Thermodynamics in $F(R)$ gravity with phantom crossing

Kazuharu Bamba* and Chao-Qiang Geng†

Department of Physics, National Tsing Hua University, Hsinchu, Taiwan 300

Abstract

We study thermodynamics of the apparent horizon in $F(R)$ gravity. In particular, we demonstrate that a $F(R)$ gravity model with realizing a crossing of the phantom divide can satisfy the second law of thermodynamics in the effective phantom phase as well as non-phantom one.

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* E-mail address: bamba“at”phys.nthu.edu.tw
† E-mail address: geng“at”phys.nthu.edu.tw
I. INTRODUCTION

Recently, there have been more and more evidences to support that the current expansion of the universe is accelerating [1, 2]. The scenarios to explain the current accelerated expansion of the universe fall into two broad categories [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. One is to introduce “dark energy” in the framework of general relativity. The other is to study a modified gravitational theory, such as $F(R)$ gravity, in which the action is described by an arbitrary function $F(R)$ of the scalar curvature $R$ (for reviews, see [8, 9, 10, 11, 12]).

On the other hand, various observational data [13] imply that the ratio of the effective pressure to the effective energy of the universe, i.e., the effective equation of state (EoS) $w_{\text{eff}} \equiv p_{\text{eff}}/\rho_{\text{eff}}$, may evolve from larger than $-1$ (non-phantom phase) to less than $-1$ (phantom phase [14]). Namely, it crosses $-1$ (the phantom divide) at the present time or in near future. Recently, an explicit model of $F(R)$ gravity with realizing a crossing of the phantom divide has been constructed in Ref. [15]. We note that the phantom crossing in the framework of general relativity has also been studied in the literature, e.g., “quintom” model [16].

It is believed that a modified gravitational theory must pass cosmological bounds and solar system tests because it corresponds to an alternative theory of gravitation to general relativity. However, at the initial studies of $F(R)$ gravity, models proposed in Refs. [17, 18, 19, 20] with the powers of the scalar curvature are strongly constrained. In recent years, various investigations for viable models of $F(R)$ gravity [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] have been executed. A curvature singularity problem in $F(R)$ gravity has also been discussed in Refs. [34, 35].

As another touchstone of modified gravity, it is interesting to examine whether the second law of thermodynamics can be satisfied in the models of $F(R)$ gravity. The connection between gravitation and thermodynamics was examined by following black hole thermodynamics (black hole entropy [36] and temperature [37]) and its application to the cosmological event horizon of de Sitter space [38]. It was shown that Einstein equation is derived from the proportionality of the entropy to the horizon area together with the fundamental thermodynamic relation, such as the Clausius relation [39]. This idea has been employed in a cosmological context [40, 41, 42, 43]. It was demonstrated that if the entropy of the apparent horizon in the Friedmann-Robertson-Walker (FRW) spacetime is proportional to the appar-
ent horizon area, Friedmann equations follow from the first law of thermodynamics \[43\]. The equivalent considerations for the FRW universe with the viscous fluid have also been studied \[44\].

In addition, it was proposed \[45\] that in $F(R)$ gravity, a non-equilibrium thermodynamic treatment should be required in order to derive the corresponding gravitational field equation by using the procedure in Ref. \[39\]. It was reconfirmed in Ref. \[46\] in $F(R)$ gravity as well as Ref. \[47\] in scalar-tensor theories. The first \[48\] and second \[49\] laws of thermodynamics on the apparent horizon in generalized theories of gravitation have recently been analyzed by taking into account the non-equilibrium thermodynamic treatment. Reinterpretations of the non-equilibrium correction \[45\] through the introduction of a mass-like function \[50\] and other approaches \[51, 52\] have also been explored. Incidentally, the horizon entropy in four-dimensional modified gravity \[53\] and a quantum logarithmic correction to the expression of the horizon entropy in a cosmological context \[54, 55, 56\] have been examined. Moreover, the first law of the ordinary equilibrium thermodynamics in $F(R)$ gravity, scalar-tensor theories, the Gauss-Bonnet gravity and more general Lovelock gravity have been discussed in Refs. \[57, 58, 59, 60\], while the corresponding studies on the second law in the accelerating universe, $F(R)$ gravity, the Gauss-Bonnet gravity and the Lovelock gravity have been done in Refs. \[61, 62, 63\], respectively. Studies of thermodynamics in braneworld scenario \[64, 65, 66\] as well as its properties of dark energy \[67\] have also been performed.

