Halving the cost of quantum multiplexed rotations

Guang Hao Low

Microsoft Quantum, Redmond, WA 98052, USA

We improve the number of T gates needed for a b-bit approximation of a multiplexed quantum gate with c controls applying n single-qubit arbitrary phase rotations from $4nb + O(\sqrt{cnb})$ to $2nb + O(\sqrt{cnb})$, and reduce the number of qubits needed by up to a factor of two. This generic quantum circuit primitive is found in many quantum algorithms, and our results roughly halve the cost of state-of-art electronic structure simulations based on qubitization of double-factorized or tensor-hypercontracted representations. We achieve this by extending recent ideas on stochastic compilation of quantum circuits to classical data and discuss space-time trade-offs and concentration of measure in its implementation.

1 Introduction

Many quantum algorithms with an exponential runtime advantage over their classical counterparts require large numbers of qubits and quantum gates. Applications at scientifically or industrially interesting scales include estimating energy levels of molecules with hundreds of spin-orbitals and electrons [13, 26] and factoring RSA integers with thousands of bits [8]. Solving these problems require at least that many qubits just to encode the input, upon which billions and trillions of elementary quantum gates are applied.

At large scale, quantum computation on noisy physical hardware requires fault-tolerant quantum gates on logical qubits in a quantum error correcting code [25]. Although Clifford gates (HADAMARD, PHASE, CNOT) can be implemented transversally and thus fault-tolerantly on many error correcting codes, they must be augmented by non-Clifford gates, typically the T gate, to achieve universal quantum computation. As a simultaneous transversal implementation of the T gate is impossible [28], fault-tolerant T gates are realized by sophisticated techniques such as magic state distillation [2] or gauge fixing [20] that are orders of magnitude more costly. The total number of T gates is thus a good heuristic for understanding the real-world cost of fault-tolerant quantum algorithms.

Optimizing the decomposition of arbitrary quantum algorithms into the fewest number of T gates is

| Result | Number of TOFFOLI gates |
|--------|--------------------------|
| [12, 24]| $O\left(cn \log\left(\frac{n}{\epsilon}\right)\right)$ |
| [6]    | $O\left(cn \log\left(\frac{1}{\epsilon}\right)\right)$ |
| [26]   | $n \left\lceil \log_2\left(\frac{n\pi^2}{2\epsilon}\right)\right\rceil + O\left(\sqrt{cn \log_2\left(\frac{n}{\epsilon}\right)}\right)$ |
| This work | $n \left\lceil \frac{1}{2} \log_2\left(\frac{n\pi^2}{2\epsilon}\right)\right\rceil + O\left(\sqrt{cn \log_2\left(\frac{n}{\epsilon}\right)}\right)$ |

Table 1: Cost comparison of implementing fault-tolerant multiplexed rotation Equation (1). Further variations realizing a T gate – qubit tradeoffs are in Table 3.

NP-hard. Hence algorithms are typically expressed through higher level subroutines for which optimal decompositions are known such as table-lookup [16], arithmetic operators [7], rotations [23], and Fourier transforms [17].

We focus on the multiplexed rotation gate

$$U_\theta = \sum_{j=0}^{c-1} |j\rangle \langle j| \otimes e^{i2\pi\theta_jZ},$$

(1)

which is another common subroutine. This subroutine is a large fraction of the cost of important quantum algorithms such as arbitrary quantum state preparation and unitary synthesis [24], and more recently, state-of-art simulations of electronic structure [13, 15, 26]. As arbitrary unitary quantum algorithms also decompose into products $V = \prod_k e^{i2\pi\theta_kZ} \cdot V_k$ of rotations interleaved by some unitaries $V$, the intuitive application of Equation (1) lies in executing a superposition of quantum algorithms

$$U_{\theta^*} = \left(\prod_{k=0}^{n-1} U_{\theta_{k,j}} \cdot V_k\right),$$

(2)

with potentially very different parameters $\theta_{i,j}$, but otherwise all with the same structure $V$.

