Strains of plates of nonlinear anisotropic materials

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Abstract. A general approach to constructing nonlinear variants of connection between stresses and strains in anisotropic materials with different types of symmetry of properties is considered. This approach is based on the concept of elastic proper subspaces of anisotropic materials introduced in the mechanics of solids by J. Rychlewski and on the partial postulate of isotropy proposed by A. A. Il’yushin. The generalization of the partial postulate on the case of nonlinear anisotropic materials is formulated.

Some variants of nonlinear constitutive relations for anisotropic materials written in the space of stresses are offered. The analysis of these relations from the point of view of their satisfaction to the generalization of partial isotropy postulate on anisotropic materials is performed.

The comparison of calculations using the offered relations with experimental data on stretching of plates made of laminated composites is fulfilled.

1. Introduction
Works [1, 2] consider the general approach to establishing nonlinear stress-to-strain relations in anisotropic materials with various types of properties symmetry. This approach relies on the concept of elastic eigenstates of anisotropic materials and A. A. Il’yushin’s partial isotropy postulate. The generalization of this postulate for nonlinear isotropic materials is formulated.

The systems of strain invariants are developed as the length projections of the strain vector to eigen subspaces. Several variants of defining relations to describe the nonlinear stress-to-strain ratio in anisotropic materials are proposed. These relations are based on determining specific free energy as the function of the strain tensor and temperature.

Another possible approach to establishing defining relations is to use the Gibbs potential. It allows deriving nonlinear defining relations in an acceptable form relative to strains, which makes it much easier to process tensile, compression, and torsion test data.

2. Elastic eigen subspaces
Assume that the strain process \( \mathbf{\varepsilon}(t) \) is assigned to its image in a six-dimensional space.

This image is the strain path drawn as the time-distance plot of the six-dimensional vector \( \mathbf{\varepsilon}(t) \) with the six-dimensional stress vector \( \mathbf{\sigma}(t) \) built in each point. The connection between the coordinates of \( \mathbf{\varepsilon}, \mathbf{\sigma} \), and the components of the strain vector \( \mathbf{\varepsilon} \) and stress vector \( \mathbf{S} \) is used hereafter and provided in the work [3] as

\[
\mathbf{\varepsilon} = e_0 \mathbf{\varepsilon}_0 + e_1 \mathbf{\varepsilon}_1 + e_2 \mathbf{\varepsilon}_2 + e_3 \mathbf{\varepsilon}_3 + e_4 \mathbf{\varepsilon}_4 + e_5 \mathbf{\varepsilon}_5,
\]
where $\vec{e}_\alpha$, $\alpha = 0, \ldots, 5$ are the basis vectors of the six-dimensional strain space,

$$
e_0 = \frac{1}{\sqrt{3}} (\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}), \quad e_1 = \frac{1}{\sqrt{6}} (2\varepsilon_{33} - \varepsilon_{11} - \varepsilon_{22}), \quad e_2 = \frac{1}{\sqrt{2}} (\varepsilon_{11} - \varepsilon_{22}),$$

$$
e_3 = \frac{1}{\sqrt{2}} (\varepsilon_{12} - \varepsilon_{21}), \quad e_4 = \frac{1}{\sqrt{2}} (\varepsilon_{23} + \varepsilon_{32}), \quad e_5 = \frac{1}{\sqrt{2}} (\varepsilon_{31} + \varepsilon_{13}).$$

The reverse relations are

$$
\varepsilon_{11} = \frac{1}{\sqrt{3}} e_0 - \frac{1}{\sqrt{6}} e_1 + \frac{1}{\sqrt{2}} e_2, \quad \varepsilon_{22} = \frac{1}{\sqrt{3}} e_0 - \frac{1}{\sqrt{2}} e_1 - \frac{1}{\sqrt{2}} e_2, \quad \varepsilon_{33} = \frac{1}{\sqrt{3}} e_0 + \frac{\sqrt{2}}{3} e_1,
$$

$$
\varepsilon_{12} = \frac{e_3}{\sqrt{2}}, \quad \varepsilon_{23} = \frac{e_4}{\sqrt{2}}, \quad \varepsilon_{31} = \frac{e_5}{\sqrt{2}}.
$$

In case of anisotropic materials it is important to define the respective strain and stress tensor components $\varepsilon_{ij}$ and $S_{ij}$ relative to the Cartesian reference system the axes of which are directed along the main anisotropy axes $\vec{a}_1, \vec{a}_2, \vec{a}_3$. The basis vectors $\vec{e}_\alpha$ are images of tensors of the generalized canonical basis

$$I^0 = \frac{1}{\sqrt{3}} (\vec{a}_1 \vec{a}_1 + \vec{a}_2 \vec{a}_2 + \vec{a}_3 \vec{a}_3); \quad I^1 = \frac{1}{\sqrt{6}} (2\vec{a}_3 \vec{a}_3 - \vec{a}_1 \vec{a}_1 - \vec{a}_2 \vec{a}_2); \quad I^2 = \frac{1}{\sqrt{6}} (\vec{a}_1 \vec{a}_1 - \vec{a}_2 \vec{a}_2);$$

$$I^3 = \frac{1}{\sqrt{2}} (\vec{a}_1 \vec{a}_2 + \vec{a}_2 \vec{a}_1); \quad I^4 = \frac{1}{\sqrt{2}} (\vec{a}_2 \vec{a}_3 + \vec{a}_3 \vec{a}_2); \quad I^5 = \frac{1}{\sqrt{2}} (\vec{a}_3 \vec{a}_1 + \vec{a}_1 \vec{a}_3).$$

