MOMENTS OF THE WEIGHTED CANTOR MEASURES

STEVEN N. HARDING AND ALEXANDER W. N. RIASANOVSKY

Abstract. Based on the seminal work of Hutchinson [5], we investigate properties of \( \alpha \)-Cantor measures whose support is a fractal contained in the unit interval. Here, \( \alpha \) is a vector of nonnegative weights summing to 1, and \( \mu^\alpha \) is the unique Borel probability measure on \([0, 1]\) such that
\[
\mu^\alpha(E) = \sum_{n=0}^{N-1} \alpha_n \mu^\alpha(\varphi_n^{-1}(E))
\]
where \( \varphi_n : x \mapsto (x + n)/N \). In Sections 1 and 2 we examine several general properties of the measure \( \mu^\alpha \) and the associated Legendre polynomials in \( L_2^\alpha[0, 1] \). In Section 3, we (1) compute the Laplacian and moment generating function of \( \mu^\alpha \), (2) characterize precisely when the moments \( I_m = \int_{[0,1]} x^m \, d\mu^\alpha \) exhibit either polynomial or exponential decay, and (3) describe an algorithm which estimates the first \( m \) moments within uniform error \( \varepsilon \) in \( O((\log \log(1/\varepsilon)) \cdot m \log m) \). We also state analogous results in the natural case where \( \alpha \) is palindromic for the measure \( \nu^\alpha \) formed by shifting \( \mu^\alpha \) to \([−1/2, 1/2]\).

1. Introduction and Motivation

In the seminal paper [5], Hutchinson realized a fractal as the invariant compact set, called the attractor, of an iterated function system (IFS), i.e. a family of contraction maps on a complete metric space. Specifically, given an IFS \( \{\varphi_n\}_{n=0}^{N-1} \) on a complete metric space \( X \), the attractor of the IFS is the unique compact set \( K \subset X \) satisfying
\[
K = \bigcup_{n=0}^{N-1} \varphi_n(K).
\]

Hutchinson showed the existence and uniqueness of a self-similar Borel probability measure supported on the attractor of an IFS. We denote by \( \Delta_N \) the standard simplex in \( \mathbb{R}^N \) and \( \Delta_N^* \subseteq \Delta_N \) consisting of \( \alpha = (\alpha_0, \alpha_1, ..., \alpha_{N-1}) \in \Delta_N \) such that \( \alpha_n < 1 \) for all \( n \) and call elements of \( \Delta_N \) weight vectors. We now paraphrase Hutchinson’s result.

Theorem 1.1 (Hutchinson, [5]). Suppose \( \{\varphi_n\}_{n=0}^{N-1} \) is an IFS on a complete metric space \( X \) with attractor \( K \), and let \( \alpha \in \Delta_N \). There exists a unique Borel regular measure \( \mu^\alpha \) on \( X \) supported on \( K \) such that
\[
\mu^\alpha(E) = \sum_{n=0}^{N-1} \alpha_n \mu^\alpha(\varphi_n^{-1}(E))
\]
for all Borel-measurable \( E \subseteq X \).

Using the terminology of [5], we refer to the measure \( \mu^\alpha \) as the \( \alpha \)-equilibrium measure when \( X = \mathbb{R}^n \) or \( X = \mathbb{C} \). We will call the measure \( \mu^\alpha \) an \( \alpha \)-weighted Cantor measure when the
an equilibrium measure is described as having maximal entropy if the associated weights are uniform, i.e. \( \alpha_n \) is either 0 or \( 1/k \) for each \( n \). An equilibrium measure that has attracted a lot of interest in the non-smooth harmonic analysis community is the ternary Cantor measure which arises from the weight vector \( \alpha = (1/2, 0, 1/2) \). In [7], Jorgensen and Pedersen addressed the question of when a maximal entropy equilibrium measure \( \mu^\alpha \) is spectral, that is, if there exists some countable set \( \Lambda \subset \mathbb{R} \) so that the complex exponential functions \( \{e^{2\pi i \lambda x}\}_{\lambda \in \Lambda} \) form an orthonormal basis for the Hilbert space \( L^2_{\mu^\alpha}[0,1] \). Jorgensen and Pedersen found that, while the quaternary Cantor measure corresponding to \( \alpha = (1/2, 0, 1/2, 0) \) is spectral, the ternary Cantor measure is not.

Much effort has been made to remedy this artifact of the ternary Cantor measure. In [2], Dutkay, Picioroaga, and Song constructed an orthonormal basis consisting of piecewise exponentials on the ternary Cantor set. Strichartz in [9] posed the question of the existence of a frame, which is a generalization of an orthonormal basis, on the ternary Cantor set; however, this problem remains open. Polynomial function systems provide a tempting alternative. To this end, we define the Legendre polynomials in \( L^2_{\mu^\alpha}[0,1] \) to be the result of applying the Gram-Schmidt algorithm to the basis of re-centered monomials \( \{1, x - 1/2, (x - 1/2)^2, \ldots\} \). At each step, it becomes necessary to compute inner products of the form \( \int_{[0,1]} x^m d\mu^\alpha(x) \). These quantities, better known as the moments of the measure \( \mu^\alpha \), have elicited a lot of attention. Dovgoshey, Martio, Ryazanov and Vuorinen provide a fairly comprehensive survey of the ternary Cantor function, including moments of the measure for which it is the distribution, in [3]; Jorgensen, Kornelson and Shuman in [6] study the moments of equilibrium measures through an operator theory perspective using infinite matrices.

Our main results are as follows. In Section 2, we make the connection of these measures to a result by Pei, showing that the weighted Cantor measures are singular except in the trivial case of \( \alpha_n = 1/N \) for all \( n \) when the measure is Lebesgue. We then provide more content in the way of characterizing these measures. In Proposition 2.9, we prove a generalization of Bonnet’s recursion formula for orthogonal polynomial systems. In Theorem 3.3, we derive an explicit infinite product formula for the Laplacian (and thus the moment generating function) of \( \mu^\alpha \) and estimate in Theorem 3.5 the rapid convergence of the coefficients of the partial product. This leads to Remark 3.7 which outlines a \( O(\log \log(1/\varepsilon) \cdot m \log m) \) algorithm for estimating the first \( m \) moments to uniform error at most \( \varepsilon > 0 \).

2. Properties of the weighted Cantor Measure

Our first observation motivates the distinction of \( \bigtriangleup^*_N \) from the simplex \( \bigtriangleup_N \). It is a direct consequence of the uniqueness of a Borel measure satisfying the invariance relation (1.1), and the proof is omitted.

**Proposition 2.1.** Suppose \( \alpha \in \bigtriangleup_N \) with \( \alpha_n = 1 \) for some \( n \). Then \( \mu^\alpha \) is the Dirac measure centered at \( n/(N-1) \).

