TIME-OPTIMAL CONTROL OF SPIN SYSTEMS

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Introduction

This diploma thesis, written under the supervision of Prof. Uwe Helmke at Würzburg University, discusses some aspects of time-optimal control theory of bilinear systems

\[ (*) \quad \dot{U} = \left( X_0 + \sum_{i=1}^{m} v_i X_i \right) U, \]

\( U \in G \), where \( G \) is a Lie group, \( X_0, ..., X_m \) are fixed elements of its Lie algebra \( g \), and \( v_i : [0, T] \to \mathbb{R} \) are the control functions.

The resurgence of such systems in recent years has been caused by applications in quantum computing and nuclear magnetic resonance spectroscopy, which are related to the question of manipulating effectively ensembles of coupled spin-particles. The dynamics of such spin systems are governed by a Schrödinger equation which takes the form of equation (\( * \)) with \( G = SU(2^n) \).

In the focus of this work is the problem of time-optimal control of such systems, i.e. the question of how to steer the system from a given initial state \( U(0) = U_0 \) to a prescribed terminal state \( U_F \) in least possible time. This is not quite a classical optimal control problem, since the control variables \( v_i \) can be chosen to be arbitrarily large, so that there will be a whole subgroup \( K \) of \( G \), all of whose points being reachable from identity within arbitrarily small time \( t \).

Non-linear optimal control problems like this play a crucial role in control theory and a number of tools have been developed in order to solve them, most notably the maximum principle of Pontrjagin. On the other hand, under additional assumptions on the class of system (\( * \)), an explicit solution can be obtained using methods from the theory of Lie groups and Lie algebras, such as the Cartan decomposition and Riemannian symmetric spaces. This is one approach followed in current research on this subject, cf. [2], [16] and [17].

The goal of this thesis is to take up the geometric ideas as formulated in the paper [16] and to present them in a setting which is both mathematically rigorous and accessible without assuming too many prerequisities.

The first chapter is aimed to give an overview on the relevant parts of Lie Theory and Geometric Control Theory which serve as the framework for the subsequent analysis of the time-optimal control problem associated with system (\( * \)).
In Chapter 2 we follow the geometric approach of [16] with its idea of replacing the original system on the Lie group $G$ by a reduced system on a homogeneous space $G/K$. This idea is given a precise formulation in terms of the equivalence theorem of Section 2.3. Although this result lacks, in contrast to Pontrjagin’s maximum principle, a recipe of how to compute time-optimal controls explicitly, it nevertheless contributes towards a solution of the control problem. This is mainly due to the following facts.

1. The equivalence theorem allows for a reduction of the dimension of the state space of the control system.
2. The space of control parameters of the reduced system is, in contrast to that of the unreduced system, compact. This guarantees the existence of controls which meet the maximality condition of Pontrjagin’s maximum principle.
3. The passage from $(\ast)$ to the reduced system is the first step towards the complete solution of the time-optimal control problem in the special case where the Lie algebra $\mathfrak{g}$ enjoys additional geometric properties, see Section 2.5.

As indicated in (2), the problem of time-optimal control of system $(\ast)$ becomes, after replacing it suitably by a system with bounded controls, approachable via the maximum principle. Our main result in Section 2.4 is a computation of those trajectories of the reduced systems that are, under certain additional assumptions, extremal in the sense of the maximum principle. However, this yields only a large family of candidates for a time-optimal solution of the control problem, and it is not evident of how to determine amongst those the actually optimal trajectories.

In Section 2.5 we are then specializing to the situation where the Lie algebra $\mathfrak{g}$ has the additional property of being semisimple and belonging to a symmetric Lie algebra pair $(\mathfrak{g}, \mathfrak{t})$ (cf. Definition 1.2.5). This makes the reduced time-optimal control problem on the homogeneous space $G/K$ accessible to a geometric solution. The reason for this is that now any point $[U_F]$ of the homogeneous space $G/K$ is contained in the projection $[A] \subseteq G/K$ of a suitable abelian subgroup $A$ of $G$, and one always can steer the reduced system between any two points of $[A]$ along a sequence of geodesics of $[A]$. An application of Kostant’s convexity theorem then shows that such a choice of controls is indeed time-optimal. The original problem on the group $G$ is thus reduced to a control problem on a so-called flat submanifold $[A]$ of $G/K$. The control problem reduced this way involves only commuting vector fields, which makes it possible to solve it explicitly.

Optimal controls for the original control system $(\ast)$ may in a subsequent step be obtained from those for the reduced system on $G/K$ by again utilising the equivalence theorem as derived in Section 2.3.
The final chapter is devoted to a discussion of low-dimensional examples of spin-systems such as one- and two-particle systems. These are well suited for explicit computations, but are at the same time general enough objects to illustrate the theory developed in the second chapter.

I am very grateful to my supervisor Professor Uwe Helmke for constantly supporting me in writing this thesis. Also, I would like to thank Martin Kleinsteuber for a number of helpful comments on this subject. Finally, I am indebted to Dr. Gunther Dirr for all his commitment in reading and discussing various aspects of this diploma thesis.
CHAPTER 1

Basic Notions from Lie Theory and Geometric Control

1.1. Lie Groups and Lie Algebras

The problem of steering a quantum mechanical spin system we are interested in can be formulated as a control problem on a Lie group, or a homogeneous space. Its solution involves (amongst others) methods from the theory of Lie groups, Lie algebras, and homogeneous spaces. In this section I shall state only those definitions and theorems that will be used later. I have nevertheless tried to make this exposition as self-contained as possible. The results that will be mentioned are all standard. They can be found in the book [6] and will therefore be stated without proof.

**Definition 1.1.1.** A *Lie group* \((G, \cdot)\) is a smooth manifold \(G\) endowed with the operations of group multiplication \(\cdot\) and group inversion such that the map

\[
G \times G \rightarrow G, \quad (g_1, g_2) \mapsto g_1 \cdot g_2^{-1}
\]

is smooth.

**Example 1.1.2.** The general linear group \(G = Gl_n\mathbb{R}\) of invertible, real \((n \times n)\)-matrices, and closed subgroups of this such as \(Sl_n\mathbb{R}\) and \(SO_n\mathbb{R}\).

The group \(G = SU_n \subseteq Gl_n\mathbb{C}\) of unitary \((n \times n)\)-matrices of determinant 1 will be the most important example to us.

**Definition 1.1.3.** A *Lie algebra* is a (real or complex) vector space \(\mathfrak{g}\) together with a skew symmetric bilinear operation \([\cdot, \cdot]\) such that *Jacobi’s identity* holds:

\[
[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \text{ for all } X, Y, Z \in \mathfrak{g}.
\]

The importance of Lie theory in many fields of mathematics and physics arises from the fact that there is a natural linearization of both the manifold structure (i.e. tangent spaces) and the conjugation map \((g, h) \mapsto hgh^{-1}\) (giving the tangent spaces the structure of Lie algebra) that allows to study nonlinear problems on the group by translating them into linear problems on the Lie algebra level. These two structures are closely related by the exponential map.

In order to associate a Lie algebra structure to a Lie group we make the following definition.
DEFINITION 1.1.4. A vector field \( \chi \) on \( G \) is called right-invariant if it satisfies

\[
\chi(g) = D_1 R_g(\chi(1)) \quad \text{for all} \quad g \in G
\]

Here \( R_g : h \mapsto hg \) denotes right-translation by \( g \).

LEMMA 1.1.5. Any right-invariant vector field \( \chi \in \Gamma(TG) \) is smooth and complete. The set of right-invariant vector fields is closed under the Lie bracket \([\cdot, \cdot]\) on \( \Gamma(TG) \). Any \( X \in T_1 G \) can be extended uniquely to a right-invariant vector field \( \hat{X} \) with \( \hat{X}(1) = X \). In particular, the space of right-invariant vector fields has dimension equal to \( \dim G \).

With these preparations in mind we are in position to endow the tangent space at identity \( 1 \) of the Lie group \( G \) with a Lie algebra structure.

PROPOSITION 1.1.6. Set \( \mathfrak{g} := T_1 G \). Then \( \mathfrak{g} \) is a Lie algebra with bracket

\[
[X, Y] := - \left[ \hat{X}, \hat{Y} \right](1).
\]

In view of Lemma 1.1.5 the following definition makes sense.

DEFINITION 1.1.7. Let \( X \in \mathfrak{g} \) and \( \gamma_X \) be the integral curve of \( \hat{X} \) starting at \( \gamma_X(0) = 1 \). Then define the exponential map

\[
\exp : \mathfrak{g} \longrightarrow G
\]

by

\[
\exp(X) := \gamma_X(1).
\]

LEMMA 1.1.8. The so defined map \( \exp \) is smooth, its differential at \( X = 0 \) being \( D_0 \exp = \text{id}_\mathfrak{g} \). In particular, \( \exp \) is a diffeomorphism near \( X = 0 \).

EXAMPLE 1.1.9. \( G = \text{Gl}_n \mathbb{R} \) has Lie algebra \( \mathfrak{g} = \mathbb{R}^{n \times n} \) and the exponential map is given by

\[
\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.
\]

There is a natural smooth Lie group action of \( G \) on itself given by

\[
\alpha : G \times G \longrightarrow G, \quad (g, h) \mapsto ghg^{-1}
\]

with the identity as a fixed point. Thus differentiating \( \alpha \) at the identity with respect to the second variable yields the smooth group homomorphism

\[
\text{Ad} : G \longrightarrow \text{Gl}(\mathfrak{g}), \quad g \mapsto \text{Ad}_g
\]

with

\[
\text{Ad}_g(X) = \left. \frac{d}{dt} \alpha(g, \exp(tX)) \right|_{t=0}.
\]

We call this homomorphism adjoint representation of \( G \) on \( \mathfrak{g} \). Differentiating \( \text{Ad} \) at the identity \( g = 1 \) yields the linear map

\[
ad : \mathfrak{g} \longrightarrow \text{End}(X), \quad X \mapsto \text{ad}(X)
\]
with
\begin{align}
(1.1.11) \quad \text{ad}X(Y) &= \left. \frac{d}{dt} \text{Ad}_{\exp tX}(Y) \right|_{t=0}.
\end{align}

This map \( \text{ad} \) is a homomorphism of Lie algebras, i.e.
\begin{align}
(1.1.12) \quad \text{ad} [X,Y] &= \text{ad}X \circ \text{ad}Y - \text{ad}Y \circ \text{ad}X
\end{align}

holds for all \( X,Y \in g \). For this reason the map \( \text{ad} \) is called \textit{adjoint Lie algebra representation}.

\textbf{Theorem 1.1.10.} \textit{For all} \( X,Y \in g \),
\begin{align}
(1.1.13) \quad \text{ad}X(Y) &= [X,Y].
\end{align}

For later use we make the following important definition.

\textbf{Definition 1.1.11.} On the Lie algebra \( g \) define the \textit{Killing form} \( \kappa \) to be the bilinear form
\begin{align}
(1.1.14) \quad (X,Y) \mapsto \langle X,Y \rangle = \text{tr} (\text{ad}X \circ \text{ad}Y).
\end{align}

It is not difficult to see that the Killing form is symmetric and \( \text{ad} \)-invariant, i.e.
\begin{align}
(1.1.15) \quad \kappa ([X,Y],Z) &= \kappa (X,[Y,Z])
\end{align}

holds for all \( X,Y,Z \in g \).

\textbf{Example 1.1.12.} The Killing form on \( g = \text{su}(n) \) is given by
\begin{align}
(1.1.16) \quad \kappa(X,Y) &= 2n\text{tr}(XY).
\end{align}

\section{1.2. Homogeneous Spaces, Riemannian Symmetric Spaces, and Maximal Tori}

Throughout this section \( G \) denotes a Lie group and \( K \subseteq G \) a closed subgroup (which is known also to be a Lie group in the induced topology). We first of all collect some facts concerning the space of left cosets
\begin{align}
(1.2.1) \quad G/K := \{ (gK) | g \in G \},
\end{align}

which we also call a \textit{G-homogeneous space}.

\textbf{Proposition 1.2.1.} \( G/K \) equipped with the quotient topology is a \textit{Hausdorff topological space}. It inherits from \( G \) the structure of smooth manifold. This manifold structure can be characterized as the unique differentiable structure such that the canonical projection
\begin{align}
(1.2.2) \quad \pi : G \longrightarrow G/K
\end{align}

is a smooth submersion. In the language of fibre bundles, the triple \((\pi, G, G/K)\) is a principal \( K \)-fibre bundle over \( G/K \).
There is a natural action of the Lie group $G$ on the homogeneous space $G/K$:

$$\varphi : G \times G/K \rightarrow G/K, \quad (g, hK) \mapsto ghK.$$  

This action is smooth and transitive on $G/K$. There is an important converse to this observation:

**Theorem 1.2.2.** (Theorem on transitive Lie group actions). Let $\sigma : G \times M \rightarrow M$ be a smooth and transitive action of $G$ on the manifold $M$, and $p \in M$ arbitrary. Then $\sigma$ is equivalent to the natural action $\varphi$ of $G$ on $G/\text{Stab}_p \sigma$ in the sense that there is a diffeomorphism $\psi : M \rightarrow G/\text{Stab}_p \sigma$ which satisfies

$$\psi(\sigma(g, x)) = \varphi(g, \psi(x))$$

for all $g \in G$ and $x \in M$.

**Proof.** [27], p. 33.

We next give a criterion for the existence of a $G$- (left-) invariant Riemannian metric on the homogeneous space $G/K$. Here a metric $\langle \cdot, \cdot \rangle$ is called $G$-invariant if for all $x \in G/K$, $h \in G$ and $X, Y \in T_x(G/K)$ the following holds:

$$\langle X, Y \rangle_x = \langle D_x \varphi(h, \cdot)(X), D_x \varphi(h, \cdot)(Y) \rangle_{\varphi(h, x)},$$

i.e. any diffeomorphism $\varphi(h, \cdot)$ of $G/K$ is an isometry of $(G/K, \langle \cdot, \cdot \rangle)$.

**Theorem 1.2.3.** Let $G$ and $K$ as before, denote by $\mathfrak{g}$ and $\mathfrak{k}$ its Lie algebras, and let $\mathfrak{p} := \mathfrak{g} / \mathfrak{k} = \{ [X] | X \in \mathfrak{g} \}$ the quotient of the vector spaces $\mathfrak{g}$ and $\mathfrak{k}$. Then there exists a $G$-invariant metric on the homogeneous space $G/K$ if and only if the closure of the set

$$\{ \text{Ad}_k : \mathfrak{p} \rightarrow \mathfrak{p}, [X] \mapsto [\text{Ad}_k X] | k \in K \}$$

is compact in $\text{End}(\mathfrak{p})$.

**Proof.** [7], p. 67.

**Example 1.2.4.** Theorem 1.2.3 can be applied in the situation of a compact Lie group $G$. Consider the action $\sigma$ of $G \times G$ on $G$ which is defined by

$$\sigma((g_1, g_2), g) := g_1 gg_2^{-1}.$$ 

$G \times G$ acts transitively on $G$ with stabilizer

$$K := \text{Stab}_2(1) = \{ (g, g) | g \in G \} \cong G.$$ 

Thus $G$ is $(G \times G)$-equivariantly (in the sense of equation (1.2.4)) diffeomorphic to the homogeneous space $(G \times G)/K$. Theorem 1.2.3 applies to this space because $K$ is compact and thus has compact image under the continuous map $\text{Ad} : K \rightarrow \text{End}(\mathfrak{p}), \quad k \mapsto \text{Ad}_k$. Here $\mathfrak{p}$ denotes the quotient of $\mathfrak{g} \times \mathfrak{g}$ with
\[ k = \{ (X, X) \mid X \in g \}. \] So the manifold \( G \) can be endowed with a metric \( \langle \cdot, \cdot \rangle \) which is invariant under the action \( \sigma \). In particular, the subgroups \( G \times 1 \) and \( 1 \times G \) act on \( G \) by isometries. This means that the metric \( \langle \cdot, \cdot \rangle \) is invariant under both left and right translations, and for this reason will be called bi-invariant.

Another invariance property of the metric \( \langle \cdot, \cdot \rangle \) is the following. Since \( K \) is the stabilizer subgroup of the identity element, we see that for each \( k \in K \) the map \( D_1 \sigma(k) \), which we define by

\[
D_1 \sigma(k)(X) := \frac{d}{dt} \sigma(k, \exp tX)_{t=0}
\]

for all \( X \in g \), is an endomorphism of \( g \). By the definition of an invariant metric it follows that \( \langle D_1 \sigma(k)(X), D_1 \sigma(k)(Y) \rangle_1 = \langle X, Y \rangle_1 \) holds for all \( X, Y \in g \) and \( k \in K \). Now

\[
D_1 \sigma(k)(X) = \frac{d}{dt} \sigma(k, \exp tX)_{t=0} = \text{Ad}_k X,
\]

and therefore

\[
\langle \text{Ad}_k X, \text{Ad}_k Y \rangle_1 = \langle X, Y \rangle_1
\]

for all \( X, Y \in g \) and \( k \in K \). Furthermore, let \( Z \in \mathfrak{t} \) arbitrary. Then for all \( t \in \mathbb{R} \),

\[
\langle \text{Ad}_{\exp tZ} X, \text{Ad}_{\exp tZ} Y \rangle_1 = \langle X, Y \rangle_1.
\]

Differentiating this equation with respect to \( t \) at \( t = 0 \) and using Theorem 1.1.10 yields

\[(1.2.7) \quad \langle [X, Z], Y \rangle_1 = \langle X, [Z, Y] \rangle_1 , \]

i.e. the so-called \( \text{ad} \)-invariance property of the metric \( \langle \cdot, \cdot \rangle \).

**Definition 1.2.5.** Let \( g \) a Lie algebra (over \( \mathbb{R} \) or \( \mathbb{C} \)) and \( \mathfrak{t} \subseteq g \) a subalgebra. Then \( (g, \mathfrak{t}) \) is called a symmetric Lie algebra pair, if there exists a Lie algebra automorphism \( \theta : g \rightarrow g \), which is involutive, i.e. \( \theta^2 = \text{id} \), and which has \( \mathfrak{t} \) as its 1-eigenspace. Such an automorphism \( \theta \) is called Cartan involution.

**Lemma 1.2.6.** Let \( (g, \mathfrak{t}) \) and \( \theta \) as in the previous definition and denote by \( \mathfrak{p} \) the \( -1 \)-eigenspace of \( \theta \). Then \( g = \mathfrak{t} \oplus \mathfrak{p} \). This direct sum decomposition will in the following be named Cartan-like decomposition. It has the following properties:

(i) (commutator relations)

\[ [\mathfrak{t}, \mathfrak{t}] \subseteq \mathfrak{t}, \]
\[ [\mathfrak{t}, \mathfrak{p}] \subseteq \mathfrak{p}, \]
\[ [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{t}. \]

(ii) With respect to the Killing form \( \kappa \) on \( g \), we have that \( \mathfrak{p} \subseteq \mathfrak{t}^\perp \) and \( \mathfrak{t} \subseteq \mathfrak{p}^\perp \). If moreover \( g \) is a semisimple Lie algebra (cf. Definition 1.3.3), then \( \mathfrak{p} = \mathfrak{t}^\perp \) and \( \mathfrak{t} = \mathfrak{p}^\perp \), i.e. \( g \) is the orthogonal sum of \( \mathfrak{t} \) and \( \mathfrak{p} \).
Proof. (i) Let $X$ and $Y$ eigenvectors of $\theta$ with eigenvalues $\lambda, \mu \in \{\pm 1\}$. $\theta$ is a Lie algebra homomorphism, so that

$$\theta [X, Y] = [\theta(X), \theta(Y)] = [\lambda X, \mu Y] = \lambda \mu [X, Y],$$

which implies the result.

(ii) Let $X \in k, Y \in p$. Then, with respect to a basis of $g$ adapted to the direct sum decomposition $g = k \oplus p$, the endomorphisms $adX$ and $adY$ are (by (i)) represented by matrices

$$adX = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad \text{and} \quad adY = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix},$$

So

$$adX \circ adY = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix},$$

which has trace equal to 0, hence $\kappa(X, Y) = 0$. This shows $p \subseteq k^\perp$ and $k \subseteq p^\perp$.

Now let $g$ be semisimple. By Definition 1.3.3, this means that the Killing form is non-degenerate on $g$. So for any subspace $V$ of $g$ it follows that

$$\dim V + \dim V^\perp = \dim g.$$ 

In particular, $\dim k^\perp = \dim p$. Now $p \subseteq t^\perp$, and therefore $p = t^\perp$. The same conclusion applies to give $t = p^\perp$. \hfill \Box

Symmetric Lie algebra pairs are closely related to a special class of homogeneous spaces $G/K$, called Riemannian symmetric spaces. Those can be characterized as Riemannian manifolds which are endowed with a $G$-invariant metric $\langle \cdot, \cdot \rangle$ such that there exists an isometry $\Phi$ which fixes the point $K \in G/K$ and reverses all the geodesics through $K$. Any such Riemannian symmetric space $(G/K, \langle \cdot, \cdot \rangle)$ gives rise to a symmetric Lie algebra pair $(g, t)$.

There is a natural group homomorphism from $G$ to the Lie group of isometries $Isom(G/K)$ of $(G/K, \langle \cdot, \cdot \rangle)$, which is given by

$$\varphi : g \mapsto \varphi(g, \cdot),$$

with $\varphi(g, \cdot)$ as in equation (1.2.3). In the case where the Lie group $G$ is connected and semisimple, one can show that $\varphi$ is indeed an isomorphism between $G$ and the connected component of identity of $Isom(G/K)$, cf. [21], p. 143. The Cartan involution $\theta$ can then be viewed as an “infinitesimal isometry” which is related to $\Phi$ as follows:

$$\theta(X) = \frac{d}{dt} \varphi^{-1} (\Phi \circ \varphi (\exp tX) \circ \Phi)|_{t=0},$$

cf. [13], p. 227-228. Thus the expression $t \mapsto \varphi^{-1} (\Phi \circ \varphi (\exp tX) \circ \Phi)$ can be regarded as an 1-parameter subgroup of $G$. Equation (1.2.9) then states that this 1-parameter subgroup is generated by $\theta(X)$.

Symmetric Lie algebra pairs will be of importance in the sequel, since the
control problem we are going to discuss in Section 2.2 can be reformulated as a problem on a homogeneous space $G/K$ (see Section 2.3) and then be solved, provided this space $G/K$ happens to be symmetric (see Section 2.5).

**Example 1.2.7.** (cf. [10], Ch. X). For the following pairs $(G, K)$ of Lie groups the homogeneous spaces $G/K$ are all symmetric. Pairs in the left columns lead to noncompact examples, those in the middle columns to compact ones.

| Type | $G$         | $K$         | $G$         | $K$         | dim                       |
|------|-------------|-------------|-------------|-------------|---------------------------|
| A I  | $SL_n(\mathbb{R})$ | $SO(n)$     | $SU(n)$     | $SO(n)$     | $\frac{1}{2}(n-1)(n+2)$  |
| A II | $SU^*(2n)$  | $Sp(n)$     | $SU(2n)$    | $Sp(n)$     | $(n-1)(2n+1)$             |
| A III| $SU(p,q)$   | $SU(p) \times U(q)$ | $SU(p+q)$  | $SU(p) \times U(q)$ | $2pq$                    |
| BD I | $SO_0(p,q)$ | $SO(p) \times SO(q)$ | $SO(p+q)$  | $SO(p) \times SO(q)$ | $pq$                     |
| D III| $SO^*(2n)$  | $U(n)$      | $SO(2n)$    | $U(n)$      | $n(n-1)$                 |
| C I  | $Sp_n(\mathbb{R})$ | $U(n)$      | $Sp(n)$     | $U(n)$      | $n(n+1)$                 |
| C II | $Sp(p,q)$   | $Sp(p) \times Sp(q)$ | $Sp(p+q)$  | $Sp(p) \times Sp(q)$ | $4pq$                    |

These examples give rise to the following table of symmetric Lie algebra pairs $(\mathfrak{g}, \mathfrak{t})$. The action of the Cartan involution on $X \in \mathfrak{g}$ is described under $\theta$, while the rightmost column shows the rank of the algebra $\mathfrak{g}$, which will be defined in Theorem 1.2.8.

| Type | $\mathfrak{g}$ | $\mathfrak{t}$ | $\mathfrak{g}$ | $\mathfrak{t}$ | $\theta$ | $\text{rk}$ |
|------|----------------|----------------|----------------|----------------|-----------|------------|
| A I  | $\mathfrak{sl}_n(\mathbb{R})$ | $\mathfrak{so}(n)$ | $\mathfrak{su}(n)$ | $\mathfrak{so}(n)$ | $-X^T$     | $n-1$     |
| A II | $\mathfrak{su}^*(2n)$ | $\mathfrak{sp}(n)$ | $\mathfrak{su}(2n)$ | $\mathfrak{sp}(n)$ | $-J_nX^TJ_n^{-1}$ | $n-1$   |
| A III| $\mathfrak{su}(p,q)$ | $\mathfrak{su}(p) \times \mathfrak{u}(q)$ | $\mathfrak{su}(p+q)$ | $\mathfrak{su}(p) \times \mathfrak{u}(q)$ | $I_{p,q}X_1I_{p,q}$ | $\min(p,q)$ |
| BD I | $\mathfrak{so}(p,q)$ | $\mathfrak{so}(p) \times \mathfrak{so}(q)$ | $\mathfrak{so}(p+q)$ | $\mathfrak{so}(p) \times \mathfrak{so}(q)$ | $I_{p,q}X_1I_{p,q}$ | $\min(p,q)$ |
| D III| $\mathfrak{so}^*(2n)$ | $\mathfrak{u}(n)$ | $\mathfrak{so}(2n)$ | $\mathfrak{u}(n)$ | $J_nX_1J_n^{-1}$ | $\frac{1}{2}n$ |
| C I  | $\mathfrak{sp}_n(\mathbb{R})$ | $\mathfrak{u}(n)$ | $\mathfrak{sp}(n)$ | $\mathfrak{u}(n)$ | $J_nX_1J_n^{-1}$ | $1$ |
| C II | $\mathfrak{sp}(p,q)$ | $\mathfrak{sp}(p) \times \mathfrak{sp}(q)$ | $\mathfrak{sp}(p+q)$ | $\mathfrak{sp}(p) \times \mathfrak{sp}(q)$ | $K_{p,q}X_1K_{p,q}$ | $\min(p,q)$ |

All those examples lead to irreducible symmetric spaces $G/K$. By this we mean that

- the Lie algebra $\mathfrak{g}$ is semisimple (cf. Definition 1.3.3) and $\mathfrak{p} \subseteq \mathfrak{g}$ contains no ideal of $\mathfrak{g}$ other than the zero ideal, and
- the Lie algebra representation
  
  $$\mathfrak{t} \rightarrow \text{End}(\mathfrak{p}), \quad X \rightarrow \text{ad}X$$

is irreducible.

