Approximate Deconvolution Model in a bounded domain with a vertical regularization

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Abstract

We study the global existence issue for a three-dimensional Approximate Deconvolution Model with a vertical filter. We consider this model in a bounded cylindrical domain where we construct a unique global weak solution. The proof is based on a refinement of the energy method given by Berselli in \cite{3}.

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1 Introduction and notation

The Large eddy simulation model (LES) with anisotropic regularization is introduced in \cite{3} in order to study flows in bounded domains. Instead of choosing the classical filter

\[ \mathbb{A} := I - (\alpha_1^2 \partial_1^2 + \alpha_2^2 \partial_2^2 + \alpha_3^2 \partial_3^2), \tag{1.1} \]

the author in \cite{3}, considers a horizontal filter

\[ \mathbb{A}_h := I - (\alpha_1^2 \partial_1^2 + \alpha_2^2 \partial_2^2). \tag{1.2} \]

This filter is less memory consuming than the classical one \cite{7, 6, 8}. Moreover, there is no need to introduce artificial boundary conditions for Helmholtz operator. However, the smoothing created by both of these filters is unnecessary strong. For this reason one can replace the classical Laplace operator by the Laplace operator with fractional regularization $\theta$ and seek for the limiting case where we can prove global existence and uniqueness of regular solutions. In a series of papers \cite{10, 4, 1, 2}, it was shown that the LES models which are derived by using instead of the operator (1.1), the operator $\mathbb{A}_\theta = I + \alpha^{2\theta} (-\Delta)^{\theta}$ for suitable (large enough) $\theta < 1$, are still well-posed.

In this paper, we consider LES models with fractional filter acting only in one variable

\[ \mathbb{A}_{3,\theta} := I + \alpha_3^{2\theta} (-\partial_3)^{2\theta}, \quad 0 \leq \theta \leq 1. \tag{1.3} \]

In particular, we study the global existence and uniqueness of solutions to the vertical LES model on a bounded product domain of the type $D = \Omega \times (-\pi, \pi)$, where $\Omega$ is a smooth domain, with homogeneous Dirichlet boundary conditions on the lateral boundary $\partial \Omega \times (-\pi, \pi)$, and with periodic boundary conditions in the vertical variable. For simplicity, we consider the domain $D = \{ x \in \mathbb{R}^3, x_1^2 + x_2^2 < d, -\pi < x_3 < \pi \}$ with $2\pi$ periodicity with respect to $x_3$. More precisely, we consider the system of equations
\[
\begin{cases}
\partial_t w + \nabla \cdot (v \otimes w) - \nu \Delta w + \nabla q = f, \quad \text{in } D, \ t > 0, \\
\nabla \cdot w = 0, \quad \text{in } D, \ t > 0, \\
w|_{\partial \Omega \times (-\pi, \pi)} = 0, \ t > 0, \\
w(x_1, x_2, x_3) = w(x_1, x_2, x_3 + 2\pi), \ t > 0, \\
w(x, 0) = w_0(x) = \overline{w}_0, 
\end{cases}
\] (1.4)

We recall that, in the above equations, the symbol \(\overline{w}\) denotes the vertical filter (1.3), applied component-by-component to the various tensor fields. Given that the filter is acting only in the vertical variable, it is possible to require the periodicity only in \(x_3\). Moreover, we consider the filtered function with homogeneous Dirichlet boundary conditions on the lateral boundary \(\partial \Omega \times (-\pi, \pi)\). These boundary conditions of the filtered function are supposed to be the same as the unfiltered ones, in order to prevent from introducing artificial boundary conditions.

The use of the vertical filter can be extended to a family of Approximate Deconvolution models. The deconvolution family including (1.4) as the zeroth order case, is given by

\[
\begin{cases}
\partial_t w + \nabla \cdot D_{N,\theta}(w) \otimes D_{N,\theta}(w) - \nu \Delta w + \nabla q = f, \quad \text{in } D, \ t > 0, \\
\nabla \cdot w = 0, \quad \text{in } D, \ t > 0, \\
w|_{\partial \Omega \times (-\pi, \pi)} = 0, \ t > 0, \\
w(x_1, x_2, x_3) = w(x_1, x_2, x_3 + 2\pi), \ t > 0, \\
w(x, 0) = w_0(x) = \overline{w}_0, 
\end{cases}
\] (1.5)

where \(D_{N,\theta}\) is a deconvolution operator of order \(N\) and is given by

\[
D_{N,\theta} = \sum_{i=0}^N (I - A_{3,\theta}^{-1})^i, \quad 0 \leq \theta \leq 1.
\] (1.6)

In order to state our main result pertaining to System (1.5), let us introduce our notation. We set:

\[
L^2(D) := \left\{ v \in L^2(D)^3, \text{2\pi-periodic in } x_3, \text{and } \int_{-\pi}^\pi v dx_3 = 0 \right\},
\] (1.7)

endowed with the norm \(\|v\|_2 := \int_D v \cdot v \, dx\) which is the usual norm in \(L^2(D)^3\).

