Online Learning of Commission Avoidant Portfolio Ensembles

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May 31, 2016

Abstract

We present a novel online ensemble learning strategy for portfolio selection. The new strategy controls and exploits any set of commission-oblivious portfolio selection algorithms. The strategy handles transaction costs using a novel commission avoidance mechanism. We prove a logarithmic regret bound for our strategy with respect to optimal mixtures of the base algorithms. Numerical examples validate the viability of our method and show significant improvement over the state-of-the-art.

1 Introduction

Online portfolio selection is a challenging sequential investment problem introduced by Cover [1991]. The problem naturally generalizes prediction under the logarithmic loss\(^1\) and has become central to online learning research. One of the major hurdles, abstracted away in the basic formulation (see Section 2), is transaction costs, which can defy many portfolio selection algorithms. A number of transaction cost aware strategies and regret bounds have been developed, and the two main techniques used to increase robustness to transaction costs were either to reduce such costs by diluting the number of transaction rounds, as in semiconstant-rebalanced portfolios [Helmbold et al., 1998; Kozat and Singer, 2011], which rebalance only on a subset of the possible trading days, or by penalizing (regularizing) costly rebalancing actions within an online (convex) optimization framework [Das et al., 2013; Das et al., 2014]. All known commission-aware online (adversarial) strategies have been designed to track the best (semi) constant rebalanced portfolio (CRP). Empirical studies (validated in this paper as well) show that these CRP-centric strategies are extremely resilient to very high commission rates. However, numerous empirical studies also indicate that, without commissions, these CRP-driven strategies, and in fact the best CRP in hindsight itself, achieve

\(^1\)From a certain minimax perspective portfolio selection isn’t any harder than logarithmic loss prediction [Cesa-Bianchi and Lugosi, 2006].
inferior performance compared to strategies that are not CRP driven, such as those used by some mean-reversion algorithms (see, e.g., [Huang et al., 2013; Li and Hoi, 2013]). To complicate matters, the mean-reverting algorithms, which can achieve phenomenal results without commissions, are known to be extremely sensitive to commissions (a fact that is also validated here), and some of these methods can crash on some datasets even with moderate commissions.

The main question we tackle in this paper is: can we devise a principled method that will utilize the better algorithms so as to be more resilient to transaction costs? We answer this question in the affirmative and propose an ensemble strategy for controlling and exploiting any set of commission-oblivious portfolio selection base algorithms. Our ensemble strategy is designed to track the best convex combination of base algorithms while systematically avoiding costly rebalancing activity.

After presenting the strategy and proving a logarithmic regret bound with respect to the best in hindsight convex combination (appropriately constrained to reduce commissions), we present extensive empirical study of our strategy implemented as an ensemble over known online portfolio selection algorithms such as OLMAR [Li and Hoi, 2012] and Anticor [Borodin et al., 2004]. The strategy can effectively handle a range of rates (including 1% proportional transaction costs on almost all the common datasets), and exhibits a graceful performance degradation with commission cost rates. Moreover, it consistently outperforms the known commission-aware strategies.

Our learning algorithm, together with its analysis, extends the composite objective mirror descent (COMID) framework of Duchi et al. [2010], so as to handle exp-concave loss function (rather than only convex and strongly convex loss functions), which allows our algorithm to achieve $O(\log T)$ regret with respect to the best possible choice in hindsight. While COMID can be applied as well, it would result in a significantly worse $O(\sqrt{T})$ regret bound.

2 Online Portfolio Selection

In Cover’s classic portfolio selection setting [Cover, 1991], we are given a market with $n$ stocks and consider an online game between an algorithm and an adversary played through $T$ rounds (say, days). On each day $t$ the market is represented by a market vector $X_t$ of relative prices, $X_t \triangleq (x_t^1, x_t^2, \ldots, x_t^n)$, where for each $i = 1, \ldots, n$, $x_t^i \geq 0$ is the relative price of stock $i$, defined to be the ratio of its closing price on day $t$ relative to its closing price on day $t-1$. We denote by $X \triangleq X_1, \ldots, X_T$ the sequence of $T$ market vectors for the entire game. The algorithm’s portfolio for day $t$ is $b_t \triangleq (b_t^1, b_t^2, \ldots, b_t^n)$, where $b_t^i \geq 0$ is the wealth allocation for stock $i$. We require that the portfolio satisfy $\sum_{i=1}^n b_t^i = 1$. Thus, $b_t$ specifies the online player’s wealth allocation for each of the $n$ stocks on day $t$, and $b_t^i$ is the fraction of total current wealth invested in stock $i$ on that day. We denote by $B \triangleq b_1, \ldots, b_T$ the sequence of $T$ portfolios played by the algorithm for the entire game. The portfolio sequence where all $b_t$ equal the same fixed portfolio is called a constant rebalanced portfolio (CRP).

