Robust adaptive efficient estimation for a
semi-Markov continuous time regression from
discrete data *

Vlad Stefan Barbu†, Slim Beltaief‡ and Serguei Pergamenshchikov§

Abstract

In this article we consider the nonparametric robust estimation problem
for regression models in continuous time with semi-Markov noises observed
in discrete time moments. An adaptive model selection procedure is pro-
posed. A sharp non-asymptotic oracle inequality for the robust risks is ob-
tained. We obtain sufficient conditions on the frequency observations under
which the robust efficiency is shown. It turns out that for the semi-Markov
models the robust minimax convergence rate may be faster or slower than the
classical one.

MSC: primary 62G08, secondary 62G05

Keywords: Non-asymptotic estimation; Robust risk; Model selection; Sharp oracle
inequality; Asymptotic efficiency.

*This work was done under partial financial support of the grant of RSF number 14-49-00079
(National Research University "MPEI" 14 Krasnokazarmennaya, 111250 Moscow, Russia) and by
RFBR Grant 16-01-00121 .
†Laboratoire de Mathématiques Raphaël Salem, UMR 6085 CNRS-Université de Rouen Nor-
mandie, France, e-mail: barbu@univ-rouen.fr
‡Laboratoire de Mathématiques Raphaël Salem, UMR 6085 CNRS-Université de Rouen Nor-
mandie, France, e-mail: slim.beltaief1@univ-rouen.fr
§Laboratoire de Mathématiques Raphaël Salem, UMR 6085 CNRS-Université de Rouen Nor-
mandie, France and International Laboratory of Statistics of Stochastic Processes and Quantitative Fi-
nance of National Research Tomsk State University, Russia, e-mail: Serge.Pergamenshchikov@univ-
rouen.fr
1 Introduction

In this paper we consider the semi-Markov regression model in continuous time introduced in [1], i.e.

\[ dy_t = S(t) dt + d \xi_t, \quad 0 \leq t \leq n, \quad (1.1) \]

where \( S(\cdot) \) is an unknown 1-periodic function defined on \( \mathbb{R} \) with values on \( \mathbb{R} \), \((\xi_t)_{t \geq 0}\) is the unobserved noise process defined through a certain semi-Markov process in Section 2.

Our problem in the present paper is to estimate the unknown function \( S \) in the model (1.1) on the basis of observations \((y_{t_j})_{0 \leq j \leq np}, \quad t_j = j \Delta, \quad \Delta = \frac{1}{p}, \quad (1.2)\)

where the integer \( p \geq 1 \) is the observation frequency. Firstly, this problem was considered in the framework “signal+white noise” (see, for example, [6] or [22]). Later, to introduce a dependence in the continuous time regression model in [11], [9], [8], [14], the Ornstein-Uhlenbeck processes has been used to model the “color noise”. Moreover, in order to introduce the dependence and the jumps in the regression model (1.1), the papers [15] and [16] use the non Gaussian Ornstein-Uhlenbeck processes defined in [2]. The problem in all these papers is that the introduced Ornstein-Uhlenbeck type of dependence decreases with a geometric rate. So, asymptotically when the duration of observations goes to infinity, we obtain the same “signal+white noise” model very quick. To keep the dependence for sufficiently large duration of observations, in [1] it was proposed the model (1.1) with a semi-Markov component in the jumps of the noise process \((\xi_t)_{t \geq 0}\).

The main goal of this paper is to develop adaptive robust method from [1], that was based on continuous observations, to the estimation problem based on discrete observations given in (1.2). In this paper we use quadratic risk defined as

\[ R_Q(\tilde{S}_n, S) = E_{Q,S} \| \tilde{S}_n - S \|^2, \quad (1.3) \]

where \( \tilde{S}_n(\cdot) \) is some estimate (i.e. any periodical function measurable with respect to the observations \( \sigma\{y_{t_0}, \ldots, y_{t_n}\} \)), \( \| f \|^2 = \int_0^1 f^2(s) ds \) and \( E_{Q,S} \) is the expectation with respect to the distribution \( P_{Q,S} \) of the process (1.1) corresponding to the unknown noise distribution \( Q \) in the Skorokhod space \( D[0, n] \). We assume that this distribution belongs to some distribution family \( Q_n \) specified in Section 2.

To study the properties of the estimators uniformly over the noise distribution (what is really needed in practice), we use the robust risk defined as

\[ R^*_n(\tilde{S}_n, S) = \sup_{Q \in Q_n} R_Q(\tilde{S}_n, S). \quad (1.4) \]
Thus the goal of this paper is to develop a robust efficient model selection method based on the observations (1.2) for the model (1.1) with the semi-Markov components in the jumps of the noise \((\xi_t)_{t \geq 0}\). We use the approach proposed by Konev and Pergamenshchikov in [16] for continuous-time regression models observed in the discrete time moments. Unfortunately, we cannot use directly this method for semi-Markov regression models, since their tool essentially uses the fact that the Ornstein-Uhlenbeck dependence decreases with geometrical rate and obtain sufficiently quickly the “white noise” case. In the present paper, in order to obtain the sharp non-asymptotic oracle inequalities, we use the renewal methods from [1] developed for the model (1.1). As a consequence, we can obtain the constructive sufficient conditions that provide the robust efficiency for proposed model selection procedures.

The rest of the paper is organized as follows. In Section 2 we state the main conditions under which we consider the model (1.1). In Section 3 we construct the model selection procedure on the basis of weighted least squares estimates, here we also specify the set of admissible weight sequences in the model selection procedure. In Section 4 we state the main results in the form of oracle inequalities for the quadratic risk and the robust risk. In Section 5 we study some properties of the regression model (1.1). Section 6 is devoted to some numerical results. In section A.2 we study some properties of the stochastic integral. Section 7 gives the proofs of the oracle inequalities for the regression model (1.1) with the noises introduced in Section 2. Some auxiliary are given in an Appendix.

## 2 Main conditions

First, we assume that the noise process \((\xi_t)_{t \geq 0}\) in the model (1.1) is defined as

\[
\xi_t = \varrho_1 L_t + \varrho_2 z_t,
\]

where \(\varrho_1\) and \(\varrho_2\) are unknown coefficients, \((L_t)_{t \geq 0}\) is a Levy process defined as

\[
L_t = \tilde{\varrho} w_t + \sqrt{1 - \tilde{\varrho}^2} \tilde{\mu}_t, \quad \tilde{\mu}_t = x \ast (\mu - \tilde{\mu})_t,
\]

where, \(0 \leq \tilde{\varrho} \leq 1\) is some unknown constant, \((w_t)_{t \geq 0}\) is a standard Brownian motion, \(\mu(ds \, dx)\) is the jump measure with deterministic compensator \(\tilde{\mu}(ds \, dx) = ds \Pi(dx)\), \(\Pi(\cdot)\) is some positive measure on \(\mathbb{R}\) (see, for example [10, 7] for details), with

\[
\Pi(x^2) = 1 \quad \text{and} \quad \Pi(x^8) < \infty.
\]
Here we use the usual notations for \( \Pi(x|m) = \int_{\mathbb{R}} |z|^m \Pi(dz) \). Note that \( \Pi(x) \) may be equal to \(+\infty\). In this paper we assume that the “dependent part” in the noise (2.1) is modeled by the semi-Markov process \((z_t)_{t\geq 0}\) defined as

\[
    z_t = \sum_{i=1}^{N_t} Y_i,
\]

where \((Y_i)_{i\geq 1}\) is an i.i.d. sequence of random variables with

\[
    \mathbb{E} Y_i = 0, \quad \mathbb{E} Y_i^2 = 1 \quad \text{and} \quad \mathbb{E} Y_i^4 < \infty.
\]

(2.4)

Here \(N_t\) is a general counting process (see, for example, [18]) defined as

\[
    N_t = \sum_{k=1}^{\infty} 1 \{ T_k \leq t \} \quad \text{and} \quad T_k = \sum_{l=1}^{k} \tau_l,
\]

(2.6)

with \((\tau_l)_{l \geq 1}\) an i.i.d. sequence of positive integrated random variables with the distribution \(\eta\) and mean \(\tilde{\tau} = \mathbb{E} \tau_1 > 0\). We assume that the processes \((N_t)_{t \geq 0}\) and \((Y_i)_{i \geq 1}\) are independent between them and are also independent of \((L_t)_{t \geq 0}\). Note that the process \((z_t)_{t \geq 0}\) is a special case of a semi-Markov process (see, e.g., [3] and [4]).

Remark 2.1. It should be noted that, if \(\tau_j\) is an Exponential random variable, i.e. \(g\) is the Exponential density, then \((N_t)_{t \geq 0}\) is a Poisson process and, in this case, \((\xi_t)_{t \geq 0}\) is a Lévy process for which this model is studied in [12], [13] and [15].

But, in the general case when the process (2.4) is not a Lévy process, this process has a memory and cannot be treated in the framework of semi-martingales with independent increments. One needs to develop a new tool based on the renewal theory arguments.

Let us denote by \(\rho\) the density of the renewal measure \(\tilde{\eta}\) defined as

\[
    \tilde{\eta} = \sum_{l=1}^{\infty} \eta^{(l)},
\]

(2.7)

where \(\eta^{(l)}\) is the \(l\)th convolution power of the measure \(\eta\). As to the parameters in (2.1), we assume that

\[
    \sigma_Q = \varrho_1^2 + \varrho_2^2/\tilde{\tau} \leq \varsigma^*,
\]

(2.8)

where the unknown bound \(\varsigma^*\) is a function of \(n\), i.e. \(\varsigma^* = \varsigma^*(n)\), such that for any \(\bar{\epsilon} > 0\)

\[
    \lim_{n \to \infty} n^{\bar{\epsilon}} \varsigma^*(n) = +\infty \quad \text{and} \quad \lim_{n \to \infty} \frac{\varsigma^*(n)}{n^{\bar{\epsilon}}} = 0.
\]

(2.9)
We denote by $Q_n$ the family of all distributions of the process (2.1) in $D[0, n]$ satisfying the properties (2.8) – (2.9).

**Remark 2.2.** As we will see later, the parameter $\sigma_Q$ is the limit of the Fourier transform of the noise process (2.1). Such a limit is called variance proxy (see [15]).

We assume that the distribution $\eta$ has a density $g$ that satisfies the following conditions.

