Bifurcation control and universal unfolding for Hopf-zero singularities with leading solenoidal terms

Majid Gazor† and Nasrin Sadri‡

Department of Mathematical Sciences, Isfahan University of Technology
Isfahan 84156-83111, Iran
December 18, 2014

Abstract

In this paper we define the universal asymptotic unfolding normal forms for nonlinear singular systems. Next, we propose an approach to find the parameters of a parametric singular system that they can play the role of universal unfolding parameters. Thus, these parameters may effectively influence the local dynamics of the system. We show how we can locate local bifurcations in terms of these parameters. This approach is useful in designing efficient feedback controllers (single or multiple inputs) for possible local bifurcation control in engineering problems. Here, we apply the proposed approach only on Hopf-zero singularities whose certain first few low degree terms are incompressible. In this direction, we first obtain the orbital normal forms of such families by assuming a nonzero quadratic condition. Next, we give a truncated universal asymptotic unfolding for the simplest orbital normal form family and prove the finite determinacy of the steady-state bifurcations for (a subfamily of) the associated amplitude systems. Accordingly, we do the local bifurcation analysis of equilibria and limit cycles. Finally, the results are implemented to a control system (on a three dimensional central manifold) with two imaginary uncontrollable modes and a multiple-parametric quadratic state feedback controller. The effective control (universal unfolding) parameters are distinguished. Then, the (estimated) transition varieties and bifurcation control analysis are drawn and concluded in terms of these parameters. This illustrates that our approach allows to design effective controllers for engineering applications involving singularities. The results are successfully implemented and verified using Maple.

Keywords: Bifurcation control; Orbital normal form; Universal asymptotic unfolding; Hopf-zero singularity; Solenoidal vector fields.

2010 Mathematics Subject Classification: 34C20; 34A34.

†Corresponding author. Phone: (98-31) 33913634; Fax: (98-31) 33912602; Email: mgazor@cc.iut.ac.ir;
‡Email: n.sadri@math.iut.ac.ir.
1 Introduction

Any small perturbation of a singular differential system may substantially change the qualitative dynamics of the system. Therefore for such engineering singular problems, a practical approach is to design a controller to inhibit its real world dynamics rather than being dominated by the mathematical modeling imperfections. This can be achieved by computing the universal unfolding of the differential system and be used for a bifurcation controller design. Bifurcation control has many applications in engineering problems such as power, electronics, and mechanical systems; predicting and preventing voltage collapse and oscillation in power networks, high-performance circuits and oscillator designs; e.g., see [7, 8]. The idea here is to design a controller for a nonlinear system so that the system follows a certain bifurcation branch and thus, it behaves as desired. Recently, normal form theory has been used for local bifurcation control; see [7, 19–22]. In this paper we describe how parametric normal form theory can propose effective feedback controller designs for an engineering problem. We apply it to the Hopf-zero singularity, i.e.,

\[
\begin{align*}
\dot{x} &= f(x, y, z), \\
\dot{y} &= z + g(x, y, z), \\
\dot{z} &= -y + h(x, y, z),
\end{align*}
\]

(1.1) for \((x, y, z) \in \mathbb{R}^3\), where \(f, g, h\) do not have linear and constant terms. Additionally, we assume that certain first few low degree terms of Equation (1.1) constitute a solenoidal vector field (that is, the case \(I\) in (3.2)). Throughout this paper, we will interchangeably use the terms vector field \(v\), denoted by \(f \frac{\partial}{\partial x} + (z + g) \frac{\partial}{\partial y} + (-y + h) \frac{\partial}{\partial z}\) or \((f, z + g, -y + h)\), and the differential system (1.1).

The dynamics of a Hopf-zero singular system given by (1.1) may not be finitely determined and many dynamical properties such as heteroclinic orbit breakdowns and Šil’nikov bifurcations cannot be detected through truncated normal form computations; e.g., see [5, 6, 11]. However, normal form computations are still useful for the analysis of finitely determined dynamical properties through an \(n\)-equivalence relation and the \(n\)-universal asymptotic unfolding. These properties may include bifurcations of equilibria and small limit cycles.

Murdock [29, 30, 32] defines two systems as \(n\)-equivalent when they share all their \(n + 1\)-jet (Taylor expansion up to degree \(n + 1\)) determined properties. The \(n\)-asymptotic unfolding is defined based on the \(n\)-equivalence relation and it seems the most natural way of defining a versal unfolding amenable to computation and normal form analysis; e.g., see [32, Theorem 1]. We slightly modify versal asymptotic unfolding and call it versal asymptotic unfolding normal form, that is, a versal asymptotic unfolding for our simplest (orbital) normal form system. This also exhibits the \(n + 1\)-jet determined properties of all small perturbations of the original system.

A parametric vector field \(\tilde{v}(x, \mu)\) (for \(\mu \in \mathbb{R}^p\) from a small neighborhood of the origin and \(x \in \mathbb{R}^3\)) is called a perturbation or an unfolding for \(v(x)\) when \(\tilde{v}(x, 0) = v(x)\). The parameterized
family of perturbations
\[ \tilde{v}(x, \mu) := v(x) + \sum_{\{i | n_i \leq n\}} \mu_i x^{n_i}, \] for \( n_i = (m_1, m_2, m_3) \in (\mathbb{N} \cup \{0\})^3, |n_i| = m_1 + m_2 + m_3, \)
is called the \( n+1 \)-jet general perturbation for \( v(x) \), that is \( v(x) \) plus all monomial perturbations of degree less than or equal to \( n + 1 \). In order to simplify such systems, we consider a smooth locally invertible changes of state variables
\[ x = \phi(y, \mu), \] where the Jacobian \( D_\phi \phi \) at the origin is invertible, (1.2)
and time rescaling
\[ \tau = T(y, \mu) t, \] with \( T(0, 0) \neq 0 \), (1.3)
where \( t \) and \( \tau \) denote old and new time indices. Using \( (1.2) \) and \( (1.3) \), we may transform \( \tilde{v}(x, \mu) \) into an orbitally equivalent vector field as
\[ \hat{v}(y, \mu) := T(y, \mu)(D_\phi \phi)^{-1}v(\phi(y, \mu), \mu). \] (1.4)
For our convenience we shall use \( \hat{v}(x, \mu) \) instead of \( \hat{v}(y, \mu) \). The idea is to use these to transform a given vector field into its simplest orbital normal form, that is, an orbitally equivalent vector field with the least possible number of (monomial- or \( F, E \) and \( \Theta \)-) terms in its \( n \)-grade truncated (Taylor- or \( F, E \) and \( \Theta \)-) expansion for any \( n \). It is known that the simplest orbital normal form is unique (i.e., the normalized coefficients are uniquely determined in terms of the original system) when a formal basis style is chosen; a formal basis style basically determines the priority of elimination between different alternative omittable terms; see [16, 17, 30]. The \( F, E \) and \( \Theta \)-terms and associated expansion are defined in Section 2. We assume that a formal basis style has been fixed.

**Definition 1.1.** A parametric vector field \( w(x, \nu) \) is called an \( n \)-versal asymptotic unfolding normal form for \( v(x) \) if

a. for any small perturbation \( \tilde{v}(x, \epsilon) \) of \( v(x) \), there exist a vector field \( \hat{v}(x, \epsilon) \) and a polynomial map \( \nu(\epsilon) \) such that some transformations \( (1.2) \) and \( (1.3) \) transform \( \tilde{v}(x, \epsilon) \) to \( \hat{v}(x, \epsilon) \), and the \( n \)-grade truncated \( (n+1 \)-jet) vector fields of \( \hat{v} \) and \( w(x, \nu(\epsilon)) \) are identical.

b. the \( n+1 \)-jet of \( w(x, 0) \) is an \( n+1 \)-jet truncated simplest orbital normal form for \( v(x) \).

We call a parametric system \( w(x, \nu) \), an \( n \)-universal asymptotic unfolding normal form for \( v(x) \) when

- \( w(x, \nu) \) is an \( n \)-versal asymptotic unfolding normal form for \( v(x) \).
- The \( n+1 \)-jet \( (n \)-grade Taylor expansion) of \( w(x, \nu) \) is the simplest \( (n+1 \)-jet) truncated orbital normal form for the \( n+1 \)-jet general perturbation of \( v(x) \).
• For any perturbation \( \tilde{v}(x, \epsilon) \), the \( n + 1 \)-jet of the map \( \nu(\epsilon) \) in item (a) is unique.

Then, we refer to the parameters \( \nu \) by universal unfolding parameters.

For any vector field \( v(x) \) and any given natural number \( n \), there always exists an \( n \)-universal asymptotic unfolding normal form \( w(x, \nu) \). Furthermore, the orbital normal form computation of the \( n + 1 \)-jet general perturbation gives rise to \( w(x, \nu) \) and then, the orbital normal form of a given perturbation \( \tilde{v}(x, \epsilon) \) readily gives the polynomial map \( \nu(\epsilon) \).

The universal asymptotic unfolding normal form facilitates the (finitely determined) local bifurcation analysis in terms of the unfolding parameters. However in practical engineering problems, the mathematical models are mostly involved with parameters such as control parameters that they may be used for possible bifurcation control; see [7]. Therefore, a practically useful approach needs to locate the bifurcations in terms of the original (control) parameters of a parametric (control) system. This has been rarely performed in the existing normal form literature; see [34] and [7] Pages 99-126. Hence, a useful normal form analysis needs to compute the relations between the unfolding parameters and the parameters of the original system. This reveals the impact of control parameters on our nonlinear control system. More details have been given in Section 4 and the results have successfully been implemented for Hopf-zero singularities with dominant solenoidal terms in our Maple program. In order to achieve this goal, we first need to derive orbital normal form of Equation (1.1) and then, orbital and parametric normal forms of its (small) multiple parametric perturbations.

