Invariant functions on $p$-divisible groups and the $p$-adic Corona problem

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1 Introduction

In this note we are concerned with $p$-divisible groups $G = (G_\nu)$ over a complete discrete valuation ring $R$. We assume that the fraction field $K$ of $R$ has characteristic zero and that the residue field $k = R/\pi R$ is perfect of positive characteristic $p$.

Let $C$ be the completion of an algebraic closure of $K$ and denote by $\mathfrak{o} = \mathfrak{o}_C$ its ring of integers. The group $G_\nu(\mathfrak{o})$ acts on $G_\nu \otimes \mathfrak{o}$ by translation. Since $G_\nu \otimes K$ is étale the $G_\nu(C)$-invariant functions on $G_\nu \otimes C$ are just the constants. Using the counit it follows that the natural inclusion

$$\mathfrak{o} \xrightarrow{\sim} \Gamma(G_\nu \otimes \mathfrak{o}, \mathcal{O})^{G_\nu(\mathfrak{o})}$$

is an isomorphism. We are interested in an approximate mod $\pi^n$-version of this statement. Set $\mathfrak{o}_n = \mathfrak{o}/\pi^n \mathfrak{o}$ for $n \geq 1$. The group $G_\nu(\mathfrak{o})$ acts by translation on $G_\nu \otimes \mathfrak{o}_n$ for all $n$.

**Theorem 1** Assume that the dual $p$-divisible group $G'$ is at most one-dimensional and that the connected-étale exact sequence for $G'$ splits over $\mathfrak{o}$. Then there is an integer $t \geq 1$ such that the cokernel of the natural inclusion

$$\mathfrak{o}_n \hookrightarrow \Gamma(G_\nu \otimes \mathfrak{o}_n, \mathcal{O})^{G_\nu(\mathfrak{o})}$$

is annihilated by $p^t$ for all $\nu$ and $n$. 

The example of $G_m = (\mu_{q^n})$ in section 2 may be helpful to get a feeling for the statement.

We expect the theorem to hold without any restriction on the dimension of $G$ as will be explained later. Its assertion is somewhat technical but the proof may be of interest because it combines some of the main results of Tate on $p$-divisible groups with van der Put’s solution of his one-dimensional $p$-adic Corona problem.

The classic corona problem concerns the Banach algebra $H^\infty(D)$ of bounded analytic functions on the open unit disc $D$. The points of $D$ give maximal ideals in $H^\infty(D)$ and hence points of the Gelfand spectrum $\hat{D} = \text{sp} H^\infty(D)$. The question was whether $D$ was dense in $\hat{D}$, (the set $\hat{D} \setminus D$ being the “corona”). This was settled affirmatively by Carleson [C]. The analogous question for the polydisc $D^d$ is still open for $d \geq 2$. An equivalent condition for $D^d$ to be dense in $\text{sp} H^\infty(D^d)$ is the following one, [H], Ch. 10:

**Condition 2** If $f_1, \ldots, f_n$ are bounded analytic functions in $D^d$ such that for some $\delta > 0$ we have

$$\max_{1 \leq i \leq n} |f_i(z)| \geq \delta \quad \text{for all } z \in D^d,$$

then $f_1, \ldots, f_n$ generate the unit ideal of $H^\infty(D^d)$.

In [P] van der Put considered the analogue of condition 2 with $H^\infty(D^d)$ replaced by the algebra of bounded analytic $\mathbb{C}$-valued functions on the $p$-adic open polydisc $\Delta^d$ in $\mathbb{C}^d$, i.e. by the algebra

$$\mathcal{C} \langle X_1, \ldots, X_d \rangle = \mathfrak{o}[[X_1, \ldots, X_d]] \otimes_{\mathfrak{o}} \mathbb{C}.$$ 

He called this $p$-adic version of condition 2 the $p$-adic Corona problem and verified it for $d = 1$. The general case $d \geq 1$ was later treated by Bartenwerfer [B] using his earlier results on rigid cohomology with bounds.

In the proof of theorem 1 applying Tate’s results from [T] we are led to a question about certain ideals in $\mathcal{C} \langle X_1, \ldots, X_d \rangle$, which for $d = 1$ can be reduced to van der Put’s $p$-adic Corona problem. For $d \geq 2$, I did not succeed in such a reduction. However it seems possible that a generalization of Bartenwerfer’s theory might settle that question.
It should be mentioned that van der Put’s term “$p$-adic Corona problem” for the $p$-adic analogue of condition 2 is somewhat misleading. Namely as pointed out in [EM] a more natural analogue would be the question whether $\Delta^d$ was dense in the Berkovich space of $C(X_1, \ldots, X_d)$. This is not known, even for $d = 1$.

The difference between the classic and the $p$-adic cases comes from the fact discovered by van der Put that contrary to $H^\infty(D^d)$ the algebra $C(X_1, \ldots, X_d)$ contains maximal ideals of infinite codimension.

