A SIMPLE CONSTRUCTION OF RECURSION OPERATORS FOR MULTIDIMENSIONAL DISPERSIONLESS INTEGRABLE SYSTEMS

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ABSTRACT. We present a novel construction of recursion operators for integrable multidimensional dispersionless systems that admit a Lax representation whose operators are linear in the spectral parameter and do not involve the derivatives with respect to the latter. The examples of hitherto unknown recursion operators obtained using our technique include inter alia those for the general heavenly equation and the Ferapontov–Khusnutdinova equation.

INTRODUCTION

Recursion operators (ROs) for multidimensional integrable dispersionless systems play a crucial role in the construction of the associated integrable hierarchies, see e.g. [22] for details, and are now a subject of intense research, cf. e.g. [16, 22, 24, 25, 26] and references therein.

There are several methods for construction of the ROs for the systems in question: the partner symmetry approach [16], the method based on adjoint action of the Lax operators [22], and the approach using the Cartan equivalence method [24, 25, 26].

Below we present another method for construction of ROs for integrable multidimensional dispersionless systems. Roughly speaking, it consists in constructing a Lax-type representation for the system under study with the following additional properties: i) the Lax operators are linear in the spectral parameter and ii) the solution of the associated linear system is a (nonlocal) symmetry of the nonlinear system under study. Taking the coefficients at the powers of spectral parameter in the operators constituting the newly constructed Lax-type representation yields, under certain technical conditions, the RO. A similar approach, with symmetries replaced by cosymmetries and Proposition 1 by Proposition 2), allows one to look for the adjoint ROs.

To the best of author’s knowledge, the key idea of our approach, i.e., that the Lax-type representation employed for the construction of the recursion operator should be constructed from the original Lax-type representation but is by no means obliged to be identical with the latter, has not yet appeared in the literature. On the other hand, the extraction of the RO from a Lax-type
representation which is linear in the spectral parameter was already explored to some extent e.g. in [29].

The paper is organized as follows. In Section 1 we recall some basic facts concerning the geometric theory of PDEs. In Section 2 we state some auxiliary results on the construction of recursion operators. Section 3, which is the core of the paper, explores the construction of special Lax-type representations that give rise to ROs through the results of Section 2. Section 4 gives a selection of examples illustrating our approach, and Section 5 provides a brief discussion.

1. PRELIMINARIES

Let \( F_I = 0, I = 1, \ldots, m \) (or, more concisely, \( \mathcal{F} = 0 \)) be a system of PDEs in \( d \) independent variables \( x^i, i = 1, \ldots, d \) (we put \( \vec{x} = (x^1, \ldots, x^d) \)), for the unknown \( N \)-component vector function \( \mathbf{u} = (u^1, \ldots, u^N)^T \), where the superscript ‘\( T \)’ indicates the transposed matrix. Let

\[
\mathbf{u}^\alpha_{i_1 \ldots i_n} = \partial_{i_1 + \ldots + i_n} u^\alpha / \partial(x^1)^{i_1} \ldots \partial(x^n)^{i_n}
\]

and \( \mathbf{u}_{00 \ldots 0}^\alpha \equiv u^\alpha \).

As e.g. in [?, ?], \( x^i \) and \( u^\alpha_{i_1 \ldots i_n} \) are considered here as independent quantities and can be viewed as coordinates on an abstract infinite-dimensional space (a jet space). By a local function we shall mean a (smooth) function of a finite number of \( x^i \) and \( \mathbf{u} \) and their derivatives.

We denote by

\[
D_x^j = \frac{\partial}{\partial x^j} + \sum_{\alpha=1}^{N} \sum_{i_1, \ldots, i_n=0}^{\infty} u^\alpha_{i_1 \ldots j \ldots i_n} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_n}}
\]

the total derivatives. For simplicity the extensions of \( D_x^j \) to nonlocal variables are again denoted by \( D_x^j \), as e.g. in [22]. The condition \( \mathcal{F} = 0 \) along with its differential consequences \( D_x^{i_1} \cdots D_x^{i_n} \mathcal{F} = 0 \) determines the diffiety \( \text{Sol}_\mathcal{F} \), a submanifold of the infinite jet space. This submanifold can be informally thought of as a set of all formal solutions of \( \mathcal{F} = 0 \), which motivates the choice of notation. In what follows all equations will be required to hold on \( \text{Sol}_\mathcal{F} \), or a certain differential covering thereof, only (rather than e.g. on the whole jet space) unless otherwise explicitly stated.

