Hitchin Functionals and Nonsupersymmetric Supersymmetry Breaking

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Abstract
A new mechanism of supersymmetry breaking involving a dynamical parameter is introduced. It is independent of particle phenomenology and gauge groups. An explicit realization of this mechanism takes place in Type II superstring compactifications which admit eight supercharges (also known as generalized $SU(3)$ structures). The resulting Kähler potentials are expressed in terms of Hitchin functionals. Specifically, every transversally regular generalized Calabi-Yau manifold in the moduli space of complex or symplectic structures deforms into compactifications with nonsupersymmetric broken supersymmetry. Furthermore, all such deformations can be represented by elements of the first Poisson cohomology groups of their moduli spaces, those two groups are anti-isomorphic, and deformations of mirror pairs of transversally regular generalized Calabi-Yau manifolds are identified via this anti-isomorphism.

1 Introduction
The particles predicted by supersymmetric field theories failed to appear in experiments, so that within the accessible energy range there is no supersymmetry. Yet supersymmetry provides the only known resolution of the hierarchy problem, and ensures that the Standard Model serves as a low-energy approximation to some futuristic unified field theory [1]. That necessitates the search for ways to make supersymmetry breakable (and restorable). Hitherto, this search has produced several mechanisms of spontaneous supersymmetry breaking [2, 3, 4, 5].

Generally speaking, it occurs only when the variation of some field under supersymmetry transformations yields nonzero vacuum expectation values:

$$\langle \text{VAC} | \delta(\text{field}) | \text{VAC} \rangle \neq 0.$$ 

That implies a necessary condition for spontaneous supersymmetry breaking: the generators of supersymmetry algebra must not annihilate vacuum. However, there still remains the possibility of all the variations having zero expectation value while the supersymmetry algebra generators fail to satisfy the

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defining identities. Such a failure would constitute nonspontaneous supersymmetry breaking. And the above-mentioned identities have to do with space-time properties of the generators. As far as we know, the first results linking supersymmetry algebras to space-time symmetries were published by Nahm [6]. The anti-de-Sitter space with its $O(3, 2)$ symmetries supports all conceivable supersymmetry algebras, whereas the de-Sitter space having $O(4, 1)$ as the symmetry group has only $N = 2$ supersymmetry. Thus this Universe evolving from the anti-de-Sitter to the de-Sitter regime may provide a toy model of nonspontaneous $N \neq 2$ supersymmetry breaking. It is very instructive to expose the fatal flaw of this model. There is no smooth direct parametric transition from $O(3, 2)$ to $O(4, 1)$ because $\mathfrak{o}(3, 2) \not\cong \mathfrak{o}(4, 1)$, and for some value of the parameter space-time symmetries collapse even infinitesimally.

Therefore to make such a theory work one needs a family of locally isomorphic Lie groups, smoothly depending on a parameter, and differing in their facility to support supersymmetry algebras. Then the parameter may be interpreted as the energy scale, pre- and post-unification values separated by an interval. In the Minkowski $\mathbb{R}^4$ one also requires Lorentz invariance. That could only be satisfied for families of Lie groups locally isomorphic to the Lorentz group, and containing that group as a member. In what follows we find one such family containing, at one extreme Spin(1, 3), and at the other a compact Lie group $G$ which, while maintaining local Lorentz invariance, does not support any supersymmetry algebras.

Nonspontaneous supersymmetry breaking allows for consistent pointwise gauging in curved space-time as long as the time components of the curvature tensor vanish: $R^0_{\alpha\beta\xi} = R^0_{\beta\alpha\xi} = 0$. This limits its applicability, or, rather, underscores its purely local character.

Naturally, one wonders whether our mechanism is realized in physically relevant models. Looking at typical instances of spontaneous supersymmetry breaking, we notice that it relies on field couplings in the Lagrangian. Therefore, it would seem logical to study situations wherein some supersymmetry breaking depends on the moduli instead of the couplings. One such situation arises in Type II superstring backgrounds with fluxes - non-trivial values for the NS-NS and R-R field strengths in the six-dimensional manifolds used to compactify the theory. Introducing fluxes one leaves the realm of Calabi-Yau compactifications and enters into $SU(3) \times SU(3)$ structures. They are based on the notion of so-called generalized geometry due to Hitchin [7]. The background manifolds $Y$ which replace Calabi-Yau manifolds are no longer Ricci-flat. There are two globally defined $SU(3)$ spinors, $\eta^1$ and $\eta^2$. The holonomy group of $Y$ is, generally speaking, not $SU(3)$ because we do not require Ricci-flatness. Consequently, $\eta^1$ and $\eta^2$ are not necessarily covariantly constant. The underlying real even degree forms $J^1$, $J^2$ and complex odd degree forms $\Omega^1$, $\Omega^2$ are not closed. Their moduli spaces $M_J$, and $M_\Omega$ are quotients of linear groups [8], and as such both are endowed with natural linear actions of $SO(6, 6)$. The intrinsic torsion classes of $dJ$ and $d\Omega$ [9] can be used to classify all possible $SU(3) \times SU(3)$ structures. Among those, a prominent role is played by the generalized Calabi-Yau manifolds [7]. They are precisely the backgrounds supporting supersymme-
try \cite{10}. Of generalized Calabi-Yau manifolds, one particular subclass happens to have some remarkable deformation properties. Namely, the transversally regular generalized Calabi-Yau manifolds \cite{7} turn out to support supersymmetry while allowing transversal deformations that break supersymmetry but maintain zero expectation values of the variations of all the fields constructed with $\eta^1$ and $\eta^2$. Thus within a neighborhood of a transversally regular generalized Calabi-Yau manifold in the moduli space, supersymmetry appears to be broken nonspontaneously. To make its breaking mechanism manifest, and to positively identify it as the above-proposed one, we need to study the group-theoretical properties of transversal deformations in the moduli spaces. The extension of mirror symmetry inaugurated by Graña et al. \cite{10} only matches pairs of generalized Calabi-Yau manifolds (i.e. supersymmetric backgrounds). Consequently, the transversal (supersymmetry breaking) deformations ought to commute with mirror symmetry. This translates into a requirement that there be a correspondence between the transversal deformations in the two moduli spaces. In the absence of the $T^3$-fibration of Strominger, Yau, and Zaslow \cite{11}, the search for this correspondence forces us to utilize some other structure. More specifically, we apply the notion of Morita equivalence of Poisson manifolds, first described by Weinstein \cite{12}, and Xu \cite{13}, to $\mathcal{M}_J$ and $\mathcal{M}_\Omega$. Using the linear $SO(6,6)$ actions we construct two Poisson structures $\pi_J$ and $\pi_\Omega$ in such a way as to generate the leaves of their symplectic foliations via the linear $SO(6,6)$ diffeomorphisms and actions by closed two-forms ($B$-fields). Then the Casimir functions of $\pi_J$ and $\pi_\Omega$ can be represented by smooth functions of the intrinsic torsion classes. The sheaves of Casimir functions and the first Čech cohomology groups are isomorphic via Morita equivalence of $(\mathcal{M}_J, \pi_J)$ and $(\mathcal{M}_\Omega, \pi_\Omega)$ due to a theorem of Ginzburg and Golubev \cite{14}, and from that isomorphism we are able to deduce anti-isomorphism of the first Poisson cohomology groups:

$$H^1_{\pi_\Omega}(\mathcal{M}_\Omega) \cong H^1_{\pi_J}(\mathcal{M}_J).$$

Given that, we assign each one-parameter family of transversal deformations to an element of $H^1_{\pi_J}(\cdot)$. This way they match once the mirror pairs of transversally regular generalized Calabi-Yau manifolds have been chosen. As a consequence, all the sufficiently small transversal deformations of the transversally regular generalized Calabi-Yau manifolds realize a representation of $G$, or, put in more technical terms, the representation of transversal deformations reduces from the most general $\text{Spin}(6)$ to one of its subgroups $G \subseteq \text{SU}(4) \cong \text{Spin}(6)$.

