Investigation of a geometrically and physically nonlinear three-point bending problem for a sandwich plate with transversally soft core

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Abstract. The geometrically and physically nonlinear problem of the mechanics of deforming a sandwich plate with a transversally soft core under three-point bending is considered. It is shown that with this type of loading in test samples, it is possible to implement a mixed flexural buckling form, which is one of the reasons for their destruction with certain mechanical characteristics of the core. A method for determining the bifurcation point, a method for finding the supercritical branch of the solution are proposed, and the supercritical behaviour of the plate is studied. The method is based on the method of parameter continuation, in which quality is proposed to use the work of external forces due to its strongly increasing. Numerical experiments were carried out to study the process of geometrically and physically nonlinear deformation of the plate.

1. Introduction

Multilayer structures are widely used in various branches of engineering and industry [1–8]. In this article, the problem of three-point bending of a sandwich plate with a transversally soft core is studied. For statement of the problem, the previously proposed equations [9–12] are used, based on the description of the stress-strain state of the outer layers by classical relations of the theory of average bending of the plates, and the equations of the theory of elasticity, simplified by the model of a transversally soft layer, are used for the core. The latter, after integration over the transverse coordinate and satisfying the kinematic conjugation conditions of the outer layers and the core along the deflections, allow one to establish two-dimensional kinematic relations for the core by introducing two two-dimensional unknowns. Based on the previously obtained results [13–17], the classical Kirchhoff Love model is used to describe the mechanics of deformation of carrier layers at their average bend, according to which the displacement components and tensor components of the tangential deformations are used to represent the points of these layers

\[ \mathbf{U}^{(k)} = \mathbf{u}^{(k)} - z^{(k)} \mathbf{w}^{(k)} = (u_i^{(k)} - z^{(k)} \omega_i^{(k)}) \mathbf{e}_i^{(k)} + w_i^{(k)} \mathbf{m}_i^{(k)} , \quad -h_{i(k)} \leq z_{i(k)} \leq h_{i(k)}, \quad k = 1,2 \]

The kinematic relations are also used

\[ \varepsilon_{ij}^{(k)} = \varepsilon_{ij}^{(k)} - z \chi_{ij}^{(k)} , \quad 2 \varepsilon_{ij}^{(k)} = \nabla_i u_j^{(k)} + \nabla_j u_i^{(k)} + \alpha_i^{(k)} \alpha_j^{(k)} , \]

\[ 2 \chi_{ij}^{(k)} = \nabla_i \omega_j^{(k)} + \nabla_j \omega_i^{(k)} . \]

Here \( \mathbf{e}_i^{(k)} , \mathbf{e}_i^{(k)} , \mathbf{e}_i^{(k)} \) are the vectors of the coordinate of orthonormal basis on the median planes of the carrier layers, the tangential displacements and deflections of which are
denoted by \( u_1^{(k)}(x_1,x_2) \), \( u_2^{(k)}(x_1,x_2) \), \( w^{(k)}(x_1,x_2) \); \( \omega_j^{(k)}(x_1,x_2) = \partial w^{(k)}/\partial x_j \) are the corners of normals \( m^{(k)} \) turns around the axes \( x_1, x_2 \) \( (i, j = 1, 2) \). For an approximate solution of the problem, the finite sum method with the use of Gaussian nodes was used. To resolve the geometric and physical nonlinearity, a two-layer iterative process was used with a preconditioner, which is the linear part of the operator of the constructed scheme. A method based on the globally incremental Lagrange theory is proposed to study the behavior of the plate in unstable equilibrium positions. A software package was developed in the Matlab environment, on the basis of which numerical simulation of the processes under consideration was carried out. Numerical experimental investigations have shown that to adequately describe the three-point bending process of a sandwich plate with a transversally soft core, it is necessary to take into account physical nonlinearity in the form of a nonlinear dependence of the transverse shear modulus of the core on shear strain. Note that geometrically and physically nonlinear problems of the plates theory were considered in [18-26].

2. Statement of the problem

Taking into account that, in the case under consideration, the plate can be considered semi-infinite, we will describe the problem as the following system of five ordinary differential equations (hereafter \( \delta_1 = 1 \), \( \delta_2 = -1 \))

\[
\frac{dS_1^{(k)}}{dx} + \delta_1^{(k)} \frac{E_3}{2h} (w^{(2)} - w^{(1)}) + X_3^{(k)} = 0, \quad \frac{dT_1^{(k)}}{dx} + \delta_1^{(k)} q^1 + X_1^{(k)} = 0,
\]

\[
u^{(1)} - u^{(2)} - H(1) \frac{d w^{(1)}}{dx} - H(2) \frac{d w^{(2)}}{dx} + \frac{2h}{G_{13}} q^1 - \frac{2h^3 d^2 q^1}{3E_3} \frac{d}{dx^2} = 0,
\]

