DOMINANT ENERGY CONDITION AND CAUSALITY FOR SKYRME-LIKE GENERALIZATIONS OF THE WAVE-MAP EQUATION

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Abstract. It is shown in this note that a class of Lagrangian field theories closely related to the wave-map equation and the Skyrme model obeys the dominant energy condition, and hence by Hawking’s theorem satisfies finite speed of propagation. The subject matter is a generalization of a recent result of Gibbons.

1. Introduction

Recently Gibbons showed [Gib03] that the Skyrme model obeys the dominant energy condition, and thus settling the problem of causality for that equation. In this note we will give a different proof of the same fact that easily generalizes to a class of Lagrangian field theories that includes, as special cases, the wave-map equation, the Skyrme model, and the Born-Infeld model.

Let \((M, g)\) be an \(m+1\) dimensional Lorentzian manifold, where sign convention is taken to be \((-,+,+,\cdots)\), and let \((N, h)\) be an \(n\) dimensional Riemannian manifold. Let \(\phi : M \rightarrow N\) be a \(C^1\) map. Then the action of \(\phi\) can be used to pull back the metric \(h\) onto \(M\) as a positive semi-definite quadratic form on \(T_M\), we write it as

\[ \phi^* h(X,Y) = h(d\phi \cdot X, d\phi \cdot Y) \]

where the left hand side is evaluated at a point \(p \in M\) and the right hand side at the point \(\phi(p) \in N\) for \(X,Y \in T_p M\). Composing with the inverse metric \(g^{-1}\) we obtain the so-called strain tensor \(D^\phi\), a section of \(T^1_1 M\):

\[ D^\phi = g^{-1} \circ \phi^* h, \]

thus at every point \(p\), \(D^\phi\) is a linear transformation of \(T_p M\). Now, if \(g\) were a Riemannian metric, then for a fixed basis of \(T_p M\), the matrix \((D^\phi)\) is positive semi-definite. This is, unfortunately, no longer true in the Lorentzian case, and thus the eigenvalues of \((D^\phi)\) are in general complex.

Let \(\{\lambda_1, \ldots, \lambda_k\}\) denote the non-zero eigenvalues, counted with multiplicity, of \((D^\phi)\). Note that by elementary linear algebra, using that \(g\) is non-degenerate and \(h\) is positive definite, one easily sees that

\[ k \leq \text{rank}(d\phi) \leq \min(m + 1, n). \]

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Recall the elementary symmetric polynomials \( s_j(\{\lambda_1, \ldots, \lambda_k\}) \) given by

\[
(3) \quad s_j(\{\lambda_1, \ldots, \lambda_k\}) = \sum_{1 \leq \alpha_1 < \cdots < \alpha_j \leq k} \prod_{i=1}^{j} \lambda_{\alpha_i}
\]

with \( s_0 = 1 \) and \( s_j = 0 \) for all \( j > k \). Observe that for the \((m+1) \times (m+1)\) matrix \((D^\phi)\), the elementary symmetric polynomials correspond to the coefficients of the characteristic polynomial, and specifically \( s_1 = tr(D^\phi) \) and \( s_{m+1} = \det(D^\phi) \). By abuse of notation, we will write \( s_j(D^\phi) \) when we mean the symmetric polynomials on the eigenvalues of \((D^\phi)\). Note that \( s_j(D^\phi) \) is independent of a basis chosen for the vector space \( T_p M \).

For a given class \( \mathcal{A} \) of maps \( \phi : M \rightarrow N \), we write

\[
U_\mathcal{A} := \{ v \in \mathbb{R}^{m+1} \mid v = (s_1, \ldots, s_{m+1})(D^\phi), \phi \in \mathcal{A} \}.
\]

**Definition 1.** For a given class \( \mathcal{A} \), let \( U_\mathcal{A} \subset \mathbb{R}^{m+1} \) be an open set that contains \( U_\mathcal{A} \cup \{0\} \). An admissible function \( F : U_\mathcal{A} \rightarrow \mathbb{R} \) for the class \( \mathcal{A} \) is a sub-additive, concave function, that is \( C^1 \) on the interior of \( U_\mathcal{A} \) and continuous up to the boundary.

