Jordan blocks and generalized bi-orthogonal bases: realizations in open wave systems

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Abstract

Dissipative systems can be described in terms of non-hermitian hamiltonians $H$, whose left eigenvectors $\langle f^j |$ and right eigenvectors $|f^j \rangle$ form a bi-orthogonal system. Bi-orthogonal systems could suffer from two difficulties. (a) If the eigenvectors do not span the whole space, then $H$ can only be diagonalized to blocks (the Jordan-block problem). (b) Normalization would not be possible and many familiar-looking formulas would fail if $\langle f^j | f^j \rangle = 0$ for some $j$ (the orthonormalization problem). Waves in open systems provide a well-founded realization of a bi-orthogonal system, and it is shown that these two problems can indeed occur and are both related to higher-order poles in the frequency-domain Green’s function. The resolution is then given by introducing a generalized duality transformation involving extra basis vectors, whose time evolution is modified by polynomials in the time $t$. One thus obtains a nontrivial extension of the bi-orthogonal formalism for dissipative systems.

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I. INTRODUCTION

A. Dissipative systems and bi-orthogonal bases

Dissipative systems are often discussed, in a phenomenological way, by postulating a non-hermitian hamiltonian (NHH) $H$, whose left eigenvectors $|f^j\rangle$ and right eigenvectors $\langle f^j|$ form a bi-orthogonal system (BS). It is usually assumed that these eigenvectors are complete; the BS then constitutes a bi-orthogonal basis (BB). These NHHs with discrete BBs can sometimes be obtained from a full microscopic theory, but usually under some approximations. BBs allow dissipative systems to be placed into a familiar framework; the advantages are obvious and need not be enumerated. As a consequence, there is a substantial literature on both the mathematics and the physical applications of BBs (e.g., chemical bonding, solid mechanics).

To introduce the formalism, consider a Hilbert space with an NHH $H$, supposed to have a complete basis of eigenvectors $|f^j\rangle$, with eigenvalues $\omega^j$. In terms of the standard inner product $\langle \cdot | \cdot \rangle$, introduce the unique dual basis $\{ |f^j\rangle \}$ with

$$\langle f^k | f^j \rangle = \delta_{jk}; \quad (1.1)$$

one finds $\langle f^k | H^\dagger | f^j \rangle = \langle f^j | H | f^k \rangle^* = \omega^*_k \delta_{jk}$, so that $H^\dagger |f^j\rangle = \omega^*_j |f^j\rangle$. Now define a duality transformation $D$ by $D|f^j\rangle = |f^j\rangle$, extended to the whole space by conjugate linearity:

$$D (\alpha |\chi\rangle + \beta |\psi\rangle) = \alpha^* D|\chi\rangle + \beta^* D|\psi\rangle. \quad (1.2)$$

The symmetry of (1.1) shows that the duality transformation satisfies $D^2 = 1$, justifying its name $D$. Since $\langle f^k | D H | f^j \rangle = \omega^*_k \delta_{jk} = \langle f^k | H^\dagger D | f^j \rangle$, one has

$$DH = H^\dagger D. \quad (1.3)$$

Any vector can now be expanded as

$$|\phi\rangle = \sum_j a_j f^j, \quad (1.4)$$

$$a_j = \frac{\langle f^j | \phi \rangle}{\langle f^j | f^j \rangle}, \quad (1.5)$$

where for application in the following Sections it is convenient not to fix the normalization $\langle f^j | f^j \rangle$ by (1.1). This leads immediately to the resolution of the identity and of the time-evolution operator

$$\mathbb{I} = \sum_j |f^j\rangle \langle f^j| \quad (1.6)$$

$$e^{-iHt} = \sum_j \frac{|f^j\rangle e^{-i\omega_j t} \langle f^j|}{\langle f^j | f^j \rangle}, \quad (1.7)$$

which in principle solves all the dynamics. For example, the wavefunction at time $t > 0$ with (1.4) as initial data would be
\[ |\phi(t)\rangle = \sum_j a_j e^{-i\omega_j t} |f_j\rangle. \]  

(1.8)

It is emphasized that (1.3)–(1.8) have been obtained only for the case in which \{\(|f_j\rangle\}\) constitutes a complete basis.

This brief account exposes two related lurking difficulties that are usually brushed aside: the Jordan block problem and the orthonormalization problem.

### B. Jordan blocks and orthonormalization

Because \(H\) is not hermitian, there is no guarantee that its eigenvectors \(|f_j\rangle\) form a complete set, so a BB may not exist. There are in fact two fundamentally different reasons why the eigenvectors may fail to be complete \[12\].

The first reason can be characterized heuristically by saying that each eigenvector with a complex \(\omega_j\) is a discrete resonance, and there may be “background” under the resonances. A more formal statement is that there may be contributions to the Green’s function other than from the discrete poles \(\omega_j\) in the complex frequency plane; this possibility will be further explained below, but will not be the focus of the present paper.

The second scenario is more intriguing—even though the discrete poles may give all the dynamics, the associated eigenvectors may nevertheless be incomplete. The simplest case is for two (in general any \(M \geq 2\)) discrete poles to coalesce into a double (in general \(M\)-th order) pole, e.g., upon tuning some system parameter(s). We shall show below that, at least for 1-d wave systems, an \(M\)-th-order pole is not associated with \(M\) degenerate eigenvectors, but with only one. With the number of eigenvectors reduced by \(M - 1\), \{\(|f_j\rangle\}\) must become incomplete, and \(H\) can only be reduced to block form (in this case an \(M \times M\) block) rather than diagonal form. This is known as the Jordan block problem \[13\]. If it were to occur, the above formulas (1.3), (1.7), and (1.8), derived assuming an eigenvector basis, would not hold. Such problems can be handled as the limiting case of several nearby simple poles, but since some eigenvectors are “lost”, the limit is singular and needs to be treated with care.

For a non-hermitian operator, in general there is no positivity condition to guarantee that \(\langle f_j | f_j \rangle \neq 0\) for its left and right eigenvectors \(\langle f_j | \) and \(| f_j \rangle\) corresponding to the same eigenvalue \(\omega_j\), even if both these vectors are unique up to a constant; in Section I A this problem did not occur only because \(| f_j \rangle\) was orthogonal to all other eigenvectors \(| f_k \rangle\) \((k \neq j)\) of a set which was supposed complete. If \(\langle f_j | f_j \rangle\) should vanish for some \(j\), then the above formulas would also fail. This is the orthonormalization problem. (So long as \(\langle f_j | f_j \rangle \neq 0\), one can adopt a normalization convention that it is unity.)

### C. Outline of paper

In most applications, it is implicitly assumed that the Jordan block problem does not occur, and the BB formalism has not been extended to handle such problems if and when they do. In part, this is because phenomenological NHs do not exhibit these problems in a natural and convincing way, through which a possible extension of the formalism could be explored.
Recently it has been shown that waves in certain 1-d open systems provide an exact realization of BBs [14]. In these systems, dissipation occurs by the leakage of waves out of the system (through the imposition of the outgoing condition at the boundary). Under some simple conditions, the dynamics is completely controlled by the discrete poles in the frequency plane [13,14] and an analogy with conservative systems can be developed [17], such that the formalism can be cast in the language of BBs. In terms of these physical realizations, we shall show that the question at hand can be reduced to investigating the existence and properties of higher-order poles in the frequency-domain Green’s function.

In Section II, waves in open systems are described, focusing on the representation of the dynamics by the discrete poles [18] and the relation to a BS. We shall deal with both the wave equation and the Klein–Gordon equation. The duality transformation, central to the BS formalism, is shown to arise naturally from the dynamics. In Section III, we construct examples where higher-order poles do occur; the examples correspond to the critical damping of an oscillator. We then show that these higher-order poles relate to the Jordan block problem (“losing” one or more eigenvectors from a finite subspace). But since the dynamics is contained in the Green’s function, a systematic study of the residue at a higher-order pole should reveal how the situation must be handled. We therefore first analyze, in Section IV, the finite-dimensional subspace associated with the higher-order pole, i.e., the matrix representation of one Jordan block. Subsequently, in Section V we solve for the field evolution, which leads to an extension of the BB formalism with a generalized duality transformation. Then, in Section VII the perturbation theory associated with such a block is presented [19]. Unlike the degenerate perturbation theory of a conservative system, $M - 1$ of the basis vectors are not eigenvectors. Concluding remarks are given in Section VIII.

The overall results of this paper are then (a) nontrivial examples of the Jordan block problem in some exact treatments of dissipative systems, (b) an extended BB formalism for dealing with such situations, and (c) a matrix representation of such Jordan blocks including their perturbation theory.

II. WAVES IN OPEN SYSTEMS

A. Wave equation

We consider waves in 1 d described by

$$\left[ \rho(x)\partial_t^2 - \partial_x^2 \right] \phi(x, t) = 0$$

(2.1)

on the half line $[0, \infty)$, with $\phi(x=0, t) = 0$ and $\phi(x, t)$ approaching zero rapidly as $x \to \infty$ [20]. Let the system $S$ be the interval $[0, a]$, and the bath $B$ be $(a, \infty)$, with $\rho(x>a) = 1$. Energy is exchanged between $S$ and $B$ only through the boundary $x = a$. The outgoing-wave condition $\partial_t \phi(x, t) = -\partial_x \phi(x, t)$ is imposed for $x > a$.

The physical relevance of this model in describing strings [21], electromagnetism [22] and gravitational waves [23], as well as the mathematical formalism, have been discussed elsewhere [14,15,16,17,24].

