A Riemann-Roch theorem for the noncommutative two torus

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Abstract

We prove the analogue of the Riemann-Roch formula for the noncommutative two torus \( A_0 = C(T^2_\theta) \) equipped with an arbitrary translation invariant complex structure and a Weyl factor represented by a positive element \( k \in C^\infty(T^2_\theta) \). We consider a topologically trivial line bundle equipped with a general holomorphic structure and the corresponding twisted Dolbeault Laplacians. We define a spectral triple \((A_0, H, D)\) that encodes the twisted Dolbeault complex of \( A_0 \) and whose index gives the left hand side of the Riemann-Roch formula. Using Connes’ pseudodifferential calculus and heat equation techniques, we explicitly compute the \( b_2 \) terms of the asymptotic expansion of \( \text{Tr}(e^{-tD^2}) \). We find that the curvature term on the right hand side of the Riemann-Roch formula coincides with the scalar curvature of the noncommutative torus recently defined and computed in \textsuperscript{8} and \textsuperscript{13}.

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1 Introduction

Let \( M \) be a closed, connected, and oriented surface equipped with a Riemannian metric and hence with a canonically defined complex structure. The complex structure is invariant under conformal perturbations of the metric. Let \( E \) be a holomorphic line bundle on \( M \) equipped with a hermitian metric, and let \( \overline{\partial}_E : C^\infty(M, E) \to \Omega^{0,1}(M, E) \) denote the unique holomorphic connection on \( E \) compatible with its hermitian metric. It is an elliptic operator and the Riemann-Roch formula computes its index as follows. Let \( \Delta_0 = \overline{\partial}_E^* \overline{\partial}_E \) and \( \Delta_1 = \overline{\partial}_E \overline{\partial}_E^* \) denote the corresponding Laplacians acting on smooth sections of \( E \) and smooth \((0,1)\)-forms with coefficients in \( E \), respectively. We have asymptotic expansion for heat traces

\[
\text{Tr} e^{-t\Delta} \sim t^{-1} \sum_{n=0}^{\infty} \left( \int_M b_{2n}(x, \Delta) \, dV \right) t^n \quad (t \to 0^+),
\]
where $\Delta = \Delta_i, i=0,1$, and $dV$ is the volume form of $M$. Then we have

$$\text{index}(\bar{\partial}_E) = \int_M (b_2(x,\Delta_0) - b_2(x,\Delta_1)) \, dV = \frac{1}{4\pi} \int_M R + \frac{1}{2\pi i} \int_M R_E,$$

where $R = KdV$ is the Gaussian curvature form of $M$, and $R_E$ is the curvature 2-form of $E$ [14].

In this paper we prove the analogue of the Riemann-Roch formula for the noncommutative two torus $A_\theta = C(T^2_\theta)$ equipped with an arbitrary translation invariant complex structure and Weyl factor. For the line bundle we consider a topologically trivial line bundle equipped with a general, non-trivial, holomorphic structure. Extending the spectral triple of [11], we define a spectral triple $(A_\theta, H, D)$ that encodes the twisted Dolbeault complex of $A_\theta$ and whose index gives the left hand side of the Riemann-Roch formula. Using Connes’ pseudodifferential calculus [11] and heat equation techniques, we explicitly compute the $b_2$ terms of the asymptotic expansion of $\text{Tr}(e^{-1D^2})$. We find that the $R$-term on the right hand side of the Riemann-Roch formula coincides with the scalar curvature of the noncommutative torus recently defined and computed in [8] and [13]. We also show that the topological term $R_E$, the Chern form, vanishes. For the trivial holomorphic structure, we recover the Gauss-Bonnet theorem of [11] and [12]. In the last section we consider the nontrivial projective module $\mathcal{S}(\mathbb{R})$ equipped with a holomorphic structure and by a variational argument verify the statement of the Riemann-Roch theorem for the twisted Dolbeault operator.

Following the pioneering work of Connes and Tretkoff on the Gauss-Bonnet theorem for the noncommutative two torus [11], and its extension and refinement in [12], the question of defining and computing the scalar curvature for the noncommutative two torus was eventually settled in the paper of Connes and Moscovici in [8], and, independently, by Fathizadeh and Khalkhali in [13]. So the question of a Riemann-Roch formula for $A_\theta$ posed itself in a natural way at this stage. M. K. would like to thank Farzad Fathizadeh for continued collaboration and many informative discussions.

2 Conformal structures on the irrational rotation algebra.

For an irrational number $\theta$, the $C^*$-algebra $A_\theta$ is, by definition, the universal unital $C^*$-algebra generated by two unitaries $U, V$ satisfying

$$VU = e^{2\pi i\theta}UV.$$

There is a continuous action of $T^2$, $T$, on $A_\theta$ given by

$$\alpha_x(U^mV^n) = e^{i\langle x, (m,n) \rangle} U^mV^n,$$

and the space of smooth elements of $A_\theta$ under this action will be denoted by $A_\theta^\infty$. This algebra is also can be described as

$$A_\theta^\infty = \{ \sum_{m,n \in \mathbb{Z}} a_{m,n}U^mV^n; (1 + |m|^k + |n|^q)|a_{m,n}| < \infty, \forall k,q \in \mathbb{Z} \}.$$

There is a unique normalized trace $\tau_0$ on $A_\theta$ that on smooth elements is given by

$$\tau_0 \left( \sum_{m,n \in \mathbb{Z}} a_{m,n}U^mV^n \right) = a_{0,0}.$$

There are two derivations denoted by $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$ induced by the action of $T^2$ on $A_\theta$. On the generators they are defined by

$$\delta_1(U) = U, \quad \delta_1(V) = 0, \quad \delta_2(U) = 0, \quad \delta_2(V) = V.$$

These derivations anti-commute with the $*$-operator of $A_\theta$, i.e. one has $\delta_j(a^*) = -\delta_j(a)^*$ for $j = 1,2$ and $a \in A_\theta^\infty$ and also they are invariant under the trace.

$$\tau_0 \circ \delta_j = 0, \quad \text{for} \quad j = 1,2.$$
This yields
\[ \tau_0 (a \delta_j(b)) = -\tau_0 (\delta_j(a)b), \quad \forall a, b \in A_\theta^\infty. \]

There exists an inner product on \( A_\theta \) given by
\[ \langle a, b \rangle = \tau_0 (b^* a), \quad a, b \in A_\theta. \]

The Hilbert space completion of \( A_\theta \) under this inner product will be denoted by \( H_0 \). The derivations \( \delta_1, \delta_2 \), as unbounded operators on \( H_0 \), are formally selfadjoint and have unique extensions to selfadjoint operators.