It was pointed out in Ref. \[68\] that thermodynamics in the phantom phase usually leads to a negative entropy. Moreover, it was noted \[69\] that in the framework of general relativity the horizon entropy decreases in phantom models. However, the conditions that the black hole entropy can be positive in the $F(R)$ gravity models \[21, 22, 23\] with the solar-system tests have been analyzed in Ref. \[70\]. These recent studies have motivated us to explore whether in the framework of $F(R)$ gravity the second law of thermodynamics can be satisfied in the phantom phase. To illustrate the point, in the present paper we consider a $F(R)$ gravity model with a crossing of the phantom divide \[15\] as it contains an effective phantom phase. We use units of $k_B = c = h = 1$ and denote the gravitational constant $8\pi G$ by $\kappa^2 \equiv 8\pi / M_{\text{Pl}}^2$ with the Planck mass of $M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19}\text{GeV}$.

The paper is organized as follows. In Sec. II, we explain the first and second laws of thermodynamics in $F(R)$ gravity. In Sec. III, we demonstrate that the model of $F(R)$ gravity with the phantom crossing \[15\] can satisfy the generalized second law of the thermodynamics.
Finally, conclusions are given in Sec. IV.

II. THERMODYNAMICS IN $F(R)$ GRAVITY

In this section, we study the first and second laws of thermodynamics of the apparent horizon in $F(R)$ gravity. We consider the four-dimensional flat spacetime.

A. $F(R)$ gravity

The action of $F(R)$ gravity with matter is written as

$$I = \int d^4x \sqrt{-g} \left[ \frac{F(R)}{2\kappa^2} + \mathcal{L}_{\text{matter}} \right], \quad (2.1)$$

where $g$ is the determinant of the metric tensor $g_{\mu\nu}$ and $\mathcal{L}_{\text{matter}}$ is the matter Lagrangian. From the action in Eq. (2.1), the field equation of modified gravity is given by

$$F'(R)R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} F(R) + g_{\mu\nu} \Box F'(R) - \nabla_{\mu} \nabla_{\nu} F'(R) = \kappa^2 T_{\mu\nu}^{(\text{matter})}, \quad (2.2)$$

where the prime denotes differentiation with respect to $R$, $\nabla_{\mu}$ is the covariant derivative operator associated with $g_{\mu\nu}$, $\Box \equiv g^{\mu\nu} \nabla_{\mu} \nabla_{\nu}$ is the covariant d’Alembertian for a scalar field, $R_{\mu\nu}$ is the Ricci curvature tensor, and $T_{\mu\nu}^{(\text{matter})} = \text{diag}(\rho, p, p, p)$ is the contribution to the energy-momentum tensor from all ordinary matters with $\rho$ and $p$ being the energy density and pressure of all ordinary matters, respectively.

We assume the flat Friedmann-Robertson-Walker (FRW) space-time with the metric,

$$ds^2 = -dt^2 + a^2(t) \gamma_{ij}dx^i dx^j, \quad (2.3)$$

$$\gamma_{ij}dx^i dx^j = dr^2 + r^2 d\Omega^2, \quad (2.4)$$

where $a(t)$ is the scale factor and $d\Omega^2$ is the metric of two-dimensional sphere with unit radius. In the FRW background, from $(\mu, \nu) = (0, 0)$ and the trace part of $(\mu, \nu) = (i, j)$ ($i, j = 1, \cdots, 3$) components in Eq. (2.2), we obtain the gravitational field equations:

$$H^2 = \frac{\kappa^2}{3F'(R)} (\rho + \rho_c), \quad (2.5)$$

$$\dot{H} = -\frac{\kappa^2}{2F'(R)} (\rho + p + \rho_c + p_c), \quad (2.6)$$
where $\rho_c$ and $p_c$ can be regarded as the energy density and pressure generated due to the difference of $F(R)$ gravity from general relativity, given by

