Direct determination of the gauge coupling derivatives for the energy density in lattice QCD *

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By matching Wilson loop ratios on anisotropic lattices we measure the coefficients $c_\sigma$ and $c_\tau$, which are required for the calculation of the energy density. The results are compared to that of an indirect method of determination. We find similar behaviour, the differences are attributed to different discretization errors.

1. Different ways to calculate the energy density

From statistical mechanics we know the energy density as $\epsilon = \frac{1}{Z} \langle \frac{\partial \ln Z}{\partial (\beta/T)} \rangle$. In lattice calculations the straightforward approach to calculate this quantity is to introduce different lattice spacings $a_\sigma, a_\tau$ to perform the derivatives with respect to $1/T = N_\tau a_\tau$ at fixed physical volume $(N_\sigma a_\sigma)^3$. Therefore one introduces the lattice anisotropy $\xi = a_\sigma/a_\tau$ and two gauge couplings in the action,

$$ S = \frac{2N}{g_\sigma^2(\xi)} \frac{1}{\xi} \sum_{\mu<\nu<4} S_{\mu,\nu} + 2N g_\tau^2(\xi) \frac{1}{\xi} \sum_{\mu<\nu=4} S_{\mu,4} . \quad (1) $$

The energy density $\epsilon$ contains then the derivatives of these gauge couplings,

$$ \epsilon = T^4 \left( \frac{N_\tau}{N_\sigma} \right)^3 \frac{2N}{g_\tau^2} \left( (S_\sigma - S_\tau) + g_\tau^2 \langle S_\sigma - S_0 \rangle + c_\tau \langle S_\tau - S_0 \rangle \right) , \quad (2) $$

where $S_0$ - action on a symmetric lattice, $T=0$ contribution, $S_\sigma$ - spatial part of the action, $S_\tau$ - temporal part of the action, finite temperature $\{\}$ simulation, and

$$ c_\sigma \equiv \frac{\partial g_\sigma^{-2}(a,\xi)}{\partial \xi} \bigg|_{\xi=1} , \quad c_\tau \equiv \frac{\partial g_\tau^{-2}(a,\xi)}{\partial \xi} \bigg|_{\xi=1} . \quad (3) $$

The coefficients $c_\sigma$ and $c_\tau$ were calculated pertubatively in [3].

A different way to determine the energy density was used by the Bielefeld group [2]. The free energy density $f$ is obtained by integrating the difference of the plaquettes,

$$ f/T^4 \bigg|_{\beta_0}^{\beta} = -N_\tau^4 \int_{\beta_0}^\beta d\tilde{\beta} \left[ 2S_0 - (S_\sigma + S_\tau) \right] . \quad (4) $$

In large systems, where $p = -f$, $\epsilon$ can be found from the pressure,

$$ \epsilon - 3p = T \frac{d}{dT} \frac{p}{T^4} = 12N N_\tau^4 (c_\sigma + c_\tau) \left[ 2S_0 - (S_\sigma + S_\tau) \right] , \quad (5) $$

Using the relation $c_\sigma + c_\tau = -\frac{1}{12} \left( a \frac{\partial \beta}{\partial a} \right)$ and the $\beta$-function from $T_c/\sqrt{\sigma}$ measurements, we can calculate $c_\sigma$ and $c_\tau$ from (2), (3) and (4). In Figure 1 the results for $c_\sigma$ and $c_\tau$ are shown as solid lines. They deviate clearly from the pertubative results (broken lines). However, this is expected, as also $c_\sigma + c_\tau$ deviate from the pertubative result. We also note, that the method works only above $\beta_{\text{critical}}(N_\tau)$, because both $p$ and $\epsilon$ become very small below the critical point.

