Scaling Functions for Baby Universes in Two-Dimensional Quantum Gravity

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Abstract

We apply the recently proposed transfer matrix formalism to 2-dimensional quantum gravity coupled to $(2, 2k - 1)$ minimal models. We find that the propagation of a parent universe in geodesic (Euclidean) time is accompanied by continual emission of baby universes and derive a distribution function describing their sizes. The $k \to \infty$ ($c \to -\infty$) limit is generally thought to correspond to classical geometry, and we indeed find a classical peak in the universe distribution function. However, we also observe dramatic quantum effects associated with baby universes at finite length scales.

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1 Introduction

One fascinating aspect of quantum gravity is the possibility of topology changing processes where a compact connected 3-geometry (a “baby universe”) splits off or joins a large “parent universe.” Such processes, peculiar as they may seem, are quite natural at the Planckian distance scales, where geometry undergoes large quantum fluctuations. Even if the topology changing processes are confined to the smallest distance scales, they may drastically affect the observed physics. As shown in ref. [1], summation over all possible emissions and absorptions of the Planck-size baby universes effectively gives rise to averaging over the fundamental constants of nature. If the weight in the average is sharply peaked, then the fundamental constants are determined [2]. The wormhole theory of fundamental constants shows that, via some unexpected connections, the short-distance properties of quantum gravity can affect the universe at large in crucial ways. The status of this theory when applied to 4-dimensional quantum gravity is somewhat uncertain due to the instability of the Euclidean path integral. It may be, for instance, that the weight in the average over the coupling constants is not sharply peaked [3]. If, on the other hand, the weight is sharply peaked, then it is not quite clear what prevents the emission and absorption of macroscopic baby universes [4]. While these puzzles await their resolution through a better understanding of 4-dimensional quantum gravity, we may try to gain some intuition about the topology changing processes from low-dimensional models.

The 2-dimensional Euclidean quantum gravity coupled to $c \leq 1$ conformal matter appears to provide some promising toy models which are not plagued by any perturbative instabilities. Our understanding of these theories has dramatically improved recently, in part due to the success of the matrix model techniques [5]. With their help we can perform summations over discretized random surfaces and extract the exact results in the universal continuum limit. Until recently, however, the available exact solutions have not shed any light on the physics of topology changing processes. The problem is that the matrix models do not give any direct information on the internal geometry of 2-dimensional space-times. Some first pieces of information were extracted via direct Monte Carlo studies of triangulated random surfaces [6]. It was noted that the set of points at a given geodesic distance $D$ from some point $P$ typically consists of many disconnected loops, whose number grows rapidly with $D$. The study in ref. [6] gave a first indication that, as a 1-dimensional universe (a string) propagates forward in geodesic (Euclidean) time, it is very likely to continually emit baby universes. This phenomenon received a detailed quantitative confirmation in a recent very interesting paper by Kawai, Kawamoto, Mogami and Watabiki [7]. Relying on new combinatorial techniques that they invented, these authors derived a formula for the average number of loops with lengths between $L$ and $L + dL$ located at a geodesic distance $D$ from some point on the surface,

$$
\rho(L,D)dL = \frac{3}{7\sqrt{\pi}D^2} \left( x^{-5/2} + \frac{1}{2} x^{-3/2} + \frac{14}{3} x^{1/2} \right) e^{-x} dL
$$
where \( x = L/D^2 \). A number of beautiful conclusions follow from this formula. First of all, \( \rho(L, D) \) is not normalizable at small \( L \), which means that the microscopic baby universes are overwhelmingly likely to be emitted. Secondly, apart from an overall dimensionful factor, \( \rho \) is a function of the dimensionless scaling variable \( x \). Introducing the moments,

\[
⟨L^n⟩_D = \int_0^∞ \rho(L, D)L^n dL ,
\]

one finds that for \( n < 2 \) they are dominated by the non-universal short-distance cut-off. But for \( n \geq 2 \) there are simple scaling relations \( ⟨L^n⟩_D \sim D^{2n} \), whose form follows essentially from \( D \) having the dimension of (Length)\(^{-1/2}\).

The distribution function \( \rho(L, D) \) is very important because it quantifies the effects of baby universes in pure Euclidean 2-dimensional quantum gravity. The next question is how the coupling to matter affects the topology changing processes. In this paper we begin to address this question by considering the emission of baby universes in 2-dimensional quantum gravity coupled to the \((2, 2k - 1)\) minimal models, whose central charges are \( c = 1 - 3(2k - 3)/(2k - 1) \). These theories have been identified \([3]\) with the multicritical points of the one-matrix model \([4]\), \([5]\). The \( k = 2 \) theory corresponds to pure gravity, where it suffices to consider the discretizations of random surfaces with squares only. For \( k = 3 \) \((c = -22/5)\) one needs squares and hexagons; for \( k = 4 \) one needs squares, hexagons and octagons; etc.

For \( k \geq 3 \) some polygons enter with negative weights, which is not surprising given that these models are not unitary. We find that, for odd \( k \), the negative weights have so much effect that there is no sensible positive \( \rho(L, D) \). For even \( k \), however, there does exist a positive \( \rho(L, D) \), whose calculation constitutes the main result of this paper. We find that it depends on the scaling variable \( x = L/D^{1/(k-3/2)} \) and for small \( x \) diverges as \( x^{-k-1/2} \). Thus, the emission of microscopic baby universes becomes more enhanced with increasing \( k \). For large \( k \), \( c \to -∞ \) and the sum over surfaces is expected to be dominated by classical geometry. In this limit we indeed find that \( \rho(L, D) \) exhibits a sharp peak that corresponds to the macroscopic classical geometry. Surprisingly, we also find that baby universes remain prevalent for all length scales less than a fixed constant, \( s_c \), times the macroscopic (classical) length scale, and calculate the critical value \( s_c \). As a further application of these ideas, we discuss the production of baby universes by a very large parent universe of length \( L_0 \). We find that their distribution is governed by a very simple scaling law,

\[
\rho(L_0 \to ∞, L, D) \sim L_0D^{-k-1/2} ,
\]

which applies to both even and odd \( k \).

The structure of our paper is as follows. In section \([2]\) we review and rederive the necessary matrix model results. In section \([3]\) we extend the transfer matrix formalism of ref. \([6]\) to general discretizations, which are necessary to describe the theories with matter. In section \([4]\) we calculate the transfer matrix in some limiting cases, and in section \([5]\) we study the branching structure of space-time by deriving the distribution functions for baby universes. In section \([6]\) we conclude with a few remarks.
The Disk Amplitudes

For the calculation of the transfer matrix, the only result needed from the matrix models is the disk amplitude—that is, the partition function for surfaces with one boundary loop. It was shown in [10] that in the $k$th multicritical theory, the universal part of the disk amplitude is

$$F_k(L, \tau) = \frac{1}{L} (\sqrt{\tau})^{k-1/2} K_{k-1/2}(\sqrt{\tau}L)$$  \hspace{1cm} (4)$$

where $\tau$ is the cosmological constant and $K_{k-1/2}$ is a modified Bessel function. We will need to keep track of certain non-singular parts of the disk amplitudes because they will be important in the calculation of the transfer matrix. Thus, we present our own calculation of the disk amplitudes.

