Algorithmic Complexity of Power Law Networks

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Abstract

It was experimentally observed that the majority of real-world networks are scale-free and follow power law degree distribution. The aim of this paper is to study the algorithmic complexity of such “typical” networks. The contribution of this work is twofold.

First, we define a deterministic condition for checking whether a graph has a power law degree distribution and experimentally validate it on real-world networks. This definition allows us to derive interesting properties of power law networks. We observe that for exponents of the degree distribution in the range [1, 2] such networks exhibit double power law phenomenon that was observed for several real-world networks. Our observation indicates that this phenomenon could be explained by just pure graph theoretical properties.

The second aim of our work is to give a novel theoretical explanation why many algorithms run faster on real-world data than what is predicted by algorithmic worst-case analysis. We show how to exploit the power law degree distribution to design faster algorithms for a number of classical P-time problems including transitive closure, maximum matching, determinant, PageRank and matrix inverse. Moreover, we deal with the problems of counting triangles and finding maximum clique. Previously, it has been only shown that these problems can be solved very efficiently on power law graphs when these graphs are random, e.g., drawn at random from some distribution. However, it is unclear how to relate such a theoretical analysis to real-world graphs, which are fixed. Instead of that, we show that the randomness assumption can be replaced with a simple condition on the degrees of adjacent vertices, which can be used to obtain similar results. Again, we experimentally validate that many real-world graphs satisfy our property. As a result, in some range of power law exponents, we are able to solve the maximum clique problem in polynomial time, although in general power law networks the problem is NP-complete.

In contrast to previously done average-case analyses, we believe that this is the first “waterproof” argument that explains why many real-world networks are easier. Moreover, an interesting aspect of this study is the existence of structure oblivious algorithms, i.e., algorithms that run faster on power law networks without explicit knowledge of this fact or explicit knowledge of the parameters of the degree distribution, e.g., algorithms for maximum clique or triangle counting.

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1 Introduction

One of the most interesting observations in our understanding of complex networks is that for most large networks the degree distribution closely resembles a power law distribution [2], i.e., the number of nodes of degree \(d\) is proportional to \(d^{-\alpha}\) for some \(\alpha > 1\). Such networks are called scale-free and many models explaining their emergence have been proposed – the most important one being the preferential attachment model [4]. The aim of this work is to study the algorithmic complexity of such “typical” networks and its contribution is twofold.

First, we define a deterministic condition for checking whether a graph has a power law degree distribution and show that many real-world networks satisfy it. Graphs satisfying this condition are called power law bounded networks (PLB). This definition allows us to derive new interesting properties of power law networks. We observe that for \(\alpha \in [1, 2]\) PLB graphs with no parallel edges (simple graphs) need to exhibit double power law phenomenon. This means that the degree distribution of vertices with sufficiently high degrees is different and has higher exponent. This faster decay in the distribution was observed for some existing simple graphs and usually was attributed to some complex processes [31]. Our results indicate that this phenomenon may have a basic explanation that uses only pure graph theoretical properties. Essentially, we show that when \(\alpha \in [1, 2]\) there are not enough low degree vertices that can be connected to high degree vertices, and so the number of high degree vertices needs to be lower and cannot be proportional to \(d^{-\alpha}\).

The second contribution of this paper is the attempt to reduce the dichotomy in current research in algorithms, where two rarely interacting directions are pursued. On one hand, theoreticians work on optimizing the performance of algorithms in the worst-case model. This is an important line of research that has given us some beautiful algorithms and solutions. There are many success stories: a number of practically efficient algorithms have been developed only thanks to this rigorous worst-case model, e.g., Dijkstra shortest paths algorithm. On the other hand, there are problems where the best solutions that are used in practice have nothing in common with the state-of-art algorithms proposed by theoreticians. This is clearly visible in the case of the Steiner tree problem, as exemplified by last year’s DIMACS implementations challenge. As shown, e.g., in [17] the algorithm of Byrka et al. [13] with the best known theoretical approximation ratio, cannot be used on instances of larger size, because it is too inefficient. Moreover, even on instances of smaller size it delivers worse results than the best metaheuristic approach based on local search [50]. The number of examples where heuristic approaches outperform “worst-case” algorithms is enormous. Intuitively, this is due to the fact that when one prepares for the worst case then the typical case will be handled in suboptimal way. Standard ways of overcoming this shortcoming are to work with stochastic models or random graphs, or use smoothed analysis. For example, in online stochastic models it is sometimes possible to obtain better bounds on expected cost of the algorithm than what is implied by worst-case competitive ratio [25, 29]. On the other hand, there are cases where smoothed analysis allows us to obtain polynomial running time in expectation instead of exponential one [48].

However, the answers given by these stochastic models are still far from being satisfactory. Consider the rumor spreading process in a social network, e.g., Twitter. It was observed that rumors spread extremely fast in such networks. The paper [20] tries to give the following explanation for this observation. Social networks have properties similar to networks obtained from preferential attachment model [4], so one tries to argue that fast spread of rumors in such random networks explains the rapid spread of rumors in real-world networks. This explanation has the following shortcomings. First, it has been observed that although many properties of social networks are explained well by this model, there are some properties that are not captured by it. For example a better model is to use affiliation networks [28]. Even if social networks were random we would never know that we have a precise model for them, as we might always miss some important property. Hence, this argument is far from explaining the observations. Second, there exists just one instantiation of any social network and there is no way we can see distribution of all random Twitter networks that is needed for this argument. Besides, as there is just one example of a social network it might be the unlucky one for

\footnote{For formal definition see Definition 3.1}
the stochastic model that lies outside the whp statement. Finally and most importantly, social networks are not random at all! They represent real-world ties, e.g., friendships which are far from begin random.

In this paper we introduce the concept of a PLB network, which gives a novel “waterproof” worst-case approach that overcomes the aforementioned problems and explains why many real-world networks are easier. We prove that on PLB networks many problems have lower complexity than what is implied by classical solutions. The problems that we are able to solve faster include basic P-time problems: transitive closure, perfect matching, PageRank and counting triangles. Additionally, we show that the NP-hard problem of finding maximum clique allows a subexponential time algorithm in PLB networks. An interesting aspect of this study is the existence of structure oblivious algorithms, i.e., algorithms that run faster on PLB networks without explicit knowledge of this fact. These structure oblivious algorithms shed some light on why some existing heuristic approaches are so efficient in practice, e.g., sorting vertices by degrees is the first step in many heuristic approaches to maximum clique problem [14].

Explaining why many algorithms work faster on real-world instances than what is predicted by worst-case analysis is one of the grand questions in algorithm that did not receive a plausible answer so far. A notable example is the SAT problem [28]. Our paper gives a possible answer to this grand challenge and calls for further research in this direction. On one hand, we shall search for faster solutions to other problems. On the other hand, we believe that real-world power law networks have more worst-case graph properties that can be exploited in the design and analysis of algorithms. In particular, we have observed that in a number of power law graphs with $\alpha > 2$, every vertex of degree $k$ has $o(k)$ neighbors of degree at least $k$ (we say that the graph has PLB neighborhoods [7]).

We have experimentally confirmed that this property is present in a number of real-world networks. This property can effectively replace the randomness assumption about the graph that has been introduced in previous works and we use it to obtain faster algorithms for counting triangles and maximum clique problem. In particular it implies that for $\alpha > 3$ our maximum clique algorithm works in polynomial time. This observation clearly contrasts with the proof that the clique problem is NP-hard on power law networks for any $\alpha > 1$ [24], and implies that it should be possible to efficiently find maximum cliques in numerous real-world networks, in which $\alpha > 3$.

1.1 Our Results and Related Work

We study the algorithmic complexity of power law networks in a worst-case model. Our work is somewhat related to the area of average-case analysis of algorithms, which tries to explain why some algorithm are fast on real-world data. However, we do not use the randomness of the data. Instead, we identify graph properties that can be exploited to give efficient algorithms. We stress that we are only interested in properties that can be decided deterministically. We also show the our model is general, by proving that one of the basic random power law network model generates PLB graphs with high probability.

**Counting Triangles** The problem of finding or counting triangles in a graph can be solved in $O(n^2)$ time or in $O(m^{2/3})$ time using fast matrix multiplication [3]. There has been some work that tried to show faster algorithms for counting triangles in power law graphs. Latapy [34] has shown two $O(mn^{1/\alpha})$ time algorithms, where $m$ is the number of edges in the graph. Moreover, Berry et. al [7] have shown that in random power law graphs, generated by erased configuration model, triangles can be counted in $O(n^{7-3\alpha})$ time, where $\Delta$ is the maximum vertex degree in the graph. Since the model assumes that $\Delta/\sqrt{m} \leq 1/2$, for $\alpha \in (2, 7/3)$ this gives a $O(n^{3/2-3/2\alpha})$ time algorithm ($\alpha > 2$ implies $m = O(n)$) and a linear time algorithm for $\alpha > 7/3$. However, as the authors admit this algorithm requires the graph to be random and does not fully apply to real-world graphs. In addition, the assumption that $\Delta/\sqrt{m} \leq 1/2$ may be unrealistic, as it is satisfied in only few of the real-world networks that we have analyzed (see Table 1).

We show that a very basic and widely used triangle counting algorithm works faster than what has been demonstrated by Latapy. This simple algorithm processes nodes in increasing order of their degrees, computes the number of triangles incident to each vertex, and then removes the processed vertex. A simple

\footnote{For formal statement see Definition 3.9}
analysis shows that this algorithm runs in $O(n^{3/\alpha})$ time for $1 < \alpha < 3$, $O(n \log n)$ time for $\alpha = 3$, and $O(n)$ time for $\alpha > 3$. Additionally, for graphs with PLB neighborhoods this algorithm runs in $O(n^{9/2 - 3/2\alpha})$ time for $2 < \alpha < 7/3$, and $O(n)$ time for $\alpha \geq 7/3$. These bounds visibly improve the running time of Latapy’s algorithm for $\alpha > 2$ and match the results of Berry et. al \[2\] (up to logarithmic factors) that have been obtained under full-randomness assumption. Moreover, when applied to random networks as in \[2\], our framework implies stronger whp bounds instead of bounds in expectations. We note that our algorithms are structure oblivious and do not need to know that the graph is PLB or has PLB neighborhoods to run in the above bounds. These running times are shown in Fig. 3 and can be slightly improved by using fast matrix multiplication.

**Maximum Clique** The fastest algorithm for finding maximum cliques in general graphs runs in $O(1.2125^n)$ time \[11\]. Moreover, Chen et. al \[15\] have shown that maximum clique cannot be solved in subexponential time unless exponential time hypothesis fails. We note that the maximum clique problem is NP-hard on power law graphs \[24\]. Janson, Łuczak and Norros \[33\] have shown that for $\alpha > 2$ maximal clique in a power law graph can be found in polynomial time and approximated for any $\alpha$. However, they assume that the graph is created using random Poissonian model. In this paper we show that on PLB graphs the problem can be solved in subexponential $\exp(O(n^{1/\alpha}))$ time. Additionally, when the graph has PLB neighborhoods our algorithm runs in $\exp(O(n^{3/2 - \alpha/2} \log n))$ time for $2 < \alpha < 3$ and $O(\text{poly}(n))$ time for $\alpha > 3$.

**Transitive Closure** The transitive closure of a graph $G$ can be either computed in $O(nm)$ time by executing $n$ graph searches, or in $O(n^\omega)$ time using block recursion and fast matrix multiplication. We show that this running time can be improved when $1 < \alpha < 2$ – see Fig. 4.

**Algebraic Matrix Algorithms** There are two complexity results for the computation of the determinant of a $n \times n$ matrix $A$ over a finite fields\[2\]: (i) fast matrix multiplication to obtain $O(n^\omega)$ time algorithm \[3\] or (ii) Wiedemann’s approach that works in $O(nm)$ time, where $m$ is the number of nonzero entries in a matrix. We note that there are many heuristic approaches that are used in practice to speed up matrix computations, e.g., minimum degree algorithm \[26\], but these ideas do not improve the worst-case complexities that are stated above. Here, we are only interested in obtaining a worst-case bound on the arithmetic complexity of these problems and therefore we will not review this rich body of literature. We note that our approach is related to minimum degree algorithm, because as the first step we partition the matrix into dense and sparse part according to the number of nonzero entries in each row or column. However, after this step novel algorithms are proposed that exploit the structure of the matrix.

We will assume that the non-zero structure of $A$ corresponds to an PLB graph $G$, i.e., $a_{ij} \neq 0$ if and only if $ij \in E(G)$. We are able to show faster algorithms for the case when $1 < \alpha < 2$. In particular our algorithm in the case of symmetric matrices works in $O(n^{2+\omega} / \alpha(2-n\alpha + 2))$ time – see Fig. 4 for the running time in the case of symmetric and general matrices.

Additionally, we show that with the same complexities it is possible to solve linear system with matrix $A$, invert matrix $A$, and compute PageRank of a graph represented by $A$. PageRank is a very simple version of the eigenproblem and our results could indicate that a general eigenproblem could be solved faster on PLB graphs. Developing such faster algorithms for eigenproblem, characteristic polynomial or even matrix rank is left as an intriguing open problem.

**Perfect Matching** There are several algorithms known for finding perfect matching in general graphs: $O(\sqrt{nm})$ time algorithm \[14\], $O(n^\omega)$ time algorithm \[15\] and $O(m^{10/7})$ time algorithm \[12\]. Here, basing on our results for computing matrix determinant we show an algorithm that improves over these results when $\alpha < 1.09$ – see Fig. 4. We conjecture, however, that an improvement is possible for $\alpha \in [1, 2]$.

