On the High $T$ Phase Transition in the Gauged $SU(2)$ Higgs Model

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Abstract

We study the effective field theory of a weakly coupled 3+1d gauged $\phi^4$ type model at high temperature. Our model has $4N$ real scalars ($N$ complex Higgs doublets) and a gauge group $SU(2)$ which is spontaneously broken by a nonzero scalar field $vev$ at zero temperature. We find, for sufficiently large $N$, that the transition from the high temperature symmetry restoring phase to the low temperature phase can be either first order or second order depending on the ratio of the gauge coupling to the scalar self coupling.
Recently there has been much interest in the nature of the high temperature electroweak phase transition. It has been known for some time that at high enough temperature the ground state of the electroweak model is symmetry preserving [1], even if the zero temperature ground state is not. What is not well understood is how the transition from the high temperature phase to the low temperature phase proceeds. If the phase transition is first order, it may be possible to generate the baryon asymmetry of the universe (BAU) within the electroweak model itself [2],[3],[4].

It is currently believed that the phase transition is first order, but too weakly first order for the purposes of generating the BAU [6],[7]. A one–loop high temperature calculation suggests that the electroweak phase transition is first order [8]. However, the high temperature field theory suffers from infrared divergence problems. A partial resummation of higher loop contributions implies that the strength of the first order transition is reduced relative to the one–loop result [9],[6],[7],[10].

In this letter we would like to point out that higher loop contributions may not only reduce the strength of the first order transition but actually result in a second order phase transition. We study not the full electroweak theory but a weakly coupled gauged $\phi^4$ model. The gauge group is $SU(2)$ and we work with $4N$ real scalars ($N$ complex Higgs doublets). Each of the $SU(2)$ Higgs doublets is coupled to the same gauge fields.

Our analysis follows [11]. The 3+1d model is, at high temperature, formally equivalent to a euclidean field theory with one compact dimension. For physics at scales less than $O(T)$ it is sufficient to study the effective 3d Lagrangian that results from integrating out the compact dimension. Vacuum polarization effects can be computed exactly in three dimensions, in contrast to four dimensions. Furthermore, the explicit $\phi^4$ term can be removed by introducing a dimension two auxiliary field $\chi$ [12],[13],[14]. The auxiliary field allows for a straightforward computation of all the quantum corrections that survive in the limit of arbitrarily large $N$. The corrections that survive depend on how the gauge coupling and scalar self–coupling are held fixed as $N$ increases.

We find that if $g^2/\lambda \leq O(1)$, where $g$ is the 4d gauge coupling and $\lambda$ is the scalar self–coupling, the effective scalar potential $V_{eff}$ admits only a 2nd order phase transition at large $N$. This is because vacuum polarization effects at nonzero external momentum prevent any $T\phi^3$ term from appearing in $V_{eff}$. This is in contrast to the case $g^2/\lambda \sim O(N)$ for which a first order
phase transition is still possible.

These conclusions are the same as for the high temperature abelian Higgs model studied by one of the current authors in [11]. The results are the same because at large $N$ and weak coupling the nonabelian nature of the $SU(2)$ group is not important and the three $SU(2)$ gauge fields interact with the scalars like three abelian gauge fields.

In the abelian case, it was shown that the next-to-leading corrections compete with the leading corrections in the $1/N$ expansion when $N = 1$ and $g^2/\lambda \sim 1$. Therefore, one may question the applicability of the our results here to the physically interesting situation of one Higgs doublet, especially since the number of $SU(2)$ gauge fields is just one less than the number of scalars in this case. However, the leading order and next-to-leading order result for $V_{eff}$ just corresponds to resuming some infinite classes of Feynman diagrams in a way that avoids overcounting any of them. For example, for $g^2/\lambda \leq O(1)$, the $O(1)$ result for $V_{eff}$ that we derive below includes the diagrams of Figure 1. The fact that in a systematic approach the phase transition is second order at $O(1)$ in $V_{eff}$ when $g^2/\lambda \leq O(1)$ means that any first order behaviour comes from at best $O(1/N)$ terms in $V_{eff}$. Therefore, all $O(1/N)$ terms must be computed in a systematic way in order to make a reliable statement about the phase transition in the $N = 1$ case. Such a computation does not currently exist in the literature, even for the abelian Higgs model. We are currently trying to develop a different $1/N$ expansion that we feel will be more suited to the electroweak model. Namely, to work with $N$ real scalars and the four $SU(2) \times U(1)$ gauge fields in $N$ dimensions. For $N$ bigger than four, such a model is nonrenormalizable but can be treated as an effective low energy theory.

