ON STEADY STATE OF SOME LOTKA-VOLTERRA
COMPETITION-DIFFUSION-ADVECTION MODEL

Qi Wang∗
College of Science, University of Shanghai for Science and Technology
Shanghai, 200093, China
(Communicated by Yuan Lou)

Abstract. In this paper, we study a shadow system of a two species Lotka-Volterra competition-diffusion-advection system, where the ratio of diffusion and advection rates are supposed to be a positive constant. We show that for any given migration, if the product of interspecific competition coefficients of competitors is small, then the shadow system has coexistence state; otherwise we can always find some migration such that it has no coexistence state. Moreover, these findings can be applied to steady state of the two-species Lotka-Volterra competition-diffusion-advection model. Particularly, we show that if the interspecific competition coefficient of the invader is sufficiently small, then rapid diffusion of the invader can drive to coexistence state.

1. Introduction and statement of the main results. The semilinear parabolic Lotka-Volterra competition model

\[
\begin{aligned}
  u_t &= \mu \Delta u + u(r(x) - u - bv), \quad \text{in } \Omega \times (0, +\infty), \\
  v_t &= \nu \Delta v + v(r(x) - cu - v), \quad \text{in } \Omega \times (0, +\infty), \\
  \frac{\partial u}{\partial n} &= \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial \Omega \times (0, +\infty), \\
  u(x, 0) &= u_0(x) \geq 0, \quad \text{in } \Omega, \\
  v(x, 0) &= v_0(x) \geq 0, \quad \text{in } \Omega,
\end{aligned}
\]  

models two competing species. Here \(u(x, t)\) and \(v(x, t)\) denote respectively the population densities of two competing species at location \(x \in \Omega\) and time \(t > 0\), and \(\mu, \nu > 0\) are random diffusion rates of species \(u\) and \(v\) respectively. The habitat \(\Omega\) is a bounded region in \(\mathbb{R}^N\), with smooth boundary \(\partial \Omega\), \(n\) denotes the unit outer normal vector on \(\partial \Omega\), and the no flux boundary condition means that no individuals cross the boundary. The function \(r(x)\) represents their common intrinsic growth rate or local carrying capacity, \(b > 0\) and \(c > 0\) are interspecific competition coefficients. Then the maximum principle [34] yields that \(u(x, t) > 0, v(x, t) > 0\) for every \(x \in \bar{\Omega}\) and every \(t > 0\). Moreover \(u(x, t)\) and \(v(x, t)\) are classical solutions of (1) and exist for all \(t > 0\). By both mathematicians and ecologists, particular interests in two-species Lotka-Volterra competition models with spatially homogeneous or heterogeneous interactions are the dynamics and coexistence states of (1). See

2010 Mathematics Subject Classification. Primary: 92B05, 35B35, 35B40; Secondary: 35B30, 35J47.

Key words and phrases. Lotka-Volterra, competition-diffusion-advection model, shadow system, coexistence state.

∗ Corresponding author: Qi Wang.
Assumption 1.1.

\( r(x) \) is non-constant, Hölder continuous, and \( r(x) > 0 \) in \( \overline{\Omega} \).

Assumption 1.2.

\[
0 < b, c \leq 1, bc < 1.
\]  

(2)

Under Assumption 1.1, the equation

\[
\begin{align*}
\mu \Delta \Theta + \Theta(r(x) - \Theta) &= 0, \quad \text{in } \Omega, \\
\frac{\partial \Theta}{\partial n} &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

(3)

has a unique positive solution, denoted by \( \Theta(x, \mu) \), which indicates that (1) has two semi-trivial states, denoted by \((\Theta(x, \mu); 0)\) and \((0, \Theta(x, \nu))\), for every \( \mu > 0 \) and \( \nu > 0 \). Clearly, Assumption 1.2 contains the weak competition case

\[
\{ (b, c) : 0 < b, c < 1 \},
\]

(4)

which was considered in \([5, 11, 17, 18, 19, 20, 21, 32]\), etc.

For the special case of identical competition ability \( b = c = 1 \), it is shown in \([11]\) that if \( \mu < \nu \), then \((\Theta(x, \mu), 0)\) is globally asymptotically stable among all non-negative nontrivial initial data. In other words, the slower diffuser wins. By symmetry, a similar conclusion holds for \( \mu > \nu \). In particular, (1) has no coexistence states if \( \mu \neq \nu \). For the case when \( \mu = \nu = 0 \), (1) has a family of coexistence states, which is the global attractor for all non-negative non-trivial initial data.

For the weak competition case \( 0 < b, c < 1 \), it is shown in \([25]\) that if \( \mu \) and \( \nu \) are sufficiently small, then (1) has a unique, globally asymptotically stable positive steady state \((u^*, v^*)\). Moreover, \((u^*, v^*)\) converges to \((\frac{1-b}{1-bc}r(x), \frac{1-c}{1-bc}r(x))\) in \( L^\infty(\Omega) \) as \( \mu \to 0 \) and \( \nu \to 0 \). When both \( \mu \) and \( \nu \) are sufficiently large, it is not difficult to see that (1) again has a unique, globally asymptotically stable positive steady state \((u^{**}, v^{**})\), which converges to \((\frac{1-b}{(1-bc)(\Omega)} \int_{\Omega} r(x)dx, \frac{1-c}{(1-bc)(\Omega)} \int_{\Omega} r(x)dx)\) as \( \mu \to +\infty \) and \( \nu \to +\infty \).

Under above conditions and assumptions, Lou \([32]\) verified that there exists some constant \( c_\ast \in (0, 1) \) such that (i) for any \( c \in (0, c_\ast) \), the steady state \((\Theta, 0)\) is unstable when \( \mu > 0 \) and \( \nu > 0 \); (ii) for any \( c \in (c_\ast, 1) \), there exist \( \bar{\mu} > 0 \) and \( \bar{\nu} > 0 \) such that \((\Theta, 0)\) is unstable. Meanwhile, when \( c \in (c_\ast, 1) \), there exists \( b_\ast \in (0, 1) \) such that for some \((\mu, \nu) \in (0, +\infty) \times (0, +\infty) \), \((\Theta, 0)\) is globally asymptotically stable provided \( b \in (0, b_\ast] \). Furthermore He and Ni \([19]\) provided a complete classification on the global dynamics of system (1), which says that either one of the two semi-trivial steady states is globally asymptotically stable, or there is a unique co-existence steady state which is globally asymptotically stable, or the system is degenerate in the sense that there is a compact global attractor consisting of a continuum of steady states which connect the two semi-trivial steady states (see \([19]\), Theorems 1.3 and 3.4). We refer the interested readers to \([17, 18]\) for some investigations on system (1) and \([20, 21]\) for some developments.

It seems reasonable to argue that besides random dispersal, it is also plausible that species could move upward along the resource gradient (see \([2, 6, 10]\) for
Example and the references therein). In this paper we deal with a general problem

\[
\begin{aligned}
    &u_t = \mu \Delta u - \alpha \nabla \cdot (u \nabla P(x)) + u(r(x) - u - bv), \quad \text{in } \Omega \times (0, +\infty), \\
    &v_t = \nu \Delta v - \beta \nabla \cdot (v \nabla P(x)) + v(r(x) - cu - v), \quad \text{in } \Omega \times (0, +\infty), \\
    &\frac{\partial u}{\partial n} - \alpha u \frac{\partial P}{\partial n} = \nu \frac{\partial v}{\partial n} - \beta v \frac{\partial P}{\partial n} = 0, \quad \text{on } \partial \Omega \times (0, +\infty), \\
    &u(x, 0) = u_0(x) \geq 0, \quad \text{in } \Omega, \\
    &v(x, 0) = v_0(x) \geq 0, \quad \text{in } \Omega,
\end{aligned}
\]

where the non-constant function \( P(x) \in C^2(\Omega) \) is used to specify the advective direction, and the advection rates of two species are denoted by \( \alpha, \beta > 0 \), respectively. Here the movement strategies, growth rates and competition abilities of two species are taken into account and allowed to be different. Throughout this paper, besides Assumption 1.1 and Assumption 1.2 we also make the following basic hypothesis.

