Abstract. In this paper we study Seshadri constants of $l$-very ample line bundles on blow ups of $\mathbb{P}^n$ on $s$ number of general points with some restrictions on $s$. We compute the nef cone of $\mathbb{P}^3$ blown up at $r \leq 6$ number of general lines and study Seshadri constants on them. We also study the Seshadri constants of an ample line bundles on blow ups of $\mathbb{P}^4$ and $\mathbb{P}^5$ at $r$ number of general lines for $r \leq 7$ and $r \leq 5$ respectively.

1. INTRODUCTION

Let $X$ be a projective variety over an algebraically closed field $k$. For a nef line bundle $L$ on $X$, the Seshadri constant of $L$ at a point $x \in X$ is defined as

$$\epsilon(X, L, x) = \inf_{x \in C} \frac{L \cdot C}{\text{mult}_x C},$$

where the infimum is taken over all curves $C$ in $X$ passing through $x$, and $L \cdot C$ and $\text{mult}_x C$ are the intersection number and multiplicity of $C$ at $x$ respectively. Sometimes we write $\epsilon(L, x)$ instead of $\epsilon(X, L, x)$ if there is no confusion about the variety $X$.

Seshadri constant measures the local positivity of a nef line bundle on a projective variety around a given point. Motivated by Seshadri’s criterion for ampleness of a line bundle, the Seshadri constant was introduced by Demailly in 1992 mainly to tackle the Fujita conjecture. But over the years it has emerged as an interesting invariant and grew on its own as an important research topic in algebraic geometry. Most of the research on Seshadri constant has been in the case of surfaces. One of the very important results is due to Ein and Lazarsfeld (see [EL]), who proved that for any line bundle $L$ on a surface $X$ the Seshadri constant $\epsilon(X, L, x) \geq 1$ for a general point $x \in X$. Another very fundamental result on Seshadri constant was shown by Miranda (see [L1], Example 5.2.1), which says that given a real number $\delta > 0$, there exists an algebraic surface $X$ and an ample line bundle $L$ on $X$ such that $\epsilon(X, L) < \delta$. This result proves that Seshadri constant can be arbitrarily small.

Studying Seshadri constant on higher dimensional varieties hasn’t yet been as successful as in the case of surfaces. In few cases where the Seshadri constant has been studied on higher dimensional varieties are Fano varieties (see [Lee]), Toric varieties (see [I]), Abelian varieties (see [B], [L2], [N]), Grassmann varieties over curves (see [BHNN]). For a detailed exposition about the works on Seshadri constant, see [BDHKSS].

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Computing Seshadri constant precisely is more often than not a very difficult task in hand. So, most of the works on Seshadri constant has been on giving bounds to them and sharpening the already known bounds when precise computation is difficult.

In this paper we will study Seshadri constants on blow ups of projective spaces. In Section 3, we consider $X^n_{0,s}$, blow up of $\mathbb{P}^n$ at $s$ general points. We show that for a $l$-very ample line bundle (see [DP]) $L$ on $X^n_{0,s}$ and for any point $x \in X$, $\varepsilon(L,x) \geq l$ for $s < 3n$ (with some extra condition for $s > 2n$).

Let $X^r_{r,0}$ be the blow up of $\mathbb{P}^n$ at $r$ number of general lines in $\mathbb{P}^n$. In Section 4, carrying some results from [DPU], we compute the nef cones of line bundles on $X^3_{r,0}$ for $r \leq 6$ and study Seshadri constants on them. In Section 5, we borrow some results on the positive cones of $X^r_{r,0}$ for $r \leq 7$ and $X^5_{r,0}$ for $r \leq 5$ from [PP] and establish results on the Seshadri constants of nef line bundles based on that.

2. Preliminaries

Let $\pi_{r,s} : X^n_{r,s} \to \mathbb{P}^n$ be the blow ups of $\mathbb{P}^n$ at $r$ general lines $l_1, ..., l_r$ and $s$ general points $p_1, ..., p_s$. In this article our focus will be on $X^n_{0,s}$, the blow up of $\mathbb{P}^n$ at $s$ general points $p_1, ..., p_s$ and on $X^n_{0,s}$, the blow up of $\mathbb{P}^n$ at $r$ general lines $l_1, ..., l_r$. For $r = 0$, $X^n_{0,s}$ is the blow up of $\mathbb{P}^n$ at $s$ general points $p_1, ..., p_s$ and for $s = 0$, $X^n_{r,0}$ is the blow up of $\mathbb{P}^n$ at $r$ general lines $l_1, ..., l_r$. Let $\pi_{0,s} : X^n_{0,s} \to \mathbb{P}^n$ and $\pi_{r,0} : X^n_{r,0} \to \mathbb{P}^n$ be the respective blow up maps.

The real Néron-Severi groups $N^1(X^n_{r,s})$ are generated by $H, e_i, i = 1, ..., s$ and $E_i, i = 1, ..., r$, where $H$ is the pullback of the hyperplane class in $\mathbb{P}^n$, $e_i$’s are the exceptional divisors of the blow ups of $p_i$’s and $E_i$’s are the exceptional divisors of the blow ups of $l_i$’s.

Let $X$ be a projective variety over any field $k$. A line bundle $L$ on $X$(or a Cartier divisor with $\mathbb{Z}$ or $\mathbb{R}$ coefficients) is nef if

$$\int_C c_1(L) \geq 0 \quad (or \quad (D \cdot C) \geq 0)$$

for all irreducible curves $C$ in $X$.

