Energy dependent ergodicity and scarring phenomena in two component Bose-Josephson junction

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We consider a Bose-Josephson junction (BJJ) formed by binary mixture of ultracold atoms to investigate the manifestation of coherent collective dynamics on ergodicity and quantum scars, unfolding the connection between them. By tuning the inter and intra-species interaction, we reveal a rich structure of steady states, which has a significant influence on the overall ergodic behavior, although its degree varies across the energy band. The signature of underlying classicality can be revealed from the entanglement spectrum, which is used to identify the quantum scars of unstable steady states and periodic orbits, leading to athermal behavior. The energy dependent ergodicity and scarring phenomena can be probed from the auto-correlation function as well from the phase fluctuation of the condensates, which has relevance in cold atom experiments.

Introduction.- Coherent collective dynamics in a quantum many body system is a fascinating phenomena. It attracts considerable interest after the realization of Bose-Josephson junction (BJJ) formed by coupling two atomic Bose-Einstein condensates (BEC) in a double well trap [1–7]. An interplay between phase coherence of the BECs and inter-particle interaction strength can lead to various non-linear oscillations, quantum transitions [5, 7–13], phase diffusion [6, 14–16] and onset of chaos [17, 18]. Moreover, such ultracold atomic systems can serve as a test bed to study the non equilibrium dynamics, entanglement and ergodic properties of an interacting many body system due to the easy tunability of parameters and powerful imaging techniques [19–24].

Till date, a complete understanding of ergodicity and its deviation in interacting quantum systems remains a challenging issue. To shed light on it, the eigenstate thermalization hypothesis (ETH) [25–27] has been put forward, and its connections with the random matrix theory (RMT) has also been explored [28–32]. However, deviation from ergodicity and the violation of ETH has also drawn a lot of interest in the recent years [33, 34]. In this context, a recent experiment on a chain of strongly interacting Rydberg atoms reveals, even in the ergodic regime, certain special choice of initial states exhibit athermal revival phenomena [35]. Such behavior has been attributed to many body quantum scars (MBQS) [36–41], which has also been theoretically studied in other interacting models [42–55]. Originally, the quantum scar of a single particle wavefunction in a chaotic system was identified as reminiscence of the unstable classical motion [56]. However, such correspondence in a generic many-body system is not obvious, since the underlying classical picture is obscured due to the absence of phase space description.

This raises the question of a possible connection between underlying classicality and ergodicity of an interacting system, that can unveil an alternate route to ergodicity as well formation of scars. To address this issue, we consider an experimentally realizable set-up of a two-component BJJ [57–59], where its phase coherence and collective nature can pave the way to explore such connection. In this work, we demonstrate how a rich variety of steady states can influence the overall ergodicity of BJJ, which is summarized in Fig.1. Also, the underlying classicality is unfolded from the entanglement spectrum, which elucidates the quantum scarring phenomena as well its deviation from ergodicity.

The model.- The BJJ formed by a binary mixture of ultracold bosons with equal population $N$ of each component, can be described by the Hamiltonian within the single mode approximation [8],

$$\hat{H} = \sum_{i,\alpha} \left[ -\frac{j}{2} \hat{a}_{i,\alpha}^\dagger \hat{a}_{i,\alpha} + \frac{U}{2N} (\hat{n}_{i,\alpha} - 1)^2 + \frac{V}{2N} \hat{n}_{i,\alpha} \hat{n}_{i,\alpha} \right]$$

(1)

The first two terms represent a two-site Bose-Hubbard model with onsite interaction strength $U$ and hopping amplitude $J$ between the two sites denoted by $\alpha \in \{L, R\}$ ($\bar{\alpha} \neq \alpha$), where, $\hat{a}_{i,\alpha}^\dagger (\hat{a}_{i,\alpha})$ represents the annihilation (creation) operator of the two species of bosons indexed by $i \in \{1, 2\}$ ($i \neq i$). The third term describes the inter-species interaction of strength $V$. We set $h = 1$ and scale energy (time) by $J(1/J)$.

The effective Hamiltonian of two component BJJ can be written as a generalized coupled top model [54, 55, 60–65] describing two interacting large spins,

$$\hat{\mathcal{H}} = -\hat{S}_{1z} - \hat{S}_{2z} + \frac{U}{2S} \left( \hat{S}_{1z}^2 + \hat{S}_{2z}^2 \right) + \frac{V}{S} \hat{S}_{1z} \hat{S}_{2z}$$

(2)

where the spin components of each species with $S = N/2$ are written as, $\hat{S}_{z} = \sum_{\alpha} \hat{\alpha}_{\alpha} \hat{n}_{\alpha}/2$ and $\hat{S}_{z} = (\hat{n}_{L} - \hat{n}_{R})/2$, within the Schwinger-Boson representation.

