Upper Measure Bounds of Nodal Sets of Solutions to the Bi-Harmonic Equations on $C^\infty$ Riemannian Manifolds

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Abstract: In this paper, we consider the nodal set of a bi-harmonic function $u$ on an $n$ dimensional $C^\infty$ Riemannian manifold $M$, that is, $u$ satisfies the equation $\Delta_M^2 u = 0$ on $M$, where $\Delta_M$ is the Laplacian operator on $M$. We first define the frequency function and the doubling index for the bi-harmonic function $u$, and then establish their monotonicity formulae and doubling conditions. Following the argument in [13], with the help of establishing the smallness propagation and partitions for $u$, we show that, for some ball $B_r(x_0) \subseteq M$ with $r$ small enough, an upper bound for the measure of nodal set of the bi-harmonic function $u$ can be controlled by $N^\alpha$, that is,

$$\mathcal{H}^{n-1}(\{x \in B_{r/2}(x_0) | u(x) = 0\}) \leq C N^\alpha r^{n-1},$$

where $N = \max \{C_0, N(x_0, r)\}$, $\alpha$, $C$ and $C_0$ both are positive constants depending only on $n$ and $M$. Here $N(x_0, r)$ is the frequency function of $u$ centered at $x_0$ with radius $r$. Among others, we also show that the $n - 1$ dimensional Hausdorff measures of nodal sets of such solutions are bounded by the frequency function, which is independently interesting.

Key Words: Bi-harmonic function, Nodal set, Frequency function, Doubling index, Measure estimate.

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1 Introduction

In this paper, we focus on establishing upper measure bounds of nodal sets of a bi-harmonic function $u$ on $M$, i.e., $u$ satisfies $\Delta_M^2 u = 0$, where $M$ is an $n$-dimensional $C^\infty$ Riemannian manifold and $\Delta_M$ is the Laplacian operator on $M$. The main result of this paper is as follows.

**Theorem 1.1.** Let $M$ be an $n$-dimensional $C^\infty$ Riemannian manifold and $u$ be a bi-harmonic function on $M$. Then for any fixed $x_0 \in M$, there exists a positive number $\alpha$ depending only on $n$ and $M$, such that

$$\mathcal{H}^{n-1}(\{x \in B_{r/2}(x_0)|u(x) = 0\}) \leq C N^\alpha r^{n-1}$$

holds for any $r \leq R_0$ with $N = \max \{C_0, N(x_0, r)\}$, where $N(x_0, r)$ is the frequency function of $u$ centered at $x_0$ with radius $r$; $R_0$, $C$ and $C_0$ are positive constants depending only on $n$ and $M$.

In [2], F.J. Almgren first introduced the frequency for harmonic functions. Then in [5, 6], N. Garofalo and F.H. Lin gave the monotonicity formula for the frequency functions and the doubling conditions for solutions of second order linear uniformly elliptic equations, and obtained the unique continuation property for the solutions. In 1991, F.H. Lin in [12] gave the measure estimates of nodal sets of solutions to the second order linear uniformly elliptic equations with analytic coefficients by the frequency functions. Such a conclusion was also given in [10] but the arguments in these two papers are different.

In [17], S.T. Yau conjectured that, the $n - 1$ dimensional Hausdorff measure bounds of nodal sets of the eigenfunctions of the Laplacian operator on $n$ dimensional compact $C^\infty$ Riemannian manifolds without boundaries are comparable to $\sqrt{\lambda}$, where $\lambda$ is the corresponding eigenvalue. Then in [4], H. Donnelly and C. Fefferman proved the upper bounds of the Yau’s conjecture for analytic manifolds. In 1989, R. Hardt and L. Simon in
[10] showed that, an upper bound for the measures of nodal sets of the eigenfunctions for $\mathcal{C}^{\infty}$ Riemannian manifolds is $C\lambda^{C\sqrt{\lambda}}$ for some positive constant $C$. In [3], R.T. Dong showed an upper bound for the measures of nodal sets of eigenfunctions in the two dimensional case is $C\lambda^{3/4}$. Such a bound for the two dimensional case was improved to $C\lambda^{3/4-\epsilon}$ for some small $\epsilon > 0$ in [4] by A. Logunov and Malinnikova. Recently, A. Logunov in [13] showed that, for the general dimensional cases, the upper bounds for the measures of nodal sets of the eigenfunctions are $C \lambda^\alpha$ for some constant $\alpha > 1/2$. On the other hand, I. Kukavica in [11] obtained the upper bounds for the eigenfunctions for the higher even order elliptic operators with the analytic data.

In [9], Q. Han showed the structure of the nodal sets of solutions for the higher order linear uniformly elliptic equations. In [15], we gave a definition of the frequency function for the bi-harmonic functions and derived some measure estimates of nodal sets for bi-harmonic functions on Euclidean spaces.

In this paper, we will consider the interior measure estimates for the nodal sets of bi-harmonic functions on $\mathcal{C}^{\infty}$ Riemannian manifolds. First we give the definitions of the frequency function and doubling index for these bi-harmonic functions. Then we establish the “almost monotonicity formula” and doubling conditions for these two quantities. Through the above preparations, based on proving that the bound of $n - 1$ dimensional Hausdorff measures of nodal sets of these bi-harmonic functions is controlled by the frequency function, following the argument in [13], with the help of showing a variant of the smallness propagation and partitions for $u$, we drive a measure upper bound for nodal sets of bi-harmonic functions on $\mathcal{C}^{\infty}$ Riemannian manifolds. Actually, among others, we also prove that the (n-1)-dimensional Hausdorff measures of nodal sets of these bi-harmonic functions are controlled by their frequency function, which is independently interesting.

The rest for this paper is organized as follows. In the second part, we give the definition of the frequency function, and show the “almost monotonicity formula” and
doubling conditions for the frequency function. In the third part, we give the definition of the doubling index, and obtain the relationship between the doubling index and the frequency function. In the fourth part, we first introduce a small Cauchy data propagation lemma and then give one variant of it. Furthermore we derive some estimates on the doubling index during partitions by applying the propagation of the small Cauchy data. In the fifth part, by employing partitions and establishing some estimates related to the frequency and doubling index over the course of partitions, we give a proof of Theorem 1 under a claim that the nodal set of \( u \) have already controlled by some constant depending on the frequency function of \( u \). Finally in the last part, we will prove the claim in the fifth part by the similar argument as in [9], Chapter 5.

2 The frequency function and the almost monotonicity formula

Let \( x_0 \) be an interior point of \( M \). Then there exists \( \bar{R} > 0 \) such that \( B_{\bar{R}}(x_0) \) is contained in a local coordinate system card. We define the frequency function of a bi-harmonic function \( u \) as follows.

**Definition 2.1.** Let \( u \) be a bi-harmonic function on \( M \). For \( r \leq \bar{R} \), denote

\[
D_M(x_0, r) = \int_{B_r(x_0)} \left( |\nabla_M u|^2 + |\nabla_M v|^2 + uv \right) dV_M
\]

and

\[
H_M(x_0, r) = \int_{\partial B_r(x_0)} \left( u^2 + v^2 \right) dV_{\partial B_r(x_0)},
\]

here \( v = \Delta_M u \). Then we define the frequency function of \( u \) centered at \( x_0 \) with radius \( r \) as

\[
N_M(x_0, r) = r \frac{D_M(x_0, r)}{H_M(x_0, r)}.
\]

We first show an “almost monotonicity formula”.
Theorem 2.1. Fix a point $x_0$ on $M$ and for any positive constant $C_0$, there exist positive constants $C(n, M, C_0)$ and $R_0(n, M) \leq \bar{R}$, such that if $N_M(x_0, r) \geq C_0$ and $r \leq R_0$, such that
\[
\frac{N'_M(x_0, r)}{N_M(x_0, r)} \geq -C.
\] (2.2)

Proof. Without loss of generality, we may assume that $x_0$ is the origin. And we set $v = \Delta_M u$. By using the polar coordinate system, a metric tensor $g_M$ on $M$ can be written as
\[
g_M = dr \otimes dr + r^2 b_{ij}(r, \theta) d\theta_i \otimes d\theta_j.
\]
From the argument in [9], we have that $b_{ij}(0, 0) = \delta_{ij}$, $i, j = 1, 2, \cdots, n - 1$, $|\partial_r b_{ij}(r, \theta)| \leq \Lambda$. Set $b(r, \theta) = |\det(b_{ij}(r, \theta))|$. Then $dV_{\partial B_r} = r^{n-1} \sqrt{b(r, \theta)} d\theta$, and
\[
H_M(0, r) = r^{n-1} \int_{\partial B_1} \left( u^2 + v^2 \right) \sqrt{b(r, \theta)} d\theta.
\]
Thus
\[
H'_M(0, r) = \frac{n-1}{r} H_M(0, r) + \int_{\partial B_r} \frac{1}{\sqrt{b}} \partial_r (\sqrt{b}) \left( u^2 + v^2 \right) dV_{\partial B_r} + 2 \int_{\partial B_r} (uu_r + vv_r) dV_{\partial B_r}
\]
\[
= \left( \frac{n-1}{r} + C \right) H_M(0, r) + 2 \int_{\partial B_r} (uu_r + vv_r) dV_{\partial B_r},
\]
where $C$ is a positive constant depending only on $M$ and $n$. From the equation $\Delta_M^2 u = 0$, we have, by direct computation, that
\[
D_M(0, r) = \int_{\partial B_r} (uu_r + vv_r) dV_{\partial B_r}.
\]
Thus
\[
H'_M(0, r) = \left( \frac{n-1}{r} + O(1) \right) H_M(0, r) + 2 D_M(0, r).
\]

Now we consider the derivative of the quantity $D_M(0, r)$. For any fixed $r$, we separate the function $u$ into two parts, i.e., $u = \bar{u} + \underline{u}$, where $\bar{u}$ and $\underline{u}$ satisfy the following equations, respectively:
\[
\begin{aligned}
\begin{cases}
\Delta_M \overline{u} = 0 & \text{in } B_r, \\
\overline{u} = u & \text{on } \partial B_r;
\end{cases}
\begin{cases}
\Delta_M \underline{u} = v & \text{in } B_r, \\
\underline{u} = 0 & \text{on } \partial B_r.
\end{cases}
\end{aligned}
\]

Denote

\[
D_1(r) = \int_{B_r} (|\nabla_M \overline{u}|^2 + |\nabla_M v|^2) dV_M; \quad D_2(r) = \int_{B_r} |\nabla_M \overline{u}|^2 dV_M;
\]

and

\[
D_3(r) = 2 \int_{B_r} \nabla_M \overline{u} \cdot \nabla_M \underline{u} dV_M; \quad D_4(r) = \int_{B_r} uv dV_M.
\]

