Partial DP-Coloring

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Abstract

In 1980 Albertson and Berman introduced partial coloring and then in 2000, Albertson, Grossman, and Haas introduced partial list coloring. Here we initiate the study of partial coloring for DP-coloring (aka, correspondence coloring), a recent insightful generalization of list coloring introduced in 2015 by Dvořák and Postle. The partial t-chromatic number of a graph G, denoted αt(G), is the maximum number of vertices that can be colored with t colors. Clearly, αt(G) ≥ t|V(G)|/χ(G) for each t ∈ {1, . . . , χ(G)}. Given a list assignment L for graph G – meaning that each vertex v ∈ V(G) is assigned a list L(v) of available “colors” – let αL(G) be the maximum number of vertices that can be colored from those lists. The partial t-choice number of a graph G, denoted αL(G), is the minimum of αL(G) taken over all assignments L for which |L(v)| = t for each v ∈ V(G). The Partial List Coloring Conjecture states that for any graph G, αt(G) ≥ t|V(G)|/χt(G) whenever t ∈ {1, . . . , χ(G)} where χt(G) is the list chromatic number of G. We show that while the DP-coloring analogue of the Partial List Coloring Conjecture does not hold, several results on partial list coloring can be extended to the DP-coloring context. We also study partial DP-coloring of the join of a graph with a complete graph, and we present several interesting open questions.

Keywords. graph coloring, list coloring, partial list coloring, DP-coloring.

Mathematics Subject Classification. 05C15, 05C69

1 Introduction

In this paper all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking we follow West [30] for terminology and notation. The set of natural numbers is N = {1, 2, 3, . . .}. Given a set A, P(A) is the power set of A. For m ∈ N, we write [m] for the set {1, . . . , m}. If G is a graph and S, U ⊆ V(G), we use G[S] for the subgraph of G induced by S, and we use E_G(S, U) for the subset of E(G) with at least one endpoint in S.

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and at least one endpoint in \( U \). If an edge in \( E(G) \) connects the vertices \( u \) and \( v \), the edge can be represented by \( uv \) or \( vu \). We use \( \alpha(G) \) and \( \omega(G) \) for the size of the largest independent set and the size of the largest clique in \( G \) respectively. For \( v \in V(G) \), we write \( d_G(v) \) for the degree of vertex \( v \) in the graph \( G \), and we use \( \Delta(G) \) and \( \delta(G) \) for the maximum and minimum degree of a vertex in \( G \) respectively. For vertex disjoint graphs \( G_1 \) and \( G_2 \), we write \( G_1 \lor G_2 \) for their join.

### 1.1 Partial List Coloring

In the classical vertex coloring problem, colors must be assigned to the vertices of a graph \( G \) so that adjacent vertices receive different colors. The assignment is a proper \( m \)-coloring if colors come from an \( m \)-set such as \([m]\); the smallest such \( m \) is the chromatic number, denoted \( \chi(G) \). Given fewer than \( \chi(G) \) colors, we might instead try to color as many vertices as possible: this is partial coloring, introduced by Albertson and Berman [1] in 1980. The partial \( t \)-chromatic number of a graph \( G \), denoted \( \alpha_t(G) \), is the maximum number of vertices that can be colored with \( t \) colors. Then \( \alpha_1(G) = \alpha(G) \), \( \alpha_t(G) = |V(G)| \) for \( t \geq \chi(G) \), and \( \alpha_t(G) \geq \ell |V(G)|/\chi(G) \) for \( t \in [\chi(G)] \) by taking the largest \( t \) color classes from a proper \( \chi(G) \)-coloring.

List coloring is a well known variation of graph coloring, introduced independently by Vizing [28] and Erdős, Rubin, and Taylor [14] in the 1970s. Each vertex \( v \) has a list \( L(v) \) of allowable colors; \( L \) is called a list assignment (or \( m \)-assignment if every list has size \( m \)), and an \( L \)-coloring (or proper \( L \)-coloring) is a proper coloring where each vertex \( v \) is assigned a color from its list \( L(v) \). A graph \( G \) is \( m \)-choosable if there is an \( L \)-coloring whenever \( L \) is an \( m \)-assignment; the minimum such \( m \) for a graph \( G \) is the list chromatic number of a graph \( G \), denoted \( \chi_L(G) \). As the lists could all be identical, \( \chi_L(G) \geq \chi(G) \).

Albertson, Grossman, and Haas [2] introduced partial list coloring with a “frankly mischievous” intent of inciting further work. Indeed, this has received attention in several papers [2, 12, 13, 16, 17, 18, 29]. Given a list assignment \( L \), we want to properly \( L \)-color as many vertices as possible. Let \( \alpha_L(G) \) be the maximum size of a subset of vertices of \( G \) that may be properly \( L \)-colored (i.e., the maximum order of an induced subgraph \( G[S] \) that has an \( L \)-coloring, where \( L \) is the list assignment \( L \) restricted to \( S \) ). The partial \( t \)-choice number of a graph \( G \), denoted \( \alpha_t^L(G) \), is the minimum of \( \alpha_L(G) \) over all \( t \)-assignments \( L \) for \( G \)\footnote{Elsewhere in the literature, the partial \( t \)-choice number of \( G \) is denoted \( \lambda_t(G) \).}. The lists could all be the same set \([t]\), so \( \alpha_t^L(G) \leq \alpha_t(G) \).

A question that has received considerable attention is whether the simple bound \( \alpha_t(G) \geq \ell |V(G)|/\chi(G) \) can be extended to partial list coloring:

**Conjecture 1** (Partial List Coloring Conjecture [2]). For any graph \( G \) and \( t \in [\chi_L(G)] \),

\[
\alpha_t^L(G) \geq \frac{\ell |V(G)|}{\chi_t(G)}.
\]

Since \( \alpha_t^L(G) = |V(G)| \) for \( t \geq \chi_t(G) \), Conjecture [1] is true for \( t = \chi_t(G) \). One can easily verify that \( \alpha_t^L(G) = \alpha_t(G) \); hence Conjecture [1] holds for \( t = 1 \) as well. Some weaker general lower bounds are known: for all graphs \( G \) and \( t \in [\chi_t(G)] \), we have \( \alpha_t^L(G) \geq \ldots \)
for every \( u \), \( v \) among the requirements:

Furthermore, \( H \) consisting of a graph with \(|\alpha_t'(G)| \geq \frac{6}{7}(t|V(G)|/\chi_t'(G)) \) (see [12]), and \( \alpha_t'(G) \geq |V(G)|/(\lceil |\chi_t(G)/t| \rceil) \) (see [15] [16]). The Partial List Coloring Conjecture has been proven for bipartite graphs (by the bound from [2] [29]), graphs \( G \) with \( \Delta(G) \leq \chi_t(G) \) (see [17]), claw-free graphs, chordless graphs, chordal graphs, series parallel graphs, and graphs \( G \) satisfying \(|V(G)| \leq 2\chi(G) + 1 \) (see [18]). Iradmusa [16] also showed that for every graph \( G \), the inequality in Conjecture [11] holds for at least half the values of \( t \) in \([\chi_t(G) - 1]\).

