On a class of recursive estimators for spatially dependent observations

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Abstract: We investigate the mean squared error and the asymptotic normality for a class of recursive kernel estimators based on a sample of spatially dependent observations. Our main result provides sufficient conditions for a spatial version of a recursive estimator introduced by Hall and Patil (1994) to satisfy a central limit theorem. The results are stated for strongly mixing random fields in the sense of Rosenblatt (1956) and for weakly dependent random fields in the sense of Wu (2005).

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1. Introduction

In statistics, the nonparametric estimation of probability density functions of continuous random variables is a basic and central problem. From a given sample of observations, the main goal for a practitioner is to understand the mechanism from which the observations have been generated. In the last several decades, this question has attracted much attention among statisticians since it is of considerable interest in many applied fields such as forecasting, computer vision and machine learning. Among the plethora of nonparametric density estimators is the kernel density estimator introduced by Parzen [31] and Rosenblatt [34]...
which received considerable attention in nonparametric estimation for time series. More precisely, if \((X_1, \ldots, X_n)\) is a sample (observations) drawn from some univariate distribution with an unknown probability density \(f\) with respect to the Lebesgue measure on \(\mathbb{R}\) then the Parzen-Rosenblatt density estimator of \(f\) is defined for any positive integer \(n\) and any \(x\) in \(\mathbb{R}\) by

\[
f_n^{PR}(x) = \frac{1}{n h_n} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}\right)
\]

where \(K\) is a density function and the bandwidth \(h_n\) is a positive parameter which converges to zero such that \(n h_n\) goes to infinity. The bandwidth \(h_n\) is the most dominant parameter in the kernel density estimator since it controls its amount of smoothness. In fact, if \(h_n\) is small then the variance of the estimator is large while the bias is small. This leads to a nonsmooth estimated density. On the other, if \(h_n\) is large then the estimated density will be much smoother (small variance) but with a large bias leading to an unsatisfactory estimation. So, in practice, a trade-off between the variance and the bias must be found and the number of publications which are devoted to this crucial question in the literature is very extensive and is still a subject of many works in the statistic community (see for example \[8\], \[17\], \[22\], \[38\]). From a theoretical but also practical point of view, it is important to investigate asymptotic properties of density estimators when the number \(n\) of observations goes to infinity. For example, the consistency and the asymptotic normality of the estimator are very important in order to get pointwise estimation and confidence intervals for the target density \(f\). In his seminal paper, Parzen \[31\] proved that when the observations \((X_1, \ldots, X_n)\) are i.i.d. and the bandwidth \(h_n\) goes to zero such that \(n h_n\) goes to infinity then \((n h_n)^{1/2} \left(f_n^{PR}(x_0) - \mathbb{E}[f_n^{PR}(x_0)]\right)\) converges in distribution to the normal law with zero mean and variance \(f(x_0) \int \limits_{\mathbb{R}} K^2(t) dt\) as \(n\) goes to infinity and this result was extended by Wu and Mielenzuk \[44\] for causal linear processes with i.i.d. innovations and by Dedecker and Merlevède \[11\] for strongly mixing sequences. Previously, Bosq, Merlevède and Peligrad \[6\] established a central limit theorem for the kernel density estimator \(f_n\) when the sequence \((X_i)_{i \in \mathbb{Z}}\) is assumed to be strongly mixing but the bandwidth parameter \(h_n\) is assumed to satisfy \(h_n \geq C n^{-1/3} \log n\) (for some positive constant \(C\)) which is stronger than the bandwidth parameter assumption in \[11\], \[31\] and \[44\].

In many situations, practitioners are also interested by the relationship between some predictors and a response. This is a natural question and a very important task in statistics. The objective is to find a relation between a pair of random variables \(X\) (predictor) and \(Y\) (response) using a given sample \((X_i, Y_i)_{1 \leq i \leq n}\) drawn from the unknown law of \((X, Y)\). A very popular tool to handle this problem is the kernel regression estimator introduced by Nadaraya \[30\] and Watson \[41\]. More formally, let \(N\) be a positive integer and assume that \((X_i, Y_i)_{1 \leq i \leq n}\) are identically distributed \(\mathbb{R}^N \times \mathbb{R}\)-valued sequence of random variables such that \(Y_i = R(X_i, \eta_i)\) where \(R\) is an unknown functional and \((\eta_i)_{i \in \mathbb{Z}}\) are i.i.d. \(\mathbb{R}^N\)-valued random variables with zero mean and finite variance and independent of \((X_i)_{i \in \mathbb{Z}}\). Let \(f\) be the marginal density function of \(X_0\). If \(r\) is the (unknown)
regression function defined for any \( x \in \mathbb{R}^N \) by \( r(x) = \mathbb{E}[R(x, \eta_0)] \) if \( f(x) \neq 0 \) and \( r(x) = \mathbb{E}[Y_0] \) if \( f(x) = 0 \) then the Nadaraya-Watson regression estimator \( r_n^{NW} \) of \( r \) is defined for any \( x \in \mathbb{R}^N \) by

\[
 r_n^{NW}(x) = \begin{cases} 
 \sum_{i=1}^{n} Y_i K((x - X_i)/h_n) / \sum_{i=1}^{n} K((x - X_i)/h_n) & \text{if } \sum_{i=1}^{n} K((x - X_i)/h_n) \neq 0 \\
 n^{-1} \sum_{i=1}^{n} Y_i & \text{else.} 
\end{cases}
\]

The literature on the asymptotic properties of \( r_n^{NW} \) for time series is very expansive. One can refer to Lu and Cheng [24], Masry and Fan [26], Robinson [33], Roussas [36] and many references therein. Kernel nonparametric methods are still very popular and fairly well established in the statistical community but despite their power, the data streams problem, which refers to data sets that continuously and rapidly grow over time, present new challenges. In order to handle such data sets, several recursive versions of the Parzen-Rosenblatt estimator (1.1) have been introduced (see for example [3], [12], [20], [43], [47]).

For example, if \((w_k)_{k \geq 1}\) is a nonincreasing sequence of positive real numbers satisfying \( \sum_{k=1}^{\infty} w_k = \infty \) and \((h_k)_{k \geq 1}\) is a sequence of positive real numbers going to 0 as \( n \) goes to infinity (bandwidth parameters) then the recursive kernel density estimator \( f_{HP}^n \) of Hall and Patil [20] is defined by

\[
 f_{HP}^n(x) = \frac{1}{\sum_{k=1}^{n} w_k} \sum_{i=1}^{n} w_i K \left( \frac{x - X_i}{h_i} \right). 
\]

This estimator is recursive in the sense that it satisfies

\[
 f_{n+1}^{HP}(x) = (1 - \gamma_n) f_n^{HP}(x) + \gamma_n f_{n+1}(x) 
\]

where \( \gamma_n := \frac{w_n}{\sum_{i=1}^{n} w_i} \) and \( f_{n+1}(x) := \frac{1}{h_{n+1}} K \left( \frac{x - X_{n+1}}{h_{n+1}} \right) \).

Such a property endows recursive estimators with a decisive computational advantage because they can be easily updated as new data items arrive over time. More precisely, in order to obtain the estimation \( f_{n+1}^{HP}(x) \) at time \( n + 1 \), using the recursive equation (1.4), it is sufficient to combine the estimation \( f_n^{HP}(x) \) at time \( n \) (which is known at time \( n + 1 \)) with the estimation \( f_{n+1}(x) \) at time \( n + 1 \) based on the single observation \( X_{n+1} \). In fact, a non-recursive estimator must be fully recomputed whenever a new observation is collected. This clearly represents a drawback in a data stream context compared to the recursive approach. The class (1.3) contains the recursive estimators introduced by Wolverton and Wagner [43] and Deheuvels [12] but also a renormalized version of the one introduced by Wegman and Davies [42] and another class of estimators introduced by Amiri [3]. It contains also the (non-recursive) Parzen-Rosenblatt estimator (1.1) when \( h_i = h_n \) and \( w_i = 1 \) for any \( 1 \leq i \leq n \). In this work, our aim is to investigate asymptotic properties for a spatial version of the Hall and Patil estimator (1.3) in terms of mean squared error and asymptotic normality under weak and strong dependence conditions. The first studies
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dependent random fields indexed by \( \mathbb{Z}^d \) where \( d \) is a positive integer. Actually, many papers in the literature are devoted to the asymptotic properties of recursive kernel density and regression estimators for i.i.d. observations. There are also some published papers on the asymptotic properties of recursive kernel density and regression estimators for dependent (weakly dependent and strongly mixing) data. One can refer for example to [4], [7], [18], [19], [25], [36], [39], [40] and others.

In our context, we deal with spatial data which is modeled using finite realizations of dependent random fields indexed by \( \mathbb{Z}^d \) where \( d \) is a positive integer. More precisely, let \( N \) be a positive integer and let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. We consider a stationary \( \mathbb{R}^N \)-valued random field \((X_k)_{k \in \mathbb{Z}^d}\) such that the law \( \mu_0 \) of \( X_0 \) is absolutely continuous with respect to the Lebesgue measure \( \lambda_N \) on \( \mathbb{R}^N \) and we denote by \( f \) the probability density function of \( \mu_0 \) with respect to \( \lambda_N \). Given two sub-\( \sigma \)-algebras \( \mathcal{U} \) and \( \mathcal{V} \) of \( \mathcal{F} \), recall that the \( \alpha \)-mixing coefficient introduced by Rosenblatt [35] is defined by

\[
\alpha(\mathcal{U}, \mathcal{V}) = \sup\{||\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)||, \ A \in \mathcal{U}, \ B \in \mathcal{V}\}.
\]

Let \( p \) be fixed in \([1, +\infty]\). The strong mixing coefficients \((\alpha_{1,p}(n))_{n \geq 0}\) associated to \((X_k)_{k \in \mathbb{Z}^d}\) are defined by

\[
\alpha_{1,p}(n) = \sup \{\alpha(\sigma(X_k), \mathcal{F}_\Gamma), \ k \in \mathbb{Z}^d, \ |\Gamma| \leq p, \ \rho(\Gamma, \{k\}) \geq n\}
\]

where \(|\Gamma|\) is the number of elements in \( \Gamma \), the collection \( \mathcal{F}_\Gamma \) is the \( \sigma \)-algebra \( \sigma(X_k : k \in \Gamma) \) and the distance \( \rho \) is defined for any subsets \( \Gamma_1 \) and \( \Gamma_2 \) of \( \mathbb{Z}^d \) by

\[
\rho(\Gamma_1, \Gamma_2) = \min\{|u - v, |, \ u \in \Gamma_1, \ v \in \Gamma_2\} \text{ with } |u - v| = \max_{1 \leq \ell \leq d} |u_\ell - v_\ell| \text{ for any } u = (u_1, \ldots, u_d) \text{ and } v = (v_1, \ldots, v_d) \text{ in } \mathbb{Z}^d.
\]

We say that the random field \((X_k)_{k \in \mathbb{Z}^d}\) is strongly mixing if \( \lim_{n \to \infty} \alpha_{1,p}(n) = 0 \). Moreover, we are going to consider also Bernoulli fields defined for any \( k \in \mathbb{Z}^d \) by

\[
X_k = G (\varepsilon_{k-u} ; u \in \mathbb{Z}^d)
\]

where \( G : (\mathbb{R}^m)^{\mathbb{Z}^d} \to \mathbb{R}^N \) is a measurable function, \((\varepsilon_k)_{k \in \mathbb{Z}^d}\) are i.i.d. \( \mathbb{R}^m \)-valued random variables and \( m \) is a positive integer. The class of random fields that (1.5) represents is huge and it includes many commonly used linear and nonlinear processes (see Wu [46] for a review). Let \((\varepsilon'_k)_{k \in \mathbb{Z}^d}\) be an i.i.d. copy of \((\varepsilon_k)_{k \in \mathbb{Z}^d}\) and let \( X_k^* \) be the coupled version of \( X_k \) defined by \( X_k^* = G (\varepsilon_k^*; u \in \mathbb{Z}^d) \) where \( \varepsilon_k^* = \varepsilon_k \) if \( k \neq 0 \) and \( \varepsilon_k^* = \varepsilon'_0 \). Note that \( X_k^* \) is obtained from \( X_k \) by replacing \( \varepsilon_0 \) by its copy \( \varepsilon'_0 \). For any positive integer \( \ell \) and any \( \mathbb{R}^\ell \)-valued random variable \( Z \in L^p(\Omega, \mathcal{F}, \mathbb{P}) \) with \( p > 0 \), we denote \( ||Z||_p := \mathbb{E}(||Z||^p)^{1/p} \) where \( ||.|| \) is the Euclidian norm on \( \mathbb{R}^\ell \). Following Wu [45] and El Machkouri et al. [16], we define the physical dependence measure coefficient \( \delta_{k,p} := ||X_k - X_k^*||_p \) as soon as \( X_k \) is \( p \)-integrable for \( p \geq 2 \). Physical dependence measure should be seen as a measure of the dependence of the function \( G \) (defined in (1.5)) in the coordinate zero. In some sense, it quantifies the degree of dependence of outputs on inputs in physical systems and provide a natural framework for a limit
theory for stationary random fields (see [16]). In particular, it gives mild and easily verifiable conditions (see condition (H2)(ii) below) because it is directly related to the data-generating mechanism.

2. Main results

Let \( \Lambda_0 = 0 \), \( s_0 = 0 \in \mathbb{Z}^d \) and \( \Lambda_n = \{s_1, ..., s_n\} \subset \mathbb{Z}^d \) for \( n \geq 1 \). Let \((w_{s_n})_{n \geq 1}\) and \((h_{s_n})_{n \geq 1}\) be two nonincreasing sequences of positive real numbers such that \((w_{s_n}, h_{s_n}^{-N})_{n \geq 1}\) is nondecreasing, \( h_{s_n} \) goes to 0 as \( n \) goes to infinity and \( \sum_{n \geq 1} w_{s_n} = \infty \). Let also \( K : \mathbb{R}^N \to \mathbb{R}_+ \) be a function (called a kernel) such that \( \int_{\mathbb{R}^N} K(t) dt = 1 \) and \( \sup_{x \in \mathbb{R}^N} K(x) < \infty \). Assume that \( K \) is Lipschitz and satisfies \( \lim_{\|x\| \to \infty} \|x||K(x) = 0 \), \( \int_{\mathbb{R}^N} \|u\|^2 K(u) du < \infty \) where \( \| \cdot \| \) is the usual norm on \( \mathbb{R}^N \) and \( \int_{\mathbb{R}^N} u_i K(u) du = 0 \) for any \( 1 \leq i \leq N \). Let \( \Phi : \mathbb{R} \to \mathbb{R} \) be a measurable function such that \( \mathbb{E}[|\Phi(Y_0)|^{2+\theta}] < \infty \) and \( \mathbb{E}[|\Phi(Y_0)|^{2+\theta}K_{s_n}(x, X_0)] \leq C h_{s_n}^{\alpha} \) for some \( \alpha > 0 \) and \( C > 0 \) and assume that \( u \mapsto \mathbb{E}[|\Phi(Y_0)|^2 X_0 = u] \) is continuous. One can notice that \( \mathbb{E}[|\Phi(Y_0)|^{2+\theta}K_{s_n}(x, X_0)] \leq C h_{s_n}^{\alpha} \) is satisfied when \( u \mapsto \mathbb{E}[|\Phi(Y_0)|^{2+\theta}X_0 = u] \) is continuous (see Lemma 2 below). Let \((\eta_k)_{k \in \mathbb{Z}^d}\) be i.i.d. \( \mathbb{R}^N \)-valued random variables with zero mean and finite variance and independent of \((X_k)_{k \in \mathbb{Z}^d}\) and consider the regression model \( Y_n = R(X_{s_n}, \eta_n) \) for any \( 1 \leq i \leq n \) where \( R \) is an unknown functional. For any \( x \in \mathbb{R}^N \), we denote \( f_{\Phi}(x) = r_{\Phi}(x)f(x) \) where \( r_{\Phi}(x) = \mathbb{E}[\Phi(Y_0)|X_0 = x] = \mathbb{E}[\Phi(R(x, \eta_0))] \) if \( f(x) \neq 0 \) and \( r_{\Phi}(x) = \mathbb{E}[\Phi(Y_0)] \) if \( f(x) = 0 \) and we consider the estimator \( f_{n, \Phi} \) of \( f_{\Phi} \) defined by

\[
f_{n, \Phi}(x) = \left( \frac{n}{w_{s_n}} \right)^{-1} \sum_{j=1}^{n} w_{s_j} h_{s_j}^{-N} \Phi(Y_{s_j}) K_{s_j}(x, X_{s_j})
\]

(2.1)

where \( K_{s_j}(x, v) = K((x - v)/h_{s_j}) \) for any \( v \in \mathbb{R}^N \) and any \( 1 \leq j \leq n \). One can notice that if \( \Phi(u) = 1 \) for any \( u \in \mathbb{R} \) then \( f_{n, \Phi} \) reduces to the spatial version \( f_{n, 1} \) of the recursive kernel density estimator of \( f \) introduced by Hall and Patil [20] and defined for any \( x \in \mathbb{R}^N \) by

\[
f_{n, 1}(x) = \left( \frac{n}{w_{s_n}} \right)^{-1} \sum_{j=1}^{n} w_{s_j} h_{s_j}^{-N} K_{s_j}(x, X_{s_j}).
\]

(2.2)

Moreover, for particular choices of the weights \((w_{s_n})_{n \geq 1}\), the estimator \( f_{2.2} \) reduces to the recursive estimators introduced by [3], [4], [12] or [43]. In particular, one can check that \( f_{n, \Phi} \) satisfies the following recursive equation

\[
f_{n, \Phi}(x) = (1 - \rho_n)f_{n-1, \Phi}(x) + \rho_n h_{s_n}^{-N} \Phi(Y_{s_n}) K_{s_n}(x, X_{s_n})
\]

(2.3)

where \( \rho_n = \frac{w_{s_n}}{\sum_{i=1}^{w_{s_n}}} \). Equation (2.3) is the spatial version of the recursive equation (1.4). It lays emphasis on that the update of \( f_{n, \Phi} \) at time \( n \) can be done from \( f_{n-1, \Phi} \) at time \( n - 1 \) and the new single observation \( X_{s_n} \). This is a definitive advantage over the spatial version of the non-recursive Parzen-Rosenblatt
estimator \( f_n^{PR} \) defined by (1.1) since it is necessary to consider the whole sample \((X_n, \ldots, X_{s_n})\) in order to compute \( f_n^{PR} \) at any time \( n \). In this work, we consider also the following class of spatial semi-recursive kernel regression estimator \( r_{n, \Phi} \) of \( r_{\Phi} \) defined for any \( x \in \mathbb{R}^N \) by

\[
\begin{align*}
    r_{n, \Phi}(x) &= \begin{cases} 
        \frac{f_{n, \Phi}(x)}{f_{n, 1}(x)} & \text{if } \sum_{j=1}^{n} w_s h_s^{-N} K_{s_j}(x, X_{s_j}) \neq 0 \\
        n^{-1} \sum_{i=1}^{n} Y_{s_i} & \text{else}
    \end{cases} \tag{2.4}
\end{align*}
\]

which contains the first two semi-recursive kernel regression estimators introduced by Ahmad and Lin [2] and Devroye and Wagner [13] for time series (i.e. for \( d = 1 \)) but also the class of semi-recursive kernel regression estimators considered by Amiri [4]. Since \( r_{n, \Phi} \) is defined from \( f_{n, \Phi} \) and \( f_{n, 1} \), it inherits the good properties in term of computation time of the recursive estimators \( f_{\Phi} \) and \( f_{n, 1} \) and consequently, in a data stream setting, it has a decisive advantage over the spatial version of the non-recursive Nadaraya-Watson estimator defined by (1.2).

