A Slight Improvement to the Colored Bárány’s Theorem

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Abstract

Suppose \(d + 1\) absolutely continuous probability measures \(m_0, \ldots, m_d\) on \(\mathbb{R}^d\) are given. In this paper, we prove that there exists a point of \(\mathbb{R}^d\) that belongs to the convex hull of \(d + 1\) points \(v_0, \ldots, v_d\) with probability at least \(\frac{2d}{(d+1)(d+1)!}\), where each point \(v_i\) is sampled independently according to probability measure \(m_i\).

1 Introduction

Let \(P \subset \mathbb{R}^d\) be a set of \(n\) points. Every \(d + 1\) of them span a simplex, for a total of \(\binom{n}{d+1}\) simplices. The point selection problem asks for a point contained in as many simplices as possible. Boros and Füredi [BF84] showed for \(d = 2\) that there always exists a point in \(\mathbb{R}^2\) contained in at least \(\frac{2}{3} \binom{n}{3} - O(n^2)\) simplices. A short and clever proof of this result was given by Bukh [Buk06]. Bárany [Bár82] generalized this result to higher dimensions:

**Theorem 1 (Bárany [Bár82]).** There exists a point in \(\mathbb{R}^d\) that is contained in at least \(c_d \binom{n}{d+1} - O(n^d)\) simplices, where \(c_d > 0\) is a constant depending only on the dimension \(d\).

This general result, the Bárány’s theorem, is also known as the first selection lemma. We will henceforth denote by \(c_d\) the largest possible constant for which the Bárány’s theorem holds true. Bukh, Matoušek and Nivasch [BMN10] used a specific construction called the stretched grid to prove that the constant \(c_2 = \frac{2}{3}\) in the planar case found by Boros and Füredi [BF84] is the best possible. In fact, they proved that \(c_d \leq \frac{d^2}{(d+1)!}\). On the other hand, Bárány’s proof in [Bár82] implies that \(c_d \geq (d+1)^{-d}\), and Wagner [Wag03] improved it to \(c_d \geq \frac{d^2+1}{(d+1)(d+1)!}\).

Gromov [Gro10] further improved the lower bound on \(c_d\) by topological means. His method gives \(c_d \geq \frac{2d}{(d+1)(d+1)!}\). Matoušek and Wagner [MW11] provided an exposition of the combinatorial component of Gromov’s approach in a combinatorial language, while Karasev [Kar12] found a very elegant proof of Gromov’s bound, which he described as a “decoded and refined” version of Gromov’s proof.

The exact value of \(c_d\) has been the subject of ongoing research and is unknown, except for the planar case. Basit, Mustafa, Ray and Raza [BMRR10] and successively Matoušek and Wagner [MW11]...
improved the Bárány’s theorem in $\mathbb{R}^3$. Král’, Mach and Sereni [KMS12] used flag algebras from extremal combinatorics and managed to further improve the lower bound on $c_3$ to more than 0.07480, whereas the best upper bound known is 0.09375.

However, in this paper, we are concerned with a colored variant of the point selection problem. Let $P_0, \ldots, P_d$ be $d + 1$ disjoint finite sets in $\mathbb{R}^d$. A colorful simplex is the convex hull of $d + 1$ points each of which comes from a distinct $P_i$. For the colored point selection problem, we are concerned with the point(s) contained in many colorful simplices. Karasev proved:

**Theorem 2** (Karasev [Kar12]). Given a family of $d + 1$ absolutely continuous probability measures $m = (m_0, \ldots, m_d)$ on $\mathbb{R}^d$, an $m$-simplex is the convex hull of $d + 1$ points $v_0, \ldots, v_d$ with each point $v_i$ sampled independently according to probability measure $m_i$. There exists a point of $\mathbb{R}^d$ that is contained in an $m$-simplex with probability $p_d \geq \frac{1}{(d+1)!}$. In addition, if two probability measures coincide, then the probability can be improved to $p_d \geq \frac{2d}{(d+1)(d+1)!}$.

