Quantum walks on hypergraphs

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Abstract

In this work we introduce the concept of a quantum walk on a hypergraph. We show that the staggered quantum walk model is a special case of a quantum walk on a hypergraph.

1 Introduction

Quantum walks may be seen as an extension of the classical random walks into the quantum realm. There is, however, one key difference. In the classical setting the randomness is built-into the process. In the quantum case, the entire process is unitary, hence deterministic and even reversible. The randomness comes only from the random nature of quantum measurements.

During the last two decades the field of quantum walks has received a lot of attention from the scientific community. One of the earliest studies are the works by Aharonov [1] and Kempe [2]. Soon afterwards the possibility for algorithmic applications was shown [3]. One notable application is the fact that Grover’s search algorithm [4] can be represented as a quantum walk. Another approach to database lookup is the quantum spatial search algorithm [5]. Finally, nontrivial results in the field of quantum games can be obtained even with a simple walk on a cycle [6], and some more exotic problems like the Parrondo paradox can be modeled as a quantum walk [7].

Since these seminal works a lot of different approaches to the concept of a quantum walk have emerged. We should note here the open quantum walk model [8, 9]. This model can be summarized as follows. Imagine we have a particle moving on a graph. The particle has a quantum state associated with it. With each transition from one vertex to another, the state is modified according

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Quantum walks summary

| Range  | Unitary | Reflection | Arbitrary |
|--------|---------|------------|-----------|
| Edges  |         | Szegedy    | coined    |
| Cliques|         | staggered  | ?         |

Table 1: Summary of existing quantum walk models. The aim of this work is to find the model which fills the gap denoted by the question mark. By “range” we mean where the unitaries involved in the model act.

to some quantum operation. The only restriction here is that all the operations associated with some vertex must sum to a proper quantum channel. There was a lot effort put into studying this approach. We should mention here various asymptotic results for this model [10, 11], hitting times studies [12] and potential applications of this model in quantum modeling of biological structures [13].

Another model which deserves mention is the quantum stochastic walk [14]. This approach is based on the Gorini-Kossakowski-Sudarshan-Lindblad [15, 16] master equation. It allows to smoothly interpolate between classical and quantum walks as well as gives raise to some new dynamics. The asymptotic behavior of this model has been extensively studied [17, 18].

Finally, there has been a lot of effort put into the extension of the standard unitary quantum walk. Let us note here the Szegedy walk model [19] which allows for quantization of arbitrary Markov chain based algorithms. One of the most prominent example of usage of this model is the quantum Page Rank algorithm [20]. Another example of such modification is the staggered walk model introduced by Portugal et al. [21, 22, 23]. It has applications in quantum search algorithms [24].

In this work we introduce a novel concept - quantum walks on hypergraphs. Our main motivation is presented in Table 1. In there, we show how the currently developed quantum walk models are constructed. The goal of this work is to fill the part represented by the question mark.

This work is organized as follows. In Section 2 we introduce the concept of a hypergraph along with some accompanying definitions. Next, in Section 3 we recall well-established quantum walk models. Section 4 introduces our model – quantum walk on a hypergraph, or hyperwalk. Next, in Section 5 how our model relates to other quantum walks. Finally, in Section 6 final conclusions are drawn.

2 Graphs and hypergraphs

In this section we provide basic definitions used throughout this work. We start with the definition of a graph

Definition 1. A graph $G$ is a pair $(V, E)$, where $V$ is a set of vertices and
$E \subseteq V \times V$ is a set of edges. We say an element $i \in V$ is connected with element $j \in V$ when $(i, j) \in E$. We will denote this by $i \sim j$. We call $G$ a directed graph if we consider the elements of $E$ as ordered pairs. Otherwise, $G$ is said to be undirected. If all vertices have the same degree, then such a graph is called a regular graph.

Next we introduce the concept of a hypergraph.

**Definition 2.** An undirected hypergraph $H$ is a pair $(V, E)$, where $V$ is a set of vertices as in the traditional graph case and $E$ is the set of edges defined as a collection of subsets of vertices $E \subseteq 2^V$. If for every $e \in E$ we have $|e| = k$ we call the hypergraph $k$-regular.

Note that any 2-regular hypergraph is an ordinary graph. An example of a hypergraph is shown in Fig. 1.

