Fidelity and entanglement close to quantum phase transition in a two-leg $XXZ$ spin ladder

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Abstract

The fidelity susceptibility and entanglement entropy in a system of two-leg $XXZ$ spin ladder with rung coupling is investigated by using exact diagonalization of the system. The effects of rung coupling on fidelity susceptibility, entanglement entropy and quantum phase transition are analyzed. It is found that the quantum phase transition between two different $XY$ phases can be well characterized by the fidelity susceptibility. Though the quantum phase transition from $XY$ phase to rung singlet phase can be hardly detected by fidelity susceptibility, it can be predicted by the first derivative of the entanglement entropy of the system.

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I. INTRODUCTION

The quantum phase transition has attracted much attention in low-dimensional quantum systems. It implies fluctuations which happened at zero temperature [1]. When a controlling parameter changes across critical point, some properties of the many-body system will change dramatically. Many results show that entanglement exists naturally in the spin chain when the temperature is at zero. The quantum entanglement of a many-body system has been paid much attention since the entanglement is considered as the heart in quantum information and computation [2, 3]. As the bipartite entanglement measurement in a pure state, the von Neumann entropy [4] in the antiferromagnetic anisotropic and isotropic spin chains [5, 6] were investigated respectively. By using the quantum many-body theory and quantum-information theory, von Neumann entropy was applied to detect quantum critical behaviors [7–10]. Meanwhile, the ground state fidelity was used to qualify quantum phase transition in the last few years [11–24]. It is shown that the fidelity and the entanglement entropy have similar predictive power for identifying quantum phase transitions in the spin systems.

Spin ladders are examples of interacting many-body systems which exhibit many novel phenomena. The spin ladders exist in the Cu$_2$O$_3$ subsystem of the compound Sr$_{14}$Cu$_2$4O$_3$ [25] and real crystals (C$_6$H$_{11}$NH$_3$)CuBr$_3$(CHAB) [26]. Recently, Bose et al. [27] studied the fidelity, the entanglement and the quantum phase transition of a spin-1/2 antiferromagnetic Heisenberg spin ladder in an external magnetic field. The effects of magnetic field on variation of fidelity and entanglement measures were investigated. It was found that the variation of fidelity and entanglement measured points close to the quantum criticality. It would be interesting to investigate the effects of rung interaction on the variation of the fidelity, the entanglement and the quantum phase transition.

In this paper, the fidelity and entanglement in a two-leg XXZ spin ladder system is investigated using exact diagonalization. In section II, the Hamiltonian of the two-leg XXZ spin ladder system is presented. In section III, the effect of rung interaction on ground state fidelity is investigated. Its relation with quantum phase transition is analyzed. The effect of rung interaction on the entanglement entropy is also calculated and analyzed in section IV. A discussion concludes the paper.
II. HAMILTONIAN OF SPIN LADDER

The Hamiltonian of a two-leg antiferromagnetic Heisenberg spin ladder is given by

\[ H = H_{\text{leg}}^1 + H_{\text{leg}}^2 + H_{\text{rung}}, \]

where the Hamiltonian \( H_{\text{leg}}^\alpha \) for leg \( \alpha \) (\( \alpha = 1 \) or 2) is given by

\[ H_{\text{leg}}^\alpha = \sum_{i=1}^{N} J(S_{\alpha,i,i}^x S_{\alpha,i+1}^x + S_{\alpha,i,i}^y S_{\alpha,i+1}^y + \Delta S_{\alpha,i,i}^z S_{\alpha,i+1}^z), \]

and the inter-leg coupling is given by

\[ H_{\text{rung}} = \sum_{i=1}^{N} J_{\text{rung}} (S_{1,i,i}^x S_{2,i,i}^x + S_{1,i,i}^y S_{2,i,i}^y + S_{1,i,i}^z S_{2,i,i}^z), \]