The directional derivative along \( \mathbf{U} = (U^1, \ldots, U^N)^T \) is the vector field on the jet space

\[
\partial \mathbf{U} = \sum_{\alpha=1}^{N} \sum_{i_1, \ldots, i_n=0}^{\infty} \left( D_x^{i_1} \cdots D_x^{i_n} U^\alpha \right) \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_n}}.
\]
The total derivatives as well as the directional derivative can be applied to (possibly vector or matrix) local functions \( P \).

Recall that a local \( N \)-component vector function \( U \) is a (characteristic of a) symmetry for the system \( F = 0 \) if \( U \) satisfies the linearized version of our system, namely, \( \partial_U F \equiv \ell_F(U) = 0 \) (recall that by our blanket assumption this is required to hold on \( \text{Sol}_F \) rather than on the whole jet space). Informally, this means that the flow \( \mathbf{u}_\tau = \mathbf{U} \) leaves \( \text{Sol}_F \) invariant.

Here

\[
\ell_f = \sum_{\alpha=1}^{N} \sum_{i_1,\ldots,i_n=0}^{\infty} \frac{\partial f}{\partial u^\alpha_{i_1\ldots i_n}} D_{x_1}^{i_1} \cdots D_{x_n}^{i_n}
\]

is the operator of linearization and

\[
\ell_F = (\ell_{F_1}, \ldots, \ell_{F_m})^T.
\]

The notation \( f(\vec{x}, [u]) \) means that \( f \) is a local function, i.e., \( f \) depends on \( \vec{x}, u \) and a finite number of the derivatives of \( u \). Note an important identity \( \ell_f(U) = \partial_U(f) \).

Finally, recall that if we have an operator in total derivatives

\[
Q = Q^0 + \sum_{k=1}^{q} \sum_{j_1=1}^{d} \cdots \sum_{j_k=1}^{d} Q^{j_1\ldots j_k} D_{x_{j_1}} \cdots D_{x_{j_k}},
\]

where \( Q^0 \) and \( Q^{j_1\ldots j_k} \) are \( s \times s' \)-matrix-valued local functions, its formal adjoint reads

\[
Q^\dag = (Q^0)^T + \sum_{k=1}^{q} \sum_{j_1=1}^{d} \cdots \sum_{j_k=1}^{d} D_{x_{j_1}} \cdots D_{x_{j_k}} \circ (Q^{j_1\ldots j_k})^T.
\]

For integrable systems in more than two independent variables their symmetries usually depend, in addition to local variables, described above, also on nonlocal variables. In other words, we are forced to consider solutions \( U \) of \( \ell_F(U) = 0 \) involving nonlocal variables. A number of authors refers to such objects as to shadows of nonlocal symmetries, see e.g. [5, 13] and references therein for more details on this terminology. However, for the sake of simplicity in what follows we shall refer to solutions \( U \) of \( \ell_F(U) = 0 \) involving nonlocal variables just as to nonlocal symmetries.

A dual object to symmetry is a cosymmetry \( \gamma \) which has \( m \) rather than \( N \) components satisfies the adjoint linearized system

\[
\ell_F^\dag(\gamma) = 0.
\]
A cosymmetry has $m$ rather than $N$ components. Note that cosymmetries, just like symmetries, may also depend on nonlocal variables.

A recursion operator (RO) for our system is then, naively, an operator that sends any symmetry of $\mathcal{F} = 0$ into another symmetry thereof [27].

However, in order to better handle nonlocal terms in ROs it is more appropriate (Papachristou, Harrison, Marvan, Guthrie) to view the RO as a Bäcklund auto-transformation for $\ell_{\mathcal{F}}(\mathcal{G}) = 0$.

Likewise, an adjoint recursion operator is then a Bäcklund auto-transformation for $\ell^\dagger_{\mathcal{F}}(\gamma) = 0$.

