Spin coherent states have been realized in Bose-Einstein condensates (BECs) with the ability of coherent control [1] and spin squeezing [2–4]. One of the main applications of such states is thought to be quantum metrology, where spin squeezed states are created to reduce quantum noise in a particular spin direction, to go beyond the standard quantum limit [5]. Spin squeezing is itself a kind of entanglement [1,6], but is a type that is localized in one spatial location, since it manifests itself as a correlation between bosons within the same BEC. Other types of entanglement of a non-local variety are also possible, as demonstrated in the experiment of Ref. [7]. Here, a single atom was entangled with magnon states within a BEC via optical fiber at two distant locations. Using similar techniques, methods were proposed for generating remote entanglement between two BECs, such that entanglement between two BECs would appear to be within experimental possibility [8,12]. Such entangling gates, typically of the form $S^z_1 S^z_2$, form a fundamental interaction that is necessary for quantum information processing, where in analogy to standard qubits, one and two qubit gates form a universal set [8]. The $S^z_1 S^z_2$ interaction forms the analogue of the CNOT gate between two BECs, a fundamental gate for many quantum circuits. Tasks such as Deutsch’s algorithm [10], Grover’s algorithm [9], and quantum teleportation using BECs [11] also rely on the creation of entanglement using the $S^z_1 S^z_2$ interaction, and thus it is of both fundamental and practical interest to understand at a deeper level the types of states that are produced by this interaction.

In this paper we show that such entangling gates have a surprisingly rich structure in terms of the entanglement generated. In addition to potential quantum information processing applications, it is interesting from a physical perspective as it is one of the rare models in quantum physics where a fractal structure is produced, which is remarkable considering the relative simplicity of the model. We also calculate the effects of decoherence on the entangled state and show that a particular set of states that should be robust despite the macroscopic nature of the BECs. The types of states formed are reminiscent optical coherent states with a self-Kerr interaction, where Schrodinger cat state are formed at particular evolution times [13,14].

**Entangling BECs** We define the spin coherent state, suitable for BECs, to be

$$|\alpha, \beta\rangle \equiv \frac{1}{\sqrt{N!}} (\alpha a^\dagger + \beta b^\dagger)^N |0\rangle,$$

where $a^\dagger, b^\dagger$ are bosonic creation operators for two orthogonal quantum states, $\alpha, \beta$ are arbitrary complex coefficients such that $|\alpha|^2 + |\beta|^2 = 1$, and we consider the total number of atoms to be fixed to $N$. Such spins may be manipulated on the Bloch sphere in precisely the same way as standard qubits using Schwinger boson operators $S^x = a^\dagger b + b^\dagger a, S^y = -ia^\dagger b + ib^\dagger a, S^z = a^\dagger a - b^\dagger b$ [8]. This analogy to standard qubits suggests that the two-component BEC may be thought as a “BEC qubit”, where the spin coherent state encodes one qubit worth of information. Let us now consider two such spin coherent states, and consider an entangling operation $JS^z_1 S^z_2$ between the two BEC qubits (Figure 1a). Starting from two BEC qubits in maximal $x$-eigenstates we obtain

$$e^{-iS^z_1 S^z_2 \tau} \left( \frac{1}{\sqrt{2}} |1\rangle \langle 1| \frac{1}{\sqrt{2}} |1\rangle \langle 1| \right) \frac{1}{\sqrt{2}} |0\rangle \langle 0|.$$  

$$= \frac{1}{\sqrt{2^N}} \sum_k \sqrt{\binom{N}{k}} \left( e^{i(N - 2k)\tau} \frac{1}{\sqrt{2}} |1\rangle \langle 1| \frac{1}{\sqrt{2}} |0\rangle \langle 0| \right) \frac{1}{\sqrt{2}} |k\rangle \langle k|, \quad (2)$$

where $|k\rangle_i = (a^\dagger)^k (b^\dagger)^{N-k}/\sqrt{k!(N-k)!}$ are eigenstates of $S^z$ (Fock states), and time is measured in units of $\hbar/J$. The sub-
scripts $i = 1, 2$ label the two BECs. As a first glimpse into the unusual behavior of the entangled state \( \rho \), let us plot the von Neumann entropy \( S = -\text{Tr}(\rho \log_2 \rho) \) where \( \rho \) is the density matrix of \( \rho \) with the partial trace over BEC qubit 2 taken. Figure 1a shows the entanglement generated between the two BEC qubits, comparing to the standard qubit case \((N = 1)\) and for BEC qubits with \( N = 100 \). For \( N = 1 \) we see the expected behavior, where a maximally entangled state is reached at \( \tau = \pi/4 \). For \( N = 100 \) the entanglement shows a repeating structure with period \( \pi/2 \), but otherwise no apparent periodicity in between. The fluctuations become increasingly at finer timescales as \( N \) grows, where the timescale of the fluctuations occur at an order of \( \sim 1/N \). The basic behavior can be summarised as follows. The entanglement increases monotonically until a characteristic time \( \tau = 1/\sqrt{2N} \), after which very fast fluctuations occur, bounded from above by a “ceiling” in the entanglement. In a realistic BEC with \( N = 10^3\cdot10^6 \) the fluctuations occur even faster than the results of the numerical simulations shown here.