Let us record Hooke’s law in the six-dimensional space as

$$\vec{\sigma} = \mathbf{n} \cdot \vec{e},$$

where $\mathbf{n}$ is the material elasticity tensor image in the six-dimensional space, and find the eigenvalues of $\lambda_\alpha$ and eigenvectors $\vec{\omega}_\alpha$ of the tensor $\mathbf{n}$ as

$$\mathbf{n} \cdot \vec{\omega}_\alpha = \lambda_\alpha \vec{\omega}_\alpha.$$

The highest number of various eigenvalues for the second-order tensor in the six-dimensional space is six. Each single-valued root $\lambda_\alpha$ can be assigned to a one-dimensional eigen subspace with the basis tensor $\Omega_\alpha = \vec{\omega}_\alpha \vec{\omega}_\alpha$. Each root $\lambda_\alpha$ of multiplicity $k$ can be assigned to its eigen subspace of dimensionality $k$ with the basis vector

$$\Omega_\alpha = \sum_{\beta = \alpha}^{\beta = \alpha + k - 1} \vec{\omega}_\beta \vec{\omega}_\beta.$$

Because of the invariability of eigenvalues and eigenvectors, the basis tensors $\Omega_\alpha$ are invariant to the group of orthogonal eigen conversions.

According to J. Rychlewski [4], elastic eigenstates of materials are strains $\varepsilon_\alpha$ corresponding to six-dimensional vectors $\vec{\omega}_\alpha$. Elastic eigenstates for materials referred to various crystal systems are defined in works by N.I. Ostrosablin [5]. In works [1, 3, 6] this problem was solved in the considered six-dimensional space. In particular, as established by [1, 3, 6], non-one-dimensional
eigen subspaces are typical not for all anisotropic materials but only for single-axis crystals (tetragonal, trigonal, hexagonal) and cubical and isotropic media (in terms of elastic properties a transverse isotropic material behaves as a hexagonal crystal).

Now we shall formulate expressions for basis eigen tensors $\Omega_\alpha$ (3), that characterize anisotropic materials of various types, and refer the vectors and tensors of the six-dimensional space to the basis $\vec{i}_\alpha$.

**Isotropic materials** have two eigen subspaces: a one-dimensional subspace with the basis tensor $\Omega_1 = \vec{i}_0\vec{i}_0$ and a five-dimensional subspace with the basis tensor $\Omega_2 = \vec{i}_1\vec{i}_1 + \vec{i}_2\vec{i}_2 + \ldots + \vec{i}_5\vec{i}_5$. In the six-dimensional space the elasticity tensor is expressed as

$$\mathbf{n} = \lambda_1 \Omega_1 + \lambda_2 \Omega_2,$$

where $\lambda_1 = 3K$, $\lambda_2 = 2G$ are the elastic modules.

**Cubical materials** have three eigen subspaces: a one-dimensional subspace with the basis tensor $\Omega_1 = \vec{i}_0\vec{i}_0$, a two-dimensional subspace with the basis tensor $\Omega_2 = \vec{i}_1\vec{i}_1 + \vec{i}_2\vec{i}_2$, and a three-dimensional subspace with the basis tensor $\Omega_3 = \vec{i}_3\vec{i}_3 + \vec{i}_4\vec{i}_4 + \vec{i}_5\vec{i}_5$.

**Hexagonal and trigonal materials** have four eigen subspaces: two one-dimensional subspaces with the bases $\Omega_1 = \vec{\omega}_1\vec{\omega}_1$ and $\Omega_2 = \vec{\omega}_2\vec{\omega}_2$, where the eigen and the basis vector are bound by the relations

$$\vec{\omega}_1 = \vec{i}_0 \cos \varphi + \vec{i}_1 \sin \varphi, \quad \vec{\omega}_2 = -\vec{i}_0 \sin \varphi + \vec{i}_1 \cos \varphi$$

and the angle $\varphi$ is found via the components $n_{\alpha\beta} = \vec{i}_\alpha \cdot \mathbf{n} \cdot \vec{i}_\beta$ as

$$\tan \varphi = \frac{2n_{01}}{n_{00} - n_{11} + \sqrt{(n_{00} - n_{11})^2 + 4n_{01}^2}}$$

and two two-dimensional eigen subspaces with the bases $\Omega_3 = \vec{i}_2\vec{i}_2 + \vec{i}_3\vec{i}_3$ and $\Omega_4 = \vec{i}_4\vec{i}_4 + \vec{i}_5\vec{i}_5$ for hexagonal materials and with the bases $\Omega_3 = \vec{\omega}_3\vec{\omega}_3 + \vec{\omega}_4\vec{\omega}_4$ and $\Omega_4 = \vec{\omega}_4\vec{\omega}_4 + \vec{\omega}_5\vec{\omega}_5$ for trigonal materials. In the latter case the position of the eigenvectors relative to the basis of the six-dimensional space is defined by the following relations:

$$\vec{\omega}_3 = \vec{i}_2 \cos \psi + \vec{i}_4 \sin \psi, \quad \vec{\omega}_4 = -\vec{i}_2 \sin \psi + \vec{i}_4 \cos \psi,$$

$$\vec{\omega}_5 = \vec{i}_3 \cos \psi + \vec{i}_5 \sin \psi, \quad \vec{\omega}_6 = -\vec{i}_3 \sin \psi + \vec{i}_5 \cos \psi,$$

$$\tan \psi = \frac{2n_{24}}{n_{22} - n_{44} + \sqrt{(n_{22} - n_{44})^2 + 4n_{24}^2}}$$

