Does the nontrivially deformed field-antifield formalism exist?

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Abstract

We reformulate the Lagrange deformed field-antifield BV -formalism suggested, in terms of the general Euler vector field $N$ generated by the antisymplectic potential. That $N$ generalizes, in a natural anticanonically-invariant manner, the usual power-counting operator. We provide for the "usual" gauge-fixing mechanism as applied to the deformed BV -formalism.

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1 Introduction

In the field-antifield formalism \[1, 2, 3\], the concept of deformations based on a nilpotent higher-order operator $\Delta$ was developed in a series of articles \[4, 5, 6, 7, 8, 9, 10, 11, 12\]. Such deformations usually modify the Jacobi identity with BRST exact terms. In contrast to that, one can, with no assumptions a'priory on underlying $\Delta$ operator, consider "local" deformations of the antibracket with a Boson deformation parameter, such that the Jacobi identity holds strongly.

Historically, the deformed antibracket has been studied in the articles \[13, 14, 15, 16\]. In \[17\], the deformed $\Delta$-operator has been found that differentiates the deformed antibracket, and the first attempt has been made to understand actually possible role of the deformed antibracket and $\Delta$-operator in the construction of the $W - X$ version \[18, 19, 4, 20, 21, 8, 22, 23\] of the Lagrange deformed field-antifield BV -formalism \[17\]. If one believes that the deformed BV -formalism still describes gauge-invariant field systems, then there appears a difficult problem of how the deformation can do coexist with the usual gauge-fixing mechanism. Or, in other words, if that is possible to provide for a proper solution to the deformed classical/quantum master equation. In the present article, we will try to pay attention enough to seek for a possible way to resolve the mentioned problem.

In principle, our present consideration is based essentially on the logic and mathematics of the article \[17\] of Batalin and Bering. Regrettably, these authors did not accomplish their task of construction of the nontrivially deformed field-antifield formalism based on the deformed antibracket and $\Delta$ operator. The main idea was to extend the original antisymplectic phase space with a single extra field-antifield pair just controlling the scale of deformation. Then, one defines a trivial deformation within the extended phase space, and then one should reduce effectively the scale of trivial deformation in such a way that the latter becomes nontrivial in a consistent manner. That idea seems promising, the same as before. However, there remains a difficult unresolved problem of consistent coexistence between the properness principle and non-triviality of the deformation. We would like to try again to attack that problem.

2 Extended $\Delta$-operator

We begin with the standard odd Laplacian operator,

$$\Delta = \frac{1}{2}(-1)^{A_A} \partial A E^{AB} \partial B, \quad \partial A = \frac{\partial}{\partial Z^A}, \quad \varepsilon(\Delta) = 1, \quad \Delta^2 = 0, \quad (2.1)$$

where $Z^A$, $\varepsilon_A = \varepsilon(Z^A)$, are original Darboux coordinates of the field-antifield formalism, and $E^{AB}$ is a constant invertible antisymplectic metric with the usual statistics and dual-antisymmetry properties,

$$\varepsilon(E^{AB}) = \varepsilon_A + \varepsilon_B + 1, \quad E^{AB} = -E^{BA}(-1)^{(\varepsilon_A + 1)(\varepsilon_B + 1)}. \quad (2.2)$$
In what follows below, typical functions, depending only on the $Z^A$ variables of the original sector, will be denoted in small letters. Now, let us extend the original $Z$-sector by including two new variables, a Boson $t$ and a Fermion $\theta$, to extend the original odd Laplacian, to become \cite{17} (see also Appendix A)

$$\Delta_\tau = t^2 \Delta + N_\tau \partial_\theta, \quad \partial_\theta = \frac{\partial}{\partial \theta}, \quad \epsilon(\Delta_\tau) = 1, \quad \Delta_\tau^2 = 0,$$

(2.3)

where

$$N_\tau = N + t \partial_t, \quad N = N^A \partial_A.$$

(2.4)

In what follows below, typical functions depending on the full set of variables, $Z^A, t, \theta$, will be denoted in capital letters. Nilpotency of $\Delta_\tau$ requires

$$[\Delta, N] = 2\Delta,$$

(2.5)

or, in more detail,

$$[\Delta, N] = [\Delta, N^A] \partial_A = [(\Delta N^A) + \text{ad}(N^A) (-1)^{\varepsilon_A}] \partial_A =$$

$$= (\Delta N^A) \partial_A + \frac{1}{2} [(N^A, Z^B) - (A \leftrightarrow B) (-1)^{\varepsilon_A+1} (-1)^{\varepsilon_B+1}] (-1)^{\varepsilon_A} \partial_B \partial_A =$$

$$= 2\Delta = E^{AB} (-1)^{\varepsilon_A} \partial_B \partial_A,$$

(2.6)

which, in turn, implies

$$\Delta N^A = 0,$$

(2.7)

and

$$E^{AB} = \frac{1}{2} [(N^A, Z^B) - (A \leftrightarrow B) (-1)^{\varepsilon_A+1} (-1)^{\varepsilon_B+1}].$$

(2.8)

Here and below the notation

$$\text{ad}(F)(...) = (F, (...))$$

(2.9)

for the left adjoint of the antibracket is used.

It follows immediately from (2.8) that

$$(N^A, N^B) = \frac{1}{2} \left[N^A \overset{\leftarrow}{\partial} C(N^C, N^B) - (A \leftrightarrow B) (-1)^{\varepsilon_A+1} (-1)^{\varepsilon_B+1}\right].$$

(2.10)

Here, in (2.6), (2.8), (2.9), (2.10), the usual antibracket, generated by the operator $\Delta$, is used

$$(f, g) = (-1)^{\varepsilon(f)} [\Delta, f], g \cdot 1 = f \overset{\leftarrow}{\partial} A E^{AB} \overset{\rightarrow}{\partial}_{BG}. $$

(2.11)
Thus, the coefficients $N^A$ of the vector field $N$ should satisfy the equations (2.7), (2.8). Of course, the simplest solution is obvious,

$$N^A = Z^A, \quad N = Z^A \partial_A.$$  

(2.12)

That was exactly the simplest ansatz used in the article [17] from the very beginning. Here, in the present article, we do not restrict ourselves with any a-priory choice of a special solution to the equations (2.7), (2.8). Only these equations themselves will be used in our further reasoning, nothing else. It can be shown that the general solution to the equation (2.8) is

$$N^A = Z^A + 2(F, Z^A), \quad \varepsilon(F) = 1$$

(2.13)

with $F$ being arbitrary Fermion. In that case, the equations (2.7), (2.8) are satisfied as follows

$$E^{AB} = \frac{1}{2} \left[2E^{AB} + 2(F, E^{AB})\right] = E^{AB},$$

(2.14)

$$\Delta N^A = 2(\Delta F, Z^A) = 0,$$

(2.15)

$$\Delta F = \text{const}(Z).$$

(2.16)

Thus, we find

$$N = Z^A \partial_A + 2 \text{ad}(F).$$

(2.17)

One can rewrite the equation (2.8) in its natural form

$$\delta^A_D - \frac{1}{2} N^A \partial_D = \frac{1}{2} E^{AB}(\partial_B N^C) E_{CD},$$

(2.18)

where $E_{AB}$ is the inverse to $E^{AB}$,

$$E^{AB} E_{BC} = \delta^A_C.$$  

(2.19)

Now, the super trace of (2.18) yields

$$\text{sTr } I = \delta^A_A (-1)^{\varepsilon_A} = 0,$$

(2.20)

that is fulfilled identically due to equal number of Bosons and Fermions among the variables $Z^A$. The supertrace imposes no restrictions for the divergence of vector field $N$,

$$\text{div } N = \partial_A N^A (-1)^{\varepsilon_A},$$

(2.21)

Notice that $E^{AB}$ and $E_{AB}$ in the right-hand side of (2.18) do enter in the form of a similarity transformation (that is just the meaning of the naturalness of (2.18)) and, therefore, they are
canceled each other when taking the supertrace or superdeterminant. Notice also, that (2.18) is a "bridge" between the initial equation (2.8) and its "dual" form,

\[ E_{AB} = \frac{1}{2} \left[ E_{AC}(N^C \partial_B) - (A \leftrightarrow B)(-1)^{\varepsilon_A \varepsilon_B} \right]. \]