By a standard argument which we will provide immediately, a result on the colored point selection problem follows:

**Corollary 3.** If $P_0, \ldots, P_d$ each contains $n$ points, then there exists a point that is contained in at least $\frac{1}{(d+1)!} \cdot n^{d+1}$ colorful simplices.

Our result drops the additional assumption in theorem 2 hence improves corollary 3:

**Main Theorem.** There is a point in $\mathbb{R}^d$ that belongs to an $m$-simplex with probability $p_d \geq \frac{2d}{(d+1)(d+1)!}$.

**Corollary 4.** There exists a point that is contained in at least $\frac{2d}{(d+1)(d+1)!} \cdot n^{d+1}$ colorful simplices.

Figure 1: 3 red points, 3 green points and 3 blue points are placed in the plane. The point marked by a square is contained in 6 ($= \frac{2}{3} \cdot 3^3$) colorful triangles.

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1 An $m$-simplex is actually a simplex-valued random variable.
Proof of corollary 4 from the main theorem. Given \( d + 1 \) sets \( P_0, \ldots, P_d \) in \( \mathbb{R}^d \) each of which contains \( n \) points. Let \( \Psi: \mathbb{R}^d \to \mathbb{R} \) be the bump function defined by \( \Psi(x_1, \ldots, x_d) = \prod_{i=1}^{d} e^{-1/(1-x^2)}1_{|x|<1} \), and set \( \Psi_n(x_1, \ldots, x_d) = n^d \Psi(nx_1, \ldots, nx_d) \) for \( n \in \mathbb{N} \). It is a standard fact that \( \Psi \) and \( \Psi_n \) are absolutely continuous probability measures supported on \([-1,1]^d\) and \([-1/n,1/n]^d\) respectively.

For each \( n \in \mathbb{N} \) and \( 0 \leq k \leq d \), define \( m_k^{(n)}(x) := \frac{1}{n} \sum_{p \in P_k} \Psi_n(x-p) \) for \( x \in \mathbb{R}^d \). Note that \( m_k^{(n)} \) is an absolutely continuous probability measure supported on the Minkowski sum of \( P_k \) and \([-1/n,1/n]^d\). Let \( m^{(n)} \) be the family of \( d + 1 \) probability measures \( m_0^{(n)}, \ldots, m_d^{(n)} \). By the main theorem, there is a point \( p^{(n)} \) of \( \mathbb{R}^d \) that belongs to an \( m^{(n)} \)-simplex with probability at least \( \frac{2d}{(d+1)(d+1)!} \).

Because no point in a certain neighborhood of infinity is contained in any \( m^{(n)} \)-simplex, the set \( \{p^{(n)}: n \in \mathbb{N}\} \) is bounded, and consequently the set has a limit point \( p \). Suppose \( p \) is contained in \( N \) colorfull simplices. Let \( \epsilon > 0 \) be the distance from \( p \) to all the colorful simplices that do not contain \( p \). Choose \( n \) large enough such that \( 1/n \ll \epsilon \) and \( |p^{(n)}-p| \ll \epsilon \). By the choice of \( n \), if \( p \) is not contained in a colorful simplex spanned by \( v_0, \ldots, v_d \), then \( p^{(n)} \) is not contained in the convex hull of \( v_0', \ldots, v_d' \) for all \( v_i' \in v_i + [-1/n,1/n]^d \). This implies that the probability that \( p^{(n)} \) is contained in an \( m^{(n)} \)-simplex is at most \( \frac{N}{n^{d+1}} \). Hence \( p \) is the desired point contained in \( N \geq \frac{2d}{(d+1)(d+1)!} \cdot n^{d+1} \) colorfull simplices.

Readers who are familiar with Karasev’s work [Kar12] would notice that our proof of the main theorem heavily relies on his arguments. The author is deeply in debt to him.