**Tetragonal materials** have five eigen subspaces: four one-dimensional subspaces with the bases $\Omega_1 = \vec{\omega}_1\vec{\omega}_1$, $\Omega_2 = \vec{\omega}_2\vec{\omega}_2$, $\Omega_3 = \vec{i}_2\vec{i}_2$, $\Omega_4 = \vec{i}_3\vec{i}_3$, where the eigenvectors $\vec{\omega}_1$ and $\vec{\omega}_2$ and the basis vectors are bound by the same relations as in hexagonal or trigonal materials, and the only two-dimensional eigen subspace with the basis $\Omega_5 = \vec{i}_4\vec{i}_4 + \vec{i}_5\vec{i}_5$.

It follows from the equations provided for the basis eigen tensors that the operators $\mathbf{n}$ are invariant to their eigen transformations. For example, isotropic material operator (4) is invariant relative to the rotation and reflection of the six-dimensional basis relative to the vector $\vec{i}_0$. One can treat this conversion as the rotation of A.A. Il’yushin’s basis round the $\vec{i}_0$ axis orthogonal to the $\vec{i}_1, \vec{i}_2$ deviator plane. That said, the strained mode angle proportionate to the rotation angle changes, which does not, however, affect the stress-and-strain linear connection described by Hooke’s law.

Thus the operator $\mathbf{n}$ of linearly elastic materials meets the condition of invariability relative to the subgroup of orthogonal eigen conversions. This condition stems from the linear connection between stress and strain tensors and is defined by its uniqueness.
3. Partial isotropy postulate and its generalizations

According to A. A. Il’yushin’s partial postulate, the six-dimensional image of the strain process of an originally isotropic body is invariant not only relative to the group of orthogonal conversions related to choosing the initial frame of reference but also to the random conversions of rotation and reflection in the five-dimensional deviator subspace. Since these conversions change the third invariants of stress and strain tensors, the partial postulate makes it necessary that the invariants of this type are not explicitly included in the number of functions used to determine scalar-valued functionals defining properties of materials. The partial isotropy postulate is the hypothesis making it possible to formulate quasi-linear constitutive relations; that said, the linear relations of Hooke’s law identically meet the conditions of this postulate.

The partial isotropy postulate has been many times validated experimentally. In his book [7] A. A. Il’yushin mentions the possibility of generalizing this postulate for originally anisotropic materials. This generalization is formulated in the space of generalized stresses, e.g., found by multiplying stresses by anisotropic basis tensors [3]. In this space one needs to define the class of invariant conversions preserving the form of the stress-to-strain connection.

Let us assume that stress and strain vectors in each eigen subspace for linearly elastic anisotropic materials are collinear [6]. This connection is preserved upon any orthogonal conversion of the strain path in each eigen subspace that is not one-dimensional. In particular, this condition for an isotropic material is met in the five-dimensional deviatoric subspace. In a cubical crystal system the specimen isotropy condition is met in the vector plane \( \vec{\omega}_2, \vec{\omega}_3 \) and in the three-dimensional subspace with the basis \( \vec{\omega}_4, \vec{\omega}_5, \vec{\omega}_6; \) in a hexagonal (trigonal) material this condition is met in the planes \( \vec{\omega}_3, \vec{\omega}_4 (\vec{\omega}_3, \vec{\omega}_5) \) and \( \vec{\omega}_5, \vec{\omega}_6 (\vec{\omega}_4, \vec{\omega}_6); \) in a tetragonal material it holds in the plane \( \vec{\omega}_5, \vec{\omega}_6. \) Let us extend this property to all the phases of the strain process. As a result, the partial postulate generalized for nonlinear anisotropic materials will be recorded as:

The image of the thermomechanical process with a strain path in the eigen subspace of the initial elasticity tensor is invariant relative to the group of orthogonal eigen conversions.

It follows from this generalization that the loading in each non-one-dimensional eigen subspace is determined only by the internal path of strain in this subspace and does not depend on the position of this path relative to the basis vectors of this subspace. This formulation makes possible the deviation of the stress vector from the strain vector located in its eigen subspace. Moreover, this deviation is the second-order smallness effect relative to strain because the strain and stress vectors are asymptotically collinear. Suffice it to say that the kind of deviation in view of the postulate must not change at orthogonal eigen conversions. An example of such behavior is the generation of hydrostatic stresses at shear strains in an isotropic material.