In turn, by substituting

\[ N^A = -2E^{AB}V_B = 2V_BE^{BA}, \]

\[ \varepsilon(V_A) = \varepsilon_A + 1, \]

into (2.22), the latter takes the form

\[ E_{AB} = \partial_A V_B - \partial_B V_A(-1)^{\varepsilon_A \varepsilon_B}, \]

which tells us that \( V_A \) is just the antisymplectic potential, generating a constant invertible metric in its covariant components \( E_{AB} \). Thereby, one realizes that the arbitrariness in \( N^A \) is generated by the natural geometric arbitrariness in the choice of the antisymplectic potential. It can be shown that the general solution to the equation (2.25) is

\[ V_B = \frac{1}{2} Z^A E_{AB} + \partial_B F, \]

with \( F \) being arbitrary Fermion, so that (2.26) is consistent with (2.25), (2.23). Of course, it follows from (2.7), (2.23), that the antisymplectic potential \( V_A \) should satisfy the condition

\[ \Delta V_A = 0, \]

which is consistent with (2.16). The relation (2.25) is invariant under the shift,

\[ V_A = V_A' + \partial_A F', \quad \varepsilon(F') = 1. \]

On the other hand, we have,

\[ \Delta V_A = \Delta V_A' + \partial_A \Delta F'(-1)^{\varepsilon_A} = 0, \]

\[ N^A = N'^A - 2E^{AB}\partial_B F', \]

\[ \Delta N^A = \Delta N'^A + 2E^{AB}\partial_B \Delta F'(-1)^{\varepsilon_A} = 0, \]

\[ N = N' + 2 \text{ad}(F'), \]

\[ \text{div} N = \text{div} N' - 4\Delta F'. \]
So, if one chooses in (2.28) - (2.32) for the $F'$ to satisfy the relation
\[ \text{div } N' = \text{div } N + 4\Delta F' = 0, \] (2.34)
then the new value (2.34) of the div $N'$ is zero. Thereby, the new operator
\[ N' = N - 2\text{ad}(F') = -N^{\prime T}, \] (2.35)
is an antisymmetric one. The transposed operation is defined via
\[ \int[dZ](A^TF)G = (-1)^{\varepsilon(A)\varepsilon(F)} \int[dZ]F(AG). \] (2.36)
So far, the condition (2.34) seems to be the only restriction on $F'$. However, let us consider the commutator
\[ [\Delta, N'] = [\Delta, N - 2\text{ad}(F')] = 2\Delta - 2\text{ad}(\Delta F'). \] (2.37)
So, if we would like for the new operator $N'$ to maintain the relation
\[ [\Delta, N'] = 2\Delta, \] (2.38)
then there should be
\[ \Delta F' = \text{const}(Z). \] (2.39)
Due to the latter, it follows from (2.31)
\[ \Delta N^A = \Delta N'^A = 0. \] (2.40)
In turn, due to (2.39), it follows from (2.34) that
\[ \text{div } N = \text{const}(Z). \] (2.41)
Thus, we see that the deviation from zero allowed for div $N$ is not so arbitrary. That is because the condition (2.40) is rather restrictive. We see that the new antisymmetric $N'$ does maintain all the basic conditions (2.38), (2.40), provided the condition (2.41) holds. In what follows below, we do mean that our $N'$-operator is chosen just in its antisymmetrical form (2.35), "from the very beginning". For brevity, in all further formulae we omit the prime of $N'$.

There exists a crucially important consequence of (2.8), that the operator $(N - 2)$ does differentiate the antibracket,
\[ (N - 2)(f, g) = ((N - 2)f, g) + (f, (N - 2)g). \] (2.42)
That goes as follows,
\[ N(f, g) = (Nf, g) + (f, Ng) - f \frac{\partial}{\partial A} \left[ (N^A \frac{\partial}{\partial C})E^{CB} - (A \leftrightarrow B)(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} \right] \frac{\partial}{\partial B} g = \]
\[ = (Nf, g) + (f, Ng) - 2(f, g), \] (2.43)
which is equivalent to (2.42).
3 Extended antibracket

Now, let us consider the antibracket generated by the extended operator $\Delta_\tau$ \cite{17},

$$(F,G)_\tau = (-1)^{\varepsilon(F)}[[\Delta_\tau, F], G] \cdot 1 = t^2(F, G) + (N_\tau F)\partial_\theta G - F\overleftarrow{\partial_\theta N_\tau G}, \quad (3.1)$$

where $(F,G)$ is the (usual) antibracket \cite{211} in the original $Z^A$ -sector, although functions $F,G$ themselves, standing for $f,g$, respectively, do depend on $t,\theta$ as well. $N_\tau$ is defined in \cite{204}. Due to \cite{242}, the operator $(N_\tau - 2)$ does differentiate the usual antibracket $(F,G)$, as well. In fact, we will use the relation equivalent to that,

$$N_\tau(F,G) = (N_\tau F, G) + (F, N_\tau G) - 2(F,G). \quad (3.2)$$

One can state that the extended antibracket \cite{3.1} does satisfy the strong Jacobi identity, provided the usual antibracket has that property. In particular, one assumes that the strong Jacobi identity holds for any Boson $B$,

$$((B,B), B) = 0, \quad \varepsilon(B) = 0. \quad (3.3)$$

In its general form, the Jacobi identity can be reproduced from \cite{3.3} via the differential polarization procedure. To do this, one has to choose a specific form for $B$ \cite{244},

$$B = \sum_{i=1}^{3} m_in_i, \quad n_1 = F, \quad n_2 = G, \quad n_3 = H. \quad (3.4)$$

Then, the operator

$$\partial_1\partial_2\partial_3(-1)^{(\varepsilon_1+1)(\varepsilon_3+1)+\varepsilon_2}, \quad (3.5)$$

should be applied to \cite{3.3}, where $\partial_i$ are partial $m_i$ -derivatives, $\varepsilon_i$ are Grassmann parities,

$$\varepsilon_i = \varepsilon(n_i) = \varepsilon(m_i). \quad (3.6)$$

It is our task now, to prove that the extended antibracket \cite{3.1} satisfies the compact form of the strong Jacobi identities,

$$((B,B)_\tau, B)_\tau = 0, \quad \varepsilon(B) = 0, \quad (3.7)$$

provided similar compact form \cite{3.3} holds. We have

$$(B,B)_\tau = t^2(B,B) + 2(N_\tau B)(\partial_\theta B). \quad (3.8)$$

By substituting that in \cite{3.7}, one gets

$$((B,B)_\tau, B)_\tau = t^2(t^2(B,B) + 2(N_\tau B)(\partial_\theta B), B) + (N_\tau(t^2(B,B) + +2(N_\tau B)(\partial_\theta B)))\partial_\theta B - (\partial_\theta(t^2(B,B) + 2(N_\tau B)(\partial_\theta B)))\partial_\theta B, \quad (3.9)$$
where the relations (3.2), (3.3) will be used, together with

\[(\partial_B B)^2 = 0\]  (3.10)

Thus, the right-hand side of (3.9) takes the form

\[\begin{align*}
(2t^2(B, B) + 2t^2(N_\tau B, B) - 2t^2(B, B))(&\partial_B B) + 2t^2(N_\tau B)(\partial_B B, B) - \\
-2t^2(N_\tau B, B)(\partial_B B) - 2t^2(\partial_B B, B)(N_\tau B) + \\
2(N_\tau B)(N_\tau \partial_B B)(\partial_B B) - 2(\partial_B N_\tau B)(\partial_B B)(N_\tau B) = 0.
\end{align*}\]  (3.11)

Here, the first and third, second and fifth, fourth and sixth, seventh and eighth terms, compensate each other in every pair mentioned. Finally, the strong Jacobi identity (3.7) for the extended antibracket is proven.