A few words about the organization of this paper. The basic mathematical results establishing our mechanism of supersymmetry breaking are gathered in Section 2. The most general relativistic setup wherein nonspontaneous supersymmetry breaking takes place is described in detail in Section 3. An overview of Type II superstring compactifications with fluxes, transversal deformations of transversally regular generalized Calabi-Yau manifolds, their connection with nonspontaneous supersymmetry breaking, their representations in Poisson cohomology are the subject of Section 4.

Lastly, we dispense with the physical constants by setting $\hbar = c = 1$. 

1 INTRODUCTION
2 Mathematical Preliminaries

The Pauli matrices are
\[
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.1}
\]

The Dirac representation of \( SU(2) \), denoted \( SU_D(2) \) is generated by
\[
J_1 = \frac{1}{2} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix}, \quad J_2 = \frac{1}{2} \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \quad J_3 = \frac{1}{2} \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}. \tag{2.2}
\]

There still exists the twofold covering epimorphism of Lie groups:
\[
A: \quad SU_D(2) \longrightarrow \begin{bmatrix} [SO(3)] & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.3}
\]

Spin(1,3) may be viewed as a complex extension of \( SU_D(2) \):
\[
\left\{ J_i = \frac{1}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \right\} \rightarrow \left\{ J_i = \frac{1}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix}, \quad K_i^C = \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \right\}. \tag{2.4}
\]

Fortuitously, there is a class of mutually isomorphic almost complex Lie algebra extensions, of which \( so(1,3) \), generated by \( \{ J_i, K_i^C \} \) of (2.4) is a member. We are interested mainly in the following almost complex extension:
\[
\left\{ J_i = \frac{1}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \right\} \rightarrow \left\{ J_i = \frac{1}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix}, \quad K_i = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \right\}. \tag{2.5}
\]

Its relevant properties are summarized in

**Theorem 2.1.** There exists a unique compact semisimple Lie group \( G \subset SU(4) \), whose Lie algebra \( g \cong so(1,3) \) is generated by (2.5).

**Proof.** Every almost complex extension corresponds (up to a nonzero factor) to a matrix
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U(2), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

\( g \cong so(1,3) \) implies \( ad - bc = 1 \). Therefore
\[
\Re a = \Re d = 0, \quad \Im c = \Im b, \quad \Re c = -\Re b.
\]

This allows us to write the most general almost complex extension as
\[
J_i \mapsto \left( w \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + u \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} + v \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) J_i, \quad w^2 + u^2 + v^2 = 1.
\]

To ensure compactness, we must have \( \exp i \kappa^a K_a \) bounded. Whence \( w = 0, u = 0 \) is the only choice. And this is (2.5).

According to Helgason ([13], Chapter II, §2, Theorem 2.1), there exists a Lie
group $G$, whose Lie algebra is generated by $\{J_i, K_i\}$ of (2.5). Its elements are all of the form $\exp i(\theta^b J_b + \kappa^a K_a)$, which means $G$ is a Lie subgroup of $SU(4)$. Now $G$ has to be closed in the standard matrix topology of $SU(4)$. That is based on a fundamental result of Mostow [16]: any semisimple Lie subgroup $H$ of a compact Lie group $C$ is closed in the relative topology of $C$. In our case, $SU(4)$ is compact, $\mathfrak{g}$ is semisimple.

In the sequel we will work with the homogeneous space $G/SU_D(2)$.

**Lemma 2.1.**

$$\pi_1(G/SU_D(2)) = 0.$$  

**Proof.** For all Lie groups $\pi_2(.) = 0$ [17]; for $SU_D(2)$, $\pi_0(SU_D(2)) = 0$ by connectedness. Also, $SU_D(2)$ is a closed subgroup of $SU(4)$ in the ordinary matrix topology. We therefore have the following exact homotopy sequence [17]:

$$0 \rightarrow \pi_2(SU(4)/SU_D(2)) \rightarrow \pi_1(SU_D(2)) \rightarrow \pi_1(SU(4)) \rightarrow \pi_1(SU(4)/SU_D(2)) \rightarrow 0.$$  

$\pi_1(SU(4)) = 0$ [17] whence

$$\pi_1(SU(4)/SU_D(2)) \cong \pi_1(SU_D(2)) = \pi_1(S^3) = 0.$$  

Now homotopy is functorial. The embedding $\xi : G/SU_D(2) \hookrightarrow SU(4)/SU_D(2)$ induces the monomorphism of fundamental groups

$$\xi_{\pi*} : \pi_1(G/SU_D(2)) \rightarrow \pi_1(SU(4)/SU_D(2)).$$  

**Theorem 2.2.**

$$G/SU_D(2) \cong S^3.$$  

**Proof.** $\mathfrak{g}$ decomposes as a vector space into two three-dimensional subspaces,

$$\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{k},$$  

Based on this decomposition, there is an involutive automorphism

$$\vartheta : \mathfrak{g} \rightarrow \mathfrak{g}$$  

defined by

$$\vartheta(J + K) = J - K, \quad \forall J \in \mathfrak{j}, \quad \forall K \in \mathfrak{k}.$$  

$\mathfrak{j}$ is the set of fixed points of $\vartheta$. It is unique ([15], Chapter IV, §3, Proposition 3.5). The pair $(\mathfrak{g}, \vartheta)$ is an orthogonal symmetric Lie algebra ([15], Chapter IV, §3). There is a Riemannian symmetric pair $(G, SU_D(2))$ associated with $(\mathfrak{g}, \vartheta)$ so that the quotient $G/SU_D(2)$ is a complete locally symmetric Riemannian space. Furthermore, its curvature corresponding to any $G$-invariant Riemannian structure is given by ([15], Chapter IV, §4, Theorem 4.2):

$$R(K_{i_1}, K_{i_2})K_{i_3} = -[[K_{i_1}, K_{i_2}], K_{i_3}] \quad \forall K_{i_1}, K_{i_2}, K_{i_3} \in \mathfrak{k}.$$
Computing the sectional curvature we see that $R^\text{sect} \equiv 1$. Now a pedestrian version of the Sphere theorem [18] asseverates that a complete simply connected Riemannian manifold with $R^\text{sect} \equiv 1$ is isometric to a sphere of appropriate dimension. In our case the topological condition is satisfied in view of Lemma 2.1.

Consider the natural inclusions of Lie groups

$$\iota : G \hookrightarrow GL(4, \mathbb{C}), \quad \iota : \text{Spin}(1, 3) \hookrightarrow GL(4, \mathbb{C}).$$

(2.6)

Their images inside $GL(4, \mathbb{C})$ intersect:

$$\iota(G) \cap \iota(\text{Spin}(1, 3)) = \text{SU}(2).$$

(2.7)

Because of (2.7), the set

$$\text{Ad}_{\iota(G)}(\iota(\text{Spin}(1, 3))) = \bigsqcup_{U \in G} U\text{Spin}(1, 3)U^H,$$

(2.8)

the disjoint union of conjugates of $\text{Spin}(1, 3)$, has the same cardinality as the set of all boosts in $G$. Similarly, there is the natural inclusion

$$\iota : \text{SO}(4) \hookrightarrow GL(4, \mathbb{R}).$$

(2.9)

The set $\text{Ad}_{\iota(\text{SO}(4))}(\iota(\text{SO}(1, 3)^e))$ is homeomorphic to $\text{SO}(4)/\text{SO}(3) \cong S^3$. Combining this with Theorem 2.2 we arrive at:

$$\begin{align*}
\text{Ad}_{\iota(G)}(\iota(\text{Spin}(1, 3))) & \quad \cong \quad \text{G}/\text{SU}_D(2) \quad \overset{\cong}{\longrightarrow} \quad \mathbb{S}^3 \\
\text{Ad}_{\iota(\text{SO}(4))}(\iota(\text{SO}(1, 3)^e)) & \quad \cong \quad \text{SO}(4)/\text{SO}(3) \quad \overset{\cong}{\longrightarrow} \quad \mathbb{S}^3
\end{align*}
$$

(2.10)

The double horizontal lines indicate set-theoretic bijective correspondences, the upper $\cong$ is an isometry, the lower one is a diffeomorphism. Furthermore, the diagram (2.10) commutes and de facto defines the diffeomorphism $\varphi$. This diffeomorphism is utilized in the sequel to effect an action of $G$.

