CARLEMAN TYPE INEQUALITIES FOR FRACTIONAL RELATIVISTIC OPERATORS

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Abstract. In this paper we derive Carleman estimates for the fractional relativistic operator. We consider changing-sign solutions to the heat equation for such operators. We prove monotonicity inequalities and convexity of certain energy functionals to deduce Carleman estimates with linear exponential weight. Our approach is based on spectral methods and functional calculus.

1. Introduction and main result

In this paper we are interested in exponential decay estimates for strong solutions of the evolution equation

\begin{equation}
\begin{cases}
  u_t(t, x) + (-\Delta + m^2)^s u(t, x) = V(t, x)u(t, x), & x \in \mathbb{R}^N, \ t > 0, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^N,
\end{cases}
\end{equation}

where $s \in (0, \frac{1}{2})$ and $m > 0$. Here, the solution will be taken as $u : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ and the potential $V : [0, T] \times \mathbb{R}^N \to \mathbb{R}$.

We will name the operator $(-\Delta + m^2)^s$ the fractional relativistic Schrödinger operator with mass $m$ (or just fractional relativistic operator). For $s = 1/2$, $(-\Delta + m^2)^{1/2}$ is sometimes called the square root Klein–Gordon operator, see for instance [22] and to be more precise, the terminology relativistic Schrödinger operator concerns $\sqrt{-\Delta + m^2} - m + V$. The latter is motivated by the kinetic energy of a relativistic particle (that is, a particle travelling with speed close to the speed of light $c$) with mass $m$, and $V$ corresponds to the quantization of the potential energy. We refer the reader to [5], where in particular the motivation and justification for the nomenclature of this operator is explained in the Introduction. The relativistic Schrödinger operator has been extensively studied ([24, Section 7.11], [14]) as well as the evolution Problem (1.1) which involves such an operator, see e.g. [1]. We will not give an exhaustive account of the references.

Related equations to (1.1) have been also considered, namely, the boson star equation was studied in [16].

1.1. Motivation and main results. One of our main motivations is the search of lower bounds for solutions of (1.1) for large $|x|$ very much in the spirit of what is known for solutions of the heat equation. More concretely, consider the heat equation with a potential

\begin{equation}
\begin{cases}
  u_t(t, x) - \Delta u(t, x) = V(t, x)u(t, x), & x \in \mathbb{R}^N, \ t > 0, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}
\end{equation}

It was proved in [9, 10] that if the potential $V$ is bounded and $\|e^{\alpha(0)|x|^2}u_0\|_{L^2(\mathbb{R}^N)} + \|e^{\alpha(T)|x|^2}u(T, \cdot)\|_{L^2(\mathbb{R}^N)} < \infty$ with $\alpha(0) = 0$ and $\alpha(T) = 1/(4T)$, then $u \equiv 0$. This is known as a unique continuation result. The proof is obtained by contradiction after getting first lower bounds that hold for all the solutions of (1.2) and which are obtained after a three-step procedure:

\begin{itemize}
  \item [1] In this paper we derive Carleman estimates for the fractional relativistic operator. We consider changing-sign solutions to the heat equation for such operators. We prove monotonicity inequalities and convexity of certain energy functionals to deduce Carleman estimates with linear exponential weight. Our approach is based on spectral methods and functional calculus.
\end{itemize}
(1) First, it is necessary to establish a monotonicity argument that gives the persistence of the Gaussian decay for positive times if the same is assumed at the initial time. In this first step it is proved that \(\alpha(t)\) can decrease with time, as it happens for example with the fundamental solution of \([1.2]\).

(2) The second step involves convexity arguments. It is proved that if the solution has the same decay at two different times \(t_1 < t_2\) (i.e. \(\alpha(t_1) = \alpha(t_2)\)), then for \(t_1 < t < t_2\) the solution has a better gaussian decay, i.e. \(\alpha(t) > \alpha(t_1)\) for \(t_1 < t < t_2\).

(3) Finally, the last ingredient is to obtain Carleman estimates that together with a localization procedure allow to prove the desired lower bounds.

This procedure has turned out to be rather general. On the one hand, it was proved in \([8]\) that they work for local evolution equations of higher degree like the generalized Korteweg-De Vries equation, which is a third order equation in the spatial variable. In that example the right decay is superlineal for local evolution equations of higher degree like the generalized Korteweg-De Vries equation, which is a initial value problem \([1.1]\). This is stated in the theorem below, where the operator concerning backward unique continuation for the heat equation with potential.

In this article we explore up to what extent the three ingredients mentioned above hold for solutions of \([1.1]\). Let us describe the structure of the paper. In Section 2 we present several definitions of the e third order equation in the spatial variable. In that example the right decay is superlineal and it is given for local evolution equations of higher degree like the generalized Korteweg-De Vries equation, which is a initial value problem \([1.1]\). This is stated in the theorem below, where the operator concerning backward unique continuation for the heat equation with potential.

Finally, in Section 5, we show Carleman-type estimates with linear exponential weight for solutions to \([1.1]\). This is stated in the theorem below, where the operator \(H_m^s\) is defined in Proposition 2.1 as

\[
H_m^s(u, u)(t, x) := -C_{N,s,m}^{N+2s \over 2} \int_{\mathbb{R}^N} \frac{(u(t,x) - u(t,y))^2}{|x-y|^{N+2s \over 2}} K_{N+2s}(m|x-y|) dy - m^{2s} u^2(t, x).
\]

Here, \(K_\nu(z)\) denotes the Macdonald’s function of order \(\nu\), see Section 2.

**Theorem 1.1.** Let \(N \geq 1, s \in (0,1/2), m > 0\), and let \(u_0 \in \mathbb{H}^{2s}(\mathbb{R}^N) \cap L^2(e^{\lambda x} \, dx)\) for some \(\lambda \in \mathbb{R}\) with \(|\lambda| \leq m\). Assume that \(F(t,x) := V(t,x) u(t,x) \in L^2(0,T : L^2(e^{\lambda x} \, dx))\). Let \(u\) be a strong solution to the initial value problem \([1.1]\) and let \(\omega(t,x) = e^{|x|} e^{-\lambda x}\). Then, the following inequality holds

\[
((-|\lambda|^2 + m^2)^s - A)^2 \int_0^1 t(1-t) \int_{\mathbb{R}^N} \omega(t,x) u^2(t,x) \, dx \, dt + {1 \over 2} \int_0^1 \int_{\mathbb{R}^N} \omega(t,x) u^2(t,x) \, dx \, dt
\]

\[
+ {1 \over 2} \int_0^1 t(1-t) \left\{ 2 \int_{\mathbb{R}^N} \omega u_t^2 \, dx - \int_{\mathbb{R}^N} \omega H^2_m(u,u) \, dx + (A + m^2) \int_{\mathbb{R}^N} H^s_m(u,u) \omega \, dx \right\} \, dt
\]

\[
\leq \int_{\mathbb{R}^N} e^{\lambda x} (u^2_0(x) + u^2(1,x)) \, dx + C_1(N,s) \int_0^1 \int_{\mathbb{R}^N} \omega(t,x) \left( (\partial_t + (-\Delta + m^2)^s)(u(t,x)) \right)^2 \, dx \, dt.
\]
for $A + m^{2s} < 0$ sufficiently small (that is, $|A|$ sufficiently large) satisfying
\[ ((-|\lambda|^2 + m^{2})^s - A)^2 \geq C_2(N, s)m^{4s}, \]
where $C_1(N, s), C_2(N, s)$ are positive constants depending only on $N$ and $s$.

The estimate in Theorem 1.1 is a Carleman-type inequality as mentioned in (3) above, which is achieved by: (1) proving monotonicity estimates for the corresponding energy functionals for the solution $u : [0, T] \times \mathbb{R}^N \to \mathbb{R}$, see Proposition 5.1 and (2) convexity arguments, see Proposition 5.3. Only linear exponential weights are admissible, and only for $m > 0$, a fact that is strongly related to the decay of the kernel associated to the fractional relativistic operator, and in big contrast with the polynomial decay of the kernel of $(-\Delta)^s$.

Our techniques are based on functional and spectral calculus and they hold for exponents $0 < s < 1/2$. Observe that this restriction in $s$ is due to the fact that we need the operator $H_{2m}^{\frac{2s}{m}}$ to be negative to keep the corresponding energy term positive, and we are able to ensure this only by using the definition given in Proposition 2.1 which is valid for $0 < s < 1/2$. We notice a major difference with respect to the classical diffusion process: the coefficient $\alpha(t)$ in the weight $e^{\lambda t}$ is always constant, thus the spatial decay does not change with time. Log-convexity still works, thus the decays at time 0 and 1 control the decay at intermediate times. We also notice the important restriction $|\lambda| \leq m$. As it is shown in Proposition 2.3, the weight $e^{\lambda x}$ is an eigenfunction of the operator $L_{m}^{u}$ under the restriction of $|\lambda| < m$, (the case $|\lambda| = m$ is trivial). There is a deeper obstruction behind, related with the analyticity of the Fourier multiplier in the case $|\lambda| > m$, see Remark 2.4.

Unfortunately, the Carleman estimates stated in Theorem 1.1 are not sufficient to conclude any lower bounds. The problem comes from the non-local properties of the operator $(-\Delta + m^{2})^s$. As we said in (3) above in order to use the Carleman estimates, a localization procedure is necessary. When this is done in the non-local setting, the only way of closing the argument is to assume that the fundamental solution in the constant coefficient case (1.3) decays at least as a superlineal exponential, which is not the case. For example the fundamental solution (see Section 3.1) for $s = 1/2$ is known to have an explicit form (see [2, 5], also [21], p. 185–186):}
\[
K_{t}^{1/2}(x) = (2\pi)^{N/2} \left( \frac{2}{\pi} \right)^{1/2} m^{\frac{N+1}{2}} t^{\frac{N+1}{2}} K_{\frac{N+1}{2}}(m\sqrt{|x|^2 + t^2}),
\]
and $K_{t}^{1/2}(x) \sim e^{-m|x|}$ for large $|x|$, see (2.5).

