Abstract. This chapter is a short introduction to Sullivan models. In particular, we find the Sullivan model of a free loop space and use it to prove the Vigué-Poirrier-Sullivan theorem on the Betti numbers of a free loop space.

In the previous chapter, we have seen the following theorem due to Gromoll and Meyer.

**Theorem 0.1.** Let $M$ be a compact simply connected manifold. If the sequence of Betti numbers of the free loop space on $M$, $M^S_1$, is unbounded then any Riemannian metric on $M$ carries infinitely many non trivial and geometrically distinct closed geodesics.

In this chapter, using Rational homotopy, we will see exactly when the sequence of Betti numbers of $M^S_1$ over a field of characteristic 0 is bounded (See Theorem 6.1 and its converse Proposition 5.5). This was one of the first major applications of rational homotopy.

Rational homotopy associates to any rational simply connected space, a commutative differential graded algebra. If we restrict to almost free commutative differential graded algebras, that is "Sullivan models", this association is unique.

### 1 Graded differential algebra

#### 1.1 Definition and elementary properties

All the vector spaces are over $\mathbb{Q}$ (or more generally over a field $k$ of characteristic 0). We will denote by $\mathbb{N}$ the set of non-negative integers.

**Definition 1.1.** A (non-negatively upper) graded vector space $V$ is a family $\{V^n\}_{n \in \mathbb{N}}$ of vector spaces. An element $v \in V_i$ is an element of $V$ of degree $i$. The degree of $v$ is denoted $|v|$. A differential $d$ in $V$ is a sequence of linear maps
d^n : V^n → V^{n+1} such that d^{n+1}∂ d^n = 0, for all n ∈ N. A differential graded vector space or complex is a graded vector space equipped with a differential. A morphism of complexes f : V → W is a quasi-isomorphism if the induced map in homology H(f) : H(V) → H(W) is an isomorphism in all degrees.

**Definition 1.2.** A graded algebra is a graded vector space A = {A^n}_{n∈N}, equipped with a multiplication µ : A^p ⊗ A^q → A^{p+q}. The algebra is commutative if ab = (−1)^{|a||b|}ba for all a, b ∈ A.

**Definition 1.3.** A differential graded algebra or dga is a graded algebra equipped with a differential d : A^n → A^{n+1} which is also a derivation: this means that for a and b ∈ A

\[ d(ab) = (da)b + (−1)^{|a|}a(db). \]

A cdga is a commutative dga.

**Example 1.4.** 1) Let (B, dB) and (C, dC) be two cdgas. Then the tensor product B ⊗ C equipped with the multiplication

\[ (b ⊗ c)(b' ⊗ c') := (−1)^{|c||c'|}bb' ⊗ cc' \]

and the differential

\[ d(b ⊗ c) = (db) ⊗ c + (−1)^{|b|}b ⊗ dc. \]

is a cdga. The tensor product of cdgas is the sum (or coproduct) in the category of cdgas.

2) More generally, let f : A → B and g : A → C be two morphisms of cdgas. Let B ⊗_A C be the quotient of B ⊗ C by the sub graded vector spanned by elements of the form bf(a) ⊗ c − b ⊗ g(a)c, a ∈ A, b ∈ B and c ∈ C. Then B ⊗_A C is a cdga such that the quotient map B ⊗ C → B ⊗_A C is a morphism of cdgas. The cdga B ⊗_A C is the pushout of f and g in the category of cdgas:

3) Let V and W be two graded vector spaces. We denote by ΛV the free graded commutative algebra on V.

If V = Qv, i.e. is of dimension 1 and generated by a single element v, then

- ΛV is E(v) = Q ⊕ Qv, the exterior algebra on v if the degree of v is odd and
- ΛV is Q[v] = ⊕_{n∈N}Qv^n, the polynomial or symmetric algebra on v if the degree of v is even.

Since Λ is left adjoint to the forgetful functor from the category of commutative graded algebras to the category of graded vector spaces, Λ preserves sums: there
is a natural isomorphism of commutative graded algebras $\Lambda(V \oplus W) \cong \Lambda V \otimes \Lambda W$.

Therefore $\Lambda V$ is the tensor product $E(V^{\text{odd}}) \otimes S(V^{\text{even}})$ of the exterior algebra on the generators of odd degree and of the polynomial algebra on the generators of even degree.

**Definition 1.5.** Let $f : A \to B$ be a morphism of commutative graded algebras. Let $d : A \to B$ be a linear map of degree $k$. By definition, $d$ is a $(f, f)$-derivation if for $a$ and $b \in A$

$$d(ab) = (da)f(b) + (-1)^{|a|}f(a)(db).$$

**Property 1.6** (Universal properties). 1) Let $i_B : B \hookrightarrow B \otimes \Lambda V, b \mapsto b \otimes 1$ and $i_V : V \hookrightarrow B \otimes \Lambda V, v \mapsto 1 \otimes v$ be the inclusion maps. Let $\varphi : B \to C$ be a morphism of commutative graded algebras. Let $f : V \to C$ be a morphism of graded vector spaces. Then $\varphi$ and $f$ extend uniquely to a morphism $B \otimes \Lambda V \to C$ of commutative graded algebras such that the following diagram commutes

$$
\begin{array}{ccc}
B & \xrightarrow{\varphi} & C \\
\downarrow{i_B} & & \downarrow{i_V} \\
B \otimes \Lambda V & \xrightarrow{f} & V \\
\end{array}
$$

2) Let $d_B : B \to B$ be a derivation of degree $k$. Let $d_V : V \to B \otimes \Lambda V$ be a linear map of degree $k$. Then there is a unique derivation $d$ such that the following diagram commutes.

$$
\begin{array}{ccc}
B & \xrightarrow{i_B} & B \otimes \Lambda V \\
\downarrow{d_B} & \swarrow{\cong d} & \downarrow{i_V} \\
B & \xrightarrow{i_B} & B \otimes \Lambda V \\
\end{array}
$$

3) Let $f : \Lambda V \to B$ be a morphism of commutative graded algebras. Let $d_V : V \to B$ be a linear map of degree $k$. Then there exists a unique $(f, f)$-derivation $d$ extending $d_V$:

$$
\begin{array}{ccc}
V & \xrightarrow{d_V} & B \\
\downarrow{i_V} & \searrow{\exists d} & \\
\Lambda V & & \\
\end{array}
$$

**Proof.** 1) Since $\Lambda V$ is the free commutative graded algebra on $V$, $f$ can be extended to a morphism of graded algebras $\Lambda V \to C$. Since the tensor product of commutative graded algebras is the sum in the category of commutative graded algebras, we obtain a morphism of commutative graded algebras from $B \otimes \Lambda V$ to $C$.

2) Since $b \otimes v_1 \ldots v_n$ is the product $(b \otimes 1)(1 \otimes v_1) \ldots (1 \otimes v_n)$, $d(b \otimes v_1 \ldots v_n)$
is given by
\[ d_B(b) \otimes v_1 \ldots v_n + \sum_{i=1}^{n} (-1)^k(b) \otimes v_1 \ldots v_{i-1})(dv_i)(1 \otimes v_{i+1} \ldots v_n) \]

3) Similarly, \( d(v_1 \ldots v_n) \) is given by
\[ \sum_{i=1}^{n} (-1)^k(f(v_1) \ldots f(v_{i-1})dv_i)f(v_{i+1}) \ldots f(v_n) \]

\[
1.2 \textbf{Sullivan models of spheres}
\]

\textbf{Sullivan models of odd spheres} \( S^{2n+1}, n \geq 0 \).

Consider a cdga \( A(S^{2n+1}) \) whose cohomology is isomorphic as graded algebras to the cohomology of \( S^{2n+1} \) with coefficients in \( k \):
\[ H^*(A(S^{2n+1})) \cong H^*(S^{2n+1}). \]

When \( k \) is \( \mathbb{R} \), you can think of \( A \) as the De Rham algebra of forms on \( S^{2n+1} \). There exists a cycle \( v \) of degree \( 2n + 1 \) in \( A(S^{2n+1}) \) such that
\[ H^*(A(S^{2n+1})) = \Lambda[v]. \]

The inclusion of complexes \((kv, 0) \hookrightarrow A(S^{2n+1})\) extends to a unique morphism of cdgas \( m : (Av, 0) \rightarrow A(S^{2n+1}) \) (Property 1.6):

\[ (kv, 0) \longrightarrow A(S^{2n+1}) \]
\[ \downarrow \quad \exists m \]
\[ (Av, 0) \]

The induced morphism in homology \( H^*(m) \) is an isomorphism. We say that \( m : (Av, 0) \cong A(S^{2n+1}) \) is a Sullivan model of \( S^{2n+1} \).

\textbf{Sullivan models of even spheres} \( S^{2n}, n \geq 1 \).