Given a Borel measure \( \mu \) on \( \mathbb{R} \), the cumulative distribution function (CDF) \( F_\mu(x) := \mu(-\infty, x] \) is the increasing, right-continuous function which uniquely determines the measure. Therefore, to understand the weighted Cantor measure \( \mu^\alpha \), it is useful to note some basic properties
Proposition 2.2. Fix $\alpha \in \Delta_N^*$, and let $k$ be a positive integer. For $n_l \in \{0, 1, ..., N - 1\},$

\[(2.1) \quad F_{\mu^\alpha} \left( \frac{1}{N^k} \left[ 1 + \sum_{l=0}^{k-1} n_l N^l \right] \right) - F_{\mu^\alpha} \left( \frac{1}{N^k} \sum_{l=0}^{k-1} n_l N^l \right) = \prod_{l=0}^{k-1} \alpha_{n_l}. \]

Proof. From the invariance relation (1.1), the CDF satisfies

\[(2.2) \quad F_{\mu^\alpha}(x) = \sum_{n=0}^{N-1} \alpha_n F_{\mu^\alpha}(Nx - n). \]

Then, since $F_{\mu^\alpha}$ is the CDF of a measure supported on $[0, 1]$, we have $F_{\mu^\alpha}(0) = \alpha_0$ which implies that $F_{\mu^\alpha}(0) = 0$. Identity (2.1) immediately follows for $k = 1$. We proceed by induction on $k$. Applying (2.2), we have

\[
F_{\mu^\alpha} \left( \frac{1}{N^k} \left[ 1 + \sum_{l=0}^{k-1} n_l N^l \right] \right) - F_{\mu^\alpha} \left( \frac{1}{N^k} \sum_{l=0}^{k-1} n_l N^l \right) \\
= \sum_{n=0}^{N-1} \alpha_n \left( F_{\mu^\alpha} \left( \frac{1}{N^k} \left[ 1 + \sum_{l=0}^{k-1} n_l N^l \right] + n_k - n \right) - F_{\mu^\alpha} \left( \frac{1}{N^k} \sum_{l=0}^{k-1} n_l N^l \right) \right) \\
= \alpha_{n_k} \left( F_{\mu^\alpha} \left( \frac{1}{N^k} \left[ 1 + \sum_{l=0}^{k-1} n_l N^l \right] \right) - F_{\mu^\alpha} \left( \frac{1}{N^k} \sum_{l=0}^{k-1} n_l N^l \right) \right) \\
= \alpha_{n_k} \prod_{l=0}^{k-1} \alpha_{n_l}
\]

which concludes the induction.

Proposition 2.2 readily implies that the monotone functions constructed by Pei in [8] are identical to the CDF’s of the weighted Cantor measures. Pei therefore proved results pertaining to differentiability and Hölder continuity of $F_{\mu^\alpha}$. We paraphrase those results.

Theorem 2.3 (Pei, [8]). Let $\alpha \in \Delta_N$. $F_{\mu^\alpha}$ is strictly increasing unless $\alpha_n = 0$ for some $n$ and is Hölder continuous with the exponent $\log(1/r) / \log(N)$ where $r = \max \{\alpha_n\}$. Furthermore, $F_{\mu^\alpha}$ is singular continuous except when $\alpha$ is the uniform distribution $(1/N, ..., 1/N)$ in which case $F_{\mu^\alpha}(x) = x$.

Recall that the weighted Cantor measure is determined by weighting, scaling and translating under the IFS according to the invariance equation (1.1). The next proposition illustrates that this invariant condition applies as well to the weight vector. Precisely, there are $\alpha \in \Delta_M$ and $\beta \in \Delta_N$ with $M \neq N$ so that $\mu^\alpha = \mu^\beta$.

Proposition 2.4. Fix $\alpha \in \Delta_N$. Let $\beta = \alpha^{\otimes k}$, the tensor of $\alpha$ with itself $k$ times. Then $\mu^\alpha = \mu^\beta$. 
Proof. It is readily checked that the element of $\beta$ indexed by $n = n_0 + n_1 N + \ldots + n_{k-1} N^{k-1}$ where $n_i \in \{0, 1, \ldots, N - 1\}$ is
\[ \beta_n = \prod_{l=0}^{k-1} \alpha_{n_l}. \]
The associated IFS for $\mu^\beta$ is $\{\psi_n\}_{n=0}^{N-1}$ where $\psi_n(x) = (x + n)/N^k$. From the invariance relation (1.1), we find
\[ \mu^\beta(E) = \sum_{n_0, n_1, \ldots, n_{k-1} = 0}^{N-1} \left( \prod_{l=0}^{k-1} \alpha_{n_l} \right) \mu^\beta(\psi_n^{-1}(E)). \]
Since the IFS $\{\phi_n\}_{n=0}^{N-1}$ for $\mu^\alpha$ is given by $\phi_n(x) = (x + n)/N$, we have
\[ \mu^\alpha(E) = \sum_{n_{k-1} = 0}^{N-1} \alpha_{n_{k-1}} \mu^\alpha(\phi_{n_{k-1}}^{-1}(E)) \]
\[ = \sum_{n_{k-1} = 0}^{N-1} \alpha_{n_{k-1}} \sum_{n_{k-2} = 0}^{N-1} \alpha_{n_{k-2}} \mu^\alpha((\phi_{n_{k-2}}^{-1} \circ \phi_{n_{k-1}}^{-1})(E)) \]
\[ = \ldots = \sum_{n_0, n_1, \ldots, n_{k-1} = 0}^{N-1} \left( \prod_{l=0}^{k-1} \alpha_{n_l} \right) \mu^\alpha((\phi_{n_0}^{-1} \circ \ldots \circ \phi_{n_{k-2}}^{-1} \circ \phi_{n_{k-1}}^{-1})(E)) \]
\[ = \sum_{n_0, n_1, \ldots, n_{k-1} = 0}^{N-1} \left( \prod_{l=0}^{k-1} \alpha_{n_l} \right) \mu^\alpha(\psi_n^{-1}(E)). \]
By uniqueness of the measure, it follows that $\mu^\alpha = \mu^\beta$, as desired.

In the interest of plotting $F_{\mu^\alpha}$, we consider the following approximations. For convenience, we write $0.n_0 n_1 \ldots n_{k-1} = 0.n$ for the number whose coefficients in its base-$N$ expansion is $n = (n_0, \ldots, n_{k-1})$. For each positive integer $k$, let $F_{\mu^\alpha,k} : [0, 1] \to [0, 1]$ be the linear interpolation of the $N^k$ points
\[ \left\{(0.n, F_{\mu^\alpha}(0.n)) : n \in \{0, 1, \ldots, N - 1\}^k \right\}. \]
Note, from Proposition 2.4, that $F_{\mu^\alpha,k} = F_{\mu^\alpha,1}$ where $\beta = \alpha \otimes k$.

**Proposition 2.5.** Let $\alpha \in \Delta^*_N$. The sequence $\{F_{\mu^\alpha,k}\}$ converges uniformly to $F_{\mu^\alpha}$.