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2 An example and other versions of the theorem

Consider an affine group scheme $G$ over a ring $S$ with Hopf-algebra $A = \Gamma(G, \mathcal{O})$, comultiplication $\mu : A \to A \otimes_S A$ and counit $\varepsilon : A \to S$. The operation of $G(S) = \text{Hom}_S(A, S)$ on $\Gamma(G, \mathcal{O})$ by translation is given by the map

\[(1) \quad G(S) \times A \to A, \ (\chi, a) \mapsto (\chi \otimes \text{id})\mu(a)\]

where $(\chi \otimes \text{id})\mu$ is the composition

$$A \xymatrix{ \overset{\mu}{\ar[r]} & A \otimes_S A \ar[r]^-{\chi \otimes \text{id}} & S \otimes_S A = A.}$$

Given a homomorphism of groups $P \to G(S)$ we may view $A$ as a $P$-module. The composition $S \to A \xymatrix{ \overset{\varepsilon}{\ar[r]} & S}$ being the identity we have an isomorphism

\[(2) \quad \ker (A^P \xymatrix{ \overset{\varepsilon}{\ar[r]} & S}) \xymatrix{ \ar[r]^-{\sim} & } A^P/S \quad \text{mapping } a \text{ to } a + S.\]

The inverse sends $a + S$ to $a - \varepsilon(a) \cdot 1$.

Example The theorem is true for $\mathbb{G}_m = (\mu_p)$. 

Proof Set $V = o_n[X, X^{-1}]/(X^{p^\nu} - 1)$. Applying formulas (1) and (2) with $\mathcal{G} = \mu_p \otimes o_n$ and $P = \mu_p(o) \to \mathcal{G}(o_n)$ we see that the cokernel of the map

\[(3) \quad o_n \to \Gamma(\mu_p \otimes o_n, \mathcal{O})^{\mu_p(o)} \]
is isomorphic to the \( \mathfrak{o}_n \)-module:

\[
\{ \overline{Q} \in V \mid \overline{Q}(\zeta X) = \overline{Q}(X) \text{ for all } \zeta \in \mu_{p^n}(\mathfrak{o}) \text{ and } \overline{Q}(1) = 0 \}.
\]

Lift \( \overline{Q} \) to a Laurent polynomial \( Q = \sum_{\mu \in S} a_{\mu} X^{\mu} \) in \( \mathfrak{o}[X, X^{-1}] \) where \( S = \{0, \ldots, p^n - 1\} \).

Then we have:

(4) \[(\zeta^\mu - 1)a_{\mu} \equiv 0 \mod \pi^n \text{ for } \mu \in S \text{ and } \zeta \in \mu_{p^n}(\mathfrak{o})\]

and

(5) \[\sum_{\mu \in S} a_{\mu} \equiv 0 \mod \pi^n.\]

For any non-zero \( \mu \) in \( S \) choose \( \zeta \in \mu_{p^n}(\mathfrak{o}) \) such that \( \zeta^\mu - 1 \) divides \( p \) in \( \mathfrak{o} \) and hence (4) implies that \( pa_{\mu} \equiv 0 \mod \pi^n \) for all \( \mu \neq 0 \). Using (5) it follows that we have \( pa_0 \equiv 0 \mod \pi^n \) as well. Hence \( pQ \) mod \( \pi^n \) is zero and therefore \( p\overline{Q} = 0 \) as well. Thus \( p \) annihilates the \( \mathfrak{o}_n \)-module (3) for all \( \nu \geq 1 \) and \( n \geq 1 \).

Now assume that \( S = R \) and that \( \mathcal{G}/R \) is a finite, flat group scheme. Consider the Cartier dual \( \mathcal{G}' = \text{spec } \mathcal{A}' \) where \( \mathcal{A}' = \text{Hom}_R(\mathcal{A}, R) \). The perfect pairing of finite free \( \mathfrak{o}_n \)-modules

(6) \[ (\mathcal{A} \otimes \mathfrak{o}_n) \times (\mathcal{A}' \otimes \mathfrak{o}_n) \to \mathfrak{o}_n \]

induces an isomorphism

(7) \[ \text{Ker } ((\mathcal{A} \otimes \mathfrak{o}_n) \to \mathfrak{o}_n) \to \text{Hom}_{\mathfrak{o}_n}((\mathcal{A}' \otimes \mathfrak{o}_n)_{\mathfrak{G}(\mathfrak{o})}/\mathfrak{o}_n, \mathfrak{o}_n). \]

Using (2) it follows that if \( p^t \) annihilates \( (\mathcal{A}' \otimes \mathfrak{o}_n)_{\mathfrak{G}(\mathfrak{o})}/\mathfrak{o}_n \) then \( p^t \) annihilates \( (\mathcal{A} \otimes \mathfrak{o}_n)_{\mathfrak{G}(\mathfrak{o})}/\mathfrak{o}_n \) as well. (The converse is not true in general.)

Hence theorem 1 follows from the next result (applied to the dual \( p \)-divisible group).