What we present below is a straightforward generalization of the results of [11]. For technical details, the reader is referred to [11]. Related works, in the case of a pure scalar theory, are references [13], [15], [16] and [19]. The 3d effective field theory of high temperature gauged $\phi^4$ models has been previously studied using the $\epsilon$-expansion technique in ref. [17], and more recently in [18]. Recently, the abelian Higgs model has also been studied in [20] using a different method than the one considered in [11]. The authors of [20] establish a range in parameter space $(\lambda, g)$ where the phase transition is first order, although no definite prediction of a second order transition is made for parameter values outside this range.
The tree level Lagrangian we consider here is

\[ L[\phi, A] := N D_\mu \phi_A^\dagger D^\mu \phi_A - \frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} - V(\phi), \] (1)

where the tree potential is

\[ V(\phi) := \frac{N \lambda}{4} \left( \phi_A^\dagger \phi_A - v^2 \right)^2, \] (2)

and the kinetic terms can be expanded using

\[ D_\mu \phi_A := \left( \partial_\mu + \frac{ig}{\sqrt{N}} A_\mu \right) \phi_A \] (3)

and

\[ F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{ig}{\sqrt{N}} [A_\mu, A_\nu]. \] (4)

Our conventions are as follows. This is the 3+1d Minkowski space Lagrangian with metric \( \eta_{\mu\nu} = \text{diag}(+,-,-,-) \). The indices \( A, B, \text{etc.} \), run from 1 to \( N \). Each field \( \phi^A \) is a complex Higgs doublet. The gauge and scalar couplings are \( g \) and \( \lambda \), respectively. The three \( SU(2) \) gauge fields \( A^a_\mu, a = 1, 2, 3, \) are defined via \( A_\mu := A^a_\mu T^a \), where \( T^a \) are half the pauli matrices. The purely scalar part of the Lagrangian is normalized to have an overall factor of \( N \) only for later convenience; the kinetic term can be put in a canonical form by the rescaling \( \phi \rightarrow \phi/\sqrt{N} \). Finally, we remark that in order to avoid any problems associated with triviality [21] we treat our model as an effective low energy theory valid below some scale \( \tilde{\Lambda} \).

To study the high temperature phase transition of the model (1) we follow the procedure outlined in [11]. We first write the corresponding euclidean version of (1). The euclidean coordinates are \( (\tau, \vec{x}) \), where at high temperature \( \tau \) describes the compact dimension. All fields are periodic in \( \tau \) with period \( \beta = 1/T \). At sufficiently high \( T \) the model is effectively three dimensional, a fact that we exploit below.

The four dimensional fields can be expanded as

\[ \phi^A(\vec{x}, \tau) = \sum_n \phi_n^A(\vec{x}) \psi^n(\tau), \quad A^\mu(\vec{x}, \tau) = \sum_n A_\mu^n(\vec{x}) \psi^n(\tau) \] (5)

where \( \psi^n \) are a complete set of periodic functions on the circle, \( \psi^n(0) = \psi^n(\beta) \). Thus, each 4d field yields an infinite tower of 3d fields when the
compact dimension is integrated out in the action. For what we are interested
in only the zero modes \( n = 0 \), for which \( \psi^0(\tau) = \psi^0(0) \), are important in the
effective 3d model. This is because for \( n \neq 0 \) the fields \( \phi^A_n, A^\mu_n \) pick up large
nonzero masses of \( O(T) \) from the 4d kinetic term. Truncating the spectrum
to keep only the zero modes and integrating out the compact dimension gives
the effective 3d Lagrangian

\[
L[\varphi, A, \rho] := -N \varphi^\dagger \partial^2 \varphi + \frac{N \lambda}{4} (\varphi^\dagger \varphi - \tilde{v}^2)^2 \\
+ \text{tr} \left( (D_m \rho)^2 + \frac{1}{2} F_{mn} F^{mn} \right) + \tilde{g}^2 \varphi^\dagger (A_m A^m + \rho^2) \varphi \\
+ i \tilde{g} \sqrt{N} \left( \partial_m \varphi^\dagger A^m \varphi - \varphi^\dagger A^m \partial_m \varphi \right)
\]