**H$_1$)** Assume that, for any $x \in \mathbb{R}$, there exist the finite limits

$$g(x-) = \lim_{z \to x^-} g(z) \quad \text{and} \quad g(x+) = \lim_{z \to x^+} g(z)$$

and, for any $K > 0$, there exists $\delta = \delta(K) > 0$ for which

$$\sup_{|x| \leq K} \int_0^\delta \frac{|g(x + t) + g(x - t) - g(x+) - g(x-)|}{t} \, dt < \infty. \quad (2.10)$$

**H$_2$)** For any $\gamma > 0$,

$$\sup_{z \geq 0} z^\gamma |2g(z) - g(z-) - g(z+)| < \infty.$$

**H$_3$)** There exists $\beta > 0$ such that $\int_\mathbb{R} e^{\beta x} g(x) \, dx < \infty$.

**Remark 2.3.** It should be noted that Condition H$_3$ means that there exists an exponential moment for the random variable $(\tau_j)_{j \geq 1}$, i.e. these random variables are not too large. This is a natural constraint since these random variables define the intervals between jumps, i.e. the jump frequency. So, to study the influence of the jumps in the model (1.1) one needs to consider the noise process (2.1) with “small” interval between jumps or large jump frequency.

For the next condition we need the Fourier transform for any function $f : \mathbb{R} \to \mathbb{R}$ from $L_1(\mathbb{R})$ defined by

$$\hat{f}(\theta) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\theta x} f(x) \, dx. \quad (2.11)$$

**H$_4$)** There exists $t^* > 0$ such that the function $\hat{g}(\theta - it)$ belongs to $L_1(\mathbb{R})$ for any $0 \leq t \leq t^*$. 

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It is clear that Conditions $H_1$–$H_4$ hold true for any continuously differentiable function $g$ having an exponential moment, for example, for the $\Gamma$ density.

It should be noted that in view of Proposition 5.2 from [1], Conditions $H_1$–$H_4$ imply
\[ \|\Upsilon\|_1 = \int_0^{+\infty} |\Upsilon(x)| \, dx < \infty, \tag{2.12} \]
where $\Upsilon(x) = \rho(x) - 1/\bar{\tau}$.

**3 Model selection**

In this section we construct a model selection procedure for estimating the unknown function $S$ given in (1.1) starting from the discrete-time observations (1.2) and we establish the oracle inequality for the associated risk. To this end, note that for any function $f : [0,n] \rightarrow \mathbb{R}$ from $L_2[0,n]$, the integral
\[ I_n(f) = \int_0^n f(s) \, d\xi_s \tag{3.1} \]
is well defined, with $E_Q I_n(f) = 0$. Moreover, as it is shown in Lemma A.2 under the conditions $H_1$–$H_4$,
\[ E_Q I_n^2(f) \leq \kappa_Q \int_0^n f_s^2 \, ds, \tag{3.2} \]
where $\kappa_Q = \epsilon_1^2 + \epsilon_2^2 |\rho|_s$ and $|\rho|_s = \sup_{t \geq 0} |\rho(t)| < \infty$.

In this paper we will use the trigonometric basis $(\phi_j)_{j \geq 1}$ in $L_2[0,1]$ defined as
\[ \phi_1 = 1, \quad \phi_j(x) = \sqrt{2}\text{Tr}_j(2\pi[j/2]x), \quad j \geq 2, \tag{3.3} \]
where the function $\text{Tr}_j(x) = \cos(x)$ for even $j$ and $\text{Tr}_j(x) = \sin(x)$ for odd $j$, $[x]$ denotes the integer part of $x$. By making use of this basis we consider the discrete Fourier transformation of $S$
\[ S(t) = \sum_{j=1}^p \theta_{j,p} \phi_j(t), \quad t \in \{t_1, ..., t_p\}, \tag{3.4} \]
where the Fourier coefficients are defined by
\[ \theta_{j,p} = (S,\phi_j)_p = \frac{1}{p} \sum_{i=1}^p S(t_i)\phi_j(t_i). \tag{3.5} \]
In the sequel the corresponding norm will be denoted by $\|x\|_p = (x, x)_p$. These Fourier coefficients $\theta_{j,p}$ can be estimated by

$$\hat{\theta}_{j,p} = \frac{1}{n} \int_0^n \Psi_{j,p}(t) d y_t, \quad \text{and} \quad \Psi_{j,p}(t) = \sum_{l=1}^{np} \phi_j(t_l) 1_{\{t_{l-1} < t \leq t_l\}}. \quad (3.6)$$

Let us note that the system of the functions $(\Psi_{j,p})_{1 \leq j \leq p}$ is orthonormal in $L_2[0, 1]$ because

$$\int_0^1 \Psi_{j,p}(t) \Psi_{i,p}(t) d t = (\phi_j, \phi_i)_p = 1_{\{i = j\}}.$$ 

In the sequel we need the Fourier coefficients of the function $S$ with respect to the new basis $(\Psi_{j,p})_{1 \leq j \leq p}$. These coefficients can be written as

$$\bar{\theta}_{j,p} = \int_0^1 S(t) \Psi_{i,p}(t) d t = \theta_{j,p} + h_{j,p}, \quad (3.7)$$

where

$$h_{j,p}(S) = \sum_{l=1}^p \int_{t_{l-1}}^{t_l} \phi_j(t_l)(S(t) - S(t_l)) d t.$$ 

From (1.1) it follows directly that these Fourier coefficients satisfy the equation

$$\hat{\theta}_{j,p} = \bar{\theta}_{j,p} + \frac{1}{\sqrt{n}} \xi_{j,p}, \quad \text{where} \quad \xi_{j,p} = \frac{1}{\sqrt{n}} I_n(\Psi_{j,p}). \quad (3.8)$$

For any $0 \leq t \leq 1$ we estimate the function $S$ by the weighted least squares estimator

$$\hat{S}_\lambda(t) = \sum_{j=1}^n \lambda(j) \hat{\theta}_{j,p} \psi_{j,p}(t), \quad (3.9)$$

where the weight vector $\lambda = (\lambda(1), \ldots, \lambda(n))$ belongs to some finite set $\Lambda$ from $[0, 1]^n$, $\hat{\theta}_{j,p}$ was defined in (3.6) and $\phi_j$ in (3.3). Now let us consider

$$\nu = #(\Lambda) \quad \text{and} \quad |\Lambda|_* = \max_{\lambda \in \Lambda} L(\lambda), \quad (3.10)$$

where $#(\Lambda)$ is the cardinal number of $\Lambda$ and $L(\lambda) = \sum_{j=1}^n \lambda(j)$. In the sequel we assume that $|\Lambda|_* \geq 1$ and $\lambda(j) = 0$ for $j \geq p$.

In order to find a proper weight sequence $\lambda$ in the set $\Lambda$, one needs to specify a cost function. When choosing an appropriate cost function, one can use the following argument. Let as consider the empirical squared error

$$\text{Err}(\lambda) = \|\hat{S}_\lambda - S\|^2, \quad (3.11)$$
which in our case is equal to

$$\text{Err}(\lambda) = \sum_{j=1}^{n} \lambda^2(j) \tilde{\theta}_j^2 - 2 \sum_{j=1}^{n} \lambda(j) \tilde{\theta}_j \tilde{\theta}_j + ||S||^2.$$ (3.12)

Since the Fourier coefficients $\theta_j$ are unknown, the weight coefficients $(\lambda(j))_{1 \leq j \leq p}$ cannot be determined by minimizing this quality. To circumvent this difficulty, one needs to replace the terms $\tilde{\theta}_j \tilde{\theta}_j$ by their estimators $\tilde{\theta}_j$. Let us set

$$\tilde{\theta}_j = \hat{\theta}_j - \frac{\sigma_n}{n}.$$ (3.13)

Here $\sigma_n$ is an estimate for the proxy variance $\sigma_Q$ defined in (2.8). For example, we can take it as

$$\hat{\sigma}_n = \frac{n}{\hat{\rho}} \sum_{j=l}^{\hat{\rho}} \hat{\theta}_j^2 \quad \text{and} \quad \hat{\rho} = \min(p, n),$$ (3.14)

where $l = \lceil \sqrt{n} \rceil$ and we set $\hat{\sigma}_n = 0$ for $l > p$. For this change in the empirical squared error, one has to pay some penalty. Thus we obtain the cost function of the form

$$J_n(\lambda) = \sum_{j=1}^{n} \lambda^2(j) \tilde{\theta}_j^2 - 2 \sum_{j=1}^{n} \lambda(j) \tilde{\theta}_j + \delta P_n(\lambda),$$ (3.15)

where $\delta > 0$ is some threshold which will be specified later and the penalty term is

$$P_n(\lambda) = \frac{\sigma_n|\lambda|^2}{n}.$$ (3.16)

Minimizing the cost function, that is

$$\hat{\lambda} = \arg\min_{\lambda \in \Lambda} J_n(\lambda),$$ (3.17)

and substituting the obtained weight coefficients $\hat{\lambda}$ in (3.9), lead to the model selection procedure

$$\hat{S}_* = \hat{S}_{\hat{\lambda}}.$$ (3.18)

We recall that the set $\Lambda$ is finite, so $\hat{\lambda}$ exists. In the case when $\hat{\lambda}$ is not unique we take one of them.
4 Main results

4.1 Oracle inequalities

First we define the following constant which will be used to describe the rest term in the oracle inequalities. We set

$$g_{n,p} = 1 + |\Lambda| \left( \frac{\sqrt{\mu}}{p} + \frac{1}{\sqrt{p}} \right). \quad (4.1)$$

Firstly, we obtain the non asymptotic oracle inequality for the model selection procedure (3.18).

**Theorem 4.1.** Assume that Conditions $H_1$–$H_4$ hold true. Then, there exists some constant $l^* > 0$ such that, for any noise distribution $Q$, the weight vector set $\Lambda$, for any periodic function $S$ for any $n \geq 1$, $p \geq 3$ and $0 < \delta \leq 1/6$, the procedure (3.18) satisfies the following oracle inequality

$$R_Q(\hat{S}_*, S) \leq 1 + 3\delta \min_{\lambda \in \Lambda} R_Q(\hat{S}_\lambda, S) + l^* \frac{\nu}{\delta n} (\sigma_Q + |\Lambda|_s E_Q|\hat{\sigma}_n - \sigma_Q|). \quad (4.2)$$

**Corollary 4.2.** Assume that Conditions $H_1$–$H_4$ hold true and that the proxy variance $\sigma_Q$ is known. Then there exists some constant $l^* > 0$ such that for any noise distribution $Q$, the weight vectors set $\Lambda$, for any periodic function $S$ for any $n \geq 1$, $p \geq 3$ and $0 < \delta \leq 1/6$, the procedure (3.18) with $\hat{\sigma}_n = \sigma_Q$, satisfies the following oracle inequality

$$R_Q(\hat{S}_*, S) \leq 1 + 3\delta \min_{\lambda \in \Lambda} R_Q(\hat{S}_\lambda, S) + l^* \frac{\sigma_Q^\nu}{\delta n}. \quad (4.3)$$

Now we study the model selection procedure (3.18) using the proxy estimate (3.14).

