Role of $PT$-symmetry in understanding Hartman effect

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Abstract We calculate the tunneling time from a layered non-Hermitian system to examine the effect of $PT$-symmetry over tunneling time. We explicitly find that for system respecting $PT$-symmetry, the tunneling time saturates with the thickness of the $PT$-symmetric barrier and thus shows the existence of Hartman effect. For non-$PT$-symmetric case, the tunneling time depends upon the thickness of the barrier and Hartman effect is lost. We further consider the limiting case in which the non-Hermitian system reduces to the real barrier to show that the Hartman effect from a real barrier is due to $PT$-symmetry (of the corresponding non-Hermitian system).

1 Introduction

The study of non-Hermitian system in quantum mechanics started as a mathematical curiosity. In the year 1998, it was shown that a non-Hermitian system which respect $PT$-symmetry can yield real energy eigenvalues [1]. It was also found that a fully consistent quantum theory can be developed for non-Hermitian system in a modified Hilbert space through the restoration of equivalent Hermiticity and the unitary time evolution [2, 3]. These theoretical works towards the consistency of non-Hermitian quantum mechanics (NHQM) strongly paved the way forward for NHQM to be the topic of frontier research in different areas in the last two decades [4–12]. Due to the analogy of the Schrödinger equation with certain wave equation in optics, the phenomena of NHQM can also be mapped to the analogous phenomena in optics. This leads to the possibility of experimental observation of the theoretical predictions of NHQM. This has been indeed the case and some of the predictions of NHQM have been observed in optics [13–19]. The realizations of NHQM phenomena have ignited huge interest to study the subject both theoretically and experimentally. The study of non-Hermitian system in optics has become a constant theme of further research and advancement in the subject.

The advancement in NHQM is one of the most recent developments in quantum mechanics. However, one of the earliest studied problems of quantum mechanics, the quantum tunneling [20–24], suffers with a paradox till today. How much time does a particle take to tunnel through a classically forbidden potential is still an open problem both theoretically and experimentally. In the year 1962, Hartman studied the problem of tunneling time by using
stationary phase method (SPM) for metal-insulator-metal sandwich and showed that the tunneling time for opaque barrier is independent of the thickness for sufficiently thick barrier [25]. This is known as Hartman effect, i.e., the saturation of tunneling time for an opaque barrier with the barrier thickness. Soon, this was also confirmed by an independent study by Fletcher [26]. Due to this paradox, various different authors proposed new definitions of tunneling time to account for the inconsistency (see [27]). However, so far no satisfactory definition of tunneling time has been found that agrees with the experimental results.

The calculation of tunneling time by the method of SPM for multi-barrier real potential shows that tunneling time is independent of the inter-barrier separation in the limit of large thickness of the barrier [28,29]. This is called as generalized Hartman effect in which the tunneling time is also independent of the inter-barrier separation for the tunneling through sufficiently thick opaque multi-barrier. For critical comments on generalized Hartman effect, see [30–32]. Various attempts have been made to test the finding of the theoretical results of the tunneling time. Initial experiments have indicated the superluminal nature of the tunneling time and found to be insensitive to the thickness of the tunneling region [33–39]. This superluminal nature of the tunneling time is not at variance with Special Relativity, and the phenomena of this kind have been discussed in a number of papers (see [40,41]). The tunneling time found to be paradoxically short for the case of double barrier optical grating [38] and double barrier photonic band gap [42]. The reason for Hartman effect is not clear to the present day. A reshaping of the incident wave as it interacts with the barrier has been proposed as a possible reason for the occurrence of Hartman effect [43–45]. Also, Hartman effect doesn’t occur in space fractional quantum mechanics [46,47].

To the best of our knowledge, the method of SPM has always shown the existence of Hartman effect from a real barrier potential (single barrier or multi-barrier). However, for a complex, non-PT symmetric barrier potential of the form $V_1 + iV_2$, it has been shown that Hartman effect doesn’t occur and the tunneling time depends upon the barrier thickness [48]. Also it is shown in [49], when the complex potential is in the form of a layered $PT$-symmetric potential, the Hartman effect does occur for single as well for periodic multi-barrier systems. The result of [48,49] and the Hartman effect from real barrier have motivated us to study the role of $PT$-symmetry in the occurrence of Hartman effect. We study the Hartman effect from a layered $PT$-symmetric potential and show that the occurrence of Hartman effect from a real barrier can be understood as the special limiting case of Hartman effect from a $PT$-symmetric complex system. We also study the tunneling time from a non-$PT$-symmetric potential at the symmetry breaking threshold and show that Hartman effect doesn’t occur when $PT$-symmetry is broken. Hartman effect is restored when $PT$-symmetry is respected. These results give strong indication that $PT$-symmetry plays an important role for the occurrence of Hartman effect. We further have shown explicitly that PT symmetry is crucial even for the real barrier which is shown to be the special limiting case of a layered $PT$-symmetric complex system.

We organize our paper as follows: In Sect. 2, we briefly discuss about stationary phase method of calculating the tunneling time. In Sect. 3 we discuss the Hartman effect from a ‘unit’ $PT$-symmetric system and a layered $PT$-symmetric system made by the periodic repetitions of the ‘unit’ $PT$-symmetric system. In Sect. 3.3, we show that the Hartman effect from real barrier is the special limiting case of our layered $PT$-symmetric system. In Sect. 4, we calculate the tunneling time from a non-$PT$-symmetric system at the $PT$-symmetry breaking threshold to show that when $PT$-symmetry is broken, Hartman effect is lost. We discuss the results in Sect. 5. Detail mathematical steps in obtaining various results are provided in “Appendix”.
2 Tunneling time and Hartman effect

This section briefly introduces the reader about the stationary phase method (SPM) to calculate the tunneling time [50]. In SPM, the tunneling time is defined as the time difference between the peak of the incoming and outgoing wave packet as the wave packet traverse through the potential barrier. To understand this quantitatively, consider a normalized Gaussian wave packet $G_{k_0}(k)$ of mean momentum $\bar{h}k_0$. For $t > 0$, the wave packet is given by

$$\int G_{k_0}(k)e^{i(kx-Et/\bar{h})}dk.$$  \hfill (1)

In the above $k = \sqrt{2mE}$. The wave packet is propagating to positive $x$-direction and interact with the potential barrier $V(x)$ ($V(x) = V$ for $0 \leq x \leq b$ and zero elsewhere). The transmitted wave packet is given by

$$\int G_{k_0}(k)|A(k)|e^{i(kx-Et/\bar{h}+\theta(k))}dk.$$ \hfill (2)

where $A(k) = |A(k)|e^{i\theta(k)}$ is the transmission coefficient for the potential barrier $V(x)$. By the method of SPM, the tunneling time $\tau$ is given by

$$\frac{d}{dk}(kb - \frac{Et}{\bar{h}} + \theta(k)) = 0.$$ \hfill (3)

This results in the following expression of the tunneling time:

$$\tau = \bar{h} \frac{d\theta(E)}{dE} + \frac{b}{(\frac{h}{m}k)}.$$ \hfill (4)

For a square barrier potential $V(x) = V$ of width $b$, Eq. (4) results in the following expression:

$$\tau = \bar{h} \frac{d}{dE} \tan^{-1}\left(\frac{k^2-q^2}{2kq} \tanh qb\right).$$ \hfill (5)

here $q = \sqrt{2m(V-E)/\bar{h}}$. The following things are apparent from Eq. (5).

$$\lim_{b\to0} \tau = 0, \quad \lim_{b\to\infty} \tau = \frac{2m}{\bar{h}qk}.$$ \hfill (6)

The tunneling time is expected to vanish for $b \to 0$. However, the result for $b \to \infty$ is highly unexpected as the tunneling time saturates to a finite value and is also independent of $b$. This shows that for thick barriers, the tunneling time is independent of the thickness ‘$b$’ of the barriers and doesn’t increase when $b$ increases. This is the famous Hartman effect. This is also shown graphically in Fig. 1. We will use the system of units $2m = 1, \bar{h} = 1, c = 1$ throughout the article. In these units the tunneling time from the square barrier is

$$\lim_{b\to\infty} \tau = \frac{1}{qk}.$$ \hfill (7)

3 Hartman effect from PT-symmetric barrier

In this section we show that controversial Hartman effect exists for barriers arranged in $PT$-symmetric configurations. We first study the simplest $PT$-symmetric system made by the
Fig. 1 Plot shows the Hartman effect, i.e., the saturation of tunneling time with the width of the barrier. Here the potential is a rectangular potential of height $V = 1$, particle energy $E = 0.5$, and $\hbar = 1, 2m = 1$.

![Plot showing the Hartman effect](image)

Fig. 2 A $PT$-symmetric ‘unit cell’ consisting of a pair of complex conjugate barrier. $y$-axis represents the ‘complex height’ of the potential

![Unit cell diagram](image)

complex potential $u + iv$ and $u - iv$ each of thickness $b$ and arrange adjacently without intervening gap. This is shown in Fig. 2. We call this as ‘unit’ $PT$-symmetric barrier system. Next we investigate the Hartman effect when this ‘unit’ system repeats periodically to make a layered $PT$-symmetric barrier of finite repetition $N$ as shown in Fig. 3. We also present our detailed calculations to show the $N \to \infty$ limit over a finite length $L$ of our layered $PT$-symmetric system gives the same tunneling time expression as of real rectangular barrier of height $u$ and length $L$. Therefore the Hartman effect from real barrier can be due to the Hartman effect of our layered $PT$-symmetric system in the special limiting case. For the purpose of clarity we discuss all the above three cases separately.

