A new pseudo-Kaluza-Klein scheme for geometrical description of interactions

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Abstract

We illustrate the main features of a new Kaluza-Klein-like scheme (Deformed Relativity in five dimensions). It is based on a five-dimensional Riemannian space in which the four-dimensional space-time metric is deformed (i.e. it depends on the energy) and energy plays the role of the fifth dimension. We review the solutions of the five-dimensional Einstein equations in vacuum and the geodetic equations in some cases of physical relevance. The Killing symmetries of the theory for the
energy-dependent metrics corresponding to the four fundamental interactions (electromagnetic, weak, strong and gravitational) are discussed for the first time. Possible developments of the formalism are also briefly outlined.

1 Introduction

The problem of the ultimate geometrical structure of the physical world - both at a large and a small scale - is an old-debated one. After Einstein, the generally accepted view is that physical phenomena do occur in a four-dimensional manifold, with three spatial and one time dimensions, and that space-time possesses a global Riemannian structure, whereas it is locally flat (i.e. endowed with a Minkowskian geometry).

However, as is well known, many attempts at generalizing the four-dimensional Einsteinian picture have been made in this century, mainly aimed at building up unified schemes of the fundamental interactions. Such efforts can be roughly divided into two main groups. In the former, the existence of further dimensions is assumed ([1]-[9]) (by preserving the usual Einsteinian structure of the 4-d. spacetime), whereas in the latter one hypothesizes [10] global and/or local four-dimensional geometries, different from the Minkowskian or the Riemannian ones (mainly of the Finsler type [11]). The most celebrated theory of the first type is due to Kaluza [2] and Klein [3], who assumed a five-dimensional space-time, in order to unify gravitation and electromagnetism in a single geometrical structure. In their scheme, the coefficient of the fifth coordinate is constant, whereas Jordan [4] and Thiry [5] considered it a general function of the space-time coordinates. The Kaluza-Klein formalism was since then extended to even higher dimensions, in order to achieve unification of all four fundamental interactions, i.e. including weak and strong forces ([6]-[8]). Modern generalizations [8] of the Kaluza-Klein scheme require a minimum number of 11 dimensions in order to accommodate the Standard Model of electroweak and strong interactions (let us recall that 11 is also the maximum number of dimensions required by supergravity theories [9]).

In the last decade, two of us (F.C. and R.M.) introduced a generalization of Special Relativity, called Deformed Special Relativity (DSR) [12]. It was essentially aimed, in origin, at dealing in a phenomenological way with
a possible breakdown of local Lorentz invariance (LLI). Actually the experimental data of some physical processes, ruled by different fundamental interactions, seem indeed to provide evidence for local departures from the usual Minkowski metric [12]. They are: the lifetime of the (weakly decaying) \( K^0_s \) meson [13]; the Bose-Einstein correlation in (strong) pion production [14]; the superluminal propagation of electromagnetic waves in waveguides [15]. All such phenomena seemingly show a (local) breakdown of Lorentz invariance and, therefore, an inadequacy of the Minkowski metric in describing them, at different energy scales and for the three interactions involved (electromagnetic, weak and strong). On the contrary, they apparently admit of a consistent interpretation in terms of a deformed Minkowski space-time, with metric coefficients depending on the energy of the process considered [12]. Moreover, it can be shown that also the experimental results on the slowing down of clocks in a gravitational field [16] can be described in terms of a deformed energy-dependent metric [12].

DSR is just a (four-dimensional) generalization of the (local) space-time structure based on an energy-dependent deformation of the usual Minkowski geometry. What’s more, the corresponding deformed metrics obtained from the experimental data provide an effective dynamical description of the interactions ruling the phenomena considered (at least at the energy scale and in the energy range considered). Then one realizes, for all four interactions, the so-called “Solidarity Principle“, between space-time and interaction (so that the peculiar features of every interaction determine — locally — its own space-time structure), that — following B. Finzi [17] — can be stated as follows: “Space-time is solid with interactions, so that their respective properties affect mutually”.

Moreover, it was shown that the deformed Minkowski space with energy-dependent metric admits a natural embedding in a five-dimensional space-time, with energy as extra dimension ([18], [12]). Namely, the four-dimensional, deformed, energy-dependent space-time is only a manifestation (a ”shadow”, to use the famous word of Minkowski) of a larger, five-dimensional space, in which energy plays the role of the fifth dimension. The new formalism one gets in this way (Deformed Relativity in Five Dimensions, DR5) is a Kaluza-Klein-like one, the main points of departure from a standard KK scheme being the deformation of the Minkowski space-time and the use of energy as extra dimension (this last feature entails, among the others, that the DR5 formalism is noncompactified).
DR5 is therefore a generalization of Einstein’s Relativity sharing both features of a change of the 4-d. Minkowski metric and the presence of extra dimensions, and it permits also to give new intriguing insights on basic properties, such as mass, of elementary particles, relating them to fundamental geometrical quantities ([33], [34]).

The purpose of the present paper is to illustrate the DR5 formalism and to give new results on the isometries of the five-dimensional space of the theory.

The paper is organized as follows. Sect. 2 contains a brief review of the formalism of the four-dimensional deformed Minkowski space and gives the explicit expressions of the deformed metrics obtained, for the fundamental interactions, by the phenomenological analysis of the experimental data. In Sect.s 3-5 we illustrate the main features of the DR5 scheme. Its geometrical structure — based on a five-dimensional space in which the four-dimensional space-time \( \mathbb{R}_5 \) is deformed and the energy \( E \) plays the role of fifth dimension — is discussed in Sect. 3. In Sect. 4 we write down the related five-dimensional Einstein equations in general (with all five metric coefficients depending on \( E \), and including an arbitrary ”cosmological constant” \( \Lambda(E) \)), and solve them explicitly in two special cases of physical relevance and for \( \Lambda = 0 \). The solutions obtained, with their physical meaning, are discussed in Sect. 5. In Sect. 6 we derive the isometries of \( \mathbb{R}_5 \) for the four phenomenological metrics by exploiting the Killing equations of DR5. Sect. 7 contains a brief discussion of the geodetic equations. Concluding remarks and possible further developments of the formalism are put forward in Sect. 8.

2 Deformed Special Relativity

2.1 Deformed Minkowski space-time

Let us briefly review the main features of the formalism of the (four-dimensional) deformed Minkowski space [12].

If \( M(x, g_{SR}, R) \) is the usual Minkowski space of the standard Special Relativity (SR) (where \( x \) is a fixed Cartesian frame), endowed with the metric tensor

\[
g_{SR} = \text{diag}(1, -1, -1, -1),
\]  

(1)
the deformed Minkowski space $\tilde{M}(x, g_{DSR}, R)$ is the same vector space on the real field as $M$, with the same frame $x$, but with metric $g_{DSR}$ given by

$$g_{DSR}(E) = \text{diag}(b_0^2(E), -b_1^2(E), -b_2^2(E), -b_3^2(E)) =$$

$$= \delta_{\mu \nu} [b_0^2(E)\delta_{\mu 0} - b_1^2(E)\delta_{\mu 1} - b_2^2(E)\delta_{\mu 2} - b_3^2(E)\delta_{\mu 3}], \quad (2)$$

where the metric coefficients $\{b_\mu^2(E)\}$ ($\mu = 0, 1, 2, 3$) are (dimensionless) positive functions of the energy $E$ of the process considered$^2$: $b_\mu^2 = b_\mu^2(E)$.

The generalized infinitesimal metric interval in $\tilde{M}$ reads therefore

$$ds^2 = b_0^2(E)c^2 (dt)^2 - b_1^2(E)(dx)^2 - b_2^2(E)(dy)^2 - b_3^2(E)(dz)^2 =$$

$$= g_{\mu \nu, DSR} dx^\mu dx^\nu = dx * dx, \quad (3)$$

with $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$, $c$ being the usual light speed in vacuo. The last equality in (3) defines the scalar product $*$ in the deformed Minkowski space $\tilde{M}$. The relativity theory based on $\tilde{M}$ is called Deformed Special Relativity (DSR) [12].

We want to stress that — although uncommon — the use of an energy-dependent space-time metric is not new. Indeed, it can be traced back to Einstein himself. In order to account for the modified rate of a clock in presence of a gravitational field, Einstein first generalized the expression of the special-relativistic interval with metric (1), by introducing a "time curvature" as follows:

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right)c^2 (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2, \quad (4)$$

where $\phi$ is the Newtonian gravitational potential. In the present scheme, the reason whereby one considers energy as the variable upon which the metric coefficients depend is twofold. On one side, it has a phenomenological basis in the fact that we want to exploit this formalism in order to derive the deformed

$^1$In the following, we shall employ the notation "ESC on" ("ESC off") to mean that the Einstein sum convention on repeated indices is (is not) used.

$^2$E is to be understood as the energy measured by the detectors via their electromagnetic interaction in the usual Minkowski space.
metrics corresponding to physical processes, whose experimental data are just expressed in terms of the energy of the process considered. On the other hand, one expects on physical grounds that a possible deformation of the space-time to be intimately related to the energy of the concerned phenomenon (in analogy to the gravitational case, where space-time curvature is determined by the energy-matter distribution).

Let us recall that the metric (2) is supposed to hold locally, i.e. in the space-time region where the process occurs. Moreover, it is supposed to play a dynamical role, thus providing a geometric description of the interaction considered, especially as far as nonlocal, nonpotential forces are concerned. In other words, each interaction produces its own metric, formally expressed by the metric tensor $g_{\text{DSR}}$, but realized via different choices of the set of parameters $b_\mu(E)$. We refer the reader to Ref. [12] for a more detailed discussion.

It is also worth to notice that the space-time described by the interval (3) actually has zero curvature, and therefore it is not a "true" Riemannian space (whence the term "deformation" used to describe such a situation). Therefore, on this respect, the geometrical description of the fundamental interactions based on the metric (2) is different from that adopted in General Relativity to describe gravitation. Moreover, for each interaction the corresponding metric reduces to the Minkowskian one, $g_{\mu\nu,\text{SR}}$, for a suitable value of the energy, $E_0$, characteristic of the interaction considered (see below). But the energy of the process is fixed, and cannot be changed at will. Thus, although it would be in principle possible to recover the Minkowskian space by a suitable change of coordinates (e.g. by a rescaling), this would amount to a mere mathematical operation, devoid of any physical meaning. In our five-dimensional vision, in fact, the physics of the interaction considered lies in the curvature of the five-dimensional metric, which depends on energy\(^3\).

Inside the deformed space-time, a maximal causal speed $u$ can be defined, whose role is analogous to that of the light speed in vacuum for the usual Minkowski space-time. It can be shown that, for an isotropic 3-dimensional\(^3\) On the contrary, the four-dimensional sections at $E = \text{constant}$ are "mathematically flat", since they have (four-dimensional) zero curvature.
space \((b_1 = b_2 = b_3 = b)\), its expression is

\[
  u = \frac{b_0}{b}c. \tag{5}
\]

This speed \(u\) can be considered as the speed of the interaction ruling the process described by the deformation of the metric. It is easily seen that there may be maximal causal speeds which are \textit{superluminal}, depending on the interaction considered, because

\[
  u \geq c \iff \frac{b_0}{b} \geq 1. \tag{6}
\]

Starting from the deformed space-time \(\tilde{\mathcal{M}}\), one can develop the Deformed Special Relativity in a straightforward way. For instance, the generalized Lorentz transformations, i.e. those transformations which preserve the interval (3), for an isotropic three-space and for a boost, say, along the \(x\)-axis, read as follows [12]

\[
  \begin{cases}
    x' = \tilde{\gamma}(x - vt); \\
    y' = y; \\
    z' = z; \\
    t' = \tilde{\gamma} \left( t - \tilde{\beta}^2 \frac{x}{v} \right),
  \end{cases} \tag{7}
\]

where \(v\) is the relative speed of the reference frames, and

\[
  \tilde{\beta} = \frac{v}{u}; \tag{8}
\]

\[
  \tilde{\gamma} = \left( 1 - \tilde{\beta}^2 \right)^{-1/2}. \tag{9}
\]

It must be carefully noted that, like the metric, the generalized Lorentz transformations, too, depend on the energy (through the deformed rapidity parameter \(\tilde{\beta}\): see the expression (5) of the maximal speed \(u\)). This means that one gets different transformation laws for different values of \(E\), but still
with the same functional dependence on the energy, so that the invariance of
the deformed interval (3) is always ensured (provided the process considered
does always occur via the same interaction).