Then, we define the following spaces that are commonly used in the study of the NSE:

\[
H := \left\{ v \in L^2(D), \text{ such that } \nabla \cdot v = 0 \text{ and } v \cdot n = 0 \text{ on } \partial D \right\},
\] (1.8)

\[
V := \left\{ v \in H, \text{ such that } \nabla v \in L^2(D) \text{ and } v = 0 \text{ on } \partial D \right\}.
\] (1.9)

Our main result is the following.

**Theorem 1.1.** Assume \(f \in L^2(0, T; H)\) be a divergence free function and \(v_0 \in H\). let \(0 \leq N < \infty\) be a given and fixed number and let \(\theta > \frac{1}{2}\). Then problem (1.5), including problem (1.4) when \(N = 0\), has a unique regular weak solution.

This result is one of the few results that consider the LES model on a bounded domain. It holds also true on the whole space \(\mathbb{R}^3\) and on the torus \(T_3\). Other \(\alpha\) models, with partial filter, will be reported in a forthcoming paper.

The remaining part of the paper is organized as follows. In the second section, we introduce several notations and spaces. We also prove auxiliary results which will be used in the third section. In the third section, we prove Theorem 1.1. Finally, a functional inequality is given in the appendix.
2 Preliminaries and auxiliary results

Let $1 < p \leq +\infty$ and $1 < q \leq +\infty$. We denote by $L^q_0 L^p_h(D) = L^q((-\pi, +\pi); L^p(\Omega))$ the space of functions $g$ such that $\int_{-\pi}^{+\pi} \left( \int_\Omega |g(x_1, x_2, x_3)|^p dx_3 dx_2 \right)^{\frac{q}{p}} < +\infty$. In what follows, we denote by $\nabla_h$ the gradient operator with respect to the variables $x_1$ and $x_2$. Of course, we have $\|\nabla_h v\|_2 \leq \|\nabla v\|_2$, for any $v \in V$.

Next, we will give some preliminary results which will play an important role in the proof of the main theorem.

**Lemma 2.1.** For any $u \in L^2(D)^3$ satisfying homogeneous Dirichlet boundary conditions on the boundary $\partial \Omega \times (-\pi, \pi)$, such that $\nabla_h u \in L^2(D)^3$, we have the following estimate

$$\|u\|_{L^2_h L^2(D)^3} \leq C \|u\|_2 \|\nabla_h u\|_2^{\frac{3}{2}}$$

**(Proof.** See in [1].)**

**Lemma 2.2.** Let $g$ be a $2\pi$-periodic and mean zero function with $g(x) \in H^s([-\pi, \pi])$, and let $s > \frac{1}{2}$, then we have the following inequality

$$\|g\|_{L^\infty} \leq C \|g\|_2^{\frac{1}{2}} \|g\|_{H^s}.$$  

**(Proof.** The proof is given in Appendix.)

**Lemma 2.3.** Let $s > \frac{1}{2}$, and suppose that $u$ and $\partial^s u \in L^2(D)^3$, then there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty L^2(D)^3} \leq C \|u\|_2^{\frac{1}{2}} \|\partial^s u\|_2^{\frac{1}{2}}.$$  

**(Proof.** We first apply inequality (2.2) in the vertical variable and then use the Hölder inequality in the horizontal variable. We get,

$$\sup_{x_3 \in [-\pi, \pi]} \left( \int_\Omega |u(x)|^2 dx_1 dx_2 \right)^{\frac{1}{2}} \leq \left( \int_\Omega \sup_{x_3 \in [-\pi, \pi]} |u(x)|^2 dx_1 dx_2 \right)^{\frac{1}{2}} \left( \int_{-\pi}^{\pi} \left( \int_{-\pi}^{\pi} |\partial^s u(x)|^2 dx_3 \right)^{\frac{1}{2}} dx_1 dx_2 \right)^{\frac{1}{2}} \leq C \|\nabla u\|_2^{\frac{1}{2}} \|\partial^s \nabla u\|_2^{\frac{1}{2}}.$$  

The following lemma will be useful for the estimate of the nonlinear term.

**Lemma 2.4.** There exists a positive constant $C > 0$ such that, for any $s > \frac{1}{2}$ and for any smooth enough divergence-free vector fields $u$, $v$, and $w$, the following estimates hold,

(i) $$\|((u \cdot \nabla)v, w)\| \leq C \|u\|_2^{\frac{1}{2}} \|\nabla u\|_2 \|\nabla v\|_2 \|\partial^s \nabla v\|_2^\frac{3}{2} \|w\|_2^{\frac{3}{2}} \|\nabla w\|_2^{\frac{3}{2}}$$

and

(ii) $$\|((u \cdot \nabla)v, w)\| \leq C \|u\|_2^{\frac{1}{2}} \|\nabla u\|_2 \|v\|_2 \|\nabla v\|_2 \|\partial^s \nabla w\|_2^\frac{3}{2} \|\nabla w\|_2^{\frac{3}{2}} \|\nabla w\|_2^\frac{3}{2}. $$
Lemma 2.5.

