Computable conditions for the occurrence of non-uniform hyperbolicity in families of one-dimensional maps

Stefano Luzzatto\textsuperscript{1} and Hiroki Takahasi\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, Imperial College, London, UK
\textsuperscript{2} Department of Mathematics, Kyoto University, Kyoto, Japan

E-mail: stefano.luzzatto@imperial.ac.uk and takahasi@kusm.kyoto-u.ac.jp

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Abstract
We formulate and prove a Jakobson–Benedicks–Carleson-type theorem on the occurrence of non-uniform hyperbolicity (stochastic dynamics) in families of one-dimensional maps, based on computable starting conditions and providing explicit, computable, lower bounds for the measure of the set of selected parameters. As a first application of our results we show that the set of parameters corresponding to maps in the quadratic family $f_a(x) = x^2 - a$ which have an absolutely continuous invariant probability measure is at least $10^{-5000}$.

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1. Introduction

In this paper we consider families $\{f_a\}_{a \in \Omega}$ of $C^2$ interval maps with a single quadratic critical point and the parameter $a$ belonging to some interval $\Omega$. Families of interval maps can exhibit a wide variety of dynamical behaviour, ranging from the existence of attracting periodic orbits to the existence of absolutely continuous (with respect to Lebesgue) invariant measures with strong mixing properties, as well as all kinds of intermediate and pathological phenomena. These dynamical phenomena can depend very sensitively, and very discontinuously, on the parameter even in very smooth parametrized families, see [35] for a comprehensive survey.

We say that $a \in \Omega$ is a regular parameter if $f_a$ has an attracting periodic orbit; we say that $a \in \Omega$ is a stochastic parameter if $f_a$ admits an ergodic invariant probability measure $\mu$ which is absolutely continuous with respect to Lebesgue and has a positive Lyapunov exponent $\lambda(\mu) = \int \log |f'| \, d\mu$. Then we define

$$\Omega^- = \{a : a \text{ is regular}\} \quad \text{and} \quad \Omega^+ = \{a : a \text{ is stochastic}\}.$$
We note that the positivity of the Lyapunov exponent is sometimes a non-trivial but automatic consequence of the existence of an absolutely continuous invariant probability measure. This is the case, for example, for maps with a single critical point such as those considered here [17] and is probably true for more general maps with multiple critical points (it is of course false for maps without critical points such as rigid circle rotations). We include it here in the definition since it is a feature which plays a key role in giving rise to stochastic behaviour, see [1, 8, 18, 23, 48] for results and references to results concerning the precise ‘random-like’ properties of such maps.

For families of maps with a single critical point it is also known that the sets \( \Omega^- \) and \( \Omega^+ \) are disjoint, see [35]. Moreover, for generic families with a quadratic critical point both \( \Omega^- \) and \( \Omega^+ \) have positive measure and their union has full measure [3–5, 32]. These are therefore the only two ‘typical’ phenomena. The topological structure of the two sets is however very different: \( \Omega^- \) is open and dense in \( \Omega \) [12, 19, 20, 33] and thus \( \Omega^+ \), which is contained in the complement of \( \Omega^- \), is nowhere dense. The fact that it has positive measure is therefore non-trivial. This was first shown in the ground-breaking work of Jakobson [16] and was generalized, over the years, in several papers; we mention in particular [6, 10, 13, 14, 35, 38, 41, 43, 47] for smooth maps with non-degenerate critical points, [42] for maps with a degenerate (flat) critical point and [24, 25] for maps with both critical points and singularities with unbounded derivatives.

The fact that stochastic behaviour occurs for a nowhere dense set of parameters means that, notwithstanding the fact that it also occurs with positive probability, it is a difficult set to actually ‘pinpoint’ in practice. In this paper we are concerned with the problem of estimating explicitly the measure of the set \( \Omega^+ \). Notwithstanding the extensive amount of research in the area, none of the existing results provide any explicit quantitative bounds on the relative measure of \( \Omega^+ \) in \( \Omega \). The arguments are constructive to some extent: they ‘construct’ a set of stochastic parameters in a given parameter interval \( \Omega \). However they all apply only under the assumptions that \( \Omega \) is a sufficiently small neighbourhood of some sufficiently good parameter value \( a^* \). Both these assumptions are problematic in different ways which we discuss briefly in the following paragraphs.

The size of the neighbourhood \( \Omega \) of \( a^* \) to which the arguments apply, as well as the proportion of \( \Omega^+ \) in \( \Omega \), is not computed in any of the existing arguments. This is to some extent more of a technical issue than a conceptual one: explicit estimates can probably be obtained from the existing papers by more careful control of the interdependence of the constants involved. It should be noted nevertheless that this interdependence is quite subtle; the issue might be technical but this does not make it non-trivial.

A more delicate issue is the assumption that the parameter interval \( \Omega \) contains some good parameter \( a^* \). This gives rise to the problem of verifying the presence of such a parameter in \( \Omega \) and of computing the required quantitative information related to the ‘goodness’ of \( a^* \). It turns out that this is essentially impossible in general. Several conditions of various kinds have been identified which imply that a given parameter value is stochastic [1, 8, 9, 15, 36, 37] but, apart from some exceptional cases, all these conditions require information about an infinite number of iterations and thus are, to all effects and purposes, uncheckable. It can also be shown that in a formal theoretical sense the set \( \Omega^+ \) is undecidable, see [2].

The main objective of this paper is to overcome these difficulties. We present a quantitative parameter exclusion argument which gives explicit lower bounds on the proportion of \( \Omega^+ \) in some parameter intervals \( \Omega \) and, perhaps most importantly, base this argument on explicitly computable starting conditions which can be computed in finite time and with finite precision. In section 3 we give an application to the quadratic family and obtain a first ever explicit lower bound for measure of \( \Omega^+ \). In the remaining sections we prove the main theorem.
2. Statement of results

To simplify the calculations we shall consider one-parameter families of $C^2$ interval maps of the form

$$f_a(x) = f(x) - a$$

for some map $f : I \to I$ with a single quadratic critical point $c$ and the parameter $a$ belonging to some interval $\Omega$. In particular, the critical point $c \in I$ does not change with the parameter; this is not an essential condition but simplifies some of the already very technical estimates. For the same reason we shall suppose without loss of generality that the interval $I$ strictly contains the interval $[-1, 1]$, the critical point $c = 0$ and $f_a(c) > 1$ for all parameters $a \in \Omega$. These conditions always hold up to a linear rescaling which will not affect the argument.

We note that we think of $\Omega$ as being ‘small’ in a sense which will become clear. This is a natural and general setting: the estimates can easily be applied to a ‘large’ parameter interval by subdividing it into smaller subintervals and considering each such subinterval independently. For future reference we let $\varepsilon = |\Omega|$.

2.1. Computable starting conditions

We start by formulating several conditions in terms of seven constants $N, \delta, \iota, C_1, \lambda, \alpha_0, \lambda_0$.

First of all we define critical neighbourhoods $\Delta = (-\delta, \delta) \subset (-\delta\iota, \delta\iota) =: \Delta^\dagger$.

We shall assume without loss of generality that both $\log \delta$ and $\iota \log \delta$ are integers.

(A1) Uniform expansivity outside a critical neighbourhood: for all $a \in \Omega, x \in I, n \geq 1$ such that $x, f_a(x), \ldots, f_a^{n-1}(x) \notin \Delta$ we have:

$$|\left(f_a^n\right)'(x)| \geq \begin{cases} C e^{\lambda n} & \text{if } f_a^n(x) \in \Delta^\dagger \\ e^{\lambda_0} & \text{if } x \in f_a(\Delta^\dagger) \text{ and/or } f_a^n(x) \in \Delta. \end{cases}$$

(A2) Random distribution of critical orbits: there exists $\tilde{N} \geq N$ such that $|f_a^n(c)| \geq \delta'$ for all $n \leq \tilde{N}$ and

$$|\Omega_{\delta'}| := |\{f_a^n(c) : a \in \Omega\}| \geq \delta'.$$

(A3) Bounded recurrence of the critical orbit: for all $a \in \Omega$ and all $N \geq n \geq 1$ we have

$$|f_a^n(c)| \geq e^{-\alpha_0 n}.$$  

(A4) Non-resonance: there exists an integer $\tilde{N} \geq 1$ such that

$$1 + \sum_{i=1}^{k} \frac{1}{(f_a^i)'(c_0)} \neq 0 \forall k \in [1, \tilde{N}] \quad \text{and} \quad 1 - \sum_{i=1}^{\tilde{N}} \frac{1}{(f_a^i)'(c_0)} \geq \frac{e^{-\lambda_0 (\tilde{N}+1)}}{1 - e^{-\alpha_0}} > 0.$$

We emphasize that all these conditions are computable in the sense that they can be verified in a finite number of steps, using explicit numerical calculations relying only on finite precision and depending only on a finite number of iterations. This does not mean however that their verification in practice is trivial or even easy; we shall discuss below some of the computational issues which arise when applying our results to specific situations.

Condition (A1) says that some uniform expansivity estimates hold outside the critical neighbourhoods, uniformly for all parameter values. This is in some sense the most important
condition of all, the general principle being that if we have uniform expansivity on a sufficiently large region of the phase space for all parameter values then we have non-uniform expansivity on the entire phase space for a large region of the parameter space. For a single parameter value this expansivity is known to hold under extremely weak conditions [34], although in general the constants $C_1$ and $\lambda$ will depend on $\delta$. By continuity it then holds for nearby maps though, again, the size of an allowed perturbation in general will depend at least on $\delta$ and $\lambda$. Condition (A1) as stated therefore is not a strong assumption per se but becomes strong if we want it to hold for large $\lambda$, small $\delta$ and/or a large interval of parameter values. On the other hand it is important to have a ‘large enough’ parameter interval relative to the choice of $\delta$, $\iota$, otherwise (A1) could be satisfied even though all parameters in $\Omega$ have an attracting periodic orbit (this may happen, for instance, if the attracting periodic orbit always has at least one point in $\Delta$ then its attracting nature could be invisible to derivative estimates outside $\Delta$).

The appropriate condition on the size of $\Omega$ is given in (A2) which says that the size of the interval given by the images of the critical points for all parameter values at some time is large enough. Technically this gives a sufficiently ‘random’ distribution of the images to provide the first step of the probabilistic induction argument, showing that critical orbits have a small probability of returning close to the critical point. This is, conceptually, the core of the overall argument and this condition is essentially implicit in all arguments of this kind, in the language of Benedicks–Carleson [6, 7] and other papers which follows similar strategies, such as [24, 25]. Intervals of parameter values satisfying these conditions are called escaping components.

Condition (A3) has been used by Benedicks and Carleson and other people and has proved extremely useful in arguments related to shadowing the critical point. Note however that this condition only refers to an initial finite number of iterates, as part of the inductive construction we will guarantee that it continues to hold for all good parameter values for all time, see section 4. We do not feel, however, that it is as essential a condition as (A1) and (A2), but rather more of a technical simplifying assumption, albeit one that is not easy to remove in this context. Binding period arguments analogous to the ones developed here using this condition have been generalized in [8] in order to study the statistical properties of a large class of maps with several critical points.

Condition (A4) is easily checked for parameter neighbourhoods of particularly good parameter values, such as when the critical orbit is pre-periodic or non-recurrent. Indeed it is shown in [22] that some form of this condition is satisfied very frequently. A similar condition played an important role in the work of Tsujii [43] in generalizing parameter exclusion arguments to neighbourhood of quite general maps. However, we do feel again that this is more of a technical simplifying assumption rather than a deep condition. In principle it should be possible to weaken or possibly even dispense with both (A3) and (A4), although this would certainly require some non-trivial technical improvements in the argument.

### 2.2. Conditions on the constants

Our assumptions on the family $\{f_a\}$ will be that conditions (A1)–(A4) hold for a set of constants $N, \delta, \iota, C_1, \lambda, \alpha_0, \lambda_0$ satisfying certain non-trivial formal relationships (C1)–(C4) which we proceed to formulate. The first condition is formulated purely in terms of the constants introduced above:

$$\lambda > \lambda_0 > \alpha_0 \geq \log \delta^{-1/N} \quad \text{and} \quad \log \delta^{-\iota} \geq e^{-(1+\lambda_0)/2}. \quad (C1)$$

This gives a first constraint on the relative values of the constants. We note that the second expression is essentially trivial since generally speaking $\log \delta^{-\iota}$ is large and $\lambda_0$ is small;
however, it does not appear to be a formal consequence of any of the other assumptions and we do use it below; therefore we add it here as a formal assumption. Additional constraints are imposed indirectly via the definition of a set of auxiliary constants in the order given in the following list:

\[ M_1, M_2, L_1, L_2, a_1, N_1, D_1, D_2, D_3, \gamma_0, \gamma_1, \gamma_2, \gamma, \hat{D}, \tilde{D}, D, D, \Gamma_1, k_0, \tau_1, \tau_0, C_3, \tilde{C}, \tau, \alpha, \tilde{\eta}, \eta. \]

As these constants are introduced we shall define conditions which implicitly impose conditions on the original constants \( N, \delta, \iota, C_1, \lambda, \alpha_0, \lambda_0. \) The entire procedure is aimed at obtaining a value for the last constant \( \eta, \) which appears in the statement of the theorem below.

We note first that the constants \( M_1, M_2, L_1, L_2, N_1, D_2, D_3 \) require some amount of computation concerning the geometry and/or the dynamics of the family \( \{ f_a \}. \) In particular they are not defined exactly: they are ‘estimates’ which are required to satisfy some lower or upper bounds. We shall use the notation \( \gg \) to denote the fact that the constant on the left is required to be an upper bound for the expression on the right. Similarly for \( \ll. \) The four constants \( a_1, \gamma_1, \gamma_2, \alpha \) are chosen with some freedom within certain ranges depending on the previously chosen constants. The remaining constants are defined directly in terms of those defined previously.

We start with fixing constants \( M_1, M_2, L_1, L_2 \) so that

\[ M_1 \gg \max\{|f_a'(x)|\} \quad \text{and} \quad M_2 \gg \max\{|f_a''(x)|\}, \]

where the maximum is taken over all \((x, a) \in I \times \Omega\) and \( L_1 \) and \( L_2 \) are chosen such that

\[ \begin{cases}
L_1^{-1}|x - c|^2 \geq |f_a(x) - f_a(c)| \geq (x - c)^2 & \forall x \in \Delta, a \in \Omega \\
L_2^{-1}|x - c| \geq |f_a''(x)| \geq |x - c| & \forall x \in I, a \in \Omega
\end{cases} \quad (2) \]

and \( L_2^{-1}|x - c| \geq |f_a''(x)| \geq L_2|x - c| \) \( \forall x \in I, a \in \Omega. \) Note that \( L_1 \) is used in bounds in \( \Delta, \) whereas \( L_2 \) applies to the entire interval \( I. \) We then choose some

\[ \lambda_0 \gg: a_1 \gg: \alpha_0 \]

and define the following constants. First of all let

\[ N_1 \ll: \max\{|d(f^i(\Delta_0), c)| \gg 1\}, \quad (4) \]

where \( d(f^i(\Delta_0), c) \) denotes the distance of iterates of \( \Delta_0 = f(\Delta) \) from the critical point \( c, \) minimized over all parameters in \( \Omega. \) We can assume without loss of generality that \( N_1 \gg 1; \) recall the discussion at the beginning of section 2.

\[ D_1 := \exp\left(\frac{M_2}{L_2} \frac{e^{-a_1}}{1 - e^{-a_1}} + \frac{e^{-(a_1 - a_0)(N_1 + 1)}}{(1 - e^{-(a_1 - a_0)(N_1 + 1)}) (1 - e^{-a_0})}\right), \quad (5) \]

\( D_1 \) bounds the distortion during binding periods, see section 5 and sublemma 5.1.1.

\[ D_2 := \max_{\alpha \in \Omega} \max_{1 \leq k \leq N} \left\{ \frac{1 + \sum_{i=1}^{k} \frac{1}{(f_a')^i(c_0)}}{1 + \sum_{i=1}^{N} \frac{1}{(f_a')^i(c_0)}} + \frac{e^{-\lambda_0(N + 1)}}{(1 - e^{-\lambda_0})}\right\}, \quad (6) \]

\[ D_3^{-1} \ll: \min_{\alpha \in \Omega} \min_{1 \leq k \leq N} \left\{ \frac{1 + \sum_{i=1}^{k} \frac{1}{(f_a')^i(c_0)}}{1 + \frac{1}{(f_a')^i(c_0)} - \frac{e^{-\lambda_0(N + 1)}}{1 - e^{-\lambda_0}}} \right\}; \quad (7) \]

recall that \( \tilde{N} \) is given (with some freedom) by condition (A4). The constants \( D_2, D_3 \) appear in the context of the estimates which compare derivatives with respect to the parameter and derivatives with respect to the space variable, see lemma 5.2. Condition (A4) is designed
specifically to ensure that \( D_3 > 0 \). The values of \( D_2 \) and \( D_3 \), or the complexity of the calculation, can be optimized by choosing different values for \( N \). Then we define

\[
\gamma_0 := \frac{2 + \log 2 + 5 \log \delta^{-1}}{\log \delta^{-1}}.
\]

The definition of \( \gamma_0 \) comes from the combinatorial estimates in section 7.4. We choose

\[
\min \left\{ 1 - \frac{\log C_1^{-1} + 2 \log \log \delta^{-1}}{\log \delta^{-1}}, 1 - \gamma_0 \right\} >: \gamma_1 := 0.
\]

