New fusion rules and $R$-matrices for $SL(N)_q$ at roots of unity

Daniel Arnaudon *

Theory Division, CERN
CH-1211 Genève 23, Switzerland

Abstract

We derive fusion rules for the composition of $q$-deformed classical representations (arising in tensor products of the fundamental representation) with semi-periodic representations of $SL(N)_q$ at roots of unity. We obtain full reducibility into semi-periodic representations. On the other hand, heterogeneous $R$-matrices which intertwine tensor products of periodic or semi-periodic representations with $q$-deformed classical representations are given. These $R$-matrices satisfy all the possible Yang Baxter equations with one another and, when they exist, with the $R$-matrices intertwining homogeneous tensor products of periodic or semi-periodic representations. This compatibility between these two kinds of representations has never been used in physical models.
Quantum groups [Dri, Jim, F.R.T] at roots of unity [Lus, D-C.K, D-C.K.P] now play an important role in physics. When the deformation parameter $q$ of the quantum group is a root of unity, we can distinguish two kinds of irreducible representations (irreps):

- **Type A irreps**: the so-called $q$-deformed representations. They are the deformation of representations that exist for $q = 1$, and have the same structure. They are highest-weight and lowest-weight representations, and do not depend on supplementary complex continuous parameters.

- **Type B irreps**: those irreps which do not match the previous definition. These representations have been studied and almost classified (See [Skl, R.A] for $SL(2)_q$ and [A.C, D-C.K, D-C.K.P, D.J.M.M.2, Dob]). Their dimension is bounded and they are described by a set of continuous complex parameters corresponding to the eigenvalues of the generators of the augmented centre. The generators can be represented in particular by invertible matrices. We will call periodic irreps those for which all the generators related to positive and negative roots are represented by invertible matrices, and semi-periodic the lowest-weight irreps for which the generators related to positive roots only are represented by invertible matrices.

The first family appears in conformal field theories (see e.g. [A-G.G.S, M.R]) as well as in statistical physics of integrable models [Pas, P.S]. The set of type A representations is not stable under fusion (tensor product). The fusion rules for these representations involve indecomposable representations [P.S, Kel]. However, these fusion rules are generally truncated for physical purposes [A-G.G.S, F.G.P].

The type B representations are used in physics in relation with integrable models. They first appeared in the study of the eight-vertex model with the Sklyanin algebra [Skl], whose trigonometric limit is $SL(2)_q$. They are now used in relation with the so-called generalized chiral Potts model [B.S, D.J.M.M, K.M.S] (periodic irreps) and with relativistic solitons [G.S] (semi-periodic irreps). (See e.g. [G.R, A.MC.P] about the chiral Potts model.) In [D.J.M.M], the fusion rules of minimal periodic representations of $SL(N)$ (type B) are considered. The tensor products of such representations is actually studied as a representation of $SL(N)_q$ and is proved to be irreducible under the action of this quantum algebra. In [G.R-A.S, G.S], the fusion rules for semi-periodic irreps of $SL(2)$ are studied.

When $q$ is not a root of unity, there exists a universal $R$-matrix, satisfying the Yang Baxter equation (see [Ros] for an expression on the case of $SL(N)_q$). When one evaluates this universal $R$-matrix on tensor products of representations, one get intertwiners, i.e. invertible matrices that express the equivalence of differently ordered tensor products. When now $q$ is a root of unity, the universal $R$-matrix diverges. Formal evaluation of it on irreps of type A still provides intertwiners for these representations. However, differently-ordered tensor products of type B irreps are not always equivalent [D.J.M.M, G.R-A.S, G.S]. When they are equivalent, the intertwiner is not related to the ill-defined universal $R$-matrix. However it can still be used to give the Boltzmann weights of some statistical models.

Although both families of irreducible representations appear in physics, they have never been used together in the same physical model. The $q$-deformed representations
(type A) can appear in some degenerate limits of the parameters of representations of the second family (B). But the $\mathcal{R}$-matrix of models based on family B is not well defined in such limits *.

