Confinement in bilayer graphene via intra- and inter-layer interactions

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Abstract
We consider confinement of Dirac fermions in AB-stacked bilayer graphene by inhomogeneous on-site interactions, (pseudo-)magnetic field or inter-layer interaction. Working within the framework of four-band approximation, we focus on the systems where the stationary equation is reducible into two stationary equations with \(2 \times 2\) Dirac-type Hamiltonians and auxiliary interactions. We show that the localized states are given in terms of solutions of an effective Schrödinger equation with energy-dependent potential. We consider several scenarios where bilayer graphene is subject to inhomogeneous (pseudo-) magnetic field, on-site interactions or inter-layer coupling. In explicit examples, we provide analytical solutions for the states localized by local fluctuations or periodicity defects of the interactions.

Keywords: bilayer, graphene, confinement, interlayer, intralayer, energy-dependent

(Some figures may appear in colour only in the online journal)

1. Introduction

In bilayer graphene, two flakes of graphene are close each other such that their electrons can mutually interact. The relative orientation of the two layers can vary. In case of Bernal (or AB-) stacking, the two layers are relatively shifted such that some bonds are parallel in the two lattices. Denoting the atoms in the two triangular sublattices of the \(j\)th layer as \(A_j\) and \(B_j\), the atoms \(A_2\) of the upper layer sit just above the \(B_1\) atoms of the lower layer, whereas the atoms \(A_1\) are below the centers of the hexagons of the upper lattice. The effective Hamiltonian (in the
The four-band approximation (for the low-energy particles can be written as [1]

\[ H_{blg} = \begin{pmatrix} \epsilon_{A_1} & \pi^\dagger & 0 & v_3 \pi \\ \pi & \epsilon_{B_1} & \gamma & 0 \\ 0 & \gamma & \epsilon_{A_2} & \pi^\dagger \\ v_3 \pi^\dagger & 0 & \pi & \epsilon_{B_2} \end{pmatrix}, \quad \pi = -i\partial_x + \partial_y. \] (1)

The basis of the bispinors is \((A_1, B_1, A_2, B_2)\). The quantities \(\epsilon_{A_j}\) (\(B_j\), \(j \in \{1, 2\}\)) correspond to the on-site energies that can originate from an external electric potential, spin–orbit interaction or an interaction with the substrate. The parameter \(\gamma\) represents interaction of the electrons on the sites \(A_2\) and \(B_1\). The term proportional to \(v_3\) is related to the interlayer trigonal warping [1], which is frequently set to zero in the literature. The four-band Hamiltonian (1) was used e.g. in the analysis of strains and their effect on electronic [2–4] or topological properties of bilayer graphene [5]. It was used in the analysis of confinement of Dirac fermions in quantum dots formed by doping [6] or by local variation of the interlayer coupling related to local delamination of the bilayer graphene (graphene blisters) studied recently in [7, 8]. It serves well for description of other bilayer Dirac materials, see e.g. bilayer silicene [9–11]. Qualitatively the same operator with \(v_3 = 0\) appears in description of spin–orbit interaction in graphene, see e.g. [12, 13].

Electrons on the binding sites \(A_2\) and \(B_1\) form dimers. For \(E \ll \gamma\), the dynamics on the non-dimer sites \(A_1\) and \(B_2\) gets dominant, and the effective Hamiltonian can be derived from (1), see [1, 14]. The latter case is known as the two-band approximation, in which the energy operator reads as

\[ H = \begin{pmatrix} 0 & \epsilon_{A_1} & \gamma & 0 \\ \epsilon_{B_1} & 0 & \gamma & 0 \\ \gamma & 0 & 0 & \epsilon_{A_2} \\ 0 & \gamma & \epsilon_{B_2} & 0 \end{pmatrix}. \] (2)

When compared to (1), this operator can be matched with the Hamiltonian of bilayer graphene for inhomogeneous \(\epsilon_{A_1} = \epsilon_{B_2} = \epsilon_1(x)\), \(\epsilon_{B_1} = \epsilon_{A_2} = \epsilon_2(x)\) and with the longitudinal momentum \(k_y = 0\). Therefore, (2) can describe dynamics of the fermions that bounce on the potential in normal direction. The operator (2) also contains an additional potential term \(A_1 \sigma_0 \otimes \sigma_1 - A_2 \sigma_3 \otimes \sigma_2\), \(A = A_1 + iA_2\). There exist numerical methods for solving the stationary equation associated with (2), such as the Fourier grid Hamiltonian [20] and its extension for matrix Hamiltonians [21]. They provide information in a finite region (local), and the intrinsic numerical error is carried on when computing other physically relevant quantities, such as expectation.
values. In this regard, analytic solutions prove to be more reliable as they provide global information, which becomes valid on the whole considered domain. Additionally, they are error-free by definition, which makes them outstanding candidates when it comes to implementing perturbative theories. Despite their advantages, it might be a challenging and sometimes impossible task to find them explicitly. Still, there are mathematical techniques that help to find the analytical solutions.

It was discussed recently in [22] that the Hamiltonian (2) belongs to the class of reducible operators where the solution of the associated stationary equation can be found via two, lower-dimensional, dynamical equations with auxiliary interactions. This substantially simplifies the determination of exact solutions. Indeed, let us make an ansatz for the eigenstates of (2) in the following manner,

\[ \Psi = (\xi_1, \xi_2, \xi_2, \xi_1)^T, \quad \Xi = (\chi_1, \chi_2, -\chi_2, -\chi_1)^T. \]  

(3)

Then, the bispinors \( \Psi \) and \( \Xi \) satisfy\(^3\)

\[ H \Psi = E \Psi, \quad H \Xi = \mathcal{E} \Xi, \]  

(4)

provided that the spinors \( \xi = (\xi_1, \xi_2)^T \) and \( \chi = (\chi_1, \chi_2)^T \) are solutions of the following two equations,

\[ h_1 \xi = \begin{bmatrix} -i v_3 & 1 & 0 \\ 1 & 0 & \end{bmatrix} \partial_x + \begin{bmatrix} \epsilon_1(x) & A(x) \\ A^*(x) & \epsilon_2(x) + \gamma(x) \end{bmatrix} \xi = E \xi, \]  

(5)

\[ h_2 \chi = \begin{bmatrix} -i v_3 & 1 & 0 \\ 1 & 0 & \end{bmatrix} \partial_x + \begin{bmatrix} \epsilon_1(x) & A(x) \\ A^*(x) & \epsilon_2(x) - \gamma(x) \end{bmatrix} \chi = \mathcal{E} \chi, \]  

(6)

with \( E \) and \( \mathcal{E} \), in general, different from each other.

These equations resemble the 1 + 1 dimensional Dirac equation up to the matrix coefficient in the kinetic term that contains the nonvanishing constant \( v_3 \). Let us investigate how the equations (5) and (6) can be solved and what are the physical scenarios that can be studied in this way.

2. Bilayer graphene via Schrödinger equation with energy-dependent potential

For the sake of generality, let us consider the matrix Hamiltonian

\[ h = -i \begin{pmatrix} v_3 & 1 \\ 1 & 0 \end{pmatrix} \partial_x + \begin{pmatrix} V_1(x) & A(x) \\ A^*(x) & V_2(x) \end{pmatrix}, \quad V_1(x), V_2(x) : \mathbb{R} \to \mathbb{R}, \quad A(x) : \mathbb{R} \to \mathbb{C}. \]  

(7)

The stationary equation

\[ h \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \]  

(8)

gives rise to the set of coupled equations for the spinor components \( \psi_1 \) and \( \psi_2 \),

\[ -i v_3 \psi_1' - i \psi_2' + A(x) \psi_2 + (V_1(x) - E) \psi_1 = 0, \]  

(9)

\(^3\)Notice that we denote \( a^* \) as complex conjugate of \( a \) in the article.
\[-i\psi_1' + A'(x)\psi_1 + (V_2(x) - E)\psi_2 = 0, \quad (10)\]

with \(f' \equiv \partial f / \partial y\). The equations (9) and (10) can be decoupled by fixing

\[\psi_2 = -i\psi_1' + A'(x)\psi_1 \over E - V_2(x), \quad (11)\]

so that (9) turns into a second-order differential equation for \(\psi_1\). In order to bring it into the Sturm–Liouville form, we make an additional energy-dependent transformation

\[\psi_1(x) = e^{-i\phi(x)} \tilde{\psi}_1(x), \quad (12)\]

where

\[\phi(x) = \frac{i}{2} \log(E - V_2(x)) + \frac{1}{2} \int x dx' (v_3(E - V_2(x)) + 2 \Re A(x')), \quad (13)\]

Notice that the first term of \(\phi(x)\) is not a pure phase so that the normalization of \(\psi_1\) and \(\tilde{\psi}_1\) can differ. This should be kept in mind when imposing boundary conditions on the functions \(\hat{\psi}_1\) in general. The equation for \(\tilde{\psi}_1\) reads as

\[-\tilde{\psi}_1'' + \left(-\Im A'(x) + (\Im A(x))^2 - v_3(E - V_2(x)) \Re A(x) - (E - V_2(x))(E - V_2(x))\right) \tilde{\psi}_1 = 0. \quad (14)\]

Therefore, \(\tilde{\psi}_1\) can be found as the zero-mode of the Schrödinger equation with a potential given in terms of \(A(x), V_1(x),\) and \(V_2(x)\). Although it is a rather complicated task to find its solutions in general, one can identify (14) with a Schrödinger equation that possesses a solvable potential \(V_0, -\tilde{\psi}_1'' + V_0\tilde{\psi}_1 = 0\) However, by doing so, one has to solve a nonlinear differential equation for either \(\Im A\) or \(V_2\). To overcome this issue, one can fix either \(\Re A(x)\) or \(V_1(x)\) so that the potential in (14) coincides with \(V_0\). This way, one of the two quantities can be fixed as

\[V_1 = V_1(V_2, A, V_0, E) \quad \text{or} \quad \Re A = \Re A(V_2, V_1, \Im A, V_0, E). \quad (15)\]

Then we can acquire a solution \(\tilde{\psi}_1\) for a single energy level \(E\). Indeed, changing \(E\) would alter the interaction \(V_1(x)\) so that we would deal with a different physical setting. We will discuss this situation in the section 4.