This raises the question of some other possible scenarios of non-local operators that exhibit super-lineal exponential decay. There are examples of distribution densities of Lévy processes which show a “weakly super-linear” asymptotic behaviour. Let us explain this more precisely: let $(Z_t)_{t \geq 0}$ be a real-valued Lévy process with characteristic exponent $\psi$, i.e., $Ee^{itZ_t} = e^{t\psi(z)}$, $t > 0$. The function $\psi : \mathbb{R} \rightarrow \mathbb{C}$ admits the Lévy–Khinchin representation
\[
\psi(z) = iaz - b z^2 + \int_{\mathbb{R}} (e^{iaz} - 1 - izu1_{|u| \leq 1}) \mu(du), \quad a \in \mathbb{R}, \quad b \geq 0
\]
and $\mu(\cdot)$ is a Lévy measure, that is, $\int_{\mathbb{R}} (1 + u^2) \mu(du) < \infty$ (the operator $(-\Delta + m^{2})^s$ falls into this class). The function $e^{t\psi}$ is integrable under some conditions of the process $Z_t$ and hence the associated transition probability density $p_{t}(x)$ has the integral representation as the inverse Fourier transform of the characteristic function, $p_{t}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixz + i\psi(z)} dz$. It is possible to investigate this oscillatory integral, under the assumption that the characteristic exponent $\psi$ admits an analytic extension to the complex plane. A usual assumption on the Lévy measure is that it is exponentially integrable
\[
\int_{|y| \geq 1} e^{Cg} \mu(dy) < \infty, \quad \text{for all } C \in \mathbb{R}.
\]
The latter assumption is satisfied, for instance, for a generalized tempered Lévy measure defined in terms of certain $\psi$ with super-exponential decay, i.e., $e^{Cu}\psi(u) \to 0$, $u \to \infty$, for all $C \in \mathbb{R}$, see for example [21, 37]. Then we may find Lévy processes where the transition probability density satisfies a “weakly super-linear"
asymptotic behavior, namely, for constants \(c_2 < c_1\), there exists \(y = y(c_1, c_2)\) such that
\[
\exp \left( - c_1 x \log^{\frac{\beta - 1}{\beta}} \left( \frac{x}{t} \right) \right) \leq p_t(x) \leq \exp \left( - c_2 x \log^{\frac{\beta - 1}{\beta}} \left( \frac{x}{t} \right) \right), \quad x/t > y,
\]
where \(\beta > 1\), if the Lévy measure \(\mu\) satisfies certain exponential estimates (see [21 (1.18)]). It seems rather natural to look for non-local operators whose fundamental solutions have superlinear decay, alike the discrete case above mentioned, and that still have the analyticity properties mentioned above. We will explore this question in the future.

We finish this subsection with some further references on unique continuation. For elliptic nonlocal models, in [28, 30, 33, 34] the authors use Carleman estimates. Other techniques are used: in [4, 11, 12] the so-called Almgren monotonicity formulas (see [17]) are used, and lower bounds and Runge approximation results for the fractional heat equation are proved in [29].

1.2. Importance of analyticity: revisiting the fractional Landis conjecture. We make some remarks concerning the case \(m = 0\). Let \(N \geq 1\) and \(s \in (0, 1)\). Let \(u : \mathbb{R}^N \to \mathbb{R}\) be a solution to the equation
\[
(−\Delta)^s u(x) = V(x)u(x), \quad x \in \mathbb{R}^N,
\]
with \(V \in L^\infty(\mathbb{R}^N)\). In a recent work, Rüland and Wang [30] proved that, for potentials with some a priori bounds, if a solution to Problem (1.5) below decays at a rate \(e^{-|x|^{1+}}\), then this solution is trivial. On the other hand, for \(s \in (1/4, 1)\) and merely bounded non-differentiable potentials, if a solution decays at a rate \(e^{-|x|^\alpha}\) with \(\alpha > 4s/(4s - 1)\), then this solution must again be trivial. We remark that when \(s \to 1\), \(4s/(4s - 1) \to 4/3\), which is the optimal exponent for the standard Laplacian, see [25]. In fact, in [30, Theorem 3] they provide a quantitative lower bound which leads to the mentioned unique continuation principle. Their result motivates us to point out how another result on unique continuation, of qualitative nature, can be obtained as a consequence of lack of analyticity. Assume that \(u\), the solution to Problem (1.5) decays exponentially fast at infinity, i.e.
\[
|u(x)| \leq e^{-c|x|^{1+}}.
\]
Then it follows that \(u \equiv 0\). Indeed (for simplicity, we restrict ourselves to the one dimensional case), if we take the Fourier transform at both sides of equation (1.5)
\[
|\xi|^{2s} \hat{u}(\xi) = \hat{F}(\xi), \quad F := Vu,
\]
we notice that the right hand side is analytic in \(\mathbb{C}\) while the left hand side is not. This is justified as follows: observe that condition (1.6) implies that \(\hat{u}(\xi)\) is analytic (the exponential decay of \(u\) makes the Fourier transform of \(u\) to be well defined, as well as its derivatives). Moreover, since \(V\) is uniformly bounded then the right hand side term \(F\) also decays exponentially fast at infinity and thus \(\hat{F}\) is analytic. Since \(|\xi|^{2s}\) is not analytic at the origin we conclude that \(\hat{u}\) has to be identically zero.

We note that condition (1.6) can be relaxed to an \(L^2\) decay condition such as \(\|u(·)e^{c|·|^{1+}}\|_{L^2(\mathbb{R}^N)} < \infty\), which is sufficient to ensure the analyticity of \(\hat{u}\). Observe also that the result can be seen as an optimal, qualitative Landis-type conjecture, valid for all \(s \in (0, \infty) \setminus \mathbb{N}\). Finally we remark that only for \(s = \frac{1}{2}\), Landis-type conjecture for (1.5) is proved under the hypothesis of merely exponential decay, see [20].

Remark 1.2. Some comments are in order:

(i) In the case of the parabolic problem \(u_t(t,x) + (−\Delta + m^2)^s u(t,x) = V(t,x)u(t,x)\) \((m \geq 0)\) the argument above does not work anymore since the exponential decay of \(u\) assumed in (1.6) does not need to be inherited by \(u_t\).

(ii) The exponential decay is a sufficient condition that can not be improved with this technique. Less decay assumptions on \(u\) were considered for instance by Frank, Lenzmann and Silvestre [15], although their result concerns only radial solutions.
1.3. Remarks and notations. We want to emphasize that what corresponds in fact to a diffusion problem is the following equation:

\[(1.7) \quad v_t(t, x) + ((-\Delta + m^2)^s - m^{2s})v(t, x) = V(t, x)v(t, x), \quad x \in \mathbb{R}^N, \ t > 0,\]

with data \(v(0, x) = u_0(x), \ x \in \mathbb{R}^N\). It is easy to check that mass is conserved when \(V = 0\), i.e. \(\int_{\mathbb{R}^N} v(t, x)dx = \int_{\mathbb{R}^N} u_0(x)dx\) for all \(t > 0\). Moreover,

\[(1.8) \quad u(t, x) := e^{-m^{2s}t}v(t, x)\]

is a solution to Problem \([1.1]\) with the same initial data. Throughout the paper we will work with Problem \([1.1]\) . Most of the results can be reformulated in terms of the solution to Problem \([1.7]\) via the transformation \((1.8)\) or by simply adapting the definition of the operator adding the term \(m^{2s}I\). For instance, the spatial behaviour for small times is the same for both \(u\) and \(v\).

We denote, for \(m \geq 0\), the operator

\[L_m := (-\Delta + m^2)\]

(observe that \(L_0 = (-\Delta)\)). The main reason to work with \(L^s_m := (-\Delta + m^2)^s\) and not \(R^s_m := (-\Delta + m^2)^s - m^{2s}\) is that the composition law becomes simpler in the case of \(L^s_m\), namely

\[L^s_m(L^s_m) = L^{2s}_m \quad \text{unlike} \quad R^s_m(R^s_m) = R^{2s}_m - 2m^{2s}R^s_m.\]

Along the paper we will use a fairly standard notation. We will just skip the variables \((t, x)\) of the functions in many of the instances e.g., we will sometimes use \(u\) instead of \(u(t, x)\). The complete expression will be used when relevant.

2. The fractional relativistic operator. Definitions and properties

There are various equivalent definitions for \(L^s_m\), we state two of them below. The proof of the equivalence is given in Appendix \(A\). We will always consider real valued functions to avoid complex conjugates. We refer the reader to \([32, 35]\) for more information about fractional powers of relativistic operators and Bessel potentials.

1. **Definition using the Fourier transform.** For a function \(f\) in the Schwartz class \(S\) we can define its Fourier transform as

\[\hat{f}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(x)e^{-ix\cdot\xi}dx, \quad \xi \in \mathbb{R}^N.\]

The inversion formula is given, for \(f \in S\), by

\[F^{-1}(f)(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(\xi)e^{-ix\cdot\xi}d\xi, \quad x \in \mathbb{R}^N.\]

Let \(0 < s < 1, \ m \geq 0\) and \(f \in S\). The operator \(L^s_m(f)\) is defined as a pseudo-differential operator

\[(2.1) \quad \widehat{L^s_m f}(\xi) = (|\xi|^2 + m^2)^s\hat{f}(\xi) = (\xi \cdot \xi + m^2)^s\hat{f}(\xi), \quad \xi \in \mathbb{R}^N.\]

2. **Definition via a singular integral.** We first introduce some well known facts about the so called modified Bessel functions and Macdonald’s functions, that will be useful later. Let \(I_\nu(z)\) be the modified Bessel function of first kind given by the formula (see \([24\,\text{Chapter 5, Section 5.7}]\))

\[(2.2) \quad I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad |z| < \infty, \ |\arg z| < \pi\]

and let \(K_\nu\) be the Macdonald’s function of order \(\nu\) defined by (see also \([23\,\text{Chapter 5, Section 5.7}]\))

\[(2.3) \quad K_\nu(z) = \frac{\pi}{2} \frac{I_\nu(z) - I_{-\nu}(z)}{\sin \nu \pi}, \quad |\arg z| < \pi, \quad \nu \neq 0, \pm 1, \pm 2, \ldots\]

and, for integral \(\nu = n\), \(K_n(z) = \lim_{\nu \to n} K_\nu(z)\), \(n = 0, \pm 1, \pm 2, \ldots\) From \((2.2)\) and \((2.3)\) it is clear that

\[(2.4) \quad K_\nu(z) \sim \frac{\Gamma(\nu)}{2} \left(\frac{z}{2}\right)^{-\nu}, \quad (\Re \nu > 0) \quad \text{as} \ z \to 0^+.\]
Moreover, it is well known (see [23] Chapter 5, Section 5.11) that

\begin{equation}
K_\nu(z) = C e^{-z} z^{-1/2} + \tilde{R}_\nu(z), \quad |\tilde{R}_\nu(z)| \leq C e^{-z} z^{-3/2}, \quad |\arg z| \leq \pi - \delta.
\end{equation}

We have the integral representation for the Macdonald’s functions, also called Sommerfeld integral (see for instance [26] p. 407 or [23] Chapter 5, (5.10.25)),

\begin{equation}
K_\nu(z) = 2^{-\nu-1} z^\nu \int_0^\infty e^{-(t+\frac{1}{4}t^2)} t^{-1-\nu} dt.
\end{equation}

Let \( s \in (0,1) \), \( m > 0 \) and \( f \) be a real function with suitable decay at infinity, for instance \( f \in C_0^\infty(\mathbb{R}^N) \). Then \( L_m^s(f) \) has a pointwise representation as

\begin{equation}
L_m^s f(x) = C_{N,s} m^\frac{N+2s}{2} \text{ P. V. } \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x-y|^\frac{N+2s}{2}} K_{N+2s} m|x-y| dy + m^{2s} f(x), \quad \forall x \in \mathbb{R}^N,
\end{equation}

where \( C_{N,s} \) is a normalization positive constant given by

\begin{equation}
C_{N,s} = -\frac{2^{1+s-N/2}}{\pi^N \Gamma(-s)}.
\end{equation}

Following [12], we define the scalar product:

\[ \langle f, g \rangle_{\mathbb{H}_m^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \hat{f}(\xi) \overline{\hat{g}(\xi)} (|\xi|^2 + m^2)^s d\xi \]

\[ = \frac{C_{N,s}}{2} m\frac{N+2s}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(f(x) - f(y))(g(x) - g(y))}{|x-y|^\frac{N+2s}{2}} K_{N+2s} m|x-y| dy dx \]

\[ + m^{2s} \int_{\mathbb{R}^N} f(x) g(x) dx. \]

The corresponding norm is

\[ \|f\|_{\mathbb{H}_m^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 (|\xi|^2 + m^2)^s d\xi \]

\[ = \frac{C_{N,s}}{2} m\frac{N+2s}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(f(x) - f(y))^2}{|x-y|^\frac{N+2s}{2}} K_{N+2s} m|x-y| dy dx + m^{2s} \|f\|_{L^2(\mathbb{R}^N)}^2. \]

Then we define \( \mathbb{H}_m^s(\mathbb{R}^N) \) as the completion of \( C_0^\infty \) with the norm \( \| \cdot \|_{\mathbb{H}_m^s(\mathbb{R}^N)} \). If \( m > 0 \), \( \mathbb{H}_m^s(\mathbb{R}^N) \) coincides with the standard Sobolev space \( \mathbb{H}^s(\mathbb{R}^N) \). Thus, we can avoid writing the subscript \( m \). Notice that, by Plancherel’s theorem,

\[ \|L_m^s u\|_{L^2(\mathbb{R}^N)} = \|u\|_{\mathbb{H}_m^s(\mathbb{R}^N)}. \]

Thus \( L_m^s : \mathbb{H}^s(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N) \) is a bounded linear operator.

