle, near 1GeV in mass. For now I just note that a heavier sigma just makes the coupling even stronger than \( \lambda = 20 \).)

To understand the dynamics better, we also considered the model at extremely weak coupling, \( \lambda = 10^{-4} \). In working at weak coupling we tried to keep at least some of the physics constant by holding the lattice spacing \( a \) and the pion decay constant \( f_\pi \) fixed. This implies that the pion and sigma fields become light in weak coupling: for \( \lambda = 10^{-4} \) and \( f_\pi = 92.5 MeV \), we obtain \( v = 87.4 MeV \), \( h = (2.03 MeV)^3 \),
\( m_\pi = 0.3 MeV \) and \( m_\sigma = 1.8 MeV \). In other words, weak coupling means that the potential is flat.

We assume that the evolution of the system is “quenched”. The idea is that the system cools so rapidly that the system finds itself in a state typical of high temperature, even though the evolution forward in time is by means of the equations of motion at zero temperature. We do not address the question of whether or not this is realistic for heavy ion collisions; it certainly is the extreme limit of a range of possibilities. We begin with a high temperature state which is chirally symmetric, \( \langle \Phi \rangle = 0 \); to simulate the effect of fluctuations in the high temperature initial state, we distribute the fields as gaussian random variables with \( \langle \Phi \rangle = 0 \), \( \langle \Phi^2 \rangle = v^2/4 \) and \( \langle \Phi^2 \rangle = v^2/1 fm^2 \) following R\&W. Pion domains can form in a quench because the chirally symmetric initial state, \( \langle \Phi \rangle = (0, \vec{0}) \), is unstable against small fluctuations in the \( T = 0 \) potential. In essence, the system “rolls down” from the unstable local maximum of \( V(\Phi) \) towards the nearly stable values \( \Phi^2 = f_\pi^2 \). This process is known in condensed matter physics as spinoidal decomposition. Long-lived DCC field configurations with \( \vec{\pi} \neq 0 \) can develop during the roll-down period. The field will eventually settle into stable oscillations about the unique \( h \neq 0 \) vacuum, \( \langle \Phi \rangle = (f_\pi, \vec{0}) \). Oscillations continue until interactions eventually damp the motion through pion radiation.

In the the Hartree approximation the equations of motion for the Fourier components of the pion field are:

\[
\frac{d^2}{dt^2} \vec{\pi}_k = \left( \lambda(v^2 - \langle \Phi^2 \rangle) - k^2 \right) \vec{\pi}_k. \tag{2}
\]

Even in the Hartree approximation we see that field configurations with \( \langle \Phi \rangle = 0 \), \( \langle \Phi^2 \rangle < v^2 \) and momentum \( k < \sqrt{\lambda}v \) are unstable, and grow exponentially. Conversely, modes with larger momentum, \( k < \sqrt{\lambda}v \) are stable, and do not grow. Of all of the unstable modes, the constant mode, with \( k = 0 \), grows the fastest; its growth has a natural time scale, which is

\[
\tau_{sp} = \left\{ \lambda(v^2 - \langle \Phi^2 \rangle) \right\}^{-1/2} \sim \sqrt{2}/m_\sigma. \tag{3}
\]

In strong coupling, \( \lambda = 20 \), this time is \( 0.5 fm/c \). The exponential growth of the unstable modes continues until \( \langle \Phi^2 \rangle \) reaches \( v^2 \), when it begins to oscillate about the stable vacuum. Rajagopal and Wilczek found that the power \( \propto \pi^a_{\vec{k}} \pi^a_{-\vec{k}} \) in the low momentum pion modes indeed grows when the exact classical equations of motion are integrated for \( \lambda = 20 \). Following Boyanovsky et al., one can estimate how long it takes domains to grow; not surprisingly, this time scale is on the order of the spinoidal times, \( \tau_{sp} \). After this time, the fields sense that there is a stable minimum to the potential, and so from the nonlinearities in the potential, the growth of domains shuts off. Consequently, if the coupling were weak, so the sigma meson is light, then it takes long time to roll down to the bottom of the potential, and domains have plenty of time to grow. For the realistic case of strong coupling, however, the rolldown is
very rapid, and so the domains are small, $\sim 1\, fm$. This is what one would expect from a system in which the Compton wavelength of the pion at rest is of order $\sim 1\, fm$.

We confirmed this intuition through numerical simulations. An asymmetric lattice geometry of dimensions $10^2 \times 40$ was chosen in order to study the issue of domain size in the one, long direction; we introduce the average of the pion field over the transverse dimensions, $\bar{\pi}_L(z, t) = \sum_{x,y} \bar{\pi}(x, y, z, t)/(10)^2$. Initially, $\pi_L$ starts out completely random. In weak coupling, by times of $100\, fm$ domains — regions in which the field is slowly varying about some nonzero value — are evident. In contrast, in strong coupling by times of $100\, fm$ the pion field is always quite random, oscillating with small amplitude about zero. We found that the pion correlation function is long ranged in weak coupling, but short ranged in strong coupling.

The most important quantity is the distribution in $R_3$. We assume that we can extract this ratio from the classical pion fields in the most naive way possible, by computing $R_3 = \langle (\pi^3)^2 \rangle / \left( \langle (\pi^1)^2 \rangle + \langle (\pi^2)^2 \rangle + \langle (\pi^3)^2 \rangle \right)$. Histograms of $R_3$ were obtained by evolving 200 independent configurations forward in time to $t = 150\, fm$ in weak coupling, and times of $t = 30\, fm$ in strong coupling. In weak coupling the distribution is far from binomial, and is approaching the DCC value of $1/(2\sqrt{R_3})$. In strong coupling, however, the distribution is clearly binomial, peaked about the expected value of $1/3$.

We have performed numerical simulations at other couplings to see where the crossover from a binomial to a DCC distribution occurs. For our lattice of $10^2 \times 40$, it appears that a DCC distribution only arises at very weak coupling, $\lambda \sim 10^{-2}$, as expected from the domain size estimate of (2).

In summary, in a realistic two flavor model we find that following a quench, DCC’s are only produced in small, pion sized domains. There appears to be one way for large DCC’s to be produced, and that is if there is a light particle about, whose Compton wavelength automatically provides a size for the DCC. But a quench alone does not seem to give us a large distance scale.