For any complex number \( \tau \) in the upper half plane, there exists a complex structure on the noncommutative two torus given by
\[ \partial = \delta_1 + \bar{\tau} \delta_2, \quad \partial^* = \delta_1 + \tau \delta_2. \]

The associated positive Hochschild two cocycle on \( A_\theta^\infty \) is given by (cf. [11])
\[ \psi(a, b, c) = -\tau_0 (a \partial b \partial^* c). \]

The map \( \varphi \) is a positive linear functional which is a twisted trace and satisfies the KMS condition at \( \beta = 1 \) for the 1-parameter group \( \{ \sigma_t \} \), \( t \in \mathbb{R} \) of inner automorphisms
\[ \sigma_t(x) = e^{ith} xe^{-ith}. \]

We have \( \sigma_t = \Delta^{-it} \) where the modular operator for \( \varphi \) is (cf. [11])
\[ \Delta(x) = e^{-h} xe^h. \]

The 1-parameter group of automorphisms \( \sigma_t \) is generated by the derivation \(- \log \Delta \) where
\[ \log \Delta(x) = [-h, x], \quad x \in A_\theta^\infty. \]

We define an inner product \( \langle \cdot, \cdot \rangle_\varphi \) on \( A_\theta \) by
\[ \langle a, b \rangle_\varphi = \varphi(b^* a), \quad a, b \in A_\theta. \]

The Hilbert space obtained from completing \( A_\theta \) with respect to this inner product will be denoted by \( H_\varphi \).

3 Topologically trivial bundles with arbitrary holomorphic structures

It is well known that holomorphic structures on a trivial line bundle over a compact Riemann surface are parameterized by points of the Jacobian of the surface. Thus for genus one surfaces they are in one to one correspondence with points of the surface itself. Its noncommutative analogue is as follows. For a noncommutative two torus, a holomorphic structure on \( \mathcal{E} = A_\theta \), considered as a free \( A_\theta^\infty \)-module, is given by a holomorphic flat connection
\[ \nabla = \partial + w : \mathcal{E} \to \Omega^{(1,0)}(\otimes A_\theta^\infty \mathcal{E}), \]
where \( w \in \mathbb{C} \) and
\[
\partial = \delta_1 + \tau \delta_2.
\]
Considered as a densely defined unbounded operator \( \nabla : \mathcal{H}_0 \to \mathcal{H}^{(1,0)} \) has a formal adjoint given by
\[
\nabla^* = \overline{w} + \partial^*,
\]
where \( \partial^* = \delta_1 + \tau \delta_2 \). Note that for the trivial bundle \( \mathcal{E} \), the completion of \( \Omega^{(1,0)} \otimes A_\infty \mathcal{E} \) can be identified by \( \mathcal{H}^{(1,0)} \). The Laplacian on \((0,0)\)-sections is given by
\[
\Delta_0 = \nabla^* \nabla = (\overline{w} + \partial^*)(w + \partial),
\]
with \( w = |w|^2 + \overline{w} \partial + w \partial^* + \partial \partial^* \).

Let us view the operator \( \nabla \) as an unbounded operator from \( \mathcal{H}_0 \) to \( \mathcal{H}^{(1,0)} \) and denote it by \( \nabla \mathcal{H}_0 \rightarrow \mathcal{H}^{(1,0)} \).

By considering the left action of \( A_\infty \) on the Hilbert space
\[
\mathcal{H} = \mathcal{H}_\infty \oplus \mathcal{H}^{(1,0)},
\]
and the operator
\[
D = \begin{pmatrix} 0 & \overline{\nabla}^* \\ \nabla & 0 \end{pmatrix} : \mathcal{H} \rightarrow \mathcal{H}.
\]

Then the Laplacian has the following form:
\[
\Delta := D^2 = \begin{pmatrix} \overline{\nabla}^* \nabla & 0 \\ 0 & \nabla \overline{\nabla}^* \end{pmatrix},
\]
and the grading is given by
\[
\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \mathcal{H} \rightarrow \mathcal{H}.
\]

A twisted spectral triple is constructed over \( A_\infty^{op} \) in \([8]\), using the Tomita anti-linear unitary map \( J_\phi \) in \( \mathcal{H}_\phi \), and the unitary right action of \( A_\infty \) in \( \mathcal{H}_\phi \) given by \( a \mapsto J_\phi a^* J_\phi \). It is shown that \((A_\infty^{op}, \mathcal{H}, D)\) is a twisted spectral triple, see \([8]\).

One can show that changing the metric within the conformal class of a given metric gives a new Laplacian \( \Delta'_0 \) and one has

**Lemma 3.1.** The operator \( \Delta'_0 : \mathcal{H}_0 \to \mathcal{H}_0 \) is anti-unitarily equivalent to the operator \( k \Delta_0 k \), where
\[
\overline{\Delta}_0 = (-\overline{w} + \partial^*)(-w + \partial).
\]

**Proof.** First note that the map \( W : \mathcal{H}_0 \to \mathcal{H}_\phi \), given by \( W(a) = ak \) is an isometry. One has \( \nabla_\phi \circ W = \nabla \circ R_k \) and hence
\[
W^* \nabla_\phi^* \nabla_\phi W = R_k \nabla^* \nabla R_k.
\]

Therefore the operators \( \Delta_0 \) and \( R_k \Delta_0 R_k \) are unitary equivalent. On the other hand
\[
JR_k(\overline{w} + \partial^*)(w + \partial)R_kJ = JR_kJ(\overline{w} + \partial^*)(w + \partial)JR_kJ = k(-\overline{w} + \partial^*)(-w + \partial)k.
\]

On the other hand on the space of twisted \((1,0)\)-sections the twisted Dolbeault Laplacian is given by
\[
\Delta_1 = \nabla \nabla^* = (w + \partial)(\overline{w} + \partial^*) = |w|^2 + \overline{w} \partial + w \partial^* + \partial \partial^*.
\]

Perturbing the metric on its conformal class yields a new Laplacian \( \Delta'_1 \) and we have

**Lemma 3.2.** The operator \( \Delta'_1 : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)} \) is anti-unitarily equivalent to the operator
\[
(-\overline{w} + \partial^*)k^2(-w + \partial)
\]

**Proof.** The proof is similar to the previous lemma.
3.1 Connes’ pseudodifferential operators on $\mathbb{T}_0^2$.

For a non-negative integer $n$, the space of differential operators on $A^\infty_0$ of order at most $n$ is defined to be the vector space of operators of the form

$$\sum_{j_1+j_2 \leq n} a_{j_1,j_2} \nabla_1^{j_1} \nabla_2^{j_2}, \quad j_1, j_2 \geq 0, \quad a_{j_1,j_2} \in A^\infty_0,$$

where $\nabla_i = \delta_i + z_i$ for $i = 1, 2$. Here $z_1$ and $z_2$ are two complex numbers such that $w = z_1 + \overline{z}z_2$.

The notion of a differential operator on $A^\infty_0$ can be generalized to the notion of a pseudodifferential operator using operator valued symbols [3]. In fact this is achieved by considering the pseudodifferential calculus associated to $C^*$-dynamical systems [3], for the canonical dynamical system $(A^\infty_0, \{\alpha_s\})$. In the sequel, we shall use the notation $\partial_1 = \frac{\partial}{\partial \xi_1}, \partial_2 = \frac{\partial}{\partial \xi_2}$.

Definition 3.1. For an integer $n$, a smooth map $\rho : \mathbb{R}^2 \to A^\infty_0$ is said to be a symbol of order $n$, if for all non-negative integers $i_1, i_2, j_1, j_2$,

$$||\delta_1^{i_1} \delta_2^{i_2} \partial_1^{j_1} \partial_2^{j_2} \rho(\xi)|| \leq c(1 + |\xi|)^{n-j_1-j_2},$$

where $c$ is a constant, and if there exists a smooth map $k : \mathbb{R}^2 \to A^\infty_0$ such that

$$\lim_{\lambda \to \infty} \lambda^{-n} \rho(\lambda \xi_1, \lambda \xi_2) = k(\xi_1, \xi_2).$$

The space of symbols of order $n$ is denoted by $S_n$.