If to suppose that, similarly to the strain path, the loading image is located in its eigen subspace then the formulated assumption will result in representing the connection between the stress and the strain vector in each of the eigen subspaces as

\[
\vec{\sigma}(t) = \sum_{i=1}^{m} A^i(t) \mathbf{F}^i(t).
\]  

In relation (5) \( A^i(t) \) \( s(i)(\tau), k^1(i)(\tau), \ldots, k^{m-1}(\tau), T(\tau) \) are the functionals of the strain \( \vec{e}(t) \) and the law of temperature variation \( T(t) \), \( \mathbf{F}^i(t) \) is the basis of the eigen subspace of dimensionality \( m \), covariant relative to orthogonal eigen conversions, \( s(i) \) is the arc length, and \( k^l(i) \) is the eigen path curvature length. In case of smooth eigen paths it is convenient to use Frenet’s basis connected to the path of \( \vec{e}(t) \) as the \( \mathbf{F}^i(t) \) basis.

Relations (5) will be referred to as the limit formula of the partial postulate. They significantly restrict the reaction of anisotropic materials to the determined strain (loading) process, that
allows developing programs of experimental specification of relations defining material properties within the limit form.

To specify the properties of an originally anisotropic material, proceeding from (5), one needs to define the position of the eigenvectors, eigen subspaces, and their dimensionality on the basis of the elastic properties of this material. The form of the functionals \( A_{(\alpha)} \) can be defined by implementing the paths of strain (loading) in each of the eigen subspaces.

4. Using Gibbs’ potential to derive constitutive relations

As a general matter, Gibbs’ potential is found as in [8]:

\[
G = U - ST - \frac{1}{\rho_0} \mathbf{S} \cdot \mathbf{\varepsilon},
\]

(6)

where \( U \) is the specific internal energy per unit weight, \( S \) is the specific entropy, \( T \) is the temperature, \( \rho_0 \) is the density of the material in its original state; \( \mathbf{S} \) and \( \mathbf{\varepsilon} \) are the energetically conjugate stress and strain tensors.

Considering (6) for Gibbs’ potential, the main thermal mechanical relation for reversible processes in Gibbs’ form is recorded as

\[
dG = -\frac{1}{\rho_0} \mathbf{\varepsilon} \cdot d\mathbf{S} - SdT.
\]

(7)

Gibbs’ potential is treated as the stress tensor and temperature function \( G = G(\mathbf{S}, T) \); then the expression for its differential is

\[
dG = \frac{\partial G}{\partial \mathbf{S}} \cdot d\mathbf{S} + \frac{\partial G}{\partial T} dT.
\]

(8)

The equations defining the behavior of the reversely strained material meet relation (6) and can be attained by assigning concrete forms of equations to Gibbs’ potential. It follows from (7) and (8) that

\[
\mathbf{\varepsilon} = -\rho_0 \frac{\partial G}{\partial \mathbf{S}}.
\]

(9)

To specify equations (9) of the strain and entropy measure, the stress and temperature measure must be used to define the specific form of the \( G = G(\mathbf{S}, T) \) function determining Gibbs’ potential.

The power of external forces for any pair of energetically conjugate stress and strain tensors is defined in the six-dimensional space as

\[
N^{(e)} = \frac{1}{\rho_0} \mathbf{\bar{\sigma}} \cdot \mathbf{\dot{\varepsilon}}.
\]

(10)

Considering (10), Gibbs’ potential in the six-dimensional space is recorded as

\[
G = U - ST - \frac{1}{\rho_0} \mathbf{\bar{\sigma}} \cdot \mathbf{\varepsilon}.
\]

(11)

Considering (11), the main thermal mechanical relation in Gibbs’ form (7) is recorded as

\[
dG = -\frac{1}{\rho_0} \mathbf{\dot{\varepsilon}} \cdot d\mathbf{\bar{\sigma}} - SdT.
\]

(12)

Considering the formula for the potential differential \( G \)
\[ dG = \frac{\partial G}{\partial \bar{\sigma}} \cdot d\bar{\sigma} + \frac{\partial G}{\partial T} dT, \]

the strain vector and the entropy are derived from (12) as

\[ \vec{e} = -\rho_0 \frac{\partial G}{\partial \sigma} \quad \text{and} \quad S = -\frac{\partial G}{\partial T}. \quad (13) \]

To specify equations (13) of the strain and entropy measure, let us use the stress and temperature measure to define the concrete form of the \( G = G(\vec{\sigma}, T) \) Gibbs’ potential. If to suppose that the reaction of the material complies with the limit form of the partial isotropy postulate, Gibbs’ potential should be recorded as the quadratic dependence of the following kind:

\[ \rho_0 G = -\frac{1}{2} \sum_{\alpha, \beta = 0}^{m-1} c^{\alpha \beta} \sigma_\alpha \sigma_\beta - \frac{n}{\gamma = 1} D^\gamma t^2_{(\gamma)} - \sum_{\alpha = 0}^{m-1} a^\alpha \sigma_\alpha (T - T_0) + \rho_0 G^{(0)}(T), \quad (14) \]

where \( c^{\alpha \beta}, D^\gamma, a^\alpha \) are the material constants, \( \sigma_\alpha \) are the linear invariants of stresses, \( t^2_{(\gamma)} = \bar{\sigma} \cdot \Omega_\gamma \cdot \bar{\sigma} \) are the quadratic invariants of stresses defined via the basis tensors of the quadratic invariant subspaces \( \Omega_\gamma \) [7], \( \bar{\sigma}_{(\gamma)} = \bar{\sigma} \cdot \Omega_\gamma \) is the stress vector projected to the eigen subspace \( \Omega_\gamma \).