4 Non-trivial deformation in the sector of original variables

In turn, let us study the role of the operator $N$ in construction of a non-trivially deformed antibracket in the original $Z^A$-sector. So, let $\kappa$ be a deformation parameter. Consider the operator

\[K = K(N) = \kappa(N - 2), \quad K(N + 2) = \kappa N.\]  (4.1)

We have

\[K(N + 2) - K(N) = 2\kappa,\]  (4.2)

\[K(N)(fg) = (K(N + 2)f)g + f(K(N)g) = (K(N)f)g + f(K(N + 2)g),\]  (4.3)

\[K(f, g) = (Kf, g) + (f, Kg).\]  (4.4)

By using the well-known Witten formula for the usual antibracket,

\[(f, g) = \Delta(fg)(-1)^{\varepsilon(f)} - [f(\Delta g) + (\Delta f)g(-1)^{\varepsilon(f)}],\]  (4.5)

we define the deformed antibracket by the relation [17],

\[(f, g)_* = \Delta(fg)(-1)^{\varepsilon(f)} - (1 - K)[f(\Delta_* g) + (\Delta_* f)g(-1)^{\varepsilon(f)}],\]  (4.6)

where

\[\Delta_* = \Delta(1 - K)^{-1} = (1 - K(N + 2))^{-1}\Delta.\]  (4.7)

By using (4.3), (4.5), the relation (4.6) can be rewritten in the form

\[(f, g)_* = (f, g) + (Kf)(\Delta_* g) + (\Delta_* f)(Kg)(-1)^{\varepsilon(f)}.\]  (4.8)
Usually, the deformation of the antibracket is defined by just that formula. It follows from (4.6)

\[ \Delta_\ast (f, g)_\ast = -\Delta \left[ f(\Delta_\ast g) + (\Delta_\ast f)g(-1)^{\varepsilon(f)} \right]. \]  

(4.9)

\[ ((\Delta_\ast f), g)_\ast = \Delta((\Delta_\ast f)g)(-1)^{\varepsilon(f)+1} - (1 - K)[(\Delta_\ast f)(\Delta_\ast g)], \]  

(4.10)

\[ (f, (\Delta_\ast g))_\ast = \Delta(f(\Delta_\ast g))(-1)^{\varepsilon(f)} - (1 - K)[(\Delta_\ast f)(\Delta_\ast g)](-1)^{\varepsilon(f)}, \]  

(4.11)

\[ ((\Delta_\ast f), g)_\ast - (f, (\Delta_\ast g))_\ast (-1)^{\varepsilon(f)} = \Delta \left[ -(\Delta_\ast f)g(-1)^{\varepsilon(f)} - f(\Delta_\ast g) \right] = \Delta_\ast (f, g)_\ast. \]  

(4.12)

The latter means that the deformed operator (4.7) does differentiate the deformed antibracket (4.6), or (4.8) [17].

Finally, one can state that the non-trivially deformed antibracket (4.6), or (4.8), does satisfy the strong Jacobi identity. Indeed, we have

\[ ((B, B)_\ast, B)_\ast = (B, B) + 2(KB)(\Delta_\ast B) + (K((B, B) + 2(KB)(\Delta_\ast B)))(\Delta_\ast B) - 2((\Delta_\ast B), B)_\ast (KB) = 2(KB)((\Delta_\ast B), B) - 2((KB), B)(\Delta_\ast B) + 2((KB)(\Delta_\ast B))(\Delta_\ast B) - 2((\Delta_\ast B), B)(KB) - 2(K\Delta_\ast B)(\Delta_\ast B)(KB) = 0. \]  

(4.13)

Here, in the right-hand side of the last equality, the first and fifth, the second and third, the fourth and sixth terms do compensate each other in every pair mentioned. Thus, the strong Jacobi identity for the non-trivially deformed antibracket (4.6), or (4.8), in the original \( Z^4 \)-sector is proven.

5 Trivial \( \tau \)-extended deformation

Now, we have to consider a trivial \( \tau \) extended deformation in the extended phase space including the variables \( t \) and \( \theta \). Let us introduce the operator

\[ K_\tau = \kappa N_\tau, \quad [K_\tau, \Delta_\tau] = 0, \]  

(5.1)

and define trivially deformed extended operator

\[ \Delta_\tau_\ast = \Delta_\tau (1 - K_\tau)^{-1}, \quad \Delta_\tau^2_\ast = 0. \]  

(5.2)

Now, introduce the operator

\[ T = 1 + \kappa \theta \Delta_\tau_\ast, \]  

(5.3)

\[ ^3 \text{This Section represents the main results and formulae of Ref. [17], related in general to trivial deformations in } \tau \text{-extended phase space.} \]
and its inverse

$$T^{-1} = 1 - \kappa \theta \Delta_{\tau}, \quad (5.4)$$

Then, one finds that

$$\Delta_{\tau^*} = T^{-1} \Delta_{\tau} T. \quad (5.5)$$

Also, it follows that the operator $T$ does satisfy the equations (see also Appendix D)

$$[\Delta_{\tau}, T] = \Delta_{\tau} TK_{\tau}, \quad [T, K_{\tau}] = 0. \quad (5.6)$$

Together with (2.3) and (5.1), these equations constitute what we call "T-algebra".

Next, define a trivially deformed extended antibracket,

$$(F, G)^{\tau*} = T^{-1}((TF), (TG)) -
= (F, G) + (K_{\tau} F)(\Delta_{\tau} G) + (\Delta_{\tau} F)(K_{\tau} G)(-1)^{\varepsilon(F)} -
= \Delta_{\tau}(FG)(-1)^{\varepsilon(F)} - (1 - K_{\tau}) \left[ F(\Delta_{\tau} G) + (\Delta_{\tau} F)G(-1)^{\varepsilon(F)} \right]. \quad (5.7)$$

The latter formula allows for a natural rewriting in terms of the $*$- modified double-commutator formula generalizing (2.11), (3.1),

$$(F, G)^{\tau*} = (-1)^{\varepsilon(F)}[[\Delta_{\tau}, (TF)_{\tau}], (TG)_{\tau}] \cdot 1, \quad (5.8)$$

where we have used (3.1), (5.5), and

$$(TF)_{\tau} = T^{-1}(TF)T, \quad (TG)_{\tau} = T^{-1}(TG)T. \quad (5.9)$$

Here in (5.9), on the right-hand sides, the operator $T$ in the middle factors applies only to the function standing to the right within the respective round bracket.

Notice that within the class of functions,

$$F = t^{-2}f, \quad G = t^{-2}g, \quad (5.10)$$

the trivially deformed $\tau$-extended operator (5.5) and antibracket (5.7) reduces, respectively, to the non-trivially deformed operator (4.7) and antibracket (4.6) in the original sector,

$$\Delta_{\tau^*}F = \Delta_{\tau} f, \quad (F, G)_{\tau^*} = t^{-2}(f, g)_{\tau}. \quad (5.11)$$

By construction, the trivially deformed extended antibracket (5.7) does satisfy the strong Jacobi identity. In turn, define a trivial associative and commutative star-product

$$(F \ast G) = T^{-1}((TF)(TG)) = FG - \kappa \theta (F, G)_{\tau^*}(-1)^{\varepsilon(F)}. \quad (5.12)$$
It is worthy to mention here that the operators (5.9) apply to a function as to yield the left
adjoint of the symbol multiplication (5.12),

\[(TF)_*G = F * G, \quad (TF)_* = F - \kappa \text{ad}_{\tau_*}(F)(-1)^{\varepsilon(F)}. \quad (5.13)\]

Due to (5.11), within the class of functions (5.10), the star-product (5.12) reduces as follows

\[F * G = (t^{-2}f)(t^{-2}g) - \kappa \theta t^{-2}(f, g)_*(-1)^{\varepsilon(f)}. \quad (5.14)\]

Then, with respect to the star-product (5.12), we have the trivially deformed extended Witten
formula \(^4\)

\[(F, G)_{\tau_*} = \Delta_{\tau_*}(F * G)(-1)^{\varepsilon(F)} - F * (\Delta_{\tau_*}G) - (\Delta_{\tau_*}F) * G(-1)^{\varepsilon(F)}, \quad (5.15)\]

the Leibnitz rule,

\[((F * G), H)_{\tau_*} = F * (G, H)_{\tau_*} + G * (F, H)_{\tau_*}(-1)^{\varepsilon(F)\varepsilon(G)}, \quad (5.16)\]

the Getzler identity \(^{[25]}\) providing for the absence of higher antibrackets in the BV -algebra

\[
\begin{align*}
\Delta_{\tau_*}(F * G * H) - \Delta_{\tau_*}(F * G) * H - F * \Delta_{\tau_*}(G * H)(-1)^{\varepsilon(F)} -
\Delta_{\tau_*}(F * H) * G(-1)^{\varepsilon(G)\varepsilon(H)} + (\Delta_{\tau_*}F) * G * H -
-F * (\Delta_{\tau_*}G) * H(-1)^{\varepsilon(F)} + F * G * (\Delta_{\tau_*}H)(-1)^{\varepsilon(F) + \varepsilon(G)} &= 0.
\end{align*}
\quad (5.17)
\]

The star exponential is defined as (see also Appendix C)

\[
\begin{align*}
\exp_*\{B\} &= 1 + B + \frac{1}{2!}B * B + \frac{1}{3!}B * B * B + ... =
= T^{-1}\exp\{(TB)\} = \exp\left\{B - \frac{1}{2}\kappa \theta(B, B)_{\tau_*}\right\}.
\end{align*}
\quad (5.18)
\]

