Estimation of the autocovariance function with missing observations.

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Abstract

We propose a novel estimator of the autocorrelation function in presence of missing observations. We establish the consistency, the asymptotic normality, and we derive deviation bounds for various classes of weakly dependent stationary time series, including causal or non causal models. In addition, we introduce a modified version periodogram defined from these autocorrelation estimators and derive asymptotic distribution of linear functionals of this estimator.

1 Introduction

The estimation of the sample autocorrelation function (hereafter ACF) from observations of $X_1, \ldots, X_N$ is important to understand the process and allows model identification.

In the classical time series analysis, the innovations $(\epsilon_i)_{i \in \mathbb{Z}}$ in the linear process $(X_i)$ are often assumed to be independent and identically distributed (iid), see for example \cite{BrockwellDavis1991}, \cite{BoxJenkins1970}. In this case asymptotic properties of the partial sums, especially the sample ACF and the ratio of the sample covariance have been extensively studied in the literature. A summary of results about the asymptotical theory of the sample ACF of autoregressive processes can be found for instance in \cite{BrockwellDavis1991}, Chapter 7.2 and 13.3, or \cite{Embrechts1997}, Chapter 7.3.

In practice, however, frequently the time series are not fully observed, and there may often be substantial numbers of missing values for a variety of reasons. The analysis of irregularly observed time series is one of the most important problems faced by applied researchers whose data arise in the form of time series. The study of the asymptotic properties of the ACF function of a time series model in presence of missing observations is more difficult than in the complete case.

Most of the literature above asymptotic properties of time series with missing observations is concerned with linear processes with normal innovations. In addition, these perturbations are usually regarded as strict white noise. This assumption is very restrictive; this characteristic implies only linear models with homoskedastic conditional variances. As far as we know, the first study that extended the sample ACF to the case of missing observations is \cite{Parzen1963}. Their study formulated that the values of the
observed series at unequally spaced times can be represented as an amplitude modulated
time series \( Y_i = C_i X_i \) where \( (C_i)_{i \in \mathbb{Z}} \) represents the censoring process. The asymptotic
properties of this modified ACF were investigated in [Dunsmuir and Robinson, 1981] under various assumptions on the noise of the linear representation \( (\epsilon_i)_{i \in \mathbb{Z}} \). More recently, [Yajima and Nishino, 1999] compare three estimators of the autocorrelation function for a stationary process with missing observations. The first estimator is the sample ACF extended to the case with censored data proposed originally by [Parzen, 1963]. The others estimators are extensions of this first estimator. The authors derive asymptotic distribution for both short memory and long memory models for the three estimators of the ACF with missing observations. They impose the same assumptions on the innovations \( (\epsilon_i)_{i \in \mathbb{Z}} \) as those in [Dunsmuir and Robinson, 1981].

The results obtained for the weak convergence studies for sample ACF in presence of missing observation assume asymptotic stationary to fourth order for the \( (C_i)_{i \in \mathbb{Z}} \), then the central limit theorem is given by [Dunsmuir and Robinson, 1981] for \( \sqrt{N} (\hat{\gamma}_{Y,N}(\ell) - \hat{\gamma}_{C,N}(\ell)\gamma_X(\ell)) \). From this, the central limit theorem can be deduced for the \( \sqrt{N} (\hat{\rho}_{X,N}(\ell) - \rho_X(\ell)) \), where \( \hat{\rho}_{X,N}(\ell) = \hat{\gamma}_{X,N}(\ell)/\hat{\gamma}_{X,N}(0) \) is the lag-\( \ell \) serial correlation and \( \rho_X(\ell) = \gamma_X(\ell)/\gamma_X(0) \). In particular, if the \( (X_i)_{i \in \mathbb{Z}} \) are iid with finite fourth moment then \( \hat{\rho}_{X,N}(\ell) \) are asymptotically independent normal.

The asymptotic problem of the sample ACF becomes more difficult if dependence
among \( (\epsilon_i)_{i \in \mathbb{Z}} \) is allowed. Financial time series often exhibit that the conditional variance
can change over time, namely heteroskedasticity. Thus, the classical limit theorems cannot
be directly applied to process with the above condition.

Theorem 6.7 in [Hall and Heyde, 1980] (p. 188) asserts asymptotic normality of sample correlations for martingale differences \( (\epsilon_i)_{i \in \mathbb{Z}} \) for which \( \mathbb{E}(\epsilon_i^2|\mathcal{F}_{i-1}) = \) a positive constant. In the literature the above condition is widely used. However, this condition appears too restrictive and it excludes many important models. Among them the most interesting case is the ARCH model. Thus, limit theorems by [Hall and Heyde, 1980] or [Dunsmuir and Robinson, 1981] cannot be directly applied to linear processes with ARCH innovations. Our results avoid this limitation.

On the other hand, various generalizations of independence have been introduced in order to extend the theory that exists to the independence framework to the more general models. The more recent is the notion of weak dependence introduced and developed by [Doukhan and Louhichi, 1999]. Our choice is explained by numerous reasons; the frame of weak dependence includes large classes of models and can be easily used in a very large statistic problems.

We shall consider the estimator of the ACF in presence of missing observations. The asymptotic behavior of the sample ACF is examined for a very general process included for the first time process whose innovations are dependent. Central limit theorems are established under fairly mild conditions.

Two frame of weak dependence are considered in this study. The first one exploits a
causal property of dependence, the \( \theta \)-weak dependence property (see [Dedecker and Doukhan, 2003]). Under some conditions, the asymptotic normality of the covariance function with missing or censored observations is found. The second frame of weak dependence, the \( \lambda \)-weak dependence property (see [Doukhan and Wintenberger, 2007]), which includes \( \eta \) and \( \kappa \)-weak dependences. This notion is convenient for Bernoulli shifts with associated inputs.

The paper is organized as follows. In Section 2 we introduce the notation and various weak dependent coefficients. Section 3 is devoted to limit theorems for causal and non
causal weakly dependent time series. Proofs and technical results are given in the last section.

2 Notations and Main assumptions

Let \((X_i)_{i \in \mathbb{Z}}\) be a discrete-time second-order stationary time series with (zero-mean). Following [Parzen, 1963], we assume that the observations are given by

\[ Y_i = C_i X_i, \]

where \((C_i)_{i \in \mathbb{Z}}\) is a non-negative modulating process taking values in \([0, 1]\). When \(C_i\) takes values in \(\{0, 1\}\), the observations are censured, but more general modulations can be considered as well. Throughout the paper, this process is assumed to be independent from \((X_i)_{i \in \mathbb{Z}}\). This property is essential in order to allow recovery of the covariance structure of \((X_i)_{i \in \mathbb{Z}}\).