### 3.2. Three flavors

In the next subsections I analyze the phase diagram of $QCD$. This is similar to the analysis of sec. 2, except that I work not with 2 but the realistic case of 2 + 1 flavors. The price I pay is that the contribution of the vector fields is dropped. Again, implicitly I speak of nonzero temperature; what happens at nonzero baryon density (if the temperature is small) may well be very different.

From the previous discussion, remember that for three colors, the order of the phase transition appears to depend crucially on the values of the quark masses. This is definitely special to three colors. For example, consider the limit of infinitely many colors, $N_c \to \infty$. The basic physics can be understood simply by remembering that there are of order $\sim N_c^2$ gluons versus order $\sim N_c$ quarks. Now assume that the
deconfining transition is of first order at $N_c = \infty$; now while I certainly cannot prove it, the most natural possibility is that the first order deconfining transition always dominates the chiral transition, so that both take place at the same temperature. This need not be the case, for it is logically possible for the chiral transition to occur at a higher temperature than the deconfining transition. (The opposite possibility, of a deconfining temperature which is higher than the chiral transition, probably contradicts general theorems on realizations of chiral symmetry, but as of yet no strict proof has been given.) Nevertheless, as the deconfining transition the free energy goes from being of order one, from hadrons, to of order $N_c^2$, from gluons, and it is difficult to imagine that this in and of itself doesn’t trigger the chiral transition at the same time.

Thus we see that at least in this instance, three colors is not close infinity for three colors and “2 + 1” flavors, while the transition appears to be of first order for the deconfining transition without dynamical quarks, $M = \infty$ and for the chiral limit, $M = 0$, these lines do not meet — there is a gap, with $QCD$ somewhere in between.

Lines of first order transitions typically end in critical points, so it is natural to ask about the two critical points. Consider first working down from infinite quark mass. As $M$ decreases from $M = \infty$, the line of deconfining first order transitions ends in a deconfining critical point. By an analysis similar to that in sec. (3.2) below, one can show that at the deconfining critical point, correlation functions between Polyakov lines are infinite ranged, and that it lies in the universality class of the Ising model, or a $Z_2$ spin system, in three dimensions.

The other limit is to go up from zero quark mass. For three flavors the chiral phase transition is of first order at $m = 0$, so as $m$ increases, the line of first order transitions can end in a chiral critical point. In the following we show that for $2 + 1$ flavors, at the chiral critical point the only massless field is the sigma meson. Thus again the chiral critical point lies in the universality class of the Ising model, or a $Z(2)$ spin system.

We stress, however, very different fields becomes massless at each critical point; the two critical points are not naturally related to each other. At least for three colors this seems to be the case, as there is a large gap in between the two critical points, in which there is no true phase transition, only a smooth crossover. Nevertheless, our results indicate that even in the region of smooth crossover — which includes $QCD$ — that very interesting and unexpected physics can be going on.

Because we are interested in making contact with $QCD$, we begin in the next subsection by fitting the scalar and pseudoscalar mass spectrum in $QCD$ to the results found in a linear sigma model. This is an old story and relatively straightforward. The classification of the chiral critical point then follows directly. When we then try to connect the two, however, we find a surprise. Numerical simulations of lattice gauge theory indicate that as a function of the mass parameter $M$, $QCD$ is not far from the chiral critical point. We show that if $QCD$ is in fact near the chiral critical
3.2.1. The QCD spectrum at zero temperature

In this subsection we use a linear sigma model to model the full theory with three quark flavors. We ignore the contribution of vector mesons, which will be incorporated at a later date. This type of analysis was first carried out by Chan and Haymaker, recent analyses, with similar results, are given by Meyer-Ortmanns, Pirner, and Patkos, and by Metzger, Meyer-Ortmanns, and Pirner. For three quark flavors the sigma field \( \Phi \) is a complex valued, three by three matrix, proportional to the quark fields as \( \Phi \sim q_{\text{left}} q_{\text{right}} \). In flavor space it can be decomposed as

\[
\Phi = \sum_{a=0}^{8} \left( \sigma_a + i \pi_a \right) t^a .
\]

\( t_{1..8} \) are the generators of the \( SU(3) \) algebra in the fundamental representation, while \( t_0 \) is proportional to the unit matrix. Normalizing the generators as \( tr(t_a t_b) = \delta^{ab}/2 \), \( t_0 = 1/\sqrt{6} \). The fields \( \sigma_a \) are components of a scalar \( (J^P = 0^+) \) nonet, those of \( \pi_a \) a pseudoscalar \( (J^P = 0^-) \) nonet. The latter are familiar: \( \pi_1, 2, 3 \) are the three pions, denoted as \( \pi \) without subscript; the \( \pi_{4,5,6,7} \) are the four kaons, the \( K \)’s, while \( \pi_8 \) and \( \pi_0 \) mix to form the mass eigenstates of the \( \eta \) and \( \eta' \) mesons. Following our notations for two flavors, we define the components of the scalar nonet analogously: we refer to \( \sigma_{1,2,3} \) as the \( \sigma_\pi \)’s, to \( \sigma_{4,5,6,7} \) as the \( \sigma_K \)’s, while \( \sigma_8 \) and \( \sigma_0 \) mix to form the \( \sigma_\eta \) and \( \sigma_{\eta'} \). What is the correct identification of the \( \sigma \) meson for three flavors will be one of the principal questions in the following.

This multiplicity of eighteen fields is to be contrasted with the usual two flavor sigma model considered in the previous subsection, which had only three \( \pi \)’s and one \( \sigma \) meson. The increase in the number of fields is because we are allowing for \( U_a(1) \) rotations; to describe two flavors in this way requires eight fields: the three \( \pi \)’s, \( \sigma_\eta \), the three \( \sigma_\pi \)’s, and the \( \eta \). For \( n_f \) flavors, then, in general \( 2n_f^2 \) fields are required to describe the lowest lying scalar and pseudoscalar multiplets.

