How extreme are extreme black holes?

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Abstract
We examine the properties of nearly extremal black holes produced by gravitational collapse. It is shown that an observer who crosses the black hole horizon at late times rapidly encounters a singularity.

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1. Introduction

Nonrotating black holes with both mass and charge have two horizons: the outer horizon is the actual event horizon of the black hole, while the inner horizon is unstable to small perturbations and is conjectured to become singular in any realistic situation [1]. For such black holes the charge is always less than or equal to the mass, with the special case of charge equals mass called an extreme black hole. In the limit as the charge approaches the mass, the two horizons merge. Thus, as pointed out by Marolf [2], one might expect that in some sense extreme black holes have singular event horizons. And indeed it is sometimes argued on various grounds, e.g. in [3, 4], that the extreme limit of black holes is in some sense discontinuous or singular. Specifically, it is claimed in [2] that an observer crossing the event horizon encounters a singularity a short time later and that this short time goes to zero in the limit of extremality.

The treatment of [2] used an eternal black hole, whereas physical black holes are formed by gravitational collapse. In this paper, we treat the problem of nearly extreme black holes formed by gravitational collapse. For simplicity, we use the collapse of a charged thin shell. We then consider a free fall observer who falls into such a black hole at late times and calculate the time it takes for him to reach the singularity. Methods and results are presented in section 2 and conclusions are presented in section 3.
2. Methods and results

A charged, nonrotating, black hole is described by the Reissner–Nordström metric
\[
\text{d}s^2 = -F \text{d}t^2 + F^{-1} \text{d}r^2 + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2),
\]
where the quantity \(F\) is given by
\[
F = 1 - \frac{2m}{r} + \frac{q^2}{r^2}
\]
and \(m\) and \(q\) are respectively the mass and charge of the black hole. The quantity \(F\) vanishes at the outer and inner horizons of the black hole, which are located respectively at \(r = r_+\) and \(r = r_-\) where
\[
r_{\pm} = m \pm \sqrt{m^2 - q^2}.
\]
The coordinate system in equation (1) becomes singular at the horizons; thus, to describe the process of gravitational collapse and black hole formation, we need a different coordinate system. We therefore use Eddington–Finkelstein coordinates \((v, r, \theta, \phi)\) where \(v\) is given by
\[
v = t + \int F^{-1} \text{d}r.
\]
In these coordinates, the Reissner–Nordström metric becomes
\[
\text{d}s^2 = -F \text{d}v^2 + 2 \text{d}v \text{d}r + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2).
\]
The coordinate \(v\) is constant along ingoing light rays. One can also define an analogous coordinate \(u\) which is constant along outgoing light rays. With a series of coordinate patches of this type, one can cover the maximal analytic extension of the Reissner–Nordström spacetime, which is shown in figure 1. However, the extended Reissner–Nordström spacetime is too large to represent what actually happens in gravitational collapse. This is for two different reasons: first when a charged, spherical object collapses, it is only the spacetime outside the object that is described by the Reissner–Nordström metric. Thus, to describe gravitational collapse, one must find the path of the outer surface of the collapsing object, and then cut out all parts of the diagram of figure 1 that are outside that surface and replace them with something appropriate for the interior of the collapsing object. The second reason is that small perturbations of the Reissner–Nordström solution blow up as \(v \to \infty\) (which is also a null surface at \(r = r_-\)). Therefore, to get the appropriate diagram, one must replace this null surface with a null singularity and cut out all portions of the diagram to the future of this surface. Since as shown in [5] generic singularities can propagate along null hypersurfaces, the null singularity in the exterior of the object should also continue to the interior of the object. In the end, one obtains a restricted portion of the extended Reissner–Nordström spacetime, which can be covered by a single coordinate patch of the sort used in equation (5).