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\[3\]We discuss here only the finite field case as it is the most relevant case for TCS.

\[4\] $O(n^\omega)$ is the time needed for a straight-line program to multiply two $n \times n$ matrices; $\omega$ is called matrix multiplication exponent. Currently $\omega < 2.373$ \[51\].
Organization of the Paper  The following part of this paper is organized as follows. In Section 2 we introduce basic notation and show some general properties that we later use. In Section 3 we define the class of PLB graphs and show their basic properties. Then, in Section 4 we verify our definitions on real-world data. Section 5 analyses very simple algorithms for counting triangles and finding maximum clique on PLB graphs. Finally, in Section 6 we present more advanced algebraic algorithms for PLB graphs that compute the transitive closure, find the perfect matching, and compute the determinant.

2 Preliminaries

Let \( G \) be a graph. Throughout the paper we use \( n \) to denote the number of vertices in a graph, \( d_k \) to denote the number of vertices of degree \( k \), and \( d_{\geq k} \) to denote the number of vertices of degree at least \( k \). It should be clear from the context, which graph we refer to. We assume that the graphs we work with are simple, i.e., they do not contain multiple edges. In the majority of problems that we study (e.g., transitive closure or maximal clique) multiple edges are not important and can be simply removed. We assume that \( \log n \) denotes the binary logarithm function.

Lemma 2.1. Let \( 1 \leq a \leq b \), for \( a, b \in \mathbb{N} \), and let \( c \) be a constant. Then

\[
\sum_{i=a}^{b} i^c = \begin{cases} 
O(b^{c+1}) & \text{if } c > -1 \\
O(\log(b/a)) & \text{if } c = -1 \\
O(a^{c+1}) & \text{if } c < -1 
\end{cases}
\]

Note that, throughout the paper we assume that for \( b < a \), and any function \( f \), \( \sum_{i=a}^{b} f(i) = 0 \).

Proof. For \( i \geq a \geq 1 \) we have \( |i|^c = O(i^c) \). Thus,

\[
\sum_{i=a}^{b} i^c = \int_{a}^{b+1} [x]^c \, dx = O(1) \int_{a}^{b+1} x^c \, dx
\]

For \( c \neq -1 \) we have

\[
\int_{a}^{b+1} x^c \, dx = \frac{1}{c+1} ((b+1)^{c+1} - a^{c+1})
\]

If \( c > -1 \), then \( \frac{1}{c+1} > 0 \), so we can bound the expression by \( O((b+1)^{c+1}) = O(b^{c+1}) \). Otherwise, if \( c < -1 \), then \( \frac{1}{c+1} < 0 \), so we can bound it by \( O(a^{c+1}) \). It remains to consider the case when \( c = -1 \):

\[
\int_{a}^{b+1} x^{-1} \, dx = \int_{a}^{b+1} x^{-1} \, dx = \log(b+1) - \log a = O(\log(b/a))
\]

We also have a reverse relation:

Lemma 2.2. Let \( 1 \leq a \leq b/2 \), for \( a, b \in \mathbb{N} \), and let \( c > 0 \) be a constant. Then \( a^{-c} = O(\sum_{i=a}^{b} i^{-c-1}) \).

Proof.

\[
\sum_{i=a}^{b} i^{-c-1} \geq \int_{a}^{b+1} x^{-c-1} \, dx = \frac{1}{c}(a^{-c} - (b+1)^{-c}) \geq \frac{1}{c}(a^{-c} - (a/2 + 1)^{-c}) = \Omega(a^{-c})
\]
Lemma 2.3. Let $c > 0$, $\alpha > 1$, and $\delta \geq 1$. Then
\[
\sum_{i=1}^{\delta} (i + t)^{-\alpha} i^c = \begin{cases} 
O(\delta^{c+1-\alpha}) & \text{if } c > \alpha - 1 \\
O(\log \delta) & \text{if } c = \alpha - 1 \\
O((t+1)^{c+1-\alpha}) & \text{if } c < \alpha - 1 
\end{cases}
\]

Proof. If $c - \alpha \geq -1$ we simply use the fact that $(i + t)^{-\alpha} \leq i^{-\alpha}$ and obtain
\[
\sum_{i=1}^{\delta} (i + t)^{-\alpha} i^c \leq \sum_{i=1}^{\delta} i^{c-\alpha}.
\]
By Lemma 2.1 this is equal to $O(\delta^{c+1-\alpha})$ for $c > \alpha - 1$, and equal to $O(\log \delta)$ for $c = \alpha - 1$.

Now consider the case when $c < \alpha - 1$.
\[
\sum_{i=1}^{\delta} (i + t)^{-\alpha} i^c \leq \sum_{i=1}^{t} (i + t)^{-\alpha} i^c + \sum_{i=t+1}^{\delta} (i + t)^{-\alpha} i^c
\]
\[
\leq O(1) \sum_{i=1}^{t} (2t + 1)^{-\alpha} i^c + \sum_{i=t+1}^{\delta} i^{c-\alpha}
\]
\[
= O(1)(t+1)^{-\alpha} \sum_{i=1}^{t} i^c + \sum_{i=t+1}^{\delta} i^{c-\alpha}
\]
\[
= O((t + 1)^{c+1-\alpha}) + O((t+1)^{c+1-\alpha})
\]
\[
= O((t + 1)^{c+1-\alpha}).
\]

\[\square\]

Lemma 2.4. Let $G$ be a graph and $k \geq 0$. The number of edges of $G$ is at most $\sum_{i=1}^{n-1} d_{\geq i}$.

Proof. Observe that a vertex of degree $k$ is counted $k$ times in the sum. Thus, the sum is equal to the total degree of all vertices, which is twice the number of edges. \[\square\]

3 Power law bounded networks

In this section we introduce our definition of a power law bounded network. There are multiple definitions of power law networks. Some state that in a power law network the number of vertices of degree $k$ is proportional to $k^{-\alpha}$ for some parameter $\alpha$. In other cases power law is defined with respect to random graphs and only talks about expected degrees of vertices. Both these approaches may not be applied to the analysis of algorithms running on real-world networks. The first one suffers from two serious drawbacks. First, it is often not stated in a formal way. Second, it seems that it effectively disallows even a single vertex with high degree. On the other hand the stochastic definition can only be applied to graphs randomly drawn from some distribution. This is not the case for real-world graphs, which are fixed.

We introduce the concept of a power law bounded network, which captures the power law behavior of degree distribution that is necessary for the analysis of algorithms. At the same time it is weak enough to cover many real-world graphs. Note that this definition for $t = 0$ is similar to the one in [7]. The main difference is that we do not impose any lower bounds on the numbers of vertices of given degrees.

Definition 3.1. Let $G$ be an undirected $n$-vertex graph and $c_1 > 0$ be a universal constant. We say that $G$ is power law bounded (PLB) for some parameters $1 < \alpha = O(1)$ and $t \geq 0$ if for every integer $k \geq 0$, the number of vertices $v$, such that $\deg(v) \in [2^d, 2^{d+1})$ is at most
\[
c_1 n(t + 1)^{\alpha-1} \sum_{i=2^d}^{2^{d+1}-1} (i + t)^{-\alpha}.
\]
In the following we say that $G$ is a PLB graph with parameters $\alpha$ and $t$.

Note that the $(t + 1)^{\alpha - 1}$ factor in the above definition is necessary to ensure that the sum of the above upper bounds over all $k$ is $O(n)$. The above power law distribution that includes the shift by the parameter $t$ is called shifted power law and was observed in different real-world networks. In particular, the parameter $t$ allows us to better fit the degree distributions in our experiments (see Section 3). As our experiments show, in the networks that we have studied the value of $t$ is very small. However, in general it is unknown whether and how $t$ depends on other parameters of the network and we are not aware of the models that would describe such dependence. A reasonable assumptions here seems to be that $t = O(n^\epsilon)$ for every $\epsilon > 0$. However, when discussing some complexities of our algorithms we will for simplicity sometimes assume that $t = O(polylog n)$. Hence, the factors in the running time, that depend on $t$ are of secondary importance. In the introduction when discussing our results we have assumed that $t = 0$.

The exact set of graph that satisfy Definition 3.1 obviously depends on the choice of the constant $c_1$. However, as we later show, many real-world graphs satisfy this definition for a small value of $c_1$, i.e., at most 5. At the same time, the running time dependency of our algorithms on $c_1$ is only polynomial. The only exception is an algorithm for finding maximum clique, whose running time itself is super-polynomial.

Let us list some basic properties of PLB graphs.

**Lemma 3.2.** Let $G$ be a PLB graph with parameters $\alpha$ and $t$. Then, $d_{\geq k} = O(n(t + 1)^{\alpha - 1}(k + t)^{1 - \alpha}) = O((n + t)^{\alpha - 1}k^{1 - \alpha})$.

**Proof.** Observe that $k' = 2^{\lceil \log k \rceil}$ is the smallest power of 2 which is not greater than $k$, thus, $k' \leq k \leq 2k'$. We bound the number of vertices, whose degree is at least $k'$, which is an upper bound on the number of vertices of degree at least $k$.

$$c_1(n + t)^{\alpha - 1} \sum_{i=k'}^{n-1} (i + t)^{-\alpha} \leq c_1(n + t)^{\alpha - 1} \sum_{i=k'}^{n-1+\lceil t \rceil} i^{-\alpha} = O(n(t + 1)^{\alpha - 1}(k' + \lceil t \rceil)^{1 - \alpha}) = O(n(t + 1)^{\alpha - 1}(k' + t)^{1 - \alpha}) = O(n(t + 1)^{\alpha - 1}(k + t)^{1 - \alpha})$$

The following lemma is used, e.g., to bound the running times of algorithms, which take $f(k)$ time to process a vertex of degree $k$, where $f$ is at most polynomial in its parameter. Roughly speaking, it says that the running time of a polynomial algorithm running on a PLB network is asymptotically the same as the running time on a graph with an ideal power law distribution.

**Lemma 3.3.** Let $G$ be a PLB graph with parameters $\alpha$ and $t$. Let $d_i$ be the number of vertices of degree $i$ in $G$. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a nondecreasing function, such that for any $x, c \in \mathbb{N}$, $f(cx) \leq c^{O(1)}f(x)$. Then, for every $k \geq 1$ we have $\sum_{i=1}^{k} d_i f(i) = O(1)n(t + 1)^{\alpha - 1} \sum_{i=1}^{k} (i + t)^{-\alpha} f(i)$. 

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Proof. Let us first derive an auxiliary inequality.

\[ \sum_{i=2^j}^{2^{j+1}-1} d_i f(i) \leq \sum_{i=2^j}^{2^{j+1}-1} d_i f(2^j + 1) \]

\[ = f(2^j + 1) \sum_{i=2^j}^{2^{j+1}-1} d_i \]

\[ \leq O(1)n(t + 1)^{\alpha - 1} \sum_{i=2^j}^{2^{j+1}-1} (i + t)^{-\alpha} f(i) \]

Note that we introduce \( O(1) \) to hide \( c_1 \) and the constant that comes from replacing \( f(2^j + 1) \) by \( f(2^j) \). Let \( k' = 2^{\lceil \log(k+1) \rceil} - 1 \). Thus \( k \leq k' \leq 2k \) and \( k' = 2^l - 1 \) for some integer \( l \).

\[ \sum_{i=1}^{k} d_i f(i) \leq \sum_{i=1}^{k'} d_i f(i) \]

\[ = \sum_{j=0}^{l-1} \sum_{i=2^j}^{2^{j+1}-1} d_i f(i) \]

\[ = \sum_{j=0}^{l-1} O(1)n(t + 1)^{\alpha - 1} \sum_{i=2^j}^{2^{j+1}-1} (i + t)^{-\alpha} f(i) \]

\[ = O(1)n(t + 1)^{\alpha - 1} \sum_{j=0}^{l-1} \sum_{i=2^j}^{2^{j+1}-1} (i + t)^{-\alpha} f(i) \]

\[ = O(1)n(t + 1)^{\alpha - 1} \sum_{i=1}^{k'} (i + t)^{-\alpha} f(i) \]

\[ \leq O(1)n(t + 1)^{\alpha - 1} \sum_{i=1}^{2k} (i + t)^{-\alpha} f(i) \]

\[ = O(1)n(t + 1)^{\alpha - 1} \sum_{i=1}^{k} ((2i - 1 + t)^{-\alpha} f(2i - 1) + (2i + t)^{-\alpha} f(2i)) \]

\[ \leq O(1)n(t + 1)^{\alpha - 1} \sum_{i=1}^{k} (i + t)^{-\alpha} (f(2i - 1) + f(2i)) \]

\[ \leq O(1)n(t + 1)^{\alpha - 1} \sum_{i=1}^{k} (i + t)^{-\alpha} f(i) \]

By using Lemma 3.3 together with Lemma 2.1 we obtain the following bound on the number of edges touching small degree vertices.
Lemma 3.4. Let $G$ be a PLB graph with parameters $\alpha$ and $t$, where $\alpha < 2$. Then, the number of edges incident to at least one vertex of degree at most $k$ is $O(n(t + 1)^{\alpha - 1}k^{2 - \alpha})$.

Proof. The number of edges incident to at least one vertex of degree at most $k$ is $\sum_{i=1}^{k} d_i$. We use Lemma 3.3 with the identity function $f(i) = i$, obtaining

$$
\sum_{i=1}^{k} d_i = O(n(t + 1)^{\alpha - 1} \sum_{i=1}^{k} (i + t)^{-\alpha} \cdot i)
$$

$$
= O(n(t + 1)^{\alpha - 1} \sum_{i=1}^{k} i^{1-\alpha})
$$

$$
= O(n(t + 1)^{\alpha - 1}k^{2 - \alpha}).
$$

(1)

In the first transformation we use the fact that $(i + t)^{-\alpha} \leq i^{-\alpha}$, whereas in the second one we use Lemma 2.1.