### 3 Equivariant Momentum Operators

$G$ does not act on the Minkowski $\mathbb{R}^4$ by isometries. We have

$$\begin{cases}
G \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4; \\
(x, \exp(i \theta^b J_b + \kappa^a K_a), x^\mu) \rightarrow x'^\mu = \mathcal{A}(\exp i \theta^b J_b)^\mu_{\lambda} \varphi(\exp ik^a K_a)_{\eta}^\lambda x^\eta.
\end{cases}
$$

(3.1)

In fact, the metric becomes frame-dependent:

$$\varphi(\exp ik^a (\alpha) K_a) g \varphi(\exp(-ik^a (\alpha) K_a)) \neq g, \quad \alpha \in [0, 2\pi], \quad \alpha \neq \{0, 2\pi\};$$
\( \alpha \) being the group parameter here. Yet physical quantities must remain frame-independent. Therefore, instead of the standard quantum field theory substitution
\[
P_\mu \longrightarrow i\partial_\mu, \quad (3.2)
\]
we employ the rule
\[
P_\mu \longrightarrow i\nabla_\mu(\alpha) \overset{\text{def}}{=} i(\varepsilon^{\nu}_\mu(\alpha)\partial_\nu + i\kappa^a_\mu(\alpha)K_a), \quad (3.3)
\]
the exact form of \( \varepsilon^{\nu}_\mu(\alpha) \) and \( \kappa^a_\mu(\alpha) \) to be determined. \( K_a \)'s are in keeping with the \((1/2, 1/2)\) representation of \( P_\mu \)'s. This construction is an equivariant incarnation of the free spin structure due to Plymen and Westbury [19]. Briefly, let \( M \) be a 4-dimensional smooth manifold with all the obstructions to the existence of a Lorentzian metric vanishing (for instance, a parallelilizable \( M \) would do). Let
\[
\Lambda : \text{Spin}(1, 3) \rightarrow SO(1, 3)^{e}
\]
be the twofold covering epimorphism of Lie groups. A free spin structure on \( M \) consists of a principal bundle \( \zeta : \Sigma \rightarrow M \) with structure group \( \text{Spin}(1, 3) \) and a bundle map \( \tilde{\Lambda} : \Sigma \rightarrow \mathcal{F}M \) into the bundle of linear frames for \( TM \), such that
\[
\tilde{\Lambda} \circ \tilde{R}_S = \tilde{R}'_{\iota(\Lambda(S))} \circ \tilde{\Lambda} \quad \forall S \in \text{Spin}(1, 3),
\]
\( \tilde{R} \) and \( \tilde{R}' \) being the canonical right actions on \( \Sigma \) and \( \mathcal{F}M \) respectively, \( \iota : SO(1, 3)^{e} \rightarrow GL(4, \mathbb{R}) \) the natural inclusion of Lie groups, and \( \pi' : \mathcal{F}M \rightarrow M \) the canonical projection. The map \( \tilde{\Lambda} \) is called a spin-frame on \( \text{Spin}(1, 3) \). This definition of a spin structure induces metrics on \( \Sigma \). Indeed, given a spin-frame \( \tilde{\Lambda} : \Sigma \rightarrow \mathcal{F}M \), a dynamic metric \( g^{\Lambda} \) is defined to be the metric that ensures orthonormality of all frames in \( \tilde{\Lambda}(\Sigma) \subset \mathcal{F}M \). It should be emphasized that within the Plymen and Westbury’s formalism the metrics are built \textit{a posteriori}, after a spin-frame has been set by the field equations. In our formalism the metrics are obtained via the \( G \)-action, and the set of all allowable metrics is \( \text{Ad}_{\iota(SO(4))}(\iota(SO(1, 3)^{e})) \).
\( \nabla_\mu(\alpha) \) qualifies as a \( G \)-connection on the principal \( G \)-bundle over the physical space-time. Furthermore, we impose an additional condition on (3.3) to ensure validity of the relativistic impulse-energy identity:
\[
P^\mu(\alpha)P_\mu(\alpha) = g^{\rho\lambda}(\alpha)\nabla_\rho(\alpha)\nabla_\lambda(\alpha) \overset{\text{def}}{=} g^{\rho\lambda}(0)\partial_\rho\partial_\lambda = P^\mu(0)P_\mu(0). \quad (3.4)
\]
This translates to some algebraic relations among \( \kappa^a_\mu \)'s and \( \varepsilon^{\nu}_\mu \)'s. However, we still need to make the \( G \)-transformation law of (3.3) more explicit. First, these operators are natural spinors in the sense that \( SU_D(2) \) acts linearly:
\[
U^\gamma U^H \nabla_\mu U^H = U^\gamma U^H \varepsilon^{\nu}_\mu \partial_\nu + i\kappa^a_\mu U^\gamma U^H UK_a U^H \\
= M^\nu_\gamma \varepsilon^{\nu}_\mu \partial_\nu + M^\nu_\gamma i\kappa^a_\mu r^a K_a \quad \text{by } [j, \xi] = \xi.
\]
Here $M^a_{\mu}$'s realize an $SO(3)$ transformation ($U \in SU(2)$), which is at its most transparent if $\gamma^0$ is diagonal. As for $r^a_{\mu}$'s, they determine how the potentials behave:

\[
\tilde{\kappa}_\mu^a = \kappa_\mu^1 r_1^a + \kappa_\mu^2 r_2^a + \kappa_\mu^3 r_3^a, \quad \text{and} \quad |r_1^a|^2 + |r_2^a|^2 + |r_3^a|^2 = 1, \quad a = \{1, 2, 3\}. 
\]

To see how they are boosted, we treat a prototypical case - that of a boost in the $x^3$-direction. Specifically,

\[
\begin{align*}
\nabla_0 &= \varepsilon_0^0(\alpha) \partial_0 + \varepsilon_3^3(\alpha) \partial_3 + i \kappa_0(\alpha) K_3, \\
\nabla_3 &= \varepsilon_3^0(\alpha) \partial_0 + \varepsilon_3^3(\alpha) \partial_3 + i \kappa_3(\alpha) K_3, \\
\nabla_1 &= \partial_1, \\
\n\nabla_2 &= \partial_2.
\end{align*}
\]

We look for solutions of

\[
(i \gamma^\mu \nabla_\mu - m) \Psi = 0, 
\]

modelled on the free spinors

\[
\Psi(\alpha) = s(\alpha)e^{-i(p_0 x^0 + p_3 x^3)},
\]

subject to the relativistic impulse condition $p_0^2 - p_3^2 = m^2$. In the standard representation

\[
\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix},
\]

the equation (3.12) yields the following matrix:

\[
\begin{bmatrix}
\varepsilon_0(\alpha) - m(\alpha) & 0 & -\varepsilon_3(\alpha) - \kappa_0(\alpha) & 0 \\
0 & \varepsilon_0(\alpha) - m(\alpha) & 0 & \varepsilon_3(\alpha) + \kappa_0(\alpha) \\
\varepsilon_3(\alpha) - \kappa_0(\alpha) & 0 & -\varepsilon_0(\alpha) - m(\alpha) & 0 \\
0 & -\varepsilon_3(\alpha) + \kappa_0(\alpha) & 0 & -\varepsilon_0(\alpha) - m(\alpha)
\end{bmatrix},
\]

where the entries are

\[
\begin{align*}
\varepsilon_0(\alpha) &= \varepsilon_0^0(\alpha)p_0 + \varepsilon_3^3(\alpha)p_3, \\
\varepsilon_3(\alpha) &= \varepsilon_3^0(\alpha)p_0 + \varepsilon_3^3(\alpha)p_3, \\
m(\alpha) &= m + \kappa_3(\alpha).
\end{align*}
\]

Its rank has to be 2 for all values of $\alpha$, thus constraining $\kappa_0(\alpha)$ and $\kappa_3(\alpha)$:

\[
\varepsilon_0^2(\alpha) - \varepsilon_3^2(\alpha) = (m + \kappa_3(\alpha))^2 - \kappa_0^2(\alpha).
\]

Evidently $\kappa_\mu^a(\alpha)$'s are not identically zero. At the same time, $\kappa_\mu^a(0) = 0, \forall \mu = \{0, 1, 2, 3\}$. Hence, a boost entails a nonlinear change in the potentials.

