Testing the violation of the equivalence principle in the electromagnetic sector and its consequences in $f(T)$ gravity

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Abstract. A violation of the distance-duality relation is directly linked with a temporal variation of the electromagnetic fine-structure constant. We consider a number of well-studied $f(T)$ gravity models and we revise the theoretical prediction of their corresponding
induced violation of the distance-duality relationship. We further extract constraints on the involved model parameters through fine-structure constant variation data, alongside with supernovae data, and Hubble parameter measurements. Moreover, we constrain the evolution of the effective $f(T)$ gravitational constant. Finally, we compare with revised constraints on the phenomenological parametrisations of the violation of the equivalence principle in the electromagnetic sector.

**Keywords:** cosmology of theories beyond the SM, dark energy theory, gravity, modified gravity

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1 Introduction

It is well-known that the ΛCDM cosmological model is evidenced by overwhelming successes in describing the Universe at all scales where observations can be made [1, 2]. This is achieved by considering an extra cold dark matter sector which can produce stable galaxies and clusters thereof [3, 4], while the late-time cosmic acceleration can be described through the cosmological constant. However, despite extraordinary efforts, the model still retains numerous questions that still appear insurmountable in this regard [5].

On the other hand, the efficiency of the ΛCDM paradigm in explaining precision cosmology observations has been called into question in recent years. In this respect, the core criticism of the ΛCDM model appears primarily through the so-called $H_0$ tension problem which quantifies the inconsistency between the ΛCDM predicted value of $H_0$ from the measurements of the anisotropies of the cosmic microwave background (CMB) and its reported value from local observations. The problem first appeared through measurements by the Planck Collaboration [6, 7], but has since been confirmed to a greater degree by the strong lensing measurements from the H0LiCOW collaboration [8]. In the interim period, measurements on the tip of the red giant branch (TRGB, Carnegie-Chicago Hubble Program) have yielded a lower $H_0$ tension [9]. The problem may be further illuminated by future observations from gravitational wave astronomy such as through the LISA mission [10, 11] similar to the work carried out in refs. [12, 13].
There have been a plethora of theories in recent years that attempt to describe the disparate phenomena that make up observational cosmology [2, 14]. Collectively, these theories are primarily extensions of General Relativity (GR) in that they consider gravity through the prism of curvature by means of the Levi-Civita connection. The most popular of these theories is standard $f(\hat{R})$ gravity [14–17] (over-circles are used throughout to refer to quantities that are calculated using the Levi-Civita connection) which is a fourth-order theory that directly generalises the Einstein-Hilbert action. These theories have been tested to a high level of precision [18].

In this work, we consider the possibility of gravity being a manifestation of torsion through the Weitzenbök connection [19]. Teleparallel gravity (TG) is the body of theories that are based on the Weitzenbök connection [20, 21]. While the Levi-Civita connection is torsion-less, the Weitzenbök connection is curvature-less, and both satisfy the metricity condition. The Einstein-Hilbert action relies on a Lagrangian that is simply constructed by the standard Ricci scalar, $\hat{R}$, while TG can produce identical dynamical equations for a Lagrangian that consists only of the torsion scalar, $T$. This is the so-called Teleparallel equivalent of General Relativity (TEGR), and differs from GR only at the level of Lagrangian by a total divergence quantity, $B$ (boundary term). The boundary term embodies the fourth-order corrections which arise to have a covariant theory (due to the second-order derivatives in the Einstein-Hilbert action). The importance of this property is that extensions to TEGR will differ from their Levi-Civita connection counterparts. Moreover, TG has a number of interesting properties such as its similarity to Yang-Mills theory [22] giving it an added particle physics dimension, its potential to define a gravitational energy-momentum tensor [23, 24], and that it is more regular than GR in that it does not require the introduction of a Gibbons-Hawking-York boundary term in order to produce a well-defined Hamiltonian formulation [20]. As an aside, the theory can be constructed even without the weak equivalence principle (but definitely it can satisfy it if needed) unlike GR [25].

Following the same reasoning as $f(\hat{R})$ gravity [14–16], the TEGR Lagrangian can be arbitrarily generalised to produce $f(T)$ theory [26–30] which is a generally second-order theory of gravity. This last point is a result of a weaken Lovelock theorem in TG [31–33] which emerges due to the absence of the boundary term. A number of $f(T)$ gravity models have shown promising results in the cosmological regime [20, 34, 35], as well as in galactic [36] and solar system [37–41] scale physics. The boundary term can also be included in this generalisation to produce $f(T,B)$ gravity [42–48]. In this latter case, the model produces a general framework in which $f(\hat{R})$ gravity forms one subclass of possible models.

Another important aspect of any potential proposal for a modified theory of gravity is its Einstein frame features. In many cases, the Einstein frame is obtained through a conformal transformation which leaves the electromagnetic action invariant (due to the conformal invariance of that action) [49]. However, $f(T)$ gravity cannot be written in the Einstein frame by taking conformal transformations. In fact, conformal transformations produce an extra term in which the conformal scalar field and the torsional contribution are coupled [48]. Conformal transformations lead to the Einstein frame only in $f(T,B)$ gravity in the limit in which $f(\hat{R})$ gravity is reproduced. It is also the case that disformal transformations cannot either lead to the Einstein frame in $f(T)$ gravity [50, 51], which implies that if an Einstein frame exists then it may produce a non-vanishing coupling to the electromagnetic sector.

The series of works in refs. [52–54] consider the possibility of a violation of the Einstein equivalence principle (EEP), a cornerstone of GR, through the appearance of a coupling
parameter between the scalar field that transforms the gravitational action to its Einstein frame, and the matter fields. On the other hand, in refs. [55, 56] the violation of the EEP is considered through the presence of quantum effects such as the coupling of heavy fermions to photons. This may be the source of the potential violation of the EEP in TG.

It is well-known that a nonminimal multiplicative coupling between a scalar field and matter fields would break the EEP, and would further lead to the variation of fundamental constants of Nature [53]. For instance, a scalar field coupling with the electromagnetic Lagrangian would lead to a variation of the fine-structure constant, or Sommerfeld’s constant, which characterises the strength of the electromagnetic field and appears as a coupling constant in the electromagnetic action. A variation in the fundamental constants of Nature [57], which could be conservatively defined as those theoretical free parameters that could not be calculated with our present knowledge of physics, has been a long-established intriguing question [58, 59] with pertinent consequences for fundamental physics and cosmology (see, for instance, refs. [60–62]). Interestingly, when Dirac’s numerological principle [58, 59] was encapsulated in a field-theoretical framework, this led to the birth of the Jordan-Fierz-Brans-Dicke scalar-tensor theory of gravitation [63–65]. Moreover, in theories with additional space-time dimensions, fundamental constants of Nature are only effective quantities and are related to the true constants via characteristic sizes of extra dimensions [66]. Such paradigms include, for instance, Kaluza-Klein models [66, 67], superstring theories [66, 68] and brane world models [69], among others [70, 71].

A number of theoretical models have been proposed in order to explore the possibility of a dynamical fine-structure constant $\alpha \equiv e^2/\hbar c$. These models have been primarily formulated as Lagrangian theories with explicit variation of the velocity of light $c$ [72, 73], or of the charge on the electron $e$ [74, 75]. The former class of models are also known as varying speed of light theories [72, 76, 77], and have also been studied in the context of inflationary cosmology [72, 78, 79]. The latter models are commonly referred to as varying electric charge theories, which have been first formulated by Bekenstein from a generalisation of Maxwell’s equations in ref. [74], that led to the construction of the cosmological varying-$c$ Bekenstein-Sandvik-Barrow-Magueijo theory of varying $\alpha$ [80–85]. Although, at first glance, the varying-$c$ and varying-$e$ theories seem to be interchangeable, each theory is characterised by its distinct cosmological imprints [86]. Other frameworks include, for instance, a runaway dilaton [87, 88], supersymmetric generalisation of Bekenstein’s model [89] and a disformally coupled electromagnetic sector [90].

Several probes have been used for the search of any space-time dependence of the fine-structure constant, including primarily astronomical and local methods. The latter ones consist of geophysical analyses of samples from the natural nuclear reactor in Oklo [91, 92], meteorites [93], and laboratory atomic clocks [94–96]. Stringent constraints on the variation of the fine-structure constant have been inferred from the analysis of spectra from high-redshift quasar absorption systems [62, 97–107]. Other constraints have been derived with the thermal Sunyaev-Zeldovich effect and X-ray measurements of galaxy clusters [108, 109], strong gravitational lensing [110, 111], and from primordial abundances of light nuclei produced during the era of Big Bang nucleosynthesis [112]. Furthermore, upcoming gravitational wave observations [113–115] are also expected to be competitive with the currently available probes of the variation of the fine-structure constant. Moreover, the space-time dependence of fundamental constants has also been linked with the currently reported Hubble tension via the inferred effects in the ionisation history and profile of CMB anisotropies. Indeed, ref. [116] reported that a variation in fundamental constants, particularly in the effective
electron mass, could play an important role in the alleviation of the low versus high-redshift Hubble tension.