![Hypergraph Example](image)

Figure 1: A sketch of a hypergraph. Here $v_1, \ldots, v_6$ denote the vertices with two 3-hyperedges $e_0, e_1$, and one 4-hyperedges $e_2$.

### 3 Walks

In this section we present three well-established models of quantum walks on graphs.

#### 3.1 The coined quantum walk

Here we introduce the coined quantum walk model. We start with a simple walk on a line. Later we discuss the scattering walk model and move to arbitrary graphs.

In the simplest case a discrete time coined quantum walk on a line is given by a bipartite system $\mathcal{H}_C \otimes \mathcal{H}_P$, where $\mathcal{H}_C$ is a two-dimensional Hilbert space with basis $\{|0\rangle, |1\rangle\}$ and $\mathcal{H}_P = \text{span} \{ |n\rangle : n \in \{0, \ldots, N-1\} \}$ is a position space. Every step of the evolution $U$ is a composition of the coin and shift operators

$$|\psi_{t+1}\rangle = S(C \otimes I_N)|\psi_t\rangle,$$

(1)
where $\psi_0 \in \mathcal{H}_C \otimes \mathcal{H}_P$ is some initial state. The coin operator has a nice, tensor product because each vertex has the same degree $d = 2$ which is equal to the dimensionality of the coin space.

### 3.2 Scattering quantum walk

In order to model a coined quantum walk on a graph which is not regular, we must modify this simple approach considerably. This can be achieved via the scattering walk approach. In this case, we introduce separate coin operators for every vertex $v$ of a graph $G$. Here, $G$ can be either directed or not. Denoting the degree of the $i^{th}$ vertex as $d_i$, we have that each coin operator $C_i$ acts on $\mathbb{C}^{d_i}$. The entire space is

$$\mathcal{X} = \mathbb{C}^{d_1} \oplus \ldots \oplus \mathbb{C}^{d_N}. \quad (2)$$

The shift operator performs the scattering on a vertex $i$ given by formula

$$S|i,j\rangle = |j,i\rangle. \quad (3)$$

Let us consider a particle coming to vertex $j$ from some vertex $i$, i.e. moving along the edge $(i,j)$. It becomes scattered after the shift operation, meaning that with equal probability it gets transferred to all other edges outgoing from $j$ and gets reflected back to $i$ along the $(j,i)$ edge, provided it exists in $G$. Hence we have for every vertex $i$ we have

$$U|i,j\rangle = r_i|i,j\rangle + t_i \sum_{v \sim j, v \neq i} |j,v\rangle. \quad (4)$$

Of course unitarity requires $|r_i|^2 + (\text{deg}(v_i) - 1)|t_i|^2 = 1$ and the entire evolution is given by this formula. In the case of a directed graph $G$, we need to remember that each edge $(i,j)$ can be seen as two directed edges.

For regular structures this simplifies to

$$C = C_0 \oplus \mathbb{1}_N \cong C_0 \otimes \mathbb{1}_N. \quad (5)$$

Let consider as an example the simple case when $r_i = 1$ for every $i$. In this case the shift operator for each edge subspace $\mathcal{X}_{i,j} = \text{span}(|i,j\rangle, |j,i\rangle)$ acts as $\sigma_x$ operator. Using a different basis, we may write the space $\mathcal{X}$ as $\mathcal{X}_E = \bigoplus_{i \sim j} \mathcal{X}_{i,j}$. Then, the shift operator takes the form

$$S' = \sigma_x^{\oplus |E|} \cong \sigma_x \otimes \mathbb{1}_{|E|}. \quad (6)$$

The shift operator is a block operator in

$$\mathcal{X} = \bigoplus_{i \sim j} \text{span}(|i,j\rangle, |j,i\rangle) \quad (7)$$

and the coin operator is a block operator in

$$\mathcal{Y} = \bigoplus_{j \in V} \text{span}(|i,j\rangle)_{i \sim j}. \quad (8)$$

Additionally we allow cases, when operator $C$ changes in time in a cyclic manner and call such model a generalized coined walk model.
3.3 The Szegedy walk model

The Szegedy walk model was first introduced in [19] as a model which allows quantization of arbitrary Markov chain based algorithms. The model is as follows. We start with undirected graph \( G = (V, E) \) and we set a bipartite graph \( G_S = G(V \cup V', F) \) where \( V' \) is the same as \( V \) with all elements primed. As for the edges we have \((i, j') \in F \) if and only if \((i, j) \in E \). The evolution is given by the unitary operators \( U_1, U_2 U_1, U_1 U_2 U_1, \ldots \) acting on the space span\( \{ |x, y' \rangle : x \in V, y' \in V' \} \).

