Yang-Baxter $R$ operators and parameter permutations

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Abstract We present an uniform construction of the solution to the Yang- Baxter equation with the symmetry algebra $\mathfrak{sl}(2)$ and its deformations: the q-deformation and the elliptic deformation or Sklyanin algebra. The R-operator acting in the tensor product of two representations of the symmetry algebra with arbitrary spins $\ell_1$ and $\ell_2$ is built in terms of products of three basic operators $S_1, S_2, S_3$ which are constructed explicitly. They have the simple meaning of representing elementary permutations of the symmetric group $\mathfrak{S}_4$, the permutation group of the four parameters entering the RLL-relation.
1 Introduction

The Yang-Baxter equation and its solutions play a key role in the theory of the completely integrable quantum models [1, 2, 3, 4, 5]. The general solution of the Yang-Baxter equation (R-matrix) is the operator $R(u)$ acting in a tensor product $V_1 \otimes V_2$ of two linear spaces. There exists the natural construction of the solution to the Yang-Baxter equation. The solution of the problem consists of two steps:

- on the first stage we construct the solution to so called $RLL$-relation
- on the second stage we prove that operator $R_{12}(u - v)$ constructed as the solution of the $RLL$-relation obeys the general Yang-Baxter equation.

Choosing one of the three spaces involved to carry a fundamental representation of the symmetry algebra the Yang-Baxter equation is reduced to the simpler defining equation for the R-matrix [6]

$$R_{12}(u - v) L_1(u) L_2(v) = L_2(v) L_1(u) R_{12}(u - v), \quad (1.1)$$

where $L(u)$ is the Lax matrix. In the cases of the symmetry algebra $sl(2)$ and its q-deformation the R-matrix can be obtained by the following method [3, 6]: The R-matrix is a function of the Casimir operator and the defining RLL-equation is reduced to a recurrence relation for the function of one variable.

We shall give an uniform construction of the solution of the RLL-relation extending to the three cases of the symmetry algebra $sl(2)$ and its two deformations, the q-deformation and the elliptic deformation or Sklyanin algebra [7]. It is well known that the resulting R-matrices are building blocks in the construction of integrable spin chain models. Whereas in questions related to condensed matter physics the sites of the chain carry usually finite-dimensional representations the case of infinite-dimensional representations turned out to be relevant in studies of QCD and super Yang-Mills theories, in particular in calculating the anomalous dimensions of composite operators or of the Regge singularities of multi-gluon exchange (cf. [8]). It is not known so far whether the deformed symmetry cases are relevant in the Yang-Mills context. The explicit formulation of the results in analogous form for all cases may be useful to find new applications.

The main idea of our construction is quite simple: The R-matrix represents some particular element of the group $S_4$ of permutations of four parameters entering the RLL-relation and it can be constructed from the simple building blocks, the operators $S_1, S_2, S_3$ corresponding to the elementary permutations. The Lax matrix $L(u)$ depends on two parameters: the spin of representation $\ell$ and the spectral parameter $u$. It is useful to introduce two related parameters $u_1$ and $u_2$, $u_1 = \frac{u}{2\eta} + \ell; \quad u_2 = \frac{u}{2\eta} - 1 - \ell$, where $\eta$ is a free parameter in the case of the Sklyanin algebra. In the cases of no deformation and q-deformation it can be fixed as $\eta = \frac{1}{2}$ without loss of generality. After extracting the operator of permutation $P_{12}$ from the R-matrix $\mathcal{R}_{12} = P_{12} \mathcal{R}_{12}$ the defining equation is rewritten in "check formalism",

$$\mathcal{R}_{12} \cdot L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_1, v_2) L_2(u_1, u_2) \cdot \mathcal{R}_{12}. \quad (1.2)$$

This equation admits the natural interpretation: The operator $\mathcal{R}_{12}$ interchanges the set of parameters $(u_1, u_2)$ of the first Lax-matrix with the set of parameters $(v_1, v_2)$ of the second Lax-matrix. It corresponds to the special permutation $s$ in the group of permutations $S_4$ of four parameters,

$$s(u_1, u_2, v_1, v_2) \mapsto (v_1, v_2, u_1, u_2).$$
Arbitrary permutations in the group $S_4$ can be constructed from the elementary transpositions $s_1, s_2$ and $s_3$ which interchange only two nearest neighboring components in the set $(u_1, u_2, v_1, v_2)$. We look for the operators $S_i$ representing these elementary transpositions

$$S_1 L_1(u_1, u_2) = L_1(u_2, u_1) S_1, \quad S_3 L_2(v_1, v_2) = L_2(v_2, v_1) S_3,$$

$$S_2 L_1(u_1, u_2) L_2(v_1, v_2) = L_1(u_1, v_1) L_2(u_2, v_2) S_2.$$

These equations appear to be much simpler than the initial defining equation for the R-operator and the solution can be obtained in a closed form. Finally we construct the R-matrix as the composite object out of the simplest building blocks, the operators $S_1, S_2, S_3$.

Note that there are different possible levels of resolution into more elementary permutations:

- **On the first level** we have the group of permutation $S_2$ of two pairs of parameters $(u_1, u_2)$ and $(v_1, v_2)$ and the R-matrix is the elementary operator corresponding to this permutation.

- **On the second level** we allow the separate permutations $u_1 \leftrightarrow v_1$ or $u_2 \leftrightarrow v_2$ so that our group of permutations is $S_2 \times S_2$. On this level the operators $\bar{R}^{(1)}$ and $\bar{R}^{(2)}$ corresponding to these permutations are the elementary building blocks but the R-matrix is a composite object.

- **On the third level** we allow any permutations of parameters so that our group of permutations becomes $S_4$. On this level the operators $S_i, i = 1, 2, 3$ are elementary building blocks. Now the operators $\bar{R}^{(1)}, \bar{R}^{(2)}$ and R-matrix are composite objects. So we have the chain of inclusions with increasing symmetry

$$S_2 \rightarrow S_2 \times S_2 \rightarrow S_4.$$

Thus the R-matrix is factorized into operators representing more elementary permutations. We shall construct the elementary operator factors $S_i, i = 1, 2, 3$ and proof that they generate a representation of the permutation group for generic values of the representation parameters $u_1, u_2, v_1, v_2$. Then the Yang-Baxter relation for the general R-matrix composed out of these factors is just a consequence of the permutation group relations.

The solution of the Yang-Baxter relation by factorization into elementary parameter permutations has been given previously in [17] for the symmetry $sl(N)$. Here we show that the method extends to the deformations of $sl(2)$. The construction of $S_i$ and the permutation group proofs will be done uniformly for the cases of no deformation and of the quantum and elliptic deformations emphasizing the analogies. Guided by the analogies we are able to present a relatively simple solution also for the non-trivial elliptic case.

For generic values of the spins $\ell_1, \ell_2$ the operators $\bar{R}^{(i)} [16, 18, 19]$ act within the tensor product of the corresponding representation spaces, whereas the operators $S_i$ map to tensor products with changed spin values. Moreover, in the case of finite dimensional representations only the R-matrix itself leaves the tensor product space invariant. At some special values of the representation parameters the elementary operators $S_i$ become singular on parts of their spectra or develop kernels. Then the permutation group properties become invalid. In this paper we restrict ourselves to generic parameter values and postpone those more subtle questions related to representation theory to the next paper. There we plan also to consider the uniform construction of Baxter $Q$ operators in this approach.

The presentation is organized as follows. In Section 2 we give the uniform expression for the fundamental R-matrices and Lax matrices for the three cases: no deformation, q-deformation
and elliptic deformation of the symmetry algebra $s\ell(2)$. In Section 3 we discuss the connection between RLL-relations and the group of permutations $S_4$. In Section 4 we construct the operators $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ representing the elementary transpositions and in Section 5 we prove the Coxeter relations for them. In Section 6 we construct the R-matrix and prove that it obeys the Yang-Baxter relation. In Section 7 we construct the operators $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ and prove the corresponding Yang-Baxter relations for them. Finally, in Section 8 we summarize.

2 Yang Baxter $\mathcal{R}$ matrices related to $s\ell_2$

The Yang-Baxter relation,

$$\mathcal{R}_{12}(u - v)\mathcal{R}_{13}(u)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u)\mathcal{R}_{12}(u - v), \quad (2.1)$$

defined on operators acting on the tensor product $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3$ where $\mathcal{R}_{ij}$ is acting non-trivially only on $\mathcal{V}_i, \mathcal{V}_j$, has the three well known solutions in the case $\mathcal{V}_i \equiv \mathbb{C}^2$ related to $s\ell_2$

$$\mathcal{R}_{12}(u) = \frac{1}{2} \sum_{a=0}^{3} w_a(u + \eta)\sigma^a \otimes \sigma^a. \quad (2.2)$$

Here $\sigma^a$ denote the Pauli matrices, $\sigma^0 = 1$. There are the three cases of no deformation, quantum deformation and elliptic deformation. The parameter $\eta$ can be fixed in the first two cases, $2\eta = 1$, but is variable in the last case. The weight functions are depending on case,

- no deformation
  $$w_0(u) = 2u; \quad w_1(u) = w_2(u) = w_3(u) = 1, \quad (2.3)$$

- quantum deformation
  $$w_0(u) = \frac{q^u - q^{-u}}{q^2 - q^{-2}}; \quad w_3(u) = \frac{q^u + q^{-u}}{q^2 + q^{-2}}; \quad w_1(u) = w_2(u) = 1, \quad (2.4)$$

- elliptic deformation
  $$w_a(u) = \frac{\theta_{a+1}(u|\tau)}{\theta_{a+1}(\eta|\tau)}. \quad (2.5)$$