Our main contribution is roughly halving the T gate cost of implementing Equation (1) and its occurrences in Equation (2) compared to the best prior methods summarized in Table 1. Given a target error $\epsilon$ in diamond distance, we show in Section 3 that it suffices to use

$$n \left\lceil \frac{1}{2} \log_2\left(\frac{n\pi^2}{2\epsilon}\right)\right\rceil + O\left(\sqrt{cn \log_2\left(\frac{n}{\epsilon}\right)}\right)$$

(3)

TOFFOLI gates, which we refer to interchangeably as four T gates and vice-versa [7, 10]. Our approach

Guang Hao Low: guanghao.low@microsoft.com
Table 2: Cost comparison of simulating active site configurations of a nitrogen fixation enzyme (FeMoco) [22] and a carbon dioxide fixation catalyst (XVIII) [26] using a randomized implementation of Equation (2). Note that the choice of $b = \lceil \log_2 (160M/N) \rceil \approx 33$ by deterministic truncation rigorously upper bounds the error of simulation [26] and is a fair comparison with our randomized scheme. In contrast, the choice of $b = 16$ was obtained by heuristics and would be also be roughly twice as large under provable deterministic conditions. The $\lambda$ parameter controls the space-time trade-off of table-lookup as in Table 3. Due to spin symmetry in these examples, the Givens rotations of size $N$ here applies to $2N$-spin-orbital systems.

| System      | Method | $N$ | $c/10^2$ | $M/10^6$ | $\lambda$ | Deterministic truncation | Randomized truncation |
|-------------|--------|-----|----------|-----------|------------|--------------------------|-----------------------|
| FeMoco [26] | $H_{DF}$ | 54  | 240      | 4.7       | 2          | 33.2 $\times 10^{10}$ | 3.6 $\times 10^8$ | 18 | 82% | 64% |
| XVIII [26]  | $H_{DF}$ | 56  | 300      | 4.7       | 4          | 33.2 $\times 10^{10}$ | 7.4 $\times 10^8$ | 18 | 71% | 62% |
| FeMoco [13] | $H_{DF}$ | 54  | 130      | 4.6       | 2          | 16.1 $\times 10^{10}$ | 3.7 $\times 10^8$ | 18 | 114% | 127% |
| FeMoco [13] | $H_{THC}$ | 54  | 3.5      | 4.8       | 2          | 16.5 $\times 10^5$ | 2.1 $\times 10^4$ | 18 | 108% | 110% |

2 Randomization as a resource

The trace distance $\mathcal{D}$ is a good characterization of the difference between quantum states and channels. For any two quantum states $\rho$ and $\rho'$,

$$\mathcal{D}(\rho, \rho') \doteq \frac{1}{2} \| \rho - \rho' \|_1,$$

where $\| \cdot \|_1$ is the Schatten 1-norm. This equals the largest possible total variational distance of measurement outcomes. Suppose $\{E_m\}$ is a positive operator valued measurement with corresponding measurement probabilities $p_m = \text{Tr} [\rho E_m]$ for outcome $m$, and similarly for $p'_m$. The distance between these distributions is then $\max_{E_m} \frac{1}{2} \sum_m \| p_m - p'_m \| = \mathcal{D}_{tv}(\rho, \rho')$ [18]. In the case of finite dimensional quantum channels $\mathcal{V}$ and $\mathcal{V}'$, it is common to use the completely bounded trace norm, or diamond distance [11]

$$\mathcal{D}_\diamond(\mathcal{V}, \mathcal{V}') = \max_{\rho} \mathcal{D}(\mathcal{V} \otimes \mathcal{I}(\rho), \mathcal{V}' \otimes \mathcal{I}(\rho)).$$

This expression through the trace distance highlights its operational meaning. It characterizes the distinguishability of quantum channels with respect to any quantum state, even allowing for entanglement assistance.

Although Equation (5) is difficult to evaluate in general, simplifications are known. For instance, convexity of trace distance implies that maximization over density matrices $\rho$ can be replaced by that over pure states. Moreover, the dimension of $\mathcal{I}$ may be limited to that of $\mathcal{V}$ without loss of generality. In fact, when $\mathcal{V}$ and $\mathcal{V}'$ correspond to unitaries $V$ and $V'$, stabilization by $\mathcal{I}$ is unnecessary [27]. Our work relies on the following special case, which is tighter than some similar results [3, 9].

**Lemma 1** ([5, Lemmas 2]) Diamond distance of unitary ensembles. Let $V$ be unitary and let the channel $V'(\cdot) = \mathbb{E} \left[ V^ \dagger (\cdot) V \right]$ apply the randomly sampled unitary $V'$. Then

$$\mathcal{D}_\diamond(\mathcal{V}, \mathcal{V}') \leq \| V - \mathbb{E}[V'] \|.$$  (6)

Randomness is a powerful tool for approximating continuous quantities with discrete quantities. Consider the deterministic rounding of a complex phase $e^{i2\pi \theta}$. As the phase has a binary expansion $\theta = \sum_{b \in \{0, 1\}} \theta_b 2^{-b} \in [0, 1)$, where $\theta_b \in \{0, 1\}$, its $b$-bit approximation $\theta^{(b)} = \sum_{i=1}^{b} \theta_i 2^{-i} - 2^{-b+1}$ has error at most $2^{-b+1}$. Hence choosing $b = \lceil \log_2 \left( \frac{\epsilon}{\pi} \right) \rceil$ guarantees

$$\left| e^{i2\pi \theta} - e^{i2\pi \theta^{(b)}} \right| \leq 2\pi \left( \theta - \theta^{(b)} \right) \leq \frac{\pi}{2^b} \leq \epsilon.$$  (7)

We may in fact halve the bits of precision with a randomized rounding scheme that will be useful in the following.