**Remark 2.** In the definitions above, it only suffices to include terms up to \( s_{m+1} \) in view of (2). Also, observe that sub-additivity and concavity of \( F \) immediately implies that \( F(0) \geq 0 \).

**Definition 3.** A Lagrangian field theory for the class \( \mathcal{A} \) of maps \( \phi : M \rightarrow N \) is said to be a generalized wave-map \(^1\) if the Lagrangian

\[
L = F(s_1(D^\phi), s_2(D^\phi), \ldots, s_{m+1}(D^\phi))
\]

for an admissible \( F \). Furthermore, we say that the generalized wave-map is defocusing if the first partial derivatives of \( F \) are all non-negative, i.e.

\[
\partial_i F(v) \geq 0 \quad \forall i = 1, \ldots, m+1 \quad \text{and} \quad \forall v \in U_\mathcal{A}.
\]

The generalized wave-map is said to be zeroed if \( F(0) = 0 \). Also, we shall refer to a generalized wave-map for which \( \partial_i F \) is non-vanishing as non-degenerate.

The author hopes that the reason behind the nomenclature will be evident after the proof of the dominant energy condition is developed. We first give some examples of generalized wave-maps:

- Observe that if \( L \) is a linear combination of the symmetric polynomials \( L = \sum c_i s_i(D^\phi) \), then it is automatically a zeroed generalized wave-map. If in addition the coefficients \( c_i \) are all non-negative, then \( L \) is defocusing. In this case if \( c_1 > 0 \) then \( L \) is non-degenerate.
- Take \((M, g)\) to be a static space-time, i.e. \( M = \mathbb{R} \times \Sigma \) and \( g = -dt^2 \oplus \gamma \) where \( \rho \) is a positive function on \( \Sigma \) and \( \gamma \) is a Riemannian metric on \( \Sigma \). A static solution to the generalized wave-map is one for which \( \nabla_t \phi = 0 \). The static solution for \( L = s_1 \) gives rise to the harmonic map from \( \Sigma \rightarrow N \), while for the case \( n > m \), \( L = \sqrt{s_m} \) (recall that \( \dim M = m+1 \)), the equation becomes the minimal surface equation for the embedding of \( \Sigma \) into \( N \). For the minimal surface equation we take \( U_\mathcal{A} = \mathbb{R}^{m+1}_+ \).
- In the Lorentzian case, \( L = s_1 \) is simply the wave-map equation. For \( L = c_1 s_1 + c_2 s_2 \) where \( c_1, c_2 > 0 \) are coupling constants, we recover the original Skyrme model if we take \((N, h)\) to be \( SU(2) \) with the bi-invariant metric.

\(^1\)For the lack of a better name. Suggestions are welcome.
In particular, the Skyrme model is a defocusing, zeroed, non-degenerate, generalized wave-map in the terminology adopted in the present paper.

- Let \( b > 0 \) be a fixed large constant. We can restrict \( \phi \) to only consider those maps such that the real parts of the eigenvalues of \( D\phi \) are greater than \(-b\). Then letting

\[
F = \sqrt{\det(b \cdot Id + D\phi)} - \sqrt{\det(b \cdot Id)}
\]

defined on \( U_A \) being the set where \( \det(b \cdot Id + D\phi) \geq 0 \), we get the zeroed, defocusing, non-degenerate, generalized wave-map also known as the Born-Infeld model.

Before stating the main theorem, we recall the statement of the dominant energy condition. Recall that the (covariant) stress-energy tensor \( T \in \Gamma(T^0_2M) \) for a Lagrangian field theory is given by a variational derivative for the Lagrangian density relative to the inverse metric,

\[
T \sqrt{|\det g|} := \left( \frac{\delta L}{\delta g^{-1}} + \frac{1}{2} L g \right) \sqrt{|\det g|}.
\]

**Definition 4.** The stress-energy tensor \( T \) is said to obey the dominant energy condition at a point \( p \in M \) if for all \( X \in T_pM \) such that \( g(X, X) < 0 \), the following two conditions are satisfied

\[
\begin{align*}
& (5a) \quad T(X, X) > 0 \\
& (5b) \quad [T \circ g^{-1} \circ T](X, X) \leq 0
\end{align*}
\]

unless \( T \) vanishes identically.