The effective lagrangian for the chiral \( \Phi \) field is taken to be

\[
\mathcal{L} = tr \left| \partial_\mu \Phi \right|^2 - tr \left( H(\Phi + \Phi^\dagger) \right) + \mu^2 tr \left( \Phi^\dagger \Phi \right) - \sqrt{6} c \left( det(\Phi) + det(\Phi^\dagger) \right) + (g_1 - g_2) \left( tr \Phi^\dagger \Phi \right)^2 + 3 g_2 tr \left( \Phi^\dagger \Phi \right)^2 .
\]

This is very much like the lagrangian for two flavors in (1), except that now new couplings are possible. This lagrangian involves six parameters: two for the background field \( H \) (one for the up equals the down quark mass, one for the strange quark mass), the mass parameter \( \mu^2 \), an “instanton” coupling constant \( c \), and two quartic couplings, \( g_1 \) and \( g_2 \). At large \( \Phi \) the potential is bounded from below if the quartic...
couplings satisfy $g_1 \geq 0$ and $g_1 + 2g_2 \geq 0$. (The bounds on the quartic couplings can be understood by taking two extreme limits for $\Phi$: the first is $\Phi$ proportional to the unit matrix, the second is where $\Phi$ only has one diagonal element which is nonzero.)

To understand the symmetries of this lagrangian, consider the theory becomes less symmetric as various couplings increase from zero. When $H = c = g_2 = 0$, the theory only involves the quadratic invariant $tr(\Phi^\dagger \Phi) = \sum_{a=0}^{8} (\sigma_a^2 + \pi_a^2)$, and so has the very large symmetry group of $O(18)$. When $g_2 \neq 0$, the symmetry reduces to $SU(3) \times SU(3) \times U(1)$, with the $U(1)$ that for axial fermion number. The effects of instantons, or more generally topological fluctuations and the Adler-Bell-Jackiw anomaly, generate a nonzero value for the instanton coupling constant $c$, and reduces the symmetry to $SU(3) \times SU(3)$. Lastly, spontaneous symmetry breaking, where $\langle \Phi \rangle \sim 1$, occurs if the mass squared, $\mu^2$ is negative; actually, due to the determinental term, spontaneous symmetry breaking can even occur for some values of positive $\mu^2$. Since $\langle \Phi \rangle \sim 1$, the symmetry is then reduced to the usual $SU(3)$ symmetry of isospin plus strangeness.

For the background field $H$ we take $H = h_0 t_0 - \sqrt{2} h_8 t_8$. Working out the algebra, $h_0$ and $h_8$ are proportional to the current quark masses as

$$M_{\text{up}} \equiv M_{\text{down}} \sim h_0 - h_8 \ , \ M_{\text{strange}} \sim h_0 + 2h_8 . \quad (6)$$

With the octet magnetic field $h_8 \neq 0$, then, only remaining symmetry is merely the usual $SU(2)$ symmetry of isospin. Effects from isospin breaking, as from $M_{\text{up}} \neq M_{\text{down}}$, are negligible for our purposes, since every other mass scale is so much larger. (Not to mention the inclusion of electromagnetic effects.) Note that $H$ has dimensions of mass cubed; hence two powers of some fixed QCD scale enter to make up the mass dimensions in (6). I assume that these two powers of a QCD scale are truly fixed, and do not vary with, say, temperature. This appears to give a behavior with nonzero temperature which is consistent with other analyses, but I must confess that I do not know how to derive the value of the background field from the fundamental theory in a way in which these two powers of the QCD scale would be manifest.

The analysis of the effective lagrangian is uniformly at the simplest level of mean field theory. I assume that in the true vacuum there are nonzero expectation values for $\sigma_8$ and $\sigma_0$,

$$\sigma_0 \rightarrow \Sigma_0 + \sigma_0 \ , \ \sigma_8 \rightarrow - \sqrt{2} \Sigma_8 + \sigma_8 , \quad (7)$$

and then expand in powers of $\sigma_a$ and $\pi_a$. To linear order there are two equations of motions, which fixes the values the two vacuum expectation values, $\Sigma_0$ and $\Sigma_8$, as

$$h_0 = M^2 \Sigma_0 - c \left( \Sigma_0^2 - \Sigma_8^2 \right) + 4g_2 \Sigma_8 \left( \Sigma_0 + \frac{1}{2} \Sigma_8 \right) ,$$

$$h_8 = M^2 \Sigma_8 + c \Sigma_8 \left( \Sigma_0 - \Sigma_8 \right) + 2g_2 \Sigma_8 \left( \Sigma_0 + \Sigma_8 \right) \left( \Sigma_0 + \frac{1}{2} \Sigma_8 \right) . \quad (8)$$

A second mass parameter, $M^2$, has been introduced; it is related to the $\mu^2$ of (5) as

$$M^2 = \mu^2 + g_1 \left( \Sigma_0^2 + 2 \Sigma_8^2 \right) . \quad (9)$$
Expansion of the effective lagrangian to quadratic order gives the masses of the pseudoscalar octet,

\[ m_\pi^2 = M^2 - c (\Sigma_0 + 2\Sigma_8) - 2g_2 \Sigma_8 \left( \frac{1}{2} \Sigma_0 + \Sigma_8 \right) = \frac{h_0 - h_8}{\Sigma_0 - \Sigma_8}, \]

\[ m_K^2 = M^2 - c (\Sigma_0 - \Sigma_8) + g_2 \Sigma_8 (\Sigma_0 + 5\Sigma_8) = \frac{h_0 + h_8/2}{\Sigma_0 + \Sigma_8/2}, \]

\[ m_{\pi\pi}^2 = M^2 - c (\Sigma_0 - 2\Sigma_8) + 2g_2 \Sigma_8 \left( \frac{1}{2} \Sigma_0 + \Sigma_8 \right), \]

\[ m_{\pi\sigma}^2 = \sqrt{2} \Sigma_8 \left( c - 2g_2 \left( \frac{1}{2} \Sigma_0 + \Sigma_8 \right) \right), \]

\[ m_\sigma^2 = M^2 + 2c \Sigma_0. \]  

(10)

Notice that \( m_{\pi\pi}^2, m_{\pi\sigma}^2, \) and \( m_{\sigma}^2 \) refer to the octet and singlet parts of the pseudoscalar nonet, which mix to give the masses of the \( \eta \) and \( \eta' \), \( m_\eta \) and \( m_{\eta'} \). In the chiral limit, \( h_0 = h_8 = 0 \), the chiral condensate is \( SU(3) \) symmetric, \( \Sigma_0 \neq 0 \) and \( \Sigma_8 = 0 \). In this limit it is easy to see that the \( \pi, K \), and \( \eta \) mesons are all massless; the \( \eta' \) meson remains massive because of the instanton coupling \( c \).