Because \( \overline{u} \) and \( v \) both are harmonic functions, by the same arguments as in the proof of Theorem 3.1.1, Chapter 3, [9], we have that

\[
D'_1(r) = \left(\frac{n-2}{r} + O(1)\right) D_1(r) + 2 \int_{\partial B_r} \overline{u}^2 dV_{\partial B_r} + \int_{\partial B_r} v^2 dV_{\partial B_r}.
\]

Note that \( \underline{u} = 0 \) on \( \partial B_r \), it holds that \( |\nabla_M u| = |u_r| \) on \( \partial B_r \), and \( \nabla_M \overline{u} \cdot \nabla_M \underline{u} = \overline{u}_r \cdot u_r \) on \( \partial B_r \). So

\[
D'_2(r) = \int_{\partial B_r} |\nabla_M \overline{u}|^2 dV_{\partial B_r} = \int_{\partial B_r} u_r^2 dV_{\partial B_r},
\]

and

\[
D'_3(r) = 2 \int_{\partial B_r} \nabla_M \overline{u} \cdot \nabla_M \underline{u} dV_{\partial B_r} = 2 \int_{\partial B_r} \overline{u}_r \cdot u_r dV_{\partial B_r}.
\]

For \( D_4(r) \), it holds that

\[
|D'_4(r)| = \left| \int_{\partial B_r} uv dV_{\partial B_r} \right| 
\leq \frac{1}{2} \left( \int_{\partial B_r} u^2 dV_{\partial B_r} + \int_{\partial B_r} v^2 dV_{\partial B_r} \right) 
= \frac{1}{2} H_M(0, r).
\]
Note that we have already required that $N_M(0, r) \geq C_0$, So

$$H_M(0, r) \leq \frac{rD_M(0, r)}{C_0}.$$  

Thus

$$D'_M(0, r) \geq \frac{n-2}{r} D_1(r) + 2 \int_{\partial B_r} (u^2_p + u^2 + 2u_p u + \nu^2_p) dV_{\partial B_r} - C(D_1(r) + D_M(0, r))$$

$$= \frac{n-2}{r} D_1(r) + 2 \int_{\partial B_r} (u^2_p + \nu^2_p) dV_{\partial B_r} - C(D_1(r) + D_M(0, r)).$$

Now we will show that

$$D_1(r) \leq CD_M(r),$$

provided that $r$ small enough.

In fact,

$$|D_4(r)| = \left| \int_{B_r} uvdV_M \right|$$

$$= \int_{B_r} \bar{u}vdV_M + \int_{B_r} uv dV_M$$

$$\leq \frac{1}{2} \left( \int_{B_r} \bar{u}^2 dV_M + \int_{B_r} u^2 dV_M \right) + \int_{B_r} v^2 dV_M$$

$$\leq Cr^2 \int_{B_r} |\nabla_M u|^2 dV_M + CrH_M(0, r)$$

$$\leq Cr^2 D_2(r) + \frac{Cr^2}{C_0} D_M(0, r).$$

Here the second inequality used the Poincare's inequality and the fact that $v$ is a harmonic function on $M$. This shows that

$$|D_4(r)| \leq Cr^2 D_2(r) + \frac{Cr^2}{C_0} D_M(0, r). \tag{2.3}$$

On the other hand, from some direct calculational, we have

$$D_3(r) = 2 \int_{B_r} \nabla_M \bar{u} \cdot \nabla_M u dV_M = 0. \tag{2.4}$$
Then from the form (2.3) and (2.4), we have

\[
D_M(0, r) = D_1(r) + D_2(r) + D_4(r)
\]

\[
\geq D_1(r) + D_2(r) - |D_4(r)|
\]

\[
\geq D_1(r) + D_2(r) - Cr^2 D_2(r) - \frac{Cr^2}{C_0} D_M(0, r).
\]

Choose \(r\) small enough, depending only on \(n\) and \(M\), such that \(Cr^2 \leq 1/2\) and \(C_0 = 1\). Then it holds that

\[
D_M(0, r) \geq D_1(r) + D_2(r) - \frac{1}{2} D_2(r) - Cr^2 D_M(0, r),
\]

which shows that

\[
D_1(r) + \frac{1}{2} D_2(r) \leq (1 + Cr^2) D_M(0, r) \leq CD_M(0, r).
\]

From the fact that \(D_2(r) > 0\), we have

\[
D_1(r) \leq D_1(r) + \frac{1}{2} D_2(r) \leq CD_M(0, r).
\]

Thus

\[
D'_M(0, r) \geq \frac{n - 2}{r} D_1(r) + 2 \int_{\partial B_r} \left( \overline{\nabla u}^2 + \overline{u}^2 + v_r^2 \right) dV_{\partial B_r} - CD_M(0, r).
\]

Now we will use \(D_1(r)\) to control \(D_M(0, r)\). For \(D_4(r)\), we have that

\[
|D_4(r)| = \left| \int_{B_r} uv dV_M \right|
\]

\[
\leq \frac{1}{2} \int_{B_r} (\overline{u}^2 + \overline{u}^2) dV_g + \int_{B_r} v^2 dV_M
\]

\[
\leq Cr^2 \int_{B_r} |\nabla_M \overline{\mu}|^2 dV_M + Cr \int_{\partial B_r} (\overline{u}^2 + v^2) dV_{\partial B_r}
\]

\[
\leq Cr^2 D_2(r) + Cr^2 D_M(0, r).
\]

In the last inequality we have used the assumption that \(N_M(0, r) \geq C_0\). For \(D_2(r)\), we
have that

\[
D_2(r) = \int_{B_r} |\nabla M u|^2 dV_M \\
= -\int_{B_r} uv dV_M \\
\leq \frac{1}{2} \left( \int_{B_r} u^2 dV_M + \int_{B_r} v^2 dV_M \right) \\
\leq Cr^2 \int_{B_r} |\nabla M u|^2 dV_M + CrH(0, r) \\
\leq Cr^2 D_2(r) + Cr^2 D_M(0, r),
\]

and so we have

\[
D_2(r) \leq Cr^2 D_M(0, r)
\]  \hspace{1cm} (2.5)

provided that \( r > 0 \) small enough. From (2.3), (2.4) and (2.5), we have that for \( r > 0 \) small enough,

\[
D_M(0, r) = D_1(r) + D_2(r) + D_4(r) \\
\leq D_1(r) + D_2(r) + |D_4(r)| \\
\leq D_1(r) + D_2(r) + Cr^2 D_2(r) + \frac{Cr^2}{C_0} D_M(0, r) \\
\leq D_1(r) + Cr^2 D_M(0, r),
\]

and thus

\[
D_1(r) \geq (1 - Cr^2) D_M(0, r).
\]

Then

\[
\frac{D_M'(0, r)}{D_M(0, r)} \geq \frac{n - 2}{r} \frac{D_1(r)}{D_M(0, r)} + 2 \frac{\int_{\partial B_r} (u_r^2 + v_r^2) dV_{\partial B_r}}{\int_{\partial B_r} (uu_r + vv_r) dV_{\partial B_r}} - C \\
\geq \frac{n - 2}{r} (1 - Cr^2) + 2 \frac{\int_{\partial B_r} (u_r^2 + v_r^2) dV_{\partial B_r}}{\int_{\partial B_r} (uu_r + vv_r) dV_{\partial B_r}} - C \\
\geq \frac{n - 2}{r} + 2 \frac{\int_{\partial B_r} (u_r^2 + v_r^2) dV_{\partial B_r}}{\int_{\partial B_r} (uu_r + vv_r) dV_{\partial B_r}} - C.
\]

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So

\[
\frac{N'_M(0, r)}{N_M(0, r)} = \frac{1}{r} + \frac{D'_M(0, r)}{D_M(0, r)} - \frac{H'_M(0, r)}{H_M(0, r)}
\geq 2 \left( \frac{\int_{\partial B_r} (u^2 + v^2) dV_{\partial B_r}}{\int_{\partial B_r} (uu + vv) dV_{\partial B_r}} - \frac{\int_{\partial B_r} (uu + vv) dV_{\partial B_r}}{\int_{\partial B_r} (u^2 + v^2) dV_{\partial B_r}} \right) - C
\geq -C.
\]

This is the desired result.

\[\square\]

From this “almost monotonicity formula”, we can get the following estimate for the frequency function.

**Lemma 2.2.** Let \( u \) be a bi-harmonic function on \( M \) and \( N_M(x_0, r) \) is the corresponding frequency function of \( u \). Fix a point \( x_0 \) on \( M \). Then for any \( \epsilon \in (0, 1) \), there exists some positive constant \( r_0 \) depending on \( n, M \) and \( \epsilon \), such that if \( N_M(x_0, r_0) > C_0 \), then

\[
N_M(x_0, \rho) \leq (1 + \epsilon)N_M(x_0, r_0)
\tag{2.6}
\]

for any \( \rho < r_0 \). Moreover, if \( r < r_0 \), and for any \( \rho \in (r, r_0) \), it holds that \( N_M(x_0, \rho) > C_0 \), then

\[
N_M(x_0, \rho) \geq (1 - \epsilon)N_M(x_0, r).
\tag{2.7}
\]

**Proof.** From Theorem 2.1, it holds that

\[
\frac{N'_M(x_0, r)}{N_M(x_0, r)} \geq -C
\]

if \( N_M(x_0, r) > C_0 \) and \( r \leq R_0 \), where \( C \) is a positive constant depending on \( n \) and \( M \). We denote

\[
b(R) = \{ r \in (0, R) | N_M(x_0, r) > C_0 \}.
\]

Then the set \( b(R) \) is an open set of \( \mathbb{R}^+ \). So \( b(R) = \cup(a_i, b_i) \) for at most countable many open intervals \((a_i, b_i), i = 1, 2, \cdots\), where \( a_i > 0, b_i \leq R \). So for any \( r \in (0, R) \setminus b(R) \), it
holds that \( N_M(x_0, r) \leq C_0 \leq (1+\varepsilon)N_M(x_0, R) \) for any \( \varepsilon > 0 \) provided that \( N_M(x_0, R) > C_0 \).

For any \( \rho \in b(R) \) and \( r < R \in b(R) \), one of the following two cases must happen.

1) \( \rho \in (a_{i_0}, b_{i_0}) \) for some \( i_0 \) with \( N(x_0, b_{i_0}) = C_0 \) and \( b_{i_0} < R \);

2) \( \rho \in (a_{i_0}, R) \).

In the first case, we have

\[
\ln \frac{C_0}{N_M(x_0, \rho)} \leq \ln \frac{N_M(x_0, b_{i_0})}{N_M(x_0, \rho)} = \int_{\rho}^{b_{i_0}} \frac{d \ln N_M(x_0, r)}{dr} dr = \int_{\rho}^{b_{i_0}} \frac{N'_M(x_0, r)}{N_M(x_0, r)} dr \geq -C(b_{i_0} - \rho) \geq -CR,
\]

and thus

\[
N_M(x_0, \rho) \leq e^{CR}C_0 \leq e^{CR}N_M(x, R).
\]

In the second case, we can also obtain that

\[
N_M(x_0, \rho) \leq e^{CR}N_M(x_0, R).
\]

Then by choosing \( r_0 = R \) small enough, such that \( e^{C_{r_0}} \leq 1 + \varepsilon \), the inequality (2.6) is proved.