The latter satisfies

\[
\begin{align*}
\exp_*\{-B\} * \exp_*\{B\} &= 1, \quad (5.19)\\
\exp_*\{-B\} * (\Delta_{\tau_*}\exp_*\{B\}) &= (\Delta_{\tau_*}B) + \frac{1}{2}(B, B)_{\tau_*}, \quad (5.20)\\
\exp_*\{B + B'\} &= \exp_*\{B\} * \exp_*\{B'\}, \quad (5.21)\\
\delta \exp_*\{B\} &= \exp_*\{B\} * \delta B, \quad \varepsilon(B) = \varepsilon(B') = 0. \quad (5.22)
\end{align*}
\]

The trivially deformed extended quantum master equation has the form

\[
\Delta_{\tau_*} \exp_* \left\{ \frac{i}{\hbar} W \right\} = 0, \quad (5.23)
\]
or, equivalently,
\[
\frac{1}{2} (W, W)_{\tau *} = i\hbar \Delta_{\tau *} W. \tag{5.24}
\]

One has to seek for a solution to that equation in the form
\[
W = \sum_{k=-2}^{\infty} W_{(k|0)} t^k + \theta \sum_{k=1}^{\infty} W_{(k|1)} t^k, \tag{5.25}
\]
where the component \(W_{(-2|0)} = S\) is identified with the classical non-trivially deformed proper action, (see also Appendix B)
\[
(S, S)_{\tau *} = 0. \tag{5.26}
\]

The detailed form of the equations for coefficients in (5.25), together with the corresponding formal techniques, can be found in [17].

The trivially deformed path integral with a measure \(d\mu\) in the extended phase space is defined as
\[
Z = \int d\mu \exp_{\kappa} \left\{ i \frac{\hbar}{\kappa} W \right\} \exp_{(-\kappa)} \left\{ i \frac{\hbar}{\kappa} X \right\} = \int d\mu \exp \left\{ i \frac{\hbar}{\kappa} A \right\}, \tag{5.27}
\]
where
\[
A = T(\kappa) W + T(-\kappa) X, \tag{5.28}
\]

\[
\Delta_{\tau^*}(\kappa) \left( \exp_{\kappa} \left\{ i \frac{\hbar}{\kappa} W \right\} \right) = 0, \quad \Delta_{\tau^*}(-\kappa) \left( \exp_{(-\kappa)} \left\{ i \frac{\hbar}{\kappa} X \right\} \right) = 0. \tag{5.29}
\]

Here \(X\) satisfies the same equation as \(W\) does, but with the formal replacement \(\kappa \rightarrow -\kappa\). This replacement just provides for right transposition properties when integrating by part.

We proceed in (5.27) with the following integration measure
\[
d\mu = t^{-1} dt d\theta d\lambda_{[dZ]} [d\lambda]. \tag{5.30}
\]

The transposed operator \(A^T\) of the operator \(A\) is defined via
\[
\int d\mu (A^T F) G = (-1)^{\varepsilon(A)\varepsilon(F)} \int d\mu F(A G). \tag{5.31}
\]

Our main transposed operators are
\[
\Delta^T = \Delta, \quad N^T = -N, \quad \Delta^T_\tau = \Delta_\tau, \quad N^T_\tau = -N_\tau, \quad \Delta_{\tau^*}^T = \Delta_{\tau^*}(-\kappa). \tag{5.32}
\]

Let us make in (5.27) the variation of the form
\[
\delta \exp_{(-\kappa)} \left\{ i \frac{\hbar}{\kappa} X \right\} = \Delta_{\tau^*}(-\kappa) \left( \exp_{(-\kappa)} \left\{ i \frac{\hbar}{\kappa} X \right\} *_{(-\kappa)} \delta \Psi \right), \tag{5.33}
\]
with arbitrary infinitesimal Fermion \( \delta \Psi \). The (5.33) is consistent with (5.22) due to the quantum master equation for the \( X \). Then, we deduce that the path integral is independent of the gauge-fixing action \( X \),

\[
\delta_X \mathcal{Z} = \int d\mu \exp s(\kappa) \left\{ \frac{i}{\hbar} W \right\} \delta \exp s(-\kappa) \left\{ \frac{i}{\hbar} X \right\} = \\
= \int d\mu \exp s(\kappa) \left\{ \frac{i}{\hbar} W \right\} \Delta_{\tau s(-\kappa)} \left( \exp s(-\kappa) \left\{ \frac{i}{\hbar} X \right\} \ast_{(-\kappa)} \delta \Psi \right) = \\
= \int d\mu \left( \Delta_{\tau s(\kappa)} \exp s(\kappa) \right) \left( \exp s(-\kappa) \left\{ \frac{i}{\hbar} X \right\} \ast_{(-\kappa)} \delta \Psi \right) = 0. \quad (5.34)
\]

It follows from (5.5), (5.18), (5.23) that

\[
\Delta_{\tau} \exp \left\{ \frac{i}{\hbar} T(\kappa) W \right\} = 0. \quad (5.35)
\]

For similar reasons, it follows that

\[
\Delta_{\tau} \exp \left\{ \frac{i}{\hbar} T(-\kappa) X \right\} = 0. \quad (5.36)
\]

These equations tell us that in terms of the barred actions,

\[
\bar{W} = T(\kappa) W, \quad \bar{X} = T(-\kappa) X, \quad (5.37)
\]

the path integral (5.27) is just the standard \( W - X \) version of the field-antifield formalism. From the latter point of view, it is well-known that the path integral (5.27) is stable under the gauge variation

\[
\delta X = \sigma_{\tau}(\bar{X}) \bar{\Psi}, \quad (5.38)
\]

where

\[
\sigma_{\tau}(\bar{X}) = -i\hbar \Delta_{\tau} + \text{ad}_{\tau}(\bar{X}), \quad (5.39)
\]

so that

\[
\delta X = T^{-1}_{(-\kappa)} \delta \bar{X}. \quad (5.40)
\]

If one identifies \( \bar{\Psi} = T(-\kappa) \Psi \), then

\[
\delta X = \sigma_{\tau s(-\kappa)}(X) \Psi, \quad (5.41)
\]

where

\[
\sigma_{\tau s(-\kappa)}(X) = -i\hbar \Delta_{\tau s(-\kappa)} + \text{ad}_{\tau s(-\kappa)}(X), \quad (5.42)
\]

which is exactly the variation of \( X \) generated by (5.33).
Gauge-fixing in the classical extended non-deformed master equation

Here we study, if the standard gauge-fixing procedure is capable to eliminate the extra variable $t$, as applied to the classical $\tau$-extended non-deformed master equation,

$$(S, S) = t^2(S, S) + 2(N_\tau S)(\partial S) = 0.$$  \hspace{1cm} (6.1)

where we restrict ourselves to the simplest choice for $N_\tau$,

$$N_\tau = Z^A \partial_A + t \partial_t.$$  \hspace{1cm} (6.2)

We proceed with the following ansatz for $S$,

$$S = S(Z, t, \theta) = S(t^{-1}Z, \theta),$$  \hspace{1cm} (6.3)

which implies

$$N_\tau S = 0,$$  \hspace{1cm} (6.4)

so that the equation (6.1) takes the usual form of the classical master equation,

$$(S, S) = 0.$$  \hspace{1cm} (6.5)

Let $S(\phi)$ be an original action of original fields $\phi^i$, and $R^i_\alpha(\phi)$ do satisfy the Noether identities,

$$S \partial_i R^i_\alpha = 0, \quad \partial_i = \frac{\partial}{\partial \phi^i}.$$  \hspace{1cm} (6.6)

For the sake of simplicity, let the generators $R^i_\alpha$ be linearly-independent, so that the theory is irreducible. Let us expand the ansatz (6.3) in powers of antifields,

$$S = S(t^{-1}\phi) + t^{-1}\phi^*_i R^i_\alpha(t^{-1}\phi)t^{-1}C^\alpha + \theta t^{-1}C^\theta + \frac{1}{2} t^{-1}C^\gamma U^\gamma_{\alpha\beta}(t^{-1}\phi)t^{-1}C^\beta t^{-1}C^\alpha(-1)^{\varepsilon_\alpha} + + t^{-1}C^{i\alpha} t^{-1}B_\alpha + t^{-1}C^{i\alpha} t^{-1}B_t + ...,$$  \hspace{1cm} (6.7)

where the terms presented explicitly are enough for a rank-one theory, while ellipses mean terms nonlinear in antifields.

