EMBEDDINGS OF ITERATION TREES

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1 — Definition of the Embeddings

Notation. We regard an iteration tree $\mathcal{T}$ on a model $M$ as a sequence $(E_\nu : \nu < \text{domain}(\mathcal{T}))$. The iterated ultrapowers $M_\nu = \text{ult}(M, \mathcal{T}[\nu])$ of $M$ by $\mathcal{T}$, and the tree orderings $\prec_\mathcal{T}$ and $\prec'_\mathcal{T}$ on domain($\mathcal{T}$), are defined by iteration on $\nu \in \text{domain}(\mathcal{T})$. Here $\prec'$ is the complete tree ordering and $\prec$ is the subordering of those nodes where there is actually an embedding, i.e., where there is no dropping to a mouse in the interval, so $\nu \prec \nu'$ implies that there is $i_{\nu', \nu} : M_\nu \rightarrow M_{\nu'}$.

1.1 Definition. (1) We say a set $\kappa$ is a support for an iterated ultrapower using measures.

(2) The empty set is a support for any set $x$.

(3) $E_\nu \in M_\nu$.

(4) If $E_\nu$ is a extender on $M_\nu$ then $\nu^* \prec \nu + 1$, $M_{\nu+1} = \text{ult}(M_{\nu^*}, E_\nu)$, and $i_{\nu^*, \nu+1}$ is the canonical embedding.

(5) If $E_\nu$ is not a extender on $M_\nu$ then $\nu^* \prec' \nu + 1$ but $\nu^* \not\prec \nu + 1$. In this case $M_{\nu+1} = \text{ult}(m_{\nu^*}, E_\nu)$ where $m_{\nu^*}$ is the least mouse in $M_{\nu^*}$ such that there is a subset of crit($E_\nu$) which is definable in $m_{\nu^*}$ but is not a member of $M_\nu$.

We write $\gamma_\nu$ for index $E_\nu$, the ordinal such that $E_\nu = \mathcal{E}_\nu(\gamma_\nu)$. We also write $\kappa_\nu = \text{crit}(E_\nu)$ and $\rho_\nu = \text{len}(E_\nu)$.

Write $M_{\nu, \kappa}$ for the least mouse $m$ which is not in $M_{\nu+1}$ such that $\text{proj}(m) \geq \kappa$. Then $m_{\nu^*} = M_{\nu^*, \kappa_\nu}$, since $\mathcal{P}^{M_\nu}(\kappa)$ is constant for all $\xi > \nu^*$.

I will use $\infty$ to mean the main branch of a tree $\mathcal{T}$. Thus $i_{0, \infty} : M_0 \rightarrow \text{ult}(M_0, \mathcal{T})$ is the canonical embedding.

**basic definitions:** The definition of a support is a direct generalization of the notion of a support for an iterated ultraprodut using measures.

1.1 Definition. (1) We say a set $y \subset \text{domain}(\mathcal{T})$ is a support if for each $\nu \in y$ the ordinal $\nu^* \in y$ is also in $y$, $y \cap \nu$ is a support for $E_\nu$ in $\mathcal{T}[\nu]$, and if the tree dropped to a mouse at stage $\nu$ then $y \cap \nu^*$ is a support for $m_{\nu^*}$.

(2) The empty set is a support for any set $x \in M_0$. If $\alpha > 0$ then $y \subset \alpha$ is a support in $\mathcal{T}|\alpha$ for $x \in M_\alpha$ if either there is $\alpha' < \alpha$ and $x' \in M_{\alpha'}$ such that $x = i_{\alpha', \alpha}(x')$, or $\alpha = \nu + 1$ and there are $a \in \text{len}(E_\nu)]^{<\omega}$ and $f \in M_{\nu^*}$ such that $x = [f]_a$, $y \cap \nu$ is a support for $a$, and $y \cap \nu^*$ is a support for $f$.

Note that rather than require $y \cap \nu^*$ be a support for $m_{\nu^*}$, it is enough to require that it be a support for $\kappa_\nu$ and $\gamma_\nu$, since $m_{\nu^*} = M_{\nu^*, \kappa_\nu}$.
The definition of embedding of iteration trees may be more complicated than necessary because I’m trying to combine two more or less different constructions.

1.2 Definition. An embedding from an iteration tree $T$ on $M$ into an iteration tree $T'$ on $M'$ is a pair $(\sigma, i)$ such $i : M \rightarrow M'$ is an elementary embedding and $\sigma$ maps $\text{domain}(T)$ into $\text{domain}(T')$ so that

1. (1) $\sigma$ is a support in $T'$.
2. (2) For all $\nu \in \text{domain}(T)$, $i_{\nu}^{\sigma}(E_{\nu}^T) = E_{\sigma(\nu)}^{T'}$.
3. (3) For all $\nu \in \text{domain}(T)$, $\sigma(\nu) + 1 \leq_{T'} \sigma(\nu + 1)$.

Where $i_{\nu+1}^{\sigma}(\lfloor f \rfloor_{a}) = i^{T'}_{\sigma(\nu)+1,\sigma(\nu+1)}([i^{\sigma}_{\nu}(f)]_{i^{\sigma}_{\nu}(a)})$.

This recursive definition of $i_{\nu}^{\sigma}$ is justified by the following proposition:

1.3 Proposition. (1) $\sigma(\nu^*) = (\sigma(\nu))^*$.
2. (2) $m_{\nu}^*$ exists in $T$ if and only if $m_{\sigma(\nu)}^*$ exists in $T'$, and in this case $m_{\sigma(\nu)}^* = i_{\nu}^{\sigma}(m_{\nu}^*)$.
3. (3) If $\alpha < \beta$ then $i_{\beta}^{\sigma}((\rho_{\alpha})) = i_{\alpha}^{\sigma}((\rho_{\alpha}))$, and $i_{\beta}^{\sigma}((\rho_{\alpha})) \geq i_{\alpha}^{\sigma}((\rho_{\alpha}))$.
4. (4) $i_{\nu}^{\sigma}$ is an elementary embedding from $M_{\nu}^T$ into $M_{\sigma(\nu)}^{T'}$.

Proof. The proof is by induction on $\nu$. We assume that the last two clauses are true up through $\nu$ to prove the first two clauses for $\nu$, and then prove the last two clauses for $\nu + 1$.

Since $\text{crit}(E_{\nu}^{T'}) < \text{len}(E_{\nu^*}^{T'})$, clause (3) implies that

(by 0.2-2)

$$\text{crit}(E_{\sigma(\nu)}^{T'}) = i_{\nu}^{\sigma}(\text{crit}(E_{\nu}^{T'}))$$

(by 0.3-3)

$$i_{\nu}^{\sigma}(\text{crit}(E_{\nu}^{T'})) < \sigma_{\nu^*}(\text{len}(E_{\nu^*}^{T'})) = \text{len}(E_{\sigma(\nu^*)}^{T'})$$

and hence we must have $(\sigma(\nu))^* \leq \sigma(\nu^*)$. On the other hand the assumption that range $\sigma$ is a support implies that $(\sigma(\nu))^* = \sigma(\nu')$ for some $\nu' \leq \nu^*$. If $\nu' < \nu^*$ then similarly we have

$$i_{\nu}^{\sigma}(\text{crit}(E_{\nu}^{T'})) = \text{crit}(E_{\sigma(\nu)}^{T'})$$

$$< \rho'_{\sigma(\nu')} = \text{len}(E_{\sigma(\nu')}^{T'})$$

$$= i_{\nu}^{\sigma}(\text{len}(E_{\nu'}^{T'})) = i_{\nu}^{\sigma}(\rho_{\nu'})$$

but since $\text{crit}(E_{\nu}^{T'}) \geq \rho_{\nu'}$ clause (3) implies

$$i_{\nu}^{\sigma}(\kappa_{\nu}) \geq i_{\nu}^{\sigma}(\rho_{\nu'}) \geq i_{\nu}^{\sigma}(\rho_{\nu'})$$

and the contradiction shows that $(\sigma(\nu))^* = \sigma(\nu^*)$.

The proof of item (2) is easy:

$$i_{\nu}^{\sigma}(m_{\nu}^*) = i_{\nu}^{\sigma}(M_{\nu^*,\kappa_{\nu}})$$

$$= M'_{\sigma(\nu^*),i_{\nu}^{\sigma}(\kappa_{\nu})}$$

$$= M'_{\sigma(\nu^*),i_{\nu}^{\sigma}(\kappa_{\nu})} = m_{\nu}^{T'}.$$
Finally we verify clause (3). Set $i' : \text{ult}(M_\nu, E_\nu) \rightarrow \text{ult}(M'_{\sigma(\nu)}, E'_{\sigma(\nu)})$ by setting $i'([f]_a) = [i_{\sigma(\nu)}^\nu(f)]_{i^\nu(\alpha)}$. Then $i'((\gamma_a + 1) = i_{\nu}^\sigma(\gamma_a + 1)$. (If $M_\nu \models ZF$ then $i' = i_{\nu}^\sigma \upharpoonright \text{ult}(M_\nu, E_\nu)$. If $M_\nu$ is a mouse this doesn’t follow, but $M_\nu$ can calculate $\text{ult}(M_\nu, E_\nu)$ up at least to $\gamma_a + 1$, and this implies the claim.) But for $f : \kappa \rightarrow V_{\kappa+1}$ in $M_\nu$ and $a < \text{len}(E_\nu)$, if $x = [f]_a$ then

$$[i_{\nu}^\sigma(f)]_{i^\nu(\alpha)} = [i_{\nu}^\sigma(f)]_{i^\nu(\alpha)} = i_{\nu}^\sigma(x)$$

where the equivalence classes are in $\text{ult}(M'_{\sigma(\nu)}, E'_{\sigma(\nu)})$, $\text{ult}(M'_{\sigma(\nu)}, E'_{\sigma(\nu)})$, and $\text{ult}(M_\nu, E_\nu)$ respectively. The first equality follows from the induction hypothesis. Now if $x = [f]_a \leq \gamma_\nu$ then

$$(\text{def of } i_{\nu+1}^\sigma)$$

$$i_{\nu+1}^\sigma(x) = i_{\nu+1}^\sigma([f]_a) = i_{\sigma(\nu)+1,\sigma(\nu)+1}^{T'}([i_{\nu}^\sigma(f)]_{i^\nu(\alpha)})$$

$$(\text{ind hyp})$$

$$= i_{\sigma(\nu)+1,\sigma(\nu)+1}^{T'}([i_{\nu}^\sigma(f)]_{i^\nu(\alpha)}) = i_{\nu}^\sigma(x)$$

$$(\text{def of } i')$$

Now if $x < \rho_\nu$ then $i'(x) < \rho_\nu$ and hence the last line is equal to $i'(x) = i_{\nu+1}^\sigma(x)$. We have in general that the last line is at least as big as $i'(x) = i_{\nu+1}^\sigma(x)$. This establishes item (3) of the proposition.