**Theorem 4.3.** Assume that the function $S$ is continuously differentiable and that Conditions $H_1$–$H_4$ hold true. Then there exists some constant $l^* > 0$ such that for any noise distribution $Q$, the weight vectors set $\Lambda$, for any periodic function $S$ for any $n \geq 1$, $p \geq 3$ and $0 < \delta \leq 1/6$, the procedure (3.18) satisfies the following oracle inequality

$$R_Q(\hat{S}_*, S) \leq 1 + 3\delta \min_{\lambda \in \Lambda} R_Q(\hat{S}_\lambda, S) + l^* \frac{\nu}{\delta n} (1 + \sigma_Q)^3 \left(1 + \|\hat{S}\|^2\right) g_{n,p}. \quad (4.4)$$

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Let us study the robust risks (1.4) for the procedure (3.18). In this case this family consists of all distributions on the Skorokhod space $\mathcal{D}[0, n]$ of the process (2.1) with the parameters satisfying Conditions (2.8)–(2.9).

In order to obtain the efficiency property, we specify the weight coefficients $(\lambda(j))_{1 \leq j \leq n}$ in the procedure (3.18). Consider, for some fixed $0 < \varepsilon < 1$, a numerical grid of the form

$$\mathcal{A} = \{1, \ldots, k^*\} \times \{\varepsilon, \ldots, m\varepsilon\},$$

where $m = [1/\varepsilon^2]$. We assume that both parameters $k^* \geq 1$ and $\varepsilon$ are functions of $n$, i.e. $k^* = k^*(n)$ and $\varepsilon = \varepsilon(n)$, such that

$$\lim_{n \to \infty} k^*(n) = +\infty, \quad \lim_{n \to \infty} \frac{k^*(n)}{\ln n} = 0,$$

$$\lim_{n \to \infty} \varepsilon(n) = 0 \quad \text{and} \quad \lim_{n \to \infty} n^\delta \varepsilon(n) = +\infty,$$

for any $\delta > 0$. One can take, for example, for $n \geq 2$

$$\varepsilon(n) = \frac{1}{\ln n} \quad \text{and} \quad k^*(n) = k^*_0 + \sqrt{\ln n},$$

where $k^*_0 \geq 0$ is some fixed constant. For each $\alpha = (\beta, 1) \in \mathcal{A}$, we introduce the weight sequence

$$\lambda_{\alpha} = (\lambda_{\alpha}(j))_{1 \leq j \leq p}$$

with the elements

$$\lambda_{\alpha}(j) = 1_{\{1 \leq j < j^*_\alpha\}} + \left(1 - \frac{j}{\omega_{\alpha}}\right)^\beta 1_{\{j^*_\alpha \leq j \leq \omega_{\alpha}\}},$$

where $j^*_\alpha = 1 + [\ln \nu_n]$, $\omega_{\alpha} = (d_{\beta} \nu_n)^{1/(2\beta + 1)}$,

$$d_{\beta} = \frac{(\beta + 1)(2\beta + 1)}{\pi^{2\beta} \beta} \quad \text{and} \quad \nu_n = n/\varsigma^*.$$

We remind that the threshold $\varsigma^*$ is introduced in the definition of the distribution family $\mathcal{Q}_n$ in (2.8). Now we define the set $\Lambda$ as

$$\Lambda = \{\lambda_{\alpha}, \alpha \in \mathcal{A}\}.$$

These weight coefficients are used in [15, 16] for continuous time regression models to show the asymptotic efficiency. Note also that in this case the cardinal of the set $\Lambda$ is

$$\nu = k^* m.$$
Moreover, taking into account that $d_\beta < 1$ for $\beta \geq 1$ we obtain for the set (4.9)

$$|\Lambda|_* \leq 1 + \sup_{\alpha \in \mathcal{A}} \omega_\alpha \leq 1 + (v_n/\varepsilon)^{1/3}. \tag{4.11}$$

Therefore, the last condition in (4.6) yields

$$\lim_{n \to \infty} \frac{|\Lambda|_*}{n^{1/3+\bar{\epsilon}}} = 0 \quad \text{for any } \bar{\epsilon} > 0. \tag{4.12}$$

Our goal is to bound asymptotically the term (4.1) by any power of $n$. To this end, we assume the following condition on the frequency of the observations.

\textbf{H}_5) \textit{Assume that there exists } \tilde{\delta} > 0 \textit{ such that for any } n \geq 3

$$p \geq n^{5/6}. \tag{4.12}$$

Now, Theorem 4.3 implies the following oracle inequality.

\textbf{Theorem 4.4.} \textit{Assume that the unknown function } S \textit{ is continuously differentiable. Moreover, assume that Conditions } \textbf{H}_1)–\textbf{H}_5) \textit{ hold true. Then, for the robust risks defined in (1.4) through the distribution family (2.8)–(2.9), the procedure (3.18) with the coefficients (4.8) for any } n \geq 1 \textit{ and } 0 < \delta < 1/6 \textit{ satisfies the following oracle inequality}

$$\mathcal{R}^*(\tilde{S}_n, S) \leq \frac{1 + 3\delta}{1 - 3\delta} \min_{\lambda \in \Lambda} \mathcal{R}^*(\tilde{S}_\lambda, S) + \frac{U_n^*(S)}{n\delta}, \tag{4.13}$$

where the sequence $U_n^*(S) > 0$ is such that under condition (4.6) for any $r > 0$ and $\tilde{\delta} > 0$,

$$\lim_{n \to \infty} \sup_{\|\tilde{S}\| \leq r} \frac{U_n^*(S)}{n\delta} = 0. \tag{4.14}$$

\textbf{4.2 Robust asymptotic efficiency}

Now we study the asymptotically efficiency properties for the procedure (3.18), (4.8) with respect to the robust risks (1.4) defined by the distribution family (2.8)–(2.9). To this end, we assume that the unknown function $S$ in the model (1.1) belongs to the Sobolev ball

$$W_r^k = \{ f \in C^k_\text{per}[0,1], \sum_{j=0}^k \|f^{(j)}\|^2 \leq r \}, \tag{4.15}$$
where $r > 0$, $k \geq 1$ are some parameters, $C^k_{\text{per}}[0, 1]$ is the set of $k$ times continuously differentiable functions $f : [0, 1] \to \mathbb{R}$ such that $f^{(i)}(0) = f^{(i)}(1)$ for all $0 \leq i \leq k$. The function class $W^k_r$ can be written as an ellipsoid in $l_2$, i.e.

$$W^k_r = \{ f \in C^k_{\text{per}}[0, 1] : \sum_{j=1}^{\infty} a_j \theta^2_j \leq r \} \quad (4.16)$$

where $a_j = \sum_{i=0}^{k} (2\pi[j/2])^{2i}$.

Similarly to [15, 16] we will show here that the asymptotic sharp lower bound for the robust risk (1.4) is given by

$$r^*_k = l(r) = \left( (2k+1) \right)^{1/(2k+1)} \left( \frac{k}{(k+1)\pi} \right)^{2k/(2k+1)} \quad (4.17)$$

Note that this is the well-known Pinsker constant obtained for the nonadaptive filtration problem in “signal + small white noise” model (see, for example, [22]).

Let $\Pi_n$ be the set of all estimators $\hat{S}_n$ measurable with respect to the sigma-algebra $\sigma \{ y_t, 0 \leq t \leq n \}$ generated by the process (1.1).

**Theorem 4.5.** Under Conditions (2.8) and (2.9)

$$\liminf_{n \to \infty} \nu_n^{2k/(2k+1)} \inf_{S_n \in \Pi_n} \sup_{S \in W^k_r} \mathcal{R}^*_n(\hat{S}_n, S) \geq r^*_k, \quad (4.18)$$

where $\nu_n = n/\varsigma^*$. Note that, if the parameters $r$ and $k$ are known, i.e. for the non-adaptive estimation case, in order to obtain the efficient estimation for the “signal+white noise” model, Pinsker proposed in [22] to use the estimate $\hat{S}_{\lambda_0}$ defined in (3.9) with the weights (4.8) in which

$$\lambda_0 = \lambda_{\alpha_0} \quad \text{and} \quad \alpha_0 = (k, l_0), \quad (4.19)$$

where $l_0 = [r/\varepsilon].$ For the model (1.1) – (2.1) we show the same result.

**Proposition 4.6.** The estimator $\hat{S}_{\lambda_0}$ satisfies the following asymptotic upper bound

$$\lim_{n \to \infty} \nu_n^{2k/(2k+1)} \sup_{S \in W^k_r} \mathcal{R}^*_n(\hat{S}_{\lambda_0}, S) \leq r^*_k \quad (4.20)$$

For the adaptive estimation we use the model selection procedure (3.18) with the parameter $\delta$ defined as a function of $n$ satisfying

$$\lim_{n \to \infty} \delta_n = 0 \quad \text{and} \quad \lim_{n \to \infty} n^\delta \delta_n = 0 \quad (4.20)$$

for any $\tilde{\delta} > 0$. For example, we can take $\delta_n = (6 + \ln n)^{-1}$. 
Theorem 4.7. Assume that Conditions $H_1$–$H_5$ hold true. Then the robust risk defined in (1.4) through the distribution family (2.8)–(2.9) for the procedure (3.18) with the coefficients (4.8) and the parameter $\delta = \delta_n$ satisfying (4.20) has the following asymptotic upper bound

$$
\limsup_{n \to \infty} v_n^{2k/(2k+1)} \sup_{S \in W^k_r} \mathcal{R}_n^*(\hat{S}_n, S) \leq r_k^*.
$$

(4.21)

Theorem 4.5 and Theorem 4.7 imply the following result.

Corollary 4.8. Under the conditions of Theorem 4.7,

$$
\lim_{n \to \infty} v_n^{2k/(2k+1)} \inf_{\hat{S}_n \in \Pi_n} \sup_{S \in W^k_r} \mathcal{R}_n^* (\hat{S}_n, S) = r_k^*.
$$

(4.22)

Remark 4.1. It is well known that the optimal (minimax) risk convergence rate for the Sobolev ball $W^k_r$ is $n^{2k/(2k+1)}$ (see, for example, [22], [21]). We see here that the efficient robust rate is $v_n^{2k/(2k+1)}$, i.e. if the distribution upper bound $\varsigma^* \to 0$ as $n \to \infty$ we obtain a faster rate with respect to $n^{2k/(2k+1)}$, and if $\varsigma^* \to \infty$ as $n \to \infty$ we obtain a slower rate. In the case when $\varsigma^*$ is constant the robust rate is the same as the classical non robust convergence rate.