2 Proof of the Main Theorem

In this section, we provide the proof of the main theorem. The topological terms in the proof are standard, and can be found in [Mat03]. In addition to the notion of an \( m \)-simplex, in the proof, we will often refer to an \((m_k, \ldots, m_d)\)-face which means the convex hull of \( d-k+1 \) points \( v_k, \ldots, v_d \) with each point \( v_i \) sampled independently according to probability measure \( m_i \). An \( m \)-simplex and an \((m_k, \ldots, m_d)\)-face are both set-valued random variables.

Proof of the main theorem. To obtain a contradiction, we suppose that for any point \( v \) in \( \mathbb{R}^d \), the probability that \( v \) belongs to an \( m \)-simplex is less than \( p_d := \frac{2d}{(d+1)(d+1)!} \). Since this probability, as a function of point \( v \), is continuous and uniformly tends to 0 as \( v \) goes to infinity, there is an \( \epsilon > 0 \) such that \( v \) is contained in an \( m \)-simplex with probability at most \( p_d - \epsilon \) for all \( v \) in \( \mathbb{R}^d \).

Let \( S^d := \mathbb{R}^d \cup \{\infty\} \) be the one-point compactification of the Euclidean space \( \mathbb{R}^d \). Take \( \delta = \epsilon/d \). Choose a finite triangulation \( \mathcal{T} \) of \( S^d \) with one of the \( d \)-simplices containing \( \infty \) such that for \( 0 < k \leq d \), any \( k \)-face of \( \mathcal{T} \) intersects an \((m_k, \ldots, m_d)\)-face with probability less than \( \delta \) and that the measure of

\footnote{A triangulation \( \mathcal{T} \) of a topological space \( X \) is a simplicial complex \( K \), homeomorphic to \( X \), together with a homeomorphism \( h: ||K|| \to X \). Since the finite triangulation of interest is an extension of the triangulation of a \( d \)-simplex \( X \) in \( \mathbb{R}^d \) and \( h \) is an identity map, we will freely use topological notions such as “a \( k \)-face (as a subset of \( S^d \)” instead of “the image of a \( k \)-face in \( K \) under \( h \).” With such abuse of language, we can avoid going back and forth between the simplicial complex and the topological space.}
any $d$-face of $\mathcal{T}$ under $(m_{d-1} + m_d)/2$ is less than $\delta$. This can be done by taking a sufficiently fine triangulation of $S^2$ with one $d$-simplex having $\infty$ in its relative interior.

We use cone$(\cdot)$ as the cone functor with apex $O$. A triangulation $\mathcal{T}$ of $S^d$ naturally extends to a triangulation cone$(\mathcal{T})$ of cone$(S^d)$. We denote the $k$-skeleton of $\mathcal{T}$ and cone$(\mathcal{T})$ by $\mathcal{T} \leq k$ and cone$(\mathcal{T}) \leq k$ respectively.

We are going to define a continuous map $f$: cone$(\mathcal{T}) \leq d \to S^d$. Put $f(x) = x$ for all $x \in S^d = \|\mathcal{T}\| \subset \|\text{cone}(\mathcal{T}) \leq d\|$, and set $f(O) = \infty$. We proceed to define $f$ on cone$(\sigma)$ for all the $k$-faces $\sigma$ of $\mathcal{T}$ inductively on dimension $k$ of $\sigma$ while we maintain the property that the image of the boundary of cone$(\sigma)$ under $f$, that is $f(\partial\text{cone}(\sigma))$, intersects an $(m_0,\ldots,m_d)$-face with probability at most $(k+1)! (p_d - \epsilon + k\delta)$. We say $f$ is economical over a $k$-face $\sigma$ of $\mathcal{T} \leq d-1$ if $f$ and $\sigma$ satisfy the above property. Unlike Karasev [Kar12], our inductive construction of $f$ follows the same pattern until $k = d-2$ instead of $d-1$. The main innovation of this proof is a different construction for $k = d-1$, which enables us to remove the additional assumption in theorem 2.

Note that for any 0-face $\sigma$ in $\mathcal{T}$, $f(\partial\text{cone}(\sigma)) = f(\{\sigma, O\}) = \{\sigma, \infty\}$. According to the assumption at the beginning of the proof, $f(\partial\text{cone}(\sigma))$ intersects an $(m_0,\ldots,m_d)$-face, that is, an $m$-simplex, with probability at most $p_d - \epsilon$. Therefore $f$ is economical over 0-faces of $\mathcal{T}$. This finishes the first step.

