Toda Hierarchy with Indefinite Metric

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May 15, 1995

Abstract

We consider a generalization of the full symmetric Toda hierarchy where the matrix \( \tilde{L} \) of the Lax pair is given by \( \tilde{L} = LS \), with a full symmetric matrix \( L \) and a nondegenerate diagonal matrix \( S \). The key feature of the hierarchy is that the inverse scattering data includes a class of noncompact groups of matrices, such as \( O(p, q) \). We give an explicit formula for the solution to the initial value problem of this hierarchy. The formula is obtained by generalizing the orthogonalization procedure of Szegő, or the QR factorization method of Symes. The behaviors of the solutions are also studied. Generically, there are two types of solutions, having either sorting property or blowing up to infinity in finite time. The \( \tau \)-function structure for the tridiagonal hierarchy is also studied.

Mathematics Subject Classifications (1991). 58F07, 34A05

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1 Introduction

The finite non-periodic Toda lattice hierarchy can be written in the Lax form \[2\], \[5\] with variables \( t := (t_1, t_2, \cdots) \),

\[
\frac{\partial}{\partial t_n} L = [B_n, L], \quad n = 1, 2, \cdots
\]

(1.1)

where \( L \) is an \( N \times N \) symmetric “tridiagonal” matrix with real entries,

\[
L = \begin{pmatrix}
    a_1 & b_1 & 0 & \cdots & 0 \\
    b_1 & a_2 & b_2 & \cdots & 0 \\
    0 & \ddots & \ddots & \ddots & 0 \\
    0 & \cdots & a_{N-1} & b_{N-1} & \cdots \\
    0 & \cdots & b_{N-1} & a_N & \cdots
\end{pmatrix}
\]

(1.2)

and \( B_n \) is the skew symmetric matrix defined by

\[
B_n = \prod_a L^n := (L^n)_{>0} - (L^n)_{<0}.
\]

(1.3)

Here \((L^n)_{>0} (\leq 0)\) denotes the strictly upper (lower) triangular part of \( L^n \). (We formally write an infinite number of flows in (1.1), even though there are at most \( N \) independent flows.) In particular, (1.1) for \( n=1 \) with \( t := t_1 \) has the form

\[
\frac{\partial a_k}{\partial t} = 2(b_k^2 - b_{k-1}^2),
\]

(1.4)

\[
\frac{\partial b_k}{\partial t} = b_k(a_{k+1} - a_k).
\]

(1.5)

This system describes a hamiltonian system of \( N \) particles on a line interacting pairwise with exponential forces. The hamiltonian for the system is given by

\[
H = \frac{1}{2} \sum_{k=1}^{N} y_k^2 + \sum_{k=1}^{N-1} \exp(x_k - x_{k+1}),
\]

(1.6)

where the canonical variables \((x_k, y_k)\) are related to \((a_k, b_k)\) by

\[
a_k = -\frac{y_k}{2},
\]

(1.7)
and
\[ b_k = \frac{1}{2} \exp\left(\frac{x_k - x_{k+1}}{2}\right). \] (1.8)

The \( \tau \)-functions were introduced in [6] to study the Toda equation (1.4) and (1.5). Writing \( a_i \)'s with the \( \tau \)-function \( \tau_i \)'s and \( t = t_1 \),
\[ a_i = \frac{1}{2} \frac{\partial}{\partial t} \log \frac{\tau_i}{\tau_{i-1}}, \] (1.9)

\( b_i \) can be expressed as
\[ b_i^2 = \frac{1}{4} \frac{\tau_{i+1}\tau_{i-1}}{\tau_i^2}. \] (1.10)

Then (1.4) and (1.5) become
\[ \frac{1}{4} \frac{\partial^2}{\partial t^2} \log \tau_i = \frac{\tau_{i+1}\tau_{i-1}}{\tau_i^2}. \] (1.11)

These \( \tau \)-functions \( \tau_i \) have a simple structure, that is, a symmetric wroskian given by [7],
\[ \tau_i = \begin{vmatrix} g & g_1 & g_2 & \ldots & g_{i-1} \\
g_1 & g_2 & g_3 & \ldots & g_i \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
g_{i-1} & g_i & \ldots & \ldots & g_{2i-2} \end{vmatrix} \] (1.12)

where the entries \( g_n \) are defined by \( g_n := (1/2)\partial g/\partial t_n \), with a function \( g \) satisfying the linear equations
\[ \frac{1}{2} \frac{\partial g}{\partial t_n} = \frac{1}{2^n} \frac{\partial^n g}{\partial t^n}. \] (1.13)

These \( \tau \)-fuctions are also shown to be positive definite in [7]. It is a key property of their relation to the moment problem of Hamburger. The equation (1.11) is also expressed by the well-known Hirota bilinear form
\[ D^2_1 \tau_i \cdot \tau_i = 8\tau_{i+1}\tau_{i-1}, \] (1.14)
where the Hirota derivative is defined by

\[(D_t f \cdot g)(t) := \left. \frac{d}{ds} f(t + s) g(t - s) \right|_{s=0}\]

\[= \left( \frac{df}{dt} g - f \frac{dg}{dt} \right)(t). \quad (1.15)\]

There have been extensive studies on the hierarchy (1.1) and its generalizations. One of them is to extend $L$ from “tridiagonal” to “full symmetric”. This more general system, which we call “full symmetric Toda hierarchy”, was shown by Deift et. al. in [1] to be a completely integrable hamiltonian system. The inverse scattering scheme for (1.1) with $L$ being any symmetric square matrix consists of two linear equations,

\[L \Phi = \Phi \Lambda, \quad (1.16)\]

\[\frac{\partial}{\partial t_n} \Phi = B_n \Phi, \quad (1.17)\]

where $\Phi$ is the orthogonal eigenmatrix of $L$, and $\Lambda$ is $\text{diag}(\lambda_1, \cdots, \lambda_N)$. The hierarchy (1.1) then results as the compatibility of these equations with $\partial \Lambda/\partial t_n = 0$ (iso-spectral deformation). In [2], Kodama and McLaughlin solved the initial value problem of (1.16) and (1.17) using the “orthonormalization method”, and derived an explicit formula of the solution in a determinant form. They also showed that the generic solution assumes the “sorting property”. Here the sorting property means that $L(t_1) \to \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_N)$ as $t_1 \to \infty$, with the eigenvalues being ordered by $\lambda_1 > \lambda_2 > \cdots > \lambda_N$.

The hierarchy we consider in this paper is for $\tilde{L} = L S$, where $L$ is full symmetric and $S$ is a constant diagonal matrix, $S = \text{diag}(s_1, \cdots, s_N)$ with nonzero “real” entries. All the entries of $\tilde{L}$ are assumed to be real, unless otherwise stated. The hierarchy is also defined as the Lax form (1.1) with (1.3). In particular, for the case of tridiagonal $\tilde{L}$ with $n=1$, we have

\[\frac{\partial a_k}{\partial t} = 2(s_k b_k^2 - s_{k-1} b_{k-1}^2), \quad (1.18)\]

\[\frac{\partial b_k}{\partial t} = b_k(s_{k+1} a_{k+1} - s_k a_k). \quad (1.19)\]
In analogue to the nonperiodic Toda equation with (1.6), the hamiltonain $\tilde{H}$ for the above equations is given by

$$\tilde{H} = \frac{1}{2} \sum_{k=1}^{N} y_k^2 + \sum_{k=1}^{N-1} s_k s_{k+1} \exp(x_k - x_{k+1}),$$

(1.20)

with the change of variables

$$s_k a_k = -\frac{y_k}{2},$$

(1.21)

and

$$s_k s_{k+1} b_k = \frac{1}{2} \exp \left( \frac{x_k - x_{k+1}}{2} \right).$$

(1.22)

Note from (1.20) that $\tilde{H}$ includes some attractive forces, thereby is not positive definite. One then expects a “blowing up” in solutions.