3.1 Unit $PT$-symmetric barrier

In this section we calculate the tunneling time and investigate the Hartman effect from the following simple $PT$-symmetric system (shown in Fig. 2)

$$V(x) = u + iv \quad \text{for} \quad -b < x < 0$$

$$V(x) = u - iv \quad \text{for} \quad 0 < x < b$$

$$V(x) = 0 \quad \text{for} \quad |x| \geq b.$$  

(8)
Fig. 3 A locally periodic $PT$-symmetric system obtained by periodic repetitions of the ‘unit’ $PT$-symmetric system shown in Fig. 2. y-axis is the ‘complex height’ of the potential

In the above $\{u, v\} \in R^+$. It will be shown that the $PT$-symmetric potential given by Eq. (8) display Hartman effect. The transmission coefficient ($t$) for this potential can be easily calculated and given by (Derivation of this by transfer matrix approach is provided in “Appendix-A”)

$$ t = \frac{e^{i(\theta - 2kb)}}{\sqrt{\xi^2 + \chi^2}}, \quad \theta = \tan^{-1}\left(\frac{\chi}{\xi}\right). \quad (9) $$

Various symbols appearing in Eq. (9) are defined below:

$$ \xi = \frac{1}{2} (\cos 2\beta + \cosh 2\alpha) + \cos 2\phi \left(\cosh^2 \alpha \sin^2 \beta + \cos^2 \beta \sinh^2 \alpha\right), \quad (10) $$

$$ \chi = \frac{1}{2} (U_+ \sin \phi \sin 2\beta + U_- \cos \phi \sin 2\alpha). \quad (11) $$

In Eqs. (10) and (11), the quantities $\alpha, \beta, \phi$ and $U_\pm$ are given by

$$ \alpha = b \rho \cos \phi, \quad \beta = b \rho \sin \phi, \quad U_\pm = \frac{k}{\rho} \pm \frac{\rho}{k}, \quad (12) $$

and

$$ \phi = \frac{1}{2} \tan^{-1}\left(\frac{v}{u - k^2}\right), \quad \rho = \left[(u - k^2)^2 + v^2\right]^\frac{1}{2}. \quad (13) $$

Using SPM, the tunneling time ($\tau$) is given by

$$ \tau = \frac{1}{2k} \left(\frac{\xi \chi' - \chi \xi'}{\xi^2 + \chi^2}\right). \quad (14) $$

We will use ‘$'$’ (prime) to denote the derivatives with respect to the magnitude of wave vector $k = \sqrt{E}$. The expressions for $\xi'$ and $\chi'$ are provided below:

$$ \xi' = 2\alpha' \cos^2 \phi \sinh 2\alpha - 2\beta' \sin 2\beta' \sin 2\phi \sin 2\beta + \phi' \sin 2\phi (\cos 2\beta - \cosh 2\alpha). \quad (15) $$

$$ \chi' = \frac{1}{2} \sin \phi \left(U'_+ \sin 2\beta + 2\beta' U_+ \cos 2\beta - \phi' U_- \sinh 2\alpha\right) $$

$$ + \frac{1}{2} \cos \phi \left(U'_- \sinh 2\alpha + 2\alpha' U_- \cosh 2\alpha + \phi' U_+ \sin 2\beta\right). \quad (16) $$
The width dependency in tunneling time enters through $\alpha$, $\alpha'$, $\beta$ and $\beta'$. Therefore for the existence of Hartman effect $\tau$ must be independent of these four quantities in the limit $b \to \infty$. It can be shown that

$$\lim_{b \to \infty} \xi = \frac{e^{2\alpha}}{2} \cos^2 \phi,$$

$$\lim_{b \to \infty} \chi = \frac{U_-}{4} e^{2\alpha} \cos \phi,$$

$$\lim_{b \to \infty} \xi' = e^{2\alpha} \left( \alpha' \cos^2 \phi - \frac{\phi'}{2} \sin 2\phi \right),$$

$$\lim_{b \to \infty} \chi' = \frac{e^{2\alpha}}{4} \left( \cos \phi (U'_- + 2\alpha' U_-) - U_- \phi' \sin \phi \right).$$

Using the results of 17–20 in Eq.(14), we find

$$\lim_{b \to \infty} \tau = \tau_\infty = \frac{U'_- \cos \phi + \phi' U_- \sin \phi}{k (4 \cos^2 \phi + U^2_-)}.$$  \hspace{1cm} (21)

In the expression of $\tau_\infty$, $b$ dependent terms don’t appear. This proves that the $PT$-symmetric system given by Eq. (8) shows Hartman effect.

3.2 Layered PT-symmetric barrier

Next we calculate the tunneling time from a layered (locally periodic) $PT$-symmetric system obtained by periodic repetitions of the ‘unit cell’ $PT$-symmetric system of Eq.(8). The layered $PT$-symmetric system is shown in Fig. 3. The net spatial extent of the layered $PT$-symmetric system is $L = 2Nb$ where $N$ is the number of repetitions. It is easy to show that the transmission coefficient from such a system is (see “Appendix-B” for derivation)

$$t = \frac{e^{-ikL}}{H(k)},$$  \hspace{1cm} (22)

where $H(k)$ is given as

$$H(k) = (\xi - i\chi)U_{N-1}(\xi) - U_{N-2}(\xi).$$  \hspace{1cm} (23)

here $U_N(\xi)$ is the $N$th Chebyshev polynomial of the second kind. The phase of the transmission coefficient is given by

$$\Theta = \tan^{-1}(g\chi) - kL,$$  \hspace{1cm} (24)

where

$$g = \frac{U_{N-1}(\xi)}{T_N(\xi)},$$  \hspace{1cm} (25)

and $T_N(\xi)$ is the $N$th Chebyshev polynomial of the first kind. In deriving Eq. (24), we have used the following identity related to Chebyshev polynomials of first and second kind

$$T_N(\xi) = \xi U_{N-1}(\xi) - U_{N-2}(\xi).$$  \hspace{1cm} (26)

From the knowledge of the phase of the transmission coefficient, the tunneling time is calculated to be

$$\tau_N = \frac{1}{2k(1 + g^2\chi^2)} \left[ g\chi' + \chi \left( \frac{N\xi'}{\xi^2 - 1} - N\xi'g^2 - \frac{g\xi\xi'}{\xi^2 - 1} \right) \right].$$  \hspace{1cm} (27)
To investigate Hartman effect, we have to evaluate $b \to \infty$ limit of Eq. (27). The necessary steps of the calculations are provided below. It can be shown that

$$
\lim_{b \to \infty} g \sim \frac{1}{\xi (b \to \infty)}.
$$

(28)

Thus,

$$
\lim_{b \to \infty} g \chi \sim \lim_{b \to \infty} \left( \frac{\chi}{\xi} \right).
$$

(29)

The $b \to \infty$ limits for $\xi$ and $\chi$ are derived in Eqs. (17) and (18), respectively. Using these limits it can be easily shown that

$$
\lim_{b \to \infty} \frac{\chi}{\xi} = \frac{U_-}{2} \sec \phi = \eta.
$$

(30)

where the quantity $\eta$ is independent of $b$. Comparing Eqs. (29) and (30) we arrive at

$$
\lim_{b \to \infty} g \chi = \eta.
$$

(31)

In Eq. (31) we have used the equality sign as the limiting value is independent of $b$. Next we evaluate $b \to \infty$ limit of the following term appearing in the right-hand side of Eq. (27)

$$
\lim_{b \to \infty} \left( \frac{N \xi' \xi}{\xi^2 - 1} - \frac{N \xi' g^2 - \xi' g \xi' g}{\xi^2 - 1} \right) = \lim_{b \to \infty} \left( \frac{N \xi' \xi}{\xi^2} - \frac{N \xi' \xi}{\xi^2} - \frac{\xi' \xi}{\xi^2} \right) = - \lim_{b \to \infty} \left( \frac{\xi' \xi}{\xi^2} \right).
$$

(32)

where we have used Eq. (28) and $\lim_{b \to \infty} \xi > 1$ in arriving at the above result. $\lim_{b \to \infty} \xi >> 1$ is evident due to exponential dependence of $\xi$ over $b$ in the limit $b \to \infty$ [see Eq. (17)]. Now, using Eqs. (28), (31) and (32), the limiting value of $\tau_N$ can be calculated as

$$
\lim_{b \to \infty} \tau_N = \frac{1}{2k} \left( \frac{1}{1 + \eta^2} \right) \left[ \lim_{b \to \infty} \left( \frac{\chi'}{\xi} - \frac{\chi' \xi'}{\xi^2} \right) \right].
$$

(33)

Using the results from Eqs. (17–20) we can easily calculate the following limit:

$$
\lim_{b \to \infty} \left( \frac{\chi'}{\xi} - \frac{\chi' \xi'}{\xi^2} \right) = \frac{2}{\cos \phi} \left( U_- + \phi U_- \tan \phi \right).
$$

(34)

Next we use Eqs. (30) (expression for $\eta$) and (34) in Eq. (33) to finally show that

$$
\lim_{b \to \infty} \tau_N = \lim_{b \to \infty} \tau = \tau_\infty.
$$

(35)

Eq. (35) proves the Hartman effect from the layered PT-symmetric system represented by Fig. 3.

### 3.3 Real barrier

In this subsection we show that the famously known Hartman effect from a real barrier is the special limiting case of $PT$-symmetric system considered in Sect. 3.2. We first show that the transmission phase of a rectangular barrier of height $u$ and width $L$ is the limiting case of

$$
\lim_{b \to \infty} \tau_N = \lim_{b \to \infty} \tau = \tau_\infty.
$$

(35)
\( N \to \infty \) of our layered \( PT \)-symmetric system such that \( b = \frac{L}{2N} \) where \( L \) is fixed. To prove this, we first Taylor expand the quantity \( g \chi \) in power of \( b \) such that

\[
g \chi = A_0 + \sum_{j=1}^{\infty} A_j b^j.
\]  

(36)

It is found that \( A_0 = 0 \) and the coefficients of even power of \( b \) are also zero. Therefore

\[
g \chi = A_1 b + A_3 b^3 + A_5 b^5 + A_7 b^7 + A_9 b^9 + \ldots.
\]  

(37)

The coefficients of various powers of \( b \) are given by

\[
A_1 = N \rho (U_- \cos^2 \phi + U_+ \sin^2 \phi),
\]  

(38)

\[
A_3 = -\frac{1}{6} N \rho^3 \left[ 8N^2(U_+ + U_-) \cos 2\phi - (U_+ - U_-)(4N^2 - 1 + (4N^2 + 1) \cos 4\phi) \right],
\]  