Note that if every graph \( G \) contained a \( t \)-choosable induced subgraph of order at least \( t|V(G)|/\chi_t(G) \) whenever \( t \in [\chi_t(G)] \), it would immediately imply the Partial List Coloring Conjecture. However, that statement does not hold true, since it is known that there is an infinite family of \( 3 \)-choosable graphs \( G \) and Postle introduced DP-coloring (which they called correspondence coloring introduced in 2015 by Dvořák and Postle [13]. Dvořák and Postle introduced DP-coloring (which they called correspondence coloring) as part of a proof that planar graphs without cycles of lengths 4 to 8 are \( 3 \)-choosable. Intuitively, DP-coloring considers the worst-case scenario of how many colors we need in the lists if we no longer can identify the names of the colors. DP-coloring has been extensively studied over the past 5 years (see e.g., [3, 4, 5, 6, 7, 8, 9, 19, 21, 22, 23, 24, 25, 26, 27]).

DP-coloring generalizes the following model of list coloring (as explained in [7]): Given a graph \( G \) and list assignment \( L \), create a new graph \( H \) such that each list \( L(v) \) is a clique in \( H \) and for each edge \( uv \) in \( G \), there is a matching from \( L(u) \) to \( L(v) \) that joins pairs of vertices with the same color. Then independent sets in \( H \) of size \(|V(G)|\) correspond to proper \( L \)-colorings of \( G \). In particular, each edge between cliques \( L(u) \) and \( L(v) \) in \( H \) represents one color that must be forbidden from being chosen for both \( u \) and \( v \). DP-coloring generalizes that model by allowing arbitrary matchings between each pair of cliques \( L(u) \) and \( L(v) \). Then the so-called colors in the lists are no longer consistent. For example, suppose \( \{u, v, w\} \) is a clique in \( G \). Then, \( c \in L(u) \) might be matched with \( c' \in L(v) \) and \( c'' \in L(w) \), which in the original model means that \( c \) is the same color as \( c' \) and \( c'' \), and yet \( c' \) need not be matched with \( c'' \); \( c' \) could be matched to some other element of \( L(w) \) instead.

We now give the formal definition, following [7]. A cover of a graph \( G \) is a pair \( \mathcal{H} = (L, H) \) consisting of a graph \( H \) and a function \( L : V(G) \rightarrow \mathcal{P}(V(H)) \) satisfying the following four requirements:

1. the sets \( \{L(u) : u \in V(G)\} \) form a partition of \( V(H) \),
2. for every \( u \in V(G) \), the graph \( H[L(u)] \) is complete,
3. for every \( uv \in E(G) \), the edge set \( E_H(L(u), L(v)) \) is a matching (possibly empty),
4. for \( u, v \in V(G) \) with \( uv \notin E(G) \) and \( u \neq v \), \( E_H(L(u), L(v)) = \emptyset \).

Furthermore, \( \mathcal{H} \) is an \( m \)-fold cover if \(|L(u)| = m \) for each \( u \in V(G) \).

An \( \mathcal{H} \)-coloring of \( G \) is defined to be an independent set in \( H \) of size \(|V(G)|\), i.e., an
independent set in \( H \) with exactly one vertex in \( L(u) \) for each \( u \in V(G) \). The \textit{DP-chromatic number} of a graph \( G \), denoted \( \chi_{DP}(G) \), is the smallest \( m \) such that every \( m \)-fold cover \( \mathcal{H} \) of \( G \) has an \( \mathcal{H} \)-coloring.

Since \( L \)-colorings for any \( m \)-assignment \( L \) for \( G \) can be modeled by \( \mathcal{H} \)-colorings of an \( m \)-fold cover \( \mathcal{H} \), as described previously, \( \chi(G) \leq \chi_{DP}(G) \). This inequality can be strict, for example, \( \chi_{DP}(C_n) = 3 \) for any cycle \( C_n \) \footnote{The coloring number of a graph \( G \) is the smallest integer \( d \) such that there exists an ordering \( v_1, \ldots, v_n \) of the vertices of \( G \) so that \( v_i \) has at most \( d - 1 \) neighbors preceding it in the ordering for each \( i \in [n] \).} but \( \chi(L(C_n)) = 2 \) when \( n \) is even \footnote{Dvořák and Postle \cite{DP-coloring} observed that Brooks' Theorem extends to DP-coloring: \( \chi_{DP}(G) \leq \Delta(G) \) provided that \( G \) is connected and neither a cycle nor a complete graph. Also note that DP-coloring is monotone: \( \chi_{DP}(H) \leq \chi_{DP}(G) \) when \( H \) is a subgraph of \( G \).}.

We further explore these concepts in the rest of this manuscript.

for all \( t \in \mathbb{N} \) such that there exists an ordering \( v_1, \ldots, v_n \) of the vertices of \( G \) so that \( v_i \) has at most \( d - 1 \) neighbors preceding it in the ordering for each \( i \in [n] \).

We now make some basic observations. For DP-chromatic number and for partial DP-\( t \)-chromatic number, it suffices to consider coverings where \( E_H(L(u), L(v)) \) is a perfect matching for all \( uv \in E(G) \). Note that \( \alpha_{DP}^t(G) = |V(G)| \) if and only if \( t \geq \chi_{DP}(G) \), \( \alpha_{DP}^t(G) = \alpha(G) \), and \( \alpha_{DP}^t(G) \leq \alpha^t(G) \leq \alpha_t(G) \). For any induced subgraph \( G[S] \) with DP-chromatic number at most \( t \), we have \( \alpha_{DP}^t(G) \geq |S| \); if one cannot do better, then we would have a simpler interpretation of the partial DP-\( t \)-chromatic number.

