On representations of the exceptional superconformal algebra $CK_6$

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We realize the exceptional superconformal algebra $CK_6$, spanned by 32 fields, inside the Lie superalgebra of pseudodifferential symbols on the supercircle $S^{1|3}$. We obtain a one-parameter family of irreducible representations of $CK_6$ in a superspace spanned by 8 fields.

1. Introduction

A superconformal algebra is a simple complex Lie superalgebra $\mathfrak{g}$ spanned by the coefficients of a finite family of pairwise local fields $a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$, one of which is the Virasoro field $L(z)$, [3, 8, 11]. Superconformal algebras play an important role in the string theory and conformal field theory.

The Lie superalgebras $K(N)$ of contact vector fields with Laurent polynomials as coefficients (with $N$ odd variables) is a superconformal algebra which is characterized by its action on a contact 1-form, [3, 6, 8, 12]. These Lie superalgebras are also known to physicists as the $SO(N)$ superconformal algebras, [1]. Note that $K(N)$ is spanned by $2^N$ fields. It is simple if $N \neq 4$, if $N = 4$, then the derived Lie superalgebra $K'(4)$ is simple. The nontrivial central extensions of $K(1)$, $K(2)$ and $K'(4)$ are well-known: they are isomorphic to the so-called Neveu-Schwarz superalgebra, “the $N = 2$”, and “the big $N = 4$” superconformal algebra, respectively, [1].

It was discovered independently in [3] and [17] that the Lie superalgebra of contact vector fields with polynomial coefficients in 1 even and 6 odd variables contains an exceptional simple Lie superalgebra (see also [6, 9, 10, 18, 19]).

In [3] a new exceptional superconformal algebra spanned by 32 fields was constructed as a subalgebra of $K(6)$, and it was denoted by $CK_6$. It was proven that $CK_6$ has no nontrivial central extensions. It was also pointed out that $CK_6$ appears to be the only new superconformal algebra, which completes their list (see [11, 12]).
In this work we realize $CK_6$ inside the Poisson superalgebra of pseudodifferential symbols on the supercircle $S^{1|3}$. It is known that a Lie algebra of contact vector fields can be realized as a subalgebra of Poisson algebra, [2]. In particular, the Lie algebra $Vect(S^1)$ of complex polynomial vector fields on the circle has a natural embedding into the Poisson algebra $P$ of formal Laurent series on the cylinder $T^*S^1 \setminus S^1$. One can consider a family of Lie algebras $P_h$, $h \in [0,1]$, having the same underlying vector space, which contracts to $P$, [13-16].

Analogously, $K(2N)$ is embedded into the Poisson superalgebra $P(2N)$ of pseudodifferential symbols on the supercircle $S^{1|N}$, and there is a family of Lie superalgebras $P_h(2N)$, which contracts to $P(2N)$ (see [20]).

A natural question is whether there exists an embedding

$$K(2N) \subset P_h(2N). \quad (1.1)$$

Recall that the answer is “yes” if $N = 2$, more precisely, there exists an embedding of a nontrivial central extension of $K'(4) = [K(4), K(4)]$:

$$\hat{K}'(4) \subset P_h(4). \quad (1.2)$$

Associated with this embedding, there is a one-parameter family of irreducible representations of $\hat{K}'(4)$ realized on 4 fields, [20].

Note that embedding (1.1) doesn’t hold if $N > 2$, [5]. However, it is remarkable that it is possible to embed $CK_6$, which is “one half” of $K(6)$, into $P_h(6)$. In this work we construct this embedding, and obtain the corresponding one-parameter family of representations of $CK_6$ realized on 8 fields.

2. Contact superconformal algebra $K(2N)$

Let $\Lambda(2N)$ be the Grassmann algebra in $2N$ variables $\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N$, and let $\Lambda(1,2N) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(2N)$ be an associative superalgebra with natural multiplication and with the following parity of generators: $p(t) = 0$, $p(\xi_i) = p(\eta_i) = 1$ for $i = 1, \ldots, N$. Let $W(2N)$ be the Lie superalgebra of all derivations of $\Lambda(1,2N)$. Let $\partial_t$, $\partial_{\xi_i}$ and $\partial_{\eta_i}$ stand for $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial \xi_i}$ and $\frac{\partial}{\partial \eta_i}$, respectively. By definition,

$$K(2N) = \{ D \in W(2N) \mid D\Omega = f\Omega \text{ for some } f \in \Lambda(1,2N) \}, \quad (2.1)$$

where $\Omega = dt + \sum_{i=1}^{N} \xi_i d\eta_i + \eta_i d\xi_i$ is a differential 1-form, which is called a contact form (see [3, 4, 6, 7, 8, 10, 12]). The Euler operator is defined by $E = \sum_{i=1}^{N} \xi_i \partial_{\xi_i} + \eta_i \partial_{\eta_i}$. 
We also define operators $\Delta = 2 - E$ and $H_f = (-1)^{p(f)+1} \sum_{i=1}^{N} \partial_{\xi_i} f \partial_{\eta_i} + \partial_{\eta_i} f \partial_{\xi_i}$, where $f \in \Lambda(1, 2N)$.

There is a one-to-one correspondence between the differential operators $D \in K(2N)$ and the functions $f \in \Lambda(1, 2N)$. The correspondence $f \leftrightarrow D_{f}$ is given by

$$D_{f} = \Delta(f) \frac{\partial}{\partial t} + \frac{\partial f}{\partial t} E - H_{f}.$$  \hspace{1cm} (2.2)

The contact bracket on $\Lambda(1, 2N)$ is

$$\{f, g\}_{K} = \Delta(f)\partial_{t}g - \partial_{t}f\Delta(g) - \{f, g\}_{P.b},$$  \hspace{1cm} (2.3)

where

$$\{f, g\}_{P.b} = (-1)^{p(f)+1} \sum_{i=1}^{N} \partial_{\xi_i} f \partial_{\eta_i} g + \partial_{\eta_i} f \partial_{\xi_i} g$$  \hspace{1cm} (2.4)

is the Poisson bracket. Thus $[D_{f}, D_{g}] = D_{\{f, g\}_{K}}$.

