Compton scattering in the Buchholz-Roberts framework of relativistic QED.

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Dedicated to the memory of John E. Roberts

Abstract

We consider a Haag-Kastler net in a positive energy representation, admitting massive Wigner particles and asymptotic fields of massless bosons. We show that states of the massive particles are always vacua of the massless asymptotic fields. Our argument is based on the Mean Ergodic Theorem in a certain extended Hilbert space. As an application of this result we construct the outgoing isometric wave operator for Compton scattering in QED in a class of representations recently proposed by Buchholz and Roberts. In the course of this analysis we use our new technique to further simplify scattering theory of massless bosons in the vacuum sector. A general discussion of the status of the infrared problem in the setting of Buchholz and Roberts is given.

1 Introduction

In general, the term infrared problems can be understood as complications in mathematical description of quantum systems encountered at large spatio-temporal scales. However, its conventional definition is more specific and refers to difficulties in scattering theory of such systems in the presence of long range forces and/or massless particles. The simplest and well understood example is Coulomb scattering in quantum mechanics which requires the Dollard modifications of the wave operators. Infrared problems in quantum electrodynamics (QED) still evade a satisfactory solution and constitute an active field of research in mathematical physics. Among many advances of recent years [MS15, He14, CFP07, BR14], a particularly radical proposal was put forward by Buchholz and Roberts in the setting of algebraic quantum field theory (AQFT) [BR14]. In essence, these authors suggest that after restricting attention to measurements in some future lightcone $V$, infrared problems should disappear. Buchholz and Roberts adopt the general point of view on infrared problems and illustrate

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their ideas by results on superselection structure of QED. However, conventional infrared problems, understood as complications in scattering theory, are not treated in their work. It is therefore an open question if the appealing ideas of Buchholz and Roberts are helpful for analysis of collision processes in QED. We give a partial answer in this work.

Infrared problems in QED can be traced back to the fact that the spacelike asymptotic flux of the electric field

\[ \phi(n) = \lim_{r \to \infty} r^2 n E(rn), \quad n \in S^2 \]  

commutes with all local observables \cite{Bu82}. Since this flux is an arbitrary function on the unit sphere \( S^2 \), restricted only by the Gauss Law, each value of the electric charge corresponds to uncountably many disjoint irreducible representations of the algebra of observables, which are of potential physical interest. This invalidates the standard Doplicher-Haag-Roberts (DHR) theory of superselection sectors. For non-zero charges none of these representations can be Poincaré covariant, since the existence of \( \phi \) is not consistent with unitary action of Lorentz transformations. For similar reasons, charged particles cannot have sharp masses \cite{Bu86}. This latter difficulty, called the infraparticle problem, invalidates the conventional Haag-Ruelle or Lehmann-Symanzik-Zimmermann (LSZ) scattering theory for electrically charged particles. In this situation a charged particle is a composite object involving a soft-photon cloud correlated with the particle’s velocity. The cloud is needed for the purpose of ‘fine-tuning the flux’, that is, keeping it constant along the time evolution \cite{Bu82}. Such infraparticles have in fact been constructed in concrete models of non-relativistic QED \cite{CFP07}.

The above discussion involves a tacit restriction to representations of the algebra of observables of QED in which the flux (1.1) exists. Buchholz and Roberts consider instead a class of representations in which this is not the case, i.e. the fluctuations of the electric field tend to infinity under large spacelike translations. Thinking heuristically, one way to achieve this is to include highly fluctuating background radiation, emitted in very distant past. Such radiation, which should not be confused with soft photon clouds mentioned above, will ‘blur the flux’, that is prevent the existence of the limit in (1.1). On the other hand, it is clear from Figure 1a and the Huygens principle that this background radiation will stay outside any future lightcone \( V \). Thus, inside \( V \) one can follow the usual DHR strategy to pass from the defining vacuum representation \( \iota \) of the algebra of observables \( \mathfrak{A} \) to an electrically charged positive energy representation \( \pi \). To this end, consider a pair of opposite charges in a hypercone \( C \subset V \), which is a region depicted in Figure 1a and defined precisely in Subsection 2.2. Next, transport one of the charges to lightlike infinity. As argued in \cite{BRL14}, this process of charge creation in \( C \) should be only weakly correlated with operations performed in the spacelike complement of \( C \) in \( V \), denoted \( C^c \). Therefore, the resulting charged representation \( \pi \) should satisfy the following property of hypercone localization

\[ \pi \upharpoonright \mathfrak{A}(C^c) \simeq \iota \upharpoonright \mathfrak{A}(C^c), \]  

where \( \simeq \) denotes unitary equivalence and \( \mathfrak{A}(C^c) \) is the algebra of all observables measurable in \( C^c \). Since \( C^c \subset V \), this property is consistent with high fluctuations of the electric field at spacelike infinity, blurring the flux (1.1). (See again Figure 1a). As
A hypercone localized representation. (b) (Non-)existence of asymptotic fields.

Figure 1. (a) A hypercone localized representation of QED is equivalent to the vacuum representation in the causal complement $C^c \subset V$ of any hypercone $C \subset V$. This condition is consistent with the presence of highly fluctuating background radiation emitted in distant past, which is needed to blur the flux $\phi$. (b) If the approximating sequence $[1, \infty) \ni t \mapsto \hat{A}_t$ of the outgoing asymptotic photon field is localized in $C^c$, the existence of the limit $\hat{A}^{\text{out}}$ can be inferred from the corresponding result in the vacuum representation [Bu77]. However, the incoming asymptotic field is not expected to exist, since its approximating sequence $[1, \infty) \ni t \mapsto \hat{A}_{-t}$ collides with the background radiation.

$\phi$ does not exist, we may require that $\pi$ is covariant under Poincaré transformations and that charged particles have sharp masses. We adopt these assumptions in this work and study their consequences.

The problem of verifying these assumptions in some concrete models of QED is outside the scope of this work. However, the above discussion reveals certain similarity of the Buchholz-Roberts ideas to the concept of infravacua [Bu82, p.59] [Ku98, Kr82]. We recall that such states result from adding to the vacuum a sufficiently strong background field. Infravacua were constructed in QED in the external current approximation by Kraus, Polley and Reents [KPR77]. We believe that a similar analysis in more realistic theories, e.g. translation invariant models of non-relativistic QED, could bring interesting new insights into the nature of electrically charged particles.

Results. Let us now give an outline of our results in somewhat simplified terms. As mentioned above, we consider a Haag-Kastler theory $(\mathfrak{A}, U)$ in a vacuum representation, given by the algebra of observables $\mathfrak{A} \subset B(H)$ and a unitary representation of the covering group of the Poincaré group $\tilde{P}_+ \ni \lambda \mapsto U(\lambda)$. We also consider a Poincaré covariant, positive energy representation $\pi$ of $\mathfrak{A}$, satisfying the property of hypercone localization (1.2), which gives rise to a new Haag-Kastler theory $(\hat{\mathfrak{A}}, \hat{U})$ on a Hilbert space $\hat{H}$. The vacuum representation is assumed to contain massless Wigner particles (‘photons’) and the representation $\pi$ to contain massive Wigner particles (‘electrons’).

That is, there is a subspace $h_{ph} \subset H$ on which $U$ acts as a representation of $\tilde{P}_+^\dagger$ with mass $m_{ph} = 0$ and, similarly, a subspace $\hat{h}_{el} \subset \hat{H}$ on which $\hat{U}$ acts as a representation of mass $m_{el} > 0$.

Our goal is to describe Compton scattering, i.e. collision processes involving one

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Footnote 1: Poincaré covariance is used in [BR14] at a technical level. The possibility of sharp masses of charged particles is only mentioned as a problem for future investigations.
electron and some finite number of photons. To be able to add photons to vectors \( \Psi_{\text{el}} \in \hat{\mathfrak{h}}_{\text{el}} \) describing one electron, we introduce asymptotic fields of photons via the LSZ prescription. For this purpose, let \( \hat{A} \in \hat{\mathfrak{a}} \) be a suitable local operator and \( \hat{A}(t, \mathbf{x}) := \hat{U}(t, \mathbf{x}) \hat{A}(t, \mathbf{x})^* \) its spacetime translations. Moreover, let \( f \) be a solution of the wave equation with compactly supported initial data, i.e.

\[
f(t, \mathbf{x}) = (2\pi)^{-3/2} \int d^3 p e^{i p \cdot \mathbf{x}} (e^{-i|p|t} \tilde{f}_P(p) + e^{i|p|t} \tilde{f}_n(p)),
\]

where \( \tilde{f}_P(p) = \tilde{f}_1(p) - i|p|\tilde{f}_2(p), \tilde{f}_n(p) = \tilde{f}_1(p) + i|p|\tilde{f}_2(p), f_1, f_2 \in C_0^{\infty}(\mathbb{R}^3) \), determine the positive and negative energy parts of \( f \). The asymptotic photon field approximants, given by

\[
\tilde{A}_t := \frac{1}{\ln t} \int_t^{t+\ln t} dt' \int d^3 x \hat{A}(t', \mathbf{x}) f(t', \mathbf{x}),
\]

finally, give rise to the asymptotic fields via

\[
\hat{A}_{\text{out}} := \lim_{t \to \infty} \tilde{A}_t.
\]

We shall show that these fields exist as strong limits on the domain \( D_{\hat{H}} \subset \hat{\mathcal{H}} \) of vectors of polynomially bounded energy and leave this domain invariant. The first step of the proof is inspired by [Bu82]. Namely, we decompose \( \tilde{A}_t \) into a finite number of terms \( \tilde{A}_{i,t}, i = 1, \ldots, N \), which are localized in causal complements of some hypercones \( C_i \). Then, we use the hypercone localization property (1.2) and the existence of asymptotic photon fields in the vacuum representation [Bu77] to obtain the limits \( \hat{A}_{i,\text{out}} \) on some domains \( D_i \). Finally, we use the energy bounds [Bu90, He14.1]

\[
\sup_{t \geq 1} \| \tilde{A}_t (1 + \hat{H})^{-1} \| < \infty,
\]

where \( \hat{H} \) is the Hamiltonian in representation \( \pi \), to obtain the limits \( \hat{A}_{\text{out}} \) on a common domain \( D_{\hat{H}} \) on which they can be added up to \( \hat{A}_{\text{out}} \). (Cf. Figure 1b).

Given \( \hat{A}_{\text{out}} \) we define asymptotic creation and annihilation operators as follows

\[
\hat{A}_{\text{out}^+} := \int d^4 x \hat{A}_{\text{out}}(x) \eta(x), \quad \hat{A}_{\text{out}^-} := (\hat{A}_{\text{out}^+})^*,
\]

where the Fourier transform \( \tilde{\eta} \in C_0^{\infty}(\mathbb{R}^4) \) of \( \eta \) is supported outside of the backward lightcone in energy-momentum space. Since \( x \mapsto \hat{A}_{\text{out}}(x) \) is a solution of the wave equation, this smearing operation restricts the energy transfer of \( \hat{A}_{\text{out}} \) to positive values. Summing up, vectors of the form

\[
\Psi_{\text{out}} := \hat{A}_{\text{out}^+} \ldots \hat{A}_{n,\text{out}^+} \Psi_{\text{el}}
\]

are natural candidates for Compton scattering states describing \( n \) photons and one electron.
These states can now be used to construct the outgoing wave operator

\[ W^{\text{out}} : \Gamma(h_{\text{ph}}) \otimes \hat{h}_{\text{el}} \to \hat{\mathcal{H}}, \]  

with \( \Gamma(h_{\text{ph}}) \) being the symmetric Fock space over \( h_{\text{ph}} \). \( W^{\text{out}} \) maps any configuration of one electron and \( n \) independent photons into the corresponding vector of the form (1.8). However, to show that \( W^{\text{out}} \) is well defined and isometric, two ingredients are needed. Firstly, the asymptotic creation and annihilation operators \( \hat{A}^{\text{out}}_{\pm} \) must satisfy the standard canonical commutation relations. This can be shown by adapting results from \( \text{Bu77, Bu82} \) to a new geometric situation. Secondly, single-electron states must play a role of vacua of the asymptotic photon fields, i.e.

\[ \hat{A}^{\text{out}}_{-} \Psi_{\text{el}} = 0. \]  

(1.10)

Our proof of this fact, which is the main new technical result of this paper, is outlined below in this introduction. This proof relies only on the Haag-Kastler postulates. In particular, the hypercone localization of \( \pi \) is not needed to show (1.10).

To formalize the idea that single-electron states are vacua of the asymptotic photon fields, we construct the corresponding Haag-Kastler theory \((\hat{\mathcal{A}}^{\text{out}}, \hat{U})\). More precisely, for any double cone \( \mathcal{O} \) we define the corresponding local algebra

\[ \hat{\mathcal{A}}^{\text{out}}(\mathcal{O}) := \{ e^{i \hat{A}^{\text{out}}} \mid \tilde{A}_{t} = (\tilde{A}_{t})^{*} \text{ for all } t \geq 1, \tilde{A}_{t} \in \hat{\mathcal{A}}(\mathcal{O}) \text{ for small } t \geq 1 \}'' \]  

(1.11)

This definition requires the self-adjointness of \( \hat{A}^{\text{out}} \) resulting from self-adjoint approximating sequences. We show this using the Nelson commutator theorem \( \text{RS2} \) with the energy bounds (1.6) as an input. Results from \( \text{Bu77} \), with the same input, yield Weyl relations for operators of the form \( e^{i \hat{A}^{\text{out}}} \). With this information at hand and relation (1.10) we verify that states of the form \( \omega_{\text{el}}(\cdot) := \langle \Psi_{\text{el}}, \cdot \Psi_{\text{el}} \rangle, \Psi_{\text{el}} \in \hat{h}_{\text{el}}, \| \Psi_{\text{el}} \| = 1 \), induce vacuum representations of \((\hat{\mathcal{A}}^{\text{out}}, \hat{U})\). We point out that the improvements of the energy bounds made in \( \text{He14.1} \) (lower powers of the resolvent of \( \hat{H} \) than in \( \text{Bu90} \)) are important for this part of our analysis.

Outline of the proof of (1.10). Let \( \hat{\mathcal{H}}_{c} \subset \hat{\mathcal{H}} \) be the subspace of vectors of bounded energy and \( B \) be almost local operator whose energy-momentum transfer is outside of the future lightcone (cf. Subsections 3.1, 3.2). Next, we define auxiliary maps \( a_{B} \) introduced in \( \text{DG14} \) by C. Gérard and one of the present authors, namely

\[ a_{B} : \hat{\mathcal{H}}_{c} \to \hat{\mathcal{H}} \otimes L^{2}(\mathbb{R}^{3}), \]  

(1.12)

\[ (a_{B} \Psi)(x) = B(x) \Psi. \]  

(1.13)

It is not obvious that the range of \( a_{B} \) is in \( \hat{\mathcal{H}} \otimes L^{2}(\mathbb{R}^{3}) \), but it follows from \( \text{Bu90, Lemma 2.2} \), restated as Lemma 3.3 below. It is easy to see that this map has the following property

\[ a_{B} \circ f(\hat{P}) = f(\hat{P} + D_{x}) \circ a_{B}, \]  

(1.14)

where \( f \) is a bounded Borel function, \( \hat{P} \) is the momentum operator, \( D_{x} = -i \nabla_{x} \) and we use the short-hand notation \( \hat{P} + D_{x} := \hat{P} \otimes 1_{L^{2}(\mathbb{R}^{3})} + 1_{\hat{\mathcal{H}}} \otimes D_{x} \).
Let \((\tilde{A}_t(\eta))^*, t \geq 1\), be the approximating sequence of the asymptotic annihilation operator \(A^{\text{out}^*}\) and put

\[
\tilde{A}_t(\eta) = \tilde{A}_{t,p}(\eta) + \tilde{A}_{t,n}(\eta),
\]

where \(\tilde{A}_{t,p/n}\) involve the positive and negative energy parts of the wave packet (1.3). For the positive energy part we have

\[
(\tilde{A}_{t,p}(\eta))^* \Psi_{\text{el}} = \frac{1}{\ln t} \int_t^{t+\ln t} dt' (1_{\tilde{H}} \otimes \langle f_p |)(e^{i\tilde{H}t'} \otimes e^{i|D_x|t'})a_B e^{-i\omega_{\text{el}}(\mathcal{P})t'} \Psi_{\text{el}}
\]

\[
= (1_{\tilde{H}} \otimes \langle f_p |) \frac{1}{\ln t} \int_t^{t+\ln t} dt' e^{i(\tilde{H} + |D_x| - \omega_{\text{el}}(\mathcal{P}+D_x))t'} a_B \Psi_{\text{el}},
\]

where \(B := A^*(\tilde{\eta})\), \(\omega_{\text{el}}(k) = \sqrt{k^2 + m_{\text{el}}^2}\) and the map \((1_{\tilde{H}} \otimes \langle f_p |): \tilde{H} \otimes L^2(\mathbb{R}^3) \to \tilde{H}\) acts according to

\[
(1_{\tilde{H}} \otimes \langle f_p |) \Phi = \int d^3 x f_p^\dagger(x) \Phi(x).
\]

Now we are in position to apply the Mean Ergodic Theorem in \(\tilde{H} \otimes L^2(\mathbb{R}^3)\), which gives

\[
\lim_{t \to \infty} (\tilde{A}_{t,p}(\eta))^* \Psi_{\text{el}} = (1_{\tilde{H}} \otimes \langle f_p |) F(\{0\}) a_B \Psi_{\text{el}},
\]

where \(F\) is the spectral measure of \(\tilde{H} + |D_x| - \omega_{\text{el}}(\mathcal{P}+D_x)\). Since \(\tilde{H}, \mathcal{P}, D_x\) commute, spectral calculus and covariance under Lorentz transformations can be used to show \(F(\{0\}) = 0\). The analysis of \((\tilde{A}_{t,n}(\eta))^* \Psi_{\text{el}}\) is analogous.

Apart from verifying (1.10), the technique described above serves as a tool to simplify scattering theory of massless bosons in the vacuum sector. In particular, the proof of the fact that \(A^{\text{out}^*}\) satisfy canonical commutation relations can now be accomplished via a Pohlmeyer argument, without referring to the quadratic decay of the vacuum correlations of local observables. Thus, with the a priori information from [Bu90, He14.1] and the present paper, collision theory for massless bosons can be developed in a way completely parallel to the fermionic case [Bu75].

Since the argument above does not rely on strict locality, it may also be useful outside of the Haag-Kastler setting, e.g. in theories satisfying some kind of asymptotic abelianess in spacelike directions. For example, it should help to remove Assumption 4 of [He14] and Assumption 3 of [DH15]. It might also find applications in scattering theory of quantum spin systems satisfying the Lieb-Robinson bounds [BDN14].