Now, we are going to present our main contributions. For \( j \in \{2, 4\} \), we adopt the notation

\[
\nu_{j}(\theta) = \mathbb{I}_{\|\Phi\|_\infty < \infty} + \frac{\theta}{j + \theta} \mathbb{I}_{\|\Phi\|_\infty = \infty} \tag{2.5}
\]

and for any sequences \((a_n)_{n \geq 1}\) and \((b_n)_{n \geq 1}\) of real positive numbers, we denote \( a_n \leq b_n \) if and only if there exists \( \kappa > 0 \) (not depending on \( n \)) such that \( a_n \leq \kappa b_n \). Recall that \((w_s)_{n \geq 1}\) and \((h_{s_n})_{n \geq 1}\) are two nonincreasing sequences of positive real numbers such that \((w_s h_s^{-N})_{n \geq 1}\) is nondecreasing, \( h_{s_n} \) goes to 0 as \( n \) goes to infinity and \( \sum_{n \geq 1} w_s = \infty \) and keep in mind that \( K : \mathbb{R}^N \to \mathbb{R}_+ \) is a function (kernel) such that \( \int_{\mathbb{R}^N} K(t) dt = 1 \) and \( |K|_\infty := \sup_{t \in \mathbb{R}^N} K(t) < \infty \).

For any integer \( n \geq 1 \) and any \((p, q) \in \mathbb{Z}^2\), we denote also

\[
A_{n, p, q} = (n h_{s_n}^p w_{s_n}^q)^{-1} \sum_{i=1}^{n} h_{w_{s_i}}^p w_{w_{s_i}}^q
\]

and we consider the following assumptions:

(H1) There exists \((\beta_{0,1}, \beta_{-N,2}) \in (\mathbb{R}_+^*)^2\) such that \( \lim_{n \to \infty} A_{n, 0, 1} = \beta_{0,1} \) and \( \lim_{n \to \infty} A_{n, -N, 2} = \beta_{-N,2} \).

(H2) There exist \( \theta > 0 \) such that \( \mathbb{E}(|\Phi(Y_0)|^{2+\theta}) < \infty \) and

\[
\mathbb{E}(|\Phi(Y_0)|^{2+\theta} K_{s_n}(x, X_{s_n})) \leq h_{s_n}^{\theta}, \quad \tau \in [1 - 1/\nu_4(\theta), 1], \quad \text{and} \quad \lim_{n \to \infty} h_{s_n}^N \left( 1 + \frac{d \nu_2(\theta) + (1/\nu_2(\theta) + 1)}{\nu_4(\theta) + \tau - 1} \right) = \infty
\]

Moreover, one of the following condition holds:

(i) \((X_k)_{k \in \mathbb{Z}^d}\) is strongly mixing and \( \sum_{k \in \mathbb{Z}^d} |k|^{d \nu_2(\theta)} \mathbb{E}^{1/\nu_4(\theta)}(1/k^\tau) \alpha_{1,\infty}^{\nu_2(\theta)}(|k|) < \infty \),

(ii) \((X_k)_{k \in \mathbb{Z}^d}\) is of the form (1.5) and

\[
\sum_{k \in \mathbb{Z}^d} |k|^{2N(\nu_2(\theta) + \tau - 1) + d(\nu_2(\theta) + \nu_4(\theta) + 1)} \delta_{k,2}^{\nu_2(\theta)} < \infty.
\]
(H3) (i) The function \( u \mapsto E[|\Phi(Y_0)|^2 | X_0 = u] \) is continuous.

(ii) The function \( f_\Phi \) is twice differentiable with bounded second derivatives.

(iii) For any \( k \in \mathbb{Z}^d \setminus \{0\} \), the law of \( (X_0, X_k) \) is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^N \times \mathbb{R}^N \) and there exists \( c > 0 \) such that \( \sup_{k \in \mathbb{Z}^d \setminus \{0\}} |f_{0,k}(x,y) - f(x)f(y)| \leq c \) for any \( (x,y) \in \mathbb{R}^d \times \mathbb{R}^N \) where \( f_{0,k} \) is the joint density function of \( (X_0, X_k) \).

Assumptions (H1) and (H3) are classical in the context of recursive kernel estimators (see [4], [25], [28], [42] and many others). In (H2), we assume that the bandwidth parameter \( h_{sn} \) satisfies a condition slightly stronger than the usual minimal condition assumed in the non recursive i.i.d. setting (i.e. \( nh_{sn}^N \to \infty \)).

However, this fact seems to be inherent to the case of recursive estimators since a condition like \( nh_{sn}^{N(1+\varepsilon)} \to \infty \) for some \( \varepsilon > 0 \) is assumed in many contributions for dependent data (see for example [4], [1], [25], [28] or [42]).

For any \( x \in \mathbb{R}^N \), we denote

\[
\sigma^2_{\Phi}(x) := \frac{\alpha_{1,1}}{\alpha_{1,1}^{\nu_2(\theta)}} \int_{\mathbb{R}^N} K^2(t)dt. \tag{2.6}
\]

Our first result gives the asymptotic variance of the estimator \( f_{n,\Phi} \) defined by (2.1).

**Proposition 1.** Assume that (H1) and (H3) hold and there exists \( \theta > 0 \) such that \( E[|\Phi(Y_0)|^{2+\theta}] < \infty \) and \( E[|\Phi(Y_0)|^{2+\theta}K_{sn}(x,X_0)] \leq h_{sn}^N \). If there exists \( \tau \in [1-\nu_4(\theta), 1] \) such that \( h_{sn}^{N(1-\tau)} \sum_{i=1}^n w_{si}^2 \leq nh_{sn}^{2\nu_3(\theta)} \) and one of the following conditions is satisfied:

(i) \( (X_k)_{k \in \mathbb{Z}^d} \) is strongly mixing and \( \sum_{k \in \mathbb{Z}^d} |k|^{d(\theta)/(2+\theta)+1/2 - \nu_2(\theta)} \leq \infty \)

(ii) \( (X_k)_{k \in \mathbb{Z}^d} \) is of the form (1.5) and

\[
\sum_{k \in \mathbb{Z}^d} |k|^{d(\theta)/(2+\theta)+1/2 - \nu_2(\theta)} \leq \infty
\]

where \( \nu_2(\theta) \) and \( \nu_4(\theta) \) are defined by (2.5) then for any \( x \in \mathbb{R} \),

\[
\lim_{n \to \infty} |nh_{sn}^{2\nu_3(\theta)}(f_{n,\Phi}(x)) - \sigma^2_{\Phi}(x)| = 0 \tag{2.7}
\]

where \( \sigma^2_{\Phi}(x) \) is defined by (2.6).

We obtain also the convergence to zero of the mean square error of \( f_{n,\Phi} \).

**Proposition 2.** Assume that \( f_\Phi \) is twice differentiable with bounded second derivatives.

Then, for any \( x \in \mathbb{R}^N \), \( E[|f_{n,\Phi}(x) - f_\Phi(x)|] \leq (\sum_{i=1}^n w_{si})^{-1} \sum_{i=1}^n w_{si}h_{si}^2 = o(1) \). So, if \( \max\{A_{n,1,1}, A_{n,2,1}\} \leq 1 \) then \( E[|f_{n,\Phi}(x) - f_\Phi(x)|] \leq h_{sn}^2 \) and, under assumptions of Proposition 1, we get \( E[(f_{n,\Phi}(x) - f_\Phi(x))^2] \leq n^{-\frac{\nu_3(\theta)}{2+\theta}} \) for \( h_{sn} = n^{-\frac{1}{1+\tau}} \).

The main contribution of this paper is the following central limit theorem.
Theorem 1. If (H1), (H2) and (H3) hold then for any \(x \in \mathbb{R}^N\),
\[
\sqrt{nh_{s_n}^N} (f_n, \Phi(x) - \mathbb{E}[f_n, \Phi(x)]) \overset{\text{Law}}{\longrightarrow} \mathcal{N}(0, \sigma_\Phi^2(x))
\]
where \(\sigma_\Phi^2(x)\) is defined by (2.6).

One can notice that Theorem 1 is an extension of Theorem 1 in [1] where the case of strongly mixing time series is considered. In fact, with our notations, if \(d = 1\) and \(\Phi(u) = 1\) for any \(u \in \mathbb{R}\) then \(f_n, \Phi\) reduces to the recursive kernel density estimator \(f_n, 1\) introduced by Hall and Patil [20]. In this case, we have \(\nu_2(\theta) = \nu_4(\theta) = 1\) and (H2)(ii) holds as soon as \(\sum_{k>0} k^{1/\alpha_1(k)} < \infty\) and \(\lim_{n \to \infty} n h_{s_n}^{-1} = \infty\) which are exactly the conditions imposed in Theorem 1 in [1]. Using Theorem 1, we derive the asymptotic normality for the recursive estimator \(r_n, \Phi\) defined by (2.4).

Theorem 2. Assume that (H1), (H2) and (H3) hold. If \(f\) is Lipschitz and twice differentiable with bounded second derivatives then for any \(x \in \mathbb{R}^N\) such that \(f(x) > 0\),
\[
\sqrt{nh_{s_n}^N} \left( r_n, \Phi(x) - \mathbb{E}[f_n, \Phi(x)] \right) \overset{\text{Law}}{\longrightarrow} \mathcal{N}(0, \tilde{\sigma}_\Phi^2(x)),
\]
with \(\tilde{\sigma}_\Phi^2(x) = V(x) \frac{\beta_{-N} \int_{\mathbb{R}^N} K^2(t) dt}{f(x) \beta_{-N}^2} \mathbb{E}[|\Phi(Y_0)|^2|X_0 = x] - r_\Phi^2(x)\).

Theorem 2 is also an extension of Theorem 2.1 in [36] where the asymptotic normality of the semi-recursive kernel regression estimator for time series (i.e. \(d = 1\)) introduced by Ahmad and Lin [2] is obtained under more restrictive conditions on the bandwidth parameter and the strong mixing coefficients. Using Theorem 2 and Proposition 2, the condition \(nh_{s_n}^{-1} \to 0\) can be imposed for the control of the bias of the estimator and leads immediately to the following result.

Theorem 3. Assume that (H1), (H2) and (H3) hold. If \(f\) is Lipschitz and twice differentiable with bounded second derivatives, \(nh_{s_n}^{-1} \to 0\) and \(A_{n,2,1} \leq 1\), then for any \(x \in \mathbb{R}^N\) such that \(f(x) > 0\),
\[
\sqrt{nh_{s_n}^N} (r_n, \Phi(x) - r_\Phi(x)) \overset{\text{Law}}{\longrightarrow} \mathcal{N}(0, \tilde{\sigma}_\Phi^2(x)),
\]
where \(\tilde{\sigma}_\Phi^2(x)\) is defined in Theorem 2.

3. Preliminary lemmas

This section is devoted to the presentation of several technical lemmas and propositions which are key tools in the proof of the main contributions in section 4. For any real \(x\), we also define \([x] = \lfloor x\rfloor + 1\), where \([x]\) is the largest integer less or equal than \(x\).
Lemma 1. Let \((a_k)_{k \in \mathbb{Z}}\) be a family of real numbers such that \(a_{s_n}\) goes to some value \(a \in \mathbb{R}\) as \(n\) goes to infinity. If \(\lim_{n \to \infty} A_{n,-N,2} = \beta_{-N,2} \in \mathbb{R}\) then

\[
\lim_{n \to \infty} \frac{h_{s_n}^N}{nh_{s_n}^{2n}} \sum_{i=1}^{n} \frac{w_i^2 a_{s_i}}{h_{s_i}} = a\beta_{-N,2}.
\]

Proof of Lemma 1. For any positive integers \(i\) and \(n\), we denote

\[
b_{i,n} = w_i^2 h_{s_i}^N/(nh_{s_i}^N w_{s_i}^2)
\]

if \(i \leq n\) and \(b_{i,n} = 0\) otherwise. Since \((h_{s_n})_{n \geq 1}\) is nonincreasing and \((w_{s_n} h_{s_n}^{-N})_{n \geq 1}\) is nondecreasing, for \(i \leq n\), we have

\[
b_{i,n} = \frac{h_{s_i}^N}{nh_{s_i}^N} \times \frac{w_i^2}{w_{s_i}^2} \leq \frac{h_{s_i}^N}{nh_{s_i}^N} \to 0\,.
\]

Moreover, \(\sum_{i=1}^{+\infty} b_{i,n} = A_{n,-N,2} \to \beta_{-N,2} \in \mathbb{R}\) as \(n \to +\infty\). So, by Toeplitz’s lemma (see Lemma 3 in [25]), we get

\[
\lim_{n \to \infty} \sum_{i=1}^{n} b_{i,n} a_{s_i} = a\beta_{-N,2}.
\]

The proof Lemma 1 is complete. \(\square\)

The following lemma will be useful in order to compute the asymptotic variance of the estimator \(f_{n,\Phi}\) (see Propositions 1 and 4).

Lemma 2. Let \(x \in \mathbb{R}^N\) be fixed and let \(\Psi_1 : \mathbb{R} \to \mathbb{R}\) and \(\Psi_2 : \mathbb{R} \to \mathbb{R}\) be two functions. If \(u \mapsto \mathbb{E}[\Psi_1(Y_0)|X_0 = u]\) is continuous and the conditions \(\sup_{t \in \mathbb{R}^N} |\Psi_2(K(t))| < \infty, \lim_{||t|| \to \infty} ||t|| |\Psi_2(K(t))| = 0\) and \(\int_{\mathbb{R}^N} |\Psi_2(K(t))| dt < \infty\) are satisfied then

\[
\lim_{n \to \infty} h_{s_n}^{-N} \mathbb{E}[\Psi_1(Y_0)\Psi_2(K_{s_n}(x,X_0))] = \mathbb{E}[\Psi_1(Y_0)|X_0 = x] f(x) \int_{\mathbb{R}^N} \Psi_2(K(v)) dv.
\]

Proof of Lemma 2. Let \(x \in \mathbb{R}^N\) and let \(n\) be a positive integer. It is obvious that

\[
\mathbb{E}[\Psi_1(Y_0)\Psi_2(K_{s_n}(x,X_0))]
\]

\[
= h_{s_n}^N \int_{\mathbb{R}^N} \mathbb{E}[\Psi_1(Y_0)|X_0 = x - vh_{s_n}] \Psi_2(K(v)) f(x - vh_{s_n}) dv.
\]

By Theorem 1A in [31], we derive

\[
\lim_{n \to \infty} h_{s_n}^{-N} \mathbb{E}[\Psi_1(Y_0)\Psi_2(K_{s_n}(x,X_0))]
\]

\[
= \mathbb{E}[\Psi_1(Y_0)|X_0 = x] f(x) \int_{\mathbb{R}^N} \Psi_2(K(v)) dv. \quad (3.1)
\]

The proof of Lemma 2 is complete. \(\square\)
For any $\ell \in \{1, 2\}$, any $1 \leq i \leq n$ and any sequence $(m_i)_{i \geq 1}$ of positive integers, we define

$$
\Delta_{s_i}^{(\ell)} = \Phi_\ell(Y_{s_i})K_{s_i}(x, X_{s_i}) - \mathbb{E}[\Phi_\ell(Y_0)K_{s_i}(x, X_0)] \quad \text{and} \quad \Delta_{s_i}^{(\ell)} = \mathbb{E}[\Delta_{s_i}^{(\ell)}|\mathcal{H}_{i,m_i}]
$$

where $\mathcal{H}_{i,m_i} = \sigma(\eta_{s_i}, \varepsilon_{s_i-k}, |k| \leq m_i)$ and $\Phi_\ell : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function.

**Lemma 3.** Let $\ell \in \{1, 2\}$ and $\theta > 0$ such that $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}K_{s_\ell}(x, X_0)] \leq h_{s_\ell}^{N_\ell}$ then $\mathbb{E}[|\Phi_\ell(Y_0)|^pK_{s_\ell}(x, X_0)] \leq h_{s_\ell}^{N_\ell}$ for any $0 < p < 2 + \theta$. Moreover, if (H3)(iii) holds then $\sup_{1 \leq i, j \leq n} (h_{s_i}h_{s_j})^{-N_\ell}\mathbb{E}[K_{s_i}(x, X_{s_i})K_{s_j}(x, X_{s_j})] \leq 1$.

