Quasi-classical Study of Form Factors in Finite Volume.

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Abstract. We construct the quasi-classical approximation of the form factors in finite volume using the separation of variables. The latter is closely related to the Baxter equation.

1 Introduction.

There is an important open problem of describing the matrix elements of local fields taken between two eigenstates of Hamiltonian (form factors) for integrable field theory with periodical boundary conditions. There are at least two reasons why this problem is interesting. First, the knowledge of such matrix elements would allow us to study the correlators at finite temperature. Second, detailed understanding of the periodical problem would give controllable interpolation between a massive field theory and its ultra-violet, conformal, limit.

In the present paper we shall study purely conformal case, more exactly, \( c < 1 \) models of Conformal Field Theory (CFT). Certainly, in CFT the correlators can be found explicitly, so one might find our study to be rather of academic interest. However this is not quite the case. The point is that we study the conformal field theory in its integrable formulation. The latter allows deformation to non-conformal case.

The integrable structure of CFT was first discussed by Zamolodchikov [1] who constructed examples of higher local integrals. The existence of infinitely many local integrals was proven in [2]. The spectrum of the local integrals is a subject of detailed study in the series of papers [3, 4]. In particular, the paper [4] provides detailed investigation of famous Baxter equation. We are trying to combine the results of this paper with the method of separation of variables in integrable models developed by Sklyanin [6]. In the latter method the solutions of Baxter equations play a role of wave functions for separated variables. We also use intensively the relation to the classical periodical solutions which are related to Riemann surfaces on which the separated variables represent divisors.

The CFT compactified on the circle depends trivially on the length \( L \) of the circle. However, the case \( L \to \infty \) is to be considered separately. Simple reasoning shows that the limit of the matrix elements in this limit must reproduce known form factors for Sine-Gordon (more exactly restricted Sine-Gordon) model in infinite volume [7]. Here we find a relation to the paper [8] where the formulae for the form factor in the infinite volume have been explained in terms of separated variables. It should be said, however, that in the paper [8] we failed to reproduce the quasi-classical limit of form-factors completely. More careful approach of the present paper will allow to find the missed pieces which are, in fact, due to the contribution of "vacuum" particles.

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As it will be clear from the paper, though our results are quasi-classical, exact quantum formulae are to be found by similar means. How efficient these formulae would be is another question.

Finally, I would like to say that the mathematical machinery used for study of periodical solution is the theory of Riemann surfaces. In $L \to \infty$ case these surfaces turn into degenerate surfaces which correspond to soliton solution. What should we achieve by studying the quantum periodical problem is a certain deformation of Riemann surfaces. For soliton case this deformation is explained in [9], [10]; we hope to have something even more interesting in periodical case.

2 The formulation of the problem.

In this paper we shall consider the $c < 1$ CFT compactified on the circle of length $L$. We have the Virasoro algebra with generators $\mathcal{L}_n$ satisfying the commutation relations:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m - n)\mathcal{L}_{m+n} + c \delta_{m,-n} \frac{n^3 - n}{12}$$

where

$$c = 13 - 6 \left( \frac{\hbar}{\bar{\hbar}} + \frac{1}{\bar{\hbar}} \right)$$

The Virasoro generators are obtained by usual construction from the Bose field $\varphi(x)$ defined as follows

$$\varphi(x) = iQ + \frac{2\pi i x}{L} P + \sum_{n \neq 0} \frac{a_n}{n} e^{- \frac{2\pi i n x}{L}}$$

where the generators of the Heisenberg algebra satisfy the commutation relations:

$$[P, Q] = \frac{\hbar}{2i}, \quad [a_m, a_n] = \delta_{m,-n} \frac{nh}{2}$$

The field $\varphi$ is quasi-periodic:

$$\varphi(x + L) = \varphi(x) + iLP$$

This Heisenberg algebra allows the representation with highest vector $|p\rangle$:

$$a_n |p\rangle = 0, \quad n > 0; \quad P |p\rangle = p |p\rangle$$

later we shall use another notation for the zero-mode: $\phi = \frac{\bar{\hbar}}{\hbar} p$.

From this Bose field we construct the operator $T(x)$ :

$$T(x) = \frac{1}{\hbar} \left( : \varphi'(x)^2 : + (1 - \hbar) \varphi''(x) + \frac{\hbar^2}{24} \right)$$

The representation of the Virasoro algebra in the space of representation of the Heisenberg algebra is defined because the Fourier components of $T(x)$ satisfy the Virasoro commutation relations:

$$T(x) = L^{-2} \left( \sum_{n=-\infty}^{\infty} \mathcal{L}_n e^{- \frac{2\pi i n x}{L}} - \frac{c}{24} \right)$$

obviously, $T(x)$ is periodical function of $x$.

As it is shown in [1] the CFT allows integrable structure. It means that in proper completion of the universal enveloping algebra of Virasoro algebra one can find a commutative subalgebra of local integrals of motion. Namely, there exist an infinite sequence of local operators $T_{2k}(x)$ (for $k = 1, 2, \cdots$) such that the operators

$$I_{2k-1} = \int_0^L T_{2k}(x) dx$$

commute with each other. The local operators $T_{2k}(x)$ are from the module of 1, i.e. they are constructed by taking derivatives of $T(x)$, multiplying and normal ordering them. In particular

$$T_2(x) = T(x), \quad T_4(x) = : T^2(x) :$$

$$T_6(x) = : T^3(x) : + \frac{c+2}{12} : (T'(x))^2 :$$
The first local integral \( I_1 \) coincides with \( L_0 \). The spectrum of \( L_0 \) is highly degenerate, but other local integrals of motion reduce drastically this degeneracy. There are two important problems. First, one has to describe this spectrum. This problem is very complicated, but as it is conjectured in \([4]\) the spectrum can be defined from solutions of Baxter equations (we shall discuss this later).

The second interesting problem consists in calculating the matrix elements of local operators between the eigen-states. Suppose that we have two eigen-states \( |\Psi\rangle \) and \( |\Psi'\rangle \) such that

\[
I_{2k-1} |\Psi\rangle = i_{2k-1} |\Psi\rangle, \quad I_{2k-1} |\Psi'\rangle = i'_{2k-1} |\Psi'\rangle
\]

where \( i, i' \) are eigen-values. We are interested in the matrix elements of the kind

\[
\langle \Psi | O(x) | \Psi' \rangle
\]

where \( O(x) \) is certain local operator. For simplicity we can take \( O(x) \) from the module of 1, in that case, obviously, the states must have equal zero-modes, otherwise the matrix element vanishes. In this paper we shall consider the matrix elements between two states with \( \phi = n \). Obviously, it is enough to consider the matrix elements of \( O(0) \) because \( |\Psi\rangle \) and \( |\Psi'\rangle \) are eigen-states of \( L_0 \), so, the \( x \) dependence is trivial.

What can be said in general about the matrix elements \([4]\) ? There are two situations when they are known.
1. Consider the “small” states, i.e. the ones created from Virasoro vacuum by applying few rising operators. On these states the local integrals can be diagonalized explicitly, and the matrix elements can be found by brut force.
2. More interesting case is the case of “big” states. There are two equivalent definitions of these states. It is a proper place to explain why we keep the length \( L \) in all formulae. In conformal theory we can always rescale the circle to one of length \( 2\pi \) passing to the variable \( y = \frac{2\pi}{L} \). The local integrals must be rescaled as follows

\[
\tilde{I}_{2k-1} = \left( \frac{L}{2\pi} \right)^{2k-1} I_{2k-1}
\]

where \( \tilde{I} \) are the integrals on the \( 2\pi \)-circle. Consider the the states on \( L \)-circle for which the eigen-values of the local integrals remain finite in the limit \( L \to \infty \) or, equivalently the states on \( 2\pi \)-circle for which the eigen-values of \( I_{2k-1} \) are of order \( L^{2k-1} \) for certain big parameter \( L \). This is the definition of “big” states. We prefer the first interpretation that is why we keep \( L \) in our formulæ. It is rather clear that in \( L \to \infty \) limit the matrix elements between “big” states must reproduce the form factors in restricted \( SG \) theory. Indeed the large \( L \) limit corresponds to conformal thory in infinite volume. This theory can be described by massless S-matrices \([6]\) and form factors which don’t differ from those of massive theory. The states with \( \phi = n \) correspond to \( n \)-soliton states in infinite volume limit.

3 The classical periodical problem for KdV.

Let us present several facts concerning the classical KdV hierarchy with periodical boundary conditions following mostly the book \([1]\). We have the field \( u(x) \) satisfying the periodicity conditions \( u(x + L) = u(x) \), the second (Magri) Poisson structure is defined by

\[
\{ u(x_1), u(x_2) \} = (u(x_1) + u(x_2)) \delta'(x_1 - x_2) + \epsilon \delta'''(x_1 - x_2)
\]

We shall consider two ”real forms”: the case \( \epsilon = + \) corresponding to usual real solutions of KdV equation (this case is denoted by rKdV), and the case \( \epsilon = - \) which corresponds to certain class of complex solutions (this case will be denoted by cKdV). The latter case is related to the \( c < 1 \) models of CFT discussed in the previous section because in the quasi-classical limit the Poisson brackets of \( u(x) = \hbar T(x) \) coincide with \([8]\). Actually the two cases must be understood as analytical continuations of each other, see \([8]\) for more expanations.

