NON-STANDARD APPROXIMATIONS OF THE ITÔ-MAP

PETER FRIZ*, HARALD OBERHAUSER

Abstract. The Wong-Zakai theorem asserts that ODEs driven by "reasonable" (e.g. piecewise linear) approximations of Brownian motion converge to the corresponding Stratonovich stochastic differential equation. With the aid of rough path analysis, we study "non-reasonable" approximations and go beyond a well-known criterion of [Ikeda–Watanabe, North Holland 1989] in the sense that our result applies to perturbations on all levels, exhibiting additional drift terms involving any iterated Lie brackets of the driving vector fields. In particular, this applies to the approximations by McShane ('72) and Sussmann ('91). Our approach is not restricted to Brownian driving signals. At last, these ideas can be used to prove optimality of certain rough path estimates.

1. Preliminaries

1.1. Rough differential equations. Let $\alpha \in (0, 1]$. A weak geometric $\alpha$-Hölder rough path $x$ over $\mathbb{R}^d$ is a continuous path on $[0, T]$ with values in $G^{[1/\alpha]}(\mathbb{R}^d)$, the step-$[1/\alpha]$ nilpotent group over $\mathbb{R}^d$, of finite $\alpha$-Hölder regularity relative to $d$, the Carnot-Carathéodory metric on $G^{[\alpha]}(\mathbb{R}^d)$, i.e.

$$
\|x\|_{\alpha, \text{Hö}}, [0, T] = \sup_{0 \leq s < t \leq T} \frac{d(x_s, x_t)}{|t - s|^{\alpha}} < \infty.
$$

For orientation, let us discuss the case $\alpha \in (1/3, 1/2)$, which covers Brownian motion (for details see [FV05, Lyo98, LCL07, FV08b]). We realize $G^2(\mathbb{R}^d)$ as the set of all $(a, b) \in \mathbb{R}^d \oplus \mathbb{R}^{d \times d}$ for which $\text{Sym}(b) \equiv a \otimes 2 / 2$. (This point of view is natural: a smooth $\mathbb{R}^d$-valued path $x = (x_i)_{i=1,\ldots,d}$ enhanced with its iterated integrals $\int_0^t \int_s^t dx_i^j \, dx_j^i$, gives canonically rise to a $G^2(\mathbb{R}^d)$-valued path.). Given $(a, b) \in G^2(\mathbb{R}^d)$ one gets rid of the redundant $\text{Sym}(b)$ by $(a, b) \mapsto (a, b - a \otimes 2 / 2) \in \mathbb{R}^d \oplus \text{so}(d)$. Applied to $x$ enhanced with its iterated integrals over $[0, t]$ this amounts to look at the path $x$ and its (signed) areas $\int_0^t x_i^0, dx_i^j - \int_0^t x_i^j, dx_i^0$, $i, j \in \{1, \ldots, d\}$. Without going in too much detail, the group structure on $G^2(\mathbb{R}^d)$ can be identified with the (truncated) tensor multiplication and is relevant as it allows to relate algebraically the path and area increments over adjacent intervals; the mapping $(a, b) \mapsto (a, b - a \otimes 2 / 2)$ maps the Lie group $G^2(\mathbb{R}^d)$ to its Lie algebra; at last, the Carnot-Carathéodory metric is defined intrinsically as (left-)invariant metric on $G^2(\mathbb{R}^d)$ and satisfies $|a| + |b|^{1/2} \lesssim d((0, 0), (a, b)) \lesssim |a| + |b|^{1/2}$.

Key words and phrases. Iterated Lie Brackets in Limit Processes of Differential Equations, Rough Paths Analysis

*) Partially supported by a Leverhulme Research Fellowship and EPSRC Grant EP/E048609/1.

1.$[\cdot]$ gives the integer part of a real number.

2.Given an Interval $I = [a, b]$, for brevity we write $x_I \equiv x_{a,b} \equiv x_b - x_a$. 
Non-standard approximations of the Ito-map

One can then think of a geometric \( \alpha \)-Hölder rough path \( \mathbf{x} \) as a path \( x : [0, T] \to \mathbb{R}^d \) enhanced with its iterated integrals (equivalently: area integrals) although the later need not make classical sense. For instance, almost every joint realization of Brownian motion and Lévy’s area process is a geometric \( \alpha \)-Hölder rough path. Lyons’ theory of rough paths then gives deterministic meaning to the rough differential equation (RDE)

\[
dy = V(y) \, dx, \quad y(0) = y_0
\]

for \( \text{Lip}^\Gamma \)-vector fields (in the sense of Stein\[3\]), \( \Gamma > 1/\alpha \geq 1 \), and we write \( y_t = \pi \left( 0, y_0; x \right) \), for this solution. By considering the space-time rough path \( \bar{\mathbf{x}} = (t, \mathbf{x}) \) and \( \bar{V} = (V_0, V_1, ..., V_d) \) one can consider RDEs with drift. Although well studied \[LY06\], with a few towards minimal regularity assumptions on \( V_0 \), we shall need certain “Euler” estimates \[FV08a\] for RDEs with drift which are not available in the current literature. The “Doss–Sussmann method” (implemented for RDEs in sections 3) will provide a quick route to these estimates.

1.2. Standard and non-standard approximations. Assume we are given a weak geometric \( \alpha \)-Hölder rough path \( \mathbf{x} \) and a path \( \mathbf{p} \) that takes values in the center of \( \mathbb{G}^N (\mathbb{R}^d) \), \( N \) some integer \( N \geq [1/\alpha] \) (think of the path \( \mathbf{p} \) as a perturbation of our original path \( \mathbf{x} \)). Further assume that \( \mathbf{p} \) is \( \beta \)-Hölder continuous. It is a well-known \[Ly08\] that \( \mathbf{x} \) can be lifted uniquely to a \( \alpha \)-Hölder path \( S_N (\mathbf{x}) \) with values in \( \mathbb{G}^N (\mathbb{R}^d) \). Then,

\[
S_N (\mathbf{x})_{0,} \otimes \mathbf{p} \in C^{\min(\alpha, \beta)-\text{Hö}} ([0, T], \mathbb{G}^N (\mathbb{R}^d)).
\]

From general facts of such spaces (e.g. \[FV06\], relying only on the fact that \( \mathbb{G}^N (\mathbb{R}^d) \) is a geodesic space) we can find a sequence of Lipschitz continuous paths, \( x^n : [0, T] \to \mathbb{R}^d \), so that

\[
d_{\infty} \left( S_N (x^n), S_N (\mathbf{x})_{0,} \otimes \mathbf{p} \right) \to 0 \text{ as } n \to \infty \text{ and } \sup_{n \in \mathbb{N}} \| S_N (x^n)\|_{\min(\alpha, \beta)-\text{Hö}} < \infty.
\]

(The approximations \( x^n \) are constructed based on geodesics associated to the \( \mathbb{G}^N (\mathbb{R}^d) \)-valued increments of \( S_N (\mathbf{x})_{0,} \otimes \mathbf{p} \).) By interpolation, it then follows that for all \( \gamma < \min (\alpha, \beta) \),

\[
d_{\gamma-\text{Hö}} \left( S_N (x^n), S_N (\mathbf{x}) \otimes \mathbf{p} \right) \to 0 \text{ as } n \to \infty.
\]

The interest of such a construction is that the limiting behaviour of ODEs driven by the \( x^n \), provided \( [1/\gamma] \geq N \) and \( \Gamma > 1/\gamma \), exhibits additional drift behaviour in terms of the Lie brackets of the driving vector fields and the perturbation \( \mathbf{p} \) (cf. sections 3.1 and 3).

We may apply this to Brownian and Lévy area, i.e. \( \mathbf{x} = \mathbf{B} (\omega) = \exp (\mathbf{B} + \mathbf{A}) \), in which case the approximations \( x^n \) are constructed in a purely deterministic fashion based on the realization of the Brownian path and its Lévy area. Interesting as it may be, this is not fully satisfactory as it stands in contrast to (probabilistic) non-standard approximation results (McShane \[McS72\], Sussmann \[Sus91\], ...) which have the desirable property that the approximations depend only on (finitely many points of) the Brownian path. The first aim of this paper is to give a criterion that covers all these examples in a flexible frame-work of random rough paths. En

\[\text{i.e. a function is Lip}^\gamma \text{ if it is } [\gamma] \text{-times } ([\gamma] = [\gamma] - 1 \text{ on integers, otherwise equal) differentiable, the } [\gamma] \text{-th derivative is } \gamma - [\gamma] \text{-Hölder continuous and the function and all its derivatives are bounded.}\]
passant, this allows for a painfree extension to various non-Brownian driving signals (cf. remark [1]). We then give a rigorous analysis on how to translate RDEs driven by $S_N(x) \otimes p$ to RDEs driven by $x$ in which see the appearance of additional drift vector fields, obtained as contraction of iterated Lie brackets of $V = (V_1, \ldots, V_d)$ and the components of $p$. (At this stage, we need a good quantitative understanding of RDEs with drift). At last, as a spin-off of these ideas, we show in section 4.2 optimality of certain rough path estimates, answering a question that left was open in [CFV05].

