TOPOLOGICALLY 2-GENERATED GROUPS

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Abstract. We formulate and prove general conditions implying that the automorphism group \( \text{Aut}(M) \) of a countable structure \( M \), and the group \( L_0(\text{Aut}(M)) \) of measurable functions with values in \( \text{Aut}(M) \), have a cyclically dense conjugacy class, in particular, are topologically 2-generated. This provides a number of new examples of groups with this property.

1. Introduction

For a Polish group \( G \), a tuple \( (a_1, \ldots, a_n) \) in \( G \) topologically generates \( G \) if the closure of the subgroup of \( G \) generated by it, is equal to \( G \). In this situation, we say that \( G \) is topologically \( n \)-generated. There is a number of Polish groups which are topologically 2-generated in a very special way, specifically, which have a cyclically dense conjugacy class. For a Polish group \( G \) say that \( (g, h) \) (topologically) cyclically generates \( G \) if \( \{g^l h g^{-l} : l \in \mathbb{Z}\} \) is dense in \( G \). In that case, we will say that \( G \) has a cyclically dense conjugacy class.

Kechris and Rosendal [9], pages 314-315) gave examples of automorphism groups of countable structures which have a cyclically dense conjugacy class, among them was the automorphism group of the countable atomless Boolean algebra. Macpherson [11] Proposition 3.3] proved that the automorphism group of the random graph has a cyclically dense conjugacy class, and Solecki [16 Corollary 4.1 proved that the automorphism group of the rational Urysohn metric space has this property. It follows from the work of Glass, McCleary and Rubin [5, Proposition 4.1] that the automorphism group of the random poset also has a cyclically dense conjugacy class. Kaplan and Simon [8, independently from our work, recently formulated a model-theoretic condition on automorphism groups of countable structures to have a cyclically dense conjugacy class.

A similar line of research is presented in Darji and Mitchell [2], where the sets of topological 2-generators in the automorphism group of rationals, and in the automorphism group of the colored random graph, are studied. It is shown there, in particular, that in each of these groups, for every non-identity element \( f \), there is an element \( g \) such that \( (f, g) \) topologically 2-generates the whole group (see [2, Theorems 1.3 and Corollary 1.8].)

It seems that topological generators of groups of measurable functions have been barely studied at all. Glasner [4] (see also Pestov [13, Theorem 4.2.6]) showed that \( L_0(\mathbb{S}^1) \) is topologically 1-generated.

2. Definitions and Results

A Polish group is non-archimedean if it admits a neighbourhood basis at the identity consisting of open subgroups. It is well known that non-archimedean Polish groups

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are exactly those that can be realized as automorphism groups of countable structures, equipped with the pointwise convergence topology. Even more, non-archimedean Polish groups are exactly those that can be realized as automorphism groups of relational ultrahomogeneous countable structures (see [11], pages 9-10), where a structure $M$ is ultrahomogeneous if every automorphism between finite substructures of $M$ can be extended to an automorphism of the whole $M$. Therefore, without loss of generality, we will restrict our attention to automorphism groups of ultrahomogeneous structures. If a countable and locally finite structure $M$ is ultrahomogeneous, then $\mathcal{F} = \text{Age}(M)$ – the class of all finite substructures embeddable in $M$ – is a Fraïssé class, i.e., it is a countable up to isomorphism class of finite structures which has the hereditary property HP (for every $A \in \mathcal{F}$, if $B$ is a substructure of $A$, then $B \in \mathcal{F}$), the joint embedding property JEP (for any $A,B \in \mathcal{F}$ there is $C \in \mathcal{F}$ which embeds both $A$ and $B$), and the amalgamation property AP (for any $A,B_1,B_2 \in \mathcal{F}$ and any embeddings $\phi_i: A \rightarrow B_i$, $i = 1,2$, there are $C \in \mathcal{F}$ and embeddings $\psi_i: B_i \rightarrow C$, $i = 1,2$, such that $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$). Moreover, by the classical theorem due to Fraïssé, for every Fraïssé class $\mathcal{F}$ of finite structures, there is a unique up to isomorphism countable ultrahomogeneous structure $M$ such that $\mathcal{F} = \text{Age}(M)$. In that case, we call $M$ the Fraïssé limit of $\mathcal{F}$, see [7, Section 7.1].

For a structure $M$, $\text{Aut}(M)$ is the automorphism group of $M$. For $X \subseteq M$, let $\text{Aut}_X(M)$ denote the subgroup of $\text{Aut}(M)$ that pointwise stabilizes $X$. We say that a structure $M$ has no algebraicity if for any finite substructures $X,Y \subseteq M$, $X \cap Y = \emptyset$, the orbit $\text{Aut}_X(M).Y$ is infinite. It is well known (see for example [7, Theorem 7.1.8]) that an ultrahomogeneous structure $M$ has no algebraicity if and only if $\text{Age}(M)$ satisfies the strong amalgamation property, i.e., we have additionally $\psi_1[B_1] \cap \psi_2[B_2] = \psi_1 \circ \phi_1[A]$ in the definition of the amalgamation property.

We say that $g \in \text{Aut}(M)$ has no cycles if for every $x \in M$ and $n > 1$, it holds that $g^n(x) \neq x$.

Let $(X,\mu)$ be a standard non-atomic Lebesgue space (without loss of generality, $X$ is the interval $[0,1]$ and $\mu$ is the Lebesgue measure), and let $Y$ be a Polish space. We define $L_0(X,\mu;Y)$ to be the set of all ($\mu$-equivalence classes of) measurable (equivalently: Borel) functions from $X$ to $Y$. We equip this space with the convergence of measure topology. The neighborhood basis at $h \in L_0(X,\mu;Y)$ is

$$[h, \delta, \epsilon] = \{g \in L_0(X,\mu;Y): \mu(\{x \in X: d(g(x), h(x)) < \delta\}) > 1 - \epsilon\},$$

where $d$ is a fixed compatible complete metric on $Y$, and $\epsilon, \delta > 0$. This topology does not depend on the choice of the metric $d$ on $Y$. When $Y = G$ is a Polish group, then $L_0(X,\mu;G)$ is a Polish group as well, with multiplication given by the pointwise multiplication in $G$: $(fg)(x) = f(x)g(x)$. Groups of measurable functions were introduced by Hartman and Mycielski [6], where they showed that every topological group can be embedded in a path connected one. In recent years, the structure and dynamics of those groups has been frequently studied, for example in [3,10,14,15,17].

