SPECIAL VALUES OF CANONICAL GREEN’S FUNCTIONS

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Abstract. We give a precise formula for the value of the canonical Green’s function at a pair of Weierstrass points on a hyperelliptic Riemann surface. Further we express the ‘energy’ of the Weierstrass points in terms of a spectral invariant recently introduced by N. Kawazumi and S. Zhang. It follows that the energy is strictly larger than log 2. Our results generalize known formulas for elliptic curves.

1. Introduction

Let \( M \) be a compact and connected Riemann surface of genus \( h \geq 1 \). On \( M \) one has a canonical Kähler form \( \mu \) given as follows. The space \( H \) of holomorphic differentials on \( M \) is a complex vector space of dimension \( h \). It carries a natural hermitian inner product given by \( \langle \alpha, \beta \rangle = \sqrt{-1/2} \int_M \alpha \wedge \overline{\beta} \) for elements \( \alpha, \beta \) in \( H \). Let \( (\omega_1, \ldots, \omega_h) \) be an orthonormal basis of \( H \). By the Riemann-Roch theorem, the \( \omega_i \) do not simultaneously vanish at any point of \( M \), and we put

\[
\mu = \sqrt{-1/2} \sum_{i=1}^h \omega_i \wedge \overline{\omega}_i.
\]

The associated metric has been studied in many contexts, ranging from arithmetic geometry \[3\] \[9\] \[10\] \[19\] \[21\] to perturbative string theory \[1\] \[2\] \[5\] \[6\] \[13\]. The metric is known to have everywhere non-positive curvature, and the curvature vanishes at \( x \in M \) if and only if \( x \) is a Weierstrass point on a hyperelliptic Riemann surface \[20\], Main Theorem.

In this paper we study the canonical Green’s function \( g_\mu \) associated to \( \mu \) (see Section 2 for definitions), and in particular its special values \( g_\mu(w_i, w_j) \) at pairs \( (w_i, w_j) \) of distinct Weierstrass points on a hyperelliptic Riemann surface. The starting point of our discussion is the case \( h = 1 \), with \( M \) given as a 2-sheeted cover of the Riemann sphere \( \mathbb{CP}^1 \) branched at four points, say \( \alpha_1, \alpha_2, \alpha_3 \) and \( \infty \). Let \( w_1, w_2, w_3, o \) be the four critical points of \( M \) lying over \( \alpha_1, \alpha_2, \alpha_3, \infty \). It is then known \[2\] \[15\] \[22\] that the formula

\[
g_\mu(w_i, w_j) = \frac{1}{3} \log 2 + \frac{1}{12} \log \left( \frac{|\alpha_i - \alpha_j|^2}{(|\alpha_i - \alpha_k| \cdot |\alpha_j - \alpha_k|)^2} \right)
\]

holds. By the translation invariance of \( g_\mu \) we obtain the formula

\[
\sum_{i=1}^3 g_\mu(w_i, o) = \log 2.
\]

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We will return to both these formulas in the text below.

Our purpose is to generalize formulas (1.1) and (1.2) to the case of a hyperelliptic Riemann surface of genus \( h \geq 2 \). Let \( M \) be such a surface, given as the 2-sheeted cover of \( \mathbb{CP}^1 \) branched at the \( 2h + 2 \) points \( \alpha_1, \ldots, \alpha_{2h+2} \) (these may or may not include \( \infty \)). We put, for each pair \( (i, j) \) of distinct indices,

\[
\delta_{ij} = \frac{(\alpha_i - \alpha_j)^{2h(2h+1)}}{\prod_{r \neq i}(\alpha_i - \alpha_r)^{2h+1} \prod_{r \neq j}(\alpha_j - \alpha_r)^{2h+1}}.
\]

In this formula, we disregard any difference of two roots when one of the roots is \( \infty \).