As a closing remark, recall that a partial differential system is dispersionless if it can be written in the form of a quasilinear homogeneous first-order system

$$
\sum_{i=1}^{d} \sum_{\alpha=1}^{n} A^j_{i\alpha}(u) \frac{\partial u^\alpha}{\partial x^j} = 0,
$$

(1)

where $I = 1, \ldots, m$, $m \geq N$, and $u = (u^1, \ldots, u^N)^T$.

This class is quite rich: for instance, quasilinear scalar second-order PDEs

$$
\sum_{i=1}^{d} \sum_{j=i}^{d} f^{ij}(\vec{x}, u, \partial u/\partial x^1, \ldots, \partial u/\partial x^d) \partial^2 u/\partial x^i \partial x^j = 0
$$

can be brought into the above form by putting $u = (u, \partial u/\partial x^1, \ldots, \partial u/\partial x^d)^T$.

2. SOME REMARKS ON RECURSION OPERATORS

Introduce the following differential operators in total derivatives

$$
A_i = A^0_i + \sum_{j=1}^{d} A^j_i D_{x^j}, \quad B_i = B^0_i + \sum_{j=1}^{d} B^j_i D_{x^j}, \quad i = 1, 2,
$$

$$
L = L^0 + \sum_{k=1}^{q} \sum_{j_1=1}^{d} \cdots \sum_{j_k=1}^{d} L^{j_1\cdots j_k} D_{x^{j_1}} \cdots D_{x^{j_k}}, \quad I = 1, \ldots, m,
$$

$$
M = M^0 + \sum_{k=1}^{q} \sum_{j_1=1}^{d} \cdots \sum_{j_k=1}^{d} M^{j_1\cdots j_k} D_{x^{j_1}} \cdots D_{x^{j_k}}, \quad I = 1, \ldots, m,
$$

where $A^j_i = A^j_i(\vec{x}, [u])$ and $B^j_i = B^j_i(\vec{x}, [u])$ for $i = 1, 2$ and $j = 1, \ldots, d$ are (smooth) functions, $A^0_i = A^0_i(\vec{x}, [u])$ and $B^0_i = B^0_i(\vec{x}, [u])$ are $N \times N$ matrices, $L^0 = L^0(\vec{x}, [u])$ and $L^{j_1\cdots j_k} = L^{j_1\cdots j_k}(\vec{x}, [u])$ are $N \times m$ matrices, $M^0 = M^0(\vec{x}, [u])$ and $M^{j_1\cdots j_k} = M^{j_1\cdots j_k}(\vec{x}, [u])$ are $m \times N$ matrices.

Proposition 1. Let the operators $A_i, B_i, L, M$ of the above form be such that on $\text{Sol}_F$ we have

i) $[A_1, A_2] = 0$, (2)
\(ii\) \((A_1B_2 - A_2B_1) = L \circ \ell_F\) \hfill (3)

\(iii\) \(\ell_F = M \circ (B_1A_2 - B_2A_1)\), \hfill (4)

\(iv\) there exist two distinct indices \(p,q \in \{1,\ldots,d\}\) such that we can express \(D_{x^p} \tilde{U}\) and \(D_{x^q} \tilde{U}\) from the relations

\[A_i(\tilde{U}) = B_i(U), \quad i = 1,2.\] \hfill (5)

Then relations (5) define a recursion operator for \(F = 0\), i.e., whenever \(U = (U^1,\ldots,U^N)^T\) is a symmetry shadow for \(F = 0\), so is \(\tilde{U} = (\tilde{U}^1,\ldots,\tilde{U}^N)^T\) defined by (5).

**Proof.** First of all, if \(U = (U^1,\ldots,U^N)^T\) is a symmetry shadow for \(F = 0\), then the system (5) for \(\tilde{U}\) is compatible by virtue of i)–iii).

We now have \(\ell_F(\tilde{U}) = 0\) by virtue of (4), and hence \(\tilde{U}\) is a shadow of symmetry for our system. \(\square\)

In a similar fashion we can prove the counterpart of Proposition 1 for adjoint recursion operators.