This actually establishes a little more. If $\kappa < \rho_\nu$ then either $\kappa^+ \leq \rho_\nu$ in $M_{\nu+1}$ or $\kappa$ is a cardinal and $\rho_\nu = \kappa + 1$. In the second case clearly we get $i_{\nu+1}^\sigma(\rho_\nu) = i_{\nu}^\sigma(\rho_\nu)$.

2 — Two Applications

I have two major applications in mind for this machinery. The first is a generalization of the characterization of ordinary iterated ultrapowers as a direct limit of finite iterated ultrapowers, and the consequent development of the theory of iterated ultrapowers. In this direction we have

2.1 Proposition. (1) If $y$ is a support in the tree $T$ on $M$ then the pair $(\sigma, \text{id})$ is a tree embedding, where $\sigma$ takes the order type of $y$ isomorphically onto $y$.

(2) Every member of the tree iteration $\text{ult}(M, T)$ has a finite support in $T$.

(3) Any tree iteration is the direct limit of iterations by finite trees.

This gives rise to a theory of indiscernibles since if $\sigma$ and $\sigma'$ are two embeddings from $T$ to $T'$ and $x \in \text{ult}(M, T)$ then $i^\sigma(x)$ and $i^\sigma'(x)$ satisfy the same formulas in $\text{ult}(M, T')$. Thus in order to get a model with $\Sigma_2$ indiscernibles we define a tree embedding: $(E_\nu : \nu < \theta)$ of $M$ by recursion on $\nu$, together with a subset $C$ of domain($\mathcal{T}$) such that embeddings of finite trees into $C$ will give $\Sigma_2$ indiscernibles: $E_\nu$ is the least extender $E$ such that either (1) there is a embedding from a finite tree $T'$ into $T|\nu$ such that there is a one step extension $T''$ of $T'$ which cannot be extended into $T|\nu$, but can be extended into $T|\nu + 1$ if $E_\nu$ is taken to be $E$, or (2) there is such a one step extension which could be extended into $T|\nu$, but cannot be extended so that domain($T''$) is mapped to a member of $C | \nu$. In the second case $\nu$ is added to $C$.

The second application comes up when we have an embedding $i : M \rightarrow M'$ and a tree $\mathcal{T}$ on $M$. In this case we want to define a tree $i[\mathcal{T}]$ on $M'$ with embeddings...
between the ultrapowers. In this case the embedding is the pair $\sigma = (id, i)$ and $i[\mathcal{T}]$ is defined to be $(i^\sigma_\nu(E_\nu) : \nu \in \text{domain}(\mathcal{T}))$. In particular I am interested in the case $i = i^\mathcal{T} : M \to M' = \text{ult}(M, \mathcal{T})$. Then $\mathcal{T}' = i^\mathcal{T}[\mathcal{T}]$ is an iteration tree on $M'$ and $i^{\mathcal{T}'} : M' \to M'' = \text{ult}(M', i^\mathcal{T}[\mathcal{T}])$ is not the canonical embedding, but instead corresponds to the embedding $\text{ult}(M, U) \xrightarrow{i^U} \text{ult}(\text{ult}(M, U), U) \cong \text{ult}(\text{ult}(M, U), i^U(U))$. This construction is what I need for the argument at the end of my notes titled “The Minimal Model for a Woodin Cardinal” that any two mice which agree up as far as their projectums can be compared. The tree $\mathcal{T}$ here is the tree giving the original embedding $Q$ there, with $\gamma$ equal to the sup of the lengths of the extenders in $\mathcal{T}$. If $M = L(\mathcal{F})$ and $M' = L(\mathcal{F}')$ then $i^*_\infty : M' \to M''$ can be treated as an extender on $M'$, and hence on $L_\gamma(\mathcal{F}')$. This is the extender $Q$ of those notes.

Note that this leaves two problems. First, I haven’t yet defined $i^Q[Q]$. This can be done, using further iterations of the tree $\mathcal{T}$, but it needs to be checked that this is an extender on $\text{ult}(L_\gamma(\mathcal{F}'))$. This doesn’t seem likely to lead to serious problems. The second problem is to prove the iterability of the model $(L_\gamma(\mathcal{E}), \mathcal{E}, Q, F)$ in those notes. Of course no final answer to this is possible without a general proof of iterability, but it should be possible to show that if this model is not iterable then there is a regular tree with no well founded branch. The proof would first involve embedding this tree into a tree using proper internal extenders (internal including the external ones we started trying to prove are in the original mouse) with improper backtracking and then embedding this tree into a tree with proper backtracking.

Note: I don’t see any reason to think that $i^Q(Q)$ can be defined as an extender on $\text{ult}(J_\gamma(\mathcal{E}), Q)$ However it should be possible to carry out the proof using the full embedding given by $i^\sigma$, rather than using the extender embedding. In fact the interest is really only in the initial segment of $Q$ which is an extender.

3 — Normalization of Trees

A general iteration tree may represented using Steel’s notation as

$$\mathcal{T} = (T, \text{deg}, D, (E_\alpha, M^*_\alpha + 1 : \alpha < \theta)).$$

We do not assume that the lengths $\rho_\alpha$ of the extenders $E_\alpha$ are increasing, but we do require that if $\alpha^* = T\text{-pred}(\alpha + 1) \leq \nu < \alpha$ then $\rho_\nu \geq \text{crit}(E_\alpha)$. (This is necessary for the theory to make any sense: otherwise $E^{M^*_\alpha} \downarrow \text{crit}(E_\alpha) \notin M_\alpha$ so there is no reasonable hope that the power sets of $\text{crit}(E_\alpha)$ in the two models will be closely related).

The tree $\mathcal{T}$ is normal if for all $\alpha < \theta$

1. If $\alpha < \alpha'$ then $\text{index}(E_\alpha) < \text{index}(E_{\alpha'})$ (or, equivalently, $\rho_\alpha < \rho_{\alpha'}$).
2. $\alpha^* = T\text{-pred}(\alpha + 1)$ is the least ordinal $\nu$ such that $\text{crit}(E_\alpha) < len(E_\nu)$.
3. $M^*_{\alpha + 1} = \text{ult}(\mathcal{M}_{\alpha^*, \kappa_\alpha}, E_\alpha)$, where $\kappa_\alpha = \text{crit}(E_\alpha)$ (i.e.: $M^*_{\alpha + 1} = \mathcal{M}_{\alpha^*, \kappa}$, which is the largest mouse $m$ such that $E_\nu$ is an extender on $m$.

An iteration strategy is a function $s$ on trees such that if $\mathcal{T}$ is any tree $\mathcal{T}$ such that

$$s(\mathcal{T}') = \{ \nu < \lambda : \nu < \text{len}(\mathcal{T}_\lambda) \}$$

for each limit ordinal $\lambda < \text{len}(\mathcal{T})$. 
then \( s(T) \) is a maximal, cofinal branch of \( T \) such that \( M_b^T \) is well founded. The function \( s \) is an iteration strategy for normal trees if the above is true for every normal tree \( T \).

Note that if \( T \) is a normal tree following an iteration strategy \( s \) then \( T \) is determined by the sequence \( (E_\alpha : \alpha < \theta) \) of extenders of \( T \), so we will write \( T = (E_\alpha : \alpha < \theta) \) in this case.

If \( G \) is a class of premice then we will write \( s_G \) for the function defined by \( s_G(T) = b \) if and only if \( b \) is the unique branch of \( T \) such that \( M_b^T \in G \).

3.1 Theorem. Suppose that \( G \) is a class of premice such that every member of \( G \) is well founded, any \( \sigma_0 \)-elementary submodel of a member of \( G \) is also in \( G \), and \( s_G \) is an iteration strategy for normal trees. Then there is an iteration strategy \( s^*_G \) for arbitrary trees such that if \( T \) is a tree and \( b = s^*_G(T) \) then \( M_b^T \in G \).

There is also an embedding from \( T \) to the normal tree, but the embedding is a more complicated structure than in the previous two sections.

For now I'm only considering trees that never drop to a mouse. I don't see any significant extra problem in dealing with mice.

Suppose that \( T = (T, \deg, D, (E_\alpha, M_{\alpha^+} : \alpha < \theta)) \) is an iteration tree (without dropping to mice).

3.2 Definition. (1) A node \( \nu \) of \( T \) is bad for length if there is \( \gamma > \nu \) such that \( \text{index}(E_\gamma) < \text{index}(E_\nu) \).

(2) A node \( \nu \) of \( T \) is bad for critical point if there is \( \xi \) such that \( \xi + 1 \prec T \nu + 1 \) and if \( \nu^* = T-\text{pred}(\nu + 1) \) then \( \text{index}(E_{\xi+1}) < \text{index}(E_{\nu^*}) \).

(3) A node \( \nu \) of \( T \) is deadwood if there is \( \gamma > \nu \) such that \( \text{index}(E_\gamma) < \text{index}(E_\nu) \) and there is no \( \xi > \gamma \) such that \( \nu < T-\text{pred}(\xi) \leq \gamma \).

3.3 Lemma. A tree \( T \) is not harmed by removing deadwood.

Proof. The point is that since deadwood doesn't have any branches extending beyond \( \gamma \) it does nothing in the tree \( T \) except to have \( E_\gamma \) in \( M_\gamma \), which may have nodes in the interval \([\nu, \gamma) \) in its support. But \( E_\gamma \) is already in \( M_\nu \), so they can just be omitted. \( \square \)

In view of the last lemma we can weaken the definition of normality to the assumption that there are no nodes which are bad for critical point. We will use this weakened definition for the rest of the paper.

