SUPERWEYL COCYCLE IN $d=4$ AND SUPERCONFORMAL-INVARIANT OPERATOR

R.P.Manvelyan

Theoretical Physics Department,
Yerevan Physics Institute
Alikhanyan Br. st.2, Yerevan, 375036 Armenia

Abstract

The super-Weyl cocycle (effective action for supertrace anomaly) and corresponding invariant operator in nonminimal formulation of $d = 4, N = 1$ supergravity are obtained.
1 Introduction.

In previous papers \cite{1} it was shown that the Weyl cocycle in $d = 4, 6$ has a form of Liouville action in $d = 2$ \cite{2}:

$$S = \int d^d x \sqrt{g} \left[ \sigma \Delta_d \sigma + \sigma \times \text{Anomaly} \right]$$ \hspace{1cm} (1.1)

where $\Delta_d$ is the zero weight conformal-invariant operator of $d$-th order on derivatives. This is true also for $d = 2$ supersymmetric Weyl cocycle \cite{3} (Super-Liouville action). In this letter we consider the supersymmetric extension of Weyl cocycle in $d = 4$, using the set of the superspace constraints, leading to non-minimal formulation of $d = 4, N = 1$ supergravity \cite{4}. The different formulations of superfield supergravity in $d = 4$ differ by their auxiliary-field structure and are parametrized by real number parameter $n$ in the generalized superconformal constraint \cite{4} \cite{5} (we use the notation of ref. \cite{6}).

$$T_{\alpha\beta}^C = T_{\alpha\dot{\beta}}^C + 2i\sigma_{\alpha\beta}^{\dot{\gamma}} = T_{ab}^c = T_{\alpha a}^b - \frac{1}{4}\sigma_{\alpha\beta}^{\dot{\gamma}\dot{\delta}} T_{\gamma\delta} = 0;$$

$$n \neq -\frac{1}{3} \Rightarrow R - \frac{n + 1}{3n + 1} \nabla_{\dot{\alpha}} T_{\dot{\alpha}} + \left( \frac{n + 1}{3n + 1} \right)^2 T_{\alpha\beta} T_{\dot{\alpha}} = 0,$$ \hspace{1cm} (1.2)

$$n = -\frac{1}{3} \Rightarrow T_{\alpha} = 0$$

where

$$T_{\alpha} = T_{\alpha a}^a, \hspace{0.5cm} R = \frac{1}{12}\sigma_{\alpha}^{\gamma\delta}\sigma_{\beta\gamma}^{\dot{\delta}} R_{\alpha\beta}$$ \hspace{1cm} (1.3)

If $n = -1/3$ we have the minimal Wess-Zumino \cite{7} formulation of supergravity with Howe and Tucker’s \cite{8} superconformal group,

$$E_{\alpha}^M \rightarrow e^{\frac{1}{2}(2\Sigma - \Sigma)} E_{\alpha}^M$$

$$\nabla_{\dot{\alpha}} \Sigma = 0, \hspace{0.5cm} \nabla_{\alpha} \Sigma = 0$$ \hspace{1cm} (1.4)

with chiral conformal parameter $\Sigma$. The effective action for this formulation was constructed in 1988 by I.L.Buchbinder and S.M.Kuzenko \cite{9}, but this effective action is not of the form:

$$\int \text{Anomaly} \frac{1}{\Delta_d} \text{Anomaly}$$ \hspace{1cm} (1.5)
and corresponding cocycle is not second order on super-Weyl parameter. Therefore in this formulation there is no room for the supersymmetric extension of our construction for conformal invariant operator. So in our paper we use $n = -1$ formulation of supergravity constraints with superconformal transformation including linear scalar superfield parameter [4, 5].

$$E^M_\alpha \rightarrow e^L E^M_\alpha$$

$$\nabla_\alpha \nabla^{\dot{\alpha}} L = 0, \quad \nabla_\alpha \nabla^{\alpha} L = 0$$

In this formulation we can write down quadratic supercocycle and superconformal invariant fourth-order differential operator, which is the supersymmetric extension of corresponding Weyl invariant operator obtained in the work of Riegert [10].

$$\sqrt{g} \Delta_4 = \sqrt{g} (\Box^2 - 2 \hat{R}^{\mu \nu} \hat{\nabla}_\mu \hat{\nabla}_\nu + \frac{2}{3} \hat{R} \Box - \frac{1}{3} (\hat{\nabla}^{\mu} \hat{R}) \hat{\nabla}_\mu)$$

where $\hat{\nabla}$ and $\hat{R}$ are ordinary covariant derivative and curvature.