The relation (2.1) is the source of a rich algebraic structure and integrable quantum systems. In the case $\mathcal{V}_1 = \mathbb{C}^2$ and $\mathcal{V}_2$ arbitrary one calls $\mathcal{R}_{12}(u + \eta) = L(u)$ Lax matrix,

$$L(u) = \frac{1}{2} \sum_{a=0}^{3} w_a(u)\sigma^a \otimes S^a = \frac{1}{2} \begin{pmatrix} w_0(u)S^0 + w_3(u)S^3 & w_1(u)S^1 + iw_2(u)S^2 \\ w_1(u)S^1 - iw_2(u)S^2 & w_0(u)S^0 - w_3(u)S^3 \end{pmatrix}, \quad (2.6)$$

where the operators $S^a$ generate the universal enveloping algebra $U(s\ell_2)$ in the first case, its quantum deformation $U_q(s\ell_2)$ (deformation parameter $q$) in the second case and its elliptic deformation (parameters $\tau, \eta$), Sklyanin algebra, in the third case. In the first two cases $S^0, S^3$ are not independent and they are related to the generators in conventional notation

- no deformation
  $$S^0 = 1, \quad S^3 = 2S, \quad S^1 = zS^2 = S^\pm$$

where the conventional generators for the representation with spin $\ell$ are

$$S^- = -\partial, \quad S = z\partial - \ell, \quad S^+ = z^2\partial - 2\ell z, \quad (2.7)$$
After substitution in the Lax matrix \((2.6)\) the functions \((2.3)\) and generators in terms of \(z, \partial\) as in \((2.7)\) we obtain that the Lax matrix decomposes into factors depending on one of the two elementary operators \(z\) or \(\partial\) only. The spectral parameter \(u\) and spin \(\ell\) enter in the combinations \(u_1 = u + \ell\) and \(u_2 = u - 1 - \ell\),

\[
L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ z & u_1 \end{pmatrix} \begin{pmatrix} 1 & -\partial \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_2 & 0 \\ -z & 1 \end{pmatrix}.
\]

- quantum deformation

\[
S^0 = \frac{q^S + q^{-S}}{q^\frac{1}{2} + q^{-\frac{1}{2}}}; \quad S^3 = \frac{q^S - q^{-S}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}; \quad S^1 \pm iS^2 = 2S^\pm. \tag{2.8}
\]

The conventional generators for the representation with spin \(\ell\) are

\[
S^- = -\frac{1}{z}[z\partial]_q, \quad S = z\partial - \ell, \quad S^+ = z[z\partial - 2\ell]_q, \tag{2.9}
\]

where \([x]_q\) is the usual notation for q-numbers \([x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}\). Substituting the functions \((2.4)\) and \(q\) deformed generators \(S^a\) from \((2.9)\) in the Lax matrix \((2.6)\) we obtain a factorized form with features analogous to the undeformed case \([9, 10, 19]\)

\[
L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ zq^{-u_1} & 1 \end{pmatrix} \begin{pmatrix} q^{\partial+1} & 0 \\ 0 & q^{-\partial-1} \end{pmatrix} \begin{pmatrix} q^{u_2} & -z^{-1} \\ -q^{-u_2} & z^{-1} \end{pmatrix}
\]

- the generators \(S^a\) of the Sklyanin algebra can be expressed in terms of \(z, \partial\) as follows

\[
S^a = \frac{(i)^{a+2}a_{a+1}(\eta)}{\theta_1(2\eta z)} \left( \theta_{a+1} [2\eta(z - \ell)] \cdot e^\partial - \theta_a [2\eta(z - \ell)] \cdot e^{-\partial} \right). \tag{2.10}
\]

Note that in order to simplify the formulae in the elliptic case we change the variable \(z \rightarrow \eta z\) in comparison to the standard notations \([7, 10]\) and absorb the model parameter \(\eta\) which indeed plays important role in XYZ model in contrast with simpler XXX and XXZ case, but in present context we do not touch the subtle questions related to the elliptic curve parametrizing the model under consideration. The needed properties of the \(\theta\)-functions are listed in the Appendix. Also in this case an analogous factorization of the Lax matrix can be written after substituting in \((2.6)\) the generators in terms of \(z, \partial\) \([9, 10]\)

\[
L(u_1, u_2) = \frac{1}{\theta_1(2\eta z)} \left( (z - u_1)_3 - (z + u_1)_3 \right) \cdot \left( \begin{pmatrix} e^\partial & 0 \\ 0 & e^{-\partial} \end{pmatrix} \right) \left( z + u_2 \right)_4 \left( z + u_2 \right)_3
\]

where \(u_1 = \frac{u}{2\eta} + \ell\); \(u_2 = \frac{u}{2\eta} - \ell - 1\) and for simplicity we use the notation

\[
(x)_3 = \theta_3 \left( \eta x \left| \frac{1}{2} \right. \right); \quad (x)_4 = \theta_4 \left( \eta x \left| \frac{1}{2} \right. \right).
\]

We obtain the uniform expressions of Lax factorization.

**Proposition 1.** In terms of a Heisenberg pair \(z, \partial\) the Lax matrices \((2.6)\) can be written in factorized form as

\[
L_c(u_1, u_2) = [u_1]_c V_c^{-1}(u_1, z) D_c V_c(u_2, z) \tag{2.11}
\]

where the subscript \(c = o, q, e\) labels the three cases of no deformation \((o)\), quantum deformation \((q)\) and elliptic deformation \((e)\). The first factor is a usual number \([u]_o = u\), q-number \([u]_q =
or elliptic number \([u]_e = \theta_1 (2u\eta)\) and the others are \(2 \times 2\) matrices. The matrices \(D_c\) do not depend on the parameters \(u_1, u_2, u_1 = \frac{u}{2\eta_c} + \ell\); \(u_2 = \frac{u}{2\eta_c} - 1 - \ell\), with \(2\eta_0 = 2\eta_1 = 1, 2\eta_c = 2\eta\) and are given in terms of operator of differentiation \(\partial (a)\), dilatation by \(q^{\pm 1}\) \((q)\) or shift by \(\pm 1\) \((e)\)

\[
D_o = \begin{pmatrix} 1 & -\partial \\ 0 & 1 \end{pmatrix}; \quad D_q = \begin{pmatrix} q^{z\partial + 1} & 0 \\ 0 & q^{-z\partial - 1} \end{pmatrix}; \quad D_e = \begin{pmatrix} e^\partial & 0 \\ 0 & e^{-\partial} \end{pmatrix}.
\]

The matrices \(V_c\) depend on the representation parameters and on \(z\)

\[
V_o(u, z) = \begin{pmatrix} u & 0 \\ -z & 1 \end{pmatrix}; \quad V_q(u, z) = \begin{pmatrix} q^u & -z^{-1} \\ -q^{-u} & z \end{pmatrix}; \quad V_e(u, z) = \begin{pmatrix} (z + u)_4 & (z + u)_3 \\ (z - u)_4 & (z - u)_3 \end{pmatrix}
\]

3 The RLL-relation and the permutation group

Let us consider the reduction of the general Yang-Baxter equation (2.11) in the special case when \(V_2 = \mathbb{C}^2\). The operator \(R_{12}(u)\) is reduced to the Lax matrix \(L_1(u)\), operator \(R_{23}(v)\) is reduced to \(L_2(v)\) so that one obtains the defining RLL-relation for the operator \(R_{12}(u - v)\) (1.1). It is useful to extract the permutation operator from the \(R\)-matrix

\[
R_{12} = \mathcal{P}_{12} \tilde{R}_{12},
\]

where \(\mathcal{P}_{12} \Phi(z_1, z_2) = \Phi(z_2, z_1)\). For the operator \(\tilde{R}_{12}\) the RLL-relation takes the form (1.2) where \(L_1\) is to be substituted as in (2.11) with \(z, \partial\) replaced by \(z_1, \partial_1\) and \(L_2\) with \(z, \partial\) replaced by \(z_2, \partial_2\). The parameters are \(u_1 = \frac{u}{2\eta_c} + \ell_1, u_2 = \frac{u}{2\eta_c} - 1 - \ell_1\) and \(v_1 = \frac{v}{2\eta_c} + \ell_2, v_2 = \frac{v}{2\eta_c} - 1 - \ell_2\). In equation (1.2) \(\tilde{R}_{12}(u - v)\) is some operator depending on the Heisenberg pairs \(z_1, \partial_1\) and \(z_2, \partial_2\).

As it is mentioned in Introduction the operator \(\tilde{R}_{12}(u - v)\) in (1.2) corresponds to the special permutation \(s\) in the group of permutations \(S_4\) of four parameters \(u = (u_1, u_2, v_1, v_2)\).

\[
s \rightarrow \tilde{R}_{12}(u - v); \quad s(u_1, u_2, v_1, v_2) = (v_1, v_2, u_1, u_2).
\]

The arbitrary permutation from the group \(S_4\) can be constructed from the elementary transpositions \(s_1, s_2, \text{ and } s_3\)

\[
s_1 u = (u_2, u_1, v_1, v_2); \quad s_2 u = (u_1, v_1, u_1, v_2); \quad s_3 u = (u_1, u_2, v_2, v_1)
\]

which interchange only two nearest neighboring components in the set \(u = (u_1, u_2, v_1, v_2)\). For example the permutation \(s\) has the following decomposition \(s = s_2 s_1 s_3 s_2\). It is natural to search the operators \(S_i(u_1, u_2, v_1, v_2) = S_i(u)\) representing these elementary transpositions

\[
S_1(u) L_1(u_1, u_2) L_2(v_1, v_2) = S_1(u) ; \quad S_3(u) L_2(v_1, v_2) = S_3(u) \quad (3.1)
\]

\[
S_2(u) L_1(u_1, u_2) L_2(v_1, v_2) = S_2(u)
\]

and our first step will be the explicit construction of these operators in Section 4. Having these operators we can construct the \(R\)-matrix.
Proposition 2. The operator $\mathcal{R}_{12}$

$$\mathcal{R}_{12}(u - v) = S_2(s_1s_3s_2u)S_1(s_3s_2u)S_3(s_2u)S_2(u)$$  \hspace{1cm} (3.3)

obeys the relation (1.2) provided the operators $S_i$ obey the relations (3.7), (3.8).

