Abstract

We establish uniform (with respect to $x, y$) semiclassical asymptotics and estimates for the Schwartz kernel $e_h(x, y; \tau)$ of spectral projector for a second order elliptic operator inside domain under microhyperbolicity (but not $\xi$-microhyperbolicity) assumption. While such asymptotics for its restriction to the diagonal $e_h(x, x, \tau)$ and, especially, for its trace $\mathbb{N}_h(\tau) = \int e_h(x, x, \tau) \, dx$ are well-known, the out-of-diagonal asymptotics are much less explored, especially uniform ones.

Our main tools: microlocal methods, improved successive approximations and geometric optics methods.

1 Introduction

In this paper we consider a self-adjoint scalar operator which is elliptic second order differential operator with $\mathcal{C}^K$-coefficients ($K = K(d, \delta)$)

\begin{equation}
A = \sum_{j,k} (hD_j - V_j(x)) g^{jk}(x) (hD_k - V_k(x)) + V(x), \quad g^{jk} = g^{kj},
\end{equation}

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and under microhyperbolicity condition

\[(1.2) \quad |V(x) - \tau| + |\nabla V(x)| \geq \epsilon_0 \quad \forall x \in B(0, 1) \subset \mathbb{R}^d, \quad d \geq 2,\]

we establish sharp (with $O(h^{1-d})$ remainder) asymptotics of the Schwartz kernel of the spectral projector $e_h(x, y, \tau)$ as $h \to +0$.

Under $\xi$-microhyperbolicity assumption

\[(1.3) \quad |V(x) - \tau| \geq \epsilon_0 \quad \forall x \in B(0, 1) \subset \mathbb{R}^d\]

such asymptotics of $e_h(x, x, \tau)$ is well-known for much more general operators including matrix ones (with a proper definition of $\xi$-microhyperbolicity). However out-of-diagonal asymptotics, especially uniform with respect to $x, y$, are much less explored and for simple main part and sharp remainder estimate in addition to $\xi$-microhyperbolicity

\[(1.4) \quad |a(x, \xi) - \tau| + |\nabla_\xi a(x, \xi)| \geq \epsilon_0\]

where $a(x, \xi)$ is the principal symbol) require strong convexity of the energy surface \{\xi: $a(x, \xi) = \tau$\}; see [Ivr3], Theorem 1.1.

On the other hand, under only microhyperbolicity condition

\[(1.5) \quad |a(x, \xi) - \tau| + \nabla_{x, \xi} a(x, \xi) \geq \epsilon_0\]

(which could be generalized for matrix operators) sharp asymptotics of

\[(1.6) \quad \mathcal{N}_h(\tau) = \int e_h(x, x, \tau) \, dx\]

are well-known (see [Ivr2], Chapter 4) \footnote[1]{And even near the boundary (see [Ivr2], Chapter 7).}. However in this case even uniform asymptotics of $e_h(x, x, \tau)$ are much less explored and only for operator (1.1) (see Subsection 5.3.1, Subsubsection “Asymptotics without Spatial Mollification”) because while microhyperbolicity conditions does not allow short periodic trajectories, it allows short loops \footnote[2]{But $\xi$-microhyperbolicity does not allow short loops.}.

**Theorem 1.1.** Consider elliptic self-adjoint operator (1.1) with smooth coefficients. Assume that microhyperbolicity condition (1.2) is fulfilled. Then asymptotics

\[(1.7) \quad e_h(x, y, \tau) = e_h^W(x, y, \tau) + \begin{cases} O(h^{1-d}) & d \geq 3, \\ O(h^{-\frac{d}{2}}) & d = 2 \end{cases}\]
holds for all \( x, y \in B(0, \epsilon) \) with

\[
e_h^W(x, y; \tau) = (2\pi h)^{-d} \int_{\{a(\xi(x+y), \xi) < \tau\}} e^{ih^{-1}(x-y, \xi)} \, d\xi,
\]

(1.8)

\[
a(x, \xi) = \sum_{j,k} g_{jk}^+(x) \xi_j \xi_k + V(x).
\]

(1.9)

**Theorem 1.2.** Consider elliptic self-adjoint operator (1.1) with smooth coefficients. Assume that microhyperbolicity condition (1.2) is fulfilled. Let \( d = 2 \) and \( x, y \in B(0, \frac{1}{2}) \).

(i) Asymptotics

\[
e_h(x, y, \tau) = e_h^W(x, y, \tau) + O(h^{-1})
\]

(1.10)

holds if either condition (1.3) is fulfilled, or \( \ell(x, y) \geq h^\frac{3}{2} \), or \( \ell(x, y) \geq h^\frac{5}{2} \) and

\[
|4V(x - \tau)V(y - \tau) - |x - y|^2 (g_{jk})|\nabla V|^2 (g_{jk}) + \langle x - y, \nabla V \rangle^2 |
\]

\[
\geq \epsilon \ell(x, y)^2
\]

(1.11)

where

\[
\ell(x, y) = \max(|x - y|, |V(x) - \tau|, |V(y) - \tau|, h^\frac{3}{2}),
\]

(1.12)

\((g_{jk}) = (g_{jk})^{-1} \) and \( \nabla V \) is calculated at \( \frac{1}{2}(x + y) \).

(ii) Without condition (1.11) for \( h^\frac{3}{2} \geq \ell(x, y) \geq h^\frac{5}{2} \) the following estimate holds:

\[
e_h(x, y, \tau) = O(h^{-\frac{2}{3}} \ell(x, y)^{-1}).
\]

(1.13)

(iii) Further, for \( \ell(x, y) \leq h^\frac{5}{2} \) asymptotics

\[
e_h(x, y, \tau) = e_h^W(x, y, \tau) + e_{corr, k,h}(x, y, \tau) + O(h^{-1})
\]

(1.14)

holds with correction term \( e_{corr, k,h}(x, y, \tau) \) to be defined by (2.46) later if either condition (1.11) is fulfilled or \( \ell(x, y) \leq h^\frac{5}{2} \).
(iv) Without condition (1.11) for $h^{\frac{d}{2}} \geq \ell(x, y) \geq h^{\frac{3}{2}}$ the following estimate holds:

\[
e_h(x, y, \tau) = e_{\text{cor}, k, h}(x, y, \tau) + O(h^{-\frac{d}{2}}(x, y)^{\frac{3}{2}}).
\]

(v) Finally, as $x = y$ one can replace $k$ by 0 while preserving remainder estimate and define $e_{\text{cor}, 0, h}(x, y, \tau)$ as

\[
2h^{-\frac{d}{2}} \int_{-\infty}^{\tau} (\tau - \tau') \text{Ai}(-2h^{-\frac{3}{2}}(\tau' - V(x))) d\tau' - \frac{1}{2\pi h^2}(\tau - V(x))_.
\]

providing $|\nabla V|_{g^h} = 1$ where $\text{Ai}(\cdot)$ denotes Airy function.

Remark 1.3. As $d = 1, 2$ and $x = y$ asymptotics are already known: see [Ivr2], Subsection 5.3.1, Subsubsection Asymptotics without Spatial Mollification and Short Loops).

Plan of the paper. General idea. Our main method is to prove (as $d \geq 2$) Tauberian asymptotics

\[
e_h(x, y, \tau) = e_{T, h}(x, y, \tau) + O(h^{1-d})
\]

with the Tauberian expression

\[
e_{T, h}(x, y, \tau) = \frac{1}{h} \int_{-\infty}^{\tau} F_{t \to -h^{-1}T}(\bar{\chi}_T(t)u_h(x, y, t)) d\tau
\]

where $u_h(x, y, t)$ is the propagator (the Schwartz kernel of $e^{ih^{-1}At_h}$), and $\bar{\chi}_T(t) = \bar{\chi}(T^{-1}t)$ is the appropriate cut-off. Then from Tauberian expression we pass to Weyl expression and a correction term if needed. So far our arguments are similar to those of [Ivr3] where we considered points near boundary but assumed $\xi$-microhyperbolicity.

To prove Tauberian asymptotics we need to prove that Fourier transform in (1.18) with $T \asymp 1$ is $O(h^{1-d})$.

Section 2. We start from the toy-model (2.1), study Hamiltonian trajectories and then do all calculations explicitly. We arrive to the oscillatory integral which has four stationary points, corresponding to two trajectories from $x$ to $y$ and two trajectories from $y$ to $x$ on the energy level $\tau$, provided we are in the regular zone where the left-hand expression in (1.11) without
absolute value is positive: two are short trajectories and two are long ones. If the same expression is negative (\textit{shadow zone}) there are neither any trajectories nor stationary points, and if (1.11) fails (\textit{singular zone}) these points almost coincide.

Then we repeat these steps for the generalized toy-model (2.31).