We denote by \(\overline{X}_N, \overline{Y}_N\) the sample means of \((X_i)_{i=1}^n\) and \((Y_i)_{i=1}^n\) and by \(\hat{\gamma}_{X,N}(\ell)\) and \(\hat{\gamma}_{Y,N}(\ell)\) the usual estimates of the covariances \(\gamma_X(\ell) = \text{Cov}(X_0, X_\ell)\) and \(\gamma_Y(\ell) = \text{Cov}(Y_0, Y_\ell)\).

The so-called Parzen estimator of the autocovariance coefficient \(\gamma_X(\ell)\), is given by

\[ \hat{\gamma}_{X,N}(\ell) = \frac{\sum_{i=1}^{N-\ell} (Y_i - \overline{Y}_N)(Y_{i+\ell} - \overline{Y}_N)}{\sum_{i=1}^{N-\ell} C_i C_{i+\ell}} = \frac{\hat{\gamma}_{Y,N}(\ell)}{\tilde{v}_C,N(\ell)}. \]

Similarly the autocorrelation function \(\rho_X(\ell)\) is estimated by \(\hat{\rho}_{X,N} = \hat{\gamma}_{X,N}(\ell)/\hat{\gamma}_{X,N}(0)\).

We study both the consistency and asymptotic normality of the autocovariance and autocorrelation functions of time series with missing observations, and also establish non-asymptotic deviation bounds. These results are obtained under general dependence structures.

2.1 Weak-dependence measures

Let \(p\) be a positive integer. For \(f : \mathbb{R}^p \to \mathbb{R}\) a function, define \(\text{Lip } f\) the Lipschitz coefficient by:

\[ \text{Lip } f = \sup_{(x_1, \ldots, x_p) \neq (y_1, \ldots, y_p)} \frac{|f(x_1, \ldots, x_p) - f(y_1, \ldots, y_p)|}{|x_1 - y_1| + \cdots + |x_p - y_p|}. \]

Definition 1. [Doukhan and Louhichi, 1999] The vector-valued \((d \times 1)\) process \((Z_i)_{i \in \mathbb{Z}}\) is said to be weakly dependent if

\[ |\text{Cov}(f(Z_{s_1}, \ldots, Z_{s_u}), g(Z_{t_1}, \ldots, Z_{t_v}))| \leq \psi(u, v, \text{Lip } f, \text{Lip } g) \epsilon_Z(r), \]

for any real valued functions \(f\) and \(g\) defined respectively on \(\mathbb{R}^d\) and \(\mathbb{R}^d\), that satisfy \(\|f\|_{\infty}, \|g\|_{\infty} \leq 1\) and \(\text{Lip } f, \text{Lip } g < \infty\), and for any \(r \geq 0\) and any \((u + v)\)-tuples such that \(s_1 \leq \cdots \leq s_u \leq s_u + r \leq t_1 \leq \cdots \leq t_v\). Here, the sequence \((\epsilon_Z(r))_{r=0}^{\infty}\) is assumed to decrease to zero at infinity and \(\psi : \mathbb{N}^2 \times (\mathbb{R}^+)^2 \to \mathbb{R}^+\) is a function.

Specific functions \(\psi\) yield different notions of weak dependence which have been shown to be appropriate to cover various time-series settings [Dedecker et al., 2007]:

- \(\psi(u, v, a, b) = vb\) corresponds to the notion of \(\theta\)-dependence.
• $\psi(u,v,a,b) = uvab$, corresponds to the notion of $\kappa$-dependence.

• $\psi(u,v,a,b) = uvab + ua + vb$, corresponds to the notion of $\lambda$-dependence.

For simplicity the sequence $\epsilon_X(r)$ will be denoted respectively as $\theta_X(r)$, $\kappa_X(r)$, and $\lambda_X(r)$. We shall also consider strong mixing coefficients $\alpha_X(r)$ related with $\psi(u,v,a,b) \equiv 1$; in this case heredity is complete through measurable images, see [Rio, 2000] or [Doukhan, 1994].

The following simple lemma, which relies on the decomposition of the covariance of two random variables conditioned to two independent $\sigma$-algebras, is intrumental in the sequel:

**Proposition 1.** Assume that $(U_i)_{i \in \mathbb{Z}}$ and $(V_i)_{i \in \mathbb{Z}}$ are two vector-valued independent processes. Assume in addition that these two processes are $(\epsilon,\psi)$-weakly dependent with the same $\psi$-function and sequences denoted $(\epsilon_U(r))_{r \in \mathbb{Z}}$ and $(\epsilon_V(r))_{r \in \mathbb{Z}}$, respectively. Then the vector-valued process $(W_i)_{i \in \mathbb{Z}}$ with $W_i = (U_i, V_i)^T$ is also $(\epsilon_W, \psi)$-weakly dependent with $\epsilon_W(r) = \epsilon_U(r) + \epsilon_V(r)$.

As a consequence, provided that $(X_i)_{i \in \mathbb{Z}}$ and $(C_i)_{i \in \mathbb{Z}}$ are both $(\epsilon,\psi)$-weakly dependent processes, then the process $2 \times 1$-process $Z_i = (X_i, C_i)$ also: weak dependences of its coordinates are equivalent to that of this process. These coefficients have some hereditary properties. For example, heredity through Lipschitz functions is clear and this may be extended to locally Lipschitz functions [Dedecker et al., 2007].

**Proposition 2.** Let $(U_n)_{n \in \mathbb{Z}}$ be a sequence of $\mathbb{R}^k$-valued random variables. Let $m > 1$. We assume that there exists some constant $C > 0$ such that $\max_{1 \leq i \leq k} \|U_i\| \leq C$. Let $h$ be a function from $\mathbb{R}^k$ to $\mathbb{R}^d$ such that $h(0) = 0$ and for $x,y \in \mathbb{R}^k$, there exist $a$ in $[1,m[$ and $c > 0$ such that

$$|h(x) - h(y)| \leq c|x - y|(1 + |x|^{a-1} + |y|^{a-1}).$$

We define the sequence $(V_n)_{n \in \mathbb{Z}}$ by $V_n = h(U_n)$. Then, if $(U_n)_{n \in \mathbb{Z}}$ is weakly dependent then $(V_n)_{n \in \mathbb{Z}}$ is also weakly dependent and,

- $\theta_V(r) = O \left( \theta_U(r) \frac{m-a}{m+a} \right)$;
- $\kappa_V(r) = O \left( \kappa_U(r) \frac{m-a}{m+a-2} \right)$;
- $\lambda_V(r) = O \left( \lambda_U(r) \frac{m-a}{m+a-2} \right)$.

**Remark 1.** For $U_n = (X_n, X_{n+\ell}, C_n, C_{n+\ell})$ the function $h(x,x',c,c') = cc'\{xx' - \gamma_X(\ell)\}$ satisfies the previous assumptions with $a = 2$.

