Functional Response Designs via the Analytic Permutation Test

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November 11, 2021

Abstract

Vast literature on experimental design extends from Fisher and Snedecor to the modern day. When data lies beyond the assumption of univariate normality, nonparametric methods including rank based statistics and permutation tests are enlisted. The permutation test is a versatile exact nonparametric significance test that requires drastically fewer assumptions than similar parametric tests. The main downfall of the permutation test is high computational cost making this approach laborious for complex data and sophisticated experimental designs and completely infeasible in any application requiring speedy results such as high throughput streaming data. We rectify this problem through application of concentration inequalities and thus propose a computation free permutation test—i.e. a permutation-less permutation test. This general framework is applied to multivariate, matrix-valued, and functional data. We improve these concentration bounds via a novel incomplete beta transform. We extend our theory from 2-sample to $k$-sample testing through the use of weakly dependent Rademacher chaoses and modified decoupling inequalities. We test this methodology on classic functional data sets including the Berkeley growth curves and the phoneme dataset. We further consider analysis of spoken vowel sound under two experimental designs: the Latin square and the randomized block design.

Keywords: Concentration of Measure; Functional Data Analysis; Latin Squares Kahane–Khintchine Inequalities; Randomized Block Design

*The authors gratefully acknowledge the Natural Sciences and Engineering Research Council of Canada and the Pacific Institute for the Mathematical Sciences
1 Introduction

Exact significance tests date back to the very origins of statistical hypothesis testing as an alternative to parametric testing. Namely, Fisher’s exact test tests for independence between the rows or columns of a $2 \times 2$ contingency table by directly using the hypergeometric distribution instead of relying on large sample asymptotic statistics such as the chi-squared test. As a consequence, it obtains the exact p-value of the data without relying on large sample asymptotics. However, Fisher’s exact test is severely limited as extension to general $r \times c$ tables requires significant amounts of computational power to enumerate or approximate the entire discrete distribution (Good, 1956; Agresti, 1992).

Permutation tests comprise a large subclass of such exact significance tests and have been thoroughly studied (Mielke and Berry, 2007; Basso et al., 2009; Pesarin and Salmaso, 2010; Brombin and Salmaso, 2013; Good, 2013). Given a sample $X = \{X_1, \ldots, X_n\} \in \mathcal{X}$ for some measure space $\mathcal{X}$, a permutation test considers the finite sampling distribution of a test statistic $T(X)$ over a discrete group where the distribution of $T$ is invariant for any group action on the observed data (Kallenberg, 2006)—i.e. for a group $G$, $T(gX) \overset{d}{=} T(X)$ for any $g \in G$. A canonical example is one-way ANOVA; see Basso et al. (2009) Section 5.2 for more details.

The permutation test requires far fewer assumptions than standard parametric approaches—namely that of exchangability under the null hypothesis—and is thus robust against deviations from distributional assumptions like normality and provides guaranteed performance for finite samples. The main limitation is that of computation. Performing a two sample permutation test for real valued data is trivial with modern computers. What if we were to perform a $k$ sample test with $\binom{k}{2}$ post-hoc comparisons taking multiple testing into account for, say, covariance operators as in Pigoli et al. (2014); Cabassi et al. (2017) where every permutation requires computation of the singular value decomposition (SVD) of a large matrix? Furthermore, what if we desire a more sophisticated experimental design such as a randomized block, Latin square, or unreplicated factorial design with the addition of multiple testing corrections? The amount of computation required to get accurate p-values will be prohibitive. The speech data and design considered in Section 5 would, for example, require 264 SVDs per permutation and with 66 hypotheses to test at, say, 2000 permutations each requires nearly 35 million SVDs. For matrices with dimension $100 \times 100$, this would take an estimated 36 hours on an Intel Core i7-7567U CPU at 3.50GHz. For a $400 \times 400$ matrix, it would take 74 days.

In this article, we present a unified methodology for performing computation-free permutation tests for $k$ sample testing in commutative and non-commutative $L^q$ spaces, which includes multivariate and functional data and covariance matrices and operators among other data types. Specifically, we consider the distribution of a test statistic on a discrete space of invariant group actions. Instead of taking random draws from that space to get a conditional Monte Carlo estimate (Hemerik and Goeman, 2018) of the p-value, we apply recent extensions of the Kahane-Khintchine inequality for commutative and non-commutative Banach spaces (Pisier and Xu, 2003; Garling, 2007; Spektor, 2016) in order to achieve sub-Gaussian bounds on the tail probability of our test statistic. Namely, we seek a result like $P(T(\pi) \geq T^*) \leq \exp(-Ct^2)$ for some universal constant.
C > 0 depending only on the space in which the data lives irrespective of sample size and dimension. This methodology is presented in Section 2 for two sample testing within commutative $L^q$ spaces—e.g. univariate, vector valued, and functional data—as well as within non-commutative $L^q$ spaces—e.g. covariance matrices and operators. As such universal constants are often less than optimal for statistical use, we introduce an adjustment for these upper bounds based on Talagrand’s concentration inequality (Talagrand, 1996) and the incomplete beta function in Section 2.4. An extension to testing on $k$-samples is considered in Section 3 making use of Rademacher chaoses and decoupling inequalities (Kwapien, 1987; De la Peña and Giné, 2012). Section 5 extends these ideas to multi-factor designs for the analysis of spoken vowel phonemes.

Most previous work on fast or computation-free permutation testing focus on univariate data in the setting of large scale testing typically applied to testing for genomics data. The recent work of He et al. (2019) achieves this goal by using Stolarsky’s invariance principle. In Yang et al. (2019), “very small” p-values are approximated via sequential Monte Carlo and the Edgeworth expansion. In Segal et al. (2018), an asymptotic approximation and a clever partitioning/resampling scheme is used to approximate small p-values. Density approximation via Pearson curves (Solomon and Stephens, 1978) has recently reemerged for p-value approximation in machine learning (Gretton et al., 2012) and neuroimaging (Winkler et al., 2016) among other areas. While past work is focused on large scale two-sample testing, this work is motivated by $k$-sample tests and more sophisticated experimental designs with functional and operator responses. While permutation tests have been used both for pointwise and curve-wise analysis of functional data (Cox and Lee, 2008; Corain et al., 2014; Chakraborty and Chaudhuri, 2015; Pigoli et al., 2014, 2018; Cabassi et al., 2017), approaching statistical hypothesis testing via analytic estimation of a permutation test p-value in general Banach spaces has not been deeply explored as of yet.

As a proof of concept for testing within commutative and non-commutative $L^q$ spaces, we consider a variety of simulated and real data sets in Section 4. In Section 5, our bounds are applied to testing for phonological differences among twelve spoken vowel sounds performed as a complete randomized block design on covariance operators with respect to two binary blocking factors: the speaker’s country of origin \{Canada, China\} and sex \{male, female\}. We also consider a Latin square design for checking the data for within subject pronunciation changes while running the experiment. Section 5 contains more detail on the data, experimental design, and its results. Proofs of the main theorems, the necessary theoretical development, further data experiments, and a discussion of past results are contained in the supplementary material.

2 Two sample testing

2.1 Univariate data

Let $n = m_1 + m_2$ and $X_1, \ldots, X_n \in \mathbb{R}$ be independent random variables such that $\mathbb{E}X_i = \mu_1$ for $i \leq m_1$ and $\mathbb{E}X_i = \mu_2$ for $i \geq m_1 + 1$. We wish to test $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$. To test these hypotheses using a permutation test, we treat $X_1, \ldots, X_n \in$
\( \mathbb{R} \) as fixed and consider \( \pi \in S_n \) a random permutation uniformly distributed on the symmetric group on \( n \) elements. That is, \( \pi \) is a bijective map \( \pi : \{1, \ldots, n\} \to \{1, \ldots, n\} \). Thus, we can consider the randomly permuted test statistic

\[
T(\pi) = \frac{1}{s_n} \left[ \frac{1}{m_1} \sum_{i=1}^{m_1} X_{\pi(i)} - \frac{1}{m_2} \sum_{i=m_1+1}^{n} X_{\pi(i)} \right],
\]

which is normalized by the sample standard deviation \( s_n \) for the entire set \( X_1, \ldots, X_n \). The conditional tail probability is

\[
P(T(\pi) \geq t \mid X_1, \ldots, X_n) = \frac{1}{n!} \sum_{\pi \in S_n} \mathbf{1}[T(\pi) \geq t].
\]

Let \( T_0 \) be the test statistic \( T(\pi) \) when \( \pi \) is the identity—i.e. the original ordering. Then, the p-value for the above hypothesis test is \( P(T(\pi) \geq T_0) \), which is often approximated by randomly generating \( N \ll n! \) random permutations from \( S_n \) instead of exhaustively enumerating all elements of \( S_n \). This results in an overly conservative test for p-values approaching \( 1/N \).

To avoid the simulation-based approximation of equation 2.2, we instead prove a sub-Gaussian bound on the p-value.

**Theorem 2.1 (Univariate Data).** For \( T(\pi) \) from equation 2.1 with \( m_1 = \kappa m_2 \) for some \( \kappa \geq 1 \), then \( P(T(\pi) \geq t) \leq \exp(-nt^2/2[\kappa + 1]^3) \).

This theorem is extended to more advanced settings, including vectors, matrices, functional data, and operators, in the following sections. We state those subsequent theorems for balanced samples, but note that the imbalanced setting of Theorem 2.1 can also be incorporated with a similar constant \( \kappa \). The proofs are more tedious for imbalanced data, but no additional innovation is required.

### 2.2 Commutative \( L^q \) Spaces

To extend our tail bounds beyond the real valued setting, we require some definitions. Note that both of the following definitions extend to the case of compact operators on Banach spaces.

**Definition 2.1 (Matrix Square Root).** Let \( A \in \mathbb{R}^{d \times d} \) with \( d \geq 2 \) be a symmetric positive semi-definite matrix with eigen-decomposition \( A = U D U^T \) where \( U = (v_1 \ v_2 \ \ldots \ v_d) \) is the orthonormal matrix of eigenvectors and \( D \) is the diagonal matrix of eigenvalues, \((\lambda_1, \ldots, \lambda_d)\). Then, \( A^{1/2} = U D^{1/2} U^T \) where \( D^{1/2} \) is the diagonal matrix with entries \((\lambda_1^{1/2}, \ldots, \lambda_d^{1/2})\).

\(^1\)Note that \( s_n \) is invariant under permutation and is only included to make the below formulation nicer.
Definition 2.2 (q-Schatten norm for matrices). For an arbitrary matrix \( A \in \mathbb{R}^{k \times l} \) and \( q \in (1, \infty) \), the q-Schatten norm is \( \| A \|_{S_q} = \text{tr}( (A^T A)^{q/2} ) = \| \nu \|_{\ell_q}^q = \sum_{i=1}^{\min(k,l)} \nu_i^q \) where \( \nu = (\nu_1, \ldots, \nu_{\min(k,l)}) \) is the vector of singular values of \( A \) and where \( \| \cdot \|_{\ell_q} \) is the standard \( \ell^q \) norm in \( \mathbb{R}^d \). In the covariance matrix case where \( A \in \mathbb{R}^{d \times d} \) is symmetric and positive-definite, \( \| A \|_{S_q}^p = \text{tr}(A^p) = \| \lambda \|_{\ell_q}^p \) where \( \lambda \) is the vector of eigenvalues of \( A \).

When \( q = \infty \), we have the standard operator norm on \( \ell^2(\mathbb{R}^d) \), \( \| A \|_{S_\infty} = \sup_{v \in \mathbb{R}^d, \| v \|_{\ell_2} = 1} \| Av \|_{\ell_2} = \sup_{v \in \mathbb{R}^d, \| v \|_{\ell_2} = 1} v^T Av \). In the covariance matrix setting, this coincides with the maximal eigenvalue of \( A \).

Let \( X_1, \ldots, X_n \in \mathcal{X} \) where \( \{ \mathcal{X}, \| \cdot \| \} \) is a commutative \( L^q \) space. The test statistic of interest is \( T_0 = \| \sum_{i \leq m} X_i - \sum_{i > m} X_i \| \). Then, Theorem 2.1 can be extended to such settings using a version of the Kahane-Khintchine inequality under a weak dependency condition from Theorem A.7 proved in the supplementary material. For simplicity of notation, we assume that the \( X_i \) are centred about the sample mean and that the samples are balanced.

