On the Ubiquity Of Electromagnetic-Duality Rotations in

4D, $\mathcal{N} = 1$ Holoraumy Tensors for On-Shell 4D Supermultiplets

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ABSTRACT

Holoraumy is a tool being developed for dimensional enhancement (supersymmetry holography) where the goal is to build higher dimensional supersymmetric multiplets from lower dimensional supersymmetric multiplets. In this paper, for the first time we investigate holoraumy for on-shell supersymmetry. Specifically, the holoraumy tensors for a number of familiar 4D, $\mathcal{N} = 1$ multiplets are calculated. It is shown in all of these cases of on-shell theories, the holoraumy is of the form of an electromagnetic duality charge multiplying a composite transformation involving an electromagnetic duality rotation through an angle of $\pi/2$ times a space time translation. The details of our calculations can be found at the HEPTHools Data Repository.

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1 Introduction

In the works of [1,2,3], the concept of “holoraumy” in four dimensions\textsuperscript{12} was introduced. This occurred as an outgrowth of developments surrounding minimal adinkras related to the Coxeter Group $BC_4$ [4,5,6]. In this latter context it was noted the transport of any fermionic adinkra node that occurs in a valise adinkra under the action of two successive applications of supercharges in comparison to the application of the supercharges in the opposite order acting on the fermionic nodes leads to either a rotation times a one-dimensional translation or an extended R-symmetry transformation times a one-dimensional translation.

The appearance [1,2,3] of this rotation was noted as a rough analogy to the behavior of parallel transport of a vector around a closed path in the differential geometry of a higher dimensional Riemannian manifold. In the case for such a D-dimensional manifold such rotations appear and when these are considered over everytwo-symplex, one is led to a definition of the Riemann curvature tensor and an intrinsic characterization of the local curvature of the manifold.

Since ultimately, the whole goal of the adinkra program is to achieve a more rigorous understanding of SUSY representations in all dimensions, our research program moves alternately back-&-forth between one dimensional adinkra based investigations to the domains on supersymmetrical theories in higher dimensions. The current work will probe a question not under previous investigation. Namely the works in [1,2], concentrated on “off-shell” theories, but no investigations have been carried out into the structure of holoraumy in the absence of a complete set of auxiliary fields in four dimensional SUSY theories, i. e. “on-shell” theories. This is the question we pursue in the current work.

In the second chapter of this work, we simply review the spectrum and supersymmetry variations of the:

(a.) chiral and/or complex linear supermultiplets\textsuperscript{13},

(b.) vector supermultiplet,

(c.) axial- vector supermultiplet,

(d.) matter gravitino supermultiplet, and

(e.) supergravity supermultiplet.

The third chapter begins with a general discussion of the methodology for obtaining the form of the SUSY closure algebra for any of the supermultiplets discussed in the previous chapter. Using this methodology on each of the supermultiplets in the second chapter allows for a uniform derivation of non-closure functions in a systematic manner. These results have been computed many times and in many places previously. For our purposes, it is simply convenient to provide them (along with derivations) for the reader.

The fourth chapter reviews holoraumy for 4D off-shell multiplets. The fifth chapter is the beginning of presentations of results we believe have not been discussed previously in the literature for on-shell four dimensional theories. We note uniformly on each of the supermultiplets, the on-shell

\textsuperscript{12}The interested reader can find in the bibliography of these works the citations for introductions of this concept into SUSY QM models and the physics literature.

\textsuperscript{13}The on-shell spectrum as well as the SUSY variations of the chiral and/or complex linear supermultiplets are identical. Thus, it is not possible to distinguish one from the other in the limit where all auxiliary fields have been removed.
theories all involve several common structures. One of these includes a uniform appearance of an “electromagnetic duality rotation” implemented on both the bosons and fermion components when calculating the on-shell holoraumy. When one looks at supermultiplets that contain gauge bosons the electromagnetic duality rotation is precisely the usual one effecting “rotations” of electric and magnetic fields. The holoraumy tensors all utilize an angle of $\pi/2$ and depend on a electromagnetic duality charge. This charge is strictly determined by the difference of two of the integer parameters that characterize the off-shell holoraumy. We find in on-shell theories that upon enforcing the equations of motion the information contained in the sum of these two parameters, along with another two integer parameters is totally “lost.”

The sixth chapter includes our conclusions. We include one appendix that gives explicit expressions for the various algebraic structure introduced and used in our calculations.
2 4D, $\mathcal{N} = 1$ On-shell Supermultiplets

The structure of on-shell supermultiplets (i.e. ones in which the supersymmetry algebra closes only with the use of supplementary kinematic conditions) is widely known. With the notable exception of the 4D, $\mathcal{N} = 1$ double tensor supermultiplet [7], it is known how to introduce “auxiliary fields” for these theories in such a way so that these supplementary conditions can be avoided and one obtains “off-shell” representations. Since the “auxiliary fields” for the 4D, $\mathcal{N} = 1$ double tensor supermultiplet are not known, it is the exception rather than the rule for such theories.