Throughout this paper we require that

\[
f_{yy}(0) + f_{zz}(0) \neq 0
\]

(also see [13]) while other hypernormal form results have assumed that \( f_{xx}(0) \left( g_{yy}(0) + h_{zz}(0) \right) \neq 0 \); see [1, 9, 35]. Using invertible changes of state variables, Equation (1.1) can be transformed into the classical normal form

\[
\dot{x} = \rho^2 + \sum_{i=2}^{\infty} a_i x^i, \quad \dot{\rho} = \sum_{i=0}^{\infty} b_i x^{2i+1} \rho, \quad \dot{\theta} = 1 + \sum_{i=0}^{\infty} c_i x^{2i+1}.
\]

We further assume that there exists \( a_k \neq 0 \) for some \( k \). Now define

\[
r := \min \{ i \mid a_i \neq 0, i \geq 1 \}, \quad s := \min \{ j \mid b_j \neq 0, j \geq 1 \}.
\]

In this paper we compute the orbital normal forms of the system (1.1), given (1.5) and

\[
r < s,
\]

and also parametric normal forms for any of its multiple-parametric perturbations. Here, we assume that \( s < \infty \). The case \( s = \infty \) consists of all solenoidal Hopf-zero vector fields and is discussed in [14]. The family associated with assumptions (1.5) and (1.8) are large enough so
that they may appear near stagnation points associated with perturbations of incompressible fluid flows, three dimensional magnetic field lines and well-known systems such as Michelson system. The results on the other two cases \((r > s)\) and \(r = s\) are in progress and appear elsewhere.

We prove that there are time-rescaling and changes of state variables that they transform any of such systems into

\[
\dot{x} = 2\rho^2 + a_r x^{r+1} + \sum_{k=s}^{\infty} \beta_k x^{k+1}, \quad \dot{\rho} = -\frac{a_r(r+1)}{2} x^r \rho + \frac{1}{2} \sum_{k=s}^{\infty} \beta_k x^k \rho, \quad \dot{\theta} = \sum_{k=0}^{r} r \gamma_k x^k,
\]

where \(\beta_k = 0\) for \(k \equiv 2(r+1) - 1\) and \(k \equiv 2(r+1)\) \(s\). Finally, we prove that any multiple-parametric perturbation of the system \((1.6)\) (for \(r < s\)) can be transformed into the \((s+1)\)-th level parametric normal form

\[
\dot{x} = 2\rho^2 + a_r x^{r+1} + \sum_{-1 \leq i < r+1} a_{in} x^{i+1} \mu^n + \sum_{0 \leq i < s, i \neq r} b_{in} x^{i+1} \mu^n + \sum_{k=s}^{\infty} \beta_{kn} x^{k+1} \mu^n, \quad (1.9)
\]

\[
\dot{\rho} = -\frac{a_r(r+1)}{2} x^r \rho + \sum_{-1 \leq i < r+1} a_{in}(\frac{i+1}{2}) \rho \mu^n + \sum_{0 \leq i < s, i \neq r} \frac{1}{2} b_{in} x^i \rho \mu^n + \sum_{k=s}^{\infty} \frac{1}{2} \beta_{kn} x^k \rho \mu^n,
\]

\[
\dot{\theta} = \sum_{k=1}^{r} \gamma_{kn} x^k \mu^n,
\]

and its \(s+1\)-universal asymptotic unfolding is given by

\[
\dot{x} = 2\rho^2 + a_r x^{r+1} + \beta_s x^{s+1} + \sum_{1 \leq i < r} \nu_i x^{i+1} + \sum_{i=r+1}^{N} \nu_i x^{i+1}, \quad (1.10)
\]

\[
\dot{\rho} = -\frac{a_r(r+1)}{2} x^r \rho + \frac{1}{2} \beta_s x^s \rho - \sum_{1 \leq i < r} (\frac{i-1}{2}) \nu_i x^{i-2} \rho + \frac{1}{2} \sum_{i=r+1}^{N} \nu_i x^{i-1} \rho,
\]

\[
\dot{\theta} = 1 + \sum_{i=1}^{r} (\gamma_i + \omega_i) x^i.
\]

Here, \(n = (m_1, \ldots, m_p) \in (\mathbb{N} \cup \{0\})^p\), the parameters \(\mu := (\mu_1, \ldots, \mu_p) \in \mathbb{R}^p\), \(\nu_i, \omega_i \in \mathbb{R}\), \(\mu_n := \mu_1^m \ldots \mu_p^m, \gamma_i, \gamma_i \in \mathbb{R}\).

The rest of this paper is organized as follows. In Section 2 we provide the time rescaling structure and introduce our normal form style. Section 3 presents our orbital normal form results. Parametric and universal asymptotic unfolding normal forms for \(s < \infty\) are given in Section 4. In Section 5 using contact equivalence relation, we prove that steady-state bifurcations associated with the amplitude systems for \(r = 1\) are finitely determined. (Here the planar normal form system obtained by ignoring the angle coordinate of normal forms in cylindrical coordinates is called the amplitude system.) Then by assuming that \(r = 1\), we discuss a limited (not a complete) bifurcation analysis of the universal asymptotic unfolding normal forms.
In Section 6 we explain how to solve the recognition problem corresponding to the universal asymptotic unfolding, that is, to distinguish the effective parameters that they can play the role of universal unfolding parameters for the truncated simplest orbital normal forms. Roughly speaking, (cognitive choices of) these parameters may effectively control certain local dynamics of the system such as bifurcations of equilibria and limit cycles. Finally, Section 7 applies our approach to an illustrating example with two imaginary uncontrollable modes. Here, estimated transition sets are drawn in terms of the distinguished parameters and are supported with some numerical simulations. This demonstrates that our distinguished parameters can suitably control the local dynamics of a nonlinear Hopf-zero singular system and can be used for a possible engineering design.

2 Time rescaling structure and normal form style

Recall (from [13, Lemma 2.2])

$$F^l_k := (k - l + 1)x^{l+1}\rho^{2(k-l)} \frac{\partial}{\partial x} - \left(\frac{l + 1}{2}\right)x^l \rho^{2k-2l+1} \frac{\partial}{\partial \rho},$$

$$E^l_k := x^{l+1}\rho^{2(k-l)} \frac{\partial}{\partial x} + \frac{1}{2}x^l \rho^{2k-2l+1} \frac{\partial}{\partial \rho},$$

$$\Theta^l_k := -x^l \rho^{2(k-l)} \frac{\partial}{\partial \theta},$$

and the fact that any Hopf-zero system can be expanded in terms of $F, E,$ and $\Theta$-terms like

$$v := c_{0,0}\Theta^0_0 + \sum a_{i,j}E^i_j + \sum b_{i,j}E^i_j + \sum c_{i,j}\Theta^i_j.$$ 

We denote $\mathcal{L}$ for the vector space generated by all Hopf-zero normal forms expanded with respect to $F, E,$ and $\Theta$-terms. The Lie algebra structure constants follow [13, Lemma 2.3].

We define a module structure for $\mathcal{L}$ that is instrumental for computing the effect of the near-identity time rescaling. The integral domain of formal power series (denoted by $\mathcal{R}$) generated by monomials

$$Z^m_n := x^m \rho^{2(n-m)}$$ (2.1)

represents the space of time-rescaling generators; $Z^m_n$ generates $t := (1 + x^m \rho^{2(n-m)})\tau,$ where $t$ and $\tau$ denote the old and new time variables. Hence, $\mathcal{R} := \text{span}\{Z^m_n | m \leq n, m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ acts on $\mathcal{L}$ and $\mathcal{L}$ is an $\mathcal{R}$-module.

**Lemma 2.1.** The $\mathcal{R}$-module structure constants associated with time rescaling is given by

$$Z^m_n F^l_k = \frac{k + 2}{k + n + 2}F^l_{k+n} + \frac{m(k + 2) - n(l + 1)}{k + n + 2}E^l_{k+n},$$

$$Z^m_n E^l_k = E^l_{k+n},$$

$$Z^m_n \Theta^l_k = \Theta^l_{k+n}.$$
In this paper we merely apply the time rescaling space
\[ \mathcal{T} := \text{span}\{Z^n_m \in \mathcal{T} \mid \text{for } m = n, \text{ or } m + 1 = n\}, \quad (2.2) \]
and use the formulas and identical notations from [12, 13]. Our computations and Maple program suggest that other time rescaling generators (from \( R \setminus \mathcal{T} \)) do not simplify the system beyond what we present in this paper.

Any normal form computation requires a normal form style. That is, for the cases with alternative terms for elimination, a style determines the priority of elimination. This is determined by an ordering on basis terms (\( F, E, \Theta \)-terms) of \( \mathcal{L} \) in formal basis style; see [17], given a grading function \( \delta \), for any \( v, w \in \{F_{m}^{n}, E_{k}^{l}, \Theta_{p}^{q}\} \) we define \( v < w \), when
\[
\bullet \text{ Style: } \begin{cases} 
\delta(v) < \delta(w), \\
\delta(v) = \delta(w), & v = F_{m}^{n} \text{ and } w = E_{k}^{l} \text{ or } w = \Theta_{q}^{p}, \\
\delta(v) = \delta(w), & v = E_{k}^{l} \text{ and } w = \Theta_{q}^{p}.
\end{cases}
\]
This indicates that our priority of elimination is with low grade terms over higher grades and then, \( F \)-terms over \( E \)-terms. We denote \( (a)^{k}_{b} := a(a + b)(a + 2b) \cdots (a + (k-1)b) \) for any natural number \( k \) and real number \( b \), and for any integer numbers \( m, n, p \) the notation
\[ m \equiv_{p} n \]
is used when there exists an integer \( k \) such that \( m - n = kp \).