To a symbol $\rho$ of order $n$, one can associate an operator on $A^\infty_0$, denoted by $P_\rho$, given by

$$P_\rho(a) = (2\pi)^{-2} \int \int e^{-ix\xi} \rho(\xi) \alpha_s(a) \, ds \, d\xi.$$ 

The operator $P_\rho$ is said to be a pseudodifferential operator of order $n$. For example, the differential operator $\sum_{j_1+j_2 \leq n} a_{j_1,j_2} \delta_1^{j_1} \delta_2^{j_2}$ is associated with the symbol $\sum_{j_1+j_2 \leq n} a_{j_1,j_2} \ell_1^{j_1} \ell_2^{j_2}$ via the above formula.

One can define the equivalent symbols as [11] and find the multiplication and adjoint symbol formula.

Definition 3.2. Let $\rho$ be a symbol of order $n$. It is said to be elliptic if $\rho(\xi)$ is invertible for $\xi \neq 0$, and if there exists a constant $c$ such that

$$||\rho(\xi)^{-1}|| \leq c(1 + |\xi|)^{-n}$$

for sufficiently large $|\xi|$.

By the Cauchy integral formula, for $i = 0, 1$, one has

$$e^{-it\Delta_i} = \frac{1}{2\pi i} \int_C e^{-i\lambda} (\Delta_i' - \lambda)^{-1} \, d\lambda,$$

where $C$ is a curve in $\mathbb{C}$, which goes around the non-negative real axis in counter clockwise direction without touching it. From this one can obtain the asymptotic expansion

$$\text{Tr}(e^{-it\Delta_i}) \sim t^{-1} B_{2n}(\Delta_i') t^n, \quad t \to 0^+, \quad i = 0, 1.$$

By McKean-Singer formula one has the following formula for the index of the twisted Dolbeault complex

$$\text{Index}(\nabla_\varphi) = \sum_{i=0}^{1} (-1)^i \text{Tr}(e^{-it\Delta_i}).$$
This gives us the following formula for the index
\[
\text{Index}(\nabla_\varphi) = B_2(\Delta'_{\theta}) - B_2(\Delta'_{\varphi}).
\]
To find this value, one can approximate the inverse of \((\Delta'_{\theta} - \lambda)\) by a pseudodifferential operator \(B_\lambda\) with a symbol \(\sigma(B_\lambda)\) of the form
\[
b_0^i(\xi, \lambda) + b_1^i(\xi, \lambda) + b_2^i(\xi, \lambda) + \cdots,
\]
where \(b_j^i(\xi, \lambda)\) is a symbol of order \(-2 - j\) for \(i = 0, 1, \) and
\[
\sigma(B_\lambda(\Delta'_{\theta} - \lambda)) \sim 1.
\]
Therefore, one can find that
\[
B_2(\Delta'_{\theta}) = \frac{1}{2\pi i} \int \int_C e^{-\lambda \tau_0(b_2^i(\xi, \lambda))} d\xi d\lambda.
\]
As in [11, 8], one can see that the contour integration can be dropped by a homogeneity argument and therefore
\[
B_2(\Delta'_{\theta}) = \int \tau_0(b_2^i(\xi, -1)) d\xi, \quad i = 0, 1.
\]
In the next section we will compute this index and show that it is zero.

4 The computation of \(B_2(\Delta'_{\theta})\) and \(B_2(\Delta'_{\varphi}).\)

In order to find the value of the \(B_2(\Delta'_{\varphi})\) for \(\Delta'_\varphi \sim k(-\overline{w} + \partial^*)(-w + \partial)k\) and \(\Delta'_\theta \sim (-\overline{w} + \partial^*)k^2(-w + \partial),\) following the approach described in the previous section, we need to find the symbol of these operators.

For \(w = 0,\) the following lemma reduce to Lemma 4.1 of [12].

**Lemma 4.1.** The symbol of the operator \(\sigma(k\overline{\Delta}_{\theta}k) = a_0(\xi) + a_1(\xi) + a_2(\xi)\)
\[
a_2 = \xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2,
\]
\[
a_1 = 2\xi_1 k \delta_1(k) + 2|\tau|^2 \xi_2 \delta_2(k) + 2\tau_1 \xi_1 \xi_2 \delta_1(k) - 2w_1 \xi_1 k^2 - 2w_1 \tau_1 \xi_2 k^2 + 2w_2 \tau_2 \xi_2 k^2
\]
\[
a_0 = \delta_1 \delta_1(k) + |\tau|^2 \xi_2 \delta_2(k) + 2\tau_1 \delta_1 \delta_2(k) - 2w_1 \delta_1(k) - 2w_1 \tau_1 \delta_2(k) + 2w_2 \tau_2 \delta_2(k) + |w|^2 k^2
\]

**Proof.** The proof easily can be obtained from the fact that \(\sigma(-\overline{w} + \partial) = -w + \xi_1 + \tau \xi_2, \sigma(-\overline{w} + \partial^*) = -\overline{w} + \xi_1 + \tau \xi_2.\)

**Lemma 4.2.** The symbol of the operator \(\sigma((-\overline{w} + \partial^*)k^2(-w + \partial)) = a_0(\xi) + a_1(\xi) + a_2(\xi),\)
\[
a_2 = \xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2,
\]
\[
a_1 = \xi_1 \delta_1(k^2) + \tau_1 \xi_2 \delta_2(k^2) + \tau_1 \delta_1 \xi_2 \delta_2(k^2) + |\tau|^2 \xi_2 \delta_2(k^2) - 2w_1 \xi_1 k^2 - 2w_1 \tau_1 \xi_2 k^2 + 2w_2 \tau_2 \xi_2 k^2
\]
\[
a_0 = -w \delta_1(k^2) - w \tau \delta_2(k^2) + |w|^2 k^2
\]

**Proof.** The proof is similar to the previous lemma.

Using similar methods as in [11, 12], one can obtain the following formulas for \(b_2:\)
\[
b_2 = -(b_0 a_0 b_0 + b_1 a_1 b_0 + \delta_i(b_0) \delta_i(a_1) b_0 + \delta_i(b_1) \delta_i(a_2) b_0 + 1/2 \delta_i \delta_j(b_0) \delta_i \delta_j(a_2) b_0).
\]
Here we have used the summation on repeated indices.
4.1 The computation of \( B_2(\Delta'_0) \)

One can apply the method of \([11, 12, 13]\) to obtain all the terms. We just give the terms which are new compared to those which already appeared in \([13]\) (after integration over the \( \xi \) plane). The rest of terms are as those found in \([13]\).