2. (Non-Standard) Approximations

2.1. Criterion for convergence to a "non-standard" limit.

**Theorem 1.** Let $\alpha, \beta \in (0, 1)$ and $N \geq [1/\gamma]$ for $\gamma := \min (\alpha, \beta)$. Also, let $x \in C^{\alpha,\text{H"older}}([0, T], G^{[1/\alpha]}(\mathbb{R}^d))$, write $x = \pi_1 (x)$ for its projection to a path with values in $\mathbb{R}^d$, and assume that there exists a sequence of dissections $(D_n) = (t^n_i : i)$ of $[0, T]$, such that

$$\sup_{n \in \mathbb{N}} \left\| S_{[1/\alpha]} (x^{D_n}) \right\|_{\alpha,\text{H"older}} < \infty$$

and

$$d_\infty \left( S_{[1/\alpha]} (x^{D_n}) , x \right) \to n \to -\infty 0,$$

where we write $x^{D_n}$ for the usual piecewise linear approximation to $x$ based on $D_n$. Let $(x^n) \subset C^{1,\text{H"older}}([0, T], \mathbb{R}^d)$ such that, for all $t \in D_n$,

$$p^n_t := S_N (x^n)_{0,t} \otimes S_N (x^{D_n})_{0,t}^{-1}$$

takes values in the center of $G^N(\mathbb{R}^d)$.

(i) If there exists $c_1, c_2, c_3 \in [1, \infty)$ such that for all $t^n_i, t^n_{i+1} \in D_n$

$$|x^n|_{1,\text{H"older}}[t^n_i, t^n_{i+1}] \leq c_1 |x^{D_n}|_{1,\text{H"older}}[t^n_i, t^n_{i+1}] + c_2 |t^n_{i+1} - t^n_i|^{\beta - 1}$$

and

$$\|p^n_{s,t}\| \leq c_3 |t - s|^\beta$$

for all $s, t \in D_n$. Then there exists a $C = C (\alpha, \beta, c_1 \|x\|_{\alpha,\text{H"older}}, c_2, T, N)$ such that

$$\sup_{n \in \mathbb{N}} \left\| S_N (x^n) \right\|_{\gamma,\text{H"older}} \leq C \left( \sup_{n \in \mathbb{N}} \left\| S_{[1/\alpha]} (x^{D_n}) \right\|_{\alpha,\text{H"older}} + c_3 + 1 \right) < \infty.$$

(ii) If $p^n_t \to p_t$, for all $t \in \cup_n D_n$ and $\cup_n D_n$ is dense in $[0, T]$ then $p$ extends to a continuous (in fact, $\beta$-H"older continuous) path with values in the center of $G^N(\mathbb{R}^d)$ and for all $t \in [0, T]$,

$$d \left( S_N (x^n)_{0,t} , S_N (x)_{0,t} \otimes p_{0,t} \right) \leq d \left( S_N (x^{D_n})_{0,t} , S_N (x)_{0,t} \right) + d \left( p^n_{0,t} , p_{0,t} \right).$$

and converges to 0 as $n \to \infty$.

In particular, if the assumptions of both (i) and (ii) are met then

$$d_\infty \left( S_N (x^n) , S_N (x) \otimes p \right) \to n \to -\infty 0,$$

$$\sup_{n \in \mathbb{N}} \left\| S_N (x^n) \right\|_{\gamma,\text{H"older}} < \infty.$$

and by interpolation, for all $\gamma' < \gamma$,

$$d_{\gamma',\text{H"older}} \left( S_N (x^n) , S_N (x) \otimes p \right) \to n \to -\infty 0.$$
Proof. (i) Take $s < t$ in $[0,T]$. If $s,t \in [t^n_{i}, t^n_{i+1}]$ we have by our assumption on $|x^n|_{1-\text{Hol}:[t^n_{i}, t^n_{i+1}]}$

$$
\|S_N(x^n)_{s,t}\| \leq |t-s| \|S_N(x^n)\|_{1-\text{Hol}:[t^n_{i}, t^n_{i+1}]}
= |t-s| \|x^n\|_{1-\text{Hol}:[t^n_{i}, t^n_{i+1}]}
\leq |t-s| \left\{ c_1 |x|_{t^n_{i}, t^n_{i+1}}^\alpha + c_2 |t^n_{i+1} - t^n_i|^{\beta-1} \right\}
\leq |t-s| \left\{ c_1 |x|_{t^n_{i}, t^n_{i+1}}^\alpha + c_2 |t^n_{i+1} - t^n_i|^{\beta-1} \right\}
\leq |t-s|^\gamma C_0,
$$
with $C_0 = C_0(\alpha, \beta, c_1 \|x\|_{\alpha-\text{Hol}}, c_2, T)$. Otherwise we can find $t^n_{i} \leq t^n_j$ so that

$$
|t-s| \leq 2C_0 + \left\| S_N(x^n)_{t^n_{i}, t^n_j} \right\|.
$$
Using estimates for the Lyons-lift $x \mapsto S_N(x)$, [Lyo98, theorem 2.2.1], we can further estimate

$$
\|S_N(x^n)_{t^n_{i}, t^n_j} \| \leq c_{N,\alpha} \left\| S_{[1/\alpha]}(x^n) \right\|_{\alpha-\text{Hol}}^{t^n_j - t^n_i} + c_3 |t^n_j - t^n_i|^\beta
\leq (c_{N,\alpha} \left\| S_{[1/\alpha]}(x^n) \right\|_{\alpha-\text{Hol}} + c_3) |t-s|^\gamma
$$
and, since $\sup_n \left\| S_{[1/\alpha]}(x^n) \right\|_{\alpha-\text{Hol}} < \infty$ by assumption, the proof of the uniform Hölder bound is finished.

(ii) By assumption, $p^n$ is uniformly $\beta$-Hölder. By a standard Arzela-Ascoli type argument, it is clear that every pointwise limit (if only on the dense set $\bigcup D_n$) is a uniform limit. $\beta$-Hölder regularity is preserved in this limit and so $p$ is $\beta$-Hölder itself. Since $t \in \bigcup D_n, p^n_t$ take values in the step-$N$ center for all $t \in D_n$, $\bigcup D_n$ is dense, and so it is easy to see that $p$ takes values in the step-$N$ center for all $t \in [0,T]$. Now take $t \in D_n$. Since elements in the center commute with all elements in $G^N(\mathbb{R}^d)$ we have

$$
d \left( S_N(x^n)_{0,t}, S_N(x)_{0,t} \otimes p_{0,t} \right)
= \left\| S_N(x^n)_{0,t} \otimes S_N(x^n)_{0,t} \otimes p^n_0 \otimes S_N(x^n)_{0,t} \otimes (p^n_0)^{-1} \otimes p_{0,t} \right\|
\leq \left\| S_N(x^n)_{0,t} \otimes (p^n_0)^{-1} \otimes p_{0,t} \right\|
\leq d \left( S_N(x^n)_{0,t}, S_N(x)_{0,t} \right) + d \left( p^n_0, p_{0,t} \right)
$$
On the other hand, given an arbitrary element $t \in [0,T]$ we can take $t^n$ to be the closest neighbour in $D_n$ and so

$$
d \left( S_N(x^n)_{0,t}, S_N(x)_{0,t} \otimes p_{0,t} \right)
= d \left( S_N(x^n)_{0,t}, S_N(x)_{0,t} \right) + 2d \left( p^n_0, p_{0,t} \right)
\leq d \left( S_N(x^n)_{0,t}^{-1} \otimes S_N(x^n)_{0,t}, S(x)_{0,t}^{-1} \otimes S_N(x^n)_{0,t} \right).
$$
Non-standard approximations of the Itô-map

From the assumptions and Hölder (resp. uniform Hölder) continuity of $S(x)$ (resp. $S_N(x^n)$) we see that $d\left(S_N(x^n)_{0,t}, S_N(x)_{0,t} \otimes \mathbf{P}_{0,t}\right) \to 0$, as required. □

**Corollary 1.** Let $\alpha, \beta \in (0, 1]$, $N \geq 1 / \min (\alpha, \beta)$. Also, let $X(\omega) \in C_{0}^{1,\beta}(0, T), C^{[\beta]}([0, T]) (\mathbb{R}^{d})$, write $X = \pi_1(X)$ for its projection to a process with values in $\mathbb{R}^{d}$, and assume that

$$\forall q \in \mathbb{N} : \sup_{n \in \mathbb{N}} \left\| S_{[1/\alpha]}(X^{D_n}) \right\|_{\alpha, \text{Hölder}} \leq \infty$$

$$d_{\infty} (S_{[1/\alpha]}(X^{D_n}), X) \to 0 \text{ in probability as } n \to \infty.$$

Let $(X^n(\omega))_{n \in \mathbb{N}} \subset C^{1,\beta}(0, 1] (\mathbb{R}^{d})$ such that, for all $\omega$ and all $t \in D_n$

$$P_{t}^{n}(\omega) := S_{N}(X^{n})_{0,t} \otimes S_{N}(X^{D_n})^{-1}_{0,t}$$

takes values in the center of $G^{N}(\mathbb{R}^{d})$.