Proof. Let $r = \max\{\alpha_0, \alpha_1, \ldots, \alpha_{N-1}\} < 1$. Let $\varepsilon > 0$, and choose an integer $k$ such that $r^k < \varepsilon$. We show that $\|F_{\mu^\alpha,j} - F_{\mu^\alpha,k}\|_\infty < \varepsilon$ for every integer $j \geq k$. Since $|F_{\mu^\alpha,j} - F_{\mu^\alpha,k}(x)|$ is continuous on $[0, 1]$, there exists an $x \in [0, 1]$ such that
\[ \|F_{\mu^\alpha,j} - F_{\mu^\alpha,k}\|_\infty = |F_{\mu^\alpha,j}(x) - F_{\mu^\alpha,k}(x)|. \]
There are $\{n_l\}_{l=0}^{k-1} \subset \{0, 1, \ldots, N - 1\}$ such that
\[ \frac{1}{N^k} \sum_{l=0}^{k-1} n_l N^l \leq x \leq \frac{1}{N^k} \left( 1 + \sum_{l=0}^{k-1} n_l N^l \right). \]
Since \( F_{\mu^\alpha,j} \) and \( F_{\mu^\beta,j} \) are linear interpolations of points belonging to \( F_{\mu^\alpha} \), we have

\[
|F_{\mu^\alpha,j}(x) - F_{\mu^\beta,j}(x)| \leq F_{\mu^\alpha} \left( \frac{1}{N^k} \left[ 1 + \sum_{l=0}^{k-1} n_l N^l \right] \right) = \prod_{l=0}^{k-1} a_n \leq r^k < \varepsilon.
\]

Therefore the sequence \( \{F_{\mu^\alpha,k}\} \) is uniformly Cauchy and, thus, converges uniformly to some continuous function \( f \). Since \( \{F_{\mu^\alpha,k}\} \) converges pointwise to \( F_{\mu^\alpha} \) on a dense set, we have \( F_{\mu^\alpha} = f \) on a dense set. Then, because \( F_{\mu^\alpha} \) is right-continuous and \( f \) is continuous, we have \( f = F_{\mu^\alpha} \).

\[\square\]

For illustration, we can attain the graph of \( F_{\mu^\alpha,k} \) through \( F_{\mu^\beta,1} \) where \( \beta = \alpha \otimes k \). The benefit of the latter is that it is simple to tensor vectors in programs such as Mathematica, which we used to produce Figure 1.

![Figure 1](image)

**Figure 1.** Graphs of \( F_{\mu^\alpha} \) for selected \( \alpha \)

The next result shows that a small variation in \( \alpha \in \Delta^*_N \) leads to a relatively small variation in the CDF.

**Lemma 2.6.** The transform \( \alpha \mapsto F_{\mu^\alpha} : \Delta^*_N \to C[0,1] \) is continuous.

**Proof.** Suppose \( \alpha \in \Delta^*_N \). Let \( \varepsilon > 0 \). By Proposition 2.5, there exists an integer \( k \) such that \( \|F_{\mu^\alpha} - F_{\mu^\alpha,k}\|_\infty < \varepsilon/2 \) and \( \|F_{\mu^\beta} - F_{\mu^\beta,k}\|_\infty < \varepsilon/2 \) for which

\[
\|F_{\mu^\alpha} - F_{\mu^\beta}\|_\infty < \varepsilon + \|F_{\mu^\alpha,k} - F_{\mu^\beta,k}\|_\infty.
\]

Because \( F_{\mu^\alpha,k} \) and \( F_{\mu^\beta,k} \) were defined as linear interpolations of \( F_{\mu^\alpha} \) and \( F_{\mu^\beta} \), respectively, on the set \( X = \{0.n_0n_1...n_{k-1}\} \), the largest absolute difference of \( F_{\mu^\alpha,k}(x) \) and \( F_{\mu^\beta,k}(x) \) must occur at a point in \( X \), say \( x = 0.n_0n_1...n_{k-1} > 0 \) such that \( \|F_{\mu^\alpha,k} - F_{\mu^\alpha,k}\|_\infty = |F_{\mu^\alpha,k}(x) - F_{\mu^\alpha,k}(x)| \). Now suppose \( 0.n_0n_1...n_{k-1} = x - N^{-k} \). Then, from Proposition 2.2, we have

\[
\|F_{\mu^\alpha,k} - F_{\mu^\alpha,k}\|_\infty \leq \left| F_{\mu^\alpha,k} \left( x - \frac{1}{N^k} \right) - F_{\mu^\beta,k} \left( x - \frac{1}{N^k} \right) \right| + \sum_{l=0}^{k-1} \alpha_{n_l} - \sum_{l=0}^{k-1} \beta_{n_l} \leq r^k < \varepsilon.
\]
\[ \ldots \leq \sum_{0, m_0, m_1, \ldots, m_k < x} \left| \prod_{l=0}^{k-1} \alpha_{m_l} - \prod_{l=0}^{k-1} \beta_{m_l} \right| \]

\[ \leq \sum_{0, m_0, m_1, \ldots, m_k < x} \left\{ (1 + \| \alpha - \beta \|_\infty)^k - 1 \right\} \]

\[ = \left( \tilde{n}_{k-1} + \tilde{n}_{k-2}N + \ldots + \tilde{n}_0N^{k-1} \right) \left\{ (1 + \| \alpha - \beta \|_\infty)^k - 1 \right\} \]

where the last inequality follows by expanding the product

\[ \prod_{l=0}^{k-1} \alpha_{m_l} = \prod_{l=0}^{k-1} \{ \beta_{m_l} + (\alpha_{m_l} - \beta_{m_l}) \} \]

and establishing a crude upper bound with \(| \alpha_{m_l} - \beta_{m_l} | \leq \| \alpha - \beta \|_\infty \) and \( \beta_{m_l} \leq 1 \). Therefore, we have

\[ \lim sup_{\| \alpha - \beta \|_\infty \to 0} \| F_{\mu^\beta} - F_{\mu^\alpha} \|_\infty \leq \varepsilon. \]

Since \( \varepsilon > 0 \) was arbitrary, we conclude that the transform is continuous on \( \Delta_N^* \). □

We note that the transform in Lemma 2.6 is not continuous on the entire simplex \( \Delta_N \) since the CDF for the measure associated to \( \alpha \in \Delta_N \setminus \Delta_N^* \) is discontinuous. Let \( \mathcal{M} \) be the space of Borel probability measures on \([0,1]\) with the total variation norm, \( \| \mu \|_{TV} = \sup_E \mu(E) \).

We next show that the transform \( \alpha \mapsto \mu^\alpha : \Delta_N^* \to \mathcal{M} \) is continuous.

**Theorem 2.7.** Let \( \alpha \in \Delta_N^* \). Then \( \beta \to \alpha \) in \( \Delta_N \) if and only if \( \mu^\beta \to \mu^\alpha \) in the total variation norm.