In this work, we consider the potential variation of the fine-structure parameter due to modified TG effects. These constraints are then used to limit the coupling parameters of literature models of $f(T)$ gravity. This is done using several data sets in conjunction with several literature approaches to parametrising the violation of the distance-duality relation which is a natural consequence of the violation of the EEP. This work builds on the foundations laid in ref. [117] where the potential violation of the fine-structure constant was first studied in the context of TG. However, since the conformal transformations that were performed in ref. [118] were elaborated more thoroughly in ref. [48], in the present work we revisit the analysis of ref. [117] and we expand its breadth with updated data and a deeper analysis of the potential implications.

Throughout the manuscript, Latin indices are used to refer to tangent space coordinates, while Greek indices refer to general manifold coordinates. The outline of the paper is as follows. In section 2 we review TG and its extension to $f(T)$ gravity in the context of its cosmology as well as its potential predictions on the variation of the fine-structure constant. A number of $f(T)$ gravity models are constrained in section 3, in which we also discuss the cosmological implications of the inferred parameter constraints. Finally, the main results of our analyses and prospective lines of research are discussed in section 4. In appendix A, we also probe the general case of the phenomenology of a non-vanishing coupling constant in the electromagnetic Lagrangian which produces a violation of the distance-duality relation. Using literature parametrisations of this violation, we revisit and update the constraints on this potential violation.

2 $f(T)$ gravity and the fine-structure constant

2.1 Teleparallel gravity

Teleparallel gravity (TG) represents a paradigm shift in the way that gravity is expressed not through the torsion-less connection of GR, but with the curvature-less one called the Weitzenböck connection, $\Gamma^\sigma_{\mu\nu}$ [119]. In GR, curvature is calculated through the Levi-Civita connection, $\overline{\Gamma}^\sigma_{\mu\nu}$ [120, 121] (recall that we use over-circles to denote quantities determined by the Levi-Civita connection). The Riemann tensor can then be used to determine a meaningful measure of curvature on a manifold, which is used in various modifications to standard gravity. Given that the Levi-Civita connection is replaced with the Weitzenböck connection in TG, it follows that irrespective of the space-time under consideration, the Riemann tensor will always vanish due to the connection being curvature-less. It is for this reason that TG necessitates different measures to construct realistic models of gravity.

The dynamical objects in TG are the tetrads $e^a_\mu$, which act as a soldering agent between tangent spaces (Latin indices) and the general manifold (Greek indices) [22]. In this way, tetrads (and their inverses $e_a^\mu$) can be used to transform to (and from) the Minkowski metric through

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}, \quad \eta_{ab} = e^a_\mu e^b_\nu g_{\mu\nu}. \quad (2.1)$$

The tetrads satisfy the orthogonality conditions

$$e^a_\mu e^b_\mu = \delta^a_b, \quad e^a_\mu e^\nu_a = \delta^\nu_\mu, \quad (2.2)$$

for internal consistency. The Weitzenböck connection can then be defined as [19]

$$\Gamma^\sigma_{\mu\nu} := e_a^\mu \partial_\mu e^a_\nu + e_a^\sigma \omega^a_{b\mu} e^b_\nu, \quad (2.3)$$
where $\omega^a_{\ b\mu}$ represents the spin connection. This is the most general linear affine connection that is both curvature-less and satisfies the metricity condition \cite{22}. Here, the spin connection appears to preserve the covariance of the resulting equations of motion \cite{122}. To do this, it incorporates the Local Lorentz Transformation (LLT) invariance of the theory, which implies that it can be set to zero for a particular choice of Lorentz frame \cite{21}.

Spin connections also appear in GR, but they are mainly hidden into the internal structure of the theory \cite{1}. Considering the full breadth of LLTs (Lorentz boosts and rotations), $\Lambda^a_{\ b}$, the spin connection can be represented completely as $\omega^a_{\ b\mu} = \Lambda^a_{\ c}\partial_{\mu}\Lambda^c_{\ b}$ \cite{22}. For any particular metric tensor, there exist an infinite number of tetrads that satisfy eq. (2.1) due to LLT invariance. Thus, it is the combination of a tetrad choice and its associated spin connection that retain the covariance of TG.

In the framework of TG the torsion tensor is defined as \cite{20}

$$T^\sigma_{\mu\nu} := -2\Gamma^\sigma_{[\mu\nu]}, \quad (2.4)$$

where the square brackets denote the anti-symmetric operator, and where this represents the field strength of gravitation. The torsion tensor transforms covariantly under both diffeomorphisms and LLTs. TG also relies on a couple of other tensorial quantities that help render a concise representation of the ensuing gravitational models. Firstly, the contorsion tensor turns out to be a useful quantity and is defined as the difference between the Weitzenböck and Levi-Civita connections, i.e.

$$K^\sigma_{\mu\nu} := \Gamma^\sigma_{\mu\nu} - \hat{\Gamma}^\sigma_{\mu\nu} = \frac{1}{2} (T^\sigma_{\mu\nu} + \Gamma^\sigma_{\nu\mu} - T^\sigma_{\mu\nu}), \quad (2.5)$$

which plays an important role in relating TG with Levi-Civita based theories. The second central ingredient to TG is the so-called superpotential

$$S^a_{\mu\nu} := \frac{1}{2} \left( K^a_{\mu\nu} - e^a_{\nu\alpha}T^a_{\alpha\mu} + e^a_{\mu\alpha}T^a_{\alpha\nu} \right), \quad (2.6)$$

which has been linked to the gauge current representation of the gravitational energy-momentum tensor in TG \cite{123, 124}. By contracting the torsion tensor with its superpotential produces the torsion scalar

$$T := S^a_{\mu\nu}T^a_{\mu\nu}, \quad (2.7)$$

which is determined entirely by the Weitzenböck connection in the same way that the Ricci scalar depends only on the Levi-Civita connection. By constructing the torsion scalar in this way, it turns out that the Ricci and torsion scalars are related by a total divergence term \cite{35, 42}

$$R = \hat{R} + T - \frac{2}{e} \partial_\mu \left( eT^\sigma_{\sigma\mu} \right) = 0, \quad (2.8)$$

where $R$ is the Ricci scalar in terms of the Weitzenböck connection, which is zero, and $\hat{R}$ is the regular Ricci scalar from GR. This implies that the Ricci and torsion scalars are equal up to a boundary term

$$\hat{R} = -T + \frac{2}{e} \partial_\mu \left( eT^\sigma_{\sigma\mu} \right) := -T + B, \quad (2.9)$$

where $e = \det (e^a_{\mu}) = \sqrt{-g}$. This fact alone guarantees that the Ricci scalar and the torsion scalar produce the same dynamical equations. That is, we can define the TEGR action as

$$S_{\text{TEGR}} = -\frac{1}{2\kappa^2} \int d^4x \ eT + \int d^4x \ e\mathcal{L}_m, \quad (2.10)$$
where $\kappa^2 = 8\pi G$ and $\mathcal{L}_m$ is the matter Lagrangian. While both actions lead to the same dynamical equations, they differ in terms of their Lagrangian in that the TG formulation decouples the second-order derivative contributions to the field equations, and the fourth-order derivative contribution which appears as a boundary quantity. This is not relevant for comparing GR and TEGR, but becomes an active agent when modifications to gravity are considered.

Using the same reasoning that led to $f(\mathring{R})$ gravity [14, 15], the Lagrangian of TEGR can be generalised to $f(T)$ gravity [26–30], giving

$$
\mathcal{S}_{\text{TEGR}} = \frac{1}{2\kappa^2} \int d^4 x \, e f(T) + \int d^4 x \, e \mathcal{L}_m .
$$

(2.11)

This produces second-order equations, which is only possible since the Lovelock theorem is much weaker in TG [31–33]. Note that TG and thus GR, are reproduced if $f(T) = -T + \Lambda$. $f(T)$ gravity also shares a number of properties with GR such as the same gravitational wave polarisation structure [46, 125–127], and being Gauss-Ostrogradsky ghost free (since it remains second-order) [21, 121]. Finally, by performing variation of the $f(T)$ action with respect to the tetrads, we arrive at the following field equations

$$
e^{-1} \partial_\nu \left( e_a^{\rho} S^\rho_{\sigma \nu} f_T \right) - e_a^{\lambda} T^\rho_{\lambda \nu} S^\rho_{\sigma} f_T + \frac{1}{4} e_a^{\rho} f(T)
$$

$$
+ e_a^{\rho} S^\rho_{\sigma \nu} \partial_\nu (T) f_{TT} + e_b^{\lambda} \omega^b_{\alpha \nu} S^\nu_{\lambda} f_T = \kappa^2 e_a^{\rho} \Theta^n_{\rho} ,
$$

(2.12)

where subscripts denote derivatives, and $\Theta^n_{\rho}$ is the regular energy-momentum tensor.

