Abstract

The interaction between the quantum vacuum and a weak gravitational field is calculated for the vacuum fields of quantum electrodynamics. The result shows that the vacuum state is modified by the gravitational field, giving rise to a nonzero interaction energy. This suggests a model that fits in the main properties of the hypothetical dark matter in galactic haloes.

1 Introduction

In quantum theory the vacuum is not empty but filled with quantum fields. The fields have energy but in most applications of quantum mechanics that energy may be ignored using the “normal ordering rule” that effectively removes it. This fact has led some people to believe that the vacuum energy is an artifact of the quantization procedure. However the belief is difficult to reconcile with observable effects like the Lamb shift or the Casimir effect[1]. An additional problem is that a simple attempt to calculate the vacuum energy leads to infinities. The problem was shelved, if not fully solved, with the renormalization techniques developed for quantum electrodynamics (QED).
In applications of quantum mechanics not involving gravity the zero of energy can be fixed arbitrarily, and putting it at the level of the vacuum is the obvious choice. However when gravity is involved this is no longer possible because energy gravitates. An attempt to solve the problem has been the hypothesis that there is a natural cutoff making the vacuum energy (and pressure) actually finite. Thus it is assumed that the vacuum stress-energy tensor gives rise to a cosmological constant. However the apparently plausible assumption that the cutoff should be obtained via combining the universal constants $c, \hbar, G$ leads us to the Planck scale with a cosmological term too big by about 123 orders of magnitude. There is not yet a fully satisfactory solution to the difficulty usually named the Cosmological Constant Problem\cite{2}. A simple solution is to assume that there is an effective cutoff, say at the scale of a mass $m$. This solution was put forward by Zeldovich in 1976\cite{3} that proposed for the energy density of the cosmological term the following
\[
\rho_{DE} = \frac{Gm^6c^2}{\hbar^4}.
\] (1)
This led Zeldovich to assume that the cosmological term derives from the quantum vacuum fluctuations. The hypothesis has been revisited recently\cite{4} as the origin of the “dark energy” needed to explain the accelerated expansion of the universe\cite{5}. Indeed eq. (1) fits in the current density of dark energy for $m \simeq 170Mev/c^2$, not much larger than the pion mass.

According to general relativity gravitation is curvature of spacetime, consequently the problem of the infinities reappears in the study of quantum fields in curved space. Indeed renormalization has been adapted to fields in curved space and there exists a good perturbative approach to quantum gravity in the framework of low energy effective field theory. A standard reference on the subject is the book by Birrell and Davies\cite{6}, but a number of review articles and many shorter introductory lecture notes are also available (see e.g. \cite{7}) .

Following a general interpretation of quantum theory\cite{8}, in this paper it is proposed that the vacuum is not an inert system, but its state may be modified by the presence of matter, in particular by gravitational interaction. It is also proposed that for weak gravitational fields such interactions may be treated, avoiding the use of quantum gravity, via a combination of quantum field theory and Newtonian gravity. One of these cases is studied in section 2 and it suggests an alternative to (or explanation for the effects of) dark matter. A specific model is presented in section 3.
2 Effect of a gravitational field on the quantum vacuum

In this section I will study the interaction between the quantum vacuum and a gravitational field. The subject is usually treated under the name quantum fields in curved spacetime\[6\]. The most celebrated predictions are Hawking radiation from black holes and the Unruh-Davis effect. Here I study the action of a weak gravitational field on the vacuum with a method that has some analogy with the calculation of the effect of an electrostatic field, treated as classical, on the electron-positron vacuum field that gives rise to vacuum polarization.

2.1 The Newtonian approximation

In the calculation I will use the Newtonian approximation. In Newtonian gravity combined with special relativity the source of the field is any mass or energy distribution and the interaction may be described via the product of the mass (or energy) density, \( \rho (\mathbf{r}) \), times the Newtonian potential \( \phi (\mathbf{r}) \).

Of course Newtonian gravity is not Lorentz invariant and therefore it is incompatible with special relativity, but here I will study stationary systems where Lorentz invariance plays no role. If we treat the gravitational field as classical the quantum interaction Hamiltonian should be

\[
\hat{H}_{\text{vac-grav}} = \int d^3 \mathbf{r} \hat{\rho} (\mathbf{r}) \phi (\mathbf{r}),
\]

where \( \hat{\rho} (\mathbf{r}) \) is the operator for the energy density of the quantum fields. This is similar to the QED treatment of the electron-positron field in an external electromagnetic field, the latter treated as classical. In this section I will use Planck units \( \hbar = c = G = 1 \), but I will write explicitly Newton’s constant \( G \) in some cases for the sake of clarity. Thus the gravitational potential appears as dimensionless.

The connection of eq.\((2)\) with a treatment in semiclassical general relativity may be sketched as follows. Our aim would be to find the energy of the ground state of a system consisting of quantum fields in quantized spacetime. Now for the sake of clarity I consider a single scalar field, \( \chi \), with minimal coupling. Then classically the dynamics of the system would derive from the action
\[ S = \frac{1}{2} \int d^4x \sqrt{|g|} \left( g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - m^2 \chi^2 \right). \] (3)

We should quantize both the field and the metric, but this would require quantum gravity. The semiclassical theory may be developed as follows. Let us assume that the ground state of the system is \(| \Psi \rangle\) (that we will determine later with some degree of accuracy). Then the semiclassical approximation consists of calculating the metric from the following Einstein tensor

\[ G_{\mu\nu} = \langle \Psi | \hat{T}_{\mu\nu} (x) | \Psi \rangle. \] (4)

Putting that metric in eq.(3) it would be straightforward to get the Hamiltonian and quantize the field, whence getting the state \(| \Psi \rangle\) would amount to solve a standard eigenvalue problem.

The proposed method presents difficulties however. Firstly the quantity eq.(4) is infinite because it contains the product of field operators at the same point. One way to make sense of the expectation value is via the difference between the values in two different states. One of these states would be the quantum vacuum in Minkowski space and the other one may be a state with some mass distribution (e.g. corresponding to a galaxy, see below). Another difficulty is that in order to get the metric we need the quantum state \(| \Psi \rangle\) of the system in eq.(4), but in order to get the quantum state we need the metric in eq.(3). The latter difficulty may be avoided in the case of static weak fields where I may use the Newtonian approximation.

In the Newtonian approximation the metric may be approximated by

\[ g_{00} = 1 + 2\phi (r), \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{\mu\nu} = 0 \text{ if } \mu \neq \nu. \] (5)

where \(\phi (r)\) is the Newtonian potential derived from the mass-energy present in the system via Newton’s law, that now replaces Einstein equation. Hence eq.(3) leads to

\[ S = \frac{1}{2} \int dt d^3r \sqrt{1 + 2\phi} \left[ \left( \frac{\partial \chi}{\partial t} \right)^2 - (\nabla \chi)^2 - m^2 \chi^2 \right] \]

\[ \simeq \frac{1}{2} \int dt d^3r (1 + \phi) \left[ \left( \frac{\partial \chi}{\partial t} \right)^2 - (\nabla \chi)^2 - m^2 \chi^2 \right], \] (6)

where \(t\) is the time measured in the local Minkowski frame, that is

\[ dt^2 = g_{00} (dx^0)^2. \]
and the second equality takes the weak field inequality \(|\phi| \ll 1\) into account. Hence it is straightforward to get the Hamiltonian for this particular case, that I will not write. I will assume that a similar result is valid in general, thus leading to eq. (2). I point out that we study a stationary system and the (approximate) metric eq. (5) corresponds to a flat space where the vector \(\mathbf{r}\) makes sense.

