REGULAR SOLUTIONS OF CHEMOTAXIS-CONSUMPTION SYSTEMS
INVOLVING TENSOR-VALUED SENSITIVITIES AND
ROBIN TYPE BOUNDARY CONDITIONS

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Abstract. This paper deals with a parabolic-elliptic chemotaxis-consumption system with
tensor-valued sensitivity $S(x, n, c)$ under no-flux boundary conditions for $n$ and Robin-type
boundary conditions for $c$. The global existence of bounded classical solutions is established
in dimension two under general assumptions on tensor-valued sensitivity $S$. One of main
steps is to show that $\nabla c(\cdot, t)$ becomes tiny in $L^2(B_r(x) \cap \Omega)$ for every $x \in \Omega$ and $t$ when $r$ is
sufficiently small, which seems to be of independent interest. On the other hand, in the case
of scalar-valued sensitivity $S = \chi(x, n, c)I$, there exists a bounded classical solution globally
in time for two and higher dimensions provided the domain is a ball with radius $R$ and all
given data are radial. The result of the radial case covers scalar-valued sensitivity $\chi$ that
can be singular at $c = 0$.

1. Introduction

Chemotaxis-consumption systems are usually studied with scalar-valued chemotactic sen-
sitivities where the chemotactic bacteria partially orient their movement along a gradient of
a signal substance which they consume. However, according to recent modeling approaches,
we do not necessarily have to assume that the chemotactic sensitivity is a scalar value. It
has been suggested, based on the experimental findings [8, 14] (see also [33]), to use more
general, tensor-valued and spatially inhomogeneous chemotactic sensitivity. [18, 32, 34]

Taking into account tensor-valued sensitivity, in this paper, we consider the parabolic-
elliptic chemotaxis-consumption system

\[
\begin{aligned}
  n_t &= \nabla \cdot (\nabla n - nS(x, n, c) \cdot \nabla c), & x \in \Omega, \ t > 0, \\
0 &= \Delta c - nc, & x \in \Omega, \ t > 0,
\end{aligned}
\]

in a bounded smooth domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, where the sensitivity $S(x, n, c)$ attains values
in $\mathbb{R}^{d \times d}$. Here, the unknowns $n$ and $c$ denote the bacterial population density and the signal
concentration, respectively. The boundary conditions posed will be

\[
(\nabla n - nS(x, n, c) \cdot \nabla c) \cdot \nu = 0, \quad \nabla c \cdot \nu = \gamma - c, \quad x \in \partial \Omega, \ t > 0,
\]

where $\nu$ denotes the outward unit normal vector to $\partial \Omega$. We emphasize that the boundary
condition for $c$ is of Robin type.

Homogeneous Neumann boundary conditions for $c$ have been often used in mathematical
studies regarding (1.1) and its variants. [6, 15, 27, 29, 30] However, in the original version
of (1.1) by Tuval et al. [21], certain non-trivial boundary conditions for $c$ are proposed to

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take into account the effect of oxygen $c$ at the drop-air interface. Motivated by experimental observations in Tuval et al., \cite{Tuval2003} it has been suggested in \cite{Alt2003} (see also \cite{Alt2004}) to use non-homogeneous boundary conditions of the form

$$\nabla c \cdot \nu = (\gamma - c)g \quad \text{on} \quad \partial \Omega. \quad (1.3)$$

As seen in (1.2), we will impose (1.3) with $g \equiv 1$ but results in Theorem 1 are valid for more general $g$ (see Remark 2).

We compare (1.1)–(1.2) to the chemotaxis-consumption system with homogeneous Neumann boundary conditions

$$n_t = \nabla \cdot (\nabla n - \chi(c)n \nabla c), \quad c_t = \Delta c - nc, \quad x \in \Omega, \ t > 0, \quad (1.4)$$
$$\nabla n \cdot \nu = \nabla c \cdot \nu = 0, \quad x \in \partial \Omega, \ t > 0. \quad (1.5)$$

We remark that the $c$ equation should be of parabolic type since an elliptic approximation of the $c$ equation in (1.4)–(1.5) leads to $c \equiv 0$. It is known that solutions of (1.4)–(1.5) satisfy the energy-like inequality (see, e.g. \cite{Dolbeault2006, Dolbeault2007, Jerrard2008, Jerrard2009, Jerrard2010})

$$\frac{d}{dt} \left( \int_{\Omega} n \log n + \frac{1}{2} \int_{\Omega} \chi(c) \left| \frac{\nabla c}{c} \right|^2 \right) + \int_{\Omega} \frac{\nabla n^2}{n} + \frac{1}{4} \int_{\Omega} \frac{c}{\chi(c)} |D^2 \rho(c)|^2 \leq 0, \quad (1.6)$$

where $\rho(c) = \int_1^c \chi(s)/s \, ds$. The inequality (1.6) is typically deduced via a subtle cancellation caused by nonlinear structure of the system (1.4)–(1.5). In the case of the system (1.1)–(1.2), because of presence of tensor-valued sensitivity, it is not clear whether or not such energy like inequality can be derived due to loss of the cancellation effect. As a variant of (1.6), we refer to \cite{Dolbeault2006} for a chemotaxis-consumption-fluid system with constant sensitivity and Robin boundary condition.

As far as we know, there have been relatively few results dealing with tensor-valued sensitivity or Robin type boundary conditions. In particular, the only result in presence of both tensor-valued sensitivities and Robin type boundary conditions we are aware of is that bounded weak solutions to a 3D chemotaxis-Stokes system with nonlinear cell diffusion are known to exist globally in time, \cite{Winkler2017}.

In presence of constant sensitivities and Robin type boundary conditions, smooth solutions to (1.1) are known to exist globally in time for general data and any dimension \cite{Dolbeault2006} (see also \cite{Dolbeault2007, Dolbeault2009}). However, in presence of tensor-valued sensitivities and Neumann boundary conditions, even when $d = 2$, smooth solutions to the fully parabolic counterpart of (1.1) have been found to exist globally in time only under a smallness assumption on $c$ \cite{Winkler2017} (see also \cite{Dolbeault2010}), or under additional regularizing effects such as nonlinear diffusion enhancement at large densities. \cite{Dolbeault2010, Dolbeault2011, Dolbeault2012, Dolbeault2013} Without such additional effects, large data global existence results are so far available only for certain generalized weak solutions when $d \geq 1$ in \cite{Winkler2017} and when $d = 2$ in \cite{Winkler2017}. Recently, the eventual smoothness and stabilization of certain generalized solutions are also investigated when $d = 2$ by Winkler. \cite{Winkler2017}

The main motivation of the present work is to prove that smooth solutions to the two-dimensional chemotaxis-consumption system (1.1)–(1.2) exist globally in time for general tensor-valued sensitivity and arbitrary large initial data. As we mentioned earlier, it is unclear whether or not the energy-like inequality (1.6) can be derived in the case of the system (1.1)–(1.2). Instead, we derive a series of spatially localized estimates (see Proposition 1,
Lemma 4, Lemma 5, which will lead to the uniform in time bound of \( \int_\Omega n \log n \) (see Corollary 1). Especially in Proposition 1 which may be of independent interest, it is shown that for arbitrary small \( \varepsilon > 0 \), we can find \( r > 0 \), independent of \( x \in \Omega \) and \( t < T_{\text{max}} \), such that

\[
\| \nabla c(\cdot, t) \|_{L^2(\Omega \cap B_r(x))} \leq \varepsilon.
\]

We also consider (1.1)–(1.2) with scalar-valued \( \varepsilon > 0 \), independent of \( x \in \Omega \) and \( t < T_{\text{max}} \), such that

\[
\| \nabla c(\cdot, t) \|_{L^2(\Omega \cap B_r(x))} \leq \varepsilon.
\]

Now, to formulate our main results, let us specify the precise problem setting. We use the notation \( \mathbb{R}_+ := (0, \infty) \). On the tensor-valued sensitivity \( S = (S_{ij})_{i,j \in \{1, \ldots, d\}} \), we will impose the conditions

\[
S_{ij} \in C^2(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+) \quad \text{for all } i,j \in \{1, \ldots, d\} \quad \text{and}
\]

\[
|S(x, r, s)| + |\partial_r S(x, r, s)| \leq S_0(s) \quad \text{for } (x, r, s) \in \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+. 
\]

(1.7)

with some \( S_0 \in C(\mathbb{R}_+) \).

The boundary data \( \gamma \) and the initial condition \( n(\cdot, 0) = n_0 \) are assumed to satisfy

\[
\gamma \in \mathbb{R}_+, \quad 0 \leq n_0 \in L^\infty(\Omega). 
\]

Our first main result is the global existence of regular solutions to the system (1.1)–(1.2) in two dimensions.