We define reflections

\[
U_1 = 2 \sum_v |d_v\rangle \langle d_v| - 1, \\
U_2 = 2 \sum_v |\bar{d}_v\rangle \langle \bar{d}_v| - 1,
\]

and unit vectors

\[
|d_v\rangle = |v\rangle \otimes \sum_{w \sim v} a_{v,w} |w\rangle, \\
|\bar{d}_v\rangle = \sum_{w \sim v} a_{v,w} |w\rangle \otimes |v\rangle,
\]

where \( a_{v,w} \) are complex constants.

Usually, the unitary operators driving the evolution for the Szegedy walk model are chosen as presented above. In our work, we assume that unitary operators can be chosen arbitrary with only assumption of respecting the graph structure i.e. the movement between not connected vertices is forbidden.

3.4 The staggered walk

To formally introduce the staggered walk model, we first introduce the following definitions. We will follow the naming used by Portugal et al. [21]

**Definition 3.** A tessellation of a set \( A \) is a collection \( \alpha = \{ p_k \}_k \) of subsets of \( A \), \( p_k \subset A \), such that \( \bigcup_k p_k = A \) and \( p_k \cap p_{k'} = \emptyset \) for \( k \neq k' \).

**Definition 4.** A tessellation of a graph \( G = (V, E) \) is a tessellation of \( V \) such that each \( p_k \) forms a clique or is a single vertex. We will call \( p_k \) a polygon.

Note that this definition allows for a polygon to contain a single vertex. The staggered quantum walk on a graph is defined using at least one graph tessellation.

**Definition 5.** Given a graph \( G(V, E) \) and its \( n \) tessellations \( \alpha_1, \ldots, \alpha_n \), \( \alpha_k = \{ p_{k,i} \}_i \), for \( k = 1, \ldots, n \), the staggered quantum walk is defined by the evolution operator \( U \in \mathcal{U}(\mathcal{X}^V) \), where \( \mathcal{X}^V = \mathbb{C}^{|V|} \):

\[
U = U_n \ldots U_2 U_1, 
\]
where

\[ U_k = 2 \sum_{i=1}^{n_k} \alpha_k^i \langle d_{k,i} | d_{k,i} \rangle - \mathbb{1}_X. \]  

(12)

The states \( |d_{k,i} \rangle \) are:

\[ |d_{k,i} \rangle = \sum_{j \in p_{k,i}} a_{k,j} |j \rangle \]  

(13)

where \( a_{k,j} \) are complex amplitudes.

For the staggered quantum walk model we assume, that unitary operators can be also chosen arbitrarily.

4 Hyperwalk model

In this section we introduce the concept of quantum walks on hypergraph networks along with some intuitions. We will call this model the quantum hyperwalk model.

Now we want to emphasize that the coined quantum walk model can be described as a composition of two operators that take block operator form with respect to two different decompositions (tessellations) of the computational basis. The main restriction in the model is that the decomposition (tessellation) corresponding to the edges of the graph always consists of sets with two basis states. We aim at loosening this restriction and developing a quantum walk model suitable for hypergraphs, in this sense, that the tessellation of a hypergraph \( H = (V, E) \) is tessellation of basis states \( \{ |v, e \rangle : v \in V, e \in E \} \).

Definition 6. We define a hyperwalk on a hypergraph \( (V, E) \) as a composition \( U^E U^V \) of two unitary operators: \( U^V \) and \( U^E \) on the space \( X = \text{span}(\{ |v, e \rangle : e \in E \land v \in e \}) \), where

\[ U^V = \sum_{v \in V} C_v, \]  

(14)

\[ U^E = \sum_{e \in E} S_e, \]  

(15)

for \( C_v \) being a coin operator for a fixed vertex \( v \) acting on \( \text{span}(\{ |v, e \rangle : e \in E \land v \in e \}) \) and \( S_e \) being a shift operator for a fixed edge \( e \) acting on \( \text{span}(\{ |v, e \rangle : v \in e \}) \).