Theorem 2.2 (Commutative \( L^q \) Spaces). Let \( m = n/2 \), \( \| \cdot \|_{S_q} \) be the q-Schatten norm for matrices or operators, and \( \varepsilon_1, \ldots, \varepsilon_n \) be Rademacher random variables such that \( \sum_{i=1}^n \varepsilon_i = 0 \). Let \( X_1(t), \ldots, X_n(t) \) be continuous function on a compact interval with empirical covariance operator \( \hat{\Sigma}(s,t) = (n - 1)^{-1} \sum_{i=1}^n X_i(s)X_i(t) \). Let \( q \in [1, \infty) \) with norm \( \| \cdot \|_{L^q} \). For \( T(\pi) = \| \sum_{i=1}^n \varepsilon_i X_i \|_{L^q} \). Then, \( \text{P} ( T(\pi) \geq t ) \leq \exp \left( -t^2 / c \| \hat{\Sigma}^{1/2} \|_{S_q}^2 \right) \).

Remark 2.3 (On optimal constants). The optimal constant \( c \) in the above theorem follows from the optimal constant in the Kahane-Khintchine inequality, which is not currently known.\(^2\) However, it is strongly conjectured to agree with the optimal constant for the standard Khintchine inequality. In that case, we would take \( c = 64 \) in the above theorem, which is 16 from Theorem 2.1 times 4 from that fact that \( T(\pi) \) is not a symmetric random variable. For more details, see the proof and discussion in the supplementary material. We also empirically adjust the \( p \)-values in Section 2.4, which is demonstrated to give strong performance in Sections 4 and 5.

2.3 Non-Commutative \( L^q \) Spaces

Following from the previous section, we outline similar tail bounds in non-commutative \( L^q \) spaces (Pisier and Xu, 2003). This methodology encompasses matrix and operator data with emphasis on application to testing for equality of covariances. Hence, the following theorem is applied to symmetric positive definite operators in the example below and to the data in Section 5. The test statistic of interest is still \( T_0 = \| \sum_{i \leq m} X_i - \sum_{i > m} X_i \| \), but with the \( X_i \) now belonging to a non-commutative \( L^q \) space.

Theorem 2.3 (Non-Commutative \( L^q \) Spaces). Let \( \| \cdot \|_{S_q} \) be the q-Schatten norm for a matrix or operator and \( \varepsilon_1, \ldots, \varepsilon_n \) be Rademacher random variables such that \( \sum_{i=1}^n \varepsilon_i = \)

\(^2\)It took about 60 years from the advent of the original Khintchine inequality for optimal constants to be determined.
0. For \( d, d' > 1 \), let \( X_1, \ldots, X_n \in \mathbb{R}^{d \times d'} \) be a collection of \( n \) fixed matrices (or let \( X_1, \ldots, X_n \) be a collection of bounded linear operators). For \( T(\pi) = \|\sum_{i=1}^n \varepsilon_i X_i\|_{S^q} \), there exists a universal constant \( c > 0 \) such that \( P(T(\pi) > t) \leq \exp\left(-\frac{t^2}{c S^2}\right) \) where \( S = \max\{\|\left(\sum_{i=1}^n X_i X_i^*\right)^{1/2}\|_{S^q}, \|\left(\sum_{i=1}^n X_i^* X_i\right)^{1/2}\|_{S^q}\} \) with \( X_i^* \) the adjoint operator.

Remark 2.4. Of particular interest are covariance operators being compact trace-class self-adjoint operators. Consequently, we have the same bound but with \( S = \|\left(\sum_{i=1}^n X_i^2\right)^{1/2}\|_{S^q} \).

2.4 Beta and Empirical Beta Adjustment

Inequalities such as the Kahane-Khintchine inequalities are useful tools for considering the finite sample performance of a statistical method. However, the biggest impediment to the use of such inequalities, as well as other concentration inequalities, for statistical inference is the nearly inevitable loss in power to reject the null due to ‘universal constants’ that are too large for application. We thus propose a transformation based on the beta distribution to correct the p-values and recover the lost statistical power. Proposition 2.5 only applies to univariate data and requires an asymptotic argument outside of our finite sample focus. It is included nevertheless to set the stage for the non-asymptotic beta transform in Theorem 2.4. Furthermore, this simpler setting yields explicit beta parameters and demonstrates stellar performance in both the simulated data of Section 4.1 and the extremely imbalanced small sample setting explored in Kashlak and Yuan (2020) where, for example, \( m_1 = 335 \) and \( m_2 = 3 \).

For a statistical test, if the correct test size is achieved, then a random null p-value will be distributed as Uniform \([0, 1]\). However, our Kahane-Khintchine based null p-values will instead closely follow a more general Beta \((\alpha, \beta)\) distribution. Thus, identification of the parameters \( \alpha \) and \( \beta \) will allow us to adjust the p-values to the null setting to recover lost statistical power. This idea is spiritually similar to the Pearson curve method (Solomon and Stephens, 1978), but that approach requires estimation of the first 4 central moments for comparison with the family of generalized Pearson distributions compared to our more focused use of the beta distribution with Theorem 2.4 proved to justify such focus. Usage of the Edgeworth expansion (Hall, 2013) is another method with a long history, but requires some care to note whether or not a finite number of terms in such an expansion can provide a satisfactory approximation to the probability density in question (Stuart et al., 1994).

We first consider the univariate case of Section 2.1 before discussing the more general Banach space setting for the beta transform.

Proposition 2.5. Under the setting of Theorem 2.1 with \( n \) sufficiently large,

\[
P\left(\exp\left\{-nT(\pi)^2/2[\kappa + 1]^3\right\} < u\right) \leq C_0 I\left(u; \frac{[\kappa + 1]^3}{2 + \kappa + \kappa^{-1}}, \frac{1}{2}\right)
\]

where \( I(u; \alpha, \beta) \) is the regularized incomplete beta function and

\[
C_0 = \left(\frac{[\kappa + 1]^3}{2 + \kappa + \kappa^{-1}}\right)^{1/2} \Gamma\left(\frac{[\kappa + 1]^3}{2 + \kappa + \kappa^{-1}}\right) \Gamma\left(\frac{1}{2}, \frac{[\kappa + 1]^3}{2 + \kappa + \kappa^{-1}}\right)^{-1}.
\]
Proposition 2.5 allows us to adjust the p-values from Theorem 2.1 so that our test statistic achieves the desired empirical size. The refined bound is
\[
P (T(\pi) > t) \leq C_0 I \left( e^{-nt^2/2[\kappa+1]^3}, \frac{\lceil \kappa + 1 \rceil^3}{2 + \kappa + \kappa^{-1}}, \frac{1}{2} \right)
\]
This adjustment is shown to work in the simulations detailed in Figure 1. For the more general Banach space setting, we can use Talagrand’s concentration inequality (Talagrand, 1996) to prove the following theorem.

**Theorem 2.4.** Let \((\mathcal{X}, \|\cdot\|)\) be a Banach space with separable dual space \(\mathcal{X}^*\), and let \(h : \mathbb{R} \to \mathbb{R}\) be monotonically increasing. For any random variable \(X\) taking values in \(\mathcal{X}\) such that \(\mathbb{E} h(\|X\|)^2 < \infty\) and \(\sup_{X \in \mathcal{X}} h(\|X\|) < U < \infty\) and for \(u \in (0,1)\) and some constants \(C, c, \alpha, \beta > 0\), \(P \left( e^{-h(\|X\|)/c} < u \right) \leq CI(u; \alpha, \beta)\) where \(I(u; \alpha, \beta)\) is the incomplete beta function for \(c\) sufficiently large.

**Remark 2.6.** Theorem 2.4 requires the Banach space \(\mathcal{X}\) to have a separable dual. This stems from writing the norm as a countable supremum for use within Talagrand’s concentration inequality (Talagrand, 1996). We can directly apply this result to commutative and non-commutative \(L^q\) spaces for \(1 < q < \infty\). However, \(L^\infty\) is a standard example of a non-separable Banach space. For our purposes, we can avoid this issue as it is typical in functional data analysis to consider the uniform norm on the space of continuous bounded functions with compact support.

When working in commutative and non-commutative \(L^q\) spaces, we no longer have easily defined constants for the right-hand bound in Theorem 2.4. Hence, we instead propose an empirical beta transform, outlined in Algorithm 1, which estimates these constants. To do this, we must choose a small number \(r\) of permutations to draw uniformly at random from \(S_n\). In practice, we find that \(r = 10\) or \(20\) is sufficient to achieve good results on real data. From these, we compute test statistics sampled from the null setting, which will yield a collection of \(r\) p-values. These p-values can in turn be used to estimate the parameters for a beta distribution via the method of moments estimate \(\hat{\alpha}\) and \(\hat{\beta}\). Lastly, the p-value \(p_0\) produced by \(T_0\) can be adjusted by application of the incomplete beta function: \(I(p_0; \hat{\alpha}, \hat{\beta})\). This method was applied to most of the data examples detailed in Section 4. This transform is also shown to work well for the construction of wild bootstrap confidence regions for least squares and ridge regression (Burak and Kashlak, 2021).

### 3 k sample testing

For general one-way ANOVA and more complex experimental designs, we extend the above two sample tests to \(k\) level factors. The two challenges to overcome are (1) proper multiple testing correction for the \(\binom{k}{2}\) pairwise comparisons and (2) the construction of a global p-value. Classical hypothesis testing would have us first reject the global hypothesis and follow up with pairwise post-hoc testing. For permutation tests, we begin with pairwise testing and combine these tests into a global p-value.
Algorithm 1 The Empirical Beta Transform

Compute p-value $p_0$ from test statistic $T_0$ using Theorem 2.2 or 2.3.
Choose $r > 1$, the number of permutations to simulate—e.g. $r = 10$.
Draw $\pi_1, \ldots, \pi_r$ from $S_n$ uniformly at random.
Compute $p$-values $p_1, \ldots, p_r$ from test statistics $T_{\pi_1}, \ldots, T_{\pi_r}$.
Find the method of moments estimator for $\alpha$ and $\beta$.
Estimate first and second central moments of the $p_i$ by $\bar{p}$ and $s^2$.
Estimate $\hat{\alpha} = \bar{p}^2(1 - \bar{p})/s^2 - \bar{p}$.
Estimate $\hat{\beta} = [\bar{p}(1 - \bar{p})/s^2 - 1][1 - \bar{p}]$.
Return the adjusted p-value $I(p_0; \hat{\alpha}, \hat{\beta})$.

For one-way ANOVA, let $X_{i,j}$ be the $j$th observation from category $i$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$ under the model

$$X_{i,j} = \mu + \tau_i + \varepsilon_{i,j} \quad (3.1)$$

with global mean $\mu$, $i$th treatment effect $\tau_i$ with $\sum_{i=1}^k \tau_i = 0$, and exchangeable errors $\varepsilon_{i,j}$—i.e. permutationally invariant (Kallenberg, 2006). We wish to test the following:

| Pairwise | $H_0^{(ij)} : \tau_i = \tau_j$ | $H_1^{(ij)} : \tau_i \neq \tau_j$ |
|----------|--------------------------------|----------------------------------|
| Global   | $H_0 : \tau_1 = \ldots = \tau_k = 0$ | $H_1 : \exists \tau_i \neq 0$. |

Under the pairwise null $H_0^{(ij)}$, the difference in category means is $\bar{X}_i - \bar{X}_j = \bar{\varepsilon}_i - \bar{\varepsilon}_j$. Thus, the permutation test requires exchangeable errors—i.e. the distribution of $\bar{\varepsilon}_i - \bar{\varepsilon}_j$ is invariant under any random permutation. This is weaker than the standard iid setup and, most critically, does not require normality.