**Theorem 3** Assume that the \( p \)-divisible group \( \mathcal{G} \) is at most one-dimensional and that the connected-étale exact sequence for \( \mathcal{G} \) splits over \( \mathfrak{o} \). Then there is an integer \( t \geq 1 \) such that \( p^t \) annihilates the cokernel of the natural map

\[
\mathfrak{o}_n \to \Gamma(\mathcal{G}_\nu \otimes \mathfrak{o}_n, \mathcal{O})_{\mathcal{G}'(\mathfrak{o})}
\]

for all \( \nu \) and \( n \).
For a finite flat group scheme \( G = \text{spec} \mathcal{A} \) over a ring \( S \), the group
\[
G'(S) = \text{Hom}\_S(-, S) \subset \mathcal{A}
\]
consists of the group-like elements in \( \mathcal{A} \) i.e. the units \( a \) in \( \mathcal{A} \) with \( \mu(a) = a \otimes a \).
In this way \( G'(S) \) becomes a subgroup of the unit group \( \mathcal{A}^* \) and hence \( G'(S) \) acts on \( \mathcal{A} \) by multiplication. On the other hand \( G'(S) \) acts on \( G' \) by translation, hence on \( \mathcal{A}' = \Gamma(G', \mathcal{O}) \) and hence on \( \mathcal{A}'' = \mathcal{A} \). Using (1) one checks that the two actions of \( G'(S) \) on \( \mathcal{A} \) are the same. This leads to the following description of the cofixed module in theorem 4. Set
\[
\mathcal{A}_\nu = \Gamma(G_\nu \otimes_{\mathfrak{o}} \mathcal{O})
\]
and let \( J_\nu \) be the ideal in \( \mathcal{A}_\nu \otimes_{\mathfrak{o}} \mathfrak{o} \) generated by the elements \( h - 1 \) with \( h \) group-like in this Hopf-algebra over \( \mathfrak{o} \). Thus \( J_\nu \) is also the \( \mathfrak{o} \)-submodule of \( \mathcal{A}_\nu \otimes_{\mathfrak{o}} \mathfrak{o} \) generated by the elements \( ha - a \) for \( h \in G'_\nu(\mathfrak{o}) \) and \( a \in \mathcal{A}_\nu \otimes_{\mathfrak{o}} \mathfrak{o} \). Then we have the formula
\[
\Gamma(G_\nu \otimes_{\mathfrak{o}} \mathfrak{o}, \mathcal{O})_{G'_\nu(\mathfrak{o})} = (\mathcal{A}_\nu \otimes_{\mathfrak{o}} \mathfrak{o})/J_\nu(\mathcal{A}_\nu \otimes_{\mathfrak{o}} \mathfrak{o})
\]
This implies an isomorphism:
\[
\text{Coker} (\mathfrak{o} \rightarrow \Gamma(G_\nu \otimes_{\mathfrak{o}} \mathfrak{o}, \mathcal{O})_{G'_\nu(\mathfrak{o})}) = \text{Coker} (\mathfrak{o} \rightarrow (\mathcal{A}_\nu \otimes_{\mathfrak{o}} \mathfrak{o})/J_\nu) \otimes_{\mathfrak{o}} \mathfrak{o}
\]
Hence theorem 3 and therefore also theorem 1 follow from the next claim:

**Claim 4** For a \( p \)-divisible group \( G = (G_\nu) \) as in theorem 3 there exists an integer \( t \geq 1 \) such that \( p^t \) annihilates the cokernel of the natural map
\[
\mathfrak{o} \rightarrow (\mathcal{A}_\nu \otimes_{\mathfrak{o}} \mathfrak{o})/J_\nu
\]
for all \( \nu \geq 1 \).

As a first step in the proof of claim 4 we reduce to the case where \( G \) is either étale or connected. For simplicity set \( \mathcal{G} = G_\nu \otimes_{\mathfrak{o}} \mathfrak{o} = \text{spec} \mathcal{A} \) and define \( \mathcal{G}^0, \mathcal{G}^\text{ét}, \mathcal{A}^0, \mathcal{A}^\text{ét} \) similarly. By assumption we have isomorphisms \( \mathcal{G} = \mathcal{G}^0 \times_{\mathfrak{o}} \mathcal{G}^\text{ét} \) and \( \mathcal{A} = \mathcal{A}^0 \otimes_{\mathfrak{o}} \mathcal{A}^\text{ét} \) as group schemes, resp. Hopf-algebras over \( \mathfrak{o} \). There is a compatible splitting of the group-like elements over \( \mathfrak{o} \):
\[
\mathcal{G}'(\mathfrak{o}) = \mathcal{G}'^0(\mathfrak{o}) \times \mathcal{G}'^\text{ét}(\mathfrak{o})
\]
For elements
\[
h^0 \in \mathcal{G}'^0(\mathfrak{o}) \subset \mathcal{A}^0 \quad \text{and} \quad h^\text{ét} \in \mathcal{G}'^\text{ét}(\mathfrak{o}) \subset \mathcal{A}^\text{ét}
\]
consider the identity:
\[
h^0 \otimes h^\text{ét} - 1 = h^0 \otimes (h^\text{ét} - 1) + (h^0 - 1) \otimes 1 \quad \text{in} \quad \mathcal{A}
\]
It implies that we have
\[ J = \mathcal{A}^0 \otimes J^{\text{et}} + J^0 \otimes \mathcal{A}^{\text{et}} \] in \( \mathcal{A} \)