Section 3. We follow the same path, but we get semi-explicit expression: using special coordinates and $x_1$-microhyperbolicity we construct in the standard way $F_{x_1 \rightarrow h^{-1}\xi_1, y_1 \rightarrow -h^{-1}\eta_1} u_h(x, y, t)$ as an oscillatory integral and then make inverse Fourier transform. Then we prove (1.17) and either pass from Tauberian expression to Weyl expression (as $\ell(x, y) \geq h^\frac{2}{3}$) and estimate the error, or use the perturbation method (as $\ell(x, y) \leq h^\frac{2}{3}$) and estimate

\[
\left( e_{T,h}^{\xi_1}(x, y, \tau) - \bar{e}_{T,k,h}^{\xi_1}(x, y, \tau) \right) - \left( e_{h}^{\xi_1}(x, y, \tau) - \bar{e}_{k,h}^{\xi_1}(x, y, \tau) \right)
\]

where $\bar{e}_{T,k,h}^{\xi_1}(x, y, \tau)$ and $\bar{e}_{k,h}^{\xi_1}(x, y, \tau)$ correspond to an appropriate generalized toy-model. Then we define correction term as

\[
e_{\text{corr},k,h} = \left( e_{T,k,h}^{\xi_1}(x, y, \tau) - \bar{e}_{k,h}^{\xi_1}(x, y, \tau) \right).
\]

Section 4. Here we show how simple rescaling arguments allows to get rid off assumption (1.3). We also discuss easier case $d = 1$.

2 Toy-model

Let us consider a toy-model operator with $d \geq 2$, $g^{jk} = \delta_{jk}$, $V_j(x) = 0$ and $V(x) = -x_1$:

\begin{equation}
\bar{A}_h := \frac{1}{2} h^2 D_x^2 - x_1 = \frac{1}{2} h^2 D_1^2 + \frac{1}{2} h^2 D'^2 - x_1
\end{equation}

By shift $x_1 \mapsto x_3 + \tau$ we can reduce $\tau \in \mathbb{R}$ to $\tau = 0$. After this by rescaling $x \mapsto h^\frac{2}{3}$, $\tau \mapsto h^\frac{2}{3}$ we can reduce $h > 0$ to $h = 1$.

2.1 Hamiltonian trajectories

Proposition 2.1. Consider toy-model operator (2.1). Then

(i) Hamiltonian trajectories from $(\bar{x}, \bar{\xi}) = (\bar{x}, \xi_1, \xi')$, $\bar{x}_1 > 0$ on the energy level 0 (so, $\bar{\xi}_1 = \mp \sqrt{2\bar{x}_1 - |\xi'|^2}$) are

\[
\xi' = \xi', \quad x_1 = \bar{x}_1 + \frac{t^2}{2} \mp t \sqrt{2\bar{x}_1 - |\xi'|^2}, \quad x' = \bar{x}' + \xi' t, \quad \xi_1 = t \mp \sqrt{2\bar{x}_1 - |\xi'|^2}.
\]
(ii) If \( d = 2 \) and \( \pm \xi_1 < 0 \) their projections to \( x \)-space are parabolas

\[
x_1 = \frac{1}{2\xi_2^2} \left( x_2 - \bar{x}_2 \mp \xi_2 \sqrt{2\bar{x}_1 - \xi_2^2} \right)^2 + \frac{\xi_2^2}{2}
\]

which at \( t = \pm \frac{2\bar{x}_1}{\sqrt{2\bar{x}_1 - \xi_2^2}} \) are tangent to the parabola \( \Gamma = \{ x_1 = \frac{1}{4\bar{x}_1} | x' - \bar{x}_2 |^2 \} \).

(iii) Any point \( x \) above \( \Gamma \) (that is with \( x_1 > \frac{1}{4\bar{x}_1} (x_2 - \bar{x}_2)^2 \)) is covered by two such rays, one of then touches \( \Gamma \) between \( \bar{x} \) and \( x \), and another outside of this segment. Any point \( x \) below \( \Gamma \) (that is with \( x_1 < \frac{1}{4\bar{x}_1} (x_2 - \bar{x}_2)^2 \)) is not reachable from \( \bar{x} \) by rays on the energy level \( \bar{\epsilon} \), and any point on \( \Gamma \) is reachable by just one ray (when point above \( \Gamma \) tends to \( \Gamma \) both rays tend to the same limit).

(iv) The vertices of these parabolas are on the ellipse

\[
L = \{ (2x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2 = \bar{x}_1^2 \};
\]

between \( \Gamma \) and \( L \) along both rays \( \xi_1 \) have the same sign (opposite to one of \( \bar{\xi}_1 \)) while inside \( L \) (brown on Figure 1) for long rays this is true only for a long ray, while for a short ray sign of \( \xi_1 \) coincides with the sign of \( \bar{\xi}_1 \).

Proof. Easy proof by direct calculations is left to the reader. \( \square \)

Remark 2.2. Observe that for a trajectory between points \( x \) and \( y \)

(i) For a long trajectory \( \ell(x, y) \asymp |x' - y'| + \nu(x) + \nu(y) \) with \( \nu(x) = |V(x) - \tau| \) and \( T(x, y) \asymp \ell^2(x, y) \) is a time to reach.

(ii) Either \( \ell(x, y) \asymp \ell^0(x, y) := |x - y| \) and \( T^0(x, y) \asymp T(x, y) \) where \( T^0(x, y) \) is time to reach along short trajectory, or \( |\xi_{long} - \xi_{short}| \asymp \nu(x)^{1/2} \) and \( |\eta_{long} - \eta_{short}| \asymp \nu(y)^{1/2} \), where \( \xi_* \) and \( \eta_* \) correspond to these trajectories at \( x \) and \( y \) for \( |t| \asymp T(x, y) \) or \( T^0(x, y) \).

(iii) Further, since \( \xi'_{short} \ell^0 = \xi'_{long} \ell \), we conclude that

\[
\ell^0 < \epsilon \ell \implies |\xi'_{short}| > \epsilon^{-1} |\xi'_{long}|.
\]
2.2 Calculations

As usual, consider \( u_h^\pm(x, y, t) = u_h(x, y, t)\theta(\pm t) \) with Heaviside function \( \theta \); then

\[
\left( h\partial_t - \frac{1}{2} h^2 \partial_{\xi_1}^2 - \frac{1}{2} h^2 |D'|^2 + x_1 \right) u_h^\pm = \mp ih\delta(t)\delta(x - y).
\]

(2.3)

making \( h \)-Fourier transform

\[
\hat{u}_h^\pm(\xi, y, \tau) := F_{t\to h^{-1}\tau, x\to h^{-1}\xi} u_h^\pm = \int \int e^{-ih^{-1}(t\tau + \langle x, \xi \rangle)} u_h^\pm(x, y, t) \, dx \, dt
\]

we get as \( \mp \text{Im} \tau > 0 \)

\[
\left( \tau - \frac{1}{2} \xi_1^2 - \frac{1}{2} |\xi'|^2 + ih\partial_{\xi_1} \right) \hat{u}_h^\pm = \mp ie^{ih^{-1}(\gamma\xi_1 - \langle y', \xi' \rangle)};
\]

(2.4)
We can see easily that the phase function we arrive to with \( \tau \)

\[
(2.9) \quad \frac{\partial}{\partial \xi_1} \left( e^{i h^{-1}(|\xi|^2/2)\xi_1 - |\xi|^6/6)} u_h^\pm \right) = \mp e^{i h^{-1}(-y_1 x_1 - (y', x')) - (\tau - |\xi'|^2/2)\xi_1 + |\xi|^3/6)},
\]

we arrive to

\[
(2.5) \quad \hat{u}_h^\pm (\xi, y, \tau) = \mp e^{i h^{-1}(|\tau - |\xi'|^2/2| \xi_1 - |\xi|^3/6)} \int_{-\infty}^{\xi_1} e^{i h^{-1}(-y_1 n - (y', x') - (\tau - |\xi|^2/2)n + |\xi|^3/6)} d\eta_1
\]

and making partial inverse \( \mathcal{h} \)-Fourier transform

\[
(2.6) \quad F_{t \rightarrow \mathcal{h}^{-1} \tau} \hat{u}_h^\pm = (2\pi \mathcal{h})^{-d} \int \int \int_{\mathbb{R}^d} e^{i h^{-1}(x_1 - y_1 \eta + (x' - y', x'') + (\tau - |\xi'|^2/2)(\xi_1 - n) - (|\xi|^3/6)\eta_1)} d\xi_1 d\xi' d\eta_1
\]