For the scalar nonet,

\[ m_{\sigma\sigma}^2 = M^2 + c (\Sigma_0 + 2\Sigma_8) + 2g_2 \left( \Sigma_0^2 - 3\Sigma_0 \Sigma_8 + \frac{1}{2} \Sigma_8^2 \right), \]

\[ m_{\sigma\pi\pi}^2 = M^2 + c (\Sigma_0 - \Sigma_8) + 2g_2 (\Sigma_0 + \Sigma_8) \left( \frac{1}{2} \Sigma_0 + \Sigma_8 \right), \]

\[ m_{\sigma_8}^2 = \mu^2 - c (\Sigma_0 - 2\Sigma_8) + g_1 \left( \Sigma_0^2 + 6\Sigma_8^2 \right) + 2g_2 \left( \Sigma_0^2 + 3\Sigma_0 \Sigma_8 + \frac{3}{2} \Sigma_8^2 \right), \]

\[ m_{\sigma_8\pi_0}^2 = -\sqrt{2} \Sigma_8 \left( c + 2g_1 \Sigma_0 + g_2 (4\Sigma_0 + 3\Sigma_8) \right), \]

\[ m_{\sigma_0}^2 = \mu^2 - 2c \Sigma_0 + g_1 \left( 3\Sigma_0^2 + 2\Sigma_8^2 \right) + 4g_2 \Sigma_8^2. \]  

(11)

For the scalars, \( m_{\sigma_8}, m_{\sigma_8\pi_0}, \) and \( m_{\sigma_0} \) mix to give \( m_{\sigma_8}, \) and \( m_{\sigma_0} \). All particles in the scalar nonet are expected to remain massive even in the chiral limit.

We also need the pion and kaon decay constants,

\[ f_\pi = \sqrt{\frac{2}{3}} (\Sigma_0 - \Sigma_8), \quad f_K = \sqrt{\frac{2}{3}} \left( \frac{1}{2} \Sigma_0 + \Sigma_8 \right). \]  

(12)

The expressions in (10)–(12) agree with previous results, up to differences in normalization of the coupling constants and vacuum expectation values in (5) and (7). There is one unexpected detail: naively one would expect that the mass \( \mu \) and the quartic coupling \( g_1 \) would enter independently. In fact, for the equations of motion in (8), the masses of the entire pseudoscalar nonet in (10), and the masses of half the
Figure 6: The kaon mass, $m_K$, in $MeV$, versus the ratio of octet to singlet vacuum expectation values, $s = \Sigma_8/\Sigma_0$. The kaon mass is rather insensitive to $s$.

scalar nonet — for the $\sigma_\pi$ and the $\sigma_K$ in (11) — these two parameters only enter in a certain combination, through the single parameter $M^2$ of (9). This means that we can fit to the pseudoscalar spectrum, and so fix $M^2$, and yet still be free to vary $\mu^2$ (or conversely $g_1$): the only change will be to alter the masses of the $\sigma_\eta$ and the $\sigma_{\eta'}$. This technical detail will play an important role in what follows; although there must be some simple group theoretic reason for it, I don’t know what it is.

There is some freedom in deciding how to fit the parameters of the linear sigma model. Various kinds of fits are given by Meyer-Ortmanns, Pirner, and Patkos, and by Metzger, Meyer-Ortmanns, and Pirner. Following the experience of Chan and Haymaker we do not fit to the entire pseudoscalar mass spectrum for the $\pi$, $K$, $\eta$, and $\eta'$ mesons, since it turns out that the kaon mass is fairly insensitive to the ratio of vacuum expectation values, $\Sigma_8/\Sigma_0$.

Because of this, we leave the ratio $\Sigma_8/\Sigma_0$ as a free parameter, and fit just to the pion decay constant $f_\pi$ and to the masses for the $\pi$, $\eta$, and $\eta'$ mesons. I assume the values:

$$f_\pi = 93\, MeV , \quad m_\pi = 137\, MeV , \quad m_\eta = 547\, MeV , \quad m_{\eta'} = 958\, MeV . \quad (13)$$
Then one can examine the sensitivity of various quantities to $s = \Sigma_8/\Sigma_0$. This is easiest of understand graphically: in fig. (6) I show how the kaon mass changes with $s$: for small $s$ it is a little high, but not terribly so. In fig. (7) I show how the coupling for the anomaly changes with $s$; again, there is some variation but nothing dramatic.

In fig. (8) I show how the coupling constant $g_2$ of the linear sigma model changes with $s$; we see that for large $s \sim .3$ the coupling $g_2$ becomes rather small. Analysis shows that while $g_2$ becomes negative for $s > .3$, its value is very small, $\leq 1$, so that for any reasonable values of $g_1$, the potential remains bounded from below. Thus values of $s \leq 1$ cannot be excluded on such grounds.

There are two quantities, however, which are very sensitive to the value of the octet to singlet vacuum expectation values, $s$. The first is the kaon decay constant, $f_K$. This is illustrated in fig. (9), where we see that small values of $s$ are preferred: the experimental value is $f_K = 113$ MeV. Since $f_K$ isn’t too far from $f_\pi$, given the definition of $f_K$ in (12), it is obvious why a small ratio of $\Sigma_8/\Sigma_0$ is preferred. The second quantity is the mixing angle between the $\eta$ and the $\eta'$, $\theta_{\eta\eta'}$. Experimentally the situation is a bit unclear: radiative decays favor a value of $\theta_{\eta\eta'} = -20^\circ$, while fits
Figure 8: The coupling $g_2$ of the linear sigma model versus the ratio of octet to singlet vacuum expectation values, $s = \Sigma_8/\Sigma_0$. While $g_2$ becomes much smaller for large $s$, this is rather innocuous.

to the mass spectrum favor $\theta_{\eta\eta'} = -10^\circ$. From fig. (10), however, we see that this uncertainty doesn’t really affect our fit, for either value favors low values of $s$. From fig. (9) and (10), we find that a good fit is obtained for $s = .1$. The parameters of this solution are

$$M^2 = + (642 \, MeV)^2, \quad c = 1920 \, MeV, \quad g_2 = 30. ,$$

$$\Sigma_0 = 127 \, MeV, \quad \Sigma_8 = 13 \, MeV, \quad h_0 = (290 \, MeV)^3, \quad h_8 = (281 \, MeV)^3.$$

For this fit with $s = \Sigma_8/\Sigma_0 \sim .1$, while

$$m_K = 516 \, MeV, \quad f_K = 109 \, MeV, \quad \theta_{\eta\eta'} = -10.4^\circ.$$

The kaon mass is a bit high, $m_K = 516 \, MeV$ instead of the (average) experimental value of $497 \, MeV$.