What is the origin of the complex behavior of the entanglement? Figure 1a shows the Q-function [4] for the initial state \( \tau = 0 \) of one of the BECs on the surface of the normalized Bloch sphere. Evolving the system now in \( \tau \), a visualization of the states is shown in Figure 2. Keeping in mind that the spin coherent states form a quasi-orthogonal complete set of states \( \langle \langle \sqrt{2}, \sqrt{2} \mid \sqrt{2}, \sqrt{2} \rangle \rangle \approx \exp[-N(\phi - \phi')^2/8] \), we represent various spin coherent states in the sum \( \langle \langle \rangle \rangle \) each as a circle of radius \( r = \sqrt{2/N} \), which corresponds to the distance where the overlap between two spin coherent states start to diminish exponentially. As all spin coherent states in \( \langle \langle \rangle \rangle \) are along the equator of the Bloch sphere \( z = 0 \), we flatten the Bloch sphere along the \( z \)-direction, such that each spin coherent state is located at an angle \( \phi \) in the \( x-y \) plane. The color of the circle corresponds to the particular \( k \) state that the spin coherent state is entangled with (see also Figure 1a). The opacity of the circles are adjusted such that it corresponds to the coefficient in \( \langle \langle \rangle \rangle \), so that smaller terms appear more transparent.

There are several characteristic times which we consider separately. As \( \tau \) is increased, the circles fan out in both clockwise and counterclockwise directions. At \( \tau = \frac{\pi}{\sqrt{2N}} \), the extremal states \( k = 0, N \) reach the \( \pm y \) directions. For \( N = 1 \) (the standard qubit case) this gives a maximally entangled state. Figure 1b shows the amount of entanglement for times \( \tau = \frac{\pi}{\sqrt{2N}} \), which shows that it quickly approaches the asymptotic value of \( E_\infty \approx 1.26 \). A maximally entangled state would have entanglement \( E_{\text{max}} = \log_2(N + 1) \), and thus gate times of \( \tau = \frac{\pi}{\sqrt{2N}} \) correspond to relatively small amounts of entanglement, equivalent to approximately one pair of entangled qubits.

The next characteristic time is \( \tau = \frac{1}{\sqrt{2N}} \). This is the time when the spin coherent states are separated far enough such that adjacent circles have no overlap (see Figure 2b). The entanglement, as can be seen in Figure 1c, reaches its maximal value at this time, and beyond this executes fast fluctuations briefly returning to this value. It is also a characteristic time due to the weight factors in \( \langle \langle \rangle \rangle \). Approximating \( \sqrt{\frac{\pi}{N^2}} \approx (\frac{\pi}{N^2})^{1/4} \exp[-\frac{1}{2}(k - N/2)^2] \), we see that the only the terms between \( k = N/2 \pm \sqrt{N} \) have a significant weight in the summation and the other terms only contribute and exponentially small amount. At the time \( \tau = \frac{1}{\sqrt{2N}} \), these states are spread out over the unit circle, since the angular positions of the spin coherent states in \( \langle \langle \rangle \rangle \) reach order unity for the first time. These two observations give us a simple way of estimating the asymptotic value of the entanglement in the limit \( N \to \infty \). Starting from \( \langle \langle \rangle \rangle \) and discarding small weight contributing states, we have \( \sum_{k=0}^{N} \sum_{k=N/2}^{N} \langle \langle \rangle \rangle \approx \frac{2}{\sqrt{2N}} \sum_{k=N/2}^{N} \exp\left[-\frac{1}{2}(k - N/2)^2\right] \) where we have approximated the binomial factor within the range to be constant. Assuming that the spin coherent states are orthogonal, this gives an entanglement \( E_\infty \approx \log_2 \sqrt{N} \). Relative to the maximal value this gives

\[
\lim_{N \to \infty} \frac{E_S}{E_{\text{max}}} = \frac{1}{2}.
\]

This result is interesting since it shows that even in the limit of \( N \to \infty \) entanglement survives, on a macroscopic scale. It is clear from this example that classical transition does not occur simply due to the large number of particles in the system, but only when decoherence is also present. The inset of Figure 1c shows results consistent with the asymptotic result (3).

