DYNAMICS OF CONVEX COCOMPACT SUBGROUPS OF MAPPING CLASS GROUPS

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Abstract. For a convex cocompact subgroup \( G \subset \text{Mod}(S) \), and points \( x, y \in T_{\text{eich}}(S) \) we obtain asymptotic formulas as \( R \to \infty \) of \( |B_R(x) \cap G y| \) as well as the number of conjugacy classes of pseudo-Anosov elements in \( G \) of dilatation at most \( R \). We do this by developing an analogue of Patterson-Sullivan theory for the action of \( G \) on \( \text{PMF} \).

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1. Statement of Results

The study of the dynamics of the action of the mapping class group on Teichmüller space has long been influenced by analogy with the actions of discrete isometry groups of manifolds of negative curvature. Two properties of interest for a negatively curved manifold \( M \) are the growth of orbits of \( \pi_1(M) \) and the asymptotics as \( R \to \infty \) of the number \( n_M(R) \) of closed geodesics of length at most \( R \). For \( M \) compact and negatively curved and \( x, y \in M \), Margulis [12] showed in his 1970 thesis that

\[
\lim_{R \to \infty} e^{-hR} |B_R(x) \cap \Gamma y| = \Lambda(x)\Lambda(y)
\]

and

\[
\lim_{R \to \infty} hRe^{-hR} n_M(R) = 1
\]
where $\Lambda$ is a continuous function on $M$ and $h$ is the topological entropy of the geodesic flow. Roblin \[22\] generalized these results to $M = X/\Gamma$ any quotient of a contractible $\text{CAT}(-1)$ metric space by a geometrically finite group $\Gamma$. We prove an analogue of Roblin’s result for certain subgroups of mapping class groups acting on Teichmüller space. Let $S$ be a surface of genus $g \geq 2$. Let $\text{Teich}(S)$ be the associated Teichmüller space of isotopy classes of marked complex structures on $S$. Let $\text{Mod}(S) = \text{Diff}(S)/\text{Diff}_0(S)$ be the associated mapping class group, and let $d_T$ denote the Teichmüller metric on $\text{Teich}(S)$. A subgroup $G < \text{Mod}(S)$ is called convex cocompact if its orbit in $\text{Teich}(S)$ is quasiconvex. Convex-cocompact subgroups of $\text{Mod}(S)$ were introduced by Farb and Mosher \[2\] and further developed by Kent and Leininger \[14\]. We prove:

**Theorem 1.1.** Let $S$ be a closed surface of genus $g \geq 2$. Let $G < \text{Mod}(S)$ be a convex cocompact subgroup containing a pseudo-Anosov element whose axis lies in the principal stratum. Let $x, y \in \text{Teich}(S)$ and $B_R(x)$ the ball of radius $R$ about $x$ in the Teichmüller metric. Let $h$ be the exponent of convergence of $G$ with respect to the Teichmüller metric. Then

$$\lim_{R \to \infty} e^{-hR} |B_R(x) \cap \Gamma y| = \Lambda(x)\Lambda(y)$$

where $\Lambda$ is some $G$ invariant continuous function on $\text{Teich}(S)$.

**Theorem 1.2.** Let $G$ be as in Theorem 1.1 Let $n_M(R)$ be the number of conjugacy classes of primitive pseudo-Anosov mapping classes in $G$ of Teichmüller translation length at most $R$ (this translation length is the logarithm of the dilatation of the pseudo-Anosov representative). Then

$$\lim_{R \to \infty} hR e^{-hR} n_M(R) = 1.$$

Like Margulis and Roblin, we prove the counting estimate by constructing a certain measure on the unit tangent bundle and prove it is mixing. We develop an analogue of Patterson-Sullivan theory for the action of $G < \text{Mod}(S)$ on Thurston’s sphere $PMF$. We construct a unique $G$-conformal density $\nu_x, x \in \text{Teich}(S)$ supported on the limit set $\Lambda(G) \subset PMF$ of $G$, and scale the product measure $\nu_x \times \nu_x$ by a factor depending on the Busemann function to form a finite (in fact compactly supported) $G$ invariant measure $\mu$ on the unit (co)tangent bundle $Q^1(S)$, which can be considered the analogue Bowen-Margulis measure in negative curvature. We prove

**Theorem 1.3.** The measure $\mu$ associated with any convex cocompact subgroup $G < \text{Mod}(S)$ is mixing.

A difficulty faced in our setting is that $\text{Teich}(S)$ is not globally hyperbolic in any reasonable sense. It is neither $\text{CAT}(0)$ nor Gromov hyperbolic: indeed pairs of geodesic rays through the same point may fellow-travel arbitrarily far apart \[31\]. Thurston proved that $\text{Teich}(S)$ has a natural $\text{Mod}(S)$ equivariant compactification by the sphere of projective measured foliations, but not every geodesic ray converges to a limit in $PMF$, rays with the same limit point are not necessarily asymptotic, and rays with different limit points may stay a bounded distance apart. Thurston’s compactification coincides neither with the Gromov compactification (which is not Hausdorff) nor with the horofunction compactification (which contains $PMF$ as a proper subset of smaller dimension). However, the proof of Theorems 1.1 and 1.2
requires that generic (with respect to the analogue of the Bowen-Margulis measure) geodesic segments have a certain property typical of \( \text{CAT}(-1) \) spaces: namely, if two geodesics both pass within two balls of bounded radius lying far apart, they become very close somewhere in the middle. This occurs only if the segments spend a uniform proportion of time in the part of \( Q^1(S) \) with no short flat curves (if we only required the segments to spend a uniform proportion time over a compact subset of the moduli space of Riemann surfaces, we would see behavior indicative of Gromov hyperbolicity). We use some ergodic-theoretic arguments together with some hodge norm estimates from [5] to show that the asymptotics is controlled by geodesic segments which are well-behaved in this sense and use techniques analogous to Roblin’s to count these well-behaved geodesics.

In order to prove mixing of \( \mu \) in Theorem 1.3 we prove a certain nondegeneracy condition for the length spectrum of \( G \), which is in our setting the measure of maximal entropy for the Teichmüller geodesic flow over \( \text{Teich}(S)/G \).

**Theorem 1.4.** Let \( G \subset \text{Mod}(S) \) be a nonelementary subgroup. Then the logarithms of the dilatations of pseudo-Anosov elements of \( G \) generate a dense subgroup of \( \mathbb{R} \).

For subsemigroups of \( SL_n \mathbb{R} \) acting irreducibly on \( \mathbb{R}^n \) and containing a proximal element, an analogous result is proved by Guivarch and Urban in [23]. In variable negative curvature this question remains open. We prove Theorem 1.4 by using the affine and symplectic structure of \( \text{MF} \) given by train track coordinates to embed a sub semigroup of \( G \) into \( SL_n \mathbb{R} \) with the image satisfying the conditions of [23].

When \( G \) is the full mapping class group, analogues of Theorems 1.1 and 1.2 respectively were proved by Athreya-Bufetov-Eskin-Mirzakhani in [1] and Eskin-Mirzakhani in [5], in which case the Bowen-Margulis measure coincides with the Masur-Veech measure.

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2. **Background on Teichmüller Theory**

Let \( S \) be a closed surface of genus at least 2. Let \( \text{Mod}(S) \) be the mapping class group of \( S \). Let \( \text{Teich}(S) \) be the space of marked complex structures on \( S \) up to isotopy. The space \( Q(S) \) of quadratic differentials can be thought as a cotangent bundle of \( \text{Teich}(S) \). A stratum of \( Q(S) \) consists of all quadratic differentials whose zeros have the same combinatorial singularity type. The principal stratum consists of all quadratic differentials with simple zeros. Let \( Q^1(S) \) be the space of area one...
quadratic differentials, which can be identified with the unit cotangent bundle to $Teich(S)$. Let $\pi : Q^{1}(S) \to Teich(S)$ be the projection. Let $MF$ be the space of measured foliations on $S$ and $PMF$ its projectivization. For a quadratic differential $q$ let $q^+, q^- \in MF$ denote its vertical and horizontal measured foliations respectively and $[q^+], [q^-] \in PMF$ its projective classes. Let $UE \subseteq PMF$ denote the projective classes of uniquely ergodic foliations. There is a compactification due to Thurston of $Teich(S)$ by $PMF$ obtained by embedding both into $\mathbb{R}^A$ where $A$ is the set of isotopy classes of simple closed curves on $S$. Unless otherwise stated, the topology on $Teich(S) \cup PMF$ in this paper comes from the Thurston compactification of $Teich(S)$. Given a basepoint in $Teich(S)$, we can also compactify $Teich(S)$ by equivalence classes of geodesic rays through the point, called the Teichmüller compactification. Hubbard and Masur showed in [11] that there is a homeomorphism $Q(S) \to Teich(S) \times MF$ obtained by associating to a quadratic differential its projection in $Teich(S)$ and vertical (or horizontal) measured foliation. In [10] Masur shows:

**Theorem 2.1.** If $q \in Q(S)$ with $q^+$ uniquely ergodic then $\pi(g_i q)$ converges to the projective class of $q^+$ in $PMF$.

**Theorem 2.2.** If $q_1, q_2 \in Q^{1}(S)$ and $q_1^+ = q_2^+$ is uniquely ergodic then $d_T(\pi(g_i q_1), \pi(g_i q_2)) \to 0$.

The following is part of Masur’s Two Boundaries Theorem [17].

**Theorem 2.3.** The identity map on $Teich(S)$ extends to a homeomorphism between $T(S) \cup UE$ in the Teichmüller compactification and $T(S) \cup UE$ in the Thurston compactification.

**Theorem 2.4.** Let $x = x_0 \in Teich(S)$. Let $x_n \in Teich(S)$ be a sequence converging in the Thurston compactification to a uniquely ergodic $\eta \in PMF$. Then there exists a sequence of quadratic differentials $q_i \in S(x)$ and $t_i > 0$ such that $x_i = \pi(g_i q_i)$ and the $q_i$ converge to $q \in S(x)$ such that $\eta = \lim_{i \to \infty} \pi(g_i q)$. Therefore for any fixed $m > 0$, the points $\pi(g_m q_i)$ converges to $\pi(g_m q_i)$.

The following result of Klarreich is Prop 5.1 in [13].

**Proposition 2.5.** Let $F_1$ and $F_2$ be topologically inequivalent minimal foliations. Let $x_n$ and $y_n$ be sequences in $Teich(S)$ converging to $F_1$ and $F_2$ respectively. Then the geodesic segments $[x_n, y_n]$ accumulate in the Teichmüller compactification to a set $s \subseteq Teich(S) \cup PMF$ such that $s \cap Teich(S)$ is a nonempty union of geodesics whose vertical and horizontal foliations are topologically equivalent to $F_2$ and $F_1$ respectively and $s \cap PMF$ consists of foliations that are equivalent to $F_1$ or $F_2$. If $x \in Teich(S)$ is fixed, then $[x, x_n]$ accumulate to a set $s \subseteq Teich(S) \cup PMF$ such that $s \cap Teich(S)$ is a union of geodesic rays based at $x$ whose endpoints are topologically equivalent to $F_1$ and $s \cap PMF$ consists of foliations topologically equivalent to $F_1$. If $F_1$ and $F_2$ are topologically equivalent then $[x_n, y_n]$ converges to a subset of $PMF$ each element of which is topologically equivalent to $F_1$.

For uniquely ergodic foliations $F_1$ and $F_2$, the above proposition and Masur’s two boundaries theorem implies the following.

**Corollary 2.6.** $[x, x_n]$ converges uniformly on compact sets to $[x, [F_1]]$ and $[x_n, y_n]$ converges to $([F_1], [F_2])$. 
The following corollary of Proposition 2.5 is proved in [14].

**Corollary 2.7.** Let \( \eta_n, \zeta_n \) in PMF converge to uniquely ergodic \( \eta, \zeta \in PMF \). Then the accumulation points of \([\eta_n, \zeta_n]\) in the Thurston topology are contained in \{\( \eta, \zeta \)\}.

Call a subgroup \( G \) of \( Mod(S) \) non-elementary if it contains a pair of non-commuting pseudo-Anosovs. For a nonelementary subgroup \( \{PMF \) it is not equal to pseudo-Anosov mapping classes in \( G \). It is perfect and has empty interior provided \( G \) dense in \( \Lambda(G) \). Let

\[
\text{Proof.}
\]

For any \( G \) convex cocompact subgroups of \( Mod(G) \) denote its limit set in \( \Lambda(G) \). Then the accumulation points of \( \Lambda(G) \) fill \( S \), we define \( WH(G) \) to be the union of all Teichmüller geodesics whose vertical and horizontal measured foliations have projective classes in \( \Lambda(G) \). A subgroup \( G \) of \( Mod(S) \) is called convex cocompact if some \( G \)-orbit in \( Teich(S) \) is quasiconvex. The following properties of convex cocompact subgroups of \( Mod(S) \) are proved in [2] and [14].

**Theorem 2.8.**

- Every \( G \) orbit is quasi-convex.
- The weak hull \( WH(G) \) is defined and \( G \) acts cocompactly on \( WH(G) \).
- Every limit point \( \eta \) of \( G \) is conical, that is for \( x \in Teich(S) \) there is some \( D > 0 \) such that the ray \([x, \eta]\) has infinite intersection with \( D \) neighborhood of \( Gx \).
- \( G \) acts cocompactly on \( WH(G) \) \( \cup \Lambda(G) \).
- \( WH(G) \) is contained in the \( \epsilon \)-thick part of \( Teich(S) \) and \( A \)-quasiconvex for some \( \epsilon > 0 \) and \( A > 0 \).
- \( WH(G) \) \( \cap \Lambda(G) \) is closed in \( Teich(S) \) \( \cap PMF \).
- Every point of \( \Lambda(G) \) is uniquely ergodic.
- \( G \) contains a finite index subgroup all of whose nonidentity elements are pseudo-Anosov.

From now on, let \( G \) denote a nonelementary convex cocompact subgroup of \( Mod(S) \).

**Lemma 2.9.** If \( G \) is nonelementary convex cocompact, \( \Lambda(G) \) coincides with \( \overline{Gx} \cap PMF \) for any \( x \in Teich(S) \).

**Proof.** Let \( \eta \in \Lambda(G) \) be an the attracting point of a pseudo-Anosov \( g \in G \). Then \( g^n x \to \eta \) so \( \eta \in \overline{Gx} \cap PMF \). Since the fixed points of pseudo-Anosovs of \( G \) are dense in \( \Lambda(G) \), we have \( \Lambda(G) \subset \overline{Gx} \cap PMF \) (this holds for any nonelementary \( G \subset Mod(S) \)). For the other direction, it is proved by McCarthy and Papadopoulos in [13] that \( \overline{Gx} \cap PMF \) is contained in

\[
Z \Lambda(G) = \{ \lambda \in PMF | \exists \beta \in \Lambda(G) : i(\lambda, \beta) = 0 \}.
\]

Since every point of \( \Lambda(G) \) is uniquely ergodic, \( Z \Lambda(G) = \Lambda(G) \). \( \square \)

Let \( Teich_\epsilon(S) \) denote the \( \epsilon \) thick part of \( Teich(S) \), the set of hyperbolic structures on \( S \) where no closed curve has hyperbolic length less than \( \epsilon \). The following property of Teichmüller geodesics, indicative of hyperbolicity in the thick part, is proved by Rafi [21].

**Lemma 2.10.** For each \( A > 0 \) and \( \epsilon > 0 \) there exists a constant \( D > 0 \) such that for points \( x, x', y, y' \in Teich_\epsilon(S) \) with \( d_T(x, x') \leq A \) and \( d_T(y, y') \leq A \) the geodesic segments \([x, y]\) and \([x', y']\) \( D \)-fellow travel in a parametrized fashion, and
for $\eta \in \text{PMF}$ such that $[x, \eta]$ and $[x', \eta]$ are contained in $\text{Teich}_r(S)$, the geodesic rays $[x, \eta]$ and $[x', \eta]$ $D$-fellow travel in a parametrized fashion.

For a subset $W$ of a metric space and $A > 0$ let $N_A W$ denote the $A$ neighborhood of $W$.

**Corollary 2.11.** Let $G \leq \text{Mod}(S)$ be convex cocompact. For every $C > 0$ there exists an $\epsilon > 0$ such that every geodesic with endpoints in $\Lambda(G) \cup N_C WH(G)$ is contained entirely in $N_{C\epsilon} WH(G)$.