**Question 2.** For any graph \( G \) and \( t \in \mathbb{N} \), does \( \alpha_{DP}^t(G) \) always equal the largest possible order of an induced subgraph of \( G \) with DP-chromatic number at most \( t \)?

We will see that the answer to Question 2 is “no”. Another natural question is whether the bound \( \alpha_t(G) \geq t|V(G)|/\chi(G) \) has a DP-coloring analogue, similar to the Partial List Coloring Conjecture.

**Question 3.** For any graph \( G \), is it always the case that

\[
\alpha_{DP}^t(G) \geq \frac{t|V(G)|}{\chi_{DP}(G)} \quad (*)
\]

for all \( t \in [\chi_{DP}(G)] \)?

Since \( \alpha_{DP}^t(G) = |V(G)| \) for \( t \geq \chi_{DP}(G) \), (*) holds for \( t = \chi_{DP}(G) \). Since \( \alpha_{DP}^t(G) = \alpha(G) \), \( \chi_{DP}(G) \geq \chi(G) \), and \( \alpha(G) \geq |V(G)|/\chi(G) \), (*) holds for \( t = 1 \). Thus, the answer to Question 3 is yes for any graph \( G \) with \( \chi_{DP}(G) \leq 2 \), and for a graph \( G \) with \( \chi_{DP}(G) = 3 \), the answer is yes if and only if \( \alpha_{DP}^2(G) \geq \frac{2|V(G)|}{\chi_{DP}(G)} \). However, in general the answer to Question 3 is no. Hence, we will say that a graph is \textit{partially DP-nice} if (*) holds true for all \( t \in [\chi_{DP}(G)] \). We further explore these concepts in the rest of this manuscript.
1.3 Outline of the Paper and Open Questions

In Section 2 we study 2-fold covers. We will show that $\alpha_{DP}^2(V_8) = 6$ for the Wagner graph $V_8$ and that $V_8$ has no induced subgraph with DP-chromatic number 2 and order at least $\frac{2}{3}|V(V_8)|$ which answers Question 2. Additionally, this will show that $V_8$ is partially DP-nice. We answer Question 3 by presenting several examples of graphs $G$ with $\chi_{DP}(G) = 3$ and $\alpha_{DP}^2(G) < \frac{2|V(G)|}{3}$, including an infinite family of graphs on $5n$ vertices, and the cube graph $Q_3$ which is the only triangle-free subcubic graph that is not partially DP-nice. Since all our examples that violate (*) have $t = 2$ and $\chi_{DP}(G) = 3$, the following questions are natural.

**Question 4.** Are there graphs $G$ such that $\alpha_{DP}^2(G) < \frac{2|V(G)|}{\chi_{DP}(G)}$ with $\chi_{DP}(G) > 3$?

**Question 5.** For each $t \geq 4$, does there exist a graph $G$ such that $\chi_{DP}(G) = t$ and $G$ is not partially DP-nice?

We also consider planar graphs in Section 2 and observe that any nontrivial planar graph $G$ of girth at least 5 is partially DP-nice. In Section 3 we extend the ideas in [16] from list coloring to DP-coloring to make progress toward the inequality (*) and Question 3 including the following results.

**Theorem 6.** For any graph $G$ and $t \in [\chi_{DP}(G)]$,

$$\alpha_{DP}^t(G) \geq \frac{|V(G)|}{\lceil \chi_{DP}(G)/t \rceil}.$$  

It follows that (*) holds true whenever $t$ divides $\chi_{DP}(G)$. Hence Question 4 can be restricted to graphs $G$ where $\chi_{DP}(G)$ is odd.

**Theorem 7.** For any graph $G$, the inequality $\alpha_{DP}^t(G) \geq t|V(G)|/\chi_{DP}(G)$ holds true for at least half of the values of $t$ in $[\chi_{DP}(G) - 1]$.

The main tool in Section 3 is a subadditivity lemma.

**Lemma 8.** For any graph $G$ and $t_1, \ldots, t_k \in \mathbb{N}$,

$$\alpha_{DP}^t(G) \leq \sum_{i=1}^{k} \alpha_{DP}^{t_i}(G),$$

where $t = \sum_{i=1}^{k} t_i$.

In Section 4 we prove various classes of graphs are partially DP-nice, including chordal graphs and series-parallel graphs. We also consider the join of a graph with a complete graph. Specifically, we use Bernshteyn, Kostochka, and Zhu’s recent result [8] that for any graph $G$ there exists a threshold $N \leq 3|E(G)|$ such that $\chi_{DP}(G \lor K_p) = \chi(G \lor K_p)$ whenever $p \geq N$ to motivate and also help answer the question as to whether for such graphs, partial coloring and partial DP-coloring are similarly related.

**Theorem 9.** For any graph $G$, there exists a $p \in \mathbb{N}$ such that $G \lor K_p$ is partially DP-nice.
2 Two-Fold Covers

A feedback vertex set is a set of vertices in a graph whose removal yields an acyclic graph. The minimum size of a feedback vertex set in a graph $G$ is called the feedback vertex number and is denoted $\tau(G)$. An acyclic graph has coloring number at most 2, so it also has DP-chromatic number at most 2. Since every graph $G$ has an induced acyclic subgraph of order $|V(G)| - \tau(G)$ with DP-chromatic number at most 2, we get the general bound $\alpha_2^{DP}(G) \geq |V(G)| - \tau(G)$.

Combining this observation with known lower bounds on $\tau(G)$ yields immediate results. If $G$ is a planar graph then $\tau(G) \leq 3|V(G)|/5$ by [11], so $\alpha_2^{DP}(G) \geq 2|V(G)|/5$. There exist planar graphs with DP-chromatic number 5 (see [13]), so we see that such graphs satisfy inequality (*) for $t = 2$.

Any nontrivial planar graph $G$ of girth at least 5 satisfies $\tau(G) \leq (|V(G)| - 2)/3$ by [20] and $\chi_{DP}(G) \leq 3$ by [13], so $\alpha_2^{DP}(G) \geq (2|V(G)| + 1)/3 \geq 2|V(G)|/\chi_{DP}(G)$ and $G$ is partially DP-nice.

A graph is subcubic if its maximum degree is at most 3. So, $\chi_{DP}(G) \leq 3$ for any connected subcubic graph $G \neq K_4$; then $\tau(G) \leq 3(|V(G)| + 2)/8$ by [10], so $\alpha_2^{DP}(G) \geq (5|V(G)| - 2)/8$. Below we will present a 10-vertex connected subcubic graph $M$ with $\alpha_2^{DP}(M) = 6$, so this bound is sharp.