Exactly as above, we construct a morphism of cdga \( m_1 : (Av, 0) \rightarrow A(S^{2n}) \). But now, \( H(m_1) \) is not an isomorphism:
\[ H(m_1)(v) = [v]. \]

Therefore \( H(m_1)(v^2) = [v^2] = [v]^2 = 0. \) Since \([v^2] = 0\) in \( H^*(A(S^{2n}))\), there exists an element \( \psi \in A(S^{2n}) \) of degree \( 4n - 1 \) such that
\[ d\psi = v^2. \]

Let \( w \) denote another element of degree \( 4n - 1 \). The morphism of graded vector spaces \( kv \oplus kw \hookrightarrow A(S^{2n}) \), mapping \( v \) to \( v \) and \( w \) to \( \psi \) extends to a unique morphism of commutative graded algebras \( m : A(v, w) \rightarrow A(S^{2n}) \) (1) of
Property 1.6:

\[ \begin{array}{ccc}
  \mathbb{k}v \oplus \mathbb{k}w & \rightarrow & A(S^{2n}) \\
  \downarrow & & \downarrow \\
  \Lambda(v, w) & \rightarrow & \Lambda(v, w)
\end{array} \]

The linear map of degree +1, \( d_V : V := \mathbb{k}v \oplus \mathbb{k}w \rightarrow \Lambda(v, w) \) mapping \( v \) to 0 and \( w \) to \( v^2 \) extends to a unique derivation \( d : \Lambda(v, w) \rightarrow \Lambda(v, w) \) (2) of Property 1.6).

\[ \begin{array}{ccc}
  \mathbb{k}v \oplus \mathbb{k}w & \rightarrow & \Lambda(v, w) \\
  \downarrow & & \downarrow \\
  \Lambda(v, w) & \rightarrow & \Lambda(v, w)
\end{array} \]

Since \( d \) is a derivation of odd degree, \( d \circ d \) (which is equal to \( 1/2[d, d] \)) is again a derivation. The following diagram commutes

\[ \begin{array}{ccc}
  V & \xrightarrow{d_V} & \Lambda V \\
  \downarrow d & & \downarrow d \\
  \Lambda V & \xrightarrow{d} & \Lambda V
\end{array} \]

Since the composite \( d \circ d_V \) is null, by unicity (2) of Property 1.6, the derivation \( d \circ d \) is also null. Therefore \( (\Lambda V, d) \) is a cdga. This is the general method to check that \( d \circ d = 0 \).

Denote by \( d_A \) the differential on \( A(S^{2n}) \). Let’s check now that \( d_A \circ m = m \circ d \).

Since \( d_A \) and \( d \) are both \((\text{id}, \text{id})\)-derivations, \( d_A \circ m \) and \( m \circ d \) are both \((m, m)\)-derivations.

Since \( d_A(m(v)) = d_A(v) = 0 = m(0) = m(d(v)) \) and \( d_A(m(w)) = d_A(\psi) = v^2 = m(v^2) = m(d(w)) \), \( d_A \circ m \) and \( m \circ d \) coincide on \( V \). Therefore by unicity (3) of Property 1.6, \( d_A \circ m = m \circ d \). Again, this method is general. So finally, we have proved that \( m \) is a morphism of cdgas. Now we prove that \( H(m) \) is an isomorphism, by checking that \( H(m) \) sends a basis to a basis.

2 Sullivan models

2.1 Definitions

Let \( V \) be a graded vector space. Denote by \( V^+ = V^1 \geq 1 \) the sub graded vector space of \( V \) formed by the elements of \( V \) of positive degrees: \( V = V^0 \oplus V^+ \).

Definition 2.1. A relative Sullivan model (or cofibration in the category of cdgas) is a morphism of cdgas of the form

\[ (B, d_B) \leftrightarrow (B \otimes AV, d), b \mapsto b \otimes 1 \]
where
- $H^0(B) \cong \mathbf{k}$,
- $V = V^1$,  
- and $V$ is the direct sum of graded vector spaces $V(k)$:

$$\forall n, V^n = \bigoplus_{k \in \mathbb{N}} V(k)^n$$

such that $d : V(0) \to B \otimes \mathbf{k}$ and $d : V(k) \to B \otimes \Lambda(V(<k))$. Here $V(<k)$ denotes the direct sum $V(0) \oplus \cdots \oplus V(k-1)$.

Let $k \in \mathbb{N}$. Denote by $\Lambda^kV$ the sub graded vector space of $\Lambda V$ generated by elements of the form $v_1 \wedge \cdots \wedge v_k$, $v_i \in V$. Elements of $\Lambda^k V$ have by definition wordlength $k$. For example $\Lambda V = \mathbf{k} \oplus V \oplus \Lambda \geq 2 V$.

**Definition 2.2.** A relative Sullivan model $(B,d_B) \hookrightarrow (B \otimes \Lambda V,d)$ is minimal if $d : V \to B^+ \otimes \Lambda V + B \otimes \Lambda \geq 2 V$. A (minimal) Sullivan model is a (minimal) relative Sullivan model of the form $(B,d_B) = (\mathbf{k},0) \hookrightarrow (\Lambda V,d)$.

**Example 2.3.** [5, end of the proof of Lemma 23.1] Let $(\Lambda V,d)$ be cdga such that $V = V_{\geq 2}$. Then $(\Lambda V,d)$ is a Sullivan model.

**proof assuming the minimality condition.** [5, p. 144] Suppose that $d : V \to \Lambda \geq 2 V$. In this case, the $V(k)$ are easy to define: let $V^k := V^k$ for $k \in \mathbb{N}$. Let $v \in V^k$. By the minimality condition, $dv$ is equal to a sum $\sum x_i y_i$ where the non trivial elements $x_i$ and $y_i$ are both of positive length and therefore both of degree $\geq 2$. Since $|x_i| + |y_i| = |dv| = k + 1$, both $x_i$ and $y_i$ are of degree less than $k$. Therefore $dv$ belongs to $\Lambda(V(<k)) = \Lambda(V(<k))$. \hfill \square

**Property 2.4.** The composite of relative Sullivan models is again a Sullivan relative model.

**Definition 2.5.** Let $C$ be a cdga. A (minimal) Sullivan model of $C$ is a (minimal) Sullivan model $(\Lambda V,d)$ such that there exists a quasi-isomorphism of cdgas $(\Lambda V,d) \cong C$. Let $\varphi : B \to C$ be a morphism of cdgas. A (minimal) relative Sullivan model of $\varphi$ is a (minimal) relative Sullivan model $(B,d_B) \hookrightarrow (B \otimes \Lambda V,d)$ such that $\varphi$ can be decomposed as the composite of the relative Sullivan model and of a quasi-isomorphism of cdgas:

$$\begin{array}{ccc}
B & \xrightarrow{\varphi} & C \\
\downarrow & & \uparrow \\
B \otimes \Lambda V & \xrightarrow{=} & C
\end{array}$$

**Theorem 2.6.** Any morphism $\varphi : B \to C$ of cdgas admits a minimal relative Sullivan model if $H^0(B) \cong \mathbf{k}$, $H^0(\varphi)$ is an isomorphism and $H^1(\varphi)$ is injective.
This theorem is proved in general by Proposition 14.3 and Theorem 14.9 of [5]. But in practice, if $H^1(\varphi)$ is an isomorphism, we construct a minimal relative Sullivan model, by induction on degrees as in Proposition 12.2 of [5].

### 2.2 An example of relative Sullivan model

Consider the minimal Sullivan model of an odd sphere found in section 1.2

\[ (\Lambda v, 0) \xrightarrow{\sim} A(S^{2n+1}). \]

Assume that $n \geq 1$. Consider the multiplication of $\Lambda v$: the morphism of cdgas

\[ \mu : (\Lambda v_1, 0) \otimes (\Lambda v_2, 0) \to (\Lambda v, 0), v_1 \mapsto v, v_2 \mapsto v. \]

Recall that $v$, $v_1$ and $v_2$ are of degree $2n + 1$.

Denote by $sv$ an element of degree $|sv| = |s| + |v| = -1 + |v|$. The operator $s$ of degree $-1$ is called the suspension.

We construct now a minimal relative Sullivan model of $\mu$. Define $d(sv) = v_2 - v_1$. Let $m : \Lambda(v_1, v_2, sv), d \to (\Lambda v, 0)$ be the unique morphism of cdgas extending $\mu$ such that $m(sv) = 0$.

\[
\begin{array}{ccc}
(\Lambda v_1, 0) \otimes (\Lambda v_2, 0) & \xrightarrow{\mu} & (\Lambda v, 0) \\
& \searrow & \downarrow m \\
& \Lambda(v_1, v_2, sv, d) &
\end{array}
\]

**Definition 2.7.** Let $A$ be a differential graded algebra such that $A^0 = k$. The complex of indecomposables of $A$, denoted $Q(A)$, is the quotient $A^+ / \mu(A^+ \otimes A^+)$. The complex of indecomposables of $(\Lambda v, 0)$, $Q((\Lambda v, 0))$, is $(kv, 0)$ while

\[ Q(\Lambda(v_1, v_2, sv, d)) = (kv_1 \oplus kv_2 \oplus ksv, d(sv) = v_2 - v_1). \]

The morphism of complexes $Q(m) : (kv_1 \oplus kv_2 \oplus ksv, d(sv) = v_2 - v_1) \to (kv, 0)$ map $v_1$ to $v$, $v_2$ to $v$ and $sv$ to 0. It is easy to check that $Q(m)$ is a quasi-isomorphism of complexes.

By Proposition 14.13 of [5], since $m$ is a morphism of cdgas between Sullivan model, $Q(m)$ is a quasi-isomorphism of if and only if $m$ is a quasi-isomorphism. So we have proved that $m$ is a quasi-isomorphism and therefore

\[ (\Lambda v_1, 0) \otimes (\Lambda v_2, 0) \cong \Lambda(v_1, v_2, sv, d) \]

is a minimal relative Sullivan model of $\mu$. Consider the following commutative
Since \( \gamma \) is \((v,sv)\), it is easy to check that the cdga \( \Lambda \to \Lambda(v,sv) \) is isomorphic to \( \Lambda(v,sv) \). As we will explain later, we have computed in fact, the minimal Sullivan model \( \Lambda(v,sv) \) of the free loop space \((S^{2n+1})^S\). In particular, the cohomology algebra \( H^*(S^{2n+1})^S; k \) is isomorphic to \( \Lambda(v,sv) \). We can deduce easily that for \( p \in \mathbb{N} \), \( \dim H^p((S^{2n+1})^S) \leq 1 \). So we have shown that the sequence of Betti numbers of the free loop space on odd dimensional spheres is bounded.