In relations (14) \( m \) and \( n \) indicate the number of linear and quadratic invariants for a certain material, respectively.

The result according to (13) and (14) is

\[ \vec{e} = \sum_{\alpha, \beta = 0}^{m-1} c^{\alpha \beta} \sigma_\alpha \sigma_\beta \quad \text{and} \quad S = \frac{1}{m} \sum_{\alpha = 0}^{m-1} a^\alpha \sigma_\alpha - \frac{dG^{(0)}}{dT}, \quad (15) \]

\[ S = \frac{1}{\rho_0} \sum_{\alpha = 0}^{m-1} a^\alpha \sigma_\alpha - \frac{dG^{(0)}}{dT}. \quad (16) \]

Let us find the expression of the temperature-dependent summand \( G^{(0)}(T) \) as represented by Gibbs’ potential (14). To specify this summand, take into account that entropy itself is thermodynamic potential [7, 8] and can therefore be represented as the state parameters function \( S = S(\sigma, T) \), then

\[ dS = \frac{\partial S}{\partial \bar{\sigma}} \cdot d\bar{\sigma} + \frac{\partial S}{\partial T} dT. \]

It follows from (16) for entropy that \( dS = \frac{1}{\rho_0} \sum_{\alpha = 0}^{m-1} a^\alpha d\sigma_\alpha - \frac{d^2G^{(0)}}{dT^2} dT \), then

\[ \frac{\partial S}{\partial T} = -\frac{d^2G^{(0)}}{dT^2}. \]

The quantity \( T \frac{\partial S}{\partial T} \) indicates the amount of heat formed per unit weight at variations in temperature and permanent stresses, i.e., specific heat capacity at permanent stresses: \( c_\sigma = -T \frac{d^2G^{(0)}}{dT^2} \). If to treat \( c_\sigma \) as constant, the post-integration formulas will be

\[ -\frac{dG^{(0)}}{dT} = c_\sigma \ln \frac{T}{T_0}, \quad G^{(0)} = -c_\sigma T_0 \left( \frac{T}{T_0} \ln \frac{T}{T_0} - \frac{T}{T_0} + 1 \right). \]
Then entropy is found as

\[ S = \frac{1}{\rho_0} \sum_{\alpha=0}^{m-1} a^\alpha \sigma_\alpha + c_\sigma \ln \frac{T}{T_0}. \]

The relation \( c_\sigma = c_e + \bar{\alpha} \cdot C^{-1} \cdot \bar{\alpha} \) links the \( c_\sigma \) heat capacity at permanent stresses with the \( c_e \) heat capacity at permanent strains [7, 8].

Let us find the physical meaning of the constants included in constitutive relations (15), for which purpose the thermal mechanical process with no stresses will be considered. The strains generated in this process are called thermal. The thermal strains determined by (15) are recorded as

\[ \bar{e}^{(T)} = \bar{e}^{(T)}_{\sigma=0} = \sum_{\alpha=0}^{m-1} a^\alpha i_\alpha (T - T_0), \]

whence it follows that \( a^\alpha \) are the constants characterizing the thermal strains.

It follows from (17) that the decomposition of the thermal strains vector contains only the six-dimensional space basis vectors invariant to the given type of material.

If to consider the isothermal process, the linear equations for strains according to (15) will be

\[ \bar{e} = \sum_{\alpha,\beta=0}^{m-1} c^{\alpha\beta} \sigma_\beta i_\alpha + \sum_{\gamma=1}^{n} 2D^\gamma \bar{\sigma}^{(\gamma)}. \]

Let us record relations (15) for an isotropic material at \( m = 1 \), \( i_\alpha = \bar{i}_0 \), \( \sigma_0 = \bar{\sigma} \cdot \bar{i}_0 \), \( n = 1 \), \( \bar{\sigma}_{(1)} = \bar{\sigma} \cdot (\bar{i}_1 \bar{i}_1 + \bar{i}_2 \bar{i}_2 + \bar{i}_3 \bar{i}_3 + \bar{i}_4 \bar{i}_4 + \bar{i}_5 \bar{i}_5) = \bar{\tau} \) is the loading vector:

\[ \bar{e} = c^{00} \sigma_0 \bar{\tau} + 2D \bar{\tau}, \]

where \( c^{00} = \frac{1}{K} \), \( D = \frac{1}{\alpha G} \). The resulting relation at infinitesimal strains is the record of Hooke’s law.