Now, let us split $Z^A$ into fields and antifields,

$$Z^A = \{\Phi^\alpha, \Phi^*_\alpha\}.$$  \hspace{1cm} (6.8)

Then the gauge-fixing Fermion allowed has the form

$$\Psi = \Psi(\Phi, t) = \Psi(t^{-1}\Phi, \ln t),$$  \hspace{1cm} (6.9)
so that the antifields $\Phi_a^*$ should be eliminated in (6.7) by the conditions

$$
\Phi_a^* = t^2 \Psi \partial_a, \quad \partial_a = \frac{\partial}{\partial \Phi_a^*},
$$

(6.10)

$$
\theta = N_\tau \Psi.
$$

(6.11)

These conditions do correspond to the following ansatz for the gauge-fixing master action $X$,

$$
X = \left( t^{-1} \Phi_a^* - t \Psi \partial_a \right) \lambda^a + (\theta - N_\tau \Psi) \lambda^\theta,
$$

(6.12)

where $\lambda^a$ and $\lambda^\theta$ is the corresponding Lagrange multiplier. In fact, it is enough for our purposes to use the ansatz

$$
\Psi = t^{-1} \bar{C}_\alpha \chi^\alpha(t^{-1} \phi) + t^{-1} \bar{C}_t \chi^t(\ln t),
$$

(6.13)

with the following identification of fields

$$
\Phi^a = \{ \phi^i, B_\alpha, C^\alpha, C_\alpha, B_t, C^t, C_t \}.
$$

(6.14)

Thus, we arrive at the following complete gauge-fixed action

$$
S_{gauge-fixed} = S(t^{-1} \phi) + t^{-1} \bar{C}_\alpha \chi(t^{-1} \phi) \partial_i R^i_\alpha(t^{-1} \phi) C^\alpha + t^{-2} \bar{C}_t(N_\tau \chi^t) C^t + \\
\chi^\alpha(t^{-1} \phi) t^{-1} B_\alpha + \chi^t(\ln t) t^{-1} B_t + ....
$$

(6.15)

That action has, in its terms presented explicitly, the standard structure of the Faddeev-Popov action, both in the sector of the usual gauge $\chi^\alpha$ and of the extra gauge $\chi^t$. By choosing $\chi^t = \ln t$, one removes the $t$-integration at the value $t = 1$. Thereby, it is shown that the extra variable $t$ is eliminated actually via the standard gauge-fixing procedure.

7 Gauge-fixing in the extended trivially deformed classical/quantum master equation

Let us consider the extended trivially deformed classical master equation,

$$
(S, S)_{\tau_\ast} = 0,
$$

(7.1)

or in more detail,

$$
t^2(S, S) + 2(N_\tau S)(\partial_\theta S) + 2\kappa(N_\tau S)((t^2 \Delta + N_\tau \partial_\theta)(1 - \kappa N_\tau)^{-1} S) = 0.
$$

(7.2)

The same as in the previous Section 6, we have chosen $N_\tau$ in the simplest form $\bar{N}_\tau$. In contrast to the previous Section, we are not allowed to require for the operator $N_\tau$ to annihilate the $S$, as the deformation by itself would be eliminated immediately in this way. However,
if we do believe that a solution for $S$ does exist, we can try to require for $N_\tau$ to annihilate the gauge-fixing part of $S$, at least. When doing that, we should provide for the form of the equation (7.1), or (7.2), to be respected. Let us seek for $S$ in the form

$$S = S_{\text{min}} + t^{-2}(\bar{C}^{*\alpha}B_\alpha + \bar{C}^*t), \quad (7.3)$$

where the minimal action $S_{\text{min}}$ depends on the minimal, gauge-algebra generating, set of variables, only,

$$S_{\text{min}} = S_{\text{min}}(\phi, \phi^*; t, \theta; C, C^*; C^t, C^*_t). \quad (7.4)$$

By construction, the second and third term in (7.3), those are just the gauge-fixing parts of $S$, are certainly annihilated by the $N_\tau$, while the $S_{\text{min}}$ is not. In this way, one can see that the $S_{\text{min}}$ by itself does satisfy exactly the equation (7.1), or (7.2). If one chooses the gauge Fermion $\Psi$ in the simplest form (6.13), then the antifields should be eliminated by the conditions (6.10), see also (6.14). In turn, the second and third term in (7.3) take exactly the form of the fourth and fifth term in (6.15), respectively. Thereby, it is shown that the extra variable $t$ is eliminated actually via the standard gauge-fixing procedure.

In the same way, one can consider the extended trivially deformed quantum master equation, (5.24),

$$t^2(W, W) + 2(N_\tau W)(\partial_\theta W) + 2(\kappa N_\tau W - i\hbar)((t^2\Delta + N_\tau \partial_\theta)(1 - \kappa N_\tau)^{-1}W) = 0. \quad (7.5)$$

As the equation (7.2) is a classical limit to the quantum equation (7.5), all the above reasoning, as well as the final statement remains the same.

Finally, let us notice the following. In Sections 6, 7, we have used the zero mode $t^{-1}Z^A$ of the operator (6.2). Here, we mention in short how to deal with the general operator (2.4), where $N^A$ is defined by (2.23) - (2.27). Let $Z^A$ be the zero mode of the operator (2.4). Then we have formally,

$$\bar{Z}^A = \exp\{-\ln t\}Z^A, \quad N = N^A\partial_A. \quad (7.6)$$

It follows from (2.42) that

$$(Z^A, Z^B) = t^{-2}E^{AB}, \quad (Z^A, Z^C)E_{CD}(Z^D, Z^B) = (Z^A, Z^B). \quad (7.7)$$

In turn, it follows from (7.7) that the general solution for the left off-diagonal block has the form,

$$(\bar{Z}^A, Z^B) = t^{-1}S^A_C(t)E^{CB}, \quad \bar{Z}^A = t^{-1}S^A_B(t)Z^B, \quad (7.8)$$

where $S^A_B(t) = \text{const}(Z)$ is a $t$-dependent antisymplectic matrix,

$$S^A_C(t)E^{CD}S^B_D(t)(-1)^{\varepsilon_D(\varepsilon_B+1)} = E^{AB}, \quad (7.9)$$
such that

\[ S^A_B(t = 1) = \delta^A_B. \]  

(7.10)

One can get the general solution for the right off-diagonal block via the supertransposition in (7.8).

Now, let us consider some explicit formulae concerning the modified $N$-operator. First, let us choose the quadratic Fermion $F$ entering (2.13) that meets the condition (2.16),

\[ 2F = Z^A F_{AB} Z^B, \quad \varepsilon(F) = 1, \]  

(7.11)

\[ \varepsilon(F_{AB}) = \varepsilon_A + \varepsilon_B + 1, \quad F_{AB} = F_{BA}(-1)^{\varepsilon_A \varepsilon_B} = \text{const}(Z). \]  

(7.12)

We have

\[ 2\Delta F = E^{AB} F_{BA}(-1)^{\varepsilon_A} = \text{const}(Z), \]  

(7.13)

\[ N^A = Z^A + 2(F, Z^A) = Z^B(\delta^A_B + 2F_{BC}E^{CA}) = (\delta^A_B - 2E^{AC}F_{CB})Z^B. \]  

(7.14)

On the other hand, as the second in (7.8) is the zero mode of $N_\tau$, we have another expression for $N^A$,

\[ N^A = -(S^{-1})^A \tau \partial_\tau(t^{-1}S^C_B)Z^B. \]  

(7.15)

It follows from (7.14), (7.15) that the Lie equation holds

\[ t\partial_\tau S^A_B = 2S^A_C E^{CD} F_{DB}, \quad S^A_B(t = 1) = \delta^A_B, \]  

(7.16)

whose formal matrix solution is

\[ S = S(t) = \exp\{2\ln(t)EF\}. \]  

(7.17)

By $t$-differentiating the formula (7.9), and using then the equation (7.16), one confirms that the matrix (7.17) by itself does satisfies exactly the antisymplecticity equation (7.9). Thus, we see that the exponential (7.17) provides for the $t$-parametrization of a family of antisymplectic matrices, with $(EF)$ being a generator.