For constant \(V_2\), equation (14) simplifies considerably as the third line in (14) cancels out, leading to

\[-\tilde{\psi}_1'' + V_E(x)\tilde{\psi}_1 = \lambda_E\tilde{\psi}_1, \quad (16)\]

where

\[V_E(x) = -\Im A'(x) + (\Im A(x))^2 - v_3(E - V_2) \Re A(x) - (E - V_2)(E - V_2), \quad (17)\]

\[\lambda_E = \frac{V_2^2}{4}(E - V_2)^2. \quad (18)\]
The equation (16) corresponds to a stationary Schrödinger equation with an energy-dependent potential term $V_E(x)$. The energy-dependent part in (17) cancels out effectively when $V_1$ is constant and either $v_3$ vanishes or Re $A$ is a constant. In the latter case, the term $v_3(E - V_2)A_0 + (E - V_1)(E - V_2)$ adds a shift to the eigenvalue $\lambda_E$.

Notice that when $E = V_2$, the equations (9) and (10) can be solved directly with

$$
\psi_1 = e^{-i \int A'(x) dx}, \quad \psi_2 = -i \int e^{i \int (A(x) - A'(x)) dx} (V_1 - V_2 - i\partial x A(x)) dx e^{i \int A(x) dx}. \tag{19}
$$

We can recover the $2 \times 2$ bilayer Hamiltonians $h_1$ and $h_2$ in (5) and (6), respectively, through the following identification,

$$
h_1 \equiv \hbar \int_{V_1(x) = v_1(x)}^{v_2(x) + \gamma(x)} \psi_1 \psi_2, \quad h_2 \equiv \hbar \int_{V_1(x) = v_1(x)}^{v_2(x) - \gamma(x)} \psi_1 \psi_2. \tag{20}
$$

Therefore, if we have

$$
h \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{21}
$$

then there also holds

$$
h_1 x = \varepsilon \xi, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \int_{V_1(x) = v_1(x)}^{v_2(x) + \gamma(x)} \psi_1 \psi_2, \quad \varepsilon \equiv E \int_{V_1(x) = v_1(x)}^{v_2(x) + \gamma(x)} \psi_1 \psi_2, \tag{22}
$$

$$
h_2 x = \bar{\varepsilon} \chi, \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \int_{V_1(x) = v_1(x)}^{v_2(x) - \gamma(x)} \psi_1 \psi_2, \quad \bar{\varepsilon} \equiv E \int_{V_1(x) = v_1(x)}^{v_2(x) - \gamma(x)} \psi_1 \psi_2. \tag{23}
$$

The corresponding bispinor solutions of (4) are

$$
\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi, \sigma_1 \xi \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1, -i \psi'_1 + A'(x) \psi_1, -i \psi'_1 + A'(x) \psi_1 \end{pmatrix}^T \begin{pmatrix} \psi_1 \psi_2 \end{pmatrix} \int_{V_1(x) = v_1(x)}^{v_2(x) + \gamma(x)} \psi_1 \psi_2, \quad \varepsilon \equiv E \int_{V_1(x) = v_1(x)}^{v_2(x) + \gamma(x)} \psi_1 \psi_2, \tag{22}
$$

$$
\Xi = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi, -\sigma_1 \chi \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1, -i \psi'_1 + A'(x) \psi_1, -i \psi'_1 + A'(x) \psi_1 \end{pmatrix}^T \begin{pmatrix} \psi_1 \psi_2 \end{pmatrix} \int_{V_1(x) = v_1(x)}^{v_2(x) - \gamma(x)} \psi_1 \psi_2, \quad \bar{\varepsilon} \equiv E \int_{V_1(x) = v_1(x)}^{v_2(x) - \gamma(x)} \psi_1 \psi_2. \tag{23}
$$

The bispinors are normalized provided that $\psi$ and $\chi$ are normalized. Let us notice that if we can find a fundamental systems of solutions of (5) and (6) for $E = \varepsilon = \bar{\varepsilon}$, then we can find the fundamental system for (4) for $E = \varepsilon$.

When $v_3$ and $\gamma$ are constants, both $h_1$ and $h_2$ correspond to $h$ with constant $V_1$ and $V_2$ (recall that the explicit form of $V_2$ differs in $h_1$ and $h_2$). In this case, the potential $V_E$ and $\lambda_E$ in (16) acquire this form

$$
V_E(x) = -\text{Im} A'(x) + (\text{Im} A(x))^2 - v_3(E - V_2) \text{Re} A(x), \tag{25}
$$

$$
\lambda_E = \frac{v_3^2}{4}(E - V_2)^2 + (E - V_2)(E - V_1). \tag{26}
$$

For $V_2$ constant, the transformation (12) does not alter integrability, such that when $\tilde{\psi}_1$ is square-integrable, so is $\tilde{\psi}_1 = e^{-i\xi(x)} \psi_1$. Therefore, we get solutions of (5) and (6) as in (22) and (23), respectively. The two solutions (3) for the stationary equation (4) then read as in (24).
Now, if we allow $\epsilon_1(x)$, $\epsilon_2(x)$ and $\gamma(x)$ to be inhomogeneous, we can still keep $V_2$ constant in either $h_1$ or $h_2$, provided either $\epsilon_2(x) + \gamma(x)$ or $\epsilon_2(x) - \gamma(x)$ is constant, respectively. For convenience, let us fix that there holds\footnote{In the case $V_2 = \epsilon_2(x) - \gamma(x)$, $V_2 \in \mathbb{R}$, all the analysis below is applicable with minor changes.}:

$$V_2 = \epsilon_2(x) + \gamma(x), \quad V_2 \in \mathbb{R}. \quad (27)$$

In this case, we get constant $V_2 = \epsilon_2(x) + \gamma(x)$ in $h_1$ whereas $h_2$ corresponds to $h$ with inhomogeneous $V_2(x) = \epsilon_2(x) - \gamma(x)$, see (20). Notice that $V_1 = \epsilon_1(x)$ can be inhomogeneous now. The equation (5) reduces into (16) whereas (6) leads to (14). It is feasible to solve analytically only one of the two equations by fixing $\epsilon_1(x)$ appropriately. In either case, the analytical solutions of (4) are either $\Psi$ or $\Xi$ in (24) with $V_1 \rightarrow \epsilon_1(x)$. It is worth noticing that the solutions of (16) depend just on $\epsilon_2(x) + \gamma(x)$, they do not ‘feel’ the explicit form of the inter-layer interaction $\gamma(x)$ and of the on-site coupling $\epsilon_2(x)$. These interactions can be changed without altering $\widetilde{\psi}_1$ provided the change complies with (27). As we can find only part of the solutions of (4) analytically, we consider the models with inhomogeneous $\epsilon_2(x)$ and $\gamma(x)$ that satisfy (27) as quasi-exactly solvable.

Solution of (16) with the energy-dependent potential (17) is nontrivial. Identification of a complete set of solutions to an energy-dependent Schrödinger equation is, if possible, a challenging task in most cases [23, 24]. We will focus on the cases where confinement of Dirac fermions is caused either by $A(x)$, $\epsilon_1(x)$, or $\gamma(x)$. The three situations will be discussed in the following three sections separately. Our strategy will be to identify the potential (17) with the potential of a known solvable system. This way, we will be able to identify a set of square-integrable solutions and the corresponding set of eigenvalues $\lambda_E$.

3. Confinement by the vector potential

The potential term $A_x \sigma_0 \otimes \sigma_1 - A_y \sigma_3 \otimes \sigma_2$ in (1) resembles a magnetic vector potential. However, the corresponding magnetic field would have a different sign on the two layers, which might be physically unfeasible. Nevertheless, it is known that deformations of graphene layers are manifested in the form of the effective (pseudo-)magnetic vector potential [2, 25, 26]. Therefore, we can see $A$ as a combination of the magnetic and pseudo-magnetic field that can acquire different values on the two layers, see e.g. [27].

In this section, we fix $\epsilon_1$, $\epsilon_2$ as well as $\gamma$ in (2) to be constant,

$$\epsilon_1, \quad \epsilon_2, \quad \gamma \in \mathbb{R}. \quad (28)$$

It allows us to convert the task of solving both (5) and (6) into solution of the Schrödinger equation with energy-dependent potential (16). We will identify the latter equation with a stationary equation of known exactly solvable model. This way, we shall find the explicit solutions $\widetilde{\psi}_1$ and energies $E$ of (16). As a consequence, the corresponding solutions of (5) and (6) are found through the relationships (22) and (23), respectively. We will discuss two settings, the harmonic oscillator and Rosen–Morse system.
3.1. Harmonic oscillator case

Let us fix the vector potential \( A(x) \) as a complex-valued function linear in \( x \) in both its real and imaginary parts, so that

\[
A(x) = A_0 + m(w_r + i w_i)x, \quad m, w_r, w_i \in \mathbb{R}.
\]

(29)

We can interpret the vector potential term as the consequence of external homogeneous magnetic field and a mechanical strain that gives rise to homogeneous pseudo-magnetic field. Homogeneous pseudo-magnetic field in bilayer graphene was discussed in [26], whereas Dirac fermions in bilayer graphene in presence of homogeneous magnetic field were discussed in [4, 14].