### 3 Limit theorems

In this section, we study the asymptotic properties of the Parzen estimator of the ACF under the different dependence conditions mentioned above. Denote by

$$\nu(\ell) = \mathbb{E}[C_0C_\ell] \quad \text{and} \quad m(\ell,k,m) = \mathbb{E}(C_0C_\ell C_k C_m).$$

As a consequence of Slutsky lemma and the results in Dedecker et alii (2007) we immediately derive
Theorem 1. Let \( (X_i)_{i \in \mathbb{Z}} \) be a real valued, stationary sequence time series of square integrable observed, with censored data. Let the modulating process \( (C_i)_{i \in \mathbb{Z}} \) be a nonnegative bounded stationary process. We assume that \( (C_i)_{i \in \mathbb{Z}} \) be independent of the process \( (X_i)_{i \in \mathbb{Z}} \). Assume either

- \( (X_i)_{i \in \mathbb{Z}} \) and \( (C_i)_{i \in \mathbb{Z}} \) are strong mixing stationary time series, \( \mathbb{E}|X_0|^m < \infty \) for \( m > 4 \) and \( \sum_{i \geq 0} i^{m-2} \alpha(i) < \infty \).
- \( (X_i)_{i \in \mathbb{Z}} \) and \( (C_i)_{i \in \mathbb{Z}} \) are \( \theta \)-weakly dependent stationary time series, \( \mathbb{E}|X_0|^m < \infty \) for \( m > 4 \) and \( \sum_{i \geq 0} i^{m-2} \theta^{m-1} \alpha(i) < \infty \).
- \( (X_i)_{i \in \mathbb{Z}} \) and \( (C_i)_{i \in \mathbb{Z}} \) are \( \kappa \)-weakly dependent stationary time series, \( \mathbb{E}|X_0|^m < \infty \) for \( m > 4 \), \( \kappa(r) = O(r^{-\kappa}) \) as \( r \to \infty \) for \( \kappa > \frac{m}{m-2} \left( 2 + 1/(m-2) \right) \).
- \( (X_i)_{i \in \mathbb{Z}} \) and \( (C_i)_{i \in \mathbb{Z}} \) are \( \lambda \)-weakly dependent stationary time series, \( \mathbb{E}|X_0|^m < \infty \) for some \( m > 4 \), \( \lambda(r) = O(r^{-\lambda}) \) as \( r \to \infty \) for \( \lambda > \frac{m}{m-2} \left( 1 + 1/(m-2) \right) \).

Then, under any of these assumptions,

\[
\sqrt{N} \nu(\ell) (\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell) ) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_\ell^2),
\]

where

\[
\sigma_\ell^2 = \sum_{k \in \mathbb{Z}} m(\ell, k, k + \ell) \left[ \kappa_4(\ell, k, k + \ell) + \gamma_X(k + \ell) \gamma_X(k - \ell) - \gamma_X^2(\ell) \right]. (4)
\]

Remark 2. For non-causal dependent sequences (\( \lambda \) and \( \kappa \)-weak dependence cases), the assumptions need to be more restrictive and stronger than for the causal \( \theta \)-weak dependence case. For this case, indeed, the dependence condition rewrites \( \theta(r) = O(r^{-\theta}) \) as \( r \to \infty \) for \( \theta > \frac{m-1}{m-2} \left( 1 + 1/(m-2) \right) \).

Remark 3. From [Dedecker et al., 2007], we know that conditions of Theorem 1 are sufficient to obtain the weak invariance principle in the \( \theta \)-weak dependence frame. Analogously, [Doukhan and Wintenberger, 2007] show the same principle in the \( \lambda \) and \( \kappa \)-weak dependence cases.

By polarization, we simply derive the following extension of this theorem which will be used hereafter (the proof is left to the reader).

Corollary 1. Under the assumptions of Theorem 1, for all \( k \in \mathbb{N}, \ell_1 < \cdots < \ell_k \in \mathbb{N}^k \),

\[
\sqrt{N} \left( \nu(\ell_1)(\tilde{\gamma}_{X,N}(\ell_1) - \gamma_X(\ell_1)) \right)_{1 \leq i \leq k} \xrightarrow{\mathcal{D}} \mathcal{N}_k(0, \Sigma(\ell_1, \ldots, \ell_k))
\]

where \( \Sigma(\ell_1, \ldots, \ell_k) = (\sigma_{\ell_i, \ell_j})_{1 \leq i, j \leq k} \) is defined by

\[
\sigma_{\ell_i, \ell_j}^2 = \sum_{k \in \mathbb{Z}} m(\ell_i, k, k + \ell_j) \left[ \kappa_4(\ell_i, k, k + \ell_j) + \gamma_X(k + \ell_j) \gamma_X(k - \ell_i) - \gamma_X^2(k + \ell_j - \ell_i) \right]. (5)
\]

The following results provide \( a.s. \) asymptotic behaviour of the Parzen autocovariance.
Theorem 2.
\[ \hat{\gamma}_{X,N}(\ell) - \gamma_X(\ell) \xrightarrow{a.s.} 0, \]
if \( \mathbb{E}|X_0|^m < \infty \) for some \( m = 2 + \delta \) and, either the processes are \( \theta \)-dependent with \( \sum_{i \geq 0} i^{-\delta/(1+\delta)} \theta^{\delta+i} (i) < \infty \) for some \( \delta > 0 \), or they are \( \kappa \), or \( \lambda \)-dependent and satisfy assumptions from theorem \([7]\).

4 Division

Let \((U_i, V_i)_{i \in \mathbb{Z}}\) be a stationary sequence and set \( \hat{D}_n = 1/n \sum_{i=1}^n U_i \), \( \bar{N}_n = 1/n \sum_{i=1}^n U_i V_i \) then
\( N_n = N = \mathbb{E}U_1V_1, D_n = D = \mathbb{E}U_1 \) and \( \bar{R}_n = \bar{N}_n/\hat{D}_n \), \( R_n = R = N/D \).

**Theorem 3.** Let \((U_i, V_i)_{i \in \mathbb{Z}}\) be a stationary sequence with \( U_i \geq 0 \) (as.). Let \( 0 < p < q \) and assume that for \( r = \frac{pq}{q-p} \) and \( s = \frac{p(q+2)}{q-p} \):
\[ \|U_iV_i\|_r \leq c, \quad \|V_i\|_s \leq c. \]
Assume that the dependence structure of the sequence \((U_i, V_i)_{i \in \mathbb{Z}}\) is such that
\[ \|\hat{D}_n - D\|_q \leq \frac{C}{\sqrt{n}}, \quad \|\bar{N}_n - N\|_p \leq \frac{C}{\sqrt{n}} \quad \text{(6)} \]
then \( \|\bar{R}_n - R\|_p = O\left(1/\sqrt{n}\right) \).