This expression for the R-matrix corresponds to the particular decomposition of the permutation $s$: $s = s_2s_1s_3s_2$. In the last section we shall present equivalent expressions corresponding to another decompositions. We shall see that the operators $S_i$ have the special dependence on parameters

$$S_1(u) = S_1(u_1 - u_2) \ ; \ S_2(u) = S_2(u_2 - v_1) \ ; \ S_3(u) = S_3(v_1 - v_2)$$  \hspace{1cm} (3.4)

so that the operator $\mathcal{R}_{12}(u - v)$ depends on the difference of spectral parameters as it should be. We have the correspondence

$$s_i \to S_i(u) \ ; \ s_is_j \to S_i(s_j u)S_j(u)$$  \hspace{1cm} (3.5)

and to prove that we obtain the representation of the permutation group $\mathfrak{S}_4$ it remains to prove the corresponding defining (Coxeter) relations for the generators

$$s_is_i = 1 \to S_i(s_i u)S_i(u) = 1 \ ; \ s_is_3 = s_3s_1 \to S_1(s_3 u)S_i(u) = S_3(s_1 u)S_1(u)$$  \hspace{1cm} (3.6)

$$s_1s_2s_1 = s_2s_1s_2 \to S_1(s_2 s_1 u)S_2(s_1 u)S_1(u) = S_2(s_1 s_2 u)S_1(s_2 u)S_2(u)$$  \hspace{1cm} (3.7)

$$s_2s_3s_2 = s_3s_2s_3 \to S_2(s_3 s_2 u)S_3(s_2 u)S_2(u) = S_3(s_2 s_3 u)S_2(s_3 u)S_3(u)$$  \hspace{1cm} (3.8)

In Section 5 we prove that the obtained operators $S_i(u)$ obey these defining relations. After this we shall prove that Yang-Baxter relation (2.1) for the operator (3.3) is consequence of the Coxeter relations (3.6), (3.7) and (3.8).

4 Elementary permutation operators $S_1$, $S_2$ and $S_3$

4.1 The operator $S_2$

Consider the defining condition for $S_2$ (3.2) where $L_1$ is to be substituted as in (2.11) with $z, \partial$ replaced by $z_1, \partial_1$ and $L_2$ with $z, \partial$ replaced by $z_2, \partial_2$

$$S_2 \cdot[u_1] V^{-1}(u_1, z_1) D_1 V(u_2, z_1) \cdot[v_1] V^{-1}(v_1, z_2) D_2 V(v_2, z_2) =$$

$$= [u_1] V^{-1}(u_1, z_1) D_1 V(v_1, z_1) \cdot[v_2] V^{-1}(u_2, z_2) D_2 V(v_2, z_2) \cdot S_2$$

This equation suggests the ansatz $S_2 = S_2(z_1, z_2)$ as a multiplication operator independent of $\partial_1, \partial_2$. Then the operator $S_2$ commutes with the matrices $V^{-1}(u_1, z_1)$ and $V(v_2, z_2)$ so that they can be cancelled and we immediately obtain a much simpler defining equation for the function $S_2(z_1, z_2)$

$$[v_1] D_1^{-1} S_2(z_1, z_2) D_1 \cdot V(u_2, z_1) V^{-1}(v_1, z_2) = [u_2] V(v_1, z_1) V^{-1}(u_2, z_2) D_2 S_2(z_1, z_2) D_2^{-1}. \hspace{1cm} (4.1)$$

It remains to solve this equation in each case.
• In the case of no deformation the defining condition (4.1) results in
\[
\begin{pmatrix}
1 & \partial_1 \ln S_2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
u_2 \\
v_1
\end{pmatrix}
= \begin{pmatrix}u_2 & 0 \\
v_1 & v_1 \\
z_2 - z_1 & u_2
\end{pmatrix}
\begin{pmatrix}
1 & -\partial_2 \ln S_2
\end{pmatrix}
\]
The wanted function of \(z_1, z_2\) has to obey two relations,
\[
\partial_1 S_2(z_1, z_2) = -\partial_2 S_2(z_1, z_2); \quad (z_2 - z_1)\partial_2 \ln S_2(z_1, z_2) = u_2 - v_1
\]
This results in the permutation operator \(S_2\) up to normalization
\[
S_2(z_1, z_2) = (z_2 - z_1)^{u_2 - v_1}. \quad (4.2)
\]

• In the case of quantum deformation the defining condition (4.1) in matrix form is
\[
\begin{pmatrix}S_2(q^{-1}z_1, z_2) & 0 \\
0 & S_2(qz_1, z_2)\end{pmatrix}
= \begin{pmatrix}S_2(z_1, qz_2) & 0 \\
0 & S_2(z_1, q^{-1}z_2)\end{pmatrix}
\]
Equating the diagonal elements of the matrices resulting on both sides leads to the conclusion
\[
S_2(z_1, z_2) = z_1^{u_2 - v_1} \cdot \Phi \left( \frac{z_2}{z_1} \right).
\]
The resulting equations from the two off-diagonal elements are compatible and lead to the difference equation
\[
\Phi(qx)(1 - x^{q^{v_1 - u_2}}) = \Phi(q^{-1}x)(1 - x^{q^{u_2 - v_1}})
\]
Therefore, in the \(q\) deformed case the permutation operator \(S_2\) has the form
\[
S_2 = z_1^{u_2 - v_1} \cdot \frac{(\frac{z_2}{z_1})^{1-u_2+v_1} q}{(\frac{z_2}{z_1})^{1+u_2-v_1} q^2} \quad (4.3)
\]
We use the standard notation \((x; q^2) = \prod_{k=0}^{\infty}(1 - x q^{2k})\). In [20] this expression appeared as the appropriate generalization to the \(q\) deformed case of \((z_2 - z_1)^{u_2 - v_1}\) (4.2) in the role of conformal propagators.

• In the case of elliptic deformation the defining equation (4.1) results in the systems of four difference equations
\[
\begin{align*}
\theta(z_1 + z_2 + u_2 - v_1) \theta(z_1 - z_2 + u_2 + v_1) S_2(z_1 - 1, z_2) &= \theta(z_1 + z_2 + v_1 - u_2) \theta(z_1 - z_2 + v_1 + u_1) S_2(z_1, z_2 + 1), \\
\theta(z_1 + z_2 - u_2 + v_1) \theta(z_1 - z_2 - u_2 - v_1) S_2(z_1 + 1, z_2) &= \theta(z_1 + z_2 - v_1 - u_2) \theta(z_1 - z_2 - v_2 - u_2) S_2(z_1, z_2 - 1), \\
\theta(z_1 + z_2 + u_2 + v_1) \theta(z_1 - z_2 + u_2 - v_1) S_2(z_1 + 1, z_2) &= \theta(z_1 + z_2 + u_2 + v_1) \theta(z_1 - z_2 + v_1 - u_2) S_2(z_1, z_2 - 1), \\
\theta(z_1 + z_2 - u_2 - v_1) \theta(z_1 - z_2 - u_2 + v_1) S_2(z_1 + 1, z_2) &= \theta(z_1 + z_2 - u_2 - v_1) \theta(z_1 - z_2 - v_1 + u_2) S_2(z_1, z_2 + 1),
\end{align*}
\]
where \(\theta(x) \equiv \theta(\eta x | \tau)\). This leads us to look for the solution in the form
\[
S_2(z_1, z_2) = \Phi_+ (z_1 + z_2) \cdot \Phi_-(z_1 - z_2).
\]
In each equation one of the factors drops out and we obtain coinciding difference equations for these factors. The result for the permutation operator $S_2$ in the elliptic case is

$$ S_2 = e^{-2\pi i u_2 - v_1) z_1} \frac{\gamma(z_1 + z_2 + u_2 - v_1 + 1)}{\gamma(z_1 + z_2 - u_2 + v_1 + 1)} \frac{\gamma(z_1 - z_2 + u_2 - v_1 + 1)}{\gamma(z_1 - z_2 - u_2 + v_1 + 1)}, $$

where the function $\gamma(x)$ is closely related to the elliptic gamma-function and is defined in Appendix (9.8).

We emphasize the common features allowing to do analogous steps in the three cases for deriving the uniform expressions of operator of parameter permutation $S_2$.

**Proposition 3.** The permutation operator is defined by the relation with two Lax matrices

$$ S_2(u_1, u_2; v_1, v_2) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(u_1, v_1) L_2(u_2, v_2) S_2(u_1, u_2; v_1, v_2), $$

where $L_i$ involves the generators of representation with spin $\ell_i$ and is expressed in terms of the Heisenberg pair $z_i, \partial_i$ and $u_1 = \frac{u}{2\eta} + \ell_1, u_2 = \frac{u}{2\eta} - 1 - \ell_1; v_1 = \frac{v}{2\eta} + \ell_2, v_2 = \frac{v}{2\eta} - 1 - \ell_2,$ with $2\eta_0 = 2\eta_1 = 1, 2\eta_c = 2\eta.$ $S_2$ is represented as an operator of multiplication, depending on $z_1, z_2$ and on $u_2 - v_1$ as given in $4.3, 4.4, 4.5$.

Notice that the result for the undeformed case is given by a particular symmetric two-point function or conformal propagator. The results in the other cases are related to the corresponding deformed propagators.

### 4.2 The operators $S_1, S_3$ and the intertwining operator

Let us consider the defining equations (3.1) for the operators $S_1$ and $S_3$. The permutation of parameters $u_1 = \frac{u}{2\eta} + \ell_1$ and $u_2 = \frac{u}{2\eta} - 1 - \ell_1$ is equivalent to the change of the spin $\ell_1 \rightarrow -1 - \ell_1$ and similarly the permutation of parameters $v_1 = \frac{v}{2\eta} + \ell_2$ and $v_2 = \frac{v}{2\eta} - 1 - \ell_2$ is equivalent to the change of the spin $\ell_2 \rightarrow -1 - \ell_2$. In the Lax matrix (2.6) only the generators $S^a$ depend on the spin so that the defining equations (3.1) can be rewritten in terms of the generators only. The meaning of this equations is the following: The operator $S_1$ intertwines the representations of spin $\ell_1$ and of spin $-1 - \ell_1$ and the operator $S_3$ intertwines the representations of spin $\ell_2$ and of spin $-1 - \ell_2$

$$ S_1 \cdot S^a_{-1-\ell_1} = S^a_{\ell_1} \cdot S_1; \quad S_3 \cdot S^a_{-1-\ell_2} = S^a_{\ell_2} \cdot S_3. $$

Let us consider the intertwining operator $W$ for some $\ell$, then $S_1$ and $S_3$ are special cases thereof.

$$ W \cdot S^a_\ell = S^a_{-1-\ell} \cdot W $$

We study subsequently the three cases of no deformation, quantum deformation and elliptic deformation.