The first condition here guarantees the convergence of the infinite sum in sublemma 6.3.2. The second condition appears in the statement of the key proposition 7.2. It also allows us to choose some

\[
1 - (\gamma_0 + \gamma_1) >: \gamma_2 >: 0 \quad \text{and let} \quad \nu := \gamma_0 + \gamma_1 + \gamma_2 \in (0, 1),
\]

where \( \gamma_2 \) and \( \nu \) both appear in the statement of proposition 7.1. Now we define

\[
\hat{D} := 2 + \frac{2C_1^{-1}D_2D_3e^{-\lambda}}{1 - e^{-\lambda}},
\]

\[
\hat{D} := \left[ \left( 1 + \frac{(\log \delta^{-1})^2}{(\log \delta^{-1} - 1)^2} \right) \frac{(\log \delta^{-1})^2}{(\log \delta^{-1})^2 - C_1^{-1}\delta(1-\gamma)} \right] \frac{1}{\log \delta^{-1} - 1}.
\]

where \( \hat{D} \) and \( \hat{D} \) are simply shortcuts to long expressions. We then let

\[
D := D_2D_3 \exp \left( \frac{M_2}{L_2} \left( \hat{D} + \frac{C_1^{-1}D_2D_3e^{-\lambda}}{1 - e^{-\lambda}} \right) \right),
\]

where \( D \) is the global distortion bound, see lemma 6.3.

\[
\Gamma_1 := DD_1D_2D_3e^{1+\lambda}/L_1C_1,
\]

where the constant \( \Gamma_1 \) is a large ‘dummy’ constant used to formulate the estimates on the derivative growth at the end of binding periods, see (24). It allows us to cancel out the small constant arising in the estimate (69). The ‘payback’ for having such large \( \Gamma_1 \) will be in terms of a lower bound on \( \gamma_1 \), see below.

\[
k_0 := \max \left\{ \log(D/L_1) + \lambda_0 + \alpha_1, 0 \right\}, \quad \tau_1 := \frac{2}{\lambda_0 + \alpha_1} \leq 2 + k_0 \quad \text{and} \quad \tau_0 := \tau_0,
\]

where \( k_0 \) is of no particular significance, it is just used in the definition of \( \tau_0 \); its definition comes from the proof of sublemma 5.1.2, see (29). \( \tau_0, \tau_1 \) appear in the binding period estimates, sublemmas 5.1.2 and 5.1.3. We can now formulate the next condition

\[
\tau_0 \nu_0 < 1. \quad \text{(C2)}
\]

The product \( \alpha_0 \nu_0 \) is related to the length of the binding periods, see (23), lemma 5.1. Then we define

\[
C_3 := D_1^{-\lambda_0}(\log(L_1)/\lambda_0 + \alpha_1) L_1^{2+\alpha_1/(\lambda_0 + \alpha_1)} \quad \text{and} \quad \tilde{C}_3 := \frac{2\alpha_1 \nu_1 L_1^2}{2DD_1D_2D_3} C_3,
\]

where \( C_3 \) and \( \tilde{C}_3 \) have no particular significance, they are just a shorthand way of writing some complex expressions arising in the paper. They are first used in (31) and (35), respectively. We require

\[
\max \left\{ \frac{t + \log(DD_2D_3C_3^{-1}) + 2 \log \log \delta^{-1}}{\log \delta^{-1}}, \frac{\log(\Gamma D_2D_3C_3^{-1}) + 2 \log \log \delta^{-1}}{\log \delta^{-1}} \right\} \leq \gamma_1. \quad \text{(C3)}
\]
These conditions arise in (73) and (41), respectively.

\[ \tau := \frac{\tau_0}{1 - \gamma_1} \left( 1 + \frac{\log |I| + 2 \log \log \delta^{-1} - \log \Gamma_1}{\log \delta^{-1}} \right) > 0. \tag{16} \]

The constant \( \tau \) gives the total proportion of iterates which belong to binding periods, see lemma 5.3. We note that it is not directly clear from the definition that \( \tau > 0 \). However this follows a fortiori from estimates (42) and (43) at the end of the proof of sublemma 5.3.

\[ \alpha := \min \left\{ \alpha_0, \frac{\lambda - \lambda_0}{(\tau (\lambda - \frac{1 - \gamma_1}{\tau_0}) + 1)} \right\} > 0. \tag{17} \]

The requirement (17) on \( \alpha \) comes from the growth estimates in the proof of lemma 6.2. Note the fact that \( \alpha > 0 \) comes from \( \lambda - (1 - \gamma_1)/\tau_0 \geq \lambda - (1 - \gamma_1)((\lambda_0 + \alpha_1)/2) \geq \lambda - (1 - \gamma_1)\lambda_0 \geq \lambda - \lambda_0 > 0 \). The first inequality comes from the definition of \( \tau_0 \), the second from the fact that \( \alpha_1 < \lambda_0 \), the third from the fact that \( \gamma_1 < 1 \) and the fourth from (C1).

\[ \tilde{\eta} := e^{-\gamma \alpha} \left( 1 + \sum_{R \geq \log \delta^{-1}} e^{-(1-\gamma)R} \right) = e^{-\gamma \alpha} \left( 1 + \frac{\delta^{1-\gamma}}{1 - e^{-(1-\gamma)}} \right), \]

\( \tilde{\eta} \) appears in section 7.1. We can then formulate our final condition which includes the definition of the main constant which appears in the statement of the main theorem.

\[ \eta := \sum_{j=N}^{\infty} \tilde{\eta}^j = \frac{\tilde{\eta}^N}{1 - \tilde{\eta}} < 1. \tag{C4} \]

2.3. Statement of results

We are now ready to precisely state our results. First we define the set

\[ \Omega_* := \{ a \in \Omega : |(f_a^n)'(c_0)| \geq e^{\omega a} \forall n \geq 0 \}. \]

By classical results, for any \( a \in \Omega_* \), \( f_a \) admits an ergodic absolutely continuous invariant probability measure and thus \( \Omega_* \subseteq \Omega^* \).

**Main theorem.** Let \( f_a : I \to I \) be a family of \( C^2 \) unimodal maps with a quadratic critical point, of the form \( f_a(x) = f(x) - a \), for \( a \) belonging to some interval \( \Omega \) of parameter values. Suppose that there exist constants \( N, \delta, i, C_1, \lambda, \alpha_0, \lambda_0 \) such that conditions (A1)–(A4) and (C1)–(C4) hold. Then

\[ |\Omega_*| \geq (1 - \eta)|\Omega|. \]

The spirit of this result is to have a ready-made ‘formula’ for rigorously proving the existence, and for obtaining a lower bound for the probability, of stochastic dynamics in a given family of maps.

A very closely related result was proved by Jakobson in [13], where explicit conditions on a one-parameter family of maps are also formulated and a lower bound on the set of good parameters, similar to the bound given above, is obtained explicitly in terms of the given assumptions. A direct comparison between our estimate and that of [13] is not immediate as the way in which the assumptions are formulated is significantly different, also no application to an explicit family is given there, yielding a concrete bound such as the one which we obtain below. However, it would clearly be extremely interesting to understand the relation between the two results and particularly between the two approaches. It seems likely that each one might
have its own advantages and disadvantages in terms of its applicability to specific examples, and therefore they might provide complementary tools towards achieving a common goal.

As a first application of our main theorem we let $\lambda_0 = 0.61$ and

$$\Omega^* = \{a \in \Omega : |(f_n^a)'(c_0)| \geq e^{0.61n} \forall n \geq 0\} \subseteq \Omega^*.$$  

By classical results all these parameters are stochastic. Then we have the following theorem.

**Theorem 1.** Let $f_a(x) = x^2 - a$ be the quadratic family for $a \in \Omega := [2 - 10^{-4990}, 2]$. Then

$$|\Omega^*| \geq |\Omega^*| \geq 0.97|\Omega| > 10^{-5000}.$$  

This is not meant to be in any way ‘optimal’ or ‘sharp’ but rather to show that the explicit constants in the main theorem can indeed be calculated explicitly in at least one case. We emphasize that even though the interval of parameter values under consideration here is relatively small, this is the first known rigorous explicit lower bound for the set $\Omega^*$ in any context. The only other related work we are aware of is by Murray [28], who gives a heuristic argument to estimate the measure of $|\Omega|$, and by Simó and Tatjer [39, 40], who carried out careful numerical estimates of the length of some of the periodic windows, i.e. the open connected components of $\hat{\Omega}^-$ in $\hat{\Omega} = [\hat{a}, 2]$, where $\hat{a}$ is the Feigenbaum parameter at the limit of the first period-doubling cascade. They were able to compute the length of some relatively ‘large’ windows (up to lengths of the order of $10^{-30}$ or so, though not all the windows of up to this size) and their calculations suggest a lower bound for the overall proportion of $\hat{\Omega}^- \in \hat{\Omega}$ of about 10%. One expects the contribution of the smaller windows to be negligible but this is probably a hard statement to prove.

A combination of the numerical methods of Simó and Tatjer with the application of the estimates given above might form a good strategy for obtaining some global bounds for the relative measures of $\hat{\Omega}^-$ and $\hat{\Omega}^*$ in the entire parameter interval $\hat{\Omega}$. Indeed, our methods appear naturally suited to the analysis of small parameter intervals and thus, in some sense, take over precisely where the purely numerical estimates no longer work. Another possible strategy might involve taking advantage of the hyperbolicity of the renormalization operator, although even if such an argument were feasible, this strategy would probably give less concrete information about the actual location of parameters of $\hat{\Omega}^-$ and $\hat{\Omega}^*$.

### 2.4. Remarks

#### 2.4.1. Computer-assisted proofs

The main result given above is perhaps not so much a computer-assisted theorem in (non-uniformly) hyperbolic dynamics but rather a ‘standard’ theorem with computer-checkable assumptions. The general philosophy underlying this approach is that the combination of rigorous numerical/computational methods with deep geometric/analytic/probabilistic methods can be extremely powerful with each approach contributing to overcome the limitations of the other. Indeed there exist extremely powerful techniques in dynamical systems and ergodic theory which yield highly sophisticated results about the fine structure and long term behaviour of certain systems, but these methods often rely on certain geometric assumptions, usually hyperbolicity assumptions, which are highly non-trivial to verify in practice. On the other hand, numerical/computational methods are much more flexible and in principle can be applied to essentially any system but suffer from the fundamental limitation of having finite precision and being able to deal with only finite time properties. These two approaches are therefore naturally complementary and the results presented in this paper are considered a step in the direction of implementing this point of view.
There are of course several other results in dynamics which have benefited from such a combination of methods, although, as far as we know, not in the area of non-uniform hyperbolicity. We mention here just some of these: the pioneering work, also in the context of families of interval maps, of Collet et al [11] and Lanford [21] on the hyperbolicity of the renormalization operator, the development of computational implementations of ideas from Conley index theory, see [27,29], and in particular the book [26] for a comprehensive review and references. The history of the study of the Lorenz attractor [31] is particularly interesting from this point of view: for more than 20 years sophisticated geometrical/analytical methods have been applied to a ‘geometric model’ of the attractor, essentially assuming some hyperbolicity conditions of the equations. Recently these conditions have been verified by Tucker [44, 45] using a combination of numerical and analytic methods and the abstract results developed for the geometric model immediately apply to the actual Lorenz equations. Finally, we also mention the extremely interesting area of so-called constructive KAM theory which aims to give a more concrete and quantitative understanding of the classical abstract results of KAM theory (see [30] for a comprehensive survey and references). Although KAM theory deals in many ways with situations which are the complete opposite of the theory of hyperbolicity, there are some striking structural similarities such as the occurrence of certain dynamical phenomena for nowhere dense sets of positive Lebesgue measure. Analogously to the situation in KAM theory, the first 20 years of work in non-uniform hyperbolicity has been relatively abstract and the present work can perhaps be seen as an initial step in the direction of an analogous constructive theory of non-uniform hyperbolicity providing a concrete and quantitative understanding of the abstract results.

2.4.2. Computational issues. Though we have stressed the computability of our assumptions, and indeed given a concrete application, we emphasize that a systematic and extensive application of our results involves several non-trivial theoretical and computational issues.

The first, relatively obvious, issue is the development of rigorous computational algorithms to verify conditions (A1)–(A4). Of these conditions, (A1) is probably the most technically and computationally demanding requiring some non-trivial algorithm. However, even for condition (A1) and also for the other three conditions, the problem essentially boils down to having sufficient precision in the calculations and thus does not seem to present particular conceptual hurdles. Rather, it is likely that issues regarding the ‘computational cost’ and ‘efficiency’ of the algorithms will come into play, since in general the number of iterations which need to be computed will be quite large.

A second, less obvious but probably ultimately more difficult, issue is that of finding a systematic way of adjusting the constants $N, \delta, i, C_1, \lambda, \alpha_0, \lambda_0$ as well as the length of the subinterval $\Omega$ of parameters and the auxiliary constants, so that all the conditions are satisfied. A natural situation would be that conditions (A1)–(A4) have been verified for a certain set of constants $N, \delta, i, C_1, \lambda, \alpha_0, \lambda_0$, but these constants fail one of the conditions (C1)–(C4). One then has many options such as changing any one or more of the constants and try to verify (A1)–(A4) and (C1)–(C4) again. The interdependence of the various conditions is however quite subtle, non-monotone, and trial and error does not appear to be a satisfactory strategy in general. It would therefore be essential to find a systematic way of optimizing the choices in order to converge to a set of constants in which all conditions are satisfied.

2.4.3. Notation. We introduce some notation which will be used extensively below. We let $c_0 = c_0(a) = f_a(0)$ denote the critical value of $f_a$ and for $i \geq 0$, $c_i = c_i(a) = f^i(c_0)$. A key feature of the argument involves considering a family of maps from parameter space to
dynamical space which tracks the orbits of the critical points for different parameter values: for \( n \geq 0 \) and \( \omega \subseteq \Omega \) let
\[
\omega_n = \{ \epsilon_n(a); a \in \omega \} \subseteq I.
\]

3. Explicit estimates for the quadratic family

We prove theorem 1 in the next three sections. In section 3.2 we discuss the choice of the main constants and check conditions (A1)–(A4); in section 3.3 we discuss the choice of the auxiliary constants and check conditions (C1)–(C4); in section 3.4 we enclose a copy of the Maple worksheet used to carry out the explicit numerical calculations. Once the calculations are set up it is very easy to obtain the estimates for other intervals of parameter intervals. In particular, choosing smaller intervals generally yields smaller values for \( \eta \) and thus a larger proportion of stochastic parameters.

We mention that the calculations here are all completely analytic. We take advantage of the very special nature of the parameter value \( a = 2 \) to be able to obtain explicit formulae for all constants simply in terms of the constants \( \delta, \iota \) and the size of the parameter interval. The only role of the computer is the use of Maple to numerically evaluate the actual values of all intermediate constants in order to verify conditions (C1)–(C4) and to compute the value of the constant \( \eta \).

3.1. Two preliminary lemmas

In this section we prove two preliminary lemmas which motivate our choice of the constants in the following subsection.

**Lemma 3.1.** Let \( 1 \geq \delta, \iota > 0 \) and \( 2 \geq a^* > 1 \). Then, for any \( a \in [a^*, 2] \) the expansivity condition (A1) holds with constants
\[
C_1 = \sqrt{\frac{4 - \delta^2}{4 - \delta^2}} \quad \text{and} \quad \lambda = \log \left( \frac{2\delta}{\sqrt{4 - \delta^2}} \right).
\]

**Proof.** Let \( h : (0, 1) \to (-2, 2) \) be given by \( x = h(\theta) = 2 \cos \pi \theta \). For each \( a \in [1, 2] \) let \( \zeta_a = \pi^{-1} \cos^{-1}((\sqrt{2} + a)/4) \) and consider the family of intervals \( I_a = (\zeta_a, 1 - \zeta_a) \subseteq [0, 1] \) and their images \( I_2 = h(I_a) = (-\sqrt{2 + a}, \sqrt{2 + a}) \). For \( a = 2 \) we have \( f_2(I_2) = I_2 \) and for \( a \in [0, 2) \) we have \( f_a(I_a) \subset I_a \), since \( f_2(0) = -a > -\sqrt{2 + a} \). Thus, we shall always consider \( f_a : I_a \to I_a \) to be the restriction of \( f_a \) to \( I_a \) and implicitly consider \( h \) restricted to \( J_a \). We then define a family of maps \( g_a : J_a \to J_a \) by
\[
g_a(\theta) = h^{-1} \circ f_a \circ h(\theta) = \frac{1}{\pi} \cos^{-1} \left( 2 \cos^2 \pi \theta - \frac{a}{2} \right).
\]

Thus, \( h : J_a \to I_a \) defines a smooth conjugacy between \( f_a \) and \( g_a \) and, for any \( n \geq 1 \), we have \( f_a^n = h \circ g_a^n \circ h^{-1} \). Therefore, for \( x = h(\theta) \) we have
\[
Dx f_a^n(x) = Dh(g_a^n(h^{-1}(x))) \cdot Dg_a^n(h^{-1}(x)) \cdot Dh^{-1}(x)
\]
\[
= \frac{Dh(g_a^n(\theta))}{-Dh(\theta)} \cdot Dg_a^n(\theta) \cdot \frac{\sin \pi(g_a^n(\theta))}{\sin \pi(\theta)}
\]
\[
= \frac{\sin \pi(g_a^n(\theta))}{\sin \pi(\theta)} Dg_a^n(\theta) \quad \text{since} \quad Dh(\theta) = -2\pi \sin \pi \theta.
\]

(19)
Differentiating (18) and using the fact that \( \theta = \pi^{-1} \cos^{-1}(x/2) \), as well as the standard identity \( \sin(\cos^{-1}(x)) = \sqrt{1 - x^2} \), to get, for \( a^* \in (1, 2), a \in [a^*, 2] \) and \( x \notin \Delta \),

\[
|D_\theta g_a(\theta)| = \left| \frac{4 \cos \pi \theta \sin \pi \theta}{\sqrt{1 - (2 \cos^2 \pi \theta - \frac{a}{2})^2}} \right| = \left| 2x \sqrt{\frac{4 - x^2}{4 - (x^2 - a)^2}} \right| \geq 2\delta \sqrt{\frac{4 - \delta^2}{4 - (\delta^2 - a^*)^2}} = \lambda.
\]

The last inequality uses the fact that \( |D_\theta g_a(\theta)| \) is monotone increasing in \( |x| \) and \( a \). Moreover, we have

\[
\frac{\sin \pi (g_a^n(\theta))}{\sin \pi (\theta)} = \frac{\sin \pi (h^{-1}(f^n(x)))}{\sin \pi (h^{-1}(x))} = \frac{\sin(\cos^{-1}(f^n(x)/2))}{\sin(\cos^{-1}(x/2))} = \sqrt{\frac{4 - (f^n(x))^2}{4 - (x)^2}}.
\]

If \( |x| \geq \delta \) and \( |f^n(x)| \leq \delta \), substituting into (19), we get the first part of (A1). Similarly if \( |f^n_a(x)| \leq |x| \), e.g. if \( x \in f_a(\Delta^*) \), and/or \( f_a^n(x) \in \Delta \) we have the second part of (A1).

