Coherence measure: Logarithmic coherence number

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We introduce a measure of coherence, which is extended from the coherence rank via the standard convex roof construction, we call it the logarithmic coherence number. This approach is parallel to the Schmidt measure in entanglement theory. We study some interesting properties of the logarithmic coherence number, and show that this quantifier can be considered as a proper coherence measure. We also find that the logarithmic coherence number can be calculated exactly for a large class of states. We give the relationship between coherence and entanglement in bipartite and multipartite settings. Finally, we find that the creation of entanglement with bipartite incoherent operations is bounded by the logarithmic coherence number of the initial system during the process.

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I. INTRODUCTION

The fundamental that distinguish quantum states from classical states is quantum coherence, which is the most basic characteristic of quantum mechanics. Quantum coherence plays an important role in the study of quantum information and quantum multipartite systems. Baumann et al proposed a theoretical framework for quantitative study of quantum coherence from the perspective of resource theory. Various ways have been built to develop the resource-theoretic framework for understanding quantum coherence, we refer to for more discussions of resource theory of coherence.

Analogously to the Schmidt rank in entanglement theory, Killoran et al presented a framework for the conversion of nonclassicality (including coherence) into entanglement, they introduced a concept of the coherence rank. A concept related to the coherence rank was also discussed by Levi and Mintert. Soon afterwards, Chin introduced a discrete coherence monotone named the coherence number, which is a generalization of the coherence rank to mixed states. Regula et al also discussed coherence number of mixed states, they presented a general formalism for the conversion of nonclassicality into multipartite entanglement. Theurer et al employed a natural generalization of the coherent rank to superposition with respect to a finite number of linear independent basis. The coherence number is proved to be a discrete coherence monotone, but it is not a proper coherence measure because it does not satisfy convexity. To resolve this issue, in this paper, we try to extend the coherence rank to mixed states via the standard convex roof construction, this approach is parallel to the Schmidt measure. We can prove that it is not only a coherence monotone but also a proper coherence measure.

The paper is organized as follows. In Sec. we review some basic concepts about the resource theory of coherence. In Sec. we discuss the coherence rank and give a new property about it. In Sec. we introduce a new coherence measure, which is the so-called the logarithmic coherence number. Some interesting properties are given. In Sec. we focus on the relationship between coherence and entanglement in bipartite and multipartite settings. In Sec. we discuss how the interplay between coherence consumption and creation of entanglement. We summarize our results in Sec. VII.

II. BASIC CONCEPTS OF COHERENCE MEASURE

We introduce some concepts about coherence measure that can be used for our main results. Given a d-dimension Hilbert space with a fixed orthogonal basis , we denote the set of all density operators acting on by . The density operators which are diagonal in this fixed basis are called incoherent, we denote the set of all incoherent states by . Any incoherent state is of the form

where are probability distribution. Any state which cannot be written as above form is defined as a coherent state, which means the coherence is basis-dependent.

The incoherent operation is to map the incoherent states to incoherent states. The definition of incoherent operation is not unique and different choices. In this paper, we only consider the incoherent operation in . The incoherent operation (IO) is a completely positive and trace preserving (CPTP) maps that admit a Kraus operator representation

where all the Kraus operators must satisfy . In general, the Kraus operator
can always be represented as

\[ K_n = \sum_i c_i |f(i)\rangle \langle i|, \]  

where \( f \) is a function in the index set and \( c_i \in [0, 1] \). 

Baumgratz et al proposed that any proper measure of the coherence \( C \) must satisfy the following conditions [1]:

\begin{enumerate}
  \item \textbf{(C1) Nonnegativity:} \( C(\rho) \geq 0 \) for all quantum states \( \rho \), and \( C(\rho) = 0 \) if and only if \( \rho \) is incoherent.
  \item \textbf{(C2) Monotonicity:} \( C(\rho) \) is non-increasing under incoherent operation \( \Lambda \), i.e., \( C(\rho) \geq C(\Lambda(\rho)) \).
  \item \textbf{(C3) Strong monotonicity:} \( C(\rho) \) does not increase on average under selective incoherent operations, i.e., \( \sum_n q_n C(\rho_n) \leq C(\rho) \), where \( \rho_n = K_n \rho K_n^\dagger / q_n \), and \( q_n = \text{Tr}(K_n \rho K_n^\dagger) \).
  \item \textbf{(C4) Convexity:} \( C(\rho) \) is a convex function of quantum states, i.e., \( \sum_j p_j C(\rho_j) \geq C(\sum_j p_j \rho_j) \), for any ensemble \( \{p_j, \rho_j\} \).
\end{enumerate}

Following standard notions from entanglement theory, we call a quantifier \( C \) which fulfills conditions (C1) and either condition (C2) or (C3) (or both) a coherence monotone. A quantifier \( C \) is further called a coherence measure if it satisfies the four conditions: (C1)-(C4). We also know that conditions (C3) and (C4) automatically imply condition (C2) [2].

