ON SYMMETRY AND UNIQUENESS OF GROUND STATES FOR LINEAR AND NONLINEAR ELLIPTIC PDES

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Abstract. We study ground state solutions for linear and nonlinear elliptic PDEs in $\mathbb{R}^n$ with (pseudo-)differential operators of arbitrary order. We prove a general symmetry result in the nonlinear case as well as a uniqueness result for ground states in the linear case. In particular, we can deal with problems (e.g., higher order PDEs) that cannot be tackled by usual methods such as maximum principles, moving planes, or Polya-Szegö inequalities. Instead, we use arguments based on the Fourier transform and we apply a rigidity result for the Hardy-Littlewood majorant problem in $\mathbb{R}^n$ recently obtained by the last two authors of the present paper.

1. Introduction and Main Results

In this short paper, we study symmetry properties and uniqueness of ground states for linear and nonlinear elliptic PDEs posed on $\mathbb{R}^n$. In particular, we will be interested in a general class of problems (including higher-order PDEs) which cannot be studied by classical methods such as maximum principles or Polya-Szegö inequalities. Instead our approach here is based on Fourier methods together with a classification of the Hardy-Littlewood majorant problem in $\mathbb{R}^n$, which was recently obtained in [1].

For a convenient organization of this paper, we will present our results on linear and nonlinear problems in two separate subsections as follows.

1.1. Linear Results. Let $s > 0$ be a real number. We consider ground states $\psi \in H^s(\mathbb{R}^n)$ of linear equations of the form

$$P(D)\psi + V\psi = E\psi,$$

where $E \in \mathbb{R}$ is the eigenvalue and $V : \mathbb{R}^n \to \mathbb{R}$ denotes a given potential. Here $P(D)$ stands for a self-adjoint, elliptic constant coefficient pseudo-differential operator of order $2s$. More precisely, we assume the following condition.

Assumption 1. Let $s > 0$. The pseudo-differential operator $P(D)$ is given by

$$(P(D)f)(\xi) = p(\xi)\hat{f}(\xi),$$

with some continuous function $p : \mathbb{R}^n \to \mathbb{R}$ that satisfies the estimates

$$A|\xi|^{2s} + c \leq p(\xi) \leq B|\xi|^{2s} \quad \text{for all } \xi \in \mathbb{R}^n$$

with suitable constants $A > 0$, $B > 0$, and $c \in \mathbb{R}$.

Let us now suppose that $P(D)$ satisfies Assumption 1. We assume that $V : \mathbb{R}^n \to \mathbb{R}$ is a bounded potential.$^1$ Hence we can consider the well-defined minimization problem

$$E_0 = \inf \left\{ (f, (P(D) + V)f) : f \in H^s(\mathbb{R}^n), \|f\|_{L^2} = 1 \right\} > -\infty.$$  

Furthermore, if we assume that $V(x) \to 0$ as $|x| \to \infty$ in the sense that $\{|V(x)| > \varepsilon\}$ has finite Lebesgue measure for every $\varepsilon > 0$, it is easy to see that

$$E_0 \leq \inf_{\xi \in \mathbb{R}^n} p(\xi) = \inf \sigma_{ess}(H),$$

where $\sigma_{ess}(H)$ denotes the essential spectrum of the self-adjoint operator $H = P(D) + V$ defined via the quadratic form appearing in (1.2). Provided a minimizer $\psi \in H^s(\mathbb{R}^n)$ for (1.2) exists, it is easy to see $\psi$ solves (1.1) with $E = E_0$. Conversely, any solution $\psi \in H^s(\mathbb{R}^n) \setminus \{0\}$ of (1.1) with $E = E_0$ is a minimizer of problem (1.2) up to a trivial

$^1$We could relax this condition to unbounded potentials $V \in L^\infty(\mathbb{R}^n) + L^p(\mathbb{R}^n)$ with $p > \max\{n/2s, 1\}$. For the sake of simplicity, we omit this generalization here.
rescaling to ensure the normalization condition $\|\psi\|_{L^2} = 1$. Following usual nomenclature in spectral theory of Schrödinger operators, we refer to such minimizing solutions $\psi \in H^s(\mathbb{R}^n)$ as **ground states** for the linear problem $(1.1)$. To have a better contradiction for the nonlinear problems discussed below, we will also use the term **linear ground state** sometimes.

In the setting of Schrödinger operators when $P(D) = -\Delta$, we remark that uniqueness of ground states $\psi$ (up to a trivial multiplicative constant) is a classical result, which can be proven by an wide array of known methods such as maximum principles, Polya-Szegő principle, and Perron-Frobenius arguments involving the corresponding heat kernel $e^{t\Delta}$. Also, the fractional case for $P = (-\Delta)^s$ with $0 < s < 1$ can be readily tackled with such methods.

However, it is fair to say that the study of uniqueness of ground states of linear problems like $(1.1)$ becomes quite elusive in the case of operators $P(D)$ with higher order $2s > 1$. In fact, uniqueness of ground states may fail in such cases. But in certain natural cases of interest (e.g. arising from linearizations around ground states of nonlinear PDEs), the potential $V$ does have the noteworthy property of having a negative Fourier transform $\hat{V} < 0$ almost everywhere. As our first main result in this paper, we prove that ground states for $(1.1)$ are in fact unique (up to a trivial constant) under this condition on $V$.

**Theorem 1** (Uniqueness of Linear Ground States). Let $n \geq 1$, $s > 0$, and suppose that $P(D)$ satisfies Assumption 4. Assume that $V: \mathbb{R}^n \to \mathbb{R}$ has a Fourier transform $\hat{V} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $\hat{V}(\xi) < 0$ for almost every $\xi \in \mathbb{R}^n$. Finally, we suppose that $E_0 < \inf_{\xi \in \mathbb{R}^n} p(\xi)$ holds in $(1.2)$. Then we have the following properties.

(a) **Uniqueness**: The ground state solution $\psi \in H^s(\mathbb{R}^n)$ for $(1.1)$ is unique (up to a constant phase). Moreover, we have the strict positivity property of its Fourier transform

$$e^{i\theta}\hat{\psi}(\xi) > 0 \quad \text{for all } \xi \in \mathbb{R}^n,$$

where $\theta \in \mathbb{R}$ is a constant.

(b) **Symmetry**: Up to a constant phase, the ground state $\psi$ in (a) satisfies

$$\psi(-x) = \overline{\psi(x)} \quad \text{for a.e. } x \in \mathbb{R}^n.$$

If, in addition, the symbol $p(-\xi) = p(\xi)$ is even, then $\psi: \mathbb{R}^n \to \mathbb{R}$ is real-valued (up to a constant phase).

**Remarks.** 1) Under some technical assumptions, we could also treat the non-generic case when $E_0 = \inf_{\xi \in \mathbb{R}^n} p(\xi) = \inf_{\sigma_{ess}}(H)$ coincides with the bottom of the essential spectrum of $H = P(D) + V$. However, we omit this discussion here.

2) Note that $V \in L^{\infty}$ by our assumption that $\hat{V} \in L^1(\mathbb{R}^n)$. As mentioned above, we could relax our conditions to unbounded potentials $V$. But again in order to keep our focus on its simple main argument, we refrain from considering more general cases here.

3) In some sense, the result above yields a Perron-Frobenius type result (i.e. positivity and uniqueness of ground states) but when viewed in Fourier space. Of course, the ground state $\psi(x)$ may fail to be real-valued at all (let alone strictly positive) in $x$-space. In fact, a simple example arises in the linearized problem for traveling solitary waves for dispersion-generalized NLS, e.g., the linear ground state of $\psi \in H^s(\mathbb{R}^n)$ for equations of the form

$$((-\Delta)^s + i v \cdot \nabla + V)\psi = E\psi$$

with $s \geq 1/2$ and $v \in \mathbb{R}^n \setminus \{0\}$ (and $|v| < 1$ when $s = 1/2$). It is easy to see that any nontrivial solutions $\psi \in H^s(\mathbb{R}^n)$ must be complex-valued due to the presence of the ‘boost term’ $i v \cdot \nabla$. However, the result above shows that (under suitable assumptions on $V$), we always have the strict positivity $e^{i\theta}\hat{\psi}(\xi) > 0$ for all $\xi \in \mathbb{R}^n$.

