Deep Random Vortex Method for Simulation and Inference of Navier-Stokes Equations

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Abstract

Navier-Stokes equations are significant partial differential equations that describe the motion of fluids such as liquids and air. Due to the importance of Navier-Stokes equations, the development on efficient numerical schemes is important for both science and engineering. Recently, with the development of AI techniques, several approaches have been designed to integrate deep neural networks in simulating and inferring the fluid dynamics governed by incompressible Navier-Stokes equations, which can accelerate the simulation or inferring process in a mesh-free and differentiable way. In this paper, we point out that the capability of existing deep Navier-Stokes informed methods is limited to handle non-smooth or fractional equations, which are two critical situations in reality. To this end, we propose the Deep Random Vortex Method (DRVM), which combines the neural network with a random vortex dynamics system equivalent to the Navier-Stokes equation. Specifically, the random vortex dynamics motivates a Monte Carlo based loss function for training the neural network, which avoids the calculation of derivatives through auto-differentiation. Therefore, DRVM not only can efficiently solve Navier-Stokes equations involving rough path, non-differentiable initial conditions and fractional operators, but also inherits the mesh-free and differentiable benefits of the deep-learning-based solver. We conduct experiments on the Cauchy problem, parametric solver learning, and the inverse problem of both 2-d and 3-d incompressible Navier-Stokes equations. The proposed method achieves accurate results for simulation and inference of Navier-Stokes equations. Especially for the cases that include singular initial conditions, DRVM significantly outperforms existing PINN method.

1 Introduction

The ubiquitous Navier-Stokes equations are indispensable for modeling the fluids ranging from meteorology to ocean currents \cite{2, 36, 39}. Consequently, it is significant to study the Navier-
Stokes equation, which has widespread applications in scientific [15], industrial [16], or engineering areas [7]. Recently, with the growth of computer technology and data availability, the development of deep learning techniques has been incredible and shed light on new chances for scientists to press ahead with the research. Deep-learning-based methods such as Physics-Informed Neural Networks (PINNs) help solve partial differential equations with accuracy and efficiency [3, 11, 17, 24, 34, 35], revealing the promising future of combining deep learning with scientific computing. The basic idea behind PINNs is seamlessly embedding the information of observed data and physical law into neural networks via automatic differentiation regime. In detail, the loss function of PINNs contains the supervised loss constructed by observed data and the physical loss of partial differential equations, including residuals of governing equations and other extra conservation laws. Compared with conventional numerical methods, PINNs are mesh-free and can handle inverse problem efficiently due to their differentiable properties [34].

Recently, there have been several researches devoted to utilizing PINNs to solve scientific problems for Navier-Stokes equations. For instance, Raissi et al. [35] developed Hidden Fluid Mechanics (HFM) agnostic to the physical geometry and boundary conditions to solve forward and inverse fluid mechanics problems in arbitrarily complex domains. Sun et al. [37] combined PINNs with Bayesian deep learning approach to reconstruct flow fields from noisy velocity data. Jin et al. [17] proposed Navier-Stokes Flow nets (NSFnets) by considering the velocity-pressure and the vorticity-velocity formulations simultaneously to simulate both laminar and turbulent flows. Moreover, several methods were proposed to optimize the network architectures and training dynamics of PINNs, e.g., multi-scale deep neural networks (MsacleDNNs) [46], hard constraints PINNs [25] and dynamic pulling method (DPM) [18].

While the methodologies mentioned above have made remarkable progress, their applications to solve Navier-Stokes equations face the following three fundamental challenges. Firstly, they utilize derivative information directly in equations to define the loss, which require the solution to be second order continuously differentiable in general. Nevertheless, the fluid velocity field might be non-smooth in many realistic scenarios and cannot satisfy such ideal properties. Thus, one would wish to be able to handle various situations, including both smooth and non-smooth solutions. The second challenge relates to the fractional Navier-Stokes equations [5, 44, 45], which could be viewed as the interpolation between Euler equations and Navier-Stokes equations. Fractional Navier-Stokes equations have received considerable attention during the past few years because of its demonstrated applications in fluid flow, rheology, and so on [14, 21]. In this case, the Laplacian operator is replaced by the fractional Laplacian \((-\Delta)^{\frac{\alpha}{2}}\). As \(\alpha\) becomes smaller, the dissipation effect becomes weaker. So the solutions of the equations become more singular, which makes challenging to learn such PDEs. There are only few deep-learning-based models aimed at fractional equations [10, 28], and the computation of fractional Laplacian for the neural network function is based on the finite difference method, which is highly time-consuming. The third challenge relates to the inefficiency on hyperparameter tuning of PINNs [20, 31, 40]. PINNs use weighted summation of the residual of the equation and the residual of the boundary/initial conditions, where the performance is sensitive to the weight. Therefore, how to properly embed the initial/boundary conditions into the loss design is also critical.

In this work, we aim to address the aforementioned challenges by combining the techniques in deep learning and the reformulation of random vortex method (RVM) [6, 23, 26, 32], which is a mesh-free algorithm to implement the fluid mechanical equations. In detail, instead of solving the Navier-Stokes equations directly, the RVM converts the velocity field in the Navier-Stokes equations to its corresponding probabilistic representation using Feynman-Kac formula, which can be approximated by the Monte-Carlo method. In the formulation of RVM, the spatial derivation in the original formulation of NSE can be approximated by sampling from a stochas-
tic differential equation (SDE) driven by Lévy process. In this way, the RVM can efficiently handle non-smooth and fractional equations. Therefore, we propose the deep random vortex method (DRVM), which utilizes the deep neural network to represent the velocity field and then construct the loss function according to its probabilistic representation in the RVM. There are three attractive advantages of the DRVM.

1. **Easy to handle non-smooth and fractional equations.** DRVM only requires the network function to be integrable in the domain, rather than second order continuously differentiable in PINN. Thus, we can represent the non-smooth solutions and initial/boundary conditions. Furthermore, the calculation of fractional derivatives can also be replaced via the efficient sampling from the Lévy process, which does not increase the algorithm’s complexity.

2. **Easy to implement.** Instead of the loss function in PINN, which is constituted by the equation term, boundary condition term, data term, and other extra information, the initial/boundary conditions are naturally embedded into the solution to SDE and the kernel in the formulation of RVM which results in only one term in the loss function of the DRVM in general. Thus, there are less hyper-parameters in DRVM loss which saves the efforts on fine-tuning the hyper-parameters.

3. **Broader applications.** Compared with the classical RVM, which concentrates on the Cauchy problem, the DRVM has broader applications due to the deep learning regime in which we construct a continuous model that directly maps the spatial-temporal coordinates to the velocity, achieving fast inference compared to the RVM. In this paper, we utilize the proposed method to handle the following three tasks for the Navier-Stokes equations, including the Cauchy problem, parametric solver learning, and inverse problem.

We demonstrate the effectiveness of DRVM by solving Cauchy and inverse problems for various equations, including 2-d and 3-d Lamb-Oseen vortex (with singular initial condition), 2-d fractional Navier-Stokes equations, 2-d Taylor-Green vortex (with periodic boundary condition). For Cauchy problems and parametric solver learning, relative $\ell_2$ errors are around 1% for most equations. Especially for the cases that include singular initial conditions, DRVM significantly outperforms existing PINN method. For inverse problems, we utilize the learned parametric solver to infer the viscosity term $\nu$ in 2-d NSE and diffusion parameter $\alpha$ in fractional NSE, respectively. Compared with the traditional adjoint method, DRVM achieves 2 order of magnitudes improvement on the training time and achieves significantly precise estimates.

This paper is organized as follows. In Section 2, we introduce notations utilized in this paper. In Section 3, we introduce several related works, including Physics-Informed Neural Networks and random vortex method. In Section 4, we take the 2-d Navier-Stokes equations as an instance to introduce our methodology. In Section 5, we report the results of numerical experiments to demonstrate the effectiveness of the DRVM. Finally, we summarize and discuss our method in Section 6.