**Lemma 2** (Randomized rounding of phases). Let $X \sim \text{Bernoulli}(r)$ where $r = 2^b (\theta - \theta^{(b-1)})$, and let $\theta^{(0)} = \theta^{(b-1)} + 2^{-b} X$ be a random $b$-bit angle. Then $b = \left\lfloor \frac{1}{2} \log_2 \left( \frac{\pi}{\epsilon} \right) \right\rfloor$ bits suffice to bound

$$\left| e^{i2\pi \theta} - \mathbb{E} \left[ e^{i2\pi \theta^{(b)}} \right] \right| \leq \epsilon.$$  (8)

Proof. The distance between the boundary of a circle and any point on a given chord is maximized at the halfway mark $r = 1/2$. Hence Equation (8) is
bounded by \( 1 - \frac{e^{-\pi^2/2} + e^{\pi^2/2}}{2\pi} \leq 1 - \cos(\pi/2^6) \leq \frac{\pi^2}{2\pi^2} \).

In other words, the random discrete angles are designed to match the target angle in expectation \( E[\Theta^{(b)}] = \theta \), and the simplest example is the distribution on \( \theta^{(b-1)} + 2^{-b} \).

3 Multiplexed rotation circuit

The multiplexed rotation gate Equation (1) has a few well-known implementations. Our approach adds randomization on top of prior state-of-art that encodes the desired rotation angles as binary integers in a quantum lookup table [26]. Let us define the table-lookup unitary as

\[
D_{c,b} |j \rangle |z \rangle = |j \rangle |z \oplus \theta^{(b)}_j \rangle.
\]

Controlled on an integer \( j \in [c] \), table-lookup performs a bit-wise XOR of the \( b \)-bit angle \( \theta^{(b)}_j \) into the input register \( |z\rangle \), typically initialized as \( |0\rangle^\otimes b \). Subsequently, a controlled rotation

\[
R_b |\theta^{(b)}_j \rangle |\psi\rangle = |\theta^{(b)}_j \rangle e^{i2\pi\theta^{(b)}_j | \psi \rangle},
\]

is performed. With a final step of uncomputing, \( D \), this realizes the multiplexed rotation \( U_\vartheta = D_{c,b}^\dagger R_b D_{c,b} \) through the sequence

\[
|j \rangle |\psi\rangle |0\rangle \xrightarrow{D_{c,b}} |j \rangle |\psi\rangle |\theta^{(b)}_j \rangle \xrightarrow{R_b} |j \rangle e^{i2\pi\theta^{(b)}_j} |\theta^{(b)}_j \rangle \xrightarrow{D_{c,b}^\dagger} |j \rangle e^{i2\pi\theta^{(b)}_j} |\psi\rangle |0\rangle.
\]

The case \( U_\vartheta \) in Equation (2) may be trivially implemented by applying Equation (1) \( n \) times. As the \( T \) cost may depends on \( c \), a more \( T \) efficient approach [26] applies \( \lfloor n/k \rfloor \) layers of multiplexed rotation gates each using \( D_{c,k,b} \). This uses more space as \( kb \) bits are written out in parallel. In the extreme case, \( D_{c,n,b} \) may write out all \( n \) rotation angles at once.

Whereas prior approaches took a deterministic rounding of \( \theta \), we halve the bits of precision needed by applying a randomized rounding. Let us approximate \( U_\vartheta \) by \( U_{\vartheta^{(b)}} \), where each angle \( \theta_j \) is replaced by the random \( b \)-bit variable \( \Theta^{(b)}_j \) as described in Lemma 2.

Whenever \( U_\vartheta \) is required by a quantum algorithm, we randomly sample \( \vartheta \sim \Theta^{(b)} \) and instead apply \( U_{\vartheta^{(b)}} \). The following theorem bounds the error of this procedure.

