BOUNDEDNESS PROPERTIES OF SEMI-DISCRETE SAMPLING OPERATORS IN MELLIN–LEBESGUE SPACES

CARLO BARDARO* AND ILARIA MANTELLINI

Department of Mathematics and Computer Sciences
University of Perugia
via Vanvitelli 1, I-06123 Perugia, Italy

Abstract. In this paper we study boundedness properties of certain semi-discrete sampling series in Mellin–Lebesgue spaces. Also we examine some examples which illustrate the theory developed. These results pave the way to the norm-convergence of these operators.

1. Introduction. The theory of the generalized sampling series (operators) was started by the Paul Butzer School in Aachen (see [20, 21, 22, 37]) and later on developed by many authors (see for example [1, 2, 7, 11, 31, 32, 38] and the references there included). The success of this theory relies on its many concrete applications in signal and image processing. The starting point is the classical Shannon sampling theory of signal analysis. This well-known theory was well outlined in the books [30, 39]. However, in spite of its wide applicability, the Shannon sampling operator is not a good mathematical model for the prediction theory, so important in applications to signal analysis. Moreover the validity of the Shannon sampling theorem for the exact reconstruction of a signal \( f \) is obtained under severe regularity assumptions on the signal (for example bandlimiteness).

A further interesting modification of the generalized sampling series is given by the Kantorovich sampling operator, introduced in [15]. In this operator the sampled values of the signal are replaced by an integral mean between two sample points (not necessarily equally spaced over the real line). This solves the problems related to the exact determination of the sampled values, which is experimentally not possible in general, so reducing certain approximation errors due to measurements. This approach produced a wide field of concrete applications to several topics of applied sciences (see for example [3, 26, 27], and the references therein).

From a theoretical point of view, these operators were also studied in [24, 28, 29, 35, 34] and they can be considered as a semi-discrete operator defined by two kernel functions \( \varphi \) and \( \psi \), the second one generating the integral mean. Thus, in
order to obtain a general and unifying theory, one can think to replace the integral mean by an arbitrary convolution integral operator generated by a suitable kernel. This was done in [12] (see also [9]), in which pointwise and uniform approximate reconstructions are studied. Later, in [25] the modular (norm) convergence in Orlicz spaces was studied. The new operator was called “Durrmeyer-type sampling operator”.

Another approach to the sampling theory was formally introduced in [16, 23, 36], in which the sampling series is constructed using a composition between the classical “sinc” function of signal analysis and the logarithm, and the samples are now not equally spaced but exponentially spaced over the positive real axis. This approach is justified by its applications to problems related to optical physics phenomena, like for example, light scattering, diffraction, radioastronomy, and so on. A rigorous approach to the exponential sampling theory was formulated in [18] (see also [5, 6]). This was the starting point for the study of the generalized exponential sampling series, introduced in [8], and then studied in recent papers (see for example [4, 14]). In particular in [14] the norm-convergence is studied in the so-called Mellin–Lebesgue spaces (see for example [17, 19, 14]), which are defined, for \( c \in \mathbb{R} \) and \( p \in [1, \infty] \), by

\[
X_p^c := \{ f : \mathbb{R}^+ \to \mathbb{C} : f(\cdot)(\cdot)^{c-1/p} \in L^p(\mathbb{R}^+) \},
\]

where \( L^p \) denoted the usual Lebesgue space.

In the recent paper [13] we introduced the so-called Durrmeyer-type exponential sampling operator, following the same idea developed for the generalized sampling series. We replace the sampled-value of a function \( f \) by a convolution integral operator of Mellin type (see [17]). These operators are of the following form

\[
(T_{\varphi, \psi} f)(x) = \sum_{k=-\infty}^{\infty} \varphi(e^{-k} x) \int_0^\infty \psi(e^{-k} u) f(u) \frac{du}{u} \quad (x \in \mathbb{R}^+),
\]

where \( \varphi \) and \( \psi \) are suitable kernel functions and \( f \) belongs to the domain of the operator. These operators contain as a special case a Kantorovich version of the exponential sampling series, studied in [33].

In order to obtain norm-convergence theorems in Mellin–Lebesgue spaces, norm-boundedness properties of the operator \( T_{\varphi, \psi} \) are of fundamental importance. The aim of the present article is a study of these continuity properties, obtaining inequalities of the form

\[
\| T_{\varphi, \psi} f \|_{X_p^c} \leq D \| f \|_{X_p^c},
\]

where \( f \in X_p^c \) and \( D \) is an absolute positive constant.