## 2 Notations

In this section, we introduce the notations in this paper. We utilize bold-faced letters for vectors and matrices. For a matrix $A$, let $A_{i,j}$ denote its $(i,j)$-th entry. For a vector $a$, let $a_i$ and $\|a\|_2$ denote its $i$-th entry and the Euclidean norm, respectively. For a 2-d vector $a$, let $a^\perp := (-a_2, a_1)$ be its orthogonal complement. Let $I_n$ be an $n \times n$ identity matrix. We use $\langle \cdot, \cdot \rangle$ to denote the standard Euclidean inner product between two vectors.
We utilize \( \delta(x) \) to denote Dirac’s delta function follows:

\[
\delta(x) = \begin{cases} 
+\infty & x = 0 \\
0 & x \neq 0 
\end{cases}
\]  

\( \delta(\cdot) \) satisfies that \( \int_{-\infty}^{+\infty} \delta(x) \psi(x) dx = \psi(0) \) for all smoothing test function \( \psi(\cdot) \). We utilize \([\cdot]: \mathbb{R} \to \mathbb{Z}\) to denote the greatest integer function. In this paper, we use relative \( \ell_2 \) error to evaluate the difference between the ground-truth \( x \) and its prediction \( \hat{x} \) defined as \( \| \hat{x} - x \|_2 / \| x \|_2 \).

In 3-d situation, we utilize Einstein notation for simplification, and \( \varepsilon_{ijk} \) is the Levi-Civita symbol. For Navier-Stokes equation, we denote the velocity and vorticity term as \( u \) and \( \omega \), respectively. Furthermore, we utilize \( (u,v) \) and \( (u,v,w) \) to represent the velocity \( u \) in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), respectively.

\section{3 Background}

\subsection{3.1 Random Vortex Formulation}

In this section, we introduce the traditional random vortex method. For simplification, we consider the following 2-d incompressible Navier-Stokes equation defined in the domain \( \Omega \in \mathbb{R}^2 \):

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = \nu \Delta u - \nabla p \quad \text{in } \Omega,
\]

\[
\nabla \cdot u = 0,
\]

where \( u(x,t) \in \mathbb{R}^2 \) is the velocity field, \( p \) is the pressure term and \( \nu > 0 \) is the viscosity. Let \( \omega = \nabla \times u \in \mathbb{R} \) be the vorticity and it evolves according to the following vorticity equation:

\[
\frac{\partial \omega}{\partial t} = -(u \cdot \nabla) \omega + \nu \Delta \omega \quad \text{in } \Omega,
\]

\[
\omega = \nabla \times u.
\]

Eq. (3) has the form of parabolic-type of PDEs, and the probabilistic representation of velocity field in Eq. (2) \([6, 23]\) is proposed according to the Feynman-Kac formula as follows. Suppose \( B_t \) is a 2-d Brownian motion and \( X_t \) is a diffusion process which satisfies the Taylor’s Brownian motion \([23, 38]\) as follows:

\[
dX_t = u(X_t, t) dt + \sqrt{2\nu} dB_t, \quad X_0 = \xi,
\]

where \( \xi \in \Omega \) represents the spatial coordinate in \( \Omega \). Then, the solution to Eq. (2) has the following random vortex form derived by its corresponding probabilistic representation:

\[
u(x,t) = \int_{\Omega} \mathbb{E}[K(x - X_t(\xi))] \omega(\xi, 0) d\xi,
\]

where \( K(\cdot): \Omega \to \mathbb{R}^2 \) is a given kernel function that depends on the boundary condition. The convergence property of the RVM in Eq. (5) was proved in \([9, 23]\). More details about RVM and its corresponding adjoint method can be seen in A.

Fractional NSEs generalize the NSEs described above by introducing the fractional order of the derivatives, which are emerging as a powerful tool for modeling the non-local dynamics and anomalous diffusion \([14]\). The 2-d fractional NSE is given by the following equations:

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nu (-\Delta)^{\alpha/2} u - \nabla p, \quad \text{in } \mathbb{R}^2
\]

\[
\nabla \cdot u = 0,
\]

\]
where the diffusion parameter $\alpha$ is restricted to the interval $(0, 2)$ (Eq.(6) below is equivalent to Eq.(2) when $\alpha = 2$). Notice that fractional Laplacian is on the right-hand side of Eq.(6), which is defined by directional derivatives [22, 28]. Similar to the general Navier-Stokes equations, we can obtain the following vorticity equation:

$$
\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = -\nu(-\Delta)^\frac{\alpha}{2}\omega, \quad \text{in } \mathbb{R}^2
$$

$$\omega = \nabla \times \mathbf{u}.
$$

(7)

For fractional NSEs, its random vortex formulation is similar as above and the only difference is to replace the Brownian motion in the SDE in Eq.(4) by the $\alpha$-stable process [42]. (See Section 5.2 for more details.)

In Table 1, we list the corresponding kernel $K(x)$ and driven noise in the corresponding SDE for NSEs with different dimensions and boundary conditions.

| Dimension | Domain | Kernel Function $K$ | Driven noise |
|-----------|--------|---------------------|--------------|
| 2-d [27]  | $\mathbb{R}^2$ | $\frac{1}{2\pi} \frac{x}{|x|^2}$ | Brownian motion |
| 2-d fractional [14] | $\mathbb{R}^2$ | $\frac{1}{2\pi} \frac{x}{|x|^2}$ | Lévy Process |
| 2-d periodic [38] | $[0, 2\pi]^2$ | $\frac{1}{2\pi} \sum_{k \in \mathbb{Z}^2} \frac{k}{|k|} \sin((k, x))$ | Brownian motion |
| 3-d [27]  | $\mathbb{R}^3$ | $K(x)_{i,j} = -\frac{\epsilon^{ijl} x_l}{4\pi|x|^3}$ | Brownian motion |

Table 1: Four different kinds of Navier-Stokes equations studied in this paper.

### 3.2 Physics-Informed Neural Networks

Using the deep neural network as function approximator to approximate the solution of the PDEs, the frameworks of PINN optimize the neural network via the loss function that contains the residual of the PDEs and the residual of the initial/boundary conditions.

Given the initial data $\{\omega(x^{(i)}, 0), u(x^{(i)}, 0)\}_{i=1}^{N_1}$ with $x^{(i)} \in \Omega$ and the boundary data $\{\omega(\tilde{x}^{(j)}, t^{(j)}), u(\tilde{x}^{(j)}, t^{(j)})\}_{j=1}^{N_2}$ with $\tilde{x}^{(j)} \in \partial\Omega$, the loss function of PINN for vortex-velocity formulation is given as:

$$
\mathcal{L}_{\text{PINN}} = \sum_{i=1}^{N_1} \left( |\omega_{NN}(x^{(i)}, 0) - \omega(x^{(i)}, 0)| + \|u_{NN}(x^{(i)}, 0) - u(x_i, 0)\|^2 \right) + \\
\lambda_1 \left( \frac{\partial \omega_{NN}}{\partial t} + (u_{NN} \cdot \nabla)\omega - \nu\Delta \omega_{NN} \right)^2 + \|\omega_{NN} - \nabla \times u_{NN}\|^2 + \\
\lambda_2 \sum_{j=1}^{N_2} \left( |\omega_{NN}(\tilde{x}^{(j)}, t^{(j)}) - \omega(\tilde{x}^{(j)}, t^{(j)})| + \|u_{NN}(\tilde{x}^{(j)}, t^{(j)}) - u(\tilde{x}^{(j)}, t^{(j)})\|^2 \right),
$$

(8)

where the input of the neural network is $(x, t)$ and the output is $(u_{NN}, \omega_{NN})$. In practice, the value of the loss function will be evaluated on uniformly sampled grid points $\{(x, t)\}$ from the domain $\Omega \times [0, T]$. Furthermore, PINNs represent the velocity field as the partial derivative of some latent function to satisfy the divergence-free condition automatically in Navier-Stokes equation [13, 34]. Two representative implementations on Navier-Stokes equations can be referred to [17, 34].

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1The $\alpha$-stable processes are independently sampled for two spatial dimensions.
4 Methodology: Deep Random Vortex Method

Figure 1: Illustration of deep random vortex method for 2-d Navier-Stokes equation. The total time is divided into $M$ lattices uniformly, and each column for $t_m$ corresponds to a multi-layer perceptron MLP$_m$, where $1 \leq m \leq M$. Starting from input $x$, $u_m(x)$ and $\hat{u}_m(x)$ are obtained via MLP$_m$ and the probabilistic representation in Eq.(5), respectively. The loss function is constructed according to the equivalent relation between $u_m(x)$ and $\hat{u}_m(x)$.