where \( J \) is the ideal of \( \mathcal{A} \) generated by the elements \( h - 1 \) for \( h \in G'(\mathfrak{o}) \) and \( J^0, J^{\text{et}} \) are defined similarly. Hence we have natural surjections
\[ \mathcal{A}^0/J^0 \otimes \mathcal{A}^{\text{et}}/J^{\text{et}} \to \mathcal{A}/J \]

and
\[ \text{Coker} (\mathfrak{o} \to \mathcal{A}^0/J^0) \otimes \text{Coker} (\mathfrak{o} \to \mathcal{A}^{\text{et}}/J^{\text{et}}) \to \text{Coker} (\mathfrak{o} \to \mathcal{A}/J) \]

Hence it suffices to prove claim 4 in the cases where \( G \) is either connected or \( \acute{\text{e}} \text{tale} \). The \( \acute{\text{e}} \text{tale} \) case is straightforward: We have \( G \otimes_R \mathfrak{o} = ((\mathbb{Z}/p^{\nu})^h)_{\nu \geq 0} \) where for an abstract group \( A \) we denote by \( \mathcal{A} \) the corresponding \( \acute{\text{e}} \text{tale} \) group scheme. Hence \( G'_{\nu} = \mu_{p^{\nu}}^h \) and \( G'_{\nu}(\mathfrak{o}) = \mu_{p^{\nu}}(\mathfrak{o})^h \). The inclusion
\[ \mu_{p^{\nu}}(\mathfrak{o})^h = \text{Hom}((\mathbb{Z}/p^{\nu})^h, \mathfrak{o}^*) \subset \text{Maps} ((\mathbb{Z}/p^{\nu})^h, \mathfrak{o}) = A_{\nu} \otimes \mathfrak{o} \]

identifies \( \mu_{p^{\nu}}(\mathfrak{o})^h \) with the group like elements in \( A_{\nu} \otimes \mathfrak{o} \).

The ideal \( J_{\nu} \) of \( A_{\nu} \otimes \mathfrak{o} \) is given by:
\[ J_{\nu} = (\chi_\zeta - 1 \mid \zeta \in \mu_{p^{\nu}}(\mathfrak{o})^h) \]

where \( \chi_\zeta \) is the character of \( (\mathbb{Z}/p^{\nu})^h \) defined by the equation
\[ \chi_\zeta((a_1, \ldots, a_h)) = \zeta_1^{a_1} \cdots \zeta_h^{a_h} \] where \( \zeta = (\zeta_1, \ldots, \zeta_h) \).

The functions \( \delta_a \) for \( a \in (\mathbb{Z}/p^{\nu})^h \) given by \( \delta_a(a) = 1 \) and \( \delta_a(b) = 0 \) if \( b \neq a \) generate \( A_{\nu} \otimes \mathfrak{o} \) as an \( \mathfrak{o} \)-module. For \( a \neq 0 \) choose \( \zeta \in \mu_{p^{\nu}}(\mathfrak{o})^h \) with \( \zeta^a \neq 1 \). Then we have \( p = (\zeta^a - 1)\beta \) for some \( \beta \in \mathfrak{o} \). Define \( f_a \in A_{\nu} \otimes \mathfrak{o} \) by setting
\[ f_a(a) = \beta \quad \text{and} \quad f_a(b) = 0 \text{ for } b \neq a \]

We then find:
\[ f_a(\chi_\zeta - 1) = p\delta_a \text{ in } A_{\nu} \otimes \mathfrak{o} \]

Hence we have \( p\delta_a \in J_{\nu} \) for all \( a \neq 0 \) and therefore \( p \) annihilates \( \text{Coker} (\mathfrak{o} \to (A_{\nu} \otimes \mathfrak{o})/J_{\nu}) \).

The next two sections are devoted to the much more interesting case where \( G \) is connected.
3 The connected case I ($p$-adic Hodge theory)

In this section we reduce the assertion of claim [4] for connected $p$-divisible groups of arbitrary dimension to an assertion on ideals in $C(X_1, \ldots, X_d)$. For this reduction we use theorems of Tate in [T].

Thus let $G = (G_{\nu})$ be a connected $p$-divisible group of dimension $d$ over $R$ and set $A = \varprojlim_{\nu} A_{\nu}$ where $G_{\nu} = \text{spec} A_{\nu}$.