2.1. Leibniz rule and pointwise estimates. Consider the operator

\[ H_m^s(f,g) := L_m^s(fg) - f L_m^s(g) - g L_m^s(f). \]

Indeed, \( H_m^s \) is the remainder arising in the fractional Leibniz rule associated to our operator \( L_m^s \). Moreover, \( 2H_m^s(f,g) \) is known in the literature as carré du champ operator, see definition and properties in [3] Subsection 1.4.2. Most of the properties we use are proved in [3] for a general Lévy process having an infinitesimal generator \( L \). All the necessary information is stated in the proposition below, whose proof is just a consequence of the symmetry of the kernel, so we omit the details.

**Proposition 2.1.** Let \( 0 < s < 1 \). For \( f, g \in \mathbb{H}^s(\mathbb{R}^N) \) we have, for all \( x \in \mathbb{R}^N \),

\[ H_m^s(f,g)(x) = -C_{N,s} m^{\frac{N+2s}{2}} \int_{\mathbb{R}^N} \frac{(f(x) - f(y))(g(x) - g(y))}{|x-y|^{\frac{N+2s}{2}}} K_{N+2s} m|x-y| dy - m^{2s} f(x) g(x)\]
where \( C_{N,s} \) is as in \((2.8)\). In particular,
\[
H_m^s(f, f)(x) = -C_{N,s}m^{\frac{N+2s}{2}} \int_{\mathbb{R}^N} \frac{(f(x) - f(y))^2}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x-y|) \, dy - m^{2s}f^2(x) \leq 0.
\]

Moreover,
\[
L_m^s(f^2)(x) = 2f(x)L_m^s(f)(x) + H_m^s(f, f)(x), \quad L_m^s(f^2)(x) \leq 2fL_m^s(f)(x).
\]

2.2. Construction of unbounded eigenfunctions. In this subsection we will construct a special family of eigenfunctions of the operator \( L_m^s \). First we prove some integral formulas involving the Bessel functions.

Lemma 2.2. Let \( N \geq 1, s \in (0, 1), \lambda \in \mathbb{R}^N \) with \(|\lambda| < 1\). Then
\[
C_{N,s} \int_{\mathbb{R}^N} \frac{1-e^{\lambda \cdot z}}{|z|^s} K_{\frac{N+2s}{2}}(|z|) \, dz = (1 - |\lambda|^2)^s - 1,
\]
where \( C_{N,s} \) is given by \((2.8)\).

Moreover, when \( N - 2s < 1 \) and \(|\lambda| = 1\), the following also holds
\[
C_{N,s} \int_{\mathbb{R}^N} \frac{1-e^{\lambda \cdot z}}{|z|^s} K_{\frac{N+2s}{2}}(|z|) \, dz = -1.
\]

Proof. Observe that the integrals are well defined (this can be checked by using the asymptotics of the Bessel function \((2.4)\) and \((2.5)\)). The restriction for \( N \) and \( s \) for the second identity comes from the integrability of the integral near the origin.

The identities follow by using the integral representation of the Bessel function \((2.6)\) and the identity
\[
\gamma^s = \frac{1}{\Gamma(-s)} \int_0^{\infty} (e^{-\gamma t} - 1) \frac{dt}{t^{1+s}}, \quad \gamma > 0, \quad 0 < s < 1.
\]

For the proof of \((2.10)\) observe that
\[
C_{N,s} \int_{\mathbb{R}^N} \frac{1-e^{\lambda \cdot z}}{|z|^s} K_{\frac{N+2s}{2}}(|z|) \, dz = C_{N,s}2^{-\frac{N+2s}{2} - 1} \int_{\mathbb{R}^N} (1 - e^{\lambda \cdot z}) \int_0^{\infty} e^{-\frac{|y|^2}{4t}} \frac{dt}{t^{\frac{N+2s}{2}+1}} \, dz
\]
\[
= C_{N,s}2^{-\frac{N+2s}{2} - 1} \int_0^{\infty} \int_{\mathbb{R}^N} (1 - e^{\lambda \cdot z}) e^{-\frac{|y|^2}{4t}} \, dz \, dt\int_0^{\infty} e^{-\frac{|y|^2}{4t}} \, dy
\]
\[
= C_{N,s}2^{-\frac{N+2s}{2} - 1} \int_0^{\infty} \int_{\mathbb{R}^N} (1 - e^{\lambda \cdot z}) e^{-\frac{|y|^2}{4t}} \, dz \, dt\int_0^{\infty} e^{-\frac{|y|^2}{4t}} \, dy
\]
\[
= C_{N,s}2^{-\frac{N+2s}{2} - 1} \sqrt{\pi} \int_{\mathbb{R}^N} (1 - e^{\lambda \cdot z}) e^{-\frac{|y|^2}{4t}} \, dy
\]
\[
= C_{N,s}2^{-\frac{N+2s}{2} - 1} \sqrt{\pi} \int_{\mathbb{R}^N} (1 - e^{\lambda \cdot z}) e^{-\frac{|y|^2}{4t}} \, dy
\]
\[
= C_{N,s}2^{-\frac{N+2s}{2} - 1} \sqrt{\pi} \Gamma(-s) \left(1 - (1 - |\lambda|^2)^s\right) = (1 - |\lambda|^2)^s - 1.
\]

For \((2.11)\) the proof is the same, except for the last integral is \( \int_0^{\infty} (e^{-t} - 1) \frac{dt}{t^{s+1}} \), which equals \( \Gamma(-s) \) according to \((2.12)\). \(\square\)
Proposition 2.3. Let $N \geq 1$, $s \in [0, 1]$, $\lambda \in \mathbb{R}^N$ with $|\lambda| < m$. Then
\begin{equation}
L_m^s e^{\lambda x} = (-|\lambda|^2 + m^2)^s e^{\lambda x}, \quad \text{a.e. } x \in \mathbb{R}^N.
\tag{2.13}
\end{equation}
Moreover, when $N - 2s < 1$ and $|\lambda| = m$ then
\begin{equation}
L_m^s e^{\lambda x} = 0, \quad \text{a.e. } x \in \mathbb{R}^N.
\end{equation}

Proof. First, observe that the cases $s = 0$ and $s = 1$ follow trivially. Let $N \geq 1$, $s \in (0, 1)$ and $\lambda \in \mathbb{R}^N$ as in the hypothesis. Then for $x \in \mathbb{R}^N$
\begin{equation}
L_m^s(e^{\lambda(\cdot)})(x) = m^{2s} e^{\lambda x} = C_{N,s} m^{2s} e^{\lambda x} \int_{\mathbb{R}^N} \frac{1 - e^{-\frac{2}{m} \lambda z}}{|z|^{N+2s}} K_{N+2s}(|z|) \, dz.
\end{equation}
The integral is well defined, as proved in Lemma 2.2 with $\lambda/m$ as the corresponding parameter. Now we use Lemma 2.2 with $\lambda/m \in \mathbb{R}^N$, $|\lambda|/m \leq 1$ and the result follows in each of the cases. 

Remark 2.4. The identity (2.13) cannot be extended to $|\lambda| > m$ because the function $C \ni z \to (z^2 + m^2)^s$ is not well defined in \{it : $t > m$\}.

3. The heat equation for the fractional relativistic operator

We devote this section to the linear heat equation
\begin{equation}
\begin{cases}
\partial_t u(t,x) + (-\Delta + m^2)^s u(t,x) = 0, & x \in \mathbb{R}^N, \ t > 0, \\
u(0,x) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}
\tag{3.1}
\end{equation}

3.1. Fundamental solution. The fundamental solution $K^s_t$ for the heat equation involving $L_m^s$ is defined via the Fourier transform as
\begin{equation}
\hat{K}^s_t(\xi) = e^{-t(|\xi|^2 + m^2)^s}.
\end{equation}
This will correspond to the probability density function of the associated stable relativistic process. Estimates for the fundamental solution are well-known and can be found in [37, Subsection 6.4] (see also [6, Theorem 1.2] and [7, Theorem 4.1]). For our purposes, we will emphasize that $K^s_t(x)$ is a smooth function for $t > 0$ and $x \in \mathbb{R}^N$, since its Fourier transform decays exponentially. Moreover, $K^s_t$ has an exponential decay in $|x|$ for small times:
\begin{equation}
K^s_t(x) \sim c(t)|x|^{-N-2s} e^{-\tilde{c}_m|x|}, \quad x \in \mathbb{R}^N,
\end{equation}
with a positive constant $\tilde{c}$ independent of time.

3.1.1. Integral representation. The following subordination formula is shown in [31] (7), see also [18],
\begin{equation}
K^s_t(x) = \int_0^\infty \Theta^s_t(\rho) e^{-m^2 \rho} \frac{e^{-\frac{|x|^2}{4\rho}}}{(4\pi \rho)^{N/2}} \, d\rho.
\end{equation}
Here, $\Theta^s_t(\rho)$, $\rho > 0$, is the density function of the $s$-stable process whose Laplace transform is $e^{-t \lambda^s}$. In the case $s = 1/2$, $K^{1/2}_t$ has an explicit expression, as stated in the Introduction. We compute the following weighted $L^1$ norm of $K^s_t$.