In this paper, the proposed DRVM can be used to handle the following three tasks for the Navier-Stokes equation: Cauchy problem, parametric solver learning, and inverse problem.

4.1 Cauchy problem

Given the initial vortex $\omega_0$, the target of the Cauchy problem is to simulate the fluid’s velocity field governed by the Navier-Stokes equation for all $x \in \Omega$ and $t \in [0, T]$ under given initial and boundary conditions.

To simulate the velocity field governed by Eqs.(2) and (3), we parameterize $u(x, t)$ as a neural network function $u_{NN}(x, t)$. The key idea in our method is to reformulate $u_{NN}(x, t)$ as the combination of $\{u_{NN}(x, s)\}_{s \leq t}$ by Eqs.(4) and (5), and then find the optimal $u_{NN}(x, t)$ which satisfies the equivalent relationship in the RVM.

In detail, we divide the time interval $[0, T]$ into $M$ uniform lattices, i.e., $0 = t_0 < t_1 < \cdots < t_M = T$, and utilize $M$ sub-networks $u_{NN}(x, t_m)$ with parameter $\Theta_{t_m}$ to represent $u(x, t_m)$ for $m \in \{1, 2, \cdots, M\}$ respectively. Given the initial coordinate points $\{\xi^{(i)}\}_{i=1}^{I}$ distributed in $\Omega$ uniformly and the corresponding vorticity field $\{\omega(\xi^{(i)}, 0)\}_{i=1}^{I}$ at $t = t_0$, we initialize $X_{t_0}(\xi^{(i)}) = \xi^{(i)}$ for all $i \in \{1, 2, \cdots, I\}$ and the neural networks’ parameters $\{\Theta_{t_m}\}_{m=1}^{M}$. Then, we consider the following Euler discretization of Taylor’s Brownian motion in Eq.(4) to calculate the path of $X_{t_m}$ for all $\xi^{(i)}$ respectively:

$$X_{t_m}(\xi^{(i)}) - X_{t_{m-1}}(\xi^{(i)}) = u_{NN}(X_{t_{m-1}}(\xi^{(i)}), t_{m-1})\Delta t + \sqrt{2\nu}\Delta B_m,$$

where $\Delta t = t_m - t_{m-1}$ and $\Delta B_m = B_{t_m} - B_{t_{m-1}}$. To approximate the expectation of kernel function in Eq.(5), we utilize Monte-Carlo method to sample $N$ paths independently for each $\xi^{(i)}$.
in the diffusion process Eq. (9), and denote them as \( \{X^n_{t_m}(\xi^{(i)})\}_{i=1}^{N} \) for all \( m \in \{1, 2, \cdots, M\} \) respectively.

After that, we can reformulate \( u_{NN}(x, t_m) \) to its probabilistic representation \( \hat{u}_{NN}(x, t_m) \) via Eq. (5) as follows:

\[
\hat{u}_{NN}(x, t_m) = \frac{|\Omega|}{T} \sum_{i=1}^{I} \sum_{n=1}^{N} \frac{1}{N} K(x - X^n_{t_m}(\xi^{(i)})) \omega(\xi^{(i)}, 0),
\]

where \( |\Omega| \) represents the area of the domain \( \Omega \). The difference between \( u_{NN}(x, t_m) \) and \( \hat{u}_{NN}(x, t_m) \) can be utilized to construct the following loss function:

\[
\mathcal{L}(\Theta) = \sum_{b=1}^{B} \sum_{m=1}^{M} \|u_{NN}(x^{(b)}, t_m) - \hat{u}_{NN}(x^{(b)}, t_m)\|_2^2,
\]

where \( B \) denotes the batch-size per epoch, and \( \Theta = \{\Theta_{t_1}, \Theta_{t_2}, \cdots, \Theta_{t_M}\} \) denote the parameters in each sub-network respectively. We utilize Adam [19] to optimize the parameter \( \Theta \), which is a variant of stochastic gradient descent for training deep models. Fig. 4 and Algorithm 1 illustrate the framework of the DRVM for 2-d Navier-Stokes equation.

The key point is that we use probabilistic representation to define a new loss function, which does not require higher order derivatives of the solution. It is worth mentioning that conventional RVM is applied to obtain \( u(X_{t_m-1}(\xi^{(i)})) \) in Eq. (9) via its probabilistic representation, i.e. Eq. (5):

\[
u(X_{t_m-1}(\xi^{(i)})) = \int_{\Omega} \mathbb{E}[K(X_{t_m-1}(\xi^{(i)}) - X_t(\eta))] \omega(\eta, 0) d\eta,
\]

whose complexity is \( \mathcal{O}(N^2) \). Fortunately, our method allows us to substitute the inference of neural network for time and memory consuming Eq. (12).

**Algorithm 1: 2-d Deep Random Vortex Method (DRVM)**

| Input: Coordinates \( \{\xi^{(i)}\}_{i=1}^{I} \), initial vortex \( \{w(\xi^{(i)}, 0)\}_{i=1}^{I} \), neural network \( \{u_{NN}(x, t_m)\}_{m=1}^{M} \) via Xavier method; |
|------|------|
| 1 Simultaneously initialize the parameter \( \{\Theta_{t_m}\}_{m=1}^{M} \) of the neural networks \( \{u_{NN}(x, t_m)\}_{m=1}^{M} \) via Xavier method; |
| 2 for \( E \) epochs do |
| 3 Initialize \( X_{t_0}(\xi^{(i)}) = \xi^{(i)}; \) |
| 4 Sample \( \{x^{(b)}\}_{b=1}^{B} \) uniformly in \( \Omega; \) |
| 5 \( \mathcal{L} = 0; \) |
| 6 for \( M \) steps do |
| 7 \( X_{t_m}(\xi^{(i)}) = X_{t_{m-1}}(\xi^{(i)}) + u_{NN}(X_{t_{m-1}}(\xi^{(i)}), t_{m-1}) \Delta t + \sqrt{2\nu\Delta t} B_m; \) |
| 8 \( \hat{u}_{NN}(x^{(b)}, t_m) = \frac{1}{N} \sum_{i=1}^{I} \sum_{n=1}^{N} \frac{1}{N} K(x^{(b)} - X^n_{t_m}(\xi^{(i)})) \omega(\xi^{(i)}, 0); \) |
| 9 \( \mathcal{L} = \mathcal{L} + \sum_{b=1}^{B} \|u_{NN}(x^{(b)}, t_m) - \hat{u}_{NN}(x^{(b)}, t_m)\|_2^2; \) |
| 10 Update \( u_{NN} \)’s parameters: \( \Theta_{t_m} = \text{optim.Adam}(\Theta_{t_m}, \nabla \Theta_{t_m} \mathcal{L}) \); for \( m = 1, \cdots, M. \) |

### 4.2 Parametric Solver Learning

Besides solving one specific Navier-Stokes equations, we also apply the DRVM to parametric solver learning, which aims to learn a generalizable model which can output the velocity for a class of Navier-Stokes equations with different parameters simultaneously. As the Reynold number is directly related to the complexity of numerical simulation and turbulence, developing
stable solver that is capable for solving NSE for different $\nu$ is very important. In the 2-d Naiver Stokes equation, we use DRVM to obtain a neural network function $u_{NN}(x, \nu, t)$ which can be generalized to a range of $\nu$ in Eq.(2). To this end, we regard $\nu$ as an input of the neural network, then sample different $\nu$ and $x \in \Omega$ synchronously during the training dynamic. In detail, we consider the following loss function:

$$L(\Theta) = \sum_{p=1}^{P} \sum_{b=1}^{B} \sum_{m=1}^{M} \| u_{NN}(x^{(b)}(\nu_p, t_m)) - \hat{u}_{NN}(x^{(b)}(\nu_p, t_m)) \|_2^2,$$  

where $B$ and $P$ denote the amount of $x$ and $\nu$ sampled in each epoch, respectively. Furthermore, we also learn a parametric solver for different diffusion parameter $\alpha$ in the experiments of 2-d fractional NSE. The rule of forward propagation, the construction of the loss function, and the optimization algorithm are all the same as the method in the Cauchy problem in Section 4.1.

