SOLUTIONS OF MULTI-COMPONENT FRACTIONAL SYMMETRIC SYSTEMS

MOSTAFA FAZLY

Abstract. We study the following elliptic system concerning the fractional Laplacian operator
\[ (-\Delta)^{s_i} u_i = H_i(u_1, \cdots, u_m) \text{ in } \mathbb{R}^n, \]
when \( 0 < s_i < 1, \) \( u_i : \mathbb{R}^n \to \mathbb{R} \) and \( H_i \) belongs to \( C^{1,\gamma}(\mathbb{R}^m) \) for \( \gamma > \max(0,1-2\min\{s_i\}) \) \( 1 \leq i \leq m. \)

The above system is called symmetric when the matrix \( H = (\partial_i H_j(u_1, \cdots, u_m))_{i,j=1}^m \) is symmetric. The notion of symmetric systems seems crucial to study this system with a general nonlinearity \( H \) the fractional Laplacian operator on \( \partial \).

We establish De Giorgi type results for stable and \( H \)-monotone solutions of symmetric systems in lower dimensions that is either \( n = 2 \) and \( 0 < s_i < 1 \) or \( n = 3 \) and \( 1/2 \leq \min\{s_i\} < 1. \) The case that \( n = 3 \) and at least one of parameters \( s_i \) belongs to \((0,1/2)\) remains open as well as the case \( n \geq 4 \). Applying a geometric Poincaré inequality, we conclude that gradients of components of solutions are parallel in lower dimensions when the system is coupled. More precisely, we show that the angle between vectors \( \nabla u_i \) and \( \nabla u_j \) is exactly \( \arccos(\langle \partial_i H_i(u)/\partial_j H_i(u) \rangle). \) In addition, we provide Hamiltonian identities, monotonicity formulae and Liouville theorems. Lastly, we apply some of our main results to a two-component nonlinear Schrödinger system, that is a particular case of the above system, and we prove Liouville theorems and monotonicity formulae.

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1. Introduction

We examine the following system of nonlocal elliptic equations for \( 1 \leq i \leq m, \)
\[ (-\Delta)^{s_i} u_i = H_i(u_1, \cdots, u_m) \text{ in } \mathbb{R}^n, \]
when \((\Delta)^{s_i}\) stands for the fractional Laplacian operator and \( u_i : \mathbb{R}^n \to \mathbb{R}. \) We assume that each parameter \( s_i \) belongs to \((0,1)\) and \( H = (H_i)_{i=1}^m \) is a sequence of functions that each component \( H_i \) belongs to \( C^{1,\gamma}(\mathbb{R}^m) \) for \( \gamma > \max(0,1-2\min\{s_i\}). \) We also assume that the nonlinearity \( H \) satisfies

\[ \partial_i H_j(u)\partial_j H_i(u) > 0 \text{ when } \partial_j H_i(u) := \frac{\partial H_i(u)}{\partial u_j} \text{ for } 1 \leq i < j \leq m. \]

More recently, the fractional Laplacian operator \((\Delta)^{s_i}\) has been of great interests in the literature and various properties of the operator are explored. There are different mathematical approaches to define this operator. The fractional Laplacian operator on \( \mathbb{R}^n \) can be defined as a pseudo-differential operator using the Fourier transform
\[ (-\Delta)^{s_i} w(\zeta) = |\zeta|^{2s_i} \hat{w}(\zeta), \]
when the hat operator is given by
\[ \hat{w}(\zeta) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} w(x)e^{-i\zeta \cdot x} dx. \]

Suppose that each \( u_i \in C^{2\sigma}(\mathbb{R}^n) \) for \( \sigma > s_i > 0 \) and \( \int_{\mathbb{R}^n} \frac{|w_i(z)|}{(1+|z|)^{n+2\sigma}} dz < \infty. \) Then, the fractional Laplacian of \( u_i \) can be also defined as
\[ (-\Delta)^{s_i} u_i(x) := P.V. \int_{\mathbb{R}^n} \frac{u_i(x) - u_i(z)}{|x-z|^{n+2s_i}} dz, \]

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for every $x \in \mathbb{R}^n$ where P.V. stands for the principal value. It is by now standard that the fractional Laplacian $(-\Delta)^s$ operator where $s$ is any positive, noninteger number can be denoted as the Dirichlet-to-Neumann map for an extension function satisfying a higher order elliptic equation in the upper half space with one extra spatial dimension, see [11] by Caffarelli and Silvestre. To be mathematically more precise, let $s_i \in (0, 1)$ and $u_i \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1 + |z|)^{n+2s_i} dz)$ when $\sigma > s_i$ for all $i = 1, \cdots, m$. For $\bar{x} = (x, y)$ in $\mathbb{R}^{n+1}_+$, set
\begin{equation}
(1.6) \quad v_i(\bar{x}) = \int_{\mathbb{R}^n} P_i(x - z, y) u_i(z) \, dz \quad \text{when} \quad P_i(x, y) = p_{n,s_i} \frac{y^{2s_i}}{(|x|^2 + |y|^2)^{\frac{n+2s_i}{2}}},
\end{equation}
and $p_{n,s_i}$ is a normalizing constant. Then, $v = (v_i)_{i=1}^m$ satisfies the following system
\begin{equation}
(1.7) \quad \begin{cases}
\text{div}(y^{1-2s_i} \nabla v_i) = 0 & \text{in} \quad \mathbb{R}^{n+1}_+ \\
v_i = u_i & \text{on} \quad \partial \mathbb{R}^{n+1}_+ \\
- \lim_{y \to 0} y^{1-2s_i} \partial_y v_i = d_{s_i}(-\Delta)^s_i u_i & \text{on} \quad \partial \mathbb{R}^{n+1}_+,
\end{cases}
\end{equation}
for the constant
\begin{equation}
(1.8) \quad d_{s_i} = \frac{\Gamma(1-s_i)}{2^{2s_i-1} \Gamma(s_i)},
\end{equation}
and $v_i \in C^2(\mathbb{R}^{n+1}_+) \cap C(\mathbb{R}^{n+1}_+)$ and $y^{1-2s_i} \partial_y v_i \in C(\mathbb{R}^{n+1}_+)$. In short, the extension function $v = (v_i)_{i=1}^m$ satisfies the system
\begin{equation}
(1.9) \quad \begin{cases}
\text{div}(y^{s_i} \nabla v_i) = 0 & \text{in} \quad \mathbb{R}^{n+1}_+ = \{x \in \mathbb{R}^n, y > 0\}, \\
- \lim_{y \to 0} y^{a_i} \partial_y v_i = d_{s_i} H_i(v) & \text{in} \quad \partial \mathbb{R}^{n+1}_+ = \{x \in \mathbb{R}^n, y = 0\},
\end{cases}
\end{equation}
when $a_i = 1 - 2s_i$. For the sake of simplicity, we fix the following notation.

**Notation 1.1.** Suppose that $0 < s_i < 1$ for $1 \leq i \leq m$. Then, $s_* = \min_{1 \leq i \leq m} \{s_i\}$ and $s^* = \max_{1 \leq i \leq m} \{s_i\}$.

We now define the notion of stable solutions by linearizing system (1.9).\n
**Definition 1.1.** A solution $v = (v_i)_{i=1}^m$ of system (1.9) in $\mathbb{R}^{n+1}_+$ is called stable, if there exists a sequence $\phi = (\phi_i)_{i=1}^m$ in $C^\infty(\mathbb{R}^{n+1}_+)$ satisfying
\begin{equation}
(1.10) \quad \begin{cases}
\text{div}(y^{a_i} \nabla \phi_i) = 0 & \text{in} \quad \mathbb{R}^{n+1}_+, \\
- \lim_{y \to 0} y^{a_i} \partial_y \phi_i = d_{s_i} \sum_{j=1}^m \partial_j H_i(v) \phi_j & \text{in} \quad \partial \mathbb{R}^{n+1}_+,
\end{cases}
\end{equation}
when $\partial_j H_i(v) \phi_j \phi_i > 0$ for all $i, j$ and each $\phi_i$ does not change sign for all $i = 1, \ldots, m$.

In 1978, De Giorgi [20] conjectured that bounded monotone solutions of the Allen-Cahn equation
\begin{equation}
(1.11) \quad \Delta u + u - u^3 = 0 \quad \text{in} \quad \mathbb{R}^n,
\end{equation}
are one-dimensional solutions at least up to eight dimensions. This conjecture is known to be true for dimensions $n = 2, 3$ by Ghoussoub-Gui in [36] and Ambrosio-Cabré in [2], respectively. There is an example by del Pino, Kowalczyk and Wei in [21] that shows the dimension eight is the critical dimension. For dimensions $4 \leq n \leq 8$ there are various partial results under certain extra (natural) assumptions on solutions by Ghoussoub and Gui in [37], Savin in [47] and references therein. The remarkable point is that in lower dimensions, that is when $n \leq 3$, this conjecture holds for the scalar equation
\begin{equation}
(1.12) \quad -\Delta u = H(u) \quad \text{in} \quad \mathbb{R}^n,
\end{equation}
with a general nonlinearity $H \in C^1(\mathbb{R})$. In this regard, we refer interested readers to [36] by Ghoussoub and Gui for $n = 2$ and to [2] by Alberti, Ambrosio and Cabré for $n = 3$ and also to [28] by Farina, Sciunzi and Valdinoci for a geometric approach in two dimensions. When the Laplacian operator in (1.12) is replaced with the fractional Laplacian operator $(-\Delta)^s$, De Giorgi type results are known when either $n = 2$ and $0 < s < 1$ or $n = 3$ and $1/2 < s < 1$, see [8–10, 49]. However, in three dimensions and when $0 < s < 1/2$ the problem seems to be more challenging and remains open.

More recently, Ghoussoub and the author in [31] provided De Giorgi type results up to three dimensions for the following elliptic gradient systems when $u : \mathbb{R}^n \to \mathbb{R}^m$
\begin{equation}
(1.13) \quad -\Delta u = \nabla H(u) \quad \text{in} \quad \mathbb{R}^n,
\end{equation}
for a general nonlinearity $H \in C^2(\mathbb{R}^m)$. In this regard, authors introduced the notions of orientable systems and $H$-monotone solutions to adjust and to apply the mathematical techniques and ideas given for the scalar equation (1.12) to (1.13). It seems that these concepts are essential, in this context, to explore system of equations. Note also that Sire and the author in [32] studied system (1.13), when the Laplacian operator is replaced with the fractional Laplacian operator $(-\Delta)^s$, and provided De Giorgi type results for certain parameters $s = (s_1, \cdots, s_n) \in \mathbb{R}^n$ in lower dimensions.

**Definition 1.2.** System (1.9) is called orientable, if there exist nonzero functions $\theta_k \in C^1(\mathbb{R}^{n+1}_+)$, $k = 1, \cdots, m$, which do not change sign, such that for all $i, j$ with $1 \leq i < j \leq m$,

\[
(1.14) \quad \partial_j H_i(u) \theta_i(x) \theta_j(x) > 0 \quad \text{for all } x \in \mathbb{R}^n.
\]

Similarly, if the condition (1.14) holds for the extension function $v$, then we say (1.9) is orientable.

Note that the orientability imposes a combinatorial assumption on the sign of the nonlinearity $H = (H_i)_i^{m}$ and therefore on the system.

**Definition 1.3.** We say that a solution $u = (u_i)_{i=1}^m$ of (1.1) is $H$-monotone if the following hold,

1. For every $i \in \{1, \cdots, m\}$, $u_i$ is strictly monotone in the $x_n$-variable (i.e., $\partial_n u_i \neq 0$).
2. For $i < j$, we have

\[
(1.15) \quad \partial_j H_i(u) \partial_n u_i(x) \partial_n u_j(x) > 0 \quad \text{for all } x \in \mathbb{R}^n.
\]

Similarly, if above conditions hold for the extension function $v$ then we say that $v$ is $H$-monotone.

It is straightforward to observe that $H$-monotonicity implies stability. One can show this by differentiating (1.1) with respect to the variable $x_n$ and setting $\phi_i := \partial_n u_i$. As the last definition of this section, we provide the notion of symmetric systems that plays a key role in our main results.

**Definition 1.4.** We call system (1.1) symmetric if the matrix of gradient of all components of $H$ that is

\[
(1.16) \quad H := (\partial_j H_i(u))_{i,j=1}^m
\]

is symmetric. Similarly, if $H := (\partial_j H_i(v))_{i,j=1}^m$ is symmetric for the extension $v$, then we say (1.9) is symmetric.