### 2.2 $f(T)$ cosmology

We investigate the cosmology of $f(T)$ gravity through a flat homogeneous and isotropic metric. We consider a tetrad choice of the form

$$
e^a_{\mu} = \text{diag} (1, a(t), a(t), a(t)) ,
$$

(2.13)

where $a(t)$ is the scale factor, and which allows us to set the spin connection to zero, i.e. $\omega^a_{\mu \nu} = 0$ [122, 128]. Through eq. (2.1), the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric is reproduced

$$
d s^2 = -d t^2 + a^2(t) \left( d x^2 + d y^2 + d z^2 \right) .
$$

(2.14)

Straightforwardly, we can calculate the torsion scalar to be

$$
T = 6H^2 ,
$$

(2.15)

and the boundary term to be $B = 6(3H^2 + \dot{H})$, which reproduces the well-known Ricci scalar for this metric, i.e. $\mathring{R} = -T + B = 6(\dot{H} + 2H^2)$, where we denote cosmic time derivatives by an over-dot (note that we use the standard convention for the metric signature [129], instead of the one used in refs. [20, 28, 29], which leads to a sign difference in $T$). By evaluating the field equations in eq. (2.12), the resulting Friedmann equations are

$$
H^2 = \frac{\kappa^2}{3} \left( \rho + \rho_{\text{eff}} \right) ,
$$

(2.16)

$$
\dot{H} = -\frac{\kappa^2}{2} \left( \rho + \rho_{\text{eff}} + p + p_{\text{eff}} \right) ,
$$

(2.17)
where $\rho$ and $p$ represent the energy density and pressure of the matter content respectively, while $f(T)$ gives rise to an effective fluid with components

$$\rho_{\text{eff}} := \frac{1}{2\kappa^2} (T - f + 2T f_T),$$  \hspace{1cm} (2.18)$$

$$p_{\text{eff}} := -\frac{1}{2\kappa^2} \left[4\dot{H} (1 + f_T + 2T f_{TT})\right] - \rho_{\text{eff}},$$  \hspace{1cm} (2.19)$$

which also satisfies the conservation equation

$$\dot{\rho}_{\text{eff}} + 3H (\rho_{\text{eff}} + p_{\text{eff}}) = 0.$$  \hspace{1cm} (2.20)$$

In this way, we can define an equation of state (EoS) of the effective fluid as \[35\]

$$\omega_{\text{eff}} := \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -1 + \frac{4\dot{H} (1 + f_T + 2T f_{TT})}{T - f + 2T f_T}$$

$$= -1 + (1 + \omega_m) \frac{(f - 2T f_T) (1 + f_T + 2T f_{TT})}{(f_T + 2T f_{TT}) (T - f + 2T f_T)},$$  \hspace{1cm} (2.21)$$

where the last line is a result of the Friedmann equations in eqs. (2.16), (2.17), and $\omega_m$ is the EoS of matter. Notice that we recover the $\Lambda$CDM scenario ($\omega_{\text{eff}} = -1$) for $f(T) = -T + \Lambda$.

Finally, by considering scalar perturbations on the flat FLRW of eq. (2.14) together with matter perturbations, an effective Newton’s constant can be defined as in refs. [130–132] such that $G_{\text{eff}} = \frac{G_N}{|f_T|}$, where $G_N$ is Newton’s constant.

### 2.3 The fine-structure constant in Teleparallel gravity

The fine-structure constant and the luminosity distance are derived from the electromagnetic action [53, 133] which is conformally invariant [134]. Conformal transformations are important because for many theories of modified gravity, they can be used to transform between the Jordan and Einstein frames [14, 135, 136], where the extra degrees of freedom of a theory may appear as scalar fields. There exists a number of theories of gravity in which conformal transformations do not lead to the Einstein frame. This implies that the Einstein frame would be a result of another type of transformation which may produce a coupling with the electromagnetic Lagrangian [88, 90, 137–141]. This also occurs when the low-energy limit of quantum gravity theories are considered [53, 87, 133, 142], which may appear as heavy fermions for instance. In either case, the result is the introduction of a new degree of freedom, $\phi$, that arises from the transformation

$$e^a_{\mu} \rightarrow \tilde{e}^a_{\mu},$$  \hspace{1cm} (2.22)$$

where $\tilde{e}^a_{\mu}$ represents the Einstein frame tetrad. This then induces an electromagnetic coupling which takes on the form

$$S_{\text{EM}} = -\frac{1}{4} \int d^4 x \, e B_F (\phi) F_{\mu\nu} F^{\mu\nu},$$  \hspace{1cm} (2.23)$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the standard Faraday tensor, and $B_F (\phi)$ represents the nonvanishing $\phi$–coupling. The consequence of this induced coupling is that the fine-structure
constant and the luminosity distance will be altered comparing to GR [1, 133]. As in refs. [55, 89, 141, 143], this can be expanded about \( \phi(t = t_0) \), which is suitably small, to give

\[
B_F (\phi) \simeq 1 + \beta_F \frac{\phi}{M_{\text{pl}}},
\]

where \( \beta_F = \mathcal{O}(1) \) is a constant, and \( M_{\text{pl}} = 1/\kappa^2 \) is the Planck mass (\( \beta_F \phi \ll M_{\text{pl}} \)).

Given an initially uncoupled Jordan-frame electromagnetic action, the fine-structure constant turns out to be given by [89]

\[
\alpha_E (\phi) = \alpha_J (\phi) B_F (\phi),
\]

where \( \alpha_E \) and \( \alpha_J \) are the fine-structure constants in the Einstein and Jordan frames respectively. To relate a change in the fine-structure constant between these frames with the electromagnetic coupling term in eq. (2.23), consider the fractional change [53, 143]

\[
\frac{\Delta \alpha}{\alpha} := \frac{\alpha_E - \alpha_J}{\alpha_J} = \frac{1}{B_F (\phi)} - 1,
\]

which depends on redshift (or cosmic time). Since \( B_F (z = 0) := B_{F_0} \neq 1 \), we need to rescale this relation so that \( \Delta \alpha = 0 \) at present time (\( z = 0 \)). This can be conveniently done by taking \( B_F (\phi) \to B_F (\phi)/B_{F_0} \), which is a result of the Maxwell tensor transformation \( F_{\mu \nu} \to \sqrt{B_{F_0}} F_{\mu \nu} \). Hence, the fractional change in the fine-structure constant now emerges as

\[
\frac{\Delta \alpha}{\alpha} = \frac{B_{F_0}}{B_F (\phi)} - 1.
\]

In \( f(T) \) gravity, the form of this fine-structure constant dependence can be obtained by a conformal transformation of the tetrad where

\[
\tilde{e}^a \mu = \Omega e^a \mu, \quad \tilde{e}_a ^\mu = \Omega^{-1} e_a ^\mu,
\]

which results in the regular conformal transformation \( \tilde{g}_{\mu \nu} = \Omega^2 g_{\mu \nu} \), as expanded upon in ref. [48], where \( \Omega^2 = -f_T = |f_T| \) (note that since in our conventions \( T > 0 \) and \( f_T < 0 \), we have replaced \( -f_T \) by \( |f_T| \)), and that tilde denotes conformally transformed quantities. It is well-known that \( f(T) \) gravity cannot be written in the Einstein frame through conformal transformations, which implies that it will induce a dependence in its associated fine-structure constant characterised by eq. (2.24) [55]. In fact, this produces an extra \( 2\Omega^{-2} \partial^\mu (\Omega^2) T^\nu_{\mu \nu} \) term which cannot be removed. The remainder of the scalar field becomes a phantom field with the choice of \( \phi = \sqrt{3} \ln f_T \) [48], which is partially favored by recent Planck data [6] (this form of the scalar field is a correction to the one used in ref. [143] due to a typo in ref. [118]). Together these contributions form the conformally transformed \( f(T) \) gravity Lagrangian which appears as [48]