- In the case of no deformation the equations (4.5) can be rewritten in terms of conventional generators

$$ W(z\partial - \ell) = (z\partial + 1 + \ell) W; \quad Wz(z\partial - 2\ell) = z(z\partial + 2 + 2\ell) W; \quad W\partial = \partial W $$

The general solution to the first equation has the form $W = z^{-2\ell - 1} \cdot \Phi(z\partial)$. The two remaining equations are compatible and lead to the difference equation:

$$ \Phi(x + 1) = \frac{x + 1}{x - 2\ell} \cdot \Phi(x); \quad \Phi(x) = \frac{\Gamma(x + 1)}{\Gamma(x - 2\ell)}, \quad x = z\partial $$
• In the case of quantum deformation the equations (4.5) again can be rewritten in terms of conventional generators

\[ W(z\partial - \ell) = (z\partial + 1 + \ell)W; \ Wz[z\partial - 2\ell]_q = z[z\partial + 2 + 2\ell]_qW; \ W_{\frac{1}{z}}[z\partial]_q = \frac{1}{z}[z\partial]_qW \]

The general solution of the first equation is the same as in previous case \( W = z^{-2\ell-1}\Phi(z\partial) \) and two remaining equations result in the difference equation with the known solution

\[ \Phi(x + 1) = \frac{q^{x+1} - q^{-x-1}}{q^{x-2\ell} - q^{-x+2\ell}} \cdot \Phi(x); \ \Phi(x) = \frac{(q^{2x-2\ell}; q^{2})}{(q^{2x+2}; q^{2})} \cdot q^{-(2\ell+1)x}. \]

• In the elliptic case the intertwining operator \( W \) has been constructed by A. Zabrodin [10]. For the generic \( \ell \) the solution of equations (4.5) is given in [10] in terms of the very well poised elliptic hypergeometric series

\[ W = \frac{e^{2\pi i(2\ell+1)\eta z}\gamma(2z)}{\gamma(2z + 2(2\ell + 1))} \sum_{k=0}^{\infty} \frac{[-z - 2\ell - 1 + 2k][-z - 2\ell - 1]_k [-2\ell - 1]_k e^{(2\ell+1-2k)\partial}}{[-z - 2\ell - 1]_k [-z + 1]_k[k]!} \tag{4.6} \]

We emphasize the common features allowing to do analogous steps in the three cases for deriving the uniform expressions for the intertwining operator \( W \).

**Proposition 4.** The intertwining operator \( W(u_1, u_2) \) is defined by the relation with Lax matrix

\[ W(u_1, u_2) L(u_1, u_2) = L(u_2, u_1) W(u_1, u_2), \]

where \( L(u_1, u_2) \) involves the generators of representation with spin \( \ell \) and \( L(u_2, u_1) \) involves the generators of representation with spin \( -1 - \ell \) expressed in terms of the Heisenberg pair \( z, \partial \) and

\[ u_1 = \frac{u}{2i\pi \ell} + \ell, \ u_2 = \frac{u}{2i\pi \ell} - 1 - \ell; \ \ u_1 - u_2 = 2\ell + 1. \]

In all cases the operator \( W \) depends on the difference \( u_1 - u_2; \ W(u_1, u_2) = W(u_1 - u_2) \)

\[ W_0(a) = \frac{1}{z^a} \cdot \frac{\Gamma(z\partial + 1)}{\Gamma(z\partial + 1 - a)}; \ W_q(a) = \frac{q^{\frac{a^2}{2}}}{z^a} \cdot \frac{(q^{2z\partial + 1 - a}; q^2)}{(q^{2z\partial + 2}; q^2)} \cdot q^{-az\partial}, \]

\[ W_e(a) = \frac{e^{2\pi i a z + \pi i a^2 \gamma(2z)}}{\gamma(2z + 2a)} \sum_{k=0}^{\infty} \frac{[-z - a + 2k][-z - a]_k [-a]_ke^{(a-2k)\partial}}{[-z - a]_k [-z + 1]_k[k]!}. \]

Note that for future convenience we change slightly the normalization of the operators \( W_q \) and \( W_e \). The operator \( S_1 \) is obtained from the operator \( W(u_1 - u_2) \) by substitution \( z \rightarrow z_1 \) and the operator \( S_3 \) is obtained from the operator \( W(u_1 - u_2) \) by substitution \( z \rightarrow z_2 \).

### 5 Coxeter relations for the elementary permutation operators

In this section we shall prove that the obtained operators \( S \) obey the Coxeter relations (3.16), (3.7) and (3.8) for the permutation group \( S_4 \). The operators \( S \) have simple dependence on parameters

\[ S_1(u) = S_1(u_1 - u_2); \ S_2(u) = S_2(u_2 - v_1); \ S_3(u) = S_3(v_1 - v_2) \]

so that the defining equations can be represented in the more simple form

\[ S_1(-a)S_1(a) = 1; \ S_1(a)S_2(a + b)S_1(b) = S_2(b)S_1(a + b)S_2(a) \tag{5.1} \]

\[ S_1(a)S_3(b) = S_3(b)S_1(a); \ S_2(a)S_3(a + b)S_2(b) = S_3(b)S_2(a + b)S_3(a). \tag{5.2} \]

There are two evident equalities. The operator \( S_3(a) \) differs from the operator \( S_1(a) \) only by a change of variable \( z_1 \rightarrow z_2 \) and therefore the operators commute, \( S_1(a)S_3(b) = S_3(b)S_1(a) \). The equality \( S_2(a)S_2(-a) = 1 \) can be checked easily because the operator \( S_2 \) reduces to multiplication on a given function.
5.1 The case of elliptic deformation

The transposition operators have the form

\[ S_1(a) = e^{2\pi i \sigma_1 + \pi i a^2} \frac{\gamma(2z_1)}{\gamma(2z_1 + 2a)} \sum_{k=0}^{\infty} S_k(a, z_1) e^{(a-2k)\partial_1} ; S_k(a, z_1) = \frac{[-z_1 - a + 2k] \cdot [-z_1 - a] \cdot [-a]_k}{[-z_1 - a]_k \cdot [-z_1 + 1]_k} \]

where we use the following notations for the elliptic numbers

\[ [z] = \theta_1(2\eta z) ; [z]_0 = [z] , [z]_k = [z] [z+1] \cdots [z+k-1] , k = 1, 2 \cdots \]

Let us calculate the product \( S_1(a)S_1(-a) \). Multiplying two power series and using the formula (9.10) for the shifted \( \gamma \)-functions we obtain the power series of the general form with the composite coefficients

\[ S_1(a)S_1(-a) = \sum_{N=0}^{\infty} S_N \cdot e^{-2N\partial_1} ; S_N = \sum_{k=0}^{N} S_k(a, z_1)S_{N-k}(-a, z_1 + a - 2k) \cdot \frac{[-z_1 + 1]_k}{[-z_1 + a + 1]_k} . \]

We have to prove that \( S_0 = 1 \) and \( S_N = 0 \) for \( N = 1, 2, \cdots \). Using the formulae (9.12) for the elliptic numbers we transform this expression to the canonical form

\[ S_N = \frac{[-z_1 + 2N] \cdot [z]_N \cdot [1 - a - N]_N}{[z]_1 \cdot [-z_1 - 1 + a]_N \cdot [-N]_N} \cdot \sum_{k=0}^{N} \frac{[-z_1 + a + 2k] \cdot [-z_1 + a]_k}{[-z_1 + a]_k} \cdot \frac{[-a]_k}{[-1 + a]_k} \cdot \frac{[-z_1 + N]_k}{[1 - a - N]_k} \cdot \frac{[-N]_k}{[-z_1 - a + 1 + N]_k} . \]

The key formula which allows to calculate the sum of this special type is the Frenkel-Turaev summation formula [13][14]

\[ \sum_{k=0}^{N} \frac{[A + 2k]_k}{[A]_k!} \cdot \frac{[B]_k}{[A + 1 - B]_k} \cdot \frac{[C]_k}{[A + 1 - C]_k} \cdot \frac{[D]_k}{[A + 1 - D]_k} \cdot \frac{[E]_k}{[A + 1 - E]_k} \cdot \frac{[-N]_k}{[A + 1 - N]_k} = \]

\[ = \frac{[A + 1]_N}{[A + 1 - B]_N} \cdot \frac{[A + 1 - C - B]_N}{[A + 1 - C]_N} \cdot \frac{[A + 1 - D - B]_N}{[A + 1 - D]_N} \cdot \frac{[A + 1 - D - C]_N}{[A + 1 - D - C - B]_N} . \] \hspace{1cm} (5.3)

Here the coefficients are restricted by the condition \( B + C + D + E = 2A + N + 1 \). Our special case corresponds to the substitutions

\[ A = -z_1 - a ; B = -a ; C = -z_1 + N ; D = E = \frac{A + 1}{2} = -z_1 + \frac{1 - a}{2} \]

and we have

\[ \sum_{k=0}^{N} \frac{[-z_1 + a + 2k] \cdot [-z_1 + a]_k}{[-z_1 + a]_k} \cdot \frac{[-a]_k}{[-1 + a]_k} \cdot \frac{[-z_1 + N]_k}{[1 - a - N]_k} \cdot \frac{[-N]_k}{[-z_1 - a + 1 + N]_k} = \]

\[ = \frac{[-z_1 + a + 1]_N}{[-z_1 + 1]_N} \cdot \frac{[1 - N]_N}{[1 - a - N]_N} \cdot \frac{[-z_1 + 1 + a]_N}{[-z_1 + 1 + a - N]_N} \cdot \frac{[-z_1 + 1 + a]_N}{[-z_1 + 1 + a - N]_N} . \]
We obtain the needed equality \( S_N = 0 \) for \( N = 1, 2, \cdots \) because the factor \( [1 - N]_N = 0 \) for \( N = 1, 2, \cdots \) and the equality \( S_0 = 1 \) can be easily checked.

