Frobenius Functors for Corings

J. Gómez-Torrecillas* M. Zarouali Darkaoui
Departamento de Álgebra Departamento de Álgebra
Facultad de Ciencias Facultad de Ciencias
Universidad de Granada Universidad de Granada
E18071 Granada, Spain E18071 Granada, Spain
e-mail: gomezj@ugr.es

Introduction

Corings were introduced by M. Sweedler in [27] as a generalization of coalgebras over commutative rings to the case of non-commutative rings, to give a formulation of a predual of the Jacobson-Bourbaki’s theorem for intermediate extensions of division ring extensions. Thus, a coring over an associative ring with unit is a coalgebra in the monoidal category of all \( A \)-bimodules. Recently, motivated by an observation of M. Takeuchi, namely that an entwining structure (resp. an entwined module) can be viewed as a suitable coring (resp. as a comodule over a suitable coring), T. Brzeziński, has given in [5] some new examples and general properties of corings. Among them, a study of Frobenius corings is developed, extending previous results on entwining structures [4] and relative Hopf modules [11].

A motivation to the study of Frobenius functors is the observation of K. Morita [24] that a ring extension is Frobenius if and only the restriction of scalars functor has isomorphic left and right adjoint functors. Thus a pair of functors \((F, G)\) is said to be a Frobenius pair [13] if \(G\) is left and right adjoint to \(F\) at the same time. The functors \(F\) and \(G\) then known as Frobenius functors [11]. These names are now standard, and we still using them rather than the original Morita’s denomination as strongly adjoint pairs.

Frobenius corings (i.e., corings for which the functor forgetting the coaction is Frobenius), have been intensively studied in [3, 6, 7, 9]. In this paper we investigate Frobenius pairs between categories of comodules over rather general corings. Precedents for coalgebras over fields are contained in [13] and [9]. We particularize to the case of the adjoint pair of functors associated to a morphism of corings over different base rings [18], which leads to a reasonable notion of Frobenius coring extension. When applied to corings stemming from entwining structures, we obtain new results in this setting.

*Partially supported by the grant BFM2001-3141 from the Ministerio de Ciencia y Tecnología
1 Basic notations

Throughout this paper, $A$, $A'$, $A''$, and $B$ denote associative and unitary algebras over a commutative ring $k$, and except mention clarifies opposite, $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$, and $\mathcal{D}$ denote corings over $A$, $A'$, $A''$, and $B$, respectively. We recall from [27] that an $A$–coring consists of an $A$–bimodule $\mathcal{C}$ with two $A$–bimodule maps

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}, \quad \epsilon : \mathcal{C} \longrightarrow A$$

such that $(\mathcal{C} \otimes_A \Delta) \circ \Delta = (\Delta \otimes_A \mathcal{C}) \circ \Delta$ and $(\epsilon \otimes_A \mathcal{C}) \circ \Delta = (\mathcal{C} \otimes_A \epsilon) \circ \Delta = 1_{\mathcal{C}}$. A right $\mathcal{C}$–comodule is a pair $(M, \rho_M)$ consisting of a right $A$–module $M$ and an $A$–linear map $\rho_M : M \rightarrow M \otimes_A \mathcal{C}$ (the coaction) satisfying $(M \otimes_A \Delta) \circ \rho_M = (\rho_M \otimes_A \mathcal{C}) \circ \rho_M$, $(M \otimes_A \epsilon) \circ \rho_M = 1_M$. A morphism of right $\mathcal{C}$–comodules $(M, \rho_M)$ and $(N, \rho_N)$ is a right $A$–linear map $f : M \rightarrow N$ such that $(f \otimes_A \mathcal{C}) \circ \rho_M = \rho_N \circ f$; the $k$–module of all such morphisms will be denoted by $\text{Hom}_A(M, N)$. The right $\mathcal{C}$–comodules together with their morphisms form the additive category $\mathcal{M}^\mathcal{C}$, whose morphisms are defined in the obvious way.

Let $Z$ be a left $A$–module and $f : X \rightarrow Y$ a morphism in $\mathcal{M}_A$. Following [8, 40.13] we say that $f$ is $Z$–pure when the functor $- \otimes_A Z \colon \mathcal{M}_A \rightarrow \text{Mod}_A$ preserves the kernel of $f$. If $f$ is $Z$–pure for every $Z \in \mathcal{M}_A$ then we say simply that $f$ is pure in $\mathcal{M}_A$.

2 Frobenius functors between categories of comodules

Let $T$ be a $k$-algebra, and $M \in T \mathcal{M}_A$. Let $\varphi : T \rightarrow \text{End}_A(M)$ the morphism of $k$–algebras given by the right $T$-module structure of the bimodule $T M_A$. Now, suppose moreover that $M \in \mathcal{M}^\mathcal{C}$. Then $\text{End}_\mathcal{C}(M)$ is a subalgebra of $\text{End}_A(M)$. We have that $\varphi(T) \subset \text{End}_\mathcal{C}(M)$ if and only if $\rho_M$ is $T$–linear. Hence the left $T$-module structure of a $T \mathcal{C}$-bicomodule $M$ can be described as a morphism of $k$-algebras $\varphi : T \rightarrow \text{End}_\mathcal{C}(M)$. Given a $k$–linear functor $F : \mathcal{M}^\mathcal{C} \rightarrow \mathcal{M}^\mathcal{D}$, and $M \in T \mathcal{M}^\mathcal{C}$, the algebra morphism $T \xrightarrow{\varphi} \text{End}_\mathcal{C}(M) \xrightarrow{F(\varphi)} \text{End}_\mathcal{D}(F(M))$ defines a $T \mathcal{D}$-bicomodule structure on $F(M)$. We have then two $k$-linear functors

$$- \otimes_T F(-), F(- \otimes_T -) : T \mathcal{M}^\mathcal{C} \rightarrow T \mathcal{M}^\mathcal{D}.$$
Let $\Upsilon_{T,M}$ be the unique isomorphism of $\mathcal{D}$-comodules making the following diagram commutative

$$
\begin{array}{ccc}
T \otimes_T F(M) & \xrightarrow{\Upsilon_{T,M}} & F(T \otimes_T M), \\
\cong & & \cong \\
F(M) & \xrightarrow{\cong} & F(M)
\end{array}
$$

for every $M \in {}^T\mathcal{M}^\mathcal{E}$. We have $\Upsilon_{T,M}$ is natural in $T$. By the theorem of Mitchell [25, Theorem 3.6.5], there exists a unique natural transformation

$$
\Upsilon_{-,-} : - \otimes_T F(-) \to F(- \otimes_T -)
$$

extending the natural transformation $\Upsilon_{T,M}$. We refer to [18] for more details.

**Remark 2.1.** The theorem of Mitchell [25, Theorem 3.6.5] holds also if we suppose only that the category $\mathcal{C}'$ is preadditive and has coproducts, or if the category $\mathcal{C}'$ is preadditive and the functor $S$ preserves coproducts. This fact is used to go up that the natural transformation $\Upsilon$ exists for every $k$-linear functor $F : \mathcal{M}^\mathcal{E} \to \mathcal{M}^\mathcal{D}$ even if the category $\mathcal{M}^\mathcal{D}$ is not abelian. Note also that its corollary [25, Corollary 3.6.6] holds also if we suppose only that the category $\mathcal{C}'$ is preadditive.

Let $M \in \mathcal{E} \mathcal{M}^\mathcal{E}$ and $N \in \mathcal{E} \mathcal{M}^\mathcal{E}'$. The map

$$
\omega_{M,N} = \rho_M \otimes_A N - M \otimes_A \lambda_N : M \otimes_A N \to M \otimes_A \mathcal{E} \otimes_A N
$$

is a $\mathcal{C}' - \mathcal{C}''$-bicomodule map. Denote by $M \square \mathcal{E} N$ its kernel in $A' \mathcal{M} A''$. If $\omega_{M,N}$ is $\mathcal{C}' A'$-pure and $A'' \mathcal{C}''$-pure, and the following maps

$$
\ker(\omega_{M,N}) \otimes_{A''} \mathcal{E}'' \otimes_{A''} \mathcal{E}'' , \quad \mathcal{C}' \otimes_{A'} \mathcal{C}' \otimes_{A'} \ker(\omega_{M,N}) \quad \text{and} \quad \mathcal{C}' \otimes_{A'} \ker(\omega_{M,N}) \otimes_{A''} \mathcal{E}'' (1)
$$

are injective, then $M \square \mathcal{E} N$ is the kernel of $\omega_{M,N}$ in $\mathcal{E} \mathcal{M}^\mathcal{E}''$. This is the case if $\omega_{M,N}$ is $(\mathcal{C}' \otimes_{A'} \mathcal{C}') A'$-pure, $A''(\mathcal{C}'' \otimes_{A''} \mathcal{E}'')$-pure, and $\mathcal{C}' \otimes_{A'} \omega_{M,N}$ is $A'' \mathcal{E}''$-pure (e.g. if $\mathcal{C}' A'$ and $A'' \mathcal{C}''$ are flat, or if $\mathcal{C}$ is a coseparable $A$-coring).

If for every $M \in \mathcal{E} \mathcal{M}^\mathcal{E}$ and $N \in \mathcal{E} \mathcal{M}^\mathcal{E}'$, $\omega_{M,N}$ is $\mathcal{C}' A'$-pure and $A'' \mathcal{C}''$-pure, then we have a bifunctor

$$
- \square \mathcal{E} - : \mathcal{E} \mathcal{M}^\mathcal{E} \times \mathcal{E} \mathcal{M}^\mathcal{E}' \to \mathcal{E} \mathcal{M}^\mathcal{E}' (2)
$$

which is $k$-linear in each variable. If in particular $\mathcal{C}' A'$ and $A'' \mathcal{C}''$ are flat, or if $\mathcal{C}$ is a coseparable $A$-coring, then the bifunctor [2] is well defined.