Lemma 3.1. Let $0 < s < 1$. We have, for all $|\lambda| \leq m$,
\begin{equation}
\|e^{\lambda(\cdot)} K^s_t(\cdot)\|_{L^1(\mathbb{R}^N)} = e^{-(m^2 - |\lambda|^2)^s t}, \quad t > 0.
\tag{3.3}
\end{equation}

Proof. In [31] p. 3], Ryznar defines the probability density function
\begin{equation}
\Theta^s_t(\rho, m) = e^{-m^2 \rho + m^2 \rho^s} \Theta^s_t(\rho), \quad \rho > 0,
\end{equation}
so in particular this means that
\begin{equation}
\int_0^\infty e^{-m^2 \rho} \Theta^s_t(\rho) \, d\rho = e^{-t m^{2s}}.
\end{equation}
Now, by (3.2) and Fubini’s Theorem,
\begin{equation}
\int_{\mathbb{R}^N} e^{\lambda x} \int_{0}^{\infty} \Theta_t^s(\rho) e^{-m^2 \rho} \frac{e^{-|x|^2}}{4\pi \rho} d\rho dx = \int_{0}^{\infty} \Theta_t^s(\rho) e^{-m^2 \rho (4\pi \rho)^{-N/2}} \int_{\mathbb{R}} e^{\lambda x} e^{-\frac{|x|^2}{4\rho}} dx d\rho.
\end{equation}

Observe that
\begin{equation}
\int_{\mathbb{R}^N} e^{\lambda x} e^{-\frac{|x|^2}{4\rho}} dx = \prod_{i=1}^{N} \int_{\mathbb{R}} e^{\lambda_i x_i} e^{-\frac{x_i^2}{4\rho}} dx_i = e^{\lambda^2 \rho} \prod_{i=1}^{N} \int_{\mathbb{R}} e^{-\left(\frac{x_i^2}{2\sqrt{\pi} \lambda_i \sqrt{\rho}}\right)^2} dx_i = e^{\lambda^2 \rho} 2^{N} (\pi \rho)^{N/2}.
\end{equation}

Therefore, (3.5) equals
\begin{equation}
\int_{0}^{\infty} \Theta_t^s(\rho) e^{-\rho (m^2 - |\lambda|^2)} d\rho = e^{-t(m^2 - |\lambda|^2)^2},
\end{equation}
where the equality follows from (3.4).

3.2. **Energy estimates.** Let \( u \) be a solution to Problem (3.1) with \( u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \). Thus \( u \) is obtained directly from the fundamental solution and the data
\begin{equation}
\begin{aligned}
&\lambda/ \quad \text{(Decay of total mass)} \quad \text{We have} \\
&\int_{\mathbb{R}^N} u(t, x) \, dx = e^{-m^2 t} \int_{\mathbb{R}^N} u_0(x) \, dx, \quad t > 0.
\end{aligned}
\end{equation}

- **(Energy estimate)** For all \( 0 < t < T \) we have
\begin{equation}
2 \int_{0}^{t} \int_{\mathbb{R}^N} |L_m^{s/2} u(\tau, x)|^2 \, d\tau dx + \int_{\mathbb{R}^N} u^2(t, x) \, dx = \int_{\mathbb{R}^N} u_0^2(x) \, dx.
\end{equation}

- **(Decay of weighted \( L^2 \) norm)** Assume \( u_0 \in L^2(\mathbb{R}^N) \). Then for all \( 0 < t \) and \( |\lambda| \leq 2m \) we have
\begin{equation}
\int_{\mathbb{R}^N} u^2(t, x) e^{\lambda x} \, dx \leq e^{-t(m^2 - |\lambda|^2/4)t + (\int_{\mathbb{R}^N} u_0^2(x) e^{\lambda x} \, dx\). \quad t \geq 0.
\end{equation}

**Proof.** The first identity follows from (1.8) and mass conservation for Problem (1.7). On the other hand, (3.8) follows from the fact that, formally,
\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^N} u(t, x) \, dx = -2 \int_{\mathbb{R}^N} uL_m^{s} u(t, x) \, dx = -2 \int_{\mathbb{R}^N} |L_m^{s/2} u(t, x)|^2 \, dx.
\end{equation}

Notice that the identities in (3.7) and (3.8) are typical energy estimates for diffusion equations. For the inequality (3.9) we proceed as follows. Let \( \lambda \in \mathbb{R}^N \) with \( |\lambda| \leq 2m \). Then, by using the representation (3.6) and (3.3) with \( \lambda/2 \), we derive that
\begin{equation}
\int_{\mathbb{R}^N} u^2(t, x) e^{\lambda x} \, dx = \int_{\mathbb{R}^N} e^{\lambda x} ((K_t^s \ast u_0)(x))^2 \, dx
= \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} e^{\lambda/2 (x-y)} K_t^s(x-y) e^{\lambda/2 y} u_0(y) \, dy \right)^2 \, dx
= ||e^{\lambda/2 \cdot} K_t^s(\cdot) \ast (e^{\lambda/2 \cdot} u_0(\cdot))||^2_{L^2(\mathbb{R}^N)}
\leq ||e^{\lambda/2 \cdot} K_t^s(\cdot)||^2_{L^1(\mathbb{R}^N)} ||(e^{\lambda/2 \cdot} u_0(\cdot))||^2_{L^2(\mathbb{R}^N)}
= e^{-(m^2 - |\lambda|^2/4)t} \ ||(e^{\lambda/2 \cdot} u_0(\cdot))||_{L^2(\mathbb{R}^N)},
\end{equation}
3.3. Construction of separate variable solutions. With the tools we have so far, it is easy to construct explicit separate variables solutions \( \omega : (0, T) \times \mathbb{R}^N \to \mathbb{R} \) to the equation

\[
\partial_t \omega(t, x) = -L^s_m \omega(t, x), \quad x \in \mathbb{R}^N, \ t > 0.
\]

We are interested in spatial increasing solutions. By Proposition 2.3 we obtain that

\[
\omega_\lambda(t, x) = e^{-(-|\lambda|^2 + m^2)t} e^{\lambda \cdot x}, \quad x \in \mathbb{R}^N, \ t > 0,
\]

verifies the equation (3.10) for every \( \lambda \in \mathbb{R}^N \) with \(|\lambda| < m\). Moreover

\[
L^s_m \omega_\lambda(t, x) = (-|\lambda|^2 + m^2)^s \omega(t, x), \quad x \in \mathbb{R}^N, \ t > 0.
\]

3.4. Log-convexity of the weighted functional for the linear heat equation. Let \( u \) be the solution to Problem (3.1) with \( u_0 \in L^2(\mathbb{R}^N) \) and let \( \mathcal{H}(t) := \int_{\mathbb{R}^N} e^{\lambda \cdot x} u^2(t, x) \, dx \), which is well defined for \(|\lambda| \leq 2m\) according to (3.9). Moreover,

\[
\mathcal{H}(t) = \int_{\mathbb{R}^N} |e^{\lambda/2 \cdot x} u(t, x)|^2 \, dx = \int_{\mathbb{R}^N} |\hat{u}(\xi + i\frac{\lambda}{2})|^2 \, d\xi = \int_{\mathbb{R}^N} e^{-2\int((\xi + \frac{i\lambda}{2})^2 + m^2)|\hat{u}_0(\xi + \frac{i\lambda}{2})|^2} \, d\xi.
\]

**Theorem 3.3.** Let \( 0 < s < 1 \). The functional \( \mathcal{H} \) is logarithmically convex. In particular

\[
\mathcal{H}(t) \leq \mathcal{H}(0)^{1-t} \mathcal{H}(1)^t, \quad t \in [0, 1].
\]

**Proof.** Starting from (3.6), we use the Fourier representation of the solution \( u \). Then, the functional \( \mathcal{H}(t) \) can be written as follows

\[
\mathcal{H}(t) = \|e^{\lambda/2 \cdot (\cdot)} (K_t^s * u_0)(\cdot)\|^2_{L^2(\mathbb{R}^N)} = \left\| e^{-t((\cdot + \frac{i\lambda}{2})^2 + m^2)^s } \hat{u}_0(\cdot + \frac{i\lambda}{2}) \right\|^2_{L^2(\mathbb{R}^N)}.
\]

Since

\[
(\log(\mathcal{H}))'' = \frac{\hat{\mathcal{H}}(t) \mathcal{H}(t) - \hat{\mathcal{H}}(t)^2}{(\mathcal{H}(t))^2}
\]

we only need to check that the numerator is positive. Indeed, by using \( \frac{d}{dt} K_t^s(\xi) = (\xi \cdot \xi + m^2)^s e^{-t(\xi \cdot \xi + m^2)^s} \), we have

\[
(\hat{\mathcal{H}}(t))^2 = \left\| \left( \cdot + \frac{i\lambda}{2} \right)^2 + m^2 \right\|_{L^2(\mathbb{R}^N)}^s \cdot \left\| e^{-(\cdot + \frac{i\lambda}{2})^2 + m^2)^s } \hat{u}_0(\cdot + \frac{i\lambda}{2}) \right\|_{L^2(\mathbb{R}^N)}^2
\]

\[
\leq \left\| \left( \cdot + \frac{i\lambda}{2} \right)^2 + m^2 \right\|_{L^2(\mathbb{R}^N)}^{2s} \cdot \left\| e^{-t((\cdot + \frac{i\lambda}{2})^2 + m^2)^s } \hat{u}_0(\cdot + \frac{i\lambda}{2}) \right\|_{L^2(\mathbb{R}^N)}^2
\]

\[
= \hat{\mathcal{H}}(t) \mathcal{H}(t)
\]

and we conclude the proof. \( \square \)

**Remark 3.4.** The logarithmic convexity is a strong tool that might lead to an uncertainty principle result for the corresponding equation, like in [9]. However, the method developed in [9] in order to prove uncertainty principles does not work here: the reason is that the decay of \( u_0 \) given by \( \mathcal{H}(0) < \infty \) does not lead to better decay for \( u(t, x), \ t > 0 \). This can be immediately seen from the definition \( \mathcal{H}(t) := \int_{\mathbb{R}^N} e^{\lambda \cdot x} u^2(t, x) \, dx \), where the space decay is always the same (\( \lambda \) is constant). In particular, if we look at the fundamental solution, it always has the same spatial decay \( e^{-m|x|} \), thus it does not improve with time. This is in big contrast to what happens with self-similar processes (for instance, as in [9] dealing with the classical heat equation).
4. The heat equation with potential

We devote this section to the study of (any sign) solutions to Problem (1.1). We assume that $V \in L^\infty([0, \infty) \times \mathbb{R}^N)$.

**Definition 4.1.** Let $u_0 \in H^{2s}(\mathbb{R}^N)$ and $T > 0$ or $T = \infty$. A *mild solution* of Problem (1.1) is a function $u \in C([0, T] : H^{2s}(\mathbb{R}^N))$ which satisfies, for a.e. $(t, x) \in [0, T] \times \mathbb{R}^N$,

$$u(t, x) = (K_t^s * u_0)(x) + \int_0^t (K_{t-\tau}^s * (V(\tau, \cdot)u(\tau, \cdot)))(x)d\tau.$$ 

The diffusion operator $L_m^s : H^{2s}(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a bounded linear operator. The right hand side $F : [0, \infty) \times H^{2s}(\mathbb{R}^N) \to H^{2s}(\mathbb{R}^N)$ with $F(t, x) := V(t, x)u(t, x)$ is Lipschitz for $V \in L^\infty([0, \infty) \times \mathbb{R}^N)$. The hypothesis of [27, Thm. 1.2, Ch. 6.1] is satisfied, thus there exists a mild solution to the initial value problem (1.1) for some $T > 0$. Moreover, given the assumption on the potential, $u$ is in fact a strong solution: $u_t \in L^1([0, T] : H^{2s}(\mathbb{R}^N))$, see [27, Thm. 1.6, Ch. 6.1].

Since we will work with exponentially decaying solutions, we prove here that, given $u_0 \in H^{2s}(\mathbb{R}^N) \cap L^2(e^{\lambda t}dx)$, then, the corresponding strong solution $u$ to Problem (1.1) satisfies $u(t, \cdot) \in L^2(e^{\lambda t}dx)$ for all $t > 0$.