It is also the case when the auxiliary fields are unknown, then it follows the theory in question cannot be expressed in terms of a set of unconstrained superfields with the attributes of being free from any a priori constraints. In the context of the quantization for non-supersymmetrical electromagnetism, this is precisely the reason that one shifts from the formulation being written in terms of the electric and magnetic fields to a formulation expressed in terms of the four-potential as the latter is free of such a priori constraints. This emphasizes in the context of electromagnetism the formulation of the theory in terms of the “E-fields” and “B-fields” constitutes the “on-shell” formulation while the formulation in terms of the “A-field” describes the “off-shell” formulation.

It is the role of the dynamical restrictions on the fields generically that is so strikingly different for these most interesting theories that suggests there remain unanswered and largely unexplored questions about the structure of supersymmetrical field theories and beyond.

In this section we will review in one place the results that are known for the “on-shell” closure of the supersymmetry algebra for the cases of supermultiplets. The equations in (2.1) - (2.5) concisely present the case of the 4D, $\mathcal{N} = 1$ supermultiplets in their on-shell formulation for spins less than two.

2.1 Chiral Supermultiplet and Complex Linear Supermultiplet On-Shell $(A, B, \psi_c)$

\[ D_a A = \psi_a , \quad D_a B = i(\gamma^5)_a^b \psi_b , \quad D_a \psi_b = i(\gamma^\mu)_{ab} \partial_\mu A - (\gamma^5 \gamma^\mu)_{ab} \partial_\mu B . \]  
\[ \text{(2.1)} \]

2.2 Vector Supermultiplet $(A_\mu, \lambda_c)$

\[ D_a A_\mu = (\gamma_{\mu})_a^b \lambda_b , \quad D_a \lambda_b = -i\frac{1}{2}(\gamma_\mu, \gamma_\nu)[\partial_\mu A_\nu] . \]  
\[ \text{(2.2)} \]

2.3 Axial-Vector Supermultiplet $(U_\mu, \bar{\lambda}_c)$

\[ D_a U_\mu = i(\gamma^5 \gamma_\mu)_a^b \bar{\lambda}_b , \quad D_a \bar{\lambda}_b = \frac{1}{2}(\gamma^5 \gamma_\mu)_{ab} \partial_\mu U_\nu . \]  
\[ \text{(2.3)} \]

2.4 Matter-Gravitino Multiplet $(B_\mu, \psi_\mu c)$

\[ D_a B_\mu = \psi_{\mu a} , \quad D_a \psi_{\mu b} = -i\frac{1}{2}(\gamma_\mu [\gamma^\alpha, \gamma^\beta])_{ab} \partial_\alpha B_\beta . \]  
\[ \text{(2.4)} \]
2.5 Supergravity Multiplet $(h_{\mu\nu}, \psi_{\mu c})$

\begin{align*}
D_a h_{\mu\nu} &= \frac{1}{2} (\gamma_{(\mu}) a^b \psi_{\nu)b} , \quad D_a \psi_{\mu b} = -i \frac{1}{2} ([\gamma^\alpha, \gamma^\beta])_{ab} \partial_\alpha h_{\beta\mu} . \tag{2.5}
\end{align*}

where we note \( h_{\mu\nu} = h_{\nu\mu} \) and we have made used the notational convention \( A_{(\mu B_{\nu)} \equiv A_\mu B_\nu + A_\nu B_\mu . \)

We have not included the discussion of the 4D, \( \mathcal{N} = 1 \) tensor supermultiplet as the fields and their SUSY transformation laws in both the on-shell and off-shell cases are exactly the same.
3 4D, $\mathcal{N} = 1$ Supersymmetry Algebra

In this section we place the review of all the SUSY commutator algebras in the same place on a common basis for all the on-shell supermultiplets of the previous chapter.

To calculate the fermionic SUSY algebra of the $D_a$ operators, we expand the anti-commutator algebra with itself in the basis of symmetrical matrices $(\gamma^\rho)_{ab}$ and $([\gamma^\kappa, \gamma^\lambda])_{ab}$

$$\{D_a, D_b\} = (\gamma^\rho)_{ab} \chi_\rho + ([\gamma^\kappa, \gamma^\lambda])_{ab} \gamma_{\kappa\lambda}$$

in terms of quantities $\chi_\rho$ and $\gamma_{\kappa\lambda}$. Since we have the identities

$$(\gamma^\rho)_{ab} (\gamma^\rho)_{cd} = -4\eta^{\rho\rho},$$

$$(\gamma^\rho)_{ab} ([\gamma^\kappa, \gamma^\lambda])_{cd} = 0,$$

$$([\gamma^\rho, \gamma^\sigma])_{ab} ([\gamma^\kappa, \gamma^\lambda])_{cd} = 16(\eta^{\rho\kappa} \eta^{\sigma\lambda} - \eta^{\rho\lambda} \eta^{\sigma\kappa}),$$

it follows we may write

$$\chi_\rho = -\frac{1}{4} (\gamma^\rho)_{ab} \{D_a, D_b\} = -\frac{1}{2} (\gamma^\rho)_{ab} D_a D_b,$$

$$\gamma_{\kappa\lambda} = \frac{1}{32} ([\gamma^\kappa, \gamma^\lambda])_{ab} \{D_a, D_b\} = \frac{1}{16} ([\gamma^\kappa, \gamma^\lambda])_{ab} D_a D_b.$$

The two operators on the far right hand sides of (3.3) and (3.4) can be evaluated on and of the fields in (2.1) - (2.5) by iteration of the $D_a$ operators.

