RIEMANNIAN GEOMETRIC REALIZATIONS FOR RICCI TENSORS OF GENERALIZED ALGEBRAIC CURVATURE OPERATORS

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Abstract. We examine questions of geometric realizability for algebraic structures which arise naturally in affine and Riemannian geometry.

1. Introduction

Many questions in Riemannian geometry involve constructing geometric realizations of algebraic objects where the objects in question are invariant under the action of the structure group $G$. We present several examples to illustrate this point. We first review previously known results. Section 1.1 deals with Riemannian algebraic curvature tensors, Section 1.2 deals with Osserman tensors, and Section 1.3 deals with generalized algebraic curvature operators.

In Section 1.4 we present the new results of this paper that deal with a mixture of affine and Riemannian geometry; this mixture has not been considered previously. The results in the real analytic context can perhaps be considered as extensions of previous results in affine geometry; the results in the $C^*$ context are genuinely new and require additional estimates. We refer to Section 1.4 for further details. To simplify the discussion, we shall assume that the underlying dimension $m$ is at least 3 as the 2-dimensional case is a bit exceptional. We adopt the Einstein convention and sum over repeated indices henceforth.

1.1. Realizing Riemannian algebraic curvature tensors. Let $V$ be an $m$-dimensional real vector space and let $\mathfrak{r}(V) \subset \otimes^4 V^*$ be the set of all Riemannian algebraic curvature tensors; $A \in \mathfrak{r}(V)$ if and only if $A$ has the symmetries of the Riemannian curvature tensor of the Levi-Civita connection:

\begin{equation}
A(x, y, z, w) = -A(y, x, z, w), \quad A(x, y, z, w) = A(z, w, x, y),
\end{equation}

\begin{equation}
A(x, y, z, w) + A(y, z, x, w) + A(z, x, y, w) = 0.
\end{equation}

Let $A \in \mathfrak{r}(V)$ and let $\langle \cdot, \cdot \rangle$ be a non-degenerate symmetric bilinear form on $V$ of signature $(p, q)$. The triple $\mathfrak{M} := (V, \langle \cdot, \cdot \rangle, A)$ is said to be a pseudo-Riemannian algebraic curvature model; let $\Xi(V)$ be the set of such models.

Let $\mathcal{M} := (M, g)$ be a pseudo-Riemannian manifold. Let $\nabla^g$ be the associated Levi-Civita connection and let $R^g_P \in \otimes^4 T^*_P M$ be the curvature tensor at a point $P$ of $M$. Since $R^g_P$ satisfies the symmetries of Equation (1.1),

\[ \mathfrak{M}_P(M) := (T_PM, g_P, R^g_P) \in \Xi(T_PM). \]
The following result shows every $\mathcal{M} \in \Xi(V)$ is geometrically realizable; in particular, the symmetries of Equation (1.a) generate the universal symmetries of the curvature tensor of the Levi-Civita connection.

**Theorem 1.1.** Let $\mathcal{M} \in \Xi(V)$. There exists a pseudo-Riemannian manifold $\mathcal{M}$, a point $P \in M$, and an isomorphism $\phi$ from $T_PM$ to $V$ so that $\mathcal{M}_P(\mathcal{M}) = \phi^*\mathcal{M}$.

1.2. **Osserman geometry.** The relevant structure group which arises in this context is the orthogonal group $O(V, \langle \cdot, \cdot \rangle)$; one can ask geometric realization questions concerning any $O(V, \langle \cdot, \cdot \rangle)$ invariant subset of $r(V)$. If $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, A) \in \Xi(V)$, the Jacobi operator $J_{\mathcal{M}} \in \text{End}(V) \otimes V^*$ is characterized by the relation:

$$\langle J_{\mathcal{M}}(x)y, z \rangle = A(y, x, x, z).$$

If $p > 0$, then $\mathcal{M}$ is said to be **timelike Osserman** if the spectrum of $J_{\mathcal{M}}$ is constant on the pseudo-sphere of unit timelike vectors in $V$. The notion **spacelike Osserman** is defined similarly if $q > 0$. If $p > 0$ and if $q > 0$, work of N. Blažič et al.\cite{1} and of García-Río et al.\cite{2} shows these two notions are equivalent and thus we shall simply say $\mathcal{M}$ is **Osserman** in this context. As this definition is invariant under the action of the structure group $O(V, \langle \cdot, \cdot \rangle)$, it extends to the geometric setting. Thus a pseudo-Riemannian manifold $\mathcal{M}$ will be said to be **Osserman** provided that the associated model $\mathcal{M}_P(\mathcal{M})$ is Osserman for every $P \in M$.