### 2.3 The relative Sullivan model of the multiplication

#### Proposition 2.8. [6] Example 2.48]
Let \((\Lambda V,d)\) be a relative minimal Sullivan model with \( V = V^{\geq 2} \) (concentrated in degrees \( \geq 2 \)). Then the multiplication \( \mu : (\Lambda V,d) \otimes (\Lambda V,d) \to (\Lambda V,d) \) admits a minimal relative Sullivan model of the form \((\Lambda V \otimes \Lambda V \otimes \Lambda sV, D)\).

**Constructive proof.** We proceed by induction on \( n \in \mathbb{N}^+ \) to construct quasi-isomorphisms of cdgas \( \varphi_n : (\Lambda V^{\leq n} \otimes \Lambda V^{\leq n} \otimes \Lambda sV^{\leq n}, D) \xrightarrow{\cong} (\Lambda V^{\leq n}, d) \) extending the multiplication on \( \Lambda V^{\leq n} \).

Suppose that \( \varphi_n \) is constructed. We now define \( \varphi_{n+1} \) extending \( \varphi_n \) and \( \mu \), the multiplication on \( \Lambda V \). Let \( v \in V^{n+1} \). Then \( d(v) \in \Lambda^{\geq 2}(V^{\leq n}) \) and \( \varphi_n(dv \otimes 1 \otimes 1 - 1 \otimes dv \otimes 1) = 0 \). Since \( \varphi_n \) is a surjective quasi-isomorphism, by the long exact sequence associated to a short exact sequence of complexes, \( \ker \varphi_n \) is acyclic. Therefore since \( dv \otimes 1 \otimes 1 - 1 \otimes dv \otimes 1 \) is a cycle, there exists an element \( \gamma \) of degree \( n+1 \) of \( \Lambda V^{\leq n} \otimes \Lambda V^{\leq n} \otimes \Lambda sV^{\leq n} \) such that \( D(\gamma) = dv \otimes 1 \otimes 1 - 1 \otimes dv \otimes 1 \) and \( \varphi_n(\gamma) = 0 \). For degree reasons, \( \gamma \) is decomposable, i.e., has wordlength \( \geq 2 \). We define \( D(1 \otimes 1 \otimes sv) = v \otimes 1 \otimes 1 - 1 \otimes v \otimes 1 - \gamma \) and \( \varphi_{n+1}(1 \otimes 1 \otimes sv) = 0 \). Since \( D \circ D(1 \otimes 1 \otimes sv) = 0 \) and \( d \circ \varphi_{n+1}(1 \otimes 1 \otimes sv) = \varphi_{n+1} \circ d(1 \otimes 1 \otimes sv) \), by Property 1.10 the derivation \( D \) is a differential on \( \Lambda V^{\leq n+1} \otimes \Lambda V^{\leq n+1} \otimes \Lambda sV^{\leq n+1} \) and the morphism of graded algebras \( \varphi_{n+1} \) is a morphism of complexes.

The complex of indecomposables of \((\Lambda V^{\leq n+1} \otimes \Lambda V^{\leq n+1} \otimes \Lambda sV^{\leq n+1}, D)\),

\[
Q((\Lambda V^{\leq n+1} \otimes \Lambda V^{\leq n+1} \otimes \Lambda sV^{\leq n+1}, D),
\]

is \((V^{\leq n+1} \otimes V^{\leq n+1} \otimes sV^{\leq n+1}, D)\) with differential \( d \) given by \( d(v' \otimes v'' \otimes sv) = v' \otimes -v'' \otimes 0 \) for \( v', v'' \) and \( v \in V^{\leq n+1} \). Therefore it is easy to check that \( Q(\varphi_{n+1}) \) is a quasi-isomorphism. So by Proposition 14.13 of [6], \( \varphi_{n+1} \) is a quasi-isomorphism. Since \( \gamma \) is of degree \( n+1 \) and \( sV^{\leq n} \) is of degree \( < n \), this relative Sullivan model...
is minimal. We now define \( \phi : (\Lambda V \otimes \Lambda V \otimes \Lambda sV, D) \to (AV, d) \) as
\[
\lim_{\to} \phi_n = \bigcup_{n \in \mathbb{N}} \phi_n : \bigcup_{n \in \mathbb{N}} (\Lambda V \leq n \otimes \Lambda V \leq n \otimes \Lambda V \leq n) \to \bigcup_{n \in \mathbb{N}} \Lambda V \leq n.
\]
Since homology commutes with direct limits in the category of complexes [[14, Chap 4, Sect 2, Theorem 7]], \( H(\phi) = \lim_{\to} H(\phi_n) \) is an isomorphism.

### 3 Rational homotopy theory

Let \( X \) be a topological space. Denote by \( S^*(X) \) the singular cochains of \( X \) with coefficients in \( k \). The dga \( S^*(X) \) is almost never commutative. Nevertheless, Sullivan, inspired by Quillen proved the following theorem.

**Theorem 3.1.** [[5, Corollary 10.10]] For any topological space \( X \), there exists two natural quasi-isomorphisms of dgas
\[
S^*(X) \cong D(X) \cong A_{PL}(X)
\]
where \( A_{PL}(X) \) is commutative.

**Remark 3.2.** This cdga \( A_{PL}(X) \) is called the algebra of polynomial differential forms. If \( k = \mathbb{R} \) and \( X \) is a smooth manifold \( M \), you can think that \( A_{PL}(M) \) is the De Rham algebra of differential forms on \( M \), \( A_{DR}(M) \) [[5, Theorem 11.4]].

**Definition 3.3.** [[6, Definition 2.34]] Two topological spaces \( X \) and \( Y \) have the same rational homotopy type if there exists a finite sequence of continuous applications
\[
X \xrightarrow{f_0} Y_1 \xleftarrow{f_1} Y_2 \ldots Y_{n-1} \xleftarrow{f_{n-1}} Y_n \xrightarrow{f_n} Y
\]
such that the induced maps in rational cohomology
\[
H^*(X; \mathbb{Q}) \xleftarrow{H^*(f_0)} H^*(Y_1; \mathbb{Q}) \xrightarrow{H^*(f_1)} H^*(Y_2; \mathbb{Q}) \ldots H^*(Y_{n-1}; \mathbb{Q}) \xrightarrow{H^*(f_{n-1})} H^*(Y_n; \mathbb{Q}) \xleftarrow{H^*(f_n)} H^*(Y; \mathbb{Q})
\]
are all isomorphisms.

**Theorem 3.4.** Let \( X \) be a path connected topological space.

1) (Unicity of minimal Sullivan models [[5, Corollary p. 191]]) Two minimal Sullivan models of \( A_{PL}(X) \) are isomorphic.

2) Suppose that \( X \) is simply connected and \( \forall n \in \mathbb{N}, H_n(X; k) \) is finite dimensional. Let \( (AV, d) \) be a minimal Sullivan model of \( X \). Then [[5, Theorem 15.11]] for all \( n \in \mathbb{N} \), \( V^n \) is isomorphic to \( \text{Hom}_k(\pi_n(X) \otimes \mathbb{Z} k, k) \cong \text{Hom}_\mathbb{Z}(\pi_n(X), k) \). In particular [[5, Remark 1 p.208]], Dimension \( V^n = \text{Dimension} \pi_n(X) \otimes \mathbb{Z} k < \infty \).

**Remark 3.5.** The isomorphism of graded vector spaces between \( V \) and \( \text{Hom}_k(\pi_n(X) \otimes \mathbb{Z} k, k) \) is natural in some sense [[6 p. 75-6]] with respect to maps \( f : X \to Y \). The
isomorphism behaves well also with respect to the long exact sequence associated to a (Serre) fibration ([5, Proposition 15.13] or [6, Proposition 2.65]).

**Theorem 3.6.** ([5, Proposition 2.35][6, p. 139] Let $X$ and $Y$ be two simply connected topological spaces such that $H^n(X; \mathbb{Q})$ and $H^n(Y; \mathbb{Q})$ are finite dimensional for all $n \in \mathbb{N}$. Let $(\Lambda V, d)$ be a minimal Sullivan model of $X$ and let $(\Lambda W, d)$ be a minimal Sullivan model of $Y$. Then $X$ and $Y$ have the same rational homotopy type if and only if $(\Lambda V, d)$ is isomorphic to $(\Lambda W, d)$ as cdgas.

### 4 Sullivan model of a pullback

#### 4.1 Sullivan model of a product

Let $X$ and $Y$ be two topological spaces. Let $p_1 : X \times Y \to Y$ and $p_2 : X \times Y \to X$ be the projection maps. Let $m$ be the unique morphism of cdgas given by the universal property of the tensor product (Example 1.4 (b))

\[
\begin{array}{ccc}
A_{PL}(Y) & \longrightarrow & A_{PL}(X) \\
\downarrow & & \downarrow \\
A_{PL}(X) \otimes A_{PL}(Y) & \longrightarrow & A_{PL}(X \times Y).
\end{array}
\]

Assume that $H^*(X; k)$ or $H^*(Y; k)$ is finite dimensional in all degrees. Then [5, Example 2, p. 142-3] $m$ is a quasi-isomorphism. Let $m_X : \Lambda V \xrightarrow{\sim} A_{PL}(X)$ be a Sullivan model of $X$. Let $m_Y : \Lambda W \xrightarrow{\sim} A_{PL}(Y)$ be a Sullivan model of $Y$. Then by Künneth theorem, the composite

\[
\Lambda V \otimes \Lambda W \xrightarrow{m_X \otimes m_Y} A_{PL}(X) \otimes A_{PL}(Y) \xrightarrow{m} A_{PL}(X \times Y)
\]

is a quasi-isomorphism of cdgas. Therefore we have proved that “the Sullivan model of a product is the tensor product of the Sullivan models”.