(6)

where we have defined the three dimensional fields and coupling constants

\[
\varphi^A := \sqrt{\beta} \phi^A_0, \quad \lambda := \lambda / \beta, \quad \tilde{v}^2 := \beta v^2 \\
A^m := \sqrt{\beta} A^m_0, \quad \rho := \sqrt{\beta} \rho_0, \quad \tilde{g} := g / \sqrt{\beta}
\]

(7)
in terms of the zero modes of the four dimensional euclidean fields and the
four dimensional coupling constants. We have suppressed indices labeling the
number of Higgs doublets so that, for example, \( \varphi^\dagger \varphi = \varphi^\dagger_A \varphi^A \). The indices
\( m, n \) are used for the euclidean spatial coordinates. The 3d field strength is
\( F_{mn} = \partial_m A_n - \partial_n A_m + ig[A_m, A_n] / \sqrt{N} \). The 3d gauge covariant derivative
on the scalars \( \rho \) is \( D_m \rho = \partial_m \rho + ig [A_m, \rho] / \sqrt{N} \). Finally, \( A_m = A^m, \partial_m = \partial^m \).

Therefore, the three dimensional model we study is a gauged Higgs model
with an extra triplet of real scalars \( \rho := \rho^a T_a \) in the adjoint representation
of \( SU(2) \). These extra scalars are just the zero modes of the 4d longitudinal
gauge bosons, while the 3d gauge bosons are just the zero modes of the 4d
transverse gauge bosons.

Before proceeding we would like to stress some important points. As
shown in [17], if one integrates out the nonzero modes \( n \neq 0 \) at the quantum
level rather than just truncating the spectrum, there is a finite \( O(T) \) cor-
rection to the mass of \( \varphi \). This correction is very important, indeed it is the
term that gives symmetry restoration at high enough temperature. We will
obtain this term another way, in analogy with what one does in effective low
energy theories of the strongly interacting standard model or effective four
dimensional supergravity models inspired by string theory [22].
The three dimensional field theory will be divergent and it must be regulated. In fact, the only divergent integrals that will be important are linearly divergent and we will regulate them by simply using sharp momentum cutoffs. We then give a physical interpretation to the cutoffs, i.e. the scale at which the full four dimensional physics becomes important. This scale is of $O(T)$. This interpretation is in complete analogy with, for example, the effective 4d theories where the regulating scale is taken to be of order the compactification scale. In our case the identification of the cutoffs with $T$ can be made precise because the corresponding 4d corrections are well known.

In [11] it was shown for the pure scalar case that if one regulates all divergent integrals with the same cutoff $\Lambda$ then the identification $\Lambda = \pi^2 T/6$ reproduces exactly the $T^2$ results from the four dimensional integrals. In general, for a model with different types of particles and different couplings, one should use different cutoffs for the different divergent terms that appear in $V_{\text{eff}}$. This is because the 3d effective theory does not know the precise way in which the full four dimensional physics enters at scales of $O(T)$. In [11] it was also shown that, in order to obtain the correct temperature dependent electric mass for the 4d abelian gauge field, a different cutoff $\Lambda' = \pi^2 T/3$ must be used to regulate the linearly divergent vacuum polarization of the zero mode of the 4d longitudinal gauge field. All of this will be seen more explicitly below.

We remark that since the four dimensional model (1) is only valid up to a scale $\bar{\Lambda}$ we must require $\Lambda, \Lambda' < \bar{\Lambda}$, i.e. that $T$ is sufficiently small. Finally note that the 4d effective potential $V_{\text{eff}}(\phi)$ can be obtained from the three dimensional potential $V_{\text{eff}}(\varphi)$ by dividing by $\beta$ and using (7).

To calculate the dominant corrections to $V_{\text{eff}}$ it is now convenient to introduce a dimension two auxiliary (real) field $\chi$ [12], [13], [14]. This technical trick allows one to write a closed expression for the corrections to $V_{\text{eff}}$ that do not fall with $N$. Specifically, we work with the 3d Lagrangian

$$L[\varphi, \chi, A, \rho] := L[\varphi, A, \rho] - \frac{N}{\lambda} \left( \chi - \frac{\lambda}{2} (\varphi^\dagger \varphi - \bar{\nu}^2) \right)^2,$$

in which the potential piece becomes

$$V(\varphi, \chi) = -N \frac{\chi^2}{\lambda} + N \chi (\varphi^\dagger \varphi - \bar{\nu}^2).$$
The original form (3) can be recovered by using the equation of motion for \( \chi \).