**Proposition 4.2.** Let $u_0 \in H^{2s}_m(\mathbb{R}^N) \cap L^2(e^{\lambda t}dx)$. Then, the corresponding strong solution $u$ to Problem (1.1) satisfies $u(t, \cdot) \in L^2(e^{\lambda t}dx)$ for all $t \in [0, T]$.

**Proof.** The $L^2$ energy estimate can be proved for $0 \leq t \leq T$, integrating by parts:

$$\|u(t, \cdot)\|_{L^2(\mathbb{R}^N)} \leq e^{\int_0^t \|V(t, \cdot)\|_{L^\infty([0, \infty) \times \mathbb{R}^N)}dt} \|u_0\|_{L^2(\mathbb{R}^N)}.$$

Using (3.9), we have

$$\|e^{\lambda t} u(t, \cdot)\|_{L^2(\mathbb{R}^N)} \leq \left( \int_0^t \|e^{\lambda \tau} u(\tau, \cdot)\|^2_{L^2(\mathbb{R}^N)} d\tau \right)^{1/2} \leq \left( \int_0^t \|e^{\lambda \tau} u(\tau, \cdot)\|^2_{L^2(\mathbb{R}^N)} + \int_0^t \langle u(\tau, \cdot), u(\tau, \cdot) \rangle d\tau \right)^{1/2} \leq e^{\lambda t} \|u_0\|_{L^2(\mathbb{R}^N)}$$

The result follows.

**Remark 4.3.** Observe that, in the proof of Proposition 4.2, it is enough to assume only $u_0 \in L^2(\mathbb{R}^N)$. Nevertheless, since we will need $u_0 \in L^2(e^{\lambda t}dx)$ later, we have decided to state the more restrictive condition in the proposition.

4.1. Backward unique continuation for the heat equation with potential.

**Theorem 4.4** (Backward unique continuation). Let $N \geq 1$, $s \in (0, 1)$ and $m \geq 0$. Let $u$ be a solution to Problem (1.1) with initial data $u_0 \in L^2(\mathbb{R}^N)$ and $V \in L^\infty(0, T \times \mathbb{R}^N)$. Assume that $u(T, \cdot) = 0$ for some time $T > 0$. Then $u \equiv 0$.

**Proof.** Let $\mathcal{H}(t) := \int_{\mathbb{R}^N} u^2(t, x) dx$. In case the potential is $V = 0$ we have

$$\mathcal{H}(t) = \langle u, u_t \rangle + \langle u_t, u \rangle = 2 \text{Re} \langle u, L_m^s u \rangle = 2 \|L_m^s u\|^2_{L^2(\mathbb{R}^N)}$$
and
\[ \hat{H}(t) = 2 \text{Re}(u, u_{tt}) + 2 \langle u_t, u_t \rangle = 4\|L_m^s u\|_{L^2(\mathbb{R}^N)}^2. \]
Then \((\hat{H}(t))^2 \leq 4\langle u, u \rangle \cdot \langle L_m^s u, L_m^s u \rangle\) and thus \(H(t)\) is logarithmically convex. Hence
\[ H(t) \leq H(0)^\theta H(T)^{1-\theta}, \quad \theta \in [0, 1]. \]
When the potential is non trivial, the functional \(H\) is still logarithmically convex, since the operator \(L_m^s\) is symmetric. According to [9, Lemma 2, p. 6] there exists a constant \(N\) such that
\[ H(t) \leq e^{N(|V|_{\infty} + \|V\|_{\infty})}H(0)^\theta H(1)^{1-t}, \quad t \in [0, 1]. \]
Thus, up to scaling in time, if \(u(T) \equiv 0\) then \(u \equiv 0\). \(\square\)

### 5. Carleman Inequality for the Parabolic Operator with Linear Exponential Weight

This section is devoted to the study of convexity estimates for an exponential weighted norm of solutions \(u : (0, T] \times \mathbb{R}^N \to \mathbb{R}\) to the initial value problem
\begin{align}
\begin{cases}
    u_t(t, x) + (-\Delta + m^2)^s u(t, x) = F(t, x) & x \in \mathbb{R}^N, \ t > 0, \\
    u(0, x) = u_0(x), & x \in \mathbb{R}^N
\end{cases}
\end{align}
for some \(T > 0\). In particular, when \(F = Vu\) we are reduced to the Problem (1.1). We consider the functional
\begin{align}
    H(t) := \int_{\mathbb{R}^N} \omega(t, x) u^2(t, x) \, dx, \quad \omega(t, x) = e^{A t + \lambda x}, \quad t \in [0, T],
\end{align}
for \(A \in \mathbb{R}, \lambda \in \mathbb{R}^N\). Note that, by Proposition 4.2, we have \(H(t) < \infty\) for \(|\lambda| \leq 2m\).

#### 5.1. Persistence of the spatial decay: monotonicity of the energy functional.

In what follows we prove that if the initial data decays at least exponentially fast in space, then the solution \(u(t, x)\) will have a similar decay at every positive time \(t > 0\). This will be a consequence of the monotonicity of the functional \(H(t)\).

**Proposition 5.1.** Let \(0 < s < 1, m > 0\) and let \(u_0 \in H^{2s}(\mathbb{R}^N) \cap L^2(e^{\lambda x} \, dx)\) for some \(\lambda \in \mathbb{R}^N\) with \(|\lambda| \leq m\). Assume that \(F(t, x) \in L^2(0, T : L^2(e^{\lambda x} \, dx))\). Let \(u\) be a strong solution to the initial value problem (5.1). Let \(\omega(t, x)\) be as in (5.2). Then, for all \(t \in [0, T]\),
\begin{align}
    \int_{\mathbb{R}^N} \omega(t, x) u^2(t, x) \, dx + \int_0^t \int_{\mathbb{R}^N} \omega(t, x) (-H_m^s(u, u)) \, dx \, dt \\
    \leq e^{(A - (|\lambda|^2 + m^2)s)t} \int_{\mathbb{R}^N} \omega(t, x) u_0^2(x) \, dx + e^{(A - (|\lambda|^2 + m^2)s)t} \int_0^t \int_{\mathbb{R}^N} \omega(t, x) F^2(t, x) \, dx \, dt.
\end{align}

**Proof.** Let \(H(t)\) be defined as in (5.2). We have that
\begin{align}
    \dot{H}(t) &= \int_{\mathbb{R}^N} (\omega_t - L_m^s(\omega)) u^2 \, dx + \int_{\mathbb{R}^N} \omega H_m^s(u, u) \, dx + 2 \int_{\mathbb{R}^N} \omega u F \, dx \\
    &= [A - (|\lambda|^2 + m^2)s] \int_{\mathbb{R}^N} \omega u^2 \, dx + \int_{\mathbb{R}^N} \omega H_m^s(u, u) \, dx + 2 \int_{\mathbb{R}^N} \omega u F \, dx.
\end{align}
Let \(a := [A - (|\lambda|^2 + m^2)s]\). For sufficiently negative \(A\), the coefficient \(a\) will be negative and thus we could ignore the term \(a \int_{\mathbb{R}^N} \omega u^2 \, dx\). However, we keep this term to avoid imposing conditions on the parameter \(A\) in this proposition. Thus we have
\begin{align}
    \dot{H}(t) - aH(t) &\leq \int_{\mathbb{R}^N} \omega H_m^s(u, u) \, dx + \int_{\mathbb{R}^N} \omega F^2 \, dx,
\end{align}
that can be rewritten as
\begin{align}
    \frac{d}{dt} e^{-at} H(t) &\leq e^{-at} \int_{\mathbb{R}^N} \omega H_m^s(u, u) \, dx + e^{-at} \int_{\mathbb{R}^N} \omega F^2 \, dx.
\end{align}
We integrate from \( t_1 \) to \( t_2 \) in time and therefore
\[
\mathcal{H}(t_2) + e^{at_2} \int_{t_1}^{t_2} e^{-at} \int_{\mathbb{R}^N} \omega(-H_m^s(u, u)) \, dx \, dt \leq e^{at_2-t_1} \mathcal{H}(t_1) + e^{at_2} \int_{t_1}^{t_2} e^{-at} \int_{\mathbb{R}^N} \omega F^2 \, dx \, dt.
\]
This implies that, for all \( 0 \leq t_1 < t_2 \leq 1 \),
\[
\mathcal{H}(t_2) + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \omega(-H_m^s(u, u)) \, dx \, dt \leq e^{at_2} \mathcal{H}(t_1) + e^{at_2} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \omega F^2 \, dx \, dt.
\]
(5.3)

The conclusion follows just by taking \( t_1 = 0 \) and \( t_2 = t \). □

\[ \text{Remark 5.2.} \] In particular, if we take \( t_2 = 1 \) in (5.3) and renaming \( t_1 \) into \( t \), we obtain that, for all \( t \in (0, 1) \),
\[
\mathcal{H}(1) + \int_{t}^{1} \int_{\mathbb{R}^N} \omega(-H_m^s(u, u)) \, dx \, dt \leq e^{a} \mathcal{H}(t) + e^{a} \int_{t}^{1} \int_{\mathbb{R}^N} \omega F^2 \, dx \, dt.
\]
(5.4)

5.2. Convexity arguments. In this subsection we will consider a similar weight as in Subsection 3.4, but with a correction in time needed in order to absorb the effects of the potential. We will prove a convexity result related to it, that will be the key point to get a Carleman inequality for \( \partial_t + L_m^s \) in Subsection 5.3. Note that the proof carried out in Subsection 3.4 is not valid anymore due to the presence of the potential.

Let \( \omega(t, x) \) be defined as in (5.2), where \( A \) is a constant to be chosen later. Let
\[
(5.5) \quad D(t) := \int_{\mathbb{R}^N} \omega(t) u^2 \, dx - 2 \int_{\mathbb{R}^N} \omega L_m^s u \, dx = \int_{\mathbb{R}^N} \omega(t) - L_m^s \omega \, u^2 \, dx + \int_{\mathbb{R}^N} \omega H_m^s(u, u) \, dx
\]
where the second equality follows easily from (2.9). Actually, (5.5) is a formal definition, for any \( \omega \). Recall also the definition of \( \mathcal{H}(t) \) in (5.2). We will prove the following.

**Proposition 5.3.** Let \( N \geq 1 \), \( s \in (0, 1/2) \), \( m > 0 \) and let \( u_0 \in H^s(\mathbb{R}^N) \cap L^2(e^{\lambda^2 \cdot x} \, dx) \) for some \( \lambda \in \mathbb{R}^N \) with \( |\lambda| \leq m \). Assume that \( F(t, x) \in L^2(0, T; L^2(e^{\lambda^2 \cdot x} \, dx)) \). Let \( u \) be a strong solution to the initial value problem (5.1) such that \( u^2 \in \text{Dom}(L_m^s) \). Let \( \omega(t, x) \) be as in (5.2). For \( A + m^{2s} \) sufficiently small (that is, \( |A| \) sufficiently large) satisfying
\[
(5.6) \quad \frac{1}{4}((-|\lambda|^2 + m^2)^s - A)^2 \geq C_2(N, s)m^{4s},
\]
where \( C_2(N, s) \) is a positive constant depending only on \( N \) and \( s \), we obtain the following lower bound
\[
(5.7) \quad \dot{D}(t) \geq \frac{3}{4}((-|\lambda|^2 + m^2)^s - A)^2 \mathcal{H}(t) - C_1(N, s) \int_{\mathbb{R}^N} \omega F^2 \, dx + 2 \int_{\mathbb{R}^N} \omega(u_t)^2 \, dx + (A + m^{2s}) \int_{\mathbb{R}^N} H_m^s(u, u) \, dx - \int_{\mathbb{R}^N} \omega H_m^s(u, u) \, dx.
\]
where \( C_1(N, s) \) is a positive constant depending only on \( N \) and \( s \).