In fact, the above list by E. Cartan exhausts all irreducible Riemannian symmetric spaces (up to 12 compact and noncompact exceptional cases).
The notion of a maximal torus of a compact, connected Lie group $G$ will be of considerable interest in the sequel. By this we mean a Lie subgroup $T \subseteq G$ which is

1. abelian,
2. compact,
3. maximal with respect to these properties, i.e. any subgroup $T'$ that satisfies (1) and (2) and contains $T$ already equals $T$.

It is easy to see that any maximal torus is isomorphic to a product of copies of $S^1$, hence the name.

A maximal abelian subalgebra $t$ of a Lie algebra $g$ is defined to be a subalgebra satisfying

1. $[t, t] = 0$,
2. any subalgebra $t'$ that contains $t$ and satisfies (1) is equal to $t$.

The salient facts concerning maximal tori of a compact Lie group, respectively maximal abelian Lie subalgebras, are collected in the following theorem.

**Theorem 1.2.8. (Torus theorem).** Let $G$ be a compact, connected Lie group with Lie algebra $g$. Then the following holds.

1. The equation $T = \exp t$ defines a bijective correspondence between the maximal abelian subalgebras $t$ of $g$ and the maximal tori $T$ of $G$. Every connected abelian subgroup of $G$ is contained in a maximal torus in $G$, and every abelian subalgebra of $g$ is contained in a maximal abelian subalgebra of $g$.

2. All maximal tori in $G$ are conjugate to each other, and $Ad_G$ acts transitively on the set of maximal abelian subalgebras of $g$. Each element of $G$ is conjugate to an element of a given maximal torus, and $Ad_G(t) = g$ for any maximal abelian subalgebra $t$ of $g$. In particular, any two maximal abelian subalgebras have the same dimension. This dimension is called the rank of the Lie algebra $g$ (the rightmost column of the second table in Example 1.2.7).

**Proof.** [6], Theorem 3.7.1 (iii), (iv). \qed

### 1.3. Root Space Decomposition and Semisimple Lie Algebras

**1.3.1. Root space decomposition of a compact Lie algebra.** Throughout this section let $g$ denote the Lie algebra of a compact Lie group $G$. For simplicity we will refer to the Lie algebra $g$ of such a Lie group as a compact Lie algebra, although there is an intrinsic definition of compact Lie algebras in terms of the Killing form, which differs from ours. For such a Lie algebra there exists the well-known root space decomposition into a direct sum of subspaces which are simultaneously invariant under the endomorphisms $adX$, where $X$ is an element of a maximal abelian subalgebra $t$ of $g$. The properties of such a
decomposition will be widely used in the proof of the time-optimal torus theorem, cf. Section 2.5.

To make the theorem on the Jordan normal form applicable it is convenient to pass to the complexification $g_C = g \otimes \mathbb{C}$ of $g$ which we give the structure of complex Lie algebra by defining its bracket as

\[(1.3.1) \quad [A + iB, A' + iB'] := [A, A'] - [B, B'] + i([A, B'] + [B, A'])\]

for $A, A', B, B' \in g$.

There also is a unique linear extension of any $A \in \text{End}(g)$ to a complex linear endomorphism of $g_C$, which will again be denoted by $A$.

The existence of a root space decomposition is based on the following lemma.

**Lemma 1.3.1.** For each $X \in g$, the endomorphism $\text{ad}X \in \text{End}(g_C)$ is diagonalizable, with only purely imaginary eigenvalues.

The proof, for which we refer the reader to [6], uses in a crucial way the boundedness of the set $\text{Ad}G \subseteq \text{End}(g_C)$ to conclude that the invariant subspaces $g_j = c_jI + N_j$ in the Jordan decomposition of $\text{ad}X$ have nilpotent part $N_j$ equal to 0 and eigenvalues $c_j \in i\mathbb{R}$.

**Theorem 1.3.2.** (Root space decomposition of $g_C$). Let $t$ be any abelian subalgebra of $g$. Set $\alpha_0 := 0 \in t^*$. Then there is a finite set $\Sigma = \{i\alpha_1, \ldots, i\alpha_m\}$ of non-zero real-linear forms $t \to \mathbb{R}$ and a decomposition

\[(1.3.2) \quad g_C = g_0 \oplus \bigoplus_{j=1}^m g_{\alpha_j}\]

such that $g_{\alpha_j} \neq \{0\}$, and

\[(1.3.3) \quad \text{ad}X(Y) = i\alpha_j(X)Y\]

holds for all $X \in t$, $Y \in g_{\alpha_j}$, and $j = 0, \ldots, m$. Moreover, if $\alpha \in \{0\} \cup \Sigma$, then $-\alpha \in \{0\} \cup \Sigma$, and $g_{-\alpha} = \overline{g_{\alpha}}$. In particular, $g_0 = \overline{g_0}$, so that $g_0$ is of the form $a + ia$ with $a = g_0 \cap g$. Also, $t \subseteq a$.

The following commutator relations hold:

\[(1.3.4) \quad [g_{\alpha_i}, g_{\alpha_j}] \subseteq \begin{cases} g_{\alpha_i + \alpha_j}, & \text{if } \alpha_i + \alpha_j \in \{0\} \cup \Sigma, \\ \{0\}, & \text{otherwise}. \end{cases}\]

If $t$ is maximal abelian in $g$, then $h := t + it = g_0$ and $t = g_0 \cap g$. In that case there are the decompositions

\[(1.3.5) \quad g_C = h \oplus \bigoplus_{j=1}^m g_{\alpha_j},\]

and

\[(1.3.6) \quad g = t \oplus \bigoplus_{\alpha_j \in \Sigma^+} (g_{\alpha_j} \oplus g_{-\alpha_j}) \cap g,\]
where $\Sigma^+$ is any subset of $\Sigma$ which satisfies
\[(1.3.7)\quad \alpha_j \in \Sigma^+ \iff -\alpha_j \notin \Sigma^+\]
for all $j = 1, \ldots, m$. The subspaces $g_{\alpha_j}$ are then called root spaces, the linear forms $i\alpha_j$ are named roots.

**Proof.** [6], p. 145-146.

For any root $i\alpha \in \Sigma$, the map $\alpha : t \to \mathbb{R}$ defines a non-zero, real-linear form. Its kernel $\ker \alpha$ therefore defines a hyperplane in $t$, which is called the root hyperplane for $\alpha$. The connected components of the set
\[(1.3.8)\quad t \setminus \left( \bigcup_{\alpha \in \Sigma} \ker \alpha \right)\]
are called Weyl chambers; they are open, convex polyhedral cones in $t$.

We next define the Weyl group $W$ to be the quotient
\[(1.3.9)\quad W = N(t)/\text{Stab}(t),\]
with
\[(1.3.10)\quad N(t) = \{ g \in G \mid \text{Ad}_g t = t \}\]
the the normalizer of $t$ in $G$, and
\[(1.3.11)\quad \text{Stab}(t) = \{ g \in G \mid \text{Ad}_g X = X \ \forall X \in t \}\]
the pointwise stabilizer of $t$ in $G$. The group $W$ turns out to be finite. Furthermore, as a consequence of the Torus Theorem 1.2.8, it can be shown that the isomorphism type of $W$ does not depend on the choice of $t$. It is therefore justified to call $W$ the Weyl group of the Lie algebra $g$.

There is the following action of $W = \{ [g] \mid g \in N(t) \}$ on $t$:
\[(1.3.12)\quad W \times t \to t, \quad ([g], X) \mapsto gXg^{-1}.\]

The action of $W$ is transitive on the set of Weyl chambers. This statement is part of the Weyl covering theorem, cf. [6], p. 153.

At this point we conclude those general considerations on compact Lie algebras and turn to compact semisimple algebras, where a refined version of some of the statements made before can be given.

### 1.3.2. Compact semisimple Lie algebras.

**Definition 1.3.3.** A Lie algebra $g$ (over $\mathbb{C}$ or over $\mathbb{R}$) is called semisimple if the Killing form
\[(1.3.13)\quad \kappa : g \times g \to \mathbb{R}, \quad (X, Y) \mapsto \text{tr}(\text{ad}X \circ \text{ad}Y)\]
is nondegenerate on \( g \).

The Lie algebra \( g \) is called \textit{simple} if it is not abelian and does not contain any ideals other than \( \{0\} \) and \( g \).

**Remark 1.3.4.** The relation between simple and semisimple Lie algebras is such that every semisimple Lie algebra splits uniquely (up to isomorphism) into an orthogonal (with respect to the Killing form) sum of simple Lie algebras, cf. [12], p. 23.

**Example 1.3.5.** See Example 1.2.7. The Lie algebras \( g \) which appear in the second table are all simple.

For the remainder of this section let \( g \) be a semisimple, compact Lie algebra over the reals. Furthermore, let \( t \) be a maximal abelian subalgebra of \( g \), and denote by \( \Sigma \subseteq t^* \) the set of roots in the root space decomposition of \( g \) with respect to \( t \). Since the Killing form is nondegenerate on \( g \) we may identify \( t^* \) with \( t \) in the following manner:

\[
(1.3.14) \quad t^* \ni \lambda \mapsto X_{\lambda} \in t \text{ such that } \lambda(Y) = \kappa(X_{\lambda}, Y) \text{ is satisfied for all } Y \in g.
\]

We will name \( X_\alpha \in t \) a coroot, if \( i\alpha \in \Sigma \) is a root. The set of coroots is denoted by \( \Sigma^* \). One observes that the coroots are perpendicular to their respective root hyperplanes.

In addition to the results of Subsection 1.3.1, the following theorem holds.

**Theorem 1.3.6.** Let \( t \subseteq g \) be a maximal abelian subalgebra of the compact, semisimple Lie algebra \( g \). Then:

(i) There exists \( X \in t \) such that \( t = \ker(\text{ad}X) \). Conversely, the kernel of \( \text{ad}X \), \( X \in g \) arbitrary, is a maximal abelian subalgebra of \( g \) if and only if its dimension is the smallest one possible. Such an \( X \in g \) is also called a regular element. In that case the dimension of \( \ker(\text{ad}X) \) coincides with the rank of the algebra \( g \) as defined in 1.2.8.

(ii) The set \( \Sigma^* \) of coroots spans \( t \) as a vector space.

(iii) The root spaces in a root space decomposition of \( g_C \) with respect to \( t \) are all 1-dimensional.

**Proof.** Cf. [12], p. 80 for a proof of (i), and p. 39 for a proof of (ii) and (iii).

**Example 1.3.7.** Consider the Lie algebra \( g = su(n) \).

The complexification of \( g \) is the simple Lie algebra \( g_C = sl_n \mathbb{C} \). Choose

\[
t := \left\{ \text{diag}(\theta_1, \ldots, \theta_n) | \theta_j \in \mathbb{R}, \sum_j \theta_j = 0 \right\}
\]
Indeed, a calculation yields

\[ E_{ij} := (e_{rs})_{r,s=1,...,n} \] with \( e_{rs} = \begin{cases} 1, & \text{if } r = i, s = j, \\ 0, & \text{otherwise.} \end{cases} \]

Then the root space decomposition of \( g_C \) with respect to \( t \) is given by

\[ g_C = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C} E_{ij}. \]

The corresponding roots are

\[ \alpha_{ij} : t \longrightarrow i\mathbb{R}, \quad \text{id} \operatorname{diag}(\theta_1, ..., \theta_n) \longrightarrow i(\theta_i - \theta_j). \]

Indeed, a calculation yields

\[ \begin{pmatrix} i\theta_1 \\ \vdots \\ i\theta_n \end{pmatrix} \begin{pmatrix} 0 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ 1 & \cdots & 0 \end{pmatrix} = i(\theta_i - \theta_j)E_{ij}. \]

The root spaces \( g_{ij} = \mathbb{C} E_{ij} \) obey the commutator relations

\[ [g_{ij}, g_{kl}] = \begin{cases} g_{il}, & \text{if } j = k, i \neq l, \\ g_{kj}, & \text{if } i = l, j \neq k, \\ 0, & \text{else.} \end{cases} \]

>From the identification of \( \mathfrak{h}^\ast \) with \( \mathfrak{h} \) via the Killing form one obtains the coroots \( X_{\alpha_{ij}} = \text{id} \operatorname{diag}(0, ..., 1, ..., -1, ..., 0) \) \((\text{here } 1 \text{ is the entry at the } i\text{-th}, -1 \text{ at the } j\text{-th position})\). They satisfy the relations

\[ \kappa \left( X_{\alpha_{ij}}, X_{\alpha_{kl}} \right) = \begin{cases} 0, & \text{if } \{i,j\} \cap \{k,l\} = \emptyset, \\ -1, & \text{if } i = l, j \neq k \text{ or } i \neq l, j = k, \\ 1, & \text{if } i = k, j \neq l \text{ or } i \neq k, j = l, \\ -2, & \text{if } i = l, j = k, \\ 2, & \text{if } i = k, j = l, \end{cases} \]

so that the coroots include angles equal to \( 0, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{3}, \text{ or } \pi \).

The Weyl group \( W \) acts on \( X \in t \) by permuting the entries on the diagonal of \( X \) and therefore turns out to be isomorphic to the symmetric group \( S(n) \).

On \( g \) one defines the involution \( \theta : X \longmapsto -X^T \). Its \( 1 \)-eigenspace \( t \) is equal to the subalgebra \( \text{so}(n) \) of \( g \), while its \(-1 \)-eigenspace \( p \) comprises those matrices \( X \) of \( g \) which satisfy \( X = X^T \). Therefore, \( (g, t) \) is a symmetric Lie algebra pair in the sense of Definition 1.2.5. In fact, \( \theta \) arises as the Cartan involution associated with the Riemannian symmetric space \( SU(n)/SO(n) \), see Example 1.2.7. Notice also that

\[ t = \sum_{i<j} g \cap \mathbb{C}(E_{ij} + \theta E_{ij}), \]
while
\[ n := \sum_{i<j} g \cap C(E_{ij} - \theta E_{ij}) \]
is complementary to \( h \) in \( p \). This is not accidently and reflects a general correspondence between the Cartan-like decomposition \( g = \mathfrak{t} \oplus p \) and the root space decomposition of \( g_\mathbb{C} \) with respect to a maximal abelian subalgebra \( h \subseteq p \) of \( g \), cf. [10], p. 336.

We next give a modified version of the Torus Theorem 1.2.8 (ii) in the situation of a Lie group \( G \) with semisimple Lie algebra \( g \), which will be of interest later on.

**Lemma 1.3.8.** Let \( G \) be a Lie group with semisimple Lie algebra \( g \), and \( K \) a closed subgroup of \( G \) such that its Lie algebra \( \mathfrak{k} \) together with \( g \) forms a symmetric Lie algebra pair. Furthermore, let \( g = \mathfrak{k} \oplus p \) be the corresponding Cartan-like decomposition, and \( h \) any maximal abelian subalgebra of \( p \). Then
\[ p = \bigcup_{k \in K} \text{Ad}_k h. \]  

**Proof.** [10], Chapter V, Lemma 6.3 (iii).

To conclude this section we cite a theorem that relates the action of the Weyl group \( W \) on \( h \) to the adjoint action of \( K \) on \( p \).

**Theorem 1.3.9.** (Kostant’s convexity theorem). In the setting of the previous Lemma 1.3.8, let \( \Gamma : p \rightarrow h \) be the orthogonal projection with respect to the Killing form on \( g \). Then for any \( X \in h \)
\[ \Gamma(\text{Ad}_K X) = c(W \cdot X), \]  
where \( c \) denotes convex hull.

**Proof.** [20].

**Example 1.3.10.** Consider the symmetric Lie algebra pair \( (g, \mathfrak{t}) \) with \( g = \mathfrak{su}(2) \) and
\[ \mathfrak{t} = \left\{ \begin{pmatrix} iX & \cr & -iX \end{pmatrix} \bigg| X \in \mathbb{R} \right\} \subseteq \mathfrak{su}(2). \]
Its Cartan-like decomposition is \( g = \mathfrak{t} \oplus p \) with
\[ p = \left\{ \begin{pmatrix} 0 & z \cr -\bar{z} & 0 \end{pmatrix} \bigg| z \in \mathbb{C} \right\}. \]
Fix \( X_0 = \begin{pmatrix} 0 & z_0 \\ -\bar{z}_0 & 0 \end{pmatrix} \in p, \ z_0 \neq 0, \) and set \( h = \mathbb{R}X_0 \). It is easily seen that the Weyl orbit of \( Y = \lambda X_0 \in h \) is \( W \cdot Y = \{ \pm Y \} \) and the orbit of the adjoint action
of $K$ on $Y$ is

$$
\text{Ad}_K Y = \left\{ kYk^{-1} \mid k \in K \right\}
$$

$$
= \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} \mid t \in \mathbb{R} \right\}
$$

$$
= \left\{ \begin{pmatrix} 0 & \lambda e^{2it}z_0 \\ -\lambda e^{-2it}z_0 & 0 \end{pmatrix} \mid t \in \mathbb{R} \right\},
$$

i.e. a circle $C \subseteq \mathfrak{p}$ centered at the origin and passing through $Y$. The orthogonal projection of $C$ on $\mathfrak{h}$ is the set $\{ \alpha Y \mid -1 \leq \alpha \leq 1 \}$, which in fact is the convex hull of $\{ \pm Y \}$.

### 1.4. Some Definitions from Geometric Control Theory

This section is aimed to introduce some basic definitions and results from geometric control theory which serves as the appropriate framework for the kind of problem to be considered later. Geometric control theory is primarily interested in the investigation of controlled dynamical systems on a manifold $M$, their behaviour being governed by ODEs of the form

$$
\dot{x} = f(x(t), u(t)), \quad x(0) = x_0,
$$

where the parameter $u$ (the “control function”) is allowed to vary with time $t$ within a given parameter space $U \subseteq \mathbb{R}^m$ (the “control set”).

Given such a system (1.4.1) together with a control set $U$ one naturally can ask the following questions.

1. Does there exists a control function $t \mapsto u(t)$ that transfers the initial state $x_0$ of system (1.4.1) to a prescribed terminal state $x_F = x(t_F)$? Describe the set of all points in $M$ that are reachable in this sense!

2. Proof the existence of time-optimal controls and give explicit construction schemes for them.

To make things precise we introduce some terminology.

**Definition 1.4.1.** A nonlinear control system $\Sigma = (M, f_u, U)$ is a triple consisting of a smooth manifold $M$, a parameter space $U \subseteq \mathbb{R}^m$ and a family

$$
\{ f_u \in \Gamma(TM) \mid u \in U \}
$$

of vector fields on $M$. We will refer to $M$ as the state space of the control system, to $U$ as the space of control parameters, and to $u \in U$ as a control parameter. A control is a path $t \mapsto u(t)$ in the space of control parameters.

A curve $x : [0, T] \to M$ is called an integral curve for the control $u : [0, T] \to U, t \mapsto u(t)$ if it is absolutely continuous, and if

$$
\dot{x}(t) = f_u(t)(x(t))
$$
is satisfied for all $0 < t < T$.

**Notation 1.4.2.** We frequently write $f(x, u)$ rather than $f_u(x)$ for the value of the vector field $f_u$ at the point $x$.

To guarantee the existence of an integral curve as defined above, we make the following standing assumptions:

- The map $u \mapsto f(x, u)$ is Lipschitzian for any fixed $x \in M$.
- The vector field $f_u \in \Gamma(TM)$ is smooth for all $u \in U$.
- The partial derivatives of the map $(x, u) \mapsto f(x, u)$ in directions of $M$ are locally bounded in any point $(x_0, u_0) \in (M \times U)$.
- The control $t \mapsto u(t)$ is measurable and locally bounded on its interval of definition.

Under these assumptions, the existence and uniqueness of an integral curve $t \mapsto x(t)$ with prescribed initial condition $x(t_0) = x_0$ is guaranteed by the Caratheodory theorem for any control $t \mapsto u(t)$, cf. [11], p. 28-29.

**Definition 1.4.3.** Let $\Sigma = (M, f_u, U)$ be a control system, $x_0 \in M$, and $T \geq 0$. We define $R(x_0, T)$ to be the set of all $x_F \in M$ with the property that there exists a control $u : [0, T] \to U$ which generates a trajectory $t \mapsto x(t)$ such that $x(0) = x_0$ and $x(T) = x_F$. We call $R(x_0, T)$ the set of reachable points from $x_0$ at time $t$.

Define furthermore the reachable set from $x_0$ within time $T$ to be

$$R(x_0, T) = \bigcup_{0 \leq t \leq T} R(x_0, t)$$

and the reachable set for $x_0$ to be

$$R(x_0) = \bigcup_{0 \leq T < \infty} R(x_0, T).$$

The system $\Sigma$ is called controllable, if $R(x_0) = M$ holds.

In the sequel we will pay attention to control problems on Lie groups $G$ and homogeneous spaces $G/H$ only. We will therefore be confronted with a special class of control systems.

**Definition 1.4.4.** A control system $\Sigma = (G, f_u, U)$, $U \subseteq \mathbb{R}^m$, on a Lie group $G$ is called affine right-invariant if $\{f_u\}_{u \in U}$ is a family of vector fields on $G$ of the form

$$f(g, u) = X_0(g) + \sum_{i=1}^m u_i X_i(g)$$

with $X_i, i = 0, \ldots, m$, right-invariant and $u = (u_1, \ldots, u_m) \in U \subseteq \mathbb{R}^m$.

Now what about questions (1) and (2) formulated above in the context of Lie groups? There are very detailed investigations on those topics, see e.g. the
paper by V. Jurdjevic and H. J. Sussmann [14] where problem (1) is completely answered, the paper by D. Mittenhuber [23] for a treatise of question (2) as well as Jurdjevic’s book [15]. The following is a survey of the results needed to tackle the question of controllability of those quantum mechanical systems we are finally interested in.

**Theorem 1.4.5.** (Controllability of affine right-invariant systems on Lie groups). Let \( \Sigma = (G, f_u, U) \) with \( f_u(g) = X_0(g) + \sum_{i=1}^m u_i X_i(g) \) be an affine right-invariant system on the Lie group \( G \). Denote by \( \mathfrak{g} \) the Lie algebra of \( G \). Then the reachable set \( R(1) \) is always a semi-group. If \( R(1) \) happens to be a group then it coincides with \( S(X_0, ..., X_m) \), the Lie subgroup of \( G \) generated by the elements \( \exp X_0, ..., \exp X_m \in G \).

Each of the following two conditions is sufficient for \( S(X_0, ..., X_m) \) to be a Lie subgroup:

(i) \( X_0 = 0 \) (absence of a drift term).

(ii) \( S(X_0, ..., X_m) \) is compact.

If (ii) is satisfied, then there is a constant \( T > 0 \) such that \( R(1) = R(1, T) \). Furthermore, under the additional assumption that \( G \) is connected, the following criterion on controllability holds:

(1.4.7) \( \Sigma \) is controllable \iff \( \langle X_0, ..., X_m \rangle_{\text{Lie}} = \mathfrak{g} \).

**Proof.** [9], Lemma 4.5 and Theorems 5.1, 6.5.

A little bit more theory is needed to answer the remaining question (2).

### 1.5. Optimal Control and the Maximum Principle

In this section we take up the discussion of time-optimal control as formulated in question (2), Section 1.4.