3.1 Multiple Pairwise Tests

From Section 2, we can compute test statistics $T_0^{(ij)}$ for $H_0^{(ij)}$ and consider the permutation distribution of $T^{(ij)}(\pi)$ for some uniformly distributed $\pi \in S_{n_i+n_j}$ for familywise type I error control, the pairwise statistics come from independent applications of independent Rademacher vectors. Hence, we can rely on standard multiple testing corrections such as the simple Bonferroni correction as proposed in (Basso et al., 2009, Chapter 5) or the slightly more involved step-down procedure used in Cabassi et al. (2017). Other methods include Holm’s stepdown method (Holm, 1979), the approach outlined in the canonical text Westfall and Young (1993), and the more recent Romano and Wolf (2005). In experimental design, some authors even prefer to forego such corrections and report raw uncorrected p-values (Wu and Hamada, 2011). The focus of this article is on computation of the raw p-values and hence, application of one’s favourite multiple testing correction is left to the reader. For the phonological data analysis in Section 5, we will just consider the raw p-values and the Bonferroni correction.
3.2 Global Test

The k-sample global significance test statistic can be written as a combination of the pairwise statistics:

\[ T_0 = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} n_i n_j (T_{ij}^0)^2 \]  

(3.2)

To test the significance of \( T_0 \), a permutation framework can be implemented in one of three ways; see Basso et al. (2009) Chapter 5 for more details. The first is the pooled method in which the entire data set of \( N = n_1 + \ldots + n_k \) observations is permuted. The second is by aggregation of the pairwise statistics where each permutation is applied independently to each pair of samples. The third is the synchronized method which only applies to balanced designs—i.e. \( n_1 = \ldots = n_k \)—in which the same permutations are applied to each category pairing \((i, j)\). This is the preferable approach when the design is balanced (Basso et al., 2009). As we have already discussed individual pairwise testing, we focus on the synchronized test in the context of our Kahane–Khintchine methodology.

Remark 3.1. Beyond univariate data, the above test statistic \( T_0 \) can be considered on the direct sum of \( \kappa = \binom{k}{2} \) Banach spaces. That is, for a sequence of Banach spaces \( (X_i, \|\cdot\|_i) \) and elements \( X_i \in X_i \), we can define a new Banach space by the \( \ell^2 \) direct sum \( (X_i)_{i=1}^n \in (\bigoplus_{i=1}^\kappa X_i)_{\ell^2} \) with norm \( \|(X_i)_{i=1}^n\| = (\sum_{i=1}^n \|X_i\|_i^2)^{1/2} \). See any text on discussing sequences in Banach spaces such as Diestel et al. (1995) for more details.

The synchronized setting is the preferred approach for balanced designs; see, for example, Basso et al. (2009); Cabassi et al. (2017). This approach applies the same permutations to each pairing. Let \( X^{(1)}, \ldots, X^{(k)} \) be \( m \)-long column vectors containing the observations of samples \( 1, \ldots, k \), respectively. Then, let \( X \) be the \( 2m \times \binom{k}{2} \) matrix with columns

\[
X = \begin{pmatrix}
X^{(1)} & X^{(2)} & \cdots & X^{(k-1)} & X^{(k)}
\end{pmatrix}.
\]

Lastly, let \( \varepsilon^T = (\varepsilon_1, \ldots, \varepsilon_{2m}) \) such that \( \sum \varepsilon_i = 0 \). The synchronized permuted version of the global test statistic in Equation 3.2 is then \( \|X^T \varepsilon\|_{\ell^2}^2 = \sum_{i,j=1}^{2m} a_{i,j} \varepsilon_i \varepsilon_j \) for \( a_{i,j} \), the \( i, j \)th entry in \( XX^T \). This is a second order Rademacher chaos (Ledoux and Talagrand, 1991, Section 4.4) except that the \( \varepsilon_i \) are not iid. In this case, we still have a sub-Gaussian bound achievable via a decoupling argument (Kwapien, 1987) with proof in the supplementary material. See De la Pena and Giné (2012) for more on decoupling inequalities.

Theorem 3.1. Let \( T = \|X^T \varepsilon\|_{\ell^2} \) for \( X \) the above \( 2m \times \binom{k}{2} \) matrix and \( \varepsilon_i \) such that \( \sum \varepsilon_i = 0 \). Then, for some universal constant \( c \), \( \Pr(T > t) \leq \exp\left[-t^2/cS\right] \) where \( S = \|XX^T\|_2^2 \).

Remark 3.2. Up to constant \( c \), this theorem coincides with the result for a two sample test as for \( X = (x_1^{(1)}, \ldots, x_m^{(1)}, x_1^{(2)}, \ldots, x_m^{(2)})^T \), the term \( S \) equals the sample variance of the \( x_i^{(j)} \). However, the constant \( c \) emerging from the proof is very large. This universal constant problem is rectified via the empirical beta transform presented in Section 2.4.
4 Data Examples

4.1 Univariate Data

4.1.1 Two Sample Test

The performance of Theorem 2.1 on simulated data is displayed in Figure 1 for balanced and for imbalanced samples averaged over 1000 replications. In the balanced case, we simulate $m_1 = m_2 = 100$ Gaussian random variates with distributions $N(0, 1)$ and $N(\mu, 1)$ for $\mu \in [0, 1]$. We compare the classic student’s t-test to the permutation test with 1000 permutations, the bounds from Theorem 2.1 with $\kappa = 1$, and the beta adjusted bound from Proposition 2.5. Notably, the balanced Khintchine bound returns p-values just slightly larger than the standard t-test while the beta adjusted bound is even tighter. For the imbalanced case, the sample sizes are now $m_1 = 140, m_2 = 60$ and $\kappa = 2.33$. The imbalanced bound is not as sharp, but the beta adjusted bound still gives a close approximation to the t-test p-value.

4.1.2 K Sample Test

The performance of Theorem 3.1 for comparing $k$ samples of size $n$ via a synchronized permutation test is demonstrated in Figure 2. For this simulation, $k = 4, 16$ for the left and right plot, respectively, samples of size $n = 20$ were generated as random Gaussian variates with variance 1 and with mean 0 for the first $k - 1$ sets and with mean $\mu \in [0, 2]$ for the $k$th set. As $\mu$ grows, the p-value for the standard F-test, the synchronized permutation test, and the beta-adjusted p-value from Theorem 3.1 all decrease in tandem for $k = 4$ with the unadjusted bound above the others. In the
Figure 2: Univariate $k$ sample test for normal data with $k = 4$ (left) and $k = 16$ (right) balanced samples of size $n = 20$. The figure compares the standard F-test (black) to the synchronized permutation test (red) with 1000 permutations and to the unadjusted (green) and beta adjusted (blue) bounds from Theorem 3.1.

In the $k = 16$ case, the beta adjusted bound and the synchronized permutation test return the same p-values until the lines approach the permutation boundary at $-\log_2(1001)$. More notably, they slightly differ from the classic F-test as for relatively large $k$ and small $n$ the synchronized permutation test returns marginally different p-values than the F-test. A total of 1000 random permutations were generated for the synchronized permutation test, and this simulation was replicated 1000 times to create these plots.

### 4.2 Berkeley Growth Curves: Functional Means

To demonstrate Theorem 2.2, we apply it to the classic Berkeley growth curve dataset (Ramsay and Silverman, 2005). This dataset contains measurements of 93 children—39 males and 54 females—taken at 31 time points between the ages of 1 and 18 years. A set of 30 curves was randomly selected from the male curves and 30 curves from the female curves to test for a difference in the population mean curves based on those observations. This was repeated 100 times to see the resulting p-values under the $L^1$, $L^2$, and $L^\infty$ norms. Table 1 displays the percentage of rejections. Applying Theorem 2.2 results in a reasonable number of rejections under the $L^1$ topology. However, differences are not detectable in $L^2$ or $L^\infty$. This is rectified via the empirical beta adjustment.

### 4.3 Phoneme Curves: Covariance Operators

We apply Theorem 2.3 to the classic phoneme dataset (Ferraty and Vieu, 2006), which consists of 400 log-periodograms for 5 different phonemes—the vowel from ‘dark’ aa, the vowel from ‘water’ ao, the plosive d-sound dc1, the fricative sh-sound sh, the vowel from

---

1 This data is available in the R package *fda* (Ramsay et al., 2018).
Table 1: Displayed above are the percentages of rejections by using Theorem 2.2 (left) and the beta adjusted bound (right) at test sizes 5%, 1%, 0.1% for the $L^1$, $L^2$, and $L^\infty$ norms.

| Size  | Percentage of Rejections |
|-------|--------------------------|
|       | Kahane Bound             | Beta Adjusted          |
|       | 5% | 1% | 0.1% | 5% | 1% | 0.1% |
| $L^1$ | 86% | 42% | 7%   | 85% | 55% | 31% |
| $L^2$ | 0%  | 0%  | 0%   | 100%| 88% | 77% |
| $L^\infty$ | 0% | 0%  | 0%   | 100%| 100%| 98% |

Table 2: The percentage of rejected two sample tests at the 1% level comparing two different phonemes with a sample size of $m_1 = m_2 = 10$ under the trace, Hilbert-Schmidt, and operator norms.

| Trace Norm | Hilbert-Schmidt Norm | Operator Norm |
|------------|----------------------|---------------|
|            | a  c  d  f           | a  c  d  f    | a  c  d  f |
| o          | 100                   | 52            | 0           |
| d          | 100 100               | 93 86         | 15 23       |
| f          | 100 100 100           | 99 100 100    | 88 97 91    |
| i          | 100 100 100 100       | 100 100 100 100 | 100 100 100 100 |

Using the notation of the International Phonetic Alphabet (IPA), $aa$ is $a$, $ao$ is $\alpha$, $dcl$ is $d$, $sh$ is $f$, and $iy$ is $i$. To produce covariance operators for testing, we first randomly permute the order of the 400 curves, then group these curves into sets of 10 to produce a set of 40 covariance operators for each of the five phoneme classes. This is replicated 100 times with different random groupings of curves.

We apply our method after using the empirical beta adjustment from Section 2.4 to each of the 10 pairwise comparisons between phonemes resulting in Table 2. In the trace norm topology, all $100 \times 10$ pairwise tests result in rejection for a test size of 1%. The Hilbert-Schmidt norm only detects a significant difference between $a$ and $c$ about 52% of the time whereas the operator norm fails to detect any significance between those two phonemes. The difference between phonemes $a$ and $c$ is hardest to identify among the 10 pairings.

\footnote{This data is available in the R package \texttt{fds} (Shang and Hyndman, 2013).}

she $iy$—sampled at 150 frequencies. Using the notation of the International Phonetic Alphabet (IPA), $aa$ is $a$, $ao$ is $\alpha$, $dcl$ is $d$, $sh$ is $f$, and $iy$ is $i$. To produce covariance operators for testing, we first randomly permute the order of the 400 curves, then group these curves into sets of 10 to produce a set of 40 covariance operators for each of the five phoneme classes. This is replicated 100 times with different random groupings of curves.

We apply our method after using the empirical beta adjustment from Section 2.4 to each of the 10 pairwise comparisons between phonemes resulting in Table 2. In the trace norm topology, all $100 \times 10$ pairwise tests result in rejection for a test size of 1%. The Hilbert-Schmidt norm only detects a significant difference between $a$ and $c$ about 52% of the time whereas the operator norm fails to detect any significance between those two phonemes. The difference between phonemes $a$ and $c$ is hardest to identify among the 10 pairings.

\footnote{This data is available in the R package \texttt{fds} (Shang and Hyndman, 2013).}
Table 3: The twelve vowel phonemes considered in our dataset along with the 12 spoken words used to produce those vowels.

|  |  |  |  |  |
|---|---|---|---|---|
| i | weave | e | waiter | ε | wetter |
| 1 | wiggle | 9 | what | u | woohoo |
| 3 | world | o | woke | θ | wonder |
| 2 | what | u | woohoo | o | woods |
| 6 | water | o | woods | u | woohoo |

5 Phonological differences between vowels

Taking inspiration from the classic phoneme dataset (Hastie et al., 1995; Ferraty and Vieu, 2006) discussed previously in Section 4.3, we consider a new data set of log-periodograms for the phonemes of 12 spoken vowels detailed in Table 3. This data is available at https://sites.ualberta.ca/~kashlak/kashData.html.

