(In)stability of black holes in the 4D Einstein-Gauss-Bonnet and Einstein-Lovelock gravities

R. A. Konoplya\textsuperscript{1,2,*} and A. Zhidenko\textsuperscript{1,3,†}

\textsuperscript{1}Institute of Physics and Research Centre of Theoretical Physics and Astrophysics, Faculty of Philosophy and Science, Silesian University in Opava, Bezručovo nám. 13, CZ-746 01 Opava, Czech Republic
\textsuperscript{2}Peoples Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya Street, Moscow 117198, Russian Federation
\textsuperscript{3}Centro de Matemática, Computação e Cognição (CMCC), Universidade Federal do ABC (UFABC), Rua Abolição, CEP: 09210-180, Santo André, SP, Brazil

\textsuperscript{*}roman.konoplya@gmail.com
\textsuperscript{†}olexandr.zhydenko@ufabc.edu.br

A (3 + 1)-dimensional Einstein-Gauss-Bonnet theory of gravity has been recently formulated as the $D \rightarrow 4$ limit of the higher dimensional field equations after the rescaling of the coupling constant. This theory is a nontrivial generalization of General Relativity, it bypasses the Lovelock’s theorem, and avoids the Ostrogradsky instability. This approach has been extended to the four-dimensional Einstein-Lovelock gravity. Here we study the eikonal gravitational instability of asymptotically flat, de Sitter and anti-de Sitter black holes in the four dimensional Einstein-Gauss-Bonnet and Einstein-Lovelock theories. We find parametric regions of the eikonal instability for various orders of the Lovelock gravity, values of coupling and cosmological constants, and share the code which allows one to construct the instability region for an arbitrary set of parameters.

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I. INTRODUCTION

Stability of a black-hole metric against small (linear) perturbations of spacetime is the necessary condition for viability of the black-hole model under consideration. Therefore, a number of black-hole solutions in various alternative theories of gravity were tested for stability [1]. One of the most promising approaches to construction of alternative theories of gravity is related to the modification of the gravitational sector via adding higher curvature corrections to the Einstein action. This is well motivated by the low energy limit of string theory. Among higher curvature corrections, the Gauss-Bonnet term (quadratic in curvature) and its natural generalization to higher orders of curvature in the Lovelock form [2,3] play an important role.

The Lovelock theorem states that only metric tensor and the Einstein tensor are divergence free, symmetric, and concomitant of the metric tensor and its derivatives in four dimensions [2,3]. Therefore, it was concluded that the appropriate vacuum equations in $D = 4$ are the Einstein equations (with the cosmological term). In $D > 4$ the theory of gravity is generalized by adding higher curvature Lovelock terms [2,3] to the Einstein action.

Black hole in the $D > 4$ Einstein-Gauss-Bonnet gravity and its Lovelock generalization were extensively studied and various peculiar properties were observed. For example, the life-time of the black hole whose geometry is only slightly corrected by the Gauss-Bonnet term is characterized by a much longer lifetime and a few orders smaller evaporation rate [5]. The eikonal quasinormal modes in the gravitational channel break down the correspondence between the eikonal quasinormal modes and null geodesics [6,7]. However, apparently the most interesting feature of higher curvature corrected black hole is the gravitational instability: When the coupling constants are not small enough, the black holes are unstable and the instability develops at high multipoles numbers [8,10]. Therefore, it was called the eikonal instability [10,11].

Recently, there was found the way to bypass the Lovelock’s theorem [17], by performing a kind of dimensional regularization of the Gauss-Bonnet equations and obtain a four-dimensional metric theory of gravity with diffeomorphism invariance and second order equations of motion.

The approach was first formulated in $D > 4$ dimensions and then, the four-dimensional theory is defined as the limit $D \rightarrow 4$ of the higher-dimensional theory after the rescaling of the coupling constant $\alpha \rightarrow \alpha/(D - 4)$. The properties of black holes in this theory, such as (in)stability, quasinormal modes and shadows, were considered in [18], while the innermost circular orbits were analyzed in [19]. The generalization to the charged black holes and an asymptotically anti-de Sitter and de Sitter cases in the 4D Einstein-Gauss-Bonnet theory was considered in [20] and to the higher curvature corrections, that is, the 4D Einstein-Lovelock theory, in [21,22]. Some further properties of black holes for this novel theory, such as axial symmetry and thermodynamics, were considered in [23,24].

Here we will study the linear stability of asymptotically flat, de Sitter and anti-de Sitter black holes in the 4D Einstein-Lovelock gravity. We will plot the paramet-
ric regions of the eikonal instability for the 4D Einstein-Gauss-Bonnet-(anti-)de Sitter black holes and various examples for its Lovelock extension. We will show that the negative values of the coupling constants allows for a larger parametric region of stability than the positive coupling constants. Some inequalities determining the instability region are derived.