**Definition 1.5.1.** Let \( \Sigma = (M, f_u, U) \) be a nonlinear control system and \( \varphi : M \times U \to \mathbb{R} \) a continuous function. For a trajectory \( t \mapsto (x(t), u(t)) \) of \( \Sigma \) with initial point \( x(t_0) = a \) and terminal point \( x(t_1) = b \) we define the cost of transfer between \( a \) and \( b \) to be

(1.5.1) \[ \int_{t_0}^{t_1} \varphi(x(t), u(t)) \, dt. \]

A trajectory \( t \mapsto (\bar{x}(t), \bar{u}(t)) \) of \( \Sigma \) that transfers \( \bar{a} \in M \) to \( \bar{b} \in M \) is called optimal if \( \int_{t_0}^{t_1} \varphi(\bar{x}(t), \bar{u}(t)) \, dt \) is minimal amongst all costs of transfer between \( a \) and \( b \).

In the special case \( \varphi \equiv 1 \) we refer to the corresponding cost functional as time and to the respective optimal trajectories as being time-optimal.
It is convenient to implement the cost function $\varphi$ into the given control system $\Sigma$ as follows. Set $\Sigma_{\text{ext}} := (\mathbb{R} \times M, \tilde{f}_u, U)$, where $\tilde{f}_u$ is the vector field on $\mathbb{R} \times M$ given by

$$\tilde{f}_u(x_0, x) = (\varphi(x, u), f_u(x)).$$

We call $\Sigma_{\text{ext}}$ the cost-extended system for $(\Sigma, \varphi)$.

The geometric significance of the trajectories of $\Sigma_{\text{ext}}$ is that optimal trajectories $t \mapsto \bar{x}(t)$ of $\Sigma$ for the transfer of $a$ to $b$ arise as the projections on $M$ of those trajectories $t \mapsto (\bar{x}_0(t), \bar{x}(t))$ of $\Sigma_{\text{ext}}$ that transfer $(0, a)$ to $(\bar{x}_0(T), b)$ with $\bar{x}_0(T)$ minimal. Such trajectories $t \mapsto (\bar{x}_0(t), \bar{x}(t))$ of the cost-extended system necessarily have their terminal point $(\bar{x}_0(T), b)$ on the boundary of $R(0, a)$. We call this the extremality property (E) of optimal trajectories for $(\Sigma, \varphi)$.

We shall now discuss a necessary condition for a control function $u$ to generate a trajectory which enjoys the extremality property (E). This will lead us to the well-known maximum principle of Pontrjagin. To this aim we need to introduce some terminology from classical mechanics (cf. e.g. [22]). For simplicity we focus at first on the case $M = \mathbb{R}^n$, and then extend the discussion to arbitrary smooth manifolds $M$.

### 1.5.1. The case $M = \mathbb{R}^n$.

In this situation we regard the manifold $N = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ as a state space on which a family $H(x, p, u)$ of so-called Hamiltonian functions, parametrized by the elements $u$ of the space $U$ of control parameters, is given. These Hamiltonian functions are defined by

$$H(\cdot, u) : N \rightarrow \mathbb{R}, \quad (x, p) \mapsto \sum_{i=0}^{n} p_i \tilde{f}_i(x, u).$$

We here consider the tangent vector $\tilde{f}(x, u)$ as an element of $\mathbb{R}^{n+1}$.

Now any smooth function $H : N \rightarrow \mathbb{R}$ defines a Hamiltonian vector field $X_H$ on $N$, whose coordinates are given by

$$\frac{\partial H}{\partial p_0}, \ldots, \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial x_0}, \ldots, -\frac{\partial H}{\partial x_n}.$$

The Hamiltonian vector field $X_H$ can alternatively be described via the canonical symplectic form $\omega$ on $N$. This is a closed 2-form which is defined at each point $(x, p) \in N$ by

$$\omega(X, Y) := X_x Y_p - X_p Y_x \in \mathbb{R},$$

for $X = (X_x, X_p), Y = (Y_x, Y_p) \in T_{(x, p)} N \cong \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. The vector field $X_H$ can then be defined to be the unique vector field on $N$ which satisfies

$$dH(Y) = \omega(X_H, Y)$$

for every vector field $Y$ on $N$.

The Hamiltonian vector field $X_u := X_{H(\cdot, u)}$ associated with the particular
Hamiltonian function \( (1.5.3) \) is in coordinates \((x_0, \ldots, x_n, p_0, \ldots, p_n)\) of \( N \) given by

\[
(1.5.7) \quad X_u = \left( f_0(x, u), \ldots, f_n(x, u), -\sum_{j=0}^{n} p_j \frac{\partial f_j}{\partial x_0}(x, u), \ldots, -\sum_{j=0}^{n} p_j \frac{\partial f_j}{\partial x_n}(x, u) \right).
\]

**Remark 1.5.2.** The function \( \tilde{f}(\cdot, u) = (\varphi(x, u), f_u(x)) \) does by definition not depend on the variable \( x_0 \). Thus the term \(-\sum_{j=0}^{n} p_j \frac{\partial f_j}{\partial x_0}(x, u)\) in equation (1.5.7) vanishes identically. So any integral curve \( t \mapsto (x_0(t), \ldots, x_n(t), p_0(t), \ldots, p_n(t)) \) of the vector field \( X_u \) has constant coordinate \( p_0 \). The Hamiltonian function in a later formulation of Pontrjagin’s maximum principle (cf. Theorem 2.4.2) will for that reason depend on the variables \( x_0, \ldots, x_n \) and \( p_1, \ldots, p_n \) only, while \( p_0 \) will appear as a parameter. As long as we are in the situation of a Euclidian state space \( N = \mathbb{R}^n \times \mathbb{R}^n \), the relevant Hamiltonian function reads

\[
(1.5.8) \quad H(x_1, \ldots, x_n, p_1, \ldots, p_n) = p_0 \varphi(x_1, \ldots, x_n, u) + \sum_{i=1}^{n} p_i x_i,
\]

where \( p_0 \) is a constant.

Now let \( t \mapsto u(t) \) be a control function and denote by \( t \mapsto \gamma(t) := (x(t), p(t)) \) the integral curve of the time-dependent Hamiltonian vector field \( X_{u(t)} \) with initial value \((x_0, p_0)\). Equation (1.5.7) implies that the projection of \( \gamma \) on the first factor of \( N = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \) is equal to the trajectory \( t \mapsto x(t) \) that arises from the control \( t \mapsto u(t) \) of the cost-extended system and has initial value \( x_0 \). The path \( t \mapsto \gamma(t) \) is for this reason called Hamiltonian lift of the path \( t \mapsto x(t) \). The existence of such a Hamiltonian lift gives rise to the idea of expressing the extremality condition (E) suitably as a condition on the time-dependent Hamiltonian function \( H(\cdot, u(t)) \). The following definition introduces the correct extremality condition.

**Definition 1.5.3.** Let \( t \mapsto u(t) \) be a control function. The extremal Hamiltonian \( M(x(t), p(t)) \) associated with the integral curve \( t \mapsto (x(t), p(t)) \) of the time-dependent Hamiltonian vector field \( X_{u(t)} \) is defined by

\[
(1.5.9) \quad M(x(t), p(t)) := \sup_{u \in U} H(x(t), p(t), u).
\]

The statement of Pontrjagin’s maximum principle (PMP) is the following. Assume that the control \( t \mapsto u(t), t \in [0, T] \) generates a trajectory \( t \mapsto x(t) \in \mathbb{R}^{n+1} \) of the cost-extended system \( \Sigma_{\text{ext}} \) which has the extremal property (E). Then the Hamiltonian lift of \( t \mapsto x(t) \) to the path \( t \mapsto (x(t), p(t)) \) satisfies the extremality condition

\[
(1.5.10) \quad H(x(t), p(t), u(t)) = M(x(t), p(t)).
\]
almost everywhere on \([0, T]\).
We demonstrate how to make use of PMP in a concrete but typical situation
(see also [11], p. 191).

**Example 1.5.4.** On \(M = \mathbb{R}\) consider the system

\[
\ddot{x}_1 = u, \quad |u| \leq 1
\]

or equivalently

\[
\begin{cases}
\dot{x} = (x_1, x_2) \in \mathbb{R}^2, \quad |u| \leq 1, \\
\dot{x}_1 = x_2, \\
\dot{x}_2 = u.
\end{cases}
\]

We wish to steer system (1.5.12) from \(x(0) = x_0\) to \(x(t_F) = 0\) such that \(t_F\)
is minimal. Thus we take \(\varphi \equiv 1\) to serve as a cost function. The family of
admissible vector fields for our problem is

\[f_u(x) = (x_2, u).\]

The Hamiltonian function \(H(\cdot, u)\) as defined through equation (1.5.8) reads in
this example

\[H(x, p, u) = p_0 + p_1 x_2 + p_2 u, \quad p_0 \text{ constant},\]

and leads to the Hamiltonian vector field

\[
\begin{cases}
\dot{x} = \frac{\partial H(\cdot, u)}{\partial p}, \\
\dot{p} = -\frac{\partial H(\cdot, u)}{\partial x},
\end{cases}
\]

\[
\iff
\begin{cases}
\ddot{x}_1 = x_2, \\
\ddot{x}_2 = u, \\
\dot{p}_1 = 0, \\
\dot{p}_2 = -p_1.
\end{cases}
\]

The Hamiltonian system (1.5.13) that belongs to a time-optimal control \(t \mapsto \hat{u}(t)\)has by PMP a solution such that

\[H(x(t), p(t), \hat{u}(t)) = \max_{|u| \leq 1} H(x(t), p(t), u) = \max_{|u| \leq 1} (p_1(t)x_2(t) + p_2(t)u).\]

>From this it is immediate that \(\hat{u}(t) = \text{sgn}(p_2(t)), \) if \(p_2(t) \neq 0.\) Therefore,

\[
\max_{|u| \leq 1} H(x(t), p(t), u) = p_1(t)x_2(t) + |p_2(t)|.
\]

>From (1.5.13) it follows that \(p_2(t) = \alpha + \beta t.\) One now solves the equations
involving \(x\) in (1.5.13) with \(\hat{u}(t) = \text{sgn}(p_2(t))\) to obtain the extremal trajectories
of system (1.5.12). The result is that for any initial value \(x(0) = (x_1(0), x_2(0))\)there is exactly one time-optimal trajectory \(t \mapsto x(t).\) This is obtained from
choosing the control \(u\) to be \(u \equiv +1\) as long as

\[x_1 > \frac{x_2^2}{2}, \quad x_2 < 0 \quad \text{or} \quad x_1 > -\frac{x_2^2}{2}, \quad x_2 > 0\]
is satisfied, and switching to \( u \equiv -1 \) if
\[
x_1 = \frac{x_2^2}{2}, \quad x_2 < 0 \quad \text{or} \quad x_1 = -\frac{x_2^2}{2}, \quad x_2 > 0,
\]

or otherwise choosing \( u \equiv -1 \) as long as
\[
x_1 < \frac{x_2^2}{2}, \quad x_2 < 0 \quad \text{or} \quad x_1 < -\frac{x_2^2}{2}, \quad x_2 > 0,
\]
holds, and then switching to \( u \equiv +1 \).

### 1.5.2. The general case

We now discuss how the previous considerations carry over to the case of an arbitrary smooth manifold \( M \). The state space is now taken to be
\[
N := T^* (\mathbb{R} \times M),
\]
the cotangent bundle of \( \mathbb{R} \times M \). The manifold \( N \) carries in a canonical way a symplectic structure \( \omega \) (i.e. \( \omega \) is a non-degenerate closed 2-form), which is defined to be
\[
\omega := d\theta,
\]
where the 1-form \( \theta \) is given by
\[
\theta_\xi(X) := \xi(D_\xi \pi(X))
\]
for \( \xi \in N \) and \( X \in T_\xi N \). Here \( \pi \) denotes canonical projection from \( N \) onto its base manifold \( \mathbb{R} \times M \).

Using the symplectic form \( \omega \) one can repeat the construction of Hamiltonian vector fields, but now in a coordinate-free manner. For any smooth function \( H : N \to \mathbb{R} \) define the Hamiltonian vector field \( X_H \) associated with \( H \) to be the unique vector field on \( N \) with the property that
\[
dH_\xi(Y) = \omega_\xi(X_H(\xi), Y)
\]
holds for all \( \xi \in N \) and \( Y \in T_\xi N \). One can show (cf. [22]) that in suitably defined local coordinates (so-called Darboux coordinates) the Hamiltonian vector field \( X_H \) is of the same form as defined in (1.5.4) for the Euclidian case.

We again introduce a family \( H(\cdot, u) : N \to \mathbb{R} \) of Hamiltonian functions, parametrized by the controls \( u \in U \), as
\[
H(\xi, u) := \xi(f_u(\pi(\xi))).
\]
This definition can be shown to be consistent with that in (1.5.3), and one also can prove that the trajectories \( t \mapsto \xi(t) \) of the Hamiltonian vector field \( X_u := X_{H(\cdot, u)} \) are projected under \( \pi \) to those of the vector field \( \tilde{f}_u \) on \( \mathbb{R} \times M \). In complete analogy to the linear case we refer to the trajectories \( t \mapsto \xi(t) \) as the Hamiltonian lifts of the integral curves of \( \tilde{f}_u \).

We finally adapt the extremality condition of the maximum principle to the new situation of a general state space \( N \).
**Definition 1.5.5.** Let \( t \mapsto u(t) \) be a control function. The extremal Hamiltonian \( M(\xi(t)) \) associated with the integral curve \( t \mapsto (\xi(t)) \) of the time-dependent Hamiltonian vector field \( X_{u(t)} \) is defined by

\[
M(\xi(t)) := \sup_{u \in U} H(\xi(t), u).
\]

The statement of Pontrjagin’s maximum principle on the relationship between the extremality \((E)\) of trajectories of the control system \( \Sigma \) and the extremality of their Hamiltonian lifts as formulated in 1.5.1 remains valid also in the non-linear case.

For a proof and detailed discussion of PMP we refer the reader to the books [1] and [15] and give here the precise statement of the maximum principle for time-optimal control problems.

**Theorem 1.5.6.** Let \( \Sigma = (M, f_u, U) \) be a control system and \( t \mapsto \tilde{u}(t), t \in [0, T] \), a time-optimal control. For each \( u \in U \) define the Hamilton function

\[
H(\cdot, u) : T^*M \to \mathbb{R}, \quad H(\xi, u) = \xi(f_u(\pi(\xi)))
\]

and denote by \( X_u \in \Gamma(T^*M) \) the Hamiltonian vector field for \( H(\cdot, u) \). Then any trajectory of \( \dot{q} = f_{\tilde{u}}(q) \) in \( M \) for the control function \( \tilde{u} \) possesses a Hamiltonian lift to a curve \( t \mapsto \xi(t) \) in \( T^*M \) with the property that the extremality condition

\[
H(\xi(t), \tilde{u}(t)) = M(\xi(t))
\]

holds almost everywhere on \([0, T]\).

**Proof.** [1], Corollary 12.12.

---

### 1.6. Kronecker Product Formalism

In this section we develop a formalism which allows for an elegant description of linear transformations on the tensor product \( V \otimes W \) of vector spaces \( V \) and \( W \). This formalism is well-suited for calculations in quantum mechanical multi-particle systems.

**Notation 1.6.1.** In the sequel, all vector spaces are finite-dimensional over the field \( K = \mathbb{R} \) or \( K = \mathbb{C} \). For short, we will always write \( V \otimes W \) for the \( K \)-tensor product of the vector spaces \( V \) and \( W \). If \( V \) and \( W \) carry the inner product \( \langle \cdot, \cdot \rangle_V \) and respectively \( \langle \cdot, \cdot \rangle_W \), then \( V \otimes W \) will also be regarded an inner product space with the induced inner product which is given by

\[
\langle v \otimes w, v' \otimes w' \rangle_{V \otimes W} = \langle v, v' \rangle_V \langle w, w' \rangle_W.
\]

We set \( \text{End}(V) \) for the vector space of \( K \)-linear endomorphisms of \( V \). This is a \( K \)-Lie algebra with bracket \([A, B] = AB - BA\).
**Definition 1.6.2.** Let $V$, $W$ vector spaces and $A \in \text{End}(V)$, $B \in \text{End}(W)$. We define the Kronecker product $A \otimes B \in \text{End}(V \otimes W)$ by

$$(A \otimes B)(v \otimes w) = Av \otimes Bw$$

for $v \in V$, $w \in W$.

Set

$$(1.6.3) \quad X := \text{Span}_K \{ A \otimes B | A \in \text{End}(V), B \in \text{End}(W) \} \subseteq \text{End}(V \otimes W).$$

**Lemma 1.6.3.** The Kronecker product has the following properties.

(i) For all $A, B \in \text{End}(V)$, $C, D \in \text{End}(W)$ and $\lambda \in \mathbb{K}$,

$$
\begin{align*}
(A + B) \otimes C &= A \otimes C + B \otimes C, \\
A \otimes (C + D) &= A \otimes C + A \otimes D, \\
(\lambda A) \otimes B &= \lambda (A \otimes B) = A \otimes (\lambda B), \\
(A \otimes C) \circ (B \otimes D) &= (A \circ B) \otimes (C \circ D).
\end{align*}
$$

(ii) If $\{A_i\}_i$, $\{B_j\}_j$ are bases of $\text{End}(V)$, $\text{End}(W)$, then $\{A_i \otimes B_j\}_{i,j}$ is a basis of $X$. Moreover, $X = \text{End}(V \otimes W)$.

(iii) Let $\langle \cdot, \cdot \rangle_{\text{End}(V)}$ and $\langle \cdot, \cdot \rangle_{\text{End}(W)}$ be inner products on $\text{End}(V)$ and on $\text{End}(W)$. Then an inner product $\langle \cdot, \cdot \rangle$ on $\text{End}(V \otimes W)$ is defined by linear continuation of

$$(1.6.4) \quad \langle A \otimes B, A' \otimes B' \rangle := \langle A, A' \rangle_{\text{End}(V)} \langle B, B' \rangle_{\text{End}(W)}.$$

If $\{A_i\}_i$, $\{B_j\}_j$ are orthonormal bases of $\text{End}(V)$ and $\text{End}(W)$, then, with respect to the inner product as defined above, the set $\{A_i \otimes B_j\}_{i,j}$ is an orthonormal basis of $\text{End}(V \otimes W)$.

(iv) For all $A, A' \in \text{End}(V)$ and $B, B' \in \text{End}(W)$ the following formula holds:

$$(1.6.5) \quad [A \otimes B, A' \otimes B'] = [A, A'] \otimes BB' + A' A \otimes [B, B'].$$

(v) For all $A \in \text{End}(V)$, $B \in \text{End}(W)$

$$(1.6.6) \quad (A \otimes B)^H = A^H \otimes B^H.$$

(vi) With respect to ordered bases $\{v_i\}_i$ of $V$, $\{w_j\}_j$ of $W$, and $\{v_i \otimes w_j\}_{i,j}$ of $V \otimes W$ (with the indices $(i, j)$ being ordered lexicographically), the endomorphism $A \otimes B$ is represented by the matrix

$$(1.6.7) \quad (r_{ik}s_{lj})_{(i,j),(k,l)}$$

if $A$ and $B$ are represented by matrices $(r_{ij})_{(i,j)}$ and $(s_{kl})_{(k,l)}$, respectively.

(vii) For all $A, B \in \text{End}(V)$,

$$(1.6.8) \quad \text{tr}(A \otimes B) = \text{tr}(A)\text{tr}(B).$$
(viii) For all \( A, B \in \text{End}(V) \),

\[
A \otimes B = P (B \otimes A) P^{-1}
\]

with an involution \( P \in \text{Gl}(V \otimes V) \).

(ix) If \( A \in \text{Gl}(V), B \in \text{Gl}(W) \) then \( A \otimes B \in \text{Gl}(V \otimes W) \) and has inverse

\[
(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.
\]

(x) Let \( A, A' \in \text{End}(V) \) and \( B, B' \in \text{End}(W) \) equivalent endomorphisms, i.e. \( A = U A' U^{-1} \) and \( B = V B' V^{-1} \) for some \( U \in \text{Gl}(V) \) and \( V \in \text{Gl}(W) \). Then also \( A \otimes B \) and \( A' \otimes B' \) are equivalent with

\[
A \otimes B = (U \otimes V) (A' \otimes B') (U \otimes V)^{-1}.
\]

(xi) For all \( A, B \in \text{End}(V) \),

\[
\det (A \otimes B) = (\det A \det B)^n,
\]

where \( n = \text{dim} V \).

**Proof.** (i) This follows from the bilinearity of the tensor product.

(ii) Let \( \alpha_{ij} \in \mathbb{K} \) such that \( \sum_{i,j} \alpha_{ij} (A_i \otimes B_j) = 0 \). So for all \( a, v \in V, b, w \in W \), and \( z_j := \langle B_j w, b \rangle_W \) it follows that

\[
0 = \left\langle \left( \sum_{i,j} \alpha_{ij} (A_i \otimes B_j) \right) (v \otimes w), a \otimes b \right\rangle
= \left\langle \sum_{i,j} \alpha_{ij} (A_i v \otimes B_j w), a \otimes b \right\rangle
= \sum_{i,j} \alpha_{ij} \langle A_i v, a \rangle_V \langle B_j w, b \rangle_W
= \sum_{i,j} \alpha_{ij} \langle A_i v, a \rangle_V z_j
= \left\langle \left( \sum_{i,j} \alpha_{ij} z_j A_i \right) v, a \right\rangle_V.
\]

Therefore, because \( \langle \cdot, \cdot \rangle_W \) is non-degenerate,

\[
\sum_{i,j} \alpha_{ij} z_j A_i = 0.
\]

Since \( \{A_i\}_i \) is linearly independent, we find that for each \( i \)

\[
\sum_j \alpha_{ij} z_j = 0 \iff \sum_j \alpha_{ij} \langle B_j w, b \rangle_W = 0.
\]
Because \( w \) and \( b \) are arbitrary, it follows that \( \sum_j a_j B_j = 0 \), and finally, by linear independence of \( \{B_j\}_j \), that \( \alpha_{ij} = 0 \). Furthermore, the dimension of

\[
\text{End}(V) \otimes \text{End}(W) = \text{Span}_K \{ A \otimes B | A \in \text{End}(V), B \in \text{End}(W) \}
\]

is

\[
\dim \text{End}(V) \dim \text{End}(W) = (\dim V)^2 (\dim W)^2 = (\dim (V \otimes W))^2 = \dim \text{End} (V \otimes W).
\]

So

\[
\text{End}(V) \otimes \text{End}(W) = \text{End} (V \otimes W).
\]

(iii) The sesquilinearity of \( \langle \cdot, \cdot \rangle \) follows from that of \( \langle \cdot, \cdot \rangle_{\text{End}(V)} \) and \( \langle \cdot, \cdot \rangle_{\text{End}(W)} \) together with the bilinearity of the tensor product. The bilinear form \( \langle \cdot, \cdot \rangle \) is positive definite because for all \( A \in \text{End}(V) \), \( B \in \text{End}(W) \) we have that

\[
\langle A \otimes B, A \otimes B \rangle = \langle A, A \rangle_{\text{End}(V)} \langle B, B \rangle_{\text{End}(W)} \geq 0,
\]

and the last expression is equal to 0 if and only if \( A = 0 \) or \( B = 0 \), i.e. if and only if \( A \otimes B = 0 \).

(iv) The Lie bracket of two matrices \((X_{ij})_{ij}\) and \((Y_{ij})_{ij}\) has matrix elements

\[
[X,Y]_{rs} = \sum_u X_{ru} Y_{us} - Y_{ru} X_{us}.
\]

We apply this to the matrix representation of \([A \otimes B, A' \otimes B']\) and find that

\[
[A \otimes B, A' \otimes B']_{ij,kl} = \sum_{st} (A \otimes B)_{ij, st} (A' \otimes B')_{st, kl} - (A \otimes B')_{ij, st} (A' \otimes B)_{st, kl}
\]

\[
= \sum_{st} A_{is} B_{jt} A'_s B'_t - A'_{is} B'_{jt} A_s B_t
\]

\[
= \sum_s \left( \sum_t B_{jt} B'_t - B'_{jt} B_t \right) A_{is} A'_s
\]

\[
- \sum_t \left( \sum_s A'_{is} A_s - A_{is} A'_s \right) B'_{jt} B_t
\]

\[
= \sum_s [B, B']_{jl} A_{is} A'_s - \sum_t [A', A]_{ik} B'_{jt} B_t
\]

\[
= [B, B']_{jl} \sum_s A_{is} A'_s + [A, A']_{ik} \sum_t B'_{jt} B_t
\]

\[
= [B, B']_{jl} (AA')_{ik} + [A, A']_{ik} (B'B)_{jl}
\]

which proves the claim.

(v) For all \( v, v' \in V \), \( w, w' \in W \) we have that

\[
\langle v \otimes w, (A^H \otimes B^H) (v' \otimes w') \rangle_{V \otimes W} = \langle v, A^H v' \rangle_V \langle w, B^H w' \rangle_W = \langle Av, v' \rangle_V \langle Bw, w' \rangle_W = \langle (A \otimes B) (v \otimes w), v' \otimes w' \rangle_{V \otimes W}.
\]
hence \((A \otimes B)^H = A^H \otimes B^H\).