The raw data consists of 12 phonemes recorded 12 times each from 4 different speakers. The data was recorded on a Tascam DR-05 portable linear PCM audio recorder as a mono 24-bit wave file sampled at 96 kHz, which is currently considered high definition audio in contrast to the standard 16-bit 44.1 kHz audio on compact discs. The primary vowel phoneme was extracted as a 170 millisecond clip corresponding to $16384 = 2^{14}$ samples. These clips were transformed into log periodograms via the tuneR package (Ligges et al., 2018) as displayed in Figure 3 for a single speaker. As is common with functional data, the raw log-periodograms were first smoothed. In this case, cubic smoothing splines were used. However, many other smoothing methods can be and have been applied to functional data.

Two experimental designs were employed in the collection of this data and will be tested in the following subsections. First, the 12 words were vocalized 12 times in a Latin square design. Each row corresponds to a replication of speaking all of the 12 words, and each column corresponds to the order of the words within a replication. This was done to test for changes in speech during the recording period. Secondly, this Latin square design was replicated for 4 different speakers with two binary blocking factors male/female and Canadian/Chinese. Thus, we have a $12 \times 4$ complete randomized block design with functional responses. The total sample size is $576 = 12 \times 12 \times 4$ log-periodogram curves.

5.1 Latin square design for functional means

For an unreplicated Latin square design, we cannot perform a permutation test for the significance of each factor simultaneously. Exchangability under the null hypothesis for one factor requires fixing the levels of all other factors when permuting labels. However, if we fix the Latin square row and column indices then only a single observation remains leaving nothing to permute. To rectify this, a stepdown approach as in Basso et al.

\[\text{Note that this data was collected outside of a proper laboratory setting to be a proof-of-concept for the proposed methodology as opposed to an in depth study of language.}\]
Figure 3: The log-periodograms of all 12 vowel phonemes spoken 12 times by one of the speakers considered over the first 1000 frequencies.
(2009) chapter 7 for unreplicated factorial designs can be applied. As a permutation test requires exchangeable observations under the null hypothesis, test statistics for each factor are first computed. Beginning with the largest, if that null hypothesis holds, then this implies that all other null hypotheses hold and hence acts as the global null allowing for the data to be permuted. If this null is rejected, then we proceed to test the second largest test statistic while fixing the levels of the first factor. Once a null is not rejected, this method stops. Otherwise, all factors can be tested except for the last one as rejecting all other null hypotheses would leave no room for further permutations.

For the vowel data, we have 12-level row, column, and vowel factors giving the model

$$y_{ijk}(t) = \mu(t) + \text{row}_i(t) + \text{column}_j(t) + \text{vowel}_k(t) + \xi_{ijk}$$

where $y_{ijk}(t)$ is a smoothed log-periodogram, $\mu$ is the global mean, and the $\xi_{ijk}$ are mean zero exchangeable errors. For all four subjects, the vowel factor produced a much larger test statistic than the row and column effects as expected indicating consistency of the speaker during the experiment. Thus, after rejecting the null of there being no difference among the spoken vowels, the row or column factor can be considered. For all four subjects, the row and column effects were not deemed to be statistically significant—i.e. there were no detectable changes in pronunciation across the recording session. Pairwise comparison of the vowels for each subject was also performed. However, of the 66 pairwise hypotheses to test, one subject rejected 25 nulls, another rejected only 5 nulls, and the last two rejected 0 nulls after taking multiple testing into account. This is in contrast to the randomized block design discussed in the next section that, making use of the entire dataset and covariance operators, identifies 60 of the 66 pairings as significantly different.

Before computing the test statistics and p-values in the randomized block design discussed next, each log-periodogram was centred by subtracting off the row and column effects from the Latin square design. This resulted in an improvement in the reported p-values, which were larger in the case that the row and column effects were not removed.
5.2 Complete Randomized block design for functional data

A complete randomized block design (CRBD) aims to test a treatment effect as in one-way ANOVA but with the addition of blocking factors to account for sources of variation unrelated to the treatment of interest. For functional data, a CRBD can be performed by using the synchronized permutation test for two-way ANOVA from chapter 6 of Basso et al. (2009) combined with the Kahane-Khintchine based tail bound. To achieve this, a difference between the functional means or covariances is computed for each of the $\binom{12}{2}$ vowel pairings while holding the levels of the blocking factors constant. For each pairing, the test statistics can be summed over the levels of the blocking factors thus removing any influence from interaction terms even though they are generally assumed to be negligible in this setting. Theorems 2.2 and 2.3 can be applied for functional means and covariance operators respectively to bound the pairwise p-values. The computed test statistics can be aggregated using Theorem 3.1 to get a global p-value. Note that a standard permutation test would require the computation of $264 = 66 \times 4$ test statistics via simulation from the symmetric group, which in the case of covariance operators and Schatten norms implies 264 SVD calculations per permutation. This is further expanded by, say, performing $132,000 = 66 \times 2000$ permutations to be able to test each hypothesis at the 0.01 level after correcting for multiple testing. Focusing only on the approximately 35 million required SVDs, a timing test run on an Intel Core i7-7567U CPU at 3.50GHz estimates 36 hours of compute time when considering 100 dimensional matrices and an estimated run time of 74 days on 400 $\times$ 400 dimensional matrices.

This approach was applied pairwise to the sample covariance operators for each vowel as past work has emphasized that the covariance structure of speech data is the best lens to detect phonological differences (Pigoli et al., 2014, 2018). Application of Theorem 2.3 using the trace norm (1-Schatten norm) and using the empirical beta adjustment from Section 2.4 produced the 66 pairwise p-values displayed in Figure 5. The words are also grouped by p-value to display which vowel phonemes proved statistically indistinguishable using our proposed methodology. The use of other Schatten norms results in lower power—i.e. fewer null hypotheses rejected.

The blocking factors \{male,female\} and \{Canadian, Chinese\} can also be similarly tested without removing the row and column effects from the Latin square design; otherwise, the mean taken over the entire dataset will be zero. In trace norm, we get p-values of 0.0002 and 0.00003 for sex and country, respectively. In Hilbert-Schmidt norm, we get the weaker p-values 0.03 and 0.07.

6 Discussion

The p-value has stood for over a century as a pillar of frequentist statistical methodology. In this article, we approached $k$-sample testing through application of an analytic approximation to the permutation test p-value notably without relying on simulation of the permutation distribution of the test statistic. Experimental design for functional data was the main motivation for this work as standard simulation-based permutation testing can be applied but at a high computational cost. Other applications of interest include
Figure 5: $\log_{10}$ p-values for pairwise two sample tests between vowel pairs under the trace norm.
online testing where data must be processed, results returned, and decisions made in real time. The lag resulting from a classic permutation test is unacceptable in such settings. This methodology is generally applicable to other complex testing settings including other types of group invariances—e.g. rotationally invariant test statistics. Furthermore, the duality of hypothesis testing with confidence sets suggests investigation into using variants of the Kahane–Khintchine inequality to construct confidence balls for estimators with finite sample guarantees on the coverage.

**Supplementary Material**

Primarily, the supplementary material contains proofs of the main results as well as auxiliary theorems and lemmas. Secondly, it contains further exposition of this methodology on simulated and real data sets. In particular, these methods are shown to produce correct p-values in the null setting.

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A Inequalities

A.1 Khintchine-type Inequalities

Theorem A.1 (Khintchine’s Inequality (1923)). For any $p \in (0, \infty)$, there exist positive finite constants $A_p$ and $B_p$ such that for any sequence $x_1, \ldots, x_n \in \mathbb{R}$ (or $x_i \in \mathbb{C}$),

$$A_p \|x\|_{\ell^2}^p \leq \mathbb{E} \left| \sum_{i=1}^{n} \varepsilon_i x_i \right|^p \leq B_p \|x\|_{\ell^2}^p$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are iid Rademacher random variables—i.e. $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = 1/2$. 

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For this article, we are only concerned with the upper bound $B_{2p}$ for $p > 2$. In Garling (2007), $B_{2p} = [(2p)!/2^p p!]^{1/2p}$ which gives $B_{2p} < \sqrt{2p}$, but also via Stirling’s inequality $B_p \sim (p/e)^{1/2}$ as $p \to \infty$. The expectation in above theorem is with respect to the $\varepsilon_i$ corresponding to a uniform distribution on the $2^n$ vertices of the $n$-hypercube. In what follows, we consider expectation over the uniform distribution on the $n!$ elements of the symmetric group $S_n$. This will be denoted $E_x$ where $\pi \in S_n$ is treated as a uniform random permutation.

In Spektor (2016), the restricted Khintchine inequality is introduced where it is required that $\sum_{i=1}^n \varepsilon_i = 0$ introducing a weak dependency among the $\varepsilon_i$. In the proof in Spektor (2016), this weak dependency doubles the variance by comparing two sets of data. Thus, the constant becomes $B_{2p} = [(2p)!/p!]^{1/2p}$.

**Theorem A.2** (Spektor (2016) Theorem 1.1). For any $p \in [2, \infty)$, there exist positive finite constant $B_p$ such that for any sequence $x_1, \ldots, x_n \in \mathbb{R}$,

$$
E \left| \sum_{i=1}^n \varepsilon_i x_i \right|^p \leq B_p \left( \|x\|_2^2 - n\bar{x}^2 \right)^{p/2} = B_p [(n-1)s_n^2]^{p/2} \tag{A.1}
$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are Rademacher random variables such that $\sum \varepsilon_i = 0$ and $s_n^2 = (n-1)^{-1} \sum_{i=1}^n (x_i - \bar{x})^2$ is the sample variance of $x$.

**Remark A.3.** In the statistics context, if we divide Inequality A.1 by $m = n/2$, we have $E_x [ \bar{x}_1^{(\pi)} - \bar{x}_2^{(\pi)} ]^p \leq B_p (2s_n^2/m)^{p/2}$ where $\bar{x}_1^{(\pi)}$ is the average of the first $m$ of the $x_{\pi(i)}$ for some random permutation $\pi$ and similarly for $\bar{x}_2^{(\pi)}$.

The previous theorem only applies to a balanced two sample setting. In the following, we extend the ideas in Spektor (2016) to the imbalanced testing setting. Other such extensions to imbalanced Khintchine inequalities were considered in Spektor (2014). Note that in the following theorem, the bound on the right-hand-side is in terms of the smaller of the two sample sizes $m_2 < m_1$ reducing the power drastically in a highly imbalanced setting. Nevertheless, it still is seen to be an excellent statistical tool in a variety of applied settings after the beta correction is applied (Kashlak and Yuan, 2020).