**Discussion.** Let us now turn to the status of the infrared problem in the Buchholz-Roberts setting of relativistic QED. It may seem that by an analogous construction as above one could obtain the incoming wave operator \(W^\text{in}\) as well and define the scattering matrix \(S\) of the Compton scattering process in the usual way, namely by putting \(S = (W^\text{out}^*)^* W^\text{in}\). Unfortunately, the situation is less satisfactory than that. As far as we can see, the incoming wave operator \(W^\text{in}\) is not available in a representation \(\pi\) which is hypercone localized in a future lightcone. While the hypercone
localization property (1.2) allows us to establish the existence of the outgoing asymptotic photon fields, as explained below formula (1.5), it is of no help for the incoming photon fields. This is due to the fact that the approximating sequences of the incoming asymptotic photon fields are localized in regions moving to infinity in negative lightlike directions. Heuristically speaking, such regions inevitably collide with the highly fluctuating background radiation, emitted in very distant past, which must be present in $\pi$ to prevent the existence of the flux (1.1). It is therefore reasonable to expect that also the incoming asymptotic photon fields are blurred by this radiation as depicted in Figure 1b. As a possible way out, one could consider a representation $\pi'$ hypercone localized in a backward lightcone in which by obvious modifications of our discussion $W^{\text{in}}$ exists but $W^{\text{out}}$ may not exist. However, the existence of the scattering matrix remains questionable, since there is no reason to expect that $\pi$ and $\pi'$ are unitarily equivalent.

Like in the conventional approach, in the absence of the scattering matrix one may try to construct inclusive collision cross sections. This idea, implemented in AQFT by Buchholz, Porrmann and Stein [BPS91], amounts in our situation to preparation of incoming states using asymptotic observables of the form

$$C_t := \int d^3x \, h(x/t)(B^*B)(t,x). \quad (1.19)$$

Here $h \in C^\infty_0(\mathbb{R}^3)$ is supported on velocities of the desired particle and $B$ is an almost-local observable whose energy-momentum transfer is outside of the future lightcone. Due to this latter property, which cannot be imposed on strictly local observables $\hat{A}$ appearing in the definition of asymptotic photon fields (1.4), $B$ is much less sensitive to the background radiation mentioned above. Thus, the tentative inclusive collision cross sections of the form

$$\lim_{t \to -\infty} \langle \Psi^{\text{out}}, C_{1,t} \ldots C_{\ell,t} \Psi^{\text{out}} \rangle \quad (1.20)$$

are likely to exist. Although available methods allow to control such limits only in massive theories [DG14], their extension to the case of sharp masses embedded in continuous spectrum is thinkable. Another strategy may be to consider limits (1.20) in the framework of algebraic perturbative QFT. As a matter of fact, (1.20) bears some similarity to expressions studied in the book of Steinmann [St, formula (16.38)].

**Summary.** Our paper is organized as follows. In Section 2 we discuss Haag-Kastler nets and their representations. Section 3 surveys various preliminary results, most of which concern the energy-momentum transfer of observables. In Section 4 we introduce the asymptotic photon fields approximants. We collect their representation-independent properties, e.g. the uniform energy bounds (1.6) and the decomposition into creation/annihilation operators. In that section we also give the proof of relation (1.10) which is our main technical result. In Section 5 we revisit scattering theory of photons in a vacuum representation. In Section 6 we combine information from all the earlier sections to construct the outgoing wave operator of Compton scattering in a hypercone localized representation and show that it is isometric and Poincaré covariant. In Section 7 we construct the Haag-Kastler net of asymptotic photon fields.
in a hypercone localized representation and show that any single-electron state induces a vacuum representation of this net. More technical aspects of our discussion are postponed to the appendices.

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2 Framework

2.1 Haag-Kastler nets

Let $M = \mathbb{R}^4$ be the Minkowski spacetime. We denote by $\mathcal{K}$ the family of double cones $\mathcal{O} \subset M$ ordered by inclusion and write $\mathcal{O}_c$ for the causal complement of $\mathcal{O}$ in $M$. Furthermore, let $\widetilde{\mathcal{P}}_+^+ = \mathbb{R}^4 \rtimes SL(2, \mathbb{C})$ denote the covering group of the proper ortochronous Poincaré group $\mathcal{P}_+^+$. Its elements $\lambda = (x, \tilde{\Lambda})$ act on $M$ via $\lambda y = \tilde{\Lambda}y + x$, where $\Lambda \in \mathcal{L}_c^+$ is the Lorentz transformation corresponding to $\tilde{\Lambda} \in SL(2, \mathbb{C})$.

**Definition 2.1.** We say that $\mathcal{K} \ni \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \subset B(\mathcal{H})$ is a Haag-Kastler net of von Neumann algebras if the following properties hold:

(a) (Isotony) $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$ for $\mathcal{O}_1 \subset \mathcal{O}_2$.

(b) (Locality) $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0$ for $\mathcal{O}_1 \subset \mathcal{O}_2_c$.

(c) (Covariance) There is a continuous unitary representation $U$ of $\widetilde{\mathcal{P}}_+^+$ such that

$$U(\lambda)\mathfrak{A}(\mathcal{O})U(\lambda)^* = \mathfrak{A}(\lambda \mathcal{O}) \quad \text{for} \quad \lambda \in \widetilde{\mathcal{P}}_+^+.$$  \hspace{1cm} (2.1)

(d) (Positivity of energy) The joint spectrum of the generators of translations, denoted $\text{Sp}(U \upharpoonright \mathbb{R}^4)$, is contained in the closed future lightcone $\overline{V}_+$.

A Haag-Kastler net will be denoted by $(\mathfrak{A}, U)$.

**Definition 2.2.** We say that a Haag-Kastler net describes Wigner particles of mass $m \geq 0$ if there is a subspace $\mathfrak{h} \subset \mathcal{H}$ on which $U(\lambda), \lambda \in \widetilde{\mathcal{P}}_+^+$, acts like a representation of mass $m$.

Further useful definitions are as follows. For any region $\mathcal{U} \subset M$ we set

$$\mathfrak{A}_{\text{loc}}(\mathcal{U}) := \bigcup_{\mathcal{O} \subset \mathcal{U}} \mathfrak{A}(\mathcal{O}) \quad \text{and} \quad \mathfrak{A}(\mathcal{U}) := \mathfrak{A}_{\text{loc}}(\mathcal{U})^\|.$$  \hspace{1cm} (2.2)

In particular, we refer to $\mathfrak{A}_{\text{loc}} := \mathfrak{A}_{\text{loc}}(M)$ as the algebra of strictly local operators and to $\mathfrak{A} := \mathfrak{A}(M)$ as the global algebra of the net. For the unitary representation of

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2Note the distinction between the causal complements in $M$ and $V$, which is indicated by lower respectively upper indices.
translations \( U \upharpoonright \mathbb{R}^4 \) we shall write \( U(x) = e^{(Hx^0 - Px)} \) and the joint spectral measure of the energy-momentum operators \((H, P)\) shall be denoted by \( E(\cdot)\). For translated observables \( A \in \mathfrak{A} \) the notations \( \alpha_x(A) := A(x) := U(x)AU(x)^* \) are used. Moreover, we define
\[
\mathfrak{A}_{\text{loc}, 0} := \{ A \in \mathfrak{A}_{\text{loc}} \mid x \mapsto A(x) \text{ smooth in norm} \}. \tag{2.3}
\]

### 2.2 Representations

Consider a Haag-Kastler net \((\mathfrak{A}, U)\) and let \( \pi : \mathfrak{A} \to B(\mathcal{H}_\pi) \) be a (unital) representation. We say that \( \pi \) is (Poincaré) covariant, if there exists a strongly continuous unitary representation \( U_\pi \) of \( \mathcal{P}_{1+}^\dagger \) on \( \mathcal{H}_\pi \) such that
\[
U_\pi(\lambda)\pi(A)U_\pi(\lambda)^* = \pi(U(\lambda)AU(\lambda)^*), \quad A \in \mathfrak{A}. \tag{2.4}
\]
Moreover, we say that \( \pi \) has positive energy if \( \text{Sp}(U_\pi \upharpoonright \mathbb{R}^4) \subset \nabla_+ \). It is easy to see that if \( \pi \) is a covariant, positive energy representation, then,
\[
\mathcal{O} \mapsto \mathfrak{A}_\pi(\mathcal{O}) := \pi(\mathfrak{A}(\mathcal{O}))'' \tag{2.5}
\]
is again a Haag-Kastler net which will be denoted \((\mathfrak{A}_\pi, U_\pi)\).

**Definition 2.3.** If \( \pi \) is an irreducible, covariant, positive energy representation and \( \mathcal{H}_\pi \) contains a unique (up to a phase) unit vector \( \Omega \), invariant under \( U_\pi \), then we say that \( \pi \) is a vacuum representation.

In order to proceed to charged representations, we choose an open future lightcone \( V \) and denote for any region \( U \subset V \) by \( U^c \) its causal complement in \( V \). Next, we define a class of regions in \( V \) which are called *hypercones* in [BR14]. We recall here briefly their definition referring to [BR14] for more details: Choose coordinates so that \( V = \{ x \in \mathbb{R}^4 \mid x_0 > \|x\| \} \) and fix a hyperboloid \( \mathcal{H}_\tau = \{ x \in V \mid x_0^2 = \sqrt{x^2 + \tau^2} \} \) for some \( \tau > 0 \). Project \( \mathcal{H}_\tau \) through the origin onto the plane \( x_0 = 1 \) so as to identify it with the open unit ball \( B \subset \mathbb{R}^3 \). This projection is the Beltrami-Klein model of hyperbolic geometry. Consider the family of (truncated) pointed convex Euclidean cones \( \mathcal{C} \) in \( B \) with elliptical bases. It gives rise to a Lorentz invariant family of hyperbolic cones \( \mathcal{C} = \mathcal{C}(\mathcal{K}) \) in \( \mathcal{H}_\tau \). A hypercone \( \mathcal{C} = \mathcal{C}(\mathcal{K}) \) is the causal completion of such \( \mathcal{C} \), i.e. \( \mathcal{C} = \mathcal{C}^{cc} \), and the family of all hypercones as described above is denoted by \( \mathcal{F}_V \). We recall that for \( \mathcal{K} \cap \mathcal{K}' = \emptyset \) we have that \( \mathcal{C}(\mathcal{K}) \) and \( \mathcal{C}(\mathcal{K}') \) are spacelike separated.

**Definition 2.4.** Let \((\mathfrak{A}, U)\) be a Haag-Kastler net in a vacuum representation and let \( \pi \) be a covariant positive energy representation. We say that \( \pi \) is hypercone localized if for any future lightcone \( V \) and \( \mathcal{C} \in \mathcal{F}_V \) there exists a unitary \( W_\mathcal{C} : \mathcal{H} \to \mathcal{H}_\pi \) such that
\[
\pi(A) = W_\mathcal{C}AW_\mathcal{C}^* \quad \text{for} \quad A \in \mathfrak{A}(\mathcal{C}^c). \tag{2.6}
\]

**Remark 2.5.** It is easy to see that the morphisms \( \sigma_{\mathcal{C}, \mathcal{M}} : \mathfrak{A} \to B(\mathcal{H}) \) from [BR14] are irreducible, hypercone localized representations.

Note that for any hypercone localized representation \( \pi \) and any \( \mathcal{O} \in \mathcal{K} \) we have \( \pi(\mathfrak{A}(\mathcal{O})) = \pi(\mathfrak{A}(\mathcal{O}))'' \) and, therefore, \( \pi(\mathfrak{A}) = \mathfrak{A}_\pi \). It is also easy to see that any hypercone localized representation is faithful.
3 Preliminaries

In this section we consider an arbitrary Haag-Kastler net \((\mathcal{A}, U)\).

3.1 Almost local operators

The following standard class of observables shall be of particular importance in Lemma 3.3 stated below.

**Definition 3.1.** An operator \(A \in \mathcal{A}\) is called almost local if there exists a family of local observables \(A_r\), localized in standard double cones \(O_r\) of radius \(r\) centered at zero, and for any \(n \in \mathbb{N}\) there is a constant \(C_n\) such that
\[
\|A - A_r\| \leq \frac{C_n}{r^n}.
\] (3.1)

Next, we define for any \(B \in \mathcal{A}\) the smeared operators
\[
B(g) := \left\{ \int d^3x B(x)g(x) \quad \text{for} \quad g \in S(\mathbb{R}^3), \right. \quad \int d^4x B(x)g(x) \quad \text{for} \quad g \in S(\mathbb{R}^4). \tag{3.2}
\]

Since local algebras are von Neumann, \(B(g) \in \mathcal{A}\). It is easy to see that for \(B \in \mathcal{A}_{\text{loc}}\) the operators \(B(g)\), as defined above, are almost local.

3.2 Arveson spectrum

**Definition 3.2.** For \(B \in \mathcal{A}\) we define the Arveson spectrum of \(B\) as the support of the Fourier transform of \(\mathbb{R}^4 \ni x \mapsto B(x)\), understood as an operator valued distribution. That is,
\[
\text{Sp}_{B\alpha} := \bigcup_{\Psi, \Phi \in \mathcal{H}} \text{supp} \langle \Psi, \hat{B}(\cdot)\Phi \rangle. \tag{3.3}
\]

This concept is useful due to the energy-momentum transfer relation which gives [Ar82]
\[
BE(\Delta)\mathcal{H} \subset \mathcal{H}(\Delta + \text{Sp}_{B\alpha}) \tag{3.4}
\]
for any Borel \(\Delta \subset \mathbb{R}^4\). For future reference, we also note the simple fact that
\[
\text{Sp}_{B(g)\alpha} \subset \text{supp} \hat{g}, \quad g \in S(\mathbb{R}^4), \tag{3.5}
\]
which allows to construct almost local observables whose Arveson spectrum is contained in a prescribed set. We refer to Appendix \[\Box\] for our conventions concerning the Fourier transform.
3.3 Energy bounds

In this subsection we recall results from [Bu90] and [He14.1] which are important for our analysis. They are proven by combination of the energy-momentum relation (3.4) and almost locality.

The following lemma enters into the proof of Proposition 3.5 stated below which is then used to obtain our main technical result, namely Theorem 4.6.

Lemma 3.3. [Bu90] Let \( B \in \mathfrak{A} \) be almost local and such that \( \text{Sp}_B \alpha \) is a compact set which does not intersect with \( \overline{V}_+ \). Then, for any compact \( \Delta \subset \mathbb{R}^4 \) there exists a constant \( c_\Delta \) such that for any compact \( K \subset \mathbb{R}^3 \)

\[
\| E(\Delta) \int_K d^3 x \left( B^* B \right)(x) E(\Delta) \| \leq c_\Delta.
\]

(3.6)

Next, we state a result which is at the basis of Proposition 4.2 stated below, giving information about the domains of asymptotic fields.

Proposition 3.4. [Bu90,He14.1] Let \( A \in \mathfrak{A}_{\text{loc},0} \) and \( n \in \mathbb{R}^4 \) be a unit future oriented timelike vector, i.e. \( n_0 = \sqrt{1 + n^2} \). Then, for any \( g \in S(\mathbb{R}^4) \)

\[
\| A((n_\mu \partial^\mu)^3 g)(1 + H)^{-1} \| \leq c \sup_{\ell=0,1} \| \partial^0 \tilde{g}_\ell \|_2,
\]

(3.7)

where the constant \( c \) is independent of \( g \).

First bounds of this type were proven in [Bu90]. The above variant can be inferred from [He14.1] as follows. Starting with Theorem 4 of this reference, one can replace operators \( \tilde{A}_k(p) := e^{-i k \pi/2} \theta(\pm p^0) | p^0 \rangle \langle k | \tilde{A}(p) \) with \( \tilde{A}_{k,n}(p) := e^{-i k \pi/2} \theta(\pm n_\mu p^\mu) | n_\mu p^\mu \rangle \langle k | \tilde{A}(p) \), where \( k > 0 \) and \( n \) is chosen as in Proposition 3.4. Then, formula (20) of [He14.1] gives (3.7). This coordinate frame independence was actually noticed and used in the proof of Theorem 5 (ii) of [He14.1].

3.4 Auxiliary maps \( a_B \)

In this subsection we recall some concepts and facts from [DG14]. For any \( B \in \mathfrak{A} \) we have the continuous map \( a_B : \mathcal{H} \to S'(\mathbb{R}^3, \mathcal{H}) \) given by

\[
(a_B \Psi)(x) = B(x) \Psi, \quad x \in \mathbb{R}^3.
\]

(3.8)

Its dual \( a_B^* : S(\mathbb{R}^3; \mathcal{H}) \to \mathcal{H} \) is given by

\[
a_B^* \Phi = \int d^3 x B^*(x) \Phi(x).
\]

(3.9)

We identify \( S'(\mathbb{R}^3, \mathcal{H}) = \mathcal{H} \otimes S'(\mathbb{R}^3) \) and define for \( g \in S(\mathbb{R}^3) \) the functionals \( (1_{\mathcal{H}} \otimes \langle \tilde{g} \rangle) : S'(\mathbb{R}^3, \mathcal{H}) \to \mathcal{H} \) by

\[
(1_{\mathcal{H}} \otimes \langle \tilde{g} \rangle) \Psi = \int d^3 x g(x) \Psi(x).
\]

(3.10)
Their adjoints are denoted by \((1_H \otimes |g\rangle)\). For future reference, we note the identities
\begin{align}
B(g) &= (1_H \otimes \langle g|) \circ a_B, \quad (3.11) \\
B^*(g) &= a_B^* \circ (1_H \otimes |g\rangle).
\end{align}
For \(B\) as in Lemma 3.3 the maps \(a_B\) have the following useful properties.

**Proposition 3.5.** [DG14] Let \(B\) be almost local and such that \(\text{Sp}_B\alpha\) is a compact set which does not intersect with \(\nabla_+\). Furthermore, let \(\Delta \subset \mathbb{R}^4\) be compact. Then:

\(a)\) \(a_B E(\Delta) : H \to H \otimes L^2(\mathbb{R}^3)\) is bounded.

\(b)\) \(a_B E(\Delta) \circ f(P) = f(P + D_x) \circ a_B E(\Delta)\) for any \(f \in L^\infty(\mathbb{R}^3)\).

Here we set \(D_x = -i\nabla_x\) and use the shorthand notation \(P + D_x\) for \(P \otimes 1 L^2(\mathbb{R}^3) + 1_H \otimes D_x\).

**Proof.** (\(a\)) follows from the identity
\[E(\Delta) a_B^* \circ a_B E(\Delta) = E(\Delta) \int d^3x (B^* B)(x) E(\Delta) \quad (3.13)\]
and Lemma 3.3. To verify (\(b\)), one first checks that
\[a_B E(\Delta) \circ e^{-iyP} = e^{-iy(P + D_x)} \circ a_B E(\Delta). \quad (3.14)\]
Then, the fact follows from properties of the Fourier transform and approximating arguments. \(\square\)

In view of Proposition 3.5, for any compact \(\Delta\) the identity
\[B(g) E(\Delta) = (1_H \otimes \langle g|) \circ a_B E(\Delta) \quad (3.15)\]
extends by continuity to \(g \in L^2(\mathbb{R}^3)\).

### 3.5 Action of Lorentz transformations on a sphere

We conclude this preliminary section with a brief consideration about Lorentz transformations which will be used to show Poincaré covariance of our constructions.