(i) By using lemma [2.1] and lemma [2.3] we get

$$||(\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}|| \leq ||\mathbf{u}||_{L^2(L^1(D)^3)} ||\nabla \mathbf{v}||_{L^2(L^1(D)^3)} ||\mathbf{w}||_{L^2(L^1(D)^3)}$$

$$\leq C||\mathbf{u}||_2^{1/2} ||\nabla \mathbf{u}||_2^{1/2} ||\nabla \mathbf{v}||_2^{1/2} ||\partial^\alpha_3 \nabla \mathbf{v}||_2^{1/2} ||\mathbf{w}||_2^{1/2} ||\nabla \mathbf{w}||_2^{1/2}$$

(ii) We have

$$||(\mathbf{u} \cdot \nabla)\mathbf{v}, \mathbf{w}|| \leq \left| \int_D \mathbf{v} \otimes \mathbf{u} : \nabla \mathbf{w} \, dx \right|,$$

thus, in order to prove the second inequality it is sufficient to swap the roles of \( \mathbf{v} \) and \( \mathbf{w} \).

2.1 The vertical filter and the vertical deconvolution operator

Let \( \mathbf{v} \) be a smooth function of the form \( \mathbf{v} = \sum_{k_3 \in \mathbb{Z} \setminus \{0\}} c_{k_3}(x_1, x_2)e^{i k_3 x_3} \). The action of the vertical filter on \( \mathbf{v}(x) = \sum_{k_3 \in \mathbb{Z} \setminus \{0\}} c_{k_3}(x_1, x_2)e^{i k_3 x_3} \) can be written as \( \mathbb{A}_{3, \theta}(\mathbf{v}) = \sum_{k_3 \in \mathbb{Z} \setminus \{0\}} \mathbb{A}_\theta(k_3)c_{k_3}(x_1, x_2)e^{i k_3 x_3} \), where the Fourier transform with respect to \( x_3 \) of the vertical filter is given by

$$\mathbb{A}_\theta(k_3) = \left( 1 + \alpha^{2\theta} |k_3|^{2\theta} \right). \quad (2.4)$$

Therefore, by using the Parseval’s identity with respect to \( x_3 \) we get,

$$||\mathbb{A}^{1/2}_{3, \theta} \mathbf{v}||_2^2 = ||\mathbf{v}||_2^2 + \alpha^{2\theta} ||\partial^\theta_3 \mathbf{v}||_2^2 = (\mathbb{A}_{3, \theta} \mathbf{v}, \mathbf{v}). \quad (2.5)$$

Next, we recall some properties of the vertical filter. These properties are proved in [3] for the horizontal filter and hold true for the vertical filter.

Lemma 2.5. Let \( f \) be a smooth function and \( \mathbf{w} \) a smooth, space-periodic with respect to \( x_3 \), and divergence-free vector field defined on the domain \( D \) such that \( \mathbf{w} \cdot \mathbf{n} = 0 \) on \( \partial D \). Let the filtering be defined by \( \mathbb{A}_{3, \theta} \). Then, the following identities hold true:

$$\int_D \mathbf{f} \cdot \mathbf{w} = (\mathbb{F}(\mathbf{f}), \mathbf{w}), \quad (2.6)$$

$$\mathbf{w} - \alpha^{2\theta} \partial^\theta_3 \mathbf{w} = \mathbf{w} - \alpha^{2\theta} \partial^\theta_3 \mathbf{w}, \quad (2.7)$$

and

$$\left( \nabla \cdot (\mathbf{w} \otimes \mathbf{w}), \mathbb{A}_{3, \theta} \mathbf{w} \right) = \left( \nabla \cdot (\mathbf{w} \otimes \mathbf{w}), \mathbf{w} \right) = 0. \quad (2.8)$$

Proof. See in [3].

The deconvolution operator \( D_{N,\theta} = \sum_{i=0}^{N} (I - \mathbb{A}_{3, \theta}^{-1})^i \) is constructed by using the vertical filter with fractional regularization \( \mathbb{A}_{3, \theta} \). For a fixed \( N > 0 \) and for \( \theta = 1 \), we recover a vertical operator form from the Van Cittert deconvolution operator (see [14] and [3]). A straightforward calculation yields