By combining Lemma 3.2 with Lemma 2.4 we obtain the following.

Lemma 3.5. Let $G$ be a PLB graph with parameters $\alpha$ and $t$. Then, the number of edges of $G$ of is (a) $O(n^{3-\alpha}(t + 1)^{\alpha - 1})$ for $1 < \alpha < 2$, (b) $O(n \log n(t + 1))$ for $\alpha = 2$, (c) $O(n(t + 1))$ for $\alpha > 2$.

Proof. By Lemma 2.4 the number of edges is at most $\sum_{i=k+1}^{n-1} d_{\geq i}$. By Lemma 3.2 $d_{\geq i} = O(n(t + 1)^{\alpha - 1}(i + t)^{-\alpha+1})$. We have

$$
\sum_{i=1}^{n-1} O(n(t + 1)^{\alpha - 1}(i + t)^{1-\alpha}) = n(t + 1)^{\alpha - 1} \sum_{i=1}^{n-1} O((i + t)^{1-\alpha}).
$$

We now use Lemma 2.1. For $1 < \alpha < 2$ we have $1 - \alpha > -1$, so the sum can be bounded by $O((n + t)^{2-\alpha}) = O(n^{2-\alpha})$. Putting it back into Equation (2) we get $O(n^{3-\alpha}(t + 1)^{\alpha - 1})$. For $\alpha = 2$, we may bound the sum by $O((n + t)/t)) = O(\log n)$, thus obtaining $O(n \log n(t + 1))$. Finally, for $\alpha > 2$, we bound the sum by $O((t + 1)^{2-\alpha})$, so the number of edges is $O(n(t + 1))$.

What is interesting, for a PLB graph with $1 < \alpha < 2$, the bound on the number of edges given by Lemma 3.5 is not tight. In particular, the number of vertices with high degree (considerably greater than $n^{1/\alpha}$) is polynomially smaller. We say that a vertex is a high-degree vertex if its degree is more than $n^{1/\alpha}$. Each edge either connects two high-degree vertices or is incident to a low-degree vertex. The number of edges of the first type is bounded, as there are few high-degree vertices, whereas the number of the edges of the second type is bounded by simply summing the degrees of low-degree vertices. Note that this reasoning heavily depends on the fact that the graph is simple. This is formalized in the following Lemma.

Lemma 3.6. Let $G$ be a PLB graph with parameters $\alpha$ and $t$, where $1 < \alpha < 2$, and $k \geq n^{1/\alpha}(t + 1)^{1-1/\alpha}$. Then, $d_{\geq k} = O(n^{3-\alpha}(t + 1)^{(\alpha - 1)(3-\alpha)}k^{2-3\alpha+1})$.

Proof. We say that a vertex of degree at least $k$ is a high-degree vertex. By Lemma 3.2, $d_{\geq k} = O(n(t + 1)^{\alpha - 1}k^{1-\alpha})$. We will use the fact that $G$ has no multiple edges to derive a stronger upper bound on $d_{\geq k}$.

We first bound the total degree of high-degree vertices, which we denote by $S$. The edges, whose both endpoints have high degrees contribute at most $d_{\geq k}(d_{\geq k} - 1) \leq d_{\geq k}^2$ to $S$. In addition, a low-degree vertex of degree $i$ contributes at most $\min(i, d_{\geq k})$. Recall that by $d_i$ we denote the number of vertices of degree $i$. Thus, we may bound $S$ by $d_{\geq k}^2 + \sum_{i=1}^{k-1} d_i \min(i, d_{\geq k})$. We now apply Lemma 3.3 using $f(i) = \min(i, d_{\geq k})$. 

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\[
d_{\geq k}^2 + \sum_{i=1}^{k-1} d_i \min(i, d_{\geq k}) \leq d_{\geq k}^2 + O(1)n(t+1)^{\alpha-1} \sum_{i=1}^{k-1} i^{-\alpha} \min(i, d_{\geq k})
\]
\[
\leq d_{\geq k}^2 + O(1)n(t+1)^{\alpha-1} \left( \sum_{i=1}^{d_{\geq k}} i^{1-\alpha} + \sum_{i=d_{\geq k}+1}^{k} i^{-\alpha} d_{\geq k} \right)
\]
\[
= d_{\geq k}^2 + O(1)n(t+1)^{\alpha-1} \left( O(d_{\geq k}^{2-\alpha}) + O(d_{\geq k}^{2-\alpha}) \right)
\]
\[
= d_{\geq k}^2 + O(n(t+1)^{\alpha-1}d_{\geq k}^{2-\alpha})
\]

Note that when we split the sum into two sums, we use the assumed convention that for \(a > b, \sum_{i=a}^{b} f(i) = 0\).

We now bound \(d_{\geq k}\):
\[
d_{\geq k} = O(n(t+1)^{\alpha-1}k^{1-\alpha}) = O(n(t+1)^{\alpha-1}n^{1/\alpha-1}(t+1)^{(1-1/\alpha)(1-\alpha)}) = O(n^{1/\alpha}(t+1)^{(\alpha-1)/\alpha}).
\]

This gives
\[
d_{\geq k}^2 = d_{\geq k}^{2-\alpha} d_{\geq k}^1 = d_{\geq k}^{2-\alpha} O(n(t+1)^{\alpha-1}).
\]

Hence, the total degree of high-degree vertices is
\[
d_{\geq k}^2 + O(n(t+1)^{\alpha-1}d_{\geq k}^{2-\alpha}) = O(n(t+1)^{\alpha-1}d_{\geq k}^2)
\]
\[
= O(n(t+1)^{\alpha-1}n^{2-\alpha}(t+1)^{(\alpha-1)(2-\alpha)}k^{(1-\alpha)(2-\alpha)})
\]
\[
= O(n^{3-\alpha}(t+1)^{(\alpha-1)(3-\alpha)}k^{(1-\alpha)(2-\alpha)}).
\]

To obtain the bound on the number of high-degree vertices, we divide the obtained bound by \(k\), which gives
\[
O(n^{3-\alpha}(t+1)^{(\alpha-1)(3-\alpha)}ka^{2-3\alpha+1}).
\]

**Corollary 3.7.** Let \(G\) be a PLB graph with parameters \(\alpha\) and \(t\), where \(1 < \alpha < 2\), and \(k \geq n^{1/\alpha}(t+1)^{1-1/\alpha}\). Moreover, assume that \(1 < \alpha < 2\). Then, the number of vertices of degree between \(k\) and \(2k\) is \(O(n^{3-\alpha}(t+1)^{(\alpha-1)(3-\alpha)}ka^{2-3\alpha+1})\).

Let us use Lemma 3.6 to derive a stricter bound on the number of edges in a PLB graph with \(1 < \alpha < 2\).

**Lemma 3.8.** Let \(G\) be a PLB graph with parameters \(\alpha\) and \(t\), where \(1 < \alpha < 2\). Then, \(G\) has \(O(n^{2/\alpha}(t+1)^{2-2/\alpha})\) edges.

**Proof.** By Lemma 2.4, the total number of edges of \(G\) is at most \(\sum_{i=1}^{n-1} d_{\geq i}\). Let \(\delta = n^{1/\alpha}(t+1)^{1-1/\alpha}\). We split the sum into two parts.

By Lemma 3.6, for \(k \geq \delta\), \(d_{\geq k} = O(n^{3-\alpha}(t+1)^{(\alpha-1)(3-\alpha)}ka^{2-3\alpha+1})\). Moreover, observe that since \(1 < \alpha < 2\), \(\alpha^2 - 3\alpha + 1 < -1\). Then
\[
\sum_{i=\delta}^{n-1} d_{\geq i} = \sum_{i=\delta}^{n-1} O(n^{3-\alpha}(t+1)^{(\alpha-1)(3-\alpha)}i^{\alpha^2-3\alpha+1})
\]
\[
= O(n^{3-\alpha}(t+1)^{(\alpha-1)(3-\alpha)}n^{1/\alpha}(\alpha^2-3\alpha+2)(t+1)^{(1-1/\alpha)(\alpha^2-3\alpha+2)})
\]
\[
= O(n^{3-\alpha+3-2/\alpha}(t+1)^{-\alpha+4\alpha-3+\alpha^2-3\alpha+2-\alpha+3-2/\alpha})
\]
\[
= O(n^{2/\alpha}(t+1)^{2-2/\alpha}).
\]
On the other hand

\[
\sum_{i=1}^{\delta} d_{\geq i} = \sum_{i=1}^{\delta} O(n(t + 1)^{0-1} i^{1-\alpha})
\]

\[
= O(n(t + 1)^{(2-\alpha)} (t + 1)^{(1-1/2)(2-\alpha)})
\]

\[
= O(n^{2/\alpha}(t + 1)^{2-2/\alpha})
\]

Thus, \(\sum_{i=1}^{n-1} d_{\geq i} = O(n^{2/\alpha}(t + 1)^{2-2/\alpha}).\)

\[
\]

### 3.1 PLB neighborhoods

Assume that we pick a random vertex from a power law graph with parameter \(\alpha\), proportionally to its degree. Then, the degree of the chosen vertex comes from a power law distribution with parameter \(\alpha - 1\). This implies that, roughly speaking, for each vertex \(v\) in a random power law graph, the degree distribution of neighbors of \(v\) also obeys power law. This fact can be exploited to obtain better running time bounds of some algorithms. However, the algorithms that we later give actually rely on a weaker property. Namely, for a vertex of degree \(k\) it only need a bound on the number of neighbors of degree at least \(k\).

Note that if we randomly pick \(k\) vertices proportionally to their degrees, then the number of chosen vertices of degree at least \(k\) is \(O((t + 1)^{α-2} k \sum_{i=k}^{n-1} i(i + t)^{-α}).\) This motivates the following.

**Definition 3.9.** Let \(G\) be a PLB graph with parameters \(\alpha > 2\) and \(t\), and let \(c_2 > 0\) be an universal constant. We say that \(G\) has PLB neighborhoods if for every vertex \(v\) of degree \(k\), the number of neighbors of \(v\) of degree at least \(k\) is at most \(c_2 \max(\log n, (t + 1)^{α-2} k^{3/α}).\)

The log \(n\) factor in the definition comes from the fact that we assume that the graph is created in a random way. Thus, the actual numbers of neighbors may slightly deviate from the expected values.

**Lemma 3.10.** Let \(G\) be a PLB graph with parameters \(\alpha > 2\) and \(t\), and PLB neighborhoods. Then, for every vertex \(v\) of degree \(k\), the number of neighbors of \(v\) of degree at least \(k\) is \(O(\max(\log n, (t + 1)^{α-2} k^{3/α})).\)

**Proof.** We have

\[
c_2 (t + 1)^{α-2} k \sum_{i=k}^{n-1} i(i + t)^{-α} \leq O((t + 1)^{α-2} k \sum_{i=k}^{n-1} i^{1-α})
\]

\[
= O((t + 1)^{α-2} k \cdot k^{2-α})
\]

\[
= O((t + 1)^{α-2} k^{3/α}).
\]

Thus, \(c_2 \max(\log n, (t + 1)^{α-2} k^{3/α}) = O(\max(\log n, (t + 1)^{α-2} k^{3/α})).\) \(\square\)

### 3.2 Relation to other models

Definitions 3.1 and 3.9 are designed to capture the properties of power law graphs that can be easily exploited in the analysis of algorithms. At the same time there are many random graph models that produce power law graphs. In these models even giving simple bounds on the degree distributions of the produced graphs is often highly nontrivial. The analyses of some these models \(\[4, 10, 36, 30, 18\]\) only give the expected numbers of vertices of given degrees and analyze the concentration. A typical concentration statement says that (with high probability) the number of vertices of degree \(k\) differs from the expected value by some small additive error (e.g., \(\sqrt{n} \log n\)). This cannot be directly used to show that these graphs satisfy Definition 3.1. Proving that would require bounding the number of vertices of degree belonging to \([2^d, 2^{d+1})\), but if we simply sum
the approximate numbers of vertices of each degree \(d \in [2^d, 2^{d+1})\), the additive errors accumulate. At the same time we believe that many of the proposed random graph processes yield PLB graphs, but proving this is a challenging open problem.

Another models for power law graphs are based on fixing a degree sequence in the beginning. In the erased configuration model \([5, 9, 12]\) the degrees of all vertices are fixed in the very beginning to obtain an almost ideal power law distribution. Then a graph is picked uniformly at random, among all graphs that have the given degree sequence. Note that we fix an “ideal” power law degree sequence, as it is done, e.g., in \([46, 1, 7]\), but in some works on this model the degree of each vertex is picked independently at random.

**Theorem 3.11.** Let \(n\) be sufficiently large and \(G\) be a random power law graph with parameter \(\alpha > 1\) created by erased configuration model. Then \(G\) is a PLB graph with parameters \(\alpha\) and \(t = 0\). Moreover, with high probability, \(G\) has PLB neighborhoods.

These statements are true for some universal constants \(c_1\) and \(c_2\) (see definitions \([3.1\) and \(3.9\)).

The remaining part of this section gives a proof of Theorem 3.11. Let us now describe the erased configuration model in detail. First, we pick a degree for every vertex, in such a way that the number of vertices of degree \(k\) is \(\Theta(n/k^\alpha)\). Since the sum of all degrees has to be even, we add one vertex of degree 1 if necessary. For simplicity of the analysis we ignore this added vertex. Then, we build a random graph with the chosen degree sequence as follows:

1. Build a complete graph \(H\) containing \(\deg(v)\) copies of vertex \(v\).
2. Choose a random perfect matching in \(H\) and remove the edges that are not in the matching.
3. Build \(G\) from \(H\) by merging the copies of each vertex.