4) If we additionally assume that $\psi \in L^1(\mathbb{R}^n)$ or, more generally, that $\hat{\psi}$ is a finite positive measure on $\mathbb{R}^n$, then $\psi: \mathbb{R}^n \to \mathbb{C}$ is a positive definite function in the sense of Bochner. See also below.

5) Notice since $\hat{V}$ and $V$ are both assumed to be real-valued, the potential $V(-x) = V(x)$ is an even function.
1.2. Nonlinear Results. We now turn to ground state solutions of nonlinear elliptic PDEs in $\mathbb{R}^n$ with pseudo-differential operators $P(D)$ of arbitrary order. As before, let $s > 0$ be a real number. We consider solutions $Q \in H^s(\mathbb{R}^n)$ of nonlinear elliptic PDEs of the form

$$P(D)Q + \lambda Q - |Q|^{2\sigma} Q = 0. \tag{1.4}$$

Here $\sigma > 0$ is a given number, which we later assume to be an integer, and $\lambda \in \mathbb{R}$ denotes a given parameter, which plays the role of a nonlinear eigenvalue. We opted to use the letters $Q$ and $\lambda$ instead of $\psi$ and $E$ above in order to make a clear distinction between the linear and nonlinear problems considered here.

As before, we suppose that $P(D)$ denotes a pseudo-differential operator with constant coefficients defined in Fourier space as

$$(P(D)\tilde{u})(\xi) = p(\xi)\tilde{u}(\xi). \tag{1.5}$$

For the nonlinear problem $\text{(1.4)}$, we now impose the following conditions on $P(D)$, where $S_{1,0}^m$ with $m \in \mathbb{R}$ denotes the usual Hörmander class of symbols for pseudo-differential operators on $\mathbb{R}^n$.

**Assumption 2.** Let $s > 0$ be a real number. We suppose that $P(D)$ is a pseudo-differential operator of order $2s$ having a symbol $p \in S_{1,0}^{2s}$ that satisfies the following conditions.

(i) **Real-Valuedness:** The symbol $p : \mathbb{R}^n \to \mathbb{R}$ is real-valued.

(ii) **Ellipticity Condition:** There exist constants $c > 0$ and $R > 0$ such that

$$p(\xi) \geq c|\xi|^{2s} \quad \text{for} \quad |\xi| \geq R.$$

For the rest of this subsection, we will always assume that $P(D)$ satisfies Assumption 2. As a consequence, the operator $P(D) = P(D)^* = \text{self-adjoint}$ and bounded below on $L^2(\mathbb{R}^n)$ with operator domain $H^{2s}(\mathbb{R}^n)$. Furthermore, we assume the eigenvalue parameter $\lambda \in \mathbb{R}$ in $\text{(1.3)}$ satisfies the condition

$$-\lambda < \inf_{\xi \in \mathbb{R}^n} p(\xi), \tag{1.6}$$

which is equivalent to saying that $-\lambda$ lies strictly below the essential spectrum $\sigma_{\text{ess}}(P(D))$ (in analogy to the condition on $E_0$ in Theorem 1 above). As a direct consequence, we obtain the norm equivalence

$$\langle f, (P(D) + \lambda)f \rangle \simeq \|f\|_{H^{2s}}^2,$$

where $\langle f, g \rangle = \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\xi)d\xi$ denotes the standard scalar product on $L^2(\mathbb{R}^n)$. Likewise, we introduce the critical exponent $\sigma_*(n, s)$ (which is not necessarily an integer) given by

$$\sigma_*(n, s) = \begin{cases} \frac{2s}{n-2s} & \text{for } s < \frac{n}{2}, \\ +\infty & \text{for } s \geq \frac{n}{2}. \end{cases} \tag{1.7}$$

Thus exponents $\sigma < \sigma_*(n, s)$ correspond to the $H^{\alpha}$-subcritical case, which is the situation we shall consider in this paper.² Note that we have the Sobolev-type inequality

$$\|f\|_{L^{2\sigma+2}}^2 \leq C \langle f, (P(D) + \lambda)f \rangle$$

for any $f \in H^s(\mathbb{R}^n)$, where $C > 0$ denotes a suitable constant. Due to the subcriticality $\sigma < \sigma_*(n, s)$, standard variational methods yield existence of an optimal constant $C > 0$ as well as the existence of optimizers $Q \in H^s(\mathbb{R}^n)$ for $\text{(1.6)}$, which are easily seen to solve $\text{(1.4)}$ after a suitable rescaling $Q \mapsto \alpha Q$ with some constant $\alpha$. In fact, we relate this fact to our definition of ground state solutions for $\text{(1.4)}$ as follows.

**Definition 1.** With the notation and assumptions above, we say that $Q \in H^s(\mathbb{R}^n) \setminus \{0\}$ is a **ground state solution** if $Q$ solves equation $\text{(1.4)}$ and optimizes inequality $\text{(1.6)}$.

Equivalently, as shown in Lemma 2.3 below, we obtain that $Q \in H^s(\mathbb{R}^n) \setminus \{0\}$ is a ground state solution for $\text{(1.4)}$ if and only if $Q$ minimizes the action functional

$$A(f) = \frac{1}{2} \langle f, (P(D) + \lambda)f \rangle - \frac{1}{2\sigma+2}\|f\|_{L^{2\sigma+2}}^{2\sigma+2} \tag{1.8}$$

²To avoid technicalities, we shall omit the discussion of the critical case $\sigma = \sigma_*(n, s)$ in this paper.
among all its non-trivial critical points. Thus the set of ground state solutions is given by
\begin{equation}
\mathcal{J} = \{ Q \in K : A(Q) \leq A(R) \text{ for all } R \in K \},
\end{equation}
where \( K = \{ u \in H^s(\mathbb{R}^n) \setminus \{ 0 \} : A'(u) = 0 \} \).

We now turn to the question of symmetries for ground states solutions for (1.4). As consequence of the real-valuedness of the symbol \( p(\xi) \), we notice the reflection-conjugation property
\begin{equation}
(P(D)f)(-x) = \overline{(P(D)f)(x)}.
\end{equation}
Based on this observation, we may ask whether all ground state solutions \( Q \) ‘inherit’ this symmetry property by their variational characterization. In fact, we will prove the following result in this paper when the exponent \( \sigma \in \mathbb{N} \) is an integer.

**Theorem 2** (Symmetry for Nonlinear Ground States). Let \( n \geq 1 \), \( s > 0 \), and \( \sigma \in \mathbb{N} \) with \( 1 \leq \sigma < \sigma_* (s, n) \). Suppose \( Q \in H^s(\mathbb{R}^n) \setminus \{ 0 \} \) is a ground state solution of (1.4) where \( \lambda \in \mathbb{R} \) satisfies (1.8). Finally, we assume that \( e^{\alpha |Q|} Q \in L^2(\mathbb{R}^n) \) for some \( \alpha > 0 \). Then it holds that
\begin{equation}
Q(x) = e^{i\alpha} Q^*(x + x_0)
\end{equation}
with some constants \( \alpha \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^n \). Here \( Q^*: \mathbb{R}^d \to \mathbb{C} \) is a smooth, bounded, and positive definite function in the sense of Bochner. As a consequence, it holds that
\begin{equation}
Q^* (-x) = \overline{Q^*(x)} \quad \text{and} \quad Q^*(0) \geq |Q^*(x)| \quad \text{for all } x \in \mathbb{R}^n.
\end{equation}

If, in addition, the operator \( P(D) \) has an even symbol \( p(\xi) = p(-\xi) \), the function \( Q^* \) must be real-valued (up to a trivial constant complex phase). Consequently, any ground state \( Q \) for (1.4) is real and even, i.e., we have \( Q(-x) = Q(x) \) for all \( x \in \mathbb{R}^n \).