From the knowledge of the generalized Lorentz transformations in the
deformed Minkowski space $\tilde{M}$, it is easy to derive the main kinematical and
dynamical laws valid in DSR. For this topic and further features of DSR the
interested reader is referred to Ref.s [12], [19], [31] and [32].

2.2 Description of interactions by energy-dependent
metrics

We want instead to review the results obtained for the deformed metrics,
describing the four fundamental interactions - electromagnetic, weak, strong
and gravitational - , from the phenomenological analysis of the experimental
data [12]. First of all, let us stress that, in all the cases considered, one gets
evidence for a departure of the space-time metric from the Minkowskian one
(at least in the energy range examined).

The explicit functional form of the DSR metric (2) for the four interac-
tions is as follows.

1) Electromagnetic interaction. The experiments considered are those
on the superluminal propagation of e.m. waves in conducting waveguides
with variable section (first observed at Cologne in 1992) [15]. The intro-
duction, in this framework, of a deformed Minkowski space is motivated by
ascribing the superluminal speed of the signals to some nonlocal e.m. effect,
inside the narrower part of the waveguide, which can be described in terms
of an effective deformation of space-time inside the barrier region [20]. Since
we are dealing with electromagnetic forces (which are usually described by
the Minkowskian metric), we can assume $b_0^2 = 1$ (this is also justified by the
fact that all the relevant deformed quantities depend actually on the ratio
$b/b_0$). Assuming moreover an isotropically deformed three-space ($b_1 = b_2 =$
\[ b_3 = b \] \(^4\), one gets \([12]\)

\[
g_{DSR,e.m.}(E) = \text{diag} \left( 1, -b_{e.m.}^2(E), -b_{e.m.}^2(E), -b_{e.m.}^2(E) \right); \quad (10)
\]

\[
b_{e.m.}^2(E) = \begin{cases} 
    \left( \frac{E}{E_{0,e.m.}} \right)^{1/3}, & 0 < E \leq E_{0,e.m.} \\
    1, & E_{0,e.m.} < E
\end{cases} = (11)
\]

\[
= 1 + \Theta(E_{0,e.m.} - E) \left[ \left( \frac{E}{E_{0,e.m.}} \right)^{1/3} - 1 \right], E > 0, \quad (12)
\]

(where \(\Theta(x)\) is the Heaviside theta function, stressing the piecewise structure of the metric). The threshold energy \(E_{0,e.m.}\) is the energy value at which the metric parameters are constant, i.e. the metric becomes Minkowskian. The fit to the experimental data yields

\[
E_{0,e.m.} = (4.5 \pm 0.2) \mu eV. \quad (13)
\]

Notice that the value obtained for \(E_0\) is of the order of the energy corresponding to the coherence length of a photon for radio-optical waves \((E_{coh} \simeq 1\mu eV)\).

2) **Weak interaction.** The experimental input was provided by the data on the pure leptonic decay of the meson \(K_0^0\), whose lifetime \(\tau\) is known in a wide energy range \((30 \div 350 \text{ GeV})\) \([13]\) (an almost unique case). Use has been made of the deformed law of time dilation as a function of the energy, which reads \([12]\)

\[
\tau = \frac{\tau_0}{\left[ 1 - \left( \frac{b}{b_0} \right)^2 + \left( \frac{b}{b_0} \right)^2 \left( \frac{m_0}{E} \right)^2 \right]^{1/2}}. \quad (14)
\]

\(^4\)Notice that the assumption of spatial isotropy for the electromagnetic interaction in the waveguide propagation is only a matter of convenience, since waveguide experiments do not provide any physical information on space directions different from the propagation one (the axis of the waveguide). An analogous consideration holds true for the weak case, too (see below).
As in the electromagnetic case, an isotropic three-space was assumed, whereas the isochrony with the usual Minkowski metric (i.e. $b_0^2 = 1$) was derived by the fit of (14) to the experimental data. The corresponding metric is therefore given by

$$g_{DSR,weak}(E) = \text{diag} \left( 1, -b_{weak}^2(E), -b_{weak}^2(E), -b_{weak}^2(E) \right); \quad (15)$$

$$b_{weak}^2(E) = \begin{cases} (E/E_{0,weak})^{1/3}, & 0 < E \leq E_{0,weak} \\ 1, & E_{0,weak} < E \end{cases} = (16)$$

$$= 1 + \Theta(E_{0,weak} - E) \left[ \left( \frac{E}{E_{0,weak}} \right)^{1/3} - 1 \right], \quad E > 0, \quad (17)$$

with

$$E_{0,weak} = (80.4 \pm 0.2) \text{GeV}. \quad (18)$$

Two points are worth stressing. First, the value of $E_{0,weak}$—i.e. the energy value at which the weak metric becomes Minkowskian—corresponds to the mass of the $W$-boson, through which the $K^0_s$-decay occurs. Moreover, the leptonic metric (15)-(18) has the same form of the electromagnetic metric (10)-(13). Therefore, one recovers, by the DSR formalism, the well-known result of the Glashow-Weinberg-Salam model that, at the energy scale $E_{0,weak}$, the weak and the electromagnetic interactions are mixed. We want also to notice that, in both the electromagnetic and the weak case, the metric parameter exhibits a "sub-Minkowskian" behavior, i.e. $b(E)$ approaches 1 from below as energy increases.

3) **Strong interaction.** The phenomenon considered is the so-called Bose-Einstein (BE) effect in the strong production of identical bosons in high-energy collisions, which consists in an enhancement of their correlation probability [14]. The DSR formalism permits to derive a generalized BE correlation function, depending on all the four metric parameters $b_\mu(E)$ [12]. By using the experimental data on pion pair production, obtained in 1984 by
the UA1 group at CERN [21], one gets the following expression of the strong
metric for the two-pion BE phenomenon [12]:

\[ g_{DSR,\text{strong}}(E) = \text{diag} \left( b_{\text{strong}}^2(E), -b_{1,\text{strong}}^2(E), -b_{2,\text{strong}}^2(E), -b_{\text{strong}}^2(E) \right); \]

(19)

\[ b_{\text{strong}}^2(E) = \begin{cases} 
1, & 0 < E \leq E_{0,\text{strong}} \\
(E/E_{0,\text{strong}})^2, & E_{0,\text{strong}} < E
\end{cases} \quad (20) \]

\[ = 1 + \Theta(E - E_{0,\text{strong}}) \left[ \left( \frac{E}{E_{0,\text{strong}}} \right)^2 - 1 \right], E > 0; \quad (21) \]

\[ b_{1,\text{strong}}^2(E) = \left( \frac{\sqrt{2}}{5} \right)^2; \quad (22) \]

\[ b_{2,\text{strong}}^2 = (2/5)^2, \quad (23) \]

with

\[ E_{0,\text{strong}} = (367.5 \pm 0.4) \text{ GeV}. \quad (24) \]

The threshold energy \( E_{0,\text{strong}} \) is still the value at which the metric becomes
Minkowskian. Let us stress that, in this case, contrarily to the electromagnetic and the weak ones, a deformation of the time coordinate occurs; moreover, the three-space is anisotropic, with two spatial parameters constant (but different in value) and the third one variable with energy in an "over-Minkowskian" way. It is also worth to recall that the strong metric parameters \( b_{\mu} \) admit of a sensible physical interpretation: the spatial parameters are (related to) the spatial sizes of the interaction region ("fireball") where pions are produced, whereas the time parameter is essentially the mean life of the process. We refer the reader to Ref. [12] for further details.
4) **Gravitation.** It is possible to show that the gravitational interaction, too (at least on a *local* scale, i.e. in a neighborhood of Earth) can be described in terms of an energy-dependent metric, whose time coefficient was derived by fitting the experimental results on the relative rates of clocks at different heights in the gravitational field of Earth [16]. No information can be derived from the experimental data about the space parameters. Physical considerations — for whose details the reader is referred to Ref. [12] — lead to assume a gravitational metric of the same type of the strong one, i.e. *spatially anisotropic* and with one spatial parameter (say, $b_3$) equal to the time one: $b_0(E) = b_3(E) = b(E)$. The energy-dependent gravitational metric has therefore the form

$$g_{DSR,grav}(E) = \text{diag}\left(b^2_{grav}(E), -b^2_{1,grav}(E), -b^2_{2,grav}(E), -b^2_{grav}(E)\right);$$

(25)

$$b^2_{grav}(E) = \begin{cases} 
1, & 0 < E \leq E_{0,grav} \\
\frac{1}{4}(1 + E/E_{0,grav})^2, & E_{0,grav} < E
\end{cases}$$

(26)

$$= 1 + \Theta(E - E_{0,grav}) \left[\frac{1}{4}\left(1 + \frac{E}{E_{0,grav}}\right)^2 - 1\right], E > 0$$

(27)

(The coefficients $b^2_{1,grav}(E)$ and $b^2_{2,grav}(E)$ are presently *undetermined* at phenomenological level), with

$$E_{0,grav} = (20.2 \pm 0.1) \mu eV.$$  

(28)

The gravitational metric (25)-(28) is *over-Minkowskian*, asymptotically Minkowskian with decreasing energy, like the strong one. Intriguingly enough, the value of the threshold energy for the gravitational case $E_{0,grav}$ is approximately of the same order of magnitude of the thermal energy corresponding to the $2.7^oK$ cosmic background radiation in the Universe.

Moreover, the comparison of the values of the threshold energies for the four fundamental interactions yields

$$E_{0,e.m.} < E_{0,grav} < E_{0,weak} < E_{0,strong},$$

(29)
i.e. an increasing arrangement of $E_0$ from the electromagnetic to the strong interaction. Moreover

$$\frac{E_{0,\text{grav}}}{E_{0,e.m.}} = 4.49 \pm 0.02 ; \quad \frac{E_{0,\text{strong}}}{E_{0,\text{weak}}} = 4.57 \pm 0.01, \quad (30)$$

namely

$$\frac{E_{0,\text{grav}}}{E_{0,e.m.}} \sim \frac{E_{0,\text{strong}}}{E_{0,\text{weak}}}, \quad (31)$$

an intriguing result indeed.

3 Five-dimensional relativity with energy as extra dimension

It is easily seen, from the examination of the phenomenological metrics considered in the previous Section, that, in the DSR formalism, energy does play a dual role. Indeed, on one side, $E$ is to be considered as a dynamical variable, because it specifies the dynamical behavior of the process under consideration, and, via the metric coefficients, it provides us with a dynamical map - in the energy range of interest - of the interaction ruling the given process. On the other hand, it represents a parameter characteristic of the phenomenon considered (and therefore, for a given process, it cannot be changed at will, as already stressed in the previous Section). In other words, when describing a given process, the deformed geometry of space-time (in the interaction region where the process is occurring) is "frozen" at the situation described by those values of the metric coefficients corresponding to the energy value of the process considered. Otherwise speaking, from a geometrical point of view, all goes on as if we were actually working on "slices" (sections) of a five-dimensional space, in which the fifth dimension is just represented by the energy. In other words, a fixed value of the energy determines the space-time structure of the interaction region for the given process at that given energy. In this respect, therefore, $E$ is to be regarded as a geometrical quantity, intimately connected to the very geometrical structure of the physical world itself. The simplest way of taking into account such a double role of $E$ is to assume that energy does in fact represent an extra dimension — besides
the space and the time ones—, namely, to embed the deformed Minkowski space-time $\tilde{M}$ in a larger, five-dimensional space $\mathbb{R}^5$ [18].

