A Concise Review on
M5-brane in Large $C$-Field Background

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Abstract

We give a concise review of recent developments of the M5-brane theory derived from the Bagger-Lambert model. It is a 6 dimensional supersymmetric self-dual gauge theory describing an M5-brane in a large constant $C$-field background. The non-Abelian gauge symmetry corresponds to diffeomorphisms preserving an exact 3-form on the worldvolume, which defines a Nambu-Poisson bracket. Various interesting geometric and algebraic properties of the theory are discussed.
1 Introduction

The basic building blocks of M theory are membranes (also called M2-branes) and M5-branes. Membranes are fundamental objects carrying electric charges with respect to the 3-form $C$-field, and M5-branes are magnetic solitons. The purpose of this article is to give a concise summary of recent developments on a new formulation of M5-branes in a large constant $C$-field background.

The bosonic field content of an M5-brane theory should include scalars that represent M5-brane fluctuations in the transverse directions and a 2-form gauge potential with self-dual field strength. The low energy effective action has been known [1] for some time. A new formulation [2, 3] was obtained from the BLG model [4, 5] by setting the Lie 3-algebra which dictates the gauge symmetry of the BLG model to be the Nambu-Poisson bracket on a 3-manifold. \(^1\) (The 3-manifold is defined as an internal space from the M2-brane viewpoint, but becomes part of the M5-brane worldvolume.) The resulting theory is a non-Abelian self-dual gauge theory with novel geometric and algebraic structures worthy of further investigations.

The description of multiple M5-branes has been a long-standing mystery. The new formulation of M5-brane might shed some light on this question, since we expect that the multiple M5-brane theory to be a non-Abelian gauge theory. This can not be an ordinary non-Abelian gauge theory resembling Yang-Mills theory, because the entropy of coincident $N$ M5-branes should scale as $N^3$, rather than $N^2$ like Yang-Mills theories. Thus we expect that there is a new gauge symmetry to be discovered for multiple M5-branes. (The fact that the entropy of $N$ M2-branes scales as $N^{3/2}$ was recently explained in the framework of the BLG model in [7].) The novel gauge symmetry in the new formulation of M5-brane theory is very likely the first step towards the discovery of the mysterious gauge structure for multiple M5-branes.

This article is organized as follows. In sec. 2, we give some motivation for why Nambu-Poisson structure is expected to play a role in M5-brane theory when there is a large $C$-field background. Basic algebraic properties of the Nambu-Poisson bracket are reviewed at the end of the section. The action and symmetries of the new formulation of M5-brane in large $C$-field background are given in sec. 3, and in sec. 4 we comment on related topics such as Seiberg-Witten map, self-dual gauge theory, and connection with the old formulation of M5-brane. Finally, in the last subsection, we give a list of questions for future research.

\(^1\)For a review on how different choices of Lie 3-algebras for the BLG model lead to different physical systems, see [6].
2 From Noncommutative space to Nambu-Poisson structure

It is well-known that D-branes in constant $B$-field background are to be described as noncommutative gauge theories \[8, 9, 10\]. This can be understood from the viewpoint of worldsheet open string theory in terms of commutation relations of target space coordinates \[8\]. As all the fields living on D-branes are originated from degrees of freedom associated to the endpoints of open strings, the D-brane worldvolume theory exhibits the noncommutative nature of the endpoint coordinates. It is very interesting that the dynamical effect of $B$-field background (with vanishing field strength $H$) can be conveniently encoded in a geometric notion. It is tempting to conjecture that this success of geometrization of interactions admit further generalization.

An immediate generalization considered in the literature was the case of a non-vanishing field strength $H = dB$ of the $B$-field \[11\]. Since the Poisson limit of the commutation relation of target space coordinates is roughly given by the inverse of $B$, $H \neq 0$ suggests that $B$ is not a closed 2-form, or equivalently that $B^{-1}$ does not satisfy the Jacobi identity. Naively this implies that the algebra of functions on the D-brane worldvolume becomes nonassociative, and it was shown \[11\] that the naive nonassociative algebra is convenient for writing down some correlation functions. On the other hand, a proper canonical quantization would always define an associative algebra, and the correct associative algebra is at least equally good for expressing correlation functions \[12\].

Another direction for generalization is to consider the analogous problem in M theory. Type IIA superstring theory can be interpreted as the 11 dimensional M theory with one spatial dimension compactified on a circle, and fundamental strings in IIA superstring correspond to membranes in M theory, while $B$-field corresponds to $C$-field. The natural question is: what is the upgraded notion of noncommutative D-brane in M theory?

The canonical quantization of an open membrane ending on an M5-brane in a $C$-field background has been extensively studied in the literature \[13, 14\]. (In particular the large $C$-field limit was studied in \[13\].) The coordinates $X(\sigma)$ on the boundary of the open membrane, which is a closed string on M5-brane, satisfy a noncommutative relation. This algebraic structure is associated with the Poisson bracket, rather than the Nambu-Poisson bracket to be discussed below. A disadvantage of the study on canonical quantization is that there are constraints involved and the result depends on
the choice of gauge. On the other hand the relevance of Nambu-Poisson bracket and volume-preserving diffeomorphism to M5-brane theory was already anticipated [15] at that time. A review of the M5-brane physics before the recent developments that will be focused on below can be found in [16].

