Tutte’s 3-Flow Conjecture in 3-tree-connected graphs

MORTEZA HASANVAND,

Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran
morteza.hasanvand@alum.sharif.edu

Abstract

Tutte’s 3-flow conjecture says that every 4-edge-connected graph admits a nowhere-zero 3-flow. Kochol (2001) showed that it is enough to prove this conjecture for 5-edge-connected graphs. Former, Jaeger, Linial, Payan, and Tarsi (1992) conjectured that every 5-edge-connected graph is $Z_3$-connected and so it admits a nowhere-zero 3-flow. In this note, we show that if the second conjecture would be true, then every 3-tree-connected graph must also be $Z_3$-connected and so Tutte’s 3-flow conjecture can be extended to this family of graphs.

Keywords: Tutte’s 3-Flow Conjecture; $Z_3$-connectivity; modulo orientation; tree-connectivity.

Introduction

In this note, all graphs have no loop, but multiple edges are allowed. Let $G$ be a graph, let $k$ be an integer, $k \geq 2$, and let $p: V(G) \to Z_k$ be a mapping such that $|E(G)| \equiv \sum_{v \in V(G)} p(v)$, where $Z_k$ is the cyclic group of order $k$. An orientation of $G$ is called $p$-orientation, if for every vertex $v$, $d_G^+(v) \equiv p(v)$, where $d_G^+(v)$ denotes the out-degree of $v$. A graph $G$ is called $Z_3$-connected, if it admits a $p$-orientation, for every mapping $p: V(G) \to Z_3$ satisfying $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. We say that a graph $G$ admits a nowhere-zero 3-flow, if it admits a $p$-orientation in which each vertex $v$, $2p(v) \equiv d_G(v)$. According these definitions, if $G$ is $Z_3$-connected, then obviously it must admit a nowhere-zero 3-flow. Note that these definitions are equivalent to their initial well-known definitions, see [7, 8]. A graph $G$ is called $k$-tree-connected, if it contains $k$ edge-disjoint spanning trees. Note that every $2k$-edge-connected graph is also $k$-tree-connected [9, 11].

Tutte’s 3-Flow Conjecture says that every 4-edge-connected graph admits a nowhere-zero 3-
flow. Jaeger, Linial, Payan, and Tarsi (1992) proposed a stronger conjecture which says every 5-edge-connected graph is $\mathbb{Z}_3$-connected. In 2001 Kochol [6] proved that if every 5-edge-connected graph admits a nowhere-zero 3-flow, then Tutte’s 3-Flow Conjecture is true.

**Conjecture 1.** (5) Let $G$ be a graph and let $p : V(G) \to \mathbb{Z}_3$ be a mapping such that $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. If $G$ is 5-edge-connected, then it admits a $p$-orientation.

In 2012 Thomassen [10] succeeded to confirm Conjecture 1 for 8-edge-connected graphs. Later, Lovász, Thomassen, Wu, and Zhang (2013) [8] improved Thomassen’s result to the following version by pushing down the needed edge-connectivity by one.

**Theorem 1.** (8) Let $G$ be a graph and let $p : V(G) \to \mathbb{Z}_3$ be a mapping such that $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. If $G$ is 6-edge-connected, then it admits a $p$-orientation.

In [3] the authors used a stronger version of their result to confirm Conjecture 1 for 4-tree-connected graphs. In this note, we show that if Conjecture 1 would be true, then that conjecture together with Tutte’s 3-Flow Conjecture can be developed to 3-tree-connected graphs. Note that this number is sharp, because the complete graph of order 4 does not have as a nowhere-zero 3-flow, while it is 2-tree-connected.

**Theorem 2.** (3) Let $G$ be a graph and let $p : V(G) \to \mathbb{Z}_3$ be a mapping such that $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. If $G$ is 4-tree-connected, then it admits a $p$-orientation.

## 1 Orientations modulo 3 in 3-tree-connected graphs

The following theorem shows a consequence of Conjecture 1.

**Theorem 3.** Assume that Conjecture 1 is true. Let $G$ be a graph, let $p : V(G) \to \mathbb{Z}_3$ be a mapping such that $|E(G)| \equiv \sum_{v \in V(G)} p(v)$. If $G$ is 3-tree-connected, then it admits a $p$-orientation.