where \( S_{\alpha,j}^{x,y,z} \) are spin operators on the \( i \)-th rung, the index \( \alpha = 1, 2 \) denotes the leg in the ladder, \( N \) is the length of the spin ladder, \( J > 0 \) denotes the antiferromagnetic coupling, \( \Delta \) is anisotropic interaction, \( J_{\text{rung}} \) denotes rung interaction. The schematic diagram of the two-leg spin ladder is shown in Fig. 1. In the paper, the opened boundary condition is considered, \( J = 1 \) and \( \Delta = -0.5 \) are chosen for simplicity. It is predicted that a novel \( \text{XY}2 \) phase \( J_{\text{rung}}^C < J_{\text{rung}} < J_{\text{rung}}^C \) appears between the \( \text{XY}1 \) phase and the rung singlet phase. The \( \text{XY} \) phase belongs to the universality class of Tomonaga-Luttinger liquid. A number of numerical studies have shown that phase transition from \( \text{XY}1 \) to \( \text{XY}2 \) occurs at \( J_{\text{rung}}^C = 0 \), and phase transition from \( \text{XY}2 \) to rung singlet phase point \( J_{\text{rung}}^C \) depends on the anisotropy interaction in the legs. When \( \Delta = -0.5 \), \( J_{\text{rung}}^C = 0.373 \).  

III. FIDELITY SUSCEPTIBILITY

The ground state fidelity and the fidelity susceptibility can be applied to detect the existence of the quantum phase transition. A general Hamiltonian of a quantum many-body system can be written as \( H(\lambda) = H_0 + \lambda H_I \) where \( H_I \) is the driving Hamiltonian and \( \lambda \) denotes its strength. If \( \rho(\lambda) \) represents a state of the system \( H(\lambda) \), the fidelity between states \( \rho(\lambda) \) and \( \rho(\lambda + \delta) \) can be defined as

\[ F(\lambda, \delta) = \text{Tr} \left[ \sqrt{\rho^{1/2}(\lambda) \rho(\lambda + \delta) \rho^{1/2}(\lambda)} \right]. \]
If the state can be written as $\rho = |\psi\rangle\langle\psi|$, Eq. (4) can be written as $F(\lambda, \delta) = |\langle \psi(\lambda) | \psi(\lambda + \delta) \rangle|$. Because $F(\lambda, \delta)$ reaches its maximum value $F_{\text{max}} = 1$ at $\delta = 0$, on expanding the fidelity in powers of $\delta$, the first derivative $\frac{\partial F(\lambda, \delta=0)}{\partial \lambda} = 0$. Then the fidelity can written by

$$F(\lambda, \delta) \approx 1 + \frac{\partial^2 F(\lambda, \delta)}{2\partial \lambda^2}|_{\lambda=\lambda'} \delta^2,$$

(5)

where $\lambda' = \lambda$ should probably read as $\delta=0$. The average fidelity susceptibility $S(\lambda, \delta)$ can be given by

$$S(\lambda, \delta) = \lim_{\delta \to 0} \frac{2[1 - F(\lambda, \delta)]}{N\delta^2}.$$

(6)

For models that are not exactly solvable, the exact diagonalization can be used to obtain the ground state. The fidelity $F$ and the fidelity susceptibility $S$ is calculated and plotted in Fig. 2 as a function of anisotropy parameter $J_{\text{rung}}$ for different sizes. The parameters are chosen as $N = 8, 10, 12$ and $\delta = 0.001$. In Fig. 2(a), there is a valley in the fidelity. The locations of the minimal value for all sizes at $J_{\text{rung}}^C = 0$. In Fig. 2(b), one peak locates at the same point as the valley in Fig. 2(a). It means that the fidelity and fidelity susceptibility can predict the quantum phase transition between $XY_1$ and $XY_2$ phases. In the region of $J_{\text{rung}} < 0$, $\langle S_{1,j}^x S_{2,j}^x \rangle > 0$ is obtained from the Marshall-Lieb-Mattis theorem. Meanwhile, in the region of $J_{\text{rung}} > 0$, $\langle S_{1,j}^z S_{2,j}^z \rangle < 0$. These two $XY$ phases have different symmetry. So the fidelity and fidelity susceptibility can predict the quantum phase transition point. However, it is difficult to detect the phase transition between $XY_2$ and rung singlet phases since the phase transition between $XY_2$ and rung singlet phases is Berezinskii-Kosterlitz-Thouless (BKT) type. The fidelity is feeble to characterize the Berezinskii-Kosterlitz-Thouless (BKT) type phase transition.