Finally, we are in a position to deal with supersymmetry algebras. For the reminder of this section, the impulse operators and all other quantities
expressly depend on the parameters introduced in the proof of Theorem 2.1. For convenience, we bundle them into one complex parameter $z$ via stereographic projection, so that $K_a(0) = K_a^C$, $K_a(1) = K_a$, $\varepsilon_\mu^\nu(\alpha, 0) = \delta_\mu^\nu$, $\kappa_\mu^\alpha(\alpha, 0) = 0$. Should there exist such algebras, $Q_m(z)$, $\bar{Q}_m(z)$ would generate them. But they realize a linear representation of the (respective) symmetry group, and we arrive at an equality impossible for some $z \in [0, 1]$:}

$$\{Q_m(z), \bar{Q}_m(z)\} = -2i\gamma_\mu(\varepsilon_\mu^\nu(\alpha, z)\partial_\nu + i\kappa_\mu_\alpha(\alpha, z)K_a(z)).$$

(3.20)

The right-hand side transforms nonlinearly because of $\kappa_\mu_\alpha(\alpha, 1)$, whereby proving that there are no $Q_m(1), \bar{Q}_m(1)$. Adding central charges $Z_m, Z_m^*$ on the right-hand side would not remedy the situation because these charges commute with the symmetry group generators.

The next question we address is that of gauging nonspontaneous supersymmetry breaking in curved space-time. Pseudo-Riemannian geometry has two ways to account for variability of the metric. One way is to introduce the Levi-Civita connection and the curvature tensor. The alternative is the Cartan’s method of equivalence [20]. The gist of the latter consists in specifying the subgroups of local diffeomorphisms that preserve the geometry. Thus the most general local diffeomorphism induces an automorphism of the tangent bundle $a^\ast(x) \in SO(1,3)e \times \mathbb{R}^4$. In our case we further constrain local diffeomorphisms by insisting that $a^\ast$ preserve the $G$-connection. The inherent gauge freedom of the connection allows it to transform equivariantly. Hence pointwise $a^\ast \in (SO(1,3)e \cap \rho(G))$, viewed as a set of abstract automorphisms, $\rho(G)$ being some linear representation of $G$. According to (3.15),

$$\rho|_{SU_d(2)} = A \text{ of (2.3)};$$

(3.21)

and the subgroup of spatial rotations - $SO(3)$ - does afford equivariance. In the language of curvature tensor coefficients these bundle automorphisms correspond to $R_{\alpha\beta\xi}^\alpha = R_{\beta\xi}^\alpha = 0$. To name a few examples, the expanding Friedman universe can have nonspontaneously broken supersymmetry. By contrast, in the maxmaster universe supersymmetry is either intact or broken spontaneously. Finally, in the most realistic unevenly expanding universe with deviations from the spherical symmetry, there are causally disconnected patches of supersymmetry in an otherwise asupersymmetric space-time, because, on general grounds, it is hard to see how supersymmetry can be broken spontaneously in all those patches.

4 Not So Super Strings

4.1 Type II Compactifications

Our goal now is to place nonspontaneous supersymmetry breaking in the context of Type II superstring theories. They play out on the space-time background of ten dimensions. The underlying manifolds have the pseudo-Riemannian metric
of signature (1,9). We specialize $M^{1,9}$ to have a fixed tangent bundle splitting
\[ TM^{1,9} = T\mathbb{R}^{1,3} \oplus TY, \tag{4.1} \]
augmented with the requirement that there be a local smooth fibration
\[ f_O : M^{1,9} \setminus O \subset \mathbb{R}^{1,3} \tag{4.2} \]
over the physical space-time, each fiber $f_O^{-1}(x)$ being a compact 6-dimensional manifold. The splitting (4.1) implies a decomposition of the Lorentz group $\text{Spin}(1,9) \supset \text{Spin}(1,3) \times \text{Spin}(6)$ and an associated decomposition of the spinor representation $\text{Spin}(6)$ according to $\text{Spin}(6) \rightarrow (2,4) \oplus (\bar{2}, \bar{4})$. $TY$ is a $SO(6)$ vector bundle which admits a pair of distinct $SU_\eta$ representations.

Now \(\eta\) are normalized so that $\eta^\dagger \eta = 1$. The Clifford algebra of $\gamma$, now \(\eta\) is a real even-degree form, $\Omega$ is a complex odd-degree form, and the 6-dimensional chirality operator, denoted $\gamma$. It assigns signs via $\gamma \eta_{1,2} = \pm \eta_{1,2}$.

The Clifford algebra of $Y$ is a deformation of $\Lambda^*T^*Y \otimes \mathbb{C}$. Utilizing the Fierz map we identify:
\[ \eta_{+}^{1,2} \otimes \eta_{+}^{1,2} = e^{ij^{1,2}}, \tag{4.3} \]
\[ \eta_{+}^{1,2} \otimes \eta_{+}^{1,2} = \Omega^{1,2}. \tag{4.4} \]
Here $J$ is a real even-degree form, $\Omega$ is a complex odd-degree form, and the spinors are normalized so that $\eta_{1,2} \eta_{1,2} = 1$.

In the presence of fluxes these spinors are no longer covariantly constant. Therefore the attendant differential forms are not closed. Instead we have [9]:
\[ dJ = \frac{3}{4} i(W_1 \Omega - \bar{W}_1 \Omega) + W_4 \wedge J + W_3, \tag{4.5} \]
\[ d\Omega = W_1 J \wedge J + W_2 \wedge J + \bar{W}_5 \wedge \Omega, \tag{4.6} \]
\[ W_3 \wedge J = W_3 \wedge \Omega = W_2 \wedge J \wedge J = 0. \tag{4.7} \]
In the last three formulas $W_1$ is a zero-form, $W_4$, $W_5$ are one-forms, $W_2$ is a two-form, and $W_3$ is a three-form, and we omit the superscripts. In addition, we have
\[ J^{1,2} \wedge J^{1,2} \wedge J^{1,2} = \frac{3}{4} i \Omega^{1,2} \wedge \bar{\Omega}^{1,2}, \quad J^{1,2} \wedge \Omega^{1,2} = 0. \tag{4.8} \]

When $\eta^1 = \eta^2$, $dJ^{1,2} = d\Omega^{1,2} = 0$, $J$ is a symplectic form, $\Omega$ determines the complex structure, and their combination defines a Calabi-Yau manifold.

Geometrically, $\eta^1$, $\eta^2$ provide two different decompositions of the complexified tangent bundle:
\[ T_\mathbb{C} Y = TY \otimes \mathbb{C} = E^1 \oplus \bar{E} = E^2 \oplus \bar{E}^2. \tag{4.9} \]
$E^1$ and $E^2$ specify two different embeddings $SU(3) \xrightarrow{\tau_{1,2}} \text{Spin}(6)$. Now the existence of the smooth local fibration signifies that $\frac{\partial}{\partial x} \tau_{1,2} \neq 0$ and the transformation law of $SU(3) \times SU(3)$ structures takes the entire $\text{Spin}(6) \times \text{Spin}(6)$ group. In contrast, a global fibration $M^{1,9} = \mathbb{R}^{1,3} \times Y$ would have kept $\tau_{1,2}$ constant throughout $\mathbb{R}^{1,3}$.

