Robustness of Randomized Rumour Spreading

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Abstract

In this work we consider three well-studied broadcast protocols: \textit{push}, \textit{pull} and \textit{push\&pull}. A key property of all these models, which is also an important reason for their popularity, is that they are presumed to be very robust, since they are simple, randomized, and, crucially, do not utilize explicitly the global structure of the underlying graph. While sporadic results exist, there has been no systematic theoretical treatment quantifying the robustness of these models. Here we investigate this question with respect to two orthogonal aspects: (adversarial) modifications of the underlying graph and message transmission failures.

We explore in particular the following notion of local resilience: beginning with a graph, we investigate up to which fraction of the edges an adversary has to be allowed to delete at each vertex, so that the protocols need significantly more rounds to broadcast the information. Our main findings establish a separation among the three models. It turns out that \textit{pull} is robust with respect to all parameters that we consider. On the other hand, \textit{push} may slow down significantly, even if the adversary is allowed to modify the degrees of the vertices by an arbitrarily small positive fraction only. Finally, \textit{push\&pull} is robust when no message transmission failures are considered, otherwise it may be slowed down.

On the technical side, we develop two novel methods for the analysis of randomized rumour spreading protocols. First, we exploit the notion of self-bounding functions to facilitate significantly the round-based analysis: we show that for any graph the variance of the growth of informed vertices is bounded by its expectation, so that concentration results follow immediately. Second, in order to control adversarial modifications of the graph we make use of a powerful tool from extremal graph theory, namely Szemerédi’s Regularity Lemma.

1 Introduction

Randomized broadcast protocols are highly relevant for data distribution in large networks of various kinds, including technological, social and biological networks. Among many others there are three basic models in the literature, introduced in [18, 9, 26], namely \textit{push}, \textit{pull} and \textit{push\&pull} (or short \textit{pp}). Consider a connected graph in which some vertex holds a piece of information; we call this vertex (initially) informed. All three models have the common characteristic that they proceed in rounds. In the \textit{push} model, in every round every informed vertex chooses a neighbour independently and uniformly at random (iuar) and informs it; this of course has only an effect if the target vertex was previously uninformed. Contrary, in the \textit{pull} model every round every uninformed vertex chooses a neighbour iuar and asks for the information. If the asked vertex has the information, then the asking vertex becomes informed as well. The third model \textit{push\&pull} combines both worlds: in each round, each vertex chooses a neighbour iuar, and if one of both vertices is informed, then afterwards both become so. We additionally assume that each message transmission succeeds independently with probability $q \in (0, 1]$. For these algorithms, the main parameter that we consider is the random variable that counts how many rounds are needed until all vertices are informed, and we call these quantities the runtimes of the respective algorithms.

In the remainder we will denote the runtime of \textit{push} by $T_{\text{push}}(G, v, q)$ where $G$ is the underlying graph, initially the vertex $v$ is informed and we have a transmission success probability of $q \in (0, 1]$. Analogously we denote the runtimes of \textit{pull} and \textit{push\&pull} by $T_{\text{pull}}(G, v, q)$ and $T_{\text{pp}}(G, v, q)$ respectively. If the choice of $v$ does not matter we will omit it in our notation. The most basic case is when $G$ is the complete graph $K_n$.
with \( n \) vertices. Then, see for example Doerr and Kostrygin [11], it is known that for \( \mathcal{P} \in \{\text{push, pull, pp}\} \) and \( q \in (0, 1] \) in expectation and with probability tending to 1 as \( n \to \infty \)

\[
T_{\mathcal{P}}(K_n, q) = c_{\mathcal{P}}(q) \log n + o(\log n),
\]

where, for \( q \in (0, 1) \),

\[
c_{\text{push}}(q) := \frac{1}{\log(1 + q)} + \frac{1}{q}, \quad c_{\text{pull}}(q) := \frac{1}{\log(1 + q)} - \frac{1}{\log(1 - q)}, \quad c_{\text{pp}}(q) := \frac{1}{\log(1 + 2q)} + \frac{1}{q - \log(1 - q)},
\]

and where we set \( c_{\mathcal{P}}(1) := \lim_{q \to 1} c_{\mathcal{P}}(q) \). If \( q \) is clear from the context, we write \( c_{\mathcal{P}} \) instead of \( c_{\mathcal{P}}(q) \). Actually, the results in [11] and also [12] are much more precise, but the stated forms will be sufficient for what follows.

**Contribution & Related Work** In this article our focus is on quantifying the robustness of all three models. Indeed, robustness is a key property that is often attributed to them, since they are simple, randomized, and, crucially, do not exploit explicitly the structure of the underlying graph (apart, of course, from considering the neighborhoods of the vertices). Clearly, the runtime can vary tremendously between different graphs with the same number of vertices. Hence it is essential to understand which structural characteristics of a graph influence in what way the runtime of rumour spreading algorithms.

One result in this spirit for the push model was shown in [27]. Roughly speaking, in that paper it is shown that even on graphs with low density, if the edges are distributed rather uniformly, then push is as fast as on the complete graph. This can be interpreted as a robustness result: starting with a complete graph, one can delete a vast amount of edges and as long as this is done rather uniformly, the runtime of push is affected insignificantly. To state the result more precisely, we need the following notion.

**Definition 1.1** ((\( n, \delta, \Delta, \lambda \))-graph). Let \( G \) be a connected graph with \( n \) vertices that has minimum degree \( \delta \) and maximum degree \( \Delta \). Let \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \) be the eigenvalues of the adjacency matrix of \( G \), and set \( \lambda = \max_{2 \leq i \leq n} |\mu_i| = \max\{|\mu_2|, |\mu_n|\} \). We will call \( G \) an \((n, \delta, \Delta, \lambda)\)-graph.

In this paper we are interested in the case where \( G \) gets large, that is, when \( n \to \infty \). Hence all asymptotic notation in this paper is with respect to \( n \); in particular “with high probability”, or short whp, means with probability \( 1 - o(1) \) when \( n \to \infty \).

**Definition 1.2** (Expander Sequence). Let \( \mathcal{G} = (G_n)_{n \in \mathbb{N}} \) be a sequence of graphs, where \( G_n \) is a \((n, \delta_n, \Delta_n, \lambda_n)\)-graph for each \( n \in \mathbb{N} \). We say that \( \mathcal{G} \) is an expander sequence if \( \Delta_n/\delta_n = 1 + o(1) \) and \( \lambda_n = o(\Delta_n) \).

Note that if we consider any sequence \( \mathcal{G} = (G_n)_{n \in \mathbb{N}} \) of graphs this always implicitly defines \( \delta_n, \Delta_n \) and \( \lambda_n \) as in Definition 1.2. Expander graphs have found numerous applications in computer science and mathematics, see for example the survey [24]. If \( \mathcal{G} \) is an expander sequence, then intuitively this means that for \( n \) large enough, the edges of \( G_n \) are rather uniformly distributed. For a more formal statement compare Lemma 2.9. Moreover, note that our definition of expander sequences excludes the case when \( d \) is bounded; this is actually a necessary condition for our robustness results to hold, compare [13]. With all these definitions at hand we can state the result from [27] that quantifies the robustness of push with respect to the network topology; that is, the runtime is asymptotically the same as on the complete graph \( K_n \).

**Theorem 1.3.** Let \( \mathcal{G} = (G_n)_{n \in \mathbb{N}} \) be an expander sequence. Then whp

\[
T_{\text{push}}(G_n) = c_{\text{push}}(1) \log n + o(\log n).
\]

Apart from expander sequences, results in the form of Theorem 1.3 (where the asymptotic runtime is determined) were also shown for sufficiently dense Erdős-Rényi random graphs [15], random regular graphs [14] as well as hypercubes [27]. Moreover, the order of the runtime on various models that describe social networks was investigated. In [16] the Chung-Lu model was studied, [10] explored preferential attachment graphs and [17] examined geometric graphs. A somewhat different approach is to derive general runtime bounds that hold for all graphs and depend only on some graph parameter, e.g. conductance [19, 6], vertex expansion [20] or diameter [5, 22]. Furthermore, several variants of push, pull and push/pull were studied. These include vertices being restricted to answer only one pull request per round [7], vertices being allowed
to contact multiple neighbours per round [27, 11], vertices not calling the same neighbour twice [10] and asynchronous versions [4, 28, 1, 2]. Finally, besides [11], robustness of these rumor spreading algorithms with respect to message transmission failures was also studied by Elsässer and Sauerwald in [13]. It was shown for any graph that if a message fails with probability $1 - p$, then the runtime of push increases at most by a factor of $6/p$.

In this work our focus is on three subjects concerning the robustness of rumour spreading. Our first (and not unexpected) result extends the validity of Theorem 1.3 to the runtimes of pull and push$\&$pull. In particular, we show that none of the three protocols slows down or speeds up on graphs with good expansion properties compared to its runtime on the complete graph. This motivates to investigate how severely a graph with good expansion properties has to be modified to increase the respective runtimes.

In our second contribution, which is also the main result and which differs from what was treated in previous works, we propose and study an unworn approach to quantifying robustness. In particular, we investigate the impact of adversarial edge deletions, where we use the well-known concept of local resilience, see e.g. [30, 8]. To be specific, we explore up to which fraction of edges an adversary needs to be allowed to delete at each vertex to slow down the process by a significant amount of time, i.e., by $\Omega(\log n)$ rounds. Here we discover a surprising dichotomy in the following sense. On the one hand, we show that both pull and push$\&$pull cannot be slowed down by such adversarial edge deletions – in essentially all but trivial cases, where the fraction is so large that the graph may become (almost) disconnected. On the other hand, we demonstrate that even a small number of edge deletions is sufficient to slow down push by $\Omega(\log n)$ rounds. In other words, we find that in contrast to pull and push$\&$pull, the push protocol is not resilient to adversarial deletions and lacks (in this specific sense) the robustness of the other two protocols.

As our third subject, we generalise the previous results by additionally considering message transmission failures that occur independently with probability $1 - q \in [0, 1)$. On the positive side, we show that for arbitrary $q \in (0, 1]$ all three algorithms inform almost all vertices at least as fast as in an expander sequence in spite of adversarial edge deletions. However, if we want to inform all vertices, only pull is not slowed down by adversarial edge deletions for all values of $q$: push can be slowed down as before; and push$\&$pull is a mixed bag, for $q = 1$ it can not be slowed down, for $q < 1$ it can. Furthermore, in general it is also possible to speed push$\&$pull up by deleting edges, which is however not surprising as the star-graph deterministically finishes in at most 2 rounds.

Summarizing, this work enhances previous (robustness) results, particularly the ones concerning precise asymptotic runtimes and random transmission failures. Crucially, we introduce and study the concept of local resilience as a method to investigate robustness. However, apart from that, in this paper we develop two new general methods for the analysis of rumour spreading algorithms.

- The most common approach in the current literature for the study of the runtime is to determine the expected number of newly informed vertices in one or more rounds and to show concentration, for example by bounding the variance. Achieving this, however, is often quite complex and makes laborious and lengthy technical arguments necessary. Here we use the theory of self-bounding functions, see Section 2, that allows us to cleanly upper bound the variance by the expected value. The argument works for all three investigated algorithms and the bound is valid for all graphs. We are certain that this method will also facilitate future work on the analysis of rumour spreading algorithms.

- Studying the robustness of the protocols is a challenging task, as the adversary (as described previously) has various opportunities to modify the graph, for example by introducing a high variance in the degrees of the vertices; this turns out to be particularly problematic in the case of push$\&$pull. Here we demonstrate that such types of irregularities can be handled universally by applying a powerful tool from a completely different area, namely extremal graph theory. In particular, we use Szemerédi’s regularity lemma (see e.g. [29]), which allows us to partition the vertex set of a graph such that nearly all pairs of sets in the partition behave nearly like perfect regular bipartite graphs. This allows us to apply our methods on these regular pairs; eventually we obtain a linear recursion that can be solved by analysing the maximal eigenvalue of the underlying matrix.

1.1 Results

Our first result addresses the question about how fast rumours spread on expander graphs; in order to obtain a concise statement also the occurrence of independent message transmission failures is considered.


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**Theorem 1.4.** Let $G = (G_n)_{n \in \mathbb{N}}$ be an expander sequence and let $q \in (0, 1]$. Then whp

(a) $T_{\text{push}}(G_n, q) = c_{\text{push}}(q) \log n + o(\log(n))$,

(b) $T_{\text{pull}}(G_n, q) = c_{\text{pull}}(q) \log n + o(\log(n))$,

(c) $T_{\text{pp}}(G_n, q) = c_{\text{pp}}(q) \log n + o(\log(n))$.

The first statement is an extension of Theorem 1.3 and its proof is a straightforward adaption of the proof in [27]. We omit it. The contribution here is the proof of (b) and (c). Next we consider the case with edge deletions in addition to the message transmission failures.

**Theorem 1.5.** Let $0 < \varepsilon < 1/2, q \in (0, 1]$ and $G = (G_n)_{n \in \mathbb{N}}$ be an expander sequence. Let $\tilde{G} = (\tilde{G}_n)_{n \in \mathbb{N}}$ be such that each $\tilde{G}_n$ is obtained by deleting edges of $G_n$ such that each vertex keeps at least a $1/(2 + \varepsilon)$ fraction of its edges. Then whp

(a) $T_{\text{pull}}(\tilde{G}_n, q) = c_{\text{pull}}(q) \log n + o(\log(n))$.

(b) $T_{\text{pp}}(\tilde{G}_n, 1) \leq c_{\text{pp}}(1) \log n + o(\log(n))$, when additionally assuming that $\delta(G_n) \geq \alpha n$ for some $0 < \alpha \leq 1$.

This result demonstrates unconditionally the robustness of pull, and conditionally on $q = 1$ the robustness of push/pull on dense graphs, in the case of edge deletions, that is, the runtime is asymptotically the same as in the complete graph. It even shows that push/pull may potentially profit from edge deletions in contrast to being slowed down. The proof of this result, especially the statement about push/pull, is rather involved, since the original graph may become quite irregular after the edge deletions. Here we use, among many other ingredients, the aforementioned decomposition of the graph given by Szemerédi’s regularity lemma.

Note that Theorem 1.5 does not consider push and push/pull (when $q \neq 1$) at all. Indeed, our next result states that in these cases the behaviour is rather different and that the algorithms may be slowed down.

**Theorem 1.6.** Let $\varepsilon > 0$ and $q \in (0, 1]$. Then there is an expander sequence $\mathcal{G} = (G_n)_{n \in \mathbb{N}}$ and a sequence of graphs $\tilde{G} = (\tilde{G}_n)_{n \in \mathbb{N}}$ with the following properties. Each $\tilde{G}_n$ is obtained by deleting edges of $G_n$ such that each vertex keeps at least a $(1 - \varepsilon)$ fraction of its edges. Moreover, whp

(a) $T_{\text{push}}(\tilde{G}_n, q) \geq c_{\text{push}}(q) \log n + \varepsilon/(2q) \log n + o(\log(n))$.

(b) $T_{\text{pp}}(\tilde{G}_n, q) \geq c_{\text{pp}}(q) \log n + (\varepsilon/(8q) - \varepsilon q^3/5) \log n + o(\log(n))$.

Nevertheless, not all hope is lost. On the positive side, the next result states that push and push/pull are able to inform almost all vertices as fast as on the complete graph in spite of adversarial edge deletions. In this sense, we obtain an almost-robustness result for these cases.

**Theorem 1.7.** Let $0 < \varepsilon < 1/2, q \in (0, 1]$ and $G = (G_n)_{n \in \mathbb{N}}$ be an expander sequence. Let $\tilde{G} = (\tilde{G}_n)_{n \in \mathbb{N}}$ be such that each $\tilde{G}_n$ is obtained by deleting edges of $G_n$ such that each vertex keeps at least a $(1/2 + \varepsilon)$ fraction of its edges. For $P \in \{\text{push, pp}\}$ let $\tilde{T}_P$ denote the number of rounds needed to inform at least $n - n/\log n$ vertices. Then whp

(a) $\tilde{T}_{\text{push}}(\tilde{G}_n) = \log_{1+q}(n) + o(\log(n))$.

(b) $\tilde{T}_{\text{pp}}(\tilde{G}_n) \leq \log_{1+2q}(n) + o(\log(n))$, when additionally assuming that $\delta(G_n) \geq \alpha n$ for some $0 < \alpha \leq 1$.

We conjecture that there is also a version of Theorem 1.7 (b) that is true for push/pull on sparse graphs; to be precise we conjecture that in the setting of Theorem 1.7 (b) $\tilde{T}_{\text{pp}}(\tilde{G}_n) \leq \log_{1+2q}(n) + o(\log(n))$, without further restrictions on $G_n$, i.e. that push/pull can not be slowed down informing almost all vertices.

As a final remark note that Theorems 1.5 and 1.7 are tight in the sense that if an adversary is allowed to delete up to half of the edges at each vertex, then there are expander graphs that become disconnected such that their components have linear size. On those graphs a linear fraction of the vertices will remain uninformed forever.
Outline The rest of this paper is structured as follows. In Section 2 we collect and prove several important facts; this part of the paper also contains our technical contribution concerning the analysis through self-bounding functions. In Subsection 3.1 we show that push is as fast on expanders with (or without) deleted edges as it is on the complete graph. Subsection 3.2 treats push\&pull on expanders without deleted edges. In the remaining subsections we focus on the cases that may be slowed down by edge deletions. In Subsection 3.3 we show that adversarial edge deletions cannot slow down the time until push has informed almost all vertices, by giving a coupling to the case without edge deletions. Contrary in Subsection 3.4 we show that the time until push has informed all vertices can be slowed down by edge deletions, even if only few edges are deleted. Then, in Subsection 3.5 we show that push\&pull informs almost all vertices of dense graphs fast in spite of adversarial edge deletions. We utilize a version of Szemerédi's Regularity Lemma to get a well-behaved partition of the vertex set that is suitable for performing a round based analysis. However, if \( q < 1 \), adversarial edge deletions can slow down or speed up the time until push\&pull has informed all nearly all values of \( q \); see Section 3.6.

Further Notation Let \( G = (V, E) \) denote a graph with vertex set \( V \) and edge set \( E \subseteq \binom{V}{2} \). Consider \( v \in V \) and \( U, W \subseteq V \) with \( U \cap W = \emptyset \). We will denote the set of neighbours of \( v \) in \( G \) by \( N_G(v) \) or by \( N(v) \) and we will denote its degree by \( d_G(v) := |N_G(v)| \) or by \( d(v) \); \( \delta_G \) or \( \delta \) and \( \Delta_G \) or \( \Delta \) denote minimum and maximum degree of \( G \). Similarly the neighbourhood of any set of vertices \( S \subseteq V \) is defined by \( N_G(S) := \cup_{v \in S} N_G(v) \). Furthermore let \( E(U, W) = E_G(U, W) \) denote the set of edges with one vertex in \( U \) and one vertex in \( W \) and let \( e(U, W) := e_G(U, W) := |E_G(U, W)| \). With \( E_G(U) \) we denote the set of edges with both vertices in \( U \); \( e_G(U) = |E_G(U)| \). For any round \( t \in \mathbb{N} \) and \( P \in \{push, pull, pp\} \), we denote by \( I_t^{(P)}(G) \) the set of vertices of \( G \) informed by \( push, pull \) and push\&pull respectively at the beginning of round \( t \) and \( |I_t^{(P)}| = 1 \); if the underlying graph is clear from the context we will omit it; if we consider a sequence of graphs \( G = (G_n)_{n \in \mathbb{N}} \) and a sequence of times \( t = (t(n))_{n \in \mathbb{N}} \), then \( I_t^{(P)}(G_n) = (I_{t(n)}^{(P)}(G_n))_{n \in \mathbb{N}} \) is also a sequence. Similarly, \( U_t^{(P)} := V \setminus I_t^{(P)} \) denotes the set of uninformed vertices. With log we refer to the natural logarithm. For any event \( A \) we will write \( E[A|I_t] \) instead of \( E[A|I_t] \) for the conditional expectation and \( P[A|I_t] \) instead of \( P[A|I_t] \) for the conditional probability. Finally we want to clarify our use of Landau symbols. Let \( a, b \in \mathbb{R} \) and \( f \) be a function. The terms \( a \leq b + o(f) \) and \( a \geq b - o(f) \) mean that there exist positive functions \( g, h \in o(f) \) such that \( a \leq b + g \) and \( a \geq b - h \). Consequently \( a = b + o(f) \) means that there exists a positive function \( g \in o(f) \) such that \( a \in [b - g, b + g] \).