Throughout this paper, we will be working with surfaces without handles, so it is sufficient to use saddle point techniques in the matrix models. The partition function of the $k$th multicritical model (that is, the sum of the weights of all vacuum diagrams, including the disconnected ones) is

$$Z = \int d\Phi \exp \left[ -\beta \text{Tr} V(\Phi) \right]$$  \hspace{1cm} (5)$$

where $\Phi$ is a $N \times N$ matrix with $N \sim \beta \to \infty$, and

$$V(\Phi) = \sum_{m \geq 1} \frac{c_m}{2m} \Phi^{2m}$$  \hspace{1cm} (6)$$

is a potential energy function. (We restrict ourselves to even $V(\Phi)$ because they are more easily analyzed, but we believe that more general $V(\Phi)$ do not exhibit any more general behavior.)

Let $u(\lambda)$ be the density of eigenvalues of $\Phi$, normalized so that $\int d\lambda u(\lambda) = 1$. As shown in [12], $u(\lambda)$ has support $[-\sqrt{z}, \sqrt{z}]$ for some $z$, and

$$V'(\lambda) = 2\frac{N}{\beta} \int_{-\sqrt{z}}^{\sqrt{z}} d\mu \frac{u(\mu)}{\lambda - \mu}.$$  \hspace{1cm} (7)$$

For $\lambda \in \mathbb{C} - [-\sqrt{z}, \sqrt{z}]$, define

$$F(\lambda) = 2\frac{N}{\beta} \int_{-\sqrt{z}}^{\sqrt{z}} d\mu \frac{u(\mu)}{\lambda - \mu}.$$  \hspace{1cm} (8)$$

$F(\lambda)$ is analytic except for a branch cut along $[-\sqrt{z}, \sqrt{z}]$: for $\lambda \in [-\sqrt{z}, \sqrt{z}]$ and $\epsilon > 0$ infinitesimally small, $F(\lambda \pm i\epsilon) = V'(\lambda) \mp 2\pi i (N/\beta) u(\lambda)$. Also, $F(\lambda) = 2(N/\beta)/\lambda + O(\lambda^{-2})$ for large $\lambda$. It is not hard to prove that these properties uniquely determine $F(\lambda)$ and hence $u(\lambda)$. It turns out that $F(\lambda)$ has the form

$$F(\lambda) = V'(\lambda) - f(\lambda)\lambda \sqrt{1 - \frac{z}{\lambda^2}}$$  \hspace{1cm} (9)$$
where \( f(\lambda) \) is a polynomial. Both \( z \) and \( f(\lambda) \) are determined by the large \( \lambda \) expansion of \( F(\lambda) \). Defining for convenience the functions \( g(\lambda) = \lambda V'(\lambda) \) and \( h(\lambda) = \lambda^2 f(\lambda) \), we find that

\[
g(\lambda) \left(1 - \frac{z}{\lambda^2}\right)^{-1/2} = \left(\sum_{m \geq 1} c_m \lambda^{2m}\right) \left(\sum_{m \geq 0} \frac{(2m)!}{m!^2 4^m \lambda^{2m}} z^m\right) = h(\lambda) + 2\frac{N}{\beta} + O(\lambda^{-1}). \tag{10}
\]

The \( O(\lambda^0) \) terms determine \( z \):

\[
\sum_{m \geq 1} c_m \frac{(2m)!}{m!^2 4^m} z^m = 2\frac{N}{\beta}. \tag{11}
\]

We can express the partition function \( Z \) as a power series in the couplings \( c_m \) and the parameter \( N/\beta \). If we consider the potential \( V \) and hence the \( c_m \) to be fixed, then \( Z \) has a radius of convergence in \( N/\beta \); we construct the potential so that this radius of convergence is 1. For \( N/\beta > 1 \)—that is, for (renormalized) temperatures higher than a critical temperature—the series expansion of \( Z \) diverges. The behavior of \( Z \) near the critical point is what describes the quantum geometry of random surfaces, since at the critical point the behavior of \( Z \) is dominated by Feynman graphs of large order. It turns out that the singular behavior of \( Z \) at the critical point is determined by the dependence of \( z \) on \( N/\beta \) in Eq. (11). The \( k^{th} \) multicritical model is constructed by finding \( c_m \) such that Eq. (11) takes the form

\[
1 - \left(1 - \frac{z}{z_c}\right)^k = \frac{N}{\beta} \equiv 1 - \mu_0 \tag{12}
\]

for some \( z_c \). We shall use

\[
c_m = \begin{cases} (-1)^{m+1} \frac{2k! m!}{(2m)! (k-m)! k^m} & \text{for } m \leq k \\ 0 & \text{for } m > k \end{cases} \tag{13}
\]

which gives \( z_c = 4k \).

Let \( \epsilon \) be the lattice spacing of our random lattice. Then area is measured in units of \( \epsilon^2 \), which will be taken to zero as the average number of plaquettes in the discretizations of the random surface diverges, so as to yield finite area. The parameter \( \mu_0 \) corresponds to the lowest dimension operator in the \( k^{th} \) multicritical theory, and it was at first believed that if one wrote \( \mu_0 = (2\epsilon)^k t_0 \), then \( t_0 \) would be the cosmological constant. This is false for the \( k > 2 \) models, which are non-unitary: \( t_0 \) has dimensions of \( (\text{Length})^{-k} \), and is the coupling constant for the gravitationally dressed conformal field of the lowest dimension. The true cosmological constant is by definition conjugate to area, so it has dimensions of \( (\text{Length})^{-2} \).

We must consider a more general perturbation around criticality. Specifically, we perturb the potential by replacing \( c_m \rightarrow c_m + \frac{2m^2}{(2m)! k^m} \mu_m \) for all \( m > 0 \). Eq. (11) then reduces to

\[
\left(1 - \frac{z}{z_c}\right)^k = \sum_{m \geq 0} \mu_m \left(\frac{z}{z_c}\right)^m. \tag{14}
\]
Higher dimension scaling operators are introduced by choosing the \( \mu_m \) so that the right hand side becomes \( t_l e^{k-l}(1 - z/z_c)^l \) for some \( l < k \), where \( t_l \) is the coupling to the \( l \)th scaling operator. As was shown in [10], the couplings to the operators corresponding to conformal fields in the Liouville theory are analytical combinations of \( t_l \) with a definite overall dimension. The cosmological constant \( \tau \) corresponds to a perturbation with \( t_l = \alpha_l 2^{k-l} \tau^{-2l/2} \) for \( l - k \) even and \( 0 \leq l \leq k - 2 \). The \( \alpha_l \) are real numbers chosen so that the singular part of the one-loop amplitude obeys a Bessel equation, which emerges from the Wheeler de Witt equation.

\[
(1 - \frac{z}{z_c})^k = \sum_l \alpha_l (2e\sqrt{\tau})^{k-l} (1 - \frac{z}{z_c})^l,
\]

and by requiring \( \sum_l \alpha_l = 1 \) (which is equivalent to fixing a normalization for \( \tau \)) we find \( z = z_c (1 - 2e\sqrt{\tau}) \). For \( k = 2 \) and \( k = 3 \) there is only one \( \alpha_l \), so \( \tau \propto t_{k-2} \). For \( k = 4 \), it is found that \( \alpha_0 = -1/5 \) and \( \alpha_2 = 6/5 \). We do not know a general way to determine the \( \alpha_l \) except by straight calculation.