In this article, we prove De Giorgi type results for stable and $H$-monotone solutions of symmetric system (1.9) with a general nonlinearity $H$ when $n = 2$ and $0 < s_1 < 1$ or $n = 3$ and $1/2 \leq s_1 < 1$, see Theorem 4.2. We also provide Liouville theorems for stable solutions of (1.9) under some extra assumptions on the nonlinearity $H$ when $n \leq 2(1 + s_1)$, see Theorem 4.4. In addition, we establish the following geometric Poincaré inequality for bounded stable solutions of (1.9). Let $\eta = (\eta_k)_{k=1}^m \in C^1_c(\mathbb{R}^{n+1}_+)$, then

\[
(1.17) \quad \sum_{i=1}^m \frac{1}{d_{s_i}} \int_{\{|\nabla v_i| \neq 0\} \cap \mathbb{R}^{n+1}_+} y^{1-2s_i} \left( |\nabla v_i|^2 A_2^2 + |\nabla T_i | |\nabla v_i|^{2} \right) \eta_k^2 \, dx
\]

\[
(1.18) \quad + \sum_{i \neq j} \int_{\mathbb{R}^{n+1}_+} \left( \sqrt{\partial_j H_i(v)} \partial_i H_j(v) |\nabla v_i| |\nabla v_j| |\eta_i \eta_j| - \partial_j H_i(v) \nabla v_i \cdot \nabla v_j \eta_i^2 \right) \, dx
\]

\[
(1.19) \quad \leq \sum_{i=1}^m \frac{1}{d_{s_i}} \int_{\mathbb{R}^{n+1}_+} y^{1-2s_i} |\nabla v_i| |\nabla \eta_i| |^2 \, dx,
\]

where $\nabla T_i$ stands for the tangential gradient along a given level set of $v_i$ and $A_2^2$ for the sum of the squares of the principal curvatures of such a level set, see Theorem 3.2. Note that for the case of scalar equations, $m = 1$, a similar inequality was established by Sternberg and Zumbrun [50] to study phase transitions and area-minimizing surfaces. For this case, the boundary term (1.18) disappears. The idea of applying the geometric Poincaré inequality to prove De Giorgi type results was initiated by Farina, Sciunzi, Valdinoci in [28] and references therein. Ghoussoub and the author in [31] established a counterpart of this inequality for local systems of the form (1.13). Note that for symmetric systems, $m \geq 2$, the boundary term (1.18) becomes

\[
(1.20) \quad \sum_{i \neq j} \int_{\partial \mathbb{R}^{n+1}_+} \partial_j H_i(v) \left( |\nabla v_i| |\nabla v_j| |\eta_i \eta_j| - \nabla v_i \cdot \nabla v_j \eta_i^2 \right) \, dx,
\]
where the integrand has a fixed sign for an appropriate test function $\eta = (\eta_i)_{i=1}^m$. In the light of this inequality, we prove De Giorgi type results in two dimensions and we show that vectors $\nabla x v_i(x, 0)$ and $\nabla x v_j(x, 0)$ for $i \neq j$ are parallel and the angle between two vectors is arccos $\left(\frac{\partial_x H_i(v)}{\nabla x H_i(v)}\right)$.

Consider the scalar equation (1.12) when $\tilde{H} \geq 0$ for $\tilde{H}(u) = -H(u)$. Modica in [44] proved that the following pointwise estimate holds for bounded solutions

$$\|\nabla u\|^2 \leq 2\tilde{H}(u) \quad \text{in} \quad \mathbb{R}^n.$$  

This inequality has been used in the literature to study entire solutions of semilinear elliptic equations and, in particular, to establish De Giorgi type results, see [1, 2, 4, 21, 36, 37, 39, 47]. Unfortunately, a counterpart of this inequality does not hold for the system of equations of the form (1.13) with a general nonlinearity. However, assuming that $n = 1$ and $m \geq 1$ and multiplying the $i^{th}$ equation of (1.13) with $u_i'$ and integrating, it is straightforward to observe that the following Hamiltonian identity holds

$$\frac{1}{2} \sum_{i=1}^m |u_i'(x)|^2 + H(u(x)) \equiv C \quad \text{for} \quad x \in \mathbb{R},$$  

when $C$ is a constant. Gui in [39] examined system (1.13) when $n \geq 1$ and $m \geq 1$ and proved the following elegant Hamiltonian identity in higher dimensions

$$\int_{\mathbb{R}^n} \left[ \sum_{i=1}^m \frac{1}{2} \left( |\nabla x u_i(x)|^2 - |\partial_n u_i(x)|^2 \right) - H(u(x)) \right] dx' \equiv C \quad \text{for} \quad x_n \in \mathbb{R},$$  

where $x = (x', x_n) \in \mathbb{R}^n$ and $C$ is a constant. In this paper, as Theorem 3.1, we provide a counterpart of (1.23) for the fractional system (1.9) in one dimension that is

$$\sum_{i=1}^m \frac{1}{2d_{s_i}} \int_0^{\infty} y^{1-2s_i} \left[ (\partial_x v_i)^2 - (\partial_y v_i)^2 \right] dy - \tilde{H}(v(x, 0)) \equiv C \quad \text{for} \quad x \in \mathbb{R},$$  

when $C$ is a constant and $\partial_i \tilde{H}(v) = H_i(v)$ for every $i$. Proving a similar identity in higher dimensions, $n \geq 2$, remains an open problem. Our methods and ideas are strongly motivated by the ones provided by Gui in [39], Cabré and Solá-Morales in [10], Cabré and Sire in [8, 9] and Sire and the author in [32].

Lastly, we consider a two-component system of the form (1.1) with the following particular nonlinearity

$$H_1(u_1, u_2) = \mu_1 u_1^3 + \beta u_2^3 \quad \text{and} \quad H_2(u_1, u_2) = \mu_2 u_2^3 + \beta u_1^3 u_2.$$  

where $u : \mathbb{R}^n \to \mathbb{R}^2$ and $\mu_1, \mu_2, \beta$ are constants. Both local and nonlocal systems of this type, known as the nonlinear Schrödinger system, have been studied extensively in the literature. For more information, we refer interested readers to [3, 15, 29, 43, 46, 51–54] and references therein. The nonlinear Schrödinger system is a natural counterpart of the following nonlinear Schrödinger equation,

$$\Delta u - u + u^3 = 0 \quad \text{in} \quad \mathbb{R}^n.$$  

Even though equations (1.26) and (1.11) look alike, their solutions behave very differently. In this article, we provide various Liouville theorems and monotonicity formulas for solutions of the nonlinear fractional Schrödinger system, under some assumptions on parameters $s_1, s_2, \mu_1, \mu_2, \beta$.

The organization of the paper is as follows. In the next section, we provide some standard regularity results and estimates regarding fractional Laplacian operator. In Section 3, we prove various technical tools needed to establish our main results. We provide a Hamiltonian identity, a geometric Poincaré inequality and various gradient and energy estimates for system (1.9) with a general nonlinearity $H = (H_i)_{i=1}^m$. We also provide a monotonicity formula to conclude the optimality of gradient and energy estimates. In Section 4, we establish De Giorgi type results and Liouville theorems for symmetric systems in lower dimensions. In addition, applying the Hamiltonian identity we analyze directional derivatives of the extension function $v$ satisfying (1.9). Lastly, in Section 5, we consider a two-component fractional Schrödinger system, that is a particular case of (1.1), and we provide various Liouville theorems and monotonicity formulas.
2. STANDARD ELLIPTIC ESTIMATES

In this section, we provide some standard estimates regarding fractional Laplacian operator. We assume that the nonlinearity \( H = (H_i)_{i=1}^m \) and each \( H_i \) belongs to \( C^1, \gamma (\mathbb{R}^m) \) with \( \gamma > \max (0, 1 - 2s_i) \) and \( u = (u_i)_{i=1}^m \) is a sequence of bounded functions. We omit the proofs of lemmata in this section and we refer interested readers to \([6–9, 32, 49]\) and references therein. We start with the following regularity result for solutions of (1.1).

Lemma 2.1. Suppose that \( u = (u_i)_i \) is a bounded solution of (1.1). Then, each \( u_i \) is \( C^{2,\beta} (\mathbb{R}^n) \) for some \( 0 < \beta < 1 \) depending only on \( s_i \) and \( \gamma \).

For the sake of convenience, we use the following notation used frequently in the literature.

Notation 2.1. Set \( B^+_R = \{ \tilde{x} = (x, y) \in \mathbb{R}^{n+1} \mid |\tilde{x}| < R \} \), \( \partial^+ B^+_R = \partial B^+_R \cap \{ y > 0 \} \), \( \Gamma^0_R = \partial B^+_R \cap \{ y = 0 \} \) and \( C_R = B_R \times (0, R) \) for \( R > 0 \).

We now provide a regularity result for solutions of an equation in the half-space. This can be applied to extensions functions \( v \) satisfying (1.9).

Lemma 2.2. Let \( f_i \in C^\sigma (\Gamma^0_{2R}) \) for some \( \sigma \in (0, 1) \), \( R > 0 \) and \( v_i \in L(\partial B^+_R) \cap H^1(\partial B^+_R, y^{1-2s_i}) \) for some \( s_i \in (0, 1) \) be a weak solution of

\[
\begin{align*}
\text{div}(y^{1-2s_i} \nabla v_i) &= 0 \quad \text{in} \quad B^+_R \subset \mathbb{R}^{n+1}, \\
- \lim_{y \to 0} y^{1-2s_i} \frac{\partial v_i}{\partial y} &= f_i \quad \text{on} \quad \Gamma^0_{2R}.
\end{align*}
\]

Then, there exists \( \beta \in (0, 1) \) depending only on \( n, s_i, \) and \( \sigma, \) such that \( v_i \in C^{0,\beta} (\partial B^+_R) \) and \( y^{1-2s_i} \partial_y v_i \in C^{0,\beta} (\partial B^+_R) \). Furthermore, there exist constants \( C_R \) and \( D_R \) such that

\[
\| v_i \|_{C^{0,\beta} (\partial B^+_R)} \leq C_R \quad \text{and} \quad \| y^{1-2s_i} \partial_y v_i \|_{C^{0,\beta} (\partial B^+_R)} \leq D_R,
\]

where \( C_R \) only depends on \( n, a_i, R \), \( \| v_i \|_{L(\partial B^+_R)} \) and \( \| f_i \|_{L(\Gamma^0_{2R})} \) and \( D_R \) only depends on \( n, a_i, R, \)

The next lemma provides gradient estimates for bounded solutions of (1.9) and it is a consequence of Lemma 2.1 and lemma 2.2. For more information, we refer interested readers to \([7–9, 48]\) and references therein. We apply these estimates frequently in the proofs of our main results.

Lemma 2.3. Let \( v = (v_i)_{i=1}^m \) be a bounded solution of (1.9). Then, each \( v_i \) satisfies

\[
\begin{align*}
| \nabla v_i(x, y) | &\leq C \quad \text{for} \quad x \in \mathbb{R}^n \quad \text{and} \quad y \geq 0, \\
| \nabla v_i(x, y) | &\leq \frac{C}{1 + y} \quad \text{for} \quad (x, y) \in \mathbb{R}^{n+1}_+,
\end{align*}
\]

where the positive constant \( C \) is independent from \( x \) and \( y \).

3. HAMILTONIAN IDENTITY AND ANALYTIC AND GEOMETRIC ESTIMATES

We start this section with proving a Hamiltonian identity for solutions of (1.9) in one dimension. We then apply this identity, see Theorem 4.4, to study bounded stable solutions of symmetric (1.9) with a general nonlinearity \( H = (H_i)_{i=1}^m \). Hamiltonian identities are amongst the most important tools to study qualitative behaviour of entire solutions of differential equations and in some cases they lead to properties such as monotonicity formulae.

Theorem 3.1. Suppose that \( v = (v_i)_{i=1}^m \) is a solution of (1.9) in one dimension and \( 0 < s_i < 1 \) for \( 1 \leq i \leq m \). Then the following Hamiltonian identity holds

\[
\sum_{i=1}^m \frac{1}{2d_i} \int_0^\infty y^{1-2s_i} \left[ (\partial_x v_i)^2 - (\partial_y v_i)^2 \right] dy - \hat{H}(v(x, 0)) = C \quad \text{for} \quad x \in \mathbb{R},
\]

where \( C \) is a constant that is independent from \( x \) and \( \partial_i \hat{H}(v) = H_i(v) \) for every \( i \).
Proof: Let \( v = (v_i)_{i=1}^m \) be a solution of the extension problem (1.9). Set

\[
(3.2) \quad w(x) := \sum_{i=1}^m \frac{1}{2d_i} \int_0^\infty y^{1-2s} \left[ (\partial_x v_i)^2 - (\partial_y v_i)^2 \right] dy.
\]

Differentiating \( w \) in terms of \( x \), we get

\[
(3.3) \quad \partial_x w(x) := \sum_{i=1}^m \frac{1}{d_i} \int_0^\infty y^{1-2s} \left[ \partial_x v_i \partial_{xx} v_i - \partial_y v_i \partial_{xy} v_i \right] dy.
\]

From (1.9) for each \( i = 1, \ldots, m \), we have

\[
(3.4) \quad y^{1-2s} \partial_{xx} v_i + \partial_y \left( y^{1-2s} \partial_y v_i \right) = 0.
\]

Combining this and (3.3), we end up with

\[
(3.5) \quad \partial_x w(x) := \sum_{i=1}^m \frac{1}{d_i} \int_0^\infty \left[ -\partial_x v_i \partial_y \left( y^{1-2s} \partial_y v_i \right) - y^{1-2s} \partial_y v_i \partial_{xy} v_i \right] dy.
\]

Integration by parts for the first term in the above yields

\[
(3.6) \quad - \int_0^\infty \partial_x v_i \partial_y \left( y^{1-2s} \partial_y v_i \right) dy = \int_0^\infty y^{1-2s} \partial_{xy} v_i \partial_y v_i dy - \lim_{y \to 0} y^{1-2s} \partial_y v_i \partial_x v_i.
\]

Substituting this in (3.5) we get

\[
(3.7) \quad \partial_x w(x) = - \sum_{i=1}^m \frac{1}{d_i} \lim_{y \to 0} y^{1-2s} \partial_x v_i \partial_y v_i.
\]

From this and the boundary term in (1.9), we have

\[
(3.8) \quad \partial_x w(x) = \sum_{i=1}^m \partial_i H(v(x, 0)) \partial_x v_i = \partial_x \left( \bar{H}(v(x, 0)) \right).
\]

Therefore,

\[
(3.9) \quad \partial_x \left[ w(x) - \bar{H}(v(x, 0)) \right] = 0.
\]

This implies that \( w(x) - \bar{H}(v(x, 0)) \) is constant in terms of \( x \).

\[\square\]

We now prove an inequality, known as the stability inequality, for stable solutions of (1.9). This inequality plays an important role in this paper.