For the values of (14), from (6) the ratio of the strange to up (= down) quark masses is

$$\frac{M_{\text{strange}}}{M_{\text{up}}} = \frac{(h_0 + 2h_8)}{(h_0 - h_8)} = 32.$$

(16)
Figure 9: Kaon structure constant, $f_K$, in $MeV$, versus the ratio of octet to singlet vacuum expectation values, $s = \Sigma_8/\Sigma_0$. Small values of $s$ are favored, $s \sim 0.1$.

This is larger than the often quoted value of $\sim 20$ because we have explicitly allowed for a vacuum expectation value that is not $SU(3)$ symmetric, $\Sigma_8 \neq 0$.

If one fits to the kaon mass, then $\Sigma_8/\Sigma_0$ is constrained to be $= 0.19$, with $c = 1790 \, MeV$ and $g_2 = 4.98$; the results for $f_K$ and $\theta_{\eta'\eta'}$ are worse, $f_K = 127 \, MeV$, while $\theta_{\eta'\eta'} = + 1.0$. The freedom to play with the ratio $\Sigma_8/\Sigma_0$ and its effects will be discussed elsewhere.

The values in (14) give unique predictions for two masses in the scalar nonet:

$$m_{\sigma_\pi} = 1177 \, MeV \; , \; m_{\sigma_K} = 1322 \, MeV \; .$$

There are observed states with these quantum numbers: the $a_0(980)$ and the $K_0^*(1430)$, respectively.

For the purposes of illustration, in fig. (11) I illustrate how the masses of the $\sigma_\pi$ and the $\sigma_K$ change with $s$. For small values of $s$, the $\sigma_K$ is heavier than the $\sigma_\pi$, as naive intuition would suggest. What is amusing is that for $s \geq 0.2$, the $\sigma_K$, which after all is strange, becomes lighter than the $\sigma_\pi$. For other reasons, I do not favor such large values of $s$, but it is interesting that such a flip of mass values can occur.

The identification of the $a_0$ with the $\sigma_\pi$ is not obvious (see pg. VII.21 of ref. [40]).
It is reasonable to expect that the width of the $\sigma_\pi$ should be large, while the $a_0$ is observed to have a small width $\sim 57 \text{MeV}$. (The $K_0^*$ is a broad resonance, with a width $\sim 287 \text{MeV}$.) While this may result from a large mixing with other channels, such as $K\bar{K}$, alternately it is possible is that the $a_0(980)$ is not what we term the $\sigma_\pi$, but a kind of $K\bar{K}$ molecule. If true, presumably the state which we term the $\sigma_\pi$ is heavier than $1 \text{GeV}$.

There is no unique prediction for the masses of the remaining members of the scalar nonet, the $\sigma_\eta$ and the $\sigma_\eta'$. As remarked, the masses of all other fields only depends upon the parameter $M^2$ in (4). Thus at constant $M^2$ we can vary $g_1$ and change only $m_{\sigma_\eta}$ and $m_{\sigma_\eta'}$. This is illustrated in fig. (12).

Like the $\eta$ and the $\eta'$ mesons, the $\sigma_\eta$ and the $\sigma_\eta'$ mix; the mixing angles are usually of order $30^\circ$. This is close to ideal mixing, where the $\sigma_\eta'$ is primarily $\bar{u}u + \bar{d}d$, while the $\sigma_\eta$ is primarily $\bar{s}s$. This mixing helps to understand the differences in the mass spectrum between the scalars and pseudoscalars. For the pseudoscalars the $\eta$ is primarily an $SU(3)$ octet, and the $\eta'$ primarily an $SU(3)$ singlet; the singlet $\eta'$ is then much heavier than the octet $\eta$ because of the anomaly. Indeed, the $\eta$ and $\eta'$ are primarily $SU(3)$ because of the anomaly. For the scalar particles, instead of being $SU(3)$ eigenstates, the $\sigma_\eta$ and the $\sigma_\eta'$ are much better described as flavor eigenstates.
Figure 11: The masses of the scalar pion, $\sigma_\pi$, and the scalar kaon, $\sigma_K$, versus the ratio of octet to singlet vacuum expectation values, $s = \Sigma_8/\Sigma_0$.

Since the $\sigma_\eta'$ is (largely) made up of up and down quarks, then, and the $\sigma_\eta$ is (largely) made up of strange quarks, it is natural that for the scalars, the $\sigma_\eta'$ is heavier than the $\sigma_\eta$.

(I note that in strong coupling, $g_1 \to \infty$, $m_{\sigma_\eta'}$ approaches a constant value, while $m_{\sigma_\eta} \sim \sqrt{g_1} \to \infty$. This clearly a special property of taking the up and down quark masses different from the strange quark mass.)

I identify the $\sigma_\eta'$ and the $\sigma_\eta$ with the observed states with the same quantum numbers: the $f_0(975)$ and the $f_0(1476)$, respectively. With the parameters of (14), from fig. (12) requiring $m_{\sigma_\eta'} = 975\text{MeV}$ fixes

$$m_{\sigma_\eta} = 1476\text{MeV} \ , \ g_1 = 40 \ , \ \mu^2 = -(492\text{MeV})^2.$$  \hspace{1cm} (18)

For these values, the mixing angle between the $\sigma_\eta'$ and the $\sigma_\eta$ is $+28^\circ$. The mass for the $\sigma_\eta$ is a bit high, 1476MeV instead of 1400MeV for the $f_0$.

Again, the identification of the $\sigma_\eta'$ with the $f_0(975)$ is not evident (see pg. VII.192 of ref. [40]): the $f_0(975)$ has a width of only $\sim 47\text{MeV}$, while the $\sigma_\eta'$ should decay into two pions with hearty abandon. (The width of the $f_0(1400)$ is large, 150 to 400 MeV.) Either the narrow width is the result of a complicated multi-pole structure in
Figure 12: Masses of the $\sigma_\eta$ and the $\sigma_{\eta'}$, in MeV, versus the coupling $g_1$ of the linear sigma model for three flavors. Notice that $m_{\sigma_{\eta'}} = 0$ for $g_1 \sim 4$; this will be used in sec. (3.2.3).