**Proof of Lemma 3.** If $0 < p < 2 + \theta$ then

$$
\mathbb{E}[|\Phi_\ell(Y_0)|^pK_{s_\ell}(x, X_0)] \leq \mathbb{E}[K_{s_\ell}(x, X_0)] + \mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}K_{s_\ell}(x, X_0)].
$$

Since $\mathbb{E}[|\Phi_\ell(Y_0)|^rK_{s_\ell}(x, X_0)] \leq h_{s_\ell}^{N_\ell}$ for $r \in \{0, 2 + \theta\}$, we get $\mathbb{E}[|\Phi_\ell(Y_0)|^pK_{s_\ell}(x, X_0)] \leq h_{s_\ell}^{N_\ell}$. In the other part, using (H3)(iii), for any $1 \leq i, j \leq n$ such that $i \neq j$ we have

$$
\mathbb{E}[K_{s_i}(x, X_{s_i})K_{s_j}(x, X_{s_j})] \leq \int_{\mathbb{R}^N} K_{s_i}(x, u)du \int_{\mathbb{R}^N} K_{s_j}(x, v)dv + \mathbb{E}[K_{s_i}(x, X_0)]\mathbb{E}[K_{s_j}(x, X_0)] \leq (h_{s_i}h_{s_j})^{N_\ell}.
$$

The proof of Lemma 3 is complete. \qed

**Lemma 4.** If $(p, q) \in \{1, 2\}^2$ and $\theta > 0$ such that $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}] < \infty$ and $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}K_{s_\ell}(x, X_0)] \leq h_{s_\ell}^{N_\ell}$ for any $\ell \in \{p, q\}$ then

$$
\sup_{1 \leq i, j \leq n} (h_{s_i}h_{s_j})^{-N_\ell}\mathbb{E}[|\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}|] \leq 1
$$

where $\gamma = \begin{cases} 
1 & \text{if } \max(||\Phi_p||_\infty, ||\Phi_q||_\infty) < \infty \\
\frac{\theta}{1 + \theta} & \text{if } \min(||\Phi_p||_\infty, ||\Phi_q||_\infty) = \infty \\
\frac{\theta}{2 + \theta} & \text{else}.
\end{cases} \quad (3.4)$

**Proof of Lemma 4.** Let $i$ and $j$ be two positive integers such that $i \neq j$ and $(p, q) \in \{1, 2\}^2$ and let $\theta > 0$ such that $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}K_{s_\ell}(x, X_0)] \leq h_{s_\ell}^{N_\ell}$ and $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}] < \infty$ for any $\ell \in \{p, q\}$. Keeping in mind the notations $\Delta_{s_i}^{(p)}$ and $\Delta_{s_j}^{(q)}$ defined by (3.2), we have the following bound

$$
(h_{s_i}h_{s_j})^{\gamma}\mathbb{E}[|\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}|] \leq \mathbb{E}[|\Phi_p(Y_{s_i})\Phi_q(Y_{s_j})|K_{s_i}(x, X_{s_i})K_{s_j}(x, X_{s_j})] + 3\mathbb{E}[|\Phi_p(Y_0)|K_{s_i}(x, X_0)]\mathbb{E}[|\Phi_q(Y_0)|K_{s_j}(x, X_0)]. \quad (3.5)
$$
Note that the second term of the right hand side of (3.5) can be dealt with using Lemma 3. Therefore, we focus on the first part of the right hand side only. Let \( L > 1 \) be fixed then

\[
E[\Phi_p(Y_s) \Phi_q(Y_s) | K_s(x, X_s) K_s(x, X_s)]
\]

\[
= E[\Phi_p(Y_s) \Phi_q(Y_s) | 1 | \Phi_p(Y_s) | \leq L \| \Phi_q(Y_s) \| \leq L K_s(x, X_s) K_s(x, X_s)]
\]

\[
+ E[\Phi_p(Y_s) \Phi_q(Y_s) | 1 | \Phi_q(Y_s) | \geq L K_s(x, X_s) K_s(x, X_s)]
\]

\[
+ E[\Phi_p(Y_s) \Phi_q(Y_s) | 1 | \Phi_q(Y_s) | \geq L K_s(x, X_s) K_s(x, X_s)]
\]

Using Cauchy-Schwarz’s inequality, we obtain

\[
E[\Phi_p(Y_s) \Phi_q(Y_s) | K_s(x, X_s) K_s(x, X_s)]
\]

\[
\leq (L \| \Phi_p \| \infty) (L \| \Phi_q \| \infty) E[K_s(x, X_s) K_s(x, X_s)]
\]

\[
+ \sqrt{E[\Phi_p(Y_0)^2 K_s^2(x, X_0)]} \sqrt{E[\Phi_q(Y_0)^2 K_s^2(x, X_0)]}
\]

\[
+ \sqrt{E[\Phi_p(Y_0)^2 \| \Phi_q(Y_0) \| > L K_s^2(x, X_0)]} \sqrt{E[\Phi_q(Y_0)^2 K_s^2(x, X_0)]}
\]

Since \( E[\Phi_{\ell}(Y_0)^{2+\theta}] < \infty \) and \( E[\Phi_{\ell}(Y_0)^{2+\theta} K_n(x, X_0)] \leq h_n^N \) for any \( \ell \in \{p, q\} \), we apply Lemma 3 and get

\[
E[\Phi_p(Y_s) \Phi_q(Y_s) | K_s(x, X_s) K_s(x, X_s)]
\]

\[
\leq (L \| \Phi_p \| \infty) (L \| \Phi_q \| \infty) (h_n h_s)^N + L^{-\theta/2} (h_n h_s)^{N/2}.
\]

Optimizing (3.6) with respect to \( L \), we derive

\[
E[\Phi_p(Y_s) \Phi_q(Y_s) | K_s(x, X_s) K_s(x, X_s)]
\]

\[
\leq (h_n h_s)^{\frac{N}{N+\theta}} \text{ if } \| \Phi_p \| \infty < \infty \text{ and } \| \Phi_q \| \infty = \infty
\]

\[
(h_n h_s)^{\frac{N}{N+\theta}} \text{ if } \| \Phi_p \| \infty = \infty \text{ and } \| \Phi_q \| \infty < \infty
\]

\[
(h_n h_s)^{\frac{N}{N+\theta}} \text{ if } \| \Phi_p \| \infty < \infty \text{ and } \| \Phi_q \| \infty < \infty
\]

\[
(h_n h_s)^{\frac{N}{N+\theta}} \text{ if } \| \Phi_p \| \infty = \infty \text{ and } \| \Phi_q \| \infty = \infty
\]

Combining (3.5), (3.7) and Lemma 3, we obtain (3.3). The proof of Lemma 4 is complete. \( \Box \)

**Lemma 5.** Let \( (\Xi_k)_{k \in \mathbb{Z}^d} \) be a family of non negative real numbers.

If \( \sum_{k \in \mathbb{Z}^d} |k|^p \Xi_k < \infty \) for some positive constants \( \ell_1 \) and \( \ell_2 \) then there exists a sequence \( (m_n)_{n \geq 1} \) of positive integers satisfying

\[
\lim_{n \to \infty} m_n = +\infty, \quad \lim_{n \to \infty} m_n^{d+\ell_1} = 0 \quad \text{and} \quad \lim_{n \to \infty} h_n^{\ell_2} \sum_{|k| > m_n} \Xi_k = 0.
\]
Notice that if \( \ell_1 \leq N \) and \( nh_{s_n}^N \to \infty \) then \( m_{n}^{d} = o(n) \).

**Proof of Lemma 5.** Let \( \ell_1, \ell_2 \) and \( r \) be positive constants such that \( r > \ell_2/\ell_1 \) and let \( (m_{n})_{n \geq 1} \) be the sequence defined for any integer \( n \geq 1 \) by

\[
m_n = \max \left\{ v_n, \left[ h_{s_n}^{-\ell_1/d} \left( \sum_{|k|>v_n} |k|^\frac{d\ell_2}{\ell_1} \Xi_k \right)^{\frac{1}{p}} \right] \right\} \quad \text{and} \quad v_n = \lfloor h_{s_n}^{-\ell_1/(2d)} \rfloor.
\]

Since \( v_n \to \infty \), we have \( m_n \to \infty \) as \( n \) goes to infinity. Moreover,

\[
m_n^{d\ell_1} \leq \max \left\{ h_{s_n}^{\ell_1/2} \left( \sum_{|k|>v_n} |k|^\frac{d\ell_2}{\ell_1} \Xi_k \right)^{\frac{1}{2}} + h_{s_n}^{\ell_1} \right\} \to 0 \quad \text{as} \quad n \to \infty.
\]

Since \( v_n \leq m_n \), we have

\[
m_n^{d\ell_1} \geq \left( \sum_{|k|>m_n} |k|^\frac{d\ell_2}{\ell_1} \Xi_k \right)^{\frac{1}{p}}.
\]

Since \( \sum_{k \in \mathbb{Z}^d} |k|^\frac{d\ell_2}{\ell_1} \Xi_k < \infty \) and \( r > \ell_2/\ell_1 \), we get

\[
h_{s_n}^{-\ell_2} \sum_{|k|>m_n} \Xi_k \leq \left( m_n^{d\ell_1} h_{s_n}^{\ell_1} \right)^{-\frac{\ell_2}{\ell_1}} \sum_{|k|>m_n} |k|^\frac{d\ell_2}{\ell_1} \Xi_k \leq \left( \sum_{|k|>m_n} |k|^\frac{d\ell_2}{\ell_1} \Xi_k \right)^{\frac{1}{p}} \to 0 \quad \text{as} \quad n \to \infty.
\]

The proof of Lemma 5 is complete. \( \square \)

**Lemma 6.** Let \( \ell \in \{1, 2\} \) and \( \theta > 0 \) be fixed such that \( \mathbb{E}[|\Phi_t(Y_0)|^{2+\theta}] < \infty \) and \( \mathbb{E}[|\Phi_t(Y_0)|^{2+\theta} K_{s_n}(x, X_0)] \leq h_{s_n}^N \) then \( \|\Delta_{s_i}^{(\ell)}\|_{2+\theta} \leq h_{s_i}^{\frac{N\theta}{2+\theta}} \) where \( \Delta_{s_i}^{(\ell)} \) is given by (3.2).

**Proof of Lemma 6.** Let \( \theta > 0 \) and \( \ell \in \{1, 2\} \) such that \( \mathbb{E}[|\Phi_t(Y_0)|^{2+\theta}] < \infty \) and let \( 1 \leq i \leq n \), we have

\[
\|\Delta_{s_i}^{(\ell)}\|_{2+\theta}^2 \leq \frac{2 \|\Phi_t(Y_0) K_{s_n}(x, X_0)\|_{2+\theta}^2}{h_{s_n}^N} + \frac{2 \left( \mathbb{E}[|\Phi_t(Y_0)| K_{s_n}(x, X_0)] \right)^2}{h_{s_i}^N}.
\]

Since sup\( t \in \mathbb{R}^N \{ K(t) \} < \infty \) and \( \mathbb{E}[|\Phi_t(Y_0)|^{2+\theta} K_{s_n}(x, X_0)] \leq h_{s_n}^N \), we get \( \mathbb{E}[|\Phi_t(Y_0) K_{s_n}(x, X_0)|^{2+\theta}] \leq h_{s_n}^N \). Moreover, using Lemma 3, we have \( \mathbb{E}[|\Phi_t(Y_0)| K_{s_n}(x, X_0)] \leq h_{s_n}^N \) and we obtain \( \|\Delta_{s_i}^{(\ell)}\|_{2+\theta}^2 \leq h_{s_i}^{\frac{2N\theta}{2+\theta}} \). The proof of Lemma 6 is complete. \( \square \)

**Proposition 3.** Let \( M \) be a positive integer and let \( x \in \mathbb{R}^N \). If \( (X_k)_{k \in \mathbb{Z}^d} \) is of the form (1.5) and \( \Phi : \mathbb{R} \to \mathbb{R} \) is a measurable function such that \( \|\Phi(Y_0)\|_{2+\theta} < \infty \)
for some $\theta \in [0, +\infty)$ then for any positive integer $n$ and any family $(c_k)_{k \in \Lambda_n}$ of real numbers and any $(p, q) \in [2, +\infty] 	imes [0, +\infty)$ such that $p + q \leq 2 + \theta$, we have

$$
\left\| \sum_{i=1}^{n} c_n W_i \right\|_p \leq 8pM^d |K|^{\frac{p}{\infty}} C(p, q) \left( \sum_{i=1}^{n} \frac{c_i^2}{h_n^{\frac{p}{\infty}}} \right)^{\frac{1}{2}} \sum_{j>M} \delta_{j,p}^\theta,
$$

where

$$W_i := \Phi(Y_{s_i})K_{s_i}(x, X_{s_i}) - \mathbb{E}[\Phi(Y_{s_i})K_{s_i}(x, X_{s_i}) | H_{i,M}], \quad (3.8)
$$

$$C(p, q) = 2^{\frac{p+q}{\infty}} \| \Phi(Y_0) \|_{p+q} \| K \|_{lip}^\theta + |K|^{\frac{1}{\infty}} \sup_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \neq y} \frac{|\Phi(R(x, \eta_0)) - \Phi(R(y, \eta_0))|}{\|x - y\|} \|x - y\|_p
$$

and

$$H_{i,M} = \sigma (\eta_{s_i}, \varepsilon_{s_i-k} ; |k| \leq M) \quad \text{and} \quad \| K \|_{lip} = \sup_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \neq y} \frac{|K(x) - K(y)|}{\|x - y\|}.
$$

**Proof of Proposition 3.** Let $M$ be a positive integer and let $x$ in $\mathbb{R}^N$ and $1 \leq i \leq n$ be fixed. Recall that $Y_{s_i} = R(X_{s_i}, \eta_{s_i})$ and let $W_i = \Phi(Y_{s_i})K_{s_i}(x, X_{s_i}) - \mathbb{E}[\Phi(Y_{s_i})K_{s_i}(x, X_{s_i}) | H_{i,M}]$ where $H_{i,M} = \sigma (\eta_{s_i}, \varepsilon_{s_i-k} ; |k| \leq M)$. We follow the same lines as in the proof of Proposition 1 in [16]. Let $2 \leq p < 2 + \theta$ and denote by $H_i$ the measurable function such that $W_i = H_i(H_{i,\infty})$ with $H_{i,\infty} = \sigma (\eta_{s_i}, \varepsilon_{s_i-k} ; k \in \mathbb{Z}^d)$. Let $\tau$ be a bijection from $\mathbb{Z}$ to $\mathbb{Z}^d$ and $\ell$ in $\mathbb{Z}$ be fixed. We define the projection operator $P_{\ell}$ by $P_{\ell}f = \mathbb{E}[f|\mathcal{F}_\ell] - \mathbb{E}[f|\mathcal{F}_{\ell-1}]$ for any integrable function $f$, where $\mathcal{F}_\ell = \sigma (\varepsilon_{\tau(j)} ; j \leq \ell)$. One can notice that the operator $P_{\ell}$ depends on the bijection $\tau$. The proof of the following technical result is postponed to section 5.

**Lemma 7.** Almost surely, it holds that $\mathbb{E}[W_i|\mathcal{F}_{\ell-1}] = \mathbb{E}[H_i(H_{i,\infty}^{\ell})|\mathcal{F}_\ell]$ with $H_{i,\infty}^{\ell} = \sigma (\eta_{s_i}, \varepsilon_{\tau(\ell)}^{\ell}, \varepsilon_{s_i-k}^{\ell} ; k \in \mathbb{Z}^d \setminus \{s_i - \tau(\ell)\})$.

Using Lemma 7, we obtain

$$
\| P_{\ell}W_i \|_p = \| \mathbb{E}[H_i(H_{i,\infty})|\mathcal{F}_\ell] - \mathbb{E}[H_i(H_{i,\infty}^{\ell})|\mathcal{F}_\ell] \|_p \leq \| H_i(H_{i,\infty}) - H_i(H_{i,\infty}^{\ell}) \|_p.
$$

(3.9)

Now, denoting $H_{i,M}^{\ell} = \sigma (\eta_{s_i}, \varepsilon_{\tau(\ell)}^{\ell}, \varepsilon_{s_i-k}^{\ell} ; k \in \mathbb{Z}^d \setminus \{s_i - \tau(\ell)\})$ and $|k| \leq M$.

we have

$$H_i(H_{i,\infty}) = \Phi(Y_{s_i})K_{s_i}(x, X_{s_i}) - \mathbb{E}[\Phi(Y_{s_i})K_{s_i}(x, X_{s_i}) | H_{i,M} \lor H_{i,M}^{\ell}],
$$

$$H_i(H_{i,\infty}^{\ell}) = \Phi(Y_{\tau(\ell)}^{\ell})K_{s_i}(x, X_{\tau(\ell)}^{\ell}) - \mathbb{E}[\Phi(Y_{\tau(\ell)}^{\ell})K_{s_i}(x, X_{\tau(\ell)}^{\ell}) | H_{i,M}^{\ell} \lor H_{i,M}].$$
where \( X'_{i,\tau(t)} = G(\varepsilon'_{\tau(t)}, \varepsilon_{s_i - k}; k \in \mathbb{Z}^d \setminus \{s_i - \tau(t)\}) \) and \( Y'_{i,\tau(t)} = R(X'_{i,\tau(t)}, \eta_{s_i}) \).

So, we derive

\[
\| P_t W_i \|_p \leq 2\| \Phi(Y_{s_i})K_{s_i}(x, X_{s_i}) - \Phi(Y'_{i,\tau(t)})K_{s_i}(x, X'_{i,\tau(t)}) \|_p.
\]

So, for any \( L > 0 \), we get the bound

\[
\| P_t W_i \|_p \leq \frac{2L}{h_{s_i}}\| X_{s_i} - X'_{i,\tau(t)} \|_p + 4|K|_{\infty} \frac{\| \Phi(Y_{0}) \|_{p+q}}{L^{q/p}} + 2|K|_{\infty} \sup_{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d \setminus x \neq y} \frac{\| \Phi(R(x, \eta_0)) - \Phi(R(y, \eta_0)) \|}{\| x - y \|} \| X_{s_i} - X'_{i,\tau(t)} \|_p \]

where

\[
\| X_{s_i} - X'_{i,\tau(t)} \|_p = \| G(\varepsilon_{s_i - k}; k \in \mathbb{Z}^d \setminus \{s_i - \tau(t)\}) \|_p
\]

\[
= \| G(\varepsilon_{s_i - \tau(t) - k}; k \in \mathbb{Z}^d) - G(\varepsilon_{s_i - \tau(t) - k}; k \in \mathbb{Z}^d \setminus \{s_i - \tau(t)\}) \|_p
\]

\[
= \| X_{s_i - \tau(t)} - X'_{s_i - \tau(t)} \|_p
\]

\[
= \delta_{s_i - \tau(t), p}.
\]

Optimizing this last inequality in \( L \), we get the following bound

\[
\| P_t W_i \|_p \leq 2|K|_{\infty}^p C(p, q) h_{s_i} \frac{4}{L^{q/p}} \delta_{s_i - \tau(t), p},
\]

(3.10)

where

\[
C(p, q) = 2^{\frac{p+q}{p}} \| \Phi(Y_{0}) \|_{p+q} \| K \|_{Lip} \left( \frac{4}{L^{q/p}} + \sup_{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d \setminus x \neq y} \frac{\| \Phi(R(x, \eta_0)) - \Phi(R(y, \eta_0)) \|}{\| x - y \|} \right).
\]

Now, we are going to obtain another bound for \( \| P_t W_i \|_p \). Let \( \ell \geq 0 \) and \( i \geq 1 \) be two integers. We denote by \( \Gamma_{i, \ell} \) the set of all \( k \) in \( \mathbb{Z}^d \) such that \( |s_i - k| = \ell \) and we define

\[
a_\ell := \sum_{j=0}^\ell |\Gamma_{i, j}| = (2\ell + 1)^d.
\]

On the lattice \( \mathbb{Z}^d \) we define the lexicographic order as follows: if \( u = (u_1, \ldots, u_d) \) and \( v = (v_1, \ldots, v_d) \) are distinct elements of \( \mathbb{Z}^d \), the notation \( u <_{\text{lex}} v \) means that either \( u_1 < v_1 \) or for some \( k \) in \( \{2, \ldots, d\} \), \( u_k < v_k \) and \( u_\ell = v_\ell \) for \( 1 \leq \ell < k \). We consider the bijection \( \tau_i : [0, +\infty] \cap \mathbb{Z} \rightarrow \mathbb{Z}^d \) defined by \( \tau_i(1) = s_i \), \( \tau_i(u) \in \Gamma_{i, \ell} \) if \( a_\ell - 1 < u \leq a_\ell \) and \( \ell > 0 \), and \( \tau_i(u) <_{\text{lex}} \tau_i(v) \) if
\[ a_{\ell-1} < u < v \leq a_{\ell} \text{ and } \ell > 0. \] Let \( G_{i,M} = \sigma (\eta_{\ell_i}, \varepsilon_{\tau_i(j)} ; 1 \leq j \leq M) \) and recall that \( H_{i,M} = \sigma (\eta_{\ell_i}, \varepsilon_{\tau_i(-k)} ; |k| \leq M) \). Since \( 1 \leq j \leq a_M \) if and only if \(|s_i - \tau_i(j)| \leq M\), we have \( G_{i,a_M} = H_{i,M} \). Consequently,

\[ W_i = \sum_{\ell > a_M} D_{i,\ell} \]

where \( D_{i,\ell} = \mathbb{E}[\Phi(Y_{s_i})K_{s_i}(x, X_{s_i})|G_{i,\ell}] - \mathbb{E}[\Phi(Y_{s_i})K_{s_i}(x, X_{s_i})|G_{i,\ell-1}] \).