If one splits the full set $Z^A$ into minimal sector $Z_{\text{min}}$ in (7.4), except for $\{t, \theta\}$, and the rest, $Z_{\text{aux}}$, $\{Z\} = \{Z_{\text{min}}\} \oplus \{Z_{\text{aux}}\}$, then, by choosing in (2.26) $F = F(Z_{\text{min}})$, one has

\[ \tilde{Z}_{\text{aux}} = t^{-1}Z_{\text{aux}}. \]  

(7.18)

In terms of (7.6), the formula (6.3) and (6.9) takes the form,

\[ S = S(\tilde{Z}, \theta), \]  

(7.19)
and
\[ \Psi = \Psi(\bar{\Phi}, \ln t), \] (7.20)
respectively. In turn, the formula (7.3) in terms of (7.6) preserves its form
\[ S = S_{\text{min}} + (C^{* \alpha} B_\alpha + \bar{C}^{* t} B_t) = S_{\text{min}} + t^{-2}(\bar{C}^{* \alpha} B_\alpha + \bar{C}^{* t} B_t), \] (7.21)
where \( S_{\text{min}} \) is given by (7.4). For the particular case \( N^A = Z^A \), we reproduce from (7.6)
\[ Z^A = t^{-1} Z^A. \]

Due to (7.18), (7.21) one has
\[ (S_{\text{min}}, S_{\text{aux}}) = 0, \quad (S_{\text{aux}}, S_{\text{aux}}) = 0, \quad \partial_\theta S_{\text{aux}} = 0, \quad S_{\text{aux}} = S - S_{\text{min}}, \] (7.22)

then with
\[ \Delta S_{\text{aux}} = 0. \] (7.23)

The relations (7.22), (7.23) allow one to preserve the form of the equation (7.2) for the minimal
action (7.4) in the general case of (7.6).

8 Generalized Darboux coordinates [17]

The \( \tau \) -extended trivially deformed classical/quantum master equation takes its simplest
form in the so-called generalized Darboux coordinates,
\[ \tau_0 : Z^A_0 = t^{-1} Z^A, \quad t_0 = \ln t, \quad t^*_0 = \theta, \] (8.1)

with the following integration measure,
\[ d\mu = dt_0 dt^*_0 d\lambda_0 [dZ_0][d\lambda]. \] (8.2)

We have already used these coordinates partially when discussing the gauge-fixing procedure.
In the case of the master equation, we have, with the use of (8.1),
\[ (W, W)_{\tau_0} = 2i\hbar \Delta_{\tau_0} W, \] (8.3)

where
\[ (F, G)_{\tau_0} = (F, G)_{\tau_0} + (K_{\tau_0} F)(\Delta_{\tau_0} G) + (\Delta_{\tau_0} F)(K_{\tau_0} G)(-1)^{\varepsilon_F}, \] (8.4)

\[ (F, G)_{\tau_0} = F[\hat{\partial}_{A_0} E^{AB} \hat{\partial}_{B_0} + \hat{\partial}_{t_0} \hat{\partial}_{t^*_0} - \hat{\partial}_{t^*_0} \hat{\partial}_{t_0}] G, \quad \partial_{A_0} = \partial_{Z^A_0}, \] (8.5)

\[ K_{\tau_0} = \kappa \partial_{A_1}, \] (8.6)
\[ \Delta_{\tau_0^*} = \Delta_{\tau_0} (1 - K_{\tau_0})^{-1}, \]  
(8.7)

\[ \Delta_{\tau_0} = \frac{1}{2} (-1)^{\varepsilon_A} \partial_{A_0} E^{AB} \partial_{B_0} + \partial_{t_0} \partial_{t_0^*}. \]  
(8.8)

We have the two main simplifications here. The first is the absence of the \( t_0 \)-dependent factors in the square bracket in (8.5). The second is a very simple form of the operator (8.6). The latter reduces to the \( t_0 \)-derivative.

The antifields are eliminated by the conditions

\[ \Phi_{a0}^* = \Psi \overset{\leftarrow}{\partial a_0}, \quad t_0^* = \Psi \overset{\leftarrow}{\partial t_0}, \]  
(8.9)

where

\[ Z_0^A = \{ \Phi_0^a, \Phi_{a0}^* \}, \quad \partial_{a0} = \frac{\partial}{\partial \Phi_0^a}. \]  
(8.10)

Finally, let us consider in short what happens if one ignores the gauge-fixing mechanism as to eliminate the extra variable \( t \). In that case one assumes the \( W \) to be \( \theta \)-independent,

\[ \partial_{\theta} W = \partial_{\theta_0} W = 0. \]  
(8.11)

Under the assumption (8.11), the path integral (5.27) becomes as represented in coordinates (8.1),

\[ Z = \int dt_0 \int d\Phi_0 \exp \left\{ \frac{i}{\hbar} A \right\}, \]  
(8.12)

\[ A = W(\Phi, \Phi^*, t, \kappa, \hbar) = W \left( \exp\{t_0\} \Phi_0, \exp\{t_0\} \left( \Psi(\Phi_0) \overset{\leftarrow}{\partial} \frac{\Phi_0}{\Phi_0} \right), \exp\{t_0\}, \kappa, \hbar \right), \]  
(8.13)

\[ (W, W)_0 + 2 (\kappa (\partial_{t_0} W) - i\hbar) (\Delta_0 (1 - \kappa \partial_{t_0})^{-1} W) = 0, \]  
(8.14)

\[ (F, G)_0 = F \overset{\leftarrow}{\partial_{A_0}} E^{AB} \overset{\leftarrow}{\partial_{B_0}} G, \]  
(8.15)

\[ \Delta_0 = \frac{1}{2} (-1)^{\varepsilon_A} \partial_{A_0} E^{AB} \partial_{B_0}. \]  
(8.16)

In [17], it was suggested that the extra variable \( t_0 \), remaining in the path integral (8.12), plays the role of the Schwinger proper time in the field-antifield formalism. It seems rather plausible that the variable \( t \) has a non-perturbative status. If one rescales in (8.13): \( W \rightarrow \exp\{-2t_0\} W \), then in (8.13), (8.14) one should substitute: \( \hbar \rightarrow \exp\{2t_0\} \hbar, \partial_{t_0} \rightarrow \partial_{t_0} - 2. \)
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Appendix A General coordinates

Although the problem of a deformation of the antibracket on the general antisymplectic manifold still has no mathematical status established, it seems rather interesting to consider a formal generalization of our basic equations to the case of general antisymplectic coordinates. As a matter of some simple formal manipulations, such a generalization appears quite natural and "minimal", and looks very nice, as well. Here, we present the general-coordinate counterpart to the basic equation (2.5), which provides, in turn, for the nilpotency of the extended $\Delta$-operator (2.3).

Let $Z^A$ be general coordinates on an antisymplectic manifold with invertible antisymplectic metric $E^{AB}$, and measure density $\rho$. Let these objects be compatible in the sense that the odd Laplacian operator

$$\Delta = \frac{1}{2} E^B \partial_B + \frac{1}{2} (-1)^{\varepsilon_A} E^{AB} \partial_B \partial_A,$$  \hspace{1cm} (A.1)

$$E^B = (-1)^{\varepsilon_A} \rho^{-1} (\partial_A \rho E^{AB}),$$  \hspace{1cm} (A.2)

is nilpotent,

$$\Delta^2 = 0.$$  \hspace{1cm} (A.3)

Let us consider our basic equation,

$$[\Delta, N] = 2\Delta, \quad N = N^A \partial_A,$$  \hspace{1cm} (A.4)

which implies

$$2\Delta N^C = (N + 2) E^C,$$  \hspace{1cm} (A.5)

$$(N^A, Z^B) - (A \leftrightarrow B)(-1)^{\varepsilon_A + 1}(\varepsilon_B + 1) = (N + 2) E^{AB}.$$  \hspace{1cm} (A.6)

These equations are general-coordinate counterparts to our equations (2.7), (2.8). It is a remarkable fact that the equations (2.23), (2.25) remain valid in the general coordinates as well. Namely, it follows from (A.6) that its dual holds,

$$(\partial_A N^C) E_{CB} - (A \leftrightarrow B)(-1)^{\varepsilon_A \varepsilon_B} = -(N - 2) E_{AB}.$$  \hspace{1cm} (A.7)
By using the Jacobi relation,

$$\partial_A E_{BC}(-1)^{\varepsilon_{ABC}} + \text{cycle}(A,B,C) = 0,$$

(A.8)

one gets from (A.7),

$$\partial_A V_B - \partial_B V_A(-1)^{\varepsilon_A\varepsilon_B} = E_{AB},$$

(A.9)

which is exactly the relation (2.25), where the $V_A$ is defined by just (2.23),

$$N^C = 2V_B E^{BC}.$$  

(A.10)

Just when rewriting in (A.7) derivatives of $N^C$ in terms of derivatives of $V_B$ coming from (A.10), there appear the terms with derivatives of $E_{CB}$ which cancel exactly the term $(-NE_{AB})$ on the right-hand side in (A.7), due to the Jacobi relation (A.8).