In this letter, we consider the $SL(N)_q$ case. We first give new fusion rules corresponding to the composition of type A irreps with only semi-periodic (type B) irreps. On the other hand, we define $\mathcal{R}$-matrices acting on heterogeneous tensor products involving both types A and B of representations (all irreps of type B included). This letter is a generalization of [Arn] and a step toward the definition of physical models involving two kinds of states related with the two kinds of irreducible representations. As an application, the construction of new integrable quantum chains will be sketched in the conclusion.

The quantum group $SL(N)_q$ is defined by the generators $k_i$, $k_i^{-1}$, $e_i$, and $f_i$, for $i = 1, \ldots, N-1$, and the relations

$$
k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i ,
$$

$$
k_i e_j k_i^{-1} = q^{a_{ij}} e_j , \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j ,
$$

$$
[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} ,
$$

$$
[e_i, e_j] = [f_i, f_j] = 0 \quad \text{for} \quad |i - j| \geq 2 ,
$$

$$
e_i^2 e_{i+1} - (q + q^{-1}) e_i e_{i+1} e_i + e_{i+1} e_i^2 = 0 ,
$$

$$
f_i^2 f_{i+1} - (q + q^{-1}) f_i f_{i+1} f_i + f_{i+1} f_i^2 = 0 ,
$$

where $(a_{ij})_{i,j=1,\ldots,N-1}$ is the Cartan matrix of $SL(N)$, i.e.

$$
\begin{align*}
a_{ii} &= 2 \\
a_{i+1} &= -1 \\
a_{ij} &= 0 \quad \text{for} \quad |i - j| \geq 2
\end{align*}
$$

The coproduct is defined by

$$
\begin{align*}
\Delta(e_i) &= e_i \otimes 1 + k_i \otimes e_i \\
\Delta(f_i) &= f_i \otimes k_i^{-1} + 1 \otimes f_i \\
\Delta(k_i) &= k_i \otimes k_i \\
\Delta : SL(N)_q &\to SL(N)_q \otimes SL(N)_q \quad \text{homomorphism of algebras},
\end{align*}
$$

* The author thanks C. Gómez for a discussion on that point.
while the opposite coproduct $\Delta'$ is $\Delta' = P \Delta P$ where $P$ is the permutation map $Px \otimes y = y \otimes x$. (The conventions here are slightly different from those of [Arn].) Denoting by $\alpha_i \equiv \alpha_{i,i+1}$, for $i = 1, \ldots, N - 1$, the simple roots of $SL(N)$, all the positive roots can be written $\alpha_{ij} = \sum_{k=1}^{j-1} \alpha_k$ for $1 \leq i < j \leq N$. We denote by $R_+$ the set of positive roots. As usual, the root vectors $e_{ij} \equiv e_{\alpha_{ij}}$ are defined inductively by

$$e_{ij+1} = e_{ij}e_{jj+1} - q e_{jj+1}e_{ij}, \quad (4)$$

so that for $1 \leq i < j < k \leq N$

$$e_{ik} = e_{ij}e_{jk} - q e_{jk}e_{ij}. \quad (5)$$

(There are other possible choices for the definition of root vectors, depending on the choice of the way of writing the longest element of the Weyl group. This choice is the most standard.) One can similarly define root vectors $f_{ij}$ associated to negative roots.

Let $q$ be a primitive $m'$-th root of unity. Denote by $m$ either $m'$ if $m'$ is odd, or $m'/2$ if $m'$ is even. Then, as proved in [D-C.K, D-C.K.P], the dimension of the irreducible representations (type A or B) is bounded by $m^{N(N-1)/2}$. All the elements $e_{ij}^m$, $f_{ij}^m$, and $k_i^m$ belong to the centre or $SL(N)_q$, and the irreducible representations of type B are labelled by at most $N^2 - 1$ complex continuous parameters corresponding to the values of these elements of the centre.

**FUSION RULES:**

We will now consider the fusion of semi-periodic representations with the fundamental representation, and then with any type A irrep.