$$D_{N,\theta} \left( \sum_{k_3 \in \mathbb{Z} \setminus \{0\}} c_{k_3}(x_1, x_2)e^{i k_3 x_3} \right) = \sum_{k_3 \in \mathbb{Z} \setminus \{0\}} \left( 1 + \alpha^{2\theta} |k_3|^{2\theta} \right) \left( 1 - \frac{\alpha^{2\theta} |k_3|^{2\theta} \left( \frac{1}{1 + \alpha^{2\theta} |k_3|^{2\theta}} \right)}{N+1} \right) c_{k_3}(x_1, x_2)e^{i k_3 x_3}. \quad (2.9)$$
Thus
\[ D_{N,θ} \left( \sum_{k_3 \in \mathbb{Z} \setminus \{0\}} c_{k_3}(x_1, x_2) e^{i k_3 x_3} \right) = \sum_{k \in \mathbb{Z} \setminus \{0\}} D_{N,θ}(k_3) c_{k_3}(x_1, x_2) e^{i k_3 x_3}, \] (2.10)
where we have for \( k_3 \in \mathbb{Z} \setminus \{0\} \), and \( θ \geq 0 \),
\[ D_{0,θ}(k_3) = 1, \] (2.11)
\[ 1 \leq D_{N,θ}(k_3) \leq N + 1 \quad \text{for each} \ N > 0, \] (2.12)
and \( D_{N,θ}(k_3) \leq A_{3,θ} \) for a fixed \( α > 0 \). (2.13)

From the previous hypothesis, one can prove the following Lemma by adapting the results summarized in the isotropic case in \([4]\):

\textbf{Lemma 2.6.} For \( θ \geq 0 \), \( k_3 \in \mathbb{Z} \setminus \{0\} \) and for each \( N > 0 \), there exists a constant \( C > 0 \) such that for all \( v \) sufficiently smooth we have
\[ \|v\|_2 \leq \|D_{N,θ}(v)\|_2 \leq (N + 1)\|v\|_2, \] (2.14)
\[ \|v\|_2 \leq C\|D_{N,θ}(v)\|_2 \leq C\|A_{3,θ} D_{N,θ}(v)\|_2, \] (2.15)
\[ \|A_{3,θ} D_{N,θ}(v)\|_2 \leq \|v\|_2, \] (2.16)
\[ \|v\|_2^2 + α^2 \|\partial_3^2 v\|_2^2 \leq \|A_{3,θ} D_{N,θ}(v)\|_2^2. \] (2.17)

\section{Existence and uniqueness results}

In this section, we give a definition of what is called a regular weak solution to problem (1.5). Then, we give the proof of theorem 1.1.

\textbf{Definition 3.1.} Let \( f \in L^2(0, T; H) \) be a divergence free function and \( v_0 \in H \). For any \( 0 \leq θ \leq 1 \) and \( 0 \leq N < ∞ \), \( w \) is called a “regular” weak solution to problem (1.5) if the following properties are satisfied:
\[ w \in C(0, T; H) \cap L^2(0, T; V), \] (3.1)
\[ \partial_3^2 w \in C(0, T; H) \cap L^2(0, T; V), \] (3.2)
\[ \partial_t w \in L^2(0, T; V^*), \] (3.3)
and the velocity \( w \) fulfills
\[ \int_0^T (\partial_t w, φ) + (\nabla \cdot (D_{N,θ}(w) \otimes D_{N,θ}(w)), φ) + ν (\nabla w, \nabla φ) \, dt \]
\[ = \int_0^T (f, φ) \, dt \quad \text{for all} \ φ \in L^2(0, T; V). \] (3.4)
Moreover,
\[ w(0) = w_0. \] (3.5)
3.1 Proof of Theorem 1.1

The proof of Theorem 1.1 follows the classical scheme. We start by constructing approximated solution $w^n$ via Galerkin method. Then, we seek for a priori estimates that are uniform with respect to $n$. Next, we take the limit in the equations after having used compactness properties. Finally, we show that the constructed solution is unique thanks to Gronwall’s lemma [15].

Step 1 (Galerkin approximation). We denote by $\{\varphi_{k3}\}_{k3=1}^{+\infty}$ the eigenfunctions of the Stokes operator on $D$, with Dirichlet boundary conditions on $\partial D$ and with periodicity only with respect to $x_3$. The explicit expression of these eigenfunctions can be found in [12]. These eigenfunctions are linear combinations of $W_{k3}(x_1, x_2)e^{ik_3x_3}$, where $k_3 \in \mathbb{Z} \setminus \{0\}$, for certain families of smooth functions $W_{k3} : \Omega \to \mathbb{R}$ vanishing at $\partial \Omega$.

