Maxwell Optics: I. An exact matrix representation of the Maxwell equations in a medium

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Abstract
Matrix representations of the Maxwell equations are well-known. However, all these representations lack an exactness or/and are given in terms of a pair of matrix equations. We present a matrix representation of the Maxwell equation in presence of sources in a medium with varying permittivity and permeability. It is shown that such a representation necessarily requires $8 \times 8$ matrices and an explicit representation for them is presented.

1 Introduction
Matrix representations of the Maxwell equations are very well-known [1]-[3]. However, all these representations lack an exactness or/and are given in terms of a pair of matrix equations. Some of these representations are in free space. Such a representation is an approximation in a medium with space- and time-dependent permittivity $\epsilon(r, t)$ and permeability $\mu(r, t)$ respectively. Even this approximation is often expressed through a pair of equations using $3 \times 3$ matrices: one for the curl and one for the divergence which occur in the Maxwell equations. This practice of writing the divergence condition separately is completely avoidable by using $4 \times 4$ matrices [4] for Maxwell equations in free-space. A single equation using $4 \times 4$ matrices is necessary and sufficient when $\epsilon(r, t)$ and $\mu(r, t)$ are treated as ‘local’ constants [4, 5].

A treatment taking into account the variations of $\epsilon(r, t)$ and $\mu(r, t)$ has been presented in [5]. This treatment uses the Riemann-Silberstein vectors,
\( \mathbf{F}^\pm (r, t) \) to reexpress the Maxwell equations as four equations: two equations are for the curl and two are for the divergences and there is mixing in \( \mathbf{F}^+ (r, t) \) and \( \mathbf{F}^- (r, t) \). This mixing is very neatly expressed through the two derived functions of \( \epsilon(r, t) \) and \( \mu(r, t) \). These four equations are then expressed as a pair of matrix equations using \( 6 \times 6 \) matrices: again one for the curl and one for the divergence. Even though this treatment is exact it involves a pair of matrix equations.

Here, we present a treatment which enables us to express the Maxwell equations in a single matrix equation instead of a pair of matrix equations. Our approach is a logical continuation of the treatment in [3]. We use the linear combination of the components of the Riemann-Silberstein vectors, \( \mathbf{F}^\pm (r, t) \) and the final matrix representation is a single equation using \( 8 \times 8 \) matrices. This representation contains all the four Maxwell equations in presence of sources taking into account the spatial and temporal variations of the permittivity \( \epsilon(r, t) \) and the permeability \( \mu(r, t) \).

In Section-I we shall summarize the treatment for a homogeneous medium and introduce the required functions and notation. In Section-II we shall present the matrix representation in an inhomogeneous medium, in presence of sources.

## 2 Homogeneous Medium

We shall start with the Maxwell equations [4, 5] in an inhomogeneous medium with sources,

\[
\begin{align*}
\nabla \cdot \mathbf{D}(r, t) &= \rho, \\
\nabla \times \mathbf{H}(r, t) - \frac{\partial}{\partial t} \mathbf{D}(r, t) &= \mathbf{J}, \\
\nabla \times \mathbf{E}(r, t) + \frac{\partial}{\partial t} \mathbf{B}(r, t) &= 0, \\
\nabla \cdot \mathbf{B}(r, t) &= 0.
\end{align*}
\]

(1)

We assume the media to be linear, that is \( \mathbf{D} = \epsilon \mathbf{E} \), and \( \mathbf{B} = \mu \mathbf{H} \), where \( \epsilon \) is the permittivity of the medium and \( \mu \) is the permeability of the medium. In general \( \epsilon = \epsilon(r, t) \) and \( \mu = \mu(r, t) \). In this section we treat them as ‘local’ constants in the various derivations. The magnitude of the velocity
of light in the medium is given by $v(r, t) = |v(r, t)| = 1/\sqrt{\varepsilon(r, t)\mu(r, t)}$. In vacuum we have, $\varepsilon_0 = 8.85 \times 10^{-12} C^2/N.m^2$ and $\mu_0 = 4\pi \times 10^{-7} N/A^2$.

One possible way to obtain the required matrix representation is to use the Riemann-Silberstein vector given by

$$
F^+ (r, t) = \frac{1}{\sqrt{2}} \left( \sqrt{\varepsilon(r, t)} E(r, t) + i \frac{1}{\sqrt{\mu(r, t)}} B(r, t) \right)
$$

$$
F^- (r, t) = \frac{1}{\sqrt{2}} \left( \sqrt{\varepsilon(r, t)} E(r, t) - i \frac{1}{\sqrt{\mu(r, t)}} B(r, t) \right) .
$$

(2)

For any homogeneous medium it is equivalent to use either $F^+ (r, t)$ or $F^- (r, t)$. The two differ by the sign before ‘i’ and are not the complex conjugate of one another. We have not assumed any form for $E(r, t)$ and $B(r, t)$. We will be needing both of them in an inhomogeneous medium, to be considered in detail in Section-III.