Given \(\Lambda \in \mathcal{L}_+^{\uparrow}\), we divide the expression \((\Lambda x) \uparrow = \Lambda^{\mu, \nu} x^\nu\) into its time and space parts, i.e.
\begin{align}
(\Lambda x)^0 &= \langle v_\Lambda \rangle x^0 + v_\Lambda x, \\
(\Lambda x)^i &= -v^{i}_\Lambda^{-1} x^0 + [\Lambda]^i_j x^j,
\end{align}
where we set \(v^i_\Lambda := \Lambda^{0, i}, [\Lambda]^i_j := \Lambda^i_j, i, j = 1,2,3\). Note that \(v^i_\Lambda^{-1} = -\Lambda^i_0\) and \(\Lambda^0_0 = \sqrt{1 + |v_\Lambda|^2} =: \langle v_\Lambda \rangle\). Now let \(S^2 = \{ n \in \mathbb{R}^3 \ | \ |n| = 1 \}\) be the unit sphere. For \(n \in S^2\) we get
\begin{align}
\Lambda(1, n) &= (\langle v_\Lambda \rangle + v_\Lambda n)(1, g_\Lambda(n)), \quad (3.18) \\
g_\Lambda(n) &= \frac{-v_\Lambda^{-1} + [\Lambda]n}{| -v_\Lambda^{-1} + [\Lambda]n|}.
\end{align}
since \( \Lambda(1, n) \) is a lightlike vector. Here \( g_\Lambda : S^2 \to S^2 \) is a family of diffeomorphisms of \( S^2 \), which in fact forms a representation of \( L^+_\uparrow \), i.e. \( g_{\Lambda_1} \circ g_{\Lambda_2} = g_{\Lambda_1 \Lambda_2} \). It is, moreover, continuous in the following sense

\[
\lim_{\Lambda \to I} \| g_\Lambda - g_I \|_\infty = 0. \tag{3.20}
\]

Finally, we recall that by using the multiplication rules of the Poincaré group, the spectrum condition and the Stone theorem one obtains invariance of the domain of any positive power of the Hamiltonian \( D(H^n) \) under the action of \( U(\lambda), \lambda \in \tilde{P}_\uparrow \). Setting \( P := (H, P) \), the standard relation follows

\[
U(\tilde{\Lambda}) P^\mu U(\tilde{\Lambda})^* = (\Lambda^{-1})^\mu_{\nu} P^\nu, \tag{3.21}
\]

in the sense of operators on \( D(H^n) \). With the above definitions, we obtain invariance of the domain of any positive power of the Hamiltonian \( D(H^n) \) under the action of \( U(\tilde{\Lambda}) \).

Denoting by \( P_{\text{ph}} \) the projection on the single-photon subspace \( h_{\text{ph}} \), it further follows for any \( f \in C^\infty(S^2) \) that

\[
U(\tilde{\Lambda}) f \left( \frac{P}{|P|} \right) U(\tilde{\Lambda})^* P_{\text{ph}} = (f \circ g_{\Lambda^{-1}}) \left( \frac{P}{|P|} \right) P_{\text{ph}}. \tag{3.24}
\]

### 4 Asymptotic photon fields

In this section the pair \( (\mathfrak{A}, U) \) still refers to an arbitrary Haag-Kastler net.

#### 4.1 Spherical means

For further purposes we introduce the following Poincaré invariant subset of \( S(\mathbb{R}^4) \)

\[
S_s(\mathbb{R}^4) := \{(n_\mu \partial^\mu)^5 g \mid g \in S(\mathbb{R}^4), \ n_0 = \sqrt{1+n^2}\}, \tag{4.1}
\]

as well as certain Poincaré invariant subsets of \( \mathfrak{A} \), namely

\[
\mathfrak{A}_{S_s} := \{ B(g) \mid B \in \mathfrak{A}_{\text{loc},0}, \ g \in S_s(\mathbb{R}^4) \}, \tag{4.2}
\]

\[
\mathfrak{A}^{S_s} := \text{Span } \mathfrak{A}_{S_s}, \tag{4.3}
\]

where \( \text{Span} \) denotes finite linear combinations. For any \( A \in \mathfrak{A}^{S_s}, f \in C^\infty(S^2) \), we set as in [Bu82]

\[
A_t\{f\} := -2 t \int d\omega(n) f(n) \partial_0 A(t, tn).
\]

Here \( d\omega(n) = \frac{\sin \nu d\nu d\phi}{4\pi} \) is the normalized, invariant measure on \( S^2 \) and \( \partial_0 A := \partial_s(e^{isH} A e^{-isH})|_{s=0} \). In order to improve the convergence in the limit of large \( t \), we proceed to time averages of \( A_t\{f\} \), namely

\[
\bar{A}_t\{f\} := \int dt' h_t(t') A_{t'}\{f\}, \tag{4.5}
\]
where for non-negative \( h \in C^\infty_0(\mathbb{R}) \), supported in the interval \([-1, 1]\) and normalized so that \( \int dt \, h(t) = 1 \), we set \( h_t(t') = t^{-\varepsilon} h(t^{-\varepsilon}(t' - t)) \) with \( t \geq 1 \) and \( 0 < \varepsilon < 1 \).

For the discussion of asymptotic creation and annihilation operators in Subsection 4.4, we need \((n_\mu \partial^\mu)^5\) in (4.1), rather than just \((n_\mu \partial^\mu)^3\) from the energy bound (3.7). For the same purpose, it is important to use Schwartz class functions in (4.1). Since strict locality plays a crucial role in the later part of this paper, we also define the following sets

\[
C_s(\mathbb{R}^4) := \{(n_\mu \partial^\mu)^5 g \mid g \in C^\infty_0(\mathbb{R}^4), \ n_0 = \sqrt{1 + n^2} \} \subset S_s(\mathbb{R}^4), \tag{4.6}
\]
\[
\mathfrak{A}_{C_\ast} := \{ B(g) \mid B \in \mathfrak{A}_{\text{loc},0}, \ g \in C_\ast(\mathbb{R}^4) \} \subset \mathfrak{A}_{C_\ast} \cap \mathfrak{A}_{\text{loc},0}, \tag{4.7}
\]
\[
\mathfrak{A}_C := \text{Span} \mathfrak{A}_{C_\ast}, \tag{4.8}
\]
\[
\mathfrak{A}_{C_\ast}(\mathcal{O}) := \mathfrak{A}_{C_\ast} \cap \mathfrak{A}(\mathcal{O}), \quad \mathfrak{A}_C(\mathcal{O}) := \mathfrak{A}_C \cap \mathfrak{A}(\mathcal{O}), \quad \mathcal{O} \in \mathcal{K}. \tag{4.9}
\]

The linear structure of \( \mathfrak{A}_{C_\ast} \) and \( \mathfrak{A}_C \) will be important in Section 7.

### 4.2 Fourier space representation

Given \( A \in \mathfrak{A}_{C_\ast} \), a convenient representation for \( A_t\{f\} \) can be found, which will be frequently used in the remaining part of this section and in the proof of Lemma 5.1 below. This representation is stated in the following lemma.

**Lemma 4.1.** Let \( A \in \mathfrak{A}_{C_\ast} \), i.e. \( A = B(g) \), where \( B \in \mathfrak{A}_{\text{loc},0} \) and \( g \in S_\ast(\mathbb{R}^4) \). Then,

\[
A_t\{f\} = (\partial_0 B)(g \ast_3 f_t)(t), \quad \text{where} \quad f_t(x) := -\frac{1}{4\pi} \frac{2}{|x|} \delta(t - |x|) f \left( \frac{x}{|x|} \right), \tag{4.10}
\]

where \( \ast_3 \) is defined in Appendix E. Moreover, the Fourier transform of \( g \ast_3 f_t \in S(\mathbb{R}^4) \) has the following form

\[
(\tilde{g} \ast_3 \tilde{f}_t)(p) = \frac{\tilde{g}(p)}{|p|} \left( f \left( \frac{p}{|p|} \right) e^{-ip|p|} - f \left( -\frac{p}{|p|} \right) e^{ip|p|} + \int_0^\pi d\nu F(p, \nu) e^{-ip|p| \cos \nu} \right), \tag{4.11}
\]

where \( F \) is a bounded measurable function depending on \( f \). (In particular, \( F = 0 \) if \( f = \text{const.} \))

**Proof.** The equality \( A_t\{f\} = \partial_0 B(g \ast_3 f_t)(t) \), with \( f_t \) given by (4.10), is straightforward to check. Since \( (g \ast_3 f_t)(p) = (2\pi)^{3/2}\tilde{g}(p)\tilde{f}_t(p) \), it remains to compute

\[
\tilde{f}_t(p) = (2\pi)^{-3/2} \int d^3 x \, e^{-ipx} f_t(x)
\]
\[
= -\frac{2}{(2\pi)^{3/2}} \int d^3 x \, e^{-ipx} \frac{1}{4\pi} \frac{1}{|x|} \delta(t - |x|) f \left( \frac{x}{|x|} \right)
\]
\[
= \frac{-2t}{(2\pi)^{3/2}} \int d\omega(n) \, e^{-intp} f(n). \tag{4.12}
\]

A coordinate independent treatment of Fourier transforms on the sphere can be found in [DH15]. We give here an elementary coordinate dependent computation. To this
Proposition 3.4 yields

\[
\bar{f}_t(p) = \frac{-i}{(2\pi)^{5/2}} \int_0^{2\pi} d\varphi \int_0^{\pi} d\nu \sin \nu e^{-it|p|\cos \nu} f(R_p n(\varphi, \nu))
\]

\[
= \frac{-1}{(2\pi)^{5/2}|p|} \int_0^{2\pi} d\varphi \int_0^{\pi} d\nu f(R_p n(\varphi, \nu)) \partial_\varphi e^{-it|p|\cos \nu}.
\] (4.13)

Finally, integrating by parts and noting that \( n(\varphi, 0) = e_3, n(\varphi, \pi) = -e_3 \), we arrive at

\[
\bar{f}_t(p) = \frac{-1}{(2\pi)^{5/2}|p|} \int_0^{2\pi} d\varphi \left( f(R_p n(\varphi, \pi))e^{it|p|} - f(R_p n(\varphi, 0))e^{-it|p|} \right)
\]

\[
+ \frac{1}{(2\pi)^{5/2}|p|} \int_0^{2\pi} d\varphi \int_0^{\pi} d\nu \partial_\varphi f(R_p n(\varphi, \nu))e^{-it|p|\cos \nu}
\]

\[
= -\frac{(2\pi)}{(2\pi)^{5/2}|p|} \left( f \left( \frac{p}{|p|} \right) e^{it|p|} - f \left( \frac{-p}{|p|} \right) e^{-it|p|} \right)
\]

\[
+ \frac{1}{(2\pi)^{5/2}|p|} \int_0^{2\pi} d\varphi \int_0^{\pi} d\nu \partial_\varphi f(R_p n(\varphi, \nu))e^{-it|p|\cos \nu}.
\] (4.14)

This completes the proof. \(\square\)

### 4.3 Uniform energy bounds

This subsection is concerned with uniform bounds on \( t \mapsto \bar{A}_t \{f\} \). The following result holds.

**Proposition 4.2.** Let \( A \in \mathfrak{A}_{S^*}, \) i.e. \( A = B((n_\mu \partial_\mu)^5g'), B \in \mathfrak{A}_{loc,0} \) and \( g' \in S(\mathbb{R}^4) \). Then,

\[
\sup_{t \in [1, \infty)} \| \bar{A}_t \{f\}(1 + H)^{-1} \| \leq c \sup_{\ell = 0, 1} \|p|^{-\ell} \partial_0^\ell \left( (n_\mu \partial_\mu)^2 \tilde{g}' \right) \|_2 < \infty. \] (4.15)

The constant \( c \) above is independent of \( g' \).

**Proof.** By Lemma 4.1 we have that \( A_t \{f\} = (\partial_0 B)(g *_3 f_t)(t) \), where \( g = (n_\mu \partial_\mu)^5 g' \) and \( g' \in S(\mathbb{R}^4) \). Therefore, \( g *_3 f_t = (n_\mu \partial_\mu)^5 (g' *_3 f_t) \), with \( g' *_3 f_t \in S(\mathbb{R}^4) \). Thus, Proposition 3.4 yields

\[
\| \bar{A}_t \{f\}(1 + H)^{-1} \| \leq c \sup_{\ell = 0, 1} \| \partial_0^\ell \left( (n_\mu \partial_\mu)^2 \tilde{g}' \right) \|_2, \] (4.16)

where \( c \) is independent of \( t \). Now by formula (4.11) we have

\[
\left| \partial_0^\ell ((n_\mu \partial_\mu)^2 \tilde{g}' *_3 f_t)(p) \right| \leq c' |p|^{-\ell} \| \partial_0^\ell \left( (n_\mu \partial_\mu)^2 \tilde{g}' \right) \|,
\] (4.17)

where \( c' \) is independent of \( p, t \) and \( g' \). This completes the proof. \(\square\)
In the following we shall be interested in the convergence of $\bar{A}_t \{f\}, A \in \mathfrak{A}^S$, to a limit $A^\text{out} \{f\}$ as $t \to \infty$. To start with, we define $A^\text{out} \{f\}$ as an operator on the domain

$$D_{\text{max}}(A, f) := \{\Psi \in \mathcal{H} \mid A^\text{out} \{f\} \Psi := \lim_{t \to \infty} \bar{A}_t \{f\} \Psi \text{ exists}\}. \quad (4.18)$$

(For $f \equiv 1$ we will abbreviate $D_{\text{max}}(A, f)$ by $D_{\text{max}}(A)$). Note that $D_{\text{max}}(A, f)$ may depend on $A, f$, may not be Poincaré invariant and a priori may even be trivial. Another domain we shall be interested in is

$$D_H := \bigcap_{n \geq 1} D(H^n), \quad (4.19)$$

where $D(H^n)$ is the domain of self-adjointness of the $n$-th power of the Hamiltonian $H$. It is easy to see that $D_H$ is dense and Poincaré invariant. The next result can be inferred from Proposition 4.2 and the discussion in Appendix B.

**Proposition 4.3.** Let $i = 1, \ldots, n$ and suppose that the domains $D_{\text{max}}(A_i, f_i)$ and $D_{\text{max}}(A_i^*, \bar{f}_i)$ are dense. Then, we have

(a) $D_H \subset D_{\text{max}}(A_i, f_i), D_H \subset D_{\text{max}}(A_i^*, \bar{f}_i)$,

(b) $A_i^\text{out} \{f_i\} D_H \subset D_H$,

(c) $A_i^\text{out} \{f_1\} \ldots A_n^\text{out} \{f_n\} \Psi = \lim_{t \to \infty} \bar{A}_{1,t} \{f_1\} \ldots \bar{A}_{n,t} \{f_n\} \Psi$ for $\Psi \in D_H$.

The operators $A_i^\text{out} \{f_i\} \mid D_H$ are closable and uniquely determined by the values of $A_i^\text{out} \{f_i\}$ on any dense subspace of $D_{\text{max}}(A_i, f_i)$.

### 4.4 Asymptotic creation/annihilation operators

Another consequence of the uniform bounds is the existence of asymptotic creation and annihilation operators under the assumptions of Proposition 4.3. In fact a similar observation was made in [DH15].

To construct these operators we proceed as follows. Let $\theta \in C^\infty(\mathbb{R}), 0 \leq \theta \leq 1$, be supported in $(0, \infty)$ and equal to one on $(1, \infty)$. Moreover, let $\beta \in C^\infty_0(\mathbb{R}^4)$, $0 \leq \beta \leq 1$, be equal to one in some neighbourhood of zero and satisfy $\beta(-p) = \beta(p)$. Furthermore, for a parameter $1 \leq r < \infty$ and a future oriented timelike unit vector $n$ we define

$$\tilde{\eta}_{\pm, r}(p) := \theta(\pm r(n_\mu p^\mu)) \beta(r^{-1} p). \quad (4.20)$$

As $r \to \infty$ these functions approximate the characteristic functions of the positive/negative energy half planes $\{ p \in \mathbb{R}^4 \mid \pm n_\mu p^\mu \geq 0 \}$. We also have $\tilde{\eta}_{\pm, r} = \eta_{\pm, r}$. Note that the family of functions $\eta_{\pm, r}$, as specified above, is invariant under Lorentz transformations.

**Proposition 4.4.** Let $A \in \mathfrak{A}_s, f \in C^\infty(S^2)$. Suppose that $D_{\text{max}}(A, f), D_{\text{max}}(A^*, \bar{f})$ are dense and the timelike unit vectors $n$ entering the definition of $A$ and of $\eta_{\pm, r}$ coincide. Then:
(a) The limits \( A^{\text{out}}\{f\}^{\pm}\Psi := \lim_{r \to \infty} A^{\text{out}}\{f\}(\eta_{\pm,r})\Psi, \ \Psi \in D_H, \) exist and define the creation and annihilation parts of \( A^{\text{out}}\{f\} \) as operators on \( D_H. \) \( A^{\text{out}}\{f\}^{\pm} \) do not depend on the choice of functions \( \theta \) and \( \beta \) in (4.20) within the specified restrictions.

(b) \( (A^{\text{out}}\{f\})^{\pm} \uparrow D_H = A^{*\text{out}}\{\tilde{f}\}^{\mp}. \) In particular, \( A^{\text{out}}\{f\}^{\pm} \) are closable operators.

(c) \( A^{\text{out}}\{f\} = A^{\text{out}}\{f\}^{+} + A^{\text{out}}\{f\}^{-} \) on \( D_H. \)

(d) \( A^{\text{out}}\{f\} = A^{\text{out}}\{f\}^{+} + A^{\text{out}}\{f\}^{-} \) on \( D_H. \)

Remark 4.5. The proposition can be generalized to \( A \in \mathfrak{A}^{S_r} \) as follows. Consider a decomposition \( A = \sum_{i=1}^{\ell} A_i, A_i \in \mathfrak{A}^{S_{r_i}}, \) and assume that \( D_{\text{max}}(A_i, f) \) and \( D_{\text{max}}(A_i^{*}, \tilde{f}) \) are dense. Define \( A^{\text{out}}\{f\}^{\pm} := \sum_{i=1}^{\ell} A_i^{\text{out}}\{f\}^{\pm} \) on \( D_H. \) Then it is easy to see that \( A^{\text{out}}\{f\}^{\pm} \) satisfy the properties (b), (c) and (d) of the proposition.

Proof. (a) Making use of Propositions 4.2 and 4.3 we compute for \( 1 \leq r_1 \leq r_2 \) and \( \Psi \in D_H, \) that

\[
\| A^{\text{out}}\{f\}(\eta_{\pm,r_1} - \eta_{\pm,r_2})\Psi \| = \lim_{t \to \infty} \| A_t\{f\}(\eta_{\pm,r_1} - \eta_{\pm,r_2})\Psi \| \\
\leq c \sup_{\ell = 0,1} \| p^{-1} \partial_0^\ell ((n_\mu p^\mu)^2(\tilde{\eta}_{\pm,r_1} - \tilde{\eta}_{\pm,r_2})(p)\tilde{g}') \|_2 \\
\leq c \sup_{\ell = 0,1} \int_{r_1}^{r_2} dr \| p^{-1} \partial_0^\ell ((n_\mu p^\mu)^2(\partial_r \tilde{\eta}_{\pm,r}(p)\tilde{g}')) \|_2, \quad (4.21)
\]

where \( \tilde{g}' \in S(\mathbb{R}^4) \) is defined as in Proposition 4.2 and the functions of \( p \) appearing in (4.21) are to be understood as multiplication operators acting on \( \tilde{g}' \). Using the fact that \( \partial \theta \) is compactly supported, and therefore \( |n_\mu p^\mu| \leq cr^{-1} \) when multiplied by \( \partial \theta(\pm r(n_\mu p^\mu)) \), it is easy to check that

\[
|\partial_0^\ell ((n_\mu p^\mu)^2\partial_r \tilde{\eta}_{\pm,r}(p))| \leq \frac{c}{r^2} (1 + |p|^3), \quad \ell = 0, 1, \quad (4.22)
\]

for \( c \) independent of \( p \) and \( r \). This completes the proof of convergence. Independence of the choice of the functions \( \theta \) and \( \beta \) is shown by a similar computation.

