Abstract—In this work, we study the problem of index coding from graph homomorphism perspective. We show that the minimum broadcast rate of an index coding problem for different variations of the problem such as non-linear, scalar, and vector index code, can be upper bounded by the minimum broadcast rate of another index coding problem when there exists a homomorphism from the complement of the side information graph of the first problem to that of the second problem. As a result, we show that several upper bounds on scalar and vector index code problem are special cases of one of our main theorems.

For the linear scalar index coding problem, it has been shown in [1] that the binary linear index of a graph is equal to a graph theoretical parameter called minrank of the graph. For undirected graphs, in [2] it is shown that $\text{minrank}(G) = k$ if and only if there exists a homomorphism from $G$ to a predefined graph $G_k$. Combining these two results, it follows that for undirected graphs, all the digraphs with linear index of at most $k$ coincide with the graphs $G$ for which there exists a homomorphism from $G$ to $G_k$. In this paper, we give a direct proof to this result that works for digraphs as well.

We show how to use this classification result to generate lower bounds on scalar and vector index. In particular, we provide a lower bound for the scalar index of a digraph in terms of the chromatic number of its complement.

Using our framework, we show that by changing the field size, linear index of a digraph can be at most increased by a factor that is independent from the number of the nodes.

I. INTRODUCTION

The index coding problem, first introduced by Birk and Kol in the context of satellite communication [3], has received significant attention during past years (see for example [1], [4]–[13]). This problem has many applications such as satellite communication, multimedia distribution over wireless networks, and distributed caching. Despite its simple description, the index coding problem has a rich structure and it has intriguing connections to some of the information theory problems. It has been recently shown that the feasibility of any network coding problem can be reduced to an equivalent feasibility problem in the index coding problem (and vice versa) [14]. Also an interesting connection between index coding problem and interference alignment technique has been appeared in [11].

In this work, we focus on the index coding problems that can be represented by a side information graph (defined in §II), i.e., user demands are distinct and there is exactly one receiver for each message. For this case we consider the framework for studying the index coding problem that uses ideas from graph homomorphism. More precisely, we show that the minimum broadcast rate of an index coding problem (linear or non-linear) can be upper bounded by the minimum broadcast rate of another index coding problem if there exists a homomorphism from the (directed) complement of the side information graph of the first problem to that of the second problem. Consequently, we show that the chromatic and fractional chromatic number upper bound are special cases of our results (e.g., see [1], [7]).

For the case of linear scalar, we also prove the opposite direction, namely, we show that for every positive integer $k$ and prime power $q$, there exists a digraph $H^q_k$ such that the $q$-ary linear index of $H^q_k$ is at most $k$ and the complement of any digraph whose $q$-ary linear index is also at most $k$ is homomorphic to $H^q_k$. The set of graphs $H^q_k$ are analogous to the “graph family $G_k$" defined in [2] for studying a parameter of the graph called minrank. In contrast to those graphs, $H^q_k$ are defined for arbitrary finite fields as opposed to the binary field and more importantly, they can be utilised to study the linear index code even if the graphs of interest are directed. Moreover, our proof does not use the result of [1] about the equivalence between the minrank and the linear index of graphs.

Using the reduction of the scalar index coding problem to the homomorphism problem and the notion of increasing functions on the set of digraphs, we provide a family of lower bounds on the binary index of digraphs. As a particular example of such lower bounds, we extend the the previously known bound $\log_q(\chi(G)) \leq \text{lind}_q(G)$ [15] from $q = 2$ to arbitrary $q$.

As an application of our work, we show a connection between $\text{lind}_p(\cdot)$ and $\text{lind}_q(\cdot)$ when $p$ and $q$ are different prime powers.

The remainder of this paper is organised as follows. In §II we introduce notation, some preliminary concepts about graph homomorphism and give the problem statement. The main results of the paper and their proofs are presented in §III and §IV. In §V, some applications of our main results are stated. The omitted proofs can be found in [16].
II. Notation and Problem Statement

A. Notation and Preliminaries

For convenience, we use \([m:n]\) to denote for the set of natural numbers \([m,\ldots,n]\). For any set \(\mathcal{A}\), we use \(\mathcal{P}^*(\mathcal{A})\) to denote for all of the non-empty subsets of \(\mathcal{A}\). Let \(x_1,\ldots,x_n\) be a set of variables. Then for any subset \(\mathcal{A} \subseteq [1:n]\) we define \(x_{\mathcal{A}} \triangleq (x_i : i \in \mathcal{A})\).