Then as \( \tau \in \mathbb{R} \)

\[
(2.7) \quad F_{t \rightarrow \mathcal{h}^{-1} \tau} u_h = (2\pi \mathcal{h})^{-d} \times \int \int \int_{\mathbb{R}^d} e^{i h^{-1}(x_1 - y_1 \eta + (x' - y', x'') + (\tau - |\xi'|^2/2)(\xi_1 - n) - (|\xi|^3/6)\eta_1)} d\xi_1 d\xi' d\eta_1
= h^{1-2d/3} J(h^{-\frac{2}{3}} \tau, h^{-\frac{2}{3}} x_1, h^{-\frac{2}{3}} y_1, h^{-\frac{2}{3}} (x' - y'))
\]

with

\[
(2.8) \quad J(\tau, x_1, y_1, z') := (2\pi)^{-d} \times \int \int \int_{\mathbb{R}^d} e^{i (x_1 - y_1 \eta + (x' - y', x'') + (\tau - |\xi'|^2/2)(\xi_1 - n) - (|\xi|^3/6)\eta_1)} d\xi_1 d\eta_1 d\xi'.
\]

We can see easily that the phase function

\[
(2.9) \quad \Phi(x_1, y_1, z'; \xi_1, \eta_1, \xi'); \tau)
= x_1 \xi_1 - y_1 \eta_1 + (z', \xi') + (\tau - \frac{1}{2} |\xi'|^2)(\xi_1 - \eta_1) - \frac{1}{6} (\xi_1^3 - \eta_1^3))
\]

has four stationary points (with respect to \((\xi_1, \eta_1, \xi')\)): on all of them

\[
(2.10) \quad 2x_1 = -2\tau + \xi_1^2 + |\xi'|^2, \quad 2y_1 = -2\tau + \eta_1^2 + |\xi'|^2, \quad z' = \xi'(\xi_1 - \eta_1),
\]

and for corresponding trajectories on the energy \( \tau \)

\[
(2.11) \quad T(x, y) = |\xi_1 - \eta_1|
\]

but
(a) on two of them $|\xi_1 - \eta_1|$ is the smallest (short rays) and
\[ \ell^0(x, y) \asymp |x_1 - y_1| \asymp |x - y| \] with $x_1 \geq 0$, $y_1 \geq 0$.

(b) on the other two $|\xi_1 - \eta_1|$ are largest (long rays) $\xi_1$ and $\eta_1$ have opposite signs and
\[ \ell(x, y) \asymp x_1 + y_1 \asymp |x - y| + x_1 + y_1 \] with $x_1 \geq 0$, $y_1 \geq 0$.

Then we arrive easily to the following proposition:

**Proposition 2.3.** For a toy-model operator (2.1) with $d \geq 1$ the following estimate holds
\[ |e_{\tau,h}^T(x, y) - e_{\tau,h}^T(x, y, \tau)| \leq Ch^{1 - \frac{2d}{\delta}} \]
with $T = \max(\epsilon_2 \sqrt{\ell(x, y)}, h^{\frac{1}{2} - \delta})$.

In particular, the right-hand expression is $O(h^{1 - d})$ for $d \geq 3$, $O(h^{-\frac{d}{2}})$ for $d = 2$ and $O(h^{-\frac{d}{2}})$ for $d = 1$.

It shows that case $d = 2$ requires special consideration. In this case
\[ \partial_\tau J(\tau, x_1, y_1, z_2) \]
\[ = \frac{i}{4\pi^2} \iiint (\xi_1 - \eta_1) e^{i(x_1\xi_1 - y_1\eta_1 + z_2\xi_2 + (\tau - \xi_1^2/2)(\xi_1 - \eta_1) - (\xi_1^2 - \eta_1^2)/6)} d\xi_1 d\eta_1 d\xi_2 \]
which after substitution $\xi_1 = \alpha + \beta$, $\eta_1 = \alpha - \beta$ becomes
\[ \frac{i}{\pi^2} \iiint \beta e^{i(\beta(x_1 + y_1 + 2\tau) - \xi_2^2 - \alpha^2 - \beta^2/3 + i\xi_2\xi_2 + i(x_1 - y_1)\alpha} d\alpha d\beta d\xi_2 \]
and calculating integrals with respect $\alpha$ and $\xi_2$ we arrive to
\[ (2.13) \quad \partial_\tau J(\tau, x_1, y_1, z_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{i(\beta(x_1 + y_1 + 2\tau) + \frac{1}{3}(x_1 - y_1)^2 + z_2^2) - \frac{1}{2} \beta^2} d\beta. \]

One can see easily that
\[ (2.14) \quad \text{Stationary points} \]
\[ \beta^2 = \frac{1}{2} \left( x_1 + y_1 + 2\tau \pm \sqrt{4x_1y_1 - z_2^2} \right) \]
are non-degenerate as $4(x_1 + \tau)(y_1 + \tau) > z_2^2$ and there are no real stationary points as $4(x_1 + \tau)(y_1 + \tau) < z_2^2$.
Proposition 2.4. Consider toy-model operator (2.1) with \( d = 2 \). Let \( x_1 \geq 0, y_1 \geq 0, \ell(x, y) := \max(|x - y|, x_1, y_1) \geq C_0 h^{\frac{2}{3}} \).

(i) If \( \ell^0(x, y) := |x - y| \leq \epsilon \ell(x, y) \) with \( \epsilon = \epsilon' > 0 \) then the following estimate holds

\[
|e_{t_1, h}^T(x, y, 0) - e_{T, h}^T(x, y, 0)| \leq C h^{-\frac{2}{3}} \ell(x, y)^{-\frac{3}{2}}
\]

with \( T = C_0 \sqrt{\ell^0(x, y)} \).

(ii) Further, if \( \ell^0(x, y) \geq \epsilon \ell(x, y) \) then the following estimates holds:

\[
|e_{t_1, h}^T(x, y, 0)| \leq C h^{-\frac{2}{3}} \ell(x, y)^{-1}.
\]

(iii) More precisely, if \( \ell^0(x, y) \geq \epsilon \ell(x, y) \) and

\[
(x_2 - y_2)^2 \leq 4x_1y_1 - \epsilon \ell(x, y)^2
\]

then

\[
|e_{t_1, h}^T(x, y, 0)| \leq C h^{-\frac{1}{2}} \ell(x, y)^{-\frac{5}{4}}.
\]

(iv) On the other hand, if

\[
(x_2 - y_2)^2 \geq 4x_1y_1 + \epsilon \ell(x, y)^2
\]

then this estimate and all similar estimates below acquire factor \((\ell(x, y) h^{-\frac{2}{3}})^{-s}\):

\[
|e_{t_1, h}^T(x, y, 0)| \leq C h^{(2s-4)/3} \ell(x, y)^{-s}.
\]

Proof. Integration by \( \tau \) and multiplication by \( h^{-1} \) results in the formal expression

\[
\frac{1}{8\pi^2h^2i} \iint (\xi_1 - \eta_1)^{-1} e^{ih^{-1}\Phi(x_1, y_1, x_2, \xi_1, \eta_1, \xi', 0)} d\xi_1 d\eta_1 d\xi'
\]

and we need to apply the stationary phase principle counting only points (in \( \mathbb{C}^3 \)) corresponding to \( |\xi_1 - \eta_1| \gtrsim \sqrt{\ell} \). Then

(a) Due to above analysis in the regular zone described in Statements (i) and (iii) all three eigenvalues of \( \text{Hess}_{\xi_1, \eta_1, \xi_2} \Phi \) are real and \( \asymp \sqrt{\ell} \) and therefore the left-hand expressions in (2.15) and (2.18) do not exceed \( Ch^{-\frac{2}{3}}(h^{\frac{1}{2}})^{\frac{3}{2}} \).
(b) On the other hand, in the singular zone where
\begin{equation}
|4x_1y_1 - (x_2 - y_2)^2| \leq \epsilon \ell(x, y)^2
\end{equation}
this is true only for two eigenvalues and therefore the left-hand expression in (2.16) does not exceed \(Ch^{-2} \ell^{-\frac{1}{2}}(h \ell^{-\frac{1}{2}})^{3/4} \times h^{3/4}\).

(c) In the shadow zone described in Statement (iv) again all eigenvalues of Hessian are \(\approx \sqrt{\ell}\) but only two are real and the third has imaginary part \(\approx \sqrt{\ell}\) and therefore the left-hand expression in (2.18) does not exceed \(Ch^{-2}(h \ell^{-\frac{1}{2}})^3\).

Remark 2.5. (i) We can be more precise in the analysis in the singular zone.

(ii) Statements (ii)–(iv) remain true as \(|x_1| \leq h^{\frac{3}{2}}\) or \(|y_1| \leq h^{\frac{3}{2}}\).

(iii) Further, as either \(x_1 < -h^{\frac{3}{2}}\) or \(y_1 < -h^{\frac{3}{2}}\) or both \(x_1 < -h^{\frac{3}{2}}\) and \(y_1 < -h^{\frac{3}{2}}\) then stationary points move to imaginary axis and estimates in Statement (ii) and (iii) and all similar estimates below acquire correspondingly either factor \(h^{\frac{3}{2}}|x_1|^3\) or \(h^{\frac{3}{2}}|y_1|^3\) or both.

