Complete constraints on a nonminimally coupled chaotic inflationary scenario from the cosmic microwave background

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We present complete constraints imposed from observations of the cosmic microwave background radiation (CMBR) on the chaotic inflationary scenario with a nonminimally coupled inflaton field proposed by Fakir and Unruh (FU). Our constraints are complete in the sense that we investigate both the scalar density perturbation and the tensor gravitational wave in the Jordan frame, as well as in the Einstein frame. This makes the constraints extremely strong without any ambiguities due to the choice of frames. We find that the FU scenario generates tiny tensor contributions to the CMBR relative to chaotic models in minimal coupling theory, in spite of its spectral index of scalar perturbation being slightly tilted. This means that the FU scenario will be excluded if any tensor contributions to CMBR are detected by the forthcoming satellite missions. Conversely, if no tensor nature is detected despite the tilted spectrum, a minimal chaotic scenario will be hard to explain and the FU scenario will be supported.

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I. INTRODUCTION

In spite of its many successes, the standard big-bang theory has faced serious problems, namely, the horizon, flatness, and monopole problems. In the beginning of the 1980s, an epoch-making idea called the inflationary scenario was advocated to solve these cosmological puzzles [1,2]. Later it was recognized that the concept gives us not only a solution to such puzzles, but also to the origin of density perturbations [3,4,5,6].

Among the various models of the inflationary scenario, Linde’s chaotic model [7] has been regarded as a feasible and natural mechanism for the realization of inflationary expansion. This model still has a serious problem; i.e., one has to fine-tune the self-coupling constant $\lambda$ of the inflaton unacceptably small to have a reasonable amplitude of the density perturbations.

On the other hand, the feasibility of inflation has been investigated in alternative theories of gravity, e.g., the Brans-Dicke scalar tensor theory [8,9], and nonminimal coupling theories of gravity [10,11]. Fakir and Unruh (FU) [12,13] proposed a way to avoid fine-tuning $\lambda$ by introducing a relatively large nonminimal coupling constant $|\xi| > 1$ in the context of the chaotic inflationary model. According to their results, the large value of $\xi$, i.e., order of $10^3$, allows us to have a reasonable value for the coupling constant $\lambda = 10^{-2}$. Thus the FU scenario remains a reasonable model of the inflationary scenario.

Constraints on the FU scenario are discussed by some authors [14,15] using the scalar perturbations generated during the inflationary phase. We investigated the spectrum of tensor mode cosmic microwave background radiation (CMBR) anisotropy [16]. Hwang also discussed the tensor mode power spectrum from inflation based on generalized gravity theories in a unified manner [17].

In discussing the constraint on generalized gravity theories including the FU scenario, one has to be careful about ambiguities associated with the conformal transformation. Sometimes the analysis is made in the conformally transformed frame in which the gravity may be described by the Einstein action. However, it has long been known that the conformal transformation often changes the physical phenomena in different frames (e.g., Ref. [18], and references therein). Thus it is quite important to make the frame dependences of the results one obtains clear.

The purpose of the present paper is to investigate the constraints on the FU scenario by taking into account the frame dependency. Namely, we shall investigate the CMBR anisotropy caused by both the scalar and tensor perturbations in two different frames, the Jordan and Einstein frames, which seem to have special physical importance among various transformed frames. From this point of view, this work can be regarded as the complete treatment of the observational constraints on the FU scenario.

| Parameter | Fakir-Unruh scenario | minimal chaotic scenario |
|-----------|----------------------|-------------------------|
| $r$       | $2 \times 10^{-4}$   | 0.2                     |
| $n_s$     | 0.97                 | 0.96                    |
| $n_t$     | $-3.0 \times 10^{-4}$| $-2.8 \times 10^{-2}$   |
| $\lambda/\xi^2$ | $4 \times 10^{-10}$ |                         |

TABLE I. Predicted parameters based on the Fakir-Unruh scenario. For comparison, the parameters of a minimal coupling chaotic scenario are also shown. All parameters are derived at $N(t_k) = 70$. 

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A number of investigations have shown that high precision CMBR temperature anisotropy and polarization experiments, e.g., two satellite missions of NASA’s Microwave Anisotropy Probe (MAP) [14] and ESA’s Planck Surveyor [20], can be used to determine many cosmological parameters to unprecedented precision [21,22,23]. We are especially interested in amplitudes and spectral indices of scalar and tensor perturbations. It is relatively hard to determine these parameters due to the cosmic variance and the cosmic confusion [24,25,26] by means of a temperature spectrum only. Including polarization informations allows us to detect tensor contributions directly because a tensor mode can generate the magnetic mode of polarization while a scalar mode cannot [27,28,29]. However, it is still hard to detect such a magnetic mode directly because of its predicted tiny amplitude. Even Planck, with the most sensitive experiment not only for temperature anisotropy but also for polarization, can detect tensor contributions only if the tensor to scalar ratio is greater than 0.2 [28,29].