The resulting graph may have multiple edges or self-loops, which we remove.

It follows easily that the maximum degree in \(G\) is \(O(n^{1/\alpha})\). We now verify that \(G\) satisfies Definition 3.1. We have

\[
\sum_{i=2^d}^{2^{d+1}-1} d_i \leq \sum_{i=2^d}^{2^{d+1}-1} O(n/i^\alpha) = O(n) \sum_{i=2^d}^{2^{d+1}-1} i^{-\alpha} \leq c_1 n \sum_{i=2^d}^{2^{d+1}-1} i^{-\alpha}
\]

for some universal constant \(c_1\). Thus, \(G\) is a PLB graph with parameters \(t = 0\) and \(\alpha\).

The proof that, with high probability, \(G\) has PLB neighborhoods (satisfies Definition 3.9) is more involved. Let us now assume that \(\alpha > 2\) and fix a vertex \(v\) of degree \(k\). Our goal is to bound the number of neighbors of \(v\) of degree at least \(k\).

Vertex \(v\) has \(k\) copies in \(H\), that we denote by \(v_1, \ldots, v_k\). We say that a vertex of \(H\) is bad if it is a copy of a vertex of degree at least \(k\), but not a copy of \(v\). Let us define a sequence of Boolean random variables \(X_1, \ldots, X_k\), where \(X_i = 1\) iff. \(v_i\) is matched in \(H\) with a bad vertex. Note that matching \(v_i\) with another copy of \(v\) does no harm, as this creates a self loop in \(G\), which is then removed. Thus, \(\sum_{i=1}^k X_i\) is an upper bound on number of neighbors of \(v\) of degree at least \(k\) in \(G\). The number of bad vertices is bounded by

\[
\sum_{i=k}^{n-1} i O(n/i^\alpha) \leq n \sum_{i=k}^{n-1} i^{1-\alpha} = O(nk^{2-\alpha}).
\]

Thus, \(P(X_i = 1) \leq Ck^{2-\alpha}\), for some universal constant \(C\), as \(P(X_i = 1)\) is bounded by the probability of a randomly chosen vertex being bad. Define \(X = \sum_{i=1}^k X_i\). It follows that \(E(X) \leq Ck^{3-\alpha}\).

We now use Chernoff bound to bound \(X\). The variables \(X_i\) are not independent, but they are negatively associated, which suffices for the Chernoff bound to work (see e.g. \([24]\)).

**Lemma 3.12.** For any set \(I \subseteq \{1, \ldots, k\}\), \(P(\bigwedge_{i \in I} X_i = 1) \leq \prod_{i \in I} P(X_i = 1)\), that is variables \(X_i\) are negatively associated.
Proof. Assume the number of bad vertices is $b$. Then $\mathbb{P}(X_i = 1) = b/(n - 1)$.

Observe that the perfect matching in $H$ can be computed as follows. We go through the vertices in any order. For each vertex, if it is already matched, we skip it. Otherwise, we match it to a randomly chosen unmatched vertex.

For the purpose of the proof, we may assume that the first vertices that are chosen in this process are $v_1, \ldots, v_k$. In the $i$-th step, we need to compute the probability that $v_i$ is matched to a bad vertex, provided that vertices $v_1, \ldots, v_{i-1}$ have been matched to a bad vertex. This probability is clearly $(b - i + 1)/(n - 1 - 2i + 2) \leq b/(n - 1)$. Thus, $\mathbb{P}(\bigwedge_{i \in I} X_i = 1) \leq (b/(n - 1))^{\#I} = \prod_{i \in I} \mathbb{P}(X_i = 1)$. The lemma follows.

In our proof we use the following version of the Chernoff bound. By Lemma 3.12, it can be applied to the random variables $X_1, \ldots, X_k$.

**Theorem 3.13** (21). Let $X = \sum_{i=1}^k X_i$, where $0 \leq X_i \leq 1$ and $X_i$ are negatively associated. Let $t > 2e\mathbb{E}(X)$. Then $\mathbb{P}(X > t) \leq 2^{-t}$.

We now proceed with the main part of the proof of the second claim of Theorem 3.11 which states that $G$ has PLB neighborhoods (with high probability). We show that the property of Definition 3.9 holds for a $t$ that is very small compared to $n$. Some of the graphs in the data sets are directed. For such graphs we make two adjustments. Either we drop the orientations of the edges (“directed, in-degree + out-degree” in Table 1), or we slightly modify Definition 3.1 and only consider the outdegrees of vertices (“directed, out-degree” in Table 1). For some of the networks, in Fig. 1 we also show the degree distribution, as well as the bound of Definition 3.1. In order to show the data with more detail, we plot not only the numbers of vertices, whose degrees belong to $[2^k, 2^{k+1})$ (actual and the upper bounds of Definition 3.1), but also the numbers of vertices of degree belonging to $[k, 2k)$ for each $1 \leq k < n$.

In the case when $\alpha < 2$, in Fig. 1 the bound from Corollary 3.7 is marked with green line. For Epinions and WikiTalk graphs the critical degree, when the second power law starts, is predicted rather well. Note that this is not the case for Facebook graph as the maximum number of friends one can have is limited to

\[ n \leq k \leq d \]

We thank the authors of [19] for sharing with us this data.

1 We thank the authors of [19] for sharing with us this data.

5 We thank the authors of [19] for sharing with us this data.

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5000. The high degree part of the distribution is cut off at this number. While the critical degree is predicted
decently, the slope of the second power law distribution is underestimated. This is most probably due to the
worst-case form of our bounds which are overly pessimistic with respect to the actual trend.

Then, we move on to Definition 3.9. For each network we use the previously computed parameters \( \alpha \) and \( t \) and find the smallest value of \( c_2 \), for which the definition is satisfied. We skip the graphs, where \( \alpha < 2 \),
as Definition 3.9 does not apply to them. The values of \( c_2 \) obtained this way are also shown in Table 1.
Observe that for every network the computed value of \( c_2 \) is less than 8.06, and for a big majority of them it is
less than 2. This confirms that the property of Definition 3.9 is indeed present in real-world graphs.

Table 1 also contains two adjustments, in which we force the value of \( t \) to be 0. In some sense this is
similar to fitting the standard definition of a power law distribution to our data. However, this causes the
value of \( \alpha \) to increase and makes our bounds much further from the real data, as shown in Fig 2.

5 Counting Triangles and Maximal Clique

This section presents our first two algorithms for PLB graphs. The first algorithm counts triangles, whereas
the second one returns the size of the maximal clique. The algorithms themselves are easy and should be
considered folklore. However, we show that in the case of PLB graphs they perform much better than in the
case of general graphs. Then we obtain even better running time bounds for graphs with PLB neighborhoods.
This is the most important contribution of this section, as we believe that it gives the first solid explanation
of the good performance of triangle counting and maximum clique algorithms in real-world graphs.

Both our algorithms are based on the same construction. We first direct the edges of \( G \) towards vertices of
higher degree. Formally, let \( v_1, \ldots, v_n \) be all vertices of \( G \) sorted in non-decreasing order of degrees. We
define \( \bar{G} \) to be a graph obtained from \( G \) by directing each undirected edge \( v_iv_j \) towards \( v_{\max(i,j)} \). Note that
since the degrees of vertices are bounded by the number of vertices, we may sort the vertices and build \( \bar{G} \)
in linear time. Moreover, note that \( \bar{G} \) does not contain any cycles. Let \( b(k) \) be the maximum out-degree in \( \bar{G} \)
of a vertex of degree \( k \) in \( G \).

Note that the value \( b(k) \) is related to graph degeneracy. We say that a graph is \( d \)-degenerate if every
subgraph has a vertex of degree at most \( d \). In our case \( G \) is \( d \)-degenerate for \( d = \max_{i=1,\ldots,n-1} b(i) \). In \( d \)-degenerate graphs we can count triangles in \( O(dm) \) time [16]. Since PLB graphs are \( O(n^{1/\alpha}) \)-degenerate
(assuming \( t = 0 \)), this can be used to obtain a running time bound of \( O(mn^{1/\alpha}) \), which is the same as the
running time given in [37]. However, with a slightly more careful analysis, in this section we improve this
bound. While this result is simple, to the best of our knowledge it has not been previously stated explicitly.

We first use the bounds derived in Section 3 to bound \( b(k) \).

**Lemma 5.1.** Let \( G \) be a PLB graph with parameters \( \alpha \) and \( t \). Then \( b(k) = O(\min(k, d_{\geq k})) = O(\min(k, n(t+1)^{\alpha-1}k^{1-\alpha})) = O(n^{1/\alpha}(t+1)^{1-1/\alpha}) \).

*Proof.* Obviously \( b(k) \leq k \), since for every \( v \in V(G) \), we have \( \text{outdeg}_{\bar{G}}(v) \leq \deg_{G}(v). \) In addition to that, since the edges are directed towards vertices of higher degree, \( b(k) \leq d_{\geq k} \). The second inequality follows
directly from the bound on \( d_{\geq k} \) derived in Lemma 3.2

It remains to show that \( O(\min(k, n(t+1)^{\alpha-1}k^{1-\alpha})) = O(n^{1/\alpha}(t+1)^{1-1/\alpha}) \). Assume that \( k \geq n^{1/\alpha}(t+1)^{1-1/\alpha} \). Then
\[
n(t+1)^{\alpha-1}k^{1-\alpha} \leq n(t+1)^{\alpha-1}n^{1/\alpha-1}(t+1)^{2-\alpha-1/\alpha} = n^{1/\alpha}(t+1)^{1-1/\alpha},
\]
as desired. The lemma follows. \( \square \)

If our graph additionally has PLB neighborhoods (see Definition 3.9), we may obtain a better bound.

**Lemma 5.2.** Let \( G \) be a PLB graph with parameters \( \alpha > 2 \) and \( t \), and PLB neighborhoods. Then \( b(k) = O(\min(n(t+1)^{\alpha-1}k^{1-\alpha}, \log n + (t+1)^{\alpha-2}k^{3-\alpha})) = O(\log n + (t+1)^{\alpha/2-1/2}n^{3/2-\alpha/2}) \).
Proof. Observe that $b(k)$ is at most the number of neighbors of degree at least $k$ in the neighborhood. Thus, by Lemma 5.10, $b(k) = O(\max(\log n, (t+1)^{\alpha-2}k^{3-\alpha}))$. By Lemma 5.1, we also have $b(k) = O(n(t+1)^{\alpha-1}k^{1-\alpha})$. Thus, we get $b(k) = O(\min(n(t+1)^{\alpha-1}k^{1-\alpha}, \log n + (t+1)^{\alpha-2}k^{3-\alpha})$. To balance the terms, we take $k = \sqrt{(t+1)n}$. Thus, $O(\min(n(t+1)^{\alpha-1}k^{1-\alpha}, \log n + (t+1)^{\alpha-2}k^{3-\alpha})) = O(\log n + (t+1)^{\alpha-2}(t+1)n^{3/2-\alpha/2}) = O(\log n + (t+1)^{\alpha/2-1/2}n^{3/2-\alpha/2})$. \qed

5.1 Counting Triangles

We now show efficient algorithms for counting triangles in an undirected PLB graph $G$ with parameter $\alpha$. Their pseudocodes are given as Algorithms 1 and 2. The first algorithm is clearly structure-oblivious. The second one also does not use the structure of the graph explicitly. However, it takes a parameter $\delta$, which will depend on graph parameters $\alpha$ and $t$. Observe that Algorithm 1 can easily be extended to list triangles in the same running time bound.

 Algorithm 1 Structure-oblivious algorithm for counting triangles

1: function \textsc{CountTriangles}(G)
2: Construct \(G\)
3: \(\text{triangles} := 0\)
4: for \(v \in V(G)\) do
5: \(S := \text{set of endpoints of outedges of } v\)
6: for each inedge \(wv\) of \(v\) in \(G\) do
7: for each outedge \(wu\) of \(w\) in \(G\) do
8: if \(u \in S\) then
9: \(\text{triangles} := \text{triangles} + 1\)
10: return \(\text{triangles}\)

 Algorithm 2 Algebraic algorithm for counting triangles

1: function \textsc{CountTrianglesFMM}(G, \delta)
2: Construct \(G\)
3: \(\text{triangles} := 0\)
4: for \(v \in V(G)\) do
5: \(S := \text{set of endpoints of outedges of } v\)
6: for each inedge \(wv\) of \(v\) in \(G\) do
7: if \(\deg_G(w) \leq \delta\) then
8: for each outedge \(wu\) of \(w\) in \(G\) do
9: if \(u \in S\) then
10: \(\text{triangles} := \text{triangles} + 1\)
11: \(G_\delta := \text{subgraph of } G \text{ induced on vertices of degree more than } \delta\)
12: \(\text{triangles} + \text{the number of triangles in } G_\delta, \text{ counted using fast matrix multiplication}\)

Lemma 5.3. Algorithms \textsc{CountTriangles} and \textsc{CountTrianglesFMM} are correct. Their running times are $O(\sum_{i=1}^{n-1} d_i b(i)^2)$, and $O(\sum_{i=1}^{\delta} d_i b(i)^2 + d_\sum_{i=\delta}^n)$, respectively.

Proof. Let us first consider the running time of \textsc{CountTriangles}. Observe that the body of the for loop in the 6th line is run exactly once per each edge of \(G\). Thus, the for loop in the 7th line is run at most $b(\deg(w))$ times for a vertex \(w\). In other words, the 8th line is executed for each pair of vertices \(u\) and \(v\), which are endpoints of the outedges of \(w\). This requires $O(\sum_{i=1}^{n-1} d_i b(i)^2)$ time.