\[
\mathcal{S}_{\text{CT}} = \frac{1}{16 \pi G} \int d^4 x e \left[ -\tilde{T} + 2\Omega^{-2} \tilde{\partial}^\mu (\Omega^2) \tilde{\nabla}_\nu + \frac{1}{2} g^{\mu \nu} \nabla_\mu \phi \nabla_\nu \phi - U(\phi) \right],
\]

where \( U(\phi) \) is the scalar potential and \( \tilde{\partial}_\mu = \tilde{\nabla}_\mu \) is preserved under conformal transformations. The kinetic term appears as a phantom field due to an incorrect sign which may lead to instabilities at perturbative level. This remains an open issue but is not the focus of the present work since this occurs at perturbative level.
Recent constraints on observationally viable models of $f(T)$ gravity [34, 37, 117, 132, 144–146] suggest that the model Lagrangian would take the form of TEGR plus small corrections. Given that $\Omega^2 = |f_T|$, this would imply that the $\tilde{\partial}^\mu (\Omega^2)$ would be very small rendering the additional term negligible. We will revisit this reasoning against the results of the analysis. With this approximation to the Einstein frame, the variation of the fine-structure constant takes the form

$$\frac{\Delta \alpha}{\alpha} = \frac{M_{\text{pl}} + \sqrt{3} \beta_F \ln |f_T(T_0)|}{M_{\text{pl}} + \sqrt{3} \beta_F \ln |f_T|} - 1,$$

which vanishes for the $\Lambda$CDM case of $f(T) = -T + \Lambda$, as expected. Eq. (2.30) embodies the redshift dependence of the fine-structure constant in TG, since the torsion scalar depends on redshift in accordance with eq. (2.15). Another consequence of a nonvanishing scalar field coupling to the electromagnetic action is that the luminosity distance will be altered [53]. By considering the standard derivation of luminosity distance [1] with this new action, ref. [133] shows that this leads to

$$d_L = c(1 + z) \sqrt{\frac{B_{F_0}}{B_F}} \int_0^z \frac{dz}{H(z)}$$

$$= c(1 + z) \sqrt{\frac{M_{\text{pl}} + \sqrt{3} \beta_F \ln |f_T(T_0)|}{M_{\text{pl}} + \sqrt{3} \beta_F \ln |f_T|}} \int_0^z \frac{dz}{H(z)},$$

as the luminosity distance for $f(T)$ gravity, which limits to the GR formula for $B_F = 1$.

### 3 Probing $f(T)$ gravity by its induced variation in $\alpha$

In this section we present the inferred constraints on five distinct $f(T)$ gravitational models by adopting a Bayesian approach for each model under consideration. This was implemented in the Markov chain Monte Carlo (MCMC) Ensemble sampler emcee [147]. We then analysed our chains by the publicly available package ChainConsumer [148].

We consider flat priors for all the varied $f(T)$ model parameters $\Theta = \{\Upsilon, \Omega_0^m, H_0, \beta_F\}$, where $\Upsilon$ is the specific model parameter characterising each particular model which will be discussed in the next section, $\Omega_0^m$ is the dimensionless energy density of pressureless matter today, $H_0$ denotes the Hubble’s constant, and $\beta_F$ is the electromagnetic coupling constant defined in eq. (2.24).

We have independently and jointly considered the measurements of $\Delta \alpha/\alpha$ from the archival astrophysical data measurements from quasar absorption lines observed at the Keck (K) observatory [98] and with the VLT (V) [99], along with a set of 21 dedicated new measurements (N) [62, 100–107], and the constraint from the Oklo (O) natural nuclear reactor at an effective redshift of $z = 0.14$ [92]. We remark that the measurements contained in the N data set were reported from the ESO Ultraviolet and Visual Echelle Spectrograph (UVES) Large Program which was specifically developed for such measurements. In what follows, we denote the joint data sets of: N + O by NO, K + V by KV, K + V + N by KVN, and K + V + N + O by KVNO.

Additionally, we will be making use of the Supernovae Type Ia (SN) Pantheon Sample [149], and a cosmic chronometers (CC) data set [150–155] composed of Hubble parameter measurements which are determined from the differential age of old and passive evolving galaxies [156]. We further adopt a prior likelihood ($H_0^0$) on the Hubble constant of $H_0 = 74.03 \pm 1.42 \text{ km s}^{-1}\text{Mpc}^{-1}$ [157], in order to check for any model parameter dependencies on the value of the Hubble constant.
### Table 1

The mean value and the corresponding 68% limits of the model parameters of the five $f_i(T)$ models ($i \in \{1, 2, 3, 4, 5\}$), as described in section 3.1. We also report the minimum $\chi^2$ value.

| $f_1(T)$ Model | Parameter | SN + CC + $H_0^R$ | SN + CC + KVNO | SN + CC + KVNO + $H_0^R$ |
|----------------|-----------|-------------------|----------------|--------------------------|
| $b$            | $-0.16^{+0.24}_{-0.49}$ | $0.003^{+0.053}_{-0.059}$ | $-0.001^{+0.050}_{-0.048}$ |
| $\Omega_m$     | $0.281^{+0.036}_{-0.035}$ | $0.300^{+0.026}_{-0.024}$ | $0.283^{+0.023}_{-0.021}$ |
| $H_0$          | $72.8^{+1.4}_{-1.3}$ | $68.9^{+2.0}_{-1.9}$ | $72.2^{+1.2}_{-1.2}$ |
| $\beta_F$      | $0.28^{+0.32}_{-0.32}$ | $-0.003^{+0.056}_{-0.056}$ | $-0.003^{+0.074}_{-0.074}$ |
| $\chi^2_{min}$ | 1042.578 | 1365.427 | 1370.014 |

| $f_2(T)$ Model | Parameter | SN + CC + $H_0^R$ | SN + CC + KVNO | SN + CC + KVNO + $H_0^R$ |
|----------------|-----------|-------------------|----------------|--------------------------|
| $1/p$          | $0.093^{+0.171}_{-0.079}$ | $(10.8^{+33.9}_{-4.9}) \times 10^{-3}$ | $(41.7^{+9.3}_{-30.8}) \times 10^{-3}$ |
| $\Omega_0$     | $0.279^{+0.025}_{-0.031}$ | $0.300^{+0.021}_{-0.020}$ | $0.283^{+0.020}_{-0.019}$ |
| $H_0/\text{km s}^{-1}\text{Mpc}^{-1}$ | $72.2^{+1.4}_{-1.2}$ | $69.0^{+1.8}_{-1.9}$ | $72.2^{+1.2}_{-1.2}$ |
| $\beta_F$      | $-0.10^{+0.49}_{-0.56}$ | $-0.01^{+0.45}_{-0.55}$ | $-0.07^{+0.58}_{-0.55}$ |
| $\chi^2_{min}$ | 1045.741 | 1366.626 | 1371.221 |

| $f_3(T)$ Model | Parameter | SN + CC + $H_0^R$ | SN + CC + KVNO | SN + CC + KVNO + $H_0^R$ |
|----------------|-----------|-------------------|----------------|--------------------------|
| $1/q$          | $0.065^{+0.088}_{-0.045}$ | $(15.7^{+28.7}_{-4.9}) \times 10^{-3}$ | $0.029^{+0.018}_{-0.017}$ |
| $\Omega_0$     | $0.279^{+0.021}_{-0.020}$ | $0.302^{+0.022}_{-0.023}$ | $0.283^{+0.019}_{-0.019}$ |
| $H_0/\text{km s}^{-1}\text{Mpc}^{-1}$ | $72.2^{+1.3}_{-1.2}$ | $69.0^{+2.0}_{-1.9}$ | $72.2^{+1.2}_{-1.2}$ |
| $\beta_F$      | $-0.22^{+0.76}_{-0.40}$ | $-0.05^{+0.61}_{-0.53}$ | $0.00^{+0.61}_{-0.52}$ |
| $\chi^2_{min}$ | 1046.074 | 1366.661 | 1371.241 |

| $f_4(T)$ Model | Parameter | SN + CC + $H_0^R$ | SN + CC + KVNO | SN + CC + KVNO + $H_0^R$ |
|----------------|-----------|-------------------|----------------|--------------------------|
| $n$            | $1.07^{+0.04}_{-0.05}$ | $1.16^{+0.51}_{-0.07}$ | $1.16^{+0.53}_{-0.05}$ |
| $\Omega_0$     | $0.264^{+0.033}_{-0.029}$ | $0.301^{+0.020}_{-0.021}$ | $0.282^{+0.019}_{-0.017}$ |
| $H_0/\text{km s}^{-1}\text{Mpc}^{-1}$ | $72.5^{+1.4}_{-1.1}$ | $69.0^{+1.9}_{-1.8}$ | $72.2^{+1.2}_{-1.2}$ |
| $\beta_F$      | $-0.25^{+0.37}_{-0.32}$ | $(-1.6^{+2.6}_{-2.3}) \times 10^{-6}$ | $(-1.5^{+2.5}_{-2.4}) \times 10^{-6}$ |
| $\chi^2_{min}$ | 1044.388 | 1365.049 | 1369.635 |