For later convenience we summarize the relevant facts of the CMBR experiments here. The experiments can measure the angular power spectrum of the temperature or polarization correlation function $C_l$,

$$\frac{\delta T(\theta, \phi)}{T_0} = \sum_{l,m} a_{lm} Y_{lm}(\theta, \phi), \quad (1.1)$$

$$\langle a_{l'm'}^* a_{lm} \rangle = C_l \delta_{l'l} \delta_{m'm}, \quad (1.2)$$

where the angle brackets denote ensemble average. The Cosmic Background Explorer (COBE) Differential Microwave Radiometer (DMR) group expressed the observed quadrupole moment in terms of $Q_{\text{rms-PS}}$,

$$Q_{\text{rms-PS}} \equiv T_0 \sqrt{\frac{5C_2}{4\pi}}. \quad (1.3)$$

According to the COBE four-year results [30,31],

$$T_0 = 2.728 \pm 0.004 \text{ K}, \quad (1.4)$$

$$Q_{\text{rms-PS}} = 18 \pm 1.4 \mu\text{K} \quad (1.5)$$

for the Harrison-Zel’dovich spectrum. This gives $C_2^{\text{obs}} = 1.1 \times 10^{-10} \mu\text{K}^2$.

This paper is organized as follows. In Sec. II we review the background solutions of the inflationary expansion in both the Jordan and Einstein frames. In Secs. III and IV, we show the amplitude of the scalar curvature perturbation and the tensor gravitational wave generated during the de Sitter phase, and discuss the constraints by means of the observed CMBR quadrupole moment. Section V derives the predicted tensor to scalar ratio and describes the possibility of detecting tensor contributions. To compare our results with well-known results in minimal coupling theory and to interpret their physical meanings, we also discuss a consistency relation which includes the spectral index in Sec. VI. Finally, Sec. VII contains conclusions. Table I summarizes our results of the predicted observables based on the FU scenario. We shall follow Misner, Thorne, and Wheeler [12] for the definition of the Riemann tensor, Ricci tensor, and Ricci scalar, but the metric convention is chosen as $g = (+--).$

II. BACKGROUND INFLATIONARY SOLUTIONS

A. Jordan frame solutions

At first, we shall review the background inflationary solutions in the original Jordan frame. This section follows our previous paper [14]. We shall consider the following action:

$$A = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} + \frac{1}{2} \kappa^2 \phi^2 R - \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + V(\phi) \right], \quad (2.1)$$

where $\kappa^2 \equiv 8\pi G$. Our definition of $\kappa$ is the same as Fakir and Unruh [13], that is, conformal coupling means $\xi = -1/6$. Note that Futamase and Maeda [10] used an opposite sign for $\xi$. For the spatially flat Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t) \delta_{ij} dx^i dx^j, \quad (2.2)$$

we can derive the fundamental background equations

$$H^2 = \frac{\kappa^2}{3(1 + \kappa^2 \phi^2)} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) - 6\kappa H \phi \dot{\phi} \right], \quad (2.3)$$

$$\ddot{\phi} - 3H \dot{\phi} + \left[ \kappa^2 \phi^2 (1 + 6\xi) \right] \frac{\dot{\phi}^2}{\phi} = \frac{1}{1 + \kappa^2 \phi^2 (1 + 6\xi)} \left[ 4\kappa^2 \phi V(\phi) - (1 + \kappa^2 \phi^2) V_{,\phi} \right], \quad (2.4)$$

where overdots denote time derivatives in the Jordan frame and $V_{,\phi} \equiv \partial V/\partial \phi$. Now we shall employ the potential $V(\phi) = \lambda \phi^4/4$ and apply ordinary slow-roll approximations to the background equations. This gives us

$$H^2 \approx \frac{\kappa^2 \lambda \phi^4}{12(1 + \kappa^2 \phi^2)} \left[ 1 + \frac{8\xi}{1 + \kappa^2 \phi^2 (1 + 6\xi)} \right], \quad (2.5)$$

$$3H \dot{\phi} \approx - \frac{\lambda \phi^3}{1 + \kappa^2 \phi^2 (1 + 6\xi)}. \quad (2.6)$$