Consider the projective limit $A = \varprojlim_{\nu} A_{\nu}$ with the topology inherited from the product topology $\prod A_{\nu}$ where the $A_{\nu}$’s are given the $\pi$-adic topology. This topology on $A$ is the one defined by the $R$-submodules $K_{n} + \pi^k A$ for $n, k \geq 1$ where $K_{n} = \text{Ker}(A \to A_{\nu})$. Equivalently it is defined by the spaces $K_{n} + \pi^k A$ for $n \geq 1$. In [T] section (2.2) it is shown that $A$ is isomorphic to $R[[X_1, \ldots, X_d]]$ as a topological $R$-algebra. If $M$ denotes the maximal ideal of $A$, then according to [T] Lemma 0 the topology of $A$ coincides with the $M$-adic topology. Let $A \widehat{\otimes}_R \mathfrak{o}$ be the completion of $A \otimes_R \mathfrak{o}$ with respect to the linear topology on $A \otimes_R \mathfrak{o}$ given by the subspaces $M^n \otimes_R \mathfrak{o} + A \otimes_R \pi^n \mathfrak{o}$.

**Lemma 5** We have $\lim (A_n \otimes_R \mathfrak{o}) = A \widehat{\otimes}_R \mathfrak{o} = \mathfrak{o}[[X_1, \ldots, X_d]]$ as topological rings.

**Proof** Consider the isomorphisms

$$\lim_n (A_n \otimes_R \mathfrak{o}) = \lim_n (A_n \otimes_R (\lim_k (\mathfrak{o}/\pi^k \mathfrak{o})))$$

$$\overset{(1)}{=} \lim_n \lim_k (A_n \otimes_R \mathfrak{o}/\pi^k \mathfrak{o})$$

$$= \lim_n (A \otimes_R \mathfrak{o})/((K_n + \pi^k A) \otimes_R \mathfrak{o} + A \otimes_R \pi^k \mathfrak{o})$$

$$\overset{(2)}{=} \lim_n (A \otimes_R \mathfrak{o})/((K_n + \pi^k A) \otimes_R \mathfrak{o} + A \otimes_R \pi^n \mathfrak{o})$$

$$\overset{(3)}{=} \lim_n (A \otimes_R \mathfrak{o})/(M^n \otimes_R \mathfrak{o} + A \otimes_R \pi^n \mathfrak{o})$$

$$= A \widehat{\otimes}_R \mathfrak{o}$$

$$\overset{(4)}{=} \mathfrak{o}[[X_1, \ldots, X_d]].$$
Here (1) holds because \( \lim \) commutes with finite direct sums, (2) is true by cofinality, (3) holds because the topology on \( A \) can also be described as the \( M \)-adic topology. Finally (4) follows from the definition of \( \hat{A} \otimes_R o \) and the fact that \( A = R[[X_1, \ldots, X_d]] \).

The \( o \)-algebra \( \hat{A} \otimes_R o = \lim_{\leftarrow \nu} (A_\nu \otimes_R o) \) contains the ideal \( \tilde{J} = \lim_{\leftarrow \nu} J_\nu \).

**Claim 6** We have

\[
\hat{A} \otimes_R o / (o + \tilde{J}) = \lim_{\nu}(A_\nu \otimes_R o / (o + J_\nu)).
\]

**Proof** The inclusion \( G_\nu \subset G_{\nu+1} \) corresponds to a surjection of Hopf-algebras \( A_{\nu+1} \to A_\nu \). Hence \( A_{\nu+1} \otimes_R o \to A_\nu \otimes_R o \) is surjective as well and group-like elements are mapped to group-like elements. The map on group-like elements is surjective because it corresponds to the surjective map \( G'_{\nu+1}(o) \to G'_\nu(o) \). Note here that \( G'_\mu(o) = G'_\mu(C) \) for all \( \mu \). It follows that the map \( J_{\nu+1} \to J_\nu \) is surjective as well. In the exact sequence of projective systems

\[
0 \to (o + J_\nu) \to (A_\nu \otimes_R o) \to (A_\nu \otimes_R o / (o + J_\nu)) \to 0
\]

the system \( (o + J_\nu) \) is therefore Mittag–Leffler. Hence the sequence of projective limits is exact and the claim follows because the sum \( o + J_\nu \) is direct: Group-like elements of \( A_\nu \otimes_R o \) are mapped to 1 by the counit \( \varepsilon_\nu \). Therefore we have

(10) \[
J_\nu \subset I_\nu := \text{Ker} (\varepsilon_\nu : A_\nu \otimes_R o \to o) .
\]

The sum \( o + I_\nu \) being direct we are done. \( \square \)

Because of claim \( \square \) and the surjectivity of the maps \( A_{\nu+1} \otimes_R o \to A_\nu \otimes_R o \), claim \( \square \) for connected groups is equivalent to the next assertion:

**Claim 7** Let \( G \) be a connected \( p \)-divisible group with \( \dim G \leq 1 \). Then there is some \( t \geq 1 \) such that \( p^t \) annihilates

\[
\text{Coker} (o \to A \hat{\otimes} R o / \tilde{J}) .
\]
For connected $G$ of arbitrary dimension consider the Tate module of $G'$

$$TG' = \lim_{\nu} G'_\nu(C) = \lim_{\nu} G'_\nu(\mathfrak{o}) \subset \lim_{\nu}(A_\nu \otimes_R \mathfrak{o}) = A\hat{\otimes}_R \mathfrak{o}.$$ 