\section{III. COHERENCE RANK}

For a pure state on Hilbert space \( \mathcal{H} \) with the fixed orthogonal basis \( \mathcal{O} \), one can define the coherence rank

\[ R_C(|\psi\rangle) = \min \left\{ |\hat{\mathcal{O}}| \mid |\psi\rangle = \sum_{|j\rangle \in \mathcal{O}} \lambda_j |j\rangle, \hat{\mathcal{O}} \subseteq \mathcal{O} \right\}, \]  

where \( \lambda_j \) are nonzero complex coefficients.

We note that the coherence rank given in Eq. (4) characterizes the minimal number of the incoherent states in the fixed orthogonal basis \( \mathcal{O} \) in such a decomposition of \( |\psi\rangle \). This is also equivalent to the fact that the coherence rank \( R_C(|\psi\rangle) = k \) if exactly \( k \) of the coefficients \( \lambda_j \) are nonzero. Thus, we say that the coherence rank is non-increasing under incoherent operations \( \Lambda \), that is,

\[ R_C(\Lambda(|\psi\rangle)) \leq R_C(|\psi\rangle). \]  

In particular, following the results in [6, 12], we know that there exists a unitary incoherent operation \( U_{in} \) on a pure state \( |\psi\rangle \) such that the coherence rank of \( U_{in}|\psi\rangle \) is equal to the coherence rank of \( |\psi\rangle \), i.e.,

\[ R_C(U_{in}|\psi\rangle) = R_C(|\psi\rangle), \]  

where \( U_{in} = \sum_j e^{i\theta_j} |j\rangle \langle j| \) with some phases \( \theta_j \). Therefore, we say that the coherence rank is a coherence monotone.

We also consider the coherence rank of superposition of two coherent states. The following result will give the lower and upper bounds of the coherence of superposition.

\begin{proposition}
Let \( |\phi\rangle = a|\psi\rangle + b|\varphi\rangle \) with \( |a|^2 + |b|^2 = 1 \), we have

\[ |R_C(|\psi\rangle) - R_C(|\varphi\rangle)| \leq R_C(|\phi\rangle) \leq R_C(|\psi\rangle) + R_C(|\varphi\rangle). \]  

\end{proposition}

\textbf{Proof.} By the definition of the coherence rank, there exist two sets \( \hat{\mathcal{O}}_\psi \) and \( \hat{\mathcal{O}}_\varphi \) such that \( R_C(|\psi\rangle) = |\hat{\mathcal{O}}_\psi| \), \( R_C(|\varphi\rangle) = |\hat{\mathcal{O}}_\varphi| \), and one has

\[ |\psi\rangle = \sum_{|j\rangle \in \hat{\mathcal{O}}_\psi} \psi_j |j\rangle, |\varphi\rangle = \sum_{|k\rangle \in \hat{\mathcal{O}}_\varphi} \varphi_k |k\rangle. \]

Then, we will consider three cases as follows.

\textbf{Case 1.} If \( \hat{\mathcal{O}}_\psi \perp \hat{\mathcal{O}}_\varphi \), by definition, we directly obtain

\[ R_C(|\phi\rangle) = R_C(|\psi\rangle) + R_C(|\varphi\rangle). \]

\textbf{Case 2.} If \( \hat{\mathcal{O}}_\psi \cap \hat{\mathcal{O}}_\varphi \neq \emptyset \), without loss of generality, we take \( \hat{\mathcal{O}} = \hat{\mathcal{O}}_\psi \cap \hat{\mathcal{O}}_\varphi \), and

\[ |\phi\rangle = a \sum_{|j\rangle \in \hat{\mathcal{O}}_\psi} \psi_j |j\rangle + \sum_{|j\rangle \in \hat{\mathcal{O}}_\varphi} (a\psi_j + b\varphi_j) |j\rangle + b \sum_{|k\rangle \in \hat{\mathcal{O}}_\varphi} \varphi_k |k\rangle. \]