(b) We note that for \( \Phi, \Psi \in D_H \)

\[
\langle \Phi, A^{\text{out}}\{f\}^{\pm}\Psi \rangle = \lim_{r \to \infty} \langle \Phi, A^{\text{out}}\{f\}(\eta_{\pm,r})\Psi \rangle \\
= \lim_{r \to \infty} \langle A^{\text{out}}\{\tilde{f}\}(\eta_{\pm,r})\Phi, \Psi \rangle = \langle A^{*\text{out}}\{\tilde{f}\}^{\mp}\Phi, \Psi \rangle. \quad (4.23)
\]

(c) It suffices to set \( A_t := A_t\{f\}(\eta_{\pm,r}) \) in formula (B.3) and take first the limit \( t \to \infty \) and then \( r \to \infty. \)

(d) We choose a function \( \gamma \in C_0^\infty(\mathbb{R}), 0 \leq \gamma \leq 1, \) such that

\[
\theta(-k) + \gamma(k) + \theta(k) = 1, \quad k \in \mathbb{R}, \quad (4.24)
\]

and set \( \tilde{\eta}_r(p) := \gamma_r(r(n_\mu p^\mu))\beta(r^{-1}p). \) Since \( \gamma \) is compactly supported, we have for \( \Psi \in D_H \)

\[
\| A^{\text{out}}\{f\}(\eta_r)\Psi \| \leq c \sup_{\ell = 0,1} \| p^{-1} \partial_0^\ell ((n_\mu p^\mu)^2\tilde{\eta}_r(p)\tilde{g}') \|_2 \leq c' r^{-1}. \quad (4.25)
\]

Hence, \( \lim_{r \to \infty} A^{\text{out}}\{f\}(\eta_r)\Psi = 0, \) which completes the proof. \( \Box \)
4.5  Asymptotic vacuum structure

In this subsection we state and prove our main technical result which is Theorem 4.6 below. This theorem will be useful in the proof of the tensor product structure of scattering states in the vacuum representation (part (a)) and in charged representations (part (b)).

**Theorem 4.6.** Let $\eta \in S(\mathbb{R}^4)$ be such that $\tilde{\eta}$ is supported outside of $\nabla_+$. Let $A \in \mathfrak{A}_S$ and $f \in C^\infty(S^2)$. Then, for $\Psi \in E(H_m)\mathcal{H}\cap D_H$ and $H_m = \{ p \in \mathbb{R}^4 | p^0 = \sqrt{p^2 + m^2}\}$, we have:

(a) For $m = 0$, $\lim_{t \to \infty} (1 - E(\{0\})) \tilde{A}_t\{f\}(\eta) \Psi = 0$.

(b) For $m > 0$, $\lim_{t \to \infty} \tilde{A}_t\{f\}(\eta) \Psi = 0$.

**Proof.** To begin with, we assume that $\tilde{\eta}$ is compactly supported and $\Psi = E(\Delta)\Psi$ for some compact $\Delta$. Making use of Lemma 4.1, we have $A_t\{f\}(\eta) = (\partial_0 B(\eta))(g \ast_3 f_t)(t)$, where

$$
(g \ast_3 f_t)(x) = (2\pi)^{-2} \int_0^\pi d\mu(\nu) \int d^4p \, \tilde{f}_\nu(p) e^{-ipx} e^{-i\cos \nu |p| t}. \tag{4.26}
$$

Here $d\mu(\nu) := d\nu + \delta(\nu)d\nu + \delta(\nu - \pi)d\nu$, $(p, \nu) \mapsto \tilde{f}_\nu(p)$ is absolutely integrable, smooth in $p^0$ and

$$
\sup_{\nu \in [0, \pi]} (\|\tilde{f}_\nu\|_2 + \|\partial_0 \tilde{f}_\nu\|_2) < \infty. \tag{4.27}
$$

We set $B' := \partial_0 B(\eta)$ and note that it is almost local and $\text{Sp}_{B' \alpha}$ is a compact set outside of $\nabla_+$. Setting $\tilde{f}^t_\nu(p) := \tilde{f}_\nu(p)e^{-i\cos \nu |p| t}$, we have

$$
\tilde{A}_t\{f\}(\eta) \Psi = \int dt' h_t(t')e^{it' H} B'(g \ast_3 f_{t'})(t) e^{-it' \omega_\nu(P)} \Psi
= \int_0^\pi d\mu(\nu) \int dt' h_t(t')e^{it' H} B'(f_{t'}^\nu)(t) e^{-it' \omega_\nu(P)} \Psi. \tag{4.28}
$$

Now we put $B'_{\nu,0}(x) := B'(x^0, x)$, $f_{\nu,x^0}^t(x) := f_{\nu,x^0}^t(x^0, x)$ and $f_{\nu,x^0}(x) := f_{\nu}(x^0, x)$. Making use of (3.11) and Proposition 3.5, we obtain

$$
\tilde{A}_t\{f\}(\eta) \Psi = \int dx^0 \int_0^\pi d\mu(\nu) \int dt' h_t(t')e^{it' H} (1_H \otimes \langle f_{\nu,x^0}^t | ) \circ a_{B'_{\nu,0}} e^{-it' \omega_\nu(P)} \Psi
= \int dx^0 \int_0^\pi d\mu(\nu) (1_H \otimes \langle f_{\nu,x^0}^t | ) \circ \int dt' h_t(t')e^{it' (H - \cos \nu |D_\nu| - \omega_\nu(P + D_\nu))} \circ a_{B'_{\nu,0}} \Psi. \tag{4.29}
$$

By means of the Dominated Convergence Theorem, the bound (4.27) and the Mean Ergodic Theorem (Theorem A.1), we obtain

$$
\lim_{t \to \infty} \tilde{A}_t\{f\}(\eta) \Psi = \int dx^0 \int_0^\pi d\mu(\nu) (1_H \otimes \langle f_{\nu,x^0}^t | ) \circ F_S(\{0\}) \circ a_{B'_{\nu,0}} \Psi, \tag{4.30}
$$

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where \( F_S \) is the spectral measure of the operator \( S := H - \cos\nu|D_x| - \omega_m(P + D_x) \) on \( L^2(\mathbb{R}^3; \mathcal{H}) \). To determine \( F_S(\{0\}) \), we diagonalize \( D_x \) with the help of the Fourier transform. We further note that \( \|S\Phi\|^2 = 0 \) for some \( \Phi = \{\Phi_\xi\}_{\xi \in \mathbb{R}^3} \in L^2(\mathbb{R}^3; \mathcal{H}) \), implies that \( S_\xi \Phi_\xi = 0 \) for almost all \( \xi \) w.r.t. the Lebesgue measure, where

\[
S_\xi := H + |\xi| - \omega(P + \xi).
\]  

(4.31)

Suppose now that \( m = 0 \). Then, Proposition \ref{lemma:projection} gives that \( \Phi_\xi \in \text{Ran } E(\{0\}) \) for \( \xi = 0 \) or \( \nu = \pi \) and \( \Phi_\xi = 0 \) otherwise. Since \( \xi = 0 \) is of zero Lebesgue measure, only \( \nu = \pi \) contributes and we obtain

\[
\lim_{t \to \infty} \tilde{A}_t \{f\}(\eta)\Psi = \int dx^0(E(\{0\}) \otimes \langle \hat{f}_{\pi,x^0} \rangle) \circ a_{B'_{\pi}} \Psi = E(\{0\})B'(f_\pi)\Psi.
\]  

(4.32)

For \( m > 0 \) a similar and simpler reasoning gives that the above limit is zero.

It remains to relax the additional assumptions made at the beginning of the proof. Let, therefore, \( \eta \) and \( \Psi \) be specified as in the theorem. Then, by spectral calculus, Proposition \ref{prop:spectral_calculus} and the fact that \( \Psi \in D_H \), we have

\[
\tilde{A}_t \{f\}(\eta)\Psi = E(\Delta_R)\tilde{A}_t \{f\}(\eta)E(\Delta_R)\Psi + O(R^{-N}),
\]  

(4.33)

where \( \Delta_R := \{ p \in \nabla_+ | p^0 \leq R \} \) and \( O(R^{-N}) \) denotes a term whose norm is bounded by \( C_N/R^N \), with \( C_N \) independent of \( t \). Making now use of the energy-momentum transfer relation (3.4), we can replace \( \eta \) in (4.33) by \( \eta' \) such that \( \eta' \) is compactly supported outside of the future lightcone. Thus, we have

\[
\tilde{A}_t \{f\}(\eta')\Psi = E(\Delta_R)\tilde{A}_t \{f\}(\eta')E(\Delta_R)\Psi + O(R^{-N}).
\]  

(4.34)

By means of formula (4.34) we conclude the proof. \( \Box \)

**Corollary 4.7.** Let \( A \in \mathfrak{A}_{S_1} \), and \( f \in C^\infty(S^2) \). Suppose further that \( D_{\max}(A,f) \) and \( D_{\max}(A^*,\tilde{f}) \) are dense. Then, for \( \Psi \in \mathcal{E}(\mathcal{H}) \cap D_H \), we have:

(a) For \( m = 0 \), \((1 - E(\{0\}))A_{\text{out}} \{f\}^-\Psi = 0\).

(b) For \( m > 0 \), \( A_{\text{out}} \{f\}^-\Psi = 0 \).

**Remark 4.8.** The result immediately generalizes to \( A \in \mathfrak{A}^{S_1} \), cf. Remark \ref{remark:generalization}.

## 5 Scattering of photons in the vacuum sector

In this section we consider a Haag-Kastler net \( (\mathfrak{A},U) \) in a vacuum representation containing massless Wigner particles (‘photons’). We collect here some basic facts about asymptotic fields of photons in a vacuum representation, which will be needed in our discussion of charged representations in Section 6. These results were first established in \[Bu77\] and recently revisited in \[DH15\], were simpler proofs, exploiting energy bounds, were given. In another recent work asymptotic fields were constructed for all

\[\text{See }[\text{Tal}, \text{Section IV.7}] \text{ for definition and basic properties of } L^2(\mathbb{R}; \mathcal{H}) \text{ for non-separable } \mathcal{H}.\]
A ∈ 𝕊_{loc,0} \[Ta14\], but we will not explore this direction here and content ourselves with A ∈ 𝕊^{C_+}. Instead, we take the opportunity to indicate another simplification. Namely, in the analysis of commutators of asymptotic fields in Proposition 5.7 below, the clustering estimates from [Bu77, DH15] can be avoided due to Theorem 4.6.

Since the approximating sequences A_t(f) we use are different than in [Bu77, DH15], we give rather complete proofs. We start our discussion with the following standard lemma.

**Lemma 5.1.** Let A ∈ 𝕊^{C_+} and f ∈ C^∞(S^2). Then,

\[ \lim_{t \to \infty} \tilde{A}_t(f) \Omega = P_{\text{ph}} f \left( \frac{P}{|P|} \right) A \Omega, \]

where P_{\text{ph}} is the projection onto the subspace of massless one-particle states h_{\text{ph}}. Vectors on the right-hand side of (5.1) span a dense, Poincaré invariant subspace D_{\text{ph}} in h_{\text{ph}}. (The subspace D^{(1)}_{\text{ph}} ⊂ D_{\text{ph}}, spanned by vectors with f ≡ 1, is also dense and Poincaré invariant).

**Proof.** It suffices to prove the lemma for A ∈ 𝕊_{C_+} and then extend by linearity. By Lemma 4.1, we have A_t(f) = \partial_0 B(g * f_i)(t), B ∈ 𝕊_{C_+} and g ∈ C_+(\mathbb{R}^4). Thus, we have

\[ A_t(f) \Omega = \partial_0 B(g * f_i)(t) \Omega = (2\pi)^2 e^{iHt} (g * f_i)(P) \partial_0 B \Omega. \]

Making now use of formula (4.11), Theorem A.1 and Proposition A.2 (a), we obtain

\[ \lim_{t \to \infty} \tilde{A}_t(f) \Omega = \lim_{t \to \infty} (2\pi)^2 \int dt' h_i(t') e^{iHt} (g * f_i)(P) \partial_0 B \Omega \]

\[ = (2\pi)^2 E(\partial \nabla_+) \tilde{g}(P) \left( \frac{P}{|P|} \right) f \left( \frac{P}{|P|} \right) iHB \Omega \]

\[ = (2\pi)^2 P_{\text{ph}} \tilde{g}(P) f \left( \frac{P}{|P|} \right) B \Omega \]

\[ = P_{\text{ph}} f \left( \frac{P}{|P|} \right) A \Omega. \]

Here we used that HBΩ is in the domain of |P|^{-1}, as one can show using the JLD method [Bu77] p.149. We also exploited that HBΩ is orthogonal to the vacuum and thus E(∂ν^+) can be replaced with P_{ph}.

Poincaré invariance of D_{ph} follows from the relation

\[ \lim_{t \to \infty} U(\lambda) \tilde{A}_t(f) \Omega = P_{\text{ph}}(f \circ g_{\Lambda^{-1}}) \left( \frac{P}{|P|} \right) A_\lambda \Omega = \lim_{t \to \infty} \tilde{A}_{\lambda t}(f \circ g_{\Lambda^{-1}}) \Omega, \]

where \( A_\lambda := U(\lambda)AU(\lambda)^* \in 𝕊_{C_+} \) and (3.23) was taken into account.

To show density, we exploit the cyclicity of the vacuum under 𝕊 and the fact that with functions \( \tilde{g} \), where g ∈ C_+(\mathbb{R}^4), one can approximate pointwise the characteristic function of \( \mathbb{R}^4 \setminus \{0\} \).

Next, denote by O_+ the future tangent of a double cone O, i.e. the cone of all points that have a positive timelike separation from O. Following the arguments of [Bu77], based on the Huygens principle, we have:
Lemma 5.2. Let $A \in \mathfrak A^{C^*}(\mathcal O)$ and $f \in C^\infty(S^2)$. Then, the limit
\begin{equation}
A^{\text{out}}\{f\} \Psi = \lim_{t \to \infty} A_t\{f\} \Psi
\end{equation}
exists for $\Psi$ in the dense domain $D(\mathcal O) := \{ B\Omega \mid B \in \mathfrak A_{\text{loc}}(\mathcal O_+) \}$. Moreover, $A^{\text{out}}\{f\}$ depends only on the single-particle state $A^{\text{out}}\{f\} \Omega$ within the above restrictions.

Proof. Let $A$ be localized in $\mathcal O$ and $\text{supp} f$ be contained in the set $\Theta \subset S^2$. Then, by construction, $A_t\{f\}$ is localized in the region
\begin{equation}
\mathcal O_t := \bigcup_{\tau \in t + t^\text{supph}} \{ \mathcal O + \tau(1, \Theta) \}.
\end{equation}
Clearly, for sufficiently large $t$ the region $\mathcal O_t$ is spacelike separated from any given double cone $\mathcal O_1$ in $\mathcal O_+$. Thus, it follows from Lemma 5.1 and the locality property that for all $B \in \mathfrak A_{\text{loc}}(\mathcal O_+)$,
\begin{equation}
\lim_{t \to \infty} A_t\{f\} B\Omega = \lim_{t \to \infty} B A_t\{f\} \Omega = B \text{P}_{\text{ph}} f \left( \frac{\mathcal H}{|\mathcal H|} \right) A \Omega,
\end{equation}
defining $A^{\text{out}}\{f\}$ on the domain $D(\mathcal O)$. This domain is dense as shown in [Bu75].

It is manifest from the above discussion that given $A_t\{f'\}$, where $A' \in \mathfrak A^{C^*}(\mathcal O)$ and $f' \in C^\infty(S^2)$, such that $A^{\text{out}}\{f\} \Omega = A^{\text{out}}\{f'\} \Omega$ we have $A^{\text{out}}\{f\} = A^{\text{out}}\{f'\}$ as operators on $D(\mathcal O)$.

In view of Lemmas 5.1 and 5.2 and Proposition 4.2 we obtain the following result.

Proposition 5.3. Let $A, A_i \in \mathfrak A^{C^*}$ and $f, f_i \in C^\infty(S^2)$, $i = 1, \ldots, n$. Then:

(a) For any $\Psi \in D_H$ the limit $\lim_{t \to \infty} A_t\{f\} \Psi$ exists and defines a closable operator $A^{\text{out}}\{f\} \uparrow D_H$. This operator is uniquely determined by the vector $A^{\text{out}}\{f\} \Omega$.

(b) $A^{\text{out}}\{f\} D_H \subset D_H$.

(c) $A^{\text{out}}_1\{f_1\} \ldots A^{\text{out}}_n\{f_n\} \Psi = \lim_{t \to \infty} A_1\{f_1\} \ldots A_n\{f_n\} \Psi$ for $\Psi \in D_H$.

The next lemma settles the transformation rules of the asymptotic fields under the Poincaré transformations.

Lemma 5.4. Let $A \in \mathfrak A^{C^*}(\mathcal O)$ and $f \in C^\infty(S^2)$. For any $\lambda \in \text{P}_+^\dagger$ we have on $D_H$
\begin{equation}
U(\lambda) A^{\text{out}}\{f\} U(\lambda)^* = A^{\text{out}}_\lambda\{f \circ g_{\lambda^{-1}}\},
\end{equation}
where $A_\lambda := U(\lambda) A U(\lambda)^* \in \mathfrak A^{C^*}(\mathcal O)$ and $g_\lambda$ was defined in (3.19).

Proof. Recall that $D(\mathcal O) := \{ B\Omega \mid B \in \mathfrak A_{\text{loc}}(\mathcal O_+) \}$ and note the relation $D(\mathcal O) = U(\lambda)^* D(\lambda \mathcal O)$. By formula (5.4) and Lemma 5.2 we obtain that relation (5.8) holds on $D(\lambda \mathcal O)$. Next, we choose $\Phi, \Psi \in D_H$ and $\Psi_n \in D(\lambda \mathcal O)$ such that $\| \Psi - \Psi_n \| \leq 1/n$. Then, making use of the fact that $U(\lambda)^* D_H \subset D_H$ and the observation that $\Phi$ is in the intersection of domains of $(U(\lambda) A^{\text{out}}\{f\} U(\lambda)^*)^*$ and $A^{\text{out}}_\lambda\{f \circ g_{\lambda^{-1}}\}^*$, we have
\begin{equation}
\langle \Phi, U(\lambda) A^{\text{out}}\{f\} U(\lambda)^* \Psi \rangle = \langle \Phi, A^{\text{out}}_\lambda\{f \circ g_{\lambda^{-1}}\} \Psi \rangle + O(1/n).
\end{equation}
Keeping $\Phi$ and $\Psi$ fixed, we can take the limit $n \to \infty$ and drop the error term. The claim follows from the resulting relation.

Now we analyse commutators of asymptotic fields. We proceed similarly as in the case of massless fermions [Bu75]. Till the end of this section $A, A', A_i \in \mathfrak{A}^C$ and $f, f', f_i \in C^\infty(S^2)$ unless stated otherwise.