Lemma 3.1. Let \( v = (v_i)_{i=1}^m \) be a stable solution of system (1.9). Then, the following stability inequality holds

\[
(3.10) \quad \sum_{i,j=1}^m \int_{\partial R^{n+1}_+} \sqrt{d_s d_t} \partial_j H_i(v) \partial_{v_j} H_j(v) \zeta_i \zeta_j d\nu \leq \sum_{i=1}^m \int_{\mathbb{R}^{n+1}_+} y^a |\nabla \zeta_i|^2 d\bar{x},
\]

for all \( \zeta_i \in C^1(\mathbb{R}^{n+1}_+) \).

Proof: Since \( v = (v_i)_{i=1}^m \) is a stable solution of system (1.9), there exists a sequence \( \phi = (\phi_i)_{i=1}^m \) that satisfies (1.10). Multiply the \( i^{th} \) equation of (1.10) with \( \phi_i^2 \phi_i \) and do integration by parts to get

\[
(3.11) \quad - \int_{\mathbb{R}^{n+1}_+} y^a \nabla \phi_i \cdot \nabla \left( \frac{x^2}{\phi_i} \right) d\bar{x} + \int_{\partial R^{n+1}_+} y^a \nabla \phi_i \cdot \nu \left( \frac{x^2}{\phi_i} \right) d\nu = 0.
\]

From this and the boundary term of equation (1.10), we get

\[
(3.12) \quad -2 \int_{\mathbb{R}^{n+1}_+} y^a \nabla \phi_i \cdot \nabla \frac{\zeta_i}{\phi_i} d\bar{x} + \int_{\mathbb{R}^{n+1}_+} y^a |\nabla \phi_i|^2 \frac{\zeta_i^2}{\phi_i} d\bar{x} + \int_{\partial R^{n+1}_+} d_s \sum_{j=1}^m \partial_j H_i(v) \phi_j \frac{\zeta_i^2}{\phi_i} d\nu = 0.
\]
Applying the Young’s inequality, for each index \( i \), we obtain

\[
\int_{\mathbb{R}^{n+1}_+} d_s \sum_{j=1}^m \partial_j H_i(v) \frac{\phi_j}{\phi_i} \zeta_i^2 \, dx = \int_{\mathbb{R}^{n+1}_+} y^{a_i} \left( -|\nabla \phi_i|^2 \frac{\zeta_i^2}{\phi_i} + 2\nabla \phi_i \cdot \nabla \zeta_i \frac{\zeta_i}{\phi_i} \right) \, d\bar{x} \\
\leq \int_{\mathbb{R}^{n+1}_+} y^{a_i} |\nabla \zeta_i|^2 \, d\bar{x}.
\]

We now provide a lower bound for the integrand in the left-hand side of (3.13) when taking the sum on both sides of (3.13) for \( i = 1, \ldots, m \),

\[
\sum_{i,j} d_s \partial_j H_i(v) \phi_j \zeta_i^2 = \sum_i d_s \partial_i H_i(v) \zeta_i^2 + \sum_{i \neq j} d_s \partial_j H_i(v) \frac{\phi_j}{\phi_i} \zeta_i^2
\]

\[
= \sum_i d_s \partial_i H_i(v) \zeta_i^2 + \sum_{i \neq j} d_s \partial_j H_i(v) \frac{\phi_j}{\phi_i} \zeta_i^2 + \sum_{i,j} d_s \partial_j H_i(v) \frac{\phi_j}{\phi_j} \zeta_j^2
\]

\[
= \sum_i d_s \partial_i H_i(v) \zeta_i^2 + \sum_{i \neq j} (\phi_j \phi_i)^{-1} (d_s \partial_j H_i(v) \phi_j^2 \zeta_i^2 + d_s \partial_i H_j(v) \phi_i^2 \zeta_j^2)
\]

\[
\geq \sum_i d_s \partial_i H_i(v) \zeta_i^2 + 2 \sum_{i < j} \sqrt{d_s d_s \partial_j H_i(v) \partial_i H_j(v) \zeta_i \zeta_j}
\]

\[
= \sum_{i,j} \sqrt{d_s d_s \partial_j H_i(v) \partial_i H_j(v) \zeta_i \zeta_j}.
\]

This completes the proof. \( \square \)

We now apply the stability inequality (3.10) to establish a geometric Poincaré inequality. Note that this inequality for the case of local scalar equations was first proved in [50] and it was used in [28] and references therein to prove De Giorgi type results. For the fractional Laplacian case this inequality was established in [49]. We also refer interested readers to [31] and [22, 32] for the case of local and nonlocal systems, respectively.

**Theorem 3.2.** Assume that \( m, n \geq 1 \) and \( v = (v_i)_i \) is a stable solution of (1.9). Then, for any \( \eta = (\eta_k)_{k=1}^m \in C^1_0(\mathbb{R}^{n+1}_+) \), the following inequality holds,

\[
\sum_{i=1}^m \frac{1}{d_s} \int_{|\nabla v_i| \neq 0} \eta_i^1 - 2s \left( |\nabla v_i|^2 A_i^2 + |\nabla T_i| |\nabla v_i|^2 \right) \eta_i^2 \, d\bar{x} \\
+ \sum_{i \neq j} \int_{\mathbb{R}^{n+1}_+} \left( \sqrt{\partial_j H_i(v) \partial_i H_j(v) |\nabla v_i| |\nabla v_j| \eta_i \eta_j - \partial_j H_i(v) \partial_i H_j(v) \nabla v_i \cdot \nabla v_j \eta_i^2} \right) \, dx
\]

\[
\leq \sum_{i=1}^m \frac{1}{d_s} \int_{\mathbb{R}^{n+1}_+} y^{1-2s} |\nabla v_i|^2 |\nabla \eta_i|^2 \, d\bar{x},
\]

where \( \nabla T_i \) stands for the tangential gradient along a given level set of \( v_i \) and \( A_i^2 \) for the sum of the squares of the principal curvatures of such a level set.

**Proof:** Let \( v = (v_i)_{i=1}^m \) be a stable solution of (1.9). From Lemma 3.1, the stability inequality (3.10) holds. Test the stability inequality with \( \zeta_i := |\nabla v_i| \eta_i \) where each \( \eta_i \in C^1_0(\mathbb{R}^{n+1}_+) \), to get

\[
I := \sum_{i,j=1}^m \int_{\mathbb{R}^{n+1}_+} \sqrt{d_s d_s \partial_j H_i(v) \partial_i H_j(v) |\nabla v_i| |\nabla v_j| \eta_i \eta_j} \, dx
\]

\[
\leq \sum_{i=1}^m \int_{\mathbb{R}^{n+1}_+} y^{a_i} |\nabla (|\nabla v_i| \eta_i)|^2 \, d\bar{x} =: J.
\]
Simplifying the left-hand side of the above inequality, we get

\begin{equation}
I = \sum_{i=1}^{m} \int_{\partial R_{n+1}^+} d_s \partial_{v_i} H_i(v) |\nabla x v_i|^2 \eta_i^2 \, dx
+ \sum_{i \neq j} \int_{\partial R_{n+1}^+} \sqrt{d_s d_s} \partial_{j} H_i(v) \partial_{v_i} H_j(v) |\nabla x v_i| |\nabla x v_j| \eta_i \eta_j \, dx.
\end{equation}

Similarly, for \( J \) we have

\begin{equation}
J = \sum_{i=1}^{m} \int_{R_{n+1}^+} y^{a_i} \left[ |\nabla x v_i|^2 |\nabla \eta_i|^2 + |\nabla |\nabla x v_i|^2 |\eta_i|^2 + \frac{1}{2} |\nabla |\nabla x v_i| \cdot |\nabla \eta_i|^2 \right] \, dx.
\end{equation}

We now differentiate the \( i^{th} \) equation of (1.9) with respect to \( x_k \) and multiply with \( \partial_{x_k} v_i \eta_i^2 \) for \( i = 1, \ldots, m \) and \( k = 1, \ldots, n \), to get

\begin{equation}
\text{div}(y_i^a \nabla \partial_{x_k} v_i) \partial_{x_k} v_i \eta_i^2 = 0.
\end{equation}

Integrating this by parts, we obtain

\begin{equation}
\int_{R_{n+1}^+} y^{a_i} |\nabla \partial_{x_k} v_i|^2 \eta_i^2 \, dx + \frac{1}{2} \int_{R_{n+1}^+} y^{a_i} |\nabla x v_i|^2 \cdot |\nabla \eta_i|^2 \, dx = \int_{\partial R_{n+1}^+} \lim_{y \to 0} y^{a_i} (-\partial_y \partial_{x_k} v_i) \partial_{x_k} v_i \eta_i^2 \, dx.
\end{equation}

Differentiating the boundary term of (1.9) with respect to \( x_k \) yields

\begin{equation}
\lim_{y \to 0} y^{a_i} (-\partial_y \partial_{x_k} v_i) \partial_{x_k} v_i \eta_i^2 = d_s \sum_{j=1}^{m} \partial_{j} H_i(v) \partial_{x_k} v_j \partial_{x_k} v_i \eta_i^2.
\end{equation}

From this and (3.20), we have

\begin{equation}
\sum_{i=1}^{m} \int_{R_{n+1}^+} y^{a_i} |\nabla \partial_{x_k} v_i|^2 \eta_i^2 \, dx + \frac{1}{2} \sum_{i=1}^{m} \int_{R_{n+1}^+} y^{a_i} |\nabla x v_i|^2 \cdot |\nabla \eta_i|^2 \, dx
= \sum_{i=1}^{m} d_s \int_{\partial R_{n+1}^+} \partial_{v_i} H_i(v) |\nabla x v_i|^2 \eta_i^2 \, dx + \sum_{i \neq j} d_s \int_{\partial R_{n+1}^+} \partial_{j} H_i(v) \partial_{x_k} v_j \partial_{x_k} v_i \eta_i^2 \, dx.
\end{equation}

Taking sum on the index \( k = 1, \ldots, n \), we get

\begin{equation}
\sum_{i=1}^{m} d_s \int_{\partial R_{n+1}^+} \partial_{v_i} H_i(v) |\nabla x v_i|^2 \eta_i^2 \, dx = \sum_{i=1}^{m} \int_{R_{n+1}^+} y^{a_i} \sum_{k=1}^{n} |\nabla \partial_{x_k} v_i|^2 \eta_i^2 \, dx
+ \frac{1}{2} \sum_{i=1}^{m} \int_{R_{n+1}^+} y^{a_i} |\nabla x v_i|^2 \cdot |\nabla \eta_i|^2 \, dx
- \sum_{i \neq j} d_s \int_{\partial R_{n+1}^+} \partial_{j} H_i(v) \nabla x v_i \cdot \nabla x v_j \eta_i^2 \, dx.
\end{equation}

We now substitute this equality in \( I \leq J \) when \( I \) is given in (3.17) and \( J \) is given in (3.18). On see that the term \( \frac{1}{2} \sum_{i=1}^{m} \int_{R_{n+1}^+} y^{a_i} |\nabla \nabla x v_i|^2 \cdot |\nabla \eta_i|^2 \, dx \) crosses out and we end up with

\begin{equation}
\sum_{i=1}^{m} \int_{R_{n+1}^+} |\nabla x v_i|^2 \left( \sum_{k=1}^{n} |\nabla \partial_{x_k} v_i|^2 - |\nabla |\nabla x v_i|^2 |\right) \eta_i^2 \, dx
+ \sum_{i \neq j} \int_{\partial R_{n+1}^+} \left[ d_s d_s \partial_{v_i} H_i(v) \partial_{v_i} H_j(v) |\nabla x v_i| |\nabla x v_j| \eta_i \eta_j - d_s \partial_{j} H_i(v) \nabla x v_i \cdot \nabla x v_j \eta_i^2 \right] \, dx
\leq \sum_{i=1}^{m} \int_{R_{n+1}^+} y^{a_i} |\nabla x v_i|^2 \cdot |\nabla \eta_i|^2 \, dx.
\end{equation}
According to formula (2.1) given in [50], the following geometric identity between the tangential gradients and curvatures holds. For any \( w \in C^2(\Omega) \)

\[
(3.27) \quad \sum_{k=1}^{n} |\nabla \partial_k w|^2 - |\nabla w|^2 = \begin{cases} |\nabla w|^2 (\sum_{l=1}^{n-1} \kappa_l^2) + |\nabla y| |\nabla w|^2 & \text{for } x \in \{|\nabla w| > 0 \cap \Omega\}, \\ 0 & \text{for } x \in \{|\nabla w| = 0 \cap \Omega\}, \end{cases}
\]

where \( \kappa_l \) are the principal curvatures of the level set of \( w \) at \( x \) and \( \nabla y \) denotes the orthogonal projection of the gradient along this level set. Applying the above identity (3.27) to (3.24) completes the proof.

\[ \square \]

For the rest of this section, we provide energy and gradient estimates for solutions of (1.9). Then, in next sections we apply these estimates to establish De Giorgi type results and Liouville theorems. Consider the energy functional

\[
(3.28) \quad E_R(v) := \sum_{i=1}^{m} \frac{1}{2d_i} \int_{C_R} y^{1-2s_i} |\nabla v_i|^2 dx - \int_{B_R} \hat{H}(v) dx,
\]

when \( \partial_i \hat{H}(v) = H_i(v) \) for \( i = 1, \cdots m \). We finish this section by proving an energy estimate for \( H \)-monotone solutions of (1.9).

