INTEGRAL REPRESENTATIONS AND SUMMATIONS OF MODIFIED STRUVE FUNCTION

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Abstract. It is known that Struve function $H_\nu$ and modified Struve function $L_\nu$ are closely connected to Bessel function of the first kind $J_\nu$ and to modified Bessel function of the first kind $I_\nu$ and possess representations through higher transcendental functions like generalized hypergeometric $\, _1F_2$ and Meijer $G$ function. Also, the NIST project and Wolfram formula collection contain a set of Kapteyn type series expansions for $L_\nu(x)$. In this paper firstly, we obtain various another type integral representation formulae for $L_\nu(x)$ using the technique developed by D. Jankov and the authors. Secondly, we present some summation results for different kind of Neumann, Kapteyn and Schlömilch series built by $I_\nu(x)$ and $L_\nu(x)$ which are connected by a Sonin–Gubler formula, and by the associated modified Struve differential equation. Finally, solving a Fredholm type convolutional integral equation of the first kind, Bromwich–Wagner line integral expressions are derived for the Bessel function of the first kind $J_\nu$ and for an associated generalized Schlömilch series.

1. Introduction

Bessel and modified Bessel function of the first kind, Struve and modified Struve function possess power series representation of the form [63]:

$$J_\nu(z) = \sum_{n \geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{\Gamma(n+\nu+1)n!}, \quad I_\nu(z) = \sum_{n \geq 0} \frac{\left(\frac{z}{2}\right)^{2n+\nu}}{\Gamma(n+\nu+1)n!},$$

$$H_\nu(z) = \sum_{n \geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu+1}}{\Gamma(n+\nu+\frac{3}{2})\Gamma(n+\nu+\frac{3}{2})}, \quad L_\nu(z) = \sum_{n \geq 0} \frac{\left(\frac{z}{2}\right)^{2n+\nu+1}}{\Gamma(n+\nu+\frac{3}{2})\Gamma(n+\nu+\frac{3}{2})},$$

where $\nu \in \mathbb{R}$ and $z \in \mathbb{C}$. Struve [56] introduced $H_\nu$ function as the series solution of the nonhomogeneous second order Bessel type differential equation (which carries his name). However, the modified Struve function $L_\nu$ appeared into mathematical literature by Nicholson [40, p. 218]. Applications of Struve functions are manifold and include among others optical investigations [60, pp. 392–395]; general expression of the power carried by a transverse magnetic or electric beam, is given in terms of $L_{\nu+\frac{1}{2}}$ [2]; triplet phase shifts of the scattering by the singular nucleon-nucleon potentials $\propto \exp(-x)/x^n$ [20]; leakage inductance in transformer windings [26]; boundary element solutions of the two-dimensional multi-energy-group neutron diffusion equation which governs the neutronic phenomena in nuclear reactors [27]; effective isotropic potential for a pair of dipoles [36]; perturbation approximations of lee–waves in a stratified flow [37]; quantum–statistical distribution functions of a hard–sphere system [42]; scattering of plane waves by circular cylinders for the general case of oblique incidence and for both real and complex values of particle refractive index [54]; aerodynamic sensitivities for subsonic, sonic, and supersonic unsteady, non-planar lifting–surface theory [14]; stress concentration around broken filaments [19] and lift and downwash distributions of oscillating wings in subsonic and supersonic flow [61, 62].

Series of Bessel and/or Struve functions in which summation indices appear in the order of the considered function and/or twist arguments of the constituting functions, can be unified in a double lacunary

2010 Mathematics Subject Classification. Primary 33C10, 33E20, 40H05; Secondary 30B50, 40C10, 65B10.

Key words and phrases. Modified Struve function; Bessel function and modified Bessel function of the first kind; Neumann, Kapteyn and Schlömilch series of modified Bessel and Struve functions, Dirichlet series, Cahen formula, generalized hypergeometric function, Struve differential equation.
form:
\[ \mathfrak{B}_{\ell_1, \ell_2}(z) := \sum_{n \geq 1} \alpha_n \mathfrak{B}_{\ell_1(n)}(\ell_2(n)z), \]
where \( x \mapsto \ell_j(x) = \mu_j + a_j x, j \in \{1, 2\}, x \in \{0, 1, \ldots\} \), \( z \in \mathbb{C} \) and \( \mathfrak{B}_\nu \) is one of the functions \( J_\nu, I_\nu, H_\nu \) and \( L_\nu \). The classical theory of the Fourier–Bessel series of the first type is based on the case when \( \mathfrak{B}_\nu = J_\nu \), see the celebrated monograph by Watson [63]. However, varying the coefficients of \( \ell_1 \) and \( \ell_2 \), we get three different cases which have not only deep roles in describing physical models and have physical interpretations in numerous topics of natural sciences and technology, but are also of deep mathematical interest, like e.g. zero function series [63]. Hence we differ: Neumann series (when \( a_1 = 0, a_2 = 0 \)), Kapteyn series (when \( a_1 \cdot a_2 \neq 0 \)) and Schlömilch series (when \( a_1 = 0, a_2 = 0 \)). Here, all three series are of the first type (the series’ terms contain only one constituting function \( \mathfrak{B}_\mu \)); the second type series contain product terms of two (or more) members - not necessarily different ones - from \( J_\nu, I_\nu, H_\nu \) and \( L_\nu \). We also point out that the Neumann series (of the first type) of Bessel function of the second kind \( Y_\nu \), modified Bessel function of the second kind \( K_\nu \) and Hankel functions (Bessel functions of the third kind) \( H_\nu^{(1)}, H_\nu^{(2)} \) have been studied by Baricz, Jankov and Pogány [5], while Neumann series of the second type were considered by Baricz and Pogány in somewhat different purposes in [6, 7]; see also [31]. An important role has throughout of this paper the Sonin–Gubler formula which connects modified Bessel function of the first kind \( J_\nu \), modified Struve function \( L_\nu \) and a definite integral of the Bessel function of the first kind \( J_\nu \) [23, p. 424] (actually a special case of a Sonin–formula [63, p. 434]):

\[ \int_0^\infty J_\nu(ax) \frac{dx}{x^2 + n^2} = \frac{\pi}{2n^{\nu+1}} (I_\nu(an) - L_\nu(an)), \]
where \( \Re(\nu) > -\frac{1}{2}, a > 0 \) and \( \Re(n) > 0 \); see [63, p. 426] for the historical background of (1.1).

Thus, under extended Neumann series (of Bessel \( J_\nu \) see [63]) we mean the following

\[ \mathfrak{N}_{\mu, \eta}(x) := \sum_{n \geq 1} \beta_n \mathfrak{B}_{\mu n + \eta}(ax), \]
where \( \mathfrak{B}_\nu \) is one of the functions \( I_\nu \) and \( L_\nu \). Integral representation discussions began very recently with the introductory article by Pogány and Suli [47], which gives an exhaustive references list concerning physical applications too; see also [5]. In Section 2 we will concentrate to the Neumann series

\[ \mathfrak{N}_{\mu, \eta}(x) := \sum_{n \geq 1} \beta_n I_{\mu n + \eta}(ax). \]

Secondly, Kapteyn series of the first type [32, 33, 41] are of the form

\[ \mathfrak{K}_{\rho, \mu}(z) := \sum_{n \geq 1} \alpha_n \mathfrak{B}_{\rho + \mu n} ((\sigma + \nu n)z); \]

more details about Kapteyn and Kapteyn–type series for Bessel function can be found also in [4, 6, 15, 57] and the references therein. Here we will consider specific Kapteyn–type series of the following form:

\[ \mathfrak{K}_{\alpha, \mu}(x) := \sum_{n \geq 1} \frac{\alpha_n}{n^\rho} (I_{\nu n}(xn) - L_{\nu n}(xn)); \]
this series appear as auxiliary expression in the fourth section of the article. Thanks to Sonin–Gubler formula (1.1) we give an alternative proof for integral representation of \( \mathfrak{K}_{\alpha, \mu}(x) \), see Section 3. Thirdly, under Schlömilch series [52, pp. 155–158] (Schlömilch considered only cases \( \mu \in \{0, 1\} \)), we understand the functions series

\[ \mathfrak{S}_{\mu, \nu}(z) := \sum_{n \geq 1} \alpha_n \mathfrak{B}_\mu ((\nu + n)z). \]

Integral representation are recently obtained for this series in [30], summations are given in [59]. Our attention is focused currently to

\[ \mathfrak{S}_{\mu, \nu}(z) := \sum_{n \geq 1} \frac{\alpha_n}{n^\mu} (I_{\nu}(xn) - L_{\nu}(xn)). \]
The next generalization is suggested by the theory of Fourier series, and the functions which naturally come under consideration instead of the classical sine and cosine, are the Bessel functions of the first kind and Struve's functions. The next type series considered here we call generalized Schlömilch series [63, p. 622], [28, p. 1803]

\[
\frac{a_0}{2\Gamma(\nu + 1)} + \left(\frac{x}{2}\right)^{-\nu} \sum_{n=1}^\infty \frac{a_n J_\nu(nx) + b_n H_\nu(nx)}{n^{\nu}}.
\]

For further subsequent generalizations consult e.g. Bondarenko's recent article [9] and the references therein and Miller's multidimensional expansion [38]. A set of summation formulae of Schlömilch series for Bessel function of the first kind can be found in the literature, such as the Nielsen formula [63, p. 636]; further, we have [58, p. 65], also consult [21, 45, 50, 59, 64, 65]. Similar summations, for Schlömilch series of Struve function, have been given by Miller [39], consult [59] too.