It is straightforward to find the self-consistent inflationary solutions under the condition $\kappa^2 \phi^2 \gg 1$. Defining $\psi \equiv \kappa^2 \phi^2$, the above equations take the following simple forms:
\[ H^2 = \frac{\lambda \psi}{12\kappa^2 \xi^2}, \quad \frac{\dot{\psi}}{H} = -\frac{8\xi}{1 + 6\xi}. \]  

(2.7)

These solutions lead to the well-known exponential expansion in the Jordan frame. The amount of expansion from any epoch to the end of inflation is calculated as

\[ N(t) \equiv \int_t^{t_f} H dt = \int_{\psi(t)}^{\psi(t_f)} \frac{H}{\psi} d\psi = \frac{1 + 6\xi}{8\xi} [\psi(t) - \psi_f]. \]

(2.8)

Note that for the initial value of \( \psi \),

\[ N(t_i) = \frac{1 + 6\xi}{8\xi}(\psi_i - \psi_f) \approx \frac{1 + 6\xi}{8\xi} \psi_i \geq 70 \]  

(2.9)

must be held to solve the cosmological puzzles.

### B. Einstein frame solutions

We shall perform the conformal transformation to the Einstein frame

\[ \hat{g}_{\mu\nu} = \Omega g_{\mu\nu}, \quad \Omega = 1 + \kappa^2 \xi \phi^2. \]

(2.10)

Hereafter we put hats on variables defined in the Einstein frame. The conformal transformation gives

\[ A = \int d^4x \sqrt{-\hat{g}} \left[ \frac{\dot{\hat{R}}}{2\kappa^2} + \frac{3}{2} F^2(\phi) \hat{g}^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + \hat{V}(\phi) \right], \]

(2.11)

where

\[ F^2(\phi) \equiv \frac{1 + \kappa^2 \xi \phi^2(1 + 6\xi)}{(1 + \kappa^2 \xi \phi^2)^2} \]

(2.12)

and

\[ \hat{V}(\phi) \equiv \frac{V(\phi)}{(1 + \kappa^2 \xi \phi^2)^2} = \frac{\lambda \phi^4}{4(1 + \kappa^2 \xi \phi^2)^2} \approx \frac{\lambda}{4\kappa^2 \xi^2}, \]

(2.13)

where the last equality of Eq. (2.13) is derived from the condition \( \kappa^2 \xi \phi^2 \gg 1 \).

To make the kinetic term of scalar field canonical form, we redefine the scalar field as

\[ \frac{d\hat{\phi}}{d\phi} = F(\phi) = \sqrt{1 + \kappa^2 \xi \phi^2(1 + 6\xi)} \quad \frac{1 + \kappa^2 \xi \phi^2}{1 + \kappa^2 \xi \phi^2}. \]

(2.14)

Then it can be clearly seen that the new potential (2.13) is still flat enough to lead to sufficient exponential inflation. When we investigate the dynamics of the universe in the Einstein frame, we should transform our coordinate system to make the metric the Robertson-Walker form

\[ \dot{a} = \sqrt{\Omega} a, \quad \dot{t} = \sqrt{\Omega} dt, \]

(2.15)

and we obtain

\[ ds^2 = dt^2 - \hat{a}^2(t) \delta_{ij} dx^i dx^j. \]

(2.16)

Note that the physical quantities in the Einstein frame should be defined in this coordinate system. Now the Einstein equation can be derived in the usual manner under the slow-roll approximations,

\[ \hat{H}^2 = \frac{\kappa^2}{3} \left[ \left( \frac{d\hat{\phi}}{dt} \right)^2 + \hat{V}(\hat{\phi}) \right] \approx \frac{\lambda}{12\kappa^2 \xi^2}, \]

(2.17)

where

\[ \hat{H} \equiv \frac{1}{\hat{a} \hat{d}} = \frac{1}{\sqrt{\Omega}} \left( \frac{H + \frac{1}{2} \Omega}{2} \right), \]

(2.18)

\[ \frac{d\hat{\phi}}{dt} = \left( \frac{d\phi}{dt} \right) \left( \frac{dt}{d\phi} \right) = \frac{\sqrt{1 + \kappa^2 \xi \phi^2} (1 + 6\xi)}{\Omega^{3/2}} \phi. \]

(2.19)

We can conclude that the exponential behavior of the expansion is retained in both frames [10]. Note that we can put a constraint on \( \lambda/\xi^2 \) by means of requiring \( V < m^2_{\text{pl}} \),

\[ \frac{\lambda}{\xi^2} < 256\pi^2. \]

(2.20)

This constraint is too weak compared with the observational constraints discussed below.

