Moment problems and boundaries of number triangles

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Abstract

The boundary problem for graphs like Pascal’s but with general multiplicities of edges is related to a ‘backward’ problem of moments of the Hausdorff type.

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1 The extreme boundary

Let \( T_n := \{(n,0),(n,1),\ldots,(n,n)\} \) and \( T := \bigcup_{n=0}^{\infty} T_n \). We endow \( T \) with the structure of a directed graph in which every node \((n,k)\) has two outgoing edges \((n,k) \to (n+1,k)\) and \((n,k) \to (n+1,k+1)\) with multiplicities \( \ell_{nk} \) and \( r_{nk} \) (respectively), where \( \{\ell_{nk}; (n,k) \in T\} \) and \( \{r_{nk}; (n,k) \in T\} \) are given triangular arrays with (strictly) positive entries. A classical example is the Pascal graph with unit multiplicities \( \ell_{nk} = r_{nk} = 1 \).

Let \( V \) be the set of nonnegative solutions \( V = \{V_{nk}; (n,k) \in T\} \) to the backward recursion

\[
V_{nk} = \ell_{nk} V_{n+1,k} + r_{nk} V_{n+1,k+1}, \quad (n,k) \in T
\]

with normalisation \( V_{00} = 1 \). The set \( V \) is convex and compact in the product topology of functions on \( T \). By some general theory in Dynkin (1978) the extreme boundary \( \text{ext}V \), comprised of indecomposable elements of \( V \), is a Borel set. Moreover, \( V \) is a Choquet simplex, meaning that each \( V \in V \) has a unique representation as convex combination

\[
V = \int_{\text{ext}V} U \mu(dU)
\]

with some probability measure \( \mu \) supported by \( \text{ext}V \). The boundary problem for the graph \( T \) is to find some explicit description of the set of extremes, meaning, if possible, a simple parametrisation of \( \text{ext}V \) along with the kernel that is implicit in (2).

The recursion (1) for the Pascal graph appeared in the work of Hausdorff on summation methods (1921, p. 78) and the ‘small’ problem of moments on \([0,1]\). In this case the bivariate array \( V \) is completely determined by \( V_{*,0} \) according to the rule \( V_{*,k} = \nabla^k(V_{*,0}) \), where \(*\) stands for the variable \( n \), and \( \nabla^k \) is the \( k \)th iterate of the difference operator \( \nabla(U_*) := U_* - U_{*+1} \). The condition \( V \geq 0 \) means that \( V_{*,0} \) is completely monotone, hence by Hausdorff’s theorem \( V_{*,0} \) is a sequence of moments

\[
V_{n0} = \int_{[0,1]} x^n \mu(dx)
\]

of some probability measure \( \mu \). That is to say, the set of extremes \( \text{ext}V \) can be identified with the unit interval, and the extremes have the form \( V_{nk}(x) = x^{n-k}(1-x)^k \) for \( x \in [0,1] \); in particular, \( V_{n0} = x^n \).

For general multiplicities the recursion (1) is equivalent to

\[
V_{*,k} = \nabla_k(\cdots(\nabla_1(V_{*,0}))\cdots),
\]

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where $\nabla_k(U_\bullet) = (U_\bullet_0 - \ell_{\bullet,k}U_{\bullet+1})/r_{\bullet,k}$ is a generalised difference operator. By analogy with Hausdorff’s criterion, the question about positivity of the generalised iterated differences of $V_{\bullet,0}$ may be regarded as a ‘backward’ problem of moments. A direct problem of moments of the Hausdorff type appears when we determine the $V_{nk}$’s for extreme solutions as functions on the boundary, and consider the integral representation of the generic $V_{00}$ in the form (2).

A bivariate array $V \in \mathcal{V}$ could be also computed by suitable differenting the diagonal sequence $(V_{nn})$, but this leads to the same type of the moment problem by virtue of the transposition of $V$ representation of the generic $\ell$ criterion, the question about positivity of the generalised iterated differences of $V'$s for extreme solutions as functions on the boundary, and consider the integral representation of the generic $V_{00}$ in the form (2).