**Proof.** Suppose \( \beta \to \alpha \) in \( \Delta_N \). The implication of convergence in the variation norm follows by proving the result for open intervals and passing to the regularity of the measure; however, the latter details are somewhat technical, so we provide a self-contained proof. Let \( \varepsilon > 0 \). By Lemma 2.6, there exists a \( \delta > 0 \) such that \( \| F_{\mu^\beta} - F_{\mu^\alpha} \|_\infty < \varepsilon / 2 \) whenever \( \| \beta - \alpha \|_\infty < \delta \). Let \( \mathcal{O} \) be an open subset of \([0,1]\), and suppose \( \{ I_n \}_{n=1}^\infty \) is the disjoint collection of open intervals whose union is \( \mathcal{O} \). Regarding \( \mu^\beta \) and \( \mu^\alpha \) as Riemann-Stieltjes measures, given \( \eta > 0 \), there exists a partition \( \mathcal{P} = \{ x_j \} \) of \( I_n \) such that

\[ \left| \mu^\beta(I_n) - \mu^\alpha(I_n) \right| \leq \left\| \mu^\beta(I_n) - \sum_\mathcal{P} (F_{\mu^\beta}(x_{j+1}) - F_{\mu^\beta}(x_j)) \Delta x_j \right\| + \sum_\mathcal{P} (F_{\mu^\beta}(x_{j+1}) - F_{\mu^\alpha}(x_{j+1})) \Delta x_j + \sum_\mathcal{P} (F_{\mu^\alpha}(x_j) - F_{\mu^\beta}(x_j)) \Delta x_j \]

\[ + \sum_\mathcal{P} (F_{\mu^\alpha}(x_{j+1}) - F_{\mu^\alpha}(x_j)) \Delta x_j - \mu^\alpha(I_n) \]

\[ \leq \eta + \varepsilon \lambda(I_n) + \eta \]
where $\lambda$ is Lebesgue measure. Since $\eta$ was arbitrary, we have $|\mu^\beta(I_n) - \mu^\alpha(I_n)| \leq \varepsilon \lambda(I_n)$ and, thus,

$$
|\mu^\beta(O) - \mu^\alpha(O)| \leq \sum_{n=1}^{\infty} |\mu^\beta(I_n) - \mu^\alpha(I_n)| = \varepsilon \lambda(O) \leq \varepsilon.
$$

Now let $E$ be a Borel measurable subset of $[0,1]$, and let $\eta' > 0$. From the regularity of the measures, there exists an open set $O \subset [0,1]$ containing $E$ such that

$$
|\mu^\beta(E) - \mu^\alpha(E)| \leq |\mu^\beta(E) - \mu^\beta(O)| + |\mu^\beta(O) - \mu^\alpha(O)| + |\mu^\alpha(O) - \mu^\alpha(E)|
$$

$$
\leq \eta' + \varepsilon + \eta'.
$$

Since $\eta'$ was arbitrary, we have $|\mu^\beta(E) - \mu^\alpha(E)| \leq \varepsilon$. This concludes that $\mu^\beta \to \mu^\alpha$ in the total variation norm.

Conversely, suppose that $\mu^\beta \to \mu^\alpha$ in the total variation norm. Then, by Proposition 2.2, we have

$$
\beta_k = \mu^\beta\left[\frac{k}{N}, \frac{k+1}{N}\right] \to \mu^\alpha\left[\frac{k}{N}, \frac{k+1}{N}\right] = \alpha_k,
$$

from which it immediately follows that $\beta \to \alpha$ in $\Delta_N$.

We conclude this section with a discussion of symmetric weighted Cantor measures. As motivation, note that both CDF’s in Figure 1 exhibit rotational symmetry about the point $(1/2, 1/2)$. First, we need a few definitions. A Borel measure $\mu$ supported on the unit interval $[0,1]$ is said to be symmetric if $\mu(E) = \mu(1-E)$ for every Borel-measurable set $E$. Here, if $E$ is Borel-measurable, then $1-E$ is Borel-measurable since the collection of sets

$$
\{E : 1-E \text{ is Borel-measurable}\}
$$

is a $\sigma$-algebra containing the open intervals. We say that a weight vector $\alpha \in \Delta_N$ is palindromic if $\alpha_{N-1-n} = \alpha_n$ for all $n \in \{0,1,...,N-1\}$.

**Theorem 2.8.** Let $\alpha \in \Delta_N$. The measure $\mu^\alpha$ is symmetric if and only if $\alpha$ is palindromic.

**Proof.** If $\alpha_n = 1$ for some $n$, then $\mu^\alpha$ is a Dirac measure centered at $n/(N-1)$. As such, the measure is symmetric only when $N = 2n + 1$, when $\alpha$ is palindromic.

So we assume otherwise, that $\alpha_n < 1$ for all $n$. Note that, since $F_{\mu^\alpha}$ is continuous, the measure of any singleton set is zero. Suppose $\alpha$ is palindromic. Then for any positive integer $k$ and $\vec{n} = (n_0, n_1, ..., n_{k-1}) \in \{0,1,...,N-1\}^k$, let $I_{\vec{n}}$ be the open interval

$$
I_{\vec{n}} := (0, n_k \ldots n_k - 2 \ldots n_0, N^{-k} + 0, n_k \ldots n_k - 2 \ldots n_0)
$$

By Proposition 2.2 we have

$$
\mu^\alpha(1 - I_{\vec{n}}) = F_{\mu^\alpha}\left(1 - \frac{1}{N^k} \sum_{l=0}^{k-1} n_l N^l\right) - F_{\mu^\alpha}\left(1 - \frac{1}{N^k} \sum_{l=0}^{k-1} n_l N^l\right)
$$

$$
= F_{\mu^\alpha}\left(\frac{1}{N^k} \sum_{l=0}^{k-1} (N - 1 - n_l) N^l\right) - F_{\mu^\alpha}\left(\frac{1}{N^k} \sum_{l=0}^{k-1} (N - 1 - n_l) N^l\right)
$$
\[ \prod_{l=0}^{k-1} \alpha_{N-l} = \prod_{l=0}^{k-1} \alpha_{n_l} \]

\[ = F_{\mu^\alpha} \left( \frac{1}{N^k} \left[ 1 + \sum_{l=0}^{k-1} n_l N^l \right] \right) - F_{\mu^\alpha} \left( \frac{1}{N^k} \sum_{l=0}^{k-1} n_l N^l \right) \]

\[ = \mu^\alpha(I_{\vec{n}}). \]

Then, from the regularity of \( \mu^\alpha \), it follows that the measure is symmetric.

Conversely, suppose \( \mu^\alpha \) is symmetric. By Proposition 2.2, we have

\[ \alpha_{n_0} = \mu^\alpha \left( \frac{n_0}{N}, \frac{n_0 + 1}{N} \right) = \mu^\alpha \left( \frac{N - 1 - n_0}{N}, \frac{N - 1 - n_0 + 1}{N} \right) = \alpha_{N-1-n_0}. \]

Therefore, \( \alpha \) is palindromic, completing the proof.

