SOME RIGIDITY PROPERTIES FOR $\lambda$-SELF-EXPANDERS

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Abstract. $\lambda$-self-expanders $\Sigma$ in $\mathbb{R}^{n+1}$ are the solutions of the isoperimetric problem with respect to the same weighted area form as in the study of the self-expanders. In this paper, we mainly extend the results on self-expanders which we obtained in [1] to $\lambda$-self-expanders. We prove some results that characterize the hyperplanes, spheres and cylinders as $\lambda$-self-expanders. We also discuss the area growths and the finiteness of the weighted areas under the control of the growth of the mean curvature.

1. Introduction

A $\lambda$-self-expander is a hypersurface $\Sigma$ immersed in $\mathbb{R}^{n+1}$ whose mean curvature $H$ satisfies the equation

$$H = -\frac{\langle x, n \rangle}{2} + \lambda,$$

where $\lambda \in \mathbb{R}$ is a real number, $x$ is the position vector in $\mathbb{R}^{n+1}$ and $n$ is the outward unit normal field on $\Sigma$. Such hypersurfaces appear as solutions to the weighted isoperimetric problem of weight $e^{\frac{|x|^2}{4}}$ and are characterized as critical points of the weighted area functional

$$A(\Sigma) = \int_\Sigma e^{\frac{|x|^2}{4}} d\sigma,$$

for any compact normal variations $F : (\varepsilon, \varepsilon) \times \Sigma \longrightarrow \mathbb{R}^{n+1}$ which satisfies $\int_\Sigma \varphi e^{\frac{|x|^2}{4}} d\sigma = 0$, where $\varphi = \langle \partial_t F(0, x), n(x) \rangle$.

On the other hand, they can also be characterized as constant weighted mean curvature hypersurfaces of weight $e^{\frac{|x|^2}{4}}$ (see Section 2).

In the case that $\lambda = 0$, $\lambda$-self-expanders just are self-expanders of the mean curvature flows, which are critical points of the weighted area functional $\int_\Sigma e^{\frac{|x|^2}{4}} d\sigma$.  

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In recent years, as the models of singularity for the mean curvature flow, self-shrinkers, self-expanders and translating solitons have been studied very much. See, for instance [1, 15, 29], [4, 12, 17, 20, 28] and [13, 18, 25, 26, 30, 32], respectively. We would like to mention that in some sense the research of self-expanders brings out the interest on $\lambda$-self-expanders. The analogous phenomenon has already happened on the self-shrinkers and translating solitons of MCF. There are notions of constant Gaussian-weighted mean curvature hypersurfaces (also called $\lambda$-hypersurfaces) and $\lambda$-translating solitons, which are linked with self-shrinkers and translating solitons, respectively (see, for instance, the work of Q. Cheng and Wei [6], McGonagle and Ross [27], and L´ opez in [22]). Various results have been obtained on $\lambda$-hypersurfaces and $\lambda$-translating solitons. See, for instance [2, 5, 6, 14, 27, 31] and [21–23], respectively. In [24], L´ opez studied $\lambda$-self-expander surfaces in $\mathbb{R}^3$ and proved that if a $\lambda$-self-expander surface is ruled surface, then it must be a cylindrical surface.

In this paper, we study $\lambda$-self-expanders. All hyperplanes, Spheres $S^n_r(0)$ centered at the origin of radius $r > 0$, and cylinders $S^k_r(0) \times \mathbb{R}^{n-k}$, where $1 \leq k \leq n-1$ and $r > 0$ are examples of $\lambda$-self-expanders (see the details in Section 2). It is well known that there exists no closed self-expanders in $\mathbb{R}^{n+1}$ (see, for instance [4]). But the situation is different for $\lambda \neq 0$. All spheres $S^n_r(0)$ centered at the origin of radius $r > 0$ are $\lambda$-self-expanders with $\lambda = \frac{2}{r} + \frac{r}{2} \geq \sqrt{2}n$ (see Example (2.3) in Section 2).

Throughout the paper, the following convention of notation are used. The notation $A$ denotes the second fundamental form of $\lambda$-self-expander $\Sigma$. $\Sigma$ is called convex if its second fundamental form $A$ is negative definite, that is, the eigenvalues $\lambda_i$ of $A$ satisfy $\lambda_i \leq 0$, where $A(e_i) = -\nabla e_i n = \lambda_i e_i$, $e_i$ is a local orthonormal frame. $\Sigma$ is called mean convex if its mean curvature $H = -\sum_{i=1}^{n} \lambda_i$ is nonnegative. We use $x^\top$ to denote the tangent component of the position vector $x$.

We give the lower bound of $\lambda$ and obtain a gap theorem for closed $\lambda$-self-expanders as follows:

**Theorem 1.1.** Let $\Sigma$ be a closed $\lambda$-self-expander immersed in $\mathbb{R}^{n+1}$. Then

(i) $\lambda \geq \sqrt{2n}$. Moreover, the sphere $S^n_{\sqrt{2n}}(0)$ is the only closed $\lambda$-self-expander with $\lambda = \sqrt{2n}$.

(ii) If

$$|A|^2 \leq \frac{1}{2} + \frac{\lambda \left( \lambda - \sqrt{\lambda^2 - 2n} \right)}{2n},$$

then $\Sigma$ is a sphere $S^n_r(0)$ with $r = \lambda + \sqrt{\lambda^2 - 2n}$. 
Theorem 1.1(ii) gives a pinching condition of $|A|$ such that a compact $\Sigma$ is a sphere. This result can be compared with the following theorem proved by Guang [14] for $\lambda$-hypersurfaces.

**Theorem 1.2** ([14]). Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed $\lambda$-hypersurface with $\lambda \geq 0$. If $\Sigma$ satisfies

$$|A|^2 \leq \frac{1}{2} + \frac{\lambda (\lambda + \sqrt{\lambda^2 + 2n})}{2n},$$

then $\Sigma$ is a round sphere with radius $\sqrt{\lambda^2 + 2n} + \lambda$.

We also obtain a condition on $\lambda$ and the mean curvature $H$ so that a complete $\lambda$-self-expander must be a sphere (see Theorem 4.2). Besides, we prove the following result:

**Theorem 1.3.** Let $\Sigma$, $n \geq 2$, be a closed mean convex $\lambda$-self-expander immersed in $\mathbb{R}^{n+1}$. If

$$H \text{tr} A^3 + |A|^4 \leq 0,$$

then $\Sigma$ must be a sphere $S^n_r(0)$.

For a convex hypersurface, note that

$$H \text{tr} A^3 + |A|^4 = -\sum_{i<j}^n \lambda_i \lambda_j (\lambda_i - \lambda_j)^2 \leq 0.$$

Therefore, (1.4) holds and a consequence of Theorem 1.3 is as follows.

**Corollary 1.1.** Let $\Sigma$, $n \geq 2$, be a closed convex $\lambda$-self-expander immersed in $\mathbb{R}^{n+1}$. Then $\Sigma$ must be a sphere $S^n_r(0)$.

Next, we study complete $\lambda$-self-expanders with some integrability condition on the norm of the second fundamental form. Now we recall some related results. In [11, Theorem 1.4], the second author of the present paper and Zhou proved the uniqueness of hyperplanes through the origin for mean convex self-expander hypersurfaces under some condition on the square of the norm of the second fundamental form.

In [6, Theorem 2], Q. Cheng and Wei proved that a complete embedded $\lambda$-hypersurface in $\mathbb{R}^{n+1}$ with polynomial area growth (there exist a constant $C > 0$ and $r_0 > 0$ such that $\text{Area}(B_r(0) \cap \Sigma) \leq Cr^\alpha$, for all $r \geq r_0$ and for some $\alpha > 0$) satisfying $H - \lambda \geq 0$ and $\lambda((H - \lambda)\text{tr} A^3 + \frac{|A|^2}{2}) \leq 0$ has to be a cylinder $S^n_r(0) \times \mathbb{R}^{n-l}$, $0 \leq l \leq n$. Later, the same classification theorem as Q. Cheng and Wei was proved in [3, Theorem 3] by the first author of the present paper and Miranda, replacing the polynomial area growth by a weaker intrinsic property on the norm of the second fundamental form. Motivated by the above work, we obtain the following result on $\lambda$-self-expanders.
**Theorem 1.4.** Let $\Sigma$, $n \geq 2$, be a complete $\lambda$-self-expander immersed in $\mathbb{R}^{n+1}$. If $H - \lambda \leq 0$, $|A| \in L^2(\Sigma, e^{\frac{|x|^2}{2}}d\sigma)$ and

\[(1.5) \quad \lambda((H - \lambda)\text{tr}A^3 - \frac{|A|^2}{2}) \geq 0,
\]
then $\lambda \geq 0$ and $\Sigma$ must be either a hyperplane or $S^n_r(0)$.

**Remark 1.1.** The equality in (1.5) holds if $\Sigma$ is either a hyperplane or $S^l_r(0) \times \mathbb{R}^{n-l}$, $1 \leq l \leq n$, but the cylinders $S^l_r(0) \times \mathbb{R}^{n-l}$, $1 \leq l \leq n - 1$ do not satisfy the property that $|A| \in L^2(\Sigma, e^{\frac{|x|^2}{2}}d\sigma)$. This is the reason why the theorem 1.4 does not characterize the cylinders $S^l_r(0) \times \mathbb{R}^{n-l}$, $1 \leq l \leq n - 1$.

Recently, Smoczyk [29, Theorem 1] proved that complete self-expanders $\Sigma \subset \mathbb{R}^{n+1}$ with $H \neq 0$ are products of self-expander curves and flat subspaces, if and only if the function $|A|^2 \frac{H^2}{H^2 + 1}$ attains a local maximum. Motivated by this result, we obtain the following result, that characterizes generalized cylinders as $\lambda$-self-expanders.

**Theorem 1.5.** Let $\Sigma$ be a complete immersed $\lambda$-self-expander in $\mathbb{R}^{n+1}$ with $\lambda \neq 0$. Then

(i) the open set $\{x \in \Sigma; H \neq \lambda\}$ must be non-empty.

(ii) Further, if the following conditions are satisfied on the non-empty set $\{x \in \Sigma; H \neq \lambda\}$,

(a) $\frac{\lambda}{H - \lambda}((H - \lambda)\text{tr}A^3 - \frac{|A|^2}{2}) \leq 0$,

(b) the function $\frac{|A|^2}{(H - \lambda)^2}$ attains a local maximum,

then $\Sigma$ is either a hyperplane or a cylinder $S^l_r(0) \times \mathbb{R}^{n-l}$, $1 \leq l \leq n$.