We set

$$w^n(t, x) = \sum_{k3 \in \mathbb{Z} \setminus \{0\}, |k3| \leq n} c^n_{k3}(x_1, x_2, t)\varphi_{k3}(x). \quad (3.6)$$

Since we are dealing with real functions, one has to take suitable conjugation and alternatively use linear combinations of sines and cosines in the variable $x_3$. We look for $w^n(t, x)$ which is determined by the following system of equations

$$\begin{cases}
(\partial_t w^n, \varphi_{k3}) + (\nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n)), \varphi_{k3}) + \nu(\nabla w^n, \nabla \varphi_{k3}) \\
= \langle f, \varphi_{k3} \rangle, \quad |k3| = 1, 2, \ldots, n, \\
w^n(0) = P_n(w_0) = P_n(\overline{w_0}),
\end{cases} \quad (3.7)$$

where $P_n$ denotes the projection operator over $H_n := \text{Span}(\varphi^1, \ldots, \varphi^n)$. Therefore, the classical Caratheodory theory [16] implies the short-time existence of solutions to (3.7). Next, we derive estimates on $c^n$ that is uniform w.r.t. $n$. These estimates imply that the solution of (3.7), constructed on a short time interval $[0, T^n[$, exists for all $t \in [0, T]$.  

Step 2 (A priori estimates) We need to derive an energy inequality for $w^n$. This can be obtained by using $\mathbb{A}_{3,\theta} D_{N,\theta}(w^n)$ as a test function in (3.7). Thanks to the explicit expression of the eigenfunctions, and the properties of $\mathbb{A}_{3,\theta}$ and $D_{N,\theta}$, the quantity $\mathbb{A}_{3,\theta} D_{N,\theta}(w^n)$ is a legitimate test function since it still belongs to $H_n$. Standard manipulations and the use of Lemma 2.5 combined with the following identities

$$(\partial_t w^n, \mathbb{A}_{3,\theta} D_{N,\theta}(w^n)) = \frac{1}{2} \frac{d}{dt} \left( ||D_{N,\theta}^{\frac{1}{2}}(w^n)||_2^2 + \alpha^{2\theta} ||\partial_{\zeta}^{\theta} D_{N,\theta}^{\frac{1}{2}}(w^n)||_2^2 \right)$$

$$= \frac{1}{2} \frac{d}{dt} ||\mathbb{A}_{3,\theta}^{\frac{1}{2}} D_{N,\theta}^{\frac{1}{2}}(w^n)||_2^2,$$  

$$(\Delta w^n, \mathbb{A}_{3,\theta} D_{N,\theta}(w^n)) = \left( ||\nabla D_{N,\theta}^{\frac{1}{2}}(w^n)||_2^2 + ||\partial_{\zeta}^{\theta} \nabla D_{N,\theta}^{\frac{1}{2}}(w^n)||_2^2 \right)$$

$$= ||\nabla \mathbb{A}_{3,\theta}^{\frac{1}{2}} D_{N,\theta}^{\frac{1}{2}}(w^n)||_2^2,$$

and

$$\left( \mathcal{F}, \mathbb{A}_{3,\theta} D_{N,\theta}(w^n) \right) = \left( D_{N,\theta}^{\frac{1}{2}} f, D_{N,\theta}^{\frac{1}{2}}(w^n) \right) \quad (3.10)$$
lead to the a priori estimate

\[
\frac{1}{2} \| A_{3,\theta}^{\frac{1}{2}} D_{N,\theta}^{\frac{1}{2}} (w^n) \|_2^2 + \nu \int_0^t \| \nabla A_{3,\theta}^{\frac{1}{2}} D_{N,\theta}^{\frac{1}{2}} (w^n) \|_2^2 \, ds
= \int_0^t \left( D_{N,\theta}^{\frac{1}{2}} (f), D_{N,\theta}^{\frac{1}{2}} (w^n) \right) \, ds + \frac{1}{2} \| A_{3,\theta}^{\frac{1}{2}} D_{N,\theta}^{\frac{1}{2}} (\nu^n) \|_2^2.
\]  

(3.11)

By using the Cauchy-Schwartz inequality, the Poincaré inequality combined with the Young inequality and inequality (2.16), we conclude from (3.11) the following inequality

\[
\sup_{t \in [0,T^n]} \| A_{3,\theta}^{\frac{1}{2}} D_{N,\theta}^{\frac{1}{2}} (w^n) \|_2^2 + \nu \int_0^t \| \nabla A_{3,\theta}^{\frac{1}{2}} D_{N,\theta}^{\frac{1}{2}} (w^n) \|_2^2 \, ds \leq \| v_0 \|_2^2 + \frac{C(N+1)}{\nu} \int_0^T \| f \|_2^2 \, ds
\]  

(3.12)

which immediately implies that the existence time is independent of \( n \) and it is possible to take \( T = T^n \).