Everywhere below we restrict to $\rho(x) > 0$, which is necessary both physically ($\rho$ is a density [21,23] or a dielectric constant [22]) and mathematically (the equation becomes
singular if \( \rho(x_0) = 0 \) for some \( x_0 \). In particular, this means that \( \rho(x) \) can contain a (positive) \( \delta \)-function, but not \( \delta' \) or higher derivatives (for which equations such as (2.3) would be distributionally undefined anyway). Thus, \( \phi \) has to be continuous, but \( \phi' \) may have discontinuities.

The eigenfunctions or quasinormal modes (QNMs) are factorized solutions

\[
\phi(x, t) = f_j(x) e^{-i\omega_j t} .
\] (2.2)

The spatial function satisfies

\[
[\partial_x^2 + \rho(x)\omega_j^2] f_j(x) = 0 ,
\] (2.3)

with \( f_j(x) \propto e^{i\omega_j x} \) for \( x > a \). Introduce the conjugate momentum \( \hat{\phi} = \rho(x)\partial_t\phi \), and the two-component vector

\[
|\phi\rangle = \begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix} .
\] (2.4)

In terms of this, the dynamics can be cast into the Schrödinger equation \( i\partial_t|\phi\rangle = H|\phi\rangle \) with the NHH

\[
H = i \begin{pmatrix} 0 & \rho(x)^{-1} \\ \rho(x)^{-1} & 0 \end{pmatrix} .
\] (2.5)

The identification \( \hat{\phi} = \rho\partial_t\phi \) follows from the evolution equation [25]. In this two-component form, the eigenvectors are

\[
|f_j\rangle \equiv \begin{pmatrix} f_j \\ f_j \end{pmatrix} = \begin{pmatrix} f_j \\ -i\omega_j\rho f_j \end{pmatrix} .
\] (2.6)

We shall define the duality transformation only after the dynamics has been discussed.

The hamiltonian (2.5) is non-hermitian, as will be obvious once the appropriate inner product (2.17) is defined. Nevertheless, on the “universe” \([0, \Lambda]\) (with a nodal condition at \( x = \Lambda \rightarrow \infty \)) a complete real spectrum of “universe modes” is guaranteed to exist. The latter are given by (2.6) for \( \pm \omega_j \), where \( f_j \) are the eigenfunctions with eigenvalue \( \omega_j^2 \) of the operator \(-\rho(x)^{-1}\partial_x^2\), which is hermitian and positive on the “one-component” space with inner product \( \langle u, v \rangle = \int_0^\Lambda dx \rho(x)u^*(x)v(x) \). This indirect construction of the spectrum of (2.5) fails on the interval \([0, a]\) (the energy of which is not conserved), however, since the very definition of the outgoing-wave condition \( \hat{\phi}(a^+) = -\phi'(a^+) \) is not possible with only one component. Consequently, the QNM frequencies will have a nonvanishing \( \text{Im} \omega_j < 0 \).

**B. Green’s function and poles**

The dynamics of these open wave systems is best discussed in terms of the Green’s function \( G(x, y; t) \), viz.,

\[
\phi(x, t) = \int_0^\infty \left[ G(x, y; t) \hat{\phi}(y) + \partial_t G(x, y; t) \rho(y) \phi(y) \right] dy ,
\] (2.7)
where $\phi$ and $\hat{\phi}$ are the initial values of the wavefunction and conjugate momentum. The behavior of $G$ is completely given by the contributions from the singularities in the frequency domain of its Fourier transform $\tilde{G}(x, y; \omega)$. It can be shown \cite{13} that under two conditions, viz., (a) $\rho(x)$ has a discontinuity at $x = a$ to provide a natural demarcation of the system from its surroundings (discontinuity condition), and (b) $\rho(x) = 1$ for $x > a$, so that outgoing waves are not scattered back into the system (no-tail condition), the only contributions will be from the isolated poles associated with the eigenvectors, and if all poles are simple then $G$ can be represented as

$$G(x, y; t) = i \sum_j \frac{f_j(x)f_j(y)}{(f_j, f_j)} e^{-i\omega_j t}. \quad (2.8)$$

The normalization factor $(f_j, f_j)$, which will turn out to be the same as $\langle f^j|f_j \rangle$ and therefore central to the issue at hand, will be derived and discussed in detail below. Note that the numerator of $G$ goes as $f_j(x)f_j(y)$, not as, e.g., $f_j(x)f_j^*(y)$ (which indeed would violate the symmetry of $G$ in $x$ and $y$). Thus $(f_j, f_j)$ will have to go as $f_j^2$, not $|f_j^2|$: this makes a crucial difference, since QNM wavefunctions in general are not real.

The derivation of the pole contributions is given in \cite{15}, and the main elements will be reviewed below as preparation for Section \ref{V}. One starts from the defining equation for the frequency-domain Green’s function

$$D(\omega) \tilde{G}(x, y; \omega) \equiv \left[ \partial_x^2 + \rho(x)\omega^2 \right] \tilde{G}(x, y; \omega) = -\delta(x - y). \quad (2.9)$$

The solution for $0 \leq x \leq y \leq a$ is given explicitly by

$$\tilde{G}(x, y; \omega) = \frac{f(x, \omega)g(y, \omega)}{W(\omega)}, \quad (2.10)$$

where $f$ and $g$ are solutions to the homogeneous equation $D(\omega)f = D(\omega)g = 0$, with $f$ satisfying the left boundary condition $f(x=0, \omega) = 0$, and $g$ satisfying the right boundary condition $g(x, \omega) \propto e^{i\omega x}$ for $x > a$. Their wronskian is $W(\omega) = f'g - fg'$, so that the combination (2.10) is independent of the normalization of $f$ and $g$.

One next writes an inverse Fourier integral to obtain $G$ in the time domain, and distorts the integral from the real $\omega$ axis to a large semicircle in the lower half-plane. Under the conditions stated, the contribution from the large semicircle vanishes and there are no cuts (or other singularities) associated with the tail of $\rho$ \cite{13,14}; one is then left with the residues at the poles, namely the zeros of $W$ \cite{20}. (If the conditions do not hold, there will be non-pole contributions from the large semicircle and/or cuts, and the discrete eigenvectors will be incomplete in the first sense mentioned in Section \ref{I}, leading to transients for short times and power-law tails for long times \cite{27}. This scenario will not be pursued further here.)

At a zero $\omega_j$ of $W$, the functions $f$ and $g$ are linearly dependent: $f_j(x) \equiv f(x, \omega_j) = C_j g(x, \omega_j)$. Thus $f_j$ satisfies both the left and right boundary conditions, and is an eigenfunction. Assuming for the moment that $\omega_j$ is a simple zero, the residue is related to $dW(\omega_j)/d\omega$, and it is straightforward to show that

$$-C_j \frac{dW(\omega_j)}{d\omega} = 2\omega_j \int_0^{a^+} \rho(x)f_j(x)^2 dx + i f_j(a)^2$$

$$\equiv (f_j, f_j), \quad (2.11)$$

They are the initial values of the wavefunction and conjugate momentum. The behavior of $G$ is completely given by the contributions from the singularities in the frequency domain of its Fourier transform $\tilde{G}(x, y; \omega)$. It can be shown \cite{13} that under two conditions, viz., (a) $\rho(x)$ has a discontinuity at $x = a$ to provide a natural demarcation of the system from its surroundings (discontinuity condition), and (b) $\rho(x) = 1$ for $x > a$, so that outgoing waves are not scattered back into the system (no-tail condition), the only contributions will be from the isolated poles associated with the eigenvectors, and if all poles are simple then $G$ can be represented as

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They are the initial values of the wavefunction and conjugate momentum.
while in the numerator we have \( f(x, \omega)g(y, \omega) = C_j^{-1}f_j(x)f_j(y) \). The representation (2.8) for the Green’s function then follows trivially. In particular, the calculation shows why \((f_j, f_j) \propto f_j^2\) instead of \(|f_j^2|\). The definition (2.11) of \((f_j, f_j)\) is valid for eigenfunctions only, and will be shown to be a special case of (2.13).

This derivation and especially (2.11) make it clear that the orthonormalization problem is the same as the possibility of higher-order zeros in \(W\), i.e., of non-trivial Jordan blocks, as was already apparent in the general analysis of Section 1B.

C. Duality transformation

While the duality transformation is simple and has been presented elsewhere [14], it is useful to emphasize how it comes about naturally from the dynamics. For this purpose, we put (2.8) into (2.7) and find that \(|\phi(t)\rangle\) (cf. (2.4)) is given by (1.8), with the coefficients (2.8)

\[
a_j = \frac{i}{(f_j, f_j)} \left\{ \int_0^{a^+} \left[ f_j(y)\hat{\phi}(y) + \hat{f}_j(y)\phi(y) \right] dy + f_j(a)\phi(a) \right\} .
\] (2.12)

All reference to initial data on the “outside” \(x > a\) has been eliminated. Physically, the outgoing condition ensures that the outside data do not propagate in; mathematically, both the initial data and the Green’s function are annihilated on the “outside” by \(\partial_t + \partial_x\).