| $f_5(T)$ Model | Parameter | SN + CC + $H_0^R$ | SN + CC + KVNO | SN + CC + KVNO + $H_0^R$ |
|----------------|-----------|-------------------|----------------|--------------------------|
| $n$            | $1.49^{+0.20}_{-0.15}$ | $1.934^{+0.055}_{-0.331}$ | $1.941^{+0.047}_{-0.306}$ |
| $\Omega_0$     | $0.298^{+0.076}_{-0.091}$ | $0.291^{+0.051}_{-0.155}$ | $0.144^{+0.156}_{-0.028}$ |
| $H_0/\text{km s}^{-1}\text{Mpc}^{-1}$ | $72.8^{+1.4}_{-1.3}$ | $68.5^{+2.2}_{-1.9}$ | $72.2^{+1.2}_{-1.3}$ |
| $\beta_F$      | $0.039^{+0.043}_{-0.037}$ | $(-6.8^{+2.1}_{-2.4}) \times 10^{-4}$ | $(3.8^{+10.5}_{-9.1}) \times 10^{-7}$ |
| $\chi^2_{min}$ | 1043.087 | 1365.321 | 1370.359 |
3.1 Current constraints

We will be considering five distinct \( f(T) \) models, which have been extensively studied in the literature and found to be cosmologically viable (see, for instance, ref. [34]). The inferred mean values and 68\% limits are reported in table 1, and the obtained results are discussed in the below sections.

3.1.1 The \( f_1(T) \) model

We consider the power-law model [28] as the first \( f(T) \) model, specified by

\[
f_1(T) = -T + \alpha_1 T^b,
\]

where \( \alpha_1 \) and \( b \) are constant model parameters, such that

\[
\alpha_1 = \left(6H_0^2\right)^{1-b} \frac{1 - \Omega^m_0}{2b - 1},
\]

which follows from eq. (2.16). Clearly, we recover the \( \Lambda \)CDM model when \( b = 0 \), while this \( f(T) \) model can mimic the Dvali-Gabadadze-Porrati (DGP) model [158] when \( b = 1/2 \).

The inferred model parameter constraints are reported in the top panel of table 1, and in figure 1 we illustrate the marginalised two-dimensional likelihood constraints. We should remark that even without the varying fine-structure constant observational probes, we were still able to impose a robust constraint on \( \beta_F \) via the SN likelihood. As expected, the constraints on \( \beta_F \) improve significantly when we further make use of the KVNO data set, from which we find that \( \beta_F \) is compatible with zero. Consequently, there is a negligible deviation from the \( f(T) \) distance-duality relation in this model.

The derived constraint on \( b = -0.16^{+0.24}_{-0.49} \) from the SN + CC + \( H_0^R \) joint data set is consistent with the findings in previous studies [34, 159–161], which however did not consider a varying \( \beta_F \). As illustrated in figure 1, the SN + CC + KVNO and SN + CC + KVNO + \( H_0^R \) joint data sets also improve the constraints on the model parameter \( b \), which we find to be consistent with zero. Consequently, our constraints are tighter than the ones reported in a similar analysis of ref. [143]. Indeed, all results show that the \( f_1(T) \) model is in agreement with the \( \Lambda \)CDM model at the 1\( \sigma \) level, in line with refs. [34, 143, 159–161], and different from ref. [117].

3.1.2 The \( f_2(T) \) model

The second \( f(T) \) model is the square-root-exponential model [29] given by

\[
f_2(T) = -T + \alpha_2 T_0 \left(1 - e^{-p\sqrt{T/T_0}}\right),
\]

with model parameters \( \alpha_2 \) and \( p \), and we recall that \( T_0 = T(z = 0) = 6H_0^2 \) denotes the current torsion scalar. From the Friedmann eq. (2.16) we find that

\[
\alpha_2 = -\frac{1 - \Omega^m_0}{1 - (1 + p)e^{-p}}.
\]

Thus, for \( p \to +\infty \), the \( f_2(T) \) model reduces to the concordance \( \Lambda \)CDM model. In our analysis we therefore vary the parameter \( 1/p \), for which we recover the \( \Lambda \)CDM model when \( 1/p \to 0^+ \).
Figure 1. Marginalised two-dimensional likelihood constraints on the parameters of the $f_1(T)$ model of eq. (3.1).

The inferred constraints on $1/p$ from the joint data sets of SN + CC + $H^R_0$ and SN + CC + KVNO + $H^R_0$ are consistent with zero at around 1σ, while the $f_2(T)$ model is found to be in agreement with the ΛCDM model at around 2σ when the SN + CC + KVNO data set is adopted. This observation is in line with other studies, such as refs. [34, 117, 143, 159–161]. We depict the marginalised confidence contours in figure 2 and we list all the derived constraints in the second panel of table 1.

This $f(T)$ model is also found to be consistent with the distance-duality relation, although the KVNO data set did not significantly improve the constraints on $\beta_F$, which were always found to be in agreement with zero. However, the variation of the fine-structure constant relationship of eq. (2.30) led to tighter constraints on $1/p$ than those reported in ref. [143].

3.1.3 The $f_3(T)$ model

A similar model to the $f_2(T)$ model is the exponential model [29], which is also motivated by $f(R)$ gravity [162], and is given by

$$f_3(T) = -T + \alpha_3 T_0 \left( 1 - e^{-qT/T_0} \right),$$

(3.5)
with
\[
\alpha_3 = \frac{1 - \Omega_0^m}{-1 + (1 + 2q)e^{-q}},
\]
(3.6)
and \(q\) is the remaining model parameter. Again, we observe that the \(\Lambda\)CDM model is recovered when \(q \to +\infty\), or equivalently \(1/q \to 0^+\). For convenience, we will be considering \(1/q\) as our free parameter.

We report the derived parameter constraints in the third panel of table 1 and we depict the marginalised confidence contours in figure 3. Similar to the previous exponential \(f(T)\) model, the model parameter \(1/q\) is found to be consistent with the \(\Lambda\)CDM limit at around \(1\sigma\). This is compatible with the results of refs. [34, 117, 160, 161], although in these studies the possible variation of the fine-structure constant has not been taken into account.

Moreover, this exponential \(f(T)\) model is characterised by a null variation in the fine-structure constant, since \(\beta_F\) is always found to be consistent with zero. However, we note that the KVNO data set did not significantly ameliorate the constraints inferred by the SN + CC + \(H_0^R\) joint data set.
3.1.4 The $f_4(T)$ model

The next model which will be considered in our analysis is the logarithmic model [163], given by

$$f_4(T) = -T + \alpha_4 T_0 \sqrt{\frac{T}{mT_0}} \ln \left( \frac{mT_0}{T} \right),$$  \hspace{1cm} (3.7)

such that

$$\alpha_4 = -\frac{(1 - \Omega_m^0) \sqrt{m}}{2},$$  \hspace{1cm} (3.8)

and $m$ is the model parameter which will be varied in our MCMC analysis. We should remark that unlike the previously considered $f(T)$ models, this model cannot reduce to the concordance model of cosmology for any chosen value of $m$.

Interestingly enough, the background evolution of this model coincides with that of the spatially flat self-accelerating branch of the DGP braneworld model [158, 164]. However, the evolution of cosmological perturbations differ from one model to another. For instance, the functional forms of $G_{\text{eff}}/G_N$ are not identical. Obviously, the well-known significant inconsistencies of the spatially flat self-accelerating DGP model with cosmological data (see...
Figure 4. Marginalised two-dimensional likelihood constraints on the parameters of the \( f_4(T) \) model specified by eq. (3.7).

Moreover, the resulting Friedmann equation is independent from the model parameters \( \alpha_4 \) and \( m \), in contrast with all the other \( f(T) \) models considered in this section. Consequently, no constraints can be placed on the free parameter \( m \) with cosmological data sets which solely probe the background evolution of this model. Indeed, to the best of our knowledge, this is the first analysis which reports a constraint on the model parameter \( m \). We were able to place some limits on \( m \), since this parameter appears in the variation of the fine-structure constant \( f(T) \) relationship defined by eq. (2.30).