### 4.3 Inverse Problem

Given the initial vortex $\omega_0$ and the dataset $D: \{x^{(d)}, t^{(d)}; u^{(d)}\}_{d=1}^{D}$ generated from a system that obeys the rule of Navier-Stokes equations, the target of the inverse problem is to infer unknown parameters through data in the system. A naive approach is to add the data term directly to the loss function 11, which is consistent with the methodology in PINN [34]. However, this approach has the following two drawbacks. On the one hand, we need to tune the hyper-parameter carefully, which balances the equation term and the data term. When the data is corrupted heavily, the data term may influence the equation term and make the model fail to learn the equation. On the other hand, we have to retrain a neural network again if we need to infer parameters from other data, which is time-consuming.

To address the above problems, instead of directly adding the data term to the loss function, we divide the inverse problem into the following two procedures. First, we train a parametric solver network $u_{NN}(x, \phi, t)$ which can generalize to different $\phi \in \Phi$, where $\phi$ is the unknown parameter in the parameter space $\Phi$. Second, we convert the inverse problem to the following optimization problem via the pre-trained parametric solver learning network:

$$\phi^* = \arg \min_{\phi \in \Phi} \sum_{d=1}^{D} \| u_{NN}(x^{(d)}, \phi, t^{(d)}) - u^{(d)} \|_2^2.$$  

(14)

On the one hand, there is only one term in the loss function 14. Thus we do not need to turn any hyper-parameters in the loss function. In addition, due to the pre-trained parametric solver network, we do not need to worry about the information in Navier-Stokes equations being corrupted by the noise of the dataset. On the other hand, equipped with our pre-trained parametric solver network, we only need to optimize the above low-dimension optimization problem in Eq.(14) via gradient descent algorithm. Thus each inverse problem can be solved in seconds.

### 5 Experiments

In this section, we apply the DRVM to solve the following three problems: Cauchy problem, parametric solver learning, and inverse problem. For these three problems, we conduct experiments on 2-d Lamb-Oseen vortex [27] and 2-d fractional Navier-Stokes equation, respectively. To verify the generality of our method, we also simulate other Navier-Stokes flows, including 2-d periodical Taylor-Green vortex [6] and 3-d Lamb-Oseen vortex. The details of the four equations
will be introduced in the following experimental parts, respectively. In this paper, all the neural networks are initialized via Xavier method [8]. We utilize the relative $\ell_2$ error at the terminal time and the mean relative $\ell_2$ error during the whole time interval to evaluate the performance of our method and denote them as $E_T$ and $E_{[0,T]}$, respectively. To make our results more convincing, we do experiments five times in every setting with five different random number seeds and report the mean value and variance. We adopt Pytorch [29] and TensorFlow [1] to implement DRVM and PINN, respectively. All experiments are implemented based on NVIDIA GeForce RTX 3080Ti 12G, NVIDIA GeForce RTX 3090 24G and NVIDIA TESLA V100 16G. All the run time reported in this paper are evaluated on GeForce RTX 3080Ti 12G.

5.1 2-d Lamb-Oseen Vortex

Firstly, we consider the following two-dimensional vorticity equation:

$$
\frac{\partial \omega(x,t)}{\partial t} + u(x,t) \cdot \nabla \omega(x,t) = \nu \Delta \omega(x,t),
$$

where the velocity field $u(x,t)$ is given by the Biot-Savart Law:

$$
u \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - \xi)\perp}{\|x - \xi\|_2^2} \omega(\xi, t) d\xi.
$$

(16)

When the initial vorticity $\omega(x,0) = \alpha \delta(x)$, where $\delta(x)$ is Dirac’s delta function, we can obtain the unique analytical solution of the Eq.(15) as follows:

$$
\omega(x,t) = \frac{\alpha}{\nu t} G\left(\frac{x}{\sqrt{\nu t}}\right), \quad u(x,t) = \frac{\alpha}{\sqrt{\nu t}} v^G\left(\frac{x}{\sqrt{\nu t}}\right),
$$

(17)

where the vorticity and velocity profiles are given by:

$$
G(\xi) = \frac{1}{4\pi} e^{-\|\xi\|_2^2/4}, \quad v^G(\xi) = \frac{1}{2\pi \|\xi\|_2^2} \left(1 - e^{-\|\xi\|_2^2/4}\right).
$$

(18)

To verify the effectiveness and the properties mentioned before, we conduct experiments on the Cauchy problem, parametric solver learning and inverse problem on the 2-d Lamb-Oseen Vortex respectively.

5.1.1 Cauchy problem

In this section, we simulate the 2-d Lamb-Oseen vortex using our proposed method and PINN respectively. We consider a computational domain of $[-2,2] \times [-2,2]$ and a time horizon of $[0,1]$, the time step is set to be $\frac{1}{40} s$, and the Reynolds number is fixed as 10. The points used to train the neural networks are chosen randomly in the computational domain. When applying our method, we use a fully connected network who has 6 hidden layers with 512 neurons and take ReLU as the activation function. In each epoch, we set batch-size as 2000 and the number of sampled SDE trajectories $N$ as 1000, respectively. We utilize Adam to optimize the neural networks for 10000 epochs with the initial learning rate 0.001, and we decay the learning rate by a factor of 0.5 every 500 epochs.

The relative $\ell_2$ error is reported in Table 2. From the table, we can conclude that DRVM can achieve accurate simulation with average relative $\ell_2$ error 0.43% for $t = T$ and 0.35% for $t \in [0,T]$. To test how the change of $N$ affects the performance of our model, we evaluate our method with different selection of $N$, ranging from $N = 10$ to $N = 10000$. The relative $\ell_2$ errors
and training dynamics are reported in Fig. 2. As we can see, with the increase of the number of sampling points, the error decreases, but the overall fluctuation for \( N > 1000 \) of the error is smaller than 0.1%, which shows the robustness of DRVM over \( N \). The snapshots of the learned velocity fields and corresponding absolute error during \( T \in [0, 1] \) are displayed in Fig. 3. It is worth mentioning that we do not constraint the curl and divergence of \( u_{NN} \) explicitly in Eq.(2) and (3) neither, but the neural network surprisingly learned the meaningful curl and divergence (Fig. 4), which correspond to the incompressible property and vorticity-velocity formulation in Navier-Stokes equations, respectively. This experimental phenomenon indicates that the learned neural network via DRVM has good physical properties.

As for the PINN, we adopt the network architecture in NSFnet [17], which involves 7 hidden layers with 100 neurons and tanh activation functions. We consider the following two types of loss functions. The first is the same as Eq.(8) which is termed as PINN and the second is to add an additional boundary observational data to Eq.(8) which is termed as PINN+. We set \( N_1 = 256 \times 256 \) and \( N_2 = 256 \times 4 \times 40 \) in Eq.(8), and sample \( 256 \times 256 \times 40 \) data points to calculate the residual of equations in each epoch. Furthermore, we set \((\lambda_1, \lambda_2) = (100, 100)\) and \((100, 0)\) for PINN+ and PINN, respectively. The learning rate and number of training epochs are set to be \( 10^{-4} \) and \( 10^4 \), respectively. We decay the learning rate by a factor of 0.1 every 2000 epochs, and report the best performance of PINN+ and PINN during the training dynamics. Besides, as the initial condition is approaches infinity around the origin, the sampling points of PINN start from \( t = 0.025s \) to avoid extremely large values.

From Table 2, we can see both PINN and PINN+ perform poorly on 2-d Lamb-Oseen vortex. There are two main reasons why PINNs fail. On the one hand, PINNs fail to learn this ill-conditioned equation due to the singularity at \( t = 0 \), and this phenomenon is also observed in [20]. On the other hand, PINNs can not embed the initial conditions on \( \mathbb{R}^2 \) integrally and only receive the truncated initial information.

