Tightness for branching random walk in time-inhomogeneous random environment

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Abstract

We consider a branching random walk in time-inhomogeneous random environment, in which all particles at generation $k$ branch into the same random number of particles $L_{k+1} \geq 2$, where the $L_k$, $k \in \mathbb{N}$, are i.i.d., and the increments are standard normal. Let $\mathbb{P}$ denote the law of $(L_k)_{k \in \mathbb{N}}$, and let $M_n$ denote the position of the maximal particle in generation $n$. We prove that there are $m_n$, which are functions of only $(\ell_k)_{k \in \{0, \ldots, n\}}$, such that (with regard to $\mathbb{P}$) the sequence $(M_n - m_n)_{n \in \mathbb{N}}$ is tight with high probability.

1 Introduction

1.1 Model definition and main result

We study the maximum of branching random walk in (time-inhomogeneous) random environment (BRWre). Given a sequence $\mathcal{L} := (L_k)_{k \in \mathbb{N}}$ of natural numbers, which we refer to as environment, we start with a single particle in position 0 at time (respectively generation) 0. At each time $k \in \mathbb{N}_0$ all particles die after giving birth to $L_k+1$ children, which take independent $\mathcal{N}(0,1)$ distributed jumps relative to their parents. Given $L$, let $T$ denote the rooted tree in which all vertices with distance $k$ from the root are adjacent to exactly $L_k+1$ vertices with distance $k+1$ from the root. For $u \in T$ denote by $|u|$ the distance to the root and by $V(u)$ the position of the particle. Denote $M_n := \max_{k \leq n} V(u)$. Given $\mathcal{L}$ we denote the law (respectively expectation) of our process by $\mathbb{P}_{\mathcal{L}}$ (respectively $E_{\mathcal{L}}$). The law $\mathbb{P}_{\mathcal{L}}$ is called the quenched law. We will consider $(L_k)_{k \in \mathbb{N}}$ an i.i.d. sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathcal{L}_1 \geq 2$ P-a.s. and $\text{Var}_{\mathbb{P}}[\log(\mathcal{L}_1)] < \infty$ and define $P := \mathbb{P} \otimes \mathbb{P}_{\mathcal{L}}$ and $E = E \otimes E_{\mathcal{L}}$. We call $P$ the annealed law of our process.

In the case of $(L_k)_{k \in \mathbb{N}}$ constant the asymptotic behaviour of $M_n$ is well understood, see for example Theorems 2 and 4 in [22, pp. 5, 9] for proofs that in this case $M_n$ grows with ballistic speed $c$ and that there is a constant $c$ such that $E[M_n] = vn - c \log(n) + O(1)$. Furthermore, see Theorem 5.15 in [21, p. 62] for a proof that $M_n - vn + c \log(n)$ converges in distribution for appropriate $c$, and a description of the limit.

The aim of our paper is to prove, that for the (random) centering $m_n$ defined in Definitions 1.1 and 2.2, the sequence $(M_n - m_n)_{n \in \mathbb{N}}$ is tight with respect to the annealed measure $P$. We proceed by introducing some notation and defining the centering $m_n$.

Definition 1.1. For $k \in \mathbb{N}$, $\vartheta \in \mathbb{R}$, set

$$
k_k(\vartheta) := \log(L_k) + \frac{\vartheta^2}{2},
$$

$$
\kappa(\vartheta) := E[k_1(\vartheta)],
$$

$$
\vartheta^* := \arg \inf_{\vartheta > 0} \frac{\kappa(\vartheta)}{\vartheta} = \sqrt{2E[\log(L_1)]},
$$

$$
K_n := \sum_{k=1}^{n} k_k(\vartheta^*),
$$

$$
m_n := (\vartheta^*)^{-1}(K_n - \log(p_n),
$$

where $p_n$ is defined in Definition 2.2. We note that $m_n$ is a function of the environment $\mathcal{L}$, since $p_n$ and $K_n$ depend on the environment. In particular $m_n$ is a random variable whose law is determined by $\mathbb{P}$.

Next we present our main result.

Theorem 1.1. We have that $(M_n - m_n)_{n \in \mathbb{N}}$ is tight with regard to the annealed measure $P$.

Remark 1.1. (a) For the sake of ease in exposition, our model uses a concrete branching model and Gaussian increments. Our methods probably can be extended to cover non-Gaussian increments as well as the case that $(\ell_k)_{k \in \mathbb{N}}$ is an i.i.d. sequence with values in the set of probability measures on
N. However this extension poses various technical issues, which would add a significant burden to an already long paper. Thus, we chose to consider the simplest model, for which new techniques needed to be developed.

(b) It is natural to speculate that $M_n - m_n$ converges in distribution (under either the annealed or quenched laws). While we expect that improvements of our techniques might lead to such a statement, significant challenges remain, and thus this remains an open problem, compare Section 9, in particular the stronger version of (3.3) assumed in Section 3.1.

1.2 Literature

Our main motivation is [17]. They consider a more general model variant of the model we use, in which $(L_k)_{k \in \mathbb{N}}$ is a sequence of point processes instead of a sequence of natural numbers. At each time $k \in \mathbb{N}_0$ all particles die after generating children, whose placements are distributed according to $L_{k+1}$ relative to the position of their parent. They prove that under some integrability conditions on $L_1$ and $\kappa_1$, which are defined analogous to Definition 1.1 there is a constant $\varphi$ such that in $\mathbb{P}$-probability

$$\lim_{n \to \infty} \mathbb{P}_{L} \left[ M_n - \frac{1}{\varphi} K_n \geq -\beta \log n \right] = \begin{cases} 1 & \text{if } \beta > \varphi, \\ 0 & \text{if } \beta < \varphi, \end{cases}$$

in particular they prove that $\lim_{n \to \infty} \frac{M_n - \frac{1}{\varphi} K_n}{\log(n)} = -\varphi$ in $\mathbb{P}$-probability. Their proof uses a ballot theorem for a random walk in random environment, see Theorem 3.3 in [17], which is proved in [16]: this ballot theorem allows to evaluate the asymptotics of certain barrier probabilities, where the barrier itself is random. The main novel ingredient for the proof of Theorem 1.1 is a variant of this ballot theorem, see (3.3), which allows to compare different starting heights relative to the chosen barrier. The model of [17] goes back to [1] and in [9] it is proved that $M_n$ grows at a ballistic speed almost surely.

In [1] a space-inhomogeneous branching random walk in random environment is considered. They take $(\xi(x))_{x \in \mathbb{Z}}$ an i.i.d. family of random variables with $0 < \text{ess inf} \xi(0) < \text{ess sup} \xi(0) < \infty$ and consider an initial configuration $u_0 : \mathbb{Z} \rightarrow \mathbb{N}_0$. Given $\xi$ and $u_0$ at each $x \in \mathbb{Z}$ they place $\xi(x)$ particles at time zero, which all move independently according to continuous time simple random walk with jump rate 1. Furthermore, while at site $x$, a particle splits into two with rate $\xi(x)$ independently of everything else. They prove that under the right assumptions there is a velocity $v_0$ and a constant $\tilde{\sigma}_0 \in (0, \infty)$ such that the sequence of processes $(M(xt) - \text{int}(nt))_{t \geq 0}$ converges in “annealed” distribution to a standard Brownian motion. We proved in [10] that there are deterministic subsequences $(t_k)_{k \in \mathbb{N}}$ and a function $m_{t_k}$ of the environment such that $(M_{t_k} - m_{t_k})_{k \in \mathbb{N}}$ is (annealed) tight. Another motivation for the study in this paper is that we hope that the techniques used to prove Theorem 1.1 can be adapted to get tightness in the space-inhomogeneous model without the need to consider subsequences.

In [6] branching random walks allowing time dependence as well as local dependence between siblings are considered. They give assumptions under which $M_n$ is tight around the median of its distribution. The assumptions can be found in Sections 2 and 5 of [6]. This result implies $\mathbb{P}$-a.s. quenched tightness for BRWre, provided that the assumptions from [6] hold $\mathbb{P}$-almost surely. While the quenched tightness in [6] is stronger than the annealed tightness we prove in Theorem 1.1 our result has the advantage, that we explicitly determine the correct centering $m_n$. Furthermore, the assumptions stated in [6] do not cover the case that $L_1$ has unbounded support.

There are other studies of inhomogeneous branching random walks. Studies of branching random walk in deterministic time-inhomogeneous environments can be found in [2], [7], [14], [15], [18]. In [13] branching Brownian motion in an environment which is periodic in space has been analyzed, the corresponding PDE has been studied in [8]. Branching Brownian motion in deterministic space-inhomogeneous environment were studied in [11], [12].

1.3 Structure of the paper

In Section 2 we do some preliminary work, by defining barrier events and stating the many-to-one lemma in our setting.

In Section 3 we give a high-level description of the structure of the proof of Theorem 1.1. Section 4 is devoted to proving Theorem 1.1 while postponing some of the details to Sections 5 to 7. The proof of the theorem is broken down into three parts: an upper bound on the right tail of $M_n - m_n$ (proved in Sections 4.1 and 5), a lower bound on the right tail of $M_n - m_n$ (proved in Sections 4.2 and 6), and an upper bound on the left tail of $M_n - m_n$ (proved in Sections 4.3 and 7). Finally, in Section 4.4 we prove Theorem 1.1, i.e. the tightness of $M_n - m_n$, by combining the upper bounds on the right and left tails of $M_n - m_n$.

The rest of the paper contains technical estimates concerning barriers, which were used in Section 4.
In Section 8 we compare the probability that a Brownian motion starting at \( y \leq x \leq c \) stays below a barrier, with the probability that a Brownian motion starting at \( x \) stays below the same barrier.

In Section 9 we compare the probability to stay below a barrier shifted by a deterministic curve to the probability to stay below the same barrier without the added curve.

Section 10 gives a very rough lower bound for a type of barrier event we frequently use.

These barrier estimates are proved in a rather general setup for the barrier, where certain time and barrier-dependent constants are assumed to be finite. We will need these families of constants to be tight for the environment we use. That this is the case is checked in Section 11 using standard calculations.

Finally, Section 12 combines the results of Sections 8 to 11 into the exact barrier estimates we use in the proof of Theorem 1.1.

1.4 Notation

We use \( c_\varepsilon, C_\varepsilon \) to denote positive constants depending on \( \varepsilon \), which will change from line to line and \( c, C \) to denote positive constants, which will also change from line to line. Named constants will not change from line to line. We use \( \xi_0 \in \mathbb{N} \) to denote a fixed negative constant, which is smaller than \(-e-1\).

Recall that \( T \) denotes the genealogical tree of our BRWre. For \( u \in T \) we use \((u_0, u_1, \ldots, u_{|u|})\) to denote the vector of ancestors of the particle. For \( u, v \in T \) we denote by \( u \wedge v \) the last common ancestor of \( u \) and \( v \).

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2 Preliminaries: Barrier Events and a Many-to-One Lemma

We first give a rather general definition of barrier events and probabilities.

Definition 2.1 (Barrier Events). For \( I \subseteq \mathbb{R} \), \( t := \max I \), \( (Z_s)_{s \in I} \) a real valued process, \( f : I \rightarrow \mathbb{R} \) a function, \( y \in \mathbb{R} \) and \( J \subseteq \mathbb{R} \) an interval, define

\[
\mathcal{B}^u_{I,f}(Z) := \{ y + Z_s + f(s) \leq 0, y + Z_t + f(t) \in J \}.
\]

If \( f = 0 \) or \( y = 0 \) we suppress them from the notation. Also set \( J_x := [x-1,x] \) for \( x \in \mathbb{R} \). If \( J = \mathbb{R} \) we supress it from notation. We call events of this form barrier events.

We recall Definition 1.1.

Definition 2.2. For \( n \in \mathbb{N} \), let \((X_n)_{n \in \mathbb{N}}\) be i.i.d. with \( X_1 \sim \mathcal{N}(0^+,1) \) and set

\[
S_n := \sum_{k=1}^n X_k \sim \mathcal{N}(n0^+,n), \quad B_n := S_n - n0^+, \quad W_n := K_n/0^+ - n(0^+).
\]

We can realize \((B_n)_{n \in \mathbb{N}}\) as the values at integer times of a Brownian motion \((B_n)_{n \geq 0}\) independent of \( L \). Furthermore we can define \( W_s \) for all \( s \geq 0 \) by linear interpolation. For \( s \geq 0 \) set \( T_s := B_s - W_s \). For \( t \geq 0 \) define

\[
p_t := \mathbb{P}_L \left[ \mathcal{B}^{0,J}_{[0,t]}(T) \right]. \tag{2.1}
\]

Finally, denote by \( \mathbb{P}_L^k \) the measure using the environment \((L_{j+k})_{j \in \mathbb{N}}\), under \( \mathbb{P}_L^k \) we write \( S_n := \sum_{j=k+1}^{j+n} X_j, T_n = S_n - (0^+)^{-1} \sum_{j=k+1}^{j+n} \kappa_j(0^+) \).

Remark 2.1. Since \((L_k)_{k \in \mathbb{N}}\) is i.i.d., so is \((W_{k+1} - W_k)_{k \in \mathbb{N}}\). Furthermore, by choice of \( 0^+ \) we have that \( E[W_2 - W_1] = 0 \) and finally, since we assume that \( \text{Var}[(\log(L_i))] < \infty \), we have that \( \text{Var}[W_2 - W_1] < \infty \). Thus we can apply the law of iterated logarithms for \((W_k)_{k \in \mathbb{N}}\), compare Theorem 5.17 in [19].

Now we have defined all random variables necessary to state the many-to-one lemma for our model.

Lemma 2.1 (Lemma 2.1, equation (2.5) in [17]). For any \( n \in \mathbb{N} \) and \( f : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) non-negative, we have that \( \mathbb{P}_a.s. \)

\[
E_L \left[ \sum_{|u|=n} f(V(u_j) - K_j/0^+, j \leq n) \right] = E_L[e^{-\alpha^+T_n} f(T_1, \ldots, T_n)].
\]

We also have, remembering the convention that under \( \mathbb{P}_L^k \) we write \( S_n = X_{k+1} + \cdots + X_{n+k} \), the time-shifted version of this

\[
E_L^k \left[ \sum_{|u|=n} f(V(u_j) - K_j/0^+, j \leq n) \right] = E_L^k \left[ e^{-\alpha^+S_n + \sum_{j=k+1}^{j+n+1} \kappa_j(0^+)} f(T_1, \ldots, T_n) \right].
\]
Remark 2.2. Because of Theorem 3.3 in [17] we expect that in $\mathbb{P}$-probability as $n \to \infty$ we have that
\[
\log(p_n)/\log(n) \to -\lambda,
\]
with $\lambda$ as in equation (3.3) of [17]. We next state a weaker bound on $\log(p_n)$, which will be proved in Section 11

Lemma 2.2. There is a $C > 0$ such that $\mathbb{P}$-a.s.
\[
\limsup_{n \to \infty} \frac{\log(p_n)}{\log(n)} \leq C.
\]

The following lemma allows us to factorize the probability of barrier events of Markov processes into two factors. This will be useful, since we mostly do barrier computations for time intervals $[0, t]$ but want to apply them also for time intervals $[0, t_1]$ and $[t_1, t]$, which equations (2.2) and (2.3) enable. The proof of Lemma 2.3 is a straightforward application of the Markov property, which we omit.

Lemma 2.3. Let $I \subseteq \mathbb{R}, t := \max I, (Z_s)_{s \in I}$ be a real valued Markov process, $f : I \to \mathbb{R}$ a function and $z_1, z_2 \in \mathbb{R}$. Take $t_0 \in I$ and set $I_1 := \{s \in I : s \leq t_0\}, I_2 := \{s \in I : s \geq t_0\}$. We have that for $x_0 \in \mathbb{R},$
\[
\text{Prob}\left[Z_{t_1} \leq z_2|Z_{t_0} = z_1\right] \geq \text{Prob}\left[Z_{t_1} \leq z_2|Z_{t_0} = z_1\right] \inf_{x \in J_{t_0}} \text{Prob}\left[Z_{t_2} \leq z_2|Z_{t_0} = z_1\right], \tag{2.2}
\]
\[
\text{Prob}\left[Z_{t_1} \leq z_2|Z_{t_0} = z_1\right] \geq \text{Prob}\left[Z_{t_1} \leq z_2|Z_{t_0} = z_1\right] \inf_{x \in J_{t_0}} \text{Prob}\left[Z_{t_2} \leq z_2|Z_{t_0} = z_1\right]. \tag{2.3}
\]

3 Structure of the proof

The first step in the proof of Theorem 1.1 is an upper bound on $\mathbb{P}_\mathbb{L}[M_n - m_n \geq -y]$ for $y < 0$. We can get a good enough upper bound for $\mathbb{P}_\mathbb{L}[M_n - m_n \geq \xi_0 + \log(n)^2]$ by applying Lemma 2.1. Thus we need to consider $y \in [-\log(n)^2, \xi_0]$. This is done using a first moment approach. For this we introduce the barrier $((\vartheta^*)^{-1}K_k + \frac{h_n}{n^\alpha}\log(p_n) - y)_{k \in \{0, \cdots, n\}}$ and take $\tau(y)$ to be the first time at which any particle in our system breaks this barrier. It is immediate that
\[
\mathbb{P}_\mathbb{L}[M_n - m_n \in (\xi_0 - y - 1, \xi_0 - y)] \leq \mathbb{P}_\mathbb{L}[\tau(y) < n] + \mathbb{P}_\mathbb{L}[\tau(y) = n, M_n - m_n \in (\xi_0 - y - 1, \xi_0 - y)]
\]
\[
\leq \mathbb{P}_\mathbb{L}[\tau(y) < n] + \mathbb{P}_\mathbb{L}\left[\bigcup_{u, |u| = n} B^{y,J_{t_0}}_{(0, \cdots, n), -((\vartheta^*)^{-1}(\pi \log(p_n) + K)(V(u)))}\right]. \tag{3.1}
\]

Applying the Markov inequality and the many-to-one Lemma 2.1 to the second summand in (3.1) yields that
\[
\mathbb{P}_\mathbb{L}[M_n - m_n \in (\xi_0 - y - 1, \xi_0 - y)] \leq \mathbb{P}_\mathbb{L}[\tau(y) < n] + C_p^{-1}e^p\forall y \mathbb{P}\left[B^{y,J_{t_0}}_{(0, \cdots, n), \pi \log(p_n)}(T)\right]. \tag{3.2}
\]

The heart of the matter is now to prove that for all $\varepsilon > 0$ there are $c_\varepsilon, C_\varepsilon$ such that
\[
\liminf_{n \to \infty} \mathbb{P}_\mathbb{L}\left[\frac{\mathbb{P}_\mathbb{L}\left[B^{y,J_{t_0}}_{(0, \cdots, n), \pi \log(p_n)}(T)\right]}{p_n} \leq C_\varepsilon |y|^{c_\varepsilon} \right] \geq 1 - \varepsilon, \tag{3.3}
\]

This barrier computation stands in for equation (3.4) in [17]. Equation (3.3) controls how much bigger the probability to stay above the barrier can get, from starting higher. We sketch below how equation (3.3) is proved. Handling the second summand in (3.2) is immediate once (3.3) is established, the first summand can be handled using (3.3) and standard techniques comparable to what is sketched in Remark 9 in (22, p. 16). In the actual proof we need a slightly more complicated barrier, which has an additional (deterministic) curve added, this does only marginally change the proof of (3.3), however we need to then remove this curve after we have moved the starting point. This is mostly an application of Girsanov.

The second step in the proof of Theorem 1.1 is a lower bound on $\mathbb{P}_\mathbb{L}[M_n - m_n \geq \xi_0 - y - 1]$ for $y \in [-\log(n)^2, \xi_0]$. This is done using a second moment approach, for this we define $Z_n(y)$ to be the number of particles for which $(V(u_k) - ((\vartheta^*)^{-1}K_k)_{k \in \{0, \cdots, n\})$ stays below $(\frac{h_n}{n^\alpha}\log(p_n) - y)_{k \in \{0, \cdots, n\}}.$ Using Cauchy-Schwartz yields that
\[
\mathbb{P}_\mathbb{L}[M_n - m_n \geq \xi_0 - y - 1] \geq \mathbb{P}_\mathbb{L}[Z_n(y) \geq 1] \geq \frac{E\mathbb{L}[Z_n(y)]^2}{E\mathbb{L}[Z_n(y)]}.
\]
and a lower bound on the right tail can be achieved by bounding $\mathbb{E}[Z_n(y)]$ and $\mathbb{E}_L[Z_n(y)^2]$. The first is done similar to the calculation in the upper bound on the right tail of $M_n - m_n$, using that for all $\varepsilon > 0$ there are $c_\varepsilon, C_\varepsilon$ such that

$$\liminf_{n \to \infty} \mathbb{P} \left[ \mathbb{P}_L \left[ \mathcal{B}^{y_0, f_{\ln \log p_n}(T)}_{(0, \ldots, n), \pi \log p_n}(T) \right] \geq C_\varepsilon |y|^{-c_\varepsilon} \right] \geq 1 - \varepsilon \quad (3.4)$$

instead of (3.3). For the bound on the second moment we use $\Lambda_k(y)$ to denote the number of pairs of particles in generation $n$, which are both counted in $Z_n(y)$, whose last common ancestor is in generation $k$. By definition we have that

$$Z_n(y)^2 = \sum_{k=0}^{n-1} \Lambda_k(y) + Z_n(y).$$

Proving an upper bound on $\mathbb{E}[(\Lambda_k(y)]$ uses (3.3) and beyond this mostly standard computations similar to Section 4.2 in [17]. As above in the proof it will be helpful to use a curved barrier instead of the straight barrier given here.

The upper bound on the left tail of $M_n - m_n$ follows from the lower bound on the right tail by using the tree structure, compare the first half of page 17 in [22]. It is worth pointing out that this is the point at which the assumption that $L_1 \geq 2 \mathbb{P}$-a.s. is used.

Having upper bounds on both the left and the right tail of $M_n - m_n$ then quickly allows to prove Theorem 1.1 i.e. tightness of $(M_n - m_n)_{n \in \mathbb{N}}$.

We finish this section by sketching how we prove (3.3), in our opinion this is the most interesting part of our paper. We ignore the random part and the log-term of the barrier in this sketch, the difficulties added by these are either already handled by [16] or minor. We also switch to continuous time, which can be justified using standard arguments. Thus let $(B_t)_{t \geq 0}$ be a Brownian motion. As a first step we use a correlation inequality for Brownian bridges to prove that

$$\text{Prob} \left[ \mathcal{B}^{y_0, f_0}_{(0, \ldots, n)}(B) \right] \leq \text{Prob} \left[ \mathcal{B}^{y_0, -c y, \varepsilon \log y}_{(0, \ldots, n)}(B) \right] \text{Prob} \left[ \mathcal{B}^{y_0, f_0}_{(0, \ldots, n)}(B) \right].$$

From [16] we know that $\text{Prob} \left[ \mathcal{B}^{y_0, -c y, \varepsilon \log y}_{(0, \ldots, n)}(B) \right] \leq C y^t$ with high probability and it remains to give a bound on $\text{Prob} \left[ \mathcal{B}^{y_0, f_0}_{(0, \ldots, n)}(B) \right]$ / $\text{Prob} \left[ \mathcal{B}^{y_0, f_0}_{(0, \ldots, n)}(B) \right]$. Using the Markov property for Brownian motion this comes down to comparing $g_{y, y^t}(z)$ to $g_{x, y^t}(z)$, where we use $g_{x, y^t}(z)$ to denote the density of a normal distribution with mean $\mu$ and variance $\sigma^2$. For $0 \geq z \geq y \log |y|$ we know that $g_{y, y^t}(z) / g_{x, y^t}(z) \leq C y^t$ and this region is fine. The region $z \geq - \log(t)$ is negligible, since we know from [16] that $\text{Prob} \left[ \mathcal{B}^{y_0, f_0}_{(0, \ldots, n)}(B) \right]$ is polynomial in $t$. The region $z \in [- \log(t), y \log |y|]$ requires extra care, we handle this with an inductive scheme in which we recursively drop the barrier for a time interval depending on $z$ and again split up the possible locations after this time interval into three regions. A more detailed overview of this inductive scheme is given in Section 8.5.

### 3.1 Remarks on convergence in distribution

The natural next question is whether $M_n - m_n$ converges in distribution. We think that this is the case and can be proved by improving our techniques and adopting the approach from [3]. We state an in our opinion sufficient improvement of (3.3) next, after this we sketch how to use this improvement for proving convergence in distribution. For this purpose set $\mathbb{P}_L \left[ \mathcal{B}^{y_0, f_{\ln \log p_n}(T)}_{(0, \ldots, n), \pi \log p_n}(T) / p_n \right] = f_n(z), \ z \in \mathbb{R}, n \in \mathbb{N}$. One sufficient set of conditions on $f_n$, is to assume that there exists a $f : \mathbb{R} \to \mathbb{R}$ such that for all big enough $z \in \mathbb{R}, a_1 \to \infty$

$$\lim_{n \to \infty} f_n(z) = f(z), \quad (3.5)$$

$$\lim_{l \to \infty} \frac{f(z + a l)}{f(a l)} = g(z), \quad \lim_{z \to \infty} f(z) = \infty. \quad (3.6)$$

Assuming (3.5), (3.6) the first step for proving convergence in distribution is to get precise estimates for $\mathbb{P}_L [M_n - m_n > z]$. We assume in the following that it is possible to prove that

$$\mathbb{P}_L [M_n - m_n > z] \sim f(z)e^{-\theta z} \quad (3.7)$$

as $n \to \infty$, i.e. that the second moment calculation in Section 3 of [3] can be adapted to our setting. We have not carefully checked this.
Theorem 4.1. For all Equation (3.9) implies that for \( M \) large and Proposition 4.3 controls small to medium deviations of \( z \). Set \( C \) in this section we prove the following theorem, postponing certain auxiliary estimates to Sections 5 and 12. From here standard techniques allow to conclude that \( M \) large and \( \epsilon > 0 \). Take \( \epsilon/2 \) with \( \epsilon > 0 \) to prove that \( M_n - m_n \sim \) for fixed \( l \). Thus we can use Theorem 1.1 to prove that \( M_n - m_n \sim V(\mu_l) \to \infty \) in \( P \)-probability and combining (3.7) with (3.8) yields that for large \( n \) and \( l \leq n \) with high \( P \)-probability

\[
\log (P\mathbb{L}[M_n - m_n \leq z|\mathcal{F}_l]) \sim \sum_{u:|u|=l} \log (1 - f(z_{u,l})).
\]  

Equation (3.9) implies that for \( l \) large and \( n \) (much) larger in \( P \)-probability

\[
\log (P\mathbb{L}[M_n - m_n \leq z|\mathcal{F}_l]) \sim \sum_{u:|u|=l} f(z_{u,l})e^{-\sigma^* z_{u,l}}.
\]

From here standard techniques allow to conclude that \( M_n - m_n \) and \( Z \) converge in \( P \)-distribution as \( n \to \infty \) respectively \( l \to \infty \).

4 Proof of Theorem 1.1

In this section we prove Theorem 1.1, leaving quite some details and proofs to the later sections.

4.1 Upper Bound on the Right Tail of \( M_n - m_n \)

In this section we prove the following theorem, postponing certain auxiliary estimates to Sections 5 and 12.

Theorem 4.1. For all \( \epsilon > 0 \), there exists a \( y_\epsilon < 0 \) such that

\[
\liminf_{n \to \infty} P\mathbb{L}[M_n - m_n \geq -y_\epsilon] \leq \epsilon \geq 1 - \epsilon.
\]

The proof of Theorem 4.1 uses two ingredients. Lemma 4.2 controls the probability that \( M_n - m_n \) is large and Proposition 4.3 controls small to medium deviations of \( M_n - m_n \).

Lemma 4.2 (Lemma 2.3 in [17]). For any \( y < 0 \), we have that \( P \)-a.s.

\[
P\mathbb{L}[\exists u \in \mathbf{T} : V(u) + y > (\sigma^*)^{-1}K_{|u|}] \leq e^{\sigma^* y}.
\]

Proposition 4.3. For all \( \epsilon > 0 \), there are \( c_\epsilon, C_\epsilon > 0 \) such that

\[
\liminf_{n \to \infty} P\mathbb{L}\left[\bigcap_{y \in [-\log(n)^2, \log(n)^2]} \{P\mathbb{L}[M_n - m_n \in [\xi_0 - y - 1, \xi_0 - y]] \leq C_\epsilon e^{\sigma^* y} | y|^{\frac{d}{2}}\} \right] \geq 1 - \epsilon.
\]

Proof of Theorem 4.1 assuming Proposition 4.3. Take \( y_\epsilon \in \mathbb{Z} \) such that for \( c_\epsilon, C_\epsilon \) as in Proposition 4.3

\[
C_\epsilon/2 \sum_{y=-y_\epsilon}^{y_\epsilon} |y|^{d/2} e^{-\sigma^* y} \leq \epsilon/2. \quad \text{Then, by Proposition 4.3}
\]

\[
\liminf_{n \to \infty} P\mathbb{L}\left[\mathbb{L}[M_n - m_n \geq \xi_0 + \log(n)^2] \geq \epsilon/2\right] \leq \epsilon/2.
\]  

On the other hand, from Lemma 4.2, \( P \)-a.s.,

\[
P\mathbb{L}[M_n - m_n \geq \xi_0 + \log(n)^2] \leq e^{-\sigma^* \log(n)^2 - \sigma^* \xi_0 - \log(p_n)}.
\]

Combining (4.2) and Lemma 2.2 yields that

\[
\liminf_{n \to \infty} P\mathbb{L}\left[\mathbb{L}[M_n - m_n \geq \xi_0 + \log(n)^2] \leq \epsilon/2\right] \geq \epsilon/2.
\]  

Combining (4.1) and (4.3) finishes the proof.