Let $J$ be the ideal of $A\hat{\otimes}_R \mathfrak{o}$ generated by the elements $h - 1$ for $h \in TG'$. The image of $J$ under the reduction map $A\hat{\otimes}_R \mathfrak{o} \rightarrow A_\nu \otimes_R \mathfrak{o}$ lies in $J_\nu$. It follows that $J \subset \tilde{J}$. With $I_\nu$ as in (10) we set $I = \lim_{\nu} I_\nu$, an ideal in $A\hat{\otimes}_R \mathfrak{o}$. We have $J \subset \tilde{J} \subset I$ because of (10). Since $A\hat{\otimes}_R \mathfrak{o} = \mathfrak{o} \oplus I$, we get a surjection

$$I / J \rightarrow \text{Coker } (\mathfrak{o} \rightarrow A\hat{\otimes}_R \mathfrak{o} / \tilde{J}).$$

Thus claim 7 will be proved if we can show that $p^d I \subset J$ at least for $\dim G = 1$. The construction in section (2.2) shows that under the isomorphism of $\mathfrak{o}$-algebras

$$A\hat{\otimes}_R \mathfrak{o} = \mathfrak{o}[[X_1, \ldots, X_d]]$$

we have $I = (X_1, \ldots, X_d)$.

We will view the elements of $A\hat{\otimes}_R \mathfrak{o}$ and in particular those of $J$ as analytic functions on the open $d$-dimensional polydisc

$$\Delta^d = \{x \in C^d \mid |x_i| < 1 \text{ for all } i\}.$$

Because of the inclusion $J \subset I$ all functions in $J$ vanish at $0 \in \Delta^d$. There are no other common zeroes:

**Proposition 8 (Tate)** The zero set of $J$ in $\Delta^d$ consists only of the origin $\mathfrak{o} \in \Delta^d$.

**Proof** The $\mathfrak{o}$-valued points of the $p$-divisible group $G$,

$$G(\mathfrak{o}) = \lim_i \lim_{\nu} G_\nu(\mathfrak{o} / \pi^i \mathfrak{o})$$

can be identified with continuous $\mathfrak{o}$-algebra homomorphisms

$$G(\mathfrak{o}) = \text{Hom}_{\text{cont.alg}}(A, \mathfrak{o}) = \text{Hom}_{\text{cont.alg}}(A\hat{\otimes}_R \mathfrak{o}, \mathfrak{o}).$$

Moreover we have a homeomorphism

$$\Delta^d \overset{\sim}{\rightarrow} G(\mathfrak{o}) \text{ via } x \mapsto (f \mapsto f(x)).$$
Here $f \in A_{\otimes_R}^\circ$ is viewed as a formal power series over $\mathfrak{g}$. The group structure on $G(\mathfrak{g})$ induces a Lie group structure on $\Delta^d$ with $0 \in \Delta^d$ corresponding to $1 \in G(\mathfrak{g})$. Let $U$ be the group of 1-units in $\mathfrak{g}$. Proposition 11 of [T] asserts that the homomorphism of Lie groups

$$\alpha : \Delta^d = G(\mathfrak{g}) \rightarrow \text{Hom}_{\text{cont}}(TG', U), \ x \mapsto (h \mapsto h(x))$$

is injective. Note here that $TG' \subset A_{\otimes_R}^\circ$. Let $x \in \Delta^d$ be a point in the zero set of $J$. Then we have $(h - 1)(x) = 0$ i.e. $h(x) = 1$ for all $h \in TG'$. Hence $x$ maps to $1 \in \text{Hom}_{\text{cont}}(TG', U)$. Since $\alpha$ is injective, it follows that we have $x = 0$.

If a Hilbert Nullstellensatz were true in $C\langle X_1, \ldots, X_d \rangle$ we could conclude that we had $\sqrt{J \otimes C} = I \otimes C$ and with further arguments from [T] we would get $p^\prime I \subset J$. However the Nullstellensatz does not hold in the ring $C\langle X_1, \ldots, X_d \rangle$.

In the next section we will provide a replacement which is proved for $d = 1$ and conjectured for $d \geq 2$. In order to apply it to the ideal $J \otimes C$ in $C\langle X_1, \ldots, X_d \rangle$ we need to know the following assertion which is stronger than proposition 8. For $x \in C^m$ set $\|x\| = \max_i |x_i|$.