(iv) The right-hand expression of (2.15) is \(O(h^{-1})\) as \(\ell(x, y) \geq h^{\frac{3}{2}}\) and the right-hand expression of (2.16) is \(O(h^{-1})\) as \(\ell(x, y) \geq h^{\frac{5}{2}}\).

(v) Improvements similar to Proposition 2.4(i)–(iv) are possible for \(d \geq 3\) as well.

Now we want to replace \(e_{T,h}^T(x, y, 0)\) in the left-hand expression of (2.15) by \(e_{h}^W(x, y, 0)\) described by (1.8).

**Proposition 2.6.** (i) For a toy-model operator (2.1) with \(d \geq 3\) and \(\ell(x, y) \leq 1\)
\begin{equation}
e_{e_{i_{1}},h}^T(x, y, 0) = e_{h}^W(x, y, 0) + O(h^{1-d}).
\end{equation}

(ii) For a toy-model operator (2.1) with \(d = 2\)
\begin{equation}
e_{e_{i_{1}},h}^T(x, y, 0) = e_{h}^W(x, y, 0) + e_{corr,h}(x, y, 0) + O(h^{-1})
\end{equation}
with the correction term

\begin{equation}
(2.25) \quad e_{\text{corr},h}(x, y, 0) = \frac{1}{h^2} \int_{-\infty}^{0} J(h^{-\frac{2}{3}} \tau, h^{-\frac{2}{3}} x_1, h^{-\frac{2}{3}} y_1, h^{-\frac{2}{3}} (x_2 - y_2)) \, d\tau - e^W_h(x, y, 0).
\end{equation}

Proof. (a) As \( \ell(x, y) \leq h^\frac{3}{2} \) we have both \( e^T_{\ell_1,h}(x, y, 0) = O(h^{-\frac{2d}{3}}) \) and \( e^W_h(x, y, 0) = O(h^{-\frac{2d}{3}}) \). Those are \( O(h^{-\frac{d}{3}}) \) for \( d = 2 \) and \( O(h^{1-d}) \) for \( d \geq 3 \).

(b) As \( \ell(x, y) \geq h^\frac{3}{2} \) due to rescaling \( x \mapsto \ell^{-1} x, y \mapsto \ell^{-1} y, \tau \mapsto \ell^{-1} \tau \) and \( h \mapsto \tilde{h} = \ell^{-\frac{1}{3}} h \) we have

\begin{equation}
(2.26) \quad e^T_{\ell_1,h}(x, y, 0) - e^W_h(x, y, 0) = O(h^{1-d} \times \ell^{-d}) = O(h^{1-d} \ell^{\frac{d-1}{3}})
\end{equation}

which is \( O(h^{-\frac{d}{3}}) \) for \( d = 2 \), \( O(h^{-2}) \) for \( d = 3 \) and \( O(h^{1-d}) \) for \( d \geq 4 \), \( \ell \leq 1 \). Statement (i) has been proven.

(c) As \( d = 2 \) for a toy-model operator we can replace with \( O(h^d) \) error \( e^T_{\ell_1,h}(x, y, 0) \) by \( e^T_{\ell_0,h}(x, y, 0) \) which due to (2.7) is exactly the first term in the right-hand expression in (2.25); so Statement (ii) is just a definition. \( \square \)

Remark 2.7. (i) Due to (2.13) the first term in the right-hand expression of (2.25) can be rewritten as

\begin{equation}
(2.27) \quad -\frac{2}{\pi} \int_{-\infty}^{0} \int_{0}^{\infty} \tau \cos(\beta(x_1 + y_1 + 2\tau) + \frac{1}{4\beta}((x_1 - y_1)^2 + z_2^2) - \frac{1}{3} \beta^3) \, d\beta d\tau.
\end{equation}

As \( x = y \) it becomes

\begin{equation}
(2.28) \quad -\frac{2}{\pi} \int_{-\infty}^{0} \int_{0}^{\infty} \tau \cos(2\beta(x_1 + \tau) + \frac{1}{3} \beta^3) \, d\beta d\tau
= -2 \int_{-\infty}^{0} \tau \text{Ai}(-2(x_1 + \tau)) \, d\tau
\end{equation}

with Airy function \( \text{Ai}(\cdot) \).
(ii) One can prove easily that

$$e^W(x, y, \tau) = \frac{1}{4\pi^2 r^2} \int_0^{2\pi} \int_0^{h^{-1}r} \cos(\sigma \cos(\theta)) \sigma d\sigma d\theta \quad \text{with} \quad \rho = \sqrt{2(\tau V((x + y)/2))}, \quad r = |x - y|$$

and in the framework of Proposition 2.4(ii)

$$|e^W(x, y, 0)| \leq C h^{-\frac{1}{2}} l(x, y)^{-\frac{3}{4}}.$$ (2.30)

(iii) In Proposition 3.6 in much more general settings will be proven that in
the framework of Proposition 2.4(i) $$e^W(x, y, 0)$$ provides a good approxima-
tion for $$e^W_T(x, y, 0)$$ with $$T = \epsilon' \sqrt{l(x, y)}$$.

2.3 Generalized toy-model

Let us consider now a generalized toy-model operator with $$d = 2$$, $$g^{jk} = \delta_{jk}$$,
$$V_2(x) = 0$$, $$V_1 = -k\xi_1$$ and $$V(x) = -x_1 - \frac{1}{2}k^2 x_1^2$$:

$$\tilde{A}_{k,h} := \frac{1}{2} h^2 D_1^2 + \frac{1}{2} h^2 (D_2 - k\xi_1)^2 + V(x)$$

$$= \frac{1}{2} h^2 D_1^2 + \frac{1}{2} h^2 D_2 - (1 + khD_2)\xi_1$$

As in Subsection 2.2 considering equation to propagator and making $$h$$-
Fourier transform we get equation similar to (2.4):

$$\hat{\tau} - \frac{1}{2} \xi_1^2 - \frac{1}{2} \xi_2^2 + ih(1 + k\xi_2)\partial_{\xi_1} \hat{u}^\pm_h = \mp ie^{ih^{-1}(\tau \xi_1 + \xi_2 \xi_2)}$$ (2.32)

as $$\mp \text{Im} \tau > 0$$; rewriting it as

$$\partial_{\xi_1} \left( e^{-ih^{-1}(1+k\xi_2)^{-1}((\tau - \xi_2^2/2)\xi_1 - \xi_1^2/6)} \hat{u}^\pm_h \right) =$$

$$\mp (1 + k\xi_2)^{-1} e^{-ih^{-1}(1+k\xi_2)^{-1}((\tau - \xi_2^2/2)\xi_1 - \xi_1^2/6) - ih^{-1}(\eta_1 + \eta_2 \xi_2)},$$

we arrive to expression similar to (2.5)

$$\hat{u}^\pm_h(\xi, y, \tau) = \mp (1 + k\xi_2)^{-1} e^{ih^{-1}(\tau - x_2^2/2)(\xi_2 - \xi_2^2/6)}$$

$$\times \int_{\mp\infty}^{\xi_1} e^{-ih^{-1}(1+k\xi_2)^{-1}((\tau - \xi_2^2/2)\eta_1 - \eta_1^2/6) - ih^{-1}(\eta_1 + \eta_2 \xi_2)} d\eta_1$$

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and making partial inverse $h$-Fourier transform we arrive to the final answer similar to (2.6):

$$(2.34) \quad F_{t \rightarrow h^{-1} \tau}^{\pm} = \mp (2\pi h)^{-2} \int \int (1 + k \xi_2)^{-1}$$

$$\times \int_{\mp \infty}^{\xi_1} e^{ih^{-1}(1+k\xi_2)^{-1}((\tau-\xi_2^2/2)(\xi_1-\eta_1)-(\xi_1^2-\eta_1^2)/6)+ih^{-1}(x_1\xi_1-y_1\eta_1+(x_2-y_2)\xi_2)} d\eta_1 d\xi_1 d\xi_2.$$  

Then as $\tau \in \mathbb{R}$ we get (cf. (2.7))

$$(2.35) \quad F_{t \rightarrow h^{-1} \tau} = (2\pi h)^{-2} \int \int \int (1 + k \xi_2)^{-1}$$

$$e^{ih^{-1}(1+k\xi_2)^{-1}((\tau-\xi_2^2/2)(\xi_1-\eta_1)-(\xi_1^2-\eta_1^2)/6)+ih^{-1}(x_1\xi_1-y_1\eta_1+(x_2-y_2)\xi_2)} d\eta_1 d\xi_1 d\xi_2.$$  

We can see easily that for $|k| \leq C$ and $\min(|x_1|, |y_1|) \leq \epsilon$ the phase function

$$(2.36) \quad \Phi_k(x_1, y_1, z_2; \xi_1, \eta_1, \xi_2; \tau)$$

$$= x_1 \xi_1 - y_1 \eta_1 + z_2 \xi_2 + (1 + k \xi_2)^{-1}[(\tau - \frac{1}{2} \xi_2^2)(\xi_1 - \eta_1) - \frac{1}{6}(\xi_1^3 - \eta_1^3)]$$

has four stationary points (with respect to $(\xi_1, \eta_1, \xi_2) \in \mathbb{C}^3$): on all of them $|\xi_2| \leq C_0 \sqrt{\epsilon}$,

$$(2.37) \quad 2x_1 = (1 + k \xi_2)^{-1}[-2\tau + \xi_1^2 + \xi_2^2],$$

$$(2.38) \quad 2y_1 = (1 + k \xi_2)^{-1}[-2\tau + \eta_1^2 + \xi_2^2],$$

and

$$(2.39) \quad z_2 = \xi_2(1 + k \xi_2)^{-1}(\xi_1 - \eta_1)$$

$$+ k(1 + k \xi_2)^{-2}[(\tau - \frac{1}{2} \xi_2^2)(\xi_1 - \eta_1) - \frac{1}{6}(\xi_1^3 - \eta_1^3)]$$

and for corresponding trajectories on the energy level $\tau$

$$(2.40) \quad T(x, y) = (1 + k \xi_2)^{-1}|\xi_1 - \eta_1|.$$  

Then the same conclusion as for $k = 0$ holds:

(a) Proposition 2.3 remains true;
(b) All statements of Proposition 2.4 remain true with regular, singular and shadow zones defined exactly by (2.17), (2.22) and (2.19).