If $G$ is connected, subcubic and triangle-free, then $\tau(G) \leq (|V(G)| + 1)/3$ by [20], and $\tau(G) \leq |V(G)|/3$ by [31] unless $G$ is $V_5$ or $Q_3$. It follows that $\alpha_2^{DP}(G) \geq 2|V(G)|/3$ and $G$ is partially DP-nice whenever $G$ is a connected, subcubic, triangle-free graph, with the possible exceptions of $V_5$ and $Q_3$.

To get further, we will need an alternative characterization of 2-fold coverings. Consider any graph $G$ with a 2-fold cover $H = (L, H)$ such that $E_H(L(u), L(v))$ is a perfect matching for all $uv \in E(G)$. Without loss of generality, suppose that $L(v) = \{v^1, v^2\}$ for all $v \in V(G)$\footnote{From this point forward, whenever $H = (L, H)$ is a 2-fold cover, we assume the vertices of $H$ are named in this manner.}. Then for each $uv \in E(G)$, either $E_H(L(u), L(v))$ equals $\{v^1v^1, u^2v^2\}$ or $\{u^1v^2, u^2v^1\}$.

We define a twist representation of $H$ to be a function $f : E(G) \rightarrow \{0, 1\}$ such that $f(uv) = 0$ if $E_H(L(u), L(v)) = \{v^1v^1, u^2v^2\}$ and $f(uv) = 1$ if $E_H(L(u), L(v)) = \{u^1v^2, u^1v^1\}$.

We think of the second case as a “twist”. This is not unique for $H$ because the naming of elements in $L(v)$ is arbitrary; if $v^1$ and $v^2$ were switched then the value of $f$ would flip for all edges incident to $v$. Nevertheless, we obtain the following characterization.

Lemma 10. Suppose that $G$ is a graph with a 2-fold cover $H = (L, H)$ such that $E_H(L(u), L(v))$ is a perfect matching for each $uv \in E(G)$, and $f$ is a twist representation of $H$. Then $G$ has an $H$-coloring if and only if for every cycle $C$ in $G$,

$$\sum_{e \in E(C)} f(e) \equiv |E(C)| \pmod{2}.$$

Proof. Suppose that $G$ has an $H$-coloring (i.e., an independent set $S$ in $H$ such that for each $v \in V(G)$, exactly one of $v^1, v^2$ is in $S$). Let $s(v)$ represent the index of $v$ (i.e., for all $v \in V(G)$, $s(v) = 1$ if $v^1 \in S$ and $s(v) = 2$ if $v^2 \in S$). Then for each edge $uv \in E(G)$,
\( s(u) \neq s(v) \) if and only if \( f(uv) = 0 \). Hence any cycle \( C \) in \( G \) must have an even number of edges \( e \) with \( f(e) = 0 \). Then the number of edges \( e \) in \( C \) with \( f(e) = 1 \) has the same parity as \(|E(C)|\), as required.

Now let us assume that \( \sum_{e \in E(C)} f(e) \equiv |E(C)| \pmod{2} \) for every cycle \( C \) in \( G \). Let \( G' \) be obtained from \( G \) by contracting every edge \( e \) with \( f(e) = 1 \); note that \( G' \) may have multiple edges and loops. Every cycle \( C' \) in \( G' \) has a corresponding cycle \( C \) in \( G \) such that \( E(C') \subseteq E(C) \). Each edge \( e \in E(C) - E(C') \) has \( f(e) = 1 \), so \( \sum_{e \in E(C) - E(C')} f(e) = |E(C) - E(C')| \). Then \( \sum_{e \in E(C')} f(e) \equiv |E(C')| \pmod{2} \). Since \( \sum_{e \in E(C')} f(e) = 0 \), \( G' \) is an even cycle. Hence \( G' \) is bipartite.

For each \( v \in V(G') \), let \( s(v) = 1 \) or \( s(v) = 2 \) according to its partite set. We will uncontract edges to recover \( G \). Each time we uncontract a vertex \( u \) and obtain an edge \( uv \), give \( s(v) \) and \( s(w) \) the same value as \( s(u) \). At the end, \( s(v) \) is assigned a value for all \( v \in V(G) \) such that \( s(u) \neq s(v) \) whenever \( f(uv) = 0 \) and \( s(u) = s(v) \) whenever \( f(uv) = 1 \). Then \( \{v^{s(v)} : v \in V(G)\} \) is an \( \mathcal{H} \)-coloring of \( G \), as required.

We often apply Lemma \([10]\) to consider not just \( \mathcal{H} \)-colorings but partial \( \mathcal{H} \)-colorings, as follows. Start with the hypotheses of Lemma \([10]\) — that \( G \) is a graph with a 2-fold cover \( \mathcal{H} = (L, H) \) such that \( E_H(L(u), L(v)) \) is a perfect matching for each \( uv \in E(G) \), and \( f \) is a twist representation of \( \mathcal{H} \). Next, let \( G' \) be any induced subgraph of \( G \), let \( H' = H[\bigcup_{v \in V(G')} L(v)] \), let \( L' \) be \( L \) restricted to \( V(G') \), and let \( \mathcal{H}' = (L', H') \) which is a 2-fold covering of \( G' \). Then restricting \( f \) to \( E(G') \) gives a twist representation of \( \mathcal{H}' \), so \( G' \) has an \( \mathcal{H}' \)-coloring if and only if every cycle \( C \) in \( G' \) satisfies \( \sum_{e \in E(C)} f(e) \equiv |E(C)| \pmod{2} \).

We will now construct an infinite class of examples for which the answer to Question \(8\) is no. Let \( G \) be the complete graph on 4 vertices \( u, v, x, y \) with one edge \( xy \) subdivided with degree 2 vertex \( z \). Note that \( \tau(G) = 2 \), which immediately yields \( \alpha^{DP}_2(G) \geq 3 \). Let \( f(yz) = 1 \) and let \( f \) be 0 on all other edges of \( G \). Suppose \( \mathcal{H} = (L, H) \) is a 2-fold cover of \( G \) such that \( f \) is a twist representation of \( \mathcal{H} \). Note that each 3-cycle in \( G \) has even sum of \( f(e) \) over its edges. Furthermore, the 4-cycles in \( G \) with vertices in cyclic order of \( v, x, z, y \) and \( u, x, z, y \) each has odd sum of \( f(e) \) over its edges. Any induced subgraph \( G' \) of \( G \) on 4 vertices will contain one of those cycles. So applying Lemma \([10]\) to \( G' \) and the corresponding restriction of \( \mathcal{H} \), we conclude that \( \mathcal{H} \) has no independent set of size 4 (i.e., \( \alpha^{DP}_2(G) < 4 \)). So, \( \alpha^{DP}_2(G) = 3 \).