The KdV hierarchy possesses infinitely many integrals of motion \( I_{2k-1} \) \((k \geq 1)\) the first of them (the momentum) being

\[
I_1 = \int_0^L u(x) \, dx
\]
Making the rescaling
\[ y = \frac{2\pi}{L} x, \quad \tilde{u}(y) = L^2 u \left( \frac{L}{2\pi} y \right) \]
we map the periodical problem for \( u \) with arbitrary \( L \) to the one for \( \tilde{u} \) with the period equal \( 2\pi \), the integrals of motion scale as in the quantum case.

The exact solution of KdV equation is due to existence of Lax representation. The auxiliary linear problem is
\[
\left( \frac{d^2}{dx^2} - u(x) \right) \psi(x, \lambda) = \lambda^2 \psi(x, \lambda)
\]
It is convenient to rewrite this equation as matrix first order equation. To this end introduce the field \( \varphi(x) \) related to \( u(x) \) by Miura transformation: \( u = (\varphi')^2 + \varphi'' \). The field \( \varphi \) is real for rKdV and pure imaginary for cKdV. The equation (3) is equivalent to the following one:
\[
\frac{d}{dx} \Psi(x, \lambda) = \mathcal{L}(x, \lambda) \Psi(x, \lambda)
\]
where
\[
\Psi(x, \lambda) = \begin{pmatrix} e^{\varphi(x)} \psi(x, \lambda) \\ e^{-\varphi(x)} (\psi(x, \lambda)' - \psi(x, \lambda) \varphi(x)') \end{pmatrix}
\]
and
\[
\mathcal{L}(x, \lambda) = \begin{pmatrix} \varphi(x)', \frac{e^{-\varphi(x)}}{2} \\ \lambda^2 e^{\varphi(x)}, -\frac{\varphi(x)'}{2} \end{pmatrix}
\]
The fundamental role is played by the monodromy matrix:
\[
T_{x_0}(\lambda) = P \exp \left( \int_{x_0}^{x_0+L} \mathcal{L}(x, \lambda) dx \right)
\]
The trace of \( T_{x_0}(\lambda) \) which does not depend on \( x_0 \) is denoted by \( T(\lambda) \). Let us recall the main properties of \( T(\lambda) \). The function \( T(\lambda) \) is an entire function of \( \lambda^2 \) with infinitely many zeros accumulating to \( \infty \) along the negative real axis. The simplest example is given by \( u = 0 \) for which
\[
T(\lambda) = 2 \cos(L \sqrt{-\lambda^2})
\]
The graph of this function looks as follows:

![Figure 1](image1.png)

Further we shall give other examples. The monodromy matrix is a 2 × 2 matrix with unit determinant, so,
\[
T(\lambda) = \Lambda(\lambda) + \Lambda(\lambda)^{-1}
\]
where \( \Lambda, \Lambda^{-1} \) are the eigenvalues of \( T \). It is important to notice that \( \Lambda(\lambda) \) is not an entire function of \( \lambda \), it has quadratic branch points at zeros of odd order of the discriminant \( \Delta(\lambda^2) = T(\lambda)^2 - 4 \). The discriminant
is an entire function of $\lambda^2$ with asymptotic following from (4), so, it can be described by converging infinite product over its zeros

$$ \Delta(\lambda^2) = C \lambda^2 \prod \left( 1 - \frac{\lambda^2}{\nu_i^2} \right)^{k_i} $$

where $C = d\Delta/d\lambda^2(0)$, $k_j = 1, 2$. The asymptotical behaviour of $\Lambda(\lambda)$ is governed by the local integrals of motion:

$$ \log \Lambda(\lambda) \sim \lambda L + \sum_{k \geq 1} \lambda^{-2k+1} I_{2k-1} $$

$$ \lambda \to \infty, \quad \Re \lambda > 0 \quad (4) $$

The function $\Lambda(\lambda)$ is a singe-valued function on the hyper-elliptic Riemann surface $\Sigma$ (generally of infinite genus) given by the equation

$$ \mu^2 = \lambda^2 \prod_{k_i=1} \left( \lambda^2 - \nu_i^2 \right) $$

The tractable mathematically and (fortunately) at the same time most interesting physically case is when $\Sigma$ has finite genus, i.e. when $\Delta(\lambda^2)$ has only finitely many zeros of first order. The typical situation of this kind is given by the periodical analogues of $n$-soliton solutions. It that case we have simple zero at $\lambda = 0$, $2n$ real positive single zeros of $\Delta$ (we denote them by $\lambda_1^2, \cdots, \lambda_n^2$) and infinitely many negative double zeros (we denote them by $-\mu_1^2, -\mu_2^2, \cdots$). The graph of the function $T(\lambda)$ looks as follows:

$\text{Figure 2}$

The segments of the real axis of $\lambda^2$ where $|T(\lambda)| > 2$ are called forbidden zones of the periodic potential $u$ (there are no bounded wave-functions for these values of energy).

The hyper-elliptic Riemann surface $\Sigma$ of genus $n$ is described by

$$ \mu^2 = \lambda^2 P(\lambda), \quad P(\lambda) = \prod_{i=1}^{2n} (\lambda^2 - \lambda_i^2) $$

Conventionally it is realized as two-sheet covering of the $\lambda^2$ plane with cuts and canonical basis of homology chosen as follows
We shall be mostly using another model of this surface considering it as \( \lambda \)-plane with cuts:

\[
\begin{align*}
\mu_0 & \quad \mu_1 & \quad \mu_2 & \quad \mu_3 \\
\lambda_2 & \quad \lambda_3 & \quad \lambda_4 & \quad \lambda_5
\end{align*}
\]

**Figure 3**

The upper (lower) bank of the cut \([\lambda_{2j-1}, \lambda_{2j}]\) is identified with the upper (lower bank of the cut \([-\lambda_{2j-1}, -\lambda_{2j}]\). Obviously the upper (lower) half planes correspond to first (second) sheet of the surface in usual realization.

The function \( \Lambda(\lambda) \) continues to the lower half-plane as

\[
\Lambda(-\lambda) = \Lambda(\lambda)^{-1}
\]

Let us consider in some details properties of this function.

The function \( \log \Lambda(\lambda) \) is called quasi-momentum because the equation (3) has the Floquet solution \( \psi(x, \lambda) \) which satisfies quasi-periodicity condition:

\[
\psi(x + L, \lambda) = \Lambda(\lambda)\psi(x, \lambda)
\]

another name for this solution is Baker-Akhiezer (BA) function. The BA function is single-valued function on \( \Sigma \), \( \psi(x, \lambda) \) and \( \psi(x, -\lambda) \) give two linearly independent for generic \( \lambda \) quasi-periodical solutions of (3), the second one corresponds to the quasi-momentum \( -\log \Lambda(\lambda) \).

The function \( \Lambda(\lambda) \) is single-valued on \( \Sigma \) with essential singularity at infinity. Hence \( d\log \Lambda(\lambda) \) is abelian differential on \( \Sigma \) with second order pole at infinity:

\[
d\log \Lambda(\lambda) = (L + O(\lambda^{-1}))d\lambda
\]
The local parameter at $\infty$ is $\lambda^{-1}$. Moreover it must be normalized:

$$\int_{a_j} d\log \Lambda = 0 \quad \forall j$$

because as it is seen on fig.2 $\Lambda(\lambda)$ is real positive or negative function in the interval corresponding to $a$-cycle. Then in order that $\Lambda(\lambda)$ is single-valued we need that

$$\int_{b_j} d\log \Lambda = 2\pi ik_j$$

for some integer $k_j$. This requirement can not be satisfied for arbitrary $\Sigma$, it is actually a restriction on the moduli of the surface. For the periodical analogues of $n$-soliton solutions we have $k_j = n - j + 1$ which corresponds to the fact that exactly $n - j + 1$ simple zeros of $T(\lambda)$ are surrounded by $b_j$. This is exactly the situation presented on the fig.2, in more general case between two forbidden zones $T(\lambda)$ can make several oscillations from -2 to 2. Provided (5) and (6) are satisfied log $\Lambda(\lambda)$ is a function defined on $\Sigma$ with cuts along the $a$-cycles whose jumps on $a_j$ equal $2\pi ik_j$.

The dynamics of finite-zone solution is conveniently described by motion of zeros of BA function. There are exactly $n$ of them ($\gamma_0, \cdots, \gamma_{n-1}$). Let us present for completeness the equation describing the $x$-dependence of $\gamma_j$:

$$\frac{\partial}{\partial x}\gamma_j = \sqrt{P(\gamma_j)} \prod_{k \neq j} (\gamma_j^2 - \gamma_k^2)$$

The dynamics with respect to higher times is described by similar equations. The dynamics is linearized by Abel transformation of the divisor of zeros of BA function onto Jacobi variety of $\Sigma$. Recall that we consider two different real forms of KdV (rKdV and cKdV). The points of divisor corresponding to this two real forms move along topologically equivalent, but geometrically different trajectories. In rKdV case $\gamma_j$ moves along the cycle $a_j$ as it is drawn on the fig.4. In cKdV $\gamma_j$ runs along a trajectory close to the cycle $c_j$ on the fig.4. Clearly, the half-basis of $c$-cycles is equivalent to the half-basis of $a$-cycles.