Let \( T^0 \) be an arbitrary tree, and let \( C = \{ \delta_\lambda : \lambda \leq \phi \} \) be the closure of \( \{ \nu + 1 : \nu \text{ is bad for critical point in } T^0 \} \). We will define a sequence of trees \( T^\lambda \) by recursion on \( \lambda \in C \). Each of the trees \( T^\lambda \) will have domain of the form \( I_\lambda \cup \bigcup_{\alpha < \lambda} \text{domain}(T^\alpha) \), where \( I_\lambda \) is a closed interval containing \( \delta_\lambda \) as its last member and otherwise disjoint from \( \bigcup_{\alpha < \lambda} \text{domain}(T^\alpha) \). At successor ordinals \( \lambda + 1 \) the interval \( I_{\lambda + 1} \) will be used to correct the failure of \( \nu \) to be good for critical point, where \( \delta_{\lambda + 1} = \nu + 1 \), and the interval \( I_\lambda \) for \( \lambda \) a limit ordinal will be used to make...
the limit of the trees $T^\alpha$ for $\alpha < \lambda$ into a tree, $T^\lambda$. The final tree in the series, $T^\phi$, will be normal and we will simply write $T$ for this tree.

As a general notation, we will use superscripts to indicate which of the trees is being referred to. Thus, $\prec^0$ is the tree order for $T^0$ and $E^\lambda_\sigma$ is the extender for node $\sigma$ in the tree $T^\lambda$. No subscript will mean that $T = T^\phi$ is meant, and we will generally omit the superscript rather than use a subscript $\lambda$ for an entity which is constant in all trees for which it makes sense. The trees will satisfy, among other properties

$$T|\delta_\lambda + 1 = T^\gamma|\delta_\lambda + 1 = T^\lambda|\delta_\lambda + 1$$

for all $\lambda < \gamma \in C$, and

$$T^\gamma\text{-pred}(\sigma) = \begin{cases} T^0\text{-pred}(\sigma) & \text{if } \sigma \in \text{domain}(T^0) \setminus C \\ T^\gamma\text{-pred}(\sigma) & \text{if } \sigma = \delta_\lambda \text{ and } \gamma < \lambda \\ T^\lambda\text{-pred} & \text{if } \sigma \in I_\lambda \text{ and } \gamma \geq \lambda. \end{cases}$$

This means that we will normally be able to omit the superscript: if $\sigma \in I_\lambda \setminus \{\delta_\lambda\}$, for example, then $E_\sigma = E^\gamma_\sigma = E^\lambda_\sigma$ for any $\gamma \geq \lambda$, that is, for any $\gamma$ with $\sigma \in \text{domain}(T^\gamma)$. In particular we can use $\tau \prec \sigma$ instead of $\tau \prec^\lambda \sigma$ whenever $\sigma \leq \delta_\lambda$ is in $T^\lambda|\delta_\lambda + 1$, and that we can normally use either $T^0\text{-pred}(\sigma)$ or $T\text{-pred}(\sigma)$ instead of $\lambda\text{-pred}(\sigma)$.

The nodes of $I_\lambda$ other than $\delta_\lambda$ will be indexed by sequences $\sigma$ of ordinals, rather than by ordinals. All such sequences $\sigma$ will be continuous and strictly increasing, and every member of $\sigma$ except possibly the first will be in $C$. Every member $\sigma$ of $I_\lambda$ (except for $\delta_\lambda$) will be a sequence $\sigma$ with $\delta_\lambda$ as its final member. In keeping with the dual role of $\delta_\lambda$ as a member of $I_\lambda$ as well as a member of $\text{domain}(T^0)$ we will (except in one case) assign sequences as alternate names for $\delta_\lambda$. These sequences are called extended indices for $\delta_\lambda$. One of these sequences is called the fully extended index and each of the extended indices of $\delta_\lambda$ will be a terminal segment of the fully extended index.

We put the following ordering on the sequences:

$$\sigma < \tau \iff \begin{cases} \tau = \tau_0 \wedge \sigma & \text{for some sequence } \tau_0, \text{ or} \\ \tau = \tau_0 \wedge <\nu'> \wedge \sigma_1 & \text{where } \sigma = \sigma_0 \wedge <\nu'> \wedge \sigma_1 \text{ and } \nu < \nu'. \end{cases}$$

The restriction of this ordering to the domain of $T^\lambda$ will be a well ordering for each of the trees $T^\lambda$ which we construct.

We define the cut function $\sigma|\alpha$ by

$$\sigma|\alpha = \sigma|\{\xi : \sigma(\xi) < \delta_\alpha\}$$

$$\delta_\gamma|\alpha = \sigma|\alpha$$

where $\sigma$ is the fully extended index of $\delta_\gamma$

$$\xi|\alpha = \xi$$

if $\xi \in \text{domain}(T^0) \setminus C$.

Thus $\sigma|\gamma \in \bigcup_{\alpha \leq \gamma} I_\alpha$, and if $\sigma \in I_\alpha$ and $\sigma \wedge \tau \in I_\gamma$ then $\sigma \wedge \tau|\alpha = \sigma$.

If $\sigma$ is a node of any of the trees $T^\alpha$ then we will write $\sigma + 1$ for the next node after $\sigma$ in domain($T^\alpha$). If $\sigma = \sigma_0 \in \text{domain}(T^0)$ then we will write $<\sigma> + 1$ for this.
successor unless is known to be equal to the ordinal \( \alpha + 1 \). The successor will be independent of the tree \( \alpha \), except that if \( \delta_{\alpha+1} = \nu + 1 \) then

\[
<\nu> + 1 = \begin{cases} 
\nu + 1 & \text{in } T^\gamma \text{ if } \gamma < \alpha \\
\min(I_\alpha) & \text{in } T^\gamma \text{ if } \gamma \geq \alpha.
\end{cases}
\]

We assume, in constructing \( T^\lambda \), that the following proposition holds for all \( \alpha \) and \( \gamma \) less than \( \lambda \):

**3.4 Proposition.**

1. if \( \alpha < \gamma \) then \( T^\alpha | \delta_\alpha + 1 = T^\gamma | \delta_\alpha + 1 \).
2. \( T^\alpha | \delta_\alpha \) is normal, and \( M^\alpha_\sigma \in G \) for all \( \sigma < \delta_\alpha + 1 \) in domain\( (T^\alpha) \).
3. If \( \sigma \in I_\alpha \) and \( \sigma \prec^\alpha \sigma' \) then either \( \sigma' \in I_\alpha \) or \( \sigma' = \nu \in \text{domain}(T^0) \) and \( \delta_\alpha \prec^0 \nu \).

The propositions at the end of the paper are also assumed as induction hypotheses.

We will define two families of maps

\[
\begin{align*}
  j_{\sigma,\sigma'} : M_\sigma &\to M_{\sigma'} & \text{if } \sigma \text{ is a proper initial segment of } \sigma' \\
  j^{\alpha,\lambda}_\nu : M^\alpha_\nu &\to M^\lambda_\nu & \text{for } \alpha < \lambda \text{ and } \nu \in \text{domain}(T^0).
\end{align*}
\]

These are actually the same family of maps, but the first form will be independent of the trees from which the models \( M_\sigma \) and \( M_{\sigma'} \) are taken. These maps will commute with the tree embeddings and will satisfy that

\[
E_{\sigma'} = j_{\sigma,\sigma'}(E_\sigma) \quad \text{and} \quad E^\lambda_\nu = j^{\alpha,\lambda}_\nu(E^\alpha_\nu).
\]

The tree \( T^\lambda \) is thus actually determined by the definition of \( T^\alpha \) for \( \alpha < \lambda \), the choice of \( I_\lambda \), and the embeddings \( j_{\sigma,\sigma'} \) for \( \sigma' \in I_\lambda \) and \( j^\lambda_\nu \) for \( \nu \in \text{domain}(T^0) \), since \( T | \delta_\lambda + 1 \) is required to be normal and the order relation for nodes \( \nu > \delta_\lambda \) is given by

\[
\begin{align*}
T^\lambda - \text{pred}(\nu) &\equiv T^0 - \text{pred}(\nu) \\
\{ \xi : \delta_\lambda \leq \xi \ \& \ \xi \prec^\lambda \nu \} &\equiv \{ \xi : \delta_\lambda \leq \xi \ \& \ \xi \prec^0 \nu \}.
\end{align*}
\]

It will, of course, be necessary to prove that the choice of \( I_\lambda \) and the maps \( j \) do yield a tree with the required properties.

**Successor stages.** Suppose that \( T^\lambda \) has been defined. We describe how to construct \( T^{\lambda + 1} \).

By the proposition the least node of \( T^\lambda \) which is bad for critical point is larger than \( \delta_\lambda \) and hence is equal to \( <\nu> \) for some \( \nu \geq \sup \{ \delta_\alpha : \alpha \leq \lambda \} \). Set \( \delta_{\lambda+1} = \nu + 1 \). We have \( T - \text{pred}(<\nu + 1>) = T^0 - \text{pred}(<\nu + 1>) = <\nu^*> \) for some \( \nu^* \leq \nu \). Let \( \tau < \nu^* \) be the least sequence in the domain of \( T^\lambda \) such that \( i_{\tau'+1,\nu^*}(\text{len}(E^\tau_\tau)) > \text{crit}(E^\nu_\nu) \) for all \( \tau' \) such that \( \tau \leq \tau' < <\nu^*> \). Since \( T^\lambda \) is normal up to \( <\delta_\lambda>, i_{\tau'+1,\nu^*} | \text{len}(E^\tau_\tau) \) is the identity and it follows that \( \text{len}(E^\tau_\tau) > \text{crit}(E^\nu_\nu) \) for all \( \tau' \) in the interval \( \tau \leq \tau' < <\nu^*> \).
Now set
\[ I = I_\lambda = \{ \sigma \downarrow <\nu + 1>: \sigma \in \text{domain}(\mathcal{T}^\lambda) \ & \ \tau \leq \sigma < \nu^* \} \cup \{ \nu + 1 \}. \]

The sequence \(<\nu^*, \nu + 1>\) will be an extended index for \(\delta_{\lambda+1} = \nu + 1\), and the fully extended index for \(\delta_{\lambda+1}\) will be \(\sigma \downarrow \delta_{\lambda+1}\) where \(\sigma\) is the full extended index for \(\nu^*\) if \(\nu^* \in C\), or \(\sigma = <\nu^*>\) otherwise. The initial part, \(\mathcal{T}^{\lambda+1}|_{\delta_{\lambda+1}}\), of the new tree will be defined by using the extended index \(<\nu^*, \nu + 1>\) for \(\delta_{\lambda+1}\). On the other hand, the tail of the tree, \(\mathcal{T}|(\theta \setminus \delta_{\lambda+1})\), will be defined by using the standard index, \(\delta_{\lambda+1}\), for this node just as in the old tree, \(\mathcal{T}^\lambda\).