Now pick any \( n \geq 2 \) and let \( G_i \) be a copy of that same graph \( G \) for \( i \in [n] \) such that \( V(G_i) = \{u_i, v_i, x_i, y_i, z_i\} \) and the function \( f : V(G) \to V(G_i) \) given by \( f(t) = t_i \) for each \( t \in V(G) \) is a graph isomorphism. Then for all \( 1 \leq i \leq n - 1 \), add the edge \( z_i u_{i+1} \), and let \( G^* \) be the resulting graph. Clearly \( \tau(G^*) = 2n \), so \( \alpha^{DP}_2(G^*) \geq 3n \). By letting \( f(y_iz_i) = 1 \) for all \( i \in [n] \) and letting \( f \) be 0 on other edges of \( G^* \), then as above we have \( \sum_{e \in E(C)} f(e) \not\equiv |E(C)| \pmod{2} \) for each appropriate cycle in each \( G_i \). Since \( \tau(G^*) = 2n \) and \( |V(G^*)| = 5n \), any induced subgraph \( G' \) with \( 3n + 1 \) vertices will contain one of these cycles, so applying Lemma \([10]\) to \( G' \) allows us to conclude that \( \alpha^{DP}_2(G^*) < 3n + 1 \). Thus, \( \alpha^{DP}_2(G^*) = 3n \).

Note that \( G^* \) has coloring number 3, which can be seen by considering vertices in the order \( u_1, v_1, x_1, y_1, z_1, u_2, \ldots, z_n \); hence \( \chi^{DP}(G^*) \leq 3 \). Since cycles have DP-chromatic number 3 and \( G^* \) contains a cycle, \( \chi^{DP}(G^*) \geq 3 \). So, \( \chi^{DP}(G^*) = 3 \). It follows that \( G^* \) is not partially DP-nice, since \( \alpha^{DP}_2(G^*) = 3n < 2|V(G^*)|/\chi^{DP}(G^*) = 10n/3 \).
Corollary 11. For each $n \geq 1$, there is a graph $G^*$ on $5n$ vertices that is not partially DP-nice because $\alpha_{DP}^2(G^*) = 3n < 2|V(G^*)|/3 = 10n/3$.

Finally, notice that if we take $G_1$ and $G_2$ and add an edge between $z_1$ and $z_2$, we obtain a 10-vertex connected subcubic (in fact 3-regular) graph $M$ with $\alpha_{DP}^2(M) = 6$. This demonstrates the sharpness of the bound $\alpha_{DP}^2(G) \geq (5|V(G)| - 2)/8$ for any connected subcubic graph $G$ other than $K_4$ which was mentioned above.

Recall that there are two triangle-free subcubic graphs which we have not shown to be partially DP-nice: the cube $Q_3$ and the Wagner graph $V_8$.

The cube graph $Q_3$ contains a cycle $C$ so $\chi_{DP}(Q_3) \geq \chi_{DP}(C) = 3$ and it is subcubic so $\chi_{DP}(Q_3) \leq \Delta(Q_3) = 3$. Clearly $\tau(Q_3) = 3$, so $\alpha_{DP}^2(Q_3) \geq 5$. We will use a twist representation to show that $\alpha_{DP}^2(Q_3) \leq 5$. It follows that $Q_3$ is not partially DP-nice, since $\alpha_{DP}^2 = 5 < 2|V(Q_3)|/\chi_{DP}(Q_3) = 16/3$.

Proposition 12. The cube graph $Q_3$ is not partially DP-nice and $\alpha_{DP}^2(Q_3) = 5$.

Proof. By the above discussion, it remains to show that $\alpha_{DP}^2(Q_3) \leq 5$. Suppose we construct a copy of $Q_3$ from the following 4-cycles (vertices are written in cyclic order): $x, y, z, w$, and $x', y', z', w'$, and we add edges that join corresponding vertices (i.e., $x$ to $x'$, $y$ to $y'$, etc.). Define $f : E(Q_3) \rightarrow \{0, 1\}$ by letting $f(xy) = f(yz) = f(zw) = f(w'x') = 1$ and letting $f = 0$ on the other 8 edges. Suppose that $H = (L, H)$ is a 2-fold cover of $Q_3$ such that $f$ is a twist representation of $H$.

Note that $Q_3$ has six 4-cycles, each of which has an odd number of edges with $f(e) = 1$. An induced subgraph on 6 vertices can omit all those cycles only by omitting a pair of “opposite corners”: $\{x, z\}, \{y, w\}, \{x', z\}$, or $\{y', w\}$. Removing such a pair yields an induced 6-cycle, and since each pair is incident to one or three edges $e$ with $f(e) = 1$, removing that pair leaves a 6-cycle with three edges or one edge $e$ with $f(e) = 1$. We have shown that any induced subgraph $G'$ of $G$ on 6 vertices will contain a cycle $C$ with $\sum_{e \in E(C)} f(e) \neq |E(C)| \pmod{2}$.

By applying Lemma [10] to $G'$ and the corresponding restriction of $H$, we conclude that $H$ has no independent set of size 6 (i.e., $\alpha_{DP}^2(G) < 6$).

Finally, we answer Question [2] in the negative using the Wagner graph $V_8$, also known as the Möbius ladder graph on 8 vertices. We may let $V(V_8) = \{v_1, \ldots, v_8\}$ and let $E(V_8) = \{v_1v_2, v_2v_3, \ldots, v_7v_8, v_8v_1\} \cup \{v_1v_5, v_2v_6, v_3v_7, v_4v_8\}$. Note that $V_8$, like $Q_3$, is subcubic and contains a cycle so $\chi_{DP}(V_8) = 3$. It is easy to check that $\tau(V_8) = 3$ (see [10]).

Proposition 13. $V_8$ is partially DP-nice and $\alpha_{DP}^2(V_8) \geq 6 > |V(V_8)| - \tau(V_8)$.