Classical dynamics.- For large magnitude of spin $S \gg 1$, the spin operators can be treated as components of the classical spin vector $\hat{S}_{i} = S \left( \sin \theta_{i} \cos \phi_{i}, \sin \theta_{i} \sin \phi_{i}, \cos \theta_{i} \right)$. Consequently, the Hamiltonian in Eq.2 can be written in terms of the
FIG. 1. Collective dynamics and ergodicity: (a) Steady-state phase diagram as a function of interaction strengths $U$ and $V$ where the phase boundaries are obtained from the stability analysis, see the text for details. The shaded regions denote the stability of the ‘$\pi$0-mode’ (dark magenta) and the AST state (light blue). (b) Schematics of spin orientation corresponding to different steady states (FP). Degree of ergodicity, quantified from the average (c) Lyapunov exponent $\bar{\Lambda}_L$ and (d) ratio of level spacing $\langle r \rangle$ is shown as a color scale in the $U$-$V$ plane. The lines carry the same meaning as in (a). (e) Entanglement entropy $S_{en}$ of the eigenstates scaled by the page value [75] at energy density $E$, with increasing $V$ at $U=0$. The solid (dashed) lines denote stable (unstable) FPs mentioned in the figure. For (d,e), $S=40$.

canonical conjugate variables $\phi_i, z = \cos \theta$ as, $\hat{H}_c = -\sqrt{1-z_i^2} \cos \phi_1 - \sqrt{1-z_2^2} \cos \phi_2 + (U/2)(z_1^2 + z_2^2) + V z_1 z_2$ [66], and corresponding equations of motion (EOM) are given by,

$$\dot{z}_i = -\sqrt{1-z_i^2} \sin \phi_i, \quad \dot{\phi}_i = \frac{z_i \cos \phi_i}{\sqrt{1-z_i^2}} + U z_i + V z_i$$ \hspace{1cm} (3)

In the context of BJJ, $z$ and $\phi$ denote the normalized number imbalance $(n_L - n_R)/N$ and the relative phase between the condensates at the two sites (L and R). The corresponding collective Josephson dynamics can be mapped on to a Bloch sphere using the above spin representation [5]. This system exhibits a rich dynamical behavior including different types of oscillations around the steady states which we investigate by analyzing the fixed points (FPs) of the above EOM. A stability analysis performed around the FPs denoted by $\{z_1^\ast, z_2^\ast, \phi_1^\ast, \phi_2^\ast\}$ reveals transition between the steady states which we discuss below.

The symmetry broken ground state FP-I: $\{z_{1,2}^0 = 0, \phi_{1,2}^0 \}$ with energy density $E = -2$ becomes unstable across the critical line $U = V - 1$ (dotted line in Fig.1(a)) and bifurcates to the symmetry broken ground states with antiferromagnetic ordering, denoted by FP-III: $\{z_1^* = -z_2^* = \pm \sqrt{1/(U-V)^2}, \phi_{1,2}^* = 0\}$ with $E = 1/(U-V) + U-V$. This bifurcation corresponds to the quantum phase transition (QPT) of the ground state. Similarly, the highest excited state FP-II: $\{z_{1,2}^0 = 0, \phi_{1,2}^0 = \pi\}$ with $E = 2$ becomes unstable across the line $U = -V + 1$ (dashed line in Fig.1(a)) and undergoes a dynamical transition (DT), giving rise to symmetry broken ferromagnetic states, FP-IVA: $\{z_{1,2}^0 = \pm \sqrt{1/(U+V)^2}, \phi_{1,2}^0 = \pi\}$ with $E = 1/(U+V) + U+V$. In the context of BJJ, the ferromagnetic and antiferromagnetic ordering corresponds to equal but same and opposite imbalance of the two species, respectively. While FP-IVA represents a self-trapped state [5, 10, 11] with equal imbalance of both the species. Interestingly, another symmetry broken antiferromagnetic state FP-IVB emerges dynamically from FP-II, which is discussed in the supplementary material [67].

Apart from the above, there are two important class of steady states which include FP-V: $\{z_i^* = z_i, \phi_i^0 = 0, \phi_i^0 = \pi\}$ (with $E = 0$). We call these states as ‘$\pi$0-mode’, describing the inter-species phase difference $\pi$, which is represented by an angle between the two spins, see Fig.1(b). It remains stable in the region $V < U < \sqrt{V^2 + 1}$ (magenta region in Fig.1(a)) and becomes unstable across the line $U = \sqrt{V^2 + 1}$ (dash-dotted line), bifurcating to ‘asymmetric self trapped’ (AST) states, denoted by FP-VI: $\{|z_1^*| < |z_2^*| \neq 0, \phi_1^0 = 0, \phi_2^0 = \pi\}$ with unequal imbalance $|z_1| \neq |z_2|$ and $E > 0$. The spin orientations of FPs are summarized in Fig.1(b). Next we ask, how such rich steady state structure and the underlying collective dynamics influence the overall ergodicity of the system and what is the signature of emergence of such classicality in quantum dynamics?

*From classical to quantum ergodicity.* In order to quantify the degree of chaos, we compute the Lyapunov exponent (LE) [68, 69] and obtain its mean value $\bar{\Lambda}_L$ averaged over an ensemble of phase space points. The overall chaotic behavior in $U$-$V$ plane is displayed by $\bar{\Lambda}_L$ as a colorscale plot in Fig.1(c).