(c) All statements of Remark 2.5 remain true.

Again consider \( \partial_r J_k(\tau, x_1, y_1, z_2) \) this time based on (2.35):

\[
\frac{i}{4\pi^2} \iint (1 + \kappa \xi_2)^{-1} (\xi_1 - \eta_1) \times e^{i(1 + \kappa \xi_2)^{-1}(\tau - \xi_2^2/2)(\xi_1 - m) - (\xi_1^3 - \eta_1^3)/6 + i\hbar^{-1}(\xi_1 \eta_1 + 2\xi_2 - \xi_1 \alpha)} d\eta_1 d\xi_1 d\xi_2.
\]

Plugging \( \xi_1 = \alpha + \beta, \eta_1 = \alpha - \beta \) and then substituting \( \beta := (1 + \kappa \xi_2)\beta \) we arrive to

\[
(2.41) \quad \frac{i}{\pi^2} \iint d\alpha d\beta d\xi_2 \times 
\beta \exp \left[ i \left( (2\tau - \xi_2^2 - \alpha^2)\beta - \frac{1}{3}(1 + \kappa \xi_2)\beta^3 + (1 + \kappa \xi_2)(x_1 + y_1)\beta + z_2 \xi_2 + (x_1 - y_1)\alpha \right) \right].
\]

Then we can calculate integrals with respect to \( \alpha \) and \( \xi_2 \), arriving to formula similar to (2.27):

\[
(2.42) \quad - \frac{1}{\pi} \int_{-\infty}^{\infty} d\beta \times 
\exp \left[ i \left( (2\tau + x_1 + y_1)\beta - \frac{1}{3} \beta^3 + \frac{1}{4\beta}(x_1 - y_1)^2 + \frac{1}{4\beta} \left( z_2 - \frac{1}{3} \kappa \beta^3 - \kappa (x_1 + y_1)\beta \right)^2 \right) \right].
\]

(a) Consider first \( \beta \approx \sqrt{\ell} \). Observe that selected terms in the phase in (2.43) are \( O(\ell^2) \) and squared and divided by \( \beta \) they are \( O(\ell^2) \) and therefore assuming that \( \ell \leq h^{\frac{1}{2}} \) we arrive to the following conclusion: if we neglect their square the error will not exceed

\[
(2.43) \quad \begin{cases}
Ch^{-\frac{1}{2}} \ell^{-\frac{3}{2}} \times \ell^{\frac{3}{2}} h^{-1} = O(h^{-1}) & \text{in the regular zone,} \\
Ch^{-\frac{1}{2}} \ell^{-1} \times \ell^{\frac{3}{2}} h^{-1} = O(h^{-\frac{3}{2}}\ell^\frac{1}{2}) & \text{in the singular zone,} \\
Ch^{2s-2)/3} \ell^{-s} & \text{in the shadow zone.}
\end{cases}
\]

(b) On the other hand, consider \( \beta \epsilon \sqrt{\ell} \) and therefore \( \ell^0 := |z_2| + |x_1 - y_1| \leq \ell \), and then one can estimate the error in the cases \( \ell^0 \sqrt{\ell} \geq h \) and \( \ell^0 \sqrt{\ell} \leq h \) and in both cases the error will be smaller than given by the first line of (2.43).
But then (2.42) becomes

\[(2.44)\]

\[
\partial_\tau J_\kappa(\tau, x_1, y_1, z_2) \equiv -\frac{2}{\pi} \int_0^\infty \cos\left((2\tau + x_1 + y_1)\beta - \frac{1}{3}\beta^3 + \frac{1}{4}\beta^2 \left[(x_1 - y_1)^2 + z_2^2\right]\right)
\times \exp\left[i\left(-\frac{1}{6}\kappa z_2\beta^2 - \frac{1}{2}\kappa z_2(x_1 + y_1)\right)\right] d\beta
\]

and and scaling back we arrive to

**Proposition 2.8.** Consider generalized toy-model operator (2.31) in dimension \(d = 2\). Then as \(h^\frac{1}{2} \leq \ell \leq h^\frac{3}{2}\) with an error (2.42)

\[(2.45) \]

\[e_{T, h}^T(x, y, 0) \equiv e_h^W(x, y, 0) + e_{\text{corr}, k, h}(x, y, 0)\]

with the correction term

\[(2.46) \]

\[e_{\text{corr}, k, h}(x, y, 0)\]

\[
= \frac{1}{h^2} \int_{-\infty}^0 J_{h^\frac{1}{2}}(\tau) \left[h^{-\frac{3}{2}}\tau, h^{-\frac{3}{2}}x_1, h^{-\frac{3}{2}}y_1, h^{-\frac{3}{2}}(x_2 - y_2)\right] d\tau - e_h^W(x, y, 0)
\]

where \(\partial_\tau J_\kappa(\tau, x_1, y_1, z_2)\) defined by (2.44) and \(e_h^W(x, y, 0)\) by (2.29).

**Remark 2.9.** Obviously \(J_\kappa(\tau, x_1, y_1, 0) = J_0(\tau, x_1, y_1, 0)\).

### 3 General case

In the framework of Theorems 1.1 and 1.2 without any loss of the generality one can assume that

\[(3.1)_{1,2} \quad V(x) = -x_1 W(x), \quad W(0) = 1,\]

\[(3.2)_{1-3} \quad \bar{x} = (\bar{x}_1, 0) \quad \bar{y} = (\bar{y}_1, \bar{y}',) \quad \bar{y}_1 \leq \bar{x}_1,\]

\[(3.3)_{1-4} \quad g^{ij} = \delta_{ij}, \quad g^{jk}(0) = \delta_{jk}, \quad V_1 = 0, \quad V_j(0) = 0,\]

where \(\bar{x}\) and \(\bar{y}\) are two “target” points. Indeed, we can reach this by a change of variables. We will assume that these assumptions are fulfilled until the end of this section. Then according to (1.12)

\[(3.4) \quad \ell(x, y) \approx \max(|x_1|, |y_1|, |x' - y'|, h^2).\]
3.1 Preliminary remarks

Proposition 3.1. (i) Estimate holds

\[
|F_{t\to h^{-1}\tau}(\chi\tau(t)u_h(x, y, t))| \leq Ch^{1-d}\ell^{(d-2)/3}h^s
\]

with \(|\tau| \leq \epsilon T^2\), \(T = \max(C_0\ell^{\frac{1}{2}}, h^{\frac{1}{3}})\) and \(h = T^{-3}h\).

(ii) Furthermore, if \(\ell^0 = \ell^0(x, y) := |x - y| \geq \epsilon_0\ell\) and \(\ell \geq h^\frac{3}{2}\), then in the same framework

\[
|F_{t\to h^{-1}\tau}(\bar{\chi}_T(t)u_h(x, y, t))| \leq Ch^{1-d}\ell^{(d-2)/3}h^s.
\]

Here and below \(\ell = \ell(x, y)\), \(\chi \in \mathcal{C}_0^\infty([-1, -\frac{1}{2}) \cup [\frac{1}{2}, 1])\), \(\bar{\chi} \in \mathcal{C}_0^\infty([-1, 1])\) equal 1 on \([-\frac{1}{2}, \frac{1}{2}]\), \(s\) is an arbitrarily large exponent and \(\delta > 0\) is an arbitrarily small exponent.

Proof. (a) Consider first \(T\) which is a small constant: \(T = \epsilon_3\) (which we can assume is 1; otherwise we achieve it by scaling) and assume only that \(T \geq C_0\sqrt{\ell}\). Then Statement (i) follows from the fact that the propagation speed with respect to \(\xi_1\) is disjoint from 0 plus ellipticity arguments.