We might believe that the vacuum is an inert stuff with nil stress-energy. In this case the presence of a gravitational field is unable to change the vacuum energy, which leads to the assumption that the interaction energy between the field and the quantum vacuum should be given by

\[
E_{\text{vac–grav}} = - \int d^3\mathbf{r} \int d^3\mathbf{r}' \frac{G \rho_{\text{vac}}(\mathbf{r}) \rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \rho_{\text{vac}} \int d^3\mathbf{r} \phi(\mathbf{r}) = 0, \tag{7}
\]

where

\[
\rho_{\text{vac}} \equiv \langle \Psi_{\text{vac}} | \hat{\rho} | \Psi_{\text{vac}} \rangle
\]

is the vacuum expectation of the quantum energy density operator. This density does not depend on the position vector, therefore it may be put outside the space integral under the assumption that the vacuum energy density \(\rho_{\text{vac}}\) is nil whence the interaction energy \(E_{\text{vac–grav}}\) is zero, as in the latter eq. (7). Even if this hypothesis does not hold true the former eq. (7) means that the vacuum is not influenced by the gravitational field created by matter. However eq. (7) is wrong, the correct one being eq. (2) that takes into account the action of the field on the quantum vacuum.

In the following I show that nonzero contributions exist to the gravitational interaction between the vacuum fields and baryonic matter. In section 3 I will study the proposal that this interaction may be the origin of the so-called “dark matter”. Here I sketch an explicit calculation of some contributions to vacuum energy in quantum electrodynamics (QED), firstly in the absence of gravity (subsection 2.2) and then with a gravitational field present (subsection 2.3).

### 2.2 The quantum vacuum of QED

In this subsection I recall a few results of QED that, although well known, will be used in later parts in the paper. Units \(c = \hbar = 1\) will be used throughout so that \(k\) may equally represent wavevector, frequency or energy.

The properties of the vacuum electromagnetic field (commonly named zeropoint field) are summarized as follows\[1\]. The spectrum is proportional
to the cube of the frequency and it is Lorentz invariant provided that a
cutoff is not placed (whence the energy density diverges quartically with the
frequency), but any cutoff breaks that symmetry.

The quantum operator of the energy density (or Hamiltonian density) is
\[
\hat{\rho}_{\text{EM}} \equiv \frac{1}{2} \left( \hat{E}^2 + \hat{B}^2 \right) = \frac{1}{4V} \sum_{k,\varepsilon} \sum_{k',\varepsilon'} \sqrt{k k'} [\varepsilon \cdot \varepsilon' + \frac{1}{kk'} (k \times \varepsilon) \cdot (k' \times \varepsilon')] \\
\times \left[ \hat{\alpha}_{k,\varepsilon} \exp(ik \cdot r - ikt) + \hat{\alpha}^\dagger_{k,\varepsilon} \exp(-ik \cdot r + ikt) \right] \\
\times \left[ \hat{\alpha}_{k',\varepsilon'} \exp(ik' \cdot r - ik't) + \hat{\alpha}^\dagger_{k',\varepsilon'} \exp(-ik' \cdot r + ik't) \right],
\]
where \( k \equiv |k|, k' \equiv |k'|, \varepsilon \) is the polarization vector that fulfils \( \varepsilon \cdot k = 0 \),
and similar for \( \varepsilon' \). For later convenience we will rewrite eq.(8) (ignoring the
time dependence) in the form
\[
\hat{\rho}_{\text{EM}} = \hat{\rho}_{\text{EM}1} + \hat{\rho}_{\text{EM}2},
\]
\[
\hat{\rho}_{\text{EM}1} = \frac{1}{2V} \sum_{k,\varepsilon} \sum_{k',\varepsilon'} \sqrt{k k'} \varepsilon \cdot \varepsilon' \hat{\alpha}^\dagger_{k',\varepsilon'} \hat{\alpha}_{k,\varepsilon} \exp [i (k - k') \cdot r] + \frac{1}{2V} \sum_{k,\varepsilon} k,
\]
\[
\hat{\rho}_{\text{EM}2} = \frac{1}{4V} \sum_{k,\varepsilon} \sum_{k',\varepsilon'} \sqrt{k k'} \varepsilon \cdot \varepsilon' \hat{\alpha}_{k,\varepsilon} \hat{\alpha}_{k',\varepsilon'} \exp [i (k + k') \cdot r] + \text{h.c.},
\]
where \( h.c. \) means Hermitean conjugate and for notational simplicity I have
labelled
\[
\varepsilon \cdot \varepsilon' \equiv \varepsilon \cdot \varepsilon' + \frac{1}{kk'} (k \times \varepsilon) \cdot (k' \times \varepsilon').
\]
In \( \hat{\rho}_{\text{EM}1} \) I have written the operators in normal order, taking the commutation
relations into account.

The Hamiltonian is obtained by performing a space integral of the energy
density eq.(9), that is
\[
\hat{H}_{\text{EM}} = \lim_{V \to \infty} \int \hat{\rho}_{\text{EM}} (r) d^3r = \lim_{V \to \infty} \int [\hat{\rho}_{\text{EM}1} (r) + \hat{\rho}_{\text{EM}2} (r)] d^3r = \sum_{k,\varepsilon} k (\hat{\alpha}^\dagger_{k,\varepsilon} \hat{\alpha}_{k,\varepsilon} + \frac{1}{2}).
\]
The term \( \hat{\rho}_{\text{EM}2} \) does not contribute to the Hamiltonian, and therefore to the
quantum expectation value of the vacuum energy, because the space integral

\[
6
\]
of the latter eq.(9) leads to \( k' = -k \), whence

\[
\int_V \hat{\rho}_{EM2}(\mathbf{r}) \, d^3\mathbf{r} = \frac{1}{4V} \sum_{k,\epsilon,\epsilon'} k\hat{\alpha}_{k,\epsilon}\hat{\alpha}_{-k,\epsilon'} \epsilon \ast \epsilon' = 0, \tag{12}
\]

because, as may be easily proved,

\[
\sum_{\epsilon,\epsilon'} \epsilon \ast \epsilon' = \sum_{\epsilon,\epsilon'} [\epsilon \cdot \epsilon' - \frac{1}{kk'} (\mathbf{k} \times \epsilon) \cdot (\mathbf{k} \times \epsilon')] = 0. \tag{13}
\]

For the free electromagnetic field the vacuum state, \(|0\rangle\), may be defined as the state with the minimal energy amongst the eigenvectors of the operator eq.(11). It is a state with zero photons and it has the properties

\[
\alpha_{k,\epsilon} |0\rangle = 0, \langle 0 | \alpha_{k,\epsilon}^\dagger = 0,
\]

whence the vacuum expectation of the energy density is

\[
\langle 0 | \hat{\rho}_{EM} | 0 \rangle = \frac{1}{V} \sum_{k,\epsilon} \frac{1}{2} \hat{\pi}k = \frac{1}{V} \sum_k \hat{\pi}k, \tag{14}
\]

where the latter equality derives from the two possible polarizations. In the limit \( V \to \infty \) eq.(14) leads to the following result if a cutoff, \( \Lambda \), is introduced in the photon energies

\[
\rho_{EM} = \frac{1}{V} \sum_k k \to \int k (2\pi)^{-3} \, d^3k, \nonumber
\]

\[
= \frac{1}{2\pi^2} \int_0^{k_{\text{max}}} k^3 \, dk = \frac{\Lambda^4}{8\pi^2}. \tag{15}
\]