Work of Chi\cite{3} shows there are 4-dimensional Osserman Riemannian algebraic curvature tensors which are not geometrically realizable by Osserman manifolds. The field is a vast one and we refer to Nikolayevsky\cite{4} for further details in the Riemannian setting and to García-Río et al.\cite{5} for a discussion in the pseudo-Riemannian setting; it is possible to construct many examples of Osserman tensors in the algebraic context which have no corresponding geometrical analogues.

1.3. **Affine geometry.** Let $\nabla$ be a torsion free connection on $M$. The associated curvature operator $\mathcal{R} \in T^*M \otimes T^*M \otimes \text{End}(TM)$ is a $(3,1)$ tensor which has the symmetries

$$(1.b) \quad \mathcal{R}(x, y)z = -\mathcal{R}(y, x)z, \quad \mathcal{R}(x, y)z + \mathcal{R}(y, z)x + \mathcal{R}(z, x)y = 0.$$  

As we are in the affine setting, there is no analogue of the additional curvature symmetry $A(x, y, z, w) = A(z, w, x, y)$ which appears in the pseudo-Riemannian setting. In the algebraic context, let $\mathfrak{A}(V) \subset V^* \otimes V^* \otimes \text{End}(V)$ be the set of $(3,1)$ tensors satisfying the relations of Equation (1.b). An element $A \in \mathfrak{A}(V)$ is said to be a **generalized algebraic curvature operator**.

If $\nabla$ is a torsion free connection and if $P \in M$, then $\mathcal{R}_\nabla^\nabla P \in \mathfrak{A}(T_PM)$. The following geometric realizability result is closely related to Theorem 1.1. It shows that any universal symmetry of the curvature tensor of an affine connection is generated by the symmetries of Equation (1.b).

**Theorem 1.2.** Let $A \in \mathfrak{A}(V)$. There exists a torsion free connection $\nabla$ on a smooth manifold $M$, a point $P \in M$, and an isomorphism $\phi$ from $T_PM$ to $V$ so that $\mathcal{R}_\nabla^\nabla P = \phi^*A$.

We contract indices to define the **Ricci tensor** $\rho(A) \in V^* \otimes V^*$ by setting

$$\rho(A)(x, y) := \text{Trace}(z \rightarrow A(z, x)y).$$

The decomposition $V^* \otimes V^* = A^2(V^*) \oplus S^2(V^*)$ sets $\rho(A) = \rho_a(A) + \rho_s(A)$ where $\rho_a(A)$ and $\rho_s(A)$ are the antisymmetric and symmetric Ricci tensors. The natural
structure group for \( \mathfrak{A}(V) \) is the general linear group \( GL(V) \). The Ricci tensor defines a \( GL(V) \) equivariant short exact sequence

\[
0 \to \ker(\rho) \to \mathfrak{A}(V) \to V^* \otimes V^* \to 0.
\]

Strichartz \[6\] showed this short exact sequence is \( GL(V) \) equivariantly split and gives a \( GL(V) \) equivariant decomposition

\[
\mathfrak{A}(V) = \ker(\rho) \oplus \Lambda^2(V^*) \oplus S^2(V^*)
\]

into irreducible \( GL(V) \) modules. The Weyl projective curvature operator \( P(\mathcal{A}) \) is the projection of \( \mathcal{A} \) on \( \ker(\rho) \); \( \mathcal{A} \) is said to be projectively flat if \( P(\mathcal{A}) = 0 \), \( \mathcal{A} \) is said to be Ricci symmetric if \( \rho_0(\mathcal{A}) = 0 \), and \( \mathcal{A} \) is said to be Ricci antisymmetric if \( \rho_s(\mathcal{A}) = 0 \). These notions for a connection are defined similarly. There are 8 natural geometric realization questions which arise in this context and whose realizability may be summarized in the following table – the possibly non-zero components being indicated by \( \ast \):

| \( \ker(\rho) \) | \( S^2(V^*) \) | \( \Lambda^2(V^*) \) | \( \ker(\rho) \) | \( S^2(V^*) \) | \( \Lambda^2(V^*) \) |
|---|---|---|---|---|---|
| * | * | yes | 0 | * | * | yes |
| * | * | 0 | yes | 0 | * | yes |
| * | * | yes | 0 | 0 | * | no |
| * | 0 | 0 | yes | 0 | 0 | yes |

Thus, for example, if \( \mathcal{A} \) is projectively flat and Ricci symmetric, then \( \mathcal{A} \) can be geometrically realized by a projectively flat Ricci symmetric torsion free connection. But if \( \mathcal{A} \neq 0 \) is projectively flat and Ricci antisymmetric, then \( \mathcal{A} \) can not be geometrically realized by a projectively flat Ricci antisymmetric torsion free connection.