In Sections 4 and 5 we obtain the desired boundedness properties in spaces \( X_p^c \) and \( X_0^c \). Indeed, the case \( c = 0 \) represents an important special case. Section 6 contains some examples illustrating the theory.

2. Preliminaries. Let us denote by \( \mathbb{N}, \mathbb{N}_0 \) and \( \mathbb{Z} \) the sets of positive integers, nonnegative integers and integers respectively. Moreover by \( \mathbb{R} \) and \( \mathbb{R}^+ \) we denote the sets of all real and positive real numbers respectively. By \( \mathbb{C} \) we denote the set of complex numbers.

Let \( \mathcal{B}(\mathbb{R}^+) \) be the \( \sigma \)-algebra of all Lebesgue measurable subsets of \( \mathbb{R}^+ \). For \( A \in \mathcal{B}(\mathbb{R}^+) \), we denote by \( \chi_A \) the characteristic function of \( A \), that is, \( \chi_A(x) = 1 \) for \( x \in A \) and \( \chi_A(x) = 0 \) for \( x \notin A \).

By \( C(\mathbb{R}^+) \) we denote the space of all continuous and bounded functions \( f : \mathbb{R}^+ \to \mathbb{C} \), endowed with its supremum norm \( \| f \|_\infty := \sup_{x \in \mathbb{R}^+} |f(x)| \) and by \( C_{\text{comp}}(\mathbb{R}^+) \) the
subspace of $C(R^+)$ comprising all functions with compact support in $R^+$. Moreover, $C^\infty_{\text{comp}}(R^+)$ denotes the subspace of $C_{\text{comp}}(R^+)$ containing all test functions, i.e., the functions of compact support which are infinitely differentiable.

For $1 \leq p \leq \infty$, we denote by $L^p(R^+)$ the usual Lebesgue spaces comprising all Lebesgue measurable functions such that
\[
\|f\|_p := \left\{ \int_0^\infty |f(x)|^p dx \right\}^{1/p} < \infty \quad (1 \leq p < \infty)
\]
and $\|f\|_\infty := \text{ess sup}_{x \in R^+} |f(x)| < \infty$. Note that $C(R^+) \subset L^\infty(R^+)$ and the norm of the two spaces is the same.

Let $c \in R$ be fixed. For $1 \leq p < \infty$, we denote by $X^p_c$ the Mellin–Lebesgue space defined by
\[
X^p_c := \{ f : R^+ \to C : f(\cdot)(\cdot)^{c-1/p} \in L^p(R^+) \}
\]
and endowed with the norm
\[
\|f\|_{X^p_c} := \left\{ \int_0^\infty |f(x)|^p x^{c-1/p} dx \right\}^{1/p} < \infty.
\]
For $p = 1$ we will simply write $X^1_c \equiv X_c$. In an equivalent form, $X^p_c$ is the space of all functions $f$ such that $f(\cdot)(\cdot)^{c-1/p} \in L^p_A(R^+)$, where $L^p_A(R^+)$ denotes the Lebesgue space with respect to the (invariant) measure $\mu(A) = \int_A dt/t$ for any measurable set $A \subset R^+$. For details see [17, 19, 14].

3. Exponential sampling Durrmeyer operator. Let $\varphi : R^+ \to R$ be a continuous function such that the following assumptions are satisfied

(i) for every $u \in R^+$, $\sum_{k=-\infty}^{\infty} \varphi(e^{-k}u) = d_1$, for an absolute constant $d_1 \neq 0$;

(ii) we have that
\[
M_0(\varphi) := \sup_{u \in R^+} \sum_{k=-\infty}^{\infty} |\varphi(e^{-k}u)| < + \infty;
\]

We denote by $\Phi$ the class of all functions $\varphi$ satisfying the above assumptions.

Let $\psi : R^+ \to R$ be a function belonging to $X_0$ with the following conditions

j) $\int_0^\infty \frac{\psi(t)}{t} dt = d_2,$

for a constant $d_2 \neq 0$. We will use the notation
\[
\tilde{M}_0(\psi) := \int_0^\infty |\psi(t)| \frac{dt}{t} < + \infty
\]
and let us denote by $\Psi$ the class of all functions $\psi$ satisfying assumption j).

Remark 1. In what follows we will assume, without loss of generality, that $d_1 = d_2 = 1$. Indeed, we may replace the functions $\varphi$, $\psi$ by $\varphi/d_1$ and $\psi/d_2$ respectively.