2 Tools & Techniques

In this section we collect and prove statements about our protocols and properties of expander sequences. We begin with applying the previously mentioned notion of self-bounding functions to derive universal and simple-to-apply concentration results for our random variables, i.e., the number of informed vertices after a particular round. Then we extend the concentration results to more than one round. In the last part we recall the well known Expander Mixing Lemma and utilize it to derive properties (weak expansion, path enumeration) for the case where we delete edges from our graphs.

Self-bounding functions. Our main technical new result in this section is the following bound on the variance for the number of informed vertices in any given round; it is true for any graph and any set of informed vertices.

**Lemma 2.1.** Let \( G \) be a graph, \( t \in \mathbb{N} \) and \( I_t = I_t^{(P)}(G) \) for \( P \in \{push, pull, pp\} \). Then
\[
Var[|I_{t+1}| |I_t] \leq E[|I_{t+1}| |I_t].
\]

Lemma 2.1 follows directly from Lemmas 2.3 and 2.4. Before stating them we introduce the notion of self-bounding functions.

**Definition 2.2 (Self-bounding function).** Let \( X \) be a set and \( m \in \mathbb{N} \). A non-negative function \( f : X^m \to \mathbb{R} \) is self-bounding, if there exist functions \( f_i : X^{m-1} \to \mathbb{R} \) such that for all \( x_1, \ldots, x_m \in X \) and all \( i = 1, \ldots, m \)
\[
0 \leq f(x_1, \ldots, x_m) - f_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \leq 1
\]
and

\[ \sum_{1 \leq i \leq m} (f(x_1, \ldots, x_m) - f_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)) \leq f(x_1, \ldots, x_m). \]

As striking property of self-bounding function is the following bound on the variance.

**Lemma 2.3 ([3]).** For a self-bounding function \( f \) and independent random variables \( X_1, \ldots, X_m, m \in \mathbb{N} \)

\[ \text{Var}[f(X_1, \ldots, X_m)] \leq \mathbb{E}[f(X_1, \ldots, X_m)]. \]

**Lemma 2.4.** Let \( G \) be a graph, \( t \in \mathbb{N} \), and let \( I_t = I_t^{(P)}(G) \) for \( P \in \{ \text{push}, \text{pull}, \text{pp} \} \). Then, conditional on \( I_t \), there exist \( m \in \mathbb{N} \), independent random variables \( X_1, \ldots, X_m \) and a self-bounding function \( f = f^{(P)} \) such that \( |I_{t+1}| = f(X_1, \ldots, X_m) \).

**Proof.** We will prove in detail the result for \( \text{push} \), and then we show what needs to be modified in order to obtain the statement in the case of \( \text{pull} \) and \( \text{push\&pull} \). Let \( I_t = I_t^{(\text{push})} \), \( n \in \mathbb{N} \) be number of vertices of \( G \) and \( f : [n]^{||I_t||} \to \mathbb{R} \) with

\[ (x_1, \ldots, x_{|I_t|}) \mapsto |I_t| + \sum_{1 \leq k \leq |I_t|} 1[x_k \in U_t] 1[\forall \ell < k : x_k \neq x_\ell]. \]

Moreover, let \( (X_i)_{1 \leq i \leq |I_t|} \) be independent random variables, where \( X_i \) is a uniformly random neighbour of the \( i \)th vertex – according to an arbitrary ordering – in \( I_t \). We argue that \( f(X_1, \ldots, X_{|I_t|}) = |I_{t+1}| \). Consider \( v \in I_t \), then \( v \) is counted by the \( |I_t| \) term in \( f \). For \( v \in I_{t+1} \setminus I_t \) let \( v_1, \ldots, v_s \in I_t, s \in \mathbb{N} \) be the informed vertices with random neighbour \( v \) in round \( t \), i.e. \( X_{v_1} = \cdots = X_{v_s} = v \) and \( X_{v_i} \neq v \) for all other \( u \in I_t \). Assume further that \( v_1 < v_2 < \cdots < v_s \). For \( k = v_1 \) the term \( 1[X_k \in U_t] 1[\forall \ell < k : x_k \neq x_\ell] = 1 \) as \( X_{v_1} = v \in U_t \) and for all \( i \leq v_1 \) it holds that \( X_i \neq X_{v_1} \). For \( k = v_r, 2 \leq r \leq s \) the term \( 1[\forall \ell < k : x_k \neq x_\ell] = 0 \) as \( v_1 < v_r \) and \( X_{v_1} = X_{v_r} = v \). Thus every vertex \( v \in I_{t+1} \setminus I_t \) is counted exactly once by \( f \). Set further

\[ f_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{|I_t|}) = |I_t| + \sum_{k=1, k \neq i}^{|I_t|} 1[x_k \in U_t] 1[\forall j < k, j \neq i : x_j \neq x_k], \quad 1 \leq i \leq |I_t|. \]

The function \( f_i \) arises from \( f \) by leaving the \( i \)th variable out of consideration, i.e., the push of the \( i \)th vertex has no effect. Then by definition \( f - f_i \in \{0, 1\} \) for all \( 1 \leq i \leq |I_t| \), and actually we have

\[ f - f_i = 1[x_i \in U_t] 1[\forall j \neq i : x_i \neq x_j]. \]

This quantity is precisely the difference in informed vertices after round \( t \), assuming the \( i \)th vertex did not push. Furthermore

\[ \sum_{1 \leq i \leq |I_t|} (f - f_i) \leq \sum_{1 \leq i \leq |I_t|} 1[x_i \in U_t] 1[\forall j \neq i : x_i \neq x_j] \leq f. \]

Thus \( f \) has the self-bounding property, which establishes the claim in the case of \( \text{push} \). The proof for \( \text{pull} \) is completely analogous, where we use

\[ f^{(\text{pull})} : [n]^{U_t} \to \mathbb{R}, \quad (x_1, \ldots, x_{|U_t|}) \mapsto |I_t| + \sum_{k \in U_t} 1[x_k \in I_t] \]

and, similarly, for \( \text{push\&pull} \) we use \( f^{(\text{pp})} : [n]^n \to \mathbb{R} \) with

\[ (x_1, \ldots, x_n) \mapsto |I_t| + \sum_{1 \leq k \leq n} 1[k \in I_t] 1[x_k \in U_t] 1[\forall j \in \{1, \ldots, k\} \cap I_t : x_k \neq x_j] \]

\[ + \sum_{1 \leq k \leq n} 1[k \in U_t] 1[x_k \in I_t] 1[\forall w \in I_t : x_w \neq k]. \]

Here it is useful to see that the two sums in \( f^{(\text{pp})} \) are complementary, i.e. that only one of the summands for index \( k \) can be 1. Thus the functions \( f_i^{(\text{pull})} \) and \( f_i^{(\text{pp})} \) are obtained analogously to the push case. \qed
Remark 2.5. Let \( G = (V, E) \) be a graph. Lemma 2.4 also applies to subsets of \( I_{t+1} \), i.e., for any \( U \subset V \) and conditioned on \( I_t \), we have that \( |I_{t+1} \cap U| \) and \( |(I_{t+1} \cap U) \setminus I_t| \) are self-bounding.

The following lemma gives a tool that we will use in order to extend our round-wise analysis to longer phases.

**Proposition 2.6.** Let \( \mathcal{P} \in \{\text{push, pull, pp}\}, I_t = I_t^{(\mathcal{P})} \) and \( t_1 \geq t_0 \geq 1 \) such that \( |I_0| \geq \sqrt{\log n} \). Let further \( (A_t)_{t \in \mathbb{N}} \) be a sequence of events, \( c > 1 \), and \( \delta > 0 \) such that
\[
P_{t_0}[A_t \mid A_{t_0}, \ldots, A_{t_1}] \geq 1 - \delta \left(c^{-t_0} |I_{t_0}|\right)^{-1/3} \quad \text{for all } t_0 \leq t \leq t_1.
\]

Then
\[
P_{t_0} \left[ \bigcap_{t = t_0}^{t_1} A_t \right] \geq 1 - O(|I_{t_0}|^{-1/3})
\]

**Proof.** Using the definition of conditional probability we obtain, as \( c > 1 \),
\[
P_{t_0} \left[ \bigcap_{t = t_0}^{t_1} A_t \right] = \prod_{t = t_0}^{t_1} P_{t_0} [A_t \mid A_{t_0}, \ldots, A_{t_1}] \geq \prod_{t = t_0}^{t_1} \left(1 - \left(c^{-t_0} \delta |I_{t_0}|\right)^{-1/3}\right)
\]
\[
\geq 1 - \sum_{t = t_0}^{t_1} \left(c^{-t_0} \delta |I_{t_0}|\right)^{-1/3} = 1 - |I_{t_0}|^{-1/3} \sum_{t = t_0}^{t_1} \delta^{-1/3} c^{-t/3} = 1 - O(|I_{t_0}|^{-1/3}).
\]

We give two typical example applications of this lemma below. The first example addresses the case where we have a lower bound for the expected number of informed vertices after one round.

**Example 2.7.** Let \( \mathcal{P} \in \{\text{push, pull, pp}\}, I_t = I_t^{(\mathcal{P})} \). Assume that there is some \( c > 1 \) such that \( \mathbb{E}_t [\|I_{t+1}\|] \geq c |I_t| \) for all \( t \) as long as \( n_{t+1} f(n) \leq |I_t| \leq n/g(n) \) for some functions \( 1 \leq f, g \leq n, f = o(n) \). Let \( t_0 \) be such that \( |I_0| \geq n/f(n) \). Then according to Lemma 2.1 we have that \( \text{Var}_t [\|I_{t+1}\|] \leq \mathbb{E}_t [\|I_{t+1}\|] \) and applying Chebychev’s inequality gives
\[
P_t \left[ \|I_{t+1}\| - \mathbb{E}_t [\|I_{t+1}\|] \leq \mathbb{E}_t [\|I_{t+1}\|]^{2/3} \right] \geq 1 - \mathbb{E}_t [\|I_{t+1}\|]^{-1/3} \geq 1 - |I_t|^{-1/3}. \tag{2.1}
\]

Consider the events
\[
A_t = \{ \|I_t\| \geq \mathbb{E}_{t-1} [\|I_{t+1}\|] - \mathbb{E}_{t-1} [\|I_{t+1}\|]^{2/3} \text{ or } |I_t| \geq n/g(n) \}
\]

The intersection of \( A_{t_0+1}, \ldots, A_t \) implies inductively that either \( |I_t| \geq n/g(n) \) or
\[
|I_t| \geq \left(1 - \mathbb{E}_{t-1} [\|I_{t+1}\|]^{-1/3}\right) \mathbb{E}_{t-1} [\|I_t\|] \geq \left(1 - (c |I_{t-1}|)^{-1/3}\right) c |I_{t-1}| \geq \left((1 - (c |I_{t-1}|)^{-1/3}) c\right)^{t-t_0} |I_{t_0}|.
\]

We obtain with (2.1)
\[
P_{t_0}[A_{t+1} \mid A_{t_0+1}, \ldots, A_t, |I_t| < n/g(n)] \geq 1 - \left((1 - (c |I_{t_0}|)^{-1/3}) c\right)^{(t-t_0)/3} |I_{t_0}|^{-1/3},
\]
and otherwise \( P_{t_0}[A_{t+1} \mid A_{t_0+1}, \ldots, A_t, |I_t| \geq n/g(n)] = 1 \). Choose \( \tau := t - t_0 = \log_c (f(n)/g(n)) + o(\log n) \) as small as possible such that this lower bound for \( |I_{t+1}| \) is \( \geq n/g(n) \), that is, this lower bound is \( < n/g(n) \) for \( t = t_0 + \tau \). Combining the two conditional probabilities we obtain for all \( t_0 \leq t \leq t_0 + \tau \)
\[
P_{t_0}[A_{t+1} \mid A_{t_0+1}, \ldots, A_t] \geq 1 - \left((1 - (c |I_{t_0}|)^{-1/3}) c\right)^{(t-t_0)/3} |I_{t_0}|^{-1/3}.
\]

Applying Proposition 2.6 then yields whp
\[
|I_{t_0 + \tau + 1}| \geq n/g(n).
\]
In the second example we make the stronger assumption that we can determine asymptotically the expected number of informed vertices after one round. Here we assume that we begin with a “small” set of informed vertices, say of size $\sqrt{\log n}$, and want to reach a set of size nearly linear in $n$.

**Example 2.8.** Assume that there is some $c > 1$ such that $\mathbb{E}_t[|I_{t+1}|] = (1 + o(1))c |I_t|$ for all $t$ as long as $\sqrt{\log n} \leq |I_t| \leq n / \log n$. Let $A_t$ be the event “$|I_t - \mathbb{E} |I_t| | \leq \mathbb{E}_{t-1} |I_t| |^{2/3}$ and let $t_0$ be such that $|I_{t_0}| \geq \sqrt{\log n}$. There is $h(n) \in o(1)$ such that for $c^- := (1 - h(n))c$ and $c^+ := (1 + h(n))c$ we have that $\mathbb{E}_t[|I_{t+1}|] \leq c^+ |I_t|$ and $\mathbb{E}_t[|I_{t+1}|] \geq c^- |I_t|$. Using this notation, the events $A_{t_0+1}, \ldots, A_{t_1}$ imply together inductively that

$$|I_{t+1}| \leq \left(1 + \varepsilon |I_t| \right)^{-1/3} \mathbb{E}_t[|I_{t+1}|] \leq \left(1 + |c^- |I_t| \right)^{-1/3} c^+ |I_t| \leq \left(1 + (c^- |I_{t_0}|)^{-1/3} c^+ \right)^{t - t_0} |I_{t_0}|$$

for all $t$ such that the right-hand side is bounded by $n / \log n$. Moreover, for all such $t$

$$|I_{t+1}| \geq \left(1 - \varepsilon |I_t| \right)^{-1/3} \mathbb{E}_t[|I_{t+1}|] \geq \left(1 - |c^- |I_t| \right)^{-1/3} c^- |I_t| \geq \left(1 - (c^- |I_{t_0}|)^{-1/3} c^- \right)^{t - t_0} |I_{t_0}|.$$

Thus, as $A_t$ only depends on $I_t$ it follows with (2.1)

$$P_{t_0}(A_{t_0} | A_{t_0+1}, \ldots, A_{t_1}) \geq 1 - \left(1 - (c^- |I_{t_0}|)^{-1/3} c^- \right)^{- (t - t_0) / 3} |I_{t_0}|^{-1/3}.$$

Applying Proposition 2.6 then immediately gives that there is $\tau_1 = \log_c (n / |I_{t_0}|) + o(\log n)$ such that whp $|I_{t_0 + \tau_1}| \leq n / \log n$. Example 2.7, setting $f = n / \sqrt{\log n}$ and $g = \log n$, gives an additional $\tau_2 = \log_c (n / |I_{t_0}|) + o(\log n)$ such that $|\tau_1 - \tau_2| = o(\log n)$ and whp

$$|I_{t_0 + \tau_1}| \leq \frac{n}{\log n} \leq |I_{t_0 + \tau_2}|.$$

**Expander Sequences.** In this section we collect some important properties of expander sequences that we are going to use later. We start by stating a version of the well-known expander mixing lemma applied to our setting of expander sequences.

**Lemma 2.9** ([27, Cor. 2.4]). Let $G = (G_n)_{n \in \mathbb{N}} = ((V_n, E_n))_{n \in \mathbb{N}}$ be an expander sequence. Then for $S_n \subseteq V_n$ such that $1 \leq |S_n| \leq n / 2$ it is

$$e(S_n, V_n \setminus S_n) - \frac{\Delta_n |S_n| (n - |S_n|)}{n} = o(\Delta_n) |S_n|.$$

The following result is a consequence of the Expander Mixing Lemma that applies to graphs in which some edges were removed. It seems very simple but it turns out to be surprisingly useful.

**Lemma 2.10.** Let $G = (G_n)_{n \in \mathbb{N}} = ((V_n, E_n))_{n \in \mathbb{N}}$ be an expander sequence. Let $\varepsilon > 0$ and set $\tilde{G} = (\tilde{G}_n)_{n \in \mathbb{N}}$, where each $\tilde{G}_n$ is obtained from $G_n$ by deleting edges such that each vertex keeps at least a $(1/2 + \varepsilon)$ fraction of its edges. For each $n \in \mathbb{N}$ let further $S_n \subseteq V_n$, then there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$e_{\tilde{G}_n}(S_n, V_n \setminus S_n) \geq \varepsilon e_{G_n}(S_n, V_n \setminus S_n).$$

**Proof.** Without loss of generality we assume that $|S_n| \leq n / 2$. Since at most $(1/2 - \varepsilon) \Delta_n$ edges are deleted at each vertex, we immediately obtain that

$$e_{\tilde{G}_n}(S_n, V_n \setminus S_n) \geq e_{G_n}(S_n, V_n \setminus S_n) - \Delta_n (1/2 - \varepsilon) |S_n|.$$

Using Lemma 2.9 and choosing $n_0$ large enough such that $\frac{o(\Delta_n)}{\Delta_n} \frac{n}{n - |S_n|} < \varepsilon$ for all $n \geq n_0$, we obtain that

$$(1 - \varepsilon) e_{G_n}(S_n, V_n \setminus S_n) - \Delta_n (1/2 - \varepsilon) |S_n| \geq (1 - \varepsilon) \frac{\Delta_n |S_n| (n - |S_n|)}{n} - o(\Delta_n) |S_n| - \Delta_n (1/2 - \varepsilon) |S_n|$$

$$= \frac{\Delta_n |S_n| (n - |S_n|)}{n} \left(1 - \varepsilon - \frac{o(\Delta_n)}{\Delta_n} \frac{n}{n - |S_n|} \frac{n(1/2 - \varepsilon)}{n - |S_n|} \right).$$
As \( n - |S_n| \geq n/2 \) the last expression is \( > 0 \). Hence 
\[
\varepsilon G_n(S_n, V_n \setminus S_n) \geq \varepsilon G(S_n, V_n \setminus S_n) + (1 - \varepsilon)\varepsilon G(S_n, V_n \setminus S_n) - \Delta_n(1/2 - \varepsilon)|S_n| \geq \varepsilon G_n(S_n, V_n \setminus S_n).
\]

Next we give a lemma that counts the number of paths between two arbitrary vertices of a dense graph satisfying a weak expander property (as for example guaranteed by Lemma 2.10). This will later be used to give a lower bound on the probability of any vertex to be informed after a given constant number of rounds.

**Lemma 2.11.** Let \( G = (V, E), |V| = n \). Assume that there is \( \alpha > 0 \) such that \( d(v) \geq \alpha n \) for all \( v \in V \) and \( e(W, V \setminus W) \geq \alpha |W||V \setminus W| \) for all \( W \subseteq V \). Then for all \( u, w \in V \) there is \( 1 \leq d \leq 8/\alpha^2 + 2 \) such that there are at least \((\alpha^4/64)^d + 1)n^{d-1}\) paths of length \( d \) from \( u \) to \( w \).