The nef cone $\text{Nef}(X) \subset N^1(X)$ is the convex cone of all nef $\mathbb{R}$-divisor classes on $X$.

Let $X$ be a projective variety and $L$ be a line bundle on $X$, the Seshadri constants of $L$ at a point $x \in X$ if defined as

$$\varepsilon(X, L, x) := \inf_{x \in C} \left\{ \frac{L \cdot C}{\text{mult}_x C} \right\}$$

where the infimum is taken over all irreducible curves $C \subset X$ passing through $X$.

Seshadri criterion for ampleness says that $L$ is ample iff $\varepsilon(X, L, x) > 0$ for all $x \in X$. For an ample line bundle $L$ on $X$,

$$\varepsilon(X, L, 1) := \sup_{x \in X} \{ \varepsilon(X, L, x) \}$$

$$\varepsilon(X, L) := \inf_{x \in X} \varepsilon(L, x)$$
so that \(0 < \varepsilon(X, L) \leq \varepsilon(X, L, x) \leq \varepsilon(X, L, 1) \leq \sqrt[n]{L^n}\) for every point \(x \in X\).

3. Point blow ups of \(\mathbb{P}^n\)

In this section, we focus on \(X^0_{0,s}\) and use the \(l\)-very ampleness of a line bundle to prove some results about Seshadri constant of an ample line bundle. But before that we state the following definition which we use.

**Definition 3.1.** Let \(X\) be a smooth complex projective variety. For an integer \(l \geq 0\), a line bundle \(\mathcal{O}_X(D)\) on \(X\) is said to be \(l\)-very ample if for any 0-dimensional subscheme \(Z \subset X\) of weight \(h^0(Z, \mathcal{O}_Z) = l + 1\), the restriction map \(H^0(X, \mathcal{O}_X(D)) \rightarrow H^0(Z, \mathcal{O}_X(D)|_Z)\) is surjective.

Fix \(l \geq 0\), then for any \(s \geq n + 3\) we introduce the following integer:

\[b_l := \begin{cases} 
\min\{n - 1, s - n - 2\} - l - 1 & \text{if } m_1 - d - l - 1 \text{ and } m_i = 1, \forall i \geq 2, \\
\min\{n, s - n - 2\} - l - 1 & \text{otherwise}.
\end{cases}\]

For \(s \leq n + 2\) define \(b_l := -l - 1\).

**Proposition 3.2** (Theorem 2.2, [DP]). Let \(\pi_{0,s} : X^0_{0,s} \rightarrow \mathbb{P}^n\) be the blow up of \(\mathbb{P}^n\) at \(p_1, p_2, ..., p_s \in \mathbb{P}^n\) and let the points be in general positions. Assume that either \(s \leq 2n\) or \(s \geq 2n + 1\) and \(d\) large enough namely

\[\sum_{i=1}^{s} m_i - nd \leq b_l\]

where \(b_l\) defined as before. Then for any non zero integer \(l\), a line bundle \(L\) is \(l\)-very ample if and only if

\[m_i \geq l, \quad \forall i \in \{1, 2, ..., s\} \text{ and,} \]

\[d - m_i - m_j \geq l, \quad \forall i, j \in \{1, 2, ..., s\} (i \neq j).
\]

**Theorem 3.3.** Let \(\pi_{0,s} : X^0_{0,s} \rightarrow \mathbb{P}^n\) be the blow up of \(\mathbb{P}^n\) at \(p_1, p_2, ..., p_s\) in general positions and \(s \leq 2n\). Let \(L = dH - \sum_{i=1}^{i=s} m_i e_i\) be a \(l\)-ample line bundle on \(X_s\) and let \(x\) be a point in \(X_s\). Then

\[\varepsilon(L, x) \geq l.
\]

**Proof.** Let \(B\) be a reduced irreducible curve in \(X_s\) passing through \(x\) and \(m = \text{mult}_x B\). Then either \(\pi_{0,s}(B) = \text{point}\) or \(B\) is the strict transform of a curve \(C\) in \(\mathbb{P}^n\) i.e. \(B = \tilde{C}\).

**Case I:** Let \(1 \leq s \leq n\).

Let \(\pi_{0,s}(B) = \text{point}\) i.e. \(\pi_{0,s}(B) \in \{p_1, p_2, ..., p_s\}\) (say \(p = p_i\)), then the Seshadri ratio

\[
\frac{L \cdot B}{m} \geq \frac{d \deg B + m_i}{m} \geq \frac{d \deg B}{m} \geq \frac{l \deg B}{m} \geq l
\]

Now assume that \(B\) is strict transform of a curve in \(\mathbb{P}^n\) i.e., \(B = \tilde{C}\). Then the Seshadri ratio is
\[
\frac{L \cdot B}{m} = \frac{(dH - \sum_{i=1}^{s} m_i e_i) \cdot B}{m} \geq \frac{dH \cdot C - \sum_{i} m_i \mult_{p_i} C}{m} \\
\geq \frac{lH \cdot C}{m} + \frac{(d-l)H \cdot C - \sum_{i} m_i \mult_{p_i} C}{m}.
\]

Now, \(p_1, p_2, ..., p_s\) can be put inside a hyperplane \(\bar{H}\) that does not contain \(\pi(x)\), so that \(C \not\subseteq \bar{H}\). Then by Bézout’s theorem

\[
\bar{H} \cdot C \geq \sum \mult_{p_i} C
\]

So,

\[
(d-l)\bar{H} \cdot C \geq (d-l)\sum \mult_{p_i} C \geq \sum m_i \mult_{p_i} C
\]

since by proposition 3.2, \(d-l \geq m_i + m_j\) for \(i \neq j\). In particular, \(d-l \geq m_i\) for all \(i\).