**Theorem A.4** (Imbalanced Case). For $m_1 > m_2 > 0$, let $n = m_1 + m_2$ and $M = m_1 - m_2$ and let $\kappa m_2 = m_1$ for some rational $\kappa > 1$. Let $\delta_1, \ldots, \delta_n$ be weighted dependent Rademacher random variables such that marginally $P(\delta_i = 1/m_1) = P(\delta_i = -1/m_2) = 1/2$ and such that $\sum \delta_i = 0$—i.e. precisely $m_1$ of the $\delta_i$ equal $1/m_1$ and $m_2$ equal $-1/m_2$. For any $p \in [2, \infty)$, there exists a positive finite constant $B_p$ such that for any sequence $x_1, \ldots, x_n \in \mathbb{R}$,\footnote{This theorem is also valid for $x_i \in \mathbb{C}$ after standard alterations are made in the proof.}

$$
E \left| \sum_{i=1}^n \delta_i x_i \right|^p \leq B_p \left( \frac{[\kappa + 1]s_n^2}{2m_2} \right)^{p/2}
$$

where $s_n^2 = (n-1)^{-1} \sum_{i=1}^n (x_i - \bar{x})^2$ is the sample variance of $x$.\footnote{This theorem is also valid for $x_i \in \mathbb{C}$ after standard alterations are made in the proof.}
Lemma A.5. Let $\xi_1, \xi_2 \in [1, \infty]$ be such that $\xi_1^{-1} + \xi_2^{-1} = 1$, and let $X, Y$ be positive real random variables. Then,

$$\min_{\xi_1, \xi_2; \xi_1^{-1} + \xi_2^{-1} = 1} \mathbb{E} \left\{ \xi_1^{p-1} X^p + \xi_2^{p-1} Y^p \right\} = \left( (EX)^{1/p} + (EY)^{1/p} \right)^p.$$ 

Proof. We note that $\xi_2 = \xi_1/(\xi_1 - 1)$. Then,

$$0 = \frac{d}{d\xi_1} \left\{ \xi_1^{p-1} EX^p + \xi_2^{p-1} EY^p \right\} = (p - 1)\xi_1^{p-2} EX^p - (p - 1)\xi_1^{p-2} EY^p/(\xi_1 - 1)^p$$

$$= \xi_1 = 1 + (EY/E^p)^{1/p}$$

$$\xi_2 = 1 + (EX/EY)^{1/p}$$

Hence,

$$\min_{\xi_1, \xi_2; \xi_1^{-1} + \xi_2^{-1} = 1} \mathbb{E} \left\{ \xi_1^{p-1} X^p + \xi_2^{p-1} Y^p \right\}$$

$$= \left[ 1 + \left( \frac{EY}{EX} \right)^{1/p} \right]^{p-1} EX^p + \left[ 1 + \left( \frac{EX}{EY} \right)^{1/p} \right]^{p-1} EY^p$$

$$= \left[ (EX)^{1/p} + (EY)^{1/p} \right]^{p-1} (EX)^{1/p} + \left[ (EX)^{1/p} + (EY)^{1/p} \right]^{p-1} (EY)^{1/p}$$

$$= \left[ (EX)^{1/p} + (EY)^{1/p} \right]^p$$

$\square$

Proof of Theorem A.4. We first decompose the weighted Rademacher sum. Without loss of generality, assume $m_1 > m_2$ and let $n = m_1 + m_2$ and $M = m_1 - m_2$. Also, assume the $x_i$ are centred—i.e. $\sum_{i=1}^n x_i = 0$—and let $\xi_1, \xi_2 > 0$ such that $\xi_1^{-1} + \xi_2^{-1} = 1$. Thus, via convexity, we have

$$\mathbb{E} \left| \sum_{i=1}^n \delta_{\pi(i)} x_i \right|^p$$

$$= \mathbb{E}_{\pi} \left| \frac{1}{m_2} \sum_{i=1}^{m_2} x_{\pi(i)} - \frac{1}{m_1} \sum_{i=m_2+1}^n x_{\pi(i)} \right|^p$$

$$= \mathbb{E}_{\pi} \left| \frac{1}{m_2} \left\{ \sum_{i=1}^{m_2} x_{\pi(i)} - \sum_{i=m_2+1}^{2m_2} x_{\pi(i)} \right\} - \frac{1}{m_1} \sum_{i=1}^{m_2} x_{\pi(i)} + \frac{M}{m_1 m_2} \sum_{i=m_2+1}^{2m_2} x_{\pi(i)} \right|^p$$

$$\leq \frac{\xi_1^{p-1}}{m_2^p} \mathbb{E}_{\pi} \left| \sum_{i=1}^{m_2} x_{\pi(i)} - \sum_{i=m_2+1}^{2m_2} x_{\pi(i)} \right|^p + \frac{\xi_2^{p-1} M^p}{m_1^p} \mathbb{E}_{\pi} \left| \sum_{i=1}^{n} x_{\pi(i)} - \frac{1}{m_2} \sum_{i=m_2+1}^{2m_2} x_{\pi(i)} \right|^p$$

$$= \frac{\xi_1^{p-1}}{m_2^p} (I) + \frac{\xi_2^{p-1} M^p}{m_1^p} (II).$$
To bound (I), we apply the balanced weakly dependent Khintchine inequality. Let $I \subset \{1, \ldots, n\}$ with cardinality $|I| = M$. For such an index set $I$, let $\Pi_I = \{\pi \in S_n : \pi(\{2m_1 + 1, \ldots, n\}) = I\}$. That is, $\pi \in \Pi_I$ maps the final $M$ indices into $I$. Note that $|\Pi_I| = \binom{n}{M}$. As a result,

$$E_{\pi} \left| \sum_{m_2+1}^{2m_2} x_{\pi(i)} - \sum_{i=m_2+1}^{2m_2} x_{\pi(i)} \right|^p \leq \frac{1}{n!} \sum_{\pi \in S_n} \left| \sum_{i=1}^{m_2} x_{\pi(i)} - \sum_{i=m_2+1}^{2m_2} x_{\pi(i)} \right|^p$$

$$\leq \frac{(n-M)!}{n!} \sum_{|I|=M} \frac{1}{(n-M)!} \sum_{\pi \in \Pi_I} \left| \sum_{i=1}^{m_2} x_{\pi(i)} - \sum_{i=m_2+1}^{2m_2} x_{\pi(i)} \right|^p$$

$$\leq M!(n-M)! \sum_{|I|=M} \left[ B_p^2 (2m_2 - 1)^{p/2} \left( \sum_{i \notin I} x_i^2 \right)^{p/2} \right]$$

$$\leq B_p^2 (2m_2 - 1)^{p/2} s_n^p.$$

As the $x_i$ are centred, we have that $(\sum_{i \notin I} x_i^2)^{p/2} \leq s_n^p = (\sum x_i^2)^{p/2}$, and hence

$$\frac{\xi_1^{p-1}}{m_2} E_{\pi} \left| \sum_{i=1}^{m_2} x_{\pi(i)} - \sum_{i=m_2+1}^{2m_2} x_{\pi(i)} \right|^p \leq \xi_1^{p-1} B_p^2 (2m_2 - 1)^{p/2} s_n^p.$$

For (II), we first assume that $\kappa$ is a positive integer and $m_1 = \kappa m_2$ so $M = (\kappa-1)m_2$. In this case, we have

$$\frac{\xi_2^{p-1} M^p}{m_1^p} (\Pi) = \xi_2^{p-1} \left( \frac{\kappa - 1}{\kappa} \right)^p E_{\pi} \left| \sum_{i=m_2+1}^{n} \tilde{\delta}_i x_i \right|^p$$

where $\tilde{\delta}_i$ are weighted Rademacher random variables with taking values $1/M$ or $-1/m_2$ such that $\sum \tilde{\delta}_i = 0$. Applying Lemma A.5 gives

$$E_{\pi} \left| \sum_{i=1}^{n} \tilde{\delta}_i x_i \right|^p \leq \left\{ B_p^{1/p} \left( \frac{s_2}{m_2} \right)^{1/2} + \left( \frac{\kappa - 1}{\kappa} \right) \left( E_{\pi} \left| \sum_{i=m_2+1}^{n} \tilde{\delta}_i x_i \right|^p \right)^{1/p} \right\}^p.$$

Noting that $E_{\pi} \left| \sum_{i=m_2+1}^{n} \tilde{\delta}_i x_i \right|^p$ is merely the original term to be bounded but with $m_1 = \kappa m_2$ and $M = (\kappa-1)m_2$ replaced by $(\kappa-1)m_2$ and $(\kappa-2)m_2$, respectively, we apply this idea $\kappa - 1$ more times to get

$$E_{\pi} \left| \sum_{i=1}^{n} \delta_i x_i \right|^p \leq B_p \left( \frac{s_2}{m_2} \right)^{p/2} \left( 1 + \left( \frac{\kappa - 1}{\kappa} \right) \left( 1 + \left( \frac{\kappa - 2}{\kappa - 1} \right) \left( \cdots (1 + 1/2) \cdots \right) \right) \right)^p$$

$$\leq B_p \left( \frac{s_2}{m_2} \right)^{p/2} \left( 1 + \frac{\kappa - 1}{\kappa} + \frac{\kappa - 2}{\kappa} + \cdots + \frac{1}{\kappa} \right)^p$$

$$\leq B_p \left( \frac{s_2}{m_2} \frac{(\kappa + 1)^2}{2} \right)^{p/2}.$$
Noting that \( n = m_1 + m_2 = (\kappa + 1)m_2 \), we have
\[
E_{\pi} \left| \sum_{i=1}^{n} \delta_i x_i \right|^p \leq B_p s_n^p \left( \frac{(\kappa + 1)^3}{2n} \right)^{p/2}.
\]

Now, consider \( \kappa = a + r \in \mathbb{Q} \) with \( a \in \mathbb{N} \) and \( r \in [0, 1) \). Then,
\[
B_p \left( \frac{2s_n^2}{m_2} \right)^{p/2} \left\{ 1 + \frac{\kappa - 1}{\kappa} + \frac{\kappa - 2}{\kappa} + \ldots + \frac{r}{\kappa} \right\}^p \\
\leq B_p \left( \frac{2s_n^2}{m_2} \right)^{p/2} \left\{ \frac{a}{\kappa} + \frac{a - 1}{\kappa} + \frac{a - 2}{\kappa} + \ldots + \frac{1}{\kappa} + \frac{(a + 1)r}{\kappa} \right\}^p \\
\leq B_p \left( \frac{2s_n^2}{m_2} \right)^{p/2} \left\{ \frac{1}{\kappa^p} \left( \frac{a(a + 1)}{2} + (a + 1)r \right) \right\}^p \\
\leq B_p \left( \frac{s_n^2 (a + 1)^2}{m_2} \right)^{p/2} \left\{ \frac{1 + r}{\kappa} \right\}^p.
\]

Noting further that \( ra + r < a + r \) so that \( 1 + r/(a + r) < 1 + 1/(a + 1) \), we multiply by \((a + 1)\) on each side to get \((a + 1)(1 + r/(a + r)) < a + 2\). Hence,
\[
B_p \left( \frac{s_n^2}{m_2} \right)^{p/2} \left\{ 1 + \frac{\kappa - 1}{\kappa} + \frac{\kappa - 2}{\kappa} + \ldots + \frac{r}{\kappa} \right\}^p \leq B_p \left( \frac{s_n^2 (a + 2)^2}{m_2} \right)^{p/2}.
\]

Hence, for \( \kappa \in \mathbb{N} \), we have \( \kappa + 1 = \lceil \kappa + 1 \rceil \), and for \( \kappa \) a non-integer we have \( a + 2 = \lfloor \kappa \rfloor + 2 = \lceil \kappa \rceil + 1 = \lceil \kappa + 1 \rceil \).

### A.2 Kahane-Khintchine-type Inequalities

Kahane extended Khintchine’s inequality from the real line to normed spaces Kahane (1964); Latała and Oleszkiewicz (1994). The optimal value for the constant \( C_{p,p'} \) in Theorem A.6 below is not known in the case of interest for this article, \( p > p' = 2 \); however, it has been conjectured to be the same as in the real case, and as we see from the simulations and real data experiments, this conjecture seems to hold for our purposes. In what follows, let \( \mathcal{X} \) be a normed space with norm \( \| \cdot \| \). Those spaces of statistical interest include \( \mathbb{R}^d \), \( L^2(0, 1) \), and spaces of matrices and positive definite trace class operators—i.e. covariance operators.

**Theorem A.6** (Kahane-Khintchine Inequality (1964)). For any \( p, p' \in [1, \infty) \), there exists a universal finite constant \( C_{p,p'} > 0 \) such that for any sequence of \( X_1, \ldots, X_n \in \mathcal{X} \)
\[
\left\{ E \left\| \sum_{i=1}^{n} \varepsilon_i X_i \right\|^p \right\}^{1/p} \leq C_{p,p'} \left\{ E \left\| \sum_{i=1}^{n} \varepsilon_i X_i \right\|^p \right\}^{1/p'}
\]

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where \( \varepsilon_i \) are iid Rademacher random variables.

In general, we will consider the right hand side with \( p' = 2 \), which bounds the \( p \)th moments by the second moment. For statistical applications, we are interested in a few specific settings for this theorem. Namely, if \( \mathcal{X} = \mathbb{R}^d \) for \( d \geq 2 \), then for the \( \ell^q \) norm with \( q \in [1, \infty] \), we have

\[
\left\{ \mathbb{E} \left[ \sum_{i=1}^{n} \varepsilon_i X_i \right]^{p} \right\}^{1/p} \leq C_p \left\| \left( \sum_{i=1}^{n} X_i X_i^T \right)^{1/2} \right\|_{S_q} = C_p (n-1)^{1/2} \left\| \hat{\Sigma}^{1/2} \right\|_{S_q}
\]

where \( \| \cdot \|_{S_q} \) is the \( q \)-Schatten norm and \( \hat{\Sigma} \) is the empirical covariance estimator for the \( X_i \). Similarly, in the functional data setting, if \( X_i \) are continuous and in \( L^q[0,1] \), then the right hand side becomes \( C_p (n-1)^{1/2} \| \Sigma (s,s')^{1/2} \|_{S_q} \) where \( \Sigma : [0,1]^2 \to \mathbb{R} \) is the empirical covariance operator.

