Casimir amplitudes in a quantum spherical model with long-range interaction

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Abstract. A \(d\)-dimensional quantum model system confined to a general hypercubical geometry with linear spatial size \(L\) and “temporal size” \(1/T\) (\(T\) - temperature of the system) is considered in the spherical approximation under periodic boundary conditions. For a film geometry in different space dimensions \(\frac{1}{2} \sigma < d < \frac{3}{2} \sigma\), where \(0 < \sigma \leq 2\) is a parameter controlling the decay of the long-range interaction, the free energy and the Casimir amplitudes are given. We have proven that, if \(d = \sigma\), the Casimir amplitude of the model, characterizing the leading temperature corrections to its ground state, is \(\Delta = -16(3)/[3\sigma(4\pi)^{d/2} \Gamma(\sigma/2)]\). The last implies that the universal constant \(\hat{c} = 4/5\) of the model remains the same for both short, as well as long-range interactions, if one takes the normalization factor for the Gaussian model to be such that \(\hat{c} = 1\). This is a generalization to the case of long-range interaction of the well-known result due to Sachdev. That constant differs from the corresponding one characterizing the leading finite-size corrections at zero temperature which for \(d = \sigma = 1\) is \(\hat{c} = 0.606\).

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1 Introduction

The confinement of quantum mechanical vacuum fluctuations of the electromagnetic field causes long-ranged forces between two conducting uncharged plates which is known as (quantum mechanical) Casimir effect [1,2]. The confinement of critical fluctuations of an order parameter field induces long-ranged forces between the surfaces of the film [3,4]. This is known as “statistical-mechanical Casimir effect”. The Casimir force in statistical-mechanical systems is characterized by the excess free energy due to the finite-size contributions to the free energy of the system. In the case of a film geometry \(L \times \infty^2\), and under given boundary conditions \(\tau\) imposed across the direction \(L\), the Casimir force is defined as

\[ F^\tau_{\text{Casimir}}(T, L) = -\frac{\partial f^{\tau}_{\text{ex}}(T, L)}{\partial L}, \]  (1)

where \(f^{\tau}_{\text{ex}}(T, L)\) is the excess free energy

\[ f^{\tau}_{\text{ex}}(T, L) = f_{\tau}(T, L) - L f_{\text{bulk}}(T). \]  (2)

Here \(f_{\tau}(T, L)\) is the full free energy per unit area and per \(k_B T\), and \(f_{\text{bulk}}(T)\) is the corresponding bulk free energy density.

The full free energy of a \(d\)-dimensional critical system in the form of a film with thickness \(L\), area \(A\), and boundary conditions \(a\) and \(b\) on the two surfaces, at the bulk critical point \(T_c\), has the asymptotic form

\[ f_{a,b}(T_c, L | d) \cong L f_{\text{bulk}}(T_c | d) + f_{\text{surface}}^a(T_c | d) + f_{\text{surface}}^b(T_c | d) + L^{-(d-1)} \Delta_{a,b}(d) + \cdots \]  (3)

as \(A \to \infty, L \gg 1\). Here \(f_{\text{surface}}^a\) is the surface free energy contribution and \(\Delta_{a,b}(d)\) is the amplitude of the Casimir interaction. The \(L\) dependence of the Casimir term (the last one in Eq. (3)) follows from the scale invariance of the free energy and has been derived by Fisher and de Gennes [3]. The amplitude \(\Delta_{a,b}(d)\) is universal, depending on the bulk universality class and the universality classes of the boundary conditions [4,5].

Equation (3) is valid for both fluid and magnetic systems at criticality. Prominent examples are, e.g., one-component fluid at the liquid-vapour critical point, the binary fluid at the consolute point, and liquid \(^4\)He at the \(\lambda\) transition point [4]. The boundaries influence the system to a depth given by the bulk correlation length \(\xi_\infty(T) \sim |T-T_c|^{-\nu}\), where \(\nu\) is its critical exponent. When \(\xi_\infty(T) \ll L\) the Casimir force, as a fluctuation induced force between the plates, is negligible. The force becomes long-ranged when \(\xi_\infty(T)\) diverges near and below the bulk critical point \(T_c\) in an \(O(n)\), \(n \geq 2\) model system in the absence of an external magnetic field [6,7]. Therefore in statistical-mechanical systems one can turn on and off the Casimir effect merely by changing, e.g., the temperature of the system.
The temperature dependence of the Casimir force for two-dimensional systems has been investigated exactly only on the example of Ising strips [8]. In $O(n)$ models for $T > T_c$, the temperature dependence of the force has been considered in [5]. The only example where it is investigated exactly as a function of both the temperature and magnetic field scaling variables is that of the three-dimensional spherical model under periodic boundary conditions [6,7]. There exact results for the Casimir force between two walls with a finite separation in a $L \times \infty^2$ mean-spherical model have been derived. The force is consistent with an attraction of the plates confining the system.