By a proof similar to that of [11, II.1.3], we have, for every $M \in \mathcal{M}^\mathcal{E}$, that the functor $M \square \mathcal{E} -$ preserves direct limits.

The following lemma was used implicitly in the proof of [18, Proposition 3.4], and it will be useful for us in the proof of the next theorem.

**Lemma 2.2.** If $M \in \mathcal{E} \mathcal{M}^\mathcal{E}$, and $F : \mathcal{M}^\mathcal{E} \to \mathcal{M}^\mathcal{D}$ is a $M$-compatible $k$-linear functor in the sense of [18, p. 210], which preserves coproducts, then for all $X \in \mathcal{M} A'$,

$$
\Upsilon_{X \otimes_A \mathcal{E}, M}(X \otimes_A \lambda_{F(M)}) = F(X \otimes_A \lambda_M) \Upsilon_{X, M}.
$$
Proof. Let us consider the diagram

\[
\begin{array}{ccc}
X \otimes_{A'} F(M) & \xrightarrow{\lambda} & X \otimes_{A'} F(M) \\
\downarrow{\tau_{X,M}} & & \downarrow{\tau_{X,A'}M} \\
F(X \otimes_{A'} M) & \xrightarrow{\lambda} & F(X \otimes_{A'} M)
\end{array}
\]

The commutativity of the top triangle follows from the definition of \(\lambda_{F(M)}\), while the right triangle commutes by \cite{18} Lemma 3.3 (we take \(S = T = A'\), and \(Y = \mathcal{C}'\)), and the left triangle is commutative since \(\tau_{X,-}\) is natural. Therefore, the commutativity of the rectangle holds.

A closer analysis of \cite{18} Theorem 3.5] gives the following generalization of \cite{29} Proposition 2.1 and \cite{3} 23.1(1) (concerning this last, the condition \(F\) is kernel preserving\) is used in its proof). Recall from \cite{21} that a coring \(\mathfrak{C}\) is said to be coseparable if the co-multiplication map \(\Delta_{\mathfrak{C}}\) is a split monomorphism of \(\mathfrak{C}\)-bicomodules. Of course, the trivial \(A\)-coring \(\mathfrak{C} = A\) is coseparable and, henceforth, every result for comodules over coseparable corings applies in particular for modules over rings.

**Theorem 2.3.** Let \(F : \mathcal{M}^\mathfrak{C} \to \mathcal{M}^\mathfrak{D}\) be a \(k\)-linear functor, such that

(I) \(\mathfrak{D}\) is flat and \(F\) preserves the kernel of \(\rho_N \otimes_A \mathfrak{C} - N \otimes_A \Delta_{\mathfrak{C}}\) for every \(N \in \mathcal{M}^\mathfrak{C}\), or

(II) \(\mathfrak{C}\) is a coseparable \(A\)-coring and the categories \(\mathcal{M}^\mathfrak{C}\) and \(\mathcal{M}^\mathfrak{D}\) are abelian.

Assume that at least one of the following statements holds

1. \(\mathfrak{C}_A\) is projective, \(F\) preserves coproducts, and \(\Upsilon_{N,\mathfrak{C}}, \Upsilon_{N \otimes_A \mathfrak{C}, \mathfrak{C}}\) are isomorphisms for all \(N \in \mathcal{M}^\mathfrak{C}\) (e.g., if \(A\) is semisimple and \(F\) preserves coproducts), or

2. \(\mathfrak{C}_A\) is flat, \(F\) preserves direct limits, and \(\Upsilon_{N,\mathfrak{C}}, \Upsilon_{N \otimes_A \mathfrak{C}, \mathfrak{C}}\) are isomorphisms for all \(N \in \mathcal{M}^\mathfrak{C}\) (e.g., if \(A\) is a von Neumann regular ring and \(F\) preserves direct limits), or

3. \(F\) preserves inductive limits (e.g., if \(F\) has a right adjoint).

Then \(F\) is naturally equivalent to \(-\Box_{\mathfrak{C}}F(\mathfrak{C})\).

**Proof.** At first, note that if \(\mathfrak{C}_A\) is projective, then the right \(A\)-module \(\mathfrak{C} \otimes_A \mathfrak{C}\) is projective (by \cite{26} Example 3, p. 105, and Proposition VI.9.5)). Hence, if \(\mathfrak{C}_A\) is projective and \(F\) preserves coproducts, then \(F\) is \(M\)-compatible in the sense of \cite{18} p. 210, for all \(M \in \mathfrak{C}\mathcal{M}^\mathfrak{C}\). In each case, we have \(F\) is \(\mathfrak{C}\)-compatible where \(\mathfrak{C} \in \mathfrak{C}\mathcal{M}^\mathfrak{C}\). Therefore, by \cite{18} Proposition 3.4, \(F(\mathfrak{C})\) can be viewed as a \(\mathfrak{C} - \mathfrak{D}\)-bimodule. From Lemma \cite{22} and since \(\Upsilon_{-,-}\) is a
natural transformation, we have, for every $N \in \mathcal{M}^\mathcal{C}$, the commutativity of the following diagram with exact rows in $\mathcal{M}^\mathcal{D}$

\[
\begin{array}{cccccc}
0 & \longrightarrow & N \otimes_\mathcal{C} F(\mathcal{C}) & \longrightarrow & N \otimes_A F(\mathcal{C}) & \longrightarrow & N \otimes_A \mathcal{C} \otimes_A F(\mathcal{C}) \\
& & & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & F(N) & \longrightarrow & F(N \otimes_\mathcal{C} \mathcal{C}) & \longrightarrow & F(N \otimes_A \mathcal{C} \otimes_A \mathcal{C})
\end{array}
\]

(the exactness of the bottom sequence for the case (II) follows by factorizing the map $\omega_{N,\mathcal{C}} = \rho_N \otimes_A \mathcal{C} - N \otimes_A \Delta_\mathcal{C}$ through its image, and using the facts that the sequence $0 \longrightarrow N \longrightarrow N \otimes_A \mathcal{C} \longrightarrow N \otimes_A \mathcal{C} \otimes_A \mathcal{C}$ is split exact in $\mathcal{M}^\mathcal{C}$ in the sense of [8, 40.5], and that additive functors between abelian categories preserve split exactness). By the universal property of kernels, there exists a unique isomorphism $\eta_N : N \otimes_\mathcal{C} F(\mathcal{C}) \rightarrow F(N)$ in $\mathcal{M}^\mathcal{D}$ making commutative the above diagram. It easy to show that $\eta$ is natural. Hence $F \cong - \otimes_\mathcal{C} F(\mathcal{C})$.

As an immediate consequence of the last theorem we have the following generalization of Eilenberg-Watts Theorem [26, Proposition VI.10.1].

**Corollary 2.4.** Let $F : \mathcal{M}^\mathcal{C} \rightarrow \mathcal{M}^\mathcal{D}$ be a $k$-linear functor.

1. If $B \mathcal{D}$ is flat and $A$ is a semisimple ring (resp. a von Neumann regular ring), then the following statements are equivalent

   (a) $F$ is left exact and preserves coproducts (resp. left exact and preserves direct limits);

   (b) $F \cong - \otimes_\mathcal{C} M$ for some bicomodule $M \in \mathcal{C} \mathcal{M}^\mathcal{D}$.

2. If $A \mathcal{C}$ and $B \mathcal{D}$ are flat, then the following statements are equivalent

   (a) $F$ is exact and preserves inductive limits;

   (b) $F \cong - \otimes_\mathcal{C} M$ for some bicomodule $M \in \mathcal{C} \mathcal{M}^\mathcal{D}$ which is coflat in $\mathcal{C} \mathcal{M}$.

3. If $\mathcal{C}$ is a coseparable $A$-coring and the categories $\mathcal{M}^\mathcal{C}$ and $\mathcal{M}^\mathcal{D}$ are abelian, then the following statements are equivalent

   (a) $F$ preserves inductive limits;

   (b) $F$ preserves cokernels and $F \cong - \otimes_\mathcal{C} M$ for some bicomodule $M \in \mathcal{C} \mathcal{M}^\mathcal{D}$.

4. If $\mathcal{C} = A$ and the category $\mathcal{M}^\mathcal{D}$ is abelian, then the following statements are equivalent

   (a) $F$ has a right adjoint;

   (b) $F$ preserves inductive limits;

   (c) $F \cong - \otimes_A M$ for some bicomodule $M \in A \mathcal{M}^\mathcal{D}$.
A bicomodule $N \in \mathcal{EM}^D$ is said to be quasi-finite as a right $D$-comodule if the functor $- \otimes_A N : \mathcal{M}_A \to \mathcal{M}^D$ has a left adjoint $h_D(N, -) : \mathcal{M}^D \to \mathcal{M}_A$; the co-hom functor. If $\omega_{Y,N}$ is $D \otimes B \mathcal{D}$-pure for every right $\mathcal{E}$-comodule $Y$ (e.g., $B \mathcal{D}$ is flat or $\mathcal{E}$ is coseparable) then $N_D$ is quasi-finite if and only if $- \square_{\mathcal{E}} N : \mathcal{M}^\mathcal{E} \to \mathcal{M}^D$ has a left adjoint, which we still denote by $h_D(N, -)$ [18 Proposition 4.2]. The particular case of the following statement when the co-hom is exact generalizes [2 Corollary 3.12].