### 4 Hamiltonian structure of finite-zone solutions.

Let us discuss the most subtle issue in the theory of finite zone integration, namely, the Hamiltonian description of the solutions. The surface $\Sigma$ is parametrized by $2n$ parameters ($\lambda_1, \cdots, \lambda_{2n}$). These parameters are not all independent, they are subject to $n$ restrictions (6). So, we are left with $n$-dimensional sub-manifold ($M$) in the moduli space of hyper-elliptic surfaces. It is convenient to parametrize $M$ by $\tau_1, \cdots, \tau_n$ such that $\tau_j^2$ are positive zeros of $T(\lambda)$ in $\lambda^2$-plane. Earlier we have introduced the variables $\gamma_0, \cdots, \gamma_{n-1}$.

The phase space of finite-zone solution is the $2n$-dimensional manifold locally described by the coordinates $\{\tau_1, \cdots, \tau_n, \gamma_0, \cdots, \gamma_{n-1}\}$ is embedded into the infinite-dimensional phase space of KdV. Restricting the symplectic form which corresponds to Poisson structure (2) to this finite-dimensional manifold we obtain the symplectic form $\omega = d\alpha$ with 1-form $\omega$ given by

$$\alpha = \sum_{j=0}^{n-1} \log \Lambda(\gamma_j) \frac{d\gamma_j}{\gamma_j}$$

Our main concern is quantization, so, we have to ask ourselves the question whether the quantization of this finite-dimensional mechanics is relevant to the real quantization of KdV. Logically, the answer to this question is negative because restricting ourselves to the finite-dimensional sub-manifold we ignore a good deal of quantum fluctuations allowed in the infinite-dimensional phase space. However, as is shown in [10] for the case of solitons ($L \to \infty$ limit) the quantization of the finite-dimensional system gives an important piece of exact quantum answer for the matrix elements (form factors [7]). To understand why it works and how to generalize the results of [10] to periodical case we have to consider the hamiltonian structure in more details. This consideration will allow also to reproduce quasi-classically an important part of solitons form factors which we could not do in [10].
Let us analyze more general situation of which KdV provides a particular case. Take a class of classical integrable models with trigonometric R-matrix. For any such system (continuous or lattice one) the monodromy matrix

\[ T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \]

can be introduced which satisfies the famous Sklyanin’s relations

\[ \{ T(\lambda) \otimes T(\mu) \} = [r(\lambda, \mu), T(\lambda) \otimes T(\mu)] \] (8)

with trigonometric R-matrix:

\[ r(\lambda, \mu) = \frac{1}{\lambda^2 - \mu^2} \begin{pmatrix} \lambda^2 + \mu^2, & 0, & 0, & 0 \\ 0, & 0, & 2\mu^2, & 0 \\ 0, & 2\lambda^2, & 0, & 0 \\ 0, & 0, & 0, & \lambda^2 + \mu^2 \end{pmatrix} \]

For a lattice regularization of KdV [12] the monodromy matrix is a polynomial in \( \lambda^2 \) of degree \( N \). The determinant \( D(\lambda) \) of \( T(\lambda) \) is in the center of the Poisson algebra. The trace \( T(\lambda) \) of \( T(\lambda) \) is a generating function of \( N \) independent integrals of motion. To describe the phase space one has to introduce \( N \) coordinate-like variables. Following general approach developed in [13, 14] we take as such zeros of \( B(\lambda) \):

\[ B(\lambda) = B(0) \prod_{j=1}^{N} \left( 1 - \left( \frac{\lambda}{\gamma_j} \right)^2 \right) \]

Using (11) one finds:

\[ \{ \Lambda(\gamma_i), \Lambda(\gamma_j) \} = \{ \gamma_i, \gamma_j \} = 0 \quad \{ \Lambda(\gamma_i), \gamma_j \} = \delta_{i,j} \gamma_i \Lambda(\gamma_i) \]

where \( \Lambda(\lambda) \) is eigenvalue of \( T(\lambda) \):

\[ \Lambda(\lambda) = \frac{T(\lambda) + \sqrt{\Delta(\lambda)}}{2} \]

the discriminant \( \Delta(\lambda) = T(\lambda)^2 - 4D(\lambda) \). Thus the variables \( \log \gamma_j \) and \( \log \Lambda(\gamma_j) \) are canonically conjugated and the symplectic form can be written as \( \omega = d\alpha \) with

\[ \alpha = \sum_{j=1}^{N} \log \Lambda(\gamma_j) \frac{d\gamma_j}{\gamma_j} \] (9)

Generally, the dynamics is linearized on the Jacobi variety of the hyper-elliptic Riemann surface \( \Sigma \) described by \( \mu^2 = \Delta(\lambda) \). The genus of \( \Sigma \) equals \( N \), the points of the divisor \( \gamma_1, \cdots, \gamma_N \) move along certain closed curves topologically equivalent to a half-basis of \( a \)-cycles on \( \Sigma \).

Later we shall need also the Liouville measure corresponding to this symplectic form. Taking zeros of \( T(\lambda) \) \((\tau_1, \cdots, \tau_N)\) and \( \gamma_1, \cdots, \gamma_N \) as coordinates on the phase space one finds

\[ \wedge^N \omega = \prod_{j=1}^{N} \frac{1}{\Lambda(\gamma_j) - \Lambda^{-1}(\gamma_j)} \prod_{i < j} (\gamma_i^2 - \gamma_j^2) \prod_{i < j} (\tau_i^2 - \tau_j^2) \frac{d\gamma_1}{\gamma_1} \wedge \cdots \wedge \frac{d\gamma_N}{\gamma_N} \wedge d\tau_1^2 \wedge \cdots \wedge d\tau_N^2 \] (10)

What happens to all that in the continuous limit which corresponds to finite-zone solution of KdV? The degree of \( T(\lambda) \) goes to infinity, but the surface \( \Sigma \) turn into a surface of infinite genus of rather special type. Namely, only finitely many of zeros of \( \Delta(\lambda^2) \) \((0, \lambda_1^2, \cdots, \lambda_{2n}^2)\) remain simple ones while infinitely many zeros \((-\mu_{-1}^2, -\mu_{-2}^2, \cdots)\) become double zeros. The polynomial \( T(\lambda) \) looks on the lattice as
In the continuous limit it turns into the one presented on the fig. 2. In the lattice regularization of rKdV case the points $\gamma_j$ are moving inside the zones where $|T(\lambda)| > 2$. In $N \to \infty$ limit only $n$ of them ($\gamma_0, \ldots, \gamma_{n-1}$) move inside the $|T(\lambda)| > 2$ zones on $\lambda^2 > 0$ half-axis on fig. 2, while infinitely many (we shall denote them by $\gamma_{-1}, \gamma_{-2}, \cdots$) happen to be confined at the points $-\mu^2_j$. This is how the restriction of degrees of freedom takes place in the classical case. It must be emphasized that this is a dynamical procedure which can not be carried out on quantum level. The points $\gamma_{-1}, \gamma_{-2}, \cdots$ can not be kept at fixed positions, but must be distributed with certain wave function localized around their classical values.

To finish this section let us present some explicit formulae concerning zeros of $B(\lambda)$ in the continuous case. Consider the equation (3) with finite-zone potential $u(x)$. Let us fix the normalization of the BA $\psi(x, \lambda)$. If we require that $\psi(\lambda) \sim \exp(\lambda x), \lambda \to \infty$ the BA function must have $n$ simple poles in the finite part of the plane. It is convenient to put this poles to $n$ branch points, say, $\lambda_1, \cdots, \lambda_n$. We can introduce another function $\psi^\dagger(x, \lambda)$ which satisfies $\psi^\dagger(x + L, \lambda) = \Lambda^{-1}(\lambda) \psi^\dagger(x, \lambda), \psi^\dagger(\lambda) \sim \exp(-\lambda x), \lambda \to \infty$ and has poles at the complimentary set of the branch points. These two solutions satisfy the relations

$$W(\psi(\lambda), \psi^\dagger(\lambda)) = 2\lambda, \quad \psi(\lambda) \psi^\dagger(\lambda) = \frac{\prod_{j=0}^{n-1} (\lambda^2 - \gamma^2_j)}{\sqrt{P(\lambda)}}$$

where $W(\psi, \psi^\dagger)$ is the Wronskian determinant. From these two solutions to (3) one easily reconstructs the monodromy matrix finding in particular that

$$B(\lambda) = (\Lambda(\lambda) - \Lambda^{-1}(\lambda)) \psi(\lambda) \psi^\dagger(\lambda) = B(0) \prod_{j=0}^{n-1} \left( 1 - \frac{\lambda^2}{\gamma^2_j} \right) \prod_{j=1}^{\infty} \left( 1 + \frac{\lambda^2}{\mu^2_j} \right)$$

Thus we find that $B(\lambda)$ has indeed zeros not only at the moving points of the divisor, but also at the confined points $\mu_{-j}$.

