On the Sum-Rate Capacity of Poisson Multiple Access Channel with Non-Perfect Photon-Counting Receiver

Zhimeng Jiang, Chen Gong, Guanchu Wang and Zhengyuan Xu

Abstract

We first investigate two-user nonasymmetric sum-rate Poisson capacity with non-perfect photon-counting receiver under certain condition and demonstrate three possible transmission strategy, including only one active user and both active users, in sharp contrast to Gaussian multiple access channel (MAC) channel. The two-user capacity reduction due to photon-counting loss is characterized compared with that of continuous Poisson channel. We then study the symmetrical case based on two different methods, demonstrating that the optimal duty cycle for two users must be the same and unique, and the last method maybe can extend to multiple users. Furthermore, we analyze the sum-Rate capacity of Poisson multiple input single output (MISO) MAC. By converting a non-convex optimization problem with a large number of variables into a non-convex optimization problem with two variables, we show that the sum-rate capacity of the Poisson MISO MAC is equivalent to that of SISO under certain condition.

Key Words: Optical wireless communications, MISO, multiple access, capacity, dead time, finite sampling rate

I. INTRODUCTION

Due to the potential large bandwidth and no electromagnetic radiation, optical wireless communication shows great promise for the future wireless communications [2]. On some specific occasions where the conventional RF is prohibited and direct link transmission cannot...
be guaranteed, non-line-of-sight (NLOS) optical scattering communication, typically in the ultra-violet spectrum, can be adopted to guarantee communications requirement [3]. For the NLOS communication, the transmitter and receiver are not required to be perfectly aligned, which expands the application range beyond the LOS links. Hence, it is difficult to detect the received signals using a conventional continuous waveform receiver, such as photodiode (PD) and avalanche photodiode (APD). Instead, a photon-counting receiver is widely deployed, including photomultiplier tube (PMT) as well as single photon avalanche diode (SPAD).

Poisson channel, whereby the arrival of photons is recorded by photon-sensitive devices incorporated in the receivers, is often used to model free-space optical (FSO) and optical scattering communication. For perfect photon-counting receiver, recent works mainly focus on point-to-point capacity under various scenarios, such as single transmitter [4], [5], multiple transmitters [6] in continuous-time [7] and discrete-time [8], [9], [10]. For multiple users scenario, recent works focus on the Poisson broadcast channel [11], the Poisson multiple-access channel (MAC) [12], and the Poisson interference channel capacity [13]. Moreover, the communication system optimization and corresponding signal processing [14], [15], [16], [17] have also been extensively studied.

Considering the practical characterization of photon-counting devices, perfect photon-counting receiver assumption is impractical. For example, a typical photon-counting receiver applied in many optical communication scenarios [18] includes a photomultiplier tube (PMT) as well as the subsequent sampling and processing blocks [19] or single photon avalanche diode (SPAD) with quenching circuit. Specifically, when a photon is captured by receiver, the square pulses subsequently generated by pulse-holding circuits typically have positive width that incurs dead time effect [20], where a photon arriving during the pulse duration of the previous photon cannot be detected due to the merge of two pulses. The longest time difference of two unrecognized photons is defined as dead time. The photon-counting system with dead time effect for infinite sampling rate and finite sampling rate with shot noise have been investigated in optical wireless communication [21], [22], which shows sub-Poisson distribution of photon counting. In addition, the achievable rate for on-off keying (OOK) modulation and capacity with non-perfect photon-
counting receiver are investigated in [23], [24]. However, The multiuser capacity with non-perfect receiver is still unknown.

In this paper, we investigate the sum-rate capacity of multiple input single output (MISO) multiple access channel (MAC) with non-perfect receiver, assuming negligible electrical thermal noise and shot noise. We first study two-user single-transmitter nonsymmetric sum-rate capacity, which the peak power of two users are not necessary the same, and show that the optimal input signal is two-level piece-wise constant waveform. This scenario naturally arises in multiuser optical communications when the transmitters have different distances to the receiver or have different transmission powers. Similar to work [25], we resort to Karush-Kuhn-Tucker (KKT) condition to solve the non-convex duty cycle optimization problem and obtain at most four candidate solutions wherein two candidate solutions corresponds to the cases of only one active user. We further investigate the optimal transmission strategy, corresponding to different possible solutions, for different peak power of each user and give the sufficient condition of each transmission strategy. In particular, we investigate two-user symmetric Poisson channel by two methods based on KKT condition and majorization, both demonstrating that the optimal duty cycle of two users are the same and unique. The last method based on majorization maybe can extend to the case of multiple symmetric users.

We then extend the study to Poisson MAC with multiple transmitters at each user. Similarly to the Poisson SISO-MAC, the complex continuous-input discrete-output Poisson MAC can be converted to a discrete-time binary-input binary-output Poisson MAC. However, the joint distribution problem is still challenging since the exponential parameters $2^{J_1} + 2^{J_2}$, where $J_1$ and $J_2$ are the total number of transmitters for user 1 and 2, respectively. We show that Poisson MISO-MAC capacity equals to Poisson SISO-MAC capacity under certain condition by two steps. The first step is to optimize joint distribution of each user given duty cycle of each transmitter and reduce the dimension from $2^{J_1} + 2^{J_2}$ to $J_1 + J_2$. The last step is to optimize duty cycle of each transmitter and further reduce the dimension from $J_1 + J_2$ to 2. The key ingredient is to show that, all antennas at each transmitter being simultaneously on or off achieve the optimality.

The remainder of the paper is organized as follows. Section II describes the model under con-
sideration. Section III and Section IV analyzes the Poisson nonsymmetric and symmetric SISO-MAC capacity, respectively. Section V analyzes the Poisson Poisson MISO-MAC. Numerical analysis is presented in Section VI and concluding remarks are presented in Section VII.

II. SYSTEM MODEL

We introduce the following notations that will be used throughout this paper. Random variables and vectors are denoted by upper-case letters and bold upper case letters, respectively. We use notation \( X^j_t \) to denote a sequence of random variables \( \{X_i, X_{i+1}, \cdots, X_j\} \); and for \( i = 1 \), we use \( X_{[j]} = \{X_1, \cdots, X_j\} \). A continuous time random process \( \{\Lambda(t), a \leq t \leq b\} \) is denoted in short by \( \Lambda^b_a \), when \( a = 0 \), we use \( \Lambda^b = \{X(t), 0 \leq t \leq b\} \). Realizations of random variables and random processes, denoted in lowercase letters, follow the same convention.

Consider multiple users communicating to a single non-perfect receiver. Assume \( M \) users, where user \( m, m = 1, \cdots, M \), is equipped with \( J_m \) transmitters. Let \( \Lambda_{mj}(t) \) denotes the \( \mathbb{R}^+_0 \)-valued photon arrival rate at time \( t \) from the \( j^{th} \) transmitter of the \( m^{th} \) user, and \( Y(t) \) denote the Poisson photon arrival process observed at the receiver and

\[
Y(t) = \mathcal{P}\left( \sum_{m=1}^{M} \sum_{j=1}^{J_m} \Lambda_{mj}(t) + \Lambda_0 \right),
\]

where \( \Lambda_0 \) is the background radiation, and \( \mathcal{P}(\cdot) \) is the Poisson process that records the timing instants and number of photon arrivals. In particular, for any time interval \( [t - \tau, t] \), the probability of \( k \) photons arriving at the receiver is given by

\[
\mathbb{P}\{Y(t) - Y(t - \tau) = k\} = \frac{1}{k!} e^{-X_t} (X_t)^k, \quad k = 0, 1, \cdots,
\]

where \( X_t = \sum_{m=1}^{M} \sum_{j=1}^{J_m} \int_{t-\tau}^{t} \Lambda_{mj}(t') dt' \), the arrival rate \( \Lambda \) is given by \( \Lambda = \frac{P}{h\nu_0} \), where \( P, h \) and \( \nu_0 \) denote the transmitted optical power, the Planck’s constant and the optical spectrum frequency, respectively, such that the energy per photon is given by \( h\nu_0 \). Thus, the photon arrival rate \( \Lambda_{mj}(t) \) must satisfy the following peak power constraint:

\[
0 \leq \Lambda_{mj}(t) \leq A_{mj},
\]

where \( A_{mj} \) is related to the corresponding maximum power allowed by the \( j^{th} \) transmitter of the \( m^{th} \) user. In practice, LEDs or lasers are adopted as the transmitter, where the peak power is limited such that the peak constraint is more of interest than the average power constraint.
Assuming perfect photon-counting receiver, each photon and the corresponding arrival time can be detected without error. However, perfect photon-counting receiver is difficult to realize and a practical receiver with finite sampling rate consisting of a PMT detector, an one-bit ADC, and a digital signal processor (DSP) unit is considered. When a photon arrives, the PMT detector generates a pulse with certain width, which causes pulses-merge if the interval of two photons arrival time is shorter than the pulse width. The threshold of arrival time interval where the two photons are not differentiable is called dead time, denoted as \( \tau \). Denote \( T_s \) as the ADC sampling interval and assume low to medium sampling rate such that \( T_s \geq \tau \). Considering finite time input \( \Lambda^T_{[MJ]} \), the PMT sampling sequence \( Z_{[L]} \overset{\Delta}{=} \{Z_1, \cdots, Z_L\} \), where \( L \overset{\Delta}{=} \lfloor \frac{T_s}{T} \rfloor \). Note that for any \( \tau > 0 \), the number of photon arrivals \( N_\tau \) on \([0, \tau]\) together with the corresponding (ordered) arrival time instants \( T^{N_\tau} = (T_1, \cdots, T_{N_\tau}) \) are sufficient statistic for \( Y^\tau \) such that the random vector \((N_T, T^{N_\tau})\) is a complete description of random process \( Y^T \).

For the practical photon-counting receiver under consideration, assume zero shot noise, thermal noise and finite dead time. For one or multiple photons arriving at the photon-counting receiver at \((iT_s - \tau, iT_s)\), the sampling value \( Z_i \) is the same due to the self-sustaining avalanche in SPAD or the shaping circuit that converts bell-shaped response into rectangular response for photon-counting [21], [22]. According to above statement, we have

\[
Z_i = \begin{cases} 
0, & T_j \notin (iT_s - \tau, iT_s), \forall j = 1, \cdots, N_T; \\
1, & \text{otherwise};
\end{cases}
\]  

(4)

where \( \mathbb{P}(Z_i = 1) = 1 - e^{-X_iT_s} \) and \( Z_i \) and \( Z_j \) are independent identically distributed for \( i \neq j \) due to the property of independent increment for Poisson process. In other words, \( Z_i \) is an indicator on whether one or more photons arrive within \( \tau \) prior to the sampling instant.

Based on above mentioned system model, the multi-user MISO Poisson channel capacity is defined as

\[
C_{MU-MISO} = \lim_{T \to \infty} \max_{\Lambda^T_{[M]} \in [0, A_{mI}]} \frac{1}{T} I(\Lambda^T_{[M]}; Z_{[L]}),
\]  

(5)

Since \( \Lambda^T_{[M]} \to (N_T, T^{N_\tau}) \to Z_{[n]} \) forms a Markov chain, we have \( I(\Lambda^T_{[M]}; Z_{[n]}) \leq I(\Lambda^T_{[M]}; N_T, T^{N_\tau}) \), which shows that the multi-user MISO Poisson capacity with non-perfect receiver is not more than that of continuous-time multi-user MISO Poisson channel.
According to the chain rule for mutual information, we have

\[
\frac{1}{T} I(A_{[MJ]}^T; Z_l) = \frac{1}{T} \sum_{l=1}^L I(A_{[MJ],(l-1)T_s}^T; Z_l|\Lambda_{[MJ]}^{(l-1)T_s}; Z_{[l-1]}^T) = \frac{1}{T} \sum_{l=1}^L H(Z_l|\Lambda_{[MJ]}^{(l-1)T_s}; Z_{[l-1]}^T) - H(Z_l) = \frac{1}{T} \sum_{l=1}^L I(\Lambda_{[MJ]}^{T_s}; Z_l).
\]

(6)

Where equality (a) holds since \( Z_l \) is conditional independent of \( (\Lambda_{[MJ],(l-1)T_s}; Z_{[l-1]}^T) \) given \( \Lambda_{[MJ]}^{T_s} \). Thus, we have \( C_{MU-MISO} \leq \max_{\Lambda_{m_j}^{T_s} \in [0,A_m]} \frac{1}{T_s} I(\Lambda_{[MJ]}^{T_s}; Z_1) \), where the equality holds if \( \Lambda_{[MJ],(l-1)T_s} \) is dependent of each other for different \( l \). Consequently, the capacity-achieving distribution requires independent input signals for different sampling intervals, and the simplified capacity is given by,

\[
C_{MU-MISO} = \max_{\Lambda_{m_j}^{T_s} \in [0,A_m]} \frac{1}{T_s} I(\Lambda_{[MJ]}^{T_s}; Z_1).
\]

(7)

In the remainder of this paper, we omit subscript \( l \) for simplicity since we focus the achievable rate within a symbol duration to obtain the exact capacity.

### III. SISO Capacity for Two Users

We focus on the case where each user has only one transmitter, i.e., \( J_1 = 1 \) and \( J_2 = 1 \). Hence for the sake of convenience, we drop subscript \( j \) and use abbreviation \( p_1 = p(A_1 + A_2 + \Lambda_0) \), \( p_2 = p(A_2 + \Lambda_0) \), \( p_3 = p(A_1 + \Lambda_0) \), and \( p_4 = p(\Lambda_0) \).

#### A. Optimality Conditions

The sum-rate capacity is defined as \( C_{SISO-MAC} \triangleq \max_{\Lambda_{m_j}^T \in [0,A]} \frac{1}{T_s} I(\Lambda_{[MJ]}^{T_s}; Z) \). The following results show that the optimal distributions belongs to binary signal levels.

**Theorem 1.** The sum-rate capacity of a Poisson MAC with non-perfect receiver is achieved if the input signal belongs to the set \( \{0, A_m\} \) for each user \( m \).