**Corollary 2.5.** Let $N \in \mathcal{EM}^D$ be a bicomodule, quasi-finite as a right $D$-comodule, such that $A \mathcal{C}$ and $B \mathcal{D}$ are flat. If the co-hom functor $h_D(N, -)$ is exact or if $D$ is a coseparable $B$-coring, then we have

$$h_D(N, -) \simeq - \square_{\mathcal{D}} h_D(N, \mathcal{D}) : \mathcal{M}^D \to \mathcal{M}^\mathcal{E}.$$  

**Proof.** The functor $h_D(N, -)$ is $k$-linear and preserves inductive limits, since it is a left adjoint to the $k$-linear functor $- \square_{\mathcal{C}} X : \mathcal{M}_C \to \mathcal{M}^D$ (by [18 Proposition 4.2]). Hence Theorem 2.3 achieves the proof. $\square$

Now we will use the following generalization of [29 Lemma 2.2].

**Lemma 2.6.** Let $\Lambda, \Lambda'$ be bicomodules in $\mathcal{D} \mathcal{M}^\mathcal{E}$ and $G = - \square_{\mathcal{D}} \Lambda, G' = - \square_{\mathcal{D}} \Lambda'$. Suppose moreover that $A \mathcal{C}$ is flat and $B$ is a von Neumann regular ring, or $A \mathcal{C}$ is flat and $G$ and $G'$ are cokernel preserving, or $D$ is a coseparable coring. Then

$$\text{Nat}(G, G') \simeq \text{Hom}_{\mathcal{D}, \mathcal{E}}(\Lambda, \Lambda').$$

**Proof.** Let $\alpha : G \to G'$ be a natural transformation. By [18 Lemma 3.2(1)], $\alpha_D$ is left $B$-linear. For the rest of the proof it suffices to replace $\otimes$ by $\otimes_B$ in the proof of [22 Lemma 4.1]. $\square$

The following proposition generalizes well-known facts from tensor product functors for modules to cotensor product functors for comodules over corings.

**Proposition 2.7.** Suppose that $A \mathcal{C}$, $C_A$, $B \mathcal{D}$ and $D_B$ are flat. Let $X \in \mathcal{EM}^D$ and $\Lambda \in \mathcal{D} \mathcal{M}^\mathcal{E}$. Consider the following properties:

1. $- \square_{\mathcal{E}} X$ is left adjoint to $- \square_{\mathcal{D}} \Lambda$;
2. $\Lambda$ is quasi-finite as a right $\mathcal{E}$-comodule and $- \square_{\mathcal{E}} X \simeq h_\mathcal{E}(\Lambda, -)$;
3. $\Lambda$ is quasi-finite as a right $\mathcal{E}$-comodule and $X \simeq h_\mathcal{E}(\Lambda, \mathcal{C})$ in $\mathcal{EM}^D$;
4. there exist bicolinear maps

$$\psi : \mathcal{C} \to X \square_{\mathcal{D}} \Lambda \text{ and } \omega : \Lambda \square_{\mathcal{E}} X \to \mathcal{D}$$

in $\mathcal{EM}^\mathcal{E}$ and $\mathcal{D} \mathcal{M}^D$ respectively, such that

$$(\omega \square_{\mathcal{D}} \Lambda) \circ (\Lambda \square_{\mathcal{E}} \psi) = \Lambda \text{ and } (X \square_{\mathcal{D}} \omega) \circ (\psi \square_{\mathcal{E}} X) = X; \quad (3)$$

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(5) \(\Lambda \Box \epsilon -\) is left adjoint to \(X \Box \Delta -\).

Then (1) and (2) are equivalent, and they imply (3). The converse is true if \(\mathcal{C}\) is a coseparable \(A\)-coring. If \(AX \) and \(B\Lambda\) are flat, and \(\omega_{XY} = \rho_X \otimes_B Y - X \otimes_A \rho_Y\) is pure as an \(A\)-linear map and \(\omega_{YX} = \rho_Y \otimes_A A - Y \otimes_B \rho_X\) is pure as a \(B\)-linear map (e.g. if \(\epsilon X\) and \(\epsilon \Lambda\) are coflat [8, 21.5] or \(A\) and \(B\) are von Neumann regular rings), or if \(\mathcal{C}\) and \(\mathcal{D}\) are coseparable, then (4) implies (3). Conversely, if \(\epsilon X\) and \(\epsilon \Lambda\) are coflat, or if \(A\) and \(B\) are von Neumann regular rings, or if \(\mathcal{C}\) and \(\mathcal{D}\) are coseparable. Finally, if \(\mathcal{C}\) and \(\mathcal{D}\) are coseparable, or if \(X\) and \(\Lambda\) are coflat on both sides, or if \(A, B\) are von Neumann regular rings, then (1), (4) and (5) are equivalent.

**Proof.** The equivalence between (1) and (2) follows from [18, Proposition 4.2]. That (2) implies (3) is a consequence of [18, Proposition 3.4]. If \(\mathcal{C}\) is coseparable and we assume (3) then, by Corollary 2.5, \(h_{\mathcal{C}}(\Lambda, -) = -\Box \epsilon h_{\mathcal{C}}(\Lambda, \mathcal{C})\) \(~\rightsquigarrow\~ -\Box \epsilon X\). That (1) implies (4) follows from Lemma 2.6 by evaluating the unit and the counit of the adjunction at \(\mathcal{C}\) and \(\mathcal{D}\), respectively. Conversely, if we put \(F = -\Box \epsilon X\) and \(G = -\Box \Delta \Lambda\), we have \(GF \simeq -\Box \epsilon (X \Box \Delta \Lambda)\) and \(FG \simeq -\Box \Delta (\Lambda \Box \epsilon X)\) by [8, Proposition 22.6]. Define natural transformations

\[
\eta : 1_{\mathcal{M}^\mathcal{D}} \xrightarrow{\sim} -\Box \epsilon C^{-\Box \epsilon} \psi G F
\]

and

\[
\varepsilon : FG \xrightarrow{-\Box \Delta \psi} -\Box \Delta \mathcal{D} \xrightarrow{\sim} 1_{\mathcal{M}^\mathcal{D}},
\]

which become the unit and the counit of an adjunction by (3). This gives the equivalence between (1) and (4). The equivalence between (4) and (5) follows by symmetry. \(\square\)

**Definition 2.8.** Following [2] and [8], a bicomodule \(N \in \mathcal{C}\mathcal{M}^\mathcal{D}\) is called an injector (resp. an injector-cogenerator) as a right \(\mathcal{D}\)-comodule if the functor \(- \otimes \mathcal{A} : \mathcal{M}^\mathcal{A} \rightarrow \mathcal{M}^\mathcal{D}\) preserves injective (resp. injective cogenerator) objects.

**Proposition 2.9.** Suppose that \(A\mathcal{C}\) and \(B\mathcal{D}\) are flat. Let \(X \in \mathcal{C}\mathcal{M}^\mathcal{D}\) and \(\Lambda \in \mathcal{D}\mathcal{M}^\mathcal{C}\). The following statements are equivalent

(i) \(-\Box \epsilon X\) is left adjoint to \(-\Box \Delta \Lambda\), and \(-\Box \epsilon X\) is left exact (or \(A\mathcal{X}\) is flat or \(\epsilon X\) is coflat);

(ii) \(\Lambda\) is quasi-finite as a right \(\mathcal{C}\)-comodule, \(-\Box \epsilon X \simeq h_{\mathcal{C}}(\Lambda, -)\), and \(-\Box \epsilon X\) is left exact (or \(A\mathcal{X}\) is flat or \(\epsilon X\) is coflat);

(iii) \(\Lambda\) is quasi-finite and injector as a right \(\mathcal{C}\)-comodule and \(X \simeq h_{\mathcal{C}}(\Lambda, \mathcal{C})\) in \(\mathcal{C}\mathcal{M}^\mathcal{D}\).

**Proof.** First, observe that if \(\epsilon X\) is coflat, then \(A\mathcal{X}\) is flat [8, 21.6], and that if \(A\mathcal{X}\) is flat, then the functor \(-\Box \epsilon X\) is left exact. Thus, in view of Proposition 2.7 it suffices if we prove that the version of (ii) with \(-\Box \epsilon X\) left exact implies (iii), and this last implies the version of (ii) with \(\epsilon X\) coflat. Assume that \(-\Box \epsilon X \simeq h_{\mathcal{C}}(\Lambda, -)\) with \(-\Box \epsilon X\) left exact. By [18, Proposition 3.4], \(X \simeq h_{\mathcal{C}}(\Lambda, \mathcal{C})\) in \(\mathcal{C}\mathcal{M}^\mathcal{D}\). Being a left adjoint, \(h_{\mathcal{C}}(\Lambda, -)\) is right
exact and, henceforth, exact. By \[25\] Theorem 3.2.8, \(\Lambda\) is an injector and we have proved \((iii)\). Conversely, if \(\Lambda\) is a quasi-finite injector and \(X \simeq h_\mathcal{C}(\Lambda, \mathcal{C})\) as bicomodules, then \(-\square_\mathcal{E}X \simeq -\square_\mathcal{E}h_\mathcal{E}(\Lambda, \mathcal{C})\) and, by \[25\] Theorem 3.2.8, we get that \(h_\mathcal{E}(\Lambda, -)\) is an exact functor. By Corollary \[25\] \(h_\mathcal{E}(\Lambda, -) \simeq -\square_\mathcal{E}h_\mathcal{E}(\Lambda, \mathcal{C}) \simeq -\square_\mathcal{E}X\), and \(\mathcal{E}X\) is coflat.