(39)

\[
A_5 = \frac{N \rho^5}{360} \left[ 2(U_+ + U_-)(96N^4 - 5N^2 - 1) - (U_+ - U_-) \cos 2\phi(288N^4 - 25N^2 - 8) + 2(U_+ + U_-) \cos 4\phi(96N^4 + 5N^2 + 1) - (U_+ - U_-) \cos 6\phi(96N^4 + 25N^2 + 8) \right],
\]  

(40)

\[
A_7 = \frac{N \rho^7}{15120} \left[ (U_+ - U_-)(9792N^6 - 1008N^4 - 161N^2 - 34) - 4(U_+ + U_-) \cos 2\phi(4896N^6 - 168N^4 - 35N^2 - 10) + 4(U_+ - U_-) \cos 4\phi(3264N^4 - 35N^2 - 16) - 4(U_+ + U_-) \cos 6\phi(1632N^4 + 168N^4 + 35N^2 + 10) + (U_+ - U_-) \cos 8\phi(3264N^6 + 1008N^4 + 301N^2 + 98) \right],
\]  

(41)

\[
A_9 = \frac{N \rho^9}{453,600} \left[ 4(U_+ + U_-)(119,040N^8 - 6120N^6 - 1029N^4 - 215N^2 - 61) - 2(U_+ - U_-) \cos 2\phi(396,800N^8 - 28,560N^6 - 5502N^4 - 1045N^2 - 268) + 40(U_+ + U_-) \cos 4\phi(15,872N^8 - 42N^4 - 15N^2 - 5) - (U_+ - U_-) \cos 6\phi(396,800N^8 + 28,560N^6 + 1722N^4 - 655N^2 - 352) + 4(U_+ + U_-) \cos 8\phi(39,680N^8 + 6120N^6 + 1449N^4 + 365N^2 + 111) - (U_+ - U_-) \cos 10\phi(79,360N^8 + 28,560N^6 + 9282N^4 + 2745N^2 + 888) \right].
\]  

(42)

Similarly other coefficients \( A_{11}, A_{13}, A_{15}, \text{etc.}, \) can also be calculated. Next we take the limiting case \( N \to \infty \) of these coefficients to show

\[
\lim_{N \to \infty} A_1 b = \left( \frac{k^2 - q^2}{2kq} \right) (qL),
\]  

(43)

\[
\lim_{N \to \infty} A_3 b^3 = -\frac{1}{3} \left( \frac{k^2 - q^2}{2kq} \right) (qL)^3,
\]  

(44)

\[
\lim_{N \to \infty} A_5 b^5 = \frac{2}{15} \left( \frac{k^2 - q^2}{2kq} \right) (qL)^5,
\]  

(45)

\[
\lim_{N \to \infty} A_7 b^7 = -\frac{17}{315} \left( \frac{k^2 - q^2}{2kq} \right) (qL)^7.
\]  

(46)
\[
\lim_{N \to \infty} A_0 b^9 = \frac{31}{2835} \left( \frac{k^2 - q^2}{2kq} \right) (qL)^9.
\] (47)

where on deriving Eqs. (43)–(47), we have identified \( L = 2Nb \) and expanded \( \cos 4\phi, \cos 6\phi, \cos 8\phi \) and \( \cos 10\phi \) in power of \( \cos 2\phi = \frac{q^2}{\rho^2} \). The detail calculations of deriving Eqs. (43)–(47) are shown in “Appendix-C”. Thus,

\[
\lim_{N \to \infty, b \to 0} g \chi = \left( \frac{k^2 - q^2}{2kq} \right) qL - \frac{17}{315} (qL)^7 + \frac{31}{2835} (qL)^9 - \ldots .
\] (48)

And we identify

\[
\lim_{N \to \infty, b \to 0} g \chi = \left( \frac{k^2 - q^2}{2kq} \right) \tanh qL, \quad L = 2Nb
\] (49)

Using Eq. (49) in Eq. (24) we find

\[
\lim_{N \to \infty, b \to 0} \Theta = \tan^{-1} \left( \frac{k^2 - q^2}{2kq} \tanh qL \right) - kL.
\] (50)

Thus the tunneling time,

\[
\lim_{N \to \infty, b \to 0} \tau_N = \frac{d}{dE} \left[ \tan^{-1} \left( \frac{k^2 - q^2}{2kq} \tanh qL \right) \right].
\] (51)

Equation (51) yields the same value of tunneling time as Eq. (7). This result shows that Hartman effect in real barrier occurs due to the limiting case \( N \to \infty \) of our layered \( PT \)-symmetric system such that each layered structure becomes infinitely thin. It can be shown that a real barrier of height \( u \) and width \( L \) is the limiting case \( N \to \infty \) of our layered \( PT \)-symmetric system such that \( b = \frac{L}{2N} \) (we have left the complete exercise; however, one can show that it exactly gives the expression of reflection and transmission coefficient of rectangular barrier. We have checked this numerically also). Here \( L \) is the net spatial extent of the layered \( PT \)-symmetric system.

4 Non-\( PT \)-symmetric barrier system: No Hartman effect

In this section we calculate the tunneling time by SPM method from the following non-Hermitian system

\[
V(x) = u + iv, \quad \text{for} \quad -b < x < 0.
\]

\[
V(x) = u - iv, \quad \text{for} \quad 0 < x < b.
\]

\[
V(x) = 0, \quad \text{for} \quad |x| \geq b.
\] (52)

where we have \( \varepsilon \in \mathbb{R} \). The system is shown graphically in Fig. 4. When \( \varepsilon = 1 \), the non-Hermitian system represented by Eq. (52) is \( PT \)-symmetric and is identical to the system represented by Eq. (8). The transmission coefficient from this system can be calculated and is given by

\[
t = \frac{e^{-2ikb}}{Q}.
\] (53)
Fig. 4 A non-Hermitian ‘unit cell’ consisting of a pair of complex barriers. y-axis represents the ‘complex height’ of the potential. Note that the system becomes PT-symmetry when $\varepsilon = 1$ and in this case identical to the ‘unit cell’ shown in Fig. 2.

where

$$Q = P_1^+ P_2^- - S_1 S_2,$$

and the various symbols are given by

$$P_{1,2}^\pm = 2 \cos k_{1,2} b \pm i \left( \mu_{1,2} + \frac{1}{\mu_{1,2}} \right) \sin k_{1,2} b,$$

$$S_{1,2} = i \left( \mu_{1,2} - \frac{1}{\mu_{1,2}} \right) \sin k_{1,2} b.$$  

here

$$\mu_{1,2} = \frac{k_{1,2}}{k},$$

and,

$$k_{1,2} = \sqrt{E - V_{1,2}}, \ V_1 = u + iv, \ V_2 = u - i\varepsilon v.$$  

We express $k_1 = \rho_1 e^{i\phi_1}, \ k_2 = \rho_2 e^{-i\phi_2}$ and define the following quantities

$$H_{\pm} = \frac{\rho_1 \rho_2}{k^2} \pm \frac{k^2}{\rho_1 \rho_2}, \ G_{\pm} = \frac{\rho_1}{\rho_2} \pm \frac{\rho_2}{\rho_1}.$$  

$$J_{1,2}^\pm = \frac{\rho_{1,2}}{k^2} \pm \frac{k^2}{\rho_{1,2}}.$$  

$$\rho_1 = \left[ (u - k^2)^2 + v^2 \right]^{\frac{1}{2}}, \ \rho_2 = \left[ (u - k^2)^2 + \varepsilon^2 v^2 \right]^{\frac{1}{2}}.$$  

$$\phi_1 = \frac{1}{2} \tan^{-1} \left( \frac{v}{u - k^2} \right), \ \phi_2 = \frac{1}{2} \tan^{-1} \left( \frac{\varepsilon v}{u - k^2} \right).$$  

Through the use of the above quantities, we separate Q in real and imaginary parts [to obtain the phase of transmission coefficient given by Eq. (53)]

$$Q = (A_1 - A_2) + i(B_1 - B_2).$$
where

\[ A_1 = 4z_1 + 2(x_2 J_2^+ \cos \phi_2 - x_1 J_2^- \sin \phi_2) + 2(w_1 J_1^- \sin \phi_1 + w_2 J_1^+ \cos \phi_1) \]
\[ - y_1(H_+ \cos (\phi_1 - \phi_2) + G_+ \cos (\phi_1 + \phi_2)) + y_2(H_\mp \sin (\phi_1 - \phi_2) + G_\mp \sin (\phi_1 + \phi_2)) \]

(64)

\[ B_1 = 4z_2 - 2(x_1 J_2^+ \cos \phi_2 + x_2 J_2^- \sin \phi_2) - 2(w_1 J_1^+ \cos \phi_1 - w_2 J_1^- \sin \phi_1) \]
\[ - y_1(H_- \sin (\phi_1 - \phi_2) + G_- \sin (\phi_1 + \phi_2)) - y_2(H_+ \cos (\phi_1 - \phi_2) + G_+ \cos (\phi_1 + \phi_2)) \]

(65)

\[ A_2 = y_2(H_- \sin (\phi_1 - \phi_2) - G_- \sin (\phi_1 + \phi_2)) - y_1(H_+ \cos (\phi_1 - \phi_2) - G_+ \cos (\phi_1 + \phi_2)) \]

(66)

\[ B_2 = - y_2(H_+ \cos (\phi_1 - \phi_2) - G_+ \cos (\phi_1 + \phi_2)) - y_1(H_- \sin (\phi_1 - \phi_2) - G_- \sin (\phi_1 + \phi_2)) \]

(67)

In the above, the quantities \( w_{1,2}, x_{1,2}, y_{1,2}, z_{1,2} \) are due to

\[ \sin k_1 b \cos k_2 b = w_1 + i w_2, \quad \cos k_1 b \sin k_2 b = x_1 + i x_2, \]

(68)

\[ \sin k_1 b \sin k_2 b = y_1 + i y_2, \quad \cos k_1 b \cos k_2 b = z_1 + i z_2. \]

(69)

The expressions of \( w_{1,2}, x_{1,2}, y_{1,2}, z_{1,2} \) are given below:

\[ w_1 = \cos \alpha_{22} \cosh \beta_{11} \cos \beta_{22} \sin \alpha_{11} - \cos \alpha_{11} \sin \alpha_{22} \sinh \beta_{11} \sinh \beta_{22}, \]