The superalgebra $K(6)$ contains an exceptional superconformal algebra, spanned by 32 fields, as a subalgebra. This superconformal algebra is denoted by $CK_6$ in [3, 8, 11]. Other notations are also used in the literature (see [6]). Let $\Theta = \xi_1 \xi_2 \tau \eta_1 \eta_2 \eta_3$. In what follows $(i, j, k) = (1, 2, 3)$ stays for the equality of cyclic permutations.

**Proposition 1** (see [3, 6]). $CK_6$ is spanned by the following 32 fields:

$$L_{n} = t^{n+1} - (\partial_{t})^{3} t^{n+1} \Theta,$$

$$G_{n}^{i} = t^{n+1} \xi_{i} + (\partial_{t})^{2} t^{n+1} \partial_{\eta_{i}} \Theta, \quad \tilde{G}_{n}^{i} = t^{n} \eta_{i} + (\partial_{t})^{2} t^{n} \partial_{\xi_{i}} \Theta, \quad i = 1, 2, 3,$$

$$T_{n}^{ij} = t^{n} \xi_{i} \eta_{j} - (\partial_{t}) t^{n} \partial_{\eta_{i}} \partial_{\xi_{j}} \Theta, \quad i \neq j, \quad T_{n}^{i} = t^{n} \xi_{i} \eta_{i} - (\partial_{t}) t^{n} \partial_{\eta_{i}} \partial_{\xi_{i}} \Theta, \quad i = 1, 2, 3,$$

$$S_{n}^{i} = t^{n} \xi_{i} (\xi_{j} \eta_{j} + \xi_{k} \eta_{k}), \quad \tilde{S}_{n}^{i} = t^{n-1} \eta_{i} (\xi_{j} \eta_{j} - \xi_{k} \eta_{k}), \quad i = 1, 2, 3,$$

$$I_{n}^{i} = t^{n-1} \xi_{i} \eta_{j} \eta_{k}, \quad i = 1, 2, 3, \quad I_{n} = t^{n+1} \xi_{1} \xi_{2} \xi_{3},$$

$$J_{n}^{ij} = t^{n+1} \xi_{i} \xi_{j} - (\partial_{t}) t^{n+1} \partial_{\eta_{i}} \partial_{\eta_{j}} \Theta, \quad \tilde{J}_{n}^{ij} = t^{n-1} \eta_{i} \eta_{j} - (\partial_{t}) t^{n-1} \partial_{\xi_{i}} \partial_{\xi_{j}} \Theta, \quad i < j,$$

where $n \in \mathbb{Z}$, and $(i, j, k) = (1, 2, 3)$ in the formulae for $S_{n}^{i}, \tilde{S}_{n}^{i}$ and $I_{n}^{i}$.

3. **The Poisson superalgebra $P(2N)$ of pseudodifferential symbols on $S^{1|N}$**

The Poisson algebra $P$ of pseudodifferential symbols on the circle is formed by the formal series $A(t, \xi) = \sum_{-\infty}^{n} a_{i}(t) \xi^{i}$, where $a_{i}(t) \in \mathbb{C}[t, t^{-1}]$, and the variable $\xi$ corresponds to $\partial_{t}$. The Poisson bracket is defined as follows:

$$\{A(t, \xi), B(t, \xi)\} = \partial_{\xi} A(t, \xi) \partial_{t} B(t, \xi) - \partial_{t} A(t, \xi) \partial_{\xi} B(t, \xi).$$  \hspace{1cm} (3.1)
The Poisson algebra \( P \) has a deformation \( P_\hbar \), where \( \hbar \in [0, 1] \). The associative multiplication in the vector space \( P \) is determined as follows:
\[
A(t, \xi) \circ_\hbar B(t, \xi) = \sum_{n \geq 0} \frac{\hbar^n}{n!} \partial_\xi^n A(t, \xi) \partial_\xi^n B(t, \xi).
\] (3.2)

The Lie algebra structure on \( P_\hbar \) is given by \([A, B]_\hbar = A \circ_\hbar B - B \circ_\hbar A\), so that the family \( P_\hbar \) contracts to \( P \). \( P_{\hbar=1} \) is called the Lie algebra of pseudodifferential symbols on the circle, [13-16].

The Poisson superalgebra \( P(2N) \) of pseudodifferential symbols on the supercircle \( S^{1|N} \) has the underlying vector space \( P \otimes \Lambda(2N) \). The Poisson bracket is defined as follows:
\[
\{A, B\} = \partial_\xi A \partial_\xi B - \partial_\xi A \partial_\xi B + \{A, B\}_{PB}.
\] (3.3)

Let \( A_\hbar(2N) \) be an associative superalgebra with generators \( \xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N \) and relations: \( \xi_i \xi_j = -\xi_j \xi_i, \eta_i \eta_j = -\eta_j \eta_i, \eta_i \xi_j = h \delta_{i,j} - \xi_j \eta_i \). Let \( P_\hbar(2N) = P_\hbar \otimes A_\hbar(2N) \) be an associative superalgebra, where the product of \( A = A_1 \otimes X \) and \( B = B_1 \otimes Y \), where \( A_1, B_1 \in P_\hbar \), and \( X, Y \in A_\hbar(2N) \), is given by
\[
AB = \frac{1}{\hbar}(A_1 \circ_\hbar B_1) \otimes (XY).
\] (3.4)

Correspondingly, the Lie bracket in \( P_\hbar(2N) \) is \([A, B]_\hbar = AB - (-1)^{p(A)p(B)} BA\), and \( \lim_{\hbar \to 0}[A, B]_\hbar = \{A, B\} \). There exist natural embeddings: \( W(N) \subset P(2N) \) and \( W(N) \subset P_\hbar(2N) \), where \( W(N) \) is the Lie superalgebra of all derivations of \( C[t, t^{-1}] \otimes \Lambda(\xi_1, \ldots, \xi_N) \), so that the commutation relations in \( P(2N) \) and in \( P_\hbar(2N) \), when restricted to \( W(N) \), coincide with the commutation relations in \( W(N) \). \( P_{\hbar=1}(2N) \) is called the Lie superalgebra of pseudodifferential symbols on \( S^{1|N} \) (see [20]).