We are ready to give our characterization of Frobenius functors between categories of comodules over corings. We left to the reader to derive the corresponding characterizations for the particular cases of rings or of coalgebras over fields.

**Theorem 2.10.** Suppose that \(\mathcal{A}\mathcal{C}\) and \(\mathcal{B}\mathcal{D}\) are flat. Let \(X \in \mathcal{C}\mathcal{M}\mathcal{D}\) and \(\Lambda \in \mathcal{D}\mathcal{M}\mathcal{C}\). The following statements are equivalent

1. \((-\square_\mathcal{C}X, -\square_\mathcal{D}\Lambda)\) is a Frobenius pair;
2. \(-\square_\mathcal{C}X\) is a Frobenius functor, and \(h_\mathcal{D}(X, \mathcal{D}) \simeq \Lambda\) as bicomodules;
3. there is a Frobenius pair \((F, G)\) for \(\mathcal{M}\mathcal{C}\) and \(\mathcal{M}\mathcal{D}\) such that \(F(\mathcal{C}) \simeq \Lambda\) and \(G(\mathcal{D}) \simeq X\) as bicomodules;
4. \(\Lambda_\mathcal{C}, X_\mathcal{D}\) are quasi-finite injectors, and \(X \simeq h_\mathcal{C}(\Lambda, \mathcal{C})\) and \(\Lambda \simeq h_\mathcal{D}(X, \mathcal{D})\) as bicomodules;
5. \(\Lambda_\mathcal{C}, X_\mathcal{D}\) are quasi-finite, and \((-\square_\mathcal{C}X, -\square_\mathcal{D}\Lambda)\) is a Frobenius pair, with \(X \simeq h_\mathcal{C}(\Lambda, \mathcal{C})\) as bicomodules.

**Proof.** \((i) \iff (ii) \iff (iii)\) This is obvious, after Theorem 2.3 and \[18\] Proposition 3.4].

\((i) \iff (iv)\) Follows from Proposition 2.9.

\((iv) \iff (v)\). If \(X_\mathcal{D}\) and \(\Lambda_\mathcal{C}\) are quasi-finite, then \(\mathcal{A}\mathcal{X}\) and \(\mathcal{B}\Lambda\) are flat. Now, apply Proposition 2.9.

From Proposition 2.7 and Proposition 2.9 (or Theorem 2.10) we get the following

**Theorem 2.11.** Let \(X \in \mathcal{E}\mathcal{M}\mathcal{D}\) and \(\Lambda \in \mathcal{D}\mathcal{E}\mathcal{M}\). Suppose that \(\mathcal{A}\mathcal{C}\), \(\mathcal{C}\mathcal{A}\), \(\mathcal{B}\mathcal{D}\) and \(\mathcal{D}\mathcal{B}\) are flat. The following statements are equivalent

1. \((-\square_\mathcal{E}X, -\square_\mathcal{D}\Lambda)\) is a Frobenius pair, with \(X_\mathcal{D}\) and \(\Lambda_\mathcal{E}\) coflat;
2. \((\Lambda\square_\mathcal{E}-, X\square_\mathcal{D}-)\) is a Frobenius pair, with \(\mathcal{E}X\) and \(\mathcal{D}\Lambda\) coflat;
3. \(X\) and \(\Lambda\) are coflat quasi-finite injectors on both sides, and \(X \simeq h_\mathcal{E}(\Lambda, \mathcal{C})\) in \(\mathcal{E}\mathcal{M}\mathcal{D}\) and \(\Lambda \simeq h_\mathcal{D}(X, \mathcal{D})\) in \(\mathcal{D}\mathcal{M}\mathcal{E}\).

If moreover \(\mathcal{C}\) and \(\mathcal{D}\) are coseparable (resp. \(\mathcal{A}\) and \(\mathcal{B}\) are von Neumann regular rings), then the following statements are equivalent

1. \((-\square_\mathcal{E}X, -\square_\mathcal{D}\Lambda)\) is a Frobenius pair;
2. \((\Lambda\square_\mathcal{E}-, X\square_\mathcal{D}-)\) is a Frobenius pair;
3. \(X\) and \(\Lambda\) are quasi-finite (resp. quasi-finite injector) on both sides, and \(X \simeq h_\mathcal{E}(\Lambda, \mathcal{C})\) in \(\mathcal{E}\mathcal{M}\mathcal{D}\) and \(\Lambda \simeq h_\mathcal{D}(X, \mathcal{D})\) in \(\mathcal{D}\mathcal{M}\mathcal{E}\).
3 Frobenius functors between corings with a duality

We will look to Frobenius functors for corings closer to coalgebras over fields, in the sense that the categories of comodules share a fundamental duality.

An object $M$ of a Grothendieck category $C$ is said to be finitely generated [26, p. 121] if whenever $M = \sum_i M_i$ is a direct union of subobjects $M_i$, then $M = M_{i_0}$ for some index $i_0$. Alternatively, $M$ is finitely generated if the functor $\text{Hom}_C(M, -)$ preserves direct unions [26, Proposition V.3.2]. The category $C$ is locally finitely generated if it has a family of finitely generated generators. Recall from [26, p. 122] that a finitely generated object $M$ is finitely presented if every epimorphism $L \to M$ with $L$ finitely generated has finitely generated kernel. By [26, Proposition V.3.4], if $C$ is locally finitely generated, then $M$ is finitely presented if and only if $\text{Hom}_C(M, -)$ preserves direct limits. For the notion of a locally projective module we refer to [30].

Lemma 3.1. Let $C$ be a coring over a ring $A$ such that $_A C$ is flat.

(1) A comodule $M \in M^C$ is finitely generated if and only if $M_A$ is finitely generated.

(2) A comodule $M \in M^C$ is finitely presented if $M_A$ is finitely presented. The converse is true whenever $M^C$ is locally finitely generated.

(3) If $_A C$ is locally projective, then $M^C$ is locally finitely generated.

Proof. The forgetful functor $U : M^C \to M_A$ has an exact left adjoint $- \otimes_A C : M_A \to M^C$ which preserves direct limits. Thus, $U$ preserves finitely generated objects and, in case that $M^C$ is locally finitely generated, finitely presented objects. Now, if $M \in M^C$ is finitely generated as a right $A$-module, and $M = \sum_i M_i$ as a direct union of subcomodules, then $U(M) = U(\sum_i M_i) = \sum_i U(M_i)$, since $U$ is exact and preserves coproducts. Therefore, $U(M) = U(M_{i_0})$ for some index $i_0$ which implies, being $U$ a faithfully exact functor, that $M = M_{i_0}$. Thus, $M$ is a finitely generated comodule. We have thus proved (1), and the converse to (2). Now, if $M \in M^C$ is such that $M_A$ is finitely presented, then for every exact sequence $0 \to K \to L \to M \to 0$ in $M^C$ with $L$ finitely generated, we get an exact sequence $0 \to K_A \to L_A \to M_A \to 0$ with $M_A$ finitely presented. Thus, $K_A$ is finitely generated and, by (1), $K \in M^C$ is finitely generated. This proves that $M$ is a finitely presented comodule. Finally, (3) is a consequence of (1) and [3] 19.12(1)].

The notation $C_f$ stands for the full subcategory of a Grothendieck category $C$ whose objects are the finitely generated objects. The category $C$ is locally noetherian [26, p. 123] if it has a family of noetherian generators or equivalently, if $C$ is locally finitely generated and every finitely generated object of $C$ is noetherian. By [26, Proposition V.4.2, Proposition V.4.1, Lemma V.3.1(i)], in an arbitrary Grothendieck category, every finitely generated object is noetherian if and only if every finitely generated object is finitely presented. The version for categories of modules of the following result is well-known.

Lemma 3.2. Let $C$ be a locally finitely generated category.
(1) The category $\mathbf{C}_f$ is additive.

(2) The category $\mathbf{C}_f$ has cokernels, and every monomorphism in $\mathbf{C}_f$ is a monomorphism in $\mathbf{C}$.

(3) The following statements are equivalent:

(a) The category $\mathbf{C}_f$ has kernels;
(b) $\mathbf{C}$ is locally noetherian;
(c) $\mathbf{C}_f$ is abelian;
(d) $\mathbf{C}_f$ is an abelian subcategory of $\mathbf{C}$.

Proof. 1. Straightforward.

2. That $\mathbf{C}_f$ has cokernels is straightforward from [26, Lemma V.3.1(i)]. Now, let $f : M \to N$ be a monomorphism in $\mathbf{C}_f$ and $\xi : X \to M$ be a morphism in $\mathbf{C}$ such that $f \xi = 0$. Suppose that $X = \bigcup_{i \in I} X_i$, where $X_i \in \mathbf{C}_f$, and $\iota_i : X_i \to X$, $i \in I$ the canonical injections. Then $f \xi \iota_i = 0$, and $\xi \iota_i = 0$, for every $i$, and by the definition of the inductive limit, $\xi = 0$.

3. (b) $\Rightarrow$ (a) Straightforward from [26, Proposition V.4.1].

(d) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are trivial.