(i) If there exists $c_1, c_2, c_3 \in [1, \infty)$ such that for all $t^n_i, t^n_{i+1} \in D_n$, all $\omega$ and all $q \in [1, \infty)$,

$$|X^{n}_{1,\text{Höld}}[t^n_{i}, t^n_{i+1}]| \leq c_1 |X^{D_n}_{1,\text{Höld}}[t^n_{i}, t^n_{i+1}]| + c_2 |t^n_{i+1} - t^n_{i}|^{\beta - 1}$$

$$\left\| P_{s,t}^{n} \right\|_{L^{q}(\mathbb{P})} \leq c_{q} |t - s|^\beta \text{ for all } s, t \in [0, T]$$

then, for all $\gamma < \min (\alpha, \beta)$,

$$\forall q : \sup_{n \in \mathbb{N}} \left\| S_{N}(x^n) \right\|_{\gamma, \text{Höld}} \leq \infty.$$

(ii) If $P_{t}^{n} \to P_t$ in probability for all $t \in \bigcup_{n \in \mathbb{N}} D_n$ and $\bigcup_{n \in \mathbb{N}} D_n$ is dense in $[0, T]$ then

$$d\left(S_{N}(X^{n})_{0,t}, S_{N}(X)_{0,t} \otimes \mathbf{P}_{0,t}\right) \to 0 \text{ in probability}$$

In particular, if the assumptions of both (i) and (ii) are met then, for all $\gamma < \min (\alpha, \beta),

$$d_{\gamma, \text{Höld}}(S_{N}(X^{n}), S_{N}(X) \otimes \mathbf{P}) \to 0 \text{ in } L^{q} \text{ for all } q \in [1, \infty).$$

**Proof.** (i) By a standard Garsia-Rodemich-Rumsey or Kolmogorov argument, the assumption on $\left\| P_{s,t}^{n} \right\|_{L^{q}(\mathbb{P})}$ implies, for any $\beta < \beta$, the existence of $C_3 \in L^{q}$ for all $q \in [1, \infty)$ so that

$$\forall s < t \text{ in } [0, T] : \left\| P_{s,t}^{n} \right\| \leq C_{3}(\omega) |t - s|^\beta$$

We apply theorem[1] with $\tilde{\beta}$ instead of $\beta$ and learn that there exists a deterministic constant $C$ such that

$$\sup_{n} \left\| S_{N}(X^{n}) \right\|_{\min(\alpha, \beta), \text{Höld}} \leq C \left( \sup_{n} \left\| S_{[1/\alpha]}(X^{D_n}) \right\|_{\alpha, \text{Hölder}} + 1 + C_3 \right).$$

Taking $L^{q}$-norms finishes the uniform $L^{q}$-bound. (We take $\tilde{\beta}$ large enough so that $\min (\alpha, \tilde{\beta}) > \gamma$.)

(ii) From theorem[2]

$$d\left(S_{N}(X^{n})_{0,t}, S_{N}(X)_{0,t} \otimes \mathbf{P}_{0,t}\right) \leq d\left(S_{N}(X^{D_n})_{0,t}, S_{N}(X)_{0,t}\right) + d(P_{0,t}^{n}, P_{0,t})$$
which, from the assumptions, obviously converges to 0 (in probability) for every fixed \( t \in \cup_n D_n \). From general facts, of \( L^q \)-convergence of rough paths (cf. [FV07 Appendix]) this implies the claimed convergence. (Inspection of the proof shows that convergence in probability for all \( t \) in a dense set of \([0, T]\) is enough.)

**Remark 1.** Although at first sight technical, our assumptions are fairly natural: firstly, we restrict our attention to (random) Hölder rough paths \( X \) which are the limit of "their (lifted) piecewise linear approximations". This covers the bulk of stochastic processes which admit a lift to a rough path including semi-martingales [CL05] [FV06a], fractional Brownian motion with \( H > 1/4 \) and many other Gaussian processes, [CQ02] [FV07], as well as Markov processes with uniformly elliptic generator in divergence form [Str88], [Lej02], [FV06c]. Secondly, the assumptions on \( X^n \) and \( P^n \) guarantee that \( X^n \) remains, at \( \min(\alpha, \beta) \)-Hölder scale, comparable to the piecewise linear approximations. In particular, the assumption on \( [X^n]_{1-Hölder}^{[t^n, t^n_{n+1}]} = \hat{X}^n_{\infty;[t^n, t^n_{n+1}]} \) is easy to verify in all examples below. The intuition is that, if we assume that \( X^n \) runs at constant speed over any interval \( I = [t^n_i, t^n_{i+1}] \), \( D_n = (t^n_i) \), it is equivalent to saying that

\[
\text{length}(X^n_{\mid I}) \leq c_1 \text{length}(X^{D_n}_{\mid I}) + c_2 |I|^{\beta}
\]

( = \( c_1 |X_{t^n_i, t^n_{i+1}}| + c_2 |t^n_{i+1} - t^n_i|^{\beta} \))

2.2. Examples.

2.2.1. Sussmann [Sus91]. Take any sequence of dissection of \([0, T]\), say \( (D_n) \) with mesh \( |D_n| \to 0 \) and think of \( X(\omega) \) as Brownian motion plus Lévy’s area so that \( \pi_1(X) = X \) is \( d \)-dimensional Brownian motion. The piecewise linear approximation \( X^{D_n} \) is nothing but the repeated concatenation of linear chords connecting the points \((X_t : t \in D_n)\) for some fixed \( \nu \in \mathfrak{g}^N (\mathbb{R}^d) \cap (\mathbb{R}^d)^{\otimes N} \), \( N \in \{2, 3, \ldots\} \) we now construct Sussmann’s non-standard approximation \( X^n \) as (repeated) concatenation of linear chords and "geodesic loops". First, we require \( X^n(t) = X(t) \) for all \( t \in D_n = (t^n_i) \). For intermediate times, i.e. \( t \in (t^n_{i-1}, t^n_i) \) for some \( i \) we proceed as follows: For \( t \in [t^n_{i-1}, (t^n_{i-1} + t^n_i) / 2] \) we run linearly and at constant speed from \( X(t^n_{i-1}) \) such as to reach \( X(t^n_i) \) by time \((t^n_{i-1} + t^n_i) / 2\). (This is the usual linear interpolation between \( X(t^n_{i-1}) \) and \( X(t^n_i) \) but run at double speed.) This leaves us with the interval \([t^n_{i-1} + t^n_i / 2, t^n_i] \) for other purposes and we run, starting at \( x(t^n_i) \in \mathbb{R}^d \), through a "geodesic" \( \xi : [(t^n_{i-1} + t^n_i) / 2, t^n_i] \to \mathbb{R}^d \) associated to \( \exp(v / |t^n_i - t^n_{i-1}|) = 0 \) and so this geodesic path returns to its starting point in \( \mathbb{R}^d \); in particular

\[
X^n((t^n_{i-1} + t^n_i) / 2) = X^n(t^n_i) = X(t^n_i).
\]

It is easy to see (via Chen’s theorem) that this approximation satisfies the assumptions of corollary \( \square \) with

\[
P^n_{s,t} := S_N(x^n_{s,t}) \otimes S_N(x^{D_n}_{s,t})^{-1} = e^{v(t-s)} \forall s, t \in D_n,
\]

(so that \( |P^n_{s,t}|_{L^1} \ll |t-s|^{1/N} \) first for all \( s, t \in D_n \) and then, easy to see, to all \( s, t \) and deterministic limit \( P_{s,t} = e^\beta, \beta = 1/N \). Indeed, the length of \( x^n \) over

\(^4\)A path \( \xi : [a, b] \to \mathbb{R}^d \) is a geodesic associated to \( g \in G^N (\mathbb{R}^d) \) if \( S_N(\xi)(a, b) = g \) and \( \xi \) has length equal to the Carnot-Carathéodory norm of \( g \). See [FV00b] and the references therein.
any interval \( I = [t^n_{i-1}, t^n_i] \) is obviously bounded by the length of the corresponding linear chord plus the length of the geodesic associated to \( \exp \left( \frac{v}{2^n} \right) = \exp \left( \frac{v}{|I|} \right) \), which is precisely equal to

\[
\| \exp \left( \frac{v}{|I|} \right) \| = |I|^{1/N} \| \exp (v) \| =: c_2 |I|^{1/N}.
\]

Obviously, nothing here is specific to Brownian motion. An application of corollary \( \text{[1]} \) gives the following convergence result which, when applied to Brownian motion and Lévy area and in conjunction with theorem \( \text{[1]} \) below, implies Sussmann’s non-standard approximation result for stochastic differential equations.

**Proposition 1** (Rough path convergence of Sussmann’s approximation). Let \( X (\omega) \in C_0^\alpha \text{-Hölder} \left( [0, T], G^{1/\alpha} (\mathbb{R}^d) \right) \) be a (random, \( \alpha \)-Hölder rough path) which is the limit of its piecewise linear approximation (as detailed in corollary \( \text{[2]} \)). Then, for any \( \gamma < \min (\alpha, 1/N) \) we have

\[
d_{\gamma, \text{Hölder}} \left( S_N \left( X^n \right), S_N \left( X \otimes e^v \right) \right) \to 0 \text{ in } L^q \text{ for all } q \in [1, \infty).
\]

Strictly speaking, his construction did not rely on the concept of geodesics as- sociated to \( \exp \left( \frac{v}{2^n} \right) \in G^N (\mathbb{R}^d) \). Since \( \exp \left( \frac{v}{2^n} \right) \) is an element of the center of the group he can give a reasonably simple inductive construction of a piecewise linear "approximate geodesic" which is also seen to satisfy the assumptions of our theorem. We also note that he only discusses dyadic approximations and obtains on a.s. convergence result (which can be obtained by a direct application of theorem \( \text{[1]} \)).