Further, we are interested in a specific variant of generalized Schlömilch series in which \(J_\nu, H_\nu\) are exchanged by \(I_\nu\) and \(L_\nu\) respectively. Indeed, the so-called counting sum \(\tilde{T}^L_{\nu,\mu}(x)\) we perform setting \((-1)^{n-1}a_n \mapsto a_n\), where \(n \in \{0, 1, \ldots\}\):

\[
\tilde{T}^L_{\nu,\mu}(x) := \sum_{n \geq 1} \left(\frac{(-1)^{n-1}}{n}\right) (I_\nu(nx) - L_\nu(nx)).
\]

Summations of these series are one of tools in obtaining explicit expressions for integrals containing Bessel functions, such as the Nielsen formula [63, p. 603]. Simpler summations, for Schlömilch series of Struve function, have been recently derived in [29].

Finally, we mention that except the Sonin–Gubler formula (1.1) another main tool we refer to is the Cahen formula on the Laplace integral representation of Dirichlet series. Namely, the Dirichlet series

\[
\mathcal{D}_\alpha(r) = \sum_{n \geq 1} a_n e^{-r\lambda_n},
\]

where \(\Re(r) > 0\), having positive monotone increasing divergent to infinity sequence \((\lambda_n)_{n \geq 1}\), possesses Cahen's integral representation formula [13, p. 97]

\[
\mathcal{D}_\alpha(r) = r \int_0^\infty e^{-rt} \sum_{n: \lambda_n \leq t} a_n \, dt = r \int_0^\infty \int_0^{[\lambda^{-1}(t)]} \varrho u a(u) \, du \, dt,
\]

where \(\varrho_x := 1 + \{x\} \frac{d}{dx}\). Here, \([x]\) and \(\{x\} = x - [x]\) denote the integer and fractional part of \(x \in \mathbb{R}\), respectively. Indeed, the so-called counting sum

\[
\mathcal{A}_\alpha(t) = \sum_{n \leq t} a_n
\]

we find by the Euler–Maclaurin summation formula, following the procedure developed by the second author [46], see also [49]. Namely

\[
\mathcal{A}_\alpha(t) = \sum_{n=1}^{[\lambda^{-1}(t)]} a_n + \int_0^{[\lambda^{-1}(t)]} \varrho u a(u) \, du,
\]

since \(\lambda: \mathbb{R}_+ \mapsto \mathbb{R}_+\) is monotone, there exists unique inverse \(\lambda^{-1}\) for the function \(\lambda: \mathbb{R}_+ \mapsto \mathbb{R}_+, \lambda|_{\mathbb{N}} = (\lambda_n)\).
2. \( L_\nu \) as a Neumann series of modified Bessel \( I \) functions

Let us observe the well–known formulæ [44, Eqs. 11.4.18–19–20]

\[
H_\nu(z) = \begin{cases} 
\frac{4}{\sqrt{\pi}(\nu + \frac{1}{2})} \sum_{n \geq 0} \frac{(2n + \nu + 1)\Gamma(n + \nu + 1)}{n!(2n + 1)(2n + 2\nu + 1)} J_{2n+\nu+1}(z) \\
\frac{2}{\nu + \frac{1}{2}} \sum_{n \geq 0} \frac{(-\frac{z}{2})^n}{n! (n + \frac{1}{2})} I_{n+\frac{1}{2}}(z) \\
\frac{1}{\Gamma(\nu + \frac{1}{2})} \sum_{n \geq 0} \frac{(-\frac{z}{2})^n}{n! (n + \nu + \frac{1}{2})} I_{n+\nu+\frac{1}{2}}(z) 
\end{cases}
\]

where the first formula is valid for \(-\nu \notin \mathbb{N}\). So, having in mind that \( L_\nu(z) = -i^\nu H_\nu(iz) \) and \( J_\nu(iz) = i^\nu J_\nu(z) \), we immediately conclude that

\[
L_\nu(z) = \begin{cases} 
\frac{4}{\sqrt{\pi}(\nu + \frac{1}{2})} \sum_{n \geq 0} \frac{(-1)^n(2n + \nu + 1)\Gamma(n + \nu + 1)}{n!(2n + 1)(2n + 2\nu + 1)} J_{2n+\nu+1}(z) \\
\frac{2}{\nu + \frac{1}{2}} \sum_{n \geq 0} \frac{(-\frac{z}{2})^n}{n! (n + \frac{1}{2})} I_{n+\frac{1}{2}}(z) \\
\frac{1}{\Gamma(\nu + \frac{1}{2})} \sum_{n \geq 0} \frac{(-\frac{z}{2})^n}{n! (n + \nu + \frac{1}{2})} I_{n+\nu+\frac{1}{2}}(z) 
\end{cases}
\]  

(2.1)

However, all three series expansions we recognize as Neumann–series built by modified Bessel functions of the first kind. This kind of series have been intensively studied very recently by the authors and D. Jankov in [5]. Exploiting the appropriate findings of that article, we give new integral expressions for the modified Struve function \( \mathcal{L}_\nu \).

First let us modestly generalize [5, Theorem 2.1] which concerns \( \mathfrak{R}_{1,\nu}(x) \), to integral expression for \( \mathfrak{R}_{\mu,\eta}(x) \) defined by (1.2), following the same procedure as in [5].

**Theorem 1.** Let \( \beta \in C^1(\mathbb{R}_+) \), \( \beta|_{\mathbb{N}} = \{\beta_n\}_{n \in \mathbb{N}} \), \( \mu > 0 \) and assume that

\[
(2.2) \quad \lim_{n \to \infty} \left| \beta_n \frac{1}{n^\frac{1}{n}} \right| < \frac{\mu}{\epsilon}.
\]

Then, for \( \mu, \eta \) such that \( \min\{\eta + \frac{3}{2}, \mu + \eta + 1\} > 0 \) and

\[
\left. x \in \left(0, \min\left(1, \left(\frac{e/\mu}{\mu}\right) \lim_{n \to \infty} \frac{n^{-\mu/2} |\beta_n|^{1/n}}{n!} \right)^{-1}\right) \right| = \mathcal{J}_\beta,
\]

we have the integral representation

\[
(2.3) \quad \mathfrak{R}_{\mu,\eta}(x) = -\int_{0}^{\infty} \int_{0}^{[u]} \frac{\partial}{\partial u} \left( \Gamma \left( \mu u + \eta + \frac{1}{2} \right) I_{\mu u + \frac{1}{2}}(x) \right) d\alpha \left( \frac{\beta(s)}{\Gamma(\mu s + \eta + \frac{1}{2})} \right) ds du = \mathcal{J}_\beta,
\]

Proof. The proof is a copy of the proving procedure delivered for [5, Theorem 2.1]. The only exception is to refine the convergence condition upon \( \mathfrak{R}_{\mu,\eta}(x) \). By the bound [3, p. 583]:

\[
I_\nu(x) < \frac{(\frac{x}{2})^\nu}{\Gamma(\nu + 1)} e^{\frac{x^2}{4(\nu + 1)}},
\]

where \( x > 0 \) and \( \nu + 1 > 0 \), we have

\[
|\mathfrak{R}_{\mu,\eta}(x)| < \frac{(\frac{x}{2})^{\mu + \frac{3}{2}}}{\Gamma(\mu + \frac{3}{2})} e^{\frac{x^2}{4(\mu + \frac{3}{2})}} \sum_{n \geq 1} \frac{|\beta_n|}{\Gamma(\mu n + \eta + 1)},
\]
so, the absolute convergence of the right hand side series suffices for the finiteness of \( \mathcal{N}_{\nu,n}(x) \) on \( \mathcal{S}_\beta \). However, condition (2.2) ensures the absolute convergence by the Cauchy convergence criterion.