**Remarks.** 1) In Theorem 2 below, we shall give an analyticity condition on \( P(D) \) that ensures the exponential decay property \( e^{\alpha |Q|} Q \in L^2(\mathbb{R}^n) \) for some \( \alpha > 0 \). In particular, it applies to operators of the form
\begin{equation}
P(D) = c_k (-\Delta)^k + \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq m/2 - 1} c_\alpha (-i\partial_\xi)^{\alpha}
\end{equation}
with positive \( c_k > 0 \), \( k \geq 1 \), and real arbitrary coefficients \( c_\alpha \in \mathbb{R} \). For example, we could take \( P(D) = \Delta^2 - \mu \Delta \) with any \( \mu \in \mathbb{R} \). Another important class is given by the pseudo-differential operators
\begin{equation}
P(D) = (1 - \Delta)^s \quad \text{for any } s > 0.
\end{equation}

2) The proof of Theorem 2 will be based on the recent characterization [7] of the case of equality in the Hardy-Littlewood majorant problem in \( \mathbb{R}^n \). Here the topological property that the set \( \Omega = \{ \xi \in \mathbb{R}^n : |Q(\xi)| > 0 \} \) is connected in \( \mathbb{R}^n \) will enter in an essential way.

3) The function \( Q^*: \mathbb{R}^n \to \mathbb{C} \) will be obtained by taking the absolute value on the Fourier side, i.e., we set \( Q^* = |Q|^Z((|\xi| Q)) \). See Section 4 for more details.

4) If the symbol \( p = p(|\xi|) \) is radially symmetric and strictly increasing in \( |\xi| \), then we actually can show that \( Q = Q^Z \) holds (up to translation and complex phase), where \( Q^Z \) denotes the symmetric-decreasing Fourier rearrangement of \( Q \). See [7]. For symbols \( p \) with cylindrical symmetry, we refer to [7].

Next, we turn to the question whether (not necessarily ground state) solutions \( Q \in H^s(\mathbb{R}^n) \) of (1.4) satisfy the exponential decay estimate that \( e^{\alpha |Q|} Q \in L^2(\mathbb{R}^n) \) for some \( \alpha > 0 \), which is a condition imposed in Theorem 2 above. In fact, we can adapt an analytic continuation argument originally developed to study exponential decay of eigenfunctions of Schrödinger operators due to Combes and Thomas [8], building upon O’Conner’s work [9]. Here is a list of sufficient conditions on \( P(D) \) to carry out such an argument in our case.

**Assumption 3.** Suppose \( P(D) \) has a symbol \( p(\xi) \) which has an analytic continuation to the strip \( T_\delta = \{ z \in \mathbb{C}^n : |\text{Im } z| < \delta \} \) with some \( \delta > 0 \). Moreover, we assume the following conditions.

(i) For each \( \kappa \in T_\delta \), there exist constant \( \gamma \in \mathbb{R} \) and \( \theta \in [0, \pi/2) \) such that
\begin{equation}
|\arg(p(\xi + \kappa) - \gamma)| \leq \theta \quad \text{for all } \xi \in \mathbb{R}^n.
\end{equation}
Remark. It is elementary to check that any polynomial
$$f(x) = \sum_{|\alpha| \leq m} c_\alpha x^\alpha$$
with coefficients $c_\alpha \in \mathbb{R}$ and $\inf_{\xi \in \mathbb{R}^n} p(\xi) > -\infty$ satisfies the above conditions (with $m = 2s$). In particular, operators of the form
$$P(D) = \Delta^2 - \mu \Delta + iv \cdot \nabla$$
with $\mu \in \mathbb{R}, v \in \mathbb{R}^n$
fall under the scope of Assumption 3. Also, one can verify that the same is true for polynomials $p$ with
$$\forall \alpha \in \mathbb{N}^n_0 \text{ such that } s|\alpha| > 2s,$$
for some $a > 0$ and $b_0 \in \mathbb{R}$, and that
$$\forall \alpha \in \mathbb{N}^n \text{ such that } s|\alpha| > 2s,$$
for some $a > 0$ and $b_0 \in \mathbb{R}$.

We can now state the following result, which established the assumed exponential decay $e^{a|Q|} \in L^2(\mathbb{R}^n)$ for some $a > 0$ appearing in Theorem 2 above.

**Theorem 3** (Exponential Decay). Let $n, s, \sigma$ and $\alpha$ be as in Theorem 2. If $P(D)$ satisfies Assumption 3, then any solution $Q \in H^s(\mathbb{R}^n)$ of (1.3) satisfies $e^{a|Q|} \in L^2(\mathbb{R}^n)$ for some $a > 0$. As a consequence, the conclusions of Theorem 2 hold true.

Remark. For an in-depth analysis of exponential decay of eigenfunctions of $P(D)+V$ with polynomial symbol $p(\xi)$, we refer to the recent work [5]. However, for our purposes here, it is sufficient to obtain a 'coarse' exponential decay estimate saying that $e^{a|Q|} \in L^2(\mathbb{R}^n)$ for some $a > 0$.

1.3. Strategy of the Proofs. Let us briefly describe the strategy behind the proofs of our main results. The idea to prove Theorems 1 and 2 is based on taking absolute values of the Fourier transform. That is, for a given function $f \in L^2(\mathbb{R}^n)$, we define
$$|f|^* = \mathcal{F}^{-1}(|f|).$$
By Plancherel’s identity, we immediately find that $\|f^*\|_{L^2} = \|f\|_{L^2}$ and $\langle f^*, P(D)f \rangle = \langle f, P(D)f \rangle$. Moreover, for potentials $V : \mathbb{R}^n \to \mathbb{R}$ as in Theorem 1 as well as for integers $\sigma \in \mathbb{N}$ with $1 \leq \sigma < \sigma(s, n)$, we readily obtain the inequalities:
$$\|f^* V f^*\| \leq \langle f, V f \rangle \quad \text{and} \quad \|f^*\|_{L^{2s+2}} \leq \|f\|_{L^{2s+2}}$$
for any $f \in H^s(\mathbb{R}^n)$. Thus if $\psi \in H^s(\mathbb{R}^n)$ and $Q \in H^s(\mathbb{R}^n)$ are ground states for (1.1) and (1.3), respectively, so are the functions $\hat{\psi}^*$ and $\hat{Q}^*$. Therefore, the conclusions of Theorems 1 and 2 will follow once we can show that
$$\hat{\psi}(\xi) = e^{i \theta} |\hat{\psi}(\xi)| \quad \text{and} \quad \hat{Q}(\xi) = e^{i (\alpha + \beta \xi)} |\hat{Q}(\xi)|$$
with some constants $\theta, \alpha, \beta \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$. We remark that $\hat{\psi}$ and $\hat{Q}$ are easily seen to be continuous functions in our setting.

In terms of harmonic analysis, we are faced to solve a phase retrieval problem, i.e., given the modulus of the Fourier transform of a function, we try to reconstruct its phase by exploiting some additional facts. For the linear problem (1.14), this is an elementary task provided that the potential $V$ satisfies the hypothesis of Theorem 1. Not surprisingly, the nonlinear problem (1.3) is harder to analyze. Here, a rigidity result for the so-called Hardy–Littlewood majorant problem in $\mathbb{R}^n$ (recently obtained in [7]) enters in an essential way; see also Lemma 2.4 below. In order to apply this result, we must verify the topological property that
$$\Omega = \{\xi \in \mathbb{R}^n : |\hat{Q}(\xi)| > 0\}$$
is a connected set in $\mathbb{R}^n$. To prove this fact (where indeed we show that $\Omega = \mathbb{R}^n$ holds in our case), we will make use of analyticity argument: By standard Payler–Wiener arguments, the exponential decay $e^{a|Q|} \in L^2(\mathbb{R}^n)$ for some $a > 0$ will ensure that $\hat{Q}(\xi)$ is analytic in some complex strip around $\mathbb{R}^n$. The analyticity of $\hat{Q}$ together with the fact $Q$ solves (1.3) will then yield the desired result.