In the quantum domain, the signature of chaos is studied from the spectral statistics [70] by diagonalizing the Hamiltonian in Eq.2. We sort the eigenvalues $E_n$ belonging to a particular symmetry sector of the Hamiltonian [67], and compute the average level spacing ratio, namely, $\langle r \rangle = \langle \min(\delta_1, \delta_2, \delta_{n+1})/\max(\delta_1, \delta_{n+1}) \rangle$ [71], where $\delta_n = E_{n+1} - E_n$. Fig.1(d) portrays chaoticity at the quantum level in terms of $\langle r \rangle$ in the $U$-$V$ plane. In the clas-
sically regular regime with $\Lambda_L \sim 0$, the level spacing $\delta_n$ follows Poisson statistics [72] and $\langle r \rangle \sim 0.386$ [73]. As the degree of chaoticity increases, $\langle r \rangle$ deviates from Poisson value and in the completely chaotic regime, the distribution of $\delta_n$ approaches Wigner-Surmise [74] with $\langle r \rangle \sim 0.529$ [73].

Remarkably, the map of dynamical chaos based on $\Lambda_L$ in Fig.1(c) retains its fingerprints at the quantum level obtained from $\langle r \rangle$ in Fig.1(d). As expected, BJJ exhibits regular dynamics for weak interactions, whereas with increasing $V$, a crossover to chaos occurs across the critical line $V = U + 1$. Interestingly, the stability of $\pi_0$-mode has a dramatic impact on the overall ergodicity of the BJJ, which is evident from comparatively lower values of $\Lambda_L$ and $\langle r \rangle$, as shown in Fig.1(c,d). With increasing $U$, a mixed phase space behavior is observed above the region of stability of $\pi_0$-mode (see Fig.1(c)), where small regular islands form within the chaotic sea.

**Energy dependent ergodicity**.- We also investigate the ergodic behavior of different eigenstates $|\psi_n\rangle$ across the energy band from relative entanglement entropy (EE) $S_{en}/S_{\text{max}}$, where $S_{en} = -\text{Tr}(\hat{\rho}_S \text{log} \hat{\rho}_S)$ is computed from the reduced density matrix $\hat{\rho}_S = \text{Tr}_\theta |\psi_n\rangle \langle\psi_n|$, obtained by tracing out the other spin ($S \neq S'$). The degree of ergodicity is maximum at the center of the energy band with $E \approx 0$ compared to the band edges, indicating an energy dependent ergodic behavior (see Fig.1(e)). The maximum value of EE corresponding to a completely random state is given by $S_{\text{max}} = \log(2S + 1) - 1/2$ [75]. In the fully chaotic regime, EE approaches to its maximum limit ($S_{en} \simeq S_{\text{max}}$) at the the band center.

To analyze such dynamical route to the ergodic behavior more carefully, we first plot the Poincaré sections [68, 69] at the $z_2 = 0$ plane for different energies, shown in Fig.2(a,b). We study the quantum signature of such mixed phase space behavior by time evolution of the initial coherent states $|\psi_z\rangle = |z_1, \phi_1\rangle \otimes |z_2, \phi_2\rangle$ [76], representing the phase space points with $z_2 = 0$ and fixed classical energy densities. To probe the ergodic behaviour, we compute the time averaged deviation of EE $\Delta S_{en} = |\bar{S}_{en} - S_{\text{max}}|/S_{\text{max}}$ from its maximum limit, where the time averaging is done towards the end of the evolution. As evident from Fig.2(c,d), the regular regions of phase portrait give lower degree of ergodicity with higher values of $\Delta S_{en}$. Moreover, both the analysis support the energy dependent ergodic behavior.

To study the signature of classicality in quantum dynamics, we focus our discussion on the $\pi_0$-mode. We time evolve the initial coherent state $|\pi_+\rangle = \frac{1}{\sqrt{2}}(|0, 0\rangle \otimes [0, \pi\rangle + |0, \pi\rangle \otimes |0, 0\rangle$) [54] describing the $\pi_0$-mode for sufficiently long time and study the ‘entanglement spectrum (ES)’ of the final state $|\psi(t)\rangle$. The ES represents the eigenvalues ($\lambda_n$) of the reduced density matrix $\hat{\rho}_S = \sum_\alpha \lambda_\alpha |\alpha_S\rangle \langle\alpha_S|$ obtained from the Schmidt decomposition of $|\psi\rangle = \sum_\alpha \sqrt{\lambda_\alpha} |\alpha_S\rangle \otimes |\alpha_S\rangle$. In the weak interaction regime, for the stable $\pi_0$-mode, only a few eigenvalues are significantly larger compared to others with a gap $\Delta \lambda$. Such structure of ES justifies the validity of product state in the weak coupling regime, capturing the classical dynamical behavior. In contrast, the ES of an arbitrary ergodic state is extended and resembles a distribution corresponding to Random matrix theory [64], without any significant gap $\Delta \lambda$. To investigate the dynamical signature of stability of the $\pi_0$-mode, we compute the time averaged gap $\Delta \lambda$ for the final state $|\psi(t)\rangle$, with varying the inter-species interaction $V$ and compare it with the corresponding classical instability exponent $\Lambda_1$ obtained from stability analysis [67]. As seen from Fig.3(a), $\Delta \lambda$ decays with increasing $V$ in the stable regime and a dip appears at the point of instability of the $\pi_0$-mode. Even after the instability of the $\pi_0$-mode, a few significantly large eigenvalues with a gap $\Delta \lambda$ still persist in the ES (see Fig.3(a)), which retain the memory of the $\pi_0$-mode, leading to the formation of quantum scars. However, such behavior in the ES with gap $\Delta \lambda$ vanishes, as the instability exponent grows rapidly.