**Proof.** Assume \( \alpha \leq 1/2 \) as otherwise the claim is trivial (with \( d \in \{1, 2\} \)). We define sequences \((U_i)_{i \in \mathbb{N}}\) and \((H_i)_{i \in \mathbb{N}} \subseteq V \) as follows. Set \( U_1 = \{u\} \cup N(u), W = \{w\} \cup N(u) \) and \( H_1 = V \setminus (U_1 \cup W) \) and proceed for \( i \geq 1 \) as follows. Let \( \tilde{U}_{i+1} \subseteq H_i \) be the set of vertices \( v \in H_i \) with \( |N(v) \cap U_i| \geq \alpha^2 n/8 \). Set \( U_{i+1} = U_i \cup \tilde{U}_{i+1} \) and \( H_{i+1} = H_i \setminus U_{i+1} \). Then we claim that for all \( i \geq 1 \)
\[
e(U_i, W) \geq \alpha^3 n^2/2 \quad \text{or} \quad |U_{i+1}| \geq |U_i| + \alpha^2 n/8.
\]

To see this, assume that \( e(U_i, W) \leq \alpha^3 n^2/2 \); since \( |U_i|, |W| \geq \alpha n \), the weak expansion property guarantees that
\[
e(U_i, H_i) = e(U_i, H_i \cup W) - e(U_i, W) \geq |U_i||H_i \cup W| - \alpha^3 n^2/2 \geq \alpha^2(1 - \alpha)n^2 - \alpha^3 n^2/2,
\]
and using \( \alpha \leq 1/2 \) we obtain that \( e(U_i, H_i) \geq \alpha^2 n^2/4 \). To complete the proof of (2.2) we compute the size of \( \tilde{U}_{i+1} \). As \( |N(v) \cap U_i| \leq \alpha^2 n/8 \) for all \( v \in H_i \setminus U_{i+1} \) and \( |N(v) \cap U_i| \leq n \) we get
\[
\frac{\alpha^2 n^2}{4} \leq e(U_i, H_i) \leq |\tilde{U}_{i+1}| n + |H_i| \frac{\alpha^2 n}{8}.
\]

Since \( |H_i| \leq n \) we immediately get that \( |\tilde{U}_{i+1}| \geq \alpha^2 n/8 \), which shows (2.2). We next show that there are (sufficiently) many paths for each vertex in \( U_i \) to \( u \). More precisely, let \( 1 \leq j \leq 8/\alpha^2 \) be such that \( e(U_i, W) < \alpha^3 n^2/2 \) for all \( 1 \leq i \leq j \). For those \( i \) we have by (2.2) that \( |U_i| \geq i \cdot \alpha^2 n/8 \). We claim that for all \( v \in U_i \setminus \{u\} \) there is \( d \leq i \) such that \( v \) has at least \((\alpha^4/64)^d \cdot n^{d-1}\) paths of length \( d \) with endpoint \( u \). We show the claim by induction on \( i \). The base case \( v \in U_1 \setminus \{u\} \) is clear, as \( 1 \geq \alpha^4/64 \). For the induction step assume that \( v \in U_{i+1} \setminus U_i, v \neq u \). Then by construction \( |N(v) \cap U_i| \geq \alpha^2 n/8 \). Thus by induction hypothesis there is \( d \leq i \) such that \( v \) has at least \( \alpha^2 n/8 \) neighbours with at least \((\alpha^4/64)^d \cdot n^{d-1}\) paths with endpoint \( u \). Since \( i \leq 8/\alpha^2 \) this gives that \( v \) has at least \( \alpha^2 n/8 \) such \( \alpha^4/64)^d \cdot n^{d-1} \geq \alpha^4/64)^{d+1}n^{d-1} \) paths of length \( d + 1 \leq i + 1 \) with endpoint \( u \), and this accomplishes the induction step. With all these facts at hand we finally show the claim of the lemma. Let \( j < 8/\alpha^2 \) be the first index such that \( e(U_j, W) \geq \alpha^3 n^2/2 \) and let \( W' \subseteq W \) be such that \( |N(v) \cap U_j| \geq \alpha^3 n/4 \) for all \( v \in W' \). Thus
\[
\frac{\alpha^3 n^2}{4} \leq e(U_j, W) \leq |W'| n + |W| \frac{\alpha^3 n}{4},
\]
and thus \( |W'| \geq \alpha^3 n/4 \). Then there is \( d \leq j \) and \( W'' \subseteq W' \) such that \( |W''| \geq |W'|/j \) and every \( v \in W'' \) has at least \( \alpha^3 n/(4j) \) neighbours with at least \((\alpha^4/64)^d \cdot n^{d-1} \) paths of length \( d \) with endpoint \( u \). Therefore every \( v \in W'' \) has at least \((\alpha^4/64)^{d+1}n^{d-1} \cdot \alpha^3 n/(4j) \geq (\alpha^4/64)^{d+1}n^{d-1} \) paths of length \( d + 1 \) with endpoint \( u \). This in turn gives that there are at least \( |W''|/j \cdot (\alpha^4/64)^{d+1}n^{d-1} \geq \alpha^3/4 \cdot (\alpha^4/64)^{d+2}n^{d-1} \) paths of length \( d + 2 \) from \( w \) to \( u \), and the proof is completed.

Next comes a technical lemma that given a small set quantifies the number of vertices for which only a small fraction of their neighbourhood intersects that given set.

**Lemma 2.12.** Let \( \mathcal{G} = (G_n)_{n \in \mathbb{N}} = ((V_n, E_n))_{n \in \mathbb{N}} \) be an expander sequence. Let \( \varepsilon > 0 \) and let \( \tilde{\mathcal{G}} = (\tilde{G}_n)_{n \in \mathbb{N}} \), where each \( \tilde{G}_n \) it is obtained from \( G_n \) by deleting edges such that each vertex keeps at least a \((1/2 + \varepsilon)\) fraction of its edges. Let further \( A_n \subseteq V_n \) with \( |A_n| = o(n) \).
(a) There is $B_n \subseteq A_n$ with $|B_n| = (1-o(1))|A_n|$ such that for all $u \in B_n$

$$\frac{|N_{\tilde{G}_n}(u) \cap A_n|}{|N_{\tilde{G}_n}(u)|} = o(1).$$

(b) There is $B_n \subseteq V_n \setminus A_n$ with $|V_n \setminus (A_n \cup B_n)| = o(|A_n|)$ such that for all $v \in B_n$

$$\frac{|N_{\tilde{G}_n}(v) \cap A_n|}{|N_{\tilde{G}_n}(v)|} = o(1).$$

Proof. Let $\delta_n, \Delta_n$ denote the minimum and maximum degree of $G_n$. Lemma 2.9 yields that

$$e_{G_n}(A_n, V_n \setminus A_n) = \frac{\Delta_n|A_n||V_n \setminus A_n|}{n} + o(\Delta_n)|A_n| = (1 + o(1))\Delta_n|A_n|.$$  

As there are a maximum of $\Delta_n|A_n|$ edges with at least one point in $A_n$, we get that $e_{G_n}(A_n) = o(\Delta_n)|A_n|$. Since we obtain $\tilde{G}_n$ from $G_n$ by deleting edges

$$e_{\tilde{G}_n}(A_n) = o(\Delta_n)|A_n|.$$  

(2.3)

With this fact at hand we show a). Let $\eta > 0$ and call a vertex $u \in A_n$ bad if $|N_{\tilde{G}_n}(u) \cap A_n| \geq \eta|N_{\tilde{G}_n}(u)|$. Since $N_{\tilde{G}_n}(u) \geq \delta_n/2$ we obtain for any bad $u$ that $|N_{\tilde{G}_n}(u) \cap A_n| \geq \eta\delta_n/2$. As $\delta_n = (1 - o(1))\Delta_n$ we infer from (2.3) that the number of bad vertices is $o(|A_n|)$.

To see the b) let again $\eta > 0$ and call this time a vertex $v \in V_n \setminus A_n$ bad if $|N_{\tilde{G}_n}(v) \cap A_n| \geq \eta|N_{\tilde{G}_n}(v)|$. Then for any such bad $v$ we know that $|N_{\tilde{G}_n}(v) \cap A_n| \geq \eta\delta_n/2$. As before, using (2.3) we readily get that the number of bad $v$’s is $o(|A_n|)$.

We conclude our preparational section by giving a lemma that bounds crudely the time needed until at least $\omega(1)$ vertices are informed.

Lemma 2.13. Let $0 < \varepsilon \leq 1/2, q \in (0, 1]$ and $G = (G_n)_{n \in \mathbb{N}}$ be an expander sequence. Let $\tilde{G} = (\tilde{G}_n)_{n \in \mathbb{N}}$ be such that each $\tilde{G}_n$ is obtained by deleting edges of $G_n$ such that each vertex keeps at least a $(1/2 + \varepsilon)$ fraction of its edges. Let further $P \in \{\text{push, pull, pp}\}$ and suppose that $|I_t| < \sqrt{\log n}$. Then there is $\tau = o(\log n)$ such that whp $|I_{t+\tau}^{(P)}| \geq \sqrt{\log n}$.

Proof. Recall that the probability that $v \in U_t$ gets informed by pull is $q|N(v) \cap I_t|/|N(v)|$. Thus

$$P_t[|I_{t+1}^{(pull)} \setminus I_t| = 0] = \prod_{u \in N(I_t) \cap U_t} \left(1 - \frac{q|N(u) \cap I_t|}{|N(u)|}\right) \leq e^{-\varepsilon q(I_t, I_t)/\Delta_n}.$$  

Similarly we obtain for push

$$P_t[|I_{t+1}^{(push)} \setminus I_t| = 0] = \prod_{v \in I_t} \left(1 - \frac{|N(v) \cap U_t|}{|N(v)|}\right) \leq e^{-\varepsilon q(I_t, U_t)/\Delta_n}.$$  

The same bound is obviously also true for push&pull. Thus, for all $P \in \{\text{push, pull, pp}\}$

$$P_t[|I_{t+1} \setminus I_t| \geq 1] \geq 1 - e^{-\varepsilon q(I_t, I_t)/\Delta_n}.$$  

As Lemma 2.9 and Lemma 2.10 imply that $e(U_t, I_t) \geq (1 + o(1))\varepsilon \Delta_n|I_t|$, there is $c \in (0, 1)$ such that $P[|I_{t+1}^{(P)} \setminus I_t| \geq 1] > c$. Define $\tau := \lceil 2\sqrt{\log n} \rceil$ and $X = \text{Bin}(\tau, c)$ with $\mathbb{E}[X] = c\tau$ and $\text{Var}[X] = \tau(1-c)c$. Then, using Chebyshev

$$P_t \left[|I_{t+\tau}^{(P)}| \leq \sqrt{\log n} \right] \leq P_t \left[X \leq \sqrt{\log n} \right] \leq P_t \left[|X - \mathbb{E}[X]| \leq \mathbb{E}[X]/2 \right] \leq 4\text{Var}[X]/\mathbb{E}[X]^2 = o(1).$$  

\[\square\]
3 Proofs

3.1 Proof of Theorems 1.4 (b), 1.5 (a) — edge deletions do not slow down pull

Let $0 < \varepsilon \leq 1/2$. In this section we study the runtime of pull in the case in which the input graph is an expander, and where at each vertex at most an $(1/2 - \varepsilon)$ fraction of the edges is deleted. The runtime on expander sequences without edge deletions, that is, the setting in Theorem 1.4 (b), is included as the special case where we set $\varepsilon = 1/2$. In contrast to previous proofs, in the analysis of pull the ‘standard’ approach that consists of showing, for example, that $E_{t}(|I_{t} + 1| - |I_{t}|)$ fails. The main reason is that the graph between $I_{t}$ and $U_{t}$ might be quite irregular, so that, depending on the actual state, $E_{t}(|I_{t+1} \setminus I_{t}|) \approx c|I_{t}|$ for some $c < 1$. However, we discover a different invariant that is preserved, namely that the number of edges between $I_{t}$ and $U_{t}$ behaves in an exponential way. With Lemmas 2.9 and 2.10 we can then relate this to the number of informed vertices.

**Lemma 3.1.** Consider the setting of Theorem 1.5 (a) and let $I_{t} = I_{t}^{(\text{pull})}$.

(a) Let $\sqrt{\log n} \leq |I_{t}| \leq n/\log n$. Then $e(U_{t+1}, I_{t+1}) - (1 + q)e(U_{t}, I_{t}) \leq |I_{t}|^{-1/3}e(U_{t}, I_{t})$ with probability at least $1 - O((|I_{t}|)^{-1/3})$.

(b) Let $|U_{t}| \leq n/\log n$. Then $E_{t}(|U_{t+1}|) = (1 - q + o(1))|U_{t}|$.

**Proof.** We start with (a). Let $D_{t} = e(U_{t+1}, I_{t+1}) - e(U_{t}, I_{t})$ and for $u \in U_{t}$ let $X_{u}$ be the random variable that indicates whether $u$ gets informed in round $t + 1$. Then

$$E_{t}[D_{t}] = \sum_{u \in U_{t}} \sum_{v \in N(u) \cap U_{t}} E_{t}[X_{u}(1 - X_{v})] - \sum_{u \in U_{t}} E_{t}[X_{u}] \cdot |N(u) \cap I_{t}|$$

and

$$= \sum_{u \in U_{t}} q \frac{|N(u) \cap I_{t}|}{|N(u)|} \left( \sum_{v \in N(u) \cap U_{t}} 1 - q \frac{|N(v) \cap I_{t}|}{|N(v)|} \right) - \frac{|N(u) \cap I_{t}|}{|N(u)|}.$$ 

The second sum is at most $|N(u)|$, so obviously $E_{t}[D_{t}] \leq qe(U_{t}, I_{t})$. To get a lower bound consider a largest set $\hat{U} \subseteq U_{t}$ such that $|N(u) \cap I_{t}|/|N(u)| = o(1)$ for all $u \in \hat{U}$. From Lemma 2.12 (b) we obtain that $|U_{t} \setminus \hat{U}| = o(|I_{t}|)$, and so

$$E_{t}[D_{t}] \geq \sum_{u \in \hat{U}} q |N(u) \cap I_{t}| \left( \sum_{v \in N(u) \cap \hat{U}} \frac{1}{|N(u)|} - o \left( \frac{1}{|N(u)|} \right) \right) - \frac{|N(u) \cap I_{t}|}{|N(u)|}.$$ 

Consider furthermore $\hat{U} \subseteq \hat{U}$ such that $|N(u) \cap \hat{U}|/|N(u)| = 1 - o(1)$ and thus also $|N(u) \cap I_{t}|/|N(u)| = o(1)$ for all $u \in \hat{U}$. Lemma 2.12 (b) again yields that we can choose $\hat{U}$ such that $|U_{t} \setminus \hat{U}| = o(|I_{t}|)$, thus

$$E_{t}[D_{t}] \geq (1 - o(1)) \sum_{u \in \hat{U}} q |N(u) \cap I_{t}| \left( \frac{|N(u) \cap \hat{U}|}{|N(u)|} - \frac{|N(u) \cap I_{t}|}{|N(u)|} \right) - \sum_{u \in U_{t} \setminus \hat{U}} |N(u) \cap I_{t}|$$

$$\geq (q - o(1))e(U_{t}, I_{t}) - 2e(U_{t} \setminus \hat{U}, I_{t}).$$ 

According to Lemmas 2.9 and 2.10 we have that $e(U_{t}, I_{t}) = \Theta(|I_{t}|)$. But $e(U_{t} \setminus \hat{U}, I_{t}) \leq |U_{t} \setminus \hat{U}| \Delta_{n} + o(|I_{t}|)$. Thus, $E_{t}[e(U_{t+1}, I_{t+1})] = (1 + q - o(1))e(U_{t}, I_{t})$. In the next step we bound the variance. For each edge $e$ let $X_{e}$ be the indicator random variable that denotes the events that $e \in E(U_{t+1}, I_{t+1})$. Thus

$$e(U_{t+1}, I_{t+1}) = \sum_{e \in E} X_{e} = \frac{1}{2} \sum_{u \in V} \sum_{v \in N(u)} X_{(u,v)}.$$ 

Using that $X_{e}$ and $X_{e'}$ are independent for all $e, e' \in E$ with $e \cap e' = \emptyset$,

$$\operatorname{Var}[e(U_{t+1}, I_{t+1})] = \sum_{e \in E} \operatorname{Var}[X_{e}] = \sum_{e, e' \in E} \operatorname{Var}[X_{e}X_{e'}] - \operatorname{Var}[X_{e}]\operatorname{Var}[X_{e'}]$$

$$\leq \sum_{u \in V} \sum_{v, v' \in N(u)} \operatorname{Var}[X_{(u,v)}] \leq \Delta_{n} \sum_{u \in V} \sum_{v \in N(u)} \operatorname{Var}[X_{(u,v)}] = 2\Delta_{n}.$$


Since $\mathbb{E}_t[e(U_{t+1}, I_{t+1})] = (1 + q - o(1))e(U_t, I_t) = \Theta(\Delta_n|I_t|)$ by Lemmas 2.9 and 2.10 and $\text{Var}[e(U_{t+1}, I_{t+1})] \leq 2\Delta_n\mathbb{E}_t[e(U_{t+1}, I_{t+1})]$ we obtain for $|I_t| \geq \log n$ with Chebychev’s inequality immediately that

$$P \left[ |e(U_{t+1}, I_{t+1}) - \mathbb{E}_t[e(U_{t+1}, I_{t+1})]| \geq e(U_t, I_t)|I_t|^{-1/3} \right] \leq O(|I_t|^{-1/3}).$$

Next we show b).\footnote{We bound the expected number of uninformed vertices after one additional round. \textit{Lemma 2.12 (a) asserts that there is a set $\hat{U} \subseteq U_t$ such that $|\hat{U}| = (1 - o(1))|U_t|$ and $|N(u) \cap I_t|/|N(u)| = 1 - o(1)$ for all $u \in \hat{U}$. Thus,}}

$$\mathbb{E}_t[|U_{t+1}|] = \sum_{u \in U_t} 1 - q \frac{|N(u) \cap I_t|}{|N(u)|} \leq |U_t| - q \sum_{u \in \hat{U}} \frac{|N(u) \cap I_t|}{|N(u)|} = |U_t| - q(1 - o(1))|\hat{U}| = (1 - q - o(1))|U_t|.$$  

As $|N(u) \cap I_t| \leq |N(u)|$ we also have

$$\mathbb{E}_t[|U_{t+1}|] = \sum_{u \in U_t} 1 - q \frac{|N(u) \cap I_t|}{|N(u)|} \geq \sum_{u \in U_t} (1 - q) = (1 - q)|U_t|.$$ 

\[
\square
\]

\textbf{Lemma 3.2 and Lemma 2.13 give lower bounds, that together with an upper bound provided by Lemma 3.3 imply Theorems 1.4 (b) and 1.5 (a).}

\textbf{Lemma 3.2 (Upper bound in Theorem 1.5 (a)). Consider the setting of Theorem 1.5 (a) and let $I_t = I_t^{(\text{pull})}$, then the following statements hold whp.}

\begin{enumerate}[\textit{(a)}]
\item Let $\sqrt{\log n} \leq |I_t| \leq n/\log n$. Then there are $\tau_1, \tau_2 = \log_{1+q}(n/|I_t|) + o(\log n)$ such that $|I_{t+\tau_1}| < n/\log n < |I_{t+\tau_2}|$.
\item Let $n/\log n \leq |I_t| \leq n - n/\log n$. Then there is $\tau = o(\log n)$ such that $|I_{t+\tau}| > n - n/\log n$.
\item Let $|I_t| \geq n - n/\log n$.
\end{enumerate}

\begin{enumerate}
\item Case $q = 1$: Then there is $\tau = o(\log n)$ such that $|I_{t+\tau}| = n$.
\item Case $q \neq 1$: Then there is $\tau \leq -\log n/\log (1 - q) + o(\log n)$ such that $|I_{t+\tau}| = n$.
\end{enumerate}

\textit{Proof.} We start with a). Let $|I_t| \in [\log n, n/\log n]$. First note that any bound on $e(U_t, I_t)$ translates to a bound for $|I_t|$, as with Lemmas 2.9, 2.10 we obtain

$$\left(1 - o(1)\right)\Delta_n|I_t| \leq e(U_t, I_t) \leq \Delta_n|I_t|.$$ 

In particular, up to constant factors, $|I_t|$ is $e(U_t, I_t)/\Delta_n$ and vice versa. From Lemma 3.1 (a) we obtain that $e(U_{t+1}, I_{t+1}) = (1 + q - o(1)|I_t|^{-1/3})e(U_t, I_t)$ with probability $1 - O(|I_t|^{-1/3})$. Proceeding as in Examples 2.7 and 2.8, where we replace the events “$|I_t| \geq \mathbb{E}_{t-1} [\frac{|I_t|}{\mathbb{E}_{t-1} [\frac{|I_t|}{2}] / \Delta_n}$ or $|I_t| \geq n/g(n)$” and “$|I_t| - \mathbb{E}_{t-1} [\frac{|I_t|}{2}] \leq \mathbb{E}_{t-1} [\frac{|I_t|}{2}] / 3\Delta_n$” and “$e(U_t, I_t) \geq (1 + q - |I_t|^{-1/3})e(U_{t-1}, I_{t-1})$ or $|I_t| \geq n/\log n$” and “$e(U_{t+1}, I_{t+1}) = (1 + q \pm |I_t|^{-1/3})e(U_t, I_t)$” we obtain the statement.

