AdS/CFT correspondence, quasinormal modes, and thermal correlators in $\mathcal{N} = 4$ SYM

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Abstract: We use the Lorentzian AdS/CFT prescription to find the poles of the retarded thermal Green’s functions of $\mathcal{N} = 4$ $SU(N)$ SYM theory in the limit of large $N$ and large ’t Hooft coupling. In the process, we propose a natural definition for quasinormal modes in an asymptotically AdS spacetime, with boundary conditions dictated by the AdS/CFT correspondence. The corresponding frequencies determine the dispersion laws for the quasiparticle excitations in the dual finite-temperature gauge theory. Correlation functions of operators dual to massive scalar, vector and gravitational perturbations in a five-dimensional AdS-Schwarzschild background are considered. We find asymptotic formulas for quasinormal frequencies in the massive scalar and tensor cases, and an exact expression for vector perturbations. In the long-distance, low-frequency limit we recover results of the hydrodynamic approximation to thermal Yang-Mills theory.
1. Introduction

Studies of gauge theory/gravity duality at nonzero temperature and density provide interesting information about both thermal Yang-Mills theory at strong coupling and the physics of black holes/branes in asymptotically AdS space. A simple yet sufficiently rich example of such a duality in four dimensions is given by the correspondence between thermal $\mathcal{N} = 4$ $SU(N)$ supersymmetric Yang-Mills (SYM) theory at large $N$ and large 't Hooft coupling and the near horizon limit of the gravitational background created by a collection of $N$ parallel nonextremal three-branes. This background is equivalent to the one of the AdS-Schwarzschild black hole of infinitely large radius [1].

Superconformal Yang-Mills theory at finite temperature has only one scale (the temperature itself), and thus it cannot be regarded as an adequate model of realistic thermal gauge theories. Nevertheless, the dual AdS-Schwarzschild background has been explored as a good laboratory for studying non-perturbative features of finite-temperature field theory: entropy [2, 3] and transport coefficients [4, 5] were computed, and the hydrodynamic approximation was used to provide a quantitative check of the gauge theory/gravity duality in the absence of supersymmetry [6].

From a technical point of view, the five-dimensional AdS-Schwarzschild background is a difficult territory. Wave equations in this background essentially reduce to the Heun
differential equation, a second order ordinary differential equation with four regular singularities, whose generic solutions are not known explicitly. Accordingly, computing even the simplest thermal two-point function via AdS/CFT remains beyond the reach unless some simplifying approximations are made. In the absence of Lorentz symmetry at finite temperature, two natural parameters arise, given by $\omega = \omega/2\pi T$ and $q = q/2\pi T$, where $\omega$ is the frequency of fluctuations and $q$ is the magnitude of the spatial momentum. Wave equations can then be solved in the low- ($\omega, q \gg 1$) and high- ($\omega, q \ll 1$) temperature limits \[7\].

The finite-temperature correlators of $\mathcal{N} = 4$ SYM operators dual to scalar, vector and tensor fluctuations in the AdS-Schwarzschild background were computed in \[5, 6\] in the high-temperature (hydrodynamic) approximation using the Lorentzian AdS/CFT prescription proposed in \[8\] and recently justified\(^1\) and generalized in \[9\] implementing the earlier ideas of \[11\], \[12\]. In this approach, finding poles of the retarded thermal correlators is equivalent to computing quasinormal frequencies of a dual perturbation in the AdS-Schwarzschild background (see Section 2 of the present paper). In the hydrodynamic approximation, the thermal R-current and the stress-energy tensor correlators computed from gravity exhibit poles with dispersion relations $\omega = \omega(q)$ predicted by field theory (more precisely, by relativistic fluid mechanics \[13\]). For example, one of the AdS-Schwarzschild gravitational quasinormal frequencies reads \[1\]

$$\omega = \frac{q}{\sqrt{3}} - \frac{iq^2}{3} + O(q^3). \tag{1.1}$$

This dispersion relation is in perfect agreement with the one found in the low frequency, long wavelength regime of the dual thermal field theory, where the retarded correlator of the appropriate components of the stress-energy tensor has a pole at

$$\omega(q) = v_s q - \frac{i}{2} \frac{1}{\epsilon + P} \left( \zeta + \frac{4}{3} \eta \right) q^2, \tag{1.2}$$

where for $\mathcal{N} = 4$ SYM theory $\epsilon = 3$, $P = 3\pi^2 N^2 T^4 / 8$, $\eta = \pi N^2 T^3 / 8$, $\zeta = 0$, and $v_s = 1/\sqrt{3}$ is the speed of sound.

In the same regime, finite temperature Green’s functions of operators dual to the minimally coupled massless scalar do not have poles. In this last case, going beyond the high-temperature approximation reveals a presumably infinite sequence of poles in the complex $\omega$ plane corresponding to damped thermal quasiparticle excitations of $\mathcal{N} = 4$ SYM plasma \[23\]. In the low temperature limit, the poles merge, forming branch cuts exhibited by zero temperature correlators.

\(^1\)An alternative justification for the prescription was suggested in \[10\].
In this paper, we generalize the work of [25] to include R-current and stress-energy tensor correlators, as well as the correlators of gauge-invariant operators dual to massive scalar modes in the AdS-Schwarzschild background. We combine analytical and numerical methods to compute the discrete spectrum of quasinormal frequencies and obtain their asymptotic behavior as well as their dependence on the spatial momentum $q$. For a vector perturbation at vanishing $q$ we find a simple analytic solution for the modes, given by Heun polynomials. In this case, the spectrum is known exactly and the thermal R-current correlator poles are determined explicitly. Spectra of quasinormal frequencies similar to the one found in the vector case are also observed for massive scalar and gravitational perturbations. Typically, frequencies stay bounded from zero, and do not show up in the hydrodynamic regime as poles of the correlators in the dual CFT. In some cases, however, a special stand alone frequency with $\text{Re} w = 0$ appears whose $w, q \ll 1$ limit coincides with the analytic expression for the diffusion pole of the retarded correlators computed in [5] in the hydrodynamic approximation.

We organize the paper as follows. In Section 2, we propose a general definition for quasinormal modes of various perturbations in an asymptotically AdS background, with boundary conditions determined by the AdS/CFT correspondence. Quasinormal frequencies of massive scalar fluctuations in AdS-Schwarzschild geometry are determined in Section 3. In Section 4 we find vector quasinormal frequencies corresponding to the poles of thermal R-current correlators. The poles of certain components of the finite-temperature stress-energy tensor correlators are determined in Section 5. Our conclusions are presented in Section 6.