For the electron-positron field the energy density may be written

\[
\hat{\rho}_D = \frac{i}{2} \left( \hat{\psi}^\dagger \frac{d\hat{\psi}}{dt} - \frac{d\hat{\psi}^\dagger}{dt} \hat{\psi} \right), \tag{16}
\]

(the subindex D stands for Dirac). Expanding \( \hat{\psi} \) and \( \hat{\psi}^\dagger \) in plane waves we get, after some algebra, the following energy density

\[
\hat{\rho}_D(\mathbf{r},t) = \rho_{D0} + \hat{\rho}_b(\mathbf{r},t) + \hat{\rho}_d(\mathbf{r},t), \tag{17}
\]
\[ \rho_{D0} = -\frac{1}{\sqrt{V}} \sum_{p,s} \sqrt{m^2 + p^2}, \]

\[ \hat{\rho}_b (r, t) = \frac{1}{\sqrt{V}} \sum_{p, p', s, s'} B(p, s, p', s') \hat{b}^\dagger_{p, s} \hat{b}_{p, s} \exp \left[ i (p - p') \cdot r - i (E - E') t \right], \]

\[ \hat{\rho}_d (r, t) = \frac{1}{\sqrt{V}} \sum_{p, p', s, s'} D(p, s, p', s') \hat{d}^\dagger_{p, s} \hat{d}_{p, s} \exp \left[ i (p - p') \cdot r - i (E - E') t \right], \]

\[ \hat{b}_{p, s} \left( \hat{d}_{p, s} \right) \] being the annihilation operator of an electron (positron) with momentum \( p \) and spin \( s (= 1, 2) \), \( \hat{b}^\dagger_{p, s} \left( \hat{d}^\dagger_{p, s} \right) \) the corresponding creation operator.

The first term in eq.(17), \( \rho_{D0} \), is a c-number, not an operator (more properly it is proportional to the unit operator). Its negative sign is a consequence of the anticommutation rules of fermi field operators. Getting the functions \( B \) and \( D \) is straightforward but I shall not write them. Integration of eq.(17) with respect to \( r \) gives the Hamiltonian of the free electron-positron field, that is

\[ \hat{H}_D = \sum_{p, s} \sqrt{m^2 + p^2} \left( \hat{b}^\dagger_{p, s} \hat{b}_{p, s} + \hat{d}^\dagger_{p, s} \hat{d}_{p, s} - 1 \right). \] (18)

The vacuum state, \( | 0 \rangle \), of the free Dirac field corresponds to the eigenvector of the Hamiltonian eq.(18) with the smallest eigenvalue. Therefore from now on we define \( | 0 \rangle \) to be the QED free-fields vacuum state, which is a simultaneous eigenvector of both Hamiltonians eqs.(11) and (18), and consequently an eigenvector of the total free fields Hamiltonian, that is

\[ \hat{H}_0 | 0 \rangle = \left( \hat{H}_{EM} + \hat{H}_D \right) | 0 \rangle = E_0 | 0 \rangle, \] (19)

The state \( | 0 \rangle \) is defined as having zero photons, electrons and positrons. It should be distinguished from the physical vacuum state, \( \mid \text{vac} \rangle \), which is an eigenvalue of the total Hamiltonian, including the interactions.

The energy density of the free electron-positron field follows easily from eq.(18). It is negative and divergent. Introducing an energy cut-off \( \Lambda \) for the
electrons and positrons, we get

\[
\rho_D = -\frac{1}{V} \sum_{p,s} \sqrt{m^2 + p^2} \rightarrow -\pi^2 \int_0^{p_{\text{max}}} \sqrt{m^2 + p^2} p^2 dp = -\pi^2 \int_0^{\Lambda} \sqrt{E^2 - m^2} E dE
\]

\[
= -\frac{1}{4\pi^2} \left[ \Lambda (\Lambda^2 - \frac{1}{2} m^2) \sqrt{\Lambda^2 - m^2} - \frac{1}{2} m^4 \cosh^{-1}\left( \frac{\Lambda}{m} \right) \right]
\]

\[
= -\frac{1}{4\pi^2} \left[ \Lambda^4 - \Lambda^2 m^2 + \frac{1}{8} m^4 - \frac{1}{2} m^4 \ln\left( \frac{2\Lambda}{m} \right) \right] + O \left( \Lambda^{-2} \right),
\]

(20)

The negative value might be anticipated by inspection of the Hamiltonian eq. (18). The total free fields vacuum energy is the product of the integration volume, \( V \), times the energy density, \( \rho_0 = \rho_{\text{EM}} + \rho_D \).

The physical vacuum of QED, \( | \text{vac} \rangle \), is different from the free field vacuum, \( | 0 \rangle \), studied above. The latter is the eigenvector, with the smallest eigenvalue, of the Hamiltonian \( H_0 \), see eq. (19). The former is an eigenvector of the total Hamiltonian \( H = H_0 + H_{\text{int}} \) that takes the interaction into account. Finding \( | \text{vac} \rangle \) as an exact eigenvector of \( H \) is not possible in practice and it is standard to use a perturbation method, that leads to

\[
| \text{vac} \rangle = c_0 | 0 \rangle + \sum_{n=0} c_n | n \rangle,
\]

(21)

the states \( | 0 \rangle \) and \( \{ | n \rangle \} \) being eigenstates of the unperturbed Hamiltonian \( H_0 \). Usually the sum eq. (21) gives an expansion in powers of the coupling constant, the electron charge \( e \). Only even powers of \( e \) would appear and the result becomes an expansion in powers of the fine structure constant \( \alpha \equiv e^2/(4\pi\hbar c) \simeq 1/137 \). The perturbation method is sensible if

\[
\sum_n |c_n|^2 = 1, |c_{n\neq 0}|^2 << 1.
\]

(22)

Actually the former condition is not fulfilled if we take \( c_0 = 1 \) as usual, but a normalization is possible if the sum converges, that is

\[
\sum_n |c_n|^2 < \infty.
\]

(23)

The Hamiltonian (or energy) density operator for the interaction in QED may be written, in the Coulomb gauge,

\[
\hat{\rho}_{\text{int}}(r,t) = -e\hat{\psi}^\dagger \alpha \hat{\psi} \cdot \hat{A}.
\]

(24)
The operators $\hat{\psi}_1$, $\hat{\psi}_1^\dagger$ and $\hat{A}$ contain two terms each when expanded in plane waves, every term corresponding to an infinite sum. One of these terms has creation operators and the other one annihilation operators. This gives rise to 8 terms for $\hat{\rho}_{int}$, eq.(23). I will write only the two terms that will survive in the Hamiltonian. We get (ignoring the time dependence that is irrelevant in the following)

$$\hat{\rho}_{int}(\mathbf{r}) = \sum_{\mathbf{p}, \mathbf{q}, \mathbf{k}, s, s', \epsilon} \left[ \zeta_n \hat{a}_{\mathbf{k}, \epsilon} \hat{b}_{\mathbf{ps}} \hat{d}_{\mathbf{qs'}} \exp \left[ i (\mathbf{p} + \mathbf{q} + \mathbf{k}) \cdot \mathbf{r} \right] + h.c. \right]$$

$$\zeta_n \equiv -e \frac{m}{V^{1/2} \sqrt{2kEE'}} u_s^\dagger (\mathbf{p}) \alpha \cdot \epsilon v_{s'} (\mathbf{q})$$, \hspace{1cm} (25)

where $h.c.$ means Hermitean conjugate, $u_s^\dagger$ and $v_{s'}$ are spinors, 

$$E \equiv \sqrt{\mathbf{p}^2 + m^2}, E' \equiv \sqrt{\mathbf{q}^2 + m^2},$$

and $n$ stands for $\{\mathbf{p}, \mathbf{q}, \mathbf{k}, s, s', \epsilon\}$. The interaction Hamiltonian $\hat{H}_{int}$ is the space integral of $\hat{\rho}_{int}(\mathbf{r})$ within the volume $V$. One of the terms of the Hamiltonian may create triples electron-positron-photon and the other term may annihilate triples.