To calculate \( V_{\text{eff}}(\varphi) \) we proceed as follows [23],[12]. First, we expand (8) about spatially constant backgrounds \( \varphi^A, \chi \) thus

\[
\varphi^A \rightarrow \varphi^A + \frac{\hat{\varphi}^A}{\sqrt{N}}, \quad \chi \rightarrow \chi + \frac{\hat{\chi}}{\sqrt{N}},
\]

(10)
deleting all terms linear in the quantum fields \( \hat{\varphi}, \hat{\chi}, A \) and \( \rho \). We need not keep backgrounds for \( A \) and \( \rho \) because we are only interested in the effective scalar potential. To this we must add gauge fixing and ghost terms. We chose

\[
L_{\text{g.f.}} = \frac{1}{2\alpha} \left( \partial^m A^a_m + i\tilde{g}\alpha [\hat{\varphi}^\dagger T^a \varphi - \varphi^\dagger T^a \hat{\varphi}] \right)^2
\]

(11)

and

\[
L_{\text{ghost}} = \partial^m \tilde{\vartheta}^a D_m \vartheta_a + \frac{1}{2} \tilde{g}^2 \alpha \tilde{\vartheta}^a \tilde{\vartheta}^b \partial^m \varphi \vartheta_a
\]

\[
+ \frac{\tilde{g}^2 \alpha}{\sqrt{N}} (\hat{\varphi}^\dagger \tilde{\vartheta}^a \varphi + \varphi^\dagger \tilde{\vartheta}^a \hat{\varphi})
\]

(12)

where \( \alpha \) is an arbitrary (small) parameter and \( \vartheta^a \) are the grassmanian ghost fields (\( \tilde{\vartheta}^a = \vartheta^a T^a \)). The gauge index \( a \) runs from 1 to 3. Also \( D_m \vartheta = \partial_m \vartheta + i\tilde{g}[A_m, \vartheta]/\sqrt{N} \).

Altogether, this procedure defines a quantum Lagrangian given by

\[
L[\hat{\varphi}, \hat{\chi}, A, \rho, \vartheta] := -\frac{N\chi^2}{\lambda} + N\chi(\varphi^\dagger \varphi - \tilde{\vartheta}^2) + \varphi^\dagger \hat{\varphi} \hat{\chi} + \hat{\varphi}^\dagger \varphi \hat{\chi} + \frac{1}{\sqrt{N}} \hat{\varphi}^\dagger \hat{\varphi} \hat{\chi}
\]

\[
+ \hat{\varphi}^\dagger (-\tilde{\vartheta}^2 + \chi) \hat{\varphi} - \frac{N\chi^2}{\lambda}
\]

\[
+ \text{tr} \left( (D_m \rho)^2 + \frac{1}{2} \mathcal{F}_{mn} \mathcal{F}^{mn} \right) + \tilde{g}^2 \varphi^\dagger (A_m A^m + \rho^2) \varphi
\]

\[
+ \frac{1}{2\alpha} (\partial^m A^a_m)^2 - \frac{1}{2\alpha} \tilde{g}^2 (\hat{\varphi}^\dagger T^a \varphi - \varphi^\dagger T^a \hat{\varphi})^2
\]

\[
+ \frac{\tilde{g}^2}{\sqrt{N}} [\hat{\varphi}^\dagger A_m A_m \hat{\varphi} + \varphi^\dagger A^m A_m \hat{\varphi}] + \frac{\tilde{g}^2}{N} \hat{\varphi}^\dagger (A^m A_m + \rho^2) \hat{\varphi}
\]

\[
+ \frac{i\tilde{g}}{\sqrt{N}} (\partial_m \hat{\varphi}^\dagger A^m \hat{\varphi} - \hat{\varphi}^\dagger A^m \partial_m \hat{\varphi}) + L_{\text{ghost}}
\]

(13)
We then calculate $V_{\text{eff}}(\varphi, \chi)$ as a sum of the tree–level potential plus all one–particle–irreducible (1PI) diagrams of (13). The effective scalar potential $V_{\text{eff}}(\varphi)$ is found by eliminating $\chi$ through its equation of motion, $\partial V_{\text{eff}}(\varphi, \chi)/\partial \chi = 0$, which gives $\chi$ as a function of $\varphi$. Since $V_{\text{eff}}$ is the sum of 1PI diagrams it need not be convex \[24\], [25]. $V_{\text{eff}}$ will allow us to deduce if the phase transition is 1st or 2nd order (see also [26]).

