Abstract

We study the problem of assigning workers to tasks where each task has demand for a particular number of workers, and the demands are dynamically changing over time. Specifically, a worker-task assignment function $\phi$ takes a multiset of $w$ tasks $T \subseteq [t]$ and produces an assignment $\phi(T)$ from the workers $1, 2, \ldots, w$ to the tasks $T$. The assignment function $\phi$ is said to have switching cost at most $k$ if, for all task multisets $T$, changing the contents of $T$ by one task changes $\phi(T)$ by at most $k$ worker assignments. The goal of the worker-task assignment problem is to produce an assignment function $\phi$ with the minimum possible switching cost.

Prior work on this problem (SSS’17, ICALP’20) observed a simple assignment function $\phi$ with switching cost $\min(w, t - 1)$, but there has been no success in constructing $\phi$ with sublinear switching cost. We construct the first assignment function $\phi$ with sublinear, and in fact polylogarithmic, switching cost. We give a probabilistic construction for $\phi$ that achieves switching cost $O(\log w \log(wt))$ and an explicit construction that achieves switching cost $\text{polylog}(wt)$.

From the lower bounds side, prior work has used involved arguments to prove constant lower bounds on switching cost, but no super-constant lower bounds are known. We prove the first super-constant lower bound on switching cost. In particular, we show that for any value of $w$ there exists a value of $t$ for which the optimal switching cost is $w$. That is, when $w \ll t$, the trivial bound on switching cost is optimal.

We also consider an application of the worker-task assignment problem to a metric embeddings problem. In particular, we use our results to give the first low-distortion embedding from sparse binary vectors into low-dimensional Hamming space.
1 Introduction

Consider a group of workers performing together a number of tasks. Each task has a demand – a non-negative integer indicating the number of workers that should be working on the task – and each worker can work on at most one task at a time. Furthermore, the demands are changing over time: at any point the demands may increase by one for one task and decrease by one for some other, so the total demand remains constant. Whenever the demands change, we want to reassign as few workers as possible so that each task has a number of assigned workers matching its new demand. The assignment of workers to tasks must be a function only of the current demands, however, and not a function of the past history. Does there exist a worker-to-task assignment function that reassigns only a small number of workers when the demands change? This problem can be formalized as follows.

In the worker-task assignment problem, there are $w$ workers $1, 2, \ldots, w$ and $t$ tasks $1, 2, \ldots, t$. A worker-task assignment function $f$ is a function that takes as input a multiset $T$ of $w$ tasks, and produces an assignment of workers to tasks such that the number of workers assigned to a given task $\tau \in T$ is equal to the multiplicity of $\tau$ in $T$.

Two task multisets $T_1, T_2$ of size $w$ are said to be adjacent if they differ by exactly one task; that is, $|T_1 \setminus T_2| = |T_2 \setminus T_1| = 1$. The switching cost between two adjacent task multisets $T_1, T_2$ of size $w$ is defined as the number of workers whose assignment changes between $f(T_1)$ and $f(T_2)$. The switching cost of $f$ is defined to be the maximum switching cost over all pairs of adjacent task multisets. The goal of the worker-task assignment problem is to design a worker-task assignment function with the minimum possible switching cost.

When thinking about the task multiset $T$ as changing dynamically over time, the goal of the worker-task assignment function is to change the set of worker-task assignments by as little as possible whenever $T$ changes by one. What makes this problem combinatorially interesting is that the worker-task assignment function must be determined entirely by the current value of $T$, and is not permitted to adapt based on the history of how $T$ dynamically changes.

The worker-task assignment problem was first formulated in the context of ant colonies, and specifically how the ants divide themselves up among tasks as the demand for these tasks changes over time [24]. More generally, worker-task assignment is applicable for analyzing the combinatorial aspects of dynamic task allocation in any distributed system with certain properties; in particular, if the distributed agents are memoryless and lack the ability to communicate with one another [25]. The requirement that agents are memoryless can be important to applications such as swarm robotics, where an agent may abruptly fail and be replaced with a new substitute agent; the substitute agent has access to $T$, but does not necessarily have access to the history of $T$ or to the current worker-task assignment. See the introduction of [25] for further motivation for these requirements, as well as more general background on task allocation in distributed systems.

Prior work. The worker-task assignment problem has been studied in two previous works [24][25].

Whenever the multiset of tasks changes to an adjacent multiset, at least one worker must switch tasks, and at worst all $w$ workers will switch, so the optimal switching cost lies somewhere in the range $[1, w]$. However, improving either of these bounds substantially has proven difficult.

Su, Su, Dornhaus, and Lynch [24] initiated the study of the worker-task assignment problem. They observed that assigning the workers to tasks in numerical order achieves a switching cost of $t - 1$, which beats the trivial upper bound on switching cost when there are more workers than tasks. They also show that the switching cost is 2 for two workers and three tasks; simple casework shows that there is no way to

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1These works did not explicitly name the problem, instead referring to it as a problem under the umbrella of dynamic distributed task allocation.
assign two workers to each of the task sets \{1, 2\}, \{1, 3\}, \{2, 3\} and maintain a switching cost of 1. They also provide constructions showing that a switching cost of 2 is achievable when \(w \leq 6\) and \(t \leq 4\). In ICALP’20 Su and Wein [25] proved that this is optimal in terms of \(t\) by showing that for \(t \geq 5\) and \(w \geq 3\), any worker-task assignment function has switching cost at least 3. Additionally, they proved that any worker-task assignment function has switching cost at least 4 for the case where \(t\) is sufficiently large in terms of \(w\) (a tower of height \(w - 1\)).

**Our results.** We prove a polylogarithmic upper bound for switching cost, which is an exponential improvement over the previously known upper bounds, which are linear.

**Theorem 1.** There exists a worker-task assignment function that achieves switching cost \(O(\log \omega \log(\omega t))\).

Theorem 1 is non-constructive. We show that one can explicitly construct a worker-task assignment function at the cost of an extra polylog factor in switching cost.

**Theorem 2.** There is an explicit worker-task assignment function that achieves switching cost \(O(\text{polylog}(\omega t))\).

Both Theorems 1 and 2 continue to hold in the more general setting where the size of \(T\) changes over time. That is, \(T\) is permitted to be any multiset of \([t]\) of size \(w\) or smaller. Two task multisets \(T_1, T_2\) of different sizes are considered adjacent if they satisfy \(|T_1 - T_2| = 1\) and \(|(T_1 \cup T_2) \setminus (T_1 \cap T_2)| = 1\). If \(|T| < w\), then our worker-task assignment function assign workers \(1, \ldots, |T|\) to tasks, and leaves workers \(|T| + 1, \ldots, w\) unassigned.

From the lower bounds side, we prove a super-constant lower bound for worker-task assignment.

**Theorem 3.** For every \(w\) and \(t \geq \text{tow}(\Omega(w))\), every worker-task assignment function has switching cost \(w\).

Here \(\text{tow}(n)\) is defined to be a tower of twos of height \(n\) (the inverse of the \(\log^*\) function).

Theorem 3 says that if \(t\) is sufficiently larger than \(w\), any worker-task assignment function must move all of the workers for some pair of adjacent task multisets. In particular, for every \(t\) there is some \(w\) for which any worker-task assignment function has switching cost at least \(\Omega(\log^*(t))\). Therefore our bounds leave a gap between \(\log^*\) and polylog in terms of dependence on \(t\).

**An application to metric embeddings: Densification into Hamming space.** The problem of embedding one metric space \(\mathcal{M}_1\) into another metric space \(\mathcal{M}_2\) with small distortion has been widely studied in many contexts and has found many algorithmic applications [11,9,13,18,20,23].

Bourgain [4] initiated the study of metric embeddings (into normed spaces) by showing that \(O(\log |M|)\)-distortion embeddings into \(\ell_2\) are possible for any space \(M\). Much of the subsequent work has focused either on embeddings between exponentially large metric spaces [2,3,6,9,13,23], or on embeddings with sub-logarithmic distortion [3,9,18,20].

One natural question is that of **densification:** can one embed sparse vectors from a high-dimensional \(\ell_1\)-space into a low-dimensional \(\ell_1\)-space? That is, if \(V_n^k\) is the set of \(n\)-dimensional vectors with \(k\) non-zero entries, what is the smallest \(m\) for which \(V_n^k\) can be embedded into \(m\)-dimensional \(\ell_1\)-space with low distortion? Charikar and Sahai [9] were the first to consider this problem, and showed how to achieve an output dimension of \(m = O((k/\varepsilon)^2 \log n)\) with distortion \(1 + \varepsilon\). They also showed how to apply densification to the related problem of embedding arbitrary tree metrics into low-dimensional \(\ell_1\)-space [9]. Subsequently, Berinde et al. [3] used expander graphs in order to achieve \(m = O(k \log(n/k)/\varepsilon^2)\) with distortion \(1 + \varepsilon\). They then used their densification embedding as a tool to perform sparse signal recovery [3,15,17]. Both of the known densification algorithms [3,9] rely on linear sketches, in which each vector \(\vec{x} \in V_n^k\) is mapped to a vector of the form \(\sum_i x_i \vec{b}_i\) for some set of vectors \(\vec{b}_1, \ldots, \vec{b}_n\).