The partition function for surfaces with one boundary is the continuum limit of the Green’s function of the field theory above. The \( l \)-point Green’s function is

\[
G_l = \langle \mathrm{Tr} \Phi^l \rangle = \int_{-\sqrt{\tau}}^{\sqrt{\tau}} d\lambda u(\lambda) \lambda^l.
\]

It is convenient to introduce \( G(y) = \sum_{l\geq0} G_l y^l \), since it is easy to see that

\[
G(y) = \frac{\beta}{N 2y} F(1/y) = \frac{\beta}{N 2} \left( g(1/y) - h(1/y) \sqrt{1 - zy^2} \right).
\]

Since at \( z = z_c \) the radius of convergence of \( G(y) \) is \( y_c = 1/\sqrt{z_c} \), the continuum limit is taken by setting \( y = y_c \exp(-\epsilon \zeta) \), where now \( \zeta \) is conjugate to the length of the boundary. The continuum limit \((\epsilon \to 0)\) of \( G(y) \) was calculated using Mathematica, with the following results:

\[
\begin{align*}
    k = 2 : & \quad G(\zeta) = \frac{4}{3} - \frac{8}{3} \zeta \epsilon + \frac{16\sqrt{2}}{3} \epsilon^{3/2} f_2(\zeta, \tau) + O(\epsilon^2) \\
    & \quad \text{where} \quad f_2(\zeta, \tau) = 2 \left( \zeta - \frac{1}{2} \sqrt{\tau} \right) \sqrt{\zeta + \sqrt{\tau}} \\
    k = 3 : & \quad G(\zeta) = \frac{6}{5} - \frac{4}{5} \zeta \epsilon + \frac{4}{5} (-4\tau + 2\zeta^2) \epsilon^2 + \frac{96\sqrt{2}}{5} \epsilon^{5/2} f_3(\zeta, \tau) + O(\epsilon^3) \\
    & \quad \text{where} \quad f_3(\zeta, \tau) = \frac{2}{3} \left( -\zeta^2 + \frac{1}{2} \zeta \sqrt{\tau} + \frac{1}{4} \tau \right) \sqrt{\zeta + \sqrt{\tau}} \\
    k = 4 : & \quad G(\zeta) = \frac{8}{7} - \frac{16}{35} \zeta \epsilon + \frac{16}{35} (-2\tau + 3\zeta^2) \epsilon^2 + \frac{32}{525} (174\tau - 185\zeta^2) \epsilon^3 + \frac{512\sqrt{2}}{7} \epsilon^{7/2} f_4(\zeta, \tau) + O(\epsilon^4) \\
    & \quad \text{where} \quad f_4(\zeta, \tau) = \frac{2}{5} \left( \zeta^3 - \frac{3}{2} \zeta^2 \sqrt{\tau} - \frac{1}{4} \zeta \tau + \frac{1}{8} \tau^{3/2} \right) \sqrt{\zeta + \sqrt{\tau}}
\end{align*}
\]

The coefficient \( f_k(\zeta, \tau) \) of the leading nonanalytic term in \( \epsilon \) is the universal part of the disk amplitude. \( f_k(\zeta, \tau) \) is, up to a numerical factor, the Laplace transform of the disk amplitude \( F_k(L, \tau) \) in Eq. (4). The lower order analytic terms in \( \zeta \) and \( \tau \) correspond to zero length and zero area terms, and most of them can be dropped, for the following reason. We are free
to adjust \( \zeta \) by an analytic function of \( \zeta \) and \( \tau \), which, in order to avoid trivial additive and multiplicative rescalings of \( \zeta \), and to preserve the dimension of \( \zeta \), should have the form

\[
\zeta \rightarrow \zeta + r_1 \epsilon \tau + r_2 \epsilon^2 \zeta^2 + r_3 \epsilon^2 \tau \zeta + r_4 \epsilon^2 \zeta^3 + \ldots
\]  

(19)

where the \( r_i \) are \( c \)-numbers. Such a redefinition preserves the universal term \( f_k(\zeta, \tau) \) and corresponds simply to a different way of treating zero area and zero length terms, as pointed out in [10]. By an appropriate choice of the \( r_i \) in Eq. (19), we can absorb all but the first two analytic terms of the expressions in Eq. (18) into \( \zeta \). We then find the simple form

\[
G(\zeta) = \frac{2k}{2k-1} \left( 1 - \frac{\zeta \epsilon}{\sigma} \right) + \alpha \epsilon^{k-1/2} f_k(\zeta, \tau) + O(\epsilon^k)
\]  

(20)

where

\[
\sigma = k - 3/2
\]  

(21)

and \( \alpha \) is a numerical factor chosen for each \( k \) so that the leading term of a small \( \tau \) expansion of \( f_k(\zeta, \tau) \) is \((-1)^k \zeta^{\sigma+1}/\sigma\).

For later convenience, we mention one more mathematical point: since \( z/z_c = 1 + O(\epsilon) \), we can alter the perturbation Eq. (13) to read

\[
\left( 1 - \frac{z}{z_c} \right)^k = \sum_l \alpha_l (2\epsilon \sqrt{\tau})^{k-l} \left( 1 - \frac{z}{z_c} \right)^l \left( \frac{z}{z_c} \right)^n
\]  

(22)

where \( n \) is a fixed integer, and the equation \( z = z_c(1 - 2\epsilon \sqrt{\tau}) \) will receive corrections that are analytic in \( \epsilon \) and of order \( \epsilon^2 \) and higher. The non-universal terms in Eq. (18) will change, but the leading analytic and leading nonanalytic terms will be unaffected, so by making an appropriate redefinition of the form Eq. (19), we still arrive at Eq. (21). If we take \( n = 2 \), then \( \mu_0 = 0 \) and \( \mu_1 = 0 \). This is desirable because it means that the weights of planar Feynman diagrams do not depend on the number of edges.

### 3 The transfer matrix

The focus of [4] is the analysis of the evolution of a loop through some fixed geodesic distance on the surface, where on a discretized surface geodesic distance is defined as the minimal number of plaquettes one must traverse to get from one point to another. The goal is to calculate the partition function of a tube with one entrance loop and one exit loop, such that each point on the exit loop is a fixed geodesic distance \( D \) from the entrance loop.

On a discretized surface, one starts with an entrance loop \( \gamma \) (see Fig. 1), defines a “forward” direction for geodesic distance (inward in Fig. 1), and thinks of advancing the loop along the lattice one step at a time. To accomplish this “one-step deformation,” as it was called in [4], one first removes any double links that may exist on \( \gamma \) and then moves each remaining link across the plaquette it borders in the forward direction. Clearly, this process

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will sometimes split $\gamma$ into several loops. To phrase it another way, once we have removed all double links from $\gamma$, we color all the plaquettes that the remaining links border in the forward direction. Each side of a colored plaquette that is not part of the entrance loop $\gamma$ becomes a link in one of the exit loops. Since we are interested in having just one exit loop, we designate one of the loops evolved from $\gamma$ as the exit loop, and close off the other loops with disks generated by the disk amplitude of the matrix model. A series of one-step deformations gives an exit loop which is a fixed geodesic distance from the entrance loop.