We want a simple model for gravitational collapse to form the black hole, so we choose a thin shell of charged dust. Thin shells in general relativity are described by the Israel formalism [6] in which the history of the shell is a timelike hypersurface \(\Sigma\) separating two spacetimes (labeled + and − and which are respectively the exterior and the interior of the shell). The intrinsic metric \(h_{ab}\) is continuous, while the extrinsic curvature \(K_{ab}\) is discontinuous and satisfies
\[
[K_{ab}] = -8\pi \left( t_{ab} - \frac{1}{2} h_{ab} \right),
\]
where \(t_{ab}\) is the delta function stress-energy of the shell. Here, for any quantity \(\Phi\), we define \(\lbrack \Phi \rbrack = \Phi_+ - \Phi_-\) where the quantity \(\Phi_+\) is \(\Phi\) on \(\Sigma\) evaluated from the + side (and correspondingly for \(\Phi_-\)).
Figure 1. The extended Reissner–Nordström spacetime. Here the outer and inner horizons are represented by solid lines at 45 degree angles to the vertical, while the timelike singularity is denoted by a dashed line.

The worldsheet of the shell is given by $r = R(\tau)$ where $\tau$ is the proper time of an observer in the shell. The stress-energy of the dust shell takes the form

$$l_{ab} = \sigma u_a u_b,$$

where $u^a$ is the four-velocity of the shell and $\sigma$ is the shell’s mass per unit area. As shown by de la Cruz and Israel [7], Kuchar [8] and Chase [9], equation (6) can be used to find the equation of motion of the shell as follows: the exterior of the shell is a Reissner–Nordström spacetime. Spherical symmetry implies that the extrinsic curvature takes the form

$$K_{ab} = Bu_a u_b + C(u_a u_b).$$

Combining equations (6), (7) and (8) yields

$$[B] = -4\pi \sigma$$

$$[C] = -4\pi \sigma.$$  

A straightforward calculation using equation (5) shows that the quantities $B$ and $C$ are given by

$$B = - (\dot{R}^2 + F)^{-1/2} \left( \ddot{R} + \frac{1}{2} \frac{dF}{dR} \right)$$

$$C = \frac{1}{R} (\dot{R}^2 + F)^{1/2},$$

where an overdot denotes derivative with respect to $\tau$. Since the interior of the shell is Minkowski spacetime, expressions for $B$ and $C$ for the shell interior are given by
equations (11) and (12) with $F$ replaced by 1. Applying equation (12) to equation (10) yields

$$\left(R^2 + 1\right)^{1/2} - \left(R^2 + F\right)^{1/2} = 4\pi R \sigma.$$  \hspace{1cm} (13)

Then, using equations (9), (11) and (13) yields

$$\frac{d}{d\tau} (R^2 \sigma) = 0$$ \hspace{1cm} (14)

which means that there is a constant $M$ such that $4\pi \sigma = M/R^2$. Using this result in equation (13) and solving for $R^2$ yields

$$\dot{R}^2 = -1 + \left(\frac{m}{M} + \frac{M^2 - q^2}{2MR}\right)^2.$$ \hspace{1cm} (15)

We start the shell at a large radius with a speed $c_0$ in the inward direction. Then, it follows from equation (15) that

$$M = \frac{m}{\sqrt{1 + c_0^2}}.$$ \hspace{1cm} (16)

We also introduce the quantity $\epsilon$ defined by

$$\epsilon = \sqrt{1 - \frac{q^2}{m^2}}.$$ \hspace{1cm} (17)

Since we are considering nearly extremal black holes, $\epsilon \ll 1$. Expressing equation (15) in terms of $m$, $\epsilon$ and $c_0$, we obtain

$$\dot{R}^2 = -1 + \left(1 + c_0^2\right) \left(1 + \frac{m}{2R} \left[\epsilon^2 - \frac{c_0^2}{1 + c_0^2}\right]\right)^2.$$ \hspace{1cm} (18)

We have $r_\pm = m(1 \pm \epsilon)$, so $r_-$ is just slightly less than $r_+$ and therefore the shell will cross $r_-$ a short (proper) time after crossing $r_+$. It is also true that the perturbed black hole becomes singular at $r_-$. However, it is here that the use of Eddington–Finkelstein coordinates becomes essential. In the collapse process, there are two different null surfaces at $r = r_-$, an outgoing null surface where $v$ is finite, and which is not singular, and an ingoing null surface which occurs in the limit as $v \to \infty$, and which is singular. The question then becomes, which of these surfaces does the shell cross first? To answer this question, we note that there is a function $V(\tau)$ such that on the worldsheet of the shell $v = V(\tau)$. Then, from equation (5) it follows that

$$-1 = -F \dot{V}^2 + 2 \dot{V} \dot{R}$$ \hspace{1cm} (19)

and therefore that

$$\dot{V} = \frac{1}{F} (R \pm \sqrt{R^2 + F}).$$ \hspace{1cm} (20)