## 3 The orbital normal forms

Equation (1.1) can be transformed into the normal form Equation (1.6) or equivalently,
\[ v^{(2)} := \Theta_{0}^{0} + a_{0}F_{0}^{-1} + \sum a_{i}F_{i}^{-1} + \sum b_{i}E_{i}^{1} + \sum c_{i}\Theta_{i}^{1}, \quad (3.1) \]
where \( a_{i}, b_{i}, c_{i} \in \mathbb{R} \); see [13, Lemma 3.1]. Without loss of generality we may assume that \( a_{0} = 1 \); see the comments above [13, Remark 3.2]. (Note that the orbital equivalence may here include reversal of time.) Recall \( r \) and \( s \) from Equation (1.8). Orbital normal form reduction of Equation (3.1) is split into three cases (this is similar to the three cases of Bogdanov-Takens singularity [4,12,23])
\[ v^{(2)} := \Theta_{0}^{0} + a_{0}F_{0}^{-1} + \sum a_{i}F_{i}^{-1} + \sum b_{i}E_{i}^{1} + \sum c_{i}\Theta_{i}^{1}, \quad (3.1) \]
where \( a_{i}, b_{i}, c_{i} \in \mathbb{R} \); see [13, Lemma 3.1]. Without loss of generality we may assume that \( a_{0} = 1 \); see the comments above [13, Remark 3.2]. (Note that the orbital equivalence may here include reversal of time.) Recall \( r \) and \( s \) from Equation (1.8). Orbital normal form reduction of Equation (3.1) is split into three cases (this is similar to the three cases of Bogdanov-Takens singularity [4,12,23])
\[ \begin{cases} 
\text{Case I: } r < s, \\
\text{Case II: } r > s, \\
\text{Case III: } r = s.
\end{cases} \quad (3.2) \]
In this paper we only deal with the case I. Recall the grading function (see [13, Equation 4.3])
\[ \delta(F_{k}^{l}) = \delta(E_{k}^{l}) = r(k-l) + k, \quad \delta(\Theta_{k}^{l}) = r(k-l) + k + s \quad (3.3) \]
and the linear map
\[ d^{m,N}(S_{n-N+1}, \ldots, S_{n-r}, T_{n-N+1}, \ldots, T_{n-r}) := \sum_{k=r}^{N-1} |S_{n-k}, v_{k}| + T_{k}v_{n-k}, \]
for any \((S_{n-N+1}, \ldots, S_{n-1-r}; T_{n-N+1}, \ldots, T_{n-1-r}) \in \ker d^{n-1,N-1}\), the updating vector field \(v = \sum_{n=r}^{\infty} v_n, S_i \in \mathcal{L}_i, \) and \(T_i \in \mathcal{T}_i\), where \(\mathcal{L}_i, \mathcal{T}_i\) denote the \(\delta\)-homogenous subspaces of \(\mathcal{L}\) and \(\mathcal{T}\). By [16, Lemma 4.3], any Hopf-zero vector field \(v\) can be transformed into the \(N+1\)-th level normal form 
\[v^{(N+1)} = \sum w_n\] such that any \(w_n\) belongs a complement space of \(\operatorname{im} d^{n,N}\). We skip many subtleties of the subject; see [12, 16, 17] and [12, Lemma 2.5] for more information. Denote
\[F_r := F_0^{-1} + a_r F_r^r.\]

We now provide some technical formulas for obtaining orbital normal forms.

Lemma 3.1. [13, Lemma 3.3] For any nonnegative integers \(m, n, n \geq m\), there exists a vector field
\[\mathcal{T}^m_n := \sum_{l=0}^{n-m-1} a_r^l ((n-m-1)(r+1) - m-1)^l 2^{l+1} (m+1)^{l+1} \Theta_{n+l+1}^{m+l+1}\]
such that
\[[\mathcal{T}^m_n, F_r] + \Theta^m_n = \frac{a_r^{n-m} ((n-m-1)(r+1) - m-1)^{n-m}}{2^{n-m} (m+1)^{n-m}} \Theta_{n+r-m+r+n}^{m+r}.\]

Lemma 3.2. For any nonnegative integer \(m\), we have
\[Z^m_{m+1} \Theta^0_0 + \left[ \frac{1}{2(m+1)} \Theta^m_{m+1}, F_r \right] = \frac{a_r}{2} \Theta^m_{m+r+1}.\] (3.4)

Besides, for any \(m \neq 0\) there exists vector filed \(Z^m_m\) such that
\[Z^m_m F_r + [Z^m_m, F_r] = \frac{a_r (r+1)}{m+1} F^m_{m+r}.\] (3.5)

Proof. By Lemma 2.1 we have
\[Z^m_m F_0^{-1} = \frac{2}{m+2} F^{m-1}_m + \frac{2m}{m+2} F^{-1}_m,\]
\[Z^m_m F_r^r = \frac{r+2}{m+r+2} F^m_{m+r} + \frac{m}{m+r+2} F^m_{m+r}.\]

The proof is complete by defining
\[Z^m_m := \frac{1}{(m+1)^2} F^m_m + \frac{1}{(m+2)} F^m_m.\] (3.6)

Lemma 3.2 implies that \(\Theta^0_0\) should be omitted when we intend to use time rescaling terms \(Z^m_m\) to simplify \(F^m_{m+r}\) while we need to have \(\Theta^0_0\) in the normal form system in order to use \(Z^m_{m+1}\) for eliminating \(\Theta^m_{m+r+1}\)-terms. We have already used this idea in [13–15]. Since it plays a central role in efficient use of time rescaling for simplifying \(\Theta\)-terms, we here state it as a Lemma.
Lemma 3.3. There exists invertible linear change of state variables $\varphi$ (given by \cite[Equation (3.3)]{13}) such that $\varphi$ eliminates the linear part $\Theta^0_0$ from the classical normal form. In addition, $\varphi^{-1}$ adds $\Theta^0_0$ back into the normal form system.

Lemma 3.4. There exists a vector field $Z^0_1 \in \mathcal{L}$ such that

$$Z^0_1 F_r + [Z^0_1, F_r] = -\frac{a_r^2(7r^2 + 17r + 9)}{6(r + 2)(2r + 3)} F^2_{r+1} + \frac{a_r^2(r + 1)}{2(2r + 3)} E^2_{r+1}. $$

Proof. Define

$$Z^0_1 := \frac{1}{3} F^0_1 + \frac{a_r(2r^2 + 10r + 9)}{6(r + 2)(r + 3)} F^{r+1}_r - \frac{a_r}{2(r + 3)} E^{r+1}_r. $$  \hspace{1cm} (3.7)

Theorem 3.5. The $(r+1)$-th level orbital normal form of \cite[(1.6)]{14} is

$$v^{(r+1)} := F_r + \sum_{k=s}^{\infty} \beta^k E^k_k + \sum_{k=0}^{r} \gamma^k \Theta^k_k$$

and $\beta^k = 0$ for $k \equiv 2(r+1) - 1$ when $k \geq s$.

Proof. Lemma 3.2 implies that $\Theta^m_n \in \text{im} d^{m+s,r+1}$ and $F^m_n \in \text{im} d^{m,r+1}$ for any $m > r$. By \cite[Lemma 4.1, Equation (4.13)]{13} we have

$$[\mathcal{E}^0_{2k+1}, F_r] = -\frac{a_r}{(2k)!2^{2k+1}((2k + 2)(r + 1) + 1)} E^{(2k+1)(r+1)+r}_{(2k+1)(r+1)+r}$$

$$-2a_r e^{2k+1}_2 (2k+1)(r + 1) + 1) F^{(2k+1)(r+1)+r}_{(2k+1)(r+1)+r},$$

where $e^{2k+1}_2$ is nonzero. Therefore, $E^{(2k+1)(r+1)+r}_{(2k+1)(r+1)+r}$ also belongs to $\text{im} d^{(2k+1)(r+1)+r, r+1}$ and the proof is complete. \hfill \Box

Note that the number $s$ needs to be updated in the $(r+1)$-th level normal form; also see \cite[Remark 2]{12}. We recall that

$$\ker \text{ad}_{F^m_0} \circ \text{ad}_{F_r} = \text{span} \{ \mathcal{X}^k_{r}, \mathcal{F}^{-1}_{k,r}, \mathcal{E}^0_{k,r}, \mathcal{T}^0_{k,r} | k \in \mathbb{N} \},$$

and assume that $s < \infty$. Here, $\mathcal{X}^k_{r}, \mathcal{F}^{-1}_{k,r}, \mathcal{E}^0_{k,r},$ and $\mathcal{T}^0_{k,r}$ are defined by \cite[Equations 3.4]{14} and \cite[Equation 4.9]{13}.

Lemma 3.6. For any natural number $k$, there exists a $\delta$-homogeneous state solution $X^k_r \in \mathcal{L}$ such that $(\mathcal{X}^k_r, 0, X^k_r, 0) \in \ker d^{2(k-1)(r+1)+s+r,s+1}$, where the zeros belong to $\mathbb{R}^{s-r-1}$ and $\mathbb{R}^{s+1-r}$, respectively.
Proof. Define $X^k_r$ by
\[
\sum_{l=0}^{2k-m-1} \sum_{m=0}^{k} \binom{k}{m} \frac{a_r^m b_s(k) 2(2k-m)^2}{2k + mr + 2} F_{2(k-1)+r(m+l)+s}^2 \frac{F_{2(k-1)+r(m+l)+s}^{(r+1)(l+m)+s}}{2^{l+1}(2k + s + (l + m)r)(m(r+1) + s)^{l+1}_{r+1}}
\]
\[
+ \sum_{l=0}^{2k-m-1} \sum_{m=0}^{k} \frac{a_r^m b_s(k) 2(2k-m)(2k-m-1)(r+1) - s)^l_{2(r+1)}}{2^{l+1}(2k + s + (l + m)r)(m(r+1) + s)^{l+1}_{r+1}} E_{2(k-1)+r(m+l)+s}^{2(k-1)(l+m)+s}
\]
Since
\[
\sum_{m=0}^{k} \binom{k}{m} \frac{(2k-m)(2k-m-1)(r+1) - s)^{2k-m-1}_{2(r+1)}}{2^{2k-m-1}(s + m(r+1))^{2k-m-1}_{r+1}} = 0,
\]
we have
\[
[X^k_r, b_s E^s_s] + [X^k_r, F_r] = 0.
\]
\[
\]
Lemma 3.7. Let $k$ be an odd number and
\[
F^{-1}_k := \frac{a_r^{k+2}(r+1)(k+2)_{k+1}^2}{2^{k+1}(k+1)!} Z_{k(r+1)+r}^{k(r+1)+r}.
\]
where $Z_{k(r+1)+r}^{k(r+1)+r}$ is given by Equation (3.6). Then,
\[
(F^{-1}_k, a_r^{k+2}(r+1)(k+2)_{k+1}^2 2^{k+1}(k+1)! Z_{k(r+1)+r}^{k(r+1)+r}) \in \ker d^{k(r+1)+2r, r+1}.
\]
In addition, there exists a state $\delta$-homogenous solution $\mathfrak{F} \in L_{k(r+1)+s}$ such that
\[
d^{k(r+1)+s+r, s+1}(F^{-1}_k, 0, \mathfrak{F}, 0, a_r^{k+2}(r+1)(k+2)_{k+1}^2 2^{k+1}(k+1)! Z_{k(r+1)+r}^{k(r+1)+r}) = 0.
\]