Finally, we recall from above that the proof of Theorem 3 is based on a strategy for deriving exponential decay for $N$-body Schrödinger operators due to Combes and Thomas [3] based on O’Connor’s lemma [11].

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3 See also the remark following Lemma 2.4 for the case of non-integer $\sigma$. 

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2. Preliminaries

2.1. Fourier Inequalities and Hardy-Littlewood Majorant Problem in \( \mathbb{R}^n \). For a function \( f \in L^1(\mathbb{R}^n) \), we define its Fourier transform by

\[
(\mathcal{F}f)(\xi) \equiv \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi} \, dx,
\]

with the usual extension to \( f \in L^2(\mathbb{R}^n) \) by density. For \( f \in L^2(\mathbb{R}^n) \) given, we recall that the function \( f^* \in L^1(\mathbb{R}^n) \) is obtained by taking the absolute value on the Fourier side, i.e., we set

\[
f^* = |\mathcal{F}f|.
\]

From Plancherel’s identity it is clear that \( \|f\|_{L^2} = \|f^*\|_{L^2} \) holds. We record some further elementary properties of this operation.

Lemma 2.1. Let \( n \geq 1 \), \( s > 0 \), and \( \sigma \in \mathbb{N} \) with \( \sigma < \sigma_+(s,n) \).

(i) For any \( f \in H^s(\mathbb{R}^n) \), we have

\[
(\mathcal{F}f, P(D)f^*) = (f, P(D)f) \quad \text{and} \quad \|f\|_{L^{2s+2}} \leq \|f^*\|_{L^{2s+2}}.
\]

(ii) For any \( f \in L^2(\mathbb{R}^n) \), it holds that \( f^*(-x) = \hat{f}(x) \) for a.e. \( x \in \mathbb{R}^n \).

(iii) If \( f \in L^2(\mathbb{R}^n) \) and \( \hat{f} \in L^1(\mathbb{R}^n) \), then \( f^* : \mathbb{R}^n \to \mathbb{C} \) is a continuous and bounded function which is positive definite in the sense that for any points \( x_1, \ldots, x_N \in \mathbb{R}^n \) the matrix \( [f^*(x_{k} - x_l)]_{1 \leq k, l \leq N} \) is positive semi-definite, i.e.,

\[
\sum_{k,l=1}^{N} f^*(x_k - x_l) = 0 \quad \text{for all } v \in \mathbb{C}^N.
\]

In particular, the inequality \( f^*(0) \geq |f^*(x)| \) holds for all \( x \in \mathbb{R}^n \).

Remark. The inequality \( \|f\|_{L^{2s+2}} \leq \|f^*\|_{L^{2s+2}} \) for integer \( \sigma \in \mathbb{N} \) is a consequence of the so-called upper majorant property (UMP) for \( L^p \)-norms with \( p \in 2\mathbb{N} \cup \{\infty\} \). That is, for such \( p \) and \( f, g \in \mathcal{F}(L^p(\mathbb{R}^n)) \) we have the implication

\[
|\hat{f}(\xi)| \leq |\hat{g}(\xi)| \quad \text{for a.e. } \xi \in \mathbb{R}^n \quad \Rightarrow \quad \|f\|_{L^p} \leq \|g\|_{L^p}.
\]

On the other hand, it is well-known that (UMP) fails for \( L^p \)-norms when \( p \not\in 2\mathbb{N} \cup \{\infty\} \). Indeed, the known counterexamples (see e.g. [1],[8],[9]) show the failure of (UMP) in the torus case, i.e., for \( L^p(\mathbb{T}^n) \). But these examples can be easily transferred to the real line case as follows. Suppose \( p > 2 \) is not an even integer. Then, as shown in [3], there exist trigonometric polynomials \( q \) and \( Q \) with Fourier coefficients \( |\hat{q}(n)| = \hat{Q}(n) \) for all \( n \in \mathbb{Z} \) satisfying \( \|q\|_{L^p(\mathbb{T}^n)} > \|Q\|_{L^p(\mathbb{T}^n)} \). We can lift this example to Fourier transform in \( \mathbb{R}^n \) by considering the Schwartz functions

\[
q_{\lambda}(x) = \lambda^{\frac{n}{2p}} Q(x)e^{-\lambda x^2}, \quad Q_{\lambda}(x) = \lambda^{\frac{1}{2p}} Q(x)e^{-\lambda x^2}
\]

with \( \lambda > 0 \). It is elementary to check that \( \|q_{\lambda}\|_{L^p(\mathbb{R}^n)} \to \|q\|_{L^p(\mathbb{R}^n)} \) and \( \|Q_{\lambda}\|_{L^p(\mathbb{R}^n)} \to \|Q\|_{L^p(\mathbb{R}^n)} \) as \( \lambda \to 0^+ \). Furthermore, we readily check for the Fourier transforms \( |\hat{q}_{\lambda}(\xi)| \leq \hat{Q}_{\lambda}(\xi) \) for all \( \xi \in \mathbb{R}^n \). Thus by taking \( \lambda > 0 \) sufficiently small, we see that (UMP) fails for \( L^p(\mathbb{R}^n) \) with non-even integer \( p \).

Proof. First, it is evident that \( \langle f, P(D)f \rangle = \int_{\mathbb{R}^n} p(\xi)|\hat{f}(\xi)|^2 \, d\xi = \langle f^*, P(D)f^* \rangle \). Next, let \( p = 2\sigma + 2 \) with \( \sigma \in \mathbb{N} \) with \( \sigma < \sigma_+(s,n) \). By Hölder’s inequality, we note that \( f \in H^s(\mathbb{R}^n) \) implies that \( f \in \mathcal{F}(L^p(\mathbb{R}^n)) \), i.e. we have \( \hat{f} \in L^{p'}(\mathbb{R}^n) \), where \( p' = \frac{2p}{2p+2} \) denotes the dual exponent of \( p = 2\sigma + 2 \). Thus we can apply to conclude

\[
\|f\|_{L^{2\sigma+2}}^2 = (\hat{f} \ast \hat{f} \ast \cdots \ast \hat{f})(0)
\]
with $2\sigma + 1$ convolutions on the right-hand side. With the use of the autocorrelation function

$$\Psi_{\hat{f}}(\xi) = (\hat{f} \ast \hat{f})(\xi) = (\hat{f} \ast \hat{f}(-\cdot))(\xi) = \int_{\mathbb{R}^n} \hat{f}(\xi + \xi') \hat{f}(\xi') d\xi',$$

we can write

$$\|f\|_{L^{2\sigma+2}}^2 = (\Psi_{\hat{f}} \ast \cdots \ast \Psi_{\hat{f}})(0),$$

where the number of convolutions is equal to $\sigma$. Since $|\Psi_{\hat{f}}(\xi)| \leq \Psi_{\hat{f}}(\xi)$, we deduce

$$\|f\|_{L^{2\sigma+2}}^2 = (\Psi_{\hat{f}} \ast \cdots \ast \Psi_{\hat{f}})(0) = \|f\|_{L^{2\sigma+2}}^2,$$

which completes the proof of item (i).

The proof of (ii) is a direct consequence of the fact that $\hat{f} = |\hat{f}|$ is real-valued. Furthermore, item (iii) is a classical fact using that $\hat{f}^* = |\hat{f}| \geq 0$ is non-negative and assuming that $\hat{f}^* \in L^1(\mathbb{R}^n)$ (or more generally $\hat{f}^*$ is a finite measure on $\mathbb{R}^n$); see, e.g., for a discussion of positive-definite functions and Bochner’s theorem.

As a next essential fact we recall from [7] the following rigidity result.

**Lemma 2.2** (Equality in the Hardy-Littlewood Majorant Problem in $\mathbb{R}^n$). Let $n \geq 1$ and $p, q \in \mathbb{N} \cup \{\infty\}$ with $p > 2$. Suppose that $f, g \in \mathcal{S}(L^p(\mathbb{R}^n))$ with $1/p + 1/p' = 1$ satisfy the majorant condition

$$|f(\xi)| \leq g(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

In addition, we assume that $\hat{f}$ is continuous and that $\{\xi \in \mathbb{R}^n : |\hat{f}(\xi)| > 0\}$ is a connected set. Then equality

$$\|f\|_{L^p} = \|g\|_{L^p}$$

holds if and only if

$$\hat{f}(\xi) = e^{i(n+\alpha\beta)} \hat{g}(\xi) \quad \text{for all } \xi \in \mathbb{R}^n,$$

with some constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.