At the start of each trading day $t$, the algorithm chooses a portfolio $b_t$. Thus, by the end of day $t$, the player’s wealth is multiplied by $\langle b_t, X_t \rangle = \sum_{i=1}^n b_t^i x_t^i$, and assuming
initial wealth of $1, the player’s cumulative wealth by the end of the game is therefore

\[ R_T(B, X) \triangleq \prod_{t=1}^{T} \langle b_t, X_t \rangle. \]  

(1)

In the setting above, it is common to consider the logarithmic cumulative wealth, \( \log R_T(B, X) \), which can be expressed as a summation of the logarithmic daily wealth increases, \( \log(\langle b_t, X_t \rangle) \).

In the online (worst-case) approach to portfolio learning the goal is to online generate a sequence \( \{b_t\} \) of portfolios that compete with the best-in-hindsight fixed portfolio, denoted \( b_* \). Letting \( f_t(b) \) be the round \( t \) loss of portfolio \( b \) (in our case, \( f_t(b) = -\log(\langle b_t, X_t \rangle) \)), we define the regret of sequence \( \{b_t\} \) as

\[ \text{Regret} \triangleq \sum_{t=1}^{T} (f_t(b_t) - f_t(b_*)). \]

In this paper we are mainly concerned with portfolio ensembles, where the weights \( b_t \) are over trading algorithms and \( b_* \) represents the optimal-in-hindsight fixed ensemble.

2.1 Introducing Transaction Costs

The vanilla portfolio selection model presented above abstracts away transaction costs. These costs should account for several components including: commissions, slippage (a.k.a. implementation shortfall), and market impact costs. Commissions are service charges required by brokerage firms or exchanges for handling the purchase or sale of securities. Slippage is the price difference between the time we decide to buy/sell a security and the time the transaction is actually executed in the exchange. Market impact costs (which can be related to slippage) are price differentials incurred when supply and demand dynamically respond to our own orders (e.g., a large buy order on a relatively illiquid security is likely to push its price up).

As opposed to modeling brokerage commissions, which follow a fixed and known schedule (agreed upon with the broker), a precise modeling of slippage and market impact costs is extremely challenging. As a first approximation, however, it is common to apply a linear transaction cost model where each transaction incurs a cost proportional to its size [Blum and Kalai, 1999; Lobo et al., 2007]. We therefore focus on the following simple multiplicative (proportional) cost model, commonly used in the online portfolio selection literature (see, e.g., Borodin and El-Yaniv [2005], Sec. 14.5.4). In this model, commissions are specified via a fixed parameter, \( 0 < \gamma \), called the commission rate, and for buying (or selling) \$d \) worth of any stock, the player must pay commission of \$\frac{\gamma}{2}d \). Thus, the transaction cost incurred when the player rebalances a portfolio \( b \) to portfolio \( b' \) is \( \frac{\gamma}{2}||b - b'||_1 \). In the present transaction cost model we assume that commissions are self-refinanced and the player pays them immediately after

\[ 2 \] These costs depend on many factors, including, for example, the type of order used, liquidity and limit order book dynamics, or recent transactions history.

\[ 3 \] In practice, market impact costs are often considered to be a concave function of the amount traded [Lobo et al., 2007].
performing the daily transactions. Thus, on day $t$, after rebalancing to portfolio $b_t$, the market vector $X_t$ is revealed and portfolio $b_t$ becomes

$$\hat{b}_t \triangleq \frac{1}{(b_t, X_t)} (b_1 x_1, b_2 x_2, \ldots, b_n x_n).$$

(2)

Therefore, the commission incurred to rebalance to the next day’s portfolio, $b_{t+1}$, is

$$\frac{\gamma}{2} ||b_{t+1} - \hat{b}_t||_1,$$

(3)

which is paid from the current wealth, $\langle b_t, X_t \rangle$. Altogether, the cumulative wealth of a player paying commission at rate $\gamma$ is

$$R_T^\gamma(B, X) = \prod_{t=1}^{T} \left( \langle b_t, X_t \rangle \left[ 1 - \frac{\gamma}{2} ||b_{t+1} - b_t||_1 \right] \right).$$

3 Related Work and Contributions

The study of portfolio optimization with transaction costs within mainstream finance is a huge topic, beyond our scope. Such studies typically have a traditional operations research flavor where stochastic optimization is carried out under specific distributions; see, e.g., [Davis and Norman] [Konno and Wijayanayake] [Lobo et al.] [1990; 2001; 2007]. In the brief survey below we only refer to related works emerging from the online learning (adversarial) line of research initiated by [Cover] [1991].