We study the initial value problem of the hierarchy by the “inverse scattering method”. The inverse scattering scheme is also given by (1.16) and (1.17), where the eigenmatrix $\Phi$ is now normalized to satisfy

$$\Phi S^{-1} \Phi^T = S^{-1}, \quad \Phi^T S \Phi = S.$$  

(1.23)

In the case of the full symmetric Toda hierarchy, where $S$ is the identity matrix, $\Phi$ is given by an orthogonal matrix, $\Phi \in O(N)$. As a special case of (1.23), we have $\Phi \in O(p, q)$ for $S = \text{diag}(1, \cdots, 1, -1, \cdots, -1)$, with $p + q = N$. From the first quation in (1.23), we define an inner product where $S^{-1}$ gives an indefinite metric for $S$ being not positive definite. For this reason, we call the hierarchy considered in this paper the “Toda hierarchy with indefinite metric”.

The content of this paper is as follows: We start with a preliminary in Section 2 to give some background information on the Toda hierarchy and the inverse scattering scheme. First we show the compatibility of the flows in (1.1). Then we show that the eigenmatrix of $\tilde{L}$ can be chosen to satisfy (1.23), and also (1.23) is invariant under the flows generated by (1.17).

In Section 3, we show that the hierarchy can be solved using the orthonormalization method with respect to the indefinite metric. The solution of the
hierarchy turns out to be given by the same solution formula in \cite{4} with a little modification.

In Section 4, we discuss the behaviors of the solutions. Due to the indefiniteness of the metric defined by $S^{-1}$, the solution has richer behaviors than the full symmetric case. Generically, in addition to the sorting property, there are solutions blowing up to infinity in finite time. The behaviors of the solutions are characterized by $S$ and the initial conditions, *i.e.*, the eigenvalues $\lambda_i, i = 1, \cdots, N$ of $\tilde{L}$ and the initial eigenmatrix $\Phi^0$. In this paper, we obtain the following main results: If $S$ is positive definite, then generic solutions have the sorting property (Theorem 2). If some eigenvalues of $\tilde{L}$ are not real, or $\Phi^0$ is not real, then generic solutions blow up to infinity in finite time (Theorem 3).

In Section 5, we illustrate these results with explicit examples.

In Section 6, we study $\tau$-function structures of the $\tilde{L}$ hierarchy. In the case of tridiagonal $\tilde{L}$, one can also introduce $\tau$-functions in the same form as the symmetric wroskians \cite{12}. However, our $\tau$-functions are no longer positive definite as a result of the indefinite metric in our hierarchy. We also find a “bilinear identity” generating relations among $\tau_i$’s of \cite{12}. From the bilinear identity, we then derive a hierarchy written in Hirota’s bilinear form for the $\tau$-functions including \cite{14} as its first member.

\section{Preliminary}

The hierarchy considered in this paper is defined as\footnote{Note that among the infinite number of flows in the hierarchy, only the first $N$ flows are independent, and the higher order flows are related to these $N$ flows through $P(\tilde{L}) = 0$, the characteristic polynomial of $\tilde{L}$.}

\begin{equation}
\frac{\partial}{\partial t_n} \tilde{L} = \left[ \tilde{B}_n, \tilde{L} \right], \quad n = 1, 2, \cdots \tag{2.1}
\end{equation}

where $\tilde{L} = L S$ with $L$, a full symmetric matrix, and $S$, a nondegenerate diagonal matrix, *i.e.*, a symmetrizable matrix in the sense of Kac \cite{3}, and $\tilde{B}_n := \prod_a \tilde{L}^a = (\tilde{L}^n)_{>0} - (\tilde{L}^n)_{<0}$. One should note that
Let us first show the compatibility of flows in (2.1), that is, \( \partial^2 \tilde{L} / \partial t_m \partial t_n = \partial^2 \tilde{L} / \partial t_n \partial t_m \). This can be shown from the “zero curvature” condition, i.e.

\[
\frac{\partial \tilde{B}_n}{\partial t_m} - \frac{\partial \tilde{B}_m}{\partial t_n} = [\tilde{B}_m, \tilde{B}_n].
\]  

(2.2)

**Proposition 1** The flows in (2.1) commute with each other.

**Proof.** It suffices to show the zero curvature condition (2.2). From (2.1), we have

\[
\frac{\partial \tilde{L}_n}{\partial t_m} - \frac{\partial \tilde{L}_m}{\partial t_n} = [\tilde{B}_m, \tilde{L}_n] - [\tilde{B}_n, \tilde{L}_m],
\]

(2.3)

which can be also written as

\[
[\tilde{B}_m, \tilde{L}_n] - [\tilde{B}_n, \tilde{L}_m] = [\tilde{B}_m, \tilde{B}_n] + [\tilde{B}_m + \tilde{L}_m, \tilde{B}_n + \tilde{L}_n].
\]

(2.4)

Note that \( \tilde{B}_m + \tilde{L}_m \) is an upper triangular matrix. Taking the lower triangular projection of (2.3), we have

\[
\frac{\partial (\tilde{L}_n)}{\partial t_m} - \frac{\partial (\tilde{L}_m)}{\partial t_n} = ([\tilde{B}_n, \tilde{B}_m])_{<0}.
\]

(2.5)

Similarly,

\[
\frac{\partial (\tilde{L}_n)}{\partial t_m} - \frac{\partial (\tilde{L}_m)}{\partial t_n} = -([\tilde{B}_n, \tilde{B}_m])_{>0}.
\]

(2.6)

Subtracting (2.6) from (2.5), we get

\[
\frac{\partial \tilde{B}_n}{\partial t_m} - \frac{\partial \tilde{B}_m}{\partial t_n} = ([\tilde{B}_m, \tilde{B}_n])_{>0} + ([\tilde{B}_m, \tilde{B}_n])_{<0}.
\]

(2.7)

Since \([\tilde{B}_n, \tilde{B}_m]\) is skew-symmetric, \(\text{diag}([\tilde{B}_m, \tilde{B}_n]) = 0\). This completes the proof. \(\blacksquare\)

**Remark 1** Since the specific form of \( \tilde{L} \) is not used in this proof, Proposition 1 is valid for an arbitrary matrix.

In the case of \( L \) being full symmetric, \( L \) can be diagonalized by an orthogonal matrix, and the flows defined by (1.4) are compatible with the choice of skew-symmetric \( B_n \) by (1.3). To set up the inverse scattering scheme for our system, we need the following lemma from linear algebra:
Lemma 1 Let $\tilde{L}$ be a matrix given by $\tilde{L} = LS$, where $L$ is symmetric, $S = \text{diag}(s_1, \ldots, s_N)$ nondegenerate. Suppose that all the eigenvalues of $\tilde{L}$ be distinct. Then $\tilde{L}$ can be diagonalized by a matrix $\Phi$ satisfying (1.23), i.e., $\Phi S^{-1} \Phi^T = S^{-1}$, $\Phi^T S \Phi = S$.

Proof. Let $(\cdot, \cdot)$ be the usual euclidean inner product, i.e. $(x, y) := \sum_{i=1}^{N} x_i y_i$, for $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$. Let $\phi(\lambda_i)$ be the eigenvector of $\tilde{L}$ corresponding to the eigenvalue $\lambda_i$, i.e., $\tilde{L} \phi(\lambda_i) = \lambda_i \phi(\lambda_i)$. Calculating $(S \phi(\lambda_j), \tilde{L} \phi(\lambda_i))$, we have

$$(S \phi(\lambda_j), \tilde{L} \phi(\lambda_i)) = \lambda_i (S \phi(\lambda_j), \phi(\lambda_i))$$

$$= (LS \phi(\lambda_j), S \phi(\lambda_i)) = \lambda_j (S \phi(\lambda_j), \phi(\lambda_i)),$$

which leads to

$$(\lambda_j - \lambda_i) (S \phi(\lambda_j), \phi(\lambda_i)) = 0.$$ 

By the assumption, $\lambda_i \neq \lambda_j$ for $i \neq j$, we obtain $(S \phi(\lambda_j), \phi(\lambda_i)) = 0$, that is, $\Phi^T S \Phi$ is a diagonal matrix. It is also nondegenerate. We then normalize $\Phi$ to satisfy $\Phi^T S \Phi = S$. We multiply $S^{-1} \Phi^T$ on the both sides from the right, get $\Phi^T S \Phi S^{-1} \Phi^T = \Phi^T$. This leads to $\Phi S^{-1} \Phi^T = S^{-1}$. \[\square\]

Remark 2 With the normalization, the eigenmatrix $\Phi$ becomes complex in general, even in the case that all the eigenvalues are real. This is simply due to a case where the sign of $\sum_{k=1}^{N} s_k \phi_k^2(\lambda_i)$ in $\Phi^T S \Phi$ differs from that of $s_i$.