**Quantum scars.**- In the unstable regime of the $\pi_0$-mode, we identify the scarred eigenstates $|\psi_n\rangle$ from a significant overlap with the coherent state $|\pi_+\rangle$ representing the $\pi_0$-mode, $|\langle \psi_n | \pi_+ \rangle|^2 \gg 1/N$ [38, 53, 54]. To illustrate the classicality of such scarred states, we construct a truncated reduced density matrix $\hat{\rho}_S^\beta$ corresponding to a few large eigenvalues in ES and compute the Husimi distribution $Q = \frac{1}{2\pi} \langle \theta, \phi | \hat{\rho}_S^\beta | \theta, \phi \rangle$. As evident from the inset of Fig.3(b), the Husimi distribution shows localization of phase space density around the classical $\pi_0$-mode,
indicating the scarring phenomena. Apart from the π0-mode, we also analyze the scars of periodic orbits in the most ergodic regime near \( E \approx 0 \). From EOM, we identify two classes of dynamics satisfying the condition, Class-I: \( \{z_1 = -z_2, \phi_1 = -\phi_2\} \) and Class-II: \( \{z_1 = z_2, \phi_1 = \phi_2\} \), for which the dynamics is restricted in the respective sub-regions of the available phase space, containing two types of periodic orbits [54]. However, within certain regime of interactions, such orbits become unstable under small initial perturbations, violating the respective classes. In the ergodic regime, both types of orbits become unstable with increasing coupling \( V \) and the Shannon entropy of most of the states approach the ergodic limit, however a few deviate from it, bearing the scars of the unstable orbits [55]. From stability analysis, we obtained a region in the parameter space, where only the orbits corresponding to class-I become unstable. Interestingly, in this region, among the deviated states, the eigenstates bearing the scars of unstable orbits coexist with those containing the image of the stable orbits, as evident from their respective Husimi distributions shown in Fig.4(c,d). Moreover, the scarred eigenstates exhibit a few significantly large eigenvalues separated from the rest by a gap \( \Delta \lambda \) (see Fig.4(f)) in the ES, retaining the memory of the classical orbits in the phase space. To confirm the scarring due to the periodic orbits, we compute the Fourier transform of the auto-correlation function \( A(t) = \sum_{a=x,y,z} \langle \hat{S}_{1a}(t) \hat{S}_{1a} \rangle \) evaluated for such scarred eigenstates, which exhibits a sharp peak at the frequency of the corresponding orbits, which can be obtained analytically [67]. In addition, we have also identified the scar of symmetry broken antiferromagnetic state in the dynamically unstable regime [67].

Moreover, the saturation value of \( A(t) \) averaged after time evolution of an ensemble of initial coherent states \( |\psi_c\rangle \) with fixed energy densities \( E \) can be used to probe the energy dependent non ergodic behavior, as already quantified from EE, shown in Fig.1(e). As evident from Fig.5, the saturation value of \( A(t) \) vanishes for states in the most ergodic region at the center of the band, whereas it saturates to a finite value for the states near the band edges.

For BJJ, the phase coherence of the atoms between the two sites [7, 78] can also serve as a measure for degree of ergodicity across the energy band, which has relevance in experiment [6]. The enhancement of phase fluctuations occurs with increasing degree of ergodicity, which can be quantified from the width of the relative phase distribution \( (\Delta \phi)_{\text{sat}}^2/(\Delta \phi)_{\text{max}}^2 \) [67] after time evolution of the auto correlation function \( A(t) \) starting from initial coherent states \( |\psi_c\rangle \) for different energy densities \( E \). Variation of saturation value of relative phase fluctuations \( (\Delta \phi)_{\text{sat}}^2/(\Delta \phi)_{\text{max}}^2 \) and auto correlation \( A_{\text{sat}} \) across the energy band starting from \( |\psi_c\rangle \) with fixed \( E \). Parameters chosen: \( U = 0.8, V = 2.8, S = 30 \).
olution of $|\psi_c\rangle$ with energy density $E$, where $(\Delta \phi)^2_{\text{max}}$ corresponds to phase fluctuation for random state [7]. As seen from Fig.5, the phase fluctuations saturate to a maximum value in the most ergodic regime, leading to the destruction of the phase coherence in Josephson oscillations. However, a revival phenomena in the phase fluctuation is observed for initial states bearing the scar [67].

**Discussion.**—We have explored a rich variety of collective dynamics in a two-component BJJ, which unveils the classical route to ergodicity and quantum scarring phenomena in an interacting system. As a signature of classicality, we identify a few significantly large eigenvalues in the entanglement spectrum, which even exist for scarred eigenstates, retaining the memory of the unstable dynamics. The bipartite nature of BJJ makes it suitable to probe the above connection in terms of entanglement [23, 79], which further addresses the issue of investigating the athermal dynamics in a reduced Hilbert space of a generic many body system [38, 77]. We have discussed methods to experimentally detect the scars of the periodic orbits as well energy dependent degree of ergodicity from phase diffusion [6] and dynamics of auto-correlation. Present work not only sheds light on underlying classicality of MBQS leading to athermal behavior, but the model can also be realized in spin systems with application to information processing [80–82].