The fundamental representation $\pi \square$ of $SL(N)_q$ on the vector space $V \square$ spanned by $w_j$ ($j = 1, \ldots, N$) is given by

$$\begin{cases} 
\pi \square (f_i)w_j = \delta_{ij}w_{j+1} & 1 \leq j \leq N - 1 \\
\pi \square (f_i)w_N = 0 \\
\pi \square (e_i)w_j = \delta_{i,j-1}w_{j-1} & 2 \leq j \leq N \\
\pi \square (e_i)w_1 = 0 \\
\pi \square (k_i)w_j = q^{\delta_{ij}-\delta_{i,j-1}}w_j & 1 \leq j \leq N 
\end{cases} \quad (6)$$

A generic semi-periodic representation $\pi_x$ on the vector space $V$ is defined as follows: it has the maximal dimension $m^{N(N-1)/2}$ allowed for irreducible representations when $q^m = 1$. The central elements $e_{ij}^m$ take the value $x_{ij} \in C \backslash \{0\}$ on $V$. It is a lowest-weight representation, i.e. it contains a vector $v_i$ such that for all $i \in \{1, \ldots, N - 1\}$, $f_i v_0 = 0$. This vector is a common eigenvector of the $k_i$'s and $k_i v_0 = \lambda_i v_0 \equiv q^{\mu_i} v_0$, so that the central elements $k_i^m$ take the value $z_i = \lambda_i^m = q^{m\mu_i}$. The parameters characterizing this representation are then $x_{ij}$ ($1 \leq i < j \leq N - 1$) and $\lambda_i$ ($1 \leq i \leq N - 1$). Their number $N(N-1)/2 + N - 1$ is less than the maximum since the parameters corresponding to the eigenvalues of $f_{ij}^m$ vanish. For generic value of the parameters $x \equiv (x_{ij}, \lambda_i)$, this
representation $\pi_x$ is irreducible, and $v_0$ is the only singular vector [D-C.K] *. The condition for this (irreducibility and uniqueness of the singular vector) is that none of the $x_{ij}$’s vanishes, and that none of the products $\lambda_i^2 \lambda_j^2 \cdots \lambda_{j-1}^2$ ($i < j$) is a power of $q$.

Using an analogue of the P.B.W. basis of $\mathcal{U}_q(\mathfrak{g}^+)$ [Ros], a basis of this representation is given by [D-C.K],

$$v_p = \pi_x(e^p) v_0 \equiv \pi_x \left( e^{p_{N-1}N} e^{p_{N-2}N} \ldots e^{p_{N-3}N-1} \right) v_0$$

where $p = (p_{ij})_{1 \leq i < j \leq N-1}$ and $p_{ij} \in \{0, \ldots, m-1\}$. (The $e_{ij}$ are written in the inverse lexical order.)

This representation is precisely given by:

$$v_{p_{ik}+m} \equiv x_{ik} v_p,$$

and the actions of the generators of $SL(N)$ on $v_p$

$$\pi_x(k_i) v_p = q^{\mu_i + \sum_{\alpha'} p_{\alpha'}(\alpha', \alpha_i)} v_p,$$

$$\pi_x(e_i) v_p = -\sum_{\alpha > \alpha_i} p_{\alpha}(\alpha, \alpha_i) v_{p_i, i+1} + \sum_{k=i+2}^N [p_{i+1,k}] q \sum_{k' > k} \delta(p_{i+1,k'}-p_{ik'}) v_{p_{i+1k}-1, p_{ik}+1},$$

$$\pi_x(f_i) v_p = \sum_{k > i+1} [p_{ik}] q \sum_{\alpha' < \alpha_{i,k}} p_{\alpha'}(\alpha_i, \alpha') v_{p_{i+1, k+1}, p_{ik}-1}$$

$$- [p_{i,i+1}] \left[ \mu_i + p_{i,i+1} - 1 + \sum_{\alpha' < \alpha_{i,i+1}} p_{\alpha'}(\alpha_i, \alpha') \right] v_{p_{i,i+1}-1}$$

$$- \sum_{j<i} [p_{j,i+1}] q \sum_{\alpha' \leq \alpha_{j,i+1}} p_{\alpha'}(\alpha_i, \alpha') v_{p_{j,i+1}-1, p_{ji}+1}$$

where only the modified indices of $v$ are written on the right-hand side, and where the symbols $<$ and $\leq$ refer to the lexical order of the roots. ($\,\delta\,$) denotes the bilinear form defined on the root lattice by $(\alpha_i, \alpha_j) = a_{ij}$. (As usual, $[a] = (q^a - q^{-a})/(q - q^{-1})$.)