In conjunction with theorem \( \text{[2]} \) below, we then recover the main result of Sussmann \( \text{[SUS91]} \).

### 2.2.2. McShane [McS72].

Given \( x \in C \left( [0, T], \mathbb{R}^2 \right) \), an interpolation function \( \phi = (\phi^1, \phi^2) \in C^1 \left( [0, 1], \mathbb{R}^2 \right) \) with \( \phi (0) = (0, 0) \) and \( \phi (1) = (1, 1) \) and a dissection \( D = D_n = \{ t_i \} \) of \( [0, T] \) we define the McShane interpolation \( x^n \in C \left( [0, T], \mathbb{R}^2 \right) \) by

\[
x^n_{i+1} := x^n_{1D} + \phi^{\Delta(t,i)} \left( \frac{t - t_D}{t_D - t} \right) x^n_{t_D, t}, \quad i = 1, 2.
\]

The points \( t_D, tD \in D \) denote the left-, resp. right-, neighbouring points of \( t \) in the dissection and

\[
\Delta (t, i) := \begin{cases} 
  i & \text{if } x^n_{1D,tD} x^n_{tD,t} \geq 0 \\
  3 - i & \text{if } x^n_{1D,tD} x^n_{tD,t} < 0
\end{cases}
\]

As a simple consequence of this definition, for \( u < v \) in \([t_i, t_{i+1}]\)

\[
S_2 \left( x^n_{u,v} \right) = \exp \left( x^n_{u,v} + A^n_{u,v} \right) = \exp \left( x^n_{u,v} + \left| x^n_{t_i,t_{i+1}} \right| A^n \left( \frac{u - t_i}{t_{i+1} - t_i}, \frac{v - t_i}{t_{i+1} - t_i} \right) \right)
\]

where \( A^n (u, v) \equiv A^n_{u,v} \) is the area increment of \( \phi \) over \([u, v] \subset [0, 1] \).

Consider now \( X (\omega) = B (\omega) = \exp (B + A) \in C_0^\alpha \text{-Hölder} \left( [0, 1], G^{1/\alpha} (\mathbb{R}^2) \right) \) with \( \alpha \in (1/3, 1/2) \) and take any \( (D_n)_{n \in \mathbb{N}} \) with \( \| D_n \| \to 0 \). (We know, e.g. from [EVi07], that lifted piecewise linear approximations, \( S_2 \left( B^{D_n} \right) \) converges to \( B \) in \( 1/\alpha \)-Hölder rough path topology and in \( L^q \) for all \( q \)). It is easy to see (via Chen’s theorem) that McShane’s approximation to two-dimensional Brownian motion satisfies the
Non-standard approximations of the Ito-map

assumptions of corollary 1 with $\beta = 1/2$, $N = 2$. Indeed, writing $S_2(B^n) = \exp(B^n + A^n)$ it is clear that for any $s < t$

$$
P^n_{s,t} = \exp \left( \left| x^1_{t_D,t_D} \right| \left| x^2_{t_D,t_D} \right| \times A^\phi \left( \frac{s-t_D}{t_D^2-t_D}, \frac{t-t_D}{t_D^2-t_D} \right) \right)
$$

with $D = D_n$

and for $t_i < t_j$ elements of the dissection $D = D_n$,

$$
P^n_{t_i,t_j} = \exp \left( A^{\phi_1} \sum_{k=i+1}^j \left| B_{t_k,t_k+1}^1 \right| \left| B_{t_k,t_k+1}^2 \right| \right).
$$

It is straightforward to see that $\sum_{k=i+1}^j \left| B_{t_k,t_k+1}^1 \right| \left| B_{t_k,t_k+1}^2 \right|$ converges, in $L^2$ say, to its mean

$$
\frac{2}{\pi} \sum_{k=i+1}^j (t_{k+1} - t_k) = \frac{2}{\pi} \left| t_j - t_i \right|
$$

while $\left\| P^n_{t_i,t_j} \right\|_{L^q} \leq c_q \left| t_j - t_i \right|^{1/2}$ follows directly from $\left\| P^n_{t_i,t_j} \right\| \sim \left( \sum_{k=i+1}^j \left| B_{t_k,t_k+1}^1 \right| \left| B_{t_k,t_k+1}^2 \right| \right)^{1/2}$ and

$$
\sum_{k=i+1}^j \left| B_{t_k,t_k+1}^1 \right| \left| B_{t_k,t_k+1}^2 \right| \left\| L^q \right\| \leq c_q \left| t_j - t_i \right|.
$$

In fact, $\left\| P^n_{s,t} \right\|_{L^q} \leq c_q \left| t - s \right|^{1/2}$ for all $s,t$ since for $u < v$ in $[t_i, t_{i+1}]$

$$
\left\| P^n_{u,v} \right\|_{L^q} = \left( \mathbb{E} \left| x^1_{t_i,t_{i+1}} \right| \left| x^2_{t_i,t_{i+1}} \right| A^\phi \left( \frac{v-t_i}{t_{i+1}-t_i}, \frac{u-t_i}{t_{i+1}-t_i} \right) \right)^{1/2} = c_{\phi,q} \left| t_{i+1} - t_i \right|^{1/2} \left| \frac{u-v}{t_{i+1}-t_i} \right| \leq c_{\phi,q} \left| u - v \right|^{1/2}.
$$

At last, we easily see that, for any $t_i \in D_n$,

$$
\left| B^n \right|_{1;\text{Hö}} \left| [t_i, t_{i+1}] \right| \leq \left| \phi' \right| \left| B^{D_n} \right|_{1;\text{Hö}} \left| [t_i, t_{i+1}] \right|.
$$

This shows that all assumptions of the above theorem are satisfied and so we have

**Proposition 2** (Rough path convergence of McShane’s approximation). For all $\alpha \in [0,1/2)$,

$$
d_{\alpha;\text{Hö}}(S_2(B^n), \exp(B_t + A_t + (\Gamma))) \to 0 \text{ in } L^q \text{ for all } q \in [1, \infty)
$$

where $A_t$ the usual so(2) valued Lévy’s area and

$$
\Gamma = \left( \begin{array}{ccc}
0 & \frac{2}{\pi} A^\phi_{0,1} \\
-\frac{2}{\pi} A^\phi_{0,1} & 0
\end{array} \right) \in \text{so}(2).
$$

We note that corollary 1 also applies to a (minor) variation of the McShane example given in [LL02].
3. RDEs with drift: the Doss-Sussmann approach

3.1. Preliminaries on ODEs and flows. Consider an ordinary differential equation (ODE), driven by a smooth $\mathbb{R}^d$-valued signal $f(t) = (f^1(t), \ldots, f^d(t))^T$ on a time interval $[t_0, T]$ along sufficiently smooth and bounded vector fields $V = (V_1, \ldots, V_d)$ and a drift vector field $V_0$.

$$\frac{dy}{dt} = V_0(y) \, dt + V(y) \, df, \quad y(t_0) = y_0 \in \mathbb{R}^e.$$  

We also call $U^f_{t-t_0}(y_0) \equiv y_t$ the associated flow. Let $J$ denote the Jacobian of $U$. It satisfies the ODE obtained by formal differentiation w.r.t. $y_0$. More specifically,

$$a \mapsto \left\{ \frac{d}{d\varepsilon} U^f_{t-t_0} (y_0 + \varepsilon a) \right\}_{\varepsilon = 0}$$

is a linear map from $\mathbb{R}^e \to \mathbb{R}^e$ and we let $J^f_{t-t_0}(y_0)$ denote the corresponding $e \times e$ matrix. It is immediate to see that

$$\frac{d}{dt} J^f_{t-t_0}(y_0) = \left[ \frac{d}{dt} M^f \left( U^f_{t-t_0}(y_0), t \right) \right] \cdot J^f_{t-t_0}(y_0)$$

where $\cdot$ denotes matrix multiplication and

$$\frac{d}{dt} M^f (y,t) = \sum_{i=1}^d V'_i(y) \frac{d}{dt} f^i_t + V'_0(y).$$

Note that also $J^f_{t_2-t_0} = J^f_{t_2-t_1} \cdot J^f_{t_1-t_0}$ and that $J^f_{t-t_0}$ is invertible with inverse, denoted $J^f_{t_0-t}$, given as the flow of (3.1) with $f$ replaced by $f^t(.) = f(t - .)$, i.e.

$$J^f_{t_0-t}(.) = \left( J^f_{t-t_0}(.) \right)^{-1} = J^f_{t_0-t}(.).$$

Now for $V = (V_1, \ldots, V_d) \in \text{Lip}^2(\mathbb{R}^e)$ and $x \in C^1([0,T], \mathbb{R}^d)$ ODE theory tells us that $dy = V(y) \, dx_t$ has a $C^2$-flow. Note that the flow

$$y_0 \mapsto \pi_V (0, y_0, x)_t \equiv U^x_{t-t_0}(y_0)$$

is even globally Lipschitz (thanks to the boundedness which is part of the Lipschitz-definition). The associated Jacobian $J^x_{t-t_0}(.)$ is itself $C^1$ and also globally Lipschitz in $y_0$ as well as its inverse $J^x_{t_0-t}(.)$. This is more than enough to see that for $V_0 \in \text{Lip}^1$.

$$(J^x_{t_0-t}(.) \cdot V_0(.) \circ \pi_V(0,*,x)_t)$$

is Lipschitz (in $t$), uniformly in $t$ in $[0,T]$. Obviously, the above expression, as a function of $(*,t)$, is also continuous in $t$. It follows that

$$\dot{z}_t = J^x_{t_0-t}(z_t) \cdot V_0 \left( \pi_V(0, z_t, x_t)_t \right)$$

has a unique solution started from $z(0) = y_0$. An elementary computation shows that $\tilde{y}(t) \equiv \pi_V(0, z_t, x_t)_t$ satisfies

$$d\tilde{y} = V_0(\tilde{y}) \, dt + V(\tilde{y}) \, dx.$$  

This is the Doss-Sussmann method (cf. [RW00, p.180]) applied in a simple ODE context.
3.2. Doss–Sussmann method for RDEs. We return to the discussion of section 3.1 and define solutions of RDEs with drift as uniform limits of solutions of ODEs with drift.