It is readily verified that \( \delta_{ij} \) is invariant under the action of \( \text{Aut}(\mathbb{CP}^1) = \text{PGL}_2(\mathbb{C}) \) on the \( \alpha_i \). In particular \( \delta_{ij} \) defines an analytic modular invariant of the pair \( (w_i, w_j) \), where \( w_i, w_j \) are the critical points of \( M \) lying over \( \alpha_i, \alpha_j \). More intrinsically \( \delta_{ij} \) is the discriminant of the branch set that one obtains by sending \( \alpha_i, \alpha_j \) to \( 0, \infty \) using an element of \( \text{Aut}(\mathbb{CP}^1) \) and by normalizing the other \( 2h \) roots such that their product is unity \([12]\).

We note that there is a tight connection with the more familiar cross ratios on \( \alpha_i \). Put \( \eta_{ijk} = \delta_{ik}\delta_{jk}^{-1} \), and let

\[
\mu_{ijkr} = (\alpha_i - \alpha_k)(\alpha_j - \alpha_r)(\alpha_j - \alpha_k)^{-1}(\alpha_i - \alpha_r)^{-1}
\]

be the cross ratio of the 4-tuple \( (\alpha_i, \alpha_j, \alpha_k, \alpha_r) \). Then the relation

\[
\mu_{i,jkr} = \eta_{ijk}\eta_{ir}^{-1} = \delta_{ik}\delta_{jr}^{-1} \delta_{ik}\delta_{jr}^{-1}
\]

holds. It is straightforward to verify that for each index \( i \), the product \( \prod_{j \neq i} \delta_{ij} \) is equal to unity.

The energy of the Weierstrass points on \( M \) is defined to be the following invariant of \( M \):

\[
\psi(M) = \frac{1}{2h+2} \sum_{i \neq j} g_\mu(w_i, w_j),
\]

where the summation runs over all pairs \( i, j \) of distinct indices. Our first result is the following.

**Theorem A.** Let \( w_i, w_j \) be two distinct Weierstrass points of \( M \). Then the formula

\[
g_\mu(w_i, w_j) = \frac{1}{4h(2h+1)} \log |\delta_{ij}| + \frac{1}{2h+1} \psi(M)
\]

holds.

In our second result we give an expression for \( \psi(M) \). Consider for the moment an arbitrary compact and connected Riemann surface \( M \) of genus \( h \geq 1 \). Let \( \Delta_\mu \) be the Laplacian on \( L^2(M, \mu) \) given by putting \( \partial \bar{\partial} f = \pi \sqrt{-1} \Delta_\mu(f) \mu \) for \( C^\infty \) functions in \( L^2(M, \mu) \). Let \( (\phi_\ell)_{\ell \geq 0} \) be an orthonormal basis of real eigenfunctions of \( \Delta_\mu \), with corresponding eigenvalues \( 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \). We then let \( \psi(M) \) denote the invariant

\[
\psi(M) = \sum_{\ell > 0} \frac{2}{\lambda_\ell} \sum_{m,n=1}^h \left| \int_M \phi_\ell \omega_m \wedge \bar{\omega}_n \right|^2
\]

of \( (M, \mu) \). This fundamental invariant was introduced and studied recently by N. Kawazumi \([18]\) and S. Zhang \([24]\). Note that \( \psi(M) \geq 0 \), and that \( \psi(M) = 0 \) if and only if \( M \) has genus one. In \([17]\) the asymptotic behavior of the \( \psi \)-invariant is determined for surfaces degenerating into a stable curve with a single node. The
result shows in particular that \( \varphi \) can be viewed as a Weil function, with respect to the boundary, on the stable (Deligne-Mumford) compactification of the moduli space of surfaces of a fixed genus \( h \geq 2 \).