[Blum and Kalai] [1999] are perhaps the first who studied commissions in online portfolio selection and showed an elegant regret analysis for Cover and Ordentlich’s universal portfolios (UP) algorithm [Cover and Ordentlich] [1996], which pays proportional commissions. The idea of semiconstant-rebalanced portfolios (SCRP), which dilute the number of rebalancing trading days for commission reduction, was first mentioned briefly by [Helmbold et al.] [1998] and then studied in-depth by [Kozat and Singer] [Kozat and Singer] [2008; 2009], who utilized the context tree weighting (CTW) lossless compression algorithm of [Willems et al.] [1995] to track the best days for rebalancing. The resulting portfolio algorithm was shown to achieve sub-linear regret with respect to the best $k$ days of rebalancing, provided that $k = o(T)$. Recently, [Huang et al.] [2015] introduced two algorithms, SUP and SUP-$q$, which improve the SCRP algorithms in the sense that they follow the best (global) CRP (SUP) and best horizon $q$ CRP (SUP-$q$) instead of following a specific (given) CRP as SCRP does. The SUP algorithms are shown in [Huang et al.] [2015] to outperform SCRP on many random projections of the NYSE-o and SP500 datasets over two stocks.

A different approach, presented by [Das et al.] [Das et al.] [2013; 2014], is called Online Lazy Updates (OLU) and proposes to deal with transaction costs by taking $b_{t+1}$, the next round portfolio, to be the simplex vector minimizing

$$-\eta \log(\langle b_{t+1}, X_t \rangle) + \frac{1}{2} ||b_{t+1} - b_t||_2^2 + \lambda ||b_{t+1} - b_t||_1.$$

The added $\ell_1$ regularization term was introduced to encourage sparse portfolio updates. The idea is to use this norm as a proxy to the true proportional transaction cost incurred
by the update as given by Equation (3). A drawback of this result is that the $O(\sqrt{T})$ regret bound for OLU holds only for $\lambda \approx \frac{1}{\sqrt{T}}$, which means that the effectiveness of this regularizer diminishes as $T$ gets larger.

Our motivation for the present work is the empirical observation that the performance of the above methods is not consistently satisfying across common benchmark datasets. While these algorithms can handle commissions very well, their starting point in a setting without commissions is hopeless. This stems from the fact that they are all designed to track the performance of an empirically inferior comparison class, namely the class of CRP strategies. As shown in a number of empirical studies (e.g., Borodin et al., Li et al., Li and Hoi, Huang et al. [2004, 2012, 2013]), even the best CRP computed in hindsight itself is not a strong contender relative to other known algorithms such as several mean-reversion methods, which attempt to exploit recurring statistical inefficiencies in market behavior (e.g., Li and Hoi [2012]), and a family of pattern matching algorithms proposed by Algoet and Cover [1988], Györfi et al., Györfi et al. [2008, 2007], and Li et al. [2011]. See Table 3, which also validates the disadvantage of the best CRP (BCRP).

Another critical point with regard to the inferiority of CRP-driven methods is that in a commission-less setting even the most sophisticated universal algorithms do not appear to do any better than the simple uniform constant rebalancing portfolio (UCRP). This observation was first made by Borodin et al. [2004]. We also observe a similar phenomenon in a setting with commissions, where UCRP achieves comparable performance to the known commission aware (CRP-driven) methods (see Section 5).

In this paper we introduce a novel mechanism for commission avoidance combined within a new learning algorithm for ensembles applied over any set of (commission-oblivious) portfolio selection algorithms. We aim to track the best combination of those algorithms rather than the best CRP. Our regret analysis for the proposed method yields optimal logarithmic regret. We report an extensive empirical study of the proposed procedure, where we simulate its performance over the 6 (publicly available) benchmark datasets that were used in this area. The results indicate that the new ensemble algorithm outperforms all existing methods over a range of commission costs.

We note also that we succeeded to extend both the methods of Kozat and Singer; Kozat and Singer [2008, 2009] and Das et al. [Das et al. 2013, 2014] to track algorithms (rather than CRPs) and even devised appropriate regret bounds for them (not reported). The empirical results of these extensions were only marginally better than the original and we abandoned them.

4 Commission Avoidant Portfolio Ensemble

Our commission avoidant portfolio ensemble procedure (henceforth, CAPE) is constructed over a set of $d$ sub-algorithms, $A_1, \ldots, A_d$, whose portfolios at round $t$ are represented by the matrix $P_t^\prime \triangleq (P_{t,1}, \ldots, P_{t,d})$; namely, $P_{t,j}$ is the $n$-ary column vector specifying the round $t$ allocation prescribed by sub-algorithm $j = 1, \ldots, d$ to the $n$ stocks.

To introduce the new commission avoidance mechanism, we present the diagram in Figure 1 depicting the probability simplex over $d$ stocks ($d = 3$ in the diagram).
Figure 1: Inside the stock portfolios simplex (grey triangle) we see the convex hull of round \( t \) portfolios of base algorithms (blue pentagon), the convex hull of round \( t + 1 \) portfolios (red pentagon), and the artificial expert extension (pink area \( A \)).