The projection formula (2.12) suggests the definition of a bilinear map (2.13) between two vectors

\[
(\psi, \chi) = i \left[ \int_0^{a^+} (\psi\hat{\chi} + \hat{\psi}\chi) dx + \psi(a)\chi(a) \right] .
\] (2.13)

It is seen that the definition (2.13) is consistent with (2.11). Using this notation, we can now write the projection formula (2.12) compactly as

\[
a_j = \frac{(f_j, \phi)}{(f_j, f_j)} .
\] (2.14)

The map (2.13) has two important properties: (a) in the integral it cross-multiplies the two components, and (b) there is no complex conjugation of the first argument. These suggest the definition of a duality transformation \(D = \mathcal{F}\), with \(\mathcal{F}\) a flip map that (a) interchanges the two components and (b) complex conjugates (cf., e.g., [30]), viz.,

\[
\mathcal{F} \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \equiv -i \left( \begin{array}{c} \psi_2^* \\ \psi_1^* \end{array} \right) ,
\] (2.15)

in terms of which the bilinear map can be related to the standard inner product between two-component vectors

\[
(\psi, \chi) = \langle \mathcal{F}\psi | \chi \rangle ,
\] (2.16)

where the latter is defined as

\[
\langle \zeta | \chi \rangle \equiv \int_0^{\infty} (\zeta^* \chi + \hat{\zeta}^* \hat{\chi}) dx .
\] (2.17)
In showing the equivalence (2.16), one has to use the outgoing property to collapse the integral over $(a,\infty)$ in (2.17). That is, (2.16) only holds when either $|\chi\rangle$ and $\mathcal{F}|\zeta\rangle$, or $\mathcal{F}|\chi\rangle$ and $|\zeta\rangle$, are both outgoing, and only in these cases will we use (2.17) in this paper. Thus, $|\chi\rangle$ and $|\zeta\rangle$ in general belong to different spaces, and hence the structure is not precisely that of a Hilbert space. Since this latter issue is unrelated to the possibility of having higher-order poles, however, these functional-analytic details will be dealt with separately [31].

Two symbols $\mathcal{D}$ and $\mathcal{F}$ have been used in anticipation of their inequivalence in the case of higher-order poles. However, in the general case $\mathcal{F}$ remains defined by (2.13); cf. (5.7).

The property (1.3) is equivalent to the symmetry of $H$ under the bilinear map:

$$\langle \psi, H\chi \rangle = \langle H\psi, \chi \rangle . \quad (2.18)$$

To verify (2.18), one needs to integrate by parts; the surface term generated exactly cancels the surface term in (2.13). The “orthogonality” of QNMs,

$$\langle f_j, f_k \rangle = 0, \quad j \neq k \quad (2.19)$$

now follows in an immediate transcription of the standard proof.

What remains then is to show that higher-order zeros in $W$ can indeed exist (Section III) and to generalize the above formalism to those cases (Sections IV–VI).

D. Klein–Gordon equation

Also of interest in this context is the Klein–Gordon equation

$$\left[ \partial_t^2 - \partial_x^2 + V(x) \right] \phi(x,t) = 0 . \quad (2.20)$$

Among other things, this equation describes the propagation of linearized gravitational waves on the curved background of a black hole [32].

Essentially the entire formalism for the wave equation applies, *mutatis mutandis* [10]. First, the discontinuity condition now refers to $V(x)$, and the no-tail condition requires $V(x) = 0$ for $x > a$. Everywhere else we replace $\rho(x) \mapsto 1$, $-\partial_x^2 \mapsto -\partial_x^2 + V(x)$. An example will be given below in terms of this equation.

III. HIGHER-ORDER POLES

A. General remarks

For 1-d conservative systems, e.g., (2.3) with nodal boundary conditions at $x = 0$ and $x = a$, $W$ can only have simple zeros since in (2.11) the surface term is now absent, while the integral is positive definite up to an overall phase. Thus, the eigenfrequencies of conservative systems have a finite spacing $\Delta \omega$, and can be labeled by the number of nodes of the corresponding eigenfunctions. Hence, it is by no means obvious that higher-order poles can exist in the case of outgoing waves. For this reason, we will first demonstrate some examples...
of higher-order poles (cf. [33]) before studying the extension of the BB formalism. For simplicity, here we shall concentrate on second-order poles. The demonstration of the existence of third-order poles and the discussion of other possibilities are deferred to Appendix A.

Second-order poles can be obtained by allowing two first-order poles to merge. The converse is also true: any second-order pole, when suitably perturbed, splits into two first-order ones, as demonstrated explicitly in Section [4]. One is thus permitted to think generically of the coalescence of poles, and this leads to the concept of the “loss” of eigenvectors. Consider two nearby first-order zeros of \( W \), associated with two distinct eigenfunctions, say \( f_j \) and \( f_k \). But when the poles merge, there is only one eigenfunction—in contrast to the case of degeneracies in conservative systems. To see this, it suffices to notice that for a given \( \omega \), the boundary conditions \( f(x=0, \omega) = 0 \) and \( f'(x=0, \omega) = 1 \), say, uniquely specify one function. The “lost” or “missing” eigenfunction makes the possibility of higher-order poles all the more interesting.

It is best to look for second-order poles on the imaginary axis in the \( \omega \) plane. First, apart from some overall factors of \( i \), the problem is purely real and easy to handle. More physically, as the system parameter(s) are tuned, it is “unlikely” that two poles in the complex plane would collide—not only would this require the simultaneous tuning of two parameters, but one also expects level repulsion [34]. However, eigenvectors of the dissipative system (2.1) exist in pairs, with frequencies \( \omega \) and \( -\omega^* \) lying on the same horizontal line. It would require the tuning of only one parameter to make them collide; when they collide, they must do so on the imaginary axis. In fact, we expect that after they collide, the two poles will move apart along the imaginary axis, in exact analogy to an oscillator going through critical damping. This scenario is exemplified in both models shown below, and it remains an open question whether higher-order poles can exist off the imaginary axis.

### B. Example in the wave equation

With these remarks, we now look for a double zero of \( W \) in the case of the wave equation for \( \omega = -i\gamma \), with \( \gamma \) positive. The differential equation (suppressing the mode index \( j \)) then becomes real:

\[
\left[ \partial_x^2 - \rho(x) \gamma^2 \right] f(x) = 0
\]

and the eigenvalue condition is \( f'/f = \gamma \) at \( x = a^+ \), which ensures that \( W(-i\gamma) = 0 \). For \( \omega = -i\gamma \) to be a double zero, we also need \( (f, f) \propto dW/d\omega \) to vanish:

\[
i(f, f) = 2\gamma \int_0^{a^+} \rho(x) f(x)^2 dx - f(a)^2 = 0.
\]

By using (3.1) to express \( \rho(x) f(x) \) in terms of \( f'' \) and then integrating by parts, this condition can be recast as

\[
i(f, f) = -\frac{2}{\gamma} \int_0^a f'^2 dx + f(a)^2 = 0.
\]

The last term has just been reversed on account of the surface term \(-2f(a)^2\) produced in the integration by parts.
Interestingly, \( \rho(x) \) does not appear in (3.3), and this is central to the construction of examples, as follows. (a) Choose any function \( f(x) \) satisfying \( f(x=0) = 0, f'(x=0) > 0 \) and \( f''(x) > 0 \). (b) Use (3.3) to determine \( \gamma \). (c) Put these back into (3.1) to find \( \rho(x) \), which is guaranteed to be positive.

There is however one further subtlety. Such a construction gives \( f'(a^-) = \gamma f(a) \); the difference between these two must be attributed, through (3.1), to \( \rho(x) = \cdots + \mu \delta(x-a) \), with \( \mu = \gamma^{-1} - f'(a^-)/(\gamma^2 f(a)) \). One must check that \( \mu \geq 0 \), i.e., that

\[
2 \int_0^a f'^2 dx \geq f(a)f'(a^-) .
\] (3.4)

This condition is nontrivial, and for instance violated for \( f(x) = x + \alpha x^n \) and for some \( \alpha \) if \( n \geq 5 \).

Yet examples abound, e.g.,

\[
f(x < 1) = \sinh(Kx) \] (3.5a)
\[
\gamma = K \cotanh K + \frac{K^2}{\sinh^2 K} \] (3.5b)
\[
\rho(x) = \frac{K^2}{\gamma^2} \theta(1-x) + \frac{K^2}{\gamma^2 \sinh^2 K} \delta(x-1) + \theta(x-1) ,
\] (3.5c)

for any \( K > 0 \). Note that in this example always \( \rho(0<x<1) = K^2/\gamma^2 < 1 \), the case in which there is a zero-mode (i.e., a single simple pole on the negative imaginary \( \omega \) axis) even if the \( \delta \)-term in (3.5c) for \( \rho \) is absent \[17,36\]. Incidentally, this case without the \( \delta \)-term thus already shows that the open string model (2.1) can exhibit features not found in damped harmonic oscillators, as will become even clearer in Appendix A.

C. Example in the Klein–Gordon equation

We next give an example for the Klein–Gordon equation; this example may appear more natural in that the system (i.e., \( V(x) \)) is specified in advance and not obtained as an answer.

Recall the Pöschl–Teller potential \[37\]

\[ V(x) = V_0 \sech^2 x . \] (3.6)

The Klein–Gordon equation \[220\] with this potential is exactly soluble \[37\], with the eigenvalues given by \[38\]

\[
\omega_j = \begin{cases} 
\pm \sqrt{V_0 - \frac{1}{4} - i(j + \frac{1}{2})} & \text{if } V_0 \geq \frac{1}{4} \\
-\frac{i}{2} \left[ j + \frac{1}{2} \pm \sqrt{\frac{1}{4} - V_0} \right] & \text{if } V_0 \leq \frac{1}{4}
\end{cases}
\] (3.7)

\( j = 0, 1, 2, \ldots \). Each pair of poles merge at the parameter value \( V_0 = \frac{1}{4} \). Again, the merging is exactly like an oscillator going through critical damping.