It is easy to get the matrix element between the vacuum and a state with one triple $e^- e^+ \gamma$. The Hamiltonian is obtained via a space integral and the interaction energy may be calculated to second order perturbation theory giving (tentatively, see below)

$$E_{int} = V \rho_{int} = -\sum_n \left| \left< 0 \left| \hat{H}_{int} \right| n \right> \right|^2 \frac{k + E + E'}{k + E + E'}$$

$$= -\sum_{\mathbf{p}, \mathbf{q}, \mathbf{k}} \sum_{s, s', \epsilon} |\zeta_n|^2 \left| \int_V \exp \left[ i (\mathbf{p} + \mathbf{q} + \mathbf{k}) \cdot \mathbf{r} \right] d^3 \mathbf{r} \right|^2.$$

Taking the definition of $\zeta_n$, eq.(25) into account we get, after some algebra,

$$E_{int} = -\frac{e^2}{V^3} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{k}} \frac{[(p^2 + q^2)k^2 + (\mathbf{p} \cdot \mathbf{k})(\mathbf{q} \cdot \mathbf{k})]}{2k^3 E'E' (k + E + E')} \left| \int_V \exp \left[ i (\mathbf{p} + \mathbf{q} + \mathbf{k}) \cdot \mathbf{r} \right] d^3 \mathbf{r} \right|^2.$$

The integrals of $\mathbf{r}$ are trivial and give the square of the integration volume, $V$, times a Kroneker delta $\delta_{\mathbf{p}+\mathbf{q}-\mathbf{k}}$ of momentum conservation. The sums in

\hspace{1cm} (27)
and \( q \) give rise to a quartic divergence similar to those found for the free fields. To make easier the calculation we may introduce an ultraviolet cutoff via the multiplication of \( \text{eq.}(27) \) times the convergence factor

\[
\exp \left[ -\gamma \left( (k + E + E') \right) \right],
\]

where \( \varepsilon \) is of order the inverse of the parameter \( \Lambda \) of eqs. (15) and (20). We are interested in the energy density \( E_{\text{int}}/V \), see eq. (27), that after including the convergence factor \( \text{eq.}(28) \), I shall label \( A \). The calculation might be performed expanding \( E = \sqrt{p^2 + m^2} \) in powers of \( m^2 \) and similar for \( E' \) but it would be lengthy. It is not reported here but it may be realized that it should negative and proportional to \( \gamma^{-4} \), that is having the form

\[
E_{\text{int}} \equiv A = -3A_0\gamma^{-4}, \quad A_0 > 0,
\]

in the limit \( m \to 0 \), that is a good approximation because \( \gamma^{-1} >> m \). Here \( A_0 \) is a numerical parameter and the factor 3 is introduced for later convenience.

Actually the result eq. (29) is not reliable because the normalization condition, former eq. (22), is not fulfilled. This problem may be solved dividing eq. (27) by the square root of the quantity \( 1 + \sum_{n \neq 0} |c_n|^2 \). In fact as is well known the second order perturbation for the energy, eq. (27), may be written in terms of the first order perturbation of the normalized statevector as follows

\[
E_{\text{int}} = V\rho_{\text{int}} = \sum_{n \neq 0} \frac{c_n}{\sqrt{1 + \sum_{n \neq 0} |c_n|^2}} \left\langle 0 \left| \hat{H}_{\text{int}} \right| n \right\rangle = \frac{A}{\sqrt{C}}.
\]

Here we define

\[
C \equiv 1 + \sum_{n \neq 0} |c_n|^2 \simeq \sum_{n \neq 0} |c_n|^2 = \sum_{n \neq 0} \left[ \left\langle 0 \left| \hat{H}_{\text{int}} \right| n \right\rangle \right]^2 \exp \left[ -\varepsilon \left( (k + E + E') \right) \right],
\]

where I have included the regularization eq. (28). The approximate equality takes into account that \( \sum_{n \neq 0} |c_n|^2 \gg 1 \). Thus we get

\[
A = dC/d\varepsilon \Rightarrow C = A_0\gamma^{-3}.
\]

consistent with eq. (29) because, with the regularization eq. (28) included, the former eq. (31) holds true as may be realized. Eq. (30) provides a better approximation than eq. (29).
As a conclusion of this subsection we exhibit the total QED vacuum energy within a finite volume $V$ in the presence of a gravitational potential $\phi(r)$, calculated as in eq.(7). It is the following

$$E_{\text{vac}}^{\text{QED}} = \rho_{\text{vac}} \left[ V + \int d^3r \phi(r) \right] = (\rho_{\text{EM}} + \rho_D + \rho_{\text{int}}) \left[ V + \bar{\phi} \right], \bar{\phi} \equiv \int d^3r \phi(r).$$

(32)

where we assume that the field $\phi(r)$ is zero (or negligible) outside the volume $V$. The result, a straightforward consequence of eq.(7), follows from the hypothesis that the quantum vacuum is an inert system in the sense that its state is not modified by a gravitational field. In the following I prove that the result is quite different if we do not support that hypothesis but use eq.(2).

### 2.3 Interaction between the vacuum and a gravitational field in QED

The aim of this subsection is to prove that, according eq.(2) and at a difference with eq.(7), a gravitational field does modify the quantum vacuum state giving rise to a nonzero interaction energy between the vacuum fields and the gravitational field. For the proof I start rewriting both eqs.(7) and (2) separating in each the QED contribution from all other contributions, that is

$$E_{\text{inert vac-grav}} = \left[ \rho_{\text{vac}}^{\text{QED}} + \rho_{\text{vac}}^{\text{other}} \right] \int d^3r \phi(r),$$

$$\hat{H}_{\text{vac-grav}} = \int d^3r \left[ \hat{\rho}_{\text{QED}}(r) + \hat{\rho}_{\text{other}}(r) \right] \phi(r),$$

(33)

where $\rho_{\text{vac}}^{\text{QED}}$ is the quantity eq.(32).

The total quantum vacuum energy, $\rho_{\text{total}} = \rho_{\text{vac}}^{\text{QED}} + \rho_{\text{vac}}^{\text{other}}$, is known to be of order $\Lambda^4$ where $\Lambda$ is an energy (or inverse length) introduced as a cut-off, usually assumed to be at the Planck scale. With this assumption the vacuum energy density is huge but nevertheless it is possible to make calculations ignoring it (e.g. the Lamb shift or the anomalous magnetic moment of the electron). Therefore some mechanism would produce an effective cancellation that allows taking the vacuum energy as nil or small in practice. The mechanism is not yet known and the difficulty gives rise to the so-called
Cosmological Constant Problem mentioned in the introduction. In this paper I shall ignore that problem via the simple expedience of assuming that the total vacuum energy is zero (notice that for instance in QED there are positive contributions like eqs. (20) and (30) and negative ones like eq. (15)). That is I will assume

\[ \rho_{\text{total vac}} = 0 \Rightarrow \rho_{\text{other vac}} = -\rho_{\text{QED vac}}, \]  

whence from eq. (33) we get \( E_{\text{inert vac-grav}} = 0 \).