The prior work on densification [3,9] has focused on embedding into \(\ell_1\)-space. In Section 6 we consider
the same problem over Hamming space, where the distance between two vectors \(\vec{x}, \vec{y}\) is given by \(\text{Ham}(\vec{x}, \vec{y}) = |\{i \mid x_i \neq y_i\}|\). Densification over Hamming space requires new techniques due to the fact that summations of vectors (and thus linear sketches) do not behave well in Hamming space.

Let \(\mathcal{H}^k_n\) denote the set of \(n\)-dimensional binary vectors with \(k\) ones. Let \(\mathcal{H}_k(n)\) denote the set of \(k\)-dimensional vectors with entries from \([n]\). We show that \(\mathcal{H}^k_n\) can be embedded into \(\mathcal{H}_k(n)\) with distortion \(O(\log n \log k)\).

**Theorem 4.** There exists a map \(\phi : \mathcal{H}^k_n \rightarrow \mathcal{H}_k(n)\) such that, for every \(\vec{x}, \vec{y} \in \mathcal{H}^k_n\),

\[
\frac{\text{Ham}(\vec{x}, \vec{y})}{2} \leq \text{Ham}(\phi(\vec{x}), \phi(\vec{y})) \leq O(\log n \log k) \cdot \text{Ham}(\vec{x}, \vec{y}).
\]

The densification embedding is a simple application of the worker-task assignment problem. In order to embed a vector \(\vec{x} \in \mathcal{H}^k_n\) into \(\mathcal{H}_k(n)\), we simply assign the workers \(\{1, 2, \ldots, k\}\) to the task set \(T = \{i \mid x_i = 1\}\), and then we construct the vector \(\vec{y}\) whose \(i\)-th coordinate denotes the task in \(T\) to which worker \(i\) is assigned. This map transforms the switching cost in the worker-task assignment problem into the distortion of the metric embedding.

The densification embedding is optimal in two senses. First, the target space of the embedding must have \(\Omega(k)\) coordinates simply in order to allow for distances of \(\Omega(k)\). Second, when \(k \ll n\), any embedding of \(\mathcal{H}^k_n\) to \(k\)-dimensional Hamming space must use \(\Omega(\log n)\) bits per coordinate, simply in order so that the embedding is an injection. It is not clear whether the distortion achieved by our embedding is optimal, however, and it remains open whether smaller distortion can be achieved by allowing for a larger target-space dimension.

## 2 Technical Overview

This section gives an overview of the main technical ideas in the paper. For simplicity, the section will treat the task multiset \(T \subseteq [t]\) as always being a set (rather than a multiset). In fact, one can formally reduce from the multiset case to the set case, as is discussed in Section 3 at the cost of \(t\) being replaced with \(t' = wt\).

### 2.1 A Warmup: The Random-Permutation Algorithm

We begin by describing a simple assignment function that we call the **random-permutation algorithm**. The random-permutation algorithm does not necessarily achieve small switching cost, but it does have the property that for any two adjacent task sets \(T_1, T_2 \subseteq [t]\), the switching cost between \(T_1\) and \(T_2\) is \(O(\log w)\) with high probability in \(w\).

**The algorithm.** The random-permutation algorithm assigns to each worker \(i \in [w]\) a random permutation \(\sigma_i = \langle \sigma_i(1), \sigma_i(2), \ldots, \sigma_i(t)\rangle\) of the numbers \([t]\). We think of worker \(i\) as preferring task \(\sigma_i(j)\) over task \(\sigma_i(j + 1)\) for all \(j \in [t - 1]\).

Suppose we wish to assign workers to tasks \(T\). The random-permutation algorithm assigns the workers \(1, 2, \ldots, w\) to tasks \(\tau_1, \tau_2, \ldots, \tau_w \in T\) one by one, assigning worker \(i\) to task

\[
\tau_i := \min_j \{\sigma_i(j) \mid j \in T \setminus \{\tau_1, \ldots, \tau_{i-1}\}\}.
\]

That is, we assign worker \(i\) to the task that it most prefers out of the tasks in \(T\) that have not yet been assigned a worker.
For each $i \in [w]$, we define the $i$-remainder tasks to be the tasks $T \setminus \{\tau_1, \ldots, \tau_i\}$. That is, the $i$-remainder tasks are the tasks that remain after the first $i$ workers are assigned. This means that worker $i + 1$ is assigned to the $i$-remainder task that it most prefers.

**Analyzing expected switching cost.** Let $T_1, T_2 \subseteq [t]$ be adjacent task sets of size $w$. We begin by showing that the expected switching cost from $T_1$ to $T_2$ is $O(\log w)$.

Let $r$ and $s$ be such that $T_1 = (T_2 \cup r) \setminus \{s\}$. Let $\psi_1$ and $\psi_2$ denote the assignments produced by the random permutation algorithm for $T_1$ and $T_2$, respectively. Let $A_i$ and $B_i$ denote the set of $i$-remainder tasks during the constructions of $\psi_1$ and $\psi_2$, respectively.

The key to analyzing the random-permutation algorithm is to compare the $i$-remainder sets $A_i$ and $B_i$ for each $i \in [w]$. We claim that $|A_i \setminus B_i| \leq 1$ for all $i \in [w]$. We prove this by induction on $i$: suppose that $A_{i-1} = (B_{i-1} \cup \{a\}) \setminus \{b\}$, and suppose for contradiction that $|A_i \setminus B_i| \geq 2$. If either $\psi_1$ assigns worker $i$ to task $a$, or $\psi_2$ assigns worker $i$ to task $b$, then we would be guaranteed that $|A_i \setminus B_i| \leq 1$, a contradiction. Thus $\psi_1$ and $\psi_2$ must each assign worker $i$ to a task in $A_{i-1} \cap B_{i-1}$. But this means that, in both assignments, worker $i$ is assigned to the task in $A_{i-1} \cap B_{i-1}$ that worker $i$ most prefers. Thus $\psi_1$ and $\psi_2$ assign worker $i$ to the same task, again contradicting that $|A_i \setminus B_i| \geq 2$.

We now analyze the probability of $\psi_1$ and $\psi_2$ differing in their assignment of worker $i$. Since $A_i$ contains at most one element $a$ not in $B_i$, the probability that worker $i$ prefers $a$ over all elements in $B_i$ is at most $1/|B_i| = 1/(w - i + 1)$. Similarly, since $B_i$ contains at most one element $b$ not in $A_i$, the probability that worker $i$ prefers $b$ over all elements in $A_i$ is at most $1/|A_i| = 1/(w - i + 1)$. By the union bound, it follows that the probability of $\psi_1$ and $\psi_2$ assigning worker $i$ to different tasks is at most $2/(w - i + 1)$.

By linearity of expectation, the expected switching cost between $T_1$ and $T_2$ is at most

$$
\sum_{i=1}^{w} \frac{2}{w - i + 1} = O(\log w).
$$

**Why a union bound fails for worst-case switching cost.** By using Chernoff-style bounds, one can modify the above analysis of the random-permutation algorithm to show that, with high probability in $w$ (i.e., probability $1 - 1/\text{poly } w$), the switching cost between $T_1$ and $T_2$ is $O(\log w)$.

On the other hand, if a switching cost of $O(\log w)$ is to be achieved for all pairs $(T_1, T_2)$ of adjacent task sets, then there are $\binom{w+1}{2} \binom{t}{w+1}$ such pairs that must be considered. When $w = t/2$, the number of distinct pairs $(T_1, T_2)$ of adjacent task sets exceeds $2^t \geq 2^w$.

Thus the probability bounds achieved by the random-permutation algorithm are nowhere near high enough to enable a union bound over all adjacent worker-set pairs. We call this the **union-bound magnitude issue**.

### 2.2 An algorithm with small switching cost

We now describe a randomized assignment algorithm $\mathcal{A}$ that, with high probability in $t$, achieves switching cost $O(\log w \log t)$ on all adjacent task-sets $T_1, T_2 \subseteq [t]$ of size $w$. That is, with high probability, $\mathcal{A}$ produces an assignment function satisfying the requirements of Theorem 1. The algorithm $\mathcal{A}$ is called the **multi-round balls-to-bins algorithm**.

The multi-round balls-to-bins algorithm essentially flips the approach taken by the random-permutation algorithm. One can think of the random-permutation algorithm as consisting of $w$ phases in which each phase deterministically assigns exactly one worker to a task, and then the phases probabilistically incur small switching cost. In contrast, the multi-round balls-to-bins algorithm consists of $O(\log w)$ phases, where each phase probabilistically assigns some number of workers to tasks, and each phase deterministically
incurs small switching cost. Whereas the failure mode of the random-permutation algorithm is that a high-
switching cost may occur, the failure mode of the multi-round balls-to-bins algorithm is that some workers
may be left unassigned. As we shall see later, this distinction plays an important role in solving the union-
bound magnitude issue.