This example may however be regarded as slightly unsatisfactory in one way: the potential has no discontinuity but does have a tail (i.e., \( V(x) \) does not vanish outside a finite interval \( |x| \leq a \)), so the set of discrete eigenvectors would not be complete even when all the
poles are first-order. Nevertheless, we may consider a minor alteration: if \( V(x) \) is truncated at \( |x| = L \) for some large \( L \), one would expect the eigenvalues not to be much affected. Actually this is not the case in general \([39]\), but here we shall not go into this subtlety, which is not related to the present issue. It is sufficient for the present purpose that at least the first pair of modes \( j = 0 \) are not too much affected by truncation, in the sense that the \( L \to \infty \) limit recovers the pole position for the untruncated \( V(x) \). This pair of poles at critical damping then demonstrates a double zero of \( W \) in a context where the eigenvectors are otherwise complete. We have verified numerically the existence of this double pole for \( L = 5, V_0 = 0.252279109 \ldots, i\omega = 0.511109 \ldots \).

IV. JORDAN BLOCKS

In this Section, we focus on a single \( M \)th-order zero \( \omega_j \) of \( W \) and the subspace associated with it. As already mentioned in Section III A, in this subspace there is only one eigenvector, and presently \( M - 1 \) other basis vectors will be obtained. This means that \( H \) is not diagonalizable, and we consider this \( M \times M \) block in \( H \). In this and the following Section, the discussion will be confined to the wave equation (2.1).

Using the definitions below (2.10), the (position-independent) wronskian can be written as \( W(\omega) = f'(0, \omega)g(0, \omega) \). Differentiating with respect to \( \omega \), one now proves by induction that \( \partial^n g(0, \omega)|_{\omega=\omega_j} = 0 \) for \( 0 \leq n \leq M - 1 \). This means that, up to this order in \( \omega - \omega_j \), the functions \( f \) and \( g \) satisfy the same boundary conditions and hence can be normalized to be equal, i.e., if we define

\[
f(x, \omega) = \sum_{n=0}^{M-1} f_{j,n}(x)(\omega - \omega_j)^n + \mathcal{O}[(\omega - \omega_j)^M],
\]

so that \( f_{j,n}(x) \equiv (1/n!) \partial^n f(x, \omega)|_{\omega=\omega_j} \), then we also have

\[
g(x, \omega) = \sum_{n=0}^{M-1} f_{j,n}(x)(\omega - \omega_j)^n + \mathcal{O}[(\omega - \omega_j)^M].
\]

Now define the time-dependent functions

\[
f_{j,n}(x, t) \equiv \frac{1}{n!} \partial^n [f(x, \omega)e^{-i\omega t}]|_{\omega=\omega_j} = \frac{1}{n!} \partial^n [g(x, \omega)e^{-i\omega t}]|_{\omega=\omega_j} = \sum_{m=0}^{n} f_{j,n-m}(x) \frac{(-it)^m}{m!} e^{-i\omega_j t}
\]

for \( 0 \leq n \leq M - 1 \), where the last line follows using (4.2). These are not only outgoing solutions of the wave equation (since \( g(x, \omega)e^{-i\omega t} \) is such a solution for any \( \omega \)), but satisfy the nodal condition at the origin as well (since \( f(x, \omega)e^{-i\omega t} \) has this property for any \( \omega \)). The momenta associated with the \( f_{j,n} \) are

\[
\hat{f}_{j,n}(x, t) \equiv \rho(x) f_{j,n}(x, t) = -i \rho(x)[\omega_j f_{j,n}(x, t) + f_{j,n-1}(x, t)],
\]

for 0 ≤ n ≤ M – 1, where the last line follows using (4.2). These are not only outgoing solutions of the wave equation (since \( g(x, \omega)e^{-i\omega t} \) is such a solution for any \( \omega \)), but satisfy the nodal condition at the origin as well (since \( f(x, \omega)e^{-i\omega t} \) has this property for any \( \omega \)). The momenta associated with the \( f_{j,n} \) are

\[
\hat{f}_{j,n}(x, t) \equiv \rho(x) f_{j,n}(x, t) = -i \rho(x)[\omega_j f_{j,n}(x, t) + f_{j,n-1}(x, t)],
\]
so that the action of the Hamiltonian is

\[ H |f_{j,n}\rangle = \omega_j |f_{j,n}\rangle + |f_{j,n-1}\rangle, \]

with \( |f_{j,-1}\rangle \equiv 0 \).

For fixed \( j \), the functions \( f_{j,n}(x,t) \) \((n = 0, \ldots, M-1)\) are linearly independent (in the sense that a non-trivial superposition cannot vanish identically in \( x \) and \( t \)), as is obvious by looking at the highest power of \( t \) in each (the coefficient of \( t^n e^{-i\omega_j t} \) in \( f_{j,n} \) is \( \propto f_j(x) \), which by definition does not vanish identically). Therefore the initial data \( |f_{j,n}\rangle \) have to be independent as well, otherwise one would have a vanishing superposition evolving into a nonvanishing function.

Thus the set \( \{ |f_{j,n}\rangle \}_{n=0}^{M-1} \) is a basis, in which the Hamiltonian reads

\[
H = \begin{pmatrix}
\omega_j & 1 & 0 & \cdots & 0 \\
0 & \omega_j & 1 & \cdots & 0 \\
0 & 0 & \omega_j & \ddots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega_j
\end{pmatrix}, \quad (4.5)
\]

Each entry in (4.5) is really a \( 2 \times 2 \) matrix (cf. (2.5)) operating on a two-component vector such as (2.4), giving \( H \) the structure of a \( 2M \times 2M \) tensor product. However, the very fact that \( \{ |f_{j,n}\rangle \}_{n=0}^{M-1} \) is a basis for \( H \) means that each of these \( 2 \times 2 \) matrices is \( \propto \mathbb{1} \).

While it is guaranteed that \( H \) can be cast into the so-called Jordan-block form (4.5) in a subspace with precisely one eigenvector (e.g., \( \mathbb{1} \)), we have now established a basis with respect to which this is indeed the case, and related this basis to the solutions \( f(x,\omega) \)

\( g(x,\omega) \). The basis is not unique, since a rescaling \( f(x,\omega) \mapsto \mathcal{N}(\omega)f(x,\omega) \) mixes the \( f_{j,n} \), with only \( f_j = f_{j,0} \) remaining invariant up to a prefactor; in fact, this rescaling is readily checked to generate precisely those basis transformations which leave the form (4.5) for \( H \) invariant. A further—essentially unique—specification of the basis will be made in Section V.

Vectors from different blocks are “orthogonal” under the bilinear map (2.13), i.e.,

\[
(f_{j,n}, f_{k,m}) = 0, \quad j \neq k. \quad (4.6)
\]

The proof proceeds by induction with respect to \( n + m \). The case \( n + m = 0 \) is the standard one of eigenvectors given in (2.19). Now consider

\[
\omega_j (f_{j,n}, f_{k,m}) + (f_{j,n-1}, f_{k,m}) = (Hf_{j,n}, f_{k,m})
= (f_{j,n}, Hf_{k,m})
= \omega_k (f_{j,n}, f_{k,m}) + (f_{j,n}, f_{k,m-1}) . \quad (4.7)
\]

On both sides, the second terms vanish by the induction hypothesis, so one is left with

\[
(\omega_j - \omega_k)(f_{j,n}, f_{k,m}) = 0, \quad \text{proving (4.6)}.\]

V. GENERALIZED DUALITY TRANSFORMATION

Having obtained the extra non-eigenvector solutions \( f_{j,n}(x,t) \) \((n \geq 1)\) in the previous Section, we now investigate how these enter into the field expansion. In doing so we shall consider all the poles simultaneously, so that the block size \( M \) acquires an index \( j \). As with the Jordan-block form of \( H \) (cf. the remark below (4.5)), a basis dual to \( \{ |f_{j,n}\rangle \}_{n=0}^{M_j-1} \)
is known to exist on general grounds (see Appendix B for details), but it remains to find
its explicit form and, if possible, to choose the original basis (which still is subject to the
freedom pointed out below (4.5)) so that the ensuing expressions will be as simple as possible.