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Note that, the classical Hamiltonian $H_{cl}$ is scaled by $S$, and the corresponding classical energy $E$ is equivalent to the quantum mechanical energy density $E_n/S$, where $E_n$ are the quantum mechanical energy eigenvalues.
See the supplementary material for the derivation of the generalized coupled top model; details of different steady states and their stability analysis; spectral statistics and symmetries of the Hamiltonian; details of periodic orbits; scar of symmetry broken antiferromagnetic state FP-IVB; and phase diffusion dynamics.
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Energy dependent ergodicity and scarring phenomena in two component Bose-Josephson junction
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(Supplementary Material)

I. DERIVATION OF GENERALIZED COUPLED TOP MODEL

In order to derive the effective spin Hamiltonian in Eq.2 from Eq.1 in the main text, we define the spin operators with spin $S = N/2$ within the Schwinger-Boson representation as follows,

$$\hat{S}_{ix} = (a_{iR}^\dagger a_{iL} + a_{iL}^\dagger a_{iR})/2, \quad \hat{S}_{iz} = (\hat{n}_{iL} - \hat{n}_{iR})/2$$  \hspace{1cm} (S-1)

In this representation, the on-site and the inter-species interaction terms can be respectively written as,

$$U \frac{N}{2N} [\hat{n}_{iL}(\hat{n}_{iL} - 1) + \hat{n}_{iR}(\hat{n}_{iR} - 1)] = \frac{U}{4N} (N^2 + 4\hat{S}_{iz}^2)$$  \hspace{1cm} (S-2)

$$\frac{V}{N}(\hat{n}_{1L}\hat{n}_{2L} + \hat{n}_{1R}\hat{n}_{2R}) = \frac{V}{2N} (N^2 + 4\hat{S}_{1z}\hat{S}_{2z})$$  \hspace{1cm} (S-3)

Since each species have the equal number of population $\hat{n}_{iL} + \hat{n}_{iR} = N = 2S$. Neglecting the zero point energy term, we can write the Hamiltonian of the two-component BJJ as generalized version of the coupled top model [1],

$$\hat{H} = -\hat{S}_{1x} - \hat{S}_{2x} + \frac{U}{2S}(\hat{S}_{1z}^2 + \hat{S}_{2z}^2) + \frac{V}{S}\hat{S}_{1z}\hat{S}_{2z}$$  \hspace{1cm} (S-4)

II. STEADY STATES AND THEIR STABILITY ANALYSIS

Let us denote the steady states by the fixed points (FPs), $X^* = \{z_i^*, \phi_i^*, \phi_i^2\}$. They are obtained by setting $\dot{z}_i, \dot{\phi}_i = 0$ in the equations of motion (EOM) in Eq.S-4,

$$\dot{z}_i = -\sqrt{1-z_i^2} \sin \phi_i^* = 0 \hspace{1cm} (S-5a)$$

$$\dot{\phi}_i = \frac{z_i^* \cos \phi_i^*}{\sqrt{1-z_i^2}} + Uz_i^* + Vz_i^2 = 0 \hspace{1cm} (S-5b)$$

where, different species are denoted by $i \in \{1, 2\}$ with $\bar{i} \neq i$. The Various FPs obtained from the EOM are charted in Table I. Note that, in a single component BJJ one observes three types of steady states, namely, ‘0-mode’, ‘π-mode’ and self-trapped state [2]. Due to the inter-species interaction in the two-component BJJ, we observe a hybridization between the steady states and their features are discussed in the main text. In addition, in the later section, we will discuss the scarring phenomena associated with the steady state having equal but opposite imbalance in each of the components, which we call the antiferromagnetic ‘symmetric self-trapped (SST)’ state.

To investigate the stability of the steady states, we consider fluctuation around the FPs, namely, $X(t) = X^* + \delta X(t)$, where $\delta X(t) = \delta X e^{i\omega t}$. By putting this in the EOM followed by an expansion upto a linear order in $\delta X$, we obtain the fluctuation equations,

$$i\omega \delta z_i = \frac{z_i^* \sin \phi_i^*}{\sqrt{1-z_i^2}} \delta \phi_i - \sqrt{1-z_i^2} \cos \phi_i^* \delta \phi_i$$  \hspace{1cm} (S-6a)

$$i\omega \delta \phi_i = \frac{z_i^* \sin \phi_i^*}{\sqrt{1-z_i^2}} \delta \phi_i + \left[ \frac{\cos \phi_i^*}{(1-z_i^2)^{3/2}} + U \right] \delta z_i + V\delta z_i$$  \hspace{1cm} (S-6b)

The above set of equations in Eq.S-6 can also be represented as, $(\mathcal{J} - \omega I)\delta X = 0$, where $\mathcal{J}$ is the Jacobian matrix and $I$ is the identity. By solving this characteristic equation, we obtain eigenvalues $\omega$ as,

$$\omega^2 = \frac{A_1 + A_2}{2} \pm \sqrt{\left(\frac{A_1 - A_2}{2}\right)^2 + V^2 \sqrt{1-z_1^2} \sqrt{1-z_2^2} \cos \phi_1^* \cos \phi_2^*}$$  \hspace{1cm} (S-7)
where, the expression of $A_i$ is given by,

$$A_i = \left( \frac{\cos^2 \phi_i^*}{1 - z_i^*} + U \sqrt{1 - z_i^2 \cos \phi_i^*} \right) \quad (S-8)$$

Here, the eigenvalues $\omega$ can appear in complex conjugate pairs whose real part of $\omega$ represents the frequency of the small amplitude oscillations around the corresponding FP. The FP becomes unstable when the instability exponent is non-vanishing, namely, $\Lambda_i = |\text{Im}[\omega]| > 0$, indicating an exponential growth of the fluctuation $\delta X(t)$ over time.