Next we prove the second equality in (5.1) and the second equality in (5.2) can be proven in a similar way. Using (9.10) the second product can be transformed to the power series of the form

\[
S_2(b)S_1(a+b)S_2(a) = P \cdot \sum_{N=0}^{\infty} S_N(a+b, z_1) \cdot \frac{\left[ -\frac{z_1+z_2}{2} + \frac{1-b}{2} \right]_N}{\left[ -\frac{z_1-z_2}{2} + \frac{1-b}{2} \right]_N} \cdot \frac{\left[ -\frac{z_1+z_2}{2} + \frac{1+b}{2} \right]_N}{\left[ -\frac{z_1-z_2}{2} + \frac{1+b}{2} \right]_N} \cdot e^{(a+b-2N)\partial_1}
\]

where

\[
P = e^{\pi i(b^2-a^2)} \frac{\gamma(z_1 + z_2 + 1 + 2a + b)}{\gamma(z_1 - z_2 + 1 - b)} \cdot \frac{\gamma(z_1 - z_2 + 1 - b)}{\gamma(2z_1 - 2a + 2b)}
\]

To calculate the first product we multiply two power series, use the formula (9.10) and obtain the power series of the general form

\[
S_1(a)S_2(a+b)S_1(b) = P \cdot \sum_{N=0}^{\infty} [-z_1 - a - b + 2N] \cdot S_N \cdot e^{(a+b-2N)\partial_1}
\]

with the following coefficients

\[
S_N = \sum_{k=0}^{N} S_k(a, z_1)S_{N-k}(b, z_1 + a - 2k) \cdot \frac{[-z_1 - a - b + 1]_{2k}}{[-z_1 - a + 1]_{2k}} \cdot \frac{[-z_1 + z_2 + 1 + b]_{2k}}{[-z_1 - z_2 + 1 - b]_{2k}} \cdot \frac{[-z_1 - z_2 + 1 + b]_{2k}}{[-z_1 - z_2 + 1 - b]_{2k}}.
\]

Using the formulae (9.12) it is possible to transform this expression to the form

\[
S_N = \frac{[-z_1 - a - b]_N}{[-z_1 - a - b]} \cdot \frac{1 + b - N}_N \cdot \sum_{k=0}^{N} \frac{[-z_1 - a + 2k]_k}{[-z_1 - a + 1]_k} \cdot \frac{[-z_1 - a + 2k]_k}{[-z_1 - a + 1]_k} \cdot \frac{[-z_1 - a + 2k]_k}{[-z_1 - a + 1]_k} \cdot \frac{[-z_1 - a + 1 + N]_k}{[-z_1 - a + 1 + N]_k}.
\]

Next we use the Frenkel-Turaev formula (5.3) for

\[
A = -z_1 - a; \quad B = -a; \quad C = -\frac{z_1 + z_2}{2} + \frac{1 + b}{2}; \quad D = -\frac{z_1 - z_2}{2} + \frac{1 + b}{2}; \quad E = -z_1 - a - b + N
\]

in the calculation of the sum and obtain desirable relation

\[
S_N = S_N(a + b, z_1) \cdot \frac{[-z_1 + z_2 + 1 + b]_N}{[-z_1 + z_2 + 1 - b]_N} \cdot \frac{[-z_1 + z_2 + 1 + b]_N}{[-z_1 + z_2 + 1 - b]_N}.
\]

This proves that the series expansions for the operators in both sides of the equality (5.1) coincide.

### 5.2 The case of quantum deformation

The operators have the form

\[
S_1(a) = \frac{q^{a^2}}{z_1^a} \cdot \frac{q^{a^2 - 2a + 2z_1 \partial_1}; q^2}{(q^{a^2 + 2z_1 \partial_1}; q^2)} \cdot q^{-a z_1 \partial_1}; \quad S_2(a) = z_1^a \cdot \frac{q^{2a} z_1^{1-a}; q^2}{(q^{2a} z_1^{1+a}; q^2)}
\]

(5.4)
Using the q-binomial formula (9.1) we represent the operator \( S_1(a) \) as the sum
\[
S_1(a) = \frac{q^{2\frac{a^2}{2}}}{z_1^2} \cdot \sum_{k=0}^{\infty} S_k(a) q^{-(a-2k)z_1 \partial_1} \quad S_k(a) = \frac{q^{2k} \cdot (q^{-2a}; q^2)_k}{(q^2; q^2)_k}
\] (5.5)

We prove the equality \( S_1(a) S_2(a+b) S_1(b) = S_2(b) S_1(a+b) S_2(a) \) and the second defining equation of this type can be proven in a similar way. Using (9.3) the second product can be transformed to the power series of the form

\[
S_2(b) S_1(a+b) S_2(a) = \frac{q^{2\frac{a^2-b^2}{2}} \left( \frac{a}{z_1} q^{1-b}, q^2 \right)}{\left( \frac{a}{z_1} q^{1+b+2a}, q^2 \right)} \cdot \sum_{N=0}^{\infty} S_N(a+b) \cdot \left( \frac{\frac{a}{z_2} q^{1-b}; q^2 \right)_N \cdot q^{-(a+b-2N)z_1 \partial_1}
\]

To calculate the first product we multiply two power series, use the formula (9.3) and obtain the power series of the general form

\[
S_1(a) S_2(a+b) S_1(b) = \frac{q^{2\frac{a^2-b^2}{2}} \left( \frac{a}{z_1} q^{1-b}, q^2 \right)}{\left( \frac{a}{z_1} q^{1+b+2a}, q^2 \right)} \cdot \sum_{N=0}^{\infty} S_N \cdot q^{-(a+b-2N)z_1 \partial_1}
\]

with the following coefficients

\[
S_N = \frac{q^{2N} \cdot \sum_{k=0}^{N} S_k(a) S_{N-k}(b) \cdot \left( \frac{\frac{a}{z_2} q^{1+b}; q^2 \right)_k \cdot q^{-2bk}}{\left( \frac{a}{z_2} q^{1-b-2a}; q^2 \right)_k}
\]

Using the formula (9.4) it is possible to transform such expression to the form

\[
S_N = \frac{q^{2N} \cdot \left( \frac{q^{-2b}; q^2 \right)_N}{\left( \frac{q^2}; q^2 \right)_N} \cdot \sum_{k=0}^{N} \left( \frac{q^{-2a}; q^2 \right)_k \cdot \left( \frac{\frac{a}{z_2} q^{1+b}; q^2 \right)_k \cdot \left( \frac{\frac{a}{z_2} q^{1-b-2a}; q^2 \right)_k \cdot q^{k}}{\left( \frac{q^2(1+b-2N); q^2 \right)_k \cdot q^{2k}}
\]

The key formula which allows to calculate this sum is the Jackson summation formula [15]

\[
\sum_{k=0}^{N} \left( \frac{A; q^k \right)}{(B; q^k \cdot (C; q^k \cdot \left( \frac{q^{-N}; q^k \right)_k \cdot \frac{C; q^k \right)}{(C; q^k \cdot \left( \frac{C; q^k \right)}{(C; q^k \cdot q^N \right)} (5.6)
\]

We use this formula for \( q \to q^2 \) and

\[
A = q^{-2a}; B = \frac{z_1}{z_2} \cdot q^{b+1}; C = \frac{z_1}{z_2} \cdot q^{1-b-2a}; AB = \frac{A}{C} \cdot q^{2-2N} = q^{2+2b-2N}
\]

in the calculation of the sum and obtain the needed relation

\[
S_N = q^{2N} \cdot \left( \frac{q^{-2b}; q^2 \right)_N \cdot \left( \frac{\frac{a}{z_2} q^{1-b}; q^2 \right)_N \cdot \left( \frac{q^{-a-b}; q^2 \right)_N \cdot \left( \frac{q^{a-b}; q^2 \right)_N = S_N(a+b) \cdot \left( \frac{\frac{a}{z_2} q^{1-b}; q^2 \right)_N \cdot \left( \frac{\frac{a}{z_2} q^{1-b-2a}; q^2 \right)_N
\]

This proves that the series expansions for the operators in both sides of the equality (5.1) coincide.
5.3 The case of no deformation

The operators have the form \( ((a)_k = a(a + 1) \cdots (a + k - 1)) \)

\[
S_1(a) = \frac{1}{z_1^a} \cdot \frac{\Gamma(z_1 \partial_1 + 1)}{\Gamma(z_1 \partial_1 + 1 - a)} \quad ; \quad S_2(a) = (z_2 - z_1)^a = z_2^a \cdot \sum_{k=0}^{\infty} \frac{(-a)_k}{k!} \cdot \left(\frac{z_1}{z_2}\right)^k \quad (5.7)
\]

We again prove the equality \( S_1(a)S_2(a+b)S_1(b) = S_2(b)S_1(a+b)S_2(a) \) only. Using the equality

\[
\Gamma(z_1 \partial_1 + A) \cdot z_1^n = z^n \cdot \Gamma(z_1 \partial_1 + A + n) \quad (5.8)
\]

the first product can be transformed to the power series of the form

\[
S_1(a)S_2(a+b)S_1(b) = \sum_{N=0}^{\infty} \frac{(-a - b)_N}{N!} \cdot \left(\frac{z_1}{z_2}\right)^{N-a-b} \cdot \frac{\Gamma(z_1 \partial_1 + 1 - b + N)}{\Gamma(z_1 \partial_1 + 1 - a - b + N)} \cdot \frac{\Gamma(z_1 \partial_1 + 1 - a + b + N)}{\Gamma(z_1 \partial_1 + 1 - b)}
\]