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded smooth domain. Then, (1.1)–(1.2) subject to (1.7)–(1.8) admits a unique non-negative solution \( (n, c) \) satisfying

\[
\begin{aligned}
n &\in C([0, \infty); L^p(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap L^\infty(0, \infty; L^\infty(\Omega)), \\
c &\in C^{2,0}(\overline{\Omega} \times (0, \infty)) \cap L^\infty(0, \infty; W^{1,\infty}(\Omega)).
\end{aligned}
\]

(1.9)

**Remark 1.** The results in Theorem 1 can be extended to higher dimensions \( d \geq 3 \) provided \( S(x, n, c) \equiv \chi(c)\mathbb{I}_d \), \( \chi \in C^2(\mathbb{R}_+) \), and \( \chi, \chi' \geq 0 \). Indeed, a priori \( L^p \)-estimate shows that

\[
\frac{1}{p} \int_\Omega n^\frac{p}{2} \| \nabla n^\frac{p}{2} \|^2_{L^2(\Omega)} + \frac{4(p-1)}{p^2} \| \nabla n^\frac{p}{2} \|^2_{L^2(\Omega)}
\]

\[
= \frac{p-1}{p} \int_{\partial \Omega} \chi(c) n^p(\gamma - c) - \frac{p-1}{p} \int_{\Omega} n^p \chi'(c)|\nabla c|^2 - \int_{\Omega} n^{p+1} \chi(c) c
\]

\[
\leq \frac{p-1}{p} \| \chi \|_{C^1([0,\gamma])} \gamma \| n^\frac{p}{2} \|^2_{L^2(\partial \Omega)}, \quad p \geq 1,
\]

where we used the non-negativities of \( \chi \) and \( \chi' \) in the last inequality. If we further use the trace and interpolation inequalities and a Moser-type iteration argument, then we can obtain a uniform-in-time bound for \( n \) (see [10] for the case \( \chi \equiv 1 \)).

**Remark 2.** We remark that the results in Theorem 1 are still valid for more general Robin boundary conditions \( \nabla c \cdot \nu = (\gamma - c)g \) with \( 0 < g \in C^{1+\theta}(\partial \Omega) \) for some \( \theta \in (0, 1) \). This can be verified by following the same methods of proof for Theorem 1 and thus, for simplicity, all computations are performed for the case \( g \equiv 1 \).
Remark 3. In Theorem 1, applying the classical parabolic regularity theory to the no-flux boundary problem \( n_t = \nabla \cdot (\nabla n - \vec{a}) \) for \( x \in \Omega, t > 0 \) with \( \vec{a} = n S \cdot \nabla c \in L^\infty(0, \infty; L^\infty(\Omega)) \), we can further have Hölder continuity of \( n \) up to \( t = 0 \) provided that \( n_0 \) is Hölder continuous. See, e.g., [19, Thm. 1.3].

Our second main result states that in the case of scalar-valued sensitivity, (1.1) has a global smooth solution in two and higher dimensions provided the domain is a ball and all given data are radial.

Theorem 2. Let \( \Omega = B_R(0) \subset \mathbb{R}^d, d \geq 2 \). Assume that (1.8) holds and \( n_0 \) is radial. Then, (1.1)–(1.2) with the scalar sensitivity \( S(x, n, c) \equiv \chi(x, n, c)I_d, 0 \leq \chi \in C^2(\Omega \times \mathbb{R}_+ \times \mathbb{R}_+) \), admits a unique non-negative solution \( (n, c) \) satisfying (1.9) provided that for \( x, y \in \Omega, r \in \mathbb{R}_+, s \in \mathbb{R}_+ \)
\[
\chi(x, r, s) = \chi(y, r, s) \quad \text{if} \quad |x| = |y| \quad \text{and} \\
\chi(x, r, s) + |\partial_r \chi(x, r, s)| \leq \chi_0(s) \quad \text{with some} \quad \chi_0 \in C(\mathbb{R}_+).
\]

Remark 4. The proof of Theorem 2 mainly relies on the decay estimate of the cumulative mass distribution \( Q \) defined in (4.1) (see Lemma 7). This is crucially used to obtain the upper bound of \( |\nabla c| \) and the lower bound of \( c \).

Remark 5. We emphasize that unlike the case of Theorem 1, the sensitivity \( \chi(\cdot, \cdot, c) \) in Theorem 2 may allow singularities at \( c = 0 \), for example, \( \chi(x, n, c) = 1/c \). Although \( \chi \) can be singular at \( c = 0 \), no singularity, however, occurs since signal concentration \( c \) is turned out to be bounded below, independent of time, away from zero (see Lemma 7).

The outline is as follows: in Section 2, the local existence result is established; in Section 3 and Section 4, we prove Theorem 1 and Theorem 2, respectively. Throughout this paper, the surface area of \( B_1(0) \) is denoted by \( \sigma_d \).

2. Local existence

In this section, we prove a local existence result via the Banach fixed point theorem. Our local existence result reads as follows.

Lemma 1. Let \( \Omega \subset \mathbb{R}^d, d \geq 2 \), be a bounded smooth domain. Then, there exists a maximal time of existence, \( T_{\text{max}} \in (0, \infty] \), such that for \( t < T_{\text{max}} \), a unique solution \( (n, c) \) of (1.1)–(1.2) subject to (1.7)–(1.8) exists and satisfies
\[
n \in \bigcap_{p \in [1, \infty]} C([0, t); L^p(\Omega)) \cap C^2(\overline{\Omega} \times (0, t)) \cap L^\infty(0, t; L^\infty(\Omega)),
\]
\[
c \in C^2(\overline{\Omega} \times (0, t)),
\]
\[
\int_\Omega n(\cdot, t) = \int_\Omega n_0, \quad n(x, t) \geq 0, \quad 0 < c(x, t) < \gamma \quad \text{for} \quad x \in \Omega.
\]

Moreover, it holds that
\[
(2.1) \quad \text{either} \quad T_{\text{max}} = \infty \quad \text{or} \quad \limsup_{t \nearrow T_{\text{max}}} \|n(\cdot, t)\|_{L^\infty(\Omega)} = \infty.
\]

To obtain Lemma 1, we prepare the following elementary lemma.
Lemma 2. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded smooth domain, and let $p > d$. For any $u, f \in L^p(\Omega)$ with $u \geq 0$ and any constant $\eta \geq 0$, the problem

$$
(2.2) \quad \begin{cases}
-\Delta v + uv = f, & x \in \Omega, \\
\nabla v \cdot v + v = \eta, & x \in \partial \Omega
\end{cases}
$$

admits a unique solution $v \in W^{2,p}(\Omega) \cap C^1(\overline{\Omega})$ with the following properties:

(i) There exists $C = (d, \Omega, p)$ such that

$$
\|v\|_{W^{2,p}(\Omega)} \leq C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \eta(\|\partial\Omega\|^\frac{1}{2} + \|\partial\Omega\|^\frac{d}{2})).
$$

(ii) If $\eta = 0$, then there exists $C = (d, \Omega, p, \|u\|_{L^p(\Omega)})$ such that

$$
\|v\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}.
$$

(iii) If $f \equiv 0$, then $0 \leq v \leq \eta$.

Proof. To obtain the existence, we let $u$ be approximated in $L^p(\Omega)$ by a sequence of bounded non-negative functions $u_l$ and let $v_l \in W^{2,p}(\Omega)$ be a unique solution of the problem

$$
(2.3) \quad \begin{cases}
-\Delta v_l + u_l v_l = f, & x \in \Omega, \\
\nabla v_l \cdot v_l + v_l = \eta, & x \in \partial \Omega,
\end{cases}
$$

which is uniquely solvable by [12, Thm. 2.4.2.6]. Note that the elliptic regularity theory [12, Thm. 2.3.3.6] gives that there exists $C > 0$ such that

$$
\|v_l\|_{W^{2,p}(\Omega)} \leq C(\|u_l v_l\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|v_l\|_{W^{1,1\frac{d}{p}}(\partial \Omega)} + \eta|\partial\Omega|^\frac{1}{p}).
$$

We also note that Hölder’s and the Gagliardo–Nirenberg inequalities yield $C > 0$ such that

$$
\|u_l v_l\|_{L^p(\Omega)} \leq C\|u_l\|_{L^p(\Omega)}\|v_l\|_{L^2(\Omega)}^\theta_1\|v_l\|_{W^{2,p}(\Omega)}^{1-\theta_1}, \quad \theta_1 = \frac{\frac{2}{d} - \frac{1}{p}}{\frac{2}{d} - 1 + \frac{1}{2}} \in (0, 1),
$$

and by $W^{1,p}(\Omega) \hookrightarrow W^{1-\frac{d}{p},p}(\partial \Omega)$ and the Gagliardo–Nirenberg inequality, there exists $C > 0$ fulfilling

$$
\|v_l\|_{W^{1-\frac{d}{p},p}(\partial \Omega)} \leq C\|v_l\|_{L^2(\Omega)}^{\theta_2}\|v_l\|_{W^{2,p}(\Omega)}^{1-\theta_2}, \quad \theta_2 = \frac{\frac{1}{d} - \frac{1}{p} + \frac{1}{2}}{\frac{2}{d} - 1 + \frac{1}{2}} \in (0, 1).
$$

Combining above estimates, after applying Young’s inequality, we have that with some $C > 0$,

$$
(2.4) \quad \|v_l\|_{W^{2,p}(\Omega)} \leq C(\|u_l\|_{L^p(\Omega)}^{\frac{1}{p}}\|v_l\|_{L^2(\Omega)} + \|f\|_{L^p(\Omega)} + \|v_l\|_{L^2(\Omega)} + \eta|\partial\Omega|^\frac{1}{p}).
$$

Now, we multiply the $v_l$ equation by $v_l$, integrate over $\Omega$, and use integration by parts, Hölder’s inequality and $H^1(\Omega) \hookrightarrow L^p(\Omega)$ to find $C > 0$ such that

$$
\int_\Omega |\nabla v_l|^2 + \int_{\partial \Omega} |v_l|^2 + \int_\Omega u_l v_l^2 = \int_\Omega f v_l + \int_{\partial \Omega} \eta v_l \leq C(\|f\|_{L^p(\Omega)}\|v_l\|_{H^1(\Omega)}) + \int_{\partial \Omega} \eta v_l.
$$
Then, the Poincaré inequality with trace term (see e.g. [4]) and Young’s inequality yield $C > 0$ satisfying

\begin{equation}
\|v\|_{H^1(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \eta|\partial\Omega|^{\frac{1}{2}}).
\end{equation}

In view of (2.4)–(2.5), $v_\ell$ is bounded in $W^{2,p}(\Omega)$. The compactness of $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ and the weak compactness of bounded sets in $W^{2,p}(\Omega)$ allow us to extract a subsequence $v_{\ell_j}$ converging in $C^1(\overline{\Omega})$ that converges weakly in $W^{2,p}(\Omega)$. If we take the limit in the problem for $v_{\ell_j}$, then its limit $v$ is in $W^{2,p}(\Omega) \cap C^1(\overline{\Omega})$ and satisfies (2.2). This concludes the existence result.