**Proof.** Let $A > 0$ be such that $WH(G) \cup \Lambda(G)$ is $A$-quasiconvex. Let $\epsilon > 0$ be such that $N_C WH(G) \subseteq \text{Teich}_r(S)$. (Such an $\epsilon$ exists since $G$ acts cocompactly on $WH(G)$ and therefore on $N_C WH(G)$). Then for each $x, y \in N_C WH(G)$ there are $x', y' \in WH(G)$ with $d_T(x, x') \leq C$ and $d_T(y, y') \leq C$. Since $x, x', y, y'$ lie in $\text{Teich}_r(S)$ and $d_T(x, x') \leq C$ and $d_T(y, y') \leq C$, it follows from Rafi’s theorem that the geodesic segments $[x, y]$ and $[x', y']$ $D$-fellow travel. Since $WH(G)$ is $A$-quasiconvex, $[x', y']$ is contained in $N_{A+C \epsilon} WH(G)$ and so $[x, y]$ is contained in $N_{A+C+D\epsilon} WH(G)$. The proof when one of $x, y$ lies in $\Lambda(G)$ is similar. $\square$

The following is proved in [19].

**Proposition 2.12.** For every $\epsilon > 0$ there exists an $\delta > 0$ such that any triangle with vertices in $\text{Teich}(S) \cup \text{PMF}$ and sides contained in $\text{Teich}_{\epsilon}(S)$ is $\delta$ thin- each side is contained in a $\delta$ neighborhood of the other two.

From the proof of [14], Theorem 4.4 we also have the following.

**Proposition 2.13.** For each $\epsilon > 0$ there exists a $K > 0$ with the following property. Suppose $x, y, z \in \text{Teich}(S) \cup \text{PMF}$ form a triangle with sides contained in the $\epsilon$ thick part of $\text{Teich}(S)$. Let $P \in [x, y]$ minimize the distance between $[x, y]$ and $z$. Then $[x, P] \cup [P, z]$ lies in a $K$ neighborhood of $[x, z]$.

3. Fixing the Quasiconvexity Constants

Existence of the following constants is guaranteed by the above remarks. Fix $A > 0$.

Let $A'' > 0$ be such that any geodesic between two points in

$N_A WH(\Lambda(G)) \cup \Lambda(G)$

is contained in

$N_{A''} WH(\Lambda(G)) \cup \Lambda(G)$

Let $A' > 0$ be such that any geodesic between two points in

$N_{A'} WH(\Lambda(G)) \cup \Lambda(G)$

is contained in

$N_{A'} WH(\Lambda(G)) \cup \Lambda(G)$

Let $\epsilon > 0$ be such that $N_{A''} WH(\Lambda(G)) \subseteq \text{Teich}_r(S)$. Let $K > 0$ be large enough so that any triangle in Teichm"uller space with sides contained in $\text{Teich}_r(S)$ is $K$ thin, satisfies Proposition 2.13, and also large enough such that the shadow $pr_{\eta} B_K(x)$ contains an open set intersecting $\Lambda(G)$ for every $x \in N_{A''} WH(G)$ and $\eta \in \Lambda(G)$ (see section 7 below).
4. Busemann Functions for the Teichmüller Metric

If $x, y \in \text{Teich}(S)$, $\alpha \in MF$ is uniquely ergodic, and $z_n \to [\alpha]$ in the Thurston compactification, then Miyachi [20] showed
\[
d_T(x, z_n) - d_T(y, z_n) \to \frac{1}{2} \log \frac{\text{Ext}_\alpha(x)}{\text{Ext}_\alpha(y)}.
\]
In particular, the limit $\beta_\alpha(x, y) = d_T(x, z_n) - d_T(y, z_n)$ exists and varies continuously with $[\alpha] \in UE$. This gives a continuous extension of the cocycle $\beta_\alpha(x, y)$ to $\text{Teich}(S) \cup UE$.

5. Conformal Densities for $G$

A conformal density for $G$ is a family $\{\nu_x| x \in \text{Teich}(S)\}$ of borel measures on $\text{PMF}$, each supported on $\Lambda(G)$ satisfying
(1) $\gamma^* \nu_x = \nu_{\gamma x}$
for all $x \in \text{Teich}(S)$ and $\gamma \in G$ and
(2) For all $x, y \in \text{Teich}(S)$ $\nu_x$ and $\nu_y$ are absolutely continuous and satisfy
\[
\frac{d\nu_x}{d\nu_y}(\alpha) = \exp(\delta(G)\beta_\alpha(x, y)).
\]

Note, by the $G$-invariance of the Busemann cocycle if condition (2) is satisfied, it suffices to check condition (1) at a single $x$.

Proposition 5.1. A conformal density $\nu_x$ for a nonelementary convex cocompact $G < \text{Mod}(S)$ has full support on $\Lambda(G)$ and has no atoms.

Proof. Suppose $U \subseteq \text{PMF}$ is open and $U \cap \Lambda(G) \neq \emptyset$ but $\nu_x(U) = 0$. Since the limit set is the closure of the set of stable (or unstable) laminations of pseudo-Anosov mapping classes in $G$, there is some pseudo Anosov $\gamma \in G$ with axis $l^+$ with repelling fixed point $l^- \in U$. Then for each $n > 0$,
\[
\nu_x(\gamma^n U) = \nu_{\gamma^{-n} x}(U) = 0
\]
since $\nu_x$ and $\nu_{\gamma^{-n} x}$ are absolutely continuous. Note
\[
\bigcup_{n>0} \gamma^n U = \text{PMF} \setminus l^+
\]
By countable subadditivity of the measure, $\nu_x$ is concentrated on the single point $l^+$. However, since $G$ is not elementary, there is some $h \in G$ with $hl^+ \neq l^+$, and by absolute continuity we must also have

$$\nu_x(hl^+) = \nu_{h^{-1}x}(l^+) \neq 0$$

giving a contradiction. Thus we have proved that $\nu_x$ has full support on $\Lambda(G)$.

Now, suppose $\nu_x$ has an atom $\eta \in \Lambda(G)$, say of mass $r$. By [KL1] every limit point $\eta$ of $G$ is conical, that is there exists a $D > 0$ such that the $D$ neighborhood of the geodesic $[x, \eta)$ intersects the orbit $Gx$ infinitely many times. Let $\gamma_n \in G$ be such a sequence. Then by the triangle inequality, $\beta_x(\gamma_n^{-1} \eta) = \nu_{\gamma_nx}(\eta) = \exp(\delta(G)\beta_{\eta}(\gamma_nx, x)) \nu_x(\eta) \to \infty$ contradicting the finiteness of $\nu_x$. □

6. Patterson-Sullivan Construction of a Conformal density

Let $\delta_G$ be the exponent of convergence of $G$. For $s > \delta_G$ and $x, y \in \text{Teich}(S)$ let

$$f_s(x, y) = \sum_{\gamma \in G} \exp(-sd(x, \gamma(y))).$$

Fix $x \in \text{Teich}(S)$. Now let

$$\nu_{x,s} = f_s(x, x)^{-1} \sum_{\gamma \in G} \exp(-sd(x, \gamma(x))) \delta_{\gamma x}$$

where $\delta_p$ denotes the dirac measure at $p$. Now, consider a weak-* limit $\nu_x$ of the $\nu_{x,s}$ as $s \to \delta_G$. It is a probability measure on

$$\overline{\text{Teich}(S)} = \text{Teich}(S) \cup \text{PMF}.$$ 

Assume first that the Poincare series diverges at $\delta_G$. Then $f(s) \to \infty$, so by discreteness of $G$, $\nu_x$ gives zero measure to compact subsets of $\text{Teich}(S)$, and thus must be supported on $\text{PMF}$.

Furthermore, since each $\nu_{x,s}$ is supported on the $G$ orbit of $x$, it follows that $\nu_x$ is supported on its closure, so it must be supported on $\Lambda(G)$. For any other $y \in \text{Teich}(S)$ define

$$d\nu_y(\alpha) = \exp(\delta(G)\beta_\alpha(x, y))d\nu_x.$$ 

Since

$$\beta_\alpha(x, z) = \beta_\alpha(x, y) + \beta_\alpha(y, z)$$

we have that

$$\frac{d\nu_y}{d\nu_z}(\alpha) = \exp(\delta(G)\beta_\alpha(z, y)).$$

Now, we show that this gives a conformal density. Indeed, for any $g \in G$ we have,

$$g\nu_{gx,s}(z) = \exp(-s\beta_x(gx, x))\nu_{x,s}(z)$$

where $\beta$ is the Busemann cocycle, and taking limits as $s \to \delta$ we get

$$gd\nu_x(\alpha) = \exp(-\delta(G)\beta_\alpha(gx, x))d\nu_x$$

so we indeed have a conformal density.

Now, suppose the Poincare series converges at $\delta(G)$ (this case turns out to be vacuous, but the construction of a conformal density is required to show it). There exists a slowly growing function $h$ on $\mathbb{R}$ such that
$\sum_{\gamma \in \Gamma} h(x, \gamma x) \exp(-sd(x, \gamma x))$

diverges at $s = \delta(G)$ but converges for $s < \delta(G)$. We then set

$$f_s(x, x) = \sum_{\gamma \in \Gamma} h(x, \gamma x) \exp(-sd(x, \gamma x))$$

and carry out the construction as before. The existence of an appropriate function $h$ is guaranteed by application of the following result of [26] to the Radon measure $\sum_{\gamma \in \Gamma} D_d(x, \gamma x)$.

**Lemma 6.1.** Let $\lambda$ be a Radon measure on $\mathbb{R}_+$, such that the Laplace transform

$$\int_{\mathbb{R}_+} e^{-st} d\lambda(t)$$

has critical exponent $\delta \in \mathbb{R}$. Then there exists a nondecreasing function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\int_{\mathbb{R}_+} h(t) e^{-\delta t} d\lambda(t) = \infty$$

and for every $\epsilon > 0$ there exists $t_0 \geq 0$ such that, for any $u \geq 0$ and $t \geq t_0$, one has

$$h(u + t) \leq e^{\epsilon t} h(t)$$

In particular the Laplace transform of $h \lambda$ has critical exponent $\delta$.

### 7. Sectors and Closures

For $x, y \in T(S)$ and $\eta \in PMF(S)$ let $pr_y(x) \in PMF(S)$ denote the vertical projective measured foliation of the quadratic differential $q \in S(x)$ with horizontal projective measured foliation $\eta$ and $pr_y(x) \in PMF(S)$ the vertical projective measured foliation of the geodesic segment from $x$ to $y$.

**Lemma 7.1.** The function $pr_\ast(\ast)$ is continuous on $(Teich(S) \cup UE) \times Teich(S)$.

**Proof.** The continuity on $Teich(S) \times Teich(S)$ and $PMF \times Teich(S)$ follows from the fact that the map $S(x) \rightarrow PMF$ is a homeomorphism for $x \in Teich(S)$. If $x_n \rightarrow \eta$ uniquely ergodic, then $[x_n, x]$ converges to $(\eta, x)$. Let $y_n \in [x_n, x]$ be at distance one from $x$. Then $y_n \rightarrow y = \gamma_{x, \eta}(1)$ so $pr_x(x) = pr_{y_n}(x) \rightarrow pr_y(x) = pr_{\eta}(x)$. $\Box$

For $x \in Teich(S)$ and $U \subseteq PMF$ let $\text{Sect}_x(U)$ be the set of all $y \in \text{Teich}(S)$ with $pr_x(y) \in U$ For any $r > 0$ let

$$C^+_r(x, U) = N_r \bigcup_{z \in B_r(x)} \text{Sect}_z(U)$$

and

$$C^-_r(x, U) = \left\{ y \in \text{Teich}(S) | B(y, r) \subseteq \bigcap_{z \in B_r(x)} \text{Sect}_z(U) \right\}$$
**Lemma 7.2.** For any \( r > 0, D > 0 \) and \( U \subset PMF \) open the closure of \( C_{r}^{+}(x,U) \cap N_{D}WH(G) \) in the Thurston compactification is

\[
(C_{r}^{+}(x,\overline{U}^{PMF}) \cap N_{D}WH(G)) \cup (\overline{U}^{PMF} \cap \Lambda(G))
\]

and the closure of \( \text{Sect}_{x}(U) \cap N_{D}WH(G) \) in the Thurston compactification is

\[
\text{Sect}_{x}(\overline{U}^{PMF}) \cap N_{D}WH(G) \cup (\overline{U}^{PMF} \cap \Lambda(G))
\]

*Proof.* Since \( pr \) is continuous, \( \text{Sect}_{x}\overline{U}^{PMF} \) and \( C_{r}^{+}(x,\overline{U}^{PMF}) \) are the closures of \( \text{Teich}(S) \) of \( \text{Sect}_{x}(U) \) and \( C_{r}^{+}(x,U) \) respectively. Since \( N_{D}WH(G) \) is closed in \( \text{Teich}(S) \) we have that \( \text{Sect}_{x}\overline{U}^{PMF} \cap N_{D}WH(G) \) and \( C_{r}^{+}(x,\overline{U}^{PMF}) \cap N_{D}WH(G) \) are the closures in \( \text{Teich}(S) \) of \( \text{Sect}_{x}(U) \cap N_{D}WH(G) \) and \( C_{r}^{+}(x,U) \cap N_{D}WH(G) \) respectively.

Now let \( x_{n} \in C_{r}^{+}(x,U) \cap N_{D}WH(G) \) converge to some \( \eta \in PMF \). Since \( x_{n} \in N_{D}WH(G) \) we have \( \eta \in \Lambda(G) \). By definition of \( C_{r}^{+} \) there exists a sequence \( y_{n} \in B_{r}(x_{n}) \cap \text{Sect}_{x}U \). Since \( \eta \) is uniquely ergodic we have \( y_{n} \to \eta \). Now let \( z_{n} \in [x, x_{n}] \) with \( d(z_{n}, x) = 1 \) and \( z \in [x, \eta] \) with \( d(z_{n}, x) = 1 \). Since \( \eta \) is uniquely ergodic we have \( z_{n} \to z \) and by continuity of \( pr \) we have \( pr_{x}z_{n} \to pr_{x}z = \eta \). By definition \( pr_{x}z_{n} \in U \) so \( \eta \in U \).

Now, suppose \( \eta \in \Lambda(G) \cap U \). Then \( \pi g_{t}q_{x,\eta} \to \eta \) as \( t \to \infty \). Let \( D' > 0 \) be such that \( [x, \alpha] \subset N_{D'}WH(G) \) for all \( \alpha \in \Lambda(G) \). Then \( w_{t} = \pi g_{t}q_{x,\eta} \in N_{D'}WH(G) \) for all \( t \). Let \( v_{t} \in WH(G) \cap B_{D'}(w_{t}) \). As \( \eta \in UE \) we have \( z_{t} \to \eta \). We claim \( B_{r+D'}(w_{t}) \subset \text{Sect}_{x}(U) \) for large enough \( t \), whence it will follow that \( v_{t} \in C_{r}^{-}(x,U) \) for large enough \( t \). Indeed, otherwise, letting \( V \subset PMF \) be an open neighborhood of \( \eta \) with closure in \( U \) there is a sequence \( t_{n} \to \infty \) and \( p_{n} \in B_{r+D'}(w_{t_{n}}) \setminus \text{Sect}_{x}U \). Since \( \eta \in UE \) we have \( p_{n} \to \eta \) and thus \( pr_{x}p_{n} \to \eta \). Thus \( pr_{x}p_{n} \in U \) for large enough \( n \) since \( U \) is open in \( PMF \), contradicting our assumption. So \( v_{t} \in WH(G) \cap C_{r}^{-}(x,U) \) converges to \( \eta \).

\[\square\]

**Corollary 7.3.** If \( U, V \subset PMF \) with \( U \subset V^{0} \) then \( (C_{r}^{+}(x,U) \cap N_{D}WH(G)) \setminus C_{r}^{-}(y,V) \) has compact closure in \( \text{Teich}(S) \).

**Corollary 7.4.** If \( V \) is open in \( \text{Teich}(S) \cup PMF \) and \( U \subset PMF \) with \( \overline{U} \subset V \cap PMF \) then \( C_{r}^{+}(x,U) \setminus V \) has compact closure in \( \text{Teich}(S) \). If \( X \) is closed in \( \text{Teich}(S) \cup PMF \) and \( U \subset PMF \) is open in \( PMF \) with \( X \cap PMF \subset U \) then \( X \setminus C_{r}^{-}(x,U) \) has compact closure in \( \text{Teich}(S) \).

Let \( V \subset PMF \) be open with \( \overline{U} \subset V \).