For $g \in G$, we will denote by $f_g$ the constant function in $L_0(G)$ with the value $g$. For notational reasons, we assume that the natural numbers $\mathbb{N}$ start with 1.

Our first result is the following theorem, which will provide a number of examples of topological groups that have a cyclically dense conjugacy class, in particular, that are topologically 2-generated.

**Theorem 2.1.** Let $G$ be the automorphism group of an ultrahomogeneous structure $M$. Suppose that there exists $g \in G$ such that for any partial automorphisms $\phi_i: A_i \rightarrow B_i$.
and \( \psi_i : C_i \to D_i, i \leq n, \) of \( M, \) there are \( k, m \in \mathbb{Z} \) such that \( g^k \phi_i g^{-k} \cup g^m \psi_i g^{-m} \) can be extended to a single automorphism of \( M, \) for each \( i \leq n. \) Then each of the groups \( G^n, \) \( n \in \mathbb{N}, \) is cyclically generated by \(( (g, \ldots, g), b)\) for some \( b \in G^n, \) and \( L_0(G) \) is cyclically generated by \(( f_g, b)\) for some \( b \in L_0(G). \)

Moreover, if \( M \) has no algebraicity and \( g \) has no cycles, we can find such \( b \in G^n \) \((b \in L_0(G), \) respectively\) such that \((g, \ldots, g)\) and \( b \) \((f_g \) and \( b, \) respectively\) generate the free group.

Our main result is Theorem 2.2, which involves a property that we call the strong\(^+\) amalgamation property. It is a variant of amalgamation which sits between strong amalgamation and free amalgamation.

Let \( \mathcal{F} \) be a class of finite structures in a relational language \( L. \) We say that \( \mathcal{F} \) satisfies the strong\(^+\) amalgamation property if for every \( A, B_1, B_2 \in \mathcal{F}, \) and embeddings \( \phi_i : A \to B_i, i \leq 2, \) there is \( C \in \mathcal{F} \) and there are embeddings \( \psi_i : B_i \to C, i \leq 2, \) such that, for \( S = \psi_1(B_1) \setminus \psi_1 \circ \phi_1(A) \) and \( T = \psi_2(B_2) \setminus \psi_2 \circ \phi_2(A) \) we have:

1. \( \psi_1 \circ \phi_1 = \psi_2 \circ \phi_2, \)
2. \( x \neq y \) for all \( x \in S \) and \( y \in T, \)
3. The map \( p \cup q \) is an isomorphism for every isomorphisms \( p \) and \( q \) such that \( \text{dom}(p) \cup \text{rng}(p) \subseteq S \) and \( \text{dom}(q) \cup \text{rng}(q) \subseteq T. \)

Note that condition (3') below implies condition (3).

(3') For every \( n \) and every relation symbol \( R \in L \) of arity \( n, \) for all \( n\)-tuples \( (x_1, \ldots, x_n) \) and \( (y_1, \ldots, y_n) \) of elements in \( S \cup T \) such that

\[- \ x_i \in S \text{ if } y_i \in S \text{ for } i = 1, \ldots, n, \]
\[- \ \{x_1, \ldots, x_n\} \cap S \neq \emptyset, \]
\[- \ \{x_1, \ldots, x_n\} \cap T \neq \emptyset, \]

we have \( R^D(x_1, \ldots, x_n) \) iff \( R^D(y_1, \ldots, y_n). \)

In particular, if \( L \) consists only of relations of arity 2, the condition (3') says the following. For every relation symbol \( R \in L \) and for all tuples \((x_1, x_2)\) and \((y_1, y_2)\) such that either we have \( x_1, y_1 \in S \) and \( x_2, y_2 \in T \) or we have \( x_1, y_1 \in T \) and \( x_2, y_2 \in S, \) it holds that \( R^D(x_1, x_2) \) iff \( R^D(y_1, y_2). \)

This notion refines the notion of the strong amalgamation property as \( \mathcal{F} \) satisfies the strong amalgamation property if for every \( A, B_1, B_2 \in \mathcal{F}, \) and embeddings \( \phi_i : A \to B_i, i \leq 2, \) there is \( C \in \mathcal{F}, \) and embeddings \( \psi_i : B_i \to C, i \leq 2, \) such that (1) and (2) in the definition of the strong\(^+\) amalgamation property hold.

Moreover, it generalizes the notion of the free amalgamation property as \( \mathcal{F} \) satisfies the free amalgamation property if for every \( A, B_1, B_2 \in \mathcal{F}, \) and embeddings \( \phi_i : A \to B_i, i \leq 2, \) there is \( C \in \mathcal{F}, \) and embeddings \( \psi_i : B_i \to C, i \leq 2, \) such that (1) and (2) in the definition of the strong\(^+\) amalgamation property and condition (3') below hold.

(3') For every \( n \) and every relation symbol \( R \) of arity \( n, \) for every \( n\)-tuple \((x_1, \ldots, x_n)\) such that \( x_i \in S \cup T \) and \( \{x_1, \ldots, x_n\} \cap S \neq \emptyset \) and \( \{x_1, \ldots, x_n\} \cap T \neq \emptyset, \) \( R^D(x_1, \ldots, x_n) \) does not hold.

Clearly, (3') implies (3').

The random graph (respectively, the random triangle-free graph) is the Fraïssé limit of the class of all finite graphs (respectively, all finite triangle-free graphs). The random tournament is the Fraïssé limit of the class of all finite tournaments, where a tournament is a directed graph \( D \) such that for any distinct \( x, y \in D \) either there is an edge from \( x \) to \( y \) or there is an edge from \( y \) to \( x, \) but not both. Furthermore, the random poset \( P \) is
the Fraïssé limit of the class of all finite partially ordered sets, and the rational Urysohn space $\mathbb{U}_0$ is the Fraïssé limit of the class of all finite metric spaces with rational valued distances.

**Theorem 2.2.** Suppose that:

(i) $M$ is a countable relational ultrahomogeneous structure such that $\text{Age}(M)$ has the strong$^+$ amalgamation property, or

(ii) $M$ is the rational Urysohn metric space, or

(iii) $M$ is the random poset, or

(iv) $M$ is the rational numbers with ordering, or

(v) $M$ is the countable atomless Boolean algebra, or the countable atomless Boolean algebra equipped with the dyadic measure.