The set \(S\) can be implemented as a Boolean array. This way we can initialize the set each time in linear time. Moreover, we can test for membership in constant time. Moreover, as observed before \(G\) can be computed in linear time. Thus, \textsc{CountTriangles} runs in $O(\sum_{i=1}^{n-1} d_i b(i)^2)$ time.
Concerning correctness, let $v_1, \ldots, v_n$ be all vertices of $G$ sorted in non-decreasing order of degrees. Consider a triangle $T$. Let $v$ be the vertex of $T$ that comes first in the sorted order $v_1, \ldots, v_n$. Then, in $G$ the two edges of $T$ that are incident to $v$ are out-edges of $v$. Thus, the correctness of CountTriangles follows.

Using similar arguments, we may observe that the first stage of CountTrianglesFMM (lines 4–10) correctly identifies exactly the triangles that contain at least one vertex of degree at most $\delta$. Clearly, $G_\delta$ contains exactly the triangles that have not been identified yet. Since $G_\delta$ has exactly $d_{\geq \delta}$ vertices, the running time of CountTrianglesFMM follows.

We now combine the algorithms with the bounds on $b(k)$ derived in lemmas 5.1 and 5.2 to obtain four running time bounds of our algorithms, that depend on the algorithm used and on whether the graph has PLB neighborhoods. These running times are shown in Fig. 3.

**Theorem 5.4.** Let $G = (V, E)$, $n = |V|$ be a PLB graph with parameters $\alpha$ and $t$. Then, algorithm CountTriangles can compute the number of triangles in $G$ in: (a) $O(n^{3/\alpha}(t+1)^{3-3/\alpha})$ time for $1 < \alpha < 3$, (b) $O(n \log n(t+1)^2)$ time for $\alpha = 3$, (c) $O(n(t+1)^2)$ time for $\alpha > 3$.

**Proof.** By Lemma 5.3 the running time is $O(\sum_{i=1}^{n-1} d_i b(i^2))$. Let $1 \leq \delta < n$ be a parameter that we fix later. We split the sum into two pieces and first bound $\sum_{i=t+1}^{\delta} b(i^2)$. By Lemma 3.3 this can be upper bounded by

$$O(1)n(t+1)^{\alpha-1} \sum_{i=1}^{\delta} (i+t)^{-\alpha} i^2$$

There are now three cases to consider, depending on the value of $\alpha$. For $\alpha > 3$ we have

$$\sum_{i=1}^{\delta} (i+t)^{-\alpha} i^2 \leq \sum_{i=1}^{t} (i+t)^{-\alpha} i^2 + \sum_{i=t+1}^{\delta} i^{2-\alpha} \leq (t+1)^{-\alpha} \sum_{i=1}^{t} i^2 + \sum_{i=t+1}^{\delta} i^{2-\alpha} = O((t+1)^{3-\alpha}),$$

so the running time is $O(1)n(t+1)^{\alpha-1}(t+1)^{3-\alpha} = O((t+1)^2 n)$, regardless of the choice of $\delta$. Thus, we may set $\delta = n - 1$.

For $\alpha \leq 3$ we have

$$O(1)n(t+1)^{\alpha-1} \sum_{i=1}^{\delta} (i+t)^{-\alpha} i^2 \leq O(1)n(t+1)^{\alpha-1} \sum_{i=1}^{\delta} i^{2-\alpha}.$$  

For $\alpha = 3$ this gives $O(n \log(t+1)^2)$. Again, we set $\delta = n - 1$ and obtain a running time of $O(n \log n(t+1)^2)$. The last case is when $1 < \alpha < 3$. Then, the sum is equal to $O(n(t+1)^{\alpha-1} \delta^{3-\alpha})$.

In this case we set $\delta = n - 1$, so we still need to bound $\sum_{i=\delta+1}^{n-1} d_i O(b(i^2))$. We use the fact that $b(i) = O(d_{\geq i})$ (see Lemma 5.1) and $d_{\geq \delta} = O(n(t+1)^{\alpha-1} \delta^{3-\alpha})$ (see Lemma 3.2):

$$\sum_{i=\delta+1}^{n-1} d_i O(b(i^2)) = \sum_{i=\delta+1}^{n-1} d_i b(i^2) \leq \sum_{i=\delta+1}^{n-1} d_i d_{\geq i}^2 = O(d_{\geq i}^3) = O(n^3(t+1)^{3(\alpha-1)} \delta^{3(1-\alpha)})$$

The overall running time is $O(n(t+1)^{\alpha-1} \delta^{3-\alpha} + n^3(t+1)^{3(\alpha-1)} \delta^{3(1-\alpha)})$. In order to balance the summands, we set $\delta = n^1/\alpha (t+1)^{1-1/\alpha}$ and obtain the running time of $O(n^{3/\alpha}(t+1)^{3-3/\alpha})$. □

**Theorem 5.5.** Let $G$ be a PLB graph with parameters $\alpha$ and $t$, where $\alpha < 3$. Then algorithm CountTrianglesFMM with $\delta = (n(t+1)^{\alpha-1}(\omega-1)/(3-\alpha-\omega+\alpha \omega))$ can compute the number of triangles in $G$ in $O((n(t+1)^{\alpha-1})^{3.45/(0.45+\alpha)})$ time.
Proof. By Lemma 5.3, the running time is $O \left( \sum_{i=1}^{\delta} d_i b(i)^2 + d_{\geq \delta} \right)$. In the proof of Theorem 5.4, we have shown that for $1 < \alpha < 3$, $O \left( \sum_{i=1}^{\delta} d_i b(i)^2 \right) = O(n(t + 1)^{\alpha - 1} \delta^{3 - \alpha} \omega(1 - \alpha))$. By Lemma 3.2, $d_{\geq \delta} = O(n(t + 1)^{\alpha - 1} \delta^{1 - \alpha})$. The running time becomes $O(n(t + 1)^{\alpha - 1} \delta^{3 - \alpha} + n\omega(t + 1)\omega(\alpha - 1)\delta^{\omega(1 - \alpha)})$. We balance both summands:

$$n(t + 1)^{\alpha - 1} \delta^{3 - \alpha} = n^\omega(t + 1)^{\omega(1 - \alpha)} \delta^{\omega(1 - \alpha)}$$

and obtain a running time of

$$O(n(t + 1)^{\alpha - 1} \delta^{3 - \alpha}) = O((n(t + 1)^{\alpha - 1})(n(t + 1)^{\alpha - 1})^{(3 - \alpha)\omega(1 - \alpha)/\omega(3 - \alpha + \omega)})$$

$$= O((n(t + 1)^{\alpha - 1})^{(3 - \alpha)\omega(1 - \alpha) / (3 - \alpha - \omega + \omega) + 1})$$

$$= O((n(t + 1)^{\alpha - 1})^{2\omega / (3 - \alpha - \omega + \omega)})$$

Setting $\omega = 2.38$, we get

$$O((n(t + 1)^{\alpha - 1})^{4.76/(0.62 + 1.38\alpha)}) = O((n(t + 1)^{\alpha - 1})^{3.45/(0.45 + \alpha)})$$

For $\alpha = 2$, $3.45/(0.45 + \alpha) \leq 1.41$, so the running time becomes $O(n^{1.41}(t + 1)^{1.41})$.

If $G$ additionally has PLB neighborhoods, we may obtain a faster algorithm.

**Theorem 5.6.** Let $G$ be a PLB graph with parameters $\alpha$ and $t$, where $\alpha < 3$. Moreover, assume that $G$ has PLB neighborhoods. Then, algorithm COUNT_TRIANGLES can compute the number of triangles in $G$ in time (a) $O(n^{\frac{\alpha - 3}{\alpha - 2}}(t + 1)^{\frac{2\alpha - 3}{\alpha - 2}})$ for $2 < \alpha < 7/3$, (b) $O(n(t + 1)^2)$ time for $\alpha \geq 7/3$.

Proof. By Lemma 5.3, the running time is $O \left( \sum_{i=1}^{\delta} d_i b(i)^2 \right)$. By Lemma 5.2, $b(k) = O(\min(n(t + 1)^{\alpha - 1} k^{1 - \alpha}, \log n + (t + 1)^{\alpha - 2} k^{3 - \alpha})).$ In particular, $b(k) = O(\log n + (t + 1)^{\alpha - 2} k^{3 - \alpha})$.

Again, split the sum using a parameter $\delta$ that we fix later. We first bound the sum of the first $\delta$ summands (we use Lemma 3.3):

$$\sum_{i=1}^{\delta} d_i b(i)^2 = O(1) n(t + 1)^{\alpha - 1} \sum_{i=1}^{\delta} b(i)^2 (i + t)^{-\alpha}$$

$$= O(n(t + 1)^{\alpha - 1} \sum_{i=1}^{\delta} (\log^2 n + (t + 1)^{2\alpha - 4}\delta^{2 - \alpha})(i + t)^{-\alpha})$$

$$= O(n(t + 1)^{3\alpha - 5} \sum_{i=1}^{\delta} \log^2 n(i + t)^{-\alpha} + \sum_{i=1}^{\delta} \delta^{6 - 2\alpha}(i + t)^{-\alpha})$$

$$= O(n(t + 1)^{3\alpha - 5} (t + 1)^{1 - \alpha} \log^2 n + \sum_{i=1}^{\delta} \delta^{6 - 2\alpha}(i + t)^{-\alpha})$$

$$= O(\log^2 n(t + 1)^{2\alpha - 4} + n(t + 1)^{3\alpha - 5} \sum_{i=1}^{\delta} \delta^{6 - 2\alpha}(i + t)^{-\alpha})$$

We use Lemma 2.3. If $\alpha \geq 7/3$, the sum is bounded by $O((\delta + 1)^{7 - 3\alpha}).$ Thus, if we set $\delta = n - 1$, the running time becomes $O(n(t + 1)^2)$.

It remains to consider the case when $2 < \alpha < 7/3$. Then, we assume that $\delta = n^{\Omega(1)}$ (which we can do, since we are free to choose $\delta$). The sum can be bounded by $O(\delta^{7 - 3\alpha})$, so $\sum_{i=1}^{\delta} d_i b(i)^2 = O(n(t + 1)^{3\alpha - 5} \delta^{7 - 3\alpha}).$
To obtain the running time we still need to bound \( \sum_{i=0}^{n-1} d_{i}b(i)^{2} \). In the proof of Theorem 5.4 we have shown that \( \sum_{i=0}^{n-1} d_{i}b(i)^{2} = O(n^{3}(t+1)^{4(\alpha-1)\delta^{3(1-\alpha)}}) \). We balance \( n(t+1)^{3\alpha-5\delta^{7-3\alpha}} \) and \( n^{3}(t+1)^{3(\alpha-1)\delta^{3(1-\alpha)}} \):

\[
\begin{align*}
n(t+1)^{3\alpha-5\delta^{7-3\alpha}} & = n^{3}(t+1)^{3(\alpha-1)\delta^{3(1-\alpha)}} \\
\delta^{4} & = n^{2}(t+1)^{2} \\
\delta & = \sqrt{n(t+1)}
\end{align*}
\]

The running time becomes \( O(n^{9/2-3/2\alpha}(t+1)^{3/2\alpha-3/2}) \).

Thus, this algorithm generalizes and strengthens the result of Berry et al. [7] by showing whp bounds on the running time instead of bounds in expectation.

**Theorem 5.7.** Let \( G \) be a PLB graph with parameters \( 2 < \alpha < 7/3 \) and \( t \). Moreover, assume that \( G \) has PLB neighborhoods. Then algorithm \textsc{CountTrianglesFMM} for \( \delta = n^{(\omega-1)/7-\omega+(\omega-3)\alpha}(t+1)^{1-2/(7-\omega+(\omega-3)\alpha)} \) can compute the number of triangles in \( G \) in \( O(n^{23.04-7.68\alpha}/(7.46-\alpha))(t+1)(7.67(\alpha-1))/7.46-\alpha) \) time.

**Proof.** By Lemma 5.3, the running time is \( O(\sum_{i=0}^{\delta} d_{i}b(i)^{2} + c\delta) \). In the proof of Theorem 5.6 we have shown that for \( 2 < \alpha < 7/3 \) and \( \delta = n^{\Omega(1)} \), \( \sum_{i=0}^{\delta} d_{i}b(i)^{2} = O(n(t+1)^{3\alpha-5\delta^{7-3\alpha}}) \). On the other hand, as in the proof of Theorem 5.5, it takes \( O(n^{\omega}(t+1)^{\omega(\alpha-1)\delta^{\omega(1-\alpha)}}) \) to process vertices of degree at least \( \delta \). We balance both times to find the optimal choice for \( \delta \).

\[
\begin{align*}
n(t+1)^{3\alpha-5\delta^{7-3\alpha}} & = n^{\omega}(t+1)^{\omega(\alpha-1)\delta^{\omega(1-\alpha)}} \\
\delta^{7-\omega+(\omega-3)\alpha} & = n^{\omega-1}(t+1)^{(\omega-3)\alpha-\omega+5} \\
\delta & = n^{(\omega-1)/(7-\omega+(\omega-3)\alpha)}(t+1)^{1-2/(7-\omega+(\omega-3)\alpha)}
\end{align*}
\]

By plugging this back, we obtain the running time of

\[
O(n^{\omega}(t+1)^{\omega(\alpha-1)n^{((\omega-1)\omega(1-\alpha))}/(7-\omega+(\omega-3)\alpha)}(t+1)(1-2/(7-\omega+(\omega-3)\alpha))\omega(1-\alpha)))
\]

\[
= O(n^{\omega(\omega-\omega\alpha+1+\omega+(\omega-3)\alpha)/7-\omega+(\omega-3)\alpha})^{(t+1)}\omega(\omega(1-1))\omega(1-1)/7-\omega+(\omega-3)\alpha)
\]

\[
= O(n^{(2\omega(3-\alpha))/7-\omega+(\omega-3)\alpha})^{(t+1)}\omega(\omega(1-1))\omega(1-1)/7-\omega+(\omega-3)\alpha)
\]

Setting \( \omega = 2.38 \), we get \( O(n^{14.28-4.76\alpha}/(4.62-0.62\alpha)(t+1)(4.76(\alpha-1))/(4.62-0.62\alpha)) = O(n^{23.04-7.68\alpha}/(7.46-\alpha)(t+1)(7.67(\alpha-1))/7.46-\alpha) \). For \( \alpha \to 2+ \), this becomes \( O(n^{1.41}(t+1)^{1.41}) \). For \( \alpha \to 7/3 \), it is \( O(n(t+1)^{2}) \).