However, the null coordinate $v$ always increases toward the future, fixing the sign above to be plus, so we obtain

$$\dot{V} = \frac{1}{F} (R + \sqrt{R^2 + F}).$$ \hspace{1cm} (21)

Note that in this equation, if $\dot{R} < 0$ when $F$ vanishes, then $\dot{V}$ remains finite; but if $\dot{R} > 0$ when $F$ vanishes, then $\dot{V} \to \infty$. Since the shell starts out ingoing, and since it cannot change the direction between the two horizons (where $r$ is a timelike coordinate) then it follows that the shell first crosses $r_-$ at a place where $v$ remains finite, and where therefore the spacetime
remains nonsingular. After crossing $r_-$, the shell reaches a minimum radius $r_{\text{min}}$ and then re-expands. It follows from equation (18) that

$$ r_{\text{min}} = \frac{m}{2} \left( 1 + \frac{1}{\sqrt{1 + c_0^2}} \right), $$

(22)

where we have neglected any terms higher than the zeroth order in $\epsilon$. After reaching $r_{\text{min}}$ the shell re-expands, and eventually reaches $r_-$. Note, however, that this time the shell approaches $r_-$ with $\dot{R} > 0$. It then follows from equation (21) that $\upsilon$ diverges as $r \to r_-$ and thus that the shell approaches the null singularity. The path of the shell in the usual Penrose diagram of the Reissner–Nordström spacetime is shown in figure 2.

The shell clearly takes a non-negligible amount of proper time to encounter the singularity, and this will also be true of an observer who falls into the black hole shortly after it forms. However, as we will see, things are somewhat different for an observer who falls in at late times. Since for such an observer, $\upsilon \gg m$ when the outer horizon is crossed, and since $\upsilon$ increases along future-directed timelike curves, it follows that $\upsilon \gg m$ when the observer encounters the shell. Therefore, we are led to consider the behavior of the shell at large $\upsilon$. We know that as $\upsilon \to \infty$ we have $r \to r_-$ and therefore $F \to 0$. However, we will need to know how rapidly $F \to 0$ at large $\upsilon$. From equation (21) it follows that on the shell we have

$$ \frac{dv}{dr} = \frac{1}{F} \left( 1 + \sqrt{1 + \frac{F}{R^2}} \right), $$

(23)
Then using equations (2) and (23) and neglecting any terms that are higher order in the small quantities $F$ and $\epsilon$, we find that on the shell at large $v$ we have

$$\frac{d\bar{v}}{dF} = \frac{-m}{F\sqrt{\epsilon^2 + F}}. \quad (24)$$

The solution of this equation is

$$F = \frac{\epsilon^2}{\sinh^2\left(\frac{\epsilon(v + c_1)}{2m}\right)}, \quad (25)$$

where $c_1$ is a constant. Note that there are two separate regimes where this equation simplifies. For $m \ll v \ll m/\epsilon$, we have $F \approx 4m^2/(v + c_1)^2$ while for $v \gg m/\epsilon$ we have $F \approx 4\epsilon^2 \exp[-\epsilon(v + c_1)/m]$.

We now consider the fate of a free-fall observer who falls into the black hole at late times. Just as the shell has four-velocity $u^a = (\dot{V}, \dot{R})$, denote the four-velocity of the observer by $v^a = (\dot{v}, \dot{r})$. Due to the time translation symmetry of the Reissner–Nordström spacetime, the observer has a conserved energy per unit mass $E$ given by

$$E = F\dot{v} - \dot{r}. \quad (26)$$

Since $v^a$ is a unit timelike vector, the same reasoning that led to equation (21) yields

$$\dot{u} = \frac{1}{F}(\dot{\bar{r}} + \sqrt{\dot{r}^2 + \bar{F}}). \quad (27)$$

Using equations (26) and (27), we find

$$\dot{r} = -\sqrt{E^2 - F}. \quad (28)$$

At late times, the shell is approximately at $r_-\approx$ and for the region in between $r_+$ and $r_-$, we have $\dot{r} \approx -E$. Thus, the observer crosses the shell approximately a proper time $2m\epsilon/E$ after crossing the outer horizon.