We also prove that

**Theorem 1.6.** Let $\Sigma$ be a complete properly immersed $\lambda$-self-expander hypersurface in $\mathbb{R}^{n+1}$. If there exists $\alpha > 0$ such that

\[(1.6) \quad |A|^2(H - \lambda)H + \frac{1}{2}H^2 \left(1 + \frac{(\alpha + 1)^2}{8}|x^\top|^2\right) \leq 0,
\]
then $\Sigma$ must be either a hyperplane or $S^n_r(0)$.

Using the Simons' type equations, we prove Proposition 6.1 which states that if a complete immersed $\lambda$-self-expander in $\mathbb{R}^{n+1}$ has constant mean curvature, then it must be either a hyperplane or a cylinder $S^l_r(0) \times \mathbb{R}^{n-l}$, $1 \leq l \leq n$.

In the last Section 7, we mainly extend some of our previous results on self-expanders in [11] to the $\lambda$-self-expanders. Theorem 7.2 deals with finiteness of weighted areas and area growth estimate of $\lambda$-self-expanders. Theorems 7.3 and 7.4 characterize the generalized cylinders as $\lambda$-self-expanders.
It is interesting to see that unlike self-expanders, hyperplanes not through the origin or a cylinder $S^l_r(0) \times \mathbb{R}^{n-l}$, $1 \leq l \leq n$ appear in the corresponding conclusions.

The rest of the paper is organized as follows: In Section 2 we give some notations and examples of $\lambda$-self-expanders. In Section 3 we give some Simons’ type equations for $\lambda$-self-expanders. In Section 4 we prove Theorems 1.1 and 1.3. In Section 5 we prove Theorems 1.4 and 1.6. In Section 6 we prove Theorem 1.5 and Proposition 6.1. In Section 7 we prove Theorems 7.2, 7.3 and 7.4.

2. Preliminaries

In this section, we will recall some concepts and basic facts.

Let $\Sigma$ denote a hypersurface immersed in the Euclidean space $(\mathbb{R}^{n+1}, g_0)$ with the induced metric $g$. We will denote by $d\sigma$ the area form of $\Sigma$. In this paper, unless otherwise specified, the notations with a bar, for instance $\bar{\nabla}$ and $\bar{\nabla}^2$, denote the quantities corresponding the Euclidean metric $g_0$ on $\mathbb{R}^{n+1}$. On the other hand, the notations like $\nabla, \Delta$ denote the quantities corresponding the intrinsic metric $g$ on $\Sigma$.

The isometric immersion $i : (\Sigma, g) \rightarrow (\mathbb{R}^{n+1}, g_0)$ is said to be properly immersed if, for any compact subset $\Omega$ in $\mathbb{R}^{n+1}$, the pre-image $i^{-1}(\Omega)$ is compact in $\Sigma$.

Let $A$ denote the second fundamental form of $(\Sigma, g)$. At $p \in \Sigma$, $A(X) = -\nabla_X n$, where $X \in T_p \Sigma$ and $n$ is the outward unit normal field on $\Sigma$. The mean curvature $H$ of $\Sigma$ is defined as the trace of $-A$.

Given a smooth function $f$ on $\mathbb{R}^{n+1}$, it induces a smooth measure $e^{-f}d\sigma$ on $(\Sigma, g)$. Define the weighted mean curvature $H_f$ of $\Sigma$ of weight $e^{-f}$ by

$$H_f := H - \langle \nabla f, n \rangle.$$

$\Sigma$ is called constant weighted mean curvature (or simply by CWMC) hypersurface (of weight $e^{-f}$) if $H_f = \lambda$ for some $\lambda \in \mathbb{R}$, or equivalently if it satisfies

(2.1) $$H = \langle \nabla f, n \rangle + \lambda.$$

If $\lambda = 0$, $\Sigma$ is called $f$-minimal. There are very interesting examples of $f$-minimal hypersurfaces:

Example 2.1. If $f = \frac{|x|^2}{4}, -\frac{|x|^2}{4},$ and $-\langle x, w \rangle$ respectively, where $w \in \mathbb{R}^{n+1}$ is a constant vector, a $f$-minimal hypersurface $\Sigma$ is a self-shrinker, self-expander and translator for MCF in the Euclidean space $\mathbb{R}^{n+1}$ respectively.
Taking $f = -|x|^2$ in (2.1), a CWMC hypersurface $\Sigma$ of weight $e^{-\frac{|x|^2}{4}}$ in $\mathbb{R}^{n+1}$ is just a $\lambda$-self-expander defined by (1.1), that is, $\Sigma$ satisfies the equation

$$H = -\frac{\langle x, n \rangle}{2} + \lambda. \quad (2.2)$$

Here are some examples of $\lambda$-self-expanders.

**Example 2.2.** Any hyperplane in $\mathbb{R}^{n+1}$ is a $\lambda$-self-expander with $\lambda = \pm \frac{d}{2}$, where $d$ denotes the distance from the origin to the hyperplane and the sign depends on the orientation. Indeed, let $\Sigma \subset \mathbb{R}^{n+1}$ be a hyperplane. Let $n$ be the unit normal of $\Sigma$. Then $\Sigma = \{x \in \mathbb{R}^{n+1}; \langle x, n \rangle = d\}$ or $\Sigma = \{x \in \mathbb{R}^{n+1}; \langle x, n \rangle = -d\}$. In both cases, we have

$$H = -\frac{\langle x, n \rangle}{2} + \lambda.$$

This implies that $\Sigma$ is a $\lambda$-self-expander with $\lambda = \frac{d}{2}$ or $-\frac{d}{2}$.

**Example 2.3.** All spheres $S^*(p)$ centered at the origin of radius $r > 0$ are $\lambda$-self-expanders with $\lambda = \frac{n}{r} + \frac{r^2}{2} \geq \sqrt{2n}$. For each $\lambda > \sqrt{2n}$, there are two spheres $S^*(p)$ of radius $r = \lambda \pm \sqrt{\lambda^2 - 2n}$ as $\lambda$-self-expanders with this $\lambda$.

Moreover, a sphere $S^*(p)$ is $\lambda$-self-expander if and only if its center $p$ is the origin. In fact, let $n$ be the outward unit normal of the sphere $S^*(p)$. We have $H = \frac{n}{r} - \langle x - p, n \rangle = r$, and hence $H + \frac{\langle x, n \rangle}{2} = \frac{n}{r} + \frac{r}{2} - \frac{\langle p, n \rangle}{2} = \lambda$. Therefore $\langle p, n \rangle$ must be constant and thus $p$ must be the origin.

**Example 2.4.** The cylinders $S^k_r(0) \times \mathbb{R}^{n-k}$, where $1 \leq k \leq n-1$ and $r > 0$, are also $\lambda$-self-expanders with $\lambda = \frac{k}{r} + \frac{r^2}{2} \geq \sqrt{2k}$. Each $\lambda > \sqrt{2k}$ corresponds to two cylinders $S^k_r(0) \times \mathbb{R}^{n-k}$ of radius $r = \lambda \pm \sqrt{\lambda^2 - 2k}$.

Analogous to the argument in Example 2.3, the only cylinders $S^k_r(p) \times \mathbb{R}^{n-k}$ as $\lambda$-self-expanders are $S^k_r(0) \times \mathbb{R}^{n-k}$.

**Remark 2.1.** Here we mention that $\lambda$-hypersurfaces defined in [6] and $\lambda$-translating solitons defined in [22] are CWMC hypersurfaces of weight $e^{-\frac{|x|^2}{4}}$ and $e^{\langle x, w \rangle}$ respectively, where $w \in \mathbb{R}^{n+1}$ is a constant vector.

Now we state the equivalent characterization of CWMC hypersurface.

The weighted area of a measurable subset $S \subset \Sigma$ with respect to the weight $e^{-f}$ is defined by

$$A_f(S) := \int_S e^{-f} \, d\sigma. \quad (2.3)$$

It is known that an $f$-minimal hypersurface is a critical point of the weighted area functional defined in (2.3). In [27], McGonagle-Ross proved that a CWMC hypersurface $\Sigma$ of weight $e^{-f}$ is also the critical points of the weighted area functional (2.3) but for compact normal variations $F : (-\varepsilon, \varepsilon) \times \Sigma \to \mathbb{R}^{n+1}$ that preserve weighted volume, i.e. for variations
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$F$ which satisfies $\int_{\Sigma} \varphi e^{-f} d\sigma = 0$, where $\varphi = \langle \partial_t F(0, x), n(x) \rangle$.

On the smooth metric measure space $(\Sigma, g, e^{-f})$, there is a very important second-order elliptic operator: the drifted Laplacian $\Delta_f = \Delta - \langle \nabla_f, \nabla \cdot \rangle$. It is well known that $\Delta f$ is a densely defined self-adjoint operator in $L^2(\Sigma, e^{-f} d\sigma)$, i.e. for $u$ and $v$ in $C^\infty_0(\Sigma)$, it holds that

$$
\int_{\Sigma} (\Delta_f u) v e^{-f} d\sigma = -\int_{\Sigma} \langle \nabla u, \nabla v \rangle e^{-f} d\sigma.
$$

In particular, if $\Sigma$ is a $\lambda$-self-expander, we denote by $\mathcal{L}$ the corresponding drifted Laplacian on $\Sigma$, that is, $\mathcal{L} = \Delta + \frac{1}{2} \langle x, \nabla \cdot \rangle$.

3. Simons’ type equations

In this section, we give some Simons’ type equations for $\lambda$-self-expanders.

**Theorem 3.1.** If $\Sigma$ is a $\lambda$-self-expander immersed in $\mathbb{R}^{n+1}$, then

$$
\mathcal{L} H = -|A|^2 (H - \lambda) - \frac{H}{2},
$$

$$
\mathcal{L} A = -\frac{1}{2} A - |A|^2 A - \lambda A^2,
$$

$$
\mathcal{L} |A|^2 = 2|\nabla A|^2 - |A|^2 - 2|A|^4 - 2\lambda \text{tr} A^3,
$$

$$
\mathcal{L} |A| = \frac{|\nabla A|^2 - |\nabla |A| |^2}{|A|} - \frac{\lambda \text{tr} A^3}{|A|} - |A| - \frac{|A|}{2}.
$$

**Proof.** Let us fix a point $p \in \Sigma$, and choose a local orthonormal frame $e_i$, $i = 1, \ldots, n$, for $\Sigma$ such that $\nabla e_i e_j(p) = 0$. Let $h_{ij} = \langle \nabla e_i, e_j, n \rangle$. At this point $p$ we have

$$
\text{Hess}(H)(e_i, e_j) = \nabla_{e_j} \nabla_{e_i} (-\frac{\langle x, n \rangle}{2} + \lambda)
$$

$$
= -\frac{1}{2} e_j \langle x, -h_{ik} e_k \rangle
$$

$$
= -\frac{1}{2} ( -h_{ij} - \langle x, h_{ijk} e_k \rangle - \langle x, h_{ik} h_{jk} n \rangle)
$$

$$
= \frac{1}{2} \langle A(e_i), e_j \rangle + \frac{1}{2} ((\nabla_{x^\top} A)(e_i), e_j) + \frac{1}{2} \langle x, n \rangle (A^2(e_i), e_j)
$$

$$
= \frac{1}{2} \langle A(e_i), e_j \rangle + \frac{1}{2} ((\nabla_{x^\top} A)(e_i), e_j) - (H - \lambda) \langle A^2(e_i), e_j \rangle.
$$

This implies that

$$
\Delta H = -\frac{1}{2} H - \frac{1}{2} \langle x, \nabla H \rangle - (H - \lambda)|A|^2.
$$

Therefore

$$
\mathcal{L} H = -|A|^2 (H - \lambda) - \frac{H}{2}.
$$
So we have proved (3.1).