**5 Separation of variables. Baxter equation.**

Our nearest goal is to write down the quasi-classical expression for the wave function corresponding to the periodical analogue of $n$-soliton solution. As we have seen there are two types of coordinate-like variables: $\gamma_1, \cdots, \gamma_n$ which moves classically along the $a$-cycles and $\gamma_{-1}, \gamma_{-2}, \cdots$ which are classically confined. The first type of variables does not pose a problem for writing the quasi-classical wave-function. The contribution to the wave-function from the second type of variables is similar to that of a number of harmonic oscillators in the ground state. To understand this contribution we will need some pieces of exact quantum information.

Consider an operator-valued monodromy matrix $T(\lambda)$ with the same notations for the matrix elements as before which satisfies the quantum analogue of the Poisson brackets (3):

$$R(\lambda, \mu)(T(\lambda) \otimes I)(I \otimes T(\mu)) = (I \otimes T(\mu))(T(\lambda) \otimes I)R(\lambda, \mu) \quad (11)$$
The idea of Sklyanin is to consider the system in \( \gamma \)-representation. In conventional notations coming from SG theory it is fixed to be 1:

\[
\mathcal{A}(\lambda)\mathcal{D}(q\lambda) - B(\lambda)C(q\lambda) = 1
\]

where \( q = \exp(i\xi) \). The monodromy matrix \( \mathcal{T}(\lambda) \) is an entire function of \( \lambda^2 \). The trace of the monodromy matrix \( T(\lambda) \) is a generating function of integrals of motion. In quantum KdV case one can construct the monodromy matrix directly in continuous theory by proper normal ordering of the classical formula.

Let us review the method of separation of variables which was developed by Sklyanin who combined the general approach of Inverse Scattering Method with the ideas of and . In application to our particular case the results of Sklyanin can be formulated as follows. Consider the element \( B(\lambda) \) of the monodromy matrix. It defines a commutative family of operators due to the fact that

\[
[B(\lambda), B(\mu)] = 0
\]

Moreover \( B(\lambda) \) is an entire function of \( \lambda^2 \) which grows not faster than \( \exp(\lambda^pL) \) (with certain \( p \), see below) at the infinity. That is why it can be presented as an infinite product over its zeros:

\[
B(\lambda) = B(0) \prod_{j=1}^{\infty} \left( 1 - \left( \frac{\lambda}{\gamma_j} \right)^2 \right)
\]

The idea of Sklyanin is to consider the system in \( \gamma \)-representation.

The functions \( \mathcal{A}(\lambda) \) and \( \mathcal{D}(\lambda) \) are entire functions of \( \lambda^2 \) and as such they can be expanded into series of infinite radius of convergence:

\[
\mathcal{A}(\lambda) = \sum_{n=0}^{\infty} \lambda^{2n} \mathcal{A}_n, \quad \mathcal{D}(\lambda) = \sum_{n=0}^{\infty} \lambda^{2n} \mathcal{D}_n
\]

Following one introduces the operators

\[
\Lambda_j = A(\gamma_j), \quad \bar{\Lambda}_j = D(\gamma_j)
\]

where the operator \( \gamma_j \) is substituted into \( A \) and \( D \) from the left (exactly replacing \( \lambda \) in \( \gamma \)-representation). The operators \( \gamma, \Lambda, \bar{\Lambda} \) possess the following important properties:

\[
[\gamma_j, \gamma_k] = 0, \quad [\Lambda_j, \Lambda_k] = 0, \quad \Lambda_j \gamma_k = q^{\delta_{j,k}} \gamma_k \Lambda_j, \quad \Lambda_j \bar{\Lambda}_j = 1
\]

first three equalities follow directly from the commutation relations while the last one is the consequence of the commutation relations together with the quantum determinant. These commutation relations show that

\[
\Lambda_j = \epsilon_j \exp \left( i\pi\hbar \gamma_j \frac{\partial}{\partial \gamma_j} \right), \quad \bar{\Lambda}_j = \epsilon_j \exp \left( -i\pi\hbar \gamma_j \frac{\partial}{\partial \gamma_j} \right)
\]

where \( \epsilon_j = \pm 1 \), the necessity of introducing \( \epsilon_j \) comes from consideration of the \( \gamma \)-representation. The variables \( \gamma \) move (or rest) classically inside zones where \( |T(\lambda)| \geq 2 \), so \( \epsilon_j = \pm 1 \) depending on whether \( T(\lambda) \geq 2 \) or \( T(\lambda) \leq -2 \).

Our goal is to diagonalize the Hamiltonians (the operator \( T(\lambda) \)) in the \( \gamma \)-representation. This can be done if we accept the following:
Conjecture.
The function $B(\lambda)^{-1}T(\lambda)$ is a meromorphic function of $\lambda^2$ with infinitely many poles which can be expanded in convergent series:

$$B(\lambda)^{-1}T(\lambda) = \sum_{j=-\infty}^{n-1} \frac{1}{\lambda^2 - \gamma_j^2} (\Lambda_j + \tilde{\Lambda}_j)$$

Assuming that the wave-function is presented as the infinite product:

$$\Psi = \prod_{j=-\infty}^{n-1} Q_j(\gamma_j)$$

where every $Q_j$ satisfies the equation:

$$\epsilon_j t(\lambda)Q_j(\lambda) = Q_j(q\lambda) + Q_j(q^{-1}\lambda)$$

The sign $\epsilon_j$ can be taken away multiplying $Q_j$ by appropriate power of $\lambda$, and basically we have to study the famous Baxter's equation:

$$t(\lambda)Q(\lambda) = Q(q\lambda) + Q(q^{-1}\lambda)$$

where $t(\lambda)$ is the eigenvalue corresponding to $\Psi$:

$$T(\lambda)\Psi = t(\lambda)\Psi$$

All this procedure is called the separation of variable because it allows to transform the infinite-dimensional spectral problem to one-dimensional ones. The separation of variables in quantum case gives an exact quantum analogue of the classical separation of variables which is obvious in the formula (9).

The equation (14) can be considered as a finite-difference analogue of Schrödinger equation, this analogy, however, is not quite straightforward. It the case of usual Schrödinger equation the wave-function belongs to certain functional space like $L_2$. In our case the wave-function $Q(\lambda)$ is characterized by its analytical properties.

As a second order difference equation (14) must have two solutions up to multiplication by quasi-constant (a function of $\lambda^2$). Following $[4]$ we require that one of them (denoted later by $Q(\lambda)$) is an entire function of $\lambda^2$. Recall that $t(\lambda)$ is also supposed to be an entire function of $\lambda^2$. The next important piece of information is the asymptotic behaviour of these functions. The equation (14) can be rewritten as

$$t(\lambda) = \Lambda_q(q^{\frac{1}{2}}\lambda) + \Lambda_q^{-1}(q^{-\frac{1}{2}}\lambda)$$

where

$$\Lambda_q(\lambda) = \frac{Q(q^{\frac{1}{2}}\lambda)}{Q(q^{-\frac{1}{2}}\lambda)}$$

is the quantum analogue of the eigenvalue of monodromy matrix. In quantum KdV theory $\Lambda_q(\lambda)$ allows the following asymptotical expansion around infinity:

$$\log \Lambda_q(\lambda) \sim \lambda^p L + i\phi \pi + \sum_{k \geq 1} \lambda^{-(2k+1)p} I_{2k-1}$$

$$\lambda \to \infty, \quad \pi \hbar < \arg \lambda^2 < \pi(1 - \hbar)$$

where

$$p = \frac{1}{1 - \hbar}$$

$I_{2k-1}$ are quantum local integrals of motion (see $[4]$ for exact normalization of them). The fact that $T(\lambda)$ is an entire function of $\lambda^2$ that has the asymptotical expansion in terms of $\lambda^p$ is well known $[4]$. The explanation of this fact is due to the renormalization of mass. More modern and clear explanation in terms of CFT is given in $[4]$. Notice that our definition of $\lambda$ coincides with the one used in $[4]$. The phase $\phi$
equals to the topological charge, for the periodic analogue of \( n \)-soliton state \( \phi = n \). The leading terms of the asymptotics of \( Q(\lambda) \) are supposed to be:

\[
\log Q(\lambda) \sim \frac{1}{2\pi\sin \frac{\lambda}{2}} \lambda^p L - \frac{1}{\hbar} \log \lambda + O(\lambda^{-p}), \quad \lambda^2 \to \infty, \quad 0 < \arg \lambda^2 < 2\pi
\]  

(17)

In the paper \([4]\) one finds detailed description of complete asymptotical series for \( Q \).

Furthermore it is required that the function \( t(\lambda) \) has only finitely many zeros away from the axis \( \lambda^2 < 0 \) \((n \text{ real positive zeros for the periodic analogue of } n\text{-soliton state})\) while the function \( Q(\lambda) \) has only finitely many zeros away from the axis \( \lambda^2 > 0 \) \((\text{for the periodic analogue of } n\text{-soliton state they are absent})\). The basic conjecture accepted in \([4]\) is that counting the solutions of (14) one counts the vectors in the space of states of CFT. This conjecture is verified in many cases, so we have no doubt that it is true.