**Assumption 1.3.**

\[
\frac{\alpha}{\mu} = \frac{\beta}{\nu} =: \eta > 0.
\]

This paper is devoted to the existence and qualitative properties of coexistence steady states of (5) (i.e. positive solutions of the following problem).

\[
\begin{aligned}
    &\mu \Delta u - \alpha \nabla \cdot (u \nabla P(x)) + u(r(x) - u - bv) = 0, \quad \text{in } \Omega, \\
    &\nu \Delta v - \beta \nabla \cdot (v \nabla P(x)) + v(r(x) - cu - v) = 0, \quad \text{in } \Omega, \\
    &\frac{\partial u}{\partial n} - \alpha u \frac{\partial P}{\partial n} = \nu \frac{\partial v}{\partial n} - \beta v \frac{\partial P}{\partial n} = 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

1.1. **Motivation and related work.** Under Assumption 1.1 and Assumption 1.2, Li, Wang and Wang [30] studied the following non-advective version of the two-species Lotka-Volterra type competition-diffusion system (7).

\[
\begin{aligned}
    &\mu \Delta u + u(r(x) - u - v) = 0, \quad \text{in } \Omega, \\
    &\nu \Delta v + v(r(x) - cu - v) = 0, \quad \text{in } \Omega, \\
    &\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

They partially considered which cases ensure the coexistence of the two species, which cases cause one species to extinct, and whether coexistence state is stable or not.

The main results established in [30] are to explore the connection between the system (8) and the following shadow system, namely the case where \( \nu \to +\infty \), and then partially verified Lou’s conjecture.

\[
\begin{aligned}
    &\mu \Delta u + u(r(x) - u - \xi) = 0, \quad \text{in } \Omega, \\
    &\bar{r} - cu - \xi = 0, \quad \text{in } \Omega, \\
    &\frac{\partial u}{\partial n} = 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

That is for each positive function \( r(x) \), there exists \( c^* = c^*(r, \Omega) \in (0, 1) \) such that (9) has a solution \( (u, \xi) \), if \( c \in (0, c^*) \). Meanwhile, for \( c \in (c^*, 1) \), there exist \( \bar{\mu} < \mu \) such that \( \xi \) is positive outside the interval \( [\bar{\mu}, \mu] \), and non-positive for some \( \bar{\mu} \in (\mu, \mu] \). Moreover for any fixed \( \mu > 0 \), if \( c \) is small enough, then there exists \( v^* = v^*(\mu, c) \) such that (8) has a positive solution \( (u_\nu, v_\nu) \) for \( \nu > v^* \), with \( u_\nu(x) \to u_0(x), \ v_\nu(x) \to \xi \) in \( C^2(\Omega) \) as \( \nu \to +\infty \), where \( (u_0(x), \xi) \) is a positive solution of (9).
Throughout this paper, we say that a system has a positive solution if two components are both positive in \( \Omega \). Motivated by the above work, we consider whether there are parallel versions to (7).

1.2. Main results. We first introduce the following linear eigenvalue problem.

\[
\begin{aligned}
\mu \nabla \cdot (e^{\eta P(x)} \nabla \psi) + e^{2nP(x)} h(x) \psi &= \sigma \psi, & \text{in } \Omega, \\
\frac{\partial \psi}{\partial n} &= 0, & \text{on } \partial \Omega, \\
\psi > 0, & \text{in } \Omega,
\end{aligned}
\]  

(10)

where \( \eta \) is as in (6) and \( h(x) \in C(\Omega) \). By employing the Krein-Rutman Theorem \cite{m29}, it is standard to prove that problem (10) admits a principal eigenvalue \( \sigma_{1,h} \), and its corresponding eigenfunction \( \psi \) can be chosen to satisfy \( \psi > 0 \) in \( \Omega \). Here \( \sigma_{1,h} \) is defined as

\[
\sigma_{1,h} = \max_{\psi \in W^{1,2}(\Omega), \psi \neq 0} \frac{-\mu \int_{\Omega} e^{\eta P} |\nabla \psi|^2 \, dx + \int_{\Omega} e^{2nP} h(x) \psi^2 \, dx}{\int_{\Omega} \psi^2 \, dx}.
\]  

(11)

We also recall the following single-species growth model corresponding to system (5).

\[
\begin{aligned}
w_t &= \mu \Delta w - \alpha \nabla \cdot (e^{\eta P} \nabla w) + \omega(r(x) - w), & \text{in } \Omega \times (0, +\infty), \\
\frac{\partial w}{\partial n} &= 0, & \text{on } \partial \Omega \times (0, +\infty), \\
w(0,0) &= 0, & \text{in } \Omega,
\end{aligned}
\]  

(12)

where \( \mu, \alpha > 0 \) and \( r(x) \) satisfies Assumption 1.1. We take \( m(x) = r(x)e^{-\eta P(x)} \) and \( \omega = we^{-\eta P(x)} \), then we have \( m(x) > 0 \) and

\[
\begin{aligned}
e^{\eta P} \omega_t &= \mu \nabla \cdot (e^{\eta P} \nabla \omega) + e^{2nP} \omega (m(x) - \omega), & \text{in } \Omega \times (0, +\infty), \\
\frac{\partial \omega}{\partial n} &= 0, & \text{on } \partial \Omega \times (0, +\infty), \\
\omega(0,0) &= w_0 e^{-\eta P(x)} \geq 0, & \text{in } \Omega.
\end{aligned}
\]  

(13)

Due to the positivity of \( m(x) \), one can see \( \sigma_{1,m} > 0 \) and (13) admits a unique positive steady state, denoted by \( \theta \) (see \cite{m6} for example). That is

\[
\begin{aligned}
\mu \Delta \theta + \alpha \nabla P(x) \nabla \theta + e^{\eta P} \theta (m(x) - \theta) &= 0, & \text{in } \Omega, \\
\frac{\partial \theta}{\partial n} &= 0, & \text{on } \partial \Omega,
\end{aligned}
\]  

(14)

\[
\begin{aligned}
\mu \nabla \cdot (e^{\eta P} \nabla \theta) + e^{2nP} \theta (m(x) - \theta) &= 0, & \text{in } \Omega, \\
\frac{\partial \theta}{\partial n} &= 0, & \text{on } \partial \Omega.
\end{aligned}
\]  

(15)

From [1] it is not difficult to see that \( \theta \to \int_{\Omega} me^{2nP} \) in \( C^2(\overline{\Omega}) \) as \( \mu \to +\infty \), and then

\[
\lim_{\mu \to +\infty} \int_{\Omega} \theta e^{2nP} = \int_{\Omega} me^{2nP}.
\]  

(16)

The standard super-subsolution method also yields that

\[
\lim_{\mu \to 0} \int_{\Omega} \theta e^{2nP} = \int_{\Omega} me^{2nP}.
\]  

(17)

Using the similar transformation \( u = e^{-\eta P(x)} \), \( v = ve^{-\eta P(x)} \), and still denote \( u, v \) by \( u, v \), (7) can be rewritten as
Assume that Theorem 1.5. \( \theta (19) \) such that no positive solutions exist for the problem \( \Omega \). We first introduce the following lemma.

\[ \nu > \nu \]

\( \nu \) enough, then there exists \( \mu \) has a positive solution for \( \nu \), then there exists \( d \) \( \nu \) \( \mu \) \( \nu \) \( \mu \) \( \nu \)

Namely, we take \( v \) \( \Delta v + \beta \nabla P \nabla v + \epsilon \nabla P v(x) - cu - v = 0 \), \( \Omega \)

\[ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad \text{on } \partial \Omega. \] (18)

Based on the above preparations, we are now ready to consider (18) and its shadow system,

\[ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad \text{on } \partial \Omega. \] (19)

\[ \mu \Delta u + a \nabla P \nabla u + \epsilon \nabla P u(m(x) - u - bv) = 0, \quad \text{in } \Omega, \]

\[ \nu \Delta v + \beta \nabla P \nabla v + \epsilon \nabla P v(m(x) - cu - v) = 0, \quad \text{in } \Omega, \]

\[ \int_{\Omega} e^{2\eta P} dx - c \int_{\Omega} e^{2\eta P} dx - \xi = 0, \quad \text{in } \Omega, \]

\[ \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial \Omega. \]

Namely, we take \( \nu \to +\infty \) in (7), then \( \Delta v + \eta \nabla P \nabla v \to 0 \) since \( \epsilon \nabla P v(m(x) - cu - v) \)

remains bounded, which implies \( v \) tends to a constant \( \xi \).