Thus it remains to prove Proposition 4.3, which uses barrier computations. For this we need the following definition, for which we recall (1.1), (2.1) and Definition 2.1.
Definition 4.1. For \( t > 0, y, x < 0 \) and \( n \in \mathbb{N} \), define
\[
h_t \colon [0, t] \to \mathbb{R}, \quad s \mapsto -(1 + s)^{1/6} \wedge (1 + t - s)^{1/6} - 1,
\]
\[
m_{t,h}(s) := h_t(s) - \frac{s}{t} \log(p_t),
\]
\[
p_n^{\tau,x}(y) := \mathbb{P}_L \left[ B_{[0,\ldots,n],m_{n,h/2}}^{y,J}(T) \right], \quad (4.4)
\]
For \( u \) with \( |u| = n \), let
\[
\tau(u, y) := \min \{ k \in \{0, \ldots, n\} : V(u_k) - (\vartheta^*)^{-1} K_k + m_{n,h/2}(k) + y + 1 \geq 0 \}
\]
and \( \tau(y) := \min_{u:|u|=n} \tau(u, y) \wedge n \). The dependency on \( n \) of \( \tau(y) \) is omitted from notation. Finally, define
\[
A_n(y, x) := \bigcup_{n:|u|=n} B_{[0,\ldots,n],m_{n,h/2}}^{y,J}(V(u) - (\vartheta^*)^{-1} K).
\]

Directly from the definitions it follows that for all \( y < 0 \)
\[
\mathbb{P}_L [M_n - m_n \in [\xi_0 - y - 1, \xi_0 - y]] \leq \mathbb{P}_L[\tau(y) < n] + \mathbb{P}_L[A_n(y, \xi_0)]. \quad (4.5)
\]

It will be useful to be able to compare \( p_n^{\tau,x}(y) \) from (4.4) to \( p_n \) from (2.1). The next Proposition 4.4 allows just that. Proposition 4.4 will be proved in Section 5.1. The term \( y^\epsilon \) \( |x - 1|^{\epsilon} \) in Proposition (4.4) comes from moving the start- and endpoint. Going from discrete to continuous time and removing the curve of the barrier does only cost a constant factor. We recall that \( \xi_0 \) is negative.

Proposition 4.4. For all \( \epsilon > 0 \), there are \( c_c, C_c > 0 \) such that
\[
\liminf_{n \to \infty} \mathbb{P}_{\epsilon} \left[ \bigcap_{y \in [- \log(n)^2, \xi_0]} \left\{ \frac{p_n^{\tau,x}(y)}{p_n} \leq C_c |y|^{c_c} |x - 1|^{\epsilon} \right\} \right] \geq 1 - \epsilon.
\]

By (4.5), proving Proposition 4.3 is reduced to bounding \( \mathbb{P}_L[A_n(y, \xi_0)] \) and \( \mathbb{P}_L[\tau(y) < n] \), which we do in Propositions 4.5 and 4.6.

Proposition 4.5. For all \( \epsilon > 0 \), there are \( C_c, c_c > 0 \) such that
\[
\liminf_{n \to \infty} \mathbb{P}_{\epsilon} \left[ \bigcap_{y \in [- \log(n)^2, \xi_0]} \left\{ \mathbb{P}_L[A_n(y, \xi_0)] \leq C_c e^{\vartheta^* y} |y|^{c_c} \right\} \right] \geq 1 - \epsilon.
\]

Proof of Proposition 4.4 assuming Proposition 4.3. Applying the many-to-one Lemma (2.1) yields that \( \mathbb{P} \)-a.s.
\[
\mathbb{P}_L[A_n(y, \xi_0)] = \mathbb{E}_L \left[ e^{-T_\epsilon} ; B_{[0,\ldots,n],m_{n,h/2}}^{y,J}(T) \right] \leq p_n^{-1} e^{\vartheta^* (1 + y - \xi_0)} p_n^{\tau,J}(y).
\]

Applying Proposition 4.4 and pulling the factor \( |\xi_0 - 1|^{c_c} e^{\vartheta^* (1 - \xi_0)} \) into the \( C_c \), which is possible, since \( \xi_0 \) is a fixed (negative) constant, finishes the proof.

The proof of the forthcoming Proposition 4.6 appears in Section 5.1.

Proposition 4.6. For all \( \epsilon > 0 \), there are \( c_c, C_c > 0 \) such that
\[
\liminf_{n \to \infty} \mathbb{P}_{\epsilon} \left[ \bigcap_{y \in [- \log(n)^2, \xi_0]} \left\{ \mathbb{P}_L[\tau(y) < n] \leq C_c e^{\vartheta^* y} |y|^{c_c} \right\} \right] \geq 1 - \epsilon.
\]

Proof of Proposition 4.3 assuming Proposition 4.6. We combine (4.5), Proposition 4.5 and Proposition 4.6.

4.2 Lower Bound on the Right Tail of \( M_n - m_n \)

Theorem 4.7. For all \( \epsilon > 0 \) there are \( c_c, C_c > 0 \) such that for \( y \in [- \log(n)^2, \xi_0] \),
\[
\liminf_{n \to \infty} \mathbb{P}_L[M_n - m_n \geq \xi_0 - y - 1] \geq e^{\vartheta^* y} C_c |y|^{c_c} \geq 1 - \epsilon.
\]

We will prove Theorem 4.7 using a second moment approach. For this we need the following definition, for which we recall the Definitions 2.1, 1.1 and 2.1.
Definition 4.2. For $t > 0$, $s \in [0, t]$, $y < 0$ and $n \in \mathbb{N}$, define

$$h_t(s) := \left((1 + s) \wedge (t - s + 1)^{1/6} - 1\right),$$

$$m_{t,h,y}(s) := y + h_t(s) - \frac{s}{n0^*} \log(p_t), \quad m_{t,0}(s) := m_{t,0}(s),$$

$$p_{n,\xi_0}(y) := \mathbb{P}[\mathcal{B}_{n,\xi_0}(m_{n,h,y}, \mathcal{B}(T))],$$

$$Z_n(y) := \sum_{|u| = n} 1_{\mathcal{B}_{n,\xi_0}(m_{n,h,y}, (V(u) - (\nabla^* - 1)K))}.$$

The random variable $Z_n$ counts the number of particles, which stay below the barrier $-m_{n,h,y}(j)$ and end up in $[\xi_0 - y - 1, \xi_0 - y]$, in particular we have that

$$\mathbb{P}[M_n - m_n \geq \xi_0 - y - 1] \geq \mathbb{P}[Z_n(y) \geq 1] \geq \frac{\mathbb{E}[Z_n(y)]^2}{\mathbb{E}[Z_n(y)^2]},$$

where the last step uses the Cauchy-Schwartz inequality. We will establish the following two Propositions 4.8, 4.9. We prove Proposition 4.8 in Section 4.2.1. We sketch the proof of Proposition 4.9 in Section 4.2.2 and fill in the detail in Section 6.1.

Proposition 4.8. For all $\varepsilon > 0$ there are $c_\varepsilon, C_\varepsilon > 0$ such that for all $y \in [-\log(n)^2, 0]$,

$$\liminf_{n \to \infty} \mathbb{P}\left[\mathbb{E}[Z_n(y)] \geq C_\varepsilon e^{\vartheta^* y |y|^{-c_\varepsilon}}\right] \geq 1 - \varepsilon.$$

Proposition 4.9. For all $\varepsilon > 0$ there are $c_\varepsilon, C_\varepsilon > 0$ such that for all $y \in [-\log(n)^2, \xi_0]$,

$$\liminf_{n \to \infty} \mathbb{P}\left[\mathbb{E}[Z_n(y)^2] \leq C_\varepsilon e^{\vartheta^* y |y|^{c_\varepsilon}} + \mathbb{E}[Z_n(y)]\right] \geq 1 - \varepsilon.$$

Proof of Theorem 4.4 assuming Propositions 4.8 and 4.9. Combining (4.7) and Proposition 4.9 yields that there are $c_\varepsilon, C_\varepsilon > 0$ such that

$$1 - \varepsilon/2 \leq \liminf_{n \to \infty} \mathbb{P}\left[\mathbb{P}[M_n - m_n \geq \xi_0 - y - 1] \geq \frac{\mathbb{E}[Z_n(y)]}{1 + C_\varepsilon e^{\vartheta^* y |y|^{c_\varepsilon}}/\mathbb{E}[Z_n(y)]}\right].$$

Applying Proposition 4.8 finishes the proof. \hfill \square

4.2.1 Lower Bound on $\mathbb{E}[Z_n(y)]$

In this section we sketch the proof of Proposition 4.8, it closely resembles the proof of Proposition 4.5 with the forthcoming Proposition 4.10 replacing Proposition 4.2.

The proof has two steps, first applying the many-to-one Lemma 2.1 gives that

$$\mathbb{E}[Z_n(y)] \geq e^{\vartheta^* y |y|^{-c_\varepsilon}} p_n^{-1} p_n^{-\xi_0}(y) \quad \text{a.s.}$$

(4.8)

Similar to Proposition 4.4 we prove a lower bound on $p_n^{-\xi_0}(y)/p_n$, recall (2.1) and (4.6). The following Proposition 4.10 will be proved in Section 12.

Proposition 4.10. For all $\varepsilon > 0$ there are $c_\varepsilon, C_\varepsilon > 0$ such that for $y \in [-\log(n)^2, \xi_0]$,

$$\liminf_{n \to \infty} \mathbb{P}\left[p_n^{-\xi_0}(y) \geq C_\varepsilon |y|^{-c_\varepsilon}\right] \geq 1 - \varepsilon.$$

Proof of Proposition 4.8 assuming Proposition 4.10. We combine (4.8) with Proposition 4.10. \hfill \square

Remark 4.1. The $|y|^{-c_\varepsilon}$ in Proposition 4.10 is far from optimal, with more effort one can prove that the statement holds (as one expects) with $|y|^{-c_\varepsilon^*}$ instead, however we do not need that.

4.2.2 Upper Bound on $\mathbb{E}[Z_n(y)^2]$

In this section we outline the proof of Proposition 4.9.

The random variable $Z_n(y)^2$ roughly counts the number of pairs of particles $u, v$ in generation $n$, for which both particles stay below the barrier $-m_{n,h,y}(j)$ and end up in $[\xi_0 - y - 1, \xi_0 - y + 1]$. We partition with regard to $|u \wedge v|$, which leads to the following definition.
Both $q_n$ for all $n$ in $\mathbb{N}$ and $y \in [-\log(n)^2, \xi_0]$, define

$$
\Lambda_k(y) := \sum_{|u|=k} \sum_{w \in \mathbb{Z}^d \cap n \cdot w} \prod_{i=1}^2 \mathbb{B}_{(0,...,n-1),m_n,h_n,y}(V(u_{i})-(\vartheta^*)^{-1}K).$$

We have that

$$
Z_n(y)^2 = \sum_{k=0}^n \Lambda_k(y) = \sum_{k=0}^{n-1} \Lambda_k(y) + Z_n(y),
$$

where the second equality uses that $\Lambda_n(y) = Z_n(y)$. Thus all we need to do to prove Proposition 4.11 is to bound $\sum_{k=0}^{n-1} \mathbb{E}[\Lambda_k(y)]$.

To do this, we introduce some additional notation. We recall Definitions 1.1, 4.2 and the convention that under $\mathbb{P}_{\xi}$ we write $T_n = \sum_{j=k+1}^{\infty} X_j - (\vartheta^*)^{-1} \kappa_j(\vartheta^*)$.

**Definition 4.4.** For $n \in \mathbb{N}$, $k \in \{0, \ldots, n-1\}$, $y \leq 0$ and $x \in \mathbb{R}$ define

$$
f_k(x) := \mathbb{P}_{\xi} \left[ \sum_{|u|=n-k} \mathbb{B}_{(0,...,n-1),m_n,h_n,y}(V(u) - \vartheta xK) \right],
$$

$$
q_k(x) := \mathbb{P}_{\xi} \left[ \mathbb{B}_{(0,...,n-1),m_n,h_n,y}(T) \right].
$$

Both $q_k(x)$ and $f_k(x)$ depend on $n$ and $y$, which we omit from notation.

By conditioning on $\mathcal{F}_k := \sigma(u, V(u) : |u| \leq k)$ we will prove the following upper bound on $\mathbb{E}[\Lambda_k(y)]$ in Section 6.1.1.

**Proposition 4.11.** For $n \in \mathbb{N}$, $k \in \{0, \ldots, n-1\}$ and $y \leq 0$, we have that $\mathbb{P}$-a.s.

$$
\mathbb{E}[\Lambda_k(y)] \leq ce^{\vartheta y - 2\vartheta \epsilon_0} p_n^{-2} e^{-\vartheta h_n(k)} p_n^{-\varphi_0} (y) \max_{x \geq 0} e^{-\vartheta x} q_k(y) K_k/y \leq m_n,h_n(k) - x).
$$

In particular, we have that

$$
\mathbb{E}[\Lambda_k(y)] \leq ce^{\vartheta y - 2\vartheta \epsilon_0} p_n^{-2} e^{-\vartheta h_n(k)} p_n^{-\varphi_0} (y).
$$

It will again be useful to derive an upper bound on $p_n^{-\epsilon_0}(y)/p_n$, recall (4.11) and (4.6). This is done in the following proposition, which is proved in Section 12.

**Proposition 4.12.** For all $\epsilon > 0$, there are $c_\epsilon, C_\epsilon > 0$ such that

$$
\lim_{n \to \infty} \mathbb{P} \left[ \bigcup_{y \in [-\log(n)^2, \xi_0]} \left\{ p_n^{-\epsilon_0}(y)/p_n \leq C_\epsilon |y|^{c_\epsilon} \right\} \right] \geq 1 - \epsilon.
$$

**Remark 4.2.** Proposition 4.12 also holds, with minor changes in the proof, when one replaces $J_{\xi_0}$ in (4.6) by $[\xi_0 - 2, \xi_0]$, this version will be used in Section 6.1.2 for the proof of Lemma 6.1.

The next two lemmas allow to bound $\sum_{k=0}^{n-1} \mathbb{E}[\Lambda_k]$.  

**Lemma 4.13.** For all $\epsilon > 0$, there are $c_\epsilon, C_\epsilon > 0$ such that for all $y \in [-\log(n)^2, \xi_0]$,

$$
\lim_{n \to \infty} \mathbb{P} \left[ \sum_{k=\log(n)^2}^{n-1} \mathbb{E}[\Lambda_k(y)] \leq C_\epsilon e^{\vartheta_1 y} |y|^{c_\epsilon} \right] \geq 1 - \epsilon.
$$

**Proof of Lemma 4.13 assuming Proposition 4.12** Applying Proposition 4.12 to equation (4.11) of Proposition 4.11 yields that there are $c_\epsilon, C_\epsilon > 0$ such that for $y \in [-2\log(n)^2, \xi_0]$

$$
\lim_{n \to \infty} \mathbb{P} \left[ \bigcap_{k=[\log(n)^2], n-1} \mathbb{E}[\Lambda_k(y)] \leq C_\epsilon e^{\vartheta y} p_n^{-2} e^{-\vartheta h_n(k)} |y|^{c_\epsilon} \right] \geq 1 - \epsilon.
$$

By Lemma 2.2 there is a $C > 0$ such that $\mathbb{P}$-a.s. $\limsup_{n \to \infty} \frac{|\log(p_n)|}{\log(n)} \leq C$, which together with (4.13) implies that there are $c_\epsilon, C_\epsilon > 0$ such that for $y \in [-2\log(n)^2, \xi_0]$.

$$
1 - \epsilon \leq \liminf_{n \to \infty} \mathbb{P} \left[ \sum_{k=\log(n)^2}^{n-1} \mathbb{E}[\Lambda_k(y)] \leq C_\epsilon e^{\vartheta y} |y|^{c_\epsilon} \sum_{k=\log(n)^2}^{n-1} e^{-\vartheta h_n(k)} e^{C \log(n)(1 - \frac{1}{n})} \right].
$$

We can bound $\sup_n \sum_{k=\log(n)^2}^{n-1} e^{-\vartheta h_n(k)} e^{C \log(n)(1 - \frac{1}{n})} < \infty$, which together with (4.14) finishes the proof. \hfill \Box
Handling $k < \log(n)^7$ in (4.12) requires a more nuanced argument because the gain from $h_n(k)$ in 4.14 is not sufficient. We prove the following lemma in Section 6.1.2.

**Lemma 4.14.** For $\varepsilon > 0$ there are $c_\varepsilon, C_\varepsilon > 0$ such that for all $y \in [-\log(n)^2, \xi_0]$,

$$\liminf_{n \to \infty} \mathbb{P} \left[ \log(n) \sum_{k=0}^{[\log(n)^7]} \mathbb{E}_L[A_k(y)] \leq C_\varepsilon e^{0+y|y|^c_\varepsilon} \right] \geq 1 - \varepsilon.$$

**Proof of Proposition 4.1.** We combine (4.9) with Lemma 4.13 and Lemma 4.14.

4.3 **Bound on the Left Tail of $M_n - m_n$**

The aim of this section is to prove the following theorem.

**Theorem 4.15.** For all $\varepsilon > 0$, there is a $y_\varepsilon \leq 0$ such that

$$\liminf_{n \to \infty} \mathbb{P} \left[ \mathbb{P}_L [M_n - m_n \leq y_\varepsilon] \leq \varepsilon \right] \geq 1 - \varepsilon.$$

The idea is, to cut the tree at depth $l$, use that there are at least $2^l$ particles in generation $l$, and then to finish the argument by applying the lower bound on the right tail proven in Theorem 4.7.

Slightly more detailed, we note that, similarly to Lemma 4.2, we get the following lemma.

**Lemma 4.16.** For all $\varepsilon > 0$ there are $c_\varepsilon, c_\varepsilon', c_\varepsilon'' > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbb{P} \left[ \mathbb{P}_L \left[ \min_{|u| = n} V(u) \leq -c_\varepsilon l \right] \leq e^{-c_\varepsilon' l} \right] \geq 1 - \varepsilon.$$

Then by cutting the tree at depth $l$ one can make the calculation that there are $c_\varepsilon, c_\varepsilon', c_\varepsilon''$ and $C_\varepsilon$ such that

$$\liminf_{n \to \infty} \mathbb{P} \left[ \mathbb{P}_L [M_n - m_{n-l} \leq y] \leq e^{-c_\varepsilon' l} + (1 - C_\varepsilon |y + c_\varepsilon' |^{-c_\varepsilon''} )^{2l} \right] \geq 1 - \varepsilon,$$

and by choosing $l, y$ right, but importantly independent of $n$, we get that $e^{-c_\varepsilon' l} + (1 - C_\varepsilon |y + c_\varepsilon' |^{-c_\varepsilon''} )^{2l} \leq \varepsilon$, which allows us to conclude Theorem 4.15. In the final calculation we can’t use $m_{n-l}$, but will instead need to use time-shifted objects, also it will be necessary to prove, that $m_{n-l} - m_n$ isn’t too big, to actually conclude Theorem 4.15. All of this is done in Section 7.

4.4 **Proof of Theorem 4.1 assuming Theorems 4.1 and 4.15**

Let $\varepsilon > 0$ and $y \geq 0$. Set

$$A_{n,\varepsilon}(y) := \{ \mathbb{P}_L [\|M_n - m_n\| \geq y] \leq \varepsilon/2 \}.$$ 

By applying Theorems 4.1 and 4.15 there is a $y_\varepsilon \in \mathbb{R}$ such that

$$\limsup_{n \to \infty} \mathbb{P} [A_{n,\varepsilon}(y_\varepsilon)'] \leq \varepsilon/2. \tag{4.15}$$

Furthermore, we have that

$$\mathbb{P} [\|M_n - m_n\| \geq y_\varepsilon] = \mathbb{E} [\mathbb{P}_L [\|M_n - m_n\| \geq y_\varepsilon] 1_{A_{n,\varepsilon}(y_\varepsilon)}] + \mathbb{E} [\mathbb{P}_L [\|M_n - m_n\| \geq y_\varepsilon] 1_{A_{n,\varepsilon}(y_\varepsilon)'}]$$

$$\leq \varepsilon/2 + \mathbb{P} [A_{n,\varepsilon}(y_\varepsilon)'] \tag{4.16}$$

Combining (4.15) and (4.16) yields that

$$\limsup_{n \to \infty} \mathbb{P} [\|M_n - m_n\| \geq y\varepsilon] \leq \varepsilon,$$

which gives that $(M_n - m_n)_{n \geq 0}$ is tight w.r.t. $\mathbb{P}$.

5 **Details for the Proof of Theorem 4.1 – Upper Bound on the Right Tail of $M_n - m_n$**

5.1 **Proof of Proposition 4.6**

We recall Definition 4.1. Proposition 4.6 follows directly from the following two lemmata.
Lemma 5.1. For all $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that
\[
\liminf_{n \to \infty} \mathbb{P} \left[ \bigcap_{y \in [-\log(n)^2, \xi_0]} \mathbb{P}_L \left[ \tau(y) \leq n - \log(n)^7 \right] \leq C_\varepsilon e^{\vartheta^*(y+1)} \right] \geq 1 - \varepsilon.
\]

Lemma 5.2. For all $\varepsilon > 0$, there are $C_\varepsilon, c_\varepsilon > 0$ such that
\[
\liminf_{n \to \infty} \mathbb{P} \left[ \bigcap_{y \in [-\log(n)^2, \xi_0]} \mathbb{P}_L \left[ n - \log(n)^7 < \tau(y) < n \right] \leq C_\varepsilon e^{\vartheta^*(y)c_\varepsilon} \right] \geq 1 - \varepsilon,
\]

Proof of Lemma 5.1 For all $y < 0$ and all $k \leq n - \log(n)^7$ we have that $\mathbb{P}$-a.s.
\[
\mathbb{P}_L[\tau(y) = k] \leq \mathbb{P}_L \left[ \sum_{u: |u| = k} 1\{V(u -(\vartheta^*)^{-1}K_n + m_{n,h/2}(k) + y + 1 \geq 0 \} \right]
\leq \mathbb{E}_L \left[ e^{-\vartheta^*T_k; T_k + m_{n,h/2}(k) + y + 1 \geq 0} \right]
\leq e^{\vartheta^*(y+1)c_{h_n}(k)/2}p_n^{-\frac{\varepsilon}{C}}.
\]  

Combining Lemma 2.2 with (5.1) gives that
\[
1 - \varepsilon \leq \liminf_{n \to \infty} \mathbb{P} \left[ \bigcap_{y \in [-\log(n)^2, \xi_0]} \mathbb{P}_L[\tau(y) \leq n - \log(n)^7] \leq e^{\vartheta^*(y+1)} \sum_{k=0}^{n-\log(n)^7} e^{h_n(k)/2} C_n^{\frac{\varepsilon}{C}} \right].
\]  

From $\sup_{n \in \mathbb{N}} \sum_{k=1}^{n-\log(n)^7} e^{h_n(k)/2} C_n^{\frac{\varepsilon}{C}} < \infty$ and (5.2) we can conclude the claim of the lemma.

The rest of this section is devoted to the proof of Lemma 5.2. First we need an additional definition, after this we will give a brief overview over the structure of the proof of Lemma 5.2. We recall the convention that w.r.t $\mathbb{P}_L$ we write $T_n = \sum_{j=1}^{k+n} X_j - (\vartheta^*)^{-1} \kappa_j(\vartheta^*)$.

Definition 5.1. For $n \in \mathbb{N}$ and $k \leq n$ set
\[
N_k(y) := \left\{ u \in \mathcal{T}_n : |u| = n - k : \forall j \leq n - k \right\} V(u_j - (\vartheta^*)^{-1} K_j + m_{n,h/2}(j) + y + 1 \geq 0, \right\}
\]  

\[ V(u_n) - (\vartheta^*)^{-1} K_n - m_{n,h/2}(n - k) + y + 1 \geq 0 \right\}
\]  

\[ \tilde{T}_j(y) := T_j + m_{n,h/2}(j) + y + 1,
\]  

\[ q_k(y) := \mathbb{P}_L \left[ \forall j \leq n - k \right] \tilde{T}_j(y) \leq 0, \tilde{T}_j(y) \geq 0 \right].
\]  

The dependence on $n$ is omitted from the notation.

For $l \in \mathbb{N}_0, x < 0$ define
\[
\tilde{q}_k(l) := \mathbb{P}_L \left[ \mathcal{B}_{\vartheta^*(y+1),J_{\vartheta^*}}^{(0,..,n-k-1),m_{n,h/2}}(T_l) \right],
\]  

\[ q_{k,\text{end}}(x) := \mathbb{P}_L^{n-k-1} \left[ \mathcal{B}_{\vartheta^*(y+1),J_{\vartheta^*}}^{(0,..,n-k-1),m_{n,h/2}(n-k+1)}(T_l) \right].
\]  

The dependence on $n$ and $y$ is omitted from the notation, we emphasize, that $q_{k,\text{end}}(x)$ doesn’t depend on $y$.

A direct consequence of Definition 5.1 and 5.4 is that $\mathbb{P}_L[\tau(y) = n - k] \leq \mathbb{P}_L[|N_k(y)| \geq 1]$ and the Markov inequality together with the many-to-one Lemma 2.1 yield that
\[
\mathbb{P}_L[\tau(y) = n - k] \leq p_n\frac{C_{n-k}^{\frac{\varepsilon}{C}}}{n} e^{\vartheta^*(y+1)+\vartheta^*h_n(k)/2} q_k(y).
\]

The idea is to decompose $q_k(y)$ according to the position of $\tilde{T}_{n-k-1}$. Lemmata 5.3 and 5.4 will establish, that
\[
q_k(y) \leq \sum_{l=0}^{\infty} e^{-c_l^2} \tilde{q}_k(l),
\]  

i.e. that $\tilde{T}_{n-l} \approx l$ costs an $e^{-c_l^2}$ factor. The term $\tilde{q}_k(l)$ then needs to be compared to $p_n^{-l}$, recall 2.1, for this we want to use the barrier computation of Proposition 4.4. To do this we need to compute an upper bound on $\tilde{q}_k(l)(p_n^{-l}(y)^{-1})^{-1}$, recall 4.4. This is done in Corollary 5.5 which is proved using Lemma 2.3. Corollary 5.6 does handle the summation over $l$ in (5.8). Finally, we need one more barrier computation, which is stated in Lemma 5.7 and proven in Section 12.
Lemma 5.3. For all $\varepsilon > 0$ there are $c, C > 0$ such that for $x \geq 0$,
\[ P_L[(T_{n-k} - T_{n-k-1}) \geq x] \leq C e^{-cx^2}. \]

Proof. We have that
\[
P_L[(T_{n-k} - T_{n-k-1}) \geq x] = P_L \left[ X_{n-k} - (\vartheta^*)^{-1} \log(L_{n-k}) - \left( \frac{\vartheta^*}{2} \right) \geq x \right]
\leq \frac{e^{-\frac{(x + \vartheta^*)^{-1} \log(L_{n-k}) - \frac{\vartheta^*}{2}}{2^2}}}{C e^{-cx^2}}
\]
for some $c, C > 0$, since $\log(L_{n-k}) \geq 0$. The second to last step has used that $X_1 - (\vartheta^*)^2/2$ is $\mathcal{N}((\vartheta^*)^2/2, 1)$ distributed.

Lemma 5.4. For all $\varepsilon > 0$, there is an $c_{\varepsilon,1} > 0$ such that
\[
\liminf_{n \to \infty} \mathbb{P} \left\{ \forall y \in [-\log(n)^2, \xi_0] \cap \mathbb{N} \left[ q_k(y) \leq \sum_{l=0}^{\infty} \tilde{q}_k(l) e^{-c_{\varepsilon,1} l^2} \right] \right\} \geq 1 - \varepsilon.
\]

Proof. This proof is mainly partitioning with respect to the value of $\tilde{T}_{n,k-1}$ and using that by Lemma 5.3, having $\tilde{T}_{n,k-1} \approx -l$ and $\tilde{T}_{n,k} \geq 0$ costs roughly $e^{-cl^2}$, where considering $\tilde{T}_k$ instead of $T_k$ will only change the $c$.

We have that $\mathbb{P}$-a.s. for all $y \in [-\log(n)^2, \xi_0] \cap \mathbb{N}$
\[
q_k(y) = \sum_{l=0}^{\infty} \mathbb{P}_L \left[ \forall j \leq n-k-1: \tilde{T}_j \leq 0, \tilde{T}_{n-k} \geq 0, \tilde{T}_{n-k-1} \in [-l - 1, -l] \right]
\leq \sum_{l=0}^{\infty} \mathbb{P}_L \left[ \forall j \leq n-k-1: \tilde{T}_j \leq 0, \tilde{T}_{n-k} \geq 0, \tilde{T}_{n-k} - \tilde{T}_{n-k-1} \geq 1 \right]
\leq \sum_{l=0}^{\infty} \tilde{q}_k(l) e^{-c(l\log(p)/\log(\vartheta^*) + 1/2)(l\log(p)/\log(\vartheta^*) + 1)}.
\]

The statement of the lemma follows from (5.9) and Lemma 2.2.

We recall the definitions (4.4), (5.5), and (5.6).

Corollary 5.5. For $l \leq |\xi_0|$, we have that
\[
\tilde{q}_k(l) \inf_{x \in [-|\xi_0| - 1, -1]} q_{k,\text{end}}(x) \leq p_n^{-\xi_0}(y),
\]

(5.10)

For $l \geq |\xi_0|$ we have that
\[
\tilde{q}_k(l) \inf_{x \in [\xi_0 - 1, -1]} q_{k,\text{end}}(x) \leq p_n^{l-1}(y),
\]

(5.11)

Proof. First, let $l \leq |\xi_0|$. By applying (2.2) from Lemma 2.3, for $f = m_n, z_1 = y, z_2 = \xi_0, I = \{0, \ldots, n\}$, $t_0 = n - k - 1$ and $x_0 = -(m_n, n - k - 1) - l$ we get,

\[ p_n^{-\xi_0}(y) \geq \tilde{q}_k(l) \inf_{x \in \mathbb{L}_{x_0}} P \left[ B_{\tilde{T} - T_{n-k-1}}(m_n, n - 1) \right] \]

(5.12)

where the last step follows from (5.6) and a time shift. Since $l \leq |\xi_0|$ equation (5.12) implies (5.10).

The case $l \geq |\xi_0|$, i.e. proving (5.11), is analogous, but we take $z_2 = -l$ instead, thus we get
\[
p_n^{-l-1}(y) \geq \tilde{q}_k(l) \inf_{x \in [-l - 1, -l]} P \left[ B_{\tilde{T} - T_{n-k-1}}(m_n, n - 1) \right]
\]

and observing that $[\xi_0 - 2, \xi_0 - 1] \subseteq [\xi_0 - 2, -1]$ finishes the proof.