4. Realization of \( CK_6 \) inside the Poisson superalgebra

Theorem 1. The superalgebra \( CK_6 \) is spanned by the following 32 fields inside \( P(2N) \):
\[
L_{n,0} = t^{n+1} \xi,
\] (4.1)
\[
G^i_{n,0} = t^{n+1} \xi_i, \quad \tilde{G}^i_{n,0} = t^n \eta_i - nt^{n-1} \xi^{-1} \xi_j \eta_i \eta_j, \quad i = 1, 2, 3,
\]
\[
T^{ij}_{n,0} = t^n \xi_i \eta_j - nt^{n-1} \xi^{-1} \xi_k \xi_i \eta_k \eta_j, \quad i \neq j \neq k,
\]
\[
T^i_{n,0} = -t^n (\xi_i \eta_j + \xi_k \eta_k) + nt^{n-1} \xi^{-1} \xi_j \xi_k \eta_j \eta_k, \quad i = 1, 2, 3,
\]
\[
S^i_{n,0} = -t^n \xi_i (\xi_j \eta_j + \xi_k \eta_k) + nt^{n-1} \xi^{-1} \xi_j \xi_k \xi_k \eta_j \eta_k, \quad i = 1, 2, 3,
\]
\[
\tilde{S}^i_{n,0} = t^{n-1} \xi^{-1} (\xi_j \eta_j - \xi_k \eta_k) \eta_i, \quad i = 1, 2, 3,
\]
\[
I^i_{n,0} = t^{n-1} \xi^{-1} \xi_j \eta_j \eta_k, \quad i = 1, 2, 3, \quad I_{n,0} = t^{n+1} \xi_1 \xi_2 \xi_3,
\]
\[
J^{ij}_{n,0} = t^{n+1} \xi_i \xi_j, \quad \tilde{J}^{ij}_{n,0} = t^{n-1} \xi^{-1} \eta_i \eta_j, \quad i < j.
\]
where \( n \in \mathbb{Z} \), and \((i, j, k) = (1, 2, 3)\) in the formulae for \( \tilde{G}_{n,0}^i, T_{n,0}^i, S_{n,0}^i, \tilde{S}_{n,0}^i, \) and \( I_{n,0}^i \).

**Proof.** Note that there exists an embedding

\[
K(2N) \subset P(2N), \quad N \geq 0,
\]

see [20]. Consider a \( \mathbb{Z} \)-grading of the associative superalgebra

\[
P(2N) = \bigoplus_{i \in \mathbb{Z}} P_i(2N)
\]

defined by

\[
\deg \xi = \deg \eta_i = 1, \quad \text{for } i = 1, \ldots, N, \\
\deg t = \deg \xi_i = 0, \quad \text{for } i = 1, \ldots, N.
\]

With respect to the Poisson bracket,

\[
\{P_i(2N), P_j(2N)\} \subset P_{i+j-1}(2N).
\]

Thus \( P_i(2N) \) is a subalgebra of \( P(2N) \), and we will show that \( P_i(2N) \cong K(2N) \). Equivalently, \( P_i(2N) \) is singled out as the set of all (Hamiltonian) functions \( A(t, \xi, \xi_i, \eta_i) \in P(2N) \) such that the corresponding vector fields supercommute with the semi-Euler operator:

\[
[H_A, \xi \partial_{\xi} + \sum_{i=1}^{N} \eta_i \partial_{\eta_i}] = 0,
\]

where

\[
A(t, \xi, \xi_i, \eta_i) \rightarrow H_A = \partial_{\xi} A \partial_t - \partial_t A \partial_{\xi} - (-1)^{p(A)} \sum_{i=1}^{N} (\partial_{\xi_i} A \partial_{\eta_i} + \partial_{\eta_i} A \partial_{\xi_i}).
\]

To describe an isomorphism from \( K(2N) \) onto \( P_i(2N) \), we change the variable \( t \) in \( \Lambda(1|2N) \): \( t \xrightarrow{\kappa} 2t - \sum_{i=1}^{N} \xi_i \eta_i \). Correspondingly, we have the following contact bracket on \( \Lambda(1|2N) \):

\[
\{f, g\}_{\tilde{K}} = \tilde{\Delta}(f) \partial_t g - \partial_t f \tilde{\Delta}(g) - \{f, g\}_{P,b},
\]

where \( \tilde{\Delta} = 1 - \tilde{E} \) and \( \tilde{E} = \sum_{i=1}^{N} \eta_i \partial_{\eta_i} \). Note that the corresponding contact form is \( \tilde{\Omega} = dt + \sum_{i=1}^{N} \xi_i d\eta_i \). Define a map \( \varphi : \Lambda(1|2N) \rightarrow P_i(2N) \) as follows:

\[
f \xrightarrow{\varphi} A_f = (-1)^s \xi^{1-s} f,
\]
where $s$ is a scalar given by $\tilde{E}(f) = sf$. Then

$$\{A_f, A_g\} = A_{(f,g)} \tilde{K}.$$  

(4.10)

Applying the isomorphism $\psi = \varphi \circ \chi$ to the fields (2.5), we obtain the following fields:

$$\psi(L_n) = 2^{n+1}L_{n,0} - 2^{n-1}(n + 1)(T^1_{n,0} + T^2_{n,0} + T^3_{n,0}), \quad \psi(G^i_n) = 2^{n+1}G^i_{n,0} - 2^n(n + 1)S^i_{n,0},$$

$$\psi(\tilde{G}^i_n) = -2^nG^i_{n,0} + 2^{n-1}n\tilde{S}^i_{n,0}, \quad \psi(T^{ij}_n) = -2^nT^{ij}_{n,0}, \quad \psi(T^i_n) = 2^{n-1}(-T^i_{n,0} + T^j_{n,0} + T^k_{n,0}),$$

$$\psi(S^i_n) = 2^nS^i_{n,0}, \quad \psi(\tilde{S}^i_n) = 2^{n-1}\tilde{S}^i_{n,0}, \quad \psi(I^i_n) = 2^{n-1}I^i_{n,0}, \quad \psi(I_n) = 2^{n+1}I_{n,0},$$

$$\psi(J^{ij}_n) = 2^{n+1}J^{ij}_{n,0}, \quad \psi(\tilde{J}^{ij}_n) = 2^{n-1}\tilde{J}^{ij}_{n,0}. \quad (4.11)$$