Now I will compare the interaction energy calculated from the Hamiltonian latter eq. (33), that is

\[ \hat{H}_{\text{QED vac-grav}} = \int d^3r \rho_{\text{QED vac}}(r) \phi(r), \]  

with the interaction energy obtained via the hypothesis involved in the former eq. (33), that is

\[ E_{\text{inert vac-grav}} = \rho_{\text{QED vac}} \int d^3r \phi(r). \]  

The relevant result is that subtracting the energy calculated via eq. (35) minus the energy eq. (36) we will obtain a nonzero quantity. Our hypothesis is that similar calculations with all other (interacting) vacuum fields may give also a nonzero contribution to the difference. I believe that this contribution is relevant in astrophysics and proceed to calculating it for QED in the following.

I shall begin with the free electromagnetic field. In contrast with eq. (36), finding the (approximate) eigenvalue of the Hamiltonian eq. (35) is made as follows. The Hamiltonian density operator of the vacuum interacting with an external gravitational field \( \phi(r) \) is

\[ \hat{\rho}_{\text{vac-grav}}^{\text{EM}} = [\hat{\rho}_{\text{EM1}}(r) + \hat{\rho}_{\text{EM2}}(r)] [1 + \phi(r)], \]  

whence the Hamiltonian is obtained performing a space integration. Taking
into account eq.(9) we get

\begin{align}
\hat{H}_{\text{EM-vac-grav}} &= \hat{H}_{\text{EM-vac-grav}}^1 + \hat{H}_{\text{EM-vac-grav}}^2, \\
\hat{H}_{\text{EM-vac-grav}}^1 &= \frac{1}{2} \left[ 1 + \frac{1}{V} \bar{\phi} \right] \left( \sum k + \sum k \hat{\alpha}^\dagger_{k,\epsilon} \hat{\alpha}_{k,\epsilon} \right) \\
&\quad + \frac{1}{2V} \sum k \epsilon \sum k' \epsilon' \sqrt{kk' \epsilon' \epsilon} \hat{\alpha}_{k,\epsilon}^\dagger \hat{\alpha}_{k',\epsilon'} \int \phi(r) \exp \left[ i (k - k') \cdot r \right] d^3r, \\
\hat{H}_{\text{EM-vac-grav}}^2 &= \frac{1}{4V} \sum k \epsilon \sum k' \epsilon' \sqrt{kk' \epsilon' \epsilon} \hat{\alpha}_{k,\epsilon} \hat{\alpha}_{k',\epsilon'} \int \phi(r) \exp \left[ i (k + k') \cdot r \right] + h.c.,
\end{align}

(38)

In comparison with the Hamiltonian obtained in absence of gravitational field, eq.(11), we see that there are additional terms.

Now we shall find the EM contribution to the vacuum energy via solving the eigenvalue problem for the Hamiltonian \( \hat{H}_{\text{EM-vac-grav}} \). Firstly it is easy to see that the zero photon state \(| 0 \rangle \) is the ground state for the Hamiltonian \( \hat{H}_{\text{EM-vac-grav}}^1 \) with eigenvalue (compare with eq.(32))

\[
E_{\text{EM-vac-grav}}^{EM1} = \rho_{EM} \left[ V + \bar{\phi} \right].
\]

(39)

In fact the two latter terms of \( \hat{H}_{\text{EM-vac-grav}}^1 \) give zero when acting on the state \(| 0 \rangle \) because they have an annihilation operator in the right.

However

\[
\hat{H}_{\text{EM-vac-grav}}^2 | 0 \rangle \neq 0,
\]

which shows that \(| 0 \rangle \) is no longer the quantum vacuum state of the electromagnetic field. We conclude that the presence of a gravitational field modifies the quantum vacuum state. The modification consists of the vacuum state becoming

\[
| \text{vac} \rangle = c_0 | 0 \rangle + \sum_n c_n | n \rangle,
\]

where \(| n \rangle \) is a set of quantum states with even number of photons, in particular 2 photons if the coefficients \( \{ c_n \} \) are calculated to first order in the perturbation \( \hat{H}_{\text{EM-vac-grav}}^2 \), eq.(38). I will not get the coefficients \( \{ c_n \} \) but
only the energy, this to second order. We have

$$E_{\text{vac-grav}}^{EM^2} = \sum_n \left| \langle 0 | \hat{H}_{\text{vac-grav}}^{EM^2} | n \rangle \right|^2 \frac{E_0 - E_n}{E_0 - E_n}$$

$$= -\langle 0 | \int d^3 r_1 \hat{\rho}_{EM^2}^2 (r_1) \phi (r_1) \int d^3 r_2 \hat{\rho}_{EM^2}^2 (r_2) \phi (r_2) | 0 \rangle / (k + k')$$

$$= -\frac{1}{V^2} \sum_{k,k'} \frac{kk'}{k + k'} \left[ 1 + \frac{k \cdot k'}{kk'} \right]^2$$

$$\times \int d^3 r_1 \int d^3 r_2 \{ \exp \left[ i (k + k') \cdot (r_2 - r_1) \right] \phi (r_1) \phi (r_2) \}, \quad (40)$$

where I have taken the latter eq. (38) into account.

After the change of variables

$$r_1 = r_0 - r/2, r_2 = r_0 + r/2, \quad (41)$$

we get

$$E_{\text{vac-grav}}^{EM^2} = -\frac{1}{V^2} \sum_{k,k'} \frac{(kk' + k \cdot k')^2}{(k + k')kk'}$$

$$\times \int d^3 r_0 \int d^3 r \exp \left[ i (k + k') \cdot r \right] \phi (r_0 + r/2) \phi (r_0 - r/2) \} \quad (42)$$

We may approximate

$$\phi (r_0 \pm r/2) \simeq \phi (r_0) \pm \frac{1}{2} \sum_i x_i \frac{\partial}{\partial x_i} \phi (r_0) + \frac{1}{8} \sum_{ij} x_i x_j \frac{\partial^2}{\partial x_{0i} \partial x_{0j}} \phi (r_0),$$

whence, retaining only the relevant terms we may write, up to second order in the components $x_i$,

$$\phi (r_0 + r/2) \phi (r_0 - r/2) \simeq \phi (r_0)^2$$

$$+ \frac{1}{12} r^2 \left[ \phi (r_0) \nabla^2 \phi (r_0) - |\nabla \phi (r_0)|^2 \right], \quad (43)$$

As the integrals in $k, k'$ involve all directions of space we have substituted

$$\frac{1}{3} r^2 |\nabla \phi (r_0)|^2$$

for $[r \cdot \nabla \phi (r_0)]^2$ and a similar change in the term with $\nabla^2 \phi (r_0)$. 

I point out that the latter term of eq. (43) is negative because \( \phi < 0 \), \( \nabla^2 \phi > 0 \). Therefore I will use the opposite to the term in the following.