### 5.1.2 Parametric Solver Learning

In this experiment, we aim to learn a parametric solver \( u(x, \nu, t) \) which can generalize to different \( \nu \) ranging from 0.01 to 0.5. We change the input dimension to 3 in the parametric solver learning network, and the other network structure is the same as the setting in the Cauchy problem. We
Figure 3: 2-d Lamb-Oseen vortex: comparisons of velocity field between the exact solution and deep random vortex method from $T = 0.2s$ to $T = 1.0s$ in $[-2, 2]^2$.

Figure 4: 2-d Lamb-Oseen vortex. Left: comparisons of vorticity field between the exact solution and the curl of neural network at $T = 1.0s$ in $[-2, 2]^2$ (relative $\ell_2$ error = 4.15%); Right: divergence of neural network at $T = 1.0s$ in $[-2, 2]^2$ (mean absolute error = 0.0031).
Table 2: **2-d Lamb-Oseen vortex via deep random vortex method**: comparisons of relative $\ell_2$ error between DRVM and PINNs.

| Method | DRVM | PINN | PINN+ |
|--------|------|------|-------|
| $E_T$% | 0.43±0.01 | 41.21±17.50 | 12.24±4.18 |
| $E_{[0,T]}$% | 0.35±0.01 | 47.39±13.81 | 24.98±1.66 |

Table 3: **2-d Lamb-Oseen vortex**: comparisons of relative $\ell_2$ error between different $\nu$ for parametric solver learning.

| $\nu$ | 0.01 | 0.02 | 0.05 | 0.1 | 0.2 | 0.5 |
|--------|------|------|------|-----|-----|-----|
| $E_T$% | 1.53±0.08 | 1.41±0.07 | 1.07±0.04 | 0.91±0.03 | 0.80±0.02 | 0.87±0.05 |
| $E_{[0,T]}$% | 4.00±0.14 | 1.94±0.03 | 1.35±0.01 | 1.06±0.003 | 0.91±0.01 | 0.92±0.001 |

5.1.3 Inverse Problem

Equipped with the learned parametric solver network in the previous experiment, we apply it to handle inverse problems under the following three situations:

1. Given 100 clean data $\{(x^{(i)}, t^{(i)}; u(x^{(i)}, t^{(i)}))\}_{i=1}^{100}$ for training;

2. Given 100 noisy data for training whose labels are perturbed by the additive uncorrelated Gaussian noise $\{(x^{(i)}, t^{(i)}; u(x^{(i)}, t^{(i)}) + \epsilon^{(i)})\}_{i=1}^{100}$, where $\epsilon^{(i)} \sim \mathcal{N}(0, (0.01\|u\|_2)^2 \cdot I_2)$;

3. Given 1000 noisy data for training whose labels are perturbed by the additive uncorrelated Gaussian noise $\{(x^{(i)}, t^{(i)}; u(x^{(i)}, t^{(i)}) + \epsilon^{(i)})\}_{i=1}^{1000}$, where $\epsilon^{(i)} \sim \mathcal{N}(0, (0.1\|u\|_2)^2 \cdot I_2)$.

All data points are sampled uniformly in $[-2, 2]^2$ at $t = 1s$. We utilize Adam to optimize the loss function in the inverse problem for 2000 epochs with the initial learning rate 0.01, and decay the learning rate by a factor of 0.2 every 500 epochs. Due to the fact that RVM is a kind of differentiable solver, we can also utilize adjoint method [30] to solve the inverse problem. More details and experimental setting about adjoint random vortex method (ARVM) can be seen in A. The results of DRVM and ARVM are reported in Table 4, where the results for DRVM and ARVM are denoted using subscripts $D$ and $A$, respectively. From the results in Table 4, DRVM uses only 1% of the time to solve the inverse problem, and obtains remarkably higher accuracy compared with ARVM.

5.1.4 Training Trick: Gradient Stopping

During the training process of DRVM, the neural networks $\{u_{NN}(x, t_j)\}_{j=1}^{m-1}$ are all involved when calculating $X_m(\xi^{(i)})$ in Eq.(9). Thus, the back propagation can become extremely time-consuming.
and memory consuming due to the composition of several neural network functions. To make the training process more efficient and stable, we stop the gradient flows in the computational graphs when updating $X_t$ in the training process, which is a similar method in reinforcement learning [12] and contrastive learning [4].

In this subsection, we conduct the ablation experiments to evaluate the effects of above gradient stopping trick. In detail, we compare the relative error, required time and memory with and without the gradient stopping trick in Table 5, respectively. We set the number of sampling as 1000, and other settings are the same with the Cauchy problem in Section 4.1. We observe that the gradient stopping trick saves around 37% time and 36% memory, and obtains comparable precision compared with the original training technique.

| Gradient Stopping | $E_{[0,T]}$ | $E_T$ | Time (per epoch) | Memory |
|-------------------|-------------|-------|-------------------|--------|
| w/o Gradient Stopping | 0.42±0.004 | 0.31±0.006 | 0.148s | 5987MB |
| w/o Gradient Stopping | 0.43±0.01 | 0.35±0.01 | 0.093s | 3811MB |

Table 5: 2-d Lamb-Oseen vortex: ablation experiments to evaluate the effects of gradient stopping.

### 5.2 2-d fractional PDE

In this section, we consider the following 2-d fractional Navier-Stokes equation:

$$
\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nu(-\Delta)^{\alpha/2} u - \nabla p, \quad \text{in } \mathbb{R}^2
$$

$$\nabla \cdot u = 0,$$

(19)
where the diffusion parameter $\alpha$ is restricted to the interval $(0, 2)$ (Eq.(19) is equivalent to Eq.(2) when $\alpha = 2$). As described in Section 3.1, we can obtain the following vorticity equation:

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = -\nu(-\Delta)^{\frac{\alpha}{2}}\omega, \quad \text{in } \mathbb{R}^2 \quad \omega = \nabla \times u. \quad (20)$$

Compared with the diffusion process of the RVM for the general 2-d Navier-Stokes equation, we only need to replace the Brownian motion with the Lévy process for the fractional equation. In detail, its corresponding diffusion process and probabilistic representation are given by [43]:

$$dX_t = u(X_t, t)dt + (2\nu)^{\frac{\alpha}{2}}dL^\alpha_t,$$

$$u(x, t) = \int_{\mathbb{R}^2} \mathbb{E}[K(x - X_t(\xi))]\omega(\xi, 0)d\xi. \quad (21)$$

where $L^\alpha_t$ is the 2-d $\alpha$-stable Lévy process, and $K(x) := \frac{1}{2\pi \|x\|^2}x^\perp \|x\|^2$.

### 5.2.1 Problem Setups

For the fractional NSEs, we consider the flow with the non-smooth initial condition given by

$$(\xi^{(1)}, \omega(\xi^{(1)}, 0)) = ((0.5, 0.5), -0.2), (\xi^{(2)}, \omega(\xi^{(2)}, 0)) = ((-0.5, 0.5), -0.2), (\xi^{(3)}, \omega(\xi^{(3)}, 0)) = ((-0.5, -0.5), -0.2), (\xi^{(4)}, \omega(\xi^{(4)}, 0)) = ((0.5, -0.5), -0.2), (\xi^{(5)}, \omega(\xi^{(5)}, 0)) = ((0.0, 0.0), 0.2),$$

and the Reynolds number is fixed as 10. The computational domain is selected to $[-2, 2] \times [-2, 2]$ and a time horizon of $[0, 1]$. We utilize the fine-grained RVM (see A) as the ground truth to evaluate our method’s performance. In detail, we divide the time interval into 200 lattices and average results of 1000k independent paths generated by Lévy processes. We utilized CUDA to accelerate the RVM algorithm, and it takes NVIDIA GeForce RTX 3080Ti 4.5h to generate each ground truth. When applying our method, we utilize the same network architecture as shown in Section 5.1.1, which is a fully connected ReLU network with 6 hidden layers and 512 neurons per layer. Moreover, we uniformly divide the total time into 40 time intervals, i.e., $M = 40$.