**Remark.** The connectedness of the set $\Omega \subset \mathbb{R}^n$ is essential. See also [7] for a counterexample when $\Omega$ is not connected. However, as we will show below, the set $\Omega = \{\xi \in \mathbb{R}^n : |\hat{Q}(\xi)| > 0\}$ will turn out to be connected (in fact, we show $\Omega = \mathbb{R}^n$ holds) for the ground states $Q$ of (2.3) in the setting considered in this paper.

2.2. Smoothness and Exponential Decay of $Q$. Recall that we always suppose that $P(D)$ satisfies Assumptions 2.

**Proposition 2.1.** Let $n \geq 1$, $s > 0$, and $\sigma \in \mathbb{N}$ with $1 \leq \sigma \leq \sigma_s(n, s)$. Then any solution $Q \in H^s(\mathbb{R}^n)$ satisfies $Q \in H^\infty(\mathbb{R}^n) = \bigcap_{k \geq 0} H^k(\mathbb{R}^n)$.

**Proof.** This follows from Sobolev embeddings and regularity theory for pseudo-differential operators. For the reader’s convenience, we give the details. By picking a sufficiently large constant $\mu > 0$, we can assume that $\varphi(\xi) + \mu \gtrsim (\xi^{2s})$ holds. Hence $Q \in H^s(\mathbb{R}^n)$ solves

$$(P(D) + \mu)Q = (QQ)^{\alpha} Q + (\mu - \lambda)Q.$$ (2.3)

Indeed, let us first suppose that $Q \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then $(P(D) + \mu)Q = (QQ)^{\alpha} Q + (\mu - \lambda)Q \in H^s \cap L^\infty(\mathbb{R}^n)$ holds, since $\sigma$ is an integer and $H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ forms an algebra. Now since $\varphi(\xi) + \mu \gtrsim (\xi^{2s})$, we have that $(P(D) + \mu)^{-1}$ belongs to class $L^{2s}$. Therefore $(P(D) + \mu)^{-1} : H^{\infty}(\mathbb{R}^n) \to H^{m-2s}(\mathbb{R}^n)$ for any $m \in \mathbb{R}$ and we deduce that $Q \in H^\infty(\mathbb{R}^n) = \bigcap_{k \geq 0} H^k(\mathbb{R}^n)$ by iterating the equation (2.3).

It remains to show that $Q \in L^\infty(\mathbb{R}^n)$ follows from our assumptions. If $s > n/2$, this is clearly true by Sobolev embeddings. For $0 < s \leq n/2$, we need to bootstrap the equation by using the mapping properties of the inverse $(P(D) + \mu)^{-1}$. Indeed, we note that $|Q|^p Q \in L^{2s\sigma/p}(\mathbb{R}^n)$ with $p_s = 2n/(n - 2s)$ by the Sobolev embedding $H^s(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. Since $(P(D) + \mu)^{-1} : H^{m,p}(\mathbb{R}^n) \to H^{m+2s,p}(\mathbb{R}^n)$ for any $m \in \mathbb{R}$ and $1 < p < \infty$, we deduce that $Q \in H^{2s, p}(\mathbb{R}^n)$, which is a gain of regularity for $Q$. We can proceed this argument to obtain after finitely many steps that $Q \in H^{m,p}(\mathbb{R}^n)$ with $m > n/p$, which yields that $Q \in L^\infty(\mathbb{R}^n)$ by Sobolev embeddings.
2.3. On the Notion of Ground State Solutions. As remarked in the introduction, we have the following simple fact, where we assume \( n, s, \sigma, \) and \( \lambda \) satisfy the assumptions of Theorem 2. Recall the definition of the set \( \mathcal{G} \) in (1.9).

**Lemma 2.3.** \( Q \in H^s(\mathbb{R}^n) \) is a ground state solution of (1.1) if and only if \( Q \in \mathcal{G} \).

**Proof.** Let \( Q, R \in H^s(\mathbb{R}^n) \) be two non-trivial solutions of (1.1). By integrating the equation (1.1) against \( Q \) and \( R \), we find
\[
\langle Q, (P(D) + \lambda)Q \rangle = \| Q \|_{L^{2s+2}}^{2s+2}, \quad \langle R, (P(D) + \lambda)R \rangle = \| R \|_{L^{2s+2}}^{2s+2}.
\]
As a consequence, we get
\[
\mathcal{A}(Q) = \left( \frac{1}{2} - \frac{1}{2s+2} \right) \| Q \|_{L^{2s+2}}^{2s+2}, \quad \mathcal{A}(R) = \left( \frac{1}{2} - \frac{1}{2s+2} \right) \| R \|_{L^{2s+2}}^{2s+2}.
\]
Hence we have the equivalence
\[
\mathcal{A}(Q) \leq \mathcal{A}(R) \iff \| Q \|_{L^{2s+2}} \leq \| R \|_{L^{2s+2}}.
\]
Next, let \( C > 0 \) denote the optimal constant for (1.7). From (2.4) we obtain the bounds
\[
\| Q \|_{L^{2s+2}}^{2s+2} \geq \frac{1}{C^2}, \quad \| R \|_{L^{2s+2}}^{2s+2} \geq \frac{1}{C},
\]
where equality occurs if and only if \( Q \) and \( R \) are optimizers for (1.7), respectively.

Suppose now that \( Q \) is a ground state solution, which means an optimizer for (1.7) by definition. Then we must have \( \| R \|_{L^{2s+2}} \geq \| Q \|_{L^{2s+2}} \). This show that \( Q \in \mathcal{G} \).

On the other hand, let us assume that \( Q \in \mathcal{G} \). To show that \( Q \) must optimize (1.7), we argue by contradiction as follows. Suppose \( Q \) is not an optimizer. Then \( \| Q \|_{L^{2s+2}} > C^{-1} \). But by taking \( R \) to be an optimizer, we deduce that \( C^{-1} = \| R \|_{L^{2s+2}} < \| Q \|_{L^{2s+2}} \), which contradicts that we must have \( \mathcal{A}(Q) \leq \mathcal{A}(R) \).

\( \square \)

3. Proof of Theorem 1

Let \( \psi \in H^s(\mathbb{R}^n) \) be a ground state for (1.1) with \( E = E_0 < \inf_{\xi \in \mathbb{R}^n} p(\xi) \). If we set \( \lambda = -E \), we can write (1.1) in Fourier space as
\[
\hat{\psi}(\xi) = \frac{1}{p(\xi) + \lambda} (\hat{W} \ast \psi)(\xi), \quad \text{with} \quad \hat{W} = -\hat{V}.
\]
Note that \( \hat{W} \in L^2(\mathbb{R}^n) \) by assumption and hence \( \hat{W} \hat{\psi} = \hat{W} \ast \hat{\psi} \) and, moreover, this is a continuous function because it is the convolution of two \( L^2 \)-functions. Since \( p(\xi) + \lambda > 0 \) is also continuous by assumption on \( p \), we deduce that the Fourier transform \( \hat{\psi}(\xi) \) is a continuous function from (3.1).

Next, we claim that
\[
|\hat{\psi}(\xi)| > 0 \quad \text{for all} \quad \xi \in \mathbb{R}^n.
\]
To see this, we first note that
\[
\psi^* = \mathcal{F}^{-1}(\hat{\psi})
\]
is also a ground state solution for (1.1). Indeed, in view of \( \hat{V}(\xi) < 0 \) almost everywhere, we can argue as in the proof of Lemma 2.1 to conclude
\[
\langle \psi, \hat{V} \psi \rangle = \langle \hat{\psi} \ast \hat{\psi} \rangle(0) \geq \langle \hat{\psi} \ast \hat{\psi} \rangle(0) = \langle \psi^*, V \psi^* \rangle,
\]
where we recall that \( \hat{\psi}(\xi) = \int_{\mathbb{R}^n} g(\xi + \eta)g(\eta) d\eta \) is the autocorrelation function of \( g \). Thus from Lemma 2.1 (i) we readily find that
\[
\langle \psi^*, (P(D) + V)\psi^* \rangle \leq \langle \psi, (P(D) + V)\psi \rangle,
\]
whence \( \psi^* \) is also a ground state, since we trivially have \( \| \psi^* \|_{L^2} = \| \psi \|_{L^2} \).

