On Target-Space Duality in $p$–Branes

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ABSTRACT

We study the target-space duality transformations in $p$–branes as transformations which mix the worldvolume field equations with Bianchi identities. We consider an $(m + p + 1)$-dimensional spacetime with $p + 1$ dimensions compactified, and a particular form of the background fields. We find that while a $GL(2) = SL(2) \times R$ group is realized when $m = 0$, only a two parameter group is realized when $m > 0$.

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Ten dimensional string theory compactified on a six–torus has a symmetry group $O(6, 6; Z)$, called “T duality”, which mixes momentum modes with winding modes [1-4] (in the heterotic case the group is $O(6, 22; Z)$). This is also referred to as target space duality. (For an extensive review and further references, see [5]). It has been conjectured that string theory possesses also a symmetry group $SL(2, Z)$, called “S duality”, which transforms Fock states into soliton states [6-8]. This is also known as strong–weak coupling duality. (For a recent review, see [9]). Such a symmetry could only be seen in a nonperturbative approach and its existence has not yet been firmly established.

One possible way of studying the S duality of strings is via fivebranes. A fivebrane in ten dimensions has been conjectured to be “dual” to a string in a certain well defined sense [10], and it has been further conjectured that under this transformation the role of the S and T dualities would interchange [11], in the sense that the $SL(2, Z)$ S duality of the string would play the role of a T duality for the fivebrane, and similarly the $O(6, 6; Z)$ T duality of the string would play the role of a nonperturbative S duality for the fivebrane. Since a T duality symmetry can be seen perturbatively, it would seem convenient to study $SL(2, Z)$ symmetry in the context of five–branes.

All these groups have been found either directly in four dimensional $N = 4$ supergravity [3] or in the toroidal compactification of ten dimensional supergravity theories [4] which one obtains as low energy limits of strings and fivebranes. It was conjectured by Duff and proven by Cecotti et al. that the T duality group $O(6, 6; Z)$ is a symmetry of string theory, transforming the worldsheet field equations and Bianchi identities into each other [2]. Although it has been shown that an $SL(2, Z)$ duality group indeed mixes the momentum modes with the winding modes of the five-brane theory [7], so far this symmetry group has not been understood as a worldvolume duality group transforming the worldvolume equations of motion to Bianchi identities.

This is the problem that we will investigate in the present paper. Although the case of greatest interest is the five-brane in $4 + 6$ dimensions, of which 6 are compact, for the sake of generality we shall discuss arbitrary $p$-branes in $m + p + 1$ dimensions, of which $p + 1$ are compact. We will assume a specific simple form of the worldvolume action [12]. In the Lagrangian formulation, we show that while an $SL(2)$ symmetry is realized for $m = 0$, only a two parameter algebra formed by a particular set of upper triangular matrices is realized for $m > 0$.

An attempt to find duality symmetries in $p$-branes has been made before, in a somewhat different setting [13]. We shall comment on some results of this paper in the end.

The duality symmetries of (compactified) strings and the conjectured duality symmetries of five-branes form discrete groups. We assume that at a local level the duality
symmetries actually form continuous groups, and that they are broken down to discrete groups by the boundary conditions due to the compactification. This is known to be the case for the string. Therefore, in discussing the duality symmetry of the equations of motion, we can restrict ourselves to infinitesimal transformations. The duality group is determined by the consistency requirements that were first discussed in this context by Gaillard and Zumino [14].

Before considering the case of general $p$–branes, let us outline how these requirements work in the simpler model of a string on a target space of dimension $m + n$, with $n$ dimensions compactified. The dynamical variables are $x^\mu(\sigma)$, $y^\alpha(\sigma)$ and a world-sheet metric $\gamma_{ij}(\sigma)$. Here $\sigma^i (i = 0, 1)$ are the world-sheet coordinates, $x^\mu$, $\mu = 0, ..., m - 1$ are coordinates on $m$ dimensional spacetime $M$ and $y^\alpha$, $\alpha = 1, ..., n$ are coordinates on an internal $n$–dimensional manifold $N$. For simplicity, we assume that the only nonvanishing background fields are the metrics $g_{\mu\nu}(x)$ and $g_{\alpha\beta}(x)$ on $M$ and $N$ respectively and an antisymmetric tensor field $b_{\alpha\beta}(x)$. Although $N$ is supposed to be compact, we shall mostly not be concerned with the boundary conditions that are implied by this fact. At the local level at which we shall work the only distinction between the $y^\alpha$ and the $x^\mu$ coordinates is that the antisymmetric tensor $b$ has vanishing $x^\mu$ components.