We now want to ask the following combinatorial question: given that the entrance loop $\gamma$ has $l$ links and the exit loop $\gamma'$ has $l'$ links, how many ways are there to construct a discretization between $\gamma$ and $\gamma'$, using a given number of squares, hexagons, octagons, etc., such that $\gamma'$ is the exit loop evolved from $\gamma$ in the course of a one-step deformation, as defined above? Or, in terms of the Feynman diagram which is the dual graph of the discretization, how many planar diagrams can be drawn on the surface of a cylinder with $l$ external legs pointing “down” and $l'$ external legs pointing “up”, with specified numbers of vertices of order four, six, eight, etc., and such that every “up” leg is connected to the same vertex as some “down” leg (but not necessarily vice versa)? To answer this question, we assign a multiplicative weight $g_m$ to each $2m$-gon (equivalently, to each vertex of order $2m$ on the dual lattice), assign as an overall weight to each of the discretizations between $\gamma$ and $\gamma'$ the product of the weights of the plaquettes used to build it, and let $N_{l,l'}$ be the sum of the weights of all such discretizations. $N_{l,l'}$ is the generating function that answers the above combinatorial question, but of course its more interesting property is that it is the partition function of tubes, or rather ribbons, of geodesic width one (in lattice units) and with entrance and exit loops of $l$ and $l'$ links, respectively.

We propose to evaluate

$$N(y, y') = \sum_{l,l' \geq 1} N_{l,l'} y^l (y')^{l'}$$

using combinatorics and the disk amplitude from the matrix models. For calculational convenience we will designate the entrance loop as unmarked and the exit loop as marked. On the dual lattice, this corresponds to considering the $l$ external legs of the planar diagram that point “down” to be distinguishable only up to cyclic permutations and the $l'$ external legs that point “up” to be completely distinguishable. A different convention of marking would change $N_{l,l'}$ only by a factor of $l$ or $l'$. The Green’s function $G(y)$ of the matrix field theory refers to diagrams in which the external legs are distinguishable—that is, $G(y)$ is the amplitude of a disk with a marked boundary. The weight $g_m$ assigned to a $2m$-gon by the Feynman rules for the matrix field theory is

$$g_m = -c_m - \frac{2m!^2}{(2m)!^2} \mu_m.$$  

Let us first consider the case where only squares are used in the discretizations. All the discretizations contributing to $N(y, y')$ may be built up as follows (here we are paralleling...
closely the work of \cite{[7]}, only using squares instead of triangles). The marked point on the exit loop must border one of the shapes in list a) of Fig. 2: the curved loops on the last three shapes denote the insertion of a disk. As we follow the exit loop around its length, we can encounter any of the shapes in list b) in any order: the entrance and exit loops are connected from the corner of one shape to the next, and in the end the last shape is connected back to the original member of list a), closing the loop.

It is perhaps not transparent why the weighting of each shape is what it is, so let us work through the example shown in Fig. 3. The loop above the square indicates the insertion of a disk amplitude, which must be marked because something must pick out the point where the disk boundary meets the square. The disk discretizations with less than two sides on the boundary are omitted because there have to be two boundary sides that match with the two free sides of the square. Let us think now in terms of the dual lattice. Given any Feynman graph contributing to the disk amplitude, we form a graph of the desired type by attaching to an adjacent pair of its external legs another vertex. The weight of the original disk amplitude graph must be multiplied by $g_2$ to get the weight of the new graph, because we have added one vertex and tied up two external legs while adding two new external legs. Summing over all allowable disk amplitude graphs (those with at least two external legs) then gives a total weight of $g_2(G(y) - G_0 - G_1 y)$, as claimed. Incidentally, the first, “undrawable” term $g_2 y^2$ in Fig. 3 corresponds to the two free sides of the square being identified. It might be helpful for the reader to identify which disk amplitude diagrams correspond to each term in Fig. 3.

Now it is not hard to write down the weight that would be assigned to a $2m$-gon bordered on $n$ sides by the exit loop. Starting with a disk amplitude graph with at least $2m - n - 2$ external legs, we add to it one $2m$-vertex, tying up $2m - n - 2$ of its external legs and adding $n + 2$ new external legs, 2 of which are entrance legs and $n$ of which are exit legs. Thus the total weight is

$$y^m y^{n+4-2m} g_m \left( G(y) - \sum_{l=0}^{2m-3} G_l y^l \right).$$

(25)

Note that because of the planarity of the surface, the sides of the $2m$-gon bordered by the exit loop must be contiguous. When $2m$-gons are allowed in the discretization, we include all the $2m$-gon shapes in list b), and also in list a) with a multiplicity determined by the number of exit links on the shape.

The combinatorical problem is now quite simple: we construct an arbitrary discretization from one member of list a) and some sequence of members of list b). The total amplitude when only squares are used is

$$N(y, y') = \left( 3y^3 y g_2 + 2y^2 y^2 g_2 G(y) + y' y g_2 (G(y) - G_0) \right) \times$$

$$\sum_{n=0}^{\infty} \left[ y^3 y g_2 + y^2 y^2 g_2 G(y) + y' y g_2 (G(y) - G_0) + g_2 (G(y) - G_0 - G_1 y) + y^2 G(y) \right]^n$$

8
\[
\begin{align*}
S &= 1 - y'^2 G(y) - \sum_{n=0}^{m-1} \left( y'^m y^{n+2m} G(y) - \sum_{l=0}^{2m-n-3} G_l y^l \right) \\
&= 1 - y y' + \frac{y}{y'} g(y') - y'^4 \sum_{m \geq 1} \frac{g_m}{y'^{2m}} \sum_{n=0}^{2m-2} (y'/y)^n \left( G(y) - \sum_{l=0}^{2m-n-3} G_l y^l \right),
\end{align*}
\] (29)

and Eq. (28) still holds.

The continuum limit of \( N(y, y') \) is taken by expanding about the radius of convergence of each of the variables. The convergence of \( S \) is determined by the convergence of \( G(y) \), which we explained in Section 2; \( y_e = 1/\sqrt{c} \), and we set \( y = y_e \exp(-\epsilon \zeta) \). For fixed \( y \leq y_e \), \( S \) is entire in \( y' \), but \( N(y, y') \) is analytic in \( y' \) only up to the magnitude of the zero of \( S \) nearest \( y' = 0 \). It turns out that for \( \zeta = 0 \) and \( \tau = 0 \), this zero is at \( y' = y_e^{-1} \). Hence we set \( y' = y_e^{-1} \exp(-\epsilon \zeta') \). An analytic redefinition of \( \zeta' \) of the form

\[
\zeta' \rightarrow \zeta' + r_1 \epsilon \tau + r_2 \epsilon \zeta^2 + r_3 \epsilon \zeta'^3 + r_4 \epsilon \zeta'^4 + \ldots
\] (30)

is allowed, for the same reasons as for the redefinition Eq. (19). Thus when we expand \( N(y, y') \) in \( \epsilon \), we need only retain the leading analytic and leading nonanalytic terms in \( \zeta' \). Writing \( N(\zeta, \zeta') \) in place of \( N(y_e \exp(-\epsilon \zeta), y_e^{-1} \exp(-\epsilon \zeta')) \), we find remarkably simple results:

\[
N(\zeta, \zeta') = \frac{1}{\epsilon} \left( \frac{1}{\zeta + \zeta' - \alpha' \epsilon^2 f(\zeta, \tau)} + O(\epsilon^{k-1}) \right)
\] (31)

where \( \alpha' \) is another numerical factor.
Let \( N_{l,l'}(d) \) be the lattice partition function of tubes of geodesic length \( d \) with an unmarked entrance loop of \( l \) links and a marked exit loop of \( l' \) links. The great insight of [7] is that this object has a simple composition law:

\[
N_{l,l'}(d_1 + d_2) = \sum_{l''=1}^{\infty} N_{l,l''}(d_1)N_{l'',l'}(d_2).
\]

(32)