(70)

\[ w_2 = \cos \alpha_{11} \cos \alpha_{22} \cosh \beta_{22} \sin \alpha_{11} \sin \alpha_{22} \sinh \beta_{11} \sinh \beta_{22}, \]

(71)

\[ x_1 = \cos \alpha_{11} \cosh \beta_{11} \sin \beta_{22} \sin \alpha_{22} \sin \beta_{11} \cosh \beta_{22}, \]

(72)

\[ x_2 = - \cosh \beta_{22} \sin \alpha_{11} \sin \alpha_{22} \sin \beta_{11} - \cos \alpha_{11} \cos \alpha_{22} \cosh \beta_{11} \sinh \beta_{22}, \]

(73)

\[ y_1 = \cosh \beta_{11} \cosh \beta_{22} \sin \alpha_{11} \sin \alpha_{22} \sinh \beta_{11} \cosh \beta_{22}, \]

(74)

\[ y_2 = \cos \alpha_{11} \cosh \beta_{22} \sin \alpha_{22} \sin \beta_{11} - \cos \alpha_{22} \cosh \beta_{11} \sin \alpha_{11} \sin \beta_{22}, \]

(75)

\[ z_1 = \cos \alpha_{11} \cosh \beta_{11} \sin \beta_{22} \sin \alpha_{22} \cosh \beta_{11} \sin \beta_{22}, \]

(76)

\[ z_2 = - \cos \alpha_{22} \cosh \beta_{22} \sin \alpha_{11} \sin \beta_{11} + \cos \alpha_{11} \cosh \beta_{11} \sin \alpha_{22} \sinh \beta_{22}. \]

(77)

In the above

\[ \alpha_{ij} = \eta \rho_i \cos \phi_j, \quad \beta_{ij} = \eta \rho_i \sin \phi_j. \]

(78)

Now, the phase of transmission coefficient can be found as

\[ \theta = \Phi_\varepsilon - 2k b. \]

(79)

where we have

\[ \Phi_\varepsilon = \tan^{-1} \left( \frac{B_2 - B_1}{A_1 - A_2} \right). \]

(80)

Thus the tunneling time is

\[ \tau_\varepsilon = \frac{d}{dE} (\Phi_\varepsilon - 2k b) + \frac{2b}{2k}. \]

(81)

The last term of R.H.S. in the above equation is due to the free propagation time of traversing the length \( 2b \). The net tunneling time can be written as
\[ \tau_\varepsilon = \frac{d\Phi_E}{dE} = \frac{1}{2k} \frac{d\Phi_E}{dk}. \]  

(82)

In order to analyze the effect of \( PT \)– symmetry over Hartman effect, we Taylor expand \( \tau_\varepsilon \) near \( \varepsilon \sim 1 \) to first order as follows:

\[ \tau_\varepsilon = \tau_\varepsilon(\varepsilon = 1) + \left( \frac{d\tau_\varepsilon}{d\varepsilon} \right)_{\varepsilon = 1} (\varepsilon - 1). \]  

(83)

We take the limit \( \lim_{b \to \infty} \) of Eq. (83) to study Hartman effect near the symmetry breaking threshold \( \varepsilon = 1 \). Taking the limit of Eq. (83),

\[ \lim_{b \to \infty} \tau_\varepsilon = \lim_{b \to \infty} \tau_\varepsilon(\varepsilon = 1) + \lim_{b \to \infty} \left( \frac{d\tau_\varepsilon}{d\varepsilon} \right)_{\varepsilon = 1} (\varepsilon - 1). \]  

(84)

The first term of right-hand side is \( \tau_\infty \) and is independent of \( b \). Thus,

\[ \lim_{b \to \infty} \tau_\varepsilon = \tau_\infty + \lim_{b \to \infty} \left( \frac{d\tau_\varepsilon}{d\varepsilon} \right)_{\varepsilon = 1} (\varepsilon - 1). \]  

(85)

Therefore to find whether Hartman effect exists or not when \( PT \)– symmetry is broken, we investigate the second term of R.H.S about its dependency on the thickness \( b \) in the limit \( b \to \infty \). We first evaluate the following derivative in the limit \( b \to \infty \),

\[ \lim_{b \to \infty} \left( \frac{d\tau_\varepsilon}{d\varepsilon} \right) = \frac{1}{2k} \lim_{b \to \infty} \left( \frac{d}{d\varepsilon} \left( \frac{d\Phi_E}{dk} \right) \right). \]  

(86)

As \( \varepsilon, k \) and \( b \) are independent quantities, we can take the limit inside the differential sign. Thus we write

\[ \lim_{b \to \infty} \left( \frac{d\tau_\varepsilon}{d\varepsilon} \right) = \frac{1}{2k} \left[ \frac{d}{d\varepsilon} \left( \frac{d\Phi_E}{dk} \left( \lim_{b \to \infty} \Phi_e \right) \right) \right]. \]  

(87)

In the next we evaluate \( \lim_{b \to \infty} \Phi_e \). For this we evaluate the limiting values of the following quantities:

\[ \lim_{b \to \infty} z_1 = \frac{1}{4} e^{\beta_1 + \beta_2} \cos(\alpha_{11} - \alpha_{22}), \]  

(88)

\[ \lim_{b \to \infty} z_2 = \frac{1}{4} e^{\beta_1 + \beta_2} \sin(\alpha_{22} - \alpha_{11}), \]  

(89)

\[ \lim_{b \to \infty} y_1 = \frac{1}{4} e^{\beta_1 + \beta_2} \cos(\alpha_{11} - \alpha_{22}), \]  

(90)

\[ \lim_{b \to \infty} y_2 = \frac{1}{4} e^{\beta_1 + \beta_2} \sin(\alpha_{22} - \alpha_{11}), \]  

(91)

\[ \lim_{b \to \infty} x_1 = \frac{1}{4} e^{\beta_1 + \beta_2} \sin(\alpha_{22} - \alpha_{11}), \]  

(92)

\[ \lim_{b \to \infty} x_2 = -\frac{1}{4} e^{\beta_1 + \beta_2} \cos(\alpha_{11} - \alpha_{22}), \]  

(93)

\[ \lim_{b \to \infty} w_1 = \frac{1}{4} e^{\beta_1 + \beta_2} \sin(\alpha_{11} - \alpha_{22}), \]  

(94)

\[ \lim_{b \to \infty} w_2 = \frac{1}{4} e^{\beta_1 + \beta_2} \cos(\alpha_{11} - \alpha_{22}). \]  

(95)

It is observe that

\[ \lim_{b \to \infty} z_1 = \lim_{b \to \infty} y_1. \]  

(96)
\[
\lim_{b \to \infty} z_2 = \lim_{b \to \infty} y_2, \tag{97}
\]
\[
\lim_{b \to \infty} x_1 = \lim_{b \to \infty} y_2, \tag{98}
\]
\[
\lim_{b \to \infty} x_2 = -\lim_{b \to \infty} y_1, \tag{99}
\]
\[
\lim_{b \to \infty} w_1 = -\lim_{b \to \infty} y_2, \tag{100}
\]
\[
\lim_{b \to \infty} w_2 = \lim_{b \to \infty} y_1. \tag{101}
\]

We define
\[
Y_1 = \lim_{b \to \infty} y_1, \quad Y_2 = \lim_{b \to \infty} y_2, \tag{102}
\]
so that
\[
Y_2 = -Y_1 \tan (\alpha_{11} - \alpha_{22}). \tag{103}
\]

With these results we simplify the expression of \( \Phi_\varepsilon \) in the limit \( b \to \infty \) to obtain
\[
\lim_{b \to \infty} \Phi_\varepsilon = \tan^{-1} \left( \frac{Q_1 \tan \zeta - Q_2}{Q_1 + Q_2 \tan \zeta} \right). \tag{104}
\]

where \( Q_1 \) and \( Q_2 \) are given by
\[
Q_1 = 2 + J_1^+ \cos \phi_1 - J_2^+ \cos \phi_2 - G_+ \cos (\phi_1 + \phi_2), \tag{105}
\]
\[
Q_2 = J_1^- \sin \phi_1 + J_2^- \sin \phi_2 - G_- \sin (\phi_1 + \phi_2). \tag{106}
\]

For future calculations in mind, we define the quantity \( P \) as
\[
P = \frac{Q_1 \tan \zeta - Q_2}{Q_1 + Q_2 \tan \zeta}. \tag{107}
\]

so that
\[
\lim_{b \to \infty} \Phi_\varepsilon = \tan^{-1} P. \tag{108}
\]

Using Eq. (108) in Eq. (87), we find the following expression at \( \varepsilon = 1 \),
\[
\left[ \lim_{b \to \infty} \left( \frac{d\tau_\varepsilon}{d\varepsilon} \right) \right]_{\varepsilon = 1} = \frac{1}{2k} \left[ \frac{1}{(1 + P^2)} \frac{d^2 P}{d\varepsilon dk} - \frac{2P}{(1 + P^2)^2} \frac{dP}{dk} \frac{dP}{d\varepsilon} \right]_{\varepsilon = 1}. \tag{109}
\]

It is a massive calculation to evaluate the right-hand side of Eq. (109). The details of the calculations are provided in “Appendix-D”. The term in the parenthesis of the right-hand side is given by
\[
\left[ \frac{1}{(1 + P^2)} \frac{d^2 P}{d\varepsilon dk} - \frac{2P}{(1 + P^2)^2} \frac{dP}{dk} \frac{dP}{d\varepsilon} \right]_{\varepsilon = 1} = K_0 + K_1 b. \tag{110}
\]

The expressions for \( K_0 \) and \( K_1 \) are provided in the “Appendix-D”. Both \( K_0 \) and \( K_1 \), are independent of ‘\( b \)’. Now, the net tunneling time in the vicinity of \( \varepsilon \sim 1 \) for large thickness ‘\( 2b \)’ is
\[
\lim_{b \to \infty} \tau_\varepsilon = \left[ \tau_\infty + \frac{K_0}{2k} (\varepsilon - 1) \right] + \frac{K_1}{2k} (\varepsilon - 1) b. \tag{111}
\]

It is clear from Eq. (111) that the tunneling time depends upon the thickness when \( \varepsilon \neq 1 \), i.e., Hartman effect is lost when the \( PT \)-symmetry is broken. It is easily seen from Eq. (111) that Hartman effect is restored when \( \varepsilon = 1 \) i.e when the system recovers the \( PT \)-symmetry.
5 Conclusions and discussions

We have investigated the role of PT-symmetry for the occurrence of Hartman effect. We have considered a ‘unit cell’ PT-symmetric potential made by two complex potentials \( u + iv \) and \( u - iv \) without an inter barrier separation and each of equal width \( 'b' \). We found that when \( b \to \infty \), the tunneling time saturates and becomes independent of \( b \). Thus Hartman effect exists in this PT-symmetric potential. Further it was found that layered PT-symmetric potential made by an arbitrary \( N \) repetitions of this ‘unit cell’ potential also shows Hartman effect. We have analytically investigated the case of infinite \( N \) repetitions over finite spatial length \( L \) and found that \( N \to \infty \) limit results in the same analytical expression of tunneling time as that of rectangular barrier of height \( 'u' \) and width \( L \). This result shows that the Hartman effect from a real barrier can be due to the Hartman effect from our layered PT-symmetric system. Also, the real rectangular barrier of height \( u \) and \( L \) is the limiting case \( N \to \infty \) of our layered PT-symmetric system over fixed spatial length \( L \).