#### 4.2 the model of the diagonal

Let $X$ be a topological space such that $H^*(X)$ is finite dimensional in all degrees. Denote by $\Delta : X \to X \times X$, $x \mapsto (x, x)$ the diagonal map of $X$. Using the previous paragraph, since $A_{PL}(p_1 \circ \Delta) = A_{PL}(p_2 \circ \Delta) = A_{PL}(\text{id}) = \text{id}$, we have
the commutative diagram of cdgas.

\[
\begin{array}{ccccccc}
A_{PL}(X) & \longrightarrow & A_{PL}(X) \otimes A_{PL}(X) & \longleftarrow & A_{PL}(X) \\
\downarrow{A_{PL}(p_1)} & & \searrow{m} & & \nearrow{A_{PL}(p_2)} \\
A_{PL}(X \times X) & \longleftarrow & A_{PL}(X) \otimes A_{PL}(X) & \longleftarrow & A_{PL}(\Delta) \\
& & \downarrow{id} & & \downarrow{id} \\
& & A_{PL}(X) & & A_{PL}(X) \\
\end{array}
\]

Therefore the composite \( A_{PL}(X) \otimes A_{PL}(X) \xrightarrow{m} A_{PL}(X \times X) \xrightarrow{A_{PL}(\Delta)} A_{PL}(X) \) coincides with the multiplication \( \mu : A_{PL}(X) \otimes A_{PL}(X) \to A_{PL}(X) \). Therefore the following diagram of cdgas commutes

\[
\begin{array}{ccccccc}
A_{PL}(X) & \xrightarrow{A_{PL}(\Delta)} & A_{PL}(X \times X) \\
\downarrow{m} & & \searrow{m} \\
A_{PL}(X) \otimes A_{PL}(X) & \xrightarrow{m_X} & A_{PL}(X) \otimes A_{PL}(X) \\
\downarrow{\Delta} & & \downarrow{m_X \otimes m_X} \\
\Lambda V & \xrightarrow{\mu} & \Lambda V \otimes \Lambda V \\
\end{array}
\]

Here \( m_X : \Lambda V \xrightarrow{\sim} A_{PL}(X) \) denotes a Sullivan model of \( X \). Therefore we have proved that “the morphism modelling the diagonal map is the multiplication of the Sullivan model”.

### 4.3 Sullivan model of a fibre product

Consider a pullback square in the category of topological spaces

\[
\begin{array}{ccc}
P & \xrightarrow{g} & E \\
q \downarrow & & \downarrow p \\
X & \xrightarrow{f} & B \\
\end{array}
\]

where

- \( p : E \to B \) is a (Serre) fibration between two topological spaces,
- for every \( i \in \mathbb{N} \), \( H^i(X) \) and \( H^i(B) \) are finite dimensional,
- the topological spaces \( X \) and \( E \) are path-connected and \( B \) is simply-connected.

Since \( p \) is a (Serre) fibration, the pullback map \( q \) is also a (Serre) fibration. Let \( A_{PL}(B) \otimes \Lambda V \) be a relative Sullivan model of \( A(p) \). Consider the corresponding
commutative diagram of cdgas

\[
\begin{array}{c}
A_{PL}(E) \\ A_{PL}(B) \\ A_{PL}(X) \\ A_{PL}(B) \otimes \Lambda V \\
\downarrow \\
A_{PL}(B) \otimes_{A_{PL}(B)} A_{PL}(B) \otimes \Lambda V \\
\downarrow \\
A_{PL}(P) \\
\end{array}
\]

where the rectangle is a pushout and \( m' \) is given by the universal property. Explicitly, for \( x \in A_{PL}(X) \) and \( e \in A_{PL}(B) \otimes \Lambda V \), \( m'(x \otimes e) \) is the product of \( A_{PL}(q)(x) \) and \( A_{PL}(g) \circ m(e) \).

Since \( A_{PL}(B) \hookrightarrow A_{PL}(B) \otimes \Lambda V \) is a relative Sullivan model, the inclusion obtained via pullback \( A_{PL}(X) \hookrightarrow A_{PL}(X) \otimes_{A_{PL}(B)} (A_{PL}(B) \otimes \Lambda V, d) \cong (A_{PL}(X) \otimes \Lambda V, d) \) is also a relative Sullivan model (minimal if \( A_{PL}(B) \hookrightarrow A_{PL}(B) \otimes \Lambda V \) is minimal).

By [5, Proposition 15.8] (or for weaker hypothesis [6, Theorem 2.70]),

**Theorem 4.1.** The morphism of cdgas \( m' \) is a quasi-isomorphism.

We can summarize this theorem by saying that: “The push-out of a (minimal) relative Sullivan model of a fibration is a (minimal) relative Sullivan model of the pullback of the fibration.”

**Idea of the proof.** Since by [5] Lemma 14.1, \( A_{PL}(B) \otimes \Lambda V \) is a “semi-free” resolution of \( A_{PL}(E) \) as left \( A_{PL}(B) \)-modules, by definition of the differential torsion product,

\[
\text{Tor}^{A_{PL}(B)}(A_{PL}(X), A_{PL}(E)) := H(A_{PL}(X) \otimes_{A_{PL}(B)} (A_{PL}(B) \otimes \Lambda V)).
\]

By Theorem 3.1 and naturality, we have an isomorphism of graded vector spaces

\[
\text{Tor}^{A_{PL}(B)}(A_{PL}(X), A_{PL}(E)) \cong \text{Tor}^{S^*(B)}(S^*(X), S^*(E)).
\]

The Eilenberg-Moore formula gives an isomorphism of graded vector spaces

\[
\text{Tor}^{S^*(B)}(S^*(X), S^*(E)) \cong H^*(P).
\]

We claimed that the resulting isomorphism between the homology of \( A_{PL}(X) \otimes_{A_{PL}(B)} (A_{PL}(B) \otimes \Lambda V) \) and \( H^*(P) \) can be identified with \( H(m) \). Therefore \( m \) is a quasi-isomorphism.

Instead of working with \( A_{PL} \), we prefer usually to work at the level of Sullivan models. Let \( m_B : \Lambda B \xrightarrow{\cong} A_{PL}(B) \) be a Sullivan model of \( B \). Let \( m_X : \Lambda X \xrightarrow{\cong} A_{PL}(X) \) be a Sullivan model of \( X \). Let \( \varphi \) be a morphism of cdgas such the following diagram commutes exactly
Let $\Lambda B \hookrightarrow \Lambda B \otimes \Lambda V$ be a relative Sullivan model of $A_{PL}(p) \circ m_B$. Consider the corresponding commutative diagram of cdgas

\[
\begin{array}{c}
A_{PL}(B) @>A_{PL}(f)>> A_{PL}(X) \\
m_B @. m_X \\
\Lambda B @>\varphi>> \Lambda X
\end{array}
\]

where the rectangle is a pushout and $m'$ is given by the universal property. Then again, $\Lambda X \hookrightarrow \Lambda X \otimes_{AB} (\Lambda B \otimes \Lambda V)$ is a relative Sullivan model and the morphism of cdgas $m'$ is a quasi-isomorphism.

The reader should skip the following remark on his first reading.

**Remark 4.2.** 1) In the previous proof, if the composites $m_X \circ \varphi$ and $A_{PL}(f) \circ m_B$ are not strictly equal then the map $m'$ is not well defined. In general, the composites $m_X \circ \varphi$ and $A_{PL}(f) \circ m_B$ are only homotopic and the situation is more complicated; see part 2) of this remark.

2) Let $m_B : \Lambda B \to A_{PL}(B)$ be a Sullivan model of $B$. Let $m'_X : \Lambda X' \to A_{PL}(X)$ be a Sullivan model of $X$. By the lifting Lemma of Sullivan models [5, Proposition 14.6], there exists a morphism of cdgas $\varphi' : \Lambda B \to \Lambda X'$ such that the following diagram commutes only up to homotopy (in the sense of [6, Section 2.2])

\[
\begin{array}{c}
A_{PL}(B) @>A_{PL}(f)>> A_{PL}(X) \\
m_B @. m'_X \\
\Lambda B @>\varphi'>> \Lambda X'
\end{array}
\]

In general, this square is not strictly commutative. Let $\Lambda B \hookrightarrow \Lambda B \otimes \Lambda V$ be a relative Sullivan model of $A_{PL}(p) \circ m_B$. Then there exists a commutative diagram
Proof of part 2) of Remark 4.2. Let $\Lambda B \xrightarrow{\phi} \Lambda X \xrightarrow{\theta} \Lambda X'$ be a relative Sullivan model of $\varphi'$. Since the composites $m'_X \circ \theta \circ \varphi$ and $A_{PL}(f) \circ m_B$ are homotopic, by the homotopy extension property [6, Proposition 2.22] of the relative Sullivan model $\phi : \Lambda B \hookrightarrow \Lambda X$, there exists a morphism of cdgas $m_X : \Lambda X \to A_{PL}(X)$ homotopic to $m'_X \circ \theta$ such that $m_X \circ \varphi = A_{PL}(f) \circ m_B$. Therefore using diagram (1), we obtain the following commutative diagram of cdgas:

\[
\begin{array}{c}
\Lambda X \\
\downarrow \simeq \\
\Lambda X' \\
\end{array} \quad \begin{array}{c}
\Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V) \\
\downarrow \simeq \\
\Lambda X' \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V) \\
\end{array}
\]

Here, since $\theta$ is a quasi-isomorphism, the pushout morphism $\theta \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$ along the relative Sullivan model $\Lambda X \hookrightarrow \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$ is also a quasi-isomorphism [5, Lemma 14.2].