The remaining part of the proof mimicks the one performed for [5, Theorem 2.1], having in mind that \( \mu = 1 \) reduces Theorem 2 to the ancestor result [5, Theorem 2.1].

\section*{Theorem 2}
If \( \nu > 0 \) and \( x \in (0,2) \), then we have the integral representation

\begin{equation}
\mathbf{L}_\nu(x) = -\frac{2\Gamma(\nu+2)}{\sqrt{\pi} \Gamma(\nu+\frac{3}{2})} \mathcal{I}_{\nu+1}(x)
\end{equation}

(2.4)

\begin{equation}
= \int_1^\infty \int_0^u \frac{\partial}{\partial u} \left( \Gamma(2u+\nu+\frac{3}{2}) \mathcal{I}_{2u+\nu+1}(x) \right) ds \left( \frac{\beta(s)}{\Gamma(2s+\nu+\frac{3}{2})} \right) du ds,
\end{equation}

where

\[ \beta(s) = -\frac{\sqrt{\pi} \Gamma(\nu+\frac{3}{2})}{\sqrt{\pi} \Gamma(\nu+\frac{1}{2})} \Gamma(s+1) \left( s + \frac{1}{2} \right) \left( s + \frac{\nu}{2} \right). \]

\section*{Proof}
Consider the first Neumann sum expansion of \( \mathbf{L}_\nu(x) \) in (2.1), that is

\begin{equation}
\mathbf{L}_\nu(x) = \frac{4}{\sqrt{\pi} \Gamma(\nu+\frac{3}{2})} \sum_{n \geq 0} \frac{(-1)^n (2n+\nu+1) \Gamma(n+\nu+1)}{n!(2n+1)(2n+2\nu+1)} \mathcal{I}_{2n+\nu+1}(x)
\end{equation}

\begin{equation}
= \frac{2\Gamma(\nu+2)}{\sqrt{\pi} \Gamma(\nu+\frac{3}{2})} \mathcal{I}_{\nu+1}(x)
\end{equation}

\begin{equation}
- \frac{4}{\sqrt{\pi} \Gamma(\nu+\frac{3}{2})} \sum_{n \geq 1} \frac{(-1)^{n-1} (2n+\nu+1) \Gamma(n+\nu+1)}{n!(2n+1)(2n+2\nu+1)} \mathcal{I}_{2n+\nu+1}(x).
\end{equation}

Observe that

\[ \mathbf{L}_\nu(x) = \frac{2\Gamma(\nu+2)}{\sqrt{\pi} \Gamma(\nu+\frac{3}{2})} \mathcal{I}_{\nu+1}(x) - \mathcal{N}_{\nu,\nu+1}(x) \]

in which we specify

\[ \beta_n = \frac{(-1)^{n-1} (2n+\nu+1) \Gamma(n+\nu+1)}{\sqrt{\pi} \Gamma(\nu+\frac{3}{2}) \Gamma(n+1) \left( n + \frac{1}{2} \right) \left( n + \frac{\nu}{2} \right)} \]

Since

\[ |\beta_n| \sim \frac{2\Gamma(\nu+2)}{\sqrt{\pi} \Gamma(\nu+\frac{3}{2})} \quad s \to \infty, \]

we deduce (by means of Theorem 1) that (2.4) is valid for \( x \in \mathcal{S}_\beta = (0,2) \).

\section*{Theorem 3}
For \( \nu + 2 > 0 \) and \( x \in (0,2) \) we have the integral representation

\begin{equation}
\mathbf{L}_\nu(x) = \sqrt{\frac{2\pi}{\pi}} \mathcal{I}_{\nu+\frac{3}{2}}(x)
\end{equation}

(2.5)

\begin{equation}
= \int_1^\infty \int_0^u \frac{\partial}{\partial u} \left( \Gamma(u+\nu+1) \mathcal{I}_{u+\nu+\frac{3}{2}}(x) \right) ds \left( \frac{\beta(s)}{\Gamma(s+\nu+1)} \right) du ds,
\end{equation}

where

\[ \beta(s) = -\sqrt{\frac{\pi}{2\pi}} \frac{e^{\nu \pi s} (\pi s)^{\nu-2}}{\Gamma(\nu+\frac{3}{2}) \Gamma(s+1) \left( s + \frac{1}{2} \right)}. \]

\section*{Proof}
Let us observe now the second Neumann sum expansion of \( \mathbf{L}_\nu(x) \) in (2.1):

\[ \mathbf{L}_\nu(x) = \sqrt{\frac{2\pi}{\pi}} \sum_{n \geq 0} \frac{(-1)^n}{n!(n+\frac{1}{2})} \mathcal{I}_{n+\nu+\frac{3}{2}}(x)
\]

\[ = \sqrt{\frac{2\pi}{\pi}} \mathcal{I}_{\nu+\frac{3}{2}}(x) - \sqrt{\frac{2\pi}{\pi}} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!(n+\frac{1}{2})} \mathcal{I}_{n+\nu+\frac{3}{2}}(x). \]
In other words,
\[ L_\nu(x) = \sqrt{\frac{2x}{\pi}} I_{\nu+\frac{1}{2}}(x) - \Re L_{\nu+\frac{1}{2}}(x) \]
in which we specify
\[ \beta(s) = -\sqrt{\frac{x}{2\pi}} \frac{e^{i\pi s} \left( \frac{\pi}{2} \right)^s}{\Gamma(s+1) \left( s + \frac{1}{2} \right)} . \]
The convergence condition (2.2) reduces to the behavior of the auxiliary series
\[ \sum_{n \geq 0} \frac{|\beta_n|}{\Gamma(n + \nu + \frac{1}{2})} \sim \sqrt{\frac{2x}{\pi}} \Gamma(\nu + \frac{1}{2}) F_2 \left( \frac{1}{2}, \nu + \frac{1}{2} \left| \frac{|x|}{2} \right. \right) , \]
which is convergent for all bounded \( x \in \mathbb{C} \), unconditionally upon \( \nu \). Here \( F_2 \) denotes the hypergeometric function defined by series [1, p. 62]
\[ 1F_2 \left( \begin{array}{c} a \\ b_1, b_2 \end{array} \bigg| z \right) = \sum_{n \geq 0} \frac{(a)_n}{(b_1)_n(b_2)_n} \frac{z^n}{n!} . \]
However, for \( \nu > -2 \) we have the integral expression [63, p. 79]
\[ \tag{2.7} L_\nu(z) = \frac{2^{1-\nu} z^{\nu}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1 - t^2)^{\nu-\frac{1}{2}} \cosh(zt) \, dt , \]
where \( z \in \mathbb{C} \) and \( \Re(\nu) > -\frac{1}{2} \). This was used in the proof of the ancestor result (2.3), see [5, Theorem 2.1]. Now, we apply Theorem 1 and conclude that (2.5) is valid for \( x \in J_\beta = (0, 2) \).

The third formula in (2.1) one reduces to the case \( \Re L_{\nu+\frac{1}{2}}(x) \). However, we shall omit the proof, since the slightly repeating derivation procedure used for (2.5) directly gets the desired integral expression.

Theorem 4. If \( \nu + 2 > 0 \) and \( x \in (0, 2) \), then we have the integral representation
\[ L_\nu(x) = \frac{x^{\nu} \sinh x}{2^\nu \sqrt{\pi} \Gamma(\nu + \frac{1}{2})} = \int_1^\infty \int_0^1 \frac{\partial}{\partial s} \left( \Gamma(u + 1) \frac{\beta(s)}{\Gamma(s)} \right) \, ds \, du , \]
where
\[ \beta(s) = \frac{(\frac{\pi}{2})^{\nu+\frac{1}{2}}}{\Gamma(s + \frac{1}{2}) \Gamma(\nu + \frac{1}{2}) \Gamma(s + \nu + \frac{1}{2})} e^{i\pi s} . \]
Now, applying the integral representation (2.7) we derive another integral expression for \( L_\nu(x) \) in terms of hypergeometric functions in the integrand.