Then each $G^n$, $n \in \mathbb{N}$, and $L_0(G)$, where $G = \text{Aut}(M)$, has a cyclically dense conjugacy class. In fact, each of these groups is cyclically generated by a pair generating the free group.

A proof of Theorem 2.2 will be presented in Sections 3.2-3.5. Each time we will verify that the assumptions of Theorem 2.1 are satisfied.

It is easy to see that the class of graphs and the class of triangle-free graphs satisfy the free amalgamation property, while the class of finite tournaments satisfies the strong$^+$ amalgamation property, but it does not satisfy the free amalgamation property. On the other hand, the classes of finite rational metric spaces, partial orderings, linear orderings, Boolean algebras, and Boolean algebras equipped with the dyadic measure, do not satisfy the strong$^+$ amalgamation property. Still, the conclusion of Theorem 2.2 holds for those structures.

As a matter of fact, in order to show that the conclusion of Theorem 2.2 holds for structures with strong$^+$ amalgamation, we introduce a family of weaker conditions $(\triangle_n)$, $n \in \mathbb{N}$. We show that the conclusion of Theorem 2.1 holds for every structure listed in Theorem 2.2. For the case of the rational Urysohn space we use results of Solecki [16], and to deal with the random poset, we use the work of Glass, McCleary and Rubin [5].

3. Proofs

3.1. **Proof of Theorem 2.1.** For $\bar{a} = (a_1, \ldots, a^n)$, an $n$-tuple in a Polish group $G$, let $f_\bar{a}$ be the step function defined by

$$f_\bar{a}(x) = a^k \text{ iff } x \in \left[\frac{k-1}{n}, \frac{k}{n}\right].$$

**Lemma 3.1.** Let $G$ be a Polish group and suppose that there is $g \in G$ such that for any $n \in \mathbb{N}$ there is $h \in G^n$ such that $(g, \ldots, g, h)$ cyclically generates $G^n$. Then $L_0(G)$ is cyclically generated by $(f_g, b)$, for some $b \in L_0(G)$.

**Proof.** It is easy to see that for every Polish group $H$, if $(g, h)$ cyclically generates $H$, then the set

$$A = \{h' \in H : (g, h') \text{ cyclically generates } H\}$$

is dense. As $A$ is a $G_\delta$ set, it is in fact comeager. Thus, for a fixed $g \in G$ as in the statement of the lemma, and $n \in \mathbb{N}$, every set

$$A_n = \{h \in G^n : ((g, \ldots, g), h) \text{ cyclically generates } G^n\}$$

is comeager.
Regarding elements $\bar{h}$ of $A_n$ as step functions $f_h$, we identify $A_n$ with a subset of $L_0(G)$, which we will denote by $A'_n$. We have that $A'_2 \subseteq A'_2 \subseteq \ldots$, and that $\bigcup_n A'_n$ is dense in $L_0(G)$. This implies that for any non-empty open set $U$ in $L_0(G)$, the set
\[ \{ h \in L_0(G) : \exists l \in \mathbb{Z} \text{ such that } g^l h g^{-l} \in U \} \]
is open and dense. Therefore the set
\[ \{ h \in L_0(G) : (f_g, h) \text{ cyclically generates } L_0(G) \} \]
is comeager, in particular non-empty. \hfill \Box

The lemma below is a generalization to every dimension $n$ a condition due to Kechris-Rosendal ([9], page 314) on a non-archimedean Polish group to have a cyclically dense conjugacy class.

**Lemma 3.2.** Let $G$ be the automorphism group of an ultrahomogeneous structure $M$, and let $n \in \mathbb{N}$ and $\bar{g} = (g_1, \ldots, g_n) \in G^n$ be given. Suppose that for every choice of partial automorphisms $\phi_i : A_i \to B_i$ and $\psi_i : C_i \to D_i$, $i \leq n$, of $M$ there are $k, m \in \mathbb{Z}$ such that for every $i \leq n$, the partial automorphisms $g_i \psi_i g_i^{-k}$ and $g_i^m \phi_i g_i^{-m}$ have a common extension to an automorphism of $M$. Then the set
\[ Y = \{ \bar{h} \in G^n : (\bar{g}, \bar{h}) \text{ cyclically generates } G^n \} \]
is comeager in $G^n$; in particular it is non-empty.

**Proof.** Without loss of generality, we can assume that $k = 1$ in the statement of the theorem.

We fix $\phi = (\phi_1 : A_1 \to B_1, \ldots, \phi_n : A_n \to B_n)$, and we will show that the set
\[ Z_\phi = \{ (h_1, \ldots, h_n) \in G^n : \exists \psi \forall_i g_i^m \phi_i g_i^{-m} \subseteq h_i \} \]
is open and dense in $G^n$.

Clearly, $Z_\phi$ is open. To show that it is also dense, fix $\psi = (\psi_1 : C_1 \to D_1, \ldots, \psi_n : C_n \to D_n)$. The set $[\psi] \cap Z_\phi$ is non-empty, where
\[ [\psi] = \{ (f_1, \ldots, f_n) \in G^n : \forall_i \psi_i \subseteq f_i \}. \]
This follows from the assumption that there is $m$ such that for each $i \leq n$, the partial automorphisms $\psi_i : C_i \to D_i$ and $g_i^m \phi_i g_i^{-m} : g_i^m(A_i) \to g_i^m(B_i)$ have a common extension to an automorphism of $M$.

Finally, notice that $Y \supseteq \bigcap \phi Z_\phi$, hence it is comeager. \hfill \Box

We will use the following lemma in Sections 3.2-3.4 to show that the automorphism group of each structure listed in Theorem 1.2 is cyclically generated by a pair that generates the free group.

**Lemma 3.3.** Let $M$ be a countable structure with no algebraicity, and let $g \in \text{Aut}(M)$ be an automorphism with no cycles. Then for each $n \in \mathbb{N}$, the set
\[ \{ \bar{h} \in \text{Aut}(M) : (g, \ldots, g) \text{ and } \bar{h} \text{ generate the free group in } \text{Aut}(M)^n \} \]
and the set
\[ \{ h \in L_0(\text{Aut}(M)) : f_g \text{ and } h \text{ generate the free group in } L_0(\text{Aut}(M)) \} \]
are comeager.
Theorem 3.4. Let $M$ be a countable relational ultrahomogeneous structure such that $\text{Age}(M)$ has the strong$^+$ amalgamation property. Then each $G^n$, $n \in \mathbb{N}$, and $L_0(G)$, where $G = \text{Aut}(M)$, has a cyclically dense conjugacy class. In fact, each of these groups is cyclically generated by a pair generating the free group.