For any $\psi \in \Psi$, we denote by $D_\psi$ the set comprising all measurable functions $f : R^+ \to C$ such that the integrals
\[
G_k := \int_0^\infty \psi(e^{-k}u)f(u) \frac{du}{u}
\]
each as Lebesgue integrals, for every $k \in \mathbb{Z}$. 
Let \( \varphi \in \Phi \) and \( \psi \in \Psi \). For any \( f : \mathbb{R}^+ \rightarrow \mathbb{C} \), we define the exponential sampling Durrmeyer series of \( f \) as (see [13])

\[
(T^{\varphi, \psi} f)(x) := \sum_{k = -\infty}^{\infty} \varphi(e^{-k}x) \int_0^{\infty} \psi(e^{-k}u)f(u) \frac{du}{u},
\]

for \( x \in \mathbb{R}^+ \) and for any function \( f \in \text{dom} T^{\varphi, \psi} \), being \( \text{dom} T^{\varphi, \psi} \) the set of all functions \( f \in D\psi \) for which the series is absolutely convergent for almost all \( x \).

Using the conditions of the classes \( \Phi \) and \( \Psi \), it is easy to see that the above operator is well defined as an absolutely convergent series, for any function \( f \in L^\infty(\mathbb{R}^+) \).

In particular \( C(\mathbb{R}^+) \subset \text{dom} T^{\varphi, \psi} \).

We can determine larger subspaces of the domain of \( T^{\varphi, \psi} \). We admit functions which grow like a power of the logarithm. In order to do that, we have to introduce the absolute moments of order \( r \in \mathbb{N} \) of the functions \( \varphi \in \Phi \) and \( \psi \in \Psi \), on setting

\[
M_r(\varphi) := \sup_{u \in \mathbb{R}^+} \sum_{k = -\infty}^{\infty} |\varphi(e^{-k}u)||\log^r(e^k u - 1)|,
\]

\[
\tilde{M}_r(\psi) := \int_0^{\infty} |\log t|^r |\psi(t)| \frac{dt}{t}.
\]

A detailed study of the domain is given in [13, 14].

4. Boundedness in \( X^p_c \). In this and in subsequent sections, for given functions \( \varphi \in \Phi \) and \( \psi \in \Psi \), we will always assume that \( f \in \text{dom} T^{\varphi, \psi} \).

We begin with the following general result.

**Theorem 4.1.** Let \( 1 \leq p < \infty \) and \( c \in \mathbb{R} \). Let \( \varphi \in \Phi \cap X_{pc} \) and \( \psi \in \Psi \). Assume that

\[
H := \sup_{u \in \mathbb{R}^+} \sum_{k = -\infty}^{\infty} \left( e^k \right)^{pc} \frac{|\psi(e^{-k}u)|}{u} < \infty.
\]

If \( f \in X^p_c \) then \( T^{\varphi, \psi} f \in X^p_c \) and

\[
\|T^{\varphi, \psi} f\|_{X^p_c} \leq D \|f\|_{X^p_c},
\]

where

\[
D := (HM_0(\varphi))^{p-1}\|\psi\|_{X_0}^{-1}\|\varphi\|_{X_{cp}} 1/p.
\]

**Proof.** We will prove directly the estimate (3). Let us consider the integrals

\[
G_k = \int_0^{\infty} \psi(e^{-k}u)f(u) \frac{du}{u}
\]

and, for any \( N \in \mathbb{N} \), we set

\[
A_N(x) := \sum_{|k| \leq N} |\varphi(e^{-k}x)|, \quad x \in \mathbb{R}^+
\]

and

\[
T_N(x) := \sum_{|k| \leq N} |\varphi(e^{-k}x)||G_k|.
\]
From the assumption (i) we have that $A_N(x) > 0$ for sufficiently large $N$, and moreover we have $T_N(x) \geq 0$, for every $x \in \mathbb{R}^+$ and $N \in \mathbb{N}$. Thus, using Jensen’s inequality for sums, we obtain

$$\left(T_N(x)\right)^p \leq M_0(\varphi)^{p-1} \sum_{|k| \leq N} |G_k|^p |\varphi(e^{-k}x)|.$$  

Now, making the change of variable $t = e^{-k}u$ and applying again Jensen’s inequality for integrals, we obtain

$$|G_k|^p \leq \|\psi\|_{X_0}^{-p-1} \int_0^\infty \left|\psi(t)\right| |f(te^k)|^p \frac{dt}{t},$$

and so

$$\left(T_N(x)\right)^p \leq M_0(\varphi)^{p-1} \|\psi\|_{X_0}^{-p-1} \sum_{|k| \leq N} \left|\varphi(e^{-k}x)\right| \int_0^\infty \left|\psi(t)\right| |f(te^k)|^p \frac{dt}{t}.$$