Theorem 5. Let \( \nu > -\frac{1}{2} \). Then for \( x > 0 \) we have
\[ L_\nu(x) = \frac{x^{\nu+1} \Gamma(\nu + 2)}{\sqrt{\pi} 2^{\nu+\frac{1}{2}} \Gamma(\nu + \frac{1}{2}) \Gamma(\frac{\nu + 1}{2}) \Gamma(\frac{\nu + 5}{2})} \int_0^1 (1 - t^2)^{\nu+\frac{1}{2}} \cosh(zt) dt , \]
where
\[ 4F_3 \left( \begin{array}{c} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3, b_4, b_5 \end{array} \bigg| z \right) = \sum_{n \geq 0} \frac{\prod_{j=1}^4 (a_j)_n}{\prod_{j=1}^5 (b_j)_n} \frac{z^n}{n!} \]
stands for the generalized hypergeometric function with four upper and five lower parameters.
Proof. Consider the first Bessel function series expansion for $L_\nu(x)$ given in (2.1). Applying mutatis mutandis the integral representation formula (2.7), the Pochhammer symbol technique, the familiar formula $(A)\alpha(n + A) = A(A + 1)n$, $n \in \{0, 1, \ldots\}$, and the duplication formula
\[\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + \frac{1}{2})\]
to the summands, we get the chain of equivalent legitimate transformations:
\[
L_\nu(x) = \frac{8}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \sum_{n \geq 0} \frac{(-1)^n(2n + \nu + 1)\Gamma(n + \nu + 1)}{n!(2n + 1)(2n + 2 + \nu)} \frac{2 (\frac{x}{2})^{2n+\nu+1}}{\sqrt{\pi} \Gamma(2n + \nu + \frac{3}{2})} 
\times \int_0^1 (1 - t^2)^{2n+\nu+\frac{1}{2}} \cosh(xt) \, dt
\]
\[= \frac{4 (\frac{x}{2})^{\nu+1}}{\Gamma(\nu + \frac{1}{2})} \int_0^1 (1 - t^2)^{\nu+\frac{1}{2}} \cosh(xt) \, dt \times \sum_{n \geq 0} \frac{(n + \nu + 1)\Gamma(n + \nu + 1)}{(n + \frac{\nu}{2}) (n + \nu + \frac{1}{2}) \Gamma(2n + \nu + \frac{3}{2})} \frac{(-x^2)^n}{n!} \int_0^1 (1 - t^2)^{\nu+\frac{1}{2}} \cosh(xt) \, dt
\]
\[= \frac{4 (\frac{x}{2})^{\nu+1}}{\Gamma(\nu + \frac{1}{2})} \int_0^1 (1 - t^2)^{\nu+\frac{1}{2}} \cosh(xt) \, dt \times \sum_{n \geq 0} \frac{(\frac{x}{2})^n(n + \nu + 1)\Gamma(2n + \nu + \frac{3}{2})}{(\frac{1}{2} n)(\frac{1}{2} + \frac{1}{2} n)(\nu + \frac{1}{2}) \Gamma(2n + \nu + \frac{3}{2})} \frac{(-x^2)^n}{n!} \int_0^1 (1 - t^2)^{\nu+\frac{1}{2}} \cosh(xt) \, dt,
\]
which proves the assertion. \qed

By virtue of similar manipulations presented above, we conclude the following results.

**Theorem 6.** Let $\nu > -\frac{1}{2}$ and $x > 0$. Then there holds
\[
L_\nu(x) = \begin{cases} \\
\frac{x^{\nu+1}}{\sqrt{\pi} \Gamma(\nu + 1)} \int_0^1 (1 - t^2)\nu \cosh(xt) \, dt \, F_2 \left( \begin{array}{c} \frac{3}{2}, \nu + 1 \\ \frac{1}{2}, \nu + \frac{1}{2} \end{array} \right) \frac{-x^2}{4} (1 - t^2) \right) \, dt \\
\frac{x^{\nu+1}}{\sqrt{\pi} 2^{\nu+\frac{1}{2}} \Gamma(\nu + \frac{1}{2})} \int_0^1 \cosh(xt) \, dt \, F_2 \left( \begin{array}{c} \nu + \frac{1}{2} \\ 1, \nu + \frac{3}{2} \end{array} \right) \frac{-x^2}{4} (1 - t^2) \right) \, dt \\
\end{cases}
\]

The proof of Theorem 6 follows from the same proving procedure as the previous theorem but now considering the second and third series expansion results in (2.1), so we shall omit the proofs of these integral representations.

3. Integrals containing $\Omega(x)$–function and Mathieu series via summation of $\tilde{T}_{\nu, \nu}^L(x)$

By virtue of the Sonin–Gubler formula (1.1) we establish the convergence conditions for the generalized Schlömilch series $\tilde{T}_{\nu, \nu}^L(x)$ and $\tilde{\tilde{T}}_{\nu, \nu}^L(x)$. As for $n$ enough large we have
\[
L_\nu(an) - L_\nu(an) = \frac{2n^{\nu-1}}{\pi} \int_0^\infty J_\nu(ax) \, dx = O(n^{\nu-1}),
\]
we immediately conclude that the following equiconvergences hold true
\[
\tilde{T}_{\nu, \nu}^L(x) \sim \zeta(\mu - \nu + 1), \quad \tilde{\tilde{T}}_{\nu, \nu}^L(x) \sim \eta(\mu - \nu + 1),
\]
that is, $\tilde{T}_{\nu, \nu}^L(x)$ converges for $\mu > \nu > 0$, while $\tilde{\tilde{T}}_{\nu, \nu}^L(x)$ converges for $\mu + 1 > \nu > 0$. On the other hand, we connect $\tilde{T}_{\nu, \nu}^L(x)$ and the Butzer–Flocke–Haus (BFL) complete Omega function [10, Definition 7.1]
\[
\Omega(w) = 2 \int_{0_+}^{\frac{1}{2}} \sinh(wu) \cot(\pi u) \, du, \quad w \in \mathbb{C}.
\]
By the Hilbert transform terminology, $\Omega(w)$ is the Hilbert transform $\mathcal{H}(e^{-wx})_1(0)$ at 0 of the 1-periodic function $(e^{-wx})_1$, defined by the periodic continuation of the following exponential function [10, p. 67]:

$$e^{-xw}, \quad x \in [-\frac{1}{2}, \frac{1}{2}], \quad w \in \mathbb{C},$$

that is,

$$\mathcal{H}(e^{-wx})_1(0) := \text{P.V.} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{uw} \cot(\pi u) \, du \equiv \Omega(w)$$

where the integral is to be understood in the sense of Cauchy’s Principal Value at zero, see e.g. [12, 48].

On the other side by differentiating once (1.1) with respect to $n$ we get a tool to obtain Mathieu series $S(x)$ (introduced by Émile Leonard Mathieu [35]) and its alternating variant $\tilde{S}(x)$ (introduced by Pogány, Srivastava and Tomovski [49]), which are defined as follows

$$S(x) = \sum_{n \geq 1} \frac{2n}{(x^2 + n^2)^2}, \quad \tilde{S}(x) = \sum_{n \geq 1} \frac{2(-1)^{n-1} n}{(x^2 + n^2)^2}.$$

Closed integral expression for $S(r)$ was considered by Emersleben [18] and subsequently by Elbert [17], while for $\tilde{S}_\mu(x)$ integral representation has been given by Pogány, Srivastava and Tomovski [49]:

$$S(x) = \frac{1}{x} \int_0^\infty \frac{t \sin(xt)}{e^t - 1} \, dt,$$

(3.2)

$$\tilde{S}(x) = \frac{1}{x} \int_0^\infty \frac{t \sin(xt)}{e^t + 1} \, dt.$$  

(3.3)

Another kind integral expressions for underlying Mathieu series can be found in [49].