Proof. The proof for the first part is complete by Equations (3.5) and (3.6) together with [14, Equation 3.4], that is,
\[
[X^{-1}_k, F_r] = -\frac{a_r^{k+2}(r+1)(k+2)_{k+1}^2}{2^{k+1}(k+1)!} F_{k(r+1)+2r}^{k(r+1)+2r}.
\]
By [13 Lemma 4.2], we have
\[
[X^{-1}_k, b_s E^s_s] + [\mathfrak{F}, F_r] = \sum_{m=0}^{k+1} b_s a_r^{k+1}(s)^2 (k+2-m)(k+2)^m_2 ((k-2m)(r+1) - s)^{k-m}_{2(r+1)} E_{k(r+1)+s+r}^{k(r+1)+2r}
\]
where $\mathcal{F}$ is here given by

$$
\begin{align*}
\sum_{m=0}^{k+1} \sum_{l=0}^{k-m} b_s a_r^{m+l}(s)^2(k+2-m)(k+2)^m_2((k-2m)(r+1)-s)^l E_{(m+1)(r+k+s)}^{(r+1)(m+l)+l} \\
+ \sum_{m=0}^{k+1} \sum_{l=0}^{k-m} b_s a_r^{m+l}(s)^2(k+2-m)((k+2)(r+1))^m_2((k-2m)(r+1)+s-r)^l E_{(m+1)(r+k+s)}^{(r+1)(m+l)+l} \\
+ \sum_{m=0}^{k+1} \sum_{l=0}^{k-m} b_s a_r^{m+l}(s)^2(k+2-m)(k+2)^m_2((k-2m)(r+1)-s)^{k-m} E_{(m+1)(r+k+s)}^{(r+1)(m+l)+l}.
\end{align*}
$$

Then, the rest of the proof follows the identity

$$
0 = \frac{b_s a_r^{k+2}k(r+1)(k+2)^{k+1}_2(s+2)((k+1)(r+1)+s)}{2k+1}(k+1)!((k+1)(r+1)+s+1)(r+1)(k+1) \\
+ \sum_{m=0}^{k+1} b_s a_r^{k+2}(s)^2(k+2-m)(k+2)^m_2((k-2m)(r+1)-s)^{k-m} E_{(m+1)(r+k+s)}^{(r+1)(m+l)+l}.
$$

**Lemma 3.8.** Let

$$
\mathcal{E}_{2k}^0 := \left( \frac{a_r(r+1)}{2kr+2k+1} \right) \mathcal{E}_{2k}^0 + a_r^{2k+1} e_{k,2k}(2k(r+1)-r) Z_{2k(r+1)}^{2k(r+1)},
$$

for any natural number $k$. Then,

$$
\left( \mathcal{E}_{2k}^0, a_r^{2k+1} e_{2k,2k}(2k(r+1)-r) Z_{2k(r+1)}^{2k(r+1)} \right) \in \ker d^{2k(r+1)+r,r+1}.
$$

In addition, there exists a $\delta$-homogenous transformation generator $\mathcal{F} \in \mathcal{L}_{2k+s}$ such that

$$
\begin{align*}
d^{2k(r+1)+s,s+1} \left( \mathcal{E}_{2k}^0, \mathcal{F}_{r}; a_r^{2k+1} e_{2k,2k}(2k(r+1)-r) Z_{2k(r+1)}^{2k(r+1)}, 0 \right) = \\
\sum_{m=0}^{2k} b_s (k+1)^m (mr+2k-s)^m_2((k-2m)(r+1)+s-r)^{2k-m-1} E_{2k(r+1)+s,s+1}^{2k(r+1)+s,s+1} \\
+ b_s a_r^{2k+1} e_{2k,2k}(s+2)(2k(r+1)-r)^{2k(r+1)+s,s+1} \left( \frac{(2k(r+1)+1)^2}{2k(r+1)+1} \right) E_{2k(r+1)+s,s+1}^{2k(r+1)+s,s+1}.
\end{align*}
$$

**Proof.** We recall \cite{13} Equation 4.9

$$
\mathcal{E}_{2k}^0, \mathcal{F}_{r} = a_r^{2k+1} e_{2k,2k}(2k(r+1)-r) F_{2k(r+1)+r}^{2k(r+1)+r}.
$$
Thus, Equations (3.5) and (3.6) imply Equation (3.11). By [13, Lemma 4.2], for any natural number $k$ there exists a state solution $\mathfrak{F}$ such that

\begin{equation}
[\mathcal{E}^0_{2k}, b_s E^s_k] + [\mathfrak{F}, \mathcal{F}_r] = \\
\sum_{m=0}^{2k} \frac{b_s a_r^{2k}(k+1)^m}{(2(k+1)+2m)^{m+2}}(2(k-m-1)(r+1)+r-s)^{2k-m-1} E^{2k(r+1)+s}.
\end{equation}

Then, the rest of the proof is straightforward.

\textbf{Theorem 3.9.} The $(s+1)$-th level orbital normal form of (1.6) is

\begin{equation}
v^{(s+1)} := \Theta^0_0 + F_0^{-1} + \delta F_r^r + \mathcal{F}_r + \sum_{k=s}^\infty \beta_k E^k_r + \sum_{k=0}^r \gamma_k \Theta^k_r.
\end{equation}

Here $\delta := \text{sign}(a_r)$, $\beta_k = 0$ when $k \equiv 2(r+1) - 1$, and $k \equiv 2(r+1)$ for $k > s$. Furthermore, the $(s+1)$-grade truncation of $v^{(s+1)}$ gives rise to the simplest $(s+1)$-degree truncated orbital normal form.

\textbf{Proof.} Using changes of variables given in [13, Equation 4.5], we may change the coefficient $a_r$ to an arbitrary number of the same sign. Thus, we may assume that $a_r = \delta$. Since

\begin{equation}
Z_{r+1}^1 E^r_r + [Z_{r+1}^1, \mathcal{F}_r] = \frac{a_r(r+1)}{r+2} F_{2r+1}^{2r+1},
\end{equation}

Equations (3.7) and (3.8) imply that the transformation generators $Z^0_1, Z^0_1$, and $\mathcal{E}^0_1$ along with $Z_{r+1}^1$ and $Z_{r+1}^{r+1}$ generate a transformation symmetry for $\mathcal{F}_r$. On the other hand

\begin{equation}
Z^0_1 E^s_s + \left[ \frac{1}{2(s+1)} E^s_{s+1}, \mathcal{F}_r \right] = \frac{a_r(r+2)}{2(s+1)(s+r+3)} E^r_{r+s+1} - \frac{a_r(s+3)}{2(r+s+3)} E^r_{r+s+1},
\end{equation}

and

\begin{equation}
[Z^0_1, E^s_s] + \left[ \frac{(s)^2}{3(s+1)^2} E^s_{s+1} - \frac{1}{2(s+2)^2} E^s_{s+1}, \mathcal{F}_r \right] = \\
\frac{(r^2+(3-s^2)-2s+2a_r)}{2(r+s+3)(r+2)} E^r_{r+s+1} - a_r \left( \frac{(r+1)(2r^2+10r+9)}{6(r+2)^2 s_{s+1}} - \frac{(r)^2 (s)^2}{3(s+1)^2(s+r+3)} - \frac{(s-r+1)}{2(s+2)^2} \right) E^r_{r+s+1},
\end{equation}

while

\begin{equation}
[\mathcal{E}^0_1, b_s E^s_s] + \left[ \frac{b_s(s-1)}{2(s+1)} E^s_{s+1}, \mathcal{F}_r \right] = -\frac{3a_r b_s r}{2(r+s+3)} E^r_{r+s+1} - \frac{a_r b_s r(2r+3-s)}{2(r+s+3)(s+1)} E^r_{r+s+1},
\end{equation}

\begin{equation}
Z_{r+1}^1 E^s_s = E^r_{r+s+1}, \quad \text{and}
\end{equation}

\begin{equation}
[Z_{r+1}^{r+1}, E^s_s] = -\frac{r+1}{(r+2)^2 s_{s+1}} E^r_{r+s+1} + \left( s - r - 1 \frac{(r+3)}{(r+2)^2 s_{s+1}} + \frac{(s)^2}{(r+2)^2 (r+s+3)} \right) E^r_{r+s+1}.
\end{equation}
Hence,
\[
d^{r+s+1,s+1} \left( Z_{r+1}^{r+1} + Z_{1}^{0} + E_{1}, 0 \right) = \left( \frac{s b_s}{2(s+1)} \left( \frac{(s+2)^2}{(s+1)^2} \right) + \frac{1}{2(s+2)^2} \left( F_{s+1}^{s+1} - \frac{1}{2(s+2)^2} \left( E_{s+1}^{s+1} + Z_{r+1}^{r+1} + Z_{1}^{0}, 0 \right) \right) \right)
\]
due to the equality
\[
\frac{r(r-s) - (s+2)^2}{(r+2)^2 s+1} - \frac{r}{s+r+3} + \frac{(s+2)^2}{(r+2)^2 s+1} = 0.
\]

Given Equation (3.13), the equality (3.14) does not contribute into normalization of the system. The proof is complete by Lemmas 3.6, 3.7, 3.8, and Equation (3.9).

4 Universal asymptotic unfolding normal form

The conventional approach for local bifurcation analysis of singular parametric differential systems is first to fold the system by putting the parameters zero and then find the normal form of the folded system. Next, the normalized system is unfolded by adding extra parametric (unfolding) terms such that the (versal) unfolded normal form system contains qualitative properties (invariant under an equivalence relation) associated with all small perturbations of the original system; e.g., see [34]. In fact the unfolding also accommodates all possible modeling imperfections. However, this approach has two major caveats. Firstly, this approach does not provide the actual relations between the original parameters of a parametric system and the unfolding parameters. This effectively prevents its implementation to bifurcation control. The second caveat is that most singular systems do not have universal unfolding due to their infinite codimension; see [32]. The later explains the reason why we use the notion of universal asymptotic unfolding normal form. Further, we use a parametric orbital normal form computation in order to compute the parameter relations. This section is devoted to treat Hopf-zero singularities (whose the first few dominant terms are solenoidal) with any possible additional nonlinear-degeneracies. Here, the s-equivalence relation (s+1-jet determined) is used for defining universal asymptotic unfolding normal form, that is originally due to [29, 30, 32] and is amenable to finite normal form computations.