Then continuing the calculation in the top of this page, the Seshadri ratio becomes

\[
\frac{L \cdot B}{m} \geq \frac{lH \cdot C}{m} \geq l
\]

**Case II:** Assume \(n \leq s \leq 2n\). If \(\pi_{0,s}(B) = \text{point}\), then we proceed as in Case I and obtain the Seshadri ratio \(\frac{L \cdot B}{m} \geq l\). Otherwise \(B\) is the strict transform of a curve \(C\) in \(\mathbb{P}^n\) i.e., \(B = \bar{C}\). We rearrange the points such that \(\mult_{p_1} C \geq \mult_{p_2} C \geq ... \geq \mult_{p_s} C\). Since \(p_1, p_2, ..., p_s\) are in general positions, we can form a hyperplane \(H_1 = \langle p_1, p_2, ..., p_n \rangle\). Then by Bézout’s theorem

\[
H_1 \cdot C \geq \sum_{i=1}^{n} \mult_{p_i} C
\]

From that,

\[
(d-l)H_1 \cdot C \geq (d-l)\sum_{i=1}^{n} \mult_{p_i} C \geq (m_i + m_{i+n})\sum_{i=1}^{n} \mult_{p_i} C
\]

\[
\geq \sum_{i=1}^{n} m_i \mult_{p_i} C + \sum_{i+n \leq s} m_{i+n} \mult_{p_{i+n}} C + \sum_{i+n > s} m_{i+n}
\]

\[
\geq \sum_{i=1}^{s} m_i \mult_{p_i} C + \sum_{i+n > s} m_{i+n}
\]

\[
\geq \sum_{i=1}^{s} m_i \mult_{p_i} C
\]

which implies that

\[
(d-l)H_1 \cdot C - \sum_{i=1}^{s} m_i \mult_{p_i} C \geq 0.
\]

So, the Seshadri ratio

\[
\frac{L \cdot B}{m} = \frac{(dH - \sum_{i=1}^{s} m_i e_i) \cdot B}{m} \geq \frac{dH \cdot C - \sum_{i} m_i \mult_{p_i} C}{m},
\]
on $X$ which implies that

\[ \text{Theorem 4.1.} \]

Let $H$ be any divisor on $X$ be the blow up of $P^n$ at $p_1, p_2, \ldots, p_s$ in general positions and $2n < s \leq 3n - 1$. Let $L = dH - \sum_{i=1}^{s} m_i e_i$ be an $l$-ample line bundle on $X_{0,s}$ with $m_i = l$ for $i = 2n + 1, \ldots, 3n - 1$. For any point $x$ in $X_s$, the Seshadri constant

\[ \varepsilon(L, x) \geq l. \]

Therefore, from Case I and Case II we conclude that $\varepsilon(L, x) \geq l$. \qed

**Theorem 3.4.** Let $\pi_{0,s} : X^n_{0,s} \rightarrow P^n$ be the blow up of $P^n$ at $p_1, p_2, \ldots, p_s$ in general positions and $2n < s \leq 3n - 1$. Let $L = dH - \sum_{i=1}^{s} m_i e_i$ be an $l$-ample line bundle on $X^n_{0,s}$ with $m_i = l$ for $i = 2n + 1, \ldots, 3n - 1$. For any point $x$ in $X_s$, the Seshadri constant

\[ \varepsilon(L, x) \geq l. \]

**Proof.** If $\pi_{0,s}(B) = \text{point}$, then as we have shown in last theorem, $\varepsilon(L, x) \geq l$. So, let’s assume that $B = \tilde{C}$. Also rearrange the points as in Case II of previous theorem such that $\text{mult}_{p_1} C \geq \text{mult}_{p_2} C \geq \ldots \geq \text{mult}_{p_s} C$. Then the Seshadri ratio,

\[ \frac{L \cdot B}{m} = \frac{(dH - \sum_{i=1}^{s} m_i e_i) \cdot B}{m} = \frac{(lH - \sum_{i=2n+1}^{3n-1} m_i e_i) \cdot B + ((d-l)H \cdot B - \sum_{i=1}^{2n} m_i e_i) \cdot B}{m} \]

Now we can choose a hyperplane $H_2$ passing through $p_{2n+1}, \ldots, p_{3n-1}$ and $\pi(x)$ that doesn’t contain $C$. Then by Bézout’s theorem

\[ H_2 \cdot B \geq \sum_{i=2n+1}^{3n-1} \text{mult}_{p_i} C + m \]

which implies that $lH_2 \cdot B \geq \sum_{i=2n+1}^{3n-1} l \text{mult}_{p_i} C + lm \geq \sum_{i=2n+1}^{3n-1} m_i \text{mult}_{p_i} C + m$.

Since $m_i \leq l$ for $i = 2n + 1, \ldots, 3n - 1$.