**Proposition:** The tensor product $(V_x \otimes V_\square, (\pi_x \otimes \pi_\square) \circ \Delta)$ of an irreducible semi-periodic representation $\pi_x$ having a single singular (=lowest-weight) vector, by the fundamental representation is totally reducible and

$$V(x_{ij}, \lambda_i) \otimes V_\square = \bigoplus_{n=0}^{N-1} V(x_{ij} \lambda_i^n \lambda_i q^{\delta_{i,n+1}-\delta_{i,n}})$$

* Note however that these two properties are not equivalent. There are actually representations with other singular vectors (for non-generic values of the parameters) which are still irreducible.
Proof:

a) Although the expression of $\Delta(e_{ij})^p$ on the tensor product is quite complicated [Ros], it becomes very simple on $V_x \otimes V_{\square}$ for $p = m$. Actually,

$$(\pi_x \otimes \pi_{\square})(\Delta(e_{ij}))^m = \pi_x(e^m_{ij}) \otimes 1 = x_{ij}.$$ (10)

b) One can find in $V_x \otimes V_{\square}$ exactly $N$ lowest weight vectors $v_0^{(n)}$ ($n = 0, ..., N - 1$). Explicitly, these vectors read

$$v_0^{(n)} = \sum_{l=0}^{n} V_l^{(n)} \otimes w_{N-n+l}.$$ (11)

They satisfy

$$(\pi_x \otimes \pi_{\square})\Delta(f_i)v_0^{(n)} = 0$$ (12)

for $i = 1, ..., N - 1$, provided

$$\left\{ \begin{array}{l}
V_l^{(n)} = -q\pi_x(f_{N-n+l})V_{l+1}^{(n)} & \text{for } 0 \leq l < n \leq N - 1 \\
\pi_x(f_i)V_n^{(n)} = 0 & \text{for } 1 \leq i \leq n - 2
\end{array} \right.$$ (13)

Explicitly, $V_n^{(n)} = \sum_{p} a_p v_p$, where the sum is limited to $v_p$ of the type

$$e_{i_R}N e_{i_{R-1}} e_{i_{R}} ... e_{i_{r+1}} e_{i_2} e_{i_1},$$

with $N - n = i_1 < i_2 < ... < i_R < N$ and

$$a_p = \prod_{l \in \{N-n, ..., N-1\} \backslash \{i_1, ..., i_R\}} \frac{1 - q^{(\mu'_{l-1})-2}}{q - q^{-1}}$$ (14)

c) Let $V_{(x_{ij}, \lambda^{(n)}_{i})_{=\lambda_i q^{s_{i,n+1}\ldots s_{i,n}}}}$ (for $n = 0, ..., N - 1$) be the semi-periodic representations generated by the action of $(\pi_x \otimes \pi_{\square})(\Delta(SL(N)q))$ on $v_0^{(n)}$. They are irreducible and have a single lowest-weight vector. It is then easy to prove that the sum of these vector spaces is a direct sum, so that the proposition is proved.

Corollary:

a) The tensor product of a generic semi-periodic representation with any irreducible representation $(V_J, \pi_J)$ of type A is totally reducible and

$$V_{(x_{ij}, \lambda^{(n)}_{i}) \otimes V_J} = \bigoplus_{\mu' \text{ weight of } \pi_J} m(\mu')V_{(x_{ij}, \lambda^{(n)}_{i})=\lambda_i q^{\mu'_{i}}},$$ (15)
where \( m(\mu') \) denotes the multiplicity of the weight \( \mu' \) of \( \pi_J \).

b) The same holds for the tensor product of a generic semi-periodic representation with an indecomposable representation that can appear in the composition of type A irreps.

This can be proved directly as in [Arn] for \( N = 2 \) (by finding all the singular vectors) or it can also be seen as a consequence of the co-associativity of \( \Delta \).

What happens now for the composition of periodic representations (with no lowest weight) with ordinary representations? In the case \( N = 2 \), one can prove [Arn2] that the quadratic Casimir takes generically two values on the tensor product of a periodic representation with the fundamental representation. So the full reducibility stated in the proposition (and its corollary) extends to periodic representations (except for non-generic values of the parameters), in the case of \( SL(2)_q \). The case \( N > 2 \) is not treated yet.