2. A definition of quasinormal modes in asymptotically AdS space

Quasinormal modes are classical perturbations with non-vanishing damping propagating in a given gravitational background subject to specific boundary conditions. When the geometry is asymptotically flat, the choice of boundary conditions is physically well motivated: no classical radiation is supposed to emerge from the (future) horizon, and no radiation originates at spatial infinity where an observer is waiting patiently to detect an outcome of some violent gravitational event (for recent reviews and references on quasinormal modes in asymptotically flat space see [14]).

In the case of asymptotically AdS space, one has less basis for intuition. Quasinormal modes in the relevant geometry have been studied in many publications [15]-[29].

\footnote{In this paper, we do not consider correlators exhibiting the sound wave pole \cite{6}. Their treatment requires more complicated analysis.}
and various boundary conditions defining quasinormal modes in asymptotically AdS space were suggested in the literature. At the horizon, the condition is clearly the same as in the asymptotically flat case (no outgoing waves). Since AdS space effectively acts as a confining box\(^3\), a natural choice of the condition at the boundary would seem to be a Dirichlet one, at least for a scalar perturbation. However, as noted in [22] for the BTZ case, and as we shall see in Section 3 of the present paper, the Dirichlet condition is not adequate for certain values of the mass parameter. It is also not a suitable condition for vector and gravitational perturbations. Another condition, used for perturbations of the BTZ black hole in [22], [28], is the vanishing flux boundary condition. It gives the correct BTZ quasinormal frequencies, including those cases when the Dirichlet condition fails. It would be interesting to understand the meaning of the vanishing flux condition from the AdS/CFT point of view\(^4\), as well as to check it in higher-dimensional examples. An important observation made in [22] was that the BTZ quasinormal frequencies coincide with the poles (in the complex frequency plane) of the retarded correlators in the boundary CFT. An explanation of this fact as being one of the consequences of the Lorentzian AdS/CFT prescription was provided in [8]. Following the logic of [8], here we propose a pragmatic general definition of quasinormal frequencies which directly follows from the Lorentzian signature AdS/CFT correspondence:

**Quasinormal frequencies of a perturbation in an asymptotically AdS space are defined as the locations in the complex frequency plane of the poles of the retarded correlator of the operators dual to that perturbation, computed using the Minkowski AdS/CFT prescription of [8].**

We stress that the implementation of this definition involves only gravity calculations. In the following sections, we use it to find quasinormal frequencies of massive scalar, vector and gravitational perturbations in a five-dimensional AdS-Schwarzschild background.

3. Quasinormal frequencies of massive scalar perturbations

According to the definition given in the previous section, computing quasinormal frequencies of a massive scalar perturbation of the near-extremal black three-brane background is equivalent to finding the poles of the retarded Green’s function of gauge invariant operators in thermal \(\mathcal{N} = 4\) SYM dual to that perturbation. Our approach

\(^3\)This can be seen by writing the radial part of the Klein-Gordon equation in the Schrödinger form and considering the corresponding effective potential

\(^4\)This condition was originally used for quantization of a scalar field in a pure AdS space in global coordinates [30].
to solving this problem will be similar to the one used in [25] where the massless case was considered.

The asymptotically AdS five-dimensional part of the metric corresponding to the collection of \(N\) parallel black three-branes in the near-horizon limit is given by

\[
\begin{align*}
ds^2 &= \frac{r^2 R^2}{r^2 f} \left(-f \, dt^2 + d\mathbf{x}^2\right) + \frac{R^2}{r^2 f} dr^2, \\
&= r^2 R^2 \left(-f \, dt^2 + d\mathbf{x}^2\right) + R^2 r^2 f \, dr^2,
\end{align*}
\]

where \(f(r) = 1 - \frac{r_0^4}{r^4}\), \(r_0\) being the parameter of non-extremality related to the Hawking temperature \(T = \frac{r_0}{\pi R^2}\). This background is dual to the \(\mathcal{N} = 4\) SU\((N)\) SYM at finite temperature \(T\) in the limit \(N \to \infty\), \(g_N^2 N \to \infty\).

Using the coordinate \(z = 1 - \frac{r_0^2}{r^2}\) and the Fourier decomposition

\[
\phi(z, t, \mathbf{x}) = \int \frac{d^4 k}{(2\pi)^4} e^{-i(\omega t + i\mathbf{k} \cdot \mathbf{x})} \phi_k(z),
\]

one can write the wave equation for the minimally coupled massive scalar in the background (3.1) as

\[
\phi''_k + \left[\frac{1}{1 - z} + \frac{\omega^2}{2} + \frac{q^2}{z(1 - z)(2 - z)} - \frac{m^2}{z(1 - z)^2(2 - z)}\right] \phi_k = 0,
\]

where \(\omega = \omega / 2\pi T\), \(q = |\vec{k}| / 2\pi T\), \(m = m R / 2\). Eq. (3.3) has four regular singularities at \(z = 0, 1, 2, \infty\), the characteristic exponents being respectively \(-i\omega / 2\), \(i\omega / 2\); \(1 - \sqrt{1 + m^2}, 1 + \sqrt{1 + m^2}\); \(-\omega / 2, \omega / 2\); \(0, 0\). In our coordinates, the horizon is located at \(z = 0\), the boundary at \(z = 1\). The mass parameter \(m\) is related to the scaling dimension \(\Delta\) of the operator \(\mathcal{O}\) in the dual CFT via

\[
\Delta = \begin{cases} \\
\Delta_-, & \Delta \in [1, 2], \\
\Delta_+, & \Delta \in [2, \infty), \end{cases}
\]

where

\[
\Delta_\pm = 2 \left(1 \pm \sqrt{1 + m^2}\right).
\]

Eq. (3.3) can be written in the standard form of the Heun equation,

\[
y'' + \left[\gamma + \frac{\delta}{z - 1} + \frac{\epsilon}{z - 2}\right] y' + \frac{\alpha \beta z - Q}{z(z - 1)(z - 2)} y = 0,
\]

\[5\] The branch \(\Delta_-\) of scaling dimensions does not arise in \(\mathcal{N} = 4\) SYM. However, for completeness we treat \(\Delta\) as a continuous variable defined in the interval \(\Delta \in [1, \infty)\), where \(\Delta = 1\) is the scalar unitarity bound in \(d = 4\).
by making the following transformation of the dependent variable
\[
\phi(z) = z^{-\frac{w}{2}} (z - 1)^{2-\frac{w}{2}} (z - 2)^{-\frac{w}{2}} y(z). \tag{3.7}
\]

The parameters of the Heun equation are constrained by the relation \(\gamma + \delta + \epsilon = \alpha + \beta + 1\).
In the massive scalar case they are given by
\[
\alpha = \beta = -\frac{w(1+i)}{2} + 2 - \frac{\Delta}{2}, \quad \gamma = 1 - i w, \quad \delta = 3 - \Delta, \quad \epsilon = 1 - w. \tag{3.8}
\]

The “accessory parameter” \(Q\) is
\[
Q = q^2 - \frac{w(1-i)}{2} - \frac{w^2(2-i)}{2} + \left(2 - \frac{\Delta}{2}\right) \left(2 - \frac{\Delta}{2} - 2iw\right). \tag{3.9}
\]

The characteristic exponents of Eq. (3.6) are \((0, 1-\gamma)\) at \(z = 0\), and \((0, 1-\Delta)\) at \(z = 1\).
With the parameters given by Eq. (3.8), the exponents become \((0, i w)\) and \((0, \Delta - 2)\), respectively.