3.1 Chiral Supermultiplet and Complex Linear Supermultiplet On-Shell $(A, B, \psi_c)$

Given the transformation laws for the fields in this supermultiplet, we obtain the 4D algebra given below.

$$\{D_a, D_b\} A = i2(\gamma^\mu)_{ab} \partial_\mu A,$$

$$\{D_a, D_b\} B = i2(\gamma^\mu)_{ab} \partial_\mu B,$$

$$\{D_a, D_b\} \psi_c = i(\gamma^\mu)_{ab} \partial_\mu \psi_c - i\frac{1}{2} (\gamma^\rho)_{ab} ([\gamma^\rho, \gamma^\mu])_{cd} \partial_\mu \psi_d$$

$$= i2(\gamma^\mu)_{ab} \partial_\mu \psi_c - i(\gamma^\rho)_{ab} (\gamma^\rho \gamma^\mu)_{cd} \partial_\mu \psi_d$$

$$= i2(\gamma^\mu)_{ab} \partial_\mu \psi_c - Z^{(CS)}_{abc}.$$ (3.6)

To pass from the first line of Eq. (3.6) we have made use of the following identity:

$$(\gamma^\rho)_{ab} ([\gamma^\rho, \gamma^\mu])_{cd} = 2(\gamma^\rho)_{ab} (\gamma^\rho \gamma^\mu)_{cd} - 2(\gamma^\mu)_{ab} C_{cd}.$$ (3.7)

The final term in Eq. (3.6)

$$Z^{(CS)}_{abc} = i(\gamma^\rho)_{ab} (\gamma^\rho \gamma^\mu)_{cd} \partial_\mu \psi_d$$ (3.8)

has a couple of descriptors in the literature. In some places, it is called “an off-shell central charge” and in others is referred to as the “non-closure function” in the SUSY algebra. It can be seen the off-shell central charge $Z^{(CS)}_{abc}$ vanishes upon enforcing the equation of motion for the fermion $i(\gamma^\mu)_{cd} \partial_\mu \psi_d = 0$ as it must.

It is of note, however, that in more complicated supersymmetrical theories, there are known cases [8] where central charges occur in the SUSY algebra, but that these do not possess the property of vanishing when the equations of motion are enforced. To distinguish such cases, these are known as “on-shell central charges.”
3.2 Vector Supermultiplet \((A_\mu, \lambda_c)\)

Turning next to the vector supermultiplet, we obtain

\[
\{D_a, D_b\} A_\mu = i 2 (\gamma^\nu)_{ab} \partial_\nu A_\mu - \partial_\mu \lambda^{(VS)}_{ab} , \quad \lambda^{(VS)}_{ab} = i 2 (\gamma^\nu)_{ab} A_\nu ,
\]

(3.9)

for the vector potential \(A_\mu\)-field. We also see the characteristic appearance of a term that (when contracted with two SUSY parameters) describes a gauge transformation. The appearance of such terms is ubiquitous in supermultiplets that contain gauge fields. Evaluation of the anti-commutator on the spinorial field in the supermultiplet leads via a direct calculation and use of a Fierz identity to the result below

\[
\{D_a, D_b\} \lambda_c = i \left[ \frac{3}{2} (\gamma^\mu)_{ab} \delta_c^d - \frac{1}{4} (\gamma^\nu)_{ab} ([\gamma_\nu, \gamma^\mu])_c^d - \frac{1}{4} (\gamma^5 [\gamma_\nu, \gamma^\mu])_{ab} (\gamma^5 \gamma_\nu)_c^d \right] \partial_\mu \lambda_d
\]

(3.10)

where the on-shell central charge for the vector multiplet \(Z^{(VS)}_{abc}\) given by

\[
Z^{(VS)}_{abc} = i \frac{1}{4} \left[ 2 (\gamma^\nu)_{ab} \delta_c^d + (\gamma^\nu)_{ab} ([\gamma_\nu, \gamma^\mu])_c^d + ([\gamma^\nu, \gamma^\mu])_{ab} (\gamma_\nu)_c^d ) \right] \partial_\mu \lambda_d
\]

(3.11)

In simplifying the central charge, we have used the identity in Eq. (3.7) as well as the following

\[
([\gamma^\nu, \gamma^\mu])_{ab} (\gamma_\nu)_c^d + (\gamma^5 [\gamma^\nu, \gamma^\mu])_{ab} (\gamma^5 \gamma_\nu)_c^d = \frac{1}{4} ([\gamma_\alpha, \gamma_\beta])_{ab} ([\gamma_\alpha, \gamma_\beta, \gamma^\mu])_c^d .
\]

(3.12)

Notice the central charge \(Z^{(VS)}_{abc}\) vanishes upon enforcing the equation of motion for the fermion \(i (\gamma^\