(a) $\Rightarrow$ (b) Let $M \in \mathbf{C}_f$, and $K$ be a subobject of $M$. Let $\iota : L \to M$ the kernel of the canonical morphism $f : M \to M/K$ in $\mathbf{C}_f$. Suppose that $K = \bigcup_{i \in I} K_i$, where $K_i \in \mathbf{C}_f$, for every $i \in I$. By the universal property of the kernel, there exist a unique morphism $\alpha : L \to K$, and a unique morphism $\beta_i : K_i \to L$, for every $i \in I$, making commutative the diagrams

\[
\begin{array}{ccc}
L & \xrightarrow{\iota} & M \\
\downarrow{\alpha} & & \downarrow{f} \\
K & & M/K \\
\end{array}
\]

By (2), $\iota$ is a monomorphism in $\mathbf{C}$, then for every $K_i \subset K_j$, the diagram

\[
\begin{array}{ccc}
K_i & \xrightarrow{\beta_i} & L \\
\downarrow & & \downarrow{\beta_j} \\
K_j & & 
\end{array}
\]

commutes. Therefore we have the commutative diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\iota} & M \\
\downarrow{\lim_{\to} \beta_i} & & \\
K & & 
\end{array}
\]
Then $K \simeq L$, and hence $K \in C_f$. Finally, by [26 Proposition V.4.1], $M$ is noetherian in $C$.

(b) $\Rightarrow$ (d) Straightforward from [17 Theorem 3.41]. \hfill $\square$

The following is a generalization of [13 Proposition 3.1].

**Proposition 3.3.** Let $C$ and $D$ be two locally noetherian categories. Then

1. If $F : C \to D$ is a Frobenius functor, then its restriction $F_f : C_f \to D_f$ is a Frobenius functor.

2. If $H : C_f \to D_f$ is a Frobenius functor, then $H$ can be uniquely extended to a Frobenius functor $\overline{H} : C \to D$.

3. The assignment $F \mapsto F_f$ defines a bijective correspondence (up to natural isomorphisms) between Frobenius functors from $C$ to $D$ and Frobenius functors from $C_f$ to $D_f$.

4. In particular, if $C = \mathcal{M}^\mathcal{C}$ and $D = \mathcal{M}^\mathcal{D}$ are locally noetherian such that $\mathcal{A}^\mathcal{C}$ and $\mathcal{B}^\mathcal{D}$ are flat, then $F : \mathcal{M}^\mathcal{C} \to \mathcal{M}^\mathcal{D}$ is a Frobenius functor if and only if it preserves direct limits and comodules which are finitely generated as right $A$-modules, and the restriction functor $F_f : \mathcal{M}_f^\mathcal{C} \to \mathcal{M}_f^\mathcal{D}$ is a Frobenius functor.

**Proof.** The proofs of [13 Proposition 3.1 and Remark 3.2] remain valid for our situation, but with some minor modifications: to prove that $\overline{H}$ is well-defined, we use Lemma 3.2. (In the proof of the statements (1), (2) and (3) we use the Grothendieck AB-5 condition). \hfill $\square$

In order to generalize [28 Proposition A.2.1] and its proof, we need

**Lemma 3.4.** (1) Let $C$ be a locally noetherian category, let $D$ be an arbitrary Grothendieck category, $F : C \to D$ be an arbitrary functor which preserves direct limits, and $F_f : C_f \to D$ be its restriction to $C_f$. Then $F$ is exact (faithfully exact, resp. left, right exact) if and only if $F_f$ is exact (faithfully exact, resp. left, right exact). In particular, an object $M$ in $C_f$ is projective (resp. projective generator) if and only if it is projective (resp. projective generator) in $C$.

(2) Let $C$ be a locally noetherian category. For every object $M$ of $C$, the following conditions are equivalent

(a) $M$ is injective (resp. an injective cogenerator);

(b) the contravariant functor $\text{Hom}_C(-, M) : C \to \text{Ab}$ is exact (resp. faithfully exact);

(c) the contravariant functor $\text{Hom}_C(-, M)_f : C_f \to \text{Ab}$ is exact (resp. faithfully exact).

In particular, an object $M$ in $C_f$ is injective (resp. injective cogenerator) if and only if it is injective (resp. injective cogenerator) in $C$. 

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Proof. (1) The “only if” part is straightforward from the fact that the injection functor $C_f \to C$ is faithfully exact.

For the “if” part, suppose that $F_f$ is left exact. Let $f : M \to N$ be a morphism in $C$. Put $M = \bigcup_{i \in I} M_i$ and $N = \bigcup_{j \in J} N_j$, as direct union of directed families of finitely generated subobjects. For $(i, j) \in I \times J$, let $M_{i,j} = M_i \cap f^{-1}(N_j)$, and $f_{i,j} : M_{i,j} \to N_j$ be the restriction of $f$ to $M_{i,j}$. We have $f = \lim_{i,j} f_{i,j}$ and then $F(f) = \lim_{i,j} F_f(f_{i,j})$. Hence

\[
\ker F(f) = \ker \lim_{i,j} F_f(f_{i,j}) = \lim_{i,j} \ker F_f(f_{i,j}) = \lim_{i,j} F_f(\ker f_{i,j}) = \lim_{i,j} F(\ker f_{i,j}) \quad \text{(by Lemma 3.2)}
\]

Finally $F$ is left exact. Analogously, it can be proved that $F_f$ is right exact implies that $F$ is also right exact. Now, suppose that $F_f$ is faithfully exact. We have already proved that $F$ is exact. It remains to prove that $F$ is faithful. For this, let $0 \neq M = \bigcup_{i \in I} M_i$ be an object of $C$, where $M_i$ is finitely generated for every $i \in I$. We have

\[
F(M) = \lim_{i\to\bigcup} F_f(M_i) \simeq \bigoplus_i F_f(M_i)
\]

(since $F$ is exact). Since $M \neq 0$, there exists some $i_0 \in I$ such that $M_{i_0} \neq 0$. By \cite[Proposition IV.6.1]{26}, $F_f(M_{i_0}) \neq 0$, hence $F(M) \neq 0$. Also by \cite[Proposition IV.6.1]{26}, $F$ is faithful.

(2). (a) $\Rightarrow$ (b) Obvious.

(b) $\Rightarrow$ (c) Analogous to that of the “only if” part of (1).

(c) $\Rightarrow$ (a) Suppose that the functor Hom$_C(\cdot, M)_f$ is exact. Since $C$ is locally noetherian, and by \cite[Proposition V.2.9 and Proposition V.4.1]{26}, $M$ is injective. Now, suppose moreover that Hom$_C(\cdot, M)_f$ is faithful. Let $L$ be a non-zero object of $C$, and $K$ be a non-zero finitely generated subobject of $L$. By \cite[Proposition IV.6.1]{26}, there exists a non-zero morphism $K \to M$. Since $M$ is injective, there exists a non-zero morphism $L \to M$ making commutative the following diagram

\[
\begin{array}{ccc}
M & \rightarrow & L \\
\downarrow & & \downarrow \\
0 & \rightarrow & K \\
\end{array}
\]

From \cite[Proposition IV.6.5]{26}, it follows that $M$ is a cogenerator. \qed

If $\mathcal{C}_A$ is flat and $M \in \mathcal{M}^\mathcal{C}$ is finitely presented as right $A$-module, then \cite[19.19]{8} the dual left $A$-module $M^* = \text{Hom}_A(M, A)$ has a left $\mathcal{C}$-comodule structure

\[
M^* \simeq \text{Hom}_\mathcal{C}(M, \mathcal{C}) \subseteq \text{Hom}_A(M, \mathcal{C}) \simeq \mathcal{C} \otimes_A M^*.
\]

Now, if $A_M^*$ turns out to be finitely presented and $\mathcal{C}$ is flat, then $(M^*) = \text{Hom}_A(M^*, A)$ is a right $\mathcal{C}$-comodule and the canonical map $\sigma_M : M \to (M^*)$ is a homomorphism in $\mathcal{M}^\mathcal{C}$. This construction leads to a duality (i.e. a contravariant equivalence)

\[
(-)^* : \mathcal{M}_0^\mathcal{C} \rightleftharpoons \mathcal{C}_0 : *(-)
\]

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between the full subcategories $\mathcal{M}_0^e$ and $\mathcal{M}_0$ of $\mathcal{M}$ whose objects are the comodules which are finitely generated and projective over $A$ on the corresponding side (this holds even without flatness assumptions of $\mathcal{C}$). Call it basic duality (details may be found in [10]). Of course, in the case $A$ is semisimple (e.g. for coalgebras over fields) these categories are that of finitely generated comodules, and this basic duality plays a remarkable role in the study of several notions in the coalgebra setting (e.g. Morita equivalence [29], semiperfect coalgebras [22], Morita duality [19], [20], or Frobenius Functors [13]). It would be interesting to know, in the coring setting, to what extent the basic duality can be extended to the subcategories $\mathcal{M}_f^e$ and $\mathcal{C}_f^m$, since, as we will try to show in this section, this allows to obtain better results. Of course, this is the underlying idea when the ground ring $A$ is assumed to be Quasi-Frobenius (see e.g. [16]) for the case of semiperfect corings and Morita duality), but we hope future developments of the theory would have some pay off from the more general setting we propose here.

Consider contravariant functors between Grothendieck categories $H : A \hookrightarrow A' : H'$, together with natural transformations $\tau : 1_A \to H' \circ H$ and $\tau' : 1_A' \to H \circ H'$, satisfying the condition $H(\tau_X) \circ \tau'_{H(X)} = 1_{H(X)}$ and $H'(\tau'_{X'}) \circ \tau_{H'(X')}$ for $X \in A$ and $X' \in A'$. Following [14], this situation is called a right adjoint pair.