The general solution to the equation (A.9) is given by

$$V_B = Z^A \bar{E}_{AB} + \partial_B F, \quad \bar{E}_{AB} = (Z^C \partial_C + 2)^{-1} E_{AB},$$

(A.11)

with $F, \varepsilon(F) = 1$, being arbitrary Fermion. With respect to the measure $\rho[dZ]$, the antisymmetry of the $N$ requires

$$\text{div} N = (-1)^{\varepsilon_A} \rho^{-1} \partial_A(\rho N^A) = 0.$$  

(A.12)

Here, we show in short that the general solution to the equation (A.9) has actually the form (A.11). By multiplying the (A.9) by $Z^A$ from the left, we have

$$(Z^A \partial_A + 1) V_B = Z^A E_{AB} + \partial_B(Z^A V_A).$$

(A.13)

Now, it is worthy to mention the two useful operator-valued identities,

$$(Z^C \partial_C + n)^{-1} Z^A = Z^A(Z^C \partial_C + n + 1)^{-1},$$

(A.14)

$$(Z^C \partial_C + n)^{-1} \partial_A = \partial_A(Z^C \partial_C + n - 1)^{-1}.$$  

(A.15)

Due to the latter identities, it follows immediately from (A.13) that the formula (A.11) holds with $F$ defined as

$$F = Z^A(Z^C \partial_C + 1)^{-1} V_A.$$  

(A.16)

If one inserts (A.11) into (A.9), the quantity (A.16) drops out, so that (A.9) imposes no restrictions on the Fermion $F$ being thereby arbitrary. Thus, we have shown that the (A.11) is just the general solution to the (A.9). By applying the operator $(Z^D \partial_D + 3)^{-1}$ to (A.8) from the left, and using (A.15), one gets the Jacobi identity, similar to (A.8), as for $E_{AB}$ in (A.11),

$$\partial_A \bar{E}_{BC}(-1)^{\varepsilon_{ASC}} + \text{cycle}(A,B,C) = 0.$$  

(A.17)
In turn, by using the latter identity, one can confirm, in an independent way, that the solution (A.11) satisfies (A.9).

Next, let us elucidate the formal essence of the equation (A.5). By inserting therein,

\[ N^C = NZ^C, \]  

(A.18) and using then (A.1), (A.2), (A.4), we have on the left-hand side of (A.5),

\[ 2\Delta N^C = 2\Delta NZ^C = 2(N + 2)\Delta Z^C = (N + 2)E^C, \]  

(A.19) which coincides exactly with the right-hand side in (A.5). Thus, we have confirmed again that the (A.5) is consistent with (A.4).

Finally, let us consider the relation

\[ N(f, g) = (Nf, g) + (f, Ng) - f \leftarrow \partial_A \left[ N^A \partial^C E^{CB} - (A \leftrightarrow B)(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} \partial_B g + f \leftarrow \partial_A (NE^{AB}) \partial_B g = \right] \ 

= (Nf, g) + (f, Ng) - 2(f, g), \]  

(A.20) where we have used (A.6). The (A.20) tells us that the operator \((N - 2)\) does differentiate the antibracket,

\[ (N - 2)(f, g) = ((N - 2)f, g) + (f, (N - 2)g), \]  

(A.21) which is the general-coordinate counterpart to (2.42).

**Appendix B Trivially-deformed extended sigma-model**

Let

\[ \tau = \{ Z^a \} = \{ Z^A, \ln t, \theta \}, \]  

(B.1) be the extended set of antisymplectic variables we have introduced in Sections 2 and 3. Let us assume now that all the variables (B.1) are superfields depending on 2n Bosons \(u^a\) and 2n Fermions \(\xi^a, a = 1, 2, ..., 2n;\) these variables are independent arguments of superfields (B.1).

Let \(D\) be the differential of De Rham,

\[ D = \xi^a \partial_a, \quad \partial_a = \frac{\partial}{\partial u^a}, \quad \varepsilon(D) = 1, \quad D^2 = 0. \]  

(B.2) The trivially-deformed extended sigma-model is defined by the action

\[ \Sigma = \int [du][d\xi] \mathcal{L}, \]  

(B.3)\footnote{In this Appendix, we restrict ourselves with the use of the simplest power-counting operator 6.2.}
with $\mathcal{L}$ being a Lagrange density,

$$
\mathcal{L} = \frac{1}{2} t^{-2} Z^B E_{BA} DZ^A (-1)^{\varepsilon_A} + \frac{1}{2} (\theta D \ln t + \ln t D\theta) + TS,
$$

(B.4)

where $E^{AB}$ is a constant invertible antisymplectic metric, see (2.2), and $E_{AB}$ is the inverse to $E^{AB}$; the operator $T$ is defined in (5.3)

$$
T = 1 + \kappa \theta \Delta_{\tau^*};
$$

(B.5)

the Boson master action $S$ satisfies the extended trivially-deformed classical master equation (7.1)/(7.2),

$$(S, S)_{\tau^*} = T^{-1} (TS, TS)_{\tau} = 0.
$$

(B.6)

One has to seek for a solution to that equation in the form similar to (5.25),

$$
S = \sum_{k=-2}^{\infty} S_{(k|0)} t^k + \theta \sum_{k=1}^{\infty} S_{(k|1)} t^k,
$$

(B.7)

where the component $S_{(-2|0)} = S$ is identified with the classical non-trivially deformed proper action,

$$(S, S)^* = 0.
$$

(B.8)

In the right-hand side in (B.4), the kinetic part has the form usual for sigma - models [26, 27, 28, 29], while the term $TS$ is a natural counterpart to (5.28). For the action $\Sigma$, we have

$$
\frac{\delta}{\delta Z^B} \Sigma = t^{-2} E_{BA} t D t^{-1} Z^A (-1)^{\varepsilon_A} + \partial_B (TS),
$$

(B.9)

$$
\frac{\delta}{\delta \ln t} \Sigma = D\theta - t^{-2} Z^B E_{BA} DZ^A (-1)^{\varepsilon_A} + \frac{\partial}{\partial \ln t} (TS),
$$

(B.10)

$$
\frac{\delta}{\delta \theta} \Sigma = D \ln t + \partial_\theta (TS).
$$

(B.11)

Thus, we get the following classical motion equations

$$
\nabla Z^A = 0, \quad \nabla \ln t = 0, \quad \nabla \theta = 0,
$$

(B.12)

$$
\nabla = D + \text{ad}_\tau (TS),
$$

(B.13)

where $\text{ad}_\tau$ is the left adjoint of the $\tau$ extended antibracket [31],

$$
\text{ad}_\tau (X) = (X, ...)_{\tau}.
$$

(B.14)
Now, let us define the functional extended antibracket,

\[
[F,G]_\tau = \int [du][d\xi] F \frac{\delta}{\delta Z^\alpha} (Z^\alpha, Z^\beta)_\tau \frac{\delta}{\delta Z^\beta} G = \int [du][d\xi] F \left[ \frac{\delta}{\delta Z^A} t^2 E^{AB} \frac{\delta}{\delta Z^B} + \frac{\delta}{\delta Z^A} Z^A \frac{\delta}{\delta \theta} - \frac{\delta}{\delta \theta} Z^A \frac{\delta}{\delta \ln t} + \frac{\delta}{\delta \ln t \delta \theta} - \frac{\delta}{\delta \theta \delta \ln t} \right] G,
\]

where \( Z^\alpha \) is the extended set \( (B.1) \), and \( F, G \) are functionals of these variables. Then, we have for the action \( (B.3) \), that the following functional master equation holds,

\[
\frac{1}{2} [\Sigma, \Sigma]_\tau = \int [du][d\xi] \left( D\mathcal{L} + \frac{1}{2} (TS, TS)_\tau \right) = 0.
\]

**Appendix C Parametric differential equation for star-exponential**

In Section 5, we have noticed the formula \( (5.18) \) for the star-exponential as derived in \( [17] \), in its Appendix E. That derivation is very nice. However, here we would like to re-derive the star exponential just from the first principle, by resolving the basic parametric differential equation. What we mean by the first principle is the definition in \( (5.14) \) of the star-product

\[
(F * G) = T^{-1}((TF)(TG)) = FG - \kappa \theta(F,G)_{\tau*}(1)^\varepsilon(F).
\]