**Proposition 2.** Suppose that \(A_i, B_i, L, M\) are as before, but with \(A_i^0\) and \(B_i^0\) being \(m \times m\) matrices. Further assume that these operators are such that

\(i\) \([A_1, A_2] = 0\), \hfill (6)

\(ii\) \(\ell^\dagger_F = L \circ (B_1A_2 - B_2A_1)\), \hfill (7)

\(iii\) \((A_1B_2 - A_2B_1) = M \circ \ell^\dagger_F\), \hfill (8)

\(iv\) there exist two distinct indices \(p,q \in \{1,\ldots,d\}\) such that we can express \(D_{x^p} \tilde{\gamma}\) and \(D_{x^q} \tilde{\gamma}\) from the (compatible) system

\[A_i(\tilde{\gamma}) = B_i(\gamma), \quad i = 1,2.\] \hfill (9)

Then (9) define an adjoint recursion operator for \(F = 0\), i.e., whenever \(\gamma = (\gamma^1,\ldots,\gamma^m)^T\) is a cosymmetry for \(F = 0\), then so is \(\tilde{\gamma}\) defined by (9).

As an aside note that the condition (4) in a somewhat different form has appeared in \([11, 6, 10]\), and was given there a geometric interpretation.

3. **Recursion Operators from Lax-Type Representations**

We start with the following result which is readily checked by straightforward computation.

**Proposition 3.** Suppose that the operators \(A_i\) and \(B_i\) define a RO (resp. an adjoint RO) as in Proposition 1 (resp. as in Proposition 2). Then the operators \(\mathcal{L}_i = A_i - \lambda B_i, \quad i = 1,2,\) where \(\lambda\) is a spectral parameter, satisfy \([\mathcal{L}_1, \mathcal{L}_2] = 0\), i.e., they constitute a Lax-type representation for \(F = 0\).
Thus, if our system admits an (adjoint) RO, it also admits a Lax-type representation which is linear in the spectral parameter.

Hence, a natural source of \( A_i, B_i \) and \( L, M \) satisfying the conditions of Proposition \( \square \) is provided by the Lax-type representations for \( F = 0 \) of the form

\[
L_i \psi = 0, \quad i = 1, 2,
\]

with \( L_i \) linear in \( \lambda \) such that \( \psi \) is a nonlocal symmetry of \( F = 0 \), i.e., we have \( \ell_F(\psi) = 0 \), cf. e.g. [29]. Then putting \( L_i = \lambda B_i - A_i \) or \( L_i = B_i - \lambda A_i \) gives us natural candidates for \( A_i \) and \( B_i \) which then should be checked against the conditions of Proposition \( \square \), and, if the latter hold, yield a recursion operator for \( F = 0 \).

It is important to stress that the Lax representation with the operators \( L_i \) employed in the above construction does not have to be the original Lax representation of our system. In general, we should custom tailor the operators \( L_{1,2} \) constituting the Lax pair for the construction in question, so that the solutions of the associated linear problem \((\square)\) are symmetries, i.e., satisfy the linearized version of our system.

The natural building blocks for these \( L_i \) are the original Lax operators \( \mathcal{X}_i \), their formal adjoints \( \mathcal{X}_i^\dagger \), and their adjoint actions \( \text{ad}_{\mathcal{X}_i} = [\mathcal{X}_i, \cdot] \), but in general one has to twist them, cf. e.g. Example 4 below.

More explicitly, suppose the system under study, i.e., \( F = 0 \), admits a Lax representation of the form

\[
\mathcal{X}_i \psi = 0, \quad i = 1, 2,
\]

where \( \mathcal{X}_i \) are linear in the spectral parameter \( \lambda \) but such \( \psi \) does not satisfy \( \ell_F(\psi) = 0 \) on \( \text{Sol}_F \), i.e., \( \psi \) is not a (nonlocal) symmetry.

Then we can seek for a nonlocal symmetry of \( F = 0 \) of the form

\[
\Phi = \Phi(\vec{x}, [u, \psi, \chi, \vec{\zeta}]),
\]

i.e. a vector function of \( \vec{x} \), and of \( u, \psi, \chi, \) and \( \vec{\zeta} \) and a finite number of the derivatives of \( u, \psi, \chi \).