Let us specify the metric structure of the five-dimensional Riemann space $\mathbb{R}^5$. We assume that the generalized metric interval in $\mathbb{R}^5$ is given by

$$ds^2(5) \equiv b_0^2(E) c^2 (dt)^2 - b_1^2(E) (dx)^2 - b_2^2(E) (dy)^2 - b_3^2(E) (dz)^2 \pm f(E) \ell_0^2 (dE)^2 =$$

$$= g_{\mu\nu,DSR}(E) dx^\mu dx^\nu \pm f(E)(dx^5)^2 \equiv$$

$$\equiv g_{AB,DR}(E)dx^A dx^B,$$

where $A, B = 0, 1, 2, 3, 5, x^5 \equiv \ell_0 E$, with $\ell_0$ being a dimensionally-transposing constant (because of in this Riemannian framework it is worth to give $x^5$ the dimension of a length, $\ell_0$ has physical dimension [length]×[energy]$^{-1}$), and $f(E) > 0$. The coefficients $\{b_\mu^2(E)\}$ are those determining the deformation of the 4-d. DSR spacetime \textsuperscript{5}. The five-dimensional metric tensor $g_{DSR}^E$ reads therefore

$$g_{AB,DR5}(x^5) = \text{diag}(b_0^2(x^5), -b_1^2(x^5), -b_2^2(x^5), -b_3^2(x^5), \pm f(x^5)), \quad (33)$$

and it is a function of the energy: $g_{DR}^E = g_{DSR}(E)$.

Some remarks are in order. First, in analogy with the space-time metric coefficients $b_\mu$, we assumed that also the fifth metric coefficient depends only

\textsuperscript{5}Since the metric coefficients $b_\mu^2(x^5)$ and $f(x^5)$ are dimensionless, they actually do depend on the ratio $\frac{x_0}{\ell_0}$, where $x_0^5 \equiv \ell_0 E_0$ is a fundamental length, proportional (by the dimensionally-transposing constant $\ell_0$) to the threshold energy $E_0$, characteristic of the interaction considered:

$$b_\mu^2(x^5) \equiv b_\mu^2 \left( \frac{x_0}{\ell_0} \right) = b_\mu^2 \left( \frac{E}{E_0} \right),$$

$$f(x^5) \equiv f \left( \frac{x_0}{\ell_0} \right) = f \left( \frac{E}{E_0} \right).$$

For simplicity’s sake, in the following we will omit, but always understand, the cumbersome, but more rigorous notations $b_\mu^2 \left( \frac{x_0}{\ell_0} \right)$ and $f \left( \frac{x_0}{\ell_0} \right)$. 

14
on the energy: $f = f(E)$. However, one might assume that the energy coefficient is a function also of the space-time coordinates $x = (x^0, x^1, x^2, x^3)$, namely $f = f(x, E)$. At present, such a possibility will be disregarded. Moreover, we leave open the issue of considering $E$ as a timelike or a spacelike coordinate in $\mathbb{R}_5$ (the double sign in front of $f$ in Eqs (32) and (33). Actually, in the standard Kaluza-Klein scheme, the fifth dimension must necessarily be spacelike, because the number of timelike dimensions cannot exceed one, if one wants to avoid causal anomalies [22]. But — and this is just another point worth stressing — this five-dimensional scheme is not a "true" Kaluza-Klein one, due to the fact that the four-dimensional space-time is endowed with the deformed metric (2). It is therefore an open issue whether or not, in such a framework, more timelike dimensions do give rise to causal anomalies.

We shall refer to the theory based on metric (33) as Deformed Relativity in Five Dimensions (DR5). This approach is a Kaluza-Klein-like (or pseudo-Kaluza-Klein) one, since the four-dimensional space-time is endowed with the deformed metric (2) and now the extra parameter is a physically sensible dimension. Thus, on the latter respect, such a formalism belongs to the class of noncompactified KK theories, which, at the present status of experimental knowledge, cannot be ruled out (see second Ref. in [8]). In the DR5 framework, the (deformed) Minkowski space-time is recovered not by means of a compactification procedure, but instead by a dimensional reduction.

As to considering energy as a dynamical variable, the use of momentum components as dynamical variables on the same foot of the space-time ones can be traced back to Ingraham [6]. Moreover, Dirac [23], Hoyle and Narlikar [24] and Canuto et al. [25] treated mass as a dynamical variable in the context of scale-invariant theories of gravity.

On the above side, the DR5 formalism has some connection with the interesting "Space-Time-Mass" (STM) theory, in which the fifth dimension is the rest mass, proposed by Wesson [26] and studied in detail by a number of Authors. In either formalism it is assumed that all metric coefficients do in general depend on the fifth coordinate. Such a feature distinguishes both models from true Kaluza-Klein theories. However, the DR5 approach differs from the STM model (as well as from similar ones [27]) at least in the following main respects:

(i) its physical motivations are based on the phenomenological analysis of Sect. 2, and therefore are not merely speculative;
(ii) the fact of assuming energy (which is a true variable), and not rest mass (which instead is an invariant), as fifth dimension;

(iii) the local (and not global) nature of the five-dimensional space, whereby the energy-dependent deformation of the four-dimensional space-time is assumed to provide a geometrical description of the interactions [12].

The space \( \mathcal{R}_5 \) has the following "slicing property"

\[
\mathcal{R}_5\big|_{dx^5=0} = \{ \overline{M}(x^5) \}_{x^5 = \overline{x}^5}
\]

(where \( \overline{x}^5 \) is a fixed value of the fifth coordinate) or, at the level of the metric tensor:

\[
g_{AB,DR5}(x^5)\big|_{dx^5=0} = \text{diag} \left( b_0^2(\overline{x}^5), -b_1^2(\overline{x}^5), -b_2^2(\overline{x}^5), -b_3^2(\overline{x}^5), \pm f(\overline{x}^5) \right) = g_{AB,DSR}(\overline{x}^5).
\]

4 Five-dimensional Einstein equations

4.1 Solving the vacuum Einstein equations in \( \mathcal{R}_5 \)

The vacuum Einstein equations in the space \( \mathcal{R}_5 \) are [18]

\[
R_{AB} - \frac{1}{2}g_{AB,DR5}R = \Lambda g_{AB,DR5}, \tag{34}
\]

where \( R_{AB} \) and \( R = R^A_A \) are the five-dimensional Ricci tensor and scalar (intrinsic) curvature, respectively, and \( \Lambda \) is the "cosmological" constant, which may, in principle, depend on both the energy \( E \) and the space-time coordinates \( x : \Lambda = \Lambda(x, E) \). As is well known from Riemannian differential geometry, the Ricci tensor explicitly reads (ESC on)

\[
R_{AB} = \partial_I \Gamma^I_{AB} - \partial_B \Gamma^I_{AI} + \Gamma^I_{AB} \Gamma^K_{IK} - \Gamma^K_{AI} \Gamma^K_{BK}, \tag{35}
\]

with the second-kind Christoffel symbols \( \Gamma^I_{AB} = \left\{ \frac{I}{AB} \right\} \) given by

\[
2\Gamma^I_{AB} = g^{IK}_{DS5}(\partial_B g_{KA,DR5} + \partial_A g_{KB,DR5} - \partial_K g_{AB,DR5}). \tag{36}
\]

6In this respect, therefore, DR5 resembles more the formalism by Ingraham [6].
We want here to consider some special cases of the five-dimensional Einstein equations, which — on account of the discussion of Section 2 — are of a special physical relevance. They are: (i) the case of spatial isotropy; and: (ii) when all the metric coefficients are powers of the energy.

In order to simplify the notation, we write the metric tensor (33) in the form

$$g_{DR5}(E) = \text{diag}(a(E), -b(E), -c(E), -d(E), f(E)),$$  \hspace{1cm} (37)

(with \(a(E), b(E), c(E), d(E)\) positive functions) and adopt units such that \(c = \text{(velocity of light)} = 1 = \ell_0\). As can be seen by comparing the 5-d. metrics (37) and (33), here we also re-adsorb the "±" in a redefinition of \(f(E)\), which now may change in sign. As can be easily understood by looking at ODEs’ systems (39) and (42) and performing the general functional reflection \(f(E) \to -f(E)\), such a redefinition of \(f(E)\) is completely uninfluential (in the sense that it leaves the results unchanged), at least in the considered case of resolution of 5-d. Einstein equations (in the reductive-simplifying cases i and ii) with vacuum prescription (i.e. with \(\Lambda = 0\)).

We have therefore:

**Case i)** - For a spatial isotropic deformation, it is \(b(E) = c(E) = d(E)\), so that the metric becomes

$$g_{DR5}(E) = \text{diag}(a(E), -b(E), -b(E), -b(E), f(E)).$$  \hspace{1cm} (38)

The independent Einstein equations obviously reduce to the following three ones (henceforth, a prime denotes derivation with respect to \(E\); moreover, for simplicity of notation, we omit the explicit functional dependence of all quantities on \(E\)):

$$\begin{cases} 
3(-2b'' f + b' f') = 4\Lambda b f^2; \\
\left. f \left[ a^2(b')^2 - 2a a' b b' - 4a^2 bb'' - 2a a'' b^2 + b^2(a')^2 \right] + \\
+ a b f'(2a b' + a' b) = 4\Lambda a^2 b^2 f^2; \\
3b'(ab)' = -4\Lambda ab^2 f.
\end{cases}$$  \hspace{1cm} (39)
Case ii) - Since the space-time metric coefficients are dimensionless, as already pointed out in Footnote 5, it is assumed that they are functions of the ratio \(E/E_0\), where \(E_0\) is an energy scale characteristic of the interaction (and the process) considered (for instance, the energy threshold in the phenomenological metrics (10)-(28)). Precisely, for the metric \(g_{DR5}\) written in the form (37), we put ("Power Ansatz")

\[
\begin{align*}
    a(E) &= (E/E_0)^{q_0}; \\
    b(E) &= (E/E_0)^{q_1}; \\
    c(E) &= (E/E_0)^{q_2}; \\
    d(E) &= (E/E_0)^{q_3}
\end{align*}
\]  

\((q_0, q_1, q_2, q_3 \in R)\). For the fifth metric coefficient \(f(E)\) we also assume

\[
    f(E) = (E/E_0)^r, r \in R, \tag{41}
\]

being understood, as before, that \(E_0 = x_0^5/\ell_0 = x_0^5\) in the assumed units, where \(\ell_0 = 1\). Of course, the Einstein equations reduce now to the following algebraic equations in the five exponents \(q_0, q_1, q_2, q_3, r\):

\[
\begin{align*}
    (2 + r)(q_3 + q_1 + q_2) - q_1^2 - q_2^2 - q_3^2 - q_1 q_2 - q_1 q_3 - q_2 q_3 &= 4\Lambda(E/E_0)^{r+2}; \\
    (2 + r)(q_3 + q_0 + q_2) - q_2^2 - q_3^2 - q_0^2 - q_2 q_3 - q_2 q_0 - q_3 q_0 &= 4\Lambda(E/E_0)^{r+2}; \\
    (2 + r)(q_3 + q_0 + q_1) - q_1^2 - q_2^2 - q_3^2 - q_1 q_3 - q_1 q_0 - q_3 q_0 &= 4\Lambda(E/E_0)^{r+2}; \\
    (2 + r)(q_0 + q_1 + q_2) - q_1^2 - q_2^2 - q_0^2 - q_1 q_2 - q_1 q_0 - q_2 q_0 &= 4\Lambda(E/E_0)^{r+2}; \\
    q_1 q_2 + q_1 q_3 + q_1 q_0 + q_2 q_3 + q_2 q_0 + q_3 q_0 &= -4\Lambda(E/E_0)^{r+2}. \tag{42}
\end{align*}
\]

Of course, for consistency one has to impose the compatibility condition that \(\Lambda\), too, is a power of the energy, and precisely one should assume the following functional dependence:

\[
    \Lambda(E/E_0) \sim (E/E_0)^{-(r+2)}. \tag{43}
\]
needless to say, the *vacuum prescription* $\Lambda = 0$ is compatible with this hypothesis.

Solving Einstein’s equations in the five-dimensional, deformed space $\mathbb{R}_5$ in the general case is quite an impossible task. On the contrary, it is possible to show [18] that, in the two special cases considered above, some classes of solutions can be found for Eq.s (39) and (42) (respectively corresponding to *spatial isotropy* and metric coefficients which are *powers* of the energy), at least for $\Lambda = 0$. Notice that assuming a vanishing cosmological constant has the physical motivation (at least as far as gravitation is concerned and one is not interested into quantum effects) that $\Lambda$ is related to the vacuum energy; experimental evidence shows that $\Lambda \simeq 3 \cdot 10^{-52} m^{-2}$.

We recall moreover that Eq.s (34) imply $R = -\frac{10}{3} \Lambda$. Being $\Lambda = 0$ (and consequently $R = 0$) the spaces we will find are obviously *Ricci flat*. However, they differ, in general, from a 5-dimensional flat space, as it can be easily checked by showing explicitly that some components of the Riemann curvature tensor do not vanish.