Relations (15) for a transverse isotropic material at \( m = 2 \), \( i_0, i_1, \sigma_0 = \bar{\sigma} \cdot \bar{i}_0, \sigma_1 = \bar{\sigma} \cdot \bar{i}_1, n = 2 \), \( \bar{\sigma}_{(1)} = \bar{\sigma} \cdot (\bar{i}_2 \bar{i}_2 + \bar{i}_3 \bar{i}_3) = \sigma_2 \bar{i}_2 + \sigma_3 \bar{i}_3, \bar{\sigma}_{(2)} = \bar{\sigma} \cdot (\bar{i}_4 \bar{i}_4 + \bar{i}_5 \bar{i}_5) = \sigma_4 \bar{i}_4 + \sigma_5 \bar{i}_5 \) are

\[ \bar{e} = (c^{00} \sigma_0 + c^{01} \sigma_1) \bar{i}_0 + (c^{10} \sigma_0 + c^{11} \sigma_1) \bar{i}_1 + c^{22} (\sigma_2 \bar{i}_2 + \sigma_3 \bar{i}_3) + c^{44} (\sigma_4 \bar{i}_4 + \sigma_5 \bar{i}_5). \]

The more complex thermomechanical model of an isotropic material is built within the limit form of the partial postulate and derived by representing Gibbs’ potential as

\[ \rho_0 G = -\frac{1}{2} \sum_{\alpha,\beta=0}^{m-1} c^{\alpha\beta} (\sigma_\alpha \sigma_\beta) + \sum_{\gamma=1}^{n} D^\gamma \left( \bar{t}^\gamma_{(\gamma)} \right) - \sum_{\alpha=0}^{m-1} a^\alpha \sigma_\alpha (T - T_0) + \rho_0 G^{(0)}(T), \]

where the coefficients defining the elastic properties of the material are considered the functions of stress invariants, wherein \( c^{\alpha\beta} \) depend only on linear invariants and \( D^\gamma \) depend on quadratic invariants in respective non-one-dimensional eigen subspaces.

Let us define the strains and entropy proceeding from equations (13) and (19) as
\[ \vec{e} = -\rho_0 \frac{\partial G}{\partial \sigma} = \frac{1}{2} \sum_{\alpha, \beta=0}^{m-1} \left( \frac{\partial c_{\alpha \beta}}{\partial \sigma_{\alpha}} \cdot \frac{\partial \sigma_{\alpha}}{\partial \sigma} \sigma_{\alpha} \sigma_{\beta} + \frac{\partial c_{\alpha \beta}}{\partial \sigma_{\beta}} \cdot \frac{\partial \sigma_{\beta}}{\partial \sigma} \sigma_{\alpha} \sigma_{\beta} + 2c_{\alpha \beta} \frac{\partial \sigma_{\alpha}}{\partial \sigma} \right) + \]
\[ + \sum_{\gamma=1}^{n} \left( \frac{\partial D^\gamma}{\partial \sigma_{\alpha}} \cdot \frac{\partial \sigma_{\gamma}}{\partial \sigma} + D^\gamma \right) + \sum_{\alpha=0}^{m-1} a^\alpha \frac{\partial \sigma_{\alpha}}{\partial \sigma} (T - T_0), \]
\[ \vec{e} = \sum_{\alpha, \beta=0}^{m-1} \left( \frac{\partial c_{\alpha \beta}}{\partial \sigma_{\alpha}} \sigma_{\alpha} \sigma_{\beta} + c_{\alpha \beta} \sigma_{\alpha} \sigma_{\beta} + a^\alpha \right) (T - T_0) + \sum_{\gamma=1}^{n} \left( 2D^\gamma \right) \sigma_{\alpha} \sigma_{\beta} + 2\bar{c}^\gamma \sigma_{\alpha} \sigma_{\beta} \right) \vec{e} = \left( G^1 + \frac{\partial G}{\partial \sigma_{\alpha}} \sigma_{\alpha} \right) \vec{e} \]
\[ S = \frac{1}{\rho_0} \sum_{\alpha=0}^{m-1} a^\alpha \sigma_{\alpha} + c_{\alpha} \ln \frac{T}{T_0}. \]

Constitutive relations (20) comply with the limit form of the partial isotropy postulate because the loading process in each non-one-dimensional eigen subspace \( \vec{e}_{(\gamma)} \) does not affect the processes in the other eigen subspaces and nor does it depend on them: if \( \vec{e} = \vec{e}_{(\gamma)} \), then \( \vec{e} = \vec{e}_{(\gamma)} \); if \( \sigma = \sigma_{(\gamma)} \), then \( \vec{e} = \vec{e}_{(\gamma)} \) as well. For a transverse isotropic material, for example, this means that the loading in the isotropy plane, determined by the stress tensor \( S = t_{12}(\vec{e}_1 + \vec{e}_2) \) and the respective stress vector \( \vec{e} = \sigma_{33} \vec{e}_3 \), when the process develops in a non-one-dimensional eigen subspace with the basis tensor \( \Omega = t_{12} \vec{e}_2 + \vec{e}_3 \), will not result in constituent strains outside this subspace because \( \vec{e} = 2 \left( G^1 + \frac{\partial G}{\partial \sigma_{\alpha}} \sigma_{\alpha} \right) \vec{e} \). Moreover, in this case the strain and stress vectors are collinear. If to assume that in relation (19) \( c_{\alpha \beta}(\sigma_{\alpha}, \sigma_{\beta}) = c^\alpha_{\beta} + c^{\alpha \beta}(\sigma_{\alpha} + \sigma_{\beta}) \) and \( D^\gamma \left( t_{(\gamma)} \right) = D^\gamma + \bar{D}^\gamma t_{(\gamma)} \), where \( c^\alpha_{\beta}, c^{\alpha \beta}, D^\gamma, \bar{D}^\gamma \) are the constants, the Gibbs’ potential expression will be
\[ \rho_0 G = -\frac{1}{2} \sum_{\alpha, \beta=0}^{m-1} \left( c^{\alpha \beta} + c^{\alpha \beta}(\sigma_{\alpha} + \sigma_{\beta}) \right) \sigma_{\alpha} \sigma_{\beta} - \sum_{\gamma=1}^{n} \left( D^\gamma + \bar{D}^\gamma t_{(\gamma)} \right) t_{(\gamma)} - \sum_{\alpha=0}^{m-1} a^\alpha \sigma_{\alpha} (T - T_0) + \rho_0 G^{(0)}(T). \]