Note that for \( a = 2 \) we have \( |D_\theta g_2(\theta)| \equiv 2 \) for all \( \theta \neq 1/2 \) or \( x \neq 0 \) and thus in particular \( |D_\theta f_2^0(x)| \geq 2^k \) whenever \( |f_2^k(x)| \leq |x| \), which is in itself a quite remarkable and non-trivial estimate for \( f_2 \). It can also be checked directly from (18) that \( g_a \) is the standard 'top' tent map, thus \( h \) defines a smooth conjugacy between the 'top' quadratic map and the 'top' tent map. Indeed this was the main idea used by Ulam and von Neumann in their paper [46] to prove that \( f_2 \) admits an ergodic absolutely continuously invariant probability measure. Similar calculations to the ones carried out above have been used in several papers to prove similar estimates concerning the expansion outside critical neighbourhoods for maps close to \( f_2 \). For \( a < 2 \), \( g_a \) is no longer piecewise linear and has a critical point at \( \theta = 1/2 \) (corresponding to \( x = h^{-1}(\theta) = 0 \)). However, it has the advantage over the maps in the quadratic family that its fold near \( \theta = 1/2 \) is extremely 'sharp' and, outside a small neighbourhood of \( 1/2 \), the slope remains essentially close to \( \log 2 \) and even tends to infinity close to the boundaries of the intervals of definition \( J_a \).

Note also that the lemma, as stated, does not specify any relations between \( \delta \) and \( a^* \). Indeed, for a fixed \( a^* < 2 \), sufficiently small values of \( \delta \) give negative values of \( \lambda \). This corresponds to the fact that such a parameter may have an attracting periodic orbit (recall that an open and dense set of parameters have attracting periodic orbits). If at least one point of this orbit lies inside \( \Delta \) then the attracting nature of the orbit is invisible to derivative estimates outside \( \Delta \); in other words the existence of such an orbit is not incompatible to expansion estimates as in condition (A1) with \( \lambda > 0 \). However, if \( \delta \) is so small that all points of the attracting orbit lie outside \( \Delta \) then clearly (A1) can only be satisfied from some \( \lambda < 0 \).

**Lemma 3.2.** For all \( a \in \Omega \) we have

\[
f_a^n(c_0) > 1.5 > \delta'
\]

for all \( 1 \leq n \leq N := \frac{1}{\log 4} \log \frac{1}{(2 - (a^*)^2 + a^*)}.
\]

**Proof.** For each parameter value \( a \), \( f_a \) has two fixed points, one of which is given by \( p_a = (1 + \sqrt{1 + 4a})/2 \). For \( a \) close to \( 2 \) the second image of the critical point lies close to this fixed point. Indeed \( c_0(a^*) = f_0(c) = -a^* \) and \( c_1(a) = f_a(c_0(a)) = (c_0(a))^2 - a = a^2 - a \).

For \( a = 2 \) we have \( c_1(2) = 2 = p_2 \) and the critical point actually lands on the fixed point; for \( a < 2 \) and close to \( 2 \) we have \( a^2 - a < (1 + \sqrt{1 + 4a})/2 < 2 \) and thus \( c_1(a) \) lies to the left of the fixed point \( p_a \), which itself lies to the left of the fixed point \( p_2 \) for \( f_2 \).

Let \( I_1 = [c_1(a), p_2] \) and \( I_a = f_a^{-1}(I_1) = [c_a(a), p_a] \). Then a simple application of the mean value theorem, using the fact that the derivative is bounded above by \( 4 \), gives

\[
|I_a| \leq 4^{n-1} |I_1| \leq 4^{n-1} (2 - a^2 + a).
\]
for every $n \geq 1$. We have used here the fact that the length of $I_1$ is the distance between $c_1(a)$ and $p_a$ which is less than the distance between $c_1(a)$ and 2, which is precisely given by $2 - c_1(a) = 2 - (a^2 - 2)$. Since $p_a$ is very close to 2 for $a \in \Omega$ it is sufficient to show that
\[4^n - 1(2 - a^2 + a) \leq 1/4.\] (20)
In particular that $f^{n-1}$ is a bijection from $I_1$ to $I_n$ as long as (20) holds, and thus this implies in particular $c_n(a) \geq 1/2$ as required. Taking logs and solving for $n$ gives exactly the choice of $N$ above.

3.2. Condition (A1)–(A4)

In this section we discuss the choice of constants in our specific setting and show that all the necessary conditions are satisfied. As per the statement of the theorem, we have first of all
\[a^* = 2 - 10^{-4990}.\]
We note that the exponent 4990 is chosen in order to have a round figure for the lower bound for the measure of stochastic parameters after the exclusions have been carried out. We now fix the values of $\delta, \iota$ as
\[\delta = 10^{-1000} \quad \text{and} \quad \iota = 0.8\]
For these choices we have the following proposition.

**Proposition 3.1.** Conditions (A1)–(A4) are satisfied for constants

\[C_1 := \sqrt{\frac{4 - \delta^2}{4 - \delta^2}} \simeq 1, \quad \lambda := \log \left(2\delta \sqrt{\frac{4 - \delta^2}{4 - (\delta^2 - a^*)^2}}\right) \simeq 0.693,\]
\[N := \left\lfloor \frac{1}{\log 4(2 - (a^*)^2 + a^*)} \right\rfloor = 8287,\]
\[\alpha_0 := \frac{\log \delta^{-1}}{N} \simeq 0.277, \quad \lambda_0 := 0.2\alpha_0 + 0.8\lambda \simeq 0.610.\]

Proposition 3.1 says in particular that all constants can be chosen as formal functions of the three variables $a^*, \delta, \iota$. We shall show below that the same is also true for the other auxiliary functions. In the statement we have given the approximate values of the constants for our particular choices of $a^*, \delta$ and $\iota$.

**Proof.** Condition (A1) follows immediately from lemma 3.1. For (A2) let $\tilde{N}$ be the smallest integer for which
\[|\Omega_{\tilde{N}}| = |f^{\tilde{N}}_a(c_0) - 2| \geq 1/4 > \delta'.\]
For (A3) we just observe that lemma 3.2 and the choice of $\alpha_0 = \log \delta^{-1}/N$ imply
\[|f^n_a(c_0)| > 1 > \delta = e^{\log \delta} = e^{-\lambda^0} = e^{-\alpha_0 n} \quad \text{for } n \leq N.\]
It only remains to verify (A4), for which it suffices to choose $N_1 = 1$. Indeed, then we have $e^{-2\lambda_0}/(1 - e^{-\lambda_0}) \approx 0.65$ and so (A4) is easily seen to hold.
3.3. Conditions (C1)–(C4)

We proceed to compute the auxiliary constants and prove the following proposition.

**Proposition 3.2.** Conditions (C1)–(C4) are satisfied.

We emphasize that all the computations are carried out in Maple on a *purely symbolic* level and thus there is no risk of approximation errors being amplified over several calculations due to, e.g., floating point arithmetic. Maple essentially calculates the final constant $\eta$ as a formal function of the previously defined constants. Only at the final stage, the command `evalf` asks for this expression to be evaluated numerically. This command can also be used at the intermediate stages to obtain approximate values arising from the intermediate calculations, and these have been shown above and will be shown below but these approximations are not used in the calculation of $\eta$.

Condition (C1) follows immediately from the calculations given above:

\[ \lambda \simeq 0.693 > \lambda_0 \simeq 0.610 > \alpha_0 = \log \delta^{-1/N} \simeq 0.277 \quad \text{and} \quad \log \delta^{-1} - e^{-(1+\lambda_0)/2} \simeq 2302. \]

To verify the other conditions we need to compute some other constants. First of all we have

\[ M_1 = M_2 = 4 \quad \text{and} \quad L_1 = L_2 = 0.5. \]

Then we choose

\[ \alpha_1 = 0.2\lambda_0 + 0.8\alpha_0 \simeq 0.344 \]

We then set

\[ N_1 := \left\lceil \frac{1}{\log 4 \log \frac{1}{2\delta^2}} \right\rceil = 3321 < N = 8287. \]

To see that this satisfies (4) recall that from lemma 3.2 we have $|f_a'(c_0)| \geq 1.5$ for all $a \in \Omega$ and all $1 \leq i \leq N$. For these iterates, by the mean value theorem we have

\[ |f^n(\Delta_0)| \leq 4^n|\Delta_0| = 4^n\delta^2. \]

Thus, it is sufficient to solve $4^n\delta^2 < 1/2$ in terms of $n$ and this gives exactly the condition on $N_1$ above. We can then compute

\[ D_1 := \exp \left( \frac{1}{1 - e^{-\alpha_1}} + \frac{e^{-(\alpha_1 - \alpha_0)(N+1)}}{(1 - e^{-(\alpha_1 - \alpha_0)(N+1)})^2} \right) \simeq 31. \]

Note that the formal expression for $D_1$ contains a term $M_2/L_2$ in the exponent. However it is immediate from the proof of sublemma 5.1.1 that the two terms cancel out in the case of the quadratic family. The same is also true for the overall distortion constant $D$ computed below. To compute $D_2$ and $D_3$, we choose $N = N_1$ and use the fact that $(f'_a(c_0)) \geq 3'$ for all $1 \leq i \leq N$. This gives

\[ D_2 := \frac{3}{2} + \frac{e^{-\lambda_0(N+1)}}{1 - e^{-\lambda_0}} \geq 1 + \sum_{i=1}^{N} \frac{1}{3'} + \frac{e^{-\lambda_0(N+1)}}{1 - e^{-\lambda_0}} \geq 1 + \sum_{i=1}^{N} (f'_a)'(c_0(a^*)) + \frac{e^{-\lambda_0(N+1)}}{1 - e^{-\lambda_0}} \]
and similarly

\[ D_3^{-1} := \frac{1}{2} - \frac{e^{-\lambda_0(N+1)}}{1 - e^{-\lambda_0}} \leq 1 - \sum_{i=1}^{N} \frac{1}{3^i} - \frac{e^{-\lambda_0(N+1)}}{1 - e^{-\lambda_0}} \leq 1 - \sum_{i=1}^{N} \frac{1}{\left( f'_0(y) c_0(a^*) \right)^i} - \frac{e^{-\lambda_0(N+1)}}{1 - e^{-\lambda_0}}. \]

This gives values of

\[ D_2 \simeq 1.5 \quad \text{and} \quad D_3 \simeq 2. \]

By explicit computation we then get

\[ \gamma_0 \simeq 0.017 \]

and choose

\[ \gamma_1 = 0.835 \min \left\{ 1 - \frac{\log C_1^{-1} + 2 \log \log \delta^{-1}}{\log \delta}, 1 - \gamma_0 \right\} \simeq 0.982, \]

\[ \gamma_2 = 0.8(1 - \gamma_0 - \gamma_1) \simeq 0.117, \]

and so

\[ \gamma := \gamma_0 + \gamma_1 + \gamma_2 \simeq 0.970. \]

Then we get

\[ \hat{\delta} \simeq 11570, \quad \hat{\delta} \simeq 0.002, \quad \delta \simeq 10^{14}, \quad \Gamma_1 \simeq 10^{17}, \quad k_0 \simeq 0.003, \]

\[ \tau_0 \simeq 2.098, \quad \tau_1 \simeq 2.095. \]

We can therefore verify

\[ \tau_0 \alpha_0 \simeq 0.58 < 1. \quad \text{(C2)} \]

By more explicit computations we get

\[ C_3 \simeq 0.002 \quad \tilde{C}_3 \simeq 10^{-22}, \]

and we can verify (C3):

\[ \gamma_1 \simeq 0.835 > 0.821 \simeq \max \left\{ \frac{1 + \log(\hat{\delta} D_3 D_3^{-1}) + 2 \log \log \delta^{-1}}{\log \delta^{-1}}, \frac{\alpha_1 \tau_1 + \log(\Gamma_1 D_3 D_3^{-1} \tilde{C}_3^{1-e_0}) + 2 \log \log \delta^{-1}}{\alpha_1 \tau_1} \right\}. \]

We then compute

\[ \tau \simeq 12, \quad \alpha \simeq 0.009, \quad \tilde{\eta} \simeq 0.998, \]

and verify

\[ \eta = \frac{\tilde{\delta}^N}{1 - \eta} \simeq 0.082 < 1. \quad \text{(C4)} \]

This gives the value of \( \eta \) which appears in the statement of theorem 1.
3.4. Maple computations