**Remark 5.** The definition is equivalent to the classical statements (see, e.g. section 4.3 in [HE73] or chapter 9 of [Wal84]) of the dominant energy condition. Observe that (5b) gives that the vector \( g^{-1} \circ T \circ X \) is a causal vector for any time-like vector \( X \), and (5a) gives that the vector \( g^{-1} \circ T \circ X \) has opposite time-orientation as the time-like vector \( X \).

Now we state the main theorem

**Theorem 6.** A defocusing generalized wave-map obeys the dominant energy condition.

First we claim that it would suffice to prove the theorem for each \( s_i \). The following lemma is a general statement on a convexity property of Lagrangian field theories.

**Lemma 7.** Let \( F \) be a sub-additive, concave function as in Definition 3. Let \( T_i \) denote the stress-energy tensor corresponding to the Lagrangian \( L_i \). Assume that \( T_i \) obeys the dominant energy condition, or, equivalently, the vectors \( Y_i = g^{-1} \circ T_i \circ X \) are all past-causal for any fixed future time-like \( X \). Then \( L = F(L_1, \ldots, L_{m+1}) \) also obeys the dominant energy condition if \( L \) is defocusing.

**Proof.** The stress-energy tensor \( T \) can be written, using (4), as

\[
T = \sum_{i=1}^{m+1} \partial_i F \cdot \frac{\delta L_i}{\delta g^{-1}} - \frac{1}{2} F g = \sum_{i=1}^{m+1} \partial_i F \cdot T_i - \frac{1}{2} (F - \sum_{i=1}^{m+1} \partial_i F \cdot L_i) g.
\]
Now considering $g^{-1} \circ T \circ X$, the first term in the above expression contributes $\sum \partial_i F \cdot Y_i$. Since $L$ is defocusing, this is a positive linear combination of past-causal vectors, and hence by elementary Minkowskian geometry, is still past-causal. For the second term, since $g^{-1} \circ g \circ X = X$, to show that it is also past-causal it suffices to show that

$$F \geq \sum_{i=1}^{m+1} \partial_i F \cdot L_i.$$ 

But this follows from the fact that $F$ is concave and $F(0) \geq 0$. \hfill \Box

Unfortunately, it is immediately clear that the theorem may not be strong enough in certain cases for practical application. This is because the vanishing of $T$ does not guarantee that the map $\phi$ is trivial. For example, using that $s_j = 0$ if $j > \text{rank}(d\phi)$, it is immediate that if locally around the point $p$, $\phi$ is one-dimensional, then for any metric $g$, $s_j(D^p) = 0$ if $j \geq 2$. On the other hand, this failure of the dominant energy condition arises from a degeneracy which forces the stress-energy tensor to be a null stress tensor in the language of Christodoulou [Chr00], which we can “normalize” away by taking $L$ to be zeroed. We claim that this is the only possible failure.

**Proposition 8.** For $L = s_i$, $T$ obeys the dominant energy condition. Furthermore, $T = 0$ at a point $p$ if and only if $i > \text{rank}(d\phi)_p$.

From this proposition one immediately sees the following energy bound for smooth solutions of the generalized wave-map equation.

**Corollary 9.** If $\phi$ is the solution to a defocusing, non-degenerate, zeroed, generalized wave-map, and if $T = 0$ on a connected open domain $B$ of $M$, then $\phi$ is constant on $B$.

By applying Hawking’s energy conservation theorem (see section 4.3 in [HE73]) the above corollary implies that defocusing, non-degenerate, zeroed, generalized wave-maps have finite speed of propagation (also known as the domain of dependence condition).

In principle, if one has advanced knowledge on a lower bound to the rank of the map $\phi$, one can also obtain analogous statements for degenerate cases. We leave such trivial generalizations to the reader.

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**2. A FORMULA FOR THE STRESS-ENERGY TENSOR AND PROOF OF THE MAIN PROPOSITION**

In this section, we’ll derive first derive a formula for the stress-energy tensor. We will begin by making a geometric observation and obtain, almost immediately, a simple tensorial formula for the Lagrangian. Taking the formal variational derivative of the Lagrangian leads to a tensorial expression for the stress-energy tensor, from which Proposition 8 follows via simple linear algebra.