### 5.2 Finding Maximal Clique

We now show an efficient algorithm for finding the largest clique in PLB graph.

**Algorithm 3 Maximal clique algorithm**

1: \textbf{function} \textsc{MaximalClique}(\textit{G})
2: \hspace{1em} \textbf{Construct} \textit{\tilde{G}}
3: \hspace{1em} \textit{maxclique} := 0
4: \hspace{1em} \textbf{for} \textit{v} \in \textit{V}(\textit{G}) \textbf{do}
5: \hspace{2em} \textit{N}_v := \{v\} \cup \text{set of endpoints of outedges of } \textit{v} \text{ in } \textit{\tilde{G}}
6: \hspace{2em} \textbf{for} \textit{S} \subseteq \textit{N}_v \textbf{do}
7: \hspace{3em} \textbf{if} \textit{S} \text{is a clique in } \textit{G} \textbf{then} \textit{maxclique} := \max(\textit{maxclique}, |\textit{S}|)
8: \hspace{1em} \textbf{return} \textit{maxclique}

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Lemma 5.8. Algorithm MaximalClique is correct, structure oblivious and runs in \( n \sum_{i=1}^{n-1} \exp(b(i)) \) time.

Proof. Let \( C \) be a clique in \( G \). Then, \( C \) contains a vertex \( w \) such that in \( \bar{G} \) \( w \) has directed edges to every other vertex of \( C \). The correctness of the algorithm follows easily. It is also easy to see that it is structure-oblivious.

Consider the iteration of the outer for loop for a vertex \( v \). The size of \( N_v \) is bounded by \( b(\deg(v)) + 1 \), so the inner for loop runs in \( \exp(b(\deg(v))) \) time. For a single \( v \) this can be crudely upper bounded by \( \sum_{i=1}^{n-1} \exp(b(i)) \). The outer loop has \( n \) iterations, so the entire algorithm runs in \( n \sum_{i=1}^{n-1} \exp(b(i)) \) time. \( \square \)

Theorem 5.9. Let \( G \) be a PLB graph with parameters \( \alpha \) and \( t \). Then, algorithm MaximalClique can find the largest clique in \( G \) in \( \exp(O(1/\alpha(t + 1)^{1-1/\alpha})) \) time.

Proof. By Lemma 5.8, the running time is \( \exp(n \sum_{i=1}^{n-1} \exp(b(i))) \). By Lemma 5.1, \( b(k) = O(n^{1/\alpha}(t + 1)^{1-1/\alpha}) \). Thus,

\[
\sum_{i=1}^{n-1} \exp(b(i)) = n \exp(O(n^{1/\alpha}(t + 1)^{1-1/\alpha})) = \exp(O(n^{1/\alpha}(t + 1)^{1-1/\alpha})),
\]

This problem can also be solved more efficiently for a PLB graph with parameter \( \alpha > 2 \) and PLB neighborhoods.

Theorem 5.10. Let \( G \) be PLB graph with parameters \( \alpha > 2 \) and \( t \) and PLB neighborhoods. Then, algorithm MaximalClique can find the largest clique in \( G \) in (a) \( \exp(O((t + 1)^{\alpha/2 - 1/2} n^{3/2 - \alpha/2} \log n)) \) time for \( 2 < \alpha < 3 \), (b) \( n^{O(t+1)} \) time for \( \alpha = 3 \), (c) \( O(\text{poly}(n)) \) time for \( \alpha > 3 \).

Proof. By Lemma 5.8, the running time is \( \text{poly}(n) \sum_{i=1}^{n-1} \exp(b(i)) \). Moreover, by Lemma 5.2, \( b(k) = O(\log n((t + 1)^{\alpha/2 - 1/2} n^{3/2 - \alpha/2} + 1)) \). Thus,

\[
\sum_{i=1}^{n-1} \exp(b(i)) = n^2 \exp(O(\log n((t + 1)^{\alpha/2 - 1/2} n^{3/2 - \alpha/2} + 1))) = \exp(O(\log n((t + 1)^{\alpha/2 - 1/2} n^{3/2 - \alpha/2} + 1))).
\]

If \( \alpha < 3 \) this can be simplified to \( \exp(O((t + 1)^{\alpha/2 - 1/2} n^{3/2 - \alpha/2} \log n)) \). For \( \alpha = 3 \), \( n^{3/2 - \alpha/2} = O(1) \), so we get \( n^{O(t+1)} \). For \( \alpha > 3 \), we use the fact that \( t + 1 = O(n^\epsilon) \) for every \( \epsilon > 0 \). Thus, \( (t + 1)^{\alpha/2 - 1/2} n^{3/2 - \alpha/2} = O(1) \), so the running time becomes \( \exp(O(\log n)) = \text{poly}(n) \). \( \square \)

Observe that for \( \alpha > 3 \) the running time is polynomial in \( n \). Note that the analysis assumes \( t = O(n^\epsilon) \).

6 Algebraic Algorithms

In this section we will give our algebraic algorithms for computing matrix determinant and solving linear systems of equations. As already mentioned we will be working over a finite field \( \mathcal{F} \). For a warm-up we will start from the generic symmetric case and next we move on to general non-symmetric case. In this section when we talk about directed PLB graphs we assume that only the outdegrees of vertices satisfy a similar bound to the one given in Definition 3.1. Moreover, we will use fast rectangular matrix multiplication. We denote by \( \omega(n, m, k) \) the time needed to multiply an \( n \times m \) matrix by an \( m \times k \) matrix [39].
6.1 Transitive Closure

Let us start by giving our algorithms for computing transitive closure of a graph.

**Theorem 6.1.** Let $G$ be a directed PLB graph with parameters $\alpha$ and $t$, and let $1 \leq k < n^{1/\alpha}(t+1)^{1-1/\alpha}$. Then, we can compute the transitive closure of $G$ in $O(n^2(t+1)^{\alpha-1}k^2-\alpha + \omega(n,n,n(t+1)^{\alpha-1}k^{1-\alpha}))$ time.

**Proof.** Let $M$ be the adjacency matrix of $G$. We start with sorting the rows of $M$ in decreasing order according to the number of non-zero entries. Then, for a given $k \in [0,n]$, we split the matrix $M$ into 4 submatrices

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $[A \ B]$ contains rows with more than $k$ non-zero entries, $[C \ D]$ has at most $k$ non-zero entries in each row and $A$, $D$ are square matrices. Let $m_{CD}$ be the total number of non-zero entries in submatrices $C, D$. By Lemma 3.4 $m_{CD}$ is bounded $O(n(t+1)^{\alpha-1}k^{2-\alpha})$. Let $n_k$ be the dimension of $A$. By Lemma 3.2 we have that $n_k$ is bounded by $O(n(t+1)^{\alpha-1}k^{1-\alpha})$.

We can express the transitive closure of $M$ in the following block form

$$M^* = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} I & 0 \\ -D^*C & I \end{bmatrix} \begin{bmatrix} (A - BD^*C)^* & 0 \\ 0 & D^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \quad (3)$$

In order to compute the transitive closure using this equation we compute:

- $D^*$ in $O(nm_{CD})$ time executing $n$ graph searches;
- $D^*C$ in $O(nm_{CD})$ time using sparse matrix multiplication;
- $B(D^*C)$ in $O(nm_{CD}^{\omega-1})$ time using fast matrix multiplication;
- $BD^*$ in $O(n_k^\omega)$ time using fast matrix multiplication;
- $(A - BD^*C)^*$ in $O(n_k^\omega)$ time using fast matrix multiplication;
- both matrix multiplications from (3) in $O(\omega(n,n_k,n))$ time, as $BD^*$ and $D^*C$ have one dimension of size $O(n_k)$.

The theorem follows by plugging the bounds for $m_{CD}$ and $n_k$ to the list above. \hfill $\square$

6.2 Determinants of Symmetric Matrices

Let us start from the Lanczos’ algorithm, which is useful when dealing with sparse matrices.

**Theorem 6.2 (\cite{8, 27}).** There is a randomized algorithm, which for a given generic symmetric square matrix $A$ in time $O(nm)$ computes $\det(A)$ together with matrices $Q, T$, such that $A = QTQ^T$, where $T$ is a tridiagonal matrix, $n$ is the dimension of $A$ and $m$ is the number of non-zero entries in $A$.

Next, we show an algorithm computing a determinant of a matrix $M$, corresponding to a given PLB graph $G$. A symmetric matrix $M$ can be seen as corresponding to the case when $G$ is undirected.

**Theorem 6.3.** Let $G$ be a directed PLB graph with parameters $\alpha < 2$ and $t$. Let $M$ be a generic symmetric matrix, whose non-zero entries are a subset of non-zero entries of the adjacency matrix of $G$. Then, we can compute the determinant of $M$ in $O((t+1)^{(\alpha-1)+\frac{(\omega-2)\alpha(\alpha-1)(\alpha-2)}{(\omega-2)\alpha(\alpha-1)+\alpha^2}-\alpha}n^{2+\frac{(\omega-2)(2-\alpha)}{(\omega-2)\alpha(\alpha-1)+\alpha^2}})$ time.
Proof. Similarly as in the proof for transitive closure, we start with sorting the rows of $M$ in decreasing order according to the number of non-zero entries (which is upper bounded by the degree of a corresponding vertex of $G$). Then, for a given $k \in [0, n]$, we split the matrix $M$ into 4 submatrices

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $[A, B]$ contains rows with more than $k$ non-zero entries, $[C, D]$ has at most $k$ non-zero entries in each row and $A, D$ are square matrices. Let $m_{BCD}$ be the total number of non-zero entries in submatrices $B, C, D$. As the matrix is symmetric $m_{BCD}$ is at most twice the number of non-zero entries in $[C, D]$. This in turn, by Lemma 3.4, is at most $O(n(t + 1)^{\alpha - 1}k^{2 - \alpha})$.

By using the formula for the determinant of a Schur complement we obtain

$$\det(M) = \det(D)\det(A - BD^{-1}C).$$

Let $n_k$ be the dimension of $A$ (which depends on $k$). By Lemma 3.2 we infer that $n_k$ is bounded by $O(n(t + 1)^{\alpha - 1}k^{1 - \alpha})$, which in turn gives

$$n_k = O(n(t + 1)^{\alpha - 1}k^{1 - \alpha}),$$

(4)

as $\alpha \geq 1$.

By invoking Theorem 6.2 we compute $\det(D)$ as well as matrices $Q, T$ such that $D = QTQ^T$. The running time needed is $O(nm_{BCD})$. Denote $Z = A - BD^{-1}C = A - BQTQ^TC$. To compute $Z$ efficiently, we first compute $Q^TC$ and $BQ$ in time $O(nm_{BCD})$ each, as both $C$ and $B$ are sparse, i.e., have at most $m_{BCD}$ non-zero entries. As $T$ is tridiagonal computing $T(Q^TC)$ takes time proportional to the size of $(Q^TC)$, that is $n \cdot n_k$. Finally we multiply $BQ$ by $T(Q^TC)$ in time $O(n \cdot n_k^{-1})$ by partitioning the matrices into $n/n_k$ submatrices of size $n_k \times n_k$ each and invoking fast matrix multiplication on square matrices. Finally, after computing $Z$, we can compute $\det(Z)$ in time $O(n_k^2)$.

Summing up, we have to set the value of $k$ to minimize the maximum of four values

- $O(nm_{BCD})$ time used by invoking Theorem 6.2 and for computing the products $Q^TC, BQ$,
- $O(n \cdot n_k)$ time for computing the product $T \cdot (Q^TC)$,
- $O(n \cdot n_k^{-1})$ time for computing the product $(BQ) \cdot (T(Q^TC))$,
- $O(n_k^2)$ time for computing $\det(Z)$.

Note that $n \cdot n_k^{-1}$ dominates both $n \cdot n_k$ and $n_k^2$. Therefore, we need to set the value of $k$, so that $m_{BCD} = n_k^{-1}$. By using (4) and (4) we set $k = t^{\frac{\omega(\omega + 1)}{\omega + 2}}n_{1-\frac{1}{\omega + 2}}^{-\frac{1}{\omega + 2}}$, which finishes the proof of the theorem.