We prove Theorem 3.4 via Theorem 3.5 saying that if a countable relational ultrahomogeneous structure $M$ has no algebraicity and satisfies conditions $(\Delta_n)$, for every $n \in \mathbb{N}$, defined below, then the conclusion of Theorem 3.4 holds. Then, in Lemma 3.6 we show that a countable relational ultrahomogeneous structure $M$ such that $\text{Age}(M)$ has the strong$^+$ amalgamation property satisfies conditions $(\Delta_n)$, for every $n \in \mathbb{N}$.

Condition $(\Delta_n)$. Let $s_i : X_i \to Y_i$, $t_i : P_i \to Q_i$, $i \leq n$, and $h$, be partial automorphisms of $M$ such that $\bigcup_i (X_i \cup Y_i)$ is disjoint from $\text{rng}(h)$ and $\bigcup_i (P_i \cup Q_i)$ is disjoint from $\text{dom}(h)$. Then there exist $P'_i, Q'_i \subseteq M$ and partial automorphism $\alpha : \bigcup_i (P_i \cup Q_i) \to \bigcup_i (P'_i \cup Q'_i)$ with $\text{rng}(\alpha \upharpoonright P_i) = P'_i$ and $\text{rng}(\alpha \upharpoonright Q_i) = Q'_i$ such that:

1. $(\alpha \circ \alpha^{-1}) \upharpoonright P'_i \cup s_i$ is a partial automorphism for every $i$;
2. $h \cup \alpha : \text{dom}(h) \cup \bigcup_i (P_i \cup Q_i) \to \text{rng}(h) \cup \bigcup_i (P'_i \cup Q'_i)$ is a partial automorphism.

Theorem 3.5. Let $M$ be a relational ultrahomogeneous structure that has no algebraicity, and that satisfies conditions $(\Delta_n)$, for every $n \in \mathbb{N}$. Then each $G^n$, $n \in \mathbb{N}$, and $L_0(G)$, where $G = \text{Aut}(M)$, has a cyclically dense conjugacy class. In fact, each of these groups is cyclically generated by a pair generating the free group.

Lemma 3.6. Suppose that $\text{Age}(M)$ of an ultrahomogeneous structure $M$ satisfies the strong$^+$ amalgamation property. Then $M$ satisfies $(\Delta_n)$, for every $n$.

It is well-known that every ultrahomogeneous structure $M$ has the extension property, i.e., for any $X,Y \in \text{Age}(M)$, embeddings $i : X \to M$ and $\phi : X \to Y$, there is an embedding $j : Y \to M$ such that $j \circ \phi = i$. If additionally $M$ has no algebraicity, we can always choose $j$ such that $j(Y) \setminus i(X)$ is disjoint from a given finite set. Say that an ultrahomogeneous structure $M$ has the strong extension property if for any $X,Y \in \text{Age}(M)$, a finite substructure $Z \subseteq M$ disjoint from $i(X)$, embeddings $i : X \to M$ and $\phi : X \to Y$, there is an embedding $j : Y \to M$ such that $j \circ \phi = i$ and $j(Y) \cap Z = \emptyset$.

Proof of Lemma 3.6. Apply the strong$^+$ amalgamation property to $A := \text{rng}(h)$, $B := \text{rng}(h) \cup \bigcup_i (X_i \cup Y_i)$, $C := \text{dom}(h) \cup \bigcup_i (P_i \cup Q_i)$, $\phi_1 := \text{Id}_A$, $\phi_2 := h^{-1}$, and (using the notation as in the definition of the strong$^+$ amalgamation property) get $D$, $\psi_1$, and

Proof of Theorem 2.1. The theorem follows immediately from Lemmas 3.1, 3.2 and 3.3. □
Moreover, by (3) in the definition of the strong
partial automorphism (we have that $\psi_1(\bigcup_i X_i \cup Y_i) = \emptyset$, we get that
$\psi_2(\bigcup_i (P_i \cup Q_i))$ is disjoint from $\psi_1(\bigcup_i X_i \cup Y_i)$). Next apply the extension property to
$X := \text{rng}(h) \cup \bigcup_i (X_i \cup Y_i)$, $Y := D$, $i := \text{Id}_X$, and $\phi := \psi_1$, and (using the notation as
in the definition of the extension property) get $j$. Then $\alpha = (j \circ \psi_2) \upharpoonright \bigcup_i (P_i \cup Q_i)$, and $P'_i = \alpha(P_i)$, and $Q'_i = \alpha(Q_i)$ satisfy the required conditions (1) and (2) in the definition of
$(\Delta_n)$. Indeed, as $h \cup \alpha = j \circ \psi_2$, we have that $h \cup \alpha$ is a partial automorphism.
Moreover, by (3) in the definition of the strong amalgamation property, we see that for every
partial automorphism $s$ such that $\text{dom}(s) \cup \text{rng}(s) \subseteq \psi_1(\bigcup_i (X_i \cup Y_i))$ and partial
automorphism $t$ such that $\text{dom}(t) \cup \text{rng}(t) \subseteq \psi_2(\bigcup_i (P_i \cup Q_i))$, we have that $s \cup t$ is a
partial automorphism (we have that $(\text{dom}(s) \cup \text{rng}(s)) \cap (\text{dom}(t) \cup \text{rng}(t)) = \emptyset$), which
implies that $(\alpha \alpha^{-1}) \cup s$ is a partial automorphism, and this implies (2) in the definition
of $(\Delta_n)$.

Finally, we prove Theorem 3.5.

**Proof of Theorem 3.5.** By Theorem 2.1 it suffices to find $g \in \text{Aut}(M)$ such that for any $n$,
and any partial automorphisms $\phi_i: A_i \to B_i$ and $\psi_i: C_i \to D_i$, $i = 1, \ldots, n$, there is $m$ such that $g^{m+1}\phi_i g^{-(m+1)}: g^{m+1}(A_i) \to g^{m+1}(B_i)$ and $g^{-m}\psi_i g^m: g^{-m}(C_i) \to g^{-m}(D_i)$ we
can extend to a single automorphism of $M$ (equivalently: $g^{m+1}\phi_i g^{-(m+1)} \cup g^{-m}\psi_i g^m$ is a
partial automorphism). We construct such a $g$ which has no cycles.