**Definition 1.** Let \( x \in C^{\alpha,\text{Hölder}}([0, T], G^{[1/\alpha]}(\mathbb{R}^d)) \). If there exists a sequence \((x^n)_{n \in \mathbb{N}}\) of Lipschitz paths with uniform \( \alpha \)-Hölder bounds converging pointwise to \( x \) (that is \( x^n_t = S_{[1/\alpha]}(x^n) \to x \) for every \( t \in [0, T] \) and \( \sup_{n \in \mathbb{N}} \|S_{[1/\alpha]}(x^n)\|_{\alpha,\text{Hölder}} < \infty \)) such that for each \( n \in \mathbb{N} \) the RDE (in fact ODE) with drift

\[
dy^n = V_0(y^n) \, dt + V(y^n) \, dx^n, \quad y^n(0) = y_0
\]

has a unique solution \( y^n \) on \([0, T]\), then we call any limit point in uniform topology of \(\{y^n, n \in \mathbb{N}\}\) a solution of the RDE with drift

\[
dy = V_0(y) \, dt + V(y) \, dx, \quad y(0) = y_0
\]

and we also write \( y = \pi(V_{\Omega}) \) \((0, y_0; (x, t))\) for this solution.

**Proposition 3** (Doss-Sussman for RDE). Assume that

(i) \( x \in C^{\alpha,\text{Hölder}}([0, T], G^{[1/\alpha]}(\mathbb{R}^d)) \), \( \alpha \in (0, 1/2) \),

(ii) \( V_0 \in \text{Lip}^1(\mathbb{R}^e) \) and \( V = (V_1, \ldots, V_d) \in \text{Lip}^{\gamma+1}(\mathbb{R}^e) \) for a \( \gamma > 1/\alpha \),

(iii) \( y_0 \in \mathbb{R}^e \).

Then there exists a unique solution \( y \) to the RDE with drift

\[
dy = V_0(y) \, dt + V(y) \, dx, \quad y(0) = y_0
\]

Further, this solution is \( \alpha \)-Hölder continuous and given as

\[
y(t) = U_{t-0}^x(z_t),
\]

with \( z_t = W(t, z_t), \quad z(0) = y_0 \),

where \( W(t, \cdot) \equiv J_{0-t}^{x_0 \cdot} \cdot V_0(\pi_V(0, \cdot; x_0)) \), \( U_{t-0}^x(\cdot) = \pi_V(0, \cdot; x) \) is the flow of

\[
d\tilde{y} = V(y) \, dx, \quad \tilde{y}(0) = y_0
\]

and \( J_{0-t}^{x_0 \cdot} = (\partial \pi(0, \cdot; 0))^{-1} \) is the inverse of the Jacobian of \( U_{t-0}^x \).

**Proof.** We first show that \( U_{t-0}^x \) has a unique solution. Existence and uniqueness of \( \pi_V(0, \cdot; x) \) (and of a \( C^2 \) flow) follows from standard rough path theory so this boils down to check that the ODE \( (3.5) \) has a unique solution on \([0, T]\). However, the vector field \( W(t, y) \) is continuous in \( t \) and \( y \) and \( W(t, \cdot) \) is Lipschitz continuous in \( \cdot \), uniformly in \( t \):

\[
|W(t, y) - W(t, x)| = |J_{0-t}^{x_0 \cdot \cdot} \cdot V_0(\pi_V(0, y; x)) \cdot J_{0-t}^{x_0 \cdot \cdot} \cdot V_0(\pi_V(0, x; x))| \leq |J_{0-t}^{x_0 \cdot \cdot}| |V_0|_{\text{Lip}} |U_{t-0}^x|_{\text{Lip}} |x - y| + |J_{0-t}^{x_0 \cdot \cdot}||x - y| |V_0|_{\infty}.
\]

Thanks to the invariance of the Lipschitz norms \(|V(\cdot, y)|_{\text{Lip}^\gamma+1} = |V(\cdot)|_{\text{Lip}^{\gamma+1}}\) and uniform estimates follow from a routine exercise showing that \(|U_{t-0}^x|_{\text{Lip}}, \sup_y |J_{0-t}^{x_0 \cdot \cdot}|\) and \(|J_{0-t}^{x_0 \cdot \cdot}|_{\text{Lip}}\) are all bounded by a constant \( c_0 = c_0(\alpha, \gamma, |V_0|_{\text{Lip}^{\gamma+1}}, \|x\|_{\alpha,\text{Hölder}}, \|V_0\|_{\text{Lip}})\), uniformly in \( t \). The desired Lipschitz regularity of \( W \) follows which implies existence of a unique solution \( z \) on \([0, T]\).

To see that the path \( y_t = U_{t-0}^x(z_t) \) is the unique RDE solution to \( (3.5) \) let \((x^n)_{n \geq 1}\) be a sequence of Lipschitz paths with uniform \( \alpha \)-Hölder bounds converging pointwise to \( x \). For brevity set \( x^n \equiv S_{[1/\alpha]}(x^n) \). Following our discussion in section

\[\text{(3.2)}\]

\[\text{(3.3)}\]

\[\text{(3.4)}\]

\[\text{(3.5)}\]
the solutions \( y^n = \pi_{(V, V_0)} (0, y_0; (x^n, t)) \) are given by solving (3.4) and (3.5) where \( x \) is replaced by \( x^n \) (same reasoning as in the first part of this proof gives existence and uniqueness of solutions \( z^n \)).

Note that by the universal limit theorem the map \( (y, x) \rightarrow \pi_{(V)} (0, y; x) \) is uniformly continuous on bounded sets so if we can show uniform convergence of \( z^n \) to \( z \) the desired conclusion follows. A standard ODE estimate (a simple consequence of Gronwall) is

\[
\sup_{t \in [0, T]} |z^n_t - z_t| \leq \frac{M^n}{L} (e^{Lt} - 1)
\]

where \( M^n = \sup_{t \in [0, T]} |W^n (t, y) - W (t, y)| \) and \( L = \sup_{t \in [0, T]} \frac{|W (t, x) - W (t, y)|}{|x - y|} \).

From the first part of this proof, \( L < \infty \) and to see \( M^n \rightarrow 0 \) as \( n \rightarrow \infty \), recall that \( J_{x^n \rightarrow y} \rightarrow J_{x \rightarrow y} \) and \( V_0 (\pi_{(V)} (0, y; x^n) \rightarrow V_0 (\pi_{(V)} (0, y; x)) \) uniformly in \( y, t \) as \( n \rightarrow \infty \) by the universal limit theorem.

Finally, \( \alpha \)-Hölder continuity of the solution follows from the estimate

\[
|\pi_{(V)} (0, z_t; x) - \pi_{(V)} (0, z_s; x)| \leq |\pi_{(V)} (0, z_t; x) - \pi_{(V)} (0, z; x)| + |\pi_{(V)} (0, z; x) - \pi_{(V)} (0, z_s; x)|
\]

\[
(3.6)
\]

\[
\leq c_1 |t - s|^\alpha c_2 |V|_{\text{Lip}^{\gamma + 1}} |z_t - z_s| e^{c_2 \|x\|_{\text{A-Hö}} V|_{\text{Lip}^{\gamma + 1}}}
\]

where \( c_1 = c_1 (\alpha, \gamma, |V|_{\text{Lip}^{\gamma + 1}}, \|x\|_{\text{A-Hö}}; [0, T]) \), \( c_2 = c_2 (\alpha, \gamma) \).

\[\Box\]

Remark 2. The above proof can be adapted to show uniqueness and (global) existence of RDEs with linear drift term, i.e.

\[ dy = Ay \, dt + V (y) \, dx, \, y (0) = y_0, \]

A a \((e \times e)\)-matrix, same assumptions as above on \( V \) and \( x \).