**Proof:** Please refer to Appendix [A-A]
Although focusing two-users MAC channel, Theorem 1 can be extended to scenario of multiple users. Let $\mu_m$ be the duty cycle of the $m^{th}$ transmitter, $m = 1, 2$. The sum-rate Poisson MAC capacity is given by

$$C_{SISO-MAC} = \max_{0 \leq \mu_1, \mu_2 \leq 1} \frac{1}{\tau} I_{X_1^2;Z}(\mu_1, \mu_2),$$  \hspace{1cm} (8)$$

where

$$I_{X_1^2;Z}(\mu_1, \mu_2) = h_b(\hat{\rho}(\mu_1, \mu_2)) - \mu_1 \mu_2 h_b(p_1) - (1 - \mu_1) \mu_2 h_b(p_2) - \mu_1 (1 - \mu_2) h_b(p_3) - (1 - \mu_1) (1 - \mu_2) h_b(p_4),$$ \hspace{1cm} (9)

$$\hat{\rho}(\mu_1, \mu_2) = \mu_1 \mu_2 p_1 + (1 - \mu_1) \mu_2 p_2 + \mu_1 (1 - \mu_2) p_3 + (1 - \mu_1) (1 - \mu_2) p_4.$$ \hspace{1cm} (10)

For the problem (8), we have the following property.

**Lemma 1.** Assume that $\tau \leq \frac{\text{ln} 2}{A_1 + A_2 + \Lambda_0}$. For general values of $A_1, A_2$ and $\Lambda_0$, $I_{X_1^2;Z}(\mu_1, \mu_2)$ is not necessarily a concave function of $(\mu_1, \mu_2)$. In addition, the optimized joint distribution set is not convex.

**Proof:** Please refer to Appendix A-B. \hfill \blacksquare

According to Lemma 1, Problem (8) is a non-convex optimization problem in general.

We focus on solving such non-convex optimization problem. We start with the necessary KKT conditions (since the problem is not convex, these conditions are not sufficient for optimality). For convenience, we write $I_{X_1^2;Z} = I$, and thus the corresponding Lagrangian equation is given by,

$$\mathcal{L} = -I + \eta_1 (\mu_1 - 1) - \eta_2 \mu_1 + \eta_3 (\mu_2 - 1) - \eta_4 \mu_2.$$ \hspace{1cm} (11)

The optimal solution $(\hat{\mu}_1, \hat{\mu}_2)$ must satisfy the following KKT constraints:

$$\frac{\partial I}{\partial \mu_1}|_{(\hat{\mu}_1, \hat{\mu}_2)} - \eta_1 = 0, \hspace{1cm} (12)$$

$$\frac{\partial I}{\partial \mu_2}|_{(\hat{\mu}_1, \hat{\mu}_2)} - \eta_3 = 0, \hspace{1cm} (13)$$

$\eta_1 (\hat{\mu}_1 - 1) = 0, \eta_2 \hat{\mu}_1 = 0, \eta_3 (\hat{\mu}_2 - 1) = 0, \eta_4 \hat{\mu}_2 = 0.$
where

\[
\frac{\partial I}{\partial \mu_1} = \left[ \mu_2 (p_1 - p_2) + (1 - \mu_2) (p_3 - p_4) \right] \ln \frac{1 - \hat{p}}{\hat{p}} - \left[ \mu_2 (h_b (p_1) - h_b (p_2)) + (1 - \mu_2) (h_b (p_3) - h_b (p_4)) \right],
\] (14)

\[
\frac{\partial I}{\partial \mu_2} = \left[ \mu_1 (p_1 - p_3) + (1 - \mu_1) (p_2 - p_4) \right] \ln \frac{1 - \hat{p}}{\hat{p}} - \left[ \mu_1 (h_b (p_1) - h_b (p_3)) + (1 - \mu_1) (h_b (p_2) - h_b (p_4)) \right],
\] (15)

Note that \( \eta_1 \eta_2 = 0 \) and \( \eta_3 \eta_4 = 0 \), in order to further analyze the above KKT conditions, we need to consider 5 cases corresponding to different combinations of active constraints. Similar to work \([25]\), we can show that two cases are non-optimal solutions. For example, if \( \eta_1 = 0, \eta_2 = 0, \eta_3 \neq 0, \eta_4 = 0 \), we have \( I(\bar{\mu}_1, 1) < I(\bar{\mu}_1, 0) \) where \( I(\bar{\mu}_1, 0) \) corresponds to Scenario 2. Therefore, the rest three possible scenarios need further investigation.

**Scenario 1:** \( \eta_1 = 0, \eta_2 = 0, \eta_3 = 0, \) and \( \eta_4 = 0 \).

The KKT conditions can be simplified to \( \frac{\partial I}{\partial \mu_1} (\bar{\mu}_1, \bar{\mu}_2) = 0 \) and \( \frac{\partial I}{\partial \mu_2} (\bar{\mu}_1, \bar{\mu}_2) = 0 \). This scenario corresponds to the case where both users are active. As both \( \frac{\partial I}{\partial \mu_1} (\bar{\mu}_1, \bar{\mu}_2) \) and \( \frac{\partial I}{\partial \mu_2} (\bar{\mu}_1, \bar{\mu}_2) \) are nonlinear, there can be multiple possible \((\mu_1, \mu_2)\) pairs solution. However, we now show that there are at most 2 possible \((\mu_1, \mu_2)\) pairs solution. By removing the common item \( \ln \frac{1 - \hat{p}}{\hat{p}} \) in equations (12) and (13), we have

\[
\mu_1 U - \mu_2 V + W = 0,
\] (16)

where

\[
U = -h_{13} + h_{14} + h_{23} - h_{24} + h_{31} - h_{32} - h_{41} + h_{42},
\] (17)

\[
V = -h_{12} + h_{14} + h_{21} - h_{23} + h_{32} - h_{34} - h_{41} + h_{43},
\] (18)

\[
W = -h_{23} + h_{24} + h_{32} - h_{34} - h_{42} + h_{43},
\] (19)

and \( h_{ij} \triangleq p_i h_b (p_j) \) for \( i, j = 1, 2, 3, 4 \). Regarding equation (16), we have the following property on \( U, V, W \).

**Lemma 2.** For any \( A_1, A_2, A_0 \), we have \( U > 0 \) and \( V > 0 \); and we have \( W \leq 0 \) if and only if \( A_1 \preceq A_2 \).

**Proof:** Please refer to Appendix \( A-C \).
Based on equation (16) and Lemma 2, we have

\[ \mu_2 = \frac{U}{V} \mu_1 + \frac{W}{V} = f_{MAC}(\mu_1). \]  

(20)

As \( \frac{\partial f}{\partial \mu_2} = 0 \), we have

\[ \mu_2 = \left( a_M + 1 \right)^{-1} - \left[ \mu_1 p_3 + (1 - \mu_1) p_4 \right] \triangleq g_{MAC}(\mu_1), \]

(21)

where \( a_M = \exp \left( \frac{\mu_1 \left( h_3(p_1) - h_3(p_3) \right) + (1 - \mu_1) \left( h_3(p_2) - h_3(p_4) \right)}{\mu_1 (p_1 - p_3) + (1 - \mu_1) (p_2 - p_4)} \right) \). Hence, the \((\mu_1, \mu_2)\) pairs where \( f_{MAC}(\mu_1) \) and \( g_{MAC}(\mu_1) \) intersect with each other satisfy equations (20) and (21) simultaneously.

For function \( g_{MAC}(\mu_1) \), we have the following lemma 3.

**Lemma 3.** Assume \( \tau \leq \frac{\ln 2}{A_1 + A_2 + \Lambda_0} \). Then, \( g_{MAC}(\mu_1) \) is a strictly convex function with respect to \( \mu_1 \).

**Proof:** Please refer to Appendix A-D.

As \( f_{MAC}(\mu_1) \) is a linear with respect to \( \mu_1 \), and \( g_{MAC}(\mu_1) \) is a strictly convex function with respect to \( \mu_1 \), there will be at most two intersection points, denoted as \((\hat{\mu}_1, \hat{\mu}_2)\) and \((\tilde{\mu}_1, \tilde{\mu}_2)\). We then need to check whether \((\hat{\mu}_1, \hat{\mu}_2)\) is in \([0, 1]^2\) or not. If yes, we keep it. If not, then for the presentation convenience, we replace it with \((0, 0)\).

**Scenario 2:** \( \eta_1 = 0, \eta_2 = 0, \eta_3 = 0, \) and \( \eta_4 \neq 0 \).

Solving the corresponding KKT conditions, we obtain \( \tilde{\mu}_1 = \alpha_\tau(A_1, \Lambda_0) \) and \( \tilde{\mu}_2 = 0 \), where

\[ \alpha_\tau(A_1, \Lambda_0) = \frac{[1 + \exp(\frac{h_3(p_3) - h_3(p_4)}{p_3 - p_4})]^{-1} - p_4}{p_3 - p_4}, \]

(22)

It is seen that \( 0 \leq \alpha_\tau(A_1, \Lambda_0) \leq 1 \), since

\[ p_4 = [1 + \exp(h_3(p_4))]^{-1} \leq [1 + \exp(\frac{h_3(p_3) - h_3(p_4)}{p_3 - p_4})]^{-1} \leq [1 + \exp(h_3(p_3))]^{-1} = p_3. \]

This scenario corresponds to the case where only user 1 is active.

**Scenario 3:** \( \eta_1 = 0, \eta_2 \neq 0, \eta_3 = 0, \) and \( \eta_4 = 0. \)

Solving the corresponding KKT conditions, we obtain \( \tilde{\mu}_2 = \alpha_\tau(A_2, \Lambda_0) \) and \( \tilde{\mu}_1 = 0 \), where

\[ \alpha_\tau(A_2, \Lambda_0) = \frac{[1 + \exp(\frac{h_3(p_2) - h_3(p_4)}{p_2 - p_4})]^{-1} - p_4}{p_2 - p_4}. \]

(23)
It is seen that $0 \leq \alpha_{\tau}(A_2, \Lambda_0) \leq 1$, since
\[ p_4 = [1 + \exp(h_b(p_4))]^{-1} \leq [1 + \exp\left(\frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4}\right)]^{-1} \leq [1 + \exp(h'_b(p_2))]^{-1} = p_2. \]

This scenario corresponds to the case where only user 2 is active.

In summary, we have the following theorem.

**Theorem 2.** Assume that $\tau \leq \frac{\ln 2}{A_1 + A_2 + \Lambda_0}$. The sum-rate capacity of the Poisson MAC is given by
\[ C_{SISO-MAC} = \frac{1}{\tau} \max\{I_1, \hat{I}_1, I_2, \hat{I}_2, I_3, \hat{I}_3, I_4, \hat{I}_4\} \text{ for any } \mu_1 \in [0, 1], \tag{24} \]

Unlike the Gaussian MAC with an average power constraint, it can be optimal to allow only one user to transmit in order to achieve the sum-rate capacity for the Poisson MAC with a peak power constraint. More detailed discussions are presented in the following subsection.

**B. Single-User or Two-User Transmission?**

We present sufficient conditions on the optimality of a single-user transmission and two-user transmission.

Similar to work \[25\], the following simple proposition characterize the sufficient conditions where there is no intersection between equations (20) and (20) in duty cycle feasible region $[0, 1]^2$ and hence two-user transmission is not optimal.

**Proposition 1.** If $g_{MAC}(0) < f_{MAC}(0)$ and $g_{MAC}(1) < f_{MAC}(1)$, then single-user transmission is optimal to achieve the sum-rate capacity.

Even if the sufficient conditions in Proposition 1 are not satisfied, it is still possible for single-user transmission to be optimal if the corresponding rate is larger than that of the two-user transmission. We conclude that if certain $A_m$ is sufficiently large, single-user transmission is optimal.

**Lemma 4.** Functions $f_{MAC}(\mu_1)$ and $g_{MAC}(\mu_1)$ have the following properties:
\[ \lim_{A_2 \to \infty} f_{MAC}(\mu_1) = \lim_{A_2 \to \infty} f_{MAC}(0) = \lim_{A_2 \to \infty} f_{MAC}(1) = 1, \tag{25} \]
\[ \lim_{A_2 \to \infty} g_{MAC}(\mu_1) = \frac{(a_{MI} + 1)^{-1} - [\mu_1 p_3 + (1 - \mu_1)p_4]}{\mu_1(p_1 - p_3) + (1 - \mu_1)(p_2 - p_4)} < 1, \text{ for any } \mu_1 \in [0, 1], \tag{26} \]
where $a_{MI} = \exp\left(-\frac{\mu_1 h_b(p_3) + (1 - \mu_1)h_b(p_4)}{\mu_1 p_3 + (1 - \mu_1) p_4}\right)$. 
Proof: Please refer to Appendix A-F.

Lemma 4 and Proposition 1 imply that a single active user is optimal for sufficient high peak power constraint of other user given peak power constraint of certain user. Furthermore, it is seen that the sum-rate capacity is achieved when only user 2 is transmitting.

We further discuss the conditions on the optimality of two-user transmission. The following proposition characterizes sufficient conditions where single-user transmission is not optimal.

**Proposition 2.** Single user 1 transmission is not optimal if $\frac{\partial I}{\partial \mu_2}(\tilde{\mu}_1, 0) > 0$. Similarly, single user 2 transmission alone is not optimal if $\frac{\partial I}{\partial \mu_1}(0, \tilde{\mu}_2) > 0$.

Proof: Please refer to Appendix A-F.

C. Asymptotic Capacity Property for $\tau \to 0$

We further investigate the asymptotic properties of the non-perfect receiver compared with the continuous Poisson channel, summarized in Theorem 3. The main clue is to show the asymptotic properties of optimized objective function and optimal duty cycle.

**Theorem 3.** The optimal duty cycle and MAC capacity of the non-perfect receiver approach those of continuous Poisson channel for any bounded $A_1$, $A_2$ and $\Lambda_0$, respectively, as $\tau \to 0$.

Proof: Please refer to Appendix A-G.

Theorem 3 studies the asymptotic property of the non-perfect receiver for $T_s = \tau \to 0$. It shows that Theorem 2 extends the result of continuous MAC Poisson capacity [25], and provides a more general and practical results.

IV. SISO Capacity for Symmetric Two Users

Section III demonstrates SISO capacity for general two users based on KKT conditions. However, this method is hard to extend for multiple users since exponential number of Lagrangian multipliers. In this Section, we reduce the number of candidate optimal solutions from 4 to 1 for symmetric channel based on Section III and provide another method to find optimal solution based on majorization. The notation in this section is similar to Section III and $p_2 = p_3$ for symmetric channel.
A. KKT Conditions Perspective

For symmetric channel, we prove that the optimal transmission strategy is two-user transmission with the same and unique duty cycle. The proof is given by the following three steps.