In the study of D-branes in $B$-field background, there are other ways to discover D-brane noncommutativity besides quantizing target space coordinates of an open string. Alternatively one can compute 3-point correlation functions for generic vertex operators and realize that the effect of the $B$-field background is a universal exponential factor that coincides with the Moyal product of functions [9]. (There is no correction to 2-point correlation functions.) Such a derivation can be most easily carried out in a low energy limit where higher oscillation modes on the string are ignored [17]. The analogous computation of correlation functions of open membranes was carried out in [18] in a large $C$-field background. The result is again a universal exponential factor, but this time for 4-point correlation functions. For a 4-point scattering amplitude with external momenta $k_1, k_2, k_3, k_4 = -k_1 - k_2 - k_3$, the exponential factor is

$$e^{\pm \frac{iC}{2\sqrt{k_1 \cdot (k_2 \times k_3)}}}.$$  

(There is no correction to 3-point correlation functions.) The total amplitude is a sum over exponential factors of both signs in the exponent, giving

$$\cos \left( \frac{C}{2\sqrt{k_1 \cdot (k_2 \times k_3)}} \right) \simeq 1 - \frac{1}{2C} k_1 \cdot (k_2 \times k_3) + \cdots.$$  

(2)

According to this formula, we expect that the M5-brane worldvolume action, which is a low energy description of open-membrane degrees of freedom, should be modified by interaction terms involving such factors

$$\frac{1}{2C} \epsilon^{ijk} \partial_i \otimes \partial_j \otimes \partial_k$$  

(3)

due to turning on a constant $C$-field background. Interestingly, this is precisely the Nambu-Poisson structure considered long time ago by Nambu [19], and later formally defined by Takhtajan [20], as a generalization of the Poisson structure in the Hamiltonian formulation. We will see below more precisely how the Nambu-Poisson structure dictates the gauge symmetry and interactions for an M5-brane in a large $C$-field background. But let us conclude this section by giving the mathematical definition of Nambu-Poisson bracket.

For an $n$-dimensional manifold $\mathcal{M}$, a tri-linear map $\{\cdot, \cdot, \cdot\}$ that maps 3 functions on $\mathcal{M}$ to a single function on $\mathcal{M}$ is called Nambu-Poisson bracket if

$$\text{Apart from the Leibniz rule, this is the same definition for Lie 3-algebra [21, 22].}$$
1. It is totally antisymmetrized
\[ \{f_1, f_2, f_3\} = -\{f_2, f_1, f_3\} = -\{f_1, f_3, f_2\}. \] (4)

2. It satisfies the Leibniz rule
\[ \{f_1, f_2, g_1 g_2\} = \{f_1, f_2, g_1\} g_2 + g_1 \{f_1, f_2, g_2\}. \] (5)

3. It satisfies the generalized Jacobi identity (fundamental identity)
\[ \{f_1, f_2, \{g_1, g_2, g_3\}\} = \{\{f_1, f_2, g_1\}, g_2, g_3\} + \{g_1, \{f_1, f_2, g_2\}, g_3\} + \{g_1, g_2, \{f_1, f_2, g_3\}\}. \] (6)

The most important theorem about Nambu-Poisson bracket is the decomposition theorem [23], which states that any Nambu-Poisson bracket can be locally expressed as
\[ \{f, g, h\} = \epsilon^{ijk} \partial_i f \partial_j g \partial_k h \] (7)
for 3 coordinates \(x^1, x^2, x^3\) out of the \(n\) coordinates. From the viewpoint of constructing an M5-brane from infinitely many M2-branes via the BLG model, the decomposition theorem explains why there exist only M5-branes but not M8-branes nor other M-branes in M theory. For a review of the mathematical properties of Nambu-Poisson bracket, see [24].

### 3 M5-brane in large \(C\)-field background

The worldvolume coordinates for an M5-brane in a large constant \(C\)-field background are naturally divided into two sets: \(\{x^\mu = x^1, x^2, x^3\}\) and \(\{y^\mu = y^1, y^2, y^3\}\), so that \(C_{\mu\nu\lambda}\) and \(C_{\mu\nu}\) are nonzero, but those components with mixed indices, \(C_{\mu\nu\lambda}\) and \(C_{\mu\nu}\), vanish. Although there is a symmetry between \(x^\mu\) and \(y^\mu\) because \(C\) is self-dual, the formulation given below is asymmetric.

The most salient feature of M5-brane theory is that there is a self-dual 2-form gauge potential \(b\), which is composed of 3 types of components \(b_{\mu\nu}, b_{\mu\nu\dot{\mu}}\) and \(b_{\mu\dot{\nu}}\). However, due to the self-duality condition, as we will see below, it is sufficient to explicitly keep only \(b_{\mu\nu}\) and \(b_{\mu\dot{\nu}}\), with \(b_{\mu\nu}\) hidden in the formulation. The hidden components \(b_{\mu\nu}\) can be constructed on the way to solve equations of motion.

The field content of the M5-brane theory includes the transverse coordinates of the M5-brane \(X^i\) \((i = 1, 2, \cdots, 5)\), the self-dual 2-form gauge potential \(b\) and 6D chiral
fermions $\Psi$, which can be conveniently organized as a single 11D Majorana spinor satisfying the 6D chirality condition

$$\Gamma^7 \Psi = -\Psi,$$

with the chirality matrix $\Gamma^7$ defined by

$$\Gamma^{\mu\nu\rho}\Gamma^{123} = \epsilon^{\mu\nu\rho} \Gamma^7.$$  

(These are 11D $\gamma$-matrices.)

A 2-form gauge potential in 6D has $C^4_2 = 6$ polarizations. The self-duality condition reduces it to 3 independent components. Together with the 5 scalars $X^i$, there are 8 bosonic degrees of freedom. An 11D Majorana fermion has 32 real components and the 6D chirality condition reduces it to 16 independent fermionic degrees of freedom as the superpartners of the 8 bosons. (Two fermionic degrees of freedom is equivalent to one bosonic degree of freedom, since the Dirac equation is first order and Klein-Gordon equation is 2nd order.) The field content of the M5-brane theory constitutes a tensor multiplet of the 6 dimensional $\mathcal{N} = (2, 0)$ supersymmetry.