Our starting point is (2.7), with \( G(x, y; t) = \int (d\omega/2\pi) \tilde{G}(x, y; \omega) e^{-i\omega t} \), where \( \tilde{G} \) is given by (2.10). The wronskian has an \( M_j \)th-order zero \[ W(\omega) = W_{j, M_j}(\omega - \omega_j)^{M_j} \mathcal{M}(\omega) \]

\[ = W_{j, M_j}(\omega - \omega_j)^{M_j} + \mathcal{O}[(\omega - \omega_j)^{M_j+1}] \], \hspace{1cm} (5.1)

with \( \mathcal{M}(\omega_j) = 1 \). We now make use of the remaining freedom \( f(x, \omega) \mapsto N(\omega) f(x, \omega) \),

taking \[ N(\omega) = M(\omega_j) - 1/2 + \mathcal{O}[(\omega - \omega_j)^{M_j+1}] \] (which is analytic in a neighborhood of \( \omega = \omega_j \)) and similarly for \( g \), preserving (4.2). After this transformation we have \[ W(\omega) = W_{j, M_j}(\omega - \omega_j)^{M_j} + \mathcal{O}[(\omega - \omega_j)^{2M_j}] \], \hspace{1cm} (5.2)

where we draw attention to the order of the error term. Eq. (5.2) will greatly simplify the
formulas below. The contour integral for \( G \) is now straightforward, leading to

\[ G(x, y; t) = \sum_j e^{-i\omega_j t} \sum_{n=0}^{M_j-1} \sum_{m=0}^{n} f_{j,m}(y) f_{j,n-m}(x) \left( \frac{(-it)^{M_j-1-n}}{(M_j-1-n)!} \right) \] \hspace{1cm} (5.3)

for \( t \geq 0 \), which we assume throughout this Section. By symmetry, (5.3) also holds for
\( 0 \leq y < x \leq a \) even though this is not the case for the original (2.10). If \( M_j = 1 \) for all \( j \) this agrees with the known result for simple poles, which follows by combining (2.8) and (2.11). In general, one can rewrite

\[ G(x, y; t) = \sum_j \frac{1}{iW_{j, M_j}} \sum_{n=0}^{M_j-1} f_{j,M_j-1-n}(y) f_{j,n}(x, t) \] \hspace{1cm} (5.4)

in terms of the functions \( f_{j,n}(x, t) \) defined in (4.3). Insertion into (2.7) yields the time evolution

\[ \phi(x, t) = -\sum_j \frac{1}{W_{j, M_j}} \sum_{n=0}^{M_j-1} \left( f_{j,M_j-1-n}, \phi \right) f_{j,n}(x, t) \] \hspace{1cm} (5.5)

in terms of the bilinear map (2.13). In particular, this holds for \( \phi = f_{k,m} \), in which case terms with \( j \neq k \) vanish by (4.6), and the linear independence of the \( f_{k,n}(t) \) discussed above (4.3) implies that the coefficients on both sides of (5.5) are equal, i.e.,

\[ (f_{j,n}, f_{k,m}) = -\delta_{jk} \delta_{n+m, M_j-1} W_{j, M_j} \]. \hspace{1cm} (5.6)

Of course, this intra-block “orthogonality” relation is conditional on the normalization
\( W^{(n)}(\omega_j) = 0 \) for \( M_j + 1 \leq n \leq 2M_j - 1 \), imposed in (5.2); without this normalization one has (5.13) below instead. In view of (2.16), the relation (5.6) leads one to define

\[ |f^{j,n}\rangle \equiv \mathcal{D}|f_{j,n}\rangle \equiv \mathcal{F}|f_{j,M_j-1-n}\rangle \], \hspace{1cm} (5.7)

where \( \mathcal{F} \) is the flip operation in (2.15). Thus
\( \langle f_j^n | f_{j',n'} \rangle = -W_{j,M_j} \delta_{jj'} \delta_{nn'} \), \hspace{1cm} (5.8)

where the constant of proportionality \( W_{j,M_j} \) is nonzero by definition (cf. (5.1)), so that there is no normalization problem. Equation (5.8) is a significant result; it shows that, unless \( M_j = 1 \) for all \( j \), the duality map \( D \) no longer coincides with \( F \): \( D \) changes the intra-block index \( n \) of \( |f_j,n\rangle \). Since it is the flip map which obeys
\[ FH = H^\dagger F \] (5.9)
(note that the proof outlined below (2.18) does not invoke any assumptions on the block structure of \( H \)), the relation (1.3) in general is not satisfied by the operator \( D \) implicit in (1.6). Since it is an immediate consequence of (5.9) that \( F \) carries right into left eigenvectors and vice versa, the left eigenvector corresponding to \( |f_j\rangle = |f_{j,0}\rangle \) is
\[ \langle F f_j | \phi \rangle = \sum_n (f_{j,n})^\dagger \phi_n \]
for \( M_j > 1 \). While the left and right eigenvectors thus are orthogonal as stipulated in Section I B, this does not lead to orthonormalization problems as these vectors are not the dual of each other.

Using (5.6) and (5.7), one may write the final result for the generalized bi-orthogonal expansion as
\[ G(x,y;t) = i \sum_j \sum_{n=0}^{M_j-1} f_{j,M_j-1-n}(y) f_{j,n}(x,t) / (f_{j,M_j-1-n}, f_{j,n}) , \]
so that the time evolution reads
\[ |\phi(t)\rangle = \sum_j \sum_{n=0}^{M_j-1} \langle f_{j,n} | \phi \rangle / (f_{j,n})^\dagger f_{j,n}(t) \rangle . \]

The \( t \downarrow 0 \) limit of \( G(x,y;t) \) then yields the sum rule
\[ i \sum_j \sum_{n=0}^{M_j-1} f_{j,M_j-1-n}(y) / (f_{j,M_j-1-n}, f_{j,n}) \left( f_{j,n}(x) / f_{j,n}(x) \right) = \left( \delta(x-y) \right) , \]
while in the same limit, (5.11) is indeed seen to be of the form (1.6).

Equation (5.6) is the generalization of (2.11) and (2.19), while our proof is a slight simplification even in the simple-pole case, cf. [15]. The representation (5.10) generalizes (2.8), and (5.11) extends (1.8) and (2.14). Also the simple-pole counterpart of (5.12) is already known [15].

One may ask to what extent the basis we have obtained is unique. On the one hand we demand that in our basis the Hamiltonian have the block form (4.5), and below this equation it has already been remarked that this forces the functions \( f_{j,n} \) to be of the form (4.3) for some normalization of \( g(x,\omega) \). On the other hand, to ensure the simplicity of formulas like (5.11), we require that the dual to \( |f_{j,n}\rangle \) be some \( F |f_{j',n'}\rangle \), where (1.0) then forces \( j' = j \). Now for any normalization of \( f \) and \( g \) (i.e., temporarily abandoning (1.2) and (5.2)) one has
\[ \langle f_{j,n}, g_{j,m} \rangle = -W_{j,M_j-1} \]
as long as \( n, m \leq M_j - 1 \) (implying that the bilinear map vanishes if \( n + m \leq M_j - 2 \), as can be proved by operating with \( \sum_{\ell=0}[(n-\ell)!(m+\ell+1)!]^{-1} \partial_\omega^{-\ell} \partial_{\omega'}^{m+\ell+1} |_{\omega=\omega'=-\omega_j} \) on the identity
\[(\omega^2 - \omega'p^2) \int_0^{a+} \rho f(\omega)g(\omega') \, dx = [i\omega'f(a,\omega) - f'(a^+,\omega)]g(a,\omega') + f'(0,\omega)g(0,\omega') \quad (5.14)\]

Hence, the product \((f_{j,n}, f_{j,M_j-1-n})\) is always nonvanishing, and the only way of achieving bi-orthogonality by setting other products to zero as in (5.6) is to normalize \(W\) as in (5.2), which obviously fixes the \(|f_{j,n}\rangle\) up to one overall prefactor per Jordan block. Hence, our choice of basis is unique up to these prefactors. In fact, further analogy to the customary treatment [24] of the simple-pole case results if one sets \(W_{j,M_j} = -2\omega_j\) for all \(j\), so that \((f_{j,n}, f_{k,m}) = 2\omega_j \delta_{j,k} \delta_n + m, M_j - 1\). With this preferential normalization, which is convenient in applications [43], the freedom discussed in this paragraph is reduced further to one overall sign per Jordan block.

In closing this account of the generalized duality transformation, let us return to the example of Section III B with \(M = 2\). Using either (4.1) or the integral representation [40], one can find the “preferred” second basis function (i.e., the one for which \((f_0, 1^1, f_0, 1^1) = 0\)) corresponding to \(f_0 = \hat{f}\) as in (3.5a) as

\[
\begin{align*}
f_{0,1}(x) &= f_a(x) + f_b(x) \
f_a(x) &= -iK \frac{\gamma}{3} x \cosh(Kx) \
f_b(x) &= -i \left( \frac{K}{\gamma} + \frac{1}{2K} \right) \tanh(Kx)
\end{align*}
\]

(5.15a)

(5.15b)

(5.15c)

where the conjugate momenta \(\hat{f}_a\) and \(\hat{f}_b\) are given by (4.4). While (5.15) is given here for reference and further use in Sec. VI, there does not seem to be a simple physical interpretation of this result. The normalization occurring in the field expansion (5.11) is evaluated to be

\[
\langle f^0|f_0 \rangle = \langle f^0, 1^1|f_0, 1^1 \rangle = (f_0, f_0) = K^3 \frac{1}{\gamma^2} \cotanh K. \quad (5.16)
\]

Since the contribution of \(f_b\) to the product (5.16) is \(\propto (f_0, f_0)\), it vanishes. However, in a calculation in Section VI C this term in \(f_{0,1}\) will be seen to be essential for arriving at the correct result.

VI. JORDAN-BLOCK PERTURBATION THEORY

A. Formalism for the generic case

The use of BBs places dissipative systems into a framework very similar to that for conservative systems. As a result, it is straightforward to develop time-independent perturbation theory, in effect by transcribing textbook results for conservative systems, which nevertheless apply to the complex eigenvalues and eigenvalue shifts [17,14].

It will presently be investigated how this formalism, previously developed only for QNM spectra with simple poles, is modified if at least one eigenvalue \(\omega_j\) is associated with a nontrivial Jordan block. The ensuing splitting of the multiple pole into \(M_j\) distinct ones (in the generic case, to be defined below) is reminiscent of the lifting of a degeneracy by a
perturbation (typically breaking some symmetry) in hermitian systems; however, important differences need to be pointed out. In the first place, in the models here under discussion an \( M_j \)th-order pole is associated with \( M_j \) degrees of freedom, of which only one is an eigenvector, as has been emphasized before \cite{[45]}. Secondly, in the Jordan-block case the splitting generically is governed by only one complex parameter, and as a result the \( M_j \) complex frequency shifts are not independent; in fact, all their relative magnitudes and phases are predetermined, and only the overall magnitude and phase depend on the details of the perturbation, namely on the one complex parameter; cf. (6.4). While this feature may seem unusual compared to conservative systems, it actually simplifies the calculation.