At \(z = 0\), the local series solution corresponding to the exponent 0 is given by
\[
y_0(z) = \sum_{n=0}^{\infty} a_n(w, q) z^n, \tag{3.10}
\]
where \(a_0 = 1\), \(a_1 = Q/2\gamma\), and the coefficients \(a_n\) with \(n \geq 2\) obey the three-term recursion relation
\[
a_{n+2} + A_n(w, q) a_{n+1} + B_n(w, q) a_n = 0, \tag{3.11}
\]
where
\[
A_n(w, q) = -\frac{(n+1)(2\delta + \epsilon + 3(n + \gamma)) + Q}{2(n+2)(n+1+\gamma)}, \tag{3.12}
\]
\[
B_n(w, q) = \frac{(n + \alpha)(n + \beta)}{2(n+2)(n+1+\gamma)}. \tag{3.13}
\]

The local solution at \(z = 0\), Eq. (3.10), is expressed as a linear combination of the two local solutions at \(z = 1\) as
\[
y_0(z) = A y_1(z) + B y_2(z), \tag{3.14}
\]
where for integer \(\Delta\)
\[
y_1(z) = (1 - z)^{\Delta - 2}(1 + \ldots), \tag{3.15}
\]
\[
y_2(z) = 1 + \cdots + h(w, q)y_1(z) \log (1 - z), \tag{3.16}
\]
where ellipses denote terms of order $1 - z$ and higher, and $A, B$ are the elements of the monodromy matrix of the Heun equation. For noninteger $\Delta$ the logarithmic term in Eq. (3.16) is absent.

To compute the retarded Green’s function of the operator $O_{\Delta}$ dual to the perturbation $\phi(z)$, one chooses the solution of Eq. (3.6) with the exponent 0 at $z = 0$ (this corresponds to the incoming wave condition for $\phi(z)$ at the horizon) and proceeds according to [8]. The correlator is then proportional to $A/B$ and thus finding its poles is equivalent to finding zeros of the coefficient $B$. For all $\Delta \in [1, \infty)$ except $\Delta = 2$ the latter condition is in turn equivalent to the vanishing Dirichlet boundary condition at $z = 1$. Consequently, one may look for the poles (or quasinormal frequencies) simply by approximating an exact solution by a finite sum, and solving the equation

$$y_0(1) \approx \sum_{n=0}^{N} a_n(w, q) = 0$$

(3.17)

numerically, provided the series (3.10) converges at $|z| = 1$. Asymptotic analysis of the large $n$ behavior of the coefficients $a_n$ shows that for any value of $w, q$ the series (3.10) is absolutely convergent at $|z| = 1$ for $\Delta > 2$ (note that with the definition (3.4) $\Delta \in [1, 2)$ is equivalent to $\Delta \in (2, 3]$). In most cases, this approach works very well. In practice, however, one usually needs to evaluate a large number of terms to achieve a good accuracy. Another difficulty is that for $\Delta = 2$ the series is logarithmically divergent.

For integer conformal dimensions, there exists an alternative approach based on rapidly converging continued fractions. One may notice that whenever the second solution $y_2(z)$ contains logarithmic terms (i.e. whenever the coefficient $h(w, q)$ in Eq. (3.10) is nonzero), the condition $B = 0$ is equivalent to the requirement of analyticity of the solution (3.10) at $z = 1$. This condition of analyticity translates (see [25]) into the requirement for the spectral parameter $w(q)$ to obey the transcendental continued fraction equation

$$Q = -\frac{B_0(w, q)}{A_0(w, q)} \frac{B_1(w, q)}{A_1(w, q)} \frac{B_2(w, q)}{A_2(w, q)} \cdots$$

(3.18)

More generally, in this case the coefficients $a_n$ obey

$$\frac{a_{n+1}}{a_n} = -\frac{B_n(w, q)}{A_n(w, q)} \frac{B_{n+1}(w, q)}{A_{n+1}(w, q)} \frac{B_{n+2}(w)}{A_{n+2}(w, q)} \cdots$$

(3.19)

Thus, scalar quasinormal frequencies are the solutions $w = w(q)$ of the eigenvalue equation (3.18) modulo those values of $w$ for which the solution $y_2(z)$ of the Heun

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\(^6\)Actually, one can use Eq. (3.17) to estimate the values of frequencies even in that case, but the accuracy is very limited.
equation is free from logarithms. We call the latter “false frequencies” since even though they automatically appear as roots of Eq. (3.18), they do not correspond to poles of the thermal gauge theory correlators.

The real and imaginary parts of the solutions to the continued fraction equation (3.18) are shown in Figures 1 and 2 as functions of the continuously varying parameter \(\Delta\). For integer \(\Delta\), the false frequencies are indicated by blank ellipses.

The values of the false frequencies can be found analytically following the standard procedure first described by Fröbenius (see e.g. §16.33 of [31]). Here we only list the results for the lowest values of \(\Delta\). The false frequencies are the solutions to the algebraic equations

\[
\Delta = 3 \quad w^2 - q^2 = 0, \\
\Delta = 4 \quad (w^2 - q^2)^2 = 0, \\
\Delta = 5 \quad (w^2 - q^2)^3 + 3w^2 + q^2 = 0, \\
\Delta = 6 \quad w^8 - 4q^2 w^6 + 2(3 + q^4)(3w^4 - 2q^2 w^2) + (q^4 - 3)^2 = 0,
\]

and so on. There are no false frequencies for \(\Delta = 2\), and the \(\Delta = 1\) set is equivalent to that of \(\Delta = 3\). For \(q = 0\) the expressions are less cumbersome, and we list more of them:

\[
\Delta = 3 \quad w^2 = 0, \\
\Delta = 4 \quad w^4 = 0, \\
\Delta = 5 \quad w^2(3 + w^4) = 0, \\
\Delta = 6 \quad 9 + 18w^4 + w^8 = 0, \\
\Delta = 7 \quad w^2(252 + 63w^4 + w^8) = 0, \\
\Delta = 8 \quad w^4(3024 + 168w^4 + w^8) = 0, \\
\Delta = 9 \quad w^2(72900 + 21105w^4 + 378w^8 + w^{12}) = 0, \\
\Delta = 10 \quad 893025 + 1803060w^4 + 104454w^8 + 756w^{12} + w^{16} = 0.
\]