Let us end this part devoted to fusion rules with some remarks on the composition of type B irreps with type B irreps in the case of \( SL(2)_q \). As proved in [G.R-A.S, G.S], the tensor product of two semi-periodic representations is reducible into semi-periodic representations. Looking again at the quadratic Casimir \( C_2 \), we can extend this result (reducibility) to periodic representations [Arn2]: some parameters of the representations enter indeed in the characteristic polynomial of \( C_2 \) only in the constant term, which proves that for generic values of the parameters the roots of this polynomial are different. (This reducibility holds with respect to \( SL(2)_q \). Extensions of these representations to representations of \( \hat{SL}(2)_q \) generate, under tensor products, irreducible representations with respect to \( \hat{SL}(2)_q \), as already known [D.J.M.M].)

**R-MATRICES:**

Let us now consider the problem of \( R \)-matrices for the tensor product of type A and type B representations. We look for an \( R \)-matrix \( R(x, J) \) intertwining \( V_x \otimes V_J \) and \( V_J \otimes V_x \), i.e. satisfying

\[
\forall X \in SL(N)_q \quad R(x, J)(\pi_x \otimes \pi_J) \circ \Delta(X) = (\pi_x \otimes \pi_J) \circ \Delta'(X)R(x, J),
\]

(16)

where \((V_x, \pi_x)\) denotes a type B irrep. \((R)\) does not contain the permutation map with this convention, and the intertwiner is actually \(P R\).

A solution of (16) is the evaluation \((\pi_x \otimes \pi_J)(R_u)\) of the truncated universal \( R \)-matrix

\[
R_u = q^{-b_{ij}h_i \otimes h_j} \prod_{\alpha \in R_+} \left( \sum_{n=0}^{m-1} q^n \frac{(1-q^2)^n}{[n]!} q^{-n(n-1)/2} (k_{\alpha}^{-1}e_\alpha)^n \otimes (k_{\alpha}f_\alpha)^n \right),
\]

(17)

where the matrix \((b_{ij})\) is the inverse of the Cartan matrix, and where the order of the product is given by the lexical order of the positive roots. Let us denote by \( R^+(x, J) \) this solution. We define similarly \( R^+(J, x) \). Let us also denote by \( R^+(J, J') \) the evaluation \((\pi_J \otimes \pi_{J'})(R_u)\), which intertwines \( \Delta \) and \( \Delta' \) on \( V_J \otimes V_{J'} \).
There is another solution to (16), given by the evaluation of the truncated, permuted inverse of the universal $R$-matrix

$$\tilde{R}_u = q^{h_i h_j} \otimes h_j \prod_{\alpha \in R_+} \left( \sum_{n=0}^{m-1} \frac{(q-q^{-1})^n}{n!} q^{n(n-1)/2} f_\alpha^* \otimes \tilde{e}_\alpha^n \right)$$

(18)

where the $\tilde{e}_\alpha$ are the root vectors related with the reverse order of the roots, i.e.

$$\begin{cases} 
\tilde{e}_\alpha = e_i \\
\tilde{e}_{ik} = \tilde{e}_{ij} \tilde{e}_{jk} - q^{-1} \tilde{e}_{jk} \tilde{e}_{ij}
\end{cases}$$

($f_\alpha$ is defined similarly.) $\tilde{R}_u$ is equal to $P(\tilde{R}_u^{-1})$ modulo powers of the generators higher or equal to $m$. Let us denote $R^{-}(x, J) = (\pi_x \otimes \pi_J)(\tilde{R}_u)$. We define similarly $R^{-}(J, x)$ and $R^{-}(J, J')$.

**Theorem:** Let $(V_x, \pi_x)$, $(V_{x'}, \pi_{x'})$ be two representations for which there exists an intertwiner $R(x, x')$, and $(V_J, \pi_J)$ a type A irrep. Then the following Yang Baxter equations are satisfied,

a) On $V_x \otimes V_{x'} \otimes V_J$,

$$R_{12}(x, x') R_{13}^+(x, J) R_{23}^+(x', J) = R_{23}^+(x', J) R_{13}^+(x, J) R_{12}(x, x') .$$

(20)

d) One can replace in a), b) and c) above one or both of the type B representations $(V_x, \pi_x)$ and $(V_{x'}, \pi_{x'})$ by type A irreps, changing $R(x, x')$ to the corresponding $R^+$ (or also $R^-$), and the eqns. (20-22) are still valid. Furthermore, all the type A irreps can also be replaced by indecomposable representations occurring in the fusion rules of type A irreps. Finally, $R^+$ and $R^-$ can be exchanged globally in each equation.