Consider a real vector space $V$. Let $A$ be a linear transformation on $V$. Then $A$ naturally extends to a linear transformation, which we denote $A^j$, on $\Lambda^j(V)$, the space of alternating $j$-vectors over $V$. A bit of basic linear algebra (perhaps by extending $V$ to $V \otimes_\mathbb{R} \mathbb{C}$ and taking a basis of eigenvectors) shows that $s_j(A)$ is
proportional to $\text{tr}_{\Lambda^j(V)} A^{\tilde{j}}$. Now, letting $V = T_p M$ and $A = D\phi = g^{-1} \circ \phi^* h$, we observe that

$$(D\phi)^{\tilde{j}} = (g^{-1})^{\tilde{j}} \circ \phi^* (h^{\tilde{j}}),$$
or, to put it in words, $(D\phi)^{\tilde{j}}$ is obtained from first taking the induced metric $h^{\tilde{j}}$ on alternating $j$-vectors in $T_{\phi(p)} N$, pulling it back via $\phi$, and composing it with the induced metric $(g^{-1})^{\tilde{j}}$ for the alternating $j$-forms. In index notation, this can be written as

$$[(D\phi)^{\tilde{j}}]_{a_1 \ldots a_j} = g^{b_1 c_1} \ldots g^{b_j c_j} \phi^* h_{a_1 \vert c_1} \phi^* h_{a_2 \vert c_2} \ldots \phi^* h_{a_{j-1} \vert c_{j-1}} \phi^* h_{a_j \vert c_j},$$

where the bracket notation in the indices denotes full anti-symmetrization of the $\{c_1, \ldots, c_j\}$ indices. For a Lagrangian proportional to an $s_j$, we can assume

$$(6) \quad L = [(D\phi)^{\tilde{j}}]_{a_1 \ldots a_j} = g^{a_1 \vert c_1} \ldots g^{a_j \vert c_j} \phi^* h_{a_1 c_1} \ldots \phi^* h_{a_j c_j}.$$It is simple to check, using $(D\phi) = \text{diag}(-1, 1, 1, \ldots)$ that the above expression has the correct sign: that $L$ defined thus is a positive multiple of $s_j$.

One can also arrive at (6) purely from a linear algebra point of view. Let $p_j$ be the power sum

$$p_j(\{\lambda_1, \ldots, \lambda_k\}) = \sum_{i=1}^k \lambda_i^j.$$Recall that we have Newton’s identity

$$j \cdot s_j = \sum_{i=1}^j (-1)^{i-1} e_{j-i} p_i$$

which allows us to express $s_j$ as a rational polynomial in $p_i$’s. Now, by definition, it is clear that

$$p_j(D\phi) = \text{tr}[(D\phi)^{\tilde{j}}]$$

where $(D\phi)^{\tilde{j}}$ is the $j$-fold composition of $D\phi$. It is easy to check then, for some $E$

$$s_j = g^{a_1 b_1} \ldots g^{a_j b_j} E_{b_1 \ldots b_j}^{c_1 \ldots c_j} (\phi^* h)_{a_1 c_1} \ldots (\phi^* h)_{a_j c_j}.$$Newton’s identity reduces to a generating condition for $E$ based on the Kronecker $\delta$ symbols,

$$E_{b}^{c} = \delta_{b}^{c},$$

$$jE_{b_1 \ldots b_j}^{c_1 \ldots c_j} = \sum_{i=1}^j (-1)^{i-1} E_{b_1 \ldots b_{j-i}}^{c_1 \ldots c_{j-i}} \delta_{b_{j-i+1}}^{c_{j-i+1}} \ldots \delta_{b_j}^{c_j},$$

A direct computation which we omit here shows that then in fact the invariant $E_{b_1 \ldots b_j}^{c_1 \ldots c_j}$ is a positive rational multiple of the generalized Kronecker symbol $\delta_{b_1 \ldots b_j}^{c_1 \ldots c_j}$, from which we recover (6).