Our paper is organized as follows. In Sec. II we briefly describe the static black hole solution in the 4D Einstein-Lovelock theory. Sec. III discusses the gravitational perturbations and the eikonal instability. Finally, in Conclusions, we summarize the obtained results.

II. STATIC BLACK HOLES IN THE FOUR-DIMENSIONAL LOVELOCK THEORY

The Lagrangian density of the Einstein-Lovelock theory has the form [2]:

\[ \mathcal{L} = -2\Lambda + \sum_{m=1}^{\overline{m}} \frac{1}{2m} \alpha_m \delta_{\mu_1\nu_1...\mu_m\nu_m} \times R_{\mu_1\nu_1}^{\lambda_1\nu_1} R_{\mu_2\nu_2}^{\lambda_2\nu_2} \cdots \times R_{\mu_m\nu_m}^{\lambda_m\nu_m}, \]

(1)

where \( \delta_{\mu_1\nu_1...\mu_p\nu_p} \) is the generalized totally antisymmetric Kronecker delta, \( R_{\mu\nu\lambda\sigma} \) is the Riemann tensor, \( \alpha_1 = 1/8\pi G = 1 \) and \( \alpha_2, \alpha_3, \alpha_4, \ldots \) are arbitrary constants of the theory.

The Euler-Lagrange equations, corresponding to the Lagrangian density [1] read [28]:

\[ \lambda \delta_{\mu}^{\nu} = R_{\nu}^{\rho} R_{\rho}^{\mu} - \frac{R}{2} \delta_{\mu}^{\nu} + \sum_{m=2}^{\overline{m}} \frac{1}{2m+1} \alpha_m \delta_{\mu_1\nu_1...\mu_m\nu_m} \times R_{\mu_1\nu_1}^{\lambda_1\nu_1} R_{\mu_2\nu_2}^{\lambda_2\nu_2} \cdots \times R_{\mu_m\nu_m}^{\lambda_m\nu_m}. \]

(2)

The antisymmetric tensor is nonzero only when the indices \( \mu, \mu_1, \nu_1, \mu_2, \nu_2, \ldots \) are all distinct. Thus, the general Lovelock theory is such that \( 2\overline{m} < D \). In particular, for \( D = 4 \), we have \( \overline{m} = 1 \) corresponding to the Einstein theory [3]. When \( D = 5 \) or \( 6 \), \( \overline{m} = 2 \) and one has the (quadratic in curvature) Einstein-Gauss-Bonnet theory with the coupling constant \( \alpha_2 \).

Following [10], we introduce

\[ \tilde{\alpha}_m = \frac{\alpha_m}{m} \frac{(D-3)!}{(D-2m-1)!} = \frac{\alpha_m}{m} \prod_{p=1}^{2m-2} (D-2-p) \]

(3)

and consider the limit \( D \to 4 \) while \( \tilde{\alpha}_m \) remain constant. In this way, we obtain the regularized 4D Einstein-Lovelock theory formulated in [22], which generalizes the approach of [17] used for the Einstein-Gauss-Bonnet theory. In the Einstein-Gauss-Bonnet case (\( \overline{m} = 2 \)) the above equation reads

\[ \alpha_2 = \frac{2\tilde{\alpha}_2}{(D-3)(D-4)}. \]

(4)

Then, taking the limit \( D \to 4 \) we see that

\[ \alpha_2 \to \frac{2\tilde{\alpha}_2}{D-4}. \]

(5)

Notice that our units differ by a factor of 2 from those used in [17] and coincide with the units of [24]. Prior to [17] the dimensional regularization of the Einstein-Gauss-Bonnet theory was suggested by Y. Tomozawa [24].

Although the Lagrangian [1] diverges in the limit \( D \to 4 \), no singular terms appear in the Einstein-Lovelock equations for any \( D \geq 3 \). In particular, following [22] one can find the four-dimensional static and spherically symmetric metric, described by the metric

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + d\sin^2 \theta d\phi^2). \]

(6)

The metric function \( f(r) \) is defined through a new variable \( \psi(r) \),

\[ f(r) = 1 - r^2 \psi(r), \]

(7)

which satisfies the algebraic equation

\[ P[\psi(r)] \equiv \psi(r) + \sum_{m=2}^{\overline{m}} \tilde{\alpha}_m \psi(r)^m = \frac{2M}{r^3} + \frac{\Lambda}{3}, \]

(8)

where \( M \) is the asymptotic mass [31].