**Proof.** First of all, direct calculations and Proposition 2.3 give
\[
(5.8) \quad w_t = A \omega, \quad \omega_t = A^2 \omega, \quad L_m^s \omega = (-|\lambda|^2 + m^2)^s \omega, \quad L_m^s (\omega_t) = (-|\lambda|^2 + m^2)^s A \omega,
\]
(5.9) \quad \quad L_m^{2s} w = (-|\lambda|^2 + m^2)^{2s} \omega,
and
\[
H_m^s(\omega, \omega) = ((-4|\lambda|^2 + m^2)^s - 2(-|\lambda|^2 + m^2)^s) \omega^2.
\]
Let \( \mathcal{H}(t) \) be defined as in (5.2). Then
\[
\mathcal{H}(t) = \int_{\mathbb{R}^N} \omega(t) u^2 \, dx + 2 \int_{\mathbb{R}^N} \omega uu_t \, dx = \int_{\mathbb{R}^N} (\omega(t) - L_m^s \omega) u^2 \, dx + \int_{\mathbb{R}^N} \omega H_m^s(u, u) \, dx + 2 \int_{\mathbb{R}^N} \omega u F \, dx.
\]
Thus
\[
(5.10) \quad \mathcal{H}(t) = D(t) + 2 \int_{\mathbb{R}^N} \omega u F \, dx.
\]
We focus on $D(t)$. We have, by using the relation (2.9) and after tedious computations,

\[(5.11) \quad \dot{D}(t) = \int_{\mathbb{R}^N} (\omega t - 2L_m^\omega)(\omega t) + 2 \int_{\mathbb{R}^N} \omega t H_m^\omega(u, u) + 2 \int_{\mathbb{R}^N} (\omega_t - L_m^\omega) \cdot uF dx \]

\[+ 2 \int_{\mathbb{R}^N} \omega(L_m^\omega - F)^2 dx - 2 \int_{\mathbb{R}^N} \omega F^2 dx - \int_{\mathbb{R}^N} \omega H_m^{2s}(u, u) dx - 2 \int_{\mathbb{R}^N} H_m^\omega(\omega, u) F dx.\]

In view of the expression of $\omega$ in (5.2), we have that, by (5.8) and (5.9),

\[(5.12) \quad \int_{\mathbb{R}^N} (\omega_t - 2L_m^\omega) + 2 \int_{\mathbb{R}^N} \omega(L_m^\omega - F)^2 dx = ((-|\lambda|^2 + m^2)s - A)^2 \mathcal{H}(t)\]

and

\[(5.13) \quad 2 \int_{\mathbb{R}^N} \omega_t H_m^\omega(u, u) = 2A \int_{\mathbb{R}^N} \omega H_m^\omega(u, u) dx.\]

Concerning the term $2 \int_{\mathbb{R}^N} (\omega_t - L_m^\omega) \cdot uF dx$, by (5.8) and by applying arithmetic-geometric inequality (AM-GM inequality), we get

\[2 \left| \int_{\mathbb{R}^N} (\omega_t - L_m^\omega)u F dx \right| = 2 \int_{\mathbb{R}^N} (A - (-|\lambda|^2 + m^2)s)\omega u F dx \]

\[\leq \frac{1}{4}((-|\lambda|^2 + m^2)s - A)^2 \mathcal{H}(t) + 4 \int_{\mathbb{R}^N} \omega F^2 dx.\]

Let us see now how to estimate the term $-2 \int H_m^\omega(\omega, u) F$ in (5.11). By definition we have that

\[-2 \int_{\mathbb{R}^N} H_m^\omega(\omega, u) F dx \]

\[= 2C_{N,s}^m \int_{\mathbb{R}^N} f(x) \cdot \frac{e^{\lambda x} - e^{\lambda y}}{|x - y|^N} (u(t, x) - u(t, y)) K_N^{2s}(m|xy|) dy F dx + 2m^{2s} \int_{\mathbb{R}^N} F u \omega dx,\]

where the integrals $I_{|x-y|<1/m}$ and $I_{|x-y|>1/m}$ are determined by the splitting $\int_{\mathbb{R}^N} I_{|x-y|<1/m} + I_{|x-y|>1/m}$, respectively. The integral close to the origin is bounded as follows (we use the asymptotics of Macdonald’s function in (2.4) that involve constants depending on $s$ that do not blow up)

\[I_{|x-y|<1/m} \simeq c C_{N,s} \Gamma\left(\frac{N + 2s}{2}\right) 2^{N+2s} \frac{m^{N+2s}}{m^{N+2s}} \]

\[\times \int_{\mathbb{R}^N} \int_{|x-y|<1/m} e^{\lambda x} \frac{1 - e^{\lambda(y-x)}}{|x - y|^{N+2s}} (u(t, x) - u(t, y)) (m|xy|)^{-N+2s} dy F dx \]

\[= c C_{N,s} \Gamma\left(\frac{N + 2s}{2}\right) 2^{N+2s} \frac{1}{m^{N+2s}} \]

\[\times \int_{\mathbb{R}^N} \int_{|x-y|<1/m} e^{\lambda x} \frac{1 - e^{\lambda(y-x)}}{|x - y|^{N+2s}} (u(t, x) - u(t, y)) (m|xy|)^{-N+2s} dy F dx.\]

Using again the asymptotics for the Macdonald’s function, and applying Cauchy-Schwartz in the integral in the variable $y$, we get

\[|I_{|x-y|<1/m}| \leq c \left( \int_{|x|<1/m} \left( 1 - e^{\lambda z} \right)^2 |z|^{N+2s}_df(z) \right)^{1/2} \cdot C_{N,s} \Gamma\left(\frac{N + 2s}{2}\right) 2^{N+2s} \frac{m^{N+2s}}{4} \int_{\mathbb{R}^N} e^{\lambda x} |F| \]

\[\times \left( \int_{|x-y|<1/m} \frac{(u(t, x) - u(t, y))^2}{|x - y|^{N+2s}} K_N^{2s}(m|xy|) \frac{1}{\Gamma\left(\frac{N+2s}{2}\right)} 2^{N+2s+1} dy \right)^{1/2} dx \]
\[
\begin{align*}
&\leq c \left( \int_{|z|<1/m} \frac{(1-e^{\lambda z})^2}{|z|^{N+2s}} \, dz \right)^{1/2} \cdot C_{N,s} \Gamma \left( \frac{N+2s}{2} \right) \frac{2^{N+2s-1} m^{N+2s}}{2} \\
&\quad \times \frac{1}{(\Gamma(N+2s))^2} \int_{\mathbb{R}^N} e^{\lambda |x|} |F| \left(-H^s_m(u,u)m - \frac{1}{C_{N,s}} \right)^{1/2} \, dx \\
&= c \left( \int_{|z|<1/m} \frac{(1-e^{\lambda z})^2}{|z|^{N+2s}} \, dz \right)^{1/2} \cdot C_{N,s} \Gamma \left( \frac{N+2s}{2} \right) \frac{2^{N+2s-1}}{2} \\
&\quad \times \frac{1}{(\Gamma(N+2s))^2} \int_{\mathbb{R}^N} e^{\lambda |x|} |F| (-H^s_m(u,u))^{1/2} \, dx,
\end{align*}
\]
where we used Proposition 2.1. Let us estimate the quantity

\begin{equation}
(5.14) \quad c \left( \int_{|z|<1/m} \frac{(1-e^{\lambda z})^2}{|z|^{N+2s}} \, dz \right)^{1/2} \cdot C_{N,s} \Gamma \left( \frac{N+2s}{2} \right) \frac{2^{N+2s-1}}{2} \frac{1}{(\Gamma(N+2s))^2} \frac{1}{(C_{N,s})^{1/2}} \right)^2.
\end{equation}

We will prove that for $|\lambda| < m$, the positive constant (5.14) is bounded from above independently of $\lambda$ and $m$. Indeed, for $|\lambda| < m$, using the mean value theorem we obtain

\[
\int_{|z|<1/m} \frac{(1-e^{\lambda z})^2}{|z|^{N+2s}} \, dy \leq \int_{|z|<1/m} \frac{e^{2|\lambda| \frac{|\lambda|^2}{m} |z|^2}}{|z|^{N+2s}} \, dz = e^{2|\lambda| \frac{|\lambda|^2}{m} |\lambda|^2} \frac{1}{2m^{N+2s}} \omega_N \leq e^{2\frac{1}{2} \omega_N m^2}.
\]

We take into account that $C_{N,s}$ is given by the formula (2.8), thus the constant (5.14) above is bounded by

\[
c e^{2\frac{1}{4} \omega_N C_{N,s} \Gamma \left( \frac{N+2s}{2} \right)} \frac{2^{N+2s-2}}{2} = -c e^{\frac{1}{2} \omega_N \frac{4s}{2}} \frac{\Gamma(N+2s)}{\Gamma(-s)} \frac{1}{m^2} = C_1(N, s) m^2 s.
\]

Now we apply AM-GM in the variable $x$ to obtain

\begin{equation}
(5.15) \quad |I_{|x-y|<1/m}| \leq - \frac{m^2 s}{2} \int_{\mathbb{R}^N} e^{\lambda x} H^s_m(u,u) \, dx + C_1(N, s) \int_{\mathbb{R}^N} e^{\lambda x} F^2 \, dx.
\end{equation}

For the integral away from the origin we have

\[
I_{|x-y|>1/m} = C_{N,s} m^{\frac{N+2s}{2}} \int_{\mathbb{R}^N} \int_{|x-y|>1/m} e^{\lambda x} u(t,x) - e^{\lambda y} u(t,y) + e^{\lambda y} u(t,y) K_{N+2s}^{(m|x-y|)} \, dy \, dx
\]

Concerning $E_1$, we get

\[
E_1 = C_{N,s} m^{\frac{N+2s}{2}} \int_{\mathbb{R}^N} \int_{|x-y|>1/m} \frac{(e^{\lambda x} - e^{\lambda y}) u(t,x)}{|x-y|^{N+2s}} K_{N+2s}^{(m|x-y|)} \, dy \, dx
\]

Observe that

\[
\int_{|z|>1/m} \frac{1-e^{\lambda z}}{|z|^{N+2s}} K_{N+2s}^{(m|z|)} \, dz = m^{\frac{N+2s}{2}} \int_{|y|>1} \frac{1-e^{\lambda y}}{|y|^{N+2s}} K_{N+2s}^{(m|y|)} \, dy.
\]