Consider the parametric differential equation
\[
\dot{x} := f(x, y, z, \mu), \quad \dot{y} := z + g(x, y, z, \mu), \quad \dot{z} := -y + h(x, y, z, \mu), \quad (x, y, z) \in \mathbb{R}^3, \mu \in \mathbb{R}^p.
\]

Here, f, g and h are nonlinear formal functions in terms of (x, y, z, \mu) and also they are nonlinear in terms of (x, y, z) when they are evaluated at \(\mu = 0\). Remark that the results presented in this paper can be easily generalized to smooth cases using Borel–Ritt theorem [32]. Equation (4.1) represents a multiple parametric perturbation of Equation (1.1). Through a sequence of
primary shift of coordinates (shifts in \( y \) and \( z \)-variables), we may assume that \( g(0, 0, 0, \mu) = h(0, 0, 0, \mu) = 0 \) for all \( \mu \in \mathbb{R}^p \); see the primary and secondary shift of coordinates on \[30\] Page 373 and \[29\]. Next, it is easy to observe that the classical normal forms of Equation (4.1) is given by

\[
v^{(1)} := \sum c_{00n} \Theta_0^0 \mu^{n} + \sum a_{-1,-1n} F_{-1}^{-1} \mu^{n} + \sum a_{-1,0n} F_{0}^{-1} \mu^{n} + \sum a_{ijn} F_{j}^{i} \mu^{n} + \sum b_{ijn} E_{j}^{i} \mu^{n} + \sum c_{ijn} \Theta_{j}^{i} \mu^{n};
\]

where \( c_{000} = 1 \) and \( a_{-1,-10} = 0 \). Using a parametric version of Lemma 3.3 we may omit \( \sum c_{00n} \Theta_0^0 \) from the system. Now define the grading function by

\[
\delta(F_k^{i} \mu^n) = \delta(E_k^{i} \mu^n) = \delta(\Theta_k^{i} \mu^n) = k + 2|n|.
\]

By similar comments following \[14\] Lemma 3.1 and assuming that \( a_{-1,00} \neq 0 \), we can modify \( a_{-1,00} \) into 1. Since \([F_0^{0}, F_0^{-1}] = -2F_0^{-1}\), we may simplify all \( F_0^{-1} \mu^n \)-terms for nonzero \( n \). Then, the formulas given in the proof of \[14\] Lemma 3.1 imply that the vector field can be transformed into

\[
v^{(2)} := F_0^{-1} + \sum a_{in} F_{i}^{n} \mu^{n} + \sum b_{in} E_{i}^{n} \mu^{n} + \sum c_{in} \Theta_{i}^{n} \mu^{n},
\]

denoting \( a_{-1n} \) for \( a_{-1,-1n} \). Define \( r, s \) by

\[
\begin{align*}
  r := \min\{i \mid a_{i0} \neq 0\} & \quad \text{and} & \\
  s := \min\{i \mid b_{i0} \neq 0\}.
\end{align*}
\]

We assume that

\[
r < s < \infty.
\]

For further hypernormalization, we apply a new grading structure (compare with Equation (3.3)) generated by

\[
\delta(F_k^{i} \mu^n) = \delta(E_k^{i} \mu^n) = r(k - l) + k + (r + 1)|n|, \quad \delta(\Theta_k^{i} \mu^n) = r(k - l) + k + s + (r + 1)|n|.
\]

This grading facilitates the use of results from non-parametric orbital normal forms (\textit{i.e.}, Theorem 3.5) for parametric cases. Given

\[
Z_{0}^{0} F_{r}^{-1} + \left[ \frac{1}{2} F_0^{0}, F_{r}^{-1} \right] = \frac{a_{r}(r + 2)}{2} F_{r}^{-1},
\]

and a sequence of secondary shifts in \( x \)-variable using the equation

\[
F_{r}^{0}(x + c(\mu), \rho) = F_{r}^{0}(x, \rho) + (r + 1)c(\mu) F_{r}^{0}(x, \rho) + \mathcal{O}(\mu^2)
\]

where \( c(0) = 0 \), we may transform Equation (4.3) into the \((r + 1)\)-th level parametric normal form

\[
v^{(r+1)} := F_0^{-1} + \delta F_{r}^{0} + \sum_{-1 \leq i < r - 1} a_{in} F_{i}^{n} \mu^{n} + \sum_{0 \leq i < s} b_{in} E_{i}^{n} \mu^{n} + \sum_{k=s}^{\infty} \beta_{kn} E_{k}^{n} \mu^{n} + \sum_{k=1}^{r} \gamma_{kn} \Theta_{k}^{n} \mu^{n},
\]
for $\delta := \text{sign}(a_{r0})$. Further, denote $\beta_s := \beta_{s0}$. Here, $\beta_{kn} = 0$ for $k \equiv 2(r+1) - 1$ and any nonnegative integer-valued vector $n$. Equation (4.7),

$$\begin{bmatrix} E_0^0, F_r^r \end{bmatrix} = rF_r^r, \begin{bmatrix} F_0^0, E_s^s \end{bmatrix} = sE_s^s \quad \text{and} \quad \begin{bmatrix} E_0^0, E_s^s \end{bmatrix} = sE_s^s$$

imply that $E_s^s\mu^n \in \text{im} d^{s+2}|n,s+1|$ when $n \neq 0$ and $s - r \neq 0$.

For our convenience we define

$$N := r + s - \left\lfloor \frac{s}{2(r+1)} \right\rfloor,$$

and a sequence of natural numbers by

$$\{k_i \mid i \in \mathbb{N}, i > r\} := \left\{ k \mid k \neq s, k \geq 0, \text{ and } \frac{k+1}{2(r+1)}, \frac{k-s}{2(r+1)} \notin \mathbb{N} \right\}.$$  

**Theorem 4.1** (Universal unfolding). Consider a multiple-parametric perturbation of Equations (4.1) satisfying the condition (4.5). Then,

I. there exist an infinite sequence of formal parametric functions $\nu_i(\mu_j)$ and the finite sequence of formal functions $\omega_i(\mu_j)$ (for $1 \leq i \leq r$) such that they transform Equations (4.1) into

$$v := \Theta_0^0 + F_0^{-1} + \delta F_r^r + \beta_s E_s^s + \sum_{1 \leq i \leq r} \nu_i F_i^{i-2} + \sum_{i=r+1}^N \nu_i E_i^{k_i} + \sum_{i=N+1}^\infty (\beta_{k_i} + \nu_i) E_i^{k_i} + \sum_{i=1}^r (\gamma_i + \omega_i) \Theta_i^i.$$  

(4.10)

II. The differential system

$$\begin{align*}
\dot{x} &= 2\rho^2 + \delta x^{r+1} + \beta_s x^{s+1} + \sum_{1 \leq i \leq r} \nu_i x^{i-1} + \sum_{i=r+1}^N \nu_i x^{k_i+1}, \\
\dot{\rho} &= -\frac{\delta(r+1)}{2} x^{r+1} + \frac{1}{2} \beta_s x^{s+1} - \sum_{1 \leq i \leq r} \frac{(i-1)}{2} \nu_i x^{i-2} + \frac{1}{2} \sum_{i=r+1}^N \nu_i x^{k_i}, \\
\dot{\theta} &= 1 + \sum_{i=1}^r (\gamma_i + \omega_i)x^i,
\end{align*}$$  

is a $s$-universal asymptotic unfolding normal form for the differential system (4.1).

**Proof.** Given the parametric normal form Equation (4.9), the proof readily follows by deriving a parametric version of formulas in Theorem 3.13. The uniqueness of the polynomial maps $\nu_i(\mu_j)$ and $\omega_i(\mu_j)$ follow the uniqueness of the truncated orbital normal form. \qed
5 Bifurcation analysis and finite determinacy

In this section, we prove that the steady-state bifurcations of equilibria associated with the amplitude system for \( r = 1 \) is 2-determined. These steady-state bifurcations for amplitude systems are not only associated with static bifurcations of equilibria, but also local bifurcations of limit cycles for the universal asymptotic unfolding normal form. Next, we accordingly analyze the local bifurcations of a 2-jet universal asymptotic unfolding normal form.

5.1 Finite determinacy

In order to study the finite determinacy, an equivalence relation is needed and we naturally use the contact equivalence due to our purpose. Similar finitely determined results for Hopf-zero singularities of codimension two have been reported using \( C^0 \)- and weak \( C^0 \)-equivalences; e.g., see [10,33]. Throughout this subsection we follow the notations, terminologies and results of singularity theory from [18]. We recall that the three-dimensional Hopf-zero singularity may demonstrate complex dynamical behaviors such as birthes/deaths of invariant tori, phase locking, chaos, strange attractors, heteroclinic orbit breakdowns and Šil’nikov bifurcations which may not be detected by singularity theory and/or normal form methods; see [24–27]. In fact we merely address the bifurcation problem of equilibria and limit cycles. A rigorous approach may employ the Liapunov-Schmidt reduction and (equivariant) singularity theory; e.g., see [26].

It is well-known that for any smooth differential system (1.1), there always exist \( C^\infty \)-smooth changes of coordinates to transform (1.1) to a smooth differential system (1.6) modulo flat parts. We may further reduce the transformed equation by ignoring the phase component. Since further smooth changes of coordinates and time rescaling transform a smooth germ to other contact equivalent germs, we may instead work with a reduced system obtained from a universal asymptotic unfolding normal form. (Recall that the later is a truncated simplest orbital normal form for the general versal unfolding. Further, the planar reduction commutes with our orbital normal form process.) The symmetry group \( \Gamma \) is the trivial (identity) group and thus, it is removed in our notations. We may choose the distinguish parameter \( \lambda := \nu_2 \) and \( r := 1 \) while we remove the remaining parameters by setting them zero. Hence, the steady-state bifurcation problem associated with the amplitude normal form system is given by \( F(x, \rho, \lambda) = (F_1, F_2) = (0, 0) \) where

\[
F(x, \rho, \lambda) := \left( \lambda x + 2\rho^2 + \delta x^{r+1} + \beta_s x^{s+1} + \frac{\lambda \rho}{2} - \delta x^r \rho + \frac{\beta_s x^s \rho}{2} + \text{h.o.t.} \right).
\]

Lemma 5.1. The steady-state bifurcation problem \( F(x, \rho, \lambda) = 0 \) is 2-determined.