Inserting eq. (43) in eq. (42) it is easy to see that the term with \( \phi(r_0) \) is nil because the space integral leads to \( k' = -k \), whence \( kk' + k \cdot k' = 0 \). Then, after taking the continuous limit of the sums in \( k, k' \), we get for the energy

\[
E_{\text{vac-grav}}^{EM} = -\frac{1}{12 \left(8\pi^3\right)^2} \int d^3k \int d^3k' \frac{(kk' + k \cdot k')^2}{(k + k')kk'} \int r^2 d^3r \exp \left[i \left(k + k' \right) \cdot r \right]
\]

\[
\times \int d^3r_0 \left\{ - \left[ \left| \nabla \phi (r_0) \right|^2 - \phi (r_0) \nabla^2 \phi (r_0) \right] \right\},
\]

\[
= \int d^3r_0 \left[ \left| \nabla \phi (r_0) \right|^2 - \phi (r_0) \nabla^2 \phi (r_0) \right] K_{EM},
\]

where

\[
K_{EM} \equiv \frac{1}{12 \left(8\pi^3\right)^2} \int d^3k \int d^3k' \frac{(kk' + k \cdot k')^2}{(k + k')kk'} \int r^2 d^3r \exp \left[i \left(k + k' \right) \cdot r \right].
\]

The product \( K_{EM} \left[ \left| \nabla \phi (r_0) \right|^2 - \phi (r_0) \nabla^2 \phi (r_0) \right] \) is the electromagnetic part of the additional energy density of interaction between the external gravitational field and the quantum vacuum (i.e. in addition to the latter term of eq. (39)).

The integral giving \( K_{EM} \) may be calculated going from the vector variables \( k, k' \) to the new variables \( k, s = k + k' \). The Jacobian of the transformation is \( |J| = 1 \). The integral in \( r \) may be easily performed and we get

\[
I = \int r^2 d^3r \exp \left[i \left(k + k' \right) \cdot r \right] \rightarrow 4\pi \int_0^\infty r^4 dr \frac{\sin (sr)}{sr} \simeq -\frac{4\pi^2}{s^6} \delta \left(s \right),
\]

where \( \delta () \) is a Dirac’s delta. Hence after some algebra we obtain, to the lowest nontrivial order of \( s \),

\[
K_{EM} \simeq -\frac{16\pi^4}{(8\pi^3)^2} \int d^3k \int_0^\infty ds \int_{-1}^1 du \frac{\left[1 - u^2\right]^2}{2k^3} \delta \left(s \right) = -\frac{4}{\pi} \int \frac{dk}{k},
\]

where \( u = k \cdot s/ks \). The integral is logarithmically divergent both at the origin and at infinity. The infrared divergence is a consequence of the approximations made. Indeed eq. (43) is not valid for too large values of \( r \), whence the
Integral in $r$ should not be extended to $\infty$ but only to some distance $R$ of order the typical range of the gravitational field $\phi(r)$. Therefore we should put $R^{-1}$ as lower limit of the $k$ integral. In contrast the ultraviolet divergence is relevant and we propose to put a cutoff $\Lambda$ in the photon energies. Hence we get

$$K_{EM} \simeq -\frac{26}{15\pi^2} \int_{R^{-1}}^{\Lambda} \frac{dk}{k} \Rightarrow K_{EM} \simeq -\frac{26}{15\pi^2} \ln(R\Lambda).$$

Actually we should divide by $\sqrt{1 + \sum |c_n|^2}$, see discussion after eq. (29). We have

$$\sum_{n \neq 0} |c_n|^2 = \frac{1}{12(8\pi^3)^2} \int d^3k \int d^3k' \frac{(kk' + k \cdot k')^2}{(k + k')^2} \int r^2 d^3r \exp[i(k + k').r] \times \int d^3r_0 \left[ |\nabla \phi(r_0)|^2 - \phi(r_0) \nabla^2 \phi(r_0) \right]$$

$$= \frac{13}{15\pi^2} R \times 2 \int d^3r_0 |\phi(r_0)| \nabla^2 \phi(r_0) \simeq \frac{26}{15\pi} \times \frac{RMc}{\hbar} \times 10^{-6},$$

where $M$ is the total mass producing the gravitational field. The second equality follows from an integration by parts of $|\nabla \phi(r_0)|^2$. The latter equality is a rough estimate substituting a typical value for $|\phi| \sim 10^{-6}$, that makes the $r_0$ integral trivial taking into account that $\nabla^2 \phi = 4\pi \rho$ in our units. In the final expression I have included explicitly $\hbar$ and $c$ in order to show that $\sum_{n \neq 0} |c_n|^2 >> 1$, whence $|K|$ is actually many orders smaller than eq. (45).

The contribution of the free electron-positron field to the vacuum interacting with a gravitational field, eq. (35) is similar to eq. (37). We obtain, taking eq. (17) into account,

$$\hat{\rho}^{D}_{vac-grav} = [\rho_{D0} + \hat{\rho}_b(r, t) + \hat{\rho}_d(r, t)] [1 + \phi(r)] .$$

Getting the Hamiltonian is straightforward via performing the space integral but I do not write it explicitly. The relevant result is that the state $|0\rangle$ with zero electrons and zero positrons is an eigenstate of $\hat{\rho}^{D}_{vac-grav}$ because both terms $\hat{\rho}_b$ and $\hat{\rho}_d$ have an annihilation operator on the right. Hence the energy is

$$E^D_0 = \rho_D [V + \phi],$$

where $\rho_D$ was given in eq. (20).
Finally we will calculate the contribution of the interaction density eq. (24), that leads to the following Hamiltonian

\[ \hat{H}_{\text{vac-grav}} = \int d^3 r \left[ 1 + \phi (r) \right] \sum_{p,q,k,s,s',\varepsilon} [\zeta_n \hat{a}_{k,\varepsilon} \hat{b}_{p,s} \hat{d}_{q,s'} \exp \{i (p + q + k) \cdot r\}] + h.c. \]

The contribution to the vacuum energy interacting with the gravitational field is

\[ E_{\text{vac-grav}}^{\text{int}} = -\sum_n \left| \left\langle 0 \left| \hat{H}_{\text{vac-grav}} \right| n \right\rangle \right|^2 \]

\[ = -\frac{e^2}{V^3} \sum_{p,q,k} \frac{[(p^2 + q^2)k^2 + (p \cdot k)(q \cdot k)]}{2k^3 E \varepsilon (k + E + E')} \]

\[ \times \left| \int_V \left[ 1 + \phi (r) \right] \exp \{i (p + q + k) \cdot r\} d^3 r \right|^2. \]

After the change of variables eq. (41), taking eq. (43) into account we get

\[ E_{\text{vac-grav}}^{\text{int}} = \rho_{\text{int}} (V + 2\phi) + B \int d^3 r_0 \left\{ |\nabla \phi (r_0)|^2 - \phi (r_0) \nabla^2 \phi (r_0) \right\}, \]

where

\[ B \equiv \frac{e^2}{V^3} \sum_{p,q,k} \frac{[(p^2 + q^2)k^2 + (p \cdot k)(q \cdot k)]}{2k^3 E \varepsilon (k + E + E')} \]

\[ \times \left| \int_V r^2 \exp \{i (p + q + k) \cdot r - \gamma ((k + E + E'))\} d^3 r \right|, \]

after including the convergence factor eq. (28). I notice that the right side does not have a minus sign because the latter term of eq. (43) was changed in going to eq. (48). Now using arguments similar to those of the previous section, see the paragraph between eqs. (27) and (31), we should redefine the quantity \( K_{\text{int}} \) as deriving from

\[ E_{\text{int-grav}} = \frac{A (1 + 2\phi/V) + B \int d^3 r_0 \left[ |\nabla \phi (r_0)|^2 - \phi (r_0) \nabla^2 \phi (r_0) \right]}{\sqrt{C (1 + 2\phi/V) + D \int d^3 r_0 \left[ |\nabla \phi (r_0)|^2 - \phi (r_0) \nabla^2 \phi (r_0) \right]}}, \]

(50)
rather than eq.(48). It may be realized that $B$ and $D$ should be of the form (compare with eq.(31))

$$B = -B_0 \gamma^{-2} = dD/d\gamma, D = B_0 \gamma^{-1}. \quad (51)$$

I stress that both terms within the square root in eq.(50) should be positive, being of the form $\sum |c_n|^2$ each. Therefore $D > 0$ whence $B < 0, B_0 > 0$.