Proof. Let $G_1 = V_8 - \{v_3, v_8\}$, $G_2 = V_8 - \{v_6, v_8\}$, and $G_3 = V_8 - \{v_1, v_7\}$. Note that each of these contains exactly one cycle, respectively, $C_1$ with vertices (in cyclic order) of $v_1, v_2, v_6, v_5, v_2$ with vertices (in cyclic order) of $v_1, v_2, v_3, v_4, v_5$, and $C_3$ with vertices (in cyclic order) of $v_2, v_3, v_4, v_5, v_6$. Note that every edge in $C_1 \cup C_2 \cup C_3$ appears in exactly two of these cycles.

Suppose that $f$ is a twist representation of an arbitrary 2-fold cover $H = (L, H)$ of $V_8$. Let $H_1 = (L_1, H_1)$, $H_2 = (L_2, H_2)$, and $H_3 = (L_3, H_3)$ be appropriate restrictions of $H = (L, H)$ that are 2-fold covers of $G_1, G_2, G_3$, respectively. Since every edge in $C_1 \cup C_2 \cup C_3$ appears in exactly two of these cycles,

$$\sum_{e \in E(C_1)} f(e) + \sum_{e \in E(C_2)} f(e) + \sum_{e \in E(C_3)} f(e)$$
is even. Therefore it cannot be that \( \sum_{e \in E(C)} f(e) \) is odd and both \( \sum_{e \in E(C_2)} f(e) \) and \( \sum_{e \in E(C_3)} f(e) \) are even. Hence, \( \sum_{e \in E(C)} f(e) \equiv |E(C)| \pmod{2} \) for some \( C \in \{C_1, C_2, C_3\} \). Since each \( G_i \) contains only one cycle, every cycle in some \( G_i \) satisfies the equivalence, so that \( G_i \) has an \( \mathcal{H}_i \)-coloring. Thus \( \mathcal{H} \) has an independent set of size \( |V(G_i)| = 6 \). So, \( \alpha_2(V_8) \geq 6 \).

Then \( \alpha_2(V_8) \geq 6 > 16/3 = 2|V(V_8)|/\chi_{DP}(V_8) \) which means \( V_8 \) is partially DP-nice.

Thus, we can now conclude that every connected, subcubic, triangle-free graph, with the unique exception of \( Q_3 \), is partially DP-nice.

## 3 Subadditivity

In this section we extend the ideas in [16] to the context of DP-coloring, using the following subadditivity lemma.

**Lemma 8.** For any graph \( G \) and \( t_1, \ldots, t_k \in \mathbb{N} \),

\[
\alpha_t^{DP}(G) \leq \sum_{i=1}^{k} \alpha_{t_i}^{DP}(G),
\]

where \( t = \sum_{i=1}^{k} t_i \).

**Proof.** For each \( i \in [k] \), let \( \mathcal{H}_i = (L_i, H_i) \) be a \( t_i \)-fold cover of \( G \) for which \( \alpha(H_i) = \alpha_{t_i}^{DP}(G) \) such that \( H_1, \ldots, H_k \) are pairwise vertex disjoint. For each \( v \in V(G) \), let \( L(v) = \bigcup_{i=1}^{k} L_i(v) \). Let \( H \) be the union of \( H_1, \ldots, H_k \) with edges added so that each \( L(v) \) is a clique. Then \( \mathcal{H} = (L, H) \) is a \( t \)-fold cover of \( G \).

There is an independent set \( S \) in \( H \) of size \( \alpha_t^{DP}(G) \). For each \( i \in [k] \), let \( S_i = S \cap V(H_i) \). Then each \( S_i \) is an independent set in \( H_i \), so we have

\[
\alpha_t^{DP}(G) = |S| = \sum_{i=1}^{k} |S_i| \leq \sum_{i=1}^{k} \alpha(H_i) = \sum_{i=1}^{k} \alpha_{t_i}^{DP}(G).
\]

**Lemma 8.** quickly yields tools and bounds.

**Corollary 14.** Let \( G \) be a graph and \( s, t \in \mathbb{N} \) such that \( t \) divides \( s \).

If \( \alpha_s^{DP}(G) \geq \frac{s|V(G)|}{\chi_{DP}(G)} \), then \( \alpha_t^{DP}(G) \geq \frac{t|V(G)|}{\chi_{DP}(G)} \).

**Proof.** Let \( k \) be the integer such that \( s = kt \). By Lemma 8 \( \alpha_s^{DP}(G) \leq k \alpha_t^{DP}(G) \). Then \( \alpha_t^{DP}(G) \geq (1/k)s|V(G)|/\chi_{DP}(G) = t|V(G)|/\chi_{DP}(G) \) as desired.

Applying Corollary 14 with \( t = 2 \) and \( s = \chi_{DP}(G) \) shows that when \( \chi_{DP}(G) \) is even, inequality (\#) holds true. Hence, Question 4 can be restricted to graphs \( G \) where \( \chi_{DP}(G) \) is odd, something we already knew as it also follows from Theorem 6.
Theorem 6. For any graph \( G \) and \( t \in \lfloor \chi_{DP}(G) \rfloor \),
\[
\alpha_t^{DP}(G) \geq \frac{|V(G)|}{\lfloor \chi_{DP}(G)/t \rfloor}.
\]

Proof. Let \( s = t \lceil \chi_{DP}(G)/t \rceil \). Since \( s \geq \chi_{DP}(G) \), \( \alpha_s^{DP}(G) = |V(G)| \). By Lemma 2 \( \alpha_s^{DP}(G) \leq \lfloor \chi_{DP}(G)/t \rfloor \alpha_t^{DP}(G) \). The desired result follows.

The next result implies that at least half the elements \( t \in \lfloor \chi_{DP}(G) - 1 \rfloor \) satisfy inequality (*).

Corollary 15. For a graph \( G \) and an integer \( t \) with \( 1 \leq t < s = \chi_{DP}(G) \), either
\[
\alpha_t^{DP}(G) \geq \frac{t}{s}|V(G)| \quad \text{or} \quad \alpha_{s-t}^{DP}(G) \geq \frac{s-t}{s}|V(G)|.
\]

Proof. For a contradiction, suppose that \( \alpha_t^{DP}(G) < \frac{t}{s}|V(G)| \) and \( \alpha_{s-t}^{DP}(G) < \frac{s-t}{s}|V(G)| \). Then by Lemma 2 we obtain the contradiction:
\[
|V(G)| = \alpha_s^{DP}(G) \leq \alpha_t^{DP}(G) + \alpha_{s-t}(G) < \left( \frac{t}{s} + \frac{s-t}{s} \right)|V(G)| = |V(G)|.
\]

Applying Corollary 15 to each \( t \) with \( 1 \leq t \leq \lfloor (\chi_{DP}(G) - 1)/2 \rfloor \) will yield distinct \( t \)-values (either \( t \) or \( \chi_{DP}(G) - t \)) that satisfies (*), for a total of \( \lfloor (\chi_{DP}(G) - 1)/2 \rfloor \) such \( t \)-values.