Alternatively, recall that the original Type II theory is formulated on a supermanifold $M^{1,9} \mid_{16+16}$ of bosonic dimension (1,9) with a manifest local Spin(1,9) symmetry and with the Grassmann variables transforming as a pair of 16-dimensional spinor representations. The requirement that we have an $SU(3) \times SU(3)$ structure means there is a sub-supermanifold $N^{1,9} \mid_{4+4} \subset M^{1,9} \mid_{16+16}$, still of bosonic dimension (1,9), but now with only eight Grassmann variables transforming as spinors of Spin(1,3) and singlets of one or the other $SU(3)$. It is natural to reformulate the ten-dimensional theory in this sub-supermanifold.

### 4.2 Generalized Structures

The generalized structures of Hitchin [7] are associated with a subgroup of $SO(n,n)$. This is the structure group of the bundle $TY \oplus T^*Y$, which always exists. It is a rank $2n$ vector bundle with a preferred orientation and an inner product/metric of signature $(n,n)$:

\[(X + \xi, X + \xi) = -iX\xi = -X_l\xi^l, \quad l = \{1, ..., n\}. \quad (4.10)\]

In conjunction with this metric, there always is a spin structure in view of standard properties of the Stiefel-Whitney classes $w^\# \in H^\#(Y, \mathbb{Z}_2)$:

\[w_2(TY \oplus T^*Y) = w_2(TY) + w_1(T^*Y) + w_2(T^*Y) = 0\]

(using $w_2(TY) = w_2(T^*Y)$). The corresponding spinors, sections of the spin bundles $S^\pm$ of Spin($n,n$) are isomorphic to even or odd forms on $Y$. However, the isomorphism is not canonical. Rather, it is parameterized by sections of $GL(n)/SL(n)$, that is by a scalar field (dilatino) $e^{-\phi}$, $\phi \in C^\infty(Y)$:

\[S^\pm = \Lambda^{even/odd}T^*Y \otimes \sqrt{\Lambda^nY}, \quad \phi(y) : \sqrt{\Lambda^nY} \rightarrow \mathbb{R}. \quad (4.11)\]

As a matter of notation, $\Phi \in S^\pm$ is designated $\Phi = \theta \otimes e^{-\phi}$.

In our case the group is $SO(6,6)$. We write it as $SO(V \oplus V^*)$, dim $V = 6$. Its Lie algebra has the following decomposition:

\[\mathfrak{so}(V \oplus V^*) = V \otimes V^* \oplus \Lambda^2V^* \oplus \Lambda^2V. \quad (4.12)\]

$V \oplus V^*$ acts on the Clifford algebra via

\[(X + \xi) \cdot \Phi = i_X\Phi + \xi \wedge \Phi. \quad (4.13)\]

There is an invariant skew-symmetric bilinear form on $S^\pm$ known in the physics literature as the Mukai pairing:

\[\langle \Phi, \Psi \rangle = \sum_m (-1)^m \Phi_{2m} \wedge \Psi_{6-2m} \in \Lambda^6V^* \otimes (\sqrt{\Lambda^6V})^2 = \mathbb{R} \quad (4.14)\]
for even spinors, and
\[
< \Phi, \Psi > = \sum_m (-1)^m \Phi_{2m+1} \wedge \Psi_{6-2m-1} \in \Lambda^6 V^* \otimes (\sqrt{\Lambda^6 V})^2 = \mathbb{R}
\]  
(4.15)

for odd ones. \(< \cdot, \cdot >\) is symplectic and allows us to identify the linear structure of \(S^\pm\) with that of \(TS^\pm\).

Given \(\Phi \in S^\pm\), consider its annihilator, the vector space
\[
V_\Phi = \{ X + \xi \in V \oplus V^* | (X + \xi) \cdot \Phi = 0 \}.
\]  
(4.16)

A spinor \(\Phi\) for which \(\dim V_\Phi = 6\) is called a pure spinor. Any two pure spinors are related by an element of Spin(6,6).

The generalized Calabi-Yau manifolds are characterized as pairs \((Y, \Phi)\), \(\Phi\) being a pure complex spinor satisfying \(< \Phi, \bar{\Phi} > \neq 0\) at each point and such that the corresponding differential form, a section of one of the complexified exterior bundles \(\Lambda^{\text{even/odd}} T^* Y \otimes \mathbb{C}\), is closed. The class of generalized Calabi-Yau manifolds trivially includes classical Calabi-Yau as well as symplectic manifolds.

For the time being, we work with the semisimple group Spin(12, \mathbb{C}) instead of Spin(6,6) and a complex six-dimensional vector space \(V\). On each of the 32-dimensional spin spaces \(S^\pm\), the actions of both groups are symplectic. Consequently, there is the moment map \(\mu : S^\pm \to so(12, \mathbb{C})^*\):
\[
\mu(\Phi)(X) = \frac{1}{2} < \rho(X)\Phi, \Phi >,
\]  
(4.17)

where \(\Phi \in S^\pm\), \(\rho : so(12, \mathbb{C}) \to \text{End} S^\pm\) denotes the representations of \(so(12, \mathbb{C})\), and \(X \in so(12, \mathbb{C})\). Using the nondegeneracy of the Killing bilinear form we identify \(so(12, \mathbb{C})\) with its dual, so \(\mu(\Phi)\) takes values in the Lie algebra. Now we define an invariant quartic function on \(S^\pm\) via
\[
q(\Phi) \overset{\text{def}}{=} \text{tr} \mu(\Phi)^2.
\]  
(4.18)

For \(\Phi \in S^\pm\), \(q(\Phi) \neq 0\) if and only if \(\Phi = \alpha + \beta\), where \(\alpha\) and \(\beta\) are pure spinors such that \(< \alpha, \beta > \neq 0\). We are interested in the open sets
\[
S^+_q = \{ \Phi \in S^+ | q(\Phi) < 0 \}, \quad S^-_q = \{ \Phi \in S^- | q(\Phi) < 0 \}.
\]  
(4.19)

Both are acted on transitively by the real group Spin(6,6) \(\times \mathbb{R}^+\). On those sets we define the homogeneous degree two functional (also known as the Hitchin functional)
\[
H(\Phi) \overset{\text{def}}{=} \sqrt{-q(\Phi)}
\]  
(4.20)

Note that if \(\Phi = \varphi + \bar{\varphi}\) for a pure spinor \(\varphi\),
\[
iH(\Phi) = < \varphi, \bar{\varphi} >.
\]  
(4.21)

The complex structure on \(S^\pm_q\) is compatible with the Hitchin functional in the sense that
\[
i\Phi < \cdot, \cdot > = dH(\Phi), \quad \text{where} \quad \Phi + i\Phi = 2\varphi.
\]  
(4.22)
Put into words, the Hamiltonian vector field of $\mathcal{H}(\Phi)$ is $\hat{\Phi}$. Also, $\hat{\Phi} = -\Phi$. Now $S^\pm_q$ are pseudo-Kähler spaces of signature $(30, 2)$. To make them Kähler, we define our moduli spaces as follows:

$$\mathcal{M}_{J;1,2} \overset{\text{def}}{=} S^+_q / \mathbb{C}^* \cong \text{Spin}(6,6) \times \mathbb{R}^+ / SU(3,3) \times \mathbb{C}^*,$$

$$\mathcal{M}_{\Omega;1,2} \overset{\text{def}}{=} S^-_q / \mathbb{C}^* \cong \text{Spin}(6,6) \times \mathbb{R}^+ / SU(3,3) \times \mathbb{C}^*.$$  \hfill (4.23)

The local Kähler potentials are given by $\lbrack 21 \rbrack$:

$$K_{J;1,2} = -\ln \mathcal{H}(\Phi^{+1,2}) = -\ln(i < \Phi^{+1,2}, \bar{\Phi}^{+1,2} >),$$

$$K_{\Omega;1,2} = -\ln \mathcal{H}(\Phi^{-1,2}) = -\ln(i < \Phi^{-1,2}, \bar{\Phi}^{-1,2} >).$$ \hfill (4.25)

However, the resulting Kähler metric is not unique due to the dilatino field. That ambiguity will be dealt with later. For the time being we slightly modify the functionals by untwisting them so that we have $\mathcal{H}_0 : S^\pm_q \rightarrow \Lambda^6 \Omega^*$. Then the volume functional

$$\mathcal{F}(\Phi) = \int_Y \mathcal{H}_0(\Phi)$$ \hfill (4.27)

has some remarkable properties. The critical points of $\mathcal{F}(\Phi)$ are precisely the generalized Calabi-Yau manifolds. Moreover, even though those critical points are never isolated (they constitute the orbits of the action of exact 2-forms), there are some that are degenerate only along the orbits of the action of 2-forms, the so-called transversally regular generalized Calabi-Yau manifolds. For those, the rank of the Hessian is constant and always equal the difference between the dimension of the entire space and that of the orbit.