To obtain the uniqueness, we let $v$ and $\tilde{v}$ be two solutions. Since a simple integration by parts gives

$$\int_{\Omega} |\nabla (v - \tilde{v})|^2 + \int_{\partial\Omega} |v - \tilde{v}|^2 + \int_{\Omega} |v|v - \tilde{v}|^2 = 0,$$

we have $v \equiv \tilde{v}$.

Repeating similar computations as above, we can find $C = C(d, \Omega, p)$ such that

$$\|v\|_{W^{2,p}(\Omega)} \leq C \left( \|uv\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \eta(|\partial\Omega|^{\frac{1}{2}} + |\partial\Omega|^{\frac{3}{2}}) \right),$$

which yields (i).

Repeating similar computations as (2.4) and (2.5), since $\eta = 0$, there exists $C = C(d, \Omega, p)$ such that

$$\|v\|_{W^{2,p}(\Omega)} \leq C(\|u\|_{L^p(\Omega)}^{\frac{1}{p}} + 1)\|f\|_{L^p(\Omega)}.$$}

This concludes (ii).

To obtain (iii), we multiply the $v$ equation by $v_- := -\min\{0, v\}$, integrate over $\Omega$, and use integration by parts. Then, we have

$$\int_{\Omega} |\nabla v_-|^2 + \int_{\Omega} |v_-|^2 + \int_{\partial\Omega} |v_-|^2 = -\eta \int_{\partial\Omega} v_-.$$ 

Since the right-hand-side is non-positive, $v_- \equiv 0$, namely, $v \geq 0$. Using the same argument, we can deduce $v \leq \eta$. \qed

We are now ready to prove Lemma 1.

**Proof of Lemma 1.** We fix $p > d + 2$ and let $M := 2\|n_0\|_{L^p(\Omega)} + 1$. With a positive number $T < 1$ to be specified below, we introduce the Banach space

$$X_T := \{ f \in C([0, T]; L^p(\Omega)) | \|f\|_{L^\infty(0, T; L^p(\Omega))} \leq M, \ f \geq 0 \ \text{for} \ t \leq T \}.$$ 

For any given $\tilde{n} \in X_T$, we note from Lemma 1 that the problem

$$\begin{cases}
0 = \Delta c - \tilde{n}c, & x \in \Omega, \\
\nabla c \cdot \nu = \gamma - c, & x \in \partial\Omega,
\end{cases}$$

admits a unique solution $c(\cdot, t) \in W^{2,p}(\Omega) \cap C^1(\overline{\Omega})$ for $t \leq T$ such that $0 \leq c \leq \gamma$. We also note, using Lemma 1 (ii), $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, and Lemma 1 (iv), that there exists $C = C(d, \Omega, p) > 0$ satisfying

\begin{equation}
\|c\|_{L^\infty(0, T; C^1(\overline{\Omega}))} \leq C_\gamma \|\tilde{n}\|_{L^\infty(0, T; L^p(\Omega))}.
\end{equation}
With such \( c = c(\tilde{n}) \), according to [13, III. Thm. 5.1], the linear problem
\[
\begin{align*}
\begin{cases}
  n_t = \nabla \cdot (\nabla n - nS(x, \tilde{n}, c) \cdot \nabla c) , & x \in \Omega, \; t > 0, \\
  (\nabla n - nS(x, \tilde{n}, c) \cdot \nabla c) \cdot \nu = 0 , & x \in \partial \Omega, \; t > 0, \\
  n(x, 0) = n_0(x) , & x \in \Omega
\end{cases}
\end{align*}
\]
has a unique weak solution \( n \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \). If we use the weak formulation with the test function \( n_- := -\min\{0, n\} \), then after using (1.7) and Young’s and Hölder’s inequalities, we have
\[
\int_{\Omega} |n_-(\cdot, t)|^2 \leq -2 \int_0^t \int_{\Omega} |\nabla n_-|^2 + 2 \int_0^t \int_{\Omega} |n_-| |S_0(c)| |\nabla n_-| |\nabla c| \\
\leq \frac{1}{2} \|S_0\|_{L^2([0, \gamma])}^2 \|\nabla c\|_{L^2(0, T; L^\infty(\Omega))} \int_0^t \int_{\Omega} |n_-|^2 \text{ for } t \leq T
\]
and \( n \geq 0 \) follows by Grönwall’s inequality. Moreover, from [17, Thm. VI. 6.40],
\[
n \in L^\infty(0, T; L^\infty(\Omega)),
\]
and by similar computations as above, we have
\[
\int_{\Omega} n^p(\cdot, t) \\
\leq \int_{\Omega} n_0^p + \frac{p(p - 1)}{4} \|S_0\|_{C([0, \gamma])}^2 \|\nabla c\|_{L^2(0, T; L^\infty(\Omega))} \int_0^t \int_{\Omega} n^p \text{ for } t \leq T.
\]
Using Grönwall’s inequality and taking supremum over the time interval, it follows that
\[
\|n\|_{L^\infty(0, T; L^p(\Omega))} \leq \|n_0\|_{L^p(\Omega)} \exp \left( \frac{(p - 1)}{4} \|S_0\|_{C([0, \gamma])}^2 \|\nabla c\|_{L^2(0, T; L^\infty(\Omega))} T \right).
\]
Therefore, if we use (2.7) and take a sufficiently small \( T \), then the mapping \( \Phi(n) := n \) maps \( X_T \) into itself.

Next, for given \( \tilde{n}_1, \tilde{n}_2 \in X_T \), we denote \( n_i = \Phi(\tilde{n}_i), \; c_i = c(\tilde{n}_i) \) for \( i = 1, 2 \), and \( \delta f = f_1 - f_2 \).

Note that for any \( t \leq T \) and \( \xi \in L^2(0, T; W^{1,2}(\Omega)) \) with \( \xi_t \in L^2(0, T; L^2(\Omega)) \), we have
\[
\int_{\Omega} \delta n \xi(\cdot, t) - \int_0^t \int_{\Omega} \delta n \xi_t + \int_0^t \int_{\Omega} \nabla \delta n \cdot \nabla \xi \\
= \int_0^t \int_{\Omega} (\delta nS(x, \tilde{n}_1, c_1) \cdot \nabla c_1 + n_2 Z \cdot \nabla c_1 + n_2 S(x, \tilde{n}_2, c_2) \cdot \nabla c) \cdot \nabla \xi,
\]
where
\[
Z = S(x, \tilde{n}_1, c_1) - S(x, \tilde{n}_2, c_1) + S(x, \tilde{n}_2, c_1) - S(x, \tilde{n}_2, c_2).
\]
Note also that by the mean value theorem, (1.7) and \( c \leq \gamma \), there exists \( C > 0 \) satisfying
\[
|Z| \leq C(|\delta \tilde{n}| + |\delta c|) \text{ a.e. in } \Omega \times (0, T).
\]
Along with this, if we use the above weak formulation with the test function $|\delta n|^{p-2}\delta n$, (2.7), and $c \leq \gamma$, then with some $C_1 = C_1(d, \Omega, p, M) > 0$, we have

$$
\frac{1}{p} \int_0^t \int_\Omega |\delta n(\cdot, t)|^p + \frac{4(p-1)}{p^2} \int_0^t \int_\Omega |\nabla |\delta n|^\frac{p}{2}|^2 \\
\quad \leq C_1 \left( \int_0^t \int_\Omega |\nabla |\delta n|^\frac{p}{2}||\delta n||^\frac{p}{2} + \int_0^t \int_\Omega |\nabla |\delta n|^\frac{p}{2}||\delta n||^\frac{p}{2-1} n_2|\delta n| \\
\quad + \int_0^t \int_\Omega |\nabla |\delta n|^\frac{p}{2}||\delta n||^\frac{p}{2-1} n_2(|\delta c| + |\nabla \delta c|) \right).
$$

(2.7)

We apply Young’s inequality to the first term on the right-hand-side above to find $C > 0$ satisfying

$$
\int_0^t \int_\Omega |\nabla |\delta n|^\frac{p}{2}||\delta n||^\frac{p}{2} \leq \frac{p-1}{C_1 p^2} \int_0^t \int_\Omega |\nabla |\delta n|^\frac{p}{2}|^2 + C \int_0^t \int_\Omega |\delta n|^p.
$$

Similarly, applying Young’s inequality to the rightmost term, after using $W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ and $n_2 \in X_T$, we observe that with some $C > 0$,

$$
\int_0^t \int_\Omega \left( |\nabla |\delta n|^\frac{p}{2}||\delta n||^\frac{p}{2-1} n_2(|\delta c| + |\nabla \delta c|) \right) \\
\quad \leq \frac{p-1}{C_1 p^2} \int_0^t \int_\Omega |\nabla |\delta n|^\frac{p}{2}|^2 + C \left( \int_0^t \int_\Omega |\delta n|^p + \int_0^t \|\delta c\|_{W^{2,p}(\Omega)}^p \right).
$$