**Lemma 5.5.** Let $A \in \mathfrak{A}^C(\mathcal{O})$. Then, for all $B \in \mathfrak{A}_{\text{loc}}(\mathcal{O}_+)$

$$[A^{\text{out}}\{f\}, B] = 0.$$ \hspace{1cm} (5.10)

Moreover, if $A' \in \mathfrak{A}^C(\mathcal{O}')$, then

$$[A^{\text{out}}\{f\}, A'^{\text{out}}\{f'\}(x)] = 0$$ \hspace{1cm} (5.11)

provided that $\mathcal{O}' + x \subset \mathcal{O}_+$. Both equalities hold in the sense of quadratic forms on $D_H \times D_H$.

**Proof.** In view of Proposition 5.3 we can write for any $\Psi, \Phi \in D_H$

$$\langle \Psi, [A^{\text{out}}\{f\}, B]\Phi \rangle = \lim_{t \to \infty} \langle \Psi, [\bar{A}_t\{f\}, B]\Phi \rangle = 0,$$ \hspace{1cm} (5.12)

since the commutator vanishes for sufficiently large $t$. By approximating

$$\langle \Psi, [A^{\text{out}}\{f\}, A'^{\text{out}}\{f'\}(x)]\Phi \rangle = \lim_{t \to \infty} \langle \Psi, [A^{\text{out}}\{f\}, \bar{A}'_t\{f'\}(x)]\Phi \rangle,$$ \hspace{1cm} (5.13)

and noting that $\bar{A}'_t\{f'\}(x)$ is localized in $\mathcal{O}_+$ for sufficiently large $t$, we obtain relation (5.11) from (5.10). \hfill \Box

**Lemma 5.6.** $x \mapsto A^{\text{out}}\{f\}(x)$ is a solution of the wave equation. That is,

$$\Box_x A^{\text{out}}\{f\}(x)\Psi = 0 \quad \text{for} \quad \Psi \in D_H.$$ \hspace{1cm} (5.14)

**Proof.** It follows immediately from Lemma 5.1 that $\Box_x A^{\text{out}}\{f\}(x)\Omega = 0$. Hence, by Proposition 5.3 (a), $\Box_x A^{\text{out}}\{f\}(x)\Omega = 0$ for any $\Psi \in D_H$. \hfill \Box

**Proposition 5.7.** Let $A^{\text{out}}\{f\}, A'^{\text{out}}\{f'\}$ be two asymptotic fields as specified above. Then,

$$[A^{\text{out}}\{f\}, A'^{\text{out}}\{f'\}] = \langle \Omega, [A^{\text{out}}\{f\}, A'^{\text{out}}\{f'\}]\Omega \rangle 1_H$$ \hspace{1cm} (5.15)

as operators on $D_H$.

**Proof.** First, we use a method of Pohlmeyer [Po69] (applied also in the collision theory of massless fermions [Bu75]) to show that

$$[A^{\text{out}}\{f\}, A'^{\text{out}}\{f'\}]\Omega = c\Omega, \quad c \in \mathbb{C}.$$ \hspace{1cm} (5.16)

To this end, we take any vector $\Phi$ such that $\Phi = E(K_\Phi)\Phi$ for a compact set $K_\Phi$ in the interior of the future light cone and consider the function

$$F(x, y) = \langle \Phi, [A^{\text{out}}\{f\}(x), A'^{\text{out}}\{f'\}(y)]\Omega \rangle.$$ \hspace{1cm} (5.17)
Theorem 5.8. The states \( \Psi^{\text{out}} := A_1^{\text{out}+} \ldots A_n^{\text{out}+}\Omega \) have the following properties:

(a) \( \Psi^{\text{out}} \) depends only on the single-particle states \( \Phi_i = A_i^{\text{out}+}\Omega \in D_{\text{ph}} \). Therefore, we put \( \Psi^{\text{out}} = \Phi_1^{\text{out}} \times \cdots \times \Phi_n^{\text{out}} \).

4In collision theory of massless fermions \( \Psi_{\text{ph}} = 0 \) was automatic in the corresponding expression, since a bosonic operator cannot create a fermionic single-particle state from the vacuum [Bu75, Lemma 4]. In the present bosonic case we can conclude using Theorem 4.6.
(b) \( \langle \Phi_1 \times \cdots \times \Phi_n, \Phi'_1 \times \cdots \times \Phi'_n \rangle = \delta_{n,n'} \sum_{\sigma \in \mathcal{S}_n} \langle \Phi_1, \Phi'_{\sigma_1} \rangle \cdots \langle \Phi_n, \Phi'_{\sigma_n} \rangle \), where \( \mathcal{S}_n \) is the set of all permutations of \( (1, \ldots, n) \).

(c) \( U(\lambda)(\Phi_1 \times \cdots \times \Phi_n) = (U(\lambda)\Phi_1) \times \cdots \times (U(\lambda)\Phi_n) \), where \( \lambda \in \widetilde{\mathcal{P}}_+^\dagger \).

### 6 Compton scattering in hypercone localized representations

In this section we consider a Haag-Kastler net \((\mathcal{A}, U)\) in a vacuum representation, containing massless Wigner particles ('photons') and a representation \(\pi\) which is hypercone localized w.r.t. this vacuum and describes Wigner particles of mass \(m > 0\) ('electrons'). For brevity we will write \((\hat{\mathcal{A}}, \hat{U})\) for the resulting net \((\mathcal{A}_\pi, U_\pi)\) and \(\hat{\mathcal{H}} := \mathcal{H}_\pi\). We also set \(\hat{A} := \pi(A)\) for \(A \in \mathcal{A}\) and denote by \((\hat{H}, \hat{P})\) the energy-momentum operators in the representation \(\pi\).

Given \(\hat{A} \in \hat{\mathcal{A}}^C(\mathcal{O})\) and \(\text{supp} f \subset \Theta \subset S^2\), the asymptotic field approximants \(t \mapsto \bar{\hat{A}}_t\{f\}\) are localized in

\[
\mathcal{O}_t := \bigcup_{\tau \in t + t^2 \text{supp} h} \{\mathcal{O} + \tau(1, \Theta)\}, \quad t \geq 1.
\]  

(6.1)

In the course of our analysis we will also consider \(t \mapsto \bar{\hat{A}}_{\lambda,t}\{f \circ g_{\Lambda^{-1}}\}\), \(\lambda = (0, \Lambda) \in \widetilde{\mathcal{P}}_+^\dagger\), whose localization regions are

\[
\mathcal{O}_t^\Lambda := \bigcup_{\tau \in t + t^2 \text{supp} h} \{\Lambda \mathcal{O} + \tau(1, g_{\Lambda}(\Theta))\}, \quad t \geq 1.
\]  

(6.2)

The following geometric lemma will be frequently used in the subsequent discussion. Its proof can be found in Appendix C.

**Lemma 6.1.** For any \(\mathcal{O} \in \mathcal{K}\) and any open \(\Theta \subset S^2\) such that \(\Theta \subset S^2\) there is a future lightcone \(V\), a hypercone \(C \subset F_V\) and a neighbourhood \(N\) of unity in the Lorentz group such that

\[
\Lambda \mathcal{O}_t \subset C^c, \quad \mathcal{O}_t^\Lambda \subset C^c, \quad t \geq 1,
\]  

(6.3)

for all \(\Lambda \in N\).

Given Lemma 6.1, the existence of a certain family of asymptotic fields is easily obtained. The following result holds true.

**Lemma 6.2.** Let \(\hat{A} \in \hat{\mathcal{A}}^C, f \in C^\infty(S^2)\) and \(\text{supp} f \subset \Theta\), with \(\Theta\) as in Lemma 6.1.

Then, the limit

\[
\hat{A}^{\text{out}}\{f\} \Psi := \lim_{t \to \infty} \bar{\hat{A}}_t\{f\} \Psi, \quad \Psi \in D_{\hat{H}}
\]  

(6.4)

exists. It defines a closable operator on \(D_{\hat{H}}\) which is uniquely determined by \(A^{\text{out}}\{f\} \Omega\).
Proof. Let $O$ be the localization region of $\hat{A}$ and $O_t$ be given by (6.1). By Lemma 6.1 there exists a future lightcone $V$ and $C \subset \mathcal{F}_V$ such that $O_t \subset C^c$. Hence, by hypercone localization of $\pi$, there is a unitary $W_C$ such that for all $t \geq 1$

$$\tilde{A}_t\{f\} = \pi(\hat{A}_t\{f\}) = W_C(\hat{A}_t\{f\})W_C^*.$$  \hfill (6.5)

Now by Proposition 6.3 the right-hand side converges on $W_C D_H$ to an operator which is uniquely determined by $A^\text{out}\{f\} \Omega$. Then, by Proposition 4.3, the left-hand side converges on $D_H$ to an operator which is uniquely determined by $A^\text{out}\{f\} \Omega$. \hfill \square

In the next proposition we eliminate the restriction on functions $f$.

**Proposition 6.3.** Let $\hat{A}, \hat{A}_i \in \mathfrak{A}^{C^*}$ and $f, f_i \in C^\infty(S^2)$. Then:

(a) For any $\Psi \in D_H$ the limit $\lim_{t \to \infty} \hat{A}_i\{f\} \Psi$ exists and defines a closable operator $\hat{A}^\text{out}\{f\}$ on $D_H$ which is uniquely specified by $A^\text{out}\{f\} \Omega$.

(b) $\hat{A}^\text{out}\{f\} D_H \subset D_H$.

(c) $\hat{A}^\text{out}\{f_1\} \ldots \hat{A}^\text{out}\{f_n\} \Psi = \lim_{t \to \infty} \tilde{A}_{1,t}\{f_1\} \ldots \tilde{A}_{n,t}\{f_n\} \Psi$ for $\Psi \in D_H$.

**Remark 6.4.** It follows immediately from Proposition 6.3 (a) that $x \mapsto \hat{A}^\text{out}\{f\}(x) \Psi$, $\Psi \in D_H$, is a solution of the wave equation.

**Proof.** In view of Lemma 6.2 and Proposition 4.3 we obtain the statement of the proposition for $f, f_i$ supported in proper open subsets of $S^2$. To remove this restriction, we choose a partition of unity on $S^2$ consisting of $f^j \in C^\infty(S^2)$, $j = 1, 2$, such that $\text{supp } f_j \subset S^2$. Thus, we may write

$$\tilde{A}_t\{f\} = \sum_{j=1,2} \tilde{A}_t\{ff^j\}.$$  \hfill (6.6)

Now it is easy to see that $\hat{A}^\text{out}\{f\} = \lim_{t \to \infty} \tilde{A}_t\{f\}$ exists on $D_H$ and has the properties specified in the proposition. The only property which requires an argument is the last statement in part (a). To verify it, suppose that $A^\text{out}\{f\} \Omega = A^\text{out}\{f'\} \Omega$. Then, we also have $A^\text{out}\{ff^j\} \Omega = A^\text{out}\{f'f^j\} \Omega$, since $A^\text{out}\{f\} \Omega = P_{\text{ph},f}(P\mid P)A\Omega$. Therefore, by Lemma 6.2 we have that $\hat{A}^\text{out}\{ff\} = \hat{A}^\text{out}\{f'f\}$ on $D_H$. Summation over $j$ yields the claim. \hfill \square

Next, we analyze the transformation rules of the asymptotic fields under Poincaré transformations. Our result is as follows.

**Lemma 6.5.** Let $\hat{A} \in \mathfrak{A}^{C^*}(O)$ and $f \in C^\infty(S^2)$. For any $\lambda \in \mathfrak{P}_+$ we have on $D_H$

$$U(\lambda) \hat{A}^\text{out}\{f\} U(\lambda)^* = \hat{A}^\text{out}\{f \circ g_\lambda^{-1}\},$$  \hfill (6.7)

where $\hat{A}_\lambda := U(\lambda) \hat{A} U(\lambda)^* \in \mathfrak{A}^{C^*}(\lambda O)$ and $g_\lambda$ is given by (5.19).
**Proof.** First, decompose $f$ according to $f = \sum_{j=1,2} f^j$, with $f^j \in C^\infty(S^2)$, $\text{supp} f^j \subset \Theta_j$ and $\Theta_j$ as in Lemma 6.1. Next, choose $\tilde{N} \subset SL(2, \mathbb{C})$ such that its image in the Lorentz group under the canonical covering map is contained in the neighbourhood $N$ from Lemma 6.1. (We can find one $N$ for both values of $j$). Now for any $j$ Lemma 6.1 gives a future lightcone $V_j$ and a hypercone $\mathcal{C}_j \subset \mathcal{F}_V$ such that

$$U(\lambda)\tilde{A}_t \{ f^j \} U(\lambda)^* \in \mathfrak{A}(\mathcal{C}_j), \quad \tilde{A}_t \{ f^j \circ g_{\lambda^{-1}} \} \in \mathfrak{A}(\mathcal{C}_j)$$

(6.8)

for all $t \geq 1$ and $\lambda = (0, \tilde{\Lambda}), \tilde{\Lambda} \in \tilde{N}$. Thus, due to the Poincaré covariance and hypercone localization, we have unitaries $W_{\mathcal{C}_j}$ such that

$$\hat{U}(\lambda)\tilde{A}_t \{ f^j \} \hat{U}(\lambda)^* = W_{\mathcal{C}_j} (U(\lambda)\tilde{A}_t \{ f^j \} U(\lambda)^*) W_{\mathcal{C}_j}^*, \quad \tilde{A}_t \{ f^j \circ g_{\lambda^{-1}} \} = W_{\mathcal{C}_j} \tilde{A}_t \{ f^j \circ g_{\lambda^{-1}} \} W_{\mathcal{C}_j}^*. \quad (6.9)$$

(6.10)

It follows from the above relations and Lemma 5.2 that

$$W_{\mathcal{C}_j} D(\lambda \mathcal{O}) \subset \hat{U}(\lambda) D_{\text{max}}(\tilde{A}, f^j) \cap D_{\text{max}}(\tilde{A}_\lambda, f^j \circ g_{\lambda^{-1}}). \quad (6.11)$$

Hence, both sides of

$$\hat{U}(\lambda)\hat{A}_{\text{out}} \{ f^j \} \hat{U}(\lambda)^* \Psi = \hat{A}_{\text{out}} \{ f^j \circ g_{\lambda^{-1}} \} \Psi, \quad \Psi \in W_{\mathcal{C}_j} D(\lambda \mathcal{O}) \quad (6.12)$$

are well defined. To verify equality (6.12), we choose $B \in \mathfrak{A}_{\text{loc}}(\mathcal{O}_+)$, set $B_\lambda := U(\lambda) B U(\lambda)^*$ and compute

$$\hat{U}(\lambda)\hat{A}_{\text{out}} \{ f^j \} \hat{U}(\lambda)^* W_{\mathcal{C}_j} B_\lambda \Omega = \lim_{t \to \infty} \hat{U}(\lambda)\tilde{A}_t \{ f^j \} \hat{U}(\lambda)^* W_{\mathcal{C}_j} B_\lambda \Omega$$

$$= \lim_{t \to \infty} W_{\mathcal{C}_j} U(\lambda)\tilde{A}_t \{ f^j \} U(\lambda)^* B_\lambda \Omega$$

$$= \lim_{t \to \infty} W_{\mathcal{C}_j} U(\lambda)\tilde{A}_t \{ f^j \} B_\lambda \Omega$$

$$= \lim_{t \to \infty} W_{\mathcal{C}_j} \tilde{A}_t \{ f^j \circ g_{\lambda^{-1}} \} B_\lambda \Omega$$

$$= \hat{A}_{\text{out}} \{ f^j \circ g_{\lambda^{-1}} \} W_{\mathcal{C}_j} B_\lambda \Omega, \quad (6.13)$$

where in the second step we used (6.9), in the fourth step (5.4) and in the last step (6.10). Arguing as in the proof of Lemma 5.4 we conclude from (6.12) that

$$\hat{U}(\lambda)\hat{A}_{\text{out}} \{ f^j \} \hat{U}(\lambda)^* \Psi = \hat{A}_{\text{out}} \{ f^j \circ g_{\lambda^{-1}} \} \Psi, \quad \Psi \in D_H. \quad (6.14)$$

Summing the above relation over $j$ we obtain the claim for $\lambda = (0, \tilde{\Lambda}), \tilde{\Lambda} \in \tilde{N}$.

It remains to extend the result to arbitrary $\lambda \in \tilde{P}_+$. To this end, we first note that (6.7) holds trivially for $\lambda = (x, I)$. Now any element of $\tilde{P}_+$ can be written as $(x, I)(0, \tilde{\Lambda}), \tilde{\Lambda} \in SL(2, \mathbb{C})$. Since $SL(2, \mathbb{C})$ is connected, it is generated by any neighbourhood of the identity. \hfill \Box

Exploiting the hypercone localization and Proposition 5.4 we obtain that commutators of asymptotic fields are numbers.
Proposition 6.6. Let \( \hat{A}_1, \hat{A}_2 \in \mathfrak{A}^c \) and \( f_1, f_2 \in C^\infty(S^2) \). Then,
\[
[\hat{A}_1^\text{out}(f_1), \hat{A}_2^\text{out}(f_2)] = \langle \Omega, [A_1^\text{out}(f_1), A_2^\text{out}(f_2)] \Omega \rangle 1_{\mathcal{H}},
\]  
(6.15)
as operators on \( D_{\mathcal{H}} \).

Proof. We decompose \( f_i = \sum_{j=1}^\ell f_i^j, i = 1, 2 \), where \( f_i^j \in C^\infty(S^2) \) are supported in sufficiently small subsets of \( S^2 \). Thus, we have
\[
[\tilde{A}_1, \tilde{A}_2] = \sum_{j_1,j_2} [\tilde{A}_1, \tilde{A}_2] = \sum_{j_1,j_2} [\tilde{A}_1, \tilde{A}_2].
\]  
(6.16)
We divide the set of indices into two subsets, namely
\[
S := \{ (j_1, j_2) \mid \text{supp } f_1^1 \cap \text{supp } f_2^2 = \emptyset \},
\]  
(6.17)
and
\[
S' := \{ (j_1, j_2) \mid \text{supp } f_1^1 \cap \text{supp } f_2^2 = \emptyset \}.
\]  
(6.18)
If the partition is sufficiently fine, for any \( (j_1, j_2) \in S \) Lemma 6.1 gives a future lightcone \( V_{j_1,j_2} \) and a hypercone \( C_{j_1,j_2} \subset \mathcal{F}_{V_{j_1,j_2}} \) such that
\[
[\tilde{A}_1, \tilde{A}_2] = \tilde{A}_1, \tilde{A}_2 \in \mathfrak{A}(C_{j_1,j_2}^c)
\]  
(6.19)
for all \( t \geq 1 \). Thus, by the hypercone localization of \( \pi \) there is a unitary \( W_{j_1,j_2} \) such that
\[
[\tilde{A}_1, \tilde{A}_2] = W_{j_1,j_2} [\tilde{A}_1, \tilde{A}_2] W^*_{j_1,j_2}.
\]  
(6.20)
Now for \( \Psi \in D_{\mathcal{H}} \) and \( \Phi \in W_{j_1,j_2} D_{\mathcal{H}} \) we have by Propositions 5.3, 5.7 and 6.3 that
\[
\langle \Psi, [\tilde{A}_1, \tilde{A}_2] \Phi \rangle = \lim_{t \to \infty} \langle \Psi, \tilde{A}_1, \tilde{A}_2 \rangle \Phi \rangle = \lim_{t \to \infty} \langle \Psi, W_{j_1,j_2} \tilde{A}_1, \tilde{A}_2 \rangle \Phi \rangle = \langle \Psi, \Phi \rangle \langle \Omega, [A_1, A_2] \Omega \rangle.
\]  
(6.21)
Since \( \Psi \) is in the domain of \( ([\tilde{A}_1, \tilde{A}_2])^* \), the above equality extends to \( \Phi \in D_{\mathcal{H}} \).