Corollary 5.6. For all $\varepsilon > 0$, there are $C_{\varepsilon}, c_{\varepsilon} > 0$ such that
\[
\liminf_{n \to \infty} \mathbb{P} \left[ \forall y \in [-\log(n)^2, \xi_0] \cap \mathbb{N} \left[ P_n^{-1} q_k(y) \leq \left( \inf_{x \in [-|\xi_0| - 1, -1]} q_{k,\text{end}}(x) \right)^{-1} C_{\varepsilon} |y|^{c_{\varepsilon}} \right] \right] \geq 1 - \varepsilon.
\]
Proof. With \( c_{\varepsilon, 1} \) from Lemma 5.4 we have that
\[
\liminf_{n \to \infty} \mathbb{P} \left[ \forall y \in [-\log(n)^2, \xi_0] \cap \mathbb{Z} \quad p_n^{-1} q_k(y) \leq \sum_{l=0}^{\infty} p_n^{-1} \tilde{q}_k(l) C e^{-c_{\varepsilon, 1} l^2} \right] \geq 1 - \varepsilon. \tag{5.13}
\]

We first handle the summation from \( l = 0 \) to \( |\xi_0| \) in (5.13). By (5.10) we have that
\[
\sum_{l=0}^{|\xi_0|} p_n^{-1} \tilde{q}_k(l) C e^{-c_{\varepsilon, 1} l^2} \leq \left( \inf_{x \in |\xi_0-2|, -1} q_{k, \text{end}}(x) \right)^{-1} \sum_{l=0}^{|\xi_0|} p_n^{-\xi_0}(y) p_n^{-1} C e^{-c_{\varepsilon, 1} l^2}. \tag{5.14}
\]

By applying Proposition 4.4 and using that \( e^{-c_{\varepsilon, 1} l^2} \) is summable, (5.14) implies that there are \( c_{\varepsilon}, C_{\varepsilon} \) such that
\[
1 - \varepsilon/3 \leq \liminf_{n \to \infty} \mathbb{P} \left[ \forall y \in [-\log(n)^2, \xi_0] \cap \mathbb{Z} \quad \sum_{l=0}^{\sqrt{\log(n)} / c_{\varepsilon, 1}} p_n^{-1} \tilde{q}_k(l) C e^{-c_{\varepsilon, 1} l^2} \leq \left( \inf_{x \in |\xi_0-2|, -1} q_{k, \text{end}}(x) \right)^{-1} C_{\varepsilon} |y|^{c_{\varepsilon}} \right] \tag{5.15}
\]

Now we handle the summation from \( |\xi_0| \) to \( \sqrt{\log(n)} / c_{\varepsilon, 1} \) in (5.13), where \( \lambda > 0 \) is such that \( p_n^{-1} n^{-\lambda} \) converges to zero \( \mathbb{P}\text{-a.s.} \) according to Lemma 2.2.

A completely parallel argument to the proof of (5.15), but using the summability of \( e^{-c_{\varepsilon, 1} l^2} \), yields that there are \( c_{\varepsilon}, C_{\varepsilon} > 0 \) such that
\[
1 - \varepsilon/3 \leq \liminf_{n \to \infty} \mathbb{P} \left[ \forall y \in [-\log(n)^2, \xi_0] \cap \mathbb{Z} \quad \sum_{l=|\xi_0|}^{\sqrt{\log(n)} / c_{\varepsilon, 1}} p_n^{-1} \tilde{q}_k(l) C e^{-c_{\varepsilon, 1} l^2} \leq \left( \inf_{x \in |\xi_0-2|, -1} q_{k, \text{end}}(x) \right)^{-1} C_{\varepsilon} |y|^{c_{\varepsilon}} \right] \tag{5.16}
\]

Finally, we handle \( l > \sqrt{\log(n)} / c_{\varepsilon, 1} \) in (5.13). Since \( \tilde{q}_k(l) \) is a probability it is smaller than 1, which yields that
\[
\sum_{l=\sqrt{\log(n)} / c_{\varepsilon, 1}}^{\infty} p_n^{-1} \tilde{q}_k(l) C e^{-c_{\varepsilon, 1} l^2} \leq C_{\varepsilon} p_n^{-1} e^{-\lambda \log(n)} \tag{5.17}
\]

Since \( p_n^{-1} e^{-\lambda \log(n)} \to 0 \) in probability, equations (5.13), (5.15), (5.16) and (5.17) finish the proof. \( \square \)

We need one more Lemma for the proof of Lemma 5.2; the proof of this lemma involves barrier computations and thus is done in Section 12. We recall the Definition 5.6 of \( q_{k, \text{end}}(x) \).

**Lemma 5.7.** For all \( \varepsilon > 0 \), there is a \( C_{\varepsilon} > 0 \) such that
\[
\liminf_{n \to \infty} \mathbb{P} \left[ \left( \sum_{k=1}^{[\log(n)^7]} e^{\theta h_n(k)/2} \left( \inf_{x \in |\xi_0-2|, -1} q_{k, \text{end}}(x) \right)^{-1} C_{\varepsilon} \right)^{-1} C_{\varepsilon} \leq 1 - \varepsilon. \right. \tag{5.18}
\]

We have now collected all ingredients necessary to finish the proof of Lemma 5.2.

**Proof of Lemma 5.2 assuming Lemma 5.7** By (5.7) we have that \( \mathbb{P}\text{-a.s.} \) for all \( y \in [-\log(n)^2, \xi_0] \cap \mathbb{Z} \)
\[
\mathbb{P} \left[ n - \log(n)^7 < \tau(y) < n \right] \leq \sum_{k=n-\log(n)^7}^{n} e^{\theta (y+1)} p_n^{-1} q_k(y) e^{\theta h_n(k)/2}. \tag{5.18}
\]

Combining Corollary 5.6 and Lemma 5.7 implies that there are \( C_{\varepsilon}, c_{\varepsilon} > 0 \) such that
\[
1 - \varepsilon \leq \mathbb{P} \left[ \forall y \in [-\log(n)^2, \xi_0] \cap \mathbb{Z} \quad \sum_{k=n-\log(n)^7}^{n} p_n^{-1} q_k(y) e^{\theta h_n(k)/2} \leq C_{\varepsilon} |y|^{c_{\varepsilon}} \right] \tag{5.19}
\]

Equations (5.18) and (5.19) together imply the lemma. \( \square \)
6 Details for Theorem 4.7 – Lower Bound on the Right Tail of $M_n - m_n$

6.1 Proof of Proposition 4.9

6.1.1 Proof of Proposition 4.11

In this section fix $n \in \mathbb{N}$, $k \in \{0, \ldots, n - 1\}$ and $y \leq 0$. We recall Definitions 4.2 and 4.4. Furthermore, define

\[
\tilde{V}(u, j) := V(u_j) - K_j/\vartheta^* + m_{n,h,y}(j),
\]

\[
A_k(u) := \{v_j \leq k \tilde{V}(u, j) \leq 0\},
\]

\[
\bar{T}_j := T_j + m_{n,h,y}(j),
\]

\[
\mathcal{F}_k := \sigma(u, V(u) : |u| \leq k).
\]

We will use two results on $f_k$ and $q_k$ in the proof of Proposition 4.11. First as a direct consequence of the many-to-one Lemma 2.1 applied to $\mathbb{P}_L^k$ we have that for all $x \in \mathbb{R}$

\[
f_k(x) \leq ce^{\vartheta^*x+\vartheta^*y-\vartheta^*c_0-K_k p_n^{-1} q_k(x)} \quad \mathbb{P}\text{-a.s.} \quad (6.1)
\]

Furthermore, by conditioning on $\mathcal{F}_k$ it follows directly from the definitions, that for all $y < 0$

\[
p_n^{\sim c_0}(y) = \mathbb{E}_L \left[ q_k((T_k + K_k)/\vartheta^*); \forall v_j \leq k \bar{T}_j \leq 0 \right] \quad \mathbb{P}\text{-a.s.} \quad (6.2)
\]

**Proof of Proposition 4.11** We first use the tree structure by conditioning on $\mathcal{F}_k$, which gives that

\[
\mathbb{E}_L[A_k(y) | \mathcal{F}_k] \leq c \sum_{|u| = k} 1_{A_k(u)} f_k(V(u))^2.
\]

The $1_{A_k(u)}$ just means that $u$ stays below the barrier up until time $k$, and the $f_k(V(u))^2$ is an upper bound for the number of pairs of particles, which descend from $u$ for which both particles stay below the barrier in $\{k, \ldots, n\}$ and end up at the right place. In this calculation we used that given $V(u)$ and $\mathcal{L}$ the events that two different particles behave this way are independent.

Applying (6.1) to bound $f_k(V(u))$ and taking expectation gives that $\mathbb{P}$-a.s.

\[
\mathbb{E}_L[A_k(y)] \leq c \Gamma_k \mathbb{E}_L \left[ \sum_{|u| = k} 1_{A_k(u)} \left( e^{\vartheta^*V(u)-K_k q_k(V(u))} \right)^2 \right],
\]

where $\Gamma_k := e^{2\vartheta^*y-2\vartheta^*c_0} p_n^{-2}$.

If we wouldn’t have the square, applying the many-to-one Lemma 2.1 and 6.2 would allow us to bound $\mathbb{E}_L[A_k]$ by $c \Gamma_k p_n^{\sim c_0}(y)$; thus we want to pull the factor $e^{\vartheta^*V(u)-K_k q_k(V(u))}$ out of the expectation. For this we bound it by $\max_{x \geq 0} e^{-\vartheta^*x-\vartheta^*m_{n,h,y}(k)} q_k(K_k/\vartheta^* - m_{n,h,y}(k) - x)$, which gives that $\mathbb{P}$-a.s.

\[
\mathbb{E}_L[A_k(y)] \leq c \Gamma_k e^{-\vartheta^*y-\vartheta^*m_{n,h,y}(k)} p_n^{\frac{k}{2}} \max_{x \geq 0} e^{-\vartheta^*x} q_k(K_k/\vartheta^* - m_{n,h,y}(k) - x)
\]

\[
\mathbb{E} \left[ \sum_{|u| = k} 1_{A_k(u)} e^{\vartheta^*V(u)-K_k q_k(V(u))} \right].
\]

Applying Lemma 2.1 then gives that $\mathbb{P}$-a.s.

\[
\mathbb{E}_L[A_k(y)] \leq c \Gamma_k e^{-\vartheta^*y-\vartheta^*m_{n,h,y}(k)} p_n^{\frac{k}{2}} \mathbb{E}_L \left[ q_k((T_k + K_k)/\vartheta^*); \forall v_j \leq k \bar{T}_j \leq 0 \right]
\]

\[
\max_{x \leq 0} e^{-\vartheta^*x} q_k(K_k/\vartheta^* - m_{n,h,y}(k) - x),
\]

Applying (6.2) finishes the proof of Proposition 4.11. \qed

6.1.2 Proof of Lemma 4.14

We recall (2.1) and Definitions 4.2 and 4.4. In order to derive an upper bound on $\sum_{k=0}^{log(n)} \mathbb{E}_L[A_k(y)]$ we will need an upper bound on $p_n^{\sim c_0} \max_{x \geq 0} e^{-\vartheta^*x} q_k(K_k/\vartheta^* - m_{n,h,y}(k) - x)$. For this we need one more definition.
Definition 6.1. Set
\[ p_{\text{start},k} := \inf_{x \in [\xi_0, 1.0]} \mathbb{P}_\mathcal{L}\left[ B_{(0, \ldots, k), m_{n,h}}(T) \right]. \]

The proof of Lemma 4.14 is split into two further lemmata. The proof of Lemma 6.2 involves barrier computations and thus is done in Section 12.

Lemma 6.1. For all \( \varepsilon > 0 \), there is a \( C_\varepsilon > 0 \) such that
\[ \liminf_{n \to \infty} \mathbb{P}\left[ \bigcap_{k \leq \log(n)^7} \left\{ p_n^{-1} \max_{x \geq 0} e^{-\theta^* x} q_k(K_k/\theta^* - m_{n,h,y}(k) - x) \leq C_\varepsilon (p_{\text{start},k})^{-1} \right\} \right] \geq 1 - \varepsilon. \]

Lemma 6.2. For all \( \varepsilon > 0 \), there is a \( C_\varepsilon > 0 \) such that
\[ \liminf_{n \to \infty} \mathbb{P}\left[ \log(n)^7 \sum_{k=0}^{\log(n)^7} e^{-\theta^* h_n(k)} (p_{\text{start},k})^{-1} \leq C_\varepsilon \right] \geq 1 - \varepsilon. \]

Proof of Lemma 4.14 assuming Lemma 6.1, 6.2. Plugging Lemma 6.1 and Lemma 6.2 into Proposition 4.11 yields that there exists a \( C_\varepsilon > 0 \) such that
\[ 1 - \varepsilon/2 \leq \liminf_{n \to \infty} \mathbb{P}\left[ \log(n)^7 \left( \sum_{k=0}^{\log(n)^7} \mathbb{E}_\mathcal{L}[A_k(y)] \right) \leq C_\varepsilon e^{\theta^* y p_n^{-1} p_{\text{start},k}^{-1}} \right], \quad (6.3) \]
where \( e^{-2\theta^* \xi_0} \) has been pulled into the \( C_\varepsilon \), which is fine, since \( \xi_0 \) is a (negative) constant.

Applying Proposition 4.12 yields that there are \( C_{\varepsilon_1}, C_{\varepsilon_2} > 0 \) such that
\[ 1 - \varepsilon/2 \leq \liminf_{n \to \infty} \mathbb{P}\left[ p_n^{-1} p_{\text{start},k}^{-1} \leq C_{\varepsilon_2} |y|^{\alpha} \right]. \quad (6.4) \]

Combining (6.3) and (6.4) yields the claim of Lemma 4.14.

To prepare the proof of Lemma 6.1, we need a statement parallel to Corollary 5.5. The proof is analogous, but using (2.3) instead of (2.2). We recall (4.6) and (4.10).

Corollary 6.3. Fix \( k \leq n \). For \( x > |\xi_0| \) we have that
\[ q_k(K_k/\theta^* - m_{n,h,y}(k) - x) \leq \left( \sum_{j=0}^{\log(n)^7} p_n^{-\xi_0 - j} (-[x]) \right) (p_{\text{start},k})^{-1}. \quad (6.5) \]

For \( x \leq |\xi_0| \), we have that
\[ q_k(K_k/\theta^* - m_{n,h,y}(k) - x) \leq \left( \sum_{j=0}^{\log(n)^7} p_n^{-\xi_0 - j} (\xi_0) \right) (p_{\text{start},k})^{-1}. \quad (6.6) \]

Proof. We begin with the proof of (6.5). We apply (2.3) for \( f = m_{n,h}, z_1 = -[x], z_2 = \xi_0, I = \{0, \ldots, n\}, t_0 = k \) and \( x_0 = -m_{n,h}(k) - x \) and get that
\[ \left( \sum_{j=0}^{\log(n)^7} p_n^{-\xi_0 - j} (-[x]) \right) \geq \mathbb{P}_\mathcal{L}\left[ B_{[x], J_{x} + m_{n,h}(k)}(T) \right] \mathbb{P}_\mathcal{L}\left[ B_{[0, \ldots, k], m_{n,h}}(T - T_k) \right] \]
\[ = \mathbb{P}_\mathcal{L}\left[ B_{[x], J_{x}}(T) \right] q_k(K_k/\theta^* - m_{n,h,y}(k) - x) \]
\[ \geq \mathbb{P}_\mathcal{L}\left[ B_{[\xi_0, \xi_0 + [x] - x]}(T) \right] q_k(K_k/\theta^* - m_{n,h,y}(k) - x), \]
where the last step used that \( -[x] \leq \xi_0 \). Since \( \xi_0 + [x] - x \in [\xi_0 - 1.0] \) this allows to conclude (6.5).

In the case \( x \leq |\xi_0| \) we take \( z_1 = \xi_0 \) instead. The application of (2.3) yields that
\[ \left( \sum_{j=0}^{\log(n)^7} p_n^{-\xi_0 - j} (\xi_0) \right) \geq \mathbb{P}_\mathcal{L}\left[ B_{[\xi_0, \xi_0 + [x] - x]}(T) \right] q_k(K_k/\theta^* - m_{n,h,y}(k) - x) \]
and since \( -x \in [\xi_0 - 1.0] \) this allows to conclude (6.6).
These are the by
Applying Proposition 4.12 yields that there is a $C > 0$ such that

$$\lim_{n \to \infty} \mathbb{P} \left[ \bigcap_{k \leq \log(n)^7} \left\{ \pi_{j,k} \leq C \{p_{\text{start},k}\}^{-1} \right\} \right] \geq 1 - \varepsilon/3. \tag{6.7}$$

The case $j = 1$: By applying (6.6) we have that $\mathbb{P}$-a.s. for all $k \leq \log(n)^7$

$$\pi_{1,k} \leq p_n^{-1} \left( \sum_{j=0}^{1} p_n^{-\xi_n - j} (\xi_n - 1) \right) (p_{\text{start},k})^{-1} \tag{6.8}$$

Applying Proposition 4.12 yields that there is a $C > 0$ such that

$$1 - \varepsilon/3 \leq \liminf_{n \to \infty} \mathbb{P} \left[ p_n^{-1} \left( \sum_{j=0}^{1} p_n^{-\xi_n - j} (\xi_n - 1) \right) \leq C \right] \tag{6.9}$$

equations (6.8) and (6.9) yield (6.7) for $j = 1$.

The case $j = 2$: By applying (6.5) we have that $\mathbb{P}$-a.s. for all $k \leq \log(n)^7$

$$\pi_{2,k} \leq \max_{\varepsilon_0 \leq x \leq \log(n)^2} e^{-\theta^* x} \left( \sum_{j=0}^{1} p_n^{-\xi_n - j} (\lfloor x \rfloor) \right) (p_{\text{start},k})^{-1}. \tag{6.10}$$

Applying Proposition 4.12 yields that there are $c_\varepsilon, C_\varepsilon > 0$ such that

$$1 - \varepsilon/3 \leq \liminf_{n \to \infty} \mathbb{P} \left[ \forall x \in [-\log(n), \xi_0] \varepsilon \geq p_n^{-1} \left( \sum_{j=0}^{1} p_n^{-\xi_n - j} (x) \right) \leq C \varepsilon \right] \tag{6.11}$$

and since $\max_{x \geq 0} e^{-\theta^* x} [x]^{c_\varepsilon} \leq C_\varepsilon < \infty$ equations (6.10) and (6.11) yield (6.7) for $j = 2$.

The case $j = 3$: By bounding $q_k(x) \leq 1$ we have that $\pi_{3,k} \leq p_n^{-1} e^{-\log(n)^7}$, which converges to 0 $\mathbb{P}$-a.s. by Lemma 2.2. This gives (6.7) for $j = 3$.

Thus we have proved (6.7) for $j \in \{1, 2, 3\}$ which finishes the proof.

**7 Proof of Theorem 4.15 - Cutting the Tree**

We recall Definitions 1.1 and 2.2

**Proof of Lemma 4.16** Since our jumps are symmetric, we have that $\min_{u \in [a,n]} V(u) = \left\lfloor -M_n \right\rfloor$. Thus by Lemma 1.2 we have that $\mathbb{P}$-a.s.

$$\mathbb{P}_L \left[ M_n < -2K_n/\theta^* \right] \leq e^{-K_n}. \tag{6.12}$$

It now suffices to prove, that there are $c' \varepsilon, c'' \varepsilon$ such that for all $n \in \mathbb{N}$

$$\mathbb{P}[K_n \geq c' \varepsilon n] \geq 1 - \varepsilon,$$

$$\mathbb{P}[K_n \leq c'' \varepsilon n] \geq 1 - \varepsilon.$$

This is true since $K_n/n$ converges $\mathbb{P}$-a.s. because the $\kappa_k(\theta^*)$ are i.i.d. with finite variance.

**Definition 7.1.** For $n, l \in \mathbb{N}$, set

$$K_n^{(l)} := \sum_{j=1}^{n+l} \kappa_j(\theta^*),$$

$$p_n^{(l)} := \mathbb{P}_L \left[ \forall s \leq n, \xi_0 + T_{n} \leq 0, \xi_0 + T_{n} \in [\xi_0 - 1, \xi_0] \right],$$

$$m_n^{(l)} := K_n^{(l)}/\theta^* - \log(p_n^{(l)})/\theta^*.$$  

These are the by $l$ time-shifted analogues of $K_n, p_n$ and $m_n$. 

16
We note that the following version of Theorem 4.7 holds, since we only make a statement about the distribution of $P_{L}[\ldots]$, which is invariant under time-shifts.

**Corollary 7.1.** For all $\varepsilon > 0$ there are $c_\varepsilon, C_\varepsilon > 0$ such that for $y \in [-\log(n)^2, \xi_0]$,

$$
\liminf_{n \to \infty} P \left[ P_{L} [M_{n-l} - m_{n-l}^{(l)} \geq \xi_0 - y - 1] \geq e^{\varepsilon}C_\varepsilon|y|^{-c_\varepsilon} \right] \geq 1 - \varepsilon.
$$

**Lemma 7.2.** For all $\varepsilon > 0$ there are $c_\varepsilon, C_\varepsilon, c''_\varepsilon, C''_\varepsilon > 0$ such that for $l \in \mathbb{N}$ and $y \in [-c''_\varepsilon l + 1, 0]$,

$$
\liminf_{n \to \infty} P \left[ P_{L} [M_{n} - m_{n-l}^{(l)} \leq y] \leq e^{-c''_\varepsilon l + \left(1 - C_\varepsilon|\xi_0 - c''_\varepsilon l - y - 1|^{-c_\varepsilon}e^{\varepsilon}|\xi_0 - c''_\varepsilon l - y - 1|\right)^{2l}] \right] \geq 1 - \varepsilon.
$$

**Proof.** We have that $P$-a.s.

$$
P L \left[ M_{n} - m_{n-l}^{(l)} \leq y \right] \leq P L \left[ M_{n} \leq -c''_\varepsilon l, \forall v : |v| = l \max_{u : |u| = n, u_i = v} V(u) - m_{n-l}^{(l)} \leq c''_\varepsilon l + y \right]
$$

$$
= P L \left[ M_{n} \leq -c''_\varepsilon l \right] + \left(1 - P L \left[ M_{n-l} - m_{n-l}^{(l)} \leq c''_\varepsilon l + y \right] \right)^{2l}.
$$

Now applying Lemma 4.16 and Corollary 7.1 yields the claim. \qed

**Lemma 7.3.** For all $\varepsilon > 0$, there are $l_\varepsilon \in \mathbb{N}, y_\varepsilon < 0$ such that

$$
\liminf_{n \to \infty} P \left[ P_{L} [M_{n} - m_{n-l_\varepsilon}^{(l_\varepsilon)} \leq y_\varepsilon] \leq \varepsilon \right] \geq 1 - \varepsilon.
$$

**Proof.** Choosing $y = -c''_\varepsilon l + 1$ in Lemma 7.2 yields that

$$
1 - \varepsilon \leq \liminf_{n \to \infty} P \left[ P_{L} \left[ M_{n} - m_{n-l_\varepsilon}^{(l_\varepsilon)} \leq -c''_\varepsilon l + 1 \right] \leq e^{-c''_\varepsilon l + \left(1 - C_\varepsilon|\xi_0 - 2|^{-c_\varepsilon}e^{\varepsilon}|\xi_0 - 2|\right)^{2l}} \right].
$$

We can choose $l_\varepsilon$ such that $e^{-c''_\varepsilon l + \left(1 - C_\varepsilon|\xi_0 - 2|^{-c_\varepsilon}e^{\varepsilon}|\xi_0 - 2|\right)^{2l}} \leq \varepsilon$, which yields the claim. \qed

**Lemma 7.4.** For all $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that for all $l \in \mathbb{N}_{\geq 2}$,

$$
\liminf_{n \to \infty} P \left[ K_n - K_{n-l}^{(l)} \leq C_\varepsilon l \right] \geq 1 - \varepsilon,
$$

$$
\liminf_{n \to \infty} P \left[ \log(p_n) - \log(p_{n-l}) \geq -C_\varepsilon l \right] \geq 1 - \varepsilon.
$$

Furthermore, there is a $C_\varepsilon > 0$ such that for all $l \in \mathbb{N}_{\geq 2}$

$$
\liminf_{n \to \infty} P \left[ m_{n-l}^{(l)} - m_n \leq -C_\varepsilon l \right] \geq 1 - \varepsilon.
$$

**Proof.** The first statement is immediate since by definition $K_n - K_{n-l}^{(l)} = K_l$ and $K_l/l \to 0$ $P$-a.s.

For the proof of the second statement define

$$
\tilde{p}_n := P L \left[ \forall s \leq n, \xi_0 + T_s \leq 0, \xi_0 + T_n \in [\xi_0 - 2, \xi_0] \right].
$$

We can prove similarly to Proposition 4.4 that there is a $c_\varepsilon > 0$ such that

$$
\liminf_{n \to \infty} P \left[ \frac{p_n}{\tilde{p}_n} \geq \frac{C_\varepsilon}{2} \right] \geq 1 - \varepsilon / 2. \quad (7.1)
$$

Applying (2.3) yields as in Corollary 6.3 that $P$-a.s.

$$
\tilde{p}_n \geq P L \left[ B_{0, \varepsilon}^{\left[1 + \xi_0 - T_0, \xi_0\right]}(T) \right] p_{n-l}^{(l)} \varepsilon_{\text{start}}. \quad (7.2)
$$

Equations (7.1) and (7.2) imply that there is a $c_\varepsilon > 0$

$$
\liminf_{n \to \infty} P \left[ \log(p_n) - \log(p_{n-l}) \geq \log(c_\varepsilon) + \log(p_{\text{start}}) \right] \geq 1 - \varepsilon / 2. \quad (7.3)
$$
By Lemma 2.2 there is a $C > 0$ such that $\mathbb{P}$-a.s. $\limsup_{l \to \infty} \frac{|p_{n, \text{term}}(l)|}{\log(l)} \leq C$, and thus (7.3) implies that there is a $C_\epsilon > 0$ such that for $l \geq 2$

$$\liminf_{n \to \infty} \mathbb{P} \left[ \log(p_n) - \log(p_{n-l}) \geq -C_\epsilon \log(l) \right] \geq 1 - \epsilon,$$

which proves the second claim.

The third claim follows directly from the first two. 

**Proof of Theorem 4.7.** By Lemma 7.3 there are $l_\epsilon \in \mathbb{N}$, $z_\epsilon < 0$ such that

$$1 - \epsilon/2 \leq \liminf_{n \to \infty} \mathbb{P} \left[ \mathbb{P}[M_n - m_{n-l_\epsilon} \leq z_\epsilon] \leq \epsilon \right].$$

Since for all $y < 0$

$$\mathbb{P}[M_n - m_n \leq y] = \mathbb{P}[M_n - m_{n-l_\epsilon} \leq m_n - m_{n-l_\epsilon} + y],$$

Lemma 7.3 implies that there is a $C_\epsilon > 0$ such that for all $y < 0$

$$1 - \epsilon/2 \leq \liminf_{n \to \infty} \mathbb{P} \left[ \mathbb{P}[M_n - m_n \leq y] \leq \mathbb{P}[M_n - m_{n-l_\epsilon} \leq y + C_\epsilon l_\epsilon] \right].$$

Taking $y_\epsilon := z_\epsilon - C_\epsilon l_\epsilon$ in (7.5) and combining it with (7.4) yields the claim of the theorem.

---

### 8 Shift of the Starting Point – Preparation for Section 12

In this section let $B$ be a Brownian motion starting at $x$ with regard to $\mathbb{P}_x$. For $t \geq 0$ let $W(t) : [0, t] \to \mathbb{R}$ be continuous with $W_0 = 0$ and let $h_t : [0, t] \to \mathbb{R}$ be a continuous function with $h_t(0) = h_t(t) = 0$. We will use $W$ and $h$ as a shorthand for $(W(t))_{t \geq 0}$, $(h_t)_{t \geq 0}$, we will suppress the upper index of $W(t)$ and instead write $W_s$. Additional assumption on $h$ and $W$ are stated in Section 8.1 together with further definitions, in Section 11 we check that the assumptions hold for the specific $W$ we use in the first half of the paper. In Section 8.2 we will state the results of Section 8. Sections 8.3, 8.4, 8.5 contain the proofs of Theorems 8.1, 8.2 and 8.5 respectively.

#### 8.1 Definitions and Assumptions

We recall Definition 2.1

**Definition 8.1.** For $y_0, y, x \leq 0$, define

$$p_{t, h}^{(y_0)}(y) := \mathbb{P}_y \left[ B_{[0, t], h(t)}(x - y_0) \sim W(\mathbb{B}) \right],$$

$$p_{t, h}^{(y_0)}(y, x) := \mathbb{P}_y \left[ B_{[0, t], h(t)}(x - y_0) \sim W(\mathbb{B}) \right].$$

For $\mu \in \mathbb{R}$, $\sigma^2 > 0$ write

$$g_{\mu, \sigma^2}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for the Gaussian density function.

Define

$$1 \vee \sup_{s \in [c, t]} \frac{|W_s|}{(\sqrt{s} \cdot \log(s))} := C_{\log}(W, t) := C_{\log},$$

$$1 \vee \sup_{t \in \mathbb{R}} \sup_{s \in [0, t]} \frac{|h_s|}{1 + s} := C_1(h) := C_1,$$

and assume that both quantities are finite. Furthermore, set

$$\gamma := \gamma(W, h, t) := \inf \left\{ \gamma \geq 0 : \forall 2 \leq s \leq t \mathbb{P} \left[ B_{[0, s], h(s)}(x - y_0) \sim W(\mathbb{B}) \right] \geq s^{-\gamma} \right\}.$$

We assume that $\gamma > 0$. Furthermore, define

$$e^{-128 - 16C_1^2 - 80C_1^2 - 134C_1 - 96C_{\log} - 32C_1C_{\log} = C_2(W, h, t) = C_2.}$$

The constants in the definition of $C_2$ aren’t important, what matters is, that $C_2$ is a continuous function of $C_{\log}$ and $C_1$.

Finally, for $\lambda > 0$ arbitrary but fixed, set

$$c_1(\lambda) := \sqrt{8(\lambda + 1)},$$

$$C_3 := C_3(W, h, \lambda, t) := \max\{48, 128C_1^2, 64\sqrt{3}C_1, (32\gamma + 128)^4, 5c_1, 96C_{\log}\}.\hspace{1cm}(8.7)$$

---

1. Actually we have shifted the starting point by one compared to the precise statement of Lemma 2.2; this will not make a difference in the proof of the lemma we give in Section 11.
8.2 Summary of Results

Theorem 8.1. Let \(-2t^2 \leq y \leq x \leq -e, y_0 \in [-2 \log(t)^2, 0]\) and \(t \geq 64\). We have that

\[
\frac{p_t^{(y_0)}(x, y^2)}{p_t^{(y_0)}(x)} \in \left[1, C_2^{-1}|y|^{2\gamma}\right].
\] (8.8)

Theorem 8.2. For \(t > 0, y \in [-2 \log(t)^2, -e], x \in \mathbb{R}, y_0 \in (-\infty, 0]\), we have that

\[
\frac{p_t^{(y_0)}(y, y^2)}{p_t^{(y_0)}(x, y^2)} \geq ce^{-2C_1 e^{-ct_{log(y)^2}}}. 
\]

Theorem 8.3. For \(y \in [-2 \log(t)^2, -e], x \in \mathbb{R}, y_0 \in [-2 \log(t)^2, 0]\) and \(\lambda > 0\) there is a \(t_0(\lambda) > 0\) such that for \(t \geq t_0(\lambda),\)

\[
\frac{p_t^{(y_0)}(y, y^2)}{p_t^{(y_0)}(x, y^2)} \leq C_2^{-1}4^{\gamma+2}e^{4C_1}|y|^{2\gamma+3C_3} + Ce^{-\lambda \log(2)}p_t^{(y_0)}(x)^{-1}. 
\] (8.9)

8.3 Proof of Theorem 8.1

We proceed with the upper bound in (8.8). We recall Definition 2.1. In this section we set for \(0 < y \leq y_0 \leq \lambda t \leq \infty\)

\[
A_{t_1, t_2} := \mathcal{B}_{[t_1, t_2]}(\cdot)|_{\mathcal{W}(B)}. 
\]

We first condition on the endpoint. For \(x, y_0 \leq 0, \) we have that \(B_t - W_t \sim \mathcal{N}(x - W_t, t)\) under \(\mathbb{P}_x\) and thus, see (8.1).

\[
p_t^{(y_0)}(x) = \int_{y_0}^{y_0} g_{x-W_t, t}(z)\mathbb{P}_x[A_{0, t}|B_t - W_t = z] \, dz. 
\] (8.10)

Under \(\mathbb{P}_x[\cdot|B_t - W_t = z]\) the process \((B_s)_{s \in [0, t]}\) is a Brownian bridge from \(x\) to \(z + W_t\). We use that to prove that the marginals of \((B_s)_{s \in [0, t]}\) are associated w.r.t. that measure.