Theorem 3 (Randomized compilation of multiplexed rotations). The distance between the channels \( U_\vartheta \) and \( U_{\Theta^{(b)}} \) corresponding to the multiplexed rotation \( U_{\vartheta} \) and its \( b \)-bit randomization \( U_{\Theta^{(b)}} \) is

\[
D_\epsilon(U_\vartheta, U_{\Theta^{(b)}}) \leq \epsilon.
\]

Proof. From Lemma 1, Equation (12) is bounded by

\[
\sum_{j=0}^{n-1} |j\rangle \otimes \left( e^{i2\pi\theta_j} Z - E[e^{i2\pi\theta^{(b)}_j} Z] \right) \leq \max_j e^{i2\pi\theta_j} Z - E[e^{i2\pi\theta^{(b)}_j} Z] \leq \frac{\pi^2}{2\pi^2 + T}.
\]

We then apply Lemma 2 to the last line.

Some useful variations of \( U_\vartheta \) admit further optimization. For instance, a triangle inequality bounds the error of \( D_{c,b}(U_\vartheta, U_{\Theta^{(b)}}) \leq \frac{\pi^2}{2\pi^2 + T} \leq \epsilon \) by choosing \( b = \left\lceil \frac{1}{2} \log_2 \left( \frac{n\pi^2}{2\pi^2} \right) \right\rceil \). Furthermore, the same set of angles \( \vartheta \) may be used multiple times such as in

\[
U_{\vartheta^{(b)}} = \prod_{k=0}^{n-1} U_{\vartheta^{(b)}}(V_k).
\]

Whereas the \( D_{c,n,b} \) implementation of \( U_{\Theta^{(b)}} \) writes out \( n \) rotation angles, these angles are repeated and only differ in the least significant bit. Thus there are only \( n + b - 1 \) unique bits and it suffices to instead use the significantly cheaper table-lookup \( D_{c,n+b-1} \).

3.1 Cost

The multiplexed rotation has a gate cost corresponding to the sum of its components Equation (11) in Table 3, and an ancillae overhead corresponding to the maximum required across all steps.

The cost of table-look up \( D_{c,b} \) [1, 16] is parameterized by a power of 2 integer \( 1 \leq \lambda \leq c \) that governs a trade-off between number of \( T \) gates and ancillary qubits used. Moreover, the power of 2 constrain on \( \lambda \) can be dropped using modular division [26] costing \( O(\log c) \) \( T \) gates and \( O(1) \) ancillae. The minimum TOFFOLI count is thus approximately \( 2\sqrt{c/b} \lambda \approx \sqrt{c/b} \) ancillae. Nevertheless, a sub-optimal \( \lambda \) may still be chosen to limit the number of ancillae used.

Importantly, all these ancillae are immediately reset to the \( |0\rangle \) state and may be reused. This is through a measurement-based uncomputation process also parameterized by a power of 2 integer \( 1 \leq \lambda' \leq c \). This
uncomputation step has a minimum TOFFOLI cost of \( \approx 2\sqrt{c} \), and will tend to have negligible cost compared to \((c, b)\)-lookup whenever \(\lambda > 1\). We may choose any \(\lambda' \leq (\lambda - 1)b\).

The controlled rotation Equation (10) is implemented by the phase gradient technique \([7, 13]\) costing \(b\) TOFFOLI gates. Given a phase state \(|\phi\rangle = \frac{1}{\sqrt{2^{k+1}}} \sum_{k=1}^{2^{k+1}} e^{-i2\pi k/2^{k+1}} |k\rangle\) and a \(b + 1\)-bit reversible adder \(\text{ADD}\), observe that adding the \(b + 1\) bit integer \(|l\rangle\) kicks back a phase

\[
\text{ADD} (|l\rangle |\phi\rangle) = e^{i2\pi l/2^{b+1}} |l\rangle |\phi\rangle. \tag{15}
\]

As the phase state catalyzes the operation and may be reused, we ignore its one-time cost of \(O(b \log(b/c))\) T gates.

Hence the controlled adder \(\text{ADD}' = \text{ADD} \otimes |0\rangle \langle 1| + \text{ADD}' \otimes |1\rangle \langle 1|\) performs a \(Z\) rotation as

\[
\text{ADD}' (|l\rangle |x\rangle) = e^{(-1)^{n-1}i2\pi n/2^{b+1}} |l\rangle |x\rangle. \tag{16}
\]

Note that \(\text{ADD}'\) has the same TOFFOLI cost as \(\text{ADD}\) as an adder can be converted into a subtractor using only Clifford gates. This follows from applying a bit-wise complement using CNOT gates in the identity \(a - b = a + b\). Comparing Equation (10) to Equation (16), \(R_b\) is thus implemented by fixing the least significant bit of \(f\) to 1 and the remaining \(b\) bits to \(\theta^{(b)}\).