Knowing values of the false frequencies exactly allows us to check the accuracy of the continued fraction method. For example, \(w = 0\), \(w = \pm(-3)^{1/4} \approx \pm0.93060485910 \pm 0.93060485910i\) and \(w = \pm i(-3)^{1/4} \approx \mp0.93060485910 \pm 0.93060485910i\) are the solutions of Eq. (3.21c). Solving the continued fraction equation (3.18) numerically, all significant figures shown above are reproduced correctly (and higher accuracy can be achieved, if desired). These checks give us confidence that the values of quasinormal frequencies reported in [25] and in this paper are determined correctly.

Nevertheless, numerical difficulties in solving the continued fraction equation remain. For large \(|w|\) and/or \(\Delta\) our algorithm suffers from instability which we were not able to overcome.

\[\text{\footnote{Nevertheless, numerical difficulties in solving the continued fraction equation remain. For large }|w|\text{ and/or }\Delta\text{ our algorithm suffers from instability which we were not able to overcome.}}\]
Another check can be made by considering the \( \omega = 0, \, \mathbf{q} = 0 \) limit, in which an analytic solution to Eq. (3.6) is available. This case is discussed in the Appendix.

Solid lines in Figures 1 and 2 represent solutions to Eq. (3.18) for non-integer \( \Delta \). These are also false frequencies, since in this case the non-analytic part of the solution is given by Eq. (3.13) rather than by Eq. (3.10) (the logarithmic term is absent in Eq. (3.16) for non-integer \( \Delta \)) and thus the continued fraction equation determines zeros rather than poles of the correlators. The true quasinormal frequencies for non-integer \( \Delta \) can be found using Eq. (3.17).

Taking all these subtleties into account, we present our results for scalar quasinormal frequencies in Figures 3, 4 and in Table 1 for \( \mathbf{q} = 0 \), and in Figures 5 and 6 for nonzero \( \mathbf{q} \). Frequencies appear in symmetric pairs \((\pm \text{Re}\, \omega_n, \text{Im}\, \omega_n)\), as reported in Table 1. (We do not show the symmetric \( \text{Re}\, \omega_n < 0 \) branch in most of our figures.) We observe the following properties of the spectrum:

- Despite the appearance, the solid lines in Figures 3, 4 are not straight lines. For zero spatial momentum \( \mathbf{q} \), the \( n \)-th frequency is given by

  \[
  \omega_n = \left( n + \frac{\Delta - 3}{2} \right) (1 - i) + \epsilon(n, \Delta), \quad n = 1, 2, \ldots ,
  \]

  where

  \[
  \epsilon(n, \Delta) = \begin{cases} \omega_*(\Delta) + O(1/n^{\alpha_1}), & n \to \infty, \\ O(1/\Delta^{\alpha_2}), & \Delta \to \infty. \end{cases}
  \]

  Formula (3.22) generalizes the asymptotic expression found in [25] for the massless \( (\Delta = 4) \) case. The asymptotic parameters \( \omega_*(\Delta), \, \alpha_{1,2} \) can in principle be computed numerically, but certainly a genuine analytic asymptotic expansion confirming (3.22) is highly desirable.

- The magnitude of the imaginary part of quasinormal frequencies increases with \( \Delta \) increasing, in agreement with the intuitive expectation that the late time behavior of thermal excitations should be dominated by the small \( \Delta \) contributions. Note that none of the frequencies lies in the region \( \omega \ll 1, \, \mathbf{q} \ll 1 \). This means that none of the \( \mathcal{N} = 4 \) SYM operators dual to massive scalar fields exhibits hydrodynamic behavior similar to the one described in [5, 6].

Recently, a progress has been made in obtaining asymptotics of the quasinormal spectrum for the Schwarzschild black hole in asymptotically flat space [32, 33]. The approach used in [32] does not seem to work for the Heun equation, while the one employed in [33] is promising. Another analytic attempt [34] (based on Ikeda’s approximation [35]) is not quite adequate due to the lack of a small parameter.
• For all conformal dimensions, the dependence of quasinormal frequency on $q$ is qualitatively the same (see Figures 3 and 4). This dependence is interpreted as a dispersion relation for quasi-particle excitations in a strongly interacting thermal gauge theory. Strictly speaking, however, our results apply only in the limit of infinite $N$ and infinite 't Hooft coupling.

• The spectrum (3.22) bears resemblance to spectra of scalar quasinormal frequencies of the two asymptotically AdS backgrounds where the modes can be found exactly, the (2 + 1)-dimensional BTZ black hole [20], [19], [21] and the 4d massless topological black hole [27]. However, in Eq. (3.22) both the real and the imaginary parts of $w_n$ depend on $n$, whereas for the BTZ, 4d massless topological black hole, as well as for the Schwarzschild black hole in asymptotically flat space, $\text{Re } w_n$ is either independent of $n$ or reaches a finite limit as $n \to \infty$. The origin and significance of this difference and its possible role in the conjectured relation between the quasinormal spectrum and a black hole entropy in quantum gravity [36] are not clear to us. The fact that scalar quasinormal frequencies are approximately evenly spaced with $n$ was noticed by Horowitz and Hubeny [17] for the AdS-Schwarzschild black hole of large but finite radius.

The pattern of quasinormal frequencies observed for scalar modes will also manifest itself for vector and gravitational perturbations considered in subsequent sections. In that case, however, a new qualitatively different type of frequency will appear characterizing the low frequency, long wavelength behavior of a dual thermal field theory.