Taking the norm in $X_c^p$, one has

$$\|T_N\|_{X_c^p}^p = \int_0^\infty (T_N(x))^{p-x^p} dx \leq M_0(\varphi)^{p-1} \|\psi\|_{X_0}^{-p-1} \sum_{|k| \leq N} \int_0^\infty \left|\psi(t)\right| |f(te^k)|^p \frac{dt}{t} \int_0^\infty \left|\varphi(e^{-k}x)\right| e^{pc} \frac{dx}{x}.$$

Using again the change of variable in the second integral $e^{-k}x = v$, we obtain

$$\|T_N\|_{X_c^p}^p \leq M_0(\varphi)^{p-1} \|\psi\|_{X_0}^{-p-1} \sum_{|k| \leq N} e^{kpc} \int_0^\infty \left|\psi(t)\right| |f(te^k)|^p \frac{dt}{t} \int_0^\infty \left|\varphi(v)\right| e^{pc} \frac{dv}{v}.$$

Since $\varphi \in X_{pc}$, we have

$$\|T_N\|_{X_c^p}^p \leq M_0(\varphi)^{p-1} \|\psi\|_{X_0}^{-p-1} \|\varphi\|_{X_{pc}} \sum_{|k| \leq N} e^{kpc} \int_0^\infty \left|\psi(t)\right| |f(te^k)|^p \frac{dt}{t}.$$

From the above inequality, setting $e^k t = u$, yields

$$\|T_N\|_{X_c^p}^p \leq M_0(\varphi)^{p-1} \|\psi\|_{X_0}^{-p-1} \|\varphi\|_{X_{pc}} e^{kpc} \int_0^\infty \sum_{|k| \leq N} \left(\frac{e^k}{u}\right)^pc \left|\psi(e^{-k}u)\right| |f(u)|^p \frac{du}{u}.$$

From the definition of $H$ we finally have

$$\|T_N\|_{X_c^p}^p \leq M_0(\varphi)^{p-1} \|\psi\|_{X_0}^{-p-1} \|\varphi\|_{X_{pc}} H \|f\|_{X_c^p}^p. \quad (4)$$

Since (4) holds for any sufficiently large $N \in \mathbb{N}$, we obtain the assertion. \hfill \Box

The next theorem gives some conditions in order to have the boundedness of $T^\varphi;\psi$ between the spaces $X_0^p$ and $X_c^p$.

**Theorem 4.2.** Let $1 \leq p < \infty$ and $c \in \mathbb{R}$. Let $\varphi \in \Phi \cap X_{pc}$ and $\psi \in \Psi$. Assume that

$$H' := \sup_{u \in \mathbb{R}^+} \sum_{k=-\infty}^\infty e^{kpc} \left|\psi(e^{-k}u)\right| < \infty. \quad (5)$$
If \( f \in X_0^p \) then \( T^{\varphi, \psi} f \in X_0^p \) and
\[
\| T^{\varphi, \psi} f \|_{X_0^p} \leq D' \| f \|_{X_0^p},
\] (6)
where
\[
D' := (H' M_0(\varphi)^{p-1} \| \varphi \|_{X_0^p}^{p-1} \| \varphi \|_{X_p})^{1/p}.
\]

Proof. Following the same arguments in the proof of Theorem 4.1, and using the same notations, we obtain
\[
\| T_N \|_{X_0^p} \leq M_0(\varphi)^{p-1} \| \varphi \|_{X_0^p} \| \varphi \|_{X_p} \int_0^\infty \sum_{|u| \leq N} e^{kpc} |\psi(e^{-k}u)| |f(u)|^p \frac{du}{u}.
\]
Now, from the definition of \( H' \) we have
\[
\| T_N \|_{X_0^p} \leq M_0(\varphi)^{p-1} \| \varphi \|_{X_0^p} \| \varphi \|_{X_p} \int_0^\infty \sum_{|u| \leq N} e^{kpc} |\psi(e^{-k}u)| |f(u)|^p \frac{du}{u}.
\]
and so the assertion. \( \square \)

5. The particular case \( c = 0 \). When \( c = 0 \), we have
\[
H = H' = M_0(\psi) = \sup_{u \in \mathbb{R}^+} \sum_{k=-\infty}^\infty |\psi(e^{k}u)|.
\]
Thus Theorems 4.1 and 4.2 reduce to the following

Corollary 1. Let \( 1 \leq p < \infty \). Let \( \varphi \in \Phi \cap X_0 \) and \( \psi \in \Psi \) be such that \( M_0(\psi) < \infty \). If \( f \in X_0^p \) then \( T^{\varphi, \psi} f \in X_0^p \) and
\[
\| T^{\varphi, \psi} f \|_{X_0^p} \leq D \| f \|_{X_0^p},
\] (7)
where \( D := (M_0(\psi) M_0(\varphi)^{p-1} \| \varphi \|_{X_0^p}^{p-1} \| \varphi \|_{X_p})^{1/p} \).