Lemma 8.4. Let \((X_s)_{s \in [0, t]}\) be a Brownian bridge from \(x\) to \(y, x, y \in \mathbb{R}\). Let \(t_1 < \cdots < t_n \in [0, t]\) and \(A, B\) be decreasing in \((X_{t_1}, \ldots, X_{t_n})\). Then

\[
\mathbb{P}[A \cap B] \geq \mathbb{P}[A]\mathbb{P}[B]. 
\]

Proof. Let \((Y_t)_{t \geq 0}\) be a Brownian motion starting at \(x\), we have that \((X_s)_{s \in [0, t]} \overset{d}{=} (Y_t - \frac{t}{l}Y_{t} - \frac{t}{l}Y_{t})_{s \in [0, t]}\), in particular \((X_{t_1}, \ldots, X_{t_n})\) are normally distributed and a straight forward calculation yields that for \(j \leq k\)

\[
\text{Cov}[X_{t_j}, X_{t_k}] = \text{Cov}\left[ Y_{t_j} - \frac{t_j}{l}Y_{t} - \frac{t_j}{l}Y_{t}, Y_{t_k} - \frac{t_k}{l}Y_{t} - \frac{t_k}{l}Y_{t}\right] = t_j \left(\frac{t - t_k}{l}\right) \geq 0. 
\] (8.11)

By [20], equation (8.11) implies that \((X_{t_1}, \ldots, X_{t_n})\) are associated. Since \(1_A, 1_B\) are decreasing in \((X_{t_1}, \ldots, X_{t_n})\), this gives that \(\text{Cov}[1_A, 1_B] \geq 0\). \(\square\)

Lemma 8.4 and (8.10) imply that for \(t \geq 0, 0 \geq x \geq y\) and \(y_0 \leq 0,\)

\[
p_t^{(y_0)}(x) \geq \int_{y_0}^{y_0} g_{x-W_t, t}(z)\mathbb{P}_x[A_{0, y^2}|B_t - W_t = z] \, dz. 
\] (8.12)

We want to pull the factor \(\mathbb{P}_x[A_{0, y^2}|B_t - W_t = z]\) in (8.12) out of the integral and for this need to bound it uniformly for \(z \in [y_0 - 1, y_0]\). This is done in Lemma 8.5 whose proof is postponed. The main idea is, that conditioning on \(B_t - W_t = z\) does barely change the distribution of \((B_s)_{s \leq y^2}\) for \(y\) and \(z\) small enough.
Lemma 8.5. For \( t \geq 64, 0 \leq x \leq y, -2t^{\frac{1}{4}} \leq y \leq -e, y_0 \in [-2 \log(t)^2, 0] \) and \( z \in [y_0 - 1, y_0] \), we have that
\[
\mathbb{P}_x[A_{0,y^2}|B_t - W_t = z] \geq C_2 \mathbb{P}_x[A_{0,y^2}, B_{y^2} - W_{y^2} \in [4C_1y, 2C_1y]].
\]

The bound \( t \geq 64 \) is of no further significance and is just a more concrete way to say, that we need \( t \) big enough, but importantly not depending on \( W \).

Equation (8.12) and Lemma 8.5 directly imply the following corollary.

Corollary 8.6. For \( t \geq 64, \eta \in [0, 1/4), 0 \leq x \leq y \geq -2t^{\frac{1}{4}}, y_0 \in [-2 \log(t)^2, 0] \), we have that
\[
P_{t,h}^{(\eta)}(x) \geq C_2 \mathbb{P}_x[A_{0,y^2}, B_{y^2} - W_{y^2} \in [4C_1y, 2C_1y]]/p_{t,h}^{(\eta)}(x, y^2).
\]

We can control \( \mathbb{P}_x[A_{0,y^2}, B_{y^2} - W_{y^2} \in [(4C_1 - 1)y, 2C_1y]] \) for \( y \leq x \leq -1 \) by applying (8.4). This is done in the next lemma.

Lemma 8.7. We have that for \(-2t^{\frac{1}{4}} \leq y \leq x \leq -1 \) and \( t \geq 64 \),
\[
\mathbb{P}_x[A_{0,y^2}, B_{y^2} - W_{y^2} \in [4C_1y, 2C_1y]] \geq y^{-2\gamma}.
\]

Proof. We have that
\[
\begin{align*}
\mathbb{P}_x[A_{0,y^2}, B_{y^2} - W_{y^2} \in [4C_1y, 2C_1y]] \\
y \leq x \leq -1 \\
& \geq \mathbb{P}_x[A_{0,y^2} - B_{y^2} + h_1(s) - (1 + x) \leq W_s, b_{y^2} - (1 + x) - W_{y^2} \in [(4C_1 - 1)y, 2C_1y]] \\
& = \mathbb{P}_{-1}[A_{0,y^2}, B_{y^2} - W_{y^2} \in [(4C_1 - 1)y, 2C_1y]] \geq y^{-2\gamma},
\end{align*}
\]

where we note that the application of (8.4) uses that \( y^2 \leq 4t^{1/2} \leq t/2 \) for \( t \geq 64 \).

Proof of Theorem 8.1 assuming Lemma 8.5. The lower bound in (8.8) is immediate by monotonicity.

Combining Corollary 8.6 and Lemma 8.7 yields the upper bound in (8.8).

Proof of Lemma 8.5. Define
\[
\mu_1 = x + y^2/t(z + W_t + x) - W_{y^2}, \quad \mu_2 = x - W_{y^2},
\]
\[
\sigma_1^2 = y^2/t - y^2, \quad \sigma_2^2 = y^2,
\]
and recall (8.1). The relevance of these is that the random variable \( B_{y^2} - W_{y^2} \) has the density function \( g_{\mu_2, \sigma_2^2} \) under \( \mathbb{P}_x \) and the density function \( g_{\mu_1, \sigma_1^2} \) under \( \mathbb{P}_x[\cdot | B_t - W_t = z] \). The latter implies that
\[
\begin{align*}
\mathbb{P}_x[A_{0,y^2}|B_t - W_t = z] & = \int_{-\infty}^{-h_1(y^2)} g_{\mu_1, \sigma_1^2}(r)\mathbb{P}_x[A_{0,y^2}|B_t - W_t = z, B_{y^2} - W_{y^2} = r] \, dr \\
& = \int_{-\infty}^{-h_1(y^2)} g_{\mu_1, \sigma_1^2}(r)\mathbb{P}_x[A_{0,y^2}|B_{y^2} - W_{y^2} = r],
\end{align*}
\]
\[
\geq \mathbb{P}_x[A_{0,y^2}|B_{y^2} - W_{y^2} = r] \mathbb{P}_x[A_{0,y^2}|B_{y^2} - W_{y^2} = r] \, dr. \tag{8.13}
\]

We have that
\[
g_{\mu_1, \sigma_1^2}(r) = \sqrt{t} e^{-y^2 \frac{r - \mu_1}{\sigma_1^2}} \geq e^{-y^2 \frac{r - \mu_1}{\sigma_1^2}} \geq e^{-y^2 \frac{r - \mu_1}{\sigma_1^2}}. \tag{8.14}
\]

Direct calculation using that \(-2t^{1/4} \leq y \leq x \leq -e, z \in [y_0 - 1, y_0], y_0 \in [-2 \log(t)^2, 0] \) and (8.2) yields that for \( r \in [4C_1y, 2C_1y] \)
\[
e^{-y^2 \frac{r - \mu_1}{\sigma_1^2}} \geq C_3. \tag{8.15}
\]

Plugging (8.14) and the inequality (8.15) into (8.13) yields that
\[
\begin{align*}
\mathbb{P}_x[A_{0,y^2}|B_t - W_t = z] & \geq C_2 \int_{4C_1y}^{2C_1y} g_{\mu_2, \sigma_2^2}(r)\mathbb{P}_x[A_{0,y^2}|B_{y^2} - W_{y^2} = r] \, dr \\
& = C_2 \mathbb{P}_x[A_{0,y^2}, B_{y^2} - W_{y^2} \in [4C_1y, 2C_1y]].
\end{align*}
\]

where the last step uses that under \( \mathbb{P}_x \) the random variable \( B_{y^2} - W_{y^2} \) has the density \( g_{\mu_2, \sigma_2^2}(r) \).
8.4 Proof of Theorem 8.2

For $z \leq 0$ and $r \in \mathbb{R}$, define
\[
\mu_{z,y^2,h} := z + h_t(y^2) - W_y,
\]
\[
\Delta_{y^2,W,h}(s) := h_t(s + y^2) - h_t(y^2) - (W_y + y^2 - W_y),
\]
\[
p_{t;\gamma}^{(0)}(r) := \mathbb{P}_r \left[ B^{f_0}_{[0,t]} \Delta_{y^2,W,h} \right].
\]

We recall (8.1). Using the Markov property we have that
\[
p_{t;\gamma}^{(0)}(z,y^2) = \int_0^t g_{\mu_{z,y^2,h}}(r)p_{t;\gamma}^{(0)}(r) \, dr. \tag{8.16}
\]

We finish the proof by taking $z = y$ in (8.16) and bounding $g_{\mu_{y,y^2,h}}(r)g_{\mu_{y,y^2,h}}(r)^{-1}$. A direct computation yields that
\[
g_{\mu_{y,y^2,h}}(r)g_{\mu_{y,y^2,h}}(r)^{-1} = \exp \left( r(y-x) + \frac{x^2 - y^2}{2y^2} + \frac{(W_y - h_t(y^2)) (y-x)}{y^2} \right). \tag{8.17}
\]

Applying (8.16) for $z = y$ yields that
\[
p_{t;\gamma}^{(0)}(y,y^2) = \int_0^t g_{\mu_{y,y^2,h}}(r)p_{t;\gamma}^{(0)}(r) \, dr \tag{8.17}
\]

\[
e^{-\frac{x^2 - y^2}{2y^2} + \frac{(y-x)^2}{y^2}} \int_0^t g_{\mu_{y,y^2,h}}(r)p_{t;\gamma}^{(0)}(r) \, dr \tag{8.16}
\]

\[
\geq e^{-2C_1} e^{-C_0 \sqrt{\log(y^2)}} p_{t;\gamma}^{(0)}(x,y^2). \tag{8.21}
\]

8.5 Proof of Theorem 8.3

8.5.1 Definitions and Outline

As in Section 8.4, we will use (8.16), but the factor $e^{r(y-x)}$ is not a-priori controllable. To fix this we will split the domain of integration into $[-L,0],[-c_1 t^{1/3} \log(t)^{1/2},-L],(-\infty,-c_1 t^{1/3} \log(t)^{1/2}]$. On the first and the last interval we will be able to control the integrand. On the middle region we will use an analogue to (8.16), which is stated in (8.21). This will give us a double integral over $[-L,-c_1 t^{1/3} \log(t)^{1/2}] \times [-\infty,0]$, we again split up the domain of integration into three parts and iterate the process. The $L$ in the $l$-th iteration will depend on the variable of integration of the $l-1$-th iteration in a way such that the middle region is empty after finitely many iterations.

Definition 8.2. For $t \geq 0$, $u,v \in [0,t]$, set $\Delta_{u,v}(W) := W_u - W_v$, $\Delta_{u,v}(h_t) := h_t(u) - h_t(v)$ and $\Delta_{u,v} := \Delta_{u,v}(h_t) - \Delta_{u,v}(W)$.

For $\sigma,z,z' \geq 0$ with $\sigma \leq t$, $\sigma + z \leq t$ and $r \in \mathbb{R}$, define
\[
p_{t;\gamma}^{(0)}(r) := \mathbb{P}_r \left[ B^{f_0}_{[0,t]} \Delta_{\sigma,z',h} \right], \tag{8.18}
\]
\[
p_{t;\gamma}^{(0)}(z,z') := \mathbb{P}_r \left[ B^{f_0}_{[0,t]} \Delta_{\sigma,z',h} \right], \tag{8.19}
\]
\[
\mu(z,\sigma,z',h) := z + \Delta_{\sigma,z',h}. \tag{8.20}
\]

If $\sigma = 0$ it is suppressed from notation.

We recall (8.1). A direct consequence of the Markov property for the Brownian motion is that for $z \in \mathbb{R}$ and $z',\sigma \geq 0$ with $\sigma + z' \leq t$ we have that
\[
p_{t;\gamma}^{(0)}(z,z') = \int_{-\infty}^0 g_{\mu(z,\sigma,z',h)} (r)p_{t;\gamma}^{(0)}(r) \, dr. \tag{8.21}
\]

Definition 8.3. Fix $\lambda > 0$ and $k = \left\lceil \frac{2\lambda \log(t)}{3} + 1 \right\rceil$. Set $z_0 := y$. For $l \in \mathbb{N}$ and $z_1,\ldots,z_l \in \mathbb{R}$ set $\sigma_l(z_0,\ldots,z_l) := \sum_{j=1}^l z_j^2$ and for $j \leq l$ set $L_j(z_{j-1}) := C_{2,1} \log(z_{j-1})$.

Introduce the shorthand $\mu_j := \mu(z_j,\sigma_{j-1},z_j^2,h)$, using the notation introduced in (8.20).

The dependence of $L_j$ and $\sigma_j$ on $z_0,\ldots,z_l$ will be omitted in the following. Let $L^{(0)}(y) := L_1(L^{(0-1)}(y))$, $L^{(0)}(y) = y$.
We claim that under the assumption that $k c_1 t^{1/5} \log(t)^{1/2} \leq t$, which will be proved to hold for big enough $t$ in Lemma 8.8

$$p_{t,h}(y, g^2)$$

$$\leq \sum_{l=2}^{k} \int_{L_{l-1}}^{L_l} \cdots \int_{L_{l-1}}^{L_{l-1}} \prod_{i=1}^{l} \left( g_{\mu_{i-1}, z_{i-1}^2}(z_i) \right) p_{t,\sigma_{i-1},h}(z_i) \, dz_i \cdots \, dz_1$$

$$+ \sum_{l=2}^{k-1} \int_{L_{l-1}}^{L_l} \cdots \int_{L_{l-1}}^{L_{l-1}} \int_{-\infty}^{0} \prod_{i=1}^{l} \left( g_{\mu_{i-1}, z_{i-1}^2}(z_i) \right) p_{t,\sigma_{i-1},h}(z_i) \, dz_i \cdots \, dz_1$$

$$+ \int_{L_1}^{L_k} \cdots \int_{L_k}^{L_k} \prod_{i=1}^{k} \left( g_{\mu_{i-1}, z_{i-1}^2}(z_i) \right) \, dz_k \cdots \, dz_1$$

$$+ \int_{0}^{0} g_{\mu(0,0,0,0), h}(z_1) p_{t,\gamma^2, h}(z_1) \, dz_1$$

$$+ \int_{-\infty}^{L_1 \land (-c_1 t^{1/5} \log(t)^{1/2})} g_{\mu(0,0,0,0), h}(z_1) p_{t,\gamma^2, h}(z_1) \, dz_1$$

where all integrals for which the lower limit is bigger than the upper limit are defined to be 0. The previous display follows by repeatedly applying 8.21 and noticing that $p_{t,\sigma_{i-1},h}(z_i) \leq p_{t,\sigma_{i-1},h}(z_i, z_i^2)$ and that the condition $k c_1 t^{1/5} \log(t)^{1/2} \leq t$ guarantees that $\sigma_{i-1} + z_i^2 \leq t$ on the region of integration considered. Example trajectories for the summands 8.22 to 8.26 are displayed in figure 2.

**Remark 8.1.** In terms of the iteration described at the beginning of this section the summands in 8.22 correspond to the probability to stay in the “middle” region for the first $l-1$ steps and then end up in $[L_l, 0]$.

The summands in 8.23 correspond to the probability to stay in the middle region for the first $l-1$ steps and then end up very low in the $l$-th step, this can be controlled, since in the $l$-th step a big jump is made.

The term in 8.24 corresponds to a particle stays in the middle region for so long, that even though it never made a big jump, the particle has ended up very low. Here in every step we pick up a factor and will need to prove, that the product of these factors is small. In this step the choice of $k$, i.e. $k$ being large, will be relevant.

The term in 8.25 corresponds to landing in $[0,L_1]$ in the first step. Here we can directly control the integrand without needing to iterate.

The term in 8.26 corresponds to making a big jump in the first step, which has low probability.

We proceed by proving properties of the $k$ we have chosen.

**Lemma 8.8.** For $k = \left( 2 \lambda \log(t) \right)^{1/3} + 1$ and $t \geq (3c_1 \lambda)^{5/2}$, we have that

$$\frac{1}{3} \sum_{i=1}^{k} \frac{L_{(i)}(y)^2}{L_{(i-1)}(y)^2} \geq 2 \lambda \log(t).$$

(8.27)
The combination of the black and the blue line is the start of a trajectory in (8.23), the combination of the black and the green of a trajectory in (8.24) and the combination of the black and the red of a trajectory in (8.23).

Figure 2: Sketching the summands in (8.22) to (8.26) for $k = 2$ and $h = W = 0$.

as well as $kc_1 t^{1/5} \log(t)^{1/2} < t$.

Proof. It is immediate from the definitions that $L^{(l)} \geq C_3 l \cdot |y|$, where we use that $\log(|y|) \geq 1$. Thus we have that

$$\frac{L^{(l)}(y)^2}{L^{(l-1)}(y)^2} = C_3^2 \log(|L_l(y)|)^2 \geq C_3^2 (l-1)^2 \log(C_3) \geq C_3 (l-1)^2$$

where in the last step we used that $C_3 \geq e$. Thus

$$\frac{1}{8} \sum_{l=1}^{k} \frac{L^{(l)}(y)^2}{L^{(l-1)}(y)^2} \geq \frac{1}{8} C_3 \sum_{l=1}^{k-1} l^2 \geq (k-1)^3 \geq 2\lambda \log(t)$$

by the choice of $k$. Furthermore, we have that

$$kc_1 t^{1/5} \log(t)^{1/2} \leq (3\lambda \log(t))^{1/3} c_1 t^{1/5} \log(t)^{1/2} \leq t,$$

since we assume that $t \geq (3c_1 \lambda)^{1/3}$. \hfill \qed

In the following we will consider the summands in the decomposition (8.22) to (8.26) one by one and provide bounds for them.

### 8.5.2 Bounds for (8.22) to (8.26), proof of theorem 8.3 assuming these bounds

**Lemma 8.9.** For $t > 0$, $y \in [-2 \log(t)^2, -\varepsilon]$, $y_0 \in [-2 \log(t)^2, 0]$, we have that

$$\sum_{l=2}^{k} \int_{-c_1 t^{1/5} \log(t)^{1/2}}^{L_{l-1}} \ldots \int_{-c_1 t^{1/5} \log(t)^{1/2}}^{L_{l-1}} \int_{-\infty}^{0} \int_{L_l} \prod_{i=1}^{t} \left( p_{\mu_{y_i}^{-1}, z_{-1}^{-1}}^{(y_0)}(z_i) \right) p_{l, \sigma_l, z_{-1}}^{(y_0)}(x, y^2) dz_i \ldots dz_1$$

$$\leq C_2^{-1} 14^{\gamma+2} 2^{\gamma} \inf_{x \in [y, -e]} p_{l, h}^{(y_0)}(x, y^2)$$

The main step in the proof of Lemma 8.9 will be to bound $\int_{L_l} g_{\mu_{y_i}^{-1}, z_{-1}^{-1}}^{(y_0)}(z_i) dz_i$ by replacing $g_{\mu_{y_i}^{-1}, z_{-1}^{-1}}^{(y_0)}(z_i)$ by $g_{\mu(x, 0, \sigma_l, h), \sigma_{l-1}}^{(y_0)}(z_i)$, which can be done since we have $z_i \in [L_l, 0]$.

**Lemma 8.10.** For $t > 0$, $-\varepsilon \geq y \geq -2 \log(t)^2$, $y_0 \in [-2 \log(t)^2, 0]$, we have that

$$\sum_{l=2}^{k-1} \int_{-c_1 t^{1/5} \log(t)^{1/2}}^{L_{l-1}} \ldots \int_{-c_1 t^{1/5} \log(t)^{1/2}}^{L_{l-1}} \int_{-\infty}^{-c_1 t^{1/5} \log(t)^{1/2}} \int_{-\infty}^{0} \prod_{i=1}^{t} \left( g_{\mu_{y_i}^{-1}, z_{-1}^{-1}}^{(y_0)}(z_i) \right) p_{l, \sigma_l, z_{-1}}^{(y_0)}(z_i) dz_i \ldots dz_1$$

$$\leq 8ke^{-\frac{1}{8} \log(t)^{1/2}}.$$
For proving Lemma 8.10 it suffices to bound $p^{(yo)}_{t,\sigma_{t-1},A}(zt)$ by 1 and to use, that $|zt|$ is big on the region of integration of the last integral, in particular the bound only involves the last two integrals, for the other integrals all we use is that the $x_{l-1},z^2_{l-1}$ are density functions.

**Lemma 8.11.** For $t \geq (3\epsilon_1)\lambda^{30}$, $y \in [-e, -2\log(t)^2]$, $y_0 \in [-2\log(t)^2, 0]$, we have that

$$
\int_{-c_1 t^{1/5} \log(t)^{1/2}}^{L_1} \cdots \int_{-c_1 t^{1/5} \log(t)^{1/2}}^{L_{k-1}} \int_{-c_1 t^{1/5} \log(t)^{1/2}}^{L_k} \prod_{i=1}^{k} (g_{x_{l-1},z^2_{l-1}}(z_i)) \; dz_k \cdots dz_1 \leq e^{-2\lambda \log(t)}.
$$

To prove Lemma 8.11 we can again use $p^{(yo)}_{t,h}(z_i) \leq 1$ but here the last two integrals alone aren’t enough. Instead we prove by induction that in the $l$-th integral we pick up a factor $e^{-\lambda \log(t)}$ \textit{then} allows to conclude.

**Lemma 8.12.** For $t > 0$, $y \in [-2\log(t)^2, -e]$, $x \in [y,-e]$, $y_0 \in [-2\log(t)^2, 0]$, we have that

$$
\int_{-c_1 t^{1/5} \log(t)^{1/2}}^{L_1} g_{\mu(y,0,y^2,h),y^2}(z_1)p^{(yo)}_{t,y^2,h}(z_1) \; dz_1 \leq C e^{4C_1} e^{3C_2 \log(|y|)} \inf_{x \in [y,-e]} p^{(yo)}_{t,h}(x,y^2).
$$

The proof of Lemma 8.12 closely mirrors the proof of Theorem 8.2.

**Lemma 8.13.** There is a $t_0(\lambda) > 0$ such that for $t \geq t_0(\lambda)$, $y \in [-2\log(t)^2, -e]$, $y_0 \in [-2\log(t)^2, 0]$, 

$$
\int_{-\infty}^{L_1 \wedge (-c_1 t^{1/5} \log(t)^{1/2})} g_{\mu(y,0,y^2,h),y^2}(z_1)p^{(yo)}_{t,y^2,h}(z_1) \; dz_1 \leq 8e^{-\frac{t^{1/2} \log(t)}{\pi^2}}.
$$

**Lemma 8.13** is quickly proven by applying Gaussian tail estimates.

**Proof of Theorem 8.3 assuming Lemma 8.9 to Lemma 8.13** Combining Lemma 8.9 to Lemma 8.13 with \textit{8.22} to \textit{8.26} gives that

$$
p^{(yo)}_{t,h}(x,y^2) \quad \frac{p^{(yo)}_{t,h}(y,y^2)}{p^{(yo)}_{t,h}(x,y^2)} \leq C_2^{-1} 4^{2+e^{4C_1} |y|^{2+3C_2}}
+ \frac{C_3}{\pi^2} \left( t^{-1/2} e^{-\frac{t^{1/2} \log(t)}{\pi^2}} + e^{-\frac{t^{1/2} \log(t)}{\pi^2}} + e^{-2\lambda \log(t)} \right).
$$

We recall our choice $c_1 = \sqrt{8(\lambda + 1)}$ in \textit{8.8}. Furthermore, we note that by definition $k \leq t$ for $t \geq ((2\lambda)^{1/3} + 2)^{3/2}$, that for $t$ big enough $\frac{t^{1/2} \log(t)}{\pi^2} \leq \lambda \log(t)$ and that by \textit{8.31} $p^{(yo)}_{t,h}(x,y^2)^{-1} \leq (p^{(yo)}_{t,h}(x)^{-1}$. Combining these observations with \textit{8.28} finishes the proof.

\textbf{8.5.3 Details}

**Preparation** We recall \textit{8.1} and Definition \textit{8.2}. The aim of this paragraph is to prove the following lemma, which allows to replace $g_{x_{l-1},z^2_{l-1}}(z)$ by $2g_{0,4z^2_{l-1}}(z)$.

**Lemma 8.14.** Fix $l \in \mathbb{N}$. Assume that for all $r \leq l - 1$ we have that $z_r \leq C_3 z_{r-1} \log(|z_{r-1}|)$. Then we have that for all $r \leq l - 1$,

$$
g_{x_{r},z^2_{r}}(z_{r+1}) \leq 2g_{0,4z^2_{r}}(z_{r+1}).
$$

To prove Lemma 8.14 we will compare $z_r$ to $z^2_{r+1}$ and establish bounds on $\Delta_{\sigma_{r-1},\sigma_{r-2}}(f)$, $f \in \{W,h_1\}$, these are stated in the next three lemmata, whose proofs are postponed, since they are mostly calculation.

**Lemma 8.15.** Fix $l \in \mathbb{N}$. Assume that for all $r \leq l - 1$ we have that $z_r \leq C_3 z_{r-1} \log(|z_{r-1}|)$. Then we have that for all $r \leq l - 1$

$$
\sigma_r \leq 2z_r^2, \quad \sigma_r \leq 2 \log(|z_{r-1}|)^2.
$$

**Lemma 8.16.** Fix $l \in \mathbb{N}$. Assume that for all $r \leq l - 1$ we have that $z_r \leq C_3 z_{r-1} \log(|z_{r-1}|)$. Then we have that for all $r \leq l - 1$

$$
|\Delta_{\sigma_{r-1},\sigma_{r-2}}(W)| \leq \frac{1}{8}|z_r - z_{r-1}|
$$
Lemma 8.17. Fix \( l \in \mathbb{N} \). Assume that for all \( r \leq l-1 \) we have that \( z_r \leq C_3 r^{-1} \log(|z_r-1|) \). Then we have that for all \( r \leq l-1 \)

\[
\Delta_s_{r-1,s_{r-2}}(h_t) \leq \frac{1}{8}|z_r - z_{r-1}|.
\]

Proof of Lemma 8.14 assuming Lemmata 8.16, 8.17. Follows by plugging Lemma 8.16, Lemma 8.17 into the definition of \( g_{s_r,s_{r-1}}(z) \) and noticing that \( z_r \leq 4z_{r-1} \) by (8.7).

Proof of Lemma 8.15. Equation (8.30) is implied by (8.29) using that \( z_{r+1}^2 \geq \log(|z_r|) z_r^2 \). Thus it remains to prove (8.29).

We have that \( z_r \leq C_3 z_{r-1} \log(|z_r-1|) \) and since \( z_{r-1} \leq y \leq -e \) we have that \( \log(|z_{r-1}|) \geq 1 \) and thus

\[
z_r \leq C_3 z_{r-1} < 0.
\]

Squaring and iterating this bound gives that for \( k \in \{0, \ldots, r\} \) we have \( z_{r-k}^2 \leq \frac{1}{C_3^k} z_r^2 \) which yields that

\[
\sigma_r = \sum_{k=0}^{r} z_k^2 \leq \sum_{k=0}^{r} \frac{1}{C_3^k} \sigma_{r-k}^2 = \frac{1}{1-1/C_3^2} \leq 2 \sigma_r^2
\]

where the second to last step uses that \( C_3^2 > 1 \) and the last step uses that \( 1 - 1/C_3^2 \geq 1/2 \) since \( C_3^2 \geq 2 \).

Proof of Lemma 8.16. Using the triangle inequality and \( |z_i-1| \leq \frac{1}{C_3} |z_i| \) shows that it suffices to prove that for \( k \in \{1, 2\} \)

\[
|W_{\sigma_{i-k}}| \leq \frac{C_3 - 1}{16C_3} |z_i|
\]

By (8.7) we have that \( (C_3 - 1)/C_3 \geq \frac{1}{2} \) and thus it suffices to prove that

\[
\frac{|W_{\sigma_{i-k}}|}{|z_i|} \leq \frac{1}{32} \tag{8.31}
\]

for \( k \in \{1, 2\} \). We note that for \( i = 1 \) and \( k = 2 \) (8.31) is trivial, since \( W_{\sigma_{i-1}} = W_0 = 0 \). For the other pairs \( i, k \) by Lemma 8.15 we have that

\[
0 < \sigma_{i-k} \leq 2z_{i-1}^2,
\]

which implies that

\[
\frac{|W_{\sigma_{i-k}}|}{|z_i|} = \frac{|W_{\sigma_{i-k}}|}{|z_i|} \cdot \frac{\sigma_{i-k}^{1/2} \log(\sigma_{i-k})}{\sigma_{i-k}^{1/2} \log(\sigma_{i-k})} \leq C_{log} \frac{2|z_{i-1}| \log(2z_{i-1}^2)}{C_3|z_{i-1}| \log(|z_{i-1}|)} \leq \frac{1}{32},
\]

where the last step uses (8.7). This yields (8.31) and finishes the proof.

Proof of Lemma 8.17. As in Lemma 8.16 it suffices to prove that

\[
\frac{|h_t(\sigma_{i-k})|}{|z_i|} \leq \frac{1}{32}
\]

for \( k \in \{1, 2\} \).