**Theorem 3.3.** Let \( v = (v_i)_{i=1}^{m} \) be a bounded \( H \)-monotone solution of (1.9) such that

\[
(3.29) \quad \lim_{x_n \to \infty} v_i(x', x_n, y) = L_i \quad \text{for } x' \in \mathbb{R}^{n-1} \text{ and } y \in \mathbb{R}^+,
\]

when \( L = (L_i)_{i=1}^{m} \) and each \( L_i \) is a constant in \( \mathbb{R} \) and \( H(L) = 0 \). Then, the following energy estimates hold.

(i) If \( 1/2 < s_i < 1 \), then \( E_R(v) \leq CR^{n-1} \),

(ii) If \( s_i = 1/2 \), then \( E_R(v) \leq CR^{n-1} \log R \),

(iii) If \( 0 < s_i < 1/2 \), then \( E_R(v) \leq CR^{n-2s_i} \),

where the positive constant \( C \) is independent from \( R > 1 \).

**Proof:** Set the shift function \( v_i^t(x, y) := v_i(x', x_n + t, y) \) for \( (x', x_n, y) \in \mathbb{R}_+^{n+1} \) and \( t \in \mathbb{R} \). It is straightforward to see that \( v^t = (v_i^t)_{i=1}^{m} \) is a solution of (1.9) and it satisfies pointwise estimates provided in Lemma 2.3. In addition, for every parameter \( t \) and all indices \( i \), one can see that \( |v_i^t| \in L^\infty(\mathbb{R}_+^{n+1}) \). Therefore, \( v^t = (v_i^t)_{i=1}^{m} \) is a sequence of bounded functions and

\[
(3.30) \quad \lim_{t \to \infty} |v_i^t(x, y) - L_i| + |\nabla v_i^t(x, y)| = 0 \quad \text{for all } (x, y) \in \mathbb{R}_+^{n+1}.
\]

This implies that

\[
(3.31) \quad \lim_{t \to \infty} E_R(v^t) = 0.
\]

We now differentiate the energy functional in terms of parameter \( t \) to get

\[
(3.32) \quad \partial_t E_R(v^t) = \sum_{i=1}^{m} \frac{1}{d_i} \int_0^R \int_{C_R} y^{1-2s_i} \nabla v_i^t \cdot \nabla (\partial_t v_i^t) dx - \int_{B_R} H_i(v^t) \partial_t v_i^t dx
\]

\[ = - \sum_{i=1}^{m} \frac{1}{d_i} \int_0^R \int_{B_R} \text{div} \left( y^{1-2s_i} \nabla v_i^t \right) \partial_t v_i^t dx + \sum_{i=1}^{m} \frac{1}{d_i} \int_0^R \int_{C_R} y^{1-2s_i} \nabla v_i^t \cdot \nu \partial_t v_i^t dH^{n-1} dy
\]

\[ - \sum_{i=1}^{m} \int_{B_R} H_i(v^t) \partial_t v_i^t dx.
\]

From the fact that \( \text{div} \left( y^{1-2s_i} \nabla v_i^t \right) = 0 \), we can simplify the above as

\[
(3.33) \quad \partial_t E_R(v^t) = \sum_{i=1}^{m} \frac{1}{d_i} \int_0^R \int_{\partial B_R} y^{1-2s_i} \nabla v_i^t \cdot \nu \partial_t v_i^t dH^{n-1} dy + \sum_{i=1}^{m} \frac{1}{d_i} \int_{B_R \times \{y = R\}} y^{1-2s_i} \partial_y v_i^t \partial_t v_i^t dx.
\]

\[ 9 \]
Consider disjoint sets of indices $I$ and $J$ such that $I \cup J = \{1, \cdots, m\}$ and $\partial_t v^\mu_\mu > 0 > \partial_t v^\mu_\lambda$ for $\mu \in I$ and $\lambda \in J$. Applying this, we can expand $\partial_t \varphi(v^t)$ as

\begin{equation}
(3.34) \quad \partial_t \varphi(v^t) = \int_{\partial B_R} \int_0^R \left( \sum_{\mu \in I} \frac{1}{d_{s_\mu}} y_1^{1-2s_\mu} \partial_\nu v^\mu_\mu \partial_t v^\nu_\mu + \sum_{\lambda \in J} \frac{1}{d_{s_\lambda}} y_1^{1-2s_\lambda} \partial_\nu v^\mu_\lambda \partial_t v^\nu_\lambda \right) dyd\mathcal{H}^{n-1}
\end{equation}

\begin{equation}
+ \int_{B_R \times \{y = R\}} \left( \sum_{\mu \in I} \frac{1}{d_{s_\mu}} y_1^{1-2s_\mu} \partial_\nu v^\mu_\mu \partial_t v^\nu_\mu + \sum_{\lambda \in J} \frac{1}{d_{s_\lambda}} y_1^{1-2s_\lambda} \partial_\nu v^\mu_\lambda \partial_t v^\nu_\lambda \right) dx.
\end{equation}

Lemma 2.3 implies that there exists a constant $M$ such that $|\partial_\nu v^\mu_\mu| \leq \frac{M}{1+y}$ for $(x, y) \in \mathbb{R}^{n+1}_+$ and $|\partial_y v^\mu_\lambda| \leq \frac{M}{y}$ for $x \in \mathbb{R}^n$ and $y > 1$. From these estimates and (3.34), we get

\begin{equation}
(3.35) \quad \partial_t \varphi(v^t) \geq M \int_{\partial B_R} \int_0^R \sum_{\mu \in I} \frac{1}{d_{s_\mu}} y_1^{1-2s_\mu} \left( -\frac{1}{1+y} \right) \partial_t v^\mu_\mu dyd\mathcal{H}^{n-1}
\end{equation}

\begin{equation}
+ M \int_{\partial B_R} \int_0^R \sum_{\lambda \in J} \frac{1}{d_{s_\lambda}} y_1^{1-2s_\lambda} \left( \frac{1}{1+y} \right) \partial_t v^\mu_\lambda dyd\mathcal{H}^{n-1}
\end{equation}

\begin{equation}
+ M \int_{B_R \times \{y = R\}} \sum_{\mu \in I} \frac{1}{d_{s_\mu}} y_1^{1-2s_\mu} \left( \frac{1}{y} \right) \partial_\nu v^\mu_\mu dx
\end{equation}

\begin{equation}
+ M \int_{B_R \times \{y = R\}} \sum_{\lambda \in J} \frac{1}{d_{s_\lambda}} y_1^{1-2s_\lambda} \left( \frac{1}{y} \right) \partial_\nu v^\mu_\lambda dx.
\end{equation}

Note that $\varphi(v) = \varphi(v^T) - \int_0^T \partial_t \varphi(v^t) dt$ for every $T > 0$. Combining this and (3.35), we conclude

\begin{equation}
(3.36) \quad \varphi(v) \leq \varphi(v^T) + M \int_{\partial B_R} \int_0^R \sum_{\mu \in I} \frac{1}{d_{s_\mu}} \left( \frac{y_1^{1-2s_\mu}}{1+y} \right) \int_0^T \partial_t v^\mu_\mu dt dyd\mathcal{H}^{n-1}
\end{equation}

\begin{equation}
- M \int_{\partial B_R} \int_0^R \sum_{\lambda \in J} \frac{1}{d_{s_\lambda}} \left( \frac{y_1^{1-2s_\lambda}}{1+y} \right) \int_0^T \partial_t v^\mu_\lambda dt dyd\mathcal{H}^{n-1}
\end{equation}

\begin{equation}
+ M \int_{B_R \times \{y = R\}} \sum_{\mu \in I} \frac{1}{d_{s_\mu}} y_1^{1-2s_\mu} \int_0^T \partial_\nu v^\mu_\mu dt dx
\end{equation}

\begin{equation}
- M \int_{B_R \times \{y = R\}} \sum_{\lambda \in J} \frac{1}{d_{s_\lambda}} y_1^{1-2s_\lambda} \int_0^T \partial_\nu v^\mu_\lambda dt dx.
\end{equation}

Simplifying (3.36) yields

\begin{equation}
(3.37) \quad \varphi(v) \leq \varphi(v^T) + M \int_{\partial B_R} \int_0^R \sum_{\mu \in I} \frac{y_1^{1-2s_\mu}}{1+y} (v^\mu_T - v_\mu) dyd\mathcal{H}^{n-1}
\end{equation}

\begin{equation}
+ M \int_{\partial B_R} \int_0^R \sum_{\lambda \in J} \frac{y_1^{1-2s_\lambda}}{1+y} (v^\lambda - v_\lambda^T) dyd\mathcal{H}^{n-1}
\end{equation}

\begin{equation}
+ M \int_{B_R \times \{y = R\}} \sum_{\mu \in I} y_1^{2s_\mu} (v^\mu_T - v_\mu) dx + M \int_{B_R \times \{y = R\}} \sum_{\lambda \in J} y_1^{2s_\lambda} (v^\lambda - v_\lambda^T) dx.
\end{equation}

Note that for $\mu \in I$ and $\lambda \in J$, we have $v^\mu_T \geq v_\mu$ and $v^\lambda \geq v_\lambda$. Therefore,

\begin{equation}
(3.38) \quad \varphi(v) \leq \varphi(v^T) + M \int_{\partial B_R} \int_0^R \sum_{i=1}^m \frac{y_1^{1-2s_i}}{1+y} dyd\mathcal{H}^{n-1} + M \int_{B_R \times \{y = R\}} \sum_{i=1}^m y_1^{2s_i} dx.
\end{equation}
Let $T \to \infty$, then $E_R(v^T)$ approaches zero. Doing integration by parts we obtain

\begin{equation}
E_R(v) \leq M \sum_{i=1}^{m} \left[ R^{n-2s_i} \chi_{(0<s_i<1/2)} + R^{n-1} \chi_{(1/2<s_i<1)} + R^{n-1} \log R \chi_{(s_i=1/2)} + R^{n-2s_i} \right].
\end{equation}

This completes the proof. \hfill \Box

Note that ideas and techniques applied in the above proof are strongly motivated by the ones provided by Ambrosio and Cabré in [2] and by Cabré and Cinti in [6, 7] for local and nonlocal scalar equations, respectively. We would like to refer interested readers to [31] by Ghoussoub and the author and to [32] by Sire and the author for local and nonlocal system of equations, respectively. In the next two lemmata, we prove gradient estimates for bounded solutions of (1.9).

**Lemma 3.2.** Suppose that $v = (v_i)_{i=1}^{m}$ is a bounded solution of (1.9). Then, the following gradient estimate holds for any $i = 1, \ldots, m$

\begin{equation}
\int_{C_R} \psi_i |\nabla v_i|^2 \, d\bar{x} \leq CR^n,
\end{equation}

when $C$ is a positive constant independent from $R$.