In order to prove (3.2), recall the Simons’ equation, that is
(3.7) \[ \Delta A = -Hess(H) - HA^2 - |A|^2 A. \]
Combining (3.5) with (3.7) yields
\[ \Delta A = -\frac{1}{2} \nabla x^\top A - \frac{1}{2} A - |A|^2 A - \lambda A^2. \]
Hence
\[ \mathcal{L} A = -\frac{1}{2} A - |A|^2 A - \lambda A^2, \]
which is (3.8). For (3.3), we have that
\[ \mathcal{L} |A|^2 = 2\langle \mathcal{L} A, A \rangle + 2|\nabla A|^2 = 2|\nabla A|^2 - |A|^2 - 2|A|^4 - 2\lambda |A|^2, A. \]
Further \( \langle A^2, A \rangle = \text{tr} A^3 \). Therefore
\[ \mathcal{L} |A|^2 = 2|\nabla A|^2 - |A|^2 - 2|A|^4 - 2\lambda \text{tr} A^3, \]
which is just (3.3). (3.4) follows from (3.3) and the following identity
\[ \mathcal{L} |A|^2 = 2|A|\mathcal{L} |A| + 2|\nabla A|^2. \]
\[ \square \]

For \( \alpha \in \mathbb{R} \), we define the operator \( \mathcal{L}_\alpha = \Delta - \frac{\alpha}{2} \langle x, \nabla \rangle \) on the \( \lambda \)-self-expander \( \Sigma \). We get the following equations.

**Corollary 3.1.** Let \( \Sigma \) be a \( \lambda \)-self-expander immersed in \( \mathbb{R}^{n+1} \). Then, for \( \alpha \in \mathbb{R} \) it holds that
\begin{align*}
(3.9) \quad & \mathcal{L}_\alpha H = -|A|^2 (H - \lambda) - \frac{1}{2} H - \frac{\alpha + 1}{2} \langle x, \nabla H \rangle, \\
(3.10) \quad & \mathcal{L}_\alpha H = -|A|^2 (H - \lambda) - \frac{1}{2} H - \frac{\alpha + 1}{4} A(x^\top, x^\top),
\end{align*}
and
\begin{align*}
(3.11) \quad & \mathcal{L}_\alpha H^2 = -2|A|^2 (H - \lambda) H - H^2 + 2 \left| \nabla H - \frac{\alpha + 1}{4} H x^\top \right|^2 \\
& \quad - \frac{(\alpha + 1)^2}{8} H^2 |x^\top|^2.
\end{align*}

**Proof.** Since \( \mathcal{L} H = -|A|^2 (H - \lambda) - \frac{H^2}{2} \),
\[ \mathcal{L}_\alpha H = -|A|^2 (H - \lambda) - \frac{1}{2} H - \frac{\alpha + 1}{2} \langle x, \nabla H \rangle. \]
Take a local orthonormal frame \( \{ e_i \} \), \( i = 1, \ldots, n \), for \( \Sigma \). From \( H = \lambda - \frac{1}{2} \langle x, n \rangle \),

\[
2 \nabla_{e_i} H = - \langle \nabla_{e_i} x, n \rangle - \langle x, \nabla_{e_i} n \rangle = h_{ij} \langle x, e_j \rangle
\]

and hence

\[
\langle x, \nabla H \rangle = \langle x, e_i \rangle \nabla_{e_i} H = \frac{1}{2} h_{ij} \langle x, e_i \rangle \langle x, e_j \rangle = \frac{1}{2} A(x^\top, x^\top).
\]

By this and Equation (3.9), we have that

\[
\mathcal{L}_\alpha H = -|A|^2 (H - \lambda) - \frac{1}{2} H - \frac{\alpha + 1}{2} \langle x, \nabla H \rangle
\]

\[
= -|A|^2 (H - \lambda) - \frac{1}{2} H - \frac{\alpha + 1}{4} A(x^\top, x^\top).
\]

On the other hand, combining (3.9) and the equality \( \mathcal{L}_\alpha H^2 = 2H \mathcal{L}_\alpha H + 2|\nabla H|^2 \) we get

\[
\mathcal{L}_\alpha H^2 = -2|A|^2 (H - \lambda) H - H^2 - (\alpha + 1) H \langle x, \nabla H \rangle + 2|\nabla H|^2
\]

\[
= -2|A|^2 (H - \lambda) H - H^2 + 2 \left| \nabla H - \frac{\alpha + 1}{4} H x^\top \right|^2
\]

\[
- \frac{(\alpha + 1)^2}{8} H^2 |x^\top|^2.
\]

\[\square\]

### 4. Rigidity of spheres

In this section, we will prove Theorems 1.1 and 1.3 which characterize the spheres as closed \( \lambda \)-self-expanders. In order to prove Theorem 1.1, we need the following

**Lemma 4.1.** Let \( \Sigma \) be an immersed \( \lambda \)-self-expander in \( \mathbb{R}^{n+1} \). Then

\[
\mathcal{L}|x|^2 = |x|^2 - 2\langle x, n \rangle + 2n,
\]

(4.1)

\[
\mathcal{L}_1|x|^2 = (2H - \lambda)^2 + 2n - \lambda^2 - |x^\top|^2,
\]

(4.2)

where the operator \( \mathcal{L}_1 = \Delta - \frac{1}{2} \langle x^\top, \nabla \cdot \rangle \).

**Proof.** Recall that for any hypersurface, we have \( \Delta x = -H n \). Then

\[
\Delta |x|^2 = 2 \langle x, \Delta x \rangle + 2|\nabla x|^2
\]

(4.3)

\[
= -2H \langle x, n \rangle + 2n.
\]
Since $H = \lambda - \frac{\langle x, n \rangle}{2}$, we get
\[
\mathcal{L}|x|^2 = \Delta |x|^2 + \frac{1}{2} \langle x, \nabla |x|^2 \rangle = -2H \langle x, n \rangle + 2n + |x^\top|^2
\]
\[
= -2 \left( \lambda - \frac{\langle x, n \rangle}{2} \right) \langle x, n \rangle + 2n + |x^\top|^2
\]
\[
= |x|^2 - 2\lambda \langle x, n \rangle + 2n,
\]
and
\[
\mathcal{L}_1|x|^2 = \Delta |x|^2 - \frac{1}{2} \langle x, \nabla |x|^2 \rangle = -2H \langle x, n \rangle + 2n - |x^\top|^2
\]
\[
= 4H(H - \lambda) + 2n - |x^\top|^2
\]
\[
= 4H^2 - 4\lambda H + 2n - |x^\top|^2
\]
\[
= (2H - \lambda)^2 + 2n - \lambda^2 - |x^\top|^2.
\]
\[\Box\]

Using (4.1), we prove Theorem 1.1, that is

**Theorem 4.1.** (Theorem 1.1) Let $\Sigma$ be a closed $\lambda$-self-expander immersed in $\mathbb{R}^{n+1}$. Then

(i) $\lambda \geq \sqrt{2n}$. Moreover, the sphere $\mathbb{S}_\sqrt{2n}(0)$ is the only closed $\lambda$-self-expander with $\lambda = \sqrt{2n}$.

(ii) If

\[
(4.4) \quad |A|^2 \leq -\frac{1}{2} + \frac{\lambda \left( \lambda - \sqrt{\lambda^2 - 2n} \right)}{2n},
\]

then $\Sigma$ is a sphere $\mathbb{S}_r^n(0)$ with $r = \lambda + \sqrt{\lambda^2 - 2n}$.

**Proof.** Since $\Sigma$ is closed, there exists a point $p \neq 0$ where $|x|$ achieves its maximum. At $p$, $x$ and $n$ are in the same direction, and $H(p) \geq \frac{n}{|p|}$. This implies that $\lambda = H(p) + \frac{\langle p, n(p) \rangle}{2} \geq \frac{n}{|p|} + \frac{|p|}{2} \geq 2\frac{\sqrt{n}}{\sqrt{|p|}} \cdot \sqrt{\frac{1}{2}} = \sqrt{2n}$.

If $\lambda = \sqrt{2n}$, (4.1) implies

\[
\mathcal{L}|x|^2 \geq (|x| - \lambda)^2 - \lambda^2 + 2n = (|x| - \lambda)^2 \geq 0.
\]

By the maximum principle, $|x|$ must be constant. We conclude that $\Sigma = \mathbb{S}_r^n(0)$. So (i) is confirmed.

Now we prove (ii). Since $\Sigma$ is closed, we consider the point $p$ where $|x|$ achieves its minimum. We claim that $p \neq 0$. In fact, if $p = 0$, then $H(p) = \lambda$. By (4.4), we have

\[
\lambda^2 = H^2(p) \leq n|A|^2 \leq -\frac{n}{2} + \frac{\lambda \left( \lambda - \sqrt{\lambda^2 - 2n} \right)}{2n}.
\]
This implies 

$$0 \leq -\frac{n}{2} - \frac{\lambda \left( \lambda + \sqrt{\lambda^2 - 2n} \right)}{2},$$

which is a contradiction. Therefore $p \neq 0$.

Now, since $p \neq 0$, it follows that either $H(p) = \lambda + \frac{|x|(p)}{2}$ or $H(p) = \lambda - \frac{|x|(p)}{2}$.

We claim that $H(p) = \lambda - \frac{|x|(p)}{2}$, which is equivalent to say that $\langle p, n \rangle = |x|(p)$. In fact, if $H(p) = \lambda + \frac{|x|(p)}{2}$, by (4.4) we get

$$\left( \lambda + \frac{|x|(p)}{2} \right)^2 = H^2(p) \leq n|A|^2 \leq -\frac{n}{2} + \frac{\lambda \left( \lambda - \sqrt{\lambda^2 - 2n} \right)}{2}. \tag{4.6}$$

By (i), we know that $\Sigma$ satisfies $\lambda \geq \sqrt{2n}$. This property and (4.6) imply that

$$|x|(p) \leq -\lambda - \sqrt{\lambda^2 - 2n},$$

which is a contradiction. Therefore, we have confirmed the claim.