Suppose $f$ is already defined on cone$(\mathcal{T}) \leq k$ and it is economical over $k$-faces of $\mathcal{T}$. We are going to extend the domain of $f$ to cone$(\mathcal{T}) \leq k+1$. Indeed, we only need to define $f$ on cone$(\sigma)$ for every $k$-face $\sigma$ of $\mathcal{T}$.

Take any $k$-face $\sigma$ of $\mathcal{T}$. Suppose convex hull of $v_k,\ldots,v_d$, denoted by conv($v_k,\ldots,v_d$), is an

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4The cone over a space $X$ is the quotient space cone$(X) := (X \times [0,1]) / (X \times \{1\})$. The apex is the equivalence class $\{(x,1): x \in X\}$.

4The $k$-skeleton of a simplicial complex $\Delta$ consists of all simplices of $\Delta$ of dimension at most $k$. 

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Figure 2: The bird’s-eye view of a triangulation of $S^2$ with a 2-simplex containing $\infty$ and the cone over part of the triangulation.
(m_k, \ldots, m_d)-face. Notice that the following statements are equivalent:

- \( f(\partial\text{cone}(\sigma)) \) intersects \( \text{conv}(v_k, \ldots, v_d) \);
- for some \( v \in f(\partial\text{cone}(\sigma)) \), the ray with initial point \( v \) in the direction \( \overrightarrow{vk} \) intersects \( \text{conv}(v_{k+1}, \ldots, v_d) \).

We call the union of such rays the shadow of \( f(\partial\text{cone}(\sigma)) \) centered at \( v_k \). Since \( f \) is economical over \( \sigma \), the probability for an \( (m_k, \ldots, m_d) \)-face to meet \( f(\partial\text{cone}(\sigma)) \) is at most \((k+1)! (p_d - \epsilon + k\delta)\), and so there exists \( v^*_k \in \mathbb{R}^d \) such that the shadow of \( f(\partial\text{cone}(\sigma)) \) centered at \( v^*_k \) intersects \( \text{conv}(v_{k+1}, \ldots, v_d) \) with probability at most \((k+1)! (p_d - \epsilon + k\delta)\).

Now, we define \( f \) on \( \text{cone}(\sigma) \). First, let \( g \) be the homeomorphism from \( \text{cone}(\sigma) \) onto the cone over \( \partial\text{cone}(\sigma) \) with apex \( c \) such that \( g \) is an identity on \( \partial\text{cone}(\sigma) \). This can be done because \( \text{cone}(\sigma) \) is homeomorphic to a \((k+1)\)-simplex \( \Delta \) and it is easy to find a homeomorphism from \( \Delta \) to \( \text{cone}(\partial\Delta) \) that keeps \( \partial\Delta \) fixed.

![Figure 3](image3.png)

**Figure 3:** An illustration of an 1-simplex \( \Delta, \partial\Delta, \text{cone}(\partial\Delta) \) and a homeomorphism from \( \Delta \) to \( \text{cone}(\partial\Delta) \).

Next, note that every point \( w \) in \( \text{cone}(\sigma) \) except \( c \) is on a line segment \([v, c]\) for a unique point \( v \) on \( \partial\text{cone}(\sigma) \). If \( t = \frac{vw}{wc} \in [0, \infty) \), then put \( h(w) = f(v) + t \cdot \overrightarrow{v_k} f(v) \). In addition, set \( h(c) = \infty \). The function \( h \) maps \([v, c]\) onto \([f(v), v^*_k]\) linearly and then takes the inversion centered at \( v^*_k \) with radius \( v^*_k f(v) \) so that \([f(v), v^*_k]\) gets mapped onto the ray with the initial point \( f(v) \) in the direction \( \overrightarrow{v_k} f(v) \). Evidently, \( h \) is a continuous map from \( \text{cone}(\partial\text{cone}(\sigma)) \) onto the shadow of \( f(\partial\text{cone}) \) centered at \( v^*_k \) that coincides with \( f \) on \( \partial\text{cone}(\sigma) \).