Let us use this representation of Gibbs’ potential to find the strains as
\[ \vec{e} = \sum_{\alpha, \beta=0}^{m-1} \left( c^{\alpha \beta}_{\sigma_{\beta}} + 2c^{\alpha \beta} \sigma_{\alpha} \sigma_{\beta} + c^{\alpha \beta} \sigma_{\alpha} \sigma_{\beta}^2 + a^\alpha (T - T_0) \right) \vec{e} + \]
\[ + 2 \sum_{\gamma=1}^{n} \left( D^\gamma + \frac{3}{2} \bar{D}^\gamma t_{(\gamma)} \right) \sigma_{\alpha} \sigma_{\beta} \vec{e}_{(\gamma)}. \]
In case of small strains during isothermal processes these relations correspond to the generalized Hooke’s law:

\[ \tilde{e} = \sum_{\alpha,\beta=0}^{m-1} c_{0}^{\alpha\beta} \sigma_{\beta} \tilde{\sigma}_{\alpha} + 2 \sum_{\gamma=1}^{n} D^{\gamma} \tilde{\sigma}_{(\gamma)}, \]

i.e., the constants \( c_{0}^{\alpha\beta} \), \( D^{\gamma} \) form the elastic restraints tensor reverse to the material elasticity tensor.

In case of an isotropic material with one linear invariant \( \sigma_{0} \), \( e_{0} \) (\( m = 1 \)), and one quadratic invariant \( t_{(1)}^{2} = \tau^{2} = \bar{\tau} \cdot \bar{\tau} \), \( e_{1}^{(2)} = s_{(1)}^{2} = \bar{e} \cdot \bar{e} \) (\( n = 1 \)) relations (21) at a constant temperature are recorded as

\[ \tilde{e} = (c_{0}^{00} \sigma_{0} + 3c_{0}^{00} \sigma_{0}^{2} \bar{\tau}) \bar{t}_{0} + 2 \left( D^{1} + \frac{3}{2} D^{1} \tau \right) \tilde{\sigma}_{(1)} \]

and represent the four-constant quadratic model of an elastic isotropic body. That said, the volumetric strain-hydrostatic stress dependence and the dependence of the forming intensity on the stress intensity are recorded as

\[ e_{0} = c_{0}^{00} \sigma_{0} + 3c_{0}^{00} \sigma_{0}^{2}, \quad s_{(1)} = 2D^{1} \tau + 3D^{1} \tau^{2}. \]

The constants of this model can be derived from two comprehensive compression and shear tests.

If the material is orthotropic, it has six eigen subspaces: three of them are linear with the bases \( \Omega_{1} = \bar{\omega}_{1} \bar{\omega}_{1}, \Omega_{2} = \bar{\omega}_{2} \bar{\omega}_{2}, \Omega_{3} = \bar{\omega}_{3} \bar{\omega}_{3} \), with the eigenvectors \( \bar{\omega}_{1}, \bar{\omega}_{2}, \bar{\omega}_{3} \) located in a three-dimensional subspace with the basis vectors \( \tilde{t}_{0}, \tilde{t}_{1}, \tilde{t}_{2} \), and the other three are quadratic subspaces with the bases \( \Omega_{4} = \bar{\omega}_{4} \bar{\omega}_{4}, \Omega_{5} = \bar{\omega}_{5} \bar{\omega}_{5}, \Omega_{6} = \bar{\omega}_{6} \bar{\omega}_{6} \). Relations (21) for an isothermally strained orthotropic material are recorded as

\[ \tilde{e} = \sum_{\alpha,\beta=0}^{2} \left( c_{0}^{\alpha\beta} \sigma_{\beta} + 2c_{0}^{\alpha\beta} \sigma_{\beta} + c_{0}^{\alpha\beta} \sigma_{\beta}^{2} \right) \tilde{t}_{\alpha} + 2 \sum_{\gamma=1}^{3} \left( D^{\gamma} + \frac{3}{2} D^{\gamma} t_{(\gamma)} \right) \tilde{\sigma}_{(\gamma)}, \tag{22} \]

whence \( \sigma_{\alpha} = \bar{\sigma} \cdot \tilde{t}_{\alpha}, \alpha = 0, 1, 2 \) are the linear invariants of stresses, \( \tilde{\sigma}_{(1)} = \left( \bar{\sigma} \cdot \tilde{t}_{1} \right) \tilde{t}_{1}, t_{(1)} = \left| \bar{\sigma} \cdot \tilde{t}_{1} \right| \), \( \tilde{\sigma}_{(2)} = \left( \bar{\sigma} \cdot \tilde{t}_{2} \right) \tilde{t}_{2}, t_{(2)} = \left| \bar{\sigma} \cdot \tilde{t}_{2} \right| \), \( \tilde{\sigma}_{(3)} = \left( \bar{\sigma} \cdot \tilde{t}_{3} \right) \tilde{t}_{3}, t_{(3)} = \left| \bar{\sigma} \cdot \tilde{t}_{3} \right| \) are the quadratic invariants of stresses.

