Diagonal quantum Bianchi type IX models in
$N = 1$ supergravity

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Abstract

We take the general quantum constraints of $N = 1$ supergravity in the special case of a Bianchi metric, with gravitino fields constant in the invariant basis. We construct the most general possible wave function which solves the Lorentz constraints and study the supersymmetry constraints in the Bianchi Class A Models. For the Bianchi-IX cases, both the Hartle-Hawking state and wormhole state are found to exist in the middle fermion levels.

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1 Introduction

Since the discovery of supersymmetry 20 years ago, many people have been fascinated by supergravity theories. There are several reasons for this. First, supergravity theories are the only consistent theories which couple fundamental spin-3/2 particles to gravity. Second, supergravity theories are less divergent than general relativity. There are some indications that pure $N=1$ supergravity is finite. The canonical formulation of $N=1$ supergravity was presented in ref. [2] in four-component spinor notation and in ref. [3] in two-component spinor notation. In finding a physical state, it is sufficient to solve the Lorentz and supersymmetry constraints of the theory; the algebra of constraints implies that physical wave function will also obey the Hamiltonian constraints.

In the past ten years, there has been active research in supersymmetric quantum cosmology, especially in $N=1$ supergravity theory. Bianchi class A models of pure $N=1$ supergravity were studied in refs. [4, 5, 6, 7] using both triad and Ashtekar variables. These authors assumed a simple Ansatz for the wave function in the investigation of supersymmetric quantum cosmology. They found that only simple solutions were present in the bosonic and full fermionic sectors of the wave function. This curious result was joined by yet another disturbing one. When a cosmological constant was added, it appeared that there was no non-trivial physical wave function. One might think that supersymmetric quantum cosmology is not very interesting.

However, recently, Csordás and Graham [9] pointed out there exist middle fermion states in the minisuperspace models of pure $N=1$ supergravity. They showed that there is a richer structure of physical states of supersymmetric quantum cosmology than that found in previous works [4, 5, 6, 7, 8]. They rightly criticise the Ansatz for the wave function used in refs. [4, 5, 6, 7, 8] as not being general enough. One now needs to investigate these middle states in the fullest possible detail.

The wave function of the universe of supersymmetric quantum cosmology can be expanded in even numbers of gravitinos up to order 6. Since we have 6 gravitinos, there are $\binom{6}{2} = 15$ allowed terms of two fermions. In this sense,
the Ansatz used in refs. [4, 5, 6, 7, 8], which has only two degrees of freedom at the two-fermion level, is not general enough and this is the reason why refs. [4, 5, 6, 7, 8] failed to find the interesting middle fermion states. Csordás and Graham [9] constructed a new Ansatz for the wave function based on a scalar function $f(h_{pq})$, where $h_{pq}$ is the three-metric of the space-like hypersurface. For the two-fermion level, they noticed first that $\bar{S}_A^A f(h_{pq})$ automatically solves the Lorentz constraints, where $\bar{S}_A^A$ is the supersymmetry constraint operator.

They further noticed that this expression solves the $\bar{S}_A^A$ supersymmetry constraint, using the anti-commutation properties. The only constraint that remains to solve is the $S_A$ constraint. By solving this constraint, they reduce the problem to solving the Wheeler-DeWitt equation for $f$. This approach is, however, limited by being based on an Ansatz. Further information can be obtained by studying the complete set of coupled first-order partial differential constraint equations, as is done here.

We start from the wave function which is the most general solution to the Lorentz constraint. In section 2, we will briefly describe the conventions and variables to be used in the calculations. We will carry out the dimensional reduction from $3 + 1$ to $0 + 1$ dimensions. From the reduced action, the supersymmetry constraints are found. It is sufficient, in finding a physical state, to solve the Lorentz and supersymmetry constraints of the theory [3, 10]. Because of the anti-commutation relations $[S_A, S_A'] = H_{AA'}$, the supersymmetry constraints $S_A\Psi = 0$, $S_A'\Psi = 0$ on a physical wave function $\Psi$ imply the Hamiltonian constraints $H_{AA'}\Psi = 0$ [3, 10]. We study the supersymmetry constraints which are a set of coupled first-order partial differential equations for the components of the wave function. We find, for the case of a diagonal Bianchi IX model, that of the 15 possible coefficients at two-fermion level, the coefficients of 9 are zero. Only the remaining 6 are dynamical. In section 3, we will make a comparison with the work of Csordás and Graham [9]. Section 4 contains the conclusion.

1 The Ansatz $\bar{S}_A^A f(h_{pq})$ only works when there are no chiral breaking terms in the supersymmetry constraints (see section 4).
2 Dimensional Reduction and Derivation of the Supersymmetry Constraints

Using two-component spinors [3], the action [11] is

$$S = \int d^4x \left[ \frac{1}{2} (\text{det } e) R + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \left( \psi_\mu^{A'} e_{\nu AA'} D_\rho \psi_\sigma^A + \text{H.c.} \right) \right].$$  \(1\)

Here the tetrad is $e_\mu^a$ or equivalently $e_\mu^{AA'}$. The gravitino field $(\psi_\mu^A, \psi_\mu^{A'})$ is an odd (anti-commuting) Grassmann quantity. The scalar curvature $R$ and the covariant derivative $D_\rho$ include torsion.