IV. ENTANGLEMENT ENTROPY

Similarly, the ground state entanglement can also be used to detect the quantum phase transition. The entropy can be chosen as a measurement of the pairwise entanglement. The entropy can be defined as follows. Let $\rho_{AB}$ be the ground state of a chain of $N$ qubits, the reduced density matrix of part A can be written as $\rho_A = Tr_B \rho_{AB}$. The bipartite entanglement between parts $A$ and $B$ can be measured by the entanglement entropy as
\[ E_{AB} = -Tr(\rho_{A(B)} \log_2 \rho_{A(B)}). \] 

By using the method of exact diagonalization, the entropy of the ground state can be calculated. The entropy \( E_{\text{rung}} \) and the first derivative \( dE_{\text{rung}}/dJ_{\text{rung}} \) of the entropy is plotted as a function of the rung interaction \( J_{\text{rung}} \) in Fig. 3 with different size of \( N = 8, 10, 12 \). In Fig. 3(a), the entanglement entropy between central rung and rest of the system is plotted. There is a peak in \( E_{\text{rung}} \) at \( J_{\text{rung}}^{C1} = 0 \). This means that the quantum phase transition between two different XY phases can be well characterized by entropy, while the transition from XY2 to rung singlet phases can be hardly detected by the entropy. It seems that this is due to the monogamy property of the entropy\(^8\,24\). In Fig. 3(b), the first derivative of the entropy is plotted. There is a sharp change in \( dE_{\text{rung}}/dJ_{\text{rung}} \) at \( J_{\text{rung}}^{C1} = 0 \). This means that the quantum phase transition between two different XY phases occurs. There is also a peak in \( dE_{\text{rung}}/dJ_{\text{rung}} \) near \( J_{\text{rung}}^{C2} = 0.37 \). This peak indicates the quantum phase transition from XY2 phase to rung singlet phase. This is in consistent with the result of the calculation of the correlation function. It was shown that the phase transition from XY1 to XY2 occurred at \( J_{\text{rung}}^{C1} = 0 \). The phase transition from XY2 to rung singlet phase appeared at \( J_{\text{rung}}^{C2} = 0.373 \) when \( \Delta = -0.5 \)\(^28\,29\). It is clear that the critical properties can be captured by the first derivative of the entropy as a function of the rung interaction \( J_{\text{rung}} \)\(^8\,24\,34\,36\).

The entropy of the diagonal two qubits of the central rung is plotted in Fig. 4. There is a peak at \( J_{\text{rung}}^{C1} = 0 \) and a valley near \( J_{\text{rung}}^{C2} = 0.37 \). It is clear that the quantum phase transition between XY1 and XY2 phases appears at \( J_{\text{rung}}^{C1} = 0 \) while the transition from XY2 phase to rung singlet phase occurs near \( J_{\text{rung}}^{C2} = 0.37 \). It seems that the entropy of the diagonal two qubits of the central rung can also predict the two different kinds of quantum phase transitions in a two-leg ladder system.

V. DISCUSSION

The fidelity susceptibility and the entanglement entropy of a two-leg \( XXZ \) spin ladder system are studied numerically. By using the exact diagonalization, the effect of rung coupling on fidelity susceptibility and entanglement entropy is investigated. Their relations with quantum phase transition are analyzed. It is found that the quantum phase transition

\[ \Delta = -0.5 \]
between two different $XY$ phases can be well characterized by fidelity susceptibility, while the transition from $XY^2$ to rung singlet phases can be hardly detected by the fidelity susceptibility. The first derivative of the pairwise entanglement entropy and the entropy between diagonal two qubits of the central rung and the rest of the system can detect the quantum phase transition between two different $XY$ phases and the transition from $XY^2$ to rung singlet phases.

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Figure Captions

Fig. 1
The schematic diagram of a two-leg spin ladder.

Fig. 2
The fidelity $F$ and the fidelity susceptibility $S$ is plotted as a function of the interaction $J_{rung}$ for different size $N$. (a). The fidelity $F$. (b). The fidelity susceptibility $S$.

Fig. 3
The entropy $E_{rung}$ and the first derivative $dE_{rung}/dJ_{rung}$ of the entropy between two qubits of central rung is plotted as a function of the interaction $J_{rung}$ for different size $N$. (a). The entropy $E_{rung}$. (b). The first derivative $dE_{rung}/dJ_{rung}$ of the entropy.

Fig. 4
The entropy of the diagonal two qubits of central rung is plotted as a function of the interaction $J_{rung}$ for different size $N$. 
Fig. 1
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