We deduce from (3.12) and (2.17) that

\[
\sup_{t \in [0,T]} \left( \| w^n \|_2^2 + \| \partial_3^\theta w^n \|_2^2 \right) + \nu \int_0^t \left( \| \nabla w^n \|_2^2 + \| \partial_3^\theta \nabla w^n \|_2^2 \right) \, ds \leq \| v_0 \|_2^2 + \frac{C(N+1)}{\nu} \int_0^T \| f \|_2^2 \, ds.
\]  

(3.13)

It follows from the above inequality that

\[
w^n \in L^\infty(0,T;H) \cap L^2(0,T;V), \tag{3.14}
\]  

\[
\partial_3^\theta w^n \in L^\infty(0,T;H) \cap L^2(0,T;V). \tag{3.15}
\]

Thus, in one hand we get,

\[
\Delta w^n \in L^2(0,T;V^*), \tag{3.16}
\]  

\[
\partial_3^\theta \Delta w^n \in L^2(0,T;V^*). \tag{3.17}
\]

In the other hand, it follows from (2.14) that

\[
D_{N,\theta}(w^n) \in L^\infty(0,T;H) \cap L^2(0,T;V), \tag{3.18}
\]

\[
\partial_3^\theta D_{N,\theta}(w^n) \in L^\infty(0,T;H) \cap L^2(0,T;V). \tag{3.19}
\]

The inequality (3.18) allows us to find estimates on the nonlinear term \( \nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n)) \).

For that we use lemma 2.3. For all \( \varphi \in V \) we have

\[
\left| \left( \nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n)) , \varphi \right) \right| \leq \| \nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n)) \|_2 \| \varphi \|_2^{\frac{1}{2}} \| \nabla \varphi \|_2^{\frac{1}{2}} \| \partial_3^\theta \varphi \|_2^{\frac{1}{2}}.
\]  

(3.20)

We also have

\[
\left| \left( A_{3,\theta}^{\frac{1}{2}} \nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n)) , \varphi \right) \right| \leq \left| \left( \nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n)), A_{3,\theta}^{\frac{1}{2}} \varphi \right) \right| \leq C \| D_{N,\theta}(w^n) \|_2 \| \nabla D_{N,\theta}(w^n) \|_2 \| \varphi \|_2^{\frac{1}{2}} \| \nabla A_{3,\theta}^{\frac{1}{2}} \varphi \|_2^{\frac{1}{2}} \left\| \nabla \partial_3^\theta A_{3,\theta}^{\frac{1}{2}} \varphi \right\|_2^{\frac{1}{2}}.
\]  

(3.21)
Since $\|\nabla \partial_3^b \varphi\|_2^{3^b} \leq \|\nabla \varphi\|_2^{3^b}$, $\|\nabla A_{3,b} \varphi\|_2^{1-\frac{3^b}{3}} \leq \|\nabla \varphi\|_2^{1-\frac{3^b}{3}}$ and $\|\nabla \partial_3^b A_{3,b} \varphi\|_2^{3^b} \leq \|\nabla \varphi\|_2^{3^b}$ we get that
\[
\left| \left( \nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n)), \varphi \right) \right| \leq C \|D_{N,\theta}(w^n)\|_2 \|\nabla D_{N,\theta}(w^n)\|_2 \|\nabla \varphi\|_2, \quad (3.22)
\]
and
\[
\left| \left( A_{3,b} A_{3,b} \nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n)), \varphi \right) \right| \leq C \|D_{N,\theta}(w^n)\|_2 \|\nabla D_{N,\theta}(w^n)\|_2 \|\nabla \varphi\|_2. \quad (3.23)
\]

Thus we obtain
\[
\nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n)) \in L^2(0,T; \mathbf{V}^*), \quad (3.24)
\]
\[
A_{3,b} \nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n)) \in L^2(0,T; \mathbf{V}^*). \quad (3.25)
\]
From eqs. (3.7), (3.14), (3.25), we also obtain that
\[
\partial_t w^n \in L^2(0,T; \mathbf{V}^*), \quad (3.26)
\]
and
\[
\partial_t A_{3,b} w^n \in L^2(0,T; \mathbf{V}^*). \quad (3.27)
\]

**Step 3** (Limit $n \to \infty$) It follows from the estimates (3.14)-(3.27) and the Aubin-Lions compactness lemma (see [13] for example) that there exists a not relabeled subsequence of $w^n$ and $w$ such that
\[
w^n \rightharpoonup w \quad \text{weakly* in } L^\infty(0,T; \mathbf{H}), \quad (3.28)
w_{N,\theta}(w^n) \rightharpoonup w_{N,\theta}(w) \quad \text{weakly* in } L^\infty(0,T; \mathbf{H}), \quad (3.29)
w^n \rightharpoonup w \quad \text{weakly in } L^2(0,T; \mathbf{V}), \quad (3.30)
\partial_t w^n \rightharpoonup \partial_t w \quad \text{weakly in } L^2(0,T; \mathbf{V}), \quad (3.31)
w_{N,\theta}(w^n) \rightharpoonup w_{N,\theta}(w) \quad \text{weakly in } L^2(0,T; \mathbf{V}), \quad (3.32)
\partial_t w^n \rightharpoonup \partial_t w \quad \text{weakly in } L^2(0,T; \mathbf{V}), \quad (3.33)
\partial_t A_{3,b} w^n \rightharpoonup \partial_t A_{3,b} w \quad \text{weakly in } L^2(0,T; \mathbf{V}), \quad (3.34)
w^n \rightharpoonup w \quad \text{strongly in } L^2(0,T; \mathbf{H}), \quad (3.35)
w_{N,\theta}(w^n) \rightharpoonup w_{N,\theta}(w) \quad \text{strongly in } L^2(0,T; \mathbf{H}), \quad (3.36)
\]