1.4. Torsion free connections and Riemannian geometry. We now combine the settings of Sections 1.1 and 1.3. Let \( \langle \cdot, \cdot \rangle \) be a non-degenerate symmetric inner product on \( V \) of signature \( (p,q) \). Fix a basis \( \{e_i\} \) for \( V \) and let \( g_{ij} := \langle e_i, e_j \rangle \) give the components of \( \langle \cdot, \cdot \rangle \). Let \( g^{ij} \) be the inverse matrix. If \( \mathcal{A} \in \mathfrak{A}(V) \), expand \( \mathcal{A}(e_i, e_j) e_k = \mathcal{A}_{ijk}^l e_l \). The scalar curvature \( \tau \) and trace free Ricci tensor are then given, respectively, by

\[
\tau(\mathcal{A}, \langle \cdot, \cdot \rangle) := g^{ij} \mathcal{A}_{ijk}^l, \quad \rho_0(\mathcal{A}, \langle \cdot, \cdot \rangle) := \rho_0(\mathcal{A}) - \frac{\tau(\mathcal{A}, \langle \cdot, \cdot \rangle)}{m} \langle \cdot, \cdot \rangle.
\]

Let \( S_0^s(V^*, \langle \cdot, \cdot \rangle) \) be the space of trace free symmetric bilinear forms. One has an \( O(V, \langle \cdot, \cdot \rangle) \) invariant decomposition of \( V^* \otimes V^* \) into irreducible \( O(V, \langle \cdot, \cdot \rangle) \) modules

\[
V^* \otimes V^* = \Lambda^2(V^*) \oplus S_0^s(V^*, \langle \cdot, \cdot \rangle) \oplus \mathbb{R}.
\]

This decomposition leads to 8 geometric realization questions which are natural with respect to the structure group \( O(V, \langle \cdot, \cdot \rangle) \) and which can all be solved either in the real analytic category or in the \( C^s \) category of \( s \)-times differentiability for any \( s \geq 1 \). The following is the main result of this paper; as our considerations are local, we take \( M = V \) and \( P = 0 \).

**Theorem 1.3.** Let \( g \) be a \( C^s \) (resp. real analytic) pseudo-Riemannian metric on \( V \). Let \( \mathcal{A} \in \mathfrak{A}(V) \). There exists a torsion free \( C^s \) (resp. real analytic) connection \( \nabla \) defined on a neighborhood of \( 0 \) in \( V \) such that:

1. \( \mathcal{R}^V_{0} = \mathcal{A} \).
2. \( \nabla \) has constant scalar curvature.
(3) If $A$ is Ricci symmetric, then $\nabla$ is Ricci symmetric.

(4) If $A$ is Ricci antisymmetric, then $\nabla$ is Ricci antisymmetric.

(5) If $A$ is Ricci traceless, then $\nabla$ is Ricci traceless.

The subspace $\ker(\rho) \subset \mathfrak{A}(V)$ is not an irreducible $O(V, \langle \cdot, \cdot \rangle)$ module but decomposes as the direct sum of 5 additional irreducible factors – see Bokan\[8\]. This decomposition will play no role in our further discussion and studying the additional realization questions which arise from this decomposition is a topic for future investigation.

2. The proof of Theorem 1.3

We assume $s \geq 1$ and $m \geq 3$ henceforth; fix $A \in \mathfrak{A}(V)$. Introduce the following notational conventions. Choose a basis $\{e_i\}$ for $V$ to identify $M = V = \mathbb{R}^m$ and let $\{x_1, \ldots, x_m\}$ be the associated coordinates.