We now show that the choice of $\tilde{B}_n$ in (1.17) is compatible with (1.23), that is:

Lemma 2 Eqs (1.23) are invariant under the flows generated by $\tilde{B}_n = \Pi_a \tilde{L}^n$.

Proof. First, we have

$$\frac{\partial(\Phi^T S \Phi)}{\partial t_n} = \frac{\partial \Phi^T}{\partial t_n} S \Phi + \Phi^T S \frac{\partial \Phi}{\partial t_n}$$

$$= \Phi^T (\tilde{B}_n^T S + S \tilde{B}_n) \Phi.$$ 

So all we need to show is $\tilde{B}_n^T S + S \tilde{B}_n = 0$. Since $S$ is diagonal, it commutes with the anti-symmetric projection $\Pi_a$, that is,

$$\tilde{B}_n := (\tilde{L}^n)_> - (\tilde{L}^n)_< = (\tilde{L}^{n-1} L S)_> - (\tilde{L}^{n-1} L S)_<$$

$$= (\tilde{L}^{n-1} L)_> S - (\tilde{L}^{n-1} L)_< S.$$ 

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This leads to
\[
\tilde{B}_n^T S + S \tilde{B}_n = S[(\tilde{L}^{-1} - L)_0^> - (\tilde{L}^{-1} - L)_0^>]^T S + S[(\tilde{L}^{-1} - L)_0^> - (\tilde{L}^{-1} - L)_0^>] S.
\]
Since \( \tilde{L}^{-1} L \) is symmetric, \( (\tilde{L}^{-1} - L)_0^> - (\tilde{L}^{-1} - L)_0^< \) is skew-symmetric. Thus, we have \( \tilde{B}_n^T S + S \tilde{B}_n = 0 \).

The eigenmatrix \( \Phi \) consists of the eigenvectors of \( \tilde{L} \), say \( \phi(\lambda_k) \equiv (\phi_1(\lambda_k), \cdots, \phi_N(\lambda_k))^T \) for \( k = 1, 2, \cdots, N \),

\[
\Phi \equiv [\phi(\lambda_1), \cdots, \phi(\lambda_N)] = [\phi_i(\lambda_j)]_{1 \leq i,j \leq N} .
\]

Then (1.23) give the “orthogonality” relations

\[
\sum_{k=1}^N s_k^{-1} \phi_i(\lambda_k) \phi_j(\lambda_k) = \delta_{ij} s_i^{-1} ,
\]

\[
\sum_{k=1}^N s_k \phi_k(\lambda_i) \phi_k(\lambda_j) = \delta_{ij} s_i .
\]

With (2.9), we now define an inner product \( < \cdot, \cdot > \) for two function \( f \) and \( g \) of \( \lambda \) as

\[
< f, g > := \sum_{k=1}^N s_k^{-1} f(\lambda_k) g(\lambda_k) ,
\]

which we write as \( < fg > \) in the sequel. The metric in the inner product is given by

\[
d\alpha(\lambda) = \sum_{k=1}^N s_k^{-1} \delta(\lambda - \lambda_k) d\lambda ,
\]

which leads to an indefinite metric due to a choice of negative entries \( s_k \) in \( S \). The entries of \( \tilde{L}^n \) are then expressed by

\[
\tilde{a}_{ij}^{(n)} := (\tilde{L}^n)_{ij} = s_j < \lambda^n \phi_i \phi_j > .
\]
### 3 Inverse scattering method

In this section, we solve the initial value problem of the hierarchy (2.14) by using the inverse scattering method. Namely, we first solve the time evolution for the eigenmatrix $\Phi(t)$ of $\tilde{L}$ with the initial matrix $\Phi(0) := \Phi^0 = [\phi_i^0(\lambda_j)]_{1 \leq i, j \leq N}$ obtained from the eigenvalue problem $\tilde{L}(0)\Phi^0 = \Phi^0\Lambda$, and then find the solution $\tilde{L}(t)$ through (2.13) with $\Phi(t)$. Here we call the eigenvalues $\lambda_k, k = 1, \cdots, N$ and the eigenmatrix $\Phi$ the “scattering data”.

Let us first note $\tilde{B}_n = \tilde{L}_n - \text{diag}(\tilde{L}_n) - 2(\tilde{L}_n)_{<0}$, and using $\tilde{L}\Phi = \Phi\Lambda$, we write (1.17) as

$$\frac{\partial}{\partial t_n} \Phi = \Phi\Lambda^n - \left[ \text{diag}(\tilde{L}_n) + 2(\tilde{L}_n)_{<0} \right] \Phi. \quad (3.1)$$

Then the equations for the first row vector $\phi_1(\lambda_k, t), k = 1, \cdots, N$, in $\Phi(t)$ are

$$\frac{\partial \phi_1(\lambda_k)}{\partial t_n} = \left( \lambda_k^n - s_1 < \lambda^n \phi_1^2(\lambda) > \right) \phi_1(\lambda_k), \quad (3.2)$$

which can be readily solved in the form $\phi_1(\lambda_k, t) = \frac{\psi_1(\lambda_k, t)}{\sqrt{s_1 < \psi_1^2(\lambda, t) >}}$ with

$$\psi_1(\lambda_k, t) = \phi_1^0(\lambda_k)e^{\xi(\lambda_k, t)}, \quad (3.3)$$

where $e^{\xi(\lambda_k, t)} := \sum_{n=1}^{\infty} \lambda_k^n t_n$. The equations for the second row vector $\phi_2(\lambda_k, t), k = 1, \cdots, N$, are

$$\frac{\partial \phi_2(\lambda_k)}{\partial t_n} = \left( \lambda_k^n - s_2 < \lambda^n \phi_2^2(\lambda) > \right) \phi_2(\lambda_k) - 2s_1 < \lambda^n \phi_2(\lambda) \phi_1(\lambda) > \phi_1(\lambda_k). \quad (3.4)$$

The solution to (3.4) can be also found in the form $\phi_2(\lambda_k, t) = \frac{\psi_2(\lambda_k, t)}{\sqrt{s_2 < \psi_2^2(\lambda, t) >}}$ with

$$\psi_2(\lambda_k, t) = \phi_2^0(\lambda_k)e^{\xi(\lambda_k, t)} - s_1 < \phi_2^0(\lambda)e^{\xi(\lambda, t)} \phi_1(\lambda, t) > \phi_1(\lambda_k, t). \quad (3.5)$$

In general, the $i$th row vector of $\Phi$ in (3.1) satisfies

$$\frac{\partial \phi_i(\lambda_k)}{\partial t_n} = \left( \lambda_k^n - s_i < \lambda^n \phi_i^2(\lambda) > \right) \phi_i(\lambda_k) - 2 \sum_{l=1}^{i-1} s_l < \lambda^n \phi_i(\lambda) \phi_l(\lambda) > \phi_l(\lambda_k). \quad (3.6)$$
Then the solution of (3.6) is expected to have the form 
\[ \phi_i(\lambda_k, t) = \frac{\psi_i(\lambda_k, t)}{\sqrt{s_i < \psi_i^2(\lambda, t)}} \]
with
\[ \psi_i(\lambda_k, t) = \phi_i^0(\lambda_k)e^{\xi(\lambda_k, t)} - \sum_{l=1}^{i-1} s_l < \phi_i^0(\lambda)e^{\xi(\lambda, t)}\phi_l(\lambda, t) > \phi_l(\lambda_k, t) \cdot (3.7) \]

As we will see below, this is indeed the case. The above procedure suggests that the eigenmatrix \( \Phi \) can be constructed by the “orthonormalization procedure” on the row vectors \( [\phi_i^0(\lambda_k)e^{\xi(\lambda_k, t)}]_{1 \leq k \leq N, i = 1, \cdots, N} \).