### III. SYMMETRY CLASSIFICATIONS AND SPECTRAL STATISTICS

#### TABLE I: Chart of the fixed points (FPs) obtained from Eq.S-5 corresponding to different steady states. The fixed points FP-I, II and V exist for all values of $U$ and $V$, whereas FP-III, IV and VI exist in certain regions mentioned inside the brackets.

| $\phi_1^*, \phi_2^*$ | $z_1^*, z_2^*$ | FP-I: (00-mode) | FP-II: ($\pi\pi$-mode) | FP-III: (Symmetry broken Antiferromagnetic) | FP-IV: (Ferromagnetic SST) | FP-V: (F0-mode) | FP-VI: (Asymmetric self-trapped (AST)) |
|----------------------|----------------|----------------|-----------------------|---------------------------------|----------------------------|----------------|--------------------------------------|
| 0, 0                 | $z_1^* = z_2^* = 0$, $z_1^* = -z_2^* = \pm \sqrt{1 - \frac{1}{(U-V)^2}} \{\text{For } U - V < -1\}$ |                               |                       |                                 |                            |                |                                       |
| $\pi, \pi$           | $z_1^* = z_2^* = 0$, $z_1^* = z_2^* = \pm \sqrt{1 - \frac{1}{(U+V)^2}} \{\text{For } U + V > 1\}$, $z_1^* = -z_2^* = \pm \sqrt{1 - \frac{1}{(U-V)^2}} \{\text{For } U - V > 1\}$ | FP-III: (Symmetry broken Antiferromagnetic) | FP-IV: (Ferromagnetic SST) | FP-V: (F0-mode) | FP-VI: (Asymmetric self-trapped (AST)) |
| 0, $\pi$            | $z_1^* = z_2^* = 0$, $|z_1^*| < |z_2^*| \{\text{For } U > \sqrt{V^2+1}\}$ |                               |                       |                                 |                            |                |                                       |
| $\pi, 0$            | $z_1^* = z_2^* = 0$, $|z_1^*| > |z_2^*| \{\text{For } U > \sqrt{V^2+1}\}$ |                               |                       |                                 |                            |                |                                       |

FIG. S1: Spectral analysis of energy levels: (a) colormap of average level spacing $\langle r \rangle$ in the $U$-$V$ plane (see also Fig.1(d) in the main text). (b-d) Level spacing distribution $P(\delta)$ at the marked places are shown as histograms and are compared with the Poisson distribution $P_P(\delta)$ and with the Wigner-Surmise $P_{WD}(\delta)$, see the text for details. Here, we have set $S = 40$. 

(b) $P_P(\delta)$ with $U = 0.8, V = 0.5$

(c) $P_P(\delta)$ with $U = 2.2, V = 1.0$

(d) $P_P(\delta)$ with $U = 0.8, V = 2.4$
In order to study spectral statistics, we first compute the eigenspectrum of the effective spin Hamiltonian in Eq.2 by solving the following eigenvalue equation,
\[ \hat{H} |\psi_n\rangle = \mathcal{E}_n |\psi_n\rangle \]  
(S-9)
where, \( \mathcal{E}_n \) are the eigenvalues and \( |\psi_n\rangle \) are the associated eigenvectors. We note that the Hamiltonian \( \hat{H} \) has two symmetries—parity symmetry corresponding to the operator \( \hat{P} = e^{i\pi(S_1 + S_2)} \) and spin exchange symmetry \( (S_1 \leftrightarrow S_2) \) associated with operator \( \hat{O} \), that is constructed from \( \langle m_{1z}, m_{2z} | \hat{O} | m_{2z}, m_{1z} \rangle = 1 \), where \( m_{1z} \) are the quantum numbers of \( S_{1z} \). Both the operators, \( \hat{P} \) and \( \hat{O} \), have two eigenvalues, namely, \( \pm 1 \). Accordingly, we separate out the eigenmodes into different symmetry sectors as follows,
\[ \langle \psi_n | \hat{P} | \psi_n \rangle = \pm 1 \rightarrow \text{even(odd)}, \quad \langle \psi_n | \hat{O} | \psi_n \rangle = \pm 1 \rightarrow \text{even(odd)} \]  
(S-10)
We focus on the eigenmodes belonging to the even-even symmetry sector. In Fig. S1, we have plotted the distribution of the consecutive energy level spacings, \( \delta_n = \mathcal{E}_{n+1} - \mathcal{E}_n \), with mean and normalization set to one [3], for various interaction strengths. Notably, in the stability region of the \( \pi \)-mode where the underlying dynamics is regular, the level spacing distribution \( P(\delta) \) exhibits Poisson statistics, \( P_P(\delta) = e^{-\delta} \) (see Fig. S1b), as per the Berry-Tabor conjecture. While above QPT, \( P(\delta) \) agrees with the Wigner surmise, \( P_{WD}(\delta) = \pi/2 \delta e^{-\pi \delta^2/4} \), corresponding to Gaussian Orthogonal Ensemble (GOE) (see Fig. S1d) as a result of the onset of chaos in the phase space in accordance with the Bohigas-Giannoni-Schmit (BGS) conjecture. Interestingly, the phase space reveals a mixed behavior in the region of unstable ‘Asymmetric self trapped (AST)’ states (see Fig. 1(a) in the main text), which leads to an intermediate statistics of \( P(\delta) \), as depicted in Fig. S1c.