**Proposition 3.5.** Let $\mathcal{C}$ be an $A$-coring such that $A \mathcal{C}$ and $\mathcal{C} A$ are flat. Assume that $\mathcal{M}$ and $\mathcal{C}$ are locally noetherian categories. If $A M^*$ and $^* N_A$ are finitely generated modules for every $M \in \mathcal{M}^e_f$ and $N \in \mathcal{C} M_f$, then the basic duality extends to a right adjoint pair $(-)^* : \mathcal{M}_f^e \leftrightarrows \mathcal{C} M_f : *(-)$.

**Proof.** If $M \in \mathcal{M}_f^e$ then, since $\mathcal{M}$ is locally noetherian, $M^e$ is finitely presented. By Lemma 3.1, $M^e$ is finitely presented and the left $\mathcal{C}$-comodule $M^*$ makes sense. Now, the assumption $A M^*$ finitely generated implies, by Lemma 3.1, that $M^* \in \mathcal{C} M_f$. We have then the functor $(-)^* : \mathcal{M}_f^e \to \mathcal{C} M_f$. The functor $(-)^*$ is analogously defined, and the rest of the proof consists of straightforward verifications (see [3, Proposition 20.14(1)], where $\tau$ and $\tau'$ are the evaluation maps).

**Example 3.6.** The hypotheses are fulfilled if $A \mathcal{C}$ and $\mathcal{C} A$ are locally projective and $A$ is left and right noetherian (in this case the right adjoint pair already appears in [16]). But there are situations in which no finiteness condition need to be required to $A$: this is the case, for instance, of cosemisimple corings (see [15, Theorem 3.1]). In particular, if an arbitrary ring $A$ contains a division ring $B$, then, by [15, Theorem 3.1] the canonical coring $A \otimes_B A$ satisfies all hypotheses in Proposition 3.5.

**Definition 3.7.** Let $\mathcal{C}$ be a coring over $A$ satisfying the assumptions of Proposition 3.5. We will say that $\mathcal{C}$ has a duality if the basic duality extends to a duality $(-)^* : \mathcal{M}_f^e \leftrightarrows \mathcal{C} M_f : *(-)$.

We have the following examples of a coring which has a duality:

- $\mathcal{C}$ is a coring over a QF ring $A$ such that $A \mathcal{C}$ and $\mathcal{C} A$ are flat (and hence projective);
• \( C \) is a cosemisimple coring;

• \( C \) is a coring over \( A \) such that \( _A C \) and \( C_A \) are flat and semisimple, \( M^C \) and \( ^C M \) are locally noetherian categories, and the dual of every simple right (resp. left) \( A \)-module in the decomposition of \( C_A \) (resp. \( C \)) as a direct sum of simple \( A \)-modules is finitely generated and \( A \)-reflexive (in fact, every right (resp. left) \( C \)-comodule \( M \) is a submodule of the semisimple right (resp.left) \( A \)-module \( M \otimes_A C \), and hence \( _A M \) (resp. \( A M \)) is also semisimple.)

**Proposition 3.8.** Suppose that the coring \( C \) has a duality. Let \( M \in M^C \) such that \( M_A \) is flat. The following are equivalent

1. \( M \) is coflat (resp. faithfully coflat);

2. \( \hom_C(−, M)_f : M^C_f \to M_k \) is exact (resp. faithfully exact);

3. \( M \) is injective (resp. an injective cogenerator).

**Proof.** Let \( M \in M^C \) and \( N \in C^M_f \). We have the following commutative diagram (in \( M_k \))

\[
\begin{array}{ccc}
0 & \rightarrow & M \square_C N \\
\downarrow & & \downarrow \\
\hom_C(N^*, M) & \rightarrow & \hom_A(N^*, M) \\
\end{array}
\]

where the vertical maps are the canonical maps. By the universal property of the kernel, there is a unique morphism \( \eta_{M,N} : M \square_C N \to \hom_C(N^*, M) \) making commutative the above diagram. By the cube Lemma (see [23, p. 43]), \( \eta \) is a natural transformation of bifunctors. If \( M_A \) is flat then \( \eta_{M,N} \) is an isomorphism for every \( N \in C^M_f \). Finally, let \( M \in M^C \) such that \( M_A \) is flat. We have

\[
M \square_C − \simeq \hom_C(−, M)_f(−)^* : ^C M_f \to M_k.
\]

Then, \( M \) is coflat (resp. faithfully coflat) iff \( M \square_C − : ^C M_f \to M_k \) is exact (resp. faithfully exact) (by Lemma [3.4]) iff \( \hom_C(−, M)_f : M_f \to M_k \) is exact (resp. faithfully exact) iff \( M_C \) is injective (resp. an injective cogenerator) (by Lemma [3.4]). □

**Corollary 3.9.** Let \( N \in C^D \) be a bicomodule. Suppose that \( A \) is a QF ring and \( D \) has a duality. If \( N \) is injective (resp. injective cogenerator) in \( C^D \) such that \( N_B \) is flat, then \( N \) is an injector (resp. an injector-cogenerator) as a right \( D \)-comodule.

**Proof.** Let \( X_A \) be an injective (resp. an injective cogenerator) module. Since \( A \) is a QF ring, \( X_A \) is projective. We have then the natural isomorphism

\[
(X \otimes_A N) \square_D − \simeq X \otimes_A (N \square_D −) : ^D M \to M_k.
\]

By Proposition [3.8] \( N \) and \( X_A \) are coflat (resp. faithfully coflat), and then \( X \otimes_A N \) is coflat (resp. faithfully coflat). Now, since \( X \otimes_A N \) is a flat right \( B \)-module, and by Proposition [3.8] \( X \otimes_A N \) is injective (resp. injective cogenerator) in \( C^D \). □
The last two results allow to improve our general statements in Section 2 for corings having a duality.

**Proposition 3.10.** Suppose that $\mathcal{C}$ and $\mathcal{D}$ have a duality. Consider the following statements

1. $(−\square_\mathcal{C}X, −\square_\mathcal{D}\Lambda)$ is an adjoint pair of functors, with $AX$ and $\Lambda_A$ flat;
2. $\Lambda$ is quasi-finite injective as a right $\mathcal{C}$-comodule, with $AX$ and $\Lambda_A$ flat and $X \simeq h_\mathcal{C}(\Lambda, \mathcal{C})$ in $\mathcal{C}\mathcal{M}_\mathcal{D}$.

We have (1) implies (2), and the converse is true if in particular $B$ is a QF ring.

**Proof.** (1) $\Rightarrow$ (2) From Proposition 3.8, $\mathcal{D}$ is injective in $\mathcal{M}_\mathcal{D}$. Since the functor $−\square_\mathcal{C}X$ is exact, $\Lambda \simeq \mathcal{D}\square_\mathcal{D}\Lambda$ is injective in $\mathcal{C}\mathcal{M}_\mathcal{D}$.

(2) $\Rightarrow$ (1) Assume that $B$ is a QF ring. From Corollary 3.9, $\Lambda$ is quasi-finite injector as a right $\mathcal{C}$-comodule, and Proposition 2.9 achieves the proof. $\square$

**Theorem 3.11.** Suppose that $\mathcal{C}$ and $\mathcal{D}$ have a duality. Let $X \in \mathcal{C}\mathcal{M}_\mathcal{D}$ and $\Lambda \in \mathcal{D}\mathcal{M}_\mathcal{C}$. The following statements are equivalent

1. $(−\square_\mathcal{C}X, −\square_\mathcal{D}\Lambda)$ is a Frobenius pair, with $X_B$ and $\Lambda_A$ flat;
2. $(\Lambda\square_\mathcal{C} − , X\square_\mathcal{D} − )$ is a Frobenius pair, with $AX$ and $B\Lambda$ flat;
3. $X$ and $\Lambda$ are quasi-finite injector on both sides, and $X \simeq h_\mathcal{C}(\Lambda, \mathcal{C})$ in $\mathcal{C}\mathcal{M}_\mathcal{D}$ and $\Lambda \simeq h_\mathcal{D}(X, \mathcal{D})$ in $\mathcal{D}\mathcal{M}_\mathcal{C}$.

In particular, if $A$ and $B$ are QF rings, then the above statements are equivalent to

4. $X$ and $\Lambda$ are quasi-finite injective on both sides, and $X \simeq h_\mathcal{C}(\Lambda, \mathcal{C})$ in $\mathcal{C}\mathcal{M}_\mathcal{D}$ and $\Lambda \simeq h_\mathcal{D}(X, \mathcal{D})$ in $\mathcal{D}\mathcal{M}_\mathcal{C}$.

Finally, suppose that $\mathcal{C}$ and $\mathcal{D}$ are cosemisimple corings. Let $X \in \mathcal{C}\mathcal{M}_\mathcal{D}$ and $\Lambda \in \mathcal{D}\mathcal{M}_\mathcal{C}$. The following statements are equivalent

1. $(−\square_\mathcal{C}X, −\square_\mathcal{D}\Lambda)$ is a Frobenius pair;
2. $(\Lambda\square_\mathcal{C} − , X\square_\mathcal{D} − )$ is a Frobenius pair;
3. $X$ and $\Lambda$ are quasi-finite on both sides, and $X \simeq h_\mathcal{C}(\Lambda, \mathcal{C})$ in $\mathcal{C}\mathcal{M}_\mathcal{D}$ and $\Lambda \simeq h_\mathcal{D}(X, \mathcal{D})$ in $\mathcal{D}\mathcal{M}_\mathcal{C}$.