It remains to estimate the second term on the right-hand side of (2.7). Note that, due to our choice of $p$, Hölder’s and the Gagliardo-Nirenberg inequalities yield $C > 0$ satisfying

$$
\int_0^t \int_\Omega |\nabla |\delta n|^\frac{p}{2}||\delta n||^\frac{p}{2-1} n_2|\delta n| \\
\quad \leq C \int_0^t \left( \|\nabla |\delta n|^\frac{p}{4}L^2(\Omega)\|_{L^{p/2}(\Omega)}^{p-d+2} + \|\nabla |\delta n|^\frac{p}{4}L^2(\Omega)\|_{L^{p/2}(\Omega)}^{p-d+2} \right)\|n_2\|_{L^p(\Omega)} \|\delta n\|_{L^p(\Omega)}.
$$

Thus, using Young’s inequality and $n_2 \in X_T$, we can find $C > 0$ fulfilling

$$
\int_0^t \int_\Omega \left( |\nabla |\delta n|^\frac{p}{2}||\delta n||^\frac{p}{2-1} n_2|\delta n| \right) \\
\quad \leq \frac{p-1}{C_1 p^2} \int_0^t \int_\Omega |\nabla |\delta n|^\frac{p}{2}|^2 + C \left( \int_0^t \int_\Omega |\delta n|^p + \int_0^t \|\delta n\|_{L^p(\Omega)}^p \right).
$$

Combining the above computations, we have that with some $C > 0$,

$$
\int_0^t |\delta n(\cdot, t)|^p \leq C \left( \int_0^t \int_\Omega |\delta n|^p + \int_0^t (\|\delta c\|_{W^{2,p}(\Omega)}^p + \|\delta n\|_{L^p(\Omega)}^p) \right).
$$

Since applying Lemma 2 (ii) to the problem for $\delta c$,

$$
\begin{cases}
-\Delta \delta c + \tilde{n}_1 \delta c = -c_2 \delta n, & x \in \Omega, \\
\nabla \delta c \cdot \nu + \delta c = 0, & x \in \partial \Omega,
\end{cases}
$$

and using $c_2 \leq \gamma$ yields $C > 0$ such that

$$
\|\delta c\|_{L^\infty(0,T;W^{2,p}(\Omega))} \leq C \gamma \|\delta n\|_{L^\infty(0,T;L^p(\Omega))},
$$
by Grönwall’s inequality, it follows that with some \( C = C(d, \Omega, p, M) > 0, \)
\[
\|\delta n\|_{L^\infty(0,T;L^p(\Omega))} \leq CT^\frac{1}{p} \exp(CT) \|\delta \tilde{n}\|_{L^\infty(0,T;L^p(\Omega))}.
\]

Hence, for a sufficiently small choice of \( T, \) the mapping \( \Phi \) becomes contraction on \( X_T, \) and by the Banach fixed point theorem, we have a unique fixed point \( n = \Phi(n). \)

Next, we consider more regularity properties of solutions. Since \( n \) belongs to \( C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega)) \) and satisfies for every \( [t_1, t_2] \subset (0, T) \) and \( \xi \in W^{1,2}_loc(0, T; L^2(\Omega)) \cap L^2_\text{loc}(0, T; W^{1,2}(\Omega)), \)
\[
\int_\Omega \xi(\cdot, t_2) - \int_{t_1}^{t_2} \int_\Omega \xi_t + \int_{t_1}^{t_2} \int_\Omega (\nabla n - nS(x, n, c) \cdot \nabla c) \cdot \nabla \xi = \int_\Omega \xi(\cdot, t_1),
\]
for any \( \eta \in (0, T) \) we have \( n \in C^{0, \frac{\theta}{2}}(\bar{\Omega} \times [\eta, T]) \) with some \( \theta \in (0, 1) \) by the classical parabolic regularity theory \([13] \) (see, e.g., \([19] \) Thm. 1.3). Then, \( c(\cdot, t) \in C^{2,\theta}(\Omega) \) for \( t \in [\eta, T] \) by the elliptic regularity theory \([16] \) Cor. 4.41, and moreover, since we have for \( [s, t] \subset [\eta, T] \)
\[
\begin{align*}
\left\{ &-\Delta\left(c(t) - c(s)\right) + n(t)(c(t) - c(s)) = -c(s)(n(t) - n(s)), \quad x \in \Omega, \\
&\nabla\left(c(t) - c(s)\right) \cdot \nu + (c(t) - c(s)) = 0, \quad x \in \partial \Omega,
\end{align*}
\]
it follows by \([16] \) Thm. 2.26 (see also \([10] \) Lem. 2.4) that with some \( C > 0, \)
\[
\|c(t) - c(s)\|_{C^{2,\theta}(\Omega)} \leq C\|n(t) - n(s)\|_{C^\theta(\Omega)} \quad \text{for} \quad \eta \leq s \leq t \leq T.
\]

This yields Hölder regularity on the time variable, \( c \in C^{2+\theta, \frac{\theta}{2}}(\bar{\Omega} \times [\eta, T]), \) and by the standard parabolic regularity theory, \( n \in C^{2,1}(\bar{\Omega} \times [\eta, T]). \) Since \( \eta \in (0, T) \) is arbitrary, we have the desired regularity result. Note that the blow-up criteria \((2.1)\) follows by the standard extension argument, the mass conservation property of \( n \) is a consequence of integrating the \( n \) equation, and \( 0 < c < \gamma \) is the result of the elliptic maximum principle.

\[\square\]

**Remark 6.** We remark that Lemma \([7] \) provides local existence of the solutions in Theorem \([1] \). Moreover, Lemma \([6] \) can be also used to obtain Theorem \([2] \) since in radial case, a priori estimate shows that there exists \( c_* > 0 \) such that \( c \geq c_* \) independent of any regularization of \( \chi \) keeping non-negative sign, local existence of the solutions in Theorem \([2] \) is also available because singularity of \( \chi \) at \( c = 0 \) does not play any role. Since its verification is admissible, the details are omitted.

### 3. Case of tensor sensitivity in two dimensions

In this section, we prove Theorem \([1] \) via a series of spatially localized estimates. To this end, we first establish a uniform-in-time smallness of spatially localized \( L^2 \)-norm of \( \nabla c \) in the following proposition. We remark that \( \nabla c \) has a uniform-in-time \( L^2 \)-norm over \( \Omega \) from
\[
\int_\Omega |\nabla c|^2 \leq \int_\Omega |\nabla c|^2 + \int_\Omega nc^2 + \frac{1}{2} \int_{\partial \Omega} c^2
\]
\[
= \int_{\partial \Omega} \gamma c - \frac{1}{2} \int_{\partial \Omega} c^2
\]
\[
\leq \frac{1}{2} \gamma^2 |\partial \Omega|.
\]

\((3.1)\)
This bound implies that $L^2$ norm of $\nabla c$ becomes very small in a small neighborhood of each point, but it may not be uniformly small in time. In the next proposition, we prove that it is the case, namely localized norm of $\nabla c$ can be uniformly small independent of time.

**Proposition 1.** Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded smooth domain. Let $(n, c)$ be a solution given by Lemma 4. For any given $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ independent of $q \in \overline{\Omega}$ such that

$$\sup_{t < T_{\text{max}}} \|\nabla c(\cdot, t)\|_{L^2(\Omega \cap B_\delta(q))} \leq \varepsilon \quad \text{for} \quad \delta \in (0, \delta_\varepsilon).$$

**Proof.** Let $\eta \in (0, e^{-1})$ and $B_\eta(0) = \{x \in \mathbb{R}^d | |x| < \eta\}$. We introduce the non-negative radial function

$$\psi_\eta(x) := \begin{cases} \ln(-\ln|x|) - \ln(-\ln \eta), & x \in B_\eta(0) \setminus \{0\}, \\ 0, & \text{otherwise}, \end{cases}$$

and recall that the surface area of $B_1(0)$ is denoted by $\sigma_d$. Direct computations show that

$$\|\psi_\eta\|^2_{L^2(\mathbb{R}^d)} = \sigma_d \int_0^\eta \ln(-\ln r) - \ln(-\ln \eta)^2 r^{d-1} dr$$

$$\leq \sigma_d \int_0^\eta \ln(-\ln r)^2 r^{d-1} dr$$

$$= \sigma_d \int_{\ln \frac{1}{\eta}}^{\infty} (\ln \rho)^2 e^{-\rho} d\rho$$

and

$$\|\nabla \psi_\eta\|^2_{L^2(\mathbb{R}^d)} = \sigma_d \int_0^\eta \frac{1}{|r \ln r|^2} r^{d-1} dr$$

$$= \sigma_d \int_{\ln \frac{1}{\eta}}^{\infty} \frac{1}{\rho^2} e^{-(d-2)\rho} d\rho.$$ 