Given that \( (D_{i,\ell})_{\ell \geq 1} \) is a martingale difference sequence with respect to the filtration \( (G_{i,\ell})_{\ell \geq 1} \), we apply Burkholder’s inequality (\cite{10}, remark 6, page 85) and we obtain

\[
\|W_i\|_p \leq \left( 2p \sum_{\ell > a_M} \|D_{i,\ell}\|_p^2 \right)^{1/2}. \tag{3.11}
\]

Since \( X'_{i,\tau_i(\ell)} = G(\varepsilon'_{\tau_i(\ell)}, \varepsilon_{s_i-k} ; k \in \mathbb{Z}^d \setminus \{s_i - \tau_i(\ell)\}) \) and \( Y'_{i,\tau_i(\ell)} = R(X'_{i,\tau_i(\ell)}, \eta_{s_i}), \) we have \( \mathbb{E}[\Phi(Y_{s_i})K_{s_i}(x, X_{s_i})|G_{i,\ell-1}] = [\Phi(Y'_{i,\tau_i(\ell)})K_{s_i}(x, X'_{i,\tau_i(\ell)})][G_{i,\ell}] \) and

\[
\|D_{i,\ell}\|_p \leq \|\Phi(Y_{s_i})K_{s_i}(x, X_{s_i}) - \Phi(Y'_{i,\tau_i(\ell)})K_{s_i}(x, X'_{i,\tau_i(\ell)})\|_p.
\]

Arguing as before, for any \( L > 0 \), we derive

\[
\|D_{i,\ell}\|_p \leq \frac{L}{\|X_{s_i} - X'_{i,\tau_i(\ell)}\|_p} \|X_{s_i} - X'_{i,\tau_i(\ell)}\|_p + \frac{2\|L\|\|\Phi(Y_0)\|_p^{p+q}}{L^{q/p} \|\Phi(Y_0)\|_p^{p+q}}
\]

\[
+ |K|_\infty \sup_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \atop x \neq y} \left\| \frac{\Phi(R(x, \eta_0)) - \Phi(R(y, \eta_0))}{\|x - y\|} \right\|_p \|X_{s_i} - X'_{i,\tau_i(\ell)}\|_p
\]

with \( \|X_{s_i} - X'_{i,\tau_i(\ell)}\|_p = \delta_{s_i - \tau_i(\ell), p} \). Optimizing this last inequality on \( L \) and noting that \( s_i - \tau_i(\ell) = -\tau_0(\ell) \), we obtain

\[
\|D_{i,\ell}\|_p \leq |K|_\infty^{p/q} C(p, q) h_{s_i}^{-q/p} \delta_{s_i - \tau_{i-1}(\ell), p}^{-q/p}.
\]

Consequently, we get

\[
\|P_t W_i\|_p \leq 2\|W_i\|_p \leq 2 \sqrt{2p} |K|_\infty^{p/q} C(p, q) h_{s_i}^{-q/p} \sum_{\ell > a_M} \delta_{-\tau_{i-1}(\ell), p}^{-q/p}
\]

\[
\leq 2 \sqrt{2p} |K|_\infty^{p/q} C(p, q) h_{s_i}^{-q/p} \sum_{|k| > M} \delta_{k, p}^{-q/p}. \tag{3.12}
\]

Since \( (\sum_{i=1}^n c_i P_t W_i)_{t \in \mathbb{Z}} \) is a martingale difference sequence with respect to the filtration \( (\mathcal{F}_t)_{t \in \mathbb{Z}} \), the Burkholder inequality (see \cite{10}, remark 6, page 85) implies

\[
\left\| \sum_{i=1}^n c_i W_i \right\|_p \leq \left( 2p \sum_{t \in \mathbb{Z}} \left\| \sum_{i=1}^n c_i P_t W_i \right\|_p^2 \right)^{1/2}.
\]
Combining (3.13) and (3.14) with the last two bound above, we get

\[
\left\| \sum_{i=1}^{n} c_{s_i} W_i \right\|_p \leq \left( 2p \sum_{\ell \in \mathbb{Z}} \left( \sum_{i=1}^{n} |c_{s_i}| \left\| P_{\ell} W_i \right\|_p \right)^2 \right)^{\frac{1}{2}}.
\]

Moreover, by the Cauchy-Schwarz inequality, we have

\[
\left( \sum_{i=1}^{n} |c_{s_i}| \left\| P_{\ell} W_i \right\|_p \right)^2 \leq \sum_{i=1}^{n} c_{s_i}^2 \left\| P_{\ell} W_i \right\|_p \times \sum_{j=1}^{n} \left\| P_{\ell} W_j \right\|_p.
\]

Now, keeping in mind that \( P_\ell \) is defined from the bijection \( \tau \) and using (3.10) and (3.12), we have

\[
\sup_{\ell \in \mathbb{Z}} \sum_{i=1}^{n} \left\| P_{\ell} W_i \right\|_p \leq \sup_{\ell \in \mathbb{Z}} \sum_{1 \leq i \leq n, |s_i - \tau(\ell)| \leq M} \left\| P_{\ell} W_i \right\|_p + \sup_{\ell \in \mathbb{Z}} \sum_{1 \leq i \leq n, |s_i - \tau(\ell)| > M} \left\| P_{\ell} W_i \right\|_p
\]

\[
< 2 \sqrt{2p} M^d |K|^p \frac{p}{\infty} \sum_{|k| > M} \delta_{k,p}^{\frac{q}{p}}
\]

\[
+ 2 |K|^p \frac{p}{\infty} \sum_{1 \leq i \leq n, |s_i - \tau(\ell)| > M} \delta_{s_i - \tau(\ell),p}^{\frac{q}{p}}
\]

\[
< 2 \left( M^d \sqrt{2p} + 1 \right) |K|^p \frac{p}{\infty} \sum_{|k| > M} \delta_{k,p}^{\frac{q}{p}}.
\]

Similarly, we have also

\[
\sup_{1 \leq i \leq n} \sum_{\ell \in \mathbb{Z}} \left\| P_{\ell} W_i \right\|_p \leq \sup_{1 \leq i \leq n} \sum_{\ell \in \mathbb{Z}, |s_i - \tau(\ell)| \leq M} \left\| P_{\ell} W_i \right\|_p + \sup_{1 \leq i \leq n} \sum_{\ell \in \mathbb{Z}, |s_i - \tau(\ell)| > M} \left\| P_{\ell} W_i \right\|_p
\]

\[
< 2 \sqrt{2p} M^d |K|^p \frac{p}{\infty} \sum_{|k| > M} \delta_{k,p}^{\frac{q}{p}}
\]

\[
+ 2 |K|^p \frac{p}{\infty} \sum_{\ell \in \mathbb{Z}, |s_i - \tau(\ell)| > M} \delta_{s_i - \tau(\ell),p}^{\frac{q}{p}}
\]

\[
< 2 \left( M^d \sqrt{2p} + 1 \right) |K|^p \frac{p}{\infty} \sum_{|k| > M} \delta_{k,p}^{\frac{q}{p}}.
\]

Combining (3.13) and (3.14) with the last two bound above, we get

\[
\left\| \sum_{i=1}^{n} c_{s_i} W_i \right\|_p \leq \sqrt{2p} (2M^d \sqrt{2p} + 1)^2 |K|^p \frac{p}{\infty} \left( \sum_{i=1}^{n} c_{s_i}^2 \right)^{\frac{1}{2}} \sum_{|k| > M} \delta_{k,p}^{\frac{q}{p}}.
\]

Noting that \( \sqrt{2p} + 1 \leq 2 \sqrt{2p} \), we obtain

\[
\left\| \sum_{i=1}^{n} c_{s_i} W_i \right\|_p \leq 8p M^d |K|^p \frac{p}{\infty} \left( \sum_{i=1}^{n} c_{s_i}^2 \right)^{\frac{1}{2}} \sum_{|k| > M} \delta_{k,p}^{\frac{q}{p}}.
\]
The proof of Proposition 3 is complete.

4. Proofs of the main results

Now, we denote by $\mathbb{V}(Z)$ the variance of any square-integrable $\mathbb{R}$-valued random variable $Z$ and we consider a sequence $(m_n)_{n \geq 1}$ of positive integers. For any $x \in \mathbb{R}$ and any $\ell \in \{1, 2\}$, denote

$$f_n^{(\ell)}(x) = f_{n, \Phi}(x) = \left(\sum_{i=1}^{n} w_{s_i}\right)^{-1} \sum_{i=1}^{n} w_{s_i} h_{s_i}^{-N} \Phi^\ell(y_{s_i})K_{s_i}(x, X_{s_i})$$

and $\overline{f}_n^{(\ell)}(x) = \mathbb{E}[f_n^{(\ell)}(x) | \mathcal{H}_{i,m_n}]$ 

(4.1)

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a measurable function and $\mathcal{H}_{i,m_n} = \sigma(\eta_{s_i}, \varepsilon_{s_i-k}; |k| \leq m_n)$.

First, we note that Proposition 1 is a particular case of the following result.

**Proposition 4.** Assume that (H1) and (H3)(iii) hold. Let $(p, q) \in \{1, 2\}^2$ and $x \in \mathbb{R}^N$ be fixed. Let $\theta > 0$ such that $\mathbb{E}[|\Phi^p(0)|^{2+\theta}K_{s_0}(x, X_0)] \leq h_{s_0}^N$ and $\mathbb{E}[|\Phi^q(0)|^{2+\theta}] < \infty$ for any $\ell \in \{p, q\}$. Assume also that the function $u \mapsto \mathbb{E}[\Phi^p(0)\Phi^q(Y_0)|X_0 = u]$ is continuous. If there exists $\tau \in [1 - \gamma, 1]$ such that $h_{s_0}^{N(1-\tau)} \sum_{i=1}^{n} w_{s_i}^2 \lec nw_{s_n}^2$ and one of the following conditions is satisfied:

(i) $(X_k)_{k \in \mathbb{Z}^d}$ is strongly mixing and

$$\sum_{k \in \mathbb{Z}^d} |\Phi^p(0)|^{2+\theta} \frac{\alpha_{s,1}^\tau(k)}{N^{(d+2\gamma+2\tau(\gamma+\tau-1))}} \delta_{s,k} \lec \infty,$$

(ii) $(X_k)_{k \in \mathbb{Z}^d}$ is of the form (1.5) and

$$\sum_{k \in \mathbb{Z}^d} |\Phi^p(0)|^{2+\theta} \frac{\alpha_{s,1}^\tau(k)}{N^{(d+2\gamma+2\tau(\gamma+\tau-1))}} \delta_{s,k} \lec \infty.$$

where $\tilde{\gamma}$ is defined by

$$\tilde{\gamma} = \left\{ \begin{array}{ll} 1 & \text{if } \max(\|\Phi^p\|_{\infty}, \|\Phi^q\|_{\infty}) < +\infty, \\ \frac{\theta}{2+\theta} & \text{else} \end{array} \right. \quad (4.2)$$

then

$$\lim_{n \to \infty} nh_{s_n}^N \mathbb{Cov}[f_n^{(p)}(x), f_n^{(q)}(x)]$$

$$= \beta_{0,1}^2 \beta_{N,2} \mathbb{E}[\Phi^p(0)\Phi^q(0)|X_0 = x] f(x) \int_{\mathbb{R}^N} K^2(t) dt.$$

**Proof of Proposition 4.** Let $x \in \mathbb{R}^N$ and let $(p, q) \in \{1, 2\}^2$ be fixed. Using the notations (3.2) and (4.1), for $\ell \in \{p, q\}$, we have

$$f_n^{(\ell)}(x) - \mathbb{E}[f_n^{(\ell)}(x)] = \left(\sum_{i=1}^{n} w_{s_i}\right)^{-1} \sum_{i=1}^{n} w_{s_i} h_{s_i}^{-N/2} \Delta_n^{(\ell)}.$$

Keeping in mind that $A_{n,0,1} = (nw_{s_n})^{-1} \sum_{i=1}^{n} w_{s_i}$, we get

$$nh_{s_n}^N \mathbb{Cov}[f_n^{(p)}(x), f_n^{(q)}(x)]$$
Moreover, for any $i \leq n$, there exists a sequence $(\ldots)$. Using Lemma 2 and Lemma 3, we derive

$$
\lim_{i \to \infty} \left| n h_n^N \text{Cov} [f_n^{(p)}(x), f_n^{(q)}(x)] - \frac{h_n^N A_{n,0,1}^{-2}}{n w_s^2} \sum_{i=1}^{n} w_s^2 h_i^{-N} \mathbb{E} [\Delta_i^{(p)} \Delta_i^{(q)}] \right| 
\leq \frac{A_{n,0,1}^{-2}}{n w_s^2} \sum_{i,j=1}^{n} w_s w_s (h_i h_j)^{-N/2} |\mathbb{E} [\Delta_i^{(p)} \Delta_j^{(q)}]|. \tag{4.3}
$$

Moreover, for any $1 \leq i \leq n$,

$$
\mathbb{E} [\Delta_i^{(p)} \Delta_i^{(q)}] = h_i^{-N} \left( \mathbb{E} [\Phi_p(Y_0) \Phi_q(Y_0) K_i^2(x, X_0)] - \mathbb{E} [\Phi_p(Y_0) K_i(x, X_0)] \mathbb{E} [\Phi_q(Y_0) K_i(x, X_0)] \right).
$$

Using Lemma 2 and Lemma 3, we derive

$$
\lim_{i \to \infty} |\mathbb{E} [\Delta_i^{(p)} \Delta_i^{(q)}] - \mathbb{E} [\Phi_p(Y_0) \Phi_q(Y_0)|X_0 = x] f(x) \int_{\mathbb{R}^N} K^2(t) dt| = 0.
$$

So, using Lemma 1 and (H1), we derive

$$
\lim_{n \to \infty} \frac{h_n^N}{n w_s^2} \sum_{i=1}^{n} h_i^{-N} w_s^2 \mathbb{E} [\Delta_i^{(p)} \Delta_i^{(q)}] = 0.
$$

Now, we are going to prove that

$$
\lim_{n \to \infty} \frac{A_{n,0,1}^{-2}}{n w_s^2} \sum_{i,j=1}^{n} w_s w_s (h_i h_j)^{-N/2} |\mathbb{E} [\Delta_i^{(p)} \Delta_j^{(q)}]| = 0. \tag{4.5}
$$

Using Lemma 4, we have $|\mathbb{E} [\Delta_i^{(p)} \Delta_i^{(q)}]| \leq (h_i h_j)^{-\gamma} \sum_{k \in \mathbb{Z}} |k|^{\gamma} \alpha_{1,1}(k) < \infty$, using Lemma 5, there exists a sequence $(m_n)_{n \geq 1}$ of positive integers such that

$$
\lim_{n \to \infty} m_n^d h_n^N \sum_{|k| > m_n} \alpha_{1,1}(k) = 0. \tag{4.6}
$$
So, we have
\[
\frac{h_{sn}^N}{nw_{sn}^2} \sum_{i,j=1 \atop i \neq j}^n w_s(w_s(h_s, h_{s_j}))^{-N/2} |E[\Delta_{s_i}^{(p)} \Delta_{s_j}^{(q)}]| = \frac{2h_{sn}^N}{nw_{sn}^2} \sum_{1 \leq i < j \leq n} w_s(w_s(h_s, h_{s_j}))^{-N/2} |E[\Delta_{s_i}^{(p)} \Delta_{s_j}^{(q)}]| \leq E_{1,n} + E_{2,n}
\]
where
\[
E_{1,n} = \frac{h_{sn}^N}{nw_{sn}^2} \sum_{1 \leq i < j \leq n \atop |s_i-s_j| < m_n} w_s(w_s(h_s, h_{s_j}))^{-N(1-\gamma)/2}
\]
and
\[
E_{2,n} = \frac{h_{sn}^N}{nw_{sn}^2} \sum_{1 \leq i < j \leq n \atop |s_i-s_j| \geq m_n} w_s(w_s(h_s, h_{s_j}))^{-N/2} |E[\Delta_{s_i}^{(p)} \Delta_{s_j}^{(q)}]|.
\]
Since \( \gamma \leq 1 \), using the inequality \( 2ab \leq a^2 + b^2 \), we have
\[
E_{1,n} \leq \frac{h_{sn}^{N\gamma}}{nw_{sn}^2} \sum_{1 \leq i < j \leq n \atop |s_i-s_j| < m_n} w_s(w_s(h_s, h_{s_j})) \leq \frac{h_{sn}^{N\gamma}}{2nw_{sn}^2} \sum_{1 \leq i < j \leq n \atop |s_i-s_j| < m_n} (w_s^2 + w_{s_j}^2) \leq \frac{n^d h_{sn}^{N\gamma}}{nw_{sn}^2} \sum_{i=1}^n w_s^2 = n^d h_{sn}^{N(\gamma+1-\tau)} \times \frac{h_{sn}^{N(1-\tau)}}{nw_{sn}^2} \sum_{i=1}^n w_s^2.
\]
Using (4.6) and keeping in mind that \( h_{sn}^{N(1-\tau)} \sum_{i=1}^n w_s^2 \leq nw_{sn}^2 \), we get \( \lim_{n \to \infty} E_{1,n} = 0 \). Now, we are going to control the term \( E_{2,n} \) when \( (X_k)_{k \in \mathbb{Z}} \) is assumed to be strongly mixing. Using Rio’s inequality ([32], Theorem 1.1), we have for any \( 1 \leq i < j \leq n \),
\[
|E[\Delta_{s_i}^{(p)} \Delta_{s_j}^{(q)}]| \leq 2 \int_0^{2\alpha_{s_i}} (|s_i-s_j|) Q_{\Delta_{s_i}^{(p)}}(u)Q_{\Delta_{s_j}^{(q)}}(u) du
\]
where \( Q_{\Delta_{s_i}^{(p)}}(u) = \inf \{ \varepsilon > 0 \mid \Pr(|\Delta_{s_i}^{(p)}| > \varepsilon) \leq u \} \) for any \( u \in [0,1] \) and any \( \ell \in \{i,j\} \).

First, we assume that \( \|\Phi_p\|_{\infty} = \infty \) or \( \|\Phi_p\|_{\infty} = \infty \). In this case, \( \tilde{\gamma} = \theta/(2+\theta) \).