Therefore, in order to show (3.2), we can assume that \( \hat{\psi}(\xi) = |\hat{\psi}(\xi)| > 0 \) is non-negative. But from the assumption that \( \hat{W} = -\hat{V} > 0 \) almost everywhere we deduce that \( (\hat{W} \ast \hat{\psi})(\xi) > 0 \) for all \( \xi \in \mathbb{R}^n \). By the positivity \( p(\xi) + \lambda > 0 \), we immediately deduce that (3.2) holds from (3.1).

Next, we establish the following result.
Proposition 3.1. There exists a constant \( \theta \in \mathbb{R} \) such that
\[
\hat{\psi}(\xi) = e^{i\theta} |\hat{\psi}(\xi)| \quad \text{for all } \xi \in \mathbb{R}^n.
\]

Proof of Proposition 3.1. By the continuity of \( \hat{\psi} \) and the fact that \( |\hat{\psi}(\xi)| > 0 \) for all \( \xi \in \mathbb{R}^n \), there exists a continuous function \( \vartheta : \mathbb{R}^n \to \mathbb{R} \) such that
\[
\hat{\psi}(\xi) = e^{i\vartheta(\xi)} |\hat{\psi}(\xi)| \quad \text{for all } \xi \in \mathbb{R}^n.
\]

Since \( \psi \) and \( \psi^* \) are both ground state solutions for (1.4), we must have equality
\[
(W * \hat{\psi})(0) = (W * \hat{\psi}^*)(0),
\]
with the autocorrelation function \( \Psi_{\psi}(\xi) = \int_{\mathbb{R}^n} g(\xi + \eta)g(\eta)d\eta \). In view of (3.3), we conclude
\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} \hat{W}(\xi)e^{i(|\vartheta(\xi + \eta) - \vartheta(\eta)|)}|\hat{\psi}(\xi + \eta)||\hat{\psi}(\eta)|d\xi d\eta = \int_{\mathbb{R}^n \times \mathbb{R}^n} \hat{W}(\xi)|\hat{\psi}(\xi + \eta)||\hat{\psi}(\eta)|d\xi d\eta.
\]

Since \( W(\xi)|\hat{\psi}(\xi + \eta)||\hat{\psi}(\eta)| > 0 \) for all \((\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n\), we deduce that
\[
\vartheta(\xi + \eta) - \vartheta(\eta) \in 2\pi\mathbb{Z} \quad \text{for all } (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n.
\]
By the continuity of \( \vartheta \), the difference above must be locally constant. Since \( \mathbb{R}^n \times \mathbb{R}^n \) is connected, we infer that
\[
\vartheta(\xi + \eta) - \vartheta(\eta) = c \quad \text{for all } (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n,
\]
with some constant \( c \in 2\pi\mathbb{Z} \). But by choosing \( \xi = 0 \), we see that \( c = 0 \) is the only possibility. From the functional equation (3.3) with \( c = 0 \) we readily deduce that \( \vartheta(\xi) = \vartheta(0) \) for all \( \xi \in \mathbb{R}^n \). Hence \( \vartheta \) is a constant function and by taking \( \theta = \vartheta(0) \) in \( \mathbb{R} \), we complete the proof of Proposition 3.1. \( \Box \)

By applying Proposition 3.1 we complete the proof of Theorem 1 part (i).

The symmetry property in part (ii) directly follows from the fact that \( e^{i\vartheta(\xi)} > 0 \) together with the elementary property \( f(-x) = \overline{f(x)} \) holds a.e. for \( f \in L^2(\mathbb{R}^n) \) whenever \( \hat{f}(\xi) \) is real-valued. Finally, let us suppose that \( p(-\xi) = p(\xi) \) is even. Then \( H = P(D) + V \) is real operator, i.e., we have \( \text{Re}(Hf) = H\text{Re}f \). In particular, we thus choose any eigenfunction of \( H \) to be real-valued and, in particular, this applies to the ground state solution \( \psi \).

The proof of Theorem 1 is now complete. \( \Box \)

4. Proof of Theorem 2

Let \( Q \in H^s(\mathbb{R}^n) \) be a ground state solution as in Theorem 2. We define the set
\[
\Omega = \{ \xi \in \mathbb{R}^n : |\hat{Q}(\xi)| > 0 \}.
\]
This is an open set in \( \mathbb{R} \), since the function \( |\hat{Q}| \) is continuous due to analyticity of \( \hat{Q} \) is analytic by our assumption \( e^{a|\cdot|}Q \in L^2(\mathbb{R}^n) \) for some \( a > 0 \) and using standard Paley–Wiener arguments.

Lemma 4.1. It holds that \( \Omega = \mathbb{R}^n \).

Remark. For non-ground state solutions \( Q \in H^s(\mathbb{R}^n) \) of (1.4), we expect that \( \hat{Q} \) vanishes at certain points. In fact, we expect that the set \( \{|\hat{Q}(\xi)| > 0 \} \) is not connected for non-ground state solutions \( Q \).

Proof. In view of Lemma 4.1, we remark that \( Q^* \in H^s(\mathbb{R}^n) \) is also a ground state solution for (1.4). Hence we can assume that \( |\hat{Q}| \geq 0 \) is non-negative without loss of generality. Next, by applying the Fourier transform to (1.4) and using that \( \sigma \in \mathbb{N} \) is an integer, we get
\[
\hat{Q}(\xi) = \frac{1}{p(\xi) + \lambda} (\hat{Q} * \ldots * \hat{Q})(\xi)
\]
with \( k = 2\sigma + 1 \in \mathbb{N} \) convolutions appearing on the right-hand side. From this identity and Lemma A.2 and iteration, we deduce that \( \Omega \subset \mathbb{R}^n \) must be identical to its \( k \)-fold Minkowski sum, i.e.,

\[
\Omega = \bigoplus_{m=1}^k \Omega \equiv \{ \xi_1 + \ldots + \xi_k : \xi_m \in \Omega \text{ for } m = 1, \ldots, k \}.
\]

For the moment, let us now suppose that

\[
(4.3) \quad \Omega = \bigoplus_{m=1}^k B_r(0) \subset \Omega.
\]

Since \( \Omega \) is open, this implies that \( B_r(0) \subset \Omega \) for some \( r > 0 \). By (4.3) this implies that

\[
\bigoplus_{m=1}^k B_r(0) \subset \Omega.
\]

On the other hand, we readily see that \( B_{2r}(0) \subset B_r(0) \oplus B_r(0) \subset \bigoplus_{m=1}^k B_r(0) \). Iterating this argument, we conclude that

\[
B_{N2r}(0) \subset \Omega \quad \text{for all } N \in \mathbb{N},
\]

whence it follows that \( \Omega = \mathbb{R}^n \) must hold.

Thus it remains to show that (4.4) is true. We argue by contradiction as follows. Suppose that \( 0 \not\in \Omega \) and define the function \( F : \mathbb{R}^n \to \mathbb{R} \) by setting

\[
F(\xi) = \hat{Q}((k - 1)\xi)\hat{Q}(-\xi)
\]

However, we must have

\[
F(\xi) \equiv 0.
\]

Indeed, if \( F(\xi) \neq 0 \) for some \( \xi \in \mathbb{R}^n \) then \( (k - 1)\xi \in \Omega \) and \( -\xi \in \Omega \). This implies that

\[
0 = (k - 1)\xi - \sum_{m=1}^{k-1} \xi_m \in \bigoplus_{m=1}^k \Omega \quad \text{so that } 0 \in \Omega \text{ by (4.3).}
\]

Thus \( 0 \not\in \Omega \) implies that \( F(\xi) \equiv 0 \) vanishes identically. Since \( \hat{Q}((k - 1)\xi) \neq 0 \), this yields that the function \( \hat{Q}(-\xi) \) must vanish on some non-empty open set in \( \mathbb{R}^n \). By the (real) analyticity of \( \hat{Q} : \mathbb{R}^n \to \mathbb{R} \) this implies \( \hat{Q} \equiv 0 \) on \( \mathbb{R}^n \). But this is a contradiction.