We continue with b). Consider first the case $|I_t| \in [n/\log n, n/2]$. Using Lemmas 2.9, 2.10, i.e. $e(U_t, I_t) \geq e(U_t||I_t|\Delta_n/n + o(\Delta_n)|I_t|)$, together with $|I_t| \geq n/2$ implies

$$\mathbb{E}_t[|U_{t+1}\setminus I_t|] = \sum_{u \in U_t} q \frac{|N(u) \cap I_t|}{|N(u)|} \geq q \frac{e(U_t, I_t)}{\Delta_n} \geq \frac{q e(U_t||I_t|\Delta_n/n + o(\Delta_n)|I_t|)}{\Delta_n(1 + o(1))} \geq \left(\frac{2q}{2} + o(1)\right)|I_t|.$$ 

Proceeding as in Example 2.7, where we set $g = 2, f = \log n$ and $c = q/2 + o(1)$, we are finished with this part as well. Now let $|I_t| \in [n/2, n - n/\log n]$. We switch our focus to the set of uninformed vertices. Setting again that $e(U_t, I_t) \geq e(U_t||I_t|\Delta_n/n + o(\Delta_n)|I_t|)$, we have

$$\mathbb{E}_t[|U_{t+1}|] = \sum_{u \in U_t} 1 - q \frac{|N(u) \cap I_t|}{|N(u)|} = \sum_{u \in U_t} 1 - q \frac{|N(u) \cap I_t|}{\Delta_n(1 + o(1))} = |U_t| - \frac{q e(U_t, I_t) \Delta_n/n + o(\Delta_n)|I_t|}{\Delta_n(1 + o(1))} \left(1 - \frac{q}{2} + o(1)\right)|I_t|.$$ 

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Inductively we obtain for any integer $\tau \geq 1$ the bound $\mathbb{E}_t [|U_{t+\tau}]| \leq (1 - q\varepsilon/2 + o(1))^\tau |U_t|$, and so for some $\tau := 2\log \log n / \log(1/(1 - q\varepsilon/2 + o(1))) = o(\log n)$ we have

$$\mathbb{E}_t [|U_{t+\tau}]| \leq |U_t|/\log^2 n = o(n/\log n).$$

Hence, by Markov’s inequality, $P_t[|U_{t+\tau}]| \geq n/\log n] = o(1)$.

In order to show c) let $|I_t| \in [n \cdot n/\log n, n]$. As for $q = 1$ the term $1 - q$ in Lemma 3.1 (b) vanishes, we distinguish the cases $q = 1$ and $q \neq 1$. We start with $q = 1$. By induction, it follows that for any round $\tau > 0$ and suitable $f = o(1)$,

$$\mathbb{E}_t [|U_{t+\tau}]| \leq (f(n))^\tau |U_t|.$$

We choose $\tau = \log_{1/f(n)}(n) = o(\log n)$, as $1/f = \omega(1)$. Hence we obtain $\mathbb{E}_t [|U_{t+\tau}]| \leq |I_t|/n \leq 1/\log n$.

Therefore we have $P_t[|U_{t+\tau}]| \geq 1] \leq o(1)$ by Markov’s inequality. For $q \neq 1$ we have by induction, for any number of rounds $\tau \geq 1$,

$$\mathbb{E}_t [|U_{t+\tau}]| \leq (1 - q + o(1))^\tau |U_t|.$$

We choose $\tau = \log_{1/(1-q+o(1))}(n) = -\log n/\log(1-q) + o(\log n)$. Thus using Markov’s inequality, analogously to the case $q = 1$, we obtain the desired upper bound.

Note that for $q = 1$ this already implies Theorems 1.4 (b), 1.5 (a). This leaves the case for $q \neq 1$.

**Lemma 3.3.** Let $0 < \varepsilon \leq 1/2$, $q \in (0, 1]$ and $\mathcal{G} = (G_n)_{n \in \mathbb{N}}$ be an expander sequence. Let $\tilde{G} = (\tilde{G}_n)_{n \in \mathbb{N}}$ be such that each $\tilde{G}_n$ is obtained by deleting edges of $G_n$ such that each vertex keeps at least a $1/(2+\varepsilon)$ fraction of its edges and abbreviate $I_t = I_t^{(\text{push})}$. Let further $q \in (0, 1)$ and $|I_t| \leq n/2$. Then for $\tau = -\log n/\log (1 - q)$ and all $c < 1$ w.h.p $|I_{t+\tau}| < n$.

**Proof.** We consider a modified process in which vertices have a higher chance of getting informed. In particular, note that the probability that $u \in U_t$ gets informed is at most $q|N(u) \cap I_t|/|N(u)| \leq q$ and that all these events are independent; now we assume that each such $u$ gets independently informed with probability exactly $q$. Then the runtime in this modified model constitutes a lower bound for the runtime in the original model.

Let $c < 1, u \in U_t$ and $E_u$ be the event that $u$ does not get informed in $c\tau$ rounds in this model. Thus

$$P[E_u] = (1 - q)^{c\tau} = (1 - q)^{-c\log n/\log(1-q)} = n^{-c} = \omega(1/n)$$

and as the events $E_u$ are independent and $|U_t| = \Theta(n)$

$$P \left[ \bigwedge_{u \in U_t} \overline{E_u} \right] \leq \prod_{u \in U_t} P[E_u] \leq \exp \left( -\sum_{u \in U_t} P[E_u] \right) = o(1).$$

**3.2 Proof of Theorem 1.4 (c) — push&pull is fast on expanders**

As we are now in the case without edge deletions, we begin with a lemma that determines the expected number of informed vertices in one round. Intuitively we will show that push and pull do not interact badly and therefore push&pull is given as a straightforward combination of push and pull.

**Lemma 3.4.** Let $\mathcal{G}$ be an expander sequence and abbreviate $I_t = I_t^{(\text{pp})}$.

(a) Let $|I_t| \leq n/\log n$. Then $\mathbb{E}_t[|I_{t+1} \setminus I_t|] = (2q + o(1))|I_t|.$

(b) Let $|U_t| \leq n/\log n$. Then $\mathbb{E}_t[|U_{t+1}|] = (1 + o(1))e^{-q}(1-q)|U_t|.$
Proof. To begin with a). The probability that \( v \in U_t \) gets informed by pull is \( q|N(v) \cap I_t|/|N(v)| \). Thus, using Lemma 2.9

\[
\mathbb{E}_t[|f_{t+1}^{(pull)} \setminus I_t|] = \sum_{u \in U_t} q \frac{|N(u) \cap I_t|}{|N(u)|} = q \sum_{u \in U_t} \frac{|N(u) \cap I_t|}{\Delta_n(1 + o(1))}
= (q + o(1)) \frac{e(U_t, I_t)}{\Delta_n} = (q + o(1)) \frac{|U_t||I_t|/\Delta_n}{n + o(\Delta_n)|I_t|}.
\]

(3.2)

Since \(|I_t| = o(n)|I_t| = (1 - o(1))n\) and this expression simplifies to \((q + o(1))|I_t|\). The probability that \( v \in U_t \) gets informed by push is \( 1 - \prod_{i \in N(v) \cap I_t}(1 - 1/|N(v)|) \). Using \( e^{-1/n + o(1)} = 1 - 1/n, e^{-1/n} = 1 - 1/n + o(1/n) \), and \(|I_t| = o(n)\) we obtain in a similar fashion

\[
\mathbb{E}_t[|f_{t+1}^{(push)} \setminus I_t|] = \sum_{u \in U_t} 1 - \prod_{i \in N(u) \cap I_t} \left(1 - \frac{q}{|N(i)|}\right) = \sum_{u \in U_t} 1 - \exp \left(-(1 - o(1))\frac{q|N(u) \cap I_t|}{\Delta_n}\right)
= q \sum_{u \in U_t} \frac{|N(u) \cap I_t|}{\Delta_n(1 + o(1))} = (q + o(1))|I_t|.
\]

(3.3)

We express the expected number of vertices informed by push/pull after one additional round in terms of the expected values we just calculated (3.2) and (3.3)):

\[
\mathbb{E}_t[|I_{t+1} \setminus I_t|] = \mathbb{E}_t \left[|f_{t+1}^{(pull)} \setminus I_t| + |f_{t+1}^{(push)} \setminus I_t| - (|f_{t+1}^{(push)} \setminus I_t| \cap (|f_{t+1}^{(pull)} \setminus I_t|))\right]
= (2q - o(1))|I_t| - \mathbb{E}_t \left[(|f_{t+1}^{(push)} \setminus I_t| \cap (|f_{t+1}^{(pull)} \setminus I_t|))\right].
\]

(3.4)

Lemma 2.12 (a) gives a set \( I \subseteq f_{t+1}^{(push)}, |I| = (1 - o(1))|f_{t+1}^{(push)}| \), such that \(|N(u) \cap I| = o(1)|N(u)|\) for all \( u \in I \). Since push and pull happen independently

\[
\mathbb{E}_t \left[(|f_{t+1}^{(pull)} \setminus I_t| \cap (|f_{t+1}^{(push)} \setminus I_t|))\right] = \sum_{u \in I} P_t[u \in I_{t+1}^{(pull)} \setminus I_t] = \sum_{u \in |I_{t+1}^{(push)} \setminus I_t|} \frac{|N(u) \cap I_t|}{|N(u)|}
\leq \sum_{u \in I} \frac{|N(u) \cap I_t|}{|N(u)|} + \sum_{u \in f_{t+1}^{(push)} \setminus I_t} \frac{q|N(u) \cap I_t|}{|N(u)|}.
\]

Using that \(|N(u) \cap I_t| = o(|N(u)|)\) for all \( u \in I \) we obtain

\[
\mathbb{E}_t \left[(|f_{t+1}^{(pull)} \setminus I_t| \cap (|f_{t+1}^{(push)} \setminus I_t|))\right] \leq \mathbb{E}_t[|I(t)| + |I_{t+1}^{(push)} \setminus I_t|] = o(|I_t|),
\]

as \(|I| \leq |I_{t+1}^{(push)}| \leq 2|I_t|\) and \(|I_{t+1}^{(push)} \setminus I_t| = o(|I_{t+1}^{(push)}|) = o(|I_t|)\). Combining this with (3.4) we get

\[
\mathbb{E}_t[|I_{t+1} \setminus I_t|] = (2q + o(1))|I_t|, \text{ as claimed.}
\]

Next we show b). Let \( A_u \) be the event that an uninformed vertex \( u \) does not get informed by the push algorithm, let \( B_u \) be the corresponding event for pull. Then \( A_u \) and \( B_u \) are independent and \( A_u \cap B_u \) is the event that \( u \) does not get informed in the current round. We obtain

\[
P_t[A_u] = \prod_{i \in N(u) \cap I_t} \left(1 - \frac{q}{|N(i)|}\right) \leq \left(1 - q/\Delta_n\right)^{|N(u) \cap I_t|} \leq \exp \left(-q|N(u) \cap I_t|/\Delta_n\right) = \exp \left(-q|N(u) \cap I_t|\right)/(1 + o(1)|N(u)|)
\]

and

\[
P_t[B_u] = 1 - \frac{q|N(u) \cap I_t|}{|N(u)|}.
\]

According to Lemma 2.12 (a) there is a set \( U \subseteq U_t, |U| = (1 - o(1))|U_t| \) such that \(|N(u) \cap I_t| = (1 - o(1))|N(u)|\) for all \( u \in U \). As \( P_t[A_u \cap B_u] \leq 1 \) we get therefore

\[
\mathbb{E}_t[|U_{t+1}|] = \sum_{u \in U_t} P_t[A_u \cap B_u] \leq \sum_{u \in U_t} P_t[A_u] \cdot P_t[B_u] + |U_t \setminus U| \leq (1 + o(1))e^{-q}(1 - q)|U_t|.
\]
For the lower bound we need to find a lower bound on the probability of a single uninformed vertex not getting informed in one round by \textit{push}. Indeed, for any \( u \in U_t \) and sufficiently large \( n \)

\[
P_t[A_u] = \prod_{v \in N(u) \cap I_t} \left( 1 - \frac{q}{|N(v)|} \right) \geq \left( 1 - \frac{q}{\delta_n} \right)^{|N(u) \cap I_t|} \geq e^{-q\Delta_n/\delta_n}. \tag{3.5}
\]

Combining this inequality with the trivial bound \( P[B_u] \geq 1 - q \), we get a lower bound on the expected number of uninformed vertices after one round using \textit{push}\textit{\textunderscore}\textit{pull}:

\[
E_t[|U_{t+1}|] = \sum_{u \in U_t} P_t[A_u \cap B_u] = \sum_{u \in U_t} P_t[A_u] \cdot P_t[B_u] \geq e^{-q\Delta_n/\delta_n} (1 - q) |U_t| = (1 + o(1)) e^{-q}(1 - q) |U_t|.
\]

Next we show upper and lower bounds that together with Lemma 2.13 imply Theorem 1.4 (c).

**Lemma 3.5.** Let \( G \) be an expander sequence and abbreviate \( I_t = I_t^{(pp)} \). Let \( q \in (0,1] \). Then the following statements hold whp.

\begin{enumerate}[a)]
\item Let \( \sqrt{\log n} \leq |I_t| \leq n/\log n \). Then there are \( \tau_1, \tau_2 = \log_{1+2q}(n/|I_t|) + o(\log n) \) such that \( |I_{t+\tau_1}| < n/\log n < |I_{t+\tau_2}| \).
\item Let \( n/\log n \leq |I_t| \leq n - n/\log n \). Then there is \( \tau = o(\log n) \) such that \( |I_{t+\tau}| > n - n/\log n \).
\item Let \( |I_t| \geq n - n/\log n \).
\end{enumerate}

\begin{enumerate}[1.]
\item Case \( q = 1 \): Then there is \( \tau = o(\log n) \) such that \( |I_{t+\tau}| = n \).
\item Case \( q \neq 1 \): Then there is \( \tau \leq \log n/(q - \log(1 - q)) + o(\log n) \) such that \( |I_{t+\tau}| = n \).
\end{enumerate}

**Proof.** Since \( |I_t| \geq |I_t^{(pp)}| \) the statements b) and c) for \( q = 1 \) follow immediately from Lemma 3.2. To see a), note that by using Lemma 3.4 we get \( E_t[|I_{t+1}|/I_t] = (2q + o(1))|I_t| \), and proceeding as in Example 2.8 implies the claim.

Finally we show c) for \( q \neq 1 \). Let \( |I_t| \geq n - n/\log n \). By Lemma 3.4, we obtain that for any \( \tau \in \mathbb{N} \)

\[
E_t[|U_{t+\tau}|] = ((1 + o(1)) e^{-q(1 - q)})^\tau |U_t|.
\]

Thus we may choose \( \tau = \log n/(q - \log(1 - q)) + o(\log n) \) such that, say, \( E_t[|U_{t+\tau}|] \leq |U_t|/n \leq 1/\log n \). Thus \( P_t[|U_{t+\tau}| \geq 1] \leq o(1) \) by Markov’s inequality.

Note that for \( q = 1 \) this already implies Theorem 1.4 (c). This leaves the case for \( q \neq 1 \).

**Lemma 3.6.** Let \( G \) be an expander sequence and abbreviate \( I_t = I_t^{(pp)} \), let \( q \in (0,1) \) and \( |I_t| \leq n/2 \). Then for \( \tau = \log n/(q - \log(1 - q)) \) and all \( c < 1 \) whp \( |I_{t+c\tau}| < n \).

**Proof.** We consider a modified process in which vertices have a higher chance of getting informed. In particular, note that the probability that \( u \in U_t \) gets informed by \textit{pull} is at most \( q|N(u) \cap I_t|/|N(u)| \leq q \) and that all these events are independent; according to (3.5) the probability that \( u \in U_t \) gets informed by \textit{push} is at most \( 1 - e^{-q\Delta_n/\delta_n} \). Now we assume that each such \( u \) gets independently informed with probability exactly \( 1 - e^{-q\Delta_n/\delta_n} \). Then the runtime in this modified model constitutes a lower bound for the runtime in the original model. Let \( u \in U_t \) and \( E_u \) be the event that \( u \) does not get informed in this modified model in \( c\tau \) rounds. Thus for \( c < 1 \),

\[
P[E_u] \geq ((1 - q)e^{-q\Delta_n/\delta_n})^c \tau = \omega (n^{-1})
\]

and as the events \( E_u \) are independent and \( |U_t| = \Theta(n) \)

\[
P \left( \bigwedge_{u \in U_t} \overline{E_u} \right) \leq \prod_{u \in U_t} P[\overline{E_u}] \leq \exp \left( - \sum_{u \in U_t} P[E_u] \right) = o(1).
\]

\[\square\]
3.3 Proof of Theorem 1.7 (a) — push informs almost all vertices fast in spite of edge deletions

To shorten the notation let us call the setting with deleted edges “new model” and the setting without “old model”, that is, the term new model corresponds to the graphs in \( \mathcal{G} \), while old model refers to the (original) graphs in \( G \). We prove Lemma 3.7 that directly implies Theorem 1.7 (a). We write \( I_t = I_t^{(\text{push})} \) throughout.

**Lemma 3.7.** Under the assumptions of Theorem 1.7 (a) the following holds for the new model:

a) There are \( \tau, \tilde{\tau} = \log_{1+q}(n) + o(\log n) \) such that whp \( |I_\tau| < n/\log n < |I_{\tilde{\tau}}| \).

b) Assume \( |I_t| \geq n/\log n \). Then there is a \( \tau = o(\log n) \) such that whp \( |I_{t+\tau}| \geq n - n/\log n \).

For the proof of Lemma 3.7 we will use the simple observations that for any \( I_t \), we will need the following statements, the first one taken from [27].

**Lemma 3.8** (Proof of Lemma 2.5 in [27]). Consider the old model. Assume \( |I_t| < n/\log n \) and \( q = 1 \). Then

\[
P_t[|I_{t+1}| = |I_t| + (1 - o(1))|I_t|] = 1 - o(1). \tag{3.6}
\]

**Lemma 3.9.** Consider push on a sequence of graphs \( (G_n)_{n \in \mathbb{N}} \), where \( G_n \) has \( n \) vertices. Assume that \( |I_t| = \omega(1) \) and that (3.6) holds for \( q = 1 \), that is, assume that \( P_t[|I_{t+1}| = |I_t| + (1 - o(1))|I_t|] = 1 - o(1) \) for \( q = 1 \). Then for \( q \in (0, 1] \)

\[
P_t[|I_{t+1}| = |I_t| + (q - o(1))|I_t|] = 1 - o(1). \tag{3.7}
\]

Moreover, assume that whenever \( |I_t| < n/\log n \), for \( q = 1 \), (3.6) holds. Then there are \( \tau, \tilde{\tau} = \log_{1+q}(n) + o(\log n) \) such that whp

\[
|I_{t+\tau}| < n/\log n < |I_{\tilde{\tau}}|. \tag{3.8}
\]

**Proof.** For a graph \( G \) and for \( v \in I_t \) let \( X_v(G) \) denote the vertex to which \( v \) pushes in round \( t \). Let

\[
N_{t+1} := \{ X_v(G_n) \mid v \in I_t \} \cap U_t.
\]

Note that whenever \( |I_t| < n/\log n \) whp \( |N_{t+1}| = (1 - o(1))|I_t| \) from (3.6). For \( q \in (0, 1] \) each vertex in \( N_{t+1} \) has a probability of at least \( q \) to get informed and all these events are independent; thus (3.7) follows directly by applying the Chernoff bounds whenever \( |I_t| = \omega(1) \).

In order to prove the second statement we call a round \( t \) that does not satisfy (3.7) a failed round. Note that we just argued that the probability that a round fails is \( o(1) \) whenever \( |I_t| = \omega(1) \) and \( |I_t| < n/\log n \), and the events that distinct rounds fail are independent. In particular, the number of failed rounds among the next \( R \) rounds, assuming that \( |I_t| \) stays below \( n/\log n \), is whp \( o(R) \). Moreover, if a round does not fail, the number of informed vertices increases by a factor of \( (1 + q + o(1)) \) and otherwise it may increase by an arbitrary factor in the interval \([1, 2]\). Finally, Lemma 2.13 yields that there is \( t^* = o(\log n) \) such that whp \( |I_{t^*}| = \omega(1) \), which implies that after \( R + t^* \) rounds, the number of informed vertices is whp in the interval

\[
[(1 + q + o(1))^{R - o(R)}, (1 + q + o(1))^{R - o(R)} \cdot 2^{o(R)}]
\]

and choosing \( R = \log_{1+q}(n) + o(\log n) \) in two ways establishes (3.8).

