THE ASYMPTOTIC VALUE OF THE MAHLER MEASURE OF THE RUDIN-SHAPIRO POLYNOMIALS

Tamás Erdélyi

December 12, 2018

Abstract. In signal processing the Rudin-Shapiro polynomials have good autocorrelation properties and their values on the unit circle are small. Binary sequences with low autocorrelation coefficients are of interest in radar, sonar, and communication systems. In this paper we show that the Mahler measure of the Rudin-Shapiro polynomials of degree \( n - 1 \) with \( n = 2^k \) is asymptotically \((2n/e)^{1/2}\), as it was conjectured by B. Saffari in 1985. Our approach is based heavily on the Saffari and Montgomery conjectures proved recently by B. Rodgers.

1. Introduction and Notation

Let \( D := \{ z \in \mathbb{C} : |z| < 1 \} \) denote the open unit disk of the complex plane. Let \( \partial D := \{ z \in \mathbb{C} : |z| = 1 \} \) denote the unit circle of the complex plane. The Mahler measure \( M_0(f) \) is defined for bounded measurable functions \( f \) on \( \partial D \) by

\[
M_0(f) := \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| \, dt \right).
\]

It is well known, see [HL-52], for instance, that

\[
M_0(f) = \lim_{q \to 0^+} M_q(f),
\]

where

\[
M_q(f) := \left( \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^q \, dt \right)^{1/q}, \quad q > 0.
\]

It is also well known that for a function \( f \) continuous on \( \partial D \) we have

\[
M_\infty(f) := \max_{t \in [0, 2\pi]} |f(e^{it})| = \lim_{q \to \infty} M_q(f).
\]

Key words and phrases. polynomial inequalities, Mahler measure, Rudin-Shapiro polynomials, zeros of polynomials.

2010 Mathematics Subject Classifications. 11C08, 41A17, 26C10, 30C15
It is a simple consequence of the Jensen formula that

\[ M_0(f) = |c| \prod_{j=1}^{n} \max\{1, |z_j| \} \]

for every polynomial of the form

\[ f(z) = c \prod_{j=1}^{n} (z - z_j), \quad c, z_j \in \mathbb{C}. \]

See [BE-95, p. 271] or [B-02, p. 3], for instance.

Let \( P_n \) be the set of all algebraic polynomials of degree at most \( n \) with complex coefficients. Let \( T_n \) be the set of all real (that is, real-valued on the real line) trigonometric polynomials of degree at most \( n \). Finding polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. The classes

\[ \mathcal{L}_n := \left\{ f : f(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \{-1, 1\} \right\} \]

of Littlewood polynomials and the classes

\[ \mathcal{K}_n := \left\{ f : f(z) = \sum_{j=0}^{n} a_j z^j, \quad a_j \in \mathbb{C}, \quad |a_j| = 1 \right\} \]

of unimodular polynomials are two of the most important classes considered. Observe that \( \mathcal{L}_n \subset \mathcal{K}_n \) and

\[ M_0(f) \leq M_2(f) = \sqrt{n+1} \]

for every \( f \in \mathcal{K}_n \). Beller and Newman [BN-73] constructed unimodular polynomials \( f_n \in \mathcal{K}_n \) whose Mahler measure \( M_0(f_n) \) is at least \( \sqrt{n} - \frac{c}{\log n} \).

Section 4 of [B-02] is devoted to the study of Rudin-Shapiro polynomials. Littlewood asked if there were polynomials \( f_{nk} \in \mathcal{L}_{nk} \) satisfying

\[ c_1 \sqrt{n_k + 1} \leq |f_{nk}(z)| \leq c_2 \sqrt{n_k + 1}, \quad z \in \partial \mathbb{D}, \]

with some absolute constants \( c_1 > 0 \) and \( c_2 > 0 \), see [B-02, p. 27] for a reference to this problem of Littlewood. To satisfy just the lower bound, by itself, seems very hard, and no such sequence \( (f_{nk}) \) of Littlewood polynomials \( f_{nk} \in \mathcal{L}_{nk} \) is known. A sequence of Littlewood polynomials that satisfies just the upper bound is given by the Rudin-Shapiro polynomials. The Rudin-Shapiro polynomials appear in Harold Shapiro’s 1951 thesis [S-51] at MIT and are sometimes called just Shapiro polynomials. They also arise independently in Golay’s paper [G-51]. They are remarkably simple to construct and are a rich source of counterexamples to possible conjectures.
The Rudin-Shapiro polynomials are defined recursively as follows:

\[ P_0(z) := 1, \quad Q_0(z) := 1, \]
\[ P_{k+1}(z) := P_k(z) + z^{2k} Q_k(z), \]
\[ Q_{k+1}(z) := P_k(z) - z^{2k} Q_k(z), \]

for \( k = 0, 1, 2, \ldots \). Note that both \( P_k \) and \( Q_k \) are polynomials of degree \( n - 1 \) with \( n := 2^k \) having each of their coefficients in \( \{-1, 1\} \). In signal processing, the Rudin-Shapiro polynomials have good autocorrelation properties and their values on the unit circle are small. Binary sequences with low autocorrelation coefficients are of interest in radar, sonar, and communication systems.