The action for the string is

$$S = \int d^2 \sigma L = \int d^2 \sigma \left[ -\frac{1}{2} \sqrt{-\gamma} \left( \gamma^{ij} \partial_i x^\mu \partial_j x^\nu g_{\mu\nu} + \gamma^{ij} \partial_i y^\alpha \partial_j y^\beta g_{\alpha\beta} \right) + \frac{1}{2} \epsilon^{ij} \partial_i y^\alpha \partial_j y^\beta b_{\alpha\beta} \right].$$

(1)

The field equations for the variables $x^\mu$ following from the action (1) are

$$g_{\mu\nu} \partial_i \left( \sqrt{-\gamma} \gamma^{ij} \partial_j x^\nu \right) + \sqrt{-\gamma} \gamma^{ij} \partial_i x^\rho \partial_j x^\nu \Gamma_{\mu, \rho\nu} + \frac{1}{2} \left( \frac{\gamma^{ij}}{\sqrt{-\gamma}} \partial_\mu g_{\alpha\beta} - \epsilon_{ij} \partial_\mu b_{\alpha\beta} \right) J^i J^j = 0 ,$$

(2)

where $\Gamma_{\mu, \rho\nu}$ are the Christoffel symbols of the first kind for the metric $g_{\mu\nu}$ and we have defined the conserved current *

$$J^i = \epsilon^{ij} \partial_j y^\alpha .$$

(3)

The field equations for the variables $y^\alpha$ can be written in the form

$$\partial_i P^i_\alpha = 0 ,$$

(4)

* In our conventions the world-sheet Levi–Civita symbols are defined by $\epsilon^{01} = -\epsilon_{01} = 1$, the world-sheet signature is $(- +)$ and the spacetime signature $(- + \cdots +)$. The Levi–Civita symbols on $N$ have components $\epsilon^{12...n} = \epsilon_{12...n} = 1$
where
\[
P^i_\alpha = \frac{\partial L}{\partial \dot{y}^\alpha} = (-\sqrt{-\gamma} \epsilon^{ij} g_{\alpha\beta} + \epsilon^{ij} b_{\alpha\beta}) \epsilon_{jk} J^{k\beta}, \tag{5}
\]

Finally, the equations of motion of the metric $\gamma$ state that the energy–momentum tensor of $x^\mu$ and $y^\alpha$, regarded as scalar fields on the world-sheet, has to be zero. From the definition (3) follows the identity
\[
\partial_i J^{i\alpha} = 0. \tag{6}
\]

This equation will be referred to as a “Bianchi identity”. The duality transformations we seek for mix the field equation (4) and Bianchi identity (6). To this end let us consider the following infinitesimal transformations
\[
\delta P^i_\alpha = A^\alpha_\beta P^i_\beta + B_{\alpha\beta} J^{i\beta},
\delta J^{i\alpha} = C^{\alpha\beta} P^i_\beta + D^\alpha_\beta J^{i\beta}, \tag{7}
\]

where $A, B, C, D$ are constant matrices. We now have to check that these transformations together with appropriate transformation rules for the background fields, which are to be determined, actually leave invariant Eq. (5). Varying (5) we obtain
\[
A^\alpha_\beta P^i_\beta + B_{\alpha\beta} J^{i\beta} = (-\sqrt{-\gamma} \epsilon^{ij} \delta g_{\alpha\beta} + \epsilon^{ij} \delta b_{\alpha\beta}) \epsilon_{jk} J^{k\beta}
+ (-\sqrt{-\gamma} \epsilon^{ij} g_{\alpha\beta} + \epsilon^{ij} b_{\alpha\beta}) \epsilon_{jk} (C^\beta_\gamma P^k_\gamma + D^\beta_\gamma J^{k\gamma}). \tag{8}
\]