**Devil’s crevasse** We now consider the origin of the fast fluctuations in the entanglement in Figure 1a. Due to the periodicity of the spin coherent states, the circles align with high symmetry at particular \( \tau \) times as seen in Figures 2c and 2d. These points correspond to sharp dips in the entanglement as seen in Figure 1a. For example, at \( \tau = \pi/4 \), (2) can be written

\[
\frac{1}{2} \left( \langle \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \rangle + \langle \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle \rangle \right) - \frac{1}{2} e^{i\pi/4} \left( \langle \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle \rangle - \langle \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \rangle \right).
\]

For large \( N \), the states \( \langle \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \rangle \) and \( \langle \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle \rangle \) require flipping \( N \) bosons from \( +x \) to \( -x \) eigenstates, which is a Schrodinger cat state. Taking into account that \( \langle \langle \sqrt{2}, \sqrt{2} \mid \sqrt{2}, \sqrt{2} \rangle \rangle = 0 \) for any \( N \), (4) is precisely equivalent to a Bell state, which has an entanglement \( E_B = \log_2 2 = 1 \) for all \( N \), in agreement with the numerical results of Figure 1c.

Similar arguments may be made for any \( \tau \) that is a rational multiple of \( \pi/4 \). Consider a general time
\[
\tau = \frac{m \pi}{d}, \quad \text{where } m, d \text{ are integers. The angular difference between adjacent circles in Figure 2 may be deduced to be } \Delta \phi = 4\tau = \frac{m \pi}{d}. \text{ The number of circles must then be the first integer multiple of } \Delta \phi \text{ that gives a multiple of } 2\pi. \text{ This is } \text{LCM}(m/d, 2)d/m, \text{ where LCM is the least common multiple. The amount of entanglement at such times can then be written}
\]
\[
E_S = \log_2 \left[ \frac{\text{LCM}(m/d, 2)d}{m} \right]. \quad (5)
\]

The entanglement at times that are an irrational multiple of \(\pi/4\) do not have any particular alignment of the circles, and thus can be considered to be approximately randomly arranged over \(\phi\). Similar arguments to 3 may then be applied, and at these points the entanglement jumps back up to \(E_S/E_{\text{max}} = 1/2\) for \(N \to \infty\).

Since the set of rational numbers is everywhere dense, in the limit of \(N \to \infty\), the entanglement has a pathological behavior where there are an infinite number of dips between any two values of \(\tau\). Such a dependence of whether a parameter is either rational or irrational is familiar in physics contexts from models such as the long-range antiferromagnetic Ising model [17], where the filling factors equal to a rational number occupy a finite extent of the chemical potential, forming a devil’s staircase. In analogy to this, we may call the behavior of the entanglement a devil’s crevasse, where every rational multiple of \(\pi/4\) in the gate time gives a sharp dip in the entanglement. It is interesting that unlike the devil’s staircase which has origins in two competing periodicities forming either commensurate or incommensurate phases [18], here the only periodicity is that of the Bloch sphere, and the mechanism of the pathological behavior is different.

Armed with this interpretation, it is natural to extend the operation to not only rotations in the \(x\)-\(y\) plane, but the full Bloch sphere. Namely we may first perform a \(S_1^+S_2^-\) operation, then perform another entangling gate around another axis, say the \(z\)-axis (see Figure 1) to perform the operation \(e^{-iS_1^+S_2^-\tau}e^{-iS_1^-S_2^+\tau} |\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle \rangle |\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle\rangle_2\). The entanglement in the two parameter space of \((\tau, \tau')\) is shown in Figure 3. We see a highly detailed fractal-like structure, with sharp dips in the entanglement at rational multiples of \(\pi/4\) for both \(\tau\) and \(\tau'\). The fine detail is beyond the resolution of the graph shown in Figure 4, as can be seen by a zoomed in plot shown in Figure 5. The detail extends to angular intervals of \(\Delta \tau, \Delta \tau' \sim \frac{\pi}{4}\), thus in the limit of \(N \to \infty\) the fine structure extends to arbitrarily small regions of the graph.

Decoherence In any realistic system decoherence is present, and considering the large number of particles that are typically present in a BEC, a question remains of how observable such entanglement is. As a generic model of decoherence we consider Markovian dephasing.
to occur on each of the spin states
\[
\frac{dp}{d\tau} = -i[S^z_j S^z, \rho] - \frac{\Gamma_j}{2} \sum_{n=1}^{2} \left[ (S^j_n)^2 \rho - 2S^j_n \rho S^j_n + \rho S^j_n S^j_n \right],
\]
where we consider the effects of \( j = z, x \) separately. We find that there is a strong dependence on how well the entanglement survives depending on the characteristic gate times. This is summarized by the behavior of the ratio between the entanglement, as measured by the logarithmic negativity [12, 16], with and without decoherence. For the \( z \)-decoherence, we find a power law decay of the entanglement as \( E_{\text{neg}}(\Gamma_x) \sim N^{-\gamma} \), where \( 0 \leq \gamma \leq 1 \).