We report the model parameter constraints in the penultimate panel of table 1, and the marginalised two-dimensional likelihood constraints are depicted in figure 4. With the considered data sets, we were able to place a lower bound on \( m \gtrsim 0.57 \) \( (m \gtrsim 0.5) \) with the \( \text{SN + CC + } H_0^R \) \( (\text{SN + CC + KVNO/ SN + CC + KVNO + } H_0^R) \) joint data set.

In order for this model to satisfy the adopted tight limits on the variation of the fine-structure constant, \( \beta_F \) was robustly constrained to \( \sim 10^{-6} \). Thus, the \( \beta_F \) constraints imposed by the KVNO data set were found to be of a similar order to the inferred constraints on the theoretical phenomenological parametrisations of section A.3. Furthermore, this model
seems to favor slightly low values of $\Omega_0^m$, particularly when high $H_0$ values are obtained. Consequently, this model will be disfavored in light of the $H_0$ tension, as already highlighted in refs. [34, 160, 161].

### 3.1.5 The $f_5(T)$ model

Our last model is the hyperbolic-tangent model [168] which is specified as follows

$$
 f_5(T) = -T + \alpha_5 T^n \tanh \left( \frac{T_0}{T} \right),
$$

with model parameters $\alpha_5$ and $n$. From eq. (2.16), we find that

$$
 \alpha_5 = \frac{(6H_0^2)^{1-n}(1 - \Omega_0^m)}{(2n - 1) \tanh(1) - 2 \text{sech}^2(1)},
$$

and therefore $n$ will be the varying model parameter. Similar to the previous $f_4(T)$ model, the $\Lambda$CDM cosmology cannot be recovered as a limiting case of the $f_5(T)$ model for any arbitrary value of $n$. Therefore, the parameter $n$ does not characterise the deviation from the concordance model of cosmology.

The derived constraints are listed in the last panel of table 1, and the corresponding marginalised confidence contours are illustrated in figure 5. From the SN + CC + $H_R^0$ data set, we obtain a smaller value of $n$ when compared with the inferred mean value of this parameter from the other data sets which include the KVNO measurements. Given that small values of $n \lesssim 1.69$ [168] naturally give rise to the crossing of the phantom divide line, the KVNO data set restricts this possibility as higher values of $n$ are preferred. We note that our inferred constraints on $n$ agree with the reported results in refs. [159–161, 169], although in these analyses $\beta_F$ was neglected. We further observe that the higher the value of $n$, the smaller the value of $\Omega_0^m$, which is consistent with refs. [160, 161]. This will however make this model inconsistent with the CMB data.

Moreover, the $\beta_F$ electromagnetic coupling parameter is loosely constrained with the SN + CC + $H_R^0$ joint data set, however the inclusion of the KVNO measurements lead to very stringent constraints on this parameter. Indeed, the inferred constraints on $\beta_F$ with the SN + CC + KVNO and SN + CC + KVNO + $H_R^0$ joint data sets are found to be similar to the derived constraints in the phenomenological parametrisations of section A.3 and in the case of the above $f_4(T)$ model. Consequently, the $f_5(T)$ hyperbolic-tangent model is also found to be in agreement with the distance-duality relation.

### 3.2 Implications for an effective Newton’s constant

As already mentioned in section 2.2, $f(T)$ gravity gives rise to an effective gravitational constant, in similarity with the majority of modified gravitational frameworks. This variation is generically given by $G_{\text{eff}} = \frac{G_N}{f(T)}$, with $G_N$ being Newton’s gravitational constant [34, 130, 161]. Hence, the effective gravitational constant in the considered $f(T)$ models will coincide with $G_N$ at earlier times, and we expect some deviation at late-times.

In the panels of figure 6, we reconstruct the redshift evolution of the variation of the effective gravitational constant, specified by the quantity $G_{\text{eff}}/G_N - 1$, at the 1σ confidence level. We also report the inferred value of $G_{\text{eff}}/G_N - 1$ at $z = 0$ in the third column of table 2. For the power-law $f_1(T)$ model, the square-root-exponential $f_2(T)$ model and the exponential $f_3(T)$ model, we can clearly observe that $G_{\text{eff}} \simeq G_N$, particularly when we make
Figure 5. Marginalised two-dimensional likelihood constraints on the \( f_5(T) \) model parameters specified in eq. (3.9).

use of the SN + CC + KVNO + \( H_0^R \) data set. Indeed, we observe that in these models, the KVNO measurements significantly restrict the deviation of \( G_{\text{eff}} \) from \( G_N \), in agreement with current independent bounds on the time variation of the gravitational constant (see, for instance, refs. [170–175] and references therein). Moreover, a much tighter 1\( \sigma \) deviation of \( G_{\text{eff}} \) from \( G_N \) is obtained at around \( z \simeq 1 \), although this redshift is model dependent.

On the other hand, we observe a significant deviation of \( G_{\text{eff}}/G_N \) from unity at low redshifts in the logarithmic \( f_4(T) \) and hyperbolic-tangent \( f_5(T) \) scenarios. For the quantification of the model dependent deviations from the measured values of the effective gravitational constant, we consider the current time variation of the effective gravitational constant

\[
\frac{\dot{G}_{\text{eff}}}{G_N} \bigg|_0 = T_0 H_0 \frac{f_{TT}(T_0)}{f_T(T_0)} \frac{dE^2}{dz} \bigg|_0,
\]

where \( E(z) \equiv H(z)/H_0 \). We adopt two of the most stringent constraints on the time variation of the effective gravitational constant, one derived from lunar laser ranging (LLR) experiments [170] and another one inferred from big bang nucleosynthesis (BBN) [171]. We therefore compute \( d^{\text{LLR}} (\dot{G}_{\text{eff}}/G_N|_0) \) and \( d^{\text{BBN}} (\dot{G}_{\text{eff}}/G_N|_0) \), where \( d^i (\dot{G}_{\text{eff}}/G_N|_0) \) is the distance in
Figure 6. Reconstruction of $\frac{\dot{G}_{\text{eff}}}{G_N} - 1$ and its 1σ uncertainty as a function of redshift, for the five $f(T)$ models considered in this work, namely $f_1(T)$ (top left), $f_2(T)$ (top right), $f_3(T)$ (middle left), $f_4(T)$ (middle right) and $f_5(T)$ (bottom) models.

$\sigma$ units between the measured value $(\dot{G}_{\text{eff}}/G_N|_0)^i$, and the theoretical value $(\dot{G}_{\text{eff}}/G_N|_0)^{f(T)}$, of the time variation in the gravitational constant

$$d^i \left( \frac{\dot{G}_{\text{eff}}}{G_N} \bigg|_{0} \right) = \frac{(\dot{G}_{\text{eff}}/G_N|_0)^{f(T)} - (\dot{G}_{\text{eff}}/G_N|_0)^i}{\sqrt{\sigma_{f(T)}^2 + \sigma_i^2}},$$

(3.12)

with $\sigma_{i,f(T)}$ being the corresponding 1σ uncertainties. The results are presented in table 2.
Moreover, the inferred non-null variation of the effective gravitational constant in the $f_i(T)$ with the reported measurements, and hence could be considered as the most viable models. We should also remark that the other $f_i(T)$ also report the minimum $G_{\text{eff}}/G_N - 1|_0$ correspondingly. Values of the current variation of the effective gravitational constant $G_{\text{eff}}/G_N - 1|_0$, and the corresponding $1\sigma$ uncertainties for the five $f_i(T)$ models ($i \in \{1, 2, 3, 4, 5\}$) with different data set combinations. We also show in the last two columns the distances $d_{\text{LLR}} (G_{\text{eff}}/G_N |_0)$ and $d_{\text{BBN}} (G_{\text{eff}}/G_N |_0)$, as defined in eq. (3.12), between each of our determinations of $G_{\text{eff}}/G_N |_0$ and the values reported in ref. [170] and ref. [171], respectively.