Inside the simplex we see \( \text{conv}(P_t) \), the convex hull of the \( d \) portfolios in \( P_t \) (the blue pentagon, corresponding to 5 base-algorithms in this diagram). Inside \( \text{conv}(P_t) \) we see the current portfolio, \( \hat{b}_t \), which is depicted inside \( \text{conv}(P_t) \) but, of course, may reside outside this pentagon. The convex hull corresponding to the next round of portfolios, \( \text{conv}(P_{t+1}) \), is also depicted (the red pentagon). Any convex combination of the \( d \) sub-algorithms will naturally yield a stock portfolio residing inside \( \text{conv}(P_{t+1}) \), and rebalancing to this next portfolio will incur commissions proportional to at least the distance from \( \hat{b}_t \) to \( \text{conv}(P_{t+1}) \). The idea is to introduce another synthetic expert whose recommended round \( t + 1 \)st portfolio is precisely \( \hat{b}_t \). This will extend the next round pentagon to be the union of \( \text{conv}(P_{t+1}) \) and the “avoidance” area called \( A \) in the diagram (colored in pink). To this end, we define \( P_t^+ \triangleq (P_{t,1}, \ldots, P_{t,d}, \hat{b}_{t-1}) \), and therefore, the above union is simply \( \text{conv}(P_{t+1}^+) \), which defines the set of feasible choices for our ensemble algorithm for the next round. This revised choice allows \text{CAPE} \ to maintain its current holding and avoid paying commissions. \text{CAPE} \ computes in each round \( t \) the next allocation vector \( w_{t+1} \) for each of the \( d \) algorithms and the artificial expert, \( w_{t+1} \triangleq (w_{t,1}^{t+1}, \ldots, w_{d+1}^{t+1}) \), where the last \( d + 1 \)st coordinate is the weight of the artificial expert. Denote by \( w'_{t+1} \) the projection of \( w_{t+1} \) over the first \( d \) coordinates. The allocation \( w_{t+1} \) is optimized using a regularized online Newton step with the following \( \ell_1 \) regularizer:

\[
R(w'_{t+1}) = \|w_{1}^{t+1} + \ldots + w_d^{t+1}\|_1.
\]

This added penalty encourages \text{CAPE} \ to keep its current holding, thus avoiding com-
Algorithm 1 CAPE

Input: $d$ trading algorithms, Parameters: $T, \eta, \lambda, \epsilon > 0$

Initialize: $P^+_1, w_1 = (\frac{1}{d+1}, \ldots, \frac{1}{d+1}), A_0 = \epsilon I_{d+1}$.

for $i = 1$ to $T$ do
  Play $w_t$ and suffer loss $g_t(w_t) + \lambda R(w_t)$
  Compute portfolios $P^+_{t+1}$ of base algorithms
  Add the artificial expert portfolio $\tilde{b}_t$ to get $P^+_{t+1}$.
  Update: $A_t = A_{t-1} + \nabla g_t(w_t)^T \nabla g_t(w_t)$ and $w_{t+1} = \arg\min_{w \in B} \{ \langle \nabla g_t(w_t), w - w_t \rangle + \lambda R(w) + \eta D_{A_t}(w || w_t) \}$
end for

missions. Taking the following $g_t$ to be our loss function,

$$g_t(w) \triangleq -\log(\langle X_t, P^+ w \rangle),$$

the regret bound we prove for CAPE in Corollary 1 is with respect to $w^*$ which minimizes

$$\sum_t g_t(w^*) + \lambda R(w^*),$$

for a given $\lambda$. This optimal static allocation achieves the best possible return with a regularized avoidance of rebalancing. To explain the pseudo-code of CAPE listed in Algorithm 1, we require the following definitions and notation. Let $A \in \mathbb{R}^{n \times n}$ be any positive-definite matrix. For $x, w \in \mathbb{R}^n$, the Bregman divergence generated by $F_A(w) \triangleq \frac{1}{2} w^T A w$ is

$$D_A(w||x) \triangleq \frac{1}{2} ||w - x||_A^2 = \frac{1}{2} (w - x)^T A (w - x).$$

We denote by $I_n$ the unit matrix of order $n$. For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we denote by $\nabla f(w)$ its gradient (if it is differentiable) and by $f'(w)$ its subgradient.

In each round $t$, CAPE first plays (rebalances its portfolio) according to already computed allocation vector $w_t$ (line 4). In response, the adversary selects a market vector, which determines the following loss, $g_t(w_t)$, as defined in (4) (where $X_t$ is the market vector selected by the adversary for round $t$). CAPE then receives $P^+_{t+1} \in \mathbb{R}^{n \times d}$, the revised portfolios of its sub-algorithms. Next, CAPE constructs $P^+_{t+1}$, the portfolio matrix augmented with the artificial expert, which is then used to optimize its next round prediction vector using a regularized online Newton step. In order to exploit the exp-concavity of the loss function, CAPE utilizes the curvature of the loss function, as embedded in the matrix $A_t$, and then uses the Bregman divergence corresponding to $A_t$ so as to optimize its prediction based on second order information (Newton step).
4.1 Regret Analysis

Let $\alpha$ be a positive real. A convex function $f: \mathbb{R}^n \to \mathbb{R}$ is $\alpha$-exp-concave over the convex domain $B \subset \mathbb{R}^n$ if the function $e^{-\alpha f(x)}$ is concave. It is well known that the class of exp-concave functions strictly contains the class of strongly-convex functions. For example, the loss function typically used in OPS, $f_t(b) = -\log(\langle b, X_t \rangle)$, is exp-concave but not strongly convex.