### III. SCALAR PERTURBATION

The scalar curvature perturbation \( R(t, x) \) generated as the quantum noise during the de Sitter phase is well-known in the Einstein frame. Let us choose the longitudinal gauge

\[ ds^2 = [1 + 2\Psi(x)] dt^2 - \hat{a}^2(t) [1 + 2\Phi(x)] \delta_{ij} dx^i dx^j. \]

(3.1)

We can construct the gauge-invariant scalar curvature perturbation from the metric and the inflaton field perturbation as follows:

\[ \mathcal{R}(x) = \Phi(x) - \frac{H}{\dot{\phi}} \phi(x). \]

(3.2)

Since we already have the prescriptions to quantize \( \Phi \) and \( \dot{\phi} \) in the Einstein frame [33,34], we can calculate the amplitude of scalar curvature perturbation

\[ \sqrt{\hat{P}}(k) = \sqrt{\frac{4\pi k^3}{(2\pi)^3}} \int d^3x e^{i k \cdot x} \left\langle \mathcal{R}(i, o) \mathcal{R}(i, x) \right\rangle \approx \frac{\hat{H}^2}{2\pi |d\dot{\phi}/dt|}. \]

(3.3)

(3.4)
where $k$ is a comoving wave number. Note that the metric perturbations defined in the Einstein frame have to be calculated in the coordinate system $\{\hat{\xi}^\mu\}$:

$$
\hat{g}_{\mu\nu}(\hat{x}) = \frac{\partial \hat{x}^\alpha}{\partial x^\mu} \frac{\partial \hat{x}^\beta}{\partial x^\nu} \delta_{\alpha\beta}(x)
$$

and for instance,

$$
\hat{\Phi}(\hat{x}) = \Phi(x) + \frac{1}{2} \frac{\delta \Omega}{\Omega}(x).
$$

Makino and Sasaki [33] and Fakir, Habib, and Unruh [34] proved that the amplitude of scalar perturbation in the Jordan frame exactly coincides with that in the Einstein frame. We can see such conformal invariance in the simple calculation

$$
\hat{R}(\hat{x}) = \hat{\Phi}(\hat{x}) - \frac{\hat{H}}{\phi} \frac{d\phi}{dt} \Phi(x) = \hat{\Phi} \Phi(x),
$$

This proof allows us to calculate the scalar power spectrum in the Jordan frame quite easily,

$$
\sqrt{P_S(k)} = \frac{1}{2\pi \sqrt{1 + 6\xi}} \frac{H^2}{\dot{\phi}} \left| \frac{\phi}{t_k} \right| = \frac{N(t_k)}{2\pi} \sqrt{\frac{\lambda}{3\xi(1 + 6\xi)}},
$$

where we used slow-roll approximations, and we have a corrected missing factor of 2 in the original paper of Makino and Sasaki [33]. The curvature perturbation gives the Newtonian potential perturbation $\Psi$ [28],

$$
\sqrt{\langle \Psi^2 \rangle_k} = \frac{2}{3} \sqrt{P_S(k)} \quad \text{radiation-dominated era},
$$

$$
\frac{3}{5} \sqrt{P_S(k)} \quad \text{matter-dominated era}.
$$

Since the observed CMBR quadrupole anisotropy is dominated by the Sachs-Wolfe (SW) effect [37,38], we can simply estimate it as

$$
C_2^{\text{scalar}} = \left\langle \frac{\delta T}{T_0} \right\rangle_{\text{SW}} = \left\langle \frac{1}{3} \Psi \right\rangle_{k = d_H^{-1}}^2
= \frac{2}{15} P_S(k = d_H^{-1}),
$$

where $d_H$ is the present Hubble horizon scale. We can thus constrain the set of parameters

$$
\frac{\lambda}{\xi^2} < 4.0 \times 10^{-10},
$$

where the inequality takes into account the contribution from a tensor mode. If we adopt $\lambda = 10^{-2}$, it gives $\xi > 5 \times 10^3$.