This integral is finite as proved in Lemma 2.3. Indeed,

\[
\left| \int_{|z|>1/m} \frac{1-e^{\lambda z}}{|z|^{N+2s}} K_{N+2s}^{(m|z|)} \, dz \right| \leq \frac{m^{\frac{N+2s}{2}}}{2} \left( 1 - \left( \frac{|\lambda|}{m} \right)^2 \right)^s - 1 \leq 2m^{\frac{N+2s}{2}}.
\]
Thus, applying AM-GM inequality, we get

\begin{equation}
|E_1| \leq C_{N,s} m^{2s} \int_{\mathbb{R}^N} e^{\lambda x} |u| |F| \, dx \leq C_{N,s}^2 m^{4s} \int_{\mathbb{R}^N} e^{\lambda x} u^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} e^{\lambda x} F^2 \, dx.
\end{equation}

For \( E_2 \) we have

\begin{align*}
|E_2| &\leq C_{N,s} m^{\frac{N+2s}{2}} \int_{\mathbb{R}^N} \int_{|x-y|>1/m} \frac{(e^{\lambda x} - e^{\lambda y}) |u(t,y)|}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}} (m|x-y|) \, dy \, |F| \, dx \\
&= C_{N,s} m^{\frac{N+2s}{2}} \int_{\mathbb{R}^N} \int_{|x-y|>1/m} e^{\frac{1}{2} \lambda y} |u(t,y)| e^{\frac{1}{2} \lambda (x+y)} (e^{\lambda x} - e^{\lambda y}) K_{\frac{N+2s}{2}} (m|x-y|) \, dy \cdot e^{\frac{1}{2} \lambda x} |F| \, dx \\
&\leq C_{N,s} m^{\frac{N+2s}{2}} \left( \int_{\mathbb{R}^N} e^{\lambda x} F^2 \, dx \right)^{1/2} \times \left( \int_{\mathbb{R}^N} \int_{|x-y|>1/m} e^{\frac{1}{2} \lambda y} |u(t,y)| \frac{e^{\frac{1}{2} \lambda (x-y)} - e^{\frac{1}{2} \lambda (y-x)}}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}} (m|x-y|) \, dy \right)^2 \, dx \right)^{1/2} \\
&\leq C_{N,s} m^{\frac{N+2s}{2}} \left( \int_{\mathbb{R}^N} e^{\lambda x} F^2 \, dx \right)^{1/2} \cdot \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{\frac{1}{2} \lambda y} |u(t,y)| \cdot k(x-y) \, dy \right)^{1/2} \\
&\leq C_{N,s} m^{\frac{N+2s}{2}} \left( \int_{\mathbb{R}^N} e^{\lambda x} F^2 \, dx \right)^{1/2} \cdot \left( \int_{\mathbb{R}^N} e^{\lambda x} u^2 \, dx \right)^{1/2} \cdot \|k(x)\|_{L^1(\mathbb{R}^N)},
\end{align*}

where \( k(z) = \chi_{\{|z|>1/m\}} |z| \cdot e^{\frac{1}{2} \lambda z} - e^{-\frac{1}{2} \lambda z} K_{\frac{N+2s}{2}} (m|z|) \). Observe that if \(|\lambda| < m\) then

\[ \|k(x)\|_{L^1(\mathbb{R}^N)} \leq 2 \sqrt{\frac{\pi}{2}} m^{-1/2} \cdot \int_{\{|x|>1/m\}} \frac{1}{|x|^{\frac{N+2s}{2}}} e^{-\frac{m}{2}|x|} \, dx = \sqrt{\frac{\pi}{2}} m^{-\frac{N+2s}{2}} \frac{\Gamma \left( \frac{N-2s-1}{2} \right)}{m^{-\frac{N-2s}{2}}} \frac{\Gamma \left( \frac{N-2s}{2} \right)}{\Gamma \left( \frac{N-2s-1}{2} \right)}. \]

We use AM-GM to get

\begin{equation}
|E_2| \leq C_{N,s} \left( \sqrt{\frac{\pi}{2}} m^{-\frac{N-2s}{2}} \frac{\Gamma \left( \frac{N-2s-1}{2} \right)}{\Gamma \left( \frac{N-2s}{2} \right)} \right)^2 m^{4s} \int_{\mathbb{R}^N} e^{\lambda x} u^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} e^{\lambda x} F^2 \, dx.
\end{equation}

Thus using (5.15), (5.16) and (5.17) we conclude that

\begin{align*}
-2 \int H_m^s(\omega, u) F \, dx &\leq -m^{2s} \int_{\mathbb{R}^N} e^{At+\lambda x} H_m^s(u, u) \, dx + 8C_1(N,s) \int_{\mathbb{R}^N} e^{At+\lambda x} F^2 \, dx \\
&\quad + C_2(N,s) m^{4s} \int_{\mathbb{R}^N} e^{At+\lambda x} u^2 \, dx + \int_{\mathbb{R}^N} e^{At+\lambda x} F^2 \, dx \\
&\quad + C_3(N,s) m^{4s} \int_{\mathbb{R}^N} e^{At+\lambda x} u^2 \, dx + \int_{\mathbb{R}^N} e^{At+\lambda x} F^2 \, dx \\
&\quad + 4m^{4s} \int_{\mathbb{R}^N} e^{At+\lambda x} u^2 \, dx + \int_{\mathbb{R}^N} e^{At+\lambda x} F^2 \, dx,
\end{align*}

where \( C_1(N,s) \) is as above, and \( C_2(N,s) \) and \( C_3(N,s) \) are positive and depend only on \( N \) and \( s \). We rename the constants (i.e., below \( C_1(N,s) \) and \( C_2(N,s) \) mean different constants, but still positive and depending only on \( N \) and \( s \)) and we have

\begin{equation}
-2 \int H_m^s(\omega, u) F \, dx \leq -m^{2s} \int_{\mathbb{R}^N} e^{At+\lambda x} H_m^s(u, u) \, dx + C_1(N,s) \int_{\mathbb{R}^N} e^{At+\lambda x} F^2 \, dx + C_2(N,s) m^{4s} \int_{\mathbb{R}^N} e^{At+\lambda x} u^2 \, dx.
\end{equation}
Summing up, from (5.11), (5.12), (5.13) and (5.18) we have
\[
\dot{D}(t) \geq \frac{3}{4} \left( (-|\lambda|^2 + m^2)^s - A \right)^2 H(t) - C_1(N,s) \int_{\mathbb{R}^N} \omega F^2 \, dx + 2 \int_{\mathbb{R}^N} \omega (u_t)^2 \, dx
\]
\[
+ (A + m^{2s}) \int_{\mathbb{R}^N} H_s^m(u,u) \omega \, dx - \int_{\mathbb{R}^N} \omega H_{2s}^m(u,u) \, dx - C_2(N,s) m^{4s} \int_{\mathbb{R}^N} \omega u^2 \, dx.
\]
Thus if we take \(A + m^{2s} < 0\) sufficiently small (that is \(|A|\) sufficiently large) such that \(A\) satisfies (5.6), we get the conclusion.

Notice that since we want our energy terms to be positive, the presence of \(H_{2s}^m\) imposes the restriction \(s \in (0,1/2]\).

\(\square\)

5.3. Carleman inequality. Once we have Proposition 5.3 at our disposal, we will be able to prove Theorem 1.1 in the Introduction. We state the result here again, with a slightly different reformulation.

**Theorem 5.4.** Let \(N \geq 1, \ s \in (0,1/2), \ m > 0, \) and let \(u_0 \in \mathbb{H}^{2s}(\mathbb{R}^N) \cap L^2(e^{\lambda \cdot x} \, dx)\) for some \(\lambda \in \mathbb{R}^N\) with \(|\lambda| \leq m\). Assume that \(F(t,x) \in L^2(0,T : L^2(e^{\lambda \cdot x} \, dx))\). Let \(u\) be a strong solution to the initial value problem (5.1) and let \(\omega(t,x) = e^{At} e^{\lambda \cdot x}\). Then, the following inequality holds
\[
\frac{1}{2} \int_0^1 \mathcal{H}(t) \, dt + \frac{1}{2} (-|\lambda|^2 + m^2)^s - A) \int_0^1 (1 - t) \mathcal{H}(t) \, dt
\]
\[
+ \frac{1}{2} \int_0^1 t(1 - t) \left( 2 \int_{\mathbb{R}^N} \omega (u_t)^2 \, dx - \int_{\mathbb{R}^N} \omega H_{2s}^m(u,u) \, dx + (A + m^{2s}) \int_{\mathbb{R}^N} H_s^m(u,u) \omega \, dx \right) \, dt
\]
\[
\leq \frac{1}{2} \mathcal{H}(0) + \frac{1}{2} \mathcal{H}(1) + C_1(N,s) \int_0^1 \int_{\mathbb{R}^N} \omega ((\partial_t + L_s^m)(u))^2 \, dx \, dt,
\]
for \(A + m^{2s} < 0\) sufficiently small (that is \(|A|\) sufficiently large) satisfying
\[
(-|\lambda|^2 + m^2)^s - A) \geq C_2(N,s)m^{4s},
\]
where \(C_1(N,s), C_2(N,s)\) are positive constants depending only on \(N\) and \(s\).

**Proof.** We will consider the following tent function
\[
\eta(\tau) = \begin{cases} \tau, & 0 \leq \tau \leq t, \\ 0, & t \leq \tau \leq 1. \end{cases}
\]
Then \(\dot{\eta}\) is a decreasing step function
\[
\dot{\eta}(\tau) + \frac{1}{1 - t} = \begin{cases} \frac{1}{1 - \tau}, & 0 \leq \tau \leq t, \\ 0, & t \leq \tau \leq 1. \end{cases}
\]
Thus by denoting \(\delta\) the distributional derivative of the Heaviside function with values 0 and 1 we obtain that \(\dot{\eta} = -\frac{1}{t(1 - \tau)} \delta_t\) in the distributional sense. Let \(\overline{\mathcal{H}}(t) = -(1 - t) \mathcal{H}(0) - t \mathcal{H}(1) + \mathcal{H}(t)\), so that \(\overline{\mathcal{H}}(0) = 0\) and \(\overline{\mathcal{H}}(1) = 0\). Then, integrating by parts we obtain
\[
\int_0^1 \overline{\mathcal{H}}(\tau) \dot{\eta}(\tau) \, d\tau = -\int_0^1 \dot{\eta}(\tau) \overline{\mathcal{H}}(\tau) \, d\tau
\]
and thus we infer that
\[
\mathcal{H}(t) = (1 - t) \mathcal{H}(0) + t \mathcal{H}(1) + t(1 - t) \int_0^1 \dot{\mathcal{H}}(\tau) \eta(\tau) \, d\tau.
\]
Then, taking into account (5.10), we have
\[
\mathcal{H}(t) = (1 - t) \mathcal{H}(0) + t \mathcal{H}(1) + t(1 - t) \int_0^1 \dot{\eta}(\tau) D(\tau) \, d\tau + 2t(1 - t) \int_0^1 \dot{\eta}(\tau) \int_{\mathbb{R}^N} \omega u F \, dx \, d\tau.
\]
Integrating by parts,

\[ \mathcal{H}(t) = (1 - t)\mathcal{H}(0) + t\mathcal{H}(1) - t(1 - t) \int_0^1 \eta(\tau) \dot{D}(\tau) d\tau + 2t(1 - t) \int_0^1 \eta(\tau) \int_{\mathbb{R}^N} \omega uF \, dx \, d\tau. \]