Finally, we consider \( (j_1, j_2) \in S' \). In this case locality gives for sufficiently large \( t \)
\[
[\tilde{A}_1, \tilde{A}_2] = 0,
\]  
(6.22)
as one can see by a straightforward computation. This concludes the proof. \( \square \)

After these preparations we proceed to the construction of scattering states of one electron and a finite number of photons i.e. Compton scattering. It suffices to consider \( f \in C^\infty(S^2) \) which are identically equal to one, in which case we write, as in the previous section, \( A^\text{out} \) for \( A^\text{out}(f) \). Similarly as in the vacuum representation, Proposition 6.3 gives
\[
[\hat{A}_1^\text{out}, \hat{A}_2^\text{out}] = \langle A^\text{out+} \Omega, A^\text{out+} \Omega \rangle, \quad [\hat{A}_1^\text{out}, \hat{A}_2^\text{out}] = [\hat{A}_1^\text{out+}, \hat{A}_2^\text{out+}] = 0. \]  
(6.23)
Recalling that by Proposition 4.3(c) \( \hat{A}^\text{out+} D_{\mathcal{H}} \subset D_{\mathcal{H}} \), scattering states are constructed in a straightforward manner.
Theorem 6.7. The states $\Psi_{\text{out}} := \hat{A}_1^{\text{out}} \cdots \hat{A}_n^{\text{out}} \Psi_{\text{el}}, \Psi_{\text{el}} \in \hat{h}_{\text{el}} \cap D_{\hat{H}}$, have the following properties:

(a) $\Psi_{\text{out}}$ depends only on the single-photon states $\Phi_i := A_i^{\text{out}} \Omega \in D_{\text{ph}}$ and the single-electron state $\Psi_{\text{el}} \in \hat{h}_{\text{el}} \cap D_{\hat{H}}$. Therefore, we can write $\Psi_{\text{out}} = \Phi_1 \times \cdots \times \Phi_n \times \Psi_{\text{el}}$.

(b) Given $\Psi_{\text{out}}$, $\Psi'_{\text{out}}$ as above,

$$\langle \Psi_{\text{out}}, \Psi'_{\text{out}} \rangle = \delta_{n,n'} \langle \Psi_{\text{el}}, \Psi'_{\text{el}} \rangle \sum_{\sigma \in S_n} \langle \Phi_1, \Phi'_{\sigma_1} \rangle \cdots \langle \Phi_n, \Phi'_{\sigma_n} \rangle,$$

(6.24)

where $S_n$ is the set of all permutations of $(1, \ldots, n)$.

(c) $\hat{U}(\lambda)(\Phi_1 \times \cdots \times \Phi_n \times \Psi_{\text{el}}) = (U(\lambda)\Phi_1) \times \cdots \times (U(\lambda)\Phi_n) \times (U(\lambda)\Psi_{\text{el}}), \lambda \in \tilde{P}_+^d$.

Proof. Parts (a) and (b) follow directly from (6.23) and Corollary 4.7 which gives $\hat{A}_{\text{out}}^{\text{ph}} \Psi_{\text{el}} = 0$ for $\Psi_{\text{el}} \in \hat{h}_{\text{el}} \cap D_{\hat{H}}$. To prove (c), it suffices to consider $\hat{A}_i \in \hat{A}_{\text{ph}}$, the general case follows by linearity (cf. Remark 4.5). We use Lemma 6.5 and the following computation

$$\hat{U}(\lambda)\hat{A}_1^{\text{out}}(\eta_{+,r_1}) \cdots \hat{A}_n^{\text{out}}(\eta_{+,r_n}) \Psi_{\text{el}}$$

$$= \int \prod_{j=1}^n d^4x_j(\eta_{+,r_1} \otimes \cdots \otimes \eta_{+,r_n})(x_1, \ldots, x_n) \hat{U}(\lambda)\hat{A}_1^{\text{out}}(x_1)\hat{A}_2^{\text{out}}(x_2) \cdots \hat{A}_n^{\text{out}}(x_n) \Psi_{\text{el}}$$

$$= \int \prod_{j=1}^n d^4x_j(\eta_{+,r_1} \otimes \cdots \otimes \eta_{+,r_n})(x_1, \ldots, x_n) \hat{A}_1^{\text{out}}(\Lambda x_1) \cdots \hat{A}_n^{\text{out}}(\Lambda x_n) \hat{U}(\lambda) \Psi_{\text{el}}$$

$$= \hat{A}_1^{\text{out}}(\eta_{+,r_1,\Lambda}) \cdots \hat{A}_n^{\text{out}}(\eta_{+,r_n,\Lambda}) \hat{U}(\lambda) \Psi_{\text{el}}.$$  

(6.25)

We note that $\eta_{+,r,\Lambda}(x) := \eta_{+,r}(\Lambda^{-1}x)$ belongs to the class of functions defined in (4.20), $\hat{A}_{i,\Lambda} \in \hat{A}_{\text{ph}}$ and the timelike unit vectors entering into the construction of $\eta_{+,r,\Lambda}$ and $\hat{A}_{i,\Lambda}$ coincide (cf. Definitions 4.20 and 4.9). It is easy to check that

$$\lim_{r_i \to \infty} A_{i,\Lambda}^{\text{out}}(\eta_{+,r_i,\Lambda}) = U(\lambda)A_i^{\text{out}} \Omega = U(\lambda)\Phi_i,$$

(6.26)

where $\Phi_i = A_i^{\text{out}} \Omega$. Thus, by taking the limits $r_i \to \infty$ on both sides of (6.25), we conclude the proof.

Let $\Gamma(h_{\text{ph}})$ be the symmetric Fock space over $h_{\text{ph}}$ and we denote by $a^*(\cdot)$ and $a(\cdot)$ the corresponding creation and annihilation operators. Using Theorem 6.7 (a), (b), we define the outgoing wave operator of Compton scattering

$$W_{\text{out}}(\Gamma(h_{\text{ph}}) \otimes \hat{h}_{\text{el}}) \rightarrow \hat{H},$$

(6.27)

as the unique linear isometry, satisfying

$$W_{\text{out}}(a^*(\Phi_1) \cdots a^*(\Phi_n) \Omega \otimes \Psi_{\text{el}}) = \hat{A}_1^{\text{out}} \cdots \hat{A}_n^{\text{out}} \Psi_{\text{el}},$$

(6.28)

for $\Phi_i = A_i^{\text{out}} \Omega \in D_{\text{ph}}$. Setting $U_{\text{ph}}(\lambda) := \Gamma(U(\lambda) \upharpoonright h_{\text{ph}})$ and $\hat{U}_{\text{el}}(\lambda) := \hat{U}(\lambda) \upharpoonright \hat{h}_{\text{el}}$, we obtain from Theorem 6.7 (c)

$$\hat{U}(\lambda) \circ W_{\text{out}} = W_{\text{out}} \circ (U_{\text{ph}}(\lambda) \otimes \hat{U}_{\text{el}}(\lambda)), \lambda \in \tilde{P}_+^d,$$

(6.29)

which amounts to the Poincaré covariance of the wave operator.
7 Haag-Kastler net of asymptotic photon fields

In this section we construct a Haag-Kastler net of asymptotic photon fields in a hypercone localized representation $\pi$ satisfying the properties specified at the beginning of Section 6. We also show that single-electron states induce vacuum representations of this net.

We start with the following technical lemma which summarizes and extends the information about the domains of the asymptotic fields.

Lemma 7.1. Let $\hat{A} \in \hat{A}^{C_*}$ be self-adjoint. Then:

(a) $D(H) \subset D_{\text{max}}(\hat{A})$ and $\hat{A}^{\text{out}} \upharpoonright D(H)$ is a symmetric operator uniquely determined by $A^{\text{out}}\Omega$.

(b) $\|\hat{A}^{\text{out}}\Psi\| \leq c\|(1 + \hat{H})\Psi\|$, $\Psi \in D(\hat{H})$.

(c) $|\langle \hat{H}\Psi, \hat{A}^{\text{out}}\Psi \rangle - \langle \hat{A}^{\text{out}}\Psi, \hat{H}\Psi \rangle| \leq c\|(1 + H)^{1/2}\Psi\|^2$, $\Psi \in D(\hat{H})$.

Moreover, $i[\hat{H}, \hat{A}^{\text{out}}]$, defined as a quadratic form on $D(\hat{H}) \times D(\hat{H})$, extends to a symmetric operator $i[\hat{H}, \hat{A}^{\text{out}}]$ on $D(\hat{H})$ in the sense explained in Appendix D. This operator coincides with $(\hat{A}^{(1)}^{\text{out}} \upharpoonright D(\hat{H}))$, $\hat{A}^{(1)} := i[\hat{H}, \hat{A}] \in \hat{A}^{C_*}$. Thus, it satisfies properties (a),(b),(c) above.

Proof. To prove (a), we note that for any $\Psi \in D(\hat{H})$ and $\varepsilon > 0$ there exists $\Phi_\varepsilon \in D_\hat{H}$ such that $\|(1 + \hat{H})(\Psi - \Phi_\varepsilon)\| < \varepsilon$. We write

$$\tilde{A}_t\Psi = \tilde{A}_t\Phi_\varepsilon + \tilde{A}_t(1 + \hat{H})^{-1}(1 + \hat{H})(\Psi - \Phi_\varepsilon) = \tilde{A}_t\Phi_\varepsilon + O(\varepsilon). \quad (7.1)$$

Here $\|O(\varepsilon)\| \leq c\varepsilon$ uniformly in $t$ due to the energy bounds from Proposition 4.2.

Now the existence of $\hat{A}^{\text{out}}$ on $D(\hat{H})$ follows from Proposition 6.3 (a) and the Cauchy criterion. It is also clear from the above argument that $\hat{A}^{\text{out}} \upharpoonright D(\hat{H})$ is uniquely determined by $\hat{A}^{\text{out}} \upharpoonright D_\hat{H}$. (That is, if $\hat{A}^{\text{out}}\Phi = 0$ for all $\Phi \in D_\hat{H}$ then $\hat{A}^{\text{out}}\Psi = 0$ for all $\Psi \in D(\hat{H})$). Part (b) is a simple consequence of part (a) and Proposition 4.2.

To complete the proof of the lemma, we write for $\Psi \in D(\hat{H})$

$$i \left( \langle \hat{H}\Psi, \hat{A}^{\text{out}}\Psi \rangle - \langle \hat{A}^{\text{out}}\Psi, \hat{H}\Psi \rangle \right) = \lim_{t \to \infty} \langle \Psi, \tilde{A}_t^{(1)}\Psi \rangle = \langle \Psi, (\hat{A}^{(1)}^{\text{out}})\Psi \rangle. \quad (7.2)$$

Since the energy bounds give $\|\tilde{A}_t^{(1)}(1 + \hat{H})^{-1}\| \leq c$ and $\|(1 + \hat{H})^{-1}\tilde{A}_t^{(1)}\| \leq c$, uniformly in $t$, we obtain by interpolation (cf. [RS2, Appendix to IX.4])

$$\|(1 + \hat{H})^{-1/2}\tilde{A}_t^{(1)}(1 + \hat{H})^{-1/2}\| \leq c, \quad (7.3)$$

uniformly in $t$. This and the first equality in (7.2) give part (c). The second equality in (7.2) ensures that $i[\hat{H}, \hat{A}^{\text{out}}]$ is defined on $D(\hat{H})$ and coincides on this domain with $(\hat{A}^{(1)}^{\text{out}})$.

In the next lemma we collect the necessary information about the commutators of asymptotic fields.
Lemma 7.2. Let \( \hat{A}_1, \hat{A}_2 \in \hat{A}^{C^*} \) be self-adjoint. Then:

(a) \( [\hat{A}_1^{\text{out}}, \hat{A}_2^{\text{out}}] = \langle \Omega, [\hat{A}_1^{\text{out}}, \hat{A}_2^{\text{out}}] \Omega \rangle 1_H \) as quadratic forms on \( D(\hat{H}) \times D(\hat{H}) \).

(b) \( \langle \Omega, [\hat{A}_1^{\text{out}}, \hat{A}_2^{\text{out}}] \Omega \rangle = 0 \) if \( A_1, A_2 \) are localized in spacelike separated double cones.

Proof. We set \( D \) and \( A \) as in (7.3). To show covariance of \( \hat{A}_1 \) and \( \hat{A}_2 \), we consider the following proposition.

\[ \hat{A}_1^{\text{out}} \hat{\Psi}_v = \hat{A}_1^{\text{out}} \Phi_{v, \varepsilon} + O(\varepsilon), \]  

where \( \|O(\varepsilon)\| \leq c \varepsilon \). Thus, we get from Proposition 6.6

\[ \langle \Psi_1, [\hat{A}_1^{\text{out}}, \hat{A}_2^{\text{out}}] \Psi_2 \rangle = \langle \Phi_{1, \varepsilon}, [\hat{A}_1^{\text{out}}, \hat{A}_2^{\text{out}}] \Phi_{2, \varepsilon} \rangle + O(\varepsilon) \]

\[ = \langle \Omega, [\hat{A}_1^{\text{out}}, \hat{A}_2^{\text{out}}] \Omega \rangle \langle \Phi_{1, \varepsilon}, \Phi_{2, \varepsilon} \rangle + O(\varepsilon) \]

\[ = \langle \Omega, [\hat{A}_1^{\text{out}}, \hat{A}_2^{\text{out}}] \Omega \rangle \langle \Psi_1, \Psi_2 \rangle + O(\varepsilon), \]  

where the rest term \( |O(\varepsilon)| \leq c \varepsilon \) changes from line to line. Since \( \varepsilon \) was arbitrary, this completes the proof of part (a). Part (b) is a known consequence of the JLD representation (cf. [Bu77], p.160).

Making use of Lemmas 7.1, 7.2 and standard results about self-adjoint extensions of unbounded operators collected in Appendix D, we set \( N \) in (7.1), we obtain

\[ \langle \hat{A}_1^{\text{out}} \hat{\Psi}_v, \hat{\Phi}_{v, \varepsilon} \rangle = \langle \hat{A}_1^{\text{out}} \hat{\Phi}_{v, \varepsilon}, \hat{\Psi}_v \rangle + O(\varepsilon), \]  

where \( \|O(\varepsilon)\| \leq c \varepsilon \). Thus, we get from Proposition 6.6

\[ \langle \Psi_1, [\hat{A}_1^{\text{out}}, \hat{A}_2^{\text{out}}] \Psi_2 \rangle = \langle \Phi_{1, \varepsilon}, [\hat{A}_1^{\text{out}}, \hat{A}_2^{\text{out}}] \Phi_{2, \varepsilon} \rangle + O(\varepsilon) \]

\[ = \langle \Omega, [\hat{A}_1^{\text{out}}, \hat{A}_2^{\text{out}}] \Omega \rangle \langle \Phi_{1, \varepsilon}, \Phi_{2, \varepsilon} \rangle + O(\varepsilon) \]

\[ = \langle \Omega, [\hat{A}_1^{\text{out}}, \hat{A}_2^{\text{out}}] \Omega \rangle \langle \Psi_1, \Psi_2 \rangle + O(\varepsilon), \]  

Proposition 7.3. Let \( \hat{A}_1, \hat{A}_2 \in \hat{A}^{C^*} \) be self-adjoint. Then \( \hat{A}_1^{\text{out}} \) and \( \hat{A}_2^{\text{out}} \) are essentially self-adjoint on \( D(\hat{H}) \) and their self-adjoint extensions \( \hat{A}_1^{\text{out}} \) and \( \hat{A}_2^{\text{out}} \) are essentially self-adjoint on any core for \( \hat{H} \). Moreover,

\[ e^{i(\hat{A}_1 + \hat{A}_2)^{\text{out}}} = e^{\frac{i}{2} \langle \Omega, [\hat{A}_1^{\text{out}}, \hat{A}_2^{\text{out}}] \Omega \rangle} e^{i\hat{A}_1^{\text{out}}} e^{i\hat{A}_2^{\text{out}}}. \]  

(7.6)

Due to the previous results we are now in a position to define the net of asymptotic photon fields. For any \( O \) we introduce the von Neumann algebra

\[ \hat{A}^{\text{out}}(O) := \{ e^{i\hat{A}^{\text{out}}} | \hat{A} \in \hat{A}^{C^*}(O), \hat{A}^* = \hat{A} \}. \]  

(7.7)

Theorem 7.4. \( (\hat{A}^{\text{out}}, \hat{U}) \) is a Haag-Kastler net in the sense of Definition 2.1.

Proof. Locality follows from Proposition 7.3 and Lemma 7.2 (b). To show covariance under Poincaré transformations, we use Lemma 6.5 which gives on \( D_{\hat{H}} \)

\[ \hat{U}(\lambda) \hat{A}^{\text{out}}(\lambda)^* = \hat{A}^{\text{out}}(\lambda). \]  

(7.8)

Since \( \hat{A}^{\text{out}}_{\lambda} \) is essentially self-adjoint on \( D_{\hat{H}} \), which, moreover, is a core for \( \hat{H} \), all its self-adjoint extensions must coincide. In particular, we obtain

\[ \hat{U}(\lambda) \hat{A}^{\text{out}}(\lambda)^* = \hat{A}^{\text{out}}(\lambda). \]  

(7.9)

Isotony and positivity of energy are obvious.