Besides, we will use the formula similar to \( (5.20) \) (\( B \) is a Boson, \( \varepsilon(B) = 0 \)),

\[
\exp_{\ast} \{-B\} \ast (B, \exp_{\ast} B)_{\tau\ast} = (B, B)_{\tau\ast}.
\]

so that

\[
(B, \exp_{\ast} \{B\})_{\tau\ast} = (B, B)_{\tau\ast} \ast \exp_{\ast} \{B\}.
\]

Together with \( (C.1) \), the \( (C.3) \) enables us to present an alternative derivation to the third equality in \( (5.18) \). The latter goes as follows. Let us define (\( x \) is a Boson parameter, \( \varepsilon(x) = 0 \)),

\[
U(x) = \exp_{\ast} \{xB\}, \quad U(x = 0) = 1.
\]

It follows from \( (C.1) \) that

\[
\partial_x U = B * U = BU - \kappa \theta(B,U)_{\tau\ast} = BU - x \kappa \theta(B,B)_{\tau\ast} * U = (B - x \kappa \theta(B,B)_{\tau\ast}) U,
\]

where we have used \( (C.1) \) in the third equality. We have also omitted the star, \( * \), just in front of the rightmost \( U \) in the second line in \( (C.5) \), because of the explicit presence of the
\( \theta \) neighboring to the left from \((B, B)_{\tau_*}\), see, again, the second equality in (C.1). The latter equality in (C.5) is just what we mean when saying about the basic parametric differential equation. By integrating (C.5), we get

\[
U(x) = \exp \left\{ xB - \frac{1}{2} x^2 \kappa \theta (B, B)_{\tau_*} \right\}. \tag{C.6}
\]

By taking herein \(x = 1\), we obtain, finally

\[
\exp_* \{ B \} = \exp \left\{ B - \frac{1}{2} \kappa \theta (B, B)_{\tau_*} \right\}, \tag{C.7}
\]

which is exactly the last equality in (5.18). The latter generalizes to arbitrary star function \(f_*(B)\) as follows

\[
f_*(B) = \exp \left\{ -\frac{1}{2} \kappa \theta (B, B)_{\tau_*} \frac{\partial^2}{\partial B^2} \right\} f(B), \tag{C.8}
\]

where, given a regular function \(f(B)\), the corresponding star-function \(f_*(B)\) is defined similarly to the first and second equalities in (5.18),

\[
f_*(B) = T^{-1} f(TB), \tag{C.9}
\]

in terms of the operators (5.3), (5.4).

If one introduces in (C.7) a polarization similar to (3.4),

\[
B = m^i n_i, \quad \varepsilon(m^i) = \varepsilon(n_i) = \varepsilon_i, \tag{C.10}
\]

then the formula (C.8) generalizes to

\[
f_*(m) = \exp \left\{ -\frac{1}{2} \kappa \theta (-1)^{\varepsilon_i} (m^i, m^j)_{\tau_*} \frac{\partial}{\partial m^i} \frac{\partial}{\partial m^j} \right\} f(m), \tag{C.11}
\]

where \(f(m)\) is a regular function of the components \(m^i\), and

\[
f_*(m) = T^{-1} f(Tm). \tag{C.12}
\]

**Appendix D General solution to the equation (5.6)**

Let us consider the equations (5.6) for the operator \(T\),

\[
[\Delta_\tau, T] = \kappa N_\tau \Delta_\tau T, \quad [N_\tau, T] = 0, \tag{D.1}
\]

within the algebra

\[
\theta^2 = 0, \quad \Delta^2_\tau = 0, \quad [\theta, \Delta_\tau] = N_\tau, \quad [\theta, N_\tau] = 0, \quad [\Delta_\tau, N_\tau] = 0. \tag{D.2}
\]
Notice that the $T$- algebra spanned by the three basic elements $\Delta_\tau, T, N_\tau$ is generated naturally via the nilpotency condition, $\Omega^2 = 0$, imposed on the following Fermionic operator

$$\Omega = C\Delta_\tau + BT + AN_\tau + BC\kappa N_\tau \Delta_\tau \bar{B}, \quad (D.3)$$

where $C$ is a Bosonic coordinate, while $B$ and $A$ are Fermionic ones, $B^2 = 0$, $A^2 = 0$; $\bar{B}$ is a Fermionic canonical momentum to $B$, $[B, \bar{B}] = 1$, $(\bar{B})^2 = 0$, $(BB)^2 = BB$. Vice versa, let the $T$- algebra holds. Then, the nilpotency of $\Omega$ (D.3) can be easily seen via rewriting

$$\Omega = C\tilde{\Delta}_\tau + BT + AN_\tau, \quad (D.4)$$

where the operator

$$\tilde{\Delta}_\tau = \Delta_\tau (1 - B\kappa N_\tau \bar{B}), \quad (D.5)$$

does satisfy

$$\tilde{\Delta}^2_\tau = 0, \quad [\tilde{\Delta}_\tau, BT] = 0, \quad (D.6)$$

$$U^{-1}\Delta_\tau U = \tilde{\Delta}_\tau, \quad (D.7)$$

$$U = \exp\{-\kappa \theta \Delta_\tau B\bar{B}\Phi(Y)\} = T_{(\kappa\bar{B}B)}^{-1} = 1 - \kappa B\bar{B}\theta \Delta_\tau. \quad (D.9)$$

$$U^{-1} = \exp\{\kappa \theta \Delta_\tau B\bar{B}\Phi(Y)\} = T_{(\kappa\bar{B}B)} = 1 + \kappa B\bar{B}\theta \Delta_\tau (\kappa\bar{B}B), \quad (D.10)$$

$$\Phi(Y) = -Y^{-1} \ln (1 - Y), \quad Y = \kappa N_\tau. \quad (D.11)$$

It is worthy to mention here that the exponential operator (D.10) generalizes for arbitrary $\Phi(Y)$ to

$$\exp\{\kappa \theta \Delta_\tau B\bar{B}\Phi\} = 1 + \kappa \theta \Delta_\tau B\bar{B}Y^{-1}(\exp\{Y\Phi\} - 1), \quad (D.12)$$

while the operator (D.9) generalizes to (D.12) with $-\kappa$ standing for $\kappa$ (including the $\kappa$ entering the $Y$). The choice (D.11) does satisfy

$$(-Y)^{-1}(\exp\{-Y\Phi\} - 1) = 1, \quad Y^{-1}(\exp\{Y\Phi\} - 1) = (1 - Y)^{-1}. \quad (D.13)$$

It follows from (D.8) that the $U$-transformation (D.9), (D.10) does Abelianize the operator $\Omega$ (D.3), by eliminating from the latter the $T$ as well as $\bar{B}$, at all,

$$U\Omega U^{-1} = \Omega_{\text{Abelian}} = C\Delta_\tau + B + AN_\tau. \quad (D.14)$$
General solution to the equations (D.1) can be sought for in the form

\[ T = \theta \Delta_r A(N_\tau, \Delta_\tau) + B(N_\tau, \Delta_\tau). \]  

(D.15)

It follows from the first in (D.1) that

\[ N_\tau \Delta_r ((1 - \kappa N_\tau) A(N_\tau, \Delta_\tau) - \kappa B(N_\tau, \Delta_\tau)) = 0. \]  

(D.16)

In turn, it follows from (D.16) that

\[ A(N_\tau, \Delta_\tau) = \kappa (1 - \kappa N_\tau)^{-1} B(N_\tau, \Delta_\tau), \]  

(D.17)

where we have included into \( B \) the zero mode of the operator \( N_\tau \Delta_\tau \). By inserting (D.17) into (D.15), we get

\[ T = (1 + \kappa \theta \Delta_r (1 - \kappa N_\tau)^{-1}) B(N_\tau, \Delta_\tau). \]  

(D.18)

Here the overall factor \( B \) remains arbitrary; that is a natural arbitrariness in the general solution for \( T \). On the other hand, as the operator \( T \) should be equal to 1 at \( \kappa = 0 \), it follows that we should choose \( B = 1 \). Thus, we arrive at the formula (5.3).

Finally, let us consider the dual to the equations (D.1), for the inverse \( T^{-1} \),

\[ [\Delta_r, T^{-1}] = -T^{-1} \kappa N_\tau \Delta_r, \quad [N_\tau, T^{-1}] = 0. \]  

(D.19)

In principle, we could analyze these equations in the same way as we have done as to the equations (D.1). However, it is much simpler to consider directly the inverse to the general solution (D.18),

\[ T^{-1} = (B(N_\tau, \Delta_\tau))^{-1} (1 - \kappa \theta \Delta_\tau). \]  

(D.20)

For the reason mentioned just below (D.18), we have to choose \( B = 1 \), so that we arrive at the formula (5.4).

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