**Theorem B.** Let \( M \) be a hyperelliptic Riemann surface of genus \( h \geq 2 \), and let \( \varphi(M) \) be the Kawazumi-Zhang invariant of \( M \). Then for the energy \( \psi(M) \) of the set of Weierstrass points of \( M \), the formula

\[
\psi(M) = \frac{1}{2h} \varphi(M) + \log 2
\]

holds.

In particular we find that \( \psi(M) > \log 2 \). With \( h = 1 \), the formulas in Theorems A and B specialize to equations (1.1) and (1.2), respectively.

In order to prove Theorem B we proceed as follows. First we prove that, when viewed as functions on the moduli space of hyperelliptic Riemann surfaces, both \( \psi \) and \( \frac{1}{2h} \varphi \) have the same image under \( \partial \partial \). Then we study the asymptotic behavior of the difference of \( \psi \) and \( \frac{1}{2h} \varphi \) on a family of hyperelliptic Riemann surfaces degenerating into a stable curve of compact type. By using induction on \( h \) we will then deduce that the difference is a constant on the moduli space, equal to \( \log 2 \). For the degeneration argument we will rely heavily on results obtained by R. Wentworth \[23\].

## 2. Canonical Green’s functions

In this section we collect some basic results on canonical Green’s functions. The main sources are \[3\] \[10\]. Proposition 2.2, about a ‘parallel’ property of certain \((1, 1)\)-forms related to the universal Riemann surface over the moduli space of Riemann surfaces, seems not to be well known, and is possibly of independent interest.

As before, let \( M \) be a compact and connected Riemann surface of genus \( h \geq 1 \) and \( \mu \) its canonical Kähler form. Let \( \Delta \) be the diagonal on the complex manifold \( M \times M \). The canonical Green’s function \( g_{\mu} = g_{\mu}(x, y) \) is the \( C^\infty \)-function on \( M \times M \setminus \Delta \) uniquely characterized by the following conditions:

1. \( \partial \partial g_{\mu}(x, y) = \pi \sqrt{-1}(\mu(y) - \delta_x) \) for all \( x \in M \);
2. \( \int_M g_{\mu}(x, y) \mu(y) = 0 \) for all \( x \in M \);
3. \( g_{\mu}(x, y) = g_{\mu}(y, x) \) for all \( x \neq y \in M \);
4. \( g_{\mu}(x, y) - \log |z(x) - z(y)| \) is bounded for all \( x \neq y \) in a coordinate chart \( (U, z: U \sim \mathbb{D}) \) of \( M \), where \( \mathbb{D} \) is the open unit disk.

The canonical Green’s function inverts the Laplacian \( \Delta_{\mu} \) in the sense that

\[
f(x) = -\int_M g_{\mu}(x, y) \Delta_{\mu}(f)(y) \mu(y) + \int_M f \mu
\]

holds for all \( x \in M \) and all \( f \) in \( C^\infty(M) \). As a consequence we have a formal development

\[
g_{\mu}(x, y) = \sum_{\ell > 0} \frac{\phi_{\ell}(x)\overline{\phi_{\ell}(y)}}{\lambda_{\ell}}
\]

for \( g_{\mu} \), where \( \phi_{\ell} \) and \( \lambda_{\ell} \) are the eigenfunctions and eigenvalues of \( \Delta_{\mu} \) that we have introduced above.

In \[10\] Section 7 using standard harmonic analysis explicit formulas are derived for the \( \phi_{\ell} \) and \( \lambda_{\ell} \) in the case \( h = 1 \). As a corollary of these formulas \[10\] Section 7...
contains a derivation of equation (1.2) from the resulting formal expression for $g_{\mu}$. We would like to present here an alternative argument.

**Proposition 2.1.** Assume $(M, o)$ is a complex torus, and let $N$ be a positive integer. Let $x_1, \ldots, x_{N^2-1}$ be the non-trivial $N$-torsion points of $M$. Then the formula

$$\sum_{i=1}^{N^2-1} g_{\mu}(x_i, o) = \log N$$

holds.