Applying (4.4) again, we obtain

$$\left( \lambda - \frac{|x|(p)}{2} \right)^2 = H^2(p) \leq n|A|^2 \leq \left( \frac{1}{2} + \frac{\lambda \left( \lambda - \sqrt{\lambda^2 - 2n} \right)}{2n} \right), \tag{4.7}$$

which gives

$$|x|(p) \geq \frac{1}{2\lambda}(|x|^2(p) - 2\lambda|x|(p) + 2n) + \lambda + \sqrt{\lambda^2 - 2n}. \tag{4.8}$$

Further, since $|x|$ achieves its minimum at $p$, by (4.1), we have

$$\mathcal{L}|x|(p) = |x|^2(p) - 2\lambda|x|(p) + 2n \geq 0.$$

Therefore, (4.8) implies

$$\min_\Sigma |x| = |x|(p) \geq \lambda + \sqrt{\lambda^2 - 2n}. \tag{4.9}$$

Combining (4.9) with (4.1) yields

$$\mathcal{L}(|x|^2) \geq 0.$$ 

By the maximum principle, $|x|^2 = \text{constant}$ for $\Sigma$. We conclude that $\Sigma$ is $\mathbb{S}^n_r(0)$.

For complete $\lambda$-self-expanders, (4.2) in Lemma 4.1 has the following consequence:

**Theorem 4.2.** Let $\Sigma$ be a complete properly immersed $\lambda$-self-expander in $\mathbb{R}^{n+1}$. If $\Sigma$ satisfies

$$\lambda^2 \geq 2n + (\lambda - 2H)^2, \tag{4.10}$$

then $\Sigma$ is a sphere $\mathbb{S}^n_r(0)$.\[\square\]
Proof. By (4.2) and hypothesis (4.10), we have
\[ L_1|x|^2 = (2H - \lambda)^2 + 2n - \lambda^2 - |x|^2 \leq 0. \]
Since \( \Sigma \) is proper, \( |x|^2 \) achieves its minimum. By the maximum principle we conclude that \( \Sigma \) is \( S^{n}_p(0) \).
\[ \square \]

Remark 4.1. The condition (4.10) implies that \( H \) is bounded by \( \frac{1}{2}(|\lambda| + \sqrt{\lambda^2 - 2n}) \). Then, the Gaussian weighted mean curvature \( H - \langle x, n \rangle \) of \( \Sigma \) is \( 2H - \lambda \) and also bounded. In [9], the second author of the present paper, Vieira and Zhou ([9, Theorem 1.4]) proved that if \( \Sigma \) is an immersed hypersurface in \( \mathbb{R}^{n+1} \), with bounded Gaussian weighted mean curvature, then the polynomial area growth is equivalent to the properness of the hypersurface. Therefore, in Theorem 4.2 it is possible to replace condition of properness of the hypersurface by the polynomial area growth condition.

Now we prove Theorem 1.3.

Theorem 4.3. (Theorem 1.3) Let \( \Sigma, n \geq 2 \), be a closed mean convex \( \lambda \)-self-expander immersed in \( \mathbb{R}^{n+1} \). If
\[ (4.11) \quad H\text{tr}A^3 + |A|^4 \leq 0, \]
then \( \Sigma \) must be a sphere \( S^n_p(0) \).

Proof. Let us consider \( w = \sqrt{|A|^2 + \varepsilon} \), where \( \varepsilon \) is a positive constant. Then
\[ \mathcal{L}(|A|^2 + \varepsilon) = (H + \varepsilon)\mathcal{L}w + w\mathcal{L}H + 2\langle \nabla w, \nabla H \rangle \]
\[ = (H + \varepsilon)\mathcal{L}w + w(-|A|^2 - 1)H + \lambda|A|^2 + 2\langle \nabla w, \nabla H \rangle. \]
On the other hand, by (3.3), we get
\[ \mathcal{L}(|A|^2 + \varepsilon) = \frac{1}{2\sqrt{|A|^2 + \varepsilon}} \left( \mathcal{L}|A|^2 - 2\nabla \sqrt{|A|^2 + \varepsilon}^2 \right) \]
\[ = \frac{1}{\sqrt{|A|^2 + \varepsilon}} \left( |\nabla A|^2 - |\nabla \sqrt{|A|^2 + \varepsilon}|^2 \right) \]
\[ + \frac{1}{2\sqrt{|A|^2 + \varepsilon}} \left( -|A|^2 - 2\lambda\text{tr}A^3 - 2|A|^4 \right) \]
\[ \geq \frac{1}{2\sqrt{|A|^2 + \varepsilon}} \left( -|A|^2 - 2|A|^4 - 2\lambda\text{tr}A^3 \right). \]
Combining this with (4.12) yields
\[ (H + \varepsilon)^2 w\mathcal{L}w \geq -\frac{1}{2}|A|^2 - |A|^4 - \lambda\text{tr}A^3 - \frac{|A|^2 + \varepsilon}{H + \varepsilon}(-|A|^2 - \frac{1}{2}H + \lambda|A|^2) \]
\[ - 2\langle \nabla w, \nabla H \rangle(H + \varepsilon)w \]
\[ \geq -\frac{\lambda(\text{tr}A^3 + |A|^4)}{H + \varepsilon} - \frac{\varepsilon((1 + 2\lambda)|A|^2 + 2|A|^4 + 2\lambda\text{tr}A^3)}{2(H + \varepsilon)} \]
\[ - 2\langle \nabla w, \nabla H \rangle(H + \varepsilon)w. \]
Noting that $\Sigma$ is closed, by Theorem 4.1, $\lambda \geq \sqrt{2n}$. Further, from (4.11) we deduce that $\text{tr} A^3 \leq 0$ and $|A| \leq H$. Therefore

$$(H + \varepsilon)^2 w, \mathcal{L} w \geq -\varepsilon \frac{(1 + 2\lambda) |A|^2 + 2 |A|^4}{2(H + \varepsilon)} - 2(\nabla w, \nabla H)(H + \varepsilon)w$$

$$\geq -\varepsilon \left( \frac{1 + 2\lambda}{2} |A| + |A|^3 \right) - 2(\nabla w, \nabla H)(H + \varepsilon)w.$$ 

Using integration by parts, we obtain

$$\int_{\Sigma} (H + \varepsilon)^2 |\nabla w|^2 e^{\frac{|x|^2}{4}} \, d\sigma = -\int_{\Sigma} (H + \varepsilon)^2 w, \mathcal{L} (w) e^{\frac{|x|^2}{4}} \, d\sigma$$

$$- 2\int_{\Sigma} (H + \varepsilon) w(\nabla w, \nabla H) e^{\frac{|x|^2}{4}} \, d\sigma$$

$$\leq \varepsilon \int_{\Sigma} \left( \frac{1 + 2\lambda}{2} |A| + |A|^3 \right) e^{\frac{|x|^2}{4}} \, d\sigma.$$

Therefore

$$\lim_{\varepsilon \to 0} \int_{\Sigma} (H + \varepsilon)^2 |\nabla w|^2 e^{\frac{|x|^2}{4}} = 0.$$ 

Let $p \in S := \{ x \in \Sigma; \ H(x) \neq 0 \}$, and $B_{r^p}^\Sigma(p) \subset S$. Then

$$\lim_{\varepsilon \to 0} \int_{B_{r^p}^\Sigma(p)} (H + \varepsilon)^2 |\nabla w|^2 e^{\frac{|x|^2}{4}} = 0.$$ 

By the dominated convergence theorem, we conclude that $\frac{|A|}{\mu}$ is constant in $B_{r^p}^\Sigma(p)$. Since $p$ is arbitrary, $\frac{|A|}{\mu}$ must be a constant on each connected component of $S$. By (3.1), (3.4) and (4.11), we conclude that $|\nabla A| = |\nabla|A||$ on $S$. Now we use an argument similar to the one used by Huisken in [16] (See, e.g. the proof of Theorem 0.17 in [12]). Since $\Sigma$ is closed, there exists $q \in S$ such that rank $A = n \geq 2$ and then $|\nabla A| = 0$ on $D$, where $D$ denotes the connected component of $S$ containing $q$. In particular, $|A|$ and $H$ are positive constants on $D$. Using a continuity argument, we have $\Sigma \setminus S = \emptyset$ and thus $D = \Sigma$. Hence, on $\Sigma$, $|\nabla A| = 0$ and $H$ is a positive constant. By a Lawson’s result [19, Theorem 4], $\Sigma$ must be a sphere $S^n_r(0)$. 

5. RIGIDITY OF HYPERPLANES AND SPHERES

In this section, we will prove some rigidity results that characterize the hyperplanes and spheres as $\lambda$-self-expanders.

First, we prove Theorem 1.4 which is,

**Theorem 5.1 (Theorem 1.4).** Let $\Sigma$, $n \geq 2$, be a complete $\lambda$-self-expander immersed in $\mathbb{R}^{n+1}$. If $H - \lambda \leq 0$, $|A| \in L^2(\Sigma, e^{\frac{|x|^2}{4}} \, d\sigma)$ and

$$(5.1) \quad \lambda((H - \lambda)\text{tr} A^3 - \frac{|A|^2}{2}) \geq 0,$$
then $\lambda \geq 0$ and $\Sigma$ must be either a hyperplane or a sphere $S^n_r(0)$.

Proof. For $\lambda \leq 0$, from Lemma 3.1 and the hypothesis $H - \lambda \leq 0$ we have

(5.2) \[ \mathcal{L}(H - \lambda) = -|A|^2(H - \lambda) - \frac{1}{2}H \geq 0. \]

Since $H - \lambda \leq 0$, by the maximum principle we have either $H - \lambda \equiv 0$ or $H - \lambda < 0$. If $H - \lambda \equiv 0$, $\Sigma$ must be a hyperplane.

For $\lambda > 0$ and $H - \lambda = 0$ at some point $p \in \Sigma$, from hypothesis (5.1)

\[ -\frac{|A|^2}{2} = (H - \lambda)\text{tr}A^3 - \frac{|A|^2}{2} \geq 0 \]

at $p \in \Sigma$. This implies that $|A|(p) = 0$, and thus $0 = H(p) = \lambda$. But this contradicts the fact that $\lambda > 0$.

To complete the proof, we only need to consider the case $H - \lambda < 0$.