**Theorem 7.** Assume that $\Re(\nu) > 0$ and $a > 0$. Then we have

$$\int_0^\infty \frac{J_\nu(ax) \Omega(2\pi x)}{x^\nu \sinh(\pi x)} \, dx = \nu \int_0^\infty e^{-\nu t} \int_0^{[e^t]} \vartheta_a(e^{i\pi u}(L_\nu(au) - I_\nu(au))) \, du \, dt.$$  

**Proof.** When we multiply (1.1) by $(-1)^{n-1} n$ and sum up all three series with respect to $n \in \mathbb{N}$, the following partial-fraction representation of the Omega function [10, Theorem 1.3]

$$\pi \Omega(2\pi w) / \sinh(\pi w) = \sum_{n=1}^\infty \frac{2(-1)^{n-1} n}{n^2 + w^2},$$

immediately give

$$\int_0^\infty \frac{J_\nu(ax) \Omega(2\pi x)}{x^\nu \sinh(\pi x)} \, dx = \sum_{n \geq 1} (-1)^{n-1} n^{-\nu} (I_\nu(an) - L_\nu(an)) = \mathfrak{M}_\nu[a](a).$$

We recognize the right-hand-side sums as Dirichlet series of $I_\nu$ and $L_\nu$, respectively. Being

$$\sum_{n \geq 1} (-1)^{n-1} n^{-\nu} I_\nu(an) = \sum_{n \geq 1} e^{i\pi(n-1)} I_\nu(an) e^{-\nu \ln n}, \quad \Re(\nu) > 0,$$

we get

$$\sum_{n \geq 1} (-1)^{n-1} n^{-\nu} I_\nu(an) = \nu \int_0^\infty e^{-\nu t} \sum_{n \in \mathbb{N}, n \leq t} e^{i\pi(n-1)} I_\nu(an) \, dt.$$ 

So, making use of the Euler–Maclaurin summation (1.6) to the Cahen’s formula we deduce

$$\sum_{n \geq 1} (-1)^{n-1} n^{-\nu} I_\nu(an) = -\nu \int_0^\infty \int_0^{[e^t]} e^{-\nu t} \vartheta_a(e^{i\pi u} I_\nu(au)) \, du \, dt;$$

and repeating the procedure to the second Dirichlet series containing $L_\nu(an)$, the proof is complete. \( \square \)

The next result concerns a hypergeometric integral, which one we integrate by means of Schlömilch series of modified Bessel and modified Struve functions.
Theorem 8. Let \( \Re(\nu) > 0 \) and \( a > 0 \). Then we have
\[
\int_0^\infty J_\nu(ax) S(x) \frac{dx}{x^{\nu}} = \frac{\sqrt{\pi} a^{\nu+2}}{2^{\nu+1} \Gamma(\nu + \frac{1}{2})} \int_0^1 \frac{t^2}{e^{at} - 1} \, 2F_1 \left( \frac{1}{2}, \frac{1}{2} - \nu ; \frac{t^2}{a^2} \right) dt
\]
\[
+ \pi a^\nu [Li_2(e^{-a}) + a \, Li_1(e^{-a})] \frac{1}{2^{\nu+1} \Gamma(\nu + 1)},
\]
where \( Li_\alpha(z) \) stands for the dilogarithm function.

Proof. Differentiating (1.1) with respect to \( n \), we get
\[
\int_0^\infty \frac{2a J_\nu(ax)}{(x^2 + a^2)^{\frac{1}{2}}} \frac{dx}{x^{\nu+1}} = \frac{\pi(\nu + 1)}{2^{\nu+2}} (I_\nu(an) - L_\nu(an)) - \frac{a\pi}{2^{\nu+1}} (I'_\nu(an) - L'_\nu(an)) .
\]
Summing up this relation with respect to positive integers \( n \in \mathbb{N} \), we get
\[
\kappa(\alpha) := \int_0^\infty J_\nu(ax) S(x) \frac{dx}{x^{\nu}} = \frac{\pi(\nu + 1)}{2} \sum_{n \geq 1} n^{-\nu-2} (I_\nu(an) - L_\nu(an))
\]
\[
- \frac{a\pi}{2} \sum_{n \geq 1} n^{-\nu-1} (I'_\nu(an) - L'_\nu(an)) = \frac{\pi(\nu + 1)}{2} \frac{T_{\nu,\nu+2}(a)}{2^{\nu+2}}(a) - \frac{a\pi}{2} \frac{d}{da} T_{\nu,\nu+2}(a).
\]
Now, by the Emersleben–Elbert formula (3.2) we conclude that
\[
\kappa(\alpha) = \int_0^\infty J_\nu(ax) S(x) \frac{dx}{x^{\nu}} = \int_0^\infty \frac{t}{e^{\nu} - 1} \left( \int_0^\infty J_\nu(ax) \sin(xt) \frac{dx}{x^{\nu+1}} \right) dt .
\]
Expressing the sine via \( J_{\frac{1}{2}} \), we get that the inner–most integral equals
\[
(3.4) \quad \int_0^\infty J_\nu(ax) \sin(xt) \frac{dx}{x^{\nu+1}} = \frac{\sqrt{\pi}}{2} \int_0^\infty J_\nu(ax) J_{\frac{1}{2}}(tx) \frac{dx}{x^{\nu+\frac{1}{2}}} .
\]
Now, we shall apply the Weber–Sonin–Schafheitlin formula [63, §13.41] for \( \lambda = \nu + \frac{1}{2} \), which reduces to
\[
\int_0^\infty J_\nu(ax) J_{\frac{1}{2}}(tx) x^{-\nu-\frac{1}{2}} \, dx = \begin{cases} \frac{a^\nu \sqrt{\pi}}{2^{\nu+1} \sqrt{T(\nu + 1)}} & \text{if } 0 < a \leq t \\ \frac{a^{\nu-1} \sqrt{T}}{2^{\nu+1} \Gamma(\nu + \frac{1}{2})} 2F_1 \left( \frac{1}{2}, \frac{1}{2} - \nu ; \frac{t^2}{a^2} \right) & \text{if } 0 < t < a . \end{cases}
\]
Accordingly, (3.4) becomes
\[
\kappa(\alpha) = \frac{\sqrt{\pi} a^{\nu-1}}{2^{\nu+1} \Gamma(\nu + \frac{1}{2})} \int_0^a \frac{t^2}{e^{at} - 1} 2F_1 \left( \frac{1}{2}, \frac{1}{2} - \nu ; \frac{t^2}{a^2} \right) dt + \frac{\pi a^\nu}{2^{\nu+1} \Gamma(\nu + 1)} \int_a^\infty \frac{t}{e^{\nu} - 1} dt
\]
\[
= \frac{\sqrt{\pi} a^{\nu+2}}{2^{\nu+1} \Gamma(\nu + \frac{1}{2})} \int_0^1 \frac{t^2}{e^{at} - 1} 2F_1 \left( \frac{1}{2}, \frac{1}{2} - \nu ; \frac{t^2}{a^2} \right) dt + \frac{\pi a^\nu}{2^{\nu+1} \Gamma(\nu + 1)} \int_0^\infty \frac{t + a}{e^{\nu a} - 1} dt
\]
\[
= \frac{\sqrt{\pi} a^{\nu+2}}{2^{\nu+1} \Gamma(\nu + \frac{1}{2})} \int_0^1 \frac{t^2}{e^{at} - 1} 2F_1 \left( \frac{1}{2}, \frac{1}{2} - \nu ; \frac{t^2}{a^2} \right) dt
\]
\[
+ \frac{\pi a^\nu}{2^{\nu+1} \Gamma(\nu + 1)} [Li_2(e^{-a}) + a \, Li_1(e^{-a})] ,
\]
where the dilogarithm \( Li_\alpha(z) = \sum_{n \geq 1} z^n n^{-\alpha} , \, |z| \leq 1 \), has the integral representation
\[
Li_\alpha(z) = z \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1}}{e^t - z} dt , \quad \Re(\alpha) > 0.
\]
This completes the proof. \( \square \)
4. Differential equations for Kapteyn and Schlömilch series of modified Bessel and modified Struve functions

Kapteyn series of Bessel functions were introduced by Willem Kapteyn [32, 33], and were considered and discussed in details by Nielsen [41] and Watson [63], who devoted a whole section of his celebrated monograph to this theme. Recently, the present authors and Jankov obtained integral representation and ordinary differential equations descriptions and related results for real variable Kapteyn series [4, 30].

Now, we will consider the Kapteyn series built by modified Bessel functions of the first kind, and modified Struve functions

\[ R_{\alpha,\mu}(x) = \sum_{n \geq 1} \frac{\alpha_n}{\nu^n} (I_{\nu n}(x) - L_{\nu n}(x)), \]

where the parameter space includes positive \( \alpha > 0 \), while sequence \((\alpha_n)_{n \geq 1}\) ensures the convergence of \( R_{\alpha,\mu}(x) \). Our first goal is to establish double definite integral representation formula for \( R_{\alpha,\mu}(x) \). In this goal we recall the definition of the confluent Fox-Wright generalized hypergeometric function \( _1\Psi^*_1 \) (for the general case \( \rho \Psi^*_\rho \) consult [55, p. 493]):

\[
_1\Psi^*_1 \left[ \begin{array}{c} (a, \rho) \\ (b, \sigma) \end{array} \bigg| z \right] = \sum_{n=0}^{\infty} \frac{(a)_\rho}{n!} \frac{z^n}{b_\sigma},
\]

where \( a, b, \rho, \sigma > 0 \) and where, as usual, \((\lambda)_\mu\) denotes the Pochhammer symbol defined, in terms of Euler’s Gamma function, by

\[
(\lambda)_\mu := \frac{\Gamma(\lambda + \mu)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } \mu = 0; \lambda \in \mathbb{C} \setminus \{0\}, \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1), & \text{if } \mu = n \in \mathbb{N}; \lambda \in \mathbb{C}. \end{cases}
\]

The defining series in (4.1) converges in the whole complex \( \mathbb{C} \)-plane when \( \Delta = \sigma - \rho + 1 > 0 \); if \( \Delta = 0 \), then the series converges for \(|z| < \sqrt{\nabla}\), where \( \nabla := \rho - \rho' \rho' \).