(vi) Let \(\{v_i\}, \{w_j\}\) be ON-bases for \(V\) and \(W\), respectively. Then \(\{v_i \otimes w_j\}\) is an ON-basis for \(V \otimes W\) and thus the matrix element \((A \otimes B)_{ij,kl}\) is given by
\[
(A \otimes B)_{ij,kl} = \langle v_i \otimes w_j, (A \otimes B) (v_k \otimes w_l) \rangle_{V \otimes W}
= \langle v_i \otimes w_j, Av_k \otimes Bw_l \rangle_{V \otimes W}
= \langle v_i, Av_k \rangle_V \langle w_j, Bw_l \rangle_W
= A_{ik} B_{jl},
\]

(vii) This follows directly from the matrix representation of \(A\), \(B\) and \(A \otimes B\) as given in (vi):
\[
\text{tr}(A \otimes B) = \sum_{(ij), (ij)} (A \otimes B)_{ij,ij}
= \sum_{i,j} A_{ii} B_{jj}
= \left( \sum_i A_{ii} \right) \left( \sum_j B_{jj} \right)
= \text{tr}(A) \text{tr}(B).
\]

(viii) A comparison of matrix elements of \(A \otimes B\) and \(B \otimes A\) shows that
\[
(A \otimes B)_{ij,kl} = (B \otimes A)_{ji,lk}.
\]

Thus a change of basis by a suitable transposition matrix \(P\) transforms \(A \otimes B\) into \(B \otimes A\).

(ix) For all \(v \in V\), \(w \in W\),
\[
(A^{-1} \otimes B^{-1})(A \otimes B)(v \otimes w)
= A^{-1} A v \otimes B^{-1} B w
= v \otimes w
= (A \otimes B)(A^{-1} \otimes B^{-1})(v \otimes w).
\]

(x) This becomes clear from
\[
(U \otimes V)(A' \otimes B')(U \otimes V)^{-1} = (UA'U^{-1}) \otimes (VB'V^{-1}) = A \otimes B.
\]

(xi) The Kronecker product of a triangular matrix \(A \in \text{End}(V)\) and an arbitrary matrix \(B \in \text{End}(V)\) is the block triangular matrix \(A \otimes B\), the diagonal blocks consisting of the \((n \times n)\)-matrices \(C_{ij}\) with
\[
(C_{ij})_{kl} = (A \otimes B)_{ik,jl} = A_{ij} B_{kl}.
\]
Thus $A \otimes B$ has determinant
\[
\det (A \otimes B) = \prod_{i,j} \det C_{ij}
\]
\[
= \prod_{i,j} \det (A_{ij}B_{kl})_{kl}
\]
\[
= \prod_{i,j} A_{ij}^n \det (B_{kl})_{k,l}
\]
\[
= (\det (B_{kl})_{k,l})^n \prod_{i,j} A_{ij}^n
\]
\[
= \left( \det (A_{ij})_{ij} \det (B_{kl})_{kl} \right)^n,
\]
where in the last equation we have made use of the triangular form of the matrix $(A_{ij})_{ij}$. Now any $A \in \text{End}(V)$ is conjugate to a triangular matrix $A'$, so by (xi) we find that $A \otimes B$ is conjugate to the block triangular matrix $A' \otimes B$.

Thus $det (A \otimes B) = det (A' \otimes B) = (\det A' \det B)^n = (\det A \det B)^n$.

\[\square\]

**Example 1.6.4.** Let $V = W = \mathbb{C}^2$, and let $\text{End}(V) = \mathfrak{gl}_2 \mathbb{C}$ be endowed with the inner product $\langle A, B \rangle = \frac{1}{2} \text{tr} (A^H B)$. Then an orthonormal basis for $\mathfrak{gl}_2 \mathbb{C}$ is given by

\[
(1.6.13) \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, I_y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, I_z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
\]

and an orthonormal basis for the $\mathbb{R}$-vector space $\mathfrak{su}_2$ by $I_x, I_y, I_z$. According to the previous lemma, an orthonormal basis for the $\mathbb{C}$-vector space $\text{End} (\mathbb{R}^2 \otimes \mathbb{R}^2)$ is made up by the set

\[
\{1 \otimes 1, 1 \otimes I_x, 1 \otimes I_y, 1 \otimes I_z, I_x \otimes 1, I_x \otimes I_y, I_x \otimes I_z, I_y \otimes 1, I_y \otimes I_x, I_y \otimes I_z, I_z \otimes 1, I_z \otimes I_x, I_z \otimes I_y, I_z \otimes I_z\}.
\]

The special choice of the basis (1.6.13) is motivated from quantum mechanics, where the matrices $I_x, I_y,$ and $I_z$ are called Pauli spin matrices.

Another aspect will become important later on. Consider the $\mathbb{R}$-linear span of the set

\[
(1.6.14) \quad X := \bigcup_{j=1}^n X_j,
\]

where

\[
X_j := \{ i^j 1 \otimes \ldots \otimes I_{\alpha_1} \otimes \ldots \otimes I_{\alpha_j} \otimes \ldots \otimes 1 | \alpha_i \in \{x, y, z\}, i = 1, \ldots, j \},
\]

and

\[
\varepsilon_j := \begin{cases} 
1, & j \text{ even}, \\
0, & j \text{ odd}.
\end{cases}
\]
So the set $X_j$ comprises (up to a sign $1$ or $i$) the $n$-fold tensor products of elements in $\{1, I_x, I_y, I_z\}$ with exactly $j$ factors different from $1$. By construction, $X \subseteq su(2^n)$, because for each element

$$Y = i^{j/2} 1 \otimes ... \otimes I_{\alpha_1} \otimes ... \otimes I_{\alpha_j} \otimes ... \otimes 1 \in X_j,$$

the equation

$$Y + Y^H = i^{j/2} 1 \otimes ... \otimes I_{\alpha_1} \otimes ... \otimes I_{\alpha_j} \otimes ... \otimes 1 + (i^{j/2}) 1 \otimes ... \otimes (-I_{\alpha_1}) \otimes ... \otimes (-I_{\alpha_j}) \otimes ... \otimes 1 = 0$$

holds by Lemma 1.6.3 (v). Since the span of $X$ has the maximal possible dimension $4^n - 1$, it follows that $X$ is a tensor product basis of $su(2^n)$. This basis will be used for further calculations in our discussion of concrete $n$-particle spin systems, cf. Chapter 3.
CHAPTER 2

General Control Theory for Spin Systems

2.1. Quantum Mechanics of Spin Systems

We here give a short overview of the basic principles of quantum mechanics, and in particular describe the physics of spin systems whose control properties are in the focus of this work. This exposition is by no means complete but is intended to introduce all the terminology and concepts needed in the subsequent sections. We refer the reader to [25] for an exhaustive treatment of the subject.

The premise of non-relativistic quantum mechanics is that the state of physical objects like electrons, protons and neutrons as well as larger systems of those like atoms and molecules is represented by a wave-function $\psi$. This function $\psi$ carries all the information of the state of the system under consideration. The collection of the physical relevant wave-functions is given by the state space, a separable complex Hilbert space $\mathcal{H}$. This space could e.g. be the space $L_2$ of square-integrable functions $\mathbb{R}^3 \rightarrow \mathbb{C}$; a wave-function $\psi \in L_2$ would then contain information of where the particle is localized in three-space. To be a little bit more concrete,

\[
\int_Q \psi(x)\psi^*(x) \, dx
\]

(2.1.1)

gives the probability of “finding” the particle within a measurable subset $Q$ of $\mathbb{R}^3$.

It is convenient to normalize the wave-function $\psi$ to have norm

\[
\int_{\mathbb{R}^3} \psi(x)\psi^*(x) \, dx = 1;
\]

(2.1.2)

wave-functions which only differ by a non-zero scalar will be regarded equivalent.

The time-evolution of a state $\psi$ is governed by Schrödinger's equation

\[
i\hbar \dot{\psi} = H\psi
\]

(2.1.3)

with $H$ a Hermitian operator, which is called the Hamilton operator of the system and which might also be time-dependent. It models the presence of a field acting on the states and causing their dynamics.
As an example, the Hamilton operator for a single particle of mass $m$ moving in a one-dimensional harmonic potential is given by

$$(2.1.4) \quad H = -\frac{\Delta}{2m} + \alpha x^2, \quad \alpha \in \mathbb{R}^+,$$

where $\Delta = \frac{\partial^2}{\partial x^2}$ denotes the Laplace operator.

One of the principles of quantum mechanics says that it is not possible to observe the wave-function $\psi$ itself by performing an experiment and thus to gain complete information about the system. What can be observed is the spectrum of certain Hermitian operators called observables. These are for instance the operators $x$ (space), $-i\hbar \nabla$ (momentum), $H$ (energy), $-i\hbar x \times \nabla$ (angular momentum), and others like e.g. “spin”.

Let $\psi_j, \ j \in \mathbb{N}$, be a complete set of orthonormalized eigenvectors for the observable $A$ with eigenvalues $\lambda_j$ and assume the state $\psi$ at some fixed time $t_0$ to be given by

$$(2.1.5) \quad \psi = \sum_{j=1}^{\infty} \langle \psi, \psi_j \rangle \psi_j.$$

Then the measurement of $A$ at time $t_0$ will give the result $\lambda_j$ with a certain probability, which simply is given by the squared modulus $|\langle \psi, \psi_j \rangle|^2$ of the coefficient of $\psi_j$ in above Fourier expansion. Thus the expectation value of $A$ in the state $\psi(t_0)$ is expressed as

$$(2.1.6) \quad \langle A \rangle = \sum_{j=1}^{\infty} |\langle \psi, \psi_j \rangle|^2 \lambda_j = \langle \psi, A \psi \rangle.$$

The process of measuring $A$ will change the state $\psi$ to $\psi = \psi_j$, if the result of the observation was $\lambda_j$. It therefore is not possible to perform at the state $\psi$ the exact measurement of two or more non-commuting observables. This is the statement of Heisenberg’s uncertainty relation, see [25] for a quantitative discussion.

Define the time-evolution operator $U$ to be the solution of the differential equation

$$(2.1.7) \quad i\hbar \dot{U} = HU, \quad U(0) = 1.$$

This differential equation is again called Schrödinger equation. It is easily seen that the dynamics of the state $\psi$ under the influence of the Hamilton operator $H$ are given by

$$(2.1.8) \quad \psi(t) = U(t)\psi(0) \text{ for all } t \geq 0.$$

It is therefore sufficient to study $U$ in order to obtain a full description of a given quantum mechanical system.

The discussion so far applies in particular to the spin of a quantum mechanical system, a phenomenon which is without analogue in classical physics. The simplest examples of quantum mechanical systems containing spin are the
fermions, or spin-$\frac{1}{2}$-particles, like e.g. electrons, neutrons and protons. To carry spin in that cases expresses the heuristic imagination that those particles possess an angular momentum, which comes from a rotation around their own axis and which is sensible towards a magnetic field (and only for that reason is measurable).

The mathematical formulation of this phenomenon is as follows. Choose $\mathcal{H} = \mathbb{C}^2$ to serve as the state space and let $S = (S_x, S_y, S_z)$ denote the so-called operator of total spin. Thus $S$ is a 3-frame of $(2 \times 2)$-Hermitian operators $S_x$, $S_y$, $S_z$, whose components will be specified below. The spin projection in direction of $e = (e_x, e_y, e_z) \in \mathbb{R}^3$, $||e|| = 1$, is given by the Hermitian matrix

$$S \cdot e := \sum_{\alpha=x,y,z} e_\alpha S_\alpha \in \mathbb{C}^{2 \times 2}.$$  

A measurement of $S \cdot e$ in the state $\psi \in \mathcal{H}$ has outcome $\pm \frac{1}{2} \hbar$. Let $\chi^\pm$ be normalized eigenstates for $S_z = S \cdot e_z$ with eigenvalues $\pm \frac{1}{2} \hbar$. Then in the basis $\{\chi^\pm\}$ of $\mathcal{H}$ the matrix representation of the spin projection operators is as follows:

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

The dynamics of the wave function $\psi \in \mathcal{H}$ under the influence of a magnetic field $B = (B_x, B_y, B_z) \in \mathbb{R}^3$ are described by the Hamiltonian operator

$$H_{\text{magn}} = \text{const} B \cdot S = \text{const} (B_x S_x + B_y S_y + B_z S_z) \in \mathbb{C}^{2 \times 2}.$$  

The generalization to ensembles of $n$ spin-$\frac{1}{2}$-particles is as follows. The total spin in this situation is given by the operator $S = (S_x, S_y, S_z)$,

$$S_\alpha := \bigotimes_{i=1}^n S_{i,\alpha} \in \mathbb{C}^{2^n \times 2^n}, \quad \alpha \in \{x, y, z\},$$  

with $S_{i,\alpha}$ the spin projection of the $i$-th particle in direction of $e_\alpha$, and $\otimes$ denoting the Kronecker product. The projection of $S$ in direction of $e = (e_x, e_y, e_z)$, $||e|| = 1$, is the operator

$$S \cdot e := \bigotimes_{i=1}^n (S_i \cdot e) \in \mathbb{C}^{2^n \times 2^n}$$  

which is acting on the Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$. Its eigenvalues are $-\frac{n\hbar}{2}, -\frac{n\hbar}{2} + 1, \ldots, \frac{n\hbar}{2} - 1, \frac{n\hbar}{2}$. The eigenspace $\mathcal{E}_m$ to the eigenvalue $\frac{m}{2}\hbar$, $m = -n, -n+2, \ldots, n-2, n$, has dimension

$$\dim \mathcal{E}_m = \binom{n}{\frac{n+m}{2}}.$$  

If the $n$-particle system is exposed to a magnetic field $B = (B_x, B_y, B_z)$ which we assume to be equal to $B_i = (B_{ix}, B_{iy}, B_{iz})$ at the locus of the $i$-th particle,
then its spin will be described by the Hamilton operator

\begin{equation}
H = H_d + H_{\text{magn}} \in \mathbb{C}^{2^n \times 2^n},
\end{equation}

with

\begin{equation}
H_{\text{magn}} = \text{const} \bigotimes_{i=1}^{n} B_i \cdot S_i,
\end{equation}

the factor \(B_i \cdot S_i\) as given by equation (2.1.11), and an operator \(H_d\), which is fixed and describes the coupling between the spins of the individual particles. This Hamilton operator \(H\) will typically be occurring in the discussion of the control properties of Schrödinger’s equation

\begin{equation}
i \hbar \dot{U} = HU,
\end{equation}

which is the content of the following sections.

### 2.2. The Control Problem

Given the Lie group \(G = SU(2^n)\) and the following family of Hermitian operators \((\mathbb{C}^2)^{\otimes n} \rightarrow (\mathbb{C}^2)^{\otimes n}\):

\begin{equation}
H(v_1, ..., v_m) = H_d + \sum_{j=1}^{m} v_j H_j, \quad v = (v_1, ..., v_m) \in \mathbb{R}^m.
\end{equation}

Fix an element \(U_F \in G\) and consider the right-invariant control system on \(G\) given by

\begin{equation}
\dot{U} = -iH(v)U, \quad U(0) = 1
\end{equation}

with \(v\) acting as control variable. The question of interest to us is whether it is possible to steer system (2.2.2) from the initial state \(U(0)\) to the final state \(U_F\). If this is the case, what will be the minimum amount of time to achieve this?

The motivation for treating that kind of problem in time-optimal control arises from questions concerning the quantum mechanics of spin systems, such as ensembles of electrons or neutrons. Indeed, the operator \(H(v)\) acts as the Hamilton operator for systems of coupled spin particles that are under the influence of an exterior magnetic field of fixed direction and variable strength (modelled by the variable \(v\)). So (2.2.2) is just Schrödinger’s equation for the time-evolution operator \(U\) of such a system (with \(\hbar\) set equal to 1).

The desire to solve a control problem as formulated above came alongside with the development of certain experiments in nuclear magnetic resonance spectroscopy (NMR) and quantum computing. Here one needs to manipulate ensembles of coupled nuclear spins and wishes to do so in least possible time. See e.g. [8] and [26] for details on this topic.

In the discussion to follow we are going to generalize this kind of control problem from the specific case \(G = SU(2^n)\) to arbitrary compact Lie groups.
2.3. Equivalence Theorem

Throughout this section $G$ denotes a compact Lie group with Lie algebra $\mathfrak{g}$, while $K$ denotes a closed subgroup of $G$ with Lie algebra $\mathfrak{k}$. We are interested in the following affine right-invariant control system on $G$:

\[(2.3.1) \quad \dot{U} = \left( H_d + \sum_{j=1}^{m} v_j H_j \right) U, \quad U(0) = 1, \]

with $H_d \in \mathfrak{g}$ arbitrary but fixed, and $H_1, \ldots, H_m$ a fixed set of generators for the Lie algebra $\mathfrak{k}$.

2.3.1. Heuristic considerations. Consider again the evolution equation of the affine right-invariant control system (2.3.1), and let $U(0) = U_0 \in G$ arbitrary. Note that in principle the control variables $v_j$, $j = 1, \ldots, m$, may be chosen to be arbitrarily large in comparison to the norm of the fixed drift Hamiltonian $H_d$. For such a choice of $v = (v_1, \ldots, v_m)$ the control system (2.3.1) will behave roughly as

\[(2.3.2) \quad \dot{U} = \left( \sum_{j=1}^{m} v_j H_j \right) U, \quad U(0) = U_0. \]

Note also that, if we choose $v$ to be constant, the solution of ODE (2.3.2) will be

\[(2.3.3) \quad U(t) = \exp \left( t \sum_{j=1}^{m} v_j H_j \right) U_0. \]

>From our assumptions on $H_j$, $j = 1, \ldots, m$, and Theorem 1.4.5 it follows that it is possible to steer system (2.3.2) to any point $U$ in the coset $KU_0 \subseteq G$, and, by choosing $|v|$ large, to achieve this in negligible time. Thus from the point of view of time-optimal control, group elements contained in the same coset $KU_0 \subseteq G$ can be considered equivalent. The question of interest therefore is to find appropriate control strategies to steer system (2.3.1) from the identity coset $K$ to any other coset $KU_0$ in least possible time. To this aim it turns out to be useful to replace system (2.3.1) by another right-invariant system on the group $G$, whose reachable sets coincide (modulo $K$) with that of (2.3.1), but which has bounded controls so that the phenomenon of arbitrary fast movement within a coset does no longer occur.

Which kind of right-invariant control system on $G$ will be the appropriate one? To answer this question assume that (2.3.1) evolves from $U_0 \in G$ under the influence of the control variable $v$, which we still assume to be constant. So we set

\[(2.3.4) \quad H_0 := \sum_{j=1}^{m} v_j H_j, \]

while $K$ denotes a closed subgroup of $G$ with Lie algebra $\mathfrak{k}$. We are interested in the following affine right-invariant control system on $G$:
and consider on $G$ the ODE
\begin{equation}
\dot{U} = (H_d + H_0) U, \quad U(0) = U_0. \tag{2.3.5}
\end{equation}

The solution of this ODE is given by
\begin{equation}
U(t) = \exp(t(H_d + H_0)) U_0. \tag{2.3.6}
\end{equation}

We separate the flow $t \mapsto U(t)$ into two components $Q(t)$ and $P(t)$, i.e. $U(t) = Q(t) P(t)$. As we have noticed before, it is possible to steer the original system (2.3.1) within a given coset $K g$ arbitrarily fast. For this reason, the factor $Q(t)$, which merely describes motion within $K$, will be factored out. The allowed directions for steering the new system on $G$ are therefore given by the values of $\dot{P}(t)$. These are now obtained from the ansatz $U(t) = Q(t) P(t)$ with $U(t)$ as in equation (2.3.6) and $Q(t) = \exp t H_0 \in K$. A calculation yields
\begin{align*}
\dot{P}(t) &= \dot{Q}^{-1} U(t) + Q^{-1} \dot{U}(t) \\
&= \exp(-tH_0) (-H_0) U(t) + \exp(-tH_0) (H_d + H_0) U(t) \\
&= \exp(-tH_0) H_d U(t) \\
&= \exp(-tH_0) H_d \exp(tH_0) \exp(-tH_0) U(t) \\
&= (\Ad_{\exp(-tH_0) H_d}) Q^{-1}(t) U(t) \\
&= (\Ad_{\exp(-tH_0) H_d}) P(t).
\end{align*}

This gives rise to the idea of replacing the original system (2.3.1) by the following control system on $G$:
\begin{equation}
\dot{P} = XP, \quad P(0) = 1, \tag{2.3.7}
\end{equation}

where the control $X$ is taken from
\begin{equation}
\Ad_K H_d = \{ k H_d k^{-1} | k \in K \}, \tag{2.3.8}
\end{equation}

the $K$-adjoint orbit of $H_d$ in $g$.

In view of our initial considerations it would be desirable to interpret system (2.3.7) as a control system on the space of right-cosets modulo $K$. However, it turns out that the expression $XP$ can only be given a precise meaning as a tangent vector of $G/K$, if $G/K$ is taken to be the space of left-cosets, i.e. $G/K = \{ gK | g \in G \}$. Then $XP$ just means right-translation of the vector $X \in g$ by $P \in G/K$. On the other hand, the reachable sets $R(1, t)$ for both system (2.3.1) and system (2.3.7) are easily shown to be $\Ad_K$-invariant, so that the identity $KR(1, t) = R(1, t)K$ holds for all $t \geq 0$. This makes it plausible that (2.3.7) can be used to define on the left-homogeneous space $G/K$ a control system, which is equivalent to system (2.3.1) on the group $G$.

This idea will be given evidence in the subsequent section.
2.3.2. Equivalence Theorem.

**Definition 2.3.1.** The control system (2.3.1) will from now on be referred to as the *unreduced system*. We furthermore define on \( G \) the *adjoint system* to be

\[
\dot{U} = XU, \quad U(0) = 1, \quad X \in \text{Ad}_KH_d,
\]

and on \( G/K \) the *reduced system* to be

\[
\dot{P} = XP, \quad P(0) = K, \quad X \in \text{Ad}_KH_d,
\]

where the expression \( XP \) is explained as follows.

If \( P = \pi(g) \), then

\[
XP := D_1(\pi \circ R_g)(X) \in T_gK(G/K).
\]

This is well-defined: If we replace \( g \) by \( g' = gk, \ k \in K \), then we find that

\[
D_1(\pi \circ R_g')(X) = D_1(\pi \circ R_{gk})(X) = D_1(\pi \circ R_k \circ R_g)(X) = D_1(\pi \circ R_g)(X) = XP,
\]

as \( \pi \circ R_k = \pi \).

**Notation 2.3.2.** We label reachable and approximately reachable sets etc. for the unreduced, adjoint and reduced systems by lower indices 1, 2 and 3, respectively.

**Note 2.3.3.** For convenience we add to the admissible vector fields of systems 1–3 the zero field. This does not change the reachable sets \( R_i(x,t) \) and \( \bar{R}_i(x) \), \( i = 1,2,3 \), nor does it have any effect on the problem of finding time-optimal trajectories for those systems. This assumption merely has the advantage that in the remainder we need not distinguish between the sets \( R_i(x,t) \) and \( \bar{R}_i(x,t) \) and also might use the fact that the sets \( R_i(x,t) \) are monotonely increasing in \( t \).

The remainder of this section is aimed to establish a theorem which will show that all three of the systems defined above can be considered equivalent. To be able to give a precise formulation of what “equivalence” should be, we introduce some terminology.

**Definition 2.3.4.** Let \( \Sigma = (M,f_u,U) \) be a control system. Define the *set \( S(x,t_0) \) of approximately reachable points* from \( x \in M \) within time \( t_0 \geq 0 \) to be

\[
S(x,t_0) := \bigcap_{t > t_0} \overline{R}(x,t).
\]

Here \( \overline{R}(x,t) \) refers to the reachable set from \( x \) within time \( t \) as defined in 1.4.3. Thus a point \( y \in M \) is contained in \( S(x,t_0) \), if and only if for any neighbourhood
U of y and any ε > 0 there exists a point z ∈ U ∩ R(x, t + ε). Furthermore, we define the infimizing time to steer Σ from x₁ ∈ M to x₂ ∈ M to be

\[ t_{\text{inf}}(x_1, x_2) := \begin{cases} \inf \{ t \in \mathbb{R} | x_2 \in S(x_1, t) \}, & \text{if } x_2 \in S(x_1, t) \text{ for some } t \in \mathbb{R}, \\ \infty, & \text{otherwise.} \end{cases} \]

**Remark 2.3.5.** As a consequence of the boundedness of the set AdₖH_d it easily follows that

\[ S_j(1, t) = R_j(1, t) \]

holds for j = 2, 3 and for all t ≥ 0, cf. Proposition 2.3.9. On the other hand, the distinction between the closure of reachable sets and approximately reachable sets in the case of system 1 became inevitable since here the set of controls is unbounded.

The equivalence between the control systems of Definition 2.3.1 can now be stated as follows.

**Theorem 2.3.6.** (Equivalence theorem). For all t ≥ 0 the following holds:

(i) \( S_1(1, t) = K R_2(1, t) = R_2(1, t) K \),

(ii) \( \pi (S_1(1, t)) = R_3(K, t) \), where \( \pi \) denotes canonical projection \( G \to G/K \).