Define the function

\[ w = \frac{\sqrt{|A|^2 + \varepsilon}}{H - \lambda}. \]

Using (3.1), we obtain

\[ \mathcal{L} \sqrt{|A|^2 + \varepsilon} = w \mathcal{L}(H - \lambda) + (H - \lambda)\mathcal{L}w + 2\langle \nabla w, \nabla H \rangle \]

\[ = \sqrt{|A|^2 + \varepsilon} \left[ -\frac{\lambda}{2(H - \lambda)} - \left( |A|^2 + \frac{1}{2} \right) \right] + (H - \lambda)\mathcal{L}w \]

\[ + 2\langle \nabla w, \nabla H \rangle. \]

Hence

(5.3) \[ (H - \lambda)^2 \mathcal{L}w = \sqrt{|A|^2 + \varepsilon} \mathcal{L} \sqrt{|A|^2 + \varepsilon} - 2\langle \nabla w, \nabla H \rangle w(H - \lambda) \]

\[ + (|A|^2 + \varepsilon) \left( \frac{\lambda}{2(H - \lambda)} + |A|^2 + \frac{1}{2} \right). \]

On the other hand, (5.3) yields

\[ \sqrt{|A|^2 + \varepsilon} \mathcal{L} \sqrt{|A|^2 + \varepsilon} = \frac{1}{2} \left( \mathcal{L} \sqrt{|A|^2 + \varepsilon} - 2 \left| \nabla \sqrt{|A|^2 + \varepsilon} \right|^2 \right) \]

\[ = |\nabla A|^2 - \left| \nabla \sqrt{|A|^2 + \varepsilon} \right|^2 \]

\[ - \frac{1}{2} (|A|^2 + 2\lambda \text{tr}A^3 + 2|A|^4). \]

Using the inequality $|\nabla A|^2 - \left| \nabla \sqrt{|A|^2 + \varepsilon} \right|^2 \geq 0$, we obtain

(5.4) \[ \sqrt{|A|^2 + \varepsilon} \mathcal{L} \sqrt{|A|^2 + \varepsilon} \geq -\frac{|A|^2}{2} - \lambda \text{tr}A^3 - |A|^4. \]
Combining (5.3) and (5.4) it follows that

\[(H - \lambda)^2 w \mathcal{L} w \geq \frac{\lambda}{\lambda - H} \left( (H - \lambda) \text{tr} A^3 - \frac{|A|^2}{2} \right) + \varepsilon |A|^2 + \frac{\varepsilon H}{2(H - \lambda)}
- 2\langle \nabla w, \nabla H \rangle w (H - \lambda).\]

(5.5)

Since \( H - \lambda < 0 \) and \( \lambda \left( (H - \lambda) \text{tr} A^3 - \frac{|A|^2}{2} \right) \geq 0 \), (5.5) yields

\[(H - \lambda)^2 w \mathcal{L} w \geq \varepsilon H^2 (H - \lambda) - 2\langle \nabla w, \nabla H \rangle w (H - \lambda).\]

(5.6)

For a nonnegative function \( \varphi \in C^\infty_0(\Sigma) \), using integration by part and (5.6), we get

\[
\int_\Sigma \varphi^2 (H - \lambda)^2 |\nabla w|^2 e^{\frac{|x|^2}{4}} = - \int_\Sigma \varphi^2 (H - \lambda)^2 w \mathcal{L} w e^{\frac{|x|^2}{4}}
- 2 \int_\Sigma \varphi^2 \langle \nabla w, \nabla H \rangle w (H - \lambda) e^{\frac{|x|^2}{4}}
- 2 \int_\Sigma \langle \varphi (H - \lambda) \nabla w, w (H - \lambda) \nabla \varphi \rangle e^{\frac{|x|^2}{4}}
\leq \int_\Sigma \frac{\varepsilon H \varphi^2}{2(\lambda - H)} e^{\frac{|x|^2}{4}} + \frac{1}{2} \int_\Sigma \varphi^2 (H - \lambda)^2 |\nabla w|^2 e^{\frac{|x|^2}{4}}
+ 2 \int_\Sigma w^2 (H - \lambda)^2 |\nabla \varphi|^2 e^{\frac{|x|^2}{4}}.
\]

Therefore,

\[
\int_\Sigma \varphi^2 (H - \lambda)^2 |\nabla w|^2 e^{\frac{|x|^2}{4}} \leq \int_\Sigma \frac{\varepsilon H \varphi^2}{2(\lambda - H)} e^{\frac{|x|^2}{4}} + 4 \int_\Sigma |\nabla \varphi|^2 |A|^2 e^{\frac{|x|^2}{4}}
+ 4\varepsilon \int_\Sigma |\nabla \varphi|^2 e^{\frac{|x|^2}{4}}.
\]

(5.7)

Now, for the following set

\[ S = \{ p \in \Sigma; |A|(p) = 0 \}, \]

we consider two cases separately:

**Case** \((S \neq \emptyset)\). Since \( H - \lambda < 0 \) and \( S \neq \emptyset \), we get \( \lambda > 0 \). First, we claim that \( \Sigma \) is noncompact. In fact, if \( \Sigma \) is compact, then taking \( \varphi = 1 \) in (5.7), we deduce that

\[
\lim_{\varepsilon \to 0} \int_\Sigma (H - \lambda)^2 |\nabla w|^2 e^{\frac{|x|^2}{4}} = 0.
\]

Suppose that \( D := \Sigma \setminus S \) is not empty. Let \( q \in D \) and \( B^\Sigma_r(q) \subset D \). Then

\[
\lim_{\varepsilon \to 0} \int_{B^\Sigma_r(q)} (H - \lambda)^2 |\nabla w|^2 e^{\frac{|x|^2}{4}} = 0.
\]
By the dominated convergence theorem, we conclude that \( \frac{|A|}{H} \) is constant in \( B_r^\Sigma(q) \). Since \( q \) is arbitrary, it is possible to conclude that \( \frac{|A|}{H} \) is constant on each connected component of \( D \). Since \( S \neq \emptyset \), using a continuity argument, we conclude that \( |A| \equiv 0 \), but this contradicts the assumption that \( D \neq \emptyset \). Hence \( S = \Sigma \). In particular \( \Sigma \) must be minimal, which is a contradiction. Therefore, we have confirmed the claim.

Now we assume that \( \Sigma \) is noncompact. For a fixed point \( p \in \Sigma \), choose \( \varphi = \varphi_k \), where \( \varphi_k \) are nonnegative cut-off functions, such that \( \varphi_k = 1 \) on \( B_{2k}^\Sigma(p) \), \( \varphi_k = 0 \) on \( \Sigma \setminus B_{2k}^\Sigma(p) \) and \( |\nabla \varphi_k| \leq \frac{1}{k} \) for every \( k \). Substitute \( \varphi \) in (5.1):

\[
\int_{\Sigma} \varphi_k^2 (H - \lambda)^2 |\nabla w|^2 e^{\frac{|x|^2}{4}} \leq \varepsilon \int_{\Sigma} \frac{H \varphi_k^2}{\lambda - H} e^{\frac{|x|^2}{4}} + 4 \int_{\Sigma} |\nabla \varphi_k|^2 |A|^2 e^{\frac{|x|^2}{4}} + 4 \varepsilon \int_{\Sigma} |A|^2 \frac{e |x|^2}{4} + 4 \varepsilon \int_{B_{2k}^\Sigma(p) \setminus B_k^\Sigma(p)} e^{\frac{|x|^2}{4}}.
\]

Choosing \( \varepsilon = \frac{1}{k} \left( \int_{B_{2k}^\Sigma(p) \setminus B_k^\Sigma(p)} e^{\frac{|x|^2}{4}} + \int_{B_k^\Sigma(p)} \frac{|H|}{\lambda - H} e^{\frac{|x|^2}{4}} \right)^{-1} \), we have

\[
\int_{\Sigma} \varphi_k^2 (H - \lambda)^2 |\nabla w_k|^2 e^{\frac{|x|^2}{4}} \leq \frac{1}{k} + \frac{4}{k^2} \int_{B_{2k}^\Sigma(p) \setminus B_k^\Sigma(p)} |A|^2 \frac{e |x|^2}{4} + \frac{4}{k^3}.
\]

In the above, \( w_k \) denotes \( w \) in which \( \varepsilon \) has been substituted.

Since \( |A| \in L^2(\Sigma, e^{\frac{|x|^2}{4}} \, d\sigma) \), it follows that

\[
\lim_{k \to \infty} \int_{\Sigma} \varphi_k^2 (H - \lambda)^2 |\nabla w_k|^2 e^{\frac{|x|^2}{4}} = 0.
\]

Given that \( S \neq \emptyset \), consider \( D = \Sigma \setminus S \). Since \( D \) is an open set, let \( q \in D \) and \( B_r^\Sigma(q) \subset D \). For \( k \) sufficiently large, \( B_r^\Sigma(q) \subset \text{supp} \varphi_k \) and \( \varphi_k = 1 \) on \( B_r^\Sigma(q) \). Hence

\[
\lim_{k \to \infty} \int_{B_r^\Sigma(q)} (H - \lambda)^2 |\nabla w_k|^2 e^{\frac{|x|^2}{4}} = 0.
\]

By the dominated convergence theorem, we conclude that \( \frac{|A|}{H} \) is constant on \( B_r^\Sigma(q) \). Since \( q \) is arbitrary, it is possible to conclude that \( \frac{|A|}{H} \) is constant on \( D \). Since \( S \neq \emptyset \), using a continuity argument, we conclude that \( |A| = 0 \) on \( \Sigma \). We conclude that \( \Sigma \) is a hyperplane.
Case ($\mathcal{S} = \emptyset$). Choosing $\varepsilon = 0$ in (5.7), we obtain

$$\int_{\Sigma} \varphi^2 (H - \lambda)^2 |\nabla w|^2 e^{|\varphi|^2} \leq 4 \int_{\Sigma} |\nabla \varphi|^2 |A|^2 e^{|\varphi|^2}. \quad (5.8)$$

In the case $\Sigma$ is compact, choose $\varphi = 1$. Otherwise, fixing a point $p \in \Sigma$, consider $\varphi = \varphi_k$, where $\varphi_k$ are nonnegative cut-off function, such that $\varphi_k = 1$ on $B_k^\Sigma (p)$, $\varphi_k = 0$ on $\Sigma \setminus B_{2k}^\Sigma (p)$ and $|\nabla \varphi_k| \leq \frac{1}{k}$, then

$$\int_{\Sigma} \varphi_k^2 (H - \lambda)^2 |\nabla w|^2 e^{|\varphi|^2} \leq 4 \int_{\Sigma} |\nabla \varphi_k|^2 |A|^2 e^{|\varphi|^2} \leq 4 \int_{B_{2k}^\Sigma (p) \setminus B_k^\Sigma (p)} |A|^2 e^{|\varphi|^2}. \quad (5.9)$$

Since $|A| \in L^2 (\Sigma, e^{|\varphi|^2} d\sigma)$, letting $k \to \infty$ in (5.9) and using the dominated convergence theorem, we get

$$\int_{\Sigma} (H - \lambda)^2 \left| \frac{|A|}{H - \lambda} \right|^2 e^{|\varphi|^2} = 0.$$ 