The interpretation of eq.(50) is more clear if it is approximated by an expression quadratic in $\phi$, taking into account that $A, B, C, D$, eqs.(29), (31) and (51), fulfil $|B| << |A| \Rightarrow |D| << |C|$ and that $|\bar{\phi}| << V$. Thus we get

$$E_{\text{int-grav}} \simeq \frac{A}{\sqrt{C}} (1 + \bar{\phi}/V) + \left[ \frac{B}{\sqrt{C}} - \frac{AD}{2C\sqrt{C}} \right] \int d^3 r_0 \left[ |\nabla \phi(r_0)|^2 - \phi(r_0) \nabla^2 \phi(r_0) \right]$$

$$\Rightarrow \rho_{\text{int-grav}} \simeq \rho_{\text{int}} (1 + \bar{\phi}/V) + \frac{B_0}{2\sqrt{|A_0|}} \gamma^{-1/2} \left[ |\nabla \phi(r_0)|^2 - \phi(r_0) \nabla^2 \phi(r_0) \right],$$

where $B_0/\sqrt{|A_0|}$ is a dimensionless quantity. I stress that the latter term is positive.

In summary we may write the QED vacuum energy in the presence of a gravitational potential $\phi(r)$ in the form

$$E_{\text{grav-vac}} = E_{\text{QED}} V + \rho_{\text{QED}} \bar{\phi} + V K_{\text{QED}} \left[ |\nabla \phi(r_0)|^2 - \phi(r_0) \nabla^2 \phi(r_0) \right],$$

$$\rho_{\text{QED}} = \rho_{\text{EM}} + \rho_D + \rho_{\text{int}} \quad (52)$$

where

$$K_{\text{QED}} = K_{\text{EM}} + K_{\text{int}}, K_{\text{int}} \equiv \frac{B_0}{2\sqrt{|A_0|}} \gamma^{-1/2}. \quad (53)$$

The first term of eq.(52) is the standard QED vacuum energy, i.e. in the absence of external gravitational field, and the second term is the interaction of that field with the unperturbed vacuum. I propose that the latter term is an alternative to “dark matter”, see below. The dimensionless quantity $K_{\text{QED}}$ consists of a pure electromagnetic contribution $K_{\text{EM}}$ defined in eq.(45) (negative and divergent like $\log \Lambda$), associated to the production of virtual photon pairs with opposite momenta, plus the contribution $K_{\text{int}}$ (positive and divergent like $\sqrt{\Lambda}$) corresponding of the production of virtual triples photon-electron-positron with total zero momentum. As a conclusion of our calculation we see that the relevant part of the interaction energy density between the vacuum fields and an external gravitational field (i.e. the latter term of eq.(52)) is positive.
Eq. (52) is the correct substitute for the wrong eq. (32). The difference between them is the latter term of eq. (52). If we were studying the whole set of elementary particles and their interactions, I have assumed that the quantum vacuum energy is zero (or it is cancelled by some unknown mechanism), whence the generalization of eq. (52) would consist only of terms like the latter one. That is the energy density at point $r_0$ due to the interaction of the quantum vacuum with an external gravitational potential $\phi (r_0)$ would be approximately

$$\rho_{vac-grav} \simeq K \left[ |\nabla \phi (r_0)|^2 - \phi (r_0) \nabla^2 \phi (r_0) \right], \quad (54)$$

$K$ being a new constant that would generalize the parameter $K_{QED}$, eq. (52), for all quantum fields and interactions. This is valid only for weak external gravitational fields where the Newtonian approximation applies, although the divergences put a problem not to be discussed further here. In any case the divergences found are logarithmic, in eq. (45), or of order $\sqrt{\Lambda}$ in eq. (53), much weaker than the quartic divergence of the vacuum energy, see eqs. (15) and (20).

I propose that this interaction might be the origin of the so-called "dark matter", thus making unnecessary to postulate the existence of new particles not appearing in the standard model. In order to support that conjecture, in the next section I will exhibit a simple model inspired on eq. (54).

3 Simple model alternative to dark matter

3.1 The problem of the flat rotation curves in galactic haloes

The existence of dark matter of unknown origin is the current solution proposed for two different problems [9], [10] which may be called astrophysical and cosmological, respectively. The astrophysical problem is the more or less flat rotation curves of stars or gas in the halo of galaxies and clusters of galaxies. The cosmological one is the need of a large amount of non-baryonic cold matter in the universe, at the recombination time, in order to explain the formation of structures. The amount of non-baryonic matter is fairly the same in both cases, which has lead to the popular assumption that both may be solved with the same hypothesis, that is dark matter. The standard $\Lambda CDM$ cosmological model predicts that about 70% of mass-energy budget
of the Universe is composed of dark energy, about 30% is the total mass of
the Universe which is dominated with more than 5/6 by dark matter, while
ordinary baryonic matter constitutes only less than 1/6 of the total mass[11].
In order to account for the flat rotation curves of spiral galaxies, dark matter
should dominate the total mass of the galaxy and should be concentrated
in the outer baryonic regions of galactic disks, as well as in the surrounding
haloes.

The fact that none of the known particles is a good candidate to form
dark matter and the failure to discover new particles with the required
properties, in spite of the big effort made at the observational level, has lead
to alternatives consisting of a modification of current gravity theories (i. e.
general relativity or Newtonian gravity) or dynamics. The most elaborate
of these is the “modified Newtonian dynamics” (MOND)[12], but there are
also proposals resting upon f(R) gravity or its generalizations[13, 14, 15].
Actually the proposed modifications of gravity or dynamics attempt mainly
to solve the astrophyscsical problem, with less implication in the cosmological
one.

MOND[12] proposes, for galaxies approximately spherical, that the flat
(i. e. independent of the radius) rotation velocities, v, in galactic haloes is
roughly given by

\[ v^2 \approx \sqrt{a_0 GM_B}, \]  

(55)

where \( G \) is Newton constant, \( M_B \) the baryonic mass of the galaxy, and \( a_0 \) a
new universal constant given by

\[ a_0 = 2 \times 10^{-10} \, \text{m/s}^2. \]

The dependence on the square root of the total baryon mass of the galaxy fits
in the empirical baryonic Tully-Fisher law[12], that is \( v^4 \propto M_B \). Also MOND
is able to successfully explain the dynamics of galaxies outside clusters and a
discovered tight relation between the radial acceleration inferred from their
observed rotation curves and the acceleration due to the baryonic components
of their disks[16]. However the MOND predictions do not fit too well in some
observations[17].

Another proposed alternative to dark matter is extended general relativity
(see [18] for a review). In these theories a different scalar obtained from
the Riemman tensor is substituted for the Ricci scalar, \( R \), in the Hilbert-
Einstein action. The most interesting case consists of choosing a function of
\( R \), leading to the socalled f(R) theories. Then it is possible to demonstrate
that the existence of a Noether symmetry in $f(R)$ theories of gravity gives rise to a further gravitational radius, besides the standard Schwarzschild one, determining the dynamics at galactic scales\cite{15}. This radius plays an analog role, in the case of weak gravitational field at galactic scales, like the Schwarzschild radius in the case of strong gravitational field in the vicinity of compact massive objects. Such a feature emerges from symmetries that exist for any power-law $f(R)$ function. In particular, for $f(R) \propto R^{3/2}$, the MOND acceleration regime is recovered. Using this new gravitational radius, $f(R)$ theories of gravity are able to explain the baryonic Tully-Fisher relation of gas-rich galaxies in a natural way.