A directed graph (digraph) \(G\) is represented by \(G(V,E)\) where \(V\) is the set of vertices and \(E \subseteq (V \times V)\) is the set of edges. For \(v \in V(G)\) we denote by \(N^+_G(v)\) as the set of outgoing neighbours of \(v\), i.e., \(N^+_G(v) = \{u \in V : (v,u) \in E(G)\}\). For a digraph \(G\) we use \(\overline{G}\) to denote for its directional complement, i.e., \((u,v) \in E(G)\) iff \((u,v) \notin E(\overline{G})\).

Definition 1 (Homomorphism, see [17]). Let \(G\) and \(H\) be any two digraphs. A homomorphism from \(G\) to \(H\), written \(\phi : G \rightarrow H\) is a mapping \(\phi : V(G) \rightarrow V(H)\) such that \((\phi(u),\phi(v)) \in E(H)\) whenever \((u,v) \in E(G)\). If there exists a homomorphism of \(G\) to \(H\) we write \(G \rightarrow H\), and if there is no such homomorphism we shall write \(G \nrightarrow H\). In the former case we say that \(G\) is isomorphic to \(H\).

Definition 2. On the set of all loop-less digraphs \(\mathcal{G}\), we define the partial pre order “\(\preceq\)" as follows. For every pair \((G,H)\) of graphs \(G \in \mathcal{G}, G \preceq H\) if and only if there exists a homomorphism \(\phi : G \rightarrow H\). It is straightforward to see that “\(\preceq\)" is reflexive and transitive. Moreover, if \(G \preceq H\) and \(H \preceq \overline{G}\) then the digraphs \(G\) and \(\overline{G}\) are homomorphically equivalent (i.e., \(G \rightarrow \overline{G}\) and \(\overline{G} \rightarrow G\)). In this case we write \(G \sim H\).

Notice that homomorphically equivalence does not imply isomorphism between graphs (digraphs). For example, all the bipartite graphs are homomorphically equivalent to \(K_2\) and therefore are homomorphically equivalent to each other but they are not necessarily isomorphic.

Definition 3. Let \(\mathcal{D} \subseteq \mathcal{G}\) be an arbitrary set of digraphs. A mapping \(h : \mathcal{D} \rightarrow \mathbb{R}\) is called increasing over \(\mathcal{D}\) if for every pair \((G,H)\) of graphs \(G \in \mathcal{D}\) such that \(G \preceq H\) then \(h(G) \leq h(H)\).

B. Problem Statement

Consider the communication problem where a transmitter intends to recover \(x_{\mathcal{A}}\) from \(y_{\mathcal{A}}\) of messages \(x_1,\ldots,x_n \in X\) to \(m\) receivers by broadcasting \(\ell\) symbols \(y_1,\ldots,y_{\ell} \in Y\) over a public noiseless channel. We assume that for each \(j \in [1:m]\), the \(j\)th receiver has access to the side information \(x_{A_j}\), i.e., a subset \(A_j \subseteq [1:n]\setminus\{j\}\) of messages. Each receiver \(j\) intends to recover \(x_j\) from \((y_{A_j}\setminus x_{A_j})\).

This problem, which is a basic setting of the index coding problem, can be represented by a directed side information graph \(G(V,E)\) where \(V\) represents the set of receivers/messages and there is an edge from node \(v_i\) to \(v_j\), i.e., \((v_i,v_j) \in E\), if the \(i\)th receiver has packet \(x_j\) as side information. An index coding problem, as defined above, is completely characterized by the side information sets \(A_j\).

In the following definitions, we formally define validity of an index codes and some other basic concepts in index coding (see also [4], [7], and [12]).

Definition 4 (Valid Index Code). A valid index code for \(G\) over an alphabet \(X\) is a set \((\Phi,\{\Psi_i\}_{i=1}^m)\) consisting of: (i) an encoding function \(\Phi : X^m \rightarrow Y^\ell\) which maps \(m\) source messages to a transmitted sequence of length \(\ell\) of symbols from \(Y\); (ii) a set of \(m\) decoding functions \(\Psi_i\) such that for each \(i \in [1:m]\) we have \(\Psi_i(\Phi(x_1,\ldots,x_m),x_{A_j}) = x_i\).

Definition 5. Let \(G\) be a digraph, and \(X\) and \(Y\) are the source and the message alphabet, respectively.

(i) The “broadcast rate” of an index code \((\Phi,\{\Psi_i\})\) is defined as \(\inf_X \{\Phi(G,\Phi,\{\Psi_i\})\} = \frac{\log|Y|}{\log|X|}\).