To do integration over \( \xi \), one uses the following substitution rule (see \([12]\))

\[
\xi_1 = r \cos \theta - \frac{\tau_1}{\tau_2} r \sin \theta, \quad \xi_2 = \frac{r}{\tau_2} \sin \theta.
\]

Up to an overall factor \( \frac{r}{\tau_2} \), the extra terms comparing \([13]\) are

\[
- 2\pi |w|^2 b_0^2 k^2 + 4\pi r^2 |w|^2 b_0^2 k^4 + 4\pi w_1 b_0^2 k \delta_1(k) - 2\pi b_0^2 k \delta_2(k) - 4\pi \tau_1 b_0^2 \delta(k) + 4\pi \tau_2 b_0^2 \delta(k) + 8\pi r^2 b_0^2 k \delta_1(k)
\]

\[
+ 8\pi r^2 \tau_1 b_0^2 k \delta_2(k) + 8\pi r^2 \tau_2 b_0^2 k \delta_2(k) - 24\pi r^2 w_1 b_0^2 k \delta_1(k) - 8\pi r^2 b_0^2 k \delta_1(k) - 8\pi r^2 b_0^2 k \delta_1(k) - 8\pi r^2 b_0^2 k \delta_1(k)
\]

Integrating over \( r \), up to factor \( \frac{r}{\tau_2} \), gives us

\[
- k^{-1} \delta_1(k) - |r|^2 k^{-1} \delta_2(k) - 2k^{-1} \delta_1(k) + 2k^{-2} \delta_1(k) + 2|r|^2 k^{-2} \delta_2(k) + 2k^{-1} (\delta_1(k) \delta_2(k) + \delta_2(k) \delta_1(k)).
\]

Surprisingly, the holomorphic structure of the bundle does not contribute in the formula above and this exactly coincides with the terms found in Section 4.1. in \([12]\).

Indeed, we have

\[
\int_{0}^{\infty} \left( - 2\pi |w|^2 b_0^2 k^2 + 4\pi r^2 |w|^2 b_0^2 k^4 \right) \left( \frac{rdr}{\tau_2} \right) = 0,
\]

\[
\int_{0}^{\infty} \left( - 4\pi w_1 b_0^2 k \delta_1(k) + 24\pi r^4 w_1 b_0^2 k \delta_1(k) - 24\pi r^2 w_1 b_0^2 k \delta_1(k) \right) \left( \frac{rdr}{\tau_2} \right) = 0,
\]

and

\[
\int_{0}^{\infty} \left( 24\pi r^4 w_1 \tau_1 b_0^4 k^5 - 24\pi r^4 w_2 \tau_2 b_0^4 k^5 - 24\pi r^2 w_1 \tau_1 b_0^3 k^3 + 24\pi r^2 w_2 \tau_2 b_0^3 k^3 
\]

\[
+ 4\pi w_1 \tau_1 b_0^2 k - 4\pi w_2 \tau_2 b_0^2 k \right) \left( \frac{rdr}{\tau_2} \right) = 0.
\]

4.2 The computation of \( B_2(\Delta'_1) \).

Now we would like to give the computation of \( B_2(\Delta'_1) \) on \((1, 0)\)-sections. As in the previous section, we just mention the integrated (over \( \xi \) plane of) the new terms.

Up to overall factor of \( r/\tau_2 \) these terms are given by
where

\[ D_{\text{page } 20}, \text{ for the first term of} \]

Integrating over \( \tau \), we have used the summation on repeated indices. Applying the computation given in [12]

Now combining these two type terms one has\[ \int_{0}^{\infty} \left( 24 \pi r^{4} w_{1} b_{0}^{5} k^{5} - 24 \pi r^{2} w_{1} b_{0}^{3} k^{3} - 8 i \pi r^{2} w_{2} b_{0}^{3} k^{3} + 4 \pi w_{1} b_{0}^{3} k + 4 i \pi w_{2} b_{0}^{3} k \right) \frac{rd\tau}{\tau} = 0, \]

and\[ \int_{0}^{\infty} \left( 24 \pi r^{4} w_{1} \tau_{1} b_{0}^{5} k^{5} - 24 \pi r^{4} w_{2} \tau_{2} b_{0}^{5} k^{5} - 24 \pi r^{2} w_{1} \tau_{1} b_{0}^{3} k^{3} - 8 i \pi r^{2} w_{2} \tau_{2} b_{0}^{3} k^{3} - 8 i \pi r^{2} w_{1} \tau_{2} b_{0}^{3} k^{3} \right) \frac{rd\tau}{\tau} = 0. \]

One can combine the terms with \( b_{0}^{3} \) in the middle with the terms with \( b_{0} \) in the middle. First for the sake of notation we introduce\[ g^{ij} = \left( \begin{array}{c} 1 \\ \tau \end{array} \right), \quad h^{ij} = \left( \begin{array}{c} 1 \\ \tau_{1} \end{array} \right). \]

Now combining these two type terms one has\[ T = -2 \pi g^{ij} r^{3} b_{0}^{2} k^{2} \delta_{i}(k) b_{0} \delta_{j}(k) - 4 \pi g^{ij} r^{3} b_{0}^{2} k^{2} \delta_{i}(k) b_{0} \delta_{j}(k) - 2 \pi g^{ij} r^{3} b_{0}^{2} k^{2} \delta_{i}(k) b_{0} \delta_{j}(k) \]

\[ + 4 \pi g^{ij} r^{3} b_{0}^{2} k^{2} \delta_{i}(k) b_{0} \delta_{j}(k) + 8 \pi g^{ij} r^{3} b_{0}^{2} k^{2} \delta_{i}(k) b_{0} \delta_{j}(k) + 4 \pi g^{ij} r^{3} b_{0}^{2} k^{2} \delta_{i}(k) b_{0} \delta_{j}(k) \]

\[ - 4 \pi h^{ij} r^{2} b_{0}^{3} k^{2} \delta_{i}(k) b_{0} \delta_{j}(k) - 8 i \pi r^{2} b_{0}^{3} k^{2} \delta_{i}(k) b_{0} \delta_{j}(k) \]

Here we have used the summation on repeated indices. Applying the computation given in [12] page 20, for the first term of \( T \), we get\[ \int -2 \pi g^{ij} r^{3} b_{0}^{2} k^{2} \delta_{i}(k) b_{0} \delta_{j}(k) rd\tau = -\pi g^{ij} k^{2} D_{1}(\delta_{i}(k)) \delta_{j}(k), \]