where \( S_1^{(k)} = Q_1^{(k)} + H_{1k} q^1 \), \( Q_1^{(k)} = dM_{11}^{(k)}/dx + T_{11}^{(k)} \omega, \) \( T_{11}^{(k)} = B_{1k} (du^{(k)}/dx + (\omega^{(k)})^2/2)^2, \) \( M_{11}^{(k)} = -D_{1k} d^2w^{(k)}/dx^2 \). Here \( B_{1k} = 2h_{1k} E^{(k)}(1 - v_{12}^{(k)} v_{21}^{(k)}), \) \( D_{1k} = B_{1k} h_{1k}^3/13 \) are the bending stiffness and tension-compressive stiffness of the \( k \)-th layer, having a thickness \( 2h_{1k} \) and made of a material with elastic modulus \( E^{(k)} \) and Poisson’s coefficients \( v_{12}^{(k)}, v_{21}^{(k)} \); \( h_{1k} = h_{1k} + h \), \( 2h \) is the thickness of the core having the elastic modulus \( E_3 \) in the transverse direction and the transverse shear modulus \( G_{13} \); \( M_{11}^{(k)} \) is the internal bending moment in the \( k \)-th layer; \( Q_1^{(k)} \) are shear forces in the \( k \)-th carrier layer without taking into account the tangential stresses in the core.

Since the problem has symmetry with respect to a point \( x = L/2 \), then it suffices to consider the problem on the segment \([0,L/2]\). For the formulated problem in the form of five differential equations (1), the boundary conditions for \( x = 0 \) are given as a free edge, and in section \( x = L/2 \), the symmetry conditions are formulated as follows:

\[
T_{11}^{(k)}(0) = 0, \quad S_1^{(k)}(0) = 0, \quad M_{11}^{(k)}(0) = 0, \quad q^1(0) = 0, \quad w^{(k)}(L/2) = 0, \quad S_1^{(k)}(L/2) = 0, \quad X_3^{(k)}(L/2) = 0, \quad q^1(L/2) = 0,
\]

To simulate the loading roller, the distributed load \( X_3^{(2)} \) applied to the upper layer on the segment \( c_0 \) is further defined by the cosine function with amplitude value \( p \). To simulate the contact interaction of the supports and the lower carrier layer in support sections \( x_1 = l \) and \( x_1 = L - l \), the conditions of the type \( w^{(l)} = 0 \) were specified, and the unknown reactions \( X_k(p_2) = X_3^{(1)} \) from the support rollers were approximated by a trigonometric cosine function with an amplitude value \( p_2 \), which is to be determined from the solution of the problem. By virtue of this, the necessary condition
of solvability is added to the system of equations (1) in the following form of equality to zero of the deflections of the lower carrier layer at the centers of the support rollers

$$w^{(1)}(l) = w^{(1)}(L-l)$$, \hfill (3)

Thus, a geometrically nonlinear boundary-value problem of determining the stress-strain state in the form of a system of nonlinear equations (1), boundary and symmetry conditions (2), additional equation (3) with respect to the vector of unknowns \((w^{(1)}, w^{(2)}, u_1^{(1)}, u_1^{(2)}, q^1, p_z)\) is formulated.

3. Numerical method

For an approximate solving the formulated boundary value problem (1)–(3), the finite sum method was used. In accordance with this method, the original differential equations are reduced to Volterra type integral equations of the second kind with additional relations for determining the unknown integration constants. For approximation of the resulting integral equations in [27, 28], a method of collocations by Gaussian nodes and a method for constructing integrating matrices was proposed. To resolve the geometric and physical nonlinearity, a two-layer iterative process [29–36] was used with lowering the nonlinearity to the lower layer with a preconditioner, which is the linear part of the operator of the constructed scheme. A method based on the globally incremental Lagrange theory is proposed to study the behavior of the plate in unstable equilibrium positions. This method is one of the options for the implementation of the incremental algorithm for the process of continuing the decision on the load parameter, in accordance with which the deformation process is represented as a realization of a sequence of equilibrium states at the corresponding loading levels. As a parameter to continue the solution in the supercritical region, the work of external forces was chosen due to its strict increase. The software implementation of the described algorithms for the numerical solution of the formulated problems was carried out in the Matlab environment, which allowed carrying out a series of numerical experiments.

![Figure 1](image-url)

**Figure 1.** Dependence of the transverse load \(p\) (MPa) on the deflection \(w^{(2)}\) in the central part of the upper layer (cm)

In Figure 1 the dependence of the amplitude bending load \(p\) on the deflection \(w^{(2)}\) of the upper carrier layer in the section \(x = L/2\) is presented. It can be seen that two different equilibrium positions \((w^{(2)}(L/2) = 0.76\) cm and \(w^{(2)}(L/2) = 1.85\) cm) correspond to the same load value.