Since the right hand sides above are both finite, $\psi_\eta \in H^1(\mathbb{R}^d)$. Moreover, since $\ln \frac{1}{\eta}$ tends to $\infty$ as $\eta$ approach 0,

$$\|\psi_\eta\|_{H^1(\mathbb{R}^d)} \to 0 \quad \text{as} \quad \eta \to 0.$$ 

Fix $q \in \overline{\Omega}$, and we denote $\psi(x) = \psi_\eta(x - q)$ and $B_\eta = B_\eta(q)$. If we test the $c$ equation of (1.1) with $c \psi^2$ and integrate over $\Omega$, then integration by parts gives, due to $\psi = 0$ in $(B_\eta)^c$, that

$$\int_{\Omega \cap B_\eta} nc^2 \psi^2 + \int_{\Omega \cap B_\eta} |\nabla c|^2 \psi^2 = \int_{\partial \Omega} \nabla c \cdot \nu c \psi - 2 \int_{\Omega \cap B_\eta} \nabla c \cdot \nabla c \psi.$$ 

Using Young’s inequality and $c \leq \gamma$, we compute the rightmost term as

$$\left| - 2 \int_{\Omega \cap B_\eta} \nabla c \cdot \nabla c \psi \right| \leq \frac{1}{2} \int_{\Omega \cap B_\eta} |\nabla c|^2 \psi^2 + 2 \gamma^2 \int_{\Omega \cap B_\eta} |\nabla \psi|^2.$$ 

Next, to control the boundary term, we consider two cases. If $B_\eta \subset \Omega$, then $\psi = 0$ on $\partial \Omega$ and thus,

$$\int_{\partial \Omega} \nabla c \cdot \nu c \psi^2 = 0.$$
Otherwise, if \( B_\eta \not\subset \Omega \), then since \( \psi = 0 \) in \( (B_\eta)^c \), we have

\[
\int_{\partial\Omega} \nabla c \cdot \nu \psi^2 = \int_{\partial\Omega \cap B_\eta} \nabla c \cdot \nu c \psi^2.
\]

Thus, using the boundary condition and \( c \leq \gamma \), we can compute

\[
\left| \int_{\partial\Omega \cap B_\eta} \nabla c \cdot \nu c \psi^2 \right| = \int_{\partial\Omega \cap B_\eta} (\gamma - c) c \psi^2 \leq \gamma^2 \int_{\partial\Omega \cap B_\eta} \psi^2 \leq \gamma^2 \int_{\partial\Omega} \psi^2.
\]

Combining the above estimates, after using \( H^1(\Omega) \hookrightarrow L^2(\partial\Omega) \), we have that with some \( C > 0 \) independent of \( \eta \),

\[
\int_{\Omega \cap B_\eta} |\nabla c|^2 \psi^2 \leq C \|\psi\|_{H^1(\mathbb{R}^d)}^2.
\]

In view of (3.2), there exists sufficiently small \( \eta_0 > 0 \) such that the right-hand-side above is less than or equal to \( \varepsilon^2 \) for \( \eta < \eta_0 \). Moreover, since there exists \( \delta_0 > 0 \) satisfying

\[ \psi^2 \geq 1 \quad \text{a.e. in } B_\delta \quad \text{for } \delta \in (0, \delta_0), \]

we can deduce the desired result. \( \square \)

For further local-in-space estimates, we introduce a smooth cut-off function and its properties (see, e.g. [11]):

**Lemma 3.** Let \( \delta > 0 \). There is a radially decreasing function \( \varphi_\delta \in C^\infty_0(\mathbb{R}^d) \) satisfying

\[
\varphi_\delta(x) = \begin{cases} 
1, & x \in B_{\frac{\delta}{2}}(0), \\
0, & x \in \mathbb{R}^d \setminus B_\delta(0),
\end{cases}
\]

for \( 0 \leq \varphi_\delta \leq 1 \) in \( \mathbb{R}^d \),

and

\[ |\nabla \varphi_\delta| \leq K \varphi_\delta^{\frac{1}{2}} \quad \text{in } \mathbb{R}^d, \]

where \( K \) is a positive constant of order \( \mathcal{O}(\delta^{-1}) \).

We now prepare the following lemma which is used to prove Lemma 5. For computational simplicity, we use \( \varphi^3 \) as a test function.

**Lemma 4.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded smooth domain. Let \( (n, c) \) be a solution given by Lemma 1. Assume that \( \delta > 0 \) and \( \varphi_\delta \) is the function introduced in Lemma 3. Denote \( \varphi(x) = \varphi_\delta(x - q) \) and \( B_\delta = B_\delta(q) \) for \( q \in \Omega \). Then, there exist two positive constants \( C_2 \) and \( C_3 \) independent of \( \delta \) and \( q \) such that

\[
(3.3) \quad \int_{\Omega} n^2 \varphi^3 \leq C_2 \left( \int_{\Omega} \frac{|\nabla n|^2}{n} \varphi^3 + \|\varphi\|_{W^{1,\infty}(\mathbb{R}^2)}^2 \right),
\]
\[\int_\Omega |\nabla c|^4 \varphi^3 \leq C_\delta \|\nabla c\|_{L^2(\Omega \cap B_\delta)} \left( \int_\Omega \frac{|\nabla n|^2}{n} \varphi^3 + \|\varphi^3\|_{L^2(\Omega)}^2 + \|\varphi\|_{W^{1,\infty}(\Omega)}^3 \right).\] (3.4)

**Proof.** Since the Sobolev inequality yields \( C > 0 \) such that
\[\int_\Omega n^2 \varphi^3 \leq C \left( \|\nabla (n\varphi^3)\|^2_{L^2(\Omega)} + \|n\varphi^3\|^2_{L^2(\Omega)} \right),\]
and after using Hölder's inequality and \( \int_\Omega n = \int_\Omega n_0 \), we can find \( C > 0 \), independent of \( \delta \) and \( q \), satisfying
\[\int_\Omega n^2 \varphi^3 \leq C \left( \|n_0\|_{L^1(\Omega)} \int_\Omega \frac{|\nabla n|^2}{n} \varphi^3 + \|n_0\|^2_{L^1(\Omega)} \|\varphi^3\|^2_{L^1(\Omega)} \right).\]
This gives (3.3).

Next, using the Hölder inequality, direct computations, \((a + b)^3 \leq 4(a^3 + b^3)\) for \(a, b \geq 0\), and \(c \leq \gamma\), we note that
\[\int_\Omega |\nabla c|^4 \varphi^3 \leq \|\nabla c\|_{L^2(\Omega \cap B_\delta)} \left( \int_\Omega |\nabla c|^6 \varphi^6 \right)^{\frac{1}{2}} \]
\[= \|\nabla c\|_{L^2(\Omega \cap B_\delta)} \|\nabla (c\varphi) - c\nabla \varphi\|^3_{L^6(\Omega)} \leq \|\nabla c\|_{L^2(\Omega \cap B_\delta)} (\|c\varphi\|_{W^{1,6}(\Omega)} + \|c\nabla \varphi\|_{L^6(\Omega)})^3 \]
\[\leq 4\|\nabla c\|_{L^2(\Omega \cap B_\delta)} (\|c\varphi\|_{W^{1,6}(\Omega)} + \|\nabla \varphi\|_{L^6(\Omega)})^3 \].
(3.5)

Since
\[\nabla (c\varphi) \cdot \nu = \nabla c \cdot \nu \varphi + c \nabla \varphi \cdot \nu = (\gamma - c)\varphi + c \nabla \varphi \cdot \nu \quad \text{on} \quad \partial \Omega,\]
using \(W^{2,\frac{3}{2}}(\Omega) \hookrightarrow W^{1,6}(\Omega)\) and the elliptic regularity theory [12, Thm. 2.3.3.6], we can find \( C > 0 \), independent of \( \delta \) and \( q \), such that
\[\|c\varphi\|_{W^{1,6}(\Omega)} \leq C (\|\Delta (c\varphi)\|_{L^\frac{6}{2}(\Omega)} + \|c\varphi\|_{L^\frac{6}{2}(\Omega)} + \|\gamma - c\varphi\|_{W^{1,6}(\partial \Omega)} + \|c\nabla \varphi \cdot \nu\|_{W^{1,6}(\partial \Omega)}).\]

Using direct computations, Hölder's inequality, and \(c \leq \gamma\), we compute the first term on the right-hand-side above as
\[\|\Delta (c\varphi)\|_{L^\frac{6}{2}(\Omega)} \leq \|\Delta c\varphi\|_{L^\frac{6}{2}(\Omega)} + 2\|\nabla c \cdot \nabla \varphi\|_{L^\frac{6}{2}(\Omega)} + \|c\Delta \varphi\|_{L^\frac{6}{2}(\Omega)} \leq \|nc\varphi\|_{L^\frac{6}{2}(\Omega)} + 2\|\nabla c\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^6(\Omega)} + \gamma \|\Delta \varphi\|_{L^\frac{6}{2}(\Omega)}\]
Since the trace inequality and the smoothness of \(\Omega\) yield \(C > 0\) satisfying
\[\|(\gamma - c)\varphi\|_{W^{1,6}(\partial \Omega)} \leq C \|(\gamma - c)\varphi\|_{W^{1,6}(\Omega)},\]
and
\[\|c\nabla \varphi \cdot \nu\|_{W^{1,6}(\partial \Omega)} \leq C \|c\nabla \varphi\|_{W^{1,6}(\Omega)},\]
using \(c \leq \gamma\) and Hölder's inequality, we have that with some \(C > 0\), independent of \(\delta\) and \(q\),
\begin{equation*}
\| (\gamma - c) \varphi \|_{W^{1/2}_{\text{loc}}(\partial \Omega)} + \| c \nabla \varphi \cdot \nu \|_{W^{1/2}_{\text{loc}}(\partial \Omega)} \\
\leq C (\gamma \| \varphi \|_{W^{2,1/2}_{\text{loc}}(\Omega)} + \| \nabla c \|_{L^{2}(\Omega)} \| \varphi \|_{W^{1,6}_{\text{loc}}(\Omega)}).
\end{equation*}