**Lemma 7.5.** For any geodesic \( l \) with endpoints in \( \Lambda(G) \setminus V \), \( l \cap \text{Sect}_{x}(U) \) is contained in a compact subset of \( \text{Teich}(S) \). (In particular \( l \) spends only a finite amount of time in \( \text{Sect}_{x}U \)).

*Proof.* Suppose \( l \cap \text{Sect}_{x}(U) \) is not contained in a compact subset of \( \text{Teich}(S) \). Then (for a correct choice of orientation of the geodesic) there exist \( s_{n} \to \infty \) such that \( p_{n} = l_{s_{n}} \in \text{Sect}_{x}U \). Note, \( l_{s_{n}} \to l^{+} \in \Lambda(G) \setminus V \), which is uniquely ergodic. Let \( q_{n} \in S(x) \) and \( t_{n} \to 0 \) be such that \( p_{n} = \pi(g_{t_{n}}q_{n}) \). Note, as \( p_{n} \in \text{Sect}_{x}(U) \) we have \([q_{n}]+ \in \overline{U}\). Then \( q_{n} \to q \in S(x) \) with

\[
l^{+} = [q^{+}] = \lim_{t \to \infty} \pi(g_{t}q)
\]
But since the map
\[ S(x) \to PMF \]
\[ q \to [q^+] \]
is a homeomorphism, \([q^+] \in \overline{U}\) contradicting that \(l^+ \in \Lambda(G) \setminus V\).

\[ \square \]

**Lemma 7.6.** There exists a \(T > 0\) such that any geodesic \(l\) with \(l^+, l^- \in \Lambda(G) \setminus V\) and \(d(x, l) \geq T\) is disjoint from \(\text{Sect}_x U\).

**Proof.** Suppose not. Then there is a sequence \(l_n\) of geodesics with \(l^+_n, l^-_n \in \Lambda(G) \setminus V\) and \(d(x, l) > n\) and \(p_n \in l_n \cap \text{Sect}_x U\). Passing to a subsequence, we have either that \((l^+_n, l^-_n)\) converges to either a pair of distinct points \((\eta, \zeta)\) in \(\Lambda(G)\) or a single point \(\eta \in \Lambda(G)\). In the first case, we would have \(l_n\) converge in the Hausdorff topology on \(\text{Teich}(S) \cup PMF\) to the geodesic \(l\) between \(\eta\) and \(\zeta\), so \(d(l_n, x) \to d(l, x)\) which would contradict \(d(x, l) \to \infty\). Thus, we have \(l^+_n\) and \(l^-_n\) converging to the same \(\eta \in \Lambda(G) \setminus V\). Then, we have \(l_n\) converging to \(\eta\) and thus \(p_n \to \eta\). Let \(q_n \in S(x)\) be such that \(q_n = \pi(g_n q_n)\). Note, as \(p_n \in \text{Sect}_x U\) we have \([q^+_n] \in \overline{U}\). Then \(q_n \to q \in S(x)\) with
\[ \eta = [q^+] = \lim_{t \to \infty} \pi(g_t q) \]
But \([q^+] \in \overline{U}\) contradicting that \(\eta \in \Lambda(G) \setminus V\).

\[ \square \]

**Lemma 7.7.** There exists a \(D > 0\) such that for every \(\eta, \zeta \in \Lambda(G) \setminus V\) the geodesic between \(\eta, \zeta\) spends at most time \(D\) in \(\text{Sect}(U)\).

**Proof.** Suppose the contrary. Then there exist a sequence of geodesics \(l_n\) with both endpoints \(l^+_n, l^-_n\) outside of \(V\) such that \(l_n\) spends time at least \(n\) in \(\text{Sect}(U)\). We may pass to a subsequence such that one of the following holds: either \((l^+_n, l^-_n)\) converges to a pair of distinct endpoints \((\eta, \zeta)\) in \(\Lambda(G) \setminus V\) or both converge to the same \(\eta \in \Lambda(G) \setminus V\). Suppose the first case. Then, the geodesics \(l_n\) converge uniformly on compact sets to the geodesic \(l\) with endpoints \(\eta, \zeta\). Thus, this geodesic must spend an infinite amount of time inside \(\text{Sect}(U)\), which is impossible if both of its endpoints are outside of \(U\). Now suppose both endpoints of \(l_n\) converge to the same \(\eta \in \Lambda(G) \setminus V\). Then, \(l_n\) converges to \(\eta\). Let \(p_n \in l_n \cap \text{Sect}_x(U)\). Let \(q_n \in S(x)\) be such that \(p_n = \pi(g_n q_n)\). Note, as \(p_n \in \text{Sect}_x(U)\) we have \([q^+_n] \in \overline{U}\). Then \(q_n \to q \in S(x)\) with
\[ \eta = [q^+] = \lim_{t \to \infty} \pi(g_t q) \]
But \([q^+] \in \overline{U}\) contradicting that \(\eta \in \Lambda(G) \setminus V\).

\[ \square \]

8. The Bowen Margulis Measure

Let \(x \in \text{Teich}(S)\). Define a measure \(\tilde{\mu}\) on \(\Lambda(G) \times \Lambda(G)\) by
\[ d\tilde{\mu}(\eta, \zeta) = \exp(\delta(G)\rho_x(\eta, \zeta)) d\nu_x(\eta) d\nu_x(\zeta) \]
Note, every distinct pair of points in \(\Lambda(G)\) give a Teichmüller geodesic in \(WH(G)\) so we can consider \(\tilde{\mu}\) as a measure on \(Q^1(S)\), invariant under the Teichmüller geodesic flow \(g_t\) and supported on \(WH(G)\). By continuity of \(\rho\) on pairs of points in the limit set, \(\tilde{\mu}\) is locally finite.

**Lemma 8.1.** The measure \(\tilde{\mu}\) is \(G\) invariant.

**Proof.** This follows from the \(G\) equivariance property of conformal densities.
Thus $\tilde{\mu}$ descends to a measure $\mu$ on $Q^1(S)/G$. Since $G$ acts cocompactly on $WH(G)$, it follows that $\mu$ is compactly supported, and therefore finite. We call $\mu$ the Bowen-Margulis measure. By projecting $Teich$ by the Busemann function: 

$$\tilde{\mu}$$

between $W$ defined measure on strong unstable horospheres $x$. Here, $\beta$ foliation restricts to a map measured foliation as $q$. Let the weak stable (unstable) leaf associated to $q_0$, denoted by $W^{ss}(q_0)$ (resp $W^{su}(q_0)$) be the set of elements of $Q^1(S)$ with the same vertical (resp horizontal) measured foliation as $q_0$. The map sending each quadratic differential to its horizontal projective measured foliation as $q_0$. 

The map sending each quadratic differential to its horizontal projective measured foliation restricts to a map

$$P_{q_0} : W^{ss}(q_0) \to PMF$$

that is a homeomorphism between $W^{ss}(q_0)$ and $PMF \setminus V(q_0^+)$ where for a measured foliation $\alpha$, $V(\alpha)$ consists of the foliations $\theta$ such that $i(\alpha, \beta) + i(\theta, \beta) = 0$ for some $\beta \in PMF$. In particular, if $q_0^+$ is uniquely ergodic, $P_{q_0}$ is a homeomorphism between $W^{ss}(q_0)$ and $PMF \setminus q_0^+$. We can define a locally finite measure on $W^{ss}(q_0)$, denoted by $\mu_{q_0}^{ss}$ by pulling back the Patterson-Sullivan measure on $PMF$ and scaling by the Busemann function:

$$d\mu_{q_0}^{ss}(v) = \exp(-\delta \beta_{v^-}| (x, \pi(v))) d\mu_x([v^-])$$

Here, $x \in Teich(S)$ is any basepoint and $\mu^{ss}$ is independent of $x$. Similarly, we can define measure on strong unstable horospheres $W^{su}(q_0)$ by

$$d\mu_{q_0}^{su}(v) = \exp(-\delta \beta_{v^+}| (x, \pi(v))) d\mu_x([v^+])$$

We can also define measures on weak horospheres by integrating the measures on strong horospheres with respect to geodesic arclength.

$$\tilde{\mu}_{q_0}^\alpha = \mu_{q_0}^{ss} dq_t$$

$$\tilde{\mu}_{q_0}^\beta = \mu_{q_0}^{ss} dq_t$$

These project modulo $\Gamma$ to measures $\mu_{q_0}^{su}$ and $\mu_{q_0}^u$, $\mu_{q_0}^s$ and $\mu_{q_0}^s$. 

Note, whenever $q_0^+$ is uniquely ergodic, there is a map

$$h_{q_0}^s : Q^1(S) \setminus W^u(-q_0) \to W^{ss}(q_0)$$

with

$$h_{q_0}^s(q) = W^u(q) \cap W^{ss}(q_0)$$

which has one to one restrictions to any unstable horosphere. When $q^+$ is uniquely ergodic, $h_{q_0}^s$ restricts to a homeomorphism between $S(x) \setminus p$ and $W^{ss}(q_0)$ where $p^+ = q_0^+$. We can also define a map

$$h_{q_0}^u : Q^1(S) \setminus W^u(-q_0) \to W^s(q_0)$$

with

$$h_{q_0}^u(q) = W^{su}(q) \cap W^s(q_0)$$

Similarly define maps $h^u$ and $h^{su}$

**Lemma 8.2.** For any $q_1, q_2$ with $q_i^+$ uniquely ergodic, the restriction of $h_{q_i}^s$ to $W^s(q_2)$ takes $\tilde{\mu}_{q_2}^s$ to $\mu_{q_1}^s$. 

We can now define, for a fixed \( q_0 \in Q^1(S) \), a measure \( d\tilde{\nu}_{q_0} \) on \( Q^1(S) \) by
\[
d\tilde{\nu}_{q_0} = dg_w(g_t w) d\mu_v^{ss}(w) d\mu_{q_0}^{ss}(v)
\]
ie by integrating over \( g_t w \) with geodesic arclength \( dg_w \), then integrating over all \( w \) in \( W^{ss}(v) \) with respect to \( \mu_v^{ss} \), finally integrating over all \( v \) in \( W^{ss}(q_0) \) with respect to \( \mu_{q_0}^{ss} \).

**Lemma 8.3.** \( \mu_{q_0} \) is independent of \( q_0 \) and coincides with the Bowen-Margulis measure \( \hat{\mu} \).

**Proof.** This follows from the fact that \( \nu_\tau \) is supported on the uniquely ergodic part of \( PMF \) and that \( \beta_{q_{n+1}}(x, \pi(w)) = \beta_{q_n}(x, \pi(v)) \) whenever \( q \in W^{ss}(v) \) and \( v^+ \) uniquely ergodic.

\[\square\]

9. **Nonarithmeticity of the Length Spectrum**

In this section we prove

**Theorem 9.1.** Let \( G < Mod(S) \) be a nonelementary subgroup. The the logarithms of the dilatations of pseudo-Anosov elements of \( G \) generate a dense subgroup of \( \mathbb{R} \).

For a train track \( \tau \), let \( W_\tau \cong \mathbb{R}^{n-1} \) be the vector space of weights on the branches of \( \tau \) satisfying the switch condition. Let \( V_\tau \subset W_\tau \) be the open cone assigning positive measure to each branch. Each element of \( V_\tau \) corresponds to a measured foliation. Let \( \phi_\tau : V_\tau \to MF \) be this correspondence. Let \( U_\tau \) be the image of \( \phi_\tau \) in \( MF \) and let \( \psi_\tau : U_\tau \to V_\tau \) be the inverse of \( \phi_\tau \).

**Lemma 9.2.** Let \( \gamma \in Mod(S) \) be a pseudo-Anosov such that \( \gamma^+ \) is carried by the interior of the maximal recurrent train track \( \tau \) and \( \gamma^- \) is not carried by \( \tau \). Then for large enough \( n \) we have \( \gamma^n \tau \) is carried by \( \tau \) and \( \gamma^n \) acts linearly on \( W_\tau \) by a positive matrix whose largest eigenvalue is the dilatation \( \lambda(\gamma) \) of \( \gamma \).

Let \( G < Mod(S) \) be nonelementary.

Let \( \tau \) be a maximal recurrent train track and \( \gamma_1, \gamma_2 \in G \) independent pseudo-Anosovs such that \( \gamma_1^+ \) are carried by \( \tau \), and \( \gamma_i^- \) are not carried by \( \tau \); replacing \( \gamma_i \) by high enough iterates we can assume that they preserve \( U_\tau \). Let \( \Gamma \subset G \) be the semigroup freely generated by the \( \gamma_i \). Then each \( \gamma \in \Gamma \) preserves \( U_\tau \) and acts linearly on \( W_\tau \) by a positive matrix whose largest eigenvalue is the dilatation \( \lambda(\gamma) \) of \( \gamma \). Moreover, since the mapping class group preserves Thurston’s symplectic form on \( Mod(S) \) we have that each \( \gamma \in \Gamma \) acts on \( W_\tau \) by a symplectic matrix.

Note, if \( A \) represents the action of \( \gamma \in G \) then \( E_A \) cannot have any elements of \( V_\tau \), ie cannot have any vectors with all entries nonnegative. Indeed, if \( v \in E_A \) then \([A^n v] \) does not converge to \([w_A] \in PW_\tau \). However, if \( v \in V_\tau \) then \( v = \psi_\tau \alpha \) for some measured foliation \( \alpha \) with \([\alpha] \neq [\gamma^-] \). Thus, \([\gamma^n \alpha] \to [\gamma^+] \) and hence
\[
[A^n v] = [A^n \psi_\tau \alpha] = [\psi_\tau (\gamma^n \alpha)] \to [\psi_\tau \gamma^+] = v_A
\]
In particular we obtain that if independent elements \( \gamma_1, \gamma_2 \in \Gamma \) act on \( W_\tau \) by matrices \( A \) and \( B \) respectively then \( \Gamma v_B \cap E_A = \emptyset \).

Thus, it suffices to prove the following result about linear semigroup actions on projective space. For a proximal element \( A \in SL_n \mathbb{R} \) let \( v_A \) be a dominant eigenvector with corresponding eigenvalue \( \lambda(A) \) and \( E_A \) the direct sum of complementary eigenspaces. We need
Theorem 9.3. Let $\Gamma$ be a semigroup of $SL_n \mathbb{R}$ every element of which is proximal. Suppose for any $A, B \in \Gamma$ we have $\Gamma v_B \cap E_A = \emptyset$. Then the logarithms of maximal eigenvalues of matrices in $\Gamma$ generate a dense subgroup of $\mathbb{R}$.

Let the limit set of $\Gamma$, denoted by $L_\Gamma$ be the closure in $P \mathbb{R}^n$ of
\[ \{ [v_A] : A \in \Gamma \} \]
Let $\tilde{L}_\Gamma$ be the preimage of $L_\Gamma$ in $\mathbb{R}^n$.

Lemma 9.4. $L_\Gamma$ is $\Gamma$ invariant.

Proof. Suppose $A, B \in \Gamma$. We need to show that $[Av_B] \in L_\Gamma$. Consider
\[ u = \lim_{n \to \infty} \frac{1}{||B||^n} B^n \in M_N(\mathbb{R}) \]
Then $u$ is a projection onto $Rv_B$ with $keru = E_B$ and $u(v_B) = v_B$. By assumption, $Av_B \notin E_B = keru = kerAu$. Thus, $Au$ is a multiple of a projection onto $RAv_B$. Note,
\[ Au = \lim_{n \to \infty} \frac{1}{||B||^n} AB^n \]
so $[v_{AB^n}] \to [Av_B]$. Hence, $[Av_B] \in L_\Gamma$. □

Lemma 9.5. Any $\Gamma$ invariant subspace $W$ is either contained in $\bigcap_{A \in \Gamma} E_A$ or contains $v_A$ for all $A \in \Gamma$.

Proof. Suppose $\Gamma W = W$, and $B \in \Gamma$ with $v_B \notin \Gamma$. Then for any $v \notin E_B$ we have $\lim_{k \to \infty} [B^k v] = [v_B] \notin [W]$. However, for any $v \in W$ we have $B^k v \in W$ and as $[W]$ is closed, any limit point of $\{ [B^k v] : k \in \mathbb{N} \}$ is in $[W]$. Thus, if $v_B \notin W$ then $W \subset E_B$. In particular, since for any $A \in \Gamma$ we have $v_A \notin E_B$ we have that $W$ does not contain $v_A$ for any $A \in \Gamma$ and is thus contained in $E_A$ for all $A \in \Gamma$. □

Let $W_\Gamma = \oplus_{A \in \Gamma} \mathbb{R}v_A$ be the smallest subspace of $\mathbb{R}^n$ containing $\tilde{L}_\Gamma$. Since $\Gamma$ preserves $L_\Gamma$, it preserves $W_\Gamma$. Let $U_\Gamma$ be a maximal proper $\Gamma$ invariant subspace of $W_\Gamma$.