**Proof:** Let $v$ be a bounded solution of (1.9). Multiplying (1.9) by $v_i \psi_R$ where $\psi_R$ is a test function and integrating over $\mathbb{R}^{n+1}_+$, we get

\begin{equation}
\int_{\mathbb{R}^{n+1}_+} \psi_i |\nabla v_i|^2 \psi_R^2 \, d\bar{x} + 2 \int_{\mathbb{R}^{n+1}_+} \psi_i \nabla \psi_R \cdot \nabla v_i v_i \psi_R \, d\bar{x} = d_s \int_{\partial \mathbb{R}^{n+1}_+} H_i(v) v_i \psi_R^2 \, d\bar{x}.
\end{equation}

Using the Cauchy-Schwarz inequality, we conclude

\begin{equation}
2 \int_{\mathbb{R}^{n+1}_+} \psi_i \nabla \psi_R \cdot \nabla v_i v_i \psi_R \, d\bar{x} \leq \frac{1}{2} \int_{\mathbb{R}^{n+1}_+} \psi_i |\nabla v_i|^2 \psi_R^2 \, d\bar{x} + 2 \int_{\mathbb{R}^{n+1}_+} \psi_i |\nabla \psi_R|^2 v_i^2 \, d\bar{x}.
\end{equation}

From this, we obtain

\begin{equation}
\int_{\mathbb{R}^{n+1}_+} \psi_i |\nabla v_i|^2 \psi_R^2 \, d\bar{x} \leq 4 \int_{\mathbb{R}^{n+1}_+} \psi_i |\nabla \psi_R|^2 v_i^2 \, d\bar{x} + 2d_s \int_{\partial \mathbb{R}^{n+1}_+} H_i(v) v_i \psi_R^2 \, d\bar{x}.
\end{equation}

From the boundedness of $v$ and the fact that $H_i \in C^{1,\gamma}$, we obtain the following bound by choosing an appropriate test function,

\begin{equation}
\int_{C_R} \psi_i |\nabla v_i|^2 \psi_R^2 \, d\bar{x} \leq CR^{n-2s_i} + CR^n \leq CR^n,
\end{equation}

where the constant $C$ is independent from $R$ but may depend on $v = (v_i)_{i=1}^{m}$. This completes the proof. \hfill \Box

We now assume extra conditions on the sign of the nonlinearity $H$. This enables us to prove stronger gradient estimates on solutions of (1.9).

**Lemma 3.3.** Suppose that $v = (v_i)_{i=1}^{m}$ is a bounded solution of (1.9).

(i) If $H_i(v) \geq 0$ for any $1 \leq i \leq m$, then $\int_{C_R} \psi_i |\nabla v_i|^2 \, d\bar{x} \leq CR^{n-2s_i}$.

(ii) If $\sum_{i=1}^{m} \psi_i H_i(v) \leq 0$, then $\sum_{i=1}^{m} \int_{C_R} y^{1-2s_i} |\nabla v_i|^2 \, d\bar{x} \leq CR^{n-2s_i}$.

Here, the positive constant $C$ is independent from $R$.

**Proof:** We start the proof with Part (i). Suppose that for a particular index $1 \leq i \leq m$ we have $H_i(v) \geq 0$. We now multiply (1.9) with $\psi_i |v_i| \psi_R$ when $\psi_R$ is a positive test function. Applying integration by parts and using the fact that $H_i$ has a fixed sign, we obtain

\begin{equation}
\int_{\mathbb{R}^{n+1}_+} \psi_i \nabla \psi_i |v_i| \psi_R \cdot \nabla v_i \psi_R \, d\bar{x} = \int_{\partial \mathbb{R}^{n+1}_+} \lim_{y \to \partial} \psi_i (-\partial_y v_i)(v_i - |v_i|) \psi_R \, d\bar{x}
= d_s \int_{\partial \mathbb{R}^{n+1}_+} (v_i - |v_i|) H_i(v) \psi_R \, d\bar{x} \leq 0.
\end{equation}
This implies that
\begin{align*}
(3.46) \quad \int_{\mathbb{R}^n_+} y^{a_i} |\nabla v_i|^2 \psi_R d\bar{x} & \leq - \int_{\mathbb{R}^n_+} y^{a_i} (v_i - ||v_i||_\infty) \nabla \psi_R \cdot \nabla v_i d\bar{x} \\
& = - \frac{1}{2} \int_{\mathbb{R}^n_+} y^{a_i} \nabla (v_i - ||v_i||_\infty)^2 \cdot \nabla \psi_R d\bar{x} \\
& = - \frac{1}{2} \int_{\mathbb{R}^n_+} (v_i - ||v_i||_\infty)^2 \text{div}(y^{a_i} \nabla \psi_R) d\bar{x} \\
& \quad + \frac{1}{2} \int_{\partial \mathbb{R}^n_+} (- \lim_{y \to 0} y^{a_i} \partial_y \psi_R)(v_i - ||v_i||_\infty)^2 d\bar{x} \\
& =: I + J.
\end{align*}

From boundedness of $v_i$ and applying an appropriate test function $\psi_R$, we obtain
\begin{align*}
(3.47) \quad I &= - \frac{1}{2} \int_{\mathbb{R}^n_+} (v_i - ||v_i||_\infty)^2 \left( y^{a_i} \Delta_x \psi_R + a_i y^{a_i-1} \partial_y \psi_R \right) d\bar{x} \\
& \leq C \int_{C^2R} \left( \frac{y^{a_i}}{R^2} + \frac{y^{a_i-1}}{R} \right) d\bar{x} \leq CR^2 \int_{R} \left( \frac{y^{a_i}}{R^2} + \frac{y^{a_i-1}}{R} \right) dy \leq CR^{n-2s_i},
\end{align*}
when the positive constant $C$ is independent from $R$. In above, we have used the fact that the test function $\psi_R$ can be chosen such that $|\partial_y \psi_R| \leq CR^{-1}$ and $|\Delta_x \psi_R| \leq CR^{-2}$ in $C_R$. Using these properties, we conclude $|y^{a_i} \partial_y \psi_R| \leq CR^{-2s_i}$ in $C_R$. Similarly, applying the test function $\psi_R$ with the above properties, we obtain the following upper bound for $J$ given in (3.47)
\begin{align*}
(3.48) \quad J \leq CR^{n-2s_i}.
\end{align*}

This completes the proof of Part (i). We now provide a proof for Part (ii). Multiplying the $i$th equation of (1.9) with $v_i \psi_R$, when $\psi_R$ is the same test function applied in Part (i), and integrating we get
\begin{align*}
(3.49) \quad \sum_{i=1}^{m} \frac{1}{d_s} \int_{\mathbb{R}^n_+} y^{a_i} \nabla [v_i \psi_R] \cdot \nabla v_i d\bar{x} &= \sum_{i=1}^{m} \int_{\partial \mathbb{R}^n_+} \lim_{y \to 0} y^{a_i} (- \partial_y v_i) v_i \psi_R dx \\
& = \int_{\partial \mathbb{R}^n_+} v_i H_i(v) \psi_R dx.
\end{align*}

Note that the latter is nonpositive since $\sum_{i=1}^{m} v_i H_i(v) \leq 0$. The rest of the proof is similar to Part (i) and we omit it here.

We end this section with a monotonicity formula for bounded solutions of (1.9). Consider the following function for $R \geq 1$,
\begin{align*}
(3.50) \quad I(R) := R^{2s} - n \left( \frac{1}{2} \int_{B_R} \sum_{i=1}^{m} y^{a_i} |\nabla v_i|^2 dy - \int_{B_R \times \{y=0\}} \tilde{H}(v) dx \right).
\end{align*}
when $\partial_i \tilde{H}(v) = H_i(v)$ for every $i$. We show that $I(R)$ is a nondecreasing function of $R$, when $\tilde{H} \leq 0$, that is equivalent to
\begin{align*}
(3.51) \quad I(R) \geq I(1) \quad \text{for} \quad R > 1.
\end{align*}
Note that from definitions of $I(R)$ in (3.50) and $E_R(v)$ in (3.28), we have
\begin{align*}
(3.52) \quad E_R(v) \geq R^{n-2s} I(R).
\end{align*}
From (3.52) and (3.51), we conclude
\begin{align*}
(3.53) \quad \sum_{i=1}^{m} \frac{1}{2d_s} \int_{\mathbb{R}^n_+} y^{1-2s} |\nabla v_i|^2 d\bar{x} - \int_{B_R} \tilde{H}(v) dx \geq CR^{n-2s},
\end{align*}
and the positive constant $C$ is independent from $R$. This clarifies the sharpness of gradient and energy estimates provided in this section, in particular when $0 < s_\ast = s^\ast < 1/2$. 

Theorem 3.4. Let \( v = (v_i)_{i=1}^m \) be a bounded solution of (1.9). Then, \( I(R) \) is a nondecreasing function of \( R \geq 1 \) when \( \partial_i \tilde{H} = H_i(v) \) for every \( 1 \leq i \leq m \) and \( \tilde{H} \leq 0 \).

Proof: Differentiating \( I(R) \) with respect to \( R \), yields

\[
I'(R) = \frac{n - 2s^*}{2} R^{2s^* - n - 1} \int_{B_R} \sum_{i=1}^m y^{a_i} |\nabla v_i|^2 dxdy + (n - 2s^*) R^{2s^* - n - 1} \int_{B_R \times \{y=0\}} \tilde{H}(v) dx
\]

\[+ \frac{1}{2} R^{2s^* - n} \int_{\partial^+ B_R^+} \sum_{i=1}^m y^{a_i} |\nabla v_i|^2 dxdy - R^{2s^* - n} \int_{\partial B_R \times \{y=0\}} \tilde{H}(v) dx.
\]

Applying some standard arguments in regards to the Pohozaev identity implies that

\[
\frac{n}{2} \int_{B_R^+} \sum_{i=1}^m y^{a_i} |\nabla v_i|^2 = \sum_{i=1}^m s_i \int_{B_R^+} y^{a_i} |\nabla v_i|^2 + n \int_{B_R} \tilde{H}(v) dx - R \int_{\partial B_R} \tilde{H}(v) dH^{n-1}
\]

\[- R \int_{\partial^+ B_R^+} \sum_{i=1}^m y^{a_i} (\partial_r v_i)^2 dH^n + \frac{R}{2} \int_{\partial^+ B_R^+} \sum_{i=1}^m y^{a_i} |\nabla v_i|^2.
\]

Combining (3.54) and (3.55), we conclude

\[
I'(R) R^{n+1-2s^*} = R \int_{\partial^+ B_R^+} \sum_{i=1}^m y^{a_i} (\partial_r v_i)^2 dH^n
\]

\[+ \int_{B_R^+} \sum_{i=1}^m (s^* - s_i) y^{a_i} |\nabla v_i|^2 - 2s^* \int_{B_R \times \{y=0\}} \tilde{H}(v) dx.
\]

Since \( \tilde{H} \leq 0 \) and \( s^* \geq s_i \) for every \( 1 \leq i \leq m \), we have \( I'(R) \geq 0 \).

\[\square\]

4. Symmetry Results and Liouville Theorems; General Nonlinearity

In this section, we provide De Giorgi type results for stable and \( H \)-monotone solutions of symmetric system (1.9) with a general nonlinearity in lower dimensions. Just like in the proof of the classical De Giorgi’s conjecture in dimensions \( n = 2, 3 \) providing a Liouville theorem for the quotient of partial derivatives is a milestone. To be mathematically more precise, consider the case of scalar equation (1.12), it was observed by Berestycki, Caffarelli and Nirenberg in [4], by Ghoussoub and Gui in [36] and by Ambrosio and Cabrè in [2] that if \( \phi \in L_{loc}(\mathbb{R}^n) \) such that \( \phi^2 > 0 \) a.e. and \( \sigma \in H^1_{loc}(\mathbb{R}^n) \) satisfy

\[
-\sigma \text{ div}(\phi^2 \sigma) \leq 0 \quad \text{in} \quad \mathbb{R}^n,
\]

under the decay-growth assumption

\[
\int_{B_{2R}\setminus B_R} \phi^2 \sigma^2 < CR^2 \quad \text{for} \quad R > 1,
\]

then \( \sigma \) must be a constant. Eventually, \( \sigma \) is set to be \( \sigma = \frac{\nabla u}{\partial_n u} \) for an arbitrary direction \( \eta \in \mathbb{R}^n \) and \( \phi = \partial_n u \). Cabrè and Sire in [8, 9] proved a similar Liouville theorem for nonlocal scalar equations that is (1.1) for \( m = 1 \). Most recently, Ghoussoub and the author in [31] and later Sire and the author in [32] provided counterparts of this Liouville theorem for the local and nonlocal gradient system (1.13), respectively. Consider the following set of functions \( \mathcal{F} \) that is somewhat standard in this context

\[
\mathcal{F} := \left\{ F : \mathbb{R}^+ \to \mathbb{R}^+ \right\} \quad \text{for} \quad R > 1,
\]

then \( \mathcal{F} \) is nondecreasing and \( \int_{B_{2R}\setminus B_R} \phi^2 \sigma^2 < CR^2 \quad \text{for} \quad R > 1 \).

For example, \( F(r) = \ln r \) and \( F(r) \equiv \text{constant} > 0 \) belong to this class and \( F(r) = r \) does not belong to \( \mathcal{F} \). As far as we know, this set of functions was introduced by Karp in [40, 41] and used in [6–9, 30, 32, 45].