| Model  | Data set       | $G_{\text{eff}}/G_N - 1|_0$ | $d_{\text{LLR}} (G_{\text{eff}}/G_N |_0)$ | $d_{\text{BBN}} (G_{\text{eff}}/G_N |_0)$ |
|--------|----------------|----------------------------|------------------------------------------|------------------------------------------|
| $f_1(T)$ | SN + CC + KVNO | $6.86 \times 10^{-7} \pm 0.05$ | 0.02                                      | 0.18                                      |
|        | SN + CC + KVNO + $H_0^R$ | $6.89 \times 10^{-7} \pm 0.08$ | 0.02                                      | 0.14                                      |
| $f_2(T)$ | SN + CC + KVNO | $0.00 \pm 5.14 \times 10^{-6}$ | 0.93                                      | 0.35                                      |
|        | SN + CC + KVNO + $H_0^R$ | $0.00 \pm 3.10 \times 10^{-6}$ | 0.93                                      | 0.35                                      |
| $f_3(T)$ | SN + CC + KVNO | $0.00 \pm 2.00 \times 10^{-5}$ | 0.92                                      | 0.35                                      |
|        | SN + CC + KVNO + $H_0^R$ | $0.00 \pm 3.77 \times 10^{-6}$ | 0.93                                      | 0.35                                      |
| $f_4(T)$ | SN + CC + KVNO | $0.11 \pm 0.21$ | 1.98                                      | 1.99                                      |
|        | SN + CC + KVNO + $H_0^R$ | $0.11 \pm 0.21$ | 2.09                                      | 2.10                                      |
| $f_5(T)$ | SN + CC + KVNO | $1.85 \pm 0.23$ | $-1.09$                                   | $-1.07$                                   |
|        | SN + CC + KVNO + $H_0^R$ | $1.97 \pm 0.24$ | $-0.96$                                   | $-0.95$                                   |

Table 2. Values of the current variation of the effective gravitational constant $G_{\text{eff}}/G_N - 1|_0$, and the corresponding $1\sigma$ uncertainties for the five $f_i(T)$ models ($i \in \{1, 2, 3, 4, 5\}$) with different data set combinations. We also show in the last two columns the distances $d_{\text{LLR}} (G_{\text{eff}}/G_N |_0)$ and $d_{\text{BBN}} (G_{\text{eff}}/G_N |_0)$, as defined in eq. (3.12), between each of our determinations of $G_{\text{eff}}/G_N |_0$ and the values reported in ref. [170] and ref. [171], respectively.

| Parameter | SN + CC + $H_0^R$ | SN + CC + KVNO | SN + CC + KVNO + $H_0^R$ |
|-----------|-------------------|----------------|---------------------|
| $\Omega_m^0$ | $0.283^{+0.019}_{-0.017}$ | $0.300^{+0.022}_{-0.019}$ | $0.283^{+0.019}_{-0.018}$ |
| $H_0$ | $72.2^{+1.1}_{-1.2}$ | $68.9^{+1.0}_{-1.7}$ | $72.2^{+1.1}_{-1.2}$ |
| $\chi^2_{\text{min}}$ | 1046.074 | 1366.684 | 1371.271 |

Table 3. The mean value and the corresponding 68% limits of the $\Lambda$CDM model parameters. We also report the minimum $\chi^2$ value.

where we could notice that the $f_4(T)$ and $f_5(T)$ models are within the $\sim 2\sigma$ and $\sim 1\sigma$ limits, respectively. We should also remark that the other $f(T)$ models are in a better agreement with the reported measurements, and hence could be considered as the most viable models. Moreover, the inferred non-null variation of the effective gravitational constant in the $f_5(T)$ model ($G_{\text{eff}}/G_N - 1|_0 > 1$) is unequivocally forbidden by current constraints [170–175], and therefore this model could clearly be considered as a cosmologically non-viable model, consistent with the results of refs. [34, 160, 161]. Consequently, only the $f(T)$ models which possess the $\Lambda$CDM model as a limiting case are found to be in an excellent agreement with the considered constraints on the variation of the gravitational constant, and could therefore still be considered as viable cosmological models.

### 3.3 Model comparison

We are here interested in the significance of the obtained results from the statistical point of view. In order to judge the fitting quality of a given model with $\hat{k}$ parameters, to a data set having $\hat{n}$ measurements, we use the standard $\chi^2$ per degree of freedom, specified by $\chi^2_{\text{dof}} = \chi^2_{\text{min}}/\hat{n} - \hat{k}$. To compare the evidence for and against cosmological models, we can
Employ well-known information criteria to assess the considered \( f(T) \) models. We therefore adopt the Akaike Information Criterion (AIC) [176], given by

\[
AIC = \chi^2_{\text{min}} + 2\tilde{k}, \tag{3.13}
\]

and the Bayesian Information Criterion (BIC) [177], specified by

\[
BIC = \chi^2_{\text{min}} + \tilde{k}(\ln \tilde{n}). \tag{3.14}
\]

We should remark that when \( \tilde{n} \gg \tilde{k} \), the BIC approximates the logarithm of the widely known Bayes factor [178].

The differences between the derived AIC or BIC values could be used to compare the models with a reference model. In our case, we will be comparing the considered \( f(T) \) models with the \( \Lambda \)CDM model, where we present the parameter constraints of the latter model in Table 3. According to Jeffreys’ scale [179], one could then determine the model which could be

\[
\Delta \text{AIC} = \text{AIC}_2 - \text{AIC}_1,
\]

\[
\Delta \text{BIC} = \text{BIC}_2 - \text{BIC}_1,
\]

\[
\Delta \text{BIC} = \text{BIC}_2 - \text{BIC}_1.
\]

where \( \Delta \text{AIC} \) and \( \Delta \text{BIC} \) denote the difference between the derived AIC or BIC values for two models. The outcomes are considered to be: weak if \( \Delta \text{AIC} < 2 \), mildly strong whenever \( 2 < \Delta \text{AIC} < 5 \), and highly significant in the instance of \( \Delta \text{AIC} \gtrsim 5 \). Similarly, the difference \( \Delta \text{BIC} = \text{BIC}_2 - \text{BIC}_1 \), represents the extent to which model \( \mathcal{M}_1 \) is preferred over model \( \mathcal{M}_2 \), such that \( 0 < \Delta \text{BIC} < 2 \) indicates a weak evidence, \( 2 < \Delta \text{BIC} < 6 \) illustrates a positive evidence, while \( 6 < \Delta \text{BIC} < 10 \) and \( 10 \leq \Delta \text{BIC} \) denote a strong and a very strong evidence, respectively. Clearly, from the reported values in Table 4, the \( \Lambda \)CDM model is mildly favoured according to the \( \Delta \text{AIC} \) analysis, whereas the \( \Delta \text{BIC} \) inferred values indicate that the considered \( f(T) \) models are found to be very strongly disfavoured. This outcome is emerging from the fact that since current varying fine-structure constant measurements are consistent with a null variation, there is no need for more model parameters than the minimalistic \( \Lambda \)CDM model parameters in order to describe this data set.

In Table 4 we also report the unnormalised likelihood that model \( \mathcal{M}_\varphi \) is the best model out of all the considered models by adopting the Akaike weight likelihood [180, 181]

\[
\mathcal{L}_{\varphi}^{\text{AIC}} = \frac{\exp(-\text{AIC}_\varphi/2)}{\sum_{\mathcal{M}_i} \exp(-\text{AIC}_i/2)}, \tag{3.15}
\]
Similarly, we also consider the Bayes weight likelihood \([180, 182]\) of model \(\mathcal{M}_\phi\) by the relation
\[
\mathcal{L}^{\text{BIC}}_\phi = \frac{\exp(-\text{BIC}_\phi/2)}{\sum_{N=1}^6 \exp(-\text{BIC}_N/2)},
\]
which determines the model’s percentage likelihood. As expected from the above differences in information criteria, the \(\Lambda\)CDM model has a much larger percentage probability of being the preferred model with the considered data sets.

4 Conclusions

In this work, we focused our attention on the \(f(T)\) gravitational framework in section 2. After our concise discussion on TG and \(f(T)\) cosmology, we analysed the induced variation of the fine-structure constant in the \(f(T)\) gravitational scenario. Indeed, we have revisited and updated the theoretical \(f(T)\) relationship of \(\Delta \alpha/\alpha\) in eq. (2.30), and the modification of the luminosity distance in eq. (2.31).

In section 3 we proceeded to the confrontation of five \(f(T)\) models with the Supernovae Type Ia Pantheon Sample, Hubble parameter measurements and measurements of the variation of the fine-structure constant. We have considered three models \((f_1(T), f_2(T), f_3(T))\) which posses the \(\Lambda\)CDM model as a limiting case. From our inferred results of table 1, we observe that the \(f_1(T), f_2(T)\), and \(f_3(T)\) models do not exclude the \(\Lambda\)CDM paradigm. It was also found that these are cosmologically viable models, since the reconstructed deviation of their effective gravitational constant from \(G_N\) is negligible and in an excellent agreement with current experimental bounds.