We also introduce a generalized version of this model. By a generalized hyperwalk we mean an instance of a hyperwalk for which the underlying unitaries change with time.

Example 1. We define \( C_v = \mathbb{1}_{\text{deg}(v)} - 2|\psi_v \rangle \langle \psi_v | \) for \( |\psi_v \rangle = \frac{1}{\sqrt{\text{deg}(v)}} \sum_{e \in V} |v, e \rangle \) and \( S_e = \mathbb{1}_{|e|} - 2|\psi_e \rangle \langle \psi_e | \) for \( |\psi_e \rangle = \frac{1}{\sqrt{|e|}} \sum_{v \in e} |v, e \rangle \) obtaining a hypergraph generalization of the Grover’s walk. Let us note that for a 2-regular hypergraph, i.e.
an ordinary graph, we obtain $S$, which are two dimensional Grover’s diffusion operator that are equal to $\sigma_x$. This shows that our model recovers the proper behaviour for a hypergraph which reduces to an ordinary graph.

**Example 2.** The idea of a hyperwalk gives the possibility to implement walks on directed graphs. The basic way to ensure that computation performed with use of directed connections is reversible (unitary) is to ensure that for each vertex the number of directed inputs and outputs is the same. In order to satisfy this condition for a finite graph the directed connections must contain loops, which may be seen as hyperedges. Thus, we define a quantum walk with directed edges as

$$S_E = \sum_{(v_1, \ldots, v_n) \in E} \sum_{i=1}^{l_e} |v_{i+1}, e\rangle \langle v_i, e|,$$

where $E$ is a set of edges defined as ordered sequences of vertices. In the case of 2-element edges we recover the canonical shift operator. For a hyperedge we obtain cyclic shift among the loop constructed by this edge.

Additionally we allow the case when operators $U^V, U^E$ change in time in a cyclic manner and call such model a *generalized hyperwalk.*

Hyperwalk model can be seen as a natural generalization of the coined walk model. This generality comes from two facts. First of all, by using hyperedges, it is possible to construct higher dimensional space of basis states than in coined walk. Second, loosening the restriction of shift operator to be a permutation matrix, gives us additional dynamics in the constructed space.

On the other hand, sometimes, for defined graphs (hypergraphs), it is not trivial or it is not even possible to obtain the given walk evolution by using the hyperwalk model, despite the graph structures allows us to do so. The explanation of this problem and the formal comparison of introduced models is left to the next section.

## 5 Relations between models

In this section we want to compare walk models discussed in the previous sections. To do this we introduce two alternative definitions of comparing walks and next we present our results of comparing quantum walk models.

To clarify notation let $QW_A(G, |\psi\rangle, n)$ denote the state after $n$ iterations of discrete-time quantum walk model $A$ on graph $G$ with initial an state $|\psi\rangle$. We also use the notation $P$ for measurement on vertices, where the probability of finding state $|\psi\rangle$ in vertex $v$ is denoted by $P(|\psi\rangle)(v)$.

**Definition 7.** For given two models $A$ and $B$, we say that model $A$ is an instance of model $B$ ($A \leq B$) when for all graphs $G_A(V_1, E_1)$, measurements $P_A$ and initial states $|\psi_A\rangle$ there exists $G_B(V_2, E_2)$, where $V_1 \subset V_2$, measurement $P_B$ and initial state $|\psi_B\rangle$, such that for all $n_0 \in \mathbb{N}_0$, exists $n_1 \in \mathbb{N}_0$ we have

$$P_A(QW_A(G_A, |\psi_A\rangle, n_0))(v) = P_B(QW_B(G_B, |\psi_B\rangle, n_1))(v).$$
Unfortunately, this definition allows us to find equivalence between walk models, which are very loosely based on the underlying structure. Some examples of equivalence of walk models which can be derived from this definition are presented below.