Moreover, if for any \( \rho \in (r, r_0) \), the inequality \( N_M(x_0, \rho) > C_0 \) holds, then similarly, we can get that

\[
N_M(x_0, r) \leq (1 + \varepsilon)N_M(x_0, \rho),
\]

and thus

\[
N_M(x_0, \rho) \geq \frac{1}{1 + \varepsilon}N_M(x_0, r) \geq (1 - \varepsilon)N_M(x_0, r).
\]

Thus the form (2.7) holds.
Furthermore, we have the following estimation of the frequency function.

**Lemma 2.3.** Let $u$ be a bi-harmonic function on $M$ and $N_M(x, r)$ be the corresponding frequency function and $x_0$ be a fixed point on $M$. Assume that $N_M(x_0, r) \geq C'_0 > 3C_0/2$ for some $r < r_0$, where $r_0$ is a positive constant depending only on $n$ and $M$, $C_0$ is the same positive constant as in Theorem 2.1. Then for any $\rho \in (r, r_0)$, it holds that

$$N_M(x_0, \rho) \geq C_0. \tag{2.8}$$

**Proof.** If the conclusion is not true, i.e., there exists some $\rho_0 \in (r, r_0)$, such that $N_M(x_0, \rho_0) < C_0$. Then because $N_M(x_0, r) > C_0$, there must exist some points $\rho \in (r, \rho_0)$, such that $N_M(x_0, \rho) = C_0$. Let $\overline{\rho}$ be the smallest point in $(r, \rho_0)$ such that $N_M(x_0, \overline{\rho}) = C_0$. Then for any $\rho \in (r, \overline{\rho})$, $N_M(x_0, \rho) > C_0$. Thus from Lemma 2.2 and let $\epsilon = 1/2$, we have

$$N_M(x_0, r) \leq \frac{3}{2} N_M(x_0, \overline{\rho}) = \frac{3}{2} C_0 < C'_0.$$ 

On the other hand, we have already assumed that $N_M(x_0, r) > C'_0$, and that is a contradiction. So the lemma is proved.

□

**Lemma 2.4.** Let $u$ be a bi-harmonic function on $M$ and $x_0$ be a fixed point on $M$. Then for any $\epsilon \in (0, 1/2)$, there exists some positive constant $r_0$ depending on $n$, $M$, and $\epsilon$, such that if $r \leq r_0$, it holds that

$$\int_{B_r(x_0)} (u^2 + v^2) dV_M \leq t^{2(1+\epsilon)N_M(x_0, r) + C} \int_{B_{tr}(x_0)} (u^2 + v^2) dV_M, \tag{2.9}$$

for any $t > 1$. Moreover, if $N_M(x, r/t) > C'_0$, where $r \leq r_0$, then it also holds that

$$\int_{B_r(x_0)} (u^2 + v^2) dV_M \geq t^{2(1-\epsilon)N_M(x_0, r/t) - C'} \int_{B_{tr}(x_0)} (u^2 + v^2) dV_M. \tag{2.10}$$

Here $C$ and $C'$ both are positive constants depending on $n$ and $M$. 

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Proof. We also assume that \( x_0 = 0 \) without loss of generality. Then from the proof of Theorem 2.11 we have
\[
\frac{dH_M(0, r)}{dr} = \left( \frac{n - 1}{r} + O(1) \right) H_M(0, r) + 2D_M(0, r).
\]
So
\[
\ln \frac{H_M(0, r)}{H_M(0, r/t)} = \int_{r/t}^r \frac{H_M'(0, \rho)}{H_M(0, \rho)} d\rho = \int_{r/t}^r \left( \frac{n - 1}{\rho} + O(1) + 2 \frac{N_M(0, \rho)}{\rho} \right) d\rho.
\] (2.11)

From Lemma 2.2 we obtain that for any \( \rho \in (0, r) \),
\[
N_M(0, \rho) \leq (1 + \epsilon)N_M(0, r)
\] (2.12)
provided that \( N_M(0, r) > C_0 \). So
\[
\ln \frac{H_M(0, r)}{H_M(0, r/t)} \leq 2(1 + \epsilon) \max \{N_M(0, r), C_0\} \ln t,
\]
and thus
\[
H_M(0, r) \leq t^{2(1 + \epsilon)N_M(0, r) + C} H_M(0, r/t).
\]
From this inequality, it is easy to get form (2.9).

Now we focus on showing (2.10). Because we have already assume that \( N_M(0, r/t) > C'_0 \), from Lemma 2.3, we have \( N_M(0, \rho) > C_0 \) for any \( \rho \in (r/t, r) \). So from Lemma 2.2 we obtain
\[
N_M(0, \rho) \geq (1 - \epsilon)N_M(0, r/t).
\]
Insert this inequality into (2.11), we have
\[
H_M(0, r) \geq t^{2(1 - \epsilon)N_M(0, r/t) - C} H_M(0, r/t)
\]
for some positive constant \( C \) depending only on \( n \) and \( M \). Then (2.10) can be obtained.

\(\square\)
3 The doubling index

We define the doubling index for bi-harmonic functions as follows.

**Definition 3.1.** The doubling index for a bi-harmonic function $u$ is defined as

$$E(x_0, r) = \frac{1}{2} \log_2 \frac{\sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2}{\sup_{B_{r/2}(x_0)} |u|^2 + \sup_{B_{r/2}(x_0)} |v|^2},$$

where $v = \Delta_M u$.

We now show the relationship between the frequency function and the doubling index.

**Lemma 3.1.** For any $\epsilon \in (0, 1/2)$, there exists a positive constant $r_0$ depending on $n, M$ and $\epsilon$, such that for any $r \leq r_0, \eta \in (0, \eta_0)$ with $\eta_0 = e^\epsilon - 1$, if $N_M(x_0, \rho) \geq C_0$ for any $\rho \in (0, (1 + \eta)r)$, it holds that

$$(1 - \epsilon)^2 N_M(x_0, \frac{1 + \eta}{2} r) - C \leq E(x_0, r) \leq (1 + \epsilon)^2 N_M(x_0, (1 + \eta)r) + C,$$

where $C$ is a positive constant depending on $n, M$ and $x_0$, and $C_0$ is the same positive constant as in Theorem 2.1.

**Proof.** From the standard elliptic estimation, we have

$$\sup_{B_r(x_0)} |u|^2 \leq C \int_{B_{1+\eta r}(x_0)} (u^2 + v^2) dV_g,$$

and

$$\sup_{B_r(x_0)} |v|^2 \leq C \int_{B_{1+\eta r}(x_0)} v^2 dV_g.$$

Thus

$$\sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2 \leq C \int_{B_{1+\eta r}(x_0)} (u^2 + v^2) dV_g.$$

On the other hand, it is obvious that

$$\sup_{B_{r/2}(x_0)} |u|^2 + \sup_{B_{r/2}(x_0)} |v|^2 \geq \int_{B_{r/2}(x_0)} (u^2 + v^2) dV_g.$$
So from Lemma 2.4 we have
\[
2E(x_0, r) = \log_2 \left( \sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2 \right)
\leq \log_2 C \cdot \frac{\int_{B_r(x_0)} (u^2 + v^2) dV_g}{\int_{B_r(x_0)} (u^2 + v^2) dV_g}
\leq \log_2 C \cdot (2(1 + \eta))^{2(1 + \epsilon)N_M(x_0, (1 + \eta)r) + C}
\leq 2(1 + \epsilon)^2 N_M(x_0, (1 + \eta)r) + C.
\]

This is the right hand side of (3.2). Here we have used the assumption that \( \eta < \eta_0 \) with \( \eta_0 = e^{\epsilon} - 1 \).

By the similar arguments, we have
\[
\sup_{B_{r/2}(x_0)} |u|^2 + \sup_{B_{r/2}(x_0)} |v|^2 \leq C \int_{B_{r/2}(x_0)} (u^2 + v^2) dV_g,
\]
and
\[
\sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2 \geq \int_{B_r(x_0)} (u^2 + v^2) dV_g.
\]

Thus also from Lemma 2.4 we have
\[
2E(x_0, r) \geq \log_2 \left( \frac{\int_{B_r(x_0)} (u^2 + v^2) dV_g}{\int_{B_r(x_0)} (u^2 + v^2) dV_g} \right)
\geq \log_2 \left( \frac{2}{1 + \eta} \right)^{2(1 - \epsilon)N_M(x_0, \frac{1 + \eta}{2} r) - C}
\geq 2(1 - \epsilon)^2 N_M(x_0, \frac{1 + \eta}{2} r) - C,
\]

which gives the left hand side of the form (3.2).

\[\square\]

Now we will show the doubling condition with respect to the doubling index.

**Lemma 3.2.** Let \( u \) be a bi-harmonic function on \( M \). Then for any \( \epsilon \in (0, 1/2) \), there exists a positive constant \( r_0 \) depending only on \( \epsilon \), \( n \) and \( M \) such that for any \( r < r_0, t > 2 \).
it holds that
\[
\sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2 \leq t^{2(1+\epsilon)E(x_0,2r)+C} \left( \sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2 \right),
\]
and
\[
\sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2 \geq t^{2(1-\epsilon)E(x_0,r)-C} \left( \sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2 \right),
\]
where C and C' both are positive constants depending on n and M.

**Proof.** As in the proof of Lemma 3.1, it holds that
\[
\sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2 \leq \frac{C}{\eta^n} \int_{B_{(1+\eta)r}(x_0)} (u^2 + v^2)dV_g,
\]
and
\[
\sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2 \geq \int_{B_r(x_0)} (u^2 + v^2)dV_g.
\]
Thus from Lemma 2.4 and Lemma 3.1 we obtain that
\[
\ln \frac{\sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2}{\sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2} \leq \log_2 \frac{C}{\eta^n} \cdot \frac{\int_{B_{(1+\eta)r}(x_0)} (u^2 + v^2)dV_g}{\int_{B_r(x_0)} (u^2 + v^2)dV_g}
\]
\[
\leq \log_2 \frac{C}{\eta^n} \cdot ((1 + \eta)t)^{2(1+\epsilon)N(x_0,(1+\eta)r)+C}
\]
\[
\leq \log_2 \frac{C}{\eta^n} \cdot ((1 + \eta)t)^{2(1+\epsilon)^2E(x_0,2r)+C-C \log_2 \eta}
\]
\[
\leq \log_2 \frac{C}{\eta^n} \cdot t^{2(1+\epsilon)^3E(x_0,2r)+C-C \log_2 \eta}
\]
\[
\leq \log_2 t^{2(1+\epsilon)^3E(x_0,2r)+C-C \log_2 \eta}.
\]
Here in the fourth inequality we have used the assumption that \( t > 2, 1 + \eta < 2^\epsilon \) and thus \( 1 + \eta \leq t^\epsilon \), \( \ln \frac{1}{\eta} \leq t^{-C \ln \eta} \). Then the first inequality in the lemma can be obtained easily by putting \( \eta = \eta_0/2 \).
Similarly, we have that
\[
\log_2 \frac{\sup_{B_r(x_0)} |u|^2 + \sup_{B_r(x_0)} |v|^2}{\sup_{B_{(r/2)}(x_0)} |u|^2 + \sup_{B_{(r/2)}(x_0)} |v|^2} \geq \log_2 \frac{\eta^n}{C} \cdot \frac{\int_{B_r(x_0)} (u^2 + v^2) dV_g}{\int_{B_{(1+\eta)r}(x_0)} (u^2 + v^2) dV_g}
\]
\[
\geq \log_2 \frac{\eta^n}{C} \left( \frac{t}{1 + \eta} \right)^{2(1-\epsilon)^3 E(x_0, \rho) - C + C \ln \eta}
\]
\[
\geq \log_2 \frac{\eta^n}{C} t^{2(1-\epsilon)^3 E(x_0, \rho) - C + C \ln \eta}
\]
\[
\geq \log_2 t^{2(1-\epsilon)^3 E(x_0, \rho) - C + C \ln \eta}.
\]

This implies the second inequality in the lemma by putting \( \eta = \eta_0/2 \).