For the Bianchi class A models, we take the usual homogeneity conditions for the Ansatz of the triads and the spatial gravitino fields. This means that when the triad $e_\mu^a$ and the spatial gravitino $\psi_\mu^A$ field are expanded with respect to the invariant basis of the spatial hypersurface, the components are functions of time only. The $p$ indices are the invariant indices which take the values 1, 2, 3. We also assume that the time component of the gravitino field $\psi_0^A$ is a function of time only. One applies these homogeneity conditions to the above Lagrangian and carries out the 3-dimensional integration over the hypersurface and then performs the Legendre transformations. The classical supersymmetry constraints are found to be:

$$\bar{S}_{AA'} = \epsilon^{pqr} e_{pAA'}^A \omega_{qAB} \psi_{rB}^B - \frac{1}{2} i \psi_{p}^A P^p_{AA'}$$  \(2\)

and the conjugate $S_{A}$. Here $n^{AA'}$ is the spinor version of the unit future-pointing normal $n^\mu$ to the surface $t = \text{const}$. It is a function of the $e_p^{AA'}$, defined by

$$n^{AA'} e_{pAA'} = 0, \quad n^{AA'} n_{AA'} = 1$$  \(3\)

where $\omega_{pAB}$ is the torision free connection of spatial hypersurface. There is as usual a pair of second class constraints between the $\psi$, $\bar{\psi}$ and their conjugate momenta. We have to introduce the Dirac bracket to get rid of this pair of second class constraints. With the help of ref. [3], we have the following bracket relations after the elimination:

$$\left[ e_p^{AA'}, P^q_{BB'} \right] = \delta_A^B \delta_{AA'}^{BB'} \delta_q^p$$
\[
[\psi_p^A, \bar{\psi}_q^{A'}]^*_+ = -D_{pq}^{AA'}
\]  
(4)

where \(D_{pq}^{AA'} = -2ie_q^{AB'}e_{pBB'}n^{BA'}\). The rest of the brackets are zero.

Quantum mechanically, one replaces Dirac brackets by anti-commutators if both arguments are odd or commutators if otherwise:

\[
[E_1, E_2] = i\hbar [E_1, E_2]^*, \quad [O, E] = i\hbar [O, E]^*, \quad \{O_1, O_2\} = i\hbar [O_1, O_2]^* .
\]

We choose \(e_p^{AA'}\) and \(\psi_p^A\) as our position coordinates and \(p_{AA'}^q\) and \(\bar{\psi}_q^{A'}\) as our momentum operators:

\[
p_{AA'}^q \rightarrow -i\hbar \frac{\partial}{\partial e_{AA'}^q} , \quad \bar{\psi}_q^{A'} \rightarrow -i\hbar D_{pq}^{AA'} \frac{\partial}{\partial \psi_p^A} .
\]

In the general theory, the corresponding quantum constraints read:

\[
\bar{S}_A = e_{pAA'}^q \psi_r^A D_q \psi_r^A - \frac{1}{2} \hbar \psi_p^A \frac{\delta}{\delta e_{pAA'}} = 0 .
\]  
(5)

and its conjugate. At a Bianchi model, these constraints read:

\[
\bar{S}_A \Psi = e_{pAA'}^q \omega_{qB}^A \psi_r^B \Psi - \frac{1}{2} \hbar \psi_p^A \frac{\partial \Psi}{\partial e_{pAA'}} = 0
\]  
(6)

\[
S_A \Psi = -\omega_{pA}^B \frac{\partial \Psi}{\partial \psi_p^B} + \frac{1}{2} \hbar D_{pq}^{BA'} \frac{\partial}{\partial \psi_p^B} \frac{\partial \Psi}{\partial e_{qAA'}} = 0 .
\]  
(7)

We notice that different \(\omega_{pAB}\) correspond to different Bianchi class A models.

The Lorentz constraints [3] are

\[
J_{AB} = e_{p(A)}^A \frac{\partial}{\partial e_{pAB}} + \psi_{p(B)}^B \frac{\partial}{\partial \psi_p^A} ,
\]  
(8)

\[
J_{A'B'} = e_{p(B')}^A \frac{\partial}{\partial e_{p[A']}} .
\]  
(9)

These two Lorentz constraints imply that the wave function should be invariant under the rotation in the spinor indices and depend on the three-geometry \(h_{pq}\) of hypersurface only. So we can write

\[
\Psi = \phi_0(h_{mn}) + C_{pq}(h_{mn}) \psi^p A \psi^q A' + V^{pqr}(h_{mn}) n_{AA'} e_{pB'}^{A'} \psi^q A' \psi^B r
\]

\[
+ \Psi_4 + \phi_6(h_{mn}) \prod_{i=1}^3 \psi^i A \psi_i A .
\]  
(10)