$$\rho_c = \frac{1}{\kappa^2} \left[ \frac{1}{2} (-F(R) + RF'(R)) - 3H \dot{R}F''(R) \right], \quad (2.7)$$

$$p_c = \frac{1}{\kappa^2} \left[ \frac{1}{2} (F(R) - RF'(R)) + \left(2H \dot{R} + \ddot{R} \right) F''(R) + \dot{R}^2 F'''(R) \right], \quad (2.8)$$

respectively, with the scalar curvature of $R = 6 \left( \dot{H} + 2H^2 \right)$. Here, $H = \dot{a}/a$ is the Hubble parameter and the dot denotes the time derivative of $\partial/\partial t$. We define the effective energy density and pressure of the universe as $\rho_{\text{eff}} = \rho_t/F'(R)$ and $p_{\text{eff}} = p_t/F'(R)$ with $\rho_t = \rho + \rho_c$ and $p_t = p + p_c$, respectively. Hence, from Eqs. (2.5) and (2.6) we see that even in $F(R)$ gravity, the gravitational field equations are expressed as $H^2 = \kappa^2 \rho_{\text{eff}}/3$ and $\dot{H} = -\kappa^2 (\rho_{\text{eff}} + p_{\text{eff}})/2$, which are the same as those in general relativity.

The continuity equation in terms of the effective energy density and pressure of the universe is given by

$$\dot{\rho}_{\text{eff}} + 3H (\rho_{\text{eff}} + p_{\text{eff}}) = 0. \quad (2.9)$$

Similarly, the (semi-)continuity equation of ordinary matters has the form

$$\dot{\rho} + 3H (\rho + p) = q. \quad (2.10)$$

One can take $q = 0$ because the gravity is determined only by ordinary matters. Assuming that the energy fluid, generated from the modification of gravity, behaves as a perfect fluid, we have similar semi-continuity equations as

$$\dot{\rho}_c + 3H (\rho_c + p_c) = q_c, \quad (2.11)$$

$$\dot{\rho}_t + 3H (\rho_t + p_t) = q_t, \quad (2.12)$$

where $q_c$ and $q_t (= q + q_c)$ are quantities of expressing energy exchange. Using Eqs. (2.5), (2.6) and (2.12), we obtain

$$q_t = \frac{3}{\kappa^2} H^2 \frac{\partial F'(R)}{\partial t}. \quad (2.13)$$

Clearly, from Eq. (2.13), we find that $q_t = 0$ for general relativity with $F(R) = R$, whereas $q_t$ does not generally vanish in $F(R)$ gravity since there could exist some energy exchange with the horizon.
B. First law of thermodynamics

We now illustrate the first law of thermodynamics in $F(R)$ gravity. By using the spherical symmetry, the metric (2.3) can be written as

$$\begin{align*}
\text{ds}^2 &= h_{\alpha\beta}dx^\alpha dx^\beta + \tilde{r}^2d\Omega^2,
\end{align*}$$

where $\tilde{r} = a(t)r$, $x^0 = t$ and $x^1 = r$, and $h_{\alpha\beta}$ is the two-dimensional metric $h_{\alpha\beta} = \text{diag}(-1, a^2)$. The dynamical apparent horizon is determined by the relation $h^{\alpha\beta}\partial_\alpha \tilde{r}\partial_\beta \tilde{r} = 0$. The radius of the apparent horizon for the FRW spacetime is given by [43, 49]

$$\tilde{r}_A = \frac{1}{H}. \quad (2.15)$$

The associated temperature $T$ of the apparent horizon is determined through the surface gravity of

$$\kappa_{sg} = \frac{1}{2\sqrt{-h}}\partial_\alpha \left(\sqrt{-h}h^{\alpha\beta}\partial_\beta \tilde{r}\right), \quad (2.16)$$

where $h$ is the determinant of the metric $h_{\alpha\beta}$. We note that the recent type Ia Supernovae data suggests that in the accelerating universe the enveloping surface should be the apparent horizon rather than the event one from the thermodynamic point of view [61].