In the following cases, we assume that \( \|V_i\|_s \leq c \) and \( \|U_iV_i\|_r \leq c \) and prove that (6) holds. Denote \( Z_i = U_iV_i - \mathbb{E}U_iV_i \). For simplicity we will often assume \( \|U_i\|_\infty < \infty, \quad \|V_i\|_r < \infty \).

4.1 Independent case

Assume that \((U_i, V_i)\) is i.i.d. Assume that \( \|U_0\|_q \leq c \) and \( \|U_0V_0\|_p \leq c \). From the Marcinkiewicz-Zygmund inequality for independent variables, for \( 2 \leq q \leq r \), we get
\[ \mathbb{E} \left| \hat{D}_n - D \right|^q \leq C_q \mathbb{E}|U_1|^q n^{-\frac{q}{2}} \leq C n^{-\frac{q}{2}}, \]
\[ \mathbb{E} \left| \bar{N}_n - N \right|^p \leq C_p \mathbb{E}|U_1V_1|^p n^{-\frac{p}{2}} \leq C n^{-\frac{p}{2}}, \]
and (6) holds. Now Hölder inequality implies those relations if \( \|U_0\|_q \), and \( \|V_0\|_{\frac{qp}{q+p}} < \infty \).

4.2 Strong mixing case

Denote \((\alpha_i)_{i \in \mathbb{Z}}\) the strong mixing coefficient sequence of the stationary sequence \((U_i, V_i)_{i \in \mathbb{Z}}\).

**Proposition 3.** Assume that for \( r' > q \), \( \|U_0\|_r \leq c \). Relation (6) holds if \( \alpha_i = O(i^{-\alpha}) \) with
\[ \alpha > \left( \frac{p}{2} \cdot \frac{r}{r-p} \right) \lor \left( \frac{q}{2} \cdot \frac{r'}{r'-p} \right), \]

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4.3 Causal weak dependence

Define the $\gamma$ coefficient of dependence of a centered sequence $(W_i)_{i \in \mathbb{N}}$ with values in $\mathbb{R}^d$ by

$$\gamma_i = \sup_{k \geq 0} \|\mathbb{E}(W_{i+k}|M_k)\|_1$$

**Proposition 4.** Assume that $\|U_0\|_{\infty} \leq c$. Relation (6) holds if the sequence of coefficients $\gamma$ associated to the stationary sequence $(W_i)_{i \in \mathbb{N}} = (U_i, V_i)_{i \in \mathbb{N}}$ is such that $\gamma_i = \mathcal{O}(i^{-\gamma})$ with

$$\gamma > \left( \frac{p}{2} \cdot \frac{r - 1}{r - p} \right) + \frac{q}{2}.$$ 

4.4 Non causal weak dependence

Here we consider non causal weakly dependent stationary sequences of bounded variables and assume that $p$ and $q$ are integers. A sequence $(W_i)_{i \in \mathbb{N}}$ is said to be $\lambda$-weakly dependent if there exists a sequence $(\lambda(i))_{i \in \mathbb{N}}$ decreasing to zero at infinity such that:

$$\left| \text{Cov}(g_1(W_{i_1}, \ldots, W_{i_u}), g_2(W_{j_1}, \ldots, W_{j_v})) \right| \leq (u \text{Lip} g_1 + v \text{Lip} g_2 + w \text{Lip} g_1 \text{Lip} g_2) \lambda(k),$$

for any $u$-tuple $(i_1, \ldots, i_u)$ and any $v$-tuple $(j_1, \ldots, j_v)$ with $i_1 \leq \cdots \leq i_u < i_u + k \leq j_1 \leq \cdots \leq j_v$ where $g_1, g_2$ are two real functions of $\Lambda^{(1)} = \{g \in \Lambda| |g|_{\infty} \leq 1\}$ respectively defined on $\mathbb{R}^{Du}$ and $\mathbb{R}^{Dv}$ $(u, v \in \mathbb{N}^*)$. Recall here that $\Lambda$ is the set of functions with $\text{Lip} g_1 < \infty$ for some $u \geq 1$, with

$$\text{Lip} g_1 = \sup_{(x_1, \ldots, x_u) \neq (y_1, \ldots, y_u)} \frac{|g_1(y_1, \ldots, y_u) - g_1(x_1, \ldots, x_u)|}{|y_1 - x_1| + \cdots + |y_u - x_u|}.$$ 

The monograph Dedecker et al. (2007) [Dedecker et al., 2007] details weak dependence concepts, as well as extensive models and results.

**Proposition 5.** Assume that $p$ and $q \geq 2$ are even integers. Assume that the stationary sequence $(U_i, V_i)_{i \in \mathbb{N}}$ is $\lambda$-weakly dependent. Assume that $Z_0 = U_0V_0 - \mathbb{E}U_0V_0$ is bounded by $M$. Relation (7) holds if $\lambda(i) = \mathcal{O}(i^{-\lambda})$ with $\lambda > \frac{3}{2}$.

**Remarks**

- Unbounded random variables may also be considered under an additional concentration inequality ($\mathbb{P}(Z_i \in (x, x+y)) \leq Cy^a$ for some $a > 0$) and Theorem 3 and Lemma 1 from Doukhan and Louhichi (1999) [Doukhan and Louhichi, 1999], imply that the same relation holds if $\mathbb{E}|Z_i|^{q+\delta} < \infty$, $\sup_x \mathbb{P}(Z_i \in (x, x+y)) \leq Cy^a$, $(\forall y > 0)$, and

$$\sum_{n=1}^{\infty} n^{q+\delta-q}\gamma^{\frac{q+\delta}{q+\delta-q}}(n) < \infty.$$

- Non-integer moments $q \in (2, 3)$ are considered in Doukhan and Wintenberger (2007) (see [Doukhan and Wintenberger, 2007], Lemma 4), and the same inequality holds if $\mathbb{E}|Z_i|^{q'} < \infty$ with $q' = q + \delta$, and $\lambda(i) = \mathcal{O}(i^{-\lambda})$ with $\lambda > 4 + 2/q'$ for $q$ small enough:

$$q \leq 2 + \frac{1}{2} \left( \sqrt{(q' + 4 - 2\lambda)^2 + 4(\lambda - 4)(q' - 2) - 2} + q' + 4 - 2\lambda \right) \left( \leq q' \right).$$

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5 Spectral estimation

In this section we study estimation of functionals of the spectral density function from the censored time series. Using the Parzen estimator of the covariance \( \hat{\gamma}_{X,N}(\ell) \), estimates of the spectral density of \((X_i)_{i \in \mathbb{Z}} \) can be constructed. More precisely, we introduce a modified periodogram defined with the empirical covariance \( \hat{\gamma}_{X,N}(\ell) \) of the censored process

$$
\hat{I}_N(\lambda) = \sum_{\ell \in \mathbb{Z}} \hat{\gamma}_{X,N}(\ell) e^{-i\ell \lambda}.
$$