Lemma 9.6. $\Gamma$ acts irreducibly on $V_\Gamma = W_\Gamma/U_\Gamma$, and each $A \in \Gamma$ has the same largest eigenvalue in this action as in the action on $\mathbb{R}^n$.

Proof. The irreducibility follows from maximality of $U_\Gamma$. Note, by Lemma 10.4 we have $U_\Gamma \subset \bigcap_{A \in \Gamma} E_A$ does not contain any $v_A$ for $A \in \Gamma$ so $v_A + U_\Gamma$ is a dominant eigenvector for $A$ with eigenvalue $\lambda(A)$. □

Lemma 9.7. For any independent $A, B \in \Gamma$ there is an integer $M > 0$ such that $\Gamma_M = sg(A^M, B^M)$ acts strongly irreducibly on $V_{\Gamma_M} = W_{\Gamma_M}/U_{\Gamma_M}$.

Proof. Now, consider independent $A, B \in \Gamma$. Since there is no infinite nested sequence of finite dimensional subspaces, there exists an $N > 0$ and subspaces $U \subset W$ such that $W_{\Gamma_M} = W$, $U_{\Gamma_M} = U$ and $V_{\Gamma_M} = W/U = V$ for all $M \geq N$. We know that $\Gamma_N = sg(A^N, B^N)$ acts irreducibly on $V$ and consequently so does the free group $G_n = < A^N, B^N >$ generated by $A^N, B^N$. Suppose the $\Gamma_N$ action on $V$ is not strongly irreducible. Let $V_1, ..., V_n \subset V$ be a minimal collection of subspaces of $V$ such that their union is preserved by $\Gamma_N$. Note, $A^N, B^N$ permute $V_1, ..., V_n$ so there exists an $K > 0$ such that $A^M V_i = V_i$ and $B^M V_i = V_i$ for each $i$. In particular, $\Gamma_{MK} = sg(A^{MK}, B^{MK})$ preserves the proper nontrivial subspace $V_1 \subset V = W$, contradicting the fact that $\Gamma_{MK}$ must act on $V$ irreducibly. □
The following is proved in [23], Proposition 4.9.

**Lemma 9.8.** Let $\Gamma < GL_mR$ be a semigroup acting strongly irreducibly on $R^m$ and containing a proximal element. Then the logarithms of maximal eigenvalues of proximal elements of $\Gamma$ generate a dense subgroup of $R$.

Theorem 9.1 follows from Lemmas 9.7 and 9.8.

10. Ergodicity and Mixing of the Bowen Margulis Measure

**Theorem 10.1.** The geodesic flow $g_t$ on $Q^1(S)/G$ is ergodic with respect to the Bowen-Margulis measure $\mu$.

**Proof.** For $f \in C_c(Q^1(S)/G)$ continuous with compact support, consider the forward and backward Birkhoff averages:

$$f^+(q) = \lim_{T \to \infty} \sup \frac{1}{T} \int_0^T g_t q dt$$

and

$$f^-(q) = \lim_{T \to \infty} \sup \frac{1}{T} \int_0^T g_{-t} q dt$$

By the Birkhoff ergodic theorem these are finite and equal for almost every $q \in Q^1(S)/G$. Moreover, it is clear that $f^+$ and $f^-$ are invariant under geodesic flow. Furthermore, $f^+$ is invariant along $W^{ss}(q)$ whenever $q^+$ is uniquely ergodic, and $f^-$ is invariant along $W^{uu}(q)$ whenever $q^-$ is uniquely ergodic. Suppose the measure is not ergodic. Then there exists some $f \in C_c(Q^1(S)/G)$ such that $f^+$ is NOT almost everywhere constant. Let $C_1, C_2$ be disjoint sets whose union is $R$ such that $D_t = (f^+)^{-1}C_i$ has positive measure. Note, by Fubini’s theorem and the product structure of the measure $\mu$ there exists $q_0$ with $q_0^+ \in \Lambda(G)$ and a set $A \subseteq W^{uu}(q_0)$ of full measure such that $f^+(v) = f^-(v)$ and $v^+ \in \Lambda(v)$ for all $v \in A$. Furthermore there are sets $A_i(q_0) \subseteq A$ of positive measure such that for all $v \in A_i(q_0)$ $W^s(v)$ intersects $D_t$ in a set of positive measure. Note, since the $D_t$ are $g_t$ invariant and $f^+$ is constant along $W^{ss}(v)$, it follows that $W^s(v) \subseteq D_t$ for $v \in A_i(q_0)$. In particular, $A_i(q_0) \subseteq D_i$. However, as $f^+ = f^-$ on $A$ this implies that $f^-(A_i(q_0)) \subseteq C_i$. However, as $q_0^-$ is uniquely ergodic, $f^-$ is constant on $W^{uu}(q_0)$ contradicting that the $C_i$ are disjoint.

It follows that $G$ acts ergodically on PMF with $\nu_z \times \nu_z$.

**Theorem 10.2.** The geodesic flow $g_t$ on $Q^1(S)/G$ is mixing with respect to the Bowen-Margulis measure $\mu$.

Our argument is modelled on Babbillot’s argument in [24] where an analogous result was proved for general quasi-product measures on manifolds of pinched negative curvature. The following result from unitary representation theory is proved in [24].

**Theorem 10.3.** Let $(X, B, m, (T_t)_{t \in \Lambda})$ be a measure preserving dynamical system where $(X, B)$ is a Borel space, $m$ a Borel measure on $X$ and $T$ an action on $X$ of a locally compact second countable abelian group on $X$ by $m$ preserving transformations. Let $f \in L^2(X, m)$, and if $m$ is finite assume also $\int_X f dm = 0$. Then, if $f \circ T_a$ does not converge weakly to $0$ as $a \to \infty$ in $A$, there exist a sequence $s_n$, going to infinity in $A$ and a non-constant function $\psi \in L^2(X, m)$ such that $f \circ T_{s_n}$ and $f \circ T_{-s_n}$ both converge to $\psi$. 
Suppose $\mu$ is not mixing. Then there is a continuous $G$ invariant function $f$ with $\text{supp}(f)/G$ compact such that

$$\int_{Q^1} f d\tilde{\mu} = 0$$

and $f \circ g_t$ does not converge weakly to zero. Let $s_n \to \infty$ and nonconstant $\psi$ be such that $\int_{Q^1} f d\mu = 0$ and both $f \circ g_{s_n}$ and $f \circ g_{-s_n}$ converge weakly to $\psi$. By the Banach-Saks theorem, there exists a subsequence $t_n$ of $s_n$ such that the Cesaro averages

$$A_N = \frac{1}{N} \sum_{n=1}^{N} f \circ g_{t_n}$$

and

$$A_{-N} = \frac{1}{N} \sum_{n=1}^{N} f \circ g_{-t_n}$$

converge almost surely to $\psi$. We first smooth out $\psi$ by considering the function $v \to \int_0^v \psi(g_{v})d\sigma$. Choosing small enough $c$ guarantees that this function remains non-constant, and it is moreover the limit of the corresponding Cesaro averages of the smoothing of $f$. By abuse of notation, we continue to call the new functions $f$ and $\psi$. Now, there exists a set $E_0$ of full $\mu$ measure in $\Lambda(G) \times \Lambda(G)$ such that for each $v$ on a geodesic with endpoints in $E_0$, the function $t \to \psi(g_t(v))$ is well defined and continuous. Consider the closed (a priori possibly trivial) subgroup $\mathbb{R}(q)$ of $\mathbb{R}$ given by the periods of $t \mapsto \psi(g_t(q))$. It is clearly flow invariant, and thus gives a measurable map from $E_0$ into the set of closed subgroups of $\mathbb{R}$. By ergodicity of $\nu_\times \nu_\times$ it must be constant almost everywhere on $E_0$. Suppose this subgroup is $\mathbb{R}$. Then $\psi$ would be $g_t$ invariant, and thus pass to a flow invariant function on $Q^1/G$, which is not almost-everywhere constant. However, this contradicts the ergodicity of $\mu$. Thus the subgroup in question must be cyclic. Say it equals $k\mathbb{Z}$ on a full measure set $E_1 \subseteq E_0$. Let

$$\psi^+ = \limsup A_N$$

and

$$\psi^- = \limsup A_{-N}$$

By Fubini’s theorem, there is a set $E_2 \subseteq E_1$ be of full measure and such that $\psi^+ = \psi^- = \psi$ everywhere along every geodesic in $E_2$.

Now, let $E^-$ be the set of $\lambda \in \Lambda(G)$ such that for $\nu_\times$ almost everywhere $\alpha$, $(\lambda, \alpha) \in E_2$ and similarly let $E^+$ be the set of $\lambda \in \Lambda(G)$ such that for $\nu$ almost everywhere $\alpha$, $(\alpha, \lambda) \in E_2$. Again, by Fubini’s theorem, $E = E_2 \cap (E^+ \times E^-)$ has full measure.

Now, let $\eta_1, \eta_2, \zeta_1, \zeta_2 \in \Lambda(G)$. Choose $p_0 \in (\eta_1, \zeta_1)$, $p_1 \in (\zeta_1, \eta_2)$, $p_2 \in (\eta_2, \zeta_2)$, $p_3 \in (\zeta_2, \eta_1)$ and $p_4 \in (\eta_1, \zeta_1)$ such that $p_i$ and $p_{i+1}$ are on the same horosphere. We claim that the distance $\tau(\eta_1, \eta_2, \zeta_1, \zeta_2)$ between $p_0$ and $p_4$ depends only on the $\eta_i$ and $\zeta_i$ and is thus independent of the position of $p_0$ on its geodesic. It will follow that this distance is a period of $t \mapsto f(g_{t}q_0)$ where $q_0 \in S(p_0)$ with $[q_0] = \zeta_1$ and thus is contained in $k\mathbb{Z}$ for $\nu_\times$ almost every $\eta_1, \eta_2, \zeta_1, \zeta_2 \in \Lambda(G)$. Indeed, let $H_{\eta_i}$ and $H_{\zeta_i}$ be horospheres centered at $\eta_i$ and $\zeta_i$ respectively for $i = 1, 2$. Let $D_{ij}$ be the signed distance between the intersections of $(\eta_i, \zeta_i)$ with $H_{\eta_i}$ and $H_{\zeta_i}$, with the sign convention chosen in such a way that $D_{ij}$ is positive if $H_{\eta_i}$ and $H_{\zeta_i}$ are disjoint. Then since the geodesic flow takes horospheres to horospheres, the quantity $D_{1,1}+D_{2,2}-D_{1,2}-D_{2,1}$ is independent of the specific choice of horospheres.
Moreover if the horospheres are chosen in such a way that $H_{\zeta_1}$ contains $p_0$, $H_{\eta_2}$ passes through $p_1 = H_{\zeta_1} \cap (\zeta_1, \eta_2)$, $H_{\zeta_2}$ passes through $p_2 = H_{\eta_2} \cap (\eta_2, \zeta_2)$, and $H(\eta_1)$ passes through $(\eta_1, \zeta_2)$, then $D_{1,1} + D_{2,2} - D_{1,2} - D_{2,1}$ reduces to the signed distance $D_{1,1}$ between $p_0$ and $p_1 = H_{\eta_1} \cap (\eta_1, \zeta_1)$. Thus, $\tau(\eta_1, \eta_2, \zeta_1, \zeta_2) = D_{1,1} + D_{2,2} - D_{1,2} - D_{2,1}$ is well-defined and continuous on quadruples of points in $\Lambda(G)$. We call it the cross ratio of the four points in $PMF$, or the cross ratio of the geodesics $(\eta_1, \zeta_1)$ and $(\eta_2, \zeta_2)$.

**Proposition 10.4.**

$$\tau(\eta_1, \eta_2, \zeta_1, \zeta_2) = \lim_{n \to \infty} d(x_1^n, y_1^n) + d(x_2^n, y_2^n) - d(x_1^n, y_2^n) - d(y_1^n, x_2^n)$$

where $x_1^n, y_1^n \in Teich(S)$ with $x_1^n \to \eta_1$, $y_1^n \to \zeta_1$.

**Proof.** Let $H_{\eta_1}$ and $H_{\zeta_1}$ be pairwise disjoint horospheres through $\eta_1$ and $\zeta_1$ respectively, and let $M_0$ be the complement in $Teich(S)$ of the corresponding horoballs. The intersections of the geodesics $(\eta_1, \zeta_1), (\zeta_1, \eta_2), (\eta_2, \zeta_2), (\eta_2, \eta_1)$ with $M_0$ consist of disjoint segments $I_j$ of length $d_j$, $j = 1, 2, 3, 4$ respectively. Note the number $\tau' = d_1 + d_2 - d_3 - d_4$

does not depend on the specific choice of horosphere and by continuity of the Busemann function on $\Lambda(G)$ depends continuously on the $\eta_1, \zeta_1 \in \Lambda(G)$. We claim that it is equal to $\tau(\eta_1, \eta_2, \zeta_1, \zeta_2)$. Indeed, suppose $x_1^n, x_2^n, y_1^n, y_2^n$ are points in $Teich(S)$ converging to $\eta_1$ and $\zeta_1$ respectively. The segments $[x_1^n, y_1^n] \to (\eta_1, \zeta_1)$ $[x_2^n, y_2^n] \to (\eta_1, \zeta_2)$ $[x_1^n, y_1^n] \to (\eta_2, \zeta_1)$ $[x_2^n, y_2^n] \to (\eta_2, \zeta_2)$ uniformly on compact sets. In particular, their intersection with $M_0$ contains four segments $I_j$ which converge toward the $I_j$. Thus, to prove that $\tau = \tau'$ it suffices to show that the contribution to

$$d(x_1^n, y_1^n) + d(x_2^n, y_2^n) - d(x_1^n, y_2^n) - d(y_1^n, x_2^n)$$

of the parts of $[x_1^n, y_1^n]$ which are contained in the complement of $M_0$ goes to zero. By symmetry it suffices to show that if $p_1^n$ is the intersection of $[x_1^n, y_1^n]$ with $H_{\eta_1}$, $p_2^n$ is the intersection of $[x_1^n, y_2^n]$ with $H_{\eta_1}$, then

$$d(p_1^n, x_1^n) - d(p_2^n, x_1^n) \to 0$$

Note, as $n \to \infty$ we have

$$p_1^n \to p_1 = H_{\eta_1} \cap (\eta_1, \zeta_1)$$

and

$$p_2^n \to p_2 = H_{\eta_1} \cap (\eta_1, \zeta_2)$$

and $\beta_{x_1^n} \to \beta_{\eta_1}$ uniformly on compact sets. Thus

$$\beta_{x_1^n}(p_1^n, p_2^n) = d(p_1^n, x_1^n) - d(p_2^n, x_1^n) \to \beta_{\eta_1}(p_1, p_2)$$

which is zero since $p_1, p_2$ lie on the same horosphere based at $\eta_1$. \hfill \Box

From the expressions of the Busemann functions in terms of extremal length, we in fact find

$$\tau([\alpha_1], [\alpha_2], [\beta_1], [\beta_2]) = \frac{1}{2} \log \frac{i(\alpha_1, \beta_1)i(\alpha_2, \beta_2)}{i(\alpha_1, \beta_2)i(\alpha_2, \beta_1)}$$

for any $\alpha_i, \beta_i$ uniquely ergodic. Note, $\tau$ defines a continuous function on quadruples of points in $Teich(S) \cup \Lambda(G)$ From this formula we obtain
Corollary 10.5. For any pseudo-Anosov $g \in \text{Mod}(S)$, with fixed points $\eta_1, \eta_2 \in \Lambda(G)$ the translation distance of $\lambda$ is twice $\tau(\eta_1, \eta_2, \beta, g\beta)$ where $\beta$ is any uniquely ergodic point in PMF distinct from the $\eta_i$.

As noted above, $\tau(\eta_1, \eta_2, \zeta_1, \zeta_2)$ is a period of $t \mapsto f(g, q_0)$ where $q_0 \in S(p_0)$ with $[q_0^p] = \zeta_1$ and so $\tau(\eta_1, \eta_2, \zeta_1, \zeta_2) \in k\mathbb{Z}$ for $\nu^p_1$ almost every $\eta_1, \eta_2, \zeta_1, \zeta_2 \in \Lambda(G)$. By continuity of the cross ratio and the fact that $\nu$ has full support on $\Lambda(G)$, it follows that $\tau(\eta_1, \eta_2, \zeta_1, \zeta_2) \in k\mathbb{Z}$ for $\nu^p_1$ for every $\eta_1, \eta_2, \zeta_1, \zeta_2 \in \Lambda(G)$. But this implies that the translation length of every element of $G$ is in $k\mathbb{Z}$, contradicting Theorem 9.1.