We enclose a copy of the Maple worksheet used to carry out the explicit calculations. The notation should be self-explanatory.

```maple
> restart;
> delta:=10^(-1000): iota:=.8: epsilon:=10^(-4990): astar:= 2-epsilon:
> C1:=sqrt((4-delta^2*(2*iota))/(4-delta^2)): evalf[15](C1); 1.0
> lambda:=ln(2*delta*sqrt((4-delta^2)/
> (4-(delta^2-astar)^2))): evalf[15](lambda); 0.693147180559945
> N:=floor((ln(1/(2-(astar^2-astar))))/ln(4)): evalf[15](N); 8287.0
> alpha0:=-ln(delta)/N: evalf[15](alpha0); 0.27788761524283
> lambda0:=(alpha0*.2+ lambda*.8): evalf[15](lambda0); 0.610088761524283
> evalf[15](log(delta^(-1/N))); 0.277855085434300
> NR:=(exp(-2*lambda0))/(1-exp(-lambda0)): evalf[15](NR); 0.646331222657726
> C1b:=ln(delta^(-1))-exp(-(1+lambda0)/2): evalf[15](C1b);
> 2302.13802490916
> M1:=4:M2:=2:L1:=.5:L2:=.5:J:=4:
> alpha1:=(lambda0*.2+alpha0*.8): evalf[15](alpha1);
> 0.344301820623978
> N1:=floor(min(log((1-delta^iota)/delta^2)
> /log(4), N-1 )): evalf[15](N1);
> 332.0
> D1:=exp(((1-exp(-alpha1))^(-1) +
> (exp(-N1+1)*(alpha1-alpha0))/
> (1-exp(-(N1+1)*(alpha1-alpha0))))*
> ( 1-exp(-(alpha1-alpha0))))): evalf[15](D1);
> 0.971385467660
> D2:=(1.5 + ((exp(-lambda0*(N+1)))/
> (1-exp(-lambda0)))): evalf[15](D2); 1.5
> D3minus:=0.5-((exp(-lambda0*(N+1)))/(1-exp(-lambda0))):
> D3:=D3minus*(-1): evalf[15](D3); 2.0
> gamma0:=(1+ln(2) + 5*ln(-ln(delta)) ) /
> (-ln(delta)): evalf[15](gamma0);
> 0.0175464029210645
> gammamax:=min(1-gamma0,
> 1-( ln((C1^(-1))+2*ln(ln(delta^(-iota))))
> /ln(delta^(-iota)))):
> evalf[15](gammamax);
> 0.982453597078936
```
\[
\begin{align*}
\text{gamma1} & := 0.85 \times \text{gamma1max} ; \\
& \quad \text{evalf}[15](\text{gamma1}) ; \\
& \quad 0.835085557517095 \\
\text{gamma2} & := 0.8 \times (1 - (\text{gamma0} + \text{gamma1})) ; \\
& \quad \text{evalf}[15](\text{gamma2}) ; \\
& \quad 0.117894431649472 \\
\text{gammatil} & := \text{gamma0} + \text{gamma1} + \text{gamma2} ; \\
& \quad \text{evalf}[15](\text{gammatil}) ; \\
& \quad 0.970526392087632 \\
\text{Dhat} & := 2 + \frac{(2 \times C1^{-1} \times D2 \times D3 \times \exp(-\lambda))}{(1 - \exp(-\lambda))} + \\
& \quad \frac{(2 \times D1 \times D2 \times D3 \times (L1^{-2}))}{(1 - \exp(-\alpha1 - \alpha0))} ; \\
& \quad \text{evalf}[15](\text{Dhat}) ; \\
& \quad 11570.3753180732 \\
\text{logdiota} & := \log(\text{delta}^{-\iota}) ; \text{logdiota2} := \log(\text{delta}^{-\iota})^2 ; \\
\text{Dhathat} & := \begin{pmatrix}
2 + \exp(1) \times \frac{\log(\text{logdiota2})}{\log(\text{logdiota2} - 1)} \\
(\text{logdiota2} - (1 - \text{delta}^{-\iota}) \times \text{delta}^{-\iota} \times (1 - \text{gamma1})) \end{pmatrix} ; \\
& \quad \text{evalf}[15](\text{Dhathat}) ; \\
& \quad 0.00256440032673570 \\
\text{Dist} & := D2 \times D3 \times \exp((\text{Dhat} \times \text{Dhathat}) + \\
& \quad \frac{(C1^{-1} \times D2 \times D3 \times \exp(-\lambda))}{(1 - \exp(-\lambda))}) ; \\
& \quad \text{evalf}[15](\text{Dist}) ; \\
& \quad 463434666796857.0 \\
\text{Gamma1} & := \text{Dist} \times D1 \times D2 \times D3 \times \exp(1 + \lambda0)/(L1 \times C1) ; \\
& \quad \text{evalf}[15](\text{Gamma1}) ; \\
& \quad 430876756737302000 \\
\text{k0} & := \max((\ln(D1/L1) + \lambda0 + \alpha1) , 0) ; \\
& \quad \text{evalf}[15](\text{k0}) ; \\
& \quad 0.00275809649265672 \\
\text{tau1} & := \frac{2}{(\lambda0 + \alpha1)} ; \\
& \quad \text{evalf}[15](\text{tau1}) ; \\
& \quad 2.09557809691856 \\
\text{tau0} & := \frac{2 + \text{k0}}{(\lambda0 + \alpha1)} ; \\
& \quad \text{evalf}[15](\text{tau0}) ; \\
& \quad 2.09846800021815 \\
\text{evalf}[15](1 - \text{tau0} \times \alpha0) ; \\
& \quad 0.416929994518240 \\
\text{C3} & := D1^{-2} \times (\lambda0 + 2 + \alpha1)/(L1) \times (L1^{-2} + \alpha1/(\lambda0 + 2 + \alpha1)) ; \\
& \quad \text{evalf}[15](\text{C3}) ; \\
& \quad 0.00182183306992661 \\
\text{C3til} & := D1^{-2} \times (\lambda0 + \text{tau1} - 1) \times L1^{-1} \times (2 + \alpha1/(\lambda0 + 2 + \alpha1)) ; \\
& \quad \text{evalf}[15](\text{C3til}) ; \\
& \quad 1.40784264658478 \times 10^{-22} \\
\text{evalf}[10](\alpha1 \times \text{tau1}) ; \\
& \quad 0.72151113540 \\
\text{gamma1min} & := \max(\iota + ((\ln(\text{Dist} \times D2 \times D3)/(C1) + \\
& \quad 2 \times \ln(\ln(\text{delta}^{-\iota}))) / (\ln(\text{delta}^{-\iota}))) + \\
& \quad \alpha1 \times \text{tau1} + ((
\ln(\text{Gamma1} \times D2 \times D3 \times \text{C3til}^{-1} \times \exp(\alpha1 \times \text{tau1} - 1)) + \\
& \quad 2 \times \ln(\ln(\text{delta}^{-\iota}))))/ \\
& \quad (\iota \times \ln((\text{delta}^{-\iota})))) ; \\
& \quad \text{evalf}[15](\text{gamma1min}) ; \\
& \quad 0.821673721125574
\end{align*}
\]
> evalf[15](gamma1-gamma1min); 0.0134118363915209
> tau:=(tau0/(1-gamma1))*
> (1+(ln(J)-ln(Gamma1))/ln(delta^(-1)) +
> (2*ln(ln(delta^(-iota)))/ln(delta^(-1))):
> evalf[15](tau); 12.5909564129903
> alpha:=min(alpha0,(lambda-lambda0)/(tau*
> (lambda-((1-gamma1)/tau0))+1)):evalf[15](alpha);
> etatil:=exp(-gamma2*alpha)*(1+((delta^((1-gamma2))/
> (1-exp(-1+gamma2)))):evalf[15](etatil);
> eta:=(etatil^N)/(1-etatil):evalf[15](eta);
> 4. Setting up the induction

The remaining sections of the paper are devoted to proving the main theorem. In section 4 we give the formal inductive construction of the set $\Omega^\ast$. In section 5 we prove the main technical lemma concerning the shadowing of the critical orbit. In section 6 we prove the inductive step in the definition of $\Omega^\ast$ and in section 7 we obtain the lower bound on $|\Omega^\ast|$.

4.1. Inductive assumptions

Let $\Omega^{(0)} = \Omega$ and $P^{(0)} = \{\Omega^{(0)}\}$ denote the trivial partition of $\Omega$. Given $n \geq 1$ suppose that for each $k \leq n-1$ there exists a set $\Omega^{(k)} \subseteq \Omega$ satisfying the combinatorial structure described in the next paragraph and satisfying the following four bulleted properties.

There exists a partition $P^{(k)}$ of $\Omega^{(k)}$ into intervals such that each $\omega \in P^{(k)}$ has an associated itinerary constituted by the following information (for the moment we describe the combinatorial structure as abstract data, the geometrical meaning of this data will become clear in the next section). To each $\omega \in P^{(k)}$ is associated a sequence $0 = \theta_0 < \theta_1 < \cdots < \theta_r \leq k$, $r = r(\omega) \geq 0$ of escape times. Escape times are divided into three categories, i.e. substantial, essential and inessential. Inessential escapes possess no combinatorial feature and are only relevant to the analytic bounded distortion argument to be developed later. Substantial and essential escapes play a role in splitting itineraries into segments in the following sense. Let $0 = \eta_0 < \eta_1 < \cdots < \eta_s \leq k$, $s = s(\omega) \geq 0$ be the maximal sequence of substantial and essential escape times. Between any of the two $\eta_{i-1}$ and $\eta_i$ (and between $\eta_i$ and $k$) there is a sequence $\eta_{i-1} < v_1 < \cdots < v_l < \eta_i$, $l = t(\omega,i) \geq 0$ of essential return times (or essential returns) and between any two essential returns $v_{j-1}$ and $v_j$ (and between $v_j$ and $\eta_i$) there is a sequence $v_{j-1} < \mu_1 < \cdots < \mu_u < v_j$, $u = u(\omega,i,j) \geq 0$ of inessential return times (or inessential returns). Following essential and inessential return (respectively escape) there is a time interval $[v_j+1, v_j+p_j]$ (respectively $[\mu_j+1, \mu_j+p_j]$) with $p_j > 0$ called the binding period. A binding period cannot contain any return and escape times. Finally, associated with each essential and inessential return time (respectively escape) is a positive integer $r$ called the return depth (respectively escape depth).

- **Bounded recurrence.** We define the function $E^{(k)} : \Omega^{(k)} \to \mathbb{N}$ which associates with each $a \in \Omega^{(k)}$ the total sum of all essential return depths of the element $\omega \in P^{(k)}$ containing $a$
in its itinerary up to and including time \( k \). Note that \( E^{(k)} \) is constant on elements of \( \mathcal{P}^{(k)} \) by construction. Then, for all \( a \in \Omega^{(k)} \)

\[
E^{(k)}(a) \leq ak. 
\]

- Slow recurrence. For all \( a \in \Omega^{(k)} \) and all \( i \leq k \) we have

\[
|c_i(a)| = e^{a_0 i}. 
\]

- Hyperbolicity. For all \( a \in \Omega^{(k)} \)

\[
|\{(f_{a}^{k+1})^{-1}(c_0)\}| \geq e^{a_0(k+1)}. 
\]

- Bounded distortion. Critical orbits with the same combinatorics satisfy uniformly comparable derivative estimates: for every \( \omega \in \mathcal{P}^{(k)} \), every pair of parameter values \( a, b \in \omega \) and every \( j \leq v + p + 1 \), where \( v \) is the last return or escape before or equal to time \( k \) and \( p \) is the associated binding period, we have

\[
\frac{|(f_{a}^{j-1})'(c_0)|}{|(f_{b}^{j-1})'(c_0)|} \leq D \quad \text{and} \quad \frac{|c_j'(a)|}{|c_j'(b)|} \leq D. 
\]

Moreover, if \( k \) is a substantial escape a similar distortion estimate holds for all \( j < l \) (is the next chopping time) replacing \( D \) by \( \hat{D} \) and \( \omega \) by any subinterval \( \omega' \subseteq \omega \) which satisfies \( \omega'_j \subseteq \Delta^* \). In particular for \( j \leq k \), the map \( c_j : \omega \rightarrow \omega_j = \{c_j(a) : a \in \omega\} \) is a bijection.

### 4.2. Definition of \( \Omega^{(a)} \) and \( \mathcal{P}^{(a)} \)

For \( r \in \mathbb{N} \), let \( I_r = [e^{-r}, e^{-r+1}] \), \( I_{-r} = -I_r \). Then

\[
\Delta^* = [0] \cup \bigcup_{|r| \geq r_2 + 1} I_r 
\]

and

\[
\Delta = [0] \cup \bigcup_{|r| \geq r_2 + 1} I_r. 
\]

where \( r_2 = \log \delta - 1, r_3 = 1 \log \delta - 1 \). Recall that we can suppose without loss of generality that \( r_2, r_3, \in \mathbb{N} \). Subdivide each \( I_r \) into \( r^2 \) subintervals of equal length. This defines partitions \( \mathcal{I}, \mathcal{I}^+ \) of \( \Delta^* \) with \( \mathcal{I} = \mathcal{I}^+|_{\Delta} \). An interval belonging to either one of these partitions is of the form \( I_{r,m} \) with \( m \in \{1, r^2\} \). Let \( I_{r,m}^+ \) and \( I_{r,m}^- \) denote the elements of \( \mathcal{I}^+ \) adjacent to \( I_{r,m} \) and let \( \hat{I}_{r,m} = I_{r,m}^+ \cup I_{r,m}^- \). If \( I_{r,m} \) happens to be one of the extreme subintervals of \( \mathcal{I}^+ \) then let \( I_{r,m}^+ \) or \( I_{r,m}^- \), depending on whether \( I_{r,m} \) is a left or right extreme, denote the intervals \((-\delta^* - (\delta^* / (\log \delta^*)^2) \delta, -\delta^*] \) or \([\delta^*, \delta^* + (\delta^* / (\log \delta^*)^2)] \), respectively. We now use this partition to define a refinement \( \mathcal{P}^{(a)} \) of \( \mathcal{P}^{(a-1)} \). Let \( \omega \in \mathcal{P}^{(a-1)} \). We distinguish two different cases.

- **Non-chopping times.** We say that \( n \) is a non-chopping time for \( \omega \in \mathcal{P}^{(a-1)} \) if one (or more) of the following situations occur: (1) \( \omega_n \cap \Delta^* = \emptyset \); (2) \( n \) belongs to the binding period associated with some return or escape time \( v < n \) of \( \omega \) and \( \omega_n \cap \Delta^* \neq \emptyset \) but \( \omega_n \) does not intersect more than two elements of the partition \( \mathcal{I}^* \). In all three cases we let \( \omega \in \mathcal{P}^{(a)\emptyset} \). In cases (1) and (2) and in case (3) if \( \omega_n \) is not completely contained in \( \Delta^* \), no additional combinatorial information is added to the itinerary of \( \omega \). Otherwise, i.e. in case (3) with \( \omega_n \subseteq \Delta^* \), we consider two distinct possibilities: if \( \omega_n \cap (\Delta \cup I_{2r}) \neq \emptyset \) we say that \( n \) is an inessential return time for \( \omega \in \mathcal{P}^{(a)} \) and if \( \omega_n \subseteq \Delta^* \setminus (\Delta \cup I_{2r}) \) we say that \( n \) is an inessential escape time for \( \omega \in \mathcal{P}^{(a)} \). We define the corresponding depth by \( r = \max\{|r| : \omega_n \cap I_r \neq \emptyset\} \).

- **Chopping times.** In all the remaining cases, i.e. if \( \omega_n \cap \Delta^* \neq \emptyset \) and \( \omega_n \) intersects at least three elements of \( \mathcal{I}^* \), we say that \( n \) is a chopping time for \( \omega \in \mathcal{P}^{(a-1)} \). We define a natural subdivision

\[
\omega = \omega^\prime \cup \bigcup_{(r,m)} \omega^{(r,m)} \cup \omega^\rho, 
\]
so that each $\omega_{n(r,m)}$ fully contains a unique element of $\mathcal{I}_+$ (though possibly extending to intersect adjacent elements) and $\omega^l_n$ and $\omega^r_n$ are components of $\omega_n \setminus \Delta^+$ with $|\omega^l_n| \geq \delta/(\log \delta)^2$ and $|\omega^r_n| \geq \delta/(\log \delta)^2$. If the connected components of $\omega_n \setminus \Delta^+$ fail to satisfy the above condition on their length we just glue them to the adjacent interval of the form $w^{(r,m)}(n)$. By definition we let each of the resulting subintervals of $\omega$ be elements of $\hat{\mathcal{P}}(n)$. The intervals $\omega^l$, $\omega^r$ and $\omega_{n(r,m)}$ with $|r| \leq r \delta$ are called escape components and are said to have a substantial escape and essential escape, respectively, at time $n$. The corresponding values of $|r| \leq r \delta$ are the associated essential escape depths. All other intervals are said to have an essential return at time $n$ and the corresponding values of $|r|$ are the associated essential return depths. We note that partition elements $I_{\pm r\delta}$ do not belong to $\Delta$ but we still say that the associated intervals $\omega_{(\pm r\delta,m)}$ have a return rather than an escape.

This completes the definition of the partition $\hat{\mathcal{P}}(n)$ of $\Omega_1(n-1)$ and of the function $\mathcal{E}^{(n)}$ on $\Omega_1$. We define

$$\Omega^{(n)} = \{ a \in \Omega_1^{(n-1)} : \mathcal{E}^{(n)}(a) \leq an \}.$$  

Note that $\mathcal{E}^{(n)}$ is constant on elements of $\hat{\mathcal{P}}^{(n)}$. Thus, $\Omega^{(n)}$ is the union of elements of $\hat{\mathcal{P}}^{(n)}$ and we can define

$$\mathcal{P}^{(n)} = \hat{\mathcal{P}}^{(n)}|_{\Omega^{(n)}}.$$  

We shall prove the following proposition.

**Proposition 4.1.** Conditions $(BR)_n$, $(EG)_n$, $(SR)_n$, $(BD)_n$ all hold for $\Omega^{(n)}$.

Note that condition $(BR)_n$ is satisfied by construction; this is precisely the criteria by which the elements of $\Omega^{(n)}$ are chosen, see (21). Thus, the key part of the proposition is the remaining three conditions. This is the general step of an induction; it allows us to iterate the construction indefinitely and thus define the set

$$\Omega^* = \bigcap_{n \geq 0} \Omega^{(n)}.$$  

Note that, in particular, for every $a \in \Omega^*$ the map $f_a$ has an exponentially growing derivative along the critical orbit and thus exhibits stochastic dynamics. Our main theorem therefore reduces to the following proposition.

**Proposition 4.2.** $|\Omega^*| \geq (1-\eta)|\Omega|$.

We shall prove proposition 4.1 in section 6 and proposition 4.2 in section 7. First of all however, in section 5, we obtain several estimates which are used extensively in the proofs of both the propositions. First we make a few remarks concerning the conditions introduced in this section, all of which have played important roles in various results in one-dimensional dynamics.

The exponential growth condition was first introduced by Collet and Eckmann [10] who showed that it implies the existence of an absolutely continuous invariant measure for unimodal maps. Indeed, this is in some sense the most important condition for us as it guarantees that parameters in $\Omega^*$ are stochastic. For the purposes of the induction however this is not sufficient. Indeed, as often is the case with inductive arguments, it is easier to assume stronger conditions even if that means that we have to prove stronger estimates. The slow recurrence condition in the slightly different form $e^{-\sqrt{n}}$ was used by Benedicks and Carleson [6] and in the form given here, in conjunction with the exponential growth condition in [7]. The bounded recurrence condition, in the precise way in which it is stated here, is new, but it is closely related to the free period assumption of [7].
5. The binding period

In this section we make precise the definition of the *binding period* which is part of the combinatorial information given above and obtain several analytic estimates which play an important role in the following. In accordance with our inductive assumptions, we assume throughout this section that the sets $\Omega^k$ and $\mathcal{P}^k$ are defined and the conditions $(BR)^k$, $(SR)^k$, $(EG)^k$ and $(BD)^k$ hold for all $k \leq n - 1$.

Now let $k \leq n - 1$ and suppose that $\omega \in \mathcal{P}^k$ has an essential or inessential return or an essential or inessential escape at time $k$ with return depth $r$. For each $a \in \omega$ we let
\[
p(c_k(a)) = \min\{i : |c_{k+i}(a) - c_i(a)| \geq e^{-\alpha_1 i}\}. \quad (22)
\]
This is the time for which the future orbit of $c_k(a)$ can be thought of as shadowing or being bound to the orbit of the critical point (that is, in some sense, the number of iterations for which the orbit of $c(a)$ repeats its early history after the $k$th iterate). Then we define the *binding period* of $\omega_k$ as
\[
p(\omega_k) = \min_{a \in \omega}\{p(c_k(a))\}.
\]

The main result of this section is the following lemma.

**Lemma 5.1.** For every $a \in \omega$ we have
\[
p \leq \tau_0 \log |c_k(a)|^{-1} \leq \min\{\tau_0 (r + 1), \tau_0 \alpha_0 k\} < k
\]
and
\[
|\left(f^{j+1}\right)'(c_k(a))| \geq D_2 D_3 \Gamma_1 e^{(1-\gamma_1) r} r^2 \geq D_2 D_3 \Gamma_1 r^2 e^{-(1+\alpha_0)} e^{((1-\gamma_1)/\tau_0)(p+1)}. \quad (23)
\]

If $k$ is an essential return or an essential escape then
\[
|\omega_{k+p+1}| \geq \Gamma_1 e^{-\gamma r'}. \quad (24)
\]

We shall prove this result in a sequence of sublemmas. We assume the notation and the setup of the lemma throughout.