### IV. Tensor Perturbation

For completion, let us show our previous result [16] of the amplitude of tensor perturbation in the Jordan frame. In the synchronous gauge, the metric becomes

$$
ds^2 = a^2(\tau) \left[ d\tau^2 - (\delta_{ij} + h_{ij}) dx^i dx^j \right],
$$

where $\tau$ is a conformal time and $h_{ij}$ is a transverse-traceless perturbation. Note that $h_{ij}$ is invariant under any conformal transformations [34]. The power spectrum of the tensor perturbation can be derived as well as the scalar one,

$$
\sqrt{P_T(k)} = \sqrt{P_T(k)} = \frac{4 \pi k^3}{(2\pi)^3} \sum_{\lambda = +, \times} \int d^3 x e^{i k \cdot x} \langle h_\lambda(\tau, 0) h_\lambda(\tau, x) \rangle
\approx \frac{4}{\sqrt{\pi \alpha}} \frac{H}{\sqrt{1 + \psi}} \approx \frac{1}{\pi} \frac{\lambda}{6 \xi^2},
$$

where $\lambda = +, \times$ are modes of the polarization.

Since a tensor perturbation also causes temperature anisotropy via SW effect as well as a scalar one, we can constrain $\lambda/\xi^2$ from the CMBR observations [16,39],

$$
C_2^{\text{tensor}} \simeq 0.0363 P_T,
$$

gives

$$
\frac{\lambda}{\xi^2} < 1.8 \times 10^{-7},
$$

where the inequality also takes into account the contribution from the scalar mode. $\lambda = 10^{-2}$ gives $\xi > 2 \times 10^2$.

### V. Constraints from the Cosmic Microwave Background

Now we are in a position to predict the ratio of the amplitude of the tensor perturbation to that of the scalar one. It can be obtained from Eqs. (3.3) and (4.3):

$$
\frac{P_T}{P_S}(k) = \frac{0.00245}{6 \xi} \left( \frac{70}{N(t_k)} \right)^2.
$$

We should keep in mind that this result does not depend on the choice of frames and does not depend on $\xi$ directly in the limit of $\xi \gg 1$ but depends on $N(t_k)$ only. Equations (3.12) and (4.3) give the simple relation between $C_2^{\text{tensor}}/C_2^{\text{scalar}}$ and $P_T/P_S$,

$$
r \equiv \frac{C_2^{\text{tensor}}}{C_2^{\text{scalar}}} \simeq 0.9 \frac{P_T}{P_S}(k = d_H^{-1}).
$$

We already know that the FU scenario requires $\xi \gg 1$ to avoid the fine-tuning of $\lambda$, and the perturbations which
contribute to the present CMBR quadrupole moment have left the Hubble horizon scale at around \( N(t_k) = 70 \). We can thus conclude that the FU scenario predicts \( r \approx 0.002 \), which is too tiny to be detected by even the Planck Surveyor. In other words, if MAP or Planck could detect any tensor contributions to the CMBR, it would mean that the FU scenario could be excluded from good candidates of the inflationary model.

VI. SPECTRAL INDICES

In the previous sections, we discussed only the amplitudes of perturbations. Here let us consider the first-order solutions in the slow-roll approximations. The behavior of perturbations are fully governed by the simple Schrödinger type equation, and the first-order nature appears in the time-dependent mass term \( R''/R \),

\[
(R\Delta)''(k, \tau) + \left( k^2 - \frac{R''}{R} \right) (R\Delta)(k, \tau) = 0,
\]

where \( \Delta \) is a scalar or tensor perturbation \([40]\), and \( \Delta \) and \( \delta \) denote conformal time derivatives. Writing \( R = \alpha \sqrt{\mathcal{Q}} \), the usual slow-roll parameter \( \epsilon \) \([11,12]\) and a new parameter \( \alpha \) can be defined as

\[
\epsilon \equiv - \frac{\dot{H}}{H^2}, \quad \alpha \equiv \frac{\dot{\Omega}}{2H\Omega}.
\]

\( \epsilon \) and \( \alpha \) give rise to the spectral index of the scalar and the tensor mode

\[
n_s \equiv 1 + \frac{d\ln P_S}{d\ln k} = 1 - 2\epsilon - 2\alpha_s, \quad (6.3)
\]

\[
n_t \equiv \frac{d\ln P_T}{d\ln k} = -2\epsilon - 2\alpha_t, \quad (6.4)
\]

and the consistency relation \([\ref{6.5}]\)

\[
n_t = n_s - 1 + 2(\alpha_s - \alpha_t). \quad (6.5)
\]

All we should do is calculate \( \Delta \) and \( Q \) for both the scalar and tensor modes in both frames.