In rough terms, we can cut a tube in two at some geodesic time and find its amplitude by summing the products of the amplitudes of the pieces over all possible lengths of the intermediate loop (see Fig. 4). The motivation for the convention of taking the entrance loop to be unmarked and the exit loop to be marked lies in the fact that there are \( l'' \) ways to glue the exit loop of one tube to the entrance loop of another—\( l'' \) being the number of links on each loop—but we want to avoid factors of \( l'' \) in the composition law. Suppose we mark the entrance loop of the rightmost tube in Fig. 4; such tubes would have amplitude \( l'' N_{l'',l'}(d_2) \). We now can glue the two tubes together in such a way that the two marks are at any of \( l'' \) positions relative to each other, so the amplitude for the resulting surface would be \( l'' (N_{l,l'}(d_1)l''N_{l',l''}(d_2)) \). But that surface would still have the two marks on the intermediate loop, and we must delete them to get a surface of the type shown on the left side of Fig. 4. Thus we divide our last expression by \( l''^2 \) to get

\[
N_{l,l'}(d_1)N_{l'',l'}(d_2) \text{ and sum over } l''.
\]

Since Eq. (32) is just a matrix product, it is clear that \( N_{l,l'}(d) \) is the time evolution kernel of some Hamiltonian. We will find the continuum limit of this Hamiltonian and then calculate \( N_{l,l'}(d) \) by solving the corresponding Schrödinger equation.

In terms of \( N(y, y', d) = \sum_{l,l' \geq 1} N_{l,l'}(d) y^l (y')^{l'} \), Eq. (32) takes the form

\[
N(y, y', d_1 + d_2) = \frac{1}{2\pi i} \oint dx \frac{dx}{x} N(y, x, d_1)N(1/x, y', d_2)
\]

(33)

where the integral is taken along a contour around the origin. Making the change of variables \( x = y_c^{-1} \exp(-\epsilon \xi) \), we find

\[
N(\zeta, \zeta', d_1 + d_2) = \frac{\epsilon}{2\pi i} \int_{i\infty}^{i\infty} d\xi N(\zeta, \xi, d_1)N(-\xi, \zeta', d_2) \rightarrow \frac{\epsilon}{2\pi i} \int_{-i\infty}^{i\infty} d\xi N(\zeta, \xi, d_1)N(-\xi, \zeta', d_2)
\]

(34)

for small \( \epsilon \). In particular, to the first nontrivial order in \( \epsilon \),

\[
N(\zeta, \zeta', d + 1) = \frac{\epsilon}{2\pi i} \int_{-i\infty}^{i\infty} d\xi N(\zeta, \xi)N(-\xi, \zeta', d)
\]

\[
= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\xi \frac{1}{\zeta + \xi - \alpha' \epsilon \sigma f(\zeta, \tau)} N(-\xi, \zeta, d)
\]

\[
= N(\zeta - \alpha' \epsilon \sigma f(\zeta, \tau), \zeta', d) = N(\zeta, \zeta', d) - \alpha' \epsilon \sigma f(\zeta, \tau) \frac{\partial}{\partial \zeta} N(\zeta, \zeta', d)
\]

10
where we have closed the contour to the left because the function $N(-\xi, \zeta, d)$ has a branch cut for positive real $\xi > \sqrt{\tau}$. Setting $D = \alpha' \epsilon \sigma d$ and taking $\epsilon \to 0$, we find that $N(\zeta, \zeta', D)$ is a solution to the equation

$$\frac{\partial}{\partial D} \psi(\zeta, D) = -f(\zeta, \tau) \frac{\partial}{\partial \zeta} \psi(\zeta, D)$$

with the initial condition read off from Eq. (31), $N(\zeta, \zeta', 0) = 1/(\zeta + \zeta')$. In [7], this equation was derived for the case $k = 2$, but using only triangles in the discretizations, rather than squares; thus for this case we have a universality check. What is interesting is that the same equation holds for higher $k$, with the disk amplitude $f(\zeta, \tau)$ adjusted appropriately, and with a new scaling behavior for macroscopic geodesic distance $D$. The definition $D = \alpha' \epsilon \sigma d$ is very interesting, and in a way unexpected, because it means that $D$ has the dimension of $L^\sigma$ instead of just $L$ (recall $\sigma = k - 3/2$).

In what follows it is convenient to introduce $N(\zeta, L', D)$, the inverse Laplace transform of $N(\zeta, \zeta', D)$ with respect to $\zeta'$. Since Eq. (33) involves only $\zeta$ and $D$, $N(\zeta, L', D)$ is also a solution of this differential equation, only with different initial conditions, namely $N(\zeta, L', 0) = \exp(-\zeta L')$. Our next step will be to find an approximate method for solving the differential equation and to use the solutions, as was done in [7], to determine universal functions describing the way random surfaces branch. These functions directly depend on $N(L, L', D)$, the inverse Laplace transform of $N(\zeta, L', D)$ with respect to $\zeta$. $N(L, L', D)$ is the continuum amplitude for tubes of length $D$ with boundaries of length $L$ and $L'$, and it satisfies the initial condition $N(L, L', 0) = \delta(L - L')$.

4 Calculating the tube amplitude

A solution to Eq. (35) may be found by the method of characteristic curves: if the function $\zeta_0(\zeta, D)$ solves the ODE

$$\frac{d}{dD} \zeta_0(\zeta, D) = -f(\zeta_0(\zeta, D), \tau) , \quad \zeta_0(\zeta, 0) = \zeta$$

then $\psi(\zeta, D) = \psi(\zeta_0(\zeta, D), 0)$ is a solution to Eq. (35). Geometrically speaking, $\psi(\zeta, D)$ is constant along a family of (non-intersecting) characteristic curves in $\zeta$-$D$ space; $\zeta_0(\zeta, D)$ is the $\zeta$-coordinate of the point where the characteristic curve passing through $(\zeta, D)$ meets the line $D = 0$. From Eq. (36) it follows that $\zeta_0(\zeta, D)$ is implicitly determined by

$$\int_0^\zeta \frac{d\xi}{f(\xi, \tau)} = D .$$

Now $N(\zeta, L', D)$, the solution of Eq. (35) with initial conditions $N(L, L', 0) = \exp(-\zeta L')$, is simply given by

$^{1}$In order to work with finite quantities in the continuum limit, we rescale $N(\zeta, \zeta', D)$ to absorb the factor $1/\epsilon$ present in the lattice definition.
\[ N(\zeta, L', D) = \exp (-\zeta_0(\zeta, D)L') \ . \] (38)

The inverse Laplace transform of this function gives the desired tube amplitude \( N(L, L', D) \). Thus, the problem is reduced to the explicit determination of \( \zeta_0(\zeta, D) \).

For \( k = 2 \), \( \zeta_0(\zeta, D) \) can be expressed in terms of elementary functions. Indeed, after evaluating the integral in Eq. (37) with \( f(\zeta, \tau) = (2\zeta - \sqrt{\tau}) \sqrt{\zeta + \sqrt{\tau}} \), and some algebra, we arrive at

\[ \zeta_0 = -\sqrt{\tau} + \frac{3}{2} \sqrt{\tau} \coth^2 \left( \sqrt{\frac{3}{2} \tau^{1/4}}D - \frac{1}{2} \log \frac{\sqrt{\zeta + \sqrt{\tau} - \sqrt{\frac{3}{2} \tau^{1/4}}} \sqrt{\zeta + \sqrt{\tau} + \sqrt{\frac{3}{2} \tau^{1/4}}} \right) \] (39)

Unfortunately, it seems impossible to find the exact inverse Laplace transform of Eq. (38). Instead, we will look for the behavior of \( N(L, L', D) \) in the limit of a point-like entrance loop \((L \to 0)\), as well as in the limit of a very large entrance loop \((L \to \infty)\). In the first case, it is sufficient to study the dominant behavior of Eq. (39) for large \( \zeta \), and in the second case, for small \( \zeta \).