Let us now give a more systematic presentation of the above procedure, which is the essence of the approach in [4]. Because of the simple structure of \( \psi_i(\lambda_k) \) in (3.7), we first write
\[ \Phi = T\Psi \tag{3.8} \]
where \( \Psi := [\psi_i(\lambda_j)]_{1 \leq i,j \leq N} \) and
\[ T = \text{diag} \left[ (s_1 < \psi_1^2 >)^{-1/2}, \cdots, (s_N < \psi_N^2 >)^{-1/2} \right] . \]

Note that (3.8) can be regarded as a gauge transformation and includes a freedom in the choice of \( \psi \). Namely, (3.8) is invariant under the scaling of \( \psi_i \), i.e., \( \psi_i \rightarrow f_i(t)\psi_i \), with \( \{f_i\}_{i=1}^{N} \) arbitrary functions of \( t \). With (3.8), (1.16) and (1.17) become
\[ \left( T^{-1}\tilde{L}T \right) \Psi = \Psi \Lambda \tag{3.9} \]
\[ \frac{\partial}{\partial t} \Psi = \left( T^{-1}\tilde{B}_nT \right) \Psi - \left( \frac{\partial}{\partial t} \log T \right) \Psi \tag{3.10} \]

Writing as in (3.1), i.e.
\[ \left( T^{-1}\tilde{B}_nT \right) = -2 \left( T^{-1}\tilde{L}^nT \right)_{<0} + \left( T^{-1}\tilde{L}^nT \right) - \text{diag} \left( \tilde{L}^n \right) \tag{3.11} \]

(3.10) gives
\[ \frac{\partial \psi}{\partial t} = -2 \left( T^{-1}\tilde{L}^nT \right)_{<0} \psi + \lambda^n \psi - \left( \text{diag} \left( \tilde{L}^n \right) + \frac{\partial}{\partial t} \log T \right) \psi \tag{3.11} \]
We here observe that (3.11) can be split into two sets of equations by fixing the gauge freedom in the determination of $\psi$. In the components, these are

$$\frac{\partial \psi_i}{\partial t_n} = -2 \sum_{j=1}^{i-1} <\lambda^n \psi_i \psi_j> - <\psi_j^2> \psi_j + \lambda^n \psi_i , \quad (3.12)$$

$$\frac{1}{2} \frac{\partial}{\partial t_n} \log <\psi_i^2> = a_i^{(n)} . \quad (3.13)$$

It is easy to check that (3.12) implies (3.13). It is also immediate from (3.12) that we have:

**Proposition 2** The solution of (3.12) can be written in the form of separation of variables,

$$\psi(\lambda, t) = A(t) \phi^0(\lambda) e^{\xi(\lambda, t)} , \quad (3.14)$$

where $A(t)$ is a lower triangular matrix with $\text{diag}[A(t)] = A(t = 0) = I_N$, the $N \times N$ identity matrix, and $\phi^0(\lambda) = \phi(\lambda, 0)$.

Note that we have chosen the initial conditions of $\psi(\lambda, t)$ to coincide with $\phi(\lambda, t)$, i.e. $\psi(\lambda, 0) = \phi^0(\lambda)$, and thus $s_i < \psi_i \psi_j > (t = 0) = s_i < \phi_i^{(0)} \phi_j^{(0)} >= \delta_{ij}$. As a direct consequence of this proposition, and the orthogonality of the eigenvectors, (2.9), i.e. $< \psi_i \psi_j > = 0$ for $i \neq j$, we have:

**Corollary 1** (Orthogonality): For each $i \in \{2, \cdots, N\}$, we have for all $t$ with $t_m \in \mathbb{R}$,

$$< \psi_i \phi_j^0 e^{\xi(\lambda, t)} > = \sum_{k=1}^{N} s_{i-1} \psi_i(\lambda_k, t) \phi_j^0(\lambda_k) e^{\xi(\lambda_k, t)} = 0 \quad (3.15)$$

for $j = 1, 2, \cdots, i - 1$.

Now we obtain the formula for the eigenvectors of $\tilde{L}$ in terms of the initial data $\{\phi_i^0(\lambda)\}_{1 \leq i \leq N}$:

**Theorem 1** The solutions $\psi_i(\lambda, t)$ of (3.12) are given by

$$\psi_i(\lambda, t) = \frac{e^{\xi(\lambda, t)}}{D_{i-1}(t)} \begin{vmatrix} s_1 c_{11} & \cdots & s_1 c_{1i} \\ \vdots & \ddots & \vdots \\ s_{i-1} c_{i-11} & \cdots & s_{i-1} c_{i-1i} \\ \phi_i^0(\lambda) & \cdots & \phi_i^0(\lambda) \end{vmatrix} \quad (3.16)$$
where $c_{ij}(t) = \langle \phi_i^0 \phi_j^0 e^{2\xi(\lambda,t)} \rangle$, and $D_k(t)$ is the determinant of the $k \times k$ matrix with entries $s_i c_{ij}(t)$, i.e.,

$$D_k(t) = \left| (s_i c_{ij}(t))_{1 \leq i,j \leq k} \right|. \tag{3.17}$$

(Note here that $s_i c_{ij}(0) = \delta_{ij}$ and $D_k(0) = 1$.)

**Proof.** From equation (3.15) with (3.14), we have

$$s_i \sum_{k=1}^{i} A_{ik}(t) < \phi_k^0 \phi_i^0 e^{2\xi(\lambda,t)} >= 0, \text{ for } 1 \leq \ell \leq i - 1. \tag{3.18}$$

Solving (3.18) for $A_{ik}$ with $A_{ii} = 1$, we obtain

$$A_{ik}(t) = -\frac{D_{i-1}^k(t)}{D_{i-1}(t)}, \tag{3.19}$$

where $D_0(t) := 1$, and $D_{i-1}(t)$ is the determinant $D_{i-1}(t)$ in the form (3.17) with the replacement of the $k$th column $(s_1 c_{1k}, \ldots, s_i-1 c_{i-1 k})^T$ by $(s_1 c_{1i}, \ldots, s_i-1 c_{i-1 i})^T$. From (3.14), we then have

$$\psi_i = e^{\xi(\lambda,t)} \sum_{k=1}^{i} A_{ik} \phi_k^0$$

$$= -e^{\xi(\lambda,t)} \sum_{k=1}^{i-1} \frac{D_{i-1}^k(t)}{D_{i-1}} \frac{\phi_k^0}{\phi_i^0} + e^{\xi(\lambda,t)} \frac{D_{i-1}(t)}{D_{i-1}}$$

$$= \frac{e^{\xi(\lambda,t)}}{D_{i-1}} \times \begin{pmatrix}
-\phi_1^0 & s_1 c_{1i} & s_1 c_{12} & \cdots & s_1 c_{1i-1} \\
& \ddots & \ddots & \ddots & \\
& & s_i-1 c_{i-1 i} & s_i-1 c_{i-1 2} & \cdots & s_i-1 c_{i-1 i-1} \\
& & & \ddots & \ddots & \ddots & \\
& & & & s_1 c_{11} & s_1 c_{1i-1} \\
& & & & & \ddots & \ddots & \\
& & & & & & \ddots & \ddots & \\
\end{pmatrix} + \cdots$$

which is just (3.10). \( \square \)

We note from (3.16) that $< \psi_i^2 >$ can be expressed by $D_i$,

$$< \psi_i^2 > = \frac{D_i}{s_i D_{i-1}}. \tag{3.20}$$
This yields the formulae for the normalized eigenfunctions

\[
\phi_i(\lambda, t) = \frac{e^{\xi(\lambda, t)}}{\sqrt{D_i(t)D_{i-1}(t)}} \begin{vmatrix}
  s_{1c_{11}} & \cdots & s_{1c_{1i}} \\
  \vdots & \ddots & \vdots \\
  s_{i-1c_{i-11}} & \cdots & s_{i-1c_{i-1i}} \\
  \phi^0_1(\lambda) & \cdots & \phi^0_i(\lambda)
\end{vmatrix}
\]  
(3.21)

With the formula (3.21), we now have the solution (2.13) of the inverse scattering problem (1.16) and (1.17).