Our main results are as follows.

**Theorem 1.4.** Assume that \( r(x) \in C(\overline{\Omega}) \) is a non-constant positive function on \( \overline{\Omega} \), then there exists \( d^*(r, \Omega) \in (0, 1) \) such that the following results hold

(a) If \( b \in (0, d^*) \), then (19) has a positive solution for every \( \mu > 0 \).

(b) If \( b \in (d^*, 1) \), then there exists two positive numbers \( \mu < \Pi \) such that (19) has a positive solution for \( \mu \in (0, \mu] \cup [\Pi, +\infty) \). Meanwhile there exists \( \mu \in [\mu, \Pi] \) such that no positive solutions exist for the problem (19). Moreover

\[ d^* \geq \frac{b \int_{\Omega} me^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} \]

\[ \sup_{\mu > 0} \int_{\Omega} e^{2\eta P} dx \]

where \( \theta \) is the unique positive solution to (14).

**Theorem 1.5.** Assume that \( r(x) \in C(\overline{\Omega}) \) is a non-constant positive function on \( \overline{\Omega} \). For any fixed \( \mu > 0 \), let \( \tilde{u}(x, \xi) \) be a positive solution of (19). If \( c \) is small enough, then there exists \( \nu^* > 0 \) such that (18) has a positive solution \( (u_\nu, v_\nu) \) for \( \nu > \nu^* \). Furthermore, \( u_\nu(x) \to \tilde{u}(x) \), \( v_\nu(x) \to \xi \) in \( C^2(\Omega) \) as \( \nu \to +\infty \).

The rest of this paper is organized as follows. In Section 2 we establish the existence of shadow system. This section is devoted to the existence of the solution of (19), where \( \mu > 0, 0 < b < 1, 0 < c < 1 \). From Assumption 1.1, it is easy to see that \( m(x) \) is a non-constant Hölder continuous positive function. Obviously, the problem (19) is equivalent to

\[ \left\{ \begin{array}{l}
\mu \Delta u + a \nabla P \nabla u + \epsilon \nabla P u(m(x) - u - bv) = 0, \\
\int_{\Omega} me^{2\eta P} dx - b \int_{\Omega} e^{2\eta P} dx + \epsilon \int_{\Omega} \nabla P u(m(x) - u - bv) dx - u = 0, \\
\frac{\partial u}{\partial n} = 0, \\
\frac{\partial v}{\partial n} = 0, \\
\xi = \int_{\Omega} me^{2\eta P} dx - c \int_{\Omega} e^{2\eta P} dx - \int_{\Omega} e^{2\eta P} dx.
\end{array} \right. \] (21)

We first introduce the following lemma.
Lemma 2.1. If \( u \) is a positive solution to (21), then it satisfies
\[
\sup_{\Omega} u(x) \leq \frac{1}{1 - bc} \left( \sup_{\Omega} m(x) - \frac{b \int_{\Omega} me^{2\eta_P} dx}{\int_{\Omega} e^{2\eta_P} dx} \right) \tag{22}
\]

Proof. Let \( x_0 \in \Omega \) such that \( u(x_0) = \sup_{\Omega} u(x) \). Note that
\[
m(x_0) - \frac{b \int_{\Omega} me^{2\eta_P} dx}{\int_{\Omega} e^{2\eta_P} dx} + bc \int_{\Omega} e^{2\eta_P} dx \int_{\Omega} e^{2\eta_P} dx - u(x_0) \leq \sup_{\Omega} m(x) - \frac{b \int_{\Omega} me^{2\eta_P} dx}{\int_{\Omega} e^{2\eta_P} dx} + u(x_0)(bc - 1). \tag{23}
\]
If \( u(x_0) > \frac{1}{1 - bc} \left( \sup_{\Omega} m(x) - \frac{b \int_{\Omega} me^{2\eta_P} dx}{\int_{\Omega} e^{2\eta_P} dx} \right) \), then (23) satisfies
\[
m(x_0) - \frac{b \int_{\Omega} me^{2\eta_P} dx}{\int_{\Omega} e^{2\eta_P} dx} + bc \int_{\Omega} e^{2\eta_P} dx \int_{\Omega} e^{2\eta_P} dx - u(x_0) < 0, \tag{24}
\]
which implies from the first equation of (21) that
\[
\mu \Delta u(x_0) + \alpha \nabla P \nabla u(x_0) > 0. \tag{25}
\]
The maximum principle yields \( x_0 \in \partial \Omega \) and Hopf boundary lemma gives \( \frac{\partial u}{\partial n} \bigg|_{x_0} > 0 \).

A contradiction to the boundary condition of (21) follows. \( \square \)

Denote \( m(x) - \frac{b \int_{\Omega} me^{2\eta_P} dx}{\int_{\Omega} e^{2\eta_P} dx} \) by \( g(x) \), then it follows that
\[
\int_{\Omega} g(x)e^{2\eta_P} dx \geq 0, \tag{26}
\]
and the first equation of problem (21) can be changed to
\[
\begin{cases}
\mu \Delta u + \alpha \nabla P \nabla u + e^{\eta_P} u(g(x) + \int_{\Omega} \frac{ue^{2\eta_P} dx}{\int_{\Omega} e^{2\eta_P} dx} - u) = 0, & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0, & \text{on } \partial \Omega. 
\end{cases} \tag{27}
\]

We study now the existence of (27).

Lemma 2.2. There is at least one solution \((u(x), \xi)\) to (21) such that \( u(x) > 0 \) for any \( \mu > 0 \).

Proof. Consider the following problem
\[
\begin{cases}
\mu \Delta w_{k,0} + \alpha \nabla P \nabla w_{k,0} + e^{\eta_P} w_{k,0}(g(x) + k - w_{k,0}) = 0, & \text{in } \Omega, \\
\frac{\partial w_{k,0}}{\partial n} = 0, & \text{on } \partial \Omega, 
\end{cases} \tag{28}
\]
where \( k \geq 0 \). The existence and uniqueness of \( w_{k,0} \) can be deduced by [6]. Denote \( w_{1,0} \) by \( \phi_0 \).

Case \( bc = \frac{\int_{\Omega} e^{2\eta_P} dx}{\int_{\Omega} \phi_0 e^{2\eta_P} dx} \): Clearly, \( \phi_0 \) is a solution of (27).
Case $bc < \frac{\int_{\Omega} e^{2np} dx}{\int_{\Omega} \phi_0 e^{2np} dx}$: Since $bc > 0 = \frac{0 \int_{\Omega} e^{2np} dx}{\int_{\Omega} \phi_0 e^{2np} dx}$, and since $\frac{k \int_{\Omega} e^{2np} dx}{\int_{\Omega} \phi_0 e^{2np} dx}$ is continuous with respect to $k$, there exists $k_0 \in (0, 1)$ such that $bc = \frac{k_0 \int_{\Omega} e^{2np} dx}{\int_{\Omega} \phi_0 e^{2np} dx}$ and then $w_{k_0,0}$ solves (27).

Case $bc > \frac{\int_{\Omega} e^{2np} dx}{\int_{\Omega} \phi_0 e^{2np} dx}$: For $k \geq 0$, we look at the following problem

$$
\begin{cases}
\mu \Delta w_{k,1} + \alpha \nabla P \nabla w_{k,1} + e^{np} w_{k,1} (g(x) + k \frac{bc \int_{\Omega} \phi_0 e^{2np} dx}{\int_{\Omega} e^{2np} dx} - w_{k,1}) = 0, & \text{in } \Omega, \\
\frac{\partial w_{k,1}}{\partial n} = 0, & \text{on } \partial \Omega.
\end{cases}
$$

Similarly, [6] leads to the existence of $w_{k,1}$. Next we handle the following three cases successively. Denote $w_{1,1}$ by $\phi_1$.

Case $\int_{\Omega} \phi_1 e^{2np} dx = \int_{\Omega} \phi_0 e^{2np} dx$: Clearly, $\phi_1$ solves (27).