### 5.2.2 Cauchy problem

The diffusion parameter $\alpha$ is essential to the property of the fractional equation. Thus we evaluate our simulation algorithm under different diffusion parameters $\alpha$, ranging from 0.5 to 2. In each epoch, we set batch-size as 2000 and the number of sampling $N$ as 1000, respectively. We utilize Adam to optimize the neural networks for 10000 epochs with the initial learning rate 0.001, and decay the learning rate by a factor of 0.5 every 500 epochs. Each epoch needs around 0.23s to be trained. The relative $\ell_2$ errors are reported in Table 6, which increase while $\alpha$ decreases, and the error at $T = 1s$ is smaller than the whole interval. To have a deeper understanding to the influence of $\alpha$, we plot the landscapes of learned fractional equations from $T = 0.025s$ to $T = 1.00s$ in Fig. 5. On the one hand, surfaces of equations become more singular when $\alpha$ is getting smaller. Thus training the neural networks will become more difficult, which is consistent with the frequency principle of neural networks studied in [33, 41]. On the other hand, the landscapes become smooth as time goes on. Thus the error at the terminal time is relatively smaller than the whole interval. Furthermore, we study the effects of the number of sampling paths in Fig. 6. It can be seen that the error decreases if we enlarge the number of sampling $N$ increases in general.
| $\alpha$ | 0.50   | 0.75   | 1.00   | 1.25   | 1.50   | 1.75   | 2.00   |
|--------|--------|--------|--------|--------|--------|--------|--------|
| $E_{T}\%$ | 13.02±1.05 | 5.97±0.43 | 2.32±0.27 | 1.15±0.18 | 0.69±0.07 | 0.81±0.09 | 0.60±0.05 |
| $E_{[0, T]}\%$ | 12.11±0.27 | 7.84±0.28 | 4.15±0.05 | 3.00±0.05 | 1.71±0.05 | 1.10±0.01 | 0.87±0.01 |

Table 6: 2-d fractional Navier-Stokes equation: comparisons of relative $\ell_2$ error between different $\alpha$.

Figure 5: 2-d fractional Navier-Stokes equation: comparisons of learned landscapes between different $\alpha$ from $T = 0.025s$ to $T = 1.00s$ in $[-2, 2]^2$. 
5.2.3 Parametric Solver Learning

In real applications, inferring the $\alpha$ in the fractional NSEs from observation data is also important. To assist the inference on $\alpha$, we aim to learn a parametric solver $u(x, \alpha, t)$ which can generalize to different $\alpha$ ranging from 1.00 to 2.00. Due to the extra input $\alpha$, the input dimension changes to 3 in the parametric solver learning network, and the other network structure is the same as the setting in the Cauchy problem. We sample 500 different $x$ and 20 different $\alpha$ per epoch, and the range of $\alpha$ is set in [0.08, 2.20] to obtain good generalization properties at boundary. We utilize Adam to optimize the neural networks for 10000 epochs with the initial learning rate 0.001, and decay the learning rate by a factor of 0.5 every 500 epochs. It needs around 2.65s for training each epoch. The results of parametric solver learning can be seen in Table 6.

We observe that the relative $\ell_2$ errors increase while $\alpha$ decreases, the same as the situations in the Cauchy problem.

| $\alpha$ | 1.00   | 1.25 | 1.50  | 1.75  | 2.00  |
|----------|--------|------|-------|-------|-------|
| $E_T$ %  | 4.21±0.29 | 1.90±0.04 | 1.67±0.06 | 1.41±0.05 | 1.35±0.08 |
| $E_{[0,T]}$ % | 7.09±0.07 | 3.69±0.08 | 2.66±0.04 | 2.99±0.05 | 2.54±0.10 |

Table 7: 2-d fractional Navier-Stokes equation: comparisons of relative $\ell_2$ error between different $\alpha$ for parametric solver learning.

5.2.4 Inverse Problem

Similar as the settings described in Section 5.1.3, we apply the learned parametric solver network with respect to $\alpha$ to handle inverse problems under the following three situations:

1. Given 100 clean data $\{(x^{(i)}, t^{(i)}; u(x^{(i)}, t^{(i)}))\}^{100}_{i=1}$ for training;

2. Given 1000 noisy data for training whose label is perturbed by the additive uncorrelated Gaussian noise $\{(x^{(i)}, t^{(i)}; u(x^{(i)}, t^{(i)}) + \epsilon^{(i)})\}^{1000}_{i=1}$, where $\epsilon^{(i)} \sim \mathcal{N}(0, (0.01\|u\|_2)^2 \cdot I_2)$;
3. Given 2000 noisy data for training whose label is perturbed by the additive uncorrelated Gaussian noise \( \{ (x^{(i)}, t^{(i)}; u(x^{(i)}, t^{(i)}) + \epsilon^{(i)}) \}_{i=1}^{2000} \), where \( \epsilon^{(i)} \sim \mathcal{N}(0, (0.1 \| u \|_2^2 \cdot I_2) \).

All data points are sampled uniformly in \([-2, 2]^2\) at \( T = 1 \) s. We utilize Adam to optimize the loss function in the inverse problem for 2000 epochs with the initial learning rate 0.01, and decay the learning rate by a factor of 0.2 every 500 epochs. The results are reported in Table 8. We observe the relative errors from 0.001 to 0.01 under low noise levels, which are sufficiently small. Moreover, the relative errors are around 0.05 when the noise is high, indicating that more data is required to obtain a more precise estimate. It is worth mentioning that the total time for optimizing the inverse loss is less than 3s for all situations.

| \( \alpha \) | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | Time |
|----------------|------|------|------|------|------|------|
| \( \hat{\alpha} \) | 0.99±0.02 | 1.26±0.003 | 1.50±0.004 | 1.75±0.002 | 1.99±0.01 | 1.81s |
| \( E \% \) | 1.25±0.81 | 0.73±0.24 | 0.15±0.23 | 0.07±0.10 | 0.49±0.33 |

**Given 1000 noisy data with 1% Gaussian noise**

| \( \hat{\alpha} \) | 0.99±0.02 | 1.26±0.01 | 1.49±0.01 | 1.75±0.01 | 1.98±0.03 | 2.07s |
| \( E \% \) | 1.35±1.09 | 0.57±0.74 | 0.61±0.59 | 0.65±0.39 | 1.19±1.08 |

**Given 2000 noisy data with 10% Gaussian noise**

| \( \hat{\alpha} \) | 1.03±0.07 | 1.27±0.08 | 1.49±0.10 | 1.72±0.10 | 1.93±0.12 | 2.61s |
| \( E \% \) | 4.87±5.48 | 4.46±4.81 | 5.85±2.05 | 3.84±4.75 | 5.03±3.98 |

Table 8: **2-d fractional Navier-Stokes equation**: comparisons of the relative error of inverse problems under various situations. \( \alpha, \hat{\alpha} \) and \( E \) represent the ground truth, estimator and relative error, respectively. The total training time is reported in the last column.

## 5.3 2-d Taylor-Green vortex

In this section, we evaluate DRVM on Navier-Stokes equation with periodic boundary condition. The Taylor–Green vortex [6] is an exact solution to 2-d incompressible Navier–Stokes equation with periodic boundary condition in the domain of \((x_1, x_2) \in [0, 2\pi]^2\), where its velocity \((u, v)\) and vorticity \(\omega\) are given by:

\[
\begin{align*}
  u(x, t) &= \cos x_1 \sin x_2 e^{-2\nu t}, \\
  v(x, t) &= -\sin x_1 \cos x_2 e^{-2\nu t}, \\
  \omega(x, t) &= -2 \cos x_1 \cos x_2 e^{-2\nu t},
\end{align*}
\]  

and the kernel functional of probabilistic representation for periodical equation in \([0, 2\pi]^2\) is given by:

\[
K(x) = \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}^2/\{0\}} \frac{k^\perp}{\|k\|^2_2} \sin(\langle k, x \rangle).
\]  

When solving periodical Navier-Stokes equations in periodical domain \( \Omega = [0, b_1] \times [0, b_2] \), we sample data points in \( \Omega \) and then train the network. However, paths of \( X_t \) in Eq.(9) can be out of \( \Omega \) due to the randomness of the Brownian motion, and the neural network has never been trained out of \( \Omega \). To reduce the error caused by the above concerns, we utilize the periodical
the following boundary loss:

\[ f_{pp}(x) = \left( x_1 - \frac{x_1}{b_1}, x_2 - \frac{x_2}{b_2} \right). \]  

(24)

Due to the periodic property of the equation, we have \( u(f_{pp}(x), t) = u(x, t) \) for all \( x \) and \( t \). Thus, the periodical pre-processing operator project the SDE’s output inside the domain \(\Omega\) without inducing extra error.