### 3.1 Action

The action of a single M5-brane in a C-field background is derived from the Bagger-Lambert action [4] with the Lie 3-algebra chosen to be the Nambu-Poisson algebra [2]. It is found to be [3]

$$S = \frac{T_{M5}}{g^2} \left( S_{boson} + S_{ferm} + S_{CS} \right),$$  

where $T_{M5}$ is the M5-brane tension and $^3$

$$S_{boson} = \int d^3x d^3y \left[ -\frac{1}{2} \left( D_\mu X^i \right)^2 - \frac{1}{2} \left( D_\lambda X^i \right)^2 - \frac{1}{4} H_{\lambda\mu\nu}^2 - \frac{1}{12} H_{\mu\nu\rho}^2 + \frac{1}{2} g^2 \{ X^\mu, X^i, X^j \}^2 - \frac{g^4}{12} \{ X^i, X^j, X^k \}^2 \right],$$

$$S_{ferm} = \int d^3x d^3y \left[ \frac{i}{2} \bar{\Psi} \Gamma^\mu D_\mu \Psi + \frac{i}{2} \bar{\Psi} \Gamma^\rho D_\rho \Psi + \frac{ig^2}{2} \bar{\Psi} \Gamma_{\mu\lambda} \{ X^\mu, X^i, \Psi \} - \frac{ig^2}{4} \bar{\Psi} \Gamma_{ij\lambda} \{ X^i, X^j, \Psi \} \right],$$

$$S_{CS} = \int d^3x d^3y \epsilon^{\mu\nu\lambda} \epsilon^{\rho\sigma\tau} \left[ -\frac{1}{2} \partial_\rho b_{\mu\nu} \partial_\sigma b_{\lambda\lambda} + \frac{g}{6} \partial_\rho b_{\mu\nu} \partial_\sigma b_{\lambda\lambda} (\partial_\lambda b_{\mu\tau} - \partial_\tau b_{\mu\lambda}) \right].$$

$^3$\Psi here was denoted by $\Psi'$ in [3].
The covariant derivatives are defined by
\[ D_\mu \Phi \equiv \partial_\mu \Phi - g \{ b_{\mu \nu}, y^\nu, \Phi \}, \quad (\Phi = X^i, \Psi, ) \tag{14} \]
\[ D_\dot{\mu} \Phi \equiv \frac{g^2}{2} \epsilon_{\dot{\mu} \rho \dot{\nu}} \{ X^{\dot{\rho}}, X^{\dot{\nu}}, \Phi \}, \tag{15} \]
where
\[ X^{\dot{\mu}}(y) \equiv \frac{y^{\dot{\mu}}}{g} + \frac{1}{2} \epsilon^{\dot{\mu} \dot{\nu} \dot{\rho}} b_{\dot{\nu} \dot{\rho}}(y). \tag{16} \]

The field strengths are defined by
\[ H_{\lambda \dot{\mu} \dot{\nu}} = \epsilon_{\dot{\mu} \lambda \dot{\nu}} D_{\dot{\lambda}} X^{\dot{\lambda}} \]
\[ = H_{\lambda \dot{\mu} \dot{\nu}} - g \epsilon^{\dot{\gamma} \dot{\rho}}(\partial_{\dot{\gamma}} b_{\lambda \dot{\rho}}) \partial_{\dot{\rho}} b_{\mu \nu}, \tag{17} \]
\[ H_{123} = g^2 \{ X^1, X^2, X^3 \} - \frac{1}{g} \]
\[ = H_{123} + \frac{g}{2}(\partial_{\mu} b^{\dot{\nu}} \partial_{\nu} b^{\dot{\rho}} - \partial_{\mu} b^{\dot{\nu}} \partial_{\nu} b^{\dot{\rho}}) + g^2 \{ b^{\dot{1}}, b^{\dot{2}}, b^{\dot{3}} \}, \tag{18} \]
where \( H \) is the linear part of the field strength
\[ H_{\lambda \dot{\mu} \dot{\nu}} = \partial_{\lambda} b_{\mu \nu} - \partial_{\mu} b_{\lambda \nu} + \partial_{\nu} b_{\lambda \mu}, \tag{19} \]
\[ H_{\dot{\lambda} \mu \nu} = \partial_{\dot{\lambda}} b_{\dot{\mu} \dot{\nu}} + \partial_{\dot{\mu}} b_{\dot{\nu} \dot{\lambda}} + \partial_{\dot{\nu}} b_{\dot{\lambda} \dot{\mu}}. \tag{20} \]

### 3.2 Symmetries

The M5-brane action (10) respects the worldvolume translational symmetry, the global \( SO(2,1) \times SO(3) \) rotation symmetry, the gauge symmetry of volume-preserving diffeomorphisms and the 6D \( N = (2,0) \) supersymmetry.

#### 3.2.1 Gauge symmetry

The gauge transformation laws are
\[ \delta_\lambda \Phi = g \kappa^\rho \partial_\rho \Phi, \quad (\Phi = X^i, \Psi, ) \tag{21} \]
\[ \delta_\lambda b_{\kappa \dot{\lambda}} = \partial_{\kappa} \Lambda_{\dot{\lambda}} - \partial_{\dot{\lambda}} \Lambda_{\kappa} + g \kappa^{\dot{\rho}} \partial_{\dot{\rho}} b_{\kappa \dot{\lambda}}, \tag{22} \]
\[ \delta_\lambda b_{\lambda \dot{\sigma}} = \partial_{\lambda} \Lambda_{\dot{\sigma}} - \partial_{\dot{\sigma}} \Lambda_{\lambda} + g \kappa^{\dot{\tau}} \partial_{\dot{\tau}} b_{\lambda \dot{\sigma}} + g(\partial_{\dot{\sigma}} \kappa^{\dot{\tau}}) b_{\lambda \dot{\tau}}, \tag{23} \]
where
\[ \kappa^{\dot{\lambda}} = \epsilon^{\dot{\mu} \dot{\nu}} \partial_{\dot{\mu}} \Lambda_{\dot{\nu}}(x, y). \tag{24} \]
Eq.(22) and (23) can be more concisely expressed in terms of