Here, we give another estimate in the space \( X_0^p \) which makes use of a notion of convolution in Mellin sense slightly different from the well-known Mellin convolution studied in [17]. For given functions \( K \in X_0 \) and \( f \in X_0^p \), \( 1 \leq p \leq \infty \), we define
\[
(K * f)(x) := \int_0^\infty K(u) f(ux) \frac{du}{u} \quad (x \in \mathbb{R}^+)
\] (8)
The following result holds

Proposition 1. Let \( K \in X_0 \) and let \( f \in X_0^p \), with \( 1 \leq p \leq \infty \). Then \( K * f \) is well defined and \( K * f \in X_0^p \). Moreover
\[
\| K * f \|_{X_0^p} \leq \| K \|_{X_0} \| f \|_{X_0^p}.
\]

Proof. Using the change of variable \( ux = t \) we can write
\[
\int_0^\infty |K(u)| \| f(ux) \|_{X_0^p}^p \frac{dx}{x} = \int_0^\infty |K(u)| \| f \|_{X_0^p}^p \frac{dx}{x}
\]
and integrating with respect to \( u \) we obtain
\[
\int_0^\infty \int_0^\infty \left\{ \int_0^\infty |K(u)| |f(ux)| \left| \frac{dx}{x} \right| \right\} \frac{du}{u} = \| K \|_{X_0} \| f \|_{X_0^p}^p.
\]
By Fubini’s theorem we immediately deduce that the integral
\[
\int_0^\infty |K(u)| |f(ux)|^p \frac{du}{u}
\]
exists for almost everywhere \( x \in \mathbb{R}^+ \), and so the assertion holds for \( p = 1 \). Let now \( 1 < p < \infty \) and denote by \( p' \) the conjugate exponent of \( p \). By Hölder’s inequality we have

\[
| (K * f)(x) | \leq \| K \|_{X_0}^{1/p'} \left\{ \int_0^\infty |f(ux)|^p |K(u)| \frac{du}{u} \right\}^{1/p}.
\]

Taking the norm in \( X_0^p \) we finally obtain

\[
\| K * f \|_{X_0^p} \leq \| K \|_{X_0}^{(p' + p)/p'} \| f \|_{X_0^p},
\]

and so the assertion follows for \( 1 < p < \infty \). The case \( p = \infty \) is obvious.

**Remark 2.** Note that the above result is not true in general for spaces \( X_0^p \) with \( c \neq 0 \). It is not difficult to show examples. For example, let \( p = 1, c = 1, K(x) = \chi_{[0,1]}(x) \) and \( f(x) = x \chi_{[0,1]}(x) \). Then we have \( K, f \in X_1 \) but \( K * f \notin X_1 \).

Using convolutions, we can establish a link between our Durrmeyer exponential sampling type operators and the usual ones. Indeed we have

\[
(T\varphi \psi f)(x) = \sum_{k=-\infty}^{\infty} \varphi(e^{-k}x)(\psi * f)(e^k) \quad (x \in \mathbb{R}^+).
\]

We have the following lemma, which will be useful in order to obtain a further estimate for spaces \( X_0^p \). We recall here that \( \ell^p(\mathbb{Z}) \) denotes the sequence space comprising all sequences \((a_k)_{k \in \mathbb{Z}}\) such that

\[
\sum_{k=-\infty}^{\infty} |a_k|^p < \infty.
\]

**Lemma 5.1.** Let \( \psi \in \Psi \) and \( f \in X_0^p \). Then \( (\psi * f)(e^k) \) \( k \in \mathbb{Z} \) belongs to \( \ell^p(\mathbb{Z}) \).

**Proof.** The proof follows essentially the same method employed in Theorem 4.1, using the partial sums of the series involved. Here we give a less rigorous proof which is justified by the previous one. Using suitable changes of variable, we have

\[
\sum_{k=-\infty}^{\infty} |(\psi * f)(e^k)|^p = \sum_{k=-\infty}^{\infty} \left| \int_0^\infty \psi(e^{-k}u) f(u) \frac{du}{u} \right|^p = \sum_{k=-\infty}^{\infty} \left| \int_0^\infty \psi(t) f(e^k t) \frac{dt}{t} \right|^p.
\]