In the FRW spacetime, one has [43]

$$T = \frac{|\kappa_{sg}|}{2\pi} = \frac{1}{2\pi \tilde{r}_A} \left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right). \quad (2.17)$$

In general relativity, the entropy is expressed as

$$S_{GR} = \frac{A}{4G}, \quad (2.18)$$

where $A = 4\pi \tilde{r}_A^2$ is the horizon area. It follows from Eqs. (2.17) and (2.18) that

$$TdS_{GR} = \frac{1}{G} \left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right) d\tilde{r}_A$$

$$= -\frac{3V}{\kappa^2} \frac{dH^2}{dt} dt - \frac{3V}{\kappa^2} \frac{\dot{H}^2}{H} dt, \quad (2.19)$$

where $V = 4\pi \tilde{r}_A^3/3$ is the volume of the apparent horizon. From Eq. (2.5), we see that $H^2 = H^2(\rho, \rho_c, F'(R))$ and hence $dH^2/dt = (\partial H^2/\partial \rho) \dot{\rho} + (\partial H^2/\partial \rho_c) \dot{\rho}_c + (\partial H^2/\partial F'(R)) \dot{F}'(R)$. 
Substituting this equation into Eq. (2.19) and multiplying the resultant equation by the factor \((\kappa^2/3) (\partial H^2/\partial \rho)^{-1}\), we get
\[
\frac{\kappa^2}{3} \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} TdS_{\text{GR}} = -V \dot{\rho} dt - V \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} \frac{H^2}{H} dt - V \left( \frac{\partial H^2}{\partial \rho_c} \right)^{-1} \dot{\rho}_c dt
\]
\[
- V \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} \left( \frac{\partial H^2}{\partial F'(R)} \right) \dot{F}'(R) dt .
\]
(2.20)
The left-hand side of Eq. (2.20) can be rewritten as
\[
\frac{\kappa^2}{3} \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} TdS_{\text{GR}} = T d \left[ \kappa^2 \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} S_{\text{GR}} \right] - T \frac{\kappa^2}{3} S_{\text{GR}} d \left[ \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} \right] ,
\]
(2.21)
which leads to the Clausius relation
\[
TdS = \delta Q ,
\]
(2.22)
where the entropy \(S\) and the energy flux \(\delta Q\) are defined by
\[
S \equiv \frac{\kappa^2}{3} \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} S_{\text{GR}} ,
\]
(2.23)
\[
\delta Q \equiv -V \dot{\rho} dt - V \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} \frac{H^2}{H} dt - V \left( \frac{\partial H^2}{\partial \rho_c} \right)^{-1} \dot{\rho}_c dt
\]
\[
+ T \frac{\kappa^2}{3} S_{\text{GR}} d \left[ \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} \right] - V \left( \frac{\partial H^2}{\partial \rho} \right)^{-1} \left( \frac{\partial H^2}{\partial F'(R)} \right) \dot{F}'(R) dt .
\]
(2.24)
From Eqs. (2.23), (2.18) and (2.23), the entropy in \(F(R)\) gravity is expressed as \[68, 71, 72\]:
\[
S = \frac{AF'(R)}{4G} .
\]
(2.25)
We note that there is arbitrariness in the definitions of \(S\) and \(\delta Q\). The reason for using those in Eqs. (2.23) and (2.24) is that we can obtain the simple form of Eq. (2.25) as the expression of the entropy in \(F(R)\) gravity. Using \(\rho_{\text{eff}} = 3H^2/\kappa^2\) and Eq. (2.23), we find
\[
\dot{\rho}_{\text{eff}} = \frac{3}{\kappa^2} \left[ \left( \frac{\partial H^2}{\partial \rho} \right) \dot{\rho} + \left( \frac{\partial H^2}{\partial \rho_c} \right) \dot{\rho}_c + \left( \frac{\partial H^2}{\partial F'(R)} \right) \dot{F}'(R) \right]
\]
\[
= \frac{\dot{\rho}}{F'(R)} + \frac{3}{\kappa^2} \left[ \left( \frac{\partial H^2}{\partial \rho_c} \right) \dot{\rho}_c + \left( \frac{\partial H^2}{\partial F'(R)} \right) \dot{F}'(R) \right] ,
\]
(2.26)
\[
\rho_{\text{eff}} + p_{\text{eff}} = \frac{\rho_t + p_t}{F'(R)} ,
\]
(2.27)
where we have used \((\partial H^2/\partial \rho) = \kappa^2/(3F'(R))\). By combining Eqs. (2.9), (2.10) and (2.26), we obtain
\[
\left( \frac{\partial H^2}{\partial \rho_c} \right) \dot{\rho}_c + \left( \frac{\partial H^2}{\partial F'(R)} \right) \dot{F}'(R) = -\frac{\kappa^2}{3F'(R)} [3H (\rho_t + p_t - \rho - p) + q] .
\]
(2.28)
Consequently, from Eqs. (2.6), (2.10), (2.24) and (2.28), we get