The overall TOFFOLI cost of Equation (2) through \(n/k\) layers of \(D_{c,k}\)-lookup is then

\[
nb + \left[ \frac{n}{k} \right] \left( \left[ \frac{c}{\lambda} \right] + \left[ \frac{c}{\lambda'} \right] + (\lambda - 1)kb + \lambda' \right). \tag{17}
\]

Some important special cases of the TOFFOLI count for parameter choices \((k, \lambda, \lambda')\) include

\[
(n, 1, 1) : nb + 2c + 1,
\]

\[
(1, 1, 1) : n(b + 2c + 1),
\]

\[
(k', \lambda, \lambda^*) : nb + 2\sqrt{c} + 2\sqrt{cnb}. \tag{18}
\]

Minimizing this by choosing \(k' \sim n, \lambda^* \sim \sqrt{c/nb}, \lambda^* \sim \sqrt{c}\) recovers Equation (3).

### 4 Simulation of electronic structure

Our work reduces the TOFFOLI and qubit count of previous state-of-art on precisely estimating energy levels of active site configurations of a nitrogen fixation enzyme (FeMoco) \([22]\) and a carbon dioxide fixation catalyst (XVIII) \([26]\) using the quantum phase estimation algorithm wrapped around queries to quantum walk units. The percentage reduction in the space-time TOFFOLI-qubit product is roughly 50\%, as highlighted in Table 2. We obtained these numbers by subtracting the cost of deterministic multiplexed rotations from previous estimates \([13, 26]\) and adding in the cost of our randomized approach.

In previous deterministic approaches, a very small approximation error \(\epsilon \sim 10^{-5}\) is chosen for these examples in order to bound the systematic shift in eigenvalues of each quantum walk. Quantum phase estimation to error \(\delta\) then estimates the energy with an error of \(\pm (\epsilon + \delta)\) with high probability \(p\). In our analysis, we instead chose a target diamond distance \(\epsilon = 0.05\), which implies that the estimated energy will be correct to error \(\delta\) with at least a confidence of \(p - \epsilon\). Thus the cost of phase estimation to the same overall error using our approach can be further multiplied by a factor of \(\frac{\epsilon}{p - \epsilon}\), which is generally less than one for well-supported trial states. However, we omit this factor in the table to focus on changes in cost due to just changes in bits of precision.

In such electronic structure simulation problems, the goal is to accurately estimate an eigenvalue of the 'double-factorized' Hermitian operator, or Hamiltonian,

\[
H = H_{h(0)} + \sum_{r=1}^{R} h_{r(0)} P_{r,0}, H_{h(0)} = \sum_{p,q=1}^{N} h_{pq} P_{p,0} P_{q,0}, \tag{19}
\]

where \(\{ P_{r,0} \}\) are mutually anti-commuting Pauli operations and \(h_{r(0)}\) are anti-Hermitian matrices \([26]\). An alternate 'tensor-hypercontracted' representation \([13]\) is

\[
H = H_{h(0)} + \sum_{r,s=1}^{R} t_{r,s} H_{h(0)} H_{h(0)}, \tag{20}
\]

where \(\{ h_{r(0)} \}\) are additionally rank-1 and \(t\) is a symmetric matrix. These are all equivalent to Fermion Hamiltonians with one- and two-body number-conserving interactions. The basic approach applies quantum phase estimation \([18]\) on a quantum walk \(W\) encoding the spectrum of \(H\) \([15]\). Given a trial state \(|\psi\rangle\), this estimates an eigenphase \(|\lambda_j\rangle = e^{i\phi_j} |\lambda_j\rangle\) where the index \(j\) is sampled with probability \(p_j = |\langle \lambda_j |\psi\rangle|^2\), and \(\phi_j\) is estimated to a chosen error \(\delta\) with high probability in the algorithm, the most expensive component is the sequence of the controlled-walk operators applied \(M = O(1/\delta)\) times.