As mentioned, the contributions to $V_{\text{eff}}$ that survive as $N \to \infty$ depend on how $\tilde{\lambda}$ and $\tilde{g}$ are held fixed in this limit. The two interesting cases that we consider here are

\begin{enumerate}
  \item $\tilde{\lambda}, \tilde{g}$ fixed
  \item $\tilde{\lambda}N, \tilde{g}$ fixed.
\end{enumerate}

Let us consider case i) first. Then it is known [1] that there is a contribution of $O(N)$ to $V_{\text{eff}}$ from the purely scalar part of the model which comes from summing the “superdaisy” graphs in ordinary perturbation theory in $\hbar$. There are no contributions of $O(N)$ involving the gauge fields [1]. The gauge fields first contribute at $O(1)$ in $V_{\text{eff}}$ because the number of gauge degrees of freedom does not increase with $N$. From examining (13) it is clear that the $O(N)$ quantum correction comes from the part which is quadratic in $\hat{\varphi}$ — the quantum fields $\hat{\chi}, A$ and $\rho$ can be ignored. A one–loop computation gives ($\chi \geq 0$)

$$V_N(\varphi, \chi) = -N \frac{\chi^2}{\lambda} + N \chi (\varphi^\dagger \varphi - \bar{v}^2) + 2N \text{Tr} \ln(-\vec{\partial}^2 + \chi)$$

$$= -N \frac{\chi^2}{\lambda} + N \chi (\varphi^\dagger \varphi - \bar{v}^2) + N \frac{\Lambda \chi}{\pi^2} - N \frac{\chi^2}{3\pi}.$$ \hspace{1cm} (14)

where we have used a cutoff $\Lambda$ and dropped constants. The identification $\Lambda = \pi^2 T/6$ reproduces exactly the renormalized high temperature 4d results of [1] for the effective mass at the origin. The “mass–gap” equation for $\chi$ is then

$$\chi = \frac{\tilde{\lambda}}{2} \left( \varphi^\dagger \varphi - \bar{v}^2 + \frac{T}{6} - \frac{\sqrt{X}}{2\pi} \right).$$ \hspace{1cm} (15)

The physical solution for $\sqrt{X}$ which then follows from $V_N$ is [23], [13], [11], [15]

$$\sqrt{X} = \frac{\tilde{\lambda}}{8\pi} \left[ \sqrt{1 + \frac{32\pi^2}{\lambda} (\varphi^\dagger \varphi - \bar{v}^2 + \frac{T}{6}) - 1} \right].$$ \hspace{1cm} (16)
The next–to–leading corrections to $V_{\text{eff}}$ involve all the quantum fields. However, to $O(1)$, many of the terms in (13) are unimportant. First note that $\hat{\varphi}$ appears in the quantum Lagrangian quadratically. Thus these fields can be explicitly integrated out. In order to do this we must first shift $\hat{\varphi}$ in such a way as to eliminate all the terms which are linear in $\hat{\varphi}$. The effect of doing this to the order we are working in is to simply replace the mixing term

$$\varphi^\dagger \varphi \chi + \varphi^\dagger \varphi \chi \rightarrow -\frac{\varphi^\dagger \varphi}{(-\partial^2 + \chi)} \chi. \quad (17)$$

To $O(1)$, no new contributions are generated from the $O(\tilde{g}^2 \hat{\varphi})$ mixing terms. Thus we can ignore these terms.