Here \( \chi \) satisfies the system

\[
\mathcal{X}_i^* \chi = 0, \quad i = 1, 2,
\]

\[
\mathcal{E} = \sum_{j=1}^{d} \zeta^j \partial / \partial x^j \]

satisfies

\[
[\mathcal{X}_i, \mathcal{E}] = 0, \quad i = 1, 2,
\]

and \( \vec{\zeta} = (\zeta^1, \ldots, \zeta^d)^T \). Finally, \( \psi \) satisfies \((\square)\).
Moreover, we should also require that there exist the operators $\mathcal{L}_i$ which are linear in $\lambda$ and such that

$$\mathcal{L}_i \Phi = 0, \quad i = 1, 2.$$ 

Then one should extract $A_i$ and $B_i$ required for Proposition 1 to work from these $\mathcal{L}_i$ rather than from original $\mathcal{X}_i$.

A similar approach, with symmetries replaced by cosymmetries and Proposition 1 by Proposition 2 can, of course, be applied to the construction of adjoint recursion operators.

Finally, let us point out that the linear dependence of the Lax operators on $\lambda$ is not as restrictive as it seems: a great many of known today multidimensional dispersionless hierarchies include systems with this property and, moreover, by Proposition 3 if a system admits an RO of the form described in Proposition 1 then it necessarily possesses a Lax-type representation whose operators are linear in $\lambda$.

Moreover, in some cases the nonlinearity in $\lambda$ can disappear upon a proper rewriting of the Lax pair.

Consider for example the Pavlov equation [14]

$$u_{yy} + u_{xt} + u_x u_{xy} - u_y u_{xx} = 0,$$  
(12)

which possesses a Lax pair of the form [14] (cf. also [18])

$$\psi_y = (-u_x - \lambda)\psi_x, \quad \psi_t = (-\lambda^2 - \lambda u_x + u_y)\psi_x$$  
(13)

which is quadratic in $\lambda$.

However, rewriting the second equation of (13) as

$$\psi_t = \lambda(-\lambda - u_x)\psi_x + u_y\psi_x$$

and substituting $\psi_y$ for $(-\lambda - u_x)\psi_x$, we obtain

$$\psi_t = \lambda\psi_y + u_y\psi_x,$$

and thus (13) can be rewritten in the form linear in $\lambda$:

$$\psi_y = (-u_x - \lambda)\psi_x, \quad \psi_t = \lambda\psi_y + u_y\psi_x.$$  
(14)

4. Examples

**Example 1.** It is readily checked $\psi$ solving (14) is a nonlocal symmetry for (12). Guided by the form of the operators $\mathcal{L}_i'$ in Proposition 3, put $A_1 = D_x$, $A_2 = D_y$, $B_1 = -D_y + u_x D_x$, and $B_2 = D_t - u_y D_x$. Then it is readily checked that they satisfy the conditions of Proposition 1 for suitably chosen
L and M, so we arrive at the recursion operator for the Pavlov equation given by the formulas
\[ \tilde{U}_x = -U_y + u_x U_x, \quad \tilde{U}_y = U_t - u_y U_x, \]
which relates a nonlocal symmetry \( U \) to a (new) nonlocal symmetry \( \tilde{U} \). This is nothing but the recursion operator found in [17] rewritten as a Bäcklund auto-transformation for the linearized version of (12),
\[ U_{yy} + U_{xt} + u_x U_{xy} - u_y U_{xx} + u_{xy} U_x - u_{xx} U_y = 0. \]

**Example 2.** Consider the general heavenly equation
\[ au_{xy} u_{zt} + bu_{xz} u_{yt} + cu_{xt} u_{yz} = 0, \quad a + b + c = 0, \tag{15} \]
where \( a, b, c \) are constants. Here \( d = 3, \ N = 1, \ x^1 = x, \ x^2 = y, \ x^3 = z, \ x^4 = t, \ u^1 = u \). Note that this equation describes inter alia a class of anti-self-dual solutions of the Einstein field equations, as shown in [15].

By definition, a (nonlocal) symmetry \( U \) of (15) satisfies the linearized version of (15), that is,
\[ au_{xy} U_{zt} + au_{zt} U_{xy} + bu_{xz} U_{yt} + bu_{yt} U_{xz} + cu_{xt} U_{yz} + cu_{yz} U_{xt} = 0, \tag{16} \]
modulo (15) and all its differential consequences.