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where $C_{pq}$ is symmetric and $V^{pqr}$ is anti-symmetric in their last two indices. The $C_{pq}$ and $V^{pqr}$ provide 6 and 9 degrees of freedom respectively. Also,

$$\Psi_4 = E_{1122} \psi^{1A} \psi_{1A} \psi_{2B} \psi_{2B} + E_{1133} \psi^{1A} \psi_{1A} \psi^{3B} \psi_{3B} + E_{2233} \psi^{2A} \psi_{2A} \psi^{3B} \psi_{3B}$$

$$+ E_{1123} \psi^{1A} \psi_{1A} \psi_{2B} \psi_{3B} + E_{2213} \psi^{2A} \psi_{2A} \psi_{1B} \psi_{3B} + E_{3312} \psi^{3A} \psi_{3A} \psi_{1B} \psi_{2B}$$

$$+ F^p_{1233} e^p_{AB} n_{AA'} \psi^{1A} \psi_{2B} \psi^{3C} \psi_{3C} + F^p_{1323} e^p_{AB} n_{AA'} \psi^{1A} \psi^{3B} \psi_{2C} \psi_{2C}$$

$$+ F^p_{2311} e^p_{AB} n_{AA'} \psi^{2A} \psi^{3B} \psi_{1C} \psi_{1C}.$$

The E’s and V’s also provide 6 and 9 degrees of freedom respectively. These then give the most general solution to the Lorentz constraints. There is a duality relation between two fermions and four fermions [3]. By solving the two-fermion level, we can apply the Fourier transform [3] to obtain the corresponding four-fermion level.

The above supersymmetry constraint and wave function are gauge invariant. We use these gauge-invariant supersymmetry constraints to annihilate a gauge-invariant wave function and obtain all the equations of the theory. We then impose the condition of a diagonal Bianchi-IX metric in these equations. There is no loss of any physical information in the last step because the equations are derived from a gauge-invariant procedure. The $\bar{S}_{A'}$ constraint is of first order, giving the derivatives $\partial \Psi / \partial e_i^{AA'}$, evaluated in particular at a Bianchi-IX model. Then the Bianchi-IX $S_A$ constraint can be found as the hermitian adjoint of $\bar{S}_{A'}$. Indeed, the imposition of a diagonal Bianchi-IX metric isolates the true degrees of physical freedom because the isometry group of a Bianchi-IX universe is three dimensional. After completion of this work, we found that our Hamilton-Jacobi equation is equivalent to that derived in [3], giving confirmation of our approach.

We also mention that because we impose the condition of a diagonal metric, there are no off-diagonal components in our metric and hence there are no off diagonal derivatives. If off-diagonal derivatives were present, we would lose some physical information in our equations and would not get the right Hamilton-Jacobi equation. To have derived the correct Hamilton-Jacobi equation in our case is a self-consistency check. Another justification is that we get the correct result for the bosonic order (see below Eqns. (13)-(16)) where the off-diagonal derivatives are not present.
The diagonal Bianchi-IX three-metric is given in terms of the three radii $A, B, C$ by
\[ h_{ij} = A^2 E_1^i E_1^j + B^2 E_2^i E_2^j + C^2 E_3^i E_3^j \] (11)
where $E_1^i, E_2^i, E_3^i$ are a basis of unit left-invariant one-forms on the three-sphere [12]. In the calculation, we shall repeatedly need the expression, formed from the connection:
\[
\omega_{pAB} n^A_{B'} e^{qBB'} = \frac{i}{4} \left( C^2 + B^2 - A^2 \right) \delta_p^1 \delta_q^1 + \frac{i}{4} \left( A^2 + C^2 - B^2 \right) \delta_p^2 \delta_q^2 + \frac{i}{4} \left( B^2 + A^2 - C^2 \right) \delta_p^3 \delta_q^3 .
\] (12)

We now solve the supersymmetry constraints. First consider $\bar{S}_A \Psi = 0$ at the one-fermion level. One obtains
\[ e^{pqr} e_{pAA'} \omega^A_{q B} \psi^B_r \phi_0 + \hbar e_{qAA'} \psi^A_r \frac{\partial \phi_0}{\partial h_{pq}} = 0 , \] (13)
where the relations $e_{pAA'} e^{AA'} = -h_{pq}$ and $\partial/\partial e_{pAA'} = -2e_{qAA'} \partial/\partial h_{pq}$ have been used. Since it is true for all $\psi^B_r$, one can conclude
\[ e^{pqr} e_{pAA'} \omega^A_{q B} \phi_0 + \hbar e_{qBA'} \frac{\partial \phi_0}{\partial h_{r q}} = 0 . \] (14)

Multiply this equation by $e_{iBA'}$, giving
\[ i \left( h_{qi} n^A_{A'} e^{pBA'} \omega_{pAB} - n^A_{A'} e^A_{qB'} \omega_{LAB} \right) \phi_0 - \hbar h_{ps} h_{qs} \frac{\partial \phi_0}{\partial h_{pq}} = 0 . \] (15)
If $q \neq l$, this is an identity $0 = 0$. Now take $q = l = 1$, say:
\[ iA^2 \left[ \frac{i}{4} \left( C^2 + B^2 - A^2 \right) - \frac{i}{4} \left( C^2 + B^2 - A^2 \right) \right] \phi_0 - \hbar h_{p1} h_{1s} \frac{\partial \phi_0}{\partial h_{pq}} = 0 \]
\[ \Rightarrow \hbar \frac{\partial \phi_0}{\partial A} + A \phi_0 = 0 . \]
Similarly for the $B, C$ dependence:

\[
\frac{\hbar}{\partial B} \frac{\partial \phi_0}{\partial B} + B \phi_0 = 0
\]
\[
\frac{\hbar}{\partial C} \frac{\partial \phi_0}{\partial C} + C \phi_0 = 0
\]
\[
\Rightarrow \phi_0 \propto \exp \left( -\frac{1}{2\hbar} \left( A^2 + B^2 + C^2 \right) \right).
\]

This is a well known result and has been first worked out in ref. [4]. Now let us study the more interesting two-fermion level. It turns out that the real physical degrees of freedom are provided by $C_{11}, C_{22}, C_{33}, V_{123}, V_{231}, V_{312}$. The other coefficients of the two-fermion level are zero and hence not physical. We will first derive the equations relating $C_{12}, \ldots$ and $V_{112} \ldots$ and show that they are zero. After that, we will derive the equations for the physical amplitudes.