For \( x \)-decoherence, there is a strong dependence on the time of evolution
\[
\frac{E_{\text{neg}}(\Gamma_x)}{E_{\text{neg}}(\Gamma_x = 0)} \sim \begin{cases} \text{const.} & \tau \sim 1/N \\
N^{-\gamma} & \tau \sim 1/\sqrt{N} \\
e^{-N^2} & \tau \sim O(1) \end{cases}
\]

For short gate times of \( \tau \sim 1/N \) the entanglement is very robust even with decoherence. For longer gate times \( \tau \sim O(1) \), the entangled states are rather fragile, due to the appearance of Schrödinger cat-like states such as [14], with the border between these two behaviors is \( \tau \sim 1/\sqrt{N} \). We thus anticipate that in practice gate times up to \( \tau \leq 1/\sqrt{N} \) should be robust enough to observable experimentally. However, in analogy to a decoherence free subspace approach [23], by arranging the interaction to be in a commuting basis to the decoherence (such as in the \( z \)-decoherence case here), states with \( \tau > 1/\sqrt{N} \) may be prepared.

**Experimental detection** In order to experimentally detect the type of entanglement that is discussed in this paper, it is beneficial to have simple quantities that serve as signatures of entanglement between two BECs. For small times \( \tau \sim 1/N \), and initial states, with both BEC qubits in maximal \( x \)-eigenstates we may approximate the spins \( S^y \) and \( S^x \) as conjugate position and momentum variables [24] according to \( X_i = \frac{S^y}{2\sqrt{(S^z)^i}} \), \( P_i = \frac{S^x}{2\sqrt{(S^z)^i}} \), where \( i = 1, 2 \). Following the arguments of Ref. [25], for a general separable state we derive the inequality
\[
\langle [\Delta(S^y_1 - S^y_2)]^2 \rangle + \langle [\Delta(S^x_1 - S^x_2)]^2 \rangle \geq 2\langle S^z_1 \rangle + 2\langle S^z_2 \rangle.
\]

This inequality may be used to test for the presence of entanglement generated by the \( S^z_1 S^z_2 \) interaction. To see why it is this combination of operators which signals the presence of entanglement, let us explicitly derive this quantity under the assumption \( \langle S^y_i \rangle = N \) the entangling Hamiltonian in these variables is \( H = 4JNP_1 P_2 \), and we may derive the Heisenberg equations of motion
\[
\begin{align*}
X_1(\tau) &= X_1(0) + 2N\tau P_2(0) \\
X_2(\tau) &= X_2(0) + 2N\tau P_1(0) \\
\rho_1(\tau) &= \rho_1(0).
\end{align*}
\]

Although the dynamics of these operators is rather different to two-mode squeezing due to the different Hamiltonian considered here, it is still possible to form EPR-like quantities with suppressed noise fluctuations. Using [9] we derive for an initially uncorrelated state
\[
\langle [\Delta(S^y_i(\tau) - S^y_i(\tau))]^2 \rangle = \langle [\Delta(S^x_i(\tau) - S^x_i(\tau))]^2 \rangle 
\]
\[
= N(1 - 2N\tau)^2 + N.
\]

It is apparent that the above expression may violate [5] which unambiguously shows the presence of entanglement between the BECs. We note that beyond times of order \( \tau \sim 1/N \), the continuous variables approximation breaks down, so [10] is only valid in this region. We may understand this as times where the entanglement creates distributions of coherent states that are no longer on one hemisphere of the Bloch sphere (see Figure 2). Heuristically, the approximation breaks down due to the position and momentum variables assuming a “flat” underlying topology, as opposed to the “spherical” topology for spins. Such oversqueezing for two-mode squeezing is also discussed in Ref. [3].

**Conclusions** We have shown that interactions of the form \( S^z_1 S^z_2 \) between two BECs exhibit a rich fractal-like behavior in terms of the entanglement generated. There is a remarkable dependence of the entanglement upon whether the gate times are a rational or irrational multiple of \( \pi/4 \), where sharp dips in the entanglement form. While typically entanglement between macroscopic objects are highly suppressed due to accelerated decoherence, for gate times \( \tau \leq 1/\sqrt{N} \) these should be robust even in the presence of decoherence. For the observation of entanglement beyond these times, arranging for interaction to be in a commuting basis to the decoherence would seem to be necessary. We note that while entanglement between macroscopic objects have been observed before
it is always in the small entanglement limit of the order of unity. The present case goes beyond this as it is truly macroscopic entanglement \textit{i.e.} the entanglement is of the order of the maximal entanglement $E \sim E_{\text{max}}/2$. It is interesting that for quantum information purposes, typically only short gate times should be necessary \cite{9}, which allow for the possibility that such systems could be used as viable quantum memories. The large duplication of the quantum information in the spin coherent state (1) would then allow for a robust way of storing and manipulating the data.

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