Eq. (15) admits [4, 15] a Lax-type representation with the operators
\[ L_1 = (1 + c\lambda) D_x - \frac{u_{xz}}{u_{zt}} D_t - c\lambda \frac{u_{xt}}{u_{zt}} D_z, \]
\[ L_2 = (1 - b\lambda) D_y - \frac{u_{yz}}{u_{zt}} D_t + b\lambda \frac{u_{yt}}{u_{zt}} D_z. \tag{17} \]

Let
\[ A_1 = -c D_x + \frac{c u_{xt}}{u_{zt}} D_z, \quad A_2 = b D_y - \frac{b u_{yt}}{u_{zt}} D_z, \]
\[ B_1 = D_x - \frac{u_{xz}}{u_{zt}} D_t, \quad B_2 = D_y - \frac{u_{yz}}{u_{zt}} D_t. \]

Then all conditions of Proposition 1 are readily seen to be satisfied, e.g. we have \( B_1 A_2 - B_2 A_1 = (1/u_{zt}) \ell_F \), where \( F = au_{xy} u_{zt} + bu_{xz} u_{yt} + cu_{xt} u_{yz} \).

Hence by Proposition 1 the relations
\[ \tilde{U}_x = \frac{u_{xz} U_t + cu_{xt} \tilde{U}_z - u_{zt} U_x}{cu_{zt}}, \quad \tilde{U}_y = -\frac{u_{yz} U_t - bu_{yt} \tilde{U}_z - u_{zt} U_y}{bu_{zt}}, \tag{18} \]
where \( U \) is a (possibly nonlocal) symmetry of (15), define a new symmetry \( \tilde{U} \) and thus a hitherto unknown recursion operator (18) for (15), i.e., a Bäcklund auto-transformation for (16).
For instance, we can get an infinite hierarchy of nonlocal symmetries for \((15)\) if we repeatedly apply this recursion operator to a symmetry \(u_t\) of \((15)\).

Consider now the evolutionary form for \((15)\). To this end we pass from \(x, y, z, t\) to new independent variables \(\tilde{x} = x, \tilde{y} = y, \tilde{z} = (z - t)/2, \tilde{t} = (t + z)/2.\)

Upon omitting the tildes over the new variables, solving the transformed version of \((15)\) with respect to \(u_{tt}\) and introducing a new dependent variable \(v = u_t\) we arrive at the following evolutionary system:

\[
\begin{align*}
u_t &= v, \quad v_t = u_{zz} + \frac{v_x v_y}{u_{xy}} - \frac{u_{xz} u_{yz}}{u_{xy}} + \frac{(b - c)(v_x u_{yz} - u_{xz} v_y)}{au_{xy}}. \\
\end{align*}
\]

(19)

Existence of the Lagrangian \([4]\) for \((15)\) gives rise to the following local Hamiltonian structure for \((19)\):

\[
\mathfrak{P}_0 = \begin{pmatrix} 0 & u_{xy} \\ -u_{xy} & A \end{pmatrix},
\]

(20)

\[
A = \left( \frac{(b - c)u_{yz}}{a} + v_y \right) D_x + \left( \frac{(c - b)u_{xz}}{a} + v_x \right) D_y \\
+ \left( \frac{(c - b)u_{yz}}{au_{xy}} - \frac{v_y}{u_{xy}} \right) u_{xxy} + \frac{(b - c)u_{xz} u_{xyy}}{au_{xy}} - \frac{u_{xyy} v_x}{u_{xy}} + 3v_{xy},
\]

The first Hamiltonian representation for \((19)\) thus reads

\[
\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathfrak{P}_0 \begin{pmatrix} \delta H_1/\delta u \\ \delta H_1/\delta v \end{pmatrix},
\]

where

\[
H_1 = \frac{1}{2} \int (v^2 u_{xy} + u_x u_y u_{zz}) \, dxdydz.
\]

Next, the recursion operator for \((15)\) defined by \((18)\) induces a recursion operator, say \(\mathcal{R}\), for \((19)\). We conjecture that the operator \(\mathfrak{P}_1 = \mathcal{R} \circ \mathfrak{P}_0\) is a (nonlocal) Hamiltonian operator for \((19)\), and that \(\mathfrak{P}_0\) and \(\mathfrak{P}_1\) are compatible, so \((19)\) is a bihamiltonian system w.r.t. \(\mathfrak{P}_0\) and \(\mathfrak{P}_1\).