For non-commutative Banach spaces (Pisier and Xu, 2003), such as when \( X_i \) are real valued matrices, we have a slightly different bound. Let \( \mathcal{X} = \mathbb{R}^{d\times d} \). Then, with respect to the \( q \)-Schatten norm,

\[
\left\{ \mathbb{E} \left[ \sum_{i=1}^{n} \varepsilon_i X_i \right]^{p} \right\}^{1/p} \leq C_p \max \left\{ \left\| \left( \sum_{i=1}^{n} X_i X_i^T \right)^{1/2} \right\|_{S_q}, \left\| \left( \sum_{i=1}^{n} X_i^T X_i \right)^{1/2} \right\|_{S_q} \right\}
\]

The above results all have iid \( \varepsilon_i \). Applying similar methods as in Spektor (2016) and as in the previous section, we can consider the moment bounds under weak dependency conditions on the \( \varepsilon_i \). This theorem is stated for balanced samples with adjustments for imbalanced samples omitted as they follow exactly as in the previously discussed real valued setting.

**Theorem A.7** (Kahane-Khintchine with Weak Dependence). Let \( \varepsilon_i \) are Rademacher random variables such that \( \sum \varepsilon_i = 0 \). Furthermore, let \( p \in [1, \infty] \).

For commutative Banach spaces there exists a universal finite constant \( C_p > 0 \) such that for any sequence of \( X_1, \ldots, X_n \in \mathcal{X} \)

\[
\left\{ \mathbb{E} \left[ \sum_{i=1}^{n} \varepsilon_i X_i \right]^{p} \right\}^{1/p} \leq C_p 2^{1/2} \left\| \left( \sum_{i=1}^{n} X_i X_i^* \right)^{1/2} \right\|.
\]

For non-commutative Banach spaces there exists a universal finite constant \( C_p > 0 \) such that for any sequence of \( X_1, \ldots, X_n \in \mathcal{X} \)

\[
\left\{ \mathbb{E} \left[ \sum_{i=1}^{n} \varepsilon_i X_i \right]^{p} \right\}^{1/p} \leq C_p 2^{1/2} \max \left\{ \left\| \left( \sum_{i=1}^{n} X_i X_i^* \right)^{1/2} \right\|, \left\| \left( \sum_{i=1}^{n} X_i^* X_i \right)^{1/2} \right\| \right\}.
\]

Before proving this theorem, we discuss some preliminary results regarding Schatten norms. Let \( \preceq \) denote positive semi-definite ordering. For positive semi-definite \( q \)-Schatten class linear operators \( \Gamma \) and \( \Delta \) with \( 0 \preceq \Gamma \preceq \Delta \),

\[
\| \Gamma \|_{S_q} \leq \| \Delta \|_{S_q}, \text{ and } \| (\Gamma \Gamma^*)^{1/2} \|_{S_q} = \| \Gamma \|_{S_{q/2}}^{1/2}
\]
where the square root is well defined as $\Gamma \Gamma^*$ is symmetric positive semi-definite. Lastly, via direct calculation,

\[(\Gamma - \Delta)(\Gamma - \Delta)^* \preceq 2(\Gamma \Gamma^* - \Delta \Delta^*)
\]
\[(\Gamma - \Delta)^*(\Gamma - \Delta) \preceq 2(\Gamma^* \Gamma - \Delta^* \Delta).
\]

**Proof.** For $S_n$ the symmetric group on $n$ elements, let $f : S_n \to \mathbb{R}$ by

\[f(\pi) := \left\Vert \sum_{i=1}^{m} X_{\pi(i)} - \sum_{i=m+1}^{2m} X_{\pi(i)} \right\Vert
\]

For $k = 1, \ldots, m$, we define $B_{k,\pi} = X_{\pi(k)} - X_{\pi(k+m)}$ and $H_{k,\pi} = \sum_{i=k+1}^{m} B_{i,\pi} = \sum_{i=k+1}^{m} X_{\pi(i)} - \sum_{i=m+k+1}^{2m} X_{\pi(i)}$ where $H_{m,\pi} = 0$ being an empty sum.

Note that the $B_{k,\pi}$ are symmetric random variables for $\pi$ uniform on $S_n$. Thus,

\[E_{\pi} \|f(\pi)\|^p = E_{\pi} \|B_{1,\pi} + H_{1,\pi}\| = E_{\pi} \| -B_{1,\pi} + H_{1,\pi}\|\]

and furthermore, letting $\delta_1, \ldots, \delta_m$ be iid Rademacher random variables,

\[E_{\pi} \|f(\pi)\|^p = E_{\pi} E_{\delta_1} \|\delta_1 B_{1,\pi} + H_{1,\pi}\|
\]

\[= E_{\pi} E_{\delta_1} E_{\delta_2} \|\delta_1 B_{1,\pi} + \delta_2 B_{2,\pi} + H_{2,\pi}\|
\]

\[= E_{\pi} E_{\delta_1} \ldots E_{\delta_m} \left\| \sum_{i=1}^{m} \delta_i B_{i,\pi} \right\|
\]

From here, we consider separately the commutative and non-commutative settings.

For the commutative setting, we apply the facts about Schatten norms preceding this proof. Beginning with the classic Kahane-Khintchine inequality from above with $p' = 2$, we have

\[E_{\pi} \|f(\pi)\|^p_q \leq C_p \left\| \left( \sum_{i=1}^{m} B_{i,\pi} B_{i,\pi}^* \right)^{1/2} \right\|_{S_q}^p
\]

Noting that $B_{i,\pi} B_{i,\pi}^* \preceq 2(X_{\pi(k)} X_{\pi(k)}^* + X_{\pi(k+m)}^* X_{\pi(k+m)}^*)$,

\[\left\| \left( \sum_{i=1}^{m} B_{i,\pi} B_{i,\pi}^* \right)^{1/2} \right\|_{S_q}^p = \left\| \sum_{i=1}^{m} B_{i,\pi} B_{i,\pi}^* \right\|_{S_q/2}^{p/2}
\]

\[\leq 2^{p/2} \left\| \sum_{i=1}^{2m} X_i X_i^* \right\|_{S_q/2}^{p/2} = 2^{p/2} \left\| \sum_{i=1}^{2m} X_i X_i^* \right\|_{S_q}^{1/2}
\]

For the non-commutative setting, we proceed as before using the non-commutative
variant of Kahane-Khintchine and also noting that $B_{i,\pi}^* B_{i,\pi} \leq 2(X_{\pi(k)}^* X_{\pi(k)+m}^* X_{\pi(k+m)}^*)$.

$$
E_\pi \| f(\pi) \|_q^p \leq C_p \max \left\{ \left( \sum_{i=1}^m B_{i,\pi}^* B_{i,\pi} \right)^{1/2} \| \sum_{i=1}^m B_{i,\pi}^* B_{i,\pi} \|_q^p \right\}
$$

$$
= C_p \max \left\{ \left( \frac{2m}{\sqrt{\pi}} \| \sum_{i=1}^m X_{i,\pi} X_{i,\pi}^* \|_q^{p/2} + \| \sum_{i=1}^m X_{i,\pi}^* X_{i,\pi} \|_q^{p/2} \right) \right\}
$$

$$
\leq C_p 2^{p/2} \max \left\{ \left( \frac{2m}{\sqrt{\pi}} \| \sum_{i=1}^m X_{i,\pi} X_{i,\pi}^* \|_q^{p/2} + \| \sum_{i=1}^m X_{i,\pi}^* X_{i,\pi} \|_q^{p/2} \right) \right\}
$$

\[ \square \]

**A.2.1 On Optimal Constants**

For the classic Khintchine inequality, the optimal constants due to Haagerup (1981) coincide with the lower bound imposed by the central limit theorem. That is, Khintchine’s inequality states that $E|\sum x_i \varepsilon_i|^p \leq B_p \| x \|_2^p$ where

$$
B_p = 2^{p/2} \Gamma \left\{ (p + 1)/2 \right\} / \sqrt{\pi}.
$$

This coincides precisely with the $p$th absolute moment of a standard normal random variable—i.e. $E|Z|^p = B_p$ for $Z \sim \mathcal{N}(0, 1)$.

For the Kahane-Khintchine inequality, optimal constants are not currently known. However, it is strongly conjectured that they coincide with those in the standard Khintchine inequality. Moreover in the multivariate setting, due again to the central limit theorem, the optimal constant has a lower bound. Indeed, let $Z \sim \mathcal{N}(0, \Sigma)$, then

$$
E\| Z \|_q^p \Sigma^{1/2} \|_q^{-1} = 2^{p/2} \Gamma \left\{ (p + 1)/2 \right\} / \sqrt{\pi}.
$$

This can be extended into a functional data setting using the fact that the space of covariance operators arises from the closure of the set of finite rank operators—i.e. the multivariate setting.

**A.3 Sub-Gaussian Concentration**

Given upper bounded on the $p$th moments of a random permutation statistic, we want to quantify the concentration behaviour. In particular, we want as sharp an upper bound as possible to achieve the best statistical power for hypothesis testing.

\[ \text{For the lower bound, optimal constants are known due to Latala and Oleszkiewicz (1994).} \]
We first consider the standard moment bounds to achieve sub-Gaussian concentration (Boucheron et al., 2013) in Proposition A.8. This is improved if $X$ is symmetric (Garling, 2007) in Proposition A.9. Lastly, even if the moment condition is weakened as in Proposition A.11, we still have sub-Gaussian concentration.

**Proposition A.8.** For a centred univariate random variable $X \in \mathbb{R}$ such that $E|X|^{2p} \leq p!C^p$ for some constant $C > 0$. Then,

$$P \left( X > t \right) \leq e^{-t^2/8C}.$$

**Proposition A.9.** For a centred symmetric univariate random variable $X \in \mathbb{R}$ such that $E|X|^{2p} \leq p!C^p$ for some constant $C > 0$. Then,

$$P \left( X > t \right) \leq e^{-t^2/2C}.$$

**Remark A.10.** Note that the difference between the above two propositions is a factor of 4 in the denominator of the exponent. This stems from a standard symmetrization trick where one considers $X$ and $X'$, an iid copy of $X$, so that

$$E|X - X'|^{2p} \leq 2^{2p}E|X|^{2p} \leq (4C)^p p!.$$

Thus, the following results can be similarly adjusted for asymmetric random variables.

**Proposition A.11.** For a centred symmetric univariate random variable $X \in \mathbb{R}$ such that $E|X|^{2p} \leq (2p)!C^p/p!$ for some constant $C > 0$. Then,

$$P \left( X > t \right) \leq e^{-t^2/4C}.$$

**Proof.** The moment generating function is

$$Ee^{\lambda Z} = \sum_{p=0}^{\infty} \frac{\lambda^p E Z^p}{p!} = \sum_{p=0}^{\infty} \frac{\lambda^{2p} E Z^{2p}}{(2p)!} \leq \sum_{p=0}^{\infty} \frac{\lambda^{2p} C^p}{p!} \leq e^{\lambda^2 C}.$$ 

The result follows from Markov’s (Chernoff’s) Inequality. \qed

**B Proofs of main theorems**

Now that all of the results from the previous section have been established, we prove the tail bounds on the test statistics of interest by (1) applying the appropriate Khintchine-type moment bound and (2) applying the appropriate sub-Gaussian bound on the moment generating function.