Theorem 4.1. Assume that \( \phi_i \in L_{loc}^\infty(\mathbb{R}^{n+1}_+) \) is a positive function and \( \sigma_i \in H^1_{loc}(\mathbb{R}^{n+1}_+; y^{a_i}) \) satisfies

\[
\limsup_{R \to \infty} \frac{1}{R^2 F(R)} \int_{B_R} \sum_{i=1}^m y^{a_i} \phi_i^2 \sigma_i^2 d\bar{x} < \infty,
\]

\[
\sum_{i=1}^m y^{a_i} \int_{B_R^+} \sum_{i=1}^m y^{a_i} |\nabla v_i|^2 dxdy + (n - 2s^*) R^{2s^* - n - 1} \int_{B_R \times \{y=0\}} \tilde{H}(v) dx
\]

\[+ \frac{1}{2} R^{2s^* - n} \int_{\partial^+ B_R^+} \sum_{i=1}^m y^{a_i} |\nabla v_i|^2 dxdy - R^{2s^* - n} \int_{\partial B_R \times \{y=0\}} \tilde{H}(v) dx.
\]
for \( F \in \mathcal{F} \) and \( i = 1, \ldots, m \). If the sequence \( (\sigma_i)_{i=1}^m \) is a solution of

\[
\begin{aligned}
-\sigma_i \text{div}(y^a_i \phi_i^2 \nabla \sigma_i) &\leq 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \\
-\lim_{y \to 0} y^a_i \sigma_i \partial_y \sigma_i &\leq \sum_{j=1}^n h_{i,j} f(\sigma_j - \sigma_i) \sigma_i \quad \text{in} \quad \partial \mathbb{R}^{n+1}_+,
\end{aligned}
\]

when \( 0 \leq h_{i,j} \in L^1_{\text{loc}}(\mathbb{R}^n) \), \( h_{i,j} = h_{j,i} \) and \( f \in L^1_{\text{loc}}(\mathbb{R}) \) is an odd function such that \( f(t) \geq 0 \) for \( t \in \mathbb{R}^+ \).

Then, each function \( \sigma_i \) is constant for all \( i = 1, \ldots, m \).

In this article, we apply the above theorem frequently to prove our main results for solutions of (1.9). Suppose that \( v = (v_i)_{i=1}^m \) is a \( H \)-monotone solution of (1.9). Let \( \phi_i := \partial_n v_i \) and \( \psi_i := \nabla v_i \cdot \eta \) for any fixed \( \eta = (\eta', 0) \in \mathbb{R}^{n-1} \times \{0\} \). Then, sequences \( (w_i)_{i=1}^m = (\phi_i)_{i=1}^m \) and \( (w_i)_{i=1}^m = (\psi_i)_{i=1}^m \) satisfy the following linearized equation

\[
\begin{aligned}
&\text{div}(y^a_i \nabla w_i) = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+,
&-\lim_{y \to 0} y^a_i \partial_y w_i = d_s, \sum_{j=1}^n \partial_j H_i(v) w_j \quad \text{in} \quad \partial \mathbb{R}^{n+1}_+.
\end{aligned}
\]

Since \( v \) is a \( H \)-monotone solution, \( \phi_i \) does not change sign for each \( i \). Then, the sequence of functions \( \sigma = (\sigma_i)_{i=1}^m \) for \( \sigma_i := \frac{\psi_i}{\phi_i} \) satisfies the following

\[
\text{div}(y^a_i \phi_i^2 \nabla \sigma_i) = \text{div}(y^a_i [\phi_i \nabla \sigma_i - \sigma_i \nabla \phi_i]) = \phi_i \text{div}(y^a_i \nabla \psi_i) - \psi_i \text{div}(y^a_i \nabla \phi_i) = 0.
\]

On the other hand, in \( \partial \mathbb{R}^{n+1}_+ \) we have

\[
-\lim_{y \to 0} y^a_i \partial_y \sigma_i = -\lim_{y \to 0} y^a_i \partial_y \psi_i \phi_i^{-1} + \lim_{y \to 0} y^a_i \partial_y \phi_i \psi_i \phi_i^{-1} = d_s, \sum_{j=1}^n \partial_j H_i(v) \left[ \frac{\psi_j}{\phi_i} - \frac{\phi_j}{\phi_i} \right],
\]

which implies that

\[
-\lim_{y \to 0} y^a_i \phi_i^2 \partial_y \sigma_i = d_s, \sum_{j=1}^n \partial_j H_i(v) [\psi_j \phi_i - \phi_j \psi_i] \sigma_i = d_s, \sum_{j=1}^n \partial_j H_i(v) \phi_i \phi_j (\sigma_j - \sigma_i) \sigma_i.
\]

Note that if we set \( h_{i,j} = \partial_j H_i(v) \phi_i \phi_j \) and \( f \) to be the identity function, then the above equations (4.6) and (4.8) satisfy (4.4). Note that for symmetric systems we have \( h_{i,j} = h_{j,i} \) and for \( H \)-monotone solutions we have \( h_{i,j} > 0 \). The following computation for the right-hand side of (4.8) is our main observation to define symmetric systems and \( H \)-monotone solutions and to establish Theorem 4.1;

\[
\sum_{i,j=1}^m \phi_i \phi_j \partial_j H_i(v) \sigma_i (\sigma_i - \sigma_j) = \sum_{i,j=1}^m \phi_i \phi_j \partial_j H_i(v) \sigma_i (\sigma_i - \sigma_j) + \sum_{i,j=1}^m \phi_i \phi_j \partial_j H_i(v) \sigma_i (\sigma_i - \sigma_j) = \sum_{i,j=1}^m \phi_i \phi_j \partial_j H_i(v) \sigma_i (\sigma_i - \sigma_j) + \sum_{i,j=1}^m \phi_i \phi_j \partial_j H_i(v) \sigma_j (\sigma_j - \sigma_i) = \sum_{i,j=1}^m \phi_i \phi_j \partial_j H_i(v) (\sigma_i - \sigma_j)^2 \geq 0.
\]

We are now ready to state the following De Giorgi type result.

**Theorem 4.2.** Suppose that \( v = (v_i)_{i=1}^m \) is a bounded solution of the orientable symmetric system (1.9). Assume also that either \( n = 2, 0 < s_i < 1 \) and \( v \) is stable or \( n = 3, 1/2 \leq s_i < 1 \) and \( v \) is \( H \)-monotone. Then, there exist a constant \( \Gamma_i \in S^{n-1} \) and \( v_i^* : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) such that

\[
(4.10)
\]

\[ v_i(x,y) = v_i^*(\Gamma_i \cdot x, y) \quad \text{for} \quad (x,y) \in \mathbb{R}^{n+1}, \]

and \( i = 1, \ldots, m \). Moreover, for all \( i, j = 1, \ldots, m \) vectors \( \nabla_x v_i(x,0) \) and \( \nabla_x v_j(x,0) \) are parallel and the angle between two vectors is arccos \( \left( \frac{\partial_i H_i(v)}{\partial_j H_j(v)} \right) \).

**Proof:** First, let \( n = 2 \) and \( v \) be a stable solution of (1.9). We apply Lemma 3.2 when \( h_{i,j} = \partial_j H_i(v) \partial_n v_i \partial_n v_j, f \) is the identity function and \( F \) is constant. For each \( 1 \leq k \leq m \), set \( \eta_k(\bar{x}) := \rho_R(\bar{x}) \) in the geometric Poincaré inequality (3.15) when

\[
(4.11)
\]

\[ \rho_R(\bar{x}) := \begin{cases} \frac{\log R}{2}, & \text{if} \quad |\bar{x}| \leq \sqrt{R}, \\
\frac{\log R - \log |\bar{x}|}{\log R}, & \text{if} \quad \sqrt{R} < |\bar{x}| < R, \\
0, & \text{if} \quad |\bar{x}| \geq R. \end{cases} \]
From (3.15) and the fact that $|\nabla \rho_R| \leq \frac{C}{|x|}$ when $\sqrt{R} < |x| < R$, we conclude

$$|\log R|^2 \sum_{i=1}^m \frac{1}{d_{s_i}} \int_{B^+_{R} \cap \mathbb{R}^{n+1}_+} y^{1-2s_i} \left( |\nabla v_i|^2 A_i^2 + |\nabla T_i| |\nabla x_i|^2 \right) d\tilde{x}$$

$$+ |\log R|^2 \sum_{i \neq j} \int_{B^+_{R} \cap \partial \mathbb{R}^{n+1}_+} (\partial_j H_i(v)||\nabla x_i|||\nabla x_j|| - \partial_j H_i(v)\nabla x_i \cdot \nabla x_j) dx$$

$$\leq C \int_{B^+_{R} \setminus B^+_{R/2}} \frac{y^{1-2s_i} |\nabla v_i|^2}{|x|^2} d\tilde{x}.$$

Here, we have used the notion of symmetric systems that is $\sqrt{\partial_j H_i(v)\partial_j H_j(v)} = |\partial_i H_j(v)|$. On the other hand, Lemma 3.2 implies that for each index $1 \leq i \leq m$,

$$\int_{B^+_{R}} y^{1-2s_i} |\nabla v_i|^2 d\tilde{x} \leq CR^2.$$

Straightforward calculations show that for each $i$, we have

$$\int_{B^+_{R} \setminus B^+_{R/2}} \frac{y^{1-2s_i} |\nabla v_i|^2}{|x|^2} d\tilde{x}$$

$$= 2 \int_{B^+_{R} \setminus B^+_{R/2}} \int_{|\tilde{x}|}^R \tau^{-3} y^{1-2s_i} |\nabla v_i|^2 d\tau d\tilde{x} + \frac{1}{R^2} \int_{B^+_{R} \setminus B^+_{R/2}} y^{1-2s_i} |\nabla v_i|^2 d\tilde{x}$$

$$\leq 2 \int_{B^+_{R}} \int_{|\tilde{x}|}^R \tau^{-3} y^{1-2s_i} |\nabla v_i|^2 d\tau d\tilde{x} + \frac{1}{R^2} \int_{B^+_{R}} y^{1-2s_i} |\nabla v_i|^2 d\tilde{x}.$$

Combining this and (1.3), we conclude

$$\int_{B^+_{R} \setminus B^+_{R/2}} \frac{y^{1-2s_i} |\nabla v_i|^2}{|x|^2} d\tilde{x} \leq C \log R,$$

where the positive constant $C$ is independent from $R$. The above decay estimates (4.12) and (4.15) imply that $|\nabla v_i|^2 A_i^2 + |\nabla T_i||\nabla x_i|^2$ and $|\partial_j H_i(v)||\nabla x_i||\nabla x_j|| - \partial_j H_i(v)\nabla x_i \cdot \nabla x_j$ vanish on $\mathbb{R}^n$ and $\partial \mathbb{R}^{n+1}$, respectively. Therefore, each $v_i$ is a one-dimensional function and vectors $\nabla x_i(x,0)$ and $\nabla x_i(x,0)$ are parallel and they are in the same direction when $\partial_j H_i(v)$ is positive and in opposite directions when $\partial_j H_i(v)$ is negative. This proves the desired result in two dimensions.

We now suppose that $n = 3$ and $v$ is a $H$-monotone solution of (1.9). Note that $H$-monotonicity implies stability. The fact that $v = (v_i)^m_{i=1}$ is a bounded stable solution of (1.9) in $\mathbb{R}^n$ implies that the function $\bar{v} = (\bar{v}_i)^m_{i=1}$ where $\bar{v}_i(x_1, x_2, y) := \lim_{x_3 \to \infty} v_i(x_1, x_2, x_3, y)$ is also a bounded stable solution for (1.9) in $\mathbb{R}^3$. From our previous arguments regarding $\mathbb{R}^3_+$, we conclude reduction of dimension for each $\bar{v}_i$ that is $\bar{v}_i(x, y) = \tilde{v}_i^* (\Gamma_i \cdot x, y)$ for $(x, y) \in \mathbb{R}^3_+$ and for some $\Gamma_i \in S^1$. From this and Lemma 2.3, we conclude that the energy of $\bar{v}$ in $\mathbb{R}^3_+$ is bounded by

$$E_R(\bar{v}) \leq CR^2,$$

when $E_R(\bar{v}) = \sum_{i=1}^m \frac{1}{2s_i} \int_{C_R} y^{1-2s_i} |\nabla \bar{v}_i|^2 d\tilde{x} - \int_{C_R} (H(\bar{v}) - c_\phi) dx$ for $c_\phi := \sup H(\bar{v})$. Set $\sigma_i := \frac{\tilde{v}_i}{\phi_i}$ when $\phi_i := \partial_n v_i$ and $\psi_i := \nabla v_i \cdot \eta$ for any fixed $\eta = (\eta', 0) \in \mathbb{R}^{n-1} \times \{0\}$. Note that $\sigma_i = (\sigma_i)^m_{i=1}$ satisfies (4.6) and (4.8). To apply Theorem 4.2, we only need to prove the following energy estimate

$$\sum_{i=1}^m \frac{1}{2s_i} \int_{C_R} y^{n'} |\nabla v_i|^2 d\tilde{x} \leq CR^2 \chi_{\{s_i > 1/2\}} + CR^2 \log R \chi_{\{s_i = 1/2\}}.$$