Assume now that we wish to estimate linear functionals of the Parzen periodogram,

$$
J_X(g) = \int_{-\pi}^{+\pi} g(\lambda) f_X(\lambda),
$$

where \( f_X(\lambda) \) is the spectral density of \((X_i)_{i \in \mathbb{Z}} \). Using the estimates of the covariances of \((X_i)_{i \in \mathbb{Z}} \) given by \( \hat{\gamma}_{X,N}(\ell) \), estimations of the integrated periodogram of \((X_i)_{i \in \mathbb{Z}} \) can be constructed by

$$
\hat{J}_{X,N}(g) = \int_{-\pi}^{+\pi} g(\lambda) \hat{I}_N(\lambda) d\lambda.
$$

Under general conditions the integrated periodogram is a consistent estimator of the \( J_X(g) \) provided the spectral density \( f_X(\lambda) \) is well defined.

Our aim is to study the asymptotic behavior of \( \mathbb{E}|\hat{J}_{X,N}(g) - J_X(g)|^q \). We consider the Sobolev space \( \mathcal{H}_s \) for \( s > 1 \):

$$
\mathcal{H}_s = \{ g \in L^2[-\pi, \pi]; g(-x) = g(x), \|g\|_{\mathcal{H}_s}^2 < \infty \}, \quad \text{with } \|g\|_{\mathcal{H}_s}^2 = \sum_{\ell \in \mathbb{Z}} (1 + |\ell|)^s |g_\ell|^2.
$$

for \( g \) a 2\(\pi\)-periodic function such that \( g \in L^2([-2\pi, 2\pi]) \) and \( g(\lambda) = \sum_{\ell \in \mathbb{Z}} g_\ell e^{i\ell \lambda} \).

The norm of the dual space \( \mathcal{H}_s^\prime \) of \( \mathcal{H}_s \) writes

$$
\|T\|_{\mathcal{H}_s^\prime} = \sup_{\|g\|_{\mathcal{H}_s} \leq 1} |T(g)|^2 = \sum_{\ell \in \mathbb{Z}} (1 + |\ell|)^{-s} |T(e^{i\ell \lambda})|^2.
$$

Note that \( \hat{J}_{X,N}(g) \) and \( J_X(g) \in \mathcal{H}_s^\prime \). We have

**Theorem 4.** Let \((X_i)_{i \in \mathbb{Z}}\) is a real valued time series observed with censored data such that \( \sum_{\ell \in \mathbb{Z}} \gamma_X(\ell)^2 < \infty \) and \((\kappa_4(i,j,k))_{i,j,k}\) the fourth cumulants of \( X \) exist. If assumptions of Lemma 2 are satisfied, then \( \forall g \in \mathcal{H}_s \)

$$
\lim_{N \to \infty} \mathbb{E}\|\hat{J}_{X,N}(g) - J_X(g)\|^2_{\mathcal{H}_s^\prime} \longrightarrow 0.
$$

**Theorem 5.** Under assumptions of Theorem 4, the central limit theorem is satisfied if we assume that \( \|\hat{\gamma}_{C,N}(\ell) - \gamma_C(\ell)\|_q \leq v_n \) for some \( q > 2 \), and \( \|\hat{\gamma}_{Y,N}(\ell) - \gamma_Y(\ell)\|_2 \leq v_n \). Moreover if \( \|X_iX_{i+\ell}C_{i+\ell}\|_r \leq K, \) and \( \|X_iX_{i+\ell}\|_s \leq k \). Then

$$
\sqrt{N}[\hat{J}_{X,N}(g) - J_X(g)] \xrightarrow{D} N(0, \sigma^2(g)) \quad \text{in the space } \mathcal{H}_s^\prime.
$$
6 Technical results and proofs

Our proof for central limit theorems is based on a weak invariance principle under weak dependence conditions.

6.1 Proof of Theorem 1

To study the asymptotic behavior of $\tilde{\gamma}_{X,N}(\ell) = \hat{\gamma}_{Y,N}(\ell)/\hat{\nu}_N(\ell)$ we decompose the quantity of interest:

$$\sqrt{N} \{\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell)\} = \frac{\hat{\gamma}_{Y,N}(\ell)}{\hat{\nu}_N(\ell)} - \gamma_X(\ell)$$

$$= \hat{\nu}_N^{-1}(\ell) \cdot \sqrt{N} \{\hat{\gamma}_{Y,N}(\ell) - \gamma_X(\ell)\hat{\nu}_N(\ell)\}. \quad (7)$$

The second factor in the RHS of the previous expression is handled by using a Central Limit Theorem for weakly dependent sequences.

$$\sqrt{N} \{\hat{\gamma}_{Y,N}(\ell) - \gamma_X(\ell)\hat{\nu}_N(\ell)\} = N^{-1/2} \sum_{k=1}^{N-\ell} Z_k, \quad (8)$$

with $Z_k = C_kC_{k+\ell}(X_kX_{k+\ell} - \gamma_X(\ell))$. According to Propositions 1 and 2, $(Z_k)$ is weakly dependent. For the $\kappa$ and $\lambda$-dependence, Theorem 1 of [Dedecker et al., 2007, Corollary 7.5] states that if $S_N = \sqrt{N}(\hat{\gamma}_{X,N}(\ell) - \gamma_X(\ell))$ is asymptotically normally distributed by showing that $N^{-1} \sum Z_k$ satisfies a central limit theorem for $\theta$–weak dependent variables. Theorem 2 of [Dedecker and Doukhan, 2003] states that if $S_N = N^{-1} \sum Z_k$ be a strictly stationary sequence of square integrable an centered random variables, then if the condition

$$Z_0 \mathbb{E}(S_N \mid \mathcal{M}_0) \text{ converges in } \mathbb{L}_1; \quad (9)$$

holds, where $\mathcal{M}_i = \sigma(Z_j; j < i)$, then $N^{-1/2}S_N$ will be asymptotically normally distributed.