The effective potential \( V_E(x) \) in (16) takes the form

\[
V_E(x) = \frac{m}{2} w_r^2 x^2 - mw_i v_3(E - V_2)x - mw_i - v_3 A_0(E - V_2).
\]

(30)

Clearly, the latter implies that we must solve the eigenvalue equation of the well-known stationary oscillator. Its general solution can be found once we cast the eigenvalue equation into the confluent hypergeometric equation. It reads explicitly as (for details, see [28])

\[
\tilde{\psi}_1 = e^{\frac{mw_i}{4} (x - \frac{\Omega_E}{w_i})^2} \left[ \ell_1 \, _1F_1 \left[ \frac{1}{4} - \frac{\lambda_E}{4mw_i}, \frac{1}{2}; \frac{mw_i}{w_i} \left( x - \frac{\Omega_E}{w_i} \right)^2 \right] + \ell_2 \left( x - \frac{\Omega_E}{w_i} \right) \, _1F_1 \left[ \frac{3}{4} - \frac{\lambda_E}{4mw_i}, \frac{3}{2}; \frac{mw_i}{w_i} \left( x - \frac{\Omega_E}{w_i} \right)^2 \right] \right].
\]

(31)

Here, \( \ell_1 \) and \( \ell_2 \) are constant coefficients, \( \, _1F_1[a, b; z] \) stands for the Kummer or confluent hypergeometric function [29], and

\[
\Omega_E := \frac{w_r v_3}{2mw_i}(E - V_2), \quad \lambda_E = \frac{v_1^2}{4} \left( 1 + \frac{w_r^2}{w_i^2} \right) (E - V_2)^2 + (E - V_2)(E - V_1 + v_3 A_0) + mw_i.
\]

(32)

It is worth noticing that the energy-dependent terms in the potential (30) cause just shifted decentering \( \Omega_E \) of the harmonic oscillator and shift the energies \( \lambda_E \).

Now, from the asymptotic behavior of the hypergeometric functions, it is straightforward to determine the physical values \( E \) for which the function \( \tilde{\psi}_1 \) becomes square integrable. This is achieved by imposing a polynomial behavior on the confluent hypergeometric function so that the Gaussian term (31) vanishes faster than the polynomial at \( x \to \pm \infty \). This leads to an exponentially vanishing function for large \( |x| \). Such a polynomial behavior is achieved if \( a = -n \) in \( \, _1F_1[a, b; z] \), with \( n \) being a non-negative integer or zero. To simplify the discussion we consider two cases. First, the conditions \( \ell_2 = 0 \) and \( \left( \frac{1}{4} - \frac{\lambda_E}{4mw_i} \right) = -n \) lead to \( \lambda_E = 2mw_i(2n + 1/2) \), from which we obtain \( \tilde{\psi}_1 \) in terms of even Hermite polynomials \( H_{2n}(z) \). Second, the conditions \( \ell_1 = 0 \) and \( \left( \frac{3}{4} - \frac{\lambda_E}{4mw_i} \right) = -n \) allow us to obtain \( \lambda_E = 2mw_i((2n + 1) + 1/2) \), and solutions for \( \tilde{\psi}_1 \) in terms of odd Hermite polynomials \( H_{2n+1}(z) \).

We can thus unify both the even and odd solutions as

\[
\tilde{\psi}_{1,n} = e^{\frac{mw_i}{4} (x - \frac{\Omega_E}{w_i})^2} H_n \left[ \frac{\sqrt{mw_i}}{w_i} \left( x - \frac{\Omega_E}{w_i} \right) \right], \quad \lambda_E = 2mw_i \left( n + \frac{1}{2} \right).
\]

(33)
The physical energies \( E_n \) are determined after comparing \( \lambda_\ell \) in (32) with (33). We get

\[
E_{\ell}^{(\pm)} = \frac{2(V_1 + V_2) - 2v_3 A_0 + V_1 v_3^2 \left(1 + \frac{x}{w_3}\right) \pm 2 \sqrt{(V_1 - V_2 - v_3 A_0)^2 + 2mv_1 \left(4 + v_1^2 \left(1 + \frac{x}{w_1}\right)\right)}}{4 + v_1^2 \left(1 + \frac{x}{w_1}\right)}.
\]

(34)

Here, it is worth remarking that the case \( n = 0 \) should be addressed with caution, as the exact value of \( E_0^{(\pm)} \) depends on the sign of \( V_1 - V_2 - v_3 A_0 \). That is,

\[
E_{0}^{(\pm)} = \begin{cases}
V_2 + 4 \frac{V_1 - V_2 - v_3 A_0}{4 + v_3^2 \left(1 + \frac{x}{w_3}\right)}, & V_1 - V_2 - v_3 A_0 > 0, \\
V_2, & V_1 - V_2 - v_3 A_0 < 0
\end{cases}
\]

(35)

From which we see that \( E = V_2 \) for either \( E_{0}^{(\pm)} \) or \( E_{0}^{(-)} \). We can find the spinor corresponding to this energy level from (19). It reads as

\[
\psi_0^{(\pm)} \big|_{E=V_2} = N_0^{(\pm)} e^{-i(\lambda_0 x + \frac{\mu_0}{m} z^2)} e^{-\frac{\mu_0}{m} x^2} \times \left( \frac{1}{\sqrt{mv_1}} \frac{V_1 - V_2 - v_3 A_0}{2} e^{mv_1 x^2} \text{Erf}(\sqrt{mv_1} x) + i \frac{\sqrt{mv_1}}{2} \right).
\]

(36)

It is not square-integrable as its the second component diverges for \( x \to \pm \infty \). The latter means that \( E = V_2 \) is not a physical energy, and we thus identify the point spectrum of \( h \) as

\[
\text{Sp}(h) = \begin{cases}
\{ E_{n+1}^{(\pm)} \}_{n=0}^{\infty} \cup \{ E_{n+1}^{(-)} \}_{n=0}^{\infty}, & V_1 - V_2 - v_3 A_0 > 0, \\
\{ E_{n+1}^{(\pm)} \}_{n=0}^{\infty} \cup \{ E_{n+1}^{(-)} \}_{n=0}^{\infty}, & V_1 - V_2 - v_3 A_0 < 0.
\end{cases}
\]

(37)

Now, we identify the corresponding spinor to each element in \( \text{Sp}(h) \). From the general solution (31), and after some calculations, we get

\[
\psi_n^{(\pm)} = N_n^{(\pm)} e^{-i(\lambda_n x + \frac{\mu_n}{m} z^2)} e^{-\frac{\mu_n}{m} x^2} \times \left( \frac{v_3}{2} \left(1 + i \frac{w_1}{w_3}\right) H_n(x) - \frac{2im}{E_n} H_n^{-1}(x)\right),
\]

(38)

for \( n = 0, 1, \ldots, \) and \( E_0^{(\pm)} \neq V_2 \), where we have introduced the reparametrized coordinate and decentering shift

\[
\tilde{z}_n^{(\pm)}(x) = \sqrt{mv_1} \left(x - \frac{\Omega_n^{(\pm)}}{w_3}\right), \quad \Omega_n^{(\pm)} := \frac{w_1 v_3}{2mv_1^2} (E_n^{(\pm)} - V_2),
\]

(39)

respectively.
Figure 1. (a) Probability densities $P_{n}^{(\pm)}$ related to the bispinors $\Psi_{n}^{(\pm)}$ for $n = 1$ (solid-blue) and $n = 2$ (dashed-red). (b) Probability densities $P_{n}^{(\pm)}$ related to the bispinors $\Xi_{n}^{(\pm)}$ for $n = 0$ (solid-blue) and $n = 1$ (dashed-red). (c) Energy levels $\varepsilon_{n}^{(+)}$ (solid-blue), $\varepsilon_{n}^{(-)}$ (dashed-blue), $\tilde{\varepsilon}_{n}^{(+)}$ (solid-red), and $\tilde{\varepsilon}_{n}^{(-)}$ (dashed-red). In all the cases, the parameters have been fixed as $m = 1, A_{0} = 0, w_{r} = 2, w_{l} = 1.8, \epsilon_{1} = -1, \epsilon_{2} = 1.5, \nu_{3} = 0.3$, and $\gamma = 0.7$.

Although the Hermite polynomials in (38) depend explicitly on the energy, it is still possible to compute the normalization factor for each spinor. This is done by exploiting the well-known properties of the Hermite polynomials, leading to, up to a global complex-phase,

$$N_{n}^{(\pm)} = \frac{1}{\sqrt{2^{n}n!}} \sqrt{\frac{mw_{l}}{\pi}} \left( 1 + \frac{v_{l}^{2}}{4} \right)^{\frac{1}{2}} \left( 1 + \frac{w_{r}^{2}}{w_{l}^{2}} \right)^{\frac{1}{2}} + \frac{2nmw_{l}}{(E_{n}^{(\pm)} - V_{2})^{2}}, \quad n = 0, 1, \ldots,$$

which holds true only for the elements in (37).

Hereafter, we have determined the eigenvalue problem related to $h$, and now the corresponding information for the reduced Hamiltonians $h_{1}$ and $h_{2}$ may be extracted directly from the relationships given in (22) and (23). For clarity, we use the notation

$$\varepsilon_{n}^{(\pm)} = E_{n}^{(\pm)} \left|_{V_{1} = \varepsilon_{1}} \right. \quad \text{and} \quad \tilde{\varepsilon}_{n}^{(\pm)} = E_{n}^{(\pm)} \left|_{V_{2} = \varepsilon_{2}} \right. \quad \text{for} \quad n = 0, 1, \ldots$$

(41)

to denote the physical energies of $h_{1}$ and $h_{2}$, respectively. The corresponding bispinors $\Psi_{n}$ and $\Xi_{n}$ of (2) are similarly extracted via (24).