Then, we have

\[ R_C(|\phi\rangle) \leq |\hat{\mathcal{O}}_\psi| + |\hat{\mathcal{O}}_\varphi| + |\hat{\mathcal{O}}| \]

\[ = |\hat{\mathcal{O}}_\psi| + |\hat{\mathcal{O}}_\varphi| - |\hat{\mathcal{O}}| \leq R_C(|\psi\rangle) + R_C(|\varphi\rangle). \]

\textbf{Case 3.} If \( \hat{\mathcal{O}}_\psi \subseteq \hat{\mathcal{O}}_\varphi \), then we have

\[ |\phi\rangle = \sum_{|j\rangle \in \hat{\mathcal{O}}_\psi} (a\psi_j + b\varphi_j) |j\rangle + b \sum_{|k\rangle \in \hat{\mathcal{O}}_\varphi} \varphi_k |k\rangle. \]

By the definition, we obtain

\[ R_C(|\phi\rangle) \geq R_C(|\varphi\rangle) - R_C(|\psi\rangle). \]

Similarly, If \( \hat{\mathcal{O}}_\varphi \subseteq \hat{\mathcal{O}}_\psi \), we have

\[ R_C(|\phi\rangle) \geq R_C(|\psi\rangle) - R_C(|\varphi\rangle). \]

Thus, we obtain our desired result. \( \square \)
The coherence rank has been generalized to mixed states in \[6,8,9\]. It is the so-called coherence number, which is defined as

\[ R_C(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \max_i |R_C(|\psi_i\rangle)|. \]  

(15)

The coherence number is the smallest possible maximal coherence rank in any decomposition of the mixed states, and for pure states the coherence rank equals the coherence number. The coherence number only satisfies condition (C1), (C2) and (C3), but it doesn’t satisfy condition (C4) \[8,11\], so it is only a coherence monotone. In the following section, we apply the standard convex roof construction to the mixed states.

IV. LOGARITHMIC COHERENCE NUMBER

In this section, we can define logarithmic coherence rank, it is in the same way as the Schmidt rank in \[3\]. Note that Theurer et al used this way to consider the superposition in \[10\].

Definition 2. For any pure state \(|\psi\rangle\) on \(\mathcal{H}\), the logarithmic coherence rank is defined as

\[ L_C(|\psi\rangle) = \log_2 R_C(|\psi\rangle). \]  

(16)

Obviously, the logarithmic coherence rank inherits some properties of coherence rank. The logarithmic coherence rank is non-negative, that is, \(L_C(|\psi\rangle) \geq 0\) for any pure state \(|\psi\rangle\). In particular, for the maximally coherent states

\[ |\psi_M\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i\theta_j} |j\rangle, \]  

(17)

we have

\[ L_C(|\psi_M\rangle) = \log_2 d. \]  

(18)

In addition, we find that the logarithmic coherence rank is also monotone, unitarily invariant and so on. The logarithmic coherence rank can be extended to mixed states by the standard convex roof construction.

Definition 3. For any mixed state \(\rho\), the logarithmic coherence number is defined as

\[ L_C(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i L_C(|\psi_i\rangle), \]  

(19)

where the minimum is taken over all pure state decompositions of \(\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|\).

In the subsequent paragraphs we will show that the logarithmic coherence number is a proper coherence measure in the sense of Refs. \[1,2\].

Proposition 4. The logarithmic coherence number \(L_C\) is a coherence measure, which satisfies the conditions (C1)-(C4).

Proof. Obviously, condition (C1) follows immediately from the definition.

To show that \(L_C\) satisfies condition (C3), let \(\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|\) be the optimal decomposition of \(\rho\) belonging to the minimum in Eq. (19), and we take the mark \[10\], and define

\[ |\tilde{\psi}_{i,n}\rangle = \frac{K_n |\psi_i\rangle}{\sqrt{q_n}}, \]  

(20)

where \(q_n = \text{Tr}(K_n^\dagger K_n \rho)\), and \(K_n\) are incoherent Kraus operators. Then, every final state \(\rho_n\) in an incoherent Kraus operator \(K_n\) can be represented by

\[ \rho_n = \frac{K_n \rho K_n^\dagger}{q_n} = \sum_i p_i |\tilde{\psi}_{i,n}\rangle \langle \tilde{\psi}_{i,n}|. \]  