Case $\int_{\Omega} \phi_1 e^{2np} dx < \int_{\Omega} \phi_0 e^{2np} dx$: Again since

$$
0 \frac{\int_{\Omega} \phi_0 e^{2np} dx}{\int_{\Omega} \phi_{0,1} e^{2np} dx} bc < bc < \frac{1 \int_{\Omega} \phi_0 e^{2np} dx}{\int_{\Omega} \phi_1 e^{2np} dx} bc,
$$

it follows that there exists $k_1 \in (0, 1)$ such that $bc = k_1 \frac{1 \int_{\Omega} \phi_0 e^{2np} dx}{\int_{\Omega} \phi_{k,1} e^{2np} dx} bc$, and then $w_{k,1}$ solves (27).

Case $\int_{\Omega} \phi_1 e^{2np} dx > \int_{\Omega} \phi_0 e^{2np} dx$: Look at again the following problem

$$
\begin{cases}
\mu \Delta w_{k,2} + \alpha \nabla P \nabla w_{k,2} + e^{np} w_{k,2} (g(x) + k \frac{bc \int_{\Omega} \phi_0 e^{2np} dx}{\int_{\Omega} e^{2np} dx} - w_{k,2}) = 0, & \text{in } \Omega, \\
\frac{\partial w_{k,2}}{\partial n} = 0, & \text{on } \partial \Omega.
\end{cases}
$$

Denote $w_{1,2}$ by $\phi_2$. Same as the above statements, we reach the following three cases.

Case $\int_{\Omega} \phi_2 e^{2np} dx = \int_{\Omega} \phi_1 e^{2np} dx$: Clearly, $\phi_2$ solves (27).

Case $\int_{\Omega} \phi_2 e^{2np} dx < \int_{\Omega} \phi_1 e^{2np} dx$: Owing to

$$
0 \frac{\int_{\Omega} \phi_1 e^{2np} dx}{\int_{\Omega} \phi_{0,2} e^{2np} dx} bc < bc < \frac{1 \int_{\Omega} \phi_1 e^{2np} dx}{\int_{\Omega} \phi_2 e^{2np} dx} bc,
$$

we have that there exists $k_2 \in (0, 1)$ such that $bc = k_2 \frac{1 \int_{\Omega} \phi_1 e^{2np} dx}{\int_{\Omega} \phi_{k,2} e^{2np} dx} bc$, and then $w_{k,2}$ solves (27).

Case $\int_{\Omega} \phi_2 e^{2np} dx > \int_{\Omega} \phi_1 e^{2np} dx$: Look at again the following problem

$$
\begin{cases}
\mu \Delta w_{k,3} + \alpha \nabla P \nabla w_{k,3} + e^{np} w_{k,3} (g(x) + k \frac{bc \int_{\Omega} \phi_2 e^{2np} dx}{\int_{\Omega} e^{2np} dx} - w_{k,3}) = 0, & \text{in } \Omega, \\
\frac{\partial w_{k,3}}{\partial n} = 0, & \text{on } \partial \Omega.
\end{cases}
$$
Denote \( w_{1,3} \) by \( \phi_3 \).

Following the similar procedure as above, one can obtain that either we get a positive solution to (27) and we finish the proof of Lemma 2.2, or \( \int_{\Omega} \phi_3 e^{2\eta P} dx > \int_{\Omega} \phi_2 e^{2\eta P} dx \). Therefore, if we do not obtain a solution in any step, we will derive a series of functions \( \phi_l \) satisfying

\[
\begin{align*}
\mu \Delta \phi_l + \alpha \nabla P \nabla \phi_l + e^{\eta P} \phi_l (g(x) + \frac{bc \int_{\Omega} \phi_{l-1} e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} - \phi_l) &= 0, \quad \text{in } \Omega, \\
\frac{\partial \phi_l}{\partial n} &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

for any fixed \( \eta > 0 \), \( \phi_l \in C^1 \) on \( \Omega \), with \( \phi_1 = 1 \) and \( \phi_2 = 0 \). It is routine to show that \( \phi_l \) is bounded in \( \Omega \), and \( \phi_l \) is uniformly bounded up to \( \eta \). Hence by elliptic regularity, \( \phi_l \) is uniformly bounded in \( \Omega \) for any \( \eta > 0 \). By means of the proof of Lemma 2.1, we arrive at

\[
\sup_{\Omega} \phi_l(x) \leq \frac{1}{1 - bc} \left( \sup_{\Omega} m(x) - \frac{bc \int_{\Omega} m e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} \right).
\]

Thus the elliptic regularity indicates that there exist a non-negative function \( \phi \in C^{1,\gamma}(\overline{\Omega}) \) and a subsequence \( \phi_{l_j} \) such that \( \phi_{l_j} \to \phi \) in \( C^{1,\gamma} \) as \( l_j \to +\infty \), for any \( 0 < \gamma < 1 \). This yields that \( \phi \) satisfies

\[
\begin{align*}
\mu \Delta \phi + \alpha \nabla P \nabla \phi + e^{\eta P} \phi (g(x) + \frac{bc \int_{\Omega} \phi e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} - \phi) &= 0, \quad \text{in } \Omega, \\
\frac{\partial \phi}{\partial n} &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

Moreover, \( \phi \) is not trivial and it is positive according to maximum principle and Hopf boundary lemma. We complete the proof now.

Next we verify that (21) has a positive solution for large \( \mu \).

**Lemma 2.3.** Fix \( b \in (0,1] \), \( c \in (0,1) \). Then (21) has at least a positive solution \( (u, \xi) \) for sufficiently large \( \mu \).

**Proof.** We claim that for any positive solution \( u(x) \) of (27), we have

\[
u(x) \to 0 \text{ or } u(x) \to \frac{(1 - b) \int_{\Omega} e^{2\eta P} m dx}{(1 - bc) \int_{\Omega} e^{2\eta P}}
\]

in \( C^{1,\gamma}(\overline{\Omega}) \) with \( 0 < \gamma < 1 \), as \( \mu \to +\infty \). Applying Lemma 2.1 and elliptic regularity, it is routine to show that \( u(x) \) is convergent to some constant \( C > 0 \) in \( C^{1,\gamma}(\overline{\Omega}) \), with \( 0 < \gamma < 1 \). Multiplying the first equation in (27) by \( e^{\eta P} \) and making an integration over \( \Omega \), we can derive \( \int_{\Omega} (bc - 1) C^2 e^{2\eta P} dx = (b - 1) C \int_{\Omega} m(x) e^{2\eta P} dx \). Hence our claim is correct. Therefore it follows from the third equation of (21) that

\[
\xi \to \frac{(1 - c) \int_{\Omega} m e^{2\eta P} dx}{(1 - bc) \int_{\Omega} e^{2\eta P} dx} > 0
\]

as \( \mu \to +\infty \). Hence we complete the proof of this lemma combined with Lemma 2.2.

The rest of this section is to check the asymptotic behavior of solutions of (27).

**Lemma 2.4.** For any fixed \( \mu > 0 \), let \( u(x) \) be any positive solution of (27), then

\[
\frac{\int_{\Omega} w e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} \to +\infty, \text{ as } bc \to 1.
\]
Proof. Suppose to the contrary that for each \( n \geq 1 \), there exist \( b_n \in (0,1], c_n \in (0,1) \) and function \( u_n > 0 \) such that

\[
\begin{align*}
\mu \Delta u_n + \alpha \nabla P \nabla u_n + e^{\eta P} u_n \left( g(x) + \frac{bc \int_{\Omega} u_n e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} - u_n \right) &= 0, \quad \text{in } \Omega, \\
\frac{\partial u_n}{\partial n} &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

and \( \frac{\int_{\Omega} u_n e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} \to K \in [0, +\infty) \) as \( b_n c_n \to 1 \). Similarly, one can deduce from Lemma 2.1 that

\[
\sup_{\Omega} u_n \leq K + 1.
\]

Elliptic regularity implies that there exist \( u \in C^{1,\gamma}(\overline{\Omega}) \), with \( 0 < \gamma < 1 \), and a subsequence \( u_{n_j} \), such that \( u_{n_j} \to u \geq 0 \) in \( C^{1,\gamma}(\overline{\Omega}) \). Therefore \( u \) solves

\[
\begin{align*}
\mu \Delta u + \alpha \nabla P \nabla u + e^{\eta P} u (g(x) + \frac{\int_{\Omega} u e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} - u) &= 0, \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]

If \( u \geq 0 \), then it is easy to verify that

\[
\int_{\Omega} g(x) e^{2\eta P} dx = \int_{\Omega} e^{2\eta P} (g(x) + \frac{\int_{\Omega} u e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} - u) dx < 0,
\]

which contradicts to (26). Hence \( u \equiv 0 \).