Note that by repeating the computations used to derive (2.5), we can find \( C > 0 \) such that

\[ \|c\|_{H^1(\Omega)} \leq C|\partial\Omega|^{1/2}. \]

Combining above estimates gives, after using \( c \leq \gamma \), that with some \( C > 0 \), independent of \( \delta \) and \( q \),

\[ \|c\varphi\|_{W^{1,6}_{\text{loc}}(\Omega)} \leq C(\gamma \|n\varphi\|_{L^{2}_{\text{loc}}(\Omega)} + \gamma|\partial\Omega|^{1/2}\|\varphi\|_{W^{1,6}_{\text{loc}}(\Omega)} + \gamma\|\varphi\|_{W^{2,1/2}_{\text{loc}}(\Omega)}). \]

Plugging it into (3.5), since Hölder’s inequality and \( \int_{\Omega} n = \int_{\Omega} n_0 \) imply

\[ (3.6) \]

\[ \|n\varphi\|_{L^{2}_{\text{loc}}(\Omega)} \leq \|n_0\|_{L^{3}_{\text{loc}}(\Omega)} \left( \int_{\Omega} n^2 \varphi^3 \right)^{1/3}, \]

it follows that there exists \( C > 0 \), independent of \( \delta \) and \( q \), satisfying

\[ \int_{\Omega} |\nabla c|^4 \varphi^3 \leq C \|\nabla c\|_{L^{2}(\Omega \cap B_{\delta})} \left( \int_{\Omega} n^2 \varphi^3 + \|\varphi\|^3_{W^{1,6}_{\text{loc}}(\Omega)} + \|\varphi\|^3_{W^{2,1/2}_{\text{loc}}(\Omega)} \right). \]

Therefore, by (3.3) and \( W^{2,1/2}(\Omega) \hookrightarrow W^{1,6}(\Omega) \), we can conclude (3.4).

The spatially localized \( L \log L \)-norm of \( n \) is bounded uniformly in time:

**Lemma 5.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded smooth domain. Let \((n, c)\) be a solution given by Lemma 4. Assume that \( \delta > 0 \) and \( \varphi_{\delta} \) is the function introduced in Lemma 3. Denote \( \varphi(x) = \varphi_{\delta}(x - q) \) and \( B_{\delta} = B_{\delta}(q) \) for \( q \in \overline{\Omega} \). Then, there exist \( \delta_* > 0 \) independent of \( q \) such that if \( \delta < \delta_* \), then there exists \( C = C(\delta) > 0 \) independent of \( q \) satisfying

\[ \sup_{t < T_{\max}} \int_{\Omega} n \log n(\cdot, t) \varphi^3 \leq C. \]

**Proof.** We begin by noting that due to Proposition 1 there exists \( \delta_* > 0 \) independent of \( q \in \overline{\Omega} \) such that

\[ (3.7) \]

\[ \sup_{t < T_{\max}} \left( \|S_0\|_{C([0, T_{\max}])} C_2 C_3 \|\nabla c\|_{L^{2}(\Omega \cap B_{\delta})} + \frac{1}{4} C_3 \|\nabla c\|_{L^{2}(\Omega \cap B_{\delta})} \right) \leq \frac{1}{5} \quad \text{for} \quad \delta < \delta_*, \]

where \( S_0 \) is the function given in (1.7), and \( C_2 \) and \( C_3 \) are the positive numbers given in Lemma 3.

Let \( \delta < \delta_* \). From the \( n \) equation and the no-flux condition, we observe that

\begin{align*}
\frac{d}{dt} \int_{\Omega} n \log n \varphi^3 &- \frac{d}{dt} \int_{\Omega} n \varphi^3 \\
&= -\int_{\Omega} \nabla (\log n \varphi^3) \cdot [\nabla n - nS(x, n, c) \cdot \nabla c] \\
&= -\int_{\Omega} \frac{\nabla n}{n} \varphi^3 + \int_{\Omega} \nabla n \cdot (S(x, n, c) \cdot \nabla c) \varphi^3 \\
&\quad - 3 \int_{\Omega} \log n \nabla n \cdot \nabla \varphi^2 + 3 \int_{\Omega} n \log n(S(x, n, c) \cdot \nabla c) \cdot \nabla \varphi^2.
\end{align*}
By (1.7), $c \leq \gamma$, and Hölder’s inequality, we have

$$
\int_{\Omega} \nabla n \cdot (S(x, n, c) \cdot \nabla c) \varphi^3 \\
\leq \|S_0\|_{c([0, \gamma])} \left( \frac{\int_{\Omega} |\nabla n|^2}{n} \varphi^3 \right)^{\frac{1}{2}} \left( \int_{\Omega} n^2 \varphi^3 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla c|^4 \varphi^3 \right)^{\frac{1}{4}}.
$$

This gives, by Lemma 4, Young’s inequality, and (3.1), that there exists $M = M(\delta) > 0$, independent of $q$, satisfying

$$
\int_{\Omega} \nabla n \cdot (S(x, n, c) \cdot \nabla c) \varphi^3 \\
\leq \|S_0\|_{c([0, \gamma])}^4 C_2^4 C_3^4 \|\nabla c\|^4_{L^2(\Omega \cap B_\delta)} \int_{\Omega} \frac{|\nabla n|^2}{n} \varphi^3 + \frac{1}{5} \int_{\Omega} \frac{|\nabla n|^2}{n} \varphi^3 + M.
$$

Next, we use Young’s inequality, $a|\log a|^2 \leq 16e^{-2}a^2 + 4e^{-2}$ for $a \geq 0$, Lemma 3 and (3.6) to compute

$$
-3 \int_{\Omega} \log n \nabla n \cdot \nabla \varphi \varphi^2 \\
\leq \frac{1}{8} \int_{\Omega} \frac{|\nabla n|^2}{n} \varphi^3 + 18 \int_{\Omega} n|\log n|^2|\nabla \varphi|^2 \\
\leq \frac{1}{8} \int_{\Omega} \frac{|\nabla n|^2}{n} \varphi^3 + 18 \int_{\Omega} (16e^{-2}n^2 + 4e^{-2})|\nabla \varphi|^2 \\
\leq \frac{1}{8} \int_{\Omega} \frac{|\nabla n|^2}{n} \varphi^3 + 18 \cdot 16e^{-2}K^2 \int_{\Omega} n^2 \varphi^3 + 18 \cdot 4e^{-2}K^2 \int_{\Omega} \varphi^2 \\
\leq \frac{1}{8} \int_{\Omega} \frac{|\nabla n|^2}{n} \varphi^3 + 18 \cdot 16e^{-2}K^2 |n_0|^4_{L^1(\Omega)} \left( \int_{\Omega} n^2 \varphi^3 \right)^{\frac{1}{4}} + 18 \cdot 4e^{-2}K^2 \int_{\Omega} \varphi^2.
$$

It follows by Young’s inequality and (3.3) that with some $M = M(\delta) > 0$,

$$
-3 \int_{\Omega} \log n \nabla n \cdot \nabla \varphi \varphi^2 \leq \frac{1}{5} \int_{\Omega} \frac{|\nabla n|^2}{n} \varphi^3 + M.
$$
Similarly, if we use \((1.7)\), Young’s inequality, \(a^\frac{2}{3} |\log a|^\frac{1}{3} \leq 16e^{-\frac{1}{2}}a^\frac{1}{2} + e^{-\frac{3}{2}} \) for \(a \geq 0\), Lemma 3, and \((3.6)\), then we have

\[
3 \int_\Omega n \log n (S(x, n, c) \cdot \nabla c) \cdot \nabla \varphi^2 
\leq 3\|S_0\|_{C([0, \gamma])} \int_\Omega |n| \log n \|\nabla c\| \|\nabla \varphi\| \varphi^2 
\leq \frac{1}{4} \int_\Omega |\nabla c|^4 \varphi^3 + \frac{3}{4} (3\|S_0\|_{C([0, \gamma])})^{\frac{4}{3}} \int_\Omega n^\frac{4}{3} |\log n|^\frac{4}{3} \phi^\frac{4}{3} |\nabla \phi|^\frac{4}{3} 
\leq \frac{1}{4} \int_\Omega |\nabla c|^4 \varphi^3 + \frac{3}{4} (3\|S_0\|_{C([0, \gamma])})^{\frac{4}{3}} K^{\frac{4}{3}} \int_\Omega (16e^{-\frac{4}{3}n^\frac{3}{2}} + e^{-\frac{4}{3}}) \phi^\frac{3}{2} 
\leq \frac{1}{4} \int_\Omega |\nabla c|^4 \varphi^3 + 12(3\|S_0\|_{C([0, \gamma])})^{\frac{4}{3}} K^{\frac{4}{3}} e^{-\frac{4}{3}} \|n_0\|_{L^1(\Omega)} \left(\int_\Omega n^2 \varphi^3 \right)^\frac{1}{\frac{4}{3}} 
\quad + \frac{3}{4} (3\|S_0\|_{C([0, \gamma])})^{\frac{4}{3}} K^{\frac{4}{3}} e^{-\frac{4}{3}} \int_{\mathbb{R}^2} \phi^\frac{3}{2}.
\]