**Structure of the multi-round balls-to-bins algorithm.** We begin with a succinct description of the
algorithm $A$. For each $i$ from 1 to $\log_{1.1} w$, repeat the following hashing procedure $c \log t$ many times.
Initialize a hash table consisting of $w/(1.1)^i$ bins and randomly hash each unassigned worker and each
unassigned task into this table. For each bin that contains at least one worker and one task, assign the
minimum worker in that bin to the minimum task in that bin.

In more detail, the algorithm $A$ is the composition of $O(\log w)$ sub-algorithms $A_1, A_2, \ldots$. Each of
$A_1, A_2, \ldots$ are **partial-assignment** algorithms, meaning that $A_i$ assigns some subset of the workers to some
subset of the tasks in $T$, possibly leaving workers and tasks unassigned. Note that the input to algorithm $A_i$
is the set of workers/tasks that remain unassigned by $A_1, \ldots, A_{i-1}$. Thus one can think of the input to $A_i$ as
being a pair $(W, T)$ where $W \subseteq [w]$ is a set of workers, $T \subseteq [t]$ is a set of tasks, and $|W| = |T|$.

The algorithm $A_i$’s responsibility is to assign enough workers to tasks so that at most $t/1.1$ workers
remain unassigned. Algorithm $A_2$ is then executed on the remaining (i.e., not-yet-assigned) workers and
tasks, and is responsible for assigning enough workers to tasks so that at most $t/(1.1)^2$ workers remain
unassigned. Continuing like this, algorithm $A_i$ is executed on the workers/tasks that remain unassigned
by all of $A_1, \ldots, A_{i-1}$, and is responsible for assigning enough workers to tasks that at most $r_i = t/(1.1)^i$
workers in $W$ remain unassigned.

Each of the $A_i$’s are randomized algorithms, meaning that they have some probability of failure. The
failure mode for $A_i$ is not high-switching cost, however. In fact, as we shall see later, each $A_i$ deterministi-
cally contributes at most $O(\log w)$ to the switching cost. Instead, the way in which $A_i$ can fail is that it may
leave more than $r_i$ workers unassigned. This means that the failure mode for the full algorithm $A$ is that it
may fail to assign all of the workers in $W$ to tasks in $T$.

**Applying the probabilistic method to $A_1, A_2, \ldots$.** Before describing the partial-assignment algorithms $A_i$
in detail, we first describe how our analysis of algorithm $A$ overcomes the union-bound magnitude issue.

Recall that each algorithm $A_i$ is responsible for reducing the number of remaining workers to $r_i = t/(1.1)^i$. We will later see that each $A_i$ has a failure probability $p_i$ that is a function of $r_i$ and $t$, namely,

$$p_i = \frac{1}{t^{\Omega(r_i)}}.$$  

As $i$ grows, the failure probability $p_i$ of $A_i$ becomes larger, making it impossible to union-bound over
exponentially many pairs of task sets $T_1, T_2$.

An important insight is that, if all of $A_1, \ldots, A_{i-1}$ succeed (i.e., they each assign the number of workers
that they are responsible for assigning) then the number of workers and tasks that $A_{i-1}$ is executed on is
only $O(r_i)$. That is, if we think of the inputs to $A_i$ as being pairs $(W, T)$ where $W \subseteq [w]$ is a set of workers
and $T \subseteq [t]$ is a set of tasks, the set of inputs $(W, T)$ that algorithm $A_{i-1}$ must succeed on is only the inputs
for which $|W| = |T| \leq O(r_i)$. The number of such inputs is at most $t^{O(r_i)}$. In other words, even though the
failure probability $p_i$ of algorithm $A_i$ increases with $i$, the number of inputs over which we must apply a
union bound decreases. By a union bound, we can deduce that $A_i$ has a high probability in $t$ of succeeding
on all relevant inputs $(W, T)$. Combining this analysis over all of the partial-assigning algorithms $A_1, A_2, \ldots$,
we get that the full assignment algorithm $A$ also succeeds with high probability in $t$. In particular, we have
proven that there exists a deterministic assignment function with the desired switching cost, and that such a
function can be obtained with high probability by the randomized algorithm $A$. 
Designing each $\mathcal{A}_i$. Each algorithm $\mathcal{A}_i$ is a composition of $\Theta(\log t)$ algorithms $\mathcal{A}_{i,1}, \mathcal{A}_{i,2}, \mathcal{A}_{i,3}, \ldots$, each of which individually is a partial assignment algorithm.

Each algorithm $\mathcal{A}_{i,j}$ takes a simple balls-in-bins approach to assigning some subset of the remaining workers to some subset of the remaining tasks.

In particular, $\mathcal{A}_{i,j}$ places the workers into bins $1, 2, \ldots, r_i$ by hashing each worker to a bin (using a random function from $[w]$ to $[r_i]$). Similarly, the tasks are placed into bins $1, 2, \ldots, r_i$ by hashing each task to a bin. If a bin $b$ contains both at least one worker and at least one task, then the smallest-numbered worker in bin $b$ is assigned to the smallest-number task in bin $b$.

Note that each of the algorithms $\mathcal{A}_{i,1}, \mathcal{A}_{i,2}, \mathcal{A}_{i,3}, \ldots$ are identical copies of one-another, except using different random bits. Also note all of the $\mathcal{A}_i$’s are defined in the same way as each other, except the number of bins hashed to decreases as $i$ increases. As we shall see shortly, the reason for having $\mathcal{A}_i$ consist of $\Theta(t)$ sub-algorithms is to enable probability amplification later in the analysis.

Bounding the switching cost. The partial assignment algorithms $\mathcal{A}_{i,j}$ are designed to satisfy two essential properties, which can then be combined to bound the switching cost of the full algorithm $\mathcal{A}$. These two properties are:

- **Compatibility**: Let $I_1 = (W_1, T_1)$ and $I_2 = (W_2, T_2)$ be inputs to $\mathcal{A}_{i,j}$. Suppose $I_1$ and $I_2$ are unit distance, meaning that

$$|W_1 \setminus W_2| + |W_2 \setminus W_1| + |T_1 \setminus T_2| + |T_2 \setminus T_1| \leq 2.$$

Let $I'_1 = (W'_1, T'_1)$ and $I'_2 = (W'_2, T'_2)$ be the workers and tasks that remain unassigned when $\mathcal{A}_{i,j}$ is executed on each of $I_1$ and $I_2$, respectively. Then $I'_1$ and $I'_2$ are guaranteed to also be unit-distance.

- **Low Switching Cost**: The switching cost of $\mathcal{A}_{i,j}$ is $O(1)$. That is, if $I_1 = (W_1, T_2)$ and $I_2 = (W_2, T_2)$ are inputs to $\mathcal{A}_{i,j}$, and $I_1$ and $I_2$ are unit-distance, then the worker-task assignments made by $\mathcal{A}_{i,j}$ on each of $I_1$ and $I_2$ differ by at most $O(1)$ assignments.

Consider two adjacent task sets $T_1$ and $T_2$. When we execute $\mathcal{A}$ on $T_1$ and $T_2$, respectively, we use $I'_1$ and $I'_2$, respectively, to denote the worker/task input that are given to partial-assignment algorithm $\mathcal{A}_{i,j}$.

The Compatibility property of the $\mathcal{A}_{i,j}$’s guarantees by induction that, for each $\mathcal{A}_{i,j}$ the worker/task inputs $I'_1$ and $I'_2$ are unit-distance (or zero-distance). The Low-Switching-Cost property then guarantees that each $\mathcal{A}_{i,j}$ contributes at most $O(1)$ to the switching cost of $\mathcal{A}$. Since there are $O(t \log w)$ $\mathcal{A}_{i,j}$’s, this bounds the total switching cost of $\mathcal{A}$ by $O(t \log w \log w)$.

Deriving the success probabilities. Next we analyze the probability of $\mathcal{A}_t$ failing on a given worker/task input $(W, T)$. Recall that the only way in which $\mathcal{A}_t$ might fail is if more than $r_t$ workers remain unassigned after $\mathcal{A}_t$ finishes. Additionally, since we need only consider cases where $\mathcal{A}_{t-1}$ succeeds, we can assume that $r_t \leq |W|, |T| \leq 1.1 r_t$.

Let $q$ denote the number of workers that $\mathcal{A}_{t,1}$ assigns to tasks. Given that $r_t \leq |W|, |T| \leq 1.1 r_t$, a simple analysis of $\mathcal{A}_{t,1}$ shows that $\mathbb{E}[q] \geq r_t/5$. On the other hand, using McDiarmid’s inequality, one can perform a balls-in-bins style analysis in order to show that $\Pr[\mathbb{E}[q] - q > r_t/10] \leq 2^{-\Omega(r_t)}$. This means that $\mathcal{A}_{t,1}$ has probability at most $2^{-\Omega(r_t)}$ of leaving more than $r_t$ workers unassigned.

In order for $\mathcal{A}_t$ to fail (i.e., $\mathcal{A}$ leaves more than $r_t$ workers unassigned), all of sub-algorithms $\mathcal{A}_{t,1}, \mathcal{A}_{t,2}, \ldots$ would have to individually fail. Since there are $\Theta(t)$ sub-algorithms, the probability of them all failing is

$$p_t = 2^{-\Omega(t \log r_t)} = t^{-\Omega(r_t)}.$$
Theorem 1. In this section, we prove the following theorem.