where \( D_{m} = L_{m}(\Delta) \) and \( L_{m} \) is the modified logarithm function given by [11]\[ L_{m}(u) = (-1)^{m}(u - 1)^{-(m+1)} \left( \log u + \sum_{i=1}^{m} (-1)^{i+1} \frac{(u - 1)^{i}}{i} \right). \]
For the second and third terms of first line of $T$, we use Connes-Tretkoff lemma \[11\] to get as
\[
\int -4\pi g^{ij} r^3 b_0^2 k\delta_i(k)\beta_0 k\delta_j(k)dr = -2\pi g^{ij} k^{-2}\Delta_1^{1/2}(\delta_i(k))\delta_j(k),
\]
and
\[
\int -2\pi g^{ij} r^3 b_0^2 \delta_i(k)\beta_0 k^2 \delta_j(k)dr = -\pi g^{ij} k^{-2}\Delta_1(\delta_i(k))\delta_j(k).
\]
For the second line in $T$ one has
\[
\int 4\pi g^{ij} r^5 b_0^3 k^4 \delta_i(k)\beta_0 \delta_j(k)dr = 2\pi g^{ij} k^{-2}\Delta_2(\delta_i(k))\delta_j(k)
\]
\[
\int 8\pi g^{ij} r^5 b_0^3 k^3 \delta_i(k)\beta_0 k \delta_j(k)dr = 4\pi g^{ij} k^{-2}\Delta_2^{1/2}(\delta_i(k))\delta_j(k),
\]
and
\[
\int 4\pi g^{ij} r^5 b_0^3 k^2 \delta_i(k)\beta_0 k^2 \delta_j(k)dr = 2\pi g^{ij} k^{-2}\Delta_2(\delta_i(k))\delta_j(k).
\]
For the last line of $T$, we have
\[
\int -4\pi h^{ij} r^7 b_0^4 k^6 \delta_i(k)\beta_0 \delta_j(k)dr = -2\pi h^{ij} k^{-2}\Delta_3(\delta_i(k))\delta_j(k),
\]
\[
\int -8\pi h^{ij} r^7 b_0^4 k^5 \delta_i(k)\beta_0 k \delta_j(k)dr = -4\pi h^{ij} k^{-2}\Delta_3^{1/2}(\delta_i(k))\delta_j(k),
\]
\[
\int -4\pi h^{ij} r^7 b_0^4 k^4 \delta_i(k)\beta_0 k^2 \delta_j(k)dr = -2\pi h^{ij} k^{-2}\Delta_3(\delta_i(k))\delta_j(k).
\]
Hence
\[
B_2(\Delta_1') = \frac{2\pi}{3} \left( k^{-2} \delta_i(k)^2 + |r|^2 k^{-2} \delta_2(k)^2 + k^{-1} \delta_1(k) + |r|^2 k^{-1} \delta_2(k) \right)
+ 2\tau k^{-1} \delta_1(k)\delta_2(k) + \tau_1 k^{-2} \delta_1(k)\delta_2(k) + \tau_2 k^{-2} \delta_2(k)\delta_1(k)
- \pi g^{ij} k^{-2}\Delta_1(\delta_i(k))\delta_j(k) - 2\pi g^{ij} k^{-2}\Delta_1^{1/2}(\delta_i(k))\delta_j(k) - \pi g^{ij} k^{-2}\Delta_1(\delta_i(k))\delta_j(k)
+ 2\pi g^{ij} k^{-2}\Delta_2(\delta_i(k))\delta_j(k) + 4\pi g^{ij} k^{-2}\Delta_2^{1/2}(\delta_i(k))\delta_j(k) + 2\pi g^{ij} k^{-2}\Delta_2(\delta_i(k))\delta_j(k)
- 2\pi h^{ij} k^{-2}\Delta_3(\delta_i(k))\delta_j(k) - 4\pi h^{ij} k^{-2}\Delta_3^{1/2}(\delta_i(k))\delta_j(k) - 2\pi h^{ij} k^{-2}\Delta_3(\delta_i(k))\delta_j(k).
\]
Therefore
\[
B_2(\Delta_1') = 2\pi k^{-2} h^{ij} \left( \frac{1}{3} \delta_i(k)\delta_j(k) + \frac{1}{3} \Delta^{-1/2}(\delta_i(k))\delta_j(k) - \Delta_3(1 + \Delta^{1/2})^2(\delta_i(k))\delta_j(k) \right)
+ \pi k^{-2} g^{ij} \left( (-\Delta_1 + 2\Delta_2)(1 + \Delta^{1/2})^2(\delta_i(k))\delta_j(k) \right).
\]
To find a simpler formula, note that
\[
g^{ij} = h^{ij} + \epsilon^{ij},
\]
where
\[
\epsilon^{ij} := i\tau_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
Then
Let us define

\[ f(u) := \frac{2}{3} + \frac{2}{3} u^{-1/2} + \left(-L_1(u) + 2L_2(u) - 2L_3(u)\right)(1 + u^{1/2})^2, \]

and

\[ g(u) := \left((-L_1(u) + 2L_2(u))(1 + u^{1/2})^2. \]

One can see that \( f \) and \( g \) can be written as \( h(\log u) \) and \( l(\log u) \), where

\[ h(x) = \frac{2 \text{Csch}^2 \left(3x \text{Cosh}[x] + 6 \text{Sinh} \frac{x}{2} - 3 \text{Sinh}[x] - 2 \text{Sinh}^3 \frac{x}{2}\right)}{3x^2}, \]

and

\[ l(x) = \frac{2 + e^x(-2 + x) + x}{(-1 + e^{x/2})^3 (1 + e^{x/2})}. \]

The Taylor series of functions \( h \) and \( l \) up to order 9 is given by

\[
\begin{align*}
    h(x) & = -\frac{x}{15} + \frac{x^2}{30} - \frac{5x^3}{504} + \frac{11x^4}{5040} - \frac{19x^5}{67200} + \frac{x^6}{172800} + \frac{5x^7}{2128896} + \frac{29x^8}{106444800} + O[x]^9, \\
    l(x) & = \frac{2}{3} - \frac{x}{3} + \frac{7x^2}{120} - \frac{x^3}{720} - \frac{29x^4}{40320} + \frac{x^5}{80640} + \frac{79x^6}{4838400} - \frac{x^7}{9676800} - \frac{697x^8}{1703116800} + O[x]^9.
\end{align*}
\]

Using the identities

\[ k^{-1} \delta_i(k) = 2 \frac{\Delta^{1/2} - 1}{\Delta} \delta_i(\log k), \quad \delta_i(k)k^{-1} = -2 \frac{\Delta^{-1/2} - 1}{\Delta} \delta_i(\log k) \]

and the fact that for an entire function \( F \)

\[ \tau_0(aF(\log \Delta)(b)) = \tau_0(F(-\log \Delta)(a)b), \]

one can find that

\[ \varphi(f(\Delta)(\delta_i(k))) = \tau_0(K(\log k)(\delta_i(k))\delta_j(k)) \]

and

\[ \varphi(g(\Delta)(\delta_i(k))) = \tau_0(K'(\log k)(\delta_i(k))\delta_j(k)), \]

where \( \varphi(x) = \tau_0(xk^{-2}) \) and

\[ K(x) = 4x^{-2} \left( e^{x/2} - 1 \right)^2 h(x), \quad L(x) = 4x^{-2} \left( e^{x/2} - 1 \right)^2 l(x). \]

We would like to find the value of the

\[ B_2(\Delta_1) = \pi k^{-2}h^{ij}\varphi(f(\Delta)(\delta_i(k))\delta_j(k)) + \pi k^{-2}e^{ij}\varphi(g(\Delta)(\delta_i(k))\delta_j(k)). \]
The fact that $K$ is an odd function implies that
$$
\varphi(f(\Delta)(\delta_i(k))\delta_j(k)) = -\varphi(f(\Delta)(\delta_j(k))\delta_i(k)).
$$
Hence
$$
\varphi(f(\Delta)(\delta_1(k))\delta_1(k)) = \varphi(f(\Delta)(\delta_2(k))\delta_2(k)) = 0,
$$
and
$$
\varphi(f(\Delta)(\delta_1(k))\delta_2(k)) = -\varphi(f(\Delta)(\delta_2(k))\delta_1(k)).
$$
On the other hand, for the second term, the fact that $\epsilon^{12} = -1$, $\epsilon^{21} = 1$ together with $L$ being an even function implies that
$$
\pi k^{-2} \epsilon^{ij} \varphi(g(\Delta)(\delta_i(k))\delta_j(k)) = 0.
$$
Indeed
$$
\varphi(g(\Delta)(\delta_i(k))\delta_j(k)) = \varphi(g(\Delta)(\delta_j(k))\delta_i(k)).
$$
Now
$$
\pi k^{-2} \epsilon^{ij} \varphi(g(\Delta)(\delta_i(k))\delta_j(k)) = -\pi k^{-2} \varphi(g(\Delta)(\delta_1(k))\delta_2(k)) + \pi k^{-2} \varphi(g(\Delta)(\delta_2(k))\delta_1(k)) = 0.
$$
This completes the proof of the following:

**Theorem 4.1.** For any $w \in \mathbb{C}$, any irrational number $\theta$, and any positive invertible element $k \in A_\theta^\infty$, the value of $B_2(\Delta_0^i) - B_2(\Delta_1^i)$, where $\Delta_0^i \sim k(-\varpi + \partial^s)(-w + \partial)k$ and $\Delta_1^i \sim (-\varpi + \partial^s)k^2(-w + \partial)$, is given by
$$
B_2(\Delta_0^i) - B_2(\Delta_1^i) = 0
$$

5 Computing the Riemann-Roch density

In this section we find the analogue of the formula (1), for the noncommutative two torus equipped with a holomorphic structure on its trivial bundle. Seeking a formula for the Riemann-Roch densities $R - 2i\mathcal{R}_E$ and $R^\gamma - 2i\mathcal{R}_{E}^\gamma$ on $(0,0)$ and $(1,0)$ sections, we will need to work out the zeta functional $\zeta_i(a,s) = \text{Trace}(a\Delta_i^{-s})$ for $i = 0,1$ (4). One has
$$
\text{Trace}(a\Delta_0^{-s})|_{s=0} + \text{Trace}(aP) = \tau_0(a(R - 2i\mathcal{R}_E)),
$$
and
$$
\text{Trace}(a\Delta_1^{-s})|_{s=0} + \text{Trace}(aP) = \tau_0(a(R^\gamma - 2i\mathcal{R}_{E}^\gamma)),
$$
where $P$ is the projection on the kernel of $\Delta_i$ for $i = 0,1$. We would like to show that $\mathcal{R}_E = \mathcal{R}_{E}^\gamma = 0$, $\mathcal{R} = R$ and $\mathcal{R}^\gamma = R^\gamma$, where $R$ and $R^\gamma$ are the scalar curvature and chiral scalar curvature of the noncommutative two torus introduced in [8, 13]. We will give the proof for $(1,0)$-sections, i.e. $\mathcal{R}^\gamma = R^\gamma$ and $\mathcal{R}_{E}^\gamma = 0$, and the proofs for $(0,0)$-sections will be similar.

**Theorem 5.1.** The formulae for the Riemann-Roch densities are given by $\mathcal{R}_E = \mathcal{R}_{E}^\gamma = 0$, $\mathcal{R} = R$ and $\mathcal{R}^\gamma = R^\gamma$, where the (chiral) scalar curvatures $R$ and $R^\gamma$ for the noncommutative two torus, up to an overall factor of $-\pi/\tau_2$, are given by [8, 13]
$$
R = R_1(\log \Delta)\left(h^{ij}\delta_i(\log k)\delta_j(\log k)\right) + R_2(\log \Delta_1, \log \Delta_2)\left(h^{ij}\delta_i(\log k)\delta_j(\log k)\right)
-W(\log \Delta_1, \log \Delta_2)\left(-i\epsilon^{ij}\delta_i(\log k)\delta_j(\log k)\right)
$$



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where
\[
R_1(x) = \frac{1}{x} - \frac{\sinh(x/2)}{\sinh^2(x/4)},
\]
\[
R_2(s, t) = -(1 + \cosh((s + t)/2)) \times \frac{(s + t)(-t \cosh s + s \cosh t) - (s - t)(s + t + \sinh s + \sinh t - \sinh(s + t))}{st(s + t) \sinh(s/2) \sinh(t/2) \sinh^2((s + t)/2)},
\]
\[
W = \frac{-(s - t + t \cosh s + s \cosh t + \sinh s + \sinh t - \sinh(s + t))}{st(s + t) \sinh(s/2) \sinh(t/2) \sinh((s + t)/2)}.
\]

and
\[
R_1' = R_1'(\log \Delta) \left( h^{ij} \delta_i \delta_j (\log k) \right) + R_2'(\log \Delta_1, \log \Delta_2) \left( h^{ij} \delta_i (\log k) \delta_j (\log k) \right)
\]
\[
- iW(\log \Delta_1, \log \Delta_2) \left( -i e^{ij} \delta_i (\log k) \delta_j (\log k) \right)
\]
where
\[
R_1'(x) = \frac{1}{x} + \frac{\sinh(x/2)}{\cosh^2(x/4)},
\]
\[
R_2'(s, t) = -(1 - \cosh((s + t)/2)) \times \frac{(s + t)(-t \cosh s + s \cosh t) - (s - t)(s + t + \sinh s + \sinh t - \sinh(s + t))}{st(s + t) \sinh(s/2) \sinh(t/2) \sinh^2((s + t)/2)},
\]
\[
W = \frac{-(s - t + t \cosh s + s \cosh t + \sinh s + \sinh t - \sinh(s + t))}{st(s + t) \sinh(s/2) \sinh(t/2) \sinh((s + t)/2)}.
\]

Proof. The terms of \( b^1_2(\xi, -1) \) can be divided into two parts: those that do not contain \( w \) which are exactly as those that appeared in [13], and new terms that contain \( w \). One can check that these new terms with the parameter \( w \) involved, up to an overall factor of \( r/\tau_2 \), are the following:

\[
+ 2\pi w_1 b_0 k \delta_1(k) b_0 + 2i\pi w_2 b_0 k \delta_1(k) b_0 + 2\pi w_1 \tau_1 b_0 k \delta_2(k) b_0 + 2i\pi w_2 \tau_1 b_0 k \delta_2(k) b_0
+ 2i\pi w_1 \tau_2 b_0 k \delta_2(k) b_0 - 2\pi w_2 \tau_2 b_0 k \delta_2(k) b_0 + 2\pi w_1 b_0 k \delta_1(k) b_0 k + 2i\pi w_2 b_0 k \delta_1(k) b_0 k
- 2\pi^2 w_1 b_0 \delta_1(k) b_0^2 k^3 - 2\pi^2 w_2 b_0 \delta_1(k) b_0^2 k^3 + 2\pi w_1 \tau_1 b_0 \delta_2(k) b_0 k + 2i\pi w_2 \tau_1 b_0 \delta_2(k) b_0 k
+ 2i\pi w_1 \tau_2 b_0 \delta_2(k) b_0 k - 2\pi w_2 \tau_2 b_0 \delta_2(k) b_0 k - 2\pi^2 w_1 \tau_1 b_0 \delta_2(k) b_0^2 k^3 - 2\pi^2 w_2 \tau_1 b_0 \delta_2(k) b_0^2 k^3
- 2\pi^2 w_1 \tau_2 b_0 \delta_2(k) b_0^2 k^3 + 2\pi^2 w_2 \tau_2 b_0 \delta_2(k) b_0^2 k^3 - 10\pi^2 w_1 b_0^2 k^3 \delta_1(k) b_0 - 2\pi^2 w_2 b_0^2 k^3 \delta_1(k) b_0
- 10\pi^2 w_1 \tau_1 b_0^2 k^3 \delta_2(k) b_0 - 2\pi^2 w_2 \tau_1 b_0^2 k^3 \delta_2(k) b_0 - 2\pi^2 w_2 \tau_1 b_0^2 k^3 \delta_2(k) b_0
+ 10\pi^2 w_2 \tau_2 b_0^2 k^3 \delta_2(k) b_0 + 8\pi^4 w_1 b_0^2 k^3 \delta_1(k) b_0 + 8\pi^4 w_1 \tau_1 b_0^2 k^3 \delta_2(k) b_0 - 8\pi^4 w_2 \tau_2 b_0^2 k^3 \delta_2(k) b_0
- 2\pi^2 w_1 b_0 k \delta_1(k) b_0^2 k^2 - 2\pi^2 w_2 b_0 k \delta_1(k) b_0^2 k^2 - 2\pi^2 w_1 \tau_1 b_0 k \delta_2(k) b_0^2 k^2 - 2\pi^2 w_2 \tau_1 b_0 k \delta_2(k) b_0^2 k^2
- 2\pi^2 w_1 \tau_2 b_0 k \delta_2(k) b_0^2 k^2 + 2\pi^2 w_2 \tau_2 b_0 k \delta_2(k) b_0^2 k^2 - 10\pi^2 w_1 b_0^2 k^3 \delta_1(k) b_0 k - 2\pi^2 w_2 b_0^2 k^3 \delta_1(k) b_0 k
+ 4\pi^4 w_1 b_0^2 k^3 \delta_1(k) b_0^2 k^2 - 10\pi^2 w_1 \tau_1 b_0^2 k^3 \delta_2(k) b_0 k - 2\pi^2 w_2 \tau_1 b_0^2 k^3 \delta_2(k) b_0 k - 2\pi^2 w_1 \tau_2 b_0^2 k^3 \delta_2(k) b_0 k
+ 10\pi^2 w_2 \tau_2 b_0^2 k^3 \delta_2(k) b_0 k + 4\pi^4 w_1 \tau_1 b_0^2 k^3 \delta_2(k) b_0^2 k^3 - 4\pi^4 w_2 \tau_2 b_0^2 k^3 \delta_2(k) b_0^2 k^3
+ 4\pi^4 w_1 b_0^2 k^3 \delta_1(k) b_0^2 k^2 + 4\pi^4 w_1 \tau_1 b_0^2 k^3 \delta_2(k) b_0^2 k^2 - 4\pi^4 w_2 \tau_2 b_0^2 k^3 \delta_2(k) b_0^2 k^2 + 8\pi^4 w_1 b_0^4 k^4 \delta_1(k) b_0 k
+ 8\pi^4 w_1 \tau_1 b_0^4 k^4 \delta_2(k) b_0 k - 8\pi^4 w_2 \tau_2 b_0^4 k^4 \delta_2(k) b_0 k - 2\pi w^2 b_0^2 k^2 + 4\pi^2 w_1 b_0^4 k^4 + 4\pi^2 w_2 b_0^4 k^4.
\]
After integration by parts, up to overall factor of $\frac{1}{2\pi} k^{-1}$, one can find that this is equal to
\[
\left( wD_0 + wD_0 \Delta^{1/2} - wD_0 \Delta^{1/2} + wD_1 \Delta^{1/2} - 5w_1 D_1 - iw_2 D_1 + 4w_1 D_2 - wD_0 wD_1 - 5w_1 D_1 \Delta^{1/2} - iw_2 D_1 \Delta^{1/2} + 4w_1 D_2 \Delta^{1/2} + 4w_1 D_2 - 4w_1 D_2 + 4w_1 D_2 \Delta^{1/2} \right)(\delta_1(k)) + \\
\left( w\tau D_0 + w\tau D_0 \Delta^{1/2} - w\tau D_0 \Delta^{1/2} + w\tau D_1 \Delta^{1/2} - 5(w\tau)_1 D_1 - i(w\tau)_2 D_1 + 4(w\tau)_1 D_2 - w\tau D_0 wD_1 - 5(w\tau)_1 D_1 \Delta^{1/2} - i(w\tau)_2 D_1 \Delta^{1/2} + 4(w\tau)_1 D_2 \Delta^{1/2} - 4(w\tau)_1 D_2 + 4(w\tau)_1 D_2 \Delta^{1/2} \right)(\delta_2(k)),
\]
where
\[
(w\tau)_1 = Re(w\tau), \quad (w\tau)_2 = Im(w\tau).
\]
One can easily find that the above expression vanishes. Indeed, it will simplify to
\[
\left( (wD_1 - w_1 D_1 - iw_2 D_1) + (w\tau D_1 - (w\tau)_1 D_1 - i(w\tau)_2 D_1) \right)(1 + \Delta^{1/2}) = 0.
\]

\[\square\]

6 The nontrivial bundle $S(\mathbb{R})$

Let $S(\mathbb{R})$ denote the space of rapidly decreasing schwartz class functions on $\mathbb{R}$. It is a right $A^\infty_\theta$-module $[3, 4, 6]$. We fix a Powers-Rieffel projection $e$ such that $eA^\infty_\theta \simeq S(\mathbb{R})$ as right $A^\infty_\theta$-modules. Then $e\delta_1$ and $e\delta_2$ will define a Grassmannian connection on $eA^\infty_\theta$ and one has $[3]$
\[
2\pi i \tau_0(e[\delta_1(e), \delta_2(e)]) = 1.
\]

Note that the difference between this formula and the one in $[3]$ is because of our convention on derivations $\delta_i$. One can adapt Connes’ pseudodifferential calculus for this case. In fact, one has $End_{A^\infty_\theta}(eA^\infty_\theta) = eA^\infty_\theta e$. Suffices to say that symbols are smooth maps $\rho : \mathbb{R}^2 \to eA^\infty_\theta e$ with appropriate growth condition. For example for any pseudodifferential operator $P_\rho$ on $A^\infty_\theta$, one can see that $eP_\rho e$ defines a pseudodifferential operator on $eA^\infty_\theta$.

We introduce the twisted Dolbeault operator $\partial_E : eA^\infty_\theta \to eA^\infty_\theta$ as
\[
\partial_E = e(\delta_1 + i\delta_2) = e\partial.
\]

The hermitian structure of $eA^\infty_\theta$ is given by
\[
(\xi, \eta) = \eta^* \xi
\]
and the connection $\nabla_i = e\delta_i, i = 1, 2$ is compatible with this hermitian structure, i.e.
\[
(\nabla_i \xi, \eta) - (\xi, \nabla_i \eta) = \delta_i(\xi, \eta), \quad i = 1, 2.
\]

This implies that
\[
\partial_E^* = e(\delta_1 - i\delta_2) = e\partial^*.
\]

One can define the Laplacian on $(0,0)$ sections as
\[
\Delta^0_E = \partial_E^* \partial_E.
\]

Therefore
\[
\Delta^0_E = e\partial^* e\partial = e\delta_1^2 + e\delta_2^2 + ie[\delta_1(e), \delta_2(e)].
\]