Thus, by Lemma 4 and Young’s inequality, we can find \(M = M(\delta) > 0\) such that

\[
3 \int_\Omega n \log n (S(x, n, c) \cdot \nabla c) \cdot \nabla \varphi^2 
\leq \frac{1}{4} C_3 \|\nabla c\|_{L^2(\Omega \cap B_\delta)} \int_\Omega |\nabla n|^2 \frac{n}{n^3} \varphi^3 + \frac{1}{5} \int_\Omega |\nabla n|^2 \frac{n}{n^3} \varphi^3 + M.
\]

Combining above estimates gives that with some \(M = M(\delta) > 0\),

\[
\frac{d}{dt} \int_\Omega n \log n \varphi^3 - \frac{d}{dt} \int_\Omega n \varphi^3 + 2 \int_\Omega \frac{|\nabla n|^2}{n} \varphi^3 
\leq (\|S_0\|_{C([0, \gamma])})^{\frac{1}{2}} C_2 \frac{1}{2} \|\nabla c\|_{L^2(\Omega \cap B_\delta)}^{\frac{1}{2}} + \frac{1}{4} C_3 \|\nabla c\|_{L^2(\Omega \cap B_\delta)} \int_\Omega \frac{|\nabla n|^2}{n} \varphi^3 + M.
\]

We note that using \(a \log a \leq 2e^{-1}a^\frac{2}{3} \) for \(a \geq 0\), \((3.6)\), \((3.3)\), and Young’s inequality, we can find \(M = M(\delta) > 0\) such that

\[
\int_\Omega n \log n \varphi^3 - \int_\Omega n \varphi^3 \leq \int_\Omega 2e^{-1}n^\frac{2}{3} \varphi^\frac{2}{3} 
\quad \leq 2e^{-1}\|n_0\|_{L^1(\Omega)} \left(\int_\Omega n^2 \varphi^3 \right)^\frac{1}{3} 
\quad \leq \frac{1}{5} \int_\Omega \frac{|\nabla n|^2}{n} \varphi^3 + M.
\]

Thus, adding both sides of \((3.8)\) by

\[
\int_\Omega n \log n \varphi^3 - \int_\Omega n \varphi^3
\]

and using \((3.7)\), and \((3.9)\), we can deduce that with some \(M = M(\delta) > 0\),

\[
\frac{d}{dt} \mathcal{F}(t) + \mathcal{F}(t) \leq M,
\]
where
\[ F(t) = \int_\Omega n \log n(\cdot, t) \varphi^3 - \int_\Omega n(\cdot, t) \varphi^3. \]
Since solving this ordinary differential inequality gives
\[ F(t) \leq F(0) e^{-t} + M(1 - e^{-t}), \]
with \( \int_\Omega n \varphi^3 \leq \int_\Omega n_0 \), we can conclude the desired estimate. \( \square \)

As a direct consequence, \( L \log L \)-norm of \( n \) over \( \Omega \) is also bounded.

**Corollary 1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded smooth domain. There exists \( C > 0 \) such that
\[ \sup_{t < T_{\text{max}}} \int_\Omega n \log n(\cdot, t) \leq C. \]

**Proof.** Let \( \delta > 0 \), and let \( \varphi_\delta \) be a function given in Lemma 3. Denote \( \varphi(x) = \varphi_\delta(x - q) \) and \( B_\delta = B_\delta(q) \) for \( q \in \overline{\Omega} \). Using Lemma 5, \( a \log a + e^{-1} \geq 0 \) for \( a \geq 0 \), and \( \varphi = 1 \) in \( B_\frac{\delta}{2} \), we can find \( \delta > 0 \) and \( C > 0 \) both independent of \( q \) such that
\[ \sup_{t < T_{\text{max}}} \int_{\Omega \cap B_\frac{\delta}{2}(q)} (n \log n(\cdot, t) + e^{-1}) \varphi^3 \leq C. \]
Since the open covering \( \bigcup_{q \in \overline{\Omega}} B_\frac{\delta}{2}(q) \) of compact set \( \overline{\Omega} \) has a finite subcovering \( \bigcup_{i=1}^N B_\frac{\delta}{2}(q_i) \), \( q_i \in \overline{\Omega} \), we have that with some \( C > 0 \)
\[ \sup_{t < T_{\text{max}}} \int_\Omega (n \log n(\cdot, t) + e^{-1}) \leq \sum_{i=1}^N \sup_{t < T_{\text{max}}} \int_{\Omega \cap B_\frac{\delta}{2}(q_i)} (n \log n(\cdot, t) + e^{-1}) \leq C. \]
This gives the desired bound. \( \square \)

To obtain higher integrability of \( n \), we prepare the following lemma which can be seen as a generalization of [11, Lem. 2.4].

**Lemma 6.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded smooth domain. There exists \( C = C(\Omega) > 0 \) such that for any \( p \geq 1, s > 1, \varepsilon > 0 \), and non-negative \( f \in C^1(\overline{\Omega}) \),
\[ \int_\Omega f^{p+1} \leq C \frac{(p + 1)^2}{\log s} \int_\Omega (f \log f + e^{-1}) \int_\Omega f^{p-2} |\nabla f|^2 \\
+ (4C)^{1+\varepsilon} \left( \int_\Omega f^{\frac{p+1}{2}} \right)^{2(1+\varepsilon)} \varepsilon + 6s^{p+1} |\Omega|. \]

**Proof.** We recall from [11, (2.1)–(2.5)] that there exists \( C = C(\Omega) > 0 \) such that
\[ \int_\Omega f^{p+1} \leq C \frac{(p + 1)^2}{2 \log s} \int_\Omega (f \log f + e^{-1}) \int_\Omega f^{p-2} |\nabla f|^2 + 2C \|w\|_{L^1(\Omega)}^2 + 3s^{p+1} |\Omega|, \]
where
\[ w = \max\{f^{\frac{p+1}{2}} - s^{\frac{p+1}{2}}, 0\}. \]
Using a direct computation, and Hölder’s and Young’s inequalities, we compute
\[
\|w\|_{L^2(\Omega)}^2 \leq \left( \int_{\{f > s\}} f^{p+1} \right)^2 \leq \left( \int_{\Omega} f^{p+1} \right)^2 \leq \left( \int_{\Omega} f^{\frac{p+1}{2}} \right)^{2(1+\epsilon)}\frac{2(1+\epsilon)}{2+\epsilon} \leq \frac{1}{4C} \int_{\Omega} f^{p+1} + \left( \frac{8C}{2+\epsilon} \right)^{\frac{2}{2+\epsilon}} \frac{\epsilon}{2+\epsilon} \left( \int_{\Omega} f^{\frac{p+1}{2}} \right)^{\frac{2(1+\epsilon)}{2+\epsilon}} .
\]

Since \( \left( \frac{8C}{2+\epsilon} \right)^{\frac{2}{2+\epsilon}} \frac{\epsilon}{2+\epsilon} \leq (4C)^{\frac{2}{2+\epsilon}} \), we can deduce the desired result. \( \square \)

We are ready to prove Theorem \([1] \)

**Proof of Theorem \([1] \)** Let \((n, c)\) be a solution given by Lemma \([1] \). Once we have a uniform-in-time bound for \(\|n\|_{L^p(\Omega)}\) with some \(p > d = 2\), then by Lemma \([2] \) (i), \(W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})\), \([1.7] \) and \(c \leq \gamma\), we have a uniform-in-time bound of \(S(x, n, c)\). Then, applying a Moser-type iteration argument to
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} n^p + \frac{4(p-1)}{p^2} \int_{\Omega} |\nabla n|^{\frac{2}{p}}
\]

we find uniform-in-time bound for \(n\) and Theorem \([1] \) follows by Lemma \([1] \). Thus, it is enough to show that there exists \(C > 0\) satisfying
\[
(3.10) \quad \sup_{t < T_{\text{max}}} \int_{\Omega} n^3(\cdot, t) \leq C.
\]

To this end, multiplying the \(n\) equation by \(n^2\) and integrating over \(\Omega\), we observe that
\[
\frac{1}{3} \frac{d}{dt} \int_{\Omega} n^3 + \frac{8}{9} \int_{\Omega} |\nabla n|^{\frac{3}{2}} = \frac{4}{3} \int_{\Omega} n^2 \nabla n^2 \cdot (S(x, n, c) \cdot \nabla c).
\]

Using \([1.7] \), \(c \leq \gamma\), and Hölder’s inequality, we compute the right-hand-side as
\[
\frac{4}{3} \int_{\Omega} n^\frac{3}{2} \nabla n^\frac{3}{2} \cdot (S(x, n, c) \cdot \nabla c)
\leq \frac{4}{3} \|S_0\|_{C([0, \gamma])} \int_{\Omega} n^\frac{3}{2} |\nabla n^\frac{3}{2}| |\nabla c|
\leq \frac{4}{3} \|S_0\|_{C([0, \gamma])} \|n\|_{L^4(\Omega)}^\frac{3}{2} \|\nabla n^\frac{3}{2}\|_{L^2(\Omega)} \|c\|_{W^{1,4}(\Omega)}.
\]

Then, we use the Gagliardo-Nirenberg interpolation inequality,
\[
\|f\|_{W^{1,s}(\Omega)} \leq C(\|f\|_{L^\infty(\Omega)}^\frac{1}{2} \|f\|_{W^{2,4}(\Omega)}^\frac{1}{2} + \|f\|_{L^\infty(\Omega)} ) \quad \text{for all} \quad f \in C^2(\overline{\Omega}),
\]