This implies that $|A| = C (H - \lambda)$, for a constant $C < 0$. So from (3.11), it follows that

$$\mathcal{L} |A| + \left( |A|^2 + \frac{1}{2} \right) |A| = -\frac{\lambda |A|^2}{2(H - \lambda)}. \quad (5.10)$$

Combining (3.11) and (5.10), we obtain

$$\frac{|\nabla A|^2 - |\nabla |A||^2}{|A|} = \frac{\lambda}{(H - \lambda)^2} \left( (H - \lambda) \text{tr} A^3 - \frac{|A|^2}{2} \right).$$

Further, by hypothesis $\lambda \left( (H - \lambda) \text{tr} A^3 - \frac{|A|^2}{2} \right) \geq 0$, and $H - \lambda < 0$. Therefore

$$|\nabla A| = |\nabla |A||.$$

Now again, as in the proof of Theorem 1.3, we use an argument similar to the Huisken’s in [16] (See, e.g. the proof of Theorem 0.17 in [12]). Following the proof in [12], $|\nabla A| = |\nabla |A||$ implies two possible sub-cases: (I) If the rank of $A$ is greater than 2, $\nabla A = 0$. By Theorem 4 in [19], it follows that $\Sigma$ is either $S^n_\rho (0)$ or $S^n_\rho (0) \times \mathbb{R}^{n-1}$, $2 \leq l \leq n - 1$. Since $S^n_\rho (0) \times \mathbb{R}^{n-1}$, $2 \leq l \leq n - 1$, do not satisfy the hypothesis that $|A| \in L^2 (\Sigma, e^{|\varphi|^2} d\sigma)$, we conclude that $\Sigma$ must be $S^n_\rho (0)$. (II) If the rank of $A$ is 1, then $\Sigma$ is the product of a curve $\Gamma \subset \mathbb{R}^2$ and a $(n - 1)$-dimensional hyperplane, but this contradicts the fact that $|A| \in L^2 (\Sigma, e^{|\varphi|^2} d\sigma)$.

Now we prove Theorem 1.6.
Theorem 5.2 (Theorem 1.6). Let $\Sigma$ be a complete properly immersed $\lambda$-self-expander in $\mathbb{R}^{n+1}$. If there exists $\alpha > 0$ such that
\begin{equation}
|A|^2(H - \lambda)H + \frac{1}{2}H^2\left(1 + \frac{(\alpha + 1)^2}{8}|x|^2\right) \leq 0,
\end{equation}
then $\Sigma$ must be either a hyperplane or $S^n_r(0)$.

Proof. Let $\varphi \in C^\infty_0(\Sigma)$, from (3.11) and hypothesis (5.11), we have
\begin{equation}
2\int_{\Sigma} \varphi^2 H^2 \left|\nabla H - \frac{\alpha + 1}{4}x^\top H\right|^2 e^{-\frac{\alpha}{4}|x|^2} \leq \int_{\Sigma} \varphi^2 H^2 L_H e^{-\frac{\alpha}{4}|x|^2}.
\end{equation}
Further,
\begin{align*}
\int_{\Sigma} \varphi^2 H^2 L_H e^{-\frac{\alpha}{4}|x|^2} &= -2\int_{\Sigma} \varphi H^2 (\nabla \varphi, \nabla H^2)e^{-\frac{\alpha}{4}|x|^2} \\
&\quad + \int_{\Sigma} \varphi^2 |\nabla H^2|^2 e^{-\frac{\alpha}{4}|x|^2} \\
&\leq -\frac{1}{2}\int_{\Sigma} \varphi^2 |\nabla H^2|^2 e^{-\frac{\alpha}{4}|x|^2} + 2\int_{\Sigma} H^4 |\nabla \varphi|^2 e^{-\frac{\alpha}{4}|x|^2}.
\end{align*}
Therefore
\begin{equation}
\int_{\Sigma} H^4 |\nabla \varphi|^2 e^{-\frac{\alpha}{4}|x|^2} \geq \int_{\Sigma} \varphi^2 H^2 \left|\nabla H - \frac{\alpha + 1}{4}x^\top H\right|^2 e^{-\frac{\alpha}{4}|x|^2} \\
+ \frac{1}{4}\int_{\Sigma} \varphi^2 |\nabla H^2|^2 e^{-\frac{\alpha}{4}|x|^2}.
\end{equation}
On the other hand, from (5.11) it follows that $(H - \lambda)H \leq 0$. In particular $|H| \leq |\lambda|$. Thus, by Corollary 7.2, which will be proved later in this paper, we have that $\int_{\Sigma} H e^{-\frac{\alpha}{4}|x|^2} < \infty$. Choose $\varphi = \varphi_j$ in (5.12), where $\varphi_j$ are nonnegative cut-off functions satisfying that $\varphi_j = 1$ on $B_j(0)$, $\varphi_j = 0$ on $\Sigma \setminus B_{j+1}(0)$ and $|\nabla \varphi_j| \leq 1$. Using the monotone convergence theorem, it follows that
\begin{equation}
H^2 \left|\nabla H - \frac{\alpha + 1}{4}x^\top H\right|^2 = 0 \text{ and } |\nabla H^2|^2 = 0.
\end{equation}
This implies that $H$ is constant and $H^4 |x|^2 = 0$. If $H \equiv 0$, then $\Sigma$ is a hyperplane. If $H \neq 0$, then $|x^\top| = 0$, $|x|^2 = |(x, n)|^2 = 4(\lambda - H)^2$ is constant. Thus $\Sigma$ is a sphere. \hfill \Box

6. Rigidity of generalized cylinders

In this section, using the maximum principle we will prove Theorem 1.5 that characterizes the generalized cylinders as $\lambda$-self-expander hypersurfaces.

Theorem 6.1 (Theorem 1.5). Let $\Sigma$ be a complete immersed $\lambda$-self-expander in $\mathbb{R}^{n+1}$ with $\lambda \neq 0$. Then
\begin{enumerate}
\item[(i)] the open set $\{x \in \Sigma; H \neq \lambda\}$ must be non-empty.
\end{enumerate}
(ii) Further, if the following conditions are satisfied on the non-empty set \( \{ x \in \Sigma; H \neq \lambda \} \),

(a) \( \frac{\lambda}{H} \left( (H - \lambda) \text{tr} A^3 - \frac{|A|^2}{2} \right) \leq 0 \),

(b) the function \( \frac{|A|^2}{(H - \lambda)^2} \) attains a local maximum,

then \( \Sigma \) is either a hyperplane or a cylinder \( S_r(0) \times \mathbb{R}^{n-1}, 1 \leq l \leq n \).

**Proof.** First we confirm (i). In fact, by contradiction, assume that \( H - \lambda \equiv 0 \) on \( \Sigma \). From (5.1) we conclude that \( \lambda = 0 \), a contradiction. Therefore, the set \( \{ x \in \Sigma; H \neq \lambda \} \) is non-empty.

Now we prove (ii). By hypothesis that the function \( \frac{|A|^2}{(H - \lambda)^2} \) attains a local maximum on the set \( D = \{ x \in \Sigma; H \neq \lambda \} \), there is a point \( p \in D \) and a connected open neighborhood \( U \subset D \) of \( p \) such that \( \frac{|A|^2}{(H - \lambda)^2}(p) = \max_U \frac{|A|^2}{(H - \lambda)^2} \). Computing \( \mathcal{L} \frac{|A|^2}{(H - \lambda)^2} \) on \( U \) we get

\[
\mathcal{L} \frac{|A|^2}{(H - \lambda)^2} = 2 \left( \mathcal{L} \frac{A}{H - \lambda} \cdot \frac{A}{H - \lambda} \right) + 2 \left| \nabla \frac{A}{H - \lambda} \right|^2. 
\]

Further

\[
\mathcal{L} \frac{A}{H - \lambda} = \frac{1}{H - \lambda} \mathcal{L} A + A \mathcal{L} \frac{1}{H - \lambda} - \frac{2}{(H - \lambda)^2} \nabla \nabla (H - \lambda) A,
\]

and

\[
\mathcal{L} \frac{1}{H - \lambda} = -\frac{\mathcal{L}(H - \lambda)}{(H - \lambda)^2} + 2 \frac{\nabla(H - \lambda)^2}{(H - \lambda)^3}.
\]

Substituting (6.2) and (6.3) into (6.1), we obtain

\[
\mathcal{L} \frac{|A|^2}{(H - \lambda)^2} = \frac{2}{(H - \lambda)^2} \langle \mathcal{L} A, A \rangle + \frac{2|A|^2}{H - \lambda} \left( -\frac{\mathcal{L}(H - \lambda)}{(H - \lambda)^2} + 2 \frac{\nabla(H - \lambda)^2}{(H - \lambda)^3} \right)
\]

\[
- \frac{4}{(H - \lambda)^3} \langle \nabla \nabla (H - \lambda) A, A \rangle + 2 \left| \nabla \frac{A}{H - \lambda} \right|^2
\]

\[
= \frac{2}{(H - \lambda)^2} \langle \mathcal{L} A, A \rangle - \frac{2|A|^2}{(H - \lambda)^3} \mathcal{L}(H - \lambda) + \frac{4|A|^2}{(H - \lambda)^4} \nabla(H - \lambda)^2
\]

\[
- \frac{2}{(H - \lambda)^3} \langle \nabla |A|^2, \nabla (H - \lambda) \rangle + 2 \left| \nabla \frac{A}{H - \lambda} \right|^2
\]

\[
= \frac{2}{(H - \lambda)^2} \langle \mathcal{L} A, A \rangle - \frac{2|A|^2}{(H - \lambda)^3} \mathcal{L}(H - \lambda)
\]