Gibbs’ potential can be used to build even more complex models, taking account of the reciprocal influences of processes developing in various eigen subspaces.

5. Strains of orthotropic plates

Let us record relations (22) in tensor form in the three-dimensional space, using representations (2) for the tensors of the generalized canonical basis that are recorded in the main anisotropy axes:

\[ \varepsilon = \sum_{\alpha,\beta=0}^{2} \left( c_{0}^{\alpha\beta} \sigma_{\beta} + 2c_{0}^{\alpha\beta} \sigma_{\beta} + c_{0}^{\alpha\beta} \sigma_{\beta}^{2} \right) \Gamma^{\alpha} + 2 \sum_{\gamma=1}^{3} \left( D^{\gamma} + \frac{3}{2} D^{\gamma} t_{(\gamma)} \right) S_{(\gamma)}, \tag{23} \]

where \( \varepsilon \) is the strain tensor, \( S \) is the Cauchy stress tensor.

The invariants of stresses for an orthotropic material according to (1) are expressed via the stress tensor components in the main anisotropy axes according to the following relations:
Figure 1. Stress-strain dependence in a stretched orthogonal plate.

\[ \sigma_0 = \frac{1}{\sqrt{3}} (S_{11} + S_{22} + S_{33}), \quad \sigma_1 = \frac{1}{\sqrt{6}} (2S_{33} - S_{11} - S_{22}), \quad \sigma_2 = \frac{1}{\sqrt{2}} (S_{11} - S_{22}), \quad (24) \]

\[ t_{(1)}^2 = S_{12}^2, \quad t_{(2)}^2 = S_{23}^2, \quad t_{(3)}^2 = S_{31}^2. \]

Relations (23) were used to solve the problem of the isothermal loading of an orthotropic square plate with the plane-perpendicular anisotropy axis \( \vec{a}_1 \) and the axes \( \vec{a}_2 \) and \( \vec{a}_3 \) forming the angle \( \phi \) with the plate sides. The plate stretches under the stress \( t \) directed along a side of the square.

The result of proceeding from the invariant form of recording defining relations (23) to the connection among the stress components in the main anisotropy axes is

\[ \varepsilon_{22} = A_{2222} S_{22} + A_{2233} S_{33} + \alpha_{22} S_{22}^2 + 2\alpha_{32} S_{22} S_{33} + \alpha_{23} S_{33}^2, \]

\[ \varepsilon_{33} = A_{3322} S_{22} + A_{3333} S_{33} + \alpha_{32} S_{33}^2 + 2\alpha_{23} S_{22} S_{33} + \alpha_{33} S_{33}^2, \quad (25) \]

\[ \varepsilon_{23} = (A_{2323} + \alpha_{2323} |S_{23}|) S_{23}, \]

where \( A_{2222}, A_{2233} = A_{3322}, A_{3333}, A_{2323} \) are the components of the material elasticity tensor in the considered plane, \( \alpha_{22}, \alpha_{23}, \alpha_{32}, \alpha_{33}, \alpha_{2323} \) are the additional constants. These constants make it possible to consider the stress-and-strain nonlinear ratio and are related to the coefficients from (23) via nonlinear relations.

The requirement true for (25) is \( \frac{\partial \varepsilon_{ij}}{\partial \sigma_{ji}} = \frac{\partial \varepsilon_{ji}}{\partial \sigma_{ij}}. \)

Paper [9] provides the results of stretch (compression) tests of laminated composite plates treated as orthotropic. The test data allowed finding the material constants included in relations (25).

For the plotted dependence of the stress \( t \) on the longitudinal (\( \varepsilon_{xx} \)) and transverse (\( \varepsilon_{yy} \)) strains of the plate at various values of the orienting angle \( \phi \) of the main anisotropy axes see figure 1: in figures. 1a, 1b, and 1c \( \phi = 0, \ \phi = \pi/8, \) and \( \phi = \pi/4, \) respectively.

The red curves and the blue curves in the figure correspond to the longitudinal strain-stress dependence and the transverse strain-stress dependence, respectively. The points indicate the test data, the solid lines correspond to the calculation according to (25), the dashed lines are drawn according to the nonlinear anisotropic elasticity proposed in [9]. The latter variant takes
account of the nonlinear relationship between the shear strains $\varepsilon_{23}$ and the tangent stresses $S_{23}$, whereas the relationship between the normal stresses and the extension (compression) strains is considered linear. E. V. Lomakin’s variant of relations can be found from (25) at zero constants $\alpha_{22}$, $\alpha_{23}$, $\alpha_{32}$, $\alpha_{33}$. It follows from the figure that the test data are better described using (25) than the relations from [9], especially at the extension along the main anisotropy axes.

6. Conclusion
The proposed variant of nonlinear relations of anisotropic elasticity complies with the limit form of generalizing A. A. Il’yaushin’s partial postulate and can be specified in a set of tests aimed at extending (compressing) anisotropic plates located on different material planes. The relations describe the data of the known tests fairly well.

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