![Figure 4](image4.png)

**Figure 4:** The illustration shows a cone over part of \( \partial\text{cone}(\sigma) \) with apex \( c \) and a point \( v \) on the boundary, and how a point \( w \) on the line segment \([v, c]\) are mapped under \( h \).
Define $f$ on $\text{cone}(\sigma)$ to be the composition of $g$ and $h$:

$$
\begin{array}{c}
\partial\text{cone}(\sigma) \\
\downarrow \\
\text{cone}(\sigma)
\end{array}
\xrightarrow{g}
\begin{array}{c}
\partial\text{cone}(\sigma) \\
\downarrow \\
\text{cone}(\partial\text{cone}(\sigma))
\end{array}
\xrightarrow{h}
\begin{array}{c}
\text{cone}(\partial\text{cone}(\sigma))
\end{array}
\xrightarrow{f}
\begin{array}{c}
\text{cone}(\partial\text{cone}(\sigma))
\end{array}
\xrightarrow{f}
\begin{array}{c}
\text{cone}(\partial\text{cone}(\sigma))
\end{array}
$$

According to the commutative diagram above, $f$ is well-defined on $\text{cone}(\sigma)$ in the sense that it is compatible with its definition on $\text{cone}(\mathcal{T})^{\leq k}$. We use the phrase “fill in the boundary of $\text{cone}(\sigma)$ against the center $v_k^a$” to represent the above process that extends the domain of $f$ from $\partial\text{cone}(\sigma)$ to $\text{cone}(\sigma)$.

To complete the inductive step, we must demonstrate that $f$ is economical over $(k+1)$-faces of $\mathcal{T}$. Pick any $(k+1)$-face $\tau$ of $\mathcal{T}$. Let $\sigma_0, \ldots, \sigma_{k+1}$ be the $k$-faces of $\tau$. Observing that $f(\partial\text{cone}(\tau)) = f(\tau \cup \text{cone}(\partial\tau)) = \tau \cup f(\text{cone}(\sigma_0)) \cup \ldots \cup f(\text{cone}(\sigma_{k+1}))$ and that $f(\text{cone}(\sigma_i))$ is the shadow of $f(\partial\text{cone}(\sigma_i))$ centered at $v_k^a_i$ which intersects with an $(m_{k+1}, \ldots, m_d)$-face with probability at most $(k+1)!/(p_d - \epsilon + k\delta)$, we obtain that the probability for an $(m_{k+1}, \ldots, m_d)$-face to intersect $f(\partial\text{cone}(\tau))$ is dominated by $\delta + (k+2)(k+1)!/(p_d - \epsilon + (k+1)\delta)$.

We have so far defined a continuous map $f$ on $\text{cone}(\mathcal{T})^{\leq d-1}$ such that for any $(d-1)$-face $\sigma$ of $\mathcal{T}$ the probability for an $(m_{d-1}m_d)$-face to intersect $D := f(\partial\text{cone}(\sigma))$ is at most $d!/(p_d - \epsilon + (d-1)\delta)$. We write $f(X) \mod 2 := \{y \in f(X) : |f^{-1}(y) \cap X| = 1 \pmod 2\}$ for the set of points in $f(X)$ whose fibers in $X$ have an odd number of points. Set $\tilde{m} := (m_{d-1} + m_d)/2$. We are going to define $f$ on $\text{cone}(\sigma)$ such that $\tilde{m} \cdot \left(f(\text{cone}(\sigma)) \mod 2\right)$ is less than $\frac{1}{d+1}$.