Integrating out the scalars generates $V_N$ and, to $O(1)$, a gauge dependent contribution from the second term in the fourth line in (13). It also generates, to $O(1)$, terms quadratic in the $\hat{\chi}$, $\rho$, and gauge fields, as well as cubic and higher powers in these fields which appear at higher order in the $\frac{1}{N}$ expansion. The quadratic contributions can be computed in terms of the vacuum polarization terms $\Pi_{\hat{\chi} \hat{\chi}}$, $\Pi_{\rho \rho}$ and $\Pi_{ab}^{mn}$. In calculating these using the quantum Lagrangian of eq.(13) we need to consider diagrams which only involve $\hat{\varphi}$ fields in the loop. Explicitly we have

$$\Pi_{\hat{\chi} \hat{\chi}}(\vec{k}) = \lambda \int \frac{1}{\vec{p}^2 + \chi} \frac{1}{(\vec{p} + \vec{k})^2 + \chi},$$

$$\Pi_{\rho \rho}^{ab}(\vec{k}) = \tilde{g}^2 \delta^{ab} \int \frac{1}{\vec{p}^2 + \chi},$$

$$\Pi_{mn}^{ab}(\vec{k}) = \frac{1}{2} \left[ \frac{\delta_{mn}}{\vec{p}^2 + \chi} - \frac{1}{2} \frac{(2p + k)_m (2p + k)_n}{(\vec{p}^2 + \chi)(\vec{p} + \vec{k})^2 + \chi} \right]. \quad (18)$$

We recall that $m, n$ are spatial indices and $a, b$ are gauge indices.

It can be explicitly checked that all terms cubic or higher in the remaining quantum fields do not generate any Feynman diagrams which contribute at $O(1)$ to $V_{\text{eff}}$. This was also true for the abelian Higgs model [14], as well as for the purely scalar model [13,14]. The quadratic part of the action for the
remaining fields is

\[
L[\hat{\chi}, A, \rho, \vartheta] = \frac{1}{2} \rho_a \left[ (-\vec{\nabla}^2 + \frac{1}{2} \tilde{g}^2 \varphi^\dagger \varphi) \delta^{ab} + \Pi^{ab} \right] \rho_b \\
+ \frac{1}{2} A_a^m \left[ (-\vec{\nabla}^2 + \frac{1}{2} \tilde{g}^2 \varphi^\dagger \varphi) \delta^{ab} \delta_{mn} + (1 - \alpha^{-1}) \partial_m \partial_n \delta^{ab} + \Pi_{mn}^{ab} \right] A_b^n \\
+ \tilde{\vartheta}^a [-\vec{\nabla}^2 + \frac{1}{2} \tilde{g}^2 \varphi^\dagger \varphi] \vartheta_a \\
- \hat{\chi} \left[ \tilde{\lambda}^{-1} + \frac{\varphi^\dagger \varphi}{(-\vec{\nabla}^2 + \chi)} + \tilde{\lambda}^{-1} \Pi \hat{\chi} \right] \hat{\chi}.
\]

(19)

\(V_{\text{eff}}\) to \(O(1)\) is then given by \(V_N\) piece, a simple one–loop contribution from ghosts \(\vartheta\), and the contributions from quadratic integrals over \(A, \rho, \hat{\chi}\) with propagators modified by the vacuum polarization effects (18). The \(O(1)\) 4d gauge contributions correspond to summing the 1PI diagrams given in Figure 1.

It is known [1],[15] that at order \(N\), the potential above admits no 1st order phase transition. The \(O(N)\) potential is exact in the small \(\chi\) limit (up to 4d corrections). To get the high \(T\) 4d potential \(V_{\text{eff}}(\phi) = TV_{\text{eff}}(\varphi)\) we use (7). In the following \(\phi^2 := \phi_A^\dagger \phi^A\).

To \(O(N)\), \(dV_{\text{eff}}/d\phi^2 = \partial V_{\text{eff}}/d\phi^2 = N\chi\). At \(\phi^2 = 0\) this vanishes at \(T_2^2 = 6v^2\). For \(T > \sqrt{6}v\) the origin is a global minimum, and at \(T = T_2\) the potential grows away from the origin. For \(T > T_2\) the point \(\chi = 0\) is away from the origin and this has the interpretation [12] as the symmetry breaking minimum below \(T_2\).

We now argue that, to \(O(1)\), the phase transition is again second order. Note that all the expressions in (19) are diagonal in the gauge indices. Thus, to \(O(1)\), the nonabelian nature of the gauge fields is not felt; the three \(SU(2)\) gauge fields behave like three abelian gauge fields. In [11] it was shown that the phase transition in the abelian Higgs model is second order at \(O(1)\) when \(g^2/\lambda\) is held fixed in the large \(N\) limit. The reason for this is the following.