**Proposition 9** Let $h_1, \ldots, h_r$ be a $\mathbb{Z}_p$-basis of $TG' \subset \mathfrak{g}[\![X_1, \ldots, X_d]\!]$ and set $H(x) = (h_1(x), \ldots, h_r(x))$ and $1 = (1, \ldots, 1)$. Then there is a constant $\delta > 0$ such that we have:

$$\|H(x) - 1\| \geq \delta \|x\| \quad \text{for all } x \in \Delta^d.$$ 

**Proof** The $\mathbb{Z}_p$-rank $r$ of $TG'$ is the height of $G'$ and hence we have $r \geq d = \dim G$. Consider the following diagram (⋆) on p. 177 of [T]:

$$\begin{array}{ccccccc}
1 & \rightarrow & G(\mathfrak{g})_{\text{tors}} & \rightarrow & G(\mathfrak{g}) & \rightarrow & t_G(C) & \rightarrow & 0 \\
& & \alpha \downarrow & & \alpha \downarrow & & \downarrow \text{log}_{\ast} & \\
1 & \rightarrow & \text{Hom}(TG', U_{\text{tors}}) & \rightarrow & \text{Hom}(TG', U) & \rightarrow & \text{Hom}(TG', C) & \rightarrow & 0.
\end{array}$$

Here the Hom-groups refer to continuous homomorphisms and the map $\alpha$ was defined in equation (12) above. The map $L$ is the logarithm map to the tangent space $t_G(C)$ of $G$ and $\log_{\ast}$ is induced by $\log : U \rightarrow C$. According to [T] proposition 11 the maps $\alpha$ and $d\alpha$ are injective and $\alpha_0$ is bijective. It will suffice to prove the following two statements:
I) For any $\varepsilon > 0$ there is a constant $\delta(\varepsilon) > 0$ such that
\[ \|H(x) - 1\| \geq \delta(\varepsilon) \] for all $x \in \Delta^d$ with $\|x\| \geq \varepsilon$.

II) There are $\varepsilon > 0$ and $a > 0$ such that
\[ \|H(x) - 1\| \geq a\|x\| \] for all $x \in \Delta^d$ with $\|x\| \leq \varepsilon$.

Identifying $G(\mathfrak{o})$ with $\Delta^d$ where we write the induced group structure on $\Delta^d$ as $\oplus$, and identifying $T G'$ with $\mathbb{Z}_p^r$ via the choice of the basis $h_1, \ldots, h_r$, the above diagram becomes the following one where $A = dH$ and $H_0$ is the restriction of $H$ to $(\Delta^d)_{\text{tors}}$

Assume that assertion I is wrong for some $\varepsilon > 0$. Then there is a sequence $x^{(i)}$ of points in $\Delta^d$ with $\|x^{(i)}\| \geq \varepsilon$ such that $H(x^{(i)}) \to 1$ for $i \to \infty$. It follows that $A(L(x^{(i)}) = \log H(x^{(i)}) \to 0$ for $i \to \infty$. Since $A$ is an injective linear map between finite dimensional $C$-vector spaces, there exists a constant $a > 0$ such that we have
\[ \|A(v)\| \geq a\|v\| \] for all $v \in C^d$.

Hence we see that $L(x^{(i)}) \to 0$ for $i \to \infty$. Since $L$ is a local homeomorphism, there exists a sequence $y^{(i)} \to 0$ in $\Delta^d$ with $L(x^{(i)}) = L(y^{(i)})$ for all $i$. The sequence $z^{(i)} = x^{(i)} \oplus y^{(i)}$ in $\Delta^d$ satisfies $L(z^{(i)}) = 0$ and hence lies in $(\Delta^d)_{\text{tors}}$. We have $H_0(z^{(i)}) = H(x^{(i)})H(y^{(i)})^{-1}$. Moreover $H(x^{(i)}) \to 1$ by assumption and $H(y^{(i)}) \to 1$ since $y^{(i)} \to 0$. Hence $H_0(z^{(i)}) \to 1$ and therefore $H_0(z^{(i)}) = 1$ for all $i \gg 0$ since the subspace topology on $U_{\text{tors}} \subset U$ is the discrete topology. The map $H_0$ being bijective we find that $z^{(i)} = 0$ for $i \gg 0$ and therefore $x^{(i)} = y^{(i)}$ for $i \gg 0$. This implies that $x^{(i)} \to 0$ for $i \to \infty$ contradicting the assumption $\|x^{(i)}\| \geq \varepsilon$ for all $i$. Hence assertion I) is proved.

We now turn to assertion II). Set $X = (X_1, \ldots, X_d)$. Then we have
\[ H(X) = 1 + AX + (\text{deg} \geq 2) \].