The $K\bar{K}$ channel or the $f_0(975)$ is not the $\sigma_{\eta'}$, but a $K\bar{K}$ molecule. Indeed, since in this model the $\sigma_{\eta'}$ is primarily composed of up and down quarks, and very little from strange quarks, it is impossible to understand why it should decay into $K\bar{K}$ in the first place. As shall become clear from our later arguments, for our purposes all we really need to know that the $\sigma_{\eta'}$ is not light; i.e., $m_{\sigma_{\eta'}} \geq 1 GeV$.

So where does the $\sigma$ meson at 600 MeV, as we assumed in sec. II, come from? This is derived from models of nucleon-nucleon forces, assuming that the force is provided by single particle exchange. An isosinglet $0^+$ particle, the $\sigma$, is needed at 600 MeV to provide sufficient attraction. It is reasonable to view this $\sigma$ simply as an approximation to two pion exchange. Notably, there is no such resonance seen in the phase shifts of $\pi - \pi$ scattering below 1 GeV, pg. VII.37 of ref. [10]. For the sake of argument, if we do assume that $m_{\sigma_{\eta'}} = 600$ MeV, we find that the coupling
$g_1$ decreases, to $g_1 = 16.2$, whilst $m_{\sigma_0} = 1412 \text{MeV}$ and $\mu^2 = +(384 \text{MeV})^2$.

Other fits to the $\sigma_{\eta'}$ have also been given. Bertsch et al.\cite{Bertsch} find that data on kaon decays and $\pi N \rightarrow \pi \pi N$ scattering can be well fit with $m_{\sigma_{\eta'}} \sim 800 \text{MeV}$.

I now show that understanding where the $\sigma_{\eta'}$ lies is not merely to interest to afficianadoes of the linear sigma model: it is crucial to understanding the nature of the phase diagram, as a function of the current quark mass $M$.

### 3.2.2. The chiral critical point

I now turn to the nature of the chiral critical point; the above classification of the zero temperature spectrum is not required in this subsection, but will be used in the next. I deal exclusively with mean field theory, where the effects of nonzero temperature are incorporated simply by varying the mass parameter $\mu^2$. This is correct in the limit of very high temperature, but should be qualitatively correct at all temperatures.

I begin with the $SU(3)$ symmetric case when $h_8 = 0$. For a constant field $\Sigma_0$, the lagrangian reduces to the potential for $\Sigma_0$,

$$L = -h_0 \Sigma_0 + \frac{1}{2} \mu^2 \Sigma_0^2 - \frac{c}{3} \Sigma_0^3 + \frac{g_1}{4} \Sigma_0^4. \quad (19)$$

This model has precisely the same phase diagram as that for the transition between a liquid and a gas. For zero background field, $h_0 = 0$, the instanton coupling “$c$” is a cubic invariant, and so drives the transition first order. As $h_0$ increases the transition becomes more weakly first order, until at $h_0 = h_0^{\text{crit}}$ the line of first order transitions ends in a critical point. For $h_0 > h_0^{\text{crit}}$ there is no true phase transition, just a smooth crossover.

There is a critical point where the potential in $\Sigma_0 - \Sigma_0^{\text{crit}}$ is purely quartic,

$$L|_{\text{crit}} = g_1 \left( \frac{(\Sigma_0 - \Sigma_0^{\text{crit}})^4}{4} \right). \quad (20)$$

This is determined algebraically, by requiring that about the critical point, in $\Sigma_0 - \Sigma_0^{\text{crit}}$, the first, second, and third derivatives of the potential vanish. This is three conditions, which fixes the values of the magnetic field, the vacuum expectation value of the $\Sigma_0$ field, and the mass parameter,

$$h_0^{\text{crit}} = \frac{c^3}{27 g_1^2}, \quad \Sigma_0^{\text{crit}} = \frac{c}{2 g_1}, \quad \mu_0^2 = \frac{c^2}{3 g_1}. \quad (21)$$

The mass spectrum at the critical point is easy to work out by going back to the original action and recomputing:

$$m_{\Sigma_0} = m_{\Sigma_0} = \frac{c^2}{9 g_1}, \quad m_{\eta'}^2 = 10 m_{\pi}^2.$$
\[ m_\sigma^2 = m_{\sigma_K}^2 = m_\pi^2 = \left( 7 + \frac{18g_2}{g_1} \right) m_\pi^2, \quad m_{\sigma_{\eta'}}^2 = 0. \]  

(22)

Of course the \( \sigma_{\eta'} \) is massless because we have deliberately tuned ourselves to sit at the chiral critical point. As we work for now in the limit of \( SU(3) \) symmetry, there are two octets of pseudoscalars and scalar fields. It certainly is extraordinary to find even an isolated point where the \( \sigma_{\eta'} \) is lighter than any other field, even the pions!

Because only one field, the \( \sigma_{\eta'} \), is massless at the chiral critical point, the similarity to the liquid gas phase transition extends to the universality class: it is that of a \( Z(2) \) spin system, or the Ising model in three space dimensions. Thus the chiral critical point is precisely analogous to critical opalescence in a liquid gas system.

The classification of the universality class still applies when the background magnetic field is not \( SU(3) \) symmetric, \( h_8 \neq 0 \). In this case I couldn’t solve the equations analytically, and had to resort to numerical methods (which are, however, elementary). The difficulty is simply that three equations must be set to zero to find the chiral critical point, for the first, second, and third derivatives of the potential, in the direction of the massless mode. (And of course for the first derivative of the potential, in the other direction.) By solving (10) and (11), I find that at least for parameters about those of the previous section, there remains a single, massless field at the chiral critical point, the \( \sigma_{\eta'} \), with the universality class that of the Ising model. Of course for \( h_8 \neq 0 \) the \( \sigma_{\eta'} \) field does not remain a pure \( SU(3) \) singlet, but mixes to become part octet.

I expect this conclusion to hold for arbitrary values of the quark masses (background field). Indeed, I conjecture that in general, for any critical end point — be it chiral, deconfining, or whatever — that the universality class is always that of the Ising model, which is \( Z(2) \). I have no profound reason for this, but simply on algebraic grounds, at best one can get the appropriate derivatives of the potential to vanish along one direction, but not in more than one direction; if one tried in more than one direction, there would be more conditions on the potential than variables to satisfy them.