The above derivation of (3.21) is the same as the orthogonalization procedure of Szegö [9], which is equivalent to the Gram-Schmidt orthogonalization as observed in [4], except here the orthogonalization is with respect to an indefinite metric. Indeed, from the form of

\[
\phi_i(\lambda) = \psi_i \sqrt{s_i} \langle \psi_i^2 \rangle,
\]

(3.14) expresses the relations

\[
\phi_i(\lambda, t) \in \text{span} \left\{ \phi^0_1(\lambda)e^{\xi(\lambda, t)}, \ldots, \phi^0_i(\lambda)e^{\xi(\lambda, t)} \right\}, \quad i = 1, \ldots, N,
\]
(3.22)

where we are viewing \( \{ \phi_i(\lambda, t) \}_{i=1}^N \) as a collection of “orthogonal” functions defined on the eigenvalues of the matrix \( \tilde{L} \). The relations (3.22) together with the orthogonality (2.9), \( \langle \phi_i \phi_j \rangle = s_i^{-1}\delta_{ij} \), imply that \( \{ \phi_i(\lambda, t) \}_{i=1}^N \) are obtained by an orthogonalization of the sequence \( \{ \phi^0_i(\lambda)e^{\xi(\lambda, t)} \}_{i=1}^N \), and hence one may obtain (3.21) from the classical formulae as in [4].

Remark 3. The above method is a natural generalization of the QR factorization method of Symes [8]. In the case of \( L \) being symmetric, the QR factorization method is given as follows: First factorize \( e^{L(0)t} = Q(t)R(t) \), where \( Q(t) \in SO(N) \) with \( Q(0) = I_N \), and \( R(t) \) is a lower triangular matrix. Then \( L(t) \) is given by \( L(t) = Q^{-1}(t)L(0)Q(t) \). In the case of our \( \tilde{L} \), the same procedure applies where \( Q(t) \) satisfies (1.23), i.e. \( \Phi(t) = Q^{-1}(t)\Phi^0 \).

Remark 4. The above method applies directly to the following equation,

\[
\frac{\partial}{\partial t_1} \tilde{L} = [\tilde{B}_1, \tilde{L}] + f(\tilde{L}),
\]
where a “pumping” term $f(\tilde{L})$ is given by an entire function of $\tilde{L}$. Due to the presence of $f(\tilde{L})$, the eigenvalues of $\tilde{L}(t)$ are no longer time-independent. In fact, we have

$$\lambda_{t_1} = f(\lambda),$$

and the solution formula (3.21) still holds with the replacement of $e^{\lambda_{t_1}}$ by $\exp\left(\int_{t_0}^{t_1} \lambda(t) dt\right)$, and $t_i = 0$ except $i = 1$.

## 4 Behaviors of the solutions

Here we consider the first equation of the hierarchy, that is, we have only one time $t_1$, which we denote by $t$. As a consequence of the explicit formula (3.21), we can now study the behaviors of the solutions. First we note:

**Proposition 3** $D_{i}, i = 1, 2, \ldots, N$, in (3.12) are real functions.

*Proof.* In the construction of the solutions $\Phi(t)$, the “gauge” $T$ is fixed by (3.13). In terms of $D_i$, (3.13) is

$$\tilde{a}_{ii} = \frac{1}{2} \frac{\partial}{\partial t} \log \frac{D_i}{D_{i-1}}. \quad (4.1)$$

Note that $D_0 \equiv 1$, $D_i(0) = 1$ and $\tilde{a}_{ii}$ are real functions. Then we see by induction that all $D_i$ are real functions. \[ \square \]

In (4.1), suppose $D_i(t_0) = 0$ for some finite $t_0$ and some $i$. Then if $\tilde{L}(t_0)$ is a finite matrix, $D_{i-1}(t_0)$ must be also 0. By induction, $D_1(t_0) = 0$, but $D_0(t) \equiv 1$, this forces $\tilde{a}_{11}$ to be infinite, which is a contradiction. So we have:

**Proposition 4** Suppose $D_i(t_0) = 0$ for some $t_0 < \infty$ and some $i$, then $\tilde{L}(t)$ blows up to infinity at $t_0$.

We note that $D_i, i = 1, 2, \ldots, N$, are $i$th principal minors of the product of matrices $S \Phi_{\epsilon} S^{-1} \Phi_{\epsilon}^T$, where $\Phi_{\epsilon}$ is

$$\begin{pmatrix}
    e^{\lambda_1 t} \phi_1^0(\lambda_1) & \cdots & e^{\lambda_N t} \phi_1^0(\lambda_N) \\
    \vdots & \ddots & \vdots \\
    e^{\lambda_1 t} \phi_N^0(\lambda_1) & \cdots & e^{\lambda_N t} \phi_N^0(\lambda_N)
\end{pmatrix}.$$
Proposition 5 $D_i, i = 1, 2, \cdots, N$ can be expressed as

$$D_i = s_1 \cdots s_i \sum_{J_N = (j_1, \cdots, j_i)_N} \frac{1}{s_{j_1} \cdots s_{j_i}} e^{2 \sum_{k=1}^{i} \lambda_k t} \begin{vmatrix} \phi_1^0(\lambda_{j_1}) & \cdots & \phi_1^0(\lambda_{j_i}) \\ \vdots & \ddots & \vdots \\ \phi_i^0(\lambda_{j_1}) & \cdots & \phi_i^0(\lambda_{j_i}) \end{vmatrix},$$ (4.2)

where $J_N$ represents all possible combinations for $1 \leq j_1 < \cdots < j_i \leq N$. In particular $D_0(t) \equiv 1$, and $D_N(t) = \exp(2 \sum_{i=1}^{N} \lambda_i t)$.

This proposition is very useful to study the asymptotic behavior of $D_i$ for large $t$. To study the time evolution of $D_i(t)$, we need the information on the scattering data, that is, the eigenvalues $\lambda_k$ and the eigenmatrix $\Phi^0$. Here we have from the linear algebra:

**Lemma 3** Let $\tilde{L}$ be a matrix given by $\tilde{L} = LS$, where $L$ is symmetric, $S = \text{diag}(s_1, \cdots, s_N)$ positive definite. Then all the eigenvalues of $\tilde{L}$ are real, and the initial eigenmatrix $\Phi^0$ of $\tilde{L}$ satisfying (1.23) is also real.

**Proof.** Let $\phi(\lambda_i)$ be an eigenvector of $\tilde{L}$ corresponding to $\lambda_i$, i.e.,

$$\tilde{L}\phi(\lambda_i) = \lambda_i \phi(\lambda_i).$$ (4.3)

Multiplying (4.3) by the adjoint of $S\phi(\lambda_i)$, i.e., $\overline{\phi}^T(\lambda_i)S$ to the left, we get

$$\overline{\phi}^T(\lambda_i)SLS\phi(\lambda_i) = \lambda_i \overline{\phi}^T(\lambda_i)S\phi(\lambda_i).$$ (4.4)

On the other hand, if we take the adjoint of (4.3), and then multiply $S\phi(\lambda_i)$ to the right, we get

$$\overline{\phi}^T(\lambda_i)SLS\phi(\lambda_i) = \overline{\lambda_i} \overline{\phi}^T(\lambda_i)S\phi(\lambda_i).$$ (4.5)

Subtracting (4.4) from (4.5), we have $(\lambda_i - \overline{\lambda_i}) \overline{\phi}^T(\lambda_i)S\phi(\lambda_i) = 0$. Since $S$ is positive definite, this forces $\lambda_i - \overline{\lambda_i} = 0$, that is, $\lambda_i$ is real.

As for the normalization of $\phi(\lambda_i)$, since both $\overline{\phi}^T(\lambda_i)S\phi(\lambda_i)$ and $s_i$ are positive, we only need to multiply $\phi(\lambda_i)$ by some “real” factor. So $\phi(\lambda_i)$ remains real after normalization. This verifies the assertion of the lemma. $\blacksquare$

We now obtain:
Theorem 2 Let the eigenvalues of $\tilde{L}$ be ordered as $\lambda_1 > \lambda_2 > \cdots > \lambda_N$. Suppose that $S$ is positive definite and $\det \Phi_n^0 \neq 0$ for $n = 1, \ldots, N$, where $\Phi_n^0$ is the $n$th principal minor of $\Phi^0$. Then as $t \to \infty$, the eigenfunctions $\phi_i(\lambda_j, t)$ satisfy

$$\phi_i(\lambda_j, t) \to \delta_{ij} \times \text{sgn} \left( \det \Phi_i^0 \right) \text{sgn} \left( \det \Phi_{i-1}^0 \right),$$

which implies $L(t) \to \text{diag}(\lambda_1, \ldots, \lambda_N)$ from (1.17).