For $\delta > 0$, let $B_\delta := \{x \in \mathbb{R}^m : |x| < \delta\}$ where $|x|$ is the usual Euclidean norm on $\mathbb{R}^m$. Let $C^s_\delta$ be the set of functions on $B_\delta$ which are $s$-times differentiable. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a multi-index. Set

$$\partial_i := \frac{\partial}{\partial x_i}, \quad \partial^\alpha := (\partial_1)^{\alpha_1} \cdots (\partial_m)^{\alpha_m}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_m.$$ 

If $\mathfrak{g}$ is a real vector space, let $C^s(\mathfrak{g})$ be the set of $C^s$ functions on $B_\delta$ with values in $\mathfrak{g}$. Fix a basis $\{f_\sigma\}$ for $\mathfrak{g}$ and expand $P \in C^s_\delta(\mathfrak{g})$ as $P = P^\sigma f_\sigma$ for $P^\sigma \in C^s_\delta$. Let $\nu \in \mathbb{R}$. Set

$$|P| := \sup_{\sigma} |P^\sigma| \in C^0_\delta \quad \text{and} \quad ||P||_{\delta, \nu, -1} := 0.$$ 

For $0 \leq r \leq s$, define $||P||_{\delta, \nu, r} \in [0, \infty]$ by setting

$$||P||_{\delta, \nu, r} := \sup_{|\alpha| = r, |x| \leq \delta} |\partial^\alpha P(x)| \cdot |x|^{-\nu}.$$ 

Thus $||P||_{\delta, \nu, r} \leq C$ implies $|\partial^\alpha P(x)| \leq C|x|^{-\nu}$ for $|\alpha| = r$ and $|x| \leq \delta$. Let \( \mathfrak{S} := S^2((\mathbb{R}^m)^*) \otimes \mathbb{R}^m, \quad \mathfrak{S} := S^2((\mathbb{R}^m)^*), \quad \text{and} \quad \mathfrak{A} := \mathfrak{A}(\mathbb{R}^m). \)

We use the basis $\{e_i\}$ and the coordinate frame $\{\partial_i\}$ to determine the components of tensors of all types; if computing relative to some orthonormal frame $\{E_i\}$, we shall make this explicit. Thus, for example, if $S \in \mathfrak{S}$, then $S_{ij} = S(e_i, e_j)$ while if $S \in C^s_\delta(\mathfrak{S})$, then $S_{ij} := S(\partial_i, \partial_j)$. If $\Gamma, \mathcal{E} \in C^s_\delta(\mathfrak{S})$, define $\mathcal{L}(\Gamma) \in C^{s-1}(\mathfrak{A})$ and $\Gamma \star \mathcal{E} \in C^s_\delta(\mathfrak{A})$ by setting

\begin{equation}
\mathcal{L}(\Gamma)_{ijk} := \partial_i \Gamma_{jk} - \partial_j \Gamma_{ik} - \partial_k \Gamma_{ij},
\end{equation}

\begin{equation}
(\Gamma \star \mathcal{E})_{ijk} := \mathcal{E}_{in} \Gamma_{jk}^n + \Gamma_{in} \mathcal{E}_{jk}^n - \mathcal{E}_{jn} \Gamma_{ik}^n - \Gamma_{jn} \mathcal{E}_{ik}^n.
\end{equation}

If $\Gamma \in C^s_\delta(\mathfrak{S})$, let $\nabla(\Gamma)$ be the $C^s$ torsion free connection on $B_\delta$ with Christoffel symbol $\Gamma$. One has:

\begin{equation}
\mathcal{R}(\Gamma) = \mathcal{L}(\Gamma) + \frac{1}{2} \Gamma \star \Gamma,
\end{equation}

\begin{equation}
\rho(\Gamma \star \Gamma)_{jk} = 2\Gamma_{kn} \Gamma_{jk}^n - 2\Gamma_{jn} \Gamma_{ik}^n = \rho(\Gamma \star \Gamma)_{kj},
\end{equation}

\begin{equation}
\rho_a(\mathcal{R}(\Gamma))_{jk} = \rho_a(\mathcal{L}(\Gamma))_{jk} = \frac{1}{2} \{ \partial_k \Gamma_{ij} - \partial_j \Gamma_{ki} \}.
\end{equation}

One says that $\Gamma$ is normalized if

1. $\Gamma(0) = 0$ and $\mathcal{R}(\Gamma) = A + O(|x|^2)$.
2. $\rho(\mathcal{R}(\Gamma))$ is $C^s$.
3. $\rho_a(\mathcal{R}(\Gamma))(\partial_i, \partial_j) = \rho_a(A)(e_i, e_j)$ on $B_\delta$. 