In this paper I propose another hypothesis, namely that the interaction of gravitational fields with the vacuum quantum fields might explain the rotation curves, but the cosmological problem is not studied. The idea of explaining the flat rotation curves of galaxies as a quantum vacuum effect is not new. For instance a model has been proposed in order to explain eq.\,(55) under the assumption of a gravitational repulsion between particles and antiparticles which, via vacuum fluctuations, would give rise to a gravitationally polarizable vacuum\cite{19}. The author attributes the repulsion to a negative gravitational mass (but positive inertial mass) of antiparticles.

3.2 Our model

I propose a model for stationary systems whose fundamental assumption is the following equation (compare with eq.\,(54))

$$\rho_g (\mathbf{r}) = K \left[ |\nabla \phi (\mathbf{r})|^2 + a \nabla^2 \phi (\mathbf{r}) \right],$$

where $\rho_g$ is the net energy density of interaction between the quantum vacuum and a gravitational field with Newtonian potential $\phi (\mathbf{r})$. In this model $K$ and $a$ are positive parameters and I will use units $c = 4\pi G = 1$ in this section (Planck constant $\hbar$ does not enter in the model although it would enter in the calculation of $K$). The model is suggested for the interaction between a weak external gravitational field and the QED vacuum, calculated in section 2 leading to eq.\,(54). In comparison with that equation I substitute $-a$ for the Newtonian potential, $\phi (\mathbf{r})$, which produces a big simplification of the model. It is justified by the fact that the potential changes more slowly than the field, as will be checked below. The field, $-\nabla \phi (\mathbf{r})$, derives from all matter in space that we assume to consists of baryonic, $\rho_B (\mathbf{r})$, plus the
grav-vac interaction $\rho_g (r)$ itself. Consequently I include in the model another equation giving the relation between field and mass-energy density, that is

$$\nabla^2 \phi (r) = \rho_B (r) + \rho_g (r).$$  \hfill (57)

From eqs.(56) and (57) we get

$$(1 - aK) \nabla^2 \phi (r) = \rho_B (r) + K | \nabla \phi (r)|^2.$$  \hfill (58)

This is the fundamental equation of the model.

For a spherically symmetric system (galaxy or cluster) eq.\(\text{(58)}\) becomes

$$\frac{d\phi'}{dr} + 2\frac{\phi'}{r} = (1 - aK)^{-1} \left[ \rho_B (r) + K \phi'^2 \right].$$  \hfill (59)

where $-\phi'$ is the gravitational field. This equation has no regular solution (both at the origin and at infinity) if $\rho_B (r) = 0$ everywhere. This is satisfactory because it shows that “dark matter” appears only near regions with baryonic matter, in agreement with observations. In the external region, $r > R$, where $\rho_B (r) = 0$ there is a solution regular at infinity, namely

$$\phi' (r) = \frac{1 - aK}{Kr}.$$  \hfill (60)

For this field the rotation curves are flat in agreement with observations. From eq.\(\text{(60)}\) the rotation velocity $v$ fulfils

$$v^2 = \frac{1 - aK}{K} c^2.$$  \hfill (61)

Thus we fix the parameters so that it fits the observed rotation velocity in the halo of a typical galaxy, say $v \sim 10^{-3} c$, whence we get

$$\frac{1 - aK}{K} \sim 10^{-6}.$$  \hfill (62)

Taking eqs.\(\text{(58)}\) and \(\text{(60)}\) into account the dark mass density in the region without baryonic matter becomes, for any spherical body,

$$\rho_g (r) = \frac{1 - aK}{Kr^2}.$$  \hfill (63)
The gravitational potential is given by the integral of eq. (60), that is
\[ \phi(r) = -\phi_0 + \frac{(1 - aK)}{K} \ln \left(\frac{r}{r_0}\right), \]
where the parameters \( r_0 \) and \( \phi_0 \) depend on the particular galaxy. The logarithmic dependence of \( \phi(r) \) shows that the potential is slowly varying in the halo of the galaxy, as we commented after eq. (56).

For a galaxy with baryonic radius \( R \sim 10^{22} m \) the dark mass density in the halo would be, taking eqs. (62) and (63) into account,
\[ \rho_g \sim 10^{-24} kg/m^3. \]
That is of order the mean density of the galaxy if \( M \sim 10^{42} kg \) is the mass. The density \( \rho_g \) predicted by the model near individual stars is too big. For instance in the surface of the sun \( (M = 2 \cdot 10^{30} kg, R = 7 \cdot 10^8 m) \) it gives \( \rho_g \sim 3000 kg/m^3 \). Obviously the model is not valid for gravitational fields much stronger then those of galaxy haloes.

Near the center of a spherical body the gravitational field, \( -\nabla \phi(r) \), is zero or small whence eq. (58) leads to
\[ \rho_g \sim \frac{aK}{1 - aK} \rho_B = 10^6 a \rho_B \sim 10^6 |\phi(0)| \rho_B(0), \]
where I have taken into account eq. (62) and the fact that \( a \) is our model substitute for the potential \( |\phi(r)| \) (compare eqs. (54) and (56)). Therefore the model predicts a relatively high dark density at the center, a phenomenon observed and usually known as a “cusp”. Indeed if \( \phi(0) \) is larger than \( 10^{-6} \) in our units, then \( \rho_g \) is larger than \( \rho_B \) at the center of the galaxy.

As we have seen there are several predictions of the model that fit fairly well in the observations. In fact, it leads to dark mass distributions that go well beyond the radius of typical galaxies. It predicts flat rotation curves in the haloes and a maximum of the dark density at the center of the galaxy (or cluster). It is also a good feature that the fundamental equations do not have any physically sound solution in the absence of baryonic matter. Nevertheless there are difficulties that point towards the need of some modification of the model, the three most relevant being the following.

1. The model predicts a large amount of dark mass around bodies like the sun that is not observed. Indeed the ratio \( \rho_g/\rho_B \) predicted by the model
is almost independent of the mean baryonic density $\bar{\rho}_B$. A modification of the model eq. (56) of the form

$$\rho_g (r) = K \left[ |\nabla \phi (r)|^2 + a \nabla^2 \phi (r) \right] \times f \left( \nabla^2 \phi (r) \right),$$

with $f(x)$ a decreasing function (i.e. $df/dx < 0$), might solve this difficulty. The decrease should be very slow at typical galaxy densities thus not spoiling the predicted flat rotation curves. This is possible because the ratio of the Sun density to the typical density in a galaxy is about $10^{27}$.

2. The rotation velocity does not depend on the baryonic mass, $M_B$, of the associated galaxy, contradicting the baryonic Tully-Fisher empirical law, that is a velocity proportional to $M_B^{1/4}$.

3. The total dark mass is divergent because the predicted dark density behaves as $r^{-2}$ at infinity, so that $\int \rho_g d^3r$ diverges.

4 Conclusions

The change of the vacuum energy in presence of a gravitational field is calculated in the particular example of QED vacuum, the result suggesting that there is a finite interaction of the field with the quantum vacuum. There should be many contributions to that interaction, but the net result is an energy density that, if it turns out to be positive, we may interpret as the origin of the flat rotation curves in galactic haloes, commonly attributed to some “dark matter”. A model is proposed that fits in the main properties of the rotation curves, although it presents difficulties.

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