In the subsequent proof of Lemma 3.7 we will use the simple observations that for any \( n \in \mathbb{N} \)

\[
P[\text{Bin}(n, 1/2) \geq n/2] \geq 1/2 \quad \text{and} \quad P[\text{Bin}(n, 1/4) \geq n/4] \geq 1/4.
\]

see for example [21] when \( n > 4 \), and the other cases are checked easily.

**Proof of Lemma 3.7.** We first show a). We assume \( q = 1 \) and prove that, for \( |I_t| < n/\log n \), (3.6) also holds in the new model; then claim a) follows directly from Lemma 3.9. Let \( G = (V, E) \) be a graph. For \( v \in I_t \) let \( X_v(G) \) denote the vertex to which \( v \) pushes in round \( t \). For \( u \in V \) let \( c_u(G) := |\{ v \in I_t \mid X_v(G) = u \} \) denote the number of times \( u \) is pushed in round \( t \). Let

\[
\mathcal{Y}_t(G) := \{ v \in I_t \mid c_v(G) = 1 \} \quad \text{and} \quad \mathcal{H}_t(G) := \{ v \in I_t \mid c_v(G) \geq 1 \}
\]
denote the set of informed vertices that are being pushed exactly once in round \( t \) and the set of informed vertices that are being pushed at least once in round \( t \) respectively. Let
\[
Z_t(G) := \{ v \in V \mid c_v(G) \geq 2 \}
\]
denote the set of vertices that are being pushed more than once in round \( t \). Let \( Y_t(G) := |\mathcal{Y}_t(G)| \) and \( H_t(G) := |\mathcal{H}_t(G)| \) and, in slight abuse of notation, let \( Z_t(G) := \sum_{k \geq 2} (k-1) \cdot |\{ v \in V \mid c_v(G) = k \}| \) denote the number of vertices that are being pushed multiple times in round \( t \) counted with multiplicity. Note that the quantity \( Y + Z \) denotes the number of pushes that have no effect in the respective round, i.e., there are \( Y + Z \) pushes that are useless in the sense that even without them, the same number of vertices would become informed in the respective round. In the following paragraphs we condition on \( I_t \) implicitly, that is, we write \( P[\ldots] \) instead of \( P_I[\ldots] \) etc. to lighten the notation. We want to show that (3.6) does hold in the new model; for contradiction we assume that this is not the case. Hence we can infer that there is a constant \( c > 0 \) such that
\[
\limsup_{n \to \infty} P[Y_t(\tilde{G}_n) \geq c|I_t|] > 0 \quad \text{or} \quad \limsup_{n \to \infty} P[Z_t(\tilde{G}_n) \geq c|I_t|] > 0.
\]
Thus, w.l.o.g., we can assume that there is \( f^* > 0 \) and \( n_0 \in \mathbb{N} \) such that
\[
P[Y_t(\tilde{G}_n) \geq c|I_t|] > f^* \quad \text{for all} \quad n \geq n_0 \quad \text{or} \quad \quad P[Z_t(\tilde{G}_n) \geq c|I_t|] > f^* \quad \text{for all} \quad n \geq n_0;
\]
if this is not the case we can restrict ourselves to a suitable subsequence of \((n)_{n \in \mathbb{N}}\) on which it is true. Next, we describe an explicit coupling between the new and the old model. For any vertex \( v \) consider \( X_v(G_n) \). If \( X_v(G_n) \in \mathcal{N}_{\tilde{G}_n}(v) \), then set \( X_v(\tilde{G}_n) := X_v(G_n) \) and otherwise choose \( X_v(\tilde{G}_n) \) uniformly at random from \( \mathcal{N}_{\tilde{G}_n}(v) \). Note that \( X_v(G_n), X_v(\tilde{G}_n) \) have by construction the correct marginal distribution. Moreover, note that by construction, the family
\[
\left( X_v(G_n) \mid (X_u(\tilde{G}_n))_{u \in V_n} \right)_{v \in V_n}
\]
of random variables is independent, since \( X_v(G_n) \) depends only on \( X_v(\tilde{G}_n) \) for all \( v \in V_n \).

We begin with the case that \( P[Y_t(\tilde{G}_n) \geq c|I_t|] > f^* \). We will show
\[
P[H_t(G_n) \geq Y_t(\tilde{G}_n)/2 \mid Y_t(\tilde{G}_n)] \geq 1/2
\]
and then, since by assumption \( P[Y_t(\tilde{G}_n) \geq c|I_t|] > f^* \), we can infer \( P[H_t(G_n) \geq c|I_t|/2] \geq f^*/2 \) which contradicts Lemma 3.8. Let \( \mathcal{Y}_t(\tilde{G}_n) = \{ y_1, \ldots, y_{Y_t(\tilde{G}_n)} \} \), then there are distinct vertices \( v_1, \ldots, v_{Y_t(\tilde{G}_n)} \in I_t \) such that \( X_{v_i}(\tilde{G}_n) = y_i \) for all \( i \in \{1, \ldots, Y_t(\tilde{G}_n)\} \). Due to (3.10) the events \( \{ X_{v_i}(G_n) = X_{v_i}(\tilde{G}_n) \} \) are independent. Moreover, for all \( i \in \{1, \ldots, Y_t(\tilde{G}_n)\} \),
\[
P[X_{v_i}(G_n) = X_{v_i}(\tilde{G}_n) \mid Y_t(\tilde{G}_n)] = \frac{d_{\tilde{G}_n}(v_i)}{d_{G_n}(v_i)} \geq 1/2 + \varepsilon
\]
and therefore, given \( \mathcal{Y}_t(\tilde{G}_n) \), \( H_t(G_n) \) dominates a binomially distributed random variable \( \text{Bin}(Y_t(\tilde{G}_n), 1/2) \). In particular, this implies with (3.9) that \( P[H_t(G_n) \geq Y_t(\tilde{G}_n)/2 \mid Y_t(\tilde{G}_n)] \geq 1/2 \), as claimed.

We continue with the case \( P[Z_t(\tilde{G}_n) \geq c|I_t|] > f^* \). Let \( \mathcal{Z}_t(\tilde{G}_n) = \{ z_1, \ldots, z_{Z_t(\tilde{G}_n)} \} \). Then, for any \( i \in \{1, \ldots, Z_t(\tilde{G}_n)\} \) let \( n_i := c_{z_i}(\tilde{G}_n) \geq 2 \), that is, there are distinct vertices \( v_{i_1}, \ldots, v_{i_{n_i}} \) such that \( X_v(\tilde{G}_n) = z_i \) for all \( v \in \{v_{i_1}, \ldots, v_{i_{n_i}}\} \). We will show that
\[
P[Z_t(G_n) \geq Z_t(\tilde{G}_n)/2 \mid Z_t(G_n), n_1, \ldots, n_{Z_t(\tilde{G}_n)}] \geq 1/8
\]
and then, since by assumption \( P[Z_t(\tilde{G}_n) \geq c|I_t|] > f^* \), we obtain \( P[Z_t(G_n) \geq c/8|I_t|] \geq f^*/8 \) which contradicts Lemma 3.8. Due to (3.10) the events
\[
\{X_{v_{i,j}}(G_n) = X_{v_{i,j}}(\tilde{G}_n)\} \quad \text{are independent. Moreover, for all} \quad 1 \leq i \leq |Z_t(\tilde{G}_n)|, 1 \leq j \leq n_i,
\]
\[
P_X \left[ X_{v_{i,j}}(G_n) = X_{v_{i,j}}(\tilde{G}_n) \mid Z_t(\tilde{G}_n), n_1, \ldots, n_{Z_t(\tilde{G}_n)} \right] = \frac{d_{\tilde{G}_n}(v_{i,j})}{d_{G_n}(v_{i,j})} \geq 1/2 + \varepsilon.
\]
For $1 \leq i \leq |Z_t(\tilde{G}_n)|$ let $B_i \sim \text{Bin}(n_i, 1/2)$ be independent random variables. Moreover, let $M_1 := \{i \mid 1 \leq i \leq |Z_t(\tilde{G}_n)|, n_i = 2\}$ and $M_2 := \{i \mid 1 \leq i \leq |Z_t(\tilde{G}_n)|, n_i > 2\}$. Using (3.12) and (3.13), given $Z_t(\tilde{G}_n)$, $n_1, \ldots, n_{|Z_t(\tilde{G}_n)|}$, we infer that $Z_t(G_n)$ dominates

$$
\sum_{i=1}^{|Z_t(\tilde{G}_n)|} \max\{B_i - 1, 0\} \geq \sum_{i \in M_1} \max\{B_i - 1, 0\} + \sum_{i \in M_2} B_i - |M_2|.
$$

We treat the two sums individually. Note that $\sum_{i \in M_1} \max\{B_i - 1, 0\} \sim \text{Bin}(|M_1|, 1/4)$; in particular, $P[\sum_{i \in M_1} \max\{B_i - 1, 0\} \geq |M_1|/4] \geq 1/4$ by (3.9). Regarding the second sum, since $\sum_{i \in M_2} B_i \sim \text{Bin}(\sum_{i \in M_2} n_i, 1/2)$ we obtain $P[\sum_{i \in M_2} B_i \geq 1/2 \sum_{i \in M_2} n_i] \geq 1/2$. Thus, given $Z_t(\tilde{G}_n), n_1, \ldots, n_{|Z_t(\tilde{G}_n)|}$ and using $2|M_1| = \sum_{i \in M_1} n_i$ and $\sum_{i \in M_2} n_i \geq 3|M_2|$, we infer that with probability at least $1/4 \cdot 1/2 = 1/8$

$$
Z_t(G_n) \geq \frac{1}{4}|M_1| + \frac{1}{2} \sum_{i \in M_2} n_i - |M_2| = \frac{1}{8} \sum_{i \in M_1} n_i + \frac{1}{2} \sum_{i \in M_2} n_i - |M_2| \geq \frac{1}{8} \sum_{i \in M_1} n_i + \frac{1}{6} \sum_{i \in M_2} n_i
$$

This establishes (3.11). All in all, for $q = 1$ we have shown that (3.6) does also hold in the new model. Hence claim a) follows directly from Lemma 3.9.

Next we prove claim b). We write $\Delta_n := \Delta(G_n), \tilde{\Delta}_n := \Delta(\tilde{G}_n), \delta_n := \delta(G_n)$ and $\tilde{\delta}_n := \delta(\tilde{G}_n)$; moreover we write $\tilde{N}(\cdot)$ instead of $N_{G_n}(\cdot)$. We assume that $|I_t| \in [n/\log n, n - n/\log n]$. We further distinguish two cases, namely $|I_t| \in [n/\log n, n/2]$ and $|I_t| \in [n/2, n - n/\log n]$. We start with the case $|I_t| \in [n/\log n, n/2]$. Using Lemmas 2.9 and 2.10 and the assumption that $\Delta_n/\delta_n = 1 + o(1)$ we obtain, for any $0 < \bar{\varepsilon} < \varepsilon/2$, for $n$ sufficiently large,

$$
e(I_t, U_t) > \bar{\varepsilon}\delta_n|I_t|.
$$

Using that $e^{\bar{\varepsilon}} \geq (1 + x/n)^n$ for $n \in \mathbb{N}$ and $|x| \leq n$ we obtain

$$
\mathbb{E}_t[|I_{t+1}\setminus I_t|] \geq \sum_{u \in \tilde{N}(I_t)\setminus I_t} \left( 1 - \prod_{v \in \tilde{N}(u) \cap I_t} \left( 1 - \frac{q}{\Delta_n} \right) \right) \geq \sum_{u \in \tilde{N}(I_t)\setminus I_t} 1 - e^{-|\tilde{N}(u) \cap I_t|/|I_t|/\Delta_n}.
$$

Further, using that $e^{-x} \leq 1 - x/2$ for any $x \in (0, 1)$ and (3.14) yields the bound

$$
\mathbb{E}_t[|I_{t+1}\setminus I_t|] \geq \sum_{u \in \tilde{N}(I_t)\setminus I_t} \frac{q|\tilde{N}(u) \cap I_t|}{2\Delta_n} = \frac{q(e(I_t, U_t))}{2\Delta_n} \geq \frac{\bar{\varepsilon}q\delta_n}{2\Delta_n} |I_t|.
$$

For this claim the followings apply by Example 2.7, when setting $f = n/\log n, g = \log n$ and $e = \bar{\varepsilon}\delta_n/(2\Delta_n)$.

Finally we consider the case $|I_t| \in [n/2, n - n/\log n]$; here we examine the shrinking of $U_t$. Using Lemmas 2.9 and 2.10 we obtain, for any $0 < \bar{\varepsilon} < \varepsilon/2$, for $n$ sufficiently large, $e(I_t, U_t) > \bar{\varepsilon}\delta_n|U_t|$. Hence, again using that for any $x \in (0, 1)$ it holds $e^{-x} \leq 1 - x/2$ and that for $n \in \mathbb{N}$ and $|x| \leq n$ it is $e^x \geq (1 + x/n)^n$, we obtain

$$
\mathbb{E}_t[|U_{t+1}|] = \sum_{u \in U_t} \prod_{v \in \tilde{N}(u) \cap I_t} \left( 1 - \frac{q}{\delta_n(u)} \right) \leq \sum_{u \in U_t} e^{-|\tilde{N}(u) \cap I_t|/\delta_n} \leq \sum_{u \in U_t} 1 - \frac{q|\tilde{N}(u) \cap I_t|}{2\Delta_n} \leq |U_t| - \frac{\bar{\varepsilon}q\delta_n}{2\Delta_n} |U_t| \leq \left[ 1 - \frac{\bar{\varepsilon}q\delta_n}{2\Delta_n} \right] |U_t|.
$$

Using the tower property of conditional expectation we immediately get

$$
\mathbb{E}_t[|U_{t+\tau}|] \leq \left( 1 - \frac{\bar{\varepsilon}q\delta_n}{2\Delta_n} \right)^\tau |U_t|, \quad \tau \in \mathbb{N}.
$$

Thus, for $\tau := -2\log(\log n)/\log(1 - \bar{\varepsilon}q\delta_n/(2\Delta_n))$ is $o(\log n)$ we have $\mathbb{E}_t[|U_{t+\tau}|] = o(n/\log n)$. Hence by Markov’s inequality, $P[|U_{t+\tau}| \geq n/\log n] = o(1)$. 

\]
3.4 Proof of Theorem 1.6 (a) — edge deletions slow down push

Let \( I_t^{(\text{push})} := I_t \). In order to show the claim we construct an explicit sequence of graphs that has the desired property. More precisely, for any \( \varepsilon > 0 \), each \( q \in (0,1) \) and \( n \in \mathbb{N} \) we will define a graph \( G_n(\varepsilon) \) that is obtained by deleting edges from the complete graph on \( n \) vertices such that each vertex keeps at least an \((1 - \varepsilon)/2\) fraction of its edges and such that push slows down significantly.

We define \( G_n(\varepsilon) = (V_1 \cup V_2, E) \) with vertex set \( V = V_1 \cup V_2 \), where \( V_1 := \{1, \ldots, \lfloor n/2 \rfloor\} \) and \( V_2 := \{ \lceil n/2 \rceil + 1, \ldots, n \} \), as follows. We include in \( E \) all pairs of vertices that intersect \( V_1 \) and moreover, we add edges (that now have endpoints only in \( V_2 \)) such that all vertices in \( V_2 \) have degree \( 1 + 1/n \) and have degree \( 1 + 1/n \) and no further neighbours. Therefore, using that for any \( a \in \mathbb{R} \) we have \((1 + a/n)^n = e^a + O(1/n)\), we obtain for each \( u \in U'_t \)

\[
P_t[I_u] \geq \left(1 - \frac{q}{n-1}\right)^{n/2} \left(1 - \frac{q}{(1-\varepsilon)n}\right)^{(1/2-\varepsilon)n+4} = (1 + o(1)) \left( e^{-q(1/2+(1/2-\varepsilon)/(1-\varepsilon))} \right)^\tau
\]

which is at least \((1 + \omega(n^{-1}))\) for some \( \omega(n^{-1}) \). In this modified model the events \( \{ E_u \mid u \in U'_t \} \) also satisfy \( P_t[I_u] \leq 1 - p \) for all \( u \in V_2 \) and \( U \subseteq V \setminus \{ u \} \) and for some \( p = \omega(n^{-1}) \). This follows immediately from the previous calculation, as conditioning on an event like \( \{ E_v : v \in U \} \) only decreases the number of vertices that can push to \( u \). Thus as \( |U'_t| = \Theta(n) \)

\[
P_t \left[ \bigwedge_{u \in U'_t} E_u \right] \leq \prod_{u \in U'_t} (1 - p) \leq \exp \left( - \sum_{u \in U'_t} p \right) = o(1).
\]

3.5 Proof of Theorems 1.5 (b), 1.7 (b) — push\&pull informs almost all vertices fast in spite of edge deletions

Before we show the actual proof we will first present an informal argument that contains all relevant ideas and important observations. Let \( \sqrt{\log n} \leq |I_t| \leq n/\log n \) and assume \( q = 1 \). In Section 3.3 we proved that for push the informed vertices nearly double in every round for an arbitrary expander sequence with edge deletions and an arbitrary otherwise arbitrary set \( I_t \). For pull this is not true; however, we proved in Section 3.1 that the number of edges between the informed and the uninform ed vertices nearly doubles in every round. The first attempt towards the proof of Theorems 1.5 (b), 1.7 (b) then seems obvious: one would try to show that either the vertices triple every round, or the the edges do so, or for example that the product of the two quantities increases by a factor of 9. As it turns out, this is in general not the case; indeed, it is possible to choose an expander sequence, to delete edges such that each vertex keeps at least an \((1/2+\varepsilon)\)-fraction of its neighbors, and to choose a (large) set of informed vertices \( I_t \) such that after one round whp either \( |I_{t+1}| < c|I_t| \) or \( c(I_t, U_t) < c|I_t|e(I_{t+1}, U_{t+1}) \). On the other hand and although we have no explicit description of these 'malicious' sets, it seems rather unlikely that such sets will occur several times during the execution of push\&pull.

In order to show the claimed running time of push\&pull we will impose some additional structure. Let \( \varepsilon > 0 \). In the subsequent exposition we assume that our graph \( G \) — obtained from an expander by deleting edges such that each vertex keeps at least an \((1/2+\varepsilon)\)-fraction of the edges — has a very special structure. In particular, we assume that there is a partition \( \Pi = (V_i)_{i \in [k]} \) of the vertex set of \( G \) into a bounded number \( k \)
of equal parts such that $E_G(V_i) = \emptyset$ for all $1 \leq i \leq k$ and such that the induced subgraph $(V_i, V_j)$ looks like a random regular bipartite graph for all $1 \leq i < j \leq k$. Of course, not every relevant $G$ admits such a partition; however, Szemeredi’s regularity lemma guarantees that every sufficiently large graph has a partition that is in a well-defined sense almost like the one described previously, and a substantial part of our proof is concerned with showing that being ‘almost special’ does not hurt significantly.