Eliminating $P^i_\alpha$ by using (5), we express all the terms in (8) in terms of $J^{i\alpha}$. Thus, demanding that (8) is satisfied we arrive at the conditions $B_{\alpha\beta} = -B_{\beta\alpha}$, $C^{\alpha\beta} = -C^{\beta\alpha}$, $D^\alpha_\beta = -A^\alpha_\beta$ and the background field transformation rules
\[
\delta g_{\alpha\beta} = A^\alpha_\gamma g_{\gamma\beta} + A^\beta_\gamma g_{\gamma\alpha} - g_{\alpha\gamma} C^{\gamma\delta} b_{\delta\beta} - g_{\beta\gamma} C^{\gamma\delta} b_{\delta\alpha},
\delta b_{\alpha\beta} = A^\alpha_\gamma b_{\gamma\beta} - A^\beta_\gamma b_{\gamma\alpha} - g_{\alpha\gamma} C^{\gamma\delta} g_{\delta\beta} - g_{\beta\gamma} C^{\gamma\delta} g_{\delta\alpha} + B_{\alpha\beta}. \tag{9}
\]

At this point one can check that equation (2) as well as the equation of motion for $\gamma$ are also invariant under this set of transformations. It is important to observe that in the case of the string one can consistently take $\gamma$ to be inert under duality transformations. In fact, the equation of motion for $\gamma$ has the general solution $\gamma_{ij} = \Omega^2 (\partial_i x^\mu \partial_j x^\nu g_{\mu\nu} + \partial_i y^\alpha \partial_j y^\beta g_{\alpha\beta})$, where $\Omega$ is an undetermined scale factor. Using equations (3), (7) and (9) one can show that $\gamma_{ij}$ is indeed duality invariant.

The conditions on $A, B, C, D$ imply that they form the group $O(n, n; R)$. (In the heterotic case one would find $O(n_1, n_2; R)$, with $n_1$ and $n_2$ referring to the number of internal left and right movers). All this was at the local level. Taking into account the boundary conditions due to the compactness of $N$, one finds the duality group $O(n, n; Z)$.
In this paper we will be interested in the case of $p$-branes when the number of compact dimensions is $p+1$. In order to extract some more information from the string case, let us therefore see what happens in the case of the string when $n = 2$. Apart from reflections, the group $O(2, 2; R)$ is the direct product of two commuting $SL(2)$ subgroups. The Lie algebra of one of these groups consists of block diagonal matrices with $B = C = 0$, $D = -A^T$ and $A$ traceless. From (7) we see that it does not mix the field equations with the Bianchi identities, and therefore we shall refer to it as the “trivial” $SL(2)$. The other $SL(2)$ is defined by $A_{\alpha\beta} = a\delta_{\alpha\beta}$, $B_{\alpha\beta} = b\epsilon_{\alpha\beta}$, $C_{\alpha\beta} = -c\epsilon_{\alpha\beta}$ and $D_{\alpha\beta} = -a\delta_{\alpha\beta}$.

Let us now consider background fields of the special form $g_{\alpha\beta} = \lambda_2 \delta_{\alpha\beta}$, $b_{\alpha\beta} = \lambda_1 \epsilon_{\alpha\beta}$, where $\lambda_1$ and $\lambda_2$ are constants. Then, defining $J^i_{\alpha} = \epsilon_{\alpha\beta} J^{i\beta}$, the resulting nontrivial $SL(2)$ duality transformations are

\[
\begin{align}
\delta P^i_{\alpha} &= a P^i_{\alpha} + b J^i_{\alpha}, \\
\delta J^i_{\alpha} &= c P^i_{\alpha} - a J^i_{\alpha}, \\
\delta \lambda_1 &= b + 2a \lambda_1 - c \lambda_1^2 + c \lambda_2^2, \\
\delta \lambda_2 &= 2(a - c \lambda_1) \lambda_2.
\end{align}
\]

Note that (10c,d) is the infinitesimal form of a finite transformation of the complex field $\lambda = \lambda_1 + i \lambda_2$, of the form $\lambda' = \frac{A \lambda + B}{C \lambda + D}$. The celebrated $R \to 1/R$ duality transformation corresponds to $A = D = 0$, $B = -C = 1$. We will be seeking the analog of the above $SL(2)$ symmetry in theories of $p$-branes in $m + p + 1$ dimensions.