4. Poles of thermal R -current correlators in $\mathcal{N} = 4$ SYM

The retarded thermal Green’s functions of R-currents in $\mathcal{N} = 4$ SYM in the large $N$, large ’t Hooft coupling limit were computed in [3] in the so-called hydrodynamic approximation, i.e. when the frequency and the spatial momentum are much smaller than the temperature $T$. Defining the retarded Green’s function in the usual way,

\[ G^{R}_{\mu\nu}(\omega, q) = -i \int d^4x e^{-iq\cdot x} \theta(t) \langle [j_\mu(x), j_\nu(0)] \rangle, \]  

(4.1)
one finds

\[
G_{xx}^{ab} = G_{yy}^{ab} = -\frac{iN^2T^2\omega\delta^{ab}}{8} + \cdots ,
\]  

(4.2a)

\[
G_{tt}^{ab} = \frac{N^2T^2q^2\delta^{ab}}{8(i\omega - q^2)} + \cdots ,
\]  

(4.2b)

\[
G_{tz}^{ab} = G_{zt}^{ab} = -\frac{N^2T^2\omega q\delta^{ab}}{8(i\omega - q^2)} + \cdots ,
\]  

(4.2c)

\[
G_{zz}^{ab} = \frac{N^2T^2\omega^2\delta^{ab}}{8(i\omega - q^2)} + \cdots ,
\]  

(4.2d)

where ellipses denote higher order perturbative corrections in \(\omega, q\).

The appearance of the diffusion pole in the correlation functions (4.2a) - (4.2d) in the limit of small \(\omega, q\) is predicted by hydrodynamics [5]. To the next order in perturbation theory, the position of the pole is given by

\[
\omega = -iq^2(1 + q^2\ln 2) + \cdots .
\]  

(4.3)

For generic values of frequencies and momenta, however, we expect additional poles to appear. We also expect the momentum dependence of the hydrodynamic pole in Eqs. (4.2b) - (4.2d) to be modified.

The approach we are following in computing thermal \(R\)-current correlators is described in detail in [5]. The correlators are determined by using the Minkowski AdS/CFT prescription and the relevant part of the action,

\[
S = -\frac{N^2}{32\pi^2R} \int dz \, d^4x \sqrt{-g} \, g^{zz}g^{ij} \partial_z A_i \partial_z A_j + \cdots
\]  

\[
= \frac{N^2T^2}{16} \int dz \, d^4x [A_t'^2 - f(A_x'^2 + A_y'^2 + A_z'^2)] + \cdots ,
\]  

(4.4)

where the components \(A_i\) satisfy the 5d Maxwell equations in the near-extremal background (B.1).

It turns out that each of the components \(A_x\) and \(A_y\) satisfies an equation identical to the one for the minimally coupled massless scalar. We immediately conclude that the pole structure of the correlators \(G_{xx}^{ab}\) and \(G_{yy}^{ab}\) is the same as the one studied in [25].

For the component \(A_t\) we have the following third-order equation

\[
A_t''' + \frac{3(1-z)^2 - 1}{(1-z)f} A_t'' + \frac{\omega^2 - q^2f(z)}{(1-z)f^2} A_t' = 0 ,
\]  

(4.5)
where \( f(z) = z(2-z) \). The components \( A_z, A_t \) and their derivatives are related by

\[
A'_z = -\frac{w}{qf} A'_t, \tag{4.6a}
\]

\[
A_z = \frac{(1-z)f}{wq} A''_t - \frac{q}{w} A_t. \tag{4.6b}
\]

Imposing the “incoming wave” boundary condition at the horizon, we find \( A'_t = z^{-i\omega/2} F(z) \), where \( F(z) \) is regular at \( z = 0 \). The function \( F \) obeys the equation

\[
F'' + \left( \frac{3(1-z)^2 - 1}{(1-z)f} - \frac{i\omega}{z} \right) F' + \frac{i\omega (3-2z)}{2(1-z)f} F + \frac{\omega^2 [4-(1-z)(2-z)^2]}{4(1-z)f^2} F - \frac{q^2}{(1-z)f} F = 0. \tag{4.7}
\]

A generic exact solution of Eq. (4.7) is beyond reach. (In Eq. (4.7) was solved perturbatively in the high-temperature (hydrodynamic) limit \( \omega \ll 1, q \ll 1 \).) Here, however, our goal is to determine for which \( \omega \) and \( q \) the retarded \( R \)-current correlators computed via AdS/CFT exhibit poles. This goal can be translated into the well posed boundary value problem as follows.

The Lorentzian AdS/CFT prescription suggests that the retarded two-point functions of \( R \)-currents in momentum space are obtained by differentiating the expression

\[
F = \frac{N^2 T^2}{16} A'_t \left( \bar{A}_t + \frac{w}{q} \bar{A}_z \right) \tag{4.8}
\]

with respect to \( \bar{A}_t \) and \( \bar{A}_z \) representing the boundary values of the components \( A_t, A_z \). (In writing Eq. (4.8) we used the constraint (4.6a).) Thus all nontrivial information about the correlators is contained in the boundary limiting value of the solution to Eq. (4.7) (considered as a functional of \( \bar{A}_t \) and \( \bar{A}_z \)).

The characteristic exponents of Eq. (1.7) at the boundary \( z = 1 \) are \( (0,0) \). It follows that the two local solutions at \( z = 1 \) are given by

\[
F^I(z) = a_0 + a_1 (1-z) + a_2 (1-z)^2 + \ldots , \tag{4.9a}
\]

\[
F^{II}(z) = F^I \log (1-z) + b_1 (1-z) + b_2 (1-z)^2 + \ldots , \tag{4.9b}
\]

where \( a_i, b_i \) are the coefficients of the Fröbenius expansion. The solution \( F(z) \) regular at \( z = 0 \) can be expressed as a linear combination of \( F^I(z) \) and \( F^{II}(z) \),

\[
F(z) = A F^I(z) + B F^{II}(z). \tag{4.10}
\]

Then, taking the limit \( z \to 1 \) in Eq. (4.6b) we get

\[
B a_0 = \omega q \bar{A}_z + q^2 \bar{A}_t.
\]
Therefore, for $z \sim 1$ the solution is represented by
\[ A'_t = \frac{A}{B} \left( wqA_z + q^2A_z \right) + \left( wqA_z + q^2A_z \right) \log (1 - z) + O(1 - z). \tag{4.11} \]

It then follows from Eq. (4.8) that the correlators are proportional to $A/B$ (constant and contact terms are ignored). Specifically,
\[ G^{ab}_{tt} = \frac{N^2 T^2 q^2 A \delta^{ab}}{8 B}, \tag{4.12a} \]
\[ G^{ab}_{tz} = G^{ab}_{zt} = -\frac{N^2 T^2 wq A \delta^{ab}}{8 B}, \tag{4.12b} \]
\[ G^{ab}_{zz} = \frac{N^2 T^2 w^2 A \delta^{ab}}{8 B}. \tag{4.12c} \]