Enumerate $M$ into $\{m_k\}_{k \geq 2}$ and enumerate all tuples of pairs of partial automorphisms of
the same length of $M$ into $\{(P_{k,i}, q_{k,i})\}_{k \geq 2}$. We will construct a sequence $\{g_k\}_{k \geq 1}$ of
partial automorphisms of $M$ with $g_{k+1}$ extending $g_k$, for each $k$, such that for each $k$:

1. $g_k$ has no cycles;
2. $m_k$ is both in the domain and in the range of $g_k$;
3. there is $m$ (that depends on $k$, but not on $i$) such that for any $i$, $g^{m+1}_{k+1} P_{k,i} g^{-m-1}_{k+1} \cup
   g^{-m}_{2k+1} q_{k,i} g^m_{2k+1}$ is a partial automorphism.

Let $g_1$ be the empty function. Suppose that we have constructed $g_1, g_2, \ldots, g_{k-1}$ and
now we want to define $g_k$. If $m_k$ is both in $\text{dom}(g_k)$ and in $\text{rng}(g_k)$, we just take
$g_k = g_{k-1}$. If $m_k \in \text{rng}(g_{k-1}) \setminus \text{dom}(g_{k-1})$ then apply the strong extension property
to $A = \text{rng}(g_{k-1}) \subseteq M$ and $B = \text{dom}(g_{k-1}) \cup \{m_k\}$ to find $b \in M$ disjoint from
$\text{dom}(g_{k-1}) \cup \text{rng}(g_{k-1})$ such that $\text{rng}(g_{k-1}) \cup \{b\}$ is isomorphic to $\text{dom}(g_{k-1}) \cup \{m_k\}$
via the map that extends $g_{k-1}$. Then we take $g_k = g_{k-1} \cup \{(m_k, b)\}$. We similarly
proceed (by considering $g^{-1}_{k-1}$ instead of $g_{k-1}$) when $m_k \in \text{dom}(g_{k-1}) \setminus \text{rng}(g_{k-1})$. In
the case when $m_k \notin \text{rng}(g_{k-1}) \cup \text{dom}(g_{k-1})$, we do two steps. First we proceed as in the
case when $m_k \in \text{rng}(g_{k-1}) \setminus \text{dom}(g_{k-1})$ and for the resulting partial automorphism we
proceed as in the case $m_k \notin \text{dom}(g_{k-1}) \setminus \text{rng}(g_{k-1})$ (where instead of $g_{k-1}$ we put that
resulting partial automorphism). Note that $g_k$ has no cycles.

We have $p_{k,i}: A_{k,i} \to B_{k,i}$, and $q_{k,i}: C_{k,i} \to D_{k,i}$, $i = 1, \ldots, n$, and we have $g_k$. Denote
$E = \bigcup_i (A_{k,i} \cup B_{k,i})$ and $F = \bigcup_i (C_{k,i} \cup D_{k,i})$. We will extend $g_k$ to some
partial automorphism $h$ such that for some $m$, $h^m(E)$ is disjoint from $\text{dom}(h)$ and $h^{-m}(F)$
is disjoint from $\text{rng}(h)$. Pick an $m$ such that for every $x \in E$, $g_{2k}^{m+1}(x)$ is undefined and
for every $y \in F$, $g_{2k}^{-m-1}(y)$ is undefined.

We first construct a partial automorphism $f$ which extends $g_k$ such that for every
$x \in E$, $f^m(x)$ is defined but $f^{m+1}(x)$ is undefined (i.e. $f^m(E)$ is disjoint from $\text{dom}(f)$)
and for every $y \in F$, $f^{-(m+1)}(y)$ is undefined. Next, we construct a partial automorphism
$h$ which extends $f$ such that for every $y \in F$, $h^{-m}(y)$ is defined but $h^{-(m+1)}(y)$ is undefined
(i.e. \( h^{-m}(F) \) is disjoint from \( \text{rng}(h) \)) and for every \( x \in E \), \( h^{m+1}(x) \) is undefined. We show how to construct \( f \) from \( g_{2k} \). The construction of \( h^{-1} \) from \( f^{-1} \), and hence the construction of \( h \), will be the same.

We will construct a sequence of partial automorphisms \( f_0 \subseteq f_1 \subseteq \ldots \subseteq f_m \) without cycles such that for \( i = 0, 1, \ldots, m \), we have that \( f_i^{m+1}(x) \) is undefined for every \( x \in E \), but \( f_i \) is defined on \( f_{i-1}^{-1}(E) \), and \( f_i^{-(m+1)}(y) \) is undefined for every \( y \in F \). Then \( f_m \) will be the required \( f \). We set \( f_0 = g_{2k} \). Having constructed \( f_0, \ldots, f_{i-1} \) for some \( i \leq m \), we take \( f_i \) such that \( \text{dom}(f_i) = \text{dom}(f_{i-1}) \cup f_{i-1}^{-1}(E) \). Let \( X = f_{i-1}^{-1}(E) \setminus \text{dom}(f_{i-1}) \). Using the strong extension property, find \( Y \) disjoint from \( \text{dom}(f_{i-1}) \cup \text{rng}(f_{i-1}) \cup (F \setminus \text{rng}(f_0)) \) such that the sets \( \text{dom}(f_{i-1}) \cup X \) and \( \text{rng}(f_{i-1}) \cup Y \) are isomorphic via an isomorphism that extends \( f_{i-1} \), we take this isomorphism to be \( f_i \). We see that since \( f_{m+1}^{-1}(x) \) is undefined for every \( x \in E \) and \( f_m \) is defined on \( f_{m-1}^{-1}(E) \), we have that \( \text{dom}(f_m) \), and \( f_m^m(E) \) are disjoint. Moreover, \( f_m^{-(m+1)}(y) \) is undefined for every \( y \in F \).