5 Properties of the regression model (1.1)

In order to prove the oracle inequalities we need to study the conditions introduced in [15] for the general semi-martingale model (1.1). To this end, we set for any $x \in \mathbb{R}^n$ the functions

$$
B_{1,Q}(x) = \sum_{j=1}^{n} x_j \left( E_Q \xi_{j,p}^2 - \sigma_Q \right) \quad \text{and} \quad B_{2,Q}(x) = \sum_{j=1}^{n} x_j \tilde{\xi}_{j,p}^2,
$$

(5.1)

where $\sigma_Q$ is defined in (2.8) and $\tilde{\xi}_{j,p} = \xi_{j,p}^2 - E_Q \xi_{j,p}^2$.

Proposition 5.1. Assume that Conditions $H_1$–$H_4$ hold true. Then

$$
L_{1,Q} = \sup_{p \geq 3} \sup_{x \in [-1,1]^n} \left| B_{1,Q}(x) \right| < 2\bar{\tau} \|Y\|_1 \sigma_Q.
$$

(5.2)

Proof. Firstly, we set

$$
I_n^L(f) = \int_0^n f(t)dL_t \quad \text{and} \quad I_n^z(f) = \int_0^n f(t)d\zeta_t.
$$

(5.3)
In view of (2.4) the last integral can be represented as
\[ I_n^z(f) = \sum_{l=1}^{\infty} f(T_l) Y_l 1_{\{T_l \leq n\}}. \] (5.4)

Therefore,
\[ \xi_{j,n} = \frac{\theta_1}{\sqrt{n}} I_{n}^l (\Psi_{j,p}) + \frac{\theta_2}{\sqrt{n}} I_{n}^z (\Psi_{j,p}) \]
and
\[ \mathbb{E} \xi_{j,n}^2 = \frac{\theta_1^2}{n} \int_0^n \Psi_{j,p}^2(t) dt + \frac{\theta_2^2}{n} \mathbb{E} \sum_{l=1}^{\infty} \Psi_{j,p}^2(T_l) 1_{\{T_l \leq n\}}. \] (5.5)

Using Proposition 5.2 from [1] we get
\[ \mathbb{E} \sum_{l=1}^{\infty} \Psi_{j,p}^2(T_l) 1_{\{T_l \leq n\}} = \int_0^n \Psi_{j,p}^2(x) \rho(x) dx \]
\[ = \frac{1}{\tau} \int_0^n \Psi_{j,p}^2(x) dx + \int_0^n \Psi_{j,p}^2(x) \Upsilon(x) dx, \]
where \( \rho \) is the renewal density introduced in (2.7). Then we obtain,
\[ \mathbb{E} \xi_{j,n}^2 = \sigma_Q + \frac{\theta_2^2}{n} \int_0^n \Psi_{j,p}^2(x) \Upsilon(x) dx \]
and
\[ \sup_{j \geq 1} \left| \int_0^n \Psi_{j,p}^2(x) \Upsilon(x) dx \right| \leq 2 \| \Upsilon \|_1, \] (5.6)
where \( \sigma_Q = \theta_1^2 + \theta_2^2 \tau / \check{\tau} \). This directly implies the desired result. \( \square \)

To study the function \( B_{2,Q}(x) \), we have to analyze the correlation properties for the following stochastic integrals
\[ \tilde{I}_n(f) = I_n^z(f) - \mathbb{E} I_n^z(f). \] (5.7)

To do this we set
\[ \check{c}_1 = 1 + \Pi(x^4) + \| \Upsilon \|_1^2 + |\rho|_* \quad \text{and} \quad \check{c}_2 = 12 (1 + \tau)^2 (1 + \check{c}_1). \] (5.8)

Now we investigate the behavior of the integrals defined in (5.7) as functions of \( f \).

**Proposition 5.2.** For any left continuous functions \( f, g : (0, \infty) \rightarrow \mathbb{R} \) such that \( \| f \|_* \leq 1, \| g \|_* \leq 1 \), we have
\[ |\mathbb{E} \tilde{I}_n(f) \tilde{I}_n(g)| \leq 12 \sigma_Q^2 (1 + \tau)^2 \left( (f, g)_n^2 + n \check{c}_1 \right). \] (5.9)
Using these properties we can obtain the following bound.

**Proposition 5.3.** Assume that Conditions $H_1\text{)–}H_4\text{) hold true. Then, for all } n \geq 1,$

$$L_{2,Q} = \sup_{p \geq 3} \sup_{|x| \leq 1} E B_{2,Q}^2(x) \leq \tilde{c}_2 \sigma_Q^2,$$  \hspace{1cm} (5.10)

where $|x|^2 = \sum_{j=1}^{n} x_j^2.$

**Proof.** Note that

$$E \left( \sum_{j=2}^{n} x_j \tilde{\xi}_{j,p} \right)^2 \leq \frac{1}{n^2} \sum_{j=1}^{n} \sum_{l=1}^{n} |x_j| |x_l| |E \tilde{I}_n(\Psi_{j,p}) \tilde{I}_n(\Psi_{l,p})|.$$

Using here Proposition 5.2 and taking into account that

$$(\Psi_{j,p}, \Psi_{l,p})_n = \int_0^n \Psi_{j,p}(t) \Psi_{l,p}(t) dt = n \mathbf{1}_{\{j=l\}},$$

we obtain the bound (5.10). Hence we obtain the desired result. \hfill \Box

Now we can study the estimate (3.18).

**Proposition 5.4.** Assume that Conditions $H_1\text{) and } H_4\text{) hold true for the model (1.1) and that } S(\cdot) \text{ is continuously differentiable. Then, for any } n \geq 2 \text{ and } p \geq 3,$

$$E_{Q,S} |\hat{\sigma}_n - \sigma_Q| \leq \tilde{c}_3 \left( \frac{\sqrt{n}}{p} + \frac{1}{\sqrt{p}} \right) (1 + \|\dot{S}\|^2)(1 + \sigma_Q)^2,$$  \hspace{1cm} (5.11)

where $\tilde{c}_3 = 6 (14 + 2|\rho|_* + 3\sqrt{1 + \tilde{c}_1})(1 + \tilde{\tau}).$

**Remark 5.1.** Propositions 5.1 and 5.3 are used to obtain the oracle inequalities given in Section 4 (see, for example, [15]).

### 6 Simulation

In this section we report the results of a Monte Carlo experiment to assess the performance of the proposed model selection procedure (3.18). In (1.1) we chose a 1-periodic function which, for $0 \leq t \leq 1,$ is defined as

$$S(t) = \begin{cases} |t - \frac{1}{2}| & \text{if } \frac{1}{4} \leq t \leq \frac{3}{4}, \\ \frac{1}{4} & \text{elsewhere.} \end{cases}$$  \hspace{1cm} (6.1)
We simulate the model
\[ \mathrm{d}y_t = S(t) \mathrm{d}t + \mathrm{d}\xi_t, \]
where \( \xi_t = 0.5 \mathrm{d}w_t + 0.5 \mathrm{d}z_t. \) Here \( z_t \) is the semi-Markov process defined in (2.4) with a Gaussian \( \mathcal{N}(0, 1) \) sequence \( (Y_j)_{j \geq 1} \) and \( (\tau_k)_{k \geq 1} \) used in (2.6) taken as \( \tau_k \sim \chi^2_3. \)

We use the model selection procedure (3.18) with the weights (4.8) in which \( k^* = 100 + \sqrt{7 \ln(n)}, \ t_i = i/\ln(n), \ m = \lfloor \ln^2(n) \rfloor \) and \( \delta = (3 + \ln(n))^{-2}. \) We define the empirical risk as
\[
\overline{R} = \frac{1}{p} \sum_{j=1}^{p} \hat{E} \left( \hat{S}_n(t_j) - S(t_j) \right)^2 , \tag{6.2}
\]
where the observation frequency \( p = 100001 \) and the expectations was taken as an average over \( N = 10000 \) replications, i.e.
\[
\hat{E} \left( \hat{S}_n(\cdot) - S(\cdot) \right)^2 = \frac{1}{N} \sum_{l=1}^{N} \left( \hat{S}_n^l(\cdot) - S(\cdot) \right)^2.
\]

We set the relative quadratic risk as
\[
\overline{R}_* = \overline{R}/\|S\|^2_p \quad \text{and} \quad \|S\|^2_p = \frac{1}{p} \sum_{j=0}^{p} S^2(t_j). \tag{6.3}
\]

In our case \( \|S\|^2_p = 0.1883601. \) The table below gives the values for the sample risks (6.2) and (6.3) for different numbers of observations \( n. \)

| \( n \) | \( \overline{R} \) | \( \overline{R}_* \) |
|--------|----------------|------------------|
| 20     | 0.0398         | 0.211            |
| 100    | 0.0091         | 0.0483           |
| 200    | 0.0067         | 0.0355           |
| 1000   | 0.0022         | 0.0116           |

Table 1: Empirical risks
Figures 1–3 show the behavior of the regression function and its estimates by the model selection procedure (3.18) depending on the values of observation periods $n$. The black full line is the regression function (6.1) and the red dotted line is the associated estimator.