11. Controlling the Multiple Zero Locus

By ergodicity, $\mu$ gives full mass to a single stratum. In the remainder of the paper, we will assume that this is the principal stratum. Here is a sufficient condition.

Proposition 11.1. If $G$ contains a pseudo-Anosov element with axis lying in the principal stratum, then $\mu$ gives full weight to the principal stratum.

Proof. By proposition 5.1, $\nu_1$ has full support on $\Lambda(G)$ and thus $\mu$ has full support on $Q^1 WH(G)$. Thus, any open set $U \subset Q^1(S)$ intersecting any geodesic with endpoints in $\Lambda(G) \times \Lambda(G)$ has positive $\mu$ measure. If $\gamma \in G$ has axis $g_\gamma$ in the principal stratum then a point $p \in g_\gamma$ has a neighborhood $U \subset Q^1(S)$ that is also contained in the principal stratum. Thus, the principal stratum has positive $\mu$ measure and by ergodicity of $\mu$ on $Q^1(S)/G$ it has full measure. \qed

In this section, we show that the contribution to orbit growth of the multiple zero locus and thin parts of the principal stratum is asymptotically negligible. For a subset $P \subset Q^1(S)$, $c \in (0, 1)$ and $x \in \text{Teich}(S)$ let $B_R(x, P, c)$ the set of points $y \in \text{Teich}(S)$ with $d_T(x, y) \leq R$ and the segment $[x, y]$ spending a proportion at most $c$ of the time in $P$.

For $x, y \in \text{Teich}(S)$ let $N_G(x, y, P, R, c)$ denote the number of $\gamma \in G$ such that $d(x, \gamma y) \leq R$ and $[x, \gamma y]$ spending a proportion at most $c$ of the time in $P$.

Specifically, we prove:

Theorem 11.2. For each $x, y \in \text{Teich}(S)$ and $\epsilon > 0$ there exists a closed subset $P' \subset Q^1(S)$ disjoint from the multiple zero locus such that
\[
\lim_{R \to \infty} \sup_{P' \subset Q^1(S)} N_G(x, y, P', R, 1/3)/e^{\delta R} \leq \epsilon
\]

In order to prove this we will show:

Theorem 11.3. For each $\epsilon > 0$ there exists a closed subset $P \subset Q^1(S)$ disjoint from the multiple zero locus such that
\[
\lim_{R \to \infty} \sup_{P \subset Q^1(S)} e^{-\delta R} m(B_R(x, P, 1/2)) \leq \epsilon
\]

We first conclude Theorem 11.2 from Theorem 11.3. We will a lemma of Eskin and Mirzakhani from [5]

Lemma 11.4 ([5], Lemma 5.4). Suppose $K \subset M_g$ is compact. Given $s > 0$, there exists constants $L_0$ depending on $s$ and $K$, and $c_0$ depending only on $K$ with the following property. If $\gamma: [0, L] \to Q^1(S)$ is a geodesic segment (parametrized by arclength) with endpoints above $K$ and $L > L_0$, $\hat{\gamma}: [0, L'] \to Q^1(S)$ is the
geodesic segment connecting \( p_1, p_2 \in \text{Teich}(S) \) such that \( d_T(p_1, \pi(\gamma(0))) < c_0 \), \( d_T(p_2, \pi(\gamma(L))) < c_0 \), and

\[
\left| \{ s \in [0, L] \mid l_{\min}(\gamma(t)) \geq s \} \right| > \frac{L}{2}
\]

then

\[
\left| \{ s \in [0, L'] \mid l_{\min}(\hat{\gamma}(t)) \geq s/4 \} \right| > \frac{L'}{3}
\]

From this, we obtain:

**Lemma 11.5.** Let \( K \subset \text{Teich}(S) \) be a compact subset, and \( P \) a closed subset of the principal stratum. Then there exists a closed subset \( P' \) of the principal stratum containing \( P \) in its interior and an \( R_0 > 0 \) such that for \( y_1, y_2 \in K \) and \( x \in \text{Mod}(S)K \) with \( d_T(x, y_i) > R_0 \), if \([x, y_1]\) spends a proportion at most \( 1/3 \) in \( P' \) then \([x, y_2]\) spends a proportion at most \( 1/2 \) in \( P \).

In particular we have:

**Lemma 11.6.** For any \( K \subset \text{Teich}(S) \) compact and \( P \) a closed subset of the principal stratum, there exists a closed subset \( P' \) of the principal stratum containing \( P \) in its interior and an \( R_0 > 0 \) such that for any \( x \in \text{GK} \), \( y_1, y_2 \in K \) and \( R \geq R_0 \) we have

\[
N_G(x, y_1, P', R, 1/3) \leq N_G(x, y_2, P, R + \text{diam}(K)).
\]

Proof of Theorem 11.2 assuming Theorem 11.3. Let \( x \in \text{Teich}(S) \) and \( P \subset Q^1(S) \) be a subset of the principal stratum be such that the conclusion of Theorem 11.3 holds with \( \epsilon/\epsilon \delta \text{diam}(K) \) in place of \( \epsilon \), ie

\[
m(B_R(x, P, 1/2)) \leq \frac{\epsilon}{\epsilon \delta \text{diam}(K)} e^{\delta R}
\]

for all large enough \( R \).

Let \( K \subset \text{Teich}(S) \) be compact and contain both \( x \) and fundamental domain for the action of \( G \) on \( \text{WH}(G) \).

**Lemma 11.7.**

\[
m(B_R(x, P, 1/2)) = \int_{y \in K} N_G(x, y, P, R, 1/2).
\]

Proof. Note,

\[
m(B_R(x, P, 1/2)) = \sum_{g \in G} \int_{y \in gK} \chi_{B_R(x, P, 1/2)}(y) dm(y)
\]

\[
= \int_{y \in K} \sum_{g \in G} \chi_{B_R(x, P, 1/2)}(gy) dm(y)
\]

\[
= \int_{y \in K} N_G(x, y, P, R, 1/2).
\]

\[\square\]

Note, by Lemma 11.6 there exists a closed subset \( P' \) of the principal stratum containing \( P \) in its interior and a \( R_0 > 0 \) such that if \( R > R_0 \), then

\[
N_G(x, y_1, P', R, 1/3) \leq N_G(x, y_2, P, R + \text{diam}(K))
\]
for any $y_1, y_2 \in K$. Moreover, by Lemma 11.7 with $R + \text{diam}(K)$ in place of $R$, for each large enough $R$ there exists a $y_2 \in K$ such that
\[
N_G(x, y_2, P, R + \text{diam}(K)), 1/2) \leq m(B_{R+\text{diam}(K)}(x, P, 1/2)) \leq \frac{e}{e^{\text{diam}(K)}}e^{\delta(R+\text{diam}(K))} = e^{\delta R}
\]
completing the proof. □

We now consider the following measure on $MF$: Note the space of strong stable (or unstable) horospheres based at uniquely egodic points can be identified with uniquely ergodic points of $MF$. Indeed, let $o \in \text{Teich}(S)$ be a basepoint. If $\eta \in MF$ with $\text{Ext}_o\eta = 1$ then $t\eta$ is identified with the horosphere $H(t\eta) = H(t, [\eta])$ based at $\eta$ such that $\beta_{t\eta}(o, z) = t$ for each $z = z_{t,[\eta]} \in H$ (ie $\text{Ext}_z\eta = e^{2t}$). For $A \subset MF$ so that $[A] \subset PMF$ let
\[
\lambda(A) = \int_{[\eta] \in [A]} \int_{H(t, [\eta])} e^{\delta t}d\nu_o([\eta]) = \int_{[\eta] \in [A]} \int_{\text{Ext}_o\eta = 1} \int_{t \eta \in A} e^{\delta t}dtd\nu_o([\eta])
\]

Lemma 11.8. The measure $\lambda$ does not depend on choice of basepoint $o \in \text{Teich}(S)$ and is $G$ invariant. It has support precisely on foliations projecting to points on $\lambda(G)$. Moreover for all $U \subset Q^1(S)$
\[
\lambda(\eta^+(g_tU)) = e^{\delta t}\lambda(\eta^+U)
\]
and
\[
\mu(U) = \int_{\eta \in MF} \mu^{ss}(A \cap H(\eta))d\lambda(\eta)
\]

Proof. The independence of basepoint and $G$ invariance follows because the $\nu_o$ form a conformal density, the other properties are immediate from the definition. □

Denote
\[
\overline{\lambda}(U) = \lambda(\text{Cone}(U))
\]
where $\text{Cone}(U)$ is the union of segments from the origin in $MF$ to points of $U$. For $W \subset Q^1(S)$ and $s > 0$ let $W(s)$ denote the set of $q \in Q^1(S)$ such that there exists $q' \subset W$ on the same leaf of $W^{su}$ as $q$ such that $d_H(q, q') < s$. For a subset $A \subset \text{Teich}(S)$ let $A(r) = \text{Nbhd}_r(A)$ denote the $r$ neighborhood of $A$ in the Teichmüller metric.

Lemma 11.9. Let $K \subset \text{Teich}(S)$ be a fundamental domain for the action of $G$ on $WH(G)$. Let $h > 0$. Then there is a $C(h) > 0$ depending only on $K$ and $h$ such that for all $U \subset Q^1WH(G) \cap \pi^{-1}K$ and all $t > 0$ letting $W_t = g_tU$ we have
\[
m(\text{Nbhd}_{2\pi}(W_t)) \leq C(h)\overline{\lambda}(\eta^+W_t(h)).
\]

Proof. Let $h_0 = h_0(K, h)$ be a small constant to be specified later. We can decompose $U$ into pieces $U_o$ such that each piece is within Hodge distance $h_0/2$ of a single unstable leaf. The minimal number of such pieces can be bounded by a constant depending only on $K$ by the compactness of $K$ and equivalence of the Euclidean and Hodge metrics over compact sets so, we may assume without loss of generality that $U$ is within Hodge distance $h_0/2$ of a single unstable leaf. Also, as in [ABEM, Lemma 4.1], we can assume without loss of generality that $U$ has $W^{su} \times W^s$ product structure. Pick a maximal $\Delta \subset \pi(W_t)$ with $d_T(x, y) = 1$ for
any distinct $x, y \in \Delta$. Note by compactness of $K$ and $G$ equiariance of $m$ there is a constant $C(K)$ depending only on $K$ such that 

$$m(B(X, 3)) \leq C(K)$$

for all $X \in GK$ Then 

$$\text{Nhbd}_{2\pi}(W_i) \subset \bigcup_{X \in \Delta} B_T(X, 3),$$

and hence 

$$m(\text{Nhbd}_{2\pi}(W_i)) \leq \sum_{X \in \Delta} (B_T(X, 3)) \leq |\Delta|C(K).$$

Now, let $\Delta' \subset W_i$ be a set containing one element of $\pi^{-1}(X) \cap Q^1WH(G)$ for each $X \in \Delta$. Let $B_*^{su}(q, r)$ denote the elements of $W^{su}(q)$ within euclidean distance $r$ of $q$. As shown in [1], Lemma 4.1 for $h_0$ small enough we can pick $h_2$ depending only on $K$ such that the $\eta^+(B_*^{su}(q, h_2))$ for distinct $q \in \Delta'$ are disjoint viewed as subsets of $PMF$. By equivalence of Hodge and Euclidean metrics there is a $h_3 \in (0, h_2)$ such that whenever $q \in \pi^{-1}K$ with $q' \in B_*^{su}(q, h_3)$ we have $d_H(q, q') \leq h$. For each $q \in \Delta'$ consider 

$$H(q) = \eta^+(B_*^{su}(q, h_3)) \subset \eta^+(W(h)).$$

These are pairwise disjoint. Note, since $\nu_\epsilon$ has full support on $\Lambda(G)$, $\overline{\lambda}(H(q)) > 0$ for all $q \in Q^1WH(G)$. Thus, as $\overline{\lambda}$ is $G$ equivariant and $G$ acts cocompactly on $Q^1WH(G)$, there is a $c = c(K, h)$ such that 

$$\overline{\lambda}(H(q)) \geq c$$

for all $q \in Q^1WH(G)$. Thus 

$$\overline{\lambda}(\eta^+(W(h))) \geq \sum_{q \in \Delta'} \overline{\lambda}(\eta^+(H(q))) \geq c|\Delta|.$$ 

This completes the proof. \hfill \Box \hfill

Now, let $P_1 \subset Q^1(S)/G$ be compact (in our application $P_1$ we will be a subset of the principal stratum of almost full $\mu$ measure) and define $P_3 \subset P_2 \subset P_1$ and $\delta \in (0, 1]$ such that if $q \in P_1$ and $d_H(q, q') \leq h \in P_1$ where $h$ is the nonexpansion constant of the modified Hodge norm over $P_1$. By choosing $h$ small enough we can assume $\mu(P_3) > 1/2$. For $T_0 > 0$ let $U_i \subset U_i(T_0)$ be the set of $q \in Q^1WH(G)$ such that there exists $T > T_0$ so that $g_iq$ is in the complement of $P_1$ for at least half of $t \in [0, T]$. By definition $U_i \subset U_2 \subset U_3$ and by the Birkhoff ergodic theorem, for every $\theta > 0$ there is a $T_0 > 0$ such that $\mu(U'_3) < \theta$. Let $U_i = p^{-1}U_i \cap \pi^{-1}K$.

**Lemma 11.10.** In the above notation, for all $t > 0$ 

$$m(\text{Nhbd}_{2\pi}(g_iU_1))) \leq C(h)e^{\delta t}\overline{\lambda}(\eta^+(U_2))$$

and for any $\epsilon > 0$ it is possible to choose $T_0$ such that for all $t > T_0$ 

$$m(\text{Nhbd}_{2\pi}(g_iU_1))) \leq \epsilon e^{\delta t}.$$ 

**Proof.** Let $W = g_iU$. As shown in the proof of [1], Lemma 4.2 we have $W(h) \subset g_iU_2$. Now, we can apply Lemma 11.9 to $W$ and use the fact that 

$$\overline{\lambda}((\eta^+g_iU)) = e^{\delta t}\overline{\lambda}(\eta^+U)$$

to get the first claim. Moreover, as shown in the proof of [ABEM, Lemma 4.2], if $q \in U_2$ and $q' \in Q^1WH(G)$ is on the same strong stable leaf as $q$ with $d_H(q, q') < h$
then \( q' \in U_3 \). By compactness of \( Q^1 WH(G)/G \) and \( G \) equivariance of \( \mu^{ss} \), there is a \( c > 0 \) such that \( \mu^{ss}(B^{ss}(q)) > c \) for all \( q \in Q^1 WH(G) \cap p^{-1}K \). Therefore, by the product structure of \( \mu, \lambda(U_2) \leq C_1(h)\mu(U_3) \) where \( C_1(h) \) depends only on \( h \). Hence, choosing a large enough \( T_0 \) the second claim of the lemma follows. \( \square \)

**Proof of Theorem 11.3.** In the above notation, let \( P_1 \) be chosen disjoint from the multiple zero locus. Let \( T_0, U_1, U_1 \) be as in the proof of Lemma 11.10. Let \( K \subset \text{Teich}(S) \) be a fundamental domain for the action of \( G \) on \( WH(G) \) (so \( m \) is supported on \( GK = WH(G) \)). Then for \( R > T_0 \), and \( x \in K \) we have

\[
B_R(X, P_1) \cap GK \subset \bigcup_{0 \leq t \leq R} \pi(g_0U_1) \cap GK \subset \bigcup_{n=0}^{\lfloor R \rfloor} \bigcup_{n \leq t \leq n+1} \pi(g_0U_1) \cap GK
\]

Then,

\[
m(Nbhd_1 B_R(X, P)) \leq \sum_{n=0}^{\lfloor R \rfloor} m(Nbhd_2(\pi(g_0U_1)) \cap GK) \leq C_4 \sum_{n=0}^{\lfloor R \rfloor} e^{\delta n}
\]

by Lemma 11.4. This completes the proof for \( x \in WH(G) \). \( \square \)