$$\delta Q = 3H(\rho + p_t)Vdt - \frac{1}{2}(\rho + p_t)\dot{V}dt + TS_{\text{GR}}d(F'(R)) \quad (2.29)$$

$$= -dE_t + W_tdV + q_tVdt + TS_{\text{GR}}d(F'(R)) \quad (2.30)$$

where $E_t = \rho_tV$ is the total intrinsic energy and $W_t \equiv -(1/2)T^{(t)\alpha\beta}h_{\alpha\beta} = (\rho_t - p_t)/2$ is the total work density [73]. Here, $T^{(t)}_{\mu\nu} = \text{diag}(\rho_t, p_t, p_t, p_t)$ is the contribution to the energy-momentum tensor from all ordinary matters and energy fluid. This may be regarded as the work generated through the evolution of the apparent horizon [73]. It follows from Eqs. (2.17), (2.18) and (2.30) that the first law of thermodynamics in modified gravity can be constructed as

$$\delta Q = -dE_t + W_tdV - T_{d}\text{c}S, \quad (2.31)$$

where

$$d_{c}S = -\frac{1}{T}q_tVdt - S_{\text{GR}}d(F'(R)) \quad (2.32)$$

$$= -\frac{8\pi^2}{\kappa^2}\frac{4H^2 + \dot{H}}{2H^2 + \dot{H}}d(F'(R)) \quad (2.33)$$

From Eq. (2.33), it is clear that $d_{c}S$ does not vanish if $F(R)$ is not equal to $R$. Hence, the emergence of the entropy production term is a special feature of thermodynamics in $F(R)$ gravity. We note that such feature also appears in scalar-tensor theories because in those theories $F'(R)$ is not constant but some dynamical quantities in terms of scalar fields. We remark that there exists a possible singularity in Eq. (2.33) if $2H^2 + \dot{H} = 0$, which corresponds to $T = 0$ since $T = \left(2H^2 + \dot{H}\right)/(4\pi H)$ due to Eqs. (2.15) and (2.17). It is clear that the necessary condition for a positive temperature is $2H^2 + \dot{H} > 0$. In this case, no singularity appears in Eq. (2.33).

Using the Clausius relation in Eq. (2.22), the first law of thermodynamics in $F(R)$ gravity in Eq. (2.31) can be rewritten as

$$TdS + T_{d}\text{c}S = -dE_t + W_tdV \quad (2.34)$$

which characterizes the non-equilibrium thermodynamics of the apparent horizon in $F(R)$ gravity.
C. Second law of thermodynamics

Next, we investigate the second law of thermodynamics in $F(R)$ gravity. From Eq. (2.34), the first law of thermodynamics in terms of the horizon entropy $S_h$ is expressed as

$$T dS_h = -dE_t + W_t dV - T d_c S.$$  \hspace{1cm} (2.35)

The Gibbs equation in terms of all matter and energy fluid is given by

$$T_t dS_t = d (\rho_t V) + p_t dV = V d\rho_t + (\rho_t + p_t) dV,$$  \hspace{1cm} (2.36)

where $T_t$ and $S_t$ denote the temperature and entropy of total energy inside the horizon, respectively. We assume that

$$T_t = b T,$$  \hspace{1cm} (2.37)

where $b$ is a constant with $0 < b < 1$. If there is no energy exchange between the outside and inside of the apparent horizon, i.e., $q_t = 0$, thermal equilibrium realizes and therefore $b = 1$.