It is well known and easy to check by using the parallelogram law that

\[ |P_{k+1}(z)|^2 + |Q_{k+1}(z)|^2 = 2(|P_k(z)|^2 + |Q_k(z)|^2), \quad z \in \partial D. \]

Hence

(1.1) \[ |P_k(z)|^2 + |Q_k(z)|^2 = 2^{k+1} = 2n, \quad z \in \partial D. \]

It is also well known (see Section 4 of [B-02], for instance), that

(1.2) \[ |Q_k(z)| = |P_k(-z)|, \quad z \in \partial D. \]

P. Borwein’s book [B-02] presents a few more basic results on the Rudin-Shapiro polynomials. Various properties of the Rudin-Shapiro polynomials are discussed in [B-73] and [BL-76]. Obviously \( M_2(P_k) = 2^{k/2} \) by the Parseval formula. In 1968 Littlewood [L-68] evaluated \( M_4(P_k) \) and found that \( M_4(P_k) \sim (4^{n+1}/3)^{1/4} \). The \( M_4 \) norm of Rudin-Shapiro like polynomials on \( \partial D \) are studied in [BM-00]. P. Borwein and Lockhart [BL-01] investigated the asymptotic behavior of the mean value of the \( M_q \) norms of Littlewood polynomials for arbitrary \( q > 0 \). They proved that

\[ \lim_{n \to \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{(M_q(f))q}{n^{q/2}} = \Gamma \left( 1 + \frac{q}{2} \right). \]

In [C-15] we proved that

\[ \lim_{n \to \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} M_q(f) n^{1/2} = \left( \Gamma \left( 1 + \frac{q}{2} \right) \right)^{1/q} \]

for every \( q > 0 \). In [CE-15] we showed also that

\[ \lim_{n \to \infty} \frac{1}{2^{n+1}} \sum_{f \in \mathcal{L}_n} \frac{M_0(f)}{n^{1/2}} = e^{-\gamma/2}, \]

where

\[ \gamma := \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.577215 \ldots \]

is the Euler constant and \( e^{-\gamma/2} = 0.749306 \ldots \). These are analogues of the results proved earlier by Choi and Mossinghoff [CM-11] for polynomials in \( K_n \).

Let \( K := \mathbb{R} \) (mod \( 2\pi \)). Let \( m(A) \) denote the one-dimensional Lebesgue measure of \( A \subset K \). In 1980 Saffari conjectured the following.
Conjecture 1.1. Let $P_k$ and $Q_k$ be the Rudin-Shapiro polynomials of degree $n - 1$ with $n := 2^k$. We have

$$M_q(P_k) = M_q(Q_k) \sim \frac{2^{(k+1)/2}}{(q/2 + 1)^{1/q}}$$

for all real exponents $q > 0$. Equivalently, we have

$$\lim_{n \to \infty} m\left(\left\{ t \in K : \left| \frac{P_k(e^{it})}{\sqrt{2^{k+1}}} \right|^2 \in [\alpha, \beta] \right\}\right) = \lim_{n \to \infty} m\left(\left\{ t \in K : \left| \frac{Q_k(e^{it})}{\sqrt{2^{k+1}}} \right|^2 \in [\alpha, \beta] \right\}\right) = 2\pi(\beta - \alpha)$$

whenever $0 \leq \alpha < \beta \leq 1$.

This conjecture was proved for all even values of $q \leq 52$ by Doche [D-05] and Doche and Habsieger [DH-04]. Recently B. Rodgers [R-17] proved Saffari’s Conjecture 1.1 for all $q > 0$. See also [EZ-17]. An extension of Saffari’s conjecture is Montgomery’s conjecture below.

Conjecture 1.2. Let $P_k$ and $Q_k$ be the Rudin-Shapiro polynomials of degree $n - 1$ with $n := 2^k$. We have

$$\lim_{n \to \infty} m\left(\left\{ t \in K : \frac{P_k(e^{it})}{\sqrt{2^{k+1}}} \in E \right\}\right) = \lim_{n \to \infty} m\left(\left\{ t \in K : \frac{Q_k(e^{it})}{\sqrt{2^{k+1}}} \in E \right\}\right) = 2m(E)$$

for any rectangle $E \subset D := \{ z \in \mathbb{C} : |z| < 1 \}$.

B. Rodgers [R-17] proved Montgomery’s Conjecture 1.2 as well.

Despite the simplicity of their definitions not much is known about the Rudin-Shapiro polynomials. It has been shown in [E-16] fairly recently that the Mahler measure ($M_0$ norm) and the $M_\infty$ norm of the Rudin-Shapiro polynomials $P_k$ and $Q_k$ of degree $n - 1$ with $n := 2^k$ on the unit circle of the complex plane have the same size, that is, the Mahler measure of the Rudin-Shapiro polynomials of degree $n - 1$ with $n := 2^k$ is bounded from below by $cn^{1/2}$, where $c > 0$ is an absolute constant.

It is shown in this paper that the Mahler measure of the Rudin-Shapiro polynomials $P_k$ and $Q_k$ of degree $n - 1 = 2^k - 1$ is asymptotically $(2/e)^{1/2}$, as it was conjectured by B. Saffari in 1985. Note that $(2/e)^{1/2} = 0.85776388496\ldots$ is larger than $e^{-\gamma/2} = 0.749306\ldots$ in the average Mahler measure result for the class of Littlewood polynomials $L_n$ we mentioned before.

2. NEW RESULT

Let $P_k$ and $Q_k$ be the Rudin-Shapiro polynomials of degree $n - 1$ with $n := 2^k$.

Theorem 2.1. We have

$$\lim_{n \to \infty} \frac{M_0(P_k)}{n^{1/2}} = \lim_{n \to \infty} \frac{M_0(Q_k)}{n^{1/2}} = \left(\frac{2}{e}\right)^{1/2}.$$
3. **Lemmas**

Let \( D(a, r) := \{ z \in \mathbb{C} : |z - a| < r \} \) denote the open disk of the complex plane centered at \( a \in \mathbb{C} \) of radius \( r > 0 \).

To prove our theorem we need some lemmas. Our first lemma states Jensen’s formula. Its proof may be found in most of the complex analysis textbooks.