**Proof.** From properties (1) and (2) it follows that for every $x, y$ on $M$ with $Nx \neq y$ the equality

$$g_{\mu}(Nx, y) = \sum_{w : Nw = y} g_{\mu}(x, w)$$

holds. Now choose $x$ in a standard euclidean coordinate chart $z : U \to D$ around $o$, where $D$ is the open unit disk. Then we have

$$g_{\mu}(Nx, o) = \sum_{w : Nw = o} g_{\mu}(x, w)$$

$$= \sum_{i=1}^{N^2-1} g_{\mu}(x_i, x) + g_{\mu}(x, o)$$

$$= \sum_{i=1}^{N^2-1} g_{\mu}(x_i, x) + \log |z(x)| + a + o(1)$$

as $x \to o$, where $a$ is some constant, by properties (3) and (4). On the other hand we have

$$g_{\mu}(Nx, o) = \log |Nz(x)| + a + o(1)$$

$$= \log N + \log |z(x)| + a + o(1)$$

as $x \to o$, by property (4). The desired equality follows. \square

One obtains (1.2) by taking $N = 2$ in the above proposition.

Let $M$ denote the moduli space of compact and connected Riemann surfaces of genus $h \geq 2$, and denote by $\pi : C \to M$ the universal surface over $M$. Both are viewed as orbifolds. The canonical Green’s functions on the fibers of $\pi$ determine a generalized function $g$ on the fiber product $C \times_M C$ with logarithmic singularities along the diagonal $\Delta$. Let $\nu$ be the $(1,1)$-form on $C \times_M C$ determined by the condition $\partial Dw = \pi \sqrt{-1} (\nu - \delta_\Delta)$. Let $e^A$ denote the restriction of $\nu$ to $\Delta$, which we view as another copy of $C$. By [3] the form $e^A$ restricts to $2 - 2h$ times the canonical $(1,1)$-form $\mu$ in each fiber of $\pi : C \to M$.

The $(1,1)$-forms $e^A$ and $\nu$ are parallel over $M$ in the following sense.

**Proposition 2.2.** Let $s, t$ be arbitrary holomorphic sections of $\pi$ over an open subset $U$ of $M$. Then the equalities $s^* e^A = t^* e^A = (s, t)^* \nu$ hold on $U$.

**Proof.** Without loss of generality we may replace $M$ by a finite cover so that we may assume that $M$ is equipped with a universal theta characteristic $\alpha$, i.e. a consistent choice of a divisor class $\alpha$ of degree $h - 1$ on each surface $M$ such that $2\alpha$ is the canonical class. Let $J \to M$ be the universal jacobian over $M$ and let $\Phi : C \times_M C \to J$ be the map over $M$ given by sending a triple $(M, x, y)$ to the pair
consisting of the jacobian \( J(M) \) of \( M \) and the class of the divisor \( h \cdot x - y - \alpha \) in \( J(M) \). Let \( \bar{\pi} : C \times_M C \to C \) be the projection on the first factor and let \( \bar{s}, \bar{t} : \pi^{-1} U \to C \times_M C \) be the local sections of \( \bar{\pi} \) obtained by pulling back the given local sections \( s, t \) of \( \pi \). It follows from [14] Lemma 3.2 that there exists a \((1,1)\)-form \( \kappa \) on \( C \) and a \((1,1)\)-form \( w \) on \( J \) which is translation-invariant in the fibers of \( J \to M \), such that \( \Phi^* w = h \nu + \bar{\pi}^* \kappa \) holds. We deduce from this that

\[
\bar{h} s^* \nu = (\Phi \bar{s})^* \nu = \kappa \quad \text{and} \quad \bar{h} t^* \nu = (\Phi \bar{t})^* \nu = \kappa.
\]

