PROOF OF A RATIONAL RAMANUJAN-TYPE SERIES FOR $1/\pi$.
THE FASTEST ONE IN LEVEL 3

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To Bruce Berndt, in admiration of his inspirational work on Ramanujan’s Notebooks

Abstract. Using a modular equation of level 3 and degree 23 due to Chan and Liaw, we prove the fastest convergent rational Ramanujan-type series for $1/\pi$ of level 3.

1. The method

A prerequisite to understand this paper well is familiarity with the notation and the method developed in [9]. This method is based on an original idea of Wan [10], whose paper was in turn influenced by some ideas of [8]. Using the notation

$$F_s(\alpha) = \frac{1}{s-1} \binom{1}{s} \binom{1-\frac{1}{s}}{1} \alpha, \quad s \in \{6, 4, 3, 2\}, \quad \ell = 4 \sin^2 \frac{\pi}{s},$$

we proved in [9] the following result:

**Theorem 1.** Let

$$F_s(\alpha) = m(\alpha, \beta)F_s(\beta), \quad A(\alpha, \beta) = 0,$$

be a transformation of modular origin and degree 1/d, and let $\beta_0 = 1 - \alpha_0$ be a solution of $A(\alpha_0, \beta_0) = 0$, and $m_0 = m(\alpha_0, \beta_0)$. Then, if

$$m_0 = \frac{1}{\sqrt{d}}, \quad \text{or} \quad m_0 = \frac{\sqrt{4d-\ell}}{2d} + \frac{\ell}{2d} i,$$

we have

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{n}\right)_a \left(\frac{1}{n}\right)_{a-\frac{1}{s}} (1-\frac{1}{s})_n}{(1)_n^3} (a+bn) z^n = \frac{1}{\pi},$$

where

(1) \quad $z = 4\alpha_0 \beta_0$, \quad $b = 2(1-2\alpha_0) \sqrt{\frac{d}{\ell}}$, \quad $a = -2\alpha_0 \beta_0 \frac{m_0'}{\alpha_0} \frac{d}{\sqrt{\ell}}$

or

(2) \quad $z = 4\alpha_0 \beta_0$, \quad $b = 2(1-2\alpha_0) \sqrt{\frac{d}{\ell} - \frac{1}{4}}$, \quad $a = -2\alpha_0 \beta_0 \frac{m_0'}{\alpha_0} \frac{d}{\sqrt{\ell}}$

respectively. The $'$ means differentiation with respect to the variable that we choose as independent.
2. THE FORMULA $(A, 3, 23)$

In this paper we prove the Ramanujan-type formula

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^3} \left(14151n + 827\right) \frac{(-1)^n}{500^{2n}} = \frac{1500\sqrt{3}}{\pi}.
\]

It is the formula $(A, 3, 23)$ in the notation of [9], we will also refer to it as $3_{A23}$, and is the fastest convergent rational Ramanujan-type series for $1/\pi$ of level $\ell = 3$. It was discovered by Chan, Liaw and Tan [5, eq. 1.19]. As $z_0 = 4\alpha_0\beta_0$ where $\beta_0 = 1 - \alpha_0$, we get

\[
\alpha_0 = \frac{1}{2} - \frac{53\sqrt{89}}{1000}, \quad \beta_0 = 1 - \alpha_0 = \frac{1}{2} + \frac{53\sqrt{89}}{1000}.
\]

Then, with a numerical approximation of 20 digits, we have

\[
m_0 = \frac{F_3(\alpha_0)}{F_3(\beta_0)} \approx 0.20508654634905660459 + 0.037653278425410375946i,
\]

which we identify as

\[
m_0 = \frac{\sqrt{89}}{46} + \frac{\sqrt{3}}{46}i.
\]

Observe that $m_0$ is of the form

\[
m_0 = \frac{\sqrt{4d - \ell}}{2d} + \frac{\sqrt{\ell}}{2d}i, \quad |m_0| = \frac{1}{\sqrt{d}}
\]

with $d = 23$. Hence, for proving (3) with our method, we need a transformation of degree $1/d$ with $d = 23$ for the level $\ell = 3$, and with such a transformation we can prove it rigorously. This is done in next section using a modular equation in which the algebraic relation $A(\alpha, \beta) = 0$ is written using two auxiliary variables $u$ and $v$, in the following way:

\[
u^k = \alpha\beta, \quad v^k = (1 - \alpha)(1 - \beta), \quad P(u, v) = 0,
\]

where $P(u, v)$ is a polynomial in $u$ and $v$, and $k$ is a positive integer.

3. THE PROOF OF $(A, 3, 23)$

We use a modular equation of level 3 and degree 23 due to Heng Huat Chan and Wen-Chin Liaw [4, Corollary 3.7], in which we have replaced $u$ and $v$ with $u^2$ and $v^2$ respectively. It is

\[
u^{12} = \alpha\beta, \quad v^{12} = (1 - \alpha)(1 - \beta), \quad P(u, v) = 0,
\]

where

\[
P(u, v) = (u^8 + v^8) - 12\sqrt{3}(u^7v + uv^7) - 87(u^6v^2 + u^2v^6) - 84\sqrt{3}(u^5v^3 + u^3v^5)
- 160(u^4v^4) - 2(u^4 + v^4) - 15\sqrt{3}(u^3v + uv^3) - 48(u^2v^2) + 1.
\]
If we let $\beta = 1 - \alpha$, then we see that $u^{12} = v^{12}$. If we choose

$$v = \left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right) u,$$

and replace it in $P(u, v) = 0$, we have the equation

$$250(1 + \sqrt{3} i)u^8 - \frac{95}{2}(1 - \sqrt{3} i)u^4 + 1 = 0,$$

which factors as

$$\frac{1 + \sqrt{3} i}{16}(10u^4 + 1 + \sqrt{3} i)(20u^2 - \sqrt{3} - i)(20u^2 + \sqrt{3} + i) = 0.$$