3 Achieving switching cost $O(w \log(wt))$. For each hyperedge of tasks with elements from $[t]$, each color being a permutation of $w$ terms of a color $\pi$, produces an assignment of workers $f$ that if $f(i)$ assigns worker $i$ to task $\tau_{\pi(i)}$ for some permutation $\pi$ of $[w]$, then $f(\{\tau_1, \ldots, \tau_{w+1}\})$ assigns worker $i$ to task $\tau_{\pi(i)+1}$. The existence of such a configuration immediately implies that $f$ has switching cost $w$.

We use an application of the hypergraph Ramsey theorem to show that, when $t$ is large enough, a configuration of the type described in the above paragraph must exist. Let $K_t(w)$ denote the complete $w$-uniform hypergraph on $t$ vertices. This is just the set of $w$-element subsets of $[t]$, which correspond to sets of tasks. For each hyperedge $T = \{\tau_1, \ldots, \tau_w\}$, where $1 \leq \tau_1 < \cdots < \tau_w \leq t$, we color the hyperedge $T$ by a color $\pi$ where $\tau_{\pi(i)}$ is the task assigned to worker $i$. This gives a coloring of the hyperedges of $K_t(w)$ by $w!$ colors, each color being a permutation of $[w]$. By the hypergraph Ramsey theorem, if $t$ is large enough in terms of $w$, there must exist $w+1$ vertices $\tau_1, \ldots, \tau_{w+1}$ so all the hyperedges formed by the vertices have the same color $\pi$. By examining the hyperedges $\{\tau_1, \ldots, \tau_w\}$ and $\{\tau_2, \ldots, \tau_{w+1}\}$, it follows that $f(\{\tau_1, \ldots, \tau_w\})$ assigns each worker $i$ to task $\tau_{\pi(i)}$ and that $f(\{\tau_2, \ldots, \tau_{w+1}\})$ assigns each worker $i$ to task $\tau_{\pi(i)+1}$, as desired.

2.3 A lower bound on switching cost

Define $s_{w,t}$ to be the optimal switching cost for assignment functions that assign workers $1, 2, \ldots, w$ to multisets of $w$ tasks from the universe $[t]$. The upper bounds in this paper establish that $s_{w,t} \leq O(\log w \log(wt))$.

It is natural to wonder whether smaller bounds can be achieved, and in particular, whether a small switching cost that depends only on $w$ can be achieved.

It trivially holds that $s_{w,t} \leq w$. We show that when $t$ is sufficiently large relative to $w$, there is a matching lower bound of $s_{w,t} \geq w$. In fact, our lower bound only uses the evaluation of the assignment function on sets (as opposed to multisets).

Consider an assignment function $f$ that, given a multiset $T$ of tasks with elements from $[t]$ of $w$ tasks, produces an assignment of workers $[w]$ to tasks $T$. Our goal will be to find tasks $\tau_1 < \tau_2 < \cdots < \tau_{w+1}$ such that if $f(\{\tau_1, \ldots, \tau_w\})$ assigns worker $i$ to task $\tau_{\pi(i)}$ for some permutation $\pi$ of $[w]$, then $f(\{\tau_2, \ldots, \tau_{w+1}\})$ assigns worker $i$ to task $\tau_{\pi(i)+1}$. The existence of such a configuration immediately implies that $f$ has switching cost $w$.

We use an application of the hypergraph Ramsey theorem to show that, when $t$ is large enough, a configuration of the type described in the above paragraph must exist. Let $K_t(w)$ denote the complete $w$-uniform hypergraph on $t$ vertices. This is just the set of $w$-element subsets of $[t]$, which correspond to sets of tasks. For each hyperedge $T = \{\tau_1, \ldots, \tau_w\}$, where $1 \leq \tau_1 < \cdots < \tau_w \leq t$, we color the hyperedge $T$ by a color $\pi$ where $\tau_{\pi(i)}$ is the task assigned to worker $i$. This gives a coloring of the hyperedges of $K_t(w)$ by $w!$ colors, each color being a permutation of $[w]$. By the hypergraph Ramsey theorem, if $t$ is large enough in terms of $w$, there must exist $w+1$ vertices $\tau_1, \ldots, \tau_{w+1}$ so all the hyperedges formed by the vertices have the same color $\pi$. By examining the hyperedges $\{\tau_1, \ldots, \tau_w\}$ and $\{\tau_2, \ldots, \tau_{w+1}\}$, it follows that $f(\{\tau_1, \ldots, \tau_w\})$ assigns each worker $i$ to task $\tau_{\pi(i)}$ and that $f(\{\tau_2, \ldots, \tau_{w+1}\})$ assigns each worker $i$ to task $\tau_{\pi(i)+1}$, as desired.

3 Achieving switching cost $O(\log w \log(wt))$

In this section, we prove the following theorem.

**Theorem 1.** There exists a worker-task assignment function that achieves switching cost $O(\log w \log(wt))$.

We demonstrate the existence of such a function via the probabilistic method, showing that there is a randomized construction that produces a low-switching cost worker-task assignment function with nonzero probability. We will also also show how to derandomize the construction in Section 4 at the cost of a few extra log factors.

From multisets to sets. We begin by showing that, without loss of generality, we can restrict our attention to task multisets $T$ that are sets (rather than multisets). We reduce from the multiset version of the problem with $w$ workers and $t$ tasks to the set version of the problem with $w$ workers and $wt$ tasks.

**Lemma 5.** Define $n = wt$. Let $f$ be a worker-task assignment function that assigns workers $[w]$ to task sets $T \subseteq [n]$ (note that $f$ is defined only on task sets $T$, and not on multisets). Let $s$ be the switching cost of...
Designing an assignment function as an algorithm. It will be helpful to think of the function we construct assigning workers to tasks using what we call a multi-round balls-to-bins algorithm. The algorithm \( \mathcal{A} \) takes as input a set \( T \subseteq [n] \) of tasks with \( |T| = w \) and must produce a bijection from the workers \([w] \) to \( T \).

The algorithm constructs this bijection in stages. Each stage is what we call a partial assignment algorithm, which takes as input the current sets of workers and tasks that have yet to be matched and assigns some subset of these workers to some subset of the tasks. Formally, we define a partial assignment algorithm to be any function \( \psi \) which accepts as input any pair of sets \( T \subseteq [n], W \subseteq [w] \) with \( |T| = |W| \) and produces a matching between some subset of \( T \) and some subset of \( W \). After applying \( \psi \) to \((T,W)\), there may remain some unmatched elements \( T' \subseteq T, W' \subseteq W \). We call \((T,W)\) the worker-task input to \( \psi \) and \((T',W')\) the worker-task output. Since a matching must remove exactly as many elements from \( T \) as it does from \( W \), we must also have \( |W'| = |T'| \). Consequently, there is a natural notion of the composition of two partial assignment algorithms: the composition \( \psi' \circ \psi \) applies \( \psi \) and then \( \psi' \), letting the worker-task output of \( \psi \) be the worker-task input to \( \psi' \).

The algorithm. We recall the description of the algorithm \( \mathcal{A} \). For each \( i \) from 1 to \( c \log w \), repeat the following hashing procedure \( c \log n \) many times. Initialize a hash table consisting of \( w/(1.1)^i \) bins and randomly hash each unassigned worker and each unassigned task into this table. For each bin that contains at least one worker and one task, assign the minimum worker in that bin to the minimum task in that bin.

In more detail, our algorithm \( \mathcal{A} \) is the composition of \( \log_{1.1} w \) partial-assignment algorithms,

\[
\mathcal{A} = \mathcal{A}_1 \circ \mathcal{A}_2 \circ \cdots \circ \mathcal{A}_{\log_{1.1} w}.
\]

Let \( c \) be a large positive constant. Each \( \mathcal{A}_i \) is itself the composition of \( c \log n \) partial-assignment algorithms,

\[
\mathcal{A}_i = \mathcal{A}_{i,1} \circ \mathcal{A}_{i,2} \circ \cdots \circ \mathcal{A}_{i,c \log n}.
\]

Designing the parts. Each \( \mathcal{A}_{i,j} \) assigns workers to tasks using what we call a \( w/(1.1)^j \)-bin hash, which we define as follows.
For a given parameter $k$, a $k$-bin hash selects functions $h_1 : [w] \to [k]$ and $h_2 : [n] \to [k]$ independently and uniformly at random. For each worker $\omega \in [w]$, we say that $\omega$ is assigned to bin $h_1(\omega)$. Similarly, for each $\tau \in [n]$ we say $\tau$ is assigned to $h_2(\tau)$. These functions are then used to construct a partial assignment. Given a worker-task input $(W, T)$, we restrict our attention to only the assignments of workers in $W$ and tasks in $T$. In each bin $\kappa \in [k]$ with at least one worker and one task assigned, match the smallest such worker to the smallest such task. Importantly, once $h_1$ and $h_2$ are fixed, the algorithm $A_{i,j}$ uses the same pair of hash functions for every worker-task input, which (as we will see later) is what allows it to make very similar assignments for similar inputs and achieve low switching cost.