Step 1: We prove that two-user transmission is the optimal transmission strategy for \( A_1 = A_2 \). When \( (\mu_1, \mu_2) = (\bar{\mu}_1, 0) \), we have

\[
\hat{p} = \bar{\mu}_1 p_3 + (1 - \bar{\mu}_1)p_4 = [1 + \exp(\frac{h_b(p_3) - h_b(p_4)}{p_3 - p_4})]^{-1}, \tag{27}
\]

Note that \( \frac{h_b(p_1) - h_b(p_3)}{p_1 - p_3} < \frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4} \), according to lemma \([3]\) we have

\[
\frac{\mu_1(h_b(p_1) - h_b(p_3)) + (1 - \mu_1)(h_b(p_2) - h_b(p_4))}{\mu_1(p_1 - p_3) + (1 - \mu_1)(p_2 - p_4)} \leq \frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4} = \ln \frac{1 - \hat{p}}{\hat{p}}, \tag{28}
\]

which implies \( \frac{\partial I}{\partial \mu_2}(0, \bar{\mu}_2) > 0 \). Thus, single active user 1 is not optimal. Similarly, single active user 2 is not optimal.

Step 2: We prove that \( \mu_1 = \mu_2 \) is optimal for both active users. Note that in such a scenario, \( p_1 = p_3, h_2 = h_3 \), and \( h_2 = h_3 \). Thus, we have \( W = 0 \) and \( U = -V \), i.e., for the optimal \( (\mu_1, \mu_2) \) we have \( \mu_1 = \mu_2 \).

Step 3: We finally prove that there exists unique pair \( (\mu_1, \mu_2) \) that satisfies equation \( (20) \) and \( (21) \). It is easy to check that \( g_{MAC}(0) > 0 = f_{MAC}(0) \) and \( g_{MAC}(1) < 1 = f_{MAC}(1) \). Thus, there exists a single intersection between \( f_{MAC}(\mu) \) and \( g_{MAC}(\mu) \) for \( 0 \leq \mu_1 \leq 1 \).

B. Majorization Perspective

KKT-conditions-based method provides the necessary condition for the optimal solution, but it is hard to capture the specific property for the objective function and extend to multiple users. We investigate problem \([8]\) based on majorization and obtain the same result as Section \([11-17]\) in addition, majorization-based method reveals more information about the problem \([8]\) and maybe can be extended to the scenario of multiple users.

Recall the sum-rate Poisson MAC capacity \( C_{SISO-MAC} = \max_{0 \leq \mu_1, \mu_2 \leq 1} \frac{1}{2} I_{X_1:Z}^2(\mu_1, \mu_2) \), where \( I_{X_1:Z}(\mu_1, \mu_2) = h_b(\hat{p}(\mu_1, \mu_2)) - \mu_1 \mu_2 h_b(p_1) - (1 - \mu_1)\mu_2 h_b(p_2) - \mu_1 (1 - \mu_2) h_b(p_3) - (1 - \mu_1)(1 - \mu_2) h_b(p_4), \) \( \hat{p}(\mu_1, \mu_2) = \mu_1 \mu_2 p_1 + (1 - \mu_1)\mu_2 p_2 + \mu_1 (1 - \mu_2) p_3 + (1 - \mu_1)(1 - \mu_2) p_4 \) and \( p_2 = p_3 \). The
solution based on majorization consists the following two steps corresponding to the inner and outer optimization as \( C_{SISO-MAC} = \max_{0 \leq s \leq 1} \frac{1}{\tau} I_2(\mu_s), \) where \( I_2(\mu_s) = \max_{\mu_1, \mu_2: \mu_1 + \mu_2 = 2\mu_s} I_{X^2;Z}(\mu_1, \mu_2). \)

**Step 1:** Assume that \( \tau \leq \frac{\ln 2}{2A_0}. \) We optimize \( \mu_1 \) and \( \mu_2 \) with the constraint \( \mu_1 + \mu_2 = 2\mu_s \) for any given \( 0 \leq \mu_s \leq 1. \)

Firstly, we provide two critical Lemmas as follows,

**Lemma 5.** Assume that \( \tau \leq \frac{\ln 2}{2A_0}. \) Define \( G(A) = \frac{2h_b(p_2) - h_b(p_1) - h_b(p_4)}{2p_2 - p_1 - p_4}, \) then we have \( G(A) \) decreases with peak power \( A \) and \( G(A) \in (\ln(1 - p_4) + \frac{p_1}{1 - p_4} \ln p_4, \frac{1}{p_4} + \ln \frac{1 - p_4}{p_4}). \)

**Proof:** Please refer to Appendix [A-H].

**Lemma 6.** The solution \( \ln \frac{1 - \hat{p}}{p} = G(A) \) for \( (\mu_1, \mu_2) \in [0, 1]^2 \) iff \( A \geq A_{th} \), where \( A_{th} \) is the unique solution on \( \ln \frac{1 - p_4}{p_4} = G(A). \)

**Proof:** Please refer to Appendix [A-I].

We focus on the region \( \mu_1 \geq \mu_2 \) since the objective function in Equation (8) and the feasible region are symmetric for \( \mu_1 \) and \( \mu_2 \). Based on Equations (14) and (15), we have

\[
\frac{\partial I}{\partial \mu_1} - \frac{\partial I}{\partial \mu_2} = \left\{ [\mu_2(p_1 - p_2) + (1 - \mu_2)(p_3 - p_4)] \ln \frac{1 - \hat{p}}{p} - [\mu_2(h_b(p_1) - h_b(p_2)) + (1 - \mu_2)(h_b(p_3) - h_b(p_4))] \right\} - \left( [\mu_1(p_1 - p_3) + (1 - \mu_1)(p_2 - p_4)] \ln \frac{1 - \hat{p}}{p} - [\mu_1(h_b(p_1) - h_b(p_3)) + (1 - \mu_1)(h_b(p_2) - h_b(p_4))] \right) \]

\[
= (\mu_1 - \mu_2)\left\{ \ln \frac{1 - \hat{p}}{p} (2p_2 - p_1 - p_4) - [2h_b(p_2) - h_b(p_1) - h_b(p_4)] \right\} = (\mu_1 - \mu_2)(2p_2 - p_1 - p_4) \ln \frac{1 - \hat{p}}{p} - G(A) \}
\]

(29)

According to Lemma 6, we can analyze Equation (29) by two cases.

**Case 1:** \( A < A_{th} \). According to Lemma 6, we have \( \ln \frac{1 - \hat{p}}{p} - G(A) < 0 \) and \( (\mu_1 - \mu_2)(\frac{\partial I}{\partial \mu_1} - \frac{\partial I}{\partial \mu_2}) < 0 \) for \( (\mu_1, \mu_2) \in [0, 1]^2 \). According to [26, A.4. Theorem, p.84], we have that mapping \( (\mu_1, \mu_2) \mapsto I_{X^2;Z}(\mu_1, \mu_2) \) is Schur-concave for \( (\mu_1, \mu_2) \in [0, 1]^2 \) and the optimal \( (\mu_1, \mu_2) \) with the constraint \( \mu_1 + \mu_2 = 2\mu_s \) is \( (\mu_s, \mu_s) \).

**Case 2:** \( A \geq A_{th} \). Define \( \mathcal{C} \triangleq \{ (\mu_1, \mu_2) : \ln \frac{1 - \hat{p}}{p} = G(A), \mu_1 \geq \mu_2 \} \) and \( \mathcal{L}_{\mu_s} \triangleq \{ (\mu_1, \mu_2) : \mu_1 + \mu_2 = 2\mu_s, \mu_1 \geq \mu_2 \}. \) According to Lemma 6, we have \( \mathcal{C} \neq \emptyset. \) We further investigate the property of \( \mathcal{C} \) as shown in Theorem 4.
Theorem 4. Assume that $\tau \leq \frac{\ln 2}{2A-A_0}$. There exists differentiable function $f_B(\cdot)$ such that $C = \{ (\mu_1, \mu_2) : \mu_1 = f_B(\mu_2) \}$, where $0 \geq f_B(0) < 1$ and $f_B(\mu_2) < -1$ for $\mu_2 \in [0, \frac{1}{2}]$. In addition, $|C \cap L_{\mu_s}| = 1$ for $\mu'_s \leq \mu_s \leq \mu^*_s$, and $|C \cap L_{\mu_s}| = 0$ for $\mu_s \geq \mu^*_s$ and $\mu_s \leq \mu'_s$, where $\mu_s = \frac{1}{2(p_2-p_4)} < \frac{\mu_1}{2}$.

Proof: Please refer to Appendix A-J.

According to Theorem 4 and $\hat{p}$ increases with $\mu_1$ and $\mu_2$, define $C^+ = \{ (\mu_1, \mu_2) : \mu_1 \geq f_B(\mu_2) \}$ and $C^- = \{ (\mu_1, \mu_2) : \mu_1 \leq f_B(\mu_2) \}$, we have that that mapping $(\mu_1, \mu_2) \rightarrow I_{X^2}$ is Schur-concave and Schur-convex for region $C^+$ and $C^-$, respectively. Thus, $I_2(\mu_s)$ is given by

$$I_2(\mu_s) = \begin{cases} 
I_{X^2}(2\mu_s, 0), & \mu_s \leq \mu'_s, \\
\max\{I_{X^2}(2\mu_s, 0), I_{X^2}(\mu_s, \mu_s)\}, & \mu'_s < \mu_s < \mu^*_s, \\
I_{X^2}(\mu_s, \mu_s), & \mu_s \geq \mu^*_s.
\end{cases} \quad (30)$$

Step 2: We optimize $\mu_s$ to maximize $I_2(\mu_s)$ over $\mu_s \in [0, 1]$.

According to Equation (30), we have the candidate solution to maximize $I_{X^2}(\mu_1, \mu_2)$ over $\mu_1 \geq \mu_2$ are $(2\mu_s, 0)$ for $0 \leq \mu_s \leq \mu^*_s$, and $(\mu_s, \mu_s)$ for $1 \geq \mu_s \geq \mu'_s$. According to $\mu^*_s < \frac{\mu_1}{2}$ and Scenario 2 in Section III, we have $I_{X^2}(2\mu_s, 0)$ increases with $\mu_s$ over $\mu_s \leq \mu^*_s$. Note that

$$\ln \frac{1 - \hat{p}(2\mu_s^*, 0)}{\hat{p}(2\mu_s^*, 0)} = h_b'(2\mu_s^*p_3 + (1 - 2\mu_s^*)p_4) = G(A)$$

$$\geq \frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4} \geq \frac{\mu(h_b(p_1) - h_b(p_3)) + (1 - \mu)(h_b(p_2) - h_b(p_4))}{\mu(p_1 - p_3) + (1 - \mu)(p_2 - p_4)} \quad (31)$$

where (a) and (b) hold according to Lemma 12 and Lemma 13 respectively. Thus, we have

$$\frac{\partial I}{\partial \mu_2}(\mu_s, 0) = [\mu^*_s(p_1 - p_3) + (1 - \mu^*_s)(p_2 - p_4)] \ln \frac{1 - \hat{p}}{\hat{p}}$$

$$- [\mu^*_s(h_b(p_1) - h_b(p_3)) + (1 - \mu^*_s)(h_b(p_2) - h_b(p_4))] > 0. \quad (32)$$

and the optimal solution to maximize $I_{X^2}(\mu_1, \mu_2)$ is not in $(2\mu_s, 0)$ for $0 \leq \mu_s \leq \mu^*_s$.

For the rest candidate region $(\mu_s, \mu_s)$ for $1 \geq \mu_s \geq \mu'_s$, it is easy to check that $I_{X^2}(\mu_s, \mu'_s) = I_{X^2}(2\mu_s^*, 0)$ and $I_{X^2}(1, 1) = 0$. For the continuous objection function $I_{X^2}(\mu_s, \mu_s)$ over
1 ≥ μ_s ≥ μ'_s, the optimal solution must be a extreme point satisfying the following equation,

\[
0 = \frac{\partial I_{X_1^2;Z}(\mu_s, \mu_s)}{\partial \mu_s} = \left( \frac{\partial I_{X_1^2;Z}}{\partial \mu_1} - \frac{\partial I_{X_1^2;Z}}{\partial \mu_2} \right)_{(\mu_s, \mu_s)}
\]

\[
= 2 \left\{ [\mu_s(p_1 - p_2) + (1 - \mu_s)(p_2 - p_4)] \ln \frac{1 - \hat{p}}{\bar{p}} - [\mu_s(h_b(p_1) - h_b(p_2)) + (1 - \mu_s)(h_b(p_2) - h_b(p_4))] \right\},
\]

where \( \hat{p} = \mu^2_s p_1 + 2 \mu_s (1 - \mu_s) p_2 + (1 - \mu_s)^2 p_4 \). It is easy to check that Equation (33) equals to \( \mu_s = g_{MAC}(\mu_s) \) in Equation (21). According to Section IV-A, we have that there exists unique solution on Equation (33).

Section IV-B shows that the optimal solution to maximize \( I_{X_1^2;Z}(\mu_1, \mu_2) \) satisfies \( \mu_1 = \mu_2 = \mu_s \) and \( \mu_s \) is the unique solution on Equation (33), the same as the result in Section IV-A. In addition, Work [12] shows that the mutual information function over \( \mu_1 \) and \( \mu_2 \) is schur-concave for continuous time Poisson channel, while does not hold for non-perfect receiver. Section IV-B demonstrates that \( I_{X_1^2;Z}(\mu_1, \mu_2) \) is schur-concave as \( A < A_{th} \), and \( I_{X_1^2;Z}(\mu_1, \mu_2) \) is schur-concave and schur-convex for \( C^+ \) and \( C^- \), respectively, for \( A \geq A_{th} \). Furthermore, we have that \( I_{X_1^2;Z}(\mu_1, \mu_2) \) is schur-concave for any fixed peak power \( A \) as \( \tau \to 0 \), as shown in Lemma 7.

Lemma 7. For dead time \( \tau \to 0 \), we have \( \lim_{\tau \to 0} A_{th} = +\infty \), i.e., \( I_{X_1^2;Z}(\mu_1, \mu_2) \) is schur-concave for any bounded peak power \( A \).

Proof: Please refer to Appendix A-K.

The same behavior of mutual information function between continuous Poisson channel and non-perfect receiver with small enough dead time, schur-concavity over any peak power \( A \) and background radiation \( \Lambda_0 \), aligns with the intuition since small enough dead time would not cause any photon-counting loss.