1. Szegedy ≤ coined and coined ≥ Szegedy: In the coined walk model we are given a set of basis states \(|i, j\rangle\), where \(i, j \in V\) if \((i, j) \in E\) and in Szegedy walk model our basis states are \(|i, j\rangle\) if \((i, j) \in E\). There exists a bijection \(\Xi\), that is defined as \(\Xi(|i, j\rangle) = |i, j\rangle\). Now we can assume, that if \(C\) is a given operator we take \(U_1 = C\) and \(U_2\) is constructed as in definition of Szegedy walk model. If \(U_1\) is given then we put \(C = U_1\). Then, as we can observe, the equalities are satisfied

\[
\Xi(SCSC|\psi\rangle) = U_2\Xi(SC|\psi\rangle) = U_2U_1\Xi(|\psi\rangle). \tag{18}
\]

The above follows from the fact both space are the same up to labeling. This observation implies that measurements are connected by \(P_{Sz} = \Xi(P_C)\), hence they are the same.

2. hyperwalk ≤ coined: For a given hypergraph \(H(V, E)\) with the evolution operator \(U_H = U^E U^V\), an initial state \(|\psi_0\rangle_H = |v, e\rangle\) and a measurement \(\mathcal{P}_H(|\psi\rangle)(v) = \sum_{e \in E} |\langle v, e|\psi\rangle|^2\), we create a bipartite graph \(G(V_1 \cup V_2, F)\), where the sets \(V_1 = V\) and \(V_2 = E\) are the partitions of a bipartite graph, and \((v, e) \in F\) if and only if \(v \in V_1\) is contained in an edge \(e_1 \in E\). We construct a space with the basis vectors \(|v, e\rangle, |v, e\rangle : v \in V, e \in E\). We define the shift operator in standard form

\[
S|v, e\rangle = |e, v\rangle
\]
\[
S|v, e\rangle = |v, e\rangle, \tag{19}
\]

and the coin operator as

\[
C = U^V \oplus U^E. \tag{20}
\]

Consequently, the evolution in quantum coined walk model with the initial state \(|\psi_0\rangle_C = |v, e\rangle\) is given by

\[
U_C = SCSC = SC(|1\rangle \langle 0| \otimes U^V + |0\rangle \langle 1| \otimes V^E)
\]
\[
= S(|1\rangle \langle 0| \otimes U^E U^V + |0\rangle \langle 1| \otimes U^V U^E) = U^E U^V \oplus U^V U^E. \tag{21}
\]

Then the measurement should be \(\mathcal{P}_C(|\psi\rangle)(v) = \sum_{e \in E} |\langle v, e|\psi\rangle|^2\) and now it is clear that \(\mathcal{P}_H(U^n_H|\psi_0\rangle_H) = \mathcal{P}_C(U^n_C|\psi_0\rangle_C)\).

3. generalized hyperwalk ≤ staggered: We are given hypergraph \(H(V, E)\) with the evolution operator \(U_{G_H,k} = U^E_k U^V\) for \(k \in \{1, \ldots, K\}\), an initial state \(|\psi_0\rangle_{G_H} = |v, e\rangle\) and a measurement \(\mathcal{P}_{G_H}(|\psi\rangle)(v) = \sum_{e \in E} |\langle v, e|\psi\rangle|^2\).

Let us note that we can consider an \(N\) dimensional system for staggered...
walk, where $N$ is the number of basis states in generalized hyperwalk. We can set $W = \{v, e : v \in V, e \in E, v \in e\}$ as a set of vertices for graph, which defined staggered walk on it and take initial state $|\psi_0\rangle_S = |\psi_0\rangle_{GH}$. We introduce such tessellations for which the unitary matrices $U_i^F, U_j^V$ can be treated as evolution operators. The measurement on the staggered walk model works on proper group of vertices i.e. $P_S(|\psi\rangle)(v) = \sum_{|v,e\rangle \in W} |\langle v,e|\psi\rangle|^2$. It is clear that we obtain the same evolution, because as introduced is this case base states in staggered walk are the same as in generalized hyperwalk model.