The proof of the equivalence theorem is based on the subsequent propositions.

**Proposition 2.3.7.** (i) The reachable sets for the adjoint system 2 are Adₖ-invariant, i.e. the identities

\[ Ad_K (R_2(1, t)) = R_2(1, t) \]

and

\[ Ad_K (R_2(1, t)) = R_2(1, t) \]

hold for all \( k \in K \) and \( t \geq 0 \).

(ii) Any trajectory \( t \mapsto U(t), \ t \in [0, t_F], \) of system 2 is mapped under \( \pi \) to a trajectory \( t \mapsto V(t) := (\pi \circ U)(t) \) of system 3. Conversely, any trajectory \( t \mapsto V(t) \) of system 3 can be lifted to a trajectory \( t \mapsto U(t) \) of system 2. In particular,

\[ \pi (R_2(1, t)) = R_3(1, t) \]

holds for all \( t \geq 0 \).

**Proof.** (i) Let the control \( t \mapsto X(t) \) of system 2 generate the trajectory \( t \mapsto U(t) \). Then the control \( t \mapsto Ad_kX(t), \ k \in K \), generates the trajectory
t → \text{Ad}_k U(t), because

\begin{align*}
\frac{d}{dt} \text{Ad}_k U(t)|_{t=t_0} &= k \dot{U}(t_0)k^{-1} \\
&= kX(t_0)U(t_0)k^{-1} \\
&= (kX(t_0)k^{-1})(kU(t_0)k^{-1}) \\
&= \text{Ad}_k X(t_0) \cdot \text{Ad}_k U(t_0).
\end{align*}

This implies that the set \( R_2(1, t) \) is \( \text{Ad}_K \)-invariant for any \( t \geq 0 \). Since the map \( \text{Ad}_k \) is a homeomorphism, the same holds for \( R_2(1, t) \).

(ii) Let \( X : [0, t_F] \rightarrow \text{Ad}_K H_d, t \rightarrow X(t) \) be any control for the adjoint and for the reduced system. Denote by \( t \mapsto U_2(t) \in G \) and by \( t \mapsto U_3(t) \in G/K \) the resulting trajectories. Then

\begin{align*}
\frac{d}{dt} (\pi \circ U_2(t))|_{t=t_0} &= D_{U_2(t_0)} \pi (\dot{U}_2(t_0)) \\
&= D_{U_2(t_0)} \pi (X(t_0)U_2(t_0)) \\
&= D_{U_2(t_0)} \pi (D_1 R_{U_2(t_0)}(X(t_0))) \\
&= D_1 (\pi \circ R_{U_2(t_0)})(X(t_0)) \\
&= X(t_0)(\pi \circ U_2(t_0)).
\end{align*}

This shows that both \( U_3 \) and \( \pi \circ U_2 \) satisfy ODE (2.3.10) on \( G/K \) together with the initial condition \( P(0) = K \). Therefore \( U_3 = \pi \circ U_2 \) holds everywhere on \([0, t_F]\).

\begin{proposition}
For all \( t_F \geq 0 \),
\begin{equation}
R_1(1, t_F) \subseteq KR_2(1, t_F).
\end{equation}
\end{proposition}

\textbf{Proof}. Let \( U_F \in R_1(1, t_F) \) and \( t \mapsto v(t) \in \mathbb{R}^m \) a control for system 1 such that the corresponding trajectory \( t \mapsto U(t) \) satisfies \( U(t_F) = U_F \). Now let \( t \mapsto Q(t) \in K \) the solution curve of the ODE

\[ \dot{Q} = \left( \sum_{i=1}^m v_i H_i \right) Q, \quad Q(0) = 1, \]

and \( t \mapsto P(t) \in G \) the solution curve of the ODE

\[ \dot{P} = (Q^{-1} H_d Q) P, \quad P(0) = 1. \]

On \([0, t_F]\) consider the map \( t \mapsto V(t) := Q(t)P(t) \). It satisfies \( V(0) = 1 \) and

\begin{align*}
\dot{V}(t) &= \dot{Q}(t)P(t) + Q(t)\dot{P}(t) \\
&= \left( \sum_{i=1}^m v_i(t) H_i \right) Q(t)P(t) + Q(t) \left( Q^{-1}(t) H_d Q(t) \right) P(t) \\
&= \left( \sum_{i=1}^m v_i(t) H_i + H_d \right) Q(t)P(t),
\end{align*}

This completes the proof. \( \square \)
which shows, by the uniqueness part of the Caratheodory theorem, that \( V \) coincides with \( U \) on \([0, t_F]\). Now let system 2 evolve according to the control law \( t \mapsto Q^{-1}(t)H_dQ(t) \) for \( t \in [0, t_F] \). Then
\[
P(t_F) = Q^{-1}(t_F) V(t_F) = Q^{-1}(t_F) U(t_F) = Q^{-1}(t_F) U_F \in KU_F.
\]
This shows \( U_F \in KP(t_F) \subseteq KR_2(1, t_F) \), as claimed. \( \square \)

**Proposition 2.3.9.** The set \( \overline{R_2(x, t)} \) is compact for all \( x \in G \) and \( t \geq 0 \). Moreover, the following holds:
\[
R_2(x, t) = \bigcap_{n=1}^{\infty} R_2 (x, t + \frac{1}{n}).
\]

**Proof.** Let \( \langle \cdot, \cdot \rangle \) be any right-invariant metric on \( G \). The set \( R_2(x, t) \) is bounded, because the set \( Ad_{K} H_d \subseteq g \) of controls is bounded by a constant \( M \) (in the norm induced by the scalar product \( \langle \cdot, \cdot \rangle_1 \) on \( g = T_1 G \)). So, by the right-invariance of the metric \( \langle \cdot, \cdot \rangle \), we have that \( \| f_u(z) \| = \| u \| \leq M \) for all \( u \in Ad_{K} H_d \) and \( z \in G \). Hence the distance \( d(x, y) \) between \( x \) and any \( y \in R_2(x, t) \) can be estimated as follows:
\[
d(x, y) \leq \int_0^t \| u(s) \| \, ds \leq \int_0^t M \, ds = Mt.
\]
Therefore, \( R_2(x, t) \) is bounded, and \( \overline{R_2(x, t)} \) is compact.

To prove equation (2.3.19), we first observe that \( \overline{R_2(x, t)} \) is contained in any of the sets \( R_2 (x, t + \frac{1}{n}) \), \( n \in \mathbb{N} \), thanks to the convention made in 2.3.3. So \( \overline{R_2(x, t)} \subseteq \bigcap_{n=1}^{\infty} R_2 (x, t + \frac{1}{n}) \). Now assume that there exists
\[
y \in \left( \bigcap_{n=1}^{\infty} R_2 \left( x, t + \frac{1}{n} \right) \right) \setminus \overline{R_2(x, t)}.
\]
Then \( y \) has distance \( d > 0 \) from the compact set \( \overline{R_2(x, t)} \). From this and the boundedness of the controls it follows that the infimizing time needed to steer system 2 from \( R_2(x, t) \) to \( y \) is positive, i.e.
\[
\inf \left\{ \varepsilon > 0 \mid y \in \overline{R_2 (x, t + \varepsilon)} \right\} > 0,
\]
in contradiction to \( y \in \bigcap_{n=1}^{\infty} R_2 (x, t + \frac{1}{n}) \). \( \square \)

**Proposition 2.3.10.** For all \( t \geq 0 \) the following holds:
\[
R_2(1, t) \subseteq K \overline{R_1(1, t)}.
\]

**Proof.** Let \( x_0 \in R_2(1, t) \). By definition there exists a control \( X : [0, t_F] \to Ad_{K} H_d \) such that the resulting trajectory \( t \mapsto P(t) \) of system 2 satisfies \( P(t_F) = x_0 \). Theorem 1.2.2 allows us to identify the smooth manifold \( Ad_{K} H_d \) with the homogeneous space \( K/\text{Stab}_{K} H_d \). From this identification it becomes clear that the path \( t \mapsto X(t) \in K/\text{Stab}_{K} H_d \) can be lifted to a path \( t \mapsto Q^{-1}(t) \in K \) with
$Q(0) = 1$ and the same regularity properties as $t \mapsto X(t)$. We therefore have

$$X(t) = Q^{-1}(t)H_dQ(t)$$

(2.3.21)
on $[0, t_F]$. Notice that the path $t \mapsto Q(t)$ need not occur as a trajectory of the control system

$$\dot{Q} = \left( \sum_{i=1}^m v_i H_i \right)Q, \quad Q(0) = 1, \quad v = (v_1, \ldots, v_m) \in \mathbb{R}^m$$

(2.3.22)
on $K$. But as it is pointed out in [16] and proved in [9], there exists a sequence of control functions $t \mapsto v^n(t) \in \mathbb{R}^m$, $t \in [0, t_F]$, such that the resulting sequence $(Q^n)_n$ of trajectories for system (2.3.22) converges in $L_1$ against $Q$. Now define for $t \in [0, t_F]$

$$X^n(t) := (Q^n)^{-1}(t)H_dQ^n(t) \in \text{Ad}_K H_d.$$ 

Then, by equation (2.3.21) and the definition of $Q^n$, the sequence $(X^n)_n$ converges in $L_1$ against $X$. Furthermore, let $P^n$ the solution of the ODE

$$\dot{P}^n = X^n P^n, \quad P^n(0) = 1.$$ 

The convergence of $(X^n)_n$ against $X$ implies

$$\lim_{n \to \infty} P^n(t_F) = P(t_F),$$

cf. [1], p. 41-42. We finally set $U^n := Q^n P^n$. Since the function $Q^n$ satisfies the ODE (2.3.22), it follows from the same calculation as in the proof of Proposition 2.3.8 that $U^n$ solves the ODE

$$\dot{U}^n = \left( H_d + \sum_{i=1}^m v^n_i H_i \right) U^n, \quad U^n(0) = 1$$

on $[0, t_F]$ and therefore is a trajectory of system 1. So we have found that

$$x_0 = P(t_F) = \lim_{n \to \infty} P^n(t_F) = \lim_{n \to \infty} (Q^n)^{-1}(t_F) U^n(t_F) = Q^{-1}(t_F) \lim_{n \to \infty} U^n(t_F) \in K\mathbb{R}_1(1, t_F),$$
as claimed. \hfill \Box

**Proposition 2.3.11.** $K \subseteq S_1(1, 0)$. As a consequence, if $z \in S_1(1, t)$ for some $t \geq 0$, then $zk \in S_1(1, t)$ holds for all $k \in K$.

**Proof.** By assumption, the elements $H_1, \ldots, H_m \in \mathfrak{t}$ generate $\mathfrak{t}$ as a Lie algebra. This implies that the system

$$\dot{W} = \left( \sum_{i=1}^m v_i H_i \right) W, \quad W(0) = 1$$

(2.3.23)
We now turn to the proof of the Equivalence Theorem 2.3.4. Moreover, by the same theorem, there exists a constant \( T > 0 \) such that \( R(1) = R(1,T) \). Since the norm of the operator \( \sum_{i=1}^{m} v_i H_i \) may be chosen to be arbitrarily large, this equation holds for any constant \( T > 0 \).

>From now on let \( T > 0 \) and \( W_F \in K \) be arbitrary but fixed, and choose a control \( t \mapsto (v_1(t), \ldots, v_m(t)), t \in [0, t_F], t_F < T \), such that the resulting trajectory \( t \mapsto W(t) \) satisfies \( W(t_F) = W_F \). Then for all \( n \in \mathbb{N} \) the trajectory \( t \mapsto W^n(t) \) of (2.3.23), which results from the control \( t \mapsto v^n(t) := nw(t) \), satisfies \( W^n \left( \frac{1}{n} t_F \right) = W_F \). Now consider the ODE

\[
\dot{U}^n = \left( H_d + n \sum_{i=1}^{m} v_i(t) H_i \right) U^n, \quad U^n(0) = 1,
\]

and let \( \varepsilon > 0 \) be arbitrary. Then for \( n \) sufficiently large it follows that

\[
\left\| U^n \left( \frac{1}{n} t_F \right) - W_F \right\| = \left\| U^n \left( \frac{1}{n} t_F \right) - W^n \left( \frac{1}{n} t_F \right) \right\| < \varepsilon,
\]

cf. [31], p. 122. Thus

\[
W_F \in \overline{R_1(1, t_F)} \subseteq \overline{R_1(1,T)}
\]

holds for all \( T > 0 \). By definition, this implies \( W_F \in S_1(1,0) \). Since \( W_F \in K \) was chosen to be arbitrarily, it follows that \( K \subseteq S_1(1,0) \).

The addendum that \( zk \in S_1(1,t) \), if \( z \in S_1(1,t) \) and \( k \in K \), is an immediate consequence of the right-invariance of system 1. Namely, if we can steer system 1 into any neighbourhood \( Z_\varepsilon \) of \( z \) at time \( t + \varepsilon \) and into any neighbourhood \( K_\varepsilon \) of \( k \) at time \( \varepsilon \), than the system can likewise be steered at time \( t + 2\varepsilon \) into an arbitrary small neighbourhood \( Z_\varepsilon K_\varepsilon \) of \( zk \).

\[\square\]

We now turn to the proof of the Equivalence Theorem 2.3.4.

**Proof.** (i) From Proposition 2.3.7 (i) we obtain

\[
\bigcup_{k \in K} k \overline{R_2(1, t)} = \bigcup_{k \in K} k \overline{R_2(1, t)} k^{-1} = \bigcup_{k \in K} \overline{R_2(1, t)} k,
\]

so that the identity

\[
K \overline{R_2(1, t)} = \overline{R_2(1, t)} K
\]

holds for all \( t \geq 0 \).

(A) \( S_1(1, t) \subseteq K \overline{R_2(1, t)} \).

Let \( x \in S_1(1, t) \). By definition of \( S_1(1, t) \) there exists a sequence \( (y_n)_n \) in \( G \) with \( y_n \in R_1(1, t + \frac{1}{n}) \cap U_{\frac{1}{n}}(x) \) and \( \lim_{n \to \infty} y_n = x \). Here \( U_\varepsilon(x) \) denotes the open ball around \( x \) of radius \( \varepsilon \). For all \( n \in \mathbb{N} \) there exists, by Proposition 2.3.8, \( z_n \in R_2(1, t + \frac{1}{n}) \) and \( k_n \in K \) such that \( y_n = k_n z_n \). Since all \( z_n \) are contained in the compact set \( R_2(1, t + 1) \) and \( K \) also is a compact set, we find a subsequence of \( (z_n, k_n)_n \) (which we label again by \( n \)) and which satisfies

\[
\lim_{n \to \infty} (z_n, k_n) = (z_0, k_0) \in \overline{R_2(1, t + 1)} \times K.
\]
By definition of $z_n$ we have that $z_0 \in \overline{R_2(1, t + \varepsilon)}$ for all $\varepsilon > 0$. This implies together with Proposition 2.3.9 that
\[
\lim_{n \to \infty} R_2 \left( 1, t + \frac{1}{n} \right) = \overline{R_2(1, t)}.
\]
It then follows that
\[
x = \lim_{n \to \infty} y_n = \lim_{n \to \infty} (k_n z_n) = k_0 z_0 \in K \overline{R_2(1, t)}.
\]
(B) $K \overline{R_2(1, t)} \subseteq S_1(1, t)$.

Let $k \in K$, $x \in R_2(1, t)$. Then by Proposition 2.3.10, $x = k' y$ for some $k' \in K$ and $y \in \overline{R_1(1, t)}$. It follows that
\[
kx \in K \overline{R_1(1, t)} \subseteq KS_1(1, t).
\]
Now by Proposition 2.3.11, $K \subseteq S_1(1, 0)$, hence $KS_1(1, t) = S_1(1, t)$. This shows $kx \in S_1(1, t)$ and implies $KR_2(1, t) \subseteq S_1(1, t)$. Furthermore, $K$ and also $S_1(1, t)$ is closed, so that
\[
\overline{KR_2(1, t)} = \overline{R_2(1, t)} \subseteq S_1(1, t).
\]
(ii) From Proposition 2.3.7 (ii) and the fact that the images of compact sets under the continuous map $\pi$ are compact, it follows that
\[
\overline{R_3(K, t_F)} = \pi \left( \overline{R_2(1, t_F)} \right) = \pi \left( \overline{R_2(1, t_F)K} \right) = \pi \left( S_1(1, t_F) \right).
\]
Combining this with part (i) of the proof we find that
\[
\overline{R_3(K, t_F)} = \pi \left( \overline{R_2(1, t_F)} \right) = \pi \left( \overline{R_2(1, t_F)K} \right) = \pi \left( S_1(1, t_F) \right).
\]
This concludes the proof of the equivalence theorem. \[\square\]

**Corollary 2.3.12.** For all $x \in G$,

\[
t_{\inf,1}(1, x) = t_{\inf,3}(K, \pi(x)).
\]

**Proof.** Statement (ii) of the equivalence theorem implies that for all $p \in G/K$ there exists $y \in \pi^{-1}(p)$ such that $t_{\inf,3}(K, p) = t_{\inf,1}(1, y)$. Now for any $x = yk \in \pi^{-1}(p)$ we have by Proposition 2.3.11 that $t_{\inf,1}(1, x) = t_{\inf,1}(1, y)$. So $t_{\inf,1}(1, x) = t_{\inf,3}(K, \pi(x))$ holds for all $x \in G$, as claimed. \[\square\]

**Corollary 2.3.13.** Assume the set

\[
W := \bigcup_{\lambda \in [0,1]} \lambda \text{Ad}_K H_d \subseteq g
\]

is convex. Then the equivalence theorem can be restated as follows:
(i) \( S_1(1, t) = KR_2(1, t) = R_2(1, t)K \),

(ii) \( \pi(S_1(1, t)) = R_3(K, t) \).

**Proof.** Replacing the set \( \text{Ad}_K H_d \) of control parameters for system 2 by the set \( W \) will not change the reachable sets \( R_2(1, t) \) since the trajectories that can occur then are just reparametrisations of the trajectories one already had before for system 2. Also there will be no effect on the infimizing, respectively, minimizing times because the tangents \( \dot{\gamma}(t) \) of the new occurring trajectories are of equal or smaller length than before, since \( \lambda \in [0, 1] \). The same holds for the reduced system 2. Because the set \( W \) is compact and due to our assumption convex, we may apply Filippov’s Theorem (cf. [1], Theorem 10.3) to obtain the compactness of the sets \( R_2(1, t) \) and \( R_3(K, t) \). Thus in the statement of the equivalence theorem, the expression \( R_2(1, t) \) can be replaced by \( R_2(1, t) \), and \( R_3(K, t) \) can be replaced by \( R_3(K, t) \). \( \square \)

**Remark 2.3.14.** We do not know if there is a criterion of how to decide in a concrete situation (where a subgroup \( K \subseteq G \) and a vector \( H_d \in g \) are given), whether the set \( W \) of the previous corollary is convex. This union of adjoint orbits turns out to be convex for instance in the example of Section 3.2. But one can also find low-dimensional examples, where \( W \) is not a convex set.

### 2.4. The Maximum Principle for Compact Lie Groups

In Section 1.5 the maximum principle of Pontrjagin (PMP) has been discussed as a tool for determining extremal trajectories in a given optimal control situation. The application of PMP involves the optimization of functionals on the set \( U \) of controls, which take the form

\[
(2.4.1) \quad u \mapsto H(x, u),
\]

where \( x \) is an arbitrary but fixed point of the phase space \( T^*M \), while

\[
(2.4.2) \quad H(\cdot, u) : T^*M \rightarrow \mathbb{R}
\]

is the Hamiltonian function associated with the optimal control problem. An immediate application of the maximum principle to our original (unreduced) control system 1 yields in general no further information on the optimality of a given control function, since in this case the space \( U \) of control parameters is unbounded. So in general there need not be a control \( u \in U \) which maximizes the functional (2.4.1). This is one of the main differences to the adjoint system 2. Here the space \( \text{Ad}_K H_d \subseteq g \) of control parameters is compact, so that the above functional always attains its maximum. Thus the passage from system 1 to system 2 via the equivalence theorem makes the time-optimal
Our discussion will be specialized to optimal control of right-invariant systems. By this we mean a control system $\Sigma = (G, f_u, U)$, where the set $U$ of control parameters is contained in the Lie algebra $\mathfrak{g}$ of $G$, and the admissible vector fields $f_u, u \in U$, are the right-invariant extensions of $u$, see Lemma 1.1.5. The results we are now going to discuss apply in particular to the adjoint system (2.3.9). The reason why right-invariant systems allow for a significant simpler formulation of Pontrjagin’s maximum principle is due to the following facts.

- The right-invariant vector fields $f_u, u \in U$, can be considered to be contained in the finite dimensional Lie algebra $\mathfrak{g}$, not just as elements of the infinite-dimensional algebra $\Gamma(TG)$ of arbitrary vector fields on $G$.
- There is a canonical isomorphism between $T^*G$ and $G \times \mathfrak{g}^*$, which allows to describe Hamiltonian functions and Hamiltonian vector fields by globally defined coordinates.
- Moreover, if $G$ is compact, it is possible to make use of the existence of an ad-invariant inner product on $\mathfrak{g}$, which allows to further identify $T^*G$ with $G \times \mathfrak{g}$. An advantage of this identification is that Hamiltonian functions and Hamiltonian vector fields often become easier to describe in coordinates of $\mathfrak{g}$ and $G$ rather than in coordinates of $T^*G$, respectively of $G$ and $\mathfrak{g}^*$.

Our discussion follows the book [1] by Agrachev and Sachkov. Similar results can also be found in Mittenhuber’s paper [23].

**Proposition 2.4.1.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Denote by $T^*G$ the cotangent bundle of $G$. Then the map

$$\Phi : G \times \mathfrak{g}^* \longrightarrow T^*G, \quad (g, \lambda) \longmapsto (D_gR_{g^{-1}})^*(\lambda)$$

is a vector bundle isomorphism. Here $(D_gR_{g^{-1}})^* : \mathfrak{g}^* \rightarrow (T_gG)^*$ denotes the dual of the linear map $D_gR_{g^{-1}} : T_gG \rightarrow \mathfrak{g}$.

**Proof.** [23], p. 187. $\square$

For the remainder of this section we restrict our discussion to the case of a compact Lie group $G$. Such groups can be endowed with an ad-invariant Riemannian metric $\langle \cdot, \cdot \rangle$, cf. Example 1.2.4. This can now be used to first identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the scalar product $\langle \cdot, \cdot \rangle_1$. Combining this identification with the map $\Phi$ of Proposition 2.4.1 then yields an isomorphism between the vector bundles $G \times \mathfrak{g}$ and $T^*G$. The ad-invariance of the metric is certainly not necessary for the existence of a vector bundle isomorphism $T^*G \cong G \times \mathfrak{g}$. Such can be
defined by using any Riemannian metric on $G$, and thus also exists for noncompact Lie groups $G$. The point is that the equations defining the Hamiltonian vector fields which occur in the general statement of the maximum principle, cf. equation (1.5.16), become particularly simple when choosing a trivialization via an $\text{ad}$-invariant metric. This point of view is substantiated in the following theorem.

**Theorem 2.4.2.** Let $\Sigma = (G, f_u, U, \varphi)$, $U \subseteq g$, be a right-invariant control system on a compact Lie group $G$. Moreover, let $\varphi$ be a cost function, which does not depend on the position variable $g \in G$, i.e. $\varphi : g \to \mathbb{R}$. Use the above identification $T^*G \cong G \times g$ to define the Hamiltonian function

$$H^\nu(\cdot, u) : G \times g \longrightarrow \mathbb{R}, \quad (g, X) \longmapsto \langle X, u \rangle_1 + \nu \varphi(u),$$

where $u \in U$ and $\nu \in \{-1, 0\}$. Then the Hamiltonian vector field associated with $H^\nu(\cdot, u)$ reads

$$\left\{ \begin{array}{l} \dot{g} = \frac{\partial H^\nu(\cdot, u)}{\partial X} g, \\ \dot{X} = [X, \frac{\partial H^\nu(\cdot, u)}{\partial X}] \end{array} \right..$$

The Hamiltonian lift $t \mapsto (g(t), X(t)) \in G \times g$ of any optimal trajectory $t \mapsto g(t)$ for the optimal control problem $(\Sigma, \varphi)$ which results from a control $t \mapsto \tilde{u}(t)$, $t \in [0, T]$, satisfies the extremality condition

$$H^\nu(g(t), X(t), \tilde{u}(t)) = \sup_{u \in U} H^\nu(g(t), X(t), u)$$

almost everywhere on $[0, T]$.

**Proof.** [11], Theorem 12.10, and p. 264. □

**Remark 2.4.3.** See Remark 1.5.2 for the role of the parameter $\nu \in \mathbb{R}$ in the Hamilton function $H^\nu(\cdot, u)$ of the previous theorem. It can be argued that only the cases $\nu = -1$ (so-called normal case) and $\nu = 0$ (so-called abnormal case) need to be distinguished, as Hamilton functions with $\nu > 0$ lead to trajectories that maximize the cost functional, while those with $\nu < 0$ can be replaced by $H^{\nu=-1}$ after rescaling the cost function $\varphi$, cf. [11], p. 180.