We conclude this section with two basic lemmas concerning exp-concavity that will be used in the proofs of Lemma 3 and Theorem 1 that follow.

**Lemma 1** [Hazan et al. (2007)]. Let $f$ be an $\alpha$-exp-concave over $B \subset \mathbb{R}^n$ with diameter $D$, such that $\forall x \in B$, $\|\nabla f(x)\|_2 \leq G$. Then, for $\eta \leq \frac{1}{2} \min\{\alpha, \frac{1}{4GD}\}$, and for every $x, y \in B$,

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\eta}{2} (y - x)^T (\nabla f(x)\nabla f(x)^T) (y - x).$$

**Lemma 2** [Hazan et al. (2007)]. Let $f_t: \mathbb{R}^n \to \mathbb{R}$ be $\alpha$-exp-concave, and let $A_t$ be as in Algorithm 1. Then, for $\eta = \frac{1}{2} \min\{\alpha, \frac{1}{4GD}\}$ and $\epsilon_0 = \frac{1}{\eta D^2}$,

$$\sum_{t=1}^{T} \|\nabla f_t(w_t)\|_{A_t^{-1}}^2 \leq n \log T.$$

We consider a standard online convex optimization game [Zinkevich, 2003] where in each round $t$ the online player selects a point $w_t$ in a convex set $B$; then a convex payoff function $f_t$ is revealed, and the player suffers loss $f_t(w_t)$. In an adversarial setting, where $f_t$ is selected in the worst possible way, it is impossible to guarantee absolute online performance. Instead, the objective of the online player is to achieve sublinear regret relative to the best choice in hindsight, $w_* \triangleq \arg\min_{w \in B} \sum_t f_t(w)$, where regret is

$$\text{Regret} \triangleq \sum_{t=1}^{T} (f_t(w_t) - f_t(w_*)).$$

The mirror descent algorithm of Nemirovsky and Yudin [1985] and Beck and Teboulle [2003] for online convex optimization was extended by Duchi et al. [2010] as follows. Instead of solving in each round

$$w_{t+1} = \arg\min_{w \in K} \{\eta \langle \nabla f_t(w_t), w - w_t \rangle + D(w||w_t)\},$$

where $D(x||y)$ is the Bregman divergence generated by some strongly convex function $\psi$, they proposed to solve

$$w_{t+1} = \arg\min_{w \in K} \{\eta \langle \nabla f_t(w_t), w - w_t \rangle + \eta r(w) + D(w||w_t)\},$$

where $r$ is some convex function which is not necessarily smooth. They proved that their revised method guarantees $O(\sqrt{T})$ regret relative to the best choice in hindsight whenever $f$ is convex. Moreover, a sharper $O(\log T)$ regret bound was shown for
strongly convex $f$. This extension, called composite objective mirror descent (COMID), opened the door to applications in many fields and, in particular, to the possibility of using an $L_1$ regularization term, which encourages sparsity. In our regret proof we use a similar analysis and extend the COMID framework to deal with exp-concave loss functions.

We assume throughout w.l.o.g. that $r(w) \geq 0$, $r(w_1) = 0$ and that $f_t$ is exp-concave and twice differentiable.

We state without proof the following results (Lemma 3 and Theorem 1). Full proofs of these statements will be presented in the long version of this paper.

**Lemma 3.** Let $f_t$ be $\alpha$-exp-concave over $B \subset \mathbb{R}^n$ with diameter $D$, such that $\forall w \in B$, $||\nabla f(w)||_2 \leq G$. If $w_t$ is the prediction of Algorithm 1 in round $t$, then, for $\eta = \frac{1}{2} \min\{\alpha, \frac{1}{4GD}\}$ and for any $w^* \in B$,

$$\frac{1}{\eta} [f_t(w_t) - f_t(w_*) + r(w_{t+1}) - r(w_*)] \leq D_{A_{t-1}}(w_*|w_t) - D_{A_t}(w_*|w_{t+1}) + \frac{1}{2\eta^2} ||\nabla f_t(w_t)||^2_{A_{t-1}}.$$