Corollary 2. Let \( x, V_0 \) and \( V \) be as in Proposition 2 and let \( x^n \) be a sequence of weak geometric \( \alpha \)-Hölder paths converging with uniform bounds to \( x \) in supremum norm, i.e.

\[
\sup_n \|x^n\|_{\text{A-Hö}} < \infty
\]

\[
d_{\infty} (x^n, x) \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Denote by \( y \) and \( y^n \) the corresponding solutions of the RDE with drift (3.25). Then

\[
\sup_n |y^n|_{\text{A-Hö}} < \infty
\]

\[
\sup_{t \in [0, T]} |y_t - y^n_t| \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

and by interpolation for every \( \alpha' < \alpha \)

\[
|y - y^n|_{\text{A-Hö}} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

5(y, x) seen as an element in a product space of two metric spaces, i.e. with metric

\[
d_{\infty} ((y, x), (\tilde{y}, \tilde{x})) = |y - \tilde{y}| + d_{\text{A-Hö}} (x, \tilde{x}).
\]

6for the convergence of the Jacobian a localisation argument is actually needed.
Remark 3. The auxiliary differential equation for \( t \) can be written as
\[
\frac{dz}{dh} = (J_{0}^{X_{t-1}}, W) \circ \pi (0, z_{t}, x)_{t} dh
\]
where \( W = V_{0} \) and \( h (t) = t \). In fact, one can take \( W \) to be \( (W_{1}, \ldots, W_{d}) \), \( h \in C^{1-loc} \left( [0, T], \mathbb{R}^{d} \right) \) in which case \( \pi (V) \) solves
\[
d\tilde{y} = V (\tilde{y}) \, dx + W (\tilde{y}) \, dh.
\]
We could make sense of (3.7) as Young-integral equation as long as \( 1/p + 1/q > 1 \) and thus obtain RDEs with drift-vector fields driven by \( h \). In this case the pair \( (x, h) \) gives rise to a rough path (the cross-integrals of \( x \) and \( h \) are well-defined Young-integrals); the advantage of the present consideration would be to reduce the regularity assumptions on \( W \).

3.3. Euler scheme for RDEs with drift. We recall the Euler scheme for RDEs, cf. [FV08a].

Definition 2. Let \( N \in \mathbb{N} \). Given \( \text{Lip}^{N} \) vector fields \( V = (V_{i})_{i=1,\ldots,d} \) on \( \mathbb{R}^{c} \), \( g \in \text{G}^{N} (\mathbb{R}^{c}) \), \( y \in \mathbb{R}^{c} \). We call
\[
\mathcal{E}_{(V)} (y, g) := \sum_{k=1}^{N} \sum_{i_{1}, \ldots, i_{k} \in \{1, \ldots, d\}} V_{i_{1}} \cdots V_{i_{k}} I (y) g^{i_{1}, \ldots, i_{k}}
\]
the step-\( N \) Euler scheme (I denotes the identity map).

Proposition 4 (Euler-estimate for RDEs with drift). Let \( N \in \mathbb{N} \), \( N \geq 1/\alpha \). For \( x \in C_{0}^{\alpha, \text{H"{o}l}} ([0, T], \text{G}^{N} (\mathbb{R}^{d})) \), \( V_{0} \in \text{Lip}^{1}, V = (V_{i})_{i=1,\ldots,d} \in \text{Lip}^{\gamma+1} (\mathbb{R}^{c}), \gamma > N \)
\[
\left| \pi (V_{0}, V) (s, y_{s} (x, t))_{s,t} - \mathcal{E}_{(V)} (y_{s}, x_{s,t}) - V_{0} (y_{s}) \, |t - s| \right| \leq c |t - s|^{\theta}
\]
where \( \theta \geq 1 + \alpha > 1 \) and \( c = c \left( \alpha, N, |y_{s}|, ||x||_{\alpha, \text{H"{o}l}}, |V|_{\text{Lip}^{N}}, |V_{0}|_{\text{Lip}^{1}} \right) \). Here \( y_{s} \in \mathbb{R}^{c} \) is a fixed "starting" point.

Proof. This is an error estimate for RDEs with drift over the time-interval \([s, t]\). By shifting time, we may consider w.l.o.g. \( s = 0 \), and from [FV08a] we know that
\[
\left| \pi (V) (0, y_{0}, x)_{0,t} - \mathcal{E}_{(V)} (y_{0}, x_{0,t}) \right| \leq c_{0} t^{\theta}
\]
where \( c_{0} = c_{0} \left( \alpha, N, |y_{0}|, ||x||_{\alpha, \text{H"{o}l}}, |V|_{\text{Lip}^{N}} \right), \theta = (N + 1) \alpha \geq 1 + \alpha \). By the triangle inequality and our definition of "RDE with drift" it then suffices to show that
\[
\left| \pi (V_{0}, V) (0, y_{0}; (x, t))_{0,t} - \pi (V) (0, y_{0}, x)_{0,t} - V_{0} (y_{0}) \, t \right| \leq c_{1} t^{\theta}
\]
To see this, recall
\[ \pi(V, y_0; (x, t))_{0, t} = \pi(V) (0, z_t, x)_t \]
where \( z_t - z_0 = z_t - y_0 = V_0(y_0) t + O(t^2) \). We then have
\[
\begin{align*}
\pi(V) (0, z_t, x)_t - \pi(V) (0, y_0, x)_t &= \int_0^t J_{x_{t}}^{x_{s}} \pi_{(V)} (0, z_s, x)_{0, t} \, ds \\
&= \int_0^t dz + \int_0^t \left( J_{x_{t}}^{x_{s}} \pi_{(V)} (0, z_s, x)_{0, t} - \mathcal{I} \right) \, ds \\
&= V_0(y_0) t + t^{1+\alpha}
\end{align*}
\]
and the proof is finished. \( \square \)

4. Applications

4.1. Drift vector fields induced by perturbed driving signals. We now show that perturbations of a rough driving signal are picked up by the RDE as a drift term of iterated Lie brackets of the vector fields. Since the RDE solution is a continuous function of the driving signal, we also have continuity under convergence in probability (in suitable Hölder rough path metrics) of random rough driving signals. Combined with the results of section 2 we arrive at a general criterion for non-standard convergence in stochastic differential equations, more general than [IW89, GM04] in the sense that our result applies to perturbations on all levels, exhibiting additional drift terms involving any iterated Lie brackets of the driving vector fields. In particular, the examples given in section 2 allows us to recover the convergence results of McShane [McS72] and Sussmann [Sus91]. (In fact, a free benefit of the rough path approach, the respective convergence results will take place at the level of stochastic flows.)

**Theorem 2.** Let \( x : [0, T] \to G^{[1/\alpha]} (\mathbb{R}^d) \) be a weak geometric \( \alpha \)-Hölder rough path, fix \( v = (v_{i_1, \ldots, i_N})_{i_1, \ldots, i_N = 1, \ldots, d} \in V^N (\mathbb{R}^d) \), \( N \geq 2 \) and set
\[ \tilde{x}_{s, t} = \exp (\log x_{s, t} + vt) \quad \text{for } s, t \]
(This defines a weak geometric \( \min(\alpha, 1/N) \)-Hölder rough path in \( G^N (\mathbb{R}^d) \) with identical projection as \( x \) to \( \mathbb{R}^d \)). Further, assume \( V_0 \in \text{Lip}^1, V = (V_i)_{i=1, \ldots, d} \in \text{Lip}^{\gamma+1}, \gamma > \max (1/\alpha, N) \), vector fields on \( \mathbb{R}^c \). Then the unique RDE solution of
\[
\begin{align*}
dy &= V_0(y) \, dt + V(y) \, d\tilde{x}, \quad y(0) = y_0
\end{align*}
\]
coincides with the unique RDE solution of
\[
\begin{align*}
dz &= (V_0(z) + W(z)) \, dt + V(z) \, dx, \quad z(0) = y_0
\end{align*}
\]
where
\[ W(z) = \sum_{i_1, \ldots, i_{[1/\alpha]} \in \{1, \ldots, d\}} \left[ V_{i_1}, \ldots, \left[ V_{i_{[1/\alpha]-1}}, V_{i_{[1/\alpha]}} \right] \ldots \right] x^{i_1, \ldots, i_{[1/\alpha]}} \]

We prepare the proof with
Lemma 1. Let $k \in \mathbb{N}$. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{1, \ldots, d\}^k$ and $k$ \text{Lip}^k vector fields $V_1, \ldots, V_k$ on $\mathbb{R}^e$, define

$$V_\alpha = [V_{\alpha_k}, [V_{\alpha_{k-1}}, \ldots, [V_{\alpha_2}, V_{\alpha_1}]]].$$

Further let $e_1, \ldots, e_d$ denote the canonical basis of $\mathbb{R}^d$. Then $g^n(\mathbb{R}^d)$, the step-$n$ free Lie algebra, is generated by elements of form

$$e_\alpha = [e_{\alpha_1}, [e_{\alpha_1}, \ldots, [e_{\alpha_2}, e_{\alpha_1}]]] \in (\mathbb{R}^d)^k, \; k \leq n$$

with $[e_i, e_j] = e_i \otimes e_j - e_j \otimes e_i$ and

$$\sum_{i_1, \ldots, i_k \in \{1, \ldots, d\}} V_{i_k} \cdots V_{i_1} (e_\alpha)^{i_k, \ldots, i_1} = V_\alpha.$$