**However,**

e) the Yang Baxter equation

$$R_{12}^+(x, J) R_{13}^+(x, x') R_{23}^+(J, x') = R_{23}^+(J, x') R_{13}^+(x, x') R_{12}^+(x, J)$$

(23)

cannot be satisfied on $V_x \otimes V_J \otimes V_{x'}$ for generic $x$ and $x'$.
a) follows from the fact that $\mathcal{R}(x, x')$ is an intertwiner for $\Delta$ and $\Delta'$ on $V_x \otimes V_{x'}$. One proves it first with $J = \square$. [The expression

$$\Delta(e_\alpha) = e_\alpha \otimes 1 + k_\alpha \otimes e_\alpha + \sum_{\beta > \gamma} (1 - q^2)k_{\gamma}e_\beta \otimes e_\gamma$$

of the coproduct of $e_\alpha$ for any $\alpha \in R_+$ (not necessarily simple) is needed in the computation.] One then goes to any $J$ using the quasi-triangularity property of $\mathcal{R}^+(x, J)$: if $V_J$ enters in the decomposition of $V_{J_1} \otimes V_{J_2} = \bigoplus_J V_J$, then

$$\mathcal{R}^+(x, J) = (1 \otimes p)(1 \otimes CG) (\mathcal{R}^+_{1,2}(x, J_1)\mathcal{R}^+_{1,3}(x, J_2)) (1 \otimes CG^{-1})(1 \otimes i) ,$$

(24)

where $p$ is the projector on $V_J$ in the decomposition of $V_{J_1} \otimes V_{J_2}$, whereas $i$ is the injection of $V_J$ into $\bigoplus_J V_J$. $CG$ is the Clebsch–Gordan invertible maps $CG \in \text{End}(V_{J_1} \otimes V_{J_2}, \bigoplus_J V_J)$ such that

$$\forall X \in SL(N)_q, \quad p \circ CG \circ (\pi_{J_1} \otimes \pi_{J_2}) (\Delta(X)) = \pi_J(X) \circ p \circ CG .$$

(25)

Note that the explicit expression for $\mathcal{R}(x, x')$ is not necessary for the proof.

b,c,d) b, c and d are consequences of a.

e) The constraints provided by the Yang Baxter equation on $V_x \otimes V_J \otimes V_{x'}$ with this choice for $\mathcal{R}(x, J)$ and $\mathcal{R}(J, x')$ generate constraints on either $x$ or $x'$.

This theorem leads to the following

**Corollary:** Consider a set of representations $(V_x, \pi_x)$ such that each pair of them is intertwined by some $\mathcal{R}$-matrix (related to chiral Potts model or anything else), these $\mathcal{R}$-matrix satisfying all together Yang Baxter equations. Then one can add to this set all the type A irreps $(V_J, \pi_J)$ and all the indecomposable representations appearing in their tensor products. With $\mathcal{R}^+(x, J)$, $\mathcal{R}^+(J, J')$ and $\mathcal{R}^-(J, x)$, the whole set of $\mathcal{R}$-matrices satisfy all the possible Yang Baxter equations.

**Remark:** As shown by part e) of the theorem, the “canonical” choice $\{\mathcal{R}^+(x, J), \mathcal{R}^+(J, x)\}$ does not work. The intertwiner for $\Delta$ and $\Delta'$ on $V_J \otimes V_x$ has to be the inverse of the one on $V_J \otimes V_x$. This “triangularity” property is to be compared to the fact that the known solutions [D.J.M.M, G.R-A.S, G.S] for $\mathcal{R}(x, x')$ satisfy $\mathcal{R}(x', x) = (\mathcal{R}(x, x'))^{-1}$. However, the whole set of $\mathcal{R}$-matrices does not satisfy this property, since it does not apply to $\mathcal{R}^+(J, J')$.