\[
(6.4) - \left\langle \nabla \frac{|A|^2}{(H - \lambda)^2}, \nabla \log(H - \lambda)^2 \right\rangle + 2 \left| \nabla \frac{A}{H - \lambda} \right|^2.
\]
From (6.4), (3.1) and (3.2) it follows that
\[
\mathcal{L} \left( \frac{|A|^2}{(H - \lambda)^2} \right) = -\left\langle \nabla \frac{|A|^2}{(H - \lambda)^2}, \nabla \log(H - \lambda)^2 \right\rangle + 2 \left| \nabla \frac{A}{H - \lambda} \right|^2
\]
\[\quad - \frac{2\lambda}{(H - \lambda)^3} \left( (H - \lambda) \text{tr} A^3 - \frac{|A|^2}{2} \right).
\]
Combining this with the hypothesis (a), we obtain
\[
\mathcal{L} \left( \frac{|A|^2}{(H - \lambda)^2} \right) + \left\langle \nabla \frac{|A|^2}{(H - \lambda)^2}, \nabla \log(H - \lambda)^2 \right\rangle \geq 0.
\]
Since \( \frac{|A|^2}{(H - \lambda)^2} \) attains a local maximum, the maximum principle implies that
\[
\frac{|A|^2}{(H - \lambda)^2} = c,
\]
where \( c \in \mathbb{R} \) is a nonnegative constant. Since \( H - \lambda \neq 0 \), either \( |A| = \sqrt{c}(H - \lambda) \) or \( |A| = -\sqrt{c}(H - \lambda) \). If \( c = 0 \), \( U \) must be contained in a hyperplane. If \( c > 0 \), from (3.1), it follows that
\[
(6.5)
\]
\[
\mathcal{L} |A| + \left( |A|^2 + \frac{1}{2} \right) |A| = -\frac{\lambda |A|}{2(H - \lambda)}.
\]
Combining (3.4) and (6.5), we obtain
\[
|\nabla A|^2 - |\nabla |A||^2 = \frac{\lambda}{(H - \lambda)} \left( (H - \lambda) \text{tr} A^3 - \frac{|A|^2}{2} \right).
\]
Since \( |\nabla A|^2 - |\nabla |A||^2 \geq 0 \), using the hypothesis (a) and the equality above, we conclude that
\[
(6.6)
\]
\[
|\nabla A| = |\nabla |A||.
\]
Now again, as in the proof of Theorem 1.4, (6.6) implies two possibilities:
(I) If rank of \( A \) is greater than 2, \( \nabla A = 0 \). By the result of Lawson in [19], it follows that \( U \) must be contained in \( S_{l_r}(0) \times \mathbb{R}^{n-l}, 2 \leq l \leq n \). (II) If rank of \( A \) is 1, then
\[
H^2 = |A|^2 = c(H - \lambda)^2.
\]
This implies that \( H \) and \( |A| \) are constants. In particular, by (6.6), \( \nabla A = 0 \). By the result of Lawson in [19], \( U \) must be contained in \( S_{l_r}(0) \times \mathbb{R}^{n-l} \).

Therefore, we have proved that \( \Sigma \) is locally contained in either a hyperplane or a \( S_{l_r}(0) \times \mathbb{R}^{n-l} \), \( 1 \leq l \leq n \). Applying the unique continuation properties for solutions of elliptic equations, we conclude that \( \Sigma \) is either a hyperplane or a generalized cylinder \( S_{l_r}(0) \times \mathbb{R}^{n-l} \), \( 1 \leq l \leq n \).

Now we prove a result (Proposition 6.1) on the immersed complete \( \lambda \)-self-expanders with constant mean curvature. In [2] Corollary 3.2, the first author of the present article and Miranda proved that a complete \( \lambda \)-hypersurface is either a hyperplane or a cylinder \( S_{l_r}(0) \times \mathbb{R}^{n-l} \), \( 1 \leq l \leq n \). Since a \( \lambda \)-self-expander with constant mean curvature is also a \( (2H - \lambda) \)-hypersurface, Proposition 6.1 can be obtained by Corollary 3.2 in [2].
the following, we give a direct proof for \( \lambda \)-self-expanders for the sake of completeness.

**Proposition 6.1.** If \( \Sigma \) is a complete \( \lambda \)-self-expander immersed in \( \mathbb{R}^{n+1} \) with constant mean curvature, then \( \Sigma \) must be either a hyperplane or a cylinder \( S^l_r(0) \times \mathbb{R}^{n-l}, 1 \leq l \leq n \).

**Proof.** If \( H = 0 \), (3.1) implies that \( \lambda |A|^2 = 0 \). Then \( \lambda = 0 \) or \(|A|^2 \equiv 0\) on \( \Sigma \). If \( \lambda = 0 \), \( \Sigma \) is a self-expander and must be a hyperplane through the origin. If \(|A|^2 = 0\), \( \Sigma \) also be a hyperplane.

If \( H \neq 0 \), it follows that from (3.1),

\[
\frac{1}{2}H - |A|^2(H - \lambda) = 0.
\]

This implies that \( H \neq \lambda \). Then \( |A| \) is constant and \( \lambda \neq 0 \). From (3.3), we get

\[
|\nabla A|^2 = \frac{|A|^2}{2} + \lambda \text{tr} A^3 + |A|^4. \tag{6.8}
\]

On the other hand, by the following Simons’ equation

\[
\frac{1}{2}\Delta |A|^2 = |\nabla A|^2 - \langle A, Hess(H) \rangle - H \text{tr} A^3 - |A|^4,
\]

it follows that

\[
|\nabla A|^2 = |A|^4 + H \text{tr} A^3. \tag{6.9}
\]

Then, by (6.8) and (6.9), we have that

\[
|\nabla A|^2 = |A|^4 + \frac{|A|^2 H}{2(H - \lambda)}.
\]

This and (6.7) imply that

\[|\nabla A| = 0.\]

So, by a result of Lawson in [19, Theorem 4], \( \Sigma \) must be a generalized cylinder.

\[\square\]

7. Further discussion

In this section, we first prove the following result which may be compared with Theorem 4.2.

**Theorem 7.1.** Let \( \Sigma \) be a complete immersed \( \lambda \)-self-expander in \( \mathbb{R}^{n+1} \). Assume that \( H^\delta \in L^2(\Sigma, e^{H^\delta} \, d\sigma) \) for some \( \delta \in \mathbb{N} \) and

\[
|A|^2((2H - \lambda)^2 + 2n - \lambda^2) \leq 0. \tag{7.1}
\]

Then \( \Sigma \) is either a hyperplane or a sphere \( S^l_r(0) \).
Proof. From hypothesis (7.1) and (3.1) it follows that
\[ H.LH = -|A|^2(H - \lambda)H - \frac{H^2}{2} \]
\[ \geq -|A|^2(H - \lambda)H - \frac{n|A|^2}{2} \]
\[ = -\frac{1}{4}|A|^2((4(H - \lambda)H + 2n) \]
\[ = -\frac{1}{4}|A|^2((2H - \lambda)^2 - \lambda^2 + 2n) \]
\[ \geq 0. \]
Let \( \varphi \in C_0^\infty(\Sigma) \). Integrating by part, we have
\[ \int_\Sigma H^{2\delta-2}\varphi^2H(LH)e^{\frac{|x|^2}{4}} \]
\[ = -2\int_\Sigma H^{2\delta-1}\varphi(\nabla \varphi, \nabla H)e^{\frac{|x|^2}{4}} - (2\delta-1)\int_\Sigma \varphi^2 H^{2(\delta-1)}|\nabla H|^2 e^{\frac{|x|^2}{4}} \]
\[ \leq -\frac{2\delta-1}{2}\int_\Sigma \varphi^2 H^{2(\delta-1)}|\nabla H|^2 e^{\frac{|x|^2}{4}} + \frac{2}{2\delta-1}\int_\Sigma |\nabla \varphi|^2 H^{2\delta} e^{\frac{|x|^2}{4}}, \]
where \( \delta \geq 1 \). Therefore
\[ \frac{2\delta-1}{2}\int_\Sigma \varphi^2 H^{2(\delta-1)}|\nabla H|^2 e^{\frac{|x|^2}{4}} \leq \frac{2}{2\delta-1}\int_\Sigma |\nabla \varphi|^2 H^{2\delta} e^{\frac{|x|^2}{4}} \]
Choose \( \varphi = \varphi_j \), where \( \varphi_j \) are nonnegative cut-off functions satisfying that \( \varphi_j = 1 \) on \( B_j^\Sigma(p) \), \( \varphi_j = 0 \) on \( \Sigma \setminus B_{j^2}(p) \) and \( |\nabla \varphi_j| \leq \frac{1}{j} \). By the monotone convergence theorem and the hypothesis that \( H^{\delta} \in L^2(\Sigma, e^{\frac{|x|^2}{4}}d\sigma) \), it follows that, on \( \Sigma \),
\[ H^{2(\delta-1)}|\nabla H|^2 = 0. \]
This implies that \( H \) is constant. If \( H = 0 \), \( \Sigma \) is a hyperplane. If \( H \neq 0 \), by Proposition 4.2, \( \Sigma \) is at most a cylinder \( S_l^\lambda(0) \times R^{n-l} \), \( 1 \leq l \leq n \). However, the integrability hypothesis of \( H^{\delta} \) implies that \( \Sigma \) must be a sphere. □

In [1], by using properties on the finiteness of weighted areas and the area growth upper estimate for self-expanders with some restriction on mean curvature, we proved some theorems that characterize the hyperplanes through the origin as self-expanders ([1, Theorems 1.3 and 1.4]). Here, motivated by the work [1] we will prove some results that characterize the generalized cylinders as \( \lambda \)-self-expanders. In order to prove these results, we first prove the following Theorem 7.2, which deals with finiteness of weighted areas and area growth estimate of \( \lambda \)-self-expanders.

**Theorem 7.2.** Let \( \Sigma \) be a complete properly immersed \( \lambda \)-self-expander in \( R^{n+1} \). Assume that its mean curvature \( H \) satisfies \( |H|(x) \leq a|x| + b, x \in \Sigma, \)
for some constants $0 \leq a < \frac{1}{2}$ and $b > 0$. Then it holds that, for any $\alpha > \frac{4a^2}{1-4a^2}$,

(i) $\int_{\Sigma} e^{-\frac{\alpha}{4}|x|^2} d\sigma < \infty$.

(ii) The area of $B_r(0) \cap \Sigma$ satisfies

$$\text{Area}(B_r(0) \cap \Sigma) \leq C(\alpha) e^{\frac{\alpha}{4}r^2},$$

where $B_r(0)$ denotes the round ball in $\mathbb{R}^{n+1}$ of radius $r$ centered at the origin $0 \in \mathbb{R}^{n+1}$.