The entire function \( Q(\lambda) \) can be presented as infinite product with respect to its zeros:

\[
Q(\lambda) = Q(0) \prod_{j=1}^{\infty} \left(1 - \left(\frac{\lambda}{\sigma_j}\right)^2\right)
\]  

(18)

The zeros \( \sigma_j \) are subject to the Bethe Ansatz equations

\[
\frac{Q(q\sigma_j)}{Q(q^{-1}\sigma_j)} = -1
\]  

(19)

In the next section we shall consider the solutions to these equations in the quasi-classical limit.

As it has been said there must be another solution \((\tilde{Q}(\lambda))\) to the equations \([14]\). For any two solutions to \([14]\) the “quantum Wronskian”

\[
W(Q, \tilde{Q})(\lambda) = Q(\lambda)\tilde{Q}(\lambda q) - Q(\lambda q)\tilde{Q}(\lambda)
\]

is a quasi-constant:

\[
W(Q, \tilde{Q})(q\lambda) - W(Q, \tilde{Q})(\lambda) = 0
\]

So, to find the second solution we have to solve a first-order difference equation. Namely, let us put \( W(Q, \tilde{Q})(\lambda) = 1 \) then \( \tilde{Q}(\lambda) = Q(\lambda)F(\lambda) \) where \( F \) satisfies the equation

\[
F(q\lambda) - F(\lambda) = \frac{1}{Q(\lambda)Q(q\lambda)}
\]  

(20)

The solution to this equation can be always found but generally it is not a single-valued function of \( \lambda^2 \). Consider for simplicity the reflectionless case \( \hbar = \frac{1}{\pi\nu} \), with integer \( \nu \). It this case \( \tilde{Q} \) is a single-valued function of \( \lambda^2 \) described as follows. Take some polynomial \( P(a) \) of degree \( n \) then

\[
F(\lambda) = P(\lambda^{2\nu}) \left(\frac{1}{2\pi i} \int_{C_1} \frac{d\mu^2}{Q(\mu)Q(\mu q)P(\mu^{2\nu})(\mu^2 - \lambda^2)} + \frac{1}{2\pi i} \int_{C_2} \frac{d\mu^2}{P(\mu^{2\nu})}G(\lambda, \mu)\right)
\]  

(21)

where \( C_1 \) encloses zeros of \( Q(\mu) \) and \( C_2 \) encloses zeros of \( P(\mu^{2\nu}) \), \( G(\lambda, \mu) \) is the following function:

\[
G(\lambda, \mu) = \sum_{j=1}^{\nu} \sum_{l=1}^{j} \frac{1}{(\lambda^2q^{2j} - \mu^2)Q(\mu^{-l})Q(\mu^{-l+1})}
\]

The polynomial \( P \) is introduced for convergence, we shall not go into more details here. The definition of \( F \) depends on \( P \), but one easily shows that solutions with different \( P \) differ by a quasi-constant.

## 6 Quasi-classical wave functions.

Our nearest goal is to understand the quasi-classical behaviour of \( Q(\lambda) \). Consider the equations:

\[
\frac{Q(q\lambda)}{Q(q^{-1}\lambda)} = -1
\]  

(22)
In the case which we shall consider in this paper all the solutions to these equations are such that $\lambda^2$ is real. A part of solutions coincides with zeros of $Q(\lambda)$ but, obviously there are other solutions which provide zeros of $t(\lambda)$. Different solutions are counted as follows:

$$\log \left( \frac{Q(q\lambda_k)}{Q(q^{-1}\lambda_k)} \right) = (2k + 1)i$$

where $-\infty < k < \infty$. We have to share these solutions between $t(\lambda)$ and $Q(\lambda)$. For the quantization of periodical analogues of soliton solutions we do it as follows. For $-\infty < k < 0$ $\lambda_k$ are zeros of $t(\lambda)$, for $0 < k < \infty$ they are zeros of $Q(\lambda)$ except for finitely many $N_1, \ldots, N_n > 0$ which correspond to positive zeros of $t(\lambda)$. We have taken the branch of logarithm such that the border between zeros of $t$ and zeros of $Q$ lies at $k = 0$.

In quasi-classical limit $\hbar \to 0$, $N_j = O(\hbar^{-1})$ and the zeros of $Q(\lambda)$ condense in the $\lambda^2$-plane forming the cuts $[13]$. From comparing with classical picture it is clear that these cuts must coincide with the intervals: $I_1 = [0, \lambda_1^2], I_2 = [\lambda_2^2, \lambda_3^2], \ldots, I_{n+1} = [\lambda_{2n}^2, \infty]$ where $\lambda_j^2$ are the branch points defining the classical solution. There is one zero of $t(\lambda)$ in every interval between $I_j$ and $I_{j+1}$. The quasi-momentum $\log \Lambda(\lambda)$ can be considered as single-valued function on the plane with these cuts. Comparing classical formula (14) and quantum formula (13) one easily finds that when $q \to 1$

$$\log Q(\lambda) = \frac{1}{i\pi\hbar} \int_\lambda^0 \log \Lambda(\sigma) \frac{d\sigma}{\sigma} + O(\hbar^0)$$

This formula must be understood as very approximate one, $\log \Lambda(\lambda)$ is not a single-valued function, and its branch has to be taken differently for different $\gamma_j$. This "tree approximation" of $Q(\lambda)$ can be rewritten in the form:

$$\frac{1}{i\pi\hbar} \int_0^\lambda \log \Lambda(\lambda) \frac{d\lambda}{\lambda} = \frac{1}{\hbar} \sum_{j=1}^{n+1} \int_{I_j} \log \left( 1 - \left( \frac{\lambda}{\sigma} \right)^2 \right) \rho(\sigma) d\sigma$$

(23)

where inside every $I_j$ the density

$$\rho(\sigma) = \frac{1}{\sigma} \text{Re}(\log \Lambda(\sigma)),$$

notice that $\rho(\sigma) > 0$ inside $I_j$, and it vanishes as $\sqrt{\sigma - \lambda_j}$ at the ends of intervals. The logarithms in (23) have cuts in $\sigma^2$ plane from $\lambda^2$ to $\infty$. Obviously the formula (23) originates from the classical limit of the infinite product (18), and $\rho(\sigma)$ describes the density of distribution of zeros.

We need the quasi-classical correction to $Q(\lambda)$. To find it we have to investigate carefully the classical limit of Bethe Anzatz equations (19). It is done in the Appendix using certain version of Destry-de Vega equations [17], here we present the final result of these calculations:

$$Q(\lambda) = \lambda^{-\frac{1}{2}} P^{-\frac{1}{2}}(\lambda) \Gamma_S(\lambda) \exp \left( \frac{1}{i\pi\hbar} \int_0^\lambda \log \Lambda(\sigma) \frac{d\sigma}{\sigma} \right)$$

(24)

where $\Gamma_S(\lambda)$ is an analogue of gamma-function related to the surface $\Sigma$ which is defined as follows.

Let us introduce the normalized third kind differential $\rho_\mu(\lambda) = G_\mu(\lambda)d\lambda$

$$\rho_\mu(\lambda) \sim \frac{d\lambda}{(\lambda - \mu)} \quad \lambda \sim \mu, \quad \int_{b_j} \rho_\mu(\lambda) = 0 \quad \forall j$$

Then

$$d \log \Gamma_S(\lambda) = - \lim_{N \to \infty} \left( \frac{1}{2\pi i} \int_{C_N} \rho_\mu(\lambda) d\log(\Lambda(\mu)^2 - 1) - \frac{L}{\pi i} \log \frac{\pi i N}{L} \rho(\lambda) \right)$$

(25)

where the integral is taken over $\mu$, and the contour $C_N$ encloses the points $-i\mu_j$ with $1 \leq j \leq N$,

$$\rho(\lambda) = \lim_{\mu \to \infty} \mu \rho_\mu(\lambda)$$
The function $\Gamma_\Sigma(\lambda)$ is not single-valued on $\Sigma$, but it is single-valued on the plane with cuts (fig.3).

The analogy with usual $\Gamma$-function is obvious. If we take, in particular, the vacuum state the corresponding Riemann surface $\Sigma_{\text{vac}}$ is Riemann sphere and

$$\Gamma_{\Sigma_{\text{vac}}}(\lambda) = \lambda \left(\frac{\pi i}{L}\right) \frac{\partial}{\partial \lambda} \Gamma\left(\frac{L\lambda}{\pi i}\right)$$

The function $\Gamma_\Sigma$ satisfies the important functional equation:

$$\Gamma_\Sigma(\lambda)\Gamma_\Sigma(-\lambda) = 2\pi^2 \lambda \sqrt{P(\lambda)} \Lambda(\lambda) - \Lambda^{-1}(\lambda)$$ (26)

Before going further into study of the wave functions let us consider the quasi-classical analogue of the quasi-momentum. By definition

$$\Lambda_q(\lambda) = \frac{Q(\lambda^{1/2})}{Q(\lambda^{-1/2})},$$ (27)

its asymptotics are (10):

$$\log \Lambda_q(\lambda) \sim \lambda^p L + \sum_{k \geq 1} \lambda^{-(2k+1)p} I_{2k-1}$$

recall that $p = \frac{1}{\tau - \bar{\hbar}}$. Obviously one has quasi-classically:

$$\log \Lambda_q(\lambda) \sim \hbar \log \Lambda(\lambda) + \sum_{k \geq 1} \lambda^{-(2k+1)p} \delta I_{2k-1}$$ (28)

where $\delta I_{2k-1}$ are quasi-classical corrections to the integrals of motion. The non-trivial term of this formula is that containing $\log \lambda$. Let us show that our previous formulae agree with this asymptotic behaviour. From (24) one finds:

$$\log \Lambda_q(\lambda) = \log \Lambda(\lambda) + i\hbar \lambda \frac{d}{d\lambda} \log \Lambda(\lambda) + \sum_{k \geq 1} \lambda^{-(2k+1)p} \delta I_{2k-1} + O(\hbar^2)$$ (29)

To compare the formulae (29) and (28) we need to know the asymptotics of $\Gamma_\Sigma$. It is quite clear from the functional equation (23) that asymptotically

$$\log(\lambda^{1/2} P(\lambda)^{-1/2} \Gamma_\Sigma(\lambda)) = \frac{1}{\pi i} \log \lambda \log \Lambda(\lambda) + \sum_{k=1}^{\infty} c_k \lambda^{-2k+1}$$

where $c_k$ are certain coefficients. Now we see that the formula (29) has indeed the same structure as (28). Actually, the coefficient $c_k$ can be evaluated explicitly providing quasi-classical corrections to the integrals of motion.