Remark 6.1. From numerical simulations of the procedure (3.18) with various observations numbers $n$ we may conclude that the quality of the proposed procedure is good for practical needs, i.e. for reasonable (non large) number of observations. We can also add that the quality of the estimation improves as the number of observations increases.
Figure 1: Estimator of $S$ for $n = 20$

Figure 2: Estimator of $S$ for $n = 100$
Figure 3: Estimator of $S$ for $n = 200$

Figure 4: Estimator of $S$ for $n = 1000$
7 Proofs

7.1 Proof of Theorem 4.1

Using the cost function given in (3.15), we can rewrite the empirical squared error in (3.12) as follows

$$\text{Err}(\lambda) = J_n(\lambda) + 2\sum_{j=1}^{n} \lambda(j) \tilde{\theta}_{j,p} + \|S\|^2 - \rho \hat{P}_n(\lambda),$$

(7.1)

where

$$\tilde{\theta}_{j,p} = \bar{\theta}_{j,p} - \bar{\theta}_{j,p} \hat{\theta}_{j,p} = \frac{1}{\sqrt{n}} \bar{\theta}_{j,p} \xi_{j,p} + \frac{1}{n} \xi_{j,p} + \frac{1}{n} \varsigma_{j,n} + \frac{\sigma_Q - \hat{\sigma}_n}{n},$$

with $\varsigma_{j,p} = E_Q \xi_{j,p}^2 - \sigma_Q$ and $\tilde{\xi}_{j,p} = \xi_{j,p}^2 - E_Q \xi_{j,p}^2$. Setting

$$M(\lambda) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \lambda(j) \theta_j \xi_{j,p} \quad \text{and} \quad P_n^0 = \frac{\sigma_Q |\lambda|^2}{n},$$

(7.2)

we can rewrite (7.1) as

$$\text{Err}(\lambda) = J_n(\lambda) + 2 \frac{\sigma_Q - \hat{\sigma}_n}{n} L(\lambda) + 2 M(\lambda) + 2 \frac{\sigma_Q}{n} \left( B_{2,Q}(\lambda) \right)$$

$$+ 2 \sqrt{P_n^0(\lambda)} \frac{B_{2,Q}(\hat{\epsilon})}{\sqrt{\sigma_Q^n}} + \|S\|^2 - \rho P_n(\lambda),$$

(7.3)

where $e(\lambda) = \lambda/|\lambda|$ and the function $L(\cdot)$ was defined in (3.10). Let $\lambda_0 = (\lambda_0(j))_{1 \leq j \leq p}$ be a fixed sequence in $\Lambda$ and $\hat{\lambda}$ be defined as in (3.17). Substituting $\lambda_0$ and $\hat{\lambda}$ in Equation (7.3), we obtain

$$\text{Err}(\hat{\lambda}) - \text{Err}(\lambda_0) = J(\hat{\lambda}) - J(\lambda_0) + 2 \frac{\sigma_Q - \hat{\sigma}_n}{n} L(\bar{\omega}) + 2 \frac{\sigma_Q}{n} B_{2,Q}(\bar{\omega}) + 2 M(\bar{\omega})$$

$$+ 2 \sqrt{P_n^0(\hat{\lambda})} \frac{B_{2,Q}(\hat{\epsilon})}{\sqrt{\sigma_Q^n}} - 2 \sqrt{P_n^0(\lambda_0)} \frac{B_{2,Q}(e_0)}{\sqrt{\sigma_Q^n}}$$

$$- \delta P_n(\hat{\lambda}) + \delta P_n(\lambda_0),$$

(7.4)

where $\bar{\omega} = \hat{\omega} - \lambda_0$, $\hat{\epsilon} = e(\hat{\lambda})$ and $e_0 = e(\lambda_0)$. Note that, by (3.10),

$$|L(\bar{\omega})| \leq L(\hat{\lambda}) + L(\lambda) \leq 2|\Lambda|.$$
The inequality

$$2|ab| \leq \delta a^2 + \delta^{-1}b^2$$

implies that, for any $\lambda \in \Lambda$,

$$2\sqrt{P_n^0(\lambda)|B_{2,Q}(e(\lambda))|} \leq \delta P_n^0(\lambda) + \frac{B_{2,Q}^2(e(\lambda))}{\delta\sigma_Q n}.$$  

Taking into account that $0 < \delta < 1$, we get

$$\text{Err}(\hat{\lambda}) \leq \text{Err}(\lambda_0) + 2M + \frac{2L_{1,Q}}{n} + \frac{2B_{2,Q}^*}{\delta\sigma_Q n}$$

$$+ \frac{1}{n}|\hat{\sigma}_n - \sigma_Q|(|\hat{\lambda}|^2 + |\lambda_0|^2) + 2\delta P_n(\lambda_0),$$

where $B_{2,Q}^* = \sup_{\lambda \in \Lambda} B_{2,Q}(e(\lambda)).$ Moreover, noting that in view of (3.10) $\sup_{\lambda \in \Lambda} |\lambda|^2 \leq |\Lambda|_1$, we can rewrite the previous bound as

$$\text{Err}(\hat{\lambda}) \leq \text{Err}(\lambda_0) + 2M + \frac{2L_{1,Q}}{n} + \frac{2B_{2,Q}^*}{\delta\sigma_Q n}$$

$$+ \frac{4|\Lambda|}{n}|\hat{\sigma} - \sigma_Q| + 2\delta P_n(\lambda_0).$$

(7.6)

To estimate the second term in the right side of this inequality we set

$$S_x = \sum_{j=1}^{n} x(j)\overline{\varphi}_j \phi_j, \quad x = (x(j))_{1 \leq j \leq n} \in \mathbb{R}^n.$$ 

Thanks to (3.2) we estimate the term $M(x)$ for any $x \in \mathbb{R}^n$ as

$$E_{Q}M^2(x) \leq \kappa_Q \frac{1}{n} \sum_{j=1}^{n} x^2(j)\overline{\varphi}_{j,p}^2 = \kappa_Q \frac{1}{n} \|S_x\|^2.$$  

(7.7)

To estimate this function for a random vector $x \in \mathbb{R}^n$, we set

$$Z^* = \sup_{x \in \Lambda} \frac{nM^2(x)}{\|S_x\|^2}, \quad \Lambda_1 = \Lambda - \lambda_0.$$ 

So, through the Inequality (7.5), we get

$$2|M(x)| \leq \delta\|S_x\|^2 + \frac{Z^*}{n\delta}.$$  

(7.8)
It is clear that the last term here can be estimated as
\[
\mathbb{E}_Q Z^* \leq \sum_{x \in \Lambda_1} \frac{n \mathbb{E}_Q M^2(x)}{\|S_x\|^2} \leq \sum_{x \in \Lambda_1} \kappa_Q = \kappa_Q \nu, \tag{7.9}
\]
where \( \nu = \text{card}(\Lambda) \). Moreover, note that, for any \( x \in \Lambda_1 \),
\[
\|S_x\|^2 - \|\hat{S}_x\|^2 = \sum_{j=1}^n x^2(j)(\theta_{j,p}^2 - \tilde{\theta}_j^2) \leq -2M_1(x), \tag{7.10}
\]
where \( M_1(x) = n^{-1/2} \sum_{j=1}^n x^2(j)\theta_{j,p}^2 \xi_{j,n} \). Taking into account now that, for any \( x \in \Lambda_1 \), the components \(|x(j)| \leq 1\), we can estimate this term as in (7.7), i.e.
\[
\mathbb{E}_Q M^2_1(x) \leq \kappa_Q \frac{\|S_x\|^2}{n}. \tag{7.11}
\]
Similarly to the previous reasoning we set
\[
Z^*_1 = \sup_{x \in \Lambda_1} \frac{nM^2_1(x)}{\|S_x\|^2}
\]
and we get
\[
\mathbb{E}_Q Z^*_1 \leq \kappa_Q \nu. \tag{7.11}
\]
Using the same type of arguments as in (7.8), we can derive
\[
2|M_1(x)| \leq \delta \|S_x\|^2 + \frac{Z^*_1}{n\delta}. \tag{7.12}
\]
From here and (7.10), we get
\[
\|S_x\|^2 \leq \frac{\|\hat{S}_x\|^2}{1 - \delta} + \frac{Z^*_1}{n\delta(1 - \delta)} \tag{7.13}
\]
for any \( 0 < \delta < 1 \). Using this bound in (7.8) yields
\[
2M(x) \leq \delta \frac{\|\hat{S}_x\|^2}{1 - \delta} + \frac{Z^* + Z^*_1}{n\delta(1 - \delta)}. \tag{7.13}
\]
Taking into account that \( \|\hat{S}_x\|^2 \leq 2(\text{Err}(\lambda) + \text{Err}(\lambda_0)) \), we obtain
\[
2M(x) \leq \frac{2\delta(\text{Err}(\lambda) + \text{Err}(\lambda_0))}{1 - \delta} + \frac{Z^* + Z^*_1}{n\delta(1 - \delta)}. \tag{7.13}
\]
Using this bound in (7.6) we obtain
\[
\text{Err}(\hat{\lambda}) \leq \frac{1 + \delta}{1 - 3\delta} \text{Err}(\lambda_0) + \frac{Z^* + Z_1^*}{n \delta (1 - 3\delta)} + \frac{2L_{1,Q}}{n(1 - 3\delta)} + \frac{2B_{2,Q}^*}{\delta (1 - 3\delta) \sigma_{Q_n}}
\]
\[
+ \frac{(4|\Lambda|_s + 2)}{n(1 - 3\delta)} |\hat{\sigma} - \sigma_Q| + \frac{2\delta}{(1 - 3\delta)} P_0^0(\lambda_0).
\]

Moreover, for \(0 < \delta < 1/6\) we can rewrite this inequality as
\[
\text{Err}(\hat{\lambda}) \leq \frac{1 + \delta}{1 - 3\delta} \text{Err}(\lambda_0) + \frac{2(Z^* + Z_1^*)}{n \delta} + \frac{4L_{1,Q}}{n} + \frac{4B_{2,Q}^*}{\delta \sigma_{Q_n}}
\]
\[
+ \frac{(8|\Lambda|_s + 2)}{n} |\hat{\sigma} - \sigma_Q| + \frac{2\delta}{(1 - 3\delta)} P_0^0(\lambda_0).
\]

Now, in view of the condition Proposition 5.3, we estimate the expectation of the term \(B_{2,Q}^*\) in (7.6) as
\[
\mathbb{E}_Q B_{2,Q}^* \leq \sum_{\lambda \in \Lambda} \mathbb{E}_Q B_{2,Q}^2(e(\lambda)) \leq \nu L_{2,Q}.
\]

Now, taking into account that \(|\Lambda|_s \geq 1\), we get
\[
\mathcal{R}_Q(\hat{S}_*, S) \leq \frac{1 + \delta}{1 - 3\delta} \mathcal{R}_Q(\hat{S}_{\lambda_0}, S) + \frac{4\nu L_{1,Q}}{n \delta} + \frac{4L_{1,Q}}{n} + \frac{4\nu L_{2,Q}}{\delta \sigma_{Q_n}}
\]
\[
+ \frac{10|\Lambda|_s}{n} \mathbb{E}_Q |\hat{\sigma} - \sigma_Q| + \frac{2\delta}{(1 - 3\delta)} P_0^0(\lambda_0).
\]

By using the upper bound for \(P_n(\lambda_0)\) in Lemma A.1, we obtain that
\[
\mathcal{R}_Q(\hat{S}_*, S) \leq \frac{1 + 3\delta}{1 - 3\delta} \mathcal{R}_Q(\hat{S}_{\lambda_0}, S) + \frac{4\nu L_{1,Q}}{n \delta} + \frac{4L_{1,Q}}{n} + \frac{4\nu L_{2,Q}}{\delta \sigma_{Q_n}}
\]
\[
+ \frac{10|\Lambda|_s}{n} \mathbb{E}_Q |\hat{\sigma} - \sigma_Q| + \frac{2\delta L_{1,Q}}{(1 - 3\delta)n}.
\]