Particularly, we depict the energy levels structure of the bilayer Hamiltonian $H$ in figure 1, for $m = 1, A_{0} = 0, \epsilon_{1} = -1, w_{r} = 2, w_{l} = 1.8, \epsilon_{2} = 1.5, \nu_{3} = 0.3$, and $\gamma = 0.7$. In this configuration, $V_{1} - V_{2} - A_{0}\nu_{3} < 0$, and so the energies $\varepsilon_{0}^{(+)}$ and $\varepsilon_{0}^{(-)}$ are both removed from the spectrum. Similarly, the corresponding bispinors are discarded. The behavior of the probability distributions related to the bispinors $\Psi_{n+1}^{(+)}$, $\Xi_{n+1}^{(+)}$, $\Psi_{n}^{(-)}$, and $\Xi_{n}^{(-)}$ are depicted in figures 1(a) and (b) for $n = 0, 1$.

3.2. Morse–Rosen potential

Now, let us associate $A(x)$ with a smooth step-like profile, defined in terms of a purely imaginary function of the form,

$$A(x) = iU_{0} \left( \frac{\ell(\ell - 1)}{2(K - 1)} + (\kappa - 1) \tanh (U_{0}x) \right), \quad \ell, \kappa \geq 1.$$

(42)
In this case, the effective potential in (16) becomes
\[
V_E(x) = -U_0^2 \kappa (\kappa - 1)[\text{sech}(U_0 x)]^2 + U_0^2 \ell (\ell - 1) \tanh(U_0 x)
+ U_0^2 \left[ \frac{\ell^2 (\ell - 1)^2}{4(1 - \kappa)^2} + (1 - \kappa)^2 \right],
\]
which corresponds to the Morse–Rosen interaction [30] (also known as the hyperbolic Rosen–Morse potential), one of the well-known exactly solvable models in quantum mechanics [28, 31–33]. Interestingly, for \( \ell = 1 \), the potential (43) reduces to the Pöschl–Teller interaction, a particular case to be discussed in detail in the upcoming sections. Let us notice that Dirac fermions in (single-layer) graphene in presence of (42) were discussed.

In this case, the effective potential in (16) becomes (43), which we obtain a polynomial equation of second-order for \( E \). We thus get the eigenstates for \( \tilde{\psi}_1 \) into the hypergeometric form [28], and after some calculations, we get the eigenstates
\[
\tilde{\psi}_{1,n} = C_n (1 - \text{tanh}(U_0 x))^\tilde{\alpha}_n (1 + \text{tanh}(U_0 x))^\tilde{\beta}_n P^{(\alpha_n, \beta_n)}_n(\tanh(U_0 x)),
\]
where \( n = 0, 1, \ldots, n_{\text{max}} \), and \( P^{(\alpha, \beta)}_n(z) \) stands for the Jacobi polynomials, with
\[
\alpha_n := (\kappa - n - 1) - \frac{\ell (\ell - 1)}{2(n + 1 - \kappa)}, \quad \beta_n := (\kappa - n - 1) + \frac{\ell (\ell - 1)}{2(n + 1 - \kappa)},
\]
and the normalization constant [31]
\[
C_n = 2^{n+\kappa-\kappa} \frac{U_0^\kappa \Gamma(2\kappa - n - 1)}{(\kappa - n - 1) \Gamma(\alpha_n + n + 1) \Gamma(\beta_n + n + 1)}\frac{\Gamma(\alpha_n)}{\Gamma(\beta_n + n + 1)}. \tag{47}
\]
From (44), it follows that \( \tilde{\psi}_{1,n} \) is square-integrable only when both \( \alpha_n, \beta_n > 0 \). This leads us to a condition for the existence of at least one bound state and an upper bound \( n_{\text{max}} \) given by
\[
(\kappa - 1)^2 > \frac{\ell (\ell - 1)}{2}, \quad n_{\text{max}} = \left[ \kappa - 1 - \sqrt{\frac{\ell (\ell - 1)}{2}} \right], \tag{48}
\]
respectively.

The physical energies \( E_n \) are then determined by comparing \( \lambda_E \) in (45) with (26), from which we obtain a polynomial equation of second-order for \( E_n \). We thus get the energies
\[
E_{n}^{(\pm)} = \frac{2(V_1 + V_2) + v_3^2 V_2 \pm 2\sqrt{(V_1 - V_2)^2 + U_0^2(4 + v_3^2)\beta_n^2 - \beta_n^4}}{4 + v_3^2}. \tag{49}
\]
belong to the pointspectrum of

Now, the discrete energylevels associated with

The second spinor component $\psi_{2,n}$ is determined from (11), and the corresponding spinors take the form

\begin{align*}
\psi_{2,n}^{(\pm)} &= \begin{pmatrix} \psi_{2,n}^{(\pm)}_1 \\ \psi_{2,n}^{(\pm)}_2 \end{pmatrix} = e^{-\frac{i}{2} \epsilon_n(E^{(\pm)}_n - V_2) \gamma (1 - z) n \gamma (1 + z) n \gamma}
\times 
\begin{pmatrix}
\frac{v_3}{2} + \frac{i U_0}{(E^{(\pm)}_n - V_2)} \left( \frac{\ell(\ell - 1)}{2(\kappa - 1)} + \frac{\ell(\ell - 1)}{2(n + 1 - \kappa)} + n^2 \right) p_n^{(\alpha_\kappa, \beta_\kappa)}(z)
\end{pmatrix}
\times 
\left. 
\begin{multlined}
\left( \frac{-i}{2} \frac{U_0}{(E^{(\pm)}_n - V_2)} \left( \frac{\ell(\ell - 1)}{2(\kappa - 1)} + \frac{\ell(\ell - 1)}{2(n + 1 - \kappa)} + n^2 \right) p_n^{(\alpha_\kappa, \beta_\kappa)}(z) \\
\frac{1}{2(E^{(\pm)}_n - V_2)} \right)
\end{multlined}
\right)
\end{align*}

with $z \equiv z(x) := \tanh(U_0 x)$.

The exact value of $E_0^{(\pm)}$ depends on the sign of $V_1 - V_2$. In analogy to the oscillator-like interaction of the previous section, we have $E_0^{(+)} = V_2$ for $V_1 < V_2$, and $E_0^{(-)} = V_2$ for $V_1 > V_2$. The corresponding spinors (19) are not square integrable so that these values do not belong to the point spectrum of $h$.

We found the following set $\text{Sp}(h)$ of discrete energies of $h$,

\begin{align*}
\text{Sp}(h) = \begin{cases}
\{ E_n^{(+)} \}_{n=1}^{n_{\text{max}}} \cup \{ E_n^{(-)} \}_{n=0}^{n_{\text{max}}} & , \quad V_1 < V_2 \\
\{ E_n^{(+)} \}_{n=0}^{n_{\text{max}}} \cup \{ E_n^{(-)} \}_{n=1}^{n_{\text{max}}} & , \quad V_1 > V_2
\end{cases}
\end{align*}

Since $\alpha_n$ and $\beta_n$ do not depend on the energy upper-index $(\pm)$, the upper bound $n_{\text{max}}$ is the same for both energies $E_n^{(\pm)}$. For $n_{\text{max}} = 0$, the set of discrete energies is just $\{ E_0^{(-)} \}$ for $V_1 < V_2$, and $\{ E_0^{(+)\gamma} \}$ for $V_1 > V_2$. Moreover, the point spectrum may be empty if the inequality in (48) is not fulfilled.

Now, the discrete energy levels associated with $h_1$ and $h_2$ are obtained via (22) and (23),

\begin{align*}
\text{Sp}(h_1) &= \text{Sp}(h)|_{V_1 = 1}^{V_2 = 1 + \gamma} \quad , \quad \text{Sp}(h_2) = \text{Sp}(h)|_{V_1 = 1}^{V_2 = 2 + \gamma}
\end{align*}

The bispinor solutions of (4) can be obtained via (24). In figure 2, we illustrate calculated energy levels of $h_1$, $h_2$, and $H$, and probability density related to the bispinors $\Psi_n$ and $\Xi_n$. In
particular, we have considered $U_0 = 1$, $\kappa = 2.5$, $\ell = 1.1$, $\epsilon_1 = -1$, $\epsilon_2 = 1.5$, $v_3 = 0.3$, and $\gamma = 0.7$. In such a case, we obtain $n_{\text{max}} = 1$, so that we generate two physical solutions for each reduced Hamiltonian. On the other hand, for both $h_1$ and $h_2$, we get $V_1 < V_2$, which means that $E^{(+)}_0$ is discarded from the point spectrum. Each reduced Hamiltonian contributes with three physical energies, and so the bilayer Hamiltonian $H$ contains six energy levels. See figure 2(c).