Proof. Using Lemma 3, we see that all the terms in the sum (4.2) are positive, thereby $D_i$'s are positive for all $t$. From the ordering in the eigenvalues, we see that the leading order for $D_i$ is given by

$$e^{2 \sum_{k=1}^i \lambda_k t} \prod_{1 \leq k < j \leq i} \frac{\phi_j^0(\lambda_k) \cdots \phi_k^0(\lambda_j)}{\phi_k^0(\lambda_k) \cdots \phi_j^0(\lambda_j)} = e^{\lambda t} \prod_{1 \leq k < j \leq i} \frac{\phi_j^0(\lambda_k) \cdots \phi_k^0(\lambda_j)}{\phi_k^0(\lambda_k) \cdots \phi_j^0(\lambda_j)} \times \frac{\det \Phi_i^0 | \det \Phi_{i-1}^0 \exp \left( \frac{2 \sum_{n=1}^{i-1} \lambda_n + \lambda_i}{t} \right)}$$

where we have assumed $\det \Phi_0^0 \neq 0$. From (3.21) and (4.7), as $t \to \infty$,

$$\phi_n(\lambda; t) \to e^{\lambda t} \prod_{1 \leq k < j \leq i} \frac{\phi_j^0(\lambda_k) \cdots \phi_k^0(\lambda_j)}{\phi_k^0(\lambda_k) \cdots \phi_j^0(\lambda_j)} \times \frac{\det \Phi_i^0 | \det \Phi_{i-1}^0 \exp \left( \frac{2 \sum_{n=1}^{i-1} \lambda_n + \lambda_i}{t} \right)}$$

The dominant term in the determinant gives

$$e^{2 \sum_{t=1}^{n-1} \lambda_t} \sum_{\mathbf{P}_{n-1}} \phi_0^0(\lambda_{\ell_1}) \cdots \phi_{n-1}^0(\lambda_{\ell_{n-1}}) \prod_{1 \leq k < j \leq n-1} \frac{\phi_j^0(\lambda_k) \cdots \phi_k^0(\lambda_j)}{\phi_k^0(\lambda_k) \cdots \phi_j^0(\lambda_j)} = e^{2 \sum_{t=1}^{n-1} \lambda_t} \prod_{1 \leq k < j \leq n-1} \frac{\phi_j^0(\lambda_k) \cdots \phi_k^0(\lambda_j)}{\phi_k^0(\lambda_k) \cdots \phi_j^0(\lambda_j)} \times \sum_{\mathbf{P}_{n-1}} \phi_0^0(\lambda_{\ell_1}) \cdots \phi_{n-1}^0(\lambda_{\ell_{n-1}}) \sigma(\{\ell_j\}_{j=1}^{n-1}),$$
where \( \sigma(\{l_j\}_{j=1}^{n-1}) \) is the signature of the permutation \( P_{n-1} \) for \( 1 \leq l_j \leq n-1 \). We note that the determinant in (1.9) is zero for \( \lambda = \lambda_j, j = 1, \ldots, n-1 \). The ordering \( \lambda_1 > \cdots > \lambda_N \) implies the result.

Theorem 2 shows if \( S \) is positive definite, then generic solutions have the “sorting property”. It is a natural generalization of Theorem 2 in [4], where \( S = I_N \). Next theorem provides sufficient conditions for the solutions to blow up to infinity in finite time.

**Theorem 3** Suppose some eigenvalues of \( \tilde{L} \) are not real, or \( \Phi^0 \) is not real, and \( \det \Phi^0_n \neq 0 \) for \( n = 1, \cdots, N \). Then \( \tilde{L}(t) \) blows up to infinity in finite time.

**Proof.** We have two cases to consider:

a). All the eigenvalues are real, but \( \Phi^0 \) is complex. From Remark 1, each column \( \phi^0(\lambda_i) \) of \( \Phi^0 \) is either pure imaginary or real. In this case, let \( k \) be the first column such that \( \phi^0(\lambda_k) \) is pure imaginary. With the ordering \( \lambda_1 > \cdots > \lambda_N \), the leading order in the expansion (1.2) of \( D_k \) is still given by (4.7). We also assume \( \det \Phi^0_k \neq 0 \). Since \( \phi^0(\lambda_k) \) is pure imaginary, while \( \phi^0(\lambda_i), i = 1, \cdots, k-1, \) are real, we have

\[
(det \Phi^0_k)^2 = \begin{vmatrix}
\phi^0_1(\lambda_1) & \cdots & \phi^0_1(\lambda_k) \\
\vdots & \ddots & \vdots \\
\phi^0_k(\lambda_1) & \cdots & \phi^0_k(\lambda_k)
\end{vmatrix}^2 < 0
\]

Being negative of the leading order in \( D_k \) implies \( \lim_{t \to \infty} D_k(t) = -\infty \). Note \( D_k(0) = 1 \), so there is some \( t_0 < \infty \), such that \( D_k(t_0) = 0 \). Then Proposition 4 results that \( \tilde{L}(t) \) blows up to infinity in finite time \( t_0 \).

b). Some eigenvalues are not real. We order the eigenvalues of \( \tilde{L} \) by their real parts. We still assume all the eigenvalues to be distinct. Since \( \tilde{L} \) is a real matrix, the complex eigenvalues appear as pairs. For a convenience, we also assume that there is at most one pair having the same real part. Suppose \( \lambda_k + i\beta_k \) and \( \lambda_k - i\beta_k \) are the first pair of complex eigenvalues. Then from (1.2), the leading order term in \( D_k \) is

\[
e^{-2\sum_{t=1}^{k} \lambda_{it} + 2i\beta_k t} \begin{vmatrix}
\phi^0_1(\lambda_1) & \cdots & \phi^0_1(\lambda_k + i\beta_k) \\
\vdots & \ddots & \vdots \\
\phi^0_k(\lambda_1) & \cdots & \phi^0_k(\lambda_k + i\beta_k)
\end{vmatrix}^2 + \]

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Since $D_k$ is real by Proposition 1, one can write the above as

$$e^{2 \sum_{i=1}^{k} \lambda_k t - 2i \beta_k t} \begin{vmatrix} \phi_0^0(\lambda_1) & \cdots & \phi_0^0(\lambda_k - i \beta_k) \\ \vdots & \ddots & \vdots \\ \phi_k^0(\lambda_1) & \cdots & \phi_k^0(\lambda_k - i \beta_k) \end{vmatrix}^2.$$ 

where $A$ and $B$ are two real constants. The above is an oscillating function about zero. Thus by Proposition 2, we conclude that $\tilde{L}(t)$ blows up to infinity in finite time. $\Box$

This theorem implies that the complexness of the initial scattering data leads to the blowing up of the solutions. Now, only the situation left undetermined is the case where $S$ is indefinite, and the scattering data are real. We then have:

**Proposition 6** Suppose conditions as stated above are satisfied and $\det \Phi_0^i \neq 0$, $i = 1, \cdots, N$. If $\tilde{L}(t)$ doesn’t blow up to infinity in finite time, then $\tilde{L}(t)$ has the sorting property, i.e., $\tilde{L}(t) \to \text{diag}(\lambda_1, \cdots, \lambda_N)$ as $t \to \infty$.

**Proof.** If $\tilde{L}(t)$ doesn’t blow up to infinity in finite time, then $D_i(t)$’s are all positive definite, in particular, the leading order in the expansion (4.2) is positive. Then the same proof as in Theorem 3 applies here. $\Box$

### 5 Examples

We here demonstrate our results using two simple examples. The first example includes a parameter, and we show a bifurcation behavior of the solutions as the parameter changes. The second example is for a blowing up aspect in the case where $S$ is indefinite, and all eigenvalues of $\tilde{L}$ are real and so is $\Phi^0$.