Taking into account here that \(1 - 3\delta \geq 1/2\) for \(0 < \delta < 1/3\) and that \(\kappa_Q \leq (1 + \hat{\tau}|\rho|_s)\sigma_Q\) and using the bounds (5.2) and (5.10) we obtain the inequality (4.2). Hence Theorem 4.1.
7.2 Proof of Proposition 5.2

By Ito’s formula one gets
\[
dI_t^2(f) = 2I_{t-}(f)dI_t(f) + \varrho_{1}^2\varrho^2 \int_{0}^{t} f^2(t)d\tau + \sum_{0 \leq s \leq t} f^2(s)(\Delta \xi_s)^2, \tag{7.14}
\]
where \(\xi_t^d = \varrho_3 \Delta t + \varrho_2 z_t\) and \(\varrho_3 = \varrho_1 \sqrt{1 - \varrho^2}\). Taking into account that the processes \((\Delta t)_{t \geq 0}\) and \((z_t)_{t \geq 0}\) are independent and the time of jumps \(T_k\) defined in (2.6) has a density, we have \(\Delta z_s \Delta L_s = 0\) a.s. for any \(s \geq 0\). Therefore, we can rewrite the differential (7.14) as
\[
dI_t^2(f) = 2I_{t-}(f)dI_t(f) + \varrho_{1}^2\varrho^2 \int_{0}^{t} f^2(t)d\tau + \sum_{0 \leq s \leq t} f^2(s)(\Delta \xi_s)^2. \tag{7.15}
\]

Therefore, using Lemma A.3 we obtain
\[
E I_t^2(f) = \varrho_{1}^2 \|f\|^2_{t} + \varrho_{2}^2 \|f\|_{t} \rho(t),
\]
where \(\|f\|^2_{t} = \int_{0}^{t} f^2(t)d\tau\), \(\rho\) is the density of the renewal measure \(\sum_{j=1}^{\infty} \eta^{(j)}\) and with \(\eta\) the distribution of \(\tau_1\). Therefore,
\[
d\bar{I}_t(f) = 2I_{t-}(f)f(t)d\xi_t + f^2(t)d\bar{m}_t, \quad \bar{m}_t = \varrho_{1}^2\bar{m}_t + \varrho_{2}^2 m_t, \tag{7.16}
\]
where \(\bar{m}_t = \sum_{0 \leq s \leq t}(\Delta \xi_s)^2 - t\) and \(m_t = \sum_{0 \leq s \leq t}(\Delta z_s)^2 - \int_{0}^{t} \rho(s)\,ds\). By the Ito formula we get
\[
E \bar{I}_n(f) \bar{I}_n(g) = E \int_{0}^{n} \bar{I}_{t-}(f)d\bar{I}_t(g)
+ E \int_{0}^{n} \bar{I}_{t-}(g)d\bar{I}_t(f) + E \left[\bar{I}(f), \bar{I}(g)\right], \tag{7.17}
\]
First, note that the process \((\bar{m}_t)_{t \geq 0}\) is a martingale and, using Lemma A.5, we get
\[
E \int_{0}^{n} \bar{I}_{t-}(f)d\bar{I}_t(g) = \rho_{1}^2 E \int_{0}^{n} \bar{I}_{t-}(f)g^2(t)dm_t = \rho_{2}^2 E \int_{0}^{n} \bar{I}_{t-}(f)g^2(t)dm_t.
\]

The last integral can be represented as
\[
E \int_{0}^{n} I_{t-}^2(f)g^2(t)dm_t = J_1 - J_2,
\]

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where

\[ J_1 = E \sum_{k \geq 1} I_{T_k}^2(f)g^2(T_k)1_{\{T_k \leq n\}} \quad \text{and} \quad J_2 = \int_0^n E I_t^2(f)g^2(t)\rho(t)dt. \]

By Lemma A.4 we get

\[ J_1 = E \sum_{k \geq 1} E \left( I_{T_k}^2(f) | G \right) g^2(T_k)1_{\{T_k \leq n\}} = \varrho_1^2 J_{1,1} + \varrho_2^2 J_{1,2}, \]

where

\[ J_{1,1} = E \sum_{k \geq 1} \|f\|_T^2 g^2(T_k)1_{\{T_k \leq n\}} \quad \text{and} \quad J_{1,2} = E \sum_{k \geq 1} \sum_{l=1}^{k-1} f^2(T_l)g^2(T_k)1_{\{T_k \leq n\}}. \]

We obtain directly that

\[ J_{1,1} = \int_0^n \|f\|_T^2 g^2(t)\rho(t)dt \]

and

\[ J_{1,2} = E \sum_{l \geq 1} f^2(T_l) \sum_{k \geq l+1} g^2(T_k)1_{\{T_k \leq n\}} = \int_0^n f^2(x) \left( \int_0^{n-x} g^2(x+t)\rho(t)dt \right) \rho(x)dx. \]

From Lemma A.3 we obtain that

\[ J_2 = \varrho_1^2 \int_0^n \|f\|_T^2 g^2(t)\rho(t)dt + \varrho_2^2 \int_0^n \|f\|_T^2 g^2(t)\rho(t)dt. \]

Therefore,

\[ E \int_0^n I_t^2(f)g^2(t)dm_t = \varrho_2^2 \int_0^n f^2(x) \left( \int_x^n g^2(t)(\rho(t-x) - \rho(t))dt \right) \rho(x)dx. \]

Taking into account that \( \rho(t-x) - \rho(t) = \Upsilon(t-x) - \Upsilon(t) \) we can estimate the last integral as

\[ |E \int_0^n I_t^2(f)g^2(t)dm_t| \leq 2\varrho_2^2 n\|\Upsilon\|_1. \]

From this and by the symmetry arguments we obtain that

\[ |E \int_0^n \bar{I}_t(f)\bar{d}_t(g)| + |E \int_0^n \bar{I}_t(g)d\bar{I}_t(f)| \leq 4\varrho_2^2 n\|\Upsilon\|_1. \quad (7.18) \]
Note now that
\[
\left[ \tilde{I}(f), \tilde{I}(g) \right]_n = \left< \tilde{I}^c(f), \tilde{I}^c(g) \right>_n + D_n(f, g),
\]
where
\[
D_n(f, g) = \sum_{0 \leq t \leq n} \Delta \tilde{I}_t^d(f) \Delta \tilde{I}_t^d(g).
\]

It should be noted that the continuous and the discrete parts of the processes (7.16) can be represented as
\[
\tilde{I}_c^c(f) = 2 \varrho_1 \tilde{\varrho} \int_0^t I_s(f) f(s) dw_s \quad \text{and} \quad \tilde{I}_d^d(g) = 2 \int_0^t I_s(f) f(s) dt + \int_0^t f^2(s) d\tilde{m}_s.
\]

So, in view of Lemma 6.1 from [1],
\[
\mathbf{E} < \tilde{I}^c(f), \tilde{I}^c(g) > = 4 \rho_1^2 \rho_2^2 \int_0^n \mathbf{E}(I_t(f) I_t(g)) f(t) g(t) dt
\]
\[
= 4 \rho_1^2 \rho_2^2 \int_0^n (f, g)_t f(t) g(t) dt + 4 \rho_1^2 \rho_2^2 \int_0^n (f, g)_t f(t) g(t) dt
\]
\[
= 4 \rho_1^2 \rho_2^2 \sigma_Q (f, g)_t f(t) g(t) + 4 \rho_1^2 \rho_2^2 \int_0^n (f, g)_t f(t) g(t) dt,
\]
with \((f, g)_t = \int_0^t f(s) g(s) ds\). Taking into account that \(\|f\|_* \leq 1\) and \(\|g\|_* \leq 1\), we can estimate the last integral as
\[
\int_0^n (f, g)_t f(t) g(t) dt \leq n \|\gamma\|_1.
\]

Therefore,
\[
\mathbf{E} \left< \tilde{I}^c(f), \tilde{I}^c(g) \right> = 4 \sigma_Q^2 (f, g)_n + 4 \tilde{\tau} \|\gamma\|_1.
\]
(7.20)

To study the last term in (7.19) note that
\[
D_n(f, g) = \sum_{0 \leq t \leq n} \left( 2 I_{t-}(f) f(t) \Delta \xi^d_t + f^2(t) \Delta \tilde{m}_t \right) \left( 2 I_{t-}(g) g(t) \Delta \xi^d_t + g^2(t) \Delta \tilde{m}_t \right).
\]
Taking into account that for any \(t > 0\)
\[
\Delta \xi^d_t \Delta \tilde{m}_t = \varrho_3^3 (\Delta \tilde{L}_t)^3 + \varrho_2^3 (\Delta z_t)^3,
\]
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we obtain that
\[
E \sum_{0 \leq t \leq n} I_{t-}(f(t)) f(t) g^2(t) \Delta \xi^d_t \Delta \tilde{m}_t = (\varrho_3^2 \Pi(x^3) + \varrho_2^2 E Y_1^3) \int_0^n E I_t(f(t)) g^2(t) dt = 0.
\]

So, using the symmetry arguments, we find that
\[
E D_n(f, g) = 4 E D_{1,n}(f, g) + E D_{2,n}(f, g), \tag{7.22}
\]
where
\[
D_{1,n}(f, g) = \sum_{0 \leq t \leq n} I_{t-}(f) I_{t-}(g) f(t) g(t) (\Delta \xi_t^d)^2 
\]
and
\[
D_{2,n}(f, g) = \sum_{0 \leq t \leq n} f^2(t) g^2(t) (\Delta \tilde{m}_t)^2.
\]

Note that
\[
D_{1,n}(f, g) = \varrho_2^2 D_{1,n}(f, g) + \varrho_2^2 \tilde{D}_{1,n}(f, g),
\]
where
\[
\tilde{D}_{1,n}(f, g) = \sum_{0 \leq t \leq n} I_{t-}(f) I_{t-}(g) f(t) g(t) (\Delta \hat{L}_t)^2
\]
and
\[
\tilde{D}_{1,n}(f, g) = \sum_{0 \leq t \leq n} I_{t-}(f) I_{t-}(g) f(t) g(t) (\Delta z_t)^2.
\]

Now, similarly to (7.20) and taking into account that \(\Pi(x^2) = 1\), we get
\[
E \tilde{D}_{1,n}(f, g) = \int_0^n f(t) g(t) E I_t(f) I_t(g) dt = \varrho_1^2 \int_0^n f(t) g(t) (f, g)_t dt
\]
\[
+ \varrho_2^2 \int_0^n f(t) g(t) (f, g)_t dt
\]
\[
= \sigma_Q(f, g)_n^2 + \varrho_2^2 \int_0^n f(t) g(t) (f, g \gamma)_t dt.
\]

So,
\[
|E \tilde{D}_{1,n}(f, g)| \leq \sigma_Q ((f, g)_n^2 + n \tau \| \gamma \|_1).
\]