**i)** In the former case (*spatial isotropy*), by putting $\Lambda = 0$, the system of ordinary differential equations (39) takes the form

$$
\begin{align*}
-2b''f + b'f' &= 0 ; \\
f \left[ a^2(b')^2 - 2aa'bb' - 4a^2bb'' - 2aa''b^2 + b^2(a')^2 \right] + \\
+abf'(2ab' + a'b) &= 0 ; \\
b'(ab)' &= 0 .
\end{align*}
$$

If $a = \text{const.}$ (i.e. $a' = 0$), then the third equation of (43) implies $b' = 0$; it is thence easy to see that the remaining equations are identically satisfied. Hence the system (39) admits only the solution $b = \text{const.}, f(E)$ undetermined, which can be shown to correspond (modulo rescaling) to a flat 5-dimensional space. This entails, as one should suspect, that a 5-dimensional Minkowski space can be a solution of our system.

If $a$ is *not* a constant, then the third equation implies either (i.1) $b' = 0, (ab)' \neq 0$ or (i.2) $b' \neq 0, (ab)' = 0$. 

19
Let us consider these two cases.

(i.1) In this case \( b = \text{const.} \) and the system (43) admits solutions with \( a(E) \) arbitrary and \( f(E) \) determined by the only remaining non-trivial equation, namely:

\[
f[(a')^2 - 2aa''] = -aa' f'.
\]

Putting

\[
A(E) = \frac{2aa'' - (a')^2}{aa'} = \frac{f'}{f},
\]

we get then

\[
f(E) = ke^{\int E A(\xi) d\xi}
\]

where \( k \) is an integration constant. We remark that, if \( f(E) = \text{const.} \), Eq. (45) becomes

\[
(a')^2 - 2aa'' = 0.
\]

It is easy to see that this equation admits the only solution

\[
a(E) = \left( 1 + \frac{E}{E_0} \right)^2,
\]

with \( E_0 \) constant. Therefore, this shows that the gravitational metric (25) corresponds to \( f = \text{const.} \), in the case of spatial isotropy.

(i.2) In this second case, it is not difficult to get the following class of solutions:

\[
f(E) = k [b'(E)]^2 ;
\]

\[
a(E) = b(E)^{-1},
\]

where \( k \) is a constant (which fixes the sign of \( f \)) and \( b(E) \) is an arbitrary function of \( E \).
Let us now discuss the case of the metric coefficients which are pure powers of the energy. For \( \Lambda = 0 \) Eq.s (42) admit of twelve possible classes of solutions, which can be classified according to the values of the five-dimensional vector \( \alpha \equiv (q_0, q_1, q_2, q_3, r) \in \mathbb{R}^5 \) built up from the energy exponents of the metric coefficients (see Eq.s (40) and (41)). Explicitly one has [18]

- **Class (I):**
  
  \[ \alpha_I = \left( q_2, -q_2 \frac{2q_3 + q_2}{2q_2 + q_3}, q_2, q_3, \frac{q_3^2 - 2q_3 + 2q_2q_3 - 4q_2 + 3q_2^2}{2q_2 + q_3} \right); \]

- **Class (II):** \( \alpha_{II} = (0, q_1, 0, 0, q_1 - 2) \);

- **Class (III):** \( \alpha_{III} = (q_2, -q_2, q_2, q_2, -2(1 - q_2)) \);

- **Class (IV):** \( \alpha_{IV} = (0, 0, 0, q_3, q_3 - 2) \);

- **Class (V):** \( \alpha_{V} = (-q_3, -q_3, -q_3, q_3, -(1 + q_3)) \);

- **Class (VI):** \( \alpha_{VI} = (q_0, 0, 0, 0, q_0 - 2) \);

- **Class (VII):** \( \alpha_{VII} = (q_0, -q_0, -q_0, -q_0, -2 - q_0) \);

- **Class (VIII):** \( \alpha_{VIII} = (0, 0, 0, 0, r) \);

- **Class (IX):** \( \alpha_{IX} = (0, 0, q_2, 0, -2 + q_2) \);

- **Class (X):**
  
  \[ \alpha_X = \left( q_0, -\frac{q_3q_0 + q_2q_3 + q_2q_0}{q_2 + q_3 + q_0}, q_2, q_3, r_X \right), \]
with
\[ r_X = \frac{q_3^2 + q_3 q_0 - 2q_3 + q_2 q_3 - 2q_2 + q_2 q_0 + q_2^2 - 2q_0 + q_0^2}{q_2 + q_3 + q_0}; \]

- Class (XI):

\[ \alpha_{XI} = \left( q_0, -\frac{q_2 (2q_0 + q_2)}{2q_2 + q_0}, q_2, \frac{3q_2^2 - 4q_2 + 2q_2 q_0 - 2q_0 + q_0^2}{2q_2 + q_0} \right); \]

- Class (XII): \[ \alpha_{XII} = \left( q_0, q_2, q_2, -\frac{q_2 (2q_0 + q_2)}{2q_2 + q_0}, r_{XII} \right), \] with

\[ r_{XII} = \frac{q_3^2 + q_3 q_0 - 2q_3 + q_2 q_3 - 2q_2 + q_2 q_0 + q_2^2 - 2q_0 + q_0^2}{q_2 + q_3 + q_0}. \]

In the following Subsection, we shall discuss the physical relevance of the above solutions.

### 4.2 Discussion of the solutions.

As we said in the previous Subsection, in the case of spatial isotropy the analytical solution of Eq. (45), for \( f = \text{const.} \), yields immediately the gravitational metric (25).

On the other hand, the twelve classes of solutions found when assuming that the metric coefficients are powers of the energy, allow one to recover, as special cases, all the phenomenological metrics discussed in Sect. 2 [18]. Let us write explicitly the infinitesimal metric interval in \( \mathbb{R}_5 \) in such a case:

\[ ds_{(5)}^2 = \left( \frac{E}{E_0} \right)^{q_0} (dt)^2 - \left( \frac{E}{E_0} \right)^{q_1} (dx)^2 - \left( \frac{E}{E_0} \right)^{q_2} (dy)^2 - \left( \frac{E}{E_0} \right)^{q_3} (dz)^2 + \left( \frac{E}{E_0} \right)^r (dE)^2. \]

(50)
Then, it is easily seen that the Minkowski metric is recovered from all classes of solutions. Solution (VIII) corresponds directly to a Minkowskian space-time, with the exponent $r$ of the fifth coefficient undetermined. In the other cases, we have to put:

- $q_1 = 0$ for class (II);
- $q_2 = 0$ for classes (III) and (IX);
- $q_3 = 0$ for (IV) and (V);
- $q_0 = 0$ for (VI) and (VII)

(for all the previous solutions, it is $r = -2$);

- $q_2 = q_3 = 0$ for class (I);
- $q_2 = q_3 = q_0 = 0$ for class (X);
- $q_2 = q_0 = 0$ for class (XI);
- $q_2 = q_0 = 0$ for class (XII).

The latter four solutions have $r = 0$, and therefore correspond to a five-dimensional Minkowskian (and thus flat) space.

If we set:

- $q_1 = 1/3$ in class (II);
- $q_3 = 1/3$ in class (IV) or
- $q_2 = 1/3$ in class (IX)

(corresponding in all three cases to the value $r = 5/3$ for the exponent of the fifth metric coefficient), we get a metric of the "electroweak type" (see Eq.s (10)-(13), (15)-(18)), i.e. with unit time coefficient and one space coefficient behaving as $(E/E_0)^{1/3}$, but spatially anisotropic, since two of the space metric coefficients are constant and Minkowskian (precisely, the $y$, $z$ coefficients for class (II); the $x$, $y$ coefficients for class (IV); and the $x$, $z$ ones for class (IX)). Notice that such an anisotropy does not disagree with the phenomenological results; indeed, in the analysis of the experimental data one was forced to assume spatial isotropy in the electromagnetic and in the weak cases, simply because of the lack of experimental information on two of the space dimensions.

Putting $q_0 = 1$ in class (VI), we find a metric which is spatially Minkowskian, with a time coefficient linear in $E$, i.e. a (gravitational) metric of the Einstein type (4).

Class (I) allows us to find as a special case a metric of the strong type (see Eq.s (19)-(24)). This is achieved by setting $q_2 = 2$, whence we get

$$q_1 = -4(q_3 + 1)/(q_3 + 4); \quad r = (q_3^2 + 2q_3 + 4)/(q_3 + 4).$$
Moreover, for \( q_3 = 0 \), it is \( q_1 = -1; r = 1 \). In other words, we have a solution corresponding to \( a(E) = b(E) = (E/E_0)^2 \) and spatially anisotropic, i.e. a metric of the type (19)-(24).

Finally, the three classes (X)-(XII) admit as special case the gravitational metric (25)-(28), which is recovered by putting \( q_0 = 2 \) and \( q_1 = q_2 = q_3 = 0 \) (whence also \( r = 0 \)) and by a rescaling and a translation of the energy parameter \( E_0 \).

In conclusion, we can state that the formalism of DR5 permits to recover, as solutions of the vacuum Einstein equations, all the phenomenological energy-dependent metrics of the electromagnetic, weak, strong and gravitational type (and also the gravitational one of the Einstein kind, Eq. (4)).

5 Killing symmetries in the space \( \mathcal{R}_5 \).

The topics concerning Deformed Relativity in 5 dimensions we expounded in the previous Sections have been already discussed in literature [18]. In the present Section, we shall deal for the first time with the problem of the metric automorphisms (i.e. isometries) of the 5-d. Riemann space \( \mathcal{R}_5 \) of DR5 [28].

5.1 General case.

Let us discuss the Killing symmetries of the space \( \mathcal{R}_5 \) [28].

The Killing equations for metric (33) read

\[
\xi_{[A;B]} = 0 \Leftrightarrow \xi_{A;B} + \xi_{B;A} = 0,
\]

(51)

where as usual ; \( A \) denotes Riemann covariant derivative with respect to \( x^A \) and

\[
\xi_A = \xi_A(x^0, x^1, x^2, x^3, x^5) \equiv \xi_A(x^B)
\]

(52)
is the covariant Killing 5-vector of \( \mathcal{R}_5 \).

From the Christoffel symbols \( \Gamma^A_{BC} \) of the metric \( g_{AB,DR5}(x^5) \) we get the following system of 15 coupled, partial derivative differential equations (PDEs) in \( \mathcal{R}_5 \) for the Killing vector \( \xi_A(x^B) \):
\[ f(x^5)\xi_{0,0}(x^4) \pm b_0(x^5)b'_0(x^5)\xi_5(x^4) = 0 ; \] (53)

\[
\begin{align*}
\xi_{0,1}(x^4) + \xi_{1,0}(x^4) &= 0 \\
\xi_{0,2}(x^4) + \xi_{2,0}(x^4) &= 0 \\
\xi_{0,3}(x^4) + \xi_{3,0}(x^4) &= 0
\end{align*}
\] type I conditions ; (54)

\[ b_0(x^5)(\xi_{0,5}(x^4) + \xi_{5,0}(x^4)) - 2b'_0(x^5)\xi_0(x^4) = 0 \} \text{ type II condition;} \] (55)

\[ f(x^5)\xi_{1,1}(x^4) \mp b_1(x^5)b'_1(x^5)\xi_5(x^4) = 0 ; \] (56)

\[
\begin{align*}
\xi_{1,2}(x^4) + \xi_{2,1}(x^4) &= 0 \\
\xi_{1,3}(x^4) + \xi_{3,1}(x^4) &= 0
\end{align*}
\] type I conditions; (57)

\[ b_1(x^5)(\xi_{1,5}(x^4) + \xi_{5,1}(x^4)) - 2b'_1(x^5)\xi_1(x^4) = 0 \} \text{ type II condition;} \] (58)

\[ f(x^5)\xi_{2,2}(x^4) \mp b_2(x^5)b'_2(x^5)\xi_5(x^4) = 0 ; \] (59)

\[ \xi_{2,3}(x^4) + \xi_{3,2}(x^4) = 0 \} \text{ type I condition;} \] (60)

\[ b_2(x^5)(\xi_{2,5}(x^4) + \xi_{5,2}(x^4)) - 2b'_2(x^5)\xi_2(x^4) = 0 \} \text{ type II condition;} \] (61)

\[ f(x^5)\xi_{3,3}(x^4) \mp b_3(x^5)b'_3(x^5)\xi_5(x^4) = 0 ; \] (62)

\[ b_3(x^5)(\xi_{3,5}(x^4) + \xi_{5,3}(x^4)) - 2b'_3(x^5)\xi_3(x^4) = 0 \} \text{ type II condition;} \] (63)

\[ 2f(x^5)\xi_{5,5}(x^4) - f'(x^5)\xi_5(x^4) = 0. \] (64)
PDEs (53)-(64) can be divided in "fundamental" equations and "constraint" equations (of type I and II). The above system is in general over-determined, i.e. its solutions will contain numerical coefficients satisfying a given algebraic system. Its explicit solutions are given by

\[ \xi_\mu(x^A) = F_\mu(x^{A\neq\mu}) + \]

\[ \pm(-\delta_{\mu_0} + \delta_{\mu_1} + \delta_{\mu_2} + \delta_{\mu_3})b_\mu(x^5)b'_\mu(x^5)(f(x^5))^{-1/2} \int dx^\mu F_5(x^0, x^1, x^2, x^3); \]

\[ \xi_5(x^A) = (f(x^5))^{1/2}F_5(x^0, x^1, x^2, x^3). \]