To study the occurrence of Hartman effect at symmetry breaking threshold, we consider the tunneling time through a non-Hermitian potential made by two potentials of height \( u + iv \) and \( u - i \epsilon v \) each of equal width \( 'b' \). Expression of tunneling time is obtained analytically at the symmetry breaking threshold \( \epsilon \sim 1 \). It is found that when \( \epsilon \neq 1 \), i.e., when PT-symmetry is broken, the Hartman effect is lost from the system. However, when \( \epsilon = 1 \), i.e., when PT-symmetry is respected, the Hartman effect is restored. This result along with the result of our layered PT-symmetric system and its limiting case \( N \to \infty \) for fixed \( L \) indicates that PT-symmetry could be playing an important role for the occurrence of Hartman effect.

In the present study, the role of PT-symmetry in the occurrence of Hartman effect is investigated based on specific examples of PT-symmetric and non-PT-symmetric systems. Whether Hartman effect is a general property of PT-symmetric systems needs a detailed investigations. This will be worth exploring as it might lead to understand the fundamental cause behind the occurrence of Hartman effect.

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Appendix-A : Derivation of transmission coefficient from unit PT-symmetric barrier

The transfer matrices for the two barriers ‘1’ and ‘2’ as labeled in Fig. 2 are given by

\[
M_{1,2}(k) = \frac{1}{2} \begin{pmatrix}
\frac{1}{2} e^{-ikb} P_{1,2}^{1,2} & \frac{1}{2} e^{-ikb(1+2j)} S_{1,2}^{1,2} \\
\frac{1}{2} e^{ikb(1+j)} S_{1,2}^{1,2} & -\frac{1}{2} e^{ikb} P_{1,2}^{1,2}
\end{pmatrix}.
\]

In the above matrix, \( j = 0 \) for barrier-1 and \( j = 1 \) for barrier-2. Various symbols are given below:

\[
P_{1,2}^{1,2} = 2 \cos k_{1,2} b \pm i \left( \mu_{1,2} + \frac{1}{\mu_{1,2}} \right) \sin k_{1,2} b, \tag{113}
\]

\[
S_{1,2}^{1,2} = i \left( \mu_{1,2} - \frac{1}{\mu_{1,2}} \right) \sin k_{1,2} b, \tag{114}
\]

\[
\mu_{1,2} = \frac{k_{1,2}}{k}, \quad k_{1,2} = \sqrt{k^2 - V_{1,2}}. \tag{115}
\]
For the potential represented by Eq. (8) (or by Fig. 2) \( V_1 = u + i v \) and \( V_2 = u - i v \). From the composition properties of the transfer matrix, we can find the net transfer matrix \( M(k) = M_2(k).M_1(k) \). Therefore,

\[
M(k) = \frac{1}{4} \left( e^{-2ikb \left( P_+^1P_+^2 - S_1^1S_2^2 \right)} - e^{2ikb \left( P_+^2S_1^1 + P_+^1S_2^2 \right)} + e^{2ikb \left( P_-^1P_-^2 - S_1^1S_2^2 \right)} - e^{-2ikb \left( P_-^2S_1^1 + P_-^1S_2^2 \right)} \right). \tag{116}
\]

Now the transmission coefficient (inverse of the \( M_{22} \) element) can be expressed as

\[
t = \frac{e^{-2ikb}}{P_+^1P_-^2 - S_1^1S_2^2}. \tag{117}\]

We separate the denominator in real and imaginary parts. To do this we first express \( k_1 = \sqrt{k^2 - (u + iv)} = \rho e^{i\phi} \) and \( k_2 = \sqrt{k^2 - (u - iv)} = \rho e^{-i\phi} \) where

\[
\rho = \sqrt{(u - k^2)^2 + v^2}, \quad \phi = \frac{1}{2} \tan^{-1} \left( \frac{v}{u - k^2} \right). \tag{118}\]

Upon substituting \( k_{1,2} \) expressions, the denominator of Eq. (117) is simplified to

\[
P_+^1P_-^2 - S_1^1S_2^2 = \xi - i \chi = (\sqrt{\xi^2 + \chi^2}e^{-i\theta}, \tag{119}\]

where

\[
\theta = \tan^{-1} \left( \frac{\chi}{\xi} \right). \tag{120}\]

\( \xi \) and \( \chi \) are given by Eqs. (10) and (11), respectively. Substitution of Eq. (119) in Eq. (117) leads to

\[
t = \frac{e^{i(\theta - 2kb)}}{\sqrt{\xi^2 + \chi^2}}. \tag{121}\]

This is Eq. (9).

**Appendix-B : Derivation of the transmission coefficient from finite layered PT-symmetric barrier**

If the transfer matrix \( M \),

\[
M(k) = \begin{pmatrix} M_{11}(k) & M_{12}(k) \\ M_{21}(k) & M_{22}(k) \end{pmatrix} \tag{122}
\]

of a ‘unit cell’ potential is known such that

\[
\begin{pmatrix} A_+(k) \\ B_+(k) \end{pmatrix} = M(k) \begin{pmatrix} A_-(k) \\ B_-(k) \end{pmatrix}. \tag{123}
\]

Where coefficients of the asymptotic solution of the scattering wave to the right of the potential are \( A_+, B_+ \) and to the left of the potential are \( A_-, B_- \). Then the transmission coefficient (for incidence from left) of a periodic system made by \( n \) repetitions of the ‘unit cell’ is given by

\[
t_n = \frac{e^{-ikns}}{M_{22}(k)e^{-iks}U_{n-1}(\Omega) - U_{n-2}(\Omega)}. \tag{124}\]
where
\[
\Omega = \frac{1}{2} \left( M_{11} e^{iks} + M_{22} e^{-iks} \right),
\] (125)

with \( s = w + g \). Here \( w \) is width of the ‘unit cell’ potential and \( g \) is the gap between consecutive ‘unit cell’ potentials. For our present problem (Sect. 3.2), \( w = 2b \) and \( g = 0 \), thus \( s = 2b \). The procedure to derive Eq. (124) is outlined in [51]. \( M_{11} \) and \( M_{22} \) elements of our ‘unit cell’ potential are given in Eq. (116). Using Eq. (119), \( M_{22} \) element can be written as
\[
M_{22} = (\xi - i\chi) e^{2ikb}.
\] (126)

Similarly,
\[
M_{11} = (\xi + i\chi) e^{-2ikb}.
\] (127)

To arrive at Eq. (127), we have separated term \( P_1^1 P_2^2 - S_1^1 S_2^2 \) in real and imaginary parts. The expression for \( \xi \) and \( \chi \) are given by Eqs. (10) and (11), respectively. From simplified expressions of \( M_{11} \) and \( M_{22} \), we observe \( M_{22} = M_{11}^* \). This shows that the argument, \( \Omega \) of Chebyshev polynomial is real. We substitute Eqs. (126) and (127) in Eq. (125) to obtain \( \Omega = \xi \). Identifying \( ns = 2Nb = L \), where \( L \) is the net spatial extent of our layered \( PT \)-symmetric system, the final expression of transmission coefficient \( t_n = t \) is given by
\[
t = \frac{e^{-ikL}}{H(k)}
\] (128)

where \( H(k) = (\xi - i\chi)U_{N-1}(\Omega) - U_{N-2}(\Omega) \) [Eq. (22)].

**Appendix-C: Limiting values of the terms of series expansion of \( g\chi \)**

The expressions for \( A_1 \) is
\[
A_1 = N\rho(U_- \cos^2 \phi + U_+ \sin^2 \phi).
\] (129)

Thus,
\[
A_1b = \frac{L}{2} \rho(U_- \cos^2 \phi + U_+ \sin^2 \phi).
\] (130)

where we have used \( Nb = L/2 \). Upon substituting the expressions for \( U_+ \) and \( U_- \) and using trigonometric identity we arrive at
\[
A_1b = \frac{L}{2} \left( k - \frac{\rho^2}{k} \cos 2\phi \right).
\] (131)

Further substituting \( \cos 2\phi = \frac{u - k^2}{\rho^2} \) in the above, we find
\[
A_1b = \frac{k^2 - q^2}{2kq} (qL),
\] (132)

where \( q = \sqrt{u - k^2} \).
Evaluation of \( \lim_{N \to \infty} A_3 b^3 \): From Eq. (39) we can write

\[
A_3 b^3 = -\frac{1}{6} N^3 b^3 \rho^3 \left[ 8 N^2 (U_+ + U_-) \cos 2\phi - (U_+ - U_-) \{ 4 - \frac{1}{N^2} + \left( 4 + \frac{1}{N^2} \right) \cos 4\phi \} \right].
\]  

(133)

Taking \( N^2 \) out from the parenthesis, the above equation can be written as

\[
A_3 b^3 = -\frac{1}{6} N^3 b^3 \rho^3 \left[ 8 (U_+ + U_-) \cos 2\phi - (U_+ - U_-) \{ 4 + 4 \cos 4\phi \} \right].
\]  

(134)

Taking limit \( N \to \infty \) of the above equation, we get

\[
\lim_{N \to \infty} A_3 b^3 = -\left( \frac{k^2 - q^2}{2kq} \right)^3 \frac{(qL)^3}{3}.
\]  

(135)

Upon substituting the values of \( U_\pm \), \( \cos 4\phi = 2 \cos^2 2\phi - 1 \), \( \cos 2\phi = \frac{q^2}{\rho^2} \) and \( Nb = L/2 \), the above expressions is simplified to

\[
\lim_{N \to \infty} A_3 b^3 = -\left( \frac{k^2 - q^2}{2kq} \right)^3 \frac{(qL)^3}{3}.
\]  

(136)

This is the same result given in Eq. (44).