Consider $S_A \Psi = 0$ at the one-fermion level:

\[
2C_{pq} h^{qe} \omega^p_{AD} - 2V^{tuv} \omega^B_{uA} n_{BC'} e_{tD'} = 0
\]
\[
+2\hbar^t_v \frac{\partial C^{pl}}{\partial q_{r}} D_{D'}^{A'p} e_{rAA'} - 2\hbar \frac{\partial V^{tuv}}{\partial h^{pq}} D_{Aq}^{B'A} e_{pAA'} n_{BC'} e_{tD'} + \hbar V^{tuv} D_{uq}^{B'A} e_{q} B_{BC'} e_{tD'} + i\bar{h} V^{tuv} e_{AD} = 0.
\]

Contracting the indices $A$ and $D$ with $\epsilon^{AD}$:

\[
\Rightarrow -2V^{tuv} \omega^B_{uA} n_{BC'} e_{tAC'} + i\bar{h} \frac{\partial V^{tuv}}{\partial h^{pq}} (2h_{pt} h_{qu} - h_{pq} h_{ut})
\]
\[
+ \frac{i}{2} \bar{h} h_{ut} V^{tuv} = 0.
\]
\[ V^{331} + \bar{h} C^2 V^{331} + B^2 \left( A^2 + C^2 - B^2 \right) V^{221} + C^2 \left( A^2 + B^2 - C^2 \right) V^{331} = 0. \]  

(19)

and two further equations which are just given by cyclic permutations of \( A, B, C \) and 1, 2, 3. We can also multiply (16) by \( \epsilon_{x}^{AD'} n_{D'}^{D} \) and simplify the expressions. After some algebra, we obtain

\[
\bar{h} \left( 2 h_{x}^{r} \frac{\partial C_{p}}{\partial h_{r}^{p}} - h_{r}^{s} \frac{\partial C_{xw}}{\partial h_{r}^{s}} \right) + 2 C_{p}^{w} \omega_{p}^{A} n_{D'}^{D} \epsilon_{x}^{AD'}
\]

\[- \bar{h} \frac{\partial V_{tuv}}{\partial h_{pq}} h_{qu}^{v} h_{vw}^{u} \epsilon_{x}^{p} + \frac{\bar{h} \partial V_{tuv}}{2 \partial h_{pq}} h_{pq}^{v} h_{vw}^{u} \epsilon_{x}^{p} \]

\[ + i \bar{h} h_{vw}^{u} V_{tuv} \epsilon_{t}^{u} \omega_{u}^{AB} n_{D'}^{B} \epsilon_{y}^{AD'} + \frac{3}{4} \bar{h} h_{vw}^{u} V_{tuv} \epsilon_{t}^{u} \omega_{v}^{AB} n_{D'}^{B} \epsilon_{y}^{AD'} = 0. \]  

(20)

If we consider the off diagonal elements of \( (x, w) \), we get six equations which are:

\[ \left( \frac{B^2}{4} \right) \left[ h \left( C \frac{\partial}{\partial C} - B \frac{\partial}{\partial B} - A \frac{\partial}{\partial A} \right) + (A^2 + B^2 - C^2) - 3h \right] V^{232} \]

\[ + \frac{i}{2} \left[ h \left( A \frac{\partial}{\partial A} - B \frac{\partial}{\partial B} - C \frac{\partial}{\partial C} \right) + (C^2 + B^2 - A^2) \right] C_{12} = 0, \]  

(21)

\[ \left( \frac{A^2}{4} \right) \left[ h \left( C \frac{\partial}{\partial C} - B \frac{\partial}{\partial B} - A \frac{\partial}{\partial A} \right) + (A^2 + B^2 - C^2) - 3h \right] V^{113} \]

\[ + \frac{i}{2} \left[ h \left( B \frac{\partial}{\partial B} - A \frac{\partial}{\partial A} - C \frac{\partial}{\partial C} \right) + (C^2 + A^2 - B^2) \right] C_{12} = 0. \]  

(22)