Let \( T_{x,y} \) denote translation by \( [x - y] \) over \( J / \nu \). Then \( \Phi \bar{t} = T_{x,t} \Phi \bar{s} \) and since \( w \) is fiberwise translation-invariant we obtain \( s^* \nu = t^* \nu \) from the above equalities. We derive \( (s, t)^* \nu = s^* t^* \nu = s^* s^* \nu = (s, s)^* \nu = s^* e^A \). By symmetry property (3) we have \( (s, t)^* \nu = (t, s)^* \nu = t^* e^A \) as well. \( \square \)

3. Proof of Theorem A

In this section we give a proof of Theorem A. Let \( x : M \to \mathbb{CP}^1 \) be the 2-sheeted cover with branch points \( \alpha_1, \ldots, \alpha_{2h+2} \). Let \( w_1, \ldots, w_{2h+2} \) be the corresponding Weierstrass points on \( M \). For the moment we fix two distinct indices \( i, j \). Consider then the meromorphic function \( f_{ij} = (x - \alpha_i)(x - \alpha_j)^{-1} \) on \( M \). We note that \( \text{div}(f_{ij}) = 2(w_i - w_j) \). From properties (1) and (2) we therefore obtain an equality

\[
g_{ij}(w_i, z) - g_{ij}(w_j, z) = \frac{1}{2} \log |f_{ij}(z)| - \frac{1}{2} \int_M \log |f_{ij}| \cdot \mu
\]

of generalized functions on \( M \). In particular we find for any pair of indices \( k, r \)

\[
g_{ij}(w_i, w_k) - g_{ij}(w_j, w_k) + g_{ij}(w_j, w_r) - g_{ij}(w_j, w_k) = \frac{1}{2} \log |f_{ij}(w_k) f_{ij}(w_r)^{-1}| = \frac{1}{2} \log |\mu_{ijk}|
\]

\[
= \frac{1}{4h(2h + 1)} \log |\eta_{ijk} \eta_{jkr}^{-1}|
\]

where we recall that \( \eta_{ijk} = \delta_{ik} \delta_{jk}^{-1} \) with \( \delta_{ij} \) the expression from [13]. This equality implies that there exists a constant \( c_{ij} \) such that

\[
g_{ij}(w_i, w_k) - g_{ij}(w_j, w_k) = \frac{1}{4h(2h + 1)} \log |\eta_{ijk}| + c_{ij}
\]

for each index \( k \). It follows from [10], Theorem 1.4 that

\[
\sum_{k \neq i} g_{ij}(w_i, w_k) = \sum_{k \neq j} g_{ij}(w_j, w_k).
\]

From the fact that \( \prod_{k \neq i,j} \delta_{ij} \) equals unity we obtain that \( \sum_{k \neq i,j} \log |\eta_{ijk}| = 0 \). It follows that

\[
2h c_{ij} = \sum_{k \neq i,j} (g_{ij}(w_i, w_k) - g_{ij}(w_j, w_k)) = 0,
\]

hence \( c_{ij} = 0 \) and

\[
g_{ij}(w_i, w_k) - g_{ij}(w_j, w_k) = \frac{1}{4h(2h + 1)} \log |\eta_{ijk}| = \frac{1}{4h(2h + 1)} \log |\delta_{ik} \delta_{jk}^{-1}|.
\]
Now fix the index $k$ and vary the indices $i, j$. We find that there exists a constant $c_k$ such that
\[
g_k(w_i, w_k) = \frac{1}{4h(2h+1)} \log |\delta_{ik}| + c_k
\]
for all indices $i \neq k$. We obtain $c_k = \frac{1}{2h+1} \psi(M)$ by summing over $i \neq k$. The formula in Theorem A follows.

Note that a similar argument in the case $h = 1$ allows one to deduce (1.1) from (1.2).

4. Proof of Theorem B

As was announced in the Introduction, we proceed in several steps. Let $\mathcal{H}$ be the moduli space of hyperelliptic Riemann surfaces of a fixed genus $h \geq 2$. Let $\pi: \mathcal{C} \to \mathcal{H}$ be the universal surface over $\mathcal{H}$. Both are viewed as orbifolds. A first result is the following.