V. SUM-RATE MISO CAPACITY FOR TWO USERS

We extend the analysis to the case when the user \( m \) is equipped with \( J_m \) (more than one) transmitters.
A. Sum-Rate MISO-MAC Capacity Analysis

The sum-rate MISO-MAC capacity is defined as

\[
C_{\text{MISO-MAC}} \triangleq \max_{\Lambda^T_s \in [0, A]} \frac{1}{T_s} I(\Lambda^T_s; Z).
\]

Similar to Section III, the input waveform signal of each transmitter is piece-wise constant waveforms with two levels \(\{0, A_{mj}\}\) for the \(j^{th}\) transmitter of the \(m^{th}\) user. Nevertheless, it is still needed to be investigated how the \(J_m\) transmitters jointly work, which is addressed in the following result.

**Theorem 5.** For \(\tau \leq \frac{\ln 2}{\sum_{m=1}^{2} \sum_{j=1}^{J_m} A_{mj} + A_0}\), the optimal solution is that all transmitters must have the same duty cycle and must be on or off simultaneously.

Theorem 5 implies the equivalence between MISO-MAC and SISO-MAC, i.e., the sum-rate MISO-MAC capacity with peak constraints \((A_1, A_2)\) is equivalent to the SISO-MAC capacity with peak power constraint \((\sum_{j=1}^{J_1} A_{1j}, \sum_{j=1}^{J_2} A_{2j})\). Therefore, further detailed investigations on the sum-rate MISO-MAC capacity is similar to that in Section III, and thus omitted here.

The proof of Theorem 5 consists of the following two major steps.

In step 1, given duty cycle \(\mu_{[2]}\), we show how each transmitter jointly work.

**Proposition 3.** For \(\tau \leq \frac{\ln 2}{\sum_{m=1}^{2} \sum_{j=1}^{J_m} A_{mj} + A_0}\), the optimal condition is that if the transmitter with a smaller duty cycle is on then all transmitters with a larger duty cycle must also be on.

*Proof: Please refer to Appendix A-L.*

This proposition shows that the capacity-achieving transmitted signals through \(J_m\) transmitter are correlated but i.i.d. in each time interval for user \(m, m = 1, 2\). The possible PMF value can be reduced from \(2^{J_1} + 2^{J_2}\) to \(J_1 + J_2 + 2\). Similarly, for a given duty cycle, there could be infinite number of possible joint distribution. The main idea is to show that, if the transmitter with lower duty cycle is on, then the transmitter with higher duty cycle must be also on to achieve optimality. In addition, the objective function is concave and the optimization is performed over a convex compact set, such that the optimal solution clearly exists.

In step 2, we show the following proposition that characterizes the optimal duty cycle.

**Proposition 4.** For the optimal solution, all transmitters of user \(m\) must have the same duty cycle, and according to Proposition 3 they must be on and off simultaneously.

*Proof: Please refer to Appendix A-M.*
This proposition shows that for the optimal solution, the transmitters of each user must have the same duty cycle (i.e., $\mu_m = \cdots = \mu_{m,J_m}$) and must be aligned. Hence, the dimension of the optimization problem can be reduced from $J_1 + J_2$ to 2. The main idea is to show that all transmitters are simultaneously on and off. Hence, from the receiver perspective, two users with multiple transmitters can be viewed as two users each with a single transmitter with peak power constraint ($\sum_{j=1}^{J_1} A_{1j}, \sum_{j=1}^{J_2} A_{2j}$).

VI. NUMERICAL RESULTS

In this section, we provide numerical examples to illustrate results obtained in this paper. As shown in the paper, the MISO-MAC Poisson capacity can be converted to that of SISO-MAC. Hence, in the following, we provide only example related to the SISO-MAC case.

Figure 1 and Figure 2 show the case of no intersection and one intersection in $0 \leq \mu_1 \leq 1$ and $0 \leq \mu_2 \leq 1$, respectively. The dead time is set to 0.02; The peak power $(A_1, A_2)$ of transmitters 1 and 2 are set to (1, 20) and (10, 12), respectively, in Figures 1 and 2 and satisfies the condition $\tau \leq \frac{\ln 2}{A_1 + A_2 + A_0}$. Lemma 3 implies at most two intersection points between function $f_{MAC}(\cdot)$ and $g_{MAC}(\cdot)$, while can not find the case of two intersection points by brute-force search. Figure 3 illustrates the optimal $\mu_1$ and $\mu_2$ against peak power of user 2 $A_2$ for different dead time $\tau$ given $A_1 = 12.5$. $\tau = 0$ represents continuous Poisson channel. It is seen that $\tau \leq \frac{\ln 2}{A_1 + A_2 + A_0}$ is satisfied and the optimal $\mu_1$ and $\mu_2$ close to that of continuous Poisson channel as $\tau \to 0$ and the optimal $\mu_1^* = \mu_2^*$ as $A_1 = A_2$ for any dead time $\tau$, aligned with the result of Section IV.

Figure 4 shows the MAC Poisson capacity with respect to peak power $A_2$ for different dead time and it is seen that the optimal MAC Poisson capacity with non-perfect receiver approaches that of continuous Poisson channel as $\tau \to 0$, aligned with Theorem 3. Figure 5 shows the optimal transmission strategy region of $A_1$ and $A_2$. “Black”, “red”, and “Blue” regions represents the optimal transmission strategy region of only active user 2, both two active users and only active user 1, respectively. It is seen that the boundary of these three regions are almost two lines through the origin with different slope, and the optimal transmission strategy are only user 2-active, both two-user-active and only user 1-active for the case of $A_1 \ll A_2$, $A_1 = A_2$, $A_1 \gg A_2$, respectively, aligned with Section III-B and Section IV.
VII. CONCLUSION

In this paper, we have characterized the two-user asymmetric sum-rate Poisson capacity for both SISO and MISO cases. We demonstrate the equivalence of these two cases under certain condition. For both two cases, the optimal input signal of each transmitter and user must be two-level piece-wise constant and there are three possible transmission strategies, including only one active user and both active users. We provide the sufficient condition of these three strategies. In addition, we investigate the two-user symmetric sum-rate Poisson capacity based on above result and majorization method, both demonstrating that the optimal duty cycle must be the same and unique, and the majorization method maybe can be extend to multiple users case.
Fig. 2. $f_{MAC}(\mu_1)$ and $g_{MAC}(\mu_1)$ have one intersection in $0 \leq \mu_1 \leq 1$ and $0 \leq \mu_2 \leq 1$.

Fig. 3. The optimal $\mu_1$ and $\mu_2$ versus peak power of user 2 $A_2$ for different dead time $\tau$. 
\[ \tau = \begin{cases} 10^{-2} \\
5 \times 10^{-3} \\
\text{Continuous Poisson} \end{cases} \]

Fig. 4. The MAC Poisson capacity versus peak power \( A_2 \) for different dead time.

Fig. 5. Optimal transmission strategy for different peak power \( A_1 \) and \( A_2 \).
APPENDIX A
THE PROOF OF MAIN RESULTS ON MISO-MAC CAPACITY

A. Proof of Theorem

Converse part: Note that $\Lambda_{[M]}^{T_s} \rightarrow X_{[M]} \rightarrow Z$ forms a Markov chain, according to DPI, we have $I(\Lambda_{[M]}^{T_s}; Z) \leq I(X_{[M]}; Z)$. The mutual information $I(X, Z)$ is as follow,

$$I(X, Z) = h_b(\hat{p}) - \int h_b(p(\sum_{m=1}^{M} X_m + \Lambda_0))d\mu(X_{[M]}),$$

where $\hat{p} = \mathbb{E}[p(\sum_{m=1}^{M} X_m + \Lambda_0)]$. Define $S_1 = p(X_1 + \Lambda_0)$, note that the mapping $X \rightarrow S$ is a one-to-one mapping and $p(x_1 + x_2) = p(x_2) + (1 - p(x_2))p(x_1)$, hence we have

$$I(X_{[M]}; Z) = I(X_2^M, S_1; Z) = h_b(\hat{p}) + \mathbb{E}_{X_2^M}[\mathbb{E}_{S_1}[ -h_b(p(X_2^M) + (1 - p(X_2^M))S_1) ]],$$

and the following equation holds,

$$\max_{\mu(X_{[M]})} I(X_{[M]}, Z) = \max_{\mu(X_2^M, S_1)} I(X_2^M, S_1; Z)$$

$$= \max_{p(\Lambda_0) \leq \hat{p} \leq p(A + \Lambda_0)} \hat{p} + \max_{\mu(X_2^M)} \mathbb{E}_{X_2^M}[\max_{\mu(S_1) \in S_m(\hat{p}, \mu(X_2^M))} \mathbb{E}_{S_1}[ -h_b(p(X_2^M) + (1 - p(X_2^M))S_1) ]],$$

where

$$S_m(\hat{p}, \mu(X_2^M)) = \left\{ \mu(S_1) : \mathbb{E}[S] = \frac{\hat{p} - \mathbb{E}[p(\sum_{m=2}^{M} A_m)]}{1 - \mathbb{E}[p(\sum_{m=2}^{M} A_m)]} \right\},$$

where the inner optimization is performed over the class of distributions of $S_1$ with a finite support $[0, A_1]$ and fixed conditional mean set. Note that convex function $-h_b(\cdot)$ compounded linear function is still a convex function, the inner maximum is achieved if and only if $S_1$ is two-levels. Then we see that the optimal marginal PMF of $X_1$ is given by

$$\mathbb{P}(X_1 = A_1) = \mathbb{P}(S_1 = p(A_1 + \Lambda_0)) \triangleq \mu_1.$$  

By symmetry, the optimal marginal PMF of $X_m$ is $\{0, A_m\}$-valued with $\mathbb{P}(X_m = A_m) \triangleq \mu_m$, where $m = 2, \ldots, M$.

Achievability part: Let waveform $\Lambda_{[M]}^{T_s}$ in $[0, T_s]$ randomly selected from waveform set $\{0, A_m*(u(t) - u(t-T_s))\}$ with probability $\mu_m^* = \mathbb{P}\{\Lambda_{[M]}^{T_s} = A_m*(u(t) - u(t-T_s))\}$, where $u(t)$ denotes as a step function, then we have the upper bound in converse part is achievable.
B. Proof of Lemma 1

**Non-convex optimized joint distribution set:** The joint distribution of two independent variable is not closed under the linear weighted operation, i.e.,

$$\mu f_{X_2} (x_1, x_2) + (1 - \mu) \mu f_{X_2'} (x_1, x_2) \neq \int \mu f_{X_2} (x_1, x_2) + (1 - \mu) \mu f_{X_2'} (x_1, x_2) \, dx_1 \cdot \int \mu f_{X_2} (x_1, x_2) + (1 - \mu) \mu f_{X_2'} (x_1, x_2) \, dx_2 \quad (39)$$

where $f_{X_2} (x_1, x_2)$ denotes as the joint distribution of $X_{[2]}$.

**Non-concavity of** $I_{X_1, Z} (\mu_1, \mu_2)$: Prove by contradiction. Assume $I_{X_1, Z} (\mu_1, \mu_2)$ is concave, \( \nabla^2 I \) needs to be negative semi-definite. By calculating, we have

\[
\frac{\partial I}{\partial \mu_1} = \ln \frac{1 - \hat{p}}{\hat{p}} \left( \mu_2 (p_1 - p_2) + (1 - \mu_2) (p_3 - p_4) \right) - \left( \mu_2 (h_b(p_1) - h_b(p_2)) + (1 - \mu_2) (h_b(p_3) - h_b(p_4)) \right),
\]

\[
\frac{\partial I}{\partial \mu_2} = - \left( \mu_2 (p_1 - p_2) + (1 - \mu_2) (p_3 - p_4) \right)^2 \frac{\hat{p}}{1 - \hat{p}} < 0,
\]

\[
\frac{\partial^2 I}{\partial^2 \mu_1 \mu_2} = - \frac{1}{\hat{p}} \left( \mu_1 (p_1 - p_3) + (1 - \mu_1) (p_2 - p_4) \right) \left( \mu_2 (p_1 - p_2) + (1 - \mu_2) (p_3 - p_4) \right) - \ln \frac{1 - \hat{p}}{\hat{p}} (p_1 - p_2 - p_3 + p_4) - \left( h_b(p_1) - h_b(p_2) - h_b(p_3) + h_b(p_4) \right),
\]

\[
\frac{\partial^2 I}{\partial^2 \mu_2} = - \left( \mu_1 (p_1 - p_3) + (1 - \mu_1) (p_2 - p_4) \right)^2 \frac{\hat{p}}{1 - \hat{p}} < 0,
\]

Thus \( |\nabla^2 I| \) is given by

\[
|\nabla^2 I| = \frac{\partial I^2}{\partial^2 \mu_1 \mu_2} - \left( \frac{\partial I^2}{\partial \mu_1 \partial \mu_2} \right)^2 = \left[ \ln \frac{1 - \hat{p}}{\hat{p}} (p_1 - p_2 - p_3 + p_4) - \left( h_b(p_1) - h_b(p_2) - h_b(p_3) + h_b(p_4) \right) - \frac{\partial I^2}{\partial^2 \mu_1 \mu_2} \right]^2 - \left( \frac{\partial I^2}{\partial^2 \mu_1 \mu_2} \right)^2 = I_b (2I_a - I_b),
\]

\[
(44)
\]
Note that
\[
\lim_{(\mu_1, \mu_2) \to 0} I_b(2I_a - I_b) = (p_1 - p_2 - p_3 + p_4) \left( \ln \frac{1 - p_2}{p_4} - \bar{G}(A_1, A_2) \right) \left( \frac{(p_2 - p_4)(p_3 - p_4)}{p_4(1 - p_4)} \right)
- (p_1 - p_2 - p_3 + p_4) \left( \ln \frac{1 - p_2}{p_4} - \bar{G}(A_1, A_2) \right)
\]
\[
= (p_1 - p_2 - p_3 + p_4)^2 \left( \ln \frac{1 - p_2}{p_4} - \bar{G}(A_1, A_2) \right) \left( \bar{G}(A_1, A_2) + \frac{2}{p_4} - \ln \frac{1 - p_2}{p_4} \right)
\]
where \( \bar{G}(A_1, A_2) \triangleq \frac{h_b(p_1) - h_b(p_2) - h_b(p_3) + h_b(p_4)}{p_1 - p_2 - p_3 + p_4} > 0 \), equality (a) holds since \((p_2 - p_4)(p_3 - p_4) = -(p_2 - p_4)(1 - p_4)p(A_1) = -(1 - p_4)(p_1 - p_2 - p_3 + p_4) = \bar{G}(A_1, A_2) + \frac{2}{p_4} - \ln \frac{1 - p_2}{p_4} > 0 \) since \( \frac{1}{p_4} > \frac{1}{p_4} \geq \frac{1}{p_4} \). Set \( \tau = 0.02, A_0 = 0.001 \) and \( A_1 = A_2 = 10 \), we have \( \tau = 0.02 \leq \frac{\ln 2}{A_1 + A_2 + A_0} \simeq 0.0347 \) and \( \bar{G}(10, 10) \simeq 9.51 < \ln \frac{1 - p_2}{p_4} \simeq 10.8198 \), i.e., there exists certain \((A_1, A_2, A_0)\) such that \( \lim_{(\mu_1, \mu_2) \to 0} I_b(2I_a - I_b) > 0 \).