The last result of this section is a recursive formula for an orthogonal polynomial system attained via the Gram Schmidt algorithm. Even for general measures, we refer to the elements of the orthogonal system as Legendre polynomials.

**Proposition 2.9.** Let \( \mu \) be a symmetric Borel probability measure supported on \([0, 1]\) and let \( \{ p_n \}_{n=0}^{\infty} \) be the orthonormal polynomials attained by the Gram Schmidt algorithm applied to \( \{(x - 1/2)^n\}_{n=0}^{\infty} \) in \( L_2^\mu[0, 1] \) with corresponding monic polynomials \( \{ m_n \}_{n=0}^{\infty} \). Then for all nonnegative integers \( n \),

\[ m_{n+2} = \left( x - \frac{1}{2} \right) m_{n+1} - \frac{\| m_{n+1} \|^2}{\| m_n \|^2} m_n. \]

**Proof.** We directly calculate the first three polynomials by hand, and find

\[ m_0(x) = 1 \]
\[ m_1(x) = x - \frac{1}{2} \]
\[ m_2(x) = \left( x - \frac{1}{2} \right)^2 - \left\| x - \frac{1}{2} \right\|^2. \]

The identity is therefore valid for \( n = 0 \).

Now suppose \( n \geq 1 \). Expressing \( p_n \) as a linear combination of terms of the form \((x - 1/2)^k\), we note as a consequence of the Gram Schmidt process in \( L_2^\mu[0, 1] \) that the degree of each term in \( p_n \) is even if \( n \) is even and is odd if \( n \) is odd. From this observation, we have \((x - 1/2)p_{n+1} \in \text{span}\{p_{n+2}, p_n, p_{n-2}, \ldots\}\), say

\[ (x - \frac{1}{2}) p_{n+1} = A_{n+2} p_{n+2} + A_n p_n + A_{n-2} p_{n-2} + \ldots + A_m p_m \]
where \( m = 0 \) if \( n \) is even and \( m = 1 \) if \( n \) is odd. Likewise \((x - 1/2)p_{n-2} \in \text{span}\{p_{n-1}, p_{n-3}, p_{n-5}, \ldots\}\), and it follows that

\[
A_{n-2} = \left\langle \left( x - \frac{1}{2} \right) p_{n+1}, p_{n-2} \right\rangle = \left\langle p_{n+1}, \left( x - \frac{1}{2} \right) p_{n-2} \right\rangle = 0.
\]

Similar arguments yield \( A_k = 0 \) for \( k \leq n - 2 \). Therefore we have

\[
\left( x - \frac{1}{2} \right) p_{n+1} = A_{n+2}p_{n+2} + A_n p_n.
\]

Note that it must be the case that \( A_{n+2} \neq 0 \), so we may rewrite the identity as

\[
m_{n+2} = \left( x - \frac{1}{2} \right) m_{n+1} + B_n m_n
\]

for some constant \( B_n \) to be determined. Indeed, by taking the inner-product of \( m_{n+2} \) with \( m_n \), we find

\[
B_n = -\frac{1}{\|m_n\|^2} \left\langle \left( x - \frac{1}{2} \right) m_{n+1}, m_n \right\rangle.
\]

To proceed, note that

\[
m_{n+1} = \left( x - \frac{1}{2} \right) m_n + B_{n-1} m_{n-1}
\]

for some constant \( B_{n-1} \), and it immediately follows that

\[
B_n = -\frac{1}{\|m_n\|^2} \left\langle \left( x - \frac{1}{2} \right) m_{n+1}, m_n \right\rangle = -\frac{1}{\|m_n\|^2} \left\langle m_{n+1}, \left( x - \frac{1}{2} \right) m_n \right\rangle = -\frac{\|m_{n+1}\|^2}{\|m_n\|^2},
\]

as desired.

\[\square\]

Up to a translation factor, Proposition 2.9 is a reproduction of Bonnet’s recurrence formula where the measure is Lebesgue. The drawback of Theorem 2.9 is that the algorithm is dependent on the norm of the monic polynomials. One method to compute the norm of a polynomial is through the moments of the measure, which is the focus of Section 3. In Figure 2, we provide the graph of the first six Legendre polynomials for the ternary Cantor measure.

3. Moments of the Weighted Cantor Measure

As previously observed, if \( \alpha \in \Delta_N \) is a standard basis vector, then \( \mu^\alpha \) is a Dirac measure, and \( L^2_{\mu^\alpha}[0,1] \) is 1-dimensional. Therefore, throughout this section, we focus mainly on \( \alpha \in \Delta_N^* \), but several results remain most general. In this case, integration with respect to \( \mu^\alpha \) presents a difficult calculation. One method is to interpret the problem as a Riemann-Stieltjes integral: for \( f \) continuous on \([0,1]\),

\[
\int_0^1 f(x) \, d\mu^\alpha(x) = \int_0^1 f(x) \, dF_{\mu^\alpha}(x).
\]

By considering a uniform mesh size of \( 1/N^k \), we obtain the left-endpoint approximation

\[
\sum_{n_0, n_1, \ldots, n_{k-1}=0}^{N-1} \left( \prod_{l=0}^{k-1} \alpha_{n_l} \right) f \left( \frac{1}{N^k} \sum_{j=0}^{k-1} n_j N^j \right).
\]
Figure 2. Selected Legendre polynomials for the ternary Cantor measure

For any $\alpha \in \triangle_N$ and any nonnegative integer $m$, we define the $m$-th moment of $\mu^\alpha$ to be

$$I_m^\alpha := \int_0^1 x^m \, d\mu^\alpha(x).$$

When the weight vector $\alpha$ is understood, we suppress the superscript on the moment notation.

From a bootstrapping argument using the invariance relation (1.1), we obtain the useful identity, for any continuous function $f : [0, 1] \to \mathbb{R}$,

$$\int_0^1 f(x) \, d\mu^\alpha(x) = \sum_{n=0}^{N-1} \alpha_n \int_0^1 (f \circ \phi_n)(x) \, d\mu^\alpha(x)$$

where $\{\phi_n\}_{n=0}^{N-1}$ is the associated IFS. We first derive a recurrence relation for these moments which exhibits the approximation (3.1). Although identity (3.4) can be found in [6], the proof of Theorem 3.1 as presented in this paper is original.