A special feature of $T$, as compared with more complicated graphs like Young’s lattice (see Kerov (2003), Borodin and Olshanski (2000)), is a natural total order on the extreme boundary. In this note we extend the argument of Gnedin and Pitman (2006) to show that the total order allows $\text{ext V}$ to be embedded into $[0,1]$. We shall also survey the connection of the boundary problem with asymptotic properties of some classical arrays of combinatorial numbers.

2 Markov chain approach

The weight of a path in $T$ joining the root $(0,0)$ and some other node $(n,k)$ is defined as the product of multiplicities of edges along the path (for instance the weight of $(0,0) \to (1,0) \to (2,1)$ is $\ell_{00}r_{10}$). The dimension $D_{nk}$ of $(n,k) \in T$ is defined to be the sum of weights of all paths from $(0,0)$ to $(n,k)$. The dimensions are computable from the forward recursion

$$D_{nk} = r_{n-1,k-1}D_{n-1,k-1} + \ell_{n-1,k}D_{n-1,k}, \quad (n,k) \in T,$$

(where the first term in the right-hand side is absent for $k = 0$ and the second term is absent for $k = n$), with the initial condition $D_{00} = 1$. The number triangle associated with $T$ is the array $\{D_{nk}; (n,k) \in T\}$.

Each $V \in \mathcal{V}$ determines the law $\mathbb{P}_V$ of an inhomogeneous Markov chain $K_\bullet$, whose backward transition probabilities for $0 \leq k \leq n$, $n > 0$ are

$$\mathbb{P}_V(K_{n-1} = j | K_n = k) = \frac{D_{n-1,j}}{D_{nk}}(\ell_{n-1,j}\delta_{jk} + r_{n-1,j}\delta_{j,k-1}),$$

and whose distribution at time $n$ is $\mathbb{P}_V(K_n = k) = D_{nk}V_{nk}$. It is important that the probabilities (4) are determined solely by the multiplicities of edges and do not depend on $V$. Hence $\mathcal{V}$ is in essence a class of distributions for Markov chains on $T$ with given backward transition probabilities.

For each fixed integer $\nu$ and $0 \leq \kappa \leq \nu$ let $V^{\nu,\kappa}$ be the function on $T$ which satisfies the recursion (1) for $n < \nu$, satisfies $V^{\nu,\kappa}_{\nu,k} = \delta_{\kappa,k}$, and equals 0 on $\cup_{n>\nu}T_n$. Such $V^{\nu,\kappa}$ determines the probability law of a finite Markov chain $(K_0, \ldots, K_\nu)$ conditioned on $K_\nu = \kappa$.

We define the sequential boundary $\partial_\nu \mathcal{V}$ to be the set of elements of $\mathcal{V}$ representable as limits $V = \lim_{\nu \to \infty} V^{\nu,\kappa(\nu)}$ taken along infinite paths $\{\kappa(\nu); \nu = 0, 1, \ldots\}$ in $T$. The sequential boundary $\partial_\nu \mathcal{V}$ may be smaller than the set of all accumulation points for $\{V^{\nu,\kappa}; (\nu, \kappa) \in T\}$ (the Martin boundary), but it is large enough to cover $\text{ext V}$, as is seen from the following lemma, which is a variation on the theme of sufficiency (see Diaconis and Freedman (1984)).

Lemma 1. If $V \in \text{ext V}$ then the random functions

$$V^{\nu,K_\nu} := \sum_{\kappa=0}^{\nu} 1\{K_\nu = \kappa\}V^{\nu,\kappa}$$

satisfy $V^{\nu,K_\nu} \to V$, as $\nu \to \infty$, $\mathbb{P}_V$-almost surely.