Using Lemma 6, we have \( \|\Delta_{s_i}^{(p)}\|_{2+\theta} \leq h_{s_i}' \) for any \( \ell \in \{p,q\} \) and any \( 1 \leq r \leq n \). So, we derive \( Q_{\Delta_{s_i}^{(p)}}(u) \leq u^{-\tilde{\gamma}/2} h_{s_i}^{-\tilde{\gamma}/2} \) and
\[
|E[\Delta_{s_i}^{(p)} \Delta_{s_j}^{(q)}]| \leq 2 (h_{s_i} h_{s_j})^{-\tilde{\gamma}/2} \int_0^{2\alpha_{s_i}} (|s_i-s_j|) u^{-\tilde{\gamma}/2} du
\]
for any $\ell$. Keeping in mind (3.2) and (4.1), we have

$$\sum |C_{k,n}| = \sum_{1 \leq i < j \leq n} w_{s_i} w_{s_j} (h_{s_i} h_{s_j})^{-\frac{N_2}{2}} \alpha_{1,1}^\varsigma (|s_i - s_j|).$$

Consequently, we get

$$E_{2,n} \leq \frac{h_{s_n}^N}{n w_{s_n}^2} \sum_{1 \leq i < j \leq n, |s_i - s_j| > m_n} w_{s_i} w_{s_j} (h_{s_i} h_{s_j})^{-\frac{N_2}{2}} \alpha_{1,1}^\varsigma (|s_i - s_j|).$$

Using again the inequality $2ab \leq a^2 + b^2$, we derive

$$E_{2,n} \leq \frac{h_{s_n}^N}{n w_{s_n}^2} \sum_{1 \leq i < j \leq n, |s_i - s_j| > m_n} \left( w_{s_i}^2 h_{s_i}^{-N_2(\varsigma + 1)} + w_{s_j}^2 h_{s_j}^{-N_2(\varsigma + 1)} \right) \alpha_{1,1}^\varsigma (|s_i - s_j|)$$

$$\leq \left( \frac{h_{s_n}^N}{n w_{s_n}^2} \sum_{i=1}^n w_{s_i}^2 h_{s_i}^{-N_2} \right) \sum_{k \in \mathbb{Z}_+^d, |k| > m_n} \alpha_{1,1}^\varsigma (|k|).$$

Using (H1) and (4.6), we get

$$E_{2,n} \leq h_{s_n}^{-N_2} \sum_{k \in \mathbb{Z}_+^d, |k| > m_n} \alpha_{1,1}^\varsigma (|k|) \to 0 \quad (4.7)$$

and finally, we obtain (4.5). Now, we deal with the case $\|\Phi_p\|\infty < \infty$ and $\|\Phi_q\|\infty < \infty$. In this case, we have $\varsigma = 1$. So, noting that $|\Delta_{s_n}^{(p)}(u)| \leq h_{r_n}^{-N_2/2}$ for any $\ell \in \{p, q\}$ and any $1 \leq r \leq n$, we derive $Q_{\Delta_{s_n}^{(p)}(u)} \leq h_{r_n}^{-N_2/2}$ and $|E[\Delta_{s_n}^{(p)}(\Delta_{s_n}^{(q)})] \leq (h_{s_n} h_{s_{j'}})^{-N_2/2} \alpha_{1,1}^\varsigma (|s_i - s_j|)$. Arguing as before, we obtain (4.7) and consequently (4.5) holds.

Finally, combining (4.3), (4.4) and (4.5), we obtain

$$\lim_{n \to \infty} nh_{s_n}^N \text{Cov}[f_n^{(p)}(x), f_n^{(q)}(x)]$$

$$= \beta_0^{-N_2} \mathbb{E}[\Phi_p(Y_0)\Phi_q(Y_0)|X_0 = x]f(x) \int_{\mathbb{R}^N} K^2(t)dt.$$

Now, we assume that $(X_k)_{k \in \mathbb{Z}^d}$ is of the form (1.5). Since

$$\sum_{k \in \mathbb{Z}_+^d, |k| \leq m_n} \frac{|k|^{d(\gamma + \tau - 1)}}{|k|^{d(\gamma + \tau - 1)} + \frac{d(\gamma + \tau - 1)}{m_n}} \gamma_{k,2} \leq \infty$$

and using Lemma 5, there exists a sequence $(m_n)_{n \geq 1}$ of positive integers such that

$$\lim_{n \to \infty} m_n^d h_{s_n}^{N_2(\gamma + \tau - 1)} = \lim_{n \to \infty} h_{s_n}^{-\frac{N_2}{2} + \varsigma} \sum_{k \in \mathbb{Z}_+^d, |k| > m_n} |k|^{d\gamma_{k,2}} = 0.$$

Keeping in mind (3.2) and (4.1), we have

$$nh_{s_n}^N \text{Cov}[f_n^{(p)}(x), f_n^{(q)}(x)] = C_{1,n} + C_{2,n} + C_{3,n} + C_{4,n},$$

where

$$C_{1,n} = nh_{s_n}^N \text{Cov}[\tilde{f}_n^{(p)}(x) - \tilde{f}_n^{(p)}(x), f_n^{(q)}(x) - \tilde{f}_n^{(q)}(x)].$$
Moreover, we have

\[ C_{2,n} = nh_n^N \text{Cov}[f^{(p)}_n(x) - f^{(p)}_n(x), \overline{f}^{(q)}_n(x)] \]
\[ C_{3,n} = nh_n^N \text{Cov}[f^{(p)}_n(x), f^{(q)}_n(x) - \overline{f}^{(q)}_n(x)] \]
\[ C_{4,n} = nh_n^N \text{Cov}[f^{(p)}_n(x), f^{(q)}_n(x)]. \]

First, we assume that \( \|\Phi_p\|_\infty = \infty \) or \( \|\Phi_q\|_\infty = \infty \). In this case, \( \hat{\gamma} = \theta/(2 + \theta) \).

Moreover, we have

\[ |C_{1,n}| \leq \left\| \frac{h_n^{N/2} A_{n,0,1}^{-1}}{\sqrt{nw_n}} \sum_{i=1}^{n} w_i h_s^{-N/2} (\Delta_s^{(p)} - \overline{\Delta}_s^{(p)}) \right\|_2 \]
\[ \times \left\| \frac{h_n^{N/2} A_{n,0,1}^{-1}}{\sqrt{nw_n}} \sum_{i=1}^{n} w_i h_s^{-N/2} (\Delta_s^{(q)} - \overline{\Delta}_s^{(q)}) \right\|_2. \]

Using (H1) and Proposition 3, we obtain for any \( \ell \in \{p, q\} \),

\[ \left\| \frac{h_n^{N/2} A_{n,0,1}^{-1}}{\sqrt{nw_n}} \sum_{i=1}^{n} w_i h_s^{-N/2} (\Delta_s^{(\ell)} - \overline{\Delta}_s^{(\ell)}) \right\|_2 \]
\[ \leq m_n^d \left( \frac{h_n^N}{nw_n^2} \sum_{i=1}^{n} \frac{w_i^2}{h_s} \right)^{1/2} h_s^{-\left(\frac{\gamma}{2} + \delta\right)} \sum_{k \in \mathbb{Z}^d \setminus |k| \geq m_n} \delta_k \leq h_s^{-\left(\frac{\gamma}{2} + \delta\right)} \sum_{k \in \mathbb{Z}^d \setminus |k| \geq m_n} |k|^d \delta_k. \]

(4.8)

So, we derive

\[ |C_{1,n}| \leq \left( h_s^{-\left(\frac{\gamma}{2} + \delta\right)} \sum_{k \in \mathbb{Z}^d \setminus |k| \geq m_n} |k|^d \delta_k \right)^2 \xrightarrow{n \to \infty} 0. \]

Similarly, we have

\[ |C_{2,n}| \leq \left\| \frac{h_n^{N/2} A_{n,0,1}^{-1}}{\sqrt{nw_n}} \sum_{i=1}^{n} w_i h_s^{-N/2} (\Delta_s^{(p)} - \overline{\Delta}_s^{(p)}) \right\|_2 \]
\[ \times \left\| \frac{h_n^{N/2} A_{n,0,1}^{-1}}{\sqrt{nw_n}} \sum_{i=1}^{n} w_i h_s^{-N/2} \overline{\Delta}_s^{(p)} \right\|_2. \]

From (4.8), we know that

\[ \left\| \frac{h_n^{N/2} A_{n,0,1}^{-1}}{\sqrt{nw_n}} \sum_{i=1}^{n} w_i h_s^{-N/2} (\Delta_s^{(p)} - \overline{\Delta}_s^{(p)}) \right\|_2 \leq h_s^{-\left(\frac{\gamma}{2} + \delta\right)} \sum_{k \in \mathbb{Z}^d \setminus |k| \geq m_n} |k|^d \delta_k. \]
Moreover, since $\overline{\Delta}(q_i)$ and $\overline{\Delta}(q_j)$ are independent as soon as $|s_i - s_j| > 2m_n$, we have

\[
\left\| \frac{h_{s_n}^{N/2}A_{n,0.1}^{-1}}{\sqrt{n}w_{s_n}} \sum_{i=1}^{n} w_{s_i}h_{s_i}^{-N/2}\overline{\Delta}(q_i) \right\|_2^2 = \frac{h_{s_n}^{N}A_{n,0.1}^{-2}}{n w_{s_n}^2} \sum_{i=1}^{n} w_{s_i}^{-2}h_{s_i}^{-N}||\overline{\Delta}(q_i)||_2^2 + \frac{2h_{s_n}^{N}A_{n,0.1}^{-2}}{n w_{s_n}^2} \sum_{1 \leq i < j \leq n \atop |s_i - s_j| \leq 2m_n} w_{s_i} w_{s_j} h_{s_i}^{-N/2} h_{s_j}^{-N/2} E[\overline{\Delta}(q_i) \overline{\Delta}(q_j)]
\]

and, keeping in mind $||\Delta(q)||_2 \leq 1$ (see Lemma 3) and (H1), we have also

\[
\frac{h_{s_n}^{N}A_{n,0.1}^{-2}}{n w_{s_n}^2} \sum_{i=1}^{n} w_{s_i}^{-2}h_{s_i}^{-N}||\overline{\Delta}(q_i)||_2^2 \leq \frac{h_{s_n}^{N}A_{n,0.1}^{-2}}{n w_{s_n}^2} \sum_{i=1}^{n} w_{s_i}^{-N}||\Delta(q_i)||_2^2 \leq 1.
\]

Denoting $W_i^{(\ell)} = \Phi(x_s)K_s(x, X_s) - E[\Phi(x_s)K_s(x, X_s)|\mathcal{H}_{i,m_n}]$ where $\mathcal{H}_{i,m_n} = \sigma(\eta_{i}, \xi_{s_i-k}; |k| \leq m_n)$ for any $1 \leq i \leq n$ and any $\ell \in \{p, q\}$ and noting that $\Delta(q)_i - \overline{\Delta}(q)_i = h_{s_i}^{-N/2} W_i^{(\ell)}$, we have

\[
|E[\overline{\Delta}(q)_i \overline{\Delta}(q)_j] - E[\Delta(q)_i \Delta(q)_j]| \leq h_{s_i}^{-N/2}||\Delta(q)||_2 W_j^{(q)}_i + h_{s_i}^{-N/2}||\Delta(q)||_2 W_i^{(q)}_j
\]

\[
\leq h_{s_i}^{-N/2} \left( ||W^{(q)}_j||_2 + ||W^{(q)}_i||_2 \right).
\]

Using (3.12), we obtain

\[
|E[\overline{\Delta}(q)_i \overline{\Delta}(q)_j] - E[\Delta(q)_i \Delta(q)_j]| \leq h_{s_n}^{N/2} \sum_{k \in \mathbb{Z}^d \atop |k| > m_n} \delta_{k,2}^2.
\] (4.9)

Consequently, using Lemma 4, we obtain for any $i \neq j$,

\[
|E[\overline{\Delta}(q)_i \overline{\Delta}(q)_j]| \leq (h_{s_i} h_{s_j})^{N/2} + h_{s_n}^{N/2} \sum_{k \in \mathbb{Z}^d \atop |k| > m_n} \delta_{k,2}^2
\]

where $\gamma$ is defined by (3.4). Since $\gamma \leq 1$ and using the inequality $2ab \leq a^2 + b^2$, we derive

\[
\frac{h_{s_n}^{N}A_{n,0.1}^{-2}}{n w_{s_n}^2} \sum_{1 \leq i < j \leq n \atop |s_i - s_j| \leq 2m_n} w_{s_i} w_{s_j} h_{s_i}^{-N/2} h_{s_j}^{-N/2} E[\overline{\Delta}(q)_i \overline{\Delta}(q)_j] \leq \frac{h_{s_n}^{N}A_{n,0.1}^{-2}}{n w_{s_n}^2} \sum_{1 \leq i < j \leq n \atop |s_i - s_j| \leq 2m_n} w_{s_i} w_{s_j} (h_{s_i} h_{s_j})^{-N/2}
\]
Using (H1) and keeping in mind that \( h_{s_{n}}^{N(1-\tau)} \sum_{i=1}^{n} w_{s_{i}}^{2} \leq nw_{s_{n}}^{2} \), we get

\[
\frac{h_{s_{n}}^{N} A_{n,0.1}^{-2}}{nw_{s_{n}}^{2}} \sum_{1 \leq i < j \leq n, |s_{i} - s_{j}| \leq 2m_{n}} w_{s_{i}} w_{s_{j}} h_{s_{i}}^{-N/2} h_{s_{j}}^{-N/2} \|E[\Delta_{s_{i}}^{(q)} \Delta_{s_{j}}^{(q)}]\| \\
\leq m_{d}^{2} h_{s_{n}}^{N(\gamma + \tau - 1)} + h_{s_{n}}^{-\left(\frac{\gamma}{2} + \tilde{\gamma}\right)} \sum_{k \in \mathbb{Z}_{d}^{\mathbb{R}}} |k|^{d} \delta_{k,2}^{\gamma} \xrightarrow{n \to \infty} 0. \tag{4.10}
\]

Consequently, we derive

\[
\left\| \frac{h_{s_{n}}^{N/2} A_{n,0.1}^{-1}}{\sqrt{nw_{s_{n}}}} \sum_{i=1}^{n} w_{s_{i}} h_{s_{i}}^{-N/2} \Delta_{s_{i}}^{(q)} \right\|_{2} \leq 1
\]

and

\[
|C_{2,n}| \leq h_{s_{n}}^{-\left(\frac{\gamma}{2} + \tilde{\gamma}\right)} \sum_{k \in \mathbb{Z}_{d}^{\mathbb{R}}, \ |k| > m_{n}} |k|^{d} \delta_{k,2}^{\gamma} \xrightarrow{n \to \infty} 0.
\]

Similarly, one can notice that

\[
|C_{3,n}| \leq h_{s_{n}}^{-\left(\frac{\gamma}{2} + \tilde{\gamma}\right)} \sum_{k \in \mathbb{Z}_{d}^{\mathbb{R}}, \ |k| > m_{n}} |k|^{d} \delta_{k,2}^{\gamma} \xrightarrow{n \to \infty} 0.
\]

Now, we have to control the last term

\[
C_{4,n} = \frac{A_{n,0.1} h_{s_{n}}^{N}}{nw_{s_{n}}^{2}} \sum_{i=1}^{n} w_{s_{i}} h_{s_{i}}^{-N} E[\Delta_{s_{i}}^{(n)} \Delta_{s_{i}}^{(q)}] \\
+ 2 \frac{A_{n,0.1} h_{s_{n}}^{N}}{nw_{s_{n}}^{2}} \sum_{1 \leq i < j \leq n, |s_{i} - s_{j}| \leq 2m_{n}} w_{s_{i}} w_{s_{j}} (h_{s_{i}} h_{s_{j}})^{-N/2} E[\Delta_{s_{i}}^{(n)} \Delta_{s_{j}}^{(q)}]
\]
We are going to prove that

\[
\lim_{n \to \infty} A_{n,0.1}^{-2} h_{\gamma,n}^N \sum_{i=1}^{n} w_{s_i}^2 h_{s_i}^{-N} \mathbb{E}[\Delta_{s_i}^{(p)} \Delta_{s_i}^{(q)}]
\]

\[
= \frac{\beta_{\gamma,N}}{\beta_{0,1}^2} \mathbb{E}[\Phi_p(Y_0) \Phi_q(Y_0)|X_0 = x] f(x) \int_{\mathbb{R}^N} K^2(t) dt
\]

and

\[
\lim_{n \to \infty} 2A_{n,0.1}^{-2} h_{\gamma,n}^N \sum_{1 \leq i < j \leq n \atop |s_i - s_j| \leq 2m_n} w_{s_i} w_{s_j} (h_{s_i} h_{s_j})^{-N/2} \mathbb{E}[\Delta_{s_i}^{(p)} \Delta_{s_j}^{(q)}] = 0.
\]

Keeping in mind (3.12) and arguing as in (4.9), we have

\[
|\mathbb{E}[\Delta_{s_i}^{(p)} \Delta_{s_i}^{(q)}] - \mathbb{E}[\Delta_{s_i}^{(p)} \Delta_{s_i}^{(q)}]| \leq h_{s_i}^{-N/2} \left( \|W_i^{(p)}\|_2 + \|W_i^{(q)}\|_2 \right)
\]

\[
\leq h_{s_i}^{-1} \left( \frac{N}{2} + \gamma \right) \sum_{|k| \in \mathbb{Z}^d \atop |k| > m_n} |k|^d \delta^k_{\gamma/2}.
\]

Using (H1), we obtain

\[
\frac{A_{n,0.1}^{-2} h_{\gamma,n}^N}{nw_{s_i}^2} \sum_{i=1}^{n} w_{s_i}^2 h_{s_i}^{-N} |\mathbb{E}[\Delta_{s_i}^{(p)} \Delta_{s_i}^{(q)}] - \mathbb{E}[\Delta_{s_i}^{(p)} \Delta_{s_i}^{(q)}]| \leq h_{s_i}^{-1} \left( \frac{N}{2} + \gamma \right) \sum_{|k| \in \mathbb{Z}^d \atop |k| > m_n} |k|^d \delta^k_{\gamma/2} \quad \quad \quad \quad \quad \quad \rightarrow 0.
\]

Using (4.4), we obtain (4.11). Now, arguing as in (4.10), we get

\[
\frac{h_{s_i}^N A_{n,0.1}^{-2}}{nw_{s_i}^2} \sum_{1 \leq i < j \leq n \atop |s_i - s_j| \leq 2m_n} w_{s_i} w_{s_j} (h_{s_i} h_{s_j})^{-N/2} |\mathbb{E}[\Delta_{s_i}^{(p)} \Delta_{s_j}^{(q)}]| \leq m_n h_{s_i}^{N(\gamma + \tau - 1)} + h_{s_i}^{-1} \left( \frac{N}{2} + \gamma \right) \sum_{|k| \in \mathbb{Z}^d \atop |k| > m_n} |k|^d \delta^k_{\gamma/2} \rightarrow 0.
\]

So, (4.12) holds.

Finally, if \(|\Phi_p|_\infty < \infty\) and \(|\Phi_q|_\infty < \infty\) then \(\tilde{\gamma} = 1\) and the proof follows exactly the same lines as above. The proof of Proposition 4 is complete. \(\square\)

**Proof of Proposition 2.** Let \(x \in \mathbb{R}^d\) and let \(n\) be a positive integer. We have

\[
\mathbb{E}[(f_{n,\gamma}(x) - f_\gamma(x))^2] = \mathbb{V}(f_{n,\gamma}(x)) + \mathbb{E}[f_{n,\gamma}(x)] - f_\gamma(x)^2.
\]

Moreover,

\[
|\mathbb{E}[f_{n,\gamma}(x)] - f_\gamma(x)| = \left| \left( \sum_{i=1}^{n} w_{s_i} \right)^{-1} \sum_{i=1}^{n} w_{s_i} \int_{\mathbb{R}^N} K(v) (f_\gamma(x - vh_{s_i}) - f_\gamma(x)) dv \right|.
\]
Using Taylor’s formula, we derive
\[
|E[f_n, \Phi(x)] - f_\Phi(x)| \leq \left( \sum_{i=1}^{n} w_i \right)^{-1} \sum_{i=1}^{n} w_i h_{i}^2.
\]

Since \( \max \{ A_n^{-1} \} \leq 1 \), we obtain
\[
|E[f_n, \Phi(x)] - f_\Phi(x)| \leq A_n^{-1} A_n h_{i}^2 \leq h_{i}^2. 
\]

Finally, using Proposition 1, we have
\[
\forall [f_n, \Phi(x)] \leq \left( n h_{i}^N \right)^{-1} 
\]
and for \( h_{i} = n^{\frac{1}{2-n}} \), we get
\[
\mathbb{E}[ (f_n, \Phi(x) - f_\Phi(x))^2] \leq n^{\frac{1}{2-n}}. 
\]
The proof of Proposition 2 is complete.