The five unknown functions \( F_A(x^{B\neq A}) \) are restricted by the two following types of conditions:

I) Type I (Cardinality 4, \( \mu \neq \nu \neq \rho \neq \sigma \)):

\[ \pm A_\mu(x^5)G_{\nu\rho\sigma}(x^0, x^1, x^2, x^3) + B_\mu(x^5)G_{\mu\nu\rho\sigma}(x^0, x^1, x^2, x^3) + \]

\[ + b_\mu(x^5)F_{\mu,5}(x^{A\neq\mu}) - 2b'_\mu(x^5)F_\mu(x^{A\neq\mu}) = 0; \]

II) Type II (Cardinality 6, symm. in \( \mu, \nu, \mu \neq \nu \neq \rho \neq \sigma \)):

\[ F_{\mu,\nu}(x^{A\neq\mu}) + F_{\nu,\mu}(x^{A\neq\nu}) + \]

\[ \pm(-\delta_{\nu_0} + \delta_{\nu_1} + \delta_{\nu_2} + \delta_{\nu_3})b_\nu(x^5)b'_\nu(x^5)(f(x^5))^{-1/2}G_{\nu\nu\rho\sigma}(x^0, x^1, x^2, x^3) + \]

\[ \pm(-\delta_{\nu_0} + \delta_{\nu_1} + \delta_{\nu_2} + \delta_{\nu_3})b_\nu(x^5)b'_\nu(x^5)(f(x^5))^{-1/2}G_{\nu\mu\rho\sigma}(x^0, x^1, x^2, x^3) = 0. \]
where:

\[ A_\mu(x^5) \equiv (-\delta_{\mu0} + \delta_{\mu1} + \delta_{\mu2} + \delta_{\mu3})b_\mu(x^5)(f(x^5))^{-1/2}. \]

\[ B_\mu(x^5) \equiv b_\mu(x^5)(f(x^5))^{1/2}; \]  

\( G(x^0, x^1, x^2, x^3) \equiv \int dx^0 dx^1 dx^2 dx^3 F_5(x^0, x^1, x^2, x^3). \)  

5.2 The hypothesis \( \Upsilon \) of functional independence.

Let us consider the derivative with respect to \( x^\mu \) of Type I conditions (ESC off):

\[ \partial_\mu I : \pm A_\mu(x^5)G_{\mu
u\rho\sigma}(x^0, x^1, x^2, x^3) + B_\mu(x^5)G_{\mu\mu\nu\rho\sigma}(x^0, x^1, x^2, x^3) = 0 \Leftrightarrow \]

\[ \Leftrightarrow \pm A_\mu(x^5)F_5(x^0, x^1, x^2, x^3) + B_\mu(x^5)F_{5,\mu\mu}(x^0, x^1, x^2, x^3) = 0. \]  

(72)

If \( G(x^0, x^1, x^2, x^3) \) satisfies the Schwarz lemma at any order, since \( \mu \neq \nu \neq \rho \neq \sigma \), one gets

\[ G_{\mu\nu\rho\sigma}(x^0, x^1, x^2, x^3) = G_{\sigma0123}(x^0, x^1, x^2, x^3)(= F_5(x^0, x^1, x^2, x^3)), \]  

(73)

namely the function \( F_5(x^0, x^1, x^2, x^3) \) is in \( \frac{\partial}{\partial x^\mu} I \) \( \forall \mu = 0, 1, 2, 3. \)

It is therefore sufficient to assume that at least a special index

\[ \overline{\mu} \in \{0, 1, 2, 3\} : \{ \begin{align*} & \exists c_\overline{\mu} \in R_0 : \pm A_\overline{\mu}(x^5) = c_\overline{\mu} B_\overline{\mu}(x^5)(\forall x^5 \in R_0^+), \\ & A_\overline{\mu}(x^5) \neq 0, B_\overline{\mu}(x^5) \neq 0 \end{align*} \]  

(74)
exists, such that \((\forall x^0, x^1, x^2, x^3 \in R)\)

\[
(F_5(x^0, x^1, x^2, x^3) =) G_{0123}(x^0, x^1, x^2, x^3) = 0 =
\]

\[
= G_{\mu\mu\mu\mu 123}(x^0, x^1, x^2, x^3)(= F_{5,\mu\mu}(x^0, x^1, x^2, x^3)) \quad \Rightarrow \quad (\text{in gen.})
\]

\[
\Rightarrow \quad G_{\mu\mu\mu\mu 123}(x^0, x^1, x^2, x^3)(= F_{5,\mu\mu}(x^0, x^1, x^2, x^3)) = 0, \ \forall \mu = 0, 1, 2, 3.
\]

\(\text{(75)}\)

In the following the existence hypothesis

\[
\exists \text{ (at least one) } \mu \in \{0, 1, 2, 3\} : \begin{cases}
\exists \mu \in R_0 : \pm A_\mu(x^5) = c_\mu B_\mu(x^5)(\forall x^5 \in R_0^+)

A_\mu(x^5) \neq 0, B_\mu(x^5) \neq 0
\end{cases}
\]

\(\text{(76)}\)

will be called "\(\Upsilon\) hypothesis" of functional independence.

### 5.3 Solving Killing equations in \(\mathcal{R}_5\) in the hypothesis \(\Upsilon\) of functional independence.

In the hypothesis \(\Upsilon\) of functional independence the contravariant Killing 5-vector has the form (ESC off)

\[
\xi_A(x^B) = \left( b^2_\mu(x^5) \tilde{F}_\mu(x^{\nu\neq\mu}), 0 \right)
\]

\(\text{(77)}\)

where the 4 unknown real functions of 3 real variables \(\left\{ \tilde{F}_\mu(x^{\nu\neq\mu}) \right\} \) are solutions of the following system of 6 (due to the symmetry in \(\mu\) and \(\nu\)) non-linear PDEs:

\[
b^2_\mu(x^5) \frac{\partial \tilde{F}_\mu(x^{\nu\neq\mu})}{\partial x^\nu} + b^2_\nu(x^5) \frac{\partial \tilde{F}_\nu(x^{\mu\neq\nu})}{\partial x^\mu} = 0, \ \mu, \nu = 0, 1, 2, 3, (\mu \neq \nu),
\]

\(\text{(78)}\)

which is in general overdetermined.
Solving system (78) yields the following expressions for the components of the contravariant Killing 5-vector $\xi^A(x^0, x^1, x^2, x^3, x^5)$ satisfying the 15 Killing PDEs (53)-(64) in the hypothesis $\Upsilon$ of functional independence (76):

$$\xi^0(x^1, x^2, x^3) = \tilde{F}_0(x^1, x^2, x^3) =$$

$$= d_8 x^1 x^2 x^3 + d_7 x^1 x^2 + d_6 x^1 x^3 + d_4 x^2 x^3 +$$

$$(d_5 + a_2) x^1 + d_3 x^2 + d_2 x^3 + (a_1 + d_1 + K_0); \quad (79)$$

$$\xi^1(x^0, x^2, x^3) = -\tilde{F}_1(x^0, x^2, x^3) =$$

$$= -h_2 x^0 x^2 x^3 - h_1 x^0 x^2 - h_8 x^0 x^3 - h_4 x^2 x^3 -$$

$$-(h_7 + e_2) x^0 - h_3 x^2 - h_6 x^3 - (K_1 + h_5 + e_1); \quad (80)$$

$$\xi^2(x^0, x^1, x^3) = -\tilde{F}_2(x^0, x^1, x^3) =$$

$$= -l_2 x^0 x^1 x^3 - l_1 x^0 x^1 - l_6 x^0 x^3 - l_4 x^1 x^3 -$$

$$-(l_5 + e_4) x^0 - l_3 x^1 - l_8 x^3 - (l_7 + K_2 + e_3); \quad (81)$$

$$\xi^3(x^0, x^1, x^2) = -\tilde{F}_3(x^0, x^1, x^2) =$$

$$= -m_8 x^0 x^1 x^2 - m_7 x^0 x^1 - m_6 x^0 x^2 - m_4 x^1 x^2 -$$

$$-(m_5 + g_2) x^0 - m_3 x^1 - m_2 x^2 - (m_1 + g_1 + c); \quad (82)$$

$$\xi^5 = 0 \neq \xi^5(x^0, x^1, x^2, x^3, x^5). \quad (83)$$
where (some of) the real parameters satisfy the algebraic system

\[
\begin{align*}
&b^2_6(x^5) \left[ d_8 x^2 x^3 + d_7 x^2 + d_6 x^3 + (d_5 + a_2) \right] + \\
&+ b^2_1(x^5) \left[ h_2 x^2 x^3 + h_1 x^2 + h_8 x^3 + (h_7 + c_2) \right] = 0; \\
&b^2_6(x^5) (d_8 x^1 x^3 + d_7 x^1 + d_4 x^3 + d_3) + \\
&+ b^2_2(x^5) \left[ l_2 x^1 x^3 + l_1 x^1 + l_6 x^3 + (l_5 + e_4) \right] = 0; \\
&b^2_6(x^5) (d_8 x^1 x^2 + d_6 x^1 + d_4 x^2 + d_2) + \\
&+ b^2_3(x^5) \left[ m_8 x^1 x^2 + m_7 x^1 + m_6 x^2 + (m_5 + g_2) \right] = 0; \\
&b^2_2(x^5) (h_2 x^0 x^3 + h_1 x^0 + h_4 x^3 + h_3) + \\
&+ b^2_2(x^5) \left[ l_2 x^0 x^3 + l_1 x^0 + l_4 x^3 + l_3 \right] = 0; \\
&b^2_2(x^5) (h_2 x^0 x^2 + h_8 x^0 + h_4 x^2 + h_6) + \\
&+ b^2_3(x^5) \left[ m_8 x^0 x^2 + m_7 x^0 + m_4 x^2 + m_3 \right] = 0; \\
&b^2_2(x^5) \left[ l_2 x^0 x^1 + l_6 x^0 + l_4 x^1 + l_8 \right] + \\
&+ b^2_3(x^5) \left[ m_8 x^0 x^1 + m_6 x^0 + m_4 x^1 + m_2 \right] = 0.
\end{align*}
\]

(84)

5.4 The "Power Ansatz" and the reductivity of the hypothesis \( \Upsilon \) of functional independence.

We want now to investigate if and when the simplifying \( \Upsilon \) hypothesis (76) — we exploited in order to solve the Killing equations in \( \mathbb{R}_5 \) — is reductive. To
this aim, one needs to consider explicit forms of the 5-d. Riemannian metric \( g_{AB,DR5}(x^5) \). As we have seen in Section 4, the "Power Ansatz" allows one to recover all the phenomenological metrics derived for the four fundamental interactions. So it is worth considering such a case, corresponding to a 5-d. metric of the form

\[
g_{AB,DR5power}(x^5) = \text{diag} \left( \left( \frac{x^5}{x_0^5} \right)^{q_0}, - \left( \frac{x^5}{x_0^5} \right)^{q_1}, - \left( \frac{x^5}{x_0^5} \right)^{q_2}, - \left( \frac{x^5}{x_0^5} \right)^{q_3}, \pm \left( \frac{x^5}{x_0^5} \right)^r \right),
\]

(85)

Notice that, in comparison with Eq. (37) with Ansätze (40) and (41) implemented, here we re-extracted "±" from the fifth metric coefficient, whence \( \left( \frac{x^5}{x_0^5} \right)^r > 0 \ \forall x^5 \in R_0^+ \).