4 Partially DP-nice Graphs

In this section we prove that some classes of graphs are partially DP-nice.

4.1 Chordal Graphs and Series Parallel Graphs

Recall that a graph family \( \mathcal{G} \) is a hereditary graph family if it is closed under taking induced subgraphs.

Proposition 16. Suppose that \( \mathcal{G} \) is a hereditary graph family such that for each \( G \in \mathcal{G} \), \( \chi(G) = \chi_{DP}(G) \). Then every \( G \in \mathcal{G} \) is partially DP-nice.

Proof. Let \( \mathcal{G} \) be such a graph family. Suppose that \( G \in \mathcal{G} \) and let \( k = \chi(G) = \chi_{DP}(G) \). Consider any \( t \in [k] \) and any \( t \)-fold cover \( \mathcal{H} = (L, H) \) of \( G \).

Consider a proper \( k \)-coloring of \( G \). Let \( S \) be the union of \( t \) of the largest color classes. Then \( S \) is a subset of \( V(G) \) with \( |S| \geq (t/k)|V(G)| \). Let \( \mathcal{H}' = (L', H') \) be the corresponding \( t \)-fold cover of \( G[S] \), i.e., let \( L' \) be \( L \) restricted to \( S \) and let \( H' = H[\bigcup_{v \in S} L(v)] \). Note that \( \chi(G[S]) = \chi_{DP}(G[S]) \) since \( G \in \mathcal{G} \). So, \( H' \) contains an independent set \( S' \) of size \( |S'| \). Since \( H' \) is an induced subgraph of \( H \), \( S' \) is also an independent set in \( H \). Hence \( \alpha_t^{DP}(G) \geq |S'| \geq (t/k)|V(G)| \) and since \( t \) was arbitrarily chosen from \( [\chi_{DP}(G)] \), we have that \( G \) is partially DP-nice.
Chordal graphs are such a hereditary graph family, as \( \chi(G) = \omega(G) \leq \chi_{DP}(G) \leq \chi(G) \) for any chordal graph (see [19]). Recall that a graph \( G \) is chordal if every cycle \( C \) in \( G \) has a chord, which is an edge with endpoints on nonconsecutive vertices of \( C \). Thus, we get the following result.

**Corollary 17.** Chordal graphs are partially DP-nice.

Given any graph \( G \), its treewidth \( tw(G) \) can be defined in terms of a chordal graph \( M \) formed by adding edges to \( G \) so that \( M \) has smallest possible clique number; then \( tw(G) = \omega(M) - 1 \). For example, \( tw(G) \leq 1 \) if and only if a graph is a forest. Note that \( \chi_{DP}(G) \leq \chi_{DP}(M) = \omega(M) = tw(G) + 1 \).

**Proposition 18.** If \( G \) is a graph with \( \chi_{DP}(G) = tw(G) + 1 \), then \( G \) is partially DP-nice.

**Proof.** Let \( M \) be a chordal graph obtained by adding edges to \( G \) such that \( \omega(M) = tw(G) + 1 \). Since \( \chi_{DP}(G) \leq \chi_{DP}(M) = \omega(M) \) and \( \chi_{DP}(G) = tw(G) + 1 \), all these are equal. Let \( t \in [\chi_{DP}(G)] \) and let \( \mathcal{H} = (L,H) \) be an arbitrary \( t \)-fold cover of \( G \). Note that \( \mathcal{H} \) is also a \( t \)-fold cover of \( M \). By Corollary 17, \( H \) has an independent set of size at least \( t|V(M)|/\chi_{DP}(M) \), which equals \( t|V(G)|/\chi_{DP}(G) \) as required. \( \square \)

**Series-parallel graphs** are the graphs with treewidth at most 2. A series parallel graph \( G \) which contains a cycle \( C \) has \( \chi_{DP}(G) \geq \chi_{DP}(C) = 3 \), so Proposition 18 applies, showing that \( G \) is partially DP-nice. Any acyclic graph \( G \) has coloring number at most 2, so \( \chi_{DP}(G) \leq 2 \) which we have noted means that it must be partially DP-nice.

**Corollary 19.** Series-parallel graphs are partially DP-nice.

### 4.2 Join of a Graph with a Complete Graph

Interestingly, partial DP-coloring gets easier when we join a vertex to a graph and the DP-chromatic number of the resulting graph is higher than the DP-chromatic number of the original graph. Proposition 21 illustrates this idea. First, we need a basic result.

**Proposition 20.** If \( v \) is a vertex in a graph \( G \), then \( \chi_{DP}(G) - 1 \leq \chi_{DP}(G - v) \leq \chi_{DP}(G) \).

**Proof.** Clearly, \( \chi_{DP}(G - v) \leq \chi_{DP}(G) \). So, we must show that \( \chi_{DP}(G - v) \geq \chi_{DP}(G) - 1 \).

Let \( t = 1 + \chi_{DP}(G - v) \) and let \( \mathcal{H} = (L,H) \) be any \( t \)-fold cover of \( G \). Pick a vertex \( v' \in L(v) \). Note that \( v' \) has at most one neighbor in each \( L(u) \) with \( u \neq v \). Construct \( H' \) from \( H \) by removing all of \( L(v) \), the neighbors of \( v' \) in lists \( L(u) \) with \( u \neq v \), and one vertex from each list \( L(u) \) that has had nothing removed from it yet. Let \( L' \) be \( L \) with the same vertices removed and domain \( V(G) - v \). Let \( \mathcal{H}' = (L',H') \). Then \( \mathcal{H}' \) is a \( (t-1) \)-fold cover of \( G - v \). Since \( t - 1 = \chi_{DP}(G - v) \), there is an \( \mathcal{H}' \)-coloring of \( G' \) which is an independent set \( S' \) in \( H' \) of size \( |V(G - v)| = |V(G)| - 1 \). Then \( S' \cup \{v'\} \) is an independent set in \( H \) and it is an \( \mathcal{H} \)-coloring of \( G \). It follows that \( \chi_{DP}(G) \leq t \) as required. \( \square \)