4. Confinement by the on-site interactions

In this section, we focus on the case where $A(x)$ in (2) vanishes. The on-site interactions $\epsilon_1(x)$, $\epsilon_2(x)$ as well as the inter-layer coupling $\gamma(x)$ can be inhomogeneous. Let us suppose that $\epsilon_2(x)$ and $\gamma(x)$ are related by (27), i.e. $V_2 = \epsilon_2(x) + \gamma(x)$ is constant. It brings the equations (5) and (6) into

$$h_1 \xi = \left( -i \begin{pmatrix} v_1 & 1 \\ 1 & 0 \end{pmatrix} \right) \partial_x + \begin{pmatrix} \epsilon_1(x) & 0 \\ 0 & V_2 \end{pmatrix} \xi = E \xi,$$

$$h_2 \chi = \left( -i \begin{pmatrix} -v_3 & 1 \\ 1 & 0 \end{pmatrix} \right) \partial_x + \begin{pmatrix} \epsilon_1(x) & 0 \\ 0 & \epsilon_2(x) - \gamma(x) \end{pmatrix} \chi = E \chi. \tag{52}$$

As discussed in the section 2, the solutions of (52) can be found via the Schrödinger equation with energy-dependent potential (16). It acquires the following simple form

$$-\tilde{\psi}'' = (E - V_2)(E - \epsilon_1(x))\tilde{\psi}_1 = \frac{v_1^2}{4}(E - V_2)^2\tilde{\psi}_1|_{x = \ell = 2 + \gamma}, \tag{54}$$

where the spinor components of $\xi = (\xi_1, \xi_2)^T$ are determined from $\tilde{\psi}_1$ through

$$\xi_1 = e^{-i\frac{v_1}{2}(E - V_2)\ell}\tilde{\psi}_1, \quad \xi_2 = -\frac{i \psi_1'}{(V_2 - E)}. \tag{55}$$

see (22) and (23). The equation (53) reduces into (14). With current fixing of the quantities and denoting $B(x) = \epsilon_2(x) - \gamma(x)$, the latter equation reads as

$$-\tilde{\chi}'' \left( \frac{v_1^2}{4}(E - B(x))^2 - (E - \epsilon_1(x))(E - B(x)) \right) + \frac{3(B(x)'(x))^2}{4(E - B(x))^2} + \frac{B(x)''(x)}{2(E - B(x))} \tilde{\chi}_1 = 0, \tag{56}$$

where $\chi_1 = \sqrt{E - B(x)}e^{i(-\int B(x)dx)}\tilde{\chi}_1$, see (12) and (23).

We find it physically reasonable to consider the systems where $\epsilon_1(x)$ is bounded. We shall match either (54) or (56) with the stationary equation of a solvable quantum systems that meets these requirements. Let us consider the stationary equation of the Pöschl–Teller system

$$-\tilde{\psi}'' - \kappa(\kappa - 1)U_0^2 \text{sech}^2 U_0 x \tilde{\psi}_1 - \lambda \tilde{\psi}_1 = 0, \quad \kappa > 1. \tag{57}$$

It is worth noticing that Dirac electrons in bilayer graphene were studied in presence of Pöschl–Teller electrostatic potential in [35], see also [36]. The equation (57) is a special case ($\ell = 1$) of the Rosen–Morse equation (43) that was discussed in the previous section. Therefore, we can use (44) and write down the square integrable solutions $\tilde{\psi}_{1,n}$ of (57),

$$\tilde{\psi}_{1,n} = c_n^{(+)} \text{sech} (U_0 x)^{\alpha_1} P^{(\ell = \alpha_1)}_n \left( \tanh(U_0 x) \right), \tag{58}$$
where $n = 0, 1, \ldots, n_{\text{max}} = \lfloor \kappa - 1 \rfloor$, and $\alpha_n = (\kappa - n - 1)$. The corresponding eigenvalues $\lambda_n$ are

$$
\lambda_n = -U_0^2(1 - \kappa + n)^2 = -U_0^2 \alpha_n^2.
$$

(59)

Now, we shall identify either (54) or (56) with (57). Let us start with (54),

$$(E - V_2)(E - \epsilon_1(x)) + \frac{v_0^2}{4}(E - V_2)^2 = \kappa(\kappa - 1) U_0^2 \sech^2 U_0 x - U_0^2 (1 - \kappa + n)^2.$$

(60)

There are different ways how to fix $\epsilon_1(x)$, and each of them leads to different values of $E$. Let us discuss some of them.

### 4.1. Case I

To begin with, let us consider the inhomogeneous on-site interaction

$$
\epsilon_1(x) = A U_0^2 \sech^2 U_0 x, \quad A > 0.
$$

(61)

Then (60) is satisfied provided that we fix $\kappa_\epsilon \equiv \kappa$ and $E$ such that they solve the following two equations,

$$
\kappa_\epsilon(\kappa_\epsilon - 1) = A (V_2 - E), \quad U_0^2 (1 - \kappa_\epsilon + n)^2 = (V_2 - E) \left( \frac{v_0^2}{4}(V_2 - E) - E \right).
$$

(62)

That is, the Pöschl–Teller amplitude $\kappa_\epsilon$ depends explicitly on the energy through

$$
\kappa_\epsilon = \frac{1}{2} + \sqrt{A(V_2 - E) + \frac{1}{4}}.
$$

(63)

On the other hand, the second equation in (62) yields to a fourth-order polynomial equation for $E$ of the form

$$
(V_2 - E) \left( 1 + \frac{v_0^2}{4} \right) (V_2 - E) - V_2 = -U_0^2 \left( n + \frac{1}{2} - \sqrt{A(V_2 - E) + \frac{1}{4}} \right)^2,
$$

(64)

the solutions of which become unfeasible to obtain in the general setup. Despite such complexity, we can proceed further and obtain some additional information.

The straightforward calculational show that the square-integrable condition (59) still holds in this case, with $\kappa \rightarrow \kappa_\epsilon$ given in (63). Thus, with the current choice of parameters, the solutions of (57)

$$
\tilde{\psi}_{1,\mu} = C_n \sech(U_0 x)^2 p_n^{\alpha_n \pi \mu} (\tanh(U_0 x))
$$

(65)

$$
\overline{\alpha}_n = -n - 1 + \kappa_\epsilon = -n - 1 + \sqrt{A(V_2 - E) + \frac{1}{4}}.
$$

(66)

are square-integrable provided that $\overline{\alpha}_n$ is positive. In order to keep $\overline{\alpha}_n$ real, the term inside the square-root of $\overline{\alpha}_n$ must be positive. Since $A > 0$, we get $E < V_2 + \frac{1}{4\kappa}$. Additionally, the requirement $\overline{\alpha}_n > 0$ is satisfied provided that $E < V_2 - \frac{v_0^2}{4\kappa}$. Now, the right-hand side of (64) is negative, as it a multiple of $-\overline{\alpha}_n$. Therefore, one obtains real solutions of (64) for $(V_2 - E)$ only if the term on the left is negative as well, which is quadratic and convex on $(V_2 - E)$. We
obtain \( 0 < (V_2 - E) < \frac{4\sqrt{3}}{4 + v_3^2} \), for \( V_2 > 0 \), and \( \frac{4\sqrt{3}}{4 + v_3^2} < (V_2 - E) < 0 \), for \( V_2 < 0 \). Therefore, any real solution of (64) has to lie inside the one of the following intervals

\[
E \in \left( \frac{v_3^2}{4 + v_3^2}, V_2, V_2 \right) \cap \left( -\infty, V_2 - \frac{n(n+1)}{A} \right), \quad \text{for } V_2 > 0,
\]

\[
E \in \left( V_2, \frac{v_3^2}{4 + v_3^2} \right) \cap \left( -\infty, V_2 - \frac{n(n+1)}{A} \right), \quad \text{for } V_2 < 0,
\]

(67)

where in the latter is clear that \( \frac{4\sqrt{3}}{4 + v_3^2} < 1 \), for \( v_3 \in \mathbb{R} \). The requirement that the intersections are non-empty sets an upper bound for possible values of \( n \). Indeed, when \( V_2 > 0 \), the intersection is nonempty for \( n \leq n_{\text{max}} \) where

\[
n_{\text{max}} = \left\lceil \frac{4A\sqrt{V_2} + 1}{4} - \frac{1}{2} \right\rceil, \quad \text{for } V_2 > 0.
\]

(68)

It provides us with an upper bound for the maximum number of physical solutions which is \( n_{\text{max}} + 1 \). It is worth noticing that the trigonal warping term acts against the confinement here; the larger is \( |v_3| \), the smaller is \( n_{\text{max}} \). When \( V_2 < 0 \), it is clear that only \( n = 0 \) leads to an nonempty intersection of the energy intervals. This yields to \( E = V_2 \). Nevertheless, the expression (19) suggests that the corresponding spinor is not square integrable. Thus, such a solution is discarded, and no physical solutions are produced for \( V_2 < 0 \).

Interestingly, even if \( E \) has to be found by numerical means, we have been able to extract general information about the spectrum and number of physically allowed solutions. Furthermore, the spinor may be computed explicitly from (55) and (65) as

\[
\psi_n = \begin{pmatrix} \psi_{1,n} \\ \psi_{2,n} \end{pmatrix} = N_n e^{i\frac{\alpha}{2}(\nu_2 - \nu_1)\alpha} \left( \frac{p^{\nu_1-\nu_2}c(z)}{V_2 - E_n} \right) \begin{pmatrix} p^{\nu_1-\nu_2}c(z) \\ \frac{1}{2} + \frac{U_0}{V_2 - E_n} \text{tanh}(U_0z) \end{pmatrix} \left( -\frac{v_3}{2} + \frac{U_0\nu_2}{V_2 - E_n} \text{tanh}(U_0z) \right) \begin{pmatrix} p^{\nu_1-\nu_2}c(z) \\ \frac{1}{2} + \frac{U_0}{V_2 - E_n} \text{sech}^2(U_0z)p^{\nu_1-\nu_2}c(z) \end{pmatrix},
\]

(69)

where \( z(x) := \text{tanh}(U_0x) \), and \( n = 0, 1, \ldots, n_{\text{max}} \). The corresponding set of bispinors follow from (3), which in this case are given through

\[
\Psi_n = \frac{1}{\sqrt{2}} (\psi_{1;n}, \psi_{2;n}, \psi_{1;n}, \psi_{1;n})^T, \quad n = 0, 1, \ldots, n_{\text{max}}.
\]

(70)

To illustrate our results, we consider \( V_2 = 2.3, v_3 = 0.1, U_0 = 1, \) and \( A = 2.7 \), so that real energies lie inside the interval \( E \in (0.000735, 2.3) \). Moreover, from (68), the maximum number of physical energies is \( n_{\text{max}} + 1 = 3 \). The numerical values for \( E_n \) and \( \nu_n \) are shown in table 1. For each \( n \), we obtain two energies \( E_n \), which are all real for \( n = 0, 1, 2 \), and complex for \( n \geq 3 \). The physical energies are identified as \( E_0 = 1.42592, E_1 = 0.36825, \) and \( E_2 = 0.0063725 \) as they render \( \nu_n \) positive. The associated probability distributions are depicted in figure 3(a).