**12. Exact Asymptotics for Orbit Growth**

The goal of this section is to prove the part of Theorem 1.1 concerning orbit growth. For \( r > 0 \), \( x \in \text{Teich}(S) \) and \( A \subseteq \text{PMF} \) recall

\[
C_r^+(x, A) = N_r \bigcup_{z \in B_r(x)} \text{Sect}_z(A)
\]

and

\[
C_r^-(x, A) = \left\{ y \in \text{Teich}(S) \mid B(y, r) \subseteq \bigcap_{z \in B(x, r)} \text{Sect}_z(A) \right\}
\]

For \( t > 0 \) and \( x, y \in \text{Teich}(S) \) define a measure

\[
d\nu^t_{x,y} = \delta||\mu||e^{-\delta t} \sum_{(x, y') \leq t} D_{y'} \otimes D_{y'^{-1}y}
\]

**Proposition 12.1.** Let \( c > 0 \), \( x, y \in \text{Teich}(S) \) and \( \eta_0, \zeta_0 \in \text{PMF} \) be such that there exist \( \eta_0, \zeta_0 \in \Lambda(G) \) with \( x \in (\eta_0, \eta_0^*) \), \( y \in (\zeta_0, \zeta_0^*) \). Then there exist open neighborhoods \( V \) and \( W \) in \( \text{PMF} \) of \( \eta_0 \) and \( \zeta_0 \) respectively such that for all borel \( A \subseteq V \) and \( B \subseteq W \) with nonempty interior, as \( T \to \infty \) we have

\[
\limsup \nu^T_{x,y}(C_r^-(x, A) \times C_r^+(y, B)) \leq e^c \nu_x(A) \nu_y(B)
\]

and

\[
\liminf \nu^T_{x,y}(C_r^+(x, A) \times C_r^-(y, B)) \geq e^{-c} \nu_x(A) \nu_y(B)
\]

**Proof.** If \( \eta_0 \) is not in \( \Lambda(G) \) then we can choose a neighborhood \( U \) of \( \eta_0 \) in \( \text{PMF} \) with \( \nu_x(U) = 0 \) and \( W = \text{PMF} \) so that both sides of the desired equation are 0. Similarly if \( \zeta_0 \) is not in \( \Lambda(G) \). Assume therefore that \( \eta_0, \zeta_0 \in \Lambda(G) \). The argument is modelled on Roblin’s Theorem 4.1.1 in [22], where an analogous result is proved for manifolds of pinched negative curvature. For \( \eta, \zeta \in \text{PMF} \) filling and \( z \in \text{Teich}(S) \), let \( z_{\eta, \zeta} \) be a quadratic differential with projective vertical and horizontal measured
Note that \( \pi \zeta \) foliations and \( a \) and \( b \).

Let \( K(z, r, A) = \{ g s z \eta, \zeta | \eta \in A, d(z, (\eta, \zeta)) < r, s \in [-r/2, r/2] \} \)

and

\( K^{-}(z, r, A) = \{ g s z \eta, \zeta | \eta \in A, d(z, (\eta, \zeta)) < r, s \in [-r/2, r/2] \} \)

Let

\[ K(z, r) = K^{+}(z, r, PMF) = K^{-}(z, r, PMF) \]

Note that \( \pi(K(z, r)) \subseteq B(z, 3r/2) \). For \( a, b \in Teich(S) \) with \( d(a, b) > 2r \) let

\[ \Theta_{r}^{+}(a, b) = \bigcup_{w \in B(a, r)} pr_{w}(B(b, r)) \]

and

\[ \Theta_{r}^{-}(a, b) = \bigcap_{w \in B(a, r)} pr_{w}(B(b, r)) \]

Note, as \( a \rightarrow \eta \in E \) we have \( \Theta_{r}^{\pm}(a, b) \) converging to \( \Theta_{r}^{\pm}(\eta, b) = \Theta_{r}(\eta, b) = pr_{\eta}(B(b, r)) \).

Let \( L_{r}(a, b) \subseteq PMF \times PMF \) denote the pairs \( (\eta, \zeta) \) such that the geodesic defined by them passes first through \( B(a, r) \) and then \( B(b, r) \).

It follows immediately from the definitions that

\[ L_{r}(a, b) \subseteq \Theta_{r}^{+}(b, a) \times \Theta_{r}^{-}(a, b). \]

**Proposition 12.2.** There exists an \( r_{0} > 0 \) such that for all \( 0 < r < r_{0} \) and all \( h > 0 \) the following holds. For each \( \epsilon > 0 \) and \( \epsilon' > 0 \) there exists an \( R_{0} > 0 \) such that for every \( a, b \in Teich(S) \) with the segment \( [a, b] \subset Teich_{n}(S) \) and \( d(a, b) > R_{0} \) such that \( [a, b] \) spends at least half the time in \( Q_{\epsilon'} \) we have that

\[ \Theta_{r}^{-}(b, a) \times \Theta_{r}^{-}(a, b) \subset L_{r+h}(a, b) \]

The proof of this proposition depends on the following lemmata from [5], which say that geodesic segments that spend enough time in the thick part of stratum behave like geodesics in a CAT(1) space.

**Lemma 12.3** ([5], Lemma 5.3). Suppose \( K \subseteq M_{g} \) is compact. Given \( 1 > \beta > 0 \) there exists a \( \rho_{0} > 0 \) (depending only on \( K \) and \( \beta \)) with the following property. Given \( t > 0 \) and \( \rho > 0 \) there exists an \( L_{0} = L_{0}(K, t, \rho, \beta) \) such that if \( X, p_{0} \in Teich(S) \) lie above \( K \), \( d_{T}(p_{0}, p_{1}) < \rho_{0} \), \( d_{T}(X, p_{1}) = L > L_{0} \), and

\[ | \{ s \in [0, L] | f_{\min}(g_{s}(q_{X, p_{0}})) \geq t \} | > \beta L \]

then

\[ d_{E}(q, q_{X, p_{0}}) < \rho \]

where \( d_{E} \) denotes the Euclidean norm and \( q \) is the unique quadratic differential in \( W^{uu}(q_{X, p_{0}}) \cap W^{s}(q_{X, p_{1}}) \).

By the equivalence of the Euclidean and Teichmüller metrics over compact subsets of \( M_{g} \), we also have
Lemma 12.4. Suppose \( K \subseteq M_g \) is compact. Given \( 1 > \beta > 0 \) there exists a \( \rho_0 > 0 \) (depending only on \( K \) and \( \beta \)) with the following property. Given \( t > 0 \) and \( \rho > 0 \) there exists an \( L_0 = L_0(K, t, \rho, \beta) \) such that if \( X, p_0 \in \text{Teich}(S) \) lie above \( K \), \( d_T(p_0, p_1) < \rho_0 \), \( d_T(X, p_1) = L > L_0 \), and

\[
| \{ s \in [0, L] | l_{\text{min}}(g_t(q_{X,p_0})) \geq t \} | > \beta L
\]

then

\[
d_T(\pi(q), X) < \rho
\]

where \( d_E \) denotes the Euclidean norm and \( q \) is the unique quadratic differential in \( W^{u}(q_{X,p_0}) \cap W^{s}(q_{X,p_1}) \).

Lemma 12.5 ([5], Lemma 5.4). Suppose \( K \subset M_g \) is compact. Given \( s > 0 \), there exists constants \( L_0 \) depending on \( s \) and \( K \), and \( c_0 \) depending only on \( K \) with the following property. If \( \gamma : [0, L] \to Q^1(S) \) is a geodesic segment (parametrized by arclength) with endpoints above \( K \) and \( L > L_0 \), \( \hat{\gamma} : [0, L'] \to Q^1(S) \) is the geodesic segment connecting \( p_1, p_2 \in \text{Teich}(S) \) such that \( d_T(p_1, \pi(\gamma(0))) < c_0 \), \( d_T(p_2, \pi(\gamma(L))) < c_0 \) and

\[
| \{ s \in [0, L] | l_{\text{min}}(\gamma(t)) \geq s \} | > \frac{L}{2}
\]

then

\[
| \{ s \in [0, L'] | l_{\text{min}}(\hat{\gamma}(t)) \geq s/4 \} | > \frac{L'}{3}
\]

Proof of Proposition 12.2. Let \( K = \text{Teich}_c(S)/\text{Mod}(S) \). Let \( L_1 \) (depending on \( K \) and \( \epsilon' \)) and \( c_0 \) (depending on \( K \)) be the \( L_0 \) in Lemma 12.4 corresponding to \( s = \epsilon' \) and \( K = \text{Teich}_c(S)/\text{Mod}(S) \). Then, if \( (x, y) \subseteq \text{Teich}_c(S) \) and \( d_T(x, y) = L > L_0 \) with

\[
| \{ s \in [0, L] | l_{\text{min}}(g_t q_{x,y}) \geq \epsilon' \} | > \frac{L}{2}
\]
it follows from Lemma 12.5 that for every $p_1 \in B_{r_0}(x)$ and $p_2 \in B_{r_0}(y)$ we have that

$$| \{ s \in [0, L] | | g_t q_{x,y} | \geq \epsilon' / 4 \} | > \frac{L}{3}$$

Now let $\rho_0$ be as in Lemma 12.3, corresponding to $K = \text{Teich}_r(S) / \text{Mod}(S)$ and $\beta = 1/3$. Let $r_0 = \min \{ \epsilon_0, \rho_0 \}$. Suppose $r < r_0$ and $h > 0$ arbitrary. Let $L_2$ be the $L_0$ in Lemma 12.4 corresponding to $K = \text{Teich}_r(S) / \text{Mod}(S)$ and $\beta = 1/3$, $t = \epsilon'/4$, and $\rho = h$. Let $R_0 > \max \{ L_1 + 2r_0, L_2 + 2r_0 \}$. Suppose $a, b \in \text{Teich}(S)$ with the segment $[a, b] \subset \text{Teich}_r(S)$ and $d(a, b) > R_0$ such that $[a, b]$ spends at least half the time outside of $Q_r$ and $\eta \in \Theta_r^c(b, a)$, $\zeta \in \Theta_r^c(a, b)$. Let $b' \in B_r(b)$ be a point on the geodesic containing $[a, \zeta]$ and $a' \in B_r(a)$ a point on the geodesic containing $[b', \eta]$. We will apply Lemma 12.4 with $X = b'$, $p_0 = a'$, $p_1 = a$ to obtain that the quadratic differential $q \in W^{\omega}(q_{\theta', \eta}) \cap W^s(q_{\theta', \zeta})$ satisfies $d_T(\pi(q), b') < h$, so $\pi(q) \in B_{r+h}(b)$. By definition $\pi(q) \in (\eta, \zeta)$ so $(\eta, \zeta)$ intersects $B_{r+h}(b)$. Similarly, $(\eta, \zeta)$ intersects $B_{r+h}(a)$ completing the proof.

We now continue with the proof of Proposition 12.1. From now on, fix $r \in [0, \epsilon/120]$ smaller than the $r_0$ in Proposition 12.2 such that

$$\nu_x(\partial \Theta_r(\eta_0, x)) = \nu_y(\partial \Theta_r(\zeta_0, y)) = 0$$

(this last condition only excludes countably many values of $r$). Since $\eta_0 \in \Theta_r(\eta_0, x)$ and $\zeta_0 \in \Theta_r(\zeta_0, y)$ and the conformal densities have full support on $\Lambda(G)$ we have that

$$\nu_x(\Theta_{r-h}(\eta_0, y)) > 0$$

Now, fix $h > 0$ such that

$$\nu_x(\Theta_{r-h}(\eta_0, x)) \geq e^{-c/120} \nu_x(\Theta_r(\eta_0, x))$$

and

$$\nu_y(\Theta_{r-h}(\zeta_0, y)) \geq e^{-c/120} \nu_y(\Theta_r(\zeta_0, y))$$

and also

$$\nu_x(\partial \Theta_{r-h}(\eta_0, x)) = \nu_y(\partial \Theta_{r-h}(\zeta_0, y)) = 0$$

Let $\widehat{V}$ and $\widehat{W}$ be open neighborhoods in $\text{Teich}(S) \cup \text{PMF}$ of $\eta_0$ and $\zeta_0$ respectively such that for all $(a, b) \in (\widehat{V} \times \widehat{W})$ with $a, b \in N_{DWH(G)} \cup \Lambda(G)$ with $D = d(x, WH(G)) + d(y, WH(G)) + 1$ we have

$$e^{-c/120} \nu_x(\Theta_r(\eta_0, x)) \leq \nu_x(\Theta_{r-h}(a, x)) \leq e^{c/120} \nu_x(\Theta_r(\eta_0, x))$$

and

$$e^{-c/120} \nu_y(\Theta_r(\zeta_0, y)) \leq \nu_y(\Theta_{r-h}(b, y)) \leq e^{c/120} \nu_y(\Theta_r(\zeta_0, y))$$

and

$$e^{-c/120} \nu_r(\Theta_{r-h}(\eta_0, x)) \leq \nu_r(\Theta_{r-h}(a, x)) \leq e^{c/120} \nu_r(\Theta_r(\eta_0, x))$$

and

$$e^{-c/120} \nu_r(\Theta_{r-h}(\zeta_0, y)) \leq \nu_r(\Theta_{r-h}(b, y)) \leq e^{c/120} \nu_r(\Theta_r(\zeta_0, y))$$

It then follows that

$$e^{-c/60} \nu_x(\Theta_r(\eta_0, x)) \leq \nu_x(\Theta_{r-h}(a, x)) \leq \nu_x(\Theta_r(\eta_0, x)) \leq e^{c/60} \nu_x(\Theta_r(\eta_0, x))$$

and

$$e^{-c/60} \nu_y(\Theta_r(\zeta_0, y)) \leq \nu_y(\Theta_{r-h}(b, y)) \leq \nu_y(\Theta_r(\zeta_0, y)) \leq e^{c/60} \nu_y(\Theta_r(\zeta_0, y))$$
Let $V$ and $W$ be open neighborhoods of $\eta_0$ and $\zeta_0$ respectively in $PMF$ such that $\overline{V} \subset \hat{V} \cap PMF$ and $\overline{W} \subset \hat{W} \cap PMF$. Consider open subsets $A \subset V$ and $B \subset W$. Let

$$K^+ = K^+(x, r, A)$$

and

$$K^+ = K^+(y, r, B)$$

We will estimate as $T \to \infty$ the quantity

$$\int_0^T e^{\delta t} \sum_{\gamma \in G} \mu(K^+ \cap g_{-t}\gamma K^-)dt$$

From the definitions, it follows that for $\gamma \in G$ and for $d(x, y) > 2r$ we have

$$\mu(K^+ \cap g_{-t}\gamma K^-)dt = \int_{L_r(x, y)(\gamma(B \times A))} e^{\delta_{\nu_x(\eta, \zeta)}d\nu_x(\eta)d\nu_y(\zeta)} \int_{-r/2}^{r/2} \chi_{K(\gamma y, \gamma)}(g_t s x, \zeta)ds$$

We first find an upper bound for

$$\int_0^{T-3r} e^{\delta t} \sum_{\gamma \in G} \mu(K^+ \cap g_{-t}\gamma K^-)dt$$

First, note that for $(\eta, \zeta) \in L_r(x, y)$ we have

$$\rho_x(\eta, \zeta) \leq 2r \leq c/30$$

Now, suppose $L_r(x, y) \cap (\gamma(B \times A)$ is nonempty. Then from the definitions it follows that $\gamma y \in C_1^+(x, A) \subset C_1^+(x, A)$. Since $\gamma$ is an isometry, it also holds that $L_r(\gamma^{-1}x, y) \cap (B \times \gamma^{-1}A)$ is nonempty and thus $\gamma^{-1}x \in C_1^+(y, B)$.