**Proposition 21.** Suppose that a graph \( G \) is partially DP-nice. Let \( G' = G \cup K_1 \). If \( \chi_{DP}(G') > \chi_{DP}(G) \), then \( G' \) is partially DP-nice.
Proof. Suppose that \( \chi_{DP}(G) = m \). By Proposition 20, it must be that \( \chi_{DP}(G') = m + 1 \). Suppose that \( t \in \mathbb{N} \) satisfies \( 2 \leq t \leq m \). Also, suppose that \( \mathcal{H} = (L, H) \) is a \( t \)-fold cover of \( G' \). Let \( H' = H[\bigcup_{v \in V(G)} L(v)] \). Since \( G \) is partially DP-nice, we know there is an independent set \( I \) in \( H' \) of size at least \( tn/m \), where \( n = |V(G)| \). Since \( n \geq m \), we know that
\[
\frac{tn}{m} \geq \frac{t(n+1)}{m+1}.
\]
Consequently, \( I \) is an independent set in \( H \) of size at least \( t(n+1)/(m+1) \). The desired result follows.

We know from [8] that for any graph \( G \), there exists \( \mu \in \mathbb{N} \) such that \( \chi_{DP}(G \lor K_\mu) = \chi(G \lor K_\mu) = \chi(G) + \mu \), which is what led us to study whether for such graphs, partial DP-niceness might behave like its ordinary coloring analogue.

**Proposition 22.** For any graph \( G \) and any \( p \geq \mu \), \( \chi_{DP}(G \lor K_p) = \chi(G) + p \).

**Proof.** We know that \( \chi_{DP}(G \lor K_\mu) = \chi(G) + \mu \). For a proof by induction, suppose that \( \chi_{DP}(G \lor K_p) = \chi(G) + p \) for some \( p \geq \mu \). Then
\[
\chi_{DP}(G \lor K_{p+1}) \geq \chi(G \lor K_{p+1}) = \chi(G) + p + 1 = \chi_{DP}(G \lor K_p) + 1.
\]
We get \( \chi_{DP}(G \lor K_{p+1}) - 1 \leq \chi_{DP}(G \lor K_p) \) by Proposition 20. Therefore, \( \chi_{DP}(G \lor K_{p+1}) = \chi(G) + p + 1 \), as required.

The rest of this manuscript will be devoted to proving Theorem 9. If \( G \) is a complete graph, then \( G \lor K_p \) is itself a complete graph \( K_q \). Since \( \beta_{DP}(K_q) = t = |V(K_q)|/\chi_{DP}(K_q) \), \( K_q \) is partially DP-nice. Thus, we may assume that \( G \) is an \( n \)-vertex graph that is not complete. Note that then \( \chi(G) < n \).

Let \( G_0 = G \) and for each \( p \geq 1 \), let \( G_p = G_{p-1} \lor K_1 \). Then \( G_{p-1} \) is a subgraph of \( G_p \), and \( G_p \) is a copy of \( G \lor K_p \) for all \( p \geq 1 \). We may fix \( \mu \in \mathbb{N} \) as in [8] and Proposition 22. Then for all \( p \geq \mu \), we have \( |V(G_p)|/\chi_{DP}(G_p) = (n+p)/(k+p) \), where we have let \( k = \chi(G) \).

For each \( p \geq \mu \), let \( B_p \) be the set of \( t \in [k+p] \) for which \( \alpha_{DP}^{\mu}(G_p) < t(n+p)/(k+p) \). In other words, \( t \in B_p \) if and only if there is a \( t \)-fold cover \( \mathcal{H} = (L, H) \) of \( G_p \) such that \( H \) has no independent set \( S \) with \( |S| \geq t(n+p)/(k+p) \). We know that \( B_p \) does not contain \( 1 \) or \( k + p \). Note that \( G_p \) is partially DP-nice if and only if \( B_p = \emptyset \).

**Proposition 23.** If \( p \geq \mu \), \( B_{p+1} \subseteq B_p \).

**Proof.** Suppose for the sake of contradiction that there exists \( t \in B_{p+1} - B_p \). We know that \( 1 < t < k + p + 1 \). There must be a \( t \)-fold cover \( \mathcal{H} = (L, H) \) of \( G_{p+1} \) such that \( H \) has no independent set \( S \) with \( |S| \geq t(n+p+1)/(k+p+1) \). Let \( H' = H[\bigcup_{v \in V(G_p)} L(v)] \), let \( L' \) be \( L \) restricted to \( V(G_p) \), and let \( \mathcal{H}' = (L', H') \); then \( \mathcal{H}' \) is a \( t \)-fold cover of \( G_p \). Since \( t \notin B_p \) and \( t \leq k + p \), there must be an independent set \( S \) of size at least \( t(n+p)/(k+p) \) in \( H' \). Note that \( S \) is an independent set in \( H \) as well and
\[
\frac{t(n+p)}{k+p} \geq \frac{t(n+p+1)}{k+p+1}
\]
since \( k < n \) which is a contradiction. □

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4This also follows from Theorem 2.1 of [8], but we include an argument, including Proposition 20 and its proof, for completeness.
Now we show that there exists $p$ such that $B_p = \emptyset$ which will complete the proof of Theorem 9.

Proof. If not, then by Proposition 23 there exists $t \in B_p$ for all $p \geq \mu$. Since $\lim_{p \to \infty} \frac{(n + p)}{(k + p)} = 1$, there exists $p \geq \mu$ such that

$$\frac{t(n + p)}{k + p} < t + 1.$$

Since $t \in B_p$, $\alpha^D_{t}(G_p) < t + 1$. So, there is a $t$-fold cover $\mathcal{H} = (L, H)$ of $G_p$ with no independent set of size $t + 1$. Since $1 < t < k + p + 1$ and $k < n$, we have $t + 1 \leq k + p + 1 \leq n + p = |V(G_p)|$. Since $G_p$ is not a complete graph, $G_p$ has an induced subgraph $G'$ on $t + 1$ vertices that is not a complete graph. The coloring number of $G'$ is at most $t$, so for every $t$-fold cover $\mathcal{H}'$ of $G'$, there is an $\mathcal{H}'$-coloring. In particular, $H$ has an independent set of size $t + 1$ which is a contradiction. 

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