To find the amount of proper time that it takes for the observer to encounter the singularity, we must also find the behavior of the observer in the interior of the shell. To begin with, we calculate the quantity $v^au^a$ when the observer crosses the shell. From equation (1) we find

$$v^au^a = -F\dot{V}\dot{v} + \dot{V}\dot{r} + \dot{v}\dot{R}. \quad (29)$$

Then, using equations (21), (27) and (29), we obtain

$$v^au^a = F^{-1}(\dot{R}\dot{r} - \sqrt{\dot{R}^2 + F}\sqrt{\dot{r}^2 + \bar{F}}). \quad (30)$$

Since when the observer crosses the shell we have $\dot{R} > 0$ and $\dot{r} < 0$ and it follows that to lowest order in $\epsilon$

$$v^au^a = -2F^{-1}|\dot{R}\dot{r}| \quad (31)$$

and then using equation (18) we obtain

$$v^au^a = -F^{-1}\frac{E\epsilon^2_0}{\sqrt{1 + c_0^2}}. \quad (32)$$

Since $F$ is very small when the observer crosses the shell, it follows that $v^au^a$ is very large there. However, since $v^a$ satisfies the geodesic equation, and since the Christoffel symbols in the thin shell spacetimes have only jumps rather than delta functions, it follows that $v^a$ is continuous across the shell. Since the thin shell formalism also requires that the metric and $u^a$ are continuous across the shell, it follows that immediately upon entering the interior of
the shell $\nu^a u_a$ is still given by the expression in equation (32). But the interior of the shell is Minkowski spacetime, and the shell has

$$R = \frac{c_0^2}{2\sqrt{1 + c_0^2}}.$$  

(33)

So it follows from equation (32) that in the rest frame of the center of the shell, the observer is an ingoing geodesic with

$$\gamma = F^{-1} \frac{E c_0^2}{1 + c_0^2}.$$  

(34)

This is a very large $\gamma$ factor, so even though the observer must travel a distance of $2m$ (to the center and back) to encounter the singularity, this only takes a proper time of

$$\Delta \tau = 2m F \frac{1 + c_0^2}{E c_0^2}.$$  

(35)

Since $F$ is given by equation (25), it follows that this is a very small proper time. The motion of the observer is shown in figure 3.

3. Conclusion

Though the pictures drawn are somewhat different, our calculations agree with the main conclusion of [2]. For a nearly extremal black hole formed by gravitational collapse an observer crossing the event horizon at late times almost immediately encounters a curvature
singularity. Note however, that this statement involves two different small numbers: $\epsilon$ which expresses how close to extremality the black hole is, and $m/v_h$ which measures how late the observer crosses the black hole event horizon. (Here $v_h$ is the value of $v$ at which the observer crosses the horizon). In all cases, the proper time from the event horizon to the shell is of order $m\epsilon$. For $m/v_h \gg \epsilon$, the proper time from shell to singularity is of order $m^3/v_h^2$, while for $m/v_h \ll \epsilon$ the proper time from shell to singularity is completely negligible.

We now consider the effect the nature of the singularity could have on this result. Throughout, we have assumed that the singularity is null. However, it is generally thought that when black holes form the singularity starts out null near the horizon and then becomes spacelike further inside the black hole. However, any such change would only serve to decrease the amount of time it would take for an observer to encounter the singularity, so the conclusion that the observer encounters the singularity almost immediately after crossing the event horizon remains true.

Finally note that the collapsing thin shell is only one simple model for the formation of a charged black hole by gravitational collapse. It would be interesting to consider other models (e.g. the collapse of a charged scalar field) to see whether the features found in the shell model continue to hold.

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References

[1] Poisson E and Israel W 1990 Phys. Rev. D 41 1796
[2] Marolf D 2010 Gen. Rel. Grav. 42 2337
[3] Hawking S W, Horowitz G and Ross S 1995 Phys. Rev. D 51 4302
[4] Edery A and Constantineau B 2011 Class. Quantum Grav. 28 045003
[5] Ori A and Flanagan E 1996 Phys. Rev. D 53 1754
[6] Israel W 1966 Nuovo Cimento B 44 1
[7] de la Cruz V and Israel W 1967 Nuovo Cimento A 51 744
[8] Kuchar K 1968 Czech. J. Phys. B 18 435
[9] Chase J 1970 Nuovo Cimento B 67 136