6.3 Determinant of General Matrices

In the general case we use the following results due to Eberly [22], who showed how the Frobenius normal form of a sparse matrix can be computed. Frobenius normal form $F_A$ of a matrix $A$ is a block diagonal matrix with companion matrices of monic polynomials $f_1, \ldots, f_k$ on the diagonal, where $f_i$ is divisible by $f_{i+1}$, for $1 \leq i \leq k - 1$ and $VAV^{-1} = F_A$. The companion matrix of a monic polynomial $x^d + g_{d-1}x^{d-1} + \ldots + g_1x + g_0 \in \mathcal{F}[x]$ is a $d \times d$ matrix defined as

$$C_g = \begin{bmatrix} 0 & \ldots & 0 & -g_0 \\ 1 & \ldots & 0 & -g_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & -g_{d-1} \end{bmatrix}.$$

The polynomials $f_1, \ldots, f_k$ are the invariant factors of $A$ and $k$ is the number of invariant factors.
Theorem 6.4 ([22]). There exists an algorithm for computing Frobenius normal form $F$ of the matrix $A$ together with the transition matrix $V$ and its inverse with use of $O(n)$ matrix-vector products and $O(kn^2)$ additional operations, where $k$ is the number of invariant factors of $A$. The algorithm is randomized and may fail with arbitrarily small probability.

We will use the following preconditioning due to [34] to make sure that there is just one invariant factor with high probability. An $n \times n$ Hankel matrix $H$ is constructed from $2n - 1$ elements $h_0, \ldots, h_{2n-2}$ in the following way

$$H = \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-2} & h_{n-1} \\ h_1 & h_2 & \cdots & h_{n-1} & h_n \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n-3} & h_{2n-2} \end{bmatrix}.$$ 

We note that multiplication of a matrix by Hankel matrix $H$ may fail with arbitrarily small probability. An $k$ additional operations, where $k$ is the number of invariant factors of $A$. The algorithm is randomized and may fail with arbitrarily small probability.

Theorem 6.5 (Theorem 2 from [34]). Let $A$ be the non-singular square matrix $A$, let $H$ be a Hankel matrix with elements selected randomly and uniformly from $F$, then all leading (or trailing) principal submatrices of $\hat{A} = AH$ are non-singular with high probability.

Theorem 6.6 (Equation (1) from [34]). Let $\hat{A}$ be matrix such that all its leading principal submatrices are non-singular, let $J$ be a diagonal matrix with elements selected randomly and uniformly from $F$, then $\hat{A} = \hat{A}J$ has one invariant factor with high probability.

Theorem 6.7. Let $G$ be a directed PLB graph with parameters $\alpha$ and $t$, and let $M$ be a matrix, whose non-zero entries are a subset of non-zero entries of the adjacency matrix of $G$. Moreover, let $1 \leq k < n^{1/\alpha}(t+1)^{1-1/\alpha}$. Then, one can compute the determinant of $M$ in $O(n^2(t+1)^{\alpha-1}k^{2-\alpha} + \omega(n, n(t+1)^{\alpha-1}k^{1-\alpha}))$ randomized time.

Proof. Similarly as in the symmetric case, we start with sorting the rows of $M$ in decreasing order according to the number of non-zero entries. Then, for a given $k \in [0, n]$, we split the matrix $M$ into 4 submatrices

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $[A \ B]$ contains rows with more than $k$ non-zero entries, $[C \ D]$ has at most $k$ non-zero entries in each row and $A, D$ are square matrices. Let $m_{CD}$ be the total number of non-zero entries in submatrices $C, D$. By Lemma 3.4 $m_{CD}$ is bounded $O(n(t+1)^{\alpha-1}k^{2-\alpha})$.

Let $X$ be an arbitrary $n \times n$ matrix. A submatrices of $X$ obtained by performing similar split as for $M$ are denoted by

$$X = \begin{bmatrix} X_A & X_B \\ X_C & X_D \end{bmatrix}.$$ 

Let $H$ and $J$ be random matrices as given in Theorem 6.5. We cannot afford to precondition the whole matrix, so we precondition only the essential part that is needed for the Schur complement to work.

$$\hat{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & H_B \\ 0 & H_D \end{bmatrix} = \begin{bmatrix} A & AH_B + BH_D \\ C & CH_B + DH_D \end{bmatrix}.$$ 

You may observe that $AH_B + BH_D = (MH)_B$ and $CH_B + DH_D = (MH)_D$. Hence, by Theorem 6.5 all trailing principal submatrices of $CH_B + DH_D$ are non-singular. Now we apply the second part of the preconditioning.

$$\tilde{M} = \hat{M} \begin{bmatrix} I & 0 \\ 0 & J_D \end{bmatrix} = \begin{bmatrix} A & (AH_B + BH_D)J_D \\ C & (CH_B + DH_D)J_D \end{bmatrix}.$$
By Theorem 6.6 the matrix $\tilde{M}_D = (CH_B + DH_B)J_D$ has one invariant factor. By using the formula for the determinant of a Schur complement we obtain

$$\det(\tilde{M}) = \det(M) \det(A - \tilde{M}_B(\tilde{M}_D)^{-1}C).$$

Let $n_k$ be the dimension of $A$. By Lemma 3.2 we have that $n_k$ is bounded by $O(n(t + 1)^{a-1}k^{1-a})$.

By invoking Theorem 6.4 we compute matrices $V,F$ such that $\tilde{M}_D = V^{-1}FV$, as well as $\det(\tilde{M}_D) = \det(F)$ in time $\tilde{O}(nm_{CD})$.

Denote $\tilde{Z} = A - \tilde{M}_B(\tilde{M}_D)^{-1}C = A - \tilde{M}_B V^{-1}F^{-1}VC$. To compute $\tilde{Z}$ efficiently, we first compute $\tilde{M}_B = (AH_B + BH_D)J_D = (A,B)(H_B,H_D)^TJ_D$ what requires $O(n^2)$ time. Then computing $\tilde{M}_B V^{-1}$ requires $O(\omega(n,n_k,n))$ time. Next, we compute $VC$ in $O(nm_{CD})$ time because $C$ has at most $m_{CD}$ non-zero entries.

Due to special structure of $F$ we know that $F^{-1}$ has $O(n)$ non-zero entries and computing it takes $O(n)$ time using the following form for each companion matrix

$$C_g^{-1} = \frac{1}{g_0} \begin{bmatrix} -g_1 & 1 & 0 & \cdots & 0 \\ -g_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -g_{d-1} & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

As $F^{-1}$ has $O(n)$ non-zero entries so computing $F^{-1}VC$ takes time proportional to the size of $VC$, that is $n \cdot n_k$. To obtain $\tilde{Z}$ we multiply $\tilde{M}_B V^{-1}$ by $F^{-1}VC$ in time $O(n/n_k \cdot n_k^\omega)$. After computing $\tilde{Z}$, we can compute $\det(\tilde{Z})$ in time $O(n_k^\omega)$. Finally, by our preconditioning $\det(\tilde{M}) = \frac{\det(\tilde{Z}) \det(D)}{\det(H_D J_D)}$, where we need $\tilde{O}(n)$ time to compute $\det(H_D J_D)$.

Summing up, we have to set the value of $\eta$ to minimize the maximum of the following

- $\tilde{O}(n^2)$ for computing $BHJ, DHJ$ and $HJV^{-1}$,
- $\tilde{O}(nm_{CD})$ time used by invoking Theorem 6.4 and for computing the product $VC$,
- $O(\omega(n,n,n_k))$ time for computing $B(HJV^{-1})$,
- $O(nn_k)$ time for computing the product $F^{-1} \cdot VC$,
- $O(nn_k^\omega)$ time for computing the product $BHJV^{-1} \cdot F^{-1}VC$,
- $O(n_k^\omega)$ time for computing $\det(\tilde{Z})$,
- $\tilde{O}(n)$ time for computing $\det(HJ)$ and $F^{-1}$.

Note that $O(\omega(n,n,n_k))$ dominates $O(nn_k^\omega)$, $nn_k$ and $n_k^\omega$, whereas $\tilde{O}(nm_{CD})$ dominates $\tilde{O}(n^2)$. Therefore we need to set the value of $\eta$, so that $nm_{CD} = \omega(n,n,n_k)$. 

The above theorem gives a general statement that in the parameter range $1 < \alpha < 2$ it is possible to compute determinant faster than by using algorithms for general graphs. However, the statement of the theorem contains tangled equation, so in order to simplify it we assume that $t = O(\polylog n)$. Let $\omega(\beta)$ be defined such that $n^\omega(\beta) = \omega(n,n,n_k)$. 

**Corollary 6.8.** Let $G$ be a directed PLB graph with parameters $\alpha$ and $t$, and $M$ be a matrix, whose non-zero entries are a subset of non-zero entries of the adjacency matrix of $G$. Let $0 \leq \beta < 1$ be such that $2 + \beta(2 - \alpha) = \omega(1 + \beta(1 - \alpha))$, $\beta < 1/\alpha$. Moreover, assume $t = O(\polylog n)$. Then, we can compute the determinant of $M$ in $\tilde{O}(n^{2+\beta(2-\alpha)})$ randomized time.

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We observe that when $M$ is symmetric then $B$ is sparse as in the proof of Theorem 6.3. In such a case computing $B(HJV^{-1})$ takes $O(nm)$ time instead of $\omega(n,n,n_k)$ time and we obtain similar bounds as in Theorem 6.3.

**Corollary 6.9.** It is possible to drop the generic assumption from Theorem 6.3 by increasing the running time by polylogarithmic factors.

### 6.4 Linear System Solution and Matrix Inverse

In order to solve linear system with matrix $M$ we will extend the idea of the algorithm from the previous section, i.e., we first run the above algorithm to compute the determinant of $M$ and store all intermediate results of this computation. Let $v$ be an $n$ length vector, then to find a vector $x$ such that $Mx = v$ we compute

$$ x = M^{-1}v = \begin{bmatrix} I & HB \\ 0 & HD \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \tilde{M}^{-1}v. $$

(5)

Now we express inverse of a $\tilde{M}$ in the block form

$$ \tilde{M}^{-1} = \begin{bmatrix} I & 0 \\ -(\tilde{M}_D)^{-1}C & I \end{bmatrix} \begin{bmatrix} \tilde{Z}^{-1} & 0 \\ C & (\tilde{M}_D)^{-1} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} - \tilde{M}_B(\tilde{M}_D)^{-1}. $$

(6)

Now we plug in the equation $(\tilde{M}_D)^{-1} = V^{-1}F^{-1}V$ to obtain

$$ \tilde{M}^{-1}v = \begin{bmatrix} I & 0 \\ -V^{-1}F^{-1}VC & I \end{bmatrix} \begin{bmatrix} \tilde{Z}^{-1} & 0 \\ C & V^{-1}F^{-1}V \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} - \tilde{M}_BV^{-1}F^{-1}V^v. $$

Observe that all matrices in the above have been computed during the computation of the determinant, so computing $\tilde{M}^{-1}v$ takes $O(n^2)$ time. Then we compute $M^{-1}v$ using (5) in $O(n^2)$ additional time.

**Corollary 6.10.** Theorem 6.3, Theorem 6.7 and Corollary 6.9 can be extended to compute a solution to linear system at the cost of using $\tilde{O}(n^2)$ additional time.

Finally, we observe that using (6) we can compute the inverse matrix explicitly. In the case of generic matrices it takes the same time as needed for transitive closure using (3), whereas in the symmetric case the most expensive multiplication takes $O(nm)$ time instead of $O(\omega(n,n,n_k))$ time, so we obtain the following corollary.

**Corollary 6.11.** Theorem 6.3, Theorem 6.7 and Corollary 6.9 can be extended to compute an inverse matrix in the same asymptotic time.

### 6.5 Perfect Matching

As observed by Lovasz [41] in 1979 checking whether a graph contains a perfect matching can be done in $O(n^\omega)$ time using one determinant computation for an appropriately defined skew-symmetric matrix. However, an algorithm for finding such perfect matching was shown 25 years latter [45]. Here, we reuse this idea in the case of PLB graphs to check whether a graph contains a perfect matching and to find one. The running time of the resulting algorithms is the same as the running time of determinant computation for symmetric matrices.

Let us define for a graph $G$ a skew-symmetric adjacency matrix $\tilde{M}$ in the following way

$$ \tilde{M}_{i,j} = \begin{cases} z_{i,j} & \text{if } ij \in E \text{ and } i < j \\ -z_{j,i} & \text{if } ij \in E \text{ and } i > j \\ 0 & \text{otherwise} \end{cases}, $$

where for each edge $ij \in E$ the variables $z_{i,j}$ are distinct.
Theorem 6.12 ([41]). Let $M$ be a matrix obtained from $M$ by substituting uniformly at random elements from $F$ for variables. If $G$ has a perfect matching then $\det(M) \neq 0$ with high probability, whereas when $G$ has no perfect matching then $\det(M) = 0$.

We can observe that in our derivation of Corollary 6.9 we have used only the fact that non-zero structure of the matrix is symmetric, so the same time bounds hold for computation of a determinant of a skew-symmetric matrix $M$. Hence, assuming that $t = O(\text{polylog} n)$, to test whether a PLB graph contains a perfect matching we need $O(n^{2(\alpha - 1)/4})$ randomized time. This is faster than Micali-Vazirani algorithm [43], that works in $O(n^{1/2+2/\alpha})$ time, when $\alpha \leq \sqrt{n}/27m + 26w - 15 + 3w - 9 < 1.0904$. However, what is left is to construct the perfect matching when we know that the graph contains one. In order to do it we need to compute $(M^{-1})_A = Z^{-1} = (A - BD^{-1}C)^{-1}$. By our preconditioning we have

$$M^{-1} = \begin{bmatrix} I & H_B \\ 0 & H_D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J_D \end{bmatrix} \tilde{M}^{-1}. $$

Using this equation and the equality $(\tilde{M}^{-1})_A = \tilde{Z}^{-1}$ we obtain

$$M^{-1}_A = Z^{-1} = \tilde{Z}^{-1} - H_B J_D V^{-1} F^{-1} V C. $$

We need $O(n m_{BCD})$ time to compute $VC$, $O(nw)$ time for $(H_B (J_D V^{-1})) F^{-1}$ and finally $O(n \cdot n_k^{\omega - 1})$ time to compute $(H_B J_D V^{-1} F^{-1}) (VC)$. These running times do not increase the running times stated in Theorem 6.3 and Corollary 6.9. Let $G_A$ be the subgraph of $G$ represented by $M_A$, i.e., the subgraph induced only by vertices of degree at least $k$. Now we can apply the following observation that is the core idea of [32].