Fix a point $s$ in $\mathbb{R}^d \setminus D$. For any point $t$ in $\mathbb{R}^d \setminus D$, if a generic piecewise linear path from $s$ to $t$ intersects with $D$ an odd number of times, then put $t$ in $B$, otherwise put it in $A$. Here the number of intersections of a piecewise linear path $L$ and $D$ might not be the cardinality of $L \cap D$. Instead, the number of intersections is precisely $\sum_{x \in L \cap D} |f^{-1}(x) \cap \partial\text{cone}(\sigma)|$, that is, it takes the multiplicity into account. Thus we have partitioned $\mathbb{R}^d \setminus D$ into $A$ and $B$ such that any generic piecewise linear path from a point in $A$ to a point in $B$ meets $D$ an odd number of times. Suppose $a := m_{d-1}(A)$, $b := m_d(A)$ and $x := \tilde{m}(A) = (a + b)/2$. The probability that an $(m_{d-1}m_d)$-face intersects with $D$ is at least $a(1-b) + (1-a)b$. Hence $a(1-b) + (1-a)b < d!(p_d - \epsilon + (d-1)\delta) < 2\left(\frac{1-\delta}{d+1}\right)\left(1 - \frac{1-\delta}{d+1}\right)$. Because $a(1-b) + (1-a)b = (a+b) - 2ab \geq (a+b) - (a+b)^2/2 = 2x(1-x)$, either $x$ or $1-x$ is less than $\frac{1-\delta}{d+1}$. In other words, one of $\tilde{m}(A)$ and $\tilde{m}(B)$ is less than $\frac{1-\delta}{d+1}$. We may assume that $\tilde{m}(B) < \frac{1-\delta}{d+1}$.

Fix a point $c \in A$. Again, we fill in the boundary of $\text{cone}(\sigma)$ against the center $c$. For any generic point $x \in A$, the line segment $[c, x]$ intersects with $D$ an even number of times. For every $v$ on $\partial\text{cone}(\sigma)$, the ray with the initial point $f(v)$ in the direction $\overrightarrow{cf(v)}$ covers $x$ once if and only if the line segment $[c, x]$ intersects with $D$ at $f(v)$. Because $f(\text{cone}(\sigma))$ is the union of such rays, the number of times that $x$ is covered by $f(\text{cone}(\sigma))$ is exactly the number of intersections between $[c, x]$ and $D$. This implies that $x$ is not in $f(\text{cone}(\sigma)) \mod 2$. Therefore $f(\text{cone}(\sigma)) \mod 2$ is a subset of $B \cup D$ almost surely. Noticing that $\tilde{m}(D) = 0$, the extension of $f$ has the desired property $\tilde{m} \cdot f(\text{cone}(\sigma)) \mod 2 \leq \frac{1-\delta}{d+1}$.
Figure 5: An illustration of the partition, the result of filling in against $c$, and $f(\text{cone}(\sigma)) \mod 2$.

Pick any $d$-face $\tau$ of $T$. Suppose the $(d - 1)$-faces of $\tau$ are $\sigma_0, \ldots, \sigma_d$. By a parity argument, we have

$$f(\partial \text{cone}(\tau)) \mod 2 = [\tau \cup f(\text{cone}(\sigma_0)) \cup \ldots \cup f(\text{cone}(\sigma_d))] \mod 2$$

$$\subset \tau \cup f(\text{cone}(\sigma_0)) \mod 2 \cup \ldots \cup f(\text{cone}(\sigma_d)) \mod 2.$$

Therefore $\tilde{m}(f(\partial \text{cone}(\tau)) \mod 2)$ is less than $\delta + (d + 1)\frac{1-\delta}{d+1} = 1$, and so the degree of $f$ on $\partial \text{cone}(\tau)$, denoted by $\deg(f, \partial \text{cone}(\tau))$, is even. Because

$$\sum_{\tau} \deg(f, \partial \text{cone}(\tau)) = 2 \sum_{\sigma} \deg(f, \text{cone}(\sigma)) + \deg(f, T) \equiv \deg(f, T) \pmod{2},$$

where the first sum and the second sum are over all $d$-faces and all $(d - 1)$-faces of $T$ respectively, we know that $\deg(f, T)$ is even, which contradicts with the fact that $f$ is identity on $T$. \hfill $\square$

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