Multiplexed rotations are a large fraction of the cost of these walk operators in state-of-art quantum resource estimates on useful instances of this problem \([13, 26]\). These occur through so-called controlled-Givens rotation units \(G = \sum_{j=0}^{c-1} |j\rangle \langle j| \otimes G_j\) that diagonalize the quadratic Hamiltonian \(\sum_{p,q} h_{pq} P_{p,0} P_{q,0} = \sum_{p \in \Xi} \lambda_p G^\dagger_p P_{0,0} G_p\). In Equation (19), we see that \(\epsilon_{\text{DDP}} = (R+1)\Xi\) Givens rotations are required to diagonalize its \(R + 1\) quadratic Hamiltonians. Similarly Equation (20) requires \(\epsilon_{\text{THC}} = R + N\) Givens rotations. The linear-combination-units technique that encodes \(H\) into a quantum walk applies the sequence

\[
G (1 \otimes \cdots) G^\dagger \cdots G (1 \otimes \cdots) G^\dagger, \tag{21}
\]
where no operation is applied on the $|j\rangle$ register within each round bracket. As each Givens rotation decomposes into

$$G_j = \left( \prod_{k=0}^{N-2} V_{k,2} e^{i 2\pi \theta_{k,j}} Z V_{k,1} e^{i 2\pi \theta_{k,j}} Z V_{k,0} \right), \quad (22)$$

each half $G (\mathbb{1} \otimes \cdots) G^\dagger$ is a multiplexed gate with $c$ controls and $4(N - 1)$ rotations, which matches the form of Equation (2). Naively, each multiplexed rotation would require table-lookup $D_{c,4(N-1)b}$ outputting $4(N - 1)$ bits. However, each rotation angle is repeated 4 times, and following Equation (14) it suffices to randomize only their least significant bit. Hence it suffices to use roughly 4 times fewer bits through $D_{c, (N-1)(b-1)+4(N-1)}$. The diamond distance of $2M$ randomized applications of $G (\mathbb{1} \otimes \cdots) G^\dagger$ in quantum phase estimation is then from Theorem 3 at most

$$\frac{8M(N-1)\pi^2}{2^{2b+1}} \leq \epsilon$$

with the choice

$$b = \left\lfloor \frac{1}{2} \log_2 \left( \frac{4M(N-1)\pi^2}{\epsilon} \right) \right\rfloor. \quad (23)$$

5 Concentration of measure

We have focused on the diamond distance in bounding the error of our randomized protocol. However, one might wonder about the error of a given sampled instance of $V'$ rather than the channel average, that is, the quantity

$$\|V' - V\| = \max_{|\psi\rangle} \| (V' - V) |\psi\rangle \|. \quad (24)$$

We may also reason about the average error by, for instance, bounding $\mathbb{E} \max_{|\psi\rangle} \| (V' - V) |\psi\rangle \|$. However, this average is too pessimistic. It states that for any sampled $V'$, we adversarially choose the worst $|\psi\rangle$. Subsequently, we take the average error. In other words, this $V'$ is expected to perform well for all possible input states. In practice however, a sampled $V'$ is only applied once to a single input state. Future repeats would re-sample $V'$ rather than reuse the original sample. Hence the more relevant measure of error is

$$\max_{|\psi\rangle} \mathbb{E} \| (V' - V) |\psi\rangle \|. \quad (25)$$

It is even more informative to bound the probability of large deviations of the error from the mean

$$\max_{|\psi\rangle} \Pr \left[ \| (V' - V) |\psi\rangle \| \geq \epsilon \right]. \quad (26)$$

Closely following the derivation by Chen et al. [5], we prove in the appendix that the error of a random instance $\| U'_{\psi} - U_{\psi}^{\Theta} \|$ is small with high probability. In other words,

$$\max_{|\psi\rangle} \mathbb{E} \left[ \| (U'_{\psi} - U_{\psi}^{\Theta}) |\psi\rangle \| \right] \lesssim 4\pi \sqrt{n}. \quad (27)$$

Note that the dependence on $2^b$ is quadratically worse than the quantum channel analysis. In more detail, we establish tail bounds on the error distribution

$$\max_{|\psi\rangle} \Pr \left[ \| (U'_{\psi} - U_{\psi}^{\Theta}) |\psi\rangle \| \geq \epsilon \right] \leq \exp \left( -\frac{\epsilon^2 2^{2b}}{32\pi^2 n} \right). \quad (28)$$

The bits of precision required is then

$$b \geq 4.88 + \log_2 \left( \frac{\log \left( \frac{1}{p} \right)}{\epsilon} \right) + \log_2 \left( \frac{\sqrt{n}}{\epsilon} \right). \quad (29)$$

Though interesting, any useful quantum computation terminates with a measurement of the quantum state. In the context of measurement, these single-shot error bounds are thus less useful as the diamond distance directly characterizes the deviation of measurement outcomes and is also tighter.