4.1.1. Particular setup for \( h_\gamma \). As we have remarked, the solutions of (54) are insensitive with respect to the explicit form of \( \gamma \) and \( \epsilon_2 \), and thus we can impose some further restrictions in
Table 1. Numerical solutions of the characteristic equation (64). We have fixed the parameters as $A = 2.7$, $v_3 = 0.1$, $U_0 = 1$. For $h_1$ we used $V_2 = \epsilon_2 + \gamma = 2.3$, whereas for $h_2$ we have $V_2 = \epsilon_2 - \gamma = 1.7$.

| $n$ | $E_n$ | $\tau_n$ | $E_n$ | $\tau_n$ |
|-----|-------|----------|-------|----------|
| 0   | 2.3   | 0        | 1.7   | 0        |
|     | 1.42592 | 1.1155  | 1.18507 | 0.780747 |
| 1   | 2.11215 | -0.629828 | 1.47578 | -0.575127 |
|     | 0.36825 | 0.837884  | 0.217073 | 0.562499 |
| 2   | 1.61515 | -1.05117  | 0.779102 | -0.845786 |
|     | 0.0063725 | 0.038266 | 0.0792946 | -0.349208 |

$\epsilon_2, \gamma \in \mathbb{R}$, \hspace{1cm} (71)

the equation (56) brings us back to the system (62), where now $V_2 \equiv B = \epsilon_2 - \gamma$. Therefore, we can solve the equation in exactly the same manner as we did in table 1 for different values of $V_2$ now. This is illustrated in the fourth and fifth columns of table 1, where the energy levels are determined for $V_2 = \epsilon_2 - \gamma = 1.6$, $v_3 = 0.1$, $U_0 = 1$, and $A = 2.7$. The corresponding probability distributions associated with the bispinors $\Xi_n$ are depicted in figure 3(b).

4.2. Case II

Fixing of $\epsilon_1(x)$ in (61) allowed us to have the on-site interaction independent on $n$. The price we paid was that the equation (64) had to be solved numerically. Let us consider the other choice of parameters such that (60) is satisfied. We fix

$$\epsilon_1(x) = \kappa A U_0^2 \text{sech}^2 U_0 x.$$ \hspace{1cm} (72)
Additionally, there must hold
\[-A(E - V_2) = (\kappa - 1)U_0^2, \quad (E - V_2) \left( \frac{\nu_0^2}{4}(E - V_2) + E \right) = -U_0(1 - \kappa + n)^2.\] (73)

We can see that inclusion of $\kappa$ into $\epsilon_1(x)$ lowered the order of $\kappa$ in the second equation in (73) when compared to (62). We can solve (73) for $E$ and either $\kappa$, $U_0$ or $V_2$. As we require $\epsilon_1(x)$ to be independent of $n$, we solve the equation for $V_2$, and $E$,
\[V_2(A, \kappa, n) = \left( 1 + \frac{\nu_0^2}{4} \right) \frac{(\kappa - 1)}{A} + \frac{A(1 + n - \kappa)^2U_0^2}{\kappa - 1},\]
\[E(A, \kappa, n) = \frac{A U_0^2 (1 + n - \kappa)^2}{\kappa - 1} + \frac{(\kappa - 1)\nu_0^2}{4A}.\] (74)

$V_2(A, \kappa, n)$ is a parabola in $n$ with minimum $V_2(n_0) = \frac{\nu_0^2}{4} \left( 1 + \frac{\nu_0^2}{4} \right)$ at $n_0 = \kappa - 1$. For each $n$ from the allowed interval $n \in \{0, \ldots, |\kappa - 1|\}$, the on-site interaction $\epsilon_1(x)$ remains the same. Nevertheless, the value of $V_2$ gets changed correspondingly.

For each of this specific configurations, we are able to find a localized solution $\tilde{\psi}_{1,n}$. The bispinor solution of (4) is then
\[\Psi = \left( \psi_1, \frac{-i\psi'_1}{E - V_2(A, \kappa, n)} \right), \quad \psi_1 = e^{-\frac{i\nu_0}{2}(E - V_2)\psi_{1|E = E(A, \kappa, n), V_{2|V_2(A, \kappa, n)}}.\] (75)

The solution is invariant with respect to the changes of $\epsilon_2(x)$ and $\gamma(x)$ that preserve $V_2$, including the case where both $\epsilon_2(x)$ and $\gamma(x)$ are constant. When this is the case, the equation (56) reduces into the equation that coincides with (54), yet for $V_2 = \epsilon_2 - \gamma$. If $\epsilon_2$ and $\gamma$ are such that
\[\epsilon_2 + \gamma = V_2(A, \kappa, n) \quad \text{and} \quad \epsilon_2 - \gamma = V_2(A, \kappa, \bar{n}), \quad n, \bar{n} \in \{0, \ldots, |\kappa - 1|\},\] (76)
then we can get bound state solutions for each equations (52) and (53) with energies $\epsilon = E(A, \kappa, n)$ and $\bar{\epsilon} = E(A, \kappa, \bar{n})$. The bispinor $\Xi$ corresponding to the latter energy is
\[\Xi = \left( \psi_1, \frac{-i\psi'_1}{E - V_2(A, \kappa, n)} \right), \quad \psi_1 = e^{\frac{i\nu_0}{2}(E - V_2)\psi_{1|E = E(A, \kappa, \bar{n}), V_{2 = V_2(A, \kappa, \bar{n})}}.\] (77)

We show density of states of the corresponding bispinors $\Psi$ and $\Xi$ in figure 4.

4.3. Case III

Let us relax the condition (27), i.e. both $\epsilon_2 + \gamma$ and $\epsilon_2 - \gamma$ can be inhomogeneous. In this case, the equations (5) and (6) lead to (14), yet with different form of $V_2(x)$ in each case. As we mentioned in section 2, we can find configuration of $\epsilon_1(x)$, $\epsilon_2(x)$ and $\gamma(x)$ such that (14) is partially solvable. Let us consider the case with $V_2(x) = \gamma(x) + \epsilon_2(x)$. We fix $V_1(x) \equiv \epsilon_1(x)$ as
\[\epsilon_1(x) = \frac{\nu_0^2}{4}(E - V_2(x)) - \frac{3(V_2(x))^2}{4(E - V_2(x))^3} - \frac{V_0(x)}{2(E - V_2(x))} + \frac{V_0(x)}{E - V_2(x)},\] (78)
Then the equation (14) reduces into

$$-\tilde{\psi}_1'' + V_0(x)\tilde{\psi}_1 = 0.$$  

(79)

Let us identify (79) with the stationary equation of the Pöschl–Teller system again, $V_0(x) = -\kappa(\kappa - 1)U^2_0 \text{sech}^2 U_0 x + U^2_0 (1 + n - \kappa)^2$. When $n$ is a positive integer $n \in \{0, \lfloor \kappa - 1 \rfloor \}$, the equation (79) has a localized solution $\tilde{\psi}_1; n$, see (58). Notice that this solution is independent on the explicit choice of $V_2(x)$ and $E$ as they do not appear in (79). Nevertheless, both $V_2(x) = \gamma(x) + \epsilon_2(x)$ and $E$ affect the form of the bispinor solution (24) via (12),

$$\Psi(x) = \left( \psi_1(x), \frac{-i\psi'_1(x)}{E - \epsilon_2(x) - \gamma(x)}, \frac{-i\psi'_2(x)}{E - \epsilon_2(x) - \gamma(x)}, \psi_1 \right),$$

$$\psi_1(x) = \sqrt{E - \epsilon_2(x) - \gamma(x)} e^{-\frac{i}{2}(E - \epsilon_2(x) - \gamma(x))\tilde{\psi}_{1,\text{R}}(x)}.  \quad (80)$$

In the current setting, $E$ plays rather the role of an interaction parameter. We can tune the interaction $\epsilon_1(x)$ by changing $E$ such that it confines a bound state with energy equal to $E$. In order to keep $\Psi(x)$ square-integrable, we require that $\sqrt{E - V_2(x)}$ is a bounded function.

We fix $V_2(x) = \epsilon_2(x) + \gamma, \gamma \in \mathbb{R}$. When $\epsilon_2(x)$ is periodic, $\epsilon_1(x)$ shares its periodicity up to the last term in (78) that represents a periodicity defect. For instance, if we fix

$$\epsilon_2(x) = V_2(x) - \gamma = c \cos x, \quad 0 < c < E,$$

(81)

then $\epsilon_1(x)$ reads explicitly

$$\epsilon_1(x) = E + \frac{(E - c \cos x)\epsilon_2^3}{4} + \frac{-2c \cos x(c \cos x - E) - 3c^2 \sin^2 x}{4(E - c \cos x)^3},$$

$$+ \frac{(1 + n - \kappa)^2 U^2_0}{E - c \cos x} + \frac{(1 - \kappa) U^2_0 \text{sech}^2 U_0 x}{E - c \cos x}.  \quad (82)$$

We illustrate the interactions for different values of parameters in figure 5 together with density of probability of the bound state.
Figure 5. Inhomogeneous on-site energy $\epsilon_1(x)$ (dotted-blue) in (83), $\epsilon_2(x)$ (dashed-red) in (81), and the probability distribution $P_\Psi = \Psi^\dagger \Psi$ (filled-green). We fixed $\kappa = 2.2$, $v_3 = 0.2$, $U_0 = 0.2$, $c = 0.1$, $n = 0$, $\gamma = 0.5$, and $E = 0.5$ (left) and $E = 1$ (right).