By (8.3) we have that

\[
\Delta_{s_{r-1},s_{r-2}}(h_t) \leq C_1 \left( \sqrt{1+\sigma_{r-1}} + \sqrt{1+\sigma_{r-2}} \right) \leq C_1 \left( \sqrt{1+2z_{i-1}^2} + \sqrt{1+2z_{i-2}^2} \right)
\]

\[
\leq C_1 \sqrt{3} \left( |z_{i-1}| + |z_{i-2}| \right) \leq 2C_1 \sqrt{3} |z_i|.
\]

Using (8.7) this yields the claim.

Proof of Lemma 8.9. The proof of this lemma is split into two steps. First we bound the last two integrals in (8.22), which is done by controlling \( g_{\mu_{i-1},\sigma_{r-1}}(z_i) g_{\mu_{(x,0,s_{r-1},h)},s_{r-1}}(z_i)^{-1} \) for \( z_i \in [L_1, 0] \), giving the following lemma.

Lemma 8.18. Let \( t > 0, y \in [-2 \log(t)^2, -e], y_0 \in [-2 \log(t)^2, 0] \) and \( l \geq 2 \). Assume that for all \( r \leq l-1 \) we have that \( z_r \leq C_3 z_{r-1} \log(|z_{r-1}|) \). Then

\[
\int_{-C_1 t^{1/2} \log(t)^{1/2}}^{L_{l-1}} g_{\mu_{l-1},z_{l-2}^2} (z_{l-1}) \int_{L_{l-1}}^{0} g_{\mu_{l-1},z_{l-2}^2} (z_l) p^{(y_0)}_{l-1,s_{l-1},h} (z_l) \, dz_l \, dz_{l-1}
\]

\[
\leq C_2^{-1} 4r^2 |z_{r-2}|^2 e^{-c_2^2 r^2} \inf_{x \in [y,-e]} p^{(y_0)}_{l,s,h} (x, y^2)
\]

25
After this we bound the other integrals in (8.22) by induction giving the following lemma.

**Lemma 8.19.** For $y \leq 0$ arbitrary and all $l \in \mathbb{N}$, we have that
\[
\int_{-\infty}^{L_l} \cdots \int_{-\infty}^{L_l} z_1^{2\gamma} \prod_{i=1}^{l} g_{\mu_{l-1},z_{i-1}^2}(z_i) \, dz_1 \cdots dz_l \leq \sqrt{8} e^{-\frac{c_2^2}{4\pi}} |y|^{2\gamma}.
\]

*Proof of Lemma 8.19.* Combining Lemma 8.18 and Lemma 8.19 gives that
\[
\int_{-c_1^{1/5} \log(t)^{1/2}}^{L_l} \cdots \int_{-c_1^{1/5} \log(t)^{1/2}}^{L_l} \int_0^0 \prod_{i=1}^{l} \left(g_{\mu_{l-1},z_{i-1}^2}(z_i)\right) p_{t,\sigma_{l-1},h}(z_l) \, dz_1 \cdots dz_l \leq C_2^{-1} |y|^{2\gamma} e^{-\frac{(|y|^{2\gamma})^2}{4\pi}} \inf_{x \in [y, -e]} p_{t,h}(x, y^2).
\]

Summing this inequality over $l \geq 2$ directly yields the claim of Lemma 8.9 since by (8.7) we have
\[
1/ \left(1 - e^{-\frac{c_2^2}{4\pi}}\right) \leq 2.\]

*Proof of Lemma 8.18.*

**Lemma 8.20.** We have that for $y \in [-2 \log(t)^2, -e], x \in [y, -e], l \geq 2, z_l \in [L_l, 0],$
\[
g_{\mu_{l-1},z_{l-1}^2}(z_l) g_{\mu(x,0,\sigma_{l-1},h),\sigma_{l-1}}(z_l) \leq \sqrt{2} e^{C_3/2 (\log(|z_{l-1}|))^{3/2}}.
\]

*Proof.* Using the definition of $g$ in (8.1), dropping negative terms, using that $z_l \geq L_l$ and simplifying gives the bound
\[
g_{\mu_{l-1},z_{l-1}^2}(z_l) g_{\mu(x,0,\sigma_{l-1},h),\sigma_{l-1}}(z_l)^{-1} \leq \frac{\sqrt{C_3}}{|z_{l-1}|^{\frac{1}{2}}} \left(\frac{\sigma_{l-2} |W_{\sigma_{l-1}}|}{\sigma_{l-1} |z_{l-1}|}, \frac{2 |W_{\sigma_{l-1}}|}{\sigma_{l-1} |z_{l-1}|}, \frac{2 |W_{\sigma_{l-1}}|}{\sigma_{l-1} |z_{l-1}|}, \frac{2 |W_{\sigma_{l-1}}|}{\sigma_{l-1} |z_{l-1}|}, \frac{2 |W_{\sigma_{l-1}}|}{\sigma_{l-1} |z_{l-1}|}, \frac{2 |W_{\sigma_{l-1}}|}{\sigma_{l-1} |z_{l-1}|}ight).
\]

This upper bound can be simplified further by plugging in the definition of $L_l$ and bounding the terms involving $W$ and $h$. Using Assumptions (8.3), (8.2) and Lemma 8.15 direct calculation gives that
\[
\frac{|W_{\sigma_{l-1}}|}{\sigma_{l-1} |z_{l-1}|} \leq 8C_3 \log (\log(|z_{l-1}|))^{1/2},
\]
\[
\frac{|W_{\sigma_{l-1}}|}{\sigma_{l-1} |z_{l-1}|} \leq 8C_3 \log (\log(|z_{l-1}|))^{1/2},
\]
\[
\frac{|W_{\sigma_{l-1}}|}{\sigma_{l-1} |z_{l-1}|} \leq 8C_3 \log (\log(|z_{l-1}|))^{1/2},
\]
\[
\frac{|W_{\sigma_{l-1}}|}{\sigma_{l-1} |z_{l-1}|} \leq 8C_3 \log (\log(|z_{l-1}|))^{1/2}.
\]

Plugging these and the definition of $L_l$ in Definition 8.3 into (8.32) and simplifying yields the claim of the lemma, we note that while simplifying we also used Assumption (8.7) as well as that $\sqrt{|z_{l-1}|/|z_{l-1}|} \leq \sqrt{2}$ by Lemma 8.15.

Moreover, Lemma 8.20 and the inequality (8.8) allow us to bound the last integral in (8.22).

**Lemma 8.21.** For $y \in [-2 \log(t)^2, -e], y_0 \in [-2 \log(t)^2, 0], l \geq 2$ and $z_j \in [-c_1^{1/5} \log(t)^{1/2}, 0]$ we have that for all $j \leq l - 1$,
\[
\int_{-c_1^{1/5} \log(t)^{1/2}}^{L_l} g_{\mu_{l-1},z_{j-1}^2}(z_l) p_{t,\sigma_{l-1},h}(z_j) \, dz_l \leq C_2^{-1} \sqrt{2} 2^{2\gamma} C_3^{3/2} (\log(|z_{l-1}|))^{3/2} |z_{l-1}|^{2\gamma} \inf_{x \in [y, -e]} p_{t,h}(x, y^2).
\]

*Proof.* Fix $x \in [y, -e]$ arbitrary. We have that
\[
\int_{-c_1^{1/5} \log(t)^{1/2}}^{L_l} g_{\mu_{l-1},z_{j-1}^2}(z_l) p_{t,\sigma_{l-1},h}(z_j) \, dz_l
\]
\[ I(z_{l-1}) := \int_{-c_1 t^{1/5} \log(t)^{1/2}}^{L_{l-1}} g_{\mu_{-z_{l-1}} z_{l-1}}(z_{l-1}) \int_{z_l}^0 g_{\mu_{-z_{l-1}} z_{l-1}}(z) p_{L_{l-1}, h}^{(y_0)}(z_l) \, dz_l \]

\[ \frac{L_{l, 20}}{\sqrt{2} e^{C_3^2/2 (\log(|z_{l-1}|))^3/2}} \int_{-\infty}^{0} g_{\mu(z_{l-1}, 0, \sigma_{l-1}, 0)}(z_l) p_{L_{l-1}, h}^{(y_0)}(z_l) \, dz_l \]

\[ \frac{8 \cdot 21}{\sqrt{2} e^{C_3^2/2 (\log(|z_{l-1}|))^3/2}} p_{L_{l-1}, h}^{(y_0)}(x, \sigma_{l-1}) . \]

Since \( z_j \in [-c_1 t^{1/5} \log(t)^{1/2}, 0] \) for all \( j \leq l - 1 \) we have that

\[ \sqrt{\sigma_{l-1}} \leq \sqrt{2} |z_{l-1}| \leq \sqrt{2} c_1 t^{1/5} \log(t)^{1/2} . \]

Thus we can apply (8.8) to conclude

\[ p_{L_{l-1}, h}^{(y_0)}(x, \sigma_{l-1}) \leq C_2^{-1} p_{L_{l-1}, h}^{(y_0)}(x) \sigma_{l-1} \]

\[ \leq C_2^{-1} p_{L_{l-1}, h}^{(y_0)}(x, y^2) |z_{l-1}|^{2\gamma} 2^{2\gamma} . \]

Together (8.33), (8.34) and the fact that \( x \in [y, -\epsilon] \) was arbitrary, finish the proof.

A direct calculation gives that for \( s \geq C_4 \log(|z_{l-2}|) \)

\[ s^{2\gamma} e^{C_3^2/2 \log(s)^3/2} \leq e^{2\gamma} . \]

With this we have all tools needed for the proof of Lemma 8.18

**Proof of Lemma 8.18** Set

\[ I(z_{l-1}) := \int_{-c_1 t^{1/5} \log(t)^{1/2}}^{L_{l-1}} g_{\mu_{-z_{l-1}} z_{l-1}}(z_{l-1}) \int_{z_l}^0 g_{\mu_{-z_{l-1}} z_{l-1}}(z) p_{L_{l-1}, h}^{(y_0)}(z_l) \, dz_l \, dz_{l-1} \]

Using Lemma 8.21 and Lemma 8.14 yields that

\[ I(z_{l-1}) \leq \int_{-\infty}^{L_{l-1}} g_{\mu_{-z_{l-1}} z_{l-1}}(z_{l-1}) e^{C_3^2/2 (\log(|z_{l-1}|))^3/2} |z_{l-1}|^{2\gamma} \, dz_{l-1} \]

\[ \leq \int_{-\infty}^{C_3 \log(|z_{l-1}|)} |z_{l-1}|^{2\gamma} e e^{C_3^2/2 (\log(|z_{l-1}|))^3/2} \, dz_{l-1} \]

\[ \leq \int_{0}^{\infty} g_{0, s}(s) \, ds \leq 8 |z_{l-2}|^{2\gamma} e^{-C_4^2/2} . \]

where the second to last step used Gaussian tail estimates\(^2\) and the last that by our assumptions \( |z_{l-2}| \geq |y| \geq \epsilon \) Rearranging yields the claim.

**Proof of Lemma 8.19** We prove this by induction. Induction basis \( l = 1 \): By applying Lemma 8.14 and (8.35) we get analogously to the calculation in the proof of Lemma 8.18

\[ \int_{-\infty}^{L_1} |z_1|^{2\gamma} g_{\mu_0, z_0}(z_1) \, dz_1 \leq |z_0|^{2\gamma} \int_{-\infty}^{\infty} \sqrt{\delta g_{0, s}(s)} \, ds \leq \sqrt{\delta} |z_0|^{2\gamma} e^{-C_4^2/2} . \]

Induction step \( l \rightarrow l + 1 \): We have that

\[ \int_{-\infty}^{L_l} \cdots \int_{-\infty}^{L_{l+1}} |z_{l+1}|^{2\gamma} \prod_{s=1}^{L_1} g_{\mu_{-z_{l-1}} z_{l-1}}(z_s) \, dz_1 \cdots dz_{l+1} \leq \int_{-\infty}^{L_l} |z_{l+1}|^{2\gamma} e^{-C_4^2/2} g_{\mu_0, z_0}(z_1) \, dz_1 \]

\[ \leq e^{-C_4^2/2} |y|^{2\gamma} e^{-C_4^2/2} \]

\[ = |y|^{2\gamma} e^{-1/2 - C_4^2/2} . \]

\(^2\)See for example Theorem 1.2.6 in [5] p. 13;
Proof of Lemma 8.10 Let \( l \in \{2, \ldots, k - 1\} \). To keep notation slightly lighter, we will as in the proof of Lemma 8.9 handle the last two integrals in (8.23) first. Since \( p_{t, \sigma, l}^{(y_0)} (z_l) \leq 1 \) we have that

\[
\begin{align*}
\int_{-c_l t^{1/5} \log(t)^{1/5} / (2 \Lambda_l)}^{L_{l-1}} \int_{-c_l t^{1/5} \log(t)^{1/5} / (2 \Lambda_l)}^{L_{l-1}} \prod_{i=l-1}^{1} \left( g_{\mu_i, z_{i-1}^2} (z_i) \right) p_{l, \sigma, l}^{(y_0)} (z_l) \, dz_l \, dz_{l-1} \\
\leq \int_{-c_l t^{1/5} \log(t)^{1/5} / (2 \Lambda_l)}^{L_{l-1}} \int_{-c_l t^{1/5} \log(t)^{1/5} / (2 \Lambda_l)}^{L_{l-1}} \prod_{i=l-1}^{1} \left( g_{\mu_i, z_{i-1}^2} (z_i) \right) \, dz_l \, dz_{l-1}
\end{align*}
\]

(8.36)

On \([- t^{1/5}, L_{l-1}]\) we have that

\[
e^{-\frac{c_l^2 \log(t)}{8 \Lambda_l}} \leq e^{-\frac{c_l^2 \log(t)}{8 \Lambda_l}}
\]

and on \([- c_l t^{1/5} \log(t)^{1/5} / (2 \Lambda_l), - t^{1/5}]\) we have that

\[
e^{-\frac{c_l^2 \log((z_{l-1})^2)}{8 \Lambda_l}} \leq e^{-\frac{c_l^2 \log(t)}{8 \Lambda_l}}
\]

where the last step used that by (8.7) \( C_3^2 \geq 5c_l^2 \).

Plugging both of these into (8.36) and using that \( g_{\mu_{i-2}, z_{i-2}^2} \) is a density function gives that

\[
\int_{-c_l t^{1/5} \log(t)^{1/5} / (2 \Lambda_l)}^{L_{l-1}} \int_{-c_l t^{1/5} \log(t)^{1/5} / (2 \Lambda_l)}^{L_{l-1}} \prod_{i=l-1}^{1} \left( g_{\mu_i, z_{i-1}^2} (z_i) \right) p_{l, \sigma, l}^{(y_0)} (z_l) \, dz_l \, dz_{l-1} \leq 8 e^{-\frac{c_l^2 \log(t)}{8 \Lambda_l}}
\]

(8.37)

Since for \( i < l - 1 \) all \( g_{\mu_{i-1}, z_{i-1}^2} \) are density functions, (8.37) implies that

\[
\int_{-c_l t^{1/5} \log(t)^{1/5} / (2 \Lambda_l)}^{L_{l-1}} \ldots \int_{-c_l t^{1/5} \log(t)^{1/5} / (2 \Lambda_l)}^{L_{l-1}} \prod_{i=1}^{l} \left( g_{\mu_i, z_{i-1}^2} (z_i) \right) p_{l, \sigma, l}^{(y_0)} (z_l) \, dz_l \ldots \, dz_1 \\
\leq 8 e^{-\frac{c_l^2 \log(t)}{8 \Lambda_l}}.
\]

(8.38)

Finally, summing (8.38) for \( l \in \{2, \ldots, k - 1\} \) yields the statement of Lemma 8.10 \( \square \)

Proof of Lemma 8.11 One preparatory lemma is needed for this.

Lemma 8.22. For \( m \in \mathbb{N} \) arbitrary, \( y \leq 0 \)

\[
\int_{-\infty}^{L_1} \ldots \int_{-\infty}^{L_m-1} \prod_{i=1}^{m} \frac{1}{\sqrt{2 \pi |z_{i-1}|}} e^{-\frac{z_{i-1}^2}{2 \sigma_{i-1}^2}} e^{-\frac{z_{m-1}^2}{2 \sigma_{m-1}^2}} \, dz_{m-1} \ldots \, dz_1 \leq e^{-\frac{1}{4} \sum_{i=1}^{m} \frac{L_i^{(y_i,y_i)}}{L_i^{(y_i,y_i)}}^2}.
\]

Proof. We prove this using induction. Induction basis (\( m = 1 \)): Here no integrals remain and the statement trivially holds by definition of \( L^{(y,y)}(y) \) and \( L^{(0,0)}(y) \).

Induction step (\( m \to m + 1 \)): We have that

\[
\int_{-\infty}^{L_1} \ldots \int_{-\infty}^{L_m} \prod_{i=1}^{m} \frac{1}{\sqrt{2 \pi |z_{i-1}|}} e^{-\frac{z_{i-1}^2}{2 \sigma_{i-1}^2}} e^{-\frac{z_{m-1}^2}{2 \sigma_{m-1}^2}} \, dz_{m-1} \ldots \, dz_1
\]

\[
\leq \int_{-\infty}^{L_1} \frac{e^{-\frac{1}{4} \sum_{i=1}^{m} \frac{L_i^{(y_i,y_i)}}{L_i^{(y_i,y_i)}}^2}}{\sqrt{2 \pi |y|}} \, dz_1
\]

since \( L^{(y,y)}(x)^2 / L^{(y,y)}(x)^2 = C_3^2 \log((L^{(l-1)}(x))^2) \) is decreasing in \( x \) for \( x \leq -1 \) and \( z_1 \leq y \), which can be seen directly from the definition in (8.3) we have that

\[
\begin{align*}
\int_{-\infty}^{L_1} \ldots \int_{-\infty}^{L_m} \prod_{i=1}^{m} \frac{1}{\sqrt{2 \pi |z_{i-1}|}} e^{-\frac{z_{i-1}^2}{2 \sigma_{i-1}^2}} e^{-\frac{z_{m-1}^2}{2 \sigma_{m-1}^2}} \, dz_{m-1} \ldots \, dz_1 \\
\leq e^{-\sum_{i=1}^{m} \frac{L_i^{(y_i,y_i)}}{L_i^{(y_i,y_i)}}^2} \int_{-\infty}^{L_1} \frac{e^{-\frac{z_1^2}{2 \sigma_1^2}}}{\sqrt{2 \pi |y|}} \, dz_1 \leq e^{-\sum_{i=1}^{m+1} \frac{L_i^{(y_i,y_i)}}{L_i^{(y_i,y_i)}}^2} e^{-\frac{L_{m+1}^{(y)}}{8 \sigma_m}}.
\end{align*}
\]

\( \square \)
Proof of Lemma 8.11. Repeating the calculation in Lemma 8.10 we get that

\[
\int_{-c_1t^{1/5}\log(t)/2}^{L_1} \cdots \int_{-c_1t^{1/5}\log(t)/2}^{L_{k-1}} \int_{-\infty}^{L_k} \prod_{i=1}^{k} \left( g_{\mu_{i-1}, z_{i-1}^2}(z_i) \right) \, dz_k \cdots dz_1
\]

\[
\leq \int_{-\infty}^{L_1} \cdots \int_{-\infty}^{L_{k-1}} \int_{-\infty}^{L_k} \prod_{i=1}^{k} \left( g_{\mu_{i-1}, z_{i-1}^2}(z_i) \right) \, dz_k \cdots dz_1
\]

\[
\leq \int_{-\infty}^{L_1} \cdots \int_{-\infty}^{L_{k-1}} \prod_{i=1}^{k} \left( g_{\mu_{i-1}, z_{i-1}^2}(z_i) \right) \, dz_k \cdots dz_1
\]

\[
\leq e^{-2\lambda \log(t)}. \quad \square
\]

Proof of Lemma 8.12. We have that

\[
\int_{-\infty}^{0} g_{\mu(y,0,y^2,h), y^2}(z_1) p_{t, y^2, h}^{(y)}(z_1) \, dz_1
\]

\[
\leq e^{\frac{y^2}{2\nu^2} \left( \frac{(y-y^2)W_{y^2}}{\nu^2} - L_1(y^2)(y-x) \right)} \int_{-\infty}^{0} \int_{-\infty}^{0} \frac{z_1(y-x)}{\nu^2} g_{\mu(x,0,y^2,h), y^2}(z_1) p_{t, y^2, h}^{(y)}(z_1) \, dz_1
\]

\[
\leq e^{\frac{y^2}{2\nu^2} \left( \frac{(y-y^2)W_{y^2}}{\nu^2} - L_1(y^2)(y-x) \right)} e^{\frac{C_3 \log(\|y^2\|)}{\nu^2}} e^{\frac{C_5 \log(\|y\|)}{\nu^2}} p_{t, y^2, h}^{(y)}(x, y^2)
\]

\[
\leq C e^4 C_1 e^{2C_3 \log(\|y^2\|)} e^{2C_5 \log(\|y\|)} p_{t, h}^{(y)}(x, y^2). \quad \square
\]

Proof of Lemma 8.13. Using that \(|y| \leq 2\log(t)^2\) we have that \(L_1 = L_1(y) < -c_1t^{1/5}\log(t)^{1/2}\) for \(t\) big enough (independent of \(W\)), and thus get that

\[
\int_{-\infty}^{L_1 \wedge (-c_1t^{1/5}\log(t)/2)} g_{\mu(y,0,y^2,h), y^2}(z_1) p_{t, y^2, h}^{(y)}(z_1) \, dz_1 \leq \int_{-\infty}^{c_1t^{1/5}\log(t)/2} 2g_{0,4y^2}(z_1) \, dz_1 \leq 8e^{-c_1t^{1/5}\log(t)/2} \leq 8e^{-c_1t^{1/5}\log(t)/2}. \quad \square
\]

9 Getting rid of \(h_t\) — Preparation for Section 12

In all of this section assume that \(t \geq 0, y \in [-\log(t)^2, 0], y_0 \in [-\log(t)^2, 0]\) are fixed.

Furthermore, let \((X_s)_{s \geq 0}\) be the universal object in what follows and assume that under \(P_y\) it is distributed as a Brownian motion with start in \(y\). In this section we will analyze how adding a shift \(h_t\) to a barrier changes the probability that \(X_s\) stays below the barrier. In Section 9.1 the additional shift will be linear, in Sections 9.3 and 9.4 we add a positive/negative banana. The main tool in this section is the Girsanov theorem, we define measures with regard to which \((X_s + h_t(s))_{s \in [0,t]}\) is a Brownian motion and control the corresponding Radon-Nikodym derivatives.

9.1 Linear Shifts

We recall Definition 2.1

Definition 9.1. Consider \((c_t)_{t \geq 0}\) in \(\mathbb{R}\) and set

\[
A_{t, h}^{(y_0)} := \mathcal{B}_{[0,t], h_t(-W)}^0 (X_t),
\]

\[
A_{t, h, c}^{(y_0)} := \mathcal{B}_{[0,t], h_t(-W)}^0 (X_t + sc_t).
\]

In this section we prove the following theorem.

Theorem 9.1. We have that for \(t \geq 0\) and \(y, y_0 \leq 0\),

\[
\frac{P_y[A_{t, h, c}^{(y_0)}]}{P_y[A_{t, h}^{(y_0)}]} = e^{-\frac{c_t^2}{2}} e^{-y_t - h_t(t)c_t + |y_0 - 1||c_t|} e^{W_t c_t},
\]

\[
\frac{P_y[A_{t, h}^{(y_0)}]}{P_y[A_{t, h, c}^{(y_0)}]} = e^{-\frac{c_t^2}{2}} e^{-y_t - h_t(t)c_t - |y_0 - 1||c_t|} e^{W_t c_t}.
\]
Proof. Set
\[
Z_t := \frac{dQ_y}{dP_y} := \exp \left( - \int_0^t c_t \, dX_s - \frac{1}{2} \int_0^t c_t^2 \, ds \right) = \exp \left( -c_t(X_t - X_0) - \frac{ct^2}{2} \right)
\]

The Girsanov Theorem then gives
\[
P_y \left[ A^{(y_0)}_{t,h,c} \right] = \mathbb{E}_{Q_y} \left[ (Z_t)^{-1} 1_{A^{(y_0)}_{t,h,c}} \right] = e^{\frac{ct^2}{2}} \mathbb{E}_{Q_y} \left[ e^{c_tX_t - yc_0} 1_{A^{(y_0)}_{t,h,c}} \right].
\]

On \( A^{(y_0)}_{t,h,c} \) we have that
\[
W_t - tc_t + y_0 - 1 - h_t(t) \leq X_t \leq W_t - tc_t - h_t(t) + y_0.
\]

Plugging (9.2) into (9.1) yields that
\[
P_y \left[ A^{(y_0)}_{t,h,c} \right] \leq e^{\frac{ct^2}{2}} \mathbb{E}_{Q_y} \left[ e^{(W_t - tc_t - h_t(t) + y_0 - 1c_t < 0)c_t - yc_0} 1_{A^{(y_0)}_{t,h,c}} \right]
\leq e^{-\frac{ct^2}{2}} e^{-yc_t - h_t(t)c_t + |y_0| - 1} c_t \mathbb{E}_{W_t} \left[ y_0 \right],
\]

\[
P_y \left[ A^{(y_0)}_{t,h,c} \right] \geq e^{\frac{ct^2}{2}} \mathbb{E}_{Q_y} \left[ e^{(W_t - tc_t + y_0 - 1c_t > 0 - h_t(t)c_t - yc_0)} 1_{A^{(y_0)}_{t,h,c}} \right]
\geq e^{-\frac{ct^2}{2}} e^{-yc_t - h_t(t)c_t - |y_0| - 1} c_t \mathbb{E}_{W_t} \left[ y_0 \right].
\]

Rearranging yields the claim. \( \square \)

9.2 General Setup

Fix \( t > 0 \) and \( h_t : [0, t] \to \mathbb{R} \) with \( h_t(t) = 0 = h_t(0) \). Set \( \chi(t) := \log(t)^{1.5} \). Define \( \tilde{W} \) via \( \tilde{W}_s := W_{t-s} - W_t, s \in [0, t] \). We recall Section 8.1, we use \( \tilde{\mathcal{C}} \) to denote variants of constants using \( \tilde{W} \) instead of \( W \), for example we set \( \tilde{C}_{\log} := C_{\log}(\tilde{W}, t) \).

From now on we will need two additional assumptions. Namely we assume that
\[
\log(t)^{-0.5} \max\{C_{\log}, \tilde{C}_{\log}\} \leq \frac{1}{3}
\]

and that there are \( C_4 := C_4(W, t) < \infty, \tilde{C}_4 := C_4(\tilde{W}, t) < \infty, c_2 := c_2(W, t) > 0, \tilde{c}_2 := c_2(\tilde{W}, t) > 0 \) such that for all \( r \leq \chi(t) \)
\[
P_{-1} \left[ B^{(y)}_{0, r}, \tilde{W}(X) \right] \geq c_2(1 + r)^{-C_4},
\]

\[
P_{-1} \left[ B^{(y)}_{0, r}, \tilde{W}(X) \right] \geq \tilde{c}_2(1 + r)^{-\tilde{C}_4}.
\]

We will check in Section 11 that these assumptions hold for the specific \( W \) we use in the first half of the paper.

Definition 9.2. For \( t \geq 0, h_t : [0, t] \to \mathbb{R}, y \in [-\log(t)^2, 0], y_0 \in [-\log(t)^2, 0] \) define
\[
\tilde{p}^{(y_0)}_{t,h}(y) := P_y \left[ A^{(y_0)}_{t,h} \right] := P_y \left[ B^{(y_0)}_{0, t}, \tilde{W}(X + h_t') + \frac{\tilde{W}_t}{t} \right],
\]

\[
p^{(y_0)}_{t,h}(y) := P_y \left[ A^{(y_0)}_{t,h} \right] := P_y \left[ B^{(y_0)}_{0, t}, W(X + h_t') \right],
\]

\[
p^{(y_0)}_{t,h}(y) := P_y \left[ A^{(y_0)}_{t,h} \right] := P_y \left[ B^{(y_0)}_{0, t}, \tilde{W}(X) \right].
\]

Set
\[
Z_t := \frac{dQ_y}{dP_y} := \exp \left( \int_0^t -h_t'(s) - \frac{W_t}{t} \, dX_s - \frac{1}{2} \int_0^t \left( h_t'(s) + \frac{W_t}{t} \right)^2 \, ds \right)
\]

\[
= \exp \left( h_t'(t)X_t - h_t'(0)y + \frac{W_tX_t}{t} - \int_0^t h_t'(s)X_s \, ds + \frac{1}{2} \int_0^t h_t'(s)^2 + 2h_t'(s) \frac{W_t}{t} + \frac{W_t^2}{2t^2} \, ds \right).
\]

We have that
\[
\tilde{p}^{(y_0)}_{t,h}(y) := P_y \left[ A^{(y_0)}_{t,h} \right] = \mathbb{E}_{Q_y} \left[ (Z_t)^{-1} 1_{A^{(y_0)}_{t,h}} \right]
\]

(9.7)
and since Girsanov implies that \((X_s + h_t(s) + \frac{t}{2} W_t)_{s \leq t}\) is a Brownian motion with regard to \(Q_y\), reduces controlling \(\frac{p_{t,h}^{(y)}}{p_{t,h}^{(0)}}(y)\) to controlling \((Z_t)^{-1}\) on \(1_{A_t^{(y)}}\). Furthermore, Section 9.1 allows us to control \(\frac{p_{t,h}^{(y)}}{p_{t,h}^{(0)}}(y)\), since applying Theorem 9.1 with \(c_t = \frac{W_t}{t}\) yields that for \(t \geq c\), \(y, y_0 \in [-2 \log(t)^2, 0]\)

\[
\left| \frac{p_{t,h}^{(y)}}{p_{t,h}^{(0)}}(y) \right| \leq e^{\frac{w_t^2}{t^2}} e^{-\frac{3}{t} |y|} e^{\frac{|y_0|}{t}} \left| \frac{W_t}{t} \right| \leq e^{\frac{w_t^2}{t^2}} e^{-10c_0 \log t}.
\]

\[
\left| \frac{p_{t,h}^{(y)}}{p_{t,h}^{(0)}}(y) \right| \geq e^{\frac{w_t^2}{t^2}} e^{-\frac{s}{t} |y|} e^{-\frac{|y_0+1|}{t}} \left| \frac{W_t}{t} \right| \geq e^{\frac{w_t^2}{t^2}} e^{-10c_0 \log t}.
\]

### 9.3 Bound Used in the Upper Bound of the Right Tail

In this section let \(h_t : [0, t] \to \mathbb{R}, s \mapsto \left(\left((1 + s) \wedge (1 + t - s)\right)^{1/6} - 1\right)\).