We set each $A_{i,j}$ to be an independent random instance of the $k$-bin hash, where $k = w/(1.1)^i$. Formally, this means that the algorithm $A = A_{1,1} \circ \cdots \circ A_{\log_1 w, \log n}$ is a random variable whose value is a partial-assignment function. Our task is thus to prove that, with non-zero probability, $A$ fully assigns all workers to tasks and has small switching cost.

**Analyzing the algorithm.** In Section 3.2 we show that $A$ deterministically has switching cost $O(\log w \log n)$. Although $A$ always has small switching cost, the algorithm is not always a legal worker-task assignment function. This is because the algorithm may sometimes act as a partial worker-task assignment function, leaving some workers and tasks unassigned.

In Section 3.1 we show that with probability greater than 0 (and, in fact, with probability $1 - 1/\text{poly} \, n$), the algorithm $A$ succeeds at fully assigning workers to tasks for all worker-task inputs $(W, T)$. Theorem 1 follows by the probabilistic method.

### 3.1 Bounding the probability of failure

Call a partial-assignment algorithm $\psi$ **fully-assigning** if for every worker/task input $(W, T)$, $\psi$ assigns all of the workers in $W$ to tasks in $T$. That is, $\psi$ never leaves workers unassigned.

**Proposition 6.** The multi-round balls-to-bins algorithm $A$ is fully-assigning with high probability in $n$. That is, for any polynomial $p(n)$, if the constant $c$ used to define $A$ is sufficiently large, then $A$ is fully-assigning with probability at least $1 - O(1/p(n))$.

Proposition 6 tells us that with high probability in $n$, $A$ succeeds at assigning all workers on all inputs. We remark that this is a much stronger statement than saying that $A$ succeeds with high probability in $n$ on a given input $(W, T)$.

The key to proving Proposition 6 is to show that each $A_i$ performs what we call $(w/(1.1)^i)$-halving. A partial-assignment function $\psi$ is said to perform $k$-halving if for every worker/task input $(W, T)$ of size at most $1.1k$, the worker-task output $(W', T')$ for $\psi(W, T)$ has size at most $k$.

If every $A_i$ performs $w/(1.1)^i$-halving, then it follows that $A_1 \circ \cdots \circ A_{\log_1 w}$ is a fully-assigning algorithm. Thus our task is to show that each $A_i$ performs $w/(1.1)^i$-halving with high probability in $n$.

We begin by analyzing the $k$-bin hash on a given worker/task input $(W, T)$.

**Lemma 7.** Let $\psi$ a randomly selected $k$-bin hash. Let $(W, T)$ be a worker/task input satisfying $|W| = |T| \leq 1.1k$, and let $(W', T')$ be the worker/task output of $\psi(W, T)$. The probability that $(W', T')$ has size $k$ or larger is $2^{-\Omega(k)}$.

**Proof.** Let $X$ be the random variable denoting the number of worker/task assignments made by $\psi(W, T)$. Equivalently, $X$ counts the number of bins in which at least one worker is assigned and at least one task is assigned—call these the active bins. We will show that $\Pr[X < k/8] \leq 2^{-\Omega(k)}$. Since $|W| = |T| \leq 1.1k$, this
immediately implies that \( |W'| = |T'| \leq 1.1k - 0.125k \leq k \) with probability \( 1 - 2^{-\Omega(k)} \), as desired.

We begin by computing \( \mathbb{E}[X] \). For each bin \( j \in [k] \), the probability no workers are assigned to bin \( j \) is \( (1 - 1/k)^{|W|} \leq (1 - 1/k)^k \leq 1/e \). Similarly, the probability that no tasks are assigned to bin \( j \) is at most \( (1 - 1/k)^{|T|} \leq 1/e \). The probability of bin \( j \) being active is therefore at least \( 1 - 2/e \geq 1/4 \). By linearity of expectation, \( \mathbb{E}[X] \geq k/4 \).

Next we show that the random variable \( X \) is tightly concentrated around its mean. Because the bins
that are active are not independent of one-another, we cannot apply a Chernoff bound. Instead, we employ McDiarmid’s inequality:

**Theorem 8** (McDiarmid ’89 [21]). Let \( A_1, \ldots, A_m \) be independent random variables over an arbitrary probability space. Let \( F \) be a function mapping \( (A_1, \ldots, A_m) \) to \( \mathbb{R} \), and suppose \( F \) satisfies,

\[
\sup_{a_1, a_2, \ldots, a_m, \tau} |F(a_1, a_2, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_m) - F(a_1, a_2, \ldots, a_{i-1}, \tau, a_{i+1}, \ldots, a_m)| \leq R,
\]

for all \( 1 \leq i \leq m \). That is, if \( A_1, A_2, \ldots, A_{i-1}, A_i, A_{i+1}, \ldots, A_m \) are fixed, then the value of \( A_i \) can affect the value of \( F(A_1, \ldots, A_m) \) by at most \( R \). Then for all \( S > 0 \),

\[
\Pr[|F(A_1, \ldots, A_m) - \mathbb{E}[F(A_1, \ldots, A_m)]| \geq R \cdot S] \leq 2e^{-2S^2/m}.
\]

The number of active bins \( X \) is a function of at most \( 2.2 \cdot k \) independent random variables (namely, the hashes \( h_1(\omega) \) for each \( \omega \in W \) and the hashes \( h_2(\tau) \) for each \( \tau \in T \)). Each of these random variables can individually change the number of active bins by at most one. It follows that we can apply McDiarmid’s inequality with \( R = 1 \) and \( m = 2.2k \). Taking \( S = k/8 \), we obtain

\[
\Pr[X - \mathbb{E}[X] \geq k/8] \leq e^{-\Omega(k)}.
\]

Since \( \mathbb{E}[X] \geq k/4 \), we have that \( \Pr[X < k/8] \leq e^{-\Omega(k)} \), which completes the proof of the lemma. \( \square \)

Our next lemma shows that each \( \mathcal{A}_i \) is \( k \)-halving with high probability in \( n \), where \( k = w/(1.1)^t \).

**Lemma 9.** Let \( \psi_1, \ldots, \psi_{c \log n} \) be independent random \( k \)-bin hashes, and let \( \psi = \psi_1 \circ \cdots \circ \psi_{c \log n} \). With high probability in \( n \), \( \psi \) is \( k \)-halving. That is, every worker-task input \( (W, T) \) with \( |W| = |T| \leq 1.1k \) has a worker task output \( (W', T') \) with \( |W'| = |T'| \leq k \).

**Proof.** Fix an arbitrary worker-task input \( (W, T) \) with \( |W| = |T| \leq 1.1k \). Let \( (W_i, T_i) \) denote the worker-task output after applying the first \( i \) rounds, \( \psi_1 \circ \cdots \circ \psi_i \). Let \( p_i \) denote the probability that \( |W_i| = |T_i| > k \).

First, we observe that \( p_1 \leq e^{-\Omega(k)} p_{i-1} \) for all \( i > 1 \). Indeed, for \( |W_i| = |T_i| > k \), we must necessarily have \( |W_{i-1}| = |T_{i-1}| > k \), which occurs with probability \( p_{i-1} \), but in this situation, the probability that \( \psi_i \) produces a worker-task output of size greater than \( k \) is a further \( e^{-\Omega(k)} \) by Lemma 7.

The probability that \( \psi \) fails to reduce the size of \( (W, T) \) to \( k \) or smaller is thus at most

\[
p_{c \log n} \leq e^{-\Omega(c \log n)} \leq n^{-\Omega(ck)},
\]

where \( c \) is treated as a parameter.

On the other hand, the number of possibilities for input pairs \( (W, T) \) satisfying \( |W| = |T| \leq 1.1k \) is

\[
\sum_{j=0}^{1.1k} \binom{w}{j} \binom{n}{j} \leq 1.1k \cdot w^{1.1k} n^{1.1k} \leq n^{O(k)}.
\]

Combining (1) and (2), the probability that there exists any pair \( (W, T) \) of size 1.1k or smaller which fails to have its size reduced to \( k \) or smaller is at most \( n^{O(k)-c\Omega(k)} \). If \( c \) is selected to be a sufficiently large constant, then it follows that \( \psi \) performs \( k \)-halving with probability at least \( 1 - n^{-\Omega(k)} \). \( \square \)
We now prove Proposition [6]

Proof of Proposition [6] By Lemma [9] each algorithm $\mathcal{A}_i$ is $(w/(1.1)^i)$-halving with high probability in $n$. By a union bound, it follows that all of $\mathcal{A}_i \in \{\mathcal{A}_1, \ldots, \mathcal{A}_{\log_1 w}\}$ are $(w/(1.1)^i)$-halving with high probability in $n$. If this occurs, then

$$\mathcal{A} = \mathcal{A}_1 \circ \cdots \circ \mathcal{A}_{\log_1 w}$$

is fully-assigning, as desired. \qed

3.2 Bounding the switching cost

Recall that two worker/task inputs $(W_1, T_1)$ and $(W_2, T_2)$ are said to be unit distance if

$$|W_1 \setminus W_2| + |W_2 \setminus W_1| + |T_1 \setminus T_2| + |T_2 \setminus T_1| \leq 2.$$ 

A partial-assignment algorithm $\psi$ is $s$-switching-cost bounded if for all unit-distance pairs of worker/task inputs $(W_1, T_1)$ and $(W_2, T_2)$, the set of assignments made by $\psi(W_1, T_1)$ deterministically differs from the set of assignments made by $\psi(W_2, T_2)$ by at most $s$.