(21)

Since the coherence rank can never increase under the action of an incoherent Kraus operator, then we have

\[ L_C(\rho_n) \leq \sum_i p_i L_C(|\tilde{\psi}_{i,n}\rangle) \leq \sum_i p_i L_C(|\psi_i\rangle) = L_C(\rho). \]  

(22)

Thus, we have

\[ \sum_n q_n L_C(\rho_n) \leq L_C(\rho). \]  

(23)

To show (C4) we take

\[ \rho = \lambda_1 \rho_1 + \lambda_2 \rho_2, \]  

(24)

where \(\lambda_1, \lambda_2 \in [0,1]\). Let \(\rho_1 = \sum_j \mu_j |\phi_j\rangle \langle \phi_j|\) and \(\rho_2 = \sum_k \eta_k |\phi_k\rangle \langle \phi_k|\) be the two decompositions for which the respective minima in Eq. (19) are attained. Then the convex combinations \(\lambda_1 \sum_j \mu_j |\phi_j\rangle \langle \phi_j| + \lambda_2 \sum_k \eta_k |\phi_k\rangle \langle \phi_k|\) is a valid decomposition of \(\rho\), but it is not necessarily the optimal one. Thus, we have

\[ L_C(\lambda_1 \rho_1 + \lambda_2 \rho_2) \leq \lambda_1 L_C(\rho_1) + \lambda_2 L_C(\rho_2). \]  

(25)

We know that the condition (C2) can be derived from conditions (C3) and (C4), so we say the logarithmic coherence number \(L_C\) satisfies conditions (C1)-(C4).

This shows that the logarithmic coherence number can indeed be used as a coherence measure quantifying the coherence of a quantum system. Not just these nice properties, we also find that the logarithmic coherence number is additive as follows.

Proposition 5. The logarithmic coherence number \(L_C\) is additive.
Proof. Let us consider the case of pure states first. From the definition of the coherence rank, we have
\[
R_C(|\psi_1\rangle \otimes |\psi_2\rangle) = R_C(|\psi_1\rangle \cdot R_C(|\psi_2\rangle).
\]
(26)
Thus, we obtain
\[
L_C(|\psi_1\rangle \otimes |\psi_2\rangle) = \log_2(R_C(|\psi_1\rangle \cdot R_C(|\psi_2\rangle)) = L_C(|\psi_1\rangle) + L_C(|\psi_2\rangle).
\]
(27)
Then we consider the case of mixed states. Without loss of generality, the pure states decompositions of \(\rho \otimes \sigma\) is of the form
\[
\rho \otimes \sigma = \sum_a p_a |\psi_a\rangle \langle \psi_a| \otimes \sum_b p_b |\phi_b\rangle \langle \phi_b|.
\]
(28)
Then we have
\[
L_C(\rho \otimes \sigma) = \min_{a,b} p_a p_b L_C(|\psi_a\rangle \otimes |\phi_b\rangle) = \min_{a} p_a L_C(|\psi_a\rangle) \otimes \min_{b} p_b L_C(|\phi_b\rangle) = L_C(\rho) + L_C(\sigma).
\]
(29)
This completes the proof the proposition. \(\square\)

From this result, for \(n\) copies of the same state \(|\psi\rangle\), we have
\[
L_C(|\psi\rangle^{\otimes n}) = n L_C(|\psi\rangle).
\]
(30)
In particular, let \(\delta\) be an incoherent state, we have
\[
L_C(\delta^{\otimes n} \otimes |\psi\rangle \langle \psi|^{\otimes n}) = n L_C(|\psi\rangle).
\]
(31)
If the states \(|\psi_1\rangle\) and \(|\psi_2\rangle\) satisfy \(||\psi_1\rangle - |\psi_2\rangle|| < \varepsilon\), we may ask whether the logarithmic coherence number also satisfies \(|L_C(|\psi_1\rangle) - L_C(|\psi_2\rangle)|| < \varepsilon\), where \(\cdot\) is trace distance. Let \(|\psi_1\rangle\) be the state
\[
|\psi_1\rangle = \sqrt{1-\varepsilon}|0\rangle + \sqrt{\frac{\varepsilon}{d-1}} \sum_{i=1}^{d-1} |i\rangle,
\]
(32)
and \(|\psi_2\rangle = |0\rangle\). When \(\varepsilon \to 0\), which means \(|\psi_1\rangle \to |\psi_2\rangle\), but we know that \(|L_C(|\psi_1\rangle) - L_C(|\psi_2\rangle)|| = \log_2 d\). Thus, we claim that the logarithmic coherence number is not continuous.