**Proof.** Let $k = 3$ and $\lambda = 5$. By induction on $|V(G)|$. For $|V(G)| = 1$, the proof is clear. So, suppose $|V(G)| \geq 2$. If $G$ is $\lambda$-edge-connected, then it follows from the assumption. Thus, we assume $G$ is not $\lambda$-edge-connected. Let $T_1, \ldots, T_{\lambda-2}$ be $\lambda-2$ edge-disjoint spanning trees of $G$. Let
$C$ be an edge cut of $G$ with the minimum size. Note that $|C| \leq \lambda - 1$ and $G \setminus C$ is composed by two disjoint connected graphs $G_1$ and $G_2$. Let $r \in \{0, \ldots, k-1\}$ be the unique integer with
\[ r + |E(G_1)| = \sum_{v \in V(G_1)} p(v). \]

Suppose first that $|E(C)| = \lambda - 2$. This implies that for each tree $T_j$, $|E(C) \cap E(T_j)| = 1$, and hence every graph $G_j$ contains $\lambda - 2$ edge-disjoint spanning trees. Since $|E(C)| = \lambda - 2 \geq k - 1 \geq r$, one can orient $r$ edges of $C$ from $G_1$ to $G_2$ and $|E(C)| - r$ remaining edges from $G_2$ to $G_1$. Now, for every graph $G_j$, let $p_j : V(G_j) \to \mathbb{Z}_k$ be a mapping such that for each vertex $v \in V(G_j)$, $p_j(v) = p(v) - q_j(v)$, where $q_j(v)$ is the number of edges of $C$ directed away from $v$. Since $r = \sum_{v \in V(G_1)} q_1(v)$, we have $|E(G_1)| = \sum_{v \in V(G_1)} p_1(v)$. Also, since $|C| - r = \sum_{v \in V(G_2)} q_2(v)$ and $|E(G)| = \sum_{v \in V(G)} p(v)$, we have $|E(G_2)| = \sum_{v \in V(G_2)} p_2(v)$. Thus the induction hypothesis implies that every graph $G_j$ has a $p_j$-orientation. It is not hard to see that these orientations of $G_1, C$, and $G_2$ induce a $p$-orientation for $G$. In future cases, we leave some details for the reader in order to apply the induction hypothesis.

Now, suppose that $|E(C)| = \lambda - 1$. Without loss of generality assume that $|E(C) \cap E(T_1)| = 2$ and $|E(C) \cap E(T_j)| = 1$ for each tree $T_j$ with $j > 1$. Also, without loss of generality assume that the spanning graph of $G_1$ with the edge set $E(G_1) \cap E(T_1)$ is connected. Therefore, $G_1$ contains $\lambda - 2$ edge-disjoint spanning trees. In $G$, contract all vertices of $G_1$ to a single vertex (by removing loops) and call the resulting graph $G'_2$. It is easy to see that $G'_2$ contains $\lambda - 2$ edge-disjoint spanning trees. Let $p'_2 : V(G'_2) \to \mathbb{Z}_k$ be a mapping such that for each vertex $v \in V(G'_2) \cap V(G_2)$, $p'_2(v) = p(v)$ and for the vertex $u \in V(G'_2)$ corresponding to $G_1$, $p'_2(u) = r$.

Now, if $|V(G_1)| > 1$ then by the induction hypothesis, the graph $G'_2$ has a $p'_2$-orientation. This orientation of $G'_2$ induces an orientation for $C$. In this case, let $p_1 : V(G_1) \to \mathbb{Z}_k$ be a mapping such that for each vertex $v \in V(G_1)$, $p_1(v) = p(v) - q(v)$, where $q(v)$ is the number of edges of $C$ directed away from $v$. At present, by the induction hypothesis, the graph $G_1$ has a $p_1$-orientation. It is not hard to see that the $p_1$-orientation of $G_1$ and $p'_2$-orientation of $G'_2$ induce a $p$-orientation for $G$.