Thus, there exists \( \epsilon > 0 \) so that for \((\mu_1, \mu_2) \in C_{\epsilon} = \{ (\mu_1, \mu_2) : \sqrt{\mu_1^2 + \mu_2^2} \leq \epsilon \}, \) we have \( |\nabla^2 I| > 0 \), contradicted with assumption of negative semi-definite \( \nabla^2 I \).

**C. Proof of Lemma 2**

According to Lemma 8 and \( h_b(\cdot) \) is concave, we have
\[
\frac{h_b(p_1) - h_b(p_2)}{p_1 - p_2} > \frac{h_b(p_1) - h_b(p_4)}{p_1 - p_4} > \frac{h_b(p_3) - h_b(p_4)}{p_3 - p_4},
\]
\[
\frac{h_b(p_2) - h_b(p_3)}{p_2 - p_3} > \frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4}.
\]

By calculating, we can have
\[
U = (p_1 - p_2)(p_3 - p_4) \left[ \frac{h_b(p_1) - h_b(p_2)}{p_1 - p_2} - \frac{h_b(p_3) - h_b(p_4)}{p_3 - p_4} \right] > 0,
\]
similarly, we can have \( V > 0 \). As for \( W \), if \( A_2 > A_1 \), we have \( p_2 > p_3 \) and
\[
W = p_2p_3 \left[ \frac{h_b(p_2) - h_b(p_3)}{p_2 - p_3} - \frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4} + \frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4} \right] > p_2p_4 \left[ \frac{h_b(p_2) - h_b(p_3)}{p_2 - p_3} - \frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4} \right] + p_3p_4 \frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4} > 0
\]
Similarly, we have \( W \leq 0 \) if and only if \( A_1 \geq A_2 \).
D. Proof of Convexity of $g_{MAC}(\mu_1)$

Define $d_0 \triangleq p_2 - p_4 > 0$, $d_1 \triangleq p_1 - p_2 - p_3 + p_4 < 0$, $d_2 \triangleq h_b(p_1) - h_b(p_2) - h_b(p_3) + h_b(p_4) < 0$, and $d_3 \triangleq h_b(p_2) - h_b(p_4)$. Note that $\frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4} > \frac{h_b(p_1) - h_b(p_3)}{p_1 - p_3}$ and $p_2 - p_4 > p_1 - p_3 > 0$, we have

$$\frac{d_2}{d_1} = \frac{[h_b(p_2) - h_b(p_4)] - [h_b(p_1) - h_b(p_3)]}{(p_2 - p_4) - (p_1 - p_3)} > \frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4} = \frac{d_3}{d_0}, \quad (50)$$

which implies $d_2 d_0 - d_1 d_3 = d_0 d_1 (\frac{d_2}{d_1} - \frac{d_3}{d_0}) < 0$. Note that $a_M = \exp(\frac{\mu_1 d_2 + d_3}{\mu_1 d_1 + d_0})$ and $d_1 + d_0 > 0$, we have

$$a_M' = \exp(\frac{\mu_1 d_2 + d_3}{\mu_1 d_1 + d_0}) \frac{d_2 d_0 - d_1 d_3}{\mu_1 d_1 + d_0} < 0, \quad (51)$$

$$a_M'' = a_M' (\mu_1 d_1 + d_0)^{-2} \{[(d_2 d_0 - d_1 d_3) - 2d_1 (\mu_1 d_1 + d_0)] (1 - \mu_1 (1 - p(A_1))) - 2(1 - p(A_1)) (\mu_1 d_1 + d_0)^2 \} \triangleq a_M' (\mu_1 d_1 + d_0)^{-2} l(\mu_1) \quad (52)$$

Note that $\mu_1 (p_1 - p_3) + (1 - \mu_1) (p_2 - p_4) = (1 - (1 - p(A_1)) \mu_1) (p_2 - p_4)$, after rearrangement of $g_{MAC}(\mu_1)$, we have

$$g_{MAC}(\mu_1) = \frac{1}{F(\mu_1)} - \frac{\mu_1 (p_3 - p_4) + p_4}{(1 - (1 - p(A_1)) \mu_1) (p_2 - p_4)}, \quad (53)$$

where $F(\mu_1) \triangleq (a_M + 1) (1 - (1 - p(A_1)) \mu_1) (p_2 - p_4) > 0$. It is easy to check

$$F'(\mu_1) = (p_2 - p_4) \{a_M' (1 - (1 - p(A_1)) \mu_1) - (1 - p(A_1)) (a_M + 1)\} < 0, \quad (54)$$

$$F''(\mu_1) = a_M' (\mu_1 d_1 + d_0)^{-2} \left\{[(d_2 d_0 - d_1 d_3) - 2d_1 (\mu_1 d_1 + d_0)] (1 - \mu_1 (1 - p(A_1))) - 2(1 - p(A_1)) (\mu_1 d_1 + d_0)^2 \right\} \triangleq a_M' (\mu_1 d_1 + d_0)^{-2} l(\mu_1) \quad (55)$$

It is easy to check that $l(\mu_1)$ is a linear function. Note that $d_1 = -d_0 (1 - p(A_1))$, we have

$$l(1) = [(d_2 d_0 - d_1 d_3) - 2d_1 (d_1 + d_0)] p(A_1) - 2(1 - p(A_1)) (d_1 + d_0)^2 = (d_2 d_0 - d_1 d_3) p(A_1) < 0, \quad (56)$$

$$l(0) = (d_2 d_0 - d_1 d_3 - 2d_1 d_0) - 2(1 - p(A_1)) d_0^2 = d_2 d_0 - d_1 d_3 < 0, \quad (57)$$

therefore, we have $l(\mu_1) < 0$ and $F''(\mu_1) < 0$ for $\mu_1 \in [0, 1]$ and $\left[\frac{1}{F'(\mu_1)}\right]' = \frac{2}{F'(\mu_1)} [F'(\mu_1)]^2 - \frac{1}{F''(\mu_1)} F''(\mu_1) > 0.$
For the last fraction in Equation (53), we have its derivation as follows,

\[
\left[-\frac{\mu_1(p_3 - p_4) + p_4}{(1 - (1 - p(A_1))\mu_1)(p_2 - p_4)}\right]'' = \frac{-2(1 - p(A_1))[(p_3 - p_4) + p_4(1 - p(A_1))]}{(1 - (1 - p(A_1))\mu_1)^3(p_2 - p_4)} = \frac{-2(a_M + 1)^3(p_2 - p_4)^2(1 - p(A_1))[p_3 - p_4p(A_1)]}{F^3(\mu_1)}
\]

Based on Equation (53), \( g''_{MAC} \) is given by

\[
g''_{MAC} = \left[\frac{1}{F'(\mu_1)}\right]'' - \left[-\frac{\mu_1(p_3 - p_4) + p_4}{(1 - (1 - p(A_1))\mu_1)(p_2 - p_4)}\right]' \\
\frac{2}{F^3(\mu_1)} + \left[F' (\mu_1)\right]^2 \frac{-2(a_M + 1)^3(p_2 - p_4)^2(1 - p(A_1))[p_3 - p_4p(A_1)]}{F^3(\mu_1)} \\
\frac{2(p_2 - p_4)^2}{F^3(\mu_1)} \left\{a_M'(1 - (1 - p(A_1))\mu_1) - (1 - p(A_1))(a_M + 1)\right\} \\
- (a_M + 1)^3(1 - p(A_1))[p_3 - p_4p(A_1)]
\]

\[
\frac{2(p_2 - p_4)^2(a_M + 1)^2(1 - p(A_1))}{F^3(\mu_1)} \left\{(1 - p(A_1)) - (a_M + 1)[p_3 - p_4p(A_1)]\right\} \\
\frac{2(p_2 - p_4)^2(a_M + 1)^2(1 - p(A_1))}{F^3(\mu_1)} \left\{(1 - p(A_1)) - (p_2 + 1)p_3\right\} \\
\frac{2(p_2 - p_4)^2(a_M + 1)^2(1 - p(A_1))}{F^3(\mu_1)} \left\{(1 - p(A_1)) - p_3p_2 + (1 - p_3)p(A_1)\right\} > 0 \tag{58}
\]

where (a) holds by dropping out positive terms \(-\frac{1}{F'(\mu_1)}F''(\mu_1)\); (b) holds since the term \(a_M'(1 - (1 - p(A_1))\mu_1) < 0 \) based on Equation (51), and \((1 - p(A_1))(a_M + 1) > 0\); (c) holds since \(a_M \leq a_M(0) = \exp\left(\frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4}\right)\) based on Equation (51); (d) holds since \(\exp\left(\frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4}\right) > \exp(h_b'(p_2)) = p_2\), and (e) holds since \(\tau \leq \frac{\ln 2}{A_1 + A_2 + A_0}\), \(p_1 \leq \frac{1}{2}\), \(p_1 - p(A_1) = (1 - p(A_1))p_2\) and \(p_1 - p_3 = (1 - p_3)p(A_1)\). Thus, we have function \(g_{MAC}(\mu_1)\) is a strictly convex function as \(\tau \leq \frac{\ln 2}{A_1 + A_2 + A_0}\).

E. Proof of Lemma 4

Note that \(\lim_{A_2 \to \infty} p_1 = \lim_{A_2 \to \infty} p_2 = 1\), and \(\lim_{A_2 \to \infty} h_b(p_1) = \lim_{A_2 \to \infty} h_b(p_2) = 0\), we have \(\lim_{A_2 \to \infty} U = 0\), and

\[
\lim_{A_2 \to \infty} V = -\lim_{A_2 \to \infty} W = -h_b(p_4) + h_b(p_3) + p_3h_b(p_4) - p_4h_b(p_4), \tag{59}
\]
thus \( \lim_{A_2 \to \infty} f_{MAC}(\mu_1) = 1 \). As for \( \lim_{A_2 \to \infty} g_{MAC} \), it is straightforward that \((1 + a_{MI})^{-1} < 1 \), thus we have \( \lim_{A_2 \to \infty} g_{MAC} < 1 \) for any \( \mu_1 \in [0, 1] \).

\[ \text{F. Proof of Proposition 2} \]

The main proof is based on continuity of \( I_{X^2_1;Z}(\mu_1, \mu_2) \). When \( \frac{\partial I}{\partial \mu_2}(\bar{\mu}_1, 0) > 0 \), for any \( \epsilon \), there exists \( 0 \mu_2 < \epsilon \) so that \( I_{X^2_1;Z}(\bar{\mu}_1, 0) < I_{X^2_1;Z}(\bar{\mu}_1, \mu_2) \). Therefore, single user 1 transmission is not optimal. Similarly, we have single user 2 transmission is not optimal if \( \frac{\partial I}{\partial \mu_1}(0, \bar{\mu}_1) > 0 \).

\[ \text{G. Proof of Theorem 3} \]

Similar to \([25]\), define \( \phi(x) \triangleq x \ln x \) and \( \alpha(x) \triangleq \frac{1}{x} \left(e^{-\frac{1}{x}}(1 + x)^{\frac{1}{x}} - 1\right) \). We first focus on the candidate optimal duty cycle for 3 scenarios. For Scenario 2 of only active user 1, since \( h_b(x) = -x \ln(x) - (1 - x) \ln(1 - x) = x - x \ln(x) + o(x) \) and \( \ln \frac{p(x)}{\tau} = \ln(x) + o(\tau) \), we have

\[
\frac{h_b(p_3) - h_b(p_4)}{p_3 - p_4} + \ln \tau = \frac{A_1 \tau - A_1 \tau \frac{p_4}{\tau} + p_4 \ln \frac{p_4}{\tau} + o(\tau)}{p_3 - p_4}
\]

\[
= 1 + \frac{\Lambda_0 \ln \Lambda_0 - (A_1 + \Lambda_0) \ln(A_1 + \Lambda_0)}{A_1} + o(1) = 1 - \ln(A_1 + \Lambda_0) + s_1 \ln \frac{s_1}{1 + s_1} + o(1),
\]

where \( s_1 = \frac{\Lambda_0}{A_1} \). Thus, we have

\[
\lim_{\tau \to 0} \bar{\mu}_1 = \lim_{\tau \to 0} \alpha(\Lambda_1, \Lambda_0) = \lim_{\tau \to 0} \left[1 + \exp\left(\frac{h_b(p_3) - h_b(p_4)}{p_3 - p_4}\right)\right]^{-1} - p_4
\]

\[
= \lim_{\tau \to 0} \left[\tau + \exp\left(\frac{h_b(p_3) - h_b(p_4)}{p_3 - p_4} + \ln \tau\right)\right]^{-1} - \Lambda_0 = \frac{1 + s_1}{e} \left(\frac{s_1}{1 + s_1}\right)^{-s_1} - s_1 = \alpha(s_1^{-1}). \tag{61}
\]

Equation (61) implies that optimal duty cycle for Scenario 2 approaches that of continuous time as \( \tau \to 0 \) \([25]\) Equation (22)). Similar to Scenario 2, the optimal duty cycle for Scenario 3 also approaches that of continuous time as \( \tau \to 0 \).