While Jordan-block perturbations thus differ essentially from perturbations of degenerate levels in conservative systems, we will study the former by a method which can also be used for the latter: transferring part of the perturbing NHH \( H' \) to the unperturbed \( H_0 \) and treating this part exactly, upon which the remainder of \( H' \) can be dealt with using conventional non-degenerate perturbation theory.

Specifically, let \( H = H_0 + \lambda H' \), where \( H_0 \) is assumed to have a known Jordan-block structure as described in Sections IV and V, \( H' \) accounts for a change in density \( \delta (\rho - 1) \), and where \( |\lambda| \ll 1 \). For simplicity of notation, it is supposed that there is only one \( M_j \geq 2 \); the generalization to several Jordan blocks is immediate. Consider the splitting part of \( H' \):

\[
H'_s \equiv \frac{|f_{j,M_j-1}\rangle \langle f_{j,M_j-1}|}{\langle f_{j,M_j-1}| f_{j,M_j-1}\rangle} H' \frac{|f_j\rangle \langle f_j|}{\langle f_j| f_j\rangle}.
\]

This has only one nonvanishing matrix element, viz.,

\[
\alpha \equiv \frac{\langle f_{j,M_j-1}| H' | f_j\rangle}{\langle f_{j,M_j-1}| f_{j,M_j-1}\rangle} = \frac{(f_j, H' f_j)}{(f_{j,M_j-1}, f_{j,M_j-1})} = \frac{\omega_j^2 f_{j,M_j-1}^+ \delta (\rho - 1) \rho^2 f_j^2 dx}{(f_{j,M_j-1}, f_j)}.
\]

We transfer this part to the unperturbed hamiltonian: \( \tilde{H}_0 \equiv H_0 + \lambda H'_s \), so that the remaining perturbation is \( \tilde{H}' = H' - H'_s \). The perturbation is said to be generic iff \( \alpha \neq 0 \). This definition of a generic perturbation will be justified below, by showing that for sufficiently small \( \lambda \) the matrix elements of \( \tilde{H}' \) effect only a higher-order correction compared to the splitting caused by \( \alpha \neq 0 \). First of all, however, it should be noted that the representation (6.3) for an infinitesimal \( \delta \rho \) is evaluated to read \( \alpha \propto \int_0^\infty \delta \rho f_j^2 dx \). Using the method of variation of the constant (cf. the inner integrand in [40]) and this latter representation, \( \alpha \neq 0 \) is seen to be equivalent to \( \partial_\lambda g(0, \omega_j) \propto \partial_\lambda W(\omega_j) \neq 0 \). In other words, if \( \alpha \neq 0 \) the \( M_j \)th-order zero in the wronskian at \( \omega = \omega_j \) is split already in lowest order in \( \lambda \).

The eigenvalue problem for \( \tilde{H}_0 \) can be solved in the \( M_j \times M_j \) block associated with \( \omega_j \). For the characteristic polynomial in this block one has \( \det(\tilde{H}_0 - \omega \mathbb{1}) = (\omega_j - \omega)^{M_j} - (-)^{M_j} \lambda \alpha \), yielding the eigenfrequencies as

\[
\tilde{\omega}_{j,n} = \omega_j + s e^{2\pi ni/M_j}.
\]
\( (n = 0, 1, \ldots, M_j - 1) \), where \( s = \sqrt[2]{\lambda} \) is an arbitrary but fixed choice of the root. Thus, the splittings \( \tilde{\omega}_{j,n} - \omega_j \) all have the same magnitude \( \propto |s| \propto \lambda^{1/M_j} \) and are equiangular, i.e., their phases have constant differences. Both their magnitude and the overall phase are determined by \( \alpha \). The corresponding eigenvectors of \( \tilde{H}_0 \) are

\[
|\tilde{f}_{j,n}\rangle = \sum_{m=0}^{M_j-1} s^m e^{2\pi nm/M_j} |f_{j,m}\rangle ;
\]

(6.5)
since the higher-order pole has been split into first-order ones, their duals read simply

\[
|\tilde{f}^{j,n}\rangle = \mathcal{F}|\tilde{f}_{j,n}\rangle .
\]

(6.6)

It remains to account for \( \tilde{H}' \), using conventional perturbation theory, by evaluating its matrix in the new basis. The validity of this procedure is not entirely trivial, since the transformation from the basis \( \{|f_{j,n}\rangle\} \) to \( \{|\tilde{f}_{j,n}\rangle\} \), effected by the matrix \( P_{mn} = s^m e^{2\pi nm/M_j} \), is singular in the limit \( \lambda \to 0 \). For a justification, denote the matrix elements of \( \tilde{H}' \) with respect to the old basis as \( \tilde{H}'_{mn} = \langle f_{j,n}|\tilde{H}'|f_{j,m}\rangle /\langle f_{j,n}|f_{j,n}\rangle \), and evaluate the inverse transformation (in fact a discrete Fourier inversion) as

\[
(P^{-1})_{mn} = M_j^{-1} s^{-m} e^{-2\pi nm/M_j} .
\]

In the basis which diagonalizes \( \tilde{H}_0 \), the perturbation is then given as

\[
\langle \tilde{f}_{j,n}|\tilde{H}'|\tilde{f}_{j,m}\rangle = \sum_{k,\ell=0}^{M_j-1} (P^{-1})_{nk} \tilde{H}'_{k\ell} P_{\ell m} = \sum_{k,\ell=0}^{M_j-1} \frac{1}{M_j} e^{2\pi (\ell m - nk)/M_j} \tilde{H}'_{k\ell} s^{\ell-k} ,
\]

(6.7)

where it is crucial that the power \( s^{1-M_j} \) does not occur since \( \tilde{H}'_{M_j-1,n} = 0 \) on account of [51]. Thus the matrix elements of \( \lambda \tilde{H}' \) are \( O(s^{M_j s^{2-M_j}}) \), which means that the first-order frequency shifts due to \( \tilde{H}' \) are \( O(s^2) \), small compared to the lowest-order splittings \( \Delta \tilde{\omega}_{j,n} \propto s \). With energy denominators given by \( \tilde{\omega}_{j,n} - \tilde{\omega}_{j,n'} \propto s \), higher-order shifts are smaller still by successive powers of \( s \). Of course, matrix elements of \( \tilde{H}' \) connecting the block associated with the unperturbed \( \omega_j \) to other blocks, or connecting two other blocks, can be handled without difficulty.

**B. Non-generic case**

If \( \alpha = 0 \) the leading behaviour is determined by other matrix elements, and the splitting of \( \omega_j \) can be partial or, depending on the scheme of calculation, occurs only in higher order [19]. We shall not investigate the general case, but instead give an example of a non-generic perturbation which is relevant to the discussion in Appendix A. Namely, if a 4 \( \times \) 4 Jordan block is perturbed by the operator

\[
H' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \end{pmatrix}
\]

(6.8)
(in the basis \( \{ |f_{j,n}\} \)), the characteristic polynomial is found to be \( \det(H - \omega \mathbb{1}) = [(\omega - \omega_j)^2 - \lambda \alpha]^2 \). Thus, the fourth-order pole is split into two second-order poles, and the latter do not undergo further splitting to any order. Of course, this treatment does not address the question whether for some \( \delta \rho \) the perturbation \( H' \) can have the form (6.8) for the open wave system (2.1), even if a fourth-order pole is assumed to exist. However, \( H' \) as in (6.8) at least satisfies the fundamental symmetry \( \langle f_{j,n} | H' | f_{j,m} \rangle = \langle f_{j,M_j-1-m} | H' | f_{j,M_j-1-n} \rangle \) (i.e., reflection symmetry with respect to the NE–SW diagonal within one block), which follows from (2.18) and (5.7).

C. Example

Returning to the example (3.5) of Section III B, and bearing in mind the NHH action on a two-component vector given by (2.5) [46], we study the perturbation \( \rho^{-1}(x) \mapsto \rho^{-1}(x) + \lambda \theta(1-x) \).

From its definition (6.2), and using (5.16) for the normalization, one obtains

\[
\alpha = \frac{\lambda}{2} [K \tanh K - \sinh^2 K].
\] (6.9)

That is, if \( \lambda > 0 \) the frequency shifts \( \pm \sqrt{\alpha} \) are purely imaginary, while for \( \lambda < 0 \) they are real and of opposite signs. In other words, as \( \lambda \) is turned from positive values through zero to negative values, the poles move together horizontally in the complex plane, merge, and then move apart vertically, in accordance with the general observation made at the end of Section III A.