In deep random vortex simulations for the Taylor-Green vortex, we divide the domain \(\Omega\) into \(64 \times 64\) grids, and the time interval into 40 uniform lattices. The Reynolds number is set to be 1. Notice that the kernel function in Eq. (23) is represented by infinite series, thus we truncate as 100 and the number of sampling \(N\) as 2, respectively. We utilize Adam to optimize the neural networks for 20000 epochs with the initial learning rate 0.0005, and decay the learning rate by a factor of 0.5 every 1000 epochs. The snapshots of the learned velocity fields and corresponding absolute error during \(T \in [0, 1]\) are displayed in Fig. 7. As shown, our method can be used to obtain accurate solutions for the Taylor-Green vortex, and the main error is distributed at the domain boundary. Furthermore, we study the effects of the number of temporal lattices and periodical pre-processing technique in Table 9. We observe that the error decreases as the number of time intervals \(M\) increases in general, and the periodical pre-processing technique improves the performance of DRVM significantly with no additional computational cost.

| \(M\) | 5 | 10 | 20 | 40 | 100 |
|-------|---|----|----|----|-----|
| \(E_T\) \(|\%|\) | 2.43±0.15 | 1.90±0.06 | 1.71±0.09 | 1.71±0.04 | 1.67±0.04 |
| \(E_T\) \(|\%|\) w/o pp | 2.68±0.11 | 2.08±0.15 | 1.88±0.05 | 1.81±0.06 | 1.79±0.06 |
| \(E_{[0,T]}\) \(|\%|\) | 1.85±0.05 | 1.16±0.07 | 0.96±0.04 | 0.99±0.10 | 0.94±0.03 |
| \(E_{[0,T]}\) \(|\%|\) w/o pp | 1.95±0.01 | 1.28±0.09 | 1.11±0.04 | 1.01±0.04 | 1.04±0.06 |
| Time | 0.052s | 0.103s | 0.206s | 0.412s | 1.028s |
| Time w/o pp | 0.052s | 0.102s | 0.204s | 0.408s | 1.008s |

Table 9: 2-d Taylor-Green vortex: comparisons of relative \(\ell_2\) error between different number of time intervals \(M\) and ablation experiments to evaluate the effects of periodical pre-processing (pp) technique.

As for PINNs, we choose a fully connected \(tanh\) neural network which has 7 hidden layers with 500 neurons per layer [17]. We choose Adam optimizer with initial learning rate 0.0001 and total epochs 20000, and decay the learning rate by a factor of 0.1 every 10000 epochs. We sample \(N_1 = 64 \times 64\) points in \(\Omega\) at \(t = 0\), \(N_2 = 64 \times 4 \times 20\) data points in \(\partial\Omega\) during \(t \in (0, T]\) and \(64 \times 64 \times 20\) data points to calculate the residual of equations in each epoch. Due to no boundary data provided for PINN, we embed the periodical conditions of \(\Omega\) by constraining the left(lower) boundary to equal to the right(upper) one, i.e., replacing the third term in loss function 8 with the following boundary loss:

\[
L_{PINN}^{bound} = \lambda \sum_{j=1}^{N_2} \left[ |\omega_{NN}(\hat{x}_r^{(j)}, t^{(j)}) - \omega_{NN}(\hat{x}_r^{(j)}, t^{(j)})| + |\omega_{NN}(\hat{x}_r^{(j)}, t^{(j)}) - \omega_{NN}(\hat{x}_{up}^{(j)}, t^{(j)})| + \right. \\
+ \left. \|u_{NN}(\hat{x}_r^{(j)}, t^{(j)}) - u_{NN}(\hat{x}_r^{(j)}, t^{(j)})\|^2 + \|u_{NN}(\hat{x}_{up}^{(j)}, t^{(j)}) - u_{NN}(\hat{x}_{up}^{(j)}, t^{(j)})\|^2 \right],
\]  

(25)
Figure 7: **2-d Taylor-Green vortex**: comparisons of velocity field between the exact solution and deep random vortex method from $T = 0.2s$ to $T = 1.0s$ in the periodical domain $[0, 2\pi]^2$. 
where $\hat{x}_l$, $\hat{x}_r$, $\hat{x}_{low}$ and $\hat{x}_{up}$ denote data points sampled from the left, right, lower and upper boundary of $\Omega$, respectively. Furthermore, we set $(\lambda_1, \lambda_2) = (1, 1)$ for both PINN+ and PINN. It can be seen in Table 10 that PINN fails to simulate the velocity field because there’s no boundary information provided. It is also worth mentioning that DRVM is comparable with PINN+, although the loss function of DRVM does not include the boundary data during training the neural network, we sample data points in the domain $[-N, N]^2$ each epoch, we set batch-size as 100 and the number of sampling with six hidden layers and 512 neurons per layer to approximate 3-d Lamb-Oseen equations. In this experiment, we aim to simulate the velocity field for 3-d Lamb-Oseen vortex [27], which is the 3-d incompressible Navier-Stokes equation with initial vorticity $(0, 0, \delta(x_1)\delta(x_2))$ for $x \in \mathbb{R}^3$, and its corresponding exact solution of velocity field is given by:

$$u(x, t) = \frac{1}{2\pi} \left( \frac{-x_2, x_1, 0}{x_1^2 + x_2^2} \right) \left( 1 - \exp \left( -\frac{x_1^2 + x_2^2}{4\nu t} \right) \right), \quad \text{in } \mathbb{R}^3. \tag{26}$$

The diffusion process for 3-d Navier-Stokes is given in [32] as follows:

$$dX_i(t) = u(X_i(t), t)dt + \sqrt{2\nu}dB_i, \tag{27}$$

where $B_i$ denotes the 3-d Brownian motion. Then, we can obtain its corresponding probabilistic representation as follows [32]:

$$u(x, t) = \int_{\mathbb{R}^3} E \left[ \varepsilon^{ijk}X_i(\xi)k - x_i \right] \left( G(\xi, t, 0) \right)_{k,j} \omega(\xi, 0)j d\xi, \tag{28}$$

where $G(\xi, t, s) \in \mathbb{R}^{3 \times 3}$ is a symmetric matrix. The evolution of $G$ obeys the following dynamic system:

$$\frac{d}{ds} [G(x, t, s)]_{i,j} = - [G(x, t, s)]_{i,p} \int_{\mathbb{R}^3} E \left[ H^p_{j,o} (X_s(x) - X_s(\xi)) [G(\xi, s, 0)]_{o,\beta} \right] \omega(\xi, 0)\beta d\xi, \tag{29}$$

where $[G(\xi, t, t)]_{i,j} = \delta(i, j)$, $H^k_{j,i}(x) := \frac{3}{2} \frac{x_i}{x_j^2} (\varepsilon^{kl}i_k^j + \varepsilon^{jl}i_k^k)$. Compared to the 2-d cases, the calculation on the velocity is more complex due to the existence of $G$. We put the detailed algorithm for 3-d DRVM in Algorithm 2.

In our experiment, the vorticity field is initialized at $\xi^{(i)} = (0, 0, i/2)$ with $\omega^{(i)} = (0, 0, 0.5)$ for $-20 \leq i \leq 20$, and the Reynolds number is fixed as 2. We utilize the ReLU neural network with six hidden layers and 512 neurons per layer to approximate 3-d Lamb-Oseen equations. In each epoch, we set batch-size as 100 and the number of sampling $N$ as 10, respectively. When training the neural network, we sample data points in the domain $[-2, 2] \times [-2, 2] \times [-10, 10]$ to guarantee the neural networks can learn the information around the initial coordinates $\xi^{(i)}$. We utilize Adam to optimize the neural networks for 20000 epochs with the initial learning rate

| $E_p\%$ | DRVM | PINN | PINN+ |
|---------|------|------|-------|
| 1.67±0.04 | 102.08±47.12 | 1.78±0.40 |
| 0.94±0.03 | 35.31±14.94 | 0.53±0.12 |

Table 10: 2-d Taylor-Green vortex: comparisons of relative $\ell_2$ error between DRVM and PINNs.