**Lemma 3.1.** Suppose \( h \) is a nonnegative integer and

\[
f(z) = \sum_{k=h}^{\infty} c_k (z - z_0)^k, \quad c_h \neq 0,
\]

is analytic on the closure of the disk \( D(z_0, r) \). Let \( a_1, a_2, \ldots, a_m \) denote the zeros of \( f \) in \( D(z_0, r) \setminus \{ z_0 \} \), where each zero is listed as many times as its multiplicity. We have

\[
\log |c_h| + h \log r + \sum_{k=1}^{m} \log \frac{r}{|a_k - z_0|} = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(z_0 + re^{i\theta})| d\theta.
\]

**Lemma 3.2.** There exists a constant \( c_1 \) depending only on \( c_2 > 0 \) such that every polynomial \( P \in \mathcal{P}_n^c \) has at most \( c_1(nr + 1) \) zeros in any open disk \( D(z_0, r) \) with \( z_0 \in \partial D \) and

\[
|P(z_0)| \geq c_2 M_\infty(P)
\]

**Proof of Lemma 3.2.** Without loss of generality we may assume that \( z_0 := 1 \) and

\[
n^{-1} \leq r \leq 1.
\]

Indeed, the case \( 0 < r < n^{-1} \) follows from the case \( r = n^{-1} \), and the case \( r > 1 \) is obvious. Let \( P \in \mathcal{P}_n^c \) satisfy (3.1). A well known polynomial inequality observed by Bernstein states that

\[
|P(\zeta)| \leq \max\{1, |\zeta|^n\} \max_{z \in \partial D} |P(z)|
\]

for any polynomials \( P \in \mathcal{P}_n^c \) and for any \( \zeta \in \mathbb{C} \). This is a simple consequence of the Maximum Principle, see [BE-95, p. 239], for instance. Using (3.2) we can deduce that

\[
\log |P(z)| \leq \log((1 + 2r)^n M_\infty(P)) \leq \log M_\infty(P) + 2nr, \quad |z| \leq 1 + 2r.
\]

Let \( m \) denote the number of zeros of \( P \) in the open disk \( D(z_0, r) \). Using Lemma 3.1 with the disk \( D(z_0, 2r) \) and \( h = 0 \), then using (3.1) and (3.3), we obtain

\[
\log c_2 + \log M_\infty(P) + m \log 2 \leq \log |P(z_0)| + m \log 2 \leq \frac{1}{2\pi} 2\pi(\log M_\infty + 2nr).
\]

This, together with \( n^{-1} \leq r \leq 1 \), implies \( \log c_2 + m \log 2 \leq 2nr \), and the lemma follows. \( \square \)

Our next lemma is stated as Lemma 3.5 in [E-16], where its proof may also be found.
Lemma 3.3. If $P_k$ and $Q_k$ are the $k$-th Rudin-Shapiro polynomials of degree $n - 1$ with $n := 2^k$, $\delta := \sin^2(\pi/8)$, and

$$z_j := e^{it_j}, \quad t_j := \frac{2\pi j}{n}, \quad j \in \mathbb{Z},$$

then

$$\max\{|P_k(z_j)|^2, |P_k(z_{j+1})|^2\} \geq \delta 2^{k+1} = 2\delta n \quad j \in \mathbb{Z}.$$

ByLemma 3.3, for every $n = 2^k$ there are

$$0 \leq \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} := \tau_1 + 2\pi$$

such that

$$\tau_j - \tau_{j-1} = \frac{2\pi l}{n}, \quad l \in \{1, 2\},$$

and with

(3.4) $$a_j := e^{i\tau_j}, \quad j = 1, 2, \ldots, m + 1,$$

we have

(3.5) $$|P_k(a_j)|^2 \geq 2\delta n, \quad j = 1, 2, \ldots, m + 1.$$ (Moreover, each $a_j$ is an $n$-th root of unity.) For the sake of brevity let $R_n \in \mathcal{T}_n$ be defined by

$$R_n(t) := |P_k(e^{it})|^2, \quad n = 2^k.$$

Using the above notation we formulate the following observation.

Lemma 3.4. There is an absolute constant $c_3 > 0$ such that

$$\mu := \left| \left\{ j \in \{2, 3, \ldots, m + 1\} : \min_{t \in [\tau_{j-1}, \tau_j]} R_n(t) \leq \varepsilon n \right\} \right| \leq c_3 n \varepsilon^{1/2}$$

for every sufficiently large $n = 2^k \geq n_\varepsilon$, $k = 1, 2, \ldots$, and $\varepsilon > 0$.

To prove Lemma 3.4 we need a consequence of the so-called Bernstein-Szegö inequality formulated by our next lemma. For its proof see [BE-95, p. 232], for instance.

Lemma 3.5. We have

$$S'(t)^2 + n^2 S(t)^2 \leq n^2 \max_{\tau \in \mathbb{R}} S(\tau)^2$$

for every $S \in \mathcal{T}_n$. 6
Lemma 3.6. We have
\[ |R_n'(t)| \leq n^{3/2} \sqrt{2R_n(t)} , \quad t \in \mathbb{R} . \]

Proof of Lemma 3.6. Let \( S \in T_n \) be defined by \( S(t) := |P_k(e^{it})|^2 - n = R_n(t) - n \). Observe that (1.1) implies that
\[ \max_{\tau \in \mathbb{R}} |S(\tau)| \leq n . \]
Combining this with Lemma 3.5 implies that
\[ |R_n'(t)| = |S'(t)| \leq n \sqrt{n^2 - S(t)^2} = \sqrt{n^2 - (R_n(t) - n)^2} \leq n \sqrt{2nR_n(t) - R_n(t)^2} \]
\[ \leq n \sqrt{2nR_n(t)} . \]
\[ \square \]

Now we are ready to prove Lemma 3.4.