Proof. We follow [18] Definition 7.1, Proposition 1.4, Theorem 7.2 and Theorem 7.4] and instead prove that \( \overrightarrow{\mathcal{M}}^3 \subseteq \mathcal{K}_s(F) \), where \( \overrightarrow{\mathcal{M}} \) is the generated module

\[
\overrightarrow{\mathcal{M}} := \left\langle \left( \begin{array}{c} x \\ 0 \end{array} \right), \left( \begin{array}{c} \rho \\ 0 \end{array} \right), \left( \begin{array}{c} \lambda \\ 0 \end{array} \right) \right\rangle
\]
over $\mathcal{E}_{x,\rho,\lambda}$, and $\mathcal{E}_{x,\rho,\lambda}$ is the local ring of all smooth germs in $(x, \rho, \lambda)$-variables. Define its unique maximal ideal by $\mathcal{M} := \langle x, \rho, \lambda \rangle$. Note that the flat vector fields fall in $\mathcal{M}^3$.

The $\mathcal{E}_{x,\rho,\lambda}$-module $\mathcal{K}_s(F)$ is generated by

$$
\mathcal{M}^2 \left( F_{1x} \right) \cdot \mathcal{M}^2 \left( F_{1\rho} \right) \cdot \mathcal{M} \left( F_1 \right) \cdot \mathcal{M} \left( F_2 \right) \cdot \mathcal{M} \left( F_1 \right) \cdot \mathcal{M} \left( F_2 \right).
$$

We choose $\delta := 1$ to simplify the formulas. For any $\mathcal{E}_{x,\rho,\lambda}$-modules $J$ and $\mathcal{K}_s$, the Nakayama’s Lemma implies that $J \subseteq \mathcal{K}_s$ if and only if $J \subseteq \mathcal{K}_s + \mathcal{M}$. Given $J := \mathcal{M}^3$, we denote $\cong$ for the equations modulo $\mathcal{M}^4$. Since

$$
\begin{align*}
\rho \left( F_1 \right) - 2x \left( F_2 \right) & \cong \left( 3x^2 \rho + 2 \rho^3 \right), \\
\rho x \left( F_{1x} \right) - 2x \left( F_1 \right) & \cong \left( 4x^2 \rho - x \rho^2 \right), \\
x \left( F_2 \right) & \cong \left( \frac{1}{2} x \lambda \rho - x^2 \rho \right), \\
\lambda \left( F_2 \right) & \cong \left( \frac{1}{2} x \lambda^2 \rho - x \lambda \rho \right), \\
\rho \left( F_{1\rho} \right) & \cong \left( 0 \right), \\
\rho \left( F_2 \right) & \cong \left( 0 \right), \\
\rho x \left( F_{2x} \right) - \rho \left( F_1 \right) & \cong \left( x^2 \rho - 2 \rho^3 \right).
\end{align*}
$$

we have $(x^2 \rho, 0, 0), (0, \rho^3, 0) \in \mathcal{K}_s + \mathcal{M}^4$. Further,

$$
\begin{align*}
\rho \left( F_{1x} \right) & \cong \left( 4x \rho - x \rho^2 \right), \\
\rho x \left( F_{2x} \right) & \cong \left( 2 \rho^2 + \frac{3}{2} x \lambda \right), \\
x \left( F_1 \right) & \cong \left( 0 \right), \\
\lambda \left( F_{1x} \right) & \cong \left( 4x \rho - x \rho^2 \right), \\
\lambda \left( F_1 \right) & \cong \left( 0 \right), \\
\rho x \left( F_{1\rho} \right) & \cong \left( 0 \right), \\
\rho x \left( F_{2\rho} \right) & \cong \left( 0 \right), \\
\rho x \left( F_{2x} \right) - \rho \left( F_1 \right) & \cong \left( x \lambda \rho \right), \\
\rho \left( F_2 \right) & \cong \left( 0 \right), \\
\rho x \left( F_{2x} \right) - \rho \left( F_1 \right) & \cong \left( x \lambda \rho \right).
\end{align*}
$$

we may imply the membership of $(0, \rho), (0, x \lambda \rho), (0, \lambda \rho^2), (0, \lambda x \rho)$ in $\mathcal{K}_s + \mathcal{M}^4$. Finally,

$$
\begin{align*}
2x \left( F_{1x} \right) & \cong \left( 4x^2 \rho + 4 \rho^2 \right), \\
\lambda \left( F_{1x} \right) & \cong \left( 4x \rho - x \rho^2 \right), \\
\rho \left( F_1 \right) & \cong \left( 0 \right), \\
\rho \left( F_{1\rho} \right) & \cong \left( 0 \right), \\
\rho \left( F_{2\rho} \right) & \cong \left( 0 \right), \\
\rho \left( F_{2x} \right) - \rho \left( F_1 \right) & \cong \left( x \lambda \rho \right), \\
\rho \left( F_2 \right) & \cong \left( 0 \right), \\
\rho \left( F_{2x} \right) - \rho \left( F_1 \right) & \cong \left( x \lambda \rho \right).
\end{align*}
$$

and $x \left( F_1 \right) \cong \left( 0 \right)$ conclude the memberships of $(x^2 \rho, 0, 0), (\lambda x \rho), (0, x \lambda \rho)$ and $(0, \lambda \rho)$ and $(0, \lambda \rho)$. This completes the proof.

### 5.2 Bifurcation analysis

In this section we discuss the steady-state bifurcation analysis for the truncated amplitude system. The analysis describes the bifurcations of equilibria and limit cycles for the three dimensional truncated normal form system. The amplitude system associated with the two-jet universal asymptotic unfolding normal form $v$ is governed by

$$
\begin{align*}
\dot{x} &= v_1 + 2 \rho^2 + \nu_2 x + a_1 x^2, \\
\dot{\rho} &= \frac{1}{2} \nu_2 \rho - a_1 x \rho.
\end{align*}
$$

Note that the $x$-axis is always an invariant line. Assume that $a_1 = \delta = \pm 1$. 

Figure 1: Bifurcation varieties (transition sets) for the truncated system (5.4): The vertical and horizontal axes stand for $\nu_1$ and $\nu_2$, respectively. Parabolic curves are associated with Equation (5.4) for $\delta = 1$.

The associated equilibria follow

$$E_\pm : (x, \rho) = \left( -\nu_2 \pm \frac{\sqrt{\nu_2^2 - 4\delta\nu_1}}{2\delta}, 0 \right) \quad \text{and} \quad C : (x, \rho) = \left( \frac{1}{2\delta} \nu_2, \sqrt{-\frac{1}{2} \nu_1 - \frac{3}{8\delta} \nu_2^2} \right).$$

The points $E_\pm$ represent equilibria while $C$ represents a limit cycle for the three dimensional system. The transition varieties associated with $E_\pm$ and $C$ (depicted in Figure 1) are governed by

$$T_{E_\pm} := \left\{ (\nu_1, \nu_2) \mid \nu_1 = \frac{1}{4\delta} \nu_2^2 \right\} \quad \text{and} \quad T_C := \left\{ (\nu_1, \nu_2) \mid \nu_1 = -\frac{3}{4\delta} \nu_2^2 \right\}.$$

The eigenvalues of the matrices $Dv(E_\pm)$ are given by $\pm \sqrt{\nu_2^2 - 4\delta\nu_1}$ and $\nu_2 \pm \frac{1}{2} \sqrt{\nu_2^2 - 4\delta\nu_1}$. When $(\nu_2 < 0)$ or $(\nu_2 > 0$ and $4\delta\nu_1 < -3\nu_2^2$) hold, the equilibrium $E_+$ is a saddle point. For $\nu_2 > 0$, $4\delta\nu_1 > -3\nu_2^2$, $E_+$ is a source. On the other hand, the conditions $\nu_2 > 0$ or $\nu_2 < 0$ and $4\delta\nu_1 < -3\nu_2^2$ imply that $E_-$ is a saddle point. However, the conditions $\nu_2 < 0$ and $4\delta\nu_1 > -3\nu_2^2$ conclude that $E_-$ is a sink.

The eigenvalues of $Dv(C)$ are $\lambda_\pm = \nu_2 \pm \frac{1}{2} \sqrt{8\delta\nu_1 + 10\nu_2^2}$. Thus, $4\delta\nu_1 < -5\nu_2^2$ infers that $C$ is a stable/unstable focus point for negative/positive values for $\nu_2$. If $-5\nu_2^2 < 4\delta\nu_1 < -3\nu_2^2$ holds, for $\nu_2 > 0$ the point $C$ is a source while $\nu_2 < 0$ concludes that $C$ is a sink. A pair of pure imaginary eigenvalues occurs at

$$T_H := \left\{ (\nu_1, \nu_2) \mid \nu_2 = 0, \delta\nu_1 < 0 \right\}.$$  \hspace{1cm} (5.5)

The truncated system (5.4) represents a nonlinear center when the parameters cross the variety $T_H$. However, this is not the case for higher degree truncated systems, i.e., the secondary Hopf (torus) bifurcation happens and it is not 2-determined. We do not address torus bifurcation and the dynamics associated with the region h in Figures 1 and 2.
6 Bifurcation control and universal asymptotic unfolding

Bifurcation control refers to designing a controller for a nonlinear system so that its dynamics gains a desirable behavior; see [7]. This has many important engineering applications and attracted many researchers. Normal form theory is a powerful tool for local bifurcation control and recently, it has been efficiently used by several authors; see [7, 8, 19–22]. The classical normal form theory is appropriately refined by Kang el. al. to include invertible changes of the control state feedbacks; see [19–22] where an efficient approach of this kind using a single input feedback controller is implemented and then, the associated local steady-state bifurcation control is discussed. Our approach can be generalized to also use invertible changes of controller variables. Although this is not pursued in this paper, our approach uses hypernormalization of the classical normal forms by applying nonlinear time rescaling and also additional nonlinear transformations taken from the symmetry transformation group of the linearized system. This is new in both theory and applications. As far as theory is concerned, this is a contribution to the orbital and parametric normal form classifications of singularities that fits in a long tradition. As a contribution to applications, the extra hyper normalization process enables the use and can propose the type of effective nonlinear state feedback multiple-input controller. The later can be achieved by finding out whether or not a parametric singularity is a universal asymptotic unfolding.