Proof. It is clear that $g^n(\mathbb{R}^d)$ is generated by the $e_\alpha$. We prove the second statement by induction: a straightforward calculation shows that it holds for $k = 2$. Now suppose it holds for $k - 1$ and denote $V_\alpha = [V_{\alpha_{k-1}}, \ldots, [V_{\alpha_2}, V_{\alpha_1}]]$. Then (using summation convention)

$$V_{i_k} \cdots V_{i_1} (e_\alpha)^{i_k, \ldots, i_1} = V_{i_k} \cdots V_{i_1} (e_{\alpha_k} \otimes [e_{\alpha_{k-1}}, \ldots, [e_{\alpha_2}, e_{\alpha_1}]]^{i_k, \ldots, i_1})$$

$$- V_{i_k} \cdots V_{i_1} ([e_{\alpha_{k-1}}, \ldots, [e_{\alpha_2}, e_{\alpha_1}]]^{i_k, \ldots, i_1} \otimes e_{\alpha_k})$$

$$= V_{i_k} \cdots V_{i_1} \delta^{\alpha_k, i_k} \otimes [e_{\alpha_{k-1}}, \ldots, [e_{\alpha_2}, e_{\alpha_1}]]^{i_k-1, \ldots, i_1}$$

$$- V_{i_k} \cdots V_{i_1} [e_{\alpha_{k-1}}, \ldots, [e_{\alpha_2}, e_{\alpha_1}]]^{i_k, \ldots, i_1} \otimes \delta^{\alpha_k, i_k},$$

$$= V_{\alpha_k} V_{i_k-1} \cdots V_{i_1} [e_{\alpha_{k-1}}, \ldots, [e_{\alpha_2}, e_{\alpha_1}]]^{i_k-1, \ldots, i_1}$$

$$- V_{i_k} \cdots V_{i_2} [e_{\alpha_{k-1}}, \ldots, [e_{\alpha_2}, e_{\alpha_1}]]^{i_k-2, \ldots, i_1} V_{\alpha_k}$$

$$= V_{\alpha_k} V_\alpha - V_0 V_{\alpha_k} = [V_{\alpha_k}, [V_{\alpha_{k-1}}, \ldots, [V_{\alpha_2}, V_{\alpha_1}]]].$$

(where we used the induction hypothesis that

$$V_{i_k-1} \cdots V_{i_1} [e_{\alpha_{k-1}}, \ldots, [e_{\alpha_2}, e_{\alpha_1}]]^{i_k-1, \ldots, i_1} = V_{i_k} \cdots V_{i_2} [e_{\alpha_{k-1}}, \ldots, [e_{\alpha_2}, e_{\alpha_1}]]^{i_k, \ldots, i_2} = V_\alpha.$$)

Proof of Theorem 4. By construction $W \in \text{Lip}^1$ and existence and uniqueness of RDE solutions $y, z$ to (4.2), (4.1) follows from proposition 3. We have to show that $y_t = z_t$ for every $t \in [0, T]$. Therefore fix $\hat{T} \in [0, T]$, take a dissection $D = (t_i)_{i=0, \ldots, |D|}$ of $[0, \hat{T}]$ with $t_0 = 0$ and $t_{|D|} = \hat{T}$ and define

$$s^i_s = \pi_{(\hat{V}_0 + W, V)} (t_i, y_{t_i}; (x, t))_s$$

for $s \in [t_i, \hat{T}] \subset [0, 1], \; i = 1, \ldots, d.$

Note that $z^0_{\hat{T}} = z_{\hat{T}}$ and $z^{|D|}_{\hat{T}} = y_{\hat{T}}$, hence

$$|z_{\hat{T}} - y_{\hat{T}}| = |z^{|D|}_{\hat{T}} - z^0_{\hat{T}}| \leq \sum_{i=1}^{|D|} |z^i_{\hat{T}} - z^{i-1}_{\hat{T}}|.$$

Using the Lipschitz continuity of RDE flows we get

$$|z^i_{\hat{T}} - z^{i-1}_{\hat{T}}| = |\pi_{(\hat{V}_0 + W, V)} (t_i, y_{t_i}; (x, t))_{\hat{T}} - \pi_{(\hat{V}_0 + W, V)} (t_{i-1}, y_{t_{i-1}}; (x, t))_{\hat{T}}|$$

$$= |\pi_{(\hat{V}_0 + W, V)} (t_i, y_{t_i}; (x, t))_{\hat{T}} - \pi_{(\hat{V}_0 + W, V)} (t_i, \pi_{(\hat{V}_0 + W, V)} (t_{i-1}, y_{t_{i-1}}; (x, t))_{\hat{T}}; (x, t))_{\hat{T}}|.$$
Theorem 3. settles a question that was left open in [CFV08].

\[ y_{t_{i-1}, t} - \pi(y_{0} + W, V) (t_{i-1}, y_{t_{i-1}}; (x,t)) \]

For brevity set \( \alpha^* = \min{(\alpha, 1/N)} \). By adding/subtracting \( E^{1/\alpha^*}_{(V)} (y_{t_{i-1}}, x_{t_{i-1}, t}) + V_{0} (y_{t_{i-1}}) |t_{i} - t_{i-1}| \) and splitting up we estimate \( z_{T}^{i} - z_{T}^{i-1} \) with

\[
\begin{align*}
(1) & = |\pi(y_{0}, V) (t_{i-1}, y_{t_{i-1}}; (x,t))_{t_{i-1}, t} - E^{1/\alpha^*}_{(V)} (y_{t_{i-1}}, x_{t_{i-1}, t}) - V_{0} (y_{t_{i-1}}) |t_{i} - t_{i-1}| \\
(2) & = E^{1/\alpha^*}_{(V)} (y_{t_{i-1}}, x_{t_{i-1}, t}) + V_{0} (y_{t_{i-1}}) |t_{i} - t_{i-1}| - \pi(y_{0} + W, V) (t_{i-1}, y_{t_{i-1}}; (x,t))_{t_{i-1}, t} |
\end{align*}
\]

From proposition 4 (1) \( \leq c_{1} |t - s|^{\theta} \), \( \theta > 1 \), and by lemma 1

\[
E^{1/\alpha^*}_{(V)} (y_{t_{i-1}}, x_{t_{i-1}, t}) = E^{1/\alpha^*}_{(V)} (y_{t_{i-1}}, x_{t_{i-1}, t}) + \sum_{i_{1}, \ldots, i_{N}} V_{i_{1}} \cdots V_{i_{N}} I (y_{t_{i-1}}) \vartheta_{i_{1}, \ldots, i_{N}} |t_{i} - t_{i-1}|
\]

Again proposition 4 applies and (2) \( \leq c_{2} |t - s|^{\theta} \). Plugging all this into (4.3) gives

\[
|z_{T}^{i} - y_{T}^{i}| \leq c_{3} \sum_{i=1}^{D} |t_{i} - t_{i-1}|^{\theta}
\]

with \( c_{3} = c_{3} (\alpha, N, \|\tilde{x}\|_{\alpha^{*}}, \|x\|_{\alpha^{*}}, \|V\|_{\text{Lip}^{1/\alpha^*}}, \|V_{0}\|_{\text{Lip^{1}}}, |W|_{\text{Lip^{1}}}) \). Since \( \theta > 1 \) the sum on the r.h.s goes to 0 as \( |D| \to 0 \) and this finishes the proof. \( \square \)

4.2. Optimality of RDE estimates. At last, we use theorem 2 to establish optimality of two important RDE estimates (proved in [Lyo98] and [FV08]) for the case of paths with \( p \)-variation, \( p \in \mathbb{N} \). The second part of the following theorem settles a question that was left open in [CPV08].

**Theorem 3.** Let \( x : [0, T] \to G^{p} (\mathbb{R}^{d}) \) be a geometric \( p \)-rough path, \( y_{0} \in \mathbb{R}^{e}, p \in \mathbb{N} \). If either

1. \( V = (V_{i})_{i=1}^{d} \in \text{Lip}^{\gamma} (\mathbb{R}^{e}), \gamma > p \) or
2. \( V = (V_{i})_{i=1}^{d} \text{ linear vector fields on } \mathbb{R}^{d}, \text{ i.e. } V_{i} (y) = A_{i} \cdot y \) with \( A_{i} \) a \((e \times e)\)-matrix

then there exists a unique RDE solution to

\[
dy = V(y) \, dx, \; y(0) = y_{0}.
\]

Further, in case (1),

\[
|y|_{\infty} \leq C \max\left(\|x\|_{p\text{-var}^{\gamma} [s,t]}, \|x\|_{p\text{-var}^{\gamma} [s,t]}^{p}\right)
\]

with \( C = C \left( y_{0}, p, |V|_{\text{Lip}^{\gamma + 1}} \right) \) and in case (2)

\[
|y|_{\infty} \leq c \exp\left( c \|x\|_{p\text{-var}}^{p} \right)
\]

with \( c = c \left( y_{0}, p, |V|_{\text{Lip}^{\gamma + 1}} \right) \) hold and both estimates are optimal (i.e. the bounds are attained).