Proof of Lemma 3.4. Let \( j \in \{2, 3, \ldots, m+1\} \) be such that
\[ \min_{t \in [\tau_{j-1}, \tau_j]} R_n(t) \leq \varepsilon n . \]
Using the notation of Lemma 3.3, without loss of generality we may assume that \( 0 < \varepsilon < \delta \).
By recalling (3.4) and (3.5) there are \( \tau_{j-1} \leq \alpha_j < \beta_j \leq \tau_j \) such that
\[ R_n(\alpha_j) = \varepsilon n , \quad R_n(\beta_j) = 2\varepsilon n , \]
and
\[ R_n(t) \leq 2\varepsilon n , \quad t \in [\alpha_j, \beta_j] . \]
Then, by the Mean Value Theorem there is \( \xi_j \in (\alpha_j, \beta_j) \) such that
\[ \varepsilon n = R_n(\beta_j) - R_n(\alpha_j) = (\beta_j - \alpha_j)R_n'(\xi_j) , \]
and hence by Lemma 3.6 we obtain
\[ \varepsilon n = (\beta_j - \alpha_j)R_n'(\xi_j) \leq (\beta_j - \alpha_j)n^{3/2} \sqrt{2R_n(\xi_j)} \]
\[ \leq (\beta_j - \alpha_j)n^{3/2} \sqrt{4\varepsilon n} , \]
that is,
\[ \beta_j - \alpha_j \geq \frac{\varepsilon^{1/2}}{2n} . \]
Hence, on one hand,
\[ m \left( \left\{ t \in K : \left| \frac{R_n(t)}{n} \right| \in [0, 2\varepsilon] \right\} \right) = m \left( \left\{ t \in K : \left| \frac{P_k(e^{it})}{\sqrt{2k+1}} \right|^2 \in [0, \varepsilon] \right\} \right) \geq \frac{\mu \varepsilon^{1/2}}{2n} , \]
where $\mu$ is defined in the statement of the lemma. On the other hand, by Conjecture 1.1 proved by B. Rodgers there is an absolute constant $c_3/2 > 0$ such that

$$m\left(\left\{ t \in K : \frac{R_n(t)}{n} \in [0, 2\varepsilon] \right\}\right) = m\left(\left\{ t \in K : \left| \frac{P_k(e^{it})}{\sqrt{2^k+1}} \right|^2 \in [0, \varepsilon] \right\}\right) \leq (c_3/2)\varepsilon$$

for every sufficiently large $n \geq n_\varepsilon$. Combining the last two inequalities we obtain

$$\mu \leq c_3 n \varepsilon^{1/2}$$

for every sufficiently large $n \geq n_\varepsilon$. □

We introduce the notation

$$A_{n, \varepsilon} := \left\{ t \in K : \frac{R_n(t)}{2n} \geq \varepsilon \right\}$$

and

$$B_{n, \varepsilon} := K \setminus A_{n, \varepsilon} = \left\{ t \in K : \frac{R_n(t)}{2n} < \varepsilon \right\}.$$

Our next lemma is an immediate consequence of Conjecture 1.1 proved by B. Rodgers.

**Lemma 3.7.** Let $\varepsilon \in (0, 1)$ be fixed. We have

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_{A_{n, \varepsilon}} \log \frac{R_n(t)}{2n} \, dt = \int_{\varepsilon}^{1} \log x \, dx.$$

**Proof of Lemma 3.7.** Let

$$F_\varepsilon(x) := \begin{cases} \log x, & \text{if } x \in [\varepsilon, 1], \\ 0, & \text{if } x \in [0, \varepsilon]. \end{cases}$$

By using the Weierstrass Approximation Theorem, it is easy to see that $F_\varepsilon$ can be approximated by polynomials in $L_1[0, 1]$ norm, and hence the lemma follows from Conjecture 1.1 proved by B. Rodgers in a standard fashion. We omit the details of this routine argument. □

The above lemma will be coupled with the following inequality.

**Lemma 3.8.** Let $\varepsilon \in (0, 1)$ be fixed. There is an absolute constant $c_4 > 0$ such that

$$\frac{1}{2\pi} \int_{B_{n, \varepsilon}} \log \frac{R_n(t)}{2n} \, dt \geq -c_4 \varepsilon^{1/2}$$

for every sufficiently large $n \geq n_\varepsilon$.

To prove Lemma 3.8 we need a few other lemmas.
Lemma 3.9. Let $f$ be a twice differentiable function on $[a, b]$. There is a $\xi \in [a, b]$ such that

$$
\int_a^b f(t) \, dt - \frac{1}{2} (f(a) + f(b))(b - a) = \frac{-(b - a)^3}{12} f''(\xi).
$$

This is the formula for the error term in the trapezoid rule. Its proof may be found in various calculus textbooks discussing numerical integration.