The Gauss-Bonnet theory (\( \overline{m} = 2 \)) leads to the two branches [20]:

\[ f(r) = 1 - \frac{r^2}{2\tilde{\alpha}_2} \left( -1 + \sqrt{1 + 4\tilde{\alpha}_2 \left( \frac{2M}{r^3} + \frac{\Lambda}{3} \right) \right), \]

one of which, corresponding to the “+” sign, is perturbative in \( \tilde{\alpha}_2 \), while for the “-“ the metric function \( f(r) \) goes to infinity when \( \tilde{\alpha}_2 \to 0 \). Notice that the above solution was also obtained in [27, 28] in a different context, when discussing quantum correction to entropy.

The higher-order Lovelock corrections result in more branches, only one of which is perturbative in \( \tilde{\alpha}_m \). Following [10], we consider here only the perturbative branch, so that we recover the Einstein theory in the limit \( \tilde{\alpha}_m \to 0 \). In particular, for \( \overline{m} = 3 \) and \( \tilde{\alpha}_3 \geq \tilde{\alpha}_2^2/3 \),

\[ f(r) = 1 - \frac{\tilde{\alpha}_2^2}{3\tilde{\alpha}_3} (A_+(r) - A_-(r) - 1), \]

(9)

where

\[ A_{\pm}(r) = 3 \sqrt{F(r)^2 + \left( \frac{3\tilde{\alpha}_3}{\tilde{\alpha}_2^2} - 1 \right)^2} \pm F(r), \]

\[ F(r) = \frac{27\tilde{\alpha}_2^2}{2\tilde{\alpha}_3} \left( \frac{2M}{r^3} + \frac{\Lambda}{3} \right) + \frac{9\tilde{\alpha}_3}{2\tilde{\alpha}_2^2} - 1. \]

It is convenient to measure all dimensional quantities in units of the horizon radius \( r_H \). For the asymptotic mass we obtain

\[ 2M = r_H \left( 1 + \sum_{m=2}^{\overline{m}} \frac{\tilde{\alpha}_m}{2m^2 \tilde{\alpha}_m} \right) - \frac{\Lambda r_H^2}{3}. \]

(10)
For $\Lambda > 0$ the perturbative branch is asymptotically de Sitter, so that $\Lambda$ can be expressed in terms of the de Sitter horizon $r_C$ as follows

$$
\frac{\Lambda}{3} = \frac{1}{r_C^2} + r_Cr_H + \frac{r_H^2}{r} + \sum_{m=2}^{\infty} \frac{(-1)^m \tilde{\alpha}_m}{r^{2m}}.
$$

When $\Lambda < 0$, we introduce the AdS radius $R$, by assuming that the metric function has the following asymptotic $f(r) \to r^2/R^2$ as $r \to \infty$, and the cosmological constant is given by

$$
\frac{\Lambda}{3} = -\frac{1}{R^2} + \sum_{m=2}^{\infty} \frac{(-1)^m \tilde{\alpha}_m}{R^{2m}}.
$$

The metric function $f(r)$ for the perturbative branch of the general Einstein-Lovelock black hole can be obtained numerically.

III. GRAVITATIONAL PERTURBATIONS AND THE EIKONAL (IN)STABILITY

In it was shown that after the decoupling of angular variables and some algebra, the gravitational perturbation equations of the higher dimensional Einstein-Gauss-Bonnet theory can be reduced to the second-order master differential equations

$$
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + V_i(r_*) \right) \Psi_i(t, r_*) = 0,
$$

where $\Psi_i$ are the wave functions, $r_*$ is the tortoise coordinate,

$$
dr_* = \frac{dr}{f(r)} = \frac{dr}{1 - r^2 \psi(r)},
$$

and $i$ stands for $t$ (tensor), $v$ (vector), and $s$ (scalar) types of gravitational perturbations, according to their transformations respectively the rotation group on a $(D-2)$-sphere.

Using the definition of constants $\tilde{\alpha}_m$ and taking the limit $D \to 4$ in the perturbation equations, one can see that the tensor-type perturbations are pure gauge, so that only the vector-type (axial) and scalar-type (polar) perturbations possess physical degrees of freedom.

The effective potentials $V_t(r), V_v(r)$ are given by the following expressions:

$$
V_t(r) = \frac{\ell(\ell + 2)}{rT(r)} \frac{d}{dr_*} T(r) + \frac{1}{r^2} \frac{d^2}{dr_*^2} \left( \frac{1}{R(r)} \right),
$$

$$
V_v(r) = \frac{\ell(\ell + 1)}{rP(r)} \frac{d}{dr_*} P(r) + \frac{P(r)}{r} \frac{d^2}{dr_*^2} \left( \frac{r}{P(r)} \right),
$$

where $\ell = 2, 3, 4, \ldots$ is the multipole number,

$$
T(r) \equiv rP'[\psi(r)] = r \left( 1 + \sum_{m=2}^{\infty} m\tilde{\alpha}_m \psi(r)^m \right),
$$

and

$$
R(r) = r \sqrt{|T'(r)|},
$$

$$
P(r) = \frac{2(\ell - 1)(\ell + 2) - 2r^2 \psi'(r)}{\sqrt{|T'(r)|}} T(r).
$$

Notice that, since we consider the perturbative branch of solutions, then we have $T(r) > 0$ for $r > r_H$.