**Theorem 1.** Let $f_t$ be $\alpha$-exp-concave over $K \subset \mathbb{R}^n$, $\eta = \frac{1}{2} \min\{\alpha, \frac{1}{4GD}\}$ and let $\epsilon_0 = \frac{1}{2\eta^2}$. If $(w_1, w_2, \ldots, w_T)$ are the predictions of Algorithm 1 then for any $w_* \in B$,

$$\sum_{t=1}^{T} (f_t(w_t) + r(w_t) - f_t(w_*) - r(w_*)) = O(\log T).$$

**Corollary 1.** For Algorithm 1, for appropriate $\epsilon, \eta > 0$, and every $\lambda \geq 0$, for any fixed point $w_* \in B$, it holds that

$$\sum_{t=1}^{T} g_t(w_t) + \lambda R(w) - g_t(w_*) - \lambda R(w_*) = O(\log T).$$

### 5 Empirical Study

In this section we present an empirical study of CAPE, examining how well CAPE controls and operates a set of base-algorithms in comparison to both the base-algorithms themselves as well as the existing competition. We selected the following set of base-algorithms, all of which are implemented in the Li et al. [2015] OLPS simulator. Unless otherwise specified, all critical parameters of the base-algorithms were set to the default parameters of the simulator in all experiments. We selected the following four base-algorithms:

- Anticor [Borodin et al., 2004]: one of the first algorithms designed to exploit mean-reversion via (anti) correlation analysis.

In our applications we used the parameter $\epsilon = \eta = 1$. In general, these parameters can be calibrated according to market variability; see Agarwal et al. [2006].
Table 1: Cumulative wealth of CAPE, its base algorithms and other known commission aware algorithms

| Commission rate | Dataset | Base Algorithms | CAPE | Other Algorithms |
|-----------------|---------|----------------|------|------------------|
|                 |         | EG  | PAMR | Anticor | OLMAR | Naive | WF | OLU | SCR | UP | SUP | UCRP |
| γ = 0.25%       | NYSE-N  | 28.34 | 1.67 | 4.18E3 | 9.8E4 | 289   | 407 | 18.06 | 18.94 | 30.70 | NA | 28.59 |
|                 | NYSE-O  | 25   | 3.9E10 | 5.8E5 | 3.3E11 | 9.4E5 | 5.4E6 | 19.14 | 18.86 | 20.95 | NA | 24.9 |
|                 | MSCI    | 0.91 | 0.14 | 1.73 | 4.67 | 1.22 | 1.4 | 0.91 | 0.902 | 0.93 | 0.66 | 0.91 |
|                 | DJIA    | 0.8  | 0.2  | 1.28 | 1.47 | 1.12 | 1.03 | 0.824 | 0.77 | 0.82 | 0.70 | 0.78 |
|                 | TSE     | 1.55 | 23.6 | 13.52 | 19.25 | 9.85 | 7.84 | 1.62 | 1.618 | 1.46 | 1.63 | 1.55 |
|                 | SP500   | 1.59 | 0.3  | 3.08 | 1.32 | 2.09 | 2.31 | 1.35 | 1.419 | 1.60 | 1.43 | 1.60 |
| γ = 0.5%        | NYSE-N  | 25.73 | 0   | 82.74 | 170.1 | 126.8 | 144.4 | 18.02 | 18.8 | 29.35 | NA | 25.9 |
|                 | NYSE-O  | 23.08 | 1.9E5 | 1.69E4 | 6.3E8 | 1E5 | 8.6E4 | 17.48 | 18.86 | 19.86 | NA | 22.9 |
|                 | MSCI    | 0.9  | 0.14 | 1.08 | 1.35 | 1.11 | 1.26 | 0.90 | 0.901 | 0.92 | 0.56 | 0.9 |
|                 | DJIA    | 0.79 | 0.08 | 1.021 | 0.88 | 0.93 | 1.01 | 0.819 | 0.77 | 0.82 | 0.67 | 0.78 |
|                 | TSE     | 1.52 | 2.09 | 6.36 | 5.96 | 6.4 | 4.33 | 1.63 | 1.617 | 1.45 | 1.61 | 1.52 |
|                 | SP500   | 1.55 | 0.02 | 1.69 | 0.38 | 1.79 | 2.16 | 1.33 | 1.418 | 1.58 | 1.39 | 1.56 |
| γ = 0.75%       | NYSE-N  | 23.44 | 0   | 1.62 | 0.2 | 55.6 | 65.66 | 18.01 | 18.65 | 28.45 | NA | 23.4 |
|                 | NYSE-O  | 21.30 | 1.14 | 465 | 1.1E6 | 1.1E4 | 4.1E3 | 17.47 | 18.86 | 19.01 | NA | 21.0 |
|                 | MSCI    | 0.89 | 0 | 0.67 | 0.36 | 0.75 | 1.16 | 0.90 | 0.90 | 0.92 | 0.54 | 0.89 |
|                 | DJIA    | 0.78 | 0.03 | 0.8 | 0.36 | 0.78 | 0.98 | 0.77 | 0.81 | 0.59 | 0.67 | 0.78 |
|                 | TSE     | 1.49 | 0.1 | 2.99 | 1.83 | 4.15 | 2.85 | 1.61 | 1.617 | 1.43 | 1.58 | 1.48 |
|                 | SP500   | 1.51 | 0.92 | 0.1 | 0.38 | 1.52 | 2.01 | 1.33 | 1.416 | 1.57 | 1.35 | 1.52 |
| γ = 1%          | NYSE-N  | 21.36 | 0   | 0.03 | 0 | 24.36 | 36 | 17.98 | 18.6 | 26.37 | NA | 21.2 |
|                 | NYSE-O  | 19.66 | 0 | 13.09 | 2.1E3 | 1.2E3 | 440.73 | 17.4 | 18.82 | 18.23 | NA | 19.4 |
|                 | MSCI    | 0.88 | 0 | 0.42 | 0.1 | 0.75 | 1.01 | 0.90 | 0.90 | 0.91 | 0.44 | 0.88 |
|                 | DJIA    | 0.77 | 0.01 | 0.64 | 0.1 | 0.65 | 0.92 | 0.76 | 0.77 | 0.81 | 0.55 | 0.78 |
|                 | TSE     | 1.45 | 0.01 | 1.41 | 0.56 | 2.5 | 1.69 | 1.61 | 1.616 | 1.43 | 1.57 | 1.45 |
|                 | SP500   | 1.47 | 0.5 | 0.03 | 1.3 | 1.82 | 1.33 | 1.415 | 1.52 | 1.34 | 1.48 |