Let $w_j \in \mathbb{C}$, $j = 1, 2, \ldots, n - 1$, denote the zeros of $P_k$. So we have

$$
\log R_n(t) = \log \left| \frac{P_k(e^{it})}{n} \right| = \sum_{j=1}^{n-1} \log |e^{it} - w_j| - \log n.
$$

It is a simple well known fact that $P_k \in \mathcal{L}_{n-1}$ implies that

$$
1/2 \leq |w_j| \leq 2, \quad j = 1, 2, \ldots, n - 1.
$$

Associated with $w \in \mathbb{C}$ we introduce $\phi \in [0, 2\pi)$ uniquely defined by $w = |w|e^{i\phi}$. For the sake of brevity let

$$
g_w(t) := \log |e^{it} - w| = \log |e^{it} - |w|e^{i\phi}|.
$$

Simple calculations show that

$$
g_w(t) = \frac{1}{2} \log(1 + |w|^2 - 2|w| \cos(t - \phi)),
$$

$$
g'_w(t) = \frac{|w| \sin(t - \phi)}{|e^{it} - w|^2},
$$

and

$$
g''_w(t) = \frac{|w| \cos(t - \phi)}{|e^{it} - w|^4} - \frac{2|w|^2 \sin^2(t - \phi)}{|e^{it} - w|^4}.
$$

The inequality of the following lemma is immediate.

**Lemma 3.10.** There is an absolute constant $c_5 > 0$ such that

$$
|g''_w(t)| \leq \frac{c_5}{|e^{it} - w|^2}
$$

for every $t \in \mathbb{R}$ and $w \in \mathbb{C}$ with $|w| \leq 2$.

Combining Lemmas 3.9 and 3.10 we get the following.

**Lemma 3.11.** Let $a_j = e^{i\tau_j}$, $j = 1, 2, \ldots, m + 1$, be as before (defined after Lemma 3.3). There are $\xi_j \in [\tau_{j-1}, \tau_j]$ and an absolute constant $c_6 > 0$ such that

$$
\int_{\tau_{j-1}}^{\tau_j} g_w(t) \, dt - \frac{1}{2} (g_w(\tau_j) + g_w(\tau_{j-1}))(\tau_j - \tau_{j-1}) \geq \frac{-c_6}{n^3 |e^{i\xi_j} - w|^2}
$$

for every $t \in \mathbb{R}$ and $w \in \mathbb{C}$ with $|w| \leq 2$.

We will also need an estimate better than the one given in Lemma 3.11 in the case when $w \in \mathbb{C}$ is close to $a_j := e^{i\tau_j}$.
Lemma 3.12. Let \( a_j = e^{i\tau_j}, j = 2, 3, \ldots, m + 1 \), be as before (defined after Lemma 3.3). There is an absolute constant \( c_7 > 0 \) such that

\[
\int_{\tau_{j-1}}^{\tau_j} g_w(t) \, dt - \frac{1}{2} g_w(\tau_j) + g_w(\tau_{j-1}) (\tau_j - \tau_{j-1}) \geq -\frac{c_7}{n}
\]

for every \( t \in \mathbb{R} \) and \( w \in \mathbb{C} \) such that \( |w - a_j| \leq 8\pi/n \).

To prove Lemma 3.12 we need the following observation.

Lemma 3.13. Let \( a_j = e^{i\tau_j}, j = 1, 2, \ldots, m + 1 \), be as before (defined after Lemma 3.3). There is an absolute constant \( c_8 > 0 \) such that

\[
|a_j - w| \geq \frac{c_8}{n}
\]

for every \( w \in \mathbb{C} \) for which \( P(w) = 0 \).

Proof of Lemma 3.13. The proof is a routine combination of (1.1), Lemma 3.3, and a couple of Bernstein’s inequalities. One of Bernstein’s polynomial inequalities asserts that

\[
\max_{z \in \partial D} |P'(z)| \leq n \max_{z \in \partial D} |P(z)|
\]

for any polynomials \( P \in \mathcal{P}_n^c \). See [BE-95, p. 232], for instance. Another polynomial inequality of Bernstein we need in this proof is (3.2).

Suppose \( w \in \mathbb{C}, |w - a_j| \leq c/n \), and \( P(w) = 0 \). Let \( \Gamma \) be the line segment connecting \( a_j = e^{i\tau_j} \) and \( w \). We have

\[
(2\delta)^{1/2}n^{1/2} \leq |P_k(a_j)| = |P_k(a_j) - P_k(w)| = \left| \int_{\Gamma} P_k'(z) \, dz \right|
\]

\[
\leq \int_{\Gamma} |P_k'(z)| \, |dz|.
\]

Hence there is a \( \zeta \in \Gamma \) such that

\[
|P_k'(\zeta)| \cdot |a_j - w| \geq (2\delta)^{1/2} n^{1/2}.
\]

Combining this with (3.6) and \( |\zeta - a_j| \leq |w - a_j| \leq c/n \), we obtain

\[
|P_k'(\zeta)| \geq \frac{(2\delta)^{1/2} n^{1/2}}{c/n} = \frac{(2\delta)^{1/2}}{c} n^{3/2}.
\]

On the other hand, combining (3.2), (3.6), and (1.1), we obtain

\[
|P_k'(\zeta)| \leq \left( \max\{1, |\zeta|^{n-1}\} \right) \left( \max_{z \in \partial D} |P_k'(z)| \right)
\]

\[
\leq \left( \max\{1, |\zeta|^{n-1}\} \right) \left( n \max_{z \in \partial D} |P_k(z)| \right) \leq \left( 1 + \frac{c}{n} \right)^n n(2n)^{1/2}
\]

\[
\leq e^c \sqrt{2n^{3/2}}
\]
Combining this with (3.7), we get $\delta^{1/2} \leq ce^c$. □

**Proof of Lemma 3.12.** Observe that Lemma 3.13 implies that there is an absolute constant $c_8 > 0$ such that

$$\frac{1}{2} (\log |e^{i\tau_j} - w| + \log |e^{i\tau_{j-1}} - w|) \leq \log(c_8/n) = \log c_8 - \log n. \tag{3.8}$$

Now we show that there is an absolute constant $c_9 > 0$ such that

$$\int_{\tau_{j-1}}^{\tau_j} \log |e^{it} - w| \, dt \geq (\tau_j - \tau_{j-1})(-c_9 - \log n). \tag{3.9}$$