□

Now we give a changing center property for the doubling index.

**Lemma 3.3.** There exist positive constants \( r_0 \) and \( E_0 \) depending only on \( n \) and \( M \), such that for any \( \rho \in (0, r_0) \), if \( x_1, x_2 \in B_{r_0}(x_0) \) with \( E(x_1, \rho) > E_0 \) and \( \text{dist}(x_1, x_2) < \rho \), then there exists a positive constant \( C > 1 \) depending on \( n \) and \( M \), such that
\[
E(x_2, C \rho) \geq \frac{99}{100} E(x_1, \rho).
\]

**Proof.** Choose the constant \( C \) large enough, depending only on \( n \) and \( M \) such that \( B_{C \rho(1-10^{-10})}(x_1) \subseteq B_{C}(x_2) \) and \( B_{C(1-10^{-9})/2}(x_2) \subseteq B_{C(1-10^{-10})/2}(x_1) \). Then from Lemma 3.2 by choosing \( t \) and \( E_0 \) properly, we have
\[
2^{2(1+10^{-3})E(x_2,C \rho)} \geq \frac{\sup_{B_{C}(x_2)} |u|^2 + \sup_{B_{C}(x_2)} |v|^2}{\sup_{B_{C(1-10^{-10})/2}(x_2)} |u|^2 + \sup_{B_{C(1-10^{-10})/2}(x_2)} |v|^2}
\]
\[
\geq \frac{\sup_{B_{C(1-10^{-10})}(x_1)} |u|^2 + \sup_{B_{C(1-10^{-10})}(x_1)} |v|^2}{\sup_{B_{C(1-10^{-10})/2}(x_1)} |u|^2 + \sup_{B_{C(1-10^{-10})/2}(x_1)} |v|^2}
\]
\[
\geq 2^{2E(x_1,\rho)(1-10^{-3})}.
\]
Thus
\[ E(x_2, C\rho) \geq \frac{999}{1001} E(x_1, \rho) \geq \frac{99}{100} E(x_1, \rho), \]
which is the desired result. \qed

4 Small Cauchy data propagation and the dividing lemmas

We first state a small Cauchy data propagation lemma and give one of its variants which is useful latter on. Such a lemma can be seen in [1] and [12]. Here we formulate the result in a form similar to that presented in [13].

**Lemma 4.1.** Let \( Q \subseteq \mathbb{R}^n \) be a cube with edge length \( r \), and \( L \) be a second order linear uniformly elliptic operator. Let \( u \) and \( v \) satisfy that \( Lu = v \) in \( Q \). Also suppose that \( |u| \leq 1 \) in \( Q \). For any \( \epsilon \in (0, 1) \), if \( |u| \leq \epsilon < 1 \), \( |\nabla u| \leq \epsilon/r \) on \( F \), where \( F \) is a face of \( Q \), and \( |v| \leq \epsilon/r^2 \) in \( Q \), then
\[
\sup_{1/2Q} |u| \leq C\epsilon^\alpha, \tag{4.1}
\]
where \( C \) and \( \alpha \) both are positive constants depending on \( n \) and the operator \( L \).

The proof of Lemma 4.1 can be found in [1]. We further have the following variant of Lemma 4.1.

**Lemma 4.2.** Assume \( Q \subseteq \mathbb{R}^n \) is a cube with edge length \( r \), and \( L \) is a second order linear uniformly elliptic operator. For any \( \epsilon \in (0, 1) \), If \( Lu = v \) in \( Q \), \( |u| < \epsilon \), \( |\nabla u| < \epsilon/r \) on \( F \) and \( |v| < \epsilon/r^2 \) in \( Q \), where \( F \) is a face of \( Q \), then it holds that
\[
\sup_T |u| \leq C\epsilon^\alpha, \tag{4.2}
\]
where \( C \) and \( \alpha \) are positive constants depending on \( n \) and \( L \). Here \( T \) is a trapezium, one of whose faces is just \( F \), the edge length of the face opposite to \( F \) is \( r/2 \), and the line connecting the centers of these two faces is vertical to \( F \).
Proof. For simplicity and clearness, we only prove the case that \( n = 2 \). For \( n > 2 \), the proof is similar.

When \( n = 2 \), \( F \) is a segment whose length is \( r \). Let \( \bar{F} \) be a segment contained in \( F \) with its length \( \sigma r \) for some \( \sigma \in (0, 1) \). Then by Lemma 4.1 for the face \( \bar{F} \), it holds that

\[
|u(x)| \leq C \varepsilon^\alpha
\]

for any \( x \in \bar{Q} \), where \( \bar{Q} \) is a cube whose side length is \( \sigma r / 2 \), the distance between \( \bar{Q} \) and \( F \) is \( \sigma r / 4 \), and the distance between \( \bar{Q} \) and any one of two sides of \( Q \) which is adjacent to \( F \) is also \( \sigma r / 4 \). Noting the facts is that \( \sigma \in (0, 1) \) is arbitrary and a trapezium in this lemma can be covered by sub-cubes \( \bar{Q} \) of the side length \( \frac{1}{2} \sigma r \), one can repeat using Lemma 4.1 to obtain the estimate in Lemma 4.2.

\( \square \)

For a cube \( Q \), we denote

\[
E(Q) = \sup_{x \in Q, r \in (0, 10 \text{diam}(Q))} E(x, r).
\]

(4.4)

It is called the doubling index of \( Q \). It is obvious that, if a cube \( q \) contained in \( Q \), then \( E(q) \leq E(Q) \); if a cube \( q \) is covered by the cubes \( \{ Q_i \} \) and \( \text{diam}(Q_i) \geq \text{diam}(q) \) for any \( i \), then there exists some \( i \) such that \( E(Q_i) \geq E(q) \).

Because \( M \) is an \( n \) dimensional \( C^\infty \) manifold, it may be locally considered as a domain of \( \mathbb{R}^n \), and the corresponding Laplacian operator \( \Delta_M \) becomes some second order linear uniformly elliptic operator \( L \). We will give some lemmas concerning estimates of the frequency and doubling index in separating a cube \( Q \) into some smaller subcubes.

From now on, we always assume that \( x_0 \) is the original point \( O \).

For a cube \( Q = [-R, R]^n \) in \( \mathbb{R}^n \) for some positive number \( R \), we can divide it into \( A^n \) equal sub-cubes with side length \( 2R/A \). Now we will show a dividing lemma.

**Lemma 4.3.** Assume that \( A \) is an odd number such that the hyperplane \( x_n = 0 \) can always intersect \( A^{n-1} \) subcubes. Let \( q_{i,0} \) denote the cubes such that the set \( q_{i,0} \cap \{ x_n = 0 \} \)
is nonempty. Suppose that \( E(q_{i,0}) > E \) for every \( q_{i,0} \) for some positive constant \( E \). Then there exist positive constants \( A_0, R_0, E_0 \) depending on \( n \) and \( L \), such that if \( A > A_0 \), \( E > E_0 \) and \( R < R_0 \), then \( E(Q) > 2E \).

Proof. Without loss of generality, we assume that \( R_0 = 1/2 \) and Lemma 3.2 can always be used. In the following proof, we use \( c, C, c_1, c_2, c_2', \cdots \) to denote the positive constants depending only on \( n \) and the operator \( L \).

Let \( B \) be the unit ball \( B_1(O) \) and we use \( kB \) to denote the ball whose center is the same as \( B \), and whose radius is \( k \) times of \( B \)’s radius. Let \( \sup_{1/8B}|u|^2 + \sup_{1/8B}|v|^2 = 1 \). For each point \( p \in 1/16B \), it holds that \( B_{1/32}(p) \subseteq 1/8B \). So \( \sup_{B_{1/32}(p)}|u|^2 + \sup_{B_{1/32}(p)}|v|^2 \leq 1 \). From the assumption, we know that for any \( q_{i,0} \), there exists a point \( p_i \in q_{i,0} \), such that \( E(p_i, r_i) > E \) for some \( r_i \in q_{i,0} \). From some simple calculation, we have that \( 2q_{i,0} \subseteq B_{3\sqrt{n}/(2A)}(p_i) \). Then by Lemma 3.2 with \( \epsilon = 1/2 \) and let \( A > A_0 \), \( E > E_0 \) large enough, we have

\[
\sup_{2q_{i,0}}|v|^2 \leq \sup_{2q_{i,0}}|u|^2 + \sup_{2q_{i,0}}|v|^2 \\
\leq C \left( \sup_{B_{3\sqrt{n}/(2A)}(p_i)}|u|^2 + \sup_{B_{3\sqrt{n}/2A}(p_i)}|v|^2 \right) \\
\leq C \left( \sup_{B_{1/32}(p_i)}|u|^2 + \sup_{B_{1/32}(p_i)}|v|^2 \right) \left( \frac{48 \sqrt{n}}{A} \right)^E \\
\leq 2^{-E \ln A}.
\]