The case \(g = 0\) is known to have a second order transition to \(O(1)\) [15],[11]. Therefore, we need only examine the gauge contributions. At one–loop (in \(h\)) and high temperature, each 4d gauge degree of freedom gives a contribution \(\sim -T\phi^3\) to \(V_{\text{eff}}\). It was subsequently realised [6],[7],[9] that vacuum polarization effects generate a large \(T\) dependent mass for the longitudinal gauge fields which eliminates their contribution to the cubic terms in \(V_{\text{eff}}\). This
corresponds to the fact that the vacuum polarizations for the zero modes of
the 4d longitudinal gauge fields in (18) are linearly divergent. However, it
was also noted that vacuum polarization effects do not generate such large $T$
dependent masses for the transverse gauge fields. This corresponds to
the fact that, due to 3d gauge and euclidean invariance, the vacuum polar-
izations of the zero modes of the 4d transverse gauge fields in (18) are not
linearly divergent. In the limit of zero external 3–momentum, $\vec{k} \to 0$, we
have $\Pi^{ab}_{mn} \to 0$. If this value is used for the $O(1)$ 3d gauge field contributions
to $V_{eff}$ then one obtains contributions to the effective potential proportional
to $\text{Tr} \ln(-\vec{\partial}^2 + \frac{1}{2} \tilde{g}^2 \varphi^+ \varphi)$, which contains a cubic term and implies a first order phase transition. However, this is incorrect.

It was argued in \cite{11}, that for the most part, the dominant contributions
from the vacuum polarization comes from using the large momentum limit
of $\Pi^{ab}_{mn}$,

$$
\Pi^{ab}_{mn}(\vec{k}) \to -\tilde{g}^2 \delta^{ab}(\delta_{mn} - k_m k_n / \vec{k}^2) \sqrt{\chi / 4\pi},
$$

(20)

not the zero momentum limit. This result is valid for $T \gg \varphi^+ \varphi - \sqrt{\chi} / 2\pi \gg \tilde{g}^2 / (64)$ and $\sqrt{\chi} > \tilde{g}^2 / (12\pi)$; for technical details the reader is referred to \cite{11}. Then the 3d gauge fields give contributions to $V_{eff}$ proportional to

$$
\text{Tr} \ln \left[-\vec{\partial}^2 + \frac{1}{2} \tilde{g}^2 (\varphi^+ \varphi - \sqrt{\chi} / 2\pi)\right].
$$

(21)

In a consistent $1/N$ approach, the critical temperature $T_2$ when the origin
changes from a local minimum to a local maximum is again given by the
leading order result, $T_2 = \sqrt{\chi} v$ \cite{11}. In addition, to $O(1)$, it is consistent
to use the leading order result (16) for $\sqrt{\chi}$. At $T = T_2$, the combination
$\varphi^+ \varphi - \sqrt{\chi} / (2\pi)$ contains no term $\sim \varphi^+ \varphi$ for small $\varphi$. Therefore, at $T = T_2$, the
effective potential to $O(1)$ contains no cubic term and the phase transition
is second order \cite{11}.

At this point we comment on a technical point. For the limit $\chi \to 0$
(where are results are not strictly valid) one can follow \cite{13}. For the case
$g = 0$, Root \cite{13} has shown for the 3d case that the point $\chi = 0$ remains
a local minimum at next–to–leading order. This analysis did not require a
computation of $V_{eff}$. Root examined $dV/d\phi^2$ in the limit of vanishing $\chi$ and
showed that the leading order gap–equation for $\chi$ (in our case eq. (15) )
was sufficient to show that $V_{eff}/d\phi^2 = 0$ still has a solution at $\chi = 0$ at
next–to–leading order. His analysis can be extended to our gauged case. We do not present a detailed analysis here but instead refer the reader to [13] and also ref. [27] where, following [13], an analysis of the small $\chi$ limit in the full 4d high temperature abelian Higgs model has been performed with the result that $\chi = 0$ remains a point of vanishing $dV_{eff}/d\phi^2$ at next–to–leading order.

This information is however insufficient to deduce a second order phase transition. As mentioned in [13], for sufficiently large $N$ the $O(1)$ corrections cannot overwhelm the leading order conclusion of a second order phase transition. For $N$ not arbitrarily large we would like to know the global properties of $V_{eff}$ away from the point $\chi = 0$, and in particular if there is a point away from the origin at $T = \sqrt{6}v$ where the $O(1)$ corrections can produce a new minimum and possibly even result in the breakdown of the $1/N$ expansion. In fact, from computing the effective potential to $O(1)$ one finds that the “next–to–leading” order corrections compete with the “leading” corrections when $g^2/\lambda \sim N$. Thus, the results here are not reliable for in this case.