**Proof of Theorem 2.1.** For the balanced case of $\kappa = 1$, let $n = 2m$ and $\varepsilon_1, \ldots, \varepsilon_n$ be Rademacher random variables such that $\sum_{i=1}^{n} \varepsilon_i = 0$—i.e. not independent. Then, we can rewrite $T(\pi)$ from equation 2.1 as

$$T(\pi) = \frac{1}{sm} \sum_{i=1}^{n} \varepsilon_i X_i.$$
Treating $X_i \in \mathbb{R}$ as fixed, we can use Theorem A.2 to bound the $p$th absolute moment of $T(\pi)$ for $\pi$ uniformly distributed on $S_n$,

$$E_\varepsilon |T(\pi)|^p = \left( \frac{1}{sm} \right)^p E_\varepsilon \left| \sum \varepsilon_i X_i \right|^p \leq B_p \left( \frac{\|X\|_2^2 - n \bar{X}^2}{s^2 m^2} \right)^{p/2}.$$  

However, the term $\|X\|_2^2 - n \bar{X}^2 = (n-1)s^2 < 2ms^2$. Hence, the result of Spektor (2016) can be equivalently rewritten as

$$E_\varepsilon |T(\pi)|^p \leq \frac{2^p (2p)!}{m^p p!}.$$  

Applying Proposition A.11 gives the desired result.

For $\kappa > 1$—i.e. the imbalanced setting—we apply Theorem A.4 to get moment bounds

$$E_\varepsilon |T(\pi)|^p \leq \left( \frac{(\kappa + 1)^2}{2m^2} \right)^p \frac{(2p)!}{p!},$$

and Proposition A.11 again to get the desired result.

**Proof of Theorem 2.2.** As with the previous proof, let $n = 2m$ and $\varepsilon_1, \ldots, \varepsilon_n$ be Rademacher random variables such that $\sum_{i=1}^n \varepsilon_i = 0$. Our permuted test statistic is $T(\pi) = \|\sum_{i=1}^n \varepsilon_i X_i\|_{q'}$. We apply Theorem A.7, our Kahane-Khintchine variant assuming the above dependency on $\varepsilon$, in the commutative Banach setting to get

$$ET(\pi)^p \leq C_p n^{p/2} \left\| \left( \sum_{i=1}^n X_i X_i^* \right)^{1/2} \right\|_p^p.$$  

Note that while the optimal constant is not known, $C_p \sim p^{1/2}$ from the central limit theorem and from the proof in Diestel et al. (1995), Chapter 11. Hence, applying the fact that $((2p)!/p!)^{1/2} \sim p^{1/2}$ and Proposition A.11. We have the desired result.

**Proof of Theorem 2.3.** This proof is identical to that for Theorem 2.2 except we apply the non-commutative variant of Kahane-Khintchine.

**Proof of Proposition 2.5.** We note first that $nT_0^2/(2 + \kappa + \kappa^{-1})$ is approximately $\chi^2(1)$ via the central limit theorem. Hence, for $Z \sim \chi^2(1)$, some $c > 0$, and some $u \in (0, 1)$,

$$P \left( e^{-Z/c} \leq u \right) = P \left( Z \geq -c \log u \right) = \left( 2\pi \right)^{-1/2} \int_{-c \log u}^{\infty} x^{-1/2} e^{-x/2} dx = \left( \frac{c}{2\pi} \right)^{1/2} \int_0^u (-\log y)^{-1/2} y^{-1/2} dy \leq \left( \frac{c}{2\pi} \right)^{1/2} \Gamma((c+1)/2) \Gamma((c+1)/2) I(u; c/2, 1/2) = \frac{(c/2)^{1/2} \Gamma(c/2)}{\Gamma((c+1)/2)} I(u; c/2, 1/2)$$
where we use the inequality $- \log y \geq 1 - y$ for $y \in (0, 1)$. The coefficient $(c/2)^{1/2} \Gamma(c/2) \Gamma((c + 1)/2)^{-1} \to 1$ as $c \to \infty$. Replacing $c$ with $2[\kappa + 1]/(2 + \kappa + \kappa^{-1})$, we conclude that

$$P \left( \{ -nT(\pi)^2/2[\kappa + 1]^3 \right) < u \right) \leq C_0 I \left( u; \frac{[\kappa + 1]^3}{(2 + \kappa + \kappa^{-1})^{1/2}} \right)$$

where $C_0 = \left( \frac{[\kappa + 1]^3}{2 + \kappa + \kappa^{-1}} \right)^{1/2} \Gamma \left( \frac{[\kappa + 1]^3}{2 + \kappa + \kappa^{-1}} \right) \Gamma \left( \frac{1}{2} + \frac{[\kappa + 1]^3}{2 + \kappa + \kappa^{-1}} \right)^{-1}$. \hfill \Box

**Proof of Theorem 2.4.** Let $Z = h(\|X\|) \in \mathbb{R}^+$, and let $B^*$ be a countable dense subset of the unit ball of the dual space $X^*$, which consists of bounded linear functionals $\phi$. Then, we can write $Z = \sup_{\phi \in B^*} h(\|X\|)$ being a countable supremum. Via application of Talagrand’s concentration inequality (Talagrand, 1996), we have that

$$P (Z \geq EZ + t) \leq \exp \left( -\frac{t^2}{a + bt} \right)$$

for positive constants $a$ and $b$ depending on $Eh(\|X\|)^2$ and $\sup_{X \in X} h(\|X\|)$.\footnote{Refined values for such constants can be found in other works (Bousquet, 2003; Klein and Rio, 2005; Giné and Nickl, 2016), but are not pertinent to this discussion.} Noting that for $t \geq 0$

$$\frac{d}{dt} \left\{ \frac{t}{1 + bt/a} \right\} = \frac{1}{(1 + bt/a)^2} \leq \frac{1}{1 + bt/a} = \frac{d}{dt} \left\{ \frac{a}{b} \log(1 + bt/a) \right\},$$

we have that

$$\exp \left( -\frac{t^2}{a + bt} \right) = \exp \left\{ -\frac{1}{b} \left( t - \frac{at}{a + bt} \right) \right\} \leq \exp \left\{ -\frac{1}{b} \left( t - \frac{a}{b} \log(1 + bt/a) \right) \right\} = e^{-t/b} \left( 1 + \frac{b}{a} t \right)^{a/b^2}.$$

If $a \notin \mathbb{N}$, then we replace $a$ with $[a]$. Then, we have that

$$\exp \left( -\frac{t^2}{a + bt} \right) \leq e^{-t/b} \left[ \left( 1 + \frac{bt}{a} \right)^{a \gamma / b^2} \right]$$

$$= e^{-t/b} \left[ \sum_{k=0}^{a} \left( \frac{a}{k} \right) \left( \frac{bt}{a} \right)^k \right]^{1/b^2} \leq e^{-t/b} \left[ \sum_{k=0}^{a} \frac{1}{k!} (bt)^k \right]^{1/b^2}.$$

If $b = 1$, then this is just the distribution function of the Erlang (gamma) distribution with shape parameter $a$ and scale parameter 1. More generally, we have that

$$\left[ \sum_{k=0}^{a} \frac{1}{k!} (bt)^k \right]^{1/b^2} = e^{t/b} \left[ 1 - e^{-bt} \sum_{k=a+1}^{\infty} \frac{1}{k!} (bt)^k \right]^{1/b^2}.$$
whose $l$th derivative for $b > 1$, denoting the Pochhammer symbol $(b^{-2})_j = \prod_{i=1}^{j}(b^{-2} - i + 1)$, can be written as

\[
\frac{d^l}{dt^l} \left[ \sum_{k=0}^{a} \frac{1}{k!} (bt)^k \right]^{1/b^2} = \frac{e^{t/b}}{b^l} \left[ 1 - e^{-bt} \sum_{k=a+1}^{\infty} \frac{1}{k!} (bt)^k \right]^{1/b^2} - e^{t/b} \sum_{j=1}^{l} b^{j-1}(b^{-2})_j \left[ 1 - e^{-bt} \sum_{k=a+1}^{\infty} \frac{1}{k!} (bt)^k \right]^{1/b^2-j} \frac{d^l}{dt^l} \left\{ e^{-bt} \sum_{k=a+1}^{\infty} \frac{1}{k!} (bt)^k \right\}
\]

Thus, for $l \leq a$, we have that

\[
\left. \frac{d^l}{dt^l} \left[ \sum_{k=0}^{a} \frac{1}{k!} (bt)^k \right]^{1/b^2} \right|_{t=0} = \frac{1}{b^l}
\]

as the second term vanishes, and for $l \geq a + 1$ and $b > 1$, we have that

\[
\left. \frac{d^l}{dt^l} \left[ \sum_{k=0}^{a} \frac{1}{k!} (bt)^k \right]^{1/b^2} \right|_{t=0} = \frac{1}{b^l} \left[ 1 - \sum_{j=a+1}^{l} b^{2j}(b^{-2})_j \sum_{i=a+1}^{j} (-1)^{j-i} \sum_{k=0}^{\infty} \frac{1}{k!} (bt)^k \right]^{1/b^2-j} \frac{d^l}{dt^l} \left\{ e^{-bt} \sum_{k=a+1}^{\infty} \frac{1}{k!} (bt)^k \right\}
\]

which is negative for odd $a$. Thus, for $a$ odd—in the case where $a \in \mathbb{R}^+$, we replace $a$ with $2\lfloor a/2 \rfloor + 1$—we can finally bound via $a$th order approximation

\[
\exp \left( -\frac{t^2}{a + bt} \right) \leq e^{-t/b} \left[ \left( 1 + \frac{bt}{a} \right)^a \right]^{1/b^2} \leq e^{-t/b} \sum_{k=0}^{a} \frac{1}{k!} \left( \frac{t}{b} \right)^k = \int_t^{\infty} x^{a-1}e^{-x/b} b^a \Gamma(a) \, dx
\]

being once again the Erlang (gamma) distribution function with shape parameter $a$ and scale parameter $b$. 

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As a result, we have for \( u \in (0, 1) \), \( C \) some positive constant, and \( I(u; a, c/b - a) \) the incomplete beta function where \( c \) is chosen large enough so that \( c/b - a > 0 \),

\[
P\left(e^{-Z/EZ}/c \leq u\right) = P\left(Z - EZ \geq -c \log u\right)
\leq \int_{-c \log u}^{\infty} \frac{x^{a-1}e^{-x/b}}{b^a \Gamma(a)} \, dx
= \frac{c^a}{b^a \Gamma(a)} \int_0^u (-\log y)^{a-1} y^{c/b-1} dy
\leq \frac{c^a}{b^a \Gamma(a)} \int_0^u (1 - y)^{a-1} y^{c/b-a-1} dy = CI(u; a, c/b - a)
\]

where the final inequality comes from \(-\log y = \sum_{k=1}^{\infty} (1 - y)^k/k \leq (1 - y)/y\) for \( 0 < y < 1 \).

\[\square\]

**Proof of Theorem 3.1.** Let \( T = \|X^T \varepsilon\|_2 \) and note for \( n = 2m \) that \( T^2 = \sum_{i=1}^n a_{i,j} \varepsilon_i \varepsilon_j \) where \( a_{ij} \) is the \( ij \)th entry of \( XX^T \). This is an homogeneous Rademacher chaos of order 2.