Although we can formally obtain expressions for the effective potentials when $T'(r) \leq 0$, the kinetic term of perturbations in such points has a wrong (negative) sign. The perturbations are linearly unstable, and this phenomenon was called the ghost instability.

Usually, we used to believe that if a gravitational instability takes place, it happens at the lowest $\ell = 2$ multipole, while higher multipoles increase the centrifugal part.
of the effective potential and make the potential barrier higher, so that, usually, higher $\ell$ are more stable. The eikonal instability we observe here is qualitatively different: higher $\ell$ leads not only to the higher height of the barrier, but also increases the depth of the negative gap near the event horizon. Then, at some sufficiently large $\ell$ the negative gap becomes so deep, that the bound state with negative energy becomes possible, which signifies the onset of instability.

For large $\ell$ the effective potential for the vector-type perturbations

$$V_v = \ell^2 \left( \frac{f(r)T'(r)}{T(r)} + O\left(\frac{1}{\ell}\right) \right)$$

becomes positive-definite in the parametric regime, which is free from the ghost instability.

Therefore, the eikonal instability exists only in the scalar channel. For large $\ell$ the effective potential reads

$$V_s = \ell^2 \left( \frac{r f(r)(2T'(r)^2 - T(r)T''(r))}{2T'(r)T(r)} + O\left(\frac{1}{\ell}\right) \right),$$

giving the following instability condition

$$2T'(r)^2 - T(r)T''(r) < 0.$$  \hspace{1cm} (19)

This condition can be test for each black-hole configuration. In the Einstein-Gauss-Bonnet theory ($m = 2$) it sufficient to test if (19) at $r = r_H$. Therefore, the problem is reduced to the polynomial inequality of fourth order in $\tilde{\alpha}_2$, which can be solved analytically. We find that the eikonal instability occurs if

$$\tilde{\alpha}_2 > \frac{\sqrt{6\sqrt{3} - 10 + \lambda^2} - \lambda}{2},$$

where $\lambda = (2\sqrt{3} - 3)\Lambda r_H^2 - 1$.

In the asymptotically flat case ($\Lambda = 0$, $\lambda = 1$), the black hole has the eikonal instability in the scalar sector for

$$\tilde{\alpha}_2 > \frac{\sqrt{6\sqrt{3} - 9} - 1}{2} \approx 0.09.$$

It is possible to show that the ghost instability takes place for larger values of $\tilde{\alpha}_2$. 
By substituting (11) and (12) into (20) one can find the instability condition in the geometrized units. We show the parametric region of the eikonal instability on Fig. 1. Indeed, we see that the asymptotically flat and (anti-)de Sitter black holes in the Einstein-Gauss-Bonnet theory are stable for the whole range of valid parameters when \( \alpha \) is negative and for sufficiently small values of positive \( \alpha \).

For the higher-order Einstein-Lovelock theory it is not sufficient to test (19) in the point \( r = r_H \). In order to obtain the region of instability we have used the Wolfram Mathematica® code developed in [10], which is attached to the arXiv preprint as an ancillary material.

From Figs. 2 we see that in the Lovelock theory of the third order (\( \overline{m} = 3 \)) the positive cosmological constant modifies the (in)stability region relatively softly. As for the four-dimensional Einstein-Gauss-Bonnet black holes, we see the \( \Lambda \)-term slightly increases the region of stability. For the asymptotically AdS black holes (Figs. 3) we see that the region of stability shrinks as we decrease their size in units of the AdS radius \( R \). It is interesting to note that for the four-dimensional black holes the ghost instability occurs for \( \overline{\alpha}_3 < \overline{\alpha}_2^2/3 \) and scalar-type eikonal instability exists for \( \overline{\alpha}_3 < 0 \).

### IV. CONCLUSIONS

Here we analyzed the (in)stability of the asymptotically flat, de Sitter and anti-de Sitter black holes in the 4D Einstein-Lovelock gravity. We showed that for all types of asymptotics the black holes are unstable unless the coupling constants are sufficiently small. Negative coupling constants allow for a much larger parametric region of stability. For the general case of the Einstein-Lovelock theory of arbitrary order and a number of particular examples the regions of instability are found as algebraic inequalities. In a similar way, our work could be extended to the case of a charged black hole.

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