One solution is

$$u_0 = \frac{\sqrt{15} + \sqrt{5}}{20} + \frac{\sqrt{15} - \sqrt{5}}{20} i, \quad v_0 = \frac{\sqrt{15} + \sqrt{5}}{20} - \frac{\sqrt{15} - \sqrt{5}}{20} i.$$

Then, from

$$\alpha_0(1 - \alpha_0) = u_0^{12} = -10^{-6}, \quad \beta_0 = 1 - \alpha_0,$$

we get

$$\alpha_0 = \frac{1}{2} - \frac{53}{1000} \sqrt{89}, \quad \beta_0 = \frac{1}{2} + \frac{53}{1000} \sqrt{89}.$$

Differentiating $P(u, v) = 0$ with respect to $u$ at $u = u_0$, we find

$$v'_0 = \frac{-294573 \sqrt{3} + 82573 i}{516854}.$$

Then, differentiating $P(u, v) = 0$ twice with respect to $u$ at $u = u_0$, we get

$$v''_0 = \frac{8674041040500000 \sqrt{5}(1 - i) + 3034180783431000 \sqrt{15}(1 + i)}{17258921684500483}.$$

From (5), we see that

$$u^{12} = \alpha \beta, \quad u^{12} - v^{12} + 1 = \alpha + \beta.$$

Differentiating (10) with respect to $u$ at $u = u_0$, we obtain $\alpha'_0$ and $\beta'_0$. Then, differentiating (10) twice with respect to $u$ at $u = u_0$ we obtain $\alpha''_0$ and $\beta''_0$. As the multiplier is given by

$$m^2 = \frac{1}{d \alpha^2 (1 - \alpha)} \frac{\beta'}{\alpha'},$$

see [9], replacing the already known values at $u = u_0$, we get

$$m_0 = \sqrt{\frac{1}{23} \frac{\alpha'_0}{\beta'_0}} = \frac{\sqrt{89}}{46} + \frac{\sqrt{3} i}{46} = \frac{\sqrt{4d - \ell}}{2d} + \frac{\sqrt{\ell}}{2d} i.$$

Taking logarithms in (10), differentiating with respect to $u$, and dividing by $\alpha'$, we get

$$\frac{m'}{\alpha'} = \frac{m}{2\alpha'} \left(\frac{\beta'}{\beta} - \frac{\beta'}{1 - \beta} - \frac{\alpha'}{\alpha} + \frac{\alpha'}{1 - \alpha} + \frac{\alpha''}{\alpha'} - \frac{\beta''}{\beta'}\right).$$
and from it, we obtain

\[ \frac{m'_0}{\alpha'_0} = \frac{827000}{69}. \]

Finally, using the formulas

\[ z = 4\alpha_0\beta_0, \quad b = 2(1-2\alpha_0)\sqrt{\frac{d}{\ell} - \frac{1}{4}}, \quad a = -2\alpha_0\beta_0 \frac{m'_0}{\alpha'_0} \frac{d}{\sqrt{\ell}}, \]

we obtain

\[ z = -\frac{1}{500^2}, \quad b = \frac{4717}{1500} \sqrt{3}, \quad a = \frac{827}{4500} \sqrt{3}, \]

and we are done.

4. **Ramanujan-type series for 1/\pi and modular equations**

In a completely similar way we can prove other Ramanujan-type series for 1/\pi using modular equations [1, 2, 3, 4], written in the form (4). For example, for proving (A, 2, 7), we can use the modular equation

\[ u^8 = \alpha\beta, \quad v^8 = (1-\alpha)(1-\beta), \quad P(u, v) = 0, \]

where

\[ P(u, v) = (u^4 + v^4) + 8\sqrt{2}(u^3v + uv^3) + 20u^2v^2 - 1, \]

see [3, 2, Theorem 10.3]. For proving (A, 3, 11), we can use

\[ u^{12} = \alpha\beta, \quad v^{12} = (1-\alpha)(1-\beta), \quad P(u, v) = 0, \]

where

\[ P(u, v) = (u^4 + v^4) + 3\sqrt{3}(u^3v + uv^3) + 6u^2v^2 - 1. \]

see [3, 2, Theorem 7.8], and we can get the proofs of the formulas (A, 3, 5) and (P, 3, 5), with the modular equation

\[ u^6 = \alpha\beta, \quad v^6 = (1-\alpha)(1-\beta), \quad u^2 + v^2 + 3uv - 1 = 0. \]

see [3, 2, Theorem 7.6]. Note that all the above examples correspond to rational series. Of course one can use the same method to prove irrational instances.

5. **Conclusion**

As we pointed out in [9], the main aspect of our method is that the modular equations used to prove the Ramanujan-type alternating series for 1/\pi have a much lower degree than those in other methods. It would be interesting to analyze in depth this phenomenon. The reader is invited to compare the values of \(d\) in the tables in [9] with the respective values of \(N\) given in the tables in [7]. We hope that the method developed here will attract others to find, for example, new modular equations of levels \(\ell \geq 5\) and apply them to prove non-hypergeometric series of Ramanujan-Sato type for 1/\pi [6, 7]. Of course a generalization of the method would be needed for that.
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