Define the sequence of functions $v^t = (v^t_i)^m_{i=1}$ when $v^t_i(x, y) := v_i(x', x_n + t, y)$ for $t \in \mathbb{R}$ and $(x, y) = (x', x_n, y) \in \mathbb{R}^n_+$. Note that $v^t$ is a bounded solution of (1.9), i.e.,

$$\begin{cases}
\text{div}(y^{1-2s_i} \nabla v^t_i) = 0 \text{ in } \mathbb{R}^n_+ \\
-\lim_{y \to 0} y^{1-2s_i} \partial_y v^t_i \rightarrow d_{s_i} H_i(v^t) \text{ in } \partial \mathbb{R}^{n+1}_+.
\end{cases}$$
Straightforward calculations show that $v^t_i$ converges to $v_i$ in $C^1_{loc}(\mathbb{R}^n)$ for all $i = 1, \cdots, m$ and

\begin{equation}
\lim_{t \to \infty} E_R(v^t) = E_R(v).
\end{equation}

On the other hand, for every $T > 0$ we have

\begin{equation}
E_R(v) = E_R(v^T) - \int_0^T \partial_t E_R(v^t) dt.
\end{equation}

Note that applying similar arguments, as the ones given in the proof of Theorem 3.3, one can get a lower bound for $\partial_t E_R(v^t)$ of the form (3.35). From this and (4.20), we have

\begin{equation}
E_R(v) \leq E_R(v^T) + M \int_{\partial B_R} \int_0^R \sum_{i=1}^m \frac{y^{1-2s_i}}{1+y} dy d\mathcal{H}^{n-1} + M \int_{B_R \times \{y=R\}} \sum_{i=1}^m y^{-2s_i} dx.
\end{equation}

Now, taking a limit when $T \to \infty$ and using (4.16) we get

\begin{equation}
E_R(v) \leq E_R(\bar{v}) + M \int_{\partial B_R} \int_0^R \sum_{i=1}^m \frac{y^{1-2s_i}}{1+y} dy d\mathcal{H}^{n-1} + M \sum_{i=1}^m R^{n-2s_i},
\end{equation}

\begin{equation}
\leq MR^2 + M \int_{\partial B_R} \int_0^R \sum_{i=1}^m \frac{y^{1-2s_i}}{1+y} dy d\mathcal{H}^{n-1} + M \sum_{i=1}^m R^{n-2s_i},
\end{equation}

\begin{equation}
\leq MR^2 + M \sum_{i=1}^m R^{n-2s_i} \chi_{\{0<s_i<1/2\}} + M \sum_{i=1}^m R^{n-1} \chi_{\{1/2<s_i<1\}}
\end{equation}

\begin{equation}
+M R^{n-1} \log R \chi_{\{s_i=1/2\}} + M \sum_{i=1}^m R^{n-2s_i}.
\end{equation}

Here, we have used estimates provided in Lemma 2.3. Note that when $1/2 < s_* < 1$ and $n = 3$, for any $1 \leq i \leq m$, we have

\begin{equation}
n - 2s_i \leq n - 2s_* < n - 1.
\end{equation}

Applying (4.22) when $n = 3$ and $1/2 < s_* < 1$, we conclude

\begin{equation}
E_R(v) \leq MR^2 + MR^{n-1} \leq CR^2,
\end{equation}

where $C$ is a positive constant that is independent from $R$. Similarly, for the case of $s_* = 1/2$ and $n = 3$, we have

\begin{equation}
E_R(v) \leq MR^2 + MR^{n-1} \log R \leq CR^2 \log R.
\end{equation}

This completes the proof.

Methods and Ideas applied in the above proof are strongly motivated by the ones given by Ambrosio and Cabré in [2], by Alberti, Ambrosio and Cabré in [1], by Farina, Scuinzì and Valdinoci in [28], by Ghoussoub and the author in [31] and Sire and the author in [32]. We now provide another consequence of Theorem 4.1. The following theorem clarifies the behaviour of derivatives of each $v_i$ in various directions when $v = (v_i)_{i=1}^m$ is a bounded stable solution of symmetric system (1.9) in lower dimensions.

**Theorem 4.3.** Suppose that $v = (v_i)_{i=1}^m$ is a bounded stable solution of symmetric system (1.9) for $n \leq 2$ when $s_*$ belong to $[\frac{1}{2}, 1)$ and for $n \leq 1 + 2s_*$ when $s_*$ belongs to $(0, \frac{1}{2})$. Then, either each $\partial_x v_i(x, y)$ vanishes in $\mathbb{R}^{n+1}_+$ or it does not change sign in $\mathbb{R}^{n+1}_+$ for every $1 \leq i \leq m$.

**Proof:** Let $v$ be a bounded stable solution of (1.9). Then, there exits a sequence of functions $\phi = (\phi_i)_i$ such that each $\phi_i$ does not change sign and it satisfies

\begin{equation}
\begin{cases}
\text{div}(y^n \nabla \phi_i) = 0 & \text{in } \mathbb{R}^{n+1}_+,
- \lim_{y \to 0} y^n \partial_y \phi_i = d_s \sum_{j=1}^m \partial_j H(v) \phi_j & \text{in } \partial \mathbb{R}^{n+1}.
\end{cases}
\end{equation}
For each index $1 \leq i \leq m$, define the quotient $s_i := \frac{\partial v_i}{\partial x_i}$ that implies $(\sigma_i \phi_i)^2 = (\partial x v_i)^2$. From Lemma 3.2, we have
\begin{equation}
(4.27) \int_{C_R} y^{s_i}(\sigma_i \phi_i)^2 \, d\bar{x} \leq \int_{C_R} y^{s_i} |\nabla v_i|^2 \, d\bar{x} \leq C R^n \int_0^R \frac{y^{s_i}}{1 + y} \, dy,
\end{equation}
where $R > 1$. Therefore, straightforward calculations show that
\begin{equation}
(4.28) \int_{C_R} y^{s_i}(\sigma_i \phi_i)^2 \, d\bar{x} \leq C \left\{ \begin{array}{ll} R^{n+1-2s_i} & \text{for } 0 < s_i < \frac{1}{2}, \\ R^n \ln R & \text{for } s_i = \frac{1}{2}, \\ R^n & \text{for } \frac{1}{2} < s_i < 1. \end{array} \right.
\end{equation}

Note that for $s_i$ and $\phi_i$ equations (4.6) and (4.8) hold. For $n \leq 2$ when all $s_i$ belong to $[\frac{1}{2}, 1)$ and for $n \leq 1 + 2s_*$ when at least one of $s_i$ belongs to $(0, \frac{1}{2})$, estimate (4.3) holds for an appropriate $F \in \mathcal{F}$. Set $h_{i,j} := \partial j H_i(v) \phi_i \phi_j$ and $f$ to be the identity function in Theorem 4.1. Note that for symmetric systems, we have $h_{i,j} = h_{j,i}$. Theorem 4.1 implies that each $\sigma_i$ is constant. Therefore, there exists a sequence of constants $C = (C_i)_i$ such that $\partial x v_i(x, y) = C_i \phi_i(x, y)$ for $(x, y) \in \mathbb{R}^{n+1}_+$. Since each $\phi_i$ does not change sign, the proof is completed.

\[\square\]

The De Giorgi’s conjecture provides a reduction of dimensions, to one-dimension, for bounded monotone solutions of the Lane-Emden equation when $n \leq 8$. The latter theorem provides a counterpart of the conjecture to multi-component fractional symmetric systems with a general nonlinearity. In what follows, we assume certain extra assumptions on the sign of the nonlinearity $H$ and we establish a Liouville theorem for bounded stable solutions of (1.9) in lower dimensions applying Theorem 4.3, Theorem 4.1 and Theorem 3.1.

**Theorem 4.4.** Suppose that $v = (v_i)_{i=1}^m$ is a bounded stable solution of (1.9) when either $H_i(v) \geq 0$ for all $1 \leq i \leq m$ or $\sum_{i=1}^m v_i H_i(v) \leq 0$. Then, each $v_i$ must be constant provided $n \leq 2(1 + s_*)$.

**Proof:** Since $v$ is a stable solution, there exists a sequence $\phi = (\phi_i)_{i=1}^m$ satisfying (1.10). On the other hand, applying Lemma 3.3, for each $i$, we have
\begin{equation}
(4.29) \int_{C_R} y^{1-2s_i} |\nabla v_i|^2 \, d\bar{x} \leq C R^{n-2s_i}.
\end{equation}
The fact that $n \leq 2 + 2s_*$ implies that $n - 2s_i \leq n - 2s_* \leq 2$. Note that $\sigma_i = \frac{\nabla v_i}{\phi}$ satisfies conditions of Theorem 4.1 for $F(r) = 1$, $h_{i,j} = \partial j H_i(v) \phi_i \phi_j$ and $f$ to be the identity function. Therefore, each $\sigma_i$ must be constant for an arbitrary direction $\eta$. This implies that there exist a constant $\Gamma_i \in S^{n-1}$ and $v_i^* : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ such that $v_i(x, y) = v_i^*(\Gamma_i, x, y)$ for $(x, y) \in \mathbb{R}^{n+1}_+$. In other words, $v = (v_i)_{i=1}^m$ is a bounded stable solution of (1.9) for $n = 1$. Applying Theorem 4.3, we conclude that $\partial x v_i$ does not change sign in $\mathbb{R}^2_+$. Here, we have used the fact that when $\partial x v_i$ vanishes, boundedness implies that each $v_i$ must be constant. Therefore, $v_i$ has to be strictly monotone in $x$ which together with boundedness of $v_i$ proves the existence of $\lim_{x \to \pm \infty} v_i(x, 0)$. Let $\lim_{x \to -\infty} v_i(x, 0) = l_i$ and $\lim_{x \to \infty} v_i(x, 0) = L_i$, where $l_i$ and $L_i$ are constants. Since $v_i$ is strictly monotone, we conclude $l_i < L_i$. We now apply the Hamiltonian identity provided in Theorem 3.1 when $x \to \pm \infty$ to get $\dot{H}(l) = \hat{H}(L)$ where $l = (l_i)_{i=1}^m$ and $L = (L_i)_{i=1}^m$. Note that this is in contradiction with the following
\begin{equation}
(4.30) 0 = \dot{H}(L) - \hat{H}(l) = \sum_{i=1}^m (L_i - l_i) H_i(t(L - l) + l),
\end{equation}
for some $t \in (0, 1)$. This completes the proof.

\[\square\]

Mathematical techniques and ideas that we applied in the above proof are strongly motivated by the ones given in [23] by Dupaigne and Farina. In [23], authors proved that any bounded stable solution of (1.12), that is when $m = 1$ and $s = 1$ in (1.1), is constant provided $n \leq 4$ and $0 \leq H \in C^1(\mathbb{R})$ is a general nonlinearity. Note that for particular nonlinearities the critical dimension is much higher than four dimensions. We also refer interested readers to [5] by Cabe and Capella, to [49] by Villegas and to [30] by the author for the case of radial stable solutions where the optimal dimension is $n = 10$ for a general nonlinearity $H \in C^1(\mathbb{R})$. So, we
expect that Theorem 4.4 could be improved. For specific nonlinearities \( H(u) = e^u \), \( H(u) = u^p \) where \( p > 1 \) and \( H(u) = -u^{-p} \) where \( p > 0 \) the equation is called Gelfand, Lane-Emden and Lane-Emden with negative exponent equations, and Liouville theorems are given for the following optimal dimensions, respectively,

- \( 1 \leq n < 10 \) by Farina in [27],
- \( 1 \leq n < 2 + \frac{4}{p-1}(p + \sqrt{p(p-1)}) \) by Farina in [26],
- \( 1 \leq n < 2 + \frac{4}{p-1}(p + \sqrt{p(p+1)}) \) by Esposito, Ghossoub and Guo in [25].

In the next section, we study a two-component nonlinear Schrödinger system, that is a particular case of (1.1), and we prove various Liouville theorems.

5. NONLINEAR SCHRODINGER SYSTEM: PARTICULAR NONLINEARITY

Consider the following two-component system

\[
\begin{aligned}
  (-\Delta)^{s_1} u_1 &= \mu_1 u_1^3 + \beta u_2^2 u_1 \quad \text{in } \mathbb{R}^n, \\
  (-\Delta)^{s_2} u_2 &= \mu_2 u_2^3 + \beta u_1^2 u_2 \quad \text{in } \mathbb{R}^n,
\end{aligned}
\]

when \( 0 < s_1, s_2 < 1 \) and \( \mu_1, \mu_2, \beta \) are parameters. This is a special case of system (1.1) when \( m = 2 \) and \( H_1(u_1, u_2) = \mu_1 u_1^3 + \beta u_2^2 u_1 \) and \( H_2(u_1, u_2) = \mu_2 u_2^3 + \beta u_1^2 u_2 \). The above system arises in Bose-Einstein condensations and it is well-studied in the literature. We refer interested readers to [3, 15, 32, 43, 46, 51–54] and references therein for more information. The extension function pair \((v_1, v_2)\) given in (1.9) satisfies

\[
\begin{aligned}
  \text{div}( y^{a_i} \nabla v_i ) &= 0 \quad \text{in } \mathbb{R}^{n+1}, \\
  -d_{s_i} \lim_{y \to 0} y^{a_i} \partial_y v_i &= H_i(v_1, v_2) \quad \text{in } \partial \mathbb{R}^{n+1},
\end{aligned}
\]

when \( a_i = 1 - 2s_i \) and \( d_{s_i} = \frac{\Gamma(1-s_i)}{\Gamma(1-s_i)^2} \gamma(s_i) \) for \( 1 \leq i \leq 2 \). Note that when \( \beta = 0 \), system (5.1) becomes decoupled and each equation in (5.1) is of the from

\[
(-\Delta)^s w = w^3 \quad \text{in } \mathbb{R}^n,
\]

when \( \mu_1, \mu_2 > 0 \) for either \( w = \sqrt{\mu_1} u_1 \) and \( \alpha = s_1 \) or \( w = \sqrt{\mu_2} u_2 \) and \( \alpha = s_2 \). The above equation (5.3) is known as the fractional Lane-Emden equation and nonnegative solutions of this equation are classified completely in the literature. It is known that whenever

\[
n < 4\alpha,
\]

the only nonnegative solution for (5.3) is the trivial solution and \( n = 4\alpha \) is the critical dimension, see [14, 38, 42]. We refer interested readers to [16, 17, 33] for the classification of stable solutions of (5.3) when \( 0 < \alpha \leq 2 \). In this article, we are interested in the case of \( \beta \neq 0 \). The following Liouville theorem addresses stable solutions of (5.2) and is a direct consequence of Theorem 4.4. The proof is straightforward and we omit it here.