4. Conclusion

For the purpose of practical verification of the developed numerical method and validation of the results obtained, an experimental study of the bending of the test sample according to the three-point scheme was carried out. From the obtained results it can be concluded that for the honeycomb core used in the experiment, the maximum bending load corresponds to the fracture of the core as a result of reaching the tensile strength of tangential and transverse normal stresses. Experimental studies have
shown that it is necessary to take into account the physical nonlinearity in the form of a nonlinear dependence of the transverse shear modulus of the core on the shear strain.

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References
[1] Bolotin V V and Novichkov Yu N 1980 Mechanics of Multilayered Structures (Moscow, Mashinostroenie) (in Russian).
[2] Badriev I B, Garipova G Z, Makarov M V, Paimushin V N and Khabibullin R F 2015 Solving physically nonlinear equilibrium problems for sandwich plates with a transversally soft core Lobachevskii Journal of Mathematics 36 (4) 474-81. DOI: 10.1134/S1995080215040216
[3] Badriev I B, Makarov M V and Paimushin V N 2017 Numerical investigation of a physically nonlinear problem of the longitudinal bending of the sandwich plate with a transversal-soft core PNRPU Mechanics Bulletin (1) 39-51. DOI: 10.15593/perm.mech/2017.1.03
[4] Badriev I B, Makarov M V and Paimushin V N 2015 Solvability of physically and geometrically nonlinear problem of the theory of sandwich plates with transversally-soft core Russian Mathematics 59 (10) 57-60. DOI: 10.3103/S1066369X15100072
[5] Badriev I B, Garipova G Z, Paimushin V N and Makarov M V 2015 Numerical solution of the issue about geometrically nonlinear behavior of sandwich plate with transversal soft filler Research Journal of Applied Sciences 10 (8) 428-35. DOI: 10.3923/rjasci.2015.428.435
[6] Badriev I B, Makarov M V and Paimushin V N 2017 Contact statement of mechanical problems of reinforced on a contour sandwich plates with transversally-soft core Russian Mathematics 61 (1) 69-75. DOI: 10.3103/S1066369X1701008X
[7] Badriev I B, Makarov M V and Paimushin V N 2018 Geometrically Nonlinear Problem of Longitudinal and Transverse Bending of a Sandwich Plate with Transversally Soft Core Lobachevskii Journal of Mathematics 39 (3) 448-57. DOI: 10.1134/S1995080218030046
[8] Badriev I B, Bandurov V V and Makarov M V 2017 Mathematical Simulation of the Problem of the Pre-Critical Sandwich Plate Bending in Geometrically Nonlinear One Dimensional Formulation IOP Conference Series: Materials Science and Engineering 208 (1) 012002. DOI: 10.1088/1757-899X/208/1/012002
[9] Paimushin V N 1987 Nonlinear theory of the central bending of three-layer shells with defects in the form of sections of bonding failure Soviet Applied Mechanics 23 (11) 1038-43
[10] Ivanov V A and Paimushin V N 1994 Refined theory of stability of three-layer constructions (Nonlinear equations of subcritical equilibrium of shells with transversal-soft aggregate) Russian Mathematics 38 (11) 26–39
[11] Paimushin V.N and Bobrov S N 2000 Refined geometric nonlinear theory of sandwich shells with a transversely soft core of medium thickness for investigation of mixed buckling forms Mechanics of Composite Materials 36 (1) 59-66.
[12] Pajmushin V N and Shalashilin V I 2004 Noncontradictory variant of solid mechanics in square approximation Doklady Akademii Nauk 396 (4) 492–5.
[13] Badriev I B, Makarov M V and Paimushin V N 2016 Longitudinal and transverse bending by a cylindrical shape of the sandwich plate stiffened in the end sections by rigid bodies IOP Conference Series-Materials Science and Engineering 158 (1) 012011. DOI: 10.1088/1757-899X/158/1/012011
[14] Badriev I B, Makarov M V and Paimushin V N 2016 Geometrically Nonlinear Problem of Longitudinal and Transverse Bending of a Sandwich Plate with Transversally Soft Core Uchenye zapiski Kazanskogo universiteta-Seriya fiziko-matematicheskie nauki 158 (4) 453-68. (in Russian).
[15] Badriev I B, Makarov M V and Paimushin V N 2017 Longitudinal and Transverse Bending on the Cylindrical Shape of a Sandwich Plate Reinforced with Absolutely Rigid Bodies in the
Front Sections Uchenye zapiski Kazanskogo universiteta-Seriya fiziko-matematicheskie nauki 159(2) 174-90. (in Russian).