**Proof.** At first we will prove the first part. In view of Theorem 2.11 and Theorem 2.10 it suffices to show that if $(−\square_\mathcal{C}X, −\square_\mathcal{D}\Lambda)$ is a Frobenius pair, the condition “$X_B$ and $\Lambda_A$ are coflat” is equivalent to “$X_B$ and $\Lambda_A$ are flat”. Indeed, the first implication is obvious, for the converse, assume that $X_B$ and $\Lambda_A$ are flat. By Proposition 3.10, $X$ and $\Lambda$ are injective in $\mathcal{M}_\mathcal{D}$ and $\mathcal{M}_\mathcal{C}$ respectively, and they are coflat by Proposition 3.8. The particular case is straightforward from Proposition 3.10 and the above equivalences.
Now we will show the second part. Since every comodule category over a cosemisimple
coring is a spectral category (see [20, p. 128]), and by Proposition 3.8, every comodule
(resp. every bicomodule) over a cosemisimple coring is coflat (resp. injector) (we can see it
directly by using the fact that every additive functor between abelian categories preserves
split exactness). The use of Theorem 2.11 finishes then proof.

Remark 3.12. 1. The equivalence “(1) ⇔ (4)” of the last theorem is a generalization
of [13, Theorem 3.3]. The proof of [13, Theorem 3.3] give an alternative proof of
“(1) ⇔ (4)” of Theorem 3.11 using Proposition 3.3.

2. The adjunction of Proposition 2.7 and Proposition 3.10 generalize the coalgebra
version of Morita’s theorem [9, Theorem 4.2].

Example 3.13. Let A be a k-algebra. Put C = A and D = k. The bicomodule A ∈ C M D
is quasi-finite as a right D-comodule. A is an injector as a right D-comodule if and only if
the k-module A is flat. If we take A = k = Z, the bicomodule A is quasi-finite and injector
as a right D-comodule but it is not injective in MD. Hence, the assertion “(− □cX, − □DΛ)
is an adjoint pair of functors” does not implies in general the assertion “Λ is quasi-finite
injective as a right C-comodule and X ≃ hC(Λ, C) in EMD”, and the following statements
are not equivalent in general:

1. (− □cX, − □DΛ) is a Frobenius pair;

2. X and Λ are quasi-finite injective on both sides, and X ≃ hC(Λ, C) in EMD and
Λ ≃ hD(X, D) in DMD.

On the other hand, there exists a commutative self-injective ring which is not coherent.
By a theorem of S.U. Chase (see for example [3, Theorem 19.20]), there exists then a k-
algebra A which is injective, but not flat as k-module. Hence, the bicomodule A ∈ EMD
is quasi-finite and injective as a right D-comodule, but not an injector as a right D-comodule.

4 Applications to induction functors

We start this section by recalling from [18], that a coring homomorphism from the coring
C to the coring D is a pair (φ, ρ), where ρ : A → B is a homomorphism of k-algebras and
φ : C → D is a homomorphism of A-bimodules such that

\[ \varepsilon_D \circ \varphi = \rho \circ \varepsilon_C \quad \text{and} \quad \Delta_D \circ \varphi = \omega_D \circ (\varphi \otimes_A \varphi) \circ \Delta_C, \]

where \( \omega_D : D \otimes_A D \to D \otimes_B D \) is the canonical map induced by \( \rho : A \to B \).

Now we will characterize when the induction functor \( - \otimes_A B : \mathcal{C} \to \mathcal{D} \) defined in
[18, Proposition 5.3] is a Frobenius functor. The coaction of \( \mathcal{D} \) over \( M \otimes_A B \) is given, when
expressed in Sweedler’s sigma notation, by

\[ \rho_{M \otimes_A B}(m \otimes_A b) = \sum m_{(0)} \otimes_A 1 \otimes_B \varphi(m_{(1)})b, \]

where \( M \) is a right \( \mathcal{C} \)-comodule with coaction \( \rho_M(m) = \sum m_{(0)} \otimes_A m_{(1)}. \)
Theorem 4.1. Let $(\varphi, \rho) : \mathcal{C} \to \mathcal{D}$ be a homomorphism of corings such that $A\mathcal{C}$ and $B\mathcal{D}$ are flat. The following statements are equivalent

(a) $- \otimes_A B : \mathcal{M}^\mathcal{C} \to \mathcal{M}^\mathcal{D}$ is a Frobenius functor;

(b) the $\mathcal{C} - \mathcal{D}$-bicomodule $\mathcal{C} \otimes_A B$ is quasi-finite and injector as a right $\mathcal{D}$-comodule and there exists an isomorphism of $\mathcal{D} - \mathcal{C}$-bicomodules $h_D(\mathcal{C} \otimes_A B, \mathcal{D}) \simeq B \otimes_A \mathcal{C}$.

Moreover, if $\mathcal{C}$ and $\mathcal{D}$ are coseparable, then the condition “injector” in (b) can be deleted.

Proof. First observe that $- \otimes_A B$ is a Frobenius functor if and only if $(- \otimes_A B, - \square_D(B \otimes_A \mathcal{C}))$ is a Frobenius pair (by [18, Proposition 5.4]). A straightforward computation shows that the map $\rho_M \otimes_A B : M \otimes_A B \to M \otimes_A \mathcal{C} \otimes_A B$ is a homomorphism of $\mathcal{D}$–comodules. We have thus a commutative diagram in $\mathcal{M}^\mathcal{C}$ with exact row

$$
\begin{array}{c}
0 & \xrightarrow{\epsilon_M} & M \square_c(\mathcal{C} \otimes_A B) & \xrightarrow{\iota} & M \otimes_A \mathcal{C} \otimes_A B & \xrightarrow{\omega_M \epsilon \otimes_A B} & M \otimes_A \mathcal{C} \otimes_A \mathcal{C} \otimes_A B, \\
& & \downarrow{\psi_M} & \downarrow{\rho_M \otimes_A B} & \downarrow{\omega_M \epsilon \otimes_A B} & \end{array}
$$

where $\psi_M$ is defined by the universal property of the kernel. Since $B\mathcal{D}$ is flat, to prove that $\psi_M$ is an isomorphism of $\mathcal{D}$–comodules it is enough to check that it is bijective, as the forgetful functor $U : \mathcal{M}^\mathcal{D} \to \mathcal{M}_B$ is faithfully exact. Some easy computations show that the map $(M \otimes_A \epsilon \otimes_A B) \circ \iota$ is the inverse in $\mathcal{M}_B$ to $\psi_M$. From this, we deduce a natural isomorphism $\psi : - \otimes_A B \simeq - \square_c(\mathcal{C} \otimes_A B)$. The equivalence between (a) and (b) is then obvious from Proposition 2.9 and Proposition 2.7.

When applied to the case where $\mathcal{C} = A$ and $\mathcal{D} = B$ are the trivial corings, Theorem 4.1 gives functorial Morita’s characterization of Frobenius ring extensions [24]. It is then reasonable to give the following definition.

Definition 4.2. Let $(\varphi, \rho) : \mathcal{C} \to \mathcal{D}$ be a homomorphism of corings such that $A\mathcal{C}$ and $B\mathcal{D}$ are flat. It is said to be a right Frobenius morphism of corings if $- \otimes_A B : \mathcal{M}^\mathcal{C} \to \mathcal{M}^\mathcal{D}$ is a Frobenius functor.

The following generalize [13, Theorem 3.5].

Theorem 4.3. Suppose that the algebras $A$ and $B$ are QF rings.
Let $(\varphi, \rho) : \mathcal{C} \to \mathcal{D}$ be a homomorphism of corings such that the modules $A\mathcal{C}$, $B\mathcal{D}$ and $D_B$ are projective. Then the following statements are equivalent

(a) $- \otimes_A B : \mathcal{M}^\mathcal{C} \to \mathcal{M}^\mathcal{D}$ is a Frobenius functor;

(b) the $\mathcal{C} - \mathcal{D}$-bicomodule $\mathcal{C} \otimes_A B$ is quasi-finite as a right $\mathcal{D}$-comodule, $(\mathcal{C} \otimes_A B)_B$ is injective and there exists an isomorphism of $\mathcal{D} - \mathcal{C}$-bicomodules $h_D(\mathcal{C} \otimes_A B, \mathcal{D}) \simeq B \otimes_A \mathcal{C}$.

Proof. Obvious from Proposition 3.10.
Now, suppose that the forgetful functor $\mathcal{M}^C \to \mathcal{M}_A$ is a Frobenius functor. Then the functor $- \otimes_A C : \mathcal{M}_A \to \mathcal{M}_A$ is also a Frobenius functor (since it is a composition of two Frobenius functors) and $\mathcal{A}C$ is finitely generated projective. On the other hand, since $- \otimes_A C : \mathcal{M}_A \to \mathcal{M}^C$ is a left adjoint to $\text{Hom}_C(C, -) : \mathcal{M}^C \to \mathcal{M}_A$. Then $\text{Hom}_C(C, -)$ is a Frobenius functor. Therefore, $C$ is finitely generated projective in $\mathcal{M}^C$, and hence in $\mathcal{M}_A$.

Lemma 4.4. Let $R$ be the opposite algebra of $^*\mathcal{C}$.