Finally, we have to require that $Q(\lambda)$ is single-valued on the plane of $\lambda^2$ with cuts (fig.3). This requirement leads to the Bohr-Sommerfeld quantization conditions:

$$\int_{a_j} d\log Q(\lambda) = 2\pi i N_j$$ (30)

obviously $N_j$ is the number of zeros of exact quantum $Q$ in corresponding interval, in the quasi-classical region $N_j$ are of order $\hbar^{-1}$. These quantization conditions are implicitly the quantization conditions on the moduli $\tau_1, \cdots, \tau_n$.

The following important circumstance must be emphasized. The function $Q$ constructed in [4] is defined in $\lambda^2$ plane, so, it has as its quasi-classical limit the function constructed above defined in $\lambda^2$ plane or, equivalently, in the upper $\lambda$ plane. However in our construction this function allows analytical continuation to the lower half-plane (second sheet). This analytical continuation is related to the second solution of the Baxter equation $\tilde{Q}$ discussed in the previous section.
Let us construct the quasi-classical wave function. We had the formula

\[ \Psi = \prod_{j=-\infty}^{n-1} Q_j(\gamma_j) \]

As it has been said the functions \( Q_j(\lambda) \) differ from \( Q(\lambda) \) by certain degree of \( \lambda \). Let us explain the origin of this difference. First, notice that the expression for \( Q(\gamma) \) is not uniquely defined on \( \Sigma \) because \( d \log \Lambda \) has non-zero \( b \)-periods (all of these periods are equal to \( 2\pi i \)), so adding them we would multiply \( Q(\lambda) \) by \( \lambda^{2k} \).

Let us fix the branch of \( \log \Lambda(\lambda) \) as follows: make cuts over the cycles \( c_j \) (fig.4) and require \( \log \Lambda(0) = 0 \) at infinity. Now if we understand the integral in (24) as

\[ \lambda \int p(\sigma) \frac{d\sigma}{\sigma} = \int_0^\lambda p(\sigma) d\sigma \]

The functions \( Q_j(\lambda) \) are defined as

\[ Q_j(\lambda) = \lambda^{j\bar{\hbar}} Q(\lambda) \quad (31) \]

This prescription is chosen for the following reasons.

1. For \( j < 0 \) it ensures the existence of saddle point of \( Q_j \) at \( i\mu_j \), the point where \( \gamma_j \) is situated in classics.
2. For \( j > 0 \) it makes the action to satisfy proper reality condition along a classical trajectory. The problems of reality are discussed in details in [8].

The prescription for quantization of cKdV explained in [8] consists in taking the analytical continuation of rKdV. In particular it corresponds to the following rule of hermitian conjugation:

\[ \Psi^\dagger(\gamma) = \overline{\Psi(\gamma)} \quad (32) \]

which is the same as

\[ \Psi^\dagger(\gamma) = \Psi(\gamma^c) \]

where \( \gamma^c = \gamma \) for classically confined coordinates and \( \gamma^c = -\gamma \) for classically moving particles.

7 Quasi-classical matrix elements.

To construct the matrix elements we need to know also the measure of integration in the space of functions of \( \gamma \). Comparing the expression for the Liouville measure (10) with the formula (26) we assume that the measure of integration is given quasi-classically by

\[ W(\gamma) = \prod_{-\infty < i < j}^{n-1} (\gamma_i^2 - \gamma_j^2) \]

Thus the matrix elements of operators in \( \gamma \)-representation are given by:

\[ \langle \Psi | O | \Psi' \rangle = \frac{1}{\mathcal{N} \mathcal{N}'} \prod_{j=-\infty}^{n-1} \int_{-\infty}^{\infty} d\gamma_j \ W(\gamma) \Psi(\gamma^c) O(\gamma) \Psi(\gamma) \quad (33) \]

where \( \mathcal{N}, \mathcal{N}' \) are norms of the wave functions, \( O(\gamma) \) is the operator \( O \) in \( \gamma \)-representation. Let us first consider the norms.

Take the wave function for the periodical \( n \)-soliton solution \( \Psi(\gamma) \) and consider

\[ \mathcal{N}^2 = \prod_{j=-\infty}^{n-1} \int_{-\infty}^{\infty} d\gamma_j \ W(\gamma) \Psi(\gamma^c) \Psi(\gamma) \]

Consider first the integral over \( \gamma_{-j} \). It is of form

\[ \int_{-\infty}^{\infty} F(\gamma) \exp \left( \frac{2}{i\pi \hbar} \gamma \int_{0}^{\gamma} \log \Lambda(\sigma) - j\pi i \frac{d\sigma}{\sigma} \right) d\gamma \]
where \( F(\lambda) \) is finite when \( \hbar \to 0 \). So, it is sitting on the stationary point \( i\mu_j \) (recall that \( \log \Lambda(\mu_j) = j\pi i \)).

Now consider the integral with respect to \( \gamma_j \) \((j > 0)\). By definition of branch of \( \log \Lambda(\lambda) \) one sees that on the real axis
\[
\log \Lambda(\lambda) + \log \Lambda(-\lambda) = \pi i k \quad \text{for} \quad |\lambda_k| < |\lambda| < |\lambda_{k+1}|
\]
where we put \( \lambda_0 = 0, \lambda_{2n+1} = \infty \). Together with (26) it gives
\[
Q(\gamma)Q(-\gamma)\frac{2^n}{\Lambda(\gamma) - \Lambda^{-1}(\gamma)} \prod_{k=1}^{2j} \lambda_k^\pm
\]
for \( |\lambda_{2j}| < |\lambda| < |\lambda_{2j+1}| \) and exponentially (with \( \frac{1}{\lambda} \) in exponent) smaller everywhere else. So, the integral with respect to \( \gamma_j \) is sitting on these two segment of real axis or, putting it differently, on the cycle \( a_j \).

After these expansion let us present the final result of calculations. We denote by \( \Lambda_q \) the quasi-classical approximation given by (29).

\[
N^2 = \Phi(\Lambda_q) \prod_{k=1}^{2n} \frac{1}{\lambda_k^\pm} \int_{a_0} a_{n-1} d\gamma_0 \cdots d\gamma_{n-1} \prod_{j=0}^{n-1} \frac{1}{\lambda(\gamma_j) - \lambda^{-1}(\gamma_j)} \prod_{k=1}^{2j} \left( 1 - \frac{\lambda_k^2}{\mu_k^2} \right) \prod_{i<j} (\gamma_i^2 - \gamma_j^2)
\]

\[
= \Phi(\Lambda_q) \prod_{k=1}^{2n} \lambda_k^\pm \Delta
\]

where
\[
\Delta = \int_{a_0} a_{n-1} d\gamma_0 \cdots d\gamma_{n-1} \prod_{j=0}^{n-1} \frac{1}{\lambda(\gamma_j) - \lambda^{-1}(\gamma_j)} \prod_{i<j} (\gamma_i^2 - \gamma_j^2)
\]

and
\[
\Phi(\Lambda_q) = \exp \left( \frac{1}{2\pi i} \int_C \log(\Lambda_q^2(\mu) - 1) \left( \frac{d}{d\mu} \log \Lambda_q(\mu) \right) \log \mu + \frac{1}{2} \frac{d^2}{d\mu^2} \log \Lambda_q(\mu) \right) d\mu - \frac{1}{(2\pi i)^2} \int_C \int \log(\Lambda_q^2(\mu_1) - 1) \log(\Lambda_q^2(\mu_2) - 1) \frac{d\mu_1 d\mu_2}{(\mu_1 - \mu_2)^2} \right),
\]

the contour \( C \) encloses the points \( i\mu_k \). The formula (34) must be interpreted as follows: \( \Delta \) gives the volume of the coordinate space of the finite-zone solution while the rest describes the measure in orthogonal, momentum, direction. Actually the integrals in (35) are divergent, one has to divide by the norm of vacuum to make them finite. This divergence does not affect our further calculations, so, we shall ignore it.