- Passive Aggressive Mean-Reversion (PAMR) [Li et al., 2012]: designed to exploit mean-reversion using “passive-aggressive” learning [Crammer et al., 2006].
- Online Moving Average Reversion (OLMAR) [Li and Hoi, 2012]: designed to exploit mean-reversion based on moving average predictions. OLMAR is known to be a strong performer in many benchmark datasets.
- Exponentiated Gradient (EG) [Helmbold et al., 1998]: one of the early universal algorithms. This algorithm is CRP-driven and as mentioned above, is not expected to serve as a useful ingredient in our ensemble. It is included to validate CAPE’s ability to avoid its portfolio recommendations.

In addition we compare performance to the following 5 algorithms, all discussed in Section 3:

- UP: universal portfolios with commissions of Cover and Ordentlich [1991]
Table 2: Some properties of the datasets

| DATASET | STARTING DAY | # DAYS | # STOCKS |
|---------|--------------|--------|----------|
| NYSE-N  | 1/1/1983     | 6431   | 23       |
| NYSE-O  | 7/3/1962     | 5651   | 36       |
| MSCI    | 4/1/2006     | 1043   | 24       |
| DJIA    | 1/14/2001    | 507    | 30       |
| TSE     | 1/4/1994     | 1258   | 88       |
| SP500   | 1/2/1998     | 1276   | 25       |

To the best of our knowledge, this is the first online portfolio selection algorithm that was considered and analyzed with commissions, by Blum and Kalai [1999].

- **SCRP**: Semi-constant rebalanced portfolios of Kozat and Singer [2011]. This algorithm is an ensemble of sequences of constant rebalancing portfolios each diluting the number of allowed trading rounds.
- **SUP**: An extension of SCRP; instead of following a fixed CRP, it follows BCRP [Huang et al., 2015].
- **OLU**: Utilizes gradient steps with an added $\ell_1$ regularization term to encourage “lazy” portfolio updates [Das et al., 2013].
- **UCRP**: The uniform CRP. This is a fixed uniform rebalancing, which is obviously a naive commission-oblivious benchmark.

We experimented with the 6 datasets that were used in the relevant literature (and appear in the public domain). These datasets span several types of market conditions, number of stocks, and total trading periods. It is worth noting that the hardest set to profitably trade (using a long-only portfolio selection algorithm) among the six is DJIA (the Dow Jones Industrial Average), which captures a bear market where 25 of the 30 DJIA stocks declined. Some properties of these sets are summarized in Table 2.

Before presenting the results with commissions, we refer the reader to the interesting performance of the base-algorithms without commissions as summarized in Table 3. With the exception of one crash of PAMR in the DJIA set, the three mean-reverting algorithms achieve unrealistically outstanding results and clearly outperform EG and BCRP by a wide margin. This heavenly success is almost completely eliminated when introducing commissions (see below).

Each dataset was considered with four commission rates ($\gamma$): 0.25%, 0.5%, 0.75%, and 1.00%. Table 1 summarizes the results of our experiments, where cumulative

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5 The performance on the NYSE-O dataset (which for years served as the only benchmark dataset in this area) remains excellent, for the most part. We note that this dataset is believed to suffer from extreme survival bias.