Statement (ii) follows from the fact that he propagation speeds with respect to \(x\) and \(\xi\) are bounded.

(b) Then we scale \(x \mapsto T^{-2}x, y \mapsto T^{-2}y, \ell \mapsto T^{-2}\ell, \tau \mapsto T^{-2}\tau\) and \(h \mapsto h = T^{-3}h\).

Proposition 3.2. The following estimates hold for \(\tau: |\tau| \leq \epsilon, T: h^\frac{3}{2} \leq T \leq \epsilon_1:\)

\[
|F_{t\to h^{-1}\tau}(\bar{\chi}_T(t)u_h(x, y, t))| \leq Ch^{1-d},
\]

\[
|e_h(x, y, \tau + h) - e_h(x, y, \tau)| \leq Ch^{1-d},
\]

and for \(T \geq T^* := C_0\max(\sqrt{\ell}, h^\frac{3}{2})\)

\[
|e_h(x, y, 0) - e_{T, h}(x, y, 0)| \leq Ch^{-2d/3}h^s + Ch^{1-d}
\]

again with \(h = T^{-3}h\).
Proof. (a) It is sufficiently to prove for $\tau = 0$ since shift of $V$ by $\tau$: $|\tau| \leq \epsilon_2$ preserves conditions. Simple scaling shows that if $\ell^0(x, y) \leq \epsilon \ell(x, y)$ then with $T = T^*$ the left-hand expression of (3.7) with $T = T^*$ does not exceed $C\hbar^{1-d}T^{*2(d-2)/3}$.

On the other hand, considering partition by $T: T \leq T \leq T^*$ we see from Proposition 3.1(i) that the same left-hand expression but with $\tilde{\chi}_{T}(t) - \tilde{\chi}_{T^*}(t)$ also does not exceed this. So, with indicated $T$ the same left-hand expression does not exceed this, which is $O(\hbar^{1-d})$.

Finally, for $\ell^0(x, y) \geq \epsilon \ell(x, y)$ the same estimate (3.7) holds again due to (3.5).

(b) In particular, taking $x = y$ and $T = \epsilon_1$ and applying standard Tauberian arguments, Part I, we conclude that (3.8) holds with $x = y$. And then it holds for $x \neq y$ due to

$$|e_h(x, y, \tau + h) - e_h(x, y, \tau)| \leq |e_h(x, x, \tau + h) - e_h(x, x, \tau)|^{\frac{1}{2}}|e_h(y, y, \tau + h) - e_h(y, y, \tau)|^{\frac{1}{2}}.$$

(c) Then (3.9) holds with $T = \epsilon_1$ due to standard Tauberian arguments, Part II. Then due to Proposition 3.1(i)

$$|e^T_{2, h}(x, y, 0) - e^T_{1, h}(x, y, 0)| \leq C T^{d-3}\hbar^s$$

and summation by partition over $[T, \epsilon_1]$ results in the same answer which implies (3.9).

Therefore

(3.10) To evaluate $e_h(x, y, \tau)$ modulo $O(\hbar^{1-d})$ it is enough to consider only $T \leq C_0 \max(\sqrt{\ell(x, y)}, \hbar^{3-d})$.

### 3.2 Propagator as an oscillatory integral

**Proposition 3.3.** Let $x, y \in B(0, \epsilon)$, $C_0 \sqrt{\ell} \leq T \leq C_0\epsilon$ and $|\tau| \leq \epsilon$. Then

$$F_{t \to h^{-1}T} \left( \tilde{\chi}_{T}(t) u_h(x, y, t) \right)$$

$$= (2\pi h)^{-d} \int e^{ih^{-1} \varphi(x, y, \xi, \eta, \xi', \eta', \tau, \tau, h)} b(x, y, \xi, \eta, \xi', \eta', \tau, h) d\xi d\eta + O(\hbar^s)$$

3) Albeit changes $x_1$ and $\ell(x, y)$.
where phase function \( \varphi(x, y, \xi_1, \eta_1, \xi', \tau) \) differs from the phase function \( \tilde{\varphi}(x, y, \xi_1, \eta_1, \xi', \tau) \) in (2.7) by \( O(T^4) \) and amplitude \( b(x, y, \xi_1, \eta_1, \xi', \tau, h) \) differs from the amplitude \( \tilde{b}_k(x, y, \xi_1, \eta_1, \xi', \tau, h) \) there by \( O(T) \).

**Proof.** (a) Let us start from the formal construction. Let \( h \)-Fourier transform with respect to \( x_1, y_1 \): 

\[
\hat{u}_h(\xi_1, x', \eta_1, y', t) = F_{x_1 \to h^{-1} \xi_1, y_1 \to h^{-1} \eta_1} u_h.
\]

Then the problem becomes

\[
(3.12) \quad (hD_x - A(x, hD_x)) u_h(x, y, t) = 0,
\]

\[
(3.13) \quad u_h(x, y, 0) = \delta(x - y)
\]

becomes

\[
(3.14) \quad (hD_x - A(-hD_{\xi_1}, x', \xi_1, hD_{\xi'})) \hat{u}_h(\xi_1, x', \eta_1, y', t) = 0,
\]

\[
(3.15) \quad \hat{u}_h(\xi_1, x', \eta_1, y', t) = 2\pi h \delta(\xi_1 - \eta_1) \delta(x' - y').
\]

This operator is \( x_1 \)-microhyperbolic where \( x_1 \) is a variable dual to \( \xi_1 \). Then using the standard arguments\(^4\) we can construct \( \hat{u}_h(\xi_1, x', \eta_1, y', t) \) near energy level 0 for \( x, y \in B(0, \epsilon_1) \) as a “simple” oscillatory integral

\[
(3.16) \quad \hat{U}_h(\xi_1, x', \eta_1, y', t) = (2\pi h)^{1-d} \int e^{ih^{-1} (\varphi(\xi_1, x', \eta_1, y', \theta, t) + a(-\theta_1, y', \theta'))} b(\xi_1, x', \eta_1, y', \theta, h) \, d\theta,
\]

with the phase satisfying

\[
(3.17) \quad a(-\partial_{\xi_1} \psi, x', \partial_{\xi'} \psi) = a(-\theta_1, y', \theta'),
\]

\[
(3.18) \quad \psi|_{\xi_1=\eta_1} = (x' - y', \theta'),
\]

\[
(3.19) \quad \nabla_{\xi_1, \xi'} \psi|_{\xi_1=\eta_1, \xi'=y'} = \theta
\]

and with amplitude \( b(\xi_1, x', \eta_1, y', \theta, h) \) decomposing into asymptotic series

\[
(3.20) \quad b(\xi_1, x', \eta_1, y', \theta, h) \sim \sum_{n \geq 0} b_n(\xi_1, x', \eta_1, y', \theta) h^n.
\]

Then for \( |\tau| \leq \epsilon \)

\[
(3.21) \quad F_{t \to h^{-1} \tau} \hat{U}_h(x, y, t) = (2\pi h)^{2-d} \int \frac{e^{ih^{-1} (\varphi(\xi_1, x', \eta_1, y', \theta, \tau))} b(\xi_1, x', \eta_1, y', \theta, h)}{\Sigma(\xi_1, y', \tau)} \, d\theta : d_0 a
\]

\(^4\) See f.e. [Shu], Theorem 20.1 or [Ivr3], Section 2.
where \( \Sigma(\xi_1, y', \tau) = \{ \theta: a(-\theta_1, y', y', \theta') = \tau \} \). Due to \( x_1 \)-microhyperbolocity we can express \( \theta_1 \) as function of \( \xi_1, \eta_1, x', y', \tau \) and rewrite the right-hand expression as an integral over \( \theta' \in \mathbb{R}^{d-1} \).

Finally, making inverse Fourier transform by \( \xi_1, \eta_1 \) and plugging \( \theta' = \xi' \) we arrive to (3.11) for \( u_h \) with

\[
\varphi(x, y, \xi, \eta, \xi', \tau) = \psi(\xi_1, x', \eta_1, y', \theta_1(\xi_1, x', \eta_1, y', \xi', \tau)), \xi, \tau) + x_1 \xi_1 - y_1 \eta_1.
\]

(b) To justify (3.11) for \( u_h \) we observe that for given \( x, y \in B(0, \epsilon) \) and \( T \leq C_0 \epsilon \) behaviour of \( a(x, \xi) \) for \( |x| \geq c \epsilon \) does not matter.

(c) Finally, since \( \varphi = O(T^3) \) and relative error of \( A_h \) in comparison to generalized toy-model operator \( \bar{A}_h \) is \( O(T) \) we conclude that the relative errors in \( \varphi \) and \( B_h \) are also \( O(T) \) and therefore absolute errors are \( O(T^4) \) and \( O(T^2) \) correspondingly.