So, $\frac{(lH_2 \cdot B - \sum_{i=2n+1}^{3n-1} m_i e_i) \cdot B}{m} \geq l$.

Also, from the proof of previous theorem we can say that $(d-l)H \cdot B - \sum_{i=1}^{2n} m_i e_i) \cdot B \geq 0$.

Therefore, the Seshadri ratio $\frac{L \cdot B}{m} \geq l \implies \varepsilon(L, x) \geq l$. \qed

4. Line blow ups of $P^3$

We now turn our attention to $X^3_{r,0}$ i.e. blow up of $P^3$ at $r$ general lines. First we compute the nef cone of divisors on $X^3_{r,0}$ for $r \leq 6$ and then study the Seshadri constants of nef line bundles on $X^3_{r,0}$.

**Theorem 4.1.** Let $X^3_{r,0}$ be the blow up of $P^3$ at $r$ general lines $l_1, l_2, \ldots, l_r$. Let $D = dH - \sum_{i=1}^{r} m_i E_i$ be any divisor on $X^3_{r,0}$. Then for $r \leq 6$, the nef cone of divisors on $X^3_{r,0}$ are described as:

\[ \text{Nef}(X^3_{r,0}) = \left\{ dH - \sum_{i=1}^{r} m_i E_i \mid d \geq \sum_{i=1}^{r} m_i \text{ for } r \leq 4, d \geq 0, m_i \geq 0 \forall r \in \{1, ..., 4\} \right\} \]

for $r = 1, ..., 4$.

\[ \text{Nef}(X^3_{r,0}) = \left\{ dH - \sum_{i=1}^{r} m_i E_i \mid d \geq (m_1 + m_2 + \ldots + m_5) - m_4 \forall r \in \{1, 2, ..., 5\}, d \geq 0, m_i \geq 0 \forall r \in \{1, ..., 5\} \right\} \]
Nef($X_{0,0}^3$) = \{dH - \sum_{i=1}^{r} m_i E_i \mid d \geq m_i + m_j + m_k + m_l, i < j < k < l, \{i, j, k, l\} \in \{1, 2, \ldots, 6\}, \ d \geq 0, \ m_i \geq 0 \ \forall r \in \{1, \ldots, 6\}\}

**Proof.** Let \( \tilde{l}_i \) be a class of a line inside the exception divisor \( E_i \), for \( i = 1, \ldots, 6 \). If \( D = dH - dH - \sum_{i=1}^{r} m_i E_i \) is a nef divisor in \( X_{r,0}^3 \), then intersecting \( D \) with \( l \) (a class of general line in \( X_{r,0}^3 \)) and \( \tilde{l}_i \) for \( i = 1, \ldots, 6 \) give \( d \geq 0 \) and \( m_i \geq 0 \) for \( i = 1, \ldots, 6 \).

- **r = 1:**
  Let \( D = dH - m_1 E_1 \) be a nef divisor on \( X_{1,0}^3 \). Consider the line \( \tilde{l} - \tilde{l}_1 \) and our assumption on \( D \) says that \((dH - m_1 E_1) \cdot (\tilde{l} - \tilde{l}_1) = d - m_1 \geq 0 \). On the other hand \( H \) and \( H - E_1 \) are nef line bundles on \( X_{1,0}^3 \). Now any divisor \( D \) on \( X_{1,0}^3 \) can be written as \( D = dH - m_1 E_1 = (d - m_1)H + m_1(H - E_1) \).

- **r = 2:**
  Let \( D = dH - \sum_{i=1}^{2} m_i E_i \) be a nef divisor on \( X_{2,0}^3 \) and let us consider the line \( \tilde{l} - \tilde{l}_1 - \tilde{l}_2 \).
  Since \( D \) if nef, \((dH - \sum_{i=1}^{2} m_i E_i) \cdot (\tilde{l} - \tilde{l}_1 - \tilde{l}_2) = d - m_1 - m_2 \geq 0 \). Conversely, any divisor on \( X_{2,0}^3 \) can be written as \( D = dH - m_1 E_1 = (d - m_1 - m_2)H + m_1(H - E_1) + m_2(H - E_2) \).

- **r = 3:**
  Let \( D = dH - \sum_{i=1}^{3} m_i E_i \) be a nef divisor on \( X_{3,0}^3 \). By Theorem 3.1 in [DPU], for any three general lines \( l_i, l_j, l_k \) in \( \mathbb{P}^3 \), there exists a unique smooth quadric surfaces \( Q_{ijk} \) containing the lines \( l_i, l_j, l_k \). If we take our lines as \( l_1, l_2, l_3 \) in this case then by generality all the three lines should belong to the same ruling of \( Q_{123} \). Then for any line \( l_0 \) in the other ruling of \( Q_{123} \), \((dH - \sum_{i=1}^{3} m_i E_i) \cdot l_0 = d - m_1 - m_2 - m_3 \geq 0 \). Conversely, we write \( dH - \sum_{i=1}^{3} m_i E_i = (d - m_1 - m_2 - m_3)H + \sum_{i=1}^{3} m_i(H - E_i) \).