If for a certain medium $\varepsilon(r, t)$ and $\mu(r, t)$ are constants (or can be treated as ‘local’ constants under certain approximations), then the vectors $F^\pm (r, t)$ satisfy

$$
i \frac{\partial}{\partial t} F^\pm (r, t) = \pm v \nabla \times F^\pm (r, t) - \frac{1}{\sqrt{2\varepsilon}} (iJ)
$$

$$
\nabla \cdot F^\pm (r, t) = \frac{1}{\sqrt{2\varepsilon}} (\rho) .
$$

(3)

Thus, by using the Riemann-Silberstein vector it has been possible to reexpress the four Maxwell equations (for a medium with constant $\varepsilon$ and $\mu$) as two equations. The first one contains the the two Maxwell equations with curl and the second one contains the two Maxwell with divergences. The first of the two equations in (3) can be immediately converted into a $3 \times 3$ matrix representation. However, this representation does not contain the divergence conditions (the first and the fourth Maxwell equations) contained in the second equation in (3). A further compactification is possible only by expressing the Maxwell equations in a $4 \times 4$ matrix representation. To this
end, using the components of the Riemann-Silberstein vector, we define,

\[
\Psi^+(r, t) = \begin{bmatrix} -F_x^+ + iF_y^+ \\ F_z^+ \\ F_x^+ + iF_y^+ \end{bmatrix}, \quad \Psi^-(r, t) = \begin{bmatrix} -F_x^- - iF_y^- \\ F_z^- \\ F_x^- - iF_y^- \end{bmatrix}.
\] (4)

The vectors for the sources are

\[
W^+ = \left( \frac{1}{\sqrt{2\epsilon}} \right) \begin{bmatrix} -J_x + iJ_y \\ J_z - \nu \rho \\ J_z + \nu \rho \\ J_x + iJ_y \end{bmatrix}, \quad W^- = \left( \frac{1}{\sqrt{2\epsilon}} \right) \begin{bmatrix} -J_x - iJ_y \\ J_z - \nu \rho \\ J_z + \nu \rho \\ J_x - iJ_y \end{bmatrix}.
\] (5)

Then we obtain

\[
\frac{\partial}{\partial t} \Psi^+ = -v \{ M \cdot \nabla \} \Psi^+ - W^+
\]

\[
\frac{\partial}{\partial t} \Psi^- = -v \{ M^* \cdot \nabla \} \Psi^- - W^-,
\] (6)

where `\*` denotes complex-conjugation and the triplet, \( M = (M_x, M_y, M_z) \) is expressed in terms of

\[
\Omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\] (7)

Alternately, we may use the matrix \( J = -\Omega \). Both differ by a sign. For our purpose it is fine to use either \( \Omega \) or \( J \). However, they have a different meaning: \( J \) is contravariant and \( \Omega \) is covariant; The matrix \( \Omega \) corresponds to the Lagrange brackets of classical mechanics and \( J \) corresponds to the Poisson brackets. An important relation is \( \Omega = J^{-1} \). The \( M \)-matrices are:

\[
M_x = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = -\beta \Omega,
\]

\[
M_y = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} = i\Omega.
\]
\[ M_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \beta. \] (8)

Each of the four Maxwell equations are easily obtained from the matrix representation in (8). This is done by taking the sums and differences of row-I with row-IV and row-II with row-III respectively. The first three give the \( y, x \) and \( z \) components of the curl and the last one gives the divergence conditions present in the evolution equation (3).

It is to be noted that the matrices \( M \) are all non-singular and all are hermitian. Moreover, they satisfy the usual algebra of the Dirac matrices, including,

\[
M_x \beta = -\beta M_x, \\
M_y \beta = -\beta M_y, \\
M_x^2 = M_y^2 = M_z^2 = I, \\
M_x M_y = -M_y M_x = iM_z, \\
M_y M_z = -M_z M_y = iM_x, \\
M_z M_x = -M_x M_z = iM_y. \] (9)

Before proceeding further we note the following: The pair \((\Psi^\pm, M)\) are not unique. Different choices of \( \Psi^\pm \) would give rise to different \( M \), such that the triplet \( M \) continues to to satisfy the algebra of the Dirac matrices in (9). We have preferred \( \Psi^\pm \) via the the Riemann-Silberstein vector (2) in [3]. This vector has certain advantages over the other possible choices. The Riemann-Silberstein vector is well-known in classical electrodynamics and has certain interesting properties and uses [3].