### 5.1. Binding period estimates for individual parameter values

We start by obtaining several estimates related to the ‘pointwise’ binding period $p(c_k(a))$. To simplify the notation we write $x = c_k(a)$ and omit the dependence on the parameter $a$ where there is no risk of confusion.

**Sublemma 5.1.1 (Distortion).** For all $a \in \omega$, all $x_0, z_0 \in [x_0, c_0]$ and all $0 \leq j \leq \min\{p, k\}$ we have
\[
\frac{|\left(f_j\right)'(z_0)|}{|\left(f_j\right)'(x_0)|} \leq D_1.
\]
We note that the distortion bound is formally calculated for iterates $j \leq \min\{p, k\}$ because we need to make use of the inductive assumption $(SR)^k$ in the proof. However, we shall show in the next lemma that we always have $p < k$ and therefore the estimates do indeed hold throughout the duration of the binding period.
**Proof.** By the mean value theorem and the standard inequality \( \log(1 + x) < x \) for \( x > 0 \), we have

\[
\log \left| \frac{(f_j')'(z_0)}{(f_j')'(y_0)} \right| = \log \left| \prod_{i=0}^{j-1} \frac{f_j'(z_i)}{f_j'(y_i)} \right| \leq \sum_{i=0}^{j-1} \log \left( 1 + \frac{|f'(z_i) - f'(y_i)|}{|f'(y_i)|} \right)
\]

By the definition of the binding period we have \( |y_i| \leq e^{-a_i} \), by the definition of \( N_1 \) in (4) we have \( |y_i| \geq 1 \) for all \( i \leq N_1 \) and by \( (SR)_k \) and the definition of the binding period in (22) we have \( |y_i| \geq e^{-a_i} - e^{-a_i} \) for \( i > N_1 \). Therefore, we have

\[
\sum_{i=0}^{j-1} \frac{|z_i - y_i|}{|y_i|} \leq \sum_{i=0}^{N_1} e^{-a_i} + e^{-a_i} \sum_{i=N_1+1}^{\infty} e^{-a_i} e^{-a_i} \sum_{i=N_1+1}^{\infty} e^{-a_i} (N_1+1) (1 - e^{-a_i}) \]

The result then follows by the definition of \( D_1 \) in (5).

**Sublemma 5.1.2 (Duration).** For all \( a \in \omega \) and \( p = p(c_k(a)) \) we have

\[
p \leq \frac{2 \log |c_k(a)|^{-1} + \log(D_1 L_1^{-1}) + \lambda_0 + \alpha_1}{\lambda_0 + \alpha_1} \leq \tau_0 \log |c_k(a)|^{-1} \tag{26}
\]

In particular,

\[
p \leq \min\{\tau_0(r + 1), \tau_0 \alpha_0 k\} < k \tag{27}
\]

**Proof.** Let \( \hat{p} = \min\{p, k\} \). We shall show that the above estimates work for \( \hat{p} \) and obtain as a corollary such that \( \hat{p} < k \) and therefore \( p = \hat{p} \). For simplicity let \( \gamma_0 = [x_0, c_0] \) and \( \gamma_i = f^i(\gamma_0) \). Then the mean value theorem and sublemma 5.1.1 imply \( |\gamma_{\hat{p}-1}| \geq D_1^{-1} |(f_{\hat{p}-1})'(c_0)||\gamma_0| \), condition \( (EG)_k \) gives \( |(f_j')'(c_0)| \geq e^{\alpha_1} \) for all \( j \leq \hat{p} \), the definition of binding gives \( |\gamma_{\hat{p}-1}| \leq e^{-a_i(\hat{p}-1)} \) and (2) gives \( |y_0| \geq L_1 |x|^2 \). Combining all these statements gives

\[
e^{-a_i(\hat{p}-1)} \geq |\gamma_{\hat{p}-1}| \geq D_1^{-1} |(f_{\hat{p}-1})'(c_0)||\gamma_0| \geq D_1^{-1} e^{\alpha_1(\hat{p}-1)} |\gamma_0| \geq D_1^{-1} L_1 e^{\lambda_0(\hat{p}-1)} |x|^2 \]

Rearranging gives \( e^{(\lambda_0 + \alpha_1)(\hat{p}-1)} \leq D_1 L_1^{-1} |x|^{-2} \) and taking logarithms on both sides we get

\[
\hat{p} - 1 \leq \frac{2 \log |x|^{-1} + \log(D_1 L_1^{-1})}{\lambda_0 + \alpha_1} \tag{28}
\]
Now, since $|x| \leq \delta'$ we can use the definition of $k_0$ in (14) to get
\[
\hat{p} \leq \frac{2 \log |x|^{-1} + \log(D_1 L_1^{-1}) + \lambda_0 + \alpha_1}{\lambda_0 + \alpha_1} \\
\leq \frac{1}{\lambda_0 + \alpha_1} \left( 2 \log |x|^{-1} + \frac{\log(D_1 L_1^{-1}) + \lambda_0 + \alpha_1}{\log \delta^{-1}} \right) \\
\leq \frac{1}{\lambda_0 + \alpha_1} \left( 2 + \frac{\log(D_1 L_1^{-1}) + \lambda_0 + \alpha_1}{\log \delta^{-1}} \right) \log |x|^{-1} \\
\leq \frac{2 + k_0}{\lambda_0 + \alpha_1} \log |x|^{-1} = \tau_0 \log |x|^{-1}.
\] (29)
By condition ((SR)$_k$) we also have $|c_\tau(a)| \geq e^{-\omega\alpha_1}$; we then get
\[
\hat{p} \leq \tau_0 \alpha_0 \alpha_1 k,
\] (30)
which is $< k$ by (C2). In particular, $p = \hat{p} < k$. Moreover, from (29) and the fact that $|c_\tau(a)| \geq e^{-r} / 2$ we also get $p \leq \tau_0 (r + \log 2)$. \hfill \Box

\textbf{Sublemma 5.1.3 (Expansion). For all $a \in \omega$ and $p = p(c_\tau(a))$ we have}
\[
|(f^{p+1})'(c_\tau(a))| \geq C_3^\alpha |c_\tau(a)|^{\alpha_1 n_1^{-1}}.
\] (31)
Recall that $\alpha_1 \tau_1 := 2\alpha_1 / (\lambda_0 + \alpha_1)$, see definition in (14), is strictly less than 1 because $\alpha_1 < \lambda_0$ by the choice of $\alpha_1$ in (3).

\textbf{Proof.} We estimate first of all the average expansion between $\gamma_0$ and $\gamma_p$. Since $|\gamma_0| \leq L_1^{-1}|x|^2$ by (2) and $|\gamma_p| \geq e^{-\alpha_1 p}$ by the definition of $p$, we have $|\gamma_p| / |\gamma_0| \geq L_1 e^{-\alpha_1} |x|^{-2}$. Applying the mean value theorem we conclude that there exists some point $\xi_0 \in \gamma_0$ for which $|(f^p)'(\xi_0)| \geq L_1 e^{-\alpha_1} |x|^{-2}$. Then, using the bounded distortion sublemma 5.1.1, it follows that $|(f^p)'(x_0)| \geq D_1^{-1} L_1 e^{-\alpha_1} |x|^{-2}$. Finally, using the fact that $|f'(x)| \geq L_1 |x|$ we obtain
\[
|(f^{p+1})'(x)| = |f'(x)| \cdot |(f^p)'(x_0)| \geq D_1^{-1} L_1^2 e^{-\alpha_1} |x|^{-1}.
\] (32)
Using the upper bound on $p$ from sublemma 5.1.2 gives
\[
e^{-\alpha_1 p} \geq e^{-\alpha_1 (2 \log |x|^{-1} \log(D_1 L_1^{-1}) / (\lambda_0 \alpha_1))} = (D_1 L_1^{-1})^{-\alpha_1 / (\lambda_0 \alpha_1)} |x|^{2\alpha_1 / (\lambda_0 \alpha_1)}.
\] Substituting this into (32) gives the result; recall definition of $C_3$ in (15). \hfill \Box

\subsection*{5.2. Parameter dependence}

We are now almost ready to prove our main lemma 5.1. This involves some estimates regarding the dependence of images of critical points on the parameter. We shall therefore need the following statement which is of intrinsic interest and of wider scope and will be used again later on in the argument.

\textbf{Lemma 5.2. For any $1 \leq k \leq n - 1$, $\omega \in \mathcal{P}^{(k)}$ and $a \in \omega$ we have}
\[
D_2 \geq \frac{|c_k^\omega(a)|}{|f_{a_k}^{k+1}(c_\omega)|} \geq D_3^{-1}.
\]
In particular, for all $1 \leq i < j \leq k + 1$, there exists $\tilde{a} \in \omega$ such that
\[
\frac{1}{D_2 D_3} |f_{\tilde{a}}^{j-i} (c_\omega(\tilde{a}))| \leq \frac{|\omega_j|}{|\omega_i|} \leq D_2 D_3 |f_{\tilde{a}}^{j-i} (c_\omega(\tilde{a}))|.
\] (33)
Proof. Recall that \( c_{k+1}(a) = f(c_k(a)) + a \). Therefore, by the chain rule we have

\[
c'_{k+1}(a) = 1 + f'(c_k)c'_k(a)
\]

\[
= 1 + f'(c_k)[1 + f'(c_{k-1})c'_{k-1}(a)]
\]

\[
= 1 + f'(c_k) + f'(c_k)f'(c_{k-1})c'_{k-1}(a)
\]

\[
= 1 + f'(c_k) + f'(c_k)f'(c_{k-1})[1 + f'(c_{k-2})c'_{k-2}(a)]
\]

\[
= 1 + f'(c_k) + f'(c_k)f'(c_{k-1}) + f'(c_k)f'(c_{k-1})f'(c_{k-2})c'_{k-2}(a)
\]

\[
= \cdots
\]

\[
= 1 + f'(c_k) + f'(c_k)f'(c_{k-1}) + \cdots + f'(c_k)f'(c_{k-1}) + \cdots + f'(c_1)f'(c_0).
\]

Dividing both sides by \((f^{k+1})'(c_0) = f'(c_k)f'(c_{k-1}), \ldots, f'(c_1)f'(c_0)\) gives

\[
\frac{c'_{k+1}(a)}{(f^{k+1})'(c_0)} = 1 + \sum_{i=1}^{k+1} \frac{1}{(f^i)'(c_0)}.
\]

(34)

Now let \( \mathcal{N} \) be as in condition (A4) and in the definition of the constants \( D_2, D_3 \) in (6) and (7). Then, for \( 1 \leq k < \mathcal{N} \) we have

\[
\left| \frac{c'_{k+1}(a)}{(f^{k+1})'(c_0)} \right| = \left| 1 + \sum_{i=1}^{k+1} \frac{1}{(f^i)'(c_0)} \right| \leq D_2.
\]

Using the fact that by \((EG)_k\) we have \( |(f^i)'(c_0)| \geq e^{\lambda i} \) for all \( 1 \leq i \leq k + 1 \), for any \( k \geq \mathcal{N} \) we have

\[
\left| \frac{c'_{k+1}(a)}{(f^{k+1})'(c_0)} \right| = \left| 1 + \sum_{i=1}^{\mathcal{N}} \frac{1}{(f^i)'(c_0)} + \sum_{i=\mathcal{N}+1}^{k+1} \frac{1}{(f^i)'(c_0)} \right|
\]

\[
\leq \left| 1 + \sum_{i=1}^{\mathcal{N}} \frac{1}{(f^i)'(c_0)} + \sum_{i=\mathcal{N}+1}^{k+1} \frac{1}{(f^i)'(c_0)} \right|
\]

\[
\leq \left| 1 + \sum_{i=1}^{\mathcal{N}} \frac{1}{(f^i)'(c_0)} + \sum_{i=\mathcal{N}+1}^{k+1} e^{-2\lambda i} \right|
\]

\[
\leq \left| 1 + \sum_{i=1}^{\mathcal{N}} \frac{1}{(f^i)'(c_0)} + \sum_{i=\mathcal{N}+1}^{\infty} e^{-2\lambda i} \right|
\]

\[
\leq \left| 1 + \sum_{i=1}^{\mathcal{N}} \frac{1}{(f^i)'(c_0)} + \frac{e^{-2\lambda(\mathcal{N}+1)}}{1 - e^{-2\lambda}} \right| \leq D_2.
\]

The lower bound is obtained similarly. If \( k < \mathcal{N} \) we have

\[
\left| \frac{c'_{k+1}(a)}{(f^{k+1})'(c_0)} \right| = \left| 1 + \sum_{i=1}^{k+1} \frac{1}{(f^i)'(c_0)} \right| \geq D_3^{-1},
\]
and if $k \geq \bar{N}$ we have
\[
\frac{c_{k+1}'(a)}{(f_{k+1}^p)'(c_0)} \geq 1 + \sum_{i=1}^{\tilde{N}} \frac{1}{(f_i')'(c_0)} + \sum_{i=\tilde{N}+1}^{k+1} \frac{1}{(f_i')'(c_0)}.
\]
This completes the proof of the first set of inequalities. For the second, consider the composition $c_j \circ c^{-1}_i: \omega_i \to \omega_j$. By the mean value theorem there exists $\tilde{a} \in \omega$ such that
\[
|\omega_j| = |(c_j \circ c^{-1}_i)'((c_i(\tilde{a})))|.
\]
Then by the chain rule and the first set of inequalities we get the desired statement.

Lemma 5.2 implies that all parameters in a given interval $\omega$ satisfy comparable derivative estimates during the binding period $p = p(\omega_k)$. This allows us to extend the expansion estimates at the end of a binding period to a parameter for which we do not necessarily have $p(c_k(a)) = p(\omega_k)$; recall that the binding period of the entire interval is defined as the minimum of these binding periods.

**Sublemma 5.2.1.** Let $p = p(\omega_k)$. Then, for all $a \in \omega$ we have
\[
|(f^{\omega+1})'(c_k(a))| \geq \tilde{C}|c_k(a)|^{p_{\omega+1} - 1}.
\]

**Proof.** For the parameter values $\tilde{a} \in \omega$ such that $p(c_k(\tilde{a})) = p(\omega_k)$, the result follows immediately from sublemma 5.1.3. Note that there must exist such an $\tilde{a}$. For a generic $a$ we argue as follows. First of all we consider the iterates coming after the return. For any $a, \tilde{a} \in \omega$, we have
\[
|(f_{\omega}^p)'(c_{k+1}(a))| \geq \frac{1}{D_1} |(f_{\omega}^p)'(c_0(a))| \quad \text{by sublemma 5.1.1},
\]
\[
\geq \frac{1}{D_1D_2} |c_{\omega}'(a)| \quad \text{by lemma 5.2},
\]
\[
\geq \frac{1}{DD_1D_2D_3} |c_{\omega}'(\tilde{a})| \quad \text{by the inductive assumption} ((BD)_{k+1}),
\]
\[
\geq \frac{1}{DD_1D_2D_3} |(f_{\omega}^p)'(c_0(\tilde{a}))| \quad \text{by lemma 5.2},
\]
\[
\geq \frac{1}{DD_1D_2D_3} |(f_{\omega}^p)'(c_{k+1}(\tilde{a}))| \quad \text{by sublemma 5.1.1}. \quad (36)
\]

Now we deal with the actual return iterate. By construction we have $|\omega_k| \leq |c_k(\tilde{a})|/2$. Therefore, using (2), we get
\[
|(f_{\omega}^p)'(c_k(a))| \geq L_1 |c_k(a)| \geq L_1 |(c_k(\tilde{a}) - |\omega_k|)| \geq L_1 |c_k(\tilde{a})|/2 \geq L_1^2 |(f_{\omega}^p)'(c_k(\tilde{a}))|/2. \quad (37)
\]
Since $|(f_{\omega}^p)'(c_k(a))| = |f'(c_k(a))| |(f_{\omega}^p)'(c_{k+1}(a))|$ and similarly for $\tilde{a}$, (36) and (37) imply
\[
|(f_{\omega}^p)'(c_k(a))| \geq \frac{L_1^2}{2DD_1^2D_2D_3} |(f_{\omega}^p)'(c_k(\tilde{a}))| \quad \text{and similarly for } \tilde{a}.
\]

(38)
for all \(a, \tilde{a} \in \omega\). Now, choosing \(\tilde{a}\) such that \(p(\omega_k) = p(c_k(\tilde{a}))\) and applying (31) and the fact that \(|c_k(a)| \leq 2|c_k(\tilde{a})|\), we get
\[
|\langle f^{p+1} \rangle(c_k(\tilde{a}))| \geq C_3|c_k(\tilde{a})|^{a_1\tau_1 - 1} \geq C_32^{a_1\tau_1 - 1}|c_k(a)|^{a_1\tau_1 - 1}.
\] (39)
Substituting (39) into (38) and using the definition of \(\tilde{C}_3\) in (15) gives the result. \(\square\)