□

5. Realization of $CK_6$ inside the Lie superalgebra of pseudodifferential symbols

Given the embedding (4.2) it is natural to ask whether there exists an embedding

$$K(2N) \subset P\hbar(2N). \quad (5.1)$$

Recall that if $N = 2$, then there is an embedding

$$\hat{K}'(4) \subset P\hbar(4), \quad (5.2)$$

where $K'(4) = [K(4), K(4)]$ is a simple ideal in $K(4)$ of codimension one defined from the exact sequence

$$0 \rightarrow K'(4) \rightarrow K(4) \rightarrow CD_{t^{-1}\xi, \xi_2, \eta_1, \eta_2} \rightarrow 0, \quad (5.3)$$

and $\hat{K}'(4)$ is a nontrivial central extension of $K'(4)$ (see [20]). The superalgebra $K'(4) \subset P(4)$ is spanned by the 12 fields:

$$f(\xi_1, \xi_2, t)\xi \text{ and } f(\xi_1, \xi_2, t)\eta_i \quad (i = 1, 2), \quad (5.4)$$

which form a subalgebra isomorphic to $W(2)$, together with 4 fields: $F^i_n$, where $i = 0, 1, 2, 3$, and $n \in \mathbb{Z}$:

$$F^0_n = t^{n-1}\xi^{-1}\eta_1\eta_2, \quad (5.5)$$

$$F^i_n = t^{n-1}\xi^{-1}\xi_i\eta_1\eta_2, \quad i = 1, 2,$$

$$F^3_n = t^{n-1}\xi^{-1}\xi_1\xi_2\eta_1\eta_2, \quad n \neq 0.$$
Proposition 2 ([20]). The superalgebra $\hat{K}'(4)$ in (5.2) is spanned by the 12 fields given in (5.4) together with 4 fields $F^i_{n,h}$:

\begin{align}
F^0_{n,h} &= (\xi^{-1} \circ_h t^{n-1})\eta_1\eta_2, \\
F^i_{n,h} &= (\xi^{-1} \circ_h t^{n-1})\eta_i\eta_2\xi_i, \quad i = 1, 2, \\
F^3_{n,h} &= (\xi^{-1} \circ_h t^{n-1})\eta_1\eta_2\xi_1\xi_2 + \frac{h}{n} t^n, \quad n \neq 0,
\end{align}

and the central element $h$, so that $\lim_{h\to 0}\hat{K}'(4) = K'(4) \subset P(4)$.

Note that we cannot obtain the embedding (5.1) if $N > 2$, [5]. However, the following theorem holds.

Theorem 2. There exists an embedding $CK_6 \subset P_h(6)$ for each $h \in [0, 1]$ such that $\lim_{h\to 0}CK_6 = CK_6 \subset P(6)$.

Proof. $CK_6$ is spanned by the following fields inside $P_h(6)$:

\begin{align}
L_{n,h} &= t^{n+1}\xi, \\
G^i_{n,h} &= t^{n+1}\xi_i, \quad \tilde{G}^i_{n,h} = t^n\eta_i - n\xi^{-1} \circ_h t^{n-1}\eta_i\eta_j\xi_j, \quad i = 1, 2, 3, \\
T^{ij}_{n,h} &= t^n\xi_i\eta_j - n\xi^{-1} \circ_h t^{n-1}\eta_k\eta_j\xi_k\xi_i, \quad i \neq j \neq k, \\
T^i_{n,h} &= -t^n(\xi_j\eta_j + \xi_k\eta_k) + n\xi^{-1} \circ_h t^{n-1}\eta_j\eta_k\xi_j(\xi_k + ht^n), \quad i = 1, 2, 3, \\
S^i_{n,h} &= -t^n(\xi_j(\xi_i\eta_j + \xi_k\eta_k)) + n\xi^{-1} \circ_h t^{n-1}\eta_j\eta_k\xi_i\xi_j\xi_k + ht^n\xi_i, \quad i = 1, 2, 3, \\
\tilde{S}^i_{n,h} &= \xi^{-1} \circ_h t^{n-1}(\eta_j\eta_i\xi_j - \eta_k\eta_i\xi_k), \quad i = 1, 2, 3, \\
I^i_{n,h} &= \xi^{-1} \circ_h t^{n-1}\eta_j\eta_k\xi_i, \quad i = 1, 2, 3, \quad I_{n,h} = t^{n+1}\xi_1\xi_2\xi_3, \\
J^{ij}_{n,h} &= t^{n+1}\xi_i\xi_j, \quad \tilde{J}^{ij}_{n,h} = \xi^{-1} \circ_h t^{n-1}\eta_i\eta_j, \quad i < j,
\end{align}

where $n \in \mathbb{Z}$, and $(i, j, k) = (1, 2, 3)$ in the formulae for $\tilde{G}^i_{n,h}, T^i_{n,h}, S^i_{n,h}, \tilde{S}^i_{n,h}$ and $I^i_{n,h}$. Let $h \in [0, 1]$. Set $J^{ij}_{n,h} = -\tilde{J}^{ij}_{n,h}$ and $\tilde{J}^{ij}_{n,h} = -\tilde{J}^{ij}_{n,h}$ for $i > j$. Given $h \in [0, 1]$, set

\begin{equation}
L_n := L_{n,h}, \quad \ldots, \quad \tilde{J}^{ij}_n := \tilde{J}^{ij}_{n,h}.
\end{equation}