Evaluation of \( \lim_{N \to \infty} A_5 b^5 \): From Eq. (40), we write

\[
A_5 b^5 = \frac{Nb^5 \rho^5}{360} \left[ 2 (U_+ + U_-) \left( 96N^4 - 5N^2 - 1 \right) - (U_+ - U_-) \cos 2\phi \left( 288N^4 - 25N^2 - 8 \right) 
+ 2 (U_+ + U_-) \cos 4\phi \left( 96N^4 + 5N^2 + 1 \right) - (U_+ - U_-) \cos 6\phi \left( 96N^4 + 25N^2 + 8 \right) \right].
\]  

(137)

This can be further written as

\[
A_5 b^5 = \frac{Nb^5 \rho^5}{360} \left[ 2 (U_+ + U_-) \left( 96 - \frac{5}{N^2} - \frac{1}{N^4} \right) - (U_+ - U_-) \cos 2\phi \left( 288 - \frac{25}{N^2} - \frac{8}{N^4} \right) 
+ 2 (U_+ + U_-) \cos 4\phi \left( 96 + \frac{5}{N^2} + \frac{1}{N^4} \right) - (U_+ - U_-) \cos 6\phi \left( 96 + \frac{25}{N^2} + \frac{8}{N^4} \right) \right].
\]  

(138)

Taking the limit \( N \to \infty \) of the above equation, all terms containing \( N \) in denominator will become zero and we get

\[
\lim_{N \to \infty} A_5 b^5 = \frac{L^5 \rho^5}{25 \cdot 360} \left[ 768 \frac{k}{\rho} \cos^2 2\phi - 192 \frac{\rho}{k} (3 \cos 2\phi + \cos 6\phi) \right]
\]  

(139)

In the above we have already used \( Nb = L/2 \) and the expressions for \( U_\pm \). Next we expand \( \cos 4\phi \) and \( \cos 6\phi \) in the power of \( \cos 2\phi \) and substitute \( \cos 2\phi = \frac{q^2}{\rho^2} \) to arrive at

\[
\lim_{N \to \infty} A_5 b^5 = \frac{2}{15} \left( \frac{k^2 - q^2}{2kq} \right)^5 (qL)^5,
\]  

(140)

which is Eq. (45).
Evaluation of \( \lim_{N \to \infty} A_7b^7 \): From Eq. (41) we can write

\[
A_7b^7 = \frac{Nb^7 \rho^7}{15120} \left[ (U_+ - U_-)(9792N^6 - 1008N^4 - 161N^2 - 34) - 4(U_+ + U_-) \cos 2\phi \left( 4896N^6 - 168N^4 - 35N^2 - 10 \right) + 4(U_+ - U_-) \cos 4\phi \left( 3264N^6 - 35N^2 - 16 \right) - 4(U_+ + U_-) \cos 6\phi \left( 1632N^6 + 168N^4 + 35N^2 + 10 \right) + (U_+ - U_-) \cos 8\phi \left( 3264N^6 + 1008N^4 + 301N^2 + 98 \right) \right].
\]  

(141)

The above equation can be further written as

\[
A_7b^7 = \frac{N^7b^7 \rho^7}{15120} \left[ (U_+ - U_-) \left( 9792 - \frac{1008}{N^2} - \frac{161}{N^4} - \frac{34}{N^6} \right) - 4(U_+ + U_-) \cos 2\phi \left( 4896 - \frac{168}{N^2} - \frac{35}{N^4} - \frac{10}{N^6} \right) + 4(U_+ - U_-) \cos 4\phi \left( 3264 - \frac{35}{N^4} - \frac{16}{N^6} \right) - 4(U_+ + U_-) \cos 6\phi \left( 1632 + \frac{168}{N^2} + \frac{35}{N^4} + \frac{10}{N^6} \right) + (U_+ - U_-) \cos 8\phi \left( 3264 + \frac{1008}{N^2} + \frac{301}{N^4} + \frac{98}{N^6} \right) \right].
\]  

(142)

Taking \( N \to \infty \) limit of the above equation and substituting \( Nb = \frac{L}{2} \), we obtain

\[
\lim_{N \to \infty} A_7b^7 = \frac{L^7 \rho^7}{2^7 \cdot 15120} \left[ 9792(U_+ - U_-) - 19584(U_+ + U_-) \cos 2\phi + 13056(U_+ - U_-) \cos 4\phi - 6528(U_+ + U_-) \cos 6\phi + 3264(U_+ - U_-) \cos 8\phi \right].
\]  

(143)

Next we expand \( \cos 4\phi, \cos 6\phi, \cos 8\phi \) in powers of \( \cos 2\phi \) and substitute the expressions for \( U_\pm \) to show

\[
\lim_{N \to \infty} A_7b^7 = \left( \frac{L}{2} \right)^7 \frac{\rho^7}{15120} \left[ \left( \frac{2\rho}{k} \right) 9792 + 3264(8 \cos^4 2\phi - 3) \right] - \left( \frac{2k}{\rho} \right) 261121 \cos^3 2\phi.
\]  

(144)

Upon substituting \( \cos 2\phi = \frac{q^2}{\rho^2} \), the above equation simplifies to

\[
\lim_{N \to \infty} A_7b^7 = -\frac{17}{315} \left( \frac{k^2 - q^2}{2kq} \right) (qL)^7.
\]  

(145)

which is Eq. (46).

Evaluation of \( \lim_{N \to \infty} A_9b^9 \): From Eq. (42) we can write

\[
A_9b^9 = \frac{Nb^9 \rho^9}{453,600} \left[ 4(U_+ + U_-)(119,040N^8 - 6120N^6 - 1029N^4 - 215N^2 - 61) - 2(U_+ - U_-) \cos 2\phi(396,800N^8 - 28560N^6 - 5502N^4 - 1045N^2 - 268) \right].
\]
Again we write the above equation as follows:

\[
A_9 b^9 = \frac{N^9 b^9 \rho^9}{453,600} \left[ 4(U_+ + U_-) \left( 119,040 - \frac{6120}{N^2} - \frac{1029}{N^4} - \frac{215}{N^6} - \frac{61}{N^8} \right) \\
- 2(U_+ - U_-) \cos 2\phi \left( 396,800 - \frac{28,560}{N^2} - \frac{5502}{N^4} - \frac{1045}{N^6} - \frac{268}{N^8} \right) \\
+ 40(U_+ + U_-) \cos 4\phi \left( 15,872 - \frac{42}{N^4} - \frac{15}{N^6} - \frac{5}{N^8} \right) \\
- (U_+ - U_-) \cos 6\phi \left( 396,800 + \frac{28,560}{N^2} + \frac{1722}{N^4} + \frac{655}{N^6} - \frac{352}{N^8} \right) \\
+ 4(U_+ + U_-) \cos 8\phi \left( 39,680 + \frac{6120}{N^2} + \frac{1449}{N^4} + \frac{365}{N^6} + \frac{111}{N^8} \right) \\
- (U_+ - U_-) \cos 10\phi \left( 79,360 + \frac{28,560}{N^2} + \frac{9282}{N^4} + \frac{2745}{N^6} + \frac{888}{N^8} \right) \right].
\]

(146)

Taking \( N \to \infty \) limit of the above equation yield,

\[
\lim_{N \to \infty} A_9 b^9 = \frac{N^9 b^9 \rho^9}{453,600} \left[ 476,160(U_+ + U_-) - 793,600(U_+ - U_-) \cos 2\phi \\
+ 634,880(U_+ + U_-) \cos 4\phi \\
- 396,800(U_+ - U_-) \cos 6\phi + 158,720(U_+ + U_-) \cos 8\phi \\
- 39,360(U_+ - U_-) \cos 10\phi \right].
\]

(147)

We expand \( \cos 10\phi, \cos 8\phi, \cos 6\phi, \cos 4\phi \) in power of \( \cos 2\phi = \frac{q^2}{\rho^2} \) and substitute the expressions for \( U_\pm \) and \( Nb = \frac{L}{2} \). It can be shown the above expression finally simplifies to

\[
\lim_{N \to \infty} A_9 b^9 = \frac{31}{2835} \left( \frac{k^2 - q^2}{2kq} \right) (q L)^9.
\]

(149)

This is Eq. (47).