Moreover, taking into account that \(E Y_1^2 = 1\) we get
\[
E \tilde{D}_{1,n}(f, g) = E \sum_{k \geq 1} I_{T_k-}(f) I_{T_k-}(g) f(T_k) g(T_k) \mathbf{1}_{\{T_k \leq n\}}.
\]
So, in view of Lemma A.4

\[ E \tilde{D}_{1,n}(f, g) = E \sum_{k \geq 1} E \left( I_{T_k - (f) T_k - (g)} | G \right) f(T_k) g(T_k) 1_{\{T_k \leq n\}} \]

\[ = \varrho_1^2 E \sum_{k \geq 1} (f, g)_{T_k} f(T_k) g(T_k) 1_{\{T_k \leq n\}} + \varrho_2^2 E D'_{1,n}(f, g) \]

\[ = \varrho_1^2 \int_0^n (f, g)_t f(t) g(t) \rho(t) dt + \varrho_2^2 E D'_{1,n}(f, g), \]

where

\[ D'_{1,n}(f, g) = \sum_{k \geq 1} \sum_{l=1}^{k-1} f(T_l) g(T_l) f(T_k) g(T_k) 1_{\{T_k \leq n\}}. \]

Noting now that

\[ \int_0^n (f, g)_t f(t) g(t) \rho(t) dt = \frac{1}{2\tau} (f, g)_n^2 + \int_0^n (f, g)_t f(t) g(t) \Upsilon(t) dt, \]

we obtain

\[ |\int_0^n (f, g)_t f(t) g(t) \rho(t) dt| \leq \frac{1}{2\tau} (f, g)_n^2 + n \| \Upsilon \|_1. \]

Furthermore, the expectation of \( D'_{1,n}(f, g) \) can be represented as

\[ E D'_{1,n}(f, g) = E \sum_{l \geq 1} f(T_l) g(T_l) \sum_{k \geq l+1} f(T_k) g(T_k) 1_{\{T_k \leq n\}} \]

\[ = \int_0^n f(x) g(x) \left( \int_0^{n-x} f(x+t) g(x+t) \rho(t) dt \right) \rho(x) dx \]

\[ = \frac{1}{2\tau} (f, g)_n^2 + D''_{1,n}(f, g), \]

where the last term in this equality can be represented as

\[ D''_{1,n}(f, g) = \int_0^n f(x) g(x) \left( \int_0^{n-x} f(x+t) g(x+t) \Upsilon(t) dt \right) \rho(x) dx \]

\[ + \frac{1}{\tau} \int_0^n f(x) g(x) \left( \int_0^{n-x} f(x+t) g(x+t) \Upsilon(t) dt \right) \Upsilon(x) dx. \]

This implies

\[ |D''_{1,n}(f, g)| \leq n(1 + \frac{1}{\tau})(1 + \| \Upsilon \|_1^2). \]
Therefore,
\[ |E \tilde{D}_{1,n}(f,g)| \leq \sigma_Q ((f,g)^2_n + n(1 + \tilde{\tau})\|\tilde{\Upsilon}\|_1^2). \quad (7.24) \]

Finally we obtain that
\[ |E D_{1,n}(f,g)| \leq \sigma_Q^2 (1 + \tilde{\tau})^2 ((f,g)^2_n + n\|\Upsilon\|_1^2). \quad (7.25) \]

As to the last term in (7.22) we can calculate directly
\[ E D_{2,n}(f,g) = 2^4 \Pi(x^4) \int_0^n f^2(t)g^2(t)dt + 2^4 \int_0^n f^2(t)g^2(t)\rho(t)dt, \]
i.e.
\[ E D_{2,n}(f,g) \leq n\sigma_Q^2 (1 + \tilde{\tau})^2(n \|\Upsilon\|_1^2 + \|\rho\|_1) (1 + \tilde{\tau}). \]

From this we obtain that
\[ |E D_{n}(f,g)| \leq \sigma_Q^2 (1 + \tilde{\tau})^2(4(f,g)^2_n + n\tilde{c}_1), \quad (7.26) \]

where \( \tilde{c}_1 \) is given in (5.8). From this and (7.21) we find
\[ E[\tilde{I}(f),\tilde{I}(g)]_n \leq 8\sigma_Q^2 (1 + \tilde{\tau})^2 ((f,g)^2_n + n\tilde{c}_1). \quad (7.27) \]

This bound and (7.18) implies (5.9). Hence Lemma 5.2.

\[ \square \]

7.3 Proof of Proposition 5.4

It is clear that the Inequality (5.11) holds true for \( l > \tilde{p} \). Let now \( l \leq \tilde{p} \). Setting \( x'_j = 1_{(\sqrt{n}\leq j \leq \tilde{p})} \) and substituting (3.8) in (3.14) yields,
\[ \hat{\sigma}_n = \frac{n}{\tilde{p}} \sum_{j=1}^{\tilde{p}} (\bar{\eta}_{j,p})^2 + \frac{2n}{\tilde{p}} M(x') + \frac{1}{\tilde{p}} \sum_{j=1}^{\tilde{p}} \xi_{j,p}^2, \quad (7.28) \]

where \( M(x') \) is defined in (7.2). Furthermore, putting \( x''_j = \tilde{p}^{-1/2} 1_{(l\leq j \leq \tilde{p})} \), one can write the last term on the right hand side of (7.28) as
\[ \frac{1}{\tilde{p}} \sum_{j=l}^{\tilde{p}} \xi_{j,p}^2 = \frac{1}{\sqrt{\tilde{p}}} B_{2,Q}(x'') + \frac{1}{\tilde{p}} B_{1,Q}(x') + \frac{(\tilde{p} - l + 1)\sigma_Q}{\tilde{p}}, \]

where the functions \( B_{1,Q} \) and \( B_{2,Q} \) are given in (5.1). Using Proposition 5.1, Proposition 5.3 and Lemma A.7, we come to the following upper bound
\[ E_Q|\hat{\sigma}_n - \sigma_Q| \leq \frac{16\|\hat{S}\|^2_n}{lp} + \frac{2n}{p} E_Q |M(x')| + \frac{L_{1,Q}}{p} + \frac{\sqrt{L_{2,Q}}}{\sqrt{p}} + \frac{\sigma_Q(l - 1)}{p}. \]
In the same way as in (7.7), we obtain
\[ E_Q |M(x')| \leq \left( \frac{\kappa_Q}{n} \sum_{j=1}^{p} \bar{\theta}_{j,p}^2 \right)^{1/2} \leq \frac{4(\kappa_Q \| \hat{S} \|^2)^{1/2}}{l}. \]

Taking into account that \( \kappa_Q \leq (1 + \bar{\tau}|\rho|_{*})\sigma_Q \) and using the bounds (5.2) and (5.10) we obtain the inequality (5.11). Hence Proposition 5.4 holds true.

7.4 Proof of Theorem 4.3
This proof directly follows from Theorem 4.1 and Proposition 5.4.

7.5 Proof of Theorem 4.5
First, we denote by \( Q_0 \) the distribution of the noise (2.1) and (2.2) with the parameter \( \varrho_1 = \varsigma^* \), \( \bar{\varrho} = 1 \) and \( \varrho_2 = 0 \), i.e., the distribution for the “signal + white noise” model. So, we can estimate as below the robust risk
\[ R^*_{Q_0}(\tilde{S}_n, S) \geq R_{Q_0}(\tilde{S}_n, S). \]

Now, Theorem 6.1 from [13] yields the lower bound (4.18). Hence this finishes the proof.

7.6 Proof of Proposition 4.6
First, we note that in view of (3.9) one can represent the quadratic risk for the empiric norm \( \| \cdot \|_p \) as
\[ E_Q \| \hat{S}_{\lambda_0} - S \|_p^2 = \frac{1}{n} \sum_{j=1}^{\hat{p}} \lambda_0^2(j) E_Q \xi_{j,p}^2 + \bar{\Theta}_p, \]
where \( \bar{\Theta}_p = \sum_{j=1}^{p} (\theta_{j,p} - \lambda_0(j) \bar{\theta}_{j,p} )^2 \). We put here \( \lambda_0(j) = 0 \) for \( j > n \) if \( p > n \). The first term can be estimated by the bound (5.2) as
\[ \sup_{Q \in Q_n} E_Q \sum_{j=1}^{\hat{p}} \lambda_0^2(j) \xi_{j,p}^2 \leq \varsigma \sum_{j=1}^{n} \lambda_0^2(j) + L_{1,Q}. \]
where \( L_{1,n} = \sup_{Q \in Q_n} L_{1,Q} \). Therefore, taking into account that \( \nu_n = n/\sigma^* \), we get
\[ \sup_{Q \in Q_n} E_Q \| \hat{S}_{\lambda_0} - S \|_p^2 \leq \frac{1}{n} \sum_{j=1}^{n} \lambda_0^2(j) + \frac{L_{1,n}^*}{\nu_n} + \bar{\Theta}_p. \]
Note that
\[
\lim_{n \to \infty} \frac{1}{v_n^{1/(2k+1)}} \sum_{j=1}^{n} \lambda_0^2(j) = \frac{2(\tau_k r)^{1/(2k+1)} k^2}{(k+1)(2k+1)}.
\] (7.29)

Furthermore, by the Inequality (7.5) for any \(0 < \bar{\varepsilon} < 1\) we get
\[
\Theta_p \leq (1 + \bar{\varepsilon}) \Theta_p + (1 + \bar{\varepsilon}^{-1}) \sum_{j=1}^{p} h_{j,p}^2,
\] (7.30)

where \(\Theta_p = \sum_{j=1}^{p} (1 - \lambda_0(j))^2 \theta_{j,p}^2\). In view of Definition (4.8), we can represent this term as
\[
\Theta_p = \sum_{j=\tau_{0\ell_0}}^{[\omega_0]} (1 - \lambda_0(j))^2 \theta_{j,p}^2 + \sum_{j=[\omega_0]+1}^{p} \theta_{j,p}^2 := \Theta_{1,p} + \Theta_{2,p},
\]

where \(\ell_0 = j^*(\alpha_0)\), \(\omega_0 = \omega_{\alpha_0} = (\tau_{k0}\nu_{\nu_n})^{1/(2k+1)}\) and \(l_0 = [r/\varepsilon]\). Applying Lemma A.9 yields
\[
\Theta_{1,p} \leq (1 + \bar{\varepsilon}) \sum_{j=\ell_0}^{[\omega_0]} (1 - \lambda_0(j))^2 \theta_{j,p}^2 + 4\pi^2 r (1 + \bar{\varepsilon}^{-1}) \omega_0^3 p^{-2}.
\]