In particular, if $0 < a < \frac{1}{2\sqrt{2}}$, then the Gaussian weighted area is finite, that is,

$$\int_{\Sigma} e^{-\frac{\alpha}{4}|x|^2} d\sigma < \infty.$$

(7.2)

Proof. Take $h = \frac{|x|^2}{4}$, $x \in \mathbb{R}^{n+1}$. Since $\Sigma$ is properly immersed in $\mathbb{R}^{n+1}$, $h$ is proper on $\Sigma$. Besides, since $H = \lambda - \frac{\langle x, n \rangle}{2}$, by (4.3) we have that, on $B_r(0) \cap \Sigma$,

$$\Delta h = -H \frac{\langle x, n \rangle}{2} + \frac{n}{2}$$

$$= H(H - \lambda) + \frac{n}{2}$$

$$\leq ar^2 + (2ab + a|\lambda|)r + (b^2 + |\lambda|b + \frac{n}{2}).$$

In the above, we also have used the hypothesis: $|H|(x) \leq a|x| + b$, $x \in \Sigma$. We also have that on $B_r(0) \cap \Sigma$,

$$\Delta h - a|\nabla h|^2 + ah = H(H - \lambda) + \frac{n}{2} + \alpha \frac{\langle x, n \rangle^2}{4}$$

$$= H(H - \lambda) + \frac{n}{2} + a(H - \lambda)^2$$

$$= (1 + \alpha)H^2 - \lambda(1 + 2\alpha)H + \frac{n}{2} + \alpha \lambda^2$$

$$\leq (1 + \alpha)a^2 r^2 + (2ab(1 + \alpha) + a|\lambda|(1 + 2\alpha))r$$

$$+ ((1 + \alpha)b^2 + |\lambda|(1 + 2\alpha)b + \frac{n}{2} + \alpha \lambda^2).$$

Let $a_2 = (1 + \alpha)a^2$, $a_1 = 2ab(1 + \alpha) + a|\lambda|(1 + 2\alpha)$, $a_0 = (1 + \alpha)b^2 + |\lambda|(1 + 2\alpha)b + \frac{n}{2} + \alpha \lambda^2$ and $\beta = \alpha$. For $\alpha > \frac{4a^2}{1-4a^2}$, where $0 \leq a < \frac{1}{2}$, it holds that $a_2 < \frac{\beta}{1}$. By applying Theorem 3.1 in [6], we obtain that $\int_{\Sigma} e^{-\frac{\alpha}{4}|x|^2} d\sigma < \infty$ for all $\alpha > \frac{4a^2}{1-4a^2}$ and the area of $B_r(0) \cap \Sigma$ satisfies that, for all $r > 0$,

$$\text{Area}(B_r(0) \cap \Sigma) \leq C(\alpha) e^{\frac{\alpha}{4}r^2}.$$

In the particular case of $a < \frac{1}{2\sqrt{2}}$, since $a < \frac{1}{2\sqrt{2}}$ implies that $\frac{4a^2}{1-4a^2} < 1$, we may take $\alpha = 1$. \hfill $\square$
By an argument analogous to the ones used in the proofs of Theorem 4.1 in [10] and Theorem 4 in [3], we may prove the following result: Let $\Sigma$ be a complete immersed $\lambda$-self-expander in $\mathbb{R}^{n+1}$. If there exists $\alpha > 0$ such that $\int_{\Sigma} e^{-\frac{\alpha}{4}|x|^2} d\sigma < \infty$, then $\Sigma$ is properly immersed on $\mathbb{R}^{n+1}$. Hence Theorem 7.2 has the following consequence.

**Corollary 7.1.** Let $\Sigma$ be a complete immersed $\lambda$-self-expander $e$ in $\mathbb{R}^{n+1}$. Assume that its mean curvature $H$ satisfies $|H|(x) \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2}$ and $b > 0$. Then for $\alpha > \frac{4a^2}{1-4a\lambda}$ the following statements are equivalent:

(i) $\Sigma$ is properly immersed on $\mathbb{R}^{n+1}$.

(ii) There exist constants $C = C(\alpha)$, $\alpha_0$, $\alpha_1$ and $\alpha_2 < \frac{4}{\alpha}$, such that

$$\text{Area}(B_r(0) \cap \Sigma) \leq C(\alpha)e^{\frac{\alpha}{2}r^2 + \frac{4}{1-4a\lambda}r + \alpha_0}.$$ (7.3)

(iii) $\int_{\Sigma} e^{-\frac{\alpha}{4}|x|^2} d\sigma < \infty$.

Another consequence of Theorem 7.2 is the following corollary, which deals of the integrable property of the powers of the norm of mean curvature $H$.

**Corollary 7.2.** Let $\Sigma$ be a complete properly immersed $\lambda$-self-expander in $\mathbb{R}^{n+1}$. Assume that $|H| \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2}$ and $b > 0$. Then for $\delta \geq 0$ and $\alpha > \frac{4a^2}{1-4a\lambda}$

$$\int_{\Sigma} |H|^\delta e^{-\frac{\alpha}{4}|x|^2} d\sigma < \infty.$$ (7.3)

**Proof.** Since $|H| \leq a|x| + b$, it is easily to see that Theorem 7.2 implies the desired conclusion. \qed

**Theorem 7.3.** Let $\Sigma$ be a complete properly immersed $\lambda$-self-expander in $\mathbb{R}^{n+1}$. Assume that its mean curvature $H$ satisfies $|H|(x) \leq a|x| + b$, $x \in \Sigma$, for some constants $0 \leq a < \frac{1}{2}$ and $b > 0$. If there exists $\alpha > \frac{4a^2}{1-4a\lambda}$ such that

$$|A|^2(H - \lambda)H + \frac{H^2}{2} + \frac{\alpha + 1}{4}A(x^\top, x^\top)H \leq 0,$$ (7.4)

then $\Sigma$ must be either a hyperplane or a cylinder $S^l(\theta) \times \mathbb{R}^{n-l}, 1 \leq l \leq n$.

**Proof.** Let $\varphi \in C^\infty_0(\Sigma)$. From (8.4) and hypothesis (7.3), we have

$$0 \leq \int_{\Sigma} \left(-|A|^2(H - \lambda)H - \frac{1}{2}H^2 - \frac{\alpha + 1}{4}A(x^\top, x^\top)H\right) \varphi^2 e^{-\frac{\alpha}{4}|x|^2}$$

$$= \int_{\Sigma} H\varphi^2(\mathcal{L}_\alpha H) e^{-\frac{\alpha}{4}|x|^2}.$$ 

Further

$$\int_{\Sigma} H\varphi^2(\mathcal{L}_\alpha H) e^{-\frac{\alpha}{4}|x|^2} = -2 \int_{\Sigma} H\varphi(\nabla \varphi, \nabla H) e^{-\frac{\alpha}{4}|x|^2} - \int_{\Sigma} \varphi^2 |\nabla H|^2 e^{-\frac{\alpha}{4}|x|^2}$$

$$\leq -\frac{1}{2} \int_{\Sigma} \varphi^2 |\nabla H|^2 e^{-\frac{\alpha}{4}|x|^2} + 2 \int_{\Sigma} |\nabla \varphi|^2 H^2 e^{-\frac{\alpha}{4}|x|^2}.$$
Therefore
\[
\frac{1}{2} \int_{\Sigma} \varphi^2 |\nabla H|^2 e^{-\frac{\alpha}{4}|x|^2} \leq 2 \int_{\Sigma} |\nabla \varphi|^2 H^2 e^{-\frac{\alpha}{4}|x|^2}.
\]

Choose \( \varphi = \varphi_j \), where \( \varphi_j \) are nonnegative cut-off functions satisfying \( \varphi_j = 1 \) on \( B_j(0) \), \( \varphi_j = 0 \) on \( \Sigma \setminus B_{j+1}(0) \) and \( |\nabla \varphi_j| \leq 1 \). By the hypothesis \( |H|(x) \leq a|x| + b, \ x \in \Sigma, \) for some constants \( 0 \leq a < \frac{1}{2} \) and \( b > 0 \), Corollary 7.2 implies that \( \int_{\Sigma} H^2 e^{-\frac{\alpha}{4}|x|^2} < \infty \), for any \( \alpha > \frac{4a^2}{1-4a^2} \). By the monotone convergence theorem it follows that
\[
|\nabla H| = 0,
\]
which implies that \( H \) is constant. By Proposition 6.1, we get that \( \Sigma \) must be a generalized cylinder. \( \square \)

**Theorem 7.4.** Let \( \Sigma \) be a complete properly immersed \( \lambda \)-self-expander in \( \mathbb{R}^{n+1} \). Assume that its mean curvature \( H \) is bounded from below and satisfies \( H(x) \leq a|x| + b, \ x \in \Sigma, \) for some constants \( 0 \leq a < \frac{1}{2} \) and \( b > 0 \). If there exists \( \alpha > \frac{4a^2}{1-4a^2} \) such that
\[
|A|^2 (H - \lambda) + \frac{H}{2} + \frac{\alpha + 1}{4} A(x^\top, x^\top) \geq 0,
\]
then \( \Sigma \) must be either a hyperplane or a cylinder \( S^l_t(0) \times \mathbb{R}^{n-l}, 1 \leq l \leq n \).

**Proof.** Let us fix \( C = \inf_{x \in \Sigma} H \). From hypothesis 7.5 and (3.10) it follows that
\[
|A|^2 (H - C) \leq 0.
\]

By the maximum principle, either \( H \equiv C \) or \( H > C \). If \( H \equiv C \), by Proposition 1.2, \( \Sigma \) is either hyperplane or a generalized cylinder. If \( H > C \), let us consider \( u := \log(H - C) \). A computing yields
\[
\Delta u = -|\nabla u|^2 + \frac{\Delta H}{H - C}.
\]

Combining (7.6) and (7.7), we get
\[
|\nabla u|^2 \leq \frac{\Delta H}{H - C}.
\]

Let us consider the sequence \( \varphi_j \) of nonnegative cut-off function satisfying that \( \varphi_j = 1 \) on \( B_j(0) \), \( \varphi_j = 0 \) on \( \Sigma \setminus B_{j+1}(0) \) and \( |\nabla \varphi_j| \leq 1 \). Multiplying (7.8) by \( \varphi_j \) and integrating by parts we obtain
\[
\int_{\Sigma} \varphi_j^2 |\nabla u|^2 e^{-\alpha \frac{|x|^2}{4}} \leq \int_{\Sigma} \varphi_j^2 (|\nabla \varphi_j|) e^{-\alpha \frac{|x|^2}{4}}
\]
\[
= \int_{\Sigma} 2 \varphi_j \nabla \varphi_j \nabla u e^{-\alpha \frac{|x|^2}{4}}
\]
\[
\leq \frac{1}{2} \int_{\Sigma} \varphi_j^2 |\nabla u|^2 e^{-\alpha \frac{|x|^2}{4}} + 2 \int_{\Sigma} |\nabla \varphi_j|^2 e^{-\alpha \frac{|x|^2}{4}}.
\]
Therefore
\begin{equation}
\int_{\Sigma} \varphi_j^2 |\nabla u|^2 e^{-\alpha |\varphi_j|^2} \leq 4 \int_{\Sigma} |\nabla \varphi_j|^2 e^{-\alpha |\varphi_j|^2}.
\end{equation}
(7.9)

From assumption on the mean curvature of $\Sigma$, it follows that $|H|(x) \leq a|x| + b_1$, for some constants $0 \leq a < \frac{1}{2}$ and $b_1 > 0$. Therefore, Theorem 7.2 implies that $\int_{\Sigma} e^{-\alpha |\varphi_j|^2} < \infty$. Applying the dominated convergence theorem in (7.9), we conclude that
\[
\int_{\Sigma} |\nabla u|^2 e^{-\alpha |x|^2/4} = 0.
\]
In particular $H$ must be a constant, but this contradicts the assumption that $H > \inf_{x \in \Sigma} H$. \hfill $\square$

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