**Theorem 9.** Let \( \mu > \nu > 0 \) and let \( \alpha \in C^2(\mathbb{R}_+), \) such that \( |\alpha|_N = (\alpha_n) \). Then for

\[
x \in \left( 0, 2 \min \left\{ 1, \frac{\nu}{\nu \limsup_{n \to \infty} |\alpha_n|^{1/\nu}} \right\} \right) := \mathcal{J}_\alpha,
\]

we have

\[
R_{\nu,\mu}(x) = -\int_1^\infty \int_0^1 \frac{\partial}{\partial t} \frac{\Gamma(\nu t + \frac{1}{2})}{\Gamma(\nu t)} \left( \frac{x}{2} \right)^{\nu t} \frac{1}{\sqrt{t}} _1\Psi^*_1 \left[ \begin{array}{c} \left( \frac{1}{2}, \frac{1}{2} \right) \\ \nu \end{array} \bigg| -xt \right] \cdot \partial_s \alpha(s)s^{\nu s - \mu} \frac{\Gamma(\nu s + \frac{1}{2})}{\Gamma(\nu s + \frac{1}{2})} \, ds \, dt.
\]

**Proof.** The Sonin–Gubler formula enables us to transform the summands of the Kapteyn series \( R_{\nu,\mu}(x) \) into

\[
R_{\nu,\mu}(x) = \frac{2}{\pi} \int_0^\infty \sum_{n \geq 1} \frac{\alpha_n}{\nu^{\nu n}} \frac{J_{\nu n}(xy)}{(y^2 + \nu^2)} y^{\nu n} \, dy.
\]

Making use of the Gegenbauer’s integral expression for \( J_\alpha \) [1, p. 204, Eq. (4.7.5)], after some algebra we get

\[
R_{\nu,\mu+1}(x) = \frac{1}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{1-t^2}} \left\{ \sum_{n \geq 1} \frac{\alpha_n}{\nu^{\nu n} \Gamma(\nu n + \frac{1}{2})} \int_0^{\infty} \frac{\cos(xy)}{y^2 + \nu^2} \, dy \right\} \, dt \]
\[
= \frac{2}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{1-t^2}} \left\{ \sum_{n \geq 1} \frac{\alpha_n}{\nu^{\nu n+1} \Gamma(\nu n + \frac{1}{2})} \right\} \, dt = \frac{2}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{1-t^2}} \mathcal{S}_\alpha(t) \, dt,
\]
where the inner sum is evidently the following Dirichlet series
\[
\mathcal{D}_n(t) = \sum_{n \geq 1} \frac{\alpha_n \exp\left\{ -n \left( xt + \nu \ln \frac{2}{x(1-t^2)} \right) \right\}}{n^{\mu-\nu n+1} \Gamma(\nu n + \frac{1}{2})},
\]
and \( p(t) = xt + \nu \ln \frac{2}{x(1-t^2)} > 0 \) for \( x \in (0, 2) \), since \( p \) is increasing on \((0, 1)\). By the Cauchy convergence test applied to \( \mathcal{D}_n(t) \) we deduce that
\[
\left( \frac{\alpha(t)}{2\nu} \right)^{\nu} e^{-xt} \limsup_{n \to \infty} |a_n|^{1/n} \leq \left( \frac{\epsilon x}{2\nu} \right)^{\nu} \limsup_{n \to \infty} |a_n|^{1/n} < 1,
\]
that is, for all \( x \in \mathcal{A} \), the series converges absolutely and uniformly. By the Cahen formula (1.6) we have
\[
\mathcal{D}_n(t) = \ln e^{xt} \left( \frac{2}{x(1-t^2)} \right)^{\nu} \int_0^\infty \int_0^{[z]} \left( \left( \frac{x}{2} (1-t^2) \right)^{\nu} e^{-xt} \right)^{z} \cdot \partial_s \frac{\alpha(s) s^{\nu s-\mu-1}}{\Gamma(\nu s + \frac{1}{2})} dz ds.
\]
Thus
\[
\mathcal{R}_{\nu,\mu+1}(x) = -\frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^{[z]} \partial_s \frac{\alpha(s) s^{\nu s-\mu-1}}{\Gamma(\nu s + \frac{1}{2})} \Phi_e(z) dz ds,
\]
where the \( t \)-integral
\[
\Phi_e(z) = \int_0^1 \frac{\ln e^{-xt} \left( \frac{\pi}{2} (1-t^2) \right)^{\nu}}{\sqrt{1-t^2}} \left( \left( \frac{x}{2} (1-t^2) \right)^{\nu} e^{-xt} \right)^{z} dt
\]
has to be evaluated. After indefinite integration, under definite integral, expanding the exponential term into Maclaurin series, legitimate termwise integration leads to
\[
\int \Phi_e(z) dz = \left( \frac{x}{2} \right)^{\nu z} \int_0^1 \left( 1-t^2 \right)^{\nu z-\frac{1}{2}} e^{-xzt} dt
\]
\[
= \sqrt{\pi} \Gamma(\nu z + \frac{1}{2}) \frac{\left( \frac{2}{2} \right)^{\nu z} \sum_{j=0}^\infty \frac{\left( \frac{1}{2} \right)^{j} (-x)^j}{j!}}{2 \Gamma(\nu z)}
\]
\[
= \sqrt{\pi} \Gamma(\nu z + \frac{1}{2}) \frac{\left( \frac{2}{2} \right)^{\nu z}}{2 \Gamma(\nu z)} \Psi_{1}^\nu \left( \left( \frac{1}{2}, \frac{1}{2} \right) \right)^{-xz}.
\]
Consequently
\[
\Phi_e(z) = \sqrt{\pi} \frac{\partial}{\partial z} \frac{\Gamma(\nu z + \frac{1}{2})}{\Gamma(\nu z)} \left( \frac{x}{2} \right)^{\nu z} \Psi_{1}^\nu \left( \left( \frac{1}{2}, \frac{1}{2} \right) \right)^{-xz},
\]
and thus
\[
\mathcal{R}_{\nu,\mu+1}(x) = -\int_1^\infty \int_0^{[t]} \frac{\partial}{\partial t} \frac{\Gamma(\nu t + \frac{1}{2})}{\Gamma(\nu t)} \left( \frac{x}{2} \right)^{\nu t}
\]
\[
\times \Psi_{1}^\nu \left( \left( \frac{1}{2}, \frac{1}{2} \right) \right)^{-xt} \cdot \partial_s \frac{\alpha(s) s^{\nu s-\mu-1}}{\Gamma(\nu s + \frac{1}{2})} dt ds.
\]
The proof is complete. \( \square \)

Now, our goal is to establish a second order nonhomogeneous ordinary differential equation which particular solution is the above introduced special kind Kaptyn series (1.3). Firstly, we introduce the modified Bessel type differential operator
\[
M[y] \equiv y'' + \frac{1}{x} y' - \left( 1 + \frac{\nu^2}{x^2} \right) y;
\]
this operator is associated with the modified Struve differential equation, reads as follows
\[
(4.3) \quad M[y] \equiv y'' + \frac{1}{x} y' - \left( 1 + \frac{\nu^2}{x^2} \right) y = \frac{\left( \frac{x}{2} \right)^{\nu-1}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})}.
\]
Theorem 10. Let \( \min(\nu, \mu) > 0 \). Then for \( x \in \mathcal{I}_\alpha \) the Kapteyn series \( R = R_{\nu, \mu}(x) \) is a particular solution of the nonhomogeneous linear second order ordinary differential equation

\[
M_\mu''[R] \equiv R'' + \frac{1}{x} R' - \left( 1 + \frac{\nu^2}{x^2} \right) R = \frac{1}{x} \Xi_{\nu, \mu}(x) + \frac{2}{x \sqrt{\pi}} \sum_{n \geq 1} \frac{\alpha_n (\frac{x}{2})^{\nu n}}{(\nu n + \frac{1}{2})^{n\nu + \nu + 1}},
\]

where

\[
\Xi_{\nu, \mu}(x) = \frac{d}{dx} \int_0^\infty \int_1^\infty \left( \frac{\partial}{\partial t} \frac{1}{\Gamma(\nu t)} \left( \frac{x}{t} \right)^\nu \right) \frac{\Gamma(\nu t + \frac{1}{2})}{\Gamma(\nu t)} \cdot 1 \Psi_1 \left( \frac{1}{\nu t}, 1 \right) \cdot 1 \Psi_1 \left( \frac{1}{\nu t}, 1 \right) \, dt \, ds.
\]