**Proof of lemma 5.1.** The upper bound (23) on \(p\) follows immediately from (26). For the expansion estimates, (35) and the fact that \(|c_k(a)| \leq e^{-r}r^{\tau_0 - 1}\) imply
\[
|\langle f^{p+1} \rangle(c_k(a))| \geq \tilde{C}_3|c_k(a)|^{a_1\tau_1 - 1} \geq \tilde{C}_3e^{-(r-1)(a_1\tau_1 - 1)}.
\]
To get the first inequality in (24) it is therefore sufficient to show that
\[
\tilde{C}_3e^{-(r-1)(a_1\tau_1 - 1)} \geq r^2D_2D_3\Gamma_1e^{(1-\gamma)p}. \tag{40}
\]
Rearranging and taking logs we get that (40) holds if and only if
\[
y_1 \geq \frac{\log(\Gamma_1D_2D_3\tilde{C}_3^{-1}e^{a_1\tau_1 - 1}) + 2\log r}{r} + \alpha_1\tau_1. \tag{41}\]
Since the expression on the right-hand side is strictly decreasing with \(r\), it is sufficient for this inequality to be verified for \(r = \iota \log \delta^{-1}\) which follows from (C3). This gives the first inequality in (24). To get the second inequality in (24) we use the fact that \(r > p\tau_0 - 1\), which comes from \(p \leq \tau_0(\tau_0 + 1)\) in (27), and get
\[
|\langle f^{p+1} \rangle(c_k(a))| \geq \Gamma_1D_2D_3e^{((1-\gamma)p_1)r^2} \geq \Gamma_1D_2D_3e^{p(1-\gamma)p_1/\tau_0}e^{-(1-\gamma)p_1}r^2 = \Gamma_1D_2D_3r^2e^{-(\tau_0 + 1)(1-\gamma)p_1/\tau_0}e^{p(1-\gamma)p_1/\tau_0}.
\]
We then use the fact that \(\tau_0 > 2/(\lambda_0 + \alpha_1)\) and \(\alpha_1 < \lambda_0\) by definition to get
\[
\frac{(\tau_0 + 1)(1-\gamma_1)}{\tau_0} = \frac{\tau_0 + 1}{\tau_0} = 1 + \frac{1}{\tau_0} < 1 + \frac{\lambda_0 + \alpha_1}{2} < 1 + \lambda_0.
\]
This completes the proof of (24). Now suppose that \(k\) is an essential return or escape, in particular \(|\omega_k| \geq e^{-r}/r^2\). Therefore lemma 5.2 implies that there exists some parameter \(a \in \omega\) such that
\[
|\omega_{k+1}| \geq \frac{1}{D_2D_3}\langle f^{p+1} \rangle(c_k(a))| |\omega_k| \geq \frac{e^{-r}}{r^2D_2D_3} |\langle f^{p+1} \rangle(c_k(a))|,
\]
and the result follows immediately by (24). \(\square\)

### 5.3. Uniform bounds on the sum of all inessential return depths

The following result is essentially a corollary of lemma 5.1. It shows that the total length of the binding periods associated with a sequence of inessential returns can be bounded in terms of the return depth of the immediately preceding essential return.

**Lemma 5.3.** Let \(\omega \in \mathcal{P}^{(n)}\) and suppose that \(\omega\) has an essential return with return depth \(\tau_0\) at some time \(\mu_0 \leq n - 1\). Let \(\mu_1, \ldots, \mu_u\) denote the following inessential returns which occur after time \(\mu_0\) and before any subsequent chopping time and before time \(n\). Let \(p_i, i = 0, \ldots, u\) denote the corresponding binding periods. Then
\[
\sum_{i=0}^{u} (p_i + 1) \leq r\tau_0.
\]
**Proof.** Since $\mu_1, \ldots, \mu_u$ are all returns (to $\Delta$), condition (A1) gives

$$|\langle f^u_{\mu_i}\rangle^u_{\mu_i}((c_{\mu_i+1}(a)))| \geq 1$$

for $0 \leq i \leq u - 1$ and $\forall a \in \omega$. Note that we do not have (and will not need) such an estimate for $i = u$ since we do not have information about the location of $\omega_u$ (nor indeed do we even know if $n \geq \mu_u + \mu_u + 1$). Equation (24) in lemma 5, together with the fact that $r_i^2 \geq r_i^2 = (\log \delta)^2 \geq e^{-(1+\delta)}$ by condition (C1), gives

$$|\langle f^u_{\mu_i+1}\rangle^u_{\mu_i+1}| \geq D_2^2 \Gamma_1 \epsilon^{((1-\gamma)/\tau_0)((p+1)^{r_0+1})}$$

for all $0 \leq i \leq u$ and $a \in \omega$. Note that this estimate does include the case $i = u$. Combining these estimates by the chain rule and applying lemma 5.2 we then get

$$\frac{|\omega_{\mu_i}|}{|\omega_{\mu_i}|} \geq \Gamma_1 \epsilon^{((1-\gamma)/\tau_0)((p+1)^{r_0+1})} \quad \text{for } i = 0, \ldots, u - 1 \quad \text{and} \quad \frac{|\omega_{\mu_i+1}|}{|\omega_{\mu_i}|} \geq \Gamma_1 \epsilon^{((1-\gamma)/\tau_0)((p+1)^{r_0+1})},$$

and therefore

$$\frac{|\omega_{\mu_i+1}|}{|\omega_{\mu_i}|} = \frac{|\omega_{\mu_i+1}|}{|\omega_{\mu_i}|} \cdots \frac{|\omega_{\mu_2}|}{|\omega_{\mu_1}|} \geq \Gamma_1^{u+1} \epsilon^{((1-\gamma)/\tau_0) \sum_{i=0}^{r_0+1}(p_i+1)}.$$

Now since $\mu_0$ is an essential return we have $|\omega_{\mu_0}| \geq e^{-\tau_0} / r_0^2$ and thus

$$|I| \geq |\omega_{\mu_0}| \geq |\omega_{\mu_0}| \Gamma_1^{u+1} \epsilon^{((1-\gamma)/\tau_0) \sum_{i=0}^{r_0+1}(p_i+1)} \geq \Gamma_1^{u+1} \epsilon^{((1-\gamma)/\tau_0) \sum_{i=0}^{r_0+1}(p_i+1)}.$$

We have replaced $\Gamma_1^{r_0+1}$ by $\Gamma_1$ in the inequality because we do not know how many inessential returns are there. There may be none, in which case we have $u = 0$. Taking logs and rearranging we get

$$\sum_{i=0}^{u} (p_i + 1) \leq \frac{r_0 - \tau_0}{1 - \gamma_i} \log \left( \frac{|I| \epsilon^{r_0} / \Gamma_1}{r_0^2} \right)$$

$$= \frac{r_0 - \tau_0}{1 - \gamma_i} \left( 1 + \frac{\log |I| + 2 \log r_0 - \log \Gamma_1}{r_0} \right)$$

$$\leq \frac{r_0 - \tau_0}{1 - \gamma_i} \left( 1 + \frac{\log |I| + 2 \log r_0 - \log \Gamma_1}{r_0} \right)$$

$$= \tau r_0. \quad (42)$$

The last inequality follows from the fact that the fraction in parentheses is decreasing in $r$; therefore, $r_0$ can be replaced by $r$ giving precisely the definition of $\tau$ in (16). Note moreover that

$$\sum_{i=0}^{u} (p_i + 1) \geq p_0 + 1 > 0, \quad (43)$$

and therefore the above inequality implies a fortiori that $\tau > 0$. \qed

### 6. Positive exponents in dynamical space

In this section we prove proposition 4.1. Note that the combinatorics and the recurrence condition $(BR)_a$ are satisfied for every $a \in \Omega^{(\alpha)}$ by construction. We therefore just need to verify the slow recurrence, exponential growth and bounded distortion conditions. We shall do this in three separate subsections.
6.1. Slow recurrence

**Lemma 6.1.** For every \( a \in \Omega^{(n)} \), \((SR)_n\) holds.

**Proof.** The statement clearly holds if \( n \) is not an escape or a return or a bound iterate for \( \omega \) containing \( a \). Now suppose that \( n \) is an escape iterate. Then it must be that \( n \geq N \) and then (1) implies that \( e^{-\alpha N} \leq e^{-\alpha_0 n} < \delta \). Since \( n \) is an escape we must have \(|c_4(a)| \geq \delta \geq e^{-\alpha_0 N} \geq e^{-\alpha_0 n}\). If \( n \) is an essential return the result follows immediately by the bounded recurrence condition \((BR)\). If \( n \) is an inessential return, it follows immediately from the binding period estimates lemma 5.3 that its return depth must be less than the return depth of the preceding inessential return depth and thus in particular satisfies the required estimate. If \( n \) belongs to a binding period the same reasoning gives the result. \( \square \)

6.2. Exponential derivative growth

**Lemma 6.2.** For every \( a \in \Omega^{(n)} \), \((EG)_n\) holds.

**Proof.** Let \( \omega \in \mathcal{P}^{(n)} \) be the element containing \( a \). If \( \omega \) has no returns before time \( n \) then this implies that \( c_4(a) \notin \Delta \) for all \( k \leq n \) and therefore we have \(|(f^{(n+1)})'(c_0(a))| \geq \lambda^{(n+1)} > e^{\lambda((n+1))}\) by condition (A1) and the fact that \( \alpha > \lambda_0 \). If \( \omega \) has a non-empty sequence of returns before time \( n \), let \( v_1 < v_2 < \cdots < v_q \leq n \) be all the free (essential and inessential) returns up to \( n \) and \( p_i \) the corresponding binding period. Lemma 5.3 and the bounded recurrence condition \((BR)\) imply

\[
\sum_{i=1}^{q} (p_i + 1) \leq \tau \sum_{\text{essential returns}} r_j(\omega) = \tau e^{(\nu_q)}(\omega) \leq \tau \alpha v_q.
\]

(44)

Splitting the orbit of \( c_0 \) into free and bound iterates and using condition (A1), the binding period estimate (24) and (44), we get

\[
|(f^{(n)})'(c_0)| \geq \Gamma_1^{\nu_q} e^{(1-v_q)(\lambda^{(n+1)} - \sum_{i=1}^{\nu_q - 1} (p_i + 1))}
\]

\[
\geq \Gamma_1^{\nu_q} e^{(1-v_q)\alpha(\lambda^{(n+1)} - \sum_{i=1}^{\nu_q - 1} (p_i + 1))}
\]

\[
\geq \Gamma_1^{\nu_q} e^{\lambda^{(n+1)} - \sum_{i=1}^{\nu_q - 1} (p_i + 1))}
\]

\[
\geq \Gamma_1^{\nu_q} e^{(\lambda^{(n+1)} - \sum_{i=1}^{\nu_q - 1} (p_i + 1))}
\]

Then condition \((BR)\) implies in particular \(|c_4| \geq e^{-(r_q + 1)} \) with \( r_q \leq \alpha v_q \) which, together with (2), implies

\[
|f'(c_4)| \geq L_1 e^{-\alpha r_q - 1}.
\]

(46)

If \( n = v_q \), equations (45) and (46) give

\[
|(f^{(n+1)})'(c_0)| \geq L_1 \Gamma_1^{\nu_q} e^{-(\lambda - \alpha)^{-1}} (\lambda - (1-v_q)/r_0 - \alpha) e^{(\lambda - \alpha)^{-1}}
\]

\[
\geq L_1 \Gamma_1^{\nu_q} e^{-(\lambda - \alpha)^{-1}} (\lambda - (1-v_q)/r_0 - \alpha) e^{(\lambda - \alpha)^{-1}}.
\]

(47)

note that

\[
\lambda - \alpha (\lambda - (1-v_q)/r_0) - \alpha \geq \lambda_0 \iff \alpha \leq (\lambda - \lambda_0) \left( \tau \left( \frac{1 - v_q}{r_0} \right) + 1 \right).
\]

(48)

This condition is therefore satisfied by the definition of \( \alpha \) in (17). In particular, from (47) we have

\[
|(f^{(n+1)})'(c_0)| \geq L_1 \Gamma_1^{\nu_q} e^{-(1-\lambda_0)(\lambda_0 + 1)} e^{\lambda_0(n+1)}.
\]

(49)

We have

\[
\Gamma_1^{\nu_q} \geq \Gamma_1 \geq L_1 e^{\epsilon_{1+\alpha}}
\]

since \( \Gamma_1^{\nu_q} \geq \Gamma_1 \geq L_1 e^{\epsilon_{1+\alpha}} \) by (13). This proves the statement in the case \( n = v_q \).
If \( n > v_q \), it remains to estimate the derivative along the remaining iterates. We claim that
\[
|f^{(n-v_q)}(c_{n+1})| \geq D_{n-v_q}^{-1} e^{\alpha n-v_q}.
\] (49)

Indeed, if \( n \leq v_q + p_q \), i.e. \( n \) belongs to the binding period following the return at time \( v_q \), then this follows immediately from the inductive assumption \((EG)_{n-v_q-1}\) and the bounded distortion during binding periods in lemma 5.1.1. Otherwise, i.e. if \( n > v_q + p_q \), we consider two possibilities. If the orbit of \( c_k \) never enters \( \Delta \) between time \( v_q + 1 \) and time \( n \) we just ignore the binding period and use (A1) to obtain \( |f^{(n-v_q-1)}(c_{n+1})| \geq e^{\alpha n-v_q} \) which clearly implies (49). Alternatively, if the orbit of \( c_k \) does enter \( \Delta \) between time \( v_q + 1 \) and time \( n \) we just apply the inductive assumption and the bounded distortion to get an estimate such as (49) up to the last time before \( n \) such that \( c_k \in \Delta \) and then apply (A1) to get an exponential growth at a rate \( \lambda \) for the remaining iterates, in this case also we obtain (49).

By the chain rule and equations (45)–(49) we then get
\[
|\left(f^{n+1}\right)'(c_0)| \geq \Gamma_1^{-1} e^{(\lambda - \tau n (1 - \gamma_1))n} L_1 e^{-\gamma_0 n} D_1^{-1} e^{\lambda n - v_q} = \Gamma_1^{-1} L_1 D_1^{-1} e^{-1 - \lambda v_q e^{(\lambda - \tau n (1 - \gamma_1))n} - \gamma_0 n} e^{\alpha n + 1} \geq e^{\lambda n + 1}.
\] (50)

The last inequality follows again by (48) and (13). This completes the proof of lemma 6.2.

\(\square\)

6.3. Bounded distortion

Lemma 6.3. Let \( \omega \in \mathcal{P}^{(n)} \). Then
\[
\frac{|c' \omega|}{|c' b|} \leq \mathcal{D}
\]
for all \( a, b \in \omega \) and all \( k \leq v_q + p_q + 1 \) where \( v_q \leq n \) is the last essential or inessential return, or the last essential or inessential escape of \( \omega \) and \( p_q \) is the corresponding binding period. If \( n > v_q + p_q + 1 \) then the same statement holds for all \( k \leq n \) restricted to any subinterval \( \bar{\omega} \subseteq \omega \) such that \( |\bar{\omega}_k| \leq \Delta^\ast \).

By lemma 5.2 it is sufficient to show that
\[
\frac{|f^{(k)}(c_0)|}{|f^{(k)}(c_0)|} \leq \frac{\mathcal{D}}{D_2 D_3}.
\] (50)

Note that, strictly speaking, lemma 5.2 is stated for \( k \leq n - 1 \), but this is not an issue here since the actual assumptions used and therefore the conclusions of the lemma clearly apply up to time \( k \). Equation (50) essentially says that critical orbits with the same combinatorics satisfy comparable derivative estimates. By standard arguments (see the proof of sublemma 5.1.1 also), we have
\[
\log \frac{|f^{(k)}(c_0)|}{|f^{(k)}(c_0)|} \leq M_2 \sum_{j=0}^{k-1} D_j \quad \text{where} \quad D_j = \frac{|\omega_j|}{d(\omega_j)}
\]
and \( d(\omega_j) = \inf_{a \in \omega_j} |f_1(a)| \). Let \( 0 < v_1 < \ldots < v_q \leq n \) be the sequence of essential and inessential returns and essential and inessential escapes of \( \omega \). By construction there is a unique element \( l_{p_i, m_i} \) in \( \mathbb{T}^\ast \) associated with each \( v_i \). Let \( p_i \) be the length of the binding period associated with \( v_i \). For notational convenience define \( v_0 \) and \( p_0 \) so that \( v_0 + p_0 + 1 = 0 \). We suppose first that \( k \leq v_q + p_q + 1 \). Then write
\[
\sum_{j=0}^{k-1} D_j \leq \sum_{j=0}^{v_q + p_q} D_j = \sum_{i=0}^{q-1} v_{i+1 + p_{i+1}} D_j.
\] (51)
Note that we have divided the itinerary of $\omega$ into a finite number of blocks corresponding to pieces of itinerary starting immediately after a binding period and going through to the end of the following binding period. In the next sublemma we obtain a bound for the sum over each individual block.