Next, we proceed to a discussion of representations of \( (\hat{A}^{\text{out}}, \hat{U}) \) induced by vector states from \( \hat{h}_{el} \).
Lemma 7.5. Let \( \hat{A} \in \mathfrak{A}^{C*} \) be self-adjoint and \( \Psi_{el} \in \mathfrak{h}_{el} \cap D_H, \|\Psi_{el}\| = 1 \). Then,
\[
\langle \Psi_{el}, e^{i\hat{A}\text{out}} \Psi_{el} \rangle = e^{-\frac{1}{2}\|\hat{A}\text{out}\Omega\|^2}.
\] (7.10)

Proof. Consider the function \( f(s) := \langle \Psi_{el}, e^{is\hat{A}\text{out}} \Psi_{el} \rangle \). Since \( \Psi_{el} \in D_H \) is contained in the domain of \( \hat{A}\text{out} \), we have by the Stone theorem
\[
(-i)\partial_s f(s) = \langle \Psi_{el}, e^{is\hat{A}\text{out}} \hat{A}\text{out} \Psi_{el} \rangle = \langle \Psi_{el}, e^{is\hat{A}\text{out}} \hat{A}\text{out} \Psi_{el} \rangle.
\] (7.11)
As \( \hat{A}\text{out} D_H \subset D_H \), we can iterate. This gives in particular
\[
(-i)^n \partial_s^n f(s)|_{s=0} = \langle \Psi_{el}, (\hat{A}\text{out})^n \Psi_{el} \rangle.
\] (7.12)
Now we use Proposition 4.4 to decompose \( \hat{A}\text{out} = \hat{A}\text{out}^+ + \hat{A}\text{out}^- \) on \( D_H \) while keeping in mind that \( \hat{A}\text{out}^\pm D_H \subset D_H \). Due to the canonical commutation relations \( \{\hat{A}, \hat{A}^\dagger\} \), the fact that \( \hat{A}\text{out}^- \Psi_{el} = 0 \) (Corollary 4.7) and standard combinatorics we, moreover, have for even \( n \geq 2 \)
\[
\langle \Psi_{el}, (\hat{A}\text{out})^n \Psi_{el} \rangle = (n-1)!! \langle \Omega, (\hat{A}\text{out})^2 \Omega \rangle^{n/2}
\] (7.13)
and zero for odd \( n \geq 1 \). Thus, we obtain
\[
\sum_{n=0}^{\infty} \frac{i^n}{n!} \langle \Psi_{el}, (\hat{A}\text{out})^n \Psi_{el} \rangle = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell s^{2\ell}}{2^{2\ell} \ell!} \langle \Omega, (\hat{A}\text{out})^2 \Omega \rangle^\ell = e^{-\frac{1}{2}s^2\|A\text{out}\Omega\|^2},
\] (7.14)
where we set \( \ell = n/2 \). Since the sum on the left-hand side above is absolutely convergent for any \( s \in \mathbb{C} \), we conclude that \( f \) extends to an entire analytic function which coincides with the function on the right-hand side of (7.14). \( \square \)

Theorem 7.6. Let \( \Psi_{el} \in \mathfrak{h}_{el} \cap D_H, \|\Psi_{el}\| = 1 \), \( \omega_{el}(\cdot) := \langle \Psi_{el}, \cdot \Psi_{el} \rangle \) be the corresponding state on \( \mathfrak{A}^{C*} \) and \( (\pi_{el}, \mathcal{H}_{\pi_{el}}, \Omega_{\pi_{el}}) \) its GNS representation. Then, \( \pi_{el} \) is a vacuum representation of \( (\mathfrak{A}^{C*}, \hat{U}) \) in the sense of Definition 2.3.

Remark 7.7. It is easy to see that the above theorem also holds if \( \pi \) is the original vacuum representation and \( \Psi_{el} \) is replaced with \( \Omega \). This gives a different proof of a result from [Bu77].

Remark 7.8. If \( U \upharpoonright \mathfrak{h}_{ph} \) is an irreducible representation of \( \mathcal{P}_{+} \) of zero mass and some finite, integer spin, one could expect that \( (\mathfrak{A}_{\pi_{el}}, \hat{U}_{\pi_{el}}) \) is just the usual massless free field of this spin on the Fock space. It turns out that this is not true under our assumptions and counter-examples can be given using the following simple fact. Let \( \mathcal{O}_r \) be the standard double cone of radius \( r \) centered at zero. Consider a Haag-Kastler net s.t. \( \mathcal{A}(\mathcal{O}_r) = \mathbb{C}I \) for \( r < 1 \) and \( \mathcal{A}(\mathcal{O}_r) \neq \mathbb{C}I \) for \( r \geq 1 \). Then also \( \mathfrak{A}^{out}(\mathcal{O}_r) = \mathbb{C}I \) for \( r < 1 \).

Proof. First, for \( \hat{A} \) as in Lemma 7.5, we have
\[
\langle \Psi_{el}, \hat{U}(\lambda)e^{i\hat{A}\text{out}} \hat{U}(\lambda^*) \Psi_{el} \rangle = \langle \Psi_{el}, e^{i\hat{A}\text{out}} \Psi_{el} \rangle = e^{-\frac{1}{2}\|\hat{A}\text{out}\Omega\|^2}
\] (7.15)
Since by Proposition 7.3 any element of $R^\text{out}$ is a strong limit of finite linear combinations of operators of the form $e^{iA_{B_1}}$, we obtain that $\omega_\pi$ is invariant under Poincaré transformations. Thus, by the GNS theorem [Ar, Theorem 2.33] we obtain a unique group of unitaries $U_\pi$ acting on $\mathcal{H}_\pi$ such that
\begin{equation}
U_\pi(\lambda)\Omega_\pi = \Omega_\pi, \quad U_\pi(\lambda)\pi_\pi(B)U_\pi(\lambda)^* = \pi_\pi(\tilde{U}(\lambda)B\tilde{U}(\lambda)^*), \tag{7.16}
\end{equation}
for all $B \in R^\text{out}$ and $\lambda \in \mathfrak{R}_+^\dagger$. Weak (and therefore strong) continuity of $U_\pi$ follows from the identity
\begin{equation}
\langle \pi_\pi(B_1)\Omega_\pi, U_\pi(\lambda)\pi_\pi(B_2)\Omega_\pi \rangle = \langle \Psi_\pi, B_1\tilde{U}(\lambda)B_2\tilde{U}(\lambda)^*\Psi_\pi \rangle, \quad B_1, B_2 \in R^\text{out}, \tag{7.17}
\end{equation}
and strong continuity of $\tilde{U}(\lambda)$. Thus, $\pi_\pi$ is a covariant representation.

Positivity of energy of $\pi_\pi$ easily follows from the above information, using ideas from [Dy08, Theorem 2.2] and [Ar, Theorem 4.5]. More precisely, let $B \in R^\text{out}$ and $f \in S(\mathbb{R}^4)$ be such that $\text{supp} \overline{f} \cap V_+ = \emptyset$. Then, we obtain from (7.17) that
\begin{equation}
\|\overline{f}(H_\pi, P_\pi)\pi_\pi(B)\Omega_\pi\|^2 = \langle \Psi_\pi, B(f)^*B(f)\Psi_\pi \rangle, \tag{7.18}
\end{equation}
where $(H_\pi, P_\pi)$ are the generators of $U_\pi$. To show that the right-hand side above is zero, we introduce compact sets $K \subset \mathbb{R}^3$, $\Delta_R = \{ p \in \mathbb{V}_+ | p^0 \leq R \}$ and write
\begin{equation}
\langle \Psi_\pi, B(f)^*B(f)\Psi_\pi \rangle = \langle E(\Delta_R)\Psi_\pi, B(f)^*B(f)E(\Delta_R)\Psi_\pi \rangle + O(R^{-N})
= \frac{1}{|K|} \int_K d^3x \langle E(\Delta_R)\Psi_\pi, (B(f)^*B(f))(x)E(\Delta_R)\Psi_\pi \rangle + O(R^{-N}), \tag{7.19}
\end{equation}
where $|O(R^{-N})| \leq C/R^N$ for any $N \in \mathbb{N}$. Here in the first step we used that $\Psi_\pi \in D_H$. In the second step we exploited the translation invariance of the functional on $R^\text{out}$, which is induced by $E(\Delta_R)\Psi_\pi \in \dot{\mathcal{H}}_\pi \cap D_H$. This invariance is proven as in (7.15). By taking first the limit $K \nearrow \mathbb{R}^3$ and making use of Lemma 3.3 and then taking the limit $R \to \infty$, we conclude the proof of positivity of energy.

It remains to show the irreducibility of $\pi_\pi$. As usually, it suffices to verify the clustering property, i.e.
\begin{equation}
\lim_{|x| \to \infty} \omega_\pi(B_1B_2(x)) = \omega_\pi(B_1)\omega_\pi(B_2), \quad B_1, B_2 \in R^\text{out}, \tag{7.20}
\end{equation}
In fact, given (7.20) and the fact that, by the Mean Ergodic Theorem,
\begin{equation}
E_{\text{inv}} = s\text{-lim}_{K \nearrow \mathbb{R}^3} \frac{1}{|K|} \int_K d^3x U_\pi(x) \tag{7.21}
\end{equation}
is a projection on invariant vectors of $U_\pi \upharpoonright \mathbb{R}^3$, we obtain that $E_{\text{inv}} = |\Omega_\pi\rangle \langle \Omega_\pi\|$. Then, [Ar, Theorem 4.6] gives irreducibility of $\pi_\pi$.

Let us verify (7.20) first for operators of the form $e^{iA_{B_1}}e^{iA_{B_2}}$, where $\hat{A}_1, \hat{A}_2$ are as in Lemma 7.5. Taking the Weyl relations and (7.10) into account, we obtain
\begin{equation}
\omega_\pi(e^{iA_{B_1}}e^{iA_{B_2}}) = e^{-\frac{1}{2}|\Omega_\pi|}e^{-\frac{1}{2}|\Omega_\pi|}e^{\frac{1}{2}(A_1+A_2)^*}\omega_\pi(e^{iA_{B_1}}e^{iA_{B_2}}) \omega_\pi(e^{iA_{B_1}}e^{iA_{B_2}})^* = \omega_\pi(e^{iA_{B_1}}e^{iA_{B_2}})e^{-\frac{1}{2}(A_1+A_2)^*}\omega_\pi(e^{iA_{B_1}}e^{iA_{B_2}})^* \tag{7.22}
\end{equation}
It is well known that in a vacuum representation \( \lim_{|x| \to \infty} U(x) = |\Omega\rangle \langle \Omega| \). (Observe that \( \langle B^* \Omega, U(x) B^* \Omega \rangle = \langle \Omega, [B, B^*(x)] \Omega \rangle \to 0 \) for all \( B \) as in Lemma [3.3].) Hence,

\[
\lim_{|x| \to \infty} \omega_{\text{el}}(e^{iA^\text{out} \cdot A^{\text{out} \ast}}) = \omega_{\text{el}}(e^{iA_1^{\text{out} \ast}}) \omega_{\text{el}}(e^{iA_2^{\text{out} \ast}}),
\]

(7.23)

and this relation extends to finite linear combinations of operators of the form \( e^{iA^{\text{out} \ast}} \).

Now for any \( B_1, B_2 \in \mathfrak{A}_{\text{loc}}^{\text{out}} \) we can find, by the Kaplansky Density Theorem, finite linear combinations \( B_{1,\varepsilon}, B_{2,\varepsilon} \) such that \( \|B_{1,\varepsilon}\| \leq c, \|B_{2,\varepsilon}\| \leq c \) uniformly in \( \varepsilon \) and \( \|(B_{1,\varepsilon} - B_{1,\varepsilon}^* ) \Psi_{\text{el}}\| \leq \varepsilon, \|(B_{2} - B_{2,\varepsilon}) \Psi_{\text{el}}\| \leq \varepsilon \). Making use of the translation invariance of \( \omega_{\text{el}} \) and relation (7.23), we write

\[
\omega_{\text{el}}(B_1 B_2(x)) = \omega_{\text{el}}(B_{1,\varepsilon} B_{2,\varepsilon}(x)) + O(\varepsilon)
\]

\[
= \omega_{\text{el}}(B_{1,\varepsilon}) \omega_{\text{el}}(B_{2,\varepsilon}) + O(\varepsilon) + o_\varepsilon(|x|^0)
\]

\[
= \omega_{\text{el}}(B_1) \omega_{\text{el}}(B_2) + O(\varepsilon) + o_\varepsilon(|x|^0),
\]

(7.24)

where \( |O(\varepsilon)| \leq c \varepsilon \) uniformly in \( \varepsilon \) and \( \lim_{|x| \to \infty} o_\varepsilon(|x|^0) = 0 \). By taking first the limit \( |x| \to \infty \) and then \( \varepsilon \to 0 \) we conclude the proof.

\[ \square \]

### A Mean Ergodic Theorem and invariant vectors

We pick \( h \) as in (4.5) and recall a variant of the abstract Mean Ergodic Theorem:

**Theorem A.1.** Let \( S \) be a self-adjoint operator on (a domain in) \( \mathcal{H} \) and \( F_S \) its spectral measure. Then,

\[
\lim_{t \to \infty} \int dt' h(t') e^{itS} = F_S(\{0\}).
\]

(A.1)

Now we determine the projection \( F_S(\{0\}) \) on the subspace of invariant vectors of \( t \mapsto e^{itS} \) for the relevant operators \( S \).

**Proposition A.2.** Let \((H, P)\) be the energy-momentum operators of a Haag-Kastler theory and \( E \) their joint spectral measure.

(a) Let \( S_\nu := H - \cos \nu |P| \) and \( F_{S_\nu} \) be the spectral measure of \( S_\nu \). Then,

\[
F_{S_\nu}(\{0\}) = \begin{cases} 
E(\overline{\nu}) & \text{for } \nu = 0, \\
E(\{0\}) & \text{for } \nu \in (0, \pi].
\end{cases}
\]

(A.2)

(b) Let \( S_{\nu, \xi} := H - |\xi| \cos \nu - \omega_m(P + \xi) \), where \( \omega_m(p) = \sqrt{p^2 + m^2} \), and \( F_{S_{\nu, \xi}} \) be the spectral measure of \( S_{\nu, \xi} \). Then, for \( \xi \neq 0 \),

\[
F_{S_{\nu, \xi}}(\{0\}) = \begin{cases} 
0 & \text{for } \nu \in [0, \pi) \text{ or } m > 0, \\
E(\{0\}) & \text{for } \nu = \pi \text{ and } m = 0.
\end{cases}
\]

(A.3)
Proof. (a) For $\Psi_0 \in \text{Ran} F_{\bar{S}_0}(\{0\})$, we have $(H - |\mathbf{P}|)\Psi_0 = 0$, hence $\Psi_0 \in \text{Ran} E(\partial \mathbf{V}_+)$.

This gives the first part of (A.2). To check the second part, we note that for $\nu \in (0, \pi]$ the set

$$\Delta_\nu := \{ (p^0, \mathbf{p}) | p^0 = \cos \nu |\mathbf{p}| \}$$

(A.4)

intersects with $\mathbf{V}_+$ only at $\{0\}$.

(b) First, we note that the set

$$\Delta_{\nu, \xi} := \{ (p^0, \mathbf{p}) | p^0 = |\xi| \cos \nu + \omega_m (\mathbf{p} + \xi) \}$$

(A.5)

describes a mass hyperboloid shifted by a spacelike or lightlike vector $(|\xi| \cos \nu, -\xi)$. Thus $\Delta_{\nu, \xi}$ contains zero only if $m = 0$ and $\nu = \pi$. Hence, it suffices to show that the relation

$$(H - \omega_m (\mathbf{P} - \xi))\Psi = |\xi| \cos \nu \Psi,$$

(A.6)

where $\Psi = E(\Delta)\Psi$, $\Delta$ compact, can only hold for $\Psi \in E(\{0\})\mathcal{H}$.

To this end, we generalize an argument from the Appendix of [Bu75]: From (A.6) we obtain

$$(H^2 - |\mathbf{P} - \xi|^2 - m^2)\Psi = |\xi| \cos \nu (H + \omega_m (\mathbf{P} - \xi))\Psi = |\xi| \cos \nu (2H - |\xi| \cos \nu)\Psi.$$ (A.7)

Setting $M^2 := H^2 - \mathbf{P}^2$, we get

$$M^2 \Psi = (2|\xi| \cos \nu - 2\mathbf{P}\xi + |\xi|^2 \sin^2 \nu + m^2)\Psi.$$ (A.8)

Now we want to apply a Lorentz transformation $U(\Lambda)$ to both sides of the above equation. We recall from Subsection 3.5 that

$$U(\Lambda)HU(\Lambda)^* = \langle \mathbf{v}_{\Lambda^{-1}} \rangle H + \mathbf{v}_{\Lambda^{-1}}\mathbf{P},$$

$$U(\Lambda)\mathbf{P}U(\Lambda)^* = -\mathbf{v}_{\Lambda}H + [\Lambda^{-1}]\mathbf{P}.$$ (A.9) (A.10)

Choosing $\Lambda = \Lambda_{\eta}$ to be a boost with rapidity $\eta$ in some direction $\mathbf{n}$ orthogonal to $\xi$, we get that $\mathbf{v}_{\Lambda_{\eta}}$ is orthogonal to $\xi$ and $[\Lambda^{-1}_0]^T\xi = \xi$. Therefore,

$$M^2 \Psi_{\Lambda_{\eta}} = (2(\langle \mathbf{v}_{\Lambda^{-1}_{\eta}} \rangle H + \mathbf{v}_{\Lambda^{-1}_{\eta}}\mathbf{P})|\xi| \cos \nu - 2\mathbf{P}\xi + |\xi|^2 \sin^2 \nu + m^2)\Psi_{\Lambda_{\eta}}.$$ (A.11)

Taking the scalar product with $\Psi$ and making use of (A.8) we get for $\xi \neq 0$ and $\cos \nu \neq 0$

$$\langle \Psi, H\Psi_{\Lambda_{\eta}} \rangle (1 - \langle \mathbf{v}_{\Lambda^{-1}_{\eta}} \rangle) = \langle \Psi, (\mathbf{v}_{\Lambda^{-1}_{\eta}}\mathbf{P})\Psi_{\Lambda_{\eta}} \rangle.$$ (A.12)

We note that the term on the left-hand side above is of order $\eta^2$ while the term on the right-hand side is of order $\eta$. Thus, dividing both sides of the equation by $\eta$ and taking the limit $\eta \to 0$, we obtain

$$\langle \Psi, \mathbf{n}\mathbf{P}\Psi \rangle = 0.$$ (A.13)
Since the above equation holds also for \( \Psi \) replaced with \( E(\Delta \pm)\Psi \), where \( \Delta \pm \) are chosen so that \( \pm E(\Delta \pm)nPE(\Delta \pm) \geq 0 \), we conclude that \( (nP)\Psi = 0 \). Substituting this to (A.12) we infer that \( \langle \Psi, H\Psi \rangle = 0 \) and, therefore, \( \Psi \in E(\{0\})\mathcal{H} \).

In the case of \( \cos \nu = 0 \) and \( \xi \neq 0 \) we choose \( \Lambda_{\eta} \) to be the boost with rapidity \( \eta \) in the direction of \( \xi \). Then, an analogous reasoning as above gives

\[
\langle \Psi, ((1 - [\Lambda_{\eta}^{-1}]^T)P)\xi \Psi_{\Lambda_{\eta}} \rangle = -\langle \Psi, H\Psi_{\Lambda_{\eta}} \rangle \nu_{\Lambda_{\eta}} \xi.
\] (A.14)

Since \( (1 - [\Lambda_{\eta}^{-1}]^T)\xi \) is of order \( \eta^2 \), we obtain \( \langle \Psi, H\Psi \rangle = 0 \), which concludes the proof. \( \square \)

\section{Admissible propagation observables}

\textbf{Definition B.1.} Let \( [1, \infty) \ni t \mapsto A_t \in B(\mathcal{H}) \) be a propagation observable, \( H \) a self-adjoint operator on a domain \( D(H) \) in \( \mathcal{H} \), and \( D, D^* \subset \mathcal{H} \) some dense domains. We say that \( A \) is admissible if:

(a) For any \( \Psi \in D(\ast) \) the limit \( \lim_{t \to \infty} A_t^{(\ast)}\Psi \) exists.

(b) \( \sup_{t \in \mathbb{R}^+} \| A_t^{(\ast)} (1 + H)^{-1} \| < \infty \).

(c) Set \( A_t(s) := e^{isH}A_t e^{-isH} \). All the derivatives \( A_t^{(n)} = \partial_s^n A_t(s)|_{s=0} \) exist in norm and satisfy (a), (b).

Here \( (\ast) \) means that the statement holds with and without all \( \ast \) symbols (correlated).

As shown in the next two propositions, limits of admissible propagation observables exist as closable operators on the following dense domain

\[
D_H := \bigcap_{n \geq 1} D(H^n).
\] (B.1)

Moreover, \( D_H \) is an invariant domain of these limits.