**Example 2.4.4.** Let $\Sigma = \{G, f_u, U\}$ be a right-invariant control system on the compact Lie group $G$, where $U := \{ u \in g | \langle u, u \rangle_1 = 1 \}$. We are interested in time-optimal trajectories of $\Sigma$, and therefore set $\varphi \equiv 1$. The Hamiltonian function (2.4.4) in this case reads

$$H^\nu(\cdot, u) : G \times g \longrightarrow \mathbb{R}, \quad (g, X) \longmapsto \langle X, u \rangle_1 + \nu,$$

and the corresponding Hamiltonian vector field is

$$\left\{ \begin{array}{l} \dot{g} = ug, \\ \dot{X} = [X, u] \end{array} \right..$$
The maximality condition of PMP implies that \( u(t) \) and \( X(t) \) have to be parallel for almost all \( t \) in order to maximize the term \( \langle X, u \rangle_1 \) in (2.4.7). But then, according to the second equation in (2.4.8), \( \dot{X} \equiv 0 \), so that \( X \) is constant. Then also the control function \( u \) is constant. So the first equation of (2.4.8) reads 
\[
\dot{g} = Zg
\]
with some \( Z \in U \), which is independent of \( t \). This equation can be integrated and yields the solution \( g(t) = \exp(tZ)g_0 \). So the fastest way to steer system \( \Sigma \) is along the integral curves of right-invariant vector fields.
Since \( G \) has been endowed with a bi-invariant metric, our time-optimal problem can be, by the choice of \( U \), considered a length-optimal problem. We thus have as a result that a length-minimizing curve necessarily is of the form \( t \mapsto \exp(tZ)g_0 \). Indeed, it can be shown that any geodesic of the Riemannian manifold \( (G, \langle \cdot, \cdot \rangle) \) is of that form.

Theorem 2.4.2 can be applied to the problem of finding time-optimal controls for the adjoint system
\[
(2.4.9) \quad \dot{U} = XU, \quad U(0) = 1, \quad X \in \Ad_K H_d.
\]
The associated Hamilton function and Hamilton vector fields to this control problem are as stated in Theorem 2.4.2, namely (with \( \varphi \equiv 1 \) as cost function)
\[
(2.4.10) \quad H(\cdot, u) : G \times g \to \mathbb{R}, \quad (g, X) \mapsto \langle X, u \rangle_1 + v
\]
and
\[
(2.4.11) \quad \begin{cases} 
\dot{g} = ug, \\
\dot{X} = [X, u].
\end{cases}
\]
The maximality condition of PMP for system (2.4.9) therefore reads
\[
(2.4.12) \quad H_{\text{max}}(g, X) = \max_{u \in \Ad_K H_d} \langle X, u \rangle_1.
\]
We next derive a necessary condition for \( u \in \Ad_K H_d \) to satisfy (2.4.12).

**Proposition 2.4.5.** Fix \( X \in g \). Then \( u_0 = \Ad_{k_0} H_d \in \Ad_K H_d \) is a local maximum of the function \( u \mapsto \langle X, u \rangle_1 \), if the following holds.

(i) \( \langle [X, u_0], Z \rangle_1 = 0 \) for all \( Z \in \mathfrak{k} \),

(ii) \( \langle [Z, X], [u_0, Z] \rangle_1 < 0 \) for all \( Z \in \mathfrak{k} \).

**Condition (i) is necessary for (2.4.12) to hold.**

**Proof.** If (2.4.12) holds, then for all \( Z \in \mathfrak{k} \)
\[
0 = \frac{d}{dt} \langle \Ad_{\exp(tZ)u_0} X \rangle_1 \big|_{t=0}
\]
\[
= \langle \ad Z(u_0), X \rangle_1
\]
\[
= \langle [Z, u_0], X \rangle_1
\]
\[
= - \langle [X, u_0], Z \rangle_1,
\]
which gives the necessity of condition (i). Now the function \( k \mapsto f(k) := \langle \text{Ad}_k H_d, X \rangle_1 \) has a local maximum in \( k_0 \) if condition (i) together with
\[
\left. \frac{d^2}{dt^2} f(\exp tZ \cdot k_0) \right|_{t=0} < 0 \quad \text{for all } Z \in \mathfrak{t}
\]
holds. But
\[
f(\exp tZ \cdot k_0) = \langle \text{Ad}_{\exp tZ \cdot k_0} H_d, X \rangle_1
\]
\[
= \langle e^{tZ u_0}, X \rangle_1
\]
\[
= \langle u_0, X \rangle_1 + t \langle [Z, u_0], X \rangle_1 + \frac{1}{2} t^2 \langle [Z, [Z, u_0]], X \rangle_1 + O(t^3),
\]
so condition \( \left. \frac{d^2}{dt^2} f(\exp tZ \cdot k_0) \right|_{t=0} < 0 \) is equivalent to (ii).

Our next goal is to derive a family of solutions of ODE (2.4.11).

**Theorem 2.4.6.** Let \( H_d \in \mathfrak{t}^\perp, A \in \text{Ad}_K H_d, \) and \( C \in \mathfrak{t}. \) Then for
\( u(t) := \text{Ad}_{\exp(-Ct)} A, \)
a solution of ODE (2.4.11) is given by
\[
\begin{align*}
g(t) &= \exp(-Ct) \exp(Ct + At), \\
X(t) &= -u(t) + C.
\end{align*}
\]
The corresponding Hamilton function is \( H(g, X, u) = \langle X, u \rangle_1. \) This Hamilton function also satisfies the maximality condition (i) of the previous proposition. Hence \( t \mapsto g(t) \) is an extremal trajectory of the time-optimal control problem associated with system (2.4.9).

**Proof.** We first notice that \( u(t) \in \text{Ad}_K H_d \) holds for all \( t \) by the choice of \( C \in \mathfrak{t}. \) A differentiation with respect to \( t \) now yields
\[
\dot{g}(t) = -C \exp(-Ct) \exp(Ct + At) + \exp(-Ct)(C + A) \exp(Ct + At)
\]
\[
= \exp(-Ct) A \exp(Ct + At)
\]
\[
= \exp(-Ct) A \exp(-Ct) \exp(Ct + At)
\]
\[
= u(t) g(t)
\]
and
\[
\dot{X}(t) = -\dot{u}(t)
\]
\[
= C \text{Ad}_{\exp(-Ct)} A - \left( \text{Ad}_{\exp(-Ct)} A \right) C
\]
\[
= [C, u(t)]
\]
\[
= [X(t), u(t)],
\]
as claimed. The fact that Hamilton system (2.4.11) arises from the Hamilton function \( H(\cdot, u) \) is part of Theorem 2.4.2. It remains to show that for \( X = X(t), \)
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\[ u = u(t), \text{ and for all } Z \in \mathfrak{k} \text{ the equation} \]

\[ 0 = \langle [X, u], Z \rangle_1 \]

holds. Since

\[ -\dot{u} = [X, u] \]

and \( \dot{u} \) is tangent to \( \text{Ad}_K H_d \), i.e. \( \dot{u} = [Y, \text{Ad}_k H_d] \) for some \( Y \in \mathfrak{k} \) and \( k \in K \), the last condition is equivalent to

\[ 0 = \langle [Y, \text{Ad}_k H_d], Z \rangle_1 \]

for all \( Z \in \mathfrak{k} \). This is satisfied, because

\[
\langle [Y, \text{Ad}_k H_d], Z \rangle_1 = \langle \text{Ad}_k [\text{Ad}_{k^{-1}} Y, H_d], Z \rangle_1 \\
= \langle [\text{Ad}_{k^{-1}} Y, H_d], \text{Ad}_{k^{-1}} Z \rangle_1 \\
= -\langle H_d, [\text{Ad}_{k^{-1}} Y, \text{Ad}_{k^{-1}} Z] \rangle_1 \\
= 0
\]

holds as a consequence of the Ad-invariance of the inner product \( \langle \cdot, \cdot \rangle_1 \) and of our assumption \( H_d \in \mathfrak{k}^\perp \). \( \square \)

**SUMMARY 2.4.7.** >From Theorem 2.4.6 we obtain a whole family of extremal trajectories associated with the control system (2.4.9). These are parametrized by real numbers \( A \) and \( C \), their role being the following. The parameter \( A = \dot{g}(0) \) determines the direction of the trajectory \( t \mapsto g(t) \) at its starting point, while the parameters \( A \) and \( C \) jointly fix the direction at \( t = 0 \) of the component \( x(t) \) of the Hamiltonian lift of \( g(t) \), as \( \dot{x}(0) = -\dot{u}(0) = [C, A] \). Thus in a subsequent step one would have to determine those pairs \((A, C)\) which actually give rise to a time-optimal trajectory.

For the special class of adjoint systems that we shall consider in the following section, it turns out that only the choice \( C = 0 \) can lead to time-optimal trajectories. So from this example one can see that the set of extremal trajectories in the sense of PMP will in general be considerably larger than that of actually time-optimal trajectories.

### 2.5. Time-Optimal Torus Theorem

In the following an explicit solution to the control problem as described in Section 2.2 and reformulated in the Equivalence Theorem 2.3.6 will be discussed under the additional assumption that the homogeneous space \( G/K \) in that theorem gives rise to a symmetric Lie algebra pair \((\mathfrak{g}, \mathfrak{k})\).

Thus in the following we fix a compact, simply connected, semisimple Lie group \( G \) together with a closed subgroup \( K \) such that their Lie algebras \( \mathfrak{g} \) and \( \mathfrak{k} \) form a symmetric Lie algebra pair \((\mathfrak{g}, \mathfrak{k})\). Let \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \) the corresponding Cartan-like decomposition, and take \( \mathfrak{h} \subseteq \mathfrak{p} \) to be a maximal abelian subalgebra of \( \mathfrak{p} \). Denote by \( A \) the torus in \( G \) with Lie algebra \( \mathfrak{h} \).
THEOREM 2.5.1. (Controllability). Let $G$ be a compact Lie group with simple Lie algebra $g$. On $G$ consider the affine right-invariant control system (2.3.1) of Section 2.3 with the Hamiltonian

$$H(v_1, ..., v_m) = H_d + \sum_{j=1}^{m} v_j H_j.$$  

Let $\mathfrak{t}$ the Lie algebra generated by $H_1, ..., H_m$ and assume $(g, \mathfrak{t})$ to be a symmetric Lie algebra pair. Denote by $\theta$ its Cartan involution and let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{p}$ that contains the projection $H_0$ of $H_d$ on $\mathfrak{p}$ (such exists in view of Lemma 1.3.8). Assume furthermore that $H_0$ is generic in the sense that it is not contained in any root hyperplane (of a root space decomposition of $g_C$ with respect to $\mathfrak{h}$).

Then system (2.3.1) has reachable set $R(1) = G$, i.e. it is controllable. The same holds for the respective reduced system on the symmetric space $G/K$.

PROOF. The result follows from Theorem 1.4.5 if we can show that the Lie algebra generated by $\mathfrak{t}$ and $H_d$ is equal to $g$. Extend $\mathfrak{h}$ to a maximal abelian subalgebra $\mathfrak{t}$ of $g$ and let

$$g_C = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Sigma} g_\alpha$$

be the root space decomposition of $g_C$ with respect to $\mathfrak{t}$. Choose as in (1.3.7) a subset $\Sigma^+ \subseteq \Sigma$ of so-called positive roots. Since $g$ is simple, it follows from Theorem 1.3.6 (iii) that the root spaces $g_\alpha$ are all 1-dimensional. We can therefore write

$$g_C = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Sigma} \mathbb{C} X_\alpha$$

with $X_\alpha \in g_\alpha \setminus \{0\}$ arbitrary. The root space decomposition (2.5.2) is related to the $-1$-eigenspace $\mathfrak{p}$ of the Cartan-like decomposition $g = \mathfrak{t} \oplus \mathfrak{p}$ in the following way:

$$\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n},$$

where

$$\mathfrak{n} := \sum_{\alpha \in \Sigma^+} \mathfrak{p} \cap \mathbb{C} (X_\alpha - \theta X_\alpha),$$

cf. [10], p. 335-336. Furthermore, the following sum is direct:

$$\sum_{\alpha \in \Sigma^+} \mathfrak{t} \cap \mathbb{C} (X_\alpha + \theta X_\alpha) \subseteq \mathfrak{t}.$$  

So for any $\alpha \in \Sigma^+$ we can choose $c \in \mathbb{C}$ such that $c(X_\alpha + \theta X_\alpha) \in \mathfrak{t} \setminus \{0\}$. We then find that

$$[H_0, c(X_\alpha + \theta X_\alpha)] = c\alpha(H_0) X_\alpha + \theta [\theta H_0, cX_\alpha]$$

$$= c\alpha(H_0) X_\alpha - c\theta [H_0, X_\alpha]$$

$$= c\alpha(H_0) (X_\alpha - \theta X_\alpha).$$
The commutator relation \([p, t] \subseteq p\) together with our assumption \(\alpha(H_0) \neq 0\) for all roots \(\alpha\) now implies that

\[ [H_0, t] = n. \]

Set \(i := \langle t, n, H_d \rangle_{\text{Lie}} = \langle t, n, H_0 \rangle_{\text{Lie}}\). It remains to show that \(\mathfrak{h} \subseteq i\). Assume \(X \in \mathfrak{h} \setminus i\). Let \(Z \in i\) arbitrary and write

\[ Z = Z_1 + Z_2 + Z_3 \text{ with } Z_1 \in t, Z_2 \in n, Z_3 \in \mathfrak{h}. \]

Repeating the calculation before yields \([X, Z_1] \in n\). Furthermore, \([\mathfrak{h}, \mathfrak{h}] = 0\) and \([\mathfrak{h}, n] \subseteq [p, p] \subseteq \mathfrak{t}\), so that

\[ [X, Z] = [X, Z_1] + [X, Z_2] \in n + \mathfrak{t} \subseteq i. \]

This shows that \(i\) is an ideal of the simple Lie algebra \(\mathfrak{g}\), which is not possible unless \(i = \mathfrak{g}\).

The statement on the reduced system follows immediately from Theorem 2.3.6 (ii).

\[ \square \]

**Remark 2.5.2.** A different proof of Theorem 2.5.1, without the assumption that \(H_0\) be not contained in any root hyperplane, can be found in [5].

We now turn to a discussion of time-optimal control. To fix ideas we initially consider the simple example of a single-particle system. Here the underlying Lie group is \(G = SU(2)\), the drift operator can be chosen to be

\[ H_d = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2), \]

and the free Hamiltonian to be

\[ H_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{su}(2). \]

Denote by \(\mathfrak{t}\) the 1-dimensional Lie algebra spanned by \(H_1\). It generates the compact Lie subgroup

\[ K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \bigg| t \in \mathbb{R} \right\} \]

of \(G\). We first observe that any element \(U_F = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\), \(a\bar{a} + b\bar{b} = 1\), of \(G\) can be decomposed as

\[ (2.5.3) \quad U_F = U_1 k \]

with \(k = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\) \(\in K\) and \(U_1 = \begin{pmatrix} r & s \\ s & \bar{r} \end{pmatrix}\) symmetric. Namely, a calculation shows that we can choose

\[ (2.5.4) \quad \tan(t) = \frac{\text{Re } b}{\text{Re } a}, \quad r = a \cos t + b \sin t, \quad s = -a \sin t + b \cos t. \]
Now $U_F$ and $U_1$, being contained in the same coset of $K$, are for all $t \geq 0$ either both contained in the approximately reachable set $S_1(1, t)$ or both not contained in $S_1(1, t)$, see Proposition 2.3.11. Therefore, in the time-optimal control problem for the unreduced system, $U_F$ can be replaced by $U_1$. Time-optimal trajectories from 1 to $U_1$ for the adjoint system

$(2.5.5) \quad \dot{U} = XU, \quad X \in \text{Ad}_K H_d$

may now be obtained as follows. The symmetric matrix $U_1$ admits a decomposition as

$(2.5.6) \quad U_1 = k (\exp t_0 H_d) k^{-1}$

where $k \in SU(2)$ is some orthogonal matrix, i.e. $kk^T = 1$, and $t_0 \in \mathbb{R}$. A calculation shows that one can choose $k \in K$. Furthermore, we choose $t_0$ such that $|t_0|$ is as small as possible. Since the set of controls for the adjoint system is invariant under conjugation by elements of $K$, $U_1$ can be replaced by the new terminal point $U_2 = \exp t_0 H_d$. The key observation is then that a time-optimal trajectory between 1 and $U_2$ is given by geodesics $t \mapsto \exp t H_d$ of a maximal torus $A$ of $G$ that contains $H_d$. This trajectory is generated by a constant control function $u(t) = H_d$ or $u(t) = -H_d$, the sign depending on that of $t_0$.

The main argument in the proof of this observation will be that a suitable projection of an arbitrary trajectory $t \mapsto U(t)$ between 1 and $U_2$ into the maximal torus $A$ leads to another trajectory $t \mapsto V(t)$ for the adjoint system, which reaches $U_2$ in the same time as $U(t)$.

This projection process reduces the original (adjoint) control system to a system on the torus $A$, which can be solved easily because the admissible vector fields are now pairwise commuting. In our particular example, those admissible vector fields are the right-invariant extensions of $H_d$ and $-H_d$. In the higher-dimensional cases, this set of vector fields will become the Weyl orbit $W \cdot H_d$ of $H_d$ as defined in Section 1.3.1. Optimal trajectories will then again be generated by choosing piecewise constant controls within the set $W \cdot H_d$, and therefore will be geodesic arcs on a maximal torus $A$ of $G$.

We next consider the general case of a compact semisimple Lie group $G$ and a symmetric Lie algebra pair $(g, t)$. Our observations made so far motivate the following theorem.

**Theorem 2.5.3.** (Time-optimal torus theorem). Let $G$ be a compact, simply connected, semisimple Lie group, and $K \subseteq G$ a closed subgroup. Assume that their respective Lie algebras form a symmetric pair $(g, t)$, its Cartan-like decomposition being $g = t \oplus p$. Let $\mathfrak{h}$ be a maximal abelian subalgebra of $p$, $H_d \in \mathfrak{h}$ a generic point in the sense of the previous theorem, and $A \subseteq G$ the maximal torus with Lie algebra $\mathfrak{h}$. Furthermore, denote by $Y_1, ..., Y_l$ the elements of the
Weyl orbit $W \cdot H_d$. Set

$$\Theta := \text{Ad}_K \bar{\Omega} = \{ k a k^{-1} | a \in \bar{\Omega}, k \in K \},$$

and let $U_F \in \Theta$ arbitrary. Here $\Omega \subseteq A$ is a certain domain in $A$, which will be specified in Lemma 2.5.7.

Then the minimal time $t_{\text{min}}(U_F)$ for steering the adjoint system

$$\dot{U} = XU, \quad U(0) = 1, \quad X \in \text{Ad}_K H_d$$

to $U_F$ is equal to $\alpha^*(U_F)$, which we define to be the smallest non-negative value of $\alpha$ such that the equation

$$U_F = k \exp \left( \alpha \sum_{i=1}^{l} \beta_i Y_i \right) k^{-1}$$

can be solved with $k \in K$, $\beta_i \in \mathbb{R}$, and $\sum_{i=1}^{l} \beta_i = 1$. Moreover, a time-optimal trajectory to $U_F$ is given by

$$U : t \mapsto \begin{cases} 
\exp \left( tk Y_1 k^{-1} \right), & t \in [0, \alpha \beta_1], \\
\vdots \\
\exp \left( tk Y_1 k^{-1} \right) \prod_{i=1}^{l-1} \exp \left( \alpha \beta_i k Y_i k^{-1} \right), & t \in \left[ \alpha \sum_{i=1}^{l-1} \beta_i, \alpha \right]. 
\end{cases}$$

The proof of this theorem needs some preparation.

To start with, we introduce the root space decomposition

$$g_C = g_0 \bigoplus_{\alpha \in \Sigma} g_\alpha$$

of $g_C$ with respect to $\mathfrak{h}$, cf. Theorem 1.3.2. Note that this is not quite the usual root-space decomposition, since $\mathfrak{h}$ was only assumed to be maximal abelian in $p$, but not in $g$. We will nevertheless call the spaces $g_\alpha$, $\alpha \in \Sigma$, root-spaces, and the linear forms $\alpha : \mathfrak{h} \to \mathbb{R}$ roots. The subset of roots which do not vanish identically on $p$ is denoted $\Sigma_p$. We define the Weyl group in the same manner as before to be the quotient

$$W := N(\mathfrak{h})/\text{Stab}(\mathfrak{h}).$$

In contrast to the previous definition, the isomorphism type of $W$ now depends on the pair $(g, t)$, not on $g$ alone.

Having fixed our notation in this way, we make now the following definition.

**Definition 2.5.4.** The following subset of $\mathfrak{h}$ is called the **diagram** $D$ associated with the symmetric Lie algebra pair $(g, t)$:

$$D := \{ X \in \mathfrak{h} | \alpha(X) \in i\pi \mathbb{Z} \text{ for some } \alpha \in \Sigma_p \}.$$
\{ X \in \mathfrak{h} | \alpha(X) = 0 \text{ for some } \alpha \in \Sigma \}, \text{ so that the arrangement of Weyl chambers in } \mathfrak{h} \text{ is now further subdivided into the system } \mathfrak{h} \setminus D \text{ of cells.} \\

The set of reflections on the hyperplanes that constitute the diagram } D \text{ generates a group } W_{\text{aff}}, \text{ the so-called } \text{affine Weyl group}. \text{ This group contains } W \text{ as a subgroup. More precisely, } W_{\text{aff}} \text{ is isomorphic to the semidirect product } W \times T \text{ of } W \text{ with the group } T \text{ of translations in } \mathfrak{h} \text{ that map } D \text{ onto itself.} \\

The affine Weyl group acts transitively on the set of cells. Furthermore, if } \Delta \text{ is a cell such that } 0 \in \bar{\Delta}, \text{ then any orbit } W_{\text{aff}} \cdot X, \ X \in \mathfrak{h}, \text{ intersects } \bar{\Delta} \text{ in exactly one point.} \\

For further details on the affine Weyl group and the diagram, see [10] and [6]. \\

The relevance of the affine Weyl group for the proof of the time-optimal torus theorem becomes clear through the following decomposition lemma.

\textbf{Lemma 2.5.5.} (KAK-decomposition). \textit{Let } G \text{ be a compact, simply connected, semisimple Lie group. Then each } g \in G \text{ yields a decomposition}

\begin{equation}
(2.5.14) \quad g = k_1 a k_2,
\end{equation}

with } k_1, k_2 \in K \text{ and } a = \exp X \in A. \text{ The factor } a = \exp X \text{ in this decomposition is unique up to an action of the affine Weyl group. So if}

\begin{equation}
(2.5.15) \quad k_1 a k_2 = k'_1 a' k'_2
\end{equation}

are two decompositions of the above type with } a = \exp X \text{ and } a' = \exp X', \text{ then there exists } w \in W_{\text{aff}} \text{ such that } X' = w \cdot X. \text{ The factor } a \text{ becomes determined uniquely, if in addition the requirement } X \in \bar{\Delta} \text{ is imposed. Here } \Delta \text{ denotes as before a cell whose closure contains } 0. \\

\textbf{Proof.} \cite{10}, p. 321-323. \ \Box

\textbf{Remark 2.5.6.} Our initial assumption on } G \text{ to be simply connected is only needed for the proof of the last lemma. We do not know a version of the KAK-decomposition lemma without this assumption.}

\textbf{Lemma 2.5.7.} \textit{Let } \Delta \in \mathfrak{h} \text{ a cell as in Lemma 2.5.5, and set } \Omega := \exp \Delta \subseteq A. \text{ Define the map}

\begin{equation}
(2.5.16) \quad \Phi : K \times A \times K \longrightarrow G, \quad (k_1, a, k_2) \longmapsto k_1 a k_2,
\end{equation}

and let

\begin{equation}
(2.5.17) \quad \pi_2 : K \times A \times K \longrightarrow A
\end{equation}

be the projection map onto } A. \text{ Then the following holds:}

(i) \textit{For each } g \in G, \text{ the set}

\begin{equation}
(2.5.18) \quad \left( \pi_2 \circ \Phi^{-1}(g) \right) \cap \bar{\Omega}
\end{equation}

consists of a single element } \pi_A(g). \text{ The map}

\begin{equation}
(2.5.19) \quad \pi_A = : G \longrightarrow \bar{\Omega}, \quad g \longmapsto \pi_A(g)
\end{equation}
is continuous. Moreover,

\[ \pi_A|_\bar{\Omega} = \text{id}. \]

(ii) Set \( G^{\text{reg}} := \pi_A^{-1}(\Omega) \). Then the restriction of \( \pi_A \) to the set \( G^{\text{reg}} \) yields a differentiable map. Its differential is for all \( g = \Phi(k_1, a, k_2) \in G^{\text{reg}} \) and for all \( X \in \mathfrak{g} \) given by the following formula:

\[ D_1 (\pi_A \circ R_g) (X) = D_1 R_{\pi_A(g)} (\Gamma (k_1^{-1} X k_1)), \]

where \( \Gamma : g \to \mathfrak{h} \) denotes orthogonal projection. In particular, as a consequence of Kostant’s Convexity Theorem 1.3.9,

\[ D_1 (\pi_A \circ R_g) (\text{Ad}_K X) = D_1 R_{\pi_A(g)} (\epsilon (W \cdot X)) \]

holds for all \( X \in \mathfrak{h} \).