In this section, we prove the following proposition.

Proposition 10. The multi-round balls-to-bins algorithm is $O(\log w \log n)$-switching-cost bounded.

We begin by showing that each of the algorithms $\mathcal{A}_{i,j}$ are $O(1)$-switching-cost bounded.

Lemma 11. For any $k$, the $k$-bin hash algorithm is $O(1)$-switching-cost bounded.

Proof. Let $\psi$ denote the $k$-bin hash algorithm. Consider unit-distance pairs of worker/task inputs $(W_1, T_1)$ and $(W_2, T_2)$. Changing $W_1$ to $W_2$ can change the assignments made by $\psi$ for at most a constant number of bins. Similarly changing $T_1$ to $T_2$ can change the assignments made by $\psi$ for at most a constant number of bins. Thus $\psi(W_1, T_1)$ differs from $\psi(W_2, T_2)$ by at most $O(1)$ assignments. \qed

Recall that $\mathcal{A}$ is the composition of the $O(\log w \log n)$ partial-assignment algorithms $\mathcal{A}_{i,j}$'s. The fact that each $\mathcal{A}_{i,j}$ is $O(1)$-switching-cost bounded does not directly imply that $\mathcal{A}$ is $O(\log w \log n)$-switching-cost bounded, however, because switching cost does not necessarily interact well with composition. In order to analyze $\mathcal{A}$, we show that each $\mathcal{A}_{i,j}$ satisfies an additional property that we call being composition-friendly.

A partial-assignment algorithm $\psi$ is composition-friendly, if for all unit-distance pairs of worker/task inputs $(W_1, T_1)$ and $(W_2, T_2)$, the corresponding worker/task outputs $(W_1', T_1')$ and $(W_2', T_2')$ are also unit-distance.

Lemma 12 shows that each $\mathcal{A}_{i,j}$ is composition-friendly.

Lemma 12. For any $k$, the $k$-bin hash is composition-friendly.

Proof. Although the algorithm $\psi$ is formally only defined on input $(W, T)$ for which $|W| = |T|$, we will abuse notation here and consider $\psi$ even on worker/task input $(W, T)$ satisfying $|W| \neq |T|$.\footnote{Indeed, the definition of the $k$-bin hash does not require a worker-task input with $|W| = |T|$. The only reason we require this equality in general is to simplify calculations, as in practice the algorithm will only be run on worker-task inputs of equal size.} Define the difference-score of a pair of worker/task inputs $I_1 = (W_1, T_1), I_2 = (W_2, T_2)$ to be the quantity

$$d(I_1, I_2) = |W_1 \setminus W_2| + |W_2 \setminus W_1| + |T_1 \setminus T_2| + |T_2 \setminus T_1|.$$ 

We will show the stronger statement that the difference-score $d(O_1, O_2)$ of the corresponding worker/task outputs $O_1 = (W_1', T_1'), O_2 = (W_2', T_2')$ satisfies

$$d(O_1, O_2) \leq d(I_1, I_2). \quad (3)$$
It suffices to consider only two special cases: the case in which \(W_2 = W_1 \cup \{\omega\}\) for some worker \(\omega\) and \(T_2 = T_1\); and the case in which \(T_2 = T_1 \cup \{\tau\}\) for some task \(\tau\) and \(W_2 = W_1\). Iteratively applying these two cases to transform \(I_1\) into \(I_2\) implies inequality \(3\).

For this purpose, the roles of \(W\) and \(T\) are identical, so suppose without loss of generality that \(W_2 = W_1 \cup \{\omega\}\) for some worker \(\omega\) and \(T_2 = T_1\). Recall that the assignment of workers and tasks to buckets is determined by some hash functions \(h_1, h_2\) and in particular is the same whether we input \(W_1\) or \(W_2\). We first assign (only) the elements of \(W_1\) and \(T_1\) to their respective buckets, and then look at how including the assignment of \(\omega\) changes the worker-task output. If \(h_1\) assigns \(\omega\) to either a bin with no tasks or a bin which already has some lexicographically smaller worker, then we will have \(W_2' = W_1' \cup \{\omega\}\) and \(T_2' = T_1\). If \(h_1\) assigns worker \(\omega\) to a bin with no other workers and at least one task, we let the smallest such task be \(\tau\) and see \(W_2' = W_1' \cup \{\tau\}\) and \(T_2' = T_1'\). Finally, if \(h_1\) assigns \(\omega\) to a bin with only larger workers and at least one task, we let the minimal such worker be \(\gamma\) and we see \(W_2' = W_1' \cup \{\gamma\}\) and \(T_2' = T_1'\). In all three cases, \(d(O_1, O_2) = 1\), as desired. \(\square\)

Next, we will show that composing composition-friendly algorithms has the effect of summing switching costs.

**Lemma 13.** Suppose that partial-assignment algorithms \(\psi_1, \psi_2, \ldots, \psi_k\) are all composition-friendly, and that each \(\psi_i\) is \(s_i\)-switching-cost bounded. Then \(\psi_1 \circ \psi_2 \circ \cdots \circ \psi_k\) is composition-friendly and is \((\sum_i s_i)\)-switching-cost bounded.

**Proof.** By induction, it suffices to prove the lemma for \(k = 2\). Let \(I_1 = (W_1, T_1)\) and \(I_2 = (W_2, T_2)\) be unit-distance worker/task inputs.

For \(i \in \{1, 2\}\), let \(I_i' = (W_i', T_i')\) be the worker/task output for \(\psi_i(W, T_i)\), and let \(I_i'' = (W_i'', T_i'')\) be the worker/task output for \(\psi_2(W_i', T_i')\).

Since \(\psi_1\) is composition friendly, its outputs \(I_1'\) and \(I_1''\) are unit distance. Since \(I_1'\) and \(I_1''\) are unit distance, and since \(\psi_2\) is composition friendly, the outputs \(I_1''\) and \(I_2''\) of \(\psi_2\) are also unit distance. Thus \(\psi_1 \circ \psi_2\) is composition friendly.

Since the inputs \(I_1\) and \(I_2\) to \(\psi_1\) are unit-distance, \(\psi_1(I_1)\) and \(\psi_1(I_2)\) differ in at most \(s_1\) worker-task assignments. Since the inputs \(I_1''\) and \(I_2''\) to \(\psi_2\) are also unit distance, \(\psi_2(I_1'')\) and \(\psi_2(I_2'')\) differ in at most \(s_2\) worker-task assignments. Thus the composition \(\psi_1 \circ \psi_2\) is \((s_1 + s_2)\)-switching-cost bounded, as desired. \(\square\)

We can now prove Proposition \(10\).

**Proof of Proposition \(10\)**. By Lemma \(11\) each \(A_{i,j}\) is \(O(1)\)-switching-cost bounded. By Lemma \(12\) each \(A_{i,j}\) is composition friendly. Since \(A\) is the composition of the \(O(\log w \log n)\) different \(A_{i,j}\)’s, it follows by Lemma \(13\) that \(A\) is \(O(\log w \log n)\)-switching-cost bounded. \(\square\)

## 4 Derandomizing the construction

In this section, we derandomize the multi-round balls-to-bins algorithm to prove the following theorem.

**Theorem 2.** There is an explicit worker-task assignment function that achieves switching cost \(O(\text{polylog}(wt))\).

To this end we use pseudorandom objects called strong dispersers. Intuitively, a disperser is a function such that the image of any not-too-small subset of its large domain (e.g. workers or tasks) is a dense subset of its small co-domain (e.g. bins). Since this requirement is hard to satisfy directly, dispersers are defined with a second argument, called the seed. For a strong disperser, the density requirement is satisfied only in expectation over the seed. The standard way to define strong dispersers (Definition \(14\) below) is in the language of random variables. We follow with an equivalent alternative Definition \(15\) more convenient for our purposes.
Definition 14 (Strong dispersers). For $k \in \mathbb{N}$, $\epsilon \in \mathbb{R}_+$, a $(k, \epsilon)$-strong disperser is a function $\text{Disp} : \{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m$ such that for any random variable $X$ over $\{0,1\}^n$ with min-entropy at least $k$ we have

$$|\text{Supp}((\text{Disp}(X,U_d),U_d))| \geq (1-\epsilon) \cdot 2^{m+d}.$$  

Here $\text{Supp}$ denotes the support of a random variable, $U_d$ denotes the uniform distribution on $\{0,1\}^d$, and the min-entropy of a random variable $X$ is defined as $\min_i (-\log_2(\Pr[X=x]))$. We will use a simple fact that any distribution which is uniform on a $2^k$-element subset of the universe and assigns zero probability elsewhere (called flat $k$-source in pseudorandomness literature) has min-entropy $k$. Interestingly, every distribution with min-entropy at least $k$ is a convex combination of such distributions (see e.g. Lemma 6.10 in [26], first proved in [10]), which makes the following definition equivalent.