It was proved in [Arn] in the case of $SL(2)_q$ that $\mathcal{R}^+(x, 1/2)$, $\mathcal{R}^-(x, 1/2)$, $\mathcal{R}^+(x, 1/2)$ and $\mathcal{R}^-(x, 1/2)$ were the only intertwiners compatible with the Yang Baxter equations involving $\pi_x$ once and $\pi_{1/2}$ twice and with the constraint that the intertwiner for $\pi_{1/2} \otimes \pi_{1/2}$ is $\mathcal{R}^+(1/2, 1/2)$. A natural conjecture is that the only solutions for $\mathcal{R}(x, J)$ and $\mathcal{R}(J, x)$ of the equations (20-22) ($\mathcal{R}(J, J')$ and $\mathcal{R}(x, x')$ being given) are precisely the evaluations on the representations of (17) and (18) and vice versa.

An application of these results will be the possibility of adding physical states corresponding to representations $(V_J, \pi_J)$, to integrable theories that do not already involve such states.
Let us for example consider the generalization of the chiral Potts model, related with minimal periodic representations of $SL(N)_q$. Date, Jimbo, Miki and Miwa [D.J.M.M] have related the intertwiner of minimal periodic representations of $SL(N)_q$ with the Boltzmann weights of the generalized chiral Potts model. Using the irreducibility (rather than indecomposability) under $SL(N)_q$ of the tensor products of these representations, they proved the Yang Baxter equation for this intertwiner. We now propose to add new states to this model, corresponding to the non-periodic representations $(V_J, \pi_J)$. Choosing the intertwiners as explained in the corollary, the integrability of the generalized Potts model will ensure that of the enlarged one.

One could also consider a generalization to $SL(N)_q$ of the context of [G.R-A.S, G.S] (where $SL(2)_q$ was considered) and add $q$-deformed classical states to the semi-periodic states of the theory.

We would like now to show an example of a one-dimensional quantum chain which is $SL(2)_q$ invariant. For $q = i$, i.e. $m = 2$, one takes ordinary spins (fundamental representation) on odd sites and periodic representations (also two-dimensional) on even sites. Let us write the periodic representation as follows:

\[
\begin{align*}
fv_0 &= v_1 & ev_0 &= y^{-1} \beta v_1 & kv_0 &= q^\mu v_0 \\
fv_1 &= yv_0 & ev_1 &= ([\mu] + \beta) v_0 & kv_1 &= -q^\mu v_1
\end{align*}
\]

where $\mu$, $y$, and $\beta$ are three independent complex parameters. The Hamiltonian is

\[H = \sum_{j=1}^{L-1} H_j\]

with

\[
H_{2j} = \frac{1}{2} q^\mu \sigma^z_{2j} - i \left( \frac{1}{2} [\mu] + \beta \right) \sigma^z_{2j+1} + iyq^\mu \sigma^+_{2j} \sigma^-_{2j+1} + y^{-1} \beta \sigma^z_{2j} \sigma^-_{2j+1} + ([\mu] + \beta) \sigma^+_{2j} \sigma^z_{2j+1} - iq^\mu \sigma^-_{2j} \sigma^+_{2j+1}
\]

and

\[
H_{2j-1} = \frac{1}{2} q^{-\mu} \sigma^z_{2j} + i \left( \frac{1}{2} [\mu] + \beta \right) \sigma^z_{2j-1} + iy^{-1} \beta \sigma^-_{2j-1} \sigma^z_{2j} + \sigma^+_{2j-1} \sigma^-_{2j} + iq^{-\mu} ([\mu] + \beta) \sigma^-_{2j-1} \sigma^+_{2j}
\]

($i = \sqrt{-1}$.) This chain is trivially integrable by a Jordan–Wigner transformation. Its interesting feature is the dependence on complex parameters: $q$ is fixed but there are three new parameters. (It is still necessary to check whether these parameters do not all disappear in equivalence transformations.) Because of the fusion rules of these two-dimensional representations into two-dimensional representations, there will be a zero-mode in the spectrum. This quantum chain (and other examples) will be studied elsewhere, following general methods of statistical mechanics [Nijs]. It will be interesting to compare...
such chains depending on parameters of representations with chains based on ordinary representations of multiparameters quantum groups [H.R].

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