Finally, we connect the $SU(3,3)$ spinors of Hitchin with the $SU(3) \times SU(3)$ structures $\lbrack 8 \rbrack$. Hereafter $\Phi^{+1,2} \in \mathcal{M}_{\Omega;1,2}$, while $\Phi^{-1,2} \in \mathcal{M}_{J;1,2}$:

$$\Phi^{+1,2} = \eta^+_{+} \otimes \eta^+_{-} \otimes e^{-\phi},$$

$$\Phi^{-1,2} = \eta^+_{-} \otimes \eta^+_{-} \otimes e^{-\phi}.$$ \hfill (4.28)

\hfill (4.29)

### 4.3 Mirror Moduli

The moduli spaces of pure Spin(6,6) spinors described, being sweeping generalizations of Calabi-Yau moduli spaces, contain many kinds of different elements. At one end of the spectrum they incorporate classical Calabi-Yau manifolds, and allow for mirror pairs thereof in $\mathcal{M}_{\Omega;1,2}$ and $\mathcal{M}_{J;1,2}$. Their matching is traditionally based on the SYZ approach $\lbrack 11 \rbrack$, and requires the existence of $T^3$ fibrations. Swapping typical fibers with corresponding dual ones effects mirror symmetry transformations $\lbrack 10 \rbrack$:

$$e^{B+iJ^{1,2}} \rightarrow \Omega^{1,2}.$$

Generalized Calabi-Yau manifolds generically do not possess any $T^3$-fibrations, so that their pairs have to be matched via a different mechanism $\lbrack 10 \rbrack$. Indeed,
Graña et al. rely on supersymmetry transformations instead, to match mirror pairs of generalized Calabi-Yau manifolds. There turn out to be representation spaces (of closed differential forms) that are swapped according to \(8 + 1 \leftrightarrow 6 + 3\). However, the bulk of \(\mathcal{M}_{\Omega_{1,2}}\) and \(\mathcal{M}_{J_{1,2}}\) consists of manifolds that are not generalized Calabi-Yau, hence not supersymmetric at all.

We study those moduli with an eye towards symmetry (and supersymmetry) properties of the ensuing 4-dimensional theories, and might need to eliminate nonphysical variability inherent in the compactifications. In the process we would greatly benefit from identifying the transversal deformations of generalized Calabi-Yau manifolds which form matching pairs in \(\mathcal{M}_{\Omega_{1,2}}\) and \(\mathcal{M}_{J_{1,2}}\). But quite apart from supersymmetry (and its breaking), the question of establishing mirror symmetry between elements of \(\mathcal{M}_{\Omega_{1,2}}\) and \(\mathcal{M}_{J_{1,2}}\) is of practical importance.

To start off, we get a grip on the moduli spaces viewing them as unions of orbits of Spin(6,6). Having described the intrinsic torsion classes associated with \(\eta^1, \eta^2\) in (4.5) - (4.7), we delineate the set of all elements of Spin(6,6) preserving intrinsic torsion. For that, we look at the group extension \(\aleph\), defined via

\[
(\Lambda^2 T^* Y)_{\text{closed}} \longrightarrow \aleph \longrightarrow \text{Diff}^c(Y),
\]

where \(\text{Diff}(Y) \subset \text{Diff}(M^{1,9})\) is a subgroup of the group of diffeomorphisms of the original manifold. As such, \(\aleph\) acts on the generic 6-dimensional manifolds \(Y\) endowed with two \(SU(3)\) spinors \(\eta^1, \eta^2\) and leaves intrinsic torsion intact. That action induces some linear actions on \(TY\) and \(T^* Y\), ergo on \(TY \oplus T^* Y\). The latter action determines a representation of \(\aleph\) on \(\text{SO}(6,6)\), which lifts to Spin(6,6). We denote its image \(\aleph\) linear \(\subset\) Spin(6,6). It is a proper subgroup because \(\mathfrak{so}(V \oplus V^*) = V \otimes V^* \oplus \Lambda^2 V^* \oplus \Lambda^2 V\), and \(\Lambda^2 V\) does not preserve intrinsic torsion. Furthermore, \(V \otimes V^*\) exponentiates to a proper subgroup of \(\aleph\) linear since there always are 1-forms that are not closed.

The orbits of \(\aleph\) linear constitute holomorphic foliations \(\mathcal{F}_{J_{1,2}}\) and \(\mathcal{F}_{\Omega_{1,2}}\). Using holomorphic Darboux coordinates, we express the symplectic form as

\[
\langle \cdot, \cdot \rangle = \frac{i}{2} \sum_{l \in I} dz_l \wedge d\bar{z}_l. \quad (4.30)
\]

Since the action of \(\aleph_{\text{linear}}\) is Kähler, there are subsets of indices \(I_0(J) \subseteq I\), \(I_0(\Omega) \subseteq I\) such that the respective symplectic forms on the leaves are

\[
\langle \cdot, \cdot \rangle |_{\mathcal{F}_{J_{1,2}}} = \frac{i}{2} \sum_{l \in I_0(J)} dz_l \wedge d\bar{z}_l, \quad (4.31)
\]

\[
\langle \cdot, \cdot \rangle |_{\mathcal{F}_{\Omega_{1,2}}} = \frac{i}{2} \sum_{l \in I_0(\Omega)} dz_l \wedge d\bar{z}_l. \quad (4.32)
\]

Therefore we define the Poisson structures as

\[
\pi_{J_{1,2}} \equiv \frac{i}{2} \sum_{l \in I_0(J)} \partial_l \wedge \bar{\partial}_l, \quad \pi_{\Omega_{1,2}} \equiv \frac{i}{2} \sum_{l \in I_0(\Omega)} \partial_l \wedge \bar{\partial}_l. \quad (4.33)
\]
Now following Weinstei\[12\] we recall that a full dual pair $(M_1, \pi_1) \xleftarrow{s_1} M \xrightarrow{s_2} (M_2, \pi_2)$ consists of two Poisson manifolds $(M_1, \pi_1)$ and $(M_2, \pi_2)$, a symplectic manifold $M$, and two submersions $s_1 : M \rightarrow M_1$ and $s_2 : M \rightarrow M_2$ such that $s_1$ is Poisson, $s_2$ is anti-Poisson and the fibers of $s_1$ and $s_2$ are symplectic orthogonal to each other. A Poisson (or anti-Poisson) mapping is said to be complete if the pull-back of a complete Hamiltonian flow under this mapping is complete. A full dual pair is called complete if both $s_1$ and $s_2$ are complete. The Poisson manifolds $M_1$ and $M_2$ are Morita equivalent if there exists a complete full dual pair such that the submersions have connected and simply connected fibers \[13\]. In the case of $\mathcal{M}_{j1,2}$ and $\mathcal{M}_{\Omega 1,2}$ we have the following full dual pair:

$$\mathcal{M}_{j1,2}, \pi_{j1,2} \xleftarrow{\mathbb{R}^{2n}} \mathcal{M}_{\Omega 1,2}, \pi_{\Omega 1,2},$$

(4.34)

for $n$ large enough, and the intermediate space equipped with the standard symplectic structure. With those conditions satisfied, (4.34) makes our moduli spaces Morita equivalent.