If \(\sigma\) and \(\sigma'\) are nodes in \(\mathcal{T}^\lambda\) and \(\sigma' \neq \delta_{\lambda+1}\) then \(\sigma \prec \lambda+1 \sigma'\) iff \(\sigma \prec \lambda \sigma'\). As stated in the proposition \(<\nu>+ 1 = \min(E_{\lambda+1}^{\nu})\) in \(\mathcal{T}^{\lambda+1}\) and \(\mathcal{T}^{\lambda+1}|_{\nu} = 1 = \mathcal{T}^\lambda|_{\nu} + 1\). The maps \(j_{\sigma, \sigma' \downarrow \delta_{\lambda+1}}\) for \(\sigma \downarrow <\nu + 1> \in I^{\lambda+1}\) will be defined later and will factor through \(i_{E_{\nu}}^{\nu}\):

\[ j_{\sigma, \sigma' \downarrow <\nu + 1>}: M_{\sigma} \xrightarrow{i_{E_{\nu}}} \text{ult}(M_{\sigma}, E_{\nu}) \xrightarrow{k_{\sigma}} M_{\sigma' \downarrow <\nu + 1>}, \]

where the critical point of \(k_{\sigma}\) is larger than \(i_{E_{\nu}}(\text{crit}(E_{\nu}))\). We will define

\[ j_{\nu + 1}^{\lambda+1} = k_{\nu^*}: M_{\nu + 1}^{\lambda} = \text{ult}(M_{\nu^*}, E_{\nu}) \rightarrow M_{<\nu^*, \nu + 1>}^{\lambda + 1} = M_{\nu + 1}^{\lambda+1} = M_{<\nu + 1>}^{\lambda+1}. \]

The definition of \(j_{\nu + 1}^{\lambda+1}\) for \(\gamma > \nu + 1\) follows immediately as in the first section of this paper, so it only remains to define \(\mathcal{T}^{\lambda+1}\) on the interval \(I_{\lambda+1}\).

We define by recursion on \(\sigma\) in the interval \(\tau \leq \sigma \leq \nu^*\) the map and the order relation:

\[ j_{\sigma, \sigma' \downarrow \delta_{\lambda+1}}: M_{\sigma} \rightarrow M_{\sigma' \downarrow \delta_{\lambda+1}} \]

\(\{ \sigma': \sigma' \prec \lambda+1 \sigma \downarrow \delta_{\lambda+1} \}\).

First, for \(\sigma = \tau\) we have \(\tau \downarrow \delta_{\lambda+1} = <\nu> + 1\) in \(\mathcal{T}^{\lambda+1}\). Since \(E_{\nu + 1}^{\lambda+1} = E_{\nu}^{\lambda}\) the requirement that \(\tau \downarrow \delta_{\lambda+1}\) be normal dictates that \(\mathcal{T}^{\lambda+1}-\text{pred}(\tau \downarrow <\nu + 1>) = \tau\). The embedding

\[ j_{\tau, \tau \downarrow \delta_{\lambda+1}}: M_{\tau} \rightarrow M_{\tau \downarrow \delta_{\lambda+1}} = \text{ult}(M_{\nu}, E_{\nu}) \]

is defined to be the canonical embedding.

**3.5 Proposition.** Suppose \(\sigma = \sigma' + 1\) is a successor node in \(I_{\gamma}\). Then

\[ \mathcal{T}^{\lambda+1}-\text{pred}(\sigma) = \begin{cases} \sigma \downarrow \lambda & \text{if } \sigma = <\nu> + 1 = \min(I_{\lambda+1}) \\ \mathcal{T}-\text{pred}(\sigma \downarrow \lambda) & \text{if } \sigma' \in I_{\lambda+1} \ \text{and } \text{crit}(E_{\sigma'}) < \text{crit}(E_{\nu}) \\ \mathcal{T}-\text{pred}(\sigma \downarrow \lambda \downarrow \delta_{\lambda+1}) & \text{otherwise}. \end{cases} \]

**Proof.** If \(\text{crit}(E_{\sigma'}) < \text{crit}(E_{\nu})\) then

\[ \text{crit}(E') = j_{\tau, \tau \downarrow \delta_{\lambda+1}}(\text{crit}(E_{\nu})) = k_{\nu^*}(\text{crit}(E_{\nu})) = i_{E_{\nu}}^{\nu}(\text{crit}(E_{\nu})) = \text{crit}(E_{\nu}). \]
so $T' \text{-pred}(\sigma^{\nu+1}) = T \text{-pred}(\sigma)$ is correct. In the third case, since $T$ is normal up to $\nu$, $\tau \leq \sigma^* = T \text{-pred}(\sigma + 1) < \nu^*$ and hence $\sigma^*^{\nu+1} \in I_{\lambda+1}$, and \( \text{crit}(E'_{\sigma^{\nu+1}}) > \text{len}(E_\nu) \) so that the $T'$ predecessor of $\sigma^{\nu+1}$ must be in $I_{\lambda+1}$. □

Suppose that $\sigma = \sigma' + 1$ is a successor node with $\sigma^{\delta_{\lambda+1}} \in I_{\lambda+1}$, and set $\sigma^* = T \text{-pred}(\sigma)$. If $x \in M_\sigma$ then $x = [a, f]_{E_\sigma'}$ for some $f \in M_{\sigma'}$ and $a < \text{len}(E_\sigma')$. Set

$$j_{\sigma, \sigma^{\nu+1}}(x) = \begin{cases} [j'(a), f]_{E'} & \text{if} \text{crit}(E_\sigma') < \text{crit}(E_\nu), \\ [j'(a), j^*(f)]_{E'} & \text{otherwise} \end{cases}$$

where $E' = E_{\sigma^{\delta_{\lambda+1}}}$, $j' = j_{\sigma', \sigma^{\delta_{\lambda+1}}}$, and $j^* = j_{\sigma^*, \sigma^{\delta_{\lambda+1}}}$.

For limit nodes $\sigma \in I_{\lambda+1}$ we have the

**3.6 Proposition.** Suppose that $\sigma^{\nu+1} \in I$ and $\sigma$ is a limit point in $T^\lambda$, with $b = \{ \sigma' : \sigma' <^\lambda \sigma \}$. Then $s_\sigma(T^{\lambda+1} | \sigma^{\delta_{\lambda+1}})$ exists, and there is $\sigma_0 \in b$ such that

$$s_\sigma(T^{\lambda+1} | \sigma^{\delta_{\lambda+1}}) = \sigma' : \sigma' <^\lambda \sigma_0 \cup \sigma^{\delta_{\lambda+1}} : \sigma_0 <^\lambda \sigma' <^\lambda \sigma.$$

*Proof.* Since $T^{\lambda+1} | \sigma^{\delta_{\lambda+1}}$ is normal, we know that $c = s_\sigma(T^{\lambda+1} | \sigma^{\delta_{\lambda+1}})$ exists. Let $c' = \{ \sigma' : \sigma \in c \text{ or } \sigma <^\lambda \delta_{\lambda+1} \in c \}$. Then $c'$ is a branch through $T^\lambda | \sigma$. Furthermore there is an embedding $j : M_\nu \rightarrow M_c$ given as the direct limit of the embeddings $j_{\sigma', \sigma^{\delta_{\lambda+1}}} : M_{\sigma'} \rightarrow M_{\sigma^{\delta_{\lambda+1}}}$ for $\sigma' <^\lambda \delta_{\lambda+1} \in c$. Since $M_c \in \mathcal{G}$ it follows that $M_{c'} \in \mathcal{G}$, and it follows that $c' = s_\sigma(T^\lambda | \sigma) = b$. □

Thus $M_{\sigma^{\delta_{\lambda+1}}} = M_c$, and the embedding $j_{\sigma, \sigma^{\nu+1}}$ is the embedding $j$ from the proof of the proposition.

This completes the definition of $T' = T^{\lambda+1}$, but in order to finish the proof that it works we have to prove

**3.7 Proposition.** The maps $j_{\sigma, \sigma^{\delta_{\lambda+1}}}$ for $\sigma^{\delta_{\lambda+1}} \in I$ can be factored

$$j_{\sigma, \sigma^{\nu+1}} : M_\sigma \xrightarrow{i_{E_\nu}} \text{ult}(M_\sigma, E_\nu) \xrightarrow{k_\sigma} M'_{\sigma^{\delta_{\lambda+1}}}$$

where $k_\sigma(i_{E_\nu}(f)(b)) = j_\sigma(f)(b)$ and $\text{crit}(k_\sigma) \geq i_{E_\nu}(\text{crit}(E_\nu))$.

*Proof.* We will prove this by induction on $\sigma$. If $\sigma$ is equal to $\tau$ then $j_{\tau^{\nu+1}} = i_{E_\nu}$ by definition, so $k_\tau$ is the identity. If $\sigma$ is a limit node then the induction step for $\sigma$ follows immediately from the definition of $j_{\sigma', \sigma^{\delta_{\lambda+1}}}$, so we can assume that $\sigma$ is a limit node, say $\sigma = \sigma' + 1$. Set $\kappa_\nu = \text{crit}(E_\nu)$ and $\sigma^* = T \text{-pred}(\sigma)$.

**Claim.** If $x \in \mathcal{P}(\kappa_\nu) \cap M_\sigma$ then $j_{\sigma, \sigma^{\delta_{\lambda+1}}}(x) = i_{E_\nu}(x)$.

*Proof.* If $\kappa \leq \text{crit}(E_\sigma')$ then $x = [\text{crit}(E_\sigma'), f_x]_{E'_\sigma}$ in $\text{ult}(M_{\sigma^*}, E_{\sigma'})$, where if $\kappa_\nu < \text{crit}(E_{\sigma'})$ then $f_x$ is the constant function, $f_x(\xi) = x$, and $\delta_\nu = \text{crit}(E_{\sigma'})$ then $f_x(\xi) = x \cap \xi$. Then

$$j_{\sigma, \sigma^{\nu+1}}(x) = [j_{\sigma', \sigma^{\delta_{\lambda+1}}}(\text{crit}(E_{\nu})), j_{\sigma^*, \sigma^{\delta_{\lambda+1}}}(f_x)]_{E_{\sigma^{\nu+1}}} = [\text{crit}(E_{\sigma^{\nu+1}}), j_{\sigma^*, \sigma^{\delta_{\lambda+1}}}(x)]_{E_{\sigma^{\nu+1}}} = [\text{crit}(E_{\sigma^{\nu+1}}), f_x]_{E_{\sigma^{\nu+1}}} = i_{E_\nu}(x).$$
Thus we can assume that \( \kappa_\nu > \text{crit}(E_\sigma) \). Then if \( x = [a, f]^{M_\sigma^*}_{E_\sigma'} \) it follows that

\[
j_{\sigma', \sigma \delta_{\lambda + 1}}(x) = j_{\sigma, \sigma \delta_{\lambda + 1}}([a, f]^{M_\sigma^*}_{E_\sigma'}) = [j_{g s', \sigma \delta_{\lambda + 1}}(a), f]^{M_\sigma^*}_{E_\sigma'} = [j_{g s', \sigma \delta_{\lambda + 1}}(a), f]^{M_\sigma'}_{E_\sigma'} = j_{\sigma', \sigma \delta_{\lambda + 1}}([a, f]_{E_\sigma'}) = i^{E_\nu}(x) \]

(1)

where step (1) follows from the fact that \( \mathcal{P}(\text{crit}(E_\sigma')) \cap M_\sigma^* = \mathcal{P}(\text{crit}(E_\sigma')) \cap M_\sigma^* \).