4. staggered ≤ generalized coined: We are given graph $G(V, E)$ with $N$ vertices, $V = \{1, \ldots, N\}$, $k$ tessellations and unitaries $U_1, \ldots, U_k$. We take an initial state $|\psi_0\rangle_S$ and measurement $P_S(|\psi\rangle)(v) = |\langle \psi|\psi\rangle|^2$. We construct a new graph by adding to the set $V$ one vertex $t_{i,j}$ for each polygon $j$ in each tessellation $i$. We connect every newly added vertex associated with some polygon to vertices included in this polygon. Let us denote the basis states by $\{|v, t_{i,j}\}, |t_{i,j}, v\\}$, where $v \leq N, i \leq k - 1$. Here, we omitted the second index in vertices $t_{i,j}$, because parameters $v$ and $i$ are sufficient to determine vertex $t_{i,j}$ uniquely in state $|v, t_{i,j}\rangle$. The generalized coined walk on this graph will be represented by $SCSC_{k-1} \ldots SCSC_0$. Here, $S$ is the standard shift operator, $C$ is defined by

$$C|v, t_i\rangle = |v, t_{i+1}\rangle, \quad C|t_0, v\rangle = |t_1, v\rangle.$$

(22)

and $C_i$ is defined as

$$C_i|v, t_j\rangle = |v, t_j\rangle, \quad C_i|t_j, v\rangle = |t_j, v\rangle, \quad j \neq i \quad (23)$$

$$C_i|t_i, v\rangle = \sum_{w \in V} \langle w|U_{i+1}|v\rangle|t_i, w\rangle.$$

The initial state can be chosen to be $|\psi_0\rangle_{GC} = |t_0, v_0\rangle$ and the measurement of vertex $v$ is $P_{GC}(|\psi\rangle)(v) = \sum_{i,v} \langle t_i, v|\psi\rangle^2$. To see the equality between both measurements, we set with $|\psi_0\rangle_S$ and after the first step we obtain $U_1|\psi_0\rangle_S$ in the staggered walk model. Assuming that the first step in the generalized coined walk is given by $SCSC_0$, we get

$$SCSC_0|t_0, v_0\rangle_{GC} = SC \sum_{w \in V} \langle w|U_1|v_0\rangle_S|t_0, w\rangle = SC \sum_{w \in V} \langle w|U_1|v_0\rangle_S|w, t_0\rangle = S \sum_{w \in V} \langle w|U_1|v_0\rangle_S|w, t_1\rangle = \sum_{w \in V} \langle w|U_1|v_0\rangle_S|t_1, w\rangle.$$

(24)

Then $\langle w|U_1|v_0\rangle_S = \langle t_1, w|SCSC_0|t_0, v_0\rangle_{GC}$, so finally we get

$$P_{GC}(SCSC_{k-1} \ldots SCSC_0|t_0, v_0\rangle_{GC})(v) = P_S(U_k \ldots U_1|v_0\rangle_S)(v).$$
5. generalized coined ≤ coined: For a given graph \( G \) with \( N \) vertices, \( \{1, \ldots, N\} \) with changing in time coins \( C_0, \ldots, C_{k-1} \), an initial state \( |\psi_0\rangle = |v, w\rangle \) and a measurement \( \mathcal{P}_{Gc}(|\psi_0\rangle)(v) = \sum_{w \in V} |\langle v, w|\psi_0\rangle|^2 \). We introduce a new graph with \( kN \) vertices \( v^{(i)} \), where \( v \leq N \), \( i \in 0, \ldots, k-1 \). In this graph we have connections only between vertices \( v^{(i)}, w^{(i+1)} \), for \( v, w \leq N, i \in 0, \ldots, k-1 \) if and only if \( v \sim w \) in \( G \). This generates new basis states \( \{|v^{(i)}, w^{(i+1)}\rangle, |w^{(i+1)}v^{(i)}\rangle\} \). In this model the initial state will be \( |v^{(0)}, w^{(k-1)}\rangle \) and the state will evolve to vertices with higher indexes and eventually come back to vertices with the index zero. This means, we define the coin operator as

\[
C|v^{(i)}, w^{(i-1)}\rangle = \sum_{z \in V} \langle v, z|C_0|v, w\rangle |v^{(i)}, z^{(i+1)}\rangle, \\
C|v^{(i)}, w^{(i+1)}\rangle = |v^{(i)}, w^{(i-1)}\rangle.
\]

The measurement on the vertex \( v \) is given on states associated with \( v^{(i)} \) i.e. \( \mathcal{P}_{C}(|\psi_0\rangle)(v) = \sum_i \sum_{w \sim v} |\langle v^{(i)}, w^{(i-1)}|\psi_0\rangle|^2 \).