The second law of thermodynamics in $F(R)$ gravity can be described by

$$\dot{S}_h + \frac{\partial}{\partial t} d_c S + \dot{S}_t \geq 0$$  \hspace{1cm} (2.38)

or

$$(1 - b) \dot{\rho}_t V + \left(1 - \frac{b}{2}\right) (\rho_t + p_t) \dot{V} \geq 0$$  \hspace{1cm} (2.39)

by using Eqs. (2.35)–(2.37). With Eqs. (2.35) and (2.36), the relation in Eq. (2.39) is reduced to

$$\frac{4\pi}{\kappa^2 H^4} J \geq 0,$$  \hspace{1cm} (2.40)

where

$$J = (1 - b) H^3 \dot{R} F''(R) + 2 (1 - b) H^2 \dot{F}'(R) + (2 - b) \dot{H}^2 F'(R).$$  \hspace{1cm} (2.41)

Thus, the condition to satisfy the second law of thermodynamics in $F(R)$ gravity is equivalent to $J \geq 0$.

We note that for $F(R) = R$, in which $q_t = 0$, $b = 1$ and $d_c S = 0$ with the realization of thermal equilibrium, we find $J = \dot{H}^2$ from Eq. (2.41). In general relativity with a pure de Sitter expansion ($\dot{H} = 0$), in which $F(R)$ is given by $F(R) = R - 2\Lambda$ with $\Lambda$ being the cosmological constant, $J = 0$. It is clear that $J$ does not vanish for $b = 1$ if $\dot{H} \neq 0$. 

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III. SECOND LAW OF THERMODYNAMICS IN A $F(R)$ GRAVITY MODEL REALIZING A CROSSING OF THE PHANTOM DIVIDE

In this section, we examine whether a $F(R)$ gravity model [15] with the phantom crossing can satisfy the second law of thermodynamics discussed in Sec. II. In the model [15], the Hubble rate $H(t)$ is given by

$$H(t) = \left(\frac{10}{t}\right) \left[\frac{\gamma + (\gamma + 1) \left(\frac{t}{t_s}\right)^{2\gamma + 1}}{1 - \left(\frac{t}{t_s}\right)^{2\gamma + 1}}\right],$$

(3.1)

where $\gamma$ is a positive constant and $t_s$ is the time when the Big Rip singularity appears. Here, we only consider the period $0 < t < t_s$. When $t \to 0$, i.e., $t \ll t_s$, $H(t)$ behaves as

$$H(t) \sim \frac{10\gamma}{t}.$$  

(3.2)

In the FRW background (2.3), the effective EoS $w_{\text{eff}}$ is given by [8] $w_{\text{eff}} = -1 - 2\dot{H}/(3H^2)$. In the limit of $t \to 0$, $w_{\text{eff}} = -1 + 1/(15\gamma) > -1$, corresponding to the non-phantom phase.

The form of $F(R)$ is given by

$$F(R) \sim \left\{ \frac{1}{t_0} \sqrt{60\gamma (20\gamma - 1)R^{-1/2}} \right\}^\gamma R^{2\gamma + 1} \times \sum_{j=\pm} \left\{ \left( \frac{5\gamma - \beta_j - 1}{20\gamma - 1} \right) \bar{p}_j [60\gamma (20\gamma - 1)]^{\beta_j/2} R^{-\beta_j/2} \right\},$$

(3.3)

where

$$\beta_{\pm} = \frac{1 \pm \sqrt{1 + 100\gamma (\gamma + 1)}}{2}.$$  

(3.4)

Here, $t_0$ is the present time and $\bar{p}_\pm$ are arbitrary constants. We remark that the stability for the obtained solutions in Eq. (3.3) under a quantum correction coming from conformal anomaly has been examined in Ref. [15]. It has been shown that the quantum correction could be small when the phantom crossing occurs, although it becomes important near the Big Rip singularity.