The other four equations are also cyclic permutations of the above on \( A, B, C \) and 1, 2, 3. Now we consider \( \mathcal{S}_{A} \Psi = 0 \) at three-fermion level. Following similar methods, we get the following equations from \( \psi_{1} \psi_{2} \psi_{2}^{*} \):

\[ \hbar \left[ \left( \frac{\hbar}{2} \right)^2 \frac{\partial C_{22}}{\partial h_{11}} \epsilon_{1BA'} - \hbar^{11} \left( \frac{\hbar}{2} \right)^2 \frac{\partial C_{12}}{\partial h_{22}} \epsilon_{2BA'} \right] \]

\[ + \left[ \left( \frac{\hbar}{2} \right)^2 C_{22} \epsilon_{pq}^{1} - \hbar^{11} \left( \frac{\hbar}{2} \right)^2 C_{12} \epsilon_{pq}^{2} \right] \epsilon_{pAA'} \omega_{a}^{A} = \frac{\hbar}{2} \frac{\partial C_{22}}{\partial h_{11}} \epsilon_{1BA'} - \hbar^{11} \left( \frac{\hbar}{2} \right)^2 \frac{\partial C_{12}}{\partial h_{22}} \epsilon_{2BA'} \]
\[
\begin{align*}
&+ \left( \hbar \frac{\partial V^{s21}}{\partial h_{22}} e^2 C'_{A'} + e^{pq2} V^{s21} e_{pA'A'} \omega_q A C \right) n_{CC'} e_{sB} C' \\
&+ \frac{h}{4} V^{s21} \left( n_{CA'} e^2_{BC'} e_{s} C' - n_{C} e^2_{CC'} e_{s} C' \right) + \frac{3}{4} \hbar V^{221} n_{BA'} = 0 \quad (23)
\end{align*}
\]

Multiplying Eq. (22) by \( n^{BA'} \), we obtain the result
\[
V^{221} = f_{221}(A, C) \frac{1}{B^3} e^{-\frac{B^2}{2\hbar}},
\]
where \( f_{221}(A, C) \) is an arbitrary function of \( A, C \). We can also multiply (22) by \( \epsilon_l^{BA'} \). In these cases for \( l = 2, l = 3 \) respectively, we obtain
\[
\begin{align*}
C_{12} &= f_{12}(A, C) e^{-\frac{B^2}{2\hbar}}, \\
V^{121} &= f_{121}(A, C) \frac{1}{B} e^{-\frac{B^2}{2\hbar}}.
\end{align*}
\]

Similarly, considering \( \psi_1 \psi_3 \psi_3, \psi_2 \psi_1 \psi_1, \ldots \), we have
\[
\begin{align*}
C_{12} &= f_{12}(C) e^{-\frac{A^2 + B^2}{2\hbar}}, \quad C_{13} = f_{13}(B) e^{-\frac{A^2 + C^2}{2\hbar}}, \quad C_{23} = f_{23}(A) e^{-\frac{B^2 + C^2}{2\hbar}}, \\
V^{112} &= f_{112}(C) \frac{1}{A^3 B} e^{-\frac{A^2 + B^2}{2\hbar}}, \quad V^{113} = f_{113}(B) \frac{1}{A^3 C} e^{-\frac{A^2 + C^2}{2\hbar}}, \\
V^{212} &= f_{212}(C) \frac{1}{A B^3} e^{-\frac{A^2 + B^2}{2\hbar}}, \quad V^{223} = f_{223}(A) \frac{1}{B^3 C} e^{-\frac{B^2 + C^2}{2\hbar}}, \\
V^{313} &= f_{313}(B) \frac{1}{A C^3} e^{-\frac{A^2 + C^2}{2\hbar}}, \quad V^{323} = f_{323}(A) \frac{1}{B C^3} e^{-\frac{B^2 + C^2}{2\hbar}}. \quad (24)
\end{align*}
\]

If we consider \( S_{A' \Psi} = 0 \) at the order corresponding to \( \psi_1^E \psi_2^F \psi_3^G \), we have three free unprimed indices \( E, F, G \) and one free primed index \( A' \). We can contract two of the three unprimed indices, say \( F, G \), to get
\[
\begin{align*}
&\left( 4h^{22} h^{33} e^{pq1} C_{23} - 2h^{11} h^{33} e^{pq2} C_{13} - 2h^{11} h^{22} e^{pq3} C_{12} \right) e_{pAA'} \omega_q A e_{sB} C' \\
&- 2 \left( V^{s12} e^{pq2} + V^{s13} e^{pq3} \right) \omega_q A F e_{sC} e_{sC'} \\
&+ \hbar \left( 4h^{22} h^{33} \frac{\partial C_{23}}{\partial h_{11}} e_{1EA'} - 2h^{11} h^{33} \frac{\partial C_{13}}{\partial h_{22}} e_{2EA'} - 2h^{11} h^{22} \frac{\partial C_{12}}{\partial h_{33}} e_{3EA'} \right) \\
&+ 2\hbar \left( \frac{\partial V^{s12}}{\partial h_{33}} e_{3GA'} e_{sC} e_{sC'} - \frac{\partial V^{s13}}{\partial h_{22}} e_{2A'} e_{sC} e_{sC'} \right)
\end{align*}
\]
\[ + \frac{1}{2} \hbar V^{13} n_f A' (\epsilon_{EC'}^2 e_{sF}^C + \epsilon_{FC'}^2 e_{sE}^C) \]
\[ + \frac{1}{2} \hbar V^{12} n_f A' (\epsilon_{EC'}^3 e_{sF}^C + \epsilon_{FC'}^3 e_{sE}^C) \]
\[ - \frac{3}{2} \hbar \left( V^{213} + V^{312} \right) n_{E A'} = 0. \] (25)