[16] Badriev I B, Makarov M V, Paimushin V N and Kholmogorov S A 2017 The Axisymmetric Problems of Geometrically Nonlinear Deformation and Stability of a Sandwich Cylindrical Shell with Contour Reinforcing Beams Uchenye zapiski Kazanskogo universiteta-Serija fiziko-matematicheskie nauki 159(4) 395-428. (in Russian)

[17] Paimushin V N, Kholmogorov S A and Badriev I B 2017 Theoretical and experimental investigations of the formation mechanisms of residual deformations of fibrous layered structure composites MATEC Web of Conferences 129 02042 DOI: 10.1051/matecconf/201712902042

[18] Badriev I B, Makarov M V and Paimushin V N 2016 Mathematical simulation of nonlinear problem of three-point composite sample bending test Procedia Engineering 150 1056-62. DOI: 10.1016/j.proeng.2016.07.214

[19] Badriev I B, Makarov M V and Paimushin V N 2016 Numerical Investigation of Physically Nonlinear Problem of Sandwich Plate Bending Procedia Engineering 150 1050-5 DOI: 10.1016/j.proeng.2016.07.213

[20] Sultanov L U 2016 Analysis of finite elasto-plastic strains. Medium kinematics and constitutive equations Lobachevskii Journal of Mathematics 37(6) 787-93. DOI: 10.1134/S1995080216060032

[21] Abdrahmanova A I, Garifullin I R, Davydov R L, Sultanov L U and Fakhruddinov L R 2015 Investigation of Strain of Solids for Incompressible Materials Applied Mathematical Sciences 9(118) 5907-14. DOI: 10.12988/ams.2015.57507

[22] Bereznoi D V, Chickrin D E, Kurchatov E Y and Galimov A F 2014 Estimation of specific energy capacity of flywheel-housing system in potential field Applied Mathematical Sciences 8(161-164) 8125-35. DOI: 10.12988/ams.2014.410831

[23] Badriev I B and Banderov V V 2014 Iterative methods for solving variational inequalities of the theory of soft shells Lobachevskii Journal of Mathematics 35(4) 371-83. DOI: 10.1134/S1995080214040015

[24] Solov'ev S I 2016 Eigen Vibrations of a beam with elastically attached load Lobachevskii Journal of Mathematics 37(5) 597-609. DOI: 10.1134/S1995080216050115.

[25] Dautov R Z, Lyashko A D and Solov'ev S I 1991 Convergence of the Bubnov-Galerkin method with perturbations for symmetric spectral problems with parameter entering nonlinearly Differential Equations 27(7) 799-806

[26] Badriev I B, Banderov V V and Zadvornov O A 2013 On the solving of equilibrium problem for the soft network shell with a load concentrated at the point PNRPU Mechanics Bulletin (3) 17-35.

[27] Dautov R Z and Paimushin V N 1996 On the method of integrating matrices for the solution of boundary value problems for fourth-order ordinary equations Russian Mathematics 40 (10) 11–23

[28] Dautov R Z, Karchevskii M M and Paimushin V N 2003 On the method of integrating matrices for systems of ordinary differential equations Russian Mathematics 47(7) P. 16-24.

[29] Lapin A, Laitinen E and Lapin S 2015 Explicit algorithms to solve a class of state constrained parabolic optimal control problems Russian Journal of Numerical Analysis and Mathematical Modelling 30(6) 351-62. DOI: 10.1515/rnam-2015-0032

[30] Badriev I B and Karchevskii M M 1994 Convergence of an iterative process in a Banach space Journal of Mathematical Sciences 71 (6) 2727-35. DOI: 10.1007/bf02110578 15

[31] Dautov R Z, Lyashko A D and Solov'ev S I 1991 Convergence of the Bubnov-Galerkin method with perturbations for symmetric spectral problems with parameter entering nonlinearly Differential Equations 27(7) 799-806

[32] Badriyev I B, Zadvornov O A, Ismagilov L N and Skvortso E V 2009 Solution of plane seepage problems for a multivalued seepage law when there is a point source Journal of
[33] Badriev I B 1983 Difference schemes for nonlinear problems of filtration theory with a discontinuous law *Soviet Mathematics* **27** (5) 1–11

[34] Badriev I and Banderov V 2014 Numerical method for solving variation problems in mathematical physics *Applied Mechanics and Materials* **668-669** 1094-97. DOI: 10.4028/www.scientific.net/AMM.668-669.1094

[35] Chebakova V J, Gerasimov A V and Kirpichnikov A P 2016 On the solving of one type of problems of mathematical physics *IOP Conference Series: Materials Science and Engineering* **158**(1) 012023. DOI: 10.1088/1757-899X/158/1/012023

[36] Badriev I B 1983 Difference-schemes for linear-problems of the filtration theory with discontinuous law *Izvestiya Vysshikh Uchebnykh Zavedenii Matematika* **5** 3-12