Lemma \([2] \) (i), and \(c \leq \gamma\) to find \(C > 0\) such that
\[
\|c\|_{W^{1,s}(\Omega)} \leq C(\|n\|_{L^4(\Omega)}^\frac{1}{2} + 1).
\]
Since Lemma 6 with \((f, p, \varepsilon) = (n, 3, 1)\) and Corollary 1 yield \(C > 0\) independent of \(s > 1\) satisfying
\[
\int_\Omega n^4 \leq \frac{C}{\log s} \int_\Omega |\nabla n^{\frac{3}{2}}|^2 + (4C)^{\frac{3}{2}} \left(\int_\Omega n_0^4\right)^4 + 6s^4|\Omega|,
\]
combining above estimates, after using Young’s inequality, we can find \(C > 0\) independent of \(s > 1\) such that
\[
\frac{1}{3} \frac{d}{dt} \int_\Omega n^3 + \frac{8}{9} \int_\Omega |\nabla n^{\frac{3}{2}}|^2 \\
\leq \frac{4}{3} \|S_0\|_{C([0,\gamma])} \|n\|_{L^2(\Omega)}^2 \|\nabla n_{\Omega}^2\|_{L^2(\Omega)} \|c\|_{W^{1,\infty}(\Omega)} \\
\leq C \left(\frac{1}{\sqrt{\log s}} \|\nabla n_{\Omega}^2\|_{L^2(\Omega)} + 1 + s^2\right) \|\nabla n_{\Omega}^2\|_{L^2(\Omega)}.
\]
If we take sufficiently large \(s\) and use Young’s inequality, then with some \(C > 0\),
\[
\frac{d}{dt} \int_\Omega n^3 + \int_\Omega |\nabla n^{\frac{3}{2}}|^2 \leq C.
\]
This implies, by the Gagliardo-Nirenberg type inequality,
\[
\|f\|_{L^3(\Omega)}^3 \leq \|\nabla f_{\Omega}^3\|_{L^2(\Omega)}^2 + C \|f\|_{L^1(\Omega)}^3 \quad \text{for all} \quad f \in C^1(\overline{\Omega}),
\]
and \(\int_\Omega n = \int_\Omega n_0\), that with some \(C > 0\),
\[
\frac{d}{dt} \int_\Omega n^3 + \int_\Omega n^3 \leq C.
\]
Therefore, we can deduce (3.10).

4. CASE OF SCALAR SENSITIVITY IN GENERAL DIMENSIONS

Throughout this section, let \(\Omega = B_R(0) = \{x \in \mathbb{R}^d \mid r = |x| < R\}\), and \(n_0\) be radial.
In the radially symmetric setting, two equations of (1.1) can be written as
\[
n_t = r^{1-d} \left(r^{d-1} n_r\right)_r - r^{1-d} \left(r^{d-1} n c e_\chi\right)_r, \quad r^{1-d} \left(r^{d-1} c r\right)_r = nc.
\]
Thus, the cumulative mass distribution \(Q\) defined by
\[
Q(r, t) := \int_{B_r(0)} n(x, t) \, dx = \sigma_d \int_0^r \rho^{d-1} n(\rho, t) \, d\rho
\]
satisfies
\[
Q_t = r^{d-1} \left(r^{1-d} Q_r\right)_r - Q_r c r(r, n, c) c r, \quad r < R, t < T_{\max}.
\]
We note that
\[
Q(R, t) = \|n_0\|_{L^1(\Omega)},
\]
and
\[
Q_r \geq 0, \quad c r = r^{1-d} \int_0^r \rho^{d-1} nc \, d\rho \geq 0, \quad r < R, t < T_{\max}.
\]
The non-negativities (4.3) and \(\chi \geq 0\) yield an upper bound for \(Q\) stated below.
Lemma 7. Let $Q$ be the cumulative mass distribution defined in (4.1). Then, there exists $M_0 = M_0(d, R, \|n_0\|_{L^1(\Omega)}, \|n_0\|_{L^\infty(\Omega)}) \geq 0$ such that

$$Q(r, t) \leq M_0 r^d \quad \text{for} \quad r < R, t < T_{\text{max}}.$$  

**Proof.** We use a comparison argument. Due to (4.3) and $\chi \geq 0$, it follows from (4.2) that

$$Q_t \leq r^{d-1} \left( r^{1-d} Q_r \right)_r.$$  

Define

$$M_0 := \max \left\{ \frac{1}{R^d} \|n_0\|_{L^1(\Omega)}, \frac{\sigma_d}{d} \|n_0\|_{L^\infty(\Omega)} \right\},$$

and

$$W(r) := M_0 r^d.$$  

Then, $Q(R, t) \leq W(R)$, $Q(r, 0) \leq W(r)$, and

$$0 = r^{d-1} \left( r^{1-d} W_r \right)_r.$$  

Let $\varepsilon > 0$ be given, and we now show that $F$ defined by

$$F(r, t) := (Q(r, t) - W(r, t)) \exp(-t)$$

can not attain value $\varepsilon$ as long as solution exists. Note that $F(0, t) = 0$, $F(R, t) \leq 0$, $F(r, 0) \leq 0$, and

$$F_t = (Q_t - W_t) \exp(-t) - F$$

$$\leq r^{d-1} \left( r^{1-d} F_r \right)_r - F$$

$$= F_{rr} + (1 - d) r^{1-d} F_r - F.$$  

Assume to the contrary that $F(r_1, t_1) = \varepsilon$ for the first time $t_1 < T_{\text{max}}$. Then, $r_1 \neq 0$ or $R$ and

$$0 \leq F_t(r_1, t_1), \quad F_{rr}(r_1, t_1) \leq 0,$$

$$(1 - d) r_1^{d-1} F(r_1, t_1) = 0, \quad -F(r_1, t_1) = -\varepsilon < 0,$$

which leads to a contradiction. Since $\varepsilon > 0$ is arbitrary, $F \leq 0$ and the desired bound follows. \(\square\)

Due to Lemma 7, for each $t < T_{\text{max}}$, there exist a radius $r_t \in [0, R]$ and a number $m_0 > 0$ satisfying $c(r_t, t) \geq m_0$:

**Lemma 8.** Let $(n, c)$ be the solution given by Lemma 7 and let $M_0$ be a number given in Lemma 7. Then, for each $t < T_{\text{max}}$, there exists a radius $r_t \in [0, R]$ such that

$$c(r_t, t) \geq m_0 := \left( \frac{M_0 R}{\sigma_d} + 1 \right)^{-1}.$$  

**Proof.** Suppose that (4.4) is false. Then, there exists $T < T_{\text{max}}$ such that $c(r, T) < m_0$ for all $r \in [0, R]$.

Fix $t = T$. Using the $c$ equation and Lemma 7, we can estimate

$$c_r = r^{1-d} \int_0^r \rho^{d-1} n c \, d\rho < \frac{M_0 m_0}{\sigma_d} r$$

for all $r \in (0, R]$. 


If we take $r = R$, then from the boundary condition and \( c < m_0 \), we have
\[
\gamma - m_0 < \gamma - c(R) = c_r(R) < \frac{M_0m_0}{\sigma_d} R.
\]
This leads to a contradiction because $m_0 < \gamma \left( \frac{M_0R}{\sigma_d} + 1 \right)^{-1}$.

As a consequence, \( c \) has the lower bound which is uniform in space and time:

**Lemma 9.** Let \((n, c)\) be the solution given by Lemma 7 and let \( M_0 \) and \( m_0 \) be numbers given in Lemma 7 and Lemma 8, respectively. Then, it holds that
\[
\min_{r \in [0, R]} c(r, t) \geq c_* := m_0 \exp \left( -\frac{1}{2} \frac{M_0 R^2}{\sigma_d} \right) \quad \text{for} \quad t < T_{\max}.
\]

**Proof.** Note that by Lemma 1, \( c > 0 \), and by \( c_r \geq 0 \) in (4.3),
\[
\min_{r \in [0, R]} c(r, t) = c(0, t).
\]
Since for each \( t < T_{\max} \), there exists \( r_t \in [0, R] \) satisfying (4.4), in view of Lemma 7 and
\[
r^{1-d} \left( r^{d-1} (\log c)_r \right)_r = \Delta \log c = n - |\nabla \log c|^2 \leq n,
\]
we have that
\[
(\log c)_r = r^{1-d} \int_0^r (\rho^{d-1} (\log c)_\rho)_\rho d\rho \leq r^{1-d} \int_0^r \rho^{d-1} n d\rho \leq \frac{M_0}{\sigma_d} r.
\]
If we integrate it from 0 to \( r_t \), then
\[
\log \frac{c(r_t, t)}{c(0, t)} \leq \int_0^{r_t} \frac{M_0}{\sigma_d} \rho d\rho = \frac{M_0}{2\sigma_d} r_t^2.
\]
Therefore, using (4.4) and \( r_t \leq R \), we can deduce the desired result.

We are ready to prove Theorem 2.

**Proof of Theorem 2**. Let \((n, c)\) be the solution given by Lemma 1. From (4.3) and \( Q(r, t) \leq M_0 r^d \) in Lemma 7, we have
\[
0 \leq c_r = r^{1-d} \int_0^r \rho^{d-1} nc d\rho \leq \frac{\gamma M_0}{\sigma_d} r \quad \text{for} \quad r \in (0, R].
\]
Thus, \( \nabla c \) has uniform-in-time pointwise bounds. Moreover, by \( c_* \leq c \leq \gamma \) and \( \chi(x, n, c) \leq \chi_0(c) \in C(\mathbb{R}^+) \), we have that \( \chi(x, n, c) \) is bounded uniformly in time. Then, the standard parabolic regularity theory gives a uniform-in-time bound for \( n \). This concludes Theorem 2 from Lemma 1.

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