Note that for $(\eta, \zeta) \in L_r(x, y)$, $|s| < r/2$ and $T > 0$ we have

$$\int_0^{T-3r} e^{\delta t} \chi_{K(\gamma y, \gamma)}(g_t s x, \zeta)dt \leq e^{\delta(3r)} e^{\delta d(x, y)} \leq e^{c/20} e^{\delta d(x, y)}$$

and moreover is zero whenever $d(x, y) > T$. Using the fact that $L_r(a, b) \subset \Theta_r^+(b, a) \times \Theta_r^+(a, b)$ it follows that

$$\int_0^{T-3r} e^{\delta t} \sum_{\gamma \in G} \mu(K^+ \cap g_{-t}\gamma K^-)dt \leq e^{c/12} r^2 \sum \nu_x(\Theta_r^+(\gamma y, x)) \nu_y(\Theta_r^+(x, y))$$

where the sum is taken over all $\gamma \in G$ such that $(x, y) \leq T$ and

$$(\gamma y, \gamma^{-1}x) \in [C_1^+(x, A) \times C_1^+(y, B)]$$

By Corollary 7.4 we see that he set

$$[C_1^+(x, A) \cap N_DWH(G) \times C_1^+(y, B) \cap N_DWH(G)] \setminus [\hat{V} \times \hat{W}]$$

has compact closure in $Teich(S)$ for any $D > 0$. If $x, y \in N_DWH(G)$ their $G$ orbits are also contained in $N_DWH(G)$. Thus, by the discreteness of the $G$ action on $Teich(S)$, for some constant $c_1$ that does not depend on $T$

$$\int_0^{T-3r} e^{\delta t} \sum_{\gamma \in G} \mu(K^+ \cap g_{-t}\gamma K^-)dt \leq e^{c/12} r^2 \sum \nu_x(\Theta_r^+(\gamma y, x)) \nu_y(\Theta_r^+(x, y)) - c_1$$

for all $T > 0$ where the sum is taken over all $\gamma \in G$ with $(x, y) \leq T$ and

$$(\gamma y, \gamma^{-1}x) \in [C_1^+(x, A) \times C_1^+(y, B)] \cap [\hat{V} \times \hat{W}].$$
Note, by the triangle inequality we have for $\eta \in \Theta^+_r(x, \gamma y))$
\[ d(x, \gamma y) - 4r \leq \beta_\eta(x, \gamma y) \leq d(x, \gamma y) \]
and by the conformality of $\nu$ it follows that
\[ \nu_{\eta}(\Theta^+_r(\gamma^{-1} x, y)) = \nu_{\gamma y}(\Theta^+_r(x, \gamma y)) \leq \nu_x(\Theta^+_r(x, \gamma y))e^{\delta d(x, \gamma y)} \leq e^{4r}\nu_y(\Theta^+_r(x, \gamma y)) \]
Since $4\delta r < c/15$ it follows that when
\[ (\gamma y, \gamma^{-1} x) \subseteq \widehat{V} \times \widehat{W} \]
we have that
\[ \nu_x(\Theta^+_r(\gamma y, x))\nu_x(\Theta^+_r(x, \gamma y)) \leq e^{c/15}\nu_x(\Theta^+_r(\gamma y, x))\nu_y(\Theta^+_r(\gamma^{-1} x, y)) \]
\[ \leq e^{c/10}\nu_x(\Theta_r(\eta_0, x))\nu_y(\Theta_r(\zeta_0, y)) \]
Thus we obtain
\[ \int_{0}^{T-3r} e^{\delta t} \sum_{\gamma \in G} \mu(K^+ \cap g_{-t}\gamma K^-)dt \leq e^{c/6}r^2|G^+(T, A, B)|\nu_x(\Theta_r(\eta_0, x))\nu_y(\Theta_r(\zeta_0, y)) + c_1 \]
where $c_1$ is independent of $T$ and $G^+(T, A, B)$ is the set of all $\gamma \in G$ such that
\[ (x, \gamma y) \leq T \]
and
\[ (\gamma y, \gamma^{-1} x) \in [C_r^+(x, A) \times C_r^+(y, B)] \cap [\widehat{V} \times \widehat{W}] \]
In a similar but more annoying manner, we will obtain a lower bound for
\[ \int_{0}^{T+3r} e^{\delta t} \sum_{\gamma \in G} \mu(K^+ \cap g_{-t}\gamma K^-)dt \]
First, note that for $\eta \in \Theta^+_{r-h}(a, b)$ we have
\[ d(a, b) - 2r \leq \beta_\eta(a, b) \leq d(a, b) \]
and thus similarly to above we have
\[ \nu_{\eta}(\Theta^+_{r-h}(\gamma^{-1} x, y)) = \nu_{\gamma y}(\Theta^+_{r-h}(x, \gamma y)) \leq \nu_x(\Theta^+_{r-h}(x, \gamma y))e^{\delta d(x, \gamma y)} \leq e^{2r}\nu_y(\Theta^+_{r-h}(x, \gamma y)) \]
Since $2\delta r < c/30$ it follows that when
\[ (\gamma y, \gamma^{-1} x) \subseteq \widehat{V} \times \widehat{W} \]
we have that
\[ \nu_x(\Theta^+_{r-h}(x, \gamma y))\nu_x(\Theta^+_{r-h}(\gamma y, x)) \geq e^{-\delta d(x, \gamma y)}\nu_x(\Theta^+_{r-h}(x, \gamma y))\nu_y(\Theta^+_{r-h}(x, \gamma y)) \]
\[ \geq e^{-c/30}\nu_x(\Theta_r(\eta_0, x))\nu_y(\Theta_r(\zeta_0, y)) \]
Now note, if $(\gamma y, \gamma^{-1} x) \in C_r^-(y, B)$ then by definition
\[ A \supset \Theta^+_r(x, \gamma y) \text{ and } B \supset \Theta^+_r(y, \gamma^{-1} x) \text{, whence } \gamma B \supset \Theta^-_r(\gamma y, x) \text{. Note for } \eta_0, \zeta_0 \text{, then } \left| \Theta_r(\eta_0, x) \right| \leq 2 \text{.} \]
\[ \int_{0}^{T+3r} e^{\delta t} \chi_K(\gamma y, r)(g_{t+s}x, \eta_0, \zeta_0)dt \geq e^{-3\delta r}r e^{\delta d(x, \gamma y)} \geq e^{-c/20}r e^{\delta d(x, \gamma y)} \]
Now fix an \( \epsilon' > 0 \) with \( \mu^{BMS}(Q_{\epsilon'}(S))/G < 1/3 \) and consider \( \gamma \in G \) such that \( [x, \gamma y] \) and \( [\gamma y, x] \) both spend less than half time in \( Q_{\epsilon'}. \) By Proposition 12.6 and the discreteness of the action of \( G, \) for all but finitely many such \( \gamma \) we have that

\[
\Theta_{\gamma}^{-1}(\gamma y, x) \times \Theta_{\gamma}^{-1}(x, \gamma y) \subset L_r(x, \gamma y)
\]

Note for \( (\eta, \zeta) \in L_r(x, y), |s| < r/2, T > 0 \) and \( 3r \leq d(x, \gamma y) \leq T \) we have

\[
\int_0^{T+3r} e^{\delta t} \chi_K(\gamma y, t)(g_{t+s}x, \eta, \zeta)dt \geq e^{-c/20} r e^{\delta d(x, \gamma y)}
\]

Thus we have that

\[
\int_0^{T+3r} e^{\delta t} \sum_{\gamma \in G} \mu(K^+ \cap g^{-t} K^-)dt \geq e^{-c/20} r^2 \sum_{\gamma \in G(T, A, B)} \nu_\gamma(\Theta_r(\eta, x)) \nu_\gamma(\Theta_r(\zeta_0, y)) - c_2
\]

where \( G^-(T, A, B) \) is the set of all \( \gamma \in G \) such that \( (x, \gamma y) \leq T \) and

\[
(\gamma y, \gamma^{-1} x) \in [C^1_r(x, A) \times C^1_r(y, B)] \cap [\hat{V} \times \hat{W}]
\]

and \( G^-(T, \epsilon', A, B) \) is the set of all \( \gamma \in G(T, A, B) \) such that the segment \( (x, \gamma y) \) spends at least half the time in the \( \epsilon' \) thin part of the principal stratum, and \( c_2 \) does not depend on \( T. \)

By mixing it follows that for all \( t \) large enough we have

\[
e^{-c/60} \mu(K^+) \mu(K^-) \leq ||\mu|| \sum_{\gamma \in G} \mu(K^+ \cap g^{-t} K^-) \leq e^{c/60} \mu(K^+) \mu(K^-)
\]

Note, that by definition

\[
\mu(K^+) = r \int_{\eta \in A} \int_{\zeta \in \Theta_r(\eta, x)} e^{\delta \rho_\gamma(\eta, \zeta)} d\nu_\gamma(\zeta) d\nu_\gamma(\eta)
\]

Since

\[
0 \leq \rho_\gamma(\eta, \zeta) \leq 2r
\]

for \( \zeta \in \Theta_r(\eta, x) \) and since \( A \subseteq V \) we obtain that

\[
e^{-c/60} r \nu_\gamma(A) \nu_\gamma(\Theta_r(\eta_0, x)) \leq \mu(K^+) \leq e^{c/20} r \nu_\gamma(A) \nu_\gamma(\Theta_r(\eta_0, x))
\]

and similarly

\[
e^{-c/60} r \nu_y(\Theta_r(\zeta_0, y)) \leq \mu(K^-) \leq e^{c/20} r \nu_y(\Theta_r(\zeta_0, y))
\]

It follows that there exists a constant \( c_2 \) independent of \( T \) such that

\[
\int_0^{T-3r} e^{\delta t} \sum_{\gamma \in G} \mu(K^+ \cap g^{-t} K^-)dt \geq e^{-c/2} e^{\delta T} M v_\gamma(A) v_y(B) - c_2
\]

and

\[
\int_0^{T+3r} e^{\delta t} \sum_{\gamma \in G} \mu(K^+ \cap g^{-t} K^-)dt \leq e^{c/2} e^{\delta T} M v_\gamma(A) v_y(B) + c_2
\]

where \( M = r^2 v_\gamma(A) v_y(B) \).

Thus, it follows that

\[
e^{-c/2} v_\gamma(A) v_y(B) \leq e^{c/3} e^{-\delta T} |G^+(T, A, B)|
\]
and

$$e^{c/2} \nu_x(A) \nu_y(B) \geq e^{-c/6} e^{-\delta T} |G^-(T, A, B) \setminus G^-(T, e'A, B)|$$

for large enough $T$. Furthermore, by Theorem 11.4, if $\epsilon'$ is chosen small enough,

$$\limsup_{T \to \infty} |G^-(T, e'A, B)|/e^{\delta t} < e^{c/12}$$

so that

$$e^{c/2} \nu_x(A) \nu_y(B) \geq e^{-\delta T} |G^-(T, A, B)|$$

for all large enough $T$.

This completes the proof of Proposition 12.1. \( \square \)

**Lemma 12.6.** Let $x, y \in \text{Teich}(S)$ and $c > 0$. For each

$$(\eta_0, \zeta_0) \in \text{PMF} \times \text{PMF}$$

there exists an $r > 0$ and neighborhoods $V$ and $W$ of $\eta_0$ and $\zeta_0$ in PMF respectively such that for all borel $A \subseteq V$ and $B \subseteq W$, with nonempty interior:

$$\limsup_{t \to \infty} \nu^t_{x, y}(C^{-}_r(x, A) \times C^{-}_r(y, B)) \leq e^{c} \nu_x(A) \nu_y(B)$$

and

$$\liminf_{t \to \infty} \nu^t_{x, y}(C^{+}_r(x, A) \times C^{+}_r(y, B)) \geq e^{-c} \nu_x(A) \nu_y(B)$$

**Proof.** If $\eta_0$ is not in $\Lambda(G)$ then we can choose a neighborhood $U$ of $\eta_0$ in PMF with $\nu_x(U) = 0$ and $W = \text{PMF}$ so that both sides of the desired equation are 0 by Corollary 9.7. Similarly if $\zeta_0$ is not in $\Lambda(G)$. Assume therefore that $\eta_0, \zeta_0 \in \Lambda(G)$. Let $\lambda_0 \in \Lambda(G)$ and $x_0 \in (\eta_0, \lambda_0)$, $y_0 \in (\zeta_0, \lambda_0)$. Let $V_0, W_0$ be open neighborhoods of $\eta_0$ and $\zeta_0$ in PMF respectively such that for all open $A \subseteq V_0$ and $B \subseteq W_0$, we have as $T \to \infty$ that

$$\limsup_{t \to \infty} \nu^T_{x_0, y_0}(C^{-}_r(x_0, A) \times C^{-}_r(y_0, B)) \leq e^{c/3} \nu_{x_0}(A) \nu_{y_0}(B)$$

and

$$\liminf_{t \to \infty} \nu^T_{x_0, y_0}(C^{+}_r(x_0, A) \times C^{+}_r(y_0, B)) \geq e^{-c/3} \nu_{x_0}(A) \nu_{y_0}(B)$$

Let $\widehat{V}_0$ and $\widehat{W}_0$ be neighborhoods in $\text{Teich}(S) \cup \text{PMF}$ of $\eta_0$ and $\zeta_0$ respectively, whose intersection with PMF are respectively contained in $V_0$ and $W_0$ and such that for all

$$a \in \widehat{V}_0 \cap N_D \text{WH}(G), b \in \widehat{W}_0 \cap N_D \text{WH}(G)$$

we have

$$|d(x_0, a) - d(x, a) - \beta_{\eta_0}(x_0, x)| \leq \frac{c}{6\delta}$$

and

$$|d(y_0, b) - d(y, b) - \beta_{\eta_0}(y_0, y)| \leq \frac{c}{6\delta}$$

and for all

$$\eta \in \widehat{V}_0 \cap \Lambda(G)$$

$$\zeta \in \widehat{W}_0 \cap \Lambda(G)$$

we have

$$|\beta_{\eta}(x_0, x) - \beta_{\eta_0}(x_0, x)| \leq \frac{c}{6\delta}$$

and

$$|\beta_{\eta}(x_0, x) - \beta_{\eta_0}(x_0, x)| \leq \frac{c}{6\delta}.$$
Let $V$ and $W$ be open neighborhoods in $PMF$ of $\eta_0$ and $\zeta_0$ respectively, with $\overline{V} \subseteq \overline{V}_0 \cap PMF$ and $\overline{W} \subseteq \overline{W}_0 \cap PMF$ and let

$$r = 1 + d(x, x_0) + d(y, y_0).$$

Consider $A \subseteq V$ and $B \subseteq W$.

Note, if $(\gamma y, \gamma^{-1} x) \in C_{r}^{-}(x, A) \times C_{r}^{-}(y, B)$ one easily checks that $(\gamma y_0, \gamma^{-1} x_0) \in C_{1}^{-}(x_0, A) \times C_{1}^{-}(y_0, B)$ by the choice of $r$. Next, note that if $d(x, \gamma y) \leq t$ and $(\gamma y, \gamma^{-1} x) \in V_{r} \times \hat{W}$, where $V_{r}$ denotes the set of points whose $r$ neighborhood is contained in $\hat{W}$, then $\gamma y_0 \in \overline{V}_0$ and $\gamma^{-1} x \in \hat{W}_0$ which implies that

$$d(x_0, \gamma y_0) \leq d(x, \gamma y_0) + \beta_{\eta_0}(x_0, x) + \frac{c}{6\delta} = d(y, \gamma^{-1} x) + \beta_{\eta}(x_0, x) + \frac{c}{6\delta}$$

$$\leq d(y, \gamma^{-1} x) + \beta_{\zeta_0}(y_0, y) + \frac{c}{3\delta} \leq t + \beta_{\eta}(x_0, x) + \beta_{\zeta_0}(y_0, y) + \frac{c}{3\delta}$$

From Corollary 7.4 we obtain

**Lemma 12.7.**

$$[C_{r}^{-}(x, A) \cap N_{D} WH(G) \times C_{r}^{-}(y, B) \cap N_{D} WH(G)] \setminus [V_{r} \times \hat{W}]$$

is relatively compact in $Teich(S)$.

Thus $[C_{r}^{-}(x, A) \times C_{r}^{-}(y, B)] \setminus [V_{r} \times \hat{W}]$ contains only finitely many points $(\gamma x, \gamma^{-1} y), \gamma \in G$.

From this, we deduce that

$$\limsup_{t \to \infty} \nu_{x,y}^{t}(C_{r}^{-}(x, A) \times C_{r}^{-}(y, B)) \leq$$

$$e^{c/3} \delta_{\eta_0}(x_0, x) + \delta_{\zeta_0}(y_0, y) \limsup_{t \to \infty} \nu_{x_0,y_0}^{t+\beta_{\eta_0}(x_0, x) + \beta_{\zeta_0}(y_0, y) + 6c} \left(C_{1}^{-}(x_0, A) \times C_{1}^{-}(y_0, B)\right)$$

and thus by Prop 12.1,

$$\limsup_{t \to \infty} \nu_{x,y}^{t}(C_{r}^{-}(x, A) \times C_{r}^{-}(y, B)) \leq e^{2c/3} \delta_{\eta_0}(x_0, x) + \delta_{\zeta_0}(y_0, y) \nu_{x_0}(A) \nu_{y_0}(B).$$

Since $e^{\delta_{\eta_0}(x_0, x)} \nu_{x_0} \leq e^{c/6} \nu_{x}$ when restricted to $V$ and $e^{\delta_{\zeta_0}(y_0, y)} \nu_{y_0} \leq e^{c/6} \nu_{y}$ when restricted to $W$, we obtain that

$$\limsup_{t \to \infty} \nu_{x,y}^{t}(C_{r}^{-}(x, A) \times C_{r}^{-}(y, B)) \leq e^{c/6} \nu_{x}(A) \nu_{y}(B)$$

The reverse estimate is proved similarly. \hfill $\square$

**Theorem 12.8.** For $x, y \in Teich(S)$ $\nu_{x,y}^{t} \converges \text{ weakly to } \nu_{x} \times \nu_{y}$ as $t \to \infty$.