We prepare the proof with two lemmas.
Non-standard approximations of the Ito-map

**Lemma 2.** For all $k \in \mathbb{N}$ there exist $k$ smooth, bounded vector fields $V_1, \ldots, V_k$ on $\mathbb{R}^e$ such that

$$[V_k, \ldots, [V_3, [V_2, V_1]] \ldots] = \frac{\partial}{\partial x_e}$$

*Proof.* Set

$$V = \sin x_e \frac{\partial}{\partial x_e}, W = -\cos x_e \frac{\partial}{\partial x_e}, E = \frac{\partial}{\partial x_e}$$

Note that $[V, W] = E$ and $[V, E] = W$. Hence,

for $k$ odd: $[V, \ldots, [V, [V, W]] \ldots] = E$
for $k$ even: $[V, \ldots, [V, [V, E]] \ldots] = E$.

**Lemma 3.** For all $k \in \mathbb{N}$ there exist $(e \times e)$-matrices $A_1, \ldots, A_k$ $(e \geq 2)$ such that

$$\begin{bmatrix} V_k, \ldots, [V_3, [V_2, A_1]] \ldots \end{bmatrix}_{i,j=1,\ldots,e} = (-\delta_{i,j}=(1,1) + \delta_{i,j}=(e,e))_{i,j=1,\ldots,e}$$

where $[\cdot, \cdot]$ denotes the usual matrix commutator.

*Proof.* Let

$$M = (\delta_{i,j}=(e,e))_{i,j=1,\ldots,e}, \quad N = (-\delta_{i,j}=(1,1) + \delta_{i,j}=(e,e))_{i,j=1,\ldots,e},$$

$$A = (\delta_{i,j}=(e,1))_{i,j=1,\ldots,e}, \quad B = \left(\frac{1}{2}\delta_{i,j}=(1,e)\right)_{i,j=1,\ldots,e}.$$  

Note that $[A, M] = N$ and $[B, N] = M$. Hence

for $k$ odd: $[A, [B, [A, \ldots, [A, [B, [A, M]]]] \ldots]] = N$
for $k$ even: $[A, [B, [A, \ldots, [A, [B, N]]]] \ldots]] = N$.

**Proof of Theorem** Existence of a unique RDE solution follows from [Lyo98]. For (3.3) use [Lyo98] Theorem 2.4.1 with control function $\omega(s, t) = \|x\|_{p\text{-var};[s,t]}$ to get

$$|y_t| \leq K |y_0| \|x\|_{p\text{-var}} \sum_{n \geq 0} K^n \|x\|_{p\text{-var}} \frac{(n/p)!}{(n/p)!}$$

where $K = \max_i (|A_i|)$. However, $\sum_{n \geq 0} \frac{a^n}{(n/p)!} \leq c_0 (p) e^{c_0(p)a}$ for $a \geq 0$ and therefore

$$|y|_{\infty,[0,T]} \leq |y_0| c_1 \exp \left(c_1 \|x\|_{p\text{-var}}^p\right).$$

Estimate (4.4) is proven in [FV08b].

To see optimality of both (4.4) and (4.5) define a geometric $1/p$-Hölder rough path $\tilde{x}$ on $[0, 1]$ in $G^p(\mathbb{R}^p)$ by

$$\tilde{x}_t = \exp(\lambda e_{1,\ldots,p})$$

where $e_{1,\ldots,p} = [e_p, \ldots, [e_3, [e_2, e_1]] \ldots] \in V^p(\mathbb{R}^p)$ for some $\lambda > 0$. Note that homogeneity of the $p$-variation norm implies

$$\|\tilde{x}\|_{p\text{-var}} = c_2 \lambda^{1/p}.\ (4.6)$$
The corresponding RDE solution
\[ dy = V(y) \, d\tilde{x}, \ y(0) = y_0. \]
then coincides with the ODE solution
\[(4.7) \quad dz = W(z) \, dt, \ z(0) = y_0 \]
where \(W\) is given by theorem 2.

Case (1): Let \( V = (V_i)_{i=1,...,p} \) be the vector fields on \( \mathbb{R}^p \) from lemma 2 with \( k = p \). Then the solution \((4.7)\) is easy to write down,
\[ y_t = (0, \ldots, 0, \lambda t)^T. \]
Clearly, \( |y|_\infty = 1/\epsilon_2^p \|\tilde{x}\|_{p\text{-var}}^p \). To see that the regime of \((4.4)\) where \( \|x\|_{p\text{-var}} \) dominates can be obtained is straightforward by looking at \( dy = (1, \ldots, 1)^T \, dx \) for any rough path \( x \).

Case (2): Let \( V = (V_i)_{i=1,...,p} \) be the linear vector fields on \( \mathbb{R}^p \) given by the matrices \((A_i)_{i=1,...,p}\) of lemma 3, i.e. \( V_i(y) = A_i \cdot y \). Using lemma 3 with \( k = p \) bracket,\(^7\)
\[ W(z) = \lambda [V_p, \ldots, [V_3, [V_2, V_1]] \ldots] J(z) \]
\[ = \lambda [A_p, \ldots, [A_3, [A_2, A_1]] \ldots] M \cdot z \]
\[ = \lambda (-\delta_{(i,j)=(1,1)} + \delta_{(i,j)=(p,p)})_{i,j=1,\ldots,e} \cdot z \]
Now choosing \( y_0 = (0, \ldots, 0, 1)^T \), the unique solution to \((4.7)\) is
\[ y_t = (0, \ldots, 0, e^{\lambda t})^T. \]
Again by \((4.6)\)
\[ \sup_{t \in [0,1]} |y_t| = e^\lambda = e^{c_2 \|x\|_{p\text{-var}}} \]
and the upper bound of estimate \((4.5)\) is attained. \( \square \)

References

[CFV08] T. Cass, P. Friz, and N. Victoir. Non-degeneracy of wiener functionals arising from rough differential equations. Accepted to Trans. of AMS (2008).

[CL05] L. Coutin and A. Lejay. Semi-martingales and rough paths theory. Electron. J. Probab. 10, no. 23, 761–785 (electronic) (2005).

[CQ02] L. Coutin and Z. Qian. Stochastic analysis, rough path analysis and fractional Brownian motions. Probab. Theory Related Fields 122(1), 108–140 (2002).

[FV05] P. Friz and N. Victoir. Approximations of the Brownian rough path with applications to stochastic analysis. Ann. Inst. H. Poincaré Probab. Statist. 41(4), 703–724 (2005).

[FV06a] P. Friz and N. Victoir. The Burkholder-Davis-Gundy Inequality for Enhanced Martingales. Preprint (2006).

[FV06b] P. Friz and N. Victoir. A note on the notion of geometric rough paths. Probab. Theory Related Fields 136(3), 395–416 (2006).

[FV06c] P. Friz and N. Victoir. On uniformly subelliptic operators and stochastic area. Preprint (2006).

[FV07] P. Friz and N. Victoir. Differential equations driven by Gaussian signals I. Preprint (2007).

\(^7\)Using the usual identification of linear maps with matrices it is easy check that we have a Lie algebra isomorphism \( \mathfrak{gl}(n, \mathbb{R}) \leftrightarrow (\mathbb{R}^n, [\cdot, \cdot]) \).
Non-standard approximations of the Ito-map

[FV08a] P. Friz and N. Victoir, Euler estimates for rough differential equations. *J. Differential Equations* **244**(2), 388–412 (2008).

[FV08b] P. Friz and N. Victoir, “Multidimensional Stochastic Processes as Rough Paths. Theory and Applications”, Cambridge University Press (2008). Forthcoming.

[GM04] I. Gyöngy and G. Michaletzky. On Wong-Zakai approximations with δ-martingales. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **460**(2041), 309–324 (2004). Stochastic analysis with applications to mathematical finance.

[IW89] N. Ikeda and S. Watanabe. “Stochastic differential equations and diffusion processes”. North-Holland Publishing Co., Amsterdam, second edition (1989).

[LCL07] T. Lyons, M. Caruana, and T. Lévy. “Differential equations driven by rough paths”, volume 1908 of “Lecture Notes in Mathematics”. Springer, Berlin (2007). Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004, With an introduction concerning the Summer School by Jean Picard.

[Lejay02] A. Lejay, Stochastic differential equations driven by a process generated by divergence form operators. Preprint (2002).

[LL02] T. Lyons and A. Lejay. On the importance of the Lévy area for systems controlled by converging stochastic processes. application to homogenization. Preprint (2002).

[LV06] A. Lejay and N. Victoir. On (p, q)-rough paths. *J. Differential Equations* **225**(1), 103–133 (2006).

[Lyo98] T. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana* **14**(2), 215–310 (1998).

[McS72] E. J. McShane. Stochastic differential equations and models of random processes. In "Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. III: Probability theory", pages 263–294, Berkeley, Calif. (1972). Univ. California Press.

[RW00] L. C. G. Rogers and D. Williams. "Diffusions, Markov processes, and martingales. Vol. 1". Cambridge Mathematical Library. Cambridge University Press, Cambridge (2000). Foundations, Reprint of the second (1994) edition.

[Str88] D. W. Stroock. Diffusion semigroups corresponding to uniformly elliptic divergence form operators. In “Séminaire de Probabilités, XXII”, volume 1321 of “Lecture Notes in Math.”, pages 316–347. Springer, Berlin (1988).

[Sus91] H. J. Sussmann. Limits of the Wong-Zakai type with a modified drift term. In “Stochastic analysis”, pages 475–493. Academic Press, Boston, MA (1991).

E-mail address: P.Friz@statslab.cam.ac.uk

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WB