The dynamical variables describing the $p$-brane are again scalar fields $x^\mu(\sigma)$, $y^\alpha(\sigma)$ and a worldvolume metric $\gamma_{ij}(\sigma)$. Here $x^i$ ($i = 0, ..., p$) are the worldvolume coordinates, $x^\mu$, $\mu = 0, ..., m - 1$ are coordinates on $m$ dimensional space $M$ and $y^\alpha$, $\alpha = 1, ..., p + 1$ are coordinates on a compact $(p + 1)$-dimensional manifold $N$. The background fields are as before the metrics $g_{\mu\nu}(x)$ and $g_{\alpha\beta}(x)$ on $M$ and $N$ respectively and an antisymmetric tensor field $b_{\alpha_1...\alpha_{p+1}}(x) = \lambda_1(x) \epsilon_{\alpha_1...\alpha_{p+1}}$.

The action for the $p$-brane is

\[
S = \int d^{p+1}\sigma L = \int d^{p+1}\sigma \left[ -\frac{1}{2} \sqrt{-\gamma} \left( \gamma^{ij} \partial_i x^\mu \partial_j x^\nu g_{\mu\nu} + \gamma^{ij} \partial_i y^\alpha \partial_j y^\beta g_{\alpha\beta} \right) + \frac{p - 1}{2} \sqrt{-\gamma} \right. \\
+ \left. \frac{1}{(p + 1)!} \epsilon^{i_1...i_{p+1}} \partial_{i_1} y^{\alpha_1} ... \partial_{i_{p+1}} y^{\alpha_{p+1}} \lambda_1 \epsilon_{\alpha_1...\alpha_{p+1}} \right].
\]

The field equations for the variables $x^\mu$ following from the action (11) are

\[
\begin{align}
&g_{\mu\nu} \partial_i (\sqrt{-\gamma} \gamma^{ij} \partial_j x^\nu) + \sqrt{-\gamma} \gamma^{ij} \partial_i x^\rho \partial_j x^\nu \Gamma_{\mu,\rho\nu} - \frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \partial_i y^\alpha \partial_j y^\beta \partial_\mu g_{\alpha\beta} \\
&+ \frac{1}{(p + 1)!} \epsilon^{i_1...i_{p+1}} \partial_{i_1} y^{\beta_1} ... \partial_{i_{p+1}} y^{\beta_{p+1}} \epsilon_{\beta_1...\beta_{p+1}} \partial_\mu \lambda_1 = 0.
\end{align}
\]
The field equations for the variables $y^\alpha$ again take the form (4) with $P^i_\alpha$ now defined by

$$P^i_\alpha = \frac{\partial L}{\partial \partial_i y^\alpha} = -\sqrt{-\gamma} \gamma^{ij} \partial_j y^\beta g_{\beta\alpha} + \lambda_1 J^i_\alpha , \quad (13)$$

$$J^i_\alpha = \frac{1}{p!} \epsilon^{ij_1\ldots j_p} \partial_{j_1} y^{j_1} \cdots \partial_{j_p} y^{j_p} \epsilon_{\alpha j_1\ldots j_p} . \quad (14)$$

From the definition (14), the Bianchi identity $\partial_i J^i_\alpha = 0$ follows. We know from ten dimensional supergravity compactified on a six–torus that under $SL(2)$ the metrics $g_{\alpha\beta}$ and $g_{\mu\nu}$ rescale. Therefore let us define

$$g_{\mu\nu} = \lambda_2^K \bar{g}_{\mu\nu} , \quad g_{\alpha\beta} = \lambda_2^L \bar{g}_{\alpha\beta} , \quad (15)$$

where $\bar{g}_{\alpha\beta}$ and $\bar{g}_{\mu\nu}$ are assumed to be inert under $SL(2)$, and $\det \bar{g}_{\alpha\beta} = 1$. Thus $\lambda_2(x) = (\det g_{\alpha\beta})^{1/(p+1)L}$. In the case $p = 5$ it is known from the $SL(2)$ duality symmetry of the effective field theory limit that $K = -1$ and $L = 1/3$ [9]. As will become clear later the most convenient choice for $L$ is $2/(p + 1)$, however, for the sake of generality we shall keep the values of these powers arbitrary most of the time.