5. Confinement by the inter-layer interaction

Up to now, the inter-layer coupling $\gamma$ had rather implicit influence on considered solutions as it was ‘hidden’ in $V_2$. Let us see whether we can get an analytical solution of confined Dirac fermions by inhomogeneous $\gamma$. It is worth noticing in this context that confinement by inhomogeneous $\gamma$ with rotational symmetry was analyzed numerically in [7, 8]. We focus on the situation where $\epsilon_1$ and $\epsilon_2$ are constant (in order to eliminate localization by on-site potentials) and $\gamma$ is inhomogeneous. We fix

$$v_3 = 0, \quad V_1 = \epsilon_1 = \text{const.}, \quad \epsilon_2 = \text{const.}, \quad V_2(x) = \epsilon_2 + \gamma(x).$$

(84)

Then the equations (5) and (6) for the spinor components $\xi_j$ and $\chi_j$, for $j = 1, 2$, can be decoupled through the relationships

$$-\frac{i\xi_2'}{E - \epsilon_1} = \xi_1, \quad -\frac{i\chi_2'}{E - \epsilon_1} = \chi_1,$$

(85)

leading to the effective energy-dependent Schrödinger equation

$$-\xi_2'' - (\epsilon_1 - E)\gamma(x)\xi_2 = (\epsilon_1 - E)(\epsilon_2 - E)\xi_2, \quad -\chi_2'' + (\epsilon_1 - E)\gamma(x)\chi_2 = (\epsilon_1 - E)(\epsilon_2 - E)\chi_2.$$

(86)

Notice that the difference between the latter equations relies on the sign of the energy-dependent potential, which both coincide qualitatively with (54), therefore, we can follow the same steps as in the previous section. We shall identify both equations in (86) with (57). We set

$$\gamma(x) = A U_0^2 \text{sech}^2 U_0 x + \gamma_0, \quad \gamma_0, A > 0, \quad U_0 \in \mathbb{R}.$$

(87)

It has similar form to $\epsilon_1(x)$ in (61), however, it acquires nonvanishing constant value $\gamma_0$ asymptotically now. In analogy to the previous section, we identify the set of solutions as

$$\psi_{2,n} = C_2 (\text{sech}(U_0 x))^{\eta_n} P_{\eta_n} (\text{tanh}(U_0 x)), \quad \chi_{2,n} = D_2 (\text{sech}(U_0 x))^{\tilde{\eta}_n} P_{\tilde{\eta}_n} (\text{tanh}(U_0 x)),$$

(88)

5 We have set, without loss of generality, $A > 0$ to simplify the conditions for the reality of the spectrum. Nevertheless, similar conclusions can be withdrawn if we allow $A < 0$. 


where $C_2$ and $D_2$ are the respective normalization factors, and
\[
\eta_\nu = \nu E - n - 1, \quad \tilde{\eta}_\nu = \tilde{\nu} E - n - 1,
\]
(89)
together with
\[
\nu E = \frac{1}{2} + \sqrt{\mathcal{A}(\epsilon_1 - E) + \frac{1}{4}}, \quad \tilde{\nu} E = \frac{1}{2} + \sqrt{-\mathcal{A}(\epsilon_1 - E) + \frac{1}{4}}.
\]
(90)
On the other hand, from the relationship
\[
(\epsilon_1 - E)(\epsilon_2 - E + \delta \gamma_0) = -U_0^2 \left( n + \frac{1}{2} - \sqrt{\delta \mathcal{A}(\epsilon_1 - E) + \frac{1}{4}} \right)^2, \quad \delta = +1, -1,
\]
(91)
we extract the energies of $h_1$ and $h_2$ after choosing $\delta = +1$ and $\delta = -1$, respectively.

It is worth to recall that an immediate solution for the energy equation (91) can be found for $n = 0$ and $E = \epsilon_1$ in both cases $\delta = \pm 1$. Nevertheless, such a solution is discarded as it is not square-integrable.

Likewise in (67), we can obtain the energy intervals in which $E$ takes real values for $h_1$ and $h_2$. First, we should guarantee that $\eta_\nu$ and $\tilde{\eta}_\nu$ in (89) are real and positive in order to get square-integrable solutions. Next, the left term in (91) should be negative as the equation would have no real solutions otherwise. Combining both results, we get the intervals where the real roots of (91) have to lie,
\[
h_1: E \in \left( \epsilon_1 + \gamma_0, \epsilon_1 - \frac{n(n + 1)}{\mathcal{A}} \right), \quad \epsilon_1 > \epsilon_2 + \gamma_0,
\]
(92)
\[
h_2: E \in \left( \epsilon_2 + \gamma_0, \epsilon_1 - \frac{n(n + 1)}{\mathcal{A}} \right), \quad \epsilon_1 < \epsilon_2 - \gamma_0.
\]
(93)
One cannot get square-integrable eigenstates of either $h_1$ or $h_2$ corresponding to real energies for other values of $\epsilon_1$. Now, the intervals (93) are nonempty for some values of $n$ only. This way, we get an upper bounds $n_{\text{max}}$ and $\tilde{n}_{\text{max}}$ for the number of bound states of $h_1$ and $h_2$, respectively, that we can obtain this way. They are
\[
n_{\text{max}} = \left\lfloor \sqrt{\mathcal{A}(\epsilon_1 - \epsilon_2 - \gamma_0) + \frac{1}{4} - \frac{1}{2}} \right\rfloor, \quad \tilde{n}_{\text{max}} = \left\lfloor \sqrt{\mathcal{A}(\epsilon_2 - \epsilon_1 - \gamma_0) + \frac{1}{4} - \frac{1}{2}} \right\rfloor.
\]
(94)
The corresponding bispinors are given by
\[
\Psi_n = \frac{1}{\sqrt{2}} \left( -i \frac{\psi_{2,n}^\dagger}{E - \epsilon_1}, \psi_{2,n}, -i \frac{\psi_{2,n}^\dagger}{E - \epsilon_1} \right), \quad \Xi_n = \frac{1}{\sqrt{2}} \left( -i \frac{\xi_{2,n}^\dagger}{E - \epsilon_1}, \xi_{2,n}, -i \frac{\xi_{2,n}^\dagger}{E - \epsilon_1} \right),
\]
(95)
for $\epsilon_1 > \epsilon_2 + \gamma_0$ and $\epsilon_1 < \epsilon_2 - \gamma_0$, respectively, with $\psi_{2,n}$ and $\chi_{2,n}$ given in (88).

To illustrate the results presented in this section, let us fix the parameters as $U_0 = \mathcal{A} = 1$, $\epsilon_1 = 1.5$, $\epsilon_2 = -1.5$, and $\gamma_0 = 0.3$. Since $\epsilon_1 > \epsilon_2 + \gamma_0$, we would expect bound states only for $h_1$, besides the non-physical solution $E = \epsilon_1 = 1.5$. Moreover, from (94), one may see that only two bound states can be generated. Such an information may be verified in table 2, where we obtain two physical energies for $h_1$ as $E_0 = -0.685308$ and $E_1 = -1.18315$. Although there are more real energies, they do not satisfy the finite-norm condition $\eta_n > 0$. The behavior for the corresponding probability densities $P_n$ is depicted in figure 6.
Table 2. Energy solutions of (91), together with the finite-norm condition $\eta_n > 0$ and
$\tilde{\eta}_n > 0$, for the reduced Hamiltonians $h_1$ and $h_2$. The parameters have been fixed as
$U_0 = A = 1$, $\epsilon_1 = 1.5$, $\epsilon_2 = -1.5$, and $\gamma_0 = 0.3$.

| $n$ | $E$  | $\eta_n$ |
|-----|------|---------|
| 0   | -0.685 308 | 1.060 55 |
| 1.5 | 0     |         |
| 1   | -1.183 15 | 0.212 643 |
| 1.245 11 | -0.789 447 |
| 2   | -0.873 539 | -0.880 266 |
| 0.381 221 | -1.330 05 |

Figure 6. Probability density related to the bispinors $\Psi_n$ of (95) for $n = 0$ (solid-blue)
and $n = 1$ (dashed-red). The parameters has been fixed as in table 2.

6. Discussion

In the article, we focused on the systems described by Dirac Hamiltonians of the form (2)
that appear in the analysis of bilayer Dirac materials. We were interested in analytical
 treatment of confined states that can appear due to local fluctuations (61), (72), (87)
or periodicity defects (83) of the involved interactions.

We have made use of the fact that equation (4) is reducible in terms of equations (5)
and (6) with $2 \times 2$ Hamiltonians. In section 2, we showed that both equations could be solved
through the solutions of the Schrödinger equation (14), whose potential is a nonlinear function
of the interactions and their derivatives. We focused on the specific case where (14) can be
significantly simplified into a Schrödinger equation with energy-dependent potential (16).