(ii) The “index” of \(G\) over \(X\), denoted by \(\ind_{\Phi}(X,G)\) is defined as \(\inf_Y \{\Phi(G,\Phi,\{\Psi_i\})\}\).

(iii) If \(X = Y = \mathbb{F}_q\) (the \(q\)-element finite field for some prime power \(q\)), the “scalar linear index” of \(G\), denoted by \(\ind_{\Phi}(G)\) is defined as \(\inf_{\Phi,\{\Psi_i\}} \{\Phi(G,\Phi,\{\Psi_i\})\}\).

(iv) The “minimum broadcast rate” of the index coding problem of \(G\) is defined as \(\inf_{\Phi,\{\Psi_i\}} \{\Phi(G,\Phi,\{\Psi_i\})\}\).

iii. Index Coding via Graph Homomorphism

In this section, we will explain a method for designing index codes from another instance of index coding problem when there exists a homomorphism from the complement of the side information graph of the first problem to that of the second one. As an application to this result, we will show in §IV that some of the previously known results about index code design are special types of our general method.

Theorem 1. Consider two instances of the index coding problems over the digraphs \(G\) and \(H\) with the source alphabet \(X\). If \(G \preceq H\) then

\[\ind_{\Phi}(G) \leq \ind_{\Phi}(H)\]

In other words, the function \(\ind_{\Phi}(\cdot)\) is a non-decreasing function on the pre order set \((\mathcal{G}, \preceq)\).

First we explain the proof idea of Theorem 1 which is as follows. If \(G \preceq H\), by definition there exists a homomorphism \(\phi : G \rightarrow H\). Notice that the function \(\phi\) maps the vertices of \(G\) to the vertices of \(H\). Thus we can also consider, \(\phi\) as a function from \(V(G)\) to \(V(H)\). For every vertex \(w \in V(H)\), we denote by \(\phi^{-1}(w)\) to be the set of all the vertices \(v \in V(G)\) such that \(\phi(v) = w\); (see Figure 1). This way, we partition the vertices of \(G\) into the classes of the form \(\phi^{-1}(w)\) where \(w \in V(H)\).

Next, we take an optimal index code for \(H\) over the source alphabet \(X\) that achieves the rate \(\ind_{\Phi}(H)\). Then, we show that we can treat every part \(\phi^{-1}(w)\) as a single node and
For every finite set $X$, a one-to-one function of the remaining variable, i.e., for every $w \in V(H)$, there exists a decoding function $\Psi^H : \mathcal{Y}^m \times \mathcal{X}^{N_H(w)} \rightarrow \mathcal{X}$ such that $\Psi^H \left( \Phi^H (x_1, \ldots, x_n), x_{N_H(w)} \right) = x_i$ for every choice of $(x_1, \ldots, x_n) \in \mathcal{X}^n$.

Finally, we construct a valid index code for $G$ over the same alphabet sets and the same transmission length $\ell$; thus it results in an index code for $G$ with the same broadcast rate. For an explicit construction see [16].

As a result of Theorem 1 we have the following corollary.

**Corollary 1.** Consider two instances of the index coding problems over the digraphs $G$ and $H$. If $G \preceq H$ then we have

1) for general multi-letter index codes: $\text{ind}(G) \leq \text{ind}(H)$,
2) for linear vector index codes: $\text{lin}_{d+}(G) \leq \text{lin}_{d+}(H)$,
3) for linear scalar index codes: $\text{lin}_d(G) \leq \text{lin}_d(H)$.

**IV. An Equivalent Formulation for Linear Scalar Index Coding Problem**

Let $G_k^q$ be the set of all the finite digraphs $G$ for which $\text{lin}_d(G) \leq k$. It is obvious to see that $G_k^q$ is an infinite family of digraphs. However, in this section, we will show that $G_k^q$ has a maximal member with respect to the pre order “$\preceq$”. We give an explicit construction for a maximal element of $G_k^q$ which we call it $H_k^q$.

In fact, we show that $\forall G \in G_k^q$, $G \preceq H_k^q$. On the other hand, by Corollary 1, Part 3, we know that if $G \preceq H_k^q$ then $\text{lin}_d(G) \leq \text{lin}_d(H_k^q) \leq k$. Thus, we can conclude the following theorem.

**Theorem 2.** For every positive integer $k$ and a prime power $q$, there exists a graph $H_k^q$ with $\frac{q^k-1}{q-1}q^{k-1}$ nodes such that if for every graph $G$, $\text{lin}_d(G) \leq k$ if and only if $G \preceq H_k^q$ or equivalently, there exists a homomorphism from $G$ to $H_k^q$.