Proof. Let $\mathcal{F}_\nu$ be the sigma-algebra generated by $K_\nu, K_{\nu+1}, \ldots$, and $\mathcal{F}_\infty = \cap \mathcal{F}_\nu$. Let $\mathbb{P}_V$ correspond to some extreme $V$. Choose any $(n,k)$ and consider random variables

$$V^{\nu,K_\nu}_{nk} = \mathbb{P}_V(K_n = (n,k) | K_\nu)/D_{nk} = \mathbb{P}_V(K_n = (n,k) | \mathcal{F}_\nu)/D_{nk}, \quad \nu \geq n.$$
where the first equality follows from the definition \[5\], and the second equality is a consequence of the Markov property. Applying Doob’s reversed martingale convergence theorem to the conditional expectations given \(\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \ldots\) we obtain
\[
V_{nk}^{\nu, K} \to \mathbb{P}_V(K_n = (n, k) \mid \mathcal{F}_\infty)/D_{nk} \quad \mathbb{P}_V \text{-a.s.}
\]
The assumption \(V \in \text{ext} \mathcal{V}\) implies that \(\mathcal{F}_\infty\) is trivial, hence
\[
\mathbb{P}_V(K_n = (n, k) \mid \mathcal{F}_\infty) = \mathbb{P}_V(K_n = (n, k)) = V_{nk}D_{nk}.
\]
Thus \(\text{ext} \mathcal{V} \subset \partial_1 \mathcal{V}\) (in general the inclusion is strict). To state this conclusion in analytical terms, define the weight of a path in \(T\) connecting two nodes \((n, k)\) and \((\nu, \kappa)\) as the product of multiplicities along the path, and define the extended dimension \(D_{nk}^{\nu, \kappa}\) as the sum of weights over all such paths (so that \(D_{00}^{\nu, \kappa} = D_{\nu, \kappa}\)). We then have a fundamental relation
\[
V_{nk}^{\nu, \kappa} = D_{nk}^{\nu, \kappa}
\]
which connects the boundary problem with asymptotic properties of \(T\). Specifically, the convergence of \(V_{nk}^{\nu, \kappa}(\nu)\) amounts to the convergence of these ratios for all \((n, k) \in T\) along the path (in fact, it is enough to focus on \(V_{\bullet, 0}\)).

### 3 Order

A special feature of \(T\) which yields the order is that the only possible increments of the variable \(k\) along any path are 0 and 1. The next lemma appeared in Gnedin and Pitman (2006) with a different proof.

**Lemma 2.** For \(\nu > n\) fixed, \(V_{n0}^{\nu, \kappa}\) is nonincreasing in \(\kappa\).

**Proof.** Choose \(0 \leq \kappa < \kappa' \leq \nu\) and consider two Markov chains \(K_\bullet, K'_\bullet\) which run in reverse time \(n = \nu, \nu - 1, \ldots, 0\) according to \[4\] and start with \(K_\nu = \kappa, K'_\nu = \kappa'\). Suppose the chain \(K'_\bullet\) jumps independently of \(K_\bullet\) as long as they are in distinct states, and suppose that \(K'_\bullet\) is coupled with \(K_\bullet\) at some random time \(0 \leq \tau < \nu\) when the states become the same. In the reverse time, only transitions \(k \to k, k \to k - 1\) for \(k > 0\) and \(0 \to 0\) are possible, hence we always have \(K'_n \geq K_n\). Therefore the event \(K'_n = 0\) occurs exactly when \(K_n = 0\) and \(\tau \geq n\), which implies
\[
\mathbb{P}(K_n = 0 \mid K_\nu = \kappa) \geq \mathbb{P}(K_n = 0 \mid K_\nu = \kappa').
\]

A minor modification of the above argument shows that if \(K_\nu\) under \(\mathbb{P}_V\) is strictly stochastically smaller than \(K_\nu\) under some other \(\mathbb{P}_{V'}\), then the same relation holds true for every \(n \leq \nu\).