**Example 1.** We take $\tilde{L}$ to be a $2 \times 2$ matrix with $S = \text{diag}(1, -1)$, that is, $\Phi \in O(1, 1)$, and $\tilde{L}$ is given by

$$
\begin{pmatrix}
a_1 & -b_1 \\
 b_1 & -a_2
\end{pmatrix}.
$$

(5.1)
Our hierarchy (2.1) then gives for $n=1$

\[
\begin{align*}
\frac{da_1}{dt} &= -2b_1^2, \\
\frac{da_2}{dt} &= -2b_1^2, \\
\frac{db_1}{dt} &= -b_1(a_1 + a_2). 
\end{align*}
\]

(5.2)

From (5.2), we can see both $a_1$ and $a_2$ are always decreasing. If $a_1 + a_2$ is initially negative, then $b_1^2$ increases faster and faster, and we expect a blowing up of the system. While, for the case where $a_1 + a_2$ is initially large positive, we expect that $b_1^2$ to decrease to 0 and $a_1 + a_2$ decreases to some positive number. This is indeed the case.

Let us take the initial conditions, $a_1 = 0$, $b_1 = 1$, and $a_2 = c$, a constant. We then show a bifurcation behavior of the solutions with the parameter $c$, i.e., blowing up at $c < c_0$ and the sorting at $c > c_0$ for certain value of $c_0$. The initial scattering data, the eigenvalues $\lambda_{1,2}$ and the normalized eigenmatrix $\Phi^0$, are given by

\[
\lambda_{1,2} = \frac{1}{2} \left[ -c \pm \sqrt{c^2 - 4} \right],
\]

(5.3)

and

\[
\Phi^0 = \begin{pmatrix}
\frac{\lambda_2}{\sqrt{\lambda_2^2 - 1}} & \frac{\lambda_1}{\sqrt{1 - \lambda_1^2}} \\
\frac{1}{\sqrt{\lambda_2^2 - 1}} & \frac{-1}{\sqrt{1 - \lambda_1^2}}
\end{pmatrix}.
\]

(5.4)

The eigenvalues $\lambda_1$ and $\lambda_2$ take the following values as the function of $c$: $0 > \lambda_1 \geq -1 \geq \lambda_2$, for $c \geq 2$; $\lambda_1 = \lambda_2$ complex for $|c| < 2$; and $\lambda_1 \geq 1 \geq \lambda_2 > 0$ for $c \leq -2$. Note in particular that $\Phi^0$ becomes complex (pure imaginary) when $c < -2$, even though the eigenvalues are real. Then from (3.17) and (3.21), we have

\[
D_1(t) = \frac{1}{\lambda_2 - \lambda_1} \left( \lambda_2 e^{2\lambda_1 t} - \lambda_1 e^{2\lambda_2 t} \right),
\]

(5.5)

and

\[
\Phi(t) = \frac{1}{\sqrt{D_1(t)}} \begin{pmatrix}
\frac{\lambda_2}{\sqrt{\lambda_2^2 - 1}} & \frac{\lambda_1}{\sqrt{1 - \lambda_1^2}} \\
\frac{1}{\sqrt{\lambda_2^2 - 1}} & \frac{-1}{\sqrt{1 - \lambda_1^2}}
\end{pmatrix} e^{\lambda_2 t}.
\]

(5.6)
With the choice of $\lambda_1$ and $\lambda_2$ in (5.3), we have $\lambda_1 > \lambda_2$ for all $|c| > 2$. In particular, notice that the dominant term in $D(t)$ for large $t$ becomes negative for the case $c < -2$. This is due to the complexness of $\Phi^0$, and implies the blowing up in the solution at the time $t = t_B$ (Theorem 3),

$$t_B = \frac{1}{2(\lambda_1 - \lambda_2)} \log \frac{\lambda_1}{\lambda_2}. \quad (5.7)$$

On the other hand, for $c > 2$ we have the sorting result. For $|c| < 2$, the eigenvalues become complex and $D_1(t)$ is expressed as

$$D_1(t) = -2(4 - c^2)^{-\frac{1}{2}} e^{-2ct} \sin \left(\sqrt{4 - c^2} t - \theta\right) \quad (5.8)$$

with $\tan \theta = \sqrt{4 - c^2}/2$. This also indicates the blowing up (Theorem 3). The solution $\tilde{L}(t)$ is obtained from (2.13),

$$\tilde{a}_{11} = a_1 = <\lambda \phi_1^2 > = \frac{e^{2\lambda_1 t} - e^{2\lambda_2 t}}{\lambda_2 e^{2\lambda_1 t} - \lambda_1 e^{2\lambda_2 t}}, \quad (5.9)$$

$$\tilde{a}_{21} = b_1 = <\lambda \phi_2 \phi_1 > = \frac{(\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2) t}}{\lambda_2 e^{2\lambda_1 t} - \lambda_1 e^{2\lambda_2 t}}. \quad (5.10)$$

It is interesting to note that for the case $c \leq -2$, the blowing up occurs at one time $t_B$ (5.7), and the solution $\tilde{L}(t)$ will be sorted as $t \to \infty$, with the asymptotic form,

$$\Phi(t) \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.11)$$

Also note that if we start with a “wrong” ordering in the eigenvalues, i.e., $\lambda_1 < \lambda_2$, then we still have the correct sorting result with

$$\Phi(t) \to \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (5.12)$$

For the case of $c = \pm 2$, we obtain by taking the limit of (5.9) and (5.10)

$$\tilde{L}(t) = \begin{pmatrix} -\frac{2t}{1+2t} & -\frac{1+2t}{1+2t} \\ \mp 2 & \pm 2 + \frac{1+2t}{1+2t} \end{pmatrix}. \quad (5.13)$$
which shows the “sorting property” as \( t \to \infty \), i.e. \( \tilde{L}(t) \to \pm \text{diag}(1, 1) \). It should be noted that \( \tilde{L}(0) \) at \( c = \pm 2 \) is not similar to \( \pm \text{diag}(1, 1) \).

In the above example, \( S \) is indefinite, when \( c > 2 \), the eigenvalues are real and so is \( \Phi^0 \), the system has the sorting property. The following example shows the blowing up aspect in that situation.

**Example 2.** Take \( S = \text{diag}(1, -1, 1) \), write

\[
\tilde{L}(0) = \Phi^0 \text{diag}(2, 1, -100) S^{-1}(\Phi^0)^T S,
\]

where

\[
\Phi^0 = \begin{pmatrix}
  1 & -2 & 2 \\
  2 & -3 & 2 \\
  -2 & 2 & -1
\end{pmatrix}.
\]

Calculating (3.17), we get \( D_1(t) = e^{4t} - 4e^{2t} + 4e^{-200t} \). Since \( D_1(0.4) = -3.95 < 0 \), we conclude \( \tilde{L}(t) \) blows up to infinity before \( t = 0.4 \).

6 \quad \tau\text{-functions for the tridiagonal hierarchy}

In this section, we only consider the case of \( \tilde{L} \) being a tridiagonal matrix, and change \( t \) to \( 2t \) for convenience. \( \tilde{L} \) can be written as:

\[
\tilde{L} = \begin{pmatrix}
  s_1a_1 & s_2b_1 & 0 & \ldots & 0 \\
  s_1b_1 & s_2a_2 & s_3b_2 & \ldots & 0 \\
  0 & \vdots & \ddots & \vdots & 0 \\
  0 & \ldots & \ldots & s_{N-1}a_{N-1} & s_{N-1}b_{N-1} \\
  0 & \ldots & \ldots & s_{N-1}b_{N-1} & s_Na_N
\end{pmatrix}.
\]

Similarly to the symmetric tridiagonal case, we can introduce \( \tau_i \)’s as in (1.9). Equations (1.9), (1.10) are now modified to,

\[
s_i a_i = \frac{\partial}{\partial t_1} \log \frac{\tau_i}{\tau_{i-1}},
\]

and

\[
s_i s_{i+1} b_i = \frac{\tau_{i+1} \tau_{i-1}}{\tau_i^2}.
\]
However the equation for $\tau_i$ remains the same as (1.11). The $\tau$-functions are also given by the symmetric wronskian (1.12) with $g = \langle (\phi_0^i(\lambda))^2 e^{\xi(\lambda,t)} \rangle$. Note in particular that the function $g$ also satisfies (1.13) with the change $t$ to $2t$, i.e.

$$\frac{\partial g}{\partial t_n} = \frac{\partial^n g}{\partial t^n_n}.$$  \hspace{1cm} (6.4)

In the symmetric tridiagonal case, it was shown in [4] that $\tau_i$’s are given by $D_i$’s in (3.17). In our case, $\tau_i$’s are related to $D_i$’s by

$$\tau_i = \frac{D_i}{s_1 \cdots s_i}.$$ \hspace{1cm} (6.5)

For instance, in example (i) of the previous section, $\tau_1$ is given by $D_1$ in (1.5) and $\tau_2 = -D_2 = - \exp(-ct)$. It should be noted that in the symmetric tridiagonal case, these $\tau$-functions are positive definite, which is important in the moment problem of Hamburger, while in our case, the $\tau$-functions are no longer positive definite, in general.