This completes the proof of the claim. It follows that the embedding \( k_\sigma : \text{ult}(M_\sigma, E_\nu) \rightarrow M_{\sigma \delta_{\lambda + 1}} \) defined by

\[
k_\sigma([a, f]_{E_\nu}) = j_{\sigma, \sigma \delta_{\lambda + 1}}(f)(a) \]

is an elementary embedding, as claimed. \( \square \)

NOTE: this gives the commutative diagram

\[
\begin{array}{ccc}
M^\lambda_\nu & \xrightarrow{i} & M^\lambda_{\delta_{\lambda + 1}} = \text{ult}(M_\nu^*, E_\nu) & \xrightarrow{i} & M^\lambda_\gamma \\
\downarrow{=} & & \downarrow{k} & & \downarrow{j} \\
M^\lambda_\nu \downarrow{j} & & M^\lambda_{\delta_{\lambda + 1}} = M^\lambda_{\nu \delta_{\lambda + 1}} & \xrightarrow{i} & M^\lambda_{\gamma + 1}
\end{array}
\]

Limit Stages. Now suppose that \( \lambda \) is a limit ordinal and \( \mathcal{T}^\alpha \) has been defined for all \( \alpha < \lambda \). Since \( \min(I_\lambda) \) will be equal to \( \sup\{ \delta_\alpha : \alpha < \lambda \} \) we can set \( \delta_\lambda = \sup_{\alpha < \lambda} \delta_\alpha \) and let \( \mathcal{T}^\lambda \downarrow \text{min} \ I_\lambda = \bigcup_{\alpha < \lambda} \mathcal{T}^\alpha \downarrow \delta_\alpha \). Since \( \mathcal{T}^\lambda \downarrow \delta_\lambda \) is normal, it has a branch \( b = s^*_G(\mathcal{T} \downarrow \delta_\lambda) \) such that \( M_b \in \mathcal{G} \).

We will deal with the easy case, in which \( b \cap \text{domain}(\mathcal{T}^0) \) is cofinal in \( \delta_\lambda \), first. In this case we set \( I_\lambda = \{ \delta_\lambda \} \). Then for each \( \alpha < \lambda \) there is a branch \( b_\alpha \) of \( \mathcal{T}^\alpha \downarrow \delta_\lambda \) containing \( b \cap \text{domain}(\mathcal{T}^\alpha) \) since \( \gamma <^\lambda \gamma' \) implies that \( \gamma <^\alpha \gamma' \) for \( \alpha < \lambda \). Furthermore there is an embedding \( j^{\alpha, \lambda} : M^\alpha_{\delta_\lambda} \rightarrow M_b \) for each \( \alpha < \lambda \), given as the direct limit of the embeddings \( j^{\xi, \lambda} \) for \( \xi \in b \). In particular it follows that \( M_{b_\lambda} \in \mathcal{G} \), so that we can define \( s^*_G(\mathcal{T} \downarrow \delta_\lambda) \) for \( \xi \in b \). This is well defined since \( b^0 \) only depends on \( \mathcal{T}^\lambda \downarrow \text{min} \ I_\delta_\lambda \), and hence only on \( \mathcal{T}^0 \downarrow \delta_\lambda \). If \( \mathcal{T}^0 \) follows the stategy \( s^*_G \) then \( M_{b_\alpha} = M^\alpha_{\delta_\lambda} \) for all \( \alpha < \lambda \) and we can let \( j^{\alpha, \lambda} \) be the map \( j^{\xi, \lambda} \) defined above.

The direct limit diagram in this case (\( b \cap \text{domain}(\mathcal{T}^0) \smallsetminus C \) is unbounded in \( \delta_\lambda \)):

\[
\begin{array}{ccc}
M^\xi_\alpha & \xrightarrow{i} & M^\xi_\delta_\lambda \\
\downarrow{j} & & \downarrow{j} \\
M^\alpha_\lambda & \xrightarrow{i} & M^\lambda_\gamma \\
\end{array}
\]

for \( \xi < \lambda \) and \( \alpha < \gamma \) in \( D \). Then \( I_\lambda = \{ \delta_\lambda \} \) and \( \delta_\lambda \) has no extended index.

Thus we can assume for the rest of the proof that \( b \) contains only boundedly many nodes \( \nu \) from \( \text{domain}(\mathcal{T}^0) \). Let \( D = \{ \alpha : \alpha_0 < \alpha \ \& \ I_\alpha \cap b \neq \emptyset \} \), where \( \alpha_0 < \lambda \) is large enough that there is no \( \mu > \delta_\lambda \) such that \( \mu \in b \).
3.8 Definition. (1) We define $I_\lambda$ to be $<\delta_\lambda>$ together with the set of closed, continuous sequences $\sigma$ such that $\delta_\lambda = \max \sigma$ and for every sufficiently large $\alpha \in D$ the cut $\sigma|\delta_\alpha$ is in $I_\alpha$ and $\delta|\delta_\alpha > \sigma'$ for every $\sigma' \in b \cap I_\alpha$.

(2) For $\sigma \in I_\lambda$ define $M_\sigma = \text{dir lim}_{\alpha \in D} M_{\sigma|\delta_\alpha}$, where the direct limit is taken along the maps $j_{\sigma|\alpha, \sigma|\alpha'}$. This direct limit also defines the embeddings $j_{\sigma|\alpha, \sigma}$ for $\sigma \in I_\lambda$ and $\alpha < \lambda$ in $D$, and hence it defines the extenders $E_\sigma = j_{\sigma|\delta_\sigma}(E_{\sigma|\delta})$.

3.9 Lemma. For every $\alpha \in D$ either there is $\sigma \in b$ such that $\sigma \cap \alpha = \delta_\alpha$, or else there is $\tau \in I_\alpha$ where $\tau = \min(I_\lambda)$ and there is $x \in I_\alpha$ such that $[\tau|\alpha, \delta_\alpha] \sim [\tau, x]$.

Proof. Suppose that $\alpha \in D$ and $\sigma|\alpha \neq \delta_\alpha$ for every $\sigma \in b$. Then set $\sigma_\gamma = \min(b \cap I_\gamma)$ for $\gamma \in D$. By the extension lemma, if $\gamma > \alpha$ is in $D$ then $\sigma_\gamma|\alpha \in I_\alpha$ and there is $x_{\alpha, \gamma} \in I_\gamma$ such that $[\sigma_\gamma|\alpha, \delta_\alpha] \sim [\sigma_\gamma, x_{\alpha, \gamma}]$.

Now note that there is no $\gamma > \alpha$ in $D$ and $\sigma \in b$ such that $\delta_\gamma = \sigma|\gamma$. Suppose to the contrary that $\sigma \in b$ and $\sigma|\gamma = \delta_\gamma$. If $\delta_\gamma|\alpha = \delta_\alpha$ then $\sigma|\alpha = \delta_\alpha$, contrary to assumption, so we can assume that $x_{\alpha, \gamma} < \delta_\gamma$, and hence $\sigma|\alpha \notin I_\alpha$. By the extension lemma this implies that there is $\sigma' \prec \sigma$ such that $\sigma'|\alpha = \delta_\alpha$, but then $\sigma' \in b$, again contradicting the assumption.

Now if $\alpha < \gamma < \gamma'$, with $\gamma$ and $\gamma'$ in $D$ then $x_{\alpha, \gamma} = x_{\alpha, \gamma'}|\gamma$ so $x = \bigcup \{x_{\alpha, \gamma} : \alpha < \gamma \& \gamma \in D\} \in I_\lambda$. Then $\tau|\gamma \in I_\gamma$ and $x_{\alpha, \gamma} \geq \tau|\gamma > \sigma_\gamma$ for sufficiently large $\gamma \in D$. Since $\sigma_\gamma|\alpha \in I_\alpha$ and $[\sigma_\gamma|\alpha, \delta_\alpha] \sim [\sigma_\gamma, x_{\alpha, \gamma}]$ it follows that $\tau|\alpha \in I_\alpha$ and $[\tau|\alpha, \delta_\alpha] \sim [\tau, x]$. □

We now have to show that $T^\lambda|\delta_\lambda$ is a normal tree, with the models $M_\sigma$ defined above and extenders $E_\sigma = j_{\sigma|\gamma, \gamma}(E_{\sigma|\gamma, \gamma})$. First consider $\sigma = \tau = \min(I_\lambda)$. We need to show that $M_\sigma = M_b$. We consider two cases: the first is that in which $b \cap I_\gamma = \{\gamma\}$ for every sufficiently large $\gamma \in D$. In this case $\tau_\alpha = \tau|\alpha$ for $\alpha < \gamma$ in $D$, and $i_{\tau_\alpha, \gamma}^\lambda = j_{\tau_\alpha, \gamma}$ so that

$$M_b = \text{dir lim}_{\alpha, \gamma \in D} (M_{\tau_\gamma}, i_{\tau_\alpha, \gamma}) = \text{dir lim}_{\alpha, \gamma \in D} (M_{\tau_\gamma}, j_{\tau_\alpha, \gamma}) = M_\tau$$

as required.