Indeed, although I did not comment on it at the time, the identification of the \( \sigma_{\eta'} \) as the massless mode of the chiral critical point could have been anticipated from the results of the previous section. Looking at fig. (12), we see that for very small values of \( g_1, g_1 \sim 4 \), that the mass of the \( \sigma_{\eta'} \) goes to zero. At the time, it seemed merely a peculiarity of the fit, of no greater significance. And yet it shows that even at zero temperature, it is possible to obtain a massless \( \sigma_{\eta'} \).

Assuming, then, that only the \( \sigma_{\eta'} \) is massless at the chiral critical point, I can flesh out the entire phase diagram as a function of \( M_{\text{up}} = M_{\text{down}} \) versus \( M_{\text{strange}} \), as proposed in fig. (1) of ref. [6]. The basic idea is that along the entire line of chiral critical points, only the \( \sigma_{\eta'} \) is massless, but that the singlet-octet ratio in the \( \sigma_{\eta'} \) changes. I just showed that at the \( SU(3) \) symmetric point, \( M_{\text{up}} = M_{\text{down}} = M_{\text{strange}} \), the \( \sigma_{\eta'} \) is pure singlet. Decreasing \( M_{\text{strange}} \) to the critical point where \( M_{\text{strange}} = 0 \) and \( M_{\text{up}} = M_{\text{down}} \neq 0 \), rather obviously the \( \sigma_{\eta'} \) must become entirely strange,
\[ \sigma_{q'} \sim \bar{s}s. \]

The opposite limit, \( M_{\text{up}} = M_{\text{down}} = 0 \), is more familiar. Assume that the chiral phase transition with two massless flavors is of second order, the universality class is that of an \( O(4) \) critical point. The chiral phase transition will remain of second order as \( M_{\text{strange}} \) decreases from \( M_{\text{strange}} = \infty \). This line of second order transitions must end in a special value of \( M_{\text{strange}} = M_{\text{crit strange}} \); for \( M_{\text{strange}} < M_{\text{crit strange}} \), the chiral transition is like that of three light flavors, and so first order. Wilczek observed that at \( M_{\text{strange}} = M_{\text{crit strange}} \), the chiral transition is in the universality class of an \( O(4) \) tri-critical point. I would say that when \( M_{\text{up}} = M_{\text{down}} = 0 \) and \( M_{\text{strange}} = M_{\text{crit strange}} \), the \( \sigma_{q'} \) is a pure \( SU(2) \) state, with \( \sigma_{q'} \sim \bar{u}u + \bar{d}d \); further, because \( M_{\text{up}} = M_{\text{down}} = 0 \), the pions are also massless, so the universality class of this point is not that of a \( Z(2) \) critical point, but an \( O(4) \) tri-critical point. This contradicts my previous argument (that only the \( \sigma_{q'} \) is massless at a chiral critical point) because when \( M_{\text{up}} = M_{\text{down}} = 0 \), that is itself the endpoint of a line of chiral critical points, and thereby special.

The analysis can be further extended to the case where there is a gap between the first order transitions for the deconfining and chiral transitions, but where the chiral transition is of first order for two, massless flavors. Then fig. (1) of ref. [35] would be modified, with a band of first order transitions about the axis \( M_{\text{up}} = M_{\text{down}} = 0 \). These first order transitions would end in chiral critical points, in the universality class of \( Z(2) \) from the presence of massless \( \sigma_{q'} \) fields. However, the latest lattice data appears to indicate that this is not the case: that for two flavors, the chiral transition is of second order.

3.2.3. The chiral critical point and QCD

To model the effects of nonzero temperature, in mean field theory it is only necessary to increase the mass parameter \( \mu^2 \) of (5). Doing so, the resulting spectrum — computed from (9)–(11) — is rather unremarkable, and look very similar to the graph in fig. (1), except that now there are many more fields about; this is illustrated in fig. (5). Remember fig. (1) refers to two flavors, including vector mesons, while fig. (5) is for three flavors, without vector mesons; also, in fig. (1) I took \( m_\sigma(0) = m_{\sigma q} = 600 \text{MeV} \), while in fig. (5) \( m_{\sigma q'} = 1 \text{GeV} \). Yet the qualitative behavior of the states is the same: the only two states which change significantly with increasing \( \mu^2 \) (which corresponds to increasing temperature \( T \)) are the pions and the \( \sigma_{q'} \). For three flavors, in fig. (5), the \( \sigma_{q'} \) comes down in mass from \( \sim 1 \text{GeV} \), while the pion moves up in mass, until they meet at about \( \sim 400 \text{MeV} \); then both increase together. Since the \( \sigma_{q'} \) is coming down in mass from \( 1 \text{GeV} \), it is natural to find that it meets up with the \( \pi \) at a much higher mass, \( \sim 400 \text{MeV} \), than before; if the \( \sigma \) starts out at 600 \text{MeV}, then it matches up with the \( \pi \) at \( \sim 200 \text{MeV} \).

The limitations of mean field theory are clear. Including the effects of nonzero temperature by only changing \( \mu^2 \) corresponds to the addition of an \( SU(3) \) symmetric mass term \( \sim T^2 \text{tr}(\Phi^\dagger \Phi) \). While this type of term is dominant at high temperature,
at low temperature things are surely much more complicated. Calculating just the thermal effects to one loop order for the model of (5) would be an arduous task. Nevertheless, we observe that mean field theory is in qualitative agreement with more sophisticated analyses in effective sigma models, instanton models, Nambu Jona-Lasinio models, etc. With one exception, these models all come to the same conclusion as mean field theory: in the chiral limit, the chiral transition is of second order for two flavors, first order for three, while for the nonzero quark masses of QCD there is only a smooth crossover. I stress that this agreement is rather remarkable, since very different methods of calculation are being employed.

The one exception is the work of ref. [48], who find a second order chiral phase transition for three massless flavors. These works, however, use an approximate solution to the Schwinger-Dyson equations. This approach does not include the axial anomaly, and so in this approximation should give a second order phase transition. Thus the one discrepancy is understandable on general grounds.

We can now use the fit to the zero temperature spectrum to ask a more detailed question: how far is QCD from the chiral critical point? The data of [35] indicates that as a function of $M = M_{up} = M_{down} = r M_{strange}$, QCD is only about a factor of two from the chiral critical point.