Let us consider now the matrix elements. We shall take the simplest operator \( T(0) \) which is classically the same as \( u(0) \). On a classical finite-zone solution \( u = \sum \gamma_j - \frac{1}{2} \sum \lambda_j^2 \). The prescription for the symbol of this operator in quasi-classical approximation is:

\[
T(0) = \sum \gamma_j^2 - \frac{1}{4} \sum (\lambda_j^2 + (\lambda_j')^2)
\]

where \( \lambda, \lambda' \) are branch points corresponding to classical solutions. We take this symmetric prescription because it looks the most natural and gives correct answer in \( L \to \infty \) limit. The calculation of the matrix element is similar to the calculation of norm. The only important point to realize is that the stationary points move to those solving the equation

\[
\Lambda(\mu)\Lambda'(\mu) = 1
\]

The final result is

\[
\langle \Psi | T(0) | \Psi' \rangle = \frac{1}{\sqrt{\Delta\delta'}} \frac{\Phi(\sqrt{\Lambda_q\Lambda_q'})}{\Phi(\Lambda_q)\Phi(\Lambda_q')}
\times \int_{-\infty}^{\infty} d\gamma_0 \cdots d\gamma_{n-1} \prod_j Q(-\gamma_j)Q'(\gamma_j) \exp \left( \frac{1}{2\pi i} \int_C \frac{d\mu}{\gamma_j^2 - \mu^2 \lambda(\gamma_j) - \lambda^{-1}(\gamma_j)} \phi(\mu, \mu') \right)
\times \prod_j \lambda_j^\pm \prod_{i<j} (\gamma_i^2 - \gamma_j^2) (\sum \gamma_j^2 - \frac{1}{4} \sum (\lambda_j^2 + (\lambda_j')^2))
\]

where the contour \( C \) encloses the zeros of \( \Lambda(\mu)\Lambda'(\mu) - 1 \) lying on positive imaginary half-axis. Notice that the exponential growth at infinity is cancelled in the combination \( Q(-\gamma_j)Q'(\gamma_j) \). Before discussing further this formula let us show that it gives correct result in the limit \( L \to \infty \).
In the classical case when $L \to \infty$ the Riemann surface degenerates: with exponential in $L$ precision \(\lambda_{2j-1} \to \tau_j \leftarrow \lambda_{2j}\). The quasi-momentum becomes an elementary function:

$$\Lambda(\lambda) = e^{L\lambda} \prod_{j=1}^{n} \left( \frac{\tau_j - \lambda}{\tau_j + \lambda} \right) \left( 1 + O(e^{-\lambda L}) \right) \quad (37)$$

In the quantum case one has \([4]\):

$$\Lambda_q(\lambda) = e^{L\lambda_p} \prod_{j=1}^{n} \left( \frac{\tau_j^p - \lambda^p}{\tau_j^p + \lambda^p} \right) \left( 1 + O(e^{-\lambda_p L}) \right) \quad (38)$$

Let us first check that our quasi-classical formula for $Q$ agrees with this result. Consider the formula \([25]\). On the degenerate surface the differential $\rho$ turns into

$$\rho_\mu(\lambda) = \left( \frac{1}{\lambda - \mu} + o(L^{-1}) \right) d\lambda$$

One finds that when $L \to \infty$

$$d \log \Gamma_\Sigma(\lambda) = -\lim_{N \to \infty} \left( \frac{1}{2\pi i} \int_C N \rho_\mu(\lambda) d\log(\Lambda(\mu)^2 - 1) - \frac{1}{\pi i} \log \frac{\pi L}{\rho(\lambda)} \right) \to \lim_{M \to \infty} \frac{1}{\pi i} \int_{-iM}^{iM} \frac{1}{\lambda - \mu} d\log \Lambda(\mu) - L \log M \right) d\lambda \quad (39)$$

where we have integrated by parts and took into account that $\Lambda(\lambda)$ is exponentially big (small) in the right (left) half plane. Calculating the integral in \([39]\) one gets:

$$d \left( \lambda^{-\frac{1}{2}} P(\lambda)^{-\frac{1}{2}} \log \Gamma_\Sigma(\lambda) \right) \to \frac{1}{\pi i} \left( \log \lambda \ d \log \Lambda(\lambda) + \sum_{i<j} \frac{2\tau_i \log \tau_j}{\tau_i^2 - \lambda^2} d\lambda \right)$$

Now we substitute this expression into the definition \([27]\). Obviously,

$$\lambda h \left( \log \lambda \ \frac{d}{d\lambda} \log \Lambda(\lambda) + \sum_{i<j} \frac{2\tau_i \log \tau_j}{\tau_i^2 - \lambda^2} \right) = \log \Lambda_q(\lambda) - \log \Lambda(\lambda) + O(h^2)$$

which proves the consistency of our quasi-classical formulae with exact quantum formula \([35]\).

The calculation of the matrix element in $L \to \infty$ limit is straightforward, but bulky. The main simplification is due to the fact that the integrals containing $\log(\Lambda^2(\mu) - 1)$ or $\log(\Lambda(\mu)\Lambda'(\mu) - 1)$ can be evaluated as it has been done in the calculation of $d \log \Gamma_\Sigma(\lambda)$. The only non-trivial part of the calculation is that concerning $\Delta$. Recall that the differential $d \log \Lambda$ is a normalized second kind differential on the surface. Requiring that $\Lambda$ is given by \([27]\) in $L \to \infty$ limit we actually fix completely the rule of degeneration of the surface in this limit. In particular, one easily finds the limiting values of the normalized holomorphic differentials, and shows that

$$\Delta \to \frac{1}{\prod_{i<j} (\tau_i^2 - \tau_j^2)} L^n$$

The appearance of $L^n$ is not surprising because in $L \to \infty$ limit the eigenstates are normalized with $\delta$-functions.

To make easier the comparison of the final result with known exact formulae \([3]\) it is convenient to introduce usual rapidity notations:

$$\beta_j = \frac{1}{p} \log \tau_j, \quad \alpha_j = \frac{1}{p} \log \gamma_j$$

Then the final result of the calculation is:

$$\langle \Psi|T(0)|\Psi' \rangle = L^n \ P \ P' \ \prod_{i<j} \zeta(\beta_i - \beta_j) \prod_{i<j} \zeta(\beta_i^\prime - \beta_j^\prime) \prod_{i,j} \zeta(\beta_i - \beta_j - \pi i) \times \int \alpha_1 \cdots \int d\alpha_n \ \prod_{i=0}^{n-1} \prod_{j=0}^{n} \varphi(\alpha_i - \beta_j - \frac{\pi i}{2}) \prod_{j=0}^{n} \varphi(\alpha_i - \beta_j^\prime + \frac{\pi i}{2}) \times \prod_j \ e^{\frac{\tau}{2}(2j-n+1)\alpha_j} \ \prod_{i<j} \sinh(\alpha_i - \alpha_j) \left( \sum e^{\alpha_j} - \frac{1}{2} \sum e^{\beta_j} - \frac{1}{2} \sum e^{\beta_j^\prime} \right) \quad (40)$$
where
\[ \int = \int - \int_{-\infty}^{-\pi i}, \]
the functions \( \zeta \) and \( \varphi \) are given (up to some constants) by
\[
\varphi(\beta) = \frac{1}{\sqrt{\cosh \beta}} \exp \left( -\frac{\pi}{2} \int_0^\infty \frac{\sinh^2 \frac{k\beta}{2}}{k^2 \cosh^2 \frac{k\pi}{2}} dk \right)
\]
\[
\zeta(\beta) = \sqrt{\sinh \frac{k\beta}{2}} \exp \left( \frac{1}{4} \int_0^\infty \frac{\sinh^2 \frac{(\beta+\pi i)}{2}}{k^2 \cosh^2 \frac{k\pi}{2}} dk \right)
\]
which is exactly the quasi-classical limit of corresponding functions used in [7]. Finally,
\[
P = \exp \left( \frac{\pi}{\xi} \sum (2j-n-1)\beta_j \right) \prod_{i<j} \sqrt{\sinh(\beta_i - \beta_j)} \varphi(\beta_i - \beta_j + \frac{\pi i}{2})
\]
and similarly for \( P' \). Notice that \(|P| = 1\).

One can show that the formula (40) basically coincides with the quasi-classical limit of modest modification (similar to one done in [3]) of usual form factors formulae [7] the only difference being due to the phases \( P, P' \). This difference is quite understandable: in quasi-classical construction we get automatically the states which are symmetric with respect to permutation of particles, while in usual form factor formulae the states are used which produce S-matrices under these permutations. This difference in normalization is responsible for presence of the phases. Notice that the presence of the functions \( \zeta \) in the quasi-classical result is due to the contribution of “vacuum” particles. This contribution could not be reproduced in more naive approach of the paper [3].

It is clear that we are not very far from the exact quantum formula for the form factors in finite volume. The main feature of both quasi-classical formula (36) and the exact formula in infinite volume [7] is that they are given by products of certain multiplier which is independent on particular local operator and finite-dimensional integral depending on the local operator. Will this structure will hold for the exact formula in finite volume? This is not clear, but this is the only chance for the formula to be efficient.