Similarly, through Lemma A.8 we have
\[
\Theta_{2,p} \leq (1 + \bar{\varepsilon}) \sum_{j=[\omega_0]+1}^{p} \theta_{j,p}^2 + (1 + \bar{\varepsilon}^{-1}) \omega_0^3 p^{-2}.
\]

Hence,
\[
\Theta_p \leq (1 + \bar{\varepsilon}) \Theta^*_\ell_0 + (1 + \bar{\varepsilon}^{-1}) \left(4\pi^2 r \omega_0^3 + r\right) p^{-2},
\]

where \(\Theta^*_\ell = \sum_{j\geq l} (1 - \lambda_0(j))^2 \theta_{j,p}^2\). Moreover, note that
\[
\sup_{S \in W^1_r} \max_{1 \leq j \leq p} h_{j,p}^2 \leq \|\hat{S}\|^2 p^{-2} \leq r p^{-2}.
\]

Moreover, \(W^k_r \subseteq W^2_r\) for any \(k \geq 2\). From here and Lemma A.10 we get
\[
\sup_{S \in W^k_r} \sum_{j=1}^{p} h_{j,p}^2 \leq r \left(p^{-1} 1_{\{k=1\}} + 3p^{-2} 1_{\{k \geq 2\}}\right).
\]
Moreover, in view of Condition $H_5$)

$$\lim_{n \to \infty} v_n^{2k/(2k+1)} \left( p^{-1} 1_{(k=1)} + \omega_0^3 p^{-2} \right) = 0.$$ 

So,

$$\limsup_{n \to \infty} v_n^{2k/(2k+1)} \sup_{S \in W_k^r} \Theta_p \leq \limsup_{n \to \infty} v_n^{2k/(2k+1)} \sup_{S \in W_k^r} \Theta_{i_0}^*.$$ 

To estimate the term $\Theta_{i_0}^*$ we set

$$U_n = v_n^{2k/(2k+1)} \sup_{j \geq i_0} (1 - \lambda_0(j))^{2/a_j},$$

where the sequence $(a_j)_{j \geq 1}$ is defined in (4.16). This leads to the inequality

$$\sup_{S \in W_k^r} v_n^{2k/(2k+1)} \Theta_{i_0}^* \leq U_n \sum_{j \geq 1} a_j \theta_j^2 \leq U_n r.$$ 

Taking into account that $\lim_{n \to \infty} t_0 = r$, we get

$$\limsup_{n \to \infty} U_n \leq \pi^{-2k} (\tau_k r)^{-2k/(2k+1)},$$

where the coefficient $\tau_k$ is given in (4.8). This implies immediately that

$$\limsup_{n \to \infty} v_n^{2k/(2k+1)} \sup_{S \in W_k^r} \Theta_p \leq \frac{r^{1/(2k+1)}}{\pi^{2k} (\tau_k)^{2k/(2k+1)}}.$$ 

(7.31)

Moreover, note that

$$R_k^* = \frac{2(\tau_k r)^{1/(2k+1)} k^2}{(k+1)(2k+1)} + \frac{r^{1/(2k+1)}}{\pi^{2k} (\tau_k)^{2k/(2k+1)}}.$$ 

So, applying (7.29) and (7.31), yields

$$\lim_{n \to \infty} v_n^{2k/(2k+1)} \sup_{S \in W_k^r} \sup_{Q \in \mathcal{Q}_n} E_Q \| \hat{S}_{\lambda_0} - S \|_2^2 \leq R_k^*.$$ 

(7.32)

Furthermore, Lemma A.6 yields that for any $\tilde{\varepsilon} > 0$

$$\sup_{S \in W_k^r} R_n^* (\hat{S}_{\lambda_0}, S) \leq (1 + \tilde{\varepsilon}) \sup_{S \in W_k^r} \sup_{Q \in \mathcal{Q}_n} E_Q \| \hat{S}_{\lambda_0} - S \|_2^2 + (1 + \tilde{\varepsilon}^{-1}) r p^{-2}.$$ 

So, in view of Condition $H_5$), we derive the desired inequality

$$\lim_{n \to \infty} v_n^{2k/(2k+1)} \sup_{S \in W_k^r} R_n^* (\hat{S}_{\lambda_0}, S) \leq R_k^*.$$ 

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Hence we obtain Proposition 4.6. □

Acknowledgments. The last author is partially supported by the RSF grant 14-49-00079 (National Research University “MPEI” 14 Krasnokazarmennaya, 111250 Moscow, Russia), the RSF grant 17-11-01049 (National Research Tomsk State University), by RFBR Grant 16-01-00121, by the Ministry of Education and Science of the Russian Federation in the framework of the research project No 2.3208.2017/PCH by the Russian Federal Professor program (Project No 1.472.2016/FPM, Ministry of Education and Science of the Russian Federation) and by the project Systèmes compleXes, intelligence TERritoriale et Mobilité XterM–Feder.

8 Appendix

A.1 Property of the penalty term

Lemma A.1. For any \( n \geq 1 \) and \( \lambda \in \Lambda \),

\[
P_n^0(\lambda) \leq R_Q(\hat{S}_\lambda, S) + \frac{L_{1,Q}}{n},
\]

where the coefficient \( P_n^0(\lambda) \) is defined in (7.2) and the \( L_{1,Q} \) is defined in (5.2).

Proof. By the definition of \( \text{Err}(\lambda) \) in (3.11) one has

\[
\text{Err}(\lambda) \geq \sum_{j=1}^{\hat{p}} \left( (\lambda(j) - 1) \bar{\theta}_{j,p} + \frac{\lambda(j)}{n} \xi_{j,n} \right)^2.
\]

In view of Proposition 5.1 we obtain that

\[
R_Q(\hat{S}_\lambda, S) = E_Q \text{Err}(\lambda) \geq \frac{1}{n} \sum_{j=1}^{n} \lambda^2(j) E_Q \xi_{j,n}^2 \geq P_n^0(\lambda) - \frac{L_{1,Q}}{n}.
\]

Hence we obtain Lemma A.1.

A.2 Properties of stochastic integrals (3.1)

In this section we give some results of stochastic calculus for the process \((\xi_t)_{t \geq 0}\) given in (2.1), needed all along this paper. As the process \(\xi_t\) is the combination of a Lévy process and a semi-Markov process, these results are not standard and need to be provided.
Lemma A.2. Assume that Conditions $H_1$–$H_4$ hold true. Then, for any $n \geq 1$ and for any non random function $f$ from $L_2[0, n]$, the stochastic integral (3.1) exists and satisfies the properties (3.2) with the coefficient $\varkappa_Q$ given in (3.2).

Lemma A.3. Let $f$ and $g$ be any non-random functions from $L_2[0, n]$ and $(I_t(f))_{t \geq 0}$ be the process defined in (3.1). Then, for any $0 \leq t \leq n$,

$$E I_t(f)I_t(g) = \varrho_1^2(f, g)_t + \varrho_2^2(f, g\rho)_t,$$

(A.1)

where $(f, g)_t = \int_0^t f(s) g(s)ds$ and $\rho$ is the density defined in (2.7).

Lemma A.4. Let $f$ and $g$ be bounded functions defined on $[0, \infty) \times \mathbb{R}$. Then, for any $k \geq 1$,

$$E \int_0^n I_{T_k-}(f)I_{T_k-}(g) \, d\xi_t = 0,$$

where $\mathcal{G}$ is the $\sigma$-field generated by the sequence $(T_l)_{l \geq 1}$, i.e., $\mathcal{G} = \sigma\{T_l, l \geq 1\}$.

Lemma A.5. Assume that Conditions $H_1$–$H_4$ hold true. Then, for any measurable bounded non-random functions $f$ and $g$, one has

$$E \int_0^n I_{T_k-}(f)I_{T_k-}(g) d\xi_t = 0.$$

Lemmas A.2 – A.5 are proved in [1].

A.3 Properties of the Fourier coefficients

Lemma A.6. Let $f$ be an absolutely continuous function, $f : [0, 1] \rightarrow \mathbb{R}$, with $\|\dot{f}\| < \infty$ and $g$ be a simple function, $g : [0, 1] \rightarrow \mathbb{R}$ of the form $g(t) = \sum_{j=1}^p c_j \chi_{(t_{j-1}, t_j]}(t)$, where $c_j$ are some constants. Then for any $\varepsilon > 0$, the function $\Delta = f - g$ satisfies the following inequalities

$$\|\Delta\|^2 \leq (1 + \varepsilon)\|\Delta\|^2_p + (1 + \varepsilon^{-1})\|\dot{f}\|^2_p, \quad \|\Delta\|^2_p \leq (1 + \varepsilon)\|\Delta\|^2 + (1 + \varepsilon^{-1})\|\dot{f}\|^2_p.$$

Lemma A.7. Let the function $S(t)$ in (1.1) be absolutely continuous and have an absolutely integrable derivative. Then the coefficients $(\bar{\theta}_{j,p})_{1 \leq j \leq p}$ defined in (3.7) satisfy the inequalities

$$|\bar{\theta}_{1,p}| \leq \|S\|_1 \quad \text{and} \quad \max_{2 \leq j \leq p} j |\bar{\theta}_{j,p}| \leq 2\sqrt{2} \|\hat{S}\|_1.$$

(A.2)
Lemma A.8. For any \( p \geq 2 \), \( 1 \leq N \leq p \) and \( r > 0 \), the coefficients \( (\theta_{j,p})_{1 \leq j \leq p} \) of functions \( S \) from the class \( W^1_r \) satisfy, for any \( \tilde{\varepsilon} > 0 \), the following inequality

\[
\sum_{j=N}^{p} \theta_{j,p}^2 \leq (1 + \tilde{\varepsilon}) \sum_{j \geq N} \theta_{j}^2 + (1 + \tilde{\varepsilon}^{-1}) r p^{-2}.
\] (A.3)

Lemma A.9. For any \( p \geq 2 \) and \( r > 0 \), the coefficients \( (\theta_{j,p})_{1 \leq j \leq p} \) of functions \( S \) from the class \( W^1_r \) satisfy the following inequality

\[
\max_{1 \leq j \leq p} \sup_{S \in W^1_r} (|\theta_{j,p} - \theta_j| - 2\pi \sqrt{r j p^{-1}}) \leq 0.
\] (A.4)

Lemma A.10. For any \( p \geq 2 \) and \( r > 0 \) the correction coefficients \( (h_{j,p})_{1 \leq j \leq p} \) for the functions \( S \) from the class \( W^2_r \) satisfy the following inequality

\[
\sup_{S \in W^2_r} \sum_{j=1}^{p} h_{j,p}^2 \leq 3 r p^{-2}.
\] (A.5)

Lemmas A.6 – A.10 are proven in [16].
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