To calculate the second product we multiply two power series, use the formula (5.8) and obtain the power series of the general form with the composite coefficients

\[
S_2(b)S_1(a+b)S_2(a) = \sum_{N=0}^{\infty} \left(\frac{z_1}{z_2}\right)^{N-a-b} \cdot S_N \quad ; \quad S_N = \sum_{k=0}^{N} \frac{(-a)_k}{k!} \cdot \frac{(-b)_{N-k}}{(N-k)!} \cdot \frac{(N)_k}{(1+b-N)_k} \cdot \frac{\Gamma(z_1 \partial_1 + 1 + k)}{\Gamma(z_1 \partial_1 + 1 - a - b + k)}
\]

Using the formula

\[
\frac{(-b)_{N-k}}{(N-k)!} = \frac{(-b)_N}{N!} \cdot \frac{(N)_k}{(1+b-N)_k} \quad (5.9)
\]

it is possible to transform such expression to the form \( d = z_1 \partial_1 \)

\[
S_N = \sum_{k=0}^{N} \frac{(-a)_k}{k!} \cdot \frac{(d + 1)_k}{(d + 1 - a - b)_k} \cdot \frac{(-N)_k}{(1+b-N)_k} \cdot \frac{(N)_k}{(d + 1 - a - b)_k} \cdot \frac{\Gamma(d + 1)}{\Gamma(d + 1 - a - b)}
\]

The key formula which allows to calculate the sum of this special type is the Pfaff-Saalschütz summation formula (5.15)

\[
\sum_{k=0}^{N} \frac{(A)_k}{(C)_k} \cdot \frac{(-N)_k}{(1+A+B-C-N)_k} = \frac{(C-A)_N}{(C)_N} \cdot \frac{(C-B)_N}{(C-A-B)_N} \quad (5.10)
\]

We use this formula for

\[
A = -a \quad ; \quad B = d + 1 \quad ; \quad C = d + 1 - a - b \quad ; \quad 1 + A + B - C - N = 1 - b - N
\]

in the calculation of the sum and obtain the needed relation

\[
S_N = \frac{(-a - b)_N}{N!} \cdot \frac{\Gamma(d + 1 - b + N)}{\Gamma(d + 1 - a - b + N)} \cdot \frac{\Gamma(d + 1)}{\Gamma(d + 1 - b)}
\]

This proves that the series expansions for the operators in both sides of the equality (5.1) coincide.
6 The permutation group and the Yang-Baxter relation

Let us return to the permutation group. The transpositions $s_1, s_2, s_3$ form the basis of the permutation group of four parameters $(u_1, u_2, v_1, v_2)$. Any permutation can be decomposed into the product of these operators $s_k$. This representation is not unique because different decompositions can represent the same permutation. The equivalence of the permutation group $S_4$ and the Coxeter group with the defining relations $s_1^2 = s_2^2 = 1$, $s_1 s_3 = s_3 s_1$ and $s_1 s_2 s_1 = s_2 s_2 s_1$, $s_3 s_2 s_3 = s_3 s_3 s_2$ guarantees that two sequences representing the same permutation can be transformed each other by using the defining relations between generators only. In the general case it is the existence theorem, but we shall follow the more constructive way and indicate the particular transformations we need each time. The operators $S_1, S_2, S_3$ represent the corresponding permutations of four parameters $(u_1, u_2, v_1, v_2)$ entering in the product of two Lax matrices $L_1(u_1, u_2) L_2(v_1, v_2)$ and obey the same defining relations. This key property of the operators $S_1$ shows that we deal with a representation of the permutation group.

Note that everything can be generalized to the product of an arbitrary number of Lax matrices. We consider the product of the three Lax matrices $L_1(u_1, u_2) L_2(v_1, v_2) L_3(w_1, w_2)$ because this example immediately leads to the Yang-Baxter equation. Let us join all six parameters in one set $u \equiv (u_1, u_2, v_1, v_2, w_1, w_2)$ and consider the group of permutations $S_6$ of six parameters. Now we have five elementary transpositions $s_i$ and corresponding operators $S_i(u)$

\[
S_1(u_1, u_2) = L_1(u_2, u_1) S_1 ; \quad S_3(v_1, v_2) = L_2(v_2, v_1) S_3 ; \quad S_5(w_1, w_2) = L_2(w_2, w_1) S_5
\]

\[
S_2(u_1, u_2) L_2(v_1, v_2) = L_1(u_1, v_1) L_2(u_2, v_2) S_2(u),
\]

\[
S_4(v_1, v_2) L_3(w_1, w_2) = L_2(v_1, w_1) L_3(v_2, w_2) S_4(u).
\]

It is evident that all is effectively reduced to the case of the product of two Lax matrices. The generators $S_1, S_2$ and $S_3$ are the same as in the case of the product of two Lax matrices $L_1(u_1, u_2) L_2(v_1, v_2)$. The generators $S_3, S_4$ and $S_5$ play the same role for the product $L_2(v_1, v_2) L_3(w_1, w_2)$ and can be obtained from the $S_1, S_2$ and $S_3$ by the simple change of variables and parameters

\[
z_1, z_2 \rightarrow z_2, z_3 ; \quad (u_1, u_2, v_1, v_2) \rightarrow (v_1, v_2, w_1, w_2).
\]

The defining relations

\[
S_i S_i = 1 \rightarrow S_i(s_i u) S_i(u) = 1 ; \quad S_i S_j = s_j S_i \rightarrow S_i(s_j u) S_j(u) = S_j(s_i u) S_i(u), \quad |i - j| > 1
\]

\[
s_i s_i s_i = s_i s_i s_i s_i = S_i(s_i s_i s_i s_i u) S_i(s_i s_i s_i s_i u) = S_i(s_i s_i s_i s_i u) S_i(s_i s_i s_i s_i u) S_i(s_i s_i s_i s_i u)
\]

are effectively reduced to the relations for the operators from the previous case also. The operator $R_{12}(u - v)$ is the solution of the equation

\[
R_{12}(u - v) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_1, v_2) L_2(u_1, u_2) R_{12}(u - v)
\]

and corresponds to the permutation $s_2 s_1 s_3 s_2$.

\[
R_{12}(u - v) = S_3(s_i s_i s_i u) S_1(s_i s_i s_i u) S_2(s_i s_i s_i u) S_2(u).
\]
The operator $\tilde{R}_{23}(v - w)$ is the solution of the equation

$$\tilde{R}_{23}(v - w)L_2(v_1, v_2)L_3(w_1, w_2) = L_2(w_1, w_2)L_3(v_1, v_2)\tilde{R}_{23}(v - w)$$

and corresponds to the permutation $s_4s_3s_5s_4$:

$$\tilde{R}_{23}(v - w) = S_4(s_3s_5s_4u)S_3(s_5s_4u)S_5(s_4u)S_4(u). \tag{6.3}$$

The operator $\tilde{R}_{23}(v - w)$ can be obtained by the same change of variables and parameters (6.1) from the operator $\hat{R}_{12}(u - v)$. There exists two equivalent decompositions of the special permutation

$$s(u_1, u_2, v_1, v_2, w_1, w_2) = (w_1, w_2, v_1, v_2, u_1, u_2)$$

in terms of permutations $s_2s_1s_3s_2$ and $s_4s_3s_5s_4$ and corresponding operators are expressed in terms of R-matrix

$$s = s_2s_1s_3s_2 \cdot s_4s_3s_5s_4 \cdot s_2s_1s_3s_2 \rightarrow \hat{R}_{12}(v - w)\hat{R}_{23}(u - w)\hat{R}_{12}(u - v),$$

$$s = s_4s_3s_5s_4 \cdot s_2s_1s_3s_2 \cdot s_4s_3s_5s_4 \rightarrow \hat{R}_{23}(u - v)\hat{R}_{12}(u - w)\hat{R}_{23}(v - w).$$

These operators correspond the same permutation and therefore there exists the transformation from the one expression to the another.

**Proposition 5.** The operators $\hat{R}_{12}$ (6.2) and $\hat{R}_{23}$ (6.3) obey the Yang-Baxter relation

$$\hat{R}_{23}(u - v)\hat{R}_{12}(u - w)\hat{R}_{23}(v - w) = \hat{R}_{12}(v - w)\hat{R}_{23}(u - w)\hat{R}_{12}(u - v).$$

It is sufficient to give an example of the chain of transformations allowing to transform the first decomposition of the permutation $s$ to the second one

$$s_2s_3s_4s_2 \cdot s_4s_3s_5s_4 \cdot s_2s_1s_3s_2 = s_2s_3s_4 \cdot s_1s_2s_3s_2s_1 \cdot s_5 \cdot s_4s_3s_2 = s_2s_3s_4s_3 \cdot s_1s_2s_3 \cdot s_5 \cdot s_3s_4s_3 =$$

$$s_2s_3s_4s_3 \cdot s_2s_1s_3s_2 \cdot s_5 \cdot s_4s_3s_2 = s_4s_3s_5 \cdot s_2s_3s_4s_3s_2 \cdot s_5s_3s_4 = s_4s_3s_5 \cdot s_2s_4s_3s_4s_2 \cdot s_5s_3s_4 = s_4s_3s_5 \cdot s_2s_3s_4s_2 \cdot s_5s_3s_4 = s_4s_3s_5 \cdot s_2s_1s_3s_2 \cdot s_4s_3s_5s_4.$$ 

Repeating step by step this chain of transformations for the operators $S_1$ it is possible to transform the operator in the right hand side of Yang-Baxter relation to the operator in the left hand side. We omit the corresponding formulae for brevity.

**7 The operators $\tilde{R}^{(1)}$ and $\tilde{R}^{(2)}$**

Recall the different levels of resolution of the Yang-Baxter operator into operators representing more elementary parameter permutations in acting on the product of Lax matrices as discussed in Introduction. Besides of the operators $S_1$ representing the elementary transpositions the following operators $\tilde{R}^{(k)}, k = 1, 2$ are important; they permute the parameters $u_k, v_k$ in acting on the product of Lax matrices.

Again everything can be generalized to the product of arbitrary number of Lax matrices. We consider the product of the three Lax matrices $L_1(u_1, u_2)L_2(v_1, v_2)L_3(w_1, w_2)$ because this example immediately leads to the analogon of the Yang-Baxter relations for the operators $R^{(i)}, i = 1, 2$. Recall that working in terms of operators $R^{(i)}$ we effectively reduce the symmetry $S_6 \to S_3 \times S_3$. 