Comparing with Eqs. (4.2b) - (4.2d), to the leading order in $w, q$ we have $B/A = iw - q^2$. In general, one looks for the poles of the correlators by demanding the condition $B/A = 0$. In other words, we should determine for which values of $w$ and $q$ the solution $F^{II}$ in Eq. (4.10) is absent or, equivalently, for which $w$ and $q$ the solution $F(z)$ is analytic at $z = 1$. This requirement provides us with the necessary boundary condition at $z = 1$. The problem thus essentially reduces to the one encountered in the scalar case in Section 3. By changing the dependent variable to $y(z) = (2 - z)^{w/2}F(z)$, Eq. (4.5) can be written in the standard form of the Heun equation (3.6) with parameters
\[ \alpha = -\frac{w(1 + i)}{2}, \quad \beta = 2 + \alpha, \quad \gamma = 1 - iw, \quad \delta = 1, \quad \epsilon = 1 - w, \tag{4.13a} \]
\[ Q = q^2 - \frac{w(1 + 3i)}{2} - \frac{w^2(2 - i)}{2}, \tag{4.13b} \]

and the boundary conditions requiring analyticity of the solution $y(z)$ at both ends of the interval $z \in [0, 1]$. As discussed in [25], the analyticity condition at $z = 1$ is satisfied for $w$ and $q$ obeying the continued fraction equation (3.18). Note that the problem of “false frequencies” encountered in Section 3 does not arise here since, the exponents at $z = 1$ being a multiple root of the indicial equation, the second solution of Eq. (4.7) is unavoidably logarithmic.

For $q = 0$, the solutions of Eq. (3.18) can be found analytically. Suppose that for some $n = n_*$ the coefficient $B_{n_*}(w)$ given by Eq. (3.13) vanishes, implying a constraint $w = w_{n_*}$. Then, as Eq. (3.13) shows, all the coefficients $a_n(w, q)$ with $n > n_*$ vanish, and the solution to our spectral problem is a polynomial of degree $n_*$, provided that the algebraic equation (with a finite continued fraction on the right hand side)
\[ \frac{Q}{2\gamma} = \frac{B_0(w)}{A_0(w) - A_1(w) - A_2(w)} \cdots \frac{B_{n_*-2}(w)}{A_{n_*-2}(w) - A_{n_*-1}(w)} \frac{B_{n_*-1}(w)}{A_{n_*-1}(w)}, \tag{4.14} \]
has $w = w_n$, among its solutions. With the parameters of the Heun equation given by Eqs. (4.13a), (4.13b), this occurs for

$$q = 0, \quad w_n = n(1 - i), \quad n = 0, 1, \ldots$$  \hspace{1cm} (4.15)

(Indeed, if $w$ and $q$ are given by Eq. (4.13), the coefficients $B_{n*}, B_{n-2}$ as well as the expression $A_{n-2} - B_{n-1}/A_{n-1}$ all vanish. Solving Eq. (4.14) “backwards”, one can see that (4.13) is in fact a solution.) In this case, the solutions to Eq. (3.6) are Heun polynomials, easily found using (3.11). The first five of them, normalized to 1 at $z = 0$, are given by

$$y_0 = 1, \quad \hspace{1cm} (4.16a)$$
$$y_1 = 1 - \frac{1 + i}{2} z, \quad \hspace{1cm} (4.16b)$$
$$y_2 = 1 - \frac{6 + 3i}{5} z + \frac{3 + 9i}{20} z^2, \quad \hspace{1cm} (4.16c)$$
$$y_3 = 1 - \frac{51 + 21i}{26} z + \frac{231 + 297i}{260} z^2 + \frac{11 - 88i}{260} z^3, \quad \hspace{1cm} (4.16d)$$
$$y_4 = 1 - \frac{68 + 26i}{25} z + \frac{108 + 111i}{50} z^2 - \frac{262 + 1179i}{850} z^3 - \frac{917 - 1441i}{6800} z^4. \quad \hspace{1cm} (4.16e)$$

The complementary sequence of solutions\(^9\) consists of the spectrum

$$q = 0, \quad w_n = -n(1 + i), \quad n = 0, 1, \ldots$$  \hspace{1cm} (4.17)

and an infinite set of polynomials

$$y_0 = 1, \quad \hspace{1cm} (4.18a)$$
$$y_1 = 1 - \frac{1 - i}{2} z, \quad \hspace{1cm} (4.18b)$$

etc. Heun polynomials can be regarded as local solutions of the Heun equation valid simultaneously at three singularities, the characteristic exponent at each singularity being zero.

We remark that the pattern $w_n \sim n(1 - i)$ had appeared in [25] and in Section 3 of the present paper as the conjectured large $n$ asymptotics of the scalar quasinormal frequencies. The above discussion suggests that Heun polynomials can be used as a

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\(^9\)Note that if $y(u)$ is a solution of Eq. (4.5) with the characteristic exponents $\nu_1 = -\mu_e/2$ at $u = 1$ and $\nu_0 = 0$ at $u = 0$ (corresponding to the spectral parameter $\mu = \mu_e \equiv i w$), then $\bar{y}(u)$ is also a solution with the same analyticity property and the spectral parameter $\mu = \bar{\mu}_e$. Thus the poles $\bar{w}_n$ come in pairs, distributed symmetrically with respect to the Im $w$ axis.
good approximation of the scalar large $n$ solution. The exact solution (4.2) also adds support to the claim [25] that the number of scalar quasinormal frequencies is infinite.

For $q \neq 0$, the solutions to our boundary value problem are Heun functions. The corresponding values of the spectral parameter $w_n(q)$ can be found numerically by solving Eq. (4.14). A typical distribution of poles in the complex $w$ plane is shown in Figure 6. In addition to the infinite symmetric sequence of poles familiar from the scalar case, there exists a special stand-alone “hydrodynamic” pole located on the negative imaginary axis. For this pole, the dispersion curve $w = w(q)$ calculated from Eq. (3.18) is shown in Figure 8 together with the analytic approximation (4.3) obtained in [3] in the limit of small $q$.

The lowest ten dispersion curves generalizing the sequence (4.15) to $q \neq 0$ are shown in Figures 9[10]. Their behavior is similar to the one observed in the case of scalar perturbations, except for the nontrivial “roton” minimum of $\text{Re} w_n(q)$ shown in detail in Figure 11.