The parametric normal form system (4.10) potentially lays the ground for applications in real life problems. Engineering problems are mostly involved with parameters such as control parameters and it is important (when it is feasible) to find explicit direct transformations that transform the asymptotic unfolding parameters to the original parameters of the system. This provides a tool to do the bifurcation analysis of the problem based on the actual controlling parameters rather than the common practice of unfolding parameters added to the nonparametric normal form. However, this is only feasible when the original system has enough parameters and has them in the right places such that they can actually play the role of the asymptotic unfolding for the system. Therefore, it is important to distinguish when and which parametric terms of the original system can effectively play the role of unfolding terms. This problem was motivated and greatly influenced by James Murdock and is our most important claimed contribution in this paper. Indeed, the remainder of this section is devoted to suggest an algorithm (similar to the early stages of Liapunov-Schmidt reduction) for performing this task; see [18, Page 27]. This approach potentially proposes certain effective controlling parameters within a parametric system for a possible engineering design; see [7]. We have implemented our suggested approach in Maple to illustrate that it is computable and successfully works.

Remark 6.1. The bifurcation analysis of the universal asymptotic unfolding normal form system provides all possible real world dynamics of an engineering problem. However, the necessity for adding extra unfolding parameters concludes that the original parametric system
may not exhibit all such possible dynamics. Hence in these circumstances, the desired dynamics may not always be produced by the existing modeling parameters and modeling refinements is required. Therefore, one needs to find other (already ignored) small parameters in the physics of the problem (our approach provides effective suggestions) to incorporate them in the model for a more comprehensive engineering design.

Denote the $s+1$-jet of the vector field (4.10) by

$$
\tilde{v}(x, \rho, \nu_1, \cdots, \nu_N) := \left( \begin{array}{c}
2\rho^2 + a_r x^{r+1} + \beta_s x^{s+1} + \sum_{1 \leq i \leq r} \nu_i x^{i-1} + \sum_{i=r+1}^{N} \nu_i x^{i+1} \\
-\frac{a_r(r+1)}{2} x^r \rho + \frac{1}{2} \beta_s x^{s+1} - \sum_{1 \leq i \leq r} \frac{i-1}{2} \nu_i x^{i-2} \rho + \frac{1}{2} \sum_{i=r+1}^{N} \nu_i x^{i+1} \rho
\end{array} \right),
$$

(6.1)

and the polynomial map $\nu(\mu)$ and matrix $J$ by

$$
\nu(\mu) := (\nu_1(\mu), \ldots, \nu_N(\mu)), \quad J := \frac{\partial(\nu_1, \ldots, \nu_N)}{\partial(\mu_1, \ldots, \mu_p)} \bigg|_{\mu=0}.
$$

(6.2)

Assume that

$$\text{rank}(J) = k \quad \text{for} \quad k \leq \min\{p, N\}.$$

Then, there always exists a linear space $M$ such that

$$\mathbb{R}^p = \ker J \oplus M.$$

Similar to the formal basis normal form style [16, Page 1006], we may choose the complement space $M$ such that $M = \text{span}\{e_{\sigma(i)} \mid i = 1, \ldots, k\}$ for some permutation $\sigma \in S_p$. Here, $e_j$ denotes the standard basis of $\mathbb{R}^p$. Let $\hat{e}_i := Je_{\sigma(i)}$, for $i = 1, \ldots, k$. Thus,

$$\text{range}(J) = \text{span}\{\hat{e}_i \mid i = 1, \ldots, k\}.$$

Define $\hat{\mu} := (\mu_{\sigma(k+1)}, \ldots, \mu_{\sigma(p)})$ and the polynomial map $\psi_{\hat{\mu}} : M \to \mathbb{R}^k$ by

$$
\psi_{\hat{\mu}}(\mu_{\sigma(1)}, \mu_{\sigma(2)}, \ldots, \mu_{\sigma(k)}) = (\nu \cdot \hat{e}_1, \nu \cdot \hat{e}_2, \ldots, \nu \cdot \hat{e}_k),
$$

(6.3)

where $\nu$ is the polynomial map given in Equation (6.2). Since the Jacobian of $\psi_0$ evaluated at the origin has the full rank and assuming that $\hat{\mu}$ is sufficiently small, the map $\psi_{\hat{\mu}}$ is locally invertible. Then,

$$
(\mu_{\sigma(1)}(\nu, \hat{\mu}), \mu_{\sigma(2)}(\nu, \hat{\mu}), \ldots, \mu_{\sigma(k)}(\nu, \hat{\mu})) = \psi_{\hat{\mu}}^{-1}(y_1(\nu), \ldots, y_k(\nu)),
$$

(6.4)

where $y_i(\nu) = \nu \cdot \hat{e}_i$ for $i = 1, \ldots, k$. Combining the map given by (6.4) and $\nu$ given by (6.2), the following proposition holds.
**Proposition 6.2.** Assume that \( \text{rank}(J) = k \). Then, there exist a permutation \( \sigma \in S_p \), near-identity parametric changes of state variables, parametric time rescaling, invertible reparametrizations \( \mu_\sigma(i)(\nu, \hat{\mu}) \) (for \( i = 1, \ldots, k \)) and polynomial functions \( \nu_\sigma(i)(\mu) \) (for \( i = k+1, k+2, \ldots, N \)) such that Equation (4.1) can be transformed into

\[
[\dot{x}, \dot{\rho}] = \tilde{v}(x, \rho, \mu_\sigma(1), \mu_\sigma(2), \ldots, \mu_\sigma(k), \nu_\sigma(k+1)(\mu), \ldots, \nu_\sigma(N)(\mu)),
\]

that is, the s-universal asymptotic unfolding planar normal form.

**Proof.** Given Theorem 4.1 and the reparametrizations given by (6.4), the proof is complete; also see the proof of [17, Lemma 3.3].

Adding extra asymptotic unfolding parameters to the system is only justified when the original control parameters of the system can not fully do the unfolding job. In this case, the control parameters may still have influences upon the added unfolding parameters and any such relation is useful for their applications in bifurcation control and needs to be computed. This is computed through Proposition 6.2. The map given by (6.4) projects the transition sets associated with the planar differential system (6.1) from the \( \nu \)-variables into the original variables \( \mu_\sigma(i) \) for \( i = 1, \ldots, k \).

**7 An illustrating example with two imaginary uncontrollable modes**

We have developed a Maple program to compute the parametric normal forms for any small perturbation of a family of Hopf-zero singularity (the case I). Further, it may take constant coefficients (different from perturbation parameters) of the parametric systems as unknown symbols rather than merely taking numerical coefficients. This greatly promotes its potential for practical applications. We here appreciate Sajjad Bakrani-Balani for writing a procedure that enhanced the efficiency of this capability. Our program will be updated as our research progresses aiming at integrating and enhancing our results [12–17] into a user-friendly Maple library for normal form analysis of singularities.

Any control system [19, Equations (2.1–2.2)] on a three dimensional central manifold with two imaginary uncontrollable modes can be transformed into

\[
\dot{x} := u + f(x, z_1, z_2, u), \quad \dot{z}_1 := z_2 + g(x, z_1, z_2, u), \quad \dot{z}_2 := -z_1 + h(x, z_1, z_2, u),
\]

(7.1)

(where \( u \) stands for the controller) using linear changes in state and feedback variables; see [19, Equation (2.3)]. We further assume that

\[
f := -d_1 x^2 + d_2 z_1^2 + d_3 x^3, \quad g := d_1 x z_1, \quad h := d_1 x z_2,
\]
Figure 2: The 2-jet estimated varieties in terms of original distinguished parameters $\mu_1, \mu_4$ in Equation (7.1). The vertical and horizontal axes are $\mu_1$ and $\mu_4$, respectively.

$u := \mu_1 + \mu_2 z_2 + \mu_3 z_1 + \mu_4 x + \mu_5 z_2^2 + \mu_6 z_1 z_2 + \mu_7 x z_2 + \mu_8 x z_1$ as a quadratic multiple-feedback controller, and $\mu_i$s stand for the control parameters. Our approach can also be applied to similar examples.

The three-jet universal asymptotic unfolding normal form is given by Equations (5.4), where

$$ u := \mu_1 + \mu_2 z_2 + \mu_3 z_1 + \mu_4 x + \mu_5 z_2^2 + \mu_6 z_1 z_2 + \mu_7 x z_2 + \mu_8 x z_1 $$

as a quadratic multiple-feedback controller, and $\mu_i$s stand for the control parameters. Our approach can also be applied to similar examples.

The three-jet universal asymptotic unfolding normal form is given by Equations (5.4), where

$$ a_1 = -\frac{4d_1}{d_2}, \quad \beta_2 = \frac{3d_3}{d_2}, $$

and

$$ \nu_1(\mu) = \frac{4}{d_2^2} \mu_1 - \frac{8}{d_2^2} \mu_1 \mu_5 - \frac{13d_1}{2d_2^2} \mu_1^2 + \frac{1}{2d_1 d_2} \mu_1^2 + \frac{1}{2d_1 d_2} \mu_4^2 + \frac{1}{2d_2 d_3} \mu_1 \mu_4 + \frac{1679 d_3^2 \mu_1^2}{3456 d_2 d_1^3} + \frac{365 d_3 \mu_1 \mu_4}{64 d_1^2 d_2} $$

and

$$ \nu_2(\mu) = \frac{2}{d_2^2} \mu_4 - \frac{2}{d_2^2} \mu_4 \mu_5 + \frac{d_1^2}{8d_2 d_3} \mu_4^2 - \frac{1}{d_2^2} \mu_1 \mu_4 + \frac{1271 d_3 \mu_1 \mu_4}{768 d_1^2 d_2} + \frac{15419 d_3^2 \mu_1 \mu_4}{13824 d_2 d_1^3} $$

Using our Maple program, the distinguished (effective) parameters are $\mu_1$ and $\mu_4$. Therefore, we choose $\mu_j = 0$ for any $j \neq 1, 4$. Then, the two-jet approximations for the inverse maps $\nu_1^{-1}$ and $\nu_2^{-1}$ are governed by

$$ \mu_1(\nu_1, \nu_2) := \frac{1}{4} d_2 \nu_1 + \frac{13d_1 d_2^2}{128} \nu_1^2 - \frac{d_2^2}{32d_2^2} \nu_1^2 - \frac{365 d_2^2 d_3 \nu_1 \nu_2}{2048 d_1^2} - \frac{1}{64} \frac{d_2^2 d_1^2 \nu_1 \nu_2}{d_3} $$

and

$$ \mu_4(\nu_1, \nu_2) := \frac{d_2}{2} \nu_2 + \frac{d_1 d_2}{8} \nu_1 \nu_2 - \frac{1271 d_2^2 d_3 \nu_1 \nu_2}{6144 d_1^2} - \frac{1}{d_3^2} \nu_1^2 - \frac{15419 d_2^2 d_3^2 \nu_1 \nu_2}{221184 d_1^3} $$

For numerical simulation, we choose

$$ d_1 := -1, \quad d_2 := 4, \quad \text{and} \quad d_3 := \frac{4}{3}. $$
(a) Region a: Asymptotic to the $x$-axis, the solution approaches infinity for $\mu_1 = 0.004, \mu_4 = 0.001$.