**Proof of Theorem 1.** We are going to split the proof in two parts. In the first part, we deal simultaneously with the strong mixing case and the weakly mixing case (see (4.15) below) whereas, in the second part, the two dependence conditions are investigated separately.

**First part**

Let \( n \) be a positive integer and \( x \in \mathbb{R}^N \) be fixed. One can notice that
\[
\sqrt{nh_{i}^N} (f_n, \Phi(x) - E[f_n, \Phi(x)]) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i
\]
where
\[
U_i = \frac{h_{i}^{N/2} w_i \Delta_i}{h_{i}^{N/2} w_i A_{n,0,1}} \quad \text{and} \quad \Delta_i = \frac{\Phi(Y_i)K_i(x,X_s) - E[\Phi(Y_0) K_i(x,X_0)]}{h_{i}^{N/2}}
\]

(4.13)

In the sequel, \((m_n)_{n \geq 1}\) is the sequence defined by Lemma 5 which satisfies \( m_n^{d} h_{i}^N (\nu_{2}(\theta)+\tau-1) \rightarrow 0 \) and either \( h_{i}^{-N \nu_{2}(\theta)} \sum_{|k|>m_n} a_{1,\infty}(k) \rightarrow 0 \) (if \( \{X_k\}_{k \in \mathbb{Z}^d} \) is strongly mixing) or \( h_{i}^{-N \nu_{2}(\theta)} \sum_{|k|>m_n} |k|^d \delta_{k,2}(\theta) \rightarrow 0 \) (if \( \{X_k\}_{k \in \mathbb{Z}^d} \) is of the form (1.5)) where \( \nu_{2}(\theta) = \Phi(\|\Phi\|_{\infty} < \infty)+\frac{\theta}{\tau} \Phi(\|\Phi\|_{\infty} = \infty) \). Moreover, if \( \{X_k\}_{k \in \mathbb{Z}^d} \) is of the form (1.5), we consider the notations
\[
\Xi_i = E[\Delta_i | \mathcal{H}_{i,m_n}], \quad \text{and} \quad \overline{U}_i = E[U_i | \mathcal{H}_{i,m_n}].
\]

where \( \mathcal{H}_{i,m_n} = \sigma(\eta_i, \varepsilon_{i-k}, |k| \leq m_n) \).

Note that \( \overline{U}_i \) and \( \overline{U}_j \) are independent if \(|s_i - s_j| > 2m_n\). Using (H1) and Proposition 3, we derive
\[
\frac{1}{\sqrt{n}} \left\| \sum_{i=1}^{n} (U_i - \overline{U}_i) \right\|_2 \leq h_{i}^{N \nu_{2}(\theta)+\frac{\theta}{\tau}} \sum_{k \in \mathbb{Z}^d} |k|^d \frac{\delta_{k,2}(\theta)}{\delta_{1,\infty}^N} \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty. \quad (4.14)
\]

From now on, we denote
\[
(Z_i, M_n) = \begin{cases} 
(U_i, m_n) & \text{if } \{X_i\}_{i \in \mathbb{Z}^d} \text{ is strongly mixing} \\
(\overline{U}_i, 2m_n) & \text{if } \{X_i\}_{i \in \mathbb{Z}^d} \text{ is of the form } (1.5)
\end{cases}
\]

(4.15)
and it suffices to prove the asymptotic normality of the partial sums 
\[ n^{-1/2} \sum_{i=1}^{n} Z_{s_i} \] as \( n \) goes to infinity. Let \((\xi_k)_{k \in \mathbb{Z}^d}\) be independent normal random variables independent of \((X_k)_{k \in \mathbb{Z}^d}\) and \((\eta_k)_{k \in \mathbb{Z}^d}\) and such that \(E[\xi_k] = 0\) and \(E[\xi_k^2] = E[Z_k^2]\). Let \(1 \leq i \leq n\) and define \(T_{s_i} = n^{-1/2} Z_{s_i}\) and \(\Xi_{s_i} = n^{-1/2} \xi_{s_i}\). One can notice that \(\sum_{i=1}^{n} \Xi_{s_i}\) is a Gaussian random variable with zero mean. If \((X_k)_{k \in \mathbb{Z}^d}\) is strongly mixing then \(Z_{s_i} = U_{s_i}\) and

\[
\mathbb{V} \left( \sum_{i=1}^{n} \Xi_{s_i} \right) = \frac{1}{n} \sum_{i=1}^{n} E[U_{s_i}^2] = \frac{h_n^2 A_{n,0,1}^{-2}}{n w_{n,1}^2} \sum_{i=1}^{n} w_{s_i}^2 \frac{\mathbb{E}[\Phi(Y_0)^2 K_{s_i}^2(x,X_0)] - \mathbb{E}[\Phi(Y_0) K_{s_i}(x,X_0)]^2}{h_{s_i}^2}.
\]

Keeping in mind (2.6) and (H1) and using Lemma 1, Lemma 2 and Lemma 3, we get

\[
\lim_{n \to +\infty} \mathbb{V} \left( \sum_{i=1}^{n} \Xi_{s_i} \right) = \sigma_\Phi^2(x). \tag{4.16}
\]

If \((X_k)_{k \in \mathbb{Z}^d}\) is of the form (1.5) then \(Z_{s_i} = \overline{U}_{s_i}\) and applying (3.12), we get

\[
|\mathbb{E}[\overline{U}_{s_i}^2] - \mathbb{E}[U_{s_i}^2]| \leq 2 \parallel U_{s_i} \parallel_2 \parallel U_{s_i} - \overline{U}_{s_i} \parallel_2 \leq \frac{h_n^2 w_{s_i}^2 A_{n,0,1}^{-2}}{h_n^2 w_{s_i}^2 A_{n,0,1}^{-2}} \times h_{s_i} \left( \frac{\xi}{s_i} + \nu_2(\theta) \right) \sum_{k \in \mathbb{Z}^d, \ |k| > m_n} \delta_{k,2}^{\nu_2(\theta)}.
\]

Using (H1), we derive

\[
\frac{1}{n} \sum_{i=1}^{n} \left| \mathbb{E}[\overline{U}_{s_i}^2] - \mathbb{E}[U_{s_i}^2] \right| \leq \frac{h_n^2 \sum_{i=1}^{n} h_{s_i}^{-N} w_{s_i}^2 A_{n,0,1}^{-2}}{n w_{n,1}^2} \times h_{s_i} \left( \frac{\nu_2(\theta) + \xi}{s_i} \right) \sum_{k \in \mathbb{Z}^d, \ |k| > m_n} \delta_{k,2}^{\nu_2(\theta)} \rightarrow 0.
\]

So, we get also (4.16) when \((X_k)_{k \in \mathbb{Z}^d}\) is of the form (1.5). Let \(\psi\) be any measurable function from \(\mathbb{R}\) to \(\mathbb{R}\). For any \(1 \leq i \leq j \leq n\), we introduce the notation

\[
\psi_{i,j} = \psi \left( \sum_{i=1}^{j} T_{s_i} + \sum_{i=1}^{j} \Xi_{s_i} \right).
\]

Let \(h : \mathbb{R} \to \mathbb{R}\) be a three times continuously differentiable function such that \(\max_{0 \leq i \leq 3} \parallel \psi^{(i)} \parallel_{\infty} \leq 1\). Keeping in mind (4.16), it suffices to prove \(\lim_{n \to \infty} \parallel L_n \parallel = 0\), where

\[
L_n := \mathbb{E} \left[ h \left( \sum_{i=1}^{n} T_{s_i} \right) \right] - \mathbb{E} \left[ h \left( \sum_{i=1}^{n} \Xi_{s_i} \right) \right].
\]
Using Lindeberg’s idea [23] (see also [9]), we have

\[ L_n = \mathbb{E}[h_{n,n+1} - h_{0,1}] = \sum_{i=1}^{n} \mathbb{E}[h_{i,i+1} - h_{i-1,i}] \]

\[ = \sum_{i=1}^{n} \left( \mathbb{E}[h_{i,i+1} - h_{i-1,i+1}] - \mathbb{E}[h_{i-1,i} - h_{i-1,i+1}] \right). \]

Applying Taylor’s formula, we get

\[ L_n = \sum_{i=1}^{n} \left( \mathbb{E}[T_i h'_{i-1,i+1} + \frac{1}{2} T_i^2 h''_{i-1,i+1} + \beta_i] - \mathbb{E}[\Xi_i h'_{i-1,i+1} + \frac{1}{2} \Xi_i^2 h''_{i-1,i+1} + \rho_i] \right), \]

where \(|\beta_i| \leq T_i^2 (1 \wedge |T_i|)\) and \(|\rho_i| \leq \Xi_i^2 (1 \wedge |\Xi_i|)\). Since \(\Xi_i^2\) and \(h''_{i-1,i+1}\) are independent, \(\mathbb{E}[\Xi_i h'_{i-1,i+1}] = 0\) and \(\mathbb{E}[\Xi_i^2] = n^{-1}\mathbb{E}[Z_{s_i}^2]\), we obtain

\[ L_n = \sum_{i=1}^{n} \left( \mathbb{E}[T_i h'_{i-1,i+1}] + \frac{1}{2} \mathbb{E}[T_i^2 - n^{-1}\mathbb{E}[Z_{s_i}^2]] h''_{i-1,i+1} + \mathbb{E}[\beta_i - \rho_i] \right). \]

Since, for any \(1 \leq i \leq n\), the random variable \(\xi_{s_i}\) is gaussian with zero mean and variance \(\mathbb{E}[Z_{s_i}^2]\), we have

\[ \mathbb{E}[|\xi_{s_i}|^3] = \sqrt{8/\pi} (\mathbb{E}[Z_{s_i}^2])^{3/2} \leq \sqrt{8/\pi} (\mathbb{E}[T_{s_i}^2])^{3/2}. \]

Moreover, since \((w_{s_n} h_{s_n}^{-N})_{n \geq 1}\) is nondecreasing, we get

\[ (\mathbb{E}[T_{s_i}^2])^{3/2} = \frac{h_{s_n}^N w_{s_n}^2}{h_{s_n}^N w_{s_n}^2 A_{n,0,1}^2} \left( \frac{h_{s_n}^N w_{s_n}^2}{h_{s_n}^N w_{s_n}^2 A_{n,0,1}^2} \right)^{1/2} (\mathbb{E}[\Delta_{s_i}^2])^{3/2} \]

\[ \leq \frac{h_{s_n}^N w_{s_n}^2}{h_{s_n}^N w_{s_n}^2 A_{n,0,1}^2} (h_{s_n}^N h_{s_n}^{-N})^{1/2} (\mathbb{E}[\Delta_{s_i}^2])^{3/2}. \]

Using (H1) and \(\mathbb{E}[\Delta_{s_i}^2] \leq 1\) (see Lemma 3), we get

\[ \sum_{i=1}^{n} \mathbb{E}[|\rho_i|] \leq \frac{1}{\sqrt{nh_{s_n}^N}} \sum_{i=1}^{n} w_{s_n}^2 h_{s_i}^{-N} (\mathbb{E}[\Delta_{s_i}^2])^{3/2} \leq (nh_{s_n}^N)^{-1/2} \xrightarrow{n \to +\infty} 0. \]

In the other part, if \(\|\Phi\|_\infty < \infty\) then \(|U_{s_i}| \leq h_{s_n}^{-N/2}\) (since \((w_{s_n} h_{s_n}^{-N})_{n \geq 1}\) is nondecreasing) and

\[ \sum_{i=1}^{n} \mathbb{E}[|\beta_i|] \leq \frac{1}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E}[|Z_{s_i}|^3] \leq \frac{1}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E}[|U_{s_i}|^3] \]

\[ \leq \frac{1}{\sqrt{nh_{s_n}^N}} \sum_{i=1}^{n} \mathbb{E}[|U_{s_i}|] \leq (nh_{s_n}^N)^{-1/2} \xrightarrow{n \to +\infty} 0. \]
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Since \((w_{sn}, h_{sn}^N)_{n \geq 1}\) is nondecreasing, if \(\|\Phi\|_\infty = \infty\) then

\[
\mathbb{E}[[U_s]^{2+\theta}] = \left( \frac{h_{sn}^N}{h_{sn}^N w_s A_{n,0,1}} \right)^{2+\theta} \mathbb{E}[[\Delta_s]^{2+\theta}] \\
\leq \frac{h_{sn}^N w_{sn}^2 A_{n,0,1}}{h_{sn}^N w_{sn}^2 A_{n,0,1}} \left( h_{sn}^N w_{sn}^2 \right)^{\theta} h_{sn}^N w_{sn}^2 = \frac{h_{sn}^N w_{sn}^2}{h_{sn}^N w_{sn}^2 A_{n,0,1}}. \tag{4.17}
\]

Consequently, if \(d_n := (n h_{sn}^N)^{\frac{\sigma}{\theta}}\) then

\[
\sum_{i=1}^{n} \mathbb{E}[|\beta_i|] \leq \frac{1}{d_n^{\theta/2}} + \frac{1}{d_n^{\theta/2}} \sum_{i=1}^{n} \mathbb{E}[U_s^2] \\
\leq d_n^\theta (n h_{sn}^N)^{-\theta/2} \times A_{n,0,1} h_{sn}^N w_{sn}^2 \sum_{i=1}^{n} \frac{w_{sn}^2}{h_{sn}^N} + d_n.
\]

Using (H1), we get

\[
\sum_{i=1}^{n} \mathbb{E}[|\beta_i|] \leq \frac{1}{d_n^{\theta/2}} + d_n = 2d_n \xrightarrow{n \to \infty} 0.
\]

Now, we have to prove that

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \left( \mathbb{E}[T_s h_{t-1,i+1}'] + (2n)^{-1} \mathbb{E}[(Z_s^2 - \mathbb{E}[Z_s^2])h_{t-1,i+1}''(M_{\Delta})] \right) = 0. \tag{4.18}
\]

For any \(1 \leq i < j \leq n\) and any function \(\psi\) from \(\mathbb{R}\) to \(\mathbb{R}\), we define also

\[
\psi^{(M_{\Delta})}_{i-1,j} = \psi \left( \sum_{\ell=1}^{i-1} T_{s\ell} + \sum_{\ell=j}^{n} \xi_{s\ell} \right).
\]

Using Taylor’s formula, we have

\[
T_s h_{t-1,i+1}' = T_s h_{t-1,i+1}'(M_{\Delta}) + T_s \left( \sum_{\ell=1}^{i-1} T_{s\ell} \right) h_{t-1,i+1}''(M_{\Delta}) + \beta_i'
\]

with

\[
|\beta_i'| \leq 2 \left| T_s \left( \sum_{\ell=1}^{i-1} T_{s\ell} \right) \right| \left( 1 \wedge \left| T_s \sum_{\ell=1}^{i-1} T_{s\ell} \right| \right). \tag{4.19}
\]

In order to obtain (4.18), we have to prove

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{E}[T_s h_{t-1,i+1}'] = 0. \tag{4.20}
\]
First, we assume that \((X_k)_{k \in \mathbb{Z}^d}\) is strongly mixing. We are going to prove (4.20).

Since \(\Xi\) is independent of \(T\), then \(\mathbb{E} \left[ T_{s_i} h^\prime \left( \sum_{r=i+1}^{n} \Xi_{s_r} \right) \right] = 0\). So, if we define \(E_i^{(n)} = \{ 1 \leq j < i \mid |s_j - s_i| > M_n \}\) and \(\pi\) is a one to one map from \([1,|E_i^{(n)}|]\) \cap \(\mathbb{Z}\) to \(E_i^{(n)}\) such that \(|s_{\pi(t)} - s_i| \leq |s_{\pi(t-1)} - s_i|\) then

\[
\mathbb{E}[T_{s_i} h_{i-1,i+1}^{(M_n)}] = \mathbb{E} \left[ T_{s_i} \left( h_{i-1,i+1}^{(M_n)} - h^\prime \left( \sum_{r=i+1}^{n} \Xi_{s_r} \right) \right) \right] = \sum_{\ell=1}^{E_i^{(n)}} \text{Cov} [T_{s_i}, t_\ell - t_{\ell-1}],
\]

where \(t_\ell = h^\prime \left( \sum_{t=1}^{\ell} T_{s_{\pi(t)}(r)} + \sum_{r=i+1}^{n} \Xi_{s_r} \right)\) and \(\sum_{r=1}^{0} T_{s_{\pi(r)}} = 0\). Since \((X_k)_{k \in \mathbb{Z}^d}\) is strongly mixing, using Rio’s inequality ([32], Theorem 1.1) and keeping in mind that \(|s_{\pi(t)} - s_i| \leq |s_{\pi(t-1)} - s_i|\), we get

\[
|\mathbb{E}[T_{s_i} h_{i-1,i+1}^{(M_n)}(u)]| \leq 2 \sum_{\ell=1}^{E_i^{(n)}} \int_0^{2\alpha_{1,\infty}(|s_{\pi(t)} - s_i|)} Q_{T_{s_i}}(u)Q_{t_\ell - t_{\ell-1}}(u) du.
\]

Assume that \(\|\Phi\|_\infty < \infty\) and let \(u \in [0,1]\) be fixed. Since \(h^\prime\) is Lipschitz and \(|U_{s_i}| \leq h_{s_i}^{-N/2}\) (because \((w_{s_i} h_{s_i}^{-N})_{n \geq 1}\) is nondecreasing), we have \(\max \{ Q_{T_{s_i}}(u), Q_{t_\ell - t_{\ell-1}}(u) \} \leq \max \{ h_{s_i}^{-N/2}, w_{s_i} \}^{-1/2}\) and we derive

\[
\sum_{i=1}^{n} |\mathbb{E}[T_{s_i} h_{i-1,i+1}^{(M_n)}(u)]| \leq \frac{1}{nh_{s_i}^{-N/2}} \sum_{i=1}^{n} \sum_{\ell=1}^{E_i^{(n)}} \alpha_{1,\infty}(|s_{\pi(t)} - s_i|) = h_{s_i}^{-N/2} \sum_{k \in \mathbb{Z}^d} \alpha_{1,\infty}(|k|).
\]

Using Lemma 6, if \(\|\Phi\|_\infty = \infty\) then

\[
Q_{T_{s_i}}(u) \leq \frac{u^{-\pi\Phi} \|U_{s_i}\|_{2+\theta}}{\sqrt{n}} \leq \frac{u^{-\pi\Phi} \|U_{s_i}\|_{2+\theta}}{\sqrt{nh_{s_i}^{2+\theta}}} \times \frac{h_{s_i}^{N/2} w_{s_i}}{h_{s_i}^{N/2} w_{s_i} A_{n,0,1}}.
\]
where
\[ \nu \] and \[ \text{It holds that} \]

**Lemma 8.** Moreover, using (4.19), we have

Finally, we proved that

Using (H1), we get

Consequently, using again the inequality \( 2ab \leq a^2 + b^2 \), we obtain

Using (H1), we get

Finally, we proved that

where \( \nu_2(\theta) = \mathbb{I}\{\|\Phi\|_\infty < \infty\} + \frac{\theta}{\gamma_2} \mathbb{I}\{\|\Phi\|_\infty = \infty\} \). So, (4.20) holds.