- **r = 4:**
  Any four general lines \( l_i, l_j, l_k, l_l \) in \( \mathbb{P}^3 \) determines two transversal lines \( t, t' \). By Lemma 3.2 in [DPU], for any nef divisor \( D \) in \( X_{4,0}^3 \), \( D \cdot t = (dH - \sum_{i=1}^{4} m_i E_i) \cdot t = d - m_1 - m_2 - m_3 - m_4 \geq 0 \). On the other hand we write \( dH - \sum_{i=1}^{4} m_i E_i = (d - m_1 - m_2 - m_3 - m_4)H + \sum_{i=1}^{4} m_i(H - E_i) \).

- **r = 5:** For each choice of four lines \( l_1, l_2, \ldots, l_5 \) there exists two transversal lines \( t_i, t'_i \). So for any nef divisor \( D \) in \( X_{5,0}^3 \), \( D \cdot t_i = (dH - \sum_{i=1}^{5} m_i E_i) \cdot t_i = d - (m_1 + \ldots + m_5) - m_i \geq 0 \forall i = 1, 2, \ldots, 5 \).
  Conversely, w.l.o.g assume \( m_1 = \min\{m_1, \ldots, m_5\} \) after re-arranging the lines if needed. Write any divisor \( D = dH - \sum_{i=1}^{5} m_i E_i \) as \( dH - \sum_{i=1}^{5} m_i E_i = m_1(4H - \sum_{i=1}^{5} E_i) + (d - 4m_1)H - \sum_{i=1}^{5} (m_i - m_1)E_i \). Now, by Theorem 4.4 in [DPU] the anti-canonical divisor \( -K_5 = 4H - \sum_{i=1}^{5} E_i \) is nef. The other part is \((d - 4m_1)H - \sum_{i=2}^{5} (m_i - m_1)E_i \), which can be considered as a divisor in \( X_{4,0}^3 \) (considered as blow up of \( \mathbb{P}^3 \) at \( l_2, \ldots, l_5 \). By hypothesis, \( d - 4m_1 \geq 0 \) since \( m_1 = \min\{m_1, \ldots, m_5\} \) and \( d \geq (m_1 + \ldots + m_5) - m_i \) for all \( i \in \{1, 2, \ldots, 5\} \) and \((d - 4m_1) \geq \sum_{i=2}^{5} (m_i - m_1)E_i \). So by \( r = 4 \) case \((d - 4m_1)H - \sum_{i=2}^{5} (m_i - m_1)E_i \) is nef in \( X_{4,0}^3 \). So, \((d - 4m_1)H - \sum_{i=2}^{5} (m_i - m_1)E_i \) is nef in \( X_{5,0}^3 \) since \( X_{5,0}^3 \) is obtained by blowing up \( l_1 \) in \( X_{4,0}^3 \).
  As a result we can conclude that \( D = dH - \sum_{i=1}^{5} m_i E_i \) is nef.

- **r = 6:**
Using similar argument as in the previous case by choosing 4 lines at a time, we can say that a nef divisor \( D = dH - \sum_{i=1}^{6} m_i E_i \) satisfies the conditions \( d \geq m_i + m_j + m_k + m_l, \ i < j < k < l, \{i,j,k,l\} \in \{1,2,\ldots,6\} \). Conversely any divisor \( D \) can be written as \( D = dH - \sum_{i=1}^{6} m_i E_i = m_1(4H - \sum_{i=1}^{6} E_i) + (d - 4m_1)H - \sum_{i=2}^{6}(m_i - m_1)E_i \). \( (4H - \sum_{i=1}^{6} E_i) \) is the anti-canonical divisor, hence nef by Theorem 4.4 [DPU]. The fact that the other part \((d - 4m_1)H - \sum_{i=2}^{6}(m_i - m_1)E_i\) if nef can be shown by the hypothesis and \( r = 5 \) case. So, \( D \) is nef in \( X^3_{6,0} \).

Next we compute Seshadri constant of an ample line bundle at a general point on \( X^3_{r,0} \).

**Theorem 4.2.** Let \( X^3_{r,0} \) denotes the blow up of \( \mathbb{P}^3 \) along \( r \leq 6 \) general lines and let \( L = dH - \sum_{i=1}^{r} m_i E_i \) be an ample line bundle on \( X^3_{r,0} \). Then

\[
\varepsilon(X^3_{r,0}, L, 1) = \begin{cases} 
    \left( d - \sum_{i=1}^{r} m_i, \ r = 1, \ldots, 4, \right. \\
    \left. \min \left\{ \left( d - \sum_{i \leq j \leq k \leq l} m_i + m_j + m_k + m_l, \ i, j, k, l \in \{1, \ldots, r\} \right) \right\}, \ r = 5, 6. \right.
\end{cases}
\]

**Proof.** Let \( x \in X^3_{r,0} \) be a general point and \( B \) be a reduced and irreducible curve in \( X^3_{r,0} \) passing through \( x \) with multiplicity \( m \). Since \( x \) is assumed to be a general point, we can assume that \( x \) does not lie on any exceptional divisors. So, there is a curve \( C \) in \( \mathbb{P}^3 \) such that \( B = \tilde{C} \). Assume that \( C \) has multiplicities \( p_i \) along \( l_i \) and \( x_i \)'s are points such that \( \text{mult}_{x_i} C = \text{mult}_{l_i} C \) for \( i = 1, \ldots, r \). Let \( \tilde{l}_i \) be the pullback of a general line in \( \mathbb{P}^3 \) and \( \tilde{l}_i \)'s be the class of a line inside \( E_i \)'s for \( i = 1, \ldots, r \).