The aim of this section is to derive an upper bound on \(p_{t,h}^{(y)}(y)/p_{t,h}^{(0)}(y)\), to formulate this upper bound we need some additional definitions. We recall (8.2), (8.4) and (8.5). We recall Definition 9.2.

**Definition 9.3.** For \(\tilde{W} := (W_{t-s} - W_t)_{s \in [0, t]}\) and \(y, y_0 \in [-\log(t)^2, 0]\) set

\[
p_t^\ast := p_t^\ast( W, t, h, s ) := e^{5\log(1 + t) / 2} \left( 1 + \chi(t) \right) e^{\frac{1}{2} \log(1 + t) / 2} \left( 1 + \frac{y}{y_0} \right).
\]

\[
p_t^\ast := p_t^\ast( W, t, h, s ) := e^{5\log(1 + t) / 2} \left( 1 + \chi(t) \right) e^{\frac{1}{2} \log(1 + t) / 2} \left( 1 + \frac{y}{y_0} \right).
\]

where \(p_t^\ast( y )\) uses \(\tilde{W}\) instead of \(W\).

Choose \(\kappa := \kappa(W, \tilde{W}, t, h, y, y_0) > 0\) such that

\[
\forall s \in [0, t] 
\kappa(1 + s) \geq h_t(s) \leq -2 \max \{C_{\log, \tilde{C}_{\log}}\}(1 + s)^{1/2}.
\]

Furthermore, fix \(1/8 \geq \delta > 0\) and choose \(\eta := \eta(W, \tilde{W}, t, h, \delta, y, y_0) \in (0, \delta)\) such that

\[
\sum_{i=1}^\infty 6 \frac{e^{-\frac{1}{2} \log(t)^2}}{e^{15\log(1 + t) / 2} \left( 1 + \chi(t) \right) e^{\frac{1}{2} \log(1 + t) / 2} \left( 1 + \frac{y}{y_0} \right)} \leq \delta.
\]

\[
\sum_{i=1}^\infty 6 \frac{e^{-\frac{1}{2} \log(t)^2}}{e^{15\log(1 + t) / 2} \left( 1 + \chi(t) \right) e^{\frac{1}{2} \log(1 + t) / 2} \left( 1 + \frac{y}{y_0} \right)} \leq \delta.
\]

In this section we will prove the following theorem.

**Theorem 9.2.** There exist \(C, C > 0\) such that for \(\lambda > 0\) arbitrary there is a \(t_0(\lambda) \geq 0\) such that for \(t \geq t_0(\lambda)\) with \(9.3\) and \(y, y_0 \in [-\log(t)^2, 0]\),

\[
\frac{p_{t,h}^{(y)}}{p_{t,h}^{(0)}}(y) - \frac{\delta}{1/8} \left( \sum_{j=0}^{2} p_{t,h}^{(y)}(y) (y - j) \right) \leq C e^{\log(1 + t) / 2} \left( 1 + t^{-\lambda} p_{t,h}^{(y)}(y)^{-1} \right) + C t^{-\lambda} p_{t,h}^{(y)}(y)^{-1} \left( p_t^\ast( y_0 ) \right) p_{t,h}^{(y)}(y)^{-1}.
\]

**Remark 9.1.** We note that \(\eta\) and \(\kappa\) can be chosen as continuous functions of \(y, y_0, C_{\log, \tilde{C}_{\log}}, C_{2}^{-1}, \tilde{C}_{2}^{-1}, \gamma, \tilde{C}_{4}, C_{4, 2}^{-1}, C_t(\tilde{W}, t), c_t(\tilde{W}, t)^{-1}, \delta\).

**Definition 9.4.** To slightly shorten the following definitions shortens \(X_h^s := X_s + h_t(s)\). Using this define

\[
\tilde{A}_{t,h,\chi}(t) := \{ \forall s \in [0, t] X_h^s \geq -\kappa(1 + s)^{1/2} + y + \eta \},
\]

\[
\tilde{A}_{t,h,\chi}(t) := \{ \forall s \in [0, t] X_h^s \geq -\kappa(1 + t - s)^{1/2} + X_t + \eta \},
\]

\[
\tilde{A}_{t,h,\chi}(t) := \tilde{A}_{t,h}(y_0) \cap \tilde{A}_{t,h,\chi,\text{start}} \cap \tilde{A}_{t,h,\text{end}}.
\]

\[
B_{t,h,\chi}(t,k) := \left\{ \frac{\min_{t \in [0, t]} X_h^s - s W_t - y}{(1 + s)^{1/2}} < k, \frac{\min_{t \in [0, t]} X_h^s - (t - s) W_t - y}{(1 + t - s)^{1/2}} < -k \right\} \in [-k, k + 1] \right\}.
\]

\[
\tilde{p}_{t,h,\chi}(y) := \mathbb{P}_y \left[ A_{t,h,\chi}(t) \right].
\]

Note that under \(Q_y\) we have that \((X_h^s + \frac{t}{2} W_t)_{s \leq t}\) is a Brownian motion starting at \(y\).

The proof of Theorem 9.2 is split into the following two propositions.
**Proposition 9.3.** There are $c, C > 0$ such that for $y, y_0 \in [-\log(t)^2, 0]$ and $\lambda > 0$ there is a $t_0(\lambda) > 0$ such that for all $t \geq t_0(\lambda)$,
\[
\frac{\hat{p}_{t,h,\chi}(y)}{p_{t,h}(y)} \leq Ce^{c(\kappa - \eta + \zeta)} \exp\left(\frac{W_2^2}{2t}\right) \left(1 + t^{-\lambda}p_{t,h}(y)^{-1}\right).
\]

**Proposition 9.4.** There is a $C > 0$ such that for $y, y_0 \in [-\log(t)^2, 0]$, $\lambda > 0$ there is a $t_0(\lambda) > 0$ such that for $t \geq t(\lambda)$, for which additionally (9.3) holds,
\[
\frac{\hat{p}_{t,h}(y) - \delta}{\hat{p}_{t,h}(y) - \delta} \sum_{j=0}^{n-1} \hat{p}_{t,h}(y - j) \leq \frac{2 + Ct^{-\lambda}(p_{t,h}(y)^{-1} + (p_{t,h}(y)^{-1})^2)}{p_{t,h,\chi}(y)}.
\]

**Proof of Theorem 9.2 assuming Propositions 9.3 and 9.4.** Combine (9.3) with Proposition 9.3 and Proposition 9.4.

**Lemma 9.5.** There is a $C > 0$ such that for $y, y_0 \in [-\log(t)^2, 0]$ and for all $k \geq 2$,
\[
Q_y \left[\hat{A}_{t,h,\chi}(t) \cap B_{t,h,\chi}(t,k)\right] \leq Ce^{-\frac{(k-1)^2}{2}(\lambda(t) - 1)}.
\]

**Proof of Proposition 9.3 assuming Lemma 9.5.**

We have that
\[
\hat{p}_{t,h,\chi}(y) = \mathbb{E}_y \left[ \hat{A}_{t,h,\chi}(y) \right] = \sum_{k=1}^{\infty} \mathbb{E}_y \left[ (Z_t)^{-1} \hat{A}_{t,h,\chi}(y) \right] \leq \sum_{k=1}^{\infty} \mathbb{E}_y \left[ (Z_t)^{-1} \hat{A}_{t,h,\chi}(y) \right].
\]

We can infer by direct calculation from (9.3) and Definition 9.4 that there are $c, C > 0$ such that on $\hat{A}_{t,h,\chi}(t) \cap B_{t,h,\chi}(t,k)$
\[
(Z_t)^{-1} \leq Ce^{c(\kappa - \eta + \zeta)} \exp\left(\frac{W_2^2}{2t}\right).
\]

In this calculation we use assumption (8.2) as well as $|y|, |y_0| \leq \log(t)^2$. Plugging (9.15) into (9.14) yields that
\[
\hat{p}_{t,h,\chi}(y) \leq Ce^{c(\kappa - \eta + \zeta)} \exp\left(\frac{W_2^2}{2t}\right) \left(\sum_{k=1}^{\infty} e^{c\chi(t)^{-1} - \frac{1}{2}} \exp\left(\frac{W_2^2}{2t}\right) \left(p_{t,h}(y) + \sum_{k=2}^{\infty} e^{c\chi(t)^{-1} - \frac{1}{2}} \exp\left(\frac{W_2^2}{2t}\right) \left(p_{t,h}(y) + e^{-\frac{1}{2}(\lambda(t) - 1)} \right)\right)\right).
\]

Since $e^{-1/2(\lambda(t) - 1)} \to 0$ for $t \to \infty$ and all $\lambda > 0$, rearranging this yields the claim of Proposition 9.3.

**Proof of Lemma 9.5.** We have that
\[
Q_y \left[ B_{t,h,\chi}(t,k)\right] \leq \sum_{l=\lfloor \chi(t) \rfloor}^{\lfloor \chi(t) \rfloor} Q_y \left[ \min_{1 \leq s \leq l} X_s^h + s/tW_t - y < -(k-1) \cdot l^2 \right] + \sum_{l=\lfloor \chi(t) \rfloor}^{\lfloor \chi(t) \rfloor} Q_y \left[ \min_{1 \leq s \leq l} X_s^h - (t-s)/tW_t - y < -(k-1)(t) \cdot l^2 \right],
\]

First handle the sum from $|\chi(t)|$ to $\lfloor \chi(t) \rfloor$ in (9.16). Since under $Q_y$ $(X_s^h + s/tW_t - y)_{s \in [0,t]}$ is a Brownian motion starting at 0 with regard to $Q_y$, we have that
\[
\sum_{l=\lfloor \chi(t) \rfloor}^{\lfloor \chi(t) \rfloor} Q_y \left[ \min_{1 \leq s \leq l} X_s^h + s/tW_t - y < -(k-1) \cdot l^2 \right] \leq \sum_{l=\lfloor \chi(t) \rfloor}^{\lfloor \chi(t) \rfloor} 4e^{-\frac{(k-1)^2}{2}} \leq Ce^{\frac{(k-1)^2}{2}(\lambda(t) - 1)}.
\]

The second sum in (9.16) can be handled analogously. Combining (9.16) and the last two facts proves the lemma.
Proof of Proposition 9.4 We recall Definitions 9.3 and 9.4 We have that
\[
\tilde{P}_{t,h,\lambda}(y) \geq \tilde{P}_{t,h}(y) - \mathbb{P}_y \left[ \tilde{A}_{t,h}^{(\gamma)} \cap \tilde{A}_{t,h,\text{start}}^c \right] - \mathbb{P}_y \left[ \tilde{A}_{t,h}^{(\gamma)} \cap \tilde{A}_{t,h,\text{end}}^c \right].
\] (9.18)

We proceed by proving upper bounds for \( \mathbb{P}_y \left[ \tilde{A}_{t,h}^{(\gamma)} \cap \tilde{A}_{t,h,\text{start}}^c \right] \) and \( \mathbb{P}_y \left[ \tilde{A}_{t,h}^{(\gamma)} \cap \tilde{A}_{t,h,\text{end}}^c \right] \), we do this in detail for the first term and handle the second term by symmetry. Thus we aim to prove the following two propositions.

Proposition 9.6. There is a \( C > 0 \) such that for \( y,y_0 \in [\log(t)^2,0], \lambda > 0 \) there is a \( t_0(\lambda) > 0 \) such that for \( t \geq t_0(\lambda) \), for which (9.3) holds,
\[
\mathbb{P}_y \left[ \tilde{A}_{t,h}^{(\gamma)} \cap \tilde{A}_{t,h,\text{start}}^c \right] \leq \frac{\tilde{P}_{t,h}^{(\gamma-1)}(y) + \tilde{P}_{t,h}^{(y)}(y)}{8} + Ct^{-\lambda}(p_t^*)^{-1}.
\]

Proposition 9.7. There is a \( C > 0 \) such that for \( y,y_0 \in [\log(t)^2,0], \lambda > 0 \) there is a \( t_0(\lambda) > 0 \) such that for \( t \geq t_0(\lambda) \), for which (9.3) holds,
\[
\mathbb{P}_y \left[ \tilde{A}_{t,h}^{(\gamma)} \cap \tilde{A}_{t,h,\text{end}}^c \right] \leq \frac{\tilde{P}_{t,h}^{(\gamma-1)}(y-2) + \tilde{P}_{t,h}^{(y)}(y-1)}{8} + Ct^{-\lambda}(p_t^*)^{-1}.
\]

Remark 9.2. We prove Proposition 9.7 by time reversal, thus essentially copying the proof of Proposition 9.6.

Proof of Proposition 9.4 assuming Proposition 9.6 and 9.7 Plugging Propositions 9.6 9.7 into equality 9.18 directly implies the claim.

We need one further definition before proceeding with the proof of Proposition 9.6

Definition 9.5. For \( r \leq \chi(t) \) set
\[
\tau := \inf \left\{ s \leq t : X_s - y \leq -2C_\log(1+s) + \eta \right\},
\]
\[
w_r := y - 2C_\log(1+r) \log \lambda + \frac{W_t r}{t} + \eta,
\]
\[
m_r(s) := W_s + h_t(s) + r.
\]
(9.19)

(9.20)

The rough strategy is to bound
\[
\mathbb{P}_y \left[ \tilde{A}_{t,h}^{(\gamma)} \cap \tilde{A}_{t,h,\text{start}}^c \right] \leq \sum_{\ell = 1}^{\lceil \chi(t) \rceil} \mathbb{P}_y [\tau \in [\ell - 1, \ell]] \mathbb{P}_{w_r \sim W_t / t} \left[ B_{[0,t-r], \sim m_r,()}^{(\gamma)} \right](X),
\]

use that \( \tau \) has exponential tails and bound
\[
\mathbb{P}_{w_r \sim W_t / t} \left[ \tau \leq t - 1 \right] \leq m_r(s), X_{t-r} - W_t \in [y_0 - 1, y_0]
\]

by applying Theorems 9.1 8.1 and 8.3. We break this up into a few lemmata, which we state next. After this we will prove Proposition 7.6 assuming the lemmata. Finally, we will prove the lemmata.

Lemma 9.8. We have that for \( t \geq \epsilon \) and \( y,y_0 \in [-\log(t)^2,0] \),
\[
\mathbb{P}_y \left[ \tilde{A}_{t,h}^{(\gamma)} \cap \tilde{A}_{t,h,\text{start}}^c \right] \leq \sum_{\ell = 1}^{\lceil \chi(t) \rceil} \mathbb{P}_y [\tau \in [\ell - 1, \ell]] \max_{r \in [\ell - 1, \ell]} \left( e^{w_r^2} \frac{w_r^2}{\gamma} \log \lambda \mathbb{P}_{w_r} \left[ B_{[0,t-r], \sim m_r,()}^{(\gamma)} \right] \right).
\]

Lemma 9.9. For \( t \in \mathbb{N} \) and \( y,y_0 \in [-\log(t)^2,0] \), we have that
\[
\mathbb{P}_y [\tau \in [t - 1, t]] \leq 2e^{-(2C_\log(2) \log(t))^2 + \eta^2 t^{-1}}.
\]

Lemma 9.10. We have that for \( t > 0 \) and \( y,y_0 \in [-\log(t)^2,0] \),
\[
\mathbb{P}_{w_r} \left[ B_{[0,t-r], \sim m_r,()}^{(\gamma)} \right] \leq \left( \rho_t^{(\gamma-1)}(w_r - W_r) + \rho_t^{(y)}(w_r - W_r) \right) \mathbb{P}^{-1} \left[ B_{[0,r], \sim W_t,()}^{(\gamma)} \right]^{-1}.
\]

Lemma 9.11. We have that for \( y,y_0 \in [-\log(t)^2,0], \lambda > 0 \) arbitrary and \( t \) for which (9.3) holds,
\[
\rho_t^{(y)}(w_r - W_r) \leq \rho_t^{(y)}(y) C_\gamma 2^{- \gamma + 2 \log(\lambda)} (|w_r - W_r|^{\gamma + 3} C_\gamma + C_\gamma^{5} \log(\lambda)^2 \log(t)^2) \rho_t^* \lambda^{-1}.
\]
Proof of Proposition 9.6 assuming Lemmata 9.8–9.11. By (9.9) we have that
\[ p_{t,h}^{(y_0)}(y) \leq e^{-\frac{w^2}{2r} + 10C\log} p_{t,h}^{(y_0)}(y). \] 

Equation (9.21), Assumption (9.4) and Lemmata 9.8–9.11 imply that for \( \lambda > 0 \) arbitrary
\[
\Pr_y \left[ \tilde{A}_{t,h}^{(y_0)} \cap \tilde{A}_{t,h,\text{start}}^c \right] 
\leq \sum_{l=1}^{\lfloor \chi(t) \rfloor} 2e^{-\frac{(2C_{\log})^2}{2} \frac{1}{r} + t^2} \max_{r \in \lfloor l-1, l \rfloor} \left( \frac{w^2}{r} - \frac{w^2}{t} - r + 5C_l \log \left( \left( p_{t,h}^{(y_0-1)}(w_t - W_t) + p_{t,h}^{(y_0)}(w_t - W_t) \right) \cdot \Pr_{-1} \left[ B_{[0,r]}^{(y_0)}(X) \right] \right)^{-1} \right) 
\leq C e^{-c_{\log}^2} \frac{w^2}{t} - \frac{w^2}{t} - l^2 \left( p_{t,h}^{(y_0-1)}(y) + p_{t,h}^{(y_0)}(y) \right) C_2 e^{4C_1 \log} e^{3C_3}.
\]

Using \( l \leq \lfloor \chi(t) \rfloor \) and the definition of \( w_r \) in (9.19) yields that
\[
\max_{r \in \lfloor l-1, l \rfloor} \left| w_r - W_r \right| \leq |y| + 3C\log(1 + l)^{\frac{3}{2}} + |\eta|,
\]
\[
\frac{W_r^2}{2l} - \frac{W_r^2}{2l} - l \leq \frac{W_r^2 \lfloor \chi(t) \rfloor}{2l^2} \leq c_{\log}^2.
\]

Plugging (9.24) and (9.25) into (9.23) and remembering the choice of \( \eta \) in (9.11) finishes the proof of Proposition 9.6. \( \square \)

Proof of Lemma 9.7. By the choice of \( \kappa \) in (9.10) we have that
\[-\kappa(1 + s)^{\frac{3}{2}} - h_t(s) \leq -2C_{\log}(1 + s)^{\frac{3}{2}} \]
for all \( s \leq \chi(t) \). Thus we have that
\[
\Pr_y \left[ \tilde{A}_{t,h}^{(y_0)} \cap \tilde{A}_{t,h,\text{start}}^c \right] 
\leq \Pr_y \left[ X \leq \chi(t) : X_s - y \leq -2C_{\log}(1 + s)^{\frac{3}{2}} \right]
\leq \Pr_y \left[ X \leq \chi(t) : X_s - y \leq -2C_{\log}(1 + s)^{\frac{3}{2}} + |\eta| \right]
\leq \sum_{l=1}^{\lfloor \chi(t) \rfloor} \Pr_y \left[ \tau \in \lfloor l-1, l \rfloor \right] \max_{r \in \lfloor l-1, l \rfloor} \Pr_{w_r} \left[ B_{[0,t-r]}^{(y_0)}(X_s + W_t(s + r)) \right]
\leq \sum_{l=1}^{\lfloor \chi(t) \rfloor} \Pr_y \left[ \tau \in \lfloor l-1, l \rfloor \right] \max_{r \in \lfloor l-1, l \rfloor} \Pr_{w_r} \left[ B_{[0,t-r]}^{(y_0)}(X_s + W_t(s + r)) \right]
\leq \sum_{l=1}^{\lfloor \chi(t) \rfloor} \Pr_y \left[ \tau \in \lfloor l-1, l \rfloor \right] \max_{r \in \lfloor l-1, l \rfloor} e^{-\frac{w^2}{r} + \frac{w^2}{t} - r + 5C_{\log} \Pr_{w_r} \left[ B_{[0,t-r],h_t(s)}(X) \right]}
\leq \sum_{l=1}^{\lfloor \chi(t) \rfloor} \Pr_y \left[ \tau \in \lfloor l-1, l \rfloor \right] \max_{r \in \lfloor l-1, l \rfloor} e^{-\frac{w^2}{r} + \frac{w^2}{t} - r + 5C_{\log}.}
\]
where in the last step we applied Theorem 9.1. \( \square \)

Proof of Lemma 9.9. We have that
\[
\Pr_y \left[ \tau \in \lfloor l-1, l \rfloor \right] \leq \Pr_y \left[ \min_{s \leq \tau} X_s - y \leq -2C_{\log}^{\frac{3}{2}} + |\eta| \right] 
= 2\Pr_y \left[ (X_\tau - y) / \sqrt{t} \leq -2C_{\log}^{1/2} + |\eta|^{1/2} \right]
\leq 2e^{-\frac{(2C_{\log})^2}{2} \frac{1}{r} + t^2}. \]

Proof of Lemma 9.10. Applying (2.3) from Lemma 2.3 for \( z_1 = w_r - W_r, z_2 = y_0, I = [0, t], t_0 = r, f(s) = h_t(s) - W_r \) and \( x_0 = w_r \) yields that
\[ p_{t,h}^{(y_0-1)}(w_r - W_r) + p_{t,h}^{(y_0)}(w_r - W_r) \geq \Pr_{w_r \sim W_r} \left[ B_{[r-I]}^{h_t(s)}(X) \right] \Pr_{w_r} \left[ B_{[0,t-r],h_t(s)}^{(y_0)}(X - X_r) \right]. \]

Using \( (X_s - X_r)_{s \geq r} \overset{d}{=} (X_s)_{s \geq 0} \) and the definition of \( m_r \) in (9.20) yields that
\[
\Pr_{w_r} \left[ B_{[r-I],h_t(s)}^{(y_0)}(X - X_r) \right] = \Pr_{w_r} \left[ B_{[0,t-r],h_t(s)}^{(y_0)}(X) \right].
\]
and thus it remains to prove that

\[
P_{w_r-W_r} \left[ B_{t,h,r,s}(X) - w_r - W_r(X) \right] \geq P_{-1} \left[ B_{t,h,r,s}(X) - W_r(X) \right]. \tag{9.26}
\]

Equation (9.26) is an immediate consequence of the facts that on the one hand \( w_r - W_r \leq -1 \) by the definition of \( C_\log \) and \( w_r \) in (8.2) and (9.19) and on the other hand \( h_t(s) \leq 0 \) for all \( s \leq 0 \).

**Proof of Lemma 9.11.** Since \( r \leq \chi(t) \), recalling (9.19) and applying (8.2) yields that

\[
|w_r - W_r| \leq \log(t)^2 + \log(t)^{1.5} + \log(t)^{1.5}C_\log \leq 2\log(t)^2
\]

by assumption (9.3). Furthermore, using the definition of \( C_\log \) and \( \eta < 0 \), yields that

\[
w_r - W_r = y - 2C_\log(1 + r)^{\frac{3}{2}} + W_r t + \eta - W_r \leq y.
\]

Thus we can apply Theorems 8.1 and 8.3 to get that there is a \( t_0(\lambda) \) such that for \( t \geq t_0(\lambda) \)

\[
p_t^{(yo)}(w_r - W_r) \leq p_t^{(yo)}(y)C_2^{-2}e^{4C_1}q^{r+2}(|w_r - W_r|)^{q+3}C_3 + C_2^{-1}log(t)^q t - \lambda p_t^{(yo)}(y)^{-1},
\]

which is what we wanted to prove.

**Proof of Proposition 9.7.** Define \( Y_s := X_{t-s} - X_t \). We have that

\[
P_y \left[ \tilde{A}_{t,h,s} \cap \tilde{A}_{t,h,s} \right] \leq P \left[ B_{t,h,s} \left( Y_s + h_t(s) + \frac{s}{t}W_t \right) \right], \quad \forall s \in [0,\chi(t)] Y_s \leq -\kappa(1 + s)^{\frac{3}{2}} - h_t(s) + y_0 + \eta \tag{9.27}
\]

Furthermore, we have that

\[
\tilde{p}_{t,h}(y) = P \left[ B_{t,h,s} \left( X_s + h_t(s) + \frac{s}{t}W_t \right) \right] \geq P \left[ B_{t,h,s} \left( Y_s + h_t(s) + \frac{s}{t}W_t \right) \right]. \tag{9.28}
\]

We notice that \((Y_s)_{s \in [0,t]} \stackrel{d}{=} (X_s)_{s \in [0,t]}\). Thus rerunning the argument proving Proposition 9.6, while considering equations (9.27) and (9.28), finishes the proof.

### 9.4 Bounds Used in the Lower Bound of the Right Tail

Consider \( h_t : [0,t] \rightarrow \mathbb{R}, s \mapsto \frac{((1 + s) \wedge (1 + t - s))^{1/6} - 1}{1} \) and \( y \in [-\log(t)^2, 0], y_0 \in [-\log(t)^2, 0] \).

We prove the following theorem.

**Theorem 9.12.** For \( \eta, \kappa \) defined analogously to Theorem 9.2, but with the \( h \) we have in this section, we have that there are \( c, C > 0 \) such that for arbitrary \( \lambda > 0 \) there is a \( t_0(\lambda) > 0 \) such that for \( t \geq t_0(\lambda) \) with (9.3) and for \( y, y_0 \in [-\log(t)^2, 0] \),

\[
\frac{p_t^{(yo)}(y)}{p_t^{(yo)}(y) - \frac{s}{t} \sum_{j=0}^{\infty} p_t^{(yo-1)}(y-j)} \geq C e^{-(C_\log + \kappa - \eta)} \left( \frac{1}{2} - Ct^{-\lambda} \left( (p_t^{(yo)})^{-1} + (p_t^{(yo)})^{-1} + 1 \right) p_t^{(yo)}(y)^{-1} \right) \tag{9.29}
\]

**Sketch of the proof of Theorem 9.12.** We recall (9.6). Furthermore, define

\[
B_{t,h,x}(t,2) := \left\{ \min_{x(\chi(t) \leq s \leq t/2} \frac{X_h^h + s/W_t - y}{(1 + s)^{\frac{3}{2}}} \leq -2 \right\} \cap \left\{ \min_{(t/2) \leq s \leq t - \chi} \frac{X_h^h - (t-s)/W_t - y}{(1 + t - s)^{\frac{3}{2}}} \geq -2 \right\}. \tag{9.29}
\]

As in (9.15) we can infer from (9.6) and (9.29) that there are \( C, c > 0 \) such that

\[
\tilde{p}_{t,h}(y) = E_{q_y} \left[ (Z_t)^{-1} \tilde{A}_{t,h,t} \right] \geq E_{q_y} \left[ (Z_t)^{-1} \tilde{A}_{t,h,h}(t) \right] B_{t,h,x}(t,2) \geq C e^{-(C_\log + \kappa - \eta)} \exp \left( \frac{W_t^2}{2t} \right) Q_y \tilde{A}_{t,h,x}(t) \cap B_{t,h,x}(t,2) \tag{9.30}
\]

We have that \( Q_y \tilde{A}_{t,h,x}(t) \cap B_{t,h,x}(t,2) \geq Q_y \tilde{A}_{t,h,x}(t) - Q_y \tilde{A}_{t,h,x}(t) \cap B_{t,h,x}(t,2). \) As in Lemma 9.5 we have that

\[
Q_y \tilde{A}_{t,h,x}(t) \cap B_{t,h,x}(t,2) \leq Ct^{-\lambda} \tag{9.31}
\]

for \( t \) big enough, depending on \( \lambda \), but not on \( W \).
Equations (9.30) and (9.31) imply that for $t$ big enough depending on $\lambda$, but not on $W$,
\[
\tilde{p}_{t,h}^{(y_0)}(y) \geq Ce^{-c(G_{\log}+\kappa-\eta)} \exp \left( \frac{W^2}{2t} \right) \left( Q_y \left[ \tilde{A}_{t,h,\lambda}(t) \right] - t^{-\lambda} \right). \tag{9.32}
\]

Furthermore, we have that
\[
Q_y \left[ \tilde{A}_{t,h,\lambda}(t) \right] = Q_y \left[ \tilde{A}_{t,h}^{(y_0)} \right] - Q_y \left[ \tilde{A}_{t,h}^{(y_0)} \cap \tilde{A}_{t,h,\text{start}} \right] - Q_y \left[ \tilde{A}_{t,h}^{(y_0)} \cap \tilde{A}_{t,h,\text{end}} \right] = p_t^{(y_0)}(y) - Q_y \left[ \tilde{A}_{t,h}^{(y_0)} \cap \tilde{A}_{t,h,\text{start}} \right] - Q_y \left[ \tilde{A}_{t,h}^{(y_0)} \cap \tilde{A}_{t,h,\text{end}} \right].
\]

From here one proceeds as in the proof of Proposition 9.4. We need to calculate the tails of $\tau$ with regard to $Q_y$ instead of $P_y$. This change results in the calculation
\[
Q_y \left[ \tau \in [l - 1, l] \right] \leq Q_y \left[ \min_{s \in [0,l]} X_s + h_t(s) + s/tW_t - y \leq -2C_{\log}(1 + l)^{\frac{3}{2}} + h_t(l) + t^2C_{\log} + \eta \right] \\
\leq Q_y \left[ \min_{s \in [0,l]} X_s + h_t(s) + s/tW_t - y \leq -C_{\log}(1 + l)^{\frac{3}{2}} + \eta \right] \leq 2e^{-\frac{C_{\log}(1 + l)^{\frac{3}{2}} - \eta}{2}}.
\]

The rest of the proof of Proposition 9.4 needs only very minor changes\footnote{Since we work with regard to $Q_y$ our Brownian motion is $(X_s + h_t(s) + s/tW_t)_{s \in [0,t]}$. Thus the analogue to $m_e$ of (9.20) won’t have an $h_t$ term, which only simplifies the situation.} to give that
\[
Q_y \left[ \tilde{A}_{t,h,\text{start}}^{(y_0)} \right] \leq \frac{p_t^{(y_0)}(y)}{8} + Ct^{-\lambda}(p_t^{(y_0)})^{-1}
\]
as well as
\[
Q_y \left[ \tilde{A}_{t,h,\text{end}}^{(y_0)} \right] \leq \frac{p_t^{(y_0)}(y) - 2 + p_t^{(y_0)}(y) - 1}{8} + Ct^{-\lambda}(p_t^{(y_0)})^{-1}
\]
which yields that
\[
Q_y \left[ \tilde{A}_{t,h,\lambda}(t) \right] = \frac{p_t^{(y_0)}(y)}{8} \sum_{j=0}^{\frac{y}{t}} \sum_{j=0}^{y_0 - 1} p_t^{(y_0)}(y) - j \geq \frac{1}{2} - Ct^{-\lambda} (p_t^{(y_0)})^{-1} + \frac{Ct^{-\lambda} (p_t^{(y_0)})^{-1}}{8}.
\tag{9.33}
\]
Combining (9.8), (9.32) and (9.33) and writing
\[
\frac{p_t^{(y_0)}(y)}{p_t^{(y_0)}(y) - \frac{y}{8} - \sum_{j=0}^{\frac{y}{t}} \sum_{j=0}^{y_0 - 1} p_t^{(y_0)}(y) - j} = \frac{p_t^{(y_0)}(y)}{\tilde{p}_{t,h}^{(y_0)}(y)} Q_y \left[ \tilde{A}_{t,h,\lambda}(t) \right] \frac{Q_y \left[ \tilde{A}_{t,h,\lambda}(t) \right]}{p_t^{(y_0)}(y) - \frac{y}{8} - \sum_{j=0}^{\frac{y}{t}} \sum_{j=0}^{y_0 - 1} p_t^{(y_0)}(y) - j}
\]
gives Theorem 9.12.