The 1-supercocycle of super-Weyl group is the super-Weyl variation of corresponding anomalous effective action for superconformal matter superfield in external supergravity:

$$S(L, E^M_\alpha) = W(e^L E^M_\alpha) - W(E^M_\alpha)$$

and has to satisfy the cocyclic property:

$$S(L_1 + L_2; E^M_\alpha) = S(L_1; e^{L_2} E^M_\alpha) + S(L_2; E^M_\alpha)$$

Finally, let’s note that from (1.9) the term of the highest-order over group parameter $L$ in the cocycle has to be super-Weyl invariant [1]. Therefore the quadratic over $L$ invariant action containing (super)conformal-invariant operator $\Delta$ leads to the following cocycle:

$$S \sim \int \left( \frac{1}{2} L \Delta L - A(E)L \right)$$

where $A(E)$ is the possible contribution to the super-Weyl anomaly [1] satisfying the well-known Wess-Zumino consistency condition [11], which is the algebraic version of cocyclic property. We can obtain the linear part $A(E)L$ solving the equation which follows from (1.9) and (1.10)

$$\Omega \Delta L = \delta_\Omega A(E)L$$

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Here $\delta_\Omega$ is super-Weyl transformation with infinitesimal linear superfield parameter $\Omega$. 

2
2 Supercocycle and superconformal-invariant operator

Firstly let’s remind that in the $n = -1$ formulation of supergravity constraints we can express all supertorsions and supercurvatures in terms of following superfields (due to (1.2), and Bianchi identities):

$$T_\alpha, \quad G_a = \frac{1}{12} \sigma^\alpha_\beta \gamma (R^\gamma_{\beta\alpha\gamma} + \tilde{R}^\gamma_{\alpha\beta\gamma}), \quad W_{\alpha\beta\gamma} = \sigma^a_{(\alpha\dot{\beta}} T_{a\gamma)}$$  \hspace{1cm} (2.1)

where

$$\nabla_\alpha T_\beta = 0, \quad G_a = \tilde{G}_a, \quad \left( \nabla_{\dot{\alpha}} - \frac{1}{2} \tilde{T}_{\dot{\alpha}} \right) W_{\alpha\beta\gamma}, \quad \sigma^a_{\alpha\dot{\beta}} T_{aD}^D = \frac{i}{4} (\nabla_{\dot{\beta}} T_a + \nabla_a T_{\dot{\beta}}), \quad T_{\alpha B}^B = T_A$$  \hspace{1cm} (2.2)

This set of superfields has the following local superscale transformations [4]:

$$\delta_\Omega W_{\alpha\beta\gamma} = 3 \Omega W_{\alpha\beta\gamma}, \quad \delta_\Omega T_a = \Omega T_a + 6 \nabla_a \Omega$$  \hspace{1cm} (2.3)

$$\delta_\Omega G_a = 2 \Omega G_a + \frac{1}{4} \sigma^a_{\alpha\dot{\beta}} \left[ \nabla_{\beta}, \nabla_a \right] \Omega$$

Beside that we can derive the infinitesimal form of transformations of the supercovariant derivatives acting on the zero weight scalar linear superfield [4, 5]:

$$\delta_\Omega \nabla_\alpha = \Omega \nabla_\alpha, \quad \delta_\Omega \nabla_{\dot{\alpha}} = \Omega \nabla_{\dot{\alpha}}$$

$$\delta_\Omega \nabla_{\alpha\dot{\alpha}} = 2 \Omega \nabla_{\alpha\dot{\alpha}} - \frac{i}{2} \left( \nabla_a \Omega \nabla_{\dot{\alpha}} + \nabla_{\dot{\alpha}} \Omega \nabla_a \right)$$  \hspace{1cm} (2.4)

where $\nabla_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}} a \nabla_a$