Definition 15 (Strong dispersers, alternative definition). For $k \in \mathbb{N}$, $\epsilon \in \mathbb{R}_+$, a $(k, \epsilon)$-strong disperser is a function $\text{Disp} : [N] \times [D] \to [M]$ such that for any subset $S \subseteq [N]$ of size $|S| \geq 2^k$ we have

$$|\{(\text{Disp}(s,d),d) : s \in S, d \in [D]\}| \geq (1-\epsilon) \cdot M \cdot D.$$  

We use efficient explicit strong dispersers constructed by Meka, Reingold and Zhou [22].

Theorem 16 (Theorem 6 in [22]). For all $N = 2^n$, $k \in \mathbb{N}$, and $\epsilon \in \mathbb{R}_+$, there exists an explicit $(k, \epsilon)$-strong disperser $\text{Disp} : [N] \times [D] \to [M]$ with $D = 2^{O(\log n)} = \text{polylog} N$ and $M = 2^{k-3\log n - O(1)} = 2^k \cdot \Omega(1/\log^3 N)$.

Designing the algorithm. We begin with applying Lemma 5 in order to be able to restrict our attention to task sets (rather than multisets), at the expense of increasing the number of tasks from $t$ to $wt$. For convenience, we round up the new number of tasks to the closest power of two $N = 2^{|\log wt|}$.

Our explicit algorithm $\mathcal{E}$ has the same structure as the randomized algorithm $\mathcal{A}$, i.e. it is the composition of $\log w$ partial assignment algorithms

$$\mathcal{E} = E_1 \circ E_2 \circ \cdots \circ E_{\log w}.$$  

Each $E_i$ is responsible for bringing down the number of unassigned workers to the next power of two, and is composed of a number of explicit sub-algorithms $E_{i,j}$. Contrary to $\mathcal{A}_{i,j}$’s, sub-algorithms $E_{i,j}$’s are not identical copies for a fixed $i$. However, the chain of distinct sub-algorithms has to be copied $O(\log^3 N)$ times. We reflect this introducing the $\widehat{E}_i$ notation:

$$E_i = \widehat{E}_i \circ \widehat{E}_i \circ \cdots \circ \widehat{E}_i,$$  
where $\widehat{E}_i = E_{i,1} \circ E_{i,2} \circ \cdots \circ E_{i,\text{polylog} N}$.

The key difference between the randomized and explicit algorithm is that $E_{i,j}$’s, instead of using random hash functions $h_1, h_2$, use explicit functions obtained from strong dispersers. Another notable difference is that $\mathcal{A}_{i,j}$’s use $k$ bins to deal with input sets of size in $[k, 1.1k]$, while $E_{i,j}$’s have to use polylogarithmically less bins, limiting the number of worker-task pairs that can be assigned by a single sub-algorithm and, as a consequence, forcing us to compose a larger number of sub-algorithms.

Let us fix $i \in [\log w]$, and denote $k_i = \lceil \log w \rceil - i$. Let $\text{Disp}_{i} : [N] \times [D_i] \to [M_i]$ be the $(k_i, 1/4)$-strong disperser given by Theorem 16. Recall that $D_i = \text{polylog} N$, $M_i = 2^{k_i} \cdot \Omega(1/\log^3 N)$, and $N$ is large enough so that all workers and all tasks are elements of $[N]$. We will have $\widehat{E}_i = E_{i,1} \circ E_{i,2} \circ \cdots \circ E_{i,D_i}$. For each $j \in [D_i]$, sub-algorithm $E_{i,j}$ assigns workers and tasks to $M_j$ bins. Each worker $\omega \in W$ is assigned to bin $\text{Disp}_i(\omega, j)$, and each task $\tau \in T$ is assigned to bin $\text{Disp}_i(\omega, j)$. Then, like in the randomized strategy, for each active bin (i.e. one which was assigned nonempty sets of workers and tasks) the smallest worker and the smallest task in that bin get assigned to each other.
Theorem 3. lower bound of cost that depends only on It is natural to wonder whether smaller bounds can be achieved, and in particular, whether a small switching tisets of balls-to-bins algorithm, we do not exploit the fact that the hash functions $h_1$, $h_2$ are random. Actually, as we already remark, our switching cost bound is deterministic and thus works for any choice of functions $h_1$, $h_2$. Therefore the same analysis works for the explicit algorithm. Namely, each sub-algorithm $\mathcal{E}_{i,j}$ is $O(1)$-switching cost bounded and composition-friendly (Lemmas 11 and 12 generalize trivially), thus the switching cost of $E$ depends only on the number of sub-algorithms, which is polylog $N = \text{polylog } w$, as desired.

Proving the algorithm is fully-assigning. We begin by analyzing the number of worker/task assignments made by $\hat{E}_i = \mathcal{E}_{i,1} \circ \cdots \circ \mathcal{E}_{i,D_i}$.

Lemma 17. Let $(W, T)$ be a worker/task input satisfying $|W| = |T| \geq 2^k$. Then $\hat{E}_i(W, T)$ makes at least $M_i/4$ worker/task assignments.

Proof. By the definition of dispersers, the two images

$$\{(\text{Disp}_i(\omega, j), j) : \omega \in W, j \in [D_i]\}, \quad \text{and} \quad \{(\text{Disp}_i(\tau, j), j) : \tau \in T, j \in [D_i]\}$$

have size at least $(3/4) \cdot M_i \cdot D_i$. Since they are both subsets of $[M_i] \times [D_i]$, their intersection has size at least $(1/2) \cdot M_i \cdot D_i$. By the pigeonhole principle, there must exist $j \in D_i$ such that

$$|\text{Disp}_i(W, j) \cap \text{Disp}_i(T, j)| \geq M_i/2. \quad (4)$$

Let us fix such $j$, and look at the execution of $\mathcal{E}_{i,j}$. For each bin $b \in \text{Disp}_i(W, j) \cap \text{Disp}_i(T, j)$, if $b$ is not active, then all workers $\{\omega \in W \mid \text{Disp}_i(\omega, j) = b\}$ or all tasks $\{\tau \in T \mid \text{Disp}_i(\tau, j) = b\}$ must have been already assigned by $(\mathcal{E}_{i,1} \circ \cdots \circ \mathcal{E}_{i,j-1})(W, T)$. Thus, each bin in $\text{Disp}_i(W, j) \cap \text{Disp}_i(T, j)$ either is active – and contributes one worker and one task to the assignment – or is inactive and testifies that at least one worker or at least one task is assigned by earlier sub-algorithms. Let $c_a$ denote the number of active bins, $c_w$ denote the number of inactive bins testifying for a worker assigned by earlier sub-algorithms, and $c_t$ denote the number of inactive bins testifying for a task. We have $c_a + c_w + c_t \geq M_i/2$, by Inequality (4). It follows that the number of worker/task assignments made by $(\mathcal{E}_{i,1} \circ \cdots \circ \mathcal{E}_{i,j})(W, T)$ is at least $c_a + \max(c_w, c_t) \geq c_a + \frac{1}{2} (c_w + c_t) \geq M_i/4$, as desired. \hfill $\Box$

Recall that $M_i = 2^k \cdot \Omega(1/ \log^3 N)$. Thus, Lemma 17 implies that each $\mathcal{E}_i$ – which is a composition of $O(\log^3 N)$ copies of $\mathcal{E}_i$ – when given a worker/task input of size at most $2 \cdot 2^k$ returns a worker/task output of size at most $2^k$. It follows that $E = \mathcal{E}_1 \circ \mathcal{E}_2 \circ \cdots \circ \mathcal{E}_{\log w}$ is fully-assigning, which concludes the proof of Theorem 2.

5 Lower bounds on switching cost

Define $s_{w,t}$ to be the optimal switching cost for assignment functions that assign workers $1, 2, \ldots, w$ to multisets of $t$ tasks from the universe $[t]$. The upper bounds in this paper establish that $s_{w,t} \leq O(\log w \log(wt))$. It is natural to wonder whether smaller bounds can be achieved, and in particular, whether a small switching cost that depends only on $w$ can be achieved.

It trivially holds that $s_{w,t} \leq w$. We show that when $t$ is sufficiently large relative to $w$, there is a matching lower bound of $s_{w,t} \geq w$.

Theorem 3. For every $w$ and $t \geq \text{tow}(\Omega(w))$, every worker-task assignment function has switching cost $w$. 

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Proof. Given any worker-task assignment function \( f \), we can actually find high switching cost between a pair of task subsets, in which all demands are 0 or 1. For each \( T \subseteq [t] \) of \( w \) tasks, \( f \) produces a bijection of workers \([w]\) to tasks \( T \). In order to lower-bound the switching cost, we produce a coloring of the complete \( w \)-uniform hypergraph with \( t \) vertices. The coloring will be designed so that, if it contains a monochromatic clique on \( w+1 \) vertices, then the assignment function \( f \) must have worst-possible switching cost \( w \). By applying the hypergraph Ramsey theorem, we deduce that, if \( t \) is large enough, then the coloring must contain a monochromatic \((w+1)\)-clique, completing the lower bound.