On the other hand, from [6], the unique positive solution \( \theta \) to (14) is globally asymptotically stable and satisfies \( u_{n_j} \geq \theta \), since

\[
\mu \Delta u_{n_j} + \alpha \nabla P \nabla u_{n_j} + e^{\eta P} u_{n_j} (g(x) - u_{n_j}) \leq 0.
\]

Letting \( n_j \to +\infty \), we have \( u_{n_j} \to u \geq \theta > 0 \), which is a contradiction. \( \square \)

Lemma 2.5. For any \( \mu > 0 \), let \( u(x) \) be a positive solution to the problem (27). Then we have \( u(x) \to z(x) \) in \( C^{1,\gamma}(\overline{\Omega}) \), with \( 0 < \gamma < 1 \), as \( bc \to 0 \), where \( z(x) \) is the unique positive solution to the following problem

\[
\begin{align*}
\mu \Delta z + \alpha \nabla P \nabla z + e^{\eta P} z (g(x) - z) &= 0, \quad \text{in } \Omega, \\
\frac{\partial z}{\partial n} &= 0, \quad \text{on } \partial \Omega.
\end{align*}
\]

Proof. For any sequence \( b_n, c_n \to 0 \) as \( n \to +\infty \), take \( u_n \) be the corresponding positive solution of (27) with \( b = b_n, c = c_n \). Applying Lemma 2.1, one can see \( \|u_n\|_{\infty} \) is uniformly bounded. Due to the elliptic regularity, \( \{u_n\} \) has a subsequence which is convergent in \( C^{1,\gamma}(\overline{\Omega}) \), with \( 0 < \gamma < 1 \), say \( u_{n_j} \to w \) in \( C^{1,\gamma}(\overline{\Omega}) \). Obviously, \( w \) solves

\[
\begin{align*}
\mu \Delta w + \alpha \nabla P \nabla w + e^{\eta P} w (g(x) - w) &= 0, \quad \text{in } \Omega, \\
\frac{\partial w}{\partial n} &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

which implies either \( w \equiv 0 \) or \( w = z \). On the other hand, since by [6] \( z \) is globally asymptotically stable and \( \mu \Delta u_{n_j} + \alpha \nabla P \nabla u_{n_j} + e^{\eta P} u_{n_j} (g(x) - u_{n_j}) \leq 0 \), we get \( u_{n_j} \geq z \). This implies that \( w = z \), that is, \( u_{n_j} \to z \). This lemma follows immediately because the sequence \( \{b_n c_n\} \) with \( b_n c_n \to 0 \) is arbitrary. \( \square \)
3. Solutions of the shadow system. In this section, we demonstrate the existence of the solutions to the shadow system (19). Let $\theta$ be the unique positive solution to (14) and $\psi_\delta$ denotes the unique positive solution to
\[
\begin{align*}
&\mu \Delta \psi + \alpha \nabla P \nabla \psi + e^{\eta P} \psi (g(x) + \delta - \psi) = 0, \quad \text{in } \Omega, \\
&\frac{\partial \psi}{\partial n} = 0, \quad \text{on } \partial \Omega.
\end{align*}
\]
(46)
Now we can begin to prove Theorem 1.4.

**Proof of Theorem 1.4:** Define
\[
d^* = \inf_{\mu > 0} \sup_{0 < \delta < \frac{b}{\int_{\Omega} e^{2\eta P} \psi_\delta dx}} \delta \int_{\Omega} e^{2\eta P} d\sigma.
\]
(47)
We first illustrate that
\[
0 < d^* < 1.
\]
(48)
On the one hand, let $\delta_0 := \frac{b\int_{\Omega} me^{2\eta P}}{2 \int_{\Omega} e^{2\eta P}}$, and then
\[
\int_{\Omega} e^{2\eta P} (g(x) + \delta_0) dx = (1 - \frac{b}{2}) \int_{\Omega} me^{2\eta P} dx > 0.
\]
(49)
Since $\int_{\Omega} \psi_\delta e^{2\eta P} dx$ is continuous with respect to $\mu$, and since
\[
\lim_{\mu \to +\infty} \int_{\Omega} \psi_\delta e^{2\eta P} dx = \lim_{\mu \to 0} \int_{\Omega} \psi_\delta e^{2\eta P} dx = \int_{\Omega} (g(x) + \delta_0) e^{2\eta P} dx > 0,
\]
(50)
similar to (16) and (17), $\sup_{\mu > 0} \int_{\Omega} \psi_\delta e^{2\eta P} dx < +\infty$ follows. Therefore for all $\mu > 0$,
\[
d^* \geq \inf_{\mu > 0} \frac{\delta_0 \int_{\Omega} e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} \psi_\delta dx} > 0.
\]
(51)
On the other hand, since $\psi_\delta$ is the positive solution of (46), direct calculation gives
\[
\int_{\Omega} e^{2\eta P} g(x) dx + \delta \int_{\Omega} e^{2\eta P} dx - \int_{\Omega} e^{2\eta P} \psi_\delta(x) dx + \mu \int_{\Omega} \frac{e^{\eta P} |\nabla \psi_\delta|^2}{\psi_\delta^2} dx = 0,
\]
(52)
which implies that $\delta \int_{\Omega} e^{2\eta P} dx < \int_{\Omega} \psi_\delta e^{2\eta P} dx$. Therefore (48) follows.

Now if $bc \in (0, d^*)$, then for any $\mu > 0$, we have that
\[
bc < \sup_{0 < \delta < \frac{b}{\int_{\Omega} me^{2\eta P} dx}} \frac{\delta \int_{\Omega} e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} \psi_\delta dx},
\]
(53)
which means $bc < \delta \int_{\Omega} e^{2\eta P} dx$ for some $0 < \delta < \frac{b}{\int_{\Omega} me^{2\eta P} dx}$. Consider next
\[
\begin{align*}
&\mu \Delta \phi_k + \alpha \nabla P \nabla \phi_k + e^{\eta P} \phi_k (g(x) + k\delta - \phi_k) = 0, \quad \text{in } \Omega, \\
&\frac{\partial \phi_k}{\partial n} = 0, \quad \text{on } \partial \Omega,
\end{align*}
\]
(54)
where \( k \geq 0 \). It follows from
\[
\frac{0}{\Omega} \int_{\Omega} e^{2n^2P} dx < bc < \frac{\delta}{\Omega} \int_{\Omega} e^{2n^2P} \phi_1 dx
\]
that there exist \( 0 < k_0 < 1 \) and a positive function \( \phi_{k_0} \) such that
\( bc = \frac{\delta}{\Omega} \int_{\Omega} e^{2n^2P} \phi_{k_0} dx \) and \( \phi_{k_0} \) solves (27). Moreover,
\[
\xi = \frac{\int_{\Omega} me^{2n^2P} dx}{\int_{\Omega} e^{2n^2P} dx} - c \int_{\Omega} \phi_{k_0} e^{2n^2P} dx = \frac{\int_{\Omega} me^{2n^2P} dx}{\int_{\Omega} e^{2n^2P} dx} - \frac{k_0 \delta}{b} > 0.
\]
Therefore, \((\phi_{k_0}, \xi)\) is a positive solution of (19).