On the other hand, the remaining \(f(T)\) models do not contain the \(\Lambda\)CDM scenario as a particular limit. The logarithmic \(f_4(T)\) model is identical to the spatially flat self-accelerating branch of the DGP model at the background level, and therefore we expected that this model will not be cosmologically viable. Indeed, the reconstructed evolution of \(G_{\text{eff}}/G_N\) significantly deviated from unity at late-times, which clearly is not consistent with current bounds on \(G_{\text{eff}}/G_N\). Moreover, this model was also found to be inconsistent with current constraints on \(G_{\text{eff}}/G_N[0]\). The remaining hyperbolic-tangent model is characterised by the crossing of the phantom divide line, although our inferred constraints did not favor this possibility due to a preference to relatively large values of the \(f_5(T)\) model parameter \(n\). Furthermore, the reconstructed evolution of \(G_{\text{eff}}/G_N\) was not found to be in agreement with the respective experimental bounds.

Also in appendix A, we explore a number of widely known theoretical parametrisations of the violation of the distance-duality relation, on which we imposed very stringent constraints \((\mathcal{O}(10^{-7}))\) by adopting several measurements of the variation of the fine-structure constant. From this analysis, we clearly illustrated that current data sets are in an excellent agreement with the distance-duality relation irrespective of the adopted phenomenological parametrisation.

A common feature of all \(f(T)\) models is that they are all in an excellent agreement with the distance-duality relation. Thus, with current measurements of the variation of the fine-structure constant, we have been able to confirm the validity of EEP in \(f(T)\) gravity. We expect that the relevant constraints will significantly improve in the era of the new generation of high-resolution ultra-stable spectrographs, such as ESPRESSO [183] and ELT-HIRES [184], which will lead to improvements in local atomic clock tests and complimentary cosmological observations.
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A Phenomenological violation of the cosmic distance-duality relation

The EEP could be easily broken by introducing a phenomenological nonminimal multiplicative coupling between a scalar field $\phi$ and matter fields. For instance, in the electromagnetic sector, the action formalism would be given by

$$S_{\text{EM}} = \int d^4x \sqrt{-g} B_F(\phi) \mathcal{L}_{\text{EM}},$$

where the electromagnetic Lagrangian is denoted by $\mathcal{L}_{\text{EM}}$, $g$ is the determinant of the space-time metric $g_{\mu\nu}$, and the scalar field dependent electromagnetic coupling is denoted by $B_F(\phi)$. We remark that the dynamical evolution of the scalar field and the metric tensor are not relevant at this point, and such dynamics are encoded in the scalar-gravitational field Lagrangian.

After the variation of the above action with respect to the electromagnetic four-potential $A_\mu$, we arrive at the homogeneous modified Maxwell equations

$$\nabla_\nu (B_F(\phi) F_{\mu\nu}) = 0,$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ being the standard antisymmetric Faraday tensor, and $\nabla_\nu$ being the regular covariant derivative calculated with the Levi-Civita connection. From eq. (A.2) we know that photons propagate on null geodesics, and therefore the reciprocity relation still holds [185]. However, the number of photons is no longer conserved, which consequently leads to a violation of Etherington’s relation [186]. Thence, one could parametrise the violation of the distance-duality relation by

$$\eta(z) = \frac{D_L(z)}{D_A(z)(1+z)^2},$$

where $D_L(z)$ and $D_A(z)$ are the luminosity distance and angular diameter distance at redshift $z$, respectively. Clearly, the distance-duality relation is recovered when $\eta(z) = 1$.

In our analyses, we will be adopting the following commonly used phenomenological parametrisations

$$\eta(z) = \eta_0,$$

$$\eta(z) = 1 + \eta_1 z,$$

$$\eta(z) = 1 + \eta_2 \frac{z}{1+z},$$

$$\eta(z) = 1 + \eta_3 \ln(1+z),$$

$$\eta(z) = (1+z)^\epsilon.$$
A.1 Induced variation of the fine-structure constant

A number of well-known cosmological consequences arising from the violation of the distance-duality relationship have been widely explored in the literature. We will be particularly interested in the induced variation of the electromagnetic fine-structure constant, which is explicitly related with the nonminimal electromagnetic coupling via \( \alpha \propto B_F^{-1}(\phi) \) \cite{87, 88, 213, 214}. Indeed, the unequivocal relationship of the redshift evolution of \( \alpha(z) \), with the nonminimal electromagnetic coupling and Etherington’s parameter \( \eta(z) \), can be expressed as follows

\[
\frac{\Delta \alpha(z)}{\alpha} = \frac{\alpha(z) - \alpha_0}{\alpha_0} = \frac{B_F(\phi_0)}{B_F(\phi)} - 1 = \eta^2(z) - 1 , \tag{A.9}
\]

where a 0-subscript indicates the current epoch values at \( z = 0 \). Thus, constraints on \( \Delta \alpha(z)/\alpha \) can be interchanged to constraints on \( \eta(z) \), and vice versa. Furthermore, the current temporal variation of \( \alpha(z) \), simplifies to the following equation

\[
\frac{\dot{\alpha}}{\alpha}|_0 = -2H_0 \frac{d\eta}{dz}|_0 \tag{A.10}
\]
Figure 7. Marginalised posterior distributions of the parameters characterising the phenomenological $\eta(z)$ theoretical parametrisations of eqs. (A.4)–(A.8).

where the Hubble constant is denoted by $H_0 = a^{-1} \frac{da}{dt}|_0 = a^{-1} \dot{a}|_0$, with $t$ being the cosmic time and $a(t)$ is the cosmic scale factor of a spatially-flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric with $a_0 = 1$.

A.2 Data sets and methodology

We will be implementing the methodology outlined in section 3 and apply it for each phenomenological parametrisation of eqs. (A.4)–(A.8). We thus make use of the MCMC
Figure 8. Best-fit $\Delta \alpha / \alpha$ evolution along with the 1$\sigma$ posterior spread for the $\eta_0$ (top left), $\eta_1$ (top right), $\eta_2$ (middle left), $\eta_3$ (middle right), and $\epsilon$ (bottom) parametrisations of eqs. (A.4)–(A.8). The illustrated data points from Keck, VLT, New and Oklo data set measurements are described in section A.2.

Ensemble sampler emcee [147], and analyse our chains by the publicly available package ChainConsumer [148].

In our constraint analyses we made use of the currently available measurements of $\Delta \alpha / \alpha$ and $\dot{\alpha} / \alpha|_0$. We remark that the constraints on the parameters ($\eta_{0,1,2,3}$, $\epsilon$) defining the $\eta(z)$ phenomenological functions (A.4)–(A.8), were transposed from the constraints on the variation in $\alpha(z)$ by using the direct relation given in eq. (A.9). We refer to section 3 for
a description on the several $\Delta\alpha/\alpha$ constraints, while the adopted atomic clocks laboratory (lab) constraint is specified by $\dot{\alpha}/\alpha|_0 = (-1.6 \pm 2.3) \times 10^{-17}\text{year}^{-1}$ [96]. When we include the latter constraint on the temporal variation of the fine-structure constant, we make use of a Hubble constant prior likelihood $H_0^R$ [157], since we then marginalise over $H_0$ to infer the constraints on the $\eta(z)$ parameters.

A.3 Current constraints

The inferred constraints on the model parameters of the theoretical functions defined in eqs. (A.4)–(A.8) are reported in table 5, in which a number of data sets have been adopted as indicated in the first column of this table. Although the derived constraints on $\eta_{0,1,2,3}$ and $\epsilon$ are all of the order of $10^{-7}$, the NO data set along with the laboratory measurement of the current temporal variation in $\alpha$, significantly tighten the constraints that are obtained from the Keck and VLT data sets. Moreover, the Keck and VLT data sets are characterised by the largest deviation from the distance-duality relation (due to a preference for non-null theoretical model parameter values) irrespective from the adopted parametrisation, as clearly illustrated by their joint posterior distribution in figure 7. Additionally, the inferred constraints from the Keck and VLT data sets lead to incompatible theoretical evolution of $\Delta\alpha/\alpha$. This is shown in figure 8, in which we depict the best-fit redshift evolution of $\Delta\alpha/\alpha$, along with the 68% uncertainty region.

From the NO, NO + lab + $H_0^R$ and KVNO + lab + $H_0^R$ joint data sets, we obtain a minute deviation from the distance-duality relation ($\eta(z) \approx 1$), and such result is independent from the adopted theoretical parametrisation. It is worth mentioning that our derived constraints are orders of magnitude more restrictive than the ones obtained from cosmological observations, such as in refs. [203, 207–211]. We remark that, as expected, the obtained results are in agreement with ref. [212], in which the parametrisation independence has been further shown with the expected data from upcoming experiments. Moreover, we observe that the laboratory measurement is complementary to the NO data set, and the inclusion of the laboratory measurement did not alter the inferred constraints from the NO data set. Thence, in the analyses of section 3.1, we exclude the laboratory measurement from our data sets, however we have verified that the final results do not change.

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