A. Scalar perturbation

Hwang calculated \( \Delta \) and \( Q \) for scalar perturbation in the Jordan frame directly \([14]\).

\[
\Delta_S = R\delta\phi = 0, \quad Q_S = \frac{\dot{\phi}^2 + (3/2)(\dot{\Omega}/\Omega)^2}{H + (1/2)(\dot{\Omega}/\Omega)^2}, \quad (6.6)
\]

where \( R\delta\phi = 0 \) is the scalar curvature perturbation in the uniform scalar field gauge, i.e., \( \delta\phi = 0 \) and \( \Omega = 1 + \psi \) is the conformal factor defined previously.

In the Einstein frame, we can use the well-known results from the minimal coupling theory

\[
\dot{Q}_S = \frac{\dot{\Omega}}{H(1+\beta)^2} \sim \epsilon + \beta, \quad (6.7)
\]

\( \dot{Q}_S \) is conformally transformed as \( \dot{Q}_S = Q_S/\Omega \), and it gives \( \dot{a}\sqrt{Q}_S = a\sqrt{Q}_S \), i.e., \( \dot{R} = R \). Defining another slow-roll parameter which appears in Eq. (6.6) as

\[
\beta \equiv \frac{\dot{\Omega}}{2H\Omega}, \quad (6.8)
\]

we can conformally transform the slow-roll parameters

\[
\dot{\epsilon} = \frac{\epsilon + \beta}{1+\beta} - \frac{\beta}{H(1+\beta)^2} \sim \epsilon + \beta, \quad (6.9)
\]

\[
\dot{\alpha}_s = \frac{\alpha_s - \beta}{1+\beta} \sim \alpha_s - \beta, \quad (6.10)
\]

and the spectral index is also transformed as

\[
n_s = 1 - 2\dot{\epsilon} - 2\dot{\alpha}_s = 1 - 2\epsilon - 2\alpha_s = n_s. \quad (6.11)
\]

Thus, we can conclude that \( n_s \) is invariant under the conformal transformation up to the first-order of the slow-roll approximations.

Now we are in a position to calculate \( n_s \) explicitly,

\[
\dot{\epsilon} \sim \frac{1}{2\kappa^2} \left( \frac{\dot{V}}{V} \right)^2 = \frac{1}{2\kappa^2} \left( \frac{\dot{V}}{V} \right)^2 \quad (6.12)
\]

\[
\dot{\alpha}_s \sim \frac{1}{\kappa^2} \left[ \left( \frac{\dot{V}}{V} \right)^2 - \left( \frac{\ddot{V}}{V} \right)^2 \right] \quad (6.13)
\]

\[
= 1.4 \times 10^{-2} \left( \frac{70}{N(t_k)} \right)^2 + 3.0 \times 10^{-4} \frac{1 + 6\xi}{6\xi} \left( \frac{70}{N(t_k)} \right)^2. \quad (6.14)
\]

With these quantities, we can rewrite the amplitude of perturbations as

\[
\sqrt{P_S(k)} = \frac{\dot{H}}{\sqrt{\pi m_{pl}} \sqrt{\epsilon}} \quad \sqrt{\dot{P}_S(k)} = \frac{4\dot{H}}{\sqrt{\pi m_{pl}}}, \quad (6.14)
\]
and Eq. (5.2) gives

\[ \hat{r} \simeq 14\hat{\epsilon} = 7(1 - \hat{n}_s - 2\hat{\alpha}_s). \] (6.15)

Note that \( \hat{r} \) depends on the potential steepness \( \hat{\epsilon} \) only.

We find that slow-roll parameters do not depend on \( \xi \) directly in the limit of \( \xi \gg 1 \) but depend on \( N(t_k) \) as well as \( r \). Here note that \( O(\epsilon) \sim O(\dot{\alpha}_s^2) \). Although it seems to be inconsistent with the first-order analysis, any higher-order terms than first-order are not important here, so it is sufficient for our purposes.

Finally, the spectral index of the scalar curvature perturbation can be calculated:

\[ n_s = 0.97, \quad r = \hat{r} = 0.002 \] (6.16)

at \( N(t_k) = 70 \). Although \( n_s \) is slightly tilted, the predicted \( r \) is still too small. Let us refer to a consistency relation here. We can see from Eq. (6.15) that the simplest relation \( \hat{r} = 7(1 - \hat{n}_s) \) is held only if \( 2\hat{\alpha}_s/(1 - \hat{n}_s) \ll 1 \), but Eq. (6.16) shows \( 2\hat{\alpha}_s/(1 - \hat{n}_s) \sim 1 \). Therefore, we must not use such a simple relation as that widely used to analyze CMBR power spectrum (e.g., Refs. [25,43]).