Using Eq. (3.3), we find $R \sim 60\gamma (20\gamma - 1)/t^2$ and $\sqrt{60\gamma (20\gamma - 1)R^{-1/2}/t_s} \sim t/t_s \ll 1$. As the denominator inside the first large braces $\{}$ on the right-hand side of Eq. (3.3) is
approximately unity, Eq. (3.3) can be simplified to
\[ F(R) \approx \left[ \frac{1}{t_0} \sqrt{60\gamma (20\gamma - 1)} \right]^{5\gamma} R^{-5\gamma/2 + 1} \]
\[ \times \sum_{j=\pm} \left\{ \left( \frac{5\gamma - \beta_j - 1}{20\gamma - 1} \right) p_j [60\gamma (20\gamma - 1)]^{\beta_j/2} R^{-\beta_j/2} \right\}. \quad (3.5) \]

In this case, we obtain
\[ J = 50\gamma^2 \left[ \frac{1}{t_0} \sqrt{60\gamma (20\gamma - 1)} \right]^{5\gamma} \frac{1}{t^4} R^{-5\gamma/2} \sum_{j=\pm} \left\{ [10\gamma (1 - b) (-5\gamma - \beta_j + 2) - (2 - b)] \right\} \]
\[ \times \left( \frac{5\gamma - \beta_j - 1}{20\gamma - 1} \right) p_j [60\gamma (20\gamma - 1)]^{\beta_j/2} (5\gamma + \beta_j - 2) R^{-\beta_j/2} \}. \quad (3.6) \]

Note that \( R > 0 \) for \( R \sim 60\gamma (20\gamma - 1)/t^2 \) and \( \gamma > 1/20 \). Using Eqs. (2.41), (3.4) and (3.6) and taking into account the fact that \( \bar{p}_\pm \) are arbitrary constants, the necessary condition to have \( J \geq 0 \) is
\[ \bar{p}_\pm \left\{ 5\gamma (1 - b) \left[ (10\gamma - 3) \pm \sqrt{1 + 100\gamma (\gamma + 1)} \right] + 2 - b \right\} \geq 0, \quad (3.7) \]
where we have assumed \( \gamma > 1/20 \). We remark that for simplicity, we have chosen positive coefficients of \( R^{-\beta_j+1/2} \) and \( R^{-\beta_j-1/2} \) in the large braces \( \{ \} \) of Eq. (3.6). By taking the values of \( \bar{p}_\pm \) so that the relation (3.7) can be met, the second law of thermodynamics, i.e., \( J \geq 0 \), can be satisfied.

It follows from \( w_{\text{eff}} = -1 - 2\dot{H}/(3H^2) \) that \( w_{\text{eff}} = -1 \) when \( \dot{H} = 0 \). Solving \( w_{\text{eff}} = -1 \) with respect to \( t \) by using Eq. (3.1), we find that \( w_{\text{eff}} \) crosses the phantom divide at the time \( t = t_c \), given by
\[ t_c = t_s \left( -2\gamma + \sqrt{4\gamma^2 + \frac{\gamma}{\gamma + 1}} \right)^{1/(2\gamma + 1)}. \quad (3.8) \]

On the other hand, when \( t \to t_s \), we have
\[ H(t) \sim \frac{10}{t_s - t}. \quad (3.9) \]

In this case, the scale factor is given by \( a(t) \sim \bar{a} (t_s - t)^{-10} \) with a constant of \( \bar{a} \). When \( t \to t_s \), \( a \to \infty \) and therefore the Big Rip singularity appears. In this limit, \( w_{\text{eff}} = -16/15 < -1 \), corresponding to the phantom phase. The form of \( F(R) \) is given by
\[ F(R) \sim \left( \frac{1}{t_0} \left[ t_s - 3\sqrt{140} R^{-1/2} \right] \right)^{5\gamma} R \sum_{j=\pm} \bar{p}_j \left[ t_s - 3\sqrt{140} R^{-1/2} \right]^{\beta_j} \]
\[ \times \left\{ 1 - \sqrt{20} \left[ \frac{15}{84} t_s + (\beta_j - 15) R^{-1/2} \right] \frac{1}{t_s - 3\sqrt{140} R^{-1/2}} \right\}, \quad (3.10) \]
which is reduced to

\[ F(R) \sim \bar{F} R^{7/2} \]  