Thus, Jensen’s inequality yields

\[
\sum_{k=-\infty}^{\infty} |(\psi * f)(e^k)|^p \leq \widehat{M}_0(\psi)^{p-1} \sum_{k=-\infty}^{\infty} \int_0^\infty |\psi(t)||f(e^k t)|^p \frac{dt}{t}
\]

\[
= \widehat{M}_0(\psi)^{p-1} \sum_{k=-\infty}^{\infty} \int_0^\infty |\psi(e^{-k}u)||f(u)|^p \frac{du}{u}
\]

\[
\leq \widehat{M}_0(\psi)^{p-1} \left\| f \right\|_{X_0^p}^p,
\]

and the assertion follows. \( \square \)
Lemma 5.1 in conjunction with Theorem 3.3 in [14] yields the following corollary, in which we set
\[
|\langle \psi * f, e^k \rangle|_{\ell^p} := \left\{ \sum_{k=-\infty}^{\infty} |\langle \psi * f, e^k \rangle|^p \right\}^{1/p}.
\]

**Corollary 2.** Let \( \varphi \in \Phi \cap X_0 \), \( \psi \in \Psi \), and assume that \( M_0(\varphi) < \infty \). If \( f \in X_0^p \) then
\[
||T^{\varphi, \psi} f||_{X_0} \leq M_0(\varphi)^{p-1}/p \| \psi \|_{X_0}^{1/p} ||\psi * f||_{\ell^p}.
\]

As a final step, using the proof of Lemma 5.1, (10) implies the following estimate in spaces \( X_0^p \).

**Theorem 5.2.** Let \( \varphi \in \Phi \cap X_0 \), \( \psi \in \Psi \), and assume that \( M_0(\varphi) < \infty \). If \( f \in X_0^p \) then
\[
||T^{\varphi, \psi} f||_{X_0} \leq \Delta_0(\psi)^{(p-1)/p} M_0(\psi)^{1/p} M_0(\varphi)^{(p-1)/p} ||\varphi||_{X_0}^{1/p} \| f \|_{X_0^p}.
\]

6. **Some examples.** In this last section, we will give some examples of kernel functions which satisfy the assumptions employed in the previous sections.

1. **Mellin splines.**

   An important class of kernel functions in \( \Phi \) and \( \Psi \) is given by the so-called Mellin splines, defined, for any \( n \in \mathbb{N} \), by (see [8]; for Fourier case see [22, 7, 9])
   \[
   B_n(x) := \frac{1}{(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left( \frac{n}{2} + \log x - j \right)^{n-1} (x \in \mathbb{R}^+),
   \]
   where we use the notation \( r_+ := (r + |r|)/2 \) for any \( r \in \mathbb{R} \). Then \( B_n \in \Phi \cap \Psi \cap X_0^p \) for every \( c \in \mathbb{R} \) and \( p \in [1, \infty] \).

   A particular case is the second order Mellin spline \( n = 2 \) defined by
   \[
   B_2(x) := (1 - |\log x|)_+ = \begin{cases} 
   1 - \log x & \text{if } 1 \leq x < e, \\
   1 + \log x & \text{if } e^{-1} < x < 1, \\
   0 & \text{otherwise.}
   \end{cases}
   \]

   We show now that \( B_2 \) satisfies also condition (2). We consider here the case \( c > 0 \). The general case is treated analogously. We have
   \[
   H = \sup_{u \in \mathbb{R}^+} \sum_{k=-\infty}^{\infty} \left( \frac{e^k}{u} \right)^{pc} B_2(e^{-k} u) \leq e^{pc} \sup_{u \in \mathbb{R}^+} \sum_{k=-\infty}^{\infty} B_2(e^{-k} u) = e^{pc} M_0(B_2).
   \]

   The same holds for every \( n \in \mathbb{N} \). Other examples of kernels with compact support for which the previous theory is satisfied, can be defined through certain linear combinations of splines and/or translates of splines (see [22, 7, 10] for the Fourier case).

   Thus, we can apply the theory developed in the previous sections taking the kernels \( \varphi \) and \( \psi \) as Mellin splines of different orders.
2. Mellin–Fejer kernel

It is defined by

\[ F_\rho(x) := \frac{\rho}{2\pi} \sin^2\left(\frac{\rho}{2\pi} \log x\right) \]  
\( \rho > 0, \ x \in \mathbb{R}^+ \).

It is not difficult to show that \( F_\rho \in \Phi \cap \Psi \cap X^p_0 \) for any \( p \in [1, \infty] \) (see [8]). Thus the theory developed in Section 5 can be applied, taking the functions \( \varphi \) and \( \psi \) as Mellin–Fejer kernels with different indices \( \rho > 0 \).