Alternatively, by Corollary 1 of [Dedecker and Doukhan, 2003], in the $\theta$-weak dependence frame, $D(2; X_0)$ implies $[\mathbb{S}]$ where

$$D(p, X) : \int_0^{\|Y\|_p} (\theta^{-1}(2u))^{p-1} Q_X^{p-1} \circ G_X(u)du < \infty. \quad (10)$$

and for $X$ a real valued random variable, $Q_X$ denotes the generalized inverse of the tail function $x \mapsto \mathbb{P}(|X| \leq x)$, and $G_X$ the inverse of $x \mapsto \int_0^x Q_X(u)du$. Theorem 7.2, p. 154] prove a C.L.T. for $\kappa$ and $\lambda$-weak dependent processes, and the proof is immediate. The situation is a bit more intricate for $\theta$-dependence. We shall show that $S_N = \sqrt{N}(\hat{\gamma}_{X,N}(\ell) - \gamma_X(\ell))$ is asymptotically normally distributed by showing that $N^{-1} \sum Z_k$ satisfies a central limit theorem for $\theta$–weak dependent variables. Theorem 2 of [Dedecker et al., 2007, Corollary 7.5] gives sufficient conditions to satisfy the condition $D(p; X)$. In particular, if $p = 2$ and $\|X_0\|_{4+\delta} \leq \infty$ then $\|Z_0\|_{2+\delta} \leq \infty$ and $\sum i > 0 i^{1/\delta}(\theta_X(i)) < \infty$ for some $\delta > 0$. The condition is satisfied and therefore the asymptotic normality in (8) follows from Corollary 7.5.

It remains to show that the limiting covariances are given by $\sigma^2_\ell = \sum_{k \in \mathbb{Z}} \mathbb{E}Z_0Z_k$.

$$\sigma^2_\ell = \sum_{k \in \mathbb{Z}} \mathbb{E}(C_0C_\ell C_k C_{k+\ell})[\mathbb{E}(X_0X_\ell X_k X_{k+\ell}) - 3\gamma_X^2(\ell)]. \quad (11)$$
We use the following identity for \((i,j,k) \in \mathbb{Z}^3\) for simplify the expression (11):

\[
\kappa_4(i,j,k) = E_0X_iX_jX_k - E_0X_iE_0E_jX_k - E_0X_jE_0E_iX_k - E_0X_kE_0E_iX_j
\]

Then the following expression, which exists for all finite \(\ell\), is equivalent to the asymptotic variance in equation (11)

\[
\sigma^2_\ell = \sum_{k \in \mathbb{Z}} E(C_0C_\ell C_{k+\ell})[\kappa_4(\ell, k + \ell) + \gamma_X(k + \ell)\gamma_X(k - \ell) - \gamma^2_X(\ell)].
\]

We note that when all the \((X_i)_{i \in \mathbb{Z}}\) are observed \(E(C_0C_\ell C_{k+\ell}) = 1\) so that \(\sigma^2_\ell\) agrees with \([Rosenblatt, 1985]\), Theorem 3, p. 58.

Finally the first factor in (7) converges in probability to \(\nu(\ell)\) because the stated assumptions ensuring a central limit theorem also imply the convergence of the empirical variances. Slutsky’s Theorem allows to complete the proof. This completes the proof of Theorem 1 □

Corollary 1 is standard from the previous result.

6.2 Proof of Theorem 2

To obtain strong laws for the sample autocorrelation function of \((X_i)_{i \in \mathbb{Z}}\) estimated like a function of \((Y_i)_{i \in \mathbb{Z}}\) and \((C_i)_{i \in \mathbb{Z}}\) in the \(\theta\)-weak dependence frame, by theorem 3 of \([Dedecker and Doukhan, 2003]\) if \(D(p, X)\) holds for some \(p \in [1, 2]\), then \(n^{-1/p} \sum_{i=1}^n (X_i - E(X_i))\) converges almost surely to 0 as \(n\) goes to infinity. We use again Lemma 2 of \([Dedecker and Doukhan, 2003]\), we need considered a \(p > 1\) and therefore, we need weaker moments conditions. If \(\|X_0\|_r < \infty\) for some \(r > 2(1+\delta)\) and \(\sum_{i \geq 0} i^{(r(\delta-1)+1)/(r-(1+\delta))} \theta_i < \infty\) for some \(\delta > 0\), then is a sufficient condition for \(\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell) \overset{a.s.}{\to} 0\).

For the strong laws for \(\lambda\) or \(\kappa\)-weak dependence cases, \([Doukhan and Wintenberger, 2007]\) proved a bound of the \((2+\delta)\)-moment of the sum of a process \(\lambda\) or \(\kappa\)-weak dependence. This bound it directly yields the strong law of large numbers using the Borel-Cantelli lemma.

6.3 Proof of Theorem 4

In order to prove that \(\|\tilde{J}_{X,N}(g) - J(g)\|_{\mathcal{H}_s'}^2\) converges, we use a bound for \(E\|\tilde{J}_{X,N}(g) - J(g)\|_{\mathcal{H}_s'}^2\) under further conditions. Then

\[
\|\tilde{J}_{X,N}(g) - J(g)\|_{\mathcal{H}_s'}^2 = \| \sum_{\ell \in \mathbb{Z}} (\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell))g_\ell \|_{\mathcal{H}_s'}^2
\]

\[
= \| \sum_{\ell \in \mathbb{Z}} (1 + |\ell|)^{-s}(\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell))(1 + |\ell|)^s g_\ell \|_{\mathcal{H}_s'}^2
\]

\[
\leq \left( \sum_{\ell \in \mathbb{Z}} (1 + |\ell|)^{-2s}(\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell))^2 \right)^{1/2} \left( \sum_{\ell \in \mathbb{Z}} (1 + |\ell|)^{2s} g_\ell^2 \right)^{1/2}
\]

\[
\leq \|g\|_{\mathcal{H}_2s} \left( \sum_{\ell \in \mathbb{Z}} (1 + |\ell|)^{-2s}(\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell))^2 \right)^{1/2}
\]
Therefore, to prove a bound for $\mathbb{E}\|\tilde{J}_{X,N}(g) - J(g)\|_{\mathcal{H}_2}^2$ it sufficient to show that $\mathbb{E}(\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell))^2$ is bounded.

$$\mathbb{E}\|\tilde{J}_{X,N}(g) - J(g)\|_{\mathcal{H}_2}^2 \leq \|g\|_{\mathcal{H}_2} \left( \sum_{\ell \in \mathbb{Z}} (1 + |\ell|)^{-2s} \mathbb{E}(\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell))^2 \right)$$

And,

$$\mathbb{E}(\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell))^2 \leq v_n \left( \frac{1}{\gamma_C(\ell)} + \frac{K}{\gamma_C(\ell)^2} + \frac{v_n^2 (Nk)^{\frac{1}{2}}}{\gamma_C(\ell)^{\alpha+1}} \right),$$

$$\left\| \max_{1 \leq i \leq n} |X_i X_{i+\ell}| \cdot \frac{|d - d|^{1+\alpha}}{|d|^{\alpha}} \right\|_2 \leq \frac{1}{|d|^{\alpha}} \max_{1 \leq i \leq n} |X_i X_{i+\ell}| \|d - d|^{1+\alpha}\|_2 \leq \frac{1}{|d|^{\alpha}} \left( \mathbb{E} \max_{1 \leq i \leq n} |X_i X_{i+\ell}|^{2a} \right)^{\frac{1}{2}} v_n^{1+\alpha}$$

if $q \geq 2b(1 + \alpha)$ or equivalently $q - 2(1 + \alpha) \geq q/a$. We need an argument of Pisier, 1978 written as follows: assume that $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is convex and non decreasing then

$$\varphi \left( \mathbb{E} \max_i |X_i X_{i+\ell}|^{2a} \right) \leq \mathbb{E} \varphi \left( \max_i |X_i X_{i+\ell}|^{2a} \right) \leq \mathbb{E} \sum_i \varphi \left( |X_i X_{i+\ell}|^{2a} \right) \leq \sum_i \mathbb{E} \varphi \left( |X_i X_{i+\ell}|^{2a} \right) \quad (12)$$