IV. CLASSICAL PERIODIC ORBITS

The spin exchange symmetry of the Hamiltonian in Eq. S-1 gives rise to integrable motion restricted on a subregion of the phase space, where the dynamical variables satisfy two classes of conditions: I. \( \{ z_1 = z_2, \phi_1 = -\phi_2 \} \) and II. \( \{ z_1 = z_2, \phi_1 = \phi_2 \} \), see also Ref. [1]. These conditions can equivalently be written as, I. \( \{ \phi_+ = 0; z_+ = 0 \} \) and II. \( \{ \phi_- = 0; z_- = 0 \} \), respectively, in terms of the new coordinates \( z_\pm = (z_1 \pm z_2)/2 \) and \( \phi_\pm = (\phi_1 \pm \phi_2)/2 \). Consequently, the dynamics of class I and II are governed by the EOM in terms of \( \{ z_-, \phi_- \} \) and \( \{ z_+, \phi_+ \} \), respectively,
\[ \dot{z}_\pm = -\sqrt{1 - z_\pm^2} \sin \phi_\pm, \quad \dot{\phi}_\pm = \frac{z_\pm}{\sqrt{1 - z_\pm^2}} \cos \phi_\pm + (U \pm V)z_\pm \]  
(S-11)
The solution of above equations can be written in terms of elliptic functions as,
\[ z_\pm(t) = C_\pm \text{cn} \left( \frac{C_\pm \mu_\pm(t + t_0), k_\pm}{2k_\pm} \right), \quad \cos(\phi_\pm(t)) = \frac{-E + \mu_\pm z_\pm^2(t)}{2\sqrt{1 - z_\pm^2(t)}} \]  
(S-12)
where, \( \text{cn} \) is the Jacobi elliptic function with elliptic modulus \( k_\pm \) and the constants are defined in the following way,
\[ C_\pm^2 = \frac{2}{\mu_\pm^2} \left[ \frac{E \mu_\pm}{2} - 1 + \Omega_\pm \right], k_\pm^2 = \frac{1}{2} \left[ 1 + \frac{E \mu_\pm/2 - 1}{\Omega_\pm} \right], t_0 = F \left( \frac{\cos^{-1}(z_\pm(0)/C_\pm), k_\pm}{\Omega_\pm^{1/2}} \right), \Omega_\pm = \sqrt{\mu_\pm^2 + 1 - E \mu_\pm} \]
where, \( \mu_\pm = (U \pm V) \) correspond to class-II and I respectively, and \( F(\phi, k_\pm) = \int_0^\phi d(1 - k_\pm^2 \sin^2 x)^{-1/2} \) is the incomplete elliptic integral of first kind. For repulsive interactions, i.e. \( U > 0 \) and \( V > 0 \), the dynamics corresponding to the class-II describes an effective antiferromagnetic Lipkin-Meshkov-Glick (LMG) model [4]. However, the dynamics of class-I can represent ferromagnetic or antiferromagnetic LMG model for \( U - V < 0 \) and \( U - V > 0 \), respectively. Interestingly, the symmetry unbroken state FP-I (FP-II) remains stable on the dynamical class-I (class-II), however it becomes unstable in presence of small fluctuation violating the conditions of the respective classes. There are periodic orbits forming around these fixed points with different energies \( E \), which can become unstable depending on the strength of the interactions. The time period of the orbits with energy \( E = 0 \) belonging to the two classes is given by,
\[ T = \frac{4K(k_\pm)}{(1 + \mu_\pm^2)^{1/4}} \]  
(S-13)
where \( K(k_\pm) = F(\pi/2, k_\pm) \). The stability analysis of such periodic orbits is performed by using the method of Monodromy matrix described in [5].
V. SCAR OF ANTIFERROMAGNETIC SYMMETRIC SELF-TRAPPED STATE

As mentioned earlier, there exists a pair of antiferromagnetic symmetric self-trapped (SST) steady state FP-IVB in the interaction regime $U \geq V + 1$ with energy $E = \frac{1}{(U-V)} + U - V$ (see Fig.S2(a)). Since the symmetry unbroken state FP-II (discussed in the main text) remains stable in the dynamical class-I (as mentioned in the previous section), it undergoes a pitchfork bifurcation at $U = V + 1$, giving rise to FP-IVB. Even though this antiferromagnetic state is stable only in class-I, it looses stability in presence of small fluctuations violating the respective class when $(U - V)^3 \leq U + V$, leading to the formation of scars. Such scarring phenomena can be analyzed quantum mechanically by the method described in the main text. We identify the corresponding scarred eigenstate $|\psi_n\rangle$ from the maximum overlap $|\langle \psi_n | \psi_c \rangle|^2 \gg 1/N$, as shown in Fig.S2(b). From the analysis of the entanglement spectrum (see the discussion in main text) for such scarred states, we find few large eigenvalues of the corresponding reduced density matrix, which are separated from the rest with a significant gap $\Delta \lambda$ (see inset of Fig.S2(b)), retaining their underlying classicality even when they become unstable. As evident from the Fig.S2(c), the Husimi distribution of the scarred state shows localization of phase space density around the point of the antiferromagnetic SST state, indicating the scarring phenomena. The quantum scar of this state can be realized experimentally in the setup of ultra-cold atoms.