5. Poles of thermal stress-energy tensor correlators.

In this Section, we shall find the poles of the retarded Green’s function for the components of the stress-energy tensor,

$$G_{\mu\nu,\lambda\rho}(w, q) = -i \int d^4 x e^{-i q \cdot x} \theta(t) \langle[T_{\mu\nu}(x), T_{\lambda\rho}(0)]\rangle.$$ (5.1)

We will focus specifically on the components of $T_{\mu\nu}$ whose correlators possess diffusion poles in the hydrodynamic regime of the theory. The procedure for computing these correlation functions from gravity is very similar to the one used in Section 4 for the $R$-current correlators. (The reader is also referred to [3] for details.) In the setup of [3], correlation functions having the diffusion pole correspond to a gravitational perturbation $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$ of the AdS-Schwarzschild background $g_{\mu\nu}^{(0)}$ with $h_{xx} \neq 0$, $h_{zz} \neq 0$. It turns out [3] that the correlators are essentially known once the solution to Eq. (3.4) with the parameters

$$\begin{align*}
\alpha &= -\frac{w(1 + i)}{2} - 1, \quad \beta = 3 + \alpha, \quad \gamma = 1 - iw, \quad \delta = 0, \quad \epsilon = 1 - w, \quad (5.2a) \\
Q &= q^2 - 2 - \frac{w(1 + i)}{2} - \frac{w^2(2 - i)}{2}. \quad (5.2b)
\end{align*}$$

The correlators are given by

\[ G_{tx,tx}(w, q) = \frac{N^2 \pi^2 T^4 q^2 A}{4 B}, \quad (5.3a) \]

\[ G_{tx,xz}(w, q) = -\frac{N^2 \pi^2 T^4 w q A}{4 B}, \quad (5.3b) \]

\[ G_{xz,xz}(w, q) = \frac{N^2 \pi^2 T^4 w^2 A}{4 B}, \quad (5.3c) \]

where \( A, B \) are the coefficients of the connection formula for the solutions of the Heun equation with parameters (5.2a), (5.2b). To the lowest order in \( w, q \) we had \( B/A = i w - q^2/2 \). For generic \( w \) and \( q \), the poles of the correlators are found by solving the eigenvalue equation (3.18). False frequencies are described by the equation \( w^2 - q^2 = 0 \).

By examining the set of coupled equations for the perturbations \( h_{tx}, h_{xz} \) (Eqs. (6.13a)-(6.13c) in [5]) we observe that in the limit \( q \to 0 \) equations decouple and, moreover, the only nontrivial equation left coincides with the one of the minimally coupled massless scalar in the background (3.1). We conclude that for \( q = 0 \) the spectrum is identical to the one of the \( \Delta = 4 \) scalar case (given in Table 1 and in [25]). Curiously, this coincidence does not seem to be obvious when comparing the parameters of the Heun equation in the two cases.

For \( q \neq 0 \), the distribution of poles in the complex \( w \) plane is qualitatively similar to the one shown in Figure 6. There is a “hydrodynamic” pole whose dispersion relation (see Figure 12) for small \( q \) is well approximated by the analytic result \( w = -i q^2/2 \). For other poles the dependence on \( q \) is shown in Figures 13, 14.

6. Conclusions

Computing quasinormal frequencies in asymptotically AdS space using the standard framework of general relativity may be interesting on its own right, but when the computation is motivated by the AdS/CFT correspondence what one really is interested in is a way to compute the Lorentzian signature correlators from gravity. In many cases this is technically difficult or impossible, and yet even in those cases computing the poles of the correlators may turn out to be relatively straightforward. In this paper we found the poles of the retarded correlators of the thermal \( \mathcal{N} = 4 \) SYM theory operators dual to scalar, vector and gravitational perturbations in the 5d AdS-Schwarzschild background.

Since our knowledge of the strong coupling regime of the theory obtained from the sources other than the AdS/CFT is very limited, the interpretation of our results is not obvious. We clearly see the emergence of a hydrodynamic behavior in the theory, but
the role of the infinite sequence of “quasi-Matsubara” frequencies $\omega_n \sim 2\pi T n (1 - i)$ is not clear. Equally, with temperature being the only scale in the theory, it is not clear whether the similarities in the spectrum for perturbations of different spins have an underlying algebraic explanation.

Our results were obtained in the limit of infinite $N$ and infinite 't Hooft coupling. The corresponding perturbative calculation in a weakly coupled gauge theory does not, to the best of our knowledge, exist in the literature. It would be interesting to compare the two regimes explicitly, as well as to compute the correction to our results appearing at large but finite 't Hooft coupling in the spirit of [3].

The singularity structure of wave equations describing scalar perturbations of a generic black $p$-brane near-horizon geometry [7] suggests that the spectrum of its quasi-normal excitations should be similar to the one observed in this paper. Studies of such a spectrum may prove to be useful in the effective description of black objects using the language of a dual real-time thermal field theory [37] (or quantum mechanics [38]).

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A. Solutions of Eq. (3.6) for $w = 0, q = 0$

Consider Eq. (3.6) with $w = 0, q = 0$. Then $Q = (2 - \Delta/2)^2 = \alpha \beta$, and the Heun equation reduces to a hypergeometric one. Changing variable to $v = (1 - z)^2$, we obtain

$$4v(v-1)y'' + 2[2v + (4 - \Delta)(v-1)]y' + \left(2 - \frac{\Delta}{2}\right)^2 y = 0,$$

(A.1)

whose formal solution is given by a linear combination

$$y(v) = C_1 y_1(v) + C_2 y_2(v),$$

(A.2)

where

$$y_1(v) = 2F_1 \left(1 - \frac{\Delta}{4}, 1 - \frac{\Delta}{4}; 2 - \frac{\Delta}{2}; v \right),$$

(A.3)

$$y_2(v) = v^{\frac{\Delta}{2} - 1} 2F_1 \left(\frac{\Delta}{4}, \frac{\Delta}{4}; \frac{\Delta}{2}; v \right).$$

(A.4)
The hypergeometric functions in Eqs. (A.3),(A.4) are degenerate. They are explicitly represented by the series expansion
\[
\mathbf{2F1}(a, a; 2a; v) = \frac{\Gamma(2a)}{\Gamma^2(a)} \sum_{k=0}^{\infty} \frac{[(a)k]^2}{k!} \left\{ 2\psi(k+1) - 2\psi(a+k) - \log (1 - v) \right\} (1 - v)^k, \quad (A.5)
\]
where \( a = 1 - \Delta / 4 \) for (A.3) and \( a = \Delta / 4 \) for (A.4), the expansion still being valid when \( a \) is a negative integer or zero (in which case an appropriate limit should be taken). However, when \( a = -1/2, -3/2, \ldots \), i.e. when \( \Delta = 2(2k+1) = 6, 10, 14, \ldots \), the correct representation for (A.3) is instead given by
\[
y_1(v) = y_2(v) \log v + v^{2k} \sum_{n=1}^{\infty} v^n \sum_{m=0}^{k} \frac{[n + (1/2) + m]}{(2k+1) n_m} \left\{ 2\psi(k + n + 1/2) - 2\psi(k + 1/2) - \psi(2k + 1 + n) \right. \\
+ \left. \psi(2k + 1) - \psi(n + 1) + \psi(1) \right\} - \sum_{n=1}^{2k} \frac{(n - 1)!(2k)_n}{(1/2 - n)_n} v^{2k-n}. \quad (A.6)
\]

Another special case is \( \Delta = 2 \) which is covered by (A.6) with \( k = 0 \) and the last sum omitted.