Recall that if $h = 0$, then (5.8) gives elements (4.1). The nonvanishing commutation relations between the elements (5.8) are as follows: let $i \neq j \neq k$, then

\begin{align}
[L_n, L_m] &= (m - n)L_{n+m}, \quad [L_n, G^i_m] = (m - n)G^i_{n+m}, \quad [L_n, \tilde{G}^i_m] = m\tilde{G}^i_{n+m}, \\
[L_n, T^i_m] &= mT^i_{n+m}, \quad [L_n, T^i_m] = mT^i_{n+m}, \quad [L_n, S^i_m] = mS^i_{n+m}, \quad [L_n, \tilde{S}^i_m] = (m + n)\tilde{S}^i_{n+m},
\end{align}
\[ [L_n, I^i_m] = (m + n)I^i_{n+m}, [L_n, J^j_m] = (m - n)J^j_{n+m}, \]
\[ [L_n, J^j_m] = (m + n)J^j_{n+m}, [G^i_n, G^j_m] = (m - n)J^j_{n+m}, [G^i_n, J^j_m] = mT^j_{n+m}, \]
\[ [G^i_n, T^j_m] = -G^i_{n+m} + mS^i_{n+m}, [G^i_n, T^j_m] = mS^i_{n+m}, [G^i_n, T^j_m] = G^i_{n+m}, \]
\[ [G^i_n, S^j_m] = J^j_{n+m}, [G^i_n, \tilde{S}^j_m] = T^j_{n+m}, [G^i_n, \tilde{S}^j_m] = (m - n)\tilde{J}^j_{n+m}, \]
\[ \mu \in \mathbb{C} \]
\[ h = 1 \]

Let \((i, j, k) = (1, 2, 3),\) then

\[ [G^i_n, \tilde{G}^i_m] = L^i_{n+m} - mT^k_{n+m}, [G^i_n, \tilde{S}^i_m] = T^j_{n+m} - T^k_{n+m}, [G^i_n, I^j_m] = T^j_{n+m}, [G^i_n, I^k_m] = -T^k_{n+m}, \]
\[ [G^i_n, J^j_m] = (m - n)J^j_{n+m}, [G^i_n, \tilde{I}^j_m] = (m + n)I^j_{n+m}, [G^i_n, \tilde{J}^j_m] = \tilde{G}^j_{n+m} - (n + m)\tilde{S}^j_{n+m}, \]
\[ [G^i_n, \tilde{J}^j_m] = \tilde{G}^j_{n+m}, [G^i_n, \tilde{T}^i_m] = \tilde{G}^j_{n+m} - n\tilde{S}^j_{n+m}, [G^i_n, \tilde{T}^i_m] = \tilde{G}^j_{n+m} - (n + m)\tilde{S}^j_{n+m}, \]
\[ [G^i_n, T^j_m] = -\tilde{G}^i_{n+m}, [G^i_n, \tilde{T}^j_m] = \tilde{G}^i_{n+m}, [G^i_n, T^j_m] = -\tilde{G}^i_{n+m} + mS^i_{n+m}, \]
\[ [G^i_n, T^j_m] = -\tilde{G}^i_{n+m}, [G^i_n, T^j_m] = \tilde{G}^i_{n+m}, [G^i_n, T^j_m] = -\tilde{G}^i_{n+m} + mS^i_{n+m}, \]
\[ [S^i_n, \tilde{J}^j_m] = \tilde{S}^i_{n+m}, [S^i_n, \tilde{J}^j_m] = 2I^j_{n+m}, [S^i_n, \tilde{J}^j_m] = S^i_{n+m}, [\tilde{S}^i_n, \tilde{J}^j_m] = -S^i_{n+m}, [\tilde{S}^i_n, \tilde{J}^j_m] = -S^i_{n+m}, \]
\[ [I_n, \tilde{J}^j_m] = S^i_{n+m}. \]

(5.10)

6. Representation of \(CK_6\) associated with its embedding into \(P_{h=1}\)

Recall that the embedding (5.2) for \(h = 1\) allows to define a one-parameter family of spinor-like representations of \(K'(4)\) in the superspace spanned by 2 even and 2 odd fields, where the central element \(h\) acts by the identity operator, [20].

**Theorem 3.** There exists a one-parameter family of irreducible representations of \(CK_6\), depending on parameter \(\mu \in \mathbb{C}\), in a superspace spanned by 4 even fields and 4 odd fields.

**Proof.** Let \(V^\mu = t^\mu \mathbb{C}[t, t^{-1}] \otimes \Lambda(3)\), where \(\Lambda(3) = \Lambda(\xi_1, \xi_2, \xi_3)\) is the Grassmann algebra,
and \( \mu \in \mathbb{R} \setminus \mathbb{Z} \). Let \( \{v^i_m, \hat{v}^i_m\} \), where \( m \in \mathbb{Z} \) and \( i = 1, 2, 3, 4 \), be the following basis in \( V^\mu \):

\[
v^i_m = \frac{t^{m+\mu}}{m+\mu} \xi_i, \quad \hat{v}^i_m = t^{m+\mu} \xi_j \xi_k, \quad i = 1, 2, 3, \quad v^4_m = \frac{t^{m+\mu}}{m+\mu}, \quad \hat{v}^4_m = -t^{m+\mu} \xi_1 \xi_2 \xi_3, \tag{6.1}
\]

where \((i, j, k) = (1, 2, 3)\) in the formulae for \( \hat{v}^i_m \). We define a representation of \( CK_6 \) in \( V^\mu \) according to the formulae (5.7), where \( h = 1 \). Namely, \( \xi_i \) is the operator of multiplication in \( \Lambda(3) \), \( \eta_i \) is identified with \( \partial \xi_i \), and \( \xi^{-1} \) is identified with the anti-derivative:

\[
\xi^{-1} g(t) = \int g(t) dt, \quad g \in t^\mu \mathbb{C}[t, t^{-1}]. \tag{6.2}
\]