Although we define the coherence measure of a mixed state via a minimization over all possible realizations of the state, it can be calculated exactly for some states. In order to calculate the logarithmic coherence number of a mixed state, the minimization over decompositions of the state is necessary. The value of \(L_C\) can be fully evaluated for some states. We consider a family of noisy maximally coherent states
\[
\rho_\lambda = \lambda |\psi_M\rangle \langle \psi_M| + (1 - \lambda) I/d,
\]
(33)
where \(\lambda \in [0,1]\). Without loss of generality, the identity operator \(I\) can be represented with the pure states \(|\psi_i\rangle\) as
\[
I = \sum_i \alpha_i |\psi_i\rangle \langle \psi_i|,
\]
(34)
where \(\alpha_i \geq 0\). Then, the pure states decompositions of \(\rho_\lambda\) is of the form
\[
\rho_\lambda = \lambda |\rangle \langle +| + \frac{1 - \lambda}{d} \sum_i \alpha_i |\psi_i\rangle \langle \psi_i|.
\]
(35)
Using the definition (19), we get
\[
L_C(\rho_\lambda) = \min_{\{\alpha_i, |\psi_i|\}} \left[ \lambda \log_2 d + \frac{1 - \lambda}{d} \sum_i \alpha_i L_C(|\psi_i|) \right].
\]
(36)
Minimizing the right-hand side of Eq. (36) over all pure states decompositions we immediately see that the minimum is achieved for every \(i\), \(L_C(|\psi_i|) = 0\). Thus, we obtain a closed expression of the logarithmic coherence number for the state \(\rho_\lambda\), i.e.,
\[
L_C(\rho_\lambda) = \lambda \log_2 d.
\]
(37)

V. MULTIPARTITE SCENARIO

Let \(\mathcal{H}^S\) and \(\mathcal{H}^A\) be two \(d\)-dimensional Hilbert spaces, and \(\mathcal{H}^{SA}\) be the Hilbert space of an ancillary system with \(\mathcal{H}^S \bowtie \mathcal{H}^A\). Without loss of generality, we take the orthogonal basis \(\{|i\rangle\}_{i=0}^{d-1}\) and \(\{|j\rangle\}_{j=0}^{d-1}\) as two fixed basis on \(\mathcal{H}^S\) and \(\mathcal{H}^A\), respectively. Then, their tensor product \(\{|i\rangle \otimes |j\rangle\}\) can be viewed as an incoherent basis for compound system \(SA\). Thus, the corresponding logarithmic coherence rank and the logarithmic coherence number can be defined just as (19) and (19). We are particularly interested in the relationship between the total coherence and coherence contained in each individual subsystem. In the following proposition, we prove that the logarithmic coherence number in the bipartite quantum states is no less than the sum between two subsystems. This relation can be viewed as the super-additivity for the logarithm coherence number.

Proposition 6. For any bipartite quantum state \(\rho^{SA}\) on \(SA\), we have
\[
L_C(\rho^S) + L_C(\rho^A) \leq L_C(\rho^{SA}),
\]
(38)
where \(\rho^S\) and \(\rho^A\) are reduced states on \(S\) and \(A\), respectively.