**Proposition 4.1.** The equality of $(1,1)$-forms
\[
\partial \bar{\partial} \psi = \frac{1}{2h} \partial \bar{\partial} \varphi
\]
holds on $\mathcal{H}$.

**Proof.** Let $\mathcal{U}$ be an open cover of $\mathcal{H}$ such that for each $U$ in $\mathcal{U}$ there exist $2h + 2$ holomorphic sections $w_1, \ldots, w_{2h+2}: U \to \pi^{-1}U$ of $\pi^{-1}U \to U$ such that each $w_i$ is a Weierstrass point in each fiber of $\pi^{-1}U \to U$. Let $U$ be an element of $\mathcal{U}$. It suffices to prove the required equality on $U$. Theorem 3.1 of [18] gives an expression for $\partial \bar{\partial} \varphi$ over the moduli space of Riemann surfaces of genus $h$ (note that [18] considers the invariant $a$ with $a = \frac{1}{2h} \varphi$). Since the ‘harmonic volume’ of a Weierstrass point on a hyperelliptic Riemann surface is trivial, this expression immediately implies that on $U$ one has the equality
\[
\partial \bar{\partial} \varphi = 2h(2h+1) \pi \sqrt{-1} w_i^* e^A,
\]
for any choice of the index $i$. On the other hand by Proposition 2.2 we have
\[
\partial \bar{\partial} \psi = \pi \sqrt{-1} \sum_{j \neq i} (w_i, w_j)^* \nu = (2h+1) \pi \sqrt{-1} w_i^* e^A
\]
for each index $i$. The proposition follows. \qed

We next determine the limit behavior of the difference $\psi - \frac{1}{2h} \varphi$ in a holomorphic family $M_t$ of hyperelliptic Riemann surfaces of genus $h$ degenerating into a stable curve $M_0$ which is the union of two hyperelliptic Riemann surfaces $M_1, M_2$ of genera $h_1, h_2 \geq 1$, respectively, with two points $p_1 \in M_1$ and $p_2 \in M_2$ identified, as in Chapter III of the book ‘Theta functions on Riemann surfaces’ by J. Fay [11]. Here $t$ runs through a small punctured open disk around 0 in the complex plane. Note that $h = h_1 + h_2$. Let $g_1, g_2$ be the canonical Green’s functions on $M_1, M_2$, respectively, and let $z_1, z_2$ be the local coordinates around $p_1, p_2$ on $M_1, M_2$ that come with the degeneration model. We put, following [23], Section 6:
\[
\log k_1 = \lim_{x \to p_1} \left[ g_1(x, p_1) - \log |z_1(x)| \right], \quad \log k_2 = \lim_{x \to p_2} \left[ g_2(x, p_2) - \log |z_2(x)| \right],
\]
and then make the reparametrization $\tau = k_1 k_2 t$. 

Theorem 4.2. For the surfaces $M_t$ in Fay's degeneration model, the limit formula

$$\lim_{t \to 0} \left[ \frac{1}{2h} \varphi(M_t) + \frac{h_1 h_2}{h^2} \log |\tau| \right] = \frac{1}{2h} \varphi(M_1) + \frac{1}{2h} \varphi(M_2)$$

holds.

Theorem 4.3. For the energy $\psi(M_t)$ of the Weierstrass points of $M_t$, the limit formula

$$\lim_{t \to 0} \left[ \psi(M_t) + \frac{h_1 h_2}{h^2} \log |\tau| \right] = \frac{h_1}{h} \psi(M_1) + \frac{h_2}{h} \psi(M_2)$$

holds.