Notice that the formula

\[
\xi^{-1} \circ_{h=1} f = \sum_{n=0}^{\infty} (-1)^n (\xi^n f) \xi^{-n-1}, \tag{6.3}
\]

where \( f \in \mathbb{C}[t, t^{-1}] \), when applied to a function \( g \in t^\mu \mathbb{C}[t, t^{-1}] \), corresponds to the formula of integration by parts:

\[
\int fg dt = f \int g dt - f' \int \int g dt^2 + f'' \int \int \int g dt^3 - \ldots. \tag{6.4}
\]

The superalgebra \( CK_6 \) acts on \( V^\mu \) as follows (see (5.8) for notations):

\[
L_n(v^i_m) = (m + n + \mu)v^i_{m+n}, \quad L_n(\hat{v}^i_m) = (m + \mu)\hat{v}^i_{m+n}, \tag{6.5}
\]

\[
G^i_n(v^i_m) = (m + n + \mu)v^i_{m+n}, \quad G^i_n(\hat{v}^i_m) = -(m + \mu)\hat{v}^i_{m+n},
\]

\[
G^i_n(\hat{v}^i_m) = \hat{v}^k_{m+n}, \quad G^i_n(v^i_m) = -\hat{v}^j_{m+n}, \quad \tilde{G}^i_n(v^i_m) = v^4_{m+n}, \quad \tilde{G}^i_n(\hat{v}^i_m) = -\hat{v}^i_{m+n},
\]

\[
\tilde{G}^i_n(\hat{v}^i_m) = -(m + \mu)v^j_{m+n}, \quad \tilde{G}^i_n(v^i_m) = (m + n + \mu)v^j_{m+n},
\]

\[
T^i_n(v^j_m) = v^i_{m+n}, \quad T^i_n(\hat{v}^i_m) = -\hat{v}^j_{m+n}, \quad T^i_n(v^i_m) = v^i_{m+n}, \quad T^i_n(\hat{v}^i_m) = v^4_{m+n},
\]

\[
T^i_n(\hat{v}^i_m) = -\hat{v}^i_{m+n}, \quad T^i_n(v^i_m) = -\hat{v}^i_{m+n}, \quad S^i_n(v^4_m) = v^i_{m+n}, \quad S^i_n(\hat{v}^i_m) = \hat{v}^4_{m+n},
\]

\[
S^i_n(v^i_m) = v^i_{m+n}, \quad S^i_n(\hat{v}^i_m) = v^j_{m+n}, \quad I^i_n(v^i_m) = -v^i_{m+n}, \quad I^i_n(\hat{v}^i_m) = -v^4_{m+n},
\]

\[
J^i_n(v^4_m) = \hat{v}^i_{m+n}, \quad J^i_n(v^i_m) = -\hat{v}^4_{m+n}, \quad \tilde{J}^i_n(v^i_m) = -\hat{v}^i_{m+n}, \quad \tilde{J}^i_n(\hat{v}^i_m) = v^k_{m+n},
\]

where \((i, j, k) = (1, 2, 3)\) in the formulae for \( \tilde{G}^i_n, S^i_n, J^i_n, \) and \( \tilde{J}^i_n \). Formulae (6.5) define a one-parameter family of representations of \( CK_6 \) in \( V^\mu =< v^i_m, \hat{v}^i_m | i = 1, \ldots, 4, \quad m \in \mathbb{Z} > \).
Remark 1. We have posed the condition $\mu \in \mathbb{R} \setminus \mathbb{Z}$ in the definition of $V^\mu$. However, formulae (6.5) actually define a representation of $CK_6$ in a superspace spanned by $v^i_m, \hat{v}^i_m$ for an arbitrary $\mu \in \mathbb{C}$. (See also section 8).

7. The second family of representations of $CK_6$

Note that the embedding of infinite-dimensional Lie superalgebras

$$CK_6 \subset K(6), \quad (7.1)$$

considered in this work, is naturally related to the embedding of finite-dimensional Lie superalgebras

$$\hat{P}(4) \subset P(0|6). \quad (7.2)$$

Recall that $P(0|6)$ is the Poisson superalgebra with 6 odd generators: $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$, and the Poisson bracket is given by (2.4). The simple Lie superalgebra $\mathcal{P}(n)$ is defined as follows. Let $\mathcal{P}(n)$ be the Lie superalgebra, which preserves the odd nondegenerate supersymmetric bilinear form antidiag $(1_n, 1_n)$ on the $(n|n)$-dimensional superspace. Thus

$$\mathcal{P}(n) = \{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A \in \mathfrak{gl}(n), B \text{ and } C \text{ are } n \times n - \text{matrices}, B^t = B, C^t = -C \}. \quad (7.3)$$

$\mathcal{P}(n)$ is a subalgebra of $\hat{P}(n)$ such that $A \in \mathfrak{sl}(n)$, [7]. A. Sergeev has proved that $\mathcal{P}(n)$ has a nontrivial central extension if and only if $N = 4$, see [18]. Note that dim $\hat{P}(4) = (16|16)$. It was pointed out in [6, 18] that $\hat{P}(4)$ has a family spin$_\lambda$ of $(4|4)$-dimensional irreducible representations. In fact, there exist two such families: they correspond to two families of embeddings of $\hat{P}(4)$ into $P(0|6)$.

For every $\lambda \neq 0$ we can realize $\hat{P}(4)$ inside $P(0|6)$ as follows:

$$\hat{P}(4) = \langle L, G^i, \tilde{G}^i, T^{ij}, T^i, S^i, \tilde{S}^i, I^i, J^{ij}, \tilde{J}^{ij} \rangle, \quad (7.4)$$

where

$$L = \lambda, G^i = \lambda \eta_i, \tilde{G}^i = \xi_i, T^{ij} = \eta_i \xi_j, T^i = -\eta_j \xi_j - \eta_k \xi_k, \quad (7.5)$$

$$S^i = -\eta_i(\eta_j \xi_j + \eta_k \xi_k), \tilde{S}^i = \frac{1}{\lambda}(\eta_j \xi_j - \eta_k \xi_k) \xi_i,$$