To see this let $w = |w|e^{i\phi}$. We have

$$|e^{it} - w| = |e^{it} - |w|e^{i\phi}| \geq |e^{it} - e^{i\phi}| = 2 \sin \left| \frac{t - \phi}{2} \right| = \frac{2}{\pi} |t - \phi|$$

whenever $|t - \phi| \leq \pi$. Hence, if

$$\phi \in \left[ \tau_{j-1} + \frac{c_8}{2n}, \tau_j - \frac{c_8}{2n} \right],$$

then

$$\int_{\tau_{j-1}}^{\tau_j} \log |e^{it} - w| \, dt \geq \int_{\tau_{j-1}}^{\tau_j} \log \left( \frac{2}{\pi} |t - \phi| \right) \, dt$$

$$= \int_{\tau_{j-1}}^{\phi} \log \left( \frac{2}{\pi} (\phi - t) \right) \, dt + \int_{\phi}^{\tau_j} \log \left( \frac{2}{\pi} (t - \phi) \right) \, dt$$

$$\geq (\tau_j - \tau_{j-1})(-c_9 - \log n)$$

with an absolute constant $c_9 > 0$, and (3.9) follows. While, if

$$\phi \notin \left[ \tau_{j-1} + \frac{c_8}{2n}, \tau_j - \frac{c_8}{2n} \right],$$

then Lemma 3.13 implies that there is an absolute constant $c_{10} > 0$ such that

$$\min_{t \in [\tau_{j-1}, \tau_j]} |e^{it} - w| \geq \frac{c_{10}}{n},$$

and hence

$$\int_{\tau_{j-1}}^{\tau_j} \log |e^{it} - w| \, dt \geq \int_{\tau_{j-1}}^{\tau_j} \log(c_{10}/n) \, dt$$

$$\geq (\tau_j - \tau_{j-1})(-c_{11} - \log n)$$

with an absolute constant $c_{11} > 0$, and (3.9) follows again. Combining (3.8) and (3.9) and recalling that $g_w(t) := \log |e^{it} - w|$ and $\tau_j - \tau_{j-1} \leq 4\pi/n$, we obtain the inequality of the lemma. □
Lemma 3.14. There is an absolute constant $c_{12} > 0$ such that

$$\int_{\tau_{j-1}}^{\tau_j} \log \left( \frac{R_n(t)}{n} \right) \geq -\frac{c_{12}}{n}$$

for every $j \in \{2, 3, \ldots, m+1\}$.

Proof of Lemma 3.14. Let, as before, $w_{\nu}, \nu = 1, 2, \ldots, n-1$, denote the zeros of $P_k$. Recall that $|w_{\nu}| \leq 2$ for each $\nu = 1, 2, \ldots, n-1$. We define the annuli

$$E_{j,q} := D(a_j, 2^{q+3/\pi} n) \setminus D(a_j, 2^{q+2/\pi} n), \quad q = 1, 2, \ldots,$$

and the disk

$$E_{j,0} := D(a_j, 8\pi / n).$$

Observe that the sets $E_{j,q}$ are pairwise disjoint and

$$C = \bigcup_{j=0}^{\infty} E_{j,q}.$$

By Lemmas 3.2 and 3.3 there is an absolute constant $c_1 > 0$ (depending only on the explicitly given value of $\delta$) such that $E_{j,q}$ contains at most $c_1(n2^{q+3/\pi}/n + 1)$ zeros of $P_k$ and $E_{j,0}$ contains at most $c_1(8\pi + 1)$ zeros of $P_k$. Hence Lemmas 3.11 and Lemma 3.12 give that

$$\int_{\tau_{j-1}}^{\tau_j} \log \left( \frac{R_n(t)}{n} \right) dt - \frac{1}{2}(\log(R_n(\tau_j)) + \log(R_n(\tau_{j-1})))$$

$$\geq \sum_{q=0}^{\infty} \sum_{w_{\nu} \in E_q} \left( \int_{\tau_{j-1}}^{\tau_j} g_{w_{\nu}}(t) dt - \frac{1}{2}(g_{w_{\nu}}(\tau_j) + g_{w_{\nu}}(\tau_{j-1})) \right)(\tau_j - \tau_{j-1})$$

$$\geq \sum_{q=1}^{\infty} \frac{c_1(8\pi + 1)}{n} - \frac{c_6}{n^3(2^{q+2/\pi}/n)^2}$$

$$\geq c_1(8\pi + 1) - \sum_{q=1}^{\infty} \frac{c_1 c_6}{2^q n}$$

$$\geq - \frac{c_{13}}{n}$$

with an absolute constant $c_{13} > 0$. Now recall that $R_n(\tau_{j-1}) \geq 2\delta n$ and $R_n(\tau_j) \geq 2\delta n$, and the result follows. □

Now we are ready to prove Lemma 3.8.
Proof of Lemma 3.8. Given $\varepsilon \in (0, 1)$, let

$$I_{n, \varepsilon} := \left\{ j \in \{2, 3, \ldots, m+1\} : \min_{t \in [\tau_{j-1}, \tau_j]} R_n(t) < \varepsilon \right\},$$

and let

$$J_{n, \varepsilon} := \bigcup_{j \in I_{n, \varepsilon}} [\tau_{j-1}, \tau_j].$$