The symbol of this operator is given by
\[
\sigma(\Delta^0_E) = a_0 + a_1 + a_2,
\]
\[13\]
where
\[ a_2 = e \xi_1^2 + e \xi_2^2, \quad a_1 = 0, \quad a_0 = ie[\delta_1(e), \delta_2(e)]. \]
The Laplacian on (1,0) sections is given by \( \Delta_E^1 = \partial_E \partial_E^* \), and its symbol is
\[ \sigma(\Delta_E^1) = a_0 + a_1 + a_2, \]
where
\[ a_2 = e \xi_1^2 + e \xi_2^2, \quad a_1 = 0, \quad a_0 = -ie[\delta_1(e), \delta_2(e)]. \]
It is not difficult to find the \( b_2 \) terms of \( \Delta^0 \) and \( \Delta^1 \) in this case. For example, for \( b_2(\Delta^1) \), after the polar change of coordinate \((\xi_1, \xi_2) \to (r, \theta)\) and integration over \( \theta \) one has:
\[
2\pi r^2 e b_0^2 \delta_1^2 (e) b_0 + 2\pi r^2 e b_0^2 \delta_2^2 (e) b_0 - 4\pi r^4 e b_0^2 \delta_1^2 (e) b_0 - 4\pi r^4 e b_0^2 \delta_2^2 (e) b_0 + 2i\pi b_0 e \delta_1 (e) e \delta_2 (e) b_0 - 2i\pi b_0 e \delta_2 (e) e \delta_1 (e) b_0 - 8\pi r^4 e b_0^2 \delta_1 (e) b_0 \delta_1 (e) b_0 - 8\pi r^4 e b_0^2 \delta_2 (e) b_0 \delta_2 (e) b_0 + 8\pi r^6 e b_0^2 \delta_1 (e) b_0 \delta_2 (e) b_0 + 4\pi r^6 e b_0^2 \delta_2 (e) b_0 \delta_1 (e) b_0 + 4\pi r^6 e b_0^2 \delta_2 (e) b_0 \delta_2 (e) b_0
\]
With a similar computation one gets
\[ B_2(\Delta_E^0) - B_2(\Delta_E^1) = -2\pi i \tau_0 (e[\delta_1(e), \delta_2(e)]) \]
The map \( W : eA^\infty \to (eA^\infty)_\varphi \), given by \( W(s) = sk \) is an isometry. Here \((eA^\infty)_\varphi\) is the completion of \( eA^\infty \) with respect to the inner product \((\xi, \eta)_\varphi = \tau_0 (\eta^* \xi k^{-2})\). One can see that
\[ \partial_E, \varphi \circ W = \partial_E \circ R_k, \]
where the operator \( \partial_E, \varphi \) is \( \partial_E \) but considered on \((eA^\infty)_\varphi\). This yields that \( W^* \Delta_E, \varphi W = R_k^* \Delta_E R_k \), that is \( \Delta_E, \varphi \) is unitarily equivalent to \( R_k^* \Delta_E R_k \). We employ a variational argument as in [8], Theorem 2.2 (cf. also [I]). Letting
\[ \Delta_E^s = e^{sh/2} \Delta_E e^{sh/2}, \]
one has
\[ \frac{d}{ds} \Delta_E^s = \frac{1}{2} (h \Delta_E^s + \Delta_E^s h). \]
We need to bear in mind that all multiplications are considered as right multiplication operators. Then
\[ \frac{d}{ds} \text{Tr} e^{-t \Delta_E^s} = -t \text{Tr} (h \Delta_E^s e^{-t \Delta_E^s}) = t \frac{d}{dt} \text{Tr} (he^{-t \Delta_E^s}). \]
Assuming that
\[ \text{Tr} (e^{-t \Delta_E^s}) \sim t^{-1} \sum_{n=0}^\infty a_n (\Delta_E^s) t^n/2, \]
and
\[ \text{Tr} (he^{-t \Delta_E^s}) \sim t^{-1} \sum_{n=0}^\infty a_n (h, \Delta_E^s) t^n/2, \]
one has
\[ \frac{d}{ds} a_j (\Delta_E^s) = \frac{j}{2} - 1 a_j (h, \Delta_E^s), \]
and hence
\[ \frac{d}{ds} a_2 (\Delta_E^s) = 0. \]
This shows that the \( B_2 \) term of \( \Delta_E, \varphi \) is independent of \( \varphi \) and therefore
\[ B_2(\Delta_E^0, \varphi) - B_2(\Delta_E^1, \varphi) = -2\pi i \tau_0 (e[\delta_1(e), \delta_2(e)]) = -1, \]
which proves the following theorem.

**Theorem 6.1.** With the above notation, the index of the operator \( \partial_E, \varphi \) is independent of the conformal class of the metric and
\[ \text{Index}(\partial_E, \varphi) = -1. \]
References

[1] A. Chamseddine, A. Connes, Scale invariance in the spectral action. J. Math. Phys. 47 (2006), N.6, 063504, 19 pp.

[2] P. B. Cohen, A. Connes, Conformal geometry of the irrational rotation algebra. Preprint MPI (92-93).

[3] A. Connes, $C^*$-algèbres et géométrie différentielle. C.R. Acad. Sc. Paris, t. 290, Série A, 599-604 (1980).

[4] A. Connes, Noncommutative geometry. Academic Press (1994).

[5] A. Connes, Noncommutative differential geometry. Inst. Hautes Etudes Sci. Publ. Math. No. 62 (1985), 257-360.

[6] A. Connes, The action functional in noncommutative geometry. Comm. Math. Phys. 117, no. 4, 673-683 (1988).

[7] A. Connes, M. Marcolli, Noncommutative Geometry, Quantum Fields and Motives. American Mathematical Society Colloquium Publications, 55 (2008).

[8] A. Connes, H. Moscovici, Modular curvature for noncommutative two-tori, arXiv:1110.3500.

[9] A. Connes, H. Moscovici, The local index formula in noncommutative geometry. Geom. Funct. Anal. 5, no. 2, 174243 (1995).

[10] A. Connes, H. Moscovici, Type III and spectral triples. Traces in number theory, geometry and quantum fields, 57-71, Aspects Math., E38, Friedr. Vieweg, Wiesbaden (2008).

[11] A. Connes, P. Tretkoff, The Gauss-Bonnet theorem for the noncommutative two torus, [arXiv:0910.0188](https://arxiv.org/abs/0910.0188). Noncommutative Geometry, Arithmetic, and Related Topics, Johns Hopkins University press, Proceedings of the Twenty-First Meeting of the Japan-U.S. Mathematics Institute edited by Caterina Consani and Alain Connes (2011).

[12] F. Fathizadeh, M. Khalkhali, The Gauss-Bonnet theorem for noncommutative two Tori with a general conformal structure, Journal of Noncommutative Geometry 6 (2012), no. 3, 457–480, [arXiv:1005.4947](https://arxiv.org/abs/1005.4947).

[13] F. Fathizadeh, M. Khalkhali, Scalar curvature for the noncommutative two torus, [arXiv:1110.3511](https://arxiv.org/abs/1110.3511) to appear in Journal of Noncommutative Geometry.

[14] P. Gilkey, Invariance theory, the heat equation, and the Atiyah-Singer index theorem. Mathematics Lecture Series, 11. Publish or Perish, Inc., Wilmington, DE (1984).

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