**Proof.** (i) Using Lemma 2.5.5 on the \( KAK \)-decomposition, we can describe the set \((\pi_2 \circ \Phi^{-1})(g)\) as

\[ (\pi_2 \circ \Phi^{-1})(g) = \exp (W_{\text{aff}} \cdot X), \]

where \( X \in \bar{\Delta} \) is the unique element such that

\[ g = k_1 \exp X k_2 \]

holds for some \( k_1, k_2 \in K \). We now claim that

\[ (\pi_2 \circ \Phi^{-1})(g) \cap \bar{\Omega} = \{ \exp X \}. \]

So suppose that \( \exp X' \in (\pi_2 \circ \Phi^{-1})(g) \cap \bar{\Omega} \). Hence \( X' = w \cdot X \) for some \( w \in W_{\text{aff}} \), and \( \exp X' = \exp Z \) for some \( Z \in \bar{\Delta} \). Thus \( \exp Z = \exp (w \cdot X) \). It follows that

\[ Z = w \cdot X + V \]

for some \( V \in \exp^{-1}(1) \).

Now \( V \) can be regarded as an element of the subgroup \( T \) of translations of \( W_{\text{aff}} \), and we can further write

\[ Z = w' \cdot X \]

for some \( w' \in W_{\text{aff}} \). This implies that

\[ Z \in W \cdot X \cap \bar{\Delta} = \{ X \}, \]

and finally \( \exp X' = \exp Z = \exp X \), as claimed. This shows in particular that the map \( \pi_A \) is well-defined.

For the proof of continuity we define the quotient space \( A/\sim_{W_{\text{aff}}} \) by

\[ a \sim_{W_{\text{aff}}} a' \iff a' = w_1 a \exp(d) w_1^{-1} \text{ for some } (w_1, d) \in W_{\text{aff}}, \]

and notice that \( \bar{\Omega} \) and \( A/\sim_{W_{\text{aff}}} \) can be identified as topological spaces. Indeed, for any \( X \in \mathfrak{h} \) there exists a unique \( w \in W_{\text{aff}} \), which we may write as \( w = \ldots \)
$(w_1, d) \in W \times T$, such that
\[ w \cdot X = w_1(X + d)w_1^{-1} \in \Delta \]
holds. Hence $\exp w \cdot X = w_1(\exp X \exp d)w_1^{-1}$, and $\exp X \sim \exp w \cdot X \in \Omega$. Now let $a, a' \in \Omega$ with $a \sim a'$. Then $a' = w_1a\exp(d)w_1^{-1}$ for some $(w_1, d) \in W \times T$, and both sides of the last equation define a $KAK$-decomposition for $a'$. By Lemma 2.5.5 it follows that $a' = a$. This shows that $\Omega$ is a complete set of representatives for the equivalence relation $\sim_{W_{aff}}$. So the projection map
\[ \text{pr} : \Omega \longrightarrow A/\sim_{W_{aff}}, \quad a \mapsto [a] \]
is bijective. It is also continuous, because the action of $W_{aff}$ on $A$ is continuous. Now that $A/\sim_{W_{aff}}$ is compact, we see that the map $\text{pr}$ is in fact a homeomorphism. Next consider an arbitrary open subset $U$ of $\Omega$. Denote the projection of $A$ onto $A/\sim_{W_{aff}}$ by $\pi_\sim$. Then the preimage of $U$ under $\pi_A$ can be described as
\[ \pi_A^{-1}(U) = \left( \pi_2 \circ \Phi^{-1} \right)^{-1}(U) = \Phi \left( K, \pi_\sim^{-1}(U), K \right). \]
As a consequence of the identification of $\Omega$ with $A/\sim_{W_{aff}}$, it follows that the set $\pi_\sim^{-1}(U)$ is open in $A$. This implies that $\Phi \left( K, \pi_\sim^{-1}(U), K \right)$ is open in $G$ (cf. [8] where such a statement is proved for general actions of compact Lie groups on manifolds), and finally proves the continuity of the map $\pi_A$.

For any $a \in A$ one has the $KAK$-decompositions $a = 1 \cdot a \cdot 1$ and $a = k_1\pi_A(a)k_2$ with suitable factors $k_1, k_2 \in K$. Now the middle factor of a $KAK$-decomposition is uniquely determined if it is in addition required to lie in $\Omega$. So for $a \in \Omega$, the elements $a$ and $\pi_A(a)$ have to coincide. This proves the identity (2.5.20).

(ii) We first consider the action $\gamma$ of $K \times K$ on $G$ given by
\[ (k_1, k_2) \cdot g = k_1 g k_2^{-1}. \]
A calculation shows that the orbit $O_\gamma(a)$ of $a \in A$ intersects the torus $A$ perpendicularly. Namely, for all $X_1, X_2 \in \mathfrak{t}$ and $Z \in \mathfrak{h}$ the following holds:

\[
\left\langle \frac{d}{dt} \left( \exp (tX_1) \cdot a \cdot \exp (tX_2) \right) \big|_{t=0} , \frac{d}{dt} \left( \exp (tZ) \cdot a \right) \big|_{t=0} \right\rangle_a = \\
= \left\langle X_1a + aX_2, Za \right\rangle_a \\
= \left\langle X_1a, Za \right\rangle_a + \left\langle aX_2^{-1}a, Za \right\rangle_a \\
= \left\langle X_1, Z \right\rangle_1 + \left\langle aX_2^{-1} , Z \right\rangle_1 \\
= \left\langle X_1, Z \right\rangle_1 + \left\langle X_2, a^{-1}Za \right\rangle_1 \\
= 0. 
\]
Here we used the bi-invariance of the metric together with the fact that the sum $g = \mathfrak{k} \oplus \mathfrak{p}$ is orthogonal. From this there follows the direct sum decomposition

\[(2.5.23)\]

\[T_a G = T_a A \oplus T_a O_\gamma(a),\]

if we can show that the dimension of $O_\gamma(a)$ is complementary to that of $A$ in $G$. This is in fact the case, provided that $a$ is contained in $\Omega$, as will be proven now.

To this aim we make use of the following result, cf. [10], p. 294-295. Denote by $M$ the centralizer of $h$ in $K$, i.e.

\[M := \{ k \in K | kXk^{-1} = X \forall X \in \mathfrak{h} \}.\]

Then the map

\[\varphi : K/M \times \mathfrak{h} \rightarrow G/K, \quad ([k], X) \mapsto [\exp (\text{Ad}_k X)]\]

is surjective. It is moreover regular on $K/M \times (\mathfrak{h} \setminus D)$, i.e. in particular regular on $K/M \times \Delta$.

Now let $U \subseteq K/M \times \Delta$ be a sufficiently small open subset such that the map

\[\varphi|_U : U \rightarrow \varphi(U)\]

is a diffeomorphism, and the set $W := \pi^{-1}(V) \subseteq G$ admits a trivialization

\[\psi : W \rightarrow V \times K\]

with

\[\psi (\exp (\text{Ad}_k X) k') = ([\exp (\text{Ad}_k X)], k^{-1})\]

for all $([k], X) \in U$ and $k' \in K$. Then, by construction, the map

\[\sigma : U \times K \rightarrow G, \quad ([k], X, k') \mapsto \psi^{-1}(\varphi([k], X), k')\]

is regular on $U \times K$. Furthermore, the map $\gamma$ is related to $\sigma$ as follows:

\[(2.5.24)\]

\[\gamma (k, k'k, \exp X) = \sigma ([k], X, k').\]

The regularity of $\sigma$ implies now that for all $X \in \text{pr}_2(U) \subseteq \Delta$ the differential of the map $\sigma (\cdot, X, \cdot)$ at the point $([k], k') \in \text{pr}_1(U) \times K \subseteq K \times K$ has maximal rank $r = \dim G - \dim \Delta = \dim G - \dim A$. From equation (2.5.24) it follows that the same holds for the map $\gamma (\cdot, \cdot, a)$, where $a = \exp X \in \exp (\text{pr}_2(U)) \subseteq \Omega$. This finally proves equation (2.5.23).

We now proceed in the proof of formula (2.5.21). By part (i) of the proof we have for all $a \in \Omega$, $Z \in \mathfrak{h}$ and $t$ sufficiently small the identity

\[\pi_A (\exp tZ) a = (\exp tZ) a,\]
and therefore
\[(2.5.25)\quad D_a \pi_A(Za) = D_1 R_a(Z).\]

On the other hand, for all \(tX \in T_a \mathcal{O}_a(a)\) it follows that
\[(2.5.26)\quad \pi_A((\exp tX) a) = \pi_A(a)\]

since \(\pi_A\) is constant on the orbits of \(\gamma\). Equations (2.5.23), (2.5.25), and (2.5.26) now imply that
\[(2.5.27)\quad D_a \pi_A(D_1 R_a(X)) = D_1 R_a(\Gamma(X))\]

holds for all \(X \in \mathfrak{g}\).

Now let \(g \in G^{\text{reg}}\) arbitrary. Then \(g = k_1 a k_2\) for \(a = \pi_A(g) \in \Omega\) and some \(k_1, k_2 \in K\). Then for all \(X \in \mathfrak{g}\),
\[\pi_A((\exp tX) g) = \pi_A((\exp tX) k_1 a k_2) = \pi_A(k_1 (\exp t k_1^{-1} X k_1) a k_2) = \pi_A((\exp t k_1^{-1} X k_1) a).\]

Differentiating both sides of this equation with respect to \(t\) at \(t = 0\) and applying equation (2.5.27) yields
\[(2.5.28)\quad D_g \pi_A(D_1 R_g(X)) = D_a \pi_A(D_1 R_a(k_1^{-1} X k_1)) = D_1 R_a(\Gamma(k_1^{-1} X k_1)),\]

as claimed.

The identity (2.5.22) is now immediate from Kostant’s Convexity Theorem 1.3.9, which for fixed \(k_1 \in K\) and \(X \in \mathfrak{t}\) implies that
\[\Gamma(k_1^{-1}(\text{Ad}_K X) k_1) = \Gamma(\text{Ad}_K X) = c(W \cdot X)\]

holds. Combining this with identity (2.5.21) we obtain (2.5.22).

We finally note that the map \(\pi_A\) is differentiable in \(g \in G^{\text{reg}}\), since equation (2.5.28) implies that the partial derivatives of \(\pi_A\) in all directions \(D_1 R_g(X)\) of \(T_g G\) exist, with continuous dependency on \(g\).

We now turn to the proof of Theorem 2.5.3.

\[\text{Proof.} \text{ First consider the special case } U_F \in \bar{\Omega}. \text{ For such an element } U_F \text{ there always exists an } \alpha \text{ which satisfies equation (2.5.9) with } k = 1. \text{ This is evident in the case where } g \text{ is a simple Lie algebra, since then } W \text{ acts irreducibly on } \mathfrak{h} \text{ (cf. [12], p. 53), so that the Weyl orbit } W \cdot H_d \text{ of any } H_d \neq 0 \text{ spans } \mathfrak{h}. \text{ In the semisimple case } \mathfrak{h} \text{ decomposes as a direct sum } \mathfrak{h} = \bigoplus_j \mathfrak{h}_j \text{ of abelian subalgebras, and } W \text{ still acts irreducibly on each summand } \mathfrak{h}_j. \text{ By our choice of } H_d, \text{ the component of } H_d \text{ in } \mathfrak{h}_j \text{ is non-zero, such that as in the simple case the set } W \cdot H_d = \{Y_1, ..., Y_l\} \text{ spans } \mathfrak{h}. \text{ Moreover, for any tuple } (k, \alpha, \beta_1, ..., \beta_l) \text{ that satisfies (2.5.9) it is clearly possible}.
\]
to steer system (2.5.8) to $U_F$ at time $t_F = \alpha$ along the trajectory as specified in (2.5.10). Let $\alpha^*(U_F)$ be the smallest non-negative number such that equation (2.5.9) can be solved with $\alpha = \alpha^*(U_F)$.

It remains to show that any trajectory of system (2.5.8) with $U(t_F) = U_F$ necessarily satisfies $t_F \geq \alpha^*(U_F)$. In order to prove this we use the projection map $\pi_A$ of Lemma 2.5.6 to replace $t \mapsto U(t)$ by another trajectory of the system which joins $U(0) = 1$ to $U(t_F) = U_F$, but is completely contained in $A$. Since the tangents $\dot{U}(t)$ of $U(t)$ are by definition right translates of the set $Ad_K H_d \subseteq g$ we can apply Lemma 2.5.6 (ii) to conclude that the path

$$ t \mapsto \pi_A(U(t)) =: V(t) $$

has tangents $\dot{V}(t) \in D_1 R_{V(t)}(t(W \cdot H_d))$. By part (i) of the same lemma,

$$ V(t_F) = \pi_A(U_F) = U_F. $$

However, $t \mapsto V(t)$ is in general not a trajectory of the control system under consideration. But for all $t \in [0, t_F]$, its tangents are of the form

$$ \dot{V}(t) = D_1 R_{V(t)} \left( \sum_{i=1}^{l} \beta_i(t) Y_i \right) $$

with $\sum_{i=1}^{l} \beta_i(t) = 1$ and $Y_i$ as stated. Therefore, since $[Y_i, Y_j] = 0$ for all $i, j = 1, ..., l$,

$$ V(t) = \prod_{i=1}^{l} \left( \exp \left( \int_{0}^{t} \beta_i(\tau) d\tau \right) Y_i \right). $$

So steering the adjoint system according to the control

$$ t \mapsto \begin{cases} Y_1, & t \in [0, \gamma_1], \\ \vdots \\ Y_l, & t \in \left[ \sum_{i=1}^{l-1} \gamma_{i-1}, \sum_{i=1}^{l} \gamma_l \right], \end{cases} $$

where

$$ \gamma_i := \int_{0}^{t_F} \beta_i(\tau) d\tau, $$

results in a trajectory $t \mapsto W(t)$ on $A$ which satisfies

$$ W \left( \sum_{i=1}^{l} \gamma_{i} \right) = W(t_F) = V(t_F) = U_F. $$

This is again immediate from the pairwise commutativity of the vector fields $Y_i$, $i = 1, ..., l$. Moreover, it follows from the definition of $\alpha^*(U_F)$ that $\alpha^*(U_F) \leq t_F$, as we have claimed. Finally, a trajectory from $U(0) = 1$ to $U(\alpha^*(U_F)) = U_F$, and therefore (by construction) a time-optimal trajectory, is given by $t \mapsto W(t)$, and this has the form as stated.

Now let $U_F \in \Theta$ be arbitrary. By definition of $\Theta$ we can choose $\tilde{k} \in K$ such that $\dot{U}_F := \tilde{k} U_F \tilde{k}^{-1} \in \tilde{\Theta}$. The trajectories joining the system from $U(0) = 1$ to $U_F$, respectively to $\dot{U}_F$ at fixed time $t_F \geq 0$ are in bijective correspondence to
each other, see the proof of Proposition 2.3.7 (i). Namely, if \( t \mapsto X(t), t \in [0, t_F] \), is a control which leads to a trajectory \( t \mapsto U(t) \) with \( U(t_F) = \tilde{U}_F \), then \( t \mapsto \tilde{k}X(t)\tilde{k}^{-1}, t \in [0, t_F] \), is a control (with values again in \( \text{Ad}_K H_d \)) which leads to a trajectory with endpoint \( \tilde{U}_F \), and vice versa.

Hence \( \alpha^*(U_F) = \alpha^*(\tilde{U}_F) \), which is, by the first part of the proof, the smallest non-negative value of \( \alpha \) such that

\[
\tilde{U}_F = k \exp \left( \alpha \sum_{i=1}^l \beta_i Y_i \right) k^{-1}
\]

can be solved for some \( k \in K \). At the same time, this \( \alpha \) is the smallest one possible such that

\[
U_F = \tilde{k}^{-1} \tilde{U}_F \tilde{k} = \tilde{k}^{-1}k \exp \left( \alpha \sum_{i=1}^l \beta_i Y_i \right) k^{-1} \tilde{k}
\]

can be solved with \( k \in K \), which means that the value of \( \alpha^*(U_F) \) is as stated.

The statement on the time-optimality of the trajectory \( t \mapsto U(t) \) with \( U(t_F) = U_F \) follows from that on time-optimal trajectories in the special case \( U_F \in \tilde{\Omega} \) by using again the correspondence of trajectories with \( K \)-conjugated endpoints as formulated before.

**Corollary 2.5.8.** Assume the Lie groups \( G \) and \( K \) to satisfy the prerequisites of Theorem 2.5.3, and let \( P_F \in G/K \) arbitrary. Then the set

\[
(2.5.29) \quad X_F := \pi^{-1}(P_F) \cap \Theta \subseteq G
\]

is non-empty. Furthermore, the canonical projection of any trajectory of type (2.5.10) with endpoint in \( X_F \) yields a time-optimal trajectory between \( \pi(1) = K \) and \( P_F \) for the reduced system 3.

**Proof.** Let \( P_F = gK \) for some \( g \in G \). Then \( g \) yields a \( KAK \)-decomposition of the form \( g = k_1a k_1^{-1}k_2 \) with \( k_1, k_2 \in K \) and \( a \in \tilde{\Omega} \). Hence \( k_1a k_1^{-1} \in \pi^{-1}(P_F) \cap \Theta = X_F \).

Now let \( t \mapsto u(t) \in \text{Ad}_K H_d, t \in [0, t_F] \), be a control for system 3, such that the corresponding trajectory \( t \mapsto P(t) \) has terminal point \( P(t_F) = P_F \). Let system 2 evolve according to the same control function \( u \), and denote the resulting trajectory by \( t \mapsto U(t) \). From the proof of Proposition 2.3.7 (ii) it follows that \( P(t) = \pi \circ U(t) \) holds on \([0, t_F]\). In particular, \( U_F := U(t_F) \) satisfies \( \pi(U_F) = P_F \).

In complete analogy to the proof of the previous Theorem 2.5.3 we can apply the projection map \( \pi_A \) together with a conjugation to the trajectory \( U \) in order to obtain a trajectory \( t \mapsto W(t), t \in [0, t_F] \), which is of the form (2.5.10) and satisfies \( W(t_F) = U_F k' \) for some \( k' \in K \). Namely, if

\[
U_F = k \pi_A(U_F)k^{-1} k'^{-1}
\]

is a \( KAK \)-decomposition of \( U_F \), then we will set \( V(t) := k \pi_A(U(t))k^{-1} \) for \( t \in [0, t_F] \). As in the proof of Theorem 2.5.3 we then obtain from the path \( t \mapsto V(t) \)
a trajectory \( t \mapsto W(t) \) of system 2 which is of the special form (2.5.10) and has endpoint \( W(t_F) = V(t_F) = U_F k' \).

We denote the control function which generates this trajectory by \( \nu \). Let system 3 evolve according to the same control function \( \nu \), and denote the corresponding trajectory by \( t \mapsto P_1(t) \). It follows as before that

\[
P_1(t) = \pi \circ W(t)
\]

holds on \([0, t_F]\). In particular,

\[
P_1(t_F) = \pi(U_F k') = \pi(U_F) = P_F.
\]

This shows that any trajectory \( t \mapsto P(t) \) which reaches \( P_F \) at time \( t_F \) can be replaced by a trajectory \( t \mapsto P_1(t) \) which also satisfies \( P_1(t_F) = P_F \), and has the special form as stated.

It remains to show that any trajectory of that type is indeed time-optimal for system 3. So let \( W'_F \in \mathcal{X}_F \) be arbitrary, and denote by \( t \mapsto W'(t), t \in [0, t'_F], \) a trajectory of type (2.5.10) which reaches \( W'_F \) at time \( t'_F \). The statement of the time-optimal torus theorem is that such a trajectory is time-optimal for system 2. Since \( \pi(W'_F) = \pi(W_F) = P_F \), we see that \( W_F = W'_F k' \) holds for some \( k' \in \mathcal{K} \). Thus \( \pi_A(W'_F) = \pi_A(W_F) = a \in \bar{\Omega} \). Furthermore, as \( W_F, W'_F \in \Theta \), it follows that both \( W_F \) and \( W'_F \) are conjugated within \( \mathcal{K} \) to \( a \). Hence \( W_F = k'' W'_F k''^{-1} \) for some \( k'' \in \mathcal{K} \). Finally, as a consequence of Proposition 2.3.7 (i), it follows that \( t_F = t'_F \), as otherwise \( t \mapsto W(t) \) and \( t \mapsto W'(t) \) could not both be time-optimal trajectories for system 2.

This shows that the projection of \( t \mapsto W'(t) \) under \( \pi \) again yields a time-optimal trajectory for system 3 between \( \mathcal{K} \) and \( P_F \).

\[\square\]

We finally describe how the combination of the Equivalence Theorem 2.3.6 with the previous Corollary 2.5.8 can be used to solve the time-optimal control problem for the unreduced system (2.3.1) we have originally been interested in. We therefore keep all the assumptions made in the time-optimal torus theorem, and let \( U_F \in G \) arbitrary. From Corollary 2.3.12 it follows that \( t_F := t_{\inf,1}(U_F) \) equals \( t_{\inf,3}(U_F K) \). Corollary 2.5.8 can now be used as follows to construct a trajectory \( t \mapsto U(t) \) for system 1, which satisfies the time-optimality condition \( U(t_F) = U_F \).

1. Decompose \( U_F \) as \( U_F = k_1 a k_1^{-1} k_2 \) with \( k_1, k_2 \in \mathcal{K} \) and \( a \in \bar{\Omega} \). This is just a \( KAK \)-decomposition as introduced in Lemma 2.5.5. By definition of \( \bar{\Omega} \), the factor \( a \) in that decomposition is uniquely determined.

2. Set \( V_F := k_1 a k_1^{-1} \in \Theta \), where \( \Theta \subseteq G \) is as defined in Theorem 2.5.3. Let \( t \mapsto V(t), t \in [0, t_F], \) be a trajectory for system 2 of type (2.5.10) such that \( V(t_F) = V_F \) holds. By Corollary 2.5.8, such a trajectory exists and is time-optimal.
(3) By construction, the trajectory \( t \mapsto V(t) \) consists of a finite number of geodesic arcs of the form

\[
t \mapsto k' \exp(tY_{j+1}) k'^{-1}V(t_j), \quad t \in [t_j, t_{j+1}],
\]

where \( k' \in K, Y_j \in W \cdot H_d, \) and \([t_j, t_{j+1}]\) is a subinterval of \([0, t_F]\) as specified in Theorem 2.5.3. Write \( Y_j \) as \( k''H_d k''^{-1} \) for some \( k'' \in K \).

(4) System 1 can be steered from \( V(t_j) \) to

\[
V(t_{j+1}) = k' \exp(t_{j+1}Y_{j+1}) k'^{-1}V(t_j)
\]

within time \( t_{j+1} - t_j \) by first producing the element \( (k'k'')^{-1}V(t_j) \) within zero infimizing time. Evolution under the influence of the drift operator \( H_d \) for time \( t_{j+1} - t_j \) transfers the system in a second step from \( (k'k'')^{-1}V(t_j) \) to \( \exp(t_{j+1}H_d)(k'k'')^{-1}V(t_j) \). The point \( V(t_{j+1}) \) is finally reached from \( \exp(t_{j+1}H_d)(k'k'')^{-1}V(t_j) \) within zero infimizing time.

(5) The iteration of such so-called pulse-drift-pulse sequences transfers the unreduced system within infimizing time \( t_F \) from \( V(0) = 1 \) to \( V_F \). The point \( U_F = V_F k_2 \) differs from \( U_F \) by only an element of \( K \) and thus can also be reached within infimizing time \( t_F \).
CHAPTER 3

Discussion of some Explicit Spin Systems

3.1. General Considerations

In this chapter we shall apply the results obtained so far to a number of concrete examples. While the Equivalence Theorem 2.3.6 and the Time-Optimal Torus Theorem 2.5.3 have been formulated for right-invariant control systems on arbitrary compact Lie groups (with additional assumptions such as semisimplicity to be satisfied in 2.5.3), we are now focussing on the particular case \( G_n = SU(2^n) \) and the time-optimal control of certain \( n \)-particle spin systems. As a typical example of that kind of control-system we want to discuss system (2.2.2) of Section 2.2, where the subgroup \( K_n \) generated by the control Hamiltonians was \( K_n = SU(2)^\otimes n \). The equivalence theorem allows us to replace this control system by the corresponding reduced system

\[
\dot{P} = XP, \quad P(0) = K, \quad X \in \text{Ad}_{K_n} H_d
\]

on the homogeneous space \( G_n/K_n \). Let \( g_n := su(2^n) \) and \( t_n := su(2)^\otimes n \) be the Lie algebras of \( G_n \), respectively of \( K_n \).