We apply the condition \((\Delta_n)\) to \( X_i = h^{-m}(C_{k,i}) \), \( Y_i = h^{-m}(D_{k,i}) \), \( s_i = h^{-m} q_{k,i} h^m \), \( p_i = h^m(A_{k,i}) \), \( Q_i = h^m(B_{k,i}) \), \( t_i = h^m p_{k,i} h^{-m} \), and \( h \). The conditions that \( \bigcup_i (X_i \cup Y_i) \) and \( \text{rng}(h) \) are disjoint and \( \bigcup_i (P_i \cup Q_i) \) and \( \text{dom}(h) \) are disjoint, are satisfied. We get \( P'_i, Q'_i \), and \( \alpha : \bigcup_i (P_i \cup Q_i) \to \bigcup_i (P'_i \cup Q'_i) \).

From (1) in the conclusions of \((\Delta_n)\), we get that \( (\alpha t_i, \alpha^{-1})^{\uparrow} \cap P'_i \cup s_i = \alpha h^m p_{k,i} h^{-m} \alpha^{-1} \cup h^{-m} q_{k,i} h^m \) is a partial automorphism.

From (2) in the conclusions of \((\Delta_n)\), we get that \( h \cup \alpha \) is a partial automorphism. Hence \( g_{2k+1} = h \cup \alpha \) is a partial automorphism. Moreover, \( g_{2k+1} \) has no cycles, as \( \bigcup_i (P_i \cup Q_i) \) is disjoint from \( \text{dom}(h) \).

Finally, note that \( \alpha h^m p_{k,i} h^{-m} \alpha^{-1} = g_{2k+1}^m p_{k,i} g_{2k+1}^{-m} \), which gives that the partial automorphism \( \alpha t_i, \alpha^{-1} \cap s_i \) is equal to \( g_{2k+1}^m p_{k,i} g_{2k+1}^{-m} \cup g_{2k+1}^m q_{k,i} g_{2k+1}^m \), as needed.

The required \( g \) is \( \bigcup_k g_k \).

We can slightly strengthen Theorem 3.5. Let \( NC(\text{Aut}(M)) \) denote the set of all automorphisms of \( \text{Aut}(M) \) without cycles. It is immediate to see that \( NC(\text{Aut}(M)) \) is a closed, and thus Polish, subspace of \( \text{Aut}(M) \).

**Corollary 3.7.** Let \( M \) be a relational ultrahomogeneous structure with no algebraicity such that for any \( n \) the condition \((\Delta_n)\) holds. If \( G = \text{Aut}(M) \), then for every \( n \) the set

\[
\{ g \in NC(G) : \text{ for some } \tilde{h} \in G^n, ((g, \ldots, g), \tilde{h}) \text{ cyclically generates } G^n \}
\]

and the set

\[
\{ g \in NC(G) : \text{ for some } h \in L_0(G), (f_g, h) \text{ cyclically generates } L_0(G) \}
\]

are dense \( G_\delta \) in \( NC(G) \).

**Proof.** In the proof of Theorem 3.5, instead of taking \( g_1 = \emptyset \), we can take an arbitrary partial automorphism which has no cycles. This implies that the set of \( g \in NC(\text{Aut}(M)) \) such that for any \( n \) and any partial automorphisms \( \phi_i : A_i \to B_i \) and \( \psi_i : C_i \to D_i \), \( i \leq n \), there are \( k, m \in \mathbb{Z} \) such that \( g^k \phi_i g^{-k} \) and \( g^m \psi_i g^{-m} \) we can extend to a single automorphism of \( M \), is a dense \( G_\delta \) subset of \( NC(\text{Aut}(M)) \).

**3.3. The rational Urysohn metric space.** Conditions \((\Delta_n)\), \( n \in \mathbb{N} \), do not seem to hold for the rational Urysohn space \( \mathbb{U}_0 \). Nevertheless, using Solecki [16, Theorem 3.2], we can, similarly as in Theorem 3.5, conclude that each \( \text{Aut}(\mathbb{U}_0)^n, n \in \mathbb{N} \), and \( L_0(\text{Aut}(\mathbb{U}_0)) \), has a cyclically dense conjugacy class. In this section, we show the following theorem.
Theorem 3.8. Each \( \text{Aut}(\mathbb{U}_0)^n \), \( n \in \mathbb{N} \), and \( L_0(\text{Aut}(\mathbb{U}_0)) \), has a cyclically dense conjugacy class. In fact, each of these groups is cyclically generated by a pair generating the free group.

We point out that Solecki [16, Corollary 4.1] already showed that each \( \text{Aut}(\mathbb{U}_0)^n \), \( n \in \mathbb{N} \), has a cyclically dense conjugacy class.

Let us first recall Solecki’s theorem:

Theorem 3.9 (Solecki, Theorem 3.2 in [16]). Let a finite metric space \( A \) and a partial isometry \( p \) of \( A \) be given. Then there exist a finite metric space \( B \) with \( A \subseteq B \) as metric spaces, an isometry \( \tilde{p} \) of \( B \) extending \( p \), and a natural number \( m \) such that:

(i) \( \tilde{p}^{2m} = \text{Id}_B \);
(ii) if \( a \in A \) has no cycles, then \( \tilde{p}^j(a) \neq a \) for all \( 0 < j < 2m \);
(iii) \( A \cup \tilde{p}^m(A) \) is the amalgam of \( A \) and \( \tilde{p}^m(A) \) over \( (Z(p), \text{id}_{Z(p)}, \tilde{p}^m | Z(p)) \), where \( Z(p) \) is the set of all cyclic points of \( p \).

Remark 3.10. If \( p \) has no cycles, then (iii) says that the distance between any point in \( A \) and any point in \( \tilde{p}^m(A) \) is equal to \( \text{diam}(A) + \text{diam}(\tilde{p}^m(A)) = 2\text{diam}(A) \). Moreover, in that case, the partial isometry

\[
p_1 = \tilde{p} \mid \left( \bigcup_{i=0}^{m-1} \tilde{p}^i(A) \right)
\]

has no cycles as well.

Proof of Theorem 3.9. We will verify that the assumptions of Theorem 2.1 are satisfied. As in the proof of Theorem 3.5 we start with enumerating \( \mathbb{U}_0 \) into \( \{m_k\}_{k \geq 2} \) and enumerating all tuples of pairs of the same length of partial automorphisms of \( \mathbb{U}_0 \) into \( \{(p_{k,i}, q_{k,i})\}_{k \geq 2} \). We then construct a sequence \( \{g_k\}_{k \geq 1} \), with \( g_{k+1} \) extending \( g_k \) for each \( k \), of partial automorphisms of \( \mathbb{U}_0 \) such that for each \( k \):

1. \( g_k \) has no cycles;
2. \( m_k \) is both in the domain and in the range of \( g_{2k} \);
3. there is \( m \) (that depends on \( k \), but not on \( i \)) such that for every \( i \), \( g_{2k+1}^m p_{k,i} g_{2k+1}^{-m} \cup q_{k,i} \) is a partial automorphism.