Let us assume that the chiral phase transition is of first order for three, massless flavors because of the presence of the instanton coupling $\sim \det(\Phi)$. Then we assert that it is very difficult to understand why QCD is (relatively) so close to the chiral critical point. Varying the current quark masses is equivalent to varying the background fields $h_0$ and $h_8$. We then take (4)–(11), and search numerically for the values of $h_0 = h_0^{\text{crit}}$ and $h_8 = h_8^{\text{crit}}$ which give a chiral critical point. In doing so we require that the ratio of up to strange quark masses is fixed; from (8), this implies that the following ratio of $h_0$ and $h_8$ is fixed to have the same value as at zero temperature:

$$\frac{(h_0 + 2h_8)}{(h_0 - h_8)} = 32 .$$  \hspace{1cm} (23)

By varying $h_0$, $h_8$, $\mu^2$, $\Sigma_0$, and $\Sigma_8$, and otherwise using the values in (14) and (18), I determine where the critical point is. There are four equations which must vanish: two equations of motion, one equation to ensure the $\sigma_{\eta'}$ is massless, and lastly, one equation so that the cubic variation of the potential, in the direction of the $\sigma_{\eta'}$, vanishes. It is this last equation that makes the task of determining the location of the critical point so difficult analytically. Numerically things are easy, using these four conditions and (23) to determine

$$h_0^{\text{crit}} = (62 \text{ MeV})^3 \, , \, h_8^{\text{crit}} = (60.4 \text{ MeV})^3 \, , \, \mu^2_{\text{crit}} = (183 \text{ MeV})^2 \, ,$$

$$\Sigma_0 = 14.5 \text{ MeV} \, , \, \Sigma_8 = 2.7 \text{ MeV} .$$  \hspace{1cm} (24)

I then compute the ratio between the current quark masses at the chiral critical point to those in QCD:

$$\frac{M_{up}^{\text{crit}}}{M_{up}} = \frac{h_0^{\text{crit}} - h_8^{\text{crit}}}{h_0 - h_8} = .01 .$$  \hspace{1cm} (25)
Now we easily confess that this ratio, computed in mean field theory, is crude at best. Even so, numerical simulations\footnote{\cite{23}} find that the ratio in (25) is not 1%, but 50% — mean field theory is off by almost two orders of magnitude!

For the purposes of discussion, the ratio in (24) is not significantly larger for a lighter $\sigma$ meson. If we take $m_{\sigma'} = 600\,\text{MeV}$, one finds that $M_{up}/M_{up} = .06$ instead of .01. This is still a long way from the value of $\sim .5$ found by the Columbia group.\footnote{\cite{25}}

Also, I note that Meyer-Ortmanns, Pirner, and Patkos\footnote{\cite{26}} had previously argued that (in our language) $QCD$ is far from the chiral critical point. They did not make a precise estimate of this distance, though, as in (25).

This leads us to discard our initial assumption. In the chiral limit of three, massless flavors, there are two mechanisms for generating a first order transition. The first is the presence of the (cubic) instanton coupling, $\sim \det(\Phi)$.\footnote{\cite{7,33}} From (25), this does not give a phase diagram that is even qualitatively correct. Therefore, perhaps the second mechanism is at work:\footnote{\cite{7}} in the chiral limit, the chiral phase transition is of first order because it is a type of Coleman-Weinberg transition.\footnote{\cite{49,50}}

In mean field theory quartic couplings are fixed and do not change with temperature. I use the term Coleman-Weinberg to describe a theory in which the quartic couplings change significantly with temperature (in condensed matter physics this is known as “fluctuation induced”). This phenomenon can be demonstrated rigorously in four and $4 - \epsilon$ dimensions; extrapolation to three dimensions, $\epsilon = 1$, may or may not be valid. Numerical simulations of $SU(3) \times SU(3)$ spin models in three dimensions suggest that they are Coleman-Weinberg like.\footnote{\cite{53,54,55}} one then has to reconcile this with the expansion up from two dimensions, which predicts a second order phase transition.\footnote{\cite{52}}

(Incidentally, the claim of Chivukula, Golden, and Simmons\footnote{\cite{51}} that the $\beta$-functions in ref. (22) is incorrect is itself incorrect: the discrepancy is only apparent, from using a different normalization of the kinetic term. I thank Y. Shen for pointing this out.)

We propose that the only way for the phase diagram presently observed in numerical simulations of $2 + 1$ flavors to be true is if the quartic couplings of the effective linear sigma model run by large amounts. In $4 - \epsilon$ dimensions,\footnote{\cite{42,43}} in the space of $g_1$ and $g_2$ the couplings tend to run most for large $g_2$, from large to small values of $g_1$ for approximately constant $g_2$. This type of flow is precisely what is necessary to obtain a massless $\sigma_{\eta'}$ at the chiral critical point. As I noted previously, this can be seen even from the parameters of the model at zero temperature, where a massless $\sigma_{\eta'}$ is obtained by letting $g_1 \rightarrow 4$. This hardly proves that such a speculative scenario is right; but at least couplings flow in the right direction, with features that are qualitatively correct.

Of course, I confess that using primitive mean field theory is a gross caricature of reality. Nevertheless, the question of how far $QCD$ is from the chiral critical point does not seem to have been asked previously. It is surprising that the question of exactly where and what the $\sigma_{\eta'}$ meson is has such a dramatic effect on the phase
diagram of QCD. If nothing else, while are predictions are probably baseless, they are precisely testable in numerical simulations of lattice gauge theory. Hopefully we won’t have to wait until the next millennium for simulations with dynamical fermions that are good enough to be believeable.

3.2.4. Three flavors and DCC’s

Given the crudeness of present day numerical simulations for 2 + 1 flavors, I conclude with wild speculation: perhaps QCD is very close to the chiral critical point. A priori, there is no reason why QCD should be very close to the chiral critical point; there would have to be some magical conspiracy of mass scales.

But if it is, then even if there is no true phase transition in QCD, there is a natural mechanism for the generation of a large distance scale at nonzero temperature, from a light $\sigma'$. As we alluded to in a previous work this large distance scale might be used to produce large domains of Disoriented Chiral Condenstates. This is not obvious, for even at the chiral critical point, only the $\sigma'$ becomes massless, not the pions. The question is then: are large domains of DCC’s generated when massive pions are coupled to a massless (or light) $\sigma'$?

One way is by the type of isospin zero states constructed by Horn and Silver and discussed recently by Kowalski and Taylor. These authors construct a state with total isospin zero for which distribution in isospin is broad, like that of a DCC. The question is then, is it reasonable for such states to be produced near the chiral critical point?

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