**Theorem 6.13** (Procedure DeleteEdgesWithin from [32]). Given a Schur complement of $M_A$ (i.e., $Z^{-1}$) one can find in $O(n_k^{\omega})$ time a set of edges $P_A \subseteq E(G_A)$ such that $P_A$ can be extended to a perfect matching $P$ of the whole graph $G$.

Hence, we first invoke the above theorem to find $P_A$ and then we are left to find matching in a graph $G - E(G_A)$, where all edges between high degree vertices have been removed. This way the degree of a vertex in $G - E(G_A)$ is bounded by $k$ and so the total number of edges is $O(m_{BCD})$. Using the algorithm by Micali and Vazirani we need $O(\sqrt{n m_{BCD}}) = O(n m_{BCD})$ time to extend $P_A$ to a perfect matching on $G$. This way we obtain the following theorem.

**Theorem 6.14.** Let $G$ be a PLB graph with parameters $\alpha < 2$ and $t$. Then, we can find a perfect matching in $G$ in $O((t + 1)^{(\alpha - 1)/\omega} n \log n + \omega - 1) + (t + 1)^{\alpha - 2/\omega} n^2 (\log n)^{\alpha - 1/2} + (t + 1)^{2/\omega} n^{\alpha - 2/\omega})$ time with high probability.

### 6.6 Complexity of PageRank

Let us now discuss the arithmetic complexity of exact PageRank computation. Computing PageRank is a simple version of the eigenproblem, where we are asked to find eigenvector for the eigenvalue which is equal to 1. Eigenproblems, in comparison with the determinant problem, is usually more challenging, because we cannot easily use preconditioning, as it can change both eigenvalues and eigenvectors. The complexity of this simple problem is either $O(n^2)$ using [49] or $O(knm)$ using [22] (where $k$ is number of invariant factors). However, we can show that the problem is easier on a directed PLB graph.

We assume that we are given a graph $G$ where out degrees satisfy power law with exponent $\alpha$. The PageRank vector $\pi$ is the eigenvector of the following $n \times n$ matrix

$$M = cP + (1 - c)1/nE \quad (7)$$

where $c$ is a damping factor which can be set between 0 and 1 (typically 0.85), matrix $P$ is an adjacency matrix of $G$ defined with rules: $P_{ij} = 0$ if there are no edges from $i$ to $j$ and $P_{ij} = 1/\text{outdeg}(j)$ otherwise.

---

6We note that the approximate iterative methods used in practice have worse theoretical running time bounds.
and $E$ is a matrix whose all entries equal one. In other words, having defined $M$, the PageRank vector is an eigenvector $\pi$ of $M$ corresponding to eigenvalue $\mu = 1$, i.e., the PageRank vector satisfies the following equations

$$
\pi^T = \pi^T M
$$

$$
\pi^T e = 1
$$

where $e$ is a vector of size $n$ whose all entries equal one.

As usual we start with sorting the rows of $P$ in decreasing order according to the number of non-zero entries, i.e., out-degrees of corresponding vertices. Let us consider the following matrix decomposition

$$
P = \begin{bmatrix} A & B \\ C & D \end{bmatrix},
$$

where $[A,B]$ contains rows with more than $k$ non-zero entries for $1 \leq k \leq n$. As previously the size of $A$ is denoted by $n_k$, whereas the number of non-zero entries in $D$ is denoted by $m_D$. We define as well the corresponding decomposition of $\pi$

$$
\pi = (\pi_A, \pi_D)
$$

Using this block form we can rewrite the equation (\ref{eq:7}) as follows

$$
M = \begin{bmatrix} cA + (1-c)/nE_A & cB + (1-c)/nE_B \\ cC + (1-c)/nE_C & cD + (1-c)/nE_D \end{bmatrix}
$$

where $E_A$, $E_B$, $E_C$ and $E_D$ are matrices of the appropriate size whose all entries equal one. In turn, the equation (\ref{eq:7}) can be rewritten as

$$
\pi^T(I - M) = \pi^T \begin{bmatrix} I - MA & MB \\ MC & I - MD \end{bmatrix} = 0.
$$

We observe that $I - M$ is an irreducible $M$-matrix\footnote{A matrix $X$ is said to be \emph{M-matrix} when it can be written as $X = aI - Y$ where all entries of $Y$ are nonnegative and $a$ is greater or equal then the spectral radius of $B$. The spectral radius of $M$ is $1$ as it is a stochastic matrix.}. This implies by Theorem 4.16 from \cite{6} that every leading or trailing principal and proper submatrix of $I - M$ is nonsingular. In particular $I - M_D$ is nonsingular. Let $H_D$ be a random Hankel matrix and let $J_D$ be a random diagonal matrix. Then by the preconditioning from Section 6.3 \ARE{I - M_D)H_DJ_D} has one invariant factor. We observe that multiplication of the matrix $I - M_D = I - cD + (1-c)/nE_D$ by a vector takes $O(m_D)$ arithmetic operations, so by Theorem 6.4 we can compute Frobenius normal $F$ and the transition matrix $V$ of $(I - M_D)H_DJ_D$ in $O(nm_D)$ arithmetic operations.

The stochastic complement of $M_A$ in $I - M$ is the following matrix

$$
S_A = MA + MB(I - M_D)^{-1}MC = MA + MBJ_DH_DV^{-1}F^{-1}VMC.
$$

Computing $S_A$ requires $O(nm_D)$ arithmetic operations to compute $VMC$, $O(nk)$ arithmetic operations to compute $F^{-1}(VMC)$ and $MBJ_DH_D$, $O(\omega(n,n,n))$ arithmetic operations to compute $(MBJ_DH_D)V^{-1}$, and finally $O(n^2)$ arithmetic operations to compute $(MBJ_DH_DV^{-1})(F^{-1}VMC)$.

Now using the equations from \cite{33} we obtain

$$
\pi_A^T S_A = \pi_A^T, \\
\pi_D^T = \pi_A^T MB(I - M_D)^{-1} = \pi_A^T MBJ_DH_DV^{-1}F^{-1}V,
$$

which means that $\pi_A$ is a stationary distribution for the smaller matrix $S_A$ and can be computed in $O(n_k^\omega)$ arithmetic operations \cite{35}. Then in order to compute $\pi_D$ we need $O(n^2)$ arithmetic operations. We note that $O(\omega(n,n,n))$ dominates $O(nn_k^{\omega-1})$, $O(nk)$ and $O(n_k^{\omega})$, whereas $O(nm_D)$ dominates $\tilde{O}(n^2)$. Therefore we obtain the following theorem.
Theorem 6.15. Let $G$ be a directed PLB graph with parameters $\alpha$ and $t$, and let $1 \leq k < n^{1/\alpha} (t + 1)^{1 - 1/\alpha}$. Then, we can compute PageRank of $G$ with $O(n^2(t + 1)^{\alpha - 1}k^2 - \alpha + \omega(n, n, n(t + 1)^{\alpha - 1}k^{1 - \alpha}))$ arithmetic operations with high probability.

We note that the above idea can be combined together with iterative methods. In such a case instead of using the stochastic complement of $M_A$ we shall use the stochastic complement of $M_D$, i.e., $S_D = M_D + M_C(I - M_A)^{-1}M_B$. However, we explicitly compute only the inverse $(I - M_A)^{-1}$ using Strassen’s fast matrix inverse, but we do not execute other multiplications and keep $S_D$ in the lazy form as given by this equation. We can apply iterative methods to compute the stationary distribution of $S_D$ using this lazy form. We have implemented this approach and on a single computer it can reduce the time needed for PageRank computation by a factor of two on graphs that have approximately 100000 nodes, e.g., WikiTalk network.

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| Graph               | \(n\) | \(m\) | \(c_1\) | \(c_2\) | \(\alpha\) | \(t\)  | \(\Delta/\sqrt{m}\) |
|--------------------|-------|-------|--------|--------|----------|-------|----------------|
| Amazon (directed, in-degree + out-degree) | 241761 | 1131217 | 5 | 0.615102 | 5.198 | 22.2094 | 0.3996 |
| AstroPh (directed, in-degree + out-degree) | 17903 | 393944 | 1.2888 | 0.208271 | 2.0189 | 21.0207 | 1.606 |
| Cities (directed, in-degree + out-degree) | 3144 | 34753 | 0.9652 | – | 1.9126 | 1.8661 | 10.6425 |
| CondMatt (undirected) | 21364 | 182572 | 4.7952 | 2.15535 | 5.2849 | 26.1942 | 0.85296 |
| Dblp (undirected) | 718115 | 5573812 | 2.6633 | 5.80802 | 3.4134 | 9.589 | 0.3808 |
| Enron (undirected) | 38096 | 361622 | 1.2549 | 0.610801 | 2.2674 | 3.4682 | 2.2998 |
| Epinions (directed, in-degree + out-degree) | 32223 | 443506 | 1.2166 | – | 1.8863 | 3.8008 | 4.1894 |
| EuAll (directed, in-degree + out-degree) | 34203 | 151132 | 3.1966 | 1.37106 | 2.4201 | 3.756 | 3.8635 |
| Facebook (undirected) | 59691 | 1456818 | 0.8077 | – | 1.6668 | 6.2728 | 0.8409 |
| HepPh (directed, in-degree + out-degree) | 12711 | 139965 | 3 | 3.60723 | 5.2231 | 10.1931 | 1.0194 |
| LiveJournal (directed, in-degree + out-degree) | 382862 | 6340587 | 2.1985 | 4.64595 | 2.3983 | 15.8438 | 2.8287 |
| NotreDame (directed, in-degree + out-degree) | 53968 | 290228 | 2.2113 | 1.56224 | 2.6274 | 9.5041 | 14.0243 |
| Slashdot (directed, in-degree + out-degree) | 71970 | 841201 | 1.4678 | 0.166008 | 2.0236 | 3.8451 | 5.5191 |
| WikiTalk (directed, in-degree + out-degree) | 111881 | 1477893 | 1.5124 | 0.177209 | 2.031 | 3.7847 | 6.6613 |
| WIW (undirected) | 29406 | 393797 | 0.5474 | – | 1.2642 | 0 | 1.9135 |
| YouTube (undirected) | 49557 | 3873496 | 1.0395 | 0.66258 | 2.2474 | 1.8672 | 12.9103 |
| AstroPh (directed, out-degree) | 17903 | 393944 | 3.1737 | 4.6716 | 3.5199 | 32.734 | – |
| Epinions (directed, out-degree) | 32223 | 443506 | 2.0569 | 1.21022 | 2.4379 | 6.3751 | – |
| EuAll (directed, out-degree) | 34204 | 151132 | 2.4122 | 0.401895 | 2.1407 | 0 | – |
| HepPh (directed, out-degree) | 12711 | 139965 | 4.3101 | 1.10021 | 4.7202 | 25.2953 | – |
| LiveJournal (directed, out-degree) | 382862 | 6340587 | 2.2261 | 8.56458 | 2.7745 | 12.0126 | – |
| NotreDame (directed, out-degree) | 53968 | 290228 | 4.9269 | 1.65359 | 2.6162 | 0.4384 | – |
| Slashdot (directed, out-degree) | 71970 | 841201 | 1.5542 | 0.376638 | 2.105 | 3.3024 | – |
| WikiTalk (directed, out-degree) | 111881 | 1477893 | 1.1869 | – | 1.9364 | 0.9833 | – |
| Amazon (directed, in-degree + out-degree) | 241761 | 1131217 | 5 | – | 1.8072 | 0 | 0.3996 |
| CondMatt (undirected) | 21363 | 182572 | 5 | 0.420346 | 2.1699 | 0 | 0.65296 |

Table 1: Adjustment of PLB universal constants.

[50] E. Uchoa and R. F. Werneck. Fast local search for the Steiner problem in graphs. *J. Exp. Algorithmics*, 17:2:2.1–2:2.22, May 2012.

[51] V. V. Williams. Multiplying matrices faster than Coppersmith-Winograd. In *Proceedings of the Forty-fourth Annual ACM Symposium on Theory of Computing*, STOC ’12, pages 887–898, New York, NY, USA, 2012. ACM.
Figure 1: Real-World networks are PLB: definition adjustment
Figure 2: Real-World networks are PLB: definition adjustment for $t=0$

Figure 3: The exponent of the running time of our algorithms for counting triangles. Here PLBN stands for PLB neighborhoods. $\#\text{edges}$ is the number of edges in a graph, and folklore is $\#\text{edges}$ multiplied by $3/2$, as the well-known algorithm for counting triangles runs in $O(m^{3/2})$ time.
Figure 4: The exponent of the running times of our algebraic algorithms for power law graphs and matrices, whose nonzero entries correspond to the edges of a power law graph. Symmetric shows the complexity of determinant algorithm for symmetric matrices as well as perfect matching algorithm. General depicts the complexity of algorithms for determinant, PageRank, matrix inverse, linear system solving and transitive closure in matrices that do not need to be symmetric. The complexities are derived using the bound on $\omega(n,n,n_k)$ given in [39]. Our results are compared to the running times of algorithms that work for arbitrary graphs and matrices. Note that the bound of $O(\sqrt{nm})$ is only known for the perfect matching algorithm.