\[ b^\mu \equiv \epsilon^{\mu \nu \lambda} b^\nu_{\lambda}; \]  
\[ B^\mu_\mu \equiv \epsilon^{\mu \nu \lambda} \partial_\nu b^\mu_{\lambda}; \]  

as

\[ \delta_\lambda b^\mu = \kappa^\mu + g\kappa^\nu \partial_\nu b^\mu; \]  
\[ \delta_\lambda B^\mu_\mu = \partial_\mu \kappa^\mu + g\kappa^\nu \partial_\nu B^\mu_\mu - g(\partial_\nu \kappa^\mu) B^\nu_\mu. \]  

In terms of \( B^\mu_\mu \), the covariant derivative \( D_\mu \) acts as

\[ D_\mu \Phi = \partial_\mu \Phi - gB^\mu_\mu \partial_\mu \Phi. \]  

While \( b^\mu \) determines \( b^\mu_{\nu \lambda} \) uniquely, \( B^\mu_\mu \) does not determine \( b_{\mu \nu} \) uniquely. Nevertheless, with the constraint

\[ \partial_\mu B^\mu_\mu = 0, \]  

\( b_{\mu \nu} \) can be determined by \( B^\mu_\mu \) up to a gauge transformation. Therefore, the physical degrees of freedom represented by \( b^\mu_{\nu \lambda} \) and \( b_{\mu \nu} \) can be equivalently represented by \( b^\mu \) and \( B^\mu_\mu \). In terms of \( X^i, \Psi, b^\mu \) and \( B^\mu_\mu \), all gauge transformations can be expressed solely in terms of \( \kappa^\mu \), without referring to \( \Lambda_{\mu} \) at all, as long as one keeps in mind the constraint

\[ \partial_\mu \kappa^\mu = 0 \]  
on the gauge transformation parameter \( \kappa^\mu \).

This gauge transformation can be interpreted as volume-preserving diffeomorphism

\[ \delta y^\mu = \kappa^\mu, \quad \text{with} \quad \partial_\mu \kappa^\mu = 0. \]  

3.2.2 Supersymmetry

The SUSY transformation laws are \(^4\)

\[ \delta^{(1)} \Psi = \chi, \quad \delta^{(1)} X^i = \delta^{(1)} b^\mu_{\nu \lambda} = \delta^{(1)} b_{\mu \nu} = 0. \]  

and

\[ \delta^{(2)} X^i = \epsilon \Gamma^i \Psi, \]  
\[ \delta^{(2)} \Psi = D_\mu X^i \Gamma^\mu \Gamma^i \epsilon + D^i_\mu X^j \Gamma^\mu \Gamma^i \epsilon \]

\(^4\epsilon\) here was denoted by \( \epsilon' \) in [3].
\[-\frac{1}{2} \mathcal{H}_{\mu\nu\rho} \Gamma^\mu \Gamma^{\hat{\nu}\hat{\rho}} \epsilon - \frac{1}{g} (1 + g \mathcal{H}_{123}) \Gamma_{123} \epsilon \]
\[-\frac{g^2}{2} \{X^{\hat{\mu}}, X^i \} \Gamma^{\hat{\mu}ij} \epsilon + \frac{g^2}{6} \{X^i, X^j, X^k \} \Gamma^{ijk} \Gamma_{123} \epsilon, \quad (35)\]
\[
\delta^{(2)}_\epsilon b_{\mu\nu} = -i (\mathbf{r} \Gamma_{\mu\nu} \Psi), \quad (36)
\]
\[
\delta^{(2)}_\epsilon b_{\mu\nu} = -i (1 + g \mathcal{H}_{123}) \mathbf{r} \Gamma_\mu \Gamma_\nu \Psi + ig (\mathbf{r} \Gamma_\mu \Gamma_\nu \Gamma_{123} \Psi) \partial_\nu X^i. \quad (37)
\]

Like \(\Psi\), the SUSY transformation parameters can be conveniently denoted as an 11D Majorana spinor satisfying the 6D chirality condition
\[
\Gamma^7 \chi = \chi, \quad \Gamma^7 \epsilon = \epsilon. \quad (38)
\]

Both \(\delta^{(1)}\) and \(\delta^{(2)}\) are nonlinear SUSY transformations. But there is a linear combination
\[
\delta^{(1)}_\chi - g \delta^{(2)}_\epsilon \quad \text{with} \quad \chi = \Gamma_{123} \epsilon \quad (39)
\]
that defines a linear SUSY transformation, a symmetry respected by the M5-brane vacuum state.

In (35) and (37), we notice the appearance of the combination \((\mathcal{H}_{123} + g^{-1})\). This should be taken as a hint that \(1/g\) is the background \(C\)-field, being reminiscent of the combination \((dB + C)\) in the old M5-brane theory. Similarly, in the action (11) there is a term \(-1/(2g^2)\), which should be interpreted as the contribution of the background \(C\)-field. This is to be compared with the kinetic term \(- (dB + C)^2/2\) of the gauge field in the old formulation, with the cross term \(CdB\) ignored as a total derivative. By comparing the M5-brane action with the noncommutative D4-brane action via double dimension reduction, one can verify the relation \(C = 1/g\) more precisely [3]. As each Nambu-Poisson bracket comes with a factor of \(g\) in the M5-brane action, this is precisely the modification due to a \(C\)-field background expected in sec. 2 (see eq.(3)).