**Sublemma 6.3.1.** For each $i = 0, \ldots, q - 1$ we have

$$\sum_{v_{i+1}^* \neq p_{i+1}} D_j \leq D |\omega_{v_{i+1}}| e^{\rho_{v_{i+1}}}.$$  \hfill (52)

**Proof.** We start first of all by subdividing the sum further into iterates corresponding to (i) the free iterates between the end of the binding period and the following return, (ii) the return and (iii) the binding period following the return.

$$\sum_{v_{i+1} \neq p_{i+1}} D_j = \sum_{v_{i+1} \neq p_{i+1}} D_j + D_{v_{i+1}} + \sum_{v_{i+1} \neq p_{i+1}} D_j.$$  \hfill (53)

We shall estimate each of the three terms in separate arguments. For the first, note that, since $\omega_{v_{i+1}} \subseteq \bar{I}_{v_{i+1}} \subset \Delta^*$, condition (A1) implies $|(f^{v_{i+1}} - f^{v_{i+1}}(a))| \geq C_i e^{\rho_{v_{i+1}}}$. Therefore, by lemma 5.2 we have $|\omega_j| \leq C_i^{-1} D_2 e^{-\rho_{v_{i+1}}} |\omega_{v_{i+1}}|$. Moreover $d(\omega_j) \geq \delta' / 2$, therefore, also using the fact that $\delta' \geq e^{-\rho_{v_{i+1}}}$, we have

$$\sum_{j = v_{i+1} + 1}^{v_{i+1} - 1} D_j \leq 2C_i^{-1} D_2 D_3 \sum_{j = v_{i+1} + 1}^{v_{i+1} - 1} e^{-\lambda \delta_{v_{i+1}} - j} |\omega_{v_{i+1}}| \delta_{v_{i+1}} \leq \frac{2C_i^{-1} D_2 D_3 e^{-\lambda}}{1 - e^{-\lambda}} |\omega_{v_{i+1}}| e^{\rho_{v_{i+1}}}.$$  \hfill (54)

For the second term we immediately have

$$D_{v_{i+1}} \leq 2 |\omega_{v_{i+1}}| e^{\rho_{v_{i+1}}}.$$  \hfill (55)

since $D_{v_{i+1}}$ is the supremum of $|\omega_{v_{i+1}}| / |c_{v_{i+1}}|$ over all points $c_{v_{i+1}}$ in $\omega_{v_{i+1}}$, and $|c_{v_{i+1}}| \geq \frac{1}{2} e^{-\rho_{v_{i+1}}}$ by definition.

The estimation for the third term is the trickiest or at least the least intuitive. By lemma 5.2 we have, for all $j \in [v_{i+1} + 1, v_{i+1} + p_{i+1}]$,

$$|\omega_j| \leq D_2 D_3 |\omega_{v_{i+1}}| \sup_{a \in \omega} |(f_j^{v_{i+1}} - f_j^{v_{i+1}}(a))|.$$  \hfill (56)

We therefore need an upper bound for

$$|(f_j^{v_{i+1}} - f_j^{v_{i+1}}(a))| = |(f_j^{v_{i+1}} - f_j^{v_{i+1}}(c_{v_{i+1}}(a)))|.$$  \hfill (57)

Thus, it only remains to get an upper bound for the derivative during the binding period. Fix $a \in \omega$ and let $\gamma_j = c_{j-1}(a) - c_j(a)$. Then we have $|\gamma_j| \geq L_1 e^{-\rho_{v_{i+1}}}$ and, by the definition of binding periods, $|\gamma_j| \leq e^{-\rho_{v_{i+1}}}$. Therefore, using the mean value theorem and lemma 5.1.1 which says that all derivatives are comparable during the binding period, we get

$$|(f_j^{v_{i+1}} - f_j^{v_{i+1}}(c_{v_{i+1}}))| \leq D_1 |\gamma_j| e^{-\rho_{v_{i+1}}}. $$  \hfill (58)
To bound $d(\omega_j)$ we just observe that the definition of binding period and the slow recurrence condition $(SR)_{j=v_{i+1}+1}$ imply, for $j > v_{i+1} + 1$,
\[
|c_j(\omega)| \geq |c_{j-v_{i+1}}(\omega) - e^{-\alpha_0(j-v_{i+1})} - e^{-\alpha_0(j-v_{i+1})}| = e^{-\alpha_0(j-v_{i+1})}(1 - e^{-\alpha_0(j-v_{i+1})}) \geq e^{-\alpha_0(j-v_{i+1})}(1 - e^{-(\alpha_1-\alpha_0)}).
\] Note that for $j = v_{i+1} + 1$ we just have $|c_j(\omega)| \geq 1 \geq e^{-\alpha_0(j-v_{i+1})}(1 - e^{-(\alpha_1-\alpha_0)})$ so, formally, the inequality holds in this case also. Thus, (58) and (59) and the fact that $\alpha_1 \geq \alpha_0$ give
\[
\frac{|\omega_j|}{d(\omega_j)} \leq 2D_1D_2L_1^{-2}e^{-\alpha_0(j-v_{i+1})}|\omega_{\nu_i}|e^{\rho_i} \leq 2D_1D_2D_3L_1^{-2}|\omega_{\nu_i}|e^{\rho_i}.
\] (59)
Substituting (54), (55) and (60) into (53) and using the definition of $\hat{D}$ in (10) gives the statement in sublemma 6.3.1.

For any $r \geq r_3^*$.
\[
\sum_{j=1}^{r} |\gamma_j| \leq \left(2 + e^{-(\log \delta^{-1})^2}/(\log \delta^{-1} - 1)^2\right) \frac{e^{-r}}{r} \sum_{j=0}^{\infty} \left(C_{i,1}^{-1}g_{(1-\gamma)}^{(1-\gamma)} \right)^{j}.
\]

Proof. Let $\mu_j = \nu_{j+1}$, $j = 1, \ldots, m$ be the subsequence of returns and escapes with return depths equal to $r$. Using lemma 5.1 and condition (A1) we have for all $a \in \omega$ and $j = 1, \ldots, m - 1$,
\[
|f_{\mu_j}^{(i+1)-\mu_j}(c_{\mu_j}(a))| \geq C_1D_2D_3g_{(1-\gamma)}^{(1-\gamma)}r_2 \geq C_1D_2D_3g_{(1-\gamma)}^{(1-\gamma)}r_2.
\]
Therefore by lemma 5.2, $|\omega_{\mu_j}|/|\omega_{\mu_{i+1}}| \leq C_{i,1}^{-1}g_{(1-\gamma)}^{(1-\gamma)}/(\log \delta)^2$ and
\[
\sum_{j=1}^{m} |\omega_{\mu_j}| \leq \sum_{j=0}^{m-1} \left(C_{i,1}^{-1}g_{(1-\gamma)}^{(1-\gamma)} \right)^{j} |\omega_{\mu_{i+1}}| \leq \sum_{j=0}^{\infty} \left(C_{i,1}^{-1}g_{(1-\gamma)}^{(1-\gamma)} \right)^{j} |\omega_{\mu_{i+1}}|.
\]
Recall that $\omega_m$ possibly spreads across three contiguous elements of $\mathcal{I}^+$. Two of these have lengths at most $e^{-r}/r^2$ and the third one has length at most $e^{-(r-1)}/(r-1)^2 \leq (er^2/(r-1)^2)(e^{-r}/r^2)$ which gives
\[
|\omega_{\mu_{i+1}}| \leq \left(2 + e^{-(\log \delta^{-1})^2}/(\log \delta^{-1} - 1)^2\right) \frac{e^{-r}}{r^2}.
\]

Note that the convergence of the sum on the right-hand side is guaranteed by the definition of $\gamma_1$ in (9). Indeed some straightforward rearrangements shows that condition (9) implies that $C_{i,1}^{-1}g_{(1-\gamma)}^{(1-\gamma)}/(\log \delta)^2 < 1$.
Substituting the estimate of sublemma 6.3.2 into (62) now gives
\[ \sum_{q=1}^{q} |\omega_{q} \epsilon^{q}| = \sum_{r \geq r_{3}}^{r_{3} - 1} \sum_{k \geq k_{r}}^{k_{r} - 1} \left( 2 + \epsilon \left( \frac{\log \delta}{\log \delta^{2} - 1} \right)^{j} \right)^{j} \sum_{j=0}^{\infty} \left( C_{j}^{-1} \delta^{(1 - \gamma_{j})} \right)^{j} \frac{1}{r^{j}}. \]
Summing over \( r \) and substituting into (61) then gives
\[ \sum_{j=0}^{v_{q} + p_{q} - 1} D_{j} \leq \hat{D} \left( 2 + \epsilon \left( \frac{\log \delta}{\log \delta^{2} - 1} \right)^{j} \sum_{j=0}^{\infty} \left( C_{j}^{-1} \delta^{(1 - \gamma_{j})} \right)^{j} \frac{1}{r^{j + 1}} - 1 \right) = \tilde{D} \hat{D}. \]
This completes the proof of lemma 6.2 for \( k \leq v_{q} + p_{q} + 1 \). If \( k > v_{q} + p_{q} + 1 \) we consider the additional terms \( D_{j} \), restricting ourselves to some subinterval \( \bar{\omega} \subset \omega \) with \( \bar{\omega} \in \Delta^{+} \). Clearly the preceding estimates are unaffected by this restriction. Using (A1) and lemma 5.2 we have
\[ \omega_{\omega_{j}} \leq C \delta \sum_{j} \left( C_{j}^{-1} \delta^{j} \right) \leq \sum_{j} \left( C_{j}^{-1} \delta^{j} \right) \],
and therefore, also using the fact that \( |c_{j}(a)| \geq \delta \) since \( \omega_{j} \cap \Delta^{+} = \emptyset \), we get
\[ \sum_{j=v_{q} + p_{q} + 1}^{k-1} D_{j} \leq \sum_{j=v_{q} + p_{q} + 1}^{k-1} \sum_{j=v_{q} + p_{q} + 1}^{k-1} \left( \frac{C_{j}^{-1} \delta^{(1 - \gamma_{j})}}{1 - e^{-\lambda}} \right) = \frac{C_{j}^{-1} \delta^{(1 - \gamma_{j})}}{1 - e^{-\lambda}}. \]

7. Positive measure in parameter space

In this section we prove proposition 4.2. We divide the proof into four sections.

7.1. Large deviations

First, recall the definition of \( \Omega^{(n)} \) in (21):
\[ \Omega^{(n)} = \{ a \in \Omega^{(n - 1)} : E^{(n)}(a) \leq an \}. \]
We will show here that the average value of \( E^{(n)} \) on \( \Omega^{(n - 1)} \) is low and therefore most parameters in \( \Omega^{(n - 1)} \) make it into \( \Omega^{(n)} \). More precisely, let \( \gamma_{1}, \gamma_{2} \) be as in section 2.2, then we have the following proposition.

**Proposition 7.1.** For every \( n \geq 1 \),
\[ \int_{\Omega^{(n - 1)}} e^{(\gamma_{1} - \gamma_{2})E^{(n)}} \leq \left( 1 + \sum_{R \geq r_{1}} e^{(\gamma_{1} - \gamma_{2})R} \right)^{n} |\Omega|. \]

We will prove this proposition in the next three sections. First we show how, by a simple large deviation argument, it implies proposition 4.2 and therefore our main theorem. The definition of \( \Omega^{(n)} \) gives
\[ |\Omega^{(n - 1)}| - |\Omega^{(n)}| = |\Omega^{(n - 1)} \setminus \Omega^{(n)}| = \left| \{ \omega \in \hat{P}^{(n)} \setminus \Omega^{(n)} : E^{(n)}(\omega) \geq \gamma_{2} an \} \right|. \]
Therefore, using Chebychev’s inequality (large deviations) and proposition 7.1 we have
\[ |\Omega^{(n - 1)}| - |\Omega^{(n)}| \leq e^{\gamma_{2} an} \int_{\Omega^{(n - 1)}} e^{\gamma_{2} E^{(n)}} \leq \left[ e^{\gamma_{2} an} \left( 1 + \sum_{R \geq r_{1}} e^{(\gamma_{1} - \gamma_{2})R} \right) \right]^{n} |\Omega|, \]
which implies, recall the definition of \( \tilde{\eta} \) in section 2.2,

\[
|\Omega^0| \geq |\Omega^{(n-1)}| - \left[ e^{-\gamma n} \left( 1 + \sum_{R \geq r_1} e^{(y-1)R} \right) \right]^n |\Omega| = |\Omega^{(n-1)}| - \tilde{\eta}|\Omega|.
\]

Iterating this expression and using the fact that no exclusions are made before time \( N \), and also using the definition of \( \eta \), we have

\[
|\Omega^0| \geq \left( 1 - \sum_{j=N}^{\infty} \left[ e^{-\gamma n} \left( 1 + \sum_{R \geq r_1} e^{(y-1)R} \right) \right]^j \right)^n |\Omega| = (1 - \eta)|\Omega|.
\]

7.2. Renormalization properties of the combinatorics

The proof of proposition 7.1 is quite subtle. In this section we give an alternative combinatorial description of the parameters in \( \Omega^{(n-1)} \) and state a key proposition 7.2 in terms of this combinatorial description. This description is crucial to the argument and highlights some remarkable renormalization properties of the construction. We shall also show how proposition 7.2 implies proposition 7.1.

Recall that \( \tilde{P}^{(n)} \) is the partition of \( \Omega^{(n-1)} \) which takes into account the dynamics at time \( n \) and which restricts to the partition \( P^{(n)} \) of \( \Omega^{(n)} \) after the exclusion of a certain elements of \( \tilde{P}^{(n)} \). To each \( \omega \in \tilde{P}^{(n)} \) is associated a sequence \( 0 = \eta_0 < \eta_1 < \cdots < \eta_s < n \), \( s = s(\omega) \geq 0 \) of escape times and a corresponding sequence of escaping components \( \omega \subseteq \omega^{(n)} \subseteq \cdots \subseteq \omega^{(n)} \) with \( \omega^{(n)} \subseteq \Omega^{(n)} \) and \( \omega^{(n)} \in P^{(n)} \). To simplify the formalism we also define some ‘fake’ escapes by letting \( \omega^{(n)} = \omega \) for all \( s + 1 \leq i \leq n \). In this way we have a well-defined parameter interval \( \omega^{(n)} \) associated with \( \omega \in \tilde{P}^{(n)} \) for each \( 0 \leq i \leq n \). Note that for two intervals \( \omega, \tilde{\omega} \in \tilde{P}^{(n)} \) and any \( 0 \leq i \leq n \), the corresponding intervals \( \omega^{(n)} \) and \( \tilde{\omega}^{(n)} \) are either disjoint or coincide. Then we define

\[
Q^{(i)} = \bigcup_{\omega \in \tilde{P}^{(n)}} \omega^{(n)}
\]

and let \( Q^{(i)} = \{ \omega^{(n)} \} \) denote the natural partition of \( Q^{(i)} \) into intervals of the form \( \omega^{(n)} \). Note that \( \Omega^{(n-1)} = Q^{(n)} \subseteq \cdots \subseteq Q^{(0)} = \Omega^{(0)} = \tilde{P}^{(n)} \), since the number \( s \) of escape times is always strictly less than \( n \) and therefore in particular \( \omega^{(n)} = \omega \) for all \( \omega \in \tilde{P}^{(n)} \). For a given \( \omega = \omega^{(n)} \in Q^{(i)} \), \( 0 \leq i \leq n - 1 \) we let

\[
Q^{(i+1)}(\omega) = \{ \omega' = \omega^{(n)}. \subseteq Q^{(i+1)} : \omega' \subseteq \omega \}
\]

denote all the elements of \( Q^{(i+1)} \) which are contained in \( \omega \) and let \( Q^{(i+1)}(\omega) \) denote the corresponding partition. Then we define a function \( \Delta E^{(i)} : Q^{(i+1)}(\omega) \to \mathbb{N} \) by

\[
\Delta E^{(i)}(\omega) = E^{(n)}(\omega) - E^{(n)}(\tilde{\omega}^{(n)}(\omega)).
\]

This gives the total sum of all essential return depths associated with the itinerary of the element \( \omega' \in Q^{(i+1)}(\omega) \) containing \( a \), between the escape at time \( \eta_i \) and the escape at time \( \eta_{i+1} \). Clearly \( \Delta E^{(i)}(\omega) \) is constant on elements of \( Q^{(i+1)}(\omega) \). Finally we let

\[
Q^{(i+1)}(\omega, R) = \{ \omega' \in Q^{(i+1)} : \omega' \subseteq \omega, \Delta E^{(i)}(\omega') = R \}.
\]

Note that the entire construction given here depends on \( n \). The main motivation for this construction is the following proposition.

**Proposition 7.2.** For all \( i \leq n - 1 \), \( \omega \in Q^{(i)} \) and \( R \geq 0 \) we have

\[
\sum_{\tilde{\omega} \in Q^{(i+1)}(\omega, R)} |\tilde{\omega}| \leq e^{(n+1)y(R-1)} |\omega|.
\]
This says that the probability of accumulating a large total return depth between one escape and the next is exponentially small. The strategy for proving this result is straightforward. We show that the size of each partition element $\omega' \in Q^{(i+1)}(\omega, R)$ is exponentially small in $R$ and then use a combinatorial argument to show that the total number of such elements cannot be too large. The proposition follows immediately from lemmas 7.1 and 7.2 in the next two sections. However, we first show how proposition 7.2 implies proposition 7.1.