Multiplying the last expression by \( e_{l E A'} \) for \( l = 1, 2, 3 \) will give
\[ \frac{2}{B^2 C^2} \left( \hbar A \frac{\partial C_{23}}{\partial A} + A^2 C_{23} \right) + i \left( \frac{\hbar}{B} \frac{\partial V^{313}}{\partial B} + \frac{\hbar}{B^2} V^{313} + V^{313} \right) \]
\[ - \frac{i}{2} \left( \frac{\hbar}{C} \frac{\partial V^{212}}{\partial C} + \frac{\hbar}{C^2} V^{212} + V^{212} \right) = 0, \] (26)
\[ \frac{1}{A^2 C^2} \left( \hbar B \frac{\partial C_{13}}{\partial B} + B^2 C_{13} \right) - \frac{i}{2} \left( \frac{\hbar}{C} \frac{\partial V^{112}}{\partial C} + \frac{\hbar}{C^2} V^{112} + V^{112} \right) = 0, \] (27)
\[ \frac{1}{A^2 B^2} \left( \hbar C \frac{\partial C_{12}}{\partial C} + C^2 C_{12} \right) + \frac{i}{2} \left( \frac{\hbar}{B} \frac{\partial V^{113}}{\partial B} + \frac{\hbar}{B^2} V^{113} + V^{113} \right) = 0. \] (28)

with their cyclic permutations. Using (25), (26), (27) and their cyclic permutations, one can easily prove that the only solutions satisfying these equations are
\[ C_{12} \propto e^{-\frac{1}{2\hbar} (A^2 + B^2 + C^2)}, \quad C_{13} \propto e^{-\frac{1}{2\hbar} (A^2 + B^2 + C^2)}, \quad C_{23} \propto e^{-\frac{1}{2\hbar} (A^2 + B^2 + C^2)}, \]
\[ V^{112} \propto \frac{1}{A^2 B C} e^{-\frac{1}{2\hbar} (A^2 + B^2 + C^2)}, \quad V^{113} \propto \frac{1}{A^2 B C} e^{-\frac{1}{2\hbar} (A^2 + B^2 + C^2)}, \]
\[ V^{212} \propto \frac{1}{A B^2 C} e^{-\frac{1}{2\hbar} (A^2 + B^2 + C^2)}, \quad V^{223} \propto \frac{1}{A B^2 C} e^{-\frac{1}{2\hbar} (A^2 + B^2 + C^2)}, \]
\[ V^{313} \propto \frac{1}{A B C^3} e^{-\frac{1}{2\hbar} (A^2 + B^2 + C^2)}, \quad V^{323} \propto \frac{1}{A B C^3} e^{-\frac{1}{2\hbar} (A^2 + B^2 + C^2)}. \] (29)

However, if we substitute the above 9 amplitudes back into (18), (20) and (21) and their cyclic permutations, they will not satisfy the equations. Hence the only solutions are zero. We can see that these 9 amplitudes are not the dynamical degrees of freedom of the theory, which are contained in the remaining 6 coefficients. Below we will derive the rest of the equations for the remaining coefficients at two-fermion level. It will be verified in the next section how these equations provide the dynamics of the theory.
We now come back to $S_A \Psi = 0$ at one-fermion level and $\bar{S}_A \Psi = 0$ at three-fermion level. We have only considered the off-diagonal elements of $(x, w)$ of (19). From the diagonal elements, one obtains coupled partial differential equations between the physical amplitudes.

\begin{align*}
A^2 \left[ \frac{\hbar}{2} \left( C \frac{\partial}{\partial C} - A \frac{\partial}{\partial A} - B \frac{\partial}{\partial B} \right) + \left( A^2 + B^2 - C^2 \right) - \frac{3\hbar}{2} \right] V^{231} \\
+ A^2 \left[ \frac{\hbar}{2} \left( C \frac{\partial}{\partial C} + A \frac{\partial}{\partial A} - B \frac{\partial}{\partial B} \right) - \left( A^2 + C^2 - B^2 \right) + \frac{3\hbar}{2} \right] V^{321} \\
+ \frac{\imath}{\hbar} \left( A \frac{\partial}{\partial A} - C \frac{\partial}{\partial C} - B \frac{\partial}{\partial B} \right) C_{11} + \imath \left( C^2 + B^2 - A^2 \right) C_{11} = 0, \quad (30)
\end{align*}

\begin{align*}
B^2 \left[ \frac{\hbar}{2} \left( -C \frac{\partial}{\partial C} + A \frac{\partial}{\partial A} + B \frac{\partial}{\partial B} \right) - \left( A^2 + B^2 - C^2 \right) + \frac{3\hbar}{2} \right] V^{132} \\
+ B^2 \left[ \frac{\hbar}{2} \left( -C \frac{\partial}{\partial C} + A \frac{\partial}{\partial A} - B \frac{\partial}{\partial B} \right) + \left( B^2 + C^2 - A^2 \right) - \frac{3\hbar}{2} \right] V^{312} \\
+ \frac{\imath}{\hbar} \left( B \frac{\partial}{\partial B} - C \frac{\partial}{\partial C} - A \frac{\partial}{\partial A} \right) C_{22} + \imath \left( C^2 + A^2 - B^2 \right) C_{22} = 0, \quad (31)
\end{align*}

\begin{align*}
C^2 \left[ \frac{\hbar}{2} \left( C \frac{\partial}{\partial C} - A \frac{\partial}{\partial A} + B \frac{\partial}{\partial B} \right) - \left( C^2 + B^2 - A^2 \right) + \frac{3\hbar}{2} \right] V^{213} \\
+ C^2 \left[ \frac{\hbar}{2} \left( -C \frac{\partial}{\partial C} - A \frac{\partial}{\partial A} + B \frac{\partial}{\partial B} \right) + \left( A^2 + C^2 - B^2 \right) - \frac{3\hbar}{2} \right] V^{123} \\
+ \frac{\imath}{\hbar} \left( -A \frac{\partial}{\partial A} - C \frac{\partial}{\partial C} + B \frac{\partial}{\partial B} \right) C_{33} + \imath \left( A^2 + B^2 - C^2 \right) C_{33} = 0. \quad (32)
\end{align*}