The superalgebra of the SUSY transformations above has been investigated in [25], where central charges of the superalgebra are derived, including the central charge for the soliton mentioned below in sec. 4.3.

4 Comments

4.1 Seiberg-Witten map

To relate the Abelian gauge symmetry in the old formulation of M5-brane and the non-Abelian gauge symmetry in the new formulation, we use the notion of Seiberg-Witten
map, which was originally proposed to connect gauge symmetries on commutative and noncommutative spaces [10]. Let the gauge transformation parameter be denoted $\lambda$ and $\hat{\lambda}$ for the Abelian and non-Abelian gauge transformations, respectively, the Seiberg-Witten map specifies the correspondence between two kinds of gauge transformations as follows

\[ \delta_\hat{\lambda}\hat{\Phi}(\Phi) = \hat{\Phi}(\Phi + \delta_\lambda\Phi) - \hat{\Phi}(\Phi), \]

where $\hat{\Phi}$ and $\Phi$ are corresponding fields in the two theories, and $\hat{\lambda}$ is determined by $\lambda$ and the gauge potential. In this notation, all the fields in previous sections should wear hats. Corresponding fields in the old formulation come without hats.

The Seiberg-Witten map was determined to the first order in $[3]$ as

\[ \hat{\Phi} = \Phi + gb^\mu \partial_\mu \Phi + \mathcal{O}(g^2), \quad (\Phi = X^i, \Psi) \]

\[ \hat{b}^\mu (b) = b^\mu + \frac{g}{2} b^\nu \partial_\nu b^\mu + \frac{g}{2} b^\nu \partial_\nu b^\mu + \mathcal{O}(g^2), \]

\[ \hat{B}_\mu^\nu (B, b) = B_\mu^\nu + gb^\nu \partial_\nu B_\mu^\nu - \frac{g}{2} b^\nu \partial_\nu \partial_\mu b^\mu + \frac{g}{2} b^\nu \partial_\nu \partial_\mu b^\mu + g\partial_\nu b^\nu B_\mu^\nu + \mathcal{O}(g^2), \]

\[ \hat{\kappa}^\mu (\kappa, b) = \kappa^\mu + \frac{g}{2} b^\nu \partial_\nu \kappa^\mu + \frac{g}{2} \partial_\nu b^\nu \kappa^\mu - \frac{g}{2} \partial_\nu b^\nu \kappa^\mu + \mathcal{O}(g^2). \]

One can check that $\hat{\kappa}^\mu$ satisfies the constraint $\partial_\mu \hat{\kappa}^\mu = 0$ automatically as long as $\kappa^\mu$ satisfies the same constraint $\partial_\mu \kappa^\mu = 0$. Similarly, $\hat{B}_\mu^\nu$ satisfies (30) automatically.

The Seiberg-Witten map connecting commutative and noncommutative gauge theories are proven to exist to all orders. It remains to be checked whether the Seiberg-Witten map defined here can also be extended to all orders in $g$.

4.2 Self-dual gauge theory

An interesting by-product of deriving M5-brane theory from the BLG model is the discovery of a new Lagrangian formulation of self-dual gauge theories. It is generally believed that there is no manifestly Lorentz-invariant Lagrangian formulation for self-dual gauge theories, unless auxiliary fields are introduced. (And the purpose of the auxiliary field is just to pick a special direction when it is gauge fixed.) In the past, there exists non-manifestly Lorentz invariant Lagrangian formulation in which one out of the $D$ dimensions of the base space is chosen to play a special role in the formulation [26]. On the other hand, the new formulation divides the $d$ coordinates into two sets of coordinates of equal number. (The spacetime dimensions of self-dual theories are always even.) The M5-brane action at the quadratic level was derived in [2] and there
the self-duality of the gauge field was explicitly checked. The complete nonlinear M5-brane theory was given in [3], including the self-dual gauge fields, but the self-duality of the gauge field strength was not explicitly checked until it was done in [27], which also gave a good introduction to the basic ideas about self-dual gauge theories. Here we give a brief review of both the old and the new formulations at the quadratic level.

In the old formulation we pick, say, the 0-th dimension (which could be either time-like or space-like) to be special, and the action of the self-dual gauge field in 6 dimensions is

\[ S = -\frac{1}{4!} \int d^6x \left[ F_{\mu\nu\lambda} F^{\mu\nu\lambda} + 3(F - \tilde{F})_{0ab}(F - \tilde{F})^{0ab} \right], \]  

(45)

where \( \mu, \nu, \lambda = 0, \cdots, 5 \), \( a, b = 1, 2, \cdots, 5 \), and \( \tilde{F} \) denotes the dual of \( F \)

\[ \tilde{F}_{\mu\nu\lambda} = \frac{1}{3!} \epsilon_{\mu\nu\lambda\alpha\beta\gamma} F^{\alpha\beta\gamma}. \]  

(46)

In terms of the gauge potential \( A_{\mu\nu} \), the field strength is

\[ F_{\mu\nu\lambda} = \partial_{\mu} A_{\nu\lambda} + \partial_{\nu} A_{\lambda\mu} + \partial_{\lambda} A_{\mu\nu}, \]  

(47)

which is invariant under the gauge transformation

\[ \delta A_{\mu\nu} = \partial_{\mu} \Lambda_{\nu\lambda} - \partial_{\nu} \Lambda_{\mu}. \]  

(48)

The action (45) depends on \( A_{0a} \) only through total derivative terms.