At even steps we proceed as in the proof of Theorem 3.5. Suppose that we constructed \( g_{2k} \) and we want to construct \( g_{2k+1} \) with the required properties. We have \( p_{k,i}: A_{k,i} \to B_{k,i} \) and \( q_{k,i}: C_{k,i} \to D_{k,i} \), \( i \leq n \). Let \( A \) be the union of all the domains and ranges of \( g_{2k} \), \( p_{k,i} \), and \( q_{k,i} \), \( i \leq n \), and let \( p = g_{2k} \). Apply Theorem 3.9 and get \( B, \tilde{p} \) and \( m \). We let

\[
g_{2k+1} = \tilde{p} \mid \left( \bigcup_{i=0}^{m-1} \tilde{p}^i(A) \right).
\]

Using observations in Remark 3.10, we conclude that \( g_{2k+1} \) has no cycles. We also see that for each \( i \leq n \), \( g_{2k+1}^m p_{k,i} g_{2k+1}^{-m}: g_{2k+1}^m(A_{k,i}) \to g_{2k+1}^m(B_{k,i}) \) and \( q_{k,i}: C_{k,i} \to D_{k,i} \), the map \( g_{2k+1}^m p_{k,i} g_{2k+1}^{-m} \cup q_{k,i} \) is a partial automorphism. Indeed, as we observed in Remark 3.10 the distance between a point in \( \bigcup_i (C_{k,i} \cup D_{k,i}) \) and a point in \( \bigcup_i (g_{2k+1}^m(A_{k,i}) \cup g_{2k+1}^m(B_{k,i})) \) is always the same. The required \( g \) is \( \bigcup_k g_k \).

\[\square\]
3.4. The random poset and other examples. In this section, we verify Theorem 2.2 (iii)-(v). We first focus on the random poset \( \mathbb{P} \) and show the following.

**Theorem 3.11.** Each \( \text{Aut}(\mathbb{P})^n, n \in \mathbb{N}, \) and \( L_0(\text{Aut}(\mathbb{P})) \), has a cyclically dense conjugacy class. In fact, each of these groups is cyclically generated by a pair generating the free group.

It already follows from the work of Glass-McCleary-Rubin [5] that \( \text{Aut}(\mathbb{P}) \) has a cyclically dense conjugacy class whose elements generate the free group.

To prove Theorem 3.11 we will need Proposition 3.12 and Lemma 3.13

**Proposition 3.12** (Corollary 3.3 [5]). There exists \( g \in \text{Aut}(\mathbb{P}) \) with the following two properties:

1. for every \( x \in \mathbb{P} \), we have \( x \leq g(x) \);
2. there exists \( a \in \mathbb{P} \) such that for any \( x \in \mathbb{P} \) there is \( n \in \mathbb{Z} \) with \( g^n(a) \leq x \leq g^{n+1}(a) \).

**Lemma 3.13.** If \( g \in \text{Aut}(\mathbb{P}) \) is such as in Proposition 3.12 then we have: For every \( a \in \mathbb{P} \) and every \( x \in \mathbb{P} \) there are \( k, n \in \mathbb{Z} \) with \( g^k(a) \leq x \leq g^n(a) \).

**Proof.** Let \( a \) and \( x \) be given, and let \( a \) be as in (2) of Proposition 3.12 Then for some \( m \), we have \( g^m(a) \leq a_1 \leq g^{m+1}(a) \). This implies

\[
\ldots g^{m-1}(a) \leq g^{-1}(a_i) \leq g^{m}(a) \leq a_1 \leq g^{m+1}(a) \leq g(a_1) \leq g^{m+2}(a) \leq \ldots
\]

Therefore, if \( n \) is such that \( g^n(a) \leq x \leq g^{n+1}(a) \), since \( g^{n-1}(a_1) \leq g^n(a) \) and \( g^{n+1}(a) \leq g^{n-m-1}(a_1) \), we get \( g^{n-m-1}(a_1) \leq x \leq g^{n-m}(a_1) \).

**Proof of Theorem 3.11.** We verify that the assumptions of Theorem 2.1 hold. Let \( g \in \text{Aut}(\mathbb{P}) \) be as in Proposition 3.12 and fix \( n \) and partial automorphisms of \( \mathbb{P} \), \( \phi_i: A_i \to B_i \) and \( \psi_i: C_i \to D_i, \) \( i \leq n \). Let \( x \in \mathbb{P} \) be such that for any \( z_1 \in \bigcup_i (A_i \cup B_i) \), \( z_1 < x \); it exists by the extension property. Then, using (1) and (2) in the properties of \( g \) multiple times, get \( m \) such that for every \( z_2 \in \bigcup_i (C_i \cup D_i), \) \( x \leq g^m(z_2) \). That implies that for every \( z_1 \in \bigcup_i (A_i \cup B_i) \) and \( z_2 \in \bigcup_i (C_i \cup D_i), \) we have \( z_1 < g^m(z_2) \). That gives that for every \( i \leq n, \phi_i \) and \( g^{-m} \psi_i g^m \) we can extend to a single automorphism of \( M \).

Let \( S_\infty \) denote the group of all permutations of integers, \( \text{Aut}(\mathbb{Q}) \) the automorphism group of rationals, \( H(2^\mathbb{Q}) \) is the homeomorphism group of the Cantor set \( 2^\mathbb{Q} \), and \( H(2^\mathbb{Q}, \mu^2) \) is the group of all \( \mu^2 \)-preserving homeomorphisms of the Cantor set, where \( \mu^2 \) is the product of the measure \( \mu \) on \( 2 = \{0, 1\} \) that assigns \( \frac{1}{2} \) to each of 0 and 1.

**Theorem 3.14.** Let \( G \) be one of the \( S_\infty, \text{Aut}(\mathbb{Q}), H(2^\mathbb{Q}) \) or \( H(2^\mathbb{Q}, \mu^2) \). Then each \( G^n, n \in \mathbb{N}, \) and \( L_0(G) \), has a cyclically dense conjugacy class. In fact, each of these groups is cyclically generated by a pair generating the free group.