2. Direct determination of $c_\sigma$ and $c_\tau$ by matching of Wilson loop ratios

An attempt to determine the lattice anisotropy was made by Burgers et al. [4]. Consider the anisotropic couplings

$$ g_\sigma^{-2}(\xi) = \frac{\xi}{2N} \beta , \quad g_\tau^{-2}(\xi) = \frac{1}{2N \xi} \beta \gamma , \quad (6) $$

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We vary the anisotropy of the couplings $\gamma$ in the action $S = \beta / \gamma S_\sigma + \beta \gamma S_\tau$ and measure the anisotropy of the lattice spacing $\xi$. Wilson loops are suitable observables, since they depend on the physical size of the lattice. Instead of matching Wilson loops, measured in spatial and in temporal direction, directly,

$$W_\sigma(x, z = \xi t) = W_\tau(x, t) ,$$

we match ratios of Wilson loops, in order to cancel corner and selfmass contributions,

$$R_1(x, t) = \frac{W(x, t)}{W(x + 1, t)} \quad \text{or} \quad R_2(x, t) = \frac{W(x + 1, t)W(x - 1, t)}{W(x, t)^2} .$$

Since these ratios contain only Wilson loops with the same extension in $t$, the following matching condition holds,

$$R_\sigma(x, z = \xi t) = R_\tau(x, t) .$$

We are using a $16^4$ lattice for $\gamma \in [0.92, 1.08]$, a $16^3 \times 32$ lattice for $\gamma \in [1.1, 2.0]$ and a $16^3 \times 48$ lattice for $\gamma = 3.0$. The link integration technique

\begin{align*}
\text{Fig. 1.} & \text{ The derivatives } c_\sigma \text{ and } c_\tau \text{ for the Wilson action (a) } (N_\tau=4,6,8) \text{ and the } 2\times2 \text{-action (b) } (N_\tau=4) \text{ versus } \beta. \text{ The broken lines show the pertubative values.} \\
\text{The dotted line shows an estimate form the different } \tau \text{-values. The single measurements are from the matching method.} \\
\text{Fig. 2.} & \text{ The ratios } R_{1,\sigma}(x, t) \text{ for fixed } x = 1, \ldots, 7 \text{ vs. } t + x, \text{ the open symbols are connected by solid lines.} \text{ The ratios } R_{1,\sigma}(x, z) \text{ correspond to the bold face symbols and are shifted in } t \text{ by } \xi = 1.63(4).}
\end{align*}

The derivatives with respect to the lattice anisotropy $\xi = a_\sigma / a_\tau$ are

$$c_{\sigma, \tau} = \pm \frac{\beta}{6} \left( 1 - \frac{\partial \gamma}{\partial \xi} \right) - \frac{1}{4} \left( a \frac{\partial g^{-2}(a)}{\partial a} \right) , \quad (7)$$

which requires the function $\gamma(\xi)$ or $\xi(\gamma)$.
of ref. [5] is applied to obtain accurate expectation values of large Wilson loops with a statistics of 2000 measurements.

The value of $\xi$ was chosen in such a way, that the square deviation of $R_{1,\sigma}(x, z = \xi t)$ from the lines connecting the $R_{1,\tau}(x, t)$ measurements was minimal. For each $\gamma$ we find in this way a value of $\xi$. Figure 3 shows an example for $\beta = 6.3$. In all cases we found a linear behaviour of $\xi(\gamma)$.

![Fig. 3. The result for $\xi$ as a function of $\gamma$ for $\beta = 6.3$. The solid line is a linear fit to $\xi(\gamma)$.](image)

To obtain the derivative $\partial \xi / \partial \gamma |_{\xi = 1}$ we have performed linear fits. The resulting $c_{\sigma}$ are shown in Figure 4.

![Fig. 4. The derivative $c_{\sigma}$ for $\beta = 6.3$ from Eq.(7) as a function of the minimal area $A$ of the smallest Wilson loops included in the ratio matching. The solid line shows the average value for $A \geq 8$, the broken lines the error band.](image)

Obviously the matching procedure requires at least loops of an area eight. For larger Wilson loops the results are consistent.

### 3. Summary and conclusion

Employing a method of matching ratios of Wilson loops on anisotropic lattices we have directly determined the derivatives $c_{\sigma}$ and $c_{\tau}$ of the gauge couplings non-pertubatively. This has been done for both, the standard Wilson action and the $2 \times 2$ improved action. We found significant deviations from the pertubative result. Qualitatively our results are in agreement with the indirect method used in [4], though they are a little lower.

We assume that the difference is due to $\mathcal{O}(a^n)$ corrections, since they enter the two methods differently.

The use of ratios of Wilson loops was crucial in order to eliminate unphysical self energy contributions. The numerical matching technique for these ratios enabled us to obtain the required accuracy for $\xi$.

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