It is worth comparing the above results with the well-known results in a minimal chaotic scenario. Employing \( V_m(\phi_m) = \lambda \phi_m^4/4 \), we obtain

\[ \epsilon_m = \frac{8}{\kappa^2 \phi_m^2} = 1.4 \times 10^{-2} \left( \frac{70}{N(t_{m,k})} \right), \] (6.17)

\[ \alpha_{m,s} = \frac{4}{\kappa^2 \phi_m^2} = 0.7 \times 10^{-2} \left( \frac{70}{N(t_{m,k})} \right), \] (6.18)

where \( N(t_m) = \kappa^2 \phi_m^2(t_m)/8 \), and \( N(t_{m,k}) = 70 \) gives

\[ n_{m,s} = 0.96, \quad r_m = 0.2. \] (6.19)

Since \( 2\hat{\alpha}_{m,s}/(1 - \hat{n}_{m,s}) \sim 0.3 \), we still should not use the simplest relation. These results are very interesting. While both the FU and minimal chaotic scenarios give similar tilted spectra, the amount of the tensor contributions to the CMBR is quite different. This is because of the difference of the order of \( \epsilon \) between each of these theories. Physically, the scalar field in the FU scenario moves much slower than in the minimal one. It can be found in the flatness of the potential

\[ \hat{V}_{FU} = \frac{\lambda \phi^4}{4(1 + \kappa^2 \xi \phi^4)^2} \simeq \frac{\lambda}{4\kappa^4 \xi^2}, \] (6.20)

\[ V_m = \frac{\lambda}{4} \phi_m^4. \] (6.21)

Equation (6.3) shows that the tilted spectrum is produced by both the steepness and curvature of the potential shape, but Eq. (6.15) shows that the tensor to scalar ratio is determined by the steepness only. This is why the tensor contributions to the CMBR are quite different between each of the theories. As a result, we can determine which theory governs our universe by means of the observation of CMBR temperature anisotropy and polarization.

### B. Tensor Perturbation

We have already derived \( \Delta \) and \( Q \) for tensor perturbation in the Jordan frame [10].

\[ \Delta_T = h_\lambda, \quad Q_T = \Omega, \] (6.22)

and in the Einstein frame

\[ \hat{\Delta}_T = \hat{h}_\lambda, \quad \hat{Q}_T = 1. \] (6.23)

We thus find \( \hat{Q}_T = Q_T/\Omega \) and \( \hat{\alpha}_t = 0 = \alpha_t - \beta \). The spectral index of the tensor mode can be calculated in both frames,

\[ \hat{n}_t = -2\hat{\epsilon} = -2\epsilon - 2\beta = n_t \simeq -3.0 \times 10^{-4}. \] (6.24)

\( n_t \) is also conformally invariant and we can see that the tensor perturbation is almost scale invariant in the FU scenario. There is another expression of the consistency relation

\[ r = \hat{r} \simeq -7\hat{n}_t \simeq 0.002. \] (6.25)

### VII. CONCLUSIONS

We have investigated the feasibility of the FU scenario, which is the chaotic inflationary scenario characterized by a large value of the nonminimal coupling constant, by means of the forthcoming CMBR experiments. We have calculated the ratio of the quadrupole contribution of the tensor mode to one of the scalar mode. As a result, if any experiment could detect the tensor gravitational wave contributions to the CMBR under current sensitivities, the FU scenario would be excluded from good candidates of the inflationary model. In addition, we discussed the spectral index of the scalar perturbation to make sure of the consistency of our results. Even if the spectral index is tilted by \( n_s = 0.97 \), the tensor contributions are still too small to be detected. This is derived from the flatness of the potential slope in the FU scenario. However, if no evident tensor contributions were detected despite the tilted spectrum, a minimal chaotic scenario would fail and the FU scenario would be more plausible. Table II shows the summary of our results.

We found that the physical observables \( r, n_s, \) and \( n_t \) do not depend on \( \xi \) in the limit of \( \xi \gg 1 \) but depend on \( N(t_k) \) only. It should be emphasized that all of them do not depend on the choice of frames, that is, they are conformally invariant, so our results can be compared to observations directly without any ambiguities.