**Theorem 5.1.** Let \( v = (v_1, v_2) \) be a bounded stable solution of (5.2) when \( n \leq 2 + \min\{s_1, s_2\} \). Assume that either \( \mu_1, \mu_2 \leq 0 \) and \( |\beta| \leq \sqrt{\mu_1 \mu_2} \) or \( \mu_1, \mu_2, \beta \geq 0 \) and solutions \((v_1, v_2)\) are nonnegative. Then, each \( v_i \equiv C_i \) where \( C_i \) is constant.

We now provide a Liouville theorem for solutions of (5.1) in the absence of stability.

**Theorem 5.2.** Suppose that \( s_1 = s_2 = \alpha \) and \( \mu_1, \mu_2 \) are nonnegative. Assume that \( u = (u_1, u_2) \) is a nonnegative solution of (5.1) when \( n \leq 3\alpha \) and \( \beta > -\sqrt{\mu_1 \mu_2} \), then \((u_1, u_2) = (0, 0)\).

**Proof:** Let \( u = (u_1, u_2) \) be a nonnegative solution of (5.1) when \( s_1 = s_2 = \alpha \). Without loss of generality assume that \( u_1 > 0 \) and \( \mu_1 > 0 \). It is straightforward to show that equivalently \((u_1, u_2)\) is a solution for the integral system

\[
\begin{aligned}
  u_1(x) &= \int_{\mathbb{R}^n} \frac{\mu_1 u_1^3(y) + \beta u_2^2(y) u_1(y)}{|x-y|^{n-2s_1}} \, dy \quad \text{for } x \in \mathbb{R}^n, \\
  u_2(x) &= \int_{\mathbb{R}^n} \frac{\mu_2 u_2^3(y) + \beta u_1^2(y) u_2(y)}{|x-y|^{n-2s_2}} \, dy \quad \text{for } x \in \mathbb{R}^n.
\end{aligned}
\]

Assume that parameter \( \beta \) is nonnegative. Since each \( u_i \) is nonnegative, system (5.5) gives

\[
\begin{aligned}
  u_1(x) &\geq \mu_1 \int_{\mathbb{R}^n} \frac{u_1^3(y)}{|x-y|^{n-2s_1}} \, dy \quad \text{for } x \in \mathbb{R}^n, \\
  u_2(x) &\geq \mu_2 \int_{\mathbb{R}^n} \frac{u_2^3(y)}{|x-y|^{n-2s_2}} \, dy \quad \text{for } x \in \mathbb{R}^n.
\end{aligned}
\]
Applying Liouville theorems given in [12], see also [13, 14, 42], to each of the above integral inequalities we conclude that \((u_1, u_2) = (0, 0)\) when \(n \leq 3\alpha\). We now let \(\beta\) be negative that is \(-\sqrt{\mu_1 \mu_2} < \beta < 0\). Multiplying the first equation in (5.5) with \(\sqrt{\mu_2/\mu_1}\) and adding both equations in (5.5) we get
\[
(5.7) \quad \sqrt{\mu_2/\mu_1} u_1(x) + u_2(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2\alpha}} \left[ \sqrt{\mu_2/\mu_1} (\mu_1 u_1^3(y) + \beta u_1(y) u_2^2(y)) + \mu_2 u_2^3(y) + \beta u_2(y) u_1^2(y) \right] dy,
\]
for \(x \in \mathbb{R}^n\). To simplify the right-hand side of above equality, we claim that there exists a positive constant \(M\) such that
\[
(5.8) \quad \sqrt{\mu_2/\mu_1} (\mu_1 u_1^3(y) + \beta u_1(y) u_2^2(y)) + \mu_2 u_2^3(y) + \beta u_2(y) u_1^2(y) \geq M \left( \sqrt{\mu_2/\mu_1} u_1 + u_2 \right)^3.
\]
In order to prove this claim, let \(\tau := u_2/u_1\) and define continuous function \(F : \mathbb{R}^+ \to \mathbb{R}\) as
\[
(5.9) \quad F(\tau) = \frac{\sqrt{\mu_2/\mu_1} (\mu_1 + \beta \tau^2) + \mu_2 \tau^3 + \beta \tau}{\left( \sqrt{\mu_2/\mu_1} + \tau \right)^3}.
\]
Note that \(F(0) = \mu_1 > 0\) and \(\lim_{\tau \to \infty} F(\tau) = \mu_2 \geq 0\). In addition, for \(\beta > -\sqrt{\mu_1 \mu_2}\), we have
\[
(5.10) \quad \left( \frac{\sqrt{\mu_2/\mu_1} + \tau}{\sqrt{\mu_2/\mu_1}} \right)^3 F(\tau) > \frac{\sqrt{\mu_2/\mu_1} (\mu_1 - \sqrt{\mu_1 \mu_2} \tau^2) + \tau (\mu_2 \tau^2 - \sqrt{\mu_1 \mu_2})}{\left( \sqrt{\mu_2/\mu_1} + \tau \right)^3} = \left( \frac{\sqrt{\mu_2 \tau} - \sqrt{\mu_1}}{\sqrt{\mu_2/\mu_1} + \tau} \right)^2 \left( \sqrt{\mu_2 \tau} + \sqrt{\mu_1} \right) \geq 0,
\]
that is \(F(\tau) > 0\) when \(\tau > 0\). This proves the claim. From (5.8) and (5.7), we get
\[
(5.12) \quad \sqrt{\mu_2/\mu_1} u_1(x) + u_2(x) \geq M \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2\alpha}} \left( \frac{\sqrt{\mu_2/\mu_1} u_1(x) + u_2(y)}{\sqrt{\mu_2/\mu_1 u_1(x) + u_2(x)}} \right)^3 dy \quad \text{for} \quad x \in \mathbb{R}^n.
\]
Let \(z(x) := \sqrt{M} \left( \frac{\sqrt{\mu_2/\mu_1} u_1(x) + u_2(x)}{\sqrt{\mu_2/\mu_1 u_1(x) + u_2(x)}} \right) > 0\) in (5.12) to obtain
\[
(5.13) \quad z(x) \geq \frac{1}{\sqrt{M}} \int_{\mathbb{R}^n} \frac{z^3(y)}{|x-y|^{n-2\alpha}} dy \quad \text{for} \quad x \in \mathbb{R}^n.
\]
We now apply Liouville theorems given in [12, 13] for the above integral inequality to conclude that \(z \equiv 0\) when \(n \leq 2\alpha\) and for \(n > 2\alpha\) when \(3 \leq \frac{n}{n-2\alpha}\). This implies that whenever \(n \leq 3\alpha\), we have \(z \equiv 0\). This completes the proof.

According to (5.4), one may expect that the critical dimension for the above theorem is \(n = 4\alpha\) and this remains an open problem. Note that methods and ideas applied in the above proof are strongly motivated by the ones given in [15] by Dancer, Wei and Weth and in [43] by Lin and Wei. We now provide a monotonicity formula for solutions of the two-component Schrödinger system (5.2). This is a direct consequence of Theorem 3.4 and we omit the proof. Note that Frank and Lenzmann in [34] and with Silvestre in [35] used similar monotonicity formulae to study uniqueness of solutions for the fractional Schrödinger operator.

**Theorem 5.3.** Let \(v = (v_1, v_2)\) be a bounded solution of (5.2). Suppose that \(\mu_1, \mu_2 \leq 0\) and \(|\beta| \leq \sqrt{\mu_1 \mu_2}\). Then
\[
(5.14) \quad I(R) = R^{-n+2s} \left[ \sum_{i=1}^2 d_s \int_{B_R^+} g^{2s} |\nabla v_i|^2 dx \right. - \left. \int_{B_R \times \{y=0\}} \left( \frac{\mu_1}{2} v_1^4 + \frac{\mu_2}{2} v_2^4 + \beta v_1^2 v_2^2 \right) dx \right],
\]
is a nondecreasing function of \(R \geq 1\).

Lastly, we provide a monotonicity formula for radial solutions of (5.2) for all parameters \(\mu_1, \mu_2, \beta \in \mathbb{R}\).

**Theorem 5.4.** Let \(v = (v_i)_{i=1}^m\) be a bounded radial solution of (5.2), i.e. \(v_i(x, y) = v_i(|x|, y)\). Then, the following function is nondecreasing in \(r > 0\),
\[
(5.15) \quad J(r) := \sum_{i=1}^2 \int_0^\infty \int_0^y y^{1-2s} \left[ (\partial_r v_i(r, y))^2 - (\partial_y v_i(r, y))^2 \right] dy - \tilde{H}(v_1(r, 0), v_2(r, 0)),
\]
where $\tilde{H}(v) = \frac{\alpha}{2}v_1^4 + \frac{\alpha}{2}v_2^4 + \beta v_1^2 v_2^2$. More precisely,

$$J'(r) = -\frac{n-1}{r} \sum_{i=1}^{2} d_{s_i} \int_{0}^{\infty} y^{1-2s_i} (\partial_r v_i)^2 dy.$$  

\textbf{Proof:} Suppose that $v = (v_i)$ is radially symmetric in $x$. For $r = |x|$ we have

$$\begin{align*}
\partial_r v_i + \frac{n-1}{r} \partial_r v_i + \partial_y v_i + \frac{\alpha}{2} \partial_y v_i &= 0 \quad \text{in} \quad (0, \infty) \times (0, \infty), \\
d_{s_i} \lim_{y \to 0} y^{\alpha s_i} \partial_y v_i &= H_1(v) \quad \text{in} \quad (0, \infty) \times (y = 0),
\end{align*}$$

where $a_i = 1 - 2s_i$. Define the following function of $r$,

$$p(r) := \sum_{i=1}^{2} d_{s_i} \int_{0}^{\infty} y^{1-2s_i} [(\partial_r v_i)^2 - (\partial_y v_i)^2] dy$$

Taking derivative of $p$ with respect to $r$ and using (5.17) to substitute values of $\partial_r v_i$, we conclude

$$p'(r) = -\frac{n-1}{r} \sum_{i=1}^{2} d_{s_i} \int_{0}^{\infty} y^{\alpha s_i} (\partial_r v_i)^2 dy - \sum_{i=1}^{2} d_{s_i} \int_{0}^{\infty} y^{\alpha s_i} (\partial_y v_i)(\partial_r v_i)dy$$

$$- \sum_{i=1}^{2} d_{s_i} a_i \int_{0}^{\infty} y^{-2s_i} (\partial_r v_i)(\partial_y v_i)dy - \sum_{i=1}^{2} d_{s_i} \int_{0}^{\infty} y^{\alpha s_i} (\partial_y v_i)(\partial_y v_i)dy.$$ 

Applying integration by parts and using the boundary term in (1.9), we have

$$\begin{align*}
d_{s_i} \int_{0}^{\infty} \left[ y^{\alpha s_i} \partial_y v_i \partial_r v_i + a_i y^{-2s_i} \partial_r v_i \partial_y v_i + y^{\alpha s_i} \partial_y v_i \partial_y v_i \right] dy &= \lim_{y \to 0} \sum_{i=1}^{2} d_{s_i} y^{\alpha s_i} \partial_r v_i \partial_y v_i \\
&= \partial_r \left( \tilde{H}(v) \right).
\end{align*}$$

Combining (5.20) and (5.19) and setting $J(r) := p(r) - \tilde{H}(v(r, 0))$ completes the proof.

We end this section with this point that in the absence of monotonicity and stability assumptions, the qualitative behaviour of solutions of elliptic and Hamiltonian systems with a general nonlinearity are studies extensively in the literature. We refer interested readers to [18, 19, 39] and references therein.

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Department of Mathematics, The University of Texas at San Antonio, San Antonio, TX 78249, USA
E-mail address: mostafa.fazly@utsa.edu

Department of Mathematics & Computer Science, University of Lethbridge, Lethbridge, AB T1K 3M4 Canada.
E-mail address: mostafa.fazly@uleth.ca