Proof. Consider the modified Struve differential equation (4.3)

\[
M[y] \equiv y'' + \frac{1}{x} y' - \left( 1 + \frac{\nu^2}{x^2} \right) y = \frac{\nu}{x^2} \sum_{n \geq 1} \frac{\alpha_n (\frac{x}{2})^{\nu n}}{(\nu n + \frac{3}{2})^{n\nu + \nu + 1}},
\]

which possesses the solution \( y(x) = c_1 I_\nu(x) + c_2 L_\nu(x) + c_3 K_\nu(x) \), where \( K_\nu \) stands for the modified Bessel function of the second kind of order \( \nu \). Being \( I_\nu \) and \( K_\nu \) independent particular solutions (the Wronskian \( W[I_\nu, K_\nu] = -x^{-1} \)) of the homogeneous modified Bessel ordinary differential equation, which appears on the left side in (4.3), the choice \( c_3 = 0 \) is legitimate. Thus \( y(x) = I_{\nu n}(x) - L_{\nu n}(x) \) is also a particular solution of (4.3). Setting \( \nu \to \nu n \), we get

\[
(I_{\nu n}(x) - L_{\nu n}(x))'' + \frac{1}{x} (I_{\nu n}(x) - L_{\nu n}(x))' - \left( 1 + \frac{\nu^2}{x^2} \right) (I_{\nu n}(x) - L_{\nu n}(x)) = \frac{(\frac{x}{2})^{\nu n-1}}{\sqrt{\pi} \Gamma(\nu n + \frac{1}{2})}.
\]

Finally, putting \( x \to xn \) multiplying the above display with \( n^{-\mu} \alpha_n \) and summing up in \( n \in \mathbb{N} \), we obtain

\[
M[R_{\nu, \mu}] = M_\mu''[R] = \frac{1}{x} \left( \frac{\nu}{x^2} \sum_{n \geq 1} \frac{\alpha_n (\frac{x}{2})^{\nu n}}{(\nu n + \frac{3}{2})^{n\nu + \nu + 1}} \right) + \frac{2}{x \sqrt{\pi}} \sum_{n \geq 1} \frac{\alpha_n (\frac{x}{2})^{\nu n}}{(\nu n + \frac{1}{2})^{n\nu + \nu + 1}},
\]

where all three right-hand side series converge uniformly inside \( \mathcal{I}_\alpha \). Applying the result (4.2) of the previous theorem to the series

\[
R_{\nu, \mu}(x) - R_{\nu, \mu+1}(x) = \sum_{n \geq 2} \frac{\alpha_n (n-1)}{n^{\mu+1}} (I_{\nu n}(xn) - L_{\nu n}(xn)),
\]

the summation begins with 2. So, the current lower integration limit in the Euler-Maclaurin summation formula related to (4.2) becomes 1. By this we clarify the stated relation (4.4). \( \square \)

In the following we concentrate on the summation of Schlömilch series

\[
\mathcal{T}_{\nu, \mu}(x) := \sum_{n \geq 1} \frac{1}{n^{\mu}} (I_{\nu}(nx) - L_{\nu}(nx))
\]

\[
\mathcal{\Xi}_{\nu, \mu}(x) := \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^{\mu}} (I_{\nu}(nx) - L_{\nu}(nx)).
\]

To unify these procedures, we consider the generalized Schlömilch series like (1.5)

\[
\mathcal{G}_{\nu, \mu}(x) := \sum_{n \geq 1} \frac{\alpha_n}{n^{\mu}} (I_{\nu}(xn) - L_{\nu}(xn));
\]

obviously \( \mathcal{T}_{\nu, \mu}(x), \mathcal{\Xi}_{\nu, \mu}(x) \) are special cases of \( \mathcal{G}_{\nu, \mu}(x) \). However, bearing in mind the asymptotics in Sonin–Gubler formula (3.1), we see that the necessary condition for the convergence of \( \mathcal{G}_{\nu, \mu}(x) \) for a fixed \( x > 0 \) becomes \( \alpha_n = o(n^{\nu+\nu+1}) \) as \( n \to \infty \).
Theorem 11. Let \( \min(\nu, \mu, x) > 0 \) and \( \alpha \in C^1(\mathbb{R}_+) \) be monotone increasing, such that \( \alpha \mid_n = (\alpha_n)_{n \geq 1} \), and that \( \sum_{n \geq 1} n^{-\mu + \nu + 1} \alpha_n \) converges. Then \( \mathfrak{D} = \mathfrak{D}^{1, L}_n(x) \) is a particular solution of the nonhomogeneous linear second order ordinary differential equation

\[
M^{\alpha}_\mu[\mathfrak{D}] = M[\mathfrak{D}^{1, L}_n(x)] = \frac{1}{x} \left( \mathcal{T}^{\alpha+1}_\mu(x) \right)' - \frac{\mu^2}{x^2} \mathcal{T}^{\alpha+2}_\nu(x) + \frac{(x/\mu - 1)}{\sqrt{\pi} \Gamma(\nu + 1/2)} \sum_{n \geq 1} \alpha_n n^{-\mu - \nu + 1},
\]

where

\[
\mathcal{T}^{\alpha+\beta}_\mu(x) = \mu \int_0^\infty e^{-\mu t} \int_1^{[e^t]} \mathcal{D}_u (\alpha(u)(u^{\beta} - 1)(I_{\nu}(xu) - L_{\nu}(xu))) \, dt \, du.
\]

Proof. Consider again the modified Struve differential equation (4.3), which possesses the solution \( y(x) = c_1 I_{\nu}(x) + c_2 L_{\nu}(x) + c_3 K_{\nu}(x) \), and choose the particular solution associated with \( c_1 = -c_2 = 1 \) and \( c_3 = 0 \). Transforming (4.3) putting \( x \mapsto xn \), multiplying it by \( \alpha_n n^{-\mu} \) and summing up the equation with respect \( n \in \mathbb{N} \), we arrive at

\[
\left( \sum_{n \geq 1} \frac{\alpha_n}{n^{\mu + 1}} y(xn) \right)'' + \frac{1}{x} \left( \sum_{n \geq 1} \frac{\alpha_n}{n^{\mu + 1}} y(xn) \right)' - \sum_{n \geq 1} \frac{\alpha_n}{n^{\mu + 1}} y(xn) - \frac{\mu^2}{x^2} \sum_{n \geq 1} \frac{\alpha_n}{n^{\mu + 2}} y(xn) = \frac{x/\mu - 1}{\sqrt{\pi} \Gamma(\nu + 1/2)} \sum_{n \geq 1} \alpha_n n^{-\mu - \nu + 1}.
\]

Thus

\[
M[\mathfrak{D}^{1, L}_n(x)] = \frac{1}{x} \left( \sum_{n \geq 2} \frac{\alpha_n(n-1)}{n^{\mu + 1}} y(xn) \right)' - \frac{\mu^2}{x^2} \sum_{n \geq 2} \frac{\alpha_n(n^2-1)}{n^{\mu + 2}} y(xn) + \frac{x/\mu - 1}{\sqrt{\pi} \Gamma(\nu + 1/2)} \sum_{n \geq 1} \alpha_n n^{-\mu - \nu + 1}.
\]

Denote

\[
\mathcal{T}^{\alpha+\beta}_\mu(x) = \sum_{n \geq 2} \frac{\alpha_n(n-1)}{n^{\mu + 1}} y(xn), \quad 0 < \nu \leq \mu, x > 0.
\]

Following the same lines of the proof of Theorem 7, by Cahen’s formula and the Euler–Maclaurin summation we immediately yield the double definite integral representation

\[
\mathcal{T}^{\alpha+\beta}_\mu(x) = \mu \int_0^\infty e^{-\mu t} \int_1^{[e^t]} \mathcal{D}_u (\alpha(u)(u^{\beta} - 1)y(xu)) \, dt \, du;
\]

which immediately lead to the stated result. \(\square\)