Assuming that $G$ is very special let us collect some easy facts. Denote the degree of $u \in V_i$ in the induced subgraph $(V_i, V_j)$ with $d_{ij}$, this immediately gives that $d_G(u) = \sum_{t=1}^k d_{it}$, and note that $d_{ii} = 0$ as there are no edges in $V_i$. Moreover, regular bipartite random graphs fulfil an expander property, that is, $e(W_i, W_j) = d_{ij}|W_i||W_j|/|V_i| + o(d_{ij})|W_i| \approx |W_i||W_j|d_{ij}k/n$ for all $W_i \subseteq V_i, W_j \subseteq V_j, 1 \leq i < j \leq k$, where we used that all $|V_i|$’s are of equal size. This is quite similar to the property that we used in our preceding analysis on expander sequences, see Lemma 2.9. As a pair in $\Pi$ behaves like a bipartite expander sequence we can easily compute the expected number of informed vertices like we did in Section 3.2. We do so now for pull. Let $|I_i^{\text{pull}}|$ be the number of vertices in $V_i$ informed after round $t+1$ by pull from vertices only in $V_j$ and set $I_i^t := I_i \cap V_i, U_i^t := U_i \cap V_i \forall 1 \leq i \leq k$. Thus, as long as $I_i^t$ is much smaller than $V_i$ (and thus also $U_i^t = |V_i| = n/k$) we get

$$E_t[|I_i^{\text{pull}}\setminus I_i^t|] = \sum_{u \in U_i^t} \frac{|N(u) \cap I_i^t|}{d(u)} = \frac{e(U_i^t, I_i^t)}{\sum_{1 \leq \ell \leq k} d_{i\ell}} \approx \frac{d_{ij}}{\sum_{1 \leq \ell \leq k} d_{i\ell}} |I_i^t|.$$  

A similar calculation, which we don’t perform in detail, yields for push

$$E_t[|I_i^{\text{push}}\setminus I_i^t|] \approx \frac{d_{ij}}{\sum_{1 \leq \ell \leq k} d_{i\ell}} |I_i^t|.$$  

Moreover, as in previous proofs it turns out that the number of vertices informed simultaneously by push as well as pull is negligible, compare with the proof of Lemma 3.4. Thus we obtain that more or less

$$E_t[|I_i^{\text{pp}}\setminus I_i^t|] \approx |I_i^t| + \left(\frac{d_{ij}}{\sum_{1 \leq \ell \leq k} d_{i\ell}} + \frac{d_{ij}}{\sum_{1 \leq \ell \leq k} d_{j\ell}}\right)|I_i^t|$$

and by linearity of expectation

$$E_t[|I_i^{\text{pp}}\setminus I_i^t|] \approx |I_i^t| + \sum_{1 \leq i \leq k} \left(\frac{d_{ij}}{\sum_{1 \leq \ell \leq k} d_{i\ell}} + \frac{d_{ij}}{\sum_{1 \leq \ell \leq k} d_{j\ell}}\right)|I_i^t|.$$  

Set $X_t = (|I_i^t|)_{i \in [k]}$ and $A = (A_{ij})_{1 \leq i \leq j \leq k}$, the matrix with entries

$$A_{ij} = \frac{d_{ij}}{\sum_{1 \leq \ell \leq k} d_{i\ell}} + \frac{d_{ij}}{\sum_{1 \leq \ell \leq k} d_{j\ell}} \quad \text{for } 1 \leq i \neq j \leq k$$

and $A_{ii} = 1$ for $1 \leq i \leq k$. With this notation we obtain the recursive relation

$$E_t[X_{t+1}] \approx A \cdot X_t,$$  

(3.15)

that is, we may expect that $X_t \approx E_t[X_t] \approx A^tX_0$. If we then denote by $\lambda_{\text{max}}$ the greatest eigenvalue of $A$, then we obtain that in leading order

$$|I_i| \approx \lambda_{\text{max}}^t.$$  

Our aim is to show that push&pull is (at least) as fast as on the complete graph, that is, $|I_i| \lesssim 3^t$, and so we take a closer look at the eigenvalues of $A$. By construction $A$ is symmetric, so that the largest eigenvalue equals $\sup_{\|x\| = 1} \|Ax\|$, and the simple choice $x = k^{-1/2} \mathbf{1}$ yields

$$\lambda_{\text{max}} \geq \sum_{i,j} A_{ij} \geq \frac{\sum_{j=1}^k 1 + \sum_{i=1}^k \sum_{j=1}^k d_{ij} / \left(\sum_{\ell=1}^k d_{i\ell}\right) + \sum_{j=1}^k \sum_{i=1}^k d_{ij} / \left(\sum_{\ell=1}^k d_{j\ell}\right)}{k} = 3.$$  

This neat property leads us to the expected result $T_{\text{pp}}(G) = (1 + o(1)) \log_{\lambda_{\text{max}}} n \leq (1 + o(1)) \log_3 n$, and it also completes the informal argument that justifies the claim made in Theorems 1.5 (b) and 1.7 (b). In the rest of this section we will turn this argument step by step into a formal proof by filling in all missing pieces.
Obtaining an Appropriate Regular Partition. An important ingredient in the previous sketch was the assumption that the given graph has a partition into a bounded number of equal parts, such that the bipartite graph induced by any two different parts looks like a random regular graph. This assumption is quite strong and very much not true in general. However, restricting ourselves to dense graphs we can actually come quite close to that. Let us begin with some definitions; the statements are taken from [29].

Definition 3.10 (Density). Given a graph \( G = (V, E) \) and two disjoint non-empty sets of vertices \( X, Y \subseteq V \), we define the density of the pair \( X, Y \) as

\[
d_G(X, Y) = \frac{e_G(X, Y)}{|X||Y|}.
\]

As usual, if the graph is clear from the context the index will be omitted. The next definition gives a partition that is close to the previously described properties; all sets in the partition have nearly the same size and nearly all pairs behave in a well-defined sense like regular bipartite random graphs.

Definition 3.11 \((\varepsilon, k_0, K_0)\)-Szemerédi partition. Let \( G = (V, E) \) and \( k \in \mathbb{N} \). We call \( \Pi = \{ V_i \}_{i \in [k]} \) an \((\varepsilon, k_0, K_0)\)-Szemerédi partition of \( G \) if the following conditions are fulfilled.

\[
a) \quad V_1 \cup \ldots \cup V_k = V,
b) \quad k_0 \leq k \leq K_0,
c) \quad |V_1| \leq \cdots \leq |V_k| \leq |V_1| + 1,
d) \quad \text{for all but at most } \varepsilon k^2 \text{ pairs } (V_i, V_j) \text{ of } \Pi \text{ with } i < j \text{ we have that for all subsets } U_i \subseteq V_i \text{ and } U_j \subseteq V_j \text{ with } |U_i| \geq \varepsilon |V_i| \text{ and } |U_j| \geq \varepsilon |V_j| \quad |d(U_i, U_j) - d(V_i, V_j)| \leq \varepsilon.
\]

A pair \((V_i, V_j)\) satisfying the last condition is called \( \varepsilon \)-regular. For pairs \((V_i, V_j)\) in \( \Pi \) we will abbreviate \( d(V_i, V_j) \) with \( d_{ij} \).

Next we state Szémeredis Regularity Lemma. It guarantees that we will have a Szemerédi partition if the underlying graph is large enough.

Lemma 3.12 ([29], The Regularity Lemma). For every \( \varepsilon > 0 \) and every \( k_0 \in \mathbb{N} \) there exist \( K_0 = K_0(\varepsilon, k_0) \) and \( n_0 \) such that every graph \( G = (V, E) \) with at least \( |V| = n \geq n_0 \) vertices admits an \((\varepsilon, k_0, K_0)\)-Szemerédi partition.

The next lemma gives a useful property of regular pairs. In particular, with the exception of a small set only, all other vertices have a degree that is close to \( dN \), where \( d \) is the density of the pair and \( N \) the number of vertices in each part. Actually, the statement also is true for arbitrary but not too small subsets of the parts.

Lemma 3.13. Let \( G = (V, E) \) be a graph, \( \varepsilon > 0 \) and \( U, U' \subseteq V \). Suppose that \((U, U')\) is an \( \varepsilon \)-regular pair, and let \( W \subseteq U' \), \( |W| \geq \varepsilon |U'| \). Let furthermore \( E(U, W) \subseteq U \) be the largest set such that \( |d(u, W) - d(U, U')| \geq \varepsilon \) for all \( u \in E(U, W) \). Then \( |E(U, W)| \leq 2\varepsilon |U| \).

Proof. We will prove this by contradiction. Assume that \( |E(U, W)| \geq 2\varepsilon |U| \). Let us write \( E(U, W) = S \cup L \), where \( S = \{ u \in E(U, W) : d(u, W) < d(U, U') - \varepsilon \} \) and \( L = \{ u \in E(U, W) : d(u, W) > d(U, U') + \varepsilon \} \). Then \( |S| \geq \varepsilon |U| \) or \( |L| \geq \varepsilon |U| \). In the former case

\[
d(S, W) = \frac{\sum_{u \in S} e(u, W)}{|S||W|} = \frac{\sum_{u \in S} d(u, W)}{|S|} < d(U, U') - \varepsilon.
\]

As \( |S| \geq \varepsilon |U|, |W| \geq \varepsilon |U'| \), this contradicts the assumption that \((U, U')\) is an \( \varepsilon \)-regular pair. The case \( |L| \geq \varepsilon |U| \) follows analogously by showing that \( d(L, W) > d(U, U') + \varepsilon \). \( \square \)
We call the set $E(U,W)$ in Lemma 3.13 the exceptional set of $U$ with respect to $W$. In particular Lemma 3.13 implies that for every $\varepsilon$-regular pair $(U,U')$ and all $W \leq U', |W| \geq (1-c\varepsilon)|U'|, c > 0$ we have

$$|d(u, W) - d(U, U')| \leq |d(u, W) - d(U, U')| + |d(U', U') - d(U, U')| \leq (c+1)\varepsilon$$

for all $u \in U \setminus E(U, U')$. (3.16)

Having done these preparations we can now determine a partition that comes close to the initially described properties.

**Lemma 3.14.** Consider the setting of Theorems 1.5 (b), 1.7 (b). Then for all $\eta > 0$ and $k_0 > 1/\sqrt{n}$ there exists $n_0, K_0 \in \mathbb{N}$ such that for all $G_n$ with $n \geq n_0$ there is a $(\eta, k_0, K_0)$-Szemerédi partition $\Pi = \{V_i\}_{i \in [k]}$ of $G_n$ with the following property. There is $F \subseteq \Pi$ with $|F| \leq \eta k$ such that for all $V_i \in \Pi \setminus F$

- there are at most $\eta k$ non-$\eta$-regular pairs $(V_i, V_j), j \in [k]$, and
- there exists an exceptional set $N_i, |N_i| \leq \eta |V_i|$ such that

$$d(u) \leq (1 + \eta)^n \frac{1}{k} \sum_{1 \leq j \leq k} d(V_i, V_j)$$

for all $u \in V_i \setminus N_i$.

**Proof.** According to Lemma 3.12, for all $\xi > 0$ and $k_0 > 1/\sqrt{n}$ there are $n_0, K_0 \in \mathbb{N}$ such that for all $G_n$ with $n \geq n_0$ there is a $k \in \mathbb{N}$ and a $(\xi, k_0, K_0)$-Szemerédi partition $\Pi = \{V_i\}_{i \in [k]}$ of $G_n$. Let $F \subseteq \Pi$ contain the parts $V_i \in \Pi$ such that there are at least $\sqrt{k}$ other parts $V_j \in \Pi$ such that the pair $(V_i, V_j)$ is not $\xi$-regular. As there are at most $\xi k^2$ non-$\xi$-regular pairs, we infer that $|F| \leq \sqrt{k}$. Let $V_i \in \Pi \setminus F$. Let further $A_i \subseteq \Pi$ be such that $(V_i, V_j)$ is a $\xi$-regular pair for all $V_j \in \Pi \setminus A_i$ and $(V_i, V_j)$ is not $\xi$-regular for all $V_j \in A_i$. The definition of $F$ implies that $|A_i| \leq \sqrt{k}$. For these $V_j \in \Pi \setminus A_i$ let $E_i(V_j) = E(V_i, V_j)$ be the exceptional set of $V_i$ with respect to $V_j$. On top of that let $N_i \subseteq V_i$ be the set of points in $V_i$ that are in at least $\sqrt{k}$ exceptional sets with respect to parts in $\Pi \setminus A_i$. As there are at most $k$ exceptional sets and by Lemma 3.13 each exceptional set has at most $2\xi |V_i|$ vertices, we get that $|N_i| \leq 2\sqrt{k} |V_i|$. Let $V_i \in \Pi \setminus F$, $u \in V_i \setminus N_i$ and let $B(u) \subseteq \Pi \setminus A_i$ be the set of parts such that $u \in E_i(V_j)$ for all $V_j \in B$. Then $|B| \leq \sqrt{k}$ and

$$d(u) = \sum_{1 \leq j \leq k} |V_j|d(u, V_j) = \left( \sum_{V_j \in A \cup B} |V_j|d(u, V_j) + \sum_{V_j \in \Pi \setminus (A \cup B)} |V_j|d(u, V_j) \right)$$

$$\leq |N(u) \cap \bigcup_{V_j \in A \cup B \setminus \{V_i\}} V_j| + \sum_{1 \leq j \leq k} |V_j|(d(V_i, V_j) + \xi).$$

By the definition of $F$ and as $u \in V_i \setminus N_i$, we get that $|\bigcup_{V_j \in A \cup B \setminus \{V_i\}} V_j| \leq (\sqrt{k} + \sqrt{k} + 1)(n/k + 1) \leq 3\sqrt{k}n$. With that at hand and by using $d(u) \geq \alpha n/2$ and that the sizes of the parts in $\Pi$ differ by at most one we obtain

$$d(u) \leq 3\sqrt{k}n + \frac{n}{k} \sum_{1 \leq j \leq k} d(V_i, V_j) + 2\xi n \leq \frac{n}{k} \sum_{1 \leq j \leq k} d(V_i, V_j) + 10\sqrt{k}d(u)/\alpha.$$

Let $\eta > 0$. Choosing $\xi$ small enough such that $\max\{\xi, 2\sqrt{k}/(1 - 10\sqrt{k}/\alpha) - 1\} \leq \eta$ implies the claim. $\Box$

**The Recursion Relation.** In this section we exploit the properties of the partition to study the expected number of informed vertices after one additional round; our aim is to establish a precise version of (3.15). In the remainder let $\|A\|_F = (\sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} |a_{i,j}|^2)^{1/2}$ denote the Frobenius norm of a matrix $A \in \mathbb{R}^{n \times n}$.

For the next lemma consider the setting of Theorems 1.5 (b), 1.7 (b), i.e., we are given an expander sequence $(G_n)_{n \in \mathbb{N}}$ with minimal degree $\delta_n \geq \alpha n$ for some $\alpha > 0$ and an $\varepsilon > 0$. We obtain a sequence of graphs $(G_n)_{n \in \mathbb{N}}$ by deleting up to a $1/\varepsilon$ fraction of the edges at each vertex in $G_n$. Let further $\eta > 0, k_0 \in \mathbb{N}$ and $\Pi = \{V_i\}_{i \in [k]}$ be the $(\eta, k_0, K_0)$-Szemerédi partition of $G_n$ as given by Lemma 3.14. For that partition define $E_i := E(V_i, V_j)$ as the exceptional set of $V_i$ with respect to $V_j$ given by Lemma 3.13, $i \neq j \in [k]$, and $N_i$ as the exceptional sets from Lemma 3.14, $i \in \Pi \setminus F$. Moreover, let $\Pi_i = \{V_j \in \Pi \setminus F : (V_i, V_j) \text{ is } \eta\text{-regular}\}$ and note that

$$|\Pi_i| \geq (1 - 2\eta)k, |N_i| \leq \eta |V_i|, |E_i,j| \leq 2\eta |V_i|$$

for all $i \in \Pi \setminus F, j \in \Pi_i$. 22
Finally, define
\[ X_{t,i,j} = |I_t \cap (V_i \cup (N_i \cup \mathcal{E}_{i,j}))|, \quad i \in \Pi \setminus F \text{ and } j \in \Pi_i \]
and
\[ X_{t,i} = \min_{j \in \Pi_i} X_{t,i,j}, \quad i \in \Pi \setminus F. \]
This definition guarantees that \( |I_t| \geq \|X_t\|_1 \). The cornerstone of our proof is the following lemma, which bounds the growth of \( X_t = (X_{t,i})_{i \in \Pi \setminus F} \) after one round.

**Lemma 3.15.** Consider the situation as described above and assume additionally that \( |X_{t,i}| \geq \log \log n \) for all \( i \in \Pi \setminus F \) and that \( |I_t| \leq n/\log n \). Then for all \( \nu > 0 \) and \( n \) large enough there exists a symmetric matrix \( A \) with biggest eigenvalue \( \lambda_{\text{max}} \geq 1 + 2q - \nu \) and an error matrix \( \Delta A \) with \( \|\Delta A\|_F \leq \nu \) such that

\[ X_{t+1} \geq (A + \Delta A)X_t. \]

**Proof.** We set \( I_{P}^{t,i,j} = I_t^{P} \cap V_i \), \( U_{P}^{t,i,j} = U_t^{P} \cap V_i \) for \( P \in \{ \text{push, pull, pp} \} \) and let

\[ I_{t+1}^{P,i,j} \setminus I_t = \{ u \in U_t \cap V_j \mid \text{there is } v \in I_t \cap V_j \text{ such that } u \text{ gets informed by } v \text{ using } P \} \]
be the vertices in \( V_i \) newly informed in round \( t + 1 \) by operations involving only vertices from \( V_i \) and \( V_j \). Let \((i,j) \in \Pi \setminus F \). For all \( u \in U_t^i \) we know that \( d(u) \geq \alpha n/2 \). Moreover, \( |I_t^i| \leq |I_t^j| \leq n/\log n \). Thus, the probability of \( u \in U_t^i \) being informed by vertices in \( I_t^i \) via pull is \( q |N(u) \cap I_t^i|/|N(u)| = o(1) \). As the events of \( u \) being informed by push and pull are independent \( \mathbb{P}[u \in I_{t+1}^{\text{push},i,j} \cap I_{t+1}^{\text{pull},i,j}] = o(1) \mathbb{P}[u \in I_{t+1}^{\text{push},i,j}] \).

Thus for any set \( U \subset V \)

\[
\mathbb{E}\left[ \left( |I_{t+1}^{\text{push},i,j} \setminus I_i \cap U| \right) \right] = (1 - o(1)) \left( \mathbb{E}\left[ \left( |I_{t+1}^{\text{pull},i,j} \setminus I_i \cap U| \right) \right] + \mathbb{E}\left[ \left( |I_{t+1}^{\text{push},i,j} \setminus I_i \cap U| \right) \right] \right). \tag{3.17}
\]

Let \( i \in \Pi \setminus F \) and \( j \in \Pi_i \). We start by determining the expected number of vertices informed by pull. Set further \( D_i = (1 + \eta) \frac{n}{k} \sum_{1 \leq \ell \leq k} d_{i\ell} \). According to Lemma 3.14 all \( v \in U^i \setminus N_i \) have degree less than \( D_i \). Let \( j' \in \Pi_i \) and set \( \mathcal{H}_{i,j'} = N_i \cup \mathcal{E}_{i,j'} \).

Then

\[
\mathbb{E}_t \left[ \left| I_{t+1}^{\text{pull},i,j} \setminus (I_i \cup \mathcal{H}_{i,j'}) \right| \right] = \sum_{u \in U_t^i \setminus \mathcal{H}_{i,j'}} q \frac{|N(u) \cap I_t^i|}{|N(u)|} \geq q \frac{e(U_t^i \setminus \mathcal{H}_{i,j'} \cup I_t^i)}{D_i}. \]

Since \( |I_t| \leq |I_t| \leq n/\log n \) we get with room to spare that \( |U_t^i \setminus \mathcal{H}_{i,j'}| \geq (1 - 5\eta) n/k \) for \( n \) large enough and all \( j' \in \Pi_i \). Applying (3.16), where we choose \( W = U_t^i \setminus \mathcal{H}_{i,j'} \), yields \( |d(U_t^i \setminus \mathcal{H}_{i,j'}, u) - d_{ij}| \leq 6\eta \) for all \( u \in V_j \setminus \mathcal{E}_{i,j'} \).