8 Appendix.

In this Appendix we give details of calculation of the quasi-classical limit of \( Q \). Recall that
\[ Q(\lambda) = Q(0) \prod_{j=1}^\infty \left( 1 - \left( \frac{\lambda}{\sigma_j} \right)^2 \right) \]
where the zeros \( \sigma_j^2 \) are real positive. Following [13] introduce
\[ a(\lambda) = \frac{Q(\lambda q)}{Q(\lambda q^{-1})} \]
Using the fundamental equation
\[ t(\lambda) Q(\lambda) = Q(\lambda q) + Q(\lambda q^{-1}) \]
one finds:
\[ \log a(\lambda) - \log a(\lambda q^{-1}) = -\frac{1}{2\pi i} \int_C \log(a(\mu) + 1) d\log \left( \frac{\lambda^2 q^2 - \mu^2}{\lambda^2 q^{-2} - \mu^2} \right) \] (41)
where the contour \( C \) encloses the points \( \sigma_j^2 \) and \( q^2 \sigma_j^2 \) in the plane of \( \mu^2 \).

We know that in the quasi-classical limit the zeros \( \sigma_j \) are dense inside \( n+1 \) intervals corresponding to cuts on the fig.3. Let us present quasi-classically
\[ a(\lambda) = a_0(\lambda) \left( 1 + \hbar \pi x(\lambda) + O(\hbar^2) \right) \]
In the order $\hbar^1$ the equations (41) give
\[
\frac{d}{d\lambda} \log a_0(\lambda) = -\frac{1}{\pi i} \int_C \log(a_0(\mu) + 1) \frac{1}{(\lambda^2 - \mu^2)^2} d\mu^2
\]  
(42)
where the contour $C$ encloses cuts on fig.3. We know in advance from classical consideration that $a_0(\lambda) = \Lambda^2(\lambda)$. Let us check, first, that this $a_0(\lambda)$ indeed satisfies the equation (42). We have
\[
\log(\Lambda^2(\mu) + 1) = \frac{1}{2} \log \Lambda(\mu) + \log T(\mu)
\]
where $T = \Lambda + \Lambda^{-1}$ is entire function without zeros on the cuts. So, the integrals in (42) with $\log T(\mu)$ disappear leaving an obvious identity. It is clear, on the other hand, that this is the only way to satisfy this equation. Notice that the equation does not have solution for arbitrary positions of the branch points $\lambda_j^2$; the equations (41) have to be satisfied.

Consider now the equation (41) in the order $\hbar^2$. After some simple transformations it can be rewritten as
\[
\frac{1 - \Lambda^2(\lambda)}{1 + \Lambda^2(\lambda)} x(\lambda) = i \frac{d}{d\lambda} \log \Lambda(\lambda) + \frac{1}{\pi i} \int_{C_+} \frac{1 - \Lambda^2(\mu)}{1 + \Lambda^2(\mu)} x(\mu) \frac{\lambda}{\mu^2 - \lambda^2} d\mu
\]
(43)
where and the contour $C_+$ encloses all zeros on $\Lambda^2 + 1$ (or $T$) in the $\mu^2$ plane. This equation shows that $x(\lambda)$ allows analytical continuation to the surface $\Sigma$ which is realized, as usual, as the plane with cuts (fig.4).

Consider the differential
\[
\varphi(\lambda) = \frac{1 - \Lambda^2(\lambda)}{1 + \Lambda^2(\lambda)} x(\lambda) \frac{d\lambda}{\lambda}
\]
The equation (43) shows that
1. $\varphi(\lambda)$ is holomorphic differential on the surface with infinitely many simple poles at the points $\pm \tau_j$ (zeros of $\Lambda^2 + 1$).
2. Since the function $x(\lambda)$ is regular on the first sheet of $\Sigma$ (in the upper half-plane) $\varphi(\lambda)$ has simple zeros at the points $i\mu_j$ (zeros of $\Lambda^2 - 1$).
3. The equation holds:
\[
\varphi(\lambda) + \varphi(-\lambda) = 2i d \log \Lambda(\lambda)
\]
(44)

4. The $\alpha$-periods of $\varphi$ vanish.

Let us show that these four conditions are sufficient to satisfy the equation (43). Consider the canonical symmetric second kind differential $\omega(\lambda, \mu) = G(\lambda, \mu) d\lambda d\mu$:
\[
\omega(\lambda, \mu) = \omega(\mu, \lambda), \quad \omega(\lambda, \mu) \sim \frac{d\lambda d\mu}{(\lambda - \mu)^2} \lambda \sim \mu, \quad \int_{a_j} G(\lambda, \mu)d\lambda = 0 \quad \forall j
\]
It is clear that on our hyper-elliptic surface
\[
G(\lambda, \mu) + G(\lambda, -\mu) = \frac{\lambda \mu}{(\lambda^2 - \mu^2)^2}
\]
(45)
Let us show that the following equation holds for $\varphi$ satisfying the conditions 1-4:
\[
\varphi(\lambda) = i \frac{d}{d\lambda} \log \Lambda(\lambda) + \frac{1}{2\pi i} \int_{C_+ + C_-} \varphi(\mu) \int_{C_+ + C_-} \omega(\lambda, \mu)
\]
(46)
where $C_+$ ($C_-$ ) enclose zeros on $\Lambda^2 + 1$ in upper (lower) half-planes. Indeed, the contour $C_+ + C_-$ encloses all the singularities of the integrand except for that at $\mu = \lambda$ and that at $\mu = \infty$. Obviously residue of the integrand at $\mu = \lambda$ equals $\varphi(\lambda)$. Thus using the Riemann bilinear relation one has
\[
\int_{C_+ + C_-} \varphi(\mu) \int_{C_+ + C_-} \omega(\lambda, \mu) = 2\pi i \varphi(\lambda) + \lim_{R \rightarrow \infty} \int_{S_+ + S_-} \varphi(\mu) \int_{S_+ + S_-} \omega(\lambda, \mu) + \sum_{i=1}^n \left( \int_{a_i} \omega(\lambda, \mu) \int_{b_i} \varphi(\mu) - \int_{a_i} \varphi(\mu) \int_{b_i} \omega(\lambda, \mu) \right)
\]
The latter sum vanishes because the differential $\omega$ and $\phi$ have vanishing $a$-periods. The contours $S_+ (S_-)$ are half-circles in the left (right) half-planes of radius $R$. Using the condition 3 one has

$$\lim_{R \to \infty} \int_{S_+ + S_-} \phi (\mu) \int_0^\mu \omega (\lambda, \mu) = \lim_{R \to \infty} 2 \int d\lambda \log \Lambda (\mu) \int_0^\mu \omega (\lambda, \mu) = 2\pi L \lim_{\mu \to \infty} \mu^2 \omega (\lambda, \mu)$$

Recall now that $\omega (\lambda, \mu)$ is normalized second kind differential with double pole at $\lambda = \mu$ while $d \log \Lambda (\lambda)$ is a normalized second kind differential with singularity at infinity where it behaves as $L d\lambda$. Hence

$$\lim_{\mu \to \infty} \mu^2 \omega (\lambda, \mu) = L^{-1} d \log \Lambda (\lambda)$$

which proves the equation (46). Using (44) and (45) and the fact that $d \log \Lambda (\lambda)$ does not have singularities inside $C_+$ one shows that (43) follows from (46). Thus $\phi$ satisfying the conditions 1-4 satisfies the equation (43).

Let us return to the function $x (\lambda)$. From (44) one gets

$$x (\lambda) + x (-\lambda) = -2i\lambda \frac{d}{d\lambda} \log (\Lambda (\lambda) - \Lambda^{-1} (\lambda))$$

Recall that

$$\Lambda (\lambda) - \Lambda^{-1} (\lambda) = \lambda \sqrt{P (\lambda)} \prod_{j=1}^{\infty} \left( 1 - \left( \frac{\lambda}{\mu - j} \right)^2 \right)$$

Let us present $x (\lambda)$ as

$$x (\lambda) = 2i\lambda \frac{d}{d\lambda} \left( -\frac{1}{2} \log (\lambda \sqrt{P (\lambda)}) + \log \Gamma_\Sigma (\lambda) \right)$$

The function $\Gamma_\Sigma (\lambda)$ must be regular in the upper half plane, so the simple poles of the RHS of (47) must be shared between $x (\lambda)$ and $x (-\lambda)$ in such a way that $d \log \Gamma_\Sigma (\lambda)$ is a differential with simple poles at the points $-i\mu - j$ with residues 1. Such a differential is not uniquely defined: one can add to it holomorphic differentials. These are zero-modes of the equation (43). The origin of these zero-modes is clear: the classical surface is parametrized by $n$ moduli $\tau_1, \cdots, \tau_n$, the zero-modes correspond to variation of these parameters. Since after all we have to impose the Bohr-Sommerfeld quantization conditions (30) one can show that the final result for $Q$ does not depend on these zero-modes. We require that the $b$-periods of $d \log \Gamma_\Sigma (\lambda)$ vanish because with this choice the quantum rapidities of solitons don’t differ from the classical ones in $L \to \infty$ limit. Thus one presents $d \log \Gamma_\Sigma (\lambda)$ in the form given by the formula (25).

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