If the first case doesn’t hold, then for $\alpha \in D$ lemma 3.14 implies that the set $\{\sigma|\alpha : \sigma \in b\}$ is an increasing sequence in $I_\alpha$ and that $\tau_\alpha = \sup_{\sigma \in b} \sigma|\alpha$ is a limit node in $I_\alpha$. Then $\tau_\alpha \leq \delta_\alpha$, and by the extension lemma $\tau_\alpha = \tau|\alpha$ for $\gamma > \alpha$. Thus $(\bigcup_{\alpha \in D} \tau_\alpha)^\gamma \cap \delta_\lambda$ is in $I_\lambda$, and clearly $\tau = \min(I_\lambda) = (\bigcup_{\alpha \in D} \tau_\alpha)^\gamma \cap \delta_\lambda$. Furthermore lemma 3.14 implies that $b_\alpha = \{\sigma|\alpha : \sigma \in b\}$ is a branch in $T^\gamma|\tau_\alpha$. Since $M_{b_\alpha}$ can be embedded into $M_b$, which is in $\mathcal{G}$, the model $M_{b_\alpha}$ is also in $\mathcal{G}$ and hence is equal to $M_{\tau|\alpha}$. Then

$$M_\tau = \text{dir lim}_{\gamma \in D} M_{\tau|\gamma} = \text{dir lim}_{\gamma \in D} \text{dir lim}_{\sigma \in b} M_{\sigma|\gamma} = \text{dir lim}_{\sigma \in b} M_{\sigma} = M_b.$$ 

Thus, in either case $M_b = M_\tau$ so that $b = \{\sigma : \sigma \prec \tau\}$. It is possible in the second case that $\tau|\alpha = \delta_\alpha$ for all sufficiently large $\alpha \in D$, so that $\tau$ is the only member of $I_\lambda$. In this case $\tau$ will be the fully expanded index for $\delta_\lambda$, and $s_\lambda^* (T^0|\delta_\lambda)$ is the branch of $T^0|\delta_\lambda$ which contains $\{\delta_\alpha : \tau|\alpha = \delta_\alpha\}$.

In this case (for every sufficiently large $\sigma \in D$ there is $\sigma_\alpha \in b$ such that $\sigma_\alpha|\sigma = \tau$).
the direct limit diagram is

\[
\begin{array}{cccc}
M_{\delta_\alpha} & \overset{i}{\longrightarrow} & M_{\delta_\gamma} & \overset{i}{\longrightarrow} & \ldots \ M_{\delta_\lambda} \\
\downarrow j & & \downarrow j & & \ldots \ \downarrow j \\
M_{\sigma_\alpha} & \overset{i}{\longrightarrow} & M_{\sigma_\gamma} & \overset{i}{\longrightarrow} & \ldots \ M_{\sigma_\lambda}
\end{array}
\]

where \( \alpha < \gamma \) are in \( D \). In this case \( I_\lambda = \{ \delta_\lambda \} \) and \( \delta_\lambda \) has no extended index.

Now we consider successor nodes \( \sigma = \sigma' + 1 \) in \( I_\lambda \). Assume that \( M_{\sigma'} \) has been defined. Then \( \sigma' + 1 = (\bigcup_{\alpha \in D} (\sigma'|\alpha) + 1) \delta_\lambda \).

3.10 Proposition. If \( \sigma \) is a successor node in \( I_\lambda \) then either

\[
\mathcal{T}-\text{pred}(\sigma) = \left( \bigcup_{\gamma \in D} \mathcal{T}-\text{pred}(\sigma|\gamma) \right) \delta_\lambda \in I_\lambda
\]

or there is \( \gamma_0 \) such that

\[
\mathcal{T}-\text{pred}(\sigma) = \mathcal{T}-\text{pred}(\sigma|\gamma) \quad \text{for any } \gamma > \gamma_0 \text{ in } D.
\]

Proof. First suppose that there is \( \alpha \in D \) and a sequence \( \chi \in b \) such that \( \text{crit}(E_{\sigma'|\alpha}) < \text{crit}(E_{\chi'|\alpha}) \). For each \( \gamma \in D \) let \( \sigma_\gamma \) be the least member of \( I_\gamma \cap b \), and pick \( \gamma_0 \in D \) large enough that \( \chi \preceq \sigma_{\gamma_0} \). We claim that \( \mathcal{T}-\text{pred}(\sigma') = \mathcal{T}-\text{pred}(\sigma'|\gamma) \) for any \( \gamma \geq \gamma_0 \). It will be enough to show that \( \mathcal{T}-\text{pred}((\sigma'|\gamma) + 1) < \min(I_{\gamma_0}) \) for any \( \gamma > \gamma_0 \) in \( D \).

Somewhere we seem to need to have a lemma that \( j_{\sigma'|\alpha,\sigma'}|\text{len}(E_{\sigma'|\alpha}) = j_{\sigma'|\alpha,\sigma'}|\text{len}(E_{\sigma'|\alpha}) \)

where \( \sigma'|\alpha < \sigma'|\alpha' \) and both are in \( I_\alpha \) and\( \sigma' \) and \( \sigma' \) are in \( I_\gamma \). This is needed for the next calculation. Should something like this be a condition for an embedding on iteration trees, and should the assertion be that these embeddings are embeddings of intervals of the iteration trees?

If \( \gamma \) is a successor member of \( D \) then

\[
\text{crit}(E_{\sigma|\gamma}) < \text{crit}(E_{\chi|\alpha}) < \text{crit}(E_{\sigma_\gamma|\alpha})
\]

and hence

\[
\text{crit}(E_{\sigma'|\gamma}) = j_{\sigma'|\alpha,\sigma'|\gamma}(\text{crit}(E_{\sigma'|\alpha}))
\]

\[
< j_{\sigma'|\alpha,\sigma'|\gamma}(\text{crit}(E_{\sigma_\gamma|\alpha}))
\]

\[
= j_{\sigma_\gamma|\alpha,\sigma_\gamma}(\text{crit}(E_{\chi|\alpha}))
\]

\[
= \text{crit}(E_{\sigma_\gamma}).
\]

It follows that

\[
\mathcal{T}-\text{pred}(\sigma|\gamma) \leq \mathcal{T}-\text{pred}(\sigma_\gamma + 1) \leq \delta_{\gamma'} \quad \text{where } \gamma' = \max(D \cap \gamma).
\]

Thus

\[
\mathcal{T}-\text{pred}(\sigma|\alpha) = \mathcal{T}-\text{pred}(\sigma|\alpha') < \min(I_{\gamma_0})
\]
using the induction hypothesis.

If \( \gamma \) is a limit member of \( D \) then for all \( \xi \in D \) in the interval \( \gamma_0 \leq \xi < \gamma \) we have \( T^-\text{pred}(\sigma|\xi) = T^-\text{pred}(\sigma|\gamma_0) \), and by this proposition for \( \lambda = \gamma \) it follows that \( T^-\text{pred}(\sigma|\gamma) = T^-\text{pred}(\sigma|\gamma_0) \), as well.

Now we consider the case in which \( \text{crit}(E_{\sigma^1|\alpha}) \geq \sup_{\chi \in b} \text{crit}(E_{\chi^1|\alpha}) \). Since \( b \) is a branch it follows that \( \text{crit}(E_{\sigma^1|\alpha}) \geq \sup_{\chi \in b} \text{len}(E_{\chi^1|\alpha}) \) as well. Thus \( T^-\text{pred}(\sigma|\gamma) \in I_{\gamma} \) for each \( \gamma \) in \( D \) since \( \text{crit}(E_{\sigma^1|\gamma}) > \text{len}(E_{\sigma^1}) > \text{crit}(E_{\nu^1}) \). It follows that \( T^-\text{pred}(E_{\sigma^1|\alpha}) = T^-\text{pred}(E_{\sigma^1|\gamma})|\alpha \) for \( \alpha < \gamma \) in \( D \), and \( T^-\text{pred}(\sigma) = \left( \bigcup_{\gamma < \lambda} T^-\text{pred}(\sigma|\gamma) \right) T^-\text{pred}(\alpha) \in I_{\lambda} \).

Now suppose that \( \sigma \) is a limit node of \( I_{\sigma} \). We have to show that \( M_{\sigma} = M_{c} \) where \( c = s_{\dot{G}}(T^\lambda|\sigma) \). For each \( \gamma \in D \) the set \( c_{\gamma} = \{ \sigma'|\gamma : \sigma' \in c \} \) is a branch through \( T^\lambda|\gamma \), and \( M_{c_{\gamma}} \) can be embedded into \( M_{c} \). Thus \( M_{c_{\gamma}} \) is in \( G \) and hence \( c_{\gamma} = s_{\dot{G}}(T^\lambda|\sigma) = \{ \sigma' : \sigma' \prec \gamma \} \) and \( M_{c_{\gamma}} = M_{\sigma^1|\gamma} \). Then

\[
M_{\sigma} = \text{dir lim}_{\gamma \in D} M_{\sigma^1|\gamma} = \text{dir lim}_{\gamma \in D} M_{c_{\gamma}} = \text{dir lim}_{\sigma' \in c} \text{dir lim}_{\gamma \in D} M_{\sigma^1|\gamma} = \text{dir lim}_{\sigma' \in c} M_{\sigma} = M_{c}.
\]

The argument for \( \sigma = \delta_{\lambda} \) is a special case of the argument for limit nodes \( \sigma \in I_{\lambda} \). We need to define the strategy \( s_{\dot{G}}^* \) at \( T^0|\delta_{\lambda} \) and then show that if \( T^0 \) followed this strategy then we can define \( M_{\delta_{\lambda}} \) with the embedding \( j_{\delta_{\lambda}}^{\alpha,\lambda} : M_{\alpha}^0 \rightarrow M_{\delta_{\lambda}} \). There are two cases, depending on whether there is a member \( \sigma \) of \( I_{\lambda} \) such that \( \sigma|\gamma = \delta_{\gamma} \) for all sufficiently large \( \gamma \in D \).

We need to show that if \( \text{crit}(E_{\sigma^1|\alpha}) \geq \sup_{\chi \in b} \text{crit}(E_{\chi^1|\alpha}) \) then \( \delta_{\alpha} \prec \gamma \). For \( \gamma = \gamma' + 1 \) we have \( T^0\text{pred}(\delta_{\gamma'}) = \nu_{\gamma' + 1} \) if \( \nu_{\gamma} = \delta_{\alpha} \) then we’re done. If not then it must be in some \( I_{\lambda} \), or else \( \delta_{\gamma} \prec \gamma \). There are no trees in \( I_{\lambda} \) and the claim follows from the induction hypothesis. For \( \gamma \) a limit ordinal this claim is established below.