Using that $0 \leq R_n(t) \leq 2n$ for every $t \in K$, and then using Lemmas 3.4 and 3.14, we get

$$\int_{B_{n, \varepsilon}} \log \frac{R_n(t)}{2n} \, dt \geq \int_{J_{n, \varepsilon}} \log \frac{R_n(t)}{2n} \, dt = \sum_{j \in I_{n, \varepsilon}} \int_{\tau_{j-1}}^{\tau_j} \log \frac{R_n(t)}{2n} \, dt \geq c_3 n \varepsilon^{1/2} (-c_{12}/n) \geq -c_4 \varepsilon^{1/2}$$

for every sufficiently large $n \geq n_\varepsilon$, where $c_4 = c_3 c_{12} > 0$, and the lemma is proved. \(\square\)

4. Proof of the Theorem

Proof of Theorem 2.1. It follows from (1.2) immediately that

$$\lim_{n \to \infty} \frac{M_0(P_k)}{n^{1/2}} = \lim_{n \to \infty} \frac{M_0(Q_n)}{n^{1/2}},$$

so it is sufficient to prove the asymptotic formula only for $M_0(P_k)$.

Let $\varepsilon \in (0, 1)$ be fixed. By Lemma 3.7 we have

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_{A_{n, \varepsilon}} \log \frac{R_n(t)}{2n} \, dt = \int_{\varepsilon}^{1} \log x \, dx,$$

while it follows from Lemma 3.8 and the inequalities $0 \leq R_n(t) \leq 2n$ that there is an absolute constant $c_4 > 0$ such that

$$-c_4 \varepsilon^{1/2} \leq \limsup_{n \to \infty} \frac{1}{2\pi} \int_{B_{n, \varepsilon}} \log \frac{R_n(t)}{2n} \, dt \leq 0$$

for every sufficiently large $n \geq n_\varepsilon$. As $K$ is the disjoint union of $A_{n, \varepsilon}$ and $B_{n, \varepsilon}$, we have

$$(4.1) \quad \limsup_{n \to \infty} \frac{1}{2\pi} \int_{K} \log \frac{R_n(t)}{2n} \, dt \leq \int_{\varepsilon}^{1} \log x \, dx$$

and

$$(4.2) \quad \liminf_{n \to \infty} \frac{1}{2\pi} \int_{K} \log \frac{R_n(t)}{2n} \, dt \geq \int_{\varepsilon}^{1} \log x \, dx - c_4 \varepsilon^{1/2}.$$
As (4.1) and (4.2) hold for an arbitrary \( \varepsilon \in (0, 1) \), it follows that
\[
\int_0^1 \log x \, dx \leq \lim \inf_{n \to \infty} \frac{1}{2\pi} \int_K \log \frac{R_n(t)}{2n} \, dt \leq \lim \sup_{n \to \infty} \frac{1}{2\pi} \int_K \log \frac{R_n(t)}{2n} \, dt \leq \int_0^1 \log x \, dx,
\]
and hence
\[
\lim_{n \to \infty} \frac{1}{2\pi} \int_K \log \frac{R_n(t)}{2n} \, dt = -1.
\]
Hence, recalling that
\[
R_n(t) := |P_k(e^{it})|^2, \quad t \in K,
\]
we obtain
\[
\lim_{n \to \infty} \frac{1}{2\pi} \int_K \log \frac{|P_k(e^{it})|}{(2n)^{1/2}} \, dt = \lim_{n \to \infty} \frac{1}{2\pi} \int_K \log \left( \frac{R_n(t)}{2n} \right)^{1/2} \, dt
\]
\[
= \lim_{n \to \infty} \frac{1}{2\pi} \frac{1}{2} \int_K \log \frac{R_n(t)}{2n} \, dt = -1/2.
\]
Hence
\[
\lim_{n \to \infty} \frac{M_0(P_k)}{(2n)^{1/2}} = \lim_{n \to \infty} \exp \left( \frac{1}{2\pi} \int_K \log \frac{|P_k(e^{it})|}{(2n)^{1/2}} \, dt \right)
\]
\[
= \exp \left( \lim_{n \to \infty} \frac{1}{2\pi} \int_K \log \frac{|P_k(e^{it})|}{(2n)^{1/2}} \, dt \right) = \exp(-1/2),
\]
which is the asymptotic formula for \( M_0(P_k) \) stated in the theorem. \( \Box \)

5. The Mahler measure of the Fekete polynomials

For a prime \( p \) the \( p \)-th Fekete polynomial is defined as
\[
f_p(z) := \sum_{k=1}^{p-1} \left( \frac{k}{p} \right) z^k,
\]
where
\[
\left( \frac{k}{p} \right) = \begin{cases} 
1, & \text{if } x^2 \equiv k \pmod{p} \text{ for an } x \not\equiv 0 \pmod{p}, \\
0, & \text{if } p \text{ divides } k, \\
-1, & \text{otherwise}
\end{cases}
\]
is the usual Legendre symbol. Since \( f_p \) has constant coefficient 0, it is not a Littlewood polynomial, but \( g_p \) defined by \( g_p(z) := f_p(z)/z \) is a Littlewood polynomial of degree \( p - 2 \), and has the same Mahler measure as \( f_p \). Fekete polynomials are examined in detail in [B-02], [CG-00], [E-11], [E-12], [EL-07], and [M-80]. In [CE-15a] and [CE-15b] the authors examined the maximal size of the Mahler measure and the \( L_q \) norms of sums of \( n \) monomials on the unit circle as well as on subarcs of the unit circles. In the constructions appearing in [CE-15a] properties of the Fekete polynomials \( f_p \) turned out to be quite useful. Montgomery [M-80] proved the following fundamental result.
**Theorem 5.1.** There are absolute constants $c_{14} > 0$ and $c_{15} > 0$ such that
\[ c_{14} \sqrt{p} \log \log p \leq \max_{z \in \partial D} |f_p(z)| \leq c_{15} \sqrt{p} \log p. \]

In [E-07] we proved the following result.