We integrate in \( t \) between 0 and 1. Notice that

\[ \int_0^1 t(1 - t)\eta(\tau) dt = \frac{1}{2}\tau(1 - \tau) \quad \text{and} \quad \int_0^1 t(1 - t)\dot{\eta}(\tau) dt = \frac{1 - 2\tau}{2}. \]

Then

\[ \int_0^1 \mathcal{H}(t) dt = \frac{1}{2}\mathcal{H}(0) + \frac{1}{2}\mathcal{H}(1) - \frac{1}{2} \int_0^1 \tau(1 - \tau) \dot{D}(\tau) d\tau + \int_0^1 (1 - 2\tau) \int_{\mathbb{R}^N} \omega uF \, dx \, d\tau. \]

By renaming the integrals in \( \tau \), this is equivalent to

\[ \int_0^1 \mathcal{H}(t) dt + \frac{1}{2} \int_0^1 t(1 - t)\dot{D}(\tau) dt = \frac{1}{2}\mathcal{H}(0) + \frac{1}{2}\mathcal{H}(1) + \int_0^1 (1 - 2t) \int_{\mathbb{R}^N} \omega uF \, dx \, dt. \]

Now we use the estimate \([5.7]\) of Proposition \([5.3]\) (under the assumptions on \( A \)), so that

\[
\begin{align*}
\int_0^1 \mathcal{H}(t) dt + \frac{1}{2} \int_0^1 t(1 - t) &\left\{ \frac{3}{4}(A - (-|\lambda|^2 + m^2)^s) \mathcal{H}(t) dt - C_1(N, s) \int_{\mathbb{R}^N} t(1 - t) \int_{\mathbb{R}^N} \omega F^2 \, dx \, dt \\
+ \frac{1}{2} \int_{\mathbb{R}^N} t(1 - t) &\left\{ 2 \int_{\mathbb{R}^N} \omega(u_t)^2 - \int_{\mathbb{R}^N} \omega H_{m}^{2s}(u, u) + (A + m^{2s}) \int_{\mathbb{R}^N} H_{m}^{s}(u, u) \omega \, dx \right\} dt \right. \\
\leq \frac{1}{2}\mathcal{H}(0) + \frac{1}{2}\mathcal{H}(1) + C_1(N, s) \int_0^1 t(1 - t) \int_{\mathbb{R}^N} \omega F^2 \, dx \, dt + \int_0^1 (1 - 2t) \int_{\mathbb{R}^N} \omega uF \, dx \, dt, 
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
\int_0^1 \mathcal{H}(t) dt + \frac{1}{2} \int_0^1 t(1 - t) &\left\{ \frac{3}{4}(A - (-|\lambda|^2 + m^2)^s) - A \right\} \mathcal{H}(t) dt \\
+ \frac{1}{2} \int_0^1 t(1 - t) &\left\{ 2 \int_{\mathbb{R}^N} \omega(u_t)^2 - \int_{\mathbb{R}^N} \omega H_{m}^{2s}(u, u) + (A + m^{2s}) \int_{\mathbb{R}^N} H_{m}^{s}(u, u) \omega \, dx \right\} dt \right.
\leq \frac{1}{2}\mathcal{H}(0) + \frac{1}{2}\mathcal{H}(1) + C_1(N, s) \int_0^1 t(1 - t)^2 \int_{\mathbb{R}^N} \omega F^2 \, dx \, dt.
\end{align*}
\]

Applying the AM-GM inequality \((1 - 2t)\omega uF \leq \frac{1}{2}(1 - 2t)^2 \omega F^2 + \omega u^2\) yields

\[
\begin{align*}
\frac{1}{2} \int_0^1 \mathcal{H}(t) dt + \frac{3}{8}(A - (-|\lambda|^2 + m^2)^s) &\left\{ \int_0^1 t(1 - t)\mathcal{H}(t) dt \\
+ \frac{1}{2} \int_0^1 t(1 - t) &\left\{ 2 \int_{\mathbb{R}^N} \omega(u_t)^2 - \int_{\mathbb{R}^N} \omega H_{m}^{2s}(u, u) + (A + m^{2s}) \int_{\mathbb{R}^N} H_{m}^{s}(u, u) \omega \, dx \right\} dt \right. \\
\leq \frac{1}{2}\mathcal{H}(0) + \frac{1}{2}\mathcal{H}(1) + \int_0^1 \left( \frac{1}{2}(1 - 2t)^2 + C_1(N, s)t(1 - t) \right) \int_{\mathbb{R}^N} \omega F^2 \, dx \, dt.
\end{align*}
\]

Finally, by considering the maximum of the weight functions in \( t \) we obtain the conclusion. \( \square \)

**Corollary 5.5.** Due to the monotonicity of \( \mathcal{H}(t) \) as a function of \( t \) (in particular, by \([5.4]\)), we have that

\[ \mathcal{H}(1) \leq \int_0^1 \mathcal{H}(t) dt + e^{A - (-|\lambda|^2 + m^2)^s} \int_0^1 \int_{\mathbb{R}^N} \omega F^2 \, dx \, dt. \]

So the term \( \frac{1}{2}\mathcal{H}(1) \) can be hidden in the left hand side into the term \( \frac{1}{2} \int \mathcal{H}(t) dt \). Hence, the resulting terms turn out to be still positive, and we derive, from the Carleman inequality \([5.19]\), that

\[ \frac{1}{2}(A - (-|\lambda|^2 + m^2)^s) \int_0^1 \mathcal{H}(t)t(1 - t) \, dt + \text{positive energy terms} \]
\[
\leq \frac{1}{2} \mathcal{H}(0) + \left( C_1(N, s) + e^{A(-|\xi|^2+m^2)} \right) \int_0^1 \int_{\mathbb{R}^N} \omega((\partial_t + L_m)(u))^2 \, dx \, dt,
\]
where \( A \) satisfies (5.6).

Remark 5.6. Observe that, from (5.20), we could also immediately deduce the following convexity inequality:
\[
(5.21) \quad \|\sqrt{t(1-t)}m^{1/2}u\|_{L^2(\mathbb{R}^N \times [0,1])} \lesssim \sup_{t \in [0,1]} \|\omega^{1/2}F\|_{L^2(\mathbb{R}^N)} + \mathcal{H}(0) + \mathcal{H}(1).
\]

The Carleman inequality in (5.21) reminds the one contained in [9, Lemma 4]. Such an inequality is used therein to obtain a convexity inequality (see [9, Theorem 3]).

APPENDIX A.

Apart from the definitions for \( L_m^s \) given in Section 3, we introduce the definition using the subordination formula. Motivated by the formula (2.12), we define the operator \( L_m^s(f) \) as follows. Let \( 0 < s < 1, \ m \geq 0 \) and \( f \in S \). The operator \( L_m^s(f) \) is obtained as a weighted integral of the associated heat semigroup, by means of the spectral theorem
\[
(\text{A.1}) \quad L_m^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta-m^2}) f(x) \, dt \, t^{1+s}.
\]

We notice that the fractional power could be also defined using functional calculus as in Kato [19, p. 286], Paży [27, p. 69] or Yosida [38, p. 260].

The following lemma is the analogous to [36, Lemma 2.1] for the fractional relativistic operator.

Lemma A.1. For \( f \in S(\mathbb{R}^N) \) and \( 0 < s < 1 \), the definitions given in (2.1), (A.1) and (2.7) are equivalent.

Proof. We will first prove that (2.1) and (A.1) are equivalent. Observe that, by the inverse Fourier transform,
\[
e^{t\Delta-m^2} f(x) - f(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} (e^{-t(|\xi|^2+m^2)} - 1) \hat{f}(\xi) e^{ix \cdot \xi} \, d\xi.
\]

With this and the change of variables \( w = t(|\xi|^2+m^2) \) we obtain
\[
\int_0^\infty \left| e^{t\Delta-m^2} f(x) - f(x) \right| \frac{dt}{t^{1+s}} \leq C_N \int_0^\infty \int_{\mathbb{R}^N} \left| e^{-t(|\xi|^2+m^2)} - 1 \right| \left| \hat{f}(\xi) \right| \, d\xi \, dt \frac{1}{t^{1+s}}
\]
\[
= C_N \int_{\mathbb{R}^N} \left| \int_0^\infty \left| e^{-w} - 1 \right| \frac{dw}{w^{1+s}} (|\xi|^2 + m^2)^s \left| \hat{f}(\xi) \right| \, d\xi \right|
\]
\[
= C_{s,N} \int_{\mathbb{R}^N} (|\xi|^2 + m^2)^s \left| \hat{f}(\xi) \right| \, d\xi < \infty,
\]

since we are considering \( f \in S(\mathbb{R}^N) \). Therefore, by Fubini’s Theorem,
\[
\frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta-m^2}) f(x) \, dt \frac{1}{t^{1+s}} \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \left( e^{-t(|\xi|^2+m^2)} - 1 \right) \frac{dt}{t^{1+s}} \left| \hat{f}(\xi) e^{ix \cdot \xi} \right| d\xi
\]
\[
= \frac{1}{\Gamma(-s)} \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \left( e^{-w} - 1 \right) \frac{dw}{w^{1+s}} (|\xi|^2 + m^2)^s \left| \hat{f}(\xi) e^{ix \cdot \xi} \right| d\xi
\]
\[
= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} (|\xi|^2 + m^2)^s \left| \hat{f}(\xi) \right| \, d\xi = F^{-1}(\cdot ; |\cdot|^2 + m^2) \hat{f}(\cdot)(x).
\]

We will check the equivalence between (A.1) and (2.7). Let us denote \( W_{t,m}(x) := e^{-tm^2} e^{-\frac{|x-y|^2}{(4\pi t)^{N/2}}} \). By Fubini’s Theorem,
\[
\int_0^\infty (e^{t\Delta-m^2}) f(x) \, dt \frac{1}{t^{1+s}} = \int_0^\infty \int_{\mathbb{R}^N} W_{t,m}(x-y) (f(y) - f(x)) \, dy \frac{dt}{t^{1+s}}
\]
\[
+ f(x) \int_0^\infty \left( \int_{\mathbb{R}^N} W_{t,m}(x-y) \, dy \right) (1 - e^{tm^2}) \frac{dt}{t^{1+s}}.
\]
The integral in the second summand boils down to
\[
\int_0^\infty \left( \int_\mathbb{R}^N e^{-\frac{|x-y|^2}{4t}} \, dy \right) (e^{-tm^2} - 1) \frac{dt}{t^{1+s}} = \Gamma(-s)m^{2s},
\]
On the other hand, the integral in the first summand reads as
\[
\int_\mathbb{R}^N (f(y) - f(x)) \int_0^\infty e^{-tm^2} e^{-\frac{|x-y|^2}{4t}} \frac{dt}{t^{1+s}} \, dy = \Gamma(-s)C_{N,s}m^{\frac{N+2s}{2}} \int_\mathbb{R}^N \frac{f(x) - f(y)}{|x-y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}} (m|x-y|) \, dy
\]
where we used the integral representation (2.6) of the Macdonald’s function $K_\nu$, after a change of variable. The applications of Fubini’s theorem can be justified following an analogous argument as in [36, Lemma 2.1], by using the asymptotics (2.4) and (2.5).

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