First suppose that \( \sigma \) is such a node. Clearly \( \sigma \) is the largest sequence in \( I_{\lambda} \), and in this case we identify \( \sigma \) with \( \delta_{\lambda} \), letting \( \sigma \) be fully expanded index for \( \delta_{\lambda} \). It follows that \( \delta_{\gamma^1|\alpha} = \delta_{\alpha} \) for sufficiently large \( \alpha < \gamma \) in \( D \), so that there is a branch \( c_{0} \) of \( T^0|\delta_{\lambda} \) containing every sufficiently large \( \delta_{\alpha} \) for \( \alpha \in D \). Then

\[
M_{c_{0}}^0 = \text{dir lim}_{\alpha, \gamma \in D} (M_{\delta_{\alpha}^1|\alpha, \gamma})
\]

, which can be embedded\(^2\) in

\[
M_{\sigma} = \text{dir lim}_{\alpha, \gamma \in D} (M_{\delta_{\alpha}^1|\alpha, \delta_{\gamma}}) \in G
\]

so \( M_{c_{0}} \in G \). Thus we can set \( s_{\dot{G}}^*(T|\delta_{\lambda}) = c_{0} \), and let \( j_{\delta_{\lambda}}^{0,\lambda} \) be this embedding. The same argument gives \( j_{\delta_{\alpha}}^{\alpha,\lambda} \) for \( \alpha < \lambda \), since \( T^\alpha \) is essentially \( T^0 \) above \( \delta_{\alpha} \).

\(^2\)This has to be proved, and will probably need a more explicit statement of the commutative diagrams involved and the structure of the embeddings and the tree. The reason is that if \( \delta_{\alpha} = T^0\text{pred}(\delta_{\alpha}) \) then \( \alpha \) factors through \( s_{\dot{G}}^* \).
The direct limit diagram in this case (There is $\sigma \upharpoonright \delta_\lambda$ in $I_\lambda$ such that $\sigma \upharpoonright \alpha = \delta_\alpha$ for every sufficiently large $\alpha \in D$):

$$
\begin{array}{c}
M_\delta^\xi \xrightarrow{i} M_\delta^\xi \xrightarrow{i} \ldots M_\delta^\xi \\
\downarrow j \quad \downarrow j \quad \ldots \downarrow j \\
M_\delta^\lambda \xrightarrow{j} M_\delta^\lambda \xrightarrow{j} \ldots M_\sigma^{-\delta_\lambda} = M_\delta^\lambda
\end{array}
$$

where $\alpha < \gamma$ in $D$. The left square is obtained via the diagram

$$
\begin{array}{c}
M_\delta^\xi \xrightarrow{i} M_\delta^\xi \\
\downarrow j \\
M_\delta^\alpha \xrightarrow{i} M_\delta^\alpha \\
\downarrow = \downarrow j \\
M_\delta^\gamma = M_\delta^\gamma \xrightarrow{j} M_\delta^\gamma \\
\downarrow = \downarrow = M_\delta^\gamma M_\delta^\gamma
\end{array}
$$

In this case $\sigma \upharpoonright \delta_\lambda$ is an extended index for $\delta_\lambda$.

If there is no such node $\sigma$ then $\delta_\lambda$ will not have an expanded index. Let $c = s_G(\mathcal{T}^\lambda | \delta_\lambda)$. For each $\gamma \in D$ there must be a sequence $\sigma \in c$ such that $\sigma \upharpoonright \gamma = \delta_\gamma$, and again it follows that there is a branch $c_0$ through $\mathcal{T}^0 | \delta_\lambda$ such that $M^0_{c_0}$ is in $G$, so that we can set $s^*_G(\mathcal{T}^0) | \delta_\lambda = c_0$. Assuming that $\mathcal{T}^0$ followed this strategy, we again have the required embeddings $i_{\delta_\lambda}^{\alpha, \lambda} : M^\alpha_{\delta_\lambda} \rightarrow M^\lambda_{\delta_\lambda} = M_\sigma$.

The direct limit diagram in this case is

$$
\begin{array}{c}
M^\xi_{\delta_\alpha} \xrightarrow{i} M^\xi_{\delta_\gamma} \xrightarrow{i} \ldots M^\xi_{\delta_\lambda} \\
\downarrow j \quad \downarrow j \quad \ldots \downarrow j \\
M^\lambda_{\sigma_\alpha} \xrightarrow{i} M^\lambda_{\sigma_\gamma} \xrightarrow{i} \ldots M^\lambda_c = M^\lambda_{\delta_\lambda}
\end{array}
$$

where the left hand square is the diagram

$$
\begin{array}{c}
M^\xi_{\delta_\alpha} \xrightarrow{i} M^\xi_{\delta_\gamma} \\
\downarrow j \\
M^\alpha_{\delta_\alpha} \xrightarrow{i} M^\alpha_{\delta_\gamma} \\
\downarrow j \\
M^\gamma_{g_{\sigma_\alpha} \gamma} \xrightarrow{i} M^\gamma_{\sigma_\gamma \gamma} = M^\gamma_{\delta_\gamma} \\
\downarrow j \\
M^\lambda_{\sigma_\alpha} \xrightarrow{i} M^\lambda_{\sigma_\gamma}
\end{array}
$$

In this case, again, there is no extended index for $\delta_\lambda$.

This completes the definition\(^3\) of $\mathcal{T}^\lambda | \delta_\lambda + 1 = \mathcal{T} | \delta_\lambda + 1$. The definition of the

---

\(^3\)Both for $\lambda$ limit or successor.
The proof is by induction on $\sigma$. Suppose first that $\gamma = \lambda + 1$, and that $\sigma|\delta = \sigma|\alpha \in I_{\alpha}$. Recall that $I_{\lambda+1} = \{ \tau : \tau \leq \lambda + 1 \}$. Since $\gamma \leq \lambda + 1$ we must have $\tau \leq \lambda + 1 \leq \sigma|\alpha \leq \delta$, and since $\nu^{*}_{\lambda+1} \in \mathrm{dom}(\mathcal{T}^{0})$ we must have $\nu^{*}_{\lambda+1} \geq \nu^{*}$, so that if $x = \delta \cap \lambda + 1$ then $[\sigma|\alpha, \delta] \sim [\sigma, x]$.

If $\gamma = \lambda + 1$ and $\sigma|\lambda \neq \sigma|\alpha$ then $[\sigma|\lambda] \in I_{\gamma'}$ for some $\gamma'$ in the interval $\alpha < \gamma' < \gamma$. By the induction hypothesis there is $x' \in I_{\gamma'}$ such that $[\sigma|\gamma', \delta_{\gamma'}] \sim [\sigma, x']$. By the last paragraph there is $x'' \in I_{\gamma}$ such that $[\sigma|\gamma', \delta_{\gamma'}] \sim [\sigma, x'']$ and it follows that if $x = x'' \cap \delta_{\gamma}$ then $x \in I_{\gamma}$ and $[\sigma|\alpha, \delta] \sim [\sigma, x]$.

Now suppose that $\gamma$ is a limit ordinal and $\sigma \in I_{\gamma}$. Then there is a cofinal subset of $\gamma$ such that $\sigma|\xi \in I_{\xi}$ for all $\xi \geq \alpha$ in $D$, and if $\xi < \xi'$ in $D$ then by the induction hypothesis there is $x_{\xi, \xi'}$ such that $[\sigma|\xi, \delta_{\xi}] \sim [\sigma|x_{\xi, \xi'}, \xi']$. Now the ordinals $\alpha < \xi < \xi'$ are in $D$ then $x_{\alpha, \xi'} \geq x_{\xi, \xi'}$ and it follows that $x_{\alpha, \xi} = x_{\alpha, \xi'}[\xi]$. It follows that $x = (\bigcup \{ x_{\alpha, \xi} : \alpha < \xi < \lambda \}) \cap \delta_{\gamma}$ is in $I_{\gamma}$, and $[\sigma|\alpha] \sim [\sigma, x]$.

3.11 Lemma. Suppose that there is $\tau < \sigma$ such that $\tau \in I_{\alpha}$. Then either (i) there is $\tau'$ such that $\tau' \leq \sigma$ and $\tau'|\alpha = \delta_{\alpha}$, or (ii) there is $\gamma > \alpha$ such that $\sigma \in I_{\gamma}$ and $[\sigma|\alpha] \sim [\sigma, x]$.

Proof. The proof is by induction on $\sigma$ and is broken into several cases.

Case (1) ($\sigma \notin I_{\gamma}$ for any $\gamma$): In this case we need to show that alternative (i) holds. If $\sigma$ is a limit node then since both $C$ and each $I_{\gamma}$ are closed there must be $\sigma'$ such that $\tau < \sigma' < \sigma$. By the induction hypothesis it follows that alternative (i) holds for the pair $\tau < \tau'$ and hence for the pair $\tau < \sigma$. Thus we can assume that $\sigma$ is a successor node. Since $\sigma$ is not in any $I_{\alpha}$ we have $T_{\alpha} \setminus \mathrm{pred}_{0}(\sigma) = T_{\alpha} \setminus \mathrm{pred}(\sigma)$, so that

$$j_{\alpha, \lambda}^{\xi, \lambda}([a, f])_{E_{\alpha}^{\xi, \lambda}} = [j_{\alpha, \lambda}^{\xi, \lambda}(a), j_{\alpha, \lambda}^{\xi, \lambda}(f)]_{E_{\alpha}^{\xi, \lambda}}.$$
\( \tau \preceq \mathcal{T}\text{-pred}^0(\sigma) \preceq \sigma \). If \( \mathcal{T}\text{-pred}^0(\sigma) = \tau \in I_\alpha \) then \( \tau \) must be equal to \( \delta_\alpha \), so that alternative (i) holds, and if \( \tau \preceq \mathcal{T}\text{-pred}(\sigma) \) and alternative (i) holds for the pair \( \tau \preceq \mathcal{T}\text{-pred}(\sigma) \) then alternative (i) holds for \( \sigma \) as well. Thus we can assume that alternative (ii) holds for \( \tau \preceq \mathcal{T}\text{-pred}^0(\sigma) \), that is, \( \mathcal{T}\text{-pred}(\sigma) \subseteq I_\lambda \) for some \( \gamma \) and \( \mathcal{T}\text{-pred}(\sigma)|\alpha \in I_\alpha \). Since \( \mathcal{T}\text{-pred}(\sigma) \subseteq I_\gamma \) we must have \( \mathcal{T}\text{-pred}(\sigma) = \delta_\gamma \); and since \( \mathcal{T}\text{-pred}(\sigma)|\alpha \in I_\alpha \) lemma 3.11 implies that there is \( x \in I_\gamma \) such that \( [\tau, \delta_\alpha] \sim [\delta_\gamma, x] \). Since \( x \) can only be equal to \( \delta_\gamma \) we must have \( \tau = \delta_\alpha \) and alternative (i) holds after all.

Thus we can assume that \( \sigma \in I_\gamma \) for some \( \gamma \).