Hence $\mathbb{E} \max_i |X_i X_{i+\ell}|^{2a} \leq (nc)^{2a/s}$ with $\varphi(x) = x^{s/2a}$. Now the bound in the right hand side of (12) can be specified as $v_n^{1+\alpha}(nk)^{\frac{1}{2}}/|d|^{\alpha}$ if $s \geq pa$ this holds because $1 - \frac{2}{q}(1 + \alpha) \geq \frac{1}{2} \geq \frac{s}{2}$ if $\alpha > 0$ is small enough with $\frac{1}{2} \geq \frac{1+\alpha}{q} + \frac{1}{s}$. We thus obtain

$$\gamma_C(\ell) \left\| \frac{\tilde{\gamma}_{X,N}(\ell)}{\gamma_C(\ell)} - \gamma_X(\ell) \right\|_2 \leq v_n + \frac{K}{\gamma_C(\ell)} v_n + v_n^{1+\alpha}(nk)^{\frac{1}{2}}/|d|^{\alpha},$$

that implies the result of the Theorem. \[ \square \]

### 6.4 Proof of Theorem 4

1. $J_N(g) - J(g) = \sum_{|\ell|<N} \tilde{\gamma}_{X,N}(\ell)g_\ell - \sum_{\ell\in N} \gamma_X(\ell)g_\ell$

   $= \sum_{|\ell|<N} \tilde{\gamma}_{X,N}(\ell)g_\ell - \sum_{\ell\geq N} \gamma_X(\ell)g_\ell - \sum_{|\ell|<N} \gamma_X(\ell)g_\ell$

   $= -\sum_{|\ell|\geq N} \gamma_X(\ell)g_\ell - \sum_{|\ell|<N} (\tilde{\gamma}_X(\ell) - \gamma_X(\ell))g_\ell$

   $= -\sum_{|\ell|\geq N} \gamma_X(\ell)g_\ell - \sum_{|\ell|<N} \tilde{\gamma}_X(\ell) - \mathbb{E}\tilde{\gamma}_X(\ell))g_\ell$

   $= -T_1 + T_3$

   $T_1 = \sum_{|\ell|\geq N} \gamma_X(\ell)g_\ell$

   $T_3 = \sum_{|\ell|<N} (\tilde{\gamma}_X(\ell) - \mathbb{E}\tilde{\gamma}_X(\ell))g_\ell$
\[\|T_1\|_{H^s}^2 \leq \sum_{|\ell| \geq N} (1 + |\ell|)^{-s} \gamma_X(\ell)^2 \leq \frac{1}{N} \sum_{|\ell| \geq N} \gamma_X(\ell)^2 = \frac{\gamma}{N} < \infty\]

\[\mathbb{E}\|T_3\|_{H^s}^2 \leq \sum_{|\ell| < N} (1 + |\ell|)^{-s} \mathbb{E}(\tilde{\gamma}_X(\ell) - \mathbb{E}\tilde{\gamma}_X(\ell))^2 \leq \sum_{|\ell| < N} (1 + |\ell|)^{-s} \text{Var}(\tilde{\gamma}_X(\ell))\]

\[\|J_N(g) - J(g)\|_{H^s}^2 \leq 2(\|T_1\|_{H^s}^2 + \mathbb{E}\|T_3\|_{H^s}^2) = \frac{2}{N}(\gamma + \sum_{|\ell| < N} (1 + |\ell|)^{-s}(\kappa_4 + 2\gamma)) = \frac{2}{N}(\gamma + c_s(1 + |\ell|)^{-s}(\kappa_4 + 2\gamma))\]

2.

\[\sigma_m^2 = n\mathbb{E}\left(\sum_{|\ell| \leq m} (\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell))\hat{g}(\ell)\right)^2 = n\text{Var}\left(\sum_{|\ell| \leq m} (\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell))\hat{g}(\ell)\right) = n\sum_{|\ell| \leq m} \text{Cov}\left(((\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell))\hat{g}(\ell)), ((\tilde{\gamma}_{X,N}(k) - \gamma_X(k))\hat{g}(k))\right)\]

Let \(A(k) = (\tilde{\gamma}_{X,N}(k) - \gamma_X(k))\) and \(A(\ell) = (\tilde{\gamma}_{X,N}(\ell) - \gamma_X(\ell))\), then

\[\sigma_m^2 = n\sum_{|\ell| \leq m} \text{Cov}(A(k), A(\ell))\hat{g}(k)\hat{g}(\ell)\]

3.

\[\sigma_m^2 - \sigma^2 \leq 2|\hat{g}|_{H^s} \left|\hat{g} - \hat{g}_m\right|_{H^s} \left|\sqrt{B} \right| \sum_{|k| > m} (1 + |k|)^{2s}\text{Var}(A(k)) \sum_{\ell} (1 + |\ell|)^{2s}\text{Var}(A(\ell)) < \infty\]

Let \(g : H^s_n \rightarrow E_k\), where \(g\) denotes the orthogonal projection on the closed linear subspace \(E_k \subset H^s_n\), generated by \((e_\ell)_{|\ell| \leq k}\) with \(e_\ell(\lambda) = e^{i\ell\lambda}\).

Suppose that \(g \in E_k = \{g_i = 0, |i| \geq k\}\). Then

\[\sqrt{N}(J_{X,N}(g) - J_X(g)) \xrightarrow{D_{N \to \infty}} N(0, \sigma^2(g))\]

because is equal to \(\sum_{|i| \leq k} g_i Z_{N(i)}\).

Also note that if \(g \in H^s_n\), and \(g^k\) is the projection on \(E_k\), then \(\sigma^2(g(k)) \rightarrow \sigma^2(g)\) and therefore

\[\sqrt{N}(\tilde{J}_{X,N}(g) - J_X(g)) \xrightarrow{D_{N \to \infty}} N(0, \sigma^2(g)) \forall g \in H^s_n. \quad \Box\]
7 Examples

We present here examples where weak dependence, as defined in Section 2.1, holds. First, we focus on some general classes of processes before and then we will study specific models. A generic sample follows

Definition 2. Let \((\epsilon_t)_{t \in \mathbb{Z}}\) be a sequence of real-valued random variables and let \(F : \mathbb{R}^\mathbb{Z} \to E\) be a measurable function. The sequence \((X_t)_{t \in \mathbb{Z}}\) defined by

\[ X_t = F(\epsilon_{t-j} ; j \in \mathbb{Z}) \]  

(13)

is called a Bernoulli shift.