$$I^i = \frac{1}{\lambda} \eta_i \xi_j \xi_k, I = \lambda \eta_1 \eta_2 \eta_3, J^{ij} = \lambda \eta_i \eta_j, \tilde{J}^{ij} = \frac{1}{\lambda} \xi_i \xi_j,$$
so that the central element is $L$. Correspondingly, there is an embedding of $\hat{\mathbf{P}}(4)$ into $P_h(0|6)$ given by

$$
L_h = \lambda, G^i_h = \lambda \eta_i, \tilde{G}^i_h = \xi_i, T^i_{ij} = \eta_i \xi_j, T^i_h = -\eta_j \xi_j - \eta_k \xi_k + h, \quad (7.6)
$$

$$
S^i_h = -\eta_i (\eta_j \xi_j + \eta_k \xi_k) + h \eta_i, \tilde{S}^i_h = \frac{1}{\lambda} (\xi_j \xi_i \eta_j - \xi_k \xi_i \eta_k),
$$

$$
I_h^i = \frac{1}{\lambda} \xi_j \xi_k \eta_i, I_h = \lambda \eta_1 \eta_3, J_{ij}^i_h = \lambda \eta_i \eta_j, J_{ij}^i = \frac{1}{\lambda} \xi_i \xi_j,
$$

and $\lim_{h \to 0} \hat{\mathbf{P}}(4) = \hat{\mathbf{P}}(4) \subset P(0|6)$. The nonvanishing commutation relations between the elements (7.5) and between the elements (7.6) are as in (5.9)-(5.10), where the indexes $m = n = 0$.

Associated to this embedding (for $h = 1$) there is a family spin$^1_\lambda$ of representations of $\hat{\mathbf{P}}(4)$ in the superspace $\Lambda(\xi_1, \xi_2, \xi_3)$. We choose the basis

$$
v^i = \xi_i, \ \tilde{v}^i = \frac{1}{\lambda} \xi_j \xi_k, \ i = 1, 2, 3, \ v^4 = 1, \ \tilde{v}^4 = -\frac{1}{\lambda} \xi_1 \xi_2 \xi_3. \quad (7.7)
$$

Explicitly,

$$
\text{spin}^1_\lambda : \left( \begin{array}{cc} A & B \\ C & -A^t \end{array} \right) + \mathbb{C} L \to \left( \begin{array}{cc} A & B - \lambda \tilde{C} \\ C & -A^t \end{array} \right) + \mathbb{C} \lambda \cdot 1_{4|4}, \quad (7.8)
$$

where $1_{4|4}$ is the identity matrix, and if $C_{ij} = E_{ij} - E_{ji}$, then $\tilde{C}_{ij} = C_{kl}$, so that the permutation $(1,2,3,4) \mapsto (i,j,k,l)$ is even, cf. [6, 18]. Formula (7.8) also gives the standard representation spin$^1_0$.

The second family of embeddings of $\hat{\mathbf{P}}(4)$ into $P(0|6)$ and into $P_h(0|6)$ is given by (7.4)-(7.6), where $\xi_i$ is interchanged with $\eta_i$ for all $i$ in all the formulae. Correspondingly, there is a family spin$^2_\lambda$ of representations of $\hat{\mathbf{P}}(4)$ associated to this embedding (for $h = 1$) in the superspace $\Lambda(\xi_1, \xi_2, \xi_3)$, so that $\Pi(\text{spin}^2_\lambda) \cong \text{spin}^1_{\lambda}$, as $\hat{\mathbf{P}}(4)$-modules, for all $\lambda$. ($\Pi$ denotes the change of parity). From Theorem 3 we have the following corollary.

**Corollary 1.** Under the restriction of the representation of $CK_6$ in $V^\mu$ to $\hat{\mathbf{P}}(4)$, $V^\mu$ decomposes into a direct sum of irreducible $(4|4)$-dimensional representations of the family spin$^2_{\lambda}$.

**Proof.** Naturally, there are embeddings:

$$
\hat{\mathbf{P}}(4) \subset CK_6, \quad P(0|6) \subset K(6). \quad (7.9)
$$

The first embedding is given as follows:

$$
\hat{\mathbf{P}}(4) = \{ x \in CK_6 \mid [L_0, x] = 0 \}, \quad (7.10)
$$
hence $L_0$ is the central element. The nontrivial 2-cocycle on $\mathcal{P}(4)$ is $(G^0_0, \tilde{G}^0_0) = \delta_{i,j} L_0$. It follows from (6.5) that $V^\mu$ is a direct sum of (4|4)-dimensional $\hat{\mathcal{P}}(4)$-submodules:

$$V^\mu = \bigoplus_{m \in \mathbb{Z}} V^\mu_m, \quad V^\mu_m = \langle v^i_m, \hat{v}^i_m | i = 1, 2, 3, 4 \rangle,$$

where $V^\mu_m \cong \text{spin}^2_{m+\mu}$. 

8. Final remarks

In Theorem 1 we realized $CK_6$ inside the deg = 1 part of the $\mathbb{Z}$-grading of $P(6)$, given by (4.3), and in Theorem 2 we realized $CK_6$ inside $P_{16}(6)$. One should note that in this realization the elements of $CK_6$ have powers $-1, 0$ and 1 with respect to $\xi$, see (4.1) and (5.7).
We will now show how to single out $CK_6$ from $P_h(6)$. Let $S$ be a subspace of $P_h(6)$ spanned by $W(3)$ (which consists of the elements of power 0 and 1 with respect to $\xi$) and the following fields $(n \in \mathbb{Z})$:

\[
\begin{align*}
\xi^{-1} \circ_h t^{n-1} \eta_i \eta_j, & \quad \xi^{-1} \circ_h t^{n-1} \eta_j \eta_k \xi_i, \\
\xi^{-1} \circ_h t^{n-1} \eta_j \eta_j \xi_i, & \quad \xi^{-1} \circ_h t^{n-1} \eta_k \eta_j \xi_i, \\
n \xi^{-1} \circ_h t^{n-1} \eta_j \eta_k \xi_j \xi_k + ht^n \xi_i.
\end{align*}
\]

(8.1)