The proof of the following lemma is postponed to section 5.

**Lemma 8.** It holds that \( \sup_{1 \leq i, j \leq n} \mathbb{E}[|U_{s_i} U_{s_j}|] \leq w_{s_i} w_{s_j} \nu_4(\theta) \)

where \( \nu_4(\theta) = \mathbb{I}\{\|\Phi\|_\infty < \infty\} + \frac{\theta}{\gamma_4} \mathbb{I}\{\|\Phi\|_\infty = \infty\} \).

Since \( (X_k)_{k \in \mathbb{Z}^d} \) is strongly mixing, we have \( Z_{s_i} = U_{s_i} \) for any \( 1 \leq i \leq n \). Moreover, using (4.19), we have

\[
\sum_{i=1}^{n} \mathbb{E}[|\beta_i'|] \leq 2 \sum_{i=1}^{n} \mathbb{E}\left[ \frac{|Z_{s_i}|}{\sqrt{n}} \left( \sum_{j=1}^{i-1} \frac{|Z_{s_j}|}{\sqrt{n}} \right)^{\nu_4(\theta)} \right] = 2 \sum_{i=1}^{n} \sum_{j=1}^{i-1} \mathbb{E}[|U_{s_i} U_{s_j}|] \]
The next result can be found in [27].

Using Lemma 8 and the inequality $2ab \leq a^2 + b^2$, we get

$$
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \mathbb{E}[|U_s, U_s_j|] \leq \frac{M_d^2 h_{s_n}^{N\nu_{\theta}(\ell)}}{nw_{s_n}^2} \sum_{i=1}^{n} u_{s_i}^2 + \frac{h_{s_n}^{N\nu_{\theta}(\ell)}}{nw_{s_n}^2} \sum_{i=1}^{n} \sum_{j=1}^{i-1} w_{s_j}^2.
$$

Moreover,

$$
\sum_{i=1}^{n} \sum_{j=1}^{i-1} w_{s_j}^2 = \sum_{j=1}^{n-1} w_{s_j} \sum_{i=j+1}^{n} 1 \leq M_d^2 \sum_{i=1}^{n} w_{s_i}^2.
$$

Consequently, using (H1), we obtain

$$
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[(U_s, U_s_j)] \leq \frac{M_d^2 h_{s_n}^{N\nu_{\theta}(\ell)}}{nw_{s_n}^2} \sum_{i=1}^{n} u_{s_i}^2.
$$

Combining (4.24), (4.25) and (4.26), we obtain (4.21) and (4.22).

Now, it suffices to prove (4.23). Let $\beta \geq 1$ be a positive integer. For any $1 \leq j \leq n$, the notation $\mathbb{E}_\beta[Z_{s_j}]$ will stand for the conditional expectation of $Z_{s_j}$ with respect to the $\sigma$-algebra $\sigma(Z_{s_j}; \ell < j$ and $|s_\ell - s_j| \geq \beta)$. Then,

$$
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[(Z_{s_i}^2 - \mathbb{E}[Z_{s_i}^2]) h_{i-1,i+1}''] \leq I_1 + I_2,
$$

where

$$
I_1 = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[(Z_{s_i}^2 - \mathbb{E}_\beta[Z_{s_i}^2]) h_{i-1,i+1}''],
$$

and

$$
I_2 = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[(\mathbb{E}_\beta[Z_{s_i}^2] - \mathbb{E}[Z_{s_i}^2]) h_{i-1,i+1}''].
$$

The next result can be found in [27].
Lemma 9. Let \( \mathcal{U} \) and \( \mathcal{V} \) be two \( \sigma \)-algebras and let \( X \) be a random variable which is measurable with respect to \( \mathcal{U} \).

If \( 1 \leq p \leq r \leq \infty \), then \( \|E[X|\mathcal{V}] - E[X]\|_p \leq 2(2^{1/p} + 1)(\alpha(\mathcal{U}, \mathcal{V}))^{\frac{1}{p} - \frac{1}{r}} \|X\|_r \).

Since \( (X_k)_{k \in \mathbb{Z}^d} \) is strongly mixing, we have \( Z_{s_i} = U_{s_i} \) for any \( 1 \leq i \leq n \). If \( \|\Phi\|_\infty = \infty \) then using Lemma 9 with \( p = 1 \) and \( r = (2 + \theta)/2 \), we derive

\[
I_2 \leq \frac{1}{n} \sum_{i=1}^{n} \|E_\beta[U_{s_i}^2] - E[U_{s_i}^2]\|_1 \leq \frac{\alpha_{1,\infty}(\beta)}{n} \sum_{i=1}^{n} \|U_{s_i}\|_{2+\theta}^2.
\]

From Lemma 6, we have

\[
\|U_{s_i}\|_{2+\theta}^2 \leq \frac{h_{s_i}^{-N\frac{2}{\theta}}}{h_{s_i}^{-N/2} A_{n,0,1}^2}.
\]

Consequently, using (H1), we get

\[
I_2 \leq h_{s_i}^{-N\frac{2}{\theta}} \alpha_{1,\infty}(\beta) \times A_{n,0,1}^{-2} \sum_{i=1}^{n} \frac{u_{s_i}^2}{h_{s_i}^N} \leq h_{s_i}^{-N\frac{2}{\theta}} \alpha_{1,\infty}(\beta).
\]

Similarly, if \( \|\Phi\|_\infty < \infty \) then \( |\Delta_{s_i}| \leq h_{s_i}^{-N/2} \) and since \( (w_{s_i} h_{s_i}^{-N})_{n \geq 0} \) is nondecreasing, we have \( |U_{s_i}| \leq h_{s_i}^{-N/2} \) and

\[
I_2 \leq \frac{6\alpha_{1,\infty}(\beta)}{n} \sum_{i=1}^{n} \|U_{s_i}\|_{2+\theta}^2 \leq h_{s_i}^{-N/2} \alpha_{1,\infty}(\beta).
\]

Finally, keeping in mind that \( \nu_2(\theta) = \frac{1}{2} (\|\Phi\|_\infty < \infty) + \frac{\theta}{2 + \theta} \frac{1}{2} (\|\Phi\|_\infty = \infty) \), it means that

\[
I_2 \leq \frac{1}{n} \sum_{i=1}^{n} \|E_\beta[U_{s_i}^2] - E[U_{s_i}^2]\|_1 \leq h_{s_i}^{-N \nu_2(\theta)} \alpha_{1,\infty}(\beta).
\]

(4.27)

Now, we make the choice

\[
\beta = \left[ \frac{h_{s_i}^{-N \nu_2(\theta)/\nu_2(\theta)(\beta) + 1}}{h_{s_i}^{-N \nu_2(\theta)/\nu_2(\theta) + 1}} \right].
\]

(4.28)

Consequently, since \( \sum_{k \in \mathbb{Z}^d} \frac{d_{\nu_2(\theta)}}{\nu_2(\theta)(\beta) + 1} \alpha_{1,\infty}(k) < \infty \)

\[
\sum_{k \in \mathbb{Z}^d} \frac{d_{\nu_2(\theta)}}{\nu_2(\theta)(\beta) + 1} \alpha_{1,\infty}(k) < \infty,
\]

we derive

\[
I_2 \leq \frac{h_{s_i}^{-N \nu_2(\theta)/\nu_2(\theta)(\beta) + 1}}{h_{s_i}^{-N \nu_2(\theta)/\nu_2(\theta) + 1}} \alpha_{1,\infty}(\beta) \rightarrow 0.
\]

Since \( h_{i-1,i+1}^{\nu_2(\beta)} \) is \( \sigma(\mathcal{E}_{s_i}, Z_{s_k}; 1 \leq k < i, |s_k - s_i| \geq \beta, \ell \in \mathbb{N}^*) \)-measurable, we find that \( E[(Z_{s_i}^2 - E_\beta[Z_{s_i}^2]) h_{i-1,i+1}^{\nu_2(\beta)}] = 0 \) and consequently

\[
E[(Z_{s_i}^2 - E_\beta[Z_{s_i}^2]) h_{i-1,i+1}^{\nu_2(\beta)}] = E[(Z_{s_i}^2 - E_\beta[Z_{s_i}^2]) (h_{i-1,i+1}^{\nu_2(\beta)} - h_{i-1,i+1}^{\nu_2(\beta)})].
\]
Keeping in mind that
\[
I_1 = \frac{1}{n} \sum_{i=1}^{n} |E[(Z_{s_i}^2 - E_\beta[Z_{s_i}^2]) h_{i-1,i+1}'']|,
\]
we obtain
\[
I_1 \leq \frac{1}{n} \sum_{i=1}^{n} E \left[ \left( 2 \wedge \sum_{j=1}^{i-1} \frac{Z_{s_j}}{\sqrt{n}} \right) (Z_{s_i}^2 + E_\beta[Z_{s_i}^2]) \right].
\]

Let \( L > 0 \), then
\[
I_1 \leq \frac{L}{n^{3/2}} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \mathbb{E}[|Z_{s_j}| | |s_j-s_i| \leq \beta] + \frac{2}{n} \sum_{i=1}^{n} E[Z_{s_i}^2 | |Z_{s_i}| > L]
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \left( 2 \wedge \sum_{j=1}^{i-1} \frac{Z_{s_j}}{\sqrt{n}} \right) (E_\beta[Z_{s_i}^2] - E[Z_{s_i}^2]) \right]
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \left( 2 \wedge \sum_{j=1}^{i-1} \frac{Z_{s_j}}{\sqrt{n}} \right) E[Z_{s_i}^2] \right].
\]

Since \( Z_{s_i} = U_{s_i} \) for any \( 1 \leq i \leq n \), we derive
\[
I_1 \leq \frac{L}{n^{3/2}} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \mathbb{E}[|U_{s_j} U_{s_i}|] + \frac{2}{n} \sum_{i=1}^{n} E[U_{s_i}^2 | |U_{s_i}| > L]
\]
\[
+ \frac{2}{n} \sum_{i=1}^{n} \mathbb{E}[|E_\beta[U_{s_i}^2] - E[U_{s_i}^2]|] + \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{U_{s_j}}{\sqrt{n}} \mathbb{E}[U_{s_i}^2].
\]

Arguing as in (4.26) and Lemma 9, we derive
\[
I_1 \leq \frac{\beta^d L h_n^{N(\nu_1(\theta) + \tau - 1)} }{\sqrt{n}} + \frac{L^{-\theta}}{n} \sum_{i=1}^{n} E[|U_{s_i}|^{2+\theta}] + h_n^{-N(\nu_2(\theta))} \alpha_{1,\infty}(\beta)
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} E[U_{s_i}^2] \left\| \sum_{j=1}^{i-1} \frac{U_{s_j}}{\sqrt{n}} \right\|_{2}.
\]
Using (4.17), (4.28) and (H1), we get

\[
I_1 \leq \frac{\beta^d L h_{s_n}^{N/(\nu_2(\theta) + \tau - 1)}}{\sqrt{n}} + L^{-\theta} h_{s_n}^{\frac{N}{\beta}} + \beta \frac{\nu_2(\theta) + (d - 1)(\nu_2(\theta) + \tau - 1)}{2} \alpha_{1,\infty}(\beta) \\
+ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[U_{s_i}^2] \left\| \sum_{j=1}^{i-1} \frac{U_{s_j}}{\sqrt{n}} \right\|_{L_2}.
\]

Moreover,

\[
\left\| \sum_{j=1}^{i-1} \frac{U_{s_j}}{\sqrt{n}} \right\|_{L_2}^2 = \frac{1}{n} \left( \sum_{j=1}^{i-1} \mathbb{E}[U_{s_j}^2] + \sum_{j=1}^{i-1} \mathbb{E}[U_{s_j} U_{s_{\ell}}] \right)
\]

If \( \| \Phi \|_{\infty} = \infty \) then using (H1) and (4.17), for \( L' = \left( \beta^{-d} n h_{s_n}^{\frac{N}{\beta}} \right)^{\frac{1}{4\pi}} \), we have

\[
\frac{1}{n} \sum_{j=1}^{i-1} \mathbb{E}[U_{s_j}^2] \leq \frac{\beta^d L'^2}{n} + L'^{-\theta} \sum_{j=1}^{i-1} \mathbb{E}[U_{s_j}]^{2+\theta} \leq \frac{\beta^d L'^2}{n} + L'^{-\theta} h_{s_n}^{-\frac{N}{\beta}}
\]

\[
= 2 \left( \frac{\beta^d}{n h_{s_n}} \right)^{\frac{N}{2\pi}}.
\]

If \( \| \Phi \|_{\infty} < \infty \) then \( |U_{s_i}| \leq h_{s_n}^{-N/2} \) (since \( |\Delta_{s_i}| \leq h_{s_i}^{-N/2} \) and \( (w_{s_i} h_{s_n}^{N})_{n \geq 1} \) is nondecreasing) and consequently

\[
\frac{1}{n} \sum_{j=1}^{i-1} \mathbb{E}[U_{s_j}^2] \leq \frac{\beta^d}{n h_{s_n}^{N/2}}.
\]

Finally, it means that

\[
\frac{1}{n} \sum_{j=1}^{i-1} \mathbb{E}[U_{s_j}^2] \leq \left( \frac{\beta^d}{n h_{s_n}^{N/2}} \right)^{\nu_2(\theta)}
\]

\[
\leq \left( n h_{s_n}^{N/4} \left( 1 + \frac{d \nu_2(\theta) \nu_2(\theta) + \tau - 1}{2} \right) \right)^{-\nu_2(\theta)} \xrightarrow{n \to \infty} 0.
\]

Moreover, we have

\[
\frac{1}{n} \sum_{j=1}^{i-1} \mathbb{E}[U_{s_j} U_{s_{\ell}}] \leq \frac{A_{n,0,1} h_{s_n}^{N/2}}{n w_{s_n}^2} \sum_{j=1}^{i-1} \sum_{j \neq \ell} \max \{ |s_j - s_i|, |s_{\ell} - s_i| \} \leq \beta \frac{w_{s_i} w_{s_{\ell}} |E[\Delta_{s_i} \Delta_{s_{\ell}}]|}{(h_{s_i} h_{s_{\ell}})^{N/2}}.
\]
Since $\nu_4(\theta) \leq 1$, we have

\[
\frac{1}{n} \sum_{j,h=1}^{i-1} |\mathbb{E}[U_{sj} U_{sh}]| \leq \sum_{k \in \mathbb{Z}^d_{|k| > M_n}} \alpha_{1,\infty}^{\nu_2(\theta)} (|k|).
\]

Using (H1) and (H2), we derive

\[
\frac{1}{n} \sum_{j,h=1}^{i-1} |\mathbb{E}[U_{sj} U_{sh}]| \leq m_n^d h_n^{N(\nu_4(\theta) + \tau - 1)} + h_n^{-N\nu_2(\theta)} \sum_{k \in \mathbb{Z}^d_{|k| > M_n}} \alpha_{1,\infty}^{\nu_2(\theta)} (|k|).
\]

Consequently, denoting

\[
\varepsilon_n^2 := \left( n h_n^{N \left(1 + \frac{\nu_4(\theta) (\nu_4(\theta) + \tau - 1)}{2 (\nu_2(\theta) + \nu_4(\theta) + \tau - 1)}\right)} \right)^{-\nu_2(\theta)} + m_n^d h_n^{N(\nu_4(\theta) + \tau - 1)}
\]

\[
+ h_n^{-N\nu_2(\theta)} \sum_{k \in \mathbb{Z}^d_{|k| > m_n}} \alpha_{1,\infty}^{\nu_2(\theta)} (|k|) \xrightarrow{n \to \infty} 0,
\]
we get
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[U_{s_j}^2] \left\| \sum_{j=1}^{i-1} \frac{U_{s_j}}{\sqrt{n}} \right\|_2 \leq A_{n,0,1}^{-2} \frac{h_n^N}{\sqrt{w}} \sum_{i=1}^{n} \frac{w_{s_i}^2 \mathbb{E}[\Delta_{s_i}^2]}{h_n^{s_i}} \times \varepsilon_n \leq \varepsilon_n
\]
and finally,
\[
I_1 \leq \frac{1}{n} \mathbb{E}[U_{s_j}^2] \left\| \sum_{j=1}^{i-1} \frac{U_{s_j}}{\sqrt{n}} \right\|_2 \leq \frac{A_{n,0,1}^{-2} h_n^N}{\sqrt{w}} \sum_{i=1}^{n} \frac{w_{s_i}^2 \mathbb{E}[\Delta_{s_i}^2]}{h_n^{s_i}} \times \varepsilon_n \leq \varepsilon_n
\]
Optimizing in \( L \), we get
\[
I_1 \leq \frac{\beta \frac{d \theta + \nu (\theta) + 1}{2(\theta + \nu)} h_{s_i}^N}{(nh_{s_i})^{2(\theta + \nu)}} + \frac{\beta \frac{d \theta + \nu (\theta) + 1}{2(\theta + \nu)} h_{s_i}^N}{(nh_{s_i})^{2(\theta + \nu)}} + \varepsilon_n \to 0.
\]
Since, \( \beta \leq h_{s_i}^N \), we derive
\[
I_1 \leq \frac{\beta \frac{d \theta + \nu (\theta) + 1}{2(\theta + \nu)} h_{s_i}^N}{(nh_{s_i})^{2(\theta + \nu)}} + \frac{\beta \frac{d \theta + \nu (\theta) + 1}{2(\theta + \nu)} h_{s_i}^N}{(nh_{s_i})^{2(\theta + \nu)}} + \varepsilon_n \to 0.
\]
So, we obtain (4.23).
Now, we assume that \( (X_k)_{k \in \mathbb{Z}^d} \) is of the form (1.5). As before, we have to prove (4.20), (4.21), (4.22) and (4.23). Now, we have \( Z_{s_i} = U_{s_i} \) for any \( 1 \leq i \leq n \). Moreover, \( U_{s_i} \) and \( U_{s_j} \) are independent as soon as \( |s_i - s_j| > M_n \) where \( M_n = 2m_n \). So, (4.20) holds since
\[
\mathbb{E}[T_{s_i} h_1^{(M_n)}] = n^{-1/2} \mathbb{E} \left[ \sum_{i=1}^{n} \frac{U_{s_i} h_1^{(M_n)}}{\sqrt{n}} \right] = 0.
\]
Arguing as in (4.24) and (4.25), we have
\[
\sum_{i=1}^{n} \mathbb{E}[|d_t|] \leq \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \mathbb{E}[|U_{s_i} U_{s_j}|]
\]
and
\[
\sum_{i=1}^{n} \mathbb{E} \left[ T_{s_i} \left( \sum_{i=1}^{n} T_{s_j} \right)^{\nu(M_n)} \right] \leq \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \mathbb{E}[|U_{s_i} U_{s_j}|].
\]
The proof of the following lemma is postponed to section 5.
Lemma 10. For any positive integer $n$,

$$
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}[|\mathcal{U}_{s_i} - \mathcal{U}_{s_j}|] \leq m_n^d h_{s_n}^{N(v_2(\theta)+\tau-1)} + h_{s_n}^{-2} \sum_{k \in \mathbb{Z}^d, |k| > n} |k|^d \delta_{k, 2}^{(\theta)} \to 0 \quad n \to \infty.
$$