3. Mellin–Jackson kernel

It is defined, in a generalized form, by (see [11], and [15] for the Fourier version)

\[ J_{\gamma, \beta}(x) := d_{\gamma, \beta} \sin^{2\beta}\left(\frac{\log x}{2\gamma \beta \pi}\right), \]

where \( x \in \mathbb{R}^+, \ \beta \in \mathbb{N}, \ \gamma \geq 1 \) and \( d_{\gamma, \beta} \) is a normalization constant, that is

\[ d_{\gamma, \beta}^{-1} := \int_0^\infty \sin^{2\beta}\left(\frac{\log u}{2\gamma \beta \pi}\right) \frac{du}{u}. \]

We have \( J_{\gamma, \beta} \in \Phi \cap \Psi \cap X^p_0 \) for any \( p \in [1, \infty] \).

4. Other interesting examples can be obtained using the so-called “multiplier technique” (see [38] for the Fourier case). In Mellin setting these examples were described and studied in [14], using the Mellin transform theory.

5. Mixed form of Durrmeyer sampling operators can be obtained taking two different kernels \( \varphi \) and \( \psi \) chosen among the previous examples.

Acknowledgments. We wish to thank very much the referees for their valuable suggestions which improved the presentation of the paper.

REFERENCES

[1] L. Angeloni, D. Costarelli and G. Vinti, A characterization of the convergence in variation for the generalized sampling series, Ann. Acad. Sci. Fenn. Math., 43 (2018), 755–767.
[2] L. Angeloni, D. Costarelli and G. Vinti, Convergence in variation for the multidimensional generalized sampling series and applications to smoothing for digital image processing, Ann. Acad. Sci. Fenn. Math., 45 (2020), 751–770.
[3] F. Asdrubali, G. Baldinelli, F. Bianchi, D. Costarelli, A. Rotili, M. Seracini and G. Vinti, Detection of thermal bridges from thermographic images by means of image processing approximation algorithms, Appl. Math. Comput., 317 (2018), 160–171.
[4] S. Balsamo and I. Mantellini, On linear combinations of general exponential sampling series, Results Math., 74 (2019), Paper No. 180, 19 pp.
[5] C. Bardaro, P. L. Butzer and I. Mantellini, The exponential sampling theorem of signal analysis and the reproduction kernel formula in the Mellin transform setting, Sampl. Theory Signal Image Process., 13 (2014), 35–66.
[6] C. Bardaro, P. L. Butzer and I. Mantellini, The Mellin-Parseval formula and its interconnections with the exponential sampling theorem of optical physics, Integral Transforms and Special Functions, 27 (2016), 17–29.
[7] C. Bardaro, P. L. Butzer, R. L. Stens and G. Vinti, Prediction by samples from the past with error estimates covering discontinuous signals, IEEE Trans. Information Theory, 56 (2010), 614–633.
[8] C. Bardaro, L.Faina and I. Mantellini, A generalization of the exponential sampling series and its approximation properties, Math. Slovaca., 67 (2017), 1481–1496.
[9] C. Bardaro, L. Faina and I. Mantellini, Quantitative Voronovskaja formulae for generalized Durrmeyer sampling type series, Math. Nachr., 289 (2016), 1702–1720.
[10] C. Bardaro and I. Mantellini, A quantitative Voronovskaja formula for generalized sampling operators, East J. Approx., 15 (2009), 459–471.
CARLO BARDARO AND ILARIA MANTELLINI