From (3.32) and (3.37), it follows that
\[
\nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n)) \rightharpoonup \nabla \cdot (D_{N,\theta}(w) \otimes D_{N,\theta}(w)) \quad \text{strongly in } L^1(0,T; L^1(D)^3), \quad (3.38)
\]
Finally, since the sequence $\nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n))$ is bounded in $L^2(0,T; \mathbf{V}^*)$, it converges weakly, up to a subsequence, to some $\chi$ in $L^2(0,T; \mathbf{V}^*)$. The previous result and the uniqueness of the limit allow us to claim that $\chi = \nabla \cdot (D_{N,\theta}(w) \otimes D_{N,\theta}(w))$. Consequently,
\[
\nabla \cdot (D_{N,\theta}(w^n) \otimes D_{N,\theta}(w^n)) \rightharpoonup \nabla \cdot (D_{N,\theta}(w) \otimes D_{N,\theta}(w)) \quad \text{weakly in } L^2(0,T; \mathbf{V}^*). \quad (3.39)
\]
The above established convergences are clearly sufficient to take the limit in (3.7) and conclude that the velocity $w$ satisfies (3.4). Moreover, from (3.30) and (3.34), one can deduce by a classical interpolation argument [9] that

$$w \in C(0, T; H).$$

(3.40)

Furthermore, from the strong continuity of $w$ with respect to the time and with values in $H$, we deduce that $w(0) = w_0$.

**Step 4** (Uniqueness) Next, we will show the continuous dependence of the solutions on the initial data and in particular the uniqueness.

Let $\theta > \frac{1}{2}$ and let $(w_1, q_1)$ and $(w_2, q_2)$ be any two solutions of (1.3) on the interval $[0, T]$, with initial values $w_1(0)$ and $w_2(0)$. Let us denote by $\delta w = w_2 - w_1$. We subtract the equations for $w_1$ from the equations for $w_2$. Then, we obtain

$$\partial_t \delta w - \nu \Delta \delta w + \nabla \cdot (D_{N, \theta} \delta w) - \nabla \cdot (D_{N, \theta}(w_1) \otimes D_{N, \theta}(w_1)) = 0,$$

(3.41)

and $\delta w = 0$ at the initial time.

Applying $A_{3, \theta}^{\frac{1}{2}}$ to (3.41) we obtain

$$A_{3, \theta}^{\frac{1}{2}} \partial_t \delta w - \nu A_{3, \theta}^{\frac{1}{2}} \Delta \delta w + A_{3, \theta}^{\frac{1}{2}} \nabla \cdot (D_{N, \theta} \delta w) - A_{3, \theta}^{\frac{1}{2}} \nabla \cdot (D_{N, \theta}(w_1) \otimes D_{N, \theta}(w_1)) = 0,$$

(3.42)

One can take $A_{3, \theta}^{\frac{1}{2}} D_{N, \theta}(\delta w) \in L^2(0, T; V)$ as test function in (3.42). Let us mention that, $\partial_t A_{3, \theta}^{\frac{1}{2}} \delta w \in L^2(0, T; V^*)$ and $A_{3, \theta}^{\frac{1}{2}} D_{N, \theta}(\delta w) \in L^2(0, T; V)$. Thus, by using Lions-Magenes Lemma [9] we have

$$\langle \partial_t A_{3, \theta}^{\frac{1}{2}} \delta w, A_{3, \theta}^{\frac{1}{2}} D_{N, \theta}(\delta w) \rangle_{V^*, V} = \frac{1}{2} \frac{d}{dt} \| A_{3, \theta}^{\frac{1}{2}} D_{N, \theta}(\delta w) \|^2_H.$$

Using lemma 2.3 and the divergence free condition we get:

$$\frac{1}{2} \frac{d}{dt} \| A_{3, \theta}^{\frac{1}{2}} D_{N, \theta}(\delta w) \|^2_H + \nu \| \nabla A_{3, \theta}^{\frac{1}{2}} D_{N, \theta}(\delta w) \|^2_H \leq \left\| (D_{N, \theta}(\delta w) \cdot \nabla) D_{N, \theta}(w_2), D_{N, \theta}(\delta w) \right\|.$$

(3.43)

We estimate now the right-hand side by using lemma 2.4 as follows,

$$\left\| ((D_{N, \theta}(\delta w) \cdot \nabla) D_{N, \theta}(w_2), D_{N, \theta}(\delta w)) \right\| \leq \| D_{N, \theta}(\delta w) \|_2 \| \nabla D_{N, \theta}(\delta w) \|_2 \| \nabla D_{N, \theta}(w_2) \|_2 \right\|_2^{1-\frac{1}{2}} \| \nabla D_{N, \theta}(w_2) \|_2^{\frac{1}{2}} \right\|.$$