![Fig. S2: Quantum scars of antiferromagnetic symmetric self-trapped state (FP-IVB): (a) Phase diagram in the $U$-$V$ plane showing the presence of FP-IVB in the colored region along with the schematic of its spin configuration. The plain (+ shaded) region indicates the region of stability (instability). (b) Overlap $|\langle \psi_c | \psi_n \rangle|^2$ of the scarred eigenstates $|\psi_n\rangle$ with the coherent state $|\psi_c\rangle$ of the corresponding FP. Inset shows the entanglement spectrum for the eigenstate with the maximum overlap marked by a red square. (c) Husimi distribution for the eigenstate with maximum overlap marked by red square in (a), plotted on the Bloch sphere.](image)

VI. PHASE DIFFUSION AND COHERENCE DYNAMICS

The phase coherence between the two sites of BJJ signifies the wave nature of the macroscopic condensate. The ergodic behavior can be characterized by studying the phase coherence from non-equilibrium dynamics. The relative phase between the two wells can be calculated by constructing an orthonormal basis of $2S+1$ phase states as following [6, 7],

$$|\phi_m\rangle = \frac{1}{\sqrt{2S+1}} \sum_{n=-S}^{S} \exp(in\phi_m)|n\rangle$$

(S-14)

with $\phi_m = \phi_0 + 2\pi m/(2S + 1)$, where $m$ is an integer $m \in [0, 2S]$ and $\phi_m \in [\pi, \pi]$. The phase distribution for a state $|\psi\rangle$ is given by $p(\phi_m) = \langle |\phi_m| \psi \rangle^2$ with $\sum_m p(\phi_m) = 1$. To study the phase diffusion dynamics, we evolve an initial coherent state with energy density $E$ and analyze its phase distribution corresponding to one of the bosonic component at different times. It is expected, the phase distribution for the states in the ergodic regime (center of the energy band with $E \approx 0$) will become flat quickly, indicating the loss of phase coherence. On the other hand, for the states in the non-ergodic regime (edge of the energy band), the phase distribution remains mostly localized and spreads comparatively less, indicating the retention of phase coherence.

To quantify the degree of ergodicity, we study the dynamics of phase fluctuations $(\Delta \phi)^2 = \sum_m (\phi_m - \langle \phi \rangle)^2 p(\phi_m)$ for initial states at different energy densities $E$. As shown in Fig.S3(a,b), starting from an initial state close to
the band edge, the growth rate of $(\Delta \phi)^2$ is small and increases as we approach the band center ($E \approx 0$), see also Fig.1(e) in the main text. In this regime, the phase fluctuation grows rapidly and saturates close to its maximal value $(\Delta \phi)^2_{\text{max}} \approx \pi^2/3$ corresponding to a random state [6], indicating complete loss of phase coherence. In addition, such behavior can also be quantified from $\langle \cos \phi \rangle = \sum m p(\phi_m) \cos \phi$, where a (non) zero value signifies the loss (persistence) of the phase coherence. Interestingly, the phase coherence factor $\langle \cos \phi \rangle$ exhibits a revival phenomena for the unstable $\pi 0$-mode during the time evolution of the corresponding coherent state $|\pi_+\rangle$ (mentioned in main text), capturing the scarring behavior. In contrast, for any other arbitrary state with same energy, $\langle \cos \phi \rangle$ decays to zero, signalling the loss of phase coherence. The energy dependent ergodicity can be probed and detection of quantum scarring of $\pi 0$-mode can be tested using relevant experiments [8].

![FIG. S3: Phase diffusion dynamics in two component BJJ: Time evolution of the relative phase fluctuations $(\Delta \phi)^2(t)/(\Delta \phi)^2_{\text{max}}$ (where $(\Delta \phi)^2_{\text{max}} \approx \pi^2/3$) starting from an arbitrary initial coherent state with energy density (a) $E = -1.9$ (non-ergodic) and (b) $E = 0.0$ (ergodic) for $V = 2.8$, indicating the energy dependent ergodic behavior. The horizontal dashed line indicates the maximum value $(\Delta \phi)^2/(\Delta \phi)^2_{\text{max}} \approx 1$. The insets show the evolution of the corresponding phase distribution at different times. Dynamics of the phase coherence factor $\langle \cos \phi \rangle$ for the scarred state of $\pi 0$-mode (dashed line) at $V = 1.2$ exhibiting revival phenomena in contrast to an arbitrary ergodic state (solid line), which decays to zero, indicating loss of phase coherence. Parameters chosen: $S = 30, U = 0.8$.]

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