Therefore we arrive immediately to the following statements:

**Proposition 3.4.** Let operator \( A_h \) satisfy (3.1)\(_{1,2} \), (3.2)\(_{1,3} \) and (3.3)\(_{1,4} \). Then

(i) In the framework of Proposition 2.3 estimate (2.12) holds.

(ii) In the framework of Proposition 2.4(i) estimate (2.15) holds.

(iii) In the framework of Proposition 2.4(ii) estimate (2.16) holds.

(iv) In the framework of Proposition 2.4(iii) estimate (2.18) holds.

(v) In the framework of Proposition 2.4(iv) estimate (2.20) holds.

Also observe that all statements of Remark 2.5 remain valid.

**Corollary 3.5.** Let \( d \geq 3 \). Then \( e_h(x, y, \tau) = e_h^w(x, y, \tau) + O(h^{1-d}). \) Furthermore \( e_h(x, y, 0) = O(h^{1-d}) \) unless \( \nu(x) \prec \nu(y) \geq h^{\frac{1}{3}} \) and \( \ell^0(x, y) \leq \epsilon \nu(x) \).

**Proof.** If \( \ell^0(x, y) \leq \epsilon \ell(x, y), \ell(x, y) \geq h^\frac{1}{3} \) we prove estimate (1.7) by simple rescaling. Otherwise the simple rescaling shows that \( |e_{f,b}^T(x, y, 0)| \leq Ch^{1-d} \) and therefore \( |e_h(x, y, 0)| \leq Ch^{1-d} \) and one can see easily that in this case also \( |e_h^w(x, y, 0)| \leq Ch^{1-d} \) due to stationary phase principle. \( \square \)
So, Theorem 1.1 has been proven. From now we consider only $d = 2$.

**Proposition 3.6.** Let $d = 2$, $\ell(x, y) \leq \epsilon_1$ and $\ell^0(x, y) \leq \epsilon \ell(x, y)$. Then

\[(3.22) \quad |e_{T, h}(x, y, 0) - e^W_h(x, y, 0)| \leq C h^{-1} + C \ell^{-2}.\]

**Proof.** Following arguments of [Ivr3], Section 2 one can prove easily that after rescaling the left-hand expression does not exceed

\[(3.23) \quad Ch^{-\frac{1}{2}} \ell^{0 - \frac{3}{2}} \min\left(1, \frac{\ell_0 h}{h}\right) + Ch^{-1} \ell + C\]

where the first term is the difference between

\[(2\pi h)^{-2} \int e^{i\hbar^{-1}(i\tau - \varphi(x, y, \theta))} B_0(x, y, \theta, t) T_{\tau} \chi T_{\tau, \tau}(t) d\theta dt\]

and $e^W_h(x, y, 0)$. Further, the second term comes from the amplitude $B_1 h$ in the same expression (and acquires factor $\ell$ due to scaling of $V_j$) and the third term comes from the amplitude $B_2 h^2$ in the same expression, which are in $e_{T, h}(x, y, 0)$ but are skipped in $e^W_h(x, y, 0)$.

The first term in (3.23) does not exceed $Ch^{-1} \ell^{\frac{1}{2}}$, so the left-hand expression of (3.22) does not exceed $Ch^{-1} \ell^{\frac{1}{2}} + C$. Scaling back (that is multiplying by $\ell^{-2}$ and plugging $h = \ell^{-\frac{3}{2}} h$) we get $Ch^{-1} + C \ell^{-2}$. \qed

Combining Propositions 3.6 and 3.4(ii) we arrive to

**Corollary 3.7.** In the framework of Proposition 3.6

\[(3.24) \quad |e_h(x, y, 0) - e^W_h(x, y, 0)| \leq Ch^{-\frac{1}{2}} \ell^{\frac{1}{2}} + C \ell^{-2} + Ch^{-1}.\]

In particular, for $\ell \geq h^2$ the right-hand expression is $Ch^{-1}$.

### 3.3 Perturbation methods

Now we compare $e^T_{T, h}(x, y, 0)$ with the same function for generalized toy-model operator (2.31) assuming that

\[(3.25)_{1-3} \quad V_{x_1}(0) = -1, \quad V_1 = 0, \quad V_{2x_1}(0) = k\]

because we can achieve (3.25)$_1$ by scaling and (3.25)$_3$ as a definition. Then

\footnote{5) These amplitudes $B_n(x, y, \theta, t)$ should not be confused with the amplitudes $b_n$ in the proof of Proposition 3.3.}
The phase function \( \phi(x, y, \xi_1, \eta_1, \xi', \tau) \) in expression (3.11) differs from the phase function \( \phi(x, y, \xi_1, \eta_1, \xi', \tau) \) in (2.35) by \( O(T^5) \) and amplitude \( b(x, y, \xi_1, \eta_1, \xi', \tau, h) \) differs from the amplitude \( \bar{b}_k(x, y, \xi_1, \eta_1, \xi', \tau, h) \) there by \( O(T^2) \).

Indeed, in comparison to the Part (c) of the proof of Proposition 3.4 now the relative error in \( A_h \) in comparison to generalized toy-model operator \( \tilde{A}_{k,h} \) is \( O(T^2) \) we conclude that the relative errors in \( \phi \) and \( B_h \) are also \( O(T^2) \).

Let us assume that

\[
\ell(x, y) \leq h^2. 
\]

**Proposition 3.8.** Let (3.25) and (3.27) be fulfilled. Then

(a) Asymptotics

\[
(3.28) \quad (e^T_{T^*, h}(x, y, 0) - e^W_h(x, y, 0)) - (\bar{e}^T_{k,h}(x, y, 0) - \bar{e}^W_k(x, y, 0)) = O(h^{-1})
\]

holds for all \( x, y \in B(0, \varepsilon) : |x - y| \leq \varepsilon \ell(x, y) \).

(b) Estimates

\[
|e^T_{T^*, h}(x, y, 0) - \bar{e}^T_{k,T^*, h}(x, y, 0)| \leq \begin{cases} \text{Ch}^{-\frac{2}{3} - \frac{s}{3}} \times \ell^2 h^{-1} = O(h^{-1}) & \text{in the regular zone,} \\ \text{Ch}^{-\frac{2}{3} - 1} \times \ell^2 h^{-1} = O(h^{-\frac{2}{3}} \ell^2) & \text{in the singular zone,} \\ \text{Ch}^{(2s-2)/3} \ell^{-s} & \text{in the shadow zone} \end{cases}
\]

and

\[
(3.30) \quad e^W_h(x, y, 0) - \bar{e}^W_{k,h}(x, y, 0) = O(h^{-1})
\]

hold for all \( x, y \in B(0, \varepsilon) : |x - y| \geq \varepsilon \ell(x, y) \) where \( \bar{e}_{k,h}(x, \tau, \tau) \), \( \bar{e}^W_{k,h}(x, \tau, \tau) \) and \( \bar{e}^T_{k,T,h}(x, \tau, \tau) \) are defined for a generalized toy-model operator (2.31).

**Proof.** (a) Observe that \( A_h - \tilde{A}_{k,h} = O(\ell^2) \). Indeed, when we replace \( V \) by \( x \) the error is \( O(\ell^2) \) due to (3.25) and when we replace \( V \) by \( k \) the error is \( O(\ell^2) \) due to (3.25) where extra factor \( \ell^2 \) comes from factor \( \xi_2 \).
Therefore as $|x - y| \leq \epsilon \ell(x, y)$ the following estimate holds

\begin{equation}
(3.31) \quad |(e^T_{T^*, h}(x, y, 0) - e^T_{T^*}(x, y, 0)) - (\tilde{e}^T_{k, T^*, h}(x, y, 0) - \tilde{e}^T_{k, T^*, h}(x, y, 0))| \leq C h^{-1}
\end{equation}

and as $|x - y| \geq \epsilon \ell(x, y)$ estimate (3.30) holds, where we recall that $T^* = C_0 \sqrt{\ell}$ and $T_* = \epsilon' \sqrt{\ell}$ with arbitrarily small constant $\epsilon'$. The proof is similar to the proof of Proposition 2.8 and the right-hand expression in (3.29) is exactly expression (2.43).

(b) Further, if $|x - y| \leq \epsilon_1 \ell(x, y)$ then combining (3.31) and (3.22) we conclude that the left-hand expression of (3.27) does not exceed $C h^{-1} + C \ell^{-2}$ which implies (3.27) for $\ell \geq h^\frac{1}{2}$.