We remark that Assertion (2) is non-trivial as $R^\nabla(\Gamma)$ need only be $C^{s-1}$. This is a technical condition used subsequently to avoid loss of smoothness.

Theorem 1.2 follows from the following observation which forms the starting point in our proof of Theorem 1.3:

**Lemma 2.1.** If $\Gamma_{ij}^w := \frac{1}{2}(A_{wvw} + A_{wvw})e^w$, then $\Gamma$ is normalized.

**Proof.** Since $\Gamma(0) = 0$, one has:

$$R^\nabla_{ij} = \{ \partial_i \Gamma_{jk}^l(0) - \partial_j \Gamma_{ik}^l(0) \} \partial_l$$

By Equation (2.1), $\rho_a(R^\nabla_{ij}) = \rho_a(\mathcal{L}(\Gamma))_{ij} = \rho_a(A)_{ij}$. □

We continue our analysis with the following basic solvability result:

**Lemma 2.2.** If $\Theta \in C^s_{\delta}(\mathfrak{g})$, then there exists $\mathcal{E} \in C^s_{\delta}(\mathfrak{g})$ so $\rho(\mathcal{L}(\mathcal{E})) = \Theta$, so $\mathcal{E}_{ij} = 0$, and so $||\mathcal{E}|r, s, r+1 \leq ||\Theta||r, s, r+1$.

**Proof.** By assumption, $m \geq 3$. For each pair of indices $\{i, j\}$, not necessarily distinct, choose $k = k(i, j) = k(j, i)$ with $k \neq i$ and $k \neq j$. Set

$$\mathcal{E}_{ij} = \left\{ \begin{array}{ll} \int_0^x \theta_{ij}(x_1, ..., x_{k-1}, u, x_{k+1}, ..., x_m)du & \text{if } \ell = k, \\ 0 & \text{if } \ell \neq k. \end{array} \right.$$ 

Since $k \neq j$, $\mathcal{E}_{ij} = 0$. Consequently Equation (2.1) yields

$$\rho(\mathcal{L}(\mathcal{E}))_{ij} = \partial_k \mathcal{E}_{ij} = \Theta_{ij}.$$

Expand $\partial^\beta_x = \partial^\mu_k \partial^\mu_j$ where $\beta$ does not involve the index $k$. Then:

$$\partial^\alpha_x \mathcal{E}_{ij} = \left\{ \begin{array}{ll} \int_0^x \partial^\mu_k \theta_{ij}(x_1, ..., x_{k-1}, u, x_{k+1}, ..., x_m)du & \text{if } \mu = 0, \\ \mu \partial^\mu_k \partial^\mu_j \theta_{ij} & \text{if } \mu > 0. \end{array} \right.$$ 

Assume that $|\partial^\alpha_x \theta_{ij}(x)| \leq C|x|^{\nu}$ for all $x \in B_3$ and all $|\alpha| = j$. Then

$$|\int_0^x \partial^\mu_k \partial^\mu_j \theta_{ij}(x_1, ..., x_{k-1}, u, x_{k+1}, ..., x_m)du|$$

$$\leq |x_k| \int_0^1 |\partial^\mu_k \theta_{ij}(x_1, ..., x_{k-1}, tx_k, x_{k+1}, ..., x_m)|dt$$

$$\leq |x_k| \cdot C|x|^{\nu} \leq C|x|^{\nu + 1}.$$

The estimates of the Lemma now follow. □

Let $g$ be a $C^s$ pseudo-Riemannian metric on $B_3$ for $\delta < 1$, let $\{e_i\}$ be a $C^s$-orthonormal frame for the tangent bundle of $B_3$, and let $e_i := E_i(0)$. Let $\Gamma \in C^s_{\delta}(\mathfrak{g})$. Define $\Theta = \Theta(\Gamma) \in C^s_{\delta}(\mathfrak{g})$ by:

(2.c) $$\Theta_{ij} := \rho_a(R^\nabla(\Gamma))(E_i, E_j) - \rho_a(A)(e_i, e_j).$$