Now, the object we are interested in, given a time-like vector $X$, is the one-form $T(X, \cdot)$. Since $T$ is tensorial, we can assume $X$ has unit length. Fix some $j$, let the Lagrangian be proportional to $s_j$ as given by (6). By the symmetry property, we can write $T(X, \cdot)$ in index notation:

$$(7) \quad T_{ab} X^b = j X^{[b} g^{a_2 \vert c_2} \ldots g^{a_j \vert c_j} (\phi^* h)_{ab} \ldots (\phi^* h)_{a_j c_j} - \frac{1}{2} g_{ab} X^b L$$
Proof of Proposition 3. Consider an orthonormal basis for $T_p M$ relative to $g$. Since we assumed $X$ unit, let $e_0 = X$ and \{$e_i$\}_{1 \leq i \leq m} are all space-like. We can take $j \leq m + 1$ as otherwise $T$ is identically 0. Then we notice that a basis for $\Lambda^j (T_p M)$ is given by

\[
\{e_0 \wedge e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{j-1}}\}_{1 \leq \alpha_1 < \cdots < \alpha_{j-1} \leq m} \cup \{e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_j}\}_{1 \leq \alpha_1 < \cdots < \alpha_j \leq m}.
\]

We write the first set as $\Lambda^j_\perp$ and the second set as $\Lambda^j_\parallel$. Using the normalization that \(v \wedge w = v \otimes w - w \otimes v\), we find that each of the element in $\Lambda^j_\perp$ has norm $-j!$ while the elements in $\Lambda^j_\parallel$ has norm $j!$.

To show that $T(X, X) > 0$ generically, we observe that under the expansion (7), the first term corresponds to

\[
\sum_{\omega \in \Lambda^j_\perp} \phi^* (h^j) (\omega, \omega),
\]

while the second term corresponds to

\[
\frac{1}{2} \left( - \sum_{\omega \in \Lambda^j_\perp} \phi^* (h^j) (\omega, \omega) + \sum_{\omega \in \Lambda^j_\parallel} \phi^* (h^j) (\omega, \omega) \right).
\]

So summing them gives

\[
\frac{1}{2} \left( \sum_{\omega \in \Lambda^j_\perp} \phi^* (h^j) (\omega, \omega) + \sum_{\omega \in \Lambda^j_\parallel} \phi^* (h^j) (\omega, \omega) \right)
\]

which is non-negative by the fact that $\phi^* (h^j)$ is a positive semi-definite quadratic form on $\Lambda^j (T_p M)$. Furthermore, observe that since $\Lambda^j_\parallel \cup \Lambda^j_\perp$ is a basis, its push-forward $\phi_* \Lambda^j_\parallel \cup \phi_* \Lambda^j_\perp$ spans $\Lambda^j (\phi_* T^M_p) \subset \Lambda^j (T_{\phi(p)} N)$. Thus by the fact that $h$ (and hence the induced metric $h^j$) is positive definite, we conclude that $\Lambda^j (\phi_* T^M_p) = \{0\}$, which proves the assertion that $T$ vanishes only when $j > \text{rank}(d\phi)$.

To show (5b), we observe that

\[
X^a T_{ac} g^{cd} T_{db} X^b = -T(X, X)^2 + \sum_{i=1}^m T(X, e_i)^2.
\]

The first thing to note is that $T(X, e_i)$ does not have any contribution from the second term in (7). For the first term, a quick computation shows that $T(X, e_i)$ corresponds to

\[
\sum_{\eta \in \Lambda^j_{\perp}} \phi^* (h^j) (e_0 \wedge \eta, e_i \wedge \eta)
\]
D.E.C. FOR GENERALIZED WAVE-MAPS

| $\sum_{i=1}^{m} T(X, e_i)^2 | \leq (\sum |T(X, e_i)|)^2$

$\leq (\sum_{i=1}^{m} \sum_{\eta \in \Lambda_i^{j-1}} |\phi^*(h^{ij})(e_0 \wedge \eta, e_i \wedge \eta)|)^2$

$\leq \frac{1}{4} (\sum_{\eta \in \Lambda_i^{j-1}} \phi^*(h^{ij})(e_0 \wedge \eta, e_0 \wedge \eta) + \sum_{i=1}^{m} \phi^*(h^{ij})(e_i \wedge \eta, e_i \wedge \eta))^2$

$= \frac{1}{4} (\sum_{\eta \in \Lambda_i^{j-1}} \sum_{i=0}^{m} \phi^*(h^{ij})(e_i \wedge \eta, e_i \wedge \eta))^2$

$= T(X, X)^2$

And therefore (5b) is satisfied. \hfill \square

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