After the first iteration in the coined walk model we have

\[
SC|\psi_0\rangle = SC|v^{(0)}, w^{(k-1)}\rangle = S \sum_{z \in V} \langle v, z|C_0|v, w\rangle |v^{(0)}, z^{(1)}\rangle \\
= \sum_{z \in V} \langle v, z|C_0|v, w\rangle |z^{(1)}, v^{(0)}\rangle.
\]  

(26)

On the other hand, considering the generalized coined walk model gives us

\[
SC_0|\psi_0\rangle = SC_0|v, w\rangle = \sum_{z \in V} \langle v, z|C_0|v, w\rangle |z, v\rangle.
\]

(27)

We can observe, that the both models give us the same evolution, which implies that the measurement outcomes will be exactly the same.

As it can be seen, according to this definition the models are equivalent. This result should not be surprising, as we are allowed to compare models \( A \) and \( B \) on graphs with different structures. For the case, when \( A \leq B \), the quantum walk model \( B \) does not have to express the idea of random walk on graph \( G_A \). For example if we want to change generalized hyperwalk with 2 different distributions on the graph shown in Figure 2 into the Szegedy walk, we need to take graph with 336 vertices. According to the previous discussion changing the generalized hyperwalk to a staggered walk costs 7 vertices. We see that if we want to put this model to generalized coined walk we should take a graph with 21 nodes and 8 coin operators. In the next step, it is necessary to model coined walk on a graph with \( 8 \times 21 = 168 \) vertices. The last step is cloning of the vertices to obtain Szegedy walk, so we end up with 336 nodes required. Of course there still can exist methods to achieve this result with a
smaller number of vertices, but this example is introduced to show problems which can appear. That is why we introduce a new concept of comparing two quantum models.

**Definition 8.** Let $A$ and $B$ be two models of a quantum walk. We say that model $A$ is strongly an instance of model $B$ ($A \prec B$) when for all graphs $G_A(V,E)$ there exists a graph $G_B$, such that the number of basis states in model $B$ is no greater then the number of basis states in model $A$. Moreover, for all initial states $|\psi_A\rangle$ there exists an initial state $|\psi_B\rangle$, such that for all $n \in \mathbb{N}_0$ and $v \in V$, we have

$$P_A(QW_A(G_A,|\psi_A\rangle,n))(v) = P_B(QW_B(G_B,|\psi_B\rangle,n))(v).$$

(28)

Based on this definition, we show that every staggered walk is an instance of a generalized hyperwalk. Furthermore, we do not need to deeply change the structure of the initial graph in order to obtain this behavior.

**Theorem 1.** According to Definition 8, every staggered walk is an instance of the generalized hyperwalk.

**Proof.** For a given graph with the staggered walk, defined by unitary matrices $U_1,\ldots,U_n$, we introduce a hypergraph with the same number of vertices and one hyperedge containing all vertices suitable for generalized hyperwalk. One can see that the spaces for both walks have the same dimensionality hence there exists a bijection between spaces on staggered and generalized coined walk models. So we can assume that the coin operator is constant and it is given by the identity matrix. The shift operator is changing in time in the same manner as the unitary operators $U_1,\ldots,U_n$ for each tessellations, namely $U^E_k := U_k$. If the measurement for staggered walk is $P_S(|\psi\rangle)(v) = |\langle v|\psi\rangle|^2$, then we take $P_{GH} = |\langle v,e|\psi\rangle|^2$. \hfill $\square$

### 6 Conclusions

In this work we introduced a model of quantum walks on hypergraphs and a generalized version of this model. By generalized we mean that the evolution
operators associated with the walk might change in time. We introduced two non-equivalent definitions of the case when one quantum walk model is an instance of another model. The first definition of this equivalence allows us to heavily manipulate the underlying graph structure of the walk. Using this definition we shown that hyperwalk model is equivalent to a coin model and the same for the generalized version.

Next, we introduced a stronger version of the equivalence of walk models. In it, we enforce the graph to be a minimal graph necessary for a given model. In this regime we were able to show that a generalized hyperwalk introduces in fact new dynamics. This result completes Table I and shows that a quantum walk on a hypergraph is a generalization of the staggered walk model.

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