To proceed in our discussion of time-optimal control, we first of all determine those examples in the family (3.1.1) of control systems which meet the requirements of Theorem 2.5.3, respectively its Corollary 2.5.8. The following theorem shows that these are satisfied for \( n = 1 \) and \( n = 2 \) only.

**THEOREM 3.1.1.** The pair \((g_n, t_n)\) is a symmetric Lie algebra pair if and only if \( n \leq 2 \).

**PROOF.** In course of proving this theorem we need to work in the tensor-product basis of \( g_n = su(2^n) \) as introduced in Example 1.6.4. This basis comprises the \( n \)-fold products of elements \( \{1, I_x, I_y, I_z\} \) with at least one factor being different from 1.

Assume \((g_n, t_n) = (su(2^n), su(2)^\otimes n), n \in \mathbb{N}, \) to be a symmetric Lie algebra pair with Cartan-like decomposition \( g_n = t_n \oplus p_n \). We claim that the set

\[
X := \bigcup_{j \geq 2} X_j
\]

with \( X_j \) as defined in Example 1.6.4 is a basis of \( p_n \). Because \( t_n \) is spanned by \( X_1 \), the \( \mathbb{R} \)-linear span of \( X \) is seen to be complementary to \( t_n \) in \( g_n \). It remains to show that \( X \) is orthogonal to \( t_n \) (cf. Lemma 1.2.6, and use that \( g_n \)
is semisimple). Therefore, let

\[ A := i^{α_1} 1 \otimes ... \otimes I_{α_1} \otimes ... \otimes I_{α_j} \otimes ... \otimes 1 \in X_j, \]

\[ j \geq 2, \text{ and} \]

\[ B := 1 \otimes ... \otimes I_β \otimes ... \otimes 1 \in X_1. \]

Let \( k \in \{1, ..., j\} \) be an index such that the position of \( I_{α_k} \) in \( A \) differs from that of \( I_β \) in \( B \). Choose \( γ, δ \in \{x, y, z\} \) such that \([I_γ, I_δ] = I_{α_j}\) and set

\[ C := 1 \otimes ... \otimes I_γ \otimes ... \otimes 1 \in X_1, \]

\( I_γ \) at the same position as \( I_{α_j} \), and

\[ D := i^{α_j} 1 \otimes ... \otimes I_{α_1} \otimes ... \otimes I_{α_j-1} \otimes ... \otimes I_δ \otimes ... \otimes 1 \in X_j, \]

\( I_δ \) at the same position as \( I_{α_j} \) and all other positions coinciding with those of \( A \). Using Lemma 1.6.3 (iv) we find that \( A = [C, D] \) and \([B, C] = 0\). From the ad-invariance of the Killing-form \( κ \) it now follows that

\[ κ(A, B) = κ([C, D], B) = κ(D, [B, C]) = κ(D, 0) = 0. \]

This shows \( A \in X_j^⊥ \). So

\[ \langle X \rangle \subseteq \langle X_j \rangle^⊥ = t_n^⊥ = p_n, \]

and finally, for dimensional reasons, \( \langle X \rangle = p_n \).

Now let \( n \geq 3 \) and consider the elements

\[ A_1 := I_x \otimes I_y \otimes I_z \otimes 1 \otimes ... \otimes 1 \in p_n \]

and

\[ A_2 := iI_y \otimes I_y \otimes 1 \otimes ... \otimes 1 \in p_n. \]

We use Lemma 1.6.3 (iv) to calculate

\[
-i[A_1, A_2] = [I_x \otimes I_y \otimes I_z \otimes 1 \otimes ... \otimes 1, I_y \otimes I_y \otimes 1 \otimes ... \otimes 1] \\
= [I_x, I_y] \otimes (I_y \otimes I_z \otimes 1 \otimes ... \otimes 1) (I_y \otimes 1 \otimes ... \otimes 1) \\
+ I_y I_x \otimes [I_y \otimes I_z \otimes 1 \otimes ... \otimes 1, I_y \otimes 1 \otimes ... \otimes 1] \\
= I_z \otimes I_y^2 \otimes I_z \otimes 1 \otimes ... \otimes 1 \\
- I_z \otimes ([I_y \otimes I_y] \otimes (I_z \otimes 1 \otimes ... \otimes 1) (1 \otimes ... \otimes 1)) \\
- I_z \otimes (I_y^2 \otimes [I_z \otimes 1 \otimes ... \otimes 1, 1 \otimes ... \otimes 1]) \\
= -I_z \otimes 1 \otimes I_z \otimes 1 \otimes ... \otimes 1 \\
- I_z \otimes (0 \otimes (I_z \otimes 1 \otimes ... \otimes 1) (1 \otimes ... \otimes 1) + I_y^2 \otimes 0) \\
\notin t_n.
\]

So \([p_n, p_n]\) is not contained in \( t_n \), and \((g_n, t_n)\) cannot be a symmetric Lie algebra pair.
In the trivial case \( n = 1 \) the assertion clearly holds. Now let \( n = 2 \). We have to check that \( g_2 \) admits an involutive Lie algebra automorphism \( \theta \) with \( t_2 \) as its 1-eigenspace. On the basis \( X_1 \cup X_2 \) as introduced before define
\[
\theta(X) = \begin{cases} 
X, & \text{if } X \in X_1, \\
-X, & \text{if } X \in X_2,
\end{cases}
\]
and extend \( \theta \) to a linear map on \( g_2 \). So \( \theta \) is by definition an involution. A calculation now shows that the commutator \( [Y_1, Y_2] \) of any two elements \( Y_1, Y_2 \in X_2 \) is contained in \( k_2 \), while for all \( Z \in k_2 \) the commutator \( [Y_1, Z] \) is in the linear span of \( X_2 \). So
\[
\theta[Y_1, Y_2] = [Y_1, Y_2] = [\theta(Y_1), \theta(Y_2)]
\]
and
\[
\theta[Y_1, Z] = -[Y_1, Z] = [\theta(Y_1), \theta(Z)],
\]
i.e. \( \theta \) is also a Lie algebra automorphism. □

**SUMMARY 3.1.2.** Our discussion so far lead to the result that the problem of time-optimal control of an \( n \)-particle spin system with Hamiltonian \( H = H_d + \sum_{j=1}^n (v_{jx}I_{jx} + v_{jy}I_{jy} + v_{jz}I_{jz}) \in su(2^n) \) can be solved by applying the results of Section 2.5 if and only if \( n \in \{1, 2\} \). This will be carried out in detail in the subsequent two sections. Although the number of cases where Theorem 2.5.3 on time-optimal control applies is quite limited as long as we are only interested in Hamiltonians of the special form above, one nevertheless could imagine other interesting right-invariant control systems on \( G = SU(n) \) or any other compact semisimple Lie group \( G \) that allow for the application of that theorem.

### 3.2. Single Particle Systems

We turn to a discussion of control system (3.1.1) in the case \( n = 1 \). Here we assume the exterior magnetic field to excite rapidly the \( x \)-component \( I_x \) of the spin \( I = (I_x, I_y, I_z) \) and consider \( I_z \) to be the drift Hamiltonian. This leads to the unreduced control system
\[
\dot{U} = (I_z + vI_x)U, \quad U(0) = 1, \quad v \in \mathbb{R}
\]
on \( G = SU(2) \). \( I_x \) generates the Lie subgroup
\[
K = \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \bigg| t \in \mathbb{R} \right\},
\]
which is isomorphic to \( U(1) \). So the corresponding reduced system is
\[
\dot{P} = PX, \quad P(0) = K, \quad X \in \text{Ad}_K I_z
\]
on the two-dimensional homogeneous space \( G/K = SU(2)/U(1) \). This space is diffeomorphic to the projective plane \( \mathbb{RP}^2 \), as will become clear later. The set
Ad_K I_z of control variables is a circle around zero, cf. Example 1.3.10.

The pair (g, t) = (su(2), u(1)) is symmetric with Cartan involution \( \theta \) defined by

\[
\theta (\alpha I_x + \beta I_y + \gamma I_z) = \alpha I_x - \beta I_y - \gamma I_z
\]

Hence Theorems 2.5.1 and 2.5.3 apply to system (3.2.3). So this system is in particular controllable.

The orthogonal complement of \( k = \mathbb{R} I_x \) with respect to the Killing form on \( su(2) \) is

\[
p = \mathbb{R} I_y + \mathbb{R} I_z = \left\{ \begin{pmatrix} i\alpha & i\beta \\ i\beta & -i\alpha \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\},
\]

which leads to the Cartan-like decomposition \( g = k \oplus p \). One can identify \( p \) with the tangent space \( T_K(G/K) \) of \( G/K \) and then argue that the geodesics emanating from \( K \in G/K \) are of the form

\[
t \mapsto (\exp tX) K, \quad X \in p,
\]

cf. [10], p. 212. Since \( G/K \) is compact, the Hopf-Rinow theorem implies that any point \( gK \in G/K \) is of the form \( gK = (\exp X) K \) for some \( X \in p \). Because \( \exp X \) is a symmetric matrix if \( X \) is symmetric, we see that the points of \( G/K \) can be represented by the symmetric unitary \((2 \times 2)\)-matrices. These comprise the set

\[
\text{Sym}(2) := \left\{ \begin{pmatrix} \cos \psi e^{i\varphi} & i \sin \psi e^{-i\varphi} \\ i \sin \psi e^{i\varphi} & \cos \psi e^{i\varphi} \end{pmatrix} \middle| \varphi, \psi \in [0, 2\pi] \right\},
\]

which is a 2-dimensional submanifold of \( G \). Moreover, the map

\[
\psi : \text{Sym}(2) \longrightarrow \mathbb{S}^2 \subseteq \mathbb{R}^3,
\]

\[
\begin{pmatrix} \cos \psi e^{i\varphi} \\ i \sin \psi \end{pmatrix} e^{i\varphi} \mapsto (\cos \varphi \cos \psi, \sin \varphi \cos \psi, \sin \psi)
\]

is a diffeomorphism between \( \text{Sym}(2) \) and the 2-sphere \( \mathbb{S}^2 \). A calculation now shows that two elements \( g_1, g_2 \in \text{Sym}(2) \) are in the same coset modulo \( K \) if and only if \( g_1 = g_2 \) or \( g_1 = -g_2 \). Hence

\[
G/K \cong \text{Sym}(2)/g \sim -g \cong \mathbb{S}^2 / g \sim -g \cong \mathbb{R} \mathbb{P}^2.
\]

In order to solve the time-optimal control problem related to the unreduced system (3.2.1) we shall proceed as outlined at the end of Section 2.5. To this aim we fix the maximal abelian subalgebra \( h := \mathbb{R} I_z \) of \( p \) and determine the Weyl orbit \( W \cdot H_d \) of \( H_d \), the action of \( W_{\text{aff}} \) on \( h \), and the sets \( \Omega, \Theta \subseteq G \).

The action of the affine Weyl group \( W_{\text{aff}} \) on \( h \) is generated by reflexions

\[
X \mapsto -X
\]

and translations

\[
X \mapsto X + \frac{\pi n}{2} I_z, \quad n \in \mathbb{Z},
\]
as is seen from the root space decomposition of \( g \) with respect to \( \mathfrak{h} \), cf. Example 1.3.7. The cells in \( \mathfrak{h} \) are then the sets
\[
\begin{align*}
L_n := \left\{ \frac{n \nu}{2} I_z \: \big| \nu \in (n, n+1) \right\}, \quad n \in \mathbb{Z}.
\end{align*}
\]
We fix the cell \( \Delta := L_0 \) and set
\[
\Omega := \exp \Delta = \left\{ \left( \begin{array}{cc}
e^{i \varphi} & 0 \\
0 & e^{-i \varphi} \end{array} \right) \bigg| \varphi \in (0, \pi/2) \right\} \subseteq A.
\]
Here \( A \) denotes as usual the maximal torus of \( G \) with Lie algebra \( \mathfrak{h} \). The set \( \Theta = \text{Ad}_K \bar{\Omega} \) of Theorem 2.5.3 is in this situation the following:
\[
\Theta = \left\{ \left( \begin{array}{cc}
\cos \psi e^{i \varphi} & i \sin \psi \\
i \sin \psi & \cos \psi e^{-i \varphi} \end{array} \right) \bigg| \varphi, \psi \in \left[ -\pi/2, \pi/2 \right] \right\}.
\]
Note that \( \text{Sym}(2) = \Theta \cup -\Theta \) so \( \Theta \) can be thought of as a hemissphere in \( \text{Sym}(2) \cong S^2 \).

Given a terminal point \( U_F \in G \), a time-optimal trajectory between \( U(0) = 1 \) and \( U_F \) is now obtained in the following manner.

1. Decompose \( U_F \) as \( U_F = U_1 k_1 \) with \( U_1 \in \Theta \) and \( k_1 \in K \). This can be accomplished by making use for instance of (2.5.4).
2. The general form of a matrix \( U_1 \subseteq \Theta \) is
\[
U_1 = \left( \begin{array}{cc}
\cos \psi e^{i \varphi} & i \sin \psi \\
i \sin \psi & \cos \psi e^{-i \varphi} \end{array} \right), \quad \varphi, \psi \in \left[ -\pi/2, \pi/2 \right].
\]
Calculate the parameters \( \varphi \) and \( \psi \) of the matrix \( U_1 \) determined in step (1) and set
\[
\alpha := \exp (\alpha I_z) \in \Omega,
\]
with
\[
\alpha := \arg \left( \cos \psi \cos \varphi + i \sqrt{1 - \cos^2 \psi \cos^2 \varphi} \right) \in [0, \pi/2].
\]
By construction, \( U_1 = kak^{-1} \) holds for some \( k \in K \).
3. Calculate \( k \in K \) such that \( U_1 = kA_1 k^{-1} \).
4. A time-optimal control sequence to generate \( U_F \) is
\[
1 \longrightarrow k^{-1}k_1 \longrightarrow ak^{-1}k_1 \longrightarrow kak^{-1}k_1 = U_F.
\]
Here the first and the last arrow mean synthesizing \( k^{-1}k_1 \) and \( k \) by so-called hard pulses (the infimizing time for accomplishing this being equal to zero), while the middle arrow denotes evolution of the system under the influence of the drift Hamiltonian \( I_z \) for time \( \alpha \).

### 3.3. Two-Particle Systems

In this section we discuss control system (3.1.1) for the special case of \( n = 2 \) spin-particles, and assume that the \( x \)- and the \( y \)-component of each of the spins
may be excited individually. The problem of controlling the spin of such a system then reads

\[(3.3.1) \quad \dot{U} = \left( H_d + \sum_{j=1}^{4} v_j H_j \right) U, \quad U(0) = 1, \quad v_j \in \mathbb{R} \]

with

\[(3.3.2) \quad H_1 = I_{1x}, \quad H_2 = I_{1y}, \quad H_3 = I_{2x}, \quad H_4 = I_{2y}. \]

The drift operator $H_d$ needs to be chosen within a maximal abelian subalgebra $h \subseteq \mathfrak{su}(4)$ (which will be specified later) subject to the restriction that it is not contained in any root hyperplane. The elements $H_j, j = 1,...,4,$ generate a subalgebra $k$ isomorphic to $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$, and we already know by Theorem 3.1.1 that the pair $(\mathfrak{su}(4), \mathfrak{su}(2) \otimes \mathfrak{su}(2))$ is a symmetric Lie algebra pair. This allows us to argue along the lines of the previous section.

We denote by $K = SU(2) \otimes SU(2)$ the connected subgroup of $G = SU(4)$ with Lie algebra $t$. So the resulting reduced system is

\[(3.3.3) \quad \dot{P} = PX, \quad P(0) = K, \quad X \in \text{Ad}_K H_d, \]

on the 9-dimensional homogeneous space $G/K$. In order to describe this space more succinctly, we first of all observe that the Lie algebras $k = \mathfrak{su}(2) \otimes \mathfrak{su}(2) \subseteq \mathfrak{su}(4)$ and $\mathfrak{so}(4) = \{ X \in \mathfrak{su}(4) | X + X^T = 0 \}$ are isomorphic. An isomorphism $\varphi : t \rightarrow \mathfrak{so}(4)$ is given by conjugation with the unitary matrix

\[U := \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix}.\]

This is for instance seen by using Lemma 1.6.3 (vi) in order to represent the elements of a basis of $t$ by $(4 \times 4)$-matrices, such as

\[I_x \otimes 1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad 1 \otimes I_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},\]

\[I_y \otimes 1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad 1 \otimes I_y = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix},\]

\[I_z \otimes 1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 \end{pmatrix}, \quad 1 \otimes I_z = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}.\]
and then by checking that the map \( \varphi \) sends this basis to a basis of \( \mathfrak{so}(4) \). The Lie algebra isomorphism \( \varphi \) can be integrated to a Lie group isomorphism \( \Phi : K \to SO(4) =: \tilde{K} \) which likewise is given by conjugation with \( U \). This also shows that the homogeneous spaces \( SU(4)/(SU(2) \otimes SU(2)) \) and \( SU(4)/SO(4) \) are diffeomorphic as \( SU(4) \)-homogeneous spaces. Namely, an equivariant diffeomorphism is given by the map

\[
(3.3.5) \quad \psi : SU(4)/(SU(2) \otimes SU(2)) \to SU(4)/SO(4), \quad gK \mapsto UgU^{-1} \tilde{K}.
\]

The map \( \psi \) is well-defined:

\[
\psi(gkK) = UkgU^{-1} \tilde{K} = Ug^{-1}UkU^{-1} \tilde{K} = UgU^{-1} \tilde{K} = \psi(gK)
\]

holds for all \( k \in K \).

We remark that the space \( SU(4)/SO(4) \) appears as a symmetric space of type A I in Example 1.2.7. It also can be shown to be diffeomorphic to the Grassmannian manifold of 3-dimensional subspaces in \( \mathbb{R}^6 \), cf. [28], p. 322. An argument analogous to that in the previous section shows that the elements of \( G/K \cong SU(4)/SO(4) \) can be represented (again not uniquely) by those of the space \( \text{Sym}(4) \) of symmetric unitary \((4 \times 4)\)-matrices. In this case a calculation yields

\[
(3.3.6) \quad G/K \cong \text{Sym}(4)/\sim,
\]

where

\[
(3.3.7) \quad g_1 \sim g_2 \iff g_1 = g_2d, \quad d \in D,
\]

with

\[
(3.3.8) \quad D := \{ \pm 1, \pm \text{diag}(1, 1, -1, -1), \pm \text{diag}(1, -1, 1, -1), \pm \text{diag}(1, -1, -1, 1) \}.
\]

We now turn to a discussion of the time-optimal control problem as formulated in Section 2.3 in the here relevant case of \( (g, \mathfrak{k}) = (\mathfrak{su}(4), \mathfrak{so}(4)) \) constituting a symmetric Lie algebra pair. The Cartan-like decomposition of the Lie algebra \( g \) is

\[
(3.3.9) \quad g = \mathfrak{t} \oplus \text{sym}(4),
\]

where \( \text{sym}(4) = \{ X \in g | X^T = X \} \). Indeed, \( p := \text{sym}(4) \) is the orthogonal complement of \( \mathfrak{t} \), because it is complementary as a vector space, and for all \( X \in p \),
Y ∈ ℓ we compute (making use of equation (1.1.16)) that
\[ \kappa(X,Y) = 8\text{tr}(XY) \]
\[ = 8 \sum_{j=1}^{4} \sum_{k=1}^{4} X_{jk}Y_{kj} \]
\[ = 8 \sum_{j \leq k} X_{jk}Y_{kj} - 8 \sum_{j > k} X_{jk}Y_{kj} \]
\[ = 8 \sum_{j < k} X_{kj}Y_{jk} - 8 \sum_{j > k} X_{jk}Y_{kj} \]
\[ = 8 \sum_{j > k} X_{jk}Y_{kj} - 8 \sum_{j > k} X_{jk}Y_{kj} \]
\[ = 0, \]
as \( X_{kj} = X_{jk} \) and \( Y_{kj} = -Y_{jk} \).

The next step is to determine a root space decomposition of \( \mathfrak{g} \). Choose \( \mathfrak{h} := \mathbb{R}H_1 + \mathbb{R}H_2 + \mathbb{R}H_3 \subseteq p \) with
\( (3.3.10) \)
\[ H_1 := \text{diag}(i,-i,0,0), \quad H_2 := \text{diag}(i,0,-i,0), \quad H_3 := \text{diag}(i,0,0,-i) \]
to serve as a maximal abelian subalgebra of \( \mathfrak{g} \). Let \( A \in G \) the maximal torus with Lie algebra \( \mathfrak{h} \). The root space decomposition of \( \mathfrak{g}_C \cong sl_4 \mathbb{C} \) with respect to \( \mathfrak{h} \) is then given by
\( (3.3.11) \)
\[ \mathfrak{g}_C = \mathfrak{h}_C \oplus \bigoplus_{i \neq j} \mathfrak{g}_{ij}, \]
\( \mathfrak{g}_{ij} \) as in Example 1.3.7. The coroots have also been determined before; they are the following
\( (3.3.12) \)
\[ Y_1 = H_1, \quad Y_2 = H_2, \quad Y_3 = H_3, \quad Y_4 = H_2 - H_1, \quad Y_5 = H_3 - H_1, \quad Y_6 = H_3 - H_2, \]
together with
\( (3.3.13) \)
\[ Y_{j+6} := -Y_j, \quad j = 1, \ldots, 6. \]
It is easily checked that reflexion in \( \mathfrak{h} \) on the hyperplane perpendicular to \( Y \in \mathfrak{h} \) is given by
\( (3.3.14) \)
\[ X \mapsto X - 2\frac{\langle X,Y \rangle}{\|Y\|^2} Y. \]
>From this we obtain the Weyl orbit \( W \cdot H_d \) of the element \( H_d = \sum_{i=1}^{3} a_i Y_i \in \mathfrak{h} \) by reflexion on the root hyperplanes \( Y_j^\perp, \ j = 1, \ldots, 6 \), which is, in coordinates
with respect to the ordered basis \((Y_1, Y_2, Y_3)\) of \(\mathfrak{h}\), the set
\[
W \cdot H_d = \{(a_1, a_2, a_3), (a_1, a_3, a_2), (a_2, a_1, a_3), (a_2, a_3, a_1), (a_3, a_1, a_2), (a_3, a_2, a_1), (-a_1 - a_2 - a_3, a_2, a_3), ..., (a_3, a_2, -a_1 - a_2 - a_3), (a_1, -a_1 - a_2 - a_3, a_3), ..., (a_3, a_1 - a_2 - a_3, a_1), (a_1, a_2, -a_1 - a_2 - a_3), ..., (a_1 - a_2 - a_3, a_2, a_1)\}.
\]

In the same manner as in 3.2 we determine the cell \(\Delta \subseteq \mathfrak{h}\) to be the convex hull of \(0, \frac{1}{2}H_1, \frac{1}{2}H_2, \) and \(\frac{1}{2}H_3\), and set \(\Omega := \exp \Delta \subseteq A\). It is easily checked that any \(a \in A\) permits a decomposition \(a = a_1 d\) with \(a_1 \in \bar{\Omega}\) and \(d \in D\), where the group \(D\) is as defined in (3.3.8).

By proceeding as described at the end of Section 2.5 we obtain a time-optimal trajectory between the identity and any given point \(U_F \in G\) in the following way.

1. Perform a polar decomposition of \(U_F\) to obtain \(k_1 \in K\) and \(U_1 \in \text{Sym}(4)\) with \(U_F = U_1 k_1\). If \(U_1 \notin \Theta\) then replace \(U_1\) suitably by \(U_1 d,\) and \(k_1\) by \(d^{-1} k_1,\) where \(d \in D\).

2. Diagonalize \(U_1\) as \(U_1 = k a k^{-1}\) with \(k \in K\) and \(a \in \bar{\Omega}\). Write \(a\) in the form
\[
a = \prod_{j=1}^{24} \exp(\alpha_j Z_j)
\]
with \(\sum_{k=1}^{24} \beta_k = 1\), and \(Z_1, ..., Z_{24}\) the elements of the Weyl orbit \(W \cdot H_d\). Choose the parameter \(\alpha \geq 0\) to be the smallest one possible. This \(\alpha\) then satisfies the minimality condition of Theorem 2.5.3.

3. Steer system (3.3.1) as depicted below:

\[
\begin{align*}
1 & \quad \rightarrow \quad k^{-1} k_1 \\
& \quad \rightarrow \quad \exp(\alpha_{24} Z_{24}) k^{-1} k_1 \\
& \quad \rightarrow \quad \exp(\alpha_{23} Z_{23}) \exp(\alpha_{24} Y Z_{24}) k^{-1} k_1 \\
& \quad \vdots \\
& \quad \rightarrow \quad \left(\prod_{j=1}^{24} \exp(\alpha_j Z_j)\right) k^{-1} k_1 \\
& \quad \rightarrow \quad k \left(\prod_{j=1}^{24} \exp(\alpha_j Z_j)\right) k^{-1} k_1 = U_F.
\end{align*}
\]

Here again the first and the last arrow means producing movement within the subgroup \(K\) of \(G\) and is realized by performing a so-called hard pulse, while the middle arrows symbolize evolution of the system in direction of \(Z_1, ..., Z_{24}\) for times \(\alpha_{1}, ..., \alpha_{24}\), respectively.
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