By the same arguments, one can also get that \( \sup_{2q_{i,0}}|u|^2 \leq 2^{-E \ln A} \). Because \( v \) satisfies the equation \( Lv = 0 \), from the standard elliptic interior estimates, we also have that

\[
\sup_{q_{i,0}}|\nabla v| \leq CA \sup_{2q_{i,0}}|v| \\
\leq CA \frac{2^{-E \ln A}}{2^{E \ln A}} \\
= 2^{-E \ln A + \log_2 C + \log_2 A} \\
\leq 2^{-c_1 E \ln A}.
\]
Note that $|v|$ and $|\nabla v|$ both are bounded by $2^{-c_1E \ln A}$ on $\frac{1}{8}B \cap \{x_n = 0\}$. So from Lemma 4.2 there exist a trapezium $T$ such that $\sup_T |v| \leq 2^{-c_1E \ln A}$. The trapezium $T$ satisfies that, its underside face is on $\{x_n = 0\}$ with side $\frac{1}{16 \sqrt{n}}$; its top face is on the hyperplane parallel to the hyperplane $\{x_n = 0\}$; the distance between these two hyperplanes is $\frac{3}{64 \sqrt{n}}$; and the side of the top face is $\frac{1}{32 \sqrt{n}}$. Such a trapezium is contained in the ball $1/8B$. Note that the cube $q'$ with side $\frac{3}{80 \sqrt{n}}$ and one face on the hyperplane $\{x_n = 0\}$ is contained in the trapezium $T$, it holds that $\sup_{q'} |v| \leq 2^{-c_1E \ln A}$. Denote by $c_1' = c_1 \alpha$, we have $\sup_{q'} |v| \leq 2^{-c_1'E \ln A}$. On the other hand, because $Lu = v$, from the standard elliptic estimate, it holds that

$$\sup_{q,\, 0} |\nabla u| \leq CA (\sup_{q,\, 0} |u| + \sup_{q,\, 0} |v|) \leq 2^{-c_2E \ln A}.$$  

Thus $|u|$ and $|\nabla u|$ both are bounded by $2^{-c_2E \ln A}$ on one of $q'$'s faces which is contained in the hyperplane $\{x_n = 0\}$. So from Lemma 4.1 we have that there exists a cube $q'$ such that $\sup_{q'} |u| \leq 2^{-c_3E \ln A}$, where $c_3 = \alpha \min \{c_1', c_2\}$. The cube $q' \subseteq \{x_n > 0\}$ satisfies that its side is $\frac{3}{160 \sqrt{n}}$, and the distance between the center of $q'$ and the face $\{x_n = 0\}$ is $\frac{3}{160 \sqrt{n}}$. Because $q' \subseteq q \subseteq T \subseteq \frac{1}{8}B$, it holds that $\sup_{q'} |v| \leq 2^{-c_1'E \ln A}$. So $\sup_{q'} |u|^2 + \sup_{q'} |v|^2 \leq 2^{-c_4E \ln A}$.

Let $p$ be the center of $q'$. Then the ball $B_{\frac{1}{120 \sqrt{n}}} (p)$ is contained in $q'$. Therefore

$$\sup_{B_{\frac{1}{120 \sqrt{n}}} (p)} |u|^2 + \sup_{B_{\frac{1}{120 \sqrt{n}}} (p)} |v|^2 \leq 2^{-c_4E \ln A}.$$  

However, because $\frac{1}{8}B \subseteq B_{1/2}(p)$, it holds that

$$\sup_{B_{1/2}(p)} |u|^2 + \sup_{B_{1/2}(p)} |v|^2 \geq 1.$$  

So

$$\sup_{B_{\frac{1}{2}\sqrt{n}} (p)} |u|^2 + \sup_{B_{\frac{1}{2}\sqrt{n}} (p)} |v|^2 \geq 2^{-c_4E \ln A}.$$  

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By Lemma 3.2 with $\epsilon = 1/2$, we also have

$$\sup_{B_{1/2}(p)} |u|^2 + \sup_{B_{1/2}(p)} |v|^2 \leq \left(160 \sqrt{n}\right)^{3E(p,1)}.$$

Hence $E(p, 1) \geq c_6 E \ln A$. Then $E(p, 1) > 2E$ for $A$ large enough. \hfill \Box

5 The simplex lemma and the proof of Theorem 1.1

Now we will give the measure estimate of the nodal set of $u$ with its frequency function by following the argument in [13]. In this section, we assume that $x_0$ is the origin $O$. We omit the details and only give the outlines of proofs of lemmas if they are similar to those in [13]

Lemma 5.1. Let $Q$ be a cube $[-R, R]^n \subseteq \mathbb{R}^n$. Suppose that $E(Q) \leq E_0$ and $R < R_0$ for some positive constants $E_0$ and $R_0$ depending on $M, n$ and $x_0$. Then for any $\epsilon > 0$, there exists a positive constant $A_1$ depending on $n$ and $\epsilon$, if we divide $Q$ into $A^n_1$ smaller equal subcubes $q_i$, and let $q_{i,0}$ denote subcubes with nonempty intersection of the hyperplane $\{x_n = 0\}$. Then the number of the subcubes $q_{i,0}$ with $E(q_{i,0}) > E_0/2$ is less than $\epsilon A^{n-1}$.

Proof. From Lemma 4.3, we can choose an integer $A_0$ such that, if we separate $Q$ into $A^n_0$ equal subcubes, then there exists at least one subcube with nonempty intersection of $\{x_n = 0\}$ having its doubling index less than $E_0/2$. We also use $q_{i,0}$ to denote the subcubes such that $q_{i,0} \cap \{x_n = 0\}$ is not empty. For the cube $q_{i,0}$ with $E(q_{i,0}) \geq E_0/2$, we can partition it into $A^n_0$ subcubes again, and then there exist at least one subcubes with nonempty intersection of $\{x_n = 0\}$ such that the doubling index for such a subcube is less than or equal to $E_0/2$. For the cube $q_{i,0}$ with $E(q_{i,0}) < E_0/2$, if we partition it into $A^n_0$ subcubes, then its each subcube’s doubling index is less than or equal to $E_0/2$. When we separate $Q$ into $A^{kn}_0$ equal subcubes, we use $T_k$ to denote the number of subcubes with
nonempty intersection with \( \{x_n = 0\} \), and whose doubling index is larger than \( E_0/2 \).

Then it holds that

\[
T_{k+1} \leq T_k(A_0^{n-1} - 1).
\]

Thus

\[
T_k \leq A_0^{k(n-1)}(1 - \frac{1}{A_0^{n-1}})^k.
\]

Choose \( k \) large enough such that \((1 - \frac{1}{A_0^{n-1}})^k \leq \epsilon\), we can get the desired result. \( \Box \)

**Remark 5.2.** The conclusion of this lemma holds if we change the hyperplane \( x_n = 0 \) into any other hyperplane, and the corresponding constant \( R_0, E_0 \) and \( A_0 \) are independent of this changing.

Let \( x_i, i = 1, 2, \cdots, n+1 \) be \( n+1 \) points on \( B_1(O) \subseteq \mathbb{R}^n \), and \( S \) be the simplex in \( \mathbb{R}^n \) with \( x_i \) as its vertices. Let \( w(S) \) denote the width of \( S \). Then we have the following lemma.

**Lemma 5.3.** Let \( \kappa \) be a given positive constant and \( w(S) > \kappa \). Then there exist a positive constant \( K \) depending only on \( n \) and \( \kappa \), and positive constants \( c, C, R_0 \) and \( E_0 \) depending on \( M, n, O \) and \( \kappa \), such that if \( S \subseteq B_{R_0}(O) \) and \( E(x_i, r) > E_0 \) for \( i = 1, 2, \cdots, n+1 \) and \( r = K\text{diam}(S)/2 \), then \( E(\overline{x}, C\text{diam}(S)) > E_0(1 + c) \), where \( \overline{x} \) is the barycenter of \( S \).

The conclusion of this lemma comes from Lemma 3.2 and the following geometric fact: there exist positive constants \( c \) and \( K \) depending on \( n \) and \( a \), such that if \( \rho = K\text{diam}(S) \), then

\[
B_{(1+c)^{\rho}}(x_0) \subseteq \bigcup_{i=1}^{n+1} B_{\rho}(x_i).
\]

Moreover, it holds that \( c \rightarrow 0 \) and \( K \rightarrow +\infty \) when \( a \rightarrow 0 \).

**Lemma 5.4.** There exist a constant \( c > 0 \), an integer \( A_2 \) depending only on \( n \), and positive numbers \( E_0 \) and \( R_0 \), such that, for any cube \( Q \subseteq B_R(O) \) with \( R < R_0 \), if \( E(Q) \geq E_0 \), and we partition \( Q \) into \( A^n \) equal subcubes with \( A \geq A_2 \), then the number of the
subcubes such that their doubling indexes are greater than \( E(Q)/(1 + c) \) is less than \( \frac{1}{2} A^{n-1} \).

**Proof.** Let \( A_1 \) be the same positive integer as in Lemma 5.1. We divide \( Q \) into \( A_1^n \) smaller equal subcubes. Then this lemma is proved by the following three steps.

**Step 1.** Let \( q \) be any one of the smaller equal subcubes. Let \( F \) be the set of points \( x \) in the cube \( q \) such that \( E(x, r) \geq E_0/(1 + c) \) with \( r \leq 10 \text{diam}(q) \). Let \( w(F) = \frac{\text{width}(F)}{\text{diam}(q)} \). Then by Lemma 5.3, we obtain that for any \( w_0 > 0 \), there exist a positive integer \( j_0 \) and a positive constant \( c_0 \), such that if \( j > j_0, \ c < c_0 \), then \( w(F) < w_0 \).

**Step 2.** Let \( w_0 = 1/A_1 \), where \( A_1 \) is the same positive integer in Lemma 5.1. Then by the conclusion as in the above step, one can show that, if \( \epsilon \) and \( c \) both are sufficiently small, then there exists a positive integer \( j_0 = j_0(\epsilon, c) \), such that if \( Q \) is separated into \( A_1^n \) smaller equal subcubes \( q \) with \( j \geq j_0 \), and \( q \) is separated into \( A_1^n \) smaller equal subcubes \( q_i \), then the number of \( q_i \) with \( E(q_i) \geq E(Q)/(1 + c) \) \( \leq \frac{1}{2} A^{n-1} \). Here we have also used Lemma 3.3.

**Step 3.** Denote by \( K_j \) the number of cubes such that its doubling index is larger than \( E(Q)/(1 + c) \) when \( Q \) is separated into \( A_1^n \) equal smaller subcubes. Then
\[
K_{j+1} \leq \frac{1}{2} A^{n-1} K_j,
\]
for \( j > j_0 \). Define \( A = A_j \), we can see that
\[
K_j \leq K_{j_0} \frac{1}{2j-j_0} A_1^{(n-1)(j-j_0)} \leq \frac{1}{2} A^{n-1},
\]
for \( j \) large enough.

\[ \square \]

**Proof of Theorem 1.1**

Define a function
\[
\mathcal{F}(E) = \sup \frac{\mathcal{H}^{n-1}(\{u = 0\} \cap Q)}{\text{diam}^{n-1}(Q)},
\] (5.1)
where the supremum is taken over the set of all the bi-harmonic functions \( u \) on \( M \), and cubes \( Q \) within \( B_R(0) \), such that its doubling index \( E(Q) \leq E \). We first claim that the function \( \mathcal{F}(E) \leq C(E) < +\infty \), where \( C(E) \) is a positive constant depending on \( n, M, Q \) and \( E \). Then the function \( \mathcal{F}(E) \) is well defined, i.e., \( \mathcal{F}(E) < +\infty \). This claim will be proved in the next section.