From Eqs. (69), (70) and (85) one thus gets:

\[
A_{\mu,\text{power}}(x^5) = - (\delta_{\mu 0} + \delta_{\mu 1} + \delta_{\mu 2} + \delta_{\mu 3}) \frac{q_\mu}{2} \left( \frac{r}{2} \right) \left( \frac{x^5}{x_0^5} \right)^{\frac{3}{2}q_\mu - \frac{1}{2}r - 2} =
\]

\[
= A_{\mu,\text{power}}(q_\mu, r; x^5);
\]

(86)

\[
B_{\mu,\text{power}}(x^5) = \left( \frac{x^5}{x_0^5} \right)^{\frac{1}{2}q_\mu + \frac{1}{2}r} = B_{\mu,\text{power}}(q_\mu, r; x^5).
\]

(87)

Therefore:

\[
\frac{\pm A_{\mu,\text{power}}(q_\mu, r; x^5)}{B_{\mu,\text{power}}(q_\mu, r; x^5)} = \pm (\delta_{\mu 0} - \delta_{\mu 1} - \delta_{\mu 2} - \delta_{\mu 3}) \frac{q_\mu}{2} \left( \frac{r}{2} \right) \left( \frac{x^5}{x_0^5} \right)^{q_\mu - r - 2}.
\]

(88)

Since \( x^5 \in R_0^+ \), one respectively gets:

\[
A_{\mu,\text{power}}(q_\mu, r; x^5) \neq 0 \iff \frac{q_\mu}{2} \left( 1 + \frac{r}{2} \right) \neq 0 \iff
\]

\[
\iff \left\{ \begin{array}{l}
q_\mu \neq 0 \\
1 + \frac{r}{2} \neq 0 \iff 2 + r \neq 0
\end{array} \right.;
\]

(89)
Therefore:

\[ B_{\mu,\text{power}}(q_\mu, r; x^5) \neq 0, \forall q_\mu, r \in R. \]  \hfill (90)

It follows that, if one assumes \( A_{\mu,\text{power}}(q_\mu, r; x^5) \neq 0 \) and \( B_{\mu,\text{power}}(q_\mu, r; x^5) \neq 0 \), in the framework of the "Power Ansatz" for \( g_{AB,DR5}(x^5) \) the hypothesis \( \Upsilon \) of functional independence (76) becomes:

\[
\exists \ (\text{at least one}) \ \bar{\mu} \in \{0, 1, 2, 3\} : \begin{cases} 
q_{\bar{\mu}} - (r + 2) \neq 0 \\
q_{\bar{\mu}} \neq 0 \\
r + 2 \neq 0
\end{cases} \iff q_{\bar{\mu}} \neq 0, r + 2 \neq 0, q_{\bar{\mu}} \neq r + 2. \]  \hfill (92)

In other words, in the framework of the "Power Ansatz" for the metric tensor the reductive nature of the \( \Upsilon \) hypothesis depends on the value of the real parameters \( q_0, q_1, q_2, q_3 \) and \( r \), exponents of the components of \( g_{AB,DR5}\text{power}(x^5) \).

The (here not explicitly considered) discussion of the possible reductivity of the \( \Upsilon \) hypothesis for the 12 classes of solutions of the 5-d. Einstein equations in vacuum derived in Subsect. 4.1 (labelled by the 5-d. real vector \( \alpha \equiv (q_0, q_1, q_2, q_3, r) \)) allows one to state that in 5 general cases such hypothesis of functional independence is reductive indeed. The Killing equations can be explicitly solved in such cases. We do not deal here with these general cases, and confine ourselves to discussing the special cases of the 5-d. phenomenological power metrics describing the four fundamental interactions (see Subsect. 2.2).

5.5 The phenomenological 5-d. metrics of fundamental interactions.

Let us now consider the 4-d. metrics of the deformed Minkowski spaces \( \tilde{M}(x^5) \) for the four fundamental interactions (e.m., weak, strong and gravitational) (see Eq.s (10)-(28)). In passing from the deformed, special-relativistic
4-d. framework of DSR to the general-relativistic 5-d. one of DR5 — geometrically corresponding to the *embedding* of the deformed 4-d. Minkowski spaces \( \{ \widetilde{M}(x^5) \}_{x^5 \in R^+_0} \) (where \( x^5 \) is a constant, non-metric parameter) in the 5-d. Riemann space \( \mathcal{R}_5 \) (where \( x^5 \) is a metric coordinate), in general the phenomenological metrics (10)-(28) take the following 5-d. form (as usual \( A, B = 0, 1, 2, 3, 5, \) and \( f(x^5) \in R^+_0 \forall x^5 \in R^+_0 \)):

\[
g_{AB,DR5,e.m.}(x^5) =
\]

\[
= \text{diag}\left(1, -\left\{1 + \Theta(x^5_{0,e.m.} - x^5) \left[\left(\frac{x^5}{x^5_{0,e.m.}}\right)^{1/3} - 1\right]\right\}\right),
\]

\[
-\left\{1 + \Theta(x^5_{0,e.m.} - x^5) \left[\left(\frac{x^5}{x^5_{0,e.m.}}\right)^{1/3} - 1\right]\right\},
\]

\[
-\left\{1 + \Theta(x^5_{0,e.m.} - x^5) \left[\left(\frac{x^5}{x^5_{0,e.m.}}\right)^{1/3} - 1\right]\right\}, \pm f(x^5)\right) \right) ; \quad (93)
\]

\[
g_{AB,DR5,weak}(x^5) =
\]

\[
= \text{diag}\left(1, -\left\{1 + \Theta(x^5_{0,weak} - x^5) \left[\left(\frac{x^5}{x^5_{0,weak}}\right)^{1/3} - 1\right]\right\}\right),
\]

\[
-\left\{1 + \Theta(x^5_{0,weak} - x^5) \left[\left(\frac{x^5}{x^5_{0,weak}}\right)^{1/3} - 1\right]\right\},
\]

\[
-\left\{1 + \Theta(x^5_{0,weak} - x^5) \left[\left(\frac{x^5}{x^5_{0,weak}}\right)^{1/3} - 1\right]\right\}, \pm f(x^5)\right) \right) ; \quad (94)
\]
\[ g_{AB,DR5,\text{strong}}(x^5) = \]
\[ = \text{diag} \left( 1 + \Theta(x^5 - x_{0,\text{strong}}^5) \left[ \left( \frac{x^5}{x_{0,\text{strong}}^5} \right)^2 - 1 \right], -\left( \frac{\sqrt{2}}{5} \right)^2, \right. \]
\[ - \left( \frac{2}{5} \right)^2, - \left\{ 1 + \Theta(x^5 - x_{0,\text{strong}}^5) \left[ \left( \frac{x^5}{x_{0,\text{strong}}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right) ; \] (95)

\[ g_{AB,DR5,\text{grav.}}(x^5) = \]
\[ = \text{diag} \left( 1 + \Theta(x^5 - x_{0,\text{grav.}}^5) \left[ \frac{1}{4} \left( 1 + \frac{x^5}{x_{0,\text{grav.}}^5} \right)^2 - 1 \right], -b_{1,\text{grav.}}^2(x^5), \right. \]
\[ -b_{2,\text{grav.}}^2(x^5), - \left\{ 1 + \Theta(x^5 - x_{0,\text{grav.}}^5) \left[ \frac{1}{4} \left( 1 + \frac{x^5}{x_{0,\text{grav.}}^5} \right)^2 - 1 \right] \right\}, \pm f(x^5) \right) . \] (96)

### 5.6 Phenomenological 5-d. metrics and the reductivity of the hypothesis $\Upsilon$ of functional independence.

We want now to investigate the possible reductivity of the hypothesis $\Upsilon$ of functional independence (76) for the 5-d. metrics (93)-(96). Due to the piecewise structure of the phenomenological metrics, we shall distinguish the two cases: $0 < x^5 < x_{0}^5$ (case a) and $x^5 \geq x_{0}^5$ (case b).

I-II - Electromagnetic and weak interactions.  
**Case a).** In this energy range the form of the metrics (93) and (94) is:

\[ g_{AB,DR5}(x^5) = \text{diag} \left( 1, -\left( \frac{x^5}{x_{0}^5} \right)^{1/3}, -\left( \frac{x^5}{x_{0}^5} \right)^{1/3}, -\left( \frac{x^5}{x_{0}^5} \right)^{1/3}, \pm f(x^5) \right) . \] (97)
Then, the $\Upsilon$ hypothesis (76) is not satisfied for $\mu = 0$ but it does for $\mu = i = 1, 2, 3$ under the following condition:

$$\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} \neq cf(x^5)(x^5)^\frac{3}{2}, c \in R. \quad (98)$$

By explicitly solving the corresponding Killing equations, one gets (under constraint (98)) the following general expression for the contravariant Killing 5-vector $\xi^A(x^0, x^1, x^2, x^3, x^5)$ of the 5-d. phenomenological electromagnetic and weak metrics ($x^5 \in R^+_0$):

$$\xi^0(x^1, x^2, x^3, x^5) =$$

$$= \Theta_R(x^5 - x^5_0) \left[ -\zeta^1 x^1 - \zeta^2 x^2 - \zeta^3 x^3 + \zeta^5 \int dx^5 f(x^5)^\frac{1}{2} \right] + T^0; \quad (99)$$

$$\xi^1(x^0, x^2, x^3, x^5) =$$

$$= \Theta_R(x^5 - x^5_0) \left[ -\zeta^1 x^0 - \Sigma^1 \int dx^5 f(x^5)^\frac{1}{2} \right] + \theta^3 x^2 - \theta^2 x^3 + T^1; \quad (100)$$

$$\xi^2(x^0, x^1, x^3, x^5) =$$

$$= \Theta_R(x^5 - x^5_0) \left[ -\zeta^2 x^0 - \Sigma^2 \int dx^5 f(x^5)^\frac{1}{2} \right] - \theta^3 x^1 + \theta^1 x^3 + T^2; \quad (101)$$

$$\xi^3(x^0, x^1, x^2, x^5) =$$

$$= \Theta_R(x^5 - x^5_0) \left[ -\zeta^3 x^0 - \Sigma^3 \int dx^5 f(x^5)^\frac{1}{2} \right] + \theta^2 x^1 - \theta^1 x^2 + T^3; \quad (102)$$
\[ \xi^5(x^0, x^1, x^2, x^3, x^5) = \]
\[ = \Theta_R(x^5 - x^5_0) \left\{ \mp f(x^5)^{-\frac{1}{2}} \left[ \xi^5 x^0 + \Sigma^1 x^1 + \Sigma^2 x^2 + \Sigma^3 x^3 - T^5 \right] \right\}, \]

where the parameters have been suitably redefined and we introduced the distribution \( \Theta_R(x^5 - x^5_0) \) (right specification of the Heaviside distribution \( \Theta(x^5 - x^5_0) \)):

\[ \Theta_R(x^5 - x^5_0) \equiv \begin{cases} 
1, & x^5 \geq x^5_0 \\
0, & 0 < x^5 < x^5_0 
\end{cases}. \]

Thus, in the energy range \( x^5 \geq x^5_0 \) the Killing group of the "slices" at \( dx^5 = 0 \) of \( \mathbb{R}_5 \) is the standard Poincaré group

\[ P(3, 1)_{\text{STD.}} = SO(3, 1)_{\text{STD.}} \otimes_s Tr.(3, 1)_{\text{STD.}}, \]

whereas for \( 0 < x^5 < x^5_0 \) the 5-d. Killing group is

\[ SO(3)_{\text{STD.}} \otimes_s Tr.(3, 1)_{\text{STD.}}. \]

In the considered energy range \( 0 < x^5 < x^5_0 \), if the hypothesis \( \Upsilon \) of functional independence (76) does not hold for any value of \( \mu \), then (98) is violated, i.e. the metric coefficient \( f(x^5) \) satisfies the following ordinary differential equation (ODE):

\[ \frac{1}{2} \frac{f'(x^5)}{f(x^5)} - cf(x^5) (x^5)^{\frac{3}{2}} + \frac{1}{x^5} = 0, c \in R. \]

Such ODE belongs to the homogeneous class of type \( G \) and to the special rational subclass of Bernoulli’s ODEs (it becomes separable for \( c = 0 \)). By solving it we get the following expression for the 5-d. metric describing e.m.
and weak interactions:

\[
g_{AB,DR5}(x^5) = \text{diag} \left( 1, -\left( \frac{x^5}{x^0} \right)^{1/3}, -\left( \frac{x^5}{x^0} \right)^{1/3}, -\left( \frac{x^5}{x^0} \right)^{1/3} \right),
\]

\[
\pm \left( 6c \left( \frac{x^5}{x^0} \right)^{\frac{5}{3}} + \gamma \left( \frac{x^5}{x^0} \right)^2 \right)^{-1}, \quad (108)
\]

with

\[
c, \gamma \in R : 6c \left( \frac{x^5}{x^0} \right)^{\frac{5}{3}} + \gamma \left( \frac{x^5}{x^0} \right)^2 > 0, \forall x^5 \in R^+_0 \Leftrightarrow c, \gamma \in R^+ \text{ (not both zero)}, \quad (109)
\]

valid in the energy range \(0 < x^5 < x^5_0\) if the hypothesis \(\Upsilon\) (76) is not satisfied.