In deriving the above \( 4 \times 4 \) matrix representation of the Maxwell equations we have ignored the spatial and temporal derivatives of \( \epsilon(\mathbf{r}, t) \) and \( \mu(\mathbf{r}, t) \) in the first two of the Maxwell equations. We have treated \( \epsilon \) and \( \mu \) as ‘local’ constants.

### 3 Inhomogeneous Medium

In the previous section we wrote the evolution equations for the Riemann-Silberstein vector in (3), for a medium, treating \( \epsilon(\mathbf{r}, t) \) and \( \mu(\mathbf{r}, t) \) as ‘local’
constants. From these pairs of equations we wrote the matrix form of the Maxwell equations. In this section we shall write the exact equations taking into account the spatial and temporal variations of $\epsilon(r, t)$ and $\mu(r, t)$. It is very much possible to write the required evolution equations using $\epsilon(r, t)$ and $\mu(r, t)$. But we shall follow the procedure in [3] of using the two derived laboratory functions.

Velocity Function: \( v(r, t) = \frac{1}{\sqrt{\epsilon(r, t)\mu(r, t)}} \)

Resistance Function: \( h(r, t) = \frac{\sqrt{\mu(r, t)}}{\sqrt{\epsilon(r, t)}}. \) \( 10 \)

The function, \( v(r, t) \) has the dimensions of velocity and the function, \( h(r, t) \) has the dimensions of resistance (measured in Ohms). We can equivalently use the Conductance Function, \( \kappa(r, t) = \frac{1}{h(r, t)} = \frac{\epsilon(r, t)}{\mu(r, t)} \) (measured in Ohms\(^{-1}\) or Mhos!) in place of the resistance function, \( h(r, t) \). These derived functions enable us to understand the dependence of the variations more transparently [3]. Moreover the derived functions are the ones which are measured experimentally. In terms of these functions, \( \epsilon = \frac{1}{\sqrt{vh}} \) and \( \mu = \sqrt{\frac{h}{v}} \). Using these functions the exact equations satisfied by \( F^{\pm}(r, t) \) are

\[
\begin{align*}
\frac{\partial}{\partial t} F^+(r, t) & = v(r, t) \left( \nabla \times F^+(r, t) \right) + \frac{1}{2} \left( \nabla v(r, t) \times F^+(r, t) \right) \\
& + \frac{v(r, t)}{2h(r, t)} \left( \nabla h(r, t) \times F^-(r, t) \right) - \frac{i}{\sqrt{2}} \sqrt{v(r, t)h(r, t)} J \\
& + \frac{i}{2} v(r, t) F^+(r, t) + \frac{i}{2} h(r, t) F^-(r, t) \\
\frac{\partial}{\partial t} F^-(r, t) & = -v(r, t) \left( \nabla \times F^-(r, t) \right) - \frac{1}{2} \left( \nabla v(r, t) \times F^-(r, t) \right) \\
& - \frac{v(r, t)}{2h(r, t)} \left( \nabla h(r, t) \times F^+(r, t) \right) - \frac{i}{\sqrt{2}} \sqrt{v(r, t)h(r, t)} J \\
& + \frac{i}{2} v(r, t) F^-(r, t) + \frac{i}{2} h(r, t) F^+(r, t) \\
\nabla \cdot F^+(r, t) & = \frac{1}{2v(r, t)} \left( \nabla v(r, t) \cdot F^+(r, t) \right)
\end{align*}
\]
\[ \nabla \cdot F^-(r, t) = \frac{1}{2v(r, t)} \left( \nabla v(r, t) \cdot F^-(r, t) \right) + \frac{1}{2h(r, t)} \left( \nabla h(r, t) \cdot F^+(r, t) \right) + \frac{1}{\sqrt{2}} \sqrt{v(r, t)} h(r, t) \rho , \]

where \( \dot{v} = \frac{\partial v}{\partial t} \) and \( \dot{h} = \frac{\partial h}{\partial t} \). The evolution equations in (11) are exact (for a linear media) and the dependence on the variations of \( \epsilon(r, t) \) and \( \mu(r, t) \) has been neatly expressed through the two derived functions. The coupling between \( F^+(r, t) \) and \( F^-(r, t) \) is via the gradient and time-derivative of only one derived function namely, \( h(r, t) \) or equivalently \( \kappa(r, t) \). Either of these can be used and both are the directly measured quantities. We further note that the dependence of the coupling is logarithmic

\[ \frac{1}{h(r, t)} \nabla h(r, t) = \nabla \left\{ \ln \left( h(r, t) \right) \right\} , \quad \frac{1}{h(r, t)} \dot{h}(r, t) = \frac{\partial}{\partial t} \left\{ \ln \left( h(r, t) \right) \right\} , \quad (12) \]

where \( \ln \) is the natural logarithm.