Use Lemma 2.2 to define $\mathcal{E} = \mathcal{E}(\Gamma) \in C^s_{\delta}(\mathfrak{g})$ so that $\rho_a(\mathcal{L}(\mathcal{E})) = -\Theta$. We use Lemma 2.1 to choose an initial Christoffel symbol $\Gamma_1 \in C^s_{\delta}(\mathfrak{g})$ which is normalized. Inductively, set

$$\Theta_{\nu} := \Theta(\Gamma_{\nu}), \quad \mathcal{E}_{\nu+1} := \mathcal{E}(\Gamma_{\nu}), \quad \Gamma_{\nu+1} := \Gamma_{\nu} + \mathcal{E}_{\nu+1}.$$
Lemma 2.3. Adopt the notation established above. Then (2.b) to compute:

\[ \Theta_{\nu+1, ij} = \rho_s(\mathcal{R}_\nu)(E_i, E_j) - \rho_s(\mathcal{A})(e_i, e_j) + \rho_s(\mathcal{L}(\mathcal{E}_{\nu+1}))_{ij} + \rho_s(\mathcal{L}(\mathcal{E}_{\nu+1}))(E_i, E_j) - \rho_s(\mathcal{L}(\mathcal{E}_{\nu+1}))_{ij} + \rho_s\{(\Gamma_\nu + \frac{s}{2}\mathcal{E}_{\nu+1}) \ast \mathcal{E}_{\nu+1}\}(E_i, E_j). \]

As \( \rho_s(\mathcal{L}(\mathcal{E}_{\nu+1})) = -\Theta_\nu \), the first line vanishes and

(2.d) \[ \Theta_{\nu+1, ij} = -\Theta_\nu(E_i, E_j) + \rho_s\{(\Gamma_\nu + \frac{s}{2}\mathcal{E}_{\nu+1}) \ast \mathcal{E}_{\nu+1}\}(E_i, E_j). \]

Choose \( \kappa \geq 1 \) so we have the following estimates for any \( x \in B_\delta \):

\[ |S(E_i, E_j) - S_{ij}| \leq \kappa |S| \cdot |x|^2 \quad \forall \quad S \in C^4(\mathcal{G}), \]
\[ |\rho_s(\Gamma \ast \mathcal{E})(E_i, E_j)| \leq \kappa |\Gamma| \cdot |\mathcal{E}| \quad \forall \quad \Gamma, \mathcal{E} \in C^4(\mathcal{G}). \]

Lemma 2.3. Adopt the notation established above. Then \( \Gamma_\nu \) is normalized for all \( \nu \). Furthermore, there exists \( \delta_0 > 0 \) and there exist constants \( C_\nu > 0 \) for \( 0 \leq \nu \leq s \) so for \( \nu = 1, 2, ... \) we have the estimates:

(1) \[ ||\Gamma_\nu||_{|x|_0, 1-r,x} \leq \frac{C}{\kappa} C_\nu, \]
(2) \[ ||\Theta_\nu||_{|x|_0, 2r-r,x} \leq C_\nu^{\nu}, \]
(3) \[ ||\mathcal{E}_{\nu+1}||_{|x|_0, 2r-1-r,x} \leq C_\nu^{\nu} + rC_\nu^{\nu-1}. \]

Proof. By assumption \( \Gamma_1 \) is normalized. We assume inductively \( \Gamma_\nu \) is normalized and show \( \Gamma_{\nu+1} \) is normalized. As \( \mathcal{E}_{\nu+1, ij} = 0 \), Equation (2.1) yields

(2.e) \[ \rho_s(\mathcal{R}_{\nu+1})_{ij} = \rho_s(\mathcal{R}_\nu)_{ij} = \rho_s(\mathcal{A})_{ij} \quad \text{on} \quad B_\delta. \]

Since \( \Theta_\nu = \rho_s(\mathcal{R}_\nu)(E_i, E_j) - \rho_s(\mathcal{A})_{ij} = O(|x|^2), \mathcal{E}_{\nu+1} = O(|x|^3) \) and

\[ \mathcal{R}_{\nu+1} = \mathcal{R}_\nu + O(|x|^2) = \mathcal{A} + O(|x|^2). \]

As \( \Gamma_\nu \) is normalized, \( \Theta_\nu \in C^4(\mathcal{G}) \). Hence \( \mathcal{E}_{\nu+1} \in C^4(\mathcal{G}) \) and \( \Gamma_{\nu+1} \in C^4(\mathcal{G}) \). Since \( \rho_s(\mathcal{L}(\mathcal{E}_{\nu+1})) = -\Theta_\nu \) is \( C^\nu \), we may conclude that \( \rho_s(\mathcal{R}_{\nu+1}) \) is \( C^\nu \) even though \( \mathcal{R}_{\nu+1} \) need only be \( C^{\nu-1} \). Thus \( \Gamma_{\nu+1} \) is normalized.