However, for $\ell(x, y) \leq h^\frac{1}{2}$ we need more subtle arguments. Namely, recall that $C \ell^{-2}$ comes from $h^2 \times \ell^{-2}$ which in turn comes from the decomposition (3.16) with $B_n = O(\ell^{-2})$. However since operators $A_h$ and $\tilde{A}_{k,h}$ after rescaling differ by $O(\ell^2)$ rather than $O(\ell)$ we can estimate $B_n - \tilde{B}_{k,n} = O(\ell^{-\frac{3n}{2}})$ and therefore

\begin{equation}
(3.32) \quad |(e^T_{T^*, h}(x, y, 0) - e^W_{h}(x, y, 0)) - (\tilde{e}^T_{k, T^*, h}(x, y, 0) - \tilde{e}^W_{k,h}(x, y, 0))| \leq C h^{-1} + C \ell^{-\frac{3}{2}} \leq C h^{-1}.
\end{equation}

(c) On the other hand, if $\ell^0(x, y) \geq \epsilon_1 \ell(x, y)$ then one can prove easily

\begin{equation}
(3.33) \quad |e^W_{h}(x, y, 0) - \tilde{e}^W_{k,h}(x, y, 0)| \leq C h^{-1} + C \ell^{-\frac{3}{2}} \leq C h^{-1}.
\end{equation}

\[ \square \]

Remark 3.9. (i) One can prove easily that

\begin{equation}
(3.34) \quad e^W_{h}(x, y, 0) - \tilde{e}^W_{k,h}(x, y, 0) = O(h^{-2} \ell^2)
\end{equation}

even as $|x - y| \geq \epsilon \ell(x, y)$. Therefore

\begin{equation}
(3.35) \quad e_{h}(x, y, 0) = \tilde{e}_{k,h}(x, y, 0) + O(h^{-1}) \quad \text{as} \quad \ell(x, y) \leq h^\frac{1}{2}.
\end{equation}

However, one can easily note that it may fail as $\ell(x, y) \gg h^\frac{1}{2}$. Thus, as $h^\frac{1}{2} \leq \ell(x, y) \leq h^\frac{3}{2}$ we need to use both terms $e^W_{h}(x, y, 0)$ and $e_{corr,k,h}(x, y, 0)$ in the asymptotics.
(ii) We can replace $k$ by $k = 0$ and preserve remainder estimate $O(h^{-1})$ if and only if $|k||x_2 - y_2| \lesssim \ell^\frac{\delta}{2}h^\frac{\delta}{2}$. In particular, we can do it as $x_2 = y_2$.

(iii) We know that in any dimension $d \geq 2$ $e_h(x, y, 0)$ is $O(h^\delta)$ if either $x_1 < -h^{\frac{\delta}{2} - \delta}$ or $y_1 < -h^{\frac{\delta}{2} - \delta}$ or $4x_1y_1 \leq (1 - \epsilon)||x' - y'||^2 - h^{\frac{\delta}{2} - \delta}$ but it is not the case for $e^W_h(x, y, 0)$!

3.4 Proof of Theorem 1.2

Proof of Theorem 1.2. Without any loss of the generality one can assume that $\tau = 0$, and assumptions (3.1)$_{1,2}$, (3.2)$_{1-3}$, (3.3)$_{1-4}$ and (3.25)$_{1-3}$ are fulfilled. Then the left-hand expression in (1.11) is $(x_1 - x_2)^2 - 4x_1y_1$ modulo $O(\ell^\delta)$.

As $\ell(x, y) \geq h^{\frac{\delta}{2} - \delta}$ we can replace in the Tauberian expressions $T = C_0\sqrt{\ell}$ by $T = \epsilon_1$ with $O(h^\delta)$ error. Then

(a) Statements (i) and (ii) of Theorem 1.2 follow immediately from Proposition 3.4.

(b) Statements (iii) and (iv) of Theorem 1.2 follow immediately from Proposition 3.8.

(c) Further, as $\ell(x, y) \leq h^{\frac{\delta}{2} - \delta}$ we can take $T^* = h^{\frac{\delta}{2} - \delta}$ and apply the same arguments.

(d) Finally, Statement (v) of Theorem 1.2 follows from (2.44) and (2.28). □

4 Generalizations and final remarks

Remark 4.1. (i) In any dimension $d \geq 1$ without assumption (1.3) the simple rescaling with the scaling function

\begin{equation}
\gamma_x = (\epsilon|V(x) - \tau| + h^\frac{\delta}{2})
\end{equation}

results in the estimate

\begin{equation}
|e_h(x, y, \tau) - e^W_h(x, y, \tau)| \leq C\ell^{1-d}\gamma_x^{(d-3)/2}
\end{equation}

as long as $\ell(x, y) \leq \gamma_x$ (and then $\gamma_y \asymp \gamma_x$) $^6$.

$^6$ As $d = 1$ Weyl expression should be modified so it contains a true eikonal rather than $(x - y, \theta) - a(x + y, \theta)$ approximation.
(ii) To explore the case $\ell(x, y) \geq \gamma_x$ observe that the standard Tauberian method, Part I, implies

\begin{equation}
\label{4.3}
e_h(x, x, \tau + h^{\gamma_x^{-\frac{1}{2}}}) - e_h(x, x, \tau) \leq Ch^{1-d\gamma_x^{(d-3)/2}}
\end{equation}

Then

\begin{equation}
\label{4.4}
|e_h(x, y, \tau + h^{\gamma_y^{-\frac{1}{2}}}) - e_h(x, y, \tau)| \leq Ch^{1-d\gamma_x^{(d-3)/4}\gamma_y^{(d-5)/4}}
\end{equation}

with $\gamma^* = \max(\gamma_x, \gamma_y)$ and $\gamma_* = \min(\gamma_x, \gamma_y)$.

Applying standard Tauberian methods, Part II, we arrive to the estimate

\begin{equation}
\label{4.5}
|e_h(x, y, \tau)| \leq CT_{x,y}^{-\frac{1}{2}} \times h^{1-d\gamma^*(d-1)/4\gamma_*^{(d-5)/4}} \\
\leq Ch^{1-d\gamma_x^{(d-3)/4}\gamma_y^{(d-3)/4}} = Ch^{1-d\gamma_x^{(d-3)/4}\gamma_y^{(d-3)/4}}
\end{equation}

where $T_{x,y} \geq |x - y|^{\frac{1}{2}} \geq \gamma_x^{\frac{1}{2}}$ is the minimal propagation time between $x$ and $y$ on the energy level $\tau$.

Therefore, in the worst case remainder estimate is $O(h^{-\frac{d}{2}})$ for $d = 1$, $O(h^{-\frac{d}{4}})$ for $d = 2$ and $O(h^{1-d})$ for $d \geq 3$.

Remark 4.2. (i) Under assumption (1.3) following arguments of Proposition 3.2 we upgrade (4.3) to

\begin{equation}
\label{4.6}
e_h(x, x, \tau + h) - e_h(x, x, \tau) \leq Ch^{1-d\gamma_x^{(d-2)/2}}
\end{equation}

and then

\begin{equation}
\label{4.7}
|e_h(x, y, \tau) - e_{ci,h}^T(x, y, \tau)| \leq Ch^{1-d\gamma_x^{(d-2)/4}\gamma_y^{(d-2)/4}}.
\end{equation}

Therefore, in the worst case remainder estimate is $O(h^{-\frac{1}{2}})$ for $d = 1$, $O(h^{-1})$ for $d = 2$ and $O(h^{1-d})$ for $d \geq 3$.

(ii) However to replace $e_{ci,h}^T(x, y, \tau)$ by $e_h^W(x, y, \tau)$ and preserve this remainder estimate we may need to add a correction term (in any dimension!) which in the regular zone is of magnitude $h^{(1-d)/2}\ell(x, y)^{-(d+3)/4}$ and in the singular zone is $O(h^{2-3d})/6\ell(x, y)^{-(d+2)/4}$ \footnote{As $d = 1$ there is no shadow zone but there still is a singular zone where $\gamma_x \neq \gamma_y$.}.

For toy-model (2.1) it is

\begin{equation}
\label{4.8}
e_{corr,h}(x, y, \tau) = \frac{2}{(2\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2} + 1)} \int_{-\infty}^{\tau} \int_0^\infty (\tau' - \tau)^{\frac{d}{2}} \\
\times \cos\left[h^{-1}\left(\beta(x_1+y_1+2\tau) + \frac{1}{4\beta}\left(\left((x_1-y_1)^2 + |z'|^2\right) - \frac{1}{3}\beta^3\right)\right) - \frac{\pi}{4}(d+2)\right] d\beta d\tau \\
- e_h^W(x, y, \tau)
\end{equation}
with \( z' = x' - y' \). Then \( e_{\text{corr}}(x, x, \tau) \) can be expressed through Airy function.

**Remark 4.3.** To get rid off condition (1.3) we can use the simple rescaling with the scaling function

\[
\gamma_{2x} = (\varepsilon|V(x) - \tau| + |\nabla V|^2 + h)^{1/2};
\]

then \( \hbar = \gamma_{2x}^{-2} h \) and the remainder estimates are not spoiled in dimensions \( d \geq 2 \), but in dimension \( d = 1 \) it becomes \( O(h^{-3/2} \gamma_{2x}^{-1}) = O(h^{-1} \gamma_{2x}^{-3/2}) \) which is \( O(h^{-3/2}) \) in the worst case.

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