For Scenario 1 of both active users, we focus on the asymptotic property of functions \( g_{MAC}(\cdot) \) and \( f_{MAC}(\cdot) \). Similar to Equation (60), we have

\[
\frac{\mu_1(h_b(p_1) - h_b(p_3)) + (1 - \mu_1)(h_b(p_2) - h_b(p_4))}{\mu_1(p_1 - p_3) + (1 - \mu_1)(p_2 - p_4)} + \ln \tau
\]

\[
\frac{\mu_1 A_2(1 + \phi(A_1 + \Lambda_0) - \phi(A_1 + A_2 + \Lambda_0)) + (1 - \mu_1) A_2(1 + \phi(A_0) - \phi(A_2 + \Lambda_0))}{\mu_1 A_2 + (1 - \mu_1) A_2} + o(1)
\]

\[
= 1 + \mu_1 A_2^{-1}(\phi(A_1 + \Lambda_0) - \phi(A_1 + A_2 + \Lambda_0)) + (1 - \mu_1) A_2^{-1}(\phi(A_0) - \phi(A_2 + \Lambda_0)) + o(1),
\]
and the function $g_{MAC}(\cdot)$ is given by
\[
\lim_{\tau \to 0} g_{MAC}(\mu_1) = \lim_{\tau \to 0} \frac{(a_M \tau + \tau)^{-1} - [\mu_1(A_1 + \Lambda_0) + (1 - \mu_1)\Lambda_0]}{\mu_1 A_2 + (1 - \mu_1) A_2} \\
= \frac{1}{A_2} \left\{ \exp \left( -1 - \mu_1 A_2^{-1}(\phi(A_1 + \Lambda_0) - \phi(A_1 + A_2 + \Lambda_0)) \right) \right. \\
- \left. (1 - \mu_1) A_2^{-1}(\phi(\Lambda_0) - \phi(A_2 + \Lambda_0)) \right\} - \frac{\mu_1 A_1 + \Lambda_0}{A_2}. \quad (63)
\]

which is aligned with [25 Equation (19)]. For the function $f_{MAC}(\cdot)$, note that 
\[(p_1 - p_2)(p_3 - p_4) = (1 - p_2)(1 - p_4)p^2(A_1) \quad \text{and} \quad (p_1 - p_3)(p_2 - p_4) = (1 - p_3)(1 - p_4)p^2(A_2),\]
we have
\[
\lim_{\tau \to 0} \frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)} = \lim_{\tau \to 0} \frac{(1 - p_2)p^2(A_1)}{(1 - p_3)p^2(A_2)} = \frac{A_1^2}{A_2^2}. \quad (64)
\]

Based on Equation (60), we have
\[
\lim_{\tau \to 0} \frac{U}{V} = \lim_{\tau \to 0} \frac{(p_1 - p_2)(h_b(p_3) - h_b(p_4)) - (p_3 - p_4)(h_b(p_1) - h_b(p_2))}{(p_1 - p_3)(h_b(p_2) - h_b(p_4)) - (p_2 - p_4)(h_b(p_1) - h_b(p_3))} \\
= \frac{A_1^{-1}(\phi(\Lambda_0) - \phi(A_2 + \Lambda_0) - \phi(A_1 + \Lambda_0) + \phi(A_1 + A_2 + \Lambda_0))}{A_2^{-1}(\phi(\Lambda_0) - \phi(A_2 + \Lambda_0) - \phi(A_1 + \Lambda_0) + \phi(A_1 + A_2 + \Lambda_0))} \lim_{\tau \to 0} \frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)} \\
= \frac{A_1}{A_2}. \quad (65)
\]

Note that 
\[-W = p_2(h_b(p_3) - h_b(p_4)) - p_3(h_b(p_2) - h_b(p_4)) + p_4(h_b(p_2) - h_b(p_3)) = (p_2 - p_4)(h_b(p_3) - h_b(p_4)) - (p_3 - p_4)(h_b(p_2) - h_b(p_4))\]
and
\[
\lim_{\tau \to 0} \frac{(p_2 - p_4)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)} = \lim_{\tau \to 0} \frac{p(A_2)p(A_1)}{p^2(A_2)} = \frac{A_1}{A_2},
\]
we have
\[
\lim_{\tau \to 0} \frac{W}{V} = \lim_{\tau \to 0} \frac{(p_2 - p_4)(h_b(p_3) - h_b(p_4)) - (p_3 - p_4)(h_b(p_2) - h_b(p_4))}{(p_1 - p_3)(h_b(p_2) - h_b(p_4)) - (p_2 - p_4)(h_b(p_1) - h_b(p_3))} \\
= \frac{A_1^{-1}(\phi(\Lambda_0) - \phi(A_1 + \Lambda_0))}{A_2^{-1}(\phi(\Lambda_0) - \phi(A_2 + \Lambda_0) - \phi(A_1 + \Lambda_0) + \phi(A_1 + A_2 + \Lambda_0))} \lim_{\tau \to 0} \frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)} \\
= \frac{\phi(\Lambda_0) - \phi(A_1 + \Lambda_0) - \frac{A_1}{A_2}(\phi(\Lambda_0) - \phi(A_2 + \Lambda_0))}{\phi(\Lambda_0) - \phi(A_2 + \Lambda_0) - \phi(A_1 + \Lambda_0) + \phi(A_1 + A_2 + \Lambda_0)}, \quad (66)
\]

which is aligned with [25 Equation (16)]. Based on equations (63), (65) and (66), all candidate optimal duty cycle approach to that of continuous time as $\tau \to 0$. 
For demonstrating the asymptotic property, we just need to show the asymptotic property of optimized objective function. According to L’Hospital’s rule, we have

\[
\lim_{\tau \to 0} \frac{I_{X_1^Z}(\mu_1, \mu_2)}{\tau} = \lim_{\tau \to 0} \frac{\partial I_{X_1^Z}(\mu_1, \mu_2)}{\partial \tau}
\]

\[
= \lim_{\tau \to 0} \mu_1 \mu_2 (1 - p_1)(A_1 + A_2 + \Lambda_0) \ln \left( \frac{1 - \hat{p}}{p(1 - p_1)} \right) + (1 - \mu_1)\mu_2 (1 - p_2)(A_2 + \Lambda_0) \ln \left( \frac{1 - \hat{p}}{p(1 - p_2)} \right)
+ \mu_1 (1 - \mu_2)(1 - p_3)(A_1 + \Lambda_0) \ln \left( \frac{1 - \hat{p}}{p(1 - p_3)} \right) + (1 - \mu_1)(1 - \mu_2)(1 - p_4)\Lambda_0 \ln \left( \frac{1 - \hat{p}}{p(1 - p_4)} \right)
\]

\[
= \mu_1 \mu_2 \phi(A_1 + A_2 + \Lambda_0) + (1 - \mu_1)\mu_2 \phi(A_2 + \Lambda_0) + \mu_1 (1 - \mu_2) \phi(A_1 + \Lambda_0)
+ (1 - \mu_1)(1 - \mu_2) \phi(\Lambda_0) - \phi(\mu_1 A_1 + \mu_2 A_2 + \Lambda_0),
\]

(67)

which is aligned with [25, Equation (8)]. Thus, the MAC Poisson capacity with non-perfect receiver approaches to that of continuous time.

**H. Proof of Lemma**

Since \( \frac{\partial p_1}{\partial A} = 2\tau(1 - p_1) \) and \( \frac{\partial p_2}{\partial A} = \tau(1 - p_2) \), the derivative of \( G(A) \) is given by

\[
G'(A) = \frac{2\tau}{(p_2 - p_1 - p_4)} \left\{ \frac{[h'_b(p_2)(1 - p_2) - h'_b(p_1)(1 - p_1)]}{p_2 - p_1 - p_4} > 0 \right\}
- \frac{(p_1 - p_2)}{p_1 - p_2} \frac{[2h_b(p_2) - h_b(p_1) - h_b(p_4)]}{p_1 - p_2} > 0
\]

\[\]

\[
= \frac{2\tau(p_1 - p_2)}{(p_2 - p_1 - p_4)(p_2 - p_4)p(A)(1 - p(A))} \left\{ [(\ln p_1 - \ln p_2) + (h_b(p_1) - h_b(p_2))]p(A)
- [2h_b(p_2) - h_b(p_1) - h_b(p_4)](1 - p(A)) \right\}
\]

\[\]

\[
= \frac{2\tau(p_1 - p_2)}{(p_2 - p_1 - p_4)(p_2 - p_4)p(A)(1 - p(A))} \left\{ [(\ln p_1 - \ln p_2)]p(A) + (h_b(p_1) - h_b(p_2))
- [h_b(p_2) - h_b(p_4)](1 - p(A)) \right\}
\]

(68)

where \( h'_b(p_2)(1 - p_2) - h'_b(p_1)(1 - p_1) < 0 \) holds since \( \frac{d}{dx}(1 - x)h'_b(x) = -\frac{1}{x} - \ln \frac{1 - x}{x} < 0, \)

\( 2p_2 - p_1 - p_4 > 0 \) holds due to concave function \( p(\cdot) \) and \( 2h_b(p_2) - h_b(p_1) - h_b(p_4) > 0 \) holds according to Lemma\[9\] equality \( (a) \) holds since \( (1 - x)h'_b = -\ln x - h_b(x) \); inequality \( (b) \) holds since \( p_1 - p_2 = (p_2 - p_4)(1 - p(A)) \) and \( 2p_2 - p_1 - p_4 = (p_2 - p_4)p(A) \). Define \( t = p(A) \), we
have $p_1 = p_4 + (1 - p_4)p(2A) = p_4 + (1 - p_4)(2t - t^2)$ and $p_2 = p_4 + (1 - p_4)t$. For obtaining $\ln \frac{p_1}{p_2} < 0$, it is sufficient to show $\hat{G}(t, p_4) = [(\ln p_1 - \ln p_2)p(A) + (h_b(p_1) - h_b(p_2)) - [h_b(p_2) - h_b(p_4)](1-p(A))] < 0$, where $0 \leq t \leq \frac{1}{2}$, and

$$
\hat{G}(t) \triangleq t \ln \frac{p_4 + (1 - p_4)(2t - t^2)}{p_4 + (1 - p_4)t} + h_b(p_4 + (1 - p_4)(2t - t^2)) - h_b(p_4 + (1 - p_4)t) - (1-t)[h_b(p_4 + (1 - p_4)t) - h_b(p_4)],
$$

(69)

It is easy to check that $\hat{G}(0, p_4) = 0$ for any $p_4$ and $\hat{G}(t, 0) = t \ln(2-t) + h_b(t(2-t)) - (2-t)h_b(t)$.

Thus, we have

$$
\frac{\partial \hat{G}(t, 0)}{\partial t} = \ln(2 - t) - \frac{t}{2 - t} + h_b'(t(2 - t))2(1 - t) - (2 - t)h_b'(t) + h_b(t)
$$

$$
= 2(1-t)(\ln \frac{(1-t)^2}{t(2-t)} - \ln \frac{1-t}{t}) - \ln(1-t) + \ln(2-t) - \frac{t}{2 - t}
$$

$$
= (1-2t)\ln \frac{1-t}{2-t} - \frac{t}{2 - t},
$$

(70)

$$
\frac{\partial^2 \hat{G}(t, 0)}{\partial t^2} = -2 \ln \frac{1-t}{2-t} - (1-2t)(\frac{1}{1-t} - \frac{1}{2-t}) - \frac{2}{(2-t)^2}
$$

$$
= 2 \ln(1 + \frac{1}{1-t}) - \frac{1-2t}{1-t} + \frac{t(2t-5)}{(2-t)^2}
$$

(71)

$$
\geq \frac{2}{2-t} - \frac{1-2t}{1-t} + \frac{t(2t-5)}{(2-t)^2} = \frac{t}{(1-t)(2-t)^2} > 0.
$$

(72)

Thus we have $\frac{\partial \hat{G}(t, 0)}{\partial t} \leq \hat{G}(\frac{1}{2}, 0) = -\frac{1}{3} < 0$ and $\hat{G}(t, 0) \leq \hat{G}(0, 0) = 0$.

Similarly, for $p_4 > 0$, we have

$$
\frac{\partial \hat{G}(t, p_4)}{\partial t} = \ln \frac{p_1}{p_2} + t(1-p_4)\left(\frac{2(1-t)}{p_1} - \frac{1}{p_2}\right) + 2(1-t)(1-p_4)h_b'(p_1) - (2-t)(1-p_4)h_b'(p_2)
$$

$$
\leq 2(1-t)(1-p_4)\ln \frac{(1-p_1)p_2}{p_1(1-p_2)} + \ln \frac{p_1}{p_2} - (1-p_4)h_b'(p_2) + (h_b(p_2) - h_b(p_4))
$$

(73)
where equality (d) holds since \(1 - p_2 = (1 - p_1)(1 - t), \) \(t(1 - p_4) = p_2 - p_4\) and \(p_2 h_b'(p_2) + h_b(p_2) = -\ln(1 - p_2), \) \(I_r < 0\) since \(\ln \frac{1 - t}{1 - p_2} + p_4 h_b'(p_2) - h_b(p_4) = \ln \frac{1 - t}{1 - p_4} + p_4 h_b'(p_4) - h_b(p_4) < 0\), inequality (e) holds since \(1 - p_2 = (1 - t)(1 - p_4),\) \((1 - t)p_2 = p_1 - t\) and \(I_r < 0\).

Thus, \(G(A)\) decreases with peak power \(A\). For peak power \(A \to \infty\), we have \(\lim_{A \to \infty} G(A) = \lim_{A \to \infty} \frac{2h_b(p_2) - h_b(p_1) - h_b(p_4)}{2p_2 - p_1 - p_4} = \frac{h_b(p_0)}{1 - p_0}.\) For peak power \(A \to 0\), we have Taylor expansion on \(p_1\) and \(p_2\) as follows,

\[
p_1 = p_4 + (1 - p_4)\tau \cdot 2A - \frac{1 - p_4}{2} \tau^2 \cdot (2A)^2 + o(A^2),
\]

\[
p_2 = p_4 + (1 - p_4)\tau \cdot A - \frac{1 - p_4}{2} \tau^2 \cdot A^2 + o(A^2),
\]

thus we have \(2p_2 - p_1 - p_4 = (1 - p_4)\tau^2 A^2 + o(A^2).\) Similarly, we have Taylor expansion on \(2h_b(p_2) - h_b(p_1) - h_b(p_4)\) as follows,

\[
2h_b(p_2) - h_b(p_1) - h_b(p_4) = -[h''_b(p_4)(1 - p_4) - h'_b(p_4)](1 - p_4)\tau^2 A^2 + o(A^2),
\]

and the limits \(\lim_{A \to 0} G(A)\) is given by

\[
\lim_{A \to 0} G(A) = \frac{-[h''_b(p_4)(1 - p_4) - h'_b(p_4)](1 - p_4)\tau^2}{(1 - p_4)\tau^2} = \frac{1}{p_4} + \ln \frac{1 - p_4}{p_4} > \ln \frac{1 - p_4}{p_4}.
\]

Thus we have \(G(A) \in (\ln(1 - p_4) + \frac{p_4}{1 - p_4} \ln p_4, \frac{1}{p_4} + \ln \frac{1 - p_4}{p_4})\).