We call a number \( E \) is bad if \( \mathcal{F}(E) \geq 4A_2^2F(E/(1 + c)) \), where \( A_2^2 \) and \( c \) both are positive constants as in Lemma 5.

4. First we will show that the set of the bad number \( E \) is bounded.

Consider a bad number \( E \) and a function \( u \) with a cube \( Q \) such that

\[
\mathcal{H}^{n-1}(\{u = 0\} \cap Q) \geq \frac{3}{4}\mathcal{F}(E). \tag{5.2}
\]

with \( E(Q) \leq E \). Divide \( Q \) into \( A_2^n \) equal small subcubes and separate them into two parts \( G_1 \) and \( G_2 \), such that for \( q_i \in G_1 \), it holds that \( E(q_i) > E/(1 + c) \), and for \( q_i \in G_2 \), \( E(q_i) \leq E/(1 + c) \). Because the number of cubes belonging to \( G_1 \) is less than \( \frac{1}{2}A_2^{n-1} \), it holds that

\[
\mathcal{H}^{n-1}(\{u = 0\} \cap Q) \leq \sum_{q_i \in G_1} \mathcal{H}^{n-1}(\{u = 0\} \cap q_i) + \sum_{q_i \in G_2} \mathcal{H}^{n-1}(\{u = 0\} \cap q_i)
\]

\[
\leq |G_1|\mathcal{F}(E)\frac{diam^{n-1}(Q)}{A_2^{n-1}} + |G_2|\mathcal{F}(E/(1 + c))\frac{diam^{n-1}(Q)}{A_2^{n-1}}
\]

\[
\leq \frac{1}{2}\mathcal{F}(E)diam^{n-1}(Q) + \frac{1}{4}\mathcal{F}(E)diam^{n-1}(Q)
\]

\[
\leq \frac{3}{4}\mathcal{F}(E)diam^{n-1}(Q),
\]

and this contradicts to (5.2). Thus the bad \( E \) is bounded by some \( E_0 \) depending only on \( M, n \) and \( O \).

From the form (5.2), we know that, for \( E > E_0 \), if \( E/(1+c)^k \leq E_0 \) and \( E/(1+c)^{k-1} > \)
$E_0$, then it holds that

$$F(E) \leq 4A_2 F(E/(1 + c))$$

$$\leq (4A_2)^2 F(E/(1 + c)^2)$$

$$\leq \cdots$$

$$\leq (4A_2)^k F(E/(1 + c)^k).$$

Let $\tilde{C}_0 = \sup_{E \leq E_0} F(E)$, and note that $k \leq \log_{1+c}(E/E_0) + 1$, the above inequalities show that

$$F(E) \leq \tilde{C}_0 (4A_2)^{1+\log_{1+c}(E)}$$

$$\leq \tilde{C}_0 \frac{E}{\log_{1+c}(4A_2)}$$

$$\leq \tilde{C}_0' E^\alpha,$$

where $\alpha = \log_{1+c}(4A_2)$ and $\tilde{C}_0' = 4A_2 \tilde{C}_0$. So

$$\mathcal{H}^{n-1}(\{u = 0\} \cap Q) \leq \tilde{C}_0' E^\alpha (diam Q)^{n-1}.$$

Then from Lemma 3.1 with $\epsilon = 1/2$ and $\eta = \eta_0/2$, we can obtain the desired result.

\[ \square \]

6 An upper bound for the nodal set of $u$

In this section we show that $F(E) < +\infty$. More precisely, we will prove that the $n-1$ dimensional Hausdorff measures of nodal sets of such solutions are bounded by the frequency function. In order to show this result, we separate the nodal set of $u$ into
the following four parts.

\[
\begin{align*}
C_1(u) &= \{ x \in M | u(x) = 0, |\nabla u|(x) \neq 0 \}, \\
C_2(u) &= \{ x \in M | u(x) = 0, |\nabla u|(x) = 0, |\nabla^2 u|(x) \neq 0 \}, \\
C_3(u) &= \{ x \in M | u(x) = 0, |\nabla u|(x) = 0, |\nabla^2 u|(x) = 0, |\nabla^3 u|(x) \neq 0 \}, \\
C_4(u) &= \{ x \in M | u(x) = 0, |\nabla u|(x) = 0, |\nabla^2 u|(x) = 0, |\nabla^3 u|(x) = 0 \}.
\end{align*}
\]

From [8], it is known that the dimension of \( C_4(u) \) is at most \( n - 2 \). Thus we only need to consider the upper bounds of the \( n - 1 \) dimensional Hausdorff measures for \( C_1(u), C_2(u) \) and \( C_3(u) \).

We first introduce an important lemma as follows.

**Lemma 6.1.** There exists an \( \eta_0 \in (0, 1/2] \) depending only on \( n \) such that for any \( \eta \in (0, \eta_0] \) and any \( w_i \in C^{1,1/2}(B_1(0)), i = 1, 2, \) with

\[
\begin{align*}
|w_i|_{C^{1,1/2}(B_1(0))} &\leq 1, \quad i = 1, 2, \\
|w_1 - w_2|_{C^{1}(B_1(0))} &\leq \frac{\eta}{2},
\end{align*}
\]

then

\[
\mathcal{H}^{n-1}(B_{1-\eta} \cap \{ w_1 = 0, |\nabla w_1| > \eta \}) \leq (1 + c \sqrt{\eta})\mathcal{H}^{n-1}(B_1 \cap \{ w_2 = 0, |\nabla w_2| > \eta/2 \}),
\]

where \( c \) is a positive constant depending only on \( n \).

This lemma can be seen in [9] and [10].

We now give some notations. Let \( L \) be the operator in \( \mathbb{R}^n \) with the form \( Lu = \sum_{i,j=1}^{n} (a_{ij}(x)u_{ij})_{ij} \). Let \( v = Lu \) and suppose that \( L^2 u = 0 \). Then we write

\[
\begin{align*}
N_1(u)(x_0, r) &= \frac{\int_{B_r(x_0)}(|\nabla v|^2 + |v|^2)dx}{\int_{B_r(x_0)}(u^2 + v^2)dx}, \\
N_2(u)(x_0, r) &= \frac{\int_{B_r(x_0)}(|\nabla^2 v|^2 + |\nabla v|^2)dx}{\int_{B_r(x_0)}(|\nabla v|^2 + |v|^2)dx}.
\end{align*}
\]

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Let \( x_0 \) be a fixed point and without loss of generality, we always assume that \( a_{ij}(x_0) = \delta_{ij} \) and \( |a_{ij}|_{C^1(B_1(0))} \leq \Gamma \), and define \( \omega(y, r) = \sup_{B_r(y)} \sum_{i,j=1}^n (r|\nabla a_{ij}|) \). Then \( \omega(y, r) \leq \Gamma r \), and \( |a_{ij}|_{C^1(B_1(0))} \leq C\omega(0, 1) \), where \( C \) is a positive constant depending only on \( n \). It is easy to see that \( N(x_0, r) \leq CN_1(u)(x_0, r) \), where \( C \) is a positive constant depending only on \( n \) and \( L \).

**Lemma 6.2.** Assume that in \( B_1(0) \) \( Lu = \sum_{i,j=1}^n (a_{ij}(x)u_{x_i})_{x_j} \) with \( a_{ij}(0) = \delta_{ij} \). Suppose that \( \omega(0, 1) \leq \epsilon \), where \( \epsilon \) is a positive constant depending only on \( n \). Also suppose that \( u \) satisfies that \( L^2 u = 0 \) and \( N_1(u)(0, 1) \leq N \) for some positive constant \( N \) large enough.

Then there exists a bi-harmonic function \( \phi \) defined in \( B_{3/4}(0) \) such that

\[
\frac{3}{4} \int_{B_{3/4}(0)} |\nabla \phi|^2 + |\nabla \psi|^2 \, dx \leq CN, \tag{6.2}
\]

and for any \( \alpha \in (0, 1) \),

\[
\begin{align*}
|u|^2_{C^{1,\alpha}(B_{3/4}(0))} + |\nabla u|^2_{C^{1,\alpha}(B_{3/4}(0))} + |\nabla \psi|^2_{C^{1,\alpha}(B_{3/4}(0))} &\leq C(\|u\|^2_{L^2(\partial B_1(0))} + \|\nabla \psi\|^2_{L^2(\partial B_1(0))}), \\
\|u - \phi\|^2_{C^{1,\alpha}(B_{3/4}(0))} + \|v - \psi\|^2_{C^{1,\alpha}(B_{3/4}(0))} &\leq C\epsilon(\|u\|^2_{L^2(\partial B_1(0))} + \|\nabla \psi\|^2_{L^2(\partial B_1(0))}),
\end{align*}
\]

where \( v = Lu, \psi = \Delta \phi \), and \( C \) is a positive constant depending only on \( n, \alpha \) and the operator \( L \).

**Proof.** Without loss of generality, assume that

\[
\int_{\partial B_1(0)} (u^2 + v^2) \, d\sigma = 1.
\]

Since \( Lu = v, Lv = 0 \), from the standard global \( L^2 \) estimate and \( N_1(u)(0, 1) \leq N \), we have

\[
\int_{B_1(0)} (u^2 + v^2) \, dx \leq C, \tag{6.3}
\]

and

\[
\int_{B_1(0)} (|\nabla u|^2 + |\nabla v|^2) \, dx \leq N. \tag{6.4}
\]
Then from (6.3) and the interior $W^{2,p}$ estimate, i.e.,

\[
\begin{align*}
\|u\|_{W^{2,p}(B_{3/4}(0))} & \leq C(\|u\|_{L^2(B_1(0))} + \|v\|_{L^2(B_1(0))}), \\
\|v\|_{W^{2,p}(B_{3/4}(0))} & \leq C\|v\|_{L^2(B_1(0))},
\end{align*}
\]

we obtain that

\[
\begin{align*}
\|u\|_{W^{2,p}(B_{3/4}(0))}^2 + \|v\|_{W^{2,p}(B_{3/4}(0))}^2 & \leq C,
\end{align*}
\]  

(6.5)

where the positive constant $C$ here depending on $n, L$ and $p$.