### 4.4 Sullivan model of a fibration

Let $p : E \to B$ be a (Serre) fibration with fibre $F := p^{-1}(b_0)$.

\[
\begin{array}{c}
F \\
\downarrow j \\
b_0 \\
\end{array} \quad \begin{array}{c}
E \\
\downarrow p \\
B \\
\end{array}
\]

Taking $X$ to be the point $b_0$, we can apply the results of the previous section. Let $m_B : (\Lambda V, d) \to A_{PL}(B)$ be a Sullivan model of $B$. Let $(\Lambda V, d) \hookrightarrow (\Lambda V \otimes \Lambda W, d)$ be a relative Sullivan model of $A_{PL}(p) \circ m_B$.

Since $A_{PL}([b_0])$ is equal to $(k, 0)$, there is a unique morphism of cdgas $m'$ such that the following diagram commutes.
Suppose that the base $B$ is a simply connected space and that the total space $E$ is path-connected. Then by the previous section, the morphism of cdga’s $m': (k,0) \otimes (AV \otimes AW, d) \xrightarrow{\simeq} (AV, d) \rightarrow A_{PL}(F)$ is a quasi-isomorphism:

"The cofiber of a relative Sullivan model of a fibration is a Sullivan model of the fiber of the fibration."

Note that the cofiber of a relative Sullivan model is minimal if and only if the relative Sullivan model is minimal.

4.5 Sullivan model of free loop spaces

Let $X$ be a simply-connected space. Consider the commutative diagram of spaces

\[
\begin{array}{c}
X_{S^1} & \xrightarrow{ev} & X \\
\sigma & \searrow & \Delta \\
X & \xrightarrow{\Delta} & X \times X
\end{array}
\]

where the square is a pullback. Here $I$ denotes the closed interval $[0,1]$, $ev$, $ev_0$, $ev_1$ are the evaluation maps and the homotopy equivalence $\sigma : X \xrightarrow{\simeq} X^I$ is the inclusion of constant paths. Let $m_X : AV \xrightarrow{\simeq} A_{PL}(X)$ be a minimal Sullivan model of $X$. By Proposition 2.8, the multiplication $\mu : AV \otimes AV \rightarrow AV$ admits a minimal relative Sullivan model of the form

$AV \otimes AV \rightarrow AV \otimes AV \otimes sV$.

Since $\mu$ is a model of the diagonal (Section 4.2) and since $\Delta = (ev_0, ev_1) \circ \sigma$, we have the commutative rectangle of cdgas

\[
\begin{array}{c}
A_{PL}(X \times X) & \xrightarrow{A_{PL}(ev_0, ev_1)} & A_{PL}(X^I) \\
\simeq & & \simeq \\
\Delta & \xrightarrow{\Delta} & AV \otimes AV \otimes sV
\end{array}
\]

Since $\sigma$ is a homotopy equivalence, $S^*(\sigma)$ is a homotopy equivalence of complexes and in particular a quasi-isomorphism. So by Theorem 3.1 and naturality, $A_{PL}(\sigma)$ is also a quasi-isomorphism. Therefore, by the lifting property of relative Sullivan models [5, Proposition 14.6], there exists a morphism of cdgas $\varphi : AV \otimes AV \otimes$
As $V \to A_{PL}(X^I)$ such that, in the diagram of cdgas

$$
\begin{align*}
A_{PL}(X \times X) &\xrightarrow{A_{PL}(\{ev_0, ev_1\})} A_{PL}(X) \\
\Lambda V \otimes \Lambda V &\xrightarrow{m_{X \times X}} A_{PL}(X \times X) \\
\Lambda V \otimes \Lambda V &\xrightarrow{\varphi \simeq m_X} \Lambda V \\
\Lambda V \otimes \Lambda V \otimes \Lambda sV &\xrightarrow{\simeq} \Lambda V
\end{align*}
$$

the left square commutes exactly and the right square commutes in homology. Therefore $\varphi$ is also a quasi-isomorphism. This means that

$$\Lambda V \otimes \Lambda V \hookrightarrow \Lambda V \otimes \Lambda V \otimes \Lambda sV.$$

is a relative Sullivan model of the composite

$$\Lambda V \otimes \Lambda V \xrightarrow{m_{X \times X}} A_{PL}(X \times X) \xrightarrow{A_{PL}(\{ev_0, ev_1\})} A_{PL}(X^I).$$

Here diagram (1) specializes to the following commutative diagram of cdgas

$$
\begin{align*}
\Lambda V \otimes \Lambda V &\xrightarrow{\mu} \Lambda V \\
\Lambda V \otimes \Lambda V \otimes \Lambda sV &\xrightarrow{\varphi \simeq m_X} \Lambda V \otimes \Lambda V \otimes \Lambda sV \\
\Lambda V \otimes \Lambda V \otimes \Lambda sV &\xrightarrow{\simeq} \Lambda V \otimes \Lambda V \otimes \Lambda sV \\
A(X^I) &\xrightarrow{\delta} A(X^{S^1})
\end{align*}
$$

where the rectangle is a pushout. Therefore

$$\Lambda V \hookrightarrow \Lambda V \otimes \Lambda V \otimes \Lambda sV \cong (\Lambda V \otimes \Lambda sV, \delta)$$

is a minimal relative Sullivan model of $A_{PL}(ev) \circ m_X$.

**Corollary 4.3.** Let $X$ be a simply-connected space. Then the free loop space cohomology of $H^*(X^{S^1}; k)$ with coefficients in a field $k$ of characteristic 0 is isomorphic to the Hochschild homology of $A_{PL}(X)$, $HH_*(A_{PL}(X), A_{PL}(X))$.

Replacing $A_{PL}(X)$ by $A_{DR}(M)$ (Remark 3.2), this Corollary is a theorem of Chen [3, 3.2.3 Theorem] when $X$ is a smooth manifold $M$.

**Proof.** The quasi-isomorphism of cdgas $m_X : \Lambda V \simeq A_{PL}(X)$ induces an isomorphism between Hochschild homologies

$$HH_*(m_X, m_X) : HH_*(\Lambda V, \Lambda V) \xrightarrow{\simeq} HH_*(A_{PL}(X), A_{PL}(X)).$$

By [5, Lemma 14.1], $\Lambda V \otimes \Lambda V \otimes \Lambda sV$ is a semi-free resolution of $\Lambda V$ as a $\Lambda V \otimes \Lambda V^{op}$-module. Therefore the Hochschild homology $HH_*(\Lambda V, \Lambda V)$ can be defined as the homology of the cdga $(\Lambda V \otimes \Lambda sV, \delta)$. We have just seen above that $H(\Lambda V \otimes \Lambda sV, \delta)$ is isomorphic to the free loop space cohomology $H^*(X^{S^1}; k)$. \qed
We have shown that a Sullivan model of $X^S_1$ is of the form $(\Lambda V \otimes \Lambda sV, \delta)$. The following theorem of Vigué-Poirrier and Sullivan gives a precise description of the differential $\delta$.

**Theorem 4.4.** ([17, Theorem p. 637] or [6, Theorem 5.11]) Let $X$ be a simply connected topological space. Let $(\Lambda V, d)$ be a minimal Sullivan model of $X$. For all $v \in V$, denote by $sv$ an element of degree $|v| - 1$. Let $s : \Lambda V \otimes \Lambda sV \to \Lambda V \otimes \Lambda sV$ be the unique derivation of (upper) degree $-1$ such that on the generators $v$, $sv$, $v \in V$, $s(v) = sv$ and $s(sv) = 0$. We have $s \circ s = 0$. Then there exists a unique Sullivan model of $X^S_1$ of the form $(\Lambda V \otimes \Lambda sV, \delta)$ such that $\delta \circ s + s \circ \delta = 0$ on $\Lambda V \otimes \Lambda sV$.

**Remark 4.5.** Consider the free loop fibration $\Omega X \hookrightarrow X^S_1 \twoheadrightarrow X$. Since $(\Lambda V, d) \hookrightarrow (\Lambda V \otimes \Lambda sV, \delta)$ is a minimal relative Sullivan model of $A_{PL}(ev) \circ m_X$, by Section 4.4, $k \otimes (\Lambda V, d) \rightarrow (\Lambda V \otimes \Lambda sV, \delta) \cong (\Lambda sV, \bar{\delta})$ is a minimal Sullivan model of $\Omega X$. Let $v \in V$. By Theorem 4.4, $\delta(sv) = -s\delta v = -sdv$. Since $dv \in \Lambda^{\geq 2}V$, $\delta(sv) \in \Lambda^{\geq 1}V \otimes \Lambda^1 sV$. Therefore $\delta = 0$. Since $\Omega X$ is a $H$-space, this follows also from Theorem 5.3 and from the unicity of minimal Sullivan models (part 1) of Theorem 3.4.

## 5 Examples of Sullivan models

### 5.1 Sullivan model of spaces with polynomial cohomology

The following proposition is a straightforward generalisation [5, p. 144] of the Sullivan model of odd-dimensional spheres (see section 1.2).

**Proposition 5.1.** Let $X$ be a path connected topological space such that its cohomology $H^*(X; k)$ is a free graded commutative algebra $\Lambda V$ (for example, polynomial). Then a Sullivan model of $X$ is $(\Lambda V, 0)$.

**Example 5.2.** Odd-dimensional spheres $S^{2n+1}$, complex or quaternionic Stiefel manifolds [6] Example 2.40 $V_k(\mathbb{C}^n)$ or $V_k(\mathbb{H}^n)$, classifying spaces $BG$ of simply connected Lie groups [6] Example 2.42, connected Lie groups $G$ as we will see in the following section.

### 5.2 Sullivan model of an $H$-space

An $H$-space is a pointed topological space $(G, e)$ equipped with a pointed continuous map $\mu : (G, e) \times (G, e) \to (G, e)$ such that the two pointed maps $g \mapsto \mu(e, g)$ and $g \mapsto \mu(g, e)$ are pointed homotopic to the identity map of $(G, e)$. 