Appendix-D : Evaluation of \( \left[ \lim_{b \to \infty} \left( \frac{d^2 g}{d\xi^2} \right) \right]_{\xi=1} \)

In this appendix, we evaluate the right-hand side of Eq. (109). The expression of \( P \) is given by Eq. (107). We express \( \frac{dP}{d\xi}, \frac{dP}{d\xi} \) and \( \frac{d^2 P}{d\xi^2} \) in terms of the derivatives of \( \alpha, Q_1 \) and \( Q_2 \) as follows:

\[
\frac{dP}{df} = \frac{1}{(Q_1 + Q_2 \tan \alpha)} \left[ Q_2 \sec^2 \alpha \frac{d\alpha}{df} + \tan \alpha \frac{dQ_1}{df} - \frac{dQ_2}{df} \right]
\]

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where \( f = k, \epsilon \). Also,

\[
\frac{d^2 P}{d\epsilon \, dk} = \frac{1}{(Q_1 + Q_2 \tan \alpha)^2} \left[ \frac{d^2 Q_1}{dk} + Q_1 \tan \alpha \frac{d^2 \alpha}{dk} + \frac{d^2 Q_2}{dk} \right] - \frac{P}{(Q_1 + Q_2 \tan \alpha)^2} \left[ \frac{d^2 Q_1}{dk} + Q_2 \tan \alpha \frac{d^2 \alpha}{dk} + \frac{d^2 Q_2}{dk} \right] - \frac{P}{(Q_1 + Q_2 \tan \alpha)^2} \left[ \frac{d^2 Q_1}{dk} + Q_2 \tan \alpha \frac{d^2 \alpha}{dk} + \frac{d^2 Q_2}{dk} \right].
\]  

(151)

Various derivatives of \( \alpha, Q_1, Q_2 \) are given by

\[
\frac{d\alpha}{dk} = bk \left[ \frac{(k^2 - u) \cos \phi_1 - v \sin \phi_1}{\rho_1^3} + \frac{v \sin \phi_2 - (k^2 - u) \cos \phi_1}{\rho_2^3} \right].
\]

(152)

\[
\frac{d\alpha}{d\epsilon} = - \frac{bv^2 \epsilon}{(2 \sqrt{2} \rho_2^2) \sqrt{\rho_2^2 - (k^2 - u)}}.
\]

(153)

\[
\frac{d^2 \alpha}{dk \, d\epsilon} = \frac{bk v}{2 \rho_2^3} \left[ \sin \phi_2 \left( (k^2 - u)^2 - v^2 \epsilon^2 \right) + 2 v \epsilon (k^2 - u) \cos \phi_2 \right].
\]

(154)

\[
\frac{dQ_1}{dk} = \frac{1}{k^2 \rho_1^5 \rho_2^5} \left[ \rho_1^5 (k^2 - \rho_2^2) \cos \phi_2 \right] \left( u (k^2 - u) \right.
\]

\[
- v^2 \epsilon^2 \right) - \rho_2^5 (k^2 - \rho_2^2) \cos \phi_1 \left( u (k^2 - u) - v^2 \right)
\]

\[
- k^2 \rho_2^5 v (k^2 + \rho_1^2) \sin \phi_1 - k^3 (\rho_1^2)
\]

\[
- \rho_2^5 v^2 (\epsilon^2 - 1) (k^2 - u) \cos \phi_1 + \phi_2
\]

\[
+ k^2 \rho_1^5 v \epsilon (k^2 + \rho_2^2) \sin \phi_2 + k^3 (\rho_1^2 + \rho_2^2) v (\epsilon + 1) \sin \phi_1
\]

\[
+ \phi_2 \left( (k^2 - u)^2 + v^2 \epsilon \right) \left].
\]

(155)

\[
\frac{dQ_1}{d\epsilon} = \frac{v}{2 k \rho_1 \rho_2^2} \left[ \rho_1 v \epsilon (k^2 - \rho_2^2) \cos \phi_2 + k (\rho_1^2 - \rho_2^2) v \epsilon \cos \phi_2 \right.
\]

\[
- (k^2 - u) (\rho_1 + \rho_2^2) \sin \phi_2 + k (\rho_1^2 + \rho_2^2) \sin \phi_2 \right)
\]

(156)

\[
\frac{d^2 Q_1}{d\epsilon \, dk} = \frac{1}{2 k^2 \rho_1^9 \rho_2^{17}} \left[ - k^2 \rho_1^9 \rho_2^8 v^2 \epsilon (k^2 + \rho_2^2) (k^2 - u) \cos \phi_2 \right.
\]

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\[ -\rho_1^0 \rho_2^0 v^2 \epsilon \cos(\phi_2) \left( -k^2 \rho_2^4 - 3k^2 \rho_2^2 (k^2 - u) + 5k^4 (k^2 - u) - \rho_2^6 \right) \\
- k^3 \rho_1^4 \rho_2^8 \left( \rho_1^2 + \rho_2^2 \right) v^2 (\epsilon + 1) (k^2 - u) \cos(\phi_1 + \phi_2) \left( (k^2 - u)^2 + v^2 \epsilon \right) \\
- 4k^2 \rho_2^8 \rho_3^8 v^3 \epsilon^2 (k^2 + \rho_2^2) \sin(\phi_2) - k^3 \rho_2^8 (5\rho_1^6 - 3\rho_2^6 \rho_1^4 \\
- \rho_2^4 \rho_1^2 - \rho_2^6) \rho_1^4 v^2 \epsilon (k^2 - u) \cos(\phi_1 + \phi_2) \\
\rho_2^8 \rho_3^8 v (k^2 - \rho_2^2) + (k^2 - u) \sin(\phi_2) (u (k^2 - u) - v^2 \epsilon^2) \\
+ k^2 \rho_2^8 \rho_1^4 v^2 \epsilon (\rho_2^2 - k^2) \sin(\phi_2) \\
+ 2k^2 \rho_1^0 \rho_2^{12} v (k^2 + \rho_2^2) \sin(\phi_2) \\
- k^3 \rho_1^4 \rho_2^8 (\rho_1^2 - \rho_2^2) v^3 \epsilon (\epsilon + 1) \sin(\phi_1 + \phi_2) \left( (k^2 - u)^2 + v^2 \epsilon \right) \\
- k^3 \rho_1^4 (\rho_1^2 - \rho_2^2) v^3 (\epsilon^2 - 1) \sin(\phi_1 + \phi_2) \left( v^2 \epsilon^2 (k^2 - u) + (k^2 - u)^3 \right)^2 \\
+ 2k^3 \rho_2^8 \rho_3^8 (\rho_1^2 + \rho_2^2) \sin(\phi_1 + \phi_2) \left( (k^2 - u)^2 - v^2 \epsilon^2 \right) \right]. \quad (157)

\[ \frac{dQ_2}{dk} = \frac{1}{k^2 \rho_1^4 \rho_2^8} \left[ k^2 \rho_2^8 v (\rho_1^2 - k^2) \cos(\phi_1) + k^2 \rho_1^4 v \epsilon (\rho_2^2 - k^2) \cos(\phi_2) \\
- \rho_2^5 (k^2 + \rho_1^2) \sin(\phi_1) (u (k^2 - u) - v^2) \\
- k^3 (\rho_1^2 - \rho_2^2) v (\epsilon + 1) \cos(\phi_1 + \phi_2) \left( (k^2 - u)^2 + v^2 \epsilon \right) \\
\rho_1^4 (k^2 + \rho_2^2) \sin(\phi_2) (u (k^2 - u) - v^2 \epsilon^2) \\
- k^3 (\rho_1^2 + \rho_2^2) v^2 (\epsilon^2 - 1) (k^2 - u) \sin(\phi_1 + \phi_2) \right]. \quad (158)

\[ \frac{dQ_2}{d\epsilon} = \frac{v}{2k \rho_1^2 \rho_2^8} \left[ \rho_1 (k^2 - \rho_2^2) (k^2 - u) \cos(\phi_2) + k (\rho_1^2 - \rho_2^2) (k^2 - u) \cos(\phi_1 + \phi_2) \\
+ v \epsilon (\rho_1 (k^2 + \rho_2^2) \sin(\phi_2) + k (\rho_1^2 + \rho_2^2) \sin(\phi_1 + \phi_2)) \right]. \quad (159)

\[ \frac{d^2 Q_2}{d\epsilon dk} = \frac{1}{2k^2 \rho_1^9 \rho_2^{17}} \left[ 4k^2 \rho_1^8 \rho_2^8 v^3 \epsilon^2 (k^2 - \rho_2^2) \cos(\phi_2) + 2k^2 \rho_1^9 \rho_2^{12} v (\rho_2^2 - k^2) \cos(\phi_2) \\
+ k^2 \rho_1^9 \rho_2^8 v^3 \epsilon^2 (k^2 + \rho_2^2) \cos(\phi_2) \\
- \rho_1^9 \rho_2^8 v (k^2 + \rho_2^2) (k^2 - u) \cos(\phi_2) (u (k^2 - u) - v^2 \epsilon^2) \\
- 2k^3 \rho_2^8 \rho_1^4 (\rho_1^2 - \rho_2^2) v \cos(\phi_1 + \phi_2) \left( (k^2 - u)^2 - v^2 \epsilon^2 \right) \\
+ k^3 \rho_1^4 \rho_2^8 (\rho_1^2 + \rho_2^2) v^3 (\epsilon^2 - 1) \cos(\phi_1 + \phi_2) \left( (k^2 - u)^2 + v^2 \epsilon \right) \\
+ k^3 \rho_1^4 (\rho_1^2 + \rho_2^2) v^3 (\epsilon^2 - 1) \cos(\phi_1 + \phi_2) \left( v^2 \epsilon^2 (k^2 - u) + (k^2 - u)^3 \right)^2 \\
\rho_1^9 \rho_2^8 (-v^2) \epsilon \sin(\phi_2) (-k^2 \rho_2^4 + 3k^2 \rho_2^2 (k^2 - u) + 5k^4 (k^2 - u) + \rho_2^6) \\
+ k^2 \rho_1^4 \rho_2^8 v^2 \epsilon (\rho_2^2 - k^2) (k^2 - u) \sin(\phi_2) \\
- k^3 \rho_1^4 \rho_2^8 (\rho_1^2 - \rho_2^2) v^2 (\epsilon + 1) (k^2 - u) \sin(\phi_1 + \phi_2) \left( (k^2 - u)^2 + v^2 \epsilon \right) \\
- k^3 \rho_1^4 \rho_2^8 (5\rho_1^6 + 3\rho_2^6 \rho_1^4 - \rho_2^4 \rho_1^2 + \rho_2^6) v^2 \epsilon (k^2 - u) \sin(\phi_1 + \phi_2) \right]. \quad (160) \]
The right-hand side of Eq. (109) is to be evaluated at $\varepsilon = 1$. Therefore, we evaluate all the above derivatives of $\alpha$, $Q_1$ and $Q_2$ at $\varepsilon = 1$. When $\varepsilon = 1$, we also have $\rho_2 = \rho_1$, $\phi_2 = \phi_1$ and $\alpha = 0$. We simplify Eqs. (153)–(160) at $\varepsilon = 1$ to obtain the following results:

\[
\frac{d\alpha}{dk}\bigg|_{\varepsilon=1} = 0. \tag{161}
\]