Define a bi-harmonic function $\phi$ satisfying that

\[
\begin{align*}
\Delta \phi &= \psi, \quad \text{in } B_{3/4}(0), \quad \phi = u, \quad \text{on } \partial B_{3/4}(0), \\
\Delta \psi &= 0, \quad \text{in } B_{3/4}(0), \quad \psi = v, \quad \text{on } \partial B_{3/4}(0).
\end{align*}
\]

Then

\[
\begin{align*}
\int_{B_{3/4}(0)} \nabla \phi \nabla (\phi - u) &= -\int_{B_{3/4}(0)} \psi (\phi - u), \\
\int_{B_{3/4}(0)} \nabla \psi \nabla (\psi - v) &= 0.
\end{align*}
\]

Then we have that

\[
\int_{B_{3/4}(0)} |\nabla \psi|^2 \, dx \leq \int_{B_{3/4}(0)} |\nabla v|^2 \, dx,
\]  

(6.6)

and

\[
\int_{B_{3/4}(0)} |\nabla \phi|^2 \, dx \leq \int_{B_{3/4}} |\nabla u|^2 \, dx + \epsilon \int_{B_{3/4}(0)} (\phi - u)^2 \, dx + \frac{1}{\epsilon} \int_{B_{3/4}(0)} \psi^2 \, dx,
\]  

(6.7)

for any $\epsilon > 0$. Note that $\phi = u$ on $\partial B_{3/4}(0)$, from the Poincare inequality, we have

\[
\int_{B_{3/4}(0)} (\phi - u)^2 \, dx \leq C \int_{B_{3/4}(0)} |\nabla (\phi - u)|^2 \, dx \leq 2C \int_{B_{3/4}(0)} |\nabla \phi|^2 \, dx + 2C \int_{B_{3/4}(0)} |\nabla u|^2 \, dx.
\]  

(6.8)

We also have that

\[
\int_{B_{3/4}(0)} \psi^2 \, dx \leq 2 \int_{B_{3/4}(0)} (\psi - v)^2 \, dx + 2 \int_{B_{3/4}(0)} v^2 \, dx.
\]  

(6.9)
Also from the Poincare inequality, we have
\[
\int_{B_{3/4}(0)} (\psi - v)^2 \, dx \leq C \int_{B_{3/4}(0)} |\nabla (\psi - v)|^2 \, dx
\]
(6.10)
\[
\leq 2C \int_{B_{3/4}(0)} |\nabla \psi|^2 \, dx + 2C \int_{B_{3/4}(0)} |v|^2 \, dx.
\]
Because \( v \) satisfies that \( Lv = 0 \) in \( B_1(0) \), we have that
\[
\int_{B_{3/4}(0)} v^2 \, dx \leq \int_{B_{1}(0)} v^2 \, dx \leq C \int_{\partial B_{1}(0)} v^2 \, d\sigma \leq C.
\]
(6.11)
Put (6.6), (6.8), (6.9), (6.10) and (6.11) into (6.7), we obtain that
\[
\int_{B_{3/4}(0)} |\nabla \phi|^2 \, dx \leq C \left( \int_{B_{3/4}(0)} |\nabla u|^2 \, dx + \int_{B_{3/4}(0)} |\nabla v|^2 \, dx + 1 \right) \leq CN,
\]
(6.12)
because we have assumed that \( N \) is large enough. Then from (6.6) and (6.12), we have
\[
\int_{B_{3/4}(0)} |\nabla \phi|^2 + |\nabla \psi|^2 \, dx \leq CN.
\]
(6.13)
This implies (6.2).

We also have that
\[
\begin{cases}
-\triangle (u - \phi) = ((a_{ij} - \delta_{ij})u_{i})_{i} - (v - \psi), & \text{in } B_{3/4}(0), u - \phi = 0 \text{ on } \partial B_{3/4}(0), \\
-\triangle (v - \psi) = ((a_{ij} - \delta_{ij})u_{i})_{j}, & \text{in } B_{3/4}(0), v - \psi = 0 \text{ on } \partial B_{3/4}(0),
\end{cases}
\]
in \( B_{3/4}(0) \).

Then from the global \( W^{2,p} \) estimate, i.e.,
\[
\begin{align*}
\|u - \phi\|_{W^{2,p}(B_{3/4}(0))} & \leq C \left( \|(a_{ij} - \delta_{ij})u_{i}\|_{L^2(B_{3/4}(0))} + \|v - \psi\|_{L^2(B_{3/4}(0))} \right), \\
\|v - \psi\|_{W^{2,p}(B_{3/4}(0))} & \leq C \|(a_{ij} - \delta_{ij})u_{i}\|_{L^2(B_{3/4}(0))},
\end{align*}
\]
and the inequality (6.5), we have that
\[
\|u - \phi\|_{W^{2,p}(B_{3/4}(0))} + \|v - \psi\|_{W^{2,p}(B_{3/4}(0))} \leq C\omega(0, 1)(\|u\|_{W^{2,2}(B_{3/4}(0))} + \|v\|_{W^{2,2}(B_{3/4}(0))}) \leq C\omega(0, 1).
\]

Then by taking \( p \) large enough and the embedding theorem, we obtain the second inequality of this lemma.  \( \Box \)
By the similar argument as in [16], we have that, if $\phi$ is a bi-harmonic function, and $N_1(\phi)(0,1) \leq N$, then it holds that

$$\mathcal{H}^{n-1}(\{x \in B_{1/2}(0)|\phi(x) = 0\}) \leq CN,$$

where $C$ is a positive constant depending only on the dimension $n$.

Define

$$H_N = \left\{ w \in H^3(B_1(0))|\Delta^2 w = 0, \int_{B_{1/2}(0)} |\nabla w|^2 + |\nabla \Delta w|^2 dx \leq N, \int_{\partial B_{1/2}(0)} w^2 + (\Delta w)^2 d\sigma = 1 \right\}.$$

Then we have the following lemma which has a similar version in $H^1$ space in [9] and [10].

**Lemma 6.3.** The set $H_N$ is compact in local $L^2$ norm and local $C^k$ norm for any $k \geq 0$.

**Proof.** For any $w \in H_N$, from the doubling conditions, we have that

$$\int_{\partial B_1} (w^2 + (\Delta w)^2) dx \leq C(N).$$

Then from the global $W^{3,2}$ estimate and the definition of $H_N$, we know that

$$\|w\|_{H^3(B_1)}^2 \leq C'(N),$$

which shows that, for any sequence $\{w_m\}$ in $H_N$ has a uniform $H^3$ bound in $B_1$. Then there exists a subsequence $\{w_{m'}\}$ and a $w_0 \in H_N$ such that $w_{m'}$ converge to $w_0$ strongly in $L^2(B_1)$; and $w_{m'}$ converge to $w_0$ weakly in $H^3(B_1)$. It is obvious that $w_0$ is a bi-harmonic function, and then from the interior estimates for bi-harmonic functions, we get that for any integer $k \geq 0$, and any $r \in (0,1)$, it holds that $w_{m'}$ converge to $w_0$ in $C^k(B_r)$. This implies that

$$\int_{B_{1/2}} (w_0^2 + (\Delta w_0)^2) d\sigma = 1, \quad \text{and} \quad \int_{B_r} (|\nabla w_0|^2 + |\nabla (\Delta w_0)|^2) dx \leq N,$$

for any $r \in (0,1)$. Thus we obtain that $w_0 \in H_N$. \qed
Lemma 6.4. For any bi-harmonic function \( w \in H_N \) in \( B_1(0) \), there exist positive constant \( \gamma(N) \) depending only on \( n \) and \( N \), and finitely many balls \( B_r(x_i) \) with \( r_i \leq \frac{1}{2} \) and \( x_i \in \{ x \in B_{1/2}(0) : w(x) = 0, |\nabla w(x)| = 0 \} \cap \overline{C}_1(w) \), such that

\[
\{ x \in B_{1/2}(0) : |\nabla w(x)| < \gamma(N) \} \cap \overline{C}_1(w) \subseteq \bigcup_i B_{r_i}(x_i),
\]

and

\[
\sum_i r_i^{n-1} < 1/2.
\]

Proof. Take any \( w_0 \in H_N \). We note that \( \{|\nabla w_0| = 0\} \cap \overline{C}_1(w_0) \subseteq \partial \overline{C}_1(w_0) \). Because the dimension of the set \( \overline{C}_1(w_0) \) is at most \( n-1 \), the dimension of \( \partial \overline{C}_1(w_0) \) is at most \( n-2 \), it holds that

\[
\mathcal{H}^{n-1}(\{|\nabla w_0| = 0\} \cap \overline{C}_1(w_0)) = 0.
\]

Then there exist countability many balls \( B_{r_i}(x_i) \) with \( r_i \leq 1/2 \) such that

\[
\{|\nabla w_0| = 0\} \cap \overline{C}_1(w_0) \subseteq \bigcup_i B_{r_i}(x_i), \tag{6.14}
\]

and

\[
\sum_i r_i^{n-1} \leq \frac{1}{2}.
\]

Set

\[
\gamma(w_0) = \frac{1}{3} \inf_{x \in B_{1/2}(0) \cup \bigcup_i B_{r_i}(x_i)} |\nabla w_0|(x).
\]

From (6.14), it is obviously that \( \gamma(w_0) > 0 \). Consider any \( w \in H_N \), such that \( |w - w_0|_{C^1(B_{1/2}(0))} \leq \gamma(w_0) \). Then

\[
\{ x \in B_{1/2}(0) : |\nabla w|(x) < \gamma(w_0) \} \cap \overline{C}_1(w) \subseteq \bigcup_i B_{r_i}(x_i).
\]

Because \( H_N \) is compact under the local \( C^1 \) norm, we know that there exist \( w_1, w_2, \ldots, w_m \in H_N \) such that \( H_N \subseteq \bigcup_{i=1}^m B_{r_i}(w_i) \). Then we obtain the desired result by setting \( \gamma(N) = \min_{1 \leq i \leq m} \gamma(w_i) \).

\[\square\]
We now give the relationship between $N_1(u(x_0, r))$ and $N_2(u(x_0, r))$ when $x_0$ satisfies that $u(x_0) = 0$.

**Lemma 6.5.** If $L^2 u = 0$ and $N_1(0, \frac{1}{2}) > N_0$, where $N_0$ is a positive constant depending on $n$ and $L$, then it holds that

$$N_2(u)(0, \frac{1}{2}) \leq 2^{C_{N_1(u)(0,1)}}. \quad (6.15)$$

Here $C$ is also a positive constant depending on $n$ and $L$.