The most of the results available at the moment are for the Casimir amplitudes. They are obtained for $d = 2$ by using conformal-invariance methods for a large class of models [4]. For $d \neq 2$ results for the amplitudes are available via field-theoretical renormalization group theory in $4-\epsilon$ dimensions [4,5,9], Migdal-Kadanoff real-space renormalization group methods [10], and, relatively recently, by Monte Carlo methods [11]. In addition to the flat geometries recently some results about the Casimir amplitudes between spherical particles in a critical fluid have been derived too [9,12]. For the purposes of experimental verification that type of geometry seems more perspective.

It should be noted that in contrast with the quantum mechanical Casimir effect, that has been tested experimentally with high accuracy [13], the statistical-mechanical Casimir effect lacks so far a satisfactory experimental verification (for comments on the specific difficulties that the experiment stacks with see, e.g. [12]).

In recent years there has been a renewed interest [14,15] in the theory of zero-temperature quantum phase transitions. In contrast to temperature driven critical phenomena, these phase transitions occur at zero temperature as a function of some non-thermal control parameter, say $g$ (or a competition between different parameters describing the basic interaction of the system), and the relevant fluctuations are of quantum rather than thermal nature. In the present article we consider a statistical-mechanical Casimir effect when critical quantum fluctuations play an essential role.

It is well-known from the theory of critical phenomena that for temperature driven phase transitions quantum effects are unimportant near critical points with $T_c > 0$. It could be expected, however, that at rather low (as compared to characteristic excitations in the system) temperatures, the leading $T$ dependence of all observables is specified by the properties of the zero-temperature critical point, say at $g_c$. The dimensional crossover rule asserts that the critical singularities with respect to $g$ of a $d$-dimensional quantum system at $T = 0$ and around $g_c$ are formally equivalent to those of a classical system with dimensionality $d + z$ ($z$ is the dynamical critical exponent) and critical temperature $T_c > 0$. This makes it possible to investigate low-temperature effects (considering an effective system with $d$ infinite spatial and $z$ finite temporal dimensions) in the framework of the theory of finite-size scaling (FSS). This theory has been applied to explore the low-temperature regime in quantum systems [14–16], when the properties of the thermodynamic observables in the finite-temperature quantum critical region have been the main focus of interest.

In this paper a theory of the scaling properties of the free energy and Casimir amplitudes of a quantum spherical model [17] with nearest-neighbor and some special cases of long-range interactions (decreasing at long distances $r$ as $1/r^{d+\sigma}$) is presented. These interactions enter the exact expressions for the free energy only through their Fourier transform which leading asymptotic is $U(q) \sim q^{\sigma^*}$, where $\sigma^* = \min(\sigma, 2)$ [18]. As it was shown for bulk systems by renormalization group arguments $\sigma \geq 2$ corresponds to the case of finite (short) range interactions, i.e. the universality class then does not depend on $\sigma$ [19]. Values satisfying $0 < \sigma < 2$ correspond to long-range interactions and the critical behaviour depends on $\sigma$. On the above reasoning one usually considers the case $\sigma > 2$ as uninteresting for critical effects, even for the finite-size treatments [20]. However recent Monte Carlo results suggest that it might well not be the case at least for continuous Ising model [21]. There Bayong and Diep state that for $d = 2$ the critical exponents do not depend on $\sigma$ and reach their short-range values for $\sigma \geq 3$. On the basis of that result it seems that for finite-size systems the case $\sigma > 2$ is nontrivial. Since, up to the authors knowledge that is the only example where $\sigma > 2$ is of interest for studying critical properties, here we will consider only the case $0 < \sigma \leq 2$.

The investigation of the Casimir effect in a classical system with long-range interaction possesses some peculiarities in comparison with the short-range system. Due to the long-range character of the interaction there exists a natural attraction between the surfaces bounding the system. One easily can estimate that in the ordered state the $L$-dependent part of the excess free energy that is raised by the direct inter-particle (spin) interaction is of order of $L^{-\sigma+1}$. In the critical region one still has some effect stemming from that interaction on the background of which develops the fluctuating induced new attraction between the surfaces which is in fact the critical Casimir force. In the definition (1) used here, that is the common one when one considers short-range systems, these both effects are superposed simultaneously. Therefore, here, generally speaking, one should expect a crossover from a regime governed by the critical Casimir force (in the sense of a fluctuation induced force; it is of the order of $L^{-d}$, see Eq. (3)) to the one govern by the direct attraction (of the order of $L^{-\sigma}$; note that if $d = \sigma$ they will of the same order being dominating in different temperature regions). An interesting case when forces of similar origin are acting simultaneously is that one of the wetting when the wetting layer is nearly critical and intrudes between two noncritical phases if one takes into account the effect of long-range correlations and that one of the long-range van der Waals forces [22,23].

In quantum systems additional new features will be observed since the “temporal direction” corresponds formally to a short-range type interaction in the corresponding classical analog of the system, i.e. one unavoidable has