10 Crude lower bound on $p_{t,h}^{(y_0)}(y)$

The aim of this section is to provide a lower bound on $p_{t,h}^{(y_0)}(y)$ of Definition 8.1. This will be used to deal with the $p_{t,h}^{(y_0)}(x)^{-1}$ terms occurring in Theorems 8.3, 9.2 and 12.2 and additionally be the main ingredient in the proof of Lemma 2.2. We recall Definition 8.1. The idea of the proof is to replace the event $\{\forall s \leq t: X_s \leq W_s - h_t(s)\}$ with $\{\forall s \leq t: X_s \leq g_t(s)\}$ with a $g_t$ for which we can apply Girsanovs theorem, to reduce the situation to a ballot theorem for Brownian motion. The $g_t(s)$ will be roughly a piecewise linear approximation of the running minimum of $W_s - h_t(s)$. The next definition defines $g_t$ precisely as well as some related quantities, compare Figure 10.

\begin{definition}
Fix $t > 0$. Set $k_1(t) := \lceil \log_2(t/3) \rceil$, $t_1(t) := 2^{k_1}$, $t_2(t) := t - t_1$, the dependence on $t$ will be omitted from notation. For $j \leq k_1 - 1$ define
\[
\Xi_j^{\text{start}}(W) := \min_{s \in [0,2^{j+1}]} W_s,
\Xi_j^{\text{end}}(W) := \min_{s \in [t-2^{j+1},t]} W_s - W_t,
\Xi_j(W) := \min \{ \Xi_j^{\text{start}}(W), \Xi_j^{\text{end}}(W) \},
\Xi_{k_1}(W) := \min \{ \min_{s \in [0,t]} W_s, \min_{s \in [t_2, t]} W_s - W_{t_1}, 1 \},
\end{definition}
Figure 3: The event in (10.1). Drawn are $(W_s)_{0 \leq s \leq t}$ (dotted line), $\Xi_j(W)$, $j \in \{0, \ldots, 5\}$ (dashed lines) and $g_t$ (solid piecewise linear curve) with parameters $h = 0$, $t = 96$, $t_1 = 32$, $t_2 = 64$, $k_1 = 5$. We note that it is less likely to stay below $(g_s)_{s \leq t}$ than below $(W_s)_{s \leq t}$.
For \( j \in \{1, \ldots, k_1 - 1\} \) define \( \Xi_j(h) \) (and analogous variables) as above, substituting \(-h_t(s)\) for \( W_s, s \in [0, t] \). Furthermore, choose \( \Xi_0(h) \) such that \( s\Xi_0(h) \leq -h_t(s) \) for all \( s \in [0, 1] \) and \((t-s)\Xi_0(h) \leq -h_t(s) \) for all \( s \in [t-1, t] \).

Set \( \Xi_j := \Xi_j(W) + \Xi_j(h) \) and \( \Delta_j = \Delta_j(W) + \Delta_j(h) \). Finally, set

\[
g_t(s) = \begin{cases} 
  s\Xi_0, & s \in [0, 1), \\
  \Xi_j + (s-2^j)\Delta_j, & s \in [2^j, 2^{j+1}] \\
  \Xi_{k_1} + (s-t_1)/(t_2-t_1)(W_t - h_t(t)), & s \in [t_1, t_2], \\
  \Xi_j + (t-2^j-s)\Delta_j + (W_t - h_t(t)), & s \in [t-2^{j+1}, t-2^j], \\
  (t-s)\Xi_0 + W_t, & s \in (t-1, t].
\end{cases}
\]

**Theorem 10.1.** For \( W \) piecewise linear and \( y, y_0 \in [-\sqrt{t}, 0] \), there is a \( \gamma_0 > 0 \) deterministic such that for \( t \geq 2 \),

\[
P_y \left[ B_{[0, t], h_t}^y - W(X) \right] \geq t^{-\gamma_0 - \log(t)^{-1}} \left( \sum_{j=0}^{k_1-1} \Delta_j 2^{j+2(1-2^{-j/2})} + \sum_{j=0}^{k_1-1} \Delta_j 2^{j/2} + \Xi_0^2 + \Xi_0 (4 + y + y_0) + 3 \frac{W_{2^{j+1}} - h_t(t)^2}{20} + \sqrt{T_n} \right).
\]

**Proof.** As illustrated in figure [0] we have that by definition \( g_t(s) \leq W_s \) for all \( s \in [0, t] \) and thus

\[
P_y \left[ B_{[0, t], h_t}^y - W(X) \right] \geq P_y \left[ \forall s \leq t, X_s \leq g_t(s), X_t - g_t(t) \in [y_0 - 1, y_0] \right]. \tag{10.1}
\]

We apply Girsanov to remove the \( g_t \). For controlling the Radon-Nikodym derivative appearing in the application of Girsanov, we need control over \( X_{2j+1} - X_{2j}, j \in \{0, \ldots, k_1 - 1\} \). Thus define

\[
A_0^{(1)} := \left\{ X_{2j+1} - g_t(2^{j+1}) \in [-2\sqrt{2^{j+1}}, -2\sqrt{2^{j+1}}] \right\}, \\
A_0^{(2)} := \left\{ X_{2j+1} - g_t(2^{j+1}) \in [-2\sqrt{2^{j+1}}, -2\sqrt{2^{j+1}}] \right\}.
\]

Furthermore, set

\[
A_{\text{start}} := \left\{ -2 \leq X_1 - g_t(1) \leq -1 \right\}, \\
A_{\text{end}} := \left\{ -2 \leq X_{t-1} - g_t(t-1) \leq -1 \right\}.
\]

By monotonicity and (10.1)

\[
P_y \left[ B_{[0, t], h_t}^y - W(X) \right] \geq P_y \left[ A_{\text{start}}, \bigcap_{i \in \{1, 2\}} \bigcap_{j=0}^{k_1-1} A_i^{(j)}, A_{\text{end}}, \forall s \leq t, X_s \leq g_t(s), X_t - g_t(t) \in [y_0 - 1, y_0] \right]
\]

\[
= P_y[A(y_0)].
\]

Consider \( Q_y \) with

\[
Z_t := \frac{dQ_y}{dP_y} = \exp \left( \int_0^t g'_t(s) \, dX_s - \frac{1}{2} \int_0^t (g'_t(s))^2 \, ds \right)
\]

\[
= \exp \left( \sum_{j=0}^{k_1-1} \Delta_j (X_{2j+1} - X_{2j}) - \Delta_j (X_{2j+1} - X_{2j+1}) - \sum_{j=0}^{k_1-1} \frac{\Delta_j 2^{j/2} - \Xi_0^2}{2} + \Xi_0 \left( X_1 - X_0 - (X_1 - X_{t-1}) + (X_{t_2} - X_{t_1}) \frac{W_t - h_t(t)}{t_2 - t_1} - \frac{(W_t - h_t(t))^2}{2(t_2 - t_1)} \right) \right).
\]

On \( A(y_0) \) we have that

\[
X_{2j+1} - X_{2j} \geq \Delta_j + 2^{j+1} \left( 1 - 2^{2^{j+1}} \right),
\]

\[
X_{2j+1} - X_{2j+1} \leq -\Delta_j + 2^{j+1} \left( 2^{2^{j+1}} - 1 \right),
\]

\[
X_{1} - X_0 \geq \Xi_0 - 2 - y,
\]

\[
X_t - X_{t-1} \leq 2 - \Xi_0 + y,
\]

\[
|X_{t_2} - X_{t_1}| \leq \sqrt{T_n} + |W_t - h_t(t)|.
\]
Thus we have that
\[
\mathbb{P}_y[A(y_0)] = \mathbb{Q}_y[Z_{\gamma_1^{-1}}; A(y_0)]
\]
\[
\geq \exp \left( -\sum_{j=0}^{k-1} \Delta_j^2 \gamma_0 \frac{t}{2} - \left( 2 - 2 \gamma_0 \right) \sum_{j=0}^{k-1} \Delta_j \gamma_0 \frac{t}{2} - \varepsilon_0^2 \frac{t}{2} - \Xi_0 (4 + y_0) - \frac{3(W_j - h(t))^2}{2(t_2 - t_1)} - \frac{\sqrt{t_1}W_t - h(t)}{t_2 - t_1} \right) \cdot \mathbb{Q}_y[A(y_0)].
\]
By the Girsanov theorem we have that \((X_s - g_t(s))_{s \in [0,t]}\) is a Brownian motion with regard to \(\mathbb{Q}_y\). This implies that there is a constant \(\gamma_0 > 0\) such that for \(t \geq 2\) arbitrary \(\mathbb{Q}_y[A(y_0)] \geq t^{-\gamma_0}.\) Thus we have that
\[
\mathbb{P}_y[A(y_0)]
\]
\[
\geq t^{-\gamma_0} \exp \left( \sum_{j=0}^{k-1} \Delta_j^2 \gamma_0 \frac{t}{2} + 2(1-2^{3/2}) \sum_{j=0}^{k-1} \Delta_j \frac{t}{2} + \Xi_0 (4 + y_0) + \frac{3(W_j - h(t))^2}{2(t_2 - t_1)} + \frac{\sqrt{t_1}W_t - h(t)}{t_2 - t_1} \right) \mathbb{Q}_y[A(y_0)]
\]
\[
\geq t^{-\gamma_0} \exp \left( \sum_{j=0}^{k-1} \Delta_j^2 \gamma_0 \frac{t}{2} + 2(1-2^{3/2}) \sum_{j=0}^{k-1} \Delta_j \frac{t}{2} + \Xi_0 (4 + y_0) + \frac{3(W_j - h(t))^2}{2(t_2 - t_1)} + \frac{\sqrt{t_1}W_t - h(t)}{t_2 - t_1} \right).
\]
\(\square\)

11 Control for the Constants Introduced in Section 8.1, Definition 9.3 and the Proof of Lemma 2.2

We recall Definitions 2.1 and 2.2. From now on \(W\) will be as in Definition 2.2. For \(t > 0\) define \(\tilde{W}\) via \(\tilde{W}_s := W_{t-s} - W_t\), where the dependence on \(t\) is not reflected in the notation. Furthermore, from now on we will mostly care about \(\mathbb{P}\) in particular Definition 9.3 and the Proof of Lemma 2.2

\[
\gamma_{a,b}(W,f,t) := \inf \left\{ \gamma > 0 : \forall t \geq 2 \mathbb{P}_\mathbb{E} \left[ \mathcal{B}_{[0,1],\gamma^{-1/\gamma}(W)}(B) \right] \geq s^{-\gamma} \right\}
\]
and \(\gamma_{a,b}(\tilde{W},f,t)\) analogous.

**Lemma 11.1.** Let \(\beta \in \{\pm 1\}\). We consider \((h_t)_{t \geq 0}\) given by \(h_t : [0,t] \rightarrow \mathbb{R}, s \mapsto \beta ((1+s) \wedge (1+t-s))^{1/6} - 1\), \(b > a \geq 0\). The families of random variables \((\gamma(W,h,t))_{t \geq 0}, (\gamma(\tilde{W},h,t))_{t \geq 0}\) are tight.

**Proof.** On \([2,t/2]\) we have that \(h_t(s) = \beta ((1+s)^{1/6} - 1) = f(s)\). Thus it suffices to prove that \((\gamma_{a,b}(\tilde{W},f,t))_{t \geq 0}\) are tight. Furthermore, by definition of \(W\) we have that \((W_s)_{s \in [0,t]} \overset{d}{=} (W_{\tilde{s}})_{s \in [0,t]}\) for all \(t \geq 0\) and thus we know that \(\gamma_{a,b}(W,f,t) \overset{d}{=} \gamma_{a,b}(\tilde{W},f,t)\) for all \(t \geq 0\) and it suffices to prove that \((\gamma_{a,b}(W,f,t))_{t \geq 0}\) is tight.

By Theorem 1.10 in [16] we know that there exists \(\gamma > 0\) independent of \(a,b\) such that \(\mathbb{P}\)-almost surely
\[
\lim_{n \rightarrow \infty} \frac{\log \left( \mathbb{P}_\mathbb{E} \left[ \mathcal{B}_{[0,1],\gamma^{-1/\gamma}(W)}(B) \right] \right)}{\log(n)} = -\gamma.
\]
in particular \(\mathbb{P}\)-a.s.
\[
\tilde{\gamma} := \inf_{n \in \mathbb{N}} \frac{\log \left( \mathbb{P}_\mathbb{E} \left[ \mathcal{B}_{[0,1],\gamma^{-1/\gamma}(W)}(B) \right] \right)}{\log(n)} > -\infty.
\]
Now take \(a',b'\) such that \(b > b' > a' > a\). We claim that there is a \(c_\varepsilon > 0\) such that for all \(t \geq 2\)
\[
\mathbb{P} \left[ \mathbb{P}_\mathbb{E} \left[ \mathcal{B}_{[0,1],\gamma^{-1/\gamma}(W)}(B) \right] \leq c_\varepsilon \right] \geq 1 - \varepsilon.
\]
Combining (11.1) with (11.2) allows to conclude the statement of the lemma, it remains to prove that (11.2) holds. We do this in two steps, first going from the time interval \([0,t]\) to \([0,[t]\), and then going from continuous time to discrete time. For this purpose define
\[
A_1 := \mathcal{B}_{[0,1],\gamma^{-1/\gamma}(W)}(B),
\]
\[
A_2 := \mathcal{B}_{[0,\ldots,[t]],\gamma^{-1/\gamma}(W)}(B).
\]
We have that
\[
\mathbb{P}_t \left( B_{[0,t],f(t),-W}(B) \right) = \mathbb{E}_t \left[ \mathbb{P}_t \left( B_{[0,t],f(t),-W}(B) \right) | \sigma(B_s, s \leq [t]) \right]
\]
\[
= \mathbb{E}_t \left[ 1_{A_t} \mathbb{P}_t \left[ \mathbb{P}_t \left( B_{[0,t],f(t),-W}(B) \right) | \sigma(B_s, s \leq [t]) \right] \right]
\]
\[
\geq \mathbb{E}_t \left[ 1_{A_t} \left( 1 - \mathbb{P}_t \left( \exists s \in [t], t : B_t - B_s > a' \sqrt{t} + W_t - W_s - f(t) - f(s) \right) \right) \right] \cup \left\{ B_t - B_s + W_t - W_s + f(t) - f(s) \not\in [-b\sqrt{t}, -a\sqrt{t}] \right\}.
\] (11.3)

Combining (11.3) with bounds on the maximum of a Brownian motion implies that there is a \( c_\varepsilon > 0 \) such that
\[
\mathbb{P} \left[ \mathbb{P}_t \left( B_{[0,t],f(t),-W}(B) \right) = c_\varepsilon \right] \geq 1 - \varepsilon.
\] (11.4)

Next we bound \( \mathbb{P}_t[A_1]/\mathbb{P}_t[A_2] \). In essence this calculation boils down to the fact, that if at times \( \{0, \ldots, [t]\} \) \( B \) is below a (curved) barrier, then it is unlikely that \( B \) crosses a multiple of the barrier in-between these times. We have that
\[
\mathbb{P}_t[A_1] = \mathbb{E}_t \left[ \mathbb{P}_t \left[ A_1 | (B_1, \ldots, B_{[t]}) \right] \right] \geq \mathbb{E}_t \left[ 1_{A_2} \mathbb{P}_t \left[ A_1 | (B_1, \ldots, B_{[t]}) \right] \right]
\]
\[
\geq \mathbb{E}_t \left[ 1_{A_2} \prod_{j=1}^{[t]} \mathbb{P}_t \left( B_{[0,j],f(j),-W}(B) | (B_{j-1}, B_j) \right) \right] \cdot
\] (11.5)

To get a lower bound on \( \mathbb{P}_t[A_2] \) it suffices to prove that there are \( c, C > 0 \) independent of \( t \) such that \( \mathbb{P} \)-a.s.
\[
1_{A_2} \mathbb{P}_t \left[ B_{[0,1],f(0),-W}(B) | (B_0, B_1) \right] \geq c \mathbb{1}_{A_2} \cdot \forall j \in \{2, \ldots, [t]\},
\] (11.6)
\[
1_{A_2} \sum_{j=2}^{[t]} \left( 1 - \mathbb{P}_t \left( B_{[0,j],f(j),-W}(B) | (B_{j-1}, B_j) \right) \right) \leq C,
\] (11.7)
\[
1_{A_2} \mathbb{P}_t \left[ B_{[0,1],f(0),-W}(B) \right] \geq c \mathbb{1}_{A_2}.
\] (11.8)

Equations (11.5) and (11.6)–(11.8) directly imply that there is a \( c > 0 \) such that
\[
\frac{\mathbb{P}_t[A_1]}{\mathbb{P}_t[A_2]} \geq c.
\] (11.9)

Equations (11.4) and (11.9) imply (11.2). Thus to finish the proof, we only need to prove that (11.6)–(11.8) hold.

We proceed towards this, by first proving (11.6) and (11.7) in one calculation. Set \( x_j := W_j + 1 - 2^a f(j) \) for \( j \in \{1, \ldots, [t]\} \). On \( A_2 \) and given \( B_{j-1}, B_j \) we couple \( (B_s)_{s \in [j-1, j]} \) with a Brownian bridge \( (E'_s)_{s \in [0,1]} \) with \( E'_0 = x_{j-1}, E'_1 = x_j, \) and \( B_s \leq E'_{s+j-1} \) for all \( j \in \{2, \ldots, [t]\}, s \in [j-1, j] \). This coupling yields that
\[
1_{A_2} \mathbb{P}_t \left[ B_{[0,1],f(0),-W}(B) \right] \geq c \mathbb{1}_{A_2} \cdot
\]
\[
1_{A_2} \mathbb{P}_t \left[ \exists s \in [j-1, j : 1 + B_s + f(s) > W_s | (B_{j-1}, B_j) \right] \leq 1_{A_2} \mathbb{P}_t \left[ \exists s \in [j-1, j : E'_s > f(s) + 1 | (j - 1, s=1]) \right] \cdot
\] (11.10)

Let \( E'_{\cdot} \) be a standard Brownian bridge. We have that
\[
(E'_{\cdot})_{s \in [0,1]} \equiv (E'_{\cdot} + s x_j + (1-s)x_{j-1})_{s \in [0,1]}.
\] (11.11)
Plugging (11.11) into (11.10) and simplifying, using that \( W \) is piecewise linear, gives that
\[
1_{A_2} \mathbb{P}_t \left[ \exists s \in [j-1, j : 1 + B_s + f(s) > W_s | (B_{j-1}, B_j) \right] \leq 1_{A_2} \mathbb{P}_t \left[ \exists s \in [j-1, j : E'_s > f(j - 1, s=1]) \right].
\] (11.12)

Direct calculation yields that there is a \( c' > 1/2 \) such that for \( j \geq 1 \)
\[
\frac{f(j)}{f(j+1)} \geq c'.
\] (11.13)
Plugging \([11.13]\) into \([11.12]\) yields that there is an \(c'' > 0\) such that
\[
1_{A_3} \mathbb{P}_L \left[ \exists s \in [j-1,j] - 1 + B_s + f(s) > W_s((B_j-1,B_j)] \right] \leq 1_{A_2} \mathbb{P}_L \left[ \exists s \in [j-1,j] E_s^c > c''((1 + j)^{1/6} - 1) \right].
\] (11.14)

Since the maximum of a standard Brownian bridge is stochastically dominated by the maximum of a standard Brownian motion, which can be proved using e.g. Slepian's lemma, \((11.14)\) yields that for \(j \in \{2, \ldots, [t]\}\)
\[
1_{A_2} \mathbb{P}_L \left[ \exists s \in [j-1,j] - 1 + B_s + f(s) > W_s((B_j-1,B_j)] \right] \leq 21 A_2 \int_{e^{(1+j)^{1/6} - 1}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.
\] (11.15)

The equations \((11.6)\) and \((11.7)\) are directly implied by \((11.15)\).

One can prove \((11.8)\) similarly taking \(x_0 = 0\) and showing by calculation, that \((1-s) - f(s) + s 2^{\beta} f(1) > 0\) is bounded away from zero.

\[\square\]

**Lemma 11.2.** We have that \((C_{\log}(W,t))_{t \geq 0}\) and \((C_{\log}(\hat{W},t))_{t \geq 0}\) are tight.

**Proof.** By construction we have that
\[C_{\log} \leq \sup_{k \in \mathbb{N}} \frac{|W_k|}{\sqrt{k} \log(k)} < \infty \quad \text{P-a.s..}\]

For the last inequality we recall that by Remark \(2.1\) we can apply the law of iterated logarithms for the random walk \((W_k)_{k \in \mathbb{N}}\). The second statement of the lemma follows from the first, since \(C_{\log}(\hat{W},t) \equiv C_{\log}(W,t)\) for all \(t \geq 0\). \[\square\]

**Lemma 11.3.** For all \(\lambda > 0\) \((C_3(W,h,\lambda,t))_{t \geq 0}\) and \((C_3(\hat{W},h,\lambda,t))_{t \geq 0}\) are tight.

**Proof.** This is an immediate consequence of Lemmata \(11.1\), \(11.2\) and the Definition of \(C_3\) in \((8.7)\). \[\square\]

**Lemma 11.4.** We have that \((C_2(W,h,t)^{-1})_{t \geq 0}\) and \((C_2(\hat{W},h,t)^{-1})_{t \geq 0}\) are tight.

**Proof.** Follows from Lemma \(11.2\), \(8.3\), and the definition of \(C_2\) in \((8.5)\). \[\square\]

**Proof of Lemma 2.2.** We recall Definitions \(8.1\), \(10.1\) and Theorem \(10.1\). We prove a more general statement than Lemma \(2.2\) for this we need one additional definition.

**Definition 11.1.** We call \((h_t)_{t \in I}, I \subseteq \{\mathbb{N}, \mathbb{R}\geq 0\}, h_t : [0,t] \to \mathbb{R}, nice, if there is a \(\eta < 1/2\) such that for all \(t \in I\)
\[
\max_{s \in [0,t]} |h_t(s)| \leq t^\eta,
\]
\[
\max_{s \in [0,t]} \frac{|h_t(s)|}{(1 + s)^{\eta}} \leq C(h),
\]
\[
\max_{s \in [0,t]} \frac{|h_t(t) - h_t(t-s)|}{(1 + s)^{\eta}} \leq C(h).
\]

**Lemma 11.5.** For \((h_n)_{n \in \mathbb{N}} nice and \(y(n), y_0(n) \in [-c \log(n), 0]\) (the arguments are omitted in what follows), there is a deterministic constant \(C_2(h) > 0\) such that \(\mathbb{P}\)-a.s.
\[
\lim_{n \to \infty} \sup_{n} \left| \frac{\log(p_{n,h}(y))}{\log(n)} \right| \leq C_2(h),
\] (11.16)

and in \(\mathbb{P}\)-probability
\[
\lim_{n \to \infty} \sup_{n} \left| \frac{\log(p_{n,h}(y))}{\log(t)} \right| \leq C_2(h).
\] (11.17)

**Remark 11.1.** Lemma \(2.2\) is an immediate corollary of Lemma \(11.5\) by taking \(h \equiv 0\) and \(y(n) = y_0(n) = \xi_0\) for all \(n \in \mathbb{N}\).
Proof. Since \(p_{n,h}^{(y)}(y)\) is a probability it is smaller than 1, thus \(\log(p_{n,h}^{(y)}(y))\) \(\leq 0\), and we only need to prove a lower bound on \(p_{n,h}^{(y)}(y)\). In the following we will use \(C(h)\) for a positive constant depending on \(h\), but on neither \(W\) nor \(n\), which may change from line to line.

Use the notation from Definition 10.1. By Theorem 10.1 we have that for \(n \geq 2\) \(P\) a.s.

\[
p_{n,h}^{(y)}(y) \geq n^{\alpha_0} \log(n)^{-1} \left( \sum_{j=0}^{k_1-1} \Delta_j^2 + 2^j (1 - 2^{-j/2}) \sum_{j=0}^{k_1-1} \Delta_j 2^{-j/2} + \sum_{j=0}^{k_1-1} \Delta_j 2^{-j/2} + \sum_{j=0}^{k_1-1} \Delta_j 2^{-j/2} + \sum_{j=0}^{k_1-1} \Delta_j 2^{-j/2} \right),
\]

where \(k_1, t_1, t_2\) are \(n\) dependent, which we omit from notation. Using that \((h_n)_{n \in \mathbb{N}}\) is nice and using the law of iterated logarithm for \((W_k)_{k \in \mathbb{N}}\), recall Remark 2.1 yields that there is a \(C(h) \in (0, \infty)\) such that \(P\) a.s.

\[
\limsup_{n \to \infty} \left( \sum_{j=0}^{k_1-1} \Delta_j^2 \log(n) + \sum_{j=0}^{k_1-1} \Delta_j \log(n) \right) \leq C(h).
\]

Thus it suffices to prove that there are \(C, C(h) > 0\) such that in \(P\) a.s.

\[
\limsup_{n \to \infty} \log(n)^{-1} \sum_{j=0}^{k_1-2} \Delta_j(h)^2 2^j + |\Delta_j(h)| 2^{j/2} \leq C(h),
\]

(11.18)

\[
\limsup_{n \to \infty} \log(n)^{-1} \sum_{j=0}^{k_1-2} \Delta_j(W)^2 2^j + |\Delta_j(W)| 2^{j/2} \leq C,
\]

(11.19)

Since \((h_n)_{n \in \mathbb{N}}\) is nice we have that for all \(j \leq k_1 - 2\)

\[
|\Delta_j(h)| \leq c t^{n-1} + 2^n(j+2)^{-1},
\]

which gives (11.18). It is left to prove (11.19). For this purpose notice, that for all \(n \in \mathbb{N}, j \leq k_1 - 2\)

\[
2^j |\Delta_j(W)| \leq \max_{s \in [2^{j/2}, 2^{j+1}]} |W_s - W_{2s+1}| \leq \max_{s \in [n-2^{j/2}, n-2^{j+1}]} |W_s - W_{2s+1}|.
\]

(11.20)

We have that \(a_{j,n,1}\) does not depend on \(n\) and \((a_{j,n,1})_{n \geq 1}\) is a sequence of independent random variables. With for example Doob's maximal inequality one can show that there is a \(c > 0\) such that for \(j \leq k_1 - 2\)

\[
\mathbb{E}[a_{j,n,1} 2^j] \leq c,
\]

\[
\mathbb{E}[|a_{j,n,1}| 2^{-j/2}] \leq c.
\]

Thus the strong law of large numbers gives that there is a \(C > 0\) such that \(P\) a.s.

\[
\limsup_{n \to \infty} \log(n)^{-1} \sum_{j=0}^{k_1-2} a_{j,n,1} 2^j + |a_{j,n,1}| 2^{-j/2} \leq C.
\]

(11.21)

For the sum involving \(a_{j,n,2}\) we want to use Borel-Cantelli, and thus want to bound \(P[\sum_{j=0}^{k_1-2} a_{j,n,2} 2^{j/2} > x]\). For this we use the exponential Chebyshev inequality to get that for \(x > 0\)

\[
P \left[ \sum_{j=0}^{k_1-2} |a_{j,n,2}| 2^{-j/2} > x \cdot \log(n) \right] \leq e^{-x \log(n)} \prod_{j=0}^{k_1-2} \mathbb{E} \left[ e^{a_{j,n,2} 2^{-j/2}} \right].
\]

(11.22)

Thus we want an upper bound on the tails of \(a_{j,n,2}\). A union bound gives that for \((\lambda_k)_{k \in \mathbb{N}} > 0\)

\[
P \left[ |a_{j,n,2}| 2^{-j/2} > x \right] \leq \sum_{k=1}^{2^j} P \left[ \sum_{t=1}^{k} W_{t} > 2^{j/2} x \right] \leq \sum_{k=1}^{2^j} e^{-\lambda k 2^{j/2} x} \left( \mathbb{E} \left[ e^{\lambda W_1} \right] \right)^k + \mathbb{E} \left[ e^{-\lambda_k W_1} \right]^k.
\]

Direct calculation yields that \(W_1\) is sub-Gaussian, in particular there are \(C, c > 0\) such that \(\mathbb{E} \left[ e^{\pm \lambda_k W_1} \right] \leq C e^{c \lambda_k^2}\). This implies that

\[
P \left[ |a_{j,n,2}| 2^{-j/2} > x \right] \leq \sum_{k=1}^{2^j} C e^{-\lambda_k 2^{j/2} x + c \lambda_k^2} \leq \sum_{k=1}^{2^j} C e^{-\frac{2^j x^2}{4 \lambda_k}} \leq 2^j C e^{-\frac{x^2}{4}}.
\]
This implies that for \( x \geq 1 \)
\[
\Pr\left[e^{\lambda a_{j,n,2}2^{-j/2}} > x\right] = \Pr\left[|a_{j,n,2}| > 2^{j/2} \log(x)\right] \leq 2e^{-\frac{\lambda^2 a_{j,n,2}^2}{2x}}.
\]

and since \(|a_{j,n,2}|2^{-j/2} \geq 0\) we have that
\[
\Pr\left[e^{\lambda a_{j,n,2}2^{-j/2}} > x\right] = 1 \text{ for } x \in [0,1].
\]

Thus
\[
\mathbb{E}\left[e^{\lambda a_{j,n,2}2^{-j/2}}\right] \leq 1 + \frac{2\lambda^2 a_{j,n,2}^2}{2y - 4c} \leq e^{\frac{\lambda^2 a_{j,n,2}^2}{2y - 4c}}.
\]

(11.23)

For \( j \) big enough, we have \(\frac{\lambda^2 a_{j,n,2}^2}{2y - 4c} \leq 8c^2\), especially by changing \( c \) we get from (11.22) and (11.23) that
\[
\Pr\left[\sum_{j=0}^{k-1} |a_{j,n,2}|2^{-j/2} > x \cdot \log(n)\right] \leq e^{-x \log(n)}e^{c \log(n)},
\]

which is summable in \( n \) for all \( x \) big enough, deterministic, which implies that there is a \( C > 0 \), such that \( \mathbb{P}\)-a.s.
\[
\limsup_{n \rightarrow \infty} (n)^{-1} \sum_{j=0}^{k-1} |a_{j,n,2}|2^{-j/2} \leq C.
\]

(11.24)

Very similar one can prove that there is a \( C > 0 \) such that \( \mathbb{P}\)-a.s.
\[
\limsup_{n \rightarrow \infty} (n)^{-1} \sum_{j=0}^{k-1} a_{j,n,2}^22^{-j} \leq C.
\]

(11.25)

Equations (11.24) and (11.25) imply that there is a \( C > 0 \) such that \( \mathbb{P}\)-a.s.
\[
\limsup_{n \rightarrow \infty} (n)^{-1} \sum_{j=0}^{k-1} a_{j,n,2}^22^{-j} + |a_{j,n,2}|2^{-j/2} \leq C.
\]

(11.26)

Equation (11.20), (11.21) and (11.21) yield (11.19), which finishes the proof of (11.16).