We establish the estimates by induction on \( r \) and then on \( \nu \). Assertion (3)\(_{\nu,r} \) follows from Assertions (2)\(_{\nu,r} \) and (2)\(_{\nu,r-1} \) and from Lemma 2.2. Suppose first that \( r = 0 \); this is, somewhat surprisingly, the most difficult case. As \( \Gamma_1 \) is normalized, \( \mathcal{R}_1 = \mathcal{A} + O(|x|^2) \). One has \( E_i(0) = \partial_i \). Thus \( \Theta_1 = O(|x|^2) \). As \( \Gamma_1 = O(|x|) \), by shrinking \( \delta \), we may choose \( C_0 \) so

\[ |\Gamma_1(x) \leq \tilde{C}_0|x| \quad \text{and} \quad |\Theta_1(x) \leq \tilde{C}_0|x|^2 \quad \text{on} \quad B_\delta. \]

Choose \( C_0 \) and \( \delta_0 < \delta < 1 \) so that

\[ \tilde{C}_0 + 1 < \frac{\kappa}{\kappa} C_0 < C_0, \quad \kappa + \frac{\kappa}{\kappa} C_0 + \frac{\kappa}{\kappa} \kappa \leq C_0, \quad \delta_0^2 C_0 < \frac{1}{\kappa}. \]

If \( \nu = 1 \), then Assertions (1) and (2) follow from the choices made. Assume the Assertions hold for \( \mu \leq \nu \) where \( \nu \geq 1 \). Then

\[ |\Gamma_{\nu+1}| \leq |\Gamma_1| + |\mathcal{E}_2| + ... + |\mathcal{E}_{\nu+1}| \leq \tilde{C}_0|x| + C_0|x|^3 + C_0^2|x|^5 + ... \]
\[ \leq \tilde{C}_0|x| + |x| \frac{C_0|x|^2}{1-C_0|x|^2} \leq (\tilde{C}_0 + 1) |x| \leq \frac{1}{1+\kappa} C_0. \]
We use Equation (2.d) to complete the induction step for \( r = 0 \) by checking
\[
|\Theta_{r+1}| \leq \kappa \{ x^2 \cdot |\Theta| + (|\Gamma| + |E_{r+1}|) |E_{r+1}| \}
\leq \kappa \{ C_0^0 |x|^{2r+2} + (C_0 + C_0^0 |x|^{2r+2}) \}
\leq C_0^1 |x|^{2r+2} \{ \kappa + \frac{1}{\kappa} C_0 \} \leq C_0^{r+1} |x|^{2r+2}.
\]

We now suppose \( r = 1 \); we get 1 less power of \(|x|\) in the decay estimates. We choose \( C_1 \) so \(|\partial_k \gamma_{1}| \leq C_1 \) and \(|\partial_k \Theta_{1}| \leq C_1 |x| \); the desired estimates then hold for \( \nu = 1 \) for \( C_1 \) sufficiently large. We proceed by induction on \( \nu \). We then have for sufficiently large \( C_1 \) and small \( \delta_0 \) that:
\[
|\partial_k \gamma_{\nu+1}| \leq |\partial_k \gamma_{1}| + |\partial_k E_3| + \ldots + |\partial_k E_{\nu+1}|
\leq \bar{C}_1 + (C_1 + C_0) |x|^2 + (C_1^2 + C_0^2) |x|^4 + \ldots \leq \frac{1}{\kappa} C_1.
\]

We differentiate Equation (2.d) to obtain
\[
\partial_k \Theta_{r+1, ij} = -(\partial_k \Theta_{ij})(E_i, E_j) + \partial_k \Theta_{r+1, ij} - \Theta_{ij}(\partial_k E_i, E_j) - \Theta_{ij}(E_i, \partial_k E_j).
\]

A crucial point is that there are no \( \nu \) for small \( \nu \) in the holomorphic setting. Theorem 1.3 follows in the real analytic context as well.

In the real analytic category, we complexify and consider the complex ball of radius \( \delta \) in \( \mathbb{C}^m \). Since \( C^0 \) convergence of holomorphic functions gives convergence in the holomorphic setting, Theorem 1.3 follows in the real analytic context as well.
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