The class of Bernoulli shifts is very general. It provides examples of processes that are weakly dependent but not mixing (see [Rosenblatt, 1985]).

7.1 Markov processes

Markov processes can be represented as Bernoulli shifts. Consider an \(\mathbb{R}^D\)-valued Markov process, driven by the recurrence equation

\[ X_t = f(X_{t-1}, \epsilon_t) \quad (t \in \mathbb{Z}) \]  

(14)

for some i.i.d. sequence \((\epsilon_t)_{t \in \mathbb{Z}}\) with \(\mathbb{E}(\epsilon_0) = 0\), \(\epsilon_t\) independent of \(\{X_s ; s < t\}\) and \(f : \mathbb{R}^D \times \mathbb{R} \to \mathbb{R}^D\). Then the function \(F\) in (13) is defined implicitly (if it exists) by the relation

\[ F(x) = f(F(x'), x_0) \]

where \(x = (x_0, x_1, x_2 \ldots)\) and \(x' = (x_1, x_2, x_3 \ldots)\).

Assume now in representation (14) that \(x_0\) is independent of the sequence \((\epsilon_t)_{t \in \mathbb{N}}\). Suppose that, for some \(0 \leq c_i < 1\),

\[ \mathbb{E}|f(0, \epsilon_1)| < \infty \text{ and } \mathbb{E}|f(u, \epsilon_1) - f(v, \epsilon_1)| \leq \sum_{i=1}^{d} c_i |u_i - v_i|, \]  

(15)

\[ c = \sum_{i=1}^{d} c_i < 1 \text{ for all } u, v \in \mathbb{R}^D. \]

Under condition (15) the Markov process \((X_t)_{t \in \mathbb{N}}\) has a stationary distribution \(\mu\) with finite first moment. Assume now in addition that \(x_0\) is distributed with \(\mu\), that is, the Markov chain is stationary. Then, if (15) holds, such a Markov chain is \(\theta\)-weak dependent with \(\theta_r = c^r \mathbb{E}|x_0|\).

7.1.1 Nonparametric AR model.

Consider the real-valued functional (nonparametric) autoregressive model

\[ X_t = r(X_{t-1}) + \epsilon_t, \]  

(16)
where \( r : \mathbb{R} \to \mathbb{R} \) and \( (\epsilon_t)_{t \in \mathbb{Z}} \) as in (13). This a special example of a Markov process as in (13). Assume that \(|r(u) - r(u')| \leq c|u - u'| \) for all \( u, u' \in \mathbb{R} \) and for some \( 0 \leq c < 1 \), and \( E|\epsilon_0|^2 < 1 \). Then (15) with \( D = 1 \) holds and implies \( \theta \)-weak dependence with \( \theta_r = c \mathbb{E}|Z_0| \).

Here is important to note that the marginal distribution of the innovations \( \epsilon_t \) can be discrete. In such a case, classical mixing properties can fail to hold. For example, consider the simple linear AR(1) model,

\[
X_t = \phi X_{t-1} + \epsilon_t = \sum_{j \geq 0} \phi^j \epsilon_{t-j}, \quad |\phi| < 1.
\]

Let \( (\epsilon_t)_{t \in \mathbb{Z}} \) be a sequence of i.i.d. Bernoulli variables with parameter \( s = \mathbb{P}[\epsilon_t = 1] = 1 - \mathbb{P}[\epsilon_t = 0] \). The AR(1) process \( (X_t)_{t \in \mathbb{Z}} \) with innovations \( (\epsilon_t)_{t \in \mathbb{Z}} \) and AR parameter \( \epsilon \in [0, \frac{1}{2}] \), is \( \theta \)-weak dependent with \( \theta_r = \phi^2 \mathbb{E}|X_0| \), but it is known to be non-mixing. In this context concentration holds. For example, \( X_t \) is uniform if \( s = \frac{1}{2} \) and it has a Cantor marginal distribution if \( s = \frac{1}{3} \). Hence, without a regularity condition on the marginal distribution of \( \epsilon_0 \), Bernoulli shifts or Markov processes may not be mixing.

7.1.2 Nonparametric ARCH model.

Consider the real-valued functional (nonparametric) ARCH model

\[
X_t = s(X_{t-1})\epsilon_t,
\]

where \( s : \mathbb{R} \to \mathbb{R}^+ \) and \( (\epsilon_t)_{t \in \mathbb{Z}} \) is defined as in (14) with \( E|\epsilon_0|^2 = 1 \). This is a special example of a Markov process as in (14) with \( f(u, v) = s(u)v \). Assume that \(|s(u) - s(u')| \leq c|u - u'| \) for all \( u, u' \in \mathbb{R} \) and for some \( 0 \leq c < 1 \). Then (15) with \( D = 1 \) holds and implies \( \theta \)-weak dependence with \( \theta_r = cr \mathbb{E}|X_0| \). Again, the innovation distribution is allowed to be discrete.

7.1.3 Nonparametric AR-ARCH model.

We interest now in the combination of AR and ARCH models. This new process have nonparametric conditional mean and variance structure,

\[
X_t = r(X_{t-1}) + s(X_{t-1})\epsilon_t,
\]

with \( r(\cdot), s(\cdot) \) and \( (\epsilon_t)_{t \in \mathbb{Z}} \) as in the examples above. Assume the Lipschitz conditions on \( r(\cdot) \) and \( s(\cdot) \) with constants \( c_r \) and \( c_s \), respectively. If \( c_r + c_s = c < 1 \), the process satisfies \( \theta \)-weak dependence with \( \theta_r = cr \mathbb{E}|X_0| \).

7.1.4 Bilinear model.

We consider the simple bilinear process with the following recurrence equation

\[
X_t = aX_{t-1} + bX_{t-1}\epsilon_{t-1} + \epsilon_t,
\]

where \( (\epsilon_t)_{t \in \mathbb{Z}} \) is as in (15). Such causal processes are associated with chaotic representation with stationary

\[
F(u) = \sum_{j=0}^{\infty} u_j \prod_{s=1}^{j} (a + bu_s), \quad u = (u_0, u_1, u_2, \ldots).
\]
If the process is stationary and \( c = E|a + b\epsilon_0| < 1 \), the process satisfies \( \theta \)-weak dependence, with \( \theta_r = \frac{c^r(r+1)}{1-c} \).

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