Proceeding to \( \mathcal{O}(\lambda) \), conventional QNM perturbation theory gives the next contributions to the shift as \( \langle f_{0,0}^0 | H'_{00} | f_{0,0}^0 \rangle / \langle f_{0,0}^0 | f_{0,0}^0 \rangle \) and \( \langle f_{0,1}^0 | H'_{11} | f_{0,1}^0 \rangle / \langle f_{0,1}^0 | f_{0,1}^0 \rangle \) respectively [35]. By (6.7), both are evaluated as \( \frac{1}{2} (H'_{00} + H'_{11}) + \mathcal{O}(\lambda^{3/2}) = H'_{00} + \mathcal{O}(\lambda^{3/2}) \), where the last equality follows from the symmetry pointed out below (6.8). In the numerator of

\[
H'_{00} = \frac{\langle f_{0,1}^0, H' f_0^0 \rangle}{\langle f_{0,1}^0, f_0^0 \rangle},
\] (6.10)

the contribution of \( f_b \) as in (5.15c) (which does not contribute to the denominator in (6.10), cf. below (5.16)) is seen to be \( \propto \alpha \) upon comparison with (6.2). For both split levels the next-order shift thus reads

\[
H'_{00} = \frac{\langle f_0^0, H' f_0^0 \rangle}{\langle f_0^0, f_0^0 \rangle} - i \left( \frac{2 K}{3 \gamma} + \frac{1}{2K} \right) \tanh(K) \alpha,
\]

\[
= i \frac{\lambda}{\gamma} \left( \frac{K}{4} \tanh K - \frac{K \sinh^3 K}{6 \cosh K} - \frac{K^2}{4} - \frac{K^2}{12} \tanh^2 K \right)
\] (6.11)

and is seen to be purely imaginary, so that if the double-pole zero-mode is split along the imaginary axis in lowest order (i.e., \( \mathcal{O}(\lambda^{1/2}) \)) the perturbed QNMs stay on this axis up to \( \mathcal{O}(\lambda) \), again consistent with the symmetry argument of Section III A.

For a check, the perturbed QNMs can be also obtained directly from the wave equation (2.3) together with the boundary conditions. For \( \rho^{-1}(x) \) having a constant value \( (\rho')^{-1} = \gamma^2/K^2 + \lambda \) on \( 0 < x < a \), the eigenvalue equation is found to be
\[ i - \sqrt{\rho'} \cotan \left( \sqrt{\rho'} \omega \right) = -\omega \frac{K^2}{\gamma^2 \sinh^2 K}, \quad (6.12) \]

in which one has to expand \( \sqrt{\rho'} = K/\gamma - (K^3/2\gamma^3)\lambda + \mathcal{O}(\lambda^2) \) and \( \omega = -i\gamma + \omega_1 \sqrt{\lambda} + \omega_2 \lambda + \omega_3 \lambda^{3/2} + \mathcal{O}(\lambda^3) \). In \( \mathcal{O}(\lambda^0) \) and \( \mathcal{O}(\sqrt{\lambda}) \), (6.12) is satisfied identically. In \( \mathcal{O}(\lambda) \) one obtains \( (\omega_1 \sqrt{\lambda})^2 = \alpha \), with \( \alpha \) as in (6.9); in \( \mathcal{O}(\lambda^{3/2}) \) one obtains \( \omega_2 \lambda = H_{00}' \), with \( H_{00}' \) as in (6.11). Hence, there is complete agreement between the Jordan-block formalism of Section VI A and direct expansion of the wave equation.

Finally, an example of a non-generic perturbation is furnished by changing \( K \) to \( K' = K + \lambda \) in (3.5c), namely by \( \delta \rho(x) = \partial_K \rho(x) \), where in the differentiation of \( \rho \) its implicit \( K \)-dependence through \( \gamma \) as in (3.5b) must also be taken into account. In lowest order, \( H' \) shifts the double pole corresponding to \( K \) to a double pole corresponding to \( K' \), and indeed \( \int_1^0 \delta \rho f_{00}^2 dx \) is found to vanish, as stipulated below (6.3). Since beyond this leading order \( H_0(K) + \lambda H' \neq H_0(K') \), the double pole will be split eventually. In line with the treatment in Section VI A, however, this is not pursued further here.

**VII. CONCLUSION**

A remark is in place on the relevance of the issue considered in this paper. The above and especially Section VII make clear that the set of systems for which non-trivial Jordan blocks occur is of measure zero in parameter space. However, this feature is shared with, among others, stationary points in the phase space of dynamical systems, critical points in phase diagrams (note the semantic coincidence with “critical damping”), and degeneracies in conservative quantum systems, all of which are worthy of study and are known to determine the global structure of a system’s parameter space to a much greater extent than one would think at first sight. In the case of degenerate quantum levels, a further motivation is their relation to a system’s physical symmetries. While the corresponding phenomenon does not yet show up on the level of this paper, further investigation reveals that two states can merge in the superpartner of a spatially symmetric open wave system [47].

The existence questions raised in Section VI I and Appendix A are closely related to the problem of spectral inversion. In conservative systems, the classic inversion problem is to determine the system (e.g., \( \rho(x) \) for the wave equation or \( V(x) \) for the Klein–Gordon equation) given all the real eigenfrequencies \( \omega_j \). The solution to this classic problem is well known [13]. The analogous problem for open systems is to determine \( \rho(x) \) or \( V(x) \) from the complex eigenfrequencies \( \omega_j \), or, more generally, from the singularity structure of \( \tilde{G}(x, y; \omega) \) in the \( \omega \)-plane. If, for example, \( \tilde{G} \) is specified to have poles of order \( M_j \) (say, \( M_j = 4 \)) at \( \omega_j \), does a corresponding \( \rho(x) \) or \( V(x) \) exist (at least for one in a class of such singularity configurations)? Assuming the general inversion problem for open systems (a topic for further investigation) to be tractable, at this stage each of the following scenarios seem conceivable.

(a) The inversion algorithm indeed yields a \( \rho(x) \) with, say, a fourth-order pole or a pair of off-axis double poles in its spectrum.

(b) The inversion problem turns out to have no solutions, yielding a non-trivial proof of the non-existence of these more exotic configurations.
This particular set of singularities points to limitations in the inversion algorithm which otherwise might have been overlooked.

Any of these possibilities would further the understanding of QNMs in open wave systems.

Recently, for the case of simple poles we have carried out the second quantization of the open wave system (2.1) using QNMs \[36\]. The QNM expansion coefficients \(a_j\) (cf. (2.14)) emerge as the pertinent quantum degrees of freedom, in terms of which it is possible to eliminate the “outside” from the equations for the cavity evolution. This relevance to the quantum problem further motivates the study of the mode structure of (2.1), and second quantization in the case for which this structure involves nontrivial Jordan blocks indeed turns out to be possible either by Hilbert-space methods or by exactly solving the associated path integral \[13\].

In closing, it may be useful to place the present work into the following context. Many wave phenomena in nature can be described by an evolution equation \(i\partial_t|\phi\rangle = H|\phi\rangle\) (see above (2.5)) and a natural question is: what are the possible forms for \(H\)—leading to various types of time evolution for \(|\phi\rangle\)—and how are these exemplified in physical systems? The most familiar examples are conservative systems, for which \(|\phi\rangle\) is expandable in a complete basis of normal modes, in terms of which \(H\) would be diagonal with real eigenvalues. Our earlier work \[14,17\] shows another realization: in a large class of outgoing wave systems, \(|\phi\rangle\) is again expandable in a complete basis (of QNMs), in terms of which \(H\) would again be diagonal, but with complex eigenvalues. The QNMs, however, are not orthogonal under the standard inner product; therefore it is convenient to introduce their duals as well, together with which they constitute a BB. The present work has identified and studied a further generalization pertaining to such open wave systems, for which \(H\) is not diagonalizable. In these circumstances we have shown that a well-defined Jordan-block structure emerges, involving a nontrivial duality transformation.

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APPENDIX A: THIRD-ORDER POLES AND BEYOND

As stated in Section IIIA, at present it is not known whether higher-order poles can exist off the imaginary axis. The analogy to damped harmonic oscillators suggests that they can not, but this does not lead to a proof directly since the harmonic oscillator picture itself is contingent on the QNM spectrum being simple, with at most a double-pole zero-mode. In fact, already at the end of Section IIIB it has been remarked that the analogy is imperfect, and also intuition which is mainly based on the conservative and WKB limits could bias one against more exotic possibilities.

A possible strategy for looking for double poles off the imaginary axis is to first construct a fourth-order zero-mode, which upon applying a suitable non-generic perturbation could be split into a pair of such double poles, cf. Sec. VII. Thus returning to the case \(\omega = -i\gamma\), (5.13) shows that for a fourth-order pole one needs \((f_0, f_{0,1}) = (f_{0,1}, f_0) = 0\) \[35\]. Expanding \(f_{0,1}\)
by its integral representation \[40\], using \(1.4\) for its momentum, and eliminating \(\rho\) and \(\gamma\) by (3.1) and (3.3) respectively, one arrives at two functional equations for \(f_0 = f\) [49]:

\[
\gamma^2 W_{0,2} = 4 \int_0^1 \frac{dx}{f^2(x)} \left[ \int_0^x dy \ f''(y) f(y) \right]^2 - \int_0^1 (f')^2 dx = 0 \quad (A1)
\]

\[
i\gamma^3 W_{0,3} = 8 \int_0^1 dx f''(x) f(x) \left[ \int_0^1 \frac{dy}{f^2(y)} \int_0^y dz f''(z) f(z) \right]^2 - 4 \int_0^1 \frac{dx}{f^2(x)} \left[ \int_0^x dy f''(y) f(y) \right]^2 = 0, \quad (A2)
\]

where without loss of generality we have chosen \(\alpha = 1\), and where (A1) alone implies a third-order pole. Solutions to these equations are to be sought among the functions \(f\) satisfying conditions (a) below (3.3) and the inequality (3.4).