As in Spektor (2016), we note the following correspondence. Let \( \Omega = \{\varepsilon \in \{\pm 1\}^n \mid \sum \varepsilon_i = 0\} \), then

\[\pi \in S_n \leftrightarrow \{\varepsilon \in \Omega \mid \varepsilon_i = 1 \text{ if } \pi(i) \leq m \text{ and } \varepsilon_i = -1 \text{ if } \pi(i) > m\} .\]

Hence, for any \( \pi \in S_n \), we can write

\[T^2(\pi) = \sum_{i \leq m,j \leq m} a_{\pi(i),\pi(j)} - \sum_{i > m,j \leq m} a_{\pi(i),\pi(j)} - \sum_{i \leq m,j > m} a_{\pi(i),\pi(j)} + \sum_{i > m,j > m} a_{\pi(i),\pi(j)}\]

and consider

\[E_\varepsilon|T^2|^p = E_\pi|T^2(\pi)|^p = E_\pi \left| \sum_{i=1}^m \{a_{\pi(i),\pi(j)} - a_{\pi(i+m),\pi(j)} - a_{\pi(i),\pi(j+m)} + a_{\pi(i+m),\pi(j+m)}\} \right|^p.
\]

Writing \( b_{k,k',\pi} = a_{\pi(k),\pi(k')} - a_{\pi(k+m),\pi(k')} - a_{\pi(k),\pi(k'+m)} + a_{\pi(k+m),\pi(k'+m)} \), and \( H_{k,\pi} = \sum_{i,j \in I_k} b_{i,j,\pi} \) where the sum is over \( I_k = \{1 \leq i, j \leq m \mid i + j > k + 1\} \) with the empty sum being zero, we note that

\[T^2(\pi) = b_{1,1,\pi} + H_{1,\pi} = b_{1,1,\pi} + b_{1,2,\pi} + b_{2,1,\pi} + b_{2,2,\pi} + H_{2,\pi} = \ldots = \sum_{i,j=1}^m b_{i,j,\pi} .\]

Then,

\[E_\pi|T^2(\pi)|^p = E_\pi|b_{1,1,\pi} + b_{1,2,\pi} + b_{2,1,\pi} + b_{2,2,\pi} + H_{2,\pi}|^p
= E_\pi|b_{1,1,\pi} - b_{1,2,\pi} - b_{2,1,\pi} + b_{2,2,\pi} + H_{2,\pi}|^p
= E_\pi E_\delta|\delta_1 b_{1,1,\pi} + \delta_1 b_{2,1,\pi} - \delta_2 b_{1,2,\pi} - \delta_2 b_{2,2,\pi} + H_{2,\pi}|^p
\]

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for $\delta_1, \delta_2$ iid Rademacher random variables. Continuing in this fashion, we have $E_\pi |T^2(\pi)|^p = E_\pi \delta_1 \delta_2 b_{i,j,\pi}$. From here we apply Corollary 3 from Kwapien (1987).

First note that as the $\delta_i$ are just iid Rademacher random variables, the standard Khintchine (or Kahane-Khintchine) inequality applies with coefficient $B_{2p} = ((2^p)!/(2^p p!)^2)^{1/2}$. Then Corollary 3 from Kwapien (1987) to this degree 2 polynomial chaos implies that

$$\left[ E_\pi \left| \sum_{i,j=1}^m \delta_i \delta_j b_{i,j,\pi} \right|^p \right]^{1/p} \leq B_{2p}^2 C \left[ \left| \sum_{i,j=1}^m \delta_i \delta_j b_{i,j,\pi} \right| \right]^{2/2},$$

where $C$ is a universal constant which for homogeneous degree $d$ polynomial chaos is $d^{3d/d!}$ or simply $25 = 32$ in our case. The expectation on the right hand side then becomes

$$\left[ E_\pi \left| \sum_{i,j=1}^m \delta_i \delta_j b_{i,j,\pi} \right|^2 \right]^{1/2} \leq \left[ \sum_{i,j=1}^m 4d^2_{i,j} \right]^{1/2} = 2\|XX^T\|_2.$$

Absorbing the 2 into $C$, we have the moment bounds

$$E_\pi |T(\pi)|^{2p} \leq B_{2p}^2 C^p \|XX^T\|^p_2.$$

To adapt these moment bounds into a tail bound, we use the standard moment generating function approach, but in preparation we first recall the Legendre duplication formula $\Gamma(2p) = 2^{2p-1}(p\Gamma(p+1/2))/\sqrt{\pi}$ and then note the following:

$$\frac{2^p(\Gamma(p+1))}{2p \Gamma(2p)(p/2)\Gamma(p/2)^2} = \frac{2^{p+1}(\Gamma(p))}{p(2e)^p} \leq \frac{\Gamma(p)}{\sqrt{\pi} p^p} \leq \frac{\Gamma(p+1/2)}{\sqrt{\pi} p^{p+1}} \leq \frac{1}{\sqrt{\pi}} + \frac{1}{\pi \sqrt{\pi}} \frac{e^{p+1}}{p!} \leq \frac{e^{p+1}}{p!},$$

because, via Watson’s formula (Watson, 1959),

$$\frac{\Gamma(p+1)}{\Gamma(p+1/2)} = \frac{\Gamma(p+1)}{p\Gamma(p+1/2)} \leq \frac{\sqrt{\pi} + \frac{1}{\sqrt{\pi}} + \frac{1}{\pi \sqrt{\pi}} + \frac{1}{\pi \sqrt{\pi}}} {p},$$

Let $\pi$ and $\pi'$ be independent uniform random permutations from $S_n$. Then, updating
as necessary,
\[ E_\pi \exp(\lambda T(\pi)) \leq E_\pi \exp(\lambda(T(\pi) - T'(\pi))) \]
\[ \leq \sum_{p=1}^{\infty} \frac{\lambda^p}{p!} E_\pi |T(\pi) - T'(\pi)|^p \]
\[ \leq \sum_{p=1}^{\infty} \frac{\lambda^{2p} 2^{2p} C_p}{(2p)!} \frac{(p!)^2}{2^p ((p/2)!)^2} \|XX^T\|^p_{S^2} \]
\[ \leq \sum_{p=1}^{\infty} \frac{\lambda^{2p} C_p}{p!} \|XX^T\|^p_{S^2} \]
\[ \leq e^{\lambda^2 \|XX^T\|_{S^2}}, \]
which gives the desired sub-Gaussian concentration as in Proposition A.11.

\[ \square \]

C  Additional Data Experiments

C.1 Multivariate Data

We test the performance of the bound in Theorem 2.2 on simulated multivariate Gaussian data in \(\ell^q(\mathbb{R}^{12})\) for \(q = 1, 2, \infty\). The sample size is \(m_1 = m_2 = 50\). Figure 6 displays the result of running such a two-sample test for each of the three norms compared to the standard permutation test approximated by sampling 1000 permutations. This was replicated 1000 times and the average log base 2 p-values are plotted. We see that the Kahane bound does not achieve as much power as the standard permutation test. However, after applying the empirical beta adjustment from Section 2.4 with moments computed via 10 permutations, the computed p-values align perfectly with the standard permutation test.

C.2 Berkeley Growth Curves Null Setting

In this section, we repeat the data analysis from Section 4.2. However, we first remove the sex labels from the Berkeley growth curve dataset. Hence, when sampling two sets of size 30, each resample will contain both male and female curves. Thus, there should be on average no statistical difference between the two sets. Over 100 replications for each of the three norms \(L^1\), \(L^2\), and \(L^\infty\) as well as the two bounds—unadjusted Kahane and beta adjusted—we have Figure 7, which plots the empirical p-values against the theoretical p-values from the uniform distribution on \([0, 1]\). We see large deviations for the unadjusted Kahane bound in the \(L^2\) and \(L^\infty\) norms yielding an overly conservative hypothesis test. Table 4 displays the results of goodness-of-fit testing for the six sets of null p-values with a similar conclusion.
Figure 6: Multivariate two sample test for normal data with balanced sample sizes $m_1 = m_2 = 50$ for $\ell^1$, $\ell^2$, and $\ell^\infty$ norms. The plots compare the permutation test (red) with 1000 permutations to the Kahane bound from Theorem 2.2 (green) and the beta adjusted Kahane bound (blue) across 1000 replications.

| Test  | $L^1$ | $L^2$ | $L^\infty$ | $L^1$ | $L^2$ | $L^\infty$ |
|-------|-------|-------|-------------|-------|-------|-------------|
| KS test | 3.6% | <0.001% | <0.001% | 8.7% | 87.9% | 9.2% |
| AD test | 3.8% | <0.001% | <0.001% | 2.2% | 79.7% | 5.1% |

Table 4: This table contains p-values from both the Kolmogorov-Smirnov and the Anderson-Darling goodness-of-fit tests for the 100 computed p-values against the Uniform[0,1] distribution.
Figure 7: 100 simulated null p-values are plotted against the theoretical values from the uniform distribution on $[0, 1]$. There are massive deviations from uniformity for unadjusted Kahane bound with the $L^2$ and $L^\infty$ norms.
Table 5: A list of p-values from the Kolmogorov-Smirnov and the Anderson-Darling tests for the two sample tests comparing two different phonemes with a sample size of \( m_1 = m_2 = 10 \) under the trace, Hilbert-Schmidt, and operator norms.

### C.3 Phoneme Curves Null Setting

Similar to Appendix C.2, we aim to test for whether or not the empirical beta adjusted p-values for the phoneme curves from Section 4.3 follow a Uniform\([0,1]\) distribution in the null setting. To do this, we repeat the test from Section 4.3 but remove the label information. Hence, the two samples of size 40 will comprise operators from both phonemes, and there should be no significant difference between the two samples.

Table 5 reports p-values from the Kolmogorov-Smirnov and Anderson-Darling goodness-of-fit tests comparing the empirical distribution of the 100 two sample test adjusted p-values to a uniform distribution on the unit interval. Most of these empirical tests yield insignificant p-values especially after taking multiple testing into account indicating no noticeable deviation from uniformity. Hence, the empirical beta adjustment is able to account for the overly conservative nature of the unadjusted Kahane bounds.

### C.4 Simulated Covariance Operator Data

In this section, we recreate the two-sample simulation study performed in Pigoli et al. (2014) Section 3 to test our methodology. Let \( \Sigma_M \) and \( \Sigma_F \) be the empirical covariance operators for the male and female Berkeley growth curves, respectively. For \( \gamma \in [0, 6] \), we generate two sets of \( n = 30 \) curves from a Gaussian process with mean zero and with covariance operator \( \Sigma_M \) for the first group and \( \Sigma(\gamma) = [\Sigma_M^{1/2} + \gamma \Delta][\Sigma_M^{1/2} + \gamma \Delta]^\ast \) where \( \Delta = \Sigma_F^{1/2} R - \Sigma_M^{1/2} \) and \( R \) is the operator that minimizes the Procrustes distance between \( \Sigma_M \) and \( \Sigma_F \). Specifically, \( R = UV^\ast \) where \( U \) and \( V \) come from the singular
Figure 8: Plotted p-values for a two sample test for equality of covariance operators coming from Pigoli et al. (2014). From left to right, the trace, Hilbert-Schmidt, and Operator norms are considered in the three plots.

For each γ, we test $H_0: \Sigma_M = \Sigma(\gamma)$ against $H_1: \Sigma_M \neq \Sigma(\gamma)$ via a standard permutation test as in Pigoli et al. (2014) with 512 permutations and via our Kahane-Khintchine bound. This is replicated 1000 times resulting in Figure 8. We see that for the trace, Hilbert-Schmidt, and operator norms, the power loss for using our upper bound is not much different from the standard permutation test. At worst, the p-values are 2-4 times larger than necessary.

D Vowel Data

D.1 Other Schatten Norms

In this section, we repeat the analysis performed in Section 5 by replacing the trace norm with both the Hilbert-Schmidt and operator norms. In Figures 9 and 10, we display results analogous to those seen previously for the trace norm. Most notably, as we consider larger values of $q$ for the $q$-Schatten norms, the number of null hypotheses that we fail to reject increases indicating less statistical power to distinguish vowel phonemes. This is in line with much past work using Schatten norms on functional data (Pigoli et al., 2014, 2018).

D.2 Null Setting

To check that our methodology, specifically the empirical beta adjustment from Section 2.4, achieves the correct empirical size and thus is neither conservative nor anti-conservative, we first randomize all of the labels within each of the Latin squares from Section 5. Then, we repeat the same analysis as before. The 66 p-values produced for each of the 1, 2, and $\infty$ Schatten norms is displayed in Figure 11. These QQ plots compare our empirical p-values to the theoretical quantiles of the Uniform[0,1] distribution.
Figure 9: $\log_{10}$ p-values for pairwise two sample tests between vowel pairs under the Hilbert-Schmidt norm.
Figure 10: $\log_{10}$ p-values for pairwise two sample tests between vowel pairs under the operator norm.
For each of the three norms, we do not see much deviation from uniformity. Furthermore, for testing goodness-of-fit with the uniform distribution, the Kolmogorov-Smirnov test returns p-values of 0.434, 0.782, and 0.290 and the Anderson-Darling test p-values 0.161, 0.511, and 0.241 for the trace, Hilbert-Schmidt, and operator norms, respectively. None of these tests are significant indicating no noticeable deviation from uniformity.
Figure 11: QQ Plots comparing the 66 null p-values from the vowel data after beta adjustment to the quantiles of the uniform distribution.