(3.44)

Hence, by using the Young inequality combined with lemma 2.6 we obtain that there exists a constant $C > 0$ that depends on $N$ and $\nu$ such that

$$\left\| ((D_{N, \theta}(\delta w) \cdot \nabla) D_{N, \theta}(w_2), D_{N, \theta}(\delta w)) \right\| \leq C \| \delta w \|_2 \| \nabla w_2 \|_2 \| \nabla D_{N, \theta}(w_2) \|_2 \right\|_2^{1-\frac{1}{2}} \| \nabla D_{N, \theta}(w_2) \|_2^{\frac{1}{2}} \right\| + \nu \| \nabla \delta w \|_2 \|^2_H.$$

(3.45)

By using (2.17) we have

$$\frac{1}{2} \frac{d}{dt} \left( \| \delta w \|_2^2 + \alpha^2 \| \delta \delta w \|_2^2 \right) \leq \frac{1}{2} \frac{d}{dt} \| A_{3, \theta}^{\frac{1}{2}} D_{N, \theta}(\delta w) \|^2_H + \nu \| \nabla A_{3, \theta}^{\frac{1}{2}} D_{N, \theta}(\delta w) \|^2_H.$$

(3.46)
From (3.44), (3.46) we get
\[
\frac{d}{dt} \left( \| \delta w \|_2^2 + \alpha^2 \| \partial_3^2 \delta w \|_2^2 \right) + \nu \left( \| \nabla \delta w \|_2^2 + \alpha^2 \| \nabla \delta w \|_2^2 \right) + \nu \left( \| \nabla \delta w \|_2^2 + \alpha^2 \| \nabla \delta w \|_2^2 \right)
\leq C \| \delta w \|_2^2 \| \nabla w \|_2^2 - \frac{1}{\theta} \| \partial_3 \nabla w \|_2^2 - \frac{\theta}{\theta}.
\]
(3.47)

Since \( \| \nabla w \|_2^2 - \frac{1}{\theta} \| \partial_3 \nabla w \|_2^2 \in L^1([0, T]) \), we conclude by using Gronwall’s inequality the continuous dependence of the solutions on the initial data in the \( L^\infty([0, T], H) \) norm. In particular, if \( \delta w_0 = 0 \) then \( \delta w = 0 \) and the solutions are unique for all \( t \in [0, T] \). In addition, since \( T > 0 \) is arbitrary chosen, this solution may be uniquely extended for all time.

This finishes the proof of Theorem 1.1.

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**Appendix: Proof of the functional estimate (2.2)**

Let \( g \) be a 2\( \pi \)-periodic and mean zero function with \( g(x) \in H^s([-\pi, \pi]) \), and let \( s > \frac{1}{2} \). If we write \( g(x) \) as the sum of its Fourier series \( g(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} g_k e^{ikx} \), then, we can estimate \( \| g \|_{L^\infty} \) by
\[
\| g \|_{L^\infty} \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} |g_k|.
\]
We then break up the sum into low and high wave-number components,
\[
\| g \|_{L^\infty} \leq \sum_{0 < |k| \leq \kappa} |g_k| + \sum_{|k| > \kappa} |g_k|.
\]
We now use the Cauchy-Schwarz inequality on each part,
\[
\| g \|_{L^\infty} \leq \left( \sum_{0 < |k| \leq \kappa} |g_k|^2 \right)^{\frac{1}{2}} \left( \sum_{0 < |k| \leq \kappa} 1 \right)^{\frac{1}{2}} + \left( \sum_{|k| > \kappa} |k|^{2s} |g_k|^2 \right)^{\frac{1}{2}} \left( \sum_{|k| > \kappa} |k|^{-2s} \right)^{\frac{1}{2}}.
\]
Since
\[
\sum_{0 < |k| \leq \kappa} 1 \leq C \kappa \quad \text{and} \quad \sum_{|k| > \kappa} |k|^{-2s} \leq C \kappa^{-2s+1},
\]
the above inequality becomes
\[
\| g \|_{L^\infty} \leq C \left( \kappa^{\frac{1}{2}} \| g \|_{L^2} + \kappa^{-s+\frac{1}{2}} \| g \|_{H^s} \right).
\]
To make both terms on the right-hand side the same, we choose
\[
\kappa = \frac{\| g \|_{H^s}^{\frac{1}{2}}}{\| g \|_{L^2}^{\frac{1}{2}}}
\]
This yields the following estimate
\[
\| g \|_{L^\infty} \leq C \| g \|_{L^2}^{\frac{1}{2}} \| g \|_{H^s}^{\frac{1}{2}},
\]
which is (2.2).
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