Let us first consider the limit of a point-like entrance loop. Substituting Eq. (39) into Eq. (38) and expanding for large \( \zeta \), we find

\[ N(\zeta, L', D) = e^{L' \sqrt{\tau}} e^{-\frac{3}{2} L' \sqrt{\tau} \coth^2 \left( \sqrt{\frac{3}{2} \tau^{1/4}}D \right)} + \frac{1}{\sqrt{\zeta}} e^{L' \sqrt{\tau}} \frac{\partial}{\partial D} e^{-\frac{3}{2} L' \sqrt{\tau} \coth^2 \left( \sqrt{\frac{3}{2} \tau^{1/4}}D \right)} + O(1/\zeta) \] (40)

Performing the inverse Laplace transform, we find

\[ N(L \to 0, L', D) = \frac{1}{\sqrt{\pi L}} e^{L' \sqrt{\tau}} \frac{\partial}{\partial D} e^{-\frac{3}{2} L' \sqrt{\tau} \coth^2 \left( \sqrt{\frac{3}{2} \tau^{1/4}}D \right)} + O(L^0) \] (41)

This formula is valid for \( L \ll D^2, \tau^{-1/2} \). Nothing is assumed, however, about the relative magnitudes of \( D^2 \) and \( \tau^{-1/2} \). In [7] the calculation was performed in the regime \( L \ll D^2 \ll \tau^{-1/2} \) so that the total area of the surface could be sent to infinity while \( D \) and \( L' \) were kept finite. Our Eq. (41) generalizes the result of [7] and reduces to it when expanded for small \( \tau \). For \( k > 3 \), however, we cannot find an explicit form of \( \zeta_0(\zeta, D) \) analogous to Eq. (39). For that reason we will work in the limit \( 1/\zeta \ll D^{1/\sigma} \ll \tau^{-1/2} \) (recall \( \sigma = k - 3/2 \)), and expand \( \zeta_0 \) in powers of \( 1/\zeta \) and \( \tau \).

Since \( f_k(\xi, \tau) = (-1)^k \frac{\xi^{\sigma + 1}}{\sigma} (1 + O(\tau/\xi^2)) \), as our first approximation we have from Eq. (37),

\[ \zeta_0(\zeta, D) = \left( (-1)^k \ z^{-\sigma} \right)^{-1/\sigma} + O(\tau) \ . \] (42)

From this formula we can see a sickness in the case of odd \( k \) as \( D \to \zeta^{-\sigma} \) from below, \( \zeta_0(\zeta, D) \to \infty \). What is happening is that the solution to Eq. (38) with \( \tau = 0 \) is running off to \( \infty \) in finite time. Eq. (42) becomes complex for \( D > \zeta^{-\sigma} \), but the real solution to Eq. (38) just stays at \( \infty \). Letting \( \tau \) become finite does not help the situation except when the initial
\( \zeta \) is less than the rightmost zero of \( f(\zeta, \tau) \), which is at \( \zeta \sim \sqrt{\tau} \). Since \( N(\zeta, L', D) \) vanishes for large enough real \( \zeta \), the inverse Laplace transform \( N(L, L', D) \) cannot be positive definite for small \( L \). This seems to lead to meaningless results, and we will not consider the limit of a point-like entrance loop in the odd \( k \) models. It may be that if we worked with fixed \( L \) from the beginning, instead of with Laplace-transformed equations like Eq. (35), we could avoid the sickness we have observed.

The even \( k \) models avoid this sickness, as is clear from Fig. 5 where we show the qualitative behavior of \( \zeta_0(\zeta, D) \) for \( k = 3 \) and \( k = 4 \) and finite \( \tau \). To determine the baby universe distribution function in the case of even \( k \), we need to continue the expansion of Eq. (42) in powers of \( \tau \) and find the coefficient of the leading fractional power. First we expand

\[
\frac{1}{f_k(\xi, \tau)} = \frac{k-3/2}{\xi^{k-1/2}} + \sum_{n=1}^{k-1} a_n \tau^n \frac{k+2n-3/2}{\xi^{k+2n-1/2}} + b \tau^{k-1/2} \frac{3k-5/2}{\xi^{3k-3/2}} + O(\tau^k),
\]

where \( a_n \) and \( b \) are numerical coefficients which can be found from Eq. (18). Substituting Eq. (43) into Eq. (37) and integrating term-by-term, we obtain the following relation:

\[
\zeta_0(\zeta, D) = (D + \zeta^{-\sigma})^{-1/\sigma} + \tau \frac{a_1}{\sigma} (D + \zeta^{-\sigma})^{-1/\sigma} - \zeta^{-\sigma} - (D + \zeta^{-\sigma})^{-1/\sigma} + \ldots + O(\tau^{k-1})
\]

\[
+ \tau^{k-1/2} \frac{b}{\sigma} (D + \zeta^{-\sigma})^{-1/\sigma} - \zeta^{-3\sigma/2} - (D + \zeta^{-\sigma})^{-1/\sigma} + O(\tau^k).
\]

Now the expansion of \( \zeta_0 \) in powers of \( \tau \) can be found iteratively. The coefficient of the leading fractional power, \( \tau^{k-1/2} \), is actually obtained after one iteration. Thus, the desired expansion has the form

\[
\zeta_0(\zeta, D) = (D + \zeta^{-\sigma})^{-1/\sigma} + \tau \frac{a_1}{\sigma} [ (D + \zeta^{-\sigma})^{-1/\sigma} - \zeta^{-\sigma} - (D + \zeta^{-\sigma})^{-1/\sigma} ] + \ldots + O(\tau^{k-1})
\]

\[
+ \tau^{k-1/2} \frac{b}{\sigma} [ (D + \zeta^{-\sigma})^{-2+1/\sigma} - \zeta^{-3\sigma/2} - (D + \zeta^{-\sigma})^{-1/\sigma} ] + O(\tau^k).
\]

Substituting this into Eq. (38), we have

\[
N(\zeta, L', D) = e^{-L'(D+\zeta^{-\sigma})^{-1/\sigma}} \left( 1 - \tau L' \frac{a_1}{\sigma} [(D + \zeta^{-\sigma})^{-1/\sigma} - \zeta^{-\sigma} - (D + \zeta^{-\sigma})^{-1/\sigma}] + \ldots + O(\tau^{k-1}) \right)
\]

\[
+ \tau^{k-1/2} \frac{b}{\sigma} [(D + \zeta^{-\sigma})^{-2+1/\sigma} - \zeta^{-3\sigma/2} - (D + \zeta^{-\sigma})^{-1/\sigma}] + O(\tau^k) \right) .
\]

Now we expand this for large \( \zeta \) and inverse Laplace transform term-by-term, discarding the \( O(\zeta^0) \) pieces, which give zero length terms. We find

\[
N(L \rightarrow 0, L', D) = \frac{L^{\sigma-1}}{\Gamma(\sigma)} e^{-x} \left( \frac{L'}{\sigma D^{1+1/\sigma}} + O(\tau) + \ldots + O(\tau^{k-1}) + \gamma_1 DL'((2k-2)D^{1/\sigma} + L')^{k-1/2} + O(\tau^k) \right) + O(L^{2\sigma-1})
\]

where \( x = L'/D^{1/\sigma} \) and \( \gamma_1 = -b/\sigma^2 \).
5 The branching structure of random surfaces