Fix $h = 1$. Let $\mu \in (0, 1)$. The action of the elements of $S$ on the spaces $V^\mu$ is well-defined. In each $V^\mu$ we defined a basis by (6.1). We will denote it now by $V^\mu = \langle v^i_m(\mu), \hat{v}^i_m(\mu) \rangle$. Let $v(\mu) \in V^\mu$ be vectors which have the same coordinates with respect to this basis for all $\mu$. Consider an odd nondegenerate superskew-symmetric form on each $V^\mu$:

\[
\begin{align*}
(v^i_m(\mu), \hat{v}^j_l(\mu))_\mu &= -(\hat{v}^i_l(\mu), v^j_m(\mu))_\mu = \delta_{m+l,0} \quad i = 1, 2, 3, \\
(v^4_m(\mu), \hat{v}^4_l(\mu))_\mu &= -(\hat{v}^4_l(\mu), v^4_m(\mu))_\mu = -\delta_{m+l,0}.
\end{align*}
\]

(8.2)

Let $V = \langle v^i_m, \hat{v}^i_m \rangle$, where $i = 1, \ldots, 4$, $m \in \mathbb{Z}$, be a superspace such that $p(v^i_m) = p(\hat{v}^i_m) = 1, p(v^4_m) = p(\hat{v}^4_m) = 0$. A superskew-symmetric form on $V$ is defined by

\[
\begin{align*}
(v^i_m, \hat{v}^j_l) &= -(\hat{v}^i_l, v^j_m) = \delta_{m+l,0} \quad i = 1, 2, 3, \\
(v^4_m, \hat{v}^4_l) &= -(\hat{v}^4_l, v^4_m) = -\delta_{m+l,0}.
\end{align*}
\]

(8.3)

**Theorem 5.**

\[
CK_6 = \{ X \in S \mid \lim_{\mu \to 0} [(Xv(\mu), w(\mu))_\mu + (-1)^{p(X)p(v(\mu))}(v(\mu), Xw(\mu))_\mu] = 0,
\]

for all $v(\mu), w(\mu) \in V^\mu$.

(8.4)

There is a representation of $CK_6$ in $V$ given by (6.5), where $\mu = 0$, and this action preserves the form (8.3).

**Remark 2.** Correspondingly, there is a representation of $CK_6$ in $V$ given by (7.13), where $\mu = 0$, and this action preserves the odd nondegenerate supersymmetric form on $V$:

\[
(v^i_m, \hat{v}^j_l) = (\hat{v}^i_l, v^j_m) = \delta_{m+l,0} \quad i = 1, 2, 3, 4.
\]

(8.5)
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References

[1] M. Ademollo, L. Brink, A. D’Adda et al., “Dual strings with $U(1)$ colour symmetry”, Nucl. Phys. B 111, 77-110 (1976).

[2] V. I. Arnold, Mathematical Methods of Classical Mechanics (Springer-Verlag, New York, 1989).

[3] S.-J. Cheng and V. G. Kac, “A new $N = 6$ superconformal algebra”, Commun. Math. Phys. 186, 219-231 (1997).

[4] B. Feigin and D. Leites, “New Lie superalgebras of string theories”, in Group-Theoretical Methods in Physics, edited by M. Markov et al., (Nauka, Moscow, 1983), Vol. 1, 269-273. [English translation Gordon and Breach, New York, 1984).

[5] B. Feigin, Private communication.

[6] P. Grozman, D. Leites, and I. Shchepochkina, “Lie superalgebras of string theories”, Acta Math. Vietnam. 26, no 1, 27-63 (2001).

[7] V. G. Kac, “Lie superalgebras”, Adv. Math. 26, 8-96 (1977).

[8] V. G. Kac, “Superconformal algebras and transitive group actions on quadrics”, Commun. Math. Phys. 186, 233-252 (1997).

[9] V. G. Kac, “Classification of infinite-dimensional simple linearly compact Lie superalgebras”, Adv. Math. 139, 1-55 (1998).

[10] V. G. Kac, “Structure of some $\mathbb{Z}$-graded Lie superalgebras of vector fields”, Transform. Groups 4, 219-272 (1999).

[11] V. G. Kac, “Vertex algebras for beginners”, University Lecture Series, Vol. 10 AMS, Providence, RI, 1996. Second edition 1998.

[12] V. G. Kac and J. W. van de Leur, “On classification of superconformal algebras”, in Strings-88, edited by S. J. Gates et al. (World Scientific, Singapore, 1989), 77-106.

[13] B. Khesin, V. Lyubashenko, and C. Roger, “Extensions and contractions of the Lie algebra of q-pseudodifferential symbols on the circle”, J. Funct. Anal. 143, 55-97 (1997).
[14] O. S. Kravchenko and B. A. Khesin, “Central extension of the algebra of pseudodifferential symbols”, Funct. Anal. Appl. 25, 83-85 (1991).

[15] V. Ovsienko and C. Roger, “Deforming the Lie algebra of vector fields on $S^1$ inside the Poisson algebra on $\hat{T}^*S^1$”, Commun. Math. Phys. 198, 97-110 (1998).

[16] V. Ovsienko and C. Roger, “Deforming the Lie algebra of vector fields on $S^1$ inside the Lie algebra of pseudodifferential symbols on $S^1$”, Am. Math. Soc. Trans. 194, 211-226 (1999).

[17] I. Shchepochkina, “The five exceptional simple Lie superalgebras of vector fields”, hep-th/9702121.

[18] I. Shchepochkina, “The five exceptional simple Lie superalgebras of vector fields”, Funkt. Anal. i Prilozhen. 33, 59-72 (1999). [Funct. Anal. Appl. 33, 208-219 (1999)].

[19] I. Shchepochkina, “The five exceptional simple Lie superalgebras of vector fields and their fourteen regradings”, Represent. Theory 3, 373-415 (1999).

[20] E. Poletaeva, A spinor-like representation of the contact superconformal algebra $K'(4)$, J. Math. Phys. 42, 526-540 (2001); hep-th/0011100 and references therein.