### 5.4 3-d Lamb-Oseen Vortex

In this experiment, we aim to simulate the velocity field for 3-d Lamb-Oseen vortex [27], which is the 3-d incompressible Navier-Stokes equation with initial vorticity $(0, 0, \delta(x_1)\delta(x_2))$ for $x \in \mathbb{R}^3$, and its corresponding exact solution of velocity field is given by:

This experiment uses Adam to optimize the neural networks for 20000 epochs with an initial learning rate.
0.0005, and decay the learning rate by a factor of 0.5 every 1000 epochs. The snapshots of the learned velocity fields and corresponding absolute error during $T \in [0, 1]$ are displayed in Fig. 5.4. Furthermore, we study the effects of the number of temporal lattices on the relative $\ell_2$ error in Table 11. Due to the singular initialization in the 3-d Lamb-Oseen vortex, the surfaces of equations change faster as time goes on. Thus unlike 2-d Taylor-Green vortex, the relative errors are more sensitive to the number of temporal lattices.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$M$ & 5 & 10 & 20 & 40 & 100 \\
\hline
$E_T \%$ & 9.21±0.07 & 3.37±0.09 & 1.58±0.04 & 1.87±0.07 & 2.38±0.09 \\
\hline
$E_{[0,T]} \%$ & 21.71±0.04 & 12.69±0.04 & 6.83±0.04 & 4.01±0.05 & 2.93±0.04 \\
\hline
Time (per epoch) & 0.014s & 0.028s & 0.052s & 0.113s & 0.278s \\
\hline
\end{tabular}
\caption{3-d Lamb-Oseen vortex. comparisons of relative $\ell_2$ error and computational time between different number of time intervals $M$.}
\end{table}

\begin{algorithm}
\caption{3-d Deep Random Vortex Method (DRVM)}
\begin{algorithmic}
\STATE \textbf{Input:} Coordinates $\{\xi^{(i)}\}_{i=1}^{I}$, initial vortex $\{w(\xi^{(i)},0)\}_{i=1}^{I}$, neural network $\{u_{NN}(x,t_m)\}_{m=1}^{M}$.
\STATE Simultaneously initialize the parameter $\{\Theta_{0,m}\}_{m=1}^{M}$ of the neural networks $\{u_{NN}(x,t_m)\}_{m=1}^{M}$ via Xavier method;
\STATE Initial $\tilde{G}^n(\xi^{(i)},t_m,t_m) = I_3$ and $\tilde{X}^n_{t_m}(\xi^{(i)}) = \xi^{(i)}$ for all $m$;
\FOR {epochs $E$}
\STATE Initial $X^0_{t_0}(\xi^{(i)}) = \xi^{(i)}$ and $G^n(\xi^{(i)},t_m,t_m) = I_3$ for all $m$;
\STATE Sample $\{x^{(b)}\}_{b=1}^{B}$ uniformly in $\Omega$;
\STATE $\mathcal{L} = 0$;
\FOR {M steps $m$ $\text{from} M - 1$ $\text{to} 0$}
\STATE $X^n_{t_m}(\xi^{(i)}) = X^n_{t_m-1}(\xi^{(i)}) + u_{NN}(X^n_{t_m-1}(\xi^{(i)}), t_m-1)\Delta t + \sqrt{2\varepsilon}\Delta B_m$;
\ENDFOR
\ENDFOR
\STATE $\tilde{u}_{NN}(x^{(b)},t_m) = \frac{[\mathcal{L}]_{x^{(b)},t_m}}{\mathcal{L}} + \frac{\sum_{b=1}^{B}k}{4\pi||\hat{X}^n_{t_m}(\xi^{(i)}) - x^{(b)}||^2} \left[ G^n(\xi^{(i)},t_m,0)k,j\omega(\xi^{(i)},0) \right]$;
\STATE $\mathcal{L} = \mathcal{L} + \sum_{b=1}^{B}||u_{NN}(x^{(b)},t_m) - \tilde{u}_{NN}(x^{(b)},t_m)||^2$;
\STATE Update $\tilde{X}^n_{t_m}(\xi^{(i)}) = X^n_{t_m}(\xi^{(i)})$ and $\tilde{G}^n(\xi^{(i)},t_m,t_m') = G^n(\xi^{(i)},t_m,t_m')$ for all $m$ and $m'$;
\STATE Update $u_{NN}$’s parameters: $\Theta_{m} = \text{optim.Adam}(\Theta_{m}, \nabla_{\Theta_{m}} \mathcal{L})$; for $m = 1, \cdots, M$.
\end{algorithmic}
\end{algorithm}

6 Conclusion

In this paper, we propose DRVM for simulating the fluids and inferring unknown parameters of Navier-Stokes equations. DRVM utilizes the probabilistic representation in random vortex formulation of NSE and substitutes Monte-Carlo sampling for the derivative calculation. Thus, DRVM can solve non-smooth and fractional Navier-Stokes equations efficiently, which expands
Figure 8: **3-d Lamb-Oseen vortex**: comparisons of velocity field between the exact solution and deep random vortex method from $T = 0.2s$ to $T = 1.0s$ in the periodical domain $[-2, 2]^3$. 
the application of the deep learning method in fluid mechanics. The numerical experiments on various equations verify our algorithm. However, DRVM still has some limitations. Firstly, the convergence rate of DRVM is non-trivial due to the non-convex propriety of neural networks and the stopping gradient technique. Secondly, we do not consider the Navier-Stokes equations with external force, which is a critical situation in control. We will investigate both the convergence of DRVM and apply DRVM to the NSEs with external force in future work.

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A Random Vortex Method

In this section, we take the 2-d Navier-Stokes equation as an example to introduce random vortex method (RVM). Recall the Eluer discretion in 4:

\[ X_{tm}(\xi^{(i)}) - X_{tm-1}(\xi^{(i)}) = u(X_{tm-1}(\xi^{(i)}), t_{m-1}) \Delta t + \sqrt{2\nu} \Delta B_m. \]  

(30)

RVM utilizes the probabilistic representation of velocity field in Eq.(5) to calculate \( u(X_{n}'(\xi^{(i)}), t) \), i.e.:

\[ u(X_{n}'(\xi^{(i)}), t) = \frac{|\Omega|}{I} \sum_{i=1}^{I} \sum_{n=1}^{N} \frac{1}{N} K(X_{n}'(\xi^{(i)}) - X_{n}(\xi^{(i)})) \omega(\xi^{(i)}, 0). \]  

(31)

For other points \( x \) out of the coordinate points \( \{\xi^{(i)}\}_{i=1}^{I} \), RVM calculates \( u(x, t) \) as follows:

\[ u(x, t) = \frac{|\Omega|}{I} \sum_{i=1}^{I} \sum_{n=1}^{N} \frac{1}{N} K(x - X_{n}(\xi^{(i)})) \omega(\xi^{(i)}, 0). \]  

(32)

Due to RVM is a kind of differentiable solver, we can utilize adjoint method to solve inverse problem. Given the initial vortex \( \omega_0 \) and the dataset \( D = \{x^{(d)}, t^{(d)}, u^{(d)}\}_{d=1}^{D} \) generated from a system that obeys the rule of Navier-Stokes equations, adjoint random vortex method (ARVM) considers the following optimization problem:

\[ \phi^* = \arg \min_{\phi \in \Phi} \sum_{d=1}^{D} \|u_{RVM}(x^{(d)}, \phi, t^{(d)}) - u^{(d)}\|^2. \]  

(33)

The calculation of \( u_{RVM} \) is differentiable, thus we directly utilize Adam to find the optimum \( \phi^* \). In detail, we optimize the loss function for 2000 epochs with the initial learning rate 0.01, and decay the learning rate by a factor of 0.2 every 500 epochs. To distinguish different \( \nu \) with small orders of magnitude and obtain stable results, we feed \( \log \sqrt{\nu} \) to the ARVM rather than \( \nu \) directly.