We now wish to extract from $N(L, L', D)$ a universal function which describes the branching structure of a large surface of planar topology and one boundary loop of length $L_0$. We will find that such surfaces have lots of little protuberances—hair, if you will. Let $R(D)$ be the part of the surface which is at a geodesic distance $D$ or less from the boundary loop. Let $\rho(L_0, L, D)$ be the distribution of boundary loop lengths for $R(D)$: on average, $R(D)$ has $\rho(L_0, L, D)dL$ boundary loops with length in the interval $(L, L + dL)$. This quantity was introduced and calculated for $k = 2$ and $L_0 \to 0$ in [7]. The strategy for calculating it is as follows. The partition function for all surfaces is the disk amplitude $F(L_0)/L_0$. Let us think for a moment of the discrete version $\rho(l_0, l, d)$ of $\rho(L_0, L, D)$. This function is the statistical average over all disks with boundary length $l_0$ of the number of loops with $l$ links at geodesic distance $d$ from the boundary. According to [7], in the limit of large disk area

$$\rho(l_0, l, d) = \lim_{\tau \to 0} \frac{\left(\frac{\partial}{\partial \tau}\right)^n [N_{l_0,t}(d)G_{l/l}]}{\left(\frac{\partial}{\partial \tau}\right)^n [G_{l_0}/l_0]} \quad (48)$$

$N_{l_0,t}(d)G_{l/l}$ generates only surfaces with at least one boundary loop of the desired sort. Moreover, if there are $p$ such boundary loops on a particular surface, $N_{l_0,t}(d)G_{l/l}$ will generate that surface $p$ times: each time, the tube generated by $N_{l_0,t}(d)$ will have a different one of its boundary loops left open for $G_{l/l}$ to plug. Hence $N_{l_0,t}(d)G_{l/l}$ is the sum over surfaces generated by $G_{l_0}/l_0$ of the weight of each surface times the number of boundary loops at distance $d$ and of length $l$ on that surface. To understand the presence of the puncture operators $\partial/\partial \tau$, think for a moment of surfaces with fixed area instead of fixed $\tau$. To get to the fixed area representation, we would separately carry out inverse Laplace transforms on the numerator and denominator. The large area behavior of the fixed area quantities would be controlled by the leading singular term in the small $\tau$ expansion of the fixed $\tau$ quantities. Inserting enough puncture operators $\partial/\partial \tau$ in the numerator and denominator of Eq. (48) to eliminate the leading analytic terms in $\tau$ thus provides a convenient way to isolate the terms that survive in the large area limit. In order to obtain a generalization of Eq. (48) to disks of finite area, we would have to replace the numerator and denominator by their inverse Laplace transforms.

The continuum limit of Eq. (48) is obviously

$$\rho(L_0, L, D) = \lim_{\tau \to 0} \frac{\left(\frac{\partial}{\partial \tau}\right)^n [N(L_0, L, D)F(L)/L]}{\left(\frac{\partial}{\partial \tau}\right)^n [F(L_0)/L_0]} \quad (49)$$

Let us first consider the limit $L_0 \to 0$ where the entrance loop is shrunk to a point.

We expand
\[
F_k(L, \tau)/L = \frac{1}{L^{k+3/2}} + O(\tau) + \ldots + O(\tau^{k-1}) + \gamma_2 L^{k-5/2} \tau^{k-1/2} + O(\tau^k),
\]

where \(\gamma_2\) is another numerical factor and we have adjusted \(F_k(L, \tau)/L\) by an overall multiplicative factor for convenience. In this expansion, as in Eq. (47), only the \(O(\tau^0)\) term and the leading nonanalytic term are relevant to \(\rho(L_0, L, D)\). This is because both the numerator \(N(L_0, L, D)F(L)/L\) and the denominator \(F(L_0)/L_0\) of Eq. (49) have \(k\) leading terms analytic in \(\tau\), followed by an \(O(\tau^{k-1/2})\) term. The leading terms are deleted by making \(k\) punctures.

In the \(\tau \to 0\), \(L_0 \to 0\) limit, the only finite physical quantities in \(\rho(L_0, L, D)\) are \(L\) and \(D\), so the only possible dimensionless scaling parameter is \(x = L/D^{1/\sigma}\). Thus \(\rho(L_0 \to 0, L, D)\) is a function only of \(x\), up to a dimensionful overall factor:

\[
\rho(L_0 \to 0, L, D) = \frac{1}{D^{1/\sigma}} \left[ \frac{\gamma_1}{\gamma_2 \Gamma(\sigma)} x^{-\sigma-2} (2\sigma + 1 + x) + \frac{x^\sigma}{\Gamma(\sigma + 1)} \right] e^{-x},
\]

where \(\sigma = k - 3/2\), and the numbers \(\gamma_1\) and \(\gamma_2\) can be calculated explicitly:

\[
\gamma_1 = \frac{2^{3-2k}}{(6k - 5)(2k - 3)}, \quad \gamma_2 = \frac{1}{(2k - 1)!!(2k - 3)!!}.
\]

The terms in square brackets in Eq. (51) separate beautifully as \(k \to \infty\): the first term gives a non-integrable divergence at \(x = 0\), and the second term gives a Poisson distribution, normalized to one and peaked at \(x = \sigma\).

The first term shows that the random surface has huge numbers of protuberances whose circumferential length is small compared to their geodesic distance from a given point; this is what we mean by the surface being hairy. The profusion of “microscopic” boundary loops suggests that as a one-dimensional universe propagates through geodesic time, it emits a divergent number of baby universes.

For large \(x\), the second term in square brackets is dominant, and it shows that there is exactly one “macroscopic” boundary loop to our region, and for large \(k\) its length is sharply peaked about \(\sigma D^{1/\sigma}\). Thinking again of a one-dimensional universe propagating through geodesic time, we would interpret the one macroscopic boundary loop as the parent universe, which survives the emission of its numerous baby universes.

To determine at what \(x\) the first term becomes significant, we note that

\[
\frac{1}{\sigma} \log \left[ \frac{\gamma_1}{\gamma_2 \Gamma(\sigma)} x^{-\sigma-2} (2\sigma + 1 + x)e^{-x} \right] \to -(1 + s + \log s)
\]

as \(\sigma = k - 3/2 \to \infty\) with \(s \equiv x/\sigma\). This is positive for \(s < s_c\) and negative for \(s > s_c\), where \(s_c\) satisfies
the solution to which is \( s_c \approx 0.2784645428 \). Let us define \( \rho(s) \) by

\[
\rho(s) \, ds = \rho(L_0 \to 0, L, D) \, dL = \left[ \frac{\gamma_1}{\gamma_2 \Gamma(\sigma)} x^{-\sigma-2} (2\sigma + 1 + x) + \frac{x^\sigma}{\Gamma(\sigma+1)} \right] e^{-x} \, dx .
\]

It follows from the above discussion that, as \( k \to \infty \), \( \rho(s) \) converges in the weak sense to a distribution which is \(+\infty\) for \( s < s_c \) and \( \delta(s-1) \) for \( s > s_c \). In Fig. 6 we show plots of \( \rho(s) \) for \( k = 2 \), where there is no separation of the “microscopic” and “macroscopic” terms; for \( k = 6 \), where the separation is significant; and for \( k = 100 \), where the convergence to the limiting case is very clear.