(b) Parameters $\mu_1 = -0.004$ and $\mu_4 = 0.001$ are taken from region b.

(c) Region c: The orbit converges to a stable equilibrium for $\mu_1 = 0.0005, \mu_4 = -0.1$, i.e., $E_-$ is a sink.

(d) Region d: For $\mu_1 = 0.0005, \mu_4 = 0.08$, the equilibrium $E_-$ is a saddle while $E_+$ is a source point.

(e) The orbit approaches a stable limit cycle (corresponding to $C$) in 3-dimension for $\mu_1 = -0.001$, the $x$-axis. Here, $C$ is a source and $\mu_1 = -0.001, \mu_4 = -0.08$ from region e.

(f) The orbit diverges to infinity while rotates around the $x$-axis. Here, $\mu_1 = -0.0015, \mu_4 = 0.08$ from region f

(g) Region g: The solution rotates and converges to a stable limit cycle, corresponding to the stable focus $C$ rotates around the $x$-axis. Here, $\mu_1 = -0.0015, \mu_4 = -0.06$

(h) Region h: The orbit diverges to infinity while it rotates around the $x$-axis. Here, $\mu_1 = -0.0015, \mu_4 = 0.06$ and $C$ is an unstable focus.

Figure 3: Time series $(x(t), y(t), z(t))$ for different choices of parameters from regions (a–h) in Figure 2.
Then, $a_1 := 1$ and $\beta_2 := 1$. Thus, a three-jet estimation in terms of $\mu_1$ and $\mu_4$ for the bifurcation varieties $T_{E\pm}$ and $T_C$ are given by

\begin{align*}
T_{E\pm} &:= \left\{ \mu_1 = \frac{3}{16} \mu_4^2 - \frac{1061}{4608} \mu_4^3 + \frac{3486907}{7077888} \mu_4^4 - \frac{31594532951}{36691771392} \mu_4^5 + \frac{29226779813459}{21134460321792} \mu_4^6 \right\}, \\
T_C &:= \left\{ \mu_1 = -\frac{1}{16} \mu_4^2 - \frac{157}{512} \mu_4^3 + \frac{22826407}{63700992} \mu_4^4 - \frac{347712683}{452984832} \mu_4^5 + \frac{1329515222107}{5405649667072} \mu_4^6 \right\}
\end{align*}

and

\begin{align*}
T_F &:= \left\{ \mu_1 = -\frac{3}{16} \mu_4^2 - \frac{1589}{4608} \mu_4^3 + \frac{4797371}{21233664} \mu_4^4 - \frac{23112314429}{36691771392} \mu_4^5 + \frac{8519077413067}{7044820107264} \mu_4^6 \right\};
\end{align*}

see Figure 2. To validate our approximated transition sets, we choose the following eight parameter values from regions (a-h) respectively:

\begin{align*}
(\mu_1, \mu_4) : &= (0.004, 0.001), (-0.004, 0.001), (5 \times 10^{-4}, -0.1), (5 \times 10^{-4}, 0.08), \\
&\quad (-0.001, -0.08), (-0.001, 0.08), (-1.5 \times 10^{-5}, -0.06), (-1.5 \times 10^{-5}, 0.06).
\end{align*}

Numerical simulations using MATHLAB and the initial point $(x, y, z) = (-0.01, 0.01, 0.01)$, we accordingly obtain Figures 3, i.e., (a)-(h). This is compatible with our anticipated bifurcations and demonstrates that cognitive choices for different values of $(\mu_1, \mu_4)$ may control the local dynamics of the system (7.1).

References

[1] A. Algaba, E. Freire, E. Gamero, *Hypernormal form for the Hopf-zero bifurcation*, Internat. J. Bifur. Chaos 8 (1998) 1857–1887.

[2] A. Baider, R.C. Churchill, *Unique normal forms for planar vector fields*, Math. Z. 199 (1988) 303–310.

[3] A. Baider, J.A. Sanders, *Further reductions of the Takens–Bogdanov normal form*, J. Differential Equations 99 (1992) 205–244.

[4] A. Baider, J.A. Sanders, *Unique normal forms: The nilpotent Hamiltonian case*, J. Differential Equations 92 (1991) 282–304.

[5] I. Baldomá, O. Castejón, T.M. Seara, *Exponentially small heteroclinic breakdown in the generic Hopf-zero singularity*, J. Dyn. Diff. Equat. 25 (2013) 335–392.

[6] H. Broer, G. Vegter, *Subordinate Šil’nikov bifurcations near some singularities of vector fields having low codimension*, Ergod. Theory Dyn. Syst. 4 (1984) 509–525.
[7] G. Chen, D.J. Hill, X. Yu, “Bifurcation Control Theory and Applications,” Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin 2003.

[8] G. Chen, J.L. Moiola, H.O. Wang, Bifurcation control: theories, methods and applications, Internat. J. Bifur. Chaos 10 (2000) 511–548.

[9] G. Chen, D. Wang, J. Yang, Unique orbital normal form for vector fields of Hopf-zero singularity, J. Dynam. Differential Equations 17 (2005) 3–20.

[10] F. Dumortier, S. Ibáñez, Singularities of vector fields on $\mathbb{R}^3$, Nonlinearity 11 (1998) 1037–1047.

[11] F. Dumortier, S. Ibáñez, H. Kokubu, C. Simó, About the unfolding of a Hopf-zero singularity, Discrete and Continuous Dynamical Systems 33 (2013) 4435–4471.

[12] M. Gazor, M. Moazeni, Parametric normal forms for Bogdanov-Takens singularity; the generalized saddle-node case, Discrete and Continuous Dynamical Systems 35 (2015) 205–224.

[13] M. Gazor, F. Mokhtari, Normal forms of Hopf-zero singularity, Nonlinearity 28 (2015) 311–330.

[14] M. Gazor, F. Mokhtari, Volume-preserving normal forms of Hopf-zero singularity, Nonlinearity 26 (2013) 2809–2832.

[15] M. Gazor, F. Mokhtari, J.A. Sanders, Normal forms for Hopf-zero singularities with non-conservative nonlinear part, J. Differential Equations 254 (2013) 1571–1581.

[16] M. Gazor, P. Yu, Spectral sequences and parametric normal forms, J. Differential Equations 252 (2012) 1003–1031.

[17] M. Gazor, P. Yu, Formal decomposition method and parametric normal forms, Internat. J. Bifur. Chaos 20 (2010) 3487–3515.

[18] M. Golubitsky, I. Stewart, D.G. Schaeffer, Singularities and Groups in Bifurcation Theory, Vol I and II, Springer, New York 1985 and 1988.

[19] B. Hamzi, W. Kang, J.P. Barbot, Analysis and control of Hopf bifurcations, SIAM J. Control and Optimization 42 (2004) 2200–2220.

[20] W. Kang, Bifurcation and normal form of nonlinear control systems, PART I and II, SIAM J. Control and Optimization 36 (1998) 193–212 and 213–232.

[21] W. Kang, A.J. Krener, Extended quadratic controller normal form and dynamic state feedback linearization of nonlinear systems, SIAM J. Control and Optimization 30 (1992) 1319–1337.
[22] W. Kang, M. Xiao, I.A. Tall, *Controllability and local accessibility: A normal form approach*, IEEE Transaction on Automatic Control **48** (2003) 1724–1736.

[23] H. Kokubu, H. Oka, D. Wang, *Linear grading function and further reduction of normal forms*, J. Differential Equations **132** (1996) 293–318.

[24] W.F. Langford, *Periodic and steady-state mode interactions lead to tori*, SIAM J. Appl. Math. **37** (1979) 649–686.

[25] W.F. Langford, *A review of interactions of Hopf and steady-state bifurcations*, Nonlinear Dynamics and Turbulence, Interaction Mech. Math. Ser., Pitman, Boston, MA (1983) 215–237.

[26] W.F. Langford, *Hopf bifurcation at a hysteresis point*, Differential Equations: Qualitative Theory, Colloq. Math. Soc. János Bolyai, **47** (North Holland), (1984) 649–686.

[27] J. Harlim, W.F. Langford, *The cusp-Hopf bifurcation*, Internat. J. Bifur. Chaos **17** (2007) 2547–2570.

[28] J. Li, L. Zhang, D. Wang, *Unique normal form of a class of 3 dimensional vector fields with symmetries*, J. Differential Equations **257** (2014) 2341–2359.

[29] J. Murdock, *Asymptotic unfoldings of dynamical systems by normalizing beyond the normal form*, J. Differential Equations **143** (1998) 151–190.

[30] J. Murdock, “Normal Forms and Unfoldings for Local Dynamical Systems,” Springer-Verlag, New York, 2003.

[31] J. Murdock, *Hypernormal form theory: foundations and algorithms*, J. Differential Equations **205** (2004) 424–465.

[32] J. Murdock, D. Malonza, *An improved theory of asymptotic unfoldings*, J. Differential Equations **247** (2009) 685–709.

[33] F. Takens, *Singularities of vector fields*, Publ. Math. IHES **43** (1974) 47–100.

[34] P. Yu, A.Y.T. Leung, *The simplest normal form of Hopf bifurcation*, Nonlinearity **16** (2003) 277–300.

[35] P. Yu, Y. Yuan, *The simplest normal form for the singularity of a pure imaginary pair and a zero eigenvalue*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **8** (2001) 219–249.