I. Proof of Lemma 5

Note that \(\hat{p} \in [p_4, p_1]\) for \((\mu_1, \mu_2) \in [0, 1]^2\), we have \(\ln \frac{1 - \hat{p}}{\hat{p}} \in [\ln \frac{1 - p_1}{p_1}, \ln \frac{1 - p_4}{p_4}].\) Based on Lemma 12 and \(p_2 - p_4 > p_1 - p_2\), we have \(G(A) > \frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4} > \ln \frac{p_4}{p_2} > \ln \frac{p_4}{p_1}.\) According to Lemma 5 and \(\lim_{A \to 0} G(A) = \frac{1}{p_4} + \ln \frac{1 - p_4}{p_4} > \ln \frac{1 - p_4}{p_4},\) there exists unique \(A_{th}\) such that \(\ln \frac{1 - p_4}{p_4} = G(A)\) and the solution \(\ln \frac{1 - \hat{p}}{\hat{p}} = G(A)\) for \((\mu_1, \mu_2) \in [0, 1]^2\) iff \(A \geq A_{th}\).

J. Proof of Theorem 4

It is easy to check that \(G(A) \leq \ln \frac{1 - p_4}{p_4}\) for \(A \geq A_{th}.\) Since \(\ln \frac{1 - \hat{p}}{\hat{p}} = G(A)\) and \(\ln \frac{1 - \hat{p}}{\hat{p}}\) decreases with \(\hat{p},\) we have \(\hat{p} = \frac{1}{1 + \exp(C(A))} > p_4.\) Note that continuous function \(\hat{p}(\mu_1, \mu_2)\) increases with \(\mu_1\) and \(\mu_2,\) thus, there exists differentiable function \(f_B(\cdot)\) such that \(C = \{(\mu_1, \mu_2) : \mu_1 = f_B(\mu_2)\}.\)
According to Lemma 12 we have \( G(A) > \frac{h_b(p_2) - h_b(p_4)}{p_2 - p_4} > h'_b(p_2) \) and \( \frac{1}{1 + \exp(G(A))} < p_2 \). Note that the solution \( \hat{\mu}(\mu_1, 0) = \mu_1 p_3 + (1 - \mu_1) p_4 = \frac{1}{1 + \exp(G(A))} \in (p_4, p_2) \) on \( \mu_1 \) exists, we have \( 0 < f_B(0) < 1 \). For region \( \mu_1 \geq \mu_2 \), we have
\[
\frac{\partial \hat{\mu}}{\partial \mu_1} - \frac{\partial \hat{\mu}}{\partial \mu_2} = \begin{bmatrix} \mu_2(p_1 - p_2) + (1 - \mu_2)(p_3 - p_4) \end{bmatrix} - \begin{bmatrix} \mu_1(p_1 - p_3) + (1 - \mu_1)(p_2 - p_4) \end{bmatrix} \\
\begin{bmatrix} (\mu_1 - \mu_2)(2p_2 - p_1 - p_4) \end{bmatrix} \geq 0
\]
(78)
Take total differential on \( \hat{\mu}(\mu_1, \mu_2) = \frac{1}{1 + \exp(G(A))} \), we have
\[
f'_B(\mu_2) = \frac{d\mu_1}{d\mu_2} = -\frac{\partial \hat{\mu}}{\partial \mu_2} \leq -1,
\]
(79)
where the last inequality holds since \( \frac{\partial \hat{\mu}}{\partial \mu_1} > \frac{\partial \hat{\mu}}{\partial \mu_2} > 0 \).

Since the cardinality of \( |\mathcal{C} \cap \mathcal{L}_{\mu_1} | \) equals the number of intersect of \( \mu_1 = f_B(\mu_2) \) and \( \mu_1 + \mu_2 = 2\mu_s \) for \( \mu_1 \geq \mu_2 \). Define \( g_B(\mu_2) = f_B(\mu_2) - (2\mu_s - \mu_2) \), according to Equation (79), we have \( g'_B(\mu_2) = f'_B(\mu_2) + 1 \leq 0 \) and the number of intersect of \( \mu_1 = f_B(\mu_2) \) and \( \mu_1 + \mu_2 = 2\mu_s \) for \( \mu_1 \geq \mu_2 \) is at most 1. Furthermore, we have \( |\mathcal{C} \cap \mathcal{L}_{\mu_s} | = 1 \) iff \( f_B(0) \geq 2\mu_s \) and \( f_B(\mu_s) \leq \mu_s \).
Define \( f_B(\mu'_s) = \mu'_s \) and \( \mu^*_s = \frac{f_B(0)}{2} = \frac{1}{1 + \exp(G(A))} - p_4 \) and \( \mu^*_s < \frac{\mu_s}{2} \) since \( G(A) > \frac{h_b(p_3) - h_b(p_4)}{p_3 - p_4} \). In addition, for \( \mu'_s \), we have
\[
(2p_2 - p_1 - p_4)(\mu'_s)^2 - 2(p_2 - p_4)\mu'_s + \left(\frac{1}{1 + \exp(G(A))} - p_4\right) = 0
\]
(80)
Since \( 2p_2 - p_1 - p_4 > 0 \), \( p_2 - p_4 > 0 \) and \( \frac{1}{1 + \exp(G(A))} - p_4 > 0 \), the two solutions on Equation (80) both are positive. Note that the summation of the two solutions equals to \( \frac{2(p_2 - p_4)}{2p_2 - p_1 - p_4} > 2 \), thus there exists unique feasible solution as follows,
\[
\mu'_s = \frac{(p_2 - p_4) - \sqrt{(p_2 - p_4)^2 - 2(p_2 - p_1 - p_4)\left[\frac{1}{1 + \exp(G(A))} - p_4\right]}}{2p_2 - p_1 - p_4}
\]
(81)

K. Proof of Lemma 7

For any fixed peak power \( A \), we need to show that there exists \( \tau > 0 \) such that \( G(A) > \ln \frac{1 - p_4}{p_4} \). The main clue is based on Taylor expansion of \( \tau \).

Note that \( p(x) = x \tau - \frac{1}{2} x^2 \tau^2 + o(\tau^2) \), we have
\[
2p_2 - p_1 - p_4 = \frac{1}{2} [(2A + \Lambda_0)^2 + \Lambda_0^2 - (A + \Lambda_0)^2] \tau^2 + o(\tau^2) = A^2 \tau^2 + o(\tau^2).
\]
(82)
Since $h_b(x) = (1 - \ln x)x + o(x)$, we have Taylor expansion of $h_b(p(x))$ on $\tau$ as follows,

$$h_b(p(x)) = [1 - \ln p(x)]p(x) + o(p(x)) = p(x) - p(x)\ln \tau + p(x)\ln(x + o(\tau)) + o(\tau) = p(x) - p(x)\ln \tau + [x \ln x]_x + o(\tau).$$

(83)

Similarly, we have

$$\ln \frac{1 - p_4}{p_4} = -\ln \tau - \ln \Lambda_0 - \Lambda_0\ln \Lambda_0 + o(\tau),$$

(84)

Based on Equations (82), (83) and (84), we have

$$G(A) - \ln \frac{1 - p_4}{p_4} = \frac{2h_b(p_2) - h_b(p_1) - h_b(p_4)}{2p_2 - p_1 - p_4} - \ln \frac{1 - p_4}{p_4} = (1 - \ln \tau) + \frac{[2(A + \Lambda_0)\ln(A + \Lambda_0) - (2A + \Lambda_0)\ln(2A + \Lambda_0) - \Lambda_0\ln \Lambda_0]_\tau}{A^2\tau^2}$$

$$-[-\ln \tau - \ln \Lambda_0] + O(1)$$

$$= \frac{[2(A + \Lambda_0)\ln(A + \Lambda_0) - (2A + \Lambda_0)\ln(2A + \Lambda_0) - \Lambda_0\ln \Lambda_0]_\tau}{A^2} + O(1)$$

(85)

Since function $x \ln x$ is convex and $2(A + \Lambda_0)\ln(A + \Lambda_0) - (2A + \Lambda_0)\ln(2A + \Lambda_0) - \Lambda_0\ln \Lambda_0 > 0$, there exists $\tau > 0$ such that $G(A) > \ln \frac{1 - p_4}{p_4}$ for any fixed peak power $A$, i.e., $\lim_{\tau \to 0} A_{th} = +\infty$.

L. Proof of Proposition

Define $b_{mj}(i_m) \in \{0, 1\}$ as $j^{th}$ bit in the binary representation of $i_m$ for user $m$, where $i_m = 0, 1, \cdots, 2^{J_m} - 1$, $i_m = \sum_{j_m=1}^{J_m} b_{mj}(i_m)2^{i-1}$. Define joint PMF $q_{[M]} = (q_1, q_2)$, where $q_m = \{q_{mim}\}_{i_m=0}^{2^{J_m}}\triangleq \mathbb{P}(X_{mj} = b_{mj}(i)A_{mj})$, $i_m = 0, 1, \cdots, 2^{J_m} - 1$. Then, $q_{[M]}$ satisfies

$$q_{mim} \geq 0, \quad i = 0, 1, \cdots, 2^{J_m} - 1;$$

$$\sum_{i_m=0}^{2^{J_m} - 1} q_{mim} = 1;$$

$$\sum_{i_m=0}^{2^{J_m} - 1} b_{mj}(i_m)q_{mim} = \mu_{mj}, j_m = 1, 2, \cdots, J_m, \quad m = 1, 2.$$
Define $\tilde{r}_{i_2i_12} = p(\sum_{m=1}^{2} \sum_{j=m}^{j_m} b_{j,m}(i_m)A_{m} + \Lambda_0)$. Noting that $\mu_{m,j} = \sum_{j_m=0}^{2^j-1} b_{j,m}(i_m)q_{m,j}$ for $j_m = 1, 2, \ldots, J_m$, we have $C_{MAC} = \max_{\mu_{M,L} \in [0,1]^{M,L}} \frac{1}{\gamma} I_{MISO}[\mu_{M,L}]$, where

$$I_{MISO[\mu_{M,L}]} = \max_{q_M} h_b \left( \sum_{i_2=0}^{2^j-1} \sum_{i_1=0}^{J_m} q_{i_2i_1} h_b(p(A_{m,j}))) - \sum_{i_2=0}^{2^j-1} \sum_{i_1=0}^{J_m} q_{i_2i_1} h_b(\tilde{r}_{i_1i_2}) \right)$$

$$= \max_{q_M} h(q_M), \quad (87)$$

Noting that

$$\sum_{i_2=0}^{2^j-1} \sum_{i_1=0}^{J_m} q_{i_1i_2} b_{m,j,m}(i_m) h_b(p(A_{m,j}))) = \sum_{m=1}^{2^j-1} \sum_{j_m=0}^{J_m} q_{i_2i_1} h_b(\tilde{r}_{i_1i_2})$$

we have

$$h(q_M) = h_b \left( \sum_{i_2=0}^{2^j-1} \sum_{i_1=0}^{J_m} q_{i_2i_1} \tilde{r}_{i_1i_2} \right) - \sum_{i_2=0}^{2^j-1} \sum_{i_1=0}^{J_m} q_{i_2i_1} h_b(\tilde{r}_{i_1i_2})$$

$$+ \sum_{i_2=0}^{2^j-1} \sum_{i_1=0}^{J_m} q_{i_1i_2} \sum_{m=1}^{J_m} b_{m,j,m}(i_m) h_b(p(A_{m,j}))) - \sum_{m=1}^{2^j-1} \sum_{j_m=0}^{J_m} \mu_{j,m} h_b(p(A_{m,j})))$$

$$\frac{\partial h(q_M)}{\partial q_{m,i_{m}}} = \sum_{i_{m}=0}^{2^j-1} \tilde{r}_{i_{m}i_m} \ln \frac{1 - \sum_{i_{m}=0}^{2^j-1} \sum_{i_{m}=0}^{2^j-1} q_{i_2i_1i_{m}} \tilde{r}_{i_1i_2}}{\sum_{i_{m}=0}^{2^j-1} \sum_{i_{m}=0}^{2^j-1} q_{i_2i_1i_{m}} \tilde{r}_{i_1i_2}} - \sum_{i_{m}=0}^{2^j-1} q_{i_{m}i_m} h_b(\tilde{r}_{i_{m}i_m})$$

$$+ \sum_{i_{m}=0}^{2^j-1} q_{i_{m}i_m} \sum_{m=1}^{J_m} b_{m,j,m}(i_m) h_b(p(A_{m,j})))$$

$$\quad, \quad (89)$$

where $m = 3 - m$. Let $I_{m,i_m} = \{ i_m \in \{1, 2, \ldots, J_m \} : b_{m,j,m}(i_m) = 1 \}$ denotes the set of nonzero bit positions in the binary representation of $i_m, i_m = 0, 1, \ldots, 2^J - 1$. Then, we will show that $I_{m,i_m} \subseteq I_{m,i'_m}$ leads to

$$\frac{\partial h(q_M)}{\partial q_{m,i'_m}} \leq \frac{\partial h(q_M)}{\partial q_{m,i_m}}, \quad \text{for} \ i_m, i'_m \in \{0, 1, \ldots, 2^J - 1\}. \quad (90)$$
For $\tau \leq \sum_{m=1}^{J_m}\ln \frac{2^{J_m}}{\sum_{m=1}^{J_m}M_m}$, we have $\ln \frac{1}{\sum_{i_{j}=0}^{2^{J_m}-1}q_{2i_j}q_{i_{j+1}}\hat{r}_{i_{j+1}}} > 0$. According to lemma 9, $h_b(p(x))$ is concave. Based on Lemma 10 and $h_b(p(0)) = 0$, we have

$$\sum_{m=0}^{2^{J_m}-1} q_{\Pi_{m}\Delta m} [h_b(\hat{r}_{\Pi_{m}\Delta m}) - h_b(\hat{r}'_{\Pi_{m}\Delta m})] \leq \sum_{m=0}^{2^{J_m}-1} q_{\Pi_{m}\Delta m} h_b(p(\sum_{m=1}^{J_m}b_m(i_m) - b_m(i'_m)A_{m(m)}) \leq \sum_{m=0}^{2^{J_m}-1} 2 \sum_{m=1}^{J_m} [b_m(i_m) - b_m(i'_m)]h_b(p(A_{m(m)})) .$$

According to equation (89) and $r_i \geq r_i'$, we have

$$\frac{\partial h(q_{[M]})}{\partial q_{m(i_m)}} - \frac{\partial h(q_{[M]})}{\partial q_{m(i'_m)}} = \sum_{i_m=0}^{2^{J_m}-1} \left( \hat{r}_{i_m} - \hat{r}'_{i_m} \right) \ln \frac{1 - \sum_{i_{j}=0}^{2^{J_m}-1} q_{2i_j}q_{i_{j+1}}\hat{r}_{i_{j+1}}}{\sum_{i_{j}=0}^{2^{J_m}-1} q_{2i_j}q_{i_{j+1}}\hat{r}_{i_{j+1}}} \geq 0 \geq 0 .$$

$$+ \sum_{i_m=0}^{2^{J_m}-1} \left\{ \sum_{m=1}^{2^{J_m}-1} b_m(i_m)h_b(p(A_{m(m)}) - [h_b(\hat{r}_{i_m}) - h_b(\hat{r}'_{i_m})] \right\} \geq 0 .$$

Similar to [27], Appendix B.2, property $\frac{\partial h(q_{[M]})}{\partial q_{m(i_m)}} \geq \frac{\partial h(q_{[M]})}{\partial q_{m(i'_m)}}$ for $I_i \subseteq I_i$ suggests the following $J+1$ steps algorithm for $m = 1, 2$ to complete the optimal PMF vector $q_{[M]}^*$:

- **Step 0:** For $\mu_{[M,J]}$ that does not satisfy $\mu_{m1} \geq \mu_{m2} \geq \cdots \geq \mu_{mJ_m}$, we can take a permutation $\Pi_m : \{1, \cdots, J_m\} \rightarrow \{1, \cdots, J_m\}$ such that $\mu_{m\Pi_m(i_m)} \geq \mu_{m\Pi_m(i_m+1)}$, $i_m = 1, \cdots, J_m - 1$.