In the final case, set $V(G_1) = \{u\}$. Let $xu$ and $uy$ be the two edges of $T_1$ incident to $u$. Now, remove the vertex $u$ from $G$ and add a new edge $xy$ to $G$. Call the resulting graph $H$. Since $T_1$ is a tree with no multiple edges, $x \neq y$ and so $H$ has no loop. Note also that $|E(H)| = |E(G)| - |E(C)| + 1$. It is not hard to see that $H$ contains $\lambda - 2$ edge-disjoint spanning trees. In $G$ orient the edge $xu$ from $x$ to $u$ and orient the edge $uy$ from $u$ to $y$. Next, orient exactly $r_u - 1$ edges of $E(C) \setminus \{xu, uy\}$ away from $u$ and orient all remaining edges of $E(C)$ in opposite direction, where
\( r_u \in \{1, \ldots, k\} \) is the unique integer with \( r_u \equiv k \mod p(u) \). Since \( |E(\mathcal{C})\setminus\{xu, uy\}| = \lambda - 3 \geq k - 1 \geq r_u - 1 \), this orientation of \( \mathcal{C} \) is possible. Now, let \( p_0: V(H) \to \mathbb{Z}_k \) be a mapping such that for each vertex \( v \in V(H) \), \( p_0(v) = p(v) - q(v) \), where \( q(v) \) is the number of edges of \( \mathcal{C} \setminus \{xu, uy\} \) directed away from \( v \). By the induction hypothesis, \( H \) has a \( p_0 \)-orientation. It is not difficult to see that the \( p_0 \)-orientation of \( H \) and the orientation of \( C \) induce a \( p \)-orientation for \( G \). Notice that if the edge \( xy \) of \( H \) oriented from \( y \) to \( x \), for inducing, the direction of two edges \( xu \) and \( uy \) in \( G \) must be reversed. This completes the proof. \( \square \)

By combining Theorem 2.1 in [4] with Theorem 3, one can derive the following corollary.

**Corollary 1.** Assume that Conjecture 1 is true. Let \( G \) be a 3-tree-connected graph. Then \( G \) does not have exactly one vertex \( z \) satisfying \( d_G(z)^3 \not\equiv 0 \) if and only if it can be edge-decomposed into three factors \( G_1, G_2, \) and \( G_3 \) such that for each \( v \in V(G_i), |d_{G_i}(v) - d_G(v)/3| < 1 \).

### 2 Conclusion: A generalization

In 2006 Barát and Thomassen [2] conjectured that for every tree \( T \) there exists a natural number \( k_T \) such that every \( k_T \)-edge-connected simple graph of size divisible by \( |E(T)| \) has a \( T \)-edge-decomposition. However, this conjecture investigates only the existence of \( k_T \), an upper bound on \( k_T \) was stated in [1] as the following conjecture. A consequence of Theorem 3 says that the following conjecture is true, if Conjecture 1 would be true (using the special case \( p = 0 \)).

**Conjecture 2.** ([1]) Let \( G \) be a simple graph of size divisible by \( k \) with \( k \geq 1 \), and let \( T \) be a tree of size \( k \). If \( G \) is \( k \)-tree-connected, then it admits a \( T \)-edge-decomposition.

When \( k \geq 3 \), the special case \( k \)-star of the above-mentioned conjecture can conclude the next conjecture (more precisely they are equivalent), using an idea that was used in [7]. To see this, for every vertex \( v \) of the graph \( G \), replace a large graph \( H_v \) containing \( k \) edge-disjoint spanning trees with \( |E(H_v)| + p(v)^k \equiv 0 \) such that after replacing the resulting graph forms a simple graph. Since the new graph contains \( k \) edge-disjoint spanning trees and its size is divisible by \( k \), by the assumption it admits a \( k \)-star-decomposition. Now, orient the edges of these stars away from their centres. It is not difficult to see that this orientation induces a \( p \)-orientation for \( G \).

**Conjecture 3.** Let \( G \) be a graph, let \( k \) be an integer, \( k \geq 3 \), and let \( p: V(G) \to \mathbb{Z}_k \) be a mapping such that \( |E(G)|^k \equiv \sum_{v \in V(G)} p(v) \). If \( G \) is \( k \)-tree-connected, then it has a \( p \)-orientation.
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