All of the results derived here are valid in both the Jordan and Einstein frames, and include both the scalar and tensor contributions one up to the first-order in the slow-roll approximations. In this sense, this work could be stated as complete.
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[1] K. Sato, Mon. Not. R. Astron. Soc. 195, 467 (1981); Phys. Lett. 99B, 66 (1981).
[2] A. Guth, Phys. Rev. D 23, 347 (1981).
[3] A. Guth and S. Y. Pi, Phys. Rev. Lett. 49, 1110 (1982).
[4] A. D. Linde, Phys. Lett. 116B, 335 (1982).
[5] S. Hawking, Phys. Lett. 115B, 295 (1982).
[6] A. A. Starovinsky, Phys. Lett. 117B, 175 (1982).
[7] A. D. Linde, Phys. Lett. 129B, 177 (1983).
[8] D. La and P. J. Steinhardt, Phys. Rev. Lett. 62, 376 (1989).
[9] P. J. Steinhardt and F. S. Accetta, Phys. Rev. Lett. 64, 2740 (1990).
[10] T. Futamase and K. Maeda, Phys. Rev. D 39, 399 (1989).
[11] F. S. Accetta, D. J. Zoller, and M. S. Turner, Phys. Rev. D 31, 3046 (1985).
[12] R. Fakir and W. G. Unruh, Phys. Rev. D 41, 1783 (1990).
[13] R. Fakir and W. G. Unruh, Phys. Rev. D 41, 1792 (1990).
[14] D. S. Salopek, Phys. Rev. Lett. 69, 3602 (1992).
[15] D. I. Kaiser, Phys. Rev. D 52, 4295 (1995).
[16] E. Komatsu and T. Futamase, Phys. Rev. D 58, 023 004 (1998); 58, 089 902 (1998).
[17] J. Hwang, Class. Quantum Grav. 15, 1401 (1998).
[18] R. Dick, Gen. Relativ. Gravit. 30, 435 (1998).
[19] C. Bennett et al., MAP home page, 1996, [http://map.gsfc.nasa.gov/](http://map.gsfc.nasa.gov/)
[20] M. Bersanelli et al., Plank home page, 1996, [http://astro.estec.esa.nl/Planck/](http://astro.estec.esa.nl/Planck/)
[21] G. Jungman, M. Kamionkowski, A. Kosowsky, and D. N. Spergel, Phys. Rev. D 54, 1332 (1996).
[22] J. R. Bond, G. Efstathiou, and M. Tegmark, Mon. Not. R. Astron. Soc., 291, L33 (1997).
[23] M. Zaldarriaga, D. N. Spergel, and U. Seljak, Astrophys. J. 488, 1 (1997).
[24] L. Knox, Phys. Rev. D 52, 4307 (1995).
[25] J. R. Bond, R. Crittenden, R. L. Davis, G. Efstathiou, and P. J. Steinhardt, Phys. Rev. Lett. 72, 13 (1994).
[26] G. Efstathiou and J. R. Bond, astro-ph/9807103, 1998.
[27] W. Hu and M. White, Phys. Rev. D 56, 596 (1997).
[28] U. Seljak and M. Zaldarriaga, Phys. Rev. Lett. 78, 2054 (1997).
[29] M. Zaldarriaga, Ph.D. thesis, Massachusetts Institute of Technology, 1998.
[30] D. J. Fixsen et al., Astrophys. J. 473, 576 (1996).
[31] C. L. Bennett et al., Astrophys. J. Lett. 464, L1 (1996).
[32] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitaion (Freeman, San Francisco, 1973).
[33] N. Makino and M. Sasaki, Prog. Theor. Phys. 86, 103 (1991).
[34] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. 215, 203 (1992).
[35] R. Fakir, S. Habib, and W. G. Unruh, Astrophys. J. 394, 396 (1992).
[36] M. Sasaki, Prog. Theor. Phys. 89, 1183 (1993).
[37] R. K. Sachs and A. M. Wolfe, Astrophys. J. 147, 73 (1967).
[38] W. Hu and N. Sugiyama, Astrophys. J. 444, 489 (1995).
[39] M. White, Phys. Rev. D 46, 4198 (1992).
[40] J. Hwang, Phys. Rev. D 53, 762 (1996).
[41] E. D. Stewart and D. H. Lyth, Phys. Lett. B 302, 171 (1993).
[42] A. R. Liddle, P. Parsons, and J. B. Barrow, Phys. Rev. D 50, 7222 (1994).
[43] R. Crittenden, J. R. Bond, R. L. Davis, G. Efstathiou, and P. J. Steinhardt, Phys. Rev. Lett. 71, 324 (1993).
[44] J. Hwang, Class. Quantum Grav. 14, 1981 (1997).