Consequently, we obtain (4.21) and (4.22). In order to finish the proof, it suffices to prove (4.23). Let $L > 0$ be fixed. Keeping in mind that $\mathcal{U}_{s_i}$ and $\mathcal{U}_{s_j}$ are independent if $|s_i - s_j| > M_n$, we have

$$
\mathbb{E} \left[ (Z_{s_i}^2 - \mathbb{E}[Z_{s_i}^2]) h''_{i-1, i+1} \right] = \mathbb{E} \left[ (\mathcal{U}_{s_i}^2 - \mathbb{E}[\mathcal{U}_{s_i}^2]) h''_{i-1, i+1} \right] = 0.
$$

So, we have

$$
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ (Z_{s_i}^2 - \mathbb{E}[Z_{s_i}^2]) h''_{i-1, i+1} \right] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ (\mathcal{U}_{s_i}^2 - \mathbb{E}[\mathcal{U}_{s_i}^2]) h''_{i-1, i+1} \right] \leq \frac{L}{n^{3/2}} \sum_{i=1}^{n} \sum_{\ell=1}^{i-1} \mathbb{E}[\mathcal{U}_{s_i} \mathcal{U}_{s_j}] + \frac{2}{n} \sum_{i=1}^{n} \mathbb{E}[\mathcal{U}_{s_i}^2 \mathbb{1}_{|\mathcal{U}_{s_i}| > L}]
$$

$$
+ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ 2 \mathbb{1}_{|s_i - s_j| \leq M_n} \right] \mathbb{E} \left[ \mathcal{U}_{s_i}^2 \right] \leq \frac{L}{\sqrt{n}} \times \frac{1}{n} \sum_{i=1}^{n} \sum_{\ell=1}^{n} \mathbb{E}[\mathcal{U}_{s_i} \mathcal{U}_{s_j}] + \frac{2L^{-\theta}}{n} \sum_{i=1}^{n} \mathbb{E}[|\mathcal{U}_{s_i}|^{2+\theta}]
$$

$$
+ \frac{1}{n} \sum_{i=1}^{n} \left\| \sum_{\ell=1}^{i-1} \frac{\mathcal{U}_{s_i}}{\sqrt{n}} \right\|_2^2 \mathbb{E}[\mathcal{U}_{s_i}^2].
$$
Moreover,
\[
\left\| \sum_{\ell=1}^{i-1} \frac{U_{s_\ell}}{\sqrt{n}} \right\|_2^2 = \frac{1}{n} \left( \sum_{\ell=1}^{i-1} \mathbb{E}[U_{s_\ell}^2] + \sum_{\ell,j=1}^{i-1} \mathbb{E}[U_{s_\ell}U_{s_j}] \right).
\]
Noting that \( \mathbb{E}[U_{s_\ell}^2] \leq \mathbb{E}[U_{s_\ell}^2] \) and using (4.29), we get
\[
\frac{1}{n} \sum_{\ell=1}^{i-1} |\mathbb{E}[U_{s_\ell}U_{s_j}]| \leq \frac{1}{n} \sum_{\ell=1}^{i-1} \sum_{j=1}^{n} |\mathbb{E}[U_{s_\ell}U_{s_j}]| \xrightarrow{n \to \infty} 0.
\]
Using Lemma 10 and keeping in mind that \( U_{s_\ell} \) and \( U_{s_j} \) are independent if \( |s_\ell - s_j| > M_n \), we have
\[
\frac{1}{n} \sum_{\ell=1}^{i-1} \max_{\ell \neq j} |\mathbb{E}[U_{s_\ell}U_{s_j}]| \leq \frac{1}{n} \sum_{\ell=1}^{i-1} \sum_{j=1}^{n} |\mathbb{E}[U_{s_\ell}U_{s_j}]| \xrightarrow{n \to \infty} 0.
\]
Consequently,
\[
\lim_{n \to \infty} \sup_{1 \leq i < n} \left\| \sum_{\ell=1}^{i-1} \frac{U_{s_\ell}}{\sqrt{n}} \right\|_2 = 0
\]
and
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left\| \sum_{\ell=1}^{i-1} \frac{U_{s_\ell}}{\sqrt{n}} \mathbb{E}[U_{s_\ell}^2] \right\|_2 \leq \lim_{n \to \infty} \frac{h_n^2 N^{-2}}{n w_2^2 h_n^N} \sum_{i=1}^{n} w_2^2 \mathbb{E}[\Delta^2_{s_i}] \times \sup_{1 \leq i < n} \left\| \sum_{\ell=1}^{i-1} \frac{U_{s_\ell}}{\sqrt{n}} \right\|_2 = 0.
\]
Using (4.17) and (H1), we obtain
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[(Z_{s_i}^2 - \mathbb{E}[Z_{s_i}^2])h_{i-1,i+1}^\nu] \leq \frac{L}{\sqrt{n}} \times \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\|U_{s_i}U_{s_i}\|] + L^{-\theta} h_{s_i}^{N\theta} + o(1).
\]
Optimizing in \( L \), we get
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[(Z_{s_i}^2 - \mathbb{E}[Z_{s_i}^2])h_{i-1,i+1}^\nu] \leq (nh_n^N)^{\lambda \nu + 1} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\|U_{s_i}U_{s_i}\|] \right)^{\frac{\theta}{1 + \theta}} + o(1).
\]
Using Lemma 10, we obtain (4.23). The proof of Theorem 1 is complete.

In the proof of Theorem 1, the asymptotic normality of the estimator \( f_{n, \Phi} \) is obtained using the Lindeberg’s method based on the stability of the standard normal law. This approach seems to be superior to the so-called Bernstein’s method (see for example [5] and [21]) since it allows us to obtain mild conditions on the weak and strong dependent coefficients of the considered random field. This fact is of theoretical importance and has already been observed in [1], [14] and [15].

**Proof of Theorem 2.** Let \( n \) be a positive integer and \( x \in \mathbb{R}^N \) such that \( f(x) > 0 \). Then,

\[
r_{n, \Phi}(x) = \frac{E[f_{n, \Phi}(x)]}{E[f_{n, 1}(x)]} - \frac{E[f_{n, \Phi}(x)]}{E[f_{n, 1}(x)]} = \frac{(f_{n, \Phi}(x) - E[f_{n, \Phi}(x)])E[f_{n, 1}(x)] - (f_{n, 1}(x) - E[f_{n, 1}(x)])E[f_{n, \Phi}(x)]}{f_{n, 1}(x)E[f_{n, 1}(x)]}.
\]

Using Proposition 1 and Proposition 2, we obtain that \( f_{n, 1}(x) \) converges in probability to \( f(x) \) and \( \frac{E[f_{n, \Phi}(x)]}{E[f_{n, 1}(x)]} \) converges to \( r_{\Phi}(x) \) as \( n \to \infty \). So, using Slutsky’s lemma, it is sufficient to prove

\[
\lambda_1 \sqrt{nh_{s_n}^N} (f_{n, \Phi}(x) - E[f_{n, \Phi}(x)]) + \lambda_2 \sqrt{nh_{s_n}^N} (f_{n, 1}(x) - E[f_{n, 1}(x)]) \xrightarrow{\text{Law}} N(0, \rho_{\lambda_1, \lambda_2}^2(x))
\]

where

\[
\rho_{\lambda_1, \lambda_2}^2(x) = (\lambda_1^2 E[|\Phi(Y_0)|^2]_X = x + 2\lambda_1 \lambda_2 r_{\Phi}(x) + \lambda_2^2) \beta_{0, 1}^{-2} \beta_{-N, 2} f(x) \int_{\mathbb{R}^N} K^2(t) dt,
\]

for any \((\lambda_1, \lambda_2) \in \mathbb{R}^2\).

Let \((\lambda_1, \lambda_2) \in \mathbb{R}^2\) be fixed. Then,

\[
\lambda_1 \sqrt{nh_{s_n}^N} (f_{n, \Phi}(x) - E[f_{n, \Phi}(x)]) + \lambda_2 \sqrt{nh_{s_n}^N} (f_{n, 1}(x) - E[f_{n, 1}(x)]) = \sqrt{nh_{s_n}^N} (f_{n, \Phi}(x) - E[f_{n, \Phi}(x)])
\]

where \( \Phi(x) = \lambda_1 \Phi(x) + \lambda_2 \) for any \( x \) in \( \mathbb{R} \). Since \( u \mapsto E[|\Phi(Y_0)|^2 | X_0 = u] \) is continuous, one can notice that \( u \mapsto E[|\Phi(Y_0)|^2 | X_0 = u] \) is continuous. Moreover, since \( E[|\Phi(Y_0)|^{2+\theta}] < \infty \) and \( E[|\Phi(Y_0)|^{2+\theta} K_{s_n}(x, X_0)] \leq h_{s_n}^{-N} \), we have \( E[|\Phi(Y_0)|^{2+\theta}] < \infty \) and \( E[|\Phi(Y_0)|^{2+\theta} K_{s_n}(x, X_0)] \leq h_{s_n}^{-N} \). Consequently, using Theorem 1, we get the result. The proof of Theorem 2 is complete.

One can notice that the asymptotic normality of the regression estimator \( r_{n, \Phi} \) obtained in the proof of Theorem 2 is a direct consequence of Theorem 1.
In some sense, it means that Theorem 1 is quite a deep result since it contains both the asymptotic normality of the kernel density estimator $f_{n,1}$ and that of the regression estimator $r_{n,1}$.

**Proof of Theorem 3.** Let $x \in \mathbb{R}^N$ such that $f(x) > 0$. Then, according to Theorem 2, we have

$$\sqrt{nh_n^N} \left( r_{n,\Phi}(x) - \frac{\mathbb{E}[f_{n,\Phi}(x)]}{\mathbb{E}[f_{n,1}(x)]} \right) \xrightarrow{\text{Law}} \mathcal{N} \left( 0, \sigma_\Phi^2(x) \right),$$

where $\sigma_\Phi^2(x) = \frac{V(x)\beta_{\alpha,2}}{f(x)\delta_0^2} \int_{\mathbb{R}^N} K^2(t)dt$ and $V(x) = \mathbb{E}[|\Phi(Y_0)|^2|X_0 = x] - r_\Phi^2(x)$. Applying Proposition 2, we have

$$\mathbb{E}[f_{n,\Phi}(x)] - f_\Phi(x) \leq h_n^2 \quad \text{and} \quad |\mathbb{E}[f_{n,1}(x)] - f(x)| \leq h_n^2.$$

Recall that $r_\Phi(x) = \frac{\ell_n(x)}{f(x)}$. Then, for $n$ sufficiently large, we have

$$\frac{\mathbb{E}[f_{n,\Phi}(x)]}{\mathbb{E}[f_{n,1}(x)]} = \frac{|(\mathbb{E}[f_{n,\Phi}(x)] - f_\Phi(x))f(x) - (\mathbb{E}[f_{n,1}(x)] - f(x))f_\Phi(x)|}{f(x)\mathbb{E}[f_{n,1}(x)]} \leq h_n^2.$$

Finally, using Slutsky’s lemma, we obtain

$$\sqrt{nh_n^N} \left( r_{n,\Phi}(x) - r_\Phi(x) \right) \xrightarrow{\text{Law}} \mathcal{N} \left( 0, \sigma_\Phi^2(x) \right).$$

The proof of Theorem 3 is complete. \( \square \)

5. Appendix

**Proof of Lemma 7.** First, we note that $\mathbb{E}[W_i|\mathcal{F}_{t-1}]$ is $\mathcal{F}_t$-measurable. It suffices to show that for every $A \in \mathcal{F}_t$, we have $\mathbb{E}[\mathbb{E}[W_i|\mathcal{F}_{t-1}]|A] = \mathbb{E} \left[ H_t(\mathcal{H}_{t,\infty}^{(f)}) I_A \right]$. One can notice that the collections $\mathcal{P} = \{ A \cap B \mid A \in \mathcal{F}_{t-1}, B \in \sigma(\varepsilon^{(f)}_{t}) \}$ and $A = \{ A \in \mathcal{F}_t \mid \mathbb{E} \left[ \mathbb{E}[W_i|\mathcal{F}_{t-1}] \mid A \right] = \mathbb{E} \left[ H_t(\mathcal{H}_{t,\infty}^{(f)}) I_A \right] \}$ are respectively a $\pi$-system and a $\lambda$-system which satisfy $\sigma(\mathcal{P}) = \mathcal{F}_t$. Since $(\varepsilon^{(f)}_{\tau(j)})_{j \in \mathbb{Z}}$ and $(\varepsilon^{(f)}_{\tau(j)})_{j \in \mathbb{Z}}$ are identically distributed where $\varepsilon^{(f)}_{\tau(j)} = \varepsilon^{(f)}_{\tau(j)}$ if $j \neq \ell$ and $\varepsilon^{(f)}_{\tau(\ell)} = \varepsilon^{(f)}_{\tau(\ell)}$, it holds for every $C \in \mathcal{F}_{t-1}$ that $\mathbb{E}[1_C W_i] = \mathbb{E} \left[ I_C H_t(\mathcal{H}_{t,\infty}^{(f)}) \right]$. Then, if $A = A_1 \cap A_2 \in \mathcal{P}$ with $A_1 \in \mathcal{F}_{t-1}$ and $A_2 \in \sigma(\varepsilon^{(f)}_{t})$, we obtain

$$\mathbb{E}[\mathbb{E}[W_i|\mathcal{F}_{t-1}]|A] = \mathbb{E} \left[ \mathbb{E}[1_{A_1} W_i|\mathcal{F}_{t-1}]|A_2 \right] = \mathbb{E} \left[ 1_{A_1} H_t(\mathcal{H}_{t,\infty}^{(f)}) I_A \right] = \mathbb{E} \left[ 1_{A_1} H_t(\mathcal{H}_{t,\infty}^{(f)}) I_{A_2} \right] = \mathbb{E} \left[ H_t(\mathcal{H}_{t,\infty}^{(f)}) I_A \right].$$
So, we obtain $A \in \Lambda$ and finally $\mathcal{P} \subset \Lambda$. Applying Dynkin’s lemma, we get the desired result. The proof of Lemma 7 is complete. 

\textbf{Proof of Lemma 8.} Let $1 \leq i, j \leq n$ such that $i \neq j$. From Lemma 4, we have $\mathbb{E}[|\Delta_s, \Delta_s|] \leq (h_s, h_s)^{N_N(\theta)}$. Keeping in mind (4.13), we have

$$
\mathbb{E}[|U_s, U_s|] = \frac{h_s^{N_s} w_s, w_s \mathbb{E}[|\Delta_s, \Delta_s|]}{(h_s, h_s)} \leq \frac{A_{n,0,1}^N w_s, w_s}{w_s^2(h_s, h_s)^{N(1-\nu_4(\theta))/2}}.
$$

Since $\nu_4(\theta) \leq 1$ and using (H1), we derive $\mathbb{E}[|U_s, U_s|] \leq w_s, w_s \frac{2^{N_N(\theta)}}{h_s^2}$. The proof of Lemma 8 is complete. 

\textbf{Proof of Lemma 10.} For any $1 \leq i, j \leq n$ such that $i \neq j$, we have

$$
|\mathbb{E}[|U_s, U_s|] - \mathbb{E}[|U_s, U_s|]| \leq \|U_s, U_s\|_2 |U_s - U_s|_2 + \|U_s, U_s\|_2 |U_s - U_s|_2
$$

and for any $1 \leq \ell \leq n$, using (3.12), we get

$$
\|U_s\|_2 = \frac{h_s^{N_s/2} w_s, ||\Delta_s||_2}{h_s^{N_s/2} w_s, A_{n,0,1}} \leq \frac{h_s^{N_s/2} w_s}{h_s^{N_s/2} w_s, A_{n,0,1}}
$$

and

$$
\|U_s - U_s\|_2 = \frac{h_s^{N_s/2} w_s, ||\Delta_s - \Delta_s||_2}{h_s^{N_s/2} w_s, A_{n,0,1}} \leq \frac{h_s^{N_s/2} w_s, h_s^{N_s/2} w_s, -\nu_2(\theta)}{h_s^{N_s/2} w_s, A_{n,0,1}} \sum_{k \in \mathbb{Z}^d} \delta_{k,2}^{\nu_2(\theta)}.
$$

Consequently, we get

$$
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} |\mathbb{E}[|U_s, U_s|] - \mathbb{E}[|U_s, U_s|]| \leq \frac{h_s^{N_s} A_{n,0,1}^{2N_s}}{n w_s^2} \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \frac{w_s, w_s, w_s, w_s}{h_s^{N_s/2} h_s} \times h_s^{N_s/2} \sum_{k \in \mathbb{Z}^d} \delta_{k,2}^{\nu_2(\theta)} \right|
$$

$$
\leq \frac{h_s^{N_s} A_{n,0,1}^{2N_s}}{n w_s^2} \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \left( \frac{w_s}{h_s^{N_s}} + \frac{w_s}{h_s^{N_s}} \right) \times h_s^{N_s/2} \sum_{k \in \mathbb{Z}^d} \delta_{k,2}^{\nu_2(\theta)} \right|
$$

$$
\leq 2M^d \left( \frac{h_s^{N_s} A_{n,0,1}^{2N_s}}{n w_s^2} \sum_{i=1}^{n} \frac{w_s}{h_s^{N_s}} \right) \sum_{k \in \mathbb{Z}^d} \delta_{k,2}^{\nu_2(\theta)}.
$$
Using (H1), we obtain
\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i \neq j \atop |s_j - s_i| \leq M_n} |E[U_{s_i} U_{s_j}] - E[U_{s_i} U_{s_j}]| \leq m_n h_n^{-\left(\frac{N}{2} + \nu_2(\theta)\right)} \sum_{k \in \mathbb{Z}^d \atop |k| > m_n} |k|^d \delta_{k,2}^\nu_2(\theta).
\]

Using Lemma 8, we obtain
\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i \neq j \atop |s_j - s_i| \leq M_n} E[U_{s_i} U_{s_j}] \leq \frac{h_n^{N\nu_4(\theta)}(\nu_4(\theta) - \nu_2(\theta))}{n} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i \neq j \atop |s_j - s_i| \leq M_n} w_{s_i} w_{s_j} + h_n^{-\left(\frac{N}{2} + \nu_2(\theta)\right)} \sum_{k \in \mathbb{Z}^d \atop |k| > m_n} |k|^d \delta_{k,2}^\nu_2(\theta)
\]

Using (H2), we derive
\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{i \neq j \atop |s_j - s_i| \leq M_n} E[U_{s_i} U_{s_j}] \leq m_n h_n^{N\nu_4(\theta) + \tau - 1} + h_n^{-\left(\frac{N}{2} + \nu_2(\theta)\right)} \sum_{k \in \mathbb{Z}^d \atop |k| > m_n} |k|^d \delta_{k,2}^\nu_2(\theta) \xrightarrow{n \to \infty} 0.
\]

The proof of Lemma 10 is complete. \hfill \Box

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