We considered confinement by a combination of external magnetic field and mechanical
deformations in section 3 where the energy-dependent Schrödinger equation was identified
with the stationary equation of the harmonic oscillator or the Rosen–Morse system.
In section 4, we focused on confinement by inhomogeneities of the on-site and inter-layer
interactions. We showed that Dirac fermions can be confined by a local fluctuation or periodicity
defect of the on-site interaction $\epsilon_1$. We demonstrated this fact on the systems with
Pöschl–Teller-type interactions (61), (72), or periodic interactions with a localized defect.
Finally, in section 5, we considered a situation where only the interlayer interaction was inhomogeneous. Here we fixed the trigonal warping term vanishing. It allowed us analytical treatment of decoupled equations with energy-dependent potential (86).

In all the scenarios, we faced the need to solve a Schrödinger equation with energy-dependent potentials. In section 3, the latter equation occurred due to the presence of the trigonal warping term, \( v_3 \neq 0 \). If such a term were absent, decoupling of (9) would produce a Schrödinger equation with energy-independent potential. In sections 4 and 5, \( \epsilon_1(x) \), \( \epsilon_2(x) \) or \( \gamma(x) \) were considered inhomogeneous functions to account for the presence of an electrostatic component in the potential term of the reduced equations (5) and (6). It is known [37] that decoupling of components in the stationary equation for \( 2 \times 2 \) Dirac Hamiltonian with electric potential leads to Schrödinger equation with energy-dependent potential. The problems related to the solution of such an equation are avoided when bound states with zero energy are of interest, see e.g. [38–41]. The zero modes in presence of inhomogeneous electric potential and an effective mass were discussed recently in [42]. In this context, the Hamiltonian (2) can be understood as two, coupled \( 2 \times 2 \) Dirac Hamiltonians with electrostatic potential accompanied by an effective mass term. For these systems, we found localized states with energies distinct from zero.

The solvability of the presented models relies on the specific form of the interactions. As much as it is rather impossible to prepare such fields in the experiments, we can see a broader application of the analytically solvable models. Besides representing a technically feasible setting that demonstrates a specific physical phenomenon (confinement of Dirac fermions in our case), they can be used to analyze physically more relevant interactions via perturbative approach where they can serve as the initial (unperturbed) system. They can also provide reference models for application of numerical methods.

In the light of the computational complexities mentioned, complete analysis of the spectral properties of the presented models would necessarily involve the use of numerical methods. Our aim in this article was different. We demonstrated confinement of Dirac fermions in bilayer graphene by different types of interactions in terms of analytical solutions. Our analytic approach is not restricted to the problems addressed in this paper and can be extended to a broader class of interactions. Indeed, in the presented models, we used exactly solvable models (e.g. harmonic oscillator, Rosen–Morse model) to find localized solutions of (16). However, to demonstrate the confinement, it would be sufficient to consider broader class of systems, where one bound state is known analytically at least. It would be also possible to analyze existence of the confined states via qualitative (e.g. asymptotic) properties of the interactions, following the same steps. Nevertheless, it goes beyond the scope of the present article and it should be addressed in the future.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] McCann E and Koshino M 2013 The electronic properties of bilayer graphene Rep. Prog. Phys. 76 056503
[2] Mariani E, Pearce A J and von Oppen F 2012 Fictitious gauge fields in bilayer graphene Phys. Rev. B 86 165448
[3] Verbeek B, Partoens B, Peeters F M and Trauzettel B 2012 Strain-induced band gaps in bilayer graphene Phys. Rev. B 85 125403
[4] Cariglia M, Giambò R and Perali A 2017 Curvature-tuned electronic properties of bilayer graphene in an effective four-dimensional spacetime Phys. Rev. B 95 245426
[5] Moulsdale C, Knothe A and Fal’ko V 2020 Engineering of the topological magnetic moment of electrons in bilayer graphene using strain and electrical bias Phys. Rev. B 101 085118
[6] Milton Pereira J, Vasilopoulos P and Peeters F M 2007 Tunable quantum dots in bilayer graphene Nano Lett. 7 946–9
[7] Abdullah H M, Bahlouli H, Peeters F M and Van Duppen B 2018 Confined states in graphene quantum blisters J. Phys.: Condens. Matter. 30 385301
[8] Solomon F and Power S R 2021 Valley current generation using biased bilayer graphene dots Phys. Rev. B 103 235435
[9] Ezawa M 2012 Quasi-topological insulator and trigonal warping in gated bilayer silicene J. Phys. Soc. Japan 81 104713
[10] Wu C-H 2018 Electronic transport and dynamical polarization in bilayer silicene-like systems Results Phys. 11 1182
[11] Ho L B and Tran L N 2019 Tunable cloaking of mexican-hat confined states in bilayer silicene Commun. Phys. 2 241
[12] Rataj M and Barnaś J 2011 Graphene p–n junctions with nonuniform Rashba spin–orbit coupling Appl. Phys. Lett. 99 162107
[13] Esmaeilzadeh M and Ahmadi S 2012 Spin-dependent electron transport in graphene junctions in the presence of Rashba spin–orbit interaction J. Appl. Phys. 112 104319
[14] McCann E and Fal’ko V I 2006 Landau-level degeneracy and quantum Hall effect in a graphite bilayer Phys. Rev. Lett. 96 086805
[15] Novoselov K S, McCann E, Morozov S V, Fal’ko V I, Katsnelson M I, Zeitler U, Jiang D, Schedin F and Geim A K 2006 Unconventional quantum Hall effect and Berry’s phase of 2π in bilayer graphene Nat. Phys. 2 177–80
[16] Katsnelson M I and Prokhorova M F 2008 Zero-energy states in corrugated bilayer graphene Phys. Rev. B 77 205424
[17] Martin I, Blanter Y M and Morpurgo A F 2008 Topological confinement in bilayer graphene Phys. Rev. Lett. 100 036804
[18] Fernández D J, García J D and O-Campa D 2020 Electron in bilayer graphene with magnetic fields leading to shape invariant potentials J. Phys. A: Math. Theor. 53 435202
[19] Fernández D J, García J D and O-Campa D 2021 Bilayer graphene in magnetic fields generated by supersymmetry J. Phys. A: Math. Theor. 54 245302
[20] Marston C C and Balint-Kurti G G 1989 The Fourier grid Hamiltonian method for bound state eigenvalues and eigenfunctions J. Chem. Phys. 91 3571
[21] Layton E and Chu S-I 1991 Generalized Fourier-grid Hamiltonian approach to the Dirac equation: variational solution without basis set Chem. Phys. Lett. 186 100
[22] Castillo-Celeita M and Jakubský V 2021 Reduction scheme for coupled Dirac systems J. Phys. A: Math. Theor. 54 455301
[23] Formáněk J, Lombard R J and Mareč J 2004 Wave equations with energy-dependent potentials Czech. J. Phys. 54 289–316
[24] García-Martínez J, García-Ravelo J, Peña JJ and Schulze-Halberg A 2009 Exactly solvable energy-dependent potentials Phys. Lett. A 373 3619
[25] Naumis G G, Barraza-Lopez S, Oliva-Leyva M and Terrones H 2017 Electronic and optical properties of strained graphene and other strained 2D materials: a review Rep. Prog. Phys. 80 096501
[26] Moldovan D and Peeters F M 2015 Strain engineering of the electronic properties of bilayer graphene quantum dots Phys. Status Solidi RRL 10 39
[27] Crosse J A 2014 Strain-dependent conductivity in biased bilayer graphene Phys. Rev. B 90 235403
[28] Nikiforov A F and Uvarov V B 1988 Special Functions of Mathematical Physics: A Unified Introduction with Applications (Basel: Birkhäuser)
[29] Olver F W J, Lozier D W, Boisvert R F and Clark C W (ed) 2010 NIST Handbook of Mathematical Functions (Cambridge: Cambridge University Press)
[30] Rosen N and Morse P M 1932 On the vibrations of polyatomic molecules Phys. Rev. 42 210
[31] Nieto M M 1978 Exact wave-function normalization constants for the $B_0 \tanh z - U_0 \cosh 2z$ and Pöschl–Teller potentials Phys. Rev. A 17 1273
[32] Barut A O, Inomata A and Wilson R 1987 Algebraic treatment of second Pöschl–Teller, Morse–Rosen and Eckart equations J. Phys. A: Math. Gen. 20 4083
[33] Garneau-Desroches S and Hussin V 2021 Ladder operators and coherent states for the Rosen–Morse system and its rational extensions J. Phys. A: Math. Theor. 54 475201
[34] Milpas E, Torres M and Murguía G 2011 Magnetic field barriers in graphene: an analytically solvable model J. Phys.: Condens. Matter. 23 245304
[35] Park C-S 2015 Two-dimensional transmission through modified Pöschl–Teller potential in bilayer graphene Phys. Rev. B 92 165422
[36] Hartmann R R, Robinson N J and Portnoi M E 2010 Smooth electron waveguides in graphene Phys. Rev. B 81 245431
[37] Ghosh P and Roy P 2016 An analysis of the zero energy states in graphene Phys. Lett. A 380 567
[38] Hartmann R R and Portnoi M E 2017 Two-dimensional Dirac particles in a Pöschl–Teller waveguide Sci. Rep. 7 111599
[39] Downing C A and Portnoi M E 2019 Zero-energy vortices in Dirac materials Phys. Status Solidi b 256 1800584
[40] Ho C-L and Roy P 2014 On zero energy states in graphene Europhys. Lett. 108 20004
[41] Schulze-Halberg A and Roy P 2017 Construction of zero-energy states in graphene through the supersymmetry formalism J. Phys. A: Math. Theor. 50 365205
[42] Schulze-Halberg A and Roy P 2021 Dirac systems with magnetic field and position-dependent mass: Darboux transformations and equivalence with generalized Dirac oscillators Ann. Phys., NY 431 168534