For the sake of simplicity, we prove the theorem for $q = 2$. For general $q$, a construction for $H_k^q$ as well as a proof of the above theorem is presented in [16].

We start by presenting a construction for $H_k^2$ as follows. Consider a $k \times (2^k - 1)$ binary matrix $B$ whose rows are labelled by numbers $1, 2, \ldots, k$ and whose columns are labelled by non-empty subsets of $[1 : k]$. For every $\emptyset \neq J \subseteq [1 : k]$, the $J$-th row of $B$ is the indicator vector of the set $J$, i.e., the $(i, J)$-th entry of $B$ is 1 iff $i \in J$.

Let $A$ be a $(2^k - 1) \times (2^k - 1)$ binary matrix whose rows and columns are indexed by non-empty subsets of $[1 : k]$ and the $I$-th row of $A$ is equal to the binary summation (xor) of the rows of $B$ corresponding to the elements of $I$. Notice that $A_{(I,J)} = 1$ iff $I \cap J$ is an odd number. Figure 3 shows an example of $A$ for $k = 2$.

Now, define the digraph $H_k^2$ as follows. The set of vertices of $H_k^2$ is the set of pairs $(I, J)$ where $\emptyset \neq I, J \subseteq [1 : k]$ and $A_{(I,J)} = 1$. We denote the vertex of $H_k^2$ associated with
whole set of \(1 : m\) such that for every message \(y_j\), either all the variables of the form \(x_i, i \in C_j\) appear, or none of them appear in \(y_j\). Hence, each message \(y_j\) can be written as a summation of \(x_{i[C_j]}\) where \(x_{i[C_j]} = \sum_{i \in E_j} x_i\). Therefore, \(y_j = \sum_{j \geq 3} x_{i[C_j]}\).

Next, we will define another partition of \([1 : m]\) as follows. By definition of linear index coding, it is guaranteed that for every vertex \(v_i\), the side information of \(v_i\) and some subset of messages, transmitted by the source, will be enough for \(v_i\) to recover \(x_i\). The set of indices of every such subset of messages is called a sufficient family for that particular receiver. Notice that for a particular receiver \(v_i\), minimal sufficient families\(^1\) are not necessarily unique. For example, it is possible that a receiver can reconstruct its bit using the first two messages and its side information, or from the third message and its side information. However, there exists at least one minimal sufficient family of the messages. Let \(\gamma : [1 : m] \mapsto \mathcal{P}([1 : k])\) be a function that to every \(i \in [1 : m]\), \(\gamma(i)\) assigns a minimal sufficient family for \(v_i\). It is also easy to observe that if the set \(\{y_j | j \in J\}\) is a minimal sufficient family of the messages for \(v_i\), then \(v_i\) is able to reconstruct \(x_i\) using its side information and \(\sum_{j \in J} y_j\).

For every \(\phi \neq I \subseteq [1 : k]\), define \(D_I\) to be \(\gamma^{-1}(I)\). Since \(\gamma\) is a function, \(D_I\)’s for different \(I\)’s will partition the set \([1 : m]\). Notice that by the definition of \(\gamma\), if \(i \in D_I\) then \(\{y_j | j \in I\}\) is a minimal sufficient family for \(v_i\). Therefore, \(v_i\) can retrieve \(x_i\) using its side information and \(\sum_{j \in I} y_j\). That is to say that in the summation \(\sum_{j \in I} y_j\), the variable \(x_i\) appears and also, if another variable \(x_{i'}\) appears, then \(v_i\) knows \(x_{i'}\) as its side information, i.e., \((v_i, v_{i'}) \in E(G)\).

At this point, we are able to define a homomorphism from \(\mathcal{G}\) to \(H^2_k\). In fact, we define a function \(\phi : V(G) \mapsto V(H^2_k)\), and show that if \((v_i, v_{i'}) \notin E(G)\) then \((\phi(v_i), \phi(v_{i'})) \notin E(H^2_k)\).

Since \(C_j\’s\) and also \(D_j\’s\) both partition the set \([1 : m]\), for every \(v_i \in V(G)\), there exists a unique pair of \((I, J)\), \(\emptyset \neq I, J \subseteq [1 : k]\) such that \(i \in D_I, i \in C_J\). Define \(\phi(v_i) = v_{(i,J)}\). In order to complete the proof, we need to show that \(\phi\) is a well-defined function, i.e., \(\phi(v)\) is a vertex of \(H^2_k\) and also \((v_i, v_{i'}) \notin E(G)\) then \((\phi(v_i), \phi(v_{i'})) \notin E(H^2_k)\).