Corollary 1. Let \( \mu - 1 > \nu > 0 \) and \( x > 0 \). Then \( \mathfrak{S} = \mathfrak{S}^{1, L}_\nu(x) \) is a particular solution of the nonhomogeneous linear second order ordinary differential equation

\[
M[\mathfrak{S}] = \frac{1}{x} \left( \mathcal{T}^{1+1}_\mu(x) \right)' - \frac{\mu^2}{x^2} \mathcal{T}^{1+2}_\nu(x) + \frac{\mu \nu (\nu + 1)}{\sqrt{\pi} \Gamma(\nu + 1/2)} \left( \frac{x}{\mu} \right)^{\nu - 1},
\]

where

\[
\mathcal{T}^{1+\beta}_\mu(x) = \mu \int_0^\infty e^{-\mu t} \int_1^{[e^t]} \mathcal{D}_u ((u^{\beta} - 1)(I_{\nu}(xu) - L_{\nu}(xu))) \, dt \, du.
\]

Corollary 2. Let \( \mu > \nu > 0 \) and \( x > 0 \). Then \( \mathfrak{S} = \mathfrak{S}^{1, L}_\nu(x) \) is a particular solution of the nonhomogeneous linear second order ordinary differential equation

\[
M^{\alpha}_\mu[\mathfrak{S}] = M[\mathfrak{S}^{1, L}_\nu(x)] = \frac{1}{x} \left( \mathcal{T}^{1+1}_\mu(x) \right)' - \frac{\mu^2}{x^2} \mathcal{T}^{1+2}_\nu(x) + \frac{\eta (\mu - \nu + 1)}{\sqrt{\pi} \Gamma(\nu + 1/2)} \left( \frac{x}{\mu} \right)^{\nu - 1},
\]
If Theorem 12. simplifies the nonhomogeneous part of related differential equation (4.7). which, in the expanded form reads has been considered by Hamburger [24, p. 130, (C)] in the slightly different form where Actually, the formula Remark 1. This arises in a Fredholm type convolutional integral equation of the first kind with degenerate kernel having nonhomogeneous partObviously, $f$ and $g$ are orthogonal a.e. with respect to the ordinary Lebesgue measure on the positive half-line when $\int_0^\infty f(x)g(x)dx$ vanishes, writing this as $f \perp g.$
Theorem 13. Let \( \nu > 0 \) and \( x > 0 \). The first kind Fredholm type convolutional integral equation with degenerate kernel (5.1) possesses particular solution \( f = J_\nu + h \), where \( h \in L^1(\mathbb{R}_+) \) and
\[
h(x) \perp x^{-\nu-1} \left( \coth(\pi x) - \frac{1}{\pi x} \right), \quad x > 0
\]
if and only if the nonhomogeneous part of the integral equation equals \( F_\nu(x) \) given by (5.2).

We mention that \( h \) as in the above theorem has been constructed in [16, Example]. To solve the integral equation (5.2) we use the Mellin integral transform technique, following some lines of a similar procedure used by Drašić–Pogany in [16]. The Mellin transform pair of certain suitable integral equation (5.2) we use the Mellin integral transform technique, following some lines of a similar fundamental strip of the inverse Mellin transform
\[
M
\]
if and only if the nonhomogeneous part of the integral equation equals \( F_\nu(x) \) given by (5.2).

Theorem 14. Let \( \nu > 0, x > 0 \). Then the following Bromwich–Wagner type line integral representation holds true
\[
J_\nu(x) = \frac{\nu + 1}{2i} \int_{c-i\infty}^{c+i\infty} M_p \left( \int_0^{\infty} \int_0^{[e^s]} e^{-(\nu+1)t} \sigma_u (I_\nu(uxu) - L_\nu(uxu)) \, ds \, du \right) x^{p-1} \, dp,
\]
where \( c \in (\nu, \nu + 1) \).

Proof. Applying \( M^{-1} \) to the equation (5.1), we get
\[
M \left( \int_0^{\infty} \int_0^{[e^s]} e^{-(\nu+1)t} \sigma_u (I_\nu(uxu) - L_\nu(uxu)) \, ds \, du \right) x^{p-1} \, dp = M_p(F_\nu).
\]
By the Mellin–convolution property
\[
M_p(f \ast g) = M_p \left( \int_0^{\infty} f(rt) \cdot g(t) \, dt \right) = M_p(f) \cdot M^{-1}_{1-p}(g),
\]
it follows that
\[
M_p \left( x^{-\nu-1}(\coth \pi x - (\pi x)^{-1}) \right) \cdot M^{-1}_{1-p}(J_\nu) = M_p(F_\nu).
\]
The fundamental analytic strip contains \( (\nu, 1+\nu) \), because the \( \coth \) behaves like
\[
\coth z = \begin{cases} \frac{1}{z} + \frac{z}{3} + O(z^3), & \text{if } z \to 0 \\ 1 + 2(e^{-2z} - 1) \frac{1}{1 + O(e^{-2z})}, & \text{if } z \to \infty \end{cases}
\]
in both cases \( \Re(z) > 0 \). Now, rewriting \( x^{-\nu-1}(\coth \pi x - (\pi x)^{-1}) \) by (C) and using termwise the Beta–function description, we conclude that
\[
M_p \left( x^{-\nu-1}(\coth \pi x - (\pi x)^{-1}) \right) = \frac{1}{\pi} B \left( \frac{p - \nu}{2}, \frac{\nu - p}{2} + 1 \right) \zeta(\nu - p + 2),
\]
for all \( p \in (\nu, \nu + 1) \). Therefore
\[
M_{1-p}(J_\nu) = \frac{\pi M_p(F_\nu)}{B \left( \frac{p - \nu}{2}, \frac{\nu - p}{2} + 1 \right) \zeta(\nu - p + 2)},
\]
which finally results in
\[
J_\nu(x) = \frac{\nu + 1}{2i} \int_{c-i\infty}^{c+i\infty} M_p \left( \int_0^{\infty} \int_0^{[e^s]} e^{-(\nu+1)t} \sigma_u (I_\nu(uxu) - L_\nu(uxu)) \, ds \, du \right) x^{p-1} \, dp,
\]
where the fundamental strip contains \( c = \nu + \frac{1}{2} \). So, the desired integral representation formula is established. \( \square \)
We note that the formula–collection [25] does not contain (5.3).

**Theorem 15.** Let $0 < \nu < \frac{3}{2}, x > 0$. Then

$$
\xi_{\nu,\nu+1}^{L}(x) = \frac{1}{2p^2 + 1} \int_{-\infty}^{\infty} \frac{\Gamma \left( \frac{\nu-p}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\nu-p}{2} + 1 \right)}{\sin \left( \frac{\pi}{2} (p-\nu) \right) \cdot \Gamma \left( \frac{\nu+1}{2} + \frac{1}{2} \right)} x^{-p} dp.
$$

Here also $c \in (\nu, \nu+1)$.

**Proof.** Consider relation (5.4). Exchanging $\mathcal{M}_{p}(J_{\nu}(ax))$ via formula [43, p.93, Eq. 10.1]

$$
\mathcal{M}_{p}(J_{\nu}(ax)) = \frac{2^{\nu-1}}{a^{\nu \pi}} \frac{\Gamma \left( \frac{\nu+\mu}{2} \right)}{\Gamma \left( \frac{\nu+\mu}{2} + 1 \right)}, \quad a > 0, \quad -\nu < p < \frac{3}{2},
$$

equality(5.4), by virtue of the Euler’s reflection formula becomes

$$
\mathcal{M}_{p}(F_{\nu}) = \frac{\Gamma \left( \frac{\nu-p}{2} + 1 \right) \Gamma \left( \frac{\nu+\mu}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\nu-p}{2} + \frac{1}{2} \right)}{2^{\nu} \pi \Gamma \left( \frac{\nu+1}{2} + 1 \right)}
$$

$$
= \frac{\Gamma \left( \frac{\nu-p}{2} + \frac{1}{2} \right) \Gamma (\nu - p + 2)}{2^{\nu} \sin \left( \frac{\pi}{2} (p-\nu) \right) \cdot \Gamma \left( \frac{\nu+1}{2} + \frac{1}{2} \right)}.
$$

Having in mind that $F_{\nu}(x)$ is the integral representation of the Schlömilch series $\xi_{\nu,\nu+1}^{L}(x)$, inverting the last display by $\mathcal{M}_{p}^{-1}$ we arrive at the asserted result. \hfill \Box

**Acknowledgements**

The authors would like to cordially thank anonymous referee for his/her constructive comments and suggestions. The research of Á. Baricz was supported by the Romanian National Research Council CNCS–UEFISCSU, project number PN-II-RU-TE_190/2013.

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