16
Let us introduce the special permutations \( r_1 = s_2s_1s_2, \ r_2 = s_4s_3s_4 \) and \( p_3 = s_2s_3s_2, \ p_5 = s_4s_5s_4. \) The permutation \( r_1 \) and \( r_3 \) interchange only the parameters \( u_1 \leftrightarrow v_1 \) and \( v_1 \leftrightarrow w_1 \) correspondingly

\[
\begin{align*}
r_1u &= s_2s_1s_2u = (v_1, u_2, u_1, v_2, w_1, w_2) \quad ; \quad r_3u &= s_4s_3s_4u = (u_1, u_2, w_1, v_2, v_1, w_2)
\end{align*}
\]

and generate the group of permutations \( \mathcal{S}_3 \) of three parameters \( (u_1, v_1, w_1) \). The corresponding operators have the form

\[
\begin{align*}
\hat{R}^{(1)}_{12}(u) &= S_2(s_1s_2u)S_1(s_2u)S_2(u) \quad ; \quad \hat{R}^{(1)}_{23}(u) &= S_4(s_3s_4u)S_3(s_4u)S_4(u) \quad (7.1)
\end{align*}
\]

The permutation \( p_3 \) and \( p_5 \) interchange in a similar way \( u_2 \leftrightarrow v_2 \) and \( v_2 \leftrightarrow w_2 \)

\[
\begin{align*}
p_3u &= s_2s_3s_2u = (u_1, v_2, u_1, v_2, w_1, w_2) \quad ; \quad p_5u &= s_4s_5s_4u = (u_1, u_2, v_2, v_1, w_1, w_2)
\end{align*}
\]

and generate the group of permutations \( \mathcal{S}_3 \) of three parameters \( (u_2, v_2, w_2) \). The corresponding operators are

\[
\begin{align*}
\hat{R}^{(2)}_{12}(u) &= S_2(s_3s_2u)S_3(s_2u)S_2(u) \quad ; \quad \hat{R}^{(2)}_{23}(u) &= S_4(s_5s_4u)S_5(s_4u)S_4(u) \quad (7.2)
\end{align*}
\]

**Proposition 6.** There are the following equivalent representations for the operator \( \hat{R}_{12} \)

\[
\begin{align*}
\hat{R}_{12}(u - v) &= \hat{R}^{(1)}_{12}(pu)\hat{R}^{(2)}_{12}(u) = \hat{R}^{(2)}_{12}(ru)\hat{R}^{(1)}_{12}(u) = S_2(s_1s_2s_2u)S_1(s_3s_2u)S_3(s_2u)S_2(u).
\end{align*}
\]

The first and second expressions correspond to the decompositions of the permutation \( s \) on the product of two commuting permutations \( r = s_2s_1s_2 \) and \( p = s_2s_3s_2 \). The third expression corresponds to the decomposition \( s = s_2s_1s_3s_2 \). The chain of the transformations is

\[
rp = s_2s_1s_2 \cdot s_2s_3s_2 = s_2s_1s_2 \cdot s_2s_3s_2 \cdot s_2s_3s_2 = s_2s_1s_3s_2 = s_2s_3s_1s_2 = s_2s_1s_2 \cdot s_2s_3s_2 = pr
\]

and there is the corresponding chain of transformations which proves the equivalence of all representations for the R-matrix

\[
\begin{align*}
\hat{R}^{(1)}_{12}(pu)\hat{R}^{(2)}_{12}(u) &= S_2(s_1s_2 \cdot s_2s_3s_2u)S_1(s_2 \cdot s_2s_3s_2u)S_2(s_2s_3s_2u) \cdot S_2(s_3s_2u)S_3(s_2u)S_2(u).
\end{align*}
\]

**Proposition 7.** The operators \( \hat{R}^{(1)} \) \( (7.1) \) and \( \hat{R}^{(2)} \) \( (7.2) \) obey the relations

\[
\begin{align*}
\hat{R}^{(2)}_{12}(r_1u)\hat{R}^{(1)}_{12}(u) &= \hat{R}^{(1)}_{12}(p_3u)\hat{R}^{(2)}_{12}(u) \quad ; \quad \hat{R}^{(2)}_{23}(r_3u)\hat{R}^{(1)}_{23}(u) &= \hat{R}^{(1)}_{23}(p_5u)\hat{R}^{(2)}_{23}(u)
\end{align*}
\]

\[
\begin{align*}
\hat{R}^{(2)}_{23}(r_1u)\hat{R}^{(1)}_{12}(u) &= \hat{R}^{(1)}_{12}(p_5u)\hat{R}^{(2)}_{23}(u) \quad ; \quad \hat{R}^{(2)}_{12}(r_3u)\hat{R}^{(1)}_{23}(u) &= \hat{R}^{(1)}_{23}(p_3u)\hat{R}^{(2)}_{12}(u)
\end{align*}
\]

\[
\begin{align*}
\hat{R}^{(2)}_{23}(r_1r_3u)\hat{R}^{(1)}_{12}(u) &= \hat{R}^{(1)}_{12}(r_3r_1u)\hat{R}^{(2)}_{23}(u) \quad ; \quad \hat{R}^{(2)}_{12}(r_3r_1u)\hat{R}^{(1)}_{23}(u) &= \hat{R}^{(1)}_{23}(r_1r_3u)\hat{R}^{(2)}_{12}(u).
\end{align*}
\]

These relations correspond to the relations for the group of permutations \( \mathcal{S}_3 \times \mathcal{S}_3 \). The first four are relations of commutativity for the transformations from two different groups \( \mathcal{S}_3 \)

\[
p_3r_1 = r_1p_3 \quad ; \quad p_5r_3 = r_3p_5 \quad ; \quad p_5r_1 = r_1p_5 \quad ; \quad p_3r_3 = r_3p_3
\]

\[17\]
The proof of the corresponding relations for the operators $\hat{R}^{(i)}$ mimics step by step the transformations for the permutation group. We collect these chains of transformations in the same order as they are listed above.

$$p_{3r1} = s_2s_3s_2 \cdot s_2s_1s_2 = s_2s_3s_1s_2 = s_2s_1s_2 \cdot s_2s_3s_2 = r_1p_3$$

$$p_{5r3} = s_4s_5s_4 \cdot s_4s_3s_4 = s_4s_5s_3s_4 = s_4s_3s_4 \cdot s_4s_5s_4 = r_3p_5$$

$$p_5r_1 = s_4s_5s_4 \cdot s_2s_1s_2 = s_2s_1s_2 \cdot s_4s_5s_4 = r_1p_5$$

$$p_3r_3 = s_2s_3s_2 \cdot s_4s_3s_4 = s_3s_2s_3s_4s_3 = s_3s_2s_4s_3 = s_3s_4s_2s_3 = s_3s_4s_3 \cdot s_3s_2s_3 = s_4s_3s_4 \cdot s_2s_3s_2 = r_3p_3$$

The last two relations are the triple defining relations of each group $S_3$

$$r_3p_1r_3 = r_1r_3p_1; p_5p_3p_5 = P_5P_5P_3$$

and the chains of transformations have the form

$$r_1r_3p_1 = s_2s_1s_2 \cdot s_4s_3s_4 \cdot s_2s_1s_2 = s_1s_2s_1 \cdot s_4s_3s_4 \cdot s_1s_2s_1 = s_4s_1 \cdot s_3s_2s_3 \cdot s_1s_4 =$$

$$= s_4s_3 \cdot s_1s_2s_1 \cdot s_3s_4 = s_4s_3s_4 \cdot s_2s_1s_2 \cdot s_4s_3s_4 = r_3p_1r_3$$

$$p_3p_5p_3 = s_2s_3s_2 \cdot s_4s_5s_4 \cdot s_3s_2s_3 = s_3s_2s_3 \cdot s_5s_4s_5 \cdot s_3s_2s_3 = s_5s_2 \cdot s_3s_4s_3 \cdot s_2s_5 = s_5s_2 \cdot s_4s_3s_4 \cdot s_2s_5 =$$

$$= s_5s_4s_5 \cdot s_2s_3s_2 \cdot s_5s_4s_5 = s_4s_5s_4 \cdot s_2s_3s_2 \cdot s_4s_5s_4 = r_3p_1r_3$$

8 Discussion and summary

In the present analysis we did not specify to a series of group representations related to the considered Lie algebra $\mathfrak{sl}_2$. This would imply a restriction on the values of the representation parameter $\ell$, but this appears unnatural here because $\ell$ enters the relevant expressions in combination with the spectral parameter.

In the undeformed and q-deformed cases the representation $\ell$ can be realized as a module spanned by the monomials $z^m, m = 0, 1, ..., n$ assuming a metric structure of a functional space. The constant function 1 represents the lowest weight vector. An invariant finite-dimensional submodul appears for positive integer values of $2\ell$. In the elliptic case the representation can be described in terms of entire even periodic functions and the finite-dimensional representation appearing in the case of positive integer $N = 2\ell$ is spanned by even products of $2N$ $\theta$-functions. These realizations are convenient for describing the generic representations and their tensor products. The Yang-Baxter operator and its factors $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}$ are operating in these classes of functions. However the operators of elementary permutations $S_i$ map to different realizations involving e.g. functions with branch points.

Also the ambiguity of the solutions of difference equations by periodic factors is to be fixed by specifying the function class.