6 Conclusion

We have presented a very simple randomized protocol for implementing the common quantum circuit subroutine of a single, or sequence of, multiplexed rotations. Our implementation uses roughly half the T gates and half the qubits of prior deterministic approaches. Whereas prior methods of stochastic compilation focus on approximating arbitrary single-qubit rotations with samples from a dense $\epsilon$-mesh of single-qubit rotations, we focus on stochastic compilation of more complex multi-qubit operations in the context of directly minimizing T count. Moreover, our randomization protocol, based only on flipping biased coins, is extremely simple. We have evaluated the relevance of our results in halving the space-time cost of important applications such as electronic structure simulation based on qubitization, and our work elucidate a path towards systematically applying stochastic compilation methods beyond [21] Trotter-based methods of simulation [4], such as quantum signal processing [14, 19], which has proven to be surprisingly elusive.
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The problem thus reduces to bounding the systematic shift of a single application of triangle inequality on a telescoping sum. The error due to the systematic component is

\[ \text{A.1 Bounding the systematic shift} \]

We now evaluate these quantities.

Note that the systematic component also upper bounds the diamond distance. As for the random component, we use \((x + y)^q \leq 2^q \max \{x^q, y^q\}\) for \(x, y \geq 0\) to bound

\[ \mathbb{E} \left[ \| (\hat{Q} - Q) |\psi\rangle \| ^q \right] \leq 2^q \max \left\{ \mathbb{E} \left[ (\hat{Q} - Q) |\psi\rangle \| ^q \right], \mathbb{E} \left[ (\hat{Q} - Q) |\psi\rangle \| ^q \right] \right\} . \]

We now evaluate these quantities.

### A.1 Bounding the systematic shift

The error due to the systematic component is \(\| (\mathbb{E} [\hat{Q}] - Q) |\psi\rangle \| \leq \| \mathbb{E} [\hat{Q}] - Q \|\). This may be bounded by a triangle inequality on a telescoping sum

\[ \| \mathbb{E} [\hat{Q}] - Q \| \leq \left\| \prod_{j=1}^{N-1} Q_j \cdot \mathbb{E} [U_j] \right\| - \left\| \prod_{j=1}^{N-1} Q_j \cdot U \right\| \leq N \| \mathbb{E} [\hat{U}] - U \|. \]

The problem thus reduces to bounding the systematic shift of a single application of \(\| \mathbb{E} [\hat{U}] - U \|\).
A.2 Bounding the random shift

We now bound the component \( \mathbb{E} \left[ \| \tilde{Q} - \mathbb{E} [\tilde{Q}] \| \psi \|^q \right] \) arising from fluctuations of \( \tilde{Q} \) from its mean. The proof is based on exploiting orthogonal vectors in a martingale difference sequence using the fact

\[
\mathbb{E} \left[ \|x + y\|^q \right] \leq \left( \mathbb{E} \|x\|^q \right)^{2/q} + (q-1) \left( \mathbb{E} \|y\|^q \right)^{2/q}
\]

for \( q \geq 2 \) and random vectors \( x, y \) that obey \( \mathbb{E} [y|x] = 0 \). Now consider a sequence of random vectors \( \psi_j = \left( \prod_{i=1}^j Q_j \cdot \tilde{U}_j \right) \cdot Q_0 |\psi\rangle \). Then \((\tilde{Q} - \mathbb{E} [\tilde{Q}]) |\psi\rangle = |\psi_N\rangle - \mathbb{E} [|\psi_N\rangle] \). By adding \( Q_N \mathbb{E} [\tilde{U}_N] |\psi_{N-1}\rangle = Q_N \mathbb{E} [\tilde{U}_N] |\psi_{N-1}\rangle \), we have

\[
(\tilde{Q} - \mathbb{E} [\tilde{Q}]) |\psi\rangle = Q_N \left( \tilde{U}_N - \mathbb{E} [\tilde{U}_N] \right) |\psi_{N-1}\rangle + Q_N \mathbb{E} \left[ \tilde{U}_N \right] \left( |\psi_{N-1}\rangle - \mathbb{E} [|\psi_{N-1}\rangle] \right).
\]