**Coloring the complete \( w \)-uniform hypergraph on \( t \) vertices.** Let \( K_{w}^{t} \) denote the complete \( w \)-uniform hypergraph on \( t \) vertices. Note that the hyperedges of \( K_{w}^{t} \) are just the \( w \)-element subsets of \([t]\), which correspond to sets of tasks.

For a task set \( T = \{\tau_1, \ldots, \tau_w\} \), where \( 1 \leq \tau_1 < \cdots < \tau_w \leq t \), we color the hyperedge \( T \) with the tuple \( \pi = \langle \pi(1), \pi(2), \ldots, \pi(w) \rangle \), where \( \tau_{R(i)} \) is the task assigned to worker \( i \). One can think of \( \pi \) as a permutation of numbers \( \{1, 2, \ldots, w\} \), and thus the coloring consists of at most \( w! \) colors.

**Monochromatic \((w+1)\)-cliques imply high switching cost.** The key property of the coloring \( C \) is that, if \( K_{w}^{t} \) contains a monochromatic \((w+1)\)-vertex clique (i.e., \( K_{w+1}^{t} \)), then \( f \) must have switching cost \( w \).

Namely, if \( K_{w}^{t} \) contains a monochromatic \((w+1)\)-clique, then we can find \( w+1 \) vertices, \( \tau_1 < \tau_2 < \cdots < \tau_{w+1} \), such that every \( w \)-element subset \( T \) of these tasks is assigned the same permutation \( \pi \) as its color. In particular, this means that for the task-set \( T_1 = \{\tau_1, \ldots, \tau_w\} \) each worker \( i \) is assigned to task \( \tau_{\pi(i)} \), but for the task-set \( T_2 = \{\tau_2, \ldots, \tau_{w+1}\} \) that same worker \( i \) is assigned to a different task \( \tau_{\pi(i)+1} \). Thus there is a pair of adjacent task sets \( T_1, T_2 \) that exhibit switching cost \( w \).

**Finding a monochromatic clique.** In order to complete the lower bound, we wish to show that, if \( t \) is sufficiently large, then the coloring contains a monochromatic \( K_{w+1}^{t} \). To do this, we employ the hypergraph Ramsey theorem.

**Theorem 18 (Theorem 1 in [12]).** Let \( k \geq 2 \) and \( N \geq n \geq 2 \) be positive integers. The hypergraph Ramsey number \( R(k, n, N) \) is defined to be the least positive integer \( M \) such that for every \( k \)-coloring of the hyperedges of \( K_{M}^{n} \), the complete \( n \)-uniform hypergraph on \( M \) vertices, contains a monochromatic copy of \( K_{M}^{n} \). This quantity satisfies

\[
R(k, n, N) \leq k^{(k^{n-1} (d^2-2) + 2^k (N-n) + 1)}.
\]

Applying Theorem 18, we see that if \( t \geq R(w!, w, w+1) \), then the \((w!)\)-coloring of \( K_{w}^{t} \) contains a monochromatic \((w+1)\)-clique, and the assignment function \( f \) must have switching cost \( w \), as desired. By Theorem 18 \( R(w!, w, w+1) \leq tw(O(w)) \), which implies that that every worker-task assignment function has switching cost \( w \) when \( t \geq tw(O(w)) \). This completes the proof of Theorem 3.

Another way of viewing this argument is that a worker-task assignment function with switching cost less than \( w \) gives rise to a proper \((w!)\)-coloring of a certain graph, with vertex set \( \binom{t}{w} \) and edges of the form \( \{\{\tau_1, \ldots, \tau_w\}, \{\tau_2, \ldots, \tau_{w+1}\}\} \) for \( \tau_1 < \tau_2 < \cdots < \tau_{w+1} \). Such graphs are studied under the name of **shift-graphs**, see e.g., [14 Section 3.4], where the definition and proofs of basic properties are attributed to [11]. In particular, the chromatic number of shift-graphs is known to be \((1 + o(1)) \cdot \log^{(w-1)} t \) (with the superscript denoting iteration). This gives an alternative way to complete the proof of Theorem 3 and it gives the same asymptotic bound on \( t \) in terms of \( w \). While the chromatic number lower bound suffices to prove the switching cost bound, the nearly matching upper bound (on chromatic number) suggests that an entirely different technique would be needed in order to asymptotically improve the switching cost bound.
6 Densification into Hamming Space

In this section, we apply our results on worker-task assignment to the problem of densification. In particular, we show how to embed sparse high-dimensional binary vectors into dense low-dimensional Hamming space.

Let \( \mathcal{H}_n^k \) denote the set of \( n \)-dimensional binary vectors with \( k \) ones. Let \( \mathcal{H}_k(n) \) denote the set of \( k \)-dimensional vectors with entries from \([n]\). We show that \( \mathcal{H}_n^k \) can be embedded into \( \mathcal{H}_k(n) \) with distortion \( O(\log n \log k) \).

**Theorem 4.** There exists a map \( \phi : \mathcal{H}_n^k \rightarrow \mathcal{H}_k(n) \) such that, for every \( \vec{x}, \vec{y} \in \mathcal{H}_n^k \),

\[
\text{Ham}(\vec{x}, \vec{y})/2 \leq \text{Ham}(\phi(\vec{x}), \phi(\vec{y})) \leq O(\log n \log k) \text{Ham}(\vec{x}, \vec{y}).
\]

**Proof.** Using Theorem 1 let \( \psi \) be a worker-task assignment function mapping workers 1, 2, \ldots, \( k \) to a task set \( T \subseteq [n] \) with switching-cost \( O(\log n \log k) \).

For \( \vec{x} \in \mathcal{H}_n^k \), define \( T(\vec{x}) = \{ i \mid \vec{x}_i = 1 \} \) to be the task set consisting of the positions in \( \vec{x} \) that are 1. Define \( \phi(\vec{x}) \) to be the \( k \)-dimensional vector whose \( i \)-th coordinate denotes the task \( i \) assigns worker \( i \). For example, if \( k = 3 \), \( \vec{x} = \langle 0, 1, 0, 1, 1, 0 \rangle \), and \( \psi(T(\vec{x})) \) assigns workers 1, 2, 3 to tasks 4, 2, 5, respectively, then \( \phi(\vec{x}) = \langle 4, 2, 5 \rangle \).

Since the coordinates of \( \phi(T(\vec{x})) \) are a permutation of the positions \( T(\vec{x}) \) in which \( \vec{x} \) is non-zero, it is necessarily the case that

\[
\text{Ham}(\phi(\vec{x}), \phi(\vec{y})) \geq |T(\vec{x}) \setminus T(\vec{y})| \geq \text{Ham}(\vec{x}, \vec{y})/2.
\]

On the other hand, since \( \psi \) has switching cost \( O(\log n \log k) \), it is also the case that \( \psi(\vec{x}) \) and \( \psi(\vec{y}) \) differ by at most \( O(\log n \log k) \text{Ham}(\vec{x}, \vec{y}) \) assignments, meaning that,

\[
\text{Ham}(\phi(\vec{x}), \phi(\vec{y})) \leq O(\log n \log k) \text{Ham}(\vec{x}, \vec{y}).
\]

This completes the proof of the theorem.

We remark that Theorem 4 can be generalized to allow for the the domain space \( \mathcal{H}_n^k \) to have non-binary entries. In particular, if \( \mathcal{L}_n^k \) is the set of vectors with non-negative integer entries that sum to \( k \), then there is an embedding \( \phi : \mathcal{L}_n^k \rightarrow \mathcal{H}_k(n) \) such that, for \( x, y \in \mathcal{L}_n^k \),

\[
\ell_1(\vec{x}, \vec{y})/2 \leq \text{Ham}(\phi(\vec{x}), \phi(\vec{y})) \leq O(\log n \log k)\ell_1(\vec{x}, \vec{y}).
\]

This follows from the same argument as Theorem 4, except that now \( T(x) \) is the multiset for which each element \( i \in [n] \) has multiplicity \( \vec{x}_i \), and now \( \psi \) is the worker-task assignment mapping workers 1, 2, \ldots, \( k \) to a task multiset \( T \subseteq [n] \).

7 Open problems

We leave open the question of closing the gap between upper and lower bounds for the worker-task assignment problem: the upper bound is \( \text{polylog} \langle \text{wt} \rangle \) and the lower bound is \( \log^* \rangle \langle t \rangle \).

One interesting parameter regime is when \( w \) and \( t \) are comparable in size (say within a polynomial factor of each other). In this regime, no super-constant lower bound is known.

Another interesting direction is the problem of densification into Hamming space. Our upper bound for the worker-task assignment problem implies an upper bound for this problem, but our lower bound does not carry over. We leave open the problem of whether there is a better upper bound or a super-constant lower bound for this problem.
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