\(\square\)

\(^1\)Here by a minimal sufficient family we refer to a sufficient family where none of its subsets is a sufficient family.
So far, we have proved that if \( \text{lind}_2(G) \leq k \) then \( G \preceq H_k^2 \). Conversely, if \( G \preceq H_k^2 \) then by Corollary 1 we have \( \text{lind}_2(G) \leq \text{lind}_2(H_k^2) \). Therefore, the following lemma will finalize the proof of Theorem 2.

**Lemma 6.** For every positive integer \( k \), \( \text{lind}_2(H_k^2) \leq k \).

### V. Application

In this section, we will demonstrate several applications of the theorems stated in the previous sections.

**A. Upper Bounds**

Here, we will show that some of the earlier upper bounds are only special cases of Corollary 1.

**Example 1.** One of the earliest upper bounds on the \( \text{lind}_2(G) \) is \( \chi(\overline{G}) \) where \( \chi(\cdot) \) is the chromatic number of a graph (e.g., see [4]). Notice that in our framework, this result is an immediate consequence of Corollary 1, Part 3. That is, if \( \chi(\overline{G}) = r \) then \( \overline{G} \rightarrow K_r \), where \( K_r \) is a complete graph with \( r \) vertices. Thus, \( \text{lind}_2(G) \leq \text{lind}_2(K_r) = r = \chi(G) \).

**Example 2.** In [7], it is shown that \( \text{lind}(G) \leq \chi_f(\overline{G}) \) where \( \chi_f(\cdot) \) is the fractional chromatic number of a graph. See [16] for the proof using our framework.

The crucial observation is that the parameters \( \chi(G) \) and \( \chi_f(\overline{G}) \) can be defined using existence of homomorphisms from \( G \) to the family of complete graphs and Kneser graphs, respectively. (See [16] for the definition of Kneser graphs).

**B. Lower Bounds**

By using Theorem 2, the following result can be proved.

**Lemma 7.** Suppose that \( h \) is an increasing function on \( (G, \preceq) \) and \( r \) is an upper bound on \( h(H_k) \). For every digraph \( G \), if \( h(G) > r \) then \( \text{lind}_2(G) > k \).

**Proof.** If \( \text{lind}_2(G) \leq k \) then by Theorem 2, \( G \preceq H_k \) and therefore \( h(G) \leq h(H_k) \leq r \) which is a contradiction.

**Theorem 3.** For every digraph \( G \), \( \text{lind}_2(G) \geq \log_q(\chi(G)) \).

**Proof.** The function \( h(G) = \chi(\overline{G}) \) is an increasing function on \( (G, \preceq) \). Suppose that \( \text{lind}_2(G) = k \). Therefore, \( \chi(\overline{G}) \leq \chi(\overline{H_k^2}) \leq q^k = q^{\text{lind}_2(G)} \). The first inequality is implied by the previous Lemma. For a proof of the second inequality see [16].

**C. Index Codes and Change of Field Size**

Existence of a certain index code for a given graph over a fixed finite field is equivalent to the existence of linear combinations of the source messages over the ground field with certain Algebraic / Combinatorial constraints. If the ground field is changed, there is no natural way of updating the index code over the new field. In other words, if for a fixed graph \( G \), an index code over a finite field \( \mathbb{F}_{q_1} \) is given, there is no natural way to construct some index code for the same graph but over a different field \( \mathbb{F}_{q_2} \). In fact, in [5], it has been shown that for every pair of finite fields \( \mathbb{F}_p \) and \( \mathbb{F}_q \) of different characteristics and for every \( 0 < \epsilon < 0.5 \), there exists a graph \( G \) with \( n \) vertices such that \( \text{lind}_2(G) < n^\epsilon, \text{lind}_2(G) > n^{1-\epsilon} \).

Here we use the results of Theorem 1 and Theorem 2 to show that if \( \text{lind}_2(G) \) is less than a fixed number then by changing the field size, the corresponding linear indices can at most differ by a factor that depends only on the field sizes and is independent from the size of the graph. More precisely the following result holds:

**Theorem 4.** Let \( G \) be a graph and \( q_1, q_2 \) are two different prime powers. Then, \( \text{lind}_2(G) \leq \text{lind}_2(H_{\text{lind}_2(G)}^{q_1}) \) and then by Theorem 1, \( \text{lind}_2(G) \leq \text{lind}_2(H_{\text{lind}_2(G)}^{q_2}) \).

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