From (2.3) and (2.4) it follows that the supertorsion component $T_a$ plays a role of the gauge field for the super-Weyl transformation. The presence of this phenomenon in $n = -1$ formulation leads us to a very clear construction of superconformal-invariant operator. In the minimal ($n = -1/3$) formulation (and in nonsupersymmetric case too) there is not such a gauge field. Owing to this it is easy to define from (2.3) and (2.4) super-Weyl covariant superderivative $D_{\alpha\dot{\beta}}$:

$$D_{\alpha\dot{\alpha}} = \nabla_{\alpha\dot{\alpha}} - \frac{i}{12} (T_a \nabla_{\dot{\alpha}} + T_{\dot{\alpha}} \nabla_a)$$  \hspace{1cm} (2.5)

$$\delta_\Omega D_{\alpha\dot{\alpha}} = 2 \Omega D_{\alpha\dot{\alpha}}$$
acting on a scalar linear superfield of zero superconformal weight. Two other
supercovariant derivatives $\nabla_{\alpha}, \nabla_{\dot{\alpha}}$ are automatically superscale covariant on
the zero weight scalar. Then we can conclude that the following expression:
\[
\frac{1}{2} E^{-1} D_{\alpha \dot{\alpha}} L D^{\alpha \dot{\alpha}} L
\]  
(2.6)
is superconformal invariant
\[
\delta_{\Omega} \left[ \frac{1}{2} E^{-1} D_{\alpha \dot{\alpha}} L D^{\alpha \dot{\alpha}} L \right] = 0
\]  
(2.7)
and has to be the highest term of our cocycle. The linear on $L$ term we can
construct from the cocyclic property (1.11) and from the following formula:
\[
\delta_{\Omega} \left[ E^{-1} \frac{1}{12} \left( \nabla_{\alpha} T_{\dot{\alpha}} + \nabla_{\dot{\alpha}} T_{\alpha} \right) D^{\alpha \dot{\alpha}} L \right] = E^{-1} D_{\alpha \dot{\alpha}} \Omega D^{\alpha \dot{\alpha}} L
\]  
(2.8)
Then we can write out super-Weyl cocycle comparing (2.8) and (2.7) to
(1.10),(1.11):
\[
S = \int d^8 z E^{-1} \left[ \frac{1}{2} D_{\alpha \dot{\alpha}} L D^{\alpha \dot{\alpha}} L - \frac{1}{12} \left( \nabla_{\alpha} T_{\dot{\alpha}} + \nabla_{\dot{\alpha}} T_{\alpha} \right) D^{\alpha \dot{\alpha}} L \right]
\]  
(2.9)
After that we can obtain the expressions for $\Delta^{SUSY}_4$ and $A^{SUSY}_4$ after
integration by parts in (2.9) using the following relation:
\[
\int d^8 z E^{-1} \nabla C V^C (-1)^C = \int d^8 z E^{-1} V^C T_{CD} (-1)^D \neq 0
\]  
(2.10)
in $n = -1$ formulation due to (2.2). It leads us to the following final results:
\[
\Delta^{SUSY}_4 = D_{\alpha \dot{\alpha}} D^{\alpha \dot{\alpha}} + \frac{1}{3} \left( \nabla_{\alpha} T_{\dot{\alpha}} + \nabla_{\dot{\alpha}} T_{\alpha} \right) D^{\alpha \dot{\alpha}}
\]  
(2.11)
\[
A^{SUSY}_4 = \frac{1}{36} \left( \nabla_{\alpha} T_{\dot{\alpha}} + \nabla_{\dot{\alpha}} T_{\alpha} \right) \left( \nabla^{\alpha \dot{\alpha}} + \nabla^{\dot{\alpha} T_{\alpha}} \right)
\]  
(2.12)
\[
- \frac{1}{12} D_{\alpha \dot{\alpha}} \left( \nabla^{\alpha \dot{\alpha}} + \nabla^{\dot{\alpha} T_{\alpha}} \right)
\]
This operator is actually fourth-order (on the component level),over space-
time derivatives, because the linear scalar multiplet $L$ includes the second
derivatives of its first component.
In conclusion let’s note that owing to (2.10) the expression (2.12) obtaining after integration by parts is not full derivative and has to include non-trivial contribution in anomaly (super-Euler characteristic density).

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