6 At the time of writing, there are deep discount brokers, such as Interactive Brokers, whose maximal rate (all included) is 0.5%. These high rates will be incurred when trading very cheap stocks. Significantly better rates can be obtained when trading highly priced stocks and through rebates received for large volumes.
Table 3: No commissions: known CRP-based and other (mean-reverting) algorithms

| Dataset | EG    | BCRP  | ANTCOR | PAMR  | OLMAR |
|---------|-------|-------|--------|-------|-------|
| NYSE-N  | 31    | 119.8 | 6.2E6  | 1.2E6 | 4E8   |
| NYSE-O  | 27.09 | 250.6 | 2.4E8  | 5E15  | 6E16  |
| MSCI    | 0.92  | 1.5   | 3.2    | 15.2  | 14.8  |
| DJIA    | 0.8   | 1.24  | 2.29   | 0.68  | 2.7   |
| TSE     | 1.59  | 6.78  | 39.36  | 264.8 | 69.9  |
| SP500   | 1.63  | 4     | 5.9    | 5.1   | 16.9  |

wealth is reported. Each row in this table corresponds to a different market and commission rate ($\gamma$), and columns correspond to algorithms. The first block of columns corresponds to the base-algorithms, the second block to CAPE, and the last block to the known (commission aware) methods.

Observing first the known methods (third block), we see that all of them are extremely resilient to commissions in this range. However, their performance is not at all superior to that of the simple uniform-CRP (UCRP), which is among the winners in several datasets. Similar observations were previously made in a setting without commissions where UCRP achieved near identical results to CRP-driven algorithms such as UP and EG [Borodin et al., 2004]. Next, consider the first block containing the (known) base-algorithms. With the exception of EG (CRP-driven), all the algorithms crash completely with increasing commission rates on a majority of the datasets.

For CAPE we present two sets of results, based on two different approaches for setting its hyper-parameter $\lambda$. The first one, called ‘Naive’, is based on a naive assignment, $\lambda = 0.005$, fixed over all datasets and all commission rates. Obviously, one cannot except to find one “universal” $\lambda$ that will fit all these datasets and commission rates. Therefore, we also considered a more pragmatic approach where we dynamically calibrated the choice of $\lambda$. We thus present another set of results for CAPE in which this parameter was dynamically optimized using a standard walk-forward procedure [Pardo, 1992], whereby $\lambda$ was sequentially optimized w.r.t. cumulative wealth for the next period over a sliding window of the previous $w$ periods. This setting is called “WF” (walk forward). The WF setting was applied with a fixed $w = 25$ for all datasets and commission rates. To assess the criticality of this choice we conducted a sensitivity analysis where we computed the sensitivity of total cumulative wealth with respect to choices of window size $w \in [10, 50]$ on two entire datasets. The resulting sensitivity graphs are shown in Figure 2. It is evident that overall performance is relatively stable as a function of $w$ over these sets.

7While risk-adjusted measures such as the Sharpe ratio are important as well, the total cumulative wealth has the same scale of the commission paid. In the full version we will also report on other performance measures.

8NYSE-X results for the SUP algorithm requires huge computational resources and over 72 computation hours were not sufficient to complete these runs.

9This rather arbitrary choice was based on a preliminary examination where we roughly estimated the dynamic range of $\lambda$ to be $[0, 0.05]$ over the first dataset.
Examining first the Naive results (with a fixed $\lambda$), we see that CAPE outperforms the existing methods with commission rates 0.25% and 0.5% but performance deteriorates with the higher rates, which are likely to require stronger regularization (larger $\lambda$). In the second WF setting (walk-forward) CAPE impressively outperforms all competing methods on all datasets. Moreover, it is evident that CAPE successfully exploits the base algorithms even in cases where all of them crash. See for example the MSCI dataset with $\gamma = 0.75\%, 1\%$.

We examined the portfolio composition of CAPE over its base algorithms throughout the runs. In all datasets and commission rates CAPE allocated nearly all its “non-static” weight (not allocated to the artificial expert) to the two top performing base algorithms, which were almost always among the three mean-reverting algorithms (OLMAR, Anticor and PAMR). The weight allocation for EG was negligible through most of the trading periods. This behavior is consistent with the underlying idea behind CAPE, which is constructed to receive its portfolio recommendations from top performing (without commission) base algorithms. Indeed, for the most part CRP-driven algorithms such as EG should not be tracked if money is to be made.

![Figure 2: Wealth sensitivity to window size on two datasets](image)

6 Concluding Remarks

We presented an ensemble learning strategy for portfolio selection algorithms. As far as we know this is the first commission-aware method designed to exploit any given set of (commission-oblivious) algorithms beyond CRPs in an online adversarial setting. Our learning algorithm extends COMID to accommodate exp-concave functions, and using Newton steps we achieve logarithmic regret bound for our procedure. The demonstrated empirical performance improves the state-of-the-art across the board, both in terms of datasets and commission rate range.

An important challenge would be to combine effective mechanisms to dynamically
control risk. However, within a regret minimization framework, like the one we consider here, this challenge is highlighted by the impossibility of achieving sub-linear regret in the adversarial setting with respect to risk adjusted measures such as the Sharpe ratio [Even-Dar et al., 2006].

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