The equations of motion derived from varying the action (45) with respect to \( A_{ab} \) are

\[ \epsilon^{abcde} \partial_c (F - \tilde{F})_{0de} = 0. \]  

(49)

This implies that locally there exist functions \( \Phi_a \) such that

\[ (F - \tilde{F})_{0ab} = \partial_a \Phi_b - \partial_b \Phi_a. \]  

(50)

Now one can redefine \( A_{0a} \) to absorb \( \Phi_a \) by the replacement

\[ A_{0a} \rightarrow A_{0a} + \Phi_a, \]  

(51)

such that for the new gauge fields \( A_{0a}, A_{ab} \), the self-duality conditions

\[ F_{0ab} = \tilde{F}_{0ab} \]  

(52)

are satisfied. (The condition \( F_{abc} = \tilde{F}_{abc} \) is equivalent to this one.)
In the new formulation [3], the base space coordinates are divided into two sets \( \{ x^\mu \} \) and \( \{ y^\dot{\mu} \} \). The action is

\[
S = \frac{1}{4} \int d^6 x \left[ F_{\mu\dot{\nu}\dot{\lambda}}(F - \tilde{F})^{\mu\dot{\nu}\dot{\lambda}} + \frac{1}{3} F_{\mu\dot{\nu}\dot{\lambda}}(F - \tilde{F})^{\mu\dot{\nu}\dot{\lambda}} \right],
\]

(53)

where \( \mu, \nu, \lambda = 1, 2, 3 \) and \( \dot{\mu}, \dot{\nu}, \dot{\lambda} = 1, 2, 3 \). (One of the dimension is time-like, but it is unnecessary to know whether it belongs to \( \{ 1, 2, 3 \} \) or \( \{ \dot{1}, \dot{2}, \dot{3} \} \).) The action above depends on \( A_{ab} \) only through total derivatives. The equations of motion derived from varying (53) with respect to \( A_{\mu\nu} \) and \( A_{\dot{\mu}\dot{\nu}} \) are

\[
\partial_\lambda (F - \tilde{F})^{\mu\dot{\nu}\dot{\lambda}} = 0,
\]

(54)

\[
\partial_\mu F^{\mu\dot{\nu}\dot{\lambda}} + \partial_{\dot{\mu}} F^{\mu\dot{\nu}\dot{\lambda}} = 0.
\]

(55)

The first equation implies that, locally,

\[
(F - \tilde{F})^{\mu\dot{\nu}\dot{\lambda}} = \frac{1}{2} \epsilon_{\dot{\mu}\dot{\nu}\dot{\lambda}} \epsilon_{\mu\nu\lambda} \partial^\rho \Phi^{\nu\lambda}
\]

(56)

for some \( \Phi^{\mu\nu} \). Via a redefinition of \( A_{\mu\nu} \)

\[
A_{\mu\nu} \rightarrow A_{\mu\nu} + \Phi_{\mu\nu},
\]

(57)

it becomes part of the self-duality conditions

\[
(F - \tilde{F})_{\mu\dot{\nu}\dot{\lambda}} = 0.
\]

(58)

This allows one to rewrite (55) as a total derivative

\[
\partial^\dot{\mu} (F_{\mu\dot{\nu}\dot{\lambda}} + \frac{1}{2} \epsilon_{\mu\dot{\nu}\dot{\lambda}} \epsilon_{\mu\nu\lambda} \partial^\rho A^{\nu\lambda}) = 0,
\]

(59)

which can be easily solved as

\[
F_{\mu\dot{\nu}\dot{\lambda}} + \frac{1}{2} \epsilon_{\mu\dot{\nu}\dot{\lambda}} \epsilon_{\mu\nu\lambda} \partial^\rho A^{\nu\lambda} = \epsilon_{\mu\dot{\nu}\dot{\lambda}} \Phi(x)
\]

(60)

for some function \( \Phi(x) \) which is independent of \( y^1, y^2, y^3 \). The function \( \Phi(x) \) can thus be absorbed by a further redefinition of \( A_{\mu\nu} \)

\[
A_{\mu\nu} \rightarrow A_{\mu\nu} + \frac{1}{3} \epsilon_{\mu\nu\lambda} \Phi_{\lambda}(x),
\]

(61)

where \( \Phi_{\lambda}(x) \) are chosen such that \( \partial_\mu \Phi^{\mu} = \Phi \). Since \( \Phi_{\lambda}(x) \) can be chosen to be independent of \( y^\mu \), this further redefinition of \( A_{\mu\nu} \) does not spoil the self-duality condition (58) obtained earlier. Thus (60) becomes the remaining self-duality conditions

\[
F_{\mu\dot{\nu}\dot{\lambda}} = \frac{1}{6} \epsilon_{\mu\dot{\nu}\dot{\lambda}} \epsilon_{\mu\nu\lambda} F^{\mu\nu\lambda}.
\]

(62)

For the self-dual gauge theory beyond the linearized equations, the readers are directed to [3] and [27].
4.3 Connection between M5-brane formulations

A flat open membrane ending on an M5-brane is described as a string soliton in the M5-brane theory. The BPS solution was found in [28], and later generalized to the case with $C$-field background in [29] for the old formulation of M5-branes. Its counterpart for the new formulation of M5-brane in $C$-field background was recently constructed in [30] to the first order in $g$.  

There the Seiberg-Witten map discussed in a previous section was shown to correctly connect the two solutions in different formulations. This is so far the only explicit evidence supporting the equivalence of the two formulations of M5-branes in the limit of a large $C$-field background. More generally we expect a connection between the two formulations of M5-branes analogous to the connection between the DBI action and the noncommutative SYM action of D-branes. Perhaps the first step is to rewrite the old M5-brane action using the new formulation of self-dual gauge theory that decomposes the base space coordinates into two sets of 3 coordinates $x^\mu$ and $y^{\dot{\mu}}$. Another direction to connecting the two formulations of M5-branes was suggested in [32, 33], where some features of the new M5-brane theory are elucidated.