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Theorem 5.3. [2] Example 3 p. 143] Let $G$ be a path connected $H$-space such that $\forall n \in \mathbb{N}$, $H_n(G; \mathbb{k})$ is finite dimensional. Then

1) its cohomology $H^*(G; \mathbb{k})$ is a free graded commutative algebra $\Lambda V$,
2) $G$ has a Sullivan model of the form $(\Lambda V, 0)$, that is with zero differential.

Proof. 1) Let $A$ be $H^*(G; \mathbb{k})$ the cohomology of $G$. By hypothesis, $A$ is a connected graded commutative graded Hopf algebra (not necessarily associative). Now the theorem of Hopf-Borel in characteristic 0 [3, VII.10.16] says that $A$ is a free graded commutative algebra.

2) By Proposition 5.1, 1) and 2) are equivalent. \qed

Example 5.4. Let $G$ be a path-connected Lie group (or more generally a $H$-space with finitely generated integral homology). Then $G$ has a Sullivan model of the form $(\Lambda V, 0)$. By Theorem 3.4, $V^n$ and $\pi_n(G) \otimes \mathbb{k}$ have the same dimension for any $n \in \mathbb{N}$. Since $H_n(G; \mathbb{k})$ is of finite (total) dimension, $V$ and therefore $\pi_n(G) \otimes \mathbb{k}$ are concentrated in odd degrees. In fact, more generally [2, Theorem 6.11], $\pi_2(G) = \{0\}$. Note, however that $\pi_4(S^3) = \mathbb{Z}/2\mathbb{Z} \neq \{0\}$.

5.3 Sullivan model of projective spaces

Consider the complex projective space $\mathbb{C}P^n$, $n \geq 1$. The construction of the Sullivan model of $\mathbb{C}P^n$ is similar to the construction of the Sullivan model of $S^2 = \mathbb{C}P^1$ done in section 1.2.

The cohomology algebra $H^*(A_{PL}(\mathbb{C}P^n)) \cong H^*(\mathbb{C}P^n)$ is the truncated polynomial algebra $\mathbb{k}[x]/x^{n+1}$ where $x$ is an element of degree 2. Let $v$ be a cycle of $A_{PL}(\mathbb{C}P^n)$ representing $x := [v]$. The inclusion of complexes $(kv, 0) \hookrightarrow A_{PL}(\mathbb{C}P^n)$ extends to a unique morphism of cdgas $m : (\Lambda v, 0) \to A_{PL}(\mathbb{C}P^n)$ (Property 1.6). Since $[v^{n+1}] = x^{n+1} = 0$, there exists an element $\psi \in A_{PL}(\mathbb{C}P^n)$ of degree $2n + 1$ such that $d\psi = v^{n+1}$. Let $w$ denote another element of degree $2n + 1$. Let $d$ be the unique derivation of $A(v, w)$ such that $d(v) = 0$ and $d(w) = v^{n+1}$. The unique morphism of graded algebras $m : (\Lambda(v, w), d) \to A_{PL}(\mathbb{C}P^n)$ such that $m(v) = v$ and $m(w) = \psi$, is a morphism of cdgas. In homology, $H(m)$ sends 1, $[v], \ldots, [v^n]$ to 1, $x, \ldots, x^n$. Therefore $m$ is a quasi-isomorphism.

More generally, let $X$ be a simply connected space such that $H^*(X)$ is a truncated polynomial algebra $\mathbb{k}[x]/x^{n+1}$ where $n \geq 1$ and $x$ is an element of even degree $d \geq 2$. Then the Sullivan model of $X$ is $(\Lambda(v, w), d)$ where $v$ is an element of degree $d$, $w$ is an element of degree $d(n+1) - 1$, $d(v) = 0$ and $d(w) = v^{n+1}$.

5.4 Free loop space cohomology for even-dimensional spheres and projective spaces

In this section, we compute the free loop space cohomology of any simply connected space $X$ whose cohomology is a truncated polynomial algebra $\mathbb{k}[x]/x^{n+1}$ where $n \geq 1$ and $x$ is an element of even degree $d \geq 2$. 

Mainly, this is the even-dimensional sphere \( S^d \) (\( n = 1 \)), the complex projective space \( \mathbb{CP}^n \) (\( d = 2 \)), the quaternionic projective space \( \mathbb{HP}^n \) (\( d = 4 \)) and the Cayley plane \( \mathbb{OP}^n \) (\( n = 2 \) and \( d = 8 \)).

In the previous section, we have seen that the minimal Sullivan model of \( X \) is \( (\Lambda(v, w), d(v) = 0, d(w) = v^{n+1}) \) where \( v \) is an element of degree \( d \) and \( w \) is an element of degree \( d(n+1) - 1 \). By the constructive proof of Proposition 2.8, the multiplications \( \mu \) of this minimal Sullivan model \( (\Lambda(v, w), d) \) admits the relative Sullivan model \( (\Lambda(v, w) \otimes \Lambda(v, w) \otimes \Lambda(sv, sw), D) \) where

\[
D(1 \otimes 1 \otimes sv) = v \otimes 1 \otimes 1 - 1 \otimes v \otimes 1 \quad \text{and}
\]

\[
D(1 \otimes 1 \otimes sw) = w \otimes 1 \otimes 1 - 1 \otimes w \otimes 1 - \sum_{i=0}^{n} v^i \otimes v^{n-i} \otimes sw.
\]

Therefore, by taking the pushout along \( \mu \) of this relative Sullivan model (diagram (2)), or simply by applying Theorem 14.4, a relative Sullivan model of \( A_{PL}(ev) \circ m_X \) is given by the inclusion of cdgas \( (\Lambda(v, w), d) \hookrightarrow (\Lambda(v, w, sv, sw), \delta) \) where \( \delta(sv) = -sd(v) = 0 \) and \( \delta(sw) = (v^{n+1}) = -(n+1)v^{n}sv \). Consider the pushout square of cdgas

\[
\begin{array}{ccc}
(\Lambda(v, w), d) & \xrightarrow{\theta} & (\Lambda(v, w, sv, sw), \delta) \\
\downarrow_{\rho} & & \downarrow_{\theta \otimes \Lambda(v, w), \Lambda(sv, sw)} \\
(\mathbb{k}, \mathbb{c})_{\mathbb{c} = 0} & \xrightarrow{(k_{d_{c+1}} \mathbb{c} = 0)} & (k_{d_{c+1}} \mathbb{c} = 0) \otimes \Lambda(sv, sw), \delta)
\end{array}
\]

Here, since \( \theta \) is a quasi-isomorphism, the pushout morphism \( \theta \otimes \Lambda(v, w), \Lambda(sv, sw) \) along the relative Sullivan model \( \Lambda(v, w) \hookrightarrow \Lambda(v, w, sv, sw) \) is also a quasi-isomorphism [5, Lemma 14.2]. Therefore, \( H^*(X; \mathbb{k}) \) is the graded vector space

\[
\mathbb{k} \oplus \bigoplus_{1 \leq p \leq n, \, i \in \mathbb{N}} \mathbb{k}v^p(sv)^i \oplus \bigoplus_{0 \leq p \leq n-1, \, i \in \mathbb{N}} \mathbb{k}v^p sv(sw)^i.
\]

(In [11] Section 8), the author extends these rational computations over any commutative ring.) Since for all \( i \in \mathbb{N} \), the degree of \( v(sw)^{i+1} \) is strictly greater than the degree of \( v^i(sw)^i \), the generators \( 1, v^p(sw)^i, 1 \leq p \leq n, \, i \in \mathbb{N} \), have all distinct (even) degrees. Since for all \( i \in \mathbb{N} \), the degree of \( sv(sw)^{i+1} \) is strictly greater than the degree of \( v^{n-1}sv(sw)^i \), the generators \( v^p sv(sw)^i, 0 \leq p \leq n-1, \, i \in \mathbb{N} \), have also distinct (odd) degrees. Therefore, for all \( p \in \mathbb{N} \), \( \dim H^p(X; \mathbb{k}) \leq 1 \).

At the end of section 2.2, we have shown the same inequalities when \( X \) is an odd-dimensional sphere, or more generally for a simply-connected space \( X \) whose cohomology \( H^*(X; \mathbb{k}) \) is an exterior algebra \( \Lambda x \) on an odd degree generator \( x \). Since every finite dimensional graded commutative algebra generated by a single element \( x \) is either \( \mathbb{k}x \) or \( \mathbb{k}x_{\mathbb{c} = 0} \), we have shown the following proposition:

**Proposition 5.5.** Let \( X \) be a simply connected topological space such that its cohomology \( H^*(X; \mathbb{k}) \) is generated by a single element and is finite dimensional. Then the sequence of Betti numbers of the free loop space on \( X \), \( b_n := \dim H^n(X; \mathbb{k}) \)
is bounded.

The goal of the following section will be to prove the converse of this proposition.

6 Vigué-Poirrier-Sullivan theorem on closed geodesics

The goal of this section is to prove (See section 6.4) the following theorem due to Vigué-Poirrier and Sullivan.

6.1 Statement of Vigué-Poirrier-Sullivan theorem and of its generalisations

**Theorem 6.1.** ([17, Theorem p. 637] or [6, Proposition 5.14]) Let $M$ be a simply connected topological space such that the rational cohomology of $M$, $H^*(M; \mathbb{Q})$ is of finite (total) dimension (in particular, vanishes in higher degrees).

If the cohomology algebra $H^*(M; \mathbb{Q})$ requires at least two generators then the sequence of Betti numbers of the free loop space on $M$, $b_n := \dim H^n(MS^1; \mathbb{Q})$ is unbounded.