\[
\frac{dQ_1}{dk}\bigg|_{\varepsilon=1} = \frac{4kv^2}{\rho_1^6}. \tag{162}
\]

\[
\frac{dQ_2}{dk}\bigg|_{\varepsilon=1} = \frac{2(k^2 + \rho_1^2) \sin(\phi_1)(u(k^2-u) - v^2)}{k^2\rho_1^3} + \frac{(2v)(\rho_1^2 - k^2) \cos(\phi_1)}{\rho_1^3} . \tag{163}
\]

\[
\frac{d\alpha}{de}\bigg|_{\varepsilon=1} = -\frac{b v^2}{2\sqrt{2}\rho_1^2} \left[ \frac{1}{\sqrt{\rho_1^2 - (k^2-u)}} \right] . \tag{164}
\]

\[
\frac{dQ_1}{de}\bigg|_{\varepsilon=1} = \left( \frac{v}{2k\rho_1^3} \right) \left[ v \left( k^2 - \rho_1^2 \right) \cos(\phi_1) - (k^2-u) \sin(\phi_1) \left( k^2 + 4k\rho_1 \cos(\phi_1) + \rho_1^2 \right) \right] . \tag{165}
\]

\[
\frac{dQ_2}{de}\bigg|_{\varepsilon=1} = \left( \frac{v}{2k\rho_1^3} \right) \left[ \left( k^2 - \rho_1^2 \right) \left( k^2 - u \right) \cos(\phi_1) + v \left( k^2 + \rho_1^2 \right) \sin(\phi_1) + \frac{2kv^2}{\rho_1^3} \right] . \tag{166}
\]

\[
\frac{d^2\alpha}{d\varepsilon k}\bigg|_{\varepsilon=1} = \frac{bkv}{2\rho_1^4} \left[ \sin(\phi_1) \left( k^2 - 2 - v^2 \right) + 2v \left( k^2 - u \right) \cos(\phi_1) \right] . \tag{167}
\]

\[
\frac{d^2Q_1}{d\varepsilon k}\bigg|_{\varepsilon=1} = \left[ \rho_1^3 \sin(\phi_1) \left( -6k^6 + 2k^4 \left( \rho_1^2 - 3u \right) + k^2 \rho_1^2 \left( \rho_1^2 - 2u \right) + \rho_1^6 \right) 
-4k^5\rho_1^4 v \left( k^2 - u \right) \cos(2\phi_1) + 4k^3\rho_1^3 \rho_1 \sin(2\phi_1) \left( k^2 - 2 \right) 
+ \rho_1^3 \sin(\phi_1) \left[ 2k^2\rho_1^4 \left( k^2 + \rho_1^2 \right) - \rho_1^2 \left( v^2 \left( 2k^2 + u \right) + u \left( k^2 - u \right)^2 \right) 
+ k^2 \left( v^2 \left( u - 6k^2 \right) + u \left( k^2 - u \right)^2 \right) \right] \right] . \tag{168}
\]

\[
\frac{d^2Q_2}{d\varepsilon k}\bigg|_{\varepsilon=1} = \frac{v}{2k^2\rho_1^4} \left[ 4k^3 v^2 \left( k^2 - u \right) \left( k^2 - u \right)^2 + 2\rho_1^4 + v^2 \right] 
+ \rho_1^5 v \sin(\phi_1) \left( -6k^6 + k^4 \left( 6u - 2\rho_1^2 \right) + k^2 \rho_1^2 \left( \rho_1^2 + 2u \right) - \rho_1^6 \right) 
- \rho_1^5 \cos(\phi_1) \left[ 2k^2\rho_1^4 \left( k^2 - \rho_1^2 \right) + \rho_1^2 \left( v^2 \left( 2k^2 + u \right) + u \left( k^2 - u \right)^2 \right) 
+ k^2 \left( v^2 \left( u - 6k^2 \right) + u \left( k^2 - u \right)^2 \right) \right] . \tag{169}
\]

For $\varepsilon = 1$, we have the following simplifications:

\[ Q_1(\varepsilon = 1) = 4 \sin^2 \phi_1, \quad Q_2(\varepsilon = 1) = 2J_1^- \sin \phi_1, \tag{170} \]

and

\[ P(\varepsilon = 1) = -\frac{1}{2} J_1^- \csc \phi_1 \tag{171} \]
From the results of Eqs. (161)–(171) we can simplify \( \frac{dP}{dk} \varepsilon = 1 \), \( \frac{dP}{dk} \varepsilon = 1 \) and \( \frac{d^2P}{dk^2} \varepsilon = 1 \) and can evaluate the right-hand side of Eq. (109). After a lengthy algebra, it can be shown that

\[
\left[ \frac{1}{(1 + P^2)} \frac{d^2P}{dk^2} - \frac{2P}{(1 + P^2)^2} \frac{dP}{dk} \frac{dP}{d\varepsilon} \right]_{\varepsilon = 1} = K_0 + K_1 b. \tag{172}
\]

where the expressions for \( K_1 \) and \( K_0 \) are given by

\[
K_1 = \frac{k u}{2 \rho_1^5} \left[ ((k^2 - u)^2 - v^2) \sin \phi_1 + 2(k^2 - u)v \cos \phi_1 \right]. \tag{173}
\]

The expression for \( K_0 \) is lengthy and expressed through the use of symbols \( C_1 \), \( C_2 \), \( C_3 \), \( C_4 \), \( C_5 \) and \( C_6 \) as given below:

\[
K_0 = \frac{v \csc^2 (\phi_1)}{2k^3 \rho_1^{13} \left( (J_1^{-})^2 \csc^2 (\phi_1) + 4 \right)^2} (C_1 + C_2 + C_3 + C_4 + C_5 - C_6). \tag{174}
\]

Various \( C_i \)’s appearing in the above equation are given by

\[
C_1 = 2k^3 v \cot (\phi_1) \left( k^4 - 2k^2 u + u^2 + v^2 \right) \left( - (J_1^{-})^2 - 2 \cos (2\phi_1) + 2 \right)
\times \cot (\phi_1) \left( (k^2 - \rho_1^2) (k^2 - v) \cot (\phi_1) + v (k^2 + \rho_1^2) + 4k\rho_1 v \cos (\phi_1) \right). \tag{175}
\]

\[
C_2 = \left[ 8 (J_1^{-}) k^3 v \cot (\phi_1) \left( k^4 - 2k^2 u + u^2 + v^2 \right) \right]
\times \left[ 4k\rho_1 (k^2 - u) \cos (\phi_1) + (k^2 + \rho_1^2) (k^2 - v) - v (k^2 - \rho_1^2) \right] \cot (\phi_1). \tag{176}
\]

\[
C_3 = \rho_1^2 \csc^2 (\phi_1) \left( 2 - \frac{1}{2} (J_1^{-})^2 \csc^2 (\phi_1) \right)
\times \left( k^2 + \rho_1^2 \right) \sin (\phi_1) \left( u (k^2 - u) - v^2 \right) + k^2 v \left( \rho_1^2 - k^2 \right) \cos (\phi_1)
\times \rho_1 v \left( k^2 - \rho_1^2 \right) \cos (\phi_1) - \rho_1 \left( k^2 - u \right) \sin (\phi_1) \left( k^2 + 4k\rho_1 \cos (\phi_1) + \rho_1^2 \right). \tag{177}
\]

\[
C_4 = 2J_1^{-} \rho_1^2 \csc^2 (\phi_1) \left( (k^2 + \rho_1^2) \sin (\phi_1) \left( u (k^2 - u) - v^2 \right) + k^2 v \left( \rho_1^2 - k^2 \right) \cos (\phi_1) \right)
\times \rho_1 \left( k^2 - \rho_1^2 \right) \cos (\phi_1) + \rho_1 v \sin (\phi_1) \left( k^2 + 4k\rho_1 \cos (\phi_1) + \rho_1^2 \right). \tag{178}
\]

\[
C_5 = 2J_1^{-} k\rho_1 \csc (\phi_1) \left( 1 - \frac{1}{4} (J_1^{-})^2 \csc^2 (\phi_1) + 1 \right)
\times \left[ \rho_1^3 v \cos (\phi_1) \left( -6k^6 + 2k^4 \left( \rho_1^2 + 3u \right) + k^2 \left( \rho_1^4 - 2\rho_1^2 u + \rho_1^6 \right) \right) \right.
\left. - 4k^3 v \left( k^2 - u \right) \cos (2\phi_1) \left( k^4 - 2k^2 u + u^2 + v^2 \right) \right]
\rho_1^3 \sin (\phi_1) \left[ k^6 u - k^4 \left( -2\rho_1^4 + 2u^2 + \rho_1^2 u + 6v^2 \right) - \rho_1^2 u (u^2 + v^2) \right]
\left. + k^2 \left( 2\rho_1^6 + u^3 + 2\rho_1^2 u^2 + u v^2 - 2\rho_1^2 v^2 \right) + 8k^3 \rho_1 \cos (\phi_1) \left( k^4 - 2k^2 u + u^2 - v^2 \right) \right] \tag{179}
\]

\[
C_6 = 4k\rho_1 \left( 1 - \frac{1}{4} (J_1^{-})^2 \csc^2 (\phi_1) + 1 \right) \left[ - \rho_1^3 \cos (\phi_1) \right]
\times \left[ k^6 u + k^4 \left( 2\rho_1^4 - 2u^2 + \rho_1^2 u - 6v^2 \right) \right]
\left. + k^2 \left( -2\rho_1^6 + u^3 - 2\rho_1^2 u^2 \right) \right]
\]
\[ \begin{align*}
&+uv^2 + 2\rho_1^2v^2) + \rho_1^2u(u^2 + v^2)) \\
&+v\left\{4k^3v \cos(2\phi_1) \left(k^4 - 2k^2u + u^2 + v^2\right) \\
&-\rho_1^3 \sin(\phi_1) \left(6k^6 + k^4 \left(2\rho_1^2 - 6u\right) \\
&-k^2 \left(\rho_4^4 + 2\rho_1^2u\right) + 16k^3\rho_1 \left(k^2 - u\right) \cos(\phi_1) + \rho_1^6\right)\right\}. \tag{180}
\end{align*} \]

\(K_0\) and \(K_1\) are independent of the thickness ‘2b’.

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