Proof. Firstly, we consider the case of pure states. Let
\[
|\psi^{SA}\rangle = \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} a_{rs} |r^S\rangle |s^A\rangle.
\]
(39)
be the optimal decomposition of $|\psi^{SA}\rangle$ belonging to the minimum in Eq. (40), it follows that
\[
R_C(|\psi^{SA}\rangle) = r_S \times r_A.
\] (40)
Further, the matrix $M$ of complex numbers $a_{ij}$ can be represented as
\[
M = \begin{pmatrix} N_{r_S \times r_A} & O \\ O & O \end{pmatrix},
\] (41)
where $N_{r_S \times r_A} = (a_{ij})_{r_S \times r_A}$, and $O$ are zero matrices. Using the singular value decomposition, $M = U \Sigma V^\dagger$, where $\Sigma$ is a diagonal matrix with non-negative elements $\lambda_m$, which are the singular values of $M$, and $U$ and $V$ are unitary matrices. Thus, it is always possible to write $|\psi^{SA}\rangle$ in the following way
\[
|\psi^{SA}\rangle = \sum_{m=0}^{r-1} \lambda_m |m^S\rangle |m^A\rangle,
\] (42)
where $r$ is the Schmidt number of the state $|\psi^{SA}\rangle$, and
\[
|m^S\rangle = \sum_{i=0}^{r_S-1} u_{im} |i^S\rangle, |m^A\rangle = \sum_{j=0}^{r_A-1} v_{mj} |j^A\rangle.
\] (43)
Here the complex number $u_{im}$ and $v_{mj}$ are matrix elements of unitary matrices $U$ and $V$. It is easy to see that the coherence rank of the states $|m^S\rangle$ and $|m^A\rangle$ can not exceed the numbers $r_S$ and $r_A$, respectively. This means that for every $m$ the following inequalities holds,
\[
L_C(|m^S\rangle) \leq \log_2 r_S, L_C(|m^A\rangle) \leq \log_2 r_A.
\] (44)
For the subsystem $S$, we know that $\rho^S = \sum_m \lambda_m^2 |m^S\rangle \langle m|\rangle$ is a valid decomposition of $\rho^S$, then we obtain
\[
L_C(\rho^S) = \sum_m \lambda_m^2 L_C(|m^S\rangle) \leq \sum_m \lambda_m^2 \log_2 r_S = \log_2 r_S,
\] (45)
and similarly that
\[
L_C(\rho^A) \leq \log_2 r_A.
\] (46)
The above inequalities together with Eq. (40) implies the following inequality,
\[
L_C(\rho^S) + L_C(\rho^A) \leq L_C(|\psi^{SA}\rangle).
\] (47)
For any mixed state $\rho^{SA}$, let $\rho^{SA} = \sum_i p_i |\psi_i^{SA}\rangle \langle \psi_i|$ be the optimal decomposition of $|\psi^{SA}\rangle$ belonging to the minimum in Eq. (19), we have
\[
L_C(\rho^{SA}) = \sum_i p_i L_C(|\psi_i^{SA}\rangle).
\] (48)
Combining Eq. (17) and Eq. (48), we obtain
\[
L_C(\rho^{SA}) = \sum_i p_i L_C(|\psi_i^{SA}\rangle) \geq \sum_i p_i L_C(\rho_i^S) + \sum_i p_i L_C(\rho_i^A) \geq L_C(\rho^S) + L_C(\rho^A).
\] (49)
This completes the proof of the proposition. \(\square\)

From the proof of the proposition, we immediately see that the Schmidt number $r$ does not exceed the numbers $r_S$ and $r_A$, i.e.,
\[
r \leq \min\{r_S, r_A\}.
\] (50)
Thus, we can obtain an interesting relation between entanglement and coherence as follows,
\[
\max\{L_C(\rho^S), L_C(\rho^A)\} + L_E(|\psi^{SA}\rangle) \leq L_C(|\psi^{SA}\rangle),
\] (51)
where $L_E(|\psi^{SA}\rangle)$ is the Schmidt number, which is defined in \(\mathbb{R}\), and $L_E(|\psi^{SA}\rangle) = \log_2 r$. Note that the equality holds if and only if the matrix $M$ is diagonal matrix.

This relation shows that the sum between the entanglement and coherence contained in one subsystem can be not more than the total coherence. This relation can be generalized to the mixed states, for any bipartite mixed state $\rho^{SA}$, we have
\[
\max\{L_C(\rho^S), L_C(\rho^A)\} + L_E(\rho^{SA}) \leq L_C(\rho^{SA}).
\] (52)
Here, $L_E(\rho^{SA})$ is the Schmidt number of mixed state, which is defined as
\[
L_E(\rho^{SA}) = \min_{\{p_i, |\psi_i^{SA}\rangle\}} \sum_i p_i L_E(|\psi_i^{SA}\rangle),
\] (53)
where the minimum is taken over all pure state decompositions of $\rho^{SA} = \sum_i p_i |\psi_i^{SA}\rangle \langle \psi_i|$.