**Example 6.2.** (Betti numbers of $(S^3 \times S^3)S^1$ over $\mathbb{Q}$)

Let $V$ and $W$ be two graded vector spaces such $\forall n \in \mathbb{N}$, $V^n$ and $W^n$ are finite dimensional. We denote by

$$P_V(z) := \sum_{n=0}^{\infty} (\text{Dim } V^n)z^n$$

the sum of the Poincaré serie of $V$. If $V$ is the cohomology of a space $X$, we denote $P_{H^*(X)}(z)$ simply by $P_X(z)$. Note that $P_{V \otimes W}(z)$ is the product $P_V(z)P_W(z)$. We saw at the end of section 2.2 that $H^*((S^3S^1); \mathbb{Q}) \cong \Lambda v \otimes \Lambda sv$ where $v$ is an element of degree 3. Therefore

$$P_{(S^3S^1S^1)}(z) = (1 + z^3)\sum_{n=0}^{\infty} z^{2n} = \frac{1 + z^3}{1 - z^2}.$$ 

Since the free loops on a product is the product of the free loops

$$H^*((S^3 \times S^3)S^1) \cong H^*((S^3S^1)S^1) \otimes H^*((S^3S^1)S^1).$$
Therefore, since \( \frac{1}{1 - z^2} = \sum_{n=0}^{+\infty} (n + 1)z^{2n} \),

\[
P_{(S^3 \times S^3) \vee 1}(z) = \left( \frac{1 + z^3}{1 - z^2} \right)^2 = 1 + 2z^2 + \sum_{n=3}^{+\infty} (n - 1)z^n.
\]

So the Betti numbers over \( \mathbb{Q} \) of the free loop space on \( S^3 \times S^3, b_n := \dim H^n((S^3 \times S^3)S^1; \mathbb{Q}) \) are equal to \( n - 1 \) if \( n \geq 3 \). In particular, they are unbounded.

**Conjecture 6.3.** The theorem of Vigué-Poirrier and Sullivan holds replacing \( \mathbb{Q} \) by any field \( \mathbb{F} \).

**Example 6.4.** (Betti numbers of \( (S^3 \times S^3)S^1 \) over \( \mathbb{F} \))

The calculation of Example 6.2 over \( \mathbb{Q} \) can be extended over any field \( \mathbb{F} \) as follows: Since \( S^3 \) is a topological group, the map \( \Omega S^3 \times S^3 \to (S^3)^S \), sending \( (w, g) \) to the free loop \( t \mapsto w(t)g \), is a homeomorphism. Using Serre spectral sequence ([13, Proposition 17] or [14, Chap 9. Sect 7. Lemma 3]) or Bott-Samelson theorem ([12, Corollary 7.3.3] or [9, Appendix 2 Theorem 1.4]), the cohomology of the pointed loops on \( S^3, H^*(\Omega S^3) \) is again isomorphic (as graded vector spaces only!) to the polynomial algebra \( \Lambda sv \) where \( sv \) is of degree 2. Therefore exactly as over \( \mathbb{Q} \), \( H^*((S^3)^S; \mathbb{F}) \cong \Lambda v \otimes \Lambda sv \) where \( v \) is an element of degree 3. Now the same proof as in Example 6.2 shows that the Betti numbers over \( \mathbb{F} \) of the free loop space on \( S^3 \times S^3, b_n := \dim H^n((S^3 \times S^3)S^1; \mathbb{F}) \) are again equal to \( n - 1 \) if \( n \geq 3 \).

In fact, the theorem of Vigué-Poirrier and Sullivan is completely algebraic:

**Theorem 6.5.** ([17] when \( \mathbb{F} = \mathbb{Q} \), [7, Theorem III p. 315] over any field \( \mathbb{F} \)) Let \( \mathbb{F} \) be a field. Let \( A \) be a cdga such that \( H^{<0}(A) = 0, H^0(A) = \mathbb{F} \) and \( H^*(A) \) is of finite (total) dimension. If the algebra \( H^*(A) \) requires at least two generators then the sequence of dimensions of the Hochschild homology of \( A, b_n := \dim HH_{-n}(A, A) \) is unbounded.

Generalising Chen’s theorem (Corollary 4.3) over any field \( \mathbb{F} \), Jones theorem [10] gives the isomorphisms of vector spaces

\[ H^n(XS^1; \mathbb{F}) \cong HH_{-n}(S^*(X; \mathbb{F}), S^*(X; \mathbb{F})), \quad n \in \mathbb{Z} \]

between the free loop space cohomology of \( X \) and the Hochschild homology of the algebra of singular cochains on \( X \). But since the algebra of singular cochains \( S^*(X; \mathbb{F}) \) is not commutative, Conjecture 6.3 does not follow from Theorem 6.5.

### 6.2 A first result of Sullivan

In this section, we start by a first result of Sullivan whose simple proof illustrates the technics used in the proof of Vigué-Poirrier-Sullivan theorem.
Theorem 6.6. [12] Let $X$ be a simply-connected space such that $H^*(X;\mathbb{Q})$ is not concentrated in degree 0 and $H^n(X;\mathbb{Q})$ is null for $n$ large enough. Then on the contrary, $H^n(X^{S^1};\mathbb{Q}) \neq 0$ for an infinite set of integers $n$.

Proof. Let $(\Lambda V, d)$ be a minimal Sullivan model of $X$. Suppose that $V$ is concentrated in even degree. Then $d = 0$. Therefore $H^*(\Lambda V, d) = \Lambda V$ is either concentrated in degree 0 or is not null for an infinite sequence of degrees. By hypothesis, we have excluded these two cases. Therefore $\dim V^{odd} \geq 1$.

Let $x_1, x_2, \ldots, x_m, y, x_{m+1}, \ldots$ be a basis of $V$ ordered by degree where $y$ denotes the first generator of odd degree ($m \geq 0$). For all $1 \leq i \leq m$, $dx_i \in \Lambda x_i$. But $dx_i$ is of odd degree and $\Lambda x_i$ is concentrated in even degree. So $dx_i = 0$.

Since $dy \in \Lambda x_{m+1}$, $dy$ is equal to a polynomial $P(x_1, \ldots, x_m)$ which belongs to $(\Lambda^2(x_1, \ldots, x_m))$.

Consider $(\Lambda V \otimes \Lambda sV, \delta)$, the Sullivan model of $X^{S^1}$, given by Theorem 4.4. We have $\forall 1 \leq i \leq m$, $\delta(sx_i) = -sdx_i = 0$ and $\delta(sy) = -sdy \in \Lambda^2(x_1, \ldots, x_m) \otimes \Lambda(sx_1, \ldots, sx_m)$. Therefore, since $sx_1, \ldots, sx_m$ are all of odd degree, $\forall p \geq 0$,

$$\delta(sx_1 \ldots sx_m(sy)^p) = sx_1 \ldots sx_m\delta(sy)(sy)^{p-1} = 0.$$ 

For all $p \geq 0$, the cocycle $sx_1 \ldots sx_m(sy)^p$ gives a non trivial cohomology class in $H^*(X^{S^1};\mathbb{Q})$, since by Remark 1.5 the image of this cohomology class in $H^*(\Omega X;\mathbb{Q}) \cong \Lambda V$ is different from 0. 

6.3 Dimension of $V^{odd} \geq 2$

In this section, we show the following proposition:

Proposition 6.7. Let $X$ be a simply connected space such that $H^*(X;\mathbb{Q})$ is of finite (total) dimension and requires at least two generators. Let $(\Lambda V, d)$ be the minimal Sullivan model of $X$. Then $\dim V^{odd} \geq 2$.

Property 6.8. (Koszul complexes) Let $A$ be a graded algebra. Let $z$ be a central element of even degree of $A$ which is not a divisor of zero. Then we have a quasi-isomorphism of dgas

$$(A \otimes \Lambda sz, d) \cong A/z.A \quad a \otimes 1 \mapsto a, a \otimes sz \mapsto 0,$$

where $d(a \otimes 1) = 0$ and $d(a \otimes sz) = (-1)^{|a|}az$ for all $a \in A$.

Proof of Proposition 6.7 (following (2) ⇒ (3) of p. 211 of [12]). As we saw in the proof of Theorem 6.6, there is at least one generator $y$ of odd degree, that is $\dim V^{odd} \geq 1$. Suppose that there is only one. Let $x_1, x_2, \ldots, x_m, y, x_{m+1}, \ldots$ be a basis of $V$ ordered by degree ($m \geq 0$).

First case: $dy = 0$. If $m \geq 1$, $dx_1 = 0$. If $m = 0$, $dx_1 \in \Lambda^2(y) = \{0\}$ and therefore again $dx_1 = 0$. Suppose that for $n \geq 1$, $x^n_1$ is a coboundary. Then $x^n_1 = d(yP(x_1, \ldots)) = yd(P(x_1, \ldots))$ where $P(x_1, \ldots)$ is a polynomial in the $x_i$'s. But this is impossible since $x^n_1$ does not belong to the ideal generated by $y$. 

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Therefore for all \( n \geq 1 \), \( x^n \) gives a non trivial cohomology class in \( H^*(X) \). But \( H^*(X) \) is finite dimensional.