Solving the relevant Killing equations yields the following expression for the contravariant Killing 5-vector \(\xi^A(x^0, x^1, x^2, x^3, x^5)\) corresponding to the e.m. and weak metrics:

\[
\xi^0 = c_0; \quad (110)
\]

\[
\xi^1(x^2, x^3) = -\left( a_2 x^2 + a_3 x^3 + a_4 \right) \left( \frac{x^5}{x^0} \right)^{1/3}; \quad (111)
\]

\[
\xi^2(x^1, x^3) = \left( a_2 x^1 b_1 x^3 - b_0 \right) \left( \frac{x^5}{x^0} \right)^{1/3}; \quad (112)
\]

\[
\xi^3(x^1, x^2) = \left( a_3 x^1 - b_1 x^2 - b_2 \right) \left( \frac{x^5}{x^0} \right)^{1/3}; \quad (113)
\]

\[
\xi^5 = 0. \quad (114)
\]

The 5-d. Killing group of isometries is therefore

\[
SO(3)_{STD(E_3)} \otimes s \ Tr.(3, 1)_{STD}. \quad (115)
\]
where $E_3$ is the 3-d. manifold with metric

$$g_{ij} = -\left(\frac{x^5}{x_0^5}\right)^{1/3} \text{diag}(1,1,1).$$  \tag{116}

**Case b).** In this energy range the 5-d. metrics (93) and (94) read:

$$g_{AB,DR}^{AB,DR}(x^5) = \text{diag}(1,-1,-1,-1,\pm f(x^5)).$$  \tag{117}

Therefore the hypothesis $\Upsilon$ of functional independence (76) is *not* satisfied $\forall \mu \in \{0,1,2,3\}$.

**III - Strong interaction.**

**Case a).** The metric (95) has the form

$$g_{AB,DR}^{AB,DR}(x^5) = \text{diag}\left(1,-\frac{2}{25},-\frac{4}{25},-1,\pm f(x^5)\right),$$  \tag{118}

which does *not* satisfy the $\Upsilon$-hypothesis (76) $\forall \mu \in \{0,1,2,3\}$.

**Case b).** The 5-d. metric (95) reads:

$$g_{AB,DR}^{AB,DR}(x^5) = \text{diag}\left(\frac{x^5}{x_0^5}^2,-\frac{2}{25},-\frac{4}{25},-\left(\frac{x^5}{x_0^5}\right)^2,\pm f(x^5)\right),$$  \tag{119}

and the hypothesis $\Upsilon$ of functional independence (76) (*not* satisfied for $\mu = 1,2$) holds true for $\mu = 0,3$ under the condition:

$$\frac{1}{x^5} + \frac{1}{2} \frac{f'(x^5)}{f(x^5)} \neq c \frac{f(x^5)}{x^5}, c \in R.$$

(120)

By solving the relevant Killing equations under condition (120), one gets (after a suitable redenomination of the parameters) the following general
form of the contravariant Killing 5-vector $\xi^A(x^0, x^1, x^2, x^3, x^5)$ for the 5-d. phenomenological metric of the strong interaction:

$$\xi^0(x^1, x^2, x^3, x^5) =$$

$$= \Theta_R(x_0^5 - x^5) \left[ -\frac{2}{25}\zeta^1 x^1 - \frac{4}{25}\zeta^2 x^2 + \zeta^5 \int dx^5 f(x^5)^{\frac{1}{2}} \right] - \zeta^3 x^3 + T^0; \quad (121)$$

$$\xi^1(x^0, x^2, x^3, x^5) =$$

$$= \Theta_R(x_0^5 - x^5) \left[ -\zeta^1 x^0 - \theta^2 x^3 - \Sigma^1 \int dx^5 f(x^5)^{\frac{1}{2}} \right] + 2\theta^3 x^2 + T^1; \quad (122)$$

$$\xi^2(x^0, x^1, x^3, x^5) =$$

$$= \Theta_R(x_0^5 - x^5) \left[ -\zeta^2 x^0 + \theta^1 x^3 - \Sigma^2 \int dx^5 f(x^5)^{\frac{1}{2}} \right] - \theta^3 x^1 + T^2; \quad (123)$$

$$\xi^3(x^0, x^1, x^2, x^5) =$$

$$= \Theta_R(x_0^5 - x^5) \left[ \frac{2}{25}\theta^2 x^1 - \frac{4}{25}\theta^3 x^2 - \Sigma^3 \int dx^5 f(x^5)^{\frac{1}{2}} \right] - \zeta^3 x^0 + T^3; \quad (124)$$

$$\xi^5(x^0, x^1, x^2, x^3, x^5) =$$

$$= \Theta_R(x_0^5 - x^5) \left\{ \mp (f(x^5))^\frac{1}{2} \left[ \zeta^5 x^0 + \frac{2}{25}\Sigma^1 x^1 + \frac{4}{25}\Sigma^2 x^2 + \Sigma^3 x^3 - T^5 \right] \right\}. \quad (125)$$
Thus, in the energy range $0 < x^5 \leq x_0^5$ the Killing group of the "slices" at $dx^5 = 0$ of $\mathbb{R}_5$ is the standard Poincaré group (suitably rescaled)

$$[P(3, 1)_{STD.} = SO(3, 1)_{STD.} \otimes_s Tr.(3, 1)_{STD.}]|_{x_1 \to \sqrt{\frac{2}{5}} x_1, x_2 \to \frac{2}{5} x_2},$$

whereas for $x^5 > x_0^5$ the 4-d Killing group is

$$\left(SO(2)_{STD.,\Pi(x^1, x^2 \to \sqrt{\frac{2}{5}} x^2)} \otimes B_{x^3,STD.}\right) \otimes_s Tr.(3, 1)_{STD.}. \quad (127)$$

Here

$$SO(2)_{STD.,\Pi(x^1, x^2 \to \sqrt{\frac{2}{5}} x^2)} = SO(2)_{STD.,\Pi(x^1 \to \sqrt{\frac{2}{5}} x^1, x^2 \to \frac{2}{5} x^2),}$$

is the 1-parameter group (generated by the usual, special-relativistic generator $S^3_{SR} \mid_{x^2 \to \sqrt{\frac{2}{5}} x^2}$) of the 2-d. rotations in the plane $\Pi(x^1, x^2)$ characterized by the coordinate contractions $x^1 \to \sqrt{\frac{2}{5}} x^1, x^2 \to \frac{2}{5} x^2$, and $B_{x^3,STD.}$ is the usual one-parameter group (generated by the special-relativistic generator $K^3_{SR} \mid_{x^2 \to \sqrt{\frac{2}{5}} x^2}$) of the standard Lorentzian boosts along $\hat{x}^3$.

In the energy range $x^5 > x_0^5$, when the hypothesis $\Upsilon$ of functional independence (76) is not satisfied for any value of $\mu$, the metric coefficient $f(x^5)$ obeys the following equation:

$$\frac{1}{2} \frac{f'(x^5)}{f(x^5)} - c \frac{f(x_5)}{x_5} + \frac{1}{x_5} = 0, \quad c \in \mathbb{R}. \quad (128)$$

Such ODE is separable $\forall c \in \mathbb{R}$. By solving it one gets the following form of the 5-d. metric of the strong interaction (for $x^5 > x_0^5$, and when the $\Upsilon$ hypothesis (76) is not satisfied):

$$g_{AB,DR5}(x^5) =$$

$$= \text{diag} \left( \left(\frac{x^5}{x_0^5}\right)^2, -\frac{2}{25}, \frac{4}{25}, -\left(\frac{x^5}{x_0^5}\right)^2, \pm \frac{1}{\gamma \left(\frac{x^5}{x_0^5}\right)^2 + c} \right),$$

$$\quad (129)$$

40
with
\[ c, \gamma \in R : \gamma \left( \frac{x^5}{x^0} \right)^2 + c > 0, \forall x^5 \in R_0^+ \Leftrightarrow c, \gamma \in R^+ \text{ (not both zero).} \]

(130)

Solving the related Killing equations yields the following contravariant Killing 5-vector \( \xi^A(x^0, x^1, x^2, x^3, x^5) \):

\[ \xi^0(x^3; c, \gamma) = (1 - \delta_c, 0) \left[ -\left( x_0^5 \right)^2 \left( (1 - \delta_\gamma, 0) d_3x^3 + T_0 \right) \right]; \]

(131)

\[ \xi^1(x^2; \gamma) = - (1 - \delta_\gamma, 0) \frac{25}{2}d_2x^2 - \frac{25}{2}T_1; \]

(132)

\[ \xi^2(x^1; \gamma) = (1 - \delta_\gamma, 0) \frac{25}{4}d_2x^1 - \frac{25}{4}T_2; \]

(133)

\[ \xi^3(x^0; c, \gamma) = -(1 - \delta_c, 0) (1 - \delta_\gamma, 0) \left( x_0^5 \right)^2 \left( d_3x^0 + T_3 \right); \]

(134)

\[ \xi^5(x^5; c, \gamma) = \pm \delta_c, 0 \frac{\gamma \alpha}{(x_0^5)^2} x^5, \]

(135)

where we evidenced the parametric dependence of \( \xi^A \) on \( c \) and \( \gamma \), and introduced the Kronecker \( \delta \).

The 4-d. Killing group (i.e. of the slices at \( dx^5 = 0 \)) is thus:

\[ \left[ Tr_{x^1, x^2} \otimes (1 - \delta_c, 0) Tr_{\bar{x}^0} \otimes (1 - \delta_\gamma, 0) (1 - \delta_\gamma, 0) B_{x^3} \right] \otimes_s \]

\[ \otimes_s \left[ (1 - \delta_\gamma, 0) SO(2) \otimes (1 - \delta_c, 0) (1 - \delta_\gamma, 0) B_{x^3} \right], \]

(136)

where \( \Pi_2 \) is the 2-d. manifold \( (x_1, x^2) \) with "metric rescaling" \( x^2 \rightarrow \sqrt{2}x^2 \) with respect to the Euclidean level. \( SO(2) \) is a 1-parameter abelian group generated by \( S_{SR}^{3} | x^2 \rightarrow \sqrt{2} x^2 \).
IV- Gravitational interaction.

**Case a).** The 5-d. metric (96) is:

\[ g_{AB,DR}^{5}(x^5) = \text{diag} \left( 1, -b_1^2(x^5), -b_2^2(x^5), -1, \pm f(x^5) \right) \]  

(137)

Therefore the validity for \( \mu = 1, 2 \) of the \( \Upsilon \) hypothesis (76) *(not satisfied for \( \mu = 0, 3 \)) depends on the nature and the functional form of the metric coefficients \( b_1^2(x^5) \) and \( b_2^2(x^5) \).