The equation of motion for the worldvolume metric $\gamma$ gives

$$\gamma_{ij} = \lambda_2^K \partial_i x^\mu \partial_j x^\nu \bar{g}_{\mu\nu} + \lambda_2^L \partial_i y^\alpha \partial_j y^\beta \bar{g}_{\alpha\beta} . \quad (16)$$

The discussion of strings in $m + 2$ dimensions leads us to postulate the following form for the infinitesimal duality transformations

$$\delta P^i_\alpha = aP^i_\alpha + bJ^i_\alpha , \quad (17a)$$

$$\delta J^i_\alpha = cP^i_\alpha + dJ^i_\alpha , \quad (17b)$$

where $a, b, c, d$ are constants. As we did in the string case, we have to show that (13) is invariant under these transformations, combined with appropriate transformation rules for the background fields $\lambda_1$ and $\lambda_2$.

Since all relevant quantities have two indices, it is convenient to use matrix notation. We define matrices $P$ and $J$ with components $P^i_\alpha$ and $J^i_\alpha$, matrices $\partial y$ with components $(\partial y)^{\alpha i} = \partial_i y^\alpha$ and $\bar{g}^{(p+1)}$ with components $\bar{g}_{\alpha\beta}$. From the definition (14) we see that the matrix $J$ is directly related to the inverse of the matrix $\partial y$ as $J = \partial y^{-1} \det \partial y$, and $\det (\partial y) = (\det J)^{1/p}$. Therefore we have the relation

$$\partial y = J^{-1} (\det J)^{1/p} . \quad (18)$$
This equation allows us to calculate the variation of $\partial y$ under the duality transformations. Using (17b), one finds
\[
\delta \partial y = \partial y \left\{ - cX + \frac{1}{p} (d + \text{tr}X) \right\},
\]
where
\[
X = P \cdot J^{-1}.
\]
Note that when $P$ and $J$ are transformed linearly, $X$ undergoes a fractional linear transformation. From (13) and (16) we find
\[
X = \lambda_1 + \lambda_2 \frac{\gamma^{-1}V}{\sqrt{-\det(\gamma^{-1}V)}},
\]
where $V = (\partial y)^T g^{(p+1)} \partial y$.

Following the same steps that led to equation (8), now taking into account also the variation of $\gamma$, we find after some algebra that the invariance of (13) under the transformations (17) requires that
\[
cX^2 + \left[ a - 2c\lambda_1 - \frac{1}{p} (d + \text{tr}X) - \left( \frac{p-1}{2}K + L \right) \lambda_2^{-1} \delta \lambda_2 \right] X
\]
\[
+ b - \frac{p-1}{p} d\lambda_1 + \frac{1}{p} c\lambda_1 \text{tr}X + \left( \frac{p-1}{2}K + L \right) \lambda_1 \lambda_2^{-1} \delta \lambda_2 - \delta \lambda_1 =
\]
\[
= \lambda_2 \gamma^{-1}V \left\{ 2cX^2 - \left[ \frac{2}{p} (d + \text{tr}X) + 2c\lambda_1 + (L - K) \lambda_2^{-1} \delta \lambda_2 \right] X
\]
\[
+ \frac{1}{p} c(\text{tr}X)^2 - \text{tr}(X^2) + \left( \frac{1}{p} c\lambda_1 + \frac{1}{p} d + \frac{1}{2} (L - K) \lambda_2^{-1} \delta \lambda_2 \right) \text{tr}X
\]
\[
- \frac{p-1}{p} d\lambda_1 - \frac{1}{2} (p-1)(L - K) \lambda_1 \lambda_2^{-1} \delta \lambda_2 \right\}.
\]
Since $\gamma^{-1}V$ can be reexpressed in terms of $X$ via Eq. (21), this is an infinite polynomial equation in the $(p+1) \times (p+1)$ matrix $X$. One is free to determine $\delta \lambda_1$ and $\delta \lambda_2$ as functions of $a, b, c, d, \lambda_1, \lambda_2$ to satisfy this equation, and also if necessary to put restrictions on the transformation parameters $a, b, c, d$. It is important to realize that these transformations have to be the same for all $X$. In order to prove that this equation has no solution it would therefore be sufficient to find two particular matrices $X$ which give incompatible values for the variations $\delta \lambda_1$ and $\delta \lambda_2$.