One property of Morita equivalent manifolds we utilize is that their respective spaces of Casimir functions are isomorphic \[12\]. That means the sheaves of Casimir functions are isomorphic too:

$$C_{j1,2} \cong C_{\Omega 1,2}.$$  

(4.35)

In the case of $\mathcal{M}_{j1,2}$ and $\mathcal{M}_{\Omega 1,2}$ these sheaves reduce to the holomorphic sheaves of functions of the appropriate intrinsic torsion classes because up to the actions of closed B-fields, the classes provide complete sets of local first-order differential invariants of the structures involved \[9\]. We denote them as

$$W_{j134} = \{f(W_1, W_3, W_4)\}^{\text{hol}}, \quad W_{125} = \{f(W_1, W_2, W_5)\}^{\text{hol}}.$$  

(4.36)

As a consequence, $W_{j134} = C_{j1,2} \cong C_{\Omega 1,2} = W_{125}$. By holomorphicity, their respective linear subsheaves are isomorphic as well, so that there is a correspondence

$$(W_1, W_3, W_4) \leftrightarrow (W_1, W_2, W_5),$$

(4.37)

which essentially is a way of swapping $8 + 1 \leftrightarrow 6 + 3$.

Now Poisson vector fields are characterized as the fields preserving the Poisson structure, i.e., for every Poisson $X$ we have $\mathcal{L}_X \pi = 0$. The first Poisson cohomology groups are defined via

$$H^1(\cdot) \overset{\text{def}}{=} \text{Poisson fields} \text{ Hamiltonian fields}.$$  

Finally, applying the results of \[14\] we obtain

$$H^1_{\pi j}(\mathcal{M}_{j1,2}) \cong H^1(\mathcal{M}_{j1,2}, W_{134}) \cong H^1(\mathcal{M}_{\Omega 1,2}, W_{125}) \cong H_{\pi \Omega}^1(\mathcal{M}_{\Omega 1,2}),$$

(4.38)

that is the first Poisson cohomology groups of our moduli spaces are iso-/anti-isomorphic to the respective first sheaf cohomology groups, and the latter are
canonically isomorphic.

Whenever there have been found classical matching Calabi-Yau elements \( Y \in \mathcal{M}_{\Omega^1,2} \), \( \hat{Y} \in \mathcal{M}_{\Omega^1,2} \), we set the initial conditions for the anti-isomorphic Poisson vector fields obtained from the Poisson cohomology classes by projecting onto the \( \pi \)-transversal subspaces (which is always possible because both \( \pi_{J^1,2} \) and \( \pi_{\Omega^1,2} \) are constant). Via mirror symmetry their integral curves hitting \( Y \) and \( \hat{Y} \) respectively must match. One curious fact meriting further exploration is that the numbers of integrable structures (i.e. of intersections of the Poisson integral curves with the leaves of \( \pi_{J^1,2} \) and \( \pi_{\Omega^1,2} \) that represent generalized Calabi-Yau manifolds) in each moduli space must be equal. By completeness, those numbers are precisely the numbers of such leaves, and the latter are bounded by the Betti and Hodge numbers:

\[
\#(\mathcal{F}_{J^1,2}|_{W=0}) \leq \sum_{i=1}^{b_Y} \frac{b_Y!}{(b_Y - i)!}, \quad b_Y = b^0(Y) + b^2(Y) + b^4(Y),
\]

\[
\#(\mathcal{F}_{\Omega^1,2}|_{W=0}) \leq \sum_{i=1}^{h_{\hat{Y}}} \frac{h_{\hat{Y}}!}{(h_{\hat{Y}} - i)!}, \quad \text{where}
\]

\[
h_{\hat{Y}} = h^{1,0}(\hat{Y}) + h^{3,0}(\hat{Y}) + h^{2,1}(\hat{Y}) + h^{3,2}(\hat{Y}).
\]

Regardless of their enumeration, some leaves among the sets \( \mathcal{F}_{J^1,2}|_{W=0} \) and \( \mathcal{F}_{\Omega^1,2}|_{W=0} \) are transversally regular.

If no mirror pairs have been identified (and the above estimates no longer apply), we can still use the isomorphism (4.38). Namely, we can match thin stacks of symplectic leaves with transversally regular generalized Calabi-Yau manifolds sandwiched in the middle. Any two of those are isomorphic. This effectively extends mirror symmetry to 6-manifolds that are gotten by transversal deformations of transversally regular generalized Calabi-Yau ones.

### 4.4 Nonspontaneous Supersymmetry Breaking

So far we have been working with general Type II compactifications, and the focus has been on their geometric properties. Now we turn to a concrete Type II theory to see the interplay between those geometric properties and supersymmetry. To fix ideas, we specialize to Type IIA and IIB gauged supergravity and its democratic formulation due to Bergshoeff at al. [22], presently grinding it down further to exhibit a mechanism of supersymmetry breaking which does not affect vanishing of the vacuum expectation values of gravitinos and other 4-dimensional fields. We are interested in the situation where the effective theory has the minimal \( N = 2 \) supersymmetry. In other words, we single out eight particular Type II supersymmetries that descend to the effective theory. The corresponding supersymmetry parameters \((\bar{\psi}^1, \bar{\psi}^2)\) are gathered in the doublet denoted \( \bar{\psi} \). Then the most general 10-dimensional gravitino supersymmetry transformation in the Einstein frame with all the R-R fluxes turned off has the
form:
\[ \delta \Psi_O = D O \tilde{g} - \frac{1}{96} e^{-\frac{1}{2}} (\Gamma^{PQR}_O H_{PQR} - 9 \Gamma^{PQ} H_{OPQ}) a^i \sigma_i \tilde{g}. \]  

(4.42)

Here \( H_{OPQ} \) are the N-S flux coefficients, \( O, P, Q \in \{0, \ldots, 9\} \), and \( a^i \)'s are theory-specific. Because of the compactification ansatz \[44\] all fields on \( M^{1,9} \) split. We write the ten-dimensional gamma matrices \( \Gamma^M = (\Gamma^\mu, \Gamma^m) \) as
\[ \Gamma^\mu = \gamma^\mu \otimes 1, \quad \mu = \{0, 1, 2, 3\}, \quad \Gamma^m = \gamma_5 \otimes \gamma^m, \quad m = \{1, \ldots, 6\}, \]  

(4.43)

where \( \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \). In keeping with this splitting the ten-dimensional gravitinos become \( \Psi_M = (\Psi_m, \psi_H) \). To simplify matters, we set \( \eta^1 = \eta^2 = \eta \), and decompose the ten-dimensional supersymmetry parameters:
\[ \begin{aligned}
\theta_{\text{IA}}^{1,2} &= \theta_+^{1,2} \otimes \eta_+ + \theta_-^{1,2} \otimes \eta_-, \\
\theta_{\text{IB}}^{1,2} &= \theta_+^{1,2} \otimes \eta_- + \theta_-^{1,2} \otimes \eta_+.
\end{aligned} \]  

(4.44)

(4.45)

Abusing notation we let the ' + ' and ' - ' signs signify both four-dimensional and six-dimensional chiralities. Then supersymmetry imposes two gravitino and two dilatino equations:
\[ (D_m \pm \frac{1}{4} i \partial_m H) \eta_\pm = 0, \]  

(4.46)

\[ (D_m - d \phi \pm \frac{1}{2} H) \eta_\pm = 0. \]  

(4.47)

Those equations narrow down the list of supersymmetric backgrounds. Thus \( D_m \eta_\pm = 0 \) is a prerequisite. Hence \( H = 0, \phi \) constant are the only choices. Therefore from now on we fix our dilatino field to support on-shell supersymmetry. Type IIA supersymmetric backgrounds are all \( B \)-transforms of symplectic manifolds, while Type IIB ones are all complex \[10\].