Let us now derive a hierarchy for $\tau_i$ in (1.12), which includes (1.14) as its first member. First we note the following: In terms of $P_i(-\tilde{\partial}_k)$, where $P_i$ are the elementary Schur polynomials defined by

$$e^{\xi(\lambda,t)} = \sum_{n=1}^{\infty} P_n(t)\lambda^n,$$ \hspace{1cm} (6.6)

(1.11) can be written as

$$\left[ P_1(-\tilde{\partial}_k)\tau_i \right] \tau_i - \left[ P_1(-\tilde{\partial}_k)\tau_i \right]^2 = \tau_{i-1}\tau_{i+1}.$$ \hspace{1cm} (6.7)

where $\tilde{\partial}_t = (\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \cdots)$. As a generalization of (6.7), we have:

**Lemma 4** For any $k, l \geq 1$, the $\tau$-functions (1.12) satisfy

$$\left[ P_k(-\tilde{\partial}_k)P_l(-\tilde{\partial}_k)\tau_i \right] \tau_i - \left[ P_k(-\tilde{\partial}_k)\tau_i \right] \left[ P_l(-\tilde{\partial}_k)\tau_i \right] = \left[ P_{k-1}(-\tilde{\partial}_k)P_{l-1}(-\tilde{\partial}_k)\tau_i \right] \tau_{i+1}.$$ \hspace{1cm} (6.8)
Proof. From the wronskian structure of the $\tau_i$ with $g$ satisfying (6.4), we have

$$P_k(-\tilde{\partial}_t)\tau_i = \tau_{i+1}(i - k + 1, i + 1),$$

(6.9)

where $\tau_{i+1}(i - k + 1, i + 1)$ is the determinant of $\tau_{i+1}$ after removing the $i - k + 1$th row and the $i + 1$th column. Note that $P_k(-\tilde{\partial}_t)\tau_i = 0$, for $k > i$. For $1 \leq k, l \leq i$, we also get from the symmetric structure of the wronskian

$$P_k(-\tilde{\partial}_t)P_l(-\tilde{\partial}_t)\tau_i = \tau_{i+1}(i - k + 1, i - l + 1).$$

(6.10)

Noting $\tau_i = \tau_{i+1}(i + 1, i + 1)$ we have

$$\left[ P_k(-\tilde{\partial}_t)P_l(-\tilde{\partial}_t) \right] \tau_i = \left[ P_k(-\tilde{\partial}_t) \right] \left[ P_l(-\tilde{\partial}_t) \right] \tau_i - \tau_{i+1}(i - k + 1, i + 1) \tau_{i+1}(i + 1, i - l + 1).$$

(6.11)

The equation (6.8) then results directly from the the Jacobi formula for a determinant $A(=\tau_{i+1})$,

$$A(i, k)A(j, l) - A(i, l)A(j, k) = A \begin{pmatrix} i & k \\ j & l \end{pmatrix} A,$$

(6.12)

where $A \begin{pmatrix} i & j & k \\ j & l \end{pmatrix}$ is the determinant of $A$ after removing the $i$th and $j$th rows and the $k$th and $l$th columns. □

We then multiply $\lambda^{-k}\mu^{-l}$ on the both sides of (6.8), and sum up $k$ and $l$ to get, from (6.6),

$$\left[ \exp(-\sum_k \frac{1}{k!x} \partial_{t_k}) \exp(-\sum_l \frac{1}{l!y} \partial_{t_l}) \right] \tau_i = \left[ \exp(-\sum_k \frac{1}{k!x} \partial_{t_k}) \right] \left[ \exp(-\sum_l \frac{1}{l!y} \partial_{t_l}) \right] \tau_i.$$

(6.13)

Thus we obtain:

**Proposition 7** The $\tau$-functions satisfy a “bilinear identity” generating the relations of (6.8),

$$\tau_i(t - \epsilon[\lambda] - \epsilon[\mu])\tau_i(t) - \tau_i(t - \epsilon[\lambda] - \epsilon[\mu])\tau_i(t - \epsilon[\mu]) = \frac{1}{\lambda\mu} \tau_{i+1}(t - \epsilon[\lambda] - \epsilon[\mu]),$$

(6.14)

where $\epsilon[\lambda] = (\lambda^{-1}, \frac{1}{2}\lambda^{-2}, \cdots)$. 24
We can now obtain a hierarchy written in the Hirota bilinear form (6.14). By setting $\mu = \lambda$, (6.14) reads

$$
\tau_i(t - 2\epsilon[\lambda])\tau_i(t) - \tau_i(t - \epsilon[\lambda])\tau_i(t - \epsilon[\lambda]) = \frac{1}{\lambda^2} \tau_i(t - 2\epsilon[\lambda])\tau_{i+1}(t).
$$

(6.15)

Then changing $t - \epsilon[\lambda]$ to $t$, we can rewrite (6.15) as

$$
\tau_i(t - \epsilon[\lambda])\tau_i(t + \epsilon[\lambda]) - \tau_i(t)\tau_i(t) = \frac{1}{\lambda^2} \tau_i(t - \epsilon[\lambda])\tau_{i+1}(t + \epsilon[\lambda]).
$$

(6.16)

This gives:

**Corollary 2** With the Hirota derivatives $\tilde{D}_n := D_n$ in (1.13), we have

$$
\left(\exp(\tilde{D}) - 1\right) \tau_i \cdot \tau_i = \frac{1}{\lambda^2} \exp(\tilde{D})\tau_{i+1} \cdot \tau_{i-1},
$$

(6.17)

where $\tilde{D} := \sum_{n=1}^{\infty} \frac{1}{n\lambda^n} \tilde{D}_n$.

Expanding (6.17) in the power of $\lambda^{-1}$, and using (6.6), we obtain equations written in Hirota’s bilinear form, for $n \geq 0$,

$$
P_{n+2}(\tilde{D})\tau_i \cdot \tau_i = P_n(\tilde{D})\tau_{i+1} \cdot \tau_{i-1}.
$$

(6.18)

The first three equations in (6.18) are

$$
\tilde{D}_1^2 \tau_i \cdot \tau_i = 2\tau_{i+1} \cdot \tau_{i-1},
$$

(6.19)

$$
(\tilde{D}_1 \tilde{D}_2)\tau_i \cdot \tau_i = 2\tilde{D}_1\tau_{i+1} \cdot \tau_{i-1},
$$

(6.20)

$$
(\tilde{D}_1^4 + 3\tilde{D}_2^2 + 8\tilde{D}_1\tilde{D}_3)\tau_i \cdot \tau_i = 12(\tilde{D}_2 + \tilde{D}_1^2)\tau_{i+1} \cdot \tau_{i-1}.
$$

(6.21)

Note here that (6.19) is just (1.14) with $t$ changes to $2t$, and the “odd” powers of $\tilde{D}_i$’s on the l.h.s. of (6.18) are suppressed by operating on $\tau_i \cdot \tau_i$.

**Acknowledgment** The work of Y. Kodama is partially supported by an NSF grant DMS9403597.

J. Ye wishes to thank Prof. Guangtian Song for his encouragement through the years.
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