We deduce Theorem B from Theorems 4.2 and 4.3 by induction on $h$. Note that the assertion in Theorem B is true for $h = 1$ by (1.2) and the fact that $\varphi$ vanishes for $h = 1$. Now assume that the assertion in Theorem B is true for all genera smaller than $h$. Consider a holomorphic family $M_t$ of hyperelliptic Riemann surfaces of genus $h$ as above. We find

$$\lim_{t \to 0} \left[ \psi(M_t) - \frac{1}{2h} \varphi(M_t) \right] = \frac{h_1}{h} \psi(M_1) - \frac{1}{2h} \varphi(M_1) + \frac{h_2}{h} \psi(M_2) - \frac{1}{2h} \varphi(M_2)$$

$$= \frac{h_1}{h} \left( \psi(M_1) - \frac{1}{2h_1} \varphi(M_1) \right) + \frac{h_2}{h} \left( \psi(M_2) - \frac{1}{2h_2} \varphi(M_2) \right)$$

$$= \frac{h_1}{h} \log 2 + \frac{h_2}{h} \log 2 = \log 2.$$

Let $\overline{H}$ be the closure of $H$ in the Deligne-Mumford compactification $\overline{\mathcal{M}}$ of the moduli space $\mathcal{M}$ of compact Riemann surfaces of genus $h$. Let $\overline{\mathcal{H}} \subset \overline{H}$ be the partial compactification of $H$ given by stable curves of compact type (i.e. whose jacobians are principally polarized abelian varieties). Write $\chi = \psi - \frac{1}{2h} \varphi$. The above calculation implies that for hyperelliptic Riemann surfaces in $H$ tending to a point in the boundary of $H$ in $\overline{H}$, the function $\chi$ tends to $\log 2$. In particular the function $\chi$ extends as a continuous function over $\overline{H}$. By Proposition 4.1 we know that $\chi$ is a pluriharmonic function on $H$. By the Riemann extension theorem we find that $\chi$ extends as a pluriharmonic function on $\overline{H}$, which is a constant function with value $\log 2$ on $\overline{H} \setminus H$.

Let $A^*$ be the Satake compactification of the moduli space $A$ of principally polarized abelian varieties of genus $h$. This is a normal projective variety $\overline{H}$. There exists a holomorphic map $t : \overline{H} \to A^*$ given by assigning to a stable curve $M$ the jacobian of its normalization. The map $t$ is injective on $H$, and the function $\chi$ descends as a pluriharmonic function on $t(\overline{H})$, which is a constant function with value $\log 2$ on $t(\overline{H} \setminus H)$.

The boundary of $H$ in $\overline{H}$ is a finite union of divisors $\Xi_i$, see [8] for example. From the moduli theoretic description of the points in the divisors $\Xi_i$ it follows that the map $t$ restricted to the boundary has positive dimensional fibers. Hence the boundary of $t(\overline{H})$ in the closure of $t(\overline{H})$ in $A^*$ has codimension at least two. Let $M$ be a point in $t(\overline{H})$. Using that $A^*$ is projective one sees from intersecting with suitable hyperplanes in a projective embedding that the variety $t(\overline{H})$ contains a complete curve $X$ passing through $M$. As $t(\overline{H})$ is affine, the curve $X$ necessarily contains a point of $t(\overline{H} \setminus H)$. Let $Y \to X$ be the normalization of $X$. The function
\( \chi \) pulls back to a pluriharmonic function on \( Y \), which is then necessarily constant. It follows that the value of \( \chi \) at \( M \) is equal to \( \log 2 \), and Theorem B is proven.

For the proof of Theorem 4.2 we refer to [17]. It remains to prove Theorem 4.3. We need the next result, which follows from [23], Theorem 6.10.