The supersymmetry transformations of \( \psi_\mu \) are \[8\]:
\[ \begin{aligned}
\delta \psi^1_{\text{IA} \mu} &= D_\mu g^1 - ie^{\frac{1}{2} (K_{J+} + K_{\Omega}) + \phi} < \Phi^+, d \Phi^- > g^1, \\
\delta \psi^2_{\text{IA} \mu} &= D_\mu g^2 + ie^{\frac{1}{2} (K_{J+} + K_{\Omega}) + \phi} < \Phi^+, d \Phi^- > g^2, \\
\delta \psi^1_{\text{IB} \mu} &= D_\mu g^1 - ie^{\frac{1}{2} (K_{J-} + K_{\Omega}) + \phi} < \Phi^-, d \Phi^+ > g^1, \\
\delta \psi^2_{\text{IB} \mu} &= D_\mu g^2 + ie^{\frac{1}{2} (K_{J-} + K_{\Omega}) + \phi} < \Phi^-, d \Phi^+ > g^2.
\end{aligned} \]  

(4.48)

(4.49)

(4.50)

(4.51)

From these formulas we glean the expression for the mass of gravitinos (maintaining the masses equal for \( \psi^1 \) and \( \psi^2 \)):
\[ m_{\text{gravitino}} = c_{\text{IA, IB}} |\langle VAC | e^{\frac{1}{2} (K_{J+} + K_{\Omega}) + \phi} < \Phi^\pm, d \Phi^\mp > | VAC \rangle|. \]  

(4.52)

Clearly, \( d \Phi^\pm = 0 \) implies \( m_{\text{gravitino}} = 0 \), but there are other possibilities to get \( < \Phi^\pm, d \Phi^\mp > = 0 \). We can always construct Lagrangian submanifolds (with respect to \( < \cdot, \cdot > \)) that transversally intersect our foliations. Unfortunately, far
away, there is no way of knowing whether the background manifold is supersymmetric, in which case $m_{\text{gravitino}} = 0$ trivially. But close to transversally regular generalized Calabi-Yau manifolds all transversally deformed Hitchin spinors are nonintegrable by transversal regularity. Therefore plenty of background manifolds realize nonspontaneous supersymmetry breaking. It must occur once the fluxes are in, *ergo* the moduli spaces are enlarged sufficiently.

To demonstrate pervasiveness of nonspontaneous supersymmetry breaking we now consider the space-time backgrounds that realize only four supercharges. Then $N = 2$ supermultiplets split. In particular, the $N = 2$ gravitational multiplet decomposes into an $N = 1$ gravitational multiplet containing the metric and one gravitino $(g_{\mu \nu}, \psi_{\mu})$, and a $N = 1$ spin-3/2 multiplet containing the second gravitino and the graviphoton $(\psi'_{\mu}, C_{\mu})$. The appearance of a standard $N = 1$-type action requires that the latter multiplet be projected out. With all those conditions in place, the $N = 1$ superpotentials $\mathcal{P}_{\text{IIA}}, \mathcal{P}_{\text{IIB}}$ have been computed from the supersymmetry transformation of the linear combination of the two $N = 2$ gravitinos which resides in the $N = 1$ gravitational multiplet [8]. All such linear combinations are parameterized by two angles that specify a particular embedding of the $N = 1$ submultiplet inside $N = 2$. They are

$$\mathcal{P}_{\text{IIA}} = \cos^2 \alpha e^{i \beta} < \Phi^+, d\Phi^- > - i \sin^2 \alpha e^{-i \beta} < \Phi^+, d\Phi^- > ,$$  

$$\mathcal{P}_{\text{IIB}} = \cos^2 \alpha e^{i \beta} < \Phi^+, d\Phi^- > - i \sin^2 \alpha e^{-i \beta} < \Phi^+, d\Phi^- > .$$

Here transversal deformations furnish the space of $D$-flat directions thus eliminating spontaneous supersymmetry breaking.

### 4.5 The $G$-Reduction

In the previous subsection we have shown that sufficiently small transversal deformations (within $\mathcal{M}_{\Omega} \times \mathcal{M}_J$) of transversally regular generalized Calabi-Yau manifolds lead to nonspontaneously broken supersymmetry. There still lingers the question whether the underlying mechanism of supersymmetry breaking is that of Section 3.

To address that, we first observe that any effective 4-dimensional theory is impervious to the actions of diffeomorphisms and closed B-fields. Whence in the physical space-time $\eta$’s transform in accordance with the transversal deformations and are governed by the (functions of) intrinsic torsion classes. Moreover, the diagram below is commutative, and its vertical arrows indicate surjective mappings squashing the dilatino fields:

$$\mathcal{M}_{\Omega} \times \mathcal{M}_J \xrightarrow{\text{Spin}(6,6)} \mathcal{M}_{\Omega} \times \mathcal{M}_J$$

$$\Lambda^* T^* Y \otimes \mathbb{C} \xrightarrow{\text{Spin}(6)} \Lambda^* T^* Y \otimes \mathbb{C}$$

Having made those points, we proceed to describe precisely how $\eta$’s behave in the effective four-dimensional theory. A convenient basis for all spinors on
subject to the differential constraint

Now let

\[
\begin{bmatrix}
\eta + i\gamma \eta \\
\eta - i\gamma \eta \\
\theta + i\gamma \theta \\
\theta - i\gamma \theta
\end{bmatrix}
\]

be a one parameter family of spinor frames stemming from a family of transversal deformations that realize nonspontaneous supersymmetry breaking. Expanding in a Taylor series we get

\[
\begin{bmatrix}
\eta + i\gamma \eta \\
\eta - i\gamma \eta \\
\theta + i\gamma \theta \\
\theta - i\gamma \theta
\end{bmatrix}^{(t)} + t \sum_i W_{il} M^i \begin{bmatrix}
\eta + i\gamma \eta \\
\eta - i\gamma \eta \\
\theta + i\gamma \theta \\
\theta - i\gamma \theta
\end{bmatrix} + O(t^2),
\]

subject to the differential constraint

\[
d \begin{bmatrix}
\eta + i\gamma \eta \\
\eta - i\gamma \eta \\
\theta + i\gamma \theta \\
\theta - i\gamma \theta
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

Here \( W_{il} \) are holomorphic functions of one of the intrinsic torsion classes, not all vanishing because of the transversality condition, and \( M^i \in \text{su}(4) \) are non-singular again by transversality.

Now by our extension of mirror symmetry \((4.38)\), there must exist a diffeomorphism/closed B-field whose action relates \((W_2, W_3)\) with \((W_3, W_4)\) so that up to a scalar function

\[
U \left( \sum_{i=\{1,2,5\}} W_{il} M^i \right) U^H = \sum_{i=\{1,3,4\}} W_{il} M^i,
\]

which is an explicit realization of the representation swap \(6 + 3 \rightarrow 8 + 1\). Using irreducibility we are forced to conclude that diffeomorphisms/closed B-fields cannot commute with \( M^i \). Therefore \( M^i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), and up to the action of diffeomorphisms/closed B-fields \( M^i = K_i \). And the smallest subgroup of \( SU(4) \) incorporating \( K_i \) is \( G \). Thus the connection matrices are determined and the connection \((4.39)\) takes the form

\[
P_{\mu} \longrightarrow i \nabla_{\mu}(x) \overset{\text{def}}{=} i(\gamma_{\mu}^{\nu}(x)\partial_{\nu} + i \sum_j W_{j\mu}^{\alpha}(x)K_{\alpha}),(4.58)
\]

where the potentials \( W_{j\mu}^{\alpha}(x) \) are defined via \((4.1)\) and \((4.2)\). For instance, within a supersymmetric region \( W_{j\mu}^{\alpha}(x) \equiv 0 \), hence \( \gamma_{\mu}^{\nu}(x) \equiv \delta_{\mu}^{\nu} \).
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