If \( bc \in (d^*, 1) \), then by the definition of \( d^* \), there exists \( \mu_0 \) such that
\[
bc > \sup_{0 < \delta < \frac{k_0 \delta}{b} < d^*} \int_{\Omega} e^{2n^2P} \psi \delta dx
\]
for \( \mu = \mu_0 \), which implies that for \( \mu = \mu_0 \), (19) has no positive solution. Hence
\[
\Lambda := \{ \mu \in (0, +\infty) | (19) \text{ has no positive solution} \} \neq \emptyset.
\]

By Lemma 2.3, we may define
\[
\bar{\mu} := \sup_{\mu \in \Lambda} \mu < +\infty.
\]
Suppose that for all \( \mu \in (0, \bar{\mu}) \), (19) has no positive solution, then by Lemma 2.2,
there exist \( \mu_n \in (0, \bar{\mu}) \) with \( \lim_{n \to \infty} \mu_n = 0 \) and the positive function \( u_n(x) \in C^2(\Omega) \)
such that \( u_n \) solves
\[
\begin{cases}
\mu_n \Delta u + \alpha \nabla P \nabla u + e^{n^2P} u (g(x) + bc \int_{\Omega} u e^{2n^2P} dx - u) = 0, & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0, & \text{on } \partial \Omega,
\end{cases}
\]
and \( \xi_n = \frac{\int_{\Omega} me^{2n^2P} dx}{\int_{\Omega} e^{2n^2P} dx} - c \int_{\Omega} u_n e^{2n^2P} dx \leq 0 \). Due to Lemma 2.1, without loss of
generality, we assume that \( \lim_{n \to \infty} \int_{\Omega} u_n e^{2n^2P} dx = \lambda \), and then \( c \lambda \geq \frac{\int_{\Omega} me^{2n^2P} dx}{\int_{\Omega} e^{2n^2P} dx} > 0 \).

Note that \( \int_{\Omega} (g(x) + bc \lambda - \varepsilon) e^{2n^2P} dx \geq (bc \lambda - \varepsilon) \int_{\Omega} e^{2n^2P} dx > 0 \) for \( \varepsilon \) small enough.
Therefore the following problem has the unique positive solution \( \phi_n, \psi_n \).
\[
\begin{cases}
\mu_n \Delta \phi + \alpha_n \nabla P \nabla \phi + e^{n^2P} \phi (g(x) + bc \lambda + \alpha - \phi) = 0, & \text{in } \Omega, \\
\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial n} = 0, & \text{on } \partial \Omega,
\end{cases}
\]

The fact that \( \lim_{n \to \infty} \int_{\Omega} u_n e^{2n^2P} dx = \lambda \) implies that there exists \( N_1 > 0 \) such that
\( bc \lambda - \varepsilon < \frac{bc \int_{\Omega} u_n e^{2n^2P} dx}{\int_{\Omega} e^{2n^2P} dx} < bc \lambda + \varepsilon \) for \( n \geq N_1 \), which implies
Moreover it is known that
\[
\begin{cases}
\psi_n \to g(x) + bc\lambda - \varepsilon \geq g(x) + \frac{b \int_{\Omega} \mu e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} - \varepsilon = m(x) - \varepsilon > 0,
\end{cases}
\]
\[\phi_n \to g(x) + bc\lambda + \varepsilon > 0.\]

Thus there exists \(N_2 > 0\) such that \(\psi_n \leq \phi_n\) for \(n > N_2\). Consequently, we can immediately deduce that \(\psi_n \leq u_n \leq \phi_n\) for \(n > \max\{N_1, N_2\}\) and that
\[
\frac{\int_{\Omega} \mu e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} \leq \frac{\int_{\Omega} u_n e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} \leq \frac{\int_{\Omega} \phi_n e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx}. \tag{63}
\]

Then by letting \(n \to \infty\), we in view of \(\frac{\int_{\Omega} g(x) e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} = (1 - b) \frac{\int_{\Omega} \mu e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx}\) obtain that
\[
\frac{(1 - b) \int_{\Omega} \mu e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} + bc\lambda - \varepsilon \leq \lambda \leq \frac{(1 - b) \int_{\Omega} \mu e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} + bc\lambda + \varepsilon. \tag{64}
\]

Recall that \(\varepsilon\) could be arbitrarily small as long as \(g(x) + bc\lambda - \varepsilon > 0\) in \(\Omega\). Hence \(\frac{(1 - b) \int_{\Omega} \mu e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} + bc\lambda = \lambda\), which implies \(\lambda = \frac{(1 - b) \int_{\Omega} \mu e^{2\eta P} dx}{(1 - bc) \int_{\Omega} e^{2\eta P} dx}\). This is impossible since \(c\lambda > \frac{\int_{\Omega} \mu e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} > 0\) and \(0 < c < 1\). Therefore we can define
\[
\mu := \inf_{\mu \in \lambda} \mu > 0. \tag{65}
\]

Finally let us estimate \(d^*\). Obviously \((16)\) and \((17)\) read that
\[
\sup_{\mu > 0} \int_{\Omega} \theta e^{2\eta P} dx < +\infty. \tag{66}
\]

Now fix any \(b, c\) satisfying
\[
0 \leq bc < \frac{\int_{\Omega} \mu e^{2\eta P} dx}{\sup_{\mu} \int_{\Omega} \theta e^{2\eta P} dx}, \tag{67}
\]
which implies that \(c < \frac{\int_{\Omega} \mu e^{2\eta P} dx}{\int_{\Omega} \theta e^{2\eta P} dx}\) for any \(\mu > 0\). Notice that \((14)\) can be rewritten as
\[
\begin{cases}
\mu \Delta \theta + \alpha \nabla P \nabla \theta + e^{\eta P} \theta (g(x) + \frac{b \int_{\Omega} \mu e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} - \theta) = 0, \quad \text{in } \Omega, \\
\frac{\partial \theta}{\partial n} = 0, \quad \text{on } \partial \Omega.
\end{cases} \tag{68}
\]
Again consider the problem

\[
\begin{align*}
\mu \Delta u + \alpha \nabla P \nabla \theta_k + e^{\eta \theta_k} (g(x) + k \theta_k / \int_{\Omega} e^{2\eta \theta_k} dx) &= 0, & \text{in } \Omega, \\
\frac{\partial \theta_k}{\partial n} &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

(69)

Since \( \frac{\int_{\Omega} me^{2\eta P} dx}{c \int_{\Omega} \theta_k e^{2\eta P} dx} > 1 \), there exists \( \theta_k \in (0, 1) \) such that \( k \theta_k \int_{\Omega} me^{2\eta P} dx / c \int_{\Omega} e^{2\eta P} dx = 1 \). This indicates that \( \theta_k \) solves (27). Together with

\[
\xi_k = \frac{\int_{\Omega} me^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} - \frac{\int_{\Omega} \theta_k e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} = (1 - k \theta_k) \frac{\int_{\Omega} me^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx} > 0,
\]

(70)

we have the existence of positive solutions \((\theta_k, \xi_k)\) to (21). Consequently,

\[
d^* \geq \sup_{\mu} \frac{b \int_{\Omega} me^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx}.
\]

(71)

The proof is now completed. \( \square \)

4. Solutions of steady state of the Lotka-Volterra competition advection system. This section is devoted to the study of solutions to the following steady state of the Lotka-Volterra competition advection system.

\[
\begin{align*}
\mu \Delta u + \alpha \nabla P \nabla u + e^{\eta u} (g(x) + b \int_{\Omega} me^{2\eta P} dx / \int_{\Omega} e^{2\eta P} dx - u) &= 0, & \text{in } \Omega, \\
\nu \Delta v + \beta \nabla P \nabla v + e^{\eta v} (m(x) - cu - v) &= 0, & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

(72)

That is, we will complete the proof of Theorem 1.5. By Lemma 2.2, the following problem

\[
\begin{align*}
\mu \Delta u + \alpha \nabla P \nabla u + e^{\eta u} (g(x) + b \int_{\Omega} me^{2\eta P} dx / \int_{\Omega} e^{2\eta P} dx - u) &= 0, & \text{in } \Omega, \\
g(x) = m(x) - \frac{b \int_{\Omega} \theta e^{2\eta P} dx}{\int_{\Omega} e^{2\eta P} dx}, & \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0, & \text{on } \partial \Omega,
\end{align*}
\]

(73)

has a positive solution \( u(x) \).