Thus

\[
\mathbb{E}_t \left[ \left| I_{t+1}^{\text{pull},i,j} \setminus (I_i \cup \mathcal{H}_{i,j'}) \right| \right] \geq q \frac{(d_{ij} - 6\eta)|U_t^i \setminus \mathcal{H}_{i,j'}|D_i}{|I_t^i \setminus \mathcal{H}_{i,j'}| \geq (1 - 5\eta)q \frac{(d_{ij} - 6\eta)|I_t^i \setminus (\mathcal{E}_{i,j} \cup N_j)|D_i k/n}{D_i k/n}. \]

As \( D_i = (1 + \eta) n/k \sum_{1 \leq \ell \leq k} d_{i\ell} \) we get for

\[
c_1 := (1 - 6\eta)(1 + \eta)^{-1} \]
with \( X_{t,i,j} = |I_t^i \setminus (\mathcal{E}_{i,j} \cup N_j)| \) that

\[
\mathbb{E}_t \left[ \left| I_{t+1}^{\text{pull},i,j} \setminus (I_i \cup \mathcal{H}_{i,j'}) \right| \right] \geq c_1 \cdot q \frac{(d_{ij} - 6\eta)X_{t,i,j}}{\sum_{1 \leq \ell \leq k} d_{i\ell}} \quad \text{for all } i \in \Pi \setminus F \text{ and } j, j' \in \Pi_i. \tag{3.18}
\]

We continue with push. Let \( i \in \Pi \setminus F \) and \( j, j' \in \Pi_i \), and set (as before) \( D_j = (1 + \eta) \frac{n}{k} \sum_{1 \leq \ell \leq k} d_{j\ell} \) and \( \mathcal{H}_{i,j'} = N_i \cup \mathcal{E}_{i,j'} \). Then

\[
\mathbb{E}_t \left[ \left| I_{t+1}^{\text{push},i,j} \setminus (I_i \cup \mathcal{H}_{i,j'}) \right| \right] = \sum_{u \in U_t^i \setminus \mathcal{H}_{i,j'}} \left( 1 - \prod_{v \in N(u) \cap I_t^i} \left( 1 - \frac{q}{|N(v)|} \right) \right) \]

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According to Lemma 3.14 all \( v \in I_t^i \setminus N_j \) have degree less than \( D_j \). Using the inequalities \((1 - 1/n)^n \leq e^{-1}
olimits \) and \( e^{-1/n} = (1 - 1/n) + o(1/n) \) we obtain the estimate

\[
E_t \left[ |I_{t+1}^{(push),i,j} \setminus (I_t \cup H_{i,j'})| \right] \geq \sum_{u \in U_t^i \setminus H_{i,j'}} \left( 1 - \frac{q (|N(u) \cap (I_t^i \setminus N_j)|)}{D_j} \right) \\
\geq \sum_{u \in U_t^i \setminus H_{i,j'}} \left( 1 - \exp \left( -q \frac{|N(u) \cap (I_t^i \setminus N_j)|)}{D_j} \right) \right) \geq (1 - o(1)) \sum_{u \in U_t^i \setminus H_{i,j'}} \frac{|N(u) \cap (I_t^i \setminus N_j)|}{D_j}. \tag{3.19}
\]

The remaining steps are similar to the previously considered case of pull. By assumption we have that \(|I_t^i \setminus H_{i,j'}| = X_{t,j,i} \) and as \(|I_t^i| \leq |I_t| \leq n/\log n\) we obtain that \(|U_t^i \setminus H_{i,j'}| \geq (1 - 5\eta)n/k \) for \( n \) large enough and all \( j' \in \Pi_i \). Using (3.16) we obtain that \(|\bar{d}(U_t^i \setminus H_{i,j'}, u) - d_{ij'}| \leq 6\eta \) for all \( u \in V_j \setminus E_{j,i} \). Thus

\[
E_t \left[ |I_{t+1}^{(push),i,j} \setminus (I_t \cup H_{i,j'})| \right] \geq (q - o(1)) \frac{e^{(U_t^i \setminus H_{i,j'}, I_t^i \setminus (N_j \cup E_{j,i}))}}{D_j} \geq (q - o(1)) \frac{(d_{ij'} - 6\eta)|U_t^i \setminus H_{i,j'}|X_{t,j,i}}{D_j}.
\]

Using that \( D_j = (1 + \eta)n/k \sum_{1 \leq \ell \leq k} d_{ij'} \), we get for the same constant \( c_1 \) as in (3.18) and \( n \) large enough

\[
E_t \left[ |I_{t+1}^{(push),i,j} \setminus (I_t \cup H_{i,j'})| \right] \geq c_1 \cdot (d_{ij'} - 6\eta)X_{t,j,i} \sum_{1 \leq \ell \leq k} d_{ij'} \text{ for all } i \in \Pi \setminus F \text{ and } j, j' \in \Pi_i. \tag{3.20}
\]

With (3.17), we can combine the results for pull, (3.18), and push, (3.20), to get for \( c_2 := c_1 - \eta \)

\[
E_t \left[ |I_{t+1}^{(push),i,j} \setminus (I_t \cup H_{i,j'})| \right] \geq c_2 \cdot q \left( \frac{d_{ij'} - 6\eta}{\sum_{1 \leq \ell \leq k} d_{ij'}} + \frac{d_{ij'} - 6\eta}{\sum_{1 \leq \ell \leq k} d_{ij'}} \right) X_{t,j,i} \text{ for all } i \in \Pi \setminus F, j, j' \in \Pi_i. \tag{3.21}
\]

Next we will show how we can exploit (3.21) to obtain (a lower bound for) \( E_t[|I_{t+1}^{(push),i} \setminus I_t|] \). Let \( i \in \Pi \setminus F \) and \( u \in U_t^i \). Using \( e^{-1/n + o(1/n)} = 1 - 1/n, e^{-1/n} = 1 - 1/n + o(1/n), \) and \( |I_t| = o(n) \) we obtain

\[
P_t[u \in I_{t+1}^{(push),i} \setminus I_t] = 1 - \prod_{i \in N(u) \cap I_t} \left( 1 - \frac{1}{|N(i)|} \right) = 1 - \exp \left( - (1 - o(1)) \sum_{i \in N(u) \cap I_t} \frac{1}{|N(i)|} \right).
\]

Let \( W \subseteq V \). Using (3.17), the previous equation and that \( \Pi \) is a partition we get

\[
E_t[|I_{t+1}^{(push),i} \setminus I_t \cap W|] = \left( 1 - o(1) \right) \sum_{u \in U_t^i \cap W} \left( \frac{|N(u) \cap I_t|}{|N(u)|} + \sum_{i \in N(u) \cap I_t} \frac{1}{|N(i)|} \right)
\]

\[
= \left( 1 - o(1) \right) \sum_{u \in U_t^i \cap W} \left( \sum_{j \in [k]} \left( \frac{|N(u) \cap I_t \cap V_j|}{|N(u)|} + \sum_{i \in N(u) \cap I_t \cap V_j} \frac{1}{|N(i)|} \right) \right)
\]

\[
= \left( 1 - o(1) \right) \sum_{j \in [k]} E_t[|I_{t+1}^{(push),i,j} \setminus I_t \cap W|].
\]

Choose \( W = V \setminus H_{i,j'} \), then the previous equation implies

\[
E_t \left[ |I_{t+1}^{(push),i} \setminus (I_t \cup H_{i,j'})| \right] \geq \left( 1 - o(1) \right) \sum_{j \in \Pi \setminus F} E_t \left[ |I_{t+1}^{(push),i,j} \setminus (I_t \cup H_{i,j'})| \right] \text{ for all } i \in \Pi \setminus F, j' \in \Pi_i,
\]

which in turn, using (3.21) and \( X_{t,j,i} \geq X_{t,j} \) for all \( i \in \Pi \setminus F \) and \( j' \in \Pi_i \), implies for \( c := c_2 - \eta \)

\[
E_t[X_{t+1,i,j'}] \geq X_{t,i} + c \cdot q \sum_{j \in \Pi_i} \left( \frac{d_{ij'} - 6\eta}{\sum_{1 \leq \ell \leq k} d_{ij'}} + \frac{d_{ij'} - 6\eta}{\sum_{1 \leq \ell \leq k} d_{ij'}} \right) X_{t,j} \text{ for all } i \in \Pi \setminus F, j' \in \Pi_i. \tag{3.22}
\]
Assume that (3.22) does not only hold in expectation but also for a slightly smaller \(c\), say \(c - \eta\), with high probability. We are going to show this at the end of the proof. Using this assumption and a union bound over \(j' \in \Pi_i\) gives

\[
X_{t+1,j} = \min_{j' \in \Pi_i} X_{t+1,i,j'} \geq \langle a_i, (X_{i,j})_{j \in \Pi_i} \rangle \quad \text{for all } i \in \Pi \setminus F,
\]

where for \(i \in \Pi \setminus F\) and \(j \in \Pi_i\) we have

\[
a_{ij} = \mathbb{1}[i = j] + c \cdot q \frac{d_{ij} - 6\eta}{\sum_{1 \leq \ell \leq k} d_{i\ell}} + \frac{d_{ij} - 6\eta}{\sum_{1 \leq \ell \leq k} d_{i\ell}}.
\]  

(3.24)

Let \(A\) be the \(|\Pi \setminus F| \times |\Pi \setminus F|\) matrix with entries as in the previous equation, i.e., \(A = (a_{ij})_{(i,j) \in (\Pi \setminus F)^2}\) is given by (3.24) for all \((i,j) \in (\Pi \setminus F)^2\). Note that \(A\) is symmetric. Then we obtain from (3.23)

\[
X_{t+1} \geq B \cdot X_t;
\]

with \(B = A + \Delta A\), where

\[
(\Delta A)_{ij} = \begin{cases} 0, & i \in \Pi \setminus F \text{ and } j \in \Pi_i \\ -a_{ij}, & i \in \Pi \setminus F \text{ and } j \in \Pi \setminus (F \cup \Pi_i) \end{cases}
\]

Set \(F' := \{(i, j) \in (\Pi \setminus F)^2 \mid j \in \Pi \setminus (F \cup \Pi_i)\}\). As \(d(u) \geq \alpha n/2\) for all \(u \in V\) and some \(\alpha > 0\), we also know that \(\sum_{1 \leq \ell \leq k} d_{i\ell} \geq k\alpha/2\). Together with \(0 \leq d_{i,j} \leq 1\) for all \((i,j) \in [k]^2\) we get that \(|(d_{ij} - 6\eta)/\sum_{1 \leq \ell \leq k} d_{i\ell}| \leq 2/(\alpha k)\). Using that \(|F'| \leq 2\eta k^2\) we obtain

\[
\|\Delta A\|_F^2 = \sum_{(i,j) \in F'} a_{ij}^2 \leq \sum_{(i,j) \in F'} \left( \frac{2}{\alpha k} \right)^2 \leq 2\eta k^2 \left( \frac{2}{\alpha k} \right)^2 = \frac{8\eta}{\alpha^2}
\]

and thus \(\|\Delta A\|_F \leq 2\sqrt{2\eta/\alpha}\). This leaves us with bounding the biggest eigenvalue \(\lambda_{\max}\) of \(A\). Using the well-known inequality for symmetric matrices \(\lambda_{\max} \geq \sum_{(i,j) \in (\Pi \setminus F)^2} A_{ij}/|\Pi \setminus F|\) we obtain

\[
\lambda_{\max} \geq \frac{1}{|\Pi \setminus F|} \sum_{(i,j) \in (\Pi \setminus F)^2} a_{ij}
\]

\[
\geq \frac{1}{|\Pi \setminus F|} \left( \sum_{(i,j) \in (\Pi \setminus F)^2} \sum_{i,j \in [k]} \frac{cq(d_{ij} - 6\eta)}{\sum_{1 \leq \ell \leq k} d_{i\ell}} + \sum_{i \in [k]} \sum_{j \in [k \setminus (\Pi \setminus F)]} \frac{cq}{\sum_{1 \leq \ell \leq k} d_{i\ell}} \right).
\]

Note that \(|\Pi \setminus F| \geq (1 - \eta)k\), \(|[k \setminus (\Pi \setminus F)]| \leq \eta k\). Moreover, \(\sum_{1 \leq \ell \leq k} d_{i\ell} \geq \alpha k/2\) for all \(j \in [k]\). Thus

\[
\lambda_{\max} \geq 1 + \frac{1}{k} \left( cqk + cqk - 12cq \sum_{i,j \in [k]} \frac{\eta}{\sum_{1 \leq \ell \leq k} d_{i\ell}} - 2cq \frac{\eta k^2}{\alpha k^2} \right) \geq 1 + 2cq(1 - 8\eta/\alpha).
\]

Choosing \(\eta\) small enough such that \(2q(1 - c(1 - 8\eta/\alpha)), 2\sqrt{2\eta/\alpha} \leq \nu\) implies the claim of this lemma.

This leaves us with proving that (3.22) also holds with high probability. As \(|I_{i+1}^{pp}\) conditioned on \(I_t\) is a self-bounding function so is \(|I_{i+1, i', j'}^{pp}\) \(i \in \Pi \setminus F\) and therefore also \(|I_{i+1, i', j'}^{pp}\) \((I_t \cup \mathcal{H}_i, j')\) for all \(j' \in \Pi_i\). Note that \(Y_{t+1, i, j'} \geq X_{t+1, i, j'} - X_{t, i}\); Lemma 2.3 yields that

\[
P_t \left[ Y_{t+1, i, j'} \geq 1 - E_t[Y_{t+1, i, j'}]^{-1/3} E_t[Y_{t+1, i, j'}] \right] \geq 1 - E_t[Y_{t+1, i, j'}]^{-1/3}
\]

and therefore setting

\[
Z_{t,i} = c \cdot q \sum_{j \in \Pi_i} \left( \frac{d_{ij} - 6\eta}{\sum_{1 \leq \ell \leq k} d_{i\ell}} + \frac{d_{ij} - 6\eta}{\sum_{1 \leq \ell \leq k} d_{i\ell}} \right) X_{t,j} \quad \text{for all } i \in \Pi \setminus F
\]
and using (3.22), i.e. \( \mathbb{E}_t[Y_{t+1,i,j'}] \geq Z_{t,i} \) for all \( i \in \Pi \setminus F \) and \( j' \in \Pi_i \) we get with probability at least 
\[ 1 - k^3 Z_{t,i}^{-1/3} \]
\[ Y_{t+1,i,j'} \geq (1 - Z_{t,i}^{-1/3})Z_{t,i} \] for all \( i \in \Pi \setminus F \) and \( j' \in \Pi_i \).

This and \(|I_i^*| \geq X_{t,i} \) for all \( i \in \Pi \setminus F \) implies that (3.22) also holds with high probability for a marginally smaller \( c \), as claimed.

\[ \square \]

**Extension.** We now solve the linear recurrence relation above and extend it to more than one round to get an upper bound on the runtime of push\&pull. We first state a Chernoff Bound that will be very useful in the next lemma.

**Lemma 3.16.** [25] Let \( \varepsilon, \delta > 0 \). Suppose that \( X_1, \ldots, X_n \) are independent geometric random variables with parameter \( \delta \), so \( \mathbb{E}[X_i] = 1/\delta \) for each \( i \). Let \( X := \sum_{1 \leq i \leq n} X_i, \mu = \mathbb{E}[X] = n/\delta \). Then

\[ P[X \geq (1 + \varepsilon)\mu] \leq e^{-n(\varepsilon \log(1 + \varepsilon))} \leq e^{-n^2/2(1 + c)} \]

Together with Lemma 2.13 the following lemma implies Theorem 1.5 (b) and Theorem 1.7 (b).

**Lemma 3.17.** Consider the setting of Theorems 1.5 (b), 1.7 (b) and let \( I_t = I_t^{(pp)} \). The following statements hold whp.

(a) Let \( I \subseteq V_n \) satisfying \( |I| = \Theta(n) \), then there is \( t = \Theta(\log \log n) \), such that whp \( |I_t| \geq |I_t \cap I| \geq \log \log n \).

(b) Let \( \log \log n \leq |I_t| \leq n/\log n \). Then there is \( \tau \leq \log_{1+2\delta}(n/|I_t|) + o(\log n) \) such that \( |I_{t+\tau}| > n/\log n \).

(c) Let \( n/\log n \leq |I_t| \leq n - n/\log n \). Then there is \( \tau = o(\log n) \) such that \( |I_{t+\tau}| > n - n/\log n \).

(d) Let \( |I_t| \geq n - n/\log n \) and \( q = 1 \). Then there is \( \tau = o(\log n) \) such that \( |I_{t+\tau}| = n \).

**Proof.** As \( |I_t^{(pp)}| \geq |I_t^{(pull)}| \) clearly c) and d) follow from Lemma 3.2. We show a) by determining a lower bound for the probability that an arbitrary vertex gets informed after a constant number of rounds. Let \( \beta = \min\{\alpha, c\} \), let \( I_0 = \{u\} \) and choose \( w \in V, w \neq u \). By Lemma 2.11 there is \( d \leq 8/\beta^2 + 2 \) and \( c = (\beta^4/64)^{8/\beta^4 + 3} \in (0, 1) \) such that there are at least \( cn^{d-1} \) paths of (edge) length \( d \) from \( u \) to \( w \). Let \( \gamma = (u, v_1, \ldots, v_{d-1}, w) \) be such a path from \( u \) to \( w \), and denote by \( A_{\gamma} \) the event that \( w \) is informed via \( \gamma \) after exactly \( d \) rounds performing only push operations, i.e., \( A_{\gamma} \) is the event that in the first round the randomly selected neighbour of \( u \) is \( v_1 \), in the second round the randomly selected neighbour of \( v_1 \) is \( v_2 \) and so forth, until in the \( d \)th round the randomly selected neighbour of \( v_{d-1} \) is \( w \). Obviously, the probability of \( A_{\gamma} \) is bounded from below by \( n^{-d} \). Let further \( \gamma' \neq \gamma \) be another path from \( u \) to \( w \) with length \( d \). As \( \gamma \) and \( \gamma' \) differ by at least one edge we readily obtain that \( P[A_{\gamma} \cap A_{\gamma}] = 0 \). Let \( \Gamma \) denote the set of all paths with length \( d \) from \( u \) to \( w \). Having done these preparations we use them to conclude for all \( w \in V \) and \( t \geq 0 \)

\[ P[w \in I_{t+d}] \geq P_t \left[ \bigcup_{\gamma \in \Gamma} A_{\gamma} \right] \geq \sum_{\gamma \in \Gamma} P_t[A_{\gamma}] \geq \sum_{\gamma \in \Gamma} n^{-d} \geq c/n. \quad (3.25) \]

We define a modified protocol as follows. Wait \( d := \lfloor 8/\beta^2 + 2 \rfloor \) rounds, after that with probability \( c \) choose one uninformed vertex uniformly at random and set it as informed. Repeat. Call the vertices informed by this algorithm \( I_t^* \). Then the probability for any vertex to be informed after \( d \) rounds is

\[ P[v \in I_{t+d} | v \notin I_t^*] = c/n. \]

Thus for any \( t \geq 0 \)

\[ P[w \in I_{t+d} | w \in U_t] \geq P_t[w \in I_{t+d}^* | w \notin I_t^*] = c/n. \]

Note that for any \( s \in \mathbb{N} \) the set \( I_t^* \) is generated by a very simple procedure: \( s \) times independently, with probability \( c \), we choose a random vertex and put it into \( I_{t+d}^* \). Thus \( |I_{t+d}^* \cap I| \) is binomially distributed with \( s \) trials, where each one has success probability \( c|I|/n = \Theta(c) \); it follows readily that \( |I_{t+d}^* \cap I| \) concentrates around a multiple of \( s \) for large \( s \), and the claim follows by choosing \( s = \Theta(\log \log n) \).
This leaves $b)$ to be shown. Part $a)$ implies that there is some $t_0 = o(\log n)$ such that $X_{t_0,i} = \Theta(\log \log n)$ for all $i \in \Pi \setminus F$ by choosing $I = V_{t_0} \setminus (N_i \cup E_{i+})$, $j \in \Pi_t$ and applying a union bound over $i$ and $j$. Thus we can apply Lemma 3.15. It gives whp, say with probability $1 - g(n) = 1 - o(1)$, that $X_{t+1} \geq (A + \Delta A)X_t$. $A$ has maximal eigenvalue $\lambda_{\max}(A) \geq 1 + 2q - \nu$ and $\|\Delta A\|_F \leq \nu$. Then $B := A + \Delta A$ has maximal eigenvalue $\lambda_{\max}(B) \geq \lambda_{\max}(A) - \|\Delta A\|_F \geq 1 + 2q - 2\nu$ (Theorem of Wielandt-Hoffmann, compare e.g. [23]).