In fact, our results \(48\) and \(42\) are also generalize to the multipartite setting. Let $\rho^{SA_1 \cdots A_N}$ be a $N + 1$-partite states, by the repeated use of the super-additivity, we have
\[
L_C(\rho^S) + \sum_{i=1}^N L_C(\rho^{A_i}) \leq L_C(\rho^{A_1 \cdots A_N}).
\] (54)
Combining Eq. (52) and Eq. (54), we have
\[
L_E(\rho^{SA_1 \cdots A_N}) + \sum_{i=1}^N L_C(\rho^{A_i}) \leq L_C(\rho^{A_1 \cdots A_N}),
\] (55)
where $L_E(\rho^{SA_1 \cdots A_N})$ is the Schmidt number with the bipartite cut $S |A_1 \cdots A_N$.

Finally, it is interesting to compare the logarithmic coherence number with the Schmidt number. We consider quantum-incoherent state, which has the following form
\[
\chi^{SA} = \sum_i p_i |i^S\rangle \langle i| \otimes \rho_i^A,
\] (56)
where \( \rho_i^A \) are arbitrary quantum states on \( A \), and the states \( \ket{i^S} \) belong to the local incoherent basis of \( S \) \cite{13}. For any quantum-incoherent state, we easily obtain that the Schmidt number is zero, i.e.,

\[
\mathcal{L}_E(\rho^{SA}) = 0. \tag{57}
\]

But, we can also obtain

\[
\mathcal{L}_C(\chi^{SA}) \leq \sum_i p_i \mathcal{L}_C(\rho_i^A). \tag{58}
\]

We note that the minimum in \( \mathcal{L}_C(\chi^{SA}) \) depends only on the pure decomposition of \( \rho_i^A \), without loss of generality, let \( \chi^{SA} = \sum_{ij} p_i q_j |i^S \rangle \langle i| \otimes |\psi_{ij}^A \rangle \langle \psi_{ij}| \) be the optimal decomposition of \( \chi^{SA} \) belonging to the minimum in Eq. \( \text{(19)} \), we have

\[
\mathcal{L}_C(\chi^{SA}) = \sum_{ij} p_i q_j \mathcal{L}_C(|\psi_{ij}^A \rangle \langle \psi_{ij}|) \\
= \sum_i p_i \sum_j q_j \mathcal{L}_C(|\psi_{ij}^A \rangle \langle \psi_{ij}|) \\
\geq \sum_i p_i \mathcal{L}_C \left( \sum_j q_j |\psi_{ij}^A \rangle \langle \psi_{ij}| \right) \\
= \sum_i p_i \mathcal{L}_C(\rho_i^A). \tag{59}
\]

Combining Eq. \( \text{(58)} \) and Eq. \( \text{(59)} \), we have

\[
\mathcal{L}_C(\chi^{SA}) = \sum_i p_i \mathcal{L}_C(\rho_i^A). \tag{60}
\]

\section{VI. CONVERTING COHERENCE TO ENTANGLEMENT}

In this section, using the logarithmic coherence number, we discuss the relation between the coherence of a mixed state \( \rho^S \) in an initial system \( S \) with the entanglement generated from \( \rho^S \) by attaching an ancilla system \( A \) and taking an incoherent operation \( \Lambda^{SA} \) on the bipartite system \( SA \). Based on different measures, some authors have been investigated as well \cite{6, 8, 13, 14}.

\textbf{Proposition 7.} The entanglement generated from a state \( \rho^S \) via an incoherent operation \( \Lambda^{SA} \) is bounded above by the logarithmic coherence number, i.e.,

\[
\mathcal{L}_C(\rho^S) \geq \mathcal{L}_E(\Lambda^{SA}(\rho^S \otimes |0^A \rangle\langle 0^A|)). \tag{61}
\]

\textbf{Proof.} Let \( |0^A \rangle\langle 0^A| \) be an incoherent state on \( A \), then we have

\[
\mathcal{L}_C(\rho^S) = \mathcal{L}_C(\rho^S \otimes |0^A \rangle\langle 0^A|) \\
\geq \mathcal{L}_C(\Lambda^{SA}(\rho^S \otimes |0^A \rangle\langle 0^A|)) \\
= \sum_k \lambda_k \mathcal{L}_C(|\psi_k^S \rangle\langle \psi_k^S|) \\
\geq \sum_k \lambda_k \mathcal{L}_E(|\psi_k^S \rangle\langle \psi_k^S|) \\
= \mathcal{L}_E(\Lambda^{SA}(\rho^S \otimes |0^A \rangle\langle 0^A|)), \tag{62}
\]

where the second equality comes from the fact that \( \Lambda^{SA}(\rho^S \otimes |0^A \rangle\langle 0^A|) = \sum_k \lambda_k |\psi_k^S \rangle\langle \psi_k^S| \) is an optimal pure states decomposition of \( \Lambda^{SA}(\rho^S \otimes |0^A \rangle\langle 0^A|) \) belonging to the minimum in Eq. \( \text{(19)} \), and the second inequality depends on the fact that the coherence rank is greater than or equal to the Schmidt rank. \( \square \)