- **\( r = 1 \):** Consider the line connecting \( \pi_{r,0}(x) \) and \( x_1 \) whose strict transform is \( \tilde{l} - \tilde{l}_1 \). If \( \tilde{C} = \tilde{l} - \tilde{l}_1 \) then Seshadri ratio is \( d - m_1 \). If \( \tilde{C} \neq \tilde{l} - \tilde{l}_1 \), then there is a hyperplane \( H \) containing \( x_1 \) and \( \pi_{r,0}(x) \) but not containing \( C \). So, by Bézout’s theorem

\[
H \cdot C \geq p_1 + m
\]

which implies that \( dH \cdot C \geq dp_1 + dm \geq m_1p_1 + dm \). So Seshadri ratio \( \frac{d - p}{m} = \frac{(d - m_1 E_1)C}{m} = \frac{dH \cdot C - m_1 p_1}{m} \geq d \)

- **\( r = 2 \):** Let \( \pi_{r,0}(x) \) be on the line joining \( x_1 \) and \( x_2 \) whose strict transform is \( \tilde{l} - \tilde{l}_1 - \tilde{l}_2 \). If \( \tilde{C} = \tilde{l} - \tilde{l}_1 - \tilde{l}_2 \), then the Seshadri ratio is \( d - \sum_{i=1}^{2} m_i \). Otherwise choose a hyperplane \( H \) that contains the line but not \( C \). If \( \pi_{r,0}(x) \) is not on the line joining \( x_1 \) and \( x_2 \), then choose a hyperplane \( H \) that contains \( x_1 \) and \( \pi_{r,0}(x) \) but not \( x_2 \). Thus in both the cases \( C \not\subseteq H \) and by Bézout’s theorem

\[
H \cdot C \geq p_1 + m
\]

After reindexing the lines \( l_1, l_2 \) if necessary we assume that \( p_1 \geq p_2 \). Then by above inequality we have

\[
dH \cdot C \geq dp_1 + dm \geq (m_1 + m_2)p_1 + dm \geq m_1p_1 + m_2p_2 + dm
\]

So the Seshadri ratio is \( \frac{dH \cdot C - \sum_{i=1}^{2} m_i p_i}{m} \geq d \).
• \( r = 3 \): Let \( \pi_{r,0}(x) \) be on the line which intersects all three lines \( l_1, l_2, l_3 \) i.e \( x \in \tilde{l} - \sum_{i=1}^{5} \tilde{l}_i \). If \( \tilde{C} = \tilde{l} - \sum_{i=1}^{3} \tilde{l}_i \), then the Seshadri ratio is \( d - \sum_{i=1}^{3} m_i \). Otherwise choose a hyperplane \( H \) that contains the line but not \( C \). If \( x \notin \tilde{l} - \sum_{i=1}^{3} \tilde{l}_i \) then a hyperplane \( H \) that contains \( x_1 \) and \( \pi_{r,0}(x) \) but misses at least one of \( x_2 \) and \( x_3 \). Applying Bézout’s theorem and reindexing the lines such that \( p_i \geq p_{i+1} \) for all \( i = 1, 2 \), we obtain in both the cases

\[
\frac{dH \cdot C}{m} \geq dp_1 + dm \geq (\sum_{i=1}^{3} m_i)p_1 + dm \geq \sum_{i=1}^{3} m_i p_i + dm
\]

So the Seshadri ratio is \( \frac{dH \cdot C - \sum_{i=1}^{3} m_i p_i}{m} \geq d \).

• \( r = 4 \): Four general lines \( l_1, \ldots, l_4 \) in \( \mathbb{P}^3 \) determines two transversal lines \( t, t' \). If \( \pi_{r,0}(x) \in t \) or \( \pi_{r,0}(x) \in t' \) and \( C \) is one of them, then the Seshadri ratio is \( d - \sum_{i=1}^{4} m_i \). If \( C \neq t, t' \) proceeding exactly like the previous case and using the inequality \( d \geq \sum_{i=1}^{4} m_i \), we can say that The Seshadri ratio is greater than equal to \( d \).

• \( r = 5 \): For every \( i = 1, \ldots, 5 \), denote by \( t_i, t'_i \) the two transversal lines determined by four general lines \( \{l_1, \ldots, l_i, l_{i+1}, l_5\} \subseteq \mathbb{P}^3 \). If \( \pi_{r,0}(x) \) is in \( t_i \) or \( t'_i \) for some \( i \in \{1, \ldots, 5\} \) and \( C \) is one of the \( t_i \) or \( t'_i \) then the Seshadri ratio is \( d - \{\sum_{j=1}^{5} m_j - m_i\} \). If \( C \) is not any of these \( t_i \) or \( t'_i \) for \( i = 1, \ldots, 5 \) and \( C \) intersects at most four of the general lines, then we can go to the previous case. So assuming that \( C \) intersects all five given general lines, we can get a hyperplane \( H \) which contain the points of intersections of \( C \) with \( l_1 \) and \( l_2 \) but does not contain \( C \). So by Bézout’s theorem

\[
H \cdot C \geq p_1 + p_2 + m
\]

which implies that

\[
\frac{dH \cdot C}{m} \geq d(p_1 + p_2) + dm \geq (m_1 + m_2 + m_3 + m_4)p_1 + m_5p_5 + dm \geq \sum_{i=1}^{5} m_i p_i + dm
\]

So, the Seshadri ratio is greater than \( d \) and

\[
\varepsilon(X_{5,0}^3, L, 1) = \min \left\{ d - \sum_{i<j<k<l} m_i + m_j + m_k + m_l, \ i, j, k, l \in \{1, \ldots, 5\} \right\}.
\]

The case \( r = 6 \) can be dealt similarly to the \( r = 5 \) case.

\[\square\]

5. Line blow ups of \( \mathbb{P}^4 \) and \( \mathbb{P}^5 \)

The following propositions are from Proposition 5.1 and Proposition 5.2 of [PP]. We manipulate and re-write the propositions to make it convenient for our next treatise on Seshadri constant.