But if we multiply (22) by $e_1^{BA'}$, we have

\begin{align*}
\hbar A \frac{\partial C_{22}}{\partial A} + A^2 C_{22} - \frac{i}{2} B^2 \left( \hbar B \frac{\partial}{\partial B} + \hbar + B^2 \right) V^{321} = 0, \quad (33)
\end{align*}

and the cyclic permutations

\begin{align*}
\hbar B \frac{\partial C_{33}}{\partial B} + B^2 C_{33} - \frac{i}{2} C^2 \left( \hbar C \frac{\partial}{\partial C} + \hbar + C^2 \right) V^{132} = 0, \quad (34)
\end{align*}
\[ \hbar C \frac{\partial C_{11}}{\partial C} + C^2 C_{11} - \frac{i}{2} A^2 \left( \hbar A \frac{\partial}{\partial A} + \hbar + A^2 \right) V^{213} = 0. \] (35)

Also multiply the equation obtained from \( \psi_1 \psi_3 \psi_3 \) by \( e_1^{BA'} \), we have with the cyclic permutations

\[ \hbar B \frac{\partial C_{11}}{\partial B} + B^2 C_{11} + \frac{i}{2} A^2 \left( \hbar A \frac{\partial}{\partial A} + \hbar + A^2 \right) V^{312} = 0, \] (36)

\[ \hbar C \frac{\partial C_{22}}{\partial C} + C^2 C_{22} + \frac{i}{2} B^2 \left( \hbar B \frac{\partial}{\partial B} + \hbar + B^2 \right) V^{123} = 0, \] (37)

\[ \hbar A \frac{\partial C_{33}}{\partial A} + A^2 C_{33} + \frac{i}{2} C^2 \left( \hbar C \frac{\partial}{\partial C} + \hbar + C^2 \right) V^{231} = 0. \] (38)

We obtain three more equations by multiplying Eq. (24) by \( n^{\bar{E}A'} \), and taking cyclic permutations:

\[ \left( \hbar B \frac{\partial}{\partial B} + B^2 + 3\hbar \right) V^{213} + \left( \hbar C \frac{\partial}{\partial C} + C^2 + 3\hbar \right) V^{312} = 0, \] (39)

\[ \left( \hbar A \frac{\partial}{\partial A} + A^2 + 3\hbar \right) V^{123} + \left( \hbar C \frac{\partial}{\partial C} + C^2 + 3\hbar \right) V^{321} = 0, \] (40)

\[ \left( \hbar A \frac{\partial}{\partial A} + A^2 + 3\hbar \right) V^{132} + \left( \hbar B \frac{\partial}{\partial B} + B^2 + 3\hbar \right) V^{231} = 0. \] (41)

We have found all the equations relating \( C_{11}, C_{22}, C_{33} \) and \( V^{123}, V^{231}, V^{312} \). In the next section, we are going to investigate the semi-classical solution of the above equations.

### 3 Semi-Classical Solutions to Supersymmetry Constraints

We have seen there are twelve equations for \( C_{11}, C_{22}, C_{33}, V^{123}, V^{231}, V^{312} \), namely (29)-(40). Here we use these equations to make a comparison with Csordás and Graham [9], who found that a Hartle-Hawking state [13] exists semi-classically. Here we check that our system of first order-partial
differential equations also admits a Hartle-Hawking state semi-classically by studying the Hamilton-Jacobi equation. We assume that the coefficients have the form

\[ C_{11} = \left( C_{(0)11} + \hbar C_{(1)11} + \hbar^2 C_{(2)11} + \cdots \right) e^{-I/\hbar}, \text{ etc.}, \]

\[ V^{123} = \left( V_{123}^{(0)} + \hbar V_{123}^{(1)} + \hbar^2 V_{123}^{(2)} + \cdots \right) e^{-I/\hbar}, \text{ etc.}, \]

(42)

where \( I \) is a classical Euclidean action.