**Proof of Theorem 1.5:** Denote

\[
f(p, q) = p(m - p - bq), \ g(p, q) = q(m - cp - q),
\]

\[
X = \{ u \in C^2(\Omega) | \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \}, Y = C(\Omega),
\]

\[
Y_1 = \{ u \in Y | \int_{\Omega} \theta e^{2\eta P} dx = 0 \}, Z = X \cap Y_1.
\]

Next we put
It is known that there exist a sequence of eigenvalues \( \{ \lambda_n \} \) for \( \phi \) where \( \psi \in Z \) and \( \mathcal{P} \) denotes the projection operator of \( Y \) onto \( Y_1 \). Then \( T = (T_1, T_2, T_3) \) is an analytic mapping from the open set \( \{ (\mu, \tau, u, \xi, \psi) | u > 0, \xi + \psi(x) > 0 \} \) of \( \mathbb{R}_+ \times \mathbb{R}_+ \times X \times \mathbb{R} \times Z \) into \( Y \times \mathbb{R} \times Y_1 \). Let \( (\mu_0, u_0, \xi_0) \) be a positive solution of (19), then \( T(\mu_0, 0, u_0, \xi_0, 0) = (0, 0, 0) \) and

\[
D_{(u, \xi, \psi)} T(\mu_0, 0, u_0, \xi_0, 0) = \begin{pmatrix}
T_{1,u} & T_{1,\xi} \\
T_{2,u} & T_{2,\xi}
\end{pmatrix}
\begin{pmatrix}
\phi \\
\varphi
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix},
\]

where

\[
\begin{pmatrix}
T_{1,u} & T_{1,\xi} \\
T_{2,u} & T_{2,\xi}
\end{pmatrix}
\begin{pmatrix}
\phi \\
\varphi
\end{pmatrix} = \begin{pmatrix}
\int_{\Omega} g_\mu(\mu_0, 0, u_0, \xi_0) e^{2\eta P} dx \\
\int_{\Omega} e^{2\eta P} dx
\end{pmatrix},
\]

\[
\begin{pmatrix}
\mu_0 \Delta \phi + \alpha \nabla P \nabla \phi + f_\phi(u_0, \xi_0) e^{\eta P} + f_\varphi(u_0, \xi_0) e^{\eta P} + \int_{\Omega} g_\mu(\mu_0, 0, u_0, \xi_0) e^{2\eta P} dx \\
\int_{\Omega} e^{2\eta P} dx
\end{pmatrix},
\]

for \( \phi \in X, \varphi \in \mathbb{R} \). In particular, consider

\[
\begin{pmatrix}
T_{1,u} & T_{1,\xi} \\
T_{2,u} & T_{2,\xi}
\end{pmatrix}
\begin{pmatrix}
\phi \\
\varphi
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix},
\]

which is equivalent to

\[
\begin{cases}
\mu_0 \Delta \phi + \alpha \nabla P \nabla \phi + \phi e^{\eta P} (m(x) - b \xi_0 - 2 u_0) + \varphi e^{\eta P} (-b u_0) = 0, & \text{in } \Omega, \\
-\int_{\Omega} \phi e^{2\eta P} dx - \varphi = 0, \\
\partial \phi / \partial n = 0, & \text{on } \partial \Omega.
\end{cases}
\]

(78)

Since \( \mu_0 \Delta + \alpha \nabla P \nabla \) is an isomorphism from \( Z \) onto \( Y_1 \), \( D_{(u, \xi, \psi)} T(\mu_0, 0, u_0, \xi_0, 0) \) is nonsingular if and only if (78) only has the trivial solution \( (\phi, \varphi) = (0, 0) \).

We now claim that (78) only has the trivial solution when \( \beta \) is sufficiently small. Define

\[
\mathcal{L}_0 = \mu_0 \Delta + \alpha \nabla P \nabla + (m - 2 u_0 - b \xi_0) e^{\eta P}.
\]

Consider

\[
\begin{cases}
e^{\eta P} \mathcal{L}_0 \Phi = \mu_0 \nabla \cdot (e^{\eta P} \nabla \Phi) + (m - 2 u_0 - b \xi_0) \Phi e^{2\eta P} = \lambda \Phi, & \text{in } \Omega, \\
\partial \Phi / \partial n = 0, & \text{on } \partial \Omega.
\end{cases}
\]

(80)

It is known that there exist a sequence of eigenvalues \( \{ \lambda_n \} \) and the corresponding eigenfunctions \( \Phi_n \) such that

\[
\lambda_1 > \lambda_2 > \lambda_3 > \cdots, \quad (\Phi_1, \Phi_2) = \frac{\int_{\Omega} \Phi_1 \Phi_2 dx}{|\Omega|}.
\]
there exist sequences \( \{ c \} \), \( \{ \lambda \} \). Suppose that for any \( n \), \( \lambda_n > 0 \), which yields that \( \mathcal{L}_0 \) is invertible. Clearly, if \( \varphi = 0 \) in (87), then \( \varphi \equiv 0 \), and our claim is verified. Otherwise without loss of generality, assume that \( \varphi = 1 \), then (87) becomes

\[
\begin{align*}
\mu_0 \Delta \phi + \alpha \nabla P \cdot \nabla \phi + \phi e^{np}(m(x) - b_0 \xi_0 - 2u_0) - bu_0 e^{np} &= 0, & \text{in } \Omega, \\
-c \int_\Omega e^{2np} dx &= 1, \\
\frac{\partial \phi}{\partial n} &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

(83)

Suppose that for any \( c > 0 \), (83) has a nontrivial solution, then by Lemma 2.5, there exist sequences \( \{ u_n \}, \{ \xi_n \}, \{ b_n \}, \{ c_n \}, \{ \phi_n \} \) with \( b_n, c_n \to 0 \) and \( u_n(x) > 0 \), such that

\[
\begin{align*}
\mu_0 \Delta u_n + \alpha \nabla P \cdot \nabla u_n + u_n e^{np}(m(x) - b_n \xi_n - u_n) &= 0, & \text{in } \Omega, \\
\xi_n e^{2np} dx &= \int_\Omega m e^{2np} dx - c_n \int_\Omega u_n e^{2np} dx, \\
\frac{\partial u_n}{\partial n} &= 0, & \text{on } \partial \Omega, \\
\mathcal{L}_0 \phi_n - b_0 u_n e^{np} &= 0, & \text{in } \Omega, \\
-c_n \int_\Omega e^{2np} dx &= 1, \\
\frac{\partial \phi_n}{\partial n} &= 0, & \text{on } \partial \Omega,
\end{align*}
\]

(84)

where \( \mathcal{L}_0 \phi_n = \mu_0 \Delta \phi_n + \alpha \nabla P \cdot \nabla \phi_n + \phi_n e^{np}(m(x) - b_n \xi_n - 2u_n) \).

Let \( \lambda_k^{(n)}, \Phi_k^{(n)} \) denote the eigenvalues and normalized eigenfunctions of

\[
\begin{align*}
e^{np} \mathcal{L}_0 \Phi_k^{(n)} &= \lambda_k^{(n)} \Phi_k^{(n)}, & \text{in } \Omega, \\
\frac{\partial \Phi_k^{(n)}}{\partial n} &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

(85)

Similar to (82), we have

\[
0 = \int_\Omega u_n \Phi_1^{(n)}(\lambda_1^{(n)} + u_n e^{2np}) dx,
\]

(86)

which yields that

\[
\lambda_1^{(n)} \leq - \inf_{\Omega} (u_n e^{2np}) \to - \inf_{\Omega} (ze^{2np}), \quad \text{as } n \to \infty,
\]

(87)

by Lemma 2.5. Therefore for \( n \) large enough,

\[
\lambda_1^{(n)} \leq - \frac{1}{2} \inf_{\Omega} (ze^{2np}) < 0,
\]

(88)

and \( \mathcal{L}_0^{(n)} \) is invertible. Set \( S = \frac{1}{2} \inf_{\Omega} (ze^{2np}) \). From the fourth equation of (84), we obtain that
\[ \phi_n = (e^{\eta P} \mathcal{L}_0^{(n)})^{-1} (b_n e^{2\eta P}) \]
\[ = \sum_{k=1}^{\infty} \left( e^{\eta P} \mathcal{L}_0^{(n)} \right)^{-1} (b_n u_n e^{2\eta P}, \Phi_k^{(n)}) \Phi_k^{(n)} \]
\[ = \sum_{k=1}^{\infty} \frac{(b_n u_n e^{2\eta P}, \Phi_k^{(n)}) \Phi_k^{(n)}}{\lambda_k^{(n)}}. \]  