The proof of (11.17) goes along the same lines defining \( a_{j,t,i}, i \in \{1,2\}, t \geq 0 \) analogous to (11.20) and noticing, that the distribution of \( a_{j,t,i} \) doesn’t depend on \( t \).

We come back to controlling three more constants which pop up in the bounds we derived, recall (9.4).

Definition 9.3 and that for given \( t \) we set \( \tilde{W}_s := W_{t-s} - W_t \)

Lemma 11.6. In (9.4) we can choose \( (C_4(W,t))_{t \geq 0}, (c_2(W,t)^{-1})_{t \geq 0}, (C_4(\tilde{W},t))_{t \geq 0}, (c_2(\tilde{W},t)^{-1})_{t \geq 0} \)

tight.

Proof. As in the proof of (11.1) using that for all \( t \geq 0 \) \( (W_s)_{s \in [0,t]} \overset{d}{=} (\tilde{W}_s)_{s \in [0,t]} \) it suffices to prove that we can choose \( (C_4(W,t))_{t \geq 0}, (c_2(W,t)^{-1})_{t \geq 0} \) tight. This can be proved similarly to (11.17) in Lemma 11.3.

Lemma 11.7. For \( y = y_0 = \xi_0 \) and \( \delta \in (0,1/8) \) fixed in Theorems 9.2 and 9.12 \((\eta(W,\tilde{W},t,h,\delta,y,y_0))_{t \geq 0} \) and \((\kappa(W,\tilde{W},t,h,\delta,y,y_0))_{t \geq 0} \) can be chosen to be tight.

Proof. As mentioned in Remark 9.1 we can choose \( \kappa \) and \( \eta \) as a continuous function of \( \gamma, C_3, C_2^{-1}, C_{log}, \) \( \tilde{\gamma}, \tilde{C}_3, \tilde{C}_2^{-1}, \tilde{C}_{log}, \) \( y, y_0, C_4, c_2^{-1}, C_4, c_2^{-1} \) which together with Lemmata 11.1, 11.4 and Lemma 11.6 yields the claim.

12 Barrier Computations – Proof of Propositions 4.4, 4.10 and 4.12 and Lemmata 5.7, 6.2

12.1 Proof of Proposition 4.4

We will change the start- and endpoint, the \( h \) and the logarithmic drift one-by-one and start by defining all the probabilities, which we encounter on the way. We recall Definition 2.1 and \( h, m_{t,h} \) from Definition 4.1. Recall \( p_{s}^{(2)}(y) \) and \( p_{h}^{(2)}(y) \) from Definition 9.2. We recall that in this section we use the \( W \) defined in Definition 2.2 and that the probability measure used is \( \mathbb{P}_\mathcal{L} \).
Definition 12.1. Let $t > 0$. For $x,y \leq 0$, define
\[ p_{t,h,barr}^{(x)}(y) := \mathbb{P}_L \left[ B^{y,J}_{[0,t],n,h} (T) \right], \]
and recall (4.4). Furthermore, define $\tilde{W}_s := W_{t-s} - W_t$, $s \in [0,t]$, and $\hat{T}_s := B_s - \tilde{W}_s$ and
\[ \hat{p}_{t,h}^{(x)}(y) := \mathbb{P}_L \left[ B^{y,J}_{[0,t],h} (\hat{T}) \right]. \]

We will now state the lemmata necessary to prove Proposition 4.3 then show the proof and after that prove the lemmata. We recall the definition of $\bar{p}_{n}^{\gamma^{-}(y)}$ in (4.4).

Lemma 12.1. There is a $C > 0$ such that for $n \in \mathbb{N}$ and all $x,y \leq \xi_0$, $\mathbb{P}$-a.s.
\[ \frac{p_{n}^{\gamma^{-}}(y)}{\bar{p}_{n}^{(x)}(y)} \leq C. \]

The proof of Lemma 12.1 is similar to the proof of (11.6)-(11.7) in Lemma 11.1 for $\beta = -1$, so we won’t give details. In words Lemma 12.1 means, that the events to be below the barrier for all $j \in \{1, \ldots, n\}$ and to be below the barrier for all $s \leq n$ do only differ by a constant, where for the discrete case we use $h_n/2$ instead of $h_n$.

Lemma 12.2. For all $\varepsilon > 0$, there is a $C_{\varepsilon} > 0$ such that
\[ \liminf_{t \to \infty} \mathbb{P} \left\{ \bigcap_{y \in [-\log(t)/2, \xi_0]} \left\{ \frac{p_{t,h}^{(x)}(y)}{\bar{p}_{t,h}^{(x)}(y)} \leq C_{\varepsilon} |y|^{c_{\varepsilon}} \right\} \right\} \geq 1 - \varepsilon. \]

In words Lemma 12.2 means, that removing the $\frac{1}{n} \log(p_n)$ factor costs a multiplicative constant.

Lemma 12.3. For all $\varepsilon > 0$, there are $c_\varepsilon, C_\varepsilon > 0$ such that
\[ \liminf_{t \to \infty} \mathbb{P} \left\{ \bigcap_{y \in [-\log(t)/2, \xi_0]} \left\{ \frac{p_{t,h}^{(x)}(y)}{\bar{p}_{t,h}^{(x)}(y)} \leq C_{\varepsilon} |y|^{c_{\varepsilon}} \right\} \right\} \geq 1 - \varepsilon. \]

\[ \liminf_{t \to \infty} \mathbb{P} \left\{ \bigcap_{y \in [-\log(t)/2, \xi_0+1]} \left\{ \frac{p_{t,h}^{(x)}(y)}{\bar{p}_{t,h}^{(x)}(y)} \leq C_{\varepsilon} |y|^{c_{\varepsilon}} \right\} \right\} \geq 1 - \varepsilon. \]

Lemma 12.3 states that moving the startpoint or endpoint of from $y$ to $\xi_0$ costs a power of $|y|$.\]

Lemma 12.4. For all $n \in \mathbb{N}$, $x,y \leq 0$, we have that $\mathbb{P}$-a.s.
\[ p_{n,h}^{(x)}(y) \leq \hat{p}_{n,h}^{(y)}(x-1), \]
\[ \hat{p}_{n,h}^{(x)}(y) \leq p_{n,h}^{(y)}(x-1). \]

Lemma 12.4 states that reversing time makes (close to) no difference.

Lemma 12.5. For all $\varepsilon > 0$, there is a $C_{\varepsilon} > 0$ such that
\[ \liminf_{t \to \infty} \mathbb{P} \left\{ \frac{(\xi_0)}{\hat{p}_{t,h}^{(x)}(\xi_0)} \leq C_{\varepsilon} \right\} \geq 1 - \varepsilon. \]

Lemma 12.5 states that when starting and ending at $\xi_0$, the curve of the barrier can be removed by paying a multiplicative constant.

Now we have stated everything we need to prove Proposition 4.4

Proof of Propositions 4.4 assuming L. 12.1-12.5 We can write
\[ \frac{p_{n}^{\gamma^{-}}(y)}{p_{n,h}^{(x)}(y)} \]
Applying Lemmata 12.1 to 12.5 then immediatly yields the claim. \qed
Now we prove Lemma 12.2 to 12.5. We note that these proofs will make heavy use of Sections 8 through 11.

**Proof of Lemma 12.3** Applying Theorem 9.1 yields that for all $t \geq 0$, $y \in [-\log(t)^2, 0]$, $x \in [-\log(t), 0]$, $\mathbb{P}$-a.s.

$$\frac{p_{t,h}(x)}{p_{t,h}(y)} \leq e^{-\frac{\log(p_{t}^{2})}{2(\log(t))^{3}}} e^{-\frac{3 \log(t)^{3} \log(p_{t})}{5 \log(t)}} e^{W} \frac{\log(p_{t})}{2n \log(t)}.$$ 

Since by Lemma 11.5 in $\mathbb{P}$-probability $\log(p_{t})/\log(t)^{2} \to 0$ and furthermore $\log(t)W_{t}/t \to 0$ $\mathbb{P}$-a.s. almost surely, the statement of the lemma follows.

**Proof of Lemma 12.3** Applying Theorems 8.1 and 8.3 yields that

$$1 = \liminf_{t \to \infty} \mathbb{P} \left[ \bigcap_{y \in [-\log(t)^2, 0]} \left\{ \frac{p_{t,h}(y)}{p_{t,h}(\xi_{0} + 1)} \leq C_{2}^{-2} 4^{\gamma + 2} e^{4C_{1}} |y|^{4\gamma + 3C_{3}} \right. \right.$$

$$\left. + C e^{-\lambda \log(t)} p_{t}^{(x)}(\xi_{0} + 1)^{-1}(2 \log(t)^{2})^{2\gamma C_{2}} \right\} \right].$$

By Lemma 11.1 to 11.4 we have that $(\gamma(W, h, t))_{t \geq 0}$, $(C_{2}(W, h, \lambda, t))_{t \geq 0}$, $(C_{2}(W, h, t^{-1}))_{t \geq 0}$ are tight, and Lemma 11.5 implies that there is a $\lambda > 0$ such that

$$\sup_{x \in [-\log(t)], \xi_{0}} C e^{-\lambda \log(t)} p_{t}^{(x)}(\xi_{0} + 1)^{-1}(2 \log(t)^{2})^{2\gamma C_{2}}$$

converges to 0 in $\mathbb{P}$-probability as $t \to \infty$. This allows to conclude (12.1).

Equation (12.2) follows the same way, only replacing $W$ by $\tilde{W}$ in the above and using that by Lemma 11.1 to 11.4 $(\gamma(W, h, t))_{t \geq 0}$, $(C_{3}(W, h, \lambda, t))_{t \geq 0}$, $(C_{2}(\tilde{W}, h, t^{-1}))_{t \geq 0}$ are tight.

**Proof of Lemma 12.4** We have that

$$\mathbb{P}_{L} \left[ \mathbb{B}_{(0, \ldots, n)}^{u,h} h_{n}(-) - W (B) \right] = \sum_{x=1}^{\infty} \mathbb{P}_{L} \left[ B_{n} = du + W_{n} - y \right] \mathbb{P}_{L} \left[ \mathbb{B}_{(0, \ldots, n)}^{u,h} h_{n}(-) - W (\tilde{B}) \mid B_{n} = u + W_{n} - y \right]$$

(12.3)

Set $\tilde{W}_{j} := W_{n-j} - W_{n}$, $\tilde{B}_{j} := B_{n-j} - B_{n}$. Since $h_{n}$ is symmetric we can reverse time in (12.3), which gives that

$$\mathbb{P}_{L} \left[ \mathbb{B}_{(0, \ldots, n)}^{u,h} h_{n}(-) - W (B) \right] = \sum_{x=1}^{\infty} \mathbb{P}_{L} \left[ B_{n} = du + W_{n} - y \right] \mathbb{P}_{L} \left[ \mathbb{B}_{(0, \ldots, n)}^{u,h} h_{n}(-) - W (\tilde{B}) \mid B_{n} = u + W_{n} - y \right]$$

(12.4)

where the last step used that $\tilde{B}_{n} = -B_{n}$ and $\tilde{W}_{n} = -W_{n}$.

Since $\mathbb{P}_{L} \left[ \mathbb{B}_{(0, \ldots, n)}^{u,h} h_{n}(-) - W (\tilde{B}) \mid B_{n} = u - W_{n} + y \right]$ is monotone in $u$ we can replace the $u$ by $-x + 1$. Furthermore, we can shift the region of integration by $y - 1 + x$ in (12.4), which yields that

$$\mathbb{P}_{L} \left[ \mathbb{B}_{(0, \ldots, n)}^{u,h} h_{n}(-) - W (B) \right] \leq \int_{y-1}^{y} \mathbb{P}_{L} \left[ \tilde{B}_{n} = dr + \tilde{W} - x + 1 \right] \mathbb{P}_{L} \left[ \mathbb{B}_{(0, \ldots, n)}^{u,h} h_{n}(-) - W (\tilde{B}) \mid \tilde{B}_{n} = r + \tilde{W} - x + 1 \right]$$

$$= \mathbb{P}_{L} \left[ \mathbb{B}_{(0, \ldots, n)}^{u,h} h_{n}(-) - W (\tilde{B}) \right].$$

(12.5)

Since $(\tilde{B}_{j}) j \in (0, \ldots, n) \overset{d}{=} (B_{j}) j \in (0, \ldots, n)$ equation (12.5) implies the claim of the lemma.

**Proof of Lemma 12.5** Denote $A_{t} := \left\{ \log(t)^{-0.5} \max(\mathcal{C}_{log}, \mathcal{C}_{log}) \leq \frac{1}{2} \right\}$. By Lemma 11.2 we have that $\log(t)^{-0.5} \max(\mathcal{C}_{log}, \mathcal{C}_{log}) \to 0$ in $\mathbb{P}$-probability and thus $1_{A_{t}} \to 1$ in $\mathbb{P}$-probability. On $A_{t}$ condition (9.3) holds, and thus we can apply Theorem 9.2 to get that there are $c, C, t_{0}(\lambda) > 0$ such that for $t \geq t_{0}(\lambda)$

$$1_{A_{t}} \frac{\hat{C}_{t,h}(\xi_{0})}{p_{t,h}(\xi_{0})} \leq \frac{\delta}{1 - \delta} \sum_{j=0}^{2} p_{t,h}^{(x-1)}(\xi_{0} - j)$$
\[ 1 - \varepsilon/2 \leq \lim_{t \to \infty} \mathbb{P} \left[ \sum_{j=0}^{2} \frac{p_{t,h}^{(\xi_{0})} (\xi_{0} - j)}{p_{t,h}^{(\xi_{0})} (\xi_{0})} \leq C_{\varepsilon,1} \right]. \]  

(12.7)

Choosing \( \delta \varepsilon := \frac{1}{2^{2\varepsilon + 1}} \wedge \frac{1}{2} \) in (12.6), applying (12.7) and using \( 1_{A_{t}} \to 1 \) in \( \mathbb{P} \)-probability implies that for all \( \lambda > 0 \)

\[ 1 - \varepsilon/2 \leq \lim_{t \to \infty} \mathbb{P} \left[ \frac{p_{t,h}^{(\xi_{0})} (\xi_{0})}{p_{t}^{(\xi_{0})} (\xi_{0})} \leq \left( C e^{c_{\lambda} \log^{2}(t^n)} + \frac{C_{\varepsilon}}{\varepsilon} \left( \eta(W,W,t,h,\varepsilon,\xi_{0},\xi_{0}) \right) \right) (1 + t^{-\lambda} p_{t}^{(\eta_{0})} (y)^{-1}) + Ct^{-\lambda} (p_{t}^{(\eta_{0})} (y)^{-1} + (p_{t}^{(\eta_{0})})^{-1}) p_{t}^{(\eta_{0})} (y)^{-1} \right]. \]  

(12.8)

By Lemma 12.3 we know that there is a \( C_{\varepsilon,1} > 0 \) such that

By Lemma 12.2 we have that \( (\eta(W,W,t,h,\varepsilon,\xi_{0},\xi_{0}))_{t \geq 0} \) is tight. By Lemma 12.17 (\( \eta(W,W,t,h,\varepsilon,\xi_{0},\xi_{0}))_{t \geq 0} \) and \( (\kappa(W,W,t,h,\varepsilon,\xi_{0},\xi_{0}))_{t \geq 0} \) are tight. By Lemma 12.15 and Definition 9.3 there is a \( \lambda > 0 \) such that the terms \( Ct^{-\lambda} (p_{t}^{(\eta_{0})} (y)^{-1} + (p_{t}^{(\eta_{0})})^{-1}) p_{t}^{(\eta_{0})} (y)^{-1} \) and \( t^{-\lambda} p_{t}^{-1} \) go to zero in \( \mathbb{P} \)-probability. Thus we can conclude the statement of Lemma 12.5 from 12.8.

12.2 Proof of Proposition 4.10.

This section parallels Section 12.1, although it is slightly simpler, since we do not need to consider endpoints other than the (negative) constant \( \xi_{0} \). Thus recall \( h \) and \( n_{t,h} \) from Definition 4.12 as well as Definition 12.1. The proof of Proposition 4.10 is split into several lemmata, we will next state those and then prove the Proposition assuming the lemmata. Since the proofs of the lemmata are very similar to Section 12.1 we won’t repeat them, but instead just reference the corresponding lemmata in Section 12.1.

**Lemma 12.6.** For all \( \varepsilon > 0 \) there is a \( C_{\varepsilon} > 0 \) such that for \( y \in [-\log(t)^2, \xi_{0}] \),

\[ \lim_{t \to \infty} \mathbb{P} \left[ \frac{p_{t,h,barr}^{(\xi_{0})} (y)}{p_{t,h}^{(\xi_{0})} (y)} \geq C_{\varepsilon} \right] \geq 1 - \varepsilon. \]

The proof of Lemma 12.6 parallels the proof of Lemma 12.2 and thus is based on Theorem 9.1.

**Lemma 12.7.** For all \( \varepsilon > 0 \), there are \( C_{\varepsilon}, c_{\varepsilon} > 0 \) such that for \( y \in [-\log(t)^2, \xi_{0}] \),

\[ \lim_{t \to \infty} \mathbb{P} \left[ \frac{p_{t,h}^{(\xi_{0})} (y)}{p_{t}^{(\xi_{0})} (\xi_{0})} \geq C_{\varepsilon} |y|^{-c_{\varepsilon}} \right] \geq 1 - \varepsilon. \]

The proof of Lemma 12.7 is parallel to the proof of Lemma 12.3 but uses Theorem 8.2 instead of Theorem 8.3.

**Lemma 12.8.** For all \( \varepsilon > 0 \), there is a \( C_{\varepsilon} > 0 \) such that

\[ \lim_{t \to \infty} \mathbb{P} \left[ \frac{p_{t,h}^{(\xi_{0})} (\xi_{0})}{p_{t}^{(\xi_{0})} (\xi_{0})} \geq C_{\varepsilon} \right] \geq 1 - \varepsilon. \]

The proof of Lemma 12.8 is parallel to the proof of Lemma 12.5 using Theorem 9.12 in place of Theorem 9.2.

**Proof of Proposition 4.10 assuming Lemma 12.6, 12.8.** We have that

\[ \frac{p_{n}^{\xi_{0}} (y)}{p_{n}} = \frac{p_{n}^{\xi_{0}} (y)}{p_{n,h,barr}^{(\xi_{0})} (y)} \frac{p_{n,h,barr}^{(\xi_{0})} (y)}{p_{n,h}^{(\xi_{0})} (\xi_{0})} \frac{p_{n,h}^{(\xi_{0})} (\xi_{0})}{p_{n}} \frac{p_{n}^{(\xi_{0})} (\xi_{0})}{p_{n}} \geq \frac{p_{n}^{(\xi_{0})} (y)}{p_{n,h}^{(\xi_{0})} (y)} \frac{p_{n,h}^{(\xi_{0})} (y)}{p_{n,h}^{(\xi_{0})} (\xi_{0})} \frac{p_{n}^{(\xi_{0})} (\xi_{0})}{p_{n}} \frac{p_{n}^{(\xi_{0})} (\xi_{0})}{p_{n}} \cdot \]

where the first factor is \( \geq 1 \) by monotonicity. Then applying Lemmata 12.6 to 12.8 immediately yields the claim of the Proposition.
12.3 Proof of Proposition 4.12

We recall \( h \) and \( m_{t,h} \) from Definition 4.2 as well as Definition 12.1. We won’t give proofs for the lemmata used in the proof of Proposition 12.11 but instead reference lemmata, which are proved similarly.

**Lemma 12.9.** For all \( \epsilon > 0 \), there is a \( C_\epsilon > 0 \) such that

\[
\liminf_{n \to \infty} P \left[ y \in [-2 \log(n)^2, \eta_0] : \frac{p_n^{-\eta_0}(y)}{p_n(y)} \leq C_\epsilon \right] \geq 1 - \epsilon.
\]

The proof is similar to the proof of Lemma 12.1.

**Lemma 12.10.** For all \( \epsilon > 0 \), there is a \( C_\epsilon > 0 \) such that

\[
\liminf_{n \to \infty} P \left[ y \in [-2 \log(n)^2, \eta_0] : \frac{p_n^{-\eta_0}(y)}{p_n(y)} \leq C_\epsilon \right] \geq 1 - \epsilon.
\]

The proof is parallel to the proof of Lemma 12.2.

**Lemma 12.11.** For all \( \epsilon > 0 \), there are \( C_\epsilon, c_\epsilon > 0 \) such that

\[
\liminf_{n \to \infty} P \left[ y \in [-2 \log(n)^2, \eta_0] : \frac{p_n^{-\eta_0}(y)}{p_n(y)} \leq C_\epsilon |y|^{c_\epsilon} \right] \geq 1 - \epsilon.
\]

The proof is analogous to the proof of Lemma 12.3.

*Proof of Proposition 4.12 assuming Lemmata 12.9, 12.10 and 12.11.* By monotonicity \( p_n^{(\eta_0)}(\eta_0)/p_n \leq 1 \). Thus writing

\[
\frac{p_n^{-\eta_0}(y)}{p_n(y)} = \frac{p_n^{-\eta_0}(y)}{p_n^{(\eta_0)}(\eta_0)} \frac{p_n^{(\eta_0)}(\eta_0)}{p_n^{(\eta_0)}(y)} \frac{p_n^{(\eta_0)}(y)}{p_n(y)}
\]

and applying Lemmata 12.9, 12.10 and 12.11 yields the claim of Proposition 4.12.

12.4 Proof of Lemma 5.7

We recall Definitions 2.1, 2.1, and (5.6). Before we can proceed with the proof, we need one additional definition.

**Definition 12.2.** Define

\[
W^{(k)}_j := W_{n-k-1+j} - W_{n-k-1},
\]

\[
C_{\text{lin},k}(W) := e^{-k(1 + \log((2+k)^{1/6} - (2+k - s)^{1/6}))} e^{-\left((2+k)^{1/6} - (2+k - s)^{1/6}\right)^2} e^{-\left((2+k)^{1/6} - (2+k - s)^{1/6}\right)^2} e^{-\theta \cdot \left((2+k)^{1/6} - (2+k - s)^{1/6}\right)^2} e^{-\theta \cdot \left((2+k)^{1/6} - (2+k - s)^{1/6}\right)^2}.
\]

**Proof of Lemma 5.7.** Shorten \((2+k)^{1/6} - (2+k - s)^{1/6}\) := \( \eta_k(s) \).

By applying Theorem 9.1, for \( \epsilon_{k+1} = \frac{(2+k)^{1/6} - (2+k - s)^{1/6}}{\log(p_n)} \), we get for all \( k \leq \log(n)^7 \)

\[
\inf_{x \in [0,1] : \eta_k(x) \leq \eta_k(s)} \frac{p_{n-k(z)}}{p_{n-k(s)}} \geq C_{\text{lin},k}(W), \quad \inf_{x \in [0,1] : \eta_k(x) \leq \eta_k(s)} \frac{p_{n-k(z)}}{p_{n-k(s)}} \geq C_{\text{lin},k}(W), \quad \inf_{x \in [0,1] : \eta_k(x) \leq \eta_k(s)} \frac{p_{n-k(z)}}{p_{n-k(s)}} \geq C_{\text{lin},k}(W).
\]

Since \( \eta_k(s) \) is convex and \( \eta_k(0) = 0 \) we have that \( \eta_k(s) \leq \frac{s}{k+1} \eta_k(k+1) \leq 0 \) for all \( s \in [0,1] \) and can just drop that term only making the probability smaller. Thus we have that for all \( k \leq \log(n)^7 \)

\[
\inf_{x \in [0,1] : \eta_k(x) \leq \eta_k(s)} \frac{p_{n-k(z)}}{p_{n-k(s)}} \geq C_{\text{lin},k}(W), \quad \inf_{x \in [0,1] : \eta_k(x) \leq \eta_k(s)} \frac{p_{n-k(z)}}{p_{n-k(s)}} \geq C_{\text{lin},k}(W), \quad \inf_{x \in [0,1] : \eta_k(x) \leq \eta_k(s)} \frac{p_{n-k(z)}}{p_{n-k(s)}} \geq C_{\text{lin},k}(W),
\]

We continue by providing lower bounds for both factors in (12.9). Applying the law of iterated logarithms for \((W_k)_{k \in N}\), which has been justified in Remark 2.1, allows to conclude that there is a \( c_\epsilon > 0 \) such that

\[
\liminf_{n \to \infty} P \left[ \left\{ \bigcap_{k \leq \log(n)^7} \frac{W_{k+1}}{(k+1)^{3/4}} \leq c_\epsilon \right\} \right] \geq 1 - \epsilon/4.
\]
Furthermore, by Lemma \[11.3\] we have that P-a.s.
\[
\frac{\left| \log(p_n) \right|}{\log(n)^7} \rightarrow 0. \tag{12.11}
\]
Plugging \((12.10)\) and \((12.11)\) into the definition of \(C_{\text{lin},k}(W)\) and using that we only care about \(k \leq \log(n)^7\) yields that there is a \(C_\epsilon > 0\) such that
\[
\liminf_{n \to \infty} \mathbb{P} \left[ \bigcap_{k \leq \log(n)^7} \left\{ C_{\text{lin},k}(W) \geq C_\epsilon \right\} \right] \geq 1 - \epsilon/2. \tag{12.12}
\]
By Lemma \[11.5\] for all \(n \in \mathbb{N}\) there are random variables \(C_1^{(n)}, C_2^{(n)}\) such that for all \(k \leq n\)
\[
\inf_{x \in [\log(n)^{-1}, 1]} -\frac{p_{n-k}}{\mathcal{L}} \left[ B_{[0,k]}^x(T) \right] \geq C_1^{(n)} k^{-c_\epsilon}.
\]
and since the environment is i.i.d. we can choose them such that for \(i \in \{1, 2\}\) the distribution of \(C_i^{(n)}\) does not depend on \(n\). This implies that there are \(C_\epsilon, c_\epsilon > 0\) such that
\[
\liminf_{n \to \infty} \mathbb{P} \left[ \bigcap_{k \leq \log(n)^7} \left\{ \inf_{x \in [\log(n)^{-1}, 1]} -\frac{p_{n-k}}{\mathcal{L}} \left[ B_{[0,k]}^x(T) \right] \geq C_\epsilon (k+1)^{-c_\epsilon} \right\} \right] \geq 1 - \epsilon/2. \tag{12.13}
\]
Plugging equations \((12.12)\) and \((12.13)\) into \((12.9)\) and the observation that
\[
\sup_n \sum_{k=1}^{\left\lfloor \log(n)^7 \right\rfloor} e^{\theta \cdot h_n(k)/2} C_\epsilon (k+1)^{c_\epsilon} < \infty
\]
finishes the proof.

\[\square\]

### 12.5 Proof of Lemma 6.2

We recall Definition 6.1.

Before we can proceed with the proof of Lemma 6.2 we need one additional definition.

**Definition 12.3.** Set
\[
C_{\text{lin},k}'(W) := e^{-h_n(k)} \frac{\log(p_n)}{\log(n)^7} + \frac{\log(p_n)}{\log(n)^7} e^{-\frac{k \log(p_n)^2}{\log(n)^7}} e^{-\frac{\log(p_n) W_k}{\log(n)^6}} e^{-\frac{\log(p_n) C_n}{\log(n)^6}}.
\]

**Proof of Lemma 6.2.** The proof is parallel to the proof of Corollary 5.7. Applying Theorem 9.1 for \(c_k = -\frac{\log(p_n)}{\log(n)^7}\) yields that P-a.s. for all \(k \leq \log(n)^7\)
\[
p_{\text{start},k} \geq C_{\text{lin},k}(W) \inf_{x \in [0, 1]} \mathbb{P}_{\mathcal{L}} \left[ B_{[0,k]}^{x,J_{\infty}}(T) \right] \tag{12.14}
\]
As in the proof of Corollary 5.7 combining Lemma \[11.5\] with the fact that there is a \(c_\epsilon > 0\) such that
\[
\liminf_{n \to \infty} \mathbb{P} \left[ \left\{ \bigcap_{k \leq \log(n)^7} \left\{ W_{k+1} \right\} \leq c_\epsilon \right\} \right] \geq 1 - \epsilon/4,
\]
yields that there is a \(C_\epsilon > 0\) such that
\[
\liminf_{n \to \infty} \mathbb{P} \left[ \bigcap_{k \leq \log(n)^7} \left\{ C_{\text{lin},k}'(W) \geq C_\epsilon \right\} \right] \geq 1 - \epsilon/2. \tag{12.15}
\]
Furthermore, by setting \(g_k : [0, k] \to \mathbb{R}, s \mapsto (1 + s)^{1/6} - 1\) we have that
\[
\mathbb{P}_{\mathcal{L}} \left[ B_{[0,k]}^{x,J_{\infty}}(T) \right] = p_{g_k}(x, \xi_k)
\]
and applying Lemma \[11.5\] yields that there are \(c_\epsilon > 0\), \(C_\epsilon > 0\) such that
\[
\liminf_{n \to \infty} \mathbb{P} \left[ \bigcap_{k \leq \log(n)^7} \left\{ \inf_{x \in [\log(n)^7, 1]} \mathbb{P}_{\mathcal{L}} \left[ B_{[0,k]}^{x,J_{\infty}}(T) \right] \geq C_\epsilon k^{-c_\epsilon} \right\} \right] \geq 1 - \epsilon/2. \tag{12.16}
\]
Plugging \((12.15)\) and \((12.16)\) into \((12.14)\) together with the observation that
\[
\sup_n \sum_{k=0}^{\log(n)^7} e^{-\theta \cdot h_n(k)} C_\epsilon k^{c_\epsilon} < \infty
\]
finishes the proof. \[\square\]
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