We focus now on \(V_{10}\). Suppose \(V \in \partial_1 \mathcal{V}\) is induced, via \[6\], by some infinite path \(\{\kappa(\nu); \nu = 0, 1, \ldots\}\), and \(V' \in \partial_1 \mathcal{V}\) is induced by some other path \(\{\kappa'(\nu); \nu = 0, 1, \ldots\}\). If \(\kappa(\nu) = \kappa'(\nu)\) for infinitely many \(\nu\) then, of course, \(V = V'\). If \(\kappa(\nu) < \kappa'(\nu)\) for infinitely many \(\nu\) and \(\kappa(\nu) > \kappa'(\nu)\) for infinitely many \(\nu\) then by Lemma 2 we have \(V_{\bullet, 0} = V'_{\bullet, 0}\) and \(V = V'\). Thus \(V \neq V'\) can only occur if the same strict inequality holds for all sufficiently large \(\nu\). To be definite, let \(\kappa(\nu) < \kappa'(\nu)\) for all large enough \(\nu\), but then \(V \neq V'\) implies that \(K_n\) under \(\mathbb{P}_V\) is strictly stochastically smaller than \(K_n\) under \(\mathbb{P}_{V'}\) for all \(n > 0\), in particular this holds for \(n = 1\) which means that \(V_{10} > V'_{10}\). We see that for \(V, V' \in \partial_1 \mathcal{V}\), the inequality \(V_{10} > V'_{10}\) holds if and only if \(K_n\) under \(\mathbb{P}_V\) is strictly stochastically smaller than \(K_n\) under \(\mathbb{P}_{V'}\) for all \(n > 0\). This defines a strict order \(<\) on \(\partial_1 \mathcal{V}\).

**Lemma 3.** The sequential boundary \(\partial_1 \mathcal{V}\) is compact.

**Proof.** Suppose \(V^j \in \partial_1 \mathcal{V}\) \((j = 1, 2, \ldots)\) is a sequence converging to some \(V \in \mathcal{V}\). We know that \(\mathcal{V}\) is a metrisable compactum with some distance function dist. Passing to a subsequence we can restrict consideration to the case of increasing or decreasing sequence, so to be definite assume that \(V^{j+1} < V^j\) for \(j = 1, 2, \ldots\) Choosing some path \(\{\kappa(\nu); \nu = 0, 1, \ldots\}\) which induces \(V^j\), the ordering implies that

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\( x_j(\nu) \to \infty \) as \( \nu \to \infty \) and \( x_j(\nu) < x_{j+1}(\nu) \) for all large enough \( \nu \). As \( \nu \) varies, define inductively in \( j \) a function \( x(\nu) \) which coincides for some \( \nu \) with \( x_j(\nu) \). Specifically, \( x(\nu) = x_j(\nu) \) until \( x_{j+1}(\nu) < x_j(\nu) \) starts to hold along with \( \text{dist}(V^{\nu, x_j(\nu)}, V) < 1/j \) and \( \text{dist}(V^{\nu, x_{j+1}(\nu)}, V) < 1/j \), then let \( x(\nu) \) decrement by 1 until it becomes equal to \( x_{j+1}(\nu) \). This defines an infinite path in \( T \), for which one can use monotonicity to show that \( V^{\nu, x(\nu)} \to V \).

Recalling that \( \ell_{00} V_{10} + r_{00} V_{11} = 1 \) we obtain:

**Theorem 4.** The function \( V \mapsto \ell_{00} V_{10} \) is an ordered homeomorphism of the sequential boundary \( \partial_1 V \) with order \( < \) into \( [0, 1] \) with order \( > \).

Two extreme cases \( \ell_{00} V_{01} = 0 \) and \( \ell_{00} V_{01} = 1 \) correspond to trivial Markov chains \( K_\bullet = 0 \) and \( K_\bullet = \bullet \), respectively.

## 4 Discrete or continuous?

In the situation covered by the following lemma, setting \( x(\nu) = m \) (for large \( \nu \)) for \( m = 1, 2, \ldots \) is the only way to induce nontrivial limits. Then \( \text{ext} V \) is discrete and coincides with the sequential boundary.

**Lemma 5.** (Gnedin and Pitman (2006)) Suppose for \( m = 0, 1, \ldots \) there are solutions \( V(m) \in \mathcal{V} \) such that \( V_{nm}(m) D_{nm} \to 1 \) as \( n \to \infty \), then each \( V(m) \) is extreme and satisfies \( K_n \rightarrow m P_{V(m)} \text{-a.s.} \). If also \( V_{10}(m) \rightarrow 0 \) as \( m \to \infty \) then \( V(m) \) converges to the trivial solution \( V(\infty) \) with \( K_\bullet = \bullet P_{V(\infty)} \text{-a.s.} \) and in this case \( \text{ext} V = \partial_1 V = \{ V(0), V(1), \ldots, V(\infty) \} \).