We refer to the particular case of the principal series of $SL(2, \mathbb{C})$ representations where all operators in question can be defined as acting on functions on the complex plane $\mathbb{C}^2$ and the mentioned difficulty does not appear. There one deals with actually two copies of $\mathfrak{sl}_2$, one represented by operators acting holomorphically and the other anti-holomorphically on functions defined on the complex plane. Correspondingly, the representations are labelled by the pair $(\ell, \tilde{\ell})$ taking values $\ell = \frac{n+1}{2} - i\nu, \tilde{\ell} = -\frac{n+1}{2} - i\nu, n$ integer and $\nu$ real. The structure of the operators being products of a holomorphic and anti-holomorphic part ensures that they do not lead beyond uniquely defined functions on the complex plane.
The simplest case of a symmetry algebra of the Yang-Baxter equation is $s\ell(2)$ and there are two possible ways of generalization. First one can deform the algebraic structure but keep the rank equal to one:

\[ U(s\ell_2) \to U_q(s\ell_2) \to\text{Sklyanin algebra} \]

In this paper we present the uniform expression for the solution of the Yang-Baxter equation connected with all three cases. The R-operator acting in the tensor product of two representations of the symmetry algebra with spins $\ell_1$ and $\ell_2$ can be constructed from the three basic operators $S_1, S_2, S_3$. The operators $S_1, S_2, S_3$ represent elementary permutations from the symmetric group $S_4$ – permutation group of four parameters entering the RLL-relation.

The second way of generalization is to increase the rank going to the algebra $s\ell(n)$. The Weyl group $W$ of $s\ell_n$ is the permutation group $S_n$ generated by elements $w_1, \ldots, w_{n-1}$ with defining relations $w_i^2 = 1$, $w_iw_{i+1}w_i = w_{i+1}w_iw_{i+1}$ and $w_iw_j = w_jw_i$ if $|i - j| > 1$. The RLL-relation has the same form [12].

\[ \mathcal{R}_{12}(u - v) L_1(u_1, u_2 \cdots u_n) L_2(v_1, v_2 \cdots v_n) = L_1(v_1, v_2 \cdots v_n) L_2(u_1, u_2 \cdots u_n) \mathcal{R}_{12}(u - v) , \]

but now the Lax matrix depends on the $n$ parameters $u_1, u_2 \cdots u_n$ [17]. There are $n - 1$ parameters which label the representation of the algebra $s\ell(n)$ (analog of spin $\ell$) and spectral parameter $u$. There are $2n$ parameters in the RLL-relation and the permutation group is now $S_{2n}$. There are $2n - 1$ elementary transpositions in the group $S_{2n}$

\[ s_1, s_2, \ldots, s_{n-1}, s_{n+1}, s_{n+2}, \ldots, s_{n-1} \quad ; \quad u_1, u_2, \ldots, u_{n-1}, u_n, v_1, v_2, \ldots, v_{n-1}, v_n \]

The R-matrix again admits the decomposition in factors of operators representing the elementary transpositions [17]. The operators, representing $s_1, \ldots, s_{n-1}$ and $s_{n+1}, \ldots, s_{n-1}$ are the well known intertwining operators and are connected to the Weyl group of the algebra $s\ell(n)$. The operator representing $s_n$ plays a special role and can be constructed explicitly [17].

It seems that the considered decomposition of the R-matrix is universal and gives some insight into a possible structure of solutions of Yang-Baxter equation with general symmetry.

In terms of the elementary parameter permutation operators we construct the factor operators $\mathcal{R}^{(k)}$. The product of these factors over $k = 1, \ldots, n$ results in the Yang-Baxter $\mathcal{R}$ operator for $s\ell(n)$. Acting on the product of two Lax operators $\mathcal{R}^{(k)}$ permutes their parameters $u_k, v_k$.

For a spin chain the products of these factor operators over the sites leads to a commuting family of operators $Q_k$ which can be identified as the Baxter operators [16, 18, 19].

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### 9 Appendix

#### 9.1 q-special functions

The standard $q$-products are

\[ (x; q) = \prod_{k=0}^{+\infty} (1 - q^k \cdot x) ; \quad (x; q)_n = \prod_{k=0}^{n-1} (1 - q^k \cdot x) = \frac{(x; q)}{(xq^n; q)} ; \quad q \in \mathbb{C} , \quad |q| < 1 \]
The q-binomial formula has the form
\[ \frac{(Az; q)}{(z; q)} = \sum_{k=0}^{\infty} \frac{(A; q)_k}{(q; q)_k} \cdot z^k \]  
(9.1)

The function \((x; q)\) obeys the recurrent relation \((qx; q) = (1 - x)^{-1}(x; q)\) so that it is used to solve the recurrent relation of the type
\[ \Phi(qx) = \frac{1 - ax}{1 - bx} \Phi(x) ; \Phi(x) = \frac{(bx; q)}{(ax; q)} \]  
(9.2)

There are useful formulae for the q-products which are used in the text
\[ \frac{(yq^{-2n}; q^2)}{(xq^{-2n}; q^2)} = \frac{y^n}{x^n} \cdot \frac{(\frac{q^2}{y}; q^2)^n}{(\frac{q^2}{x}; q^2)^n} \]  
(9.3)

\[ \frac{(q^{-2b}; q^2)_{N-k}}{(q^2; q^2)^{N-k}} = q^{2b(b+1)} \cdot \frac{(q^{-2b}; q^2)_N}{(q^2; q^2)_N} \cdot \frac{(q^{-2N}; q^2)_k}{(q^2(1+b-N); q^2)_k} \]  
(9.4)

### 9.2 Elliptic special functions

The general \(\theta\)-function with characteristics is \[\theta_{a,b}(z|\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (n+a)^2 \tau} \cdot e^{2\pi i (n+a)(z+b)}\]

and we shall use the four standard functions
\[\theta_1(z|\tau) = \theta_{1,1}(z|\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (n+\frac{1}{2})^2 \tau} \cdot e^{2\pi i (n+\frac{1}{2})(z+\frac{1}{2})}\]
\[\theta_2(z|\tau) = \theta_{1,0}(z|\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (n+\frac{1}{2})^2 \tau} \cdot e^{2\pi i (n+\frac{1}{2})z}\]
\[\theta_3(z|\tau) = \theta_{0,0}(z|\tau) = \sum_{n \in \mathbb{Z}} e^{\pi in^2 \tau} \cdot e^{2\pi inz}\]
\[\theta_4(z|\tau) = \theta_{0,1}(z|\tau) = \sum_{n \in \mathbb{Z}} e^{\pi in^2 \tau} \cdot e^{2\pi i(n+\frac{1}{2})(z+\frac{1}{2})}\]

The following identities are used to factorize the Lax matrix and for the derivation of the defining equations for the operator \(S_2\)

\[2\theta_1(x + y)\theta_1(x - y) = \bar{\theta}_4(x)\bar{\theta}_3(y) - \bar{\theta}_4(y)\bar{\theta}_3(x) ; \quad 2\theta_4(x + y)\theta_4(x - y) = \bar{\theta}_4(x)\bar{\theta}_3(y) + \bar{\theta}_4(y)\bar{\theta}_3(x)\]

\[2\theta_2(x + y)\theta_2(x - y) = \bar{\theta}_3(x)\bar{\theta}_3(y) - \bar{\theta}_4(y)\bar{\theta}_4(x) ; \quad 2\theta_3(x + y)\theta_3(x - y) = \bar{\theta}_3(x)\bar{\theta}_3(y) + \bar{\theta}_4(y)\bar{\theta}_4(x)\]

where \(\bar{\theta}_3(z) \equiv \theta_3(z|\tau)\) and \(\bar{\theta}_4(z) \equiv \theta_4(z|\tau)\).

The elliptic gamma function \[\Gamma(z|\tau, \tau')\] is defined by the double product
\[\Gamma(z|\tau, \tau') \equiv \prod_{n,m=0}^{\infty} \frac{1 - e^{2\pi i (n+\frac{1}{2})(\tau+\tau')^{-1}(m+\frac{1}{2})-z}}{1 - e^{2\pi i (n+\frac{1}{2})(\tau+\tau')^{-1}(m+\frac{1}{2})+z}}. \]  
(9.5)
We need the following properties of this function

\begin{align}
\Gamma(z + \tau \mid \tau, \tau') &= R(\tau') \cdot e^{\pi i z} \theta_1(z \mid \tau') \cdot \Gamma(z \mid \tau, \tau'), \\
\Gamma(z + \tau' \mid \tau, \tau') &= R(\tau) \cdot e^{\pi i z} \theta_1(z \mid \tau) \cdot \Gamma(z \mid \tau, \tau'),
\end{align}

where the constant \( R(\tau) \) does not depend on \( z \):

\[ R(\tau) = -ie^{-\frac{\pi i \tau}{4}} \cdot (1; e^{2\pi i \tau})^{-1}. \]

We introduce the function \( \gamma(z) \equiv \Gamma(\eta z \mid \tau, 2\eta) \) which obeys the recurrence relation

\[ \gamma(z + 2) = R(\tau) \cdot e^{\pi i \eta z} \theta(\gamma(z)) \cdot \gamma(z) ; \theta(z) \equiv \theta_1(\eta z \mid \tau) \]

and can be used to solve the more general recurrence relation

\[ \Phi(z + 2) = \theta(\gamma(z + a)) \theta(\gamma(z + b)) \cdot \Phi(z) \]

\[ \Phi(z) = e^{\pi i \eta(z + a)} \gamma(z + a) \gamma(z + b), \]

needed to obtain the operator \( S_2 \). There is the useful formula for the shifted \( \gamma \)-functions

\[ \frac{\gamma(x - 2m) \gamma(y)}{\gamma(x) \gamma(y - 2m)} = e^{-\pi i m(y - x)} \cdot \left[ \frac{\eta}{2} + 1 \right] _m \left[ -\frac{\eta}{2} + 1 \right] _m, \]

where we use the following notations for the elliptic numbers

\[ [z] = \theta_1(2\eta z) ; [z]_0 = [z] , [z]_k = [z] \cdot [z + 1] \cdots [z + k - 1] , \ k = 1, 2 \cdots \]

To transform the series with elliptic numbers one uses the transformation formulæ

\[ \frac{[A + 1]_{2k} [A + 2k]_{N-k}}{[A + 2k]} = \frac{[A]_N [A + N]_k}{[A]} ; \ [B]_{2k} [B + 2k]_{N-k} = [B]_N [B + N]_k ; \]

\[ \frac{[-b]_{N-k}}{[N-k]!} = \frac{[1 + b - N]_N [-N]_k}{[N]_N [1 + b - N]_k}. \]

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