**Proof of proposition 7.1 assuming proposition 7.2.** Note first of all that

$$\sum_{\omega \in Q^{(i+1)}} e^{\gamma i R} e^{\lambda n} = \sum_{\omega \in Q^{(i+1)}} e^{\gamma i R} |\omega| \tag{63}$$

Thus it is sufficient to bound the right-hand side. Let $0 \leq i \leq n - 1$ and $\omega \in Q^{(i)}$ and recall that $\lambda(n) = \Delta e^{(0)} + \cdots + \Delta e^{(n-1)}$ and $\Delta e^{(i)}$ is constant on elements of $Q^{(i)}$. Then we write

$$\sum_{\omega' \in Q^{(i+1)}(\omega)} e^{\gamma i R} |\omega'| = \sum_{\omega' \in Q^{(i+1)}(\omega, R)} |\omega'| + \sum_{R \geq R_3} e^{\gamma i R} \sum_{\omega' \in Q^{(i+1)}(\omega, R)} |\omega'| \tag{64}$$

For the first term in the sum we just use the trivial bound

$$\sum_{\omega' \in Q^{(i+1)}(\omega, 0)} |\omega'| \leq |\omega|. \tag{65}$$

For the second, we use proposition 7.2 and the definition of $\gamma$ to get

$$\sum_{R \geq R_3} e^{\gamma i R} \sum_{\omega' \in Q^{(i+1)}(\omega, R)} |\omega'| \leq \sum_{R \geq R_3} e^{(\gamma R - 1) R} |\omega| = \sum_{R \geq R_3} e^{(\gamma - 1) R} |\omega|. \tag{66}$$

Substituting (64) and (65) into (63) we get

$$\sum_{\omega' \in Q^{(i+1)}(\omega)} e^{\gamma i R} |\omega'| \leq \left(1 + \sum_{R \geq R_3} e^{(\gamma - 1) R}\right) |\omega|. \tag{67}$$

Now, to obtain a bound for (63) recall that by construction each $\omega^{(n)} \in Q^{(n)}$ belongs to a nested sequence of intervals $\omega^{(n)} \subseteq \omega^{(n-1)} \subseteq \cdots \subseteq \omega^{(0)} = \Omega$ in which each $\omega$ belong to $Q^{(i)}(\omega^{i-1})$ for $n \geq i \geq 1$. Therefore, we can write (63) as

$$\sum_{\omega^{(n)} \in Q^{(n)}} e^{\gamma i R} \sum_{\omega^{(n-1)} \in Q^{(n-1)}} e^{\lambda(n) R} \sum_{\omega^{(n-2)} \in Q^{(n-2)}} \cdots \sum_{\omega^{(0)} \in Q^{(0)}} e^{\gamma \lambda(n) R} |\omega|.$$

Note the nested nature of the expression. Applying (67) repeatedly gives

$$\sum_{\omega \in Q^{(n)}} e^{\gamma i R} |\omega| \leq \left(1 + \sum_{R \geq R_3} e^{(\gamma - 1) R}\right) |\omega|^n.$$

### 7.3. Metric estimates

**Lemma 7.1.** For all $0 \leq i \leq n - 1$, $\omega \in Q^{(i)}$, $R \geq 0$ and $\bar{\omega} \in Q^{(i+1)}(\omega, R)$ we have

$$|\bar{\omega}| \leq e^{(\gamma - 1) R} |\omega|. \tag{68}$$
Proof. From the construction of \( \tilde{\omega} \) there is a nested sequence of intervals
\[
\tilde{\omega} = \omega(\nu_1) \subseteq \omega(\nu_2) \subseteq \cdots \subseteq \omega(\nu_s) = \omega
\]
where \( \omega \) has an escape at time \( \nu_0 \), each \( \omega(\nu_j), j = 1, \ldots, s \), has an essential return at time \( \nu_j \) (intuitively \( \omega(\nu_j) \) is created as a consequence of the intersection of \( \omega(\nu_{j-1}) \) with \( \Delta \) at time \( \nu_j \)). We write
\[
\frac{|\tilde{\omega}|}{|\omega|} = \frac{|\omega(\nu_0)|}{|\omega(\nu_0)|} \frac{|\omega(\nu_1)|}{|\omega(\nu_0)|} \cdots \frac{|\omega(\nu_s)|}{|\omega(\nu_{s-1})|} \frac{|\tilde{\omega}|}{|\omega(\nu_s)|}.
\]
We shall estimate the right-hand side of (68) term by term. We start by considering the terms \( |\omega(\nu_j)|/|\omega(\nu_{j-1})| \) for \( j = 1, \ldots, s - 1 \) for which both \( \nu_j \) and \( \nu_{j+1} \) are essential returns. The idea is to compare the two parameter intervals by comparing their images and applying the bounded distortion condition. In this case we choose to compare their images at some intermediate time between the return at time \( \nu_j \) and the return at time \( \nu_{j+1} \), more specifically at the end of the binding period following the return at time \( \nu_j \). Thus, using the bounded distortion condition we have
\[
\frac{|\omega(\nu_{j+1})|}{|\omega(\nu_j)|} = \frac{|c_{\nu_j+\rho_{j+1}}(a)|}{|c_{\nu_j+\rho_{j+1}}(b)|} \frac{|\omega(\nu_{j+1})|}{|\omega(\nu_j)|} \leq D \frac{|\omega(\nu_{j+1})|}{|\omega(\nu_{j+1})|}.
\]
To bound the numerator we use lemma 5.1 to get
\[
|\omega(\nu_{j+1})| \geq \Gamma_1 e^{-\gamma r_j}.
\]
To bound the denominator note that there may (or may not) be a sequence of inessential returns between time \( \nu_j + \rho_j + 1 \) and time \( \nu_{j+1} \). In any case, the accumulated derivative taken over all free and bound periods is \( \geq 1 \). Therefore, using (33), condition (A1) and lemma 5.1 we get
\[
|\omega(\nu_{j+1})| \geq C_1 D_2^{-1} D_3^{-1} |\omega(\nu_{j+1})|.
\]
Thus, by the definition of \( \Gamma_1 \) in (13),
\[
\frac{|\omega(\nu_{j+1})|}{|\omega(\nu_j)|} \leq D \frac{|\omega(\nu_{j+1})|}{|\omega(\nu_{j+1})|} \leq D D_2 D_3 \frac{|\omega(\nu_{j+1})|}{|\omega(\nu_{j+1})|} e^{-r_{j+1} + \gamma r_j} \leq e^{-r_{j+1} + \gamma r_j}.
\]
For the last term in ratios we just use the trivial bound
\[
|\tilde{\omega}| \leq |\omega(\nu_0)|,
\]
and so we get
\[
\frac{|\tilde{\omega}|}{|\omega|} \leq \frac{|\omega(\nu_0)|}{|\omega(\nu_0)|} e^{\sum_{j=1}^{s-1} -r_{j+1} + \gamma r_j} \leq \frac{|\omega(\nu_0)|}{|\omega(\nu_0)|} e^{-r_0 + \gamma r_0 + \sum_{j=1}^{s} (\gamma r_j - r_{j+1})}.
\]
Estimates for \( |\omega(\nu_1)|/|\omega(\nu_0)| \) are different from the other cases, in principle, since \( \nu_0 \) is an escape and not a return time. However, if \( \nu_0 \) is an essential escape then we can actually apply the binding period estimates and, using exactly the same arguments as above, we get
\[
|\omega(\nu_1)|/|\omega(\nu_0)| = e^{-r_0 + \gamma r_0},
\]
and substituting this into (70) and using the fact that \( r_0 < r_1 \) we get the statement in the lemma in this case.

It remains to consider the case in which \( \nu_0 \) is a substantial escape, i.e. \( \omega(\nu_0) \) lies outside \( \Delta^* \) and satisfies \( |\omega(\nu_0)| \geq \delta/(\log \delta)^2 \). We consider two separate cases depending on whether its image \( \omega(\nu_1) \) satisfies \( \omega(\nu_1) \subset \Delta^* \) or not. Suppose first that \( \omega(\nu_1) \subset \Delta^* \). Then we can apply the bounded distortion as above and for some \( a \in \omega(\nu_0) \) and \( b \in \omega(\nu_1) \) we get
\[
\frac{|\omega(\nu_0)|}{|\omega(\nu_0)|} = \frac{|c_{\nu_0}(a)|}{|c_{\nu_0}(b)|} \leq D \frac{|\omega(\nu_1)|}{|\omega(\nu_1)|} \leq D e^{-r_1}.
\]
To bound the denominator we once more use condition (A1) and (33) to get

\[ |\omega^{(n)}_{\nu_1}| \geq \frac{1}{D_2D_3} \min_{a \in \omega^{(n)}} (l((f^{m(n)-m})'(c^{(n)}_{\nu_1}(a)))) |\omega^{(n)}_{\nu_1}| \geq \frac{1}{D_2D_3} \frac{\delta^i}{(\log \delta^{-1})^2}. \]  

(72)

Substituting (72) into (71) and then (71) into (70), and using the fact that \( r_s \geq r_3 = \log \delta^{-1} \), we get

\[ \frac{|\tilde{\omega}|}{|\omega|} \leq DD_2D_3 \frac{e^{-\gamma_{n+1}(\log \delta^{-1})^2}}{\delta^i} e^{(\gamma_{n+1})R} \leq \frac{DD_2D_3 \delta^2 (\log \delta^{-1})^2}{C_1} \frac{1}{\delta^i} e^{(\gamma_{n+1})R}. \]  

(73)

By a straightforward rearrangement and taking logs we have

\[ \frac{DD_2D_3 \delta^2 (\log \delta^{-1})^2}{C_1} \frac{1}{\delta^i} \leq 1 \iff \gamma_1 \geq i + \frac{\log(DD_2D_3C_1^{-1}) + 2 \log \delta^{-1}}{\log \delta^{-1}}. \]

The inequality on the right is satisfied by (C3), and thus we obtain our result in this case. Now, if \( \omega^{(n)}_{\nu_1} \not\subset \Delta^* \), the bounded distortion result applies only to the subinterval \( \overline{\omega}^{(n)}_{\nu_1} \subset \omega^{(n)}_{\nu_1} \) such that \( \overline{\omega}^{(n)}_{\nu_1} \subset \Delta^* \) and we get

\[ \frac{|\omega^{(n)}_{\nu_1}|}{|\overline{\omega}^{(n)}_{\nu_1}|} = \frac{|c'_{\nu_1}(a)|}{|c'_{\nu_1}(b)|} ; \frac{|\omega^{(n+1)}_{\nu_1}|}{|\overline{\omega}^{(n+1)}_{\nu_1}|} < DD_2 \frac{|\omega^{(n+1)}_{\nu_1}|}{|\overline{\omega}^{(n+1)}_{\nu_1}|} \leq De^{-\gamma_{n+1}}. \]

However, in this case recall that \( \omega^{(n)}_{\nu_1} \) necessarily intersects \( \Delta \) since \( \nu_1 \) is an essential return time for \( \omega^{(n)}_{\nu_1} \subset \omega^{(n)}_{\nu_1} \). Therefore, we immediately get \( |\overline{\omega}^{(n)}_{\nu_1}| \geq \delta^i/2 \) simply using the observation that \( \overline{\omega}^{(n)}_{\nu_1} \) intersects both \( \partial \Delta^* \) and \( \partial \Delta \). The final estimate follows as in the previous paragraph. \( \square \)

7.4. The counting argument

**Lemma 7.2.** For all \( 0 \leq i \leq n - 1 \), \( \omega \in Q_i^{(1)} \) and \( R \geq r_4 \), we have

\[ \#Q_i^{(1)}(\omega, R) \leq \varepsilon^{\gamma_{n+1}}. \]

Before starting the proof, recall that each \( \tilde{\omega} \in Q_i^{(i+1)}(\omega, R) \) has a combinatorial itinerary, associated with the iterates between the \( i \)th escape at time \( \eta_i \) and the \( i + 1 \)th escape at time \( \eta_{i+1} \), specified by a sequence

\[ (\pm r_1, m_1), (\pm r_2, m_2), \ldots, (\pm r_s, m_s) \]

(74)

with

\[ s \geq 1, \quad |r_1| + \cdots + |r_s| = R, \quad |r_j| \geq r_3, \quad m_j \in [1, r_j^2], \quad j = 1, \ldots, s. \]  

(75)

The combinatorics is not unique but the multiplicity of partition elements with the same combinatorics can be controlled. Indeed this is essentially the main reason for introducing the sets \( Q_i^{(1)} \).

**Sublemma 7.2.1.** The cardinality of elements of \( Q_i^{(i+1)}(\omega, R) \) having the same combinatorial itinerary is at most \( r_i^2 \).

**Proof.** The first time, after time \( \eta_i \), that \( \omega \) intersects \( \Delta^* \) in a chopping time, every subinterval which arises out of the ‘chopping’ has either an escape time, in which case the sequence above is empty or an essential return time in which case a unique pair of integers \( r_1 \) and \( m_1 \) are associated with it. Thus, no two elements created up to this time can share the same sequence. Fixing one of these subintervals which has an essential return we consider higher iterates until
the next time that it intersects $\Delta^*$. At this time it is further subdivided into subintervals. All those which have essential returns at this time have another uniquely defined pair of integers $r_2$ and $m_2$ associated with them. However, multiplicity can occur for those which have escape times: all the subintervals which fall in $\Delta^* \setminus \Delta$ have an escape at this time and therefore belong to $Q^I$, and we do not consider further iterates, but all share the same first (and only) term of the associated sequence of return depths. However, the number of such subintervals can be estimated by the number of elements of the partition $T^I|_{\Delta^* \setminus \Delta}$ plus at most two elements which can escape by falling outside $\Delta^*$. The number of such intervals is then $\leq 2(r_3 - r_{3'})r_{3'}^2 + 2 \leq r_3^3$.

In the case of the intervals which have two or more returns we repeat the same reasoning to get the result.

Proof of lemma 7.4. Sublemma 7.2.1 reduces the proof of lemma 7.4 to a purely combinatorial calculation of the cardinality of the set $S_R$ of all possible sequences of the form (74) satisfying the constraints given in (75). Let us denote by $S_R(s)$ the subset of $S_R$ given by sequences with some fixed number $s$ of terms and by $S^*_R(s)$ the subsequence of those given by considering only positive $r_j's$. Note that the crucial condition $|r_j| \geq r_3$ implies that $s \leq R/r_3$, and therefore we have

$$\#S_R \leq \sum_{s \leq R/r_3} \#S_R(s) \leq \sum_{s \leq R/r_3} 2^s \#S^*_R(s). \quad \text{(76)}$$

To obtain a bound for $\#S^*_R(s)$ observe first of all that for a given sequence $(r_1, \ldots, r_s)$ the terms $(m_1, \ldots, m_s)$ contribute an additional factor of

$$\prod_{j=1}^s r_j^{(2 \log r_j)} = e^{2 \sum_{j=1}^s \log r_j} = e^{2 \sum_{j=1}^s (\log r_j/r_j)r_j} \leq e^{2 \sum_{j=1}^s (\log r_j/r_j)r_j} = e^{(2 \log r_j/r_j)R}. \quad \text{(77)}$$

It remains therefore to only estimate the number of possible sequences $(r_1, \ldots, r_s)$ with $r_j \geq r_3$ and $r_j + \cdots + r_s = R$. This number is bounded above by

$$\left( \begin{array}{c} R - 1 \\ s - 1 \end{array} \right) \leq \left( \begin{array}{c} R \\ s \end{array} \right) = \frac{R!}{(R-s)!s!}.$$

Indeed, consider a row of $R$ objects. The number of ways of partitioning such a set into exactly $s$ non-empty subsets is equivalent to the number of ways of selecting $s$ objects out of the $R - 1$ objects following the first object in the row. Indeed, once such a set object has been chosen such that we can define the partition as being formed by the consecutive objects which follow one of the choices (including the chosen object itself) up to (and not including) the next chosen object.

Using Stirling’s approximation formula for factorials $k! \in [1, 1 + \frac{1}{\pi} \sqrt{2\pi}kk e^{-k}$ we get

$$\left( \begin{array}{c} R \\ s \end{array} \right) = \frac{R!}{(R-s)!s!} \leq \frac{R^R}{(R-s)^{R-s}s^s} = \left( \frac{R}{R-s} \right)^R \left( \frac{R-s}{s} \right)^s. \quad \text{(78)}$$

To estimate the first term we use the fact that $s \leq R/r_3$ to get

$$\left( \frac{R}{R-s} \right)^R \leq \left( \frac{R}{R-r_3} \right)^R \leq \left( \frac{R}{R(1-r_3)} \right)^R = \left( 1 - \frac{1}{r_3} \right)^{-R} \leq e^{-R \log(1+2/r_3)} \leq e^{(2/r_3)R}. \quad \text{(79)}$$

Note that the last two inequalities follow from the Taylor series $(1-x)^{-1} = 1+x+x^2+\cdots \leq 1 + 2x$ with $x = 1/r_3$, using the fact that $r_3 \gg 2$ and from the fact that $\log(1 + x) < x$ for all $x > 0$. To estimate the second term we first of all write

$$\left( \frac{R-s}{s} \right)^s \leq \left( \frac{R}{s} \right)^s \leq \left[ \left( \frac{S}{R} \right)^{-s} \right]^R \leq \left[ \left( \frac{1}{r_3} \right)^{-1/r_3} \right]^R = e^{(s/r_3)}R. \quad \text{(80)}$$
The third inequality uses the fact that \( x^{-\frac{r}{r_3}} \) monotonically decreases to 1 as \( x \to 0 \) and \( s/R \leq 1/r_3 \). Now, substituting (79) and (80) into (78) and multiplying by (77) gives

\[
\#S_R(s) \leq e^{(2(\log r_3 + r_3)R/r_3)}.
\]

Substituting this into (76) we then get

\[
\#S_R \leq \sum_{s \leq R/r_3} 2^s e^{((2+\log 2+3 \log r_3)/r_3)R} = e^{(2+3 \log r_3)/r_3)R}.
\]

Finally, taking into account the bound on the multiplicity of intervals with the same combinatorics, given by sublemma 7.2.1, we get

\[
\#Q_{n+1}(\omega, R) \leq r_3^n \#S_R \leq R r_3^2 e^{((2+3 \log r_3)/r_3)R} = e^{\log R + 2 \log r_3} e^{((2+3 \log r_3)/r_3)R}.
\]

We now use the fact that

\[
\log R = \frac{\log R}{R} \leq \frac{\log r_3}{r_3} R \quad \text{and} \quad \log r_3 = \frac{\log r_3}{r_3} R \leq \frac{\log r_3}{r_3} R,
\]

and thus, substituting into (81), we get

\[
\#S_R \leq e^{((2+3 \log r_3)/r_3)R}.
\]

By the definition of \( r_3 \) in (8) we obtain the result.

\[\square\]

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