Second case: \( dy \neq 0 \). In particular \( m \geq 1 \). Since \( dy \) is a non zero polynomial, \( dy \) is not a zero divisor, so by Property 6.8 we have a quasi-isomorphism of cdgas
\[
\Lambda(x_1, \ldots, x_m, y) \cong \Lambda(x_1, \ldots, x_m)/(dy).
\]
Consider the push out in the category of cdgas
\[
\begin{array}{ccc}
\Lambda(x_1, \ldots, x_m, y) & \xrightarrow{d} & \Lambda(x_1, \ldots, x_m, y, x_{m+1}, \ldots), \\
\Lambda(x_1, \ldots, x_m)/(dy) & \cong & \Lambda(x_1, \ldots, x_m)/(dy) \otimes \Lambda(x_{m+1}, \ldots), \tilde{d}
\end{array}
\]
Since \( \Lambda(x_1, \ldots, x_m)/(dy) \otimes \Lambda(x_{m+1}, \ldots) \) is concentrated in even degrees, \( \tilde{d} = 0 \). Since the top arrow is a Sullivan relative model and the left arrow is a quasi-isomorphism, the right arrow is also a quasi-isomorphism ([5, Lemma 14.2], or more generally the category of cdgas over \( \mathbb{Q} \) is a Quillen model category). Therefore the algebra \( H^*(X) \) is isomorphic to \( \Lambda(x_1, \ldots, x_m)/(dy) \otimes \Lambda(x_{m+1}, \ldots) \).

6.4 Proof of Vigué-Poirrier-Sullivan theorem

Lemma 6.9. ([7] Proposition 4] Let \( A \) be a dga over any field such that the multiplication by a cocycle \( x \) of any degree \( A \to A, a \mapsto xa \) is injective (Our example will be \( A = (\Lambda V, d) \) and \( x \) a non-zero element of \( V \) of even degree such that \( dx = 0 \)). If the Betti numbers \( b_n = \dim H^n(A) \) of \( A \) are bounded then the Betti numbers \( b_n = \dim H^n(A/xA) \) of \( A/xA \) are also bounded.

Proof. Since \( H^n(xA) \cong H^{n-|x|}(A) \), the short exact sequence of complexes
\[
0 \to xA \to A \to A/xA \to 0
\]
gives the long exact sequence in homology
\[
\cdots \to H^n(A) \to H^n(A/xA) \to H^{n+1-|x|}(A) \to \cdots
\]
Therefore \( \dim H^n(A/xA) \leq \dim H^n(A) + \dim H^{n+1-|x|}(A) \).

Proof of Vigué-Poirrier-Sullivan theorem (Theorem 6.1). Let \( (\Lambda V, d) \) be the minimal Sullivan model of \( X \). Let \( (\Lambda V \otimes \Lambda sV, \delta) \) be the Sullivan model of \( X^{s1} \) given by Theorem 4.3. From Proposition 6.7 we know that \( \dim V^{odd} \geq 2 \). Let \( x_1, x_2, \ldots, x_m, y, x_m+1, \ldots, x_n, z = x_{n+1}, \ldots \) be a basis of \( V \) ordered by degrees where \( x_1, \ldots, x_n \) are of even degrees and \( y, z \) are of odd degrees. Consider the
commutative diagram of cdgas where the three rectangles are push outs

\[
\begin{array}{ccc}
\Lambda(x_1, \ldots, x_n) & \longrightarrow & (\Lambda V, d) \\
\downarrow & & \downarrow \\
\mathbb{Q} & \longrightarrow & (\Lambda V \otimes \Lambda sV, \delta) \\
\downarrow & & \downarrow \\
\Lambda(y, z, \ldots) & \longrightarrow & (\Lambda(y, z, \ldots) \otimes \Lambda sV, \bar{\delta}) \\
\downarrow & & \downarrow \\
\mathbb{Q} & \longrightarrow & (\Lambda sV, 0)
\end{array}
\]

Note that by Remark 6.3, the differential on $\Lambda sV$ is 0. For all $1 \leq j \leq n + 1$,

\[
\delta x_j = dx_j \in \Lambda^{2, (x_{<j}, y)} \subset \Lambda^{2, 1} (x_{<j}) \otimes \Lambda y.
\]

Therefore

\[
\delta(sx_j) = -s \delta x_j \in \Lambda x_{<j} \otimes \Lambda^1 sx_{<j} \otimes \Lambda y + \Lambda^{2, 1} (x_{<j}) \otimes \Lambda^1 sy.
\]

Since $(sx_1)^2 = \cdots = (sx_{j-1})^2 = 0$, the product

\[
\begin{align*}
& sx_1 \cdots sx_{j-1} \delta(sx_j) \in \Lambda^{2, 1} (x_{<j}) \otimes \Lambda^1 sy.
\end{align*}
\]

So $\forall 1 \leq j \leq n + 1$, $sx_1 \cdots sx_{j-1} \delta(sx_j) = 0$. In particular $sx_1 \cdots sx_n \delta(sz) = 0$. Similarly, since $dy \in \Lambda^{2, 2} x_{<j}$, $sx_1 \cdots sx_n \delta(sy) = 0$ and so $sx_1 \cdots sx_n \delta(sy) = 0$. By induction, $\forall 1 \leq j \leq n$, $\delta(sx_1 \cdots sx_j) = 0$. In particular, $\delta(sx_1 \cdots sx_n) = 0$.

So finally, for all $p \geq 0$ and all $q \geq 0$, $\delta(sx_1 \cdots sx_n (sy)^p (sz)^q) = 0$. The cocycles $sx_1 \cdots sx_n (sy)^p (sz)^q$, $p \geq 0$, $q \geq 0$, give linearly independent cohomology classes in $H^*(\Lambda(y, z, \ldots) \otimes \Lambda sV, \bar{\delta})$ since their images in $(\Lambda sV, 0)$ are linearly independent.

For all $k \geq 0$, there is at least $k + 1$ elements of the form $sx_1 \cdots sx_n (sy)^p (sz)^q$ in degree $\lvert sx_1 \rvert + \ldots + \lvert sx_n \rvert + k \cdot \lcm(\lvert sy \rvert, \lvert sz \rvert)$ (just take $p = i \cdot \lcm(\lvert sy \rvert, \lvert sz \rvert) / \lvert sy \rvert$ and $q = (k - i) \lcm(\lvert sy \rvert, \lvert sz \rvert) / \lvert sz \rvert$ for $i$ between 0 and $k$). Therefore the Betti numbers of $H^*(\Lambda(y, z, \ldots) \otimes \Lambda sV, \bar{\delta})$ are unbounded.

Suppose that the Betti numbers of $(\Lambda V \otimes \Lambda sV, \delta)$ are bounded. Then by Lemma 6.9 applied to $A = (\Lambda V \otimes \Lambda sV, \delta)$ and $x = x_1$, the Betti numbers of the quotient cdga $(\Lambda x_{2, \ldots} \otimes \Lambda sV, \delta)$ are bounded. By continuing to apply Lemma 6.9 to $x_2, x_3, \ldots, x_n$, we obtain that the Betti numbers of the quotient cdga $(\Lambda(y, z, \ldots) \otimes \Lambda sV, \delta)$ are bounded. But we saw just above that they are unbounded. \qed

7 Further readings

In this last section, we suggest some further readings that we find appropriate for the student.

In [1, Chapter 19], one can find a very short and gentle introduction to rational homotopy that the reader should compare to our introduction.
In this introduction, we have tried to explain that rational homotopy is a functor which transforms homotopy pullbacks of spaces into homotopy pushouts of cdgas. Therefore after our introduction, we advise the reader to look at [8], a more advanced introduction to rational homotopy, which explains the model category of cdgas.

The canonical reference for rational homotopy [5] is highly readable.

In the recent book [6], you will find many geometric applications of rational homotopy. The proof of Vigué-Poirrier-Sullivan theorem we give here, follows more or less the proof given in [6].

We also like [16] recently reprinted because it is the only book where you can find the Quillen model of a space: a differential graded Lie algebra representing its rational homotopy type (instead of a commutative differential graded algebra as the Sullivan model).

References

[1] Raoul Bott and Loring W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York, 1982.

[2] William Browder, *Torsion in H-spaces*, Ann. of Math. (2) 74 (1961), 24–51.

[3] Jean-Luc Brylinski, *Loop spaces, characteristic classes and geometric quantization*, Progress in Mathematics, vol. 107, Birkhäuser Boston Inc., Boston, MA, 1993.

[4] A. Dold, *Lectures on algebraic topology*, reprint of the 1972 ed., Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[5] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, 2000.

[6] Yves Félix, John Oprea, and Daniel Tanré, *Algebraic models in geometry*, Oxford Graduate Texts in Mathematics, vol. 17, Oxford University Press, Oxford, 2008.

[7] S. Halperin and M. Vigué-Poirrier, *The homology of a free loop space*, Pacific J. Math. 147 (1991), no. 2, 311–324.

[8] Kathryn Hess, *Rational homotopy theory: a brief introduction*, Interactions between homotopy theory and algebra, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, pp. 175–202.

[9] D. Husemoller, *Fibre bundles*, third ed., Graduate Texts in Mathematics, no. 20, Springer-Verlag, New York, 1994.

[10] J. D. S. Jones, *Cyclic homology and equivariant homology*, Invent. Math. 87 (1987), no. 2, 403–423.

[11] L. Menichi, *The cohomology ring of free loop spaces*, Homology Homotopy Appl. 3 (2001), no. 1, 193–224.

[12] P. Selick, *Introduction to homotopy theory*, Fields Institute Monographs, vol. 9, Amer. Math. Soc., 1997.

[13] Jean-Pierre Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2) 54 (1951), 425–505.
[14] Edwin H. Spanier, *Algebraic topology*, Springer-Verlag, New York, 1981, Corrected reprint.

[15] D. Sullivan, *Differential forms and the topology of manifolds*, Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), Univ. Tokyo Press, Tokyo, 1975, pp. 37–49.

[16] D. Tanré, *Homotopie rationnelle: Modèles de Chen, Quillen, Sullivan*, Lecture Notes in Mathematics, vol. 1025, Springer-Verlag, Berlin-New York, 1983.

[17] M. Vigué-Poirrier and D. Sullivan, *The homology theory of the closed geodesic problem*, J. Differential Geometry **11** (1976), no. 4, 633–644.
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