**Case b).** The 5-d. metric (96) reads:

\[ g_{AB,DR}^{5}(x^5) = \text{diag} \left( \frac{1}{4} \left( 1 + \frac{x^5}{x_0^5} \right)^2, -b_1^2(x^5), -b_2^2(x^5), -\frac{1}{4} \left( 1 + \frac{x^5}{x_0^5} \right)^2, \pm f(x^5) \right) \].  

(138)

By making suitable assumptions on the functional form of the coefficients \( b_1^2(x^5) \) and \( b_2^2(x^5) \), it is possible in 11 cases (which include all cases of physical and mathematical interest) to solve the relevant Killing equations for the gravitational interaction [28].

5.6.1 The 5-d. ”\( \Upsilon \)-violating” (\( \forall \mu = 0, 1, 2, 3 \)) metrics of gravitational interaction.

The ”\( \Upsilon \)-violating” gravitational metrics can be discussed by exploiting a general treatment of such a case [28]. If the \( \Upsilon \) hypothesis (76) is *not* satisfied for \( \mu = 0, 3 \) in the energy range \( x^5 > x_0^5 \), then the metric coefficient \( f(x^5) \) obeys the following equation:

\[ f'(x^5) + \frac{2}{x^5 + x_0^5} f(x^5) - \frac{2C}{x^5 + x_0^5} \left( f(x^5) \right)^2 = 0, \quad C \in \mathbb{R}. \]  

(139)

This ODE belongs to the separable subclass of Bernoulli type \( \forall C \in \mathbb{R} \). Solving it one gets the following 5-d. metric:

\[ g_{AB,DR}^{5}(x^5) = \text{diag} \left( \frac{1}{4} \left( 1 + \frac{x^5}{x_0^5} \right)^2, -b_1^2(x^5), -b_2^2(x^5), -\frac{1}{4} \left( 1 + \frac{x^5}{x_0^5} \right)^2, \pm \left( \gamma \left( 1 + \frac{x^5}{x_0^5} \right)^2 + C \right)^{-1} \right), \]  

(140)
where in general parameters $\gamma$ and $\mathcal{C}$ are real and positive (not both zero).

The case when the hypothesis $\Upsilon$ (76) is not satisfied for $\mu = 1$ and/or 2 corresponds to metric coefficients $b_1^2(x^5), b_2^2(x^5)$ and $f(x^5)$ satisfying the following ODE (ESC off)

$$- \left( b'(x^5) \right)^2 + b_1(x^5)b''_1(x^5) - \frac{1}{2} b_1(x^5)b'_1(x^5)f'(x^5)(f(x^5))^{-1} - c_i f(x^5) = 0,$$

$c_i \in \mathbb{R}, i = 1$ and/or $2$, \hspace{1cm} (141)

whose solution in terms of $f(x^5)$ is:

$$f(x^5) = \frac{(b'_i(x^5))^2}{d_ib_i^2(x^5) - c_i} \iff$$

$$d_ib_i^2(x^5)f(x^5) - \left( b'_i(x^5) \right)^2 - c_i f(x^5) = 0, \hspace{1cm} i = 1 \text{ and/or } 2 \hspace{1cm} (142)$$

$$d_i \in \mathbb{R}^+, c_i \in \mathbb{R}^- \text{ (not both zero).} \hspace{1cm} (143)$$

The non-linear ODE (142) can be solved in all possible cases:

1) $d_i \in R_0^+, c_i \in R_0^-$;
2) $d_i = 0, c_i \in R_0^-;
3) c_i = 0, d_i \in R_0^+$,

(even in the limit case of $b_i$ constant). We refer the interested reader to Ref. [28].

The above general formalism allows one to deal with the 5-d. metrics of DR5 for the gravitational interaction which violate $\Upsilon \forall \mu = 0, 1, 2, 3$ in the energy ranges $0 < x^5 \leqslant x^5_{0(grav)}$ and $x^5 > x^5_{0(grav)}$.

In the first case ($0 < x^5 \leqslant x^5_{0(grav)}$), the functional form of $f(x^5)$ is undetermined, since in general it must only satisfy the condition $f > 0 \forall x^5 \in R_0^+$. 

43
In the second case \((x^5 > x^5_0)\), one gets \(27\) expressions for the 5-d. gravitational metrics. Their general functional form is:

\[
g_{AB,DR5(grav.)}(x^5) = \text{diag} \left( \frac{1}{4} \left(1 + \frac{x^5}{x^5_0}\right)^2, -b_1^2(x^5), -b_2^2(x^5), \right.
\]

\[
-\frac{1}{4} \left(1 + \frac{x^5}{x^5_0}\right)^2, \pm \left(\gamma \left(1 + \frac{x^5}{x^5_0}\right)^2 + C\right)^{-1},
\]

where parameters \(\gamma\) and \(C\) are real and positive (not both zero). The explicit expressions of \(b_1^2(x^5)\) and \(b_2^2(x^5)\) (and therefore of the 27 gravitational metrics) can be found in Ref. [28]. All such metrics satisfy the \(\Upsilon\)-violating equation (141) that, for fixed \(\mu \in \{0, 1, 2, 3\}\), reads

\[
- \left(b'_\mu(x^5)^2 + b_\mu(x^5)b''_\mu(x^5) - \frac{1}{2}b_\mu(x^5)b'_\mu(x^5)f'(x^5)(f(x^5))^{-1} - c_\mu f(x^5)\right) = 0,
\]

with the following ranges and conditions:

\[
f(x^5) \in R_0^+ \ \forall x^5 \in R_0^+ \ \ (146)
\]

\[
b_\mu(x^5) \in R_0 \ \forall x^5 \in R_0^+ \iff b_\mu^2(x^5) \in R_0^+ \ \forall x^5 \in R_0^+ \ \ (147)
\]

\[
c_\mu \in R. \ \ (148)
\]

The solution of non-linear ODE (145) in terms of \(f(x^5)\) is:

\[
f(x^5) = \frac{(b'_\mu(x^5))^2}{d(\mu)b^2_\mu(x^5) - c_\mu} \ \ (149)
\]

In correspondence to the above-mentioned 27 different gravitational metrics, one gets 27 systems of 15 Killing non-linearly coupled PDEs, which it is very difficult to solve explicitly.

Analogous results hold true in the \(\Upsilon\)-violating case for the gravitational interaction in the energy range \(0 < x^5 \leq x^5_0\), namely the same functional
forms of $b_i^2(x^5), i = 1, 2$ as before are obtained, but one has to put $b_i^2(x^5) = b_j^2(x^5) = 1$ and to leave $f(x^5)$ undetermined (but strictly positive $\forall x^5 \in R_0^+$). One gets therefore “$f(x^5)$-dependent”, i.e. in general “functionally parametrized”, metrics. We refer the interested reader to Ref. [28] for a deeper discussion of this case.

6 Five-dimensional geodesics

As a last topic in DR5, let us consider the geodesics in the sui generis five-dimensional Riemann manifold $\mathbb{R}_5$, in order to clarify their possible physical meaning.

The geodesic equations are

$$\frac{d^2 x^A}{d\tau^2} + \Gamma^A_{BC} \frac{dx^B}{d\tau} \frac{dx^C}{d\tau} = 0.$$  \hspace{1cm} (150)

Let us here confine ourselves to find solutions to this equation in the Power Ansatz for the metric coefficients (see case ii) of Sect. 4). In this case
Eq.s (150) explicitly read

\[
\begin{align*}
\frac{d^2 t}{d\tau^2} + \frac{q_0}{E} \frac{dt}{d\tau} \frac{dE}{d\tau} &= 0; \\
\frac{d^2 x}{d\tau^2} + \frac{q_1}{E} \frac{dx}{d\tau} \frac{dE}{d\tau} &= 0; \\
\frac{d^2 y}{d\tau^2} + \frac{q_2}{E} \frac{dy}{d\tau} \frac{dE}{d\tau} &= 0; \\
\frac{d^2 z}{d\tau^2} + \frac{q_3}{E} \frac{dz}{d\tau} \frac{dE}{d\tau} &= 0; \\
\frac{d^2 E}{d\tau^2} + \frac{r}{2E} \left( \frac{dE}{d\tau} \right)^2 - \frac{1}{2E^{r+1}} \left[ q_0 \left( \frac{E}{E_0} \right)^{q_0} \left( \frac{dt}{d\tau} \right)^2 - q_1 \left( \frac{E}{E_0} \right)^{q_1} \left( \frac{dx}{d\tau} \right)^2 - q_2 \left( \frac{E}{E_0} \right)^{q_2} \left( \frac{dy}{d\tau} \right)^2 - q_3 \left( \frac{E}{E_0} \right)^{q_3} \left( \frac{dz}{d\tau} \right)^2 \right] &= 0.
\end{align*}
\tag{151}
\]

The complete solutions of Eq.s (151) for all classes (I)-(XII) of Sect. 4 can be found in Ref.s [28] and [31].

Here we shall confine ourselves to consider the solution of Eq.s (151) for the metric class (VIII) \((\alpha_{VI} = (0,0,0,0,r))\), corresponding to a four-dimensional Minkowski space-time with undetermined energy exponent (which represents, in our framework, the electromagnetic interaction: see Ref. [12] for the phenomenological aspects of this metric). Indeed, the solution of (151) reads, in this case ([18],[28]):

\[
t = \frac{2}{C_1(2 + r)} E^{\frac{2+r}{2}} + C_2, \tag{152}
\]

where \(C_1, C_2\) are integration constants. Putting \(C_1 = C, C_2 = 0\) Eq. (152) becomes

\[
E = C \frac{2 + r}{2} t^{\frac{2+r}{2}}, \tag{153}
\]

46
whence, for \( r = -4 \):

\[
Et = -C.
\]  

By assuming \( C = -\hbar \), Eq. (154) takes a form which reminds the quantum-mechanical, Heisenberg uncertainty relation for time and energy. Otherwise stated, we can say that the geodesics in a five-dimensional space-time, embedding a standard four-dimensional Minkowski space, correspond to trajectories of minimal time-energy uncertainty. This result (first derived in Ref. [18], and rigorously analyzed and generalized in Ref. [28]), although preliminary, seemingly indicates that the five-dimensional scheme of DR5 may play a role toward understanding certain aspects of quantum mechanics in purely classical (geometrical) terms. It agrees with Wesson’s results on the connection between Heisenberg’s principle and Kaluza-Klein theory in the STM model ([29], [30]).

7 Conclusions and perspectives

The DR5 formalism lends itself to a number of possible, future developments. These include e.g. solving the general Einstein equations with a non-zero cosmological constant, \( \Lambda \neq 0 \). Further improvements of the predictive power of the theory may come from the explicit introduction of a space-time-coordinate dependence in the fifth metric coefficient \( f \) and/or in the cosmological constant \( \Lambda \), i.e. assuming

\[
f = f(E, x) \quad \text{and/or} \quad \Lambda = \Lambda(E, x).\]

As it is easily seen, this amounts to taking into account also the presence of matter in our scheme. Clearly, solving the five-dimensional Einstein equations in such a case is expected to be a quite formidable task.

A further topic deserving investigation is that of the five-dimensional action. The Einstein-Hilbert action in \( \mathbb{R} \) reads, in this case:

\[
S = -\frac{1}{16\pi \tilde{G}} \int d^5x \sqrt{\pm \tilde{g}} R, \tag{155}
\]

where \( \tilde{g} = \det g_{DR5} \), \( \tilde{G} \) is the gravitational constant and the double sign in the square root accords to that in front of \( f \). Among the problems concerning
$S$, let us quote its physical meaning (as well as that of $\tilde{G}$) and the meaning of those energy values $\tilde{E}$ such that $S(\tilde{E}) = 0$ (due to a possible degeneracy of the metric).

The Killing symmetries of DR5 deserve further investigation on many respects. Let us quote, for instance:

1 - the Lie nature of the infinitesimal symmetries derived;
2 - the passage from the infinitesimal level to the finite one;
3 - and, last but not least, the physical meaning of the symmetries obtained. As far as this last point is concerned, the results obtained seemingly show an invariance of physical laws under non-linear coordinate transformations (in particular in time and energy).

Besides the above "classical" problems, there are also what we may call the possible "quantum" aspects of the formalism. They are related to the fact that actually, in most systems of physical interest at a microscopical level, energy is quantized. How does energy quantization match in this scheme? How to account for energy jumps within an apparently completely classical framework?

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