**Proposition 5.1.** Let \( X_{r,0}^4 \) be the blow ups of \( \mathbb{P}^4 \) at \( r \) general lines \( l_1, \ldots, l_r \). Let \( D = dH - \sum_{i=1}^{r} m_i E_i \) be any divisor on \( X_{r,0}^4 \). Then for \( r \leq 7 \), the nef cone of divisors on \( X_{r,0}^3 \) are as follows:

| \( \operatorname{Nef}(X_{1,0}^4) \) | \( \{dH - m_1 E_1 \mid d \geq m_1, d \geq 0, m_1 \geq 0\} \) |
| --- | --- |
| \( \operatorname{Nef}(X_{2,0}^4) \) | \( \{dH - \sum_{i=1}^{2} m_i E_i \mid d \geq m_1 + m_2, d \geq 0, m_i \geq 0 \text{ for } r = 1, 2\} \) |
\[ \text{Nef}(X_{r,0}^4) = \{ dH - \sum_{i=1}^r m_i E_i \mid d \geq m_i + m_j + m_k, i < j < k, i, j, k \in \{1, ..., r\}, d \geq 0, m_i \geq 0 \forall i \in \{1, ..., r\} \} \text{ for } 3 \leq r \leq 7. \]

**Proof.** By the proof of Proposition 5.1 in [PP], nef cones of divisors on \( X_{r,0}^4 \) for \( r \leq 7 \) are generated by \( H, H - E_i \) for \( i = 1, ..., r \) and \( 3H - \sum_{i=1}^r E_i \) and for any 3 lines in \( \mathbb{P}^4 \), there is a line intersecting all 3 lines. So, cases for \( r = 1, 2, 3 \) are clear. For \( r = 4 \), w.l.g assume \( m_1 = \min\{m_1, ..., m_3\} \) after re-arranging the lines if needed. Write any divisor \( D = dH - \sum_{i=1}^4 m_i E_i \) as \( dH - \sum_{i=1}^4 m_i E_i = m_1(3H - \sum_{i=1}^4 E_i) + (d - 3m_1)H + \sum_{i=2}^4 (m_i - m_1)E_i \). Now \( (3H - \sum_{i=1}^4 E_i) \) is nef from Proposition 5.1 in [PP]. The other part \( (d - 3m_1)H + \sum_{i=2}^4 (m_i - m_1)E_i \) is nef by \( r = 3 \) case. So, nef cone in this case can be written as \( \text{Nef}(X_{4,0}^4) = \{ dH - \sum_{i=1}^r m_i E_i \mid d \geq m_i + m_j + m_k, i < j < k, i, j, k \in \{1, ..., 4\}, d \geq 0, m_i \geq 0 \forall i \in \{1, ..., 4\} \} \). Other (i.e \( r = 5, 6, 7 \)) cases follow similarly by iterative process. \( \square \)

**Proposition 5.2.** Let \( X_{r,0}^5 \) be the blow ups of \( \mathbb{P}^5 \) at \( r \) general lines \( l_1, ..., l_r \). Let \( D = dH - \sum_{i=1}^r m_i E_i \) be any divisor on \( X_{r,0}^5 \). Then for \( r \leq 5 \), the nef cone of divisors on \( X_{r,0}^4 \) are described as:

\[ \text{Nef}(X_{r,0}^5) = \{ dH - \sum_{i=1}^r m_i E_i \mid d \geq m_i + m_j, i \neq j, i, j \in \{1, ..., r\}, m_i \geq 0, \forall i \in \{1, ..., r\}, d \geq 0, \} \]

**Proof.** For \( r \leq 5 \), the nef cones \( \text{Nef}(X_{r,0}^5) \) are generated by \( H, H - E_i \) for \( i = 1, ..., r \) and \( 2H - \sum_{i=1}^r E_i \) (see Proposition 5.2 in [PP]). From that we can describe the nef cones as described above by following the methods of previous propositions. \( \square \)

Next we look into the Seshadri constants of ample line bundles on \( X_{r,0}^4 \).