**Case (2):** \((\sigma \in I_\gamma \text{ is a limit node in } I_\gamma)\): In this case there are nodes \( \sigma' \prec \sigma \) in \( I_\gamma \). By the induction hypothesis one of the alternatives holds for \( \tau \prec \sigma' \). If the first alternative holds for any \( \sigma' \prec \sigma \) then it also holds for \( \sigma \), so we can assume that \( \sigma'|\alpha \in I_\alpha \) for each such \( \sigma' \). Then by lemma 3.11 there is \( x \in I_\gamma \) such that \( [\sigma'|\alpha, \delta_\alpha] \sim [\sigma', x] \), and it will be sufficient to show that \( x \geq \sigma \). But \( \{\sigma' : \sigma' \prec \sigma' \prec \sigma \} \) is cofinal in \( \sigma \), and for each \( \sigma'' \) in this set \( \sigma''|\alpha \in I_\alpha \), so each such \( \sigma'' \leq x \).

It follows that \( \sigma \leq x \).

**Case (3):** \((\sigma \in I_\gamma \text{ is a successor node})\): In this case we have \( \tau \preceq \mathcal{T}\text{-pred}(\sigma) \preceq \sigma \). If \( \tau \preceq \mathcal{T}\text{-pred}(\sigma) \) and alternative (ii) holds for the pair \( \tau \preceq \mathcal{T}\text{-pred}(\sigma) \) then it also holds \( \tau \prec \sigma \), so we can assume that either \( \tau = \mathcal{T}\text{-pred}(\sigma) \) or \( \tau \prec \mathcal{T}\text{-pred}(\sigma) \) and \( \mathcal{T}\text{-pred}(\sigma)|\alpha \in I_\alpha \). Thus \( \mathcal{T}\text{-pred}(\sigma)|\alpha \in I_\alpha \setminus \{\delta_\alpha\} \) in either case, so lemma 3.13 implies that \( \sigma|\alpha \in I_\alpha \).

**3.13 Lemma.** Suppose that \( \sigma \) is a successor node and that \( \mathcal{T}\text{-pred}(\sigma)|\alpha \in I_\alpha \setminus \{\delta_\alpha\} \). Then \( \sigma|\alpha \in I_\alpha \).

**Proof.** First we show that \( \sigma \in I_\gamma \) for some \( \gamma \). If not, then \( \sigma = \xi \) for some \( \xi \), and \( \mathcal{T}\text{-pred}(\sigma) = \mathcal{T}\text{-pred}^0(\sigma) = \delta \), say, so that \( \delta|\alpha \in I_\alpha \). If \( \delta \in I_\alpha \) then \( \delta = \delta_\alpha \), contradicting the assumption that \( \mathcal{T}\text{-pred}(\sigma)|\alpha \neq \delta_\alpha \). If \( \delta \notin I_\alpha \) then \( \delta \in I_\gamma \) for some \( \gamma' \) in the interval \( \alpha < \gamma' < \gamma \), since otherwise \( \delta|\alpha \) would be empty. It follows that \( \delta = \delta_{\gamma'} \). Now lemma 3.11 implies that \( [\delta_{\gamma'}|\alpha, \delta_\alpha] \sim [\delta_{\gamma'}, x] \) for some \( x \in I_{\gamma'} \) and since \( x \) can only be \( \delta_{\gamma'} \) it follows that \( \delta_{\gamma'}|\alpha = \delta_\alpha \), contradicting the assumption that \( \mathcal{T}\text{-pred}(\sigma|\alpha) \neq \delta_\alpha \).

Thus we can assume that \( \sigma \in I_\gamma \) for some \( \gamma \), and we complete the proof by induction on \( \gamma \). Suppose first that \( \gamma = \lambda + 1 \). Then \( \mathcal{T}\text{-pred}(\sigma) \) is equal to one of

\[
\mathcal{T}\text{-pred}(\sigma|\lambda) \\
\mathcal{T}\text{-pred}(\sigma|\lambda) \upharpoonright \delta_{\lambda+1} \\
\sigma|\lambda 
\text{ (if } \sigma = \min(I_{\lambda+1})).
\]

In the third case \( \sigma|\alpha = \mathcal{T}\text{-pred}(\sigma)|\alpha \) so \( \sigma|\alpha \in I_\alpha \), while in either of the first two cases \( \mathcal{T}\text{-pred}(\sigma|\lambda)|\alpha = \mathcal{T}\text{-pred}(\sigma)|\alpha \in I_\alpha \) and by the induction hypothesis it follows that \( \sigma|\alpha = (\sigma|\lambda)|\alpha \in I_\alpha \).

Now suppose that \( \sigma \in I_\gamma \) where \( \gamma \) is a limit ordinal. In this \( \mathcal{T}\text{-pred}(\sigma) \) has one of the two forms

\[
\mathcal{T}\text{-pred}(\sigma|\xi) \quad \text{for some } \xi < \gamma \\
\bigcup_{\eta \leq \gamma} \sigma|\xi \upharpoonright \delta_\gamma \quad \text{for a cofinal subset } D \text{ of } \lambda.
\]
In either case $\mathcal{T}$-pred$(\sigma)|\alpha = \mathcal{T}$-pred$(\sigma|\xi)|\alpha$ for any sufficiently large $\xi < \gamma$, and by the induction hypothesis it follows that $\sigma|\alpha = (\sigma|\xi)|\alpha \in I_\alpha$.  

3.14 Lemma. Suppose that $\sigma < \sigma'$ and $\{\sigma''|\alpha : \sigma \leq \sigma'' \leq \sigma'\} \subset I_\alpha$. Then $\sigma|\alpha \leq \sigma'|\alpha$, with equality if and only if for all $\sigma''$ in the interval $\sigma < \sigma'' \leq \sigma$ there is $\gamma$ such that $\sigma'' = \min(I_\gamma)$.

3.15 Proposition. If $\sigma_0 < \sigma_1$ and $\sigma_1|\gamma \in I_\gamma$ then $\sigma_0|\gamma < \sigma_1|\gamma$. (Note that $\sigma_0|\gamma$ need not be in $I_\gamma$.)

3.16 Proposition. If $\sigma_0$, $\sigma_1$, and $\sigma_2$ are in $I_\alpha$ and $\sigma_0|\gamma_0 < \sigma_1|\gamma_1 < \sigma_2$ then $\gamma_1 = \alpha$.

3.17 Proposition. If $\sigma < \sigma'$ then $\sigma(0) < 0 \sigma'(0)$. (Where both $\sigma$ and $\sigma'$ are fully expanded.)

Dropping to mice. In addition to the points considered above, the tree $\mathcal{T}$ may be nonnormal because $M_{\nu+1} = \text{ult}_n(M_{\nu+1}^*, E_\nu)$ where either $M_{\nu+1}^*$ is a mouse in $M_{\mathcal{T}$-pred$(\nu+1)}$ which is smaller than necessary or $n$ is smaller than necessary. In this case $M_{\nu+1}^*$ can be embedded into $\text{ult}_n'(M', E_\nu)$ where $n'$ and $M'$ are the “right” choices, and this embedding will give an embedding from the rest of $\mathcal{T}$ to the tree using $n'$ and $M'$.

The argument given is also complicated by the fact that $M_0^\nu + 1$ may be equal to $\text{ult}(M_{\nu+1}^*, E_\nu)$ where $M_{\nu+1}^*$ is a mouse in $M_0^\nu$, or $E_\nu$ may not be an extender on $M_\tau$, making it difficult to drop to a mouse in the normalized tree. By the last paragraph we can assume that $M_{\nu+1}^*$ is as large as possible so that $E_\nu$ is an measure on $M_{\nu+1}^*$. Now suppose that there is $\sigma$ such that $\tau \leq \sigma < \nu^*$ such that $\text{index}(E_\sigma) < \text{index}(E_{\nu^*})$. In this case the power set of $\text{crit}(E_\nu)$ in $M_{\nu^*}$ is equal to the power set of $\text{crit}(E_\nu)$ in $M_{\sigma}$. In particular if $M_{\nu+1}^*$ is a mouse in $M_{\nu+1}$ then it is also a mouse in $M_{\sigma}$ and we may as well take $\sigma$, rather than $\nu^*$, to be the predecessor of $\nu + 1$ (this is like deadwood). Thus in this situation we can assume that $M_{\nu+1}^* = M_{\nu^*}$. Now take $\sigma$ least so that $E_\nu$ is an extender on $\sigma + 1$. Then $\tau \leq \sigma \leq \nu^*$, and by the argument in these notes for trees without dropping to a mouse we can embed the tree into one in which $\mathcal{T}$-pred$(\nu + 1) = \sigma + 1$.

Now the same procedure can be used once more to embed the tree into one with $\mathcal{T}$-pred$(\nu + 1) = \sigma$, with $M_{\nu+1}^* = \mathcal{M}_{\sigma, \text{crit}(E_\nu)}$. Then $\langle \nu \rangle + 1 = \sigma^\langle \nu + 1 \rangle$, and $\sigma^\langle \nu + 1 \rangle + 1 = (\sigma + 1)^\langle \nu + 1 \rangle = \nu + 1$, that is, the set corresponding to $I_{\lambda+1}$ is $\{\sigma^\langle \nu + 1 \rangle, (\sigma + 1)^\langle \nu + 1 \rangle\}$. In this case $j_{\tau, \tau^\langle \nu + 1 \rangle}$ only embeds $M_{\nu+1}^*$ into $M_{\sigma^\langle \nu + 1 \rangle}$, rather than embedding all of $M_{\sigma}$ into $M_{\sigma^\langle \nu + 1 \rangle}$, but this doesn’t matter since $M_{\sigma^\langle \nu + 1 \rangle} = \text{ult}(M_{\sigma+1}, E_{\sigma^\langle \nu + 1 \rangle})$. Thus only the extender $E_{\sigma^\langle \nu + 1 \rangle}$ is used from $M_{\sigma^\langle \nu + 1 \rangle}$ and no more is needed of that model^4.

Thus we can assume that $M_{\nu+1}^*$ is a mouse in $M_{\nu^*}$, with power set the same as that in $M_{\nu^*+1}$, and that $\text{index}(E_\sigma) > \text{index}(E_{\nu^*})$ for all $\sigma$ with $\tau \leq \sigma < \nu^*$. It follows that $M_{\nu+1}^*$ is also a mouse in $M_{\tau}$, so that we may as well assume that $\nu^* = \tau$ (again, this is the deadwood argument).

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4 A more general argument is needed to verify that this won’t cause trouble later.