**Example 3.** Consider the Martínez Alonso–Shabat \([21]\) equation

\[
u_{yt} = u_z u_{xy} - u_y u_{xz}
\]

(21)

where \(d = 4, N = 1, x^1 = x, x^2 = y, x^3 = z, x^4 = t, u^1 = u.\) Eq.\((15)\) admits \([26]\) a Lax-type representation with the operators

\[
L_1 = D_y - \lambda u_y D_x, \quad L_2 = D_z - \lambda u_z D_x + \lambda D_t.
\]

(22)
It turns out that in this case we should use the adjoint action, so we set

\[ \mathcal{L}_i = \text{ad}L_i = [L_i, \cdot] = \lambda B_i - A_i, \]

\[ A_1 = D_y, \quad A_2 = D_z, \quad B_1 = u_y D_x - u_{xy}, \quad B_2 = u_z D_x - D_t - u_{xz}. \]

All conditions of Proposition 1 are again satisfied.

By Proposition 1 the relations

\[ \tilde{U}_y = u_y U_x - u_{xy} U, \]
\[ \tilde{U}_z = u_z U_x - u_{xz} U - U_t, \]

where \( U \) is any (possibly nonlocal) symmetry of (21), define a new symmetry \( \tilde{U} \), i.e. Eq. (23) is a recursion operator for (21). This operator was already found in [26].

Applying (23) to the simplest symmetry \( U = u_x \) we obtain (modulo an arbitrary function of \( x \) and \( t \) resulting from the integration) a nonlocal symmetry of the form \( \tilde{U} = w - u_x^2/2 \), where the nonlocal variable \( w \) is defined by the relations

\[ w_y = u_y u_{xx}, \quad w_z = u_z u_{xx} - u_{xt}. \]

The above example suggests that Eq. (23) can be further simplified: a new symmetry \( \tilde{U} \) can be constructed by putting

\[ \tilde{U} = W - u_x U, \]

where \( W \) is defined by the relations

\[ W_y = u_y U_x + u_x U_y, \]
\[ W_z = u_z U_x + u_x U_z - U_t. \]

Thus, the recursion operators produced within our approach not necessarily are in the simplest possible form.

**Example 4.** Consider a system [8, 9]

\[ m_t = n_x + n m_r - m n_r, \quad n_z = m_y + m n_s - m n_s, \]

where \( x, y, z, r, s, t \) are independent and \( m, n \) are dependent variables.

Eq. (24) can be written as a condition of commutativity for the following pair of vector fields,

\[ [D_z - m D_s - \lambda D_x + \lambda m D_r, D_y - n D_s - \lambda D_t + \lambda n D_r] = 0. \]

where \( \lambda \) is a spectral parameter, and is therefore integrable.

By virtue of (24) there exists a potential \( u \) such that

\[ m = u_z / u_s, \quad n = u_y / u_s. \]
Substituting this into (24) gives a single second-order equation for $u$,

$$u_s u_{zt} - u_z u_{st} - u_s u_{xy} + u_y u_{sx} - u_y u_{rz} + u_z u_{ry} = 0 \quad (27)$$

which can be written as a commutativity condition $[L_1, L_2] = 0$ for the vector fields

$$L_1 = D_z - \frac{u_z}{u_s} D_s - \lambda D_x + \lambda \frac{u_z}{u_s} D_r, \quad L_2 = D_y - \frac{u_y}{u_s} D_s - \lambda D_t + \lambda \frac{u_y}{u_s} D_r.$$

Let $\chi$ satisfy $L_1^\dagger \chi = 0$, $L_2^\dagger \chi = 0$, where $\dagger$ indicates the formal adjoint. Then $\zeta = u_s/\chi$ is a nonlocal symmetry for (27) and we can readily obtain a linear system for $\zeta$ of the form $L_1 \zeta = 0$, $L_2 \zeta = 0$ from that for $\chi$.

Using this linear system for $\zeta$ our approach immediately produces the following RO $\mathcal{R}$ s.t. $\tilde{U} = \mathcal{R}(U)$ for (27):

$$\tilde{U}_y = \frac{u_y}{u_s} \tilde{U}_s - \frac{u_y}{u_s} U_r + U_t - \frac{(u_{st} - u_{ry})}{u_s} U,$$

$$\tilde{U}_z = \frac{u_z}{u_s} \tilde{U}_s - \frac{u_z}{u_s} U_r + U_x + \frac{(u_{rz} - u_{sx})}{u_s} U. \quad (28)$$

Here again $U$ is any (possibly nonlocal) symmetry of (27), and $\tilde{U}$ is a new symmetry for (27).