**Theorem 3.1.** Let $\alpha \in \triangle_N$ and $k$ be a positive integer. Then $I_0 = 1$ and for all $m \geq 1$,

$$I_m = \sum_{i=0}^{m-1} \binom{m}{i} \frac{\sum_{n=0}^{N-1} \alpha_n \sum_{n_0, n_1, \ldots, n_{k-1}=0}^{N-1} \prod_{l=0}^{k-1} \alpha_{n_l} \left( \frac{1}{N^k} \sum_{j=0}^{k-1} n_j N^j \right)^{m-i}}{N^k m - 1}.\]$$

In particular,

$$I_m = \frac{1}{N^m - 1} \sum_{n=0}^{N-1} \alpha_n \sum_{i=0}^{m-1} \binom{m}{i} n^{m-i} I_i.$$
Proof. Applying identity (3.2) with respect to \( \mu^\beta \) in Proposition 2.4, we have

\[
\int_0^1 x^m d\mu^\alpha(x) = \sum_{n_0, n_1, \ldots, n_{k-1} = 0}^{N-1} \left( \prod_{l=0}^{k-1} \alpha_{n_l} \right) \int_0^1 \left( \frac{1}{N^k} \left[ x + \sum_{j=0}^{k-1} n_j N^j \right] \right)^m d\mu^\alpha(x).
\]

We then expand the product in the integrand and rearrange the terms and sums to find

\[
I_m = \sum_{i=0}^{m} \left( \binom{m}{i} \frac{1}{N^{ki}} I_i \right) \sum_{n_0, n_1, \ldots, n_{k-1} = 0}^{N-1} \left( \prod_{l=0}^{k-1} \alpha_{n_l} \right) \left( \frac{1}{N^k} \sum_{j=0}^{k-1} n_j N^j \right)^{m-i}.
\]

The term in the outermost summand corresponding to \( i = m \) is \( N^{-km} I_m \). Subtracting this term from the left-hand side of the equation, we attain the desired recurrence relation.

\[\square\]

Instead of computing the moments recursively, we can individually approximate them from (3.1). The next result estimates the error of this approximation.

Corollary 3.2. Let \( \varepsilon > 0 \), and fix an integer \( m \geq 1 \). If \( k \geq \log_N \left( \frac{m}{\log(\varepsilon + 1)} \right) \), then

\[
0 \leq I_m - \sum_{n_0, n_1, \ldots, n_{k-1} = 0}^{N-1} \left( \prod_{l=0}^{k-1} \alpha_{n_l} \right) \left( \frac{1}{N^k} \sum_{j=0}^{k-1} n_j N^j \right)^m < \varepsilon.
\]

Proof. The first inequality follows from the observation that (3.1) is a lower approximation of the Riemann-Stieltjes integral. Now, rewriting (3.3), the difference above is bounded by

\[
\sum_{i=1}^{m} \left( \binom{m}{i} \frac{1}{N^{ki}} I_i \right) \sum_{n_0, n_1, \ldots, n_{k-1} = 0}^{N-1} \left( \prod_{l=0}^{k-1} \alpha_{n_l} \right) \left( \frac{1}{N^k} \sum_{j=0}^{k-1} n_j N^j \right)^{m-i} \leq \sum_{i=1}^{m} \left( \binom{m}{i} \frac{1}{N^{ki}} I_{m-i} \right)
\]

Using \( I_i < 1 \), we have

\[
\sum_{i=1}^{m} \left( \binom{m}{i} \frac{1}{N^{ki}} I_{m-i} \right) \leq \sum_{i=1}^{m} \left( \binom{m}{i} \right) \frac{1}{N^{ki}} = \left( 1 + \frac{1}{N^k} \right)^m - 1 \leq \exp \left( \frac{m}{N^k} \right) - 1 \leq \varepsilon,
\]

as desired.

\[\square\]

We define the Laplace transform of a finite measure \( \mu \) on \([0, 1] \) as the function on \( \mathbb{R} \) given by

\[
\mathcal{L}_\mu(s) = \int_0^1 e^{-sx} d\mu(x).
\]

Here, we use the Laplace transform of a weighted Cantor measure to approach the moment problem.
Theorem 3.3. Let $\alpha \in \Delta_N$. Then

$$\mathcal{L}_{\mu^\alpha}(s) = \prod_{r=1}^{N-1} \sum_{n=0}^{N-1} \alpha_n \exp\left(-\frac{ns}{N^r}\right)$$

where the infinite product converges absolutely.

Proof. Observing that

$$\sum_{n=0}^{N-1} \alpha_n \exp\left(-\frac{ns}{N^r}\right) = 1 + \sum_{n=0}^{N-1} \alpha_n \left[\exp\left(-\frac{ns}{N^r}\right) - 1\right],$$

we conclude that the infinite product converges absolutely from

$$\sum_{r=1}^{\infty} \sum_{n=0}^{N-1} \alpha_n \left[\exp\left(-\frac{ns}{N^r}\right) - 1\right] \leq \sum_{r=1}^{\infty} \sum_{n=0}^{N-1} \alpha_n \frac{ns}{N^r} = \frac{s}{N-1} \sum_{n=0}^{N-1} n\alpha_n.$$

Applying identity (3.2) to $f(x) = e^{-sx}$, we find

$$\mathcal{L}_{\mu^\alpha}(s) = \sum_{n=0}^{N-1} \alpha_n \int_0^1 \exp\left(-s\left[x + \frac{n}{N}\right]\right) d\mu^\alpha(x) = \mathcal{L}_{\mu^\alpha}\left(\frac{s}{N}\right) \sum_{n=0}^{N-1} \alpha_n \exp\left(-\frac{ns}{N}\right).$$

Then, by induction, we have

$$\mathcal{L}_{\mu^\alpha}(s) = \mathcal{L}_{\mu^\alpha}\left(\frac{s}{N^k}\right) \prod_{r=1}^{k} \sum_{n=0}^{N-1} \alpha_n \exp\left(-\frac{ns}{N^r}\right).$$

Since the Bounded Convergence Theorem implies

$$\lim_{k \to \infty} \mathcal{L}_{\mu^\alpha}\left(\frac{s}{N^k}\right) = 1,$$

the desired identity follows.

The moment generating function (MGF) $G_\alpha(s)$ is defined analogously,

(3.5) $$G_\alpha(s) := \mathcal{L}_{\mu^\alpha}(-s).$$

It can be seen that

$$G_\alpha(s) = \sum_{m=0}^{\infty} \frac{s^m}{m!}.$$

We may derive many interesting identities from $G_\alpha(s)$, such as the following recurrence relation.