**Theorem 4.4.** Let \( g_t \) be the canonical Green’s function on \( M_1 \). Denote by \( p \) both the point \( p_1 \) on \( M_1 \) and the point \( p_2 \) on \( M_2 \). Then for local distinct holomorphic sections \( x, y \) of the family \( M_t \) specializing onto \( M_1 \) we have

\[
\lim_{t \to 0} \left[ g_t(x, y) - \left( \frac{h_2}{h} \right)^2 \log |\tau| \right] = g_1(x, y) - \frac{h_2}{h} (g_1(x, p) + g_1(y, p)).
\]

For local distinct holomorphic sections \( x, y \) with \( x \) specializing onto \( M_1 \) and \( y \) specializing onto \( M_2 \) we have

\[
\lim_{t \to 0} \left[ g_t(x, y) + \frac{h_1 h_2}{h^2} \log |\tau| \right] = \frac{h_1}{h} g_1(x, p) + \frac{h_2}{h} g_2(y, p).
\]

Now let \( w_1, \ldots, w_{2h_2+2} \) be the Weierstrass points on \( M_t \). As \( t \to 0 \) a portion of \( 2h_1 + 1 \) of them degenerate onto \( M_1 \), and \( 2h_2 + 1 \) of them degenerate onto \( M_2 \). The point \( p \) is a Weierstrass point of both \( M_1, M_2 \). Let \( I \) be the set of indices such that \( i \in I \) if and only if \( w_i \) degenerates onto \( M_1 \).

**Lemma 4.5.** The equality

\[
\sum_{i,j \in I} g_1(w_i, w_j) = 2h_1 \psi(M_1)
\]

holds.

**Proof.** We calculate

\[
(2h_1 + 2) \psi(M_1) = \sum_{i,j \in I, i \neq j} g_1(w_i, w_j) = \sum_{i,j \in I, i \neq j} g_1(w_i, w_j) + 2 \psi(M_1).
\]

The required equality follows. \( \square \)

Using Theorem 4.3 we find

\[
(2h + 2) \psi(M_t) = \sum_{i \neq j} g_t(w_i, w_j)
\]

\[
= \sum_{i,j \in I} g_t(w_i, w_j) + \sum_{i,j \in I, i \neq j} g_1(w_i, w_j) + 2 \sum_{i \in I} g_t(w_i, w_j)
\]

\[
= \sum_{i,j \in I} \left( \left( \frac{h_2}{h} \right)^2 \log |\tau| + g_1(w_i, w_j) - \frac{h_2}{h} (g_1(w_i, p) + g_1(w_j, p)) \right)
\]

\[
+ \sum_{i,j \in I, i \neq j} \left( \left( \frac{h_1}{h} \right)^2 \log |\tau| + g_2(w_i, w_j) - \frac{h_1}{h} (g_2(w_i, p) + g_2(w_j, p)) \right)
\]

\[
+ 2 \sum_{i \in I} \left( -\frac{h_1 h_2}{h^2} \log |\tau| + \frac{h_1}{h} g_1(w_i, p) + \frac{h_2}{h} g_2(w_j, p) \right) + o(1)
\]
as $t \to 0$. With the help of Lemma 4.3 this can be rewritten as

$$(2h + 2) \psi(M_t) = \left( (2h_1 + 1)\frac{h_2}{h} \right)^2 + (2h_2 + 1) \frac{h_1h_2}{h^2} \log |\tau|$$

$$-2(2h_1 + 1)(2h_2 + 1)\frac{h_1h_2}{h^2} \log |\tau|$$

$$+ \left( 2h_1 - \frac{4h_1h_2}{h} + \frac{2h_1(2h_2 + 1)}{h} \right) \psi(M_1)$$

$$+ \left( 2h_2 - \frac{4h_1h_2}{h} + \frac{2h_2(2h_1 + 1)}{h} \right) \psi(M_2) + o(1)$$

$$= (2h + 2) \left( -\frac{h_1h_2}{h^2} \log |\tau| + \frac{h_1}{h} \psi(M_1) + \frac{h_2}{h} \psi(M_2) \right) + o(1)$$

as $t \to 0$. Theorem 4.3 follows.

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