We shall implement this idea by choosing a particular matrix $X$ and expanding equation (22) around it. For our background we choose the fields $x^\mu$ and $y^a$ such that $\gamma^{-1}V = \lambda_2^{-L} \eta$. For example one can have $V_{ij} = \lambda_2^{-L} \delta_{ij}$ and $\gamma_{ij} = \eta_{ij}$. Then writing
\[
\gamma^{-1}V = \lambda_2^{-L} (\eta + Y)
\]
(23)
we can expand equation (22) in powers of $Y$. The zeroth order terms determine the form of the variations

\[
\delta \lambda_1 = b + (a-d) \lambda_1 + \left( \frac{2}{p} d + \frac{2}{p+1} \frac{L-K}{L} (a-d) \right) \lambda_2^{(p+1)L/2} - c \lambda_1^2 + 2c \left( \frac{1}{p} - \frac{2}{p+1} \frac{L-K}{L} \right) \lambda_1 \lambda_2^{(p+1)L/2} + \frac{3p-2}{p} \lambda_2^{(p+1)L}, \tag{24a}
\]

\[
\delta \lambda_2 = \frac{2}{(p+1)L} ((a-d) - 2c \lambda_1) \lambda_2^{(p+1)L/2}. \tag{24b}
\]

At linear order in $Y$ one gets terms proportional to the matrices $\mathbf{1}$, $\eta$, $Y$ and $Y \eta$. Since these are linearly independent, one can put separately their coefficients to zero. In this way one finds

\[
c = 0, \quad d = p \frac{K - L}{pK + L} a, \tag{25}
\]

which inserted in (24) yield

\[
\delta \lambda_1 = b + \frac{(p+1)L}{pK + L} a \lambda_1, \tag{26}
\]

\[
\delta \lambda_2 = \frac{2}{pK + L} a \lambda_2.
\]

Remarkably, these transformations give a solution of the full equation (22). Furthermore, varying the $x^\mu$-equation of motion (12) under these transformations, we find that its form is preserved. Therefore, we have a two parameter group of duality transformations of the $p$–brane. It is easy to check that the transformation rules (17) and (26) yield the same commutator algebra. Denoting the transformations by $a$ and $b$, the only nonvanishing commutator is $[a, b] = b$. One can have $d = -a$ by choosing $K = \frac{p-1}{2p} L$. If we further choose $L = 2/(p+1)$, (26) becomes a special case of (10c,d). In this way the two parameter group appears to be a subgroup of the expected group $SL(2)$. Notice that from a solution with magnetic charge, one can obtain a solution with magnetic and electric charge. However, the two parameter group does not contain the important $R \rightarrow 1/R$ transformations mentioned earlier.

It may be of some interest to consider the special case $m = 0$, namely a $p$-brane propagating in a $p + 1$-dimensional space. This cannot immediately be obtained from the previous discussion by putting $x^\mu = 0$, since the action would be complex. Instead, we have to assume now that the metric $g_{\alpha \beta}$ is Lorentzian. Then, equation (21) is replaced by $X = \lambda_1 + \lambda_2$, where we have chosen $L = 2/(p+1)$. From (17) and (20) one easily finds that $\delta X = b + (a-d)X - cX^2$. Therefore it follows that $\delta(\lambda_1 + \lambda_2) = b + (a-d)(\lambda_1 + \lambda_2)$.
\( \lambda_2 - c(\lambda_1 + \lambda_2)^2 \). This is the infinitesimal form of a fractional linear transformation of the real variable \( \lambda_1 + \lambda_2 \) and gives a representation of the algebra \( GL(2) = SL(2) \times R \). \footnote{Unlike in equation (10c,d) it is not possible to define the variations of \( \lambda_1 \) and \( \lambda_2 \) separately. If we had kept the metric \( g_{\alpha \beta} \) Euclidean we would have found \( X = \lambda_1 + i\lambda_2 \), leading to separate transformation rules. However this choice is clearly unphysical since the action (11) would become complex.}

The subgroup \( R \), corresponding to \( d = a, b = c = 0 \), acts trivially on the background fields and does not mix the field equations with Bianchi identities. Thus, we can regard this \( R \) as “trivial”. A similar trivial factor is present also in the duality group \( GL(1, C) = U(1) \times R \) of a free Maxwell field, see footnote on p. 222 in [14]. Note that the duality group \( SL(2) \) of the \( p \)-brane in \( p + 1 \) dimensions is the generalization of the nontrivial \( SL(2) \) of the string in \( d + 2 \) dimensions discussed earlier.