The coupling can be best summarized by expressing the equations in (11) in a (block) matrix form. For this we introduce the following logarithmic function

\[ \mathcal{L}(r, t) = \frac{1}{2} \left\{ \mathbb{I} \ln \left( v(r, t) \right) + \sigma_z \ln \left( h(r, t) \right) \right\} , \quad (13) \]

where \( \sigma_x \) is one the triplet of the Pauli matrices

\[ \sigma = \begin{bmatrix} \sigma_x & = & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \sigma_y & = & \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ \sigma_z & = & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} . \quad (14) \]

Using the above notation the matrix form of the equations in (11) is

\[ i \left\{ \frac{\partial}{\partial t} - \frac{\partial}{\partial u} \mathcal{L} \right\} \begin{bmatrix} F^+(r, t) \\ F^-(r, t) \end{bmatrix} = v(r) \sigma_z \{ \mathbb{I} \nabla + \nabla \mathcal{L} \} \times \begin{bmatrix} F^+(r, t) \\ F^-(r, t) \end{bmatrix} \]
\[ (\mathbb{I} \nabla - \nabla \mathcal{L}) \cdot \left[ \begin{array}{c} F^+(r,t) \\ F^-(r,t) \end{array} \right] = -\frac{i}{\sqrt{2}} \sqrt{v(r,t)h(r,t)} J \]

where the dot-product and the cross-product are to be understood as

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A \cdot u + B \cdot v \\ C \cdot u + D \cdot v \end{bmatrix},
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A \times u + B \times v \\ C \times u + D \times v \end{bmatrix}.
\]

It is to be noted that the 6\times6 matrices in the evolution equations in (15) are either hermitian or antihermitian. Any dependence on the variations of \( \epsilon(r,t) \) and \( \mu(r,t) \) is at best ‘weak’. We further note, \( \nabla (\ln (v(r,t))) = -\nabla (\ln (n(r,t))) \) and \( \frac{\partial}{\partial t} (\ln (v(r,t))) = -\frac{\partial}{\partial t} (\ln (n(r,t))) \). In some media, the coupling may vanish (\( \nabla h(r,t) = 0 \) and \( \dot{h}(r,t) = 0 \)) and in the same medium the refractive index, \( n(r,t) = c/v(r,t) \) may vary (\( \nabla n(r,t) \neq 0 \) or/and \( \dot{n}(r,t) \neq 0 \)). It may be further possible to use the approximations \( \nabla (\ln (h(r,t))) \approx 0 \) and \( \frac{\partial}{\partial t} (\ln (h(r,t))) \approx 0 \).

We shall be using the following matrices to express the exact representation

\[
\Sigma = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \quad \alpha = \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}, \quad I = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix},
\]

where \( \Sigma \) are the Dirac spin matrices and \( \alpha \) are the matrices used in the Dirac equation. Then,

\[
\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \Psi^+ \\ \Psi^- \end{bmatrix} - \frac{i}{2} \tilde{v}(r,t) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi^+ \\ \Psi^- \end{bmatrix}
+ \frac{i}{2} \dot{h}(r,t) \begin{bmatrix} 0 & i\beta \alpha_y \\ i\beta \alpha_y & 0 \end{bmatrix} \begin{bmatrix} \Psi^+ \\ \Psi^- \end{bmatrix}
= -i \tilde{v}(r,t) \begin{bmatrix} \{M \cdot \nabla + \Sigma \cdot u\} & -i\beta (\Sigma \cdot w) \alpha_y \\ -i\beta (\Sigma^* \cdot w) \alpha_y & \{M^* \cdot \nabla + \Sigma^* \cdot u\} \end{bmatrix} \begin{bmatrix} \Psi^+ \\ \Psi^- \end{bmatrix}
- \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} W^+ \\ W^- \end{bmatrix}.
\]
The above representation contains thirteen $8 \times 8$ matrices! Ten of these are hermitian. The exceptional ones are the ones that contain the three components of $w(\mathbf{r}, t)$, the logarithmic gradient of the resistance function. These three matrices are antihermitian.

4 Concluding Remarks

We have been able to express the Maxwell equations in a matrix form in a medium with varying permittivity $\epsilon(\mathbf{r}, t)$ and permeability $\mu(\mathbf{r}, t)$, in presence of sources. We have been able to do so using a single equation instead of a pair of matrix equations. We have used $8 \times 8$ matrices and have been able to separate the dependence of the coupling between the upper components ($\Psi^+$) and the lower components ($\Psi^-$) through the two laboratory functions. Moreover the exact matrix representation has an algebraic structure very similar to the Dirac equation. We feel that this representation would be more suitable for some of the studies related to the photon wave function. This representation is the starting point for the exact formalism of Maxwell optics [6]-[8]. This formalism provides a unified treatment of beam-optics and polarization.

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