\textbf{Proposition B.2.} Let \( A \) be an admissible propagation observable. Then:

(a) For any \( \Psi \in D_H \) the limit \( \lim_{t \to \infty} A_t \Psi \) exists and defines a closable operator \( A^{\text{out}} \) on \( D_H \). This operator is uniquely specified by its values on \( D \).

(b) \( A^{\text{out}} D_H \subset D_H \).

\textbf{Proof.} Exploiting part (c) of Definition [B.1] we write

\[
A_t \Psi = (1 + H)^{-1}(-i)A_t^{(1)} \Psi + (1 + H)^{-1}A_t(1 + H)\Psi.
\] (B.2)

To prove (a), we use Definition [B.1] (b), (c) to approximate vectors \( \Psi, (1 + H)\Psi \) by elements of \( D \) uniformly in \( t \). By part (a) of Definition [B.1] \( A_t, A_t^{(1)} \) converge on \( D \) which gives the existence of \( A^{\text{out}} \) as an operator on \( D_H \). Since the above reasoning applies also to \( A_t^* \), the operator \( A^{\text{out}} \) is closable. To show that it is uniquely determined
by its values on \( D \), consider admissible propagation observables \( A_1 \) and \( A_2 \) such that 
\[
\lim_{t \to \infty} A_{1,t} \Phi = \lim_{t \to \infty} A_{2,t} \Phi \quad \text{for} \quad \Phi \in D.
\]
Then, it is clear from the above discussion that \( A_1^{\text{out}} = A_2^{\text{out}} \) as operators on \( D_H \). This completes the proof of (a).

To prove (b), we make use of a standard commutator formula (see e.g. [FGS01])
\[
[(1 + H)^\ell, A_t] = \sum_{k=1}^\ell \binom{\ell}{k} \text{ad}^k_H(A_t)(1 + H)^{\ell-k},
\]
which holds as an equality of quadratic forms on \( D_H \times D_H \). Exploiting part (c) of Definition B.1, which ensures that \( \text{ad}^k_H(A_t) = (-i)^k A_t^{(k)} \) are bounded operators, we obtain for any \( \Psi \in D_H \)
\[
A_t \Psi = (1 + H)^{-\ell} \left( \sum_{k=0}^\ell \binom{\ell}{k} (-i)^k A_t^{(k)}(1 + H)^{\ell-k} \right) \Psi,
\]
where we set by convention \( A_t^{(0)} = A_t \). Taking now the limit \( t \to \infty \) on both sides of (B.5), we obtain (b).

\begin{proof}
For \( n = 1 \) the statement follows from Proposition B.2. We suppose now it holds for some \( n > 1 \) and prove it for \( n + 1 \). Similarly as in (B.2), we write for any \( \Psi \in D_H \)
\[
A_{1,t} \ldots A_{n+1,t} \Psi = A_{1,t}(1 + H)^{-1}(-i) \sum_{\ell=2}^{n+1} A_{2,t} \ldots A_{\ell,t}^{(1)} \ldots A_{n+1,t} \Psi
+ A_{1,t}(1 + H)^{-1} A_{2,t} \ldots A_{n+1,t}(1 + H) \Psi.
\]
By the induction hypothesis and Proposition B.2, the above expression converges strongly as \( t \to \infty \). Next, we pick \( \Phi \in D_H \) and write
\[
\langle \Phi, A_{1,t} \ldots A_{n+1,t} \Psi \rangle = \langle \Phi, A_1^{\text{out}} A_2^{\text{out}} \ldots A_{n+1}^{\text{out}} A_{n+1,t} \Psi \rangle + o(t^0)
= \langle \Phi, A_1^{\text{out}} A_2^{\text{out}} \ldots A_{n+1}^{\text{out}} A_{n+1,t} \Psi \rangle + o(t^0),
\]
where in the first step we used the induction hypothesis, in the second step Proposition B.2 and \( o(t^0) \) denotes terms which tend to zero as \( t \to \infty \). This concludes the proof.
\end{proof}
C  Geometric argument

Proof of Lemma 6.1: First, we note that

$$\bigcup_{t \geq 1} O^A_t = \bigcup_{t \geq 1} \bigcup_{\tau \in t + t^\ast \text{supph}} \{ \Lambda O + \tau (1, g_A(\Theta)) \} \subset \bigcup_{\tau \in \mathbb{R}^+} \{ \Lambda O + \tau (1, g_A(\Theta)) \} =: U^A.$$  \hspace{1cm} (C.1)

On the other hand,

$$\bigcup_{t \geq 1} \Lambda O_t = \bigcup_{t \geq 1} \bigcup_{\tau \in t + t^\ast \text{supph}} \{ \Lambda O + \tau \Lambda (1, \Theta) \} \subset \bigcup_{\tau \in \mathbb{R}^+} \{ \Lambda O + \tau \Lambda (1, \Theta) \}$$

$$\subset \bigcup_{\tau \in \mathbb{R}^+} \{ \Lambda O + (\langle v_A \rangle + v_A \Theta) \tau (1, g_A(\Theta)) \} = U^A.$$  \hspace{1cm} (C.2)

Here in the third step we made use of (3.18) and (3.19) and in the last step of the fact that the prefactors $\langle v_A \rangle + v_A \Theta$ are strictly positive and, thus, they just reparametrize \( \tau \).

Let us first disregard the Lorentz transformations, i.e. show that for any double cone $O \in \mathcal{K}$ and open $\Theta \subset S^2$ with \( \mathcal{O} \subset S^2 \), there is a future lightcone $V$ and a hypercone $C \subset F_V$ such that the corresponding set $U^{A=I}$ is in $C^c$. The extension of the statement to $U^A$, where $\Lambda$ is in some neighbourhood of unity $N$ in $L^1$, will be discussed in the last part of the proof.

First, we fix a future lightcone $V$ so that $\mathcal{O} \subset V$ and choose a coordinate frame in which the origin is at the apex of $V$. Next, use the fact that there is an $\ell_0 \in S^2$ and an $1 \geq \varepsilon_0 > 0$ such that the spherical cap

$$\Theta_\varepsilon := \{ \ell \in S^2 | 1 - \varepsilon \leq \ell \ell_0 \leq 1 \}$$  \hspace{1cm} (C.3)

is contained in $S^2 \setminus \mathcal{O}$ for all $0 < \varepsilon \leq \varepsilon_0$. Let, moreover, $K_\varepsilon$ be a cone in the unit ball $B$ with apex at $u_\varepsilon := (1 - \varepsilon)\ell_0$ and the opening angle determined by $\Theta_\varepsilon$. More precisely,

$$K_\varepsilon := \{ u \in B | u = u_\varepsilon + s (\ell - u_\varepsilon), \ 0 \leq s < 1, \ \ell \in \Theta_\varepsilon \}.$$  \hspace{1cm} (C.4)

Using the Beltrami-Klein map $v : H_\varepsilon \to B$ given by $v(a) = a/a^0$, the corresponding hyperbolic cone $C(K_\varepsilon) \subset H_\varepsilon$ is given by

$$C(K_\varepsilon) = \left\{ \bar{r} \frac{(1, u)}{\sqrt{1 - u^2}} \in H_\varepsilon \left| u = u_\varepsilon + s (\ell - u_\varepsilon), \ 0 \leq s < 1, \ \ell \in \Theta_\varepsilon \right. \right\}.$$  \hspace{1cm} (C.5)

We note that as $\varepsilon \to 0$, the apex of $C(K_\varepsilon)$ tends to lightlike infinity in the direction of $\ell_0$ and the opening angle tends to zero. In fact, for all $0 \leq s < 1$ and $\ell \in \Theta_\varepsilon$ we have

$$u_\varepsilon(s, \ell) := u_\varepsilon + s (\ell - u_\varepsilon) = \ell_0 (1 - \varepsilon (1 - s)) + s (\ell - \ell_0).$$  \hspace{1cm} (C.6)

Noting that $(\ell - \ell_0)^2 = 2(1 - \ell \ell_0) \leq 2\varepsilon$ and setting $h_\varepsilon(s, \ell) := -\varepsilon^2 \ell_0 (1 - s) + s \varepsilon^{-\frac{1}{2}} (\ell - \ell_0)$, we have

$$u_\varepsilon(s, \ell) = \ell_0 + \varepsilon^2 h_\varepsilon(s, \ell),$$  \hspace{1cm} (C.7)

$$|h_\varepsilon(s, \ell)| \leq 3.$$  \hspace{1cm} (C.8)
Now a simple computation using \((C.6)\) gives
\[
1 - u_\varepsilon(s, \ell)^2 = \varepsilon(1 - s)\{2 - \varepsilon(1 - s) + 2s(1 - \varepsilon)(1 - \ell \ell_0)\varepsilon^{-1}\}.
\]
\[(C.9)\]
It is easy to see that \(1 \leq \{\ldots\} \leq 4\) and, therefore, we can find a function \((s, \ell) \mapsto g_\varepsilon(s, \ell)\) such that \(\frac{s}{\tau} \leq g_\varepsilon(s, \ell) \leq \bar{\tau}\) and
\[
\bar{\tau} \frac{1}{\sqrt{1 - u_\varepsilon(s, \ell)^2}} = \frac{g_\varepsilon(s, \ell)}{\sqrt{\varepsilon(1 - s)}}.
\]
\[(C.10)\]
Thus, skipping the arguments of \(g, h\) and setting \(M := \varepsilon^{-\frac{1}{2}}, S := g(1 - s)^{-\frac{1}{2}}\), we have
\[
\bar{\tau} \frac{(1, u_\varepsilon(s, \ell))}{\sqrt{1 - u_\varepsilon(s, \ell)^2}} = MS(1, \ell_0) + S(0, h),
\]
\[(C.11)\]
where \(M\) takes values in \([\varepsilon_0^{-1/2}, \infty)\) and \(S\) in \([\frac{\tau}{\bar{\tau}}, \infty)\).

Let us now show that there is a \(c > 0\) such that for sufficiently large \(M\)
\[
(MS(1, \ell_0) + S(0, h) - x)^2 < -c,
\]
\[(C.12)\]
for all \(x \in O, S \in [\frac{\tau}{\bar{\tau}}, \infty)\) and \(h\) within the above restrictions. Since \(\overline{O} \subset V\), there are constants \(c_O, c'O\) such that
\[
0 < c_O \leq (x^0 \pm |x|) \leq c_O',
\]
\[(C.13)\]
uniformly in \(x \in O\). Moreover, due to \((C.11)\) we have \((MS(1, \ell_0) + S(0, h))^2 = \bar{\tau}^2\).
Hence,
\[
(MS(1, \ell_0) + S(0, h) - x)^2 = \bar{\tau}^2 - 2MS(x^0 - x\ell_0) - 2S(0, h)x + x^2
\]
\[
\leq -2MSc_O + 6Sc'_O + (c'_O)^2 + \bar{\tau}^2,
\]
\[(C.14)\]
which proves \((C.12)\).

Next, let us show that there is a \(c' > 0\) such that for sufficiently large \(M\)
\[
(MS(1, \ell_0) + S(0, h) - x - \tau(1, \ell'))^2 < -c',
\]
\[(C.15)\]
for all \(x \in \mathbb{R}_+, \ell' \in \Theta, x \in O, S \in [\frac{\tau}{\bar{\tau}}, \infty)\) and \(h\) within the above restrictions. In view of \((C.12)\), it suffices to note the estimate
\[
(MS(1, \ell_0) + S(0, h) - x)(1, \ell') = S(M(1 - \ell_0 \ell') - h\ell') - x(1, \ell')
\]
\[
\geq (\tau/2)(M\varepsilon_0 - 3) - c'_O.
\]
\[(C.16)\]
Thus, we have proven that \(U^{A = I} \subset C(K_\varepsilon)^c = C(K_\varepsilon)^c\) for \(\varepsilon\) sufficiently small, depending on \(O\) and \(\Theta\).

Finally, let us choose a double cone \(O_0\), satisfying \(\overline{O}_0 \subset O\), and an open set \(\Theta_0 \subset S^2\), fulfilling \(\overline{O}_0 \subset \Theta\). \((O_0\) and \(\Theta_0\) are still arbitrary, within the restrictions of the lemma, since \(O\) and \(\Theta\) were arbitrary). Then, there is clearly a neighbourhood of unity \(N\) in the Lorentz group such that \(\Lambda O_0 \subset O\) and \(g_\Lambda(\Theta_0) \subset \Theta\) for all \(\Lambda \in N\) (cf. \((B.20)\) for the latter condition). Therefore, by the first part of the proof,
\[
U^A_0 \subset C^c, \quad \Lambda \in N,
\]
\[(C.17)\]
where \(U^A_0\) is defined as in \((C.1)\) using \(O_0\) and \(\Theta_0\). \(\square\)
D Integrating Heisenberg commutation relations to Weyl relations

We state below two known results which were used in Section 7. The first one is the Nelson commutator theorem \[^{[RS2, Theorem X.37], [Fr77, Theorem 0']}]\.

**Theorem D.1.** Let \( N \) be a self-adjoint operator on \( D(N) \) with \( N \geq 1 \). Let \( A \) be a symmetric operator on \( \mathcal{H} \) with domain \( D(A) \) which contains \( D(N) \). Suppose that

\[
\|A\Psi\| \leq c\|N\Psi\| \quad \text{and} \quad |\langle A\Psi, N\Psi \rangle - \langle N\Psi, A\Psi \rangle| \leq d\|N^{1/2}\Psi\|^2 \tag{D.18}
\]

for all \( \Psi \in D(N) \). Then, \( A \) is essentially self-adjoint on \( D(N) \) and its unique self-adjoint extension \( A^\bullet \) is essentially self-adjoint on any core for \( N \).

Before we state the second result we need some preparations. Let \( A, N \) be as in Theorem D.1. We then define \( \dot{A} = i[N, A] \) (D.19) as a quadratic form on \( D(N) \times D(N) \). The associated operator \( \dot{A}^\circ \) is given by

\[
D(\dot{A}^\circ) = \{ \Psi \in D(N) \mid \exists c_\Psi \text{ s.t. } |\langle \Phi, \dot{A}\Psi \rangle| \leq c_\Psi\|\Phi\| \text{ for all } \Phi \in D(N) \},
\]

\[
\dot{A}^\circ\Psi = \dot{A}\Psi, \quad \Psi \in D(\dot{A}^\circ), \tag{D.20}
\]

where the vector \( \dot{A}\Psi \) corresponds via the Riesz theorem to the bounded functional appearing in (D.20). It is easy to see that \( \dot{A}^\circ \) is a symmetric operator on \( D(\dot{A}^\circ) \). However, it is not guaranteed that \( D(\dot{A}^\circ) \) is dense.

Now we are in a position to state a result about integration of canonical commutation relations from \[^{[Fr77, Theorem 1]}\]. (Although separability of \( \mathcal{H} \) is assumed in \[^{[Fr77]}\], this property is not used in the proof of the following result).

**Theorem D.2.** Let \( N \) be a self-adjoint operator with \( N \geq 1 \). Let \( A_1, A_2 \) be symmetric operators with domains \( D(A_1) \) and \( D(A_2) \), containing \( D(N) \), and such that

\[
C := i[A_1, A_2], \tag{D.22}
\]

defined as a quadratic form on \( D(N) \times D(N) \), is a multiple of the identity. Assume moreover that \( D(\dot{A}_1^\circ) \supset D(N) \) and \( A_1, A_2, \dot{A}_1^\circ \) satisfy (D.18). Then, the self-adjoint extensions \( A_1^\bullet, A_2^\bullet \), given by Theorem D.1, satisfy

\[
e^{itA_1^\circ}e^{isA_2^\circ}e^{-itA_1^\circ} = e^{isA_1^\bullet}e^{istC}, \quad s, t \in \mathbb{R}. \tag{D.23}
\]

From Theorem D.2 we easily get the usual form of the Weyl relations appearing in Proposition 7.3.

**Corollary D.3.** Let \( N, A_1, A_2 \) be as in Theorem D.2. Then \( A_1 + A_2 \), defined as a symmetric operator on \( D(A_1) \cap D(A_2) \) is essentially self-adjoint on \( D(N) \) and its self-adjoint extension \( (A_1 + A_2)^\bullet \) is essentially self-adjoint on any core for \( N \). Moreover,

\[
e^{it(A_1+A_2)^\bullet} = e^{-\frac{it}{2}C}e^{itA_1^\bullet}e^{itA_2^\bullet}, \quad t \in \mathbb{R}, \tag{D.24}
\]

with \( C \) defined by (D.22).
Proof. To justify the first statement, we note that $A_1 + A_2$ satisfies the assumptions of Theorem D.1. Now we define
\[ V(t) := e^{-\frac{i}{2}(t^2 + s^2)}e^{itA_1}e^{isA_2}. \] (D.25)

Clearly, $V(0) = 1$ and, making use of (D.23) we get
\[ V(t)V(s) = e^{-\frac{i}{2}(t^2 + s^2)}e^{itA_1}e^{isA_2}e^{itA_2}e^{isA_1} = V(t + s). \] (D.26)

Thus $V$ is a one-parameter group of unitaries, whose weak (and therefore strong) continuity is obvious. By the Stone theorem $V(t) = e^{itQ}$, for a self-adjoint operator $Q$ given by
\[
D(Q) := \{ \Psi \in \mathcal{H} \mid \lim_{\tau \to 0} \frac{V(\tau) - 1}{\tau} \Psi \text{ exists} \},
\]
\[
Q\Psi = \lim_{\tau \to 0} \frac{V(\tau) - 1}{\tau} \Psi \text{ for } \Psi \in D(Q),
\]
\[
\text{cf. [RS1] Theorems VIII.7, VIII.8]. From the equality
\]
\[
(V(\tau) - 1) = (e^{-\frac{i}{2}(t^2 + s^2)} - 1)e^{itA_1}e^{isA_2} + e^{itA_1}(e^{isA_2} - 1) + (e^{itA_1} - 1)
\]
and the Stone theorem we immediately conclude that $D(N) \subset D(Q)$, (since $D(N) \subset D(A_1^*) \cap D(A_2^*)$), and that
\[
Q \upharpoonright D(N) = (A_1 + A_2) \upharpoonright D(N).
\] (D.30)

Thus, $Q$ is a self-adjoint extension of $(A_1 + A_2) \upharpoonright D(N)$ and by the first part of the theorem we obtain $Q = (A_1 + A_2)^*$. \qed

E Conventions

1. $\tilde{g}(p^0) = (2\pi)^{-1/2} \int e^{ip^0 x^0} g(x^0) d{x^0}$ for $g \in L^1(\mathbb{R})$.
2. $\tilde{g}(p) = (2\pi)^{-3/2} \int e^{-ipx} g(x) d^3x$ for $g \in L^1(\mathbb{R}^3)$.
3. $\tilde{g}(p) = (2\pi)^{-2} \int e^{ip^0 x^0 - px} g(x) d^4x$ for $g \in L^1(\mathbb{R}^4)$.
4. $\tilde{T}(p) = (2\pi)^{-2} \int e^{-ip^0 x^0 - px} T(x) d^4x$ for $T \in S(\mathbb{R}^4)$.
5. $(f * g)(x) = \int f(x - y)g(y) d^3y$ for $f, g \in L^1(\mathbb{R}^3)$.
6. $(f * g)(x) = \int f(x^0, x - y)g(y) d^3y$ for $f \in L^1(\mathbb{R}^4)$, $g \in L^1(\mathbb{R}^3)$.  

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