Consider the Hamilton-Jacobi equation. Substituting (41) into (29)-(40), and collecting all the terms of order \( \hbar^0 \), we have

\[ i \left( -A \frac{\partial I}{\partial A} + B \frac{\partial I}{\partial B} + C \frac{\partial I}{\partial C} - A^2 - B^2 - C^2 \right) C_{(0)11} \]

\[ + \frac{A^2}{2} \left( B \frac{\partial I}{\partial B} + A \frac{\partial I}{\partial A} - C \frac{\partial I}{\partial C} + A^2 + B^2 - C^2 \right) V_{231}^{(0)} \]

\[ + \frac{A^2}{2} \left( B \frac{\partial I}{\partial B} - A \frac{\partial I}{\partial A} - C \frac{\partial I}{\partial C} - A^2 + B^2 - C^2 \right) V_{321}^{(0)} = 0, \]

(43)

\[ C \left( -\frac{\partial I}{\partial C} + C \right) C_{(0)11} + \frac{i}{2} A^3 \left( -\frac{\partial I}{\partial A} + A \right) V_{231}^{(0)} = 0, \]

(44)

\[ B \left( -\frac{\partial I}{\partial B} + B \right) C_{(0)11} - \frac{i}{2} A^3 \left( -\frac{\partial I}{\partial A} + A \right) V_{321}^{(0)} = 0, \]

(45)

\[ B \left( -\frac{\partial I}{\partial B} + B \right) V_{231}^{(0)} + C \left( -\frac{\partial I}{\partial C} + C \right) V_{321}^{(0)} = 0. \]

(46)

Equations (43), (44) and (45) are consistent with each other. From (43), (44), we substitute the \( V \)'s into (42) to get an equation homogenous in \( C_{(0)11} \) only. To have a non-trivial solution of \( C_{(0)11} \), the coefficient must be zero, giving

\[ A^2 \left( \frac{\partial I}{\partial A} \right)^2 + B^2 \left( \frac{\partial I}{\partial B} \right)^2 + C^2 \left( \frac{\partial I}{\partial C} \right)^2 - 2AB \left( \frac{\partial I}{\partial A} \right) \left( \frac{\partial I}{\partial B} \right) \]

\[ - 2AC \left( \frac{\partial I}{\partial A} \right) \left( \frac{\partial I}{\partial C} \right) - 2BC \left( \frac{\partial I}{\partial B} \right) \left( \frac{\partial I}{\partial C} \right) \]

\[ - A^4 - B^4 - C^4 + 2A^2 B^2 + 2A^2 C^2 + 2B^2 C^2 = 0. \]

(47)
For the other $C$’s and $V$’s, one obtain the same Hamilton-Jacobi equation\footnote{It can be checked that the Hamilton-Jacobi equation is equivalent to the one that can be derived in \cite{9}.}

From dimensional ground, the action $I$ has the general form,

$$ I = \alpha A^2 + \beta B^2 + \gamma C^2 + \mu AB + \nu BC + \lambda AC. \quad (48) $$

Substituting this action $I$ into above Hamilton-Jacobi equation, it gives

$$
(4\alpha^2 - 1) A^4 + (4\beta^2 - 1) B^4 + (4\gamma^2 - 1) C^4 \\
+ (2 - 8\alpha\beta) A^2 B^2 + (2 - 8\gamma\alpha) A^2 C^2 + (2 - 8\beta\gamma) B^2 C^2 \\
- 4(2\alpha\nu + \mu\lambda) A^2 BC - 4(2\beta\gamma + \nu\mu) AB^2 C - 4(2\gamma\mu + \nu\lambda) ABC^2 = 0. \\
(49)
$$

Since the polynomials are independent of each other, each coefficient vanishes identically. Hence the most general solutions are

$$
\begin{align*}
\pm I &= \frac{1}{2} \left( A^2 + B^2 + C^2 \right) \\
\pm I &= \frac{1}{2} \left( A^2 + B^2 + C^2 \right) - AB - AC - BC \\
\pm I &= \frac{1}{2} \left( A^2 + B^2 + C^2 \right) + AB + AC - BC \\
\pm I &= \frac{1}{2} \left( A^2 + B^2 + C^2 \right) + AB - AC + BC \\
\pm I &= \frac{1}{2} \left( A^2 + B^2 + C^2 \right) - AB + AC + BC. \quad (50)
\end{align*}
$$

The first one is the wormhole action and the second one is the Hartle-Hawking action. At least in the semi-classical level, both wormhole and Hartle-Hawking state exist in the same fermion sector. Our results in here exactly correspond to Csordás and Graham \cite{9}.

### 4 Conclusion And Discussion

In section 2 we carried out the dimensional reduction and obtained the supersymmetry constraints; this involves writing down the most general solution to the Lorentz constraints. We then solved for the supersymmetry
constraints and found that 9 out of 15 degrees of freedom at the two-fermion level are not physical. The coupled first order partial differential equations describing the remaining 6 degrees of freedom were given in section 3. We solved the Hamilton-Jacobi equation completely and found the complete set of solutions. Both the Hartle-Hawking and wormhole actions are among the solutions.

We also mention that the Ansatz of the wave function constructed by Csordás and Graham [9] may only work if there are no chiral breaking terms in the supersymmetry constraints as in pure $N = 1$ supergravity. Supersymmetry constraints with no chiral breaking terms will preserve the number of fermions. The presence of chiral breaking terms will not conserve the number of fermions and gives mixing of different levels of fermions. This occurs (e.g.) when $N = 1$ supergravity is coupled to supermatter [11]. However, our approach can readily be generalized to non-chiral models.

In the future, we hope to study inhomogenous perturbations of a Friedmann $k = +1$ model in supersymmetric quantum cosmology, using spectral boundary conditions for gravitinos [15] (Bianchi IX models are a particular kind of distortion of a $k = +1$ model). It will be interesting to see if a Hartle-Hawking state still exists in these models.

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