**Proof.** Let $c > 0$. For each $(\eta_0, \zeta_0) \in \Lambda(G) \times \Lambda(G)$ take neighborhoods $V_{(\eta_0, \zeta_0)}$ and $W_{(\eta_0, \zeta_0)}$ of $\eta_0$ and $\zeta_0$ respectively such that the conclusion of Lemma 12.6 holds for $c$. By compactness finitely many of the $V \times W$ cover $PMF \times PMF$, say $V_i \times W_i, i = 1, \ldots, n$. Let $\overline{V}_i$ and $\overline{W}_i$ be open subsets of $Teich(S) \cup PMF$ such that $V_i = \overline{V}_i \cap PMF$ and $W_i = \overline{W}_i \cap PMF$. Let $\overline{A}$ and $\overline{B}$ be borel subsets of $Teich(S) \cup PMF$ with

$$\overline{A} \subset \overline{V}_i$$

$$\overline{B} \subset \overline{W}_i$$

and

$$(\nu_{x} \otimes \nu_{y})(\partial(\overline{A} \times \overline{B})) = 0.$$
Similarly we obtain the reverse estimate
\[ \nu_x(\hat{A}) \nu_y(\partial \hat{B}) = \nu_y(\partial \hat{A}) \times \nu_x(\hat{A}) = 0 \]
Let \( \alpha > 0 \). Let \( A^+, B^+ \subset PMF \) be open and \( A^-, B^- \subset PMF \) compact with
\( A^-, B^- \) either being empty or having nonempty interior such that
\[ A^- \subset \hat{A}^o \cap PMF \subset \hat{A} \cap PMF \subset A^+ \subset V_i \]
\[ B^- \subset \hat{B}^o \cap PMF \subset \hat{B} \cap PMF \subset B^+ \subset W_i \]
\[ \nu_x(\hat{A}^o \setminus A^-) < \alpha, \nu_x(A^+ \setminus \hat{A}) < \alpha, \nu_x(\hat{B}^o \setminus B^-) < \alpha, \nu_x(A^+ \setminus \hat{A}) < \alpha \]
Let \( D > d(x, WH(G)) + d(y, WH(G)) \) so that the \( G \) orbits of \( x, y \) are contained in
\( N_D WH(G) \). By Corollary 7.4 the sets
\[ [\hat{A} \cap N_D WH(G) \times \hat{B} \cap N_D WH(G)] \setminus [\hat{C}^- (x, A^+) \times \hat{C}^- (y, B^+)] \]
and
\[ [C^+(x, A^-) \cap N_D WH(G) \times C^+(y, B^-) \cap N_D WH(G)] \setminus [\hat{A}^o \times \hat{B}^o] \]
are relatively compact in \( \text{Teich}(S) \times \text{Teich}(S) \). Thus, by Lemma 12.6 we have
\[
\limsup \nu_x^t(\hat{A} \times \hat{B}) \leq \limsup \nu_x^t(C^-_r(x, A^+) \times C^-_r(y, B^+)) \leq e^c \nu_x(A^+) \nu_y(B^+) \leq e^c \nu_x(\hat{A}) \nu_y(\hat{B}) + \alpha e^c(||x|| + ||y||) = e^c \nu_x(\hat{A}) \nu_y(\hat{B}) + \alpha e^c(||x|| + ||y||)
\]
Since \( \alpha > 0 \) can be chosen arbitrarily small we obtain
\[ \limsup \nu_{x,y}^t(\hat{A} \times \hat{B}) \leq e^c \nu_x(\hat{A}) \nu_y(\hat{B}) \]
Similarly we obtain the reverse estimate
\[ \liminf \nu_{x,y}^t(\hat{A} \times \hat{B}) \geq e^{-c} \nu_x(\hat{A}) \nu_y(\hat{B}) \]
Indeed,
\[ \liminf \nu_{x,y}^t(\hat{A} \times \hat{B}) \geq \liminf \nu_{x,y}^t(C^+_r(x, A^-) \times C^+_r(y, B^-)) \geq e^{-c} \nu_x(\hat{A}) \nu_y(\hat{B}) - \alpha e^c(||x|| + ||y||) = e^{-c} \nu_x(\hat{A}) \nu_y(\hat{B}) - \alpha e^c(||x|| + ||y||) \]
for any \( \alpha > 0 \). Thus, for \( \phi \) a continuous function supported on \( \hat{V} \times \hat{W} \) we have
\[ e^{-c} \int \phi d\nu_x \otimes d\nu_y \leq \liminf \int \phi d\nu_{x,y} \leq \liminf \int \phi d\nu_{x,y} \leq e^{-c} \int \phi d\nu_x \otimes d\nu_y \]
Furthermore the complement \( O \) of \( \bigcup_{i=1}^n V_i \times W_i \) in \( \text{Teich}(S) \cup PMF \) is a compact subset of \( \text{Teich}(S) \), so for any function supported on \( O \) we have
\[ \int \phi d\nu_x \otimes d\nu_y = 0 \]
and
\[ \lim_{t \to \infty} \int \phi d\nu_{x,y} = 0 \]
By choosing a partition of unity subordinate to the cover \( O, \hat{V}_i \times \hat{W}_i \) we obtain that
\[ e^{-c} \int \phi d\nu_x \otimes d\nu_y \leq \liminf \int \phi d\nu_{x,y} \leq \liminf \int \phi d\nu_{x,y} \leq e^{-c} \int \phi d\nu_x \otimes d\nu_y \]
for any continuous \( \phi \) on \( (\text{Teich}(S) \cup PMF) \times (\text{Teich}(S) \cup PMF) \) and letting \( c \to 0 \) yields the desired result.
\[ \square \]
Theorem 12.9. For all \( x, y \in \text{Teich}(S) \) we have
\[
\lim_{R \to \infty} |B_R(x) \cap G |e^{-\delta R} = \frac{||\nu_x|| ||\nu_y||}{\delta ||\mu^{BM5}||}
\]

13. Counting Closed Geodesics

In this section we prove Theorem 1.2. Denote by \( G_h \) the set of pseudo-Anosov elements of \( G \). Denote by \( G_{hp} \) the set of primitive pseudo-Anosov elements of \( G \). Let \( \Omega(l) \) be the set of closed primitive geodesics on \( \text{Teich}(S)/G \) of length at most \( R \). For \( g \in \Omega(l) \) let \( D_g \) be the Lebesgue measure on \( g \) normalized to unit mass.

We will prove:

Theorem 13.1.
\[
\lim_{t \to \infty} \delta t e^{-\delta t} \sum_{g \in \Omega(l)} D_g = ||\mu||^{-1} \mu
\]

Theorem 1.2 is an immediate corollary.

Proof of Theorem 13.1. Let \( x \in \text{Teich}(S) \). Denote by \( V(x, r) \subset \text{Teich}(S)^2 \cup \text{PMF}^2 \) the set of pairs \((a, b)\) such that \([a, b] \cap B(x, r) \neq \emptyset\). Recall the measure \( \bar{\mu} \) on \( \Lambda(G) \times \Lambda(G) \) by
\[
d\bar{\mu}(\eta, \zeta) = \exp(\delta(G)\rho_x(\eta, \zeta))d\nu_x(\eta)d\nu_x(\zeta)
\]
and let
\[
d\nu_{x,1} = d\nu_{x,x} = \delta ||\mu||e^{-\delta t} \sum_{\gamma \in G, d(x, \gamma y) \leq t} D_{\gamma x} \otimes D_{\gamma^{-1} x},
\]
\[
d\nu_{x,2} = \delta ||\mu||e^{-\delta t} \sum_{\gamma \in G_h, d(x, \gamma y) \leq t} D_{\gamma x} \otimes D_{\gamma^{-1} x},
\]
\[
d\nu_{x,3} = \delta ||\mu||e^{-\delta t} \sum_{\gamma \in G_{hp}, d(x, \gamma y) \leq t} D_{\gamma x} \otimes D_{\gamma^{-1} x},
\]
where \( \gamma^\pm \in \text{PMF} \) denote the stable and unstable laminations of \( \gamma \). Note that for \( D = d(x, WHG) \) we have \( \nu_{x,1} \) and \( \nu_x \times \nu_x \) all supported on \((\text{NpWH}(G) \cup \Lambda(G))^2\) and \( V(x, r) \cap (\text{NpWH}(G) \cup \Lambda(G))^2 \) is closed in \( \text{Teich}(S) \cup \text{PMF} \).

Lemma 13.2. For every \( c > 0 \) there exists a \( t_0 = t_0(x, r, c) > 0 \) such that if \( \gamma \in G \) with \( d(x, \gamma x) > t_0 \) and \((\gamma x, \gamma^{-1} x) \in V(x, r) \) we have that \( \gamma \) is pseudo-Anosov and \( \rho_x(\gamma^{\pm 1} x, \gamma^{\pm}) > c \).

Proof. Suppose \( \gamma \in G \) is a pseudo-Anosov, \( p \) a point on the axis of \( \gamma \) and \( l \) the unit speed parametrization of the axis starting at \( p \) in direction \( \gamma^+ \). Note, \( \rho_x(\gamma x, \gamma^+) = \lim_{t \to \infty} d(x, \gamma x) + d(x, l(t)) - d(\gamma x, l(t)) = d(x, \gamma x) + d(x, l(t)) - d(x, l(t - l(\gamma))) \geq d(x, \gamma x) \)
and similarly \( \rho_x(\gamma^{-1} x, \gamma^-) \geq d(x, \gamma x) \)
Note, as \( G \) is convex cocompact it contains no parabolic elements. Since \( G \) is a hyperbolic group, it contains only finitely many conjugacy classes of finite order elements. Therefore, the fixed points of finite order elements of \( G \) are contained in finitely many \( G \) orbits of \( \text{Teich}(S) \). Let \( D \) be the maximum distance of these orbits.
from \( WH(G) \). Suppose now that \( \gamma_n \in G \) is a sequence of finite order elements with 
\( d(x, \gamma_n x) \to \infty \) and \( (\gamma_n x, \gamma_n^{-1} x) \in V(x, r) \). Let \( p_n \) be the fixed point of \( \gamma_n \). Then 
\( x, \gamma_n x, \gamma_n^{-1} x \) all lie on the same circle \( C_n \) of radius \( r_n \to \infty \) centered at \( p_n \).

Taking a subsequence, we can assume

\[
p_n \to \eta \in PMF
\]

and

\[
y_n = \gamma_n x \to \zeta \in PMF
\]

and

\[
z_n = \gamma_n^{-1} x \to \theta \in PMF
\]

Then clearly \( \zeta, \theta \in \Lambda(G) \).

Also, the \( p_n \) are all contained in \( N_D WH(G) \) and thus \( \eta \in \Lambda(G) \subset UE \). We claim \( \zeta = \eta = \theta \), which would imply that for large enough \( n \) we have \( (\gamma_n x, \gamma_n^{-1} x) \notin V(x, r) \), contradicting our assumption. Indeed,

\[
\rho_x(\eta, \zeta) = \lim_{n \to \infty} d(x, p_n) + d(x, y_n) - d(y_n, p_n) = \lim_{n \to \infty} r_n \to \infty
\]

which is impossible if \( \eta \neq \zeta \) by continuity of \( \rho \). Similarly, \( \eta = \theta \).

For the remainder of the argument, the proof of Roblin’s Theorem 5.1.1 carries through with essentially no modification. From Lemma 13.2 we obtain

**Corollary 13.3.** When restricted to \( V(x, r) \) we have \( \nu_{x,i}^t - \nu_{x,j}^t \to 0 \) for \( i, j = 1, 2, 3 \).

Note, for \( \eta, \zeta \in V(x, r) \) we have \( 0 < \rho_x(\eta, \zeta) \leq 2r \) and thus by Theorem 12.8 and Lemma 13.2 for any positive continuous \( \psi \) compactly supported on \( V(x, r) \) we have

\[
e^{-2dr} \int \psi d\bar{\mu} \leq \liminf \int \psi d\nu_{x,3}^t \leq \limsup \int \psi d\nu_{x,3}^t \leq \int \psi d\bar{\mu}
\]

as \( t \to \infty \). For \( \gamma \in G \) let \( g_\gamma \) be the axis of \( \gamma \), and denote by \( L_\gamma \) the arclength measure on \( \gamma \). Let \( l(\gamma) \) denote the translation length of \( \gamma \) in the Teichmüller metric. Let

\[
M^t_x = \delta e^{-\delta t} \sum_{\gamma \in G, l(\gamma) \leq t} L_\gamma
\]

and

\[
M^{t+2r}_x \leq e^{-2r} M_{x,3}^t
\]

Let \( \tilde{V}(x, r) \subset Q^1(S) \) denote all quadratic differentials on geodesic segments defined by elements of \( V(x, r) \). Note,

\[
M^{t+2r}_{x,3} = ||\mu^{BMS}||^{-1} \nu_{x,3}^t \otimes \mathbb{R}ds
\]

and thus for any \( \phi \in C^1_c(\tilde{V}(x, r)) \) we have

\[
e^{-2dr} ||\mu||^{-1} \int \phi d\bar{\mu} \leq \liminf \int \phi dM^t_{x,3} \leq \limsup \int \phi dM^t_{x,3} \leq ||\mu||^{-1} \int \phi d\mu
\]
Denote by $G_{hp} \subset G$ the set of primitive hyperbolic isometries so that

$$M^t_x = \delta e^{-\delta t} \sum_{\gamma \in G_{hp}, l(\gamma) \leq t} \left\lfloor \frac{t}{l(\gamma)} \right\rfloor L_{\gamma}$$

Clearly

$$M^t_x \leq E^t := \delta e^{-\delta t} \sum_{\gamma \in G_{hp}, l(\gamma) \leq t} \frac{1}{l(\gamma)} L_{\gamma}$$

Moreover, note $\left\lfloor \frac{t}{l(\gamma)} \right\rfloor \geq \frac{1}{l(\gamma)}$ whenever $1 \leq l(\gamma) \leq t$ and $t \geq 2$, so for any $\phi \in C_c^+(\hat{V}(x,r))$ we have

$$\sum_{\gamma \in G_{hp}, l(\gamma) \leq t} \frac{1}{l(\gamma)} \int \phi dL_{\gamma} = O(e^{\delta t})$$

since $\int \phi dM^t_x$ is bounded as $t \to \infty$.

Now, if $e^{-r}t < l(\gamma) \leq t$ then

$$\left\lfloor \frac{t}{l(\gamma)} \right\rfloor \geq \frac{1}{l(\gamma)} \geq e^{-r}t$$

and so

$$\int \phi dM^t_x \geq e^{-r} \delta e^{-\delta t} \sum_{\gamma \in G_{hp}, e^{-r}t < l(\gamma) \leq t} \frac{1}{l(\gamma)} \int \phi dL_{\gamma} =

e^{-r} \int \phi dE^t - e^{-r} \delta e^{-\delta t} \sum_{\gamma \in G_{hp}, l(\gamma) \leq e^{-r}t} \frac{1}{l(\gamma)} \int \phi dL_{\gamma}$$

By above remarks, the second term in the above difference is bounded above by a constant multiple of

$$te^{\delta(e^{-r}-1)t} = o(1)$$

Thus we get

$$\limsup \int \phi dM^t_x \geq e^{-r} \limsup \int \phi dE^t$$

Putting everything together we obtain as $t \to \infty$

$$e^{-2\delta r} ||\mu||^{-1} \int \phi d\mu \leq \limsup \int \phi dE^t \leq \limsup \int \phi dE^t \leq e^{(2\delta + 1)r} ||\mu||^{-1} \int \phi d\mu$$

Choosing a partition of unity subordinate to the locally finite cover of $Q^1(S)$ by the $\hat{V}(x,r)$ (where $r > 0$ is fixed) we obtain the above relation for each $\phi \in C_c^+(Q^1(S))$.

Letting $r \to 0$ completes the proof. \qed

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