**Proof.**

$$N_2(u)(x_0, r) = \frac{\int_{B_r(x_0)} (|\nabla^2 u|^2 + |\nabla^2 v|^2)dx}{\int_{\partial B_r(x_0)} (|\nabla u|^2 + |\nabla v|^2) d\sigma} = Cr^2 \frac{\int_{B_r(x_0)} (|\nabla^2 u|^2 + |\nabla^2 v|^2)dx}{\int_{\partial B_r(x_0)} (|\nabla u|^2 + |\nabla v|^2) d\sigma}.$$

Then

$$N_2(u)(0, 1) = C \frac{\int_{B_1(0)} (|\nabla^2 u|^2 + |\nabla^2 v|^2)dx}{\int_{\partial B_1(0)} (|\nabla u|^2 + |\nabla v|^2) d\sigma}.$$

From the standard $W^{2,2}$ interior estimate, we know that

$$\int_{B_{1/2}(0)} (|\nabla^2 u|^2 + |\nabla^2 v|^2)dx \leq C \int_{B_1(0)} (u^2 + v^2)dx.$$

Now we claim that

$$\int_{\partial B_{1/2}(0)} (|\nabla u|^2 + |\nabla v|^2) d\sigma \geq C \left( \int_{B_{1/2}(x_0)} (u^2 + v^2)dx \right).$$

If not, then for any $C > 0$, there exist $u$ and $v$ satisfying that $Lu = v$, $Lv = 0$, and

$$\int_{\partial B_{1/2}(0)} (|\nabla u|^2 + |\nabla v|^2) d\sigma \leq C \left( \int_{B_{1/2}(0)} (u^2 + v^2)dx \right). \quad (6.16)$$

But from the global $L^2$ estimate, we have

$$\int_{B_{1/2}(0)} (|\nabla u|^2 + |\nabla v|^2)dx \leq C \int_{\partial B_{1/2}(0)} (|\nabla u|^2 + |\nabla v|^2) d\sigma,$$

and

$$\int_{B_{1/2}(x_0)} (u^2 + v^2)dx \leq C \int_{\partial B_{1/2}(x_0)} (u^2 + v^2)dx.$$

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Thus we have that
\[ N_1(u)(0, \frac{1}{2}) \leq C', \]
where \( C' \) is a positive constant depending on \( n, L \) and the constant \( C \) in (6.16), which is a contradiction to the assumption that \( N_1(u)(0, \frac{1}{2}) \geq N_0 \) for large enough \( N_0 \) depending only on \( n \) and \( L \). Then from Lemma 2.24, we will get the desired result. \( \square \)

**Lemma 6.6.** Suppose that \( L \) is a linear operator in \( B_1(0) \) with the form \( Lw = (a_{ij}w_i)_j \) with \( a_{ij}(0) = \delta_{ij} \) and \( \omega(0, 1) \leq \epsilon \), where \( \epsilon \) is a positive constant depending only on \( n \). Let \( u \) satisfying that \( L^2u = 0 \) in \( B_1 \), and set \( v = Lu \). Also assume that \( N_2(u)(0, 1) \leq N \) for some positive constant \( N \) large enough. Then there exists a vector valued function \( \phi = (\phi_1, \phi_2, \cdots, \phi_n) \) such that \( \phi_i \) is bi-harmonic for each \( i = 1, 2, \cdots, n \), defined in \( B_{3/4}(0) \) with \( \psi = \Delta \phi \), such that
\[
3 \int_{B_{3/4}(0)} |\nabla \phi|^2 + |\nabla \Delta \phi|^2 \, dx \leq CN, \tag{6.17}
\]
and for any \( \alpha \in (0, 1) \),
\[
\left\|
\begin{array}{l}
|\nabla u|_{C^{1,\alpha}(B_{3/4}(0))}^2 + |\phi|_{C^{1,\alpha}(B_{3/4}(0))}^2 + |\nabla v|_{C^{1,\alpha}(B_{3/4}(0))}^2 + |\psi|_{C^{1,\alpha}(B_{3/4}(0))}^2 \\
|\nabla u - \phi|_{C^{1,\alpha}(B_{3/4}(0))}^2 + |\nabla v - \psi|_{C^{1,\alpha}(B_{3/4}(0))}^2 \\
\end{array}
\right\| \leq C\left(\|\nabla u\|^2_{L^2(\partial B_1(0))} + \|\nabla v\|^2_{L^2(\partial B_1(0))}\right),
\]
where \( C \) is a positive constant depending only on \( n, \alpha \) and the operator \( L \).

The proof is similar to that of Lemma 5.2; we omit it.

**Lemma 6.7.** Suppose \( N \) is a positive integer. Then for \( k = 1, 2, 3 \), there exists a positive constant \( \omega_N \) such that for any \( y \in B_{1/2} \) and \( r \leq 1/2 \) if \( \omega(y, 2r) \leq \omega_N \) where \( \omega_N \) is a positive constant depending on \( n, L \) and \( N \), then there exist finitely many balls \( \{B_k^i\} \) in
\( B_{2r}(y) \) with \( \text{rad}(B^k_i) \leq r/2 \), where \( \text{rad}(B^k_i) \) is the radius of the ball \( B^k_i \), for each \( i \) and \( k \), such that the center of \( B^k_i \) is in \( \overline{C}_k \setminus C_k \).

\[
B_r(y) \cap \overline{C}_k(u) \setminus C_k(u) \subseteq \cup_i B^k_i,
\]

\[
\mathcal{H}^{n-1}(C_k(u) \cap B_r(y) \setminus \cup_i B^k_i) \leq CNr^{n-1},
\]

and

\[
\sum_i (r^k_i)^{n-1} \leq \frac{1}{2} r^{n-1}.
\]

**Proof.** We first focus on the case that \( k = 1 \). Consider the transformation \( x \rightarrow y + 2rz \).

Then by \( L^2 u = 0 \), we have that \( \overline{L}^2 \overline{u} = 0 \) in \( B_1 \), where

\[
\overline{L} = (\overline{a}_{ij}(z) \partial_i)_j = (a_{ij}(y + 2rz) \partial_i)_j,
\]

and

\[
\overline{u}(z) = u(y + 2rz)/\|u\|_{L^2(\partial B_{2r}(y))}.
\]

Then \( \sup_{B_1} (|\nabla \overline{a}_{ij}|) = \sup_{B_{2r}(y)} 2r(\|\nabla a_{ij}\|) = \omega(y, 2r) \). By Lemma 6.2 and \( \|\overline{u}\|_{L^2(\partial B_1)} = 1 \), \( \|\nabla \overline{u}\|_{L^2(B_1)} \leq C\sqrt{N} \), we can find a function \( \phi \) such that

\[
L^2_0 \phi = 0 \quad \text{in} \quad B_{3/4},
\]

with \( L_0 = (\overline{a}_{ij} \partial_i)_j \),

\[
3 \int_{B_{3/4}(0)} \|
abla \phi\|^2 + |\nabla \Delta \phi|^2 dx \\
4 \int_{\partial B_{3/4}(0)} \phi^2 + (\Delta \phi)^2 d\sigma \leq CN,
\]

and

\[
\begin{align*}
|\overline{u}|_{C^{1,1/2}(B_{3/4})} + |\phi|_{C^{1,1/2}(B_{3/4})} & \leq C, \\
|\overline{u} - \phi|_{C^{1}(B_{3/4})} & \leq C\omega(y, 2r).
\end{align*}
\]

So by choosing \( r \) small enough, we have

\[
\mathcal{H}^{n-1}(C_1(u) \cap B_{1/2} \cap \{ |\nabla \overline{u}| > 2^{-CN} \gamma(N) \}) \leq C\mathcal{H}^{n-1}(\{ \phi = 0 \} \cap B_{2/3}).
\]
Because $\phi$ is an analytic function, we can get that
\[
\mathcal{H}^{n-1}(\phi = 0 \cap B_{2/3}) \leq CN,
\]
and thus
\[
\mathcal{H}^{n-1}(C_1(u) \cap B_{1/2} \setminus \bigcup_i B_{\bar{r}_i}(\bar{x}_i)) \leq CN.
\]

In the second step, we consider $k = 2$.

Also consider the transformation $x \rightarrow y + 2rz$. Then because $L^2u = 0$ in $B_2(y)$, we have that $\bar{L}^2(\nabla \bar{u}) = 0$ in $B_1$, where $\bar{u}(z) = u(y + 2rz)$. If $|\nabla^2 \bar{u}|(y) \geq \gamma(N)$, then by changing variables, we have that $|\nabla u_i|(y) \geq c(n)\gamma(N)$, where $u_i$ is the partial derivative of $u$ to $z_i$, $i = 1, 2, \cdots, n$. Thus from the interior estimate, we have that for any $i$, $\sup_{B_{\bar{r}_i}} |\nabla^2 \bar{u}| \geq c'(n)\gamma(N)$ for each $i = 1, 2, \cdots, n$. It is also true that there exists some $i = 1, 2, \cdots, n$, such that $\int_{B_{\bar{r}_i}} \left( |\nabla \bar{u}_i|^2 + |\nabla \bar{v}_i|^2 \right) dx \leq N$, where $\bar{r}$ is a positive constant depending on $n$ and $L$, it holds that
\[
\mathcal{H}^{n-1}(\overline{C_2 \cap B_{1/2} \cap \{|\nabla^2 \bar{u}| > 2^{CN}\gamma(N)\}}) \leq \mathcal{H}^{n-1}(\{\bar{u}_i = 0\} \cap B_{1/2} \cap \{|\nabla u_i| > 2^{-C'N}\gamma(N)\})) \leq C\mathcal{H}^{n-1}(\{\phi_i = 0\}) \leq CN.
\]

Then we obtain the desired result by transforming $B_{1/2}$ back to $B_r(y)$ by $z \rightarrow \frac{x-y}{2r}$.

For $k = 3$, we only need to note that
\[
\overline{C_3(u) \setminus C_1(u + v)} \subseteq \{v(x) = 0, |\nabla v|(x) = 0\},
\]
and
\[
\overline{C_3(u) \setminus C_1(u - v)} \subseteq \{v(x) = 0, |\nabla v|(x) = 0\}.
\]

Because $v$ satisfies that $Lv = 0$, the dimension of the set $\{v(x) = 0, |\nabla v|(x) = 0\}$ is at most $n - 2$. This conclusion can be seen in [12] or in [9]. So we only need to consider the set $C_1(u + v)$ or $C_1(u - v)$. Then from the proof of the first case, we have that
\[
\mathcal{H}^{n-1}(B_r(y) \cap C_3(u)) \leq C\bar{N}r^{n-1},
\]
where $\tilde{N} = \min(N_1(u + v)(y, 2r), N_1(u - v)(y, 2r))$. Because

$$N_1(u + v)(y, r) = \frac{\int_{B_r(y)} (|\nabla (u + v)|^2 + |v|^2) dx}{\int_{\partial B_r(y)}((u + v)^2 + v^2)d\sigma},$$

$$N_1(u - v)(y, r) = \frac{\int_{B_r(y)} (|\nabla (u - v)|^2 + |v|^2) dx}{\int_{\partial B_r(y)}((u - v)^2 + v^2)d\sigma},$$

we have that

$$\min(N_1(u + v)(y, r), N_1(u - v)(y, r)) \leq 2N_1(u)(y, r).$$

Then we have that

$$H^{n-1}(B_r(y) \cap C_3(u)) \leq CN_1(u)(y, 2r),$$

Which is the desired result for $k = 3$.

□

By an iteration argument, we can get that

**Lemma 6.8.** Suppose that $u$ is a nonzero solution of $L^2u = 0$ in $B_1(0)$ satisfying that $N(0, 1) \leq N$. Then

$$H^{n-1}(C(u) \cap B_{1/2}(0)) \leq \sum_{k=1}^{3} H^{n-1}(C_k(u) \cap B_{1/2}(0)) \leq C(N),$$

where $C(N)$ is a positive constant depending on $n, L$ and $N$.

That lemma gives an upper bound for the nodal set of $u$. And this is the desired result of this section.

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