Having found the explicit solutions, we can now use them to illustrate the reasoning adopted in Section 3. We notice that for \( \Delta \neq 2(2k+1) = 2, 6, 10, 14, \ldots \) the solution (A.2) does not contain logarithmic terms at \( v = 0 \), in agreement with Eq. (3.21). Moreover, (A.3) shows that in this case one can choose integration constants \( C_1, C_2 \) to get rid of the logarithms also at \( v = 1 \). Consequently, an analytic solution in the interval \( v \in [0, 1] \) exists, and thus \( w = 0, q = 0 \) must be among the solutions of the continued fraction equation (3.18) for \( \Delta \neq 2(2k+1) \). This is illustrated in Figures 1,2. The solutions are “false frequencies” since the absence of logarithms at \( v = 0 \) reflects the property of Eq. (A.3) rather than the requirement \( C_1 = 0 \).

On the other hand, for the “exceptional” conformal dimensions \( \Delta = 2(2k+1) = 2, 6, 10, 14, \ldots \), Eq. (A.6) shows that logarithms do appear in the second solution, and that in this case there is no nontrivial analytic solution to Eq. (3.6) for \( w = 0, q = 0 \), again in agreement with Figures 1 and 2.

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Table 1: Real (with \(\pm\) sign) and imaginary parts of the ten lowest scalar quasinormal frequencies for integer conformal dimensions \(\Delta \in [2, 10]\) at zero spatial momentum \(q\).

| \(n\) | \(\Delta = 2\) | \(\Delta = 3\) | \(\Delta = 4\) | \(\Delta = 5\) | \(\Delta = 6\) | \(\Delta = 7\) | \(\Delta = 8\) | \(\Delta = 9\) | \(\Delta = 10\) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | \(\pm 0.640759\) | \(-0.411465\) | \(\pm 1.099407\) | \(-0.879767\) | \(\pm 1.559726\) | \(-1.373338\) | \(\pm 2.028589\) | \(-1.879263\) | \(\pm 2.506053\) |
| 2 | \(\pm 1.618564\) | \(-1.393310\) | \(\pm 2.105949\) | \(-1.887444\) | \(\pm 2.584760\) | \(-2.381785\) | \(\pm 3.061998\) | \(-2.880185\) | \(\pm 3.540615\) |
| 3 | \(\pm 2.614565\) | \(-2.391212\) | \(\pm 3.107772\) | \(-2.888629\) | \(\pm 3.593965\) | \(-3.384782\) | \(\pm 4.077135\) | \(-3.882247\) | \(\pm 4.559507\) |
| 4 | \(\pm 3.613045\) | \(-3.390584\) | \(\pm 4.108584\) | \(-3.889040\) | \(\pm 4.598600\) | \(-4.386241\) | \(\pm 4.612274\) | \(-4.390299\) | \(\pm 5.109031\) |
| 5 | \(\pm 5.611817\) | \(-5.390141\) | \(\pm 6.109309\) | \(-5.889349\) | \(\pm 6.603123\) | \(-6.387619\) | \(\pm 6.611519\) | \(-6.390042\) | \(\pm 7.109497\) |
| 6 | \(\pm 7.611311\) | \(-7.389975\) | \(\pm 8.109631\) | \(-7.889468\) | \(\pm 8.605279\) | \(-8.388258\) | \(\pm 8.611159\) | \(-8.389927\) | \(\pm 9.109731\) |
| 7 | \(\pm 9.611043\) | \(-9.389892\) | \(\pm 10.109808\) | \(-9.889528\) | \(\pm 10.606513\) | \(-10.388616\) | \(\pm 10.611043\) | \(-9.389892\) | \(\pm 10.109808\) |


Figure 1: Real part of the eigenfrequencies (solutions of the continued fraction equation \((3.18)\)) at \(q = 0\) versus the conformal dimension \(\Delta\). Black dots correspond to quasinormal frequencies at integer values of \(\Delta\), while blank ellipses are the “false frequencies”. The dashed line indicates that the sequence presumably continues to infinity.

Figure 2: (Minus) imaginary part of the eigenfrequencies (solutions of the continued fraction equation \((3.18)\)) at \(q = 0\) versus the conformal dimension \(\Delta\). Black dots correspond to quasinormal frequencies at integer values of \(\Delta\), while blank ellipses are the “false frequencies”. The dashed line indicates that the sequence presumably continues to infinity.
Figure 3: \( \text{Re } w \) of the lowest eight scalar quasinormal frequencies versus the conformal dimension \( \Delta \). Dots correspond to integer conformal dimensions.

Figure 4: \( -\text{Im } w \) of the lowest eight scalar quasinormal frequencies versus the conformal dimension \( \Delta \). Dots correspond to integer conformal dimensions.
Figure 5: Re $w$ of the scalar fundamental quasinormal frequency vs $q$ for the (integer) conformal dimensions $\Delta \in [2, 10]$. The lowest curve corresponds to $\Delta = 2$.

Figure 6: $-\text{Im } w$ of the scalar fundamental quasinormal frequency vs $q$ for the (integer) conformal dimensions $\Delta \in [2, 10]$. The lowest curve corresponds to $\Delta = 2$. 
Figure 7: Poles of an $R$-current thermal correlator in the complex $w$ plane for $q = 1$.

Figure 8: $-\text{Im} w$ of the $R$-current thermal correlator’s “hydrodynamic pole” as a function of $q$. The light curve corresponds to the analytic approximation (4.3) for small $q$. 

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Figure 9: The ten lowest dispersion curves ($\text{Re } w$ vs $q$) for the R-current correlators.

Figure 10: The ten lowest dispersion curves ($-\text{Im } w$ vs $q$) for the R-current correlators.
Figure 11: Details of the lowest dispersion curve for the R-current correlators.

Figure 12: The dispersion curve for the tensor diffusion pole. The dashed line corresponds to the analytic approximation $w = -i q^2/2$ valid in the hydrodynamic regime $q \ll 1$. 
Figure 13: The ten lowest dispersion curves ($\text{Re } w$ vs $q$) for the stress-energy tensor correlators.

Figure 14: The ten lowest dispersion curves ($-\text{Im } w$ vs $q$) for the stress-energy tensor correlators.