The expression (21) appears to characterize, to $O(1)$, the correct behavior in the limit of vanishing $\chi$ even though it is not strictly valid in this limit. Assuming no pathological behaviour occurs in an exact computation of $V_{eff}$ at next–to–leading order for small but nonzero $\sqrt{\chi}$ we expect that (21) gives a good description all the way down to $\sqrt{\chi} = 0$ and also $\phi^2 = 0$ at $\sqrt{\chi} = 0$.

We now discuss case $ii)$, $g^2/(\lambda N)$ fixed for increasing $N$. This has the effect of making the pure scalar contributions lower order in $N$ and, as a result, both the gauge and scalar fields contribute at leading order, $O(1)$. To obtain the leading corrections we make the replacements $\hat{\lambda} = \hat{\lambda}/N$, $\chi \to \chi/N$, $\hat{\chi} \to \hat{\chi}/N$ in (13) and study the limit $\hat{\lambda}'$, $\tilde{g}$ fixed for increasing $N$. For the 3d gauge fields, this has the effect of making the vacuum polarization contribution in (21) lower order in $N$. Hence, to $O(1)$, the transverse components of the 4d gauge fields give contributions to $V_{eff}$ proportional to $\text{Tr} \ln(-\partial^2 + \frac{1}{2}g^2\varphi\varphi^\dagger)$. This contains a cubic term so the phase transition is first order for sufficiently large $N$.

To conclude, the high $T$ phase transition in the gauged $SU(2)$ Higgs model with $N$ complex Higgs doublets is, for sufficiently large $N$, 1st order if $i)$ $g^2/\lambda$ grows as $N$ and second order if $ii)$ $g^2/\lambda$ is fixed as $N$ increases. As mentioned, we do not believe our results can be used to make a reliable statement about the nature of the phase transition in the electroweak...
model, \( N = 1 \), because in this case the number of gauge fields is equal to the number of real scalars. In order to make a reliable statement, more sub-leading corrections than those that have been computed for either cases \( i) \) or \( ii) \) must be determined. However, our results indicate that if the phase transition is first order in the exact electroweak model, it is probably more weakly first order than what the effective potential computed as a sum of ring diagrams with vacuum polarization effects evaluated only at zero external 3-momentum would indicate. In this case it is clearly too weakly first order for generating the BAU [6],[7]. It is therefore necessary to consider extensions to the standard model in which the phase transition can be made sufficiently first order in order to generate the BAU.

The toy models we have studied here and in refs. [11],[15] demonstrate the care which must be taken in order to determine the strength of any first order phase transition. For example, let us add to the tree Lagrangian (1) an extra scalar particle with the Lagrangian

\[
L[s] := \frac{1}{2} \partial_\mu s \partial^\mu s - (m^2 + \xi^2 \phi_A^\dagger \phi_A) s^2 - \frac{\lambda_s}{N} s^4. \tag{22}
\]

Here, \( m^2, \xi \) and \( \lambda_s \) are field independent. In the 3d effective theory the one-loop contribution from \( s \) to the effective potential is \( \frac{1}{2} \text{Tr} \ln(-\partial^2 + m^2 + \xi^2 \phi_A^\dagger \phi_A) \). It contains a cubic term \( \sim \xi^3 \phi^3 \) for \( m^2 \) sufficiently small [8]. However, since this extra field for small \( m^2 \) is very similar to the zero mode of the longitudinal component of the gauge field in the abelian Higgs model, it is clear that in the large \( N \) limit vacuum polarization effects will generate a large temperature dependent mass for it and eliminate the cubic term for non-negative \( m^2 \) and \( \xi \phi^2 < O(T^2) \). This conclusion is irrespective of how \( \xi^2/\lambda \) scales in the large \( N \) limit (with \( \lambda_s \) fixed).

Our results suggest that one should try and find simple extensions of the \( SU(2) \) Higgs model studied here which admit strongly first order phase transitions in the large \( N \) limit. One might hope that, for such models, corrections to \( V_{eff} \) which fall with \( N \) cannot significantly reduce the strength of the first order phase transition, even for small \( N \).
Figure 1.
Gauge loops contributing at $O(1)$. 
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