Up to now we have only investigated third-order poles. The strategy is to seek a function \(f\) satisfying (A1) and, once one is found, to construct the corresponding \(\rho(x)\) using (3.1). We have done so within the class of functions \(f(x) = x + \alpha x^n (n > 2)\) mentioned below (3.4) [50]. It is easy to see that \(W_{0,2} < 0\) both for small and for large \(\alpha\): (a) for small \(\alpha\), \(f'' \approx 0\) so that the second term in (A1) dominates; (b) for large \(\alpha\), one can neglect the linear term and hence find that \(\gamma^2 W_{0,2} = -\alpha^2 n^2 (4n - 3)/(2n - 1)^3 < 0\). However, there exist \(n\) for which \(W_{0,2}\) can become positive for \(\alpha\) in an interval \((\alpha_1, \alpha_2)\)—so that \(W_{0,2} = 0\) at \(\alpha_1\) and at \(\alpha_2\)—for example \((\alpha_1, \alpha_2) = (2.059, 3.8209)\) for \(n = 5\), and \((\alpha_1, \alpha_2) = (1.063096, 8.30908)\) for \(n = 6\). In both cases, the inequality (3.4) is violated at \(\alpha_1\) but satisfied at \(\alpha_2\), implying that third-order poles indeed do exist. Besides being a stepping-stone in the search for fourth-order (and hence off-axis) poles, this result in itself already justifies the general (i.e., not limited to \(M_j \leq 2\)) setup in Sections \[IV, V\].

**APPENDIX B: CONSTRUCTING DUAL BASES**

In Section \[V\], one is faced with the problem of calculating the basis dual to \(\{[f_{j,n}]\}_{n=0}^{M-1}\). There is a standard result for finite-dimensional spaces \[7\], which however applies only when the dual basis is constructed within the space spanned by the original one. Here we are concerned with the original basis in \(V = \mathcal{L}[\{[f_{j,n}]\}_{n=0}^{M-1}]\) (\(\mathcal{L}\) denotes the linear span), but with the dual basis in a different space \(W\). (Guided by the simple-pole case, we expect \(W = \mathcal{F}V\) with \(\mathcal{F}\) as in (2.13), but this property will not be used below.)

Therefore we are led to consider the following problem. Let \(V\) be an \(M\)-dimensional subspace of a Hilbert space with basis \(\{v_n\}_{n=0}^{M-1}\), and let \(W\) be another \(M\)-dimensional subspace of the same Hilbert space. Under what conditions will there be a dual basis \(\{w^n\}\) in \(W\), in the sense that \(\langle w^m | v_n \rangle = \delta^m_n\)? We claim that the necessary and sufficient condition is

\[
W \cap V^\perp = \{0\}, \quad (B1)
\]

where \(V^\perp\) is the orthogonal complement to \(V\). (For example, if the whole Hilbert space is 3-d, and if \(V\) is the \(x-y\) plane, then \(W\) must not contain the \(z\)-axis.)

For a proof, let \(\{\bar{w}_n\}\) be any basis for \(W\). Then the duality of \(\{w^n\}\) and \(\{v_n\}\) is equivalent to
\[
\begin{pmatrix}
    \langle \tilde{w}_0 | v_0 \rangle & \cdots & \langle \tilde{w}_0 | v_{M-1} \rangle \\
    \vdots & \ddots & \vdots \\
    \langle \tilde{w}_{M-1} | v_0 \rangle & \cdots & \langle \tilde{w}_{M-1} | v_{M-1} \rangle
\end{pmatrix}
\begin{pmatrix}
    w^0 \\
    \vdots \\
    w^{M-1}
\end{pmatrix}
= \begin{pmatrix}
    \tilde{w}_0 \\
    \vdots \\
    \tilde{w}_{M-1}
\end{pmatrix}.
\] (B2)

The necessary and sufficient condition is that the metric matrix on the LHS is nonsingular. Singularity of this matrix would mean that a nontrivial linear superposition \( w \) of the \( \tilde{w}_m \) (i.e., a nonzero vector in \( W \)) is perpendicular to all \( v_n \), i.e., that \( w \in V^\perp \). This simple calculation not only proves our assertion but, in any finite-dimensional space, also gives a constructive algorithm.

Returning to the system of outgoing waves under discussion, the contour-integral calculation of the main text now in effect shows that (B1) is indeed satisfied for \( V_j = \mathcal{L} \{ |f_{j,n} \rangle \}_{n=0}^{M_j-1} \) and \( W_j = \mathcal{F}V_j \), solves (B2) for this case, and extends this bi-orthogonalisation to the whole Hilbert space by showing that the latter equals \( \bigoplus_j V_j \), with (4.6) holding between different blocks.
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[11] Strictly speaking, this symmetry is violated if the normalization \( \langle f_j | f_j \rangle \) is not chosen real, as is for instance the case in [14]. However, this will cause no problems here.
[12] Since our NHH \( H \) will operate on function pairs defined on a finite interval (cf. below (2.5)), we do not need to consider the possibility of a non-eigenvector continuous spectrum.
[13] Strictly speaking, Jordan blocks refer to a specific way of representing these blocks; see, e.g., [11]. However, we shall use this term here simply to mean that \( H \) cannot be fully diagonalized.
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[25] See, e.g., J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964).
[26] For open systems, the order of the zero of $W$ coincides with the order of the pole in $\tilde{G}$ not only for generic $x, y$, but in fact for all $x, y > 0$. Namely, a node $f(x_0, \omega_j) = 0$ or $g(x_0, \omega_j) \propto f(x_0, \omega_j) = 0$ would imply a vanishing energy current at position $x_0$, which is incompatible with the dissipative nature of the QNMs. This observation is also relevant to the discussion in Section $V$.

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[29] Since the arguments of the bilinear map are always two-component vectors we avoid the pedantry of $(|\psi\rangle, |\chi\rangle)$. In some of our earlier papers, the bilinear map is called a generalized inner product or norm, and sometimes the notation $\langle \psi|\chi \rangle$ is adopted for it.

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[35] Following our conventions in previous papers [17, 36], QNMs on the negative imaginary $\omega$-axis will be termed zero-modes and labeled with $j = 0$ if necessary.

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[40] That the $|f_{j,n}\rangle$ are linearly independent of course does not mean that they are unrelated. In fact, by differentiating $D(\omega_j)f(x, \omega) = 0$ with respect to $\omega$ and solving the ensuing $D(\omega_j)f_{j,1} = -2\omega_j \rho f_j$ by variation of the constant, one finds (for $M \geq 2$) $f_{j,1}(x) = 2\omega_1 f_j(x) \int dy f_j^2(y) \int_0^y dz \rho(z) f_j^2(z)$, where the undetermined constant of integration corresponds to the freedom pointed out below (14.5). For $M \geq 3$, higher derivatives

\[ 24 \]
can be calculated in a similar fashion. However, the expressions in the main text are more transparent if the \( f_{j,n} \) are left unevaluated.

[41] V. A. Ilyin and E. G. Poznyak, *Linear Algebra* (Mir, Moscow, 1986).

[42] One has \( W_{j,n} = \sum_{m=0}^{n} f'_{j,n-m}(0)g_{j,M_j+m}(0) \), and for \( n \leq M_j - 1 \) this does not depend on the choice of \( g_{j,M_j+m} \). Namely, \( g(x,\omega) \to [1+ (\omega - \omega_j)M_jN(\omega)]g(x,\omega) \) yields \( g_{j,M_j+m} \to g_{j,M_j+m} + \sum_{\ell=0}^{m} N_{j,m-\ell}g_{j,\ell} \), with \( g_{j,\ell}(0) = 0 \) since \( \ell \leq M_j - 1 \). Thus (5.2) only depends on the normalization of \( f \) and \( g \) up to order \( M_j \), as stated in the main text. In this context the reader is also referred to (5.13) where, at the expense of a slightly more involved calculation, the derivative \( W_{j,n} \) (with \( n \leq 2M_j - 1 \)) is expressed in a form which manifestly involves only \( f_{j,m}, g_{j,m} \) for \( m \leq M_j - 1 \).

[43] A. Maassen van den Brink, preprint quant-ph/9905082, submitted to Phys. Rev. E.

[44] In [17], the duality transformation is not explicitly mentioned. Rather, bilinear maps \( (\psi, \chi) \) are used; these can be cast into the BB language by (2.16).

[45] Thus, we shall avoid the terminology “degeneracy” in this context.

[46] Strictly speaking, \( \rho^{-1}(x) \) is undefined if \( \rho \) contains a \( \delta \)-function as in (3.5c). However, this can always be dealt with as a limiting case of regular \( \rho \). To be sure, in the formulas of Section VI C no problems arise.

[47] P. T. Leung, A. Maassen van den Brink, W. M. Suen, C. W. Wong, and K. Young, preprint.

[48] G. Borg, Acta Math. 78, 1 (1946); N. Levinson, Math. Tidsskr. B 25, 24 (1949); C. P. Sun, K. Young, and J. Zou, to appear in J. Phys. A.

[49] In more systematic numerical work, one should circumvent the multiple integrals in (A1) and (A2), which as they stand are very time-consuming. For instance, \( p = 4 \int_0^1 dx f^{-2}(x) \left[ \int_0^x dy f''(y)f(y) \right]^2 \) is the solution \( p(1) \) of the system \( (p, q)' = (4q^2/f^2, f''f) \) with initial conditions \( (p(0), q(0)) = (0, 0) \).

[50] In this example, \( \rho(x) \to 0 \) if \( x \to 0 \) and \( n > 3 \), but this can always be remedied by adding a small positive \( x^3 \) term and adjusting \( \alpha \) accordingly.