Observe that conditioning on \( x \) sets the value of \( |\psi_{N-1}\rangle \) but not \( \tilde{U}_N \). Hence \( \mathbb{E} [Q_N (\tilde{U}_N - \mathbb{E} [\tilde{U}_N]) |\psi_{N-1}\rangle |x\rangle] = Q_N \mathbb{E} [(\tilde{U}_N - \mathbb{E} [\tilde{U}_N]) |\psi_{N-1}\rangle] = 0 \). and we may apply inequality on \( \mathbb{E} [\|x + y\|^q] \) to obtain

\[
\mathbb{E} [\|Q_N |\psi\rangle - \mathbb{E} [Q_N |\psi\rangle]\|q/2 \leq \mathbb{E} \left[ \left\| Q_N (\tilde{U}_N - \mathbb{E} [\tilde{U}_N]) |\psi_{N-1}\rangle \right\| q/2 \right] + (q-1) \mathbb{E} \left[ \left\| Q_N (\tilde{U}_N - \mathbb{E} [\tilde{U}_N]) |\psi_{N-1}\rangle \right\| q/2 \right]
\]

\[
\leq (q-1) \sum_{i=1}^N \mathbb{E} \left[ \left\| (\tilde{U}_i - \mathbb{E} [\tilde{U}_i]) \right\| q\right]
\]

We now substitute \( \| (\tilde{U}_1 - \mathbb{E} [\tilde{U}_1]) \| \leq \frac{\pi}{2\pi} \) to obtain

\[
\mathbb{E} [\|Q_N |\psi\rangle - \mathbb{E} [Q_N |\psi\rangle]\|q/2 \leq 2\pi \sqrt{\frac{(q-1) N}{2b}}.
\]

A.3 Tail bounds on approximation error

We now combine the bounds on systematic and random errors.

\[
\mathbb{E} \left[ \| (\tilde{Q} - Q) |\psi\rangle \| \right] \leq 2^q \max \left\{ \| (\mathbb{E} [\tilde{Q}] - Q) |\psi\rangle \|, \mathbb{E} \left[ \| (\tilde{Q} - \mathbb{E} [\tilde{Q}]) |\psi\rangle \| \right] \right\}
\]

\[
\leq 2^q \max \left\{ \left( \frac{\pi^2 N}{2^{2b}} \right)^q, \left( 2\pi \sqrt{\frac{(q-1) N}{2^{2b}}} \right)^q \right\}
\]

\[
\leq \left( 4\pi \sqrt{\frac{(q-1) N}{2^{2b}}} \right)^q \leq \left( \frac{4\pi \sqrt{q} N}{2^b} \right)^q.
\]

Using Lyapunov’s inequality, \( \mathbb{E} [\|X\|^{q/4}] \leq \mathbb{E} [\|X\|^{s/4}] \) for \( 0 < q < s < \infty \). Hence the expected error

\[
\epsilon = \max_{|\psi\rangle} \mathbb{E} [\| (\tilde{Q} - Q) |\psi\rangle \|] \leq \max_{|\psi\rangle} \mathbb{E} \left[ \| (\tilde{Q} - Q) |\psi\rangle \| ^{1/2} \right] \leq \frac{4\pi \sqrt{q} N}{2^b}.
\]

\[
b \geq \log_2 \left( \frac{4\pi \sqrt{q} N}{\epsilon} \right) \approx 3.65 + \log_2 \left( \frac{\sqrt{q} N}{\epsilon} \right)
\]
Substituting into Markov’s inequality,
\[
\Pr \left[ \| (\tilde{Q} - Q) |\psi\rangle \| \geq \epsilon \right] \leq \frac{\mathbb{E} \left[ \| (\tilde{Q} - Q) |\psi\rangle \|^q \right]}{\epsilon^q} \leq \left( \frac{4\pi}{\epsilon^2 b \sqrt{qN}} \right)^q
\]
\[
= \exp \left( q \left( \frac{1}{2} \ln (q) + \ln \left( \frac{4\pi}{\epsilon^2 b \sqrt{N}} \right) \right) \right)
\]
\[
= \exp \left( q \left( \frac{1}{2} \ln (q) + \ln (z) \right) \right)
\]
(41)

The exponent is minimized by choosing \( q = \frac{1}{e^2} = \frac{e^2 2^b}{16 \pi^2 N} \). Hence
\[
p = \Pr \left[ \| (\tilde{Q} - Q) |\psi\rangle \| \geq \epsilon \right] \leq \exp \left( q \left( \ln (\sqrt{c}) \right) \right) = \exp \left( - \frac{e^2 2^b}{32 \pi^2 N} \right)
\]
(42)

The bits of precision required is then
\[
b \geq \frac{1}{2} \log_2 \left( 32e^2 \log_2 \left( \frac{1}{p} \right) \right) + \log_2 \left( \frac{\sqrt{N}}{\epsilon} \right)
\]
\[
\geq 4.88 + \log_2 \left( \log_2 \left( \frac{1}{p} \right) \right) + \log_2 \left( \frac{\sqrt{N}}{\epsilon} \right).
\]
(43)