Proposition 3.4. Let $\alpha \in \Delta_N$ be palindromic, and let $m$ be an odd integer. Then

$$I_m = \frac{1}{2} \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} I_k.$$
Proof. From Theorem 3.3 and the assumption that $\alpha$ is palindromic, we find
\[
G_\alpha(s) = \prod_{r=1}^{\infty} \sum_{n=0}^{N-1} \alpha_{N-1-n} \exp \left( \frac{(N-1-n)s}{N^r} \right) \\
= \prod_{r=1}^{\infty} \exp \left( \frac{(N-1)s}{N^r} \right) \sum_{n=0}^{N-1} \alpha_n \exp \left( -\frac{ns}{N^r} \right) \\
= e^s G_\alpha(-s).
\]
This identity, in terms of the power series expansion of $G_\alpha(s)$ and of $e^s$, is then
\[
\sum_{m=0}^{\infty} \frac{I_m}{m!} s^m = \left( \sum_{m=0}^{\infty} \frac{1}{m!} s^m \right) \left( \sum_{m=0}^{\infty} (-1)^m \frac{I_m}{m!} s^m \right) = \sum_{m=0}^{\infty} \left( \sum_{k=0}^{m} \frac{(-1)^k I_k}{k!(m-k)!} \right) s^m.
\]
From the uniqueness of the coefficients, we have
\[
I_m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} I_k
\]
from which the desired identity follows.

\[
\square
\]

Considering $G_\alpha(z)$ as a function on the complex plane, the proof of Theorem 3.3 further implies that $G_\alpha(z)$ is entire. For the purpose of estimating the moments, we consider the partial product approximations defined for all $k$ as
\[
G_{\alpha,k}(z) := \prod_{r=1}^{k} \sum_{n=0}^{N-1} \alpha_n \exp \left( \frac{nz}{N^r} \right) = \sum_{m=0}^{\infty} I_{m;k} \frac{z^m}{m!}.
\]
We first make two observations. Expanding the product, we observe that $I_m$ are nonnegative. Next we differentiate $G_{\alpha,k+1}$ and then evaluate at $z = 0$ to obtain
\[
I_{m;k+1} = \sum_{l=0}^{m} \frac{1}{N^{(k+1)(m-l)}} \binom{m}{l} I_l \sum_{n=0}^{N-1} \alpha_n n^{m-l} \geq I_{m;k}.
\]
Thus, we observe that $I_{m;k} \nearrow I_m$ as $k \to \infty$.

**Theorem 3.5.** Let $\alpha \in \triangle_N$. For any positive integers $m, k$ with $m \geq 2$,
\[
|I_m - I_{m;k}| \leq \frac{em\sqrt{m-1}}{N^k}.
\]

**Proof.** We apply the Cauchy integral formula to the expression
\[
G_\alpha(z) - G_{\alpha,k}(z) = G_{\alpha,k}(z) \left[ G_\alpha \left( \frac{z}{N^k} \right) - 1 \right].
\]
For any positive integers $m, k$ with $m \geq 2$ and all $R > 0$,
\[
|I_m - I_{m;k}| = \frac{m!}{2\pi} \left| \int_{|z|=R} \frac{G_\alpha(z) - G_{\alpha,k}(z)}{z^{m+1}} \, dz \right| \leq \frac{m!}{R^m} \max_{|z|=R} |G_{\alpha,k}(z)| \cdot \max_{|z|=R} \left| G_\alpha \left( \frac{z}{N^k} \right) - 1 \right|.
\]
= \frac{m!}{R^m} G_{\alpha;k}(R) \left[ G_\alpha \left( \frac{R}{N^k} \right) - 1 \right]
\leq \frac{m!}{R^m} G_{(0,...,0,1);k}(R) \left[ G_{(0,...,0,1)} \left( \frac{R}{N^k} \right) - 1 \right]
= \frac{m!}{R^m} \exp \left( R \left( 1 - \frac{1}{N^k} \right) \right) \left[ \exp \left( \frac{R}{N^k} \right) - 1 \right]
= \frac{m!}{R^m} R \left[ 1 - \exp \left( - \frac{R}{N^k} \right) \right]
\leq \frac{m! e^R}{R^{m-1} N^k}

Writing \( f(x) := e^x x^{1-m} \) for \( x \in (0, \infty) \), we note that \( f(x) \to +\infty \) as \( x \) tends to either 0 or \( +\infty \). Since

\[ f'(x) = e^x \left( (1 - m)x^{-m} + x^{1-m} \right), \]

it follows that \( f \) is maximized at \( x = m - 1 \). Evaluating \( f(m-1) \), we have

\[ |I_m - I_{m;k}| \leq m! \left( \frac{e}{m-1} \right)^{m-1} \frac{1}{N^k}. \]

The desired bound then follows as a consequence of Stirling’s approximation,

\[ \left( \frac{e}{m-1} \right)^{m-1} \leq \frac{e^{\sqrt{m-1}}}{(m-1)!}. \]

\[ \Box \]

**Remark 3.6.** For any palindromic weight vector \( \alpha \), it is suitable to alternatively define the moment generating function under \( \nu^\alpha \), the measure defined by shifting \( \mu^\alpha \) from \([0, 1]\) to \([-1/2, 1/2]\). Then the moment generating function with respect to \( \nu^\alpha \) satisfies

\[ \sum_{m=0}^{\infty} J_m \frac{s^m}{m!} = H_\alpha(s) := \int_{[-1/2,1/2]} e^{sx} d\nu^\alpha(x) = \int_{[0,1]} e^{s(x-1/2)} d\mu^\alpha(x) = e^{-s/2} G_\alpha(s). \]

With some careful manipulation, we may then analogously define \( H_{\alpha;k}(s) = \sum_{m=0}^{\infty} J_m \frac{k^m}{m!} \) as the partial product \( \prod_{r=1}^{k} e^{-\frac{(N-1)x}{2N^r}} \sum_{\ell=0}^{N-1} \alpha_\ell \exp \left( \frac{\ell x}{N^r} \right) \). Distributing, each product is now a weighted average of hyperbolic cosines of the form \( \cosh \left( \frac{\ell x}{N^r} \right) \), where each \( \ell \) is a half integer between 0 and \( N/2 \).

**Remark 3.7.** For any \( m \geq 0 \), we may estimate the coefficients \( I_1, \ldots, I_m \) (or \( J_1, \ldots, J_m \) for a palindromic weight vector) within uniform error at most \( \varepsilon > 0 \in O(\log \log (1/\varepsilon) \cdot m \log m) \). This is a substantial improvement when compared to the exact computation of each \( I_1, \ldots, I_m \) (or \( J_1, \ldots, J_m \)) from Proposition 3.4 which runs in \( O(n^2) \).

We describe the details for moments under \( \mu^\alpha \). First, apply Theorem 3.5 to select \( k = k(\varepsilon) = O(\log(1/\varepsilon)) \) so that \( |I_m - I_{m;k}| \leq \varepsilon \). Writing \( f(s) = \sum_{m=0}^{\infty} c_m s^m \) for the truncation of \( \sum_{\ell=0}^{N-1} \alpha_\ell \exp \left( \frac{\ell x}{N^r} \right) \) to degree \( m \), it follows that \( G_{\alpha;k}(s) \) and \( F(s) := f(s) f(s/N) \cdots f(s/N^{k-1}) \) have identical coefficients up to degree \( m \). For algorithmic simplicity, we may assume that \( k \) is a power of two, but this assumption may be circumvented with some care, or absorbed as a factor in \( O(\log(1/\varepsilon)) \). We provide the following pseudocode.
(1) \( F(s) \leftarrow \sum_{n=0}^{m} \sum_{\ell=0}^{N-1} \alpha_\ell \cdot (s\ell/N)^n/n! \).

(2) \( j \rightarrow 0 \).

(3) If \( 2^j = k \), go to step (8).

(4) \( F(s) \rightarrow F(s) \cdot F(s/N^{2^j}) \).

(5) Truncate \( F(s) \) to degree \( m \) in \( s \).

(6) \( j \rightarrow j + 1 \).

(7) Go to step (3).

(8) return \( F(s) \)

Since a successful termination performs \( \log_2(k) \) products of degree \( m \) polynomials, by using a Fast Fourier Transform, the overall complexity is reduced to \( O(\log \log(1/\varepsilon) \cdot m \log m) \), as desired.

In [4], Grabner and Prodinger investigated measures whose distributions are given by Cantor sets and are somewhat similar to the ternary Cantor measure yet in general do not arise from an IFS. The major result in their paper is the following asymptotic behavior of the corresponding moments,

\[
I_m = F(\log_{1/\theta} m) m^{-\log_{1/\theta}(2)} \left( 1 + O \left( \frac{1}{m} \right) \right)
\]

where \( F(x) \) is a periodic function of period 1 and known Fourier coefficients. In regards to our paper, the ternary Cantor measure is ascertained by letting \( \theta = 1/3 \). The final result of this paper is a lower bound approximation for the rate of decay of the moments for a weighted Cantor measure. It is intriguing that the bound that we obtain is precisely of the same order as the result of Grabner and Prodinger.

**Theorem 3.8.** Let \( \alpha \in \triangle_N \). If \( \alpha_{N-1} = 0 \), then \( I_m \leq \left( \frac{N-1}{N} \right)^m \) for \( m \geq 0 \). Else, if \( \alpha_{N-1} > 0 \), then there exists a constant \( C(\alpha) > 0 \) such that \( I_m \geq C(\alpha) m^\gamma \) for all \( m \geq 0 \) where \( \gamma = \log_N(\alpha_{N-1}) \).

**Proof.** Suppose \( \alpha_{N-1} = 0 \). From the invariance relation \( (1.1) \), we observe that the support of \( \mu^\alpha \) is contained in \([0, 1 - 1/N]\), and we have

\[
\int_0^1 x^m d\mu^\alpha(x) = \int_0^{1-1/N} x^m d\mu^\alpha(x) \leq \left( \frac{N-1}{N} \right)^m,
\]

so the first claim holds.

Now suppose \( \alpha_{N-1} > 0 \). Let \( m \) be positive integer. Note, for all positive integers \( k \),

\[
\int_0^1 x^m d\mu^\alpha(x) \geq \int_{1-N^{-k}}^1 x^m d\mu^\alpha(x) \geq (1 - N^{-k})^m (\alpha_{N-1})^k
\]

since \( \mu^\alpha[1 - N^{-k}, 1] = (\alpha_{N-1})^k \). As stipulated, we write \( \gamma := \log_N(\alpha_{N-1}) \) and also let \( f(x) := N^\gamma x (1 - N^{-x})^m \) for \( x > 0 \). We wish to optimize \( f(k) \) over the positive integers \( k \). First, we optimize \( f \) on \((0, \infty)\). Through logarithmic differentiation, we find

\[
\frac{f'(x)}{f(x)} = (\gamma x \log N + m \log (1 - N^{-x}))'
\]
Thus, $f'$ has its unique zero at $x_0 = \log_N \left( 1 - \frac{m}{\gamma} \right)$ and $f'(x) > 0$ for $0 < x < x_0$ and $f'(x) < 0$ for $x > x_0$, and it follows that $f(x)$ is maximized over $(0, \infty)$ at $x_0$ and that the optimal integer is either $\lfloor x_0 \rfloor$ or $\lceil x_0 \rceil$. Write $k_0 = k_0(m)$ for the positive integer which maximizes $f$, and let $\epsilon = \epsilon(m) := k_0 - x_0 \in (-1, 1)$. Since $N^{\gamma \epsilon} \in (N^{\gamma}, N^{-\gamma})$ and $1 - N^{-\epsilon} \in (1 - N, 1 - 1/N)$, it follows that
\[
\frac{f(k_0)}{f(x_0)} = N^{\gamma \epsilon} \left( 1 - N^{-k_0} \right)^m \\
= N^{\gamma \epsilon} \left( 1 + \frac{1 - N^{-\epsilon}}{N x_0 - 1} \right)^m \\
= N^{\gamma \epsilon} \left( 1 - \frac{\gamma (1 - N^{-\epsilon})}{m} \right)^m \\
\leq N^{\gamma \epsilon} C e^{-\gamma (1 - N^{-\epsilon})}
\]
is at most a positive constant depending only on $\alpha$. Since
\[
f(x_0) = N^{\gamma x_0} \left( 1 - N^{-x_0} \right)^m = \left( 1 - \frac{m}{\gamma} \right)^{\gamma} \left( 1 - \frac{\gamma}{\gamma - m} \right)^m \leq \left( \frac{m}{\gamma} + 1 \right)^{\gamma} C' e^{\gamma},
\]
where $C' > 0$ depends only on $\alpha$, the desired claim holds.

\begin{remark}
Under the shifted measure $\nu^\alpha$ defined in Remark 3.6, the moments decay exponentially regardless of weight vector $\alpha$. Indeed,
\[
\left| \int_{-1/2}^{1/2} x^m d\nu^\alpha(x) \right| = \left| \int_0^1 (x - 1/2)^m d\mu^\alpha(x) \right| \leq \left( \frac{1}{2} \right)^m.
\]
\end{remark}

\section*{References}
\begin{enumerate}
\item V.I. Bogachev. Measure theory. Vol. II. Springer-Verlag, Berlin, 2007.
\item Dorin Ervin Dutkay, Gabriel Picioroaga, and Myung-Sin Song. Orthonormal bases generated by Cuntz algebras. \textit{J. Math. Anal. Appl.}, 409(2):1128-1139, 2014.
\item O. Dovgoshey, O. Martio, V. Ryazanov, and M. Vuorinen. The Cantor function. \textit{Expo. Math.}, 24(1):1-37, 2006.
\item P.J. Grabner and H. Prodinger. Asymptotic analysis of the moments of the Cantor distribution. \textit{Statist. Probab. Lett.}, 26(3):243-248, 1996.
\item John E. Hutchinson. Fractals and self-similarity. \textit{Indian Univ. Math. J.}, 30(5):713-747, 1981.
\item Palle E. T. Jorgensen, Keri A. Kornelson, and Karen L. Shuman. Iterated function systems, moments, and transformations of infinite matrices. \textit{Mem. Amer. Math. Soc.}, 213(1003):x+105, 2011.
\item Palle E. T. Jorgensen and Steen Pedersen. Dense analytic subspaces in fractal $L^2$-spaces. \textit{J. Anal. Math.}, 75:185-228, 1998.
\item Elton Pei Hsu. A class of singular continuous functions. \textit{Elem. Math.}, 47(4):169-172, 1992.
\item Robert S. Strichartz. Mock fourier series and transforms associated with certain cantor measures. \textit{Journal d’Analyse Mathématique}, 81(1):209-238, Dec 2000.
\end{enumerate}