(89)

Thus
\[ \langle \phi_n, \phi_n \rangle = \sum_{k=1}^{\infty} \left( \frac{(b_n u_n e^{2\eta P}, \Phi_k^{(n)})}{\lambda_k^{(n)}} \right)^2 \]
\[ \leq \sum_{k=1}^{\infty} \frac{(b_n u_n e^{2\eta P}, \Phi_k^{(n)})^2}{S^2} \]
\[ = \frac{(b_n u_n e^{2\eta P}, b_n u_n e^{2\eta P})}{S^2}. \]  

(90)

This leads to
\[ \limsup_{n \to \infty} \langle \phi_n, \phi_n \rangle \leq \frac{S^2}{S^2} \langle e^{2\eta P}, e^{2\eta P} \rangle < +\infty. \]  

(91)

Combined with (84) one can see that there exists some constant \( c > 0 \), such that
\[ -\int_{\Omega} e^{2\eta P} d\Omega \leq \int_{\Omega} \phi_n e^{2\eta P} d\Omega \leq c \| \phi_n \|^2 < \infty. \]  

(92)

This is a contradiction to \( c_n \to 0 \). Hence our claim is correct, and then
\[ D_{(u, \xi, \psi)} T(\mu_0, u_0, \xi_0, 0) \]

is nonsingular. Notice that if \( T(\mu, \tau, u, \xi, \psi) = (0, 0, 0) \) with \( \tau > 0 \), then \( (\mu, (u, \xi + \psi)) \) is a solution to problem (72) with \( \nu = \frac{1}{\tau} \). Therefore, by the application of the implicit function theorem, we finish the proof of Theorem 1.5. \( \square \)

REFERENCES

[1] I. Averill, The Effect of Intermediate Advection on Two Competing Species, Doctoral Thesis, Ohio State University, 2012.
[2] F. Belgacem and C. Cosner, The effects of dispersal along environmental gradients on the dynamics of populations in heterogeneous environment, Can. Appl. Math. Q., 3 (1995), 379–397.
[3] R. S. Cantrell and C. Cosner, The effect of spatial heterogeneity in population dynamics, J. Math. Biol., 29 (1991), 315–338.
[4] R. S. Cantrell and C. Cosner, Should a park be an island?, SIAM J. Appl. Math., 53 (1993), 219–252.
[5] R. S. Cantrell and C. Cosner, On the effects of spatial heterogeneity on the persistence of interacting species, J. Math. Biol., 37 (1998), 103–145.
[6] R. S. Cantrell and C. Cosner, Spatial Ecology Via Reaction-Diffusion Equations, Series in Mathematical and Computational Biology, Wiley, Chichester, UK, 2003.
[7] R. S. Cantrell, C. Cosner and V. Huston, Permanence in ecological systems with diffusion, Proc. Roy. Soc. Edinburgh A, 123 (1993), 553–559.
[8] R. S. Cantrell, C. Cosner and V. Huston, Ecological models, permanence and spatial heterogeneity, Rocky Mount. J. Math., 26 (1996), 1–35.
[9] R. S. Cantrell, C. Cosner and Y. Lou, Multiple reversals of competitive dominance in ecological reserves via external habitat degradation, J. Dynam. Differential Equations, 16 (2004), 973–1010.
[10] C. Cosner and Y. Lou, When does movement toward better environment benefit a population? J. Math. Analysis Appl., 277 (2003), 489–503.
[11] J. Dockery, V. Hutson, K. Mischaikow and M. Pernarowski, The evolution of slow dispersal rates: A reaction-diffusion equations, *J. Math. Biol.*, 37 (1998), 61–83.
[12] Y. Du, Effects of a degeneracy in the competition model, Part I. Classical and generalized steady-state solutions, *J. Diff. Eqs.*, 181 (2002), 92–132.
[13] Y. Du, Effects of a degeneracy in the competition model, Part II. Perturbation and dynamical behavior, *J. Diff. Eqs.*, 181 (2002), 133–164.
[14] Y. Du, Realization of prescribed patterns in the competition model, *J. Diff. Eqs.*, 193 (2003), 147–179.
[15] J. E. Furter and J. López-Gómez, Diffusion-mediated permanence problem for a heterogeneous Lotka-Volterra competition model, *Proc. Roy. Soc. Edinburgh A*, 127 (1997), 281–336.
[16] A. Hastings, Spatial heterogeneity and ecological models, *Ecology*, 71 (1990), 426–428.
[17] X. He and W.-M. Ni, The effects of diffusion and spatial variation in Lotka-Volterra competition-diffusion system I: heterogeneity vs. homogeneity, *J. Diff. Eqs.*, 254 (2013), 528–546.
[18] X. He and W.-M. Ni, The effects of diffusion and spatial variation in Lotka-Volterra competition-diffusion system II: the general case, *J. Diff. Eqs.*, 254 (2013), 4088–4108.
[19] X. He and W.-M. Ni, Global dynamics of the Lotka-Volterra competition-diffusion system: diffusion and spatial heterogeneity I, *Comm. Pure Appl. Math.*, 69 (2016), 981–1014.
[20] X. He and W.-M. Ni, Global dynamics of the Lotka-Volterra competition-diffusion system with equal amount of total resources, II, *Calc. Var. Partial Differential Equations*, 55 (2016), Art. 25, 20 pp.
[21] X. He and W.-M. Ni, Global dynamics of the Lotka-Volterra competition-diffusion system with equal amount of total resources, III, *Calc. Var. Partial Differential Equations*, 56 (2017), Art. 132, 26 pp.
[22] E. E. Holmes, M. A. Lewis, J. E. Banks and R. R. Veit, Partial differential equations in ecology: Spatial interactions and population dynamics, *Ecology*, 75 (1994), 17–29.
[23] V. Hutson, J. López-Gómez, K. Mischaikow and G. Vickers, Limit behavior for a competing species problem with diffusion, in Dynamical Systems and applications, *World Scientific Series Applicable Analysis*, 4, World Scientific, River Edge, NJ, (1995), 343–358.
[24] V. Hutson, Y. Lou and K. Mischaikow, Spatial heterogeneity of resources versus Lotka-Volterra dynamics, *J. Diff. Eqs.*, 185 (2002), 97–136.
[25] V. Hutson, Y. Lou and K. Mischaikow, Convergence in competition models with small diffusion coefficients, *J. Diff. Eqs.*, 211 (2005), 135–161.
[26] V. Hutson, Y. Lou, K. Mischaikow and P. Poláčik, Competing species near the degenerate limit, *SIAM J. Math. Anal.*, 35 (2003), 453–491.
[27] V. Hutson, S. Martinez, K. Mischaikow and G. T. Vicker, The evolution of dispersal, *J. Math. Biol.*, 47 (2003), 483–517.
[28] V. Hutson, K. Mischaikow and P. Poláčik, The evolution of dispersal rates in a heterogeneous time-periodic environment, *J. Math. Biol.*, 43 (2001), 501–533.
[29] M. G. Krein and M. A. Rutman, Linear operators leaving invariant a cone in a Banach space, *Uspekhi Mat. Nauk*, 3 (1948), 3–95.
[30] F. Li, L. Wang and Y. Wang, On the effects of migration and inter-specific competitions in steady state of some Lotka-Volterra model, *Discrete Contin. Dyn. Syst. Ser. B*, 15 (2011), 669–686.
[31] J. López-Gómez, Coexistence and meta-coexistence for competing species, *Houston J. Math.*, 29 (2003), 483–536.
[32] Y. Lou, On the effects of migration and spatial heterogeneity on single and multiple species, *J. Diff. Eqs.*, 223 (2006), 400–426.
[33] Y. Lou, S. Martinez and P. Poláčik, Loops and branches of coexistence states in a Lotka-Volterra competition model, *J. Diff. Eqs.*, 230 (2006), 720–742.
[34] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, New York, 1984.

Received November 2018; revised May 2019.

*E-mail address: qwang@ust.edu.cn*