(1) $\mathcal{C} \in {}^C\mathcal{M}^A$ is quasi-finite (resp. quasi-finite and injector) as a right $A$-comodule if and only if $\mathcal{A}C$ is finitely generated projective (resp. $\mathcal{A}C$ is finitely generated projective and $\mathcal{A}R$ is flat). Let $h_A(C, -) = - \otimes_A R : \mathcal{M}^A \to \mathcal{M}^C$ be the co-hom functor.

(2) If $\mathcal{A}C$ is finitely generated projective and $\mathcal{A}R$ is flat, then
$$h_A(C, A)_C \cong \mathcal{A}R_C,$$
where the right $\mathcal{C}$-comodule structure of $\mathcal{A}R$ is defined as in [5, Lemma 4.3].

Proof. (1) Straightforward from [18, Example 4.3].

(2) From [5, Lemma 4.3], the forgetful functor $\mathcal{M}^C \to \mathcal{M}_A$ is the composition of functors $\mathcal{M}^C \to \mathcal{M}_R \to \mathcal{M}_A$. By [18, Proposition 4.2], $h_A(C, -)$ is a left adjoint to $- \otimes_C C : \mathcal{M}^C \to \mathcal{M}_A$ which is isomorphic to the forgetful functor $\mathcal{M}^C \to \mathcal{M}_A$. Then $h_A(C, -)$ is isomorphic to the composition of functors
$$\mathcal{M}^C \longrightarrow \mathcal{M}_R \longrightarrow \mathcal{M}_A.$$

In particular, $h_A(C, A)_C \cong A(\otimes_A R)_C \cong A\mathcal{R}_C$. \hfill \Box

Corollary 4.5. ([8, 27.10])

Let $\mathcal{C}$ be an $A$-coring and let $R$ be the opposite algebra of $^*\mathcal{C}$. Then the following statements are equivalent

(a) The forgetful functor $F : \mathcal{C} \to \mathcal{M}_A$ is a Frobenius functor;

(b) $\mathcal{A}C$ is finitely generated projective and $\mathcal{C} \cong R$ as $(A, R)$-bimodules, where $\mathcal{C}$ is a right $R$-module via $c.r = c(1).r(c(2))$, for all $c \in \mathcal{C}$ and $r \in R$.

Proof. Straightforward from Theorem 4.1 and Lemma 4.4. \hfill \Box

The following proposition gives sufficient conditions to have that a morphism of corings is right Frobenius if and only if it is left Frobenius. Note that it says in particular that the notion of Frobenius homomorphism of coalgebras over fields (by (b)) or of rings (by (d)) is independent on the side. Of course, the latter is well-known.

Proposition 4.6. Let $(\varphi, \rho) : \mathcal{C} \to \mathcal{D}$ be a homomorphism of corings such that $\mathcal{A}C$, $\mathcal{B}D$, $\mathcal{C}_A$ and $\mathcal{D}_B$ are flat. Assume that at least one of the following holds
(a) \( \mathcal{C} \) and \( \mathcal{D} \) have a duality, and \( _A B \) and \( _B A \) are flat;

(b) \( A \) and \( B \) are von Neumann regular rings;

(c) \( B \otimes_A \mathcal{C} \) is coflat in \( \mathcal{D} \mathcal{M} \) and \( \mathcal{C} \otimes_A B \) is coflat in \( \mathcal{M} \mathcal{D} \) and \( _A B \) and \( _B A \) are flat;

(d) \( \mathcal{C} \) and \( \mathcal{D} \) are coseparable corings.

Then the following statements are equivalent

1. \( - \otimes_A B : \mathcal{M} \mathcal{C} \rightarrow \mathcal{M} \mathcal{D} \) is a Frobenius functor;

2. \( B \otimes_A - : \mathcal{C} \mathcal{M} \rightarrow \mathcal{D} \mathcal{M} \) is a Frobenius functor.

Proof. Obvious from Theorem 2.11 and Theorem 3.11.

Let us finally show to derive from our results a remarkable characterization of the so called Frobenius corings.

Corollary 4.7. (\cite{8, 27.8})

The following statements are equivalent

(a) the forgetful functor \( \mathcal{M} \mathcal{C} \rightarrow \mathcal{M} \mathcal{A} \) is a Frobenius functor;

(b) the forgetful functor \( \mathcal{C} \mathcal{M} \rightarrow \mathcal{A} \mathcal{M} \) is a Frobenius functor;

(c) there exist an \((A, A)\)-bimodule map \( \eta : A \rightarrow \mathcal{C} \) and a \((\mathcal{C}, \mathcal{C})\)-bicomodule map \( \pi : \mathcal{C} \otimes_A \mathcal{C} \rightarrow \mathcal{C} \) such that \( \pi(\mathcal{C} \otimes_A \eta) = \mathcal{C} = \pi(\eta \otimes_A \mathcal{C}) \).

Proof. The proof of “(1) \( \Leftrightarrow \) (4)” in Proposition 2.7 for \( X = \mathcal{C} \in \mathcal{A} \mathcal{M} \mathcal{C} \) and \( \Lambda = \mathcal{C} \in \mathcal{E} \mathcal{M} \mathcal{A} \) remains valid for our situation. It remains to see that the condition (4) in this case is exactly the condition (c).

5 Applications to entwined modules

In this section we particularize some our results in Section 4 to the category of entwined modules. We adopt the notations of \cite{12}. We start with some remarks.

(1) Let a right-right entwining structure \((A, C, \psi) \in \mathcal{E}^\bullet(k)\) and a left-left entwining structure \((B, D, \varphi) \in \mathcal{E}^\bullet(k)\). The category of two-sided entwined modules \(\mathcal{D} \mathcal{B} \mathcal{M}(\varphi, \psi) \mathcal{C}_A \mathcal{M} \mathcal{A} \mathcal{C} \mathcal{D} \mathcal{B} \) defined in \cite{12} pp. 68–69] is isomorphic to the category of bicomodules \(\mathcal{D} \otimes \mathcal{B} \mathcal{M} \mathcal{A} \otimes \mathcal{C} \mathcal{D} \) over the associated corings.

(2) If \((A, C, \psi)\) and \((A', C', \psi')\) belong to \(\mathcal{E}^\bullet(k)\) such that \(\psi\) is an isomorphism, then \(\psi\) is an isomorphism of corings (see \cite{12} Proposition 34]], and consequently if the coalgebra \(C\) is flat as a \(k\)-module, then the modules \(\mathcal{A}(A \otimes C)\) and \((A \otimes C)_A\) are flat, and

\[
A \otimes C \mathcal{M} A' \otimes C' \simeq C \mathcal{A} \mathcal{M}(\psi^{-1}, \psi') A'.
\]
(3) Let \((\alpha, \gamma) : (A, C, \psi) \to (A', C', \psi')\) be a morphism in \(E^\bullet(k)\). We know that \((\alpha \otimes \gamma, \alpha) : A \otimes C \to A' \otimes C'\) is a morphism of corings. The functor \(F\) defined in [12, Lemma 8] satisfies the commutativity of the diagram

\[
\begin{array}{ccc}
M^{A \otimes C} & \xrightarrow{- \otimes A'} & M^{A' \otimes C'} \\
\downarrow{=} & & \downarrow{=} \\
M(\psi)_{A} & \xrightarrow{- \otimes A'} & M(\psi')_{A'}
\end{array}
\]

where \(- \otimes A' : M^{A \otimes C} \to M^{A' \otimes C'}\) is the induction functor defined in [18, Proposition 5.3].

We have the following result concerning the category of entwined modules:

**Theorem 5.1.** Let \((\alpha, \gamma) : (A, C, \psi) \to (A', C', \psi')\) be a morphism in \(E^\bullet(k)\), such that \(kC\) and \(kD\) are flat.

1. The following statements are equivalent

   (a) The functor \(F = - \otimes_A A' : M(\psi)_{A} \to M(\psi')_{A'}\) defined in [12, Lemma 8] is a Frobenius functor;

   (b) the \(A \otimes C - A' \otimes C'\)-bicomodule \((A \otimes C) \otimes_A A'\) is quasi-finite injector as a right \(A' \otimes C'\)-comodule and there exists an isomorphism of \(A' \otimes C' - A \otimes C\)-bicomodules

\[
h_{A' \otimes C'}((A \otimes C) \otimes_A A', A' \otimes C') \cong A' \otimes A (A \otimes C).
\]

Moreover, if \(A \otimes C\) and \(A' \otimes C'\) are coseparable corings, then the condition “injector” in (b) can be deleted.

2. If \(A\) and \(B\) are QF rings and the module \((A' \otimes C')_{A'}\) is projective, then the following are equivalent

   (a) The functor \(F = - \otimes_A A' : M(\psi)_{A} \to M(\psi')_{A'}\) defined in [12, Lemma 8] is a Frobenius functor;

   (b) the \(A \otimes C - A' \otimes C'\)-bicomodule \((A \otimes C) \otimes_A A'\) is quasi-finite and injective as a right \(A' \otimes C'\)-comodule and there exists an isomorphism of \(A' \otimes C' - A \otimes C\)-bicomodules

\[
h_{A' \otimes C'}((A \otimes C) \otimes_A A', A' \otimes C') \cong A' \otimes A (A \otimes C).
\]

**Proof.** Follows from Theorem 4.1 and Theorem 4.3.

**Remark 5.2.** Let a right-right entwining structure \((A, C, \psi) \in E^\bullet(k)\). The coseparability of the coring \(A \otimes C\) is characterized in [12, Theorem 38(1)] (see also [5, Corollary 3.6]).

**Acknowledgements**

We would like to thank the Professor Edgar Enochs to have communicated to us the example of a commutative self-injective ring which is not coherent.
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