Thus we have shown that (4.4) holds, which completes the proof. \( \square \)

With the result of Lemma 4.1 at hand, we are ready to finish the proof of Theorem 2. Indeed, if \( Q \in H^s(\mathbb{R}^n) \) is a ground state solution, we must necessarily have the equality

\[
\|Q\|_{L^{2s+2}} = \|Q^*\|_{L^{2s+2}}.
\]

But we can apply Lemma 2.2 with \( f = Q \) and \( g = Q^* \) to conclude that

\[
\hat{Q} = e^{i(\alpha + \beta \xi)}|\hat{Q}(\xi)|
\]

for all \( \xi \) with some constants \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R}^n \). Hence we find

\[
Q(x) = e^{i\theta}Q^*(x + x_0)
\]

with the constant \( x_0 = -\frac{1}{2\pi} \beta \in \mathbb{R}^n \). The asserted properties of \( Q^* \) now follow from Lemma 2.3 together with the fact that \( Q^* \in L^1(\mathbb{R}^n) \), since we have \( (1 + |\xi|)^m \hat{Q} \in L^2(\mathbb{R}^n) \) for \( m > n/2 \) by Proposition 2.4.

Finally, let us additionally assume that the symbol

\[
p(-\xi) = p(\xi)
\]

is even. In this case, we can adapt a trick from [3] (see also Lemma A.3) to show that any ground state \( Q \in H^s(\mathbb{R}^n) \) must be real-valued up to a trivial constant complex phase, i.e., we claim that

\[
(4.5) \quad e^{i\theta}Q(x) \in \mathbb{R} \quad \text{for all } \xi \in \mathbb{R}^n
\]

with some constant \( \theta \in \mathbb{R} \). To prove this, we decompose

\[
Q = Q_R + iQ_I
\]

into real and imaginary part. If either \( Q_R \equiv 0 \) or \( Q_I \equiv 0 \), then there is nothing is left to prove. Hence we assume that both parts are non-trivial. From Lemma A.3 we obtain

\[
(4.6) \quad \langle Q, (P(D) + \lambda)Q \rangle = \langle Q_R, (P(D) + \lambda)Q_R \rangle + \langle Q_I, (P(D) + \lambda)Q_I \rangle =: D_R + D_I,
\]

\[
(4.7) \quad \|Q\|^2_{L^{2s+2}} \leq \|Q_R\|^2_{L^{2s+2}} + \|Q_I\|^2_{L^{2s+2}} =: N_R + N_I.
\]
Now let $C > 0$ denote the optimal constant for (1.7). Since $Q$ is an optimizer, we deduce
\[ C = \frac{\|Q\|^2_{L^2(\mathbb{R}^n)}}{\langle f, (P(D) + \lambda)f \rangle} \leq \frac{N_R + N_I}{D_R + D_I} \leq \max \left( \frac{N_R}{D_R}, \frac{N_I}{D_I} \right) \leq C. \]
This shows that we must have equality in (1.7), which by Lemma 3.1 and $Q_R \neq 0 \neq Q_I$ implies that there is some constant $\alpha > 0$ such that $Q_I^2 = \alpha^2 Q_R^2$. We want to establish $Q_I = \pm \alpha Q_R$. To do so, we apply Lemma 3.1 now to the decomposition
\[ Q = e^{is/4}Q_a + ie^{is/4}Q_b \]
with real-valued functions $Q_a$ and $Q_b$. In fact, an elementary computation shows that $Q_a = \frac{1}{\sqrt{2}}(Q_R + Q_I)$ and $Q_b = \frac{1}{\sqrt{2}}(-Q_R + Q_I)$. We still have $\langle Q(x) \rangle^2 = Q_a(x)^2 + Q_b(x)^2$ and also $\langle Q, (P(D) + \lambda)Q \rangle = \langle Q_a, (P(D) + \lambda)Q_a \rangle + \langle Q_b, (P(D) + \lambda)Q_b \rangle$ by using that $p(-\xi) = p(\xi)$ is even. Now if $Q_a \equiv 0$, then we are done since $Q_I = -Q_R$ in this case. If $Q_a \neq 0$, we obtain $Q^2 = \beta^2 Q^2_R$ with some constant $\beta > 0$. Note that $\beta^2 \neq 1$ because otherwise this would imply $Q_R Q_I \equiv 0$ (which would yield $Q \equiv 0$ from using $Q_I^2 = \alpha^2 Q_R^2$).

In summary, we conclude
\[ Q_I^2 = \alpha^2 Q_R^2 \text{ and } 1 + (1 + \beta^2) Q^2_R = (1 + \beta^2) Q_R Q_I. \]
But this implies that $Q_I = \pm \alpha Q_R$, which proves that (1.5) is true.

The proof of Theorem 2 is now complete. \qed

5. Proof of Theorem 3

We will adopt a novel idea due Combes and Thomas [2] who proved exponential decay of eigenfunctions for $N$-body Schrödinger operators by an analytic continuation argument, which is based on O’Connor’s Lemma (see Lemma 3.3 below) together with standard analytic perturbation theory; see [3][4].

We define the operator $H = P(D) + V$ with $V = -Q^{2\sigma}$ acting on $L^2(\mathbb{R}^n)$. Note that $V \in L^\infty(\mathbb{R}^n)$ is bounded by Proposition 2.1. Hence, by standard theory, the operator $H$ is self-adjoint with operator domain $H^2(\mathbb{R}^n)$. In particular, we see that $Q$ is an $L^2$-eigenfunction of $H$ satisfying
\[ HQ = -\lambda Q. \]
Since $V(x) \to 0$ as $|x| \to \infty$, we have $\sigma_{ess}(H) = \sigma_{ess}(P(D)) = \inf_{\xi \in \mathbb{R}^n} p(\xi)$. By our assumption (1.6), we see that the eigenvalue $-\lambda$ lies strictly below the essential spectrum of $H$.

We shall now implement an analytic continuation argument to show that $e^{\lambda t}Q \in L^2(\mathbb{R}^n)$ must hold for some sufficiently small $a > 0$. To do so, we adapt an argument due to Combes and Thomas as follows. For real $\kappa \in \mathbb{R}^n$, we can define the unitary operators
\[ (U(\kappa) f)(x) = e^{2\pi i \kappa \cdot x} f(x) \]
acting on $L^2(\mathbb{R}^n)$. Likewise, we consider the family of unitarily equivalent operators
\[ H(\kappa) = U(\kappa) H U(\kappa)^{-1}. \]
We readily find that
\[ U(\kappa) P(D) U(\kappa)^{-1} = P_\kappa(D), \quad U(\kappa) V U(\kappa)^{-1} = V, \]
where $P_\kappa(D)$ has the shifted symbol $p(\xi + \kappa)$.

Now, by standard Paley-Wiener theory, we note that if $U(\kappa)Q$ has an analytic continuation for $|\Im \kappa| < \delta$ then $e^{\lambda t}Q \in L^2(\mathbb{R}^n)$ for all $0 < a < \delta$, which would finish the proof. To see that $U(\kappa)Q$ can be analytically continued if $|\Im \kappa| < \delta$ for some $\delta > 0$, we prove that $H(\kappa)$ is an analytic family of type (B) on the complex strip $T_\delta$. We use a form argument as follows. For any $\kappa \in T_\delta$, we can define the quadratic form
\[ q(\kappa)(f, f) = \int_{\mathbb{R}^n} p(\xi + \kappa) |\tilde{f}(\xi)|^2 d\xi + \int_{\mathbb{R}^n} |f|^2 dx \quad \text{for} \quad f \in H^2(\mathbb{R}^n). \]
We claim that $\{q(\kappa)\}_{\kappa \in T_\delta}$ is an analytic family of quadratic forms of type (b) with form domain $H^2(\mathbb{R}^n)$ (in the nomenclature of [10]). That is, we have the following properties.