There is also an analog of the trivial \( SL(2) \). In the case we are considering, with \( L = 2/(p + 1) \), Eq. (13) reduces to \( P^i_\alpha = (\lambda_1 + \lambda_2)J^i_\alpha \), so it is clear that there is a larger set of transformations which leave this equation invariant, without mixing \( P \) and \( J \), namely \( \delta \lambda_1 = 0, \delta \lambda_2 = 0, \delta P^i_\alpha = (R_\alpha^\beta + S^{(\alpha)}\delta_\alpha^\beta)P^i_\beta \) and \( \delta J^i_\alpha = (R_\alpha^\beta + S^{(\alpha)}\delta_\alpha^\beta)J^i_\beta \) where \( R_\alpha^\beta \) are real traceless matrices and \( S^{(\alpha)} \) are the real scaling parameters. These transformations form the group \( SL(p + 1, R) \times R^{p+1} \). This group has been discussed in Ref. [13] as a manifest duality symmetry of the \( p \)-brane.

The case of \( p \)-branes in \( p + 1 \) dimensions is very special since it has only a finite number of degrees of freedom [15] and therefore is in a certain sense a topological field theory. Above we have also considered the case of the \( p \)-brane in \( m + n \) dimensions, with \( n = p + 1 \) and a specific choice of background fields. In Ref. [13] \( m \) was taken to be zero, but the dimension \( n \) of the internal space was taken greater than \( p + 1 \), and the backgrounds were constant. In particular, in the case \( p = 2, n = 4 \) it was suggested that there is an \( SL(5, R) \) duality group. However, one should check the consistency of the transformation rules assigned to all objects. Specifically, in [13] transformation rules were assigned to the quantities \( F^{i\alpha} = \sqrt{-\gamma} ij \partial_j y^\alpha \) and \( \tilde{F}^{i\alpha\beta} = \varepsilon_{ijk} \partial_j y^\alpha \partial_k y^\beta \). These quantities, however, are related to each other by the relation \( F^{i\alpha} F^{j_\beta} \varepsilon_{ijk} = -\gamma_{k\ell} \tilde{F}^{i\alpha\beta} \) (we are using here our convention for the Levi-Civita tensor). The transformation rules given for \( F^{i\alpha} \) and \( \tilde{F}^{i\alpha\beta} \) are inconsistent with this relation (the inconsistency arises for the \( B^{\alpha\beta\gamma} \) transformations, which are the membrane generalization of the \( C^{\alpha\beta} \) transformations of Eq. (7)). This, and the results given earlier in this paper, seem to suggest that the intrinsic nonlinearities of \( p \)-brane theories will make it very hard to establish analogs of the “\( C \)-symmetries”.

On the other hand, the results of [7] strongly suggest the existence of an \( SL(2, Z) \) dual-
ity symmetry of five-branes. To reconcile this with our results perhaps one should consider a modification of (17). It is known that the charges $Q_J^J = \int d^p \sigma J^0_\alpha$ and $Q_P^P = \int d^p \sigma P^0_\alpha$ have to transform under $SL(2)$ as $\delta Q_J^P = aQ_J^P + bQ_J^J$ and $\delta Q_J^J = cQ_P^P + dQ_J^J$ [9], but maybe the currents themselves have a slightly different transformation compatible with this transformation of the charges. A current algebra argument shows that if the charges transform like this, the currents have to transform as in (17) up to the addition of a conserved current whose total charge is zero. We have not been able to find such a current in this theory. A more radical departure from our ansatz would be to assume that $SL(2)$ acts in a nonlocal way, as discussed in [16] in the string case. Another possible generalization would be to relax the assumption that all components of the current $J^i_\alpha$, namely, $J^0_\alpha, J^1_\alpha, ..., J^p_\alpha$ transform in the same way. The time components could transform differently from the space components. This applies also to the components of $P^i_\alpha$. It is natural to consider this idea in the Hamiltonian formalism [17]. Again, this has not led so far to any viable generalization of (17). Finally, it is possible that only $SL(2, Z)$ is a symmetry of the five-brane and not $SL(2, R)$. If this is the case, the issue will have to be settled with methods other than those considered here, which are based on infinitesimal transformations.

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