4.4 Future study

In the above, we briefly reviewed the new M5-brane theory recently derived from the BLG model. This theory introduces a novel non-Abelian self-dual gauge theory defined by a Nambu-Poisson structure, and opens a new window to understanding the system of multiple M5-branes in M theory. A lot of interesting questions remain to be answered. We conclude this article by giving an incomplete list of topics for future study.

1. the quantization of Nambu-Poisson bracket.

2. M5-brane action in finite $C$-field background.

3. the action for the system of multiple M5-branes.

4. the connection between the old and new formulations of M5-brane.

5. the Seiberg-Witten map extended to higher orders in $g$.

---

The complementary view of the string soliton from the multiple M2-branes’ viewpoint was provided in [31].
6. formulation of self-dual gauge theory in arbitrary dimensions with generic separation of coordinates into two sets.  

While some of these are of a technical nature, others might be considered breakthroughs in theoretical physics.

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References

[1] P. S. Howe and E. Sezgin, “D = 11, p = 5,” Phys. Lett. B 394, 62 (1997) [arXiv:hep-th/9611008]. P. Pasti, D. P. Sorokin and M. Tonin, “Covariant action for a D = 11 five-brane with the chiral field,” Phys. Lett. B 398, 41 (1997) [arXiv:hep-th/9701037]. I. A. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. P. Sorokin and M. Tonin, “Covariant action for the super-five-brane of M-theory,” Phys. Rev. Lett. 78, 4332 (1997) [arXiv:hep-th/9701149]. M. Aganagic, J. Park, C. Popescu and J. H. Schwarz, “World-volume action of the M-theory five-brane,” Nucl. Phys. B 496, 191 (1997) [arXiv:hep-th/9701116]. P. S. Howe, E. Sezgin and P. C. West, “Covariant field equations of the M-theory five-brane,” Phys. Lett. B 399, 49 (1997) [arXiv:hep-th/9702008]. I. A. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. P. Sorokin and M. Tonin, “On the equivalence of different formulations of the M theory five-brane,” Phys. Lett. B 408, 135 (1997) [arXiv:hep-th/9703127].

[2] P. M. Ho and Y. Matsuo, “M5 from M2,” arXiv:0804.3629 [hep-th].

[3] P. M. Ho, Y. Imamura, Y. Matsuo and S. Shiba, “M5-brane in three-form flux and multiple M2-branes,” JHEP 0808, 014 (2008) [arXiv:0805.2898 [hep-th]].

6In sec. 4.2 we mentioned decomposing 6 dimensions as 1 + 5 and as 3 + 3. It may be possible to find a formulation for 2 + 4.
[4] J. Bagger and N. Lambert, “Modeling multiple M2’s,” Phys. Rev. D 75, 045020 (2007) [arXiv:hep-th/0611108]; J. Bagger and N. Lambert, “Gauge Symmetry and Supersymmetry of Multiple M2-Branes,” Phys. Rev. D 77, 065008 (2008) [arXiv:0711.0955 [hep-th]]; J. Bagger and N. Lambert, “Comments On Multiple M2-branes,” JHEP 0802, 105 (2008) [arXiv:0712.3738 [hep-th]].

[5] A. Gustavsson, “Algebraic structures on parallel M2-branes,” arXiv:0709.1260 [hep-th]; A. Gustavsson, “Selfdual strings and loop space Nahm equations,” arXiv:0802.3456 [hep-th].

[6] P. M. Ho, “Nambu Bracket for M Theory,” arXiv:0912.0055 [hep-th].

[7] C. S. Chu, P. M. Ho, Y. Matsuo and S. Shiba, “Truncated Nambu-Poisson Bracket and Entropy Formula for Multiple Membranes,” JHEP 0808, 076 (2008) [arXiv:0807.0812 [hep-th]].

[8] C. S. Chu and P. M. Ho, “Noncommutative open string and D-brane,” Nucl. Phys. B 550, 151 (1999) [arXiv:hep-th/9812219]. C. S. Chu and P. M. Ho, “Constrained quantization of open string in background B field and noncommutative D-brane,” Nucl. Phys. B 568, 447 (2000) [arXiv:hep-th/9906192].

[9] V. Schomerus, “D-branes and deformation quantization,” JHEP 9906, 030 (1999) [arXiv:hep-th/9903205].

[10] N. Seiberg and E. Witten, “String theory and noncommutative geometry,” JHEP 9909, 032 (1999) [arXiv:hep-th/9908142].

[11] L. Cornalba and R. Schiappa, “Nonassociative star product deformations for D-brane worldvolumes in curved backgrounds,” Commun. Math. Phys. 225, 33 (2002) [arXiv:hep-th/0101219].

[12] P. M. Ho, “Making non-associative algebra associative,” JHEP 0111, 026 (2001) [arXiv:hep-th/0103024].

[13] E. Bergshoeff, D. S. Berman, J. P. van der Schaar and P. Sundell, “A noncommutative M-theory five-brane,” Nucl. Phys. B 590, 173 (2000) [arXiv:hep-th/0005026];

[14] S. Kawamoto and N. Sasakura, “Open membranes in a constant C-field background and noncommutative boundary strings,” JHEP 0007, 014 (2000) [arXiv:hep-th/0005123]; N. Ikeda, “Deformation of BF theories, topological open
membrane and a generalization of the star deformation,” JHEP 0107, 037 (2001) [arXiv:hep-th/0105286]; B. Pioline, “Comments on the topological open membrane,” Phys. Rev. D 66, 025010 (2002) [arXiv:hep-th/0201257]; D. S. Berman and B. Pioline, “Open membranes, ribbons and deformed Schild strings,” Phys. Rev. D 70, 045007 (2004) [arXiv:hep-th/0404049].