  Repeating Step 1 to Step $J$, we can have similar result by substituting $\Pi_m(j)$ to $j$.

- **Step 1:** Assume $\mu_{m1} \geq \mu_{m2} \geq \cdots \geq \mu_{mJ_m}$. Since $\frac{\partial h(q)}{\partial q_{i}} \geq \frac{\partial h(q)}{\partial q_{i'}}$ for all $i = 0, 1, \cdots, 2^J-1$, $q_{m(2^J-1)}^*$ should be assigned the biggest allowable value. Note that $q_{m(2^J-1)} \leq \mu_{mJ_m}$, $j_m = 1, \cdots, J_m$, we have

$$q_{m(2^J-1)} = \min_{j_m=1, \cdots, J_m} \mu_{mJ_m} = \mu_{mJ_m} .$$

Due to constraints equation (86), we have $q_{m(i_m)}^* = 0$ for $i_m$ with $b_{mJ_m}(i_m) = 1$ and $i_m \neq 2^{J_m} - 1$.

- **Step 2:** For all $i_m \in \{0, 1, \cdots, 2^{J_m}-1\}$ and $b_{mJ_m}(i_m) = 0$, we have $\frac{\partial h(q_{[M]})}{\partial q_{i_m}} \geq \frac{\partial h(q_{[M]})}{\partial q_{i'_m}}$, where $i'_m = \sum_{j_m=1}^{J_m-1} 2^{j_m} - 1$. Furthermore, for all $i_m$, $b_{mJ_m}(i_m) = 0$, it follows that $q_{m(i_m)} \leq \mu_{mJ_m} - \mu_{mJ_m}$ for $j_m = 1, \cdots, J_m - 1$. Summarizing these facts, we have $q_{m(2^{J_m-1}-1)}^* = \mu_{J_m-1} - \mu_{J_m}$, and $q_{m(i_m)}^* = 0$ for $i_m$ with $b_{mJ_m}(i_m) = 0$, $b_{J_m-1}(i_m) = 1$, and $i_m \neq \sum_{j_m=1}^{J_m-1} 2^{j_m} - 1$. 


• Step $k_m$, $2 < k_m < J_m$: Similar to Step 2, we get $q^{\ast}_{m(2J_m - k_m + 1 - 1)} = \mu J_m - k_m + 1 - \mu J_m - k_m + 2$ and $q^{\ast}_{mim} = 0$, for $i_m$ with $b_s(i_m) = 0$, where $s = J_m - k_m + 2, \ldots, J$, and $b_m(J_m - k_m + 1)(i_m) = 1, i_m \neq \sum_{j=1}^{J_m-k_m+1} 2j_m - 1$.

• Step $J_m$: The only remaining PMF is $q^{\ast}_{m0}$ and $q^{\ast}_{m0} = 1 - \sum_{i_m=1}^{J_m-k_m+1} q^{\ast}_{mim}$.

Thus, the right side of equation (87) is maximized for $q^{\ast}_{i_m} = \nu_{i_m}$ if there exists $k_m \in \{0, 1, \ldots, J\}$ such that $i_m = \sum_{j=1}^{k_m} \nu_{i_j}$; otherwise $q^{\ast}_{i_m} = 0$, where

$$
\nu_{i_m} \triangleq \begin{cases} 
1 - \mu \nu_{i_m(1)}, & i = 0, \\
\mu \nu_{i_m(i_m)} - \mu \nu_{i_m(i_m + 1)}, & i_m = 1, \ldots, J_m - 1, \\
\mu \nu_{i_m(J_m)}, & i = J_m;
\end{cases}
$$

(94)

and $\nu_{i_m}: \{1, \ldots, J_m\} \rightarrow \{1, \ldots, J_m\}$ is a permutation of $\{1, \ldots, J_m\}$ such that $\mu \nu_{i_m(i_m)} \geq \mu \nu_{i_m(i_m + 1)}, i_m = 1, \ldots, J_m - 1$.

**M. Proof of Proposition 4**

For simplicity, define $s_{mim} = p(\sum_{i_m=1}^{i_m} A_{mim} + \Lambda_0)$ for $m = 1, 2$, and $\hat{s}_{j_1,j_2} = p(\sum_{m=1}^{2} \sum_{i_m=1}^{i_m} A_{mim} + \Lambda_0)$. Based on symmetry, without loss of generality, assume $\mu_{m1} \geq \mu_{m2} \geq \cdots \geq \mu_{mJ_m}$. We know

$$
I_{MISO-MAC}(\mu_{M,J}) = h_b \left( \sum_{j_1=0}^{J_1} \sum_{j_2=0}^{J_2} \nu_{j_1,j_2} \hat{s}_{j_1,j_2} \right) - \sum_{j_1=0}^{J_1} \sum_{j_2=0}^{J_2} \nu_{j_1,j_2} h_b(\hat{s}_{j_1,j_2}),
$$

(95)

Note that $\sum_{j=0}^{J_m} \nu_{mj} = 1$ for $m = 1, 2$, after rearrangement we get

$$
I_{MISO-MAC}(\mu_{M,J})
\begin{align*}
&= h_b \left( \sum_{j_2=0}^{J_2} \nu_{j_2} \left[ (1 - \sum_{j_1=1}^{J_1} \nu_{j_1,j_2}) s_{2j_2} + \sum_{j_1=1}^{J_1} \nu_{j_1,j_2} \hat{s}_{j_1,j_2} \right] \right) \\
&\quad - \left( \sum_{j_2=0}^{J_2} \nu_{j_2} \left[ (1 - \sum_{j_1=1}^{J_1} \nu_{j_1,j_2}) h_b(s_{2j_2}) + \sum_{j_1=1}^{J_1} \nu_{j_1,j_2} h_b(\hat{s}_{j_1,j_2}) \right] \right) \\
&= h_b \left( \sum_{j_2=0}^{J_2} \nu_{j_2} \left[ s_{2j_2} + u_{2,j_2}(\hat{s}_{j_1,j_2} - s_{2j_2}) - \sum_{j_1=1}^{J_1-1} u_{1,j_1}(\hat{s}_{j_1,j_2} - s_{2j_2}) \right] \right) - \sum_{j_2=0}^{J_2} \nu_{j_2} \left[ h_b(s_{2j_2}) \\
&\quad + u_{2,j_2}(h_b(\hat{s}_{j_1,j_2}) - h_b(s_{2j_2})) - \sum_{j_1=1}^{J_1-1} u_{1,j_1}(h_b(\hat{s}_{j_1,j_2}) - h_b(s_{2j_2})) \right]
\end{align*}
$$

(96)
where \( u_{m_j} \) for \( j_m = 1, 2, \ldots, J_m - 1, m = 1, 2 \) and \( u_{J_m} \) \( \triangleq \sum_{j_m=1}^{J_m} u_{m_j} \), then we have \( 0 \leq u_{m_j} \leq 1 \) for \( j_m = 1, 2, \ldots, J_m - 1 \) and \( \max\{u_{m_j} \mid j_m = 1, 2, \ldots, J_m - 1\} \leq u_{J_m} \leq 1 \).

Note that \( s_{j_1,j_2} = p(\sum_{m=1}^{2} \sum_{j_m=1}^{J_m} A_{mj} + A_0) \leq \frac{1}{2} \), then for \( j_m = 1, 2, \ldots, J_m - 1 \),

\[
\frac{\partial I_{MISO-MAC}}{\partial u_{2j_2}} = -h_b'(\sum_{j_2=0}^{j_2} u_{2j_2} [s_{2j_2} + u_{2j_2}(\hat{s}_{j_1,j_2} - s_{2j_2}) - \sum_{j_1=1}^{J_1-1} u_{1j_1}(\hat{s}_{j_1,j_2} - s_{2j_2})]) \leq 0
\]

\[
\cdot \sum_{j_2} u_{2j_2}(\hat{s}_{j_1,j_2} - s_{2j_2}) = [h_b(\hat{s}_{j_1,j_2}) - h_b(s_{2j_2})] < 0
\]

(97)

and the max range of optimized \( u_{J_m} \) corresponds to \( u_{m_1} = u_{m_2} = \cdots = u_{J_m-1} \), we have the optimal \( u_{[M,J]} \) satisfies \( u_{m_1} = u_{m_2} = \cdots = u_{J_m-1} \), which implies \( \mu_{m_1} = \mu_{m_2} = \cdots = \mu_{m,J} \) for \( m = 1, 2 \).

**APPENDIX B**

**Auxiliary Lemma**

**Lemma 8.** Assume function \( f(x) \) is strictly convex and its first-order derivative exists. For \( x > y \), then we have function \( g(x, y) \) \( \triangleq \frac{f(x) - f(y)}{x - y} \) strictly monotonically increases with \( x \), strictly monotonically decreases with \( y \). To be specific, we have \( f'(y) < \frac{f(x) - f(y)}{x - y} < f'(x) \).

Proof: According to Lagrange mean value theorem, for \( x > y \), we have \( f(x) - f(y) = f'(\xi)(x - y) < f'(x)(x - y) \), where \( y < \xi < x \). Since \( g'_x = \frac{f'(x) - f'(y)}{(x-y)^2} \) > 0, function \( g(x, y) \) strictly monotonically increases with \( x \). Similarly, we have function \( g(x, y) \) strictly monotonically decreases with \( y \).

Note that function \( g(x, y) \) strictly monotonically increases with \( x \), we have \( f'(x) = \sup_{y : x > y} \frac{f(x) - f(y)}{x - y} \) for any \( y < x \). Similarly, we have \( f'(y) < \frac{f(x) - f(y)}{x - y} \).

**Lemma 9.** Assume \( \tau \leq \frac{\ln 2}{b} \). \( p(x) = 1 - e^{-x\tau} \), then we have function \( h_b(p(x)), x \in [0, b] \) is concave.

Proof: Note that \( [h_b(p(x))]'' = h_b''(p(x))[p'(x)]^2 + h_b'(p(x))g''(x), h_b''(x) < 0, h_b(x) \) monotonically increase if \( x \leq \frac{1}{2} \) and \( p''(x) < 0 \), we have \( h_b''(p(x)) < 0 \) when \( \tau \leq \frac{\ln 2}{b} \).

**Lemma 10.** Assume function \( f(x) \) is concave. For \( a < b < c < d \) and \( a + d = b + c \), then we have \( f(a) + f(d) < f(b) + f(c) \).

Proof: Note that \( b - a = d - c \) and \( f(x) \) is concave, then we have \( f(b) - f(a) > f(d) - f(c) \).
Lemma 11. For \( f(x) = \frac{dx+x+d_0}{d_1x+d_0} \), \( x \in [0, 1] \) and \( \min\{d_0, d_1+d_0\} > 0 \). if \( d_1 < 0 \) and \( d_2d_0-d_3d_1 < 0 \), then \( f(x) \) is concave (convex) and monotonically decrease (increase).

Proof: Taking the derivative of \( f(x) \), we have \( f'(x) = \frac{d_3d_0-d_2d_1}{(d_1x+d_0)^2} \) and \( f''(x) = -2d_1\frac{d_3d_0-d_2d_1}{(d_1x+d_0)^3} \). Therefore, it is obvious to complete the proof.

Lemma 12. For \( a, b, c, d > 0 \), \( a < c \), and \( \frac{b}{a} < \frac{d}{c} \), then we have \( \frac{b}{a} < \frac{d}{c} < \frac{d-b}{c-a} \) for any \( \mu \in [0, 1] \)

Proof: It is easy to check

\[
\frac{d}{c} < \frac{d-b}{c-a} \Leftrightarrow dc-da < dc-bc \Leftrightarrow bc < ad
\]

Lemma 13. For \( a, b, c, d > 0 \), \( a < c \), \( \frac{b}{a} < \frac{d}{c} \) and \( f(\mu) \triangleq \frac{\mu b + (1-\mu)d}{\mu a + (1-\mu)c} \), then we have \( \frac{\mu b}{\mu a} + (1-\mu)\frac{d}{c} < f(\mu) < \frac{d}{c} \) for any \( \mu \in (0, 1) \), \( f'(\mu) < 0 \) and \( f''(\mu) < 0 \)

Proof: For the inequality on \( f(\mu) \), we have

\[
\frac{\mu b + (1-\mu)d}{\mu a + (1-\mu)c} > \frac{b}{a} + (1-\mu)\frac{d}{c} \Leftrightarrow \frac{\mu(b-da/c)}{\mu a + (1-\mu)c} > \frac{\mu(b-da/c)}{a} \Leftrightarrow a < c,
\]

Based on Lemma 72 we have

\[
f'(\mu) = c\left(\frac{d}{c} - \frac{d-b}{c-a}\right)(c-a)[c-(c-a)\mu]^2 < 0, \quad (99)
\]

\[
f''(\mu) = 2c\left(\frac{d}{c} - \frac{d-b}{c-a}\right)(c-a)^2[c-(c-a)\mu]^3 < 0, \quad (100)
\]

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