[11] C. Bardaro and I. Mantellini, Asymptotic formulae for linear combinations of generalized sampling type operators, Z. Anal. Anwend., 32 (2013), 279–298.
[12] C. Bardaro and I. Mantellini, Asymptotic expansion of generalized Durrmeyer sampling type series, Jaen Journal on Approximation, 6 (2014), 143–165.
[13] C. Bardaro and I. Mantellini, On a Durrmeyer type modification of the Exponential sampling series, Rend. Circ. Mat. Palermo (2), 70 (2021), 1289–1304.
[14] C. Bardaro, I. Mantellini and G. Schmeisser, Exponential sampling series: Convergence in Mellin-Lebesgue spaces, Results Math., 74 (2019), Paper No. 119, 20 pp.
[15] C. Bardaro, G. Vinti, P. L. Butzer and R. L. Stens, Kantorovich-type generalized sampling series in the setting of Orlicz spaces, Sampling Theory Signal Image Processing, 6 (2007), 29–52.
[16] M. Bertero and E. R. Pike, Exponential sampling method for Laplace and other dilationally invariant transforms: I. Singular-system analysis, II. Examples in photon correction spectroscopy and Frauenhofer diffraction, Inverse Problems, 7 (1991), 1–20, 21–41.
[17] P. L. Butzer and S. Jansche, A direct approach to the Mellin transform, J. Fourier Anal. Appl., 3 (1997), 325–376.
[18] P. L. Butzer and S. Jansche, The exponential sampling theorem of signal analysis, Atti Sem. Mat. Fis. Univ. Modena, Suppl., (special issue dedicated to Professor Calogero Vinti), 46 (1998), 99–122.
[19] P. L. Butzer and S. Jansche, A self-contained approach to Mellin transform analysis for square integrable functions; applications, Integral Transform. Spec. Funct., 8 (1999), 175–198.
[20] P. L. Butzer, G. Schmeisser and R. L. Stens, An introduction to sampling analysis, In: Marvasti, F. (ed.) Nonuniform Sampling, Theory and Practice, 17–121. Kluwer Academic/Plenum Publishers, New York, (2001).
[21] P. L. Butzer, W. Splettstößer and R. L. Stens, The sampling theorem and linear prediction in signal analysis, Jahresber. Deutsch. Math.-Verem., 90 (1988), 1–70.
[22] P. L. Butzer and R. L. Stens, Linear prediction by samples from the past, In: Marks II, R.J. (ed.) Advanced Topics in Shannon Sampling and Interpolation Theory, 157–183. Springer, New York, (1993).
[23] D. Casasent, Optical signal processing, In: Casasent, D. (ed.) Optical Data Processing, 241–282. Springer, Berlin, (1978).
[24] D. Costarelli, A. M. Minotti and G. Vinti, Approximation of discontinuous signals by sampling Kantorovich series, J. Math. Anal. Appl., 450 (2017), 1083–1103.
[25] D. Costarelli, M. Piconi and G. Vinti, On the convergence properties of Durrmeyer-Sampling type operators in Orlicz spaces, to appear, arXiv:2007.02450v1, 2021.
[26] D. Costarelli, M Seracini and G. Vinti, A comparison between the sampling Kantorovich algorithm for digital image processing with some interpolation and quasi-interpolation methods, Appl. Math. Comput., 347, (2020), 125046, 18 pp.
[27] D. Costarelli, M. Seracini and G. Vinti, A segmentation procedure of the previous area of the aorta artery from CT images without contrast medium, Math. Methods Appl. Sci., 43 (2020), 114–133.
[28] D. Costarelli and G. Vinti, An inverse result of approximation by sampling Kantorovich series, Proc. Edinb. Math. Soc., 62 (2019), 265–280.
[29] D. Costarelli and G. Vinti, Saturation by the Fourier transform method for the sampling Kantorovich series based on bandlimited kernels, Anal. Math. Phys., 9 (2019), 2263–2280.
[30] J. R. Higgins, Sampling Theory in Fourier and Signal Analysis, Foundations. Oxford Univ. Press, Oxford, 1996.
[31] A. Kivinukk and G. Tamberg, Interpolating generalized Shannon sampling operators, their norms and approximation properties, Sampl. Theory Signal Image Process., 8 (2009), 77–95.
[32] A. Kivinukk and G. Tamberg, On window methods in generalized Shannon sampling operators. In New perspectives on approximation and sampling theory, 63–85, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham, (2014).
[33] A. S. Kumar and S. Bajpeyi, Direct and inverse results for Kantorovich type exponential sampling series, Results Math., 75 (2020), Paper No. 119, 17 pp.
[34] A. S. Kumar and D. Ponnaian, Approximation by generalized bivariate Kantorovich sampling type series, J. Anal., 27 (2019), 429–449.
[35] A. S. Kumar and B. Shivam, Inverse approximation and GBS of bivariate Kantorovich type sampling series, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, 114 (2020), Paper No. 82, 15 pp.
[36] N. Ostrowsky, D. Sornette, P. Parker and E. R. Pike, Exponential sampling method for light scattering polydispersity analysis, Opt. Acta, 28 (1981), 1059–1070.

[37] S. Ries and R. L. Stens, Approximation by generalized sampling series, In: Sendov, Bl., Petrushev, P., Maalev, R., Tashev, S. (eds.) Constructive Theory of Functions, pp. 746–756. Pugl. House Bulgarian Academy of Sciences, Sofia, (1984).

[38] G. Schmeisser, Interconnections between the multiplier methods and the window methods in generalized sampling, Sampl. Theory Signal Image Process., 9 (2010), 1–24.

[39] A. I. Zayed, Advances in Shannon’s Sampling Theory, CRC Press, Boca Raton, 1993.

Received August 2021; revised October 2021; early access November 2021.

E-mail address: carlo.bardaro@unipg.it
E-mail address: ilaria.mantellini@unipg.it