**Theorem 5.3.** Let \( \pi_{r,0} : X_{r,0}^4 \to \mathbb{P}^4 \) denote the the blow of \( \mathbb{P}^4 \) along \( r \leq 7 \) general lines. Let \( L = dH - \sum_{i=1}^r m_i E_i \) be an ample line bundle on \( X_{r,0}^4 \) then

\[ \varepsilon(X_{r,0}^4, L, 1) = \begin{cases} 
    d - \sum_{i=1}^r m_i, & r = 1, ..., 3. \\
    \min \left\{ d - \sum_{i<j<k} m_i + m_j + m_k, i, j, k \in \{1, ..., r\} \right\}, & 4 \leq r \leq 7.
\end{cases} \]

**Proof.** Let \( x \in X_{r,0}^4 \) be a general point and \( B \) be a reduced and irreducible curve in \( X_{r,0}^4 \) passing through \( x \) with multiplicity \( m \). Since \( x \) is assumed to be a general point, we can assume that \( x \) does not lie on any exceptional divisors. So, there is a curve \( C \) in \( \mathbb{P}^4 \) such that \( B = \tilde{C} \). Assume that \( C \) has multiplicities \( p_i \) along \( l_i \) and \( x_i \)’s are points such that \( \text{mult}_{x_i} C = \text{mult}_{l_i} C \) for \( i = 1, ..., r \). Let \( \tilde{l} \) be the pullback of a general line in \( \mathbb{P}^4 \) and \( \tilde{l}_i \)’s be the class of a line inside \( E_i \)’s for \( i = 1, ..., r \).

- \( r = 1, 2 \): The proof is similar to the \( r = 1 \) and \( r = 2 \) cases of Theorem 4.2.
- \( r = 3 \): For any three lines in \( \mathbb{P}^4 \) there is a line intersecting all of them. Let \( l_{123} \) be the line intersecting \( l_1, l_2, l_3 \). Then If \( C = l_{123} \), then the Seshadri ratio is \( d - \sum_{i=1}^3 m_i \). If \( C \neq l_{123} \), then we can find a hyperplane that contains \( x_1 \) and \( \pi_{r,0}(x) \) but misses a point of \( C \) (even if \( x_i \)’s
for $i = 1, 2, 3$ and $\pi_{r,0}(x)$ are co-linear, since $C \neq l_{123}$, we are able to find a hyperplane $H$ s.t $C \not\subset H$. So, Bézout’s theorem gives $H \cdot C \geq p_1 + m$. After re-indexing the lines such that $p_i \geq p_{i+1}$ for all $i = 1, 2, 3$, we get

$$dH \cdot C \geq dp_1 + dm \geq (m_1 + m_2 + m_3)p_1 + dm \geq \sum_{i=1}^{3} m_ip_i + dm$$

So, the Seshadri ratio is greater than $d$ and $\varepsilon(X^4_{r,0}, L, 1) = d - \sum_{i=1}^{3} m_i$.

- $r = 4, 5, 6$: There is a line $l_{ijk}$ intersecting any three lines $l_i, l_j, l_k$, where $i < j < k$ and $i, j, k \in \{1, \ldots, 6\}$. If $C = l_{ijk}$ for some $i, j, k \in \{1, \ldots, r\}$, then the Seshadri ratio is $d - (m_i + m_j + m_k)$. If $C$ is not any of these lines $l_{ijk}$ then since no four of $x_i$’s and $\pi_{r,0}(x)$ can be co-linear we can find a hyperplane $H$ containing $x_1, x_2$ and $\pi_{r,0}(x)$ but not containing $C$. So, Bézout’s Theorem gives $H \cdot C \geq p_1 + p_2 + m$.

After re-indexing the lines such that $p_i \geq p_{i+1}$ for all $i = 1, \ldots, 6$, we obtain

$$dH \cdot C \geq dp_1 + dp_2 + dm \geq (m_1 + m_2 + m_3)p_1 + (m_4 + \ldots + m_r)p_2 + dm \geq \sum_{i=1}^{4} m_ip_i + dm$$

where $4 \leq r \leq 6$. So the Seshadri ratio is greater than equal to $d$. Consequently,

$$\varepsilon(X^4_{r,0}, L, 1) = \min \left\{d - \sum_{i<j<k} m_i + m_j + m_k, \ i, j, k, l \in \{1, \ldots, r\}\right\}, \ 4 \leq r \leq 6.$$

- $r = 7$: If $C = l_{ijk}$ for some $i, j, k \in \{1, \ldots, 7\}$ then the Seshadri ratio is $d - m_4m_5m_6$. Otherwise, after re-indexing the $l_i$ as in previous cases we can choose a hyperplane $H$ that contains $x_1, x_2, x_3$ and $\pi_{r,0}(x)$ but does not contain $C$. By Bézout’s theorem then, $H \cdot C \geq \sum_{i=1}^{3} p_i + m$, which implies that

$$dH \cdot C \geq d \sum_{i=1}^{3} p_i + dm \geq \left(\sum_{i=1}^{3} m_i\right)p_1 + \left(\sum_{i=4}^{6} m_i\right)p_2 + m_7p_7 + dm \geq \sum_{i=1}^{7} m_ip_i + dm$$

So, the Seshadri ratio in this case is $\geq d$.

\[ \square \]

**Theorem 5.4.** Let $\pi_{r,0} : X^5_{r,0} \rightarrow \mathbb{P}^5$ denote the the blow of $\mathbb{P}^5$ along $r \leq 5$ general lines. Let $L = dH - \sum_{i=1}^{r} m_i E_i$ be an ample line bundle on $X^5_{r,0}$ then

$$\varepsilon(X^5_{r,0}, L, 1) = \begin{cases} d - \sum_{i=1}^{r} m_i, \ r = 1, 2 \\ \min \left\{d - \sum_{i<j} m_i + m_j, \ i, j \in \{1, \ldots, r\}\right\}, \ 3 \geq r \geq 5. \end{cases}$$

**Proof.** Using Proposition 5.2 it can be proved similarly to the last theorem. \[ \square \]

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