It already follows from the work of Kechris-Rosendal [9] that each of the groups \( S_\infty, \text{Aut}(\mathbb{Q}), H(2^\mathbb{Q}), \) and \( H(2^\mathbb{Q}, \mu^2) \) has a cyclically dense conjugacy class.

**Proof.** Again, we verify that the assumptions of Theorem 2.1 hold. When \( G = S_\infty \), we let \( g \) in Theorem 2.1 to be the shift map \( n \mapsto n + 1 \). Similarly, when \( G = \text{Aut}(\mathbb{Q}), \) we take \( g \) to be a shift by some rational number.

If \( G = H(2^\mathbb{Q}) \) (we identify \( H(2^\mathbb{Q}) \) with the automorphism group of the Boolean algebra of all clopen sets in \( 2^\mathbb{Q} \)) or \( G = H(2^\mathbb{Q}, \mu^2) \) (we identify \( H(2^\mathbb{Q}, \mu^2) \) with the group of measure preserving automorphisms of the Boolean algebra of all clopen sets in \( 2^\mathbb{Q} \)), the \( g \) equal to the Bernoulli shift will work. Indeed, for any partial automorphisms of finite
3.5. Final remarks. It is not hard to see the following.

**Proposition 3.15.** If $G$ is a Polish group and $L_0(G)$ is topologically $m$-generated, then $G^n$ is topologically $m$-generated for every $n \in \mathbb{N}$.

Proof. Let $d$ denote a metric on $G$ and $\rho$ be the corresponding metric on $L_0(G)$. Fix $n$, and suppose that $g_1, \ldots, g_m$ topologically generate $L_0(G)$. Let $(e_k)_{k \in \mathbb{N}}$ be a countable dense set in $G$. For each $s \in \mathbb{N}^n$ and $i \in \mathbb{N}$, we pick $\epsilon'_i > 0$ such that for each $s$, we have $\sum_s \epsilon'_s < \frac{1}{n}$. For each $s \in \mathbb{N}^n$ and $i \in \mathbb{N}$ now pick a word $l = l(s, i) \in F_m$, such that $\rho(l(g_1, \ldots, g_m), f_s) < \epsilon'_i$, where $f_s \in L_0(G)$ is such that $f_s(x) = e_{s(k)}$ if and only if $x \in (\frac{k}{n}, \frac{k+1}{n})$, $k = 0, 1, \ldots, n-1$. Then there is a set $A_{s,i}$ of measure $\geq 1 - \epsilon'_i$, such that for $x \in A_{s,i}$ we have $d(l(g_1(x), \ldots, g_m(x)), f_s(x)) < \epsilon'_i$. This implies that for every $k$ the set $(\frac{k}{n}, \frac{k+1}{n}) \cap \left( \bigcap_{s \in \mathbb{N}^n, i \in \mathbb{N}} A_{s,i} \right)$ is non-empty, and for each $(x_0, \ldots, x_{n-1})$, where $x_i \in (\frac{k}{n}, \frac{k+1}{n}) \cap \left( \bigcap_{s \in \mathbb{N}^n, i \in \mathbb{N}} A_{s,i} \right)$, the pair $((g_1(x_0), \ldots, g_1(x_n)), \ldots, (g_m(x_0), \ldots, g_m(x_n)))$ topologically generates $G^n$, in particular, that $G^n$ is topologically generated.

We do not know whether the converse to Proposition 3.15 holds.

**Question 3.16.** Let be $G$ a Polish group. Suppose that there exists $m$ such that for every $n$, $G^n$ is topologically $m$-generated. Is it the case that $L_0(G)$ is topologically $m$-generated?

So far we only dealt with automorphism groups of countable structures. Below we use our earlier results to conclude that certain “large” Polish groups have a cyclically dense conjugacy class.

For the Lebesgue measure $\lambda$ on $[0, 1]$ let $\text{Aut}([0, 1], \lambda)$ denote the Polish group of all measure preserving transformations of the interval $[0, 1]$. Let $\text{Iso}(U)$ be the Polish group of all isometries of the Urysohn metric space, where the Urysohn metric space is the unique Polish metric space which is ultrahomogeneous, and embeds isometrically every finite metric space.

Knowing that the groups $L_0(H(2^\mathbb{Z}, \mu_{2^\mathbb{Z}}))$ and $L_0(\text{Aut}(U_0))$ have a cyclically dense conjugacy class, we can deduce that the groups $L_0(\text{Aut}([0, 1], \lambda))$ and $L_0(\text{Iso}(U))$ have the same property.

**Proposition 3.17.** If there is a continuous 1-to-1 homomorphism with a dense image of a Polish group $G$ into a Polish group $H$, then there is a continuous 1-to-1 homomorphism with a dense image of a Polish group $L_0(G)$ into a Polish group $L_0(H)$. In that case, if $L_0(G)$ has a cyclically dense conjugacy class, the same is true for $L_0(H)$.

Proof. For a homomorphism $f : G \to H$ of groups, the function $F : L_0(G) \to L_0(H)$ given by $F(h)(x) = f(h(x))$ is also a homomorphism of groups. Moreover, if $f$ is Borel, so is $F$ (see [12], the corollary on page 7), and hence it is continuous as long as $G, H$ are Polish (see Theorem 1.2.6 in [1]). Finally, if $f$ is 1-to-1, and has a dense image, the same holds for $F$. Indeed, if $f$ has a dense image then for any $\epsilon > 0$ and a Borel function $h \in L_0(H)$, since the set

$$\{(x, y) \in [0, 1] \times H : d(h(x), y) < \epsilon \text{ and } y \in f[G]\}$$
is analytic, the density of the image of $F$ follows from the Jankov von Neumann uniformization theorem. □

**Corollary 3.18.** The groups $L_0(\text{Aut}([0,1],\lambda))$ and $L_0(\text{Iso}(\mathbb{U}))$ have a cyclically dense conjugacy class.

**Proof.** Apply Proposition 3.17 to $G = H(2^\mathbb{Z},\mu^\mathbb{Z})$ and $H = \text{Aut}([0,1],\lambda)$ and then to $G = \text{Aut}(\mathbb{U}_0)$ and $H = \text{Iso}(\mathbb{U})$. □

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