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Componentwise this gives for $1 \leq j \leq r$

$$h_j(x) - 1 = \sum_{i=1}^{d} a_{ij}x_i + (\text{deg } \geq 2)_j .$$

Let $a$ be the constant from equation (13) and choose $\varepsilon > 0$, such that for $\|x\| \leq \varepsilon$ we have

$$\|(\text{deg } \geq 2)_j\| \leq \frac{a}{2}\|x\| \quad \text{for } 1 \leq j \leq r .$$

For any $x$ with $\|x\| < \varepsilon$, according to (13) there is an index $j$ with

$$\left| \sum_{i=1}^{d} a_{ij}x_i \right| \geq a\|x\| .$$

This implies that we have

$$|h_j(x) - 1| = \left| \sum_{i=1}^{d} a_{ij}x_i + (\text{deg } \geq 2)_j \right| = \left| \sum_{i=1}^{d} a_{ij}x_i \right| \geq a\|x\|$$

and hence

$$\|H(x) - 1\| \geq a\|x\| .$$

\[\square\]

4 The connected case II (the $p$-adic Corona problem)

As remarked in the previous section we need a version of the Hilbert Nullstellensatz in $C\langle X_1, \ldots, X_d \rangle$ for the case where the zero set is $\{0\} \subset \Delta^d$. The only result for $C\langle X_1, \ldots, X_d \rangle$ in the spirit of the Nullstellensatz that I am aware of concerns an empty zero set:

\textit{P-adic Corona theorem 10 (van der Put, Bartenwerfer)} For $f_1, \ldots, f_n$ in $C\langle X_1, \ldots, X_d \rangle$ the following conditions are equivalent:
1) The functions $f_1, \ldots, f_n$ generate the $C$-algebra $C\langle X_1, \ldots, X_d \rangle$.

2) There is a constant $\delta > 0$ such that

$$\max_{1 \leq j \leq n} |f_j(x)| \geq \delta \quad \text{for all } x \in \Delta^d.$$ 

It is clear that the first condition implies the second. The non-trivial implication was proved by van der Put for $d = 1$ in [P] and by Bartenwerfer in general, c.f. [B]. Both authors give a more precise statement of the theorem where the norms of possible functions $g_j$ with $\sum_j f_j g_j = 1$ are estimated.

Consider the following conjecture which deals with the case where the zero set may contain $\{0\}$.

**Conjecture 11** For $g_1, \ldots, g_n$ in $C\langle X_1, \ldots, X_d \rangle$ the following conditions are equivalent:

1) $(g_1, \ldots, g_n) \supset (X_1, \ldots, X_d)$.

2) There is a constant $\delta > 0$ such that

$$(14) \quad \max_{1 \leq j \leq n} |g_j(x)| \geq \delta \|x\| \quad \text{for all } x \in \Delta^d.$$ 

As above, immediate estimates show that the first condition implies the second. Note also that if some $g_j$ does not vanish at $x = 0$ we have

$$\max_{1 \leq j \leq n} |g_j(x)| \geq \delta' > 0 \text{ in a neighborhood of } x = 0.$$ 

Together with (14) this implies that

$$\max_{1 \leq j \leq n} |g_j(x)| \geq \delta'' > 0 \quad \text{for all } x \in \Delta^d.$$ 

The $p$-adic Corona theorem then gives $(g_1, \ldots, g_n) = C\langle X_1, \ldots, X_d \rangle$. Thus condition 1 follows in this case.

**Proposition 12** The preceding conjecture is true for $d = 1$. 

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Proof As explained above, we may assume that all functions $g_1, \ldots, g_n$ vanish at $x = 0$. Then $f_j(X) = X^{-1}g_j(X)$ is in $C\langle X \rangle$ for every $1 \leq j \leq n$ and estimate (14) implies the estimate
\[
\max_{1 \leq j \leq n} |f_j(x)| \geq \delta \quad \text{for all } x \in \Delta^1.
\]
The $p$-adic Corona theorem for $d = 1$ now shows that
\[
(f_1, \ldots, f_n) = (1) \quad \text{and hence} \quad (g_1, \ldots, g_n) = (X).
\]
\[\square\]

Let us now return to $p$-divisible groups and recall the surjection (11):
\[
I / J \twoheadrightarrow \text{Coker } (o \to A \hat{\otimes} R / \tilde{J}).
\]
Here $I = (X_1, \ldots, X_d)$ in $o[[X_1, \ldots, X_d]]$ and $J$ is the ideal generated by the elements $h - 1$ for $h \in TG'$. Let $J_0 \subset J$ be the ideal generated by the elements $h_1 - 1, \ldots, h_r - 1$ where $h_1, \ldots, h_r$ form a $\mathbb{Z}_p$-basis of $TG'$. In proposition 9 we have seen that for some $\delta > 0$ we have
\[
\max_{1 \leq j \leq r} |h_j(x) - 1| \geq \delta \|x\| \quad \text{for all } x \in \Delta^d.
\]
Conjecture 11 (which is true for $d = 1$) would therefore imply
\[
(h_1 - 1, \ldots, h_r - 1) = (X_1, \ldots, X_d) \quad \text{in } C\langle X_1, \ldots, X_d \rangle.
\]
Thus we would find some $t \geq 1$, such that we have
\[
p^t X_i \in J_0 \subset o[[X_1, \ldots, X_d]] \quad \text{for all } 1 \leq j \leq r
\]
and hence also $p^t I \subset J_0 \subset J$. Using the surjection (15) this would prove claim 7 and hence theorem 3 without restriction on $\dim G$. Also theorem 1 would follow without restriction on $\dim G'$. As it is we have to assume $\dim G \leq 1$ resp. $\dim G' \leq 1$ in these assertions.

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