From the results in \cite{6, 8, 13, 14}, we know that a unitary operation which makes the coherence rank and the Schmidt number equal is given by

\[
U = \sum_{i=0}^{d-1} \sum_{j=i}^{d-1} |i^S \rangle \langle i| \otimes |i \oplus (j-1)^A \rangle \langle j|, \tag{63}
\]

where \( \oplus \) mens an addition modulo \( d \). Let

\[
|\psi^S \rangle = \sum_i \lambda_i |i^S \rangle \tag{64}
\]

be a pure state on \( S \). The unitary operation maps the state \( |\psi^S \rangle \otimes |0^A \rangle \) to the state

\[
U(|\psi^S \rangle \otimes |0^A \rangle) = \sum_i \lambda_i |i^S \rangle|i^A \rangle. \tag{65}
\]

Then we easily obtain

\[
\mathcal{L}_C(|\psi^S \rangle) = \mathcal{L}_E(U(|\psi^S \rangle \otimes |0^A \rangle)). \tag{66}
\]

Similar to the result \cite{6}, we can extend it to the general case of mixed states as follows.

\textbf{Proposition 8.} There exists an isometry \( W : \mathcal{H}^S \rightarrow \mathcal{H}^S \otimes \mathcal{H}^A \) such that for any state \( \rho^S \) on \( S \), we have

\[
\mathcal{L}_C(\rho^S) = \mathcal{L}_E(W \rho^SW^\dagger). \tag{67}
\]

\textbf{Proof.} Let \( \{ |i \rangle \} \) be an orthonormal basis and \( |a \rangle \) be any state in \( \mathcal{H}^A \), one can define

\[
W = \sum_i K_i \otimes |i \rangle \langle 0|. \tag{68}
\]

Then we have \( W^\dagger W = I \otimes |0 \rangle \langle 0| \). Note that there exists a unitary operation \( U \) such that \( W = U(I \otimes |0 \rangle \langle 0|) \). In particular, we take the unitary operation given in \( \text{(63)} \). Let \( \rho = \sum \lambda_i^* |\psi_i^S \rangle \langle \psi_i^S| \) be a decompositions for which the minima in Eq. \( \text{(19)} \) is attained. Since the operation \( I \otimes |0 \rangle \langle 0| \) does not effect the Schmidt number, for any state \( |\psi_i^S \rangle \), using Eq. \( \text{(60)} \), then we have

\[
\mathcal{L}_C(|\psi_i^S \rangle) = \mathcal{L}_E(W|\psi_i^S \rangle). \tag{69}
\]

We know that there exists a one-to-one correspondence between the pure states decompositions of \( \rho \) and the decompositions of \( \rho' = W \rho^SW^\dagger \) for given \( W \), then we obtain \( \{ \lambda_i^* W|\psi_i^S \rangle \} \) will form an optimal pure-state decomposition of \( \rho' \), and

\[
\mathcal{L}_C(\rho) = \sum_i \lambda_i^* \mathcal{L}_C(|\psi_i^S \rangle) \\
= \sum_i \lambda_i^* \mathcal{L}_E(W|\psi_i^S \rangle) \\
= \mathcal{L}_E(W \rho^SW^\dagger). \tag{70}
\]

This completes the proof of the proposition. \( \square \)
VII. CONCLUSIONS

We have introduced a new coherence measure of coherence, the logarithmic coherence number, which is generalized from the Schmidt measure and coherence rank. We have shown that the logarithmic coherence number is a proper coherence measure. We have also proved the logarithmic coherence number is additive but not continuous. In particular, we have found that the logarithmic coherence number is computable for a large class of states. We have shown that the logarithmic coherence number satisfies the super-additivity, and obtained the relationship between coherence and entanglement via our presented measures. The results can be also extended to multipartite setting. We have shown that the creation of entanglement with bipartite incoherent operations is bounded by the logarithmic coherence number of the initial system during the process. Some interesting results are given. We hope this measure of coherence will improve the understanding of quantum resource theory.

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