Large \( k \) corresponds to the central charge going to \(-\infty\), which is held to be the semiclassical limit where quantum fluctuations vanish and the surface is smooth, with constant scalar curvature. In the thermodynamic (infinite area) limit, the surface would be a plane, and we would have \( \rho(L_0 \to 0, L, R) = \delta(L - 2\pi R) \), so \( \rho(s) = \delta(s-1) \) with \( s = L/(2\pi R) \). It is intriguing that this classical term is present in semiclassical limit we found for \( \rho(s) \). The different scaling law,

\[
s = \frac{L}{\sigma D^{1/\sigma}} ,
\]

and the profusion of microscopic boundary loops for \( s < s_c \), are striking features of the quantum case which we cannot conceive of predicting by quasi-classical arguments. The scaling law Eq. (56) might seem to be an artifact of the combinatorics: on the discrete surface, the matter fields are incorporated into the manifold by polygons with different numbers of sides, so the notion of defining geodesic distance as the minimal number of polygons one must traverse to get from point to point is suspect. So, couldn’t we just define \( R = \sigma D^{1/\sigma}/2\pi \) and claim that \( R \) is the “real” geodesic distance? The problem with this approach is that \( D \) enjoys a linearity property that seems essential to the notion of geodesic distance. Namely, if \( \gamma \) is a loop each of whose points is a geodesic distance \( D_1 \) from a given point \( P \) on a random surface, and if \( \gamma' \) is a loop each of whose points is a geodesic distance \( D_2 \) from \( \gamma \), then each point on \( \gamma' \) will be a geodesic distance \( D_1 + D_2 \) from \( P \). Any increasing function of \( D \) with the same property would have to be linear. The authors feel that a resolution of this question will have to come from a continuum formalism where the matter fields are more easily distinguished from the metric on the manifold.

Another interesting limit where exact calculations are possible is that of an extremely long entrance loop, \( L_0 \to \infty \). Here it is appropriate to calculate \( N(\zeta, L', D) \) in the limit where \( \zeta^{-1} \) and \( \tau^{-1/2} \) are much larger than \( D^{1/\sigma} \) and \( L' \). Now the characteristic curve equation, Eq. (57), has a very simple approximate solution,

\[
\zeta_0 \approx \zeta - D f_k(\zeta, \tau) ,
\]
so that, from Eq. (38),

\[ N(\zeta, L', D) = 1 - \zeta L' + L'Df_k(\zeta, \tau) + \ldots \]  

Performing the inverse Laplace transform, we obtain

\[ N(L \to \infty, L', D) = L'DF_k(L, \tau) . \]  

Using this and Eq. (54) in Eq. (59), we arrive at a remarkably simple formula:

\[ \rho(L_0 \to \infty, L, D) = L_0DL^{-k-1/2} . \]  

This distribution is valid whenever \( \tau^{-1/2} \gg L_0 \) and both are much larger than \( L \) and \( D^{1/\sigma} \).

The fact that the number of emitted baby universes scales as \( L_0 \) could have been anticipated: this is simply due to the fact that a baby universe can split off anywhere along the parent.

It is also clear why the number grows linearly with the elapsed geodesic time \( D \), since in the discrete case the number of baby universes that split off at each step of evolution should be constant as long as the length of the parent universe does not change appreciably. Note that the distribution in \( L \) is again non-integrable for small \( L \): the emitted baby universes are overwhelmingly likely to be microscopic.

Another consequence of the preceding discussion is that

\[ N(L, L' \to 0, D \to 0) = L'DF_k(L, \tau) . \]  

This formula is interesting because it establishes a connection between the tube amplitude and the disk amplitude. \( N(L, L' \to 0, D \to 0) \) is the sum over disks of boundary length \( L \) with a marked point located at a vanishing geodesic distance \( D \to 0 \) from the boundary.

Thus, we expect that in the \( L' \to 0, D \to 0 \) limit the tube amplitude reduces to the disk amplitude with a marked boundary point. This is indeed what happens, according to Eq. (61). We speculate that Eq. (61) or some more general form of it could be used as the basis of a continuum derivation of Eq. (35).

6 Discussion

In ref. [11] a string field theory formalism for \( c = 0 \) gravity was introduced. From this formalism Eq. (35) was elegantly derived. In fact, Eq. (35) was derived first in ref. [7] via a careful combinatorial analysis of discretizations, and the string field theory was tailored to reproduce this result. In this paper we extended the combinatorial analysis to arbitrary discretizations with \( 2m \)-gons and established the validity of Eq. (35) for 2-dimensional gravity coupled to the \((2, 2k - 1)\) minimal models. This strongly suggests that the string field theory formalism of ref. [11] encompasses all these theories. We choose a particular theory only through its disk amplitude \( f(\zeta, \tau) \), which is the background value of the string field.
In solving Eq. (35) we found a drastic difference between the even and odd \( k \). In other calculations no such major differences were noted. For instance, the disk amplitudes of Eq. (4) are positive for all \( k \). Thus, we may have the first indication of a serious difference between even and odd \( k \) occurring for spherical surfaces. It would be nice to understand a deeper reason behind this.

The non-integrable divergence of \( \rho(L, D) \) for small \( L \) indicates that random surfaces are very hairy, even when restricted to spherical topology. In terms of the propagation through geodesic distance of a loop along its world-sheet, we take this to mean that tiny loops—the baby universes—are constantly splitting off the main loop. The conclusion that the microscopic baby universes are overwhelmingly more likely to split off than the macroscopic ones is quite intriguing in light of the large wormhole problem \([4]\).

One might think of the microscopic baby universes as analogous to the soft photons which create the well-known infrared problem in the bremsstrahlung cross-section. Therefore, only the inclusive probabilities, where we sum over all possible splittings, are non-vanishing. We have verified that the probability for a loop to propagate any finite \( D \) along its world sheet without splitting is zero.

The limit \( k \to \infty \) (corresponding to the central charge decreasing without bound) bears a subtle relationship to classical gravity which we do not fully understand. Suggestions of a manifold that is smooth at length scales large compared to \( D^{1/\sigma} \) emerge from the \( k \to \infty \) limit of \( \rho(L_0 \to 0, L, D) \). But baby universes still play an important role in this limit, as evidenced by the divergence of \( \rho(L_0 \to 0, L, D) \) as \( k \to \infty \) for \( L < s_c \sigma D^{1/\sigma} \). Perhaps the proliferation of baby universes up to this critical scale is related to the presence of many operators of negative dimension.

It would be very interesting to study models with unitary conformal fields coupled to gravity, to see whether the surfaces they produce are more regular or more wild. To calculate the transfer matrix with the same methods as described here, however, one would need the disk amplitudes of the model with arbitrary boundary conditions on the matter fields, and these are not available even for such simple models as the Ising model.

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**Figures**

1. Small example of a one-step evolution of a loop on a random surface tiled with squares.
2. Basic shapes for $k = 2$.
3. The first few terms contributing to one basic shape, drawn on both the polygonal and dual lattices.
4. The composition law.
5. Characteristic curves for $k = 3$ (top left) and $k = 4$ (top right) above graphs of $f_3(\zeta, \tau)$ (bottom left) and $f_4(\zeta, \tau)$ (bottom right).
6. The scaling function $\rho(s)$ for $k = 2$, $k = 6$, and $k = 100$. 
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