**Example 5.** Consider (see e.g. [2, 23] and references therein) the following Lax operators:

$$L_1 = D_y + \partial_x K - \lambda D_x, \quad L_2 = D_x + \partial_y K - \lambda D_y. \quad (29)$$

The commutativity condition $[L_1, L_2] = 0$ yields

$$\partial_x \partial_x K - \partial_y \partial_y K - [\partial_x K, \partial_y K] = 0. \quad (30)$$

Here $K$ takes values in a Lie algebra $\mathfrak{g}$ and is known as the Yang $K$-matrix. Eq.(30) is known as the equation for the $K$-matrix or the Leznov equation.

If we define the gauge potentials $A_j$ by setting $A_x = \partial_y K$, $A_y = \partial_x K$, $A_{\tilde{x}} = 0$, $A_{\tilde{y}} = 0$, then they satisfy the anti-self-dual Yang–Mills (ASDYM) equations on $\mathbb{R}^4$ with Euclidean metric with coordinates $X^i$ such that $\sqrt{2}x = X^1 + iX^2$, $\sqrt{2}\tilde{x} = X^1 - iX^2$, $\sqrt{2}y = X^3 - iX^4$, $\sqrt{2}\tilde{y} = X^3 + iX^4$, where $i = \sqrt{-1}$, cf. e.g. [23].

It is natural to assume that in the Lax pair equations $L_i \psi = 0$ we have $\psi \in G = \text{exp}(\mathfrak{g})$. However, such $\psi$ cannot be a symmetry for (30), since a symmetry must live in $\mathfrak{g}$ just like $K$. However, we can get a Lax representation for $\Phi \in \mathfrak{g}$ using the adjoint action: $[L_1, \Phi] = [L_2, \Phi] = 0$, i.e., we use the Lax operators $\mathcal{L}_i = \text{ad} L_i$. 
Then all conditions of Proposition 1 are satisfied and we obtain the following RO for (30):

$$V_x = U_y + [\partial_x K, U], \quad V_y = U_x + [\partial_y K, U],$$

(31)

where $U$ and $V$ are symmetries for (30). This RO is nothing but the RO of (30) known from the literature (see [30, 2]).

If in the above construction we put

$$K = \sum_{i=1}^{N} u_i \partial / \partial z^i,$$

i.e., take $g = \text{diff}(\mathbb{R}^N)$, where $u_i = u_i(x, y, \tilde{x}, \tilde{y}, z^1, \ldots, z^N)$ are scalar functions, then (30) becomes the Manakov–Santini system [17], an integrable system in $(N + 4)$ independent variables, and (31) provides a recursion operator for this system. This RO is precisely the one found earlier in [22] by a different method.

Further examples of recursion operators obtained using our approach can be found in [12].

Our method of constructing recursion operators apparently works for all known examples of multidimensional dispersionless integrable systems admitting recursion operators in the form of Bäcklund auto-transformation for linearized systems, including e.g. the ABC equation [33]

$$au_xu_yu_t + bu_yu_xu_t + cu_tu_{xy} = 0, \quad a + b + c = 0,$$

the simplest (2+1)-dimensional equation of the so-called universal hierarchy of Martínez Alonso and Shabat [21],

$$u_{yy} - u_yu_{tx} + u_xu_{ty} = 0,$$

first, second and modified heavenly equations, etc.

5. CONCLUDING REMARKS

We presented a construction for recursion operators and adjoint recursion operators for a broad class of multidimensional integrable systems that can be written as commutativity conditions for a pair of vector fields (or even more broadly, linear combinations of vector field with zero-order matrix operators) linear in the spectral parameter and free of derivatives in the latter.

Our method is apparently quite universal. In particular, it enabled us to find hitherto unknown recursion operators for the general heavenly equation of Doubrov and Ferapontov and for the Ferapontov–Khusnutdinova equation.
It would be interesting to find out whether it is possible to extend this approach to other classes of integrable systems.

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