[15] Y. Matsuo and Y. Shibusa, “Volume preserving diffeomorphism and noncommutative branes,” JHEP 0102, 006 (2001) [arXiv:hep-th/0010040]; M. Sakakibara, “Remarks on a deformation quantization of the canonical Nambu bracket,” Prog. Theor. Phys. 104 (2000) 1067.

[16] D. S. Berman, “M-theory branes and their interactions,” Phys. Rept. 456, 89 (2008) [arXiv:0710.1707 [hep-th]].

[17] Z. Yin, “A note on space noncommutativity,” Phys. Lett. B 466, 234 (1999) [arXiv:hep-th/9908152]. D. Bigatti and L. Susskind, “Magnetic fields, branes and noncommutative geometry,” Phys. Rev. D 62, 066004 (2000) [arXiv:hep-th/9908056].

[18] P. M. Ho and Y. Matsuo, “A toy model of open membrane field theory in constant 3-form flux,” Gen. Rel. Grav. 39, 913 (2007) [arXiv:hep-th/0701130].

[19] Y. Nambu, “Generalized Hamiltonian dynamics,” Phys. Rev. D 7, 2405 (1973);

[20] L. Takhtajan, “On Foundation Of The Generalized Nambu Mechanics (Second Version),” Commun. Math. Phys. 160, 295 (1994) [arXiv:hep-th/9301111].

[21] V. T. Filippov, ”n-Lie algebras,” Sib. Mat. Zh., 26, No. 6, 126140 (1985).

[22] P. M. Ho, R. C. Hou and Y. Matsuo, “Lie 3-Algebra and Multiple M2-branes,” JHEP 0806, 020 (2008) [arXiv:0804.2110 [hep-th]].

[23] R. Weitzenböck, “Invariantentheorie,” P. Noordhoff, Gröningen, 1923. Ph. Gaunt-theron, “Some remarks concerning Nambu mechanics,” Lett. in Math. Phys. 37 (1996), 103. D. Alekseevsky, P. Guha, “On Decomposability of Nambu-Poisson Tensor,” Acta. Math. Univ. Commenianae 65 (1996), 1. R. Ibáñez, M. de León, J. C. Marrero, D. M. de Diego, “Dynamics of generalized Poisson and Nambu-Poisson brackets,” J. of Math. Physics 38 (1997), 2332. N. Nakanishi, “On Nambu-Poisson Manifolds,” Reviews in Mathematical Physics 10 (1998), 499. G. Marmo, G. Vilasi, A. M. Vinogradov, “The local structure of n-Poisson and n-Jacobi manifolds,” J. Geom. Physics 25 (1998), 141.
[24] For a review of the Nambu-Poisson bracket, see: I. Vaisman, “A survey on Nambu-Poisson brackets,” Acta. Math. Univ. Comenianae 2 (1999), 213.

[25] A. M. Low, arXiv:0909.1941 [hep-th].

[26] D. Zwanziger, “Local Lagrangian quantum field theory of electric and magnetic charges,” Phys. Rev. D 3, 880 (1971). S. Deser and C. Teitelboim, “Duality Transformations Of Abelian And Nonabelian Gauge Fields,” Phys. Rev. D 13, 1592 (1976). R. Floreanini and R. Jackiw, “Selfdual Fields As Charge Density Solitons,” Phys. Rev. Lett. 59, 1873 (1987). M. Henneaux and C. Teitelboim, “DYNAMICS OF CHIRAL (SELF-DUAL) P FORMS,” Phys. Lett. B 206, 650 (1988). A. A. Tseytlin, “Duality Symmetric Formulation Of String World Sheet Dynamics,” Phys. Lett. B 242, 163 (1990). A. A. Tseytlin, “Duality Symmetric Closed String Theory And Interacting Chiral Scalars,” Nucl. Phys. B 350, 395 (1991). J. H. Schwarz and A. Sen, “Duality symmetric actions,” Nucl. Phys. B 411, 35 (1994) [arXiv:hep-th/9304154].

[27] P. Pasti, I. Samsonov, D. Sorokin and M. Tonin, “BLG-motivated Lagrangian formulation for the chiral two-form gauge field in D=6 and M5-branes,” Phys. Rev. D 80, 086008 (2009) [arXiv:0907.4596 [hep-th]].

[28] P. S. Howe, N. D. Lambert and P. C. West, “The self-dual string soliton,” Nucl. Phys. B 515, 203 (1998) [arXiv:hep-th/9709014].

[29] Y. Michishita, “The M2-brane soliton on the M5-brane with constant 3-form,” JHEP 0009, 036 (2000) [arXiv:hep-th/0008247]. D. Youm, “BPS solitons in M5-brane worldvolume theory with constant three-form field,” Phys. Rev. D 63, 045004 (2001) [arXiv:hep-th/0009082].

[30] K. Furunchi and T. Takimi, “String solitons in the M5-brane worldvolume action with Nambu-Poisson structure and Seiberg-Witten map,” JHEP 0908, 050 (2009) [arXiv:0906.3172 [hep-th]].

[31] C. S. Chu and D. J. Smith, “Towards the Quantum Geometry of the M5-brane in a Constant C-Field from Multiple Membranes,” JHEP 0904, 097 (2009) [arXiv:0901.1847 [hep-th]].

[32] I. A. Bandos and P. K. Townsend, “Light-cone M5 and multiple M2-branes,” Class. Quant. Grav. 25, 245003 (2008) [arXiv:0806.4777 [hep-th]].
[33] I. A. Bandos and P. K. Townsend, “SDiff Gauge Theory and the M2 Condensate,” JHEP 0902, 013 (2009) [arXiv:0808.1583 [hep-th]].