(3.11)

for \( t_s^2 R \gg 1 \), where

\[ \bar{F} = \frac{2}{7} \left[ \frac{1}{3\sqrt{140}} \frac{1}{(2\gamma + 1)} \left( \frac{t_s}{t_0} \right)^\gamma \right]^5 \left( \sum_{j=\pm} \tilde{p}_j t_s^{\beta_j} \right) t_s^5. \]  

(3.12)

In this case, we obtain

\[ J = \frac{441000 (72 - 71b) \bar{F}}{(t_s - t)^6} R^{3/2}. \]  

(3.13)

For \( \gamma > 1/20 \), because \( R > 0 \) and \( 0 < b < 1 \), the necessary condition to have \( J \geq 0 \) in Eq. (3.13) is

\[ \sum_{j=\pm} (\tilde{p}_j t_s^{\beta_j}) \geq 0, \]  

(3.14)

which can be met by choosing the appropriate values of \( \tilde{p}_\pm \). Since \( \tilde{p}_\pm \) are arbitrary integration constants, for simplicity, we choose \( \tilde{p}_+ = 0 \). When \( \gamma \sim \mathcal{O}(1) \) and \( \tilde{p}_- > 0 \), \( F(R) \) is always positive in both non-phantom and phantom phases. This is reasonable because for general relativity, \( F(R) = R > 0 \). Consequently, in this \( F(R) \) gravity model with the crossing of the phantom divide, the second law of thermodynamics can be satisfied.

For a power-law type \( F(R) \) gravity described as \( F(R) = c_1 M^2 (R/M^2)^{-n} \), where \( c_1 \) and \( n \) are dimensionless constants and \( M \) denotes a mass scale, the scale factor \( a(t) \) is given by

\[ a(t) = \bar{a} (t_s - t)^{(n+1)(2n+1)/(n+2)}, \]  

[15, 74]. The form of \( F(R) \) in Eq. (3.11) corresponds to the case with \( n = -7/2 \). Accordingly, \( a(t) = \bar{a} (t_s - t)^{-10} \), which implies \( \ddot{a} = 110 \bar{a} (t_s - t)^{-12} \geq 0 \). Thus, a late-time cosmic acceleration can be realized. This is the outcome of the \( F(R) \) gravity model in Eq. (3.11). It should be noted that whether the gravity model of \( F(R) \sim \bar{F} R^{7/2} \) in Eq. (3.11) can pass Solar System tests still needs to be examined [75].

From Eq. (3.9), we obtain \( R = 1260/(t_s - t)^2 \), which leads to \( J = [(72 - 71b) \bar{F}/4536] R^{3/2} \) based on Eq. (3.13). In the thermal equilibrium limit, i.e., \( b \sim 1 \), we find \( J \sim (\bar{F}/4536) R^{3/2} \). On the other hand, for general relativity with \( F(R) = R \), it follows from Eq. (2.41) with \( b = 1 \) that \( J = \dot{H}^2 = (1/15876) R^2 \) by assuming the same behavior of \( H \) in Eq. (3.9). The main difference between the expressions of \( J \) in the thermal equilibrium limit and that for general relativity is only the power of \( R \). This comes from
the difference of the action between the present $F(R)$ gravity $F(R) \sim \bar{F} R^{7/2}$ in Eq. (3.11) and general relativity with $F(R) = R$.

Finally, we remark that even in the effective phantom era of this $F(R)$ gravity model, the second law of thermodynamics can be satisfied due to the non-equilibrium thermodynamic treatment. Hence, this model is more similar to a phantom model with ordinary thermodynamics suggested in Ref. [76].

IV. CONCLUSION

We have investigated the first and second laws of thermodynamics of the apparent horizon in $F(R)$ gravity [49]. We have shown that in the $F(R)$ gravity model with realizing the crossing of the phantom divide proposed in Ref. [15], the second law of thermodynamics can be satisfied in not only the non-phantom phase but also the effective phantom one. In addition to cosmological constraints and solar system tests on the models of $F(R)$ gravity, such an examination whether the second law of thermodynamics can be met in those models is important. The demonstration in this work can be regarded as a meaningful step to construct a more realistic model of $F(R)$ gravity, which could correctly describe the expansion history of the universe.

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