Set $f(n) := (\log(n/\log n))^{2/3}$. Our assumptions guarantee that $f(n) = \omega(1)$ and $f(n) = o(\log n)$. Moreover, set

$$\tau := 1/(1 - g(n)) \log(n/\log n)/\log(\lambda_{\max}(B)) + f(n) = \log(n) / \log(\lambda_{\max}(B)) + o(n).$$

Let $(X_i)_{i \in \mathbb{N}}$ be i.i.d. geometric random variables with expectation $1 - g(n)$. Set $X = X_1 + X_2 + \cdots + X_\tau$ with $T = \log(n/\log n)/\log(\lambda_{\max}(B))$. We show that $P[X \leq \tau] = 1 - o(1)$. To see this, note first that by linearity of expectation $E[X] = \tau - f(n)$. Then with Lemma 3.16

$$P[X \leq \tau] = P \left[ X \leq \left( 1 + \frac{f(n)}{\tau} + f(n) \right) E[X] \right] \geq 1 - \exp \left( -O \left( \frac{(f(n))^2}{\tau} \right) \right) = 1 - o(1).$$

Thus we have whp

$$|I_{t+\tau}| \geq \|X_{t+\tau}\|_1 \geq \|B^T X_{t_0}\|_1.$$ 

Let $v$ be an eigenvector of $B$ to $\lambda_{\max}(B)$. As $v \neq 0$ there is an index $\ell$ such that $v_\ell \neq 0$. Without loss of generality we can assume that $v_\ell = 1$, as $v/v_\ell$ is also an eigenvector to $\lambda_{\max}(B)$. Thus $(B^T v)_\ell = \lambda_{\max}(B)^T$, $(B^T (X_{t_0} - v))_i \geq 0$ for all $1 \leq i \leq k$ and therefore

$$|I_{t+\tau}| \geq (B^T X_{t_0})_\ell \geq (B^T (v + X_{t_0} - v))_\ell = (B^T v)_\ell + (B^T (X_{t_0} - v))_\ell \geq (B^T v)_\ell \geq \lambda_{\max}(B)^T.$$

Our choice of $T$ yields whp $|I_{t+\tau}| \geq \lambda_{\max}(B)^T \geq n/\log n$. Note that, since $\nu > 0$ was chosen arbitrarily, we actually have that $\tau \leq \log_{1+2q}(n) + o(\log n)$, and the proof is completed. \hfill $\square$

### 3.6 Proof of Theorem 1.6 (b) — edge deletions may slow down push&pull

For any $0 < \varepsilon < 1/2$, $q \in (0, 1)$ we consider a sequence of graphs $(G_n(\varepsilon))_{n \in \mathbb{N}} = (V_n, E_n(\varepsilon))_{n \in \mathbb{N}}$ that is similar to the one studied in the proof of Theorem 1.6 (a). Let $V_n = A_n \cup B_n$ with $A_n := \{1, \ldots, [n/2]\}$, $B_n := \{\lfloor n/2 \rfloor + 1, \ldots, n\}$ and $\deg(v) = n - 1$ for all $v \in A_n$. Let the induced subgraph of $B_n$ be a random graph in which each edge is included independently with probability $p = 1 - 2\varepsilon$. We know and it is easy to show, see for example [14, Section IV], that whp this subgraph is almost regular, i.e.,

$$d_{B_n}(v) = (1 + o(1))(1 - 2\varepsilon)n/2 \quad \text{for all} \ v \in B_n,$$ 

and is an expander, which means that for every $S_n \subseteq B_n$, $1 \leq |S_n| \leq n/4$ and $d_{B_n} := (1 - 2\varepsilon)n/2$ we have

$$e(S_n, B_n \setminus S_n) = (1 + o(1)) \frac{d_{B_n}|S_n||B_n \setminus S_n|}{|B_n|} = (1 - 2\varepsilon + o(1))|S_n||B_n \setminus S_n|.$$ 

At first we give a statement that describes the expected number of informed vertices after performing one round of push&pull.

**Lemma 3.18.** Let $G_n(\varepsilon) = (A_n \cup B_n, E_n(\varepsilon))$ be as above.

(a) Let $\sqrt{\log n} \leq |I_t| \leq n/\log n$ and set

$$X_t = \left( |I_t|^{pp}(A) \cap |I_t|^{pp}(B) \right) := \left( |I_t|^{pp} \cap A_n \right) \{ |I_t|^{pp} \cap B_n \}.$$

Then $E_t[X_{t+1}] = (1 + o(1))MX_t$, where

$$M = \begin{pmatrix} 1 + q & q(1 + \varepsilon/(2 - 2\varepsilon)) \\ q(1 + \varepsilon/(2 - 2\varepsilon)) & 1 + q(1 - 2\varepsilon/(2 - 2\varepsilon)) \end{pmatrix}.$$ 

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(b) Let $|U_{t+1}^{(pp)}| \leq n/\log n$. Then $\mathbb{E}_t[|U_{t+1}^{(pp)}|] \leq (1 + o(1))e^{-q(1/2+(1/2-\varepsilon)/(1-\varepsilon))} (1 - q) |U_t|.$

Proof. For $J \in \{A, B\}, J_n \in \{A_n, B_n\}$ set $U_{t}^{(J)} := U_t \cap J_n, I_{t}^{(J)} := I_t \cap J_n$ and $I_{t+1}^{(pp), (J)} = I_{t+1}^{(pp)} \cap J_n.$ We first prove $a)$ by computing the expected number of informed vertices after a single round. Since $d(u) = \Omega(n)$ for all $u \in V_n$ and $|I_t| \leq n/\log n$, the probability of $u \in U_t$ being informed by pull is

$$P_t \left[ u \in I_{t+1}^{(pull)} \setminus I_t \right] = \frac{q |N(u) \cap I_t|}{|N(u)|} = o(1).$$

As the events of $u$ being informed by push and pull are independent we have $P_t[u \in (I_{t+1}^{(push)} \cap I_{t+1}^{(pull)}) \setminus I_t] = o(1)P_t[u \in I_{t+1}^{(push)} \setminus I_t].$ Thus

$$\mathbb{E}_t \left[ |I_{t+1}^{(pull)} \setminus I_t| \right] = (1 + o(1)) \left( \mathbb{E}_t \left[ |I_{t+1}^{(push)} \setminus I_t| \right] + \mathbb{E}_t \left[ |I_{t+1}^{(pull)} \setminus I_t| \right] \right).$$

We look at pull in detail first. Recall that $\deg(v) = n - 1$ for all $v \in A_n$ and $\deg(v) = (1 + o(1))(1 - \varepsilon)n$ for all $v \in B_n$. Moreover, using (3.27) we obtain

$$\mathbb{E}_t \left[ |I_{t+1}^{(pull)} \setminus I_t| \right] = \sum_{u \in U_t} q \frac{|N(u) \cap I_t|}{|N(u)|} = \sum_{u \in U_t^{(A)}} q \frac{|N(u) \cap I_t|}{|N(u)|} + \sum_{u \in U_t^{(B)}} q \frac{|N(u) \cap I_t|}{|N(u)|}$$

$$= (q + o(1)) \frac{n}{2} \left( \frac{|I_t^{(A)}| + |I_t^{(B)}|}{n} + (1 - 2\varepsilon) |I_t^{(B)}| \right)$$

and thus

$$\mathbb{E}_t \left[ |I_{t+1}^{(pull), (A)} \setminus I_t| \right] = (q + o(1)) \frac{|I_t^{(A)}| + |I_t^{(B)}|}{2},$$

$$\mathbb{E}_t \left[ |I_{t+1}^{(pull), (B)} \setminus I_t| \right] = (q + o(1)) \frac{|I_t^{(A)}| + (1 - 2\varepsilon) |I_t^{(B)}|}{2(1 - \varepsilon)}.$$ 

Next we consider push. We obtain by using that $(1 - 1/n)^n = e^{-1+o(1)}$

$$\mathbb{E}_t \left[ |I_{t+1}^{(push)} \setminus I_t| \right] = \sum_{u \in U_t} 1 - \prod_{i \in N(u) \cap I_t} \left( 1 - \frac{q}{|N(i)|} \right)$$

$$= \sum_{u \in U_t} 1 - \left( 1 - \frac{q}{n} \right)^{|I_t^{(A)}|} \cdot \left( 1 - \frac{(1 + o(1))q}{(1 - \varepsilon)n} \right)^{|I_t^{(B)}|} + \frac{1}{|N(u) \cap I_t^{(B)}|}$$

$$= \sum_{u \in U_t} 1 - \exp \left( -\left( q + o(1) \right) \frac{|I_t^{(A)}|}{n} + \frac{1}{|N(u) \cap I_t^{(B)}|} \right).$$

Using that $1 - 1/n = (1 + o(1))e^{-1/n}$ we get

$$\mathbb{E}_t \left[ |I_{t+1}^{(push)} \setminus I_t| \right] = (q + o(1)) \sum_{u \in U_t} \left( \frac{|I_t^{(A)}|}{n} + \frac{1}{|N(u) \cap I_t^{(B)}|} \right)$$

and thus with $|U_t^{(A)}| = (1 - o(1))n/2$ and (3.27),

$$\mathbb{E}_t \left[ |I_{t+1}^{(push), (A)} \setminus I_t| \right] = (q + o(1)) \left( \frac{|I_t^{(A)}|}{2} + \frac{|I_t^{(B)}|}{2} + \frac{\varepsilon |I_t^{(B)}|}{2(1 - \varepsilon)} \right),$$

$$\mathbb{E}_t \left[ |I_{t+1}^{(push), (B)} \setminus I_t| \right] = (q + o(1)) \left( \frac{|I_t^{(A)}|}{2} + \frac{|I_t^{(B)}|}{2} - \frac{\varepsilon |I_t^{(B)}|}{2(1 - \varepsilon)} \right).$$

accumulating the calculated expectations for pull and push yields the claim.
Next we show b). The assumption implies that \(|I_1| = (1 - o(1))n\) and therefore \(|I_1^{(A)}| = |I_1^{(B)}| = (1 - o(1))n/2\). Let \(A_u\) be the event that an uninformed vertex \(u\) does not get informed by the push algorithm, let \(B_u\) be the corresponding event for pull. Then \(A_u\) and \(B_u\) are independent and \(A_u \cap B_u\) is the event that \(u\) does not get informed in the current round. Let \(u \in U_t^{(A)}\), then

\[
P_t[A_u] = \prod_{v \in I_t^{(A)}} \left(1 - \frac{q}{|N(v)|}\right) \prod_{v \in I_t^{(B)}} \left(1 - \frac{q}{|N(v)|}\right) = (1 - o(1)) \left(1 - \frac{q}{n}\right) \left(1 - \frac{q}{(1 - \varepsilon)n}\right)^{|I_t^{(B)}|}
\]

and

\[
P_t[B_u] = 1 - \frac{q|N(u) \cap I_t^{(A)}|}{|N(u)|} = 1 - \frac{q |I_t|}{n - 1} = 1 - q + o(1).
\]

Consider now \(u \in U_t^{(B)}\), then according to (3.26) we have \(|N(u) \cap I_t^{(B)}| = |N(u) \cap B_u| - |N(u) \cap U_t^{(B)}| = (1 + o(1))(1 - 2\varepsilon)n/2\); therefore

\[
P_t[A_u] = \prod_{v \in I_t^{(A)}} \left(1 - \frac{q}{|N(v)|}\right) \prod_{v \in N(u) \cap I_t^{(B)}} \left(1 - \frac{q}{|N(v)|}\right) = (1 - o(1))e^{-q/2} \left(1 - \frac{q}{(1 - \varepsilon)n}\right)^{|N(u) \cap I_t^{(B)}|}
\]

and

\[
P_t[B_u] = 1 - \frac{q|N(u) \cap I_t^{(B)}|}{|N(u)|} = 1 - (1 + o(1)) \frac{q |I_t^{(A)}| + |N(u) \cap I_t^{(B)}|}{(1 - \varepsilon)n} = 1 - q + o(1).
\]

Combining the results for \(u \in U_t^{(A)}\) and \(u \in U_t^{(B)}\) we get

\[
E_t[|U_{t+1}|] = \sum_{u \in U_t} P_t[A_u] P_t[B_u] \leq (1 + o(1))e^{-q(1/2 + (1/2 - \varepsilon)/(1 - \varepsilon))} (1 - q) |U_t|.
\]

**Remark 3.19.** Let \(\lambda_{\max}\) be the greatest eigenvalue of \(M\) as defined in Lemma 3.18 (a). Then

\[
\lambda_{\max} = 1 + 2q + (2q\sqrt{\varepsilon^2/2 - \varepsilon + 1}) - 1 + q\varepsilon)/(2 - 2\varepsilon) > 1 + 2q.
\]

Next comes a lemma that bounds the runtime of push\textsuperscript{d}pull on \(G_n(\varepsilon)\). In particular, Lemma 3.20 a) and c) provide a lower bound on the runtime and Lemma 3.20 a), b) and d) together with Lemma 3.17 (a) provide an upper bound.

**Lemma 3.20.** Let \(I_t = I_t^{(pp)}, \varepsilon > 0\) and \(\lambda = \lambda_{\max}(M)\) be the greatest eigenvalue of \(M\) as given in Lemma 3.18 (a). Consider \(G_n(\varepsilon)\).

a) Let \(\sqrt{\log n} \leq |I_t| \leq n/\log n\). Then there are \(\tau_1, \tau_2 = \log_\lambda(n/|I_t|) + o(\log n)\) such that \(|I_{t+\tau_1}| < n/\log n < |I_{t+\tau_2}|\).

b) Let \(n/\log n \leq |I_t| \leq n - n/\log n\). Then there is \(\tau = o(\log n)\) such that \(|I_{t+\tau}| > n - n/\log n\).

c) Let \(|I_t| \leq n/\log n\). Then there is \(\tau \geq \log n/\log((1 - q)^{-1} \exp(q(1/2 + (1/2 - \varepsilon)/(1 - \varepsilon)))) - o(\log n)\) such that \(|I_{t+\tau}| < n\).

d) Let \(|I_t| \geq n - n/\log n\) and \(q \in (0, 1)\). Then there is \(\tau \leq \log n/\log((1 - q)^{-1} \exp(q(1/2 + (1/2 - \varepsilon)/(1 - \varepsilon)))) + o(\log n)\) such that \(|I_{t+\tau}| = n|\).
Proof. We do not give a proof for b) as it follows immediately from Lemma 3.17 (a). For \( J \in \{A, B\} \) set 
\( U_t^{(A)} := U_t \cap J_n, I_t^{(j)} := I_t \cap J_n \). We prove a) first. Let \( t_0 > 0 \) be the first round such that \( |I_{t_0}| \geq \log \log n \) and set \( x_t \) and \( M \) as in Lemma 3.18 (a), note that Lemma 3.17 (a) also gives that \( x_t \geq \log \log n \). Then for all \( t \geq t_0 \) such that \( |I_t| \leq n/\log n \) we obtain from Lemma 3.18 (a) that \( \mathbb{E}_t[x_{t+1}] = (1 + o(1))Mx_t \) and, in particular, \( \mathbb{E}_t[(x_{t+1})_i] = \Theta(|I_t|) \) for \( i \in \{1, 2\} \). As every component of \( x_t \) is self-bounding, Lemma 2.1 applies and we get for \( i \in \{1, 2\} \)

\[
P_t[(x_{t+1})_i - \mathbb{E}_t[(x_{t+1})_i]] \geq \mathbb{E}_t[(x_{t+1})_i]^{2/3} = O(|I_t|^{-1/3})
\]

and by union bound, provided that \( |I_t| \leq n/\log n \),

\[
P_t \left[ \bigcap_{i \in \{1, 2\}} \left((x_{t+1})_i - \mathbb{E}_t[(x_{t+1})_i] \leq \mathbb{E}_t[(x_{t+1})_i]^{2/3}\right) \right] = 1 - O(|I_t|^{-1/3}). \tag{3.28}
\]

Using (3.28) we want to find a bound on \( |I_{t+1}| \). We get as long as \( |I_t| \leq n/\log n \) that

\[
(1 - O(|I_{t_0}|^{-1/3}))M^{t+1-t_0}x_{t_0} \leq x_{t+1} \leq (1 + O(|I_{t_0}|^{-1/3}))M^{t+1-t_0}x_{t_0}.
\]

As seen in Remark 3.19, \( M \) has maximal eigenvalue \( \lambda_{\max} \geq 1 \) and as \( M \) is a positive matrix there is a positive eigenvector \( v \) to \( \lambda_{\max} \), compare [31]. This gives constants \( c_1, c_2 > 0 \) such that \( c_1v \log \log n \leq x_t \leq c_2v \log \log n \) and for \( t \) large enough

\[
\frac{c_1}{c_2} \left((1 - O(|I_{t_0}|^{-1/3}))\lambda_{\max}\right)^{t+1-t_0}x_{t_0} \leq x_{t+1} \leq \frac{c_2}{c_1} \left((1 + O(|I_{t_0}|^{-1/3}))\lambda_{\max}\right)^{t+1-t_0}x_{t_0},
\]

and therefore

\[
|I_{t+1}| \leq \frac{c_1}{c_2}((1 + o(1))\lambda_{\max})^{t-t_0}|I_{t_0}|.
\]

as long as the right hand side is bounded by \( n/\log n \). For all these \( t \) we get additionally

\[
|I_{t+1}| \geq \frac{c_2}{c_1}((1 - o(1))\lambda_{\max})^{t-t_0}|I_{t_0}|.
\]

Proceeding as in Examples 2.7 and 2.8, where we replace the events “\( |I_t| \geq \mathbb{E}_{t-1}[|I_t|] - \mathbb{E}_{t-1}[|I_t|]^{2/3} \) or \( |I_t| \geq n/g(n) \)” and “\( |I_t| - \mathbb{E}_{t-1}[|I_t|] \leq \mathbb{E}_{t-1}[|I_t|]^{2/3} \)” with “\( \bigcap_{i \in \{1, 2\}} (x_{t+1})_i \geq (1 - \mathbb{E}_t[(x_{t+1})_i])^{2/3} \)” or “\( |I_t| \geq n/\log n \)” and “\( \bigcap_{i \in \{1, 2\}} (x_{t+1})_i \leq \mathbb{E}_t[(x_{t+1})_i]^{2/3} \)” we obtain the statement.

Next we show c). The assumption guarantees that less than \( n/\log n \) vertices are informed. Thus \( |U_t^{(B)}| \geq n/2 - |I_t| \geq (1/2 - 1/\log n) \cdot n \). We consider a modified dissemination process, where in each round, each uninformed vertex always chooses an informed neighbour (but does not necessarily get informed as the message transmission may fail), and additionally each vertex chooses a neighbour iuar and after this round the chosen vertex is informed with probability \( q \); in other words, we assume that also uninformed vertices can inform other vertices. In this modified process the probability of an uninformed vertex \( u \in U_t^{(B)} \) staying uninformed after performing one round is given by the product of the probabilities of not being informed by pull or via push by a vertex in \( A_n \) or \( B_n \). Using (3.27) and \( (1 - 1/n)^n = e^{-1+o(1)} \) we get \( g(n) = o(1) \) such that

\[
P_t[u \in U_t^{(B)}] = (1 - q) \left(1 - \frac{q}{n}\right)^{n/2} \left(1 - \frac{q}{(1 - \varepsilon)n}\right)^{|N(u) \cap B_n|} = (1 - q) \exp \left(-q \left(\frac{1}{2} + \frac{1/2 - \varepsilon}{1 - \varepsilon}\right) + g(n)\right).
\]

As we have seen in the proof of Lemma 3.18 (b), the probability to be in formed by push/pull is greater for a vertex in \( A_n \) than for a vertex in \( B_n \). Therefore it is sensible to expect that some vertices in \( B_n \) will be
Finally we show $d)$. By Lemma 3.18 (b), we obtain that for any $\tau \in \mathbb{N}$,

$$\mathbb{E} \left[ |U_{t+\tau}| \right] \leq \left( (1 + o(1))e^{-q(1/(2 + (1/2 - \varepsilon)/(1 - \varepsilon))} \cdot (1 - q) \right)^\tau |U_t|.$$ 

Then for some

$$\tau := \frac{\log(n)}{\log((1 - q)^{-1} \exp(q(1/2 + (1/2 - \varepsilon)/(1 - \varepsilon))))) + o(\log(n))}$$

we obtain that, say, $\mathbb{E} \left[ |U_{t+\tau}| \right] \leq |U_t|/n \leq 1/\log n$. Thus $P_t[|U_{t+\tau}| \geq 1] \leq o(1)$ by Markov’s inequality. \hfill $\square$

Lemma 3.20 together with Lemma 2.13 give that

$$T_{pp}(G_n(\varepsilon), q) = \log \lambda n + \frac{1}{q(1 - 1.5\varepsilon)/(1 - \varepsilon) - \log(1 - q)} \log n + o(\log n)$$

where $\lambda = 1 + 2q + (2q(\sqrt{\varepsilon^2/2 - \varepsilon + 1}) - 1) + q\varepsilon)/(2 - 2\varepsilon) > 1 + 2q$. To see whether push\&pull actually slowed down (in terms of order $\log n$) one has to compare the runtime on this sequence of graphs to $c_{pp} \log n$; the runtime on expander sequences. In the figure below we can see that it slows down for nearly all values of $\varepsilon$ and $q$ in question; however, there are admissible values of $\varepsilon$ and $q$ such that the process even speeds up.

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Figure 1: Plotted values of $\Delta$ in $T_{pp}(G_n(\varepsilon), q) - c_{pp} \log n = \Delta \log n + o(\log n)$, for $0.9 < q < 1$ and $0 < \varepsilon < 1/2$.

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