1. For each $\kappa \in T_\delta$, the form $q(\kappa)$ is closed and strictly $m$-sectorial with domain $H^2(\mathbb{R}^n)$.
Thus we deduce that \( E(\kappa_0) \) of \( H(\kappa_0) \) moves analytically for \( \kappa \) close to \( \kappa_0 \). But if \( \text{Im}(\kappa - \kappa_0) = 0 \), we have that \( E(\kappa) = E(\kappa_0) \) since the operators \( H(\kappa) \) and \( H(\kappa_0) \) are unitarily equivalent in this case. Hence \( E(\kappa) \) is constant and remains an eigenvalue as long as it stays away from \( \sigma_{\text{ess}}(H(\kappa)) \).

Now, by standard perturbation theory, any discrete eigenvalue \( E(\kappa_0) \) of \( H(\kappa_0) \) moves analytically as long as \( \kappa \) remains close to \( \kappa_0 \). From the fact that \( q(\kappa) \) is an analytic family of form of type (B) it follows that the set of associated operators \( \{ H(\kappa) \}_{\kappa \in T_\kappa} \) defines an analytic family of operators of type (B).

Indeed, by Assumption 3 item (i), we see that \( r > 0 \) or

\[
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\]

(2) For each \( f \in H^s(\mathbb{R}^n) \), the function \( \kappa \mapsto q(\kappa)[f, f] \) is analytic in \( \kappa \in T_\kappa \).

Now, suppose that \( f \in H^s(\mathbb{R}^n) \) with \( f(\cdot) = e^{i \vartheta} f_R(\cdot) + i e^{i \vartheta} f_I(\cdot) \) with some constant \( \vartheta \in \mathbb{R} \) and real-valued functions \( f_R, f_I : \mathbb{R}^n \to \mathbb{R} \). Then we have

\[
(f, (P(D) + \lambda)f) = (f_R, (P(D) + \lambda)f_R) + (f_I, (P(D) + \lambda)f_I),
\]

where equality holds if and only if \( f_I = 0 \) or \( f_R = 0 \) with some constant \( \mu \geq 0 \).

Proof. By subtracting the constant \( \lambda \) from \( p(\xi) \), we can assume without loss of generality that \( \lambda = 0 \) holds. Since \( f_R, f_I : \mathbb{R}^n \to \mathbb{R} \) are real-valued, their Fourier transforms satisfy

\[
\hat{f}_R(-\xi) = \bar{f}_R(\xi) \quad \text{and} \quad \hat{f}_I(-\xi) = f_I(\xi).
\]

Using that \( p(-\xi) = p(\xi) \) is even and \( |e^{i \vartheta} z| = |z| \) for all \( z \in \mathbb{C} \), we calculate

\[
\langle f, (P(D)f) \rangle = \int_{\mathbb{R}^n} p(\xi) \hat{f}_R(\xi) + i \hat{f}_I(\xi) d\xi = \int_{\mathbb{R}^n} \left[ \hat{f}_R(\xi) \right]^2 d\xi + \int_{\mathbb{R}^n} p(\xi) |\hat{f}_I(\xi)|^2 d\xi
\]

\[
+ i \int_{\mathbb{R}^n} p(\xi) \left[ \overline{\hat{f}_R(\xi)} \hat{f}_I(\xi) - \overline{\hat{f}_R(\xi)} \hat{f}_I(\xi) \right] d\xi = (f_R, P(D)f_R) + (f_I, P(D)f_I),
\]

as claimed.

Appendix A. Auxiliary Results

Lemma A.1. Suppose \( P(D) \) satisfies Assumption 4 with some \( s > 0 \) and its multiplier \( p(-\xi) = p(\xi) \) is an even function and let \( \lambda \in \mathbb{R} \). Let \( f \in H^s(\mathbb{R}^n) \) with \( f : \mathbb{R}^n \to \mathbb{C} \) be of the form

\[
f(x) = e^{i \vartheta} f_R(x) + i e^{i \vartheta} f_I(x)
\]

with some constant \( \vartheta \in \mathbb{R} \) and real-valued functions \( f_R, f_I : \mathbb{R}^n \to \mathbb{R} \). Then we have

\[
\| f \|_{L^q}^2 \leq \| f_R \|_{L^q}^2 + \| f_I \|_{L^q}^2,
\]

where equality holds if and only if \( f_I = 0 \) or \( f_R = 0 \) with some constant \( \mu \geq 0 \).
Remark. We could also allow that \((f * g)(x) = +\infty\) for some \(x \in \mathbb{R}^n\) and the result remains valid. But since we apply this lemma iteratively in the proof of Theorem 2, we assume that \((f * g)(x) < +\infty\) for all \(x \in \mathbb{R}^n\).

Proof. The proof is elementary. For the reader’s convenience, we give the details.

Let us write \(\Omega = \{ f > 0 \}, \Omega_\theta = \{ g > 0 \}\) and \(\Omega_{f \oplus g} = \{ f * g > 0 \}\). We suppose that both \(f \not\equiv 0\) and \(g \not\equiv 0\), since otherwise the claimed result trivially follows.

First, we show that \(\Omega_{f \oplus g} = \Omega_f \ominus \Theta\). Let \(x = x_1 + x_2\) with \(x_1 \in \Omega_F\) and \(x_2 \in \Omega_g\). By continuity of \(f\) and \(g\), there exists some \(\varepsilon > 0\) such that \(f > 0\) on \(B(x_1, \varepsilon)\) and \(g > 0\) on \(B(x_2, \varepsilon)\). Thus, by using that \(f \not\equiv 0\) and \(g \not\equiv 0\) on all of \(\mathbb{R}^n\), we get

\[
(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\,dy \geq \int_{B(x_2)} f(x_1 + x_2 - y)g(y)\,dy > 0,
\]

since \(x_1 + x_2 - y \in B(x_1)\) when \(y \in B(x_2)\). This shows that \(\Omega_f \ominus \Theta \subset \Omega_{f \oplus g}\).

Next, we prove that \(\Omega_{f \oplus g} \subset \Omega_f \ominus \Theta\) holds. Indeed, for every \(x \in \mathbb{R}^n\), we can write

\[
(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\,dy = \int_{(\{x\} \ominus \Omega_f) \cap \Omega_g} f(x - y)g(y)\,dy,
\]

since \(f(x - \cdot) \equiv 0\) on \(\mathbb{R}^n \setminus \{x\} \ominus \Omega_f\) and \(g \equiv 0\) on \(\mathbb{R}^n \setminus \Omega_g\), where we denote \(A \ominus B = \{a - b : a \in A, b \in B\}\) for subsets \(A\) and \(B\) in \(\mathbb{R}^n\). However, if \(x \not\in \Omega_f \ominus \Theta\) then \((\{x\} \ominus \Omega_f) \cap \Omega_g = \emptyset\). Thus \((f * g)(x) = 0\) for any \(x \not\in \Omega_f \ominus \Theta\), whence it follows that the inclusion \(\Omega_{f \oplus g} \subset \Omega_f \ominus \Theta\) is valid.

Lemma A.3 (O’Connor’s Lemma [11]). Let \(H\) be a Hilbert space and suppose \(U(\kappa)\) are unitary operators on \(H\) parametrized by \(\kappa \in \mathbb{R}^n\). Let \(P\) be a finite-rank projection on \(H\) such that that \(P(\kappa) = U(\kappa)PU(\kappa)^{-1}\) has an analytic continuation to \(D = \{ z \in \mathbb{C}^n : \text{Im} z < a\}\) for some \(a > 0\). Then any \(f \in \text{ran} P\) has an analytic continuation from \(D \cap \mathbb{R}\) to \(D\) given by \(f(\kappa) = U(\kappa)f\).

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