**Theorem 5.2.** For every $\varepsilon > 0$ there is a constant $c_\varepsilon$ such that
\[ M_0(f_p) \geq \left( \frac{1}{2} - \varepsilon \right) \sqrt{p} \]
for all primes $p \geq c_\varepsilon$.

In [E-18] the factor $\left( \frac{1}{2} - \varepsilon \right)$ in Theorem 1.2 has been improved to an absolute constant $c > 1/2$. Namely we prove the following.

**Theorem 5.3.** There is an absolute constant $c > 1/2$ such that
\[ M_0(f_p) \geq c \sqrt{p} \]
for all sufficiently large primes.

The determine the asymptotic size of the Mahler measure $M_0(f_p)$ of the Fekete polynomials $f_p$ seems to be beyond reach at the moment. Not even a (published or unpublished) conjecture seems to be known.

6. Acknowledgement

The author thanks Stephen Choi and Bahman Saffari for checking the details of the proof in this paper and for their suggestions to make the paper more readable.

References

BN-73. E. Beller and D.J. Newman, *An extremal problem for the geometric mean of polynomials*, Proc. Amer. Math. Soc. **39** (1973), 313–317.

B-02. P. Borwein, *Computational Excursions in Analysis and Number Theory*, Springer, New York, 2002.

BE-95. P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer, New York, 1995.

BL-01. P. Borwein and R. Lockhart, *The expected $L_p$ norm of random polynomials*, Proc. Amer. Math. Soc. **129** (2001), 1463–1472.

BM-00. P. Borwein and M.J. Mossinghoff, *Rudin-Shapiro like polynomials in $L_4$*, Math. Comp. **69** (2000), 1157–1166.

B-73. J. Brillhart, *On the Rudin-Shapiro polynomials*, Duke Math. J. **40** (1973), no. 2, 335–353.

BL-76. J. Brillhart, J.S. Lemont, and P. Morton, *Cyclotomic properties of the Rudin-Shapiro polynomials*, J. Reine Angew. Math. (Crelle’s J.) **288** (1976), 37–65.

CE-15a. K.-K. S. Choi and T. Erdélyi, *Sums of monomials with large Mahler measure*, J. Approx. Theory **197** (2015), 49–61.
CE-15b. K.-K. S. Choi and T. Erdélyi, *On a problem of Bourgain concerning the $L_p$ norms of exponential sums*, Math. Zeit. **279** (2015), 577–584.

CE-15c. K.-K. S. Choi and T. Erdélyi, *On the average Mahler measures on Littlewood polynomials*, Proc. Amer. Math. Soc. Ser. B **1** (2015), 105–120.

CM-11. K.-K. S. Choi and M.J. Mossinghoff, *Average Mahler’s measure and $L_p$ norms of unimodular polynomials*, Pacific J. Math. **252** (2011), no. 1, 31–50.

CG-00. B. Conrey, A. Granville, B. Poonen, and K. Soundararajan, *Zeros of Fekete polynomials*, Ann. Inst. Fourier (Grenoble) **50** (2000), 865–884.

D-05. Ch. Doche, *Even moments of generalized Rudin-Shapiro polynomials*, Math. Comp. **74** (2005), no. 252, 1923–1935.

DH-04. Ch. Doche and L. Habsieger, *Moments of the Rudin-Shapiro polynomials*, J. Fourier Anal. Appl. **10** (2004), no. 5, 497–505.

EZ-17. S.B. Ekhad and D. Zeilberger, *Integrals involving Rudin-Shapiro polynomials and sketch of a proof of Saffari’s conjecture*, in Analytic Number Theory, Modular Forms and $q$-Hypergeometric Series, G.E. Andrews, F.G. Garvan (Eds.) **221** (2017), Springer Proceedings in Mathematics & Statistics, Springer, Cham, 253–265.

E-11. T. Erdélyi, *Sieve-type lower bounds for the Mahler measure of polynomials on subarcs*, Computational Methods and Function Theory **11** (2011), 213–228.

E-12. T. Erdélyi, *Upper bounds for the $L_q$ norm of Fekete polynomials on subarcs*, Acta Arith. **153** (2012), no. 1, 81–91.

E-16. T. Erdélyi, *The Mahler measure of the Rudin-Shapiro polynomials*, Constr. Approx. **43** (2016), no. 3, 353–569.

E-18. T. Erdélyi, *Improved lower bound for the Mahler measure of the Fekete polynomials*, Constr. Approx. **48** (2018), no. 2, 383–399.

EL-07. T. Erdélyi and D. Lubinsky, *Large sieve inequalities via subharmonic methods and the Mahler measure of Fekete polynomials*, Canad. J. Math. **59** (2007), 730–741.

G-51. M.J. Golay, *Static multislit spectrometry and its application to the panoramic display of infrared spectra*, J. Opt. Soc. America **41** (1951), 468–472.

HL-52. G.H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Univ. Press, London, 1952.

L-68. J.E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath Mathematical Monographs, Lexington, Massachusetts, 1968.

M-80. H.L. Montgomery, *An exponential polynomial formed with the Legendre symbol*, Acta Arith. **37** (1980), 375–380.

R-17. B. Rodgers, *On the distribution of Rudin-Shapiro polynomials and lacunary walks on $SU(2)$*, Adv. Math. **320** (2017), 993–1008.

S-51. H.S. Shapiro, *Extremal problems for polynomials and power series*, Master thesis, MIT, 1951.

Department of Mathematics, Texas A&M University, College Station, Texas 77843, College Station, Texas 77843 (T. Erdélyi)

E-mail address: terdelyi@math.tamu.edu