In some cases the limits can be obtained by setting \( x(\nu) \sim s c(\nu) \) with suitable scaling \( c(\nu) \to \infty \) and \( s \geq 0 \). Under conditions in the next lemma, \( \text{ext} V \) coincides with \( \partial_1 V \) and is homeomorphic to \( [0, 1] \). The scaling determines the order of growth of \( K_\bullet \) under \( P_V \)'s.

**Lemma 6.** (Gnedin and Pitman (2006)) Suppose there is a sequence of positive constants \( \{c(\nu); \nu = 0, 1, \ldots \} \) with \( c(\nu) \to \infty \), and for each \( s \in [0, \infty] \) there is a solution \( V(s) \in \mathcal{V} \) which satisfies \( K_{\nu}/c(\nu) \to s \ P_{V(s)} \text{-a.s.} \). Suppose the mapping \( s \mapsto V(s) \) is a continuous injection from \( [0, \infty] \) to \( \mathcal{V} \) with 0 and \( \infty \) corresponding to the trivial solutions. Then a path \( \{x(\nu); \nu = 0, 1, \ldots \} \) induces a limit if and only if \( x(\nu)/c(\nu) \to s \) for some \( s \in [0, \infty] \), in which case the limit is \( V(s) \). Moreover, \( \text{ext} V = \partial_1 V = \{ V(s), s \in [0, \infty] \} \).

Minor variations of the above two situations are obtained by transposing multiplicities \( \ell_{nk} \leftrightarrow r_{n,n-k} \). Still, this does not exhaust all possibilities. See Kerov (2003) (Section 1.3, Theorem 2) for examples of boundaries with both discrete and continuous components.

## 5 Number triangles

**The Pascal triangle.** For the Pascal graph the dimensions are \( D_{nk} = \binom{n}{k} \) and \( D_{nk}^{x(\nu)} = \binom{n}{k} \). The ratios \( \frac{V_{nk}^{x(\nu)}}{x(\nu)} = \frac{n - k}{x(\nu)} \) converge iff \( x(\nu)/\nu \to x \in [0, 1] \), in which case the limit is \( V_{nk}(x) = x^n \).

This identification of extremes is equivalent to de Finetti’s theorem (see Aldous (2003)), since \( V \in \mathcal{V} \) determines the law of some infinite sequence of exchangeable Bernoulli trials. A closely related type of moment problem with a monotonicity constraint have been discussed recently in Gnedin and Pitman (2007).

**The q-Pascal triangle.** This has multiplicities \( \ell_{nk} = 1, r_{nk} = q^{n-k} \) \( (n, k) \in T \), and may be seen as a parametric deformation of the Pascal graph. The extreme boundary was found in Kerov (2003) by an algebraic method and justified by Olshanski (2001) by the analysis of [3]. The dimensions are expressible through q-binomial coefficients as

\[
D_{nk} = \binom{n}{k}_q, \quad D_{nk}^{x(\nu)} = q^{(x-k)(n-k)} \frac{(\nu - n)}{(x - k)} \frac{(\nu)}{(x)}_q.
\]

Suppose first that \( 0 < q < 1 \). Lemma 5 is applicable, and all nontrivial extremes are given by

\[
V_{nk}(m) = q^{(m-k)(n-k)} \frac{(1 - q) \cdots (1 - q^n)}{(1 - q) \cdots (1 - q^{n-k})} 1_{\{0 \leq k \leq m\}}, \quad m = 1, 2, \ldots
\]
In particular, $V_{n0}(m) = q^{mn}$, $(m = 0, 1, \ldots, \infty)$.

The function $V \mapsto \ell_{00} V_{10}$ identifies the extreme boundary with $\{q^m, m = 0, 1, \ldots, \infty\}$. The decomposition 2 into extremes corresponds to a version of Hausdorff’s moment problem on $[0, 1]$ with kernel $x^n$, but subject to the constraint that the measure is to be supported by a geometric progression. That is to say, a sequence $V_{\bullet,0}$ with $V_{00} = 1$ is representable as a mixture

$$V_{n0} = \sum_{m \in \{0,1,\ldots,\infty\}} p_m q^{mn}$$

with some probability distribution $\{p_m; m = 0, 1, \ldots, \infty\}$ if and only if $V_{\bullet,k} = V_k(\cdots (V_1(\cdots (V_{\bullet,0})\cdots)) \geq 0$ for all $k \geq 0$, where $V_k(U_\bullet) = (U_\bullet - U_{\bullet+1}) / q^{k}$. In the case $q > 1$ the extreme boundary is $\{1 - q^{-m}, m = 0, 1, \ldots, \infty\}$ (this case is reducible to $q < 1$ by transposition of $T$ and replacing $q$ by $q^{-1}$). The only accumulation point of $\text{ext} V$ for $q < 1$ is 0 and for $q > 1$ is 1. A phase transition occurs at $q = 1$, when the extreme boundary is continuous.

**Stirling triangles.** Let $r_{nk} = 1$ and $\ell_{nk} = (n + 1) - \alpha(k + 1)$ for $-\infty < \alpha < 1$. For $\alpha = -\infty$ take $\ell_{nk} = k + 1$. The dimension is $D_{nk} = \left[ \begin{array}{c} n + 1 \\ k + 1 \end{array} \right] / \alpha$. The notation stands for the generalised Stirling numbers defined as connection coefficients in

$$(t)_n! = \sum_{k=1}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \alpha^n (t/\alpha)_n!,$$

(where $\uparrow$ denotes the rising factorial), with the convention that these are the Stirling numbers of the second kind for $\alpha = -\infty$. For $\alpha = 0$ these are the signed Stirling numbers of the first kind.

For $-\infty \leq \alpha < 0$ the extreme boundary is discrete, with

$$V_{n0}(m) = \frac{1}{(m|\alpha| + 1)_n} \quad \text{for} \quad -\infty < \alpha < 0, \quad V_{n0}(m) = \frac{1}{m^n} \quad \text{for} \quad \alpha = -\infty.$$  

These kernels underly a moment problem for measures on the set $\{0, 1, \ldots, \infty\}$.

A phase transition occurs at $\alpha = 0$. Lemma 5 applies with $\kappa(\nu) \sim s \log n$, the extreme boundary is continuous and the kernel is

$$V_{n0}(s) = \frac{1}{(s + 1)_n} \quad s \in [0, \infty].$$

This case is closely related to random permutations, records and Ewens’ sampling formula (see Arratia et al (2003)).

In the case $0 < \alpha < 1$ we should take $\kappa(\nu) \sim s n^\alpha$ to generate the boundary, see Gnedin and Pitman (2006) for formulas for $V_{n0}(s)$ to adjust the notation in Gnedin and Pitman (2006) to the present setting, one should replace $(n,k)$ by $(n+1,k+1)$. This family of solutions is related to Poisson-Kingman partitions, see Gnedin and Pitman (2006) and references therein.

Several results and (still open) conjectures about boundaries of more general Stirling graphs, with multiplicities of the form $\ell_{nk} = b_n + a_k$, $r_{nk} = 1$, are given in Kerov (2003).

**The Eulerian triangle.** For multiplicities $\ell_{nk} = k + 1$, $r_{nk} = n - k + 1$ the dimension is the Eulerian number $\left( \begin{array}{c} n + 1 \\ k \end{array} \right)$ (that counts permutations with a given number of descents). The boundary problem was solved in Gnedin and Olshanski (2006). The extreme solutions are given by

$$V_{nk}(m) = \frac{1}{(n+1)!} \sum_{i=-k}^{n-k} \left( 1 + \frac{i}{m} \right)^n$$

with $m \in \mathbb{Z} \cup \{\infty\}$. Note that the range of $\ell_{00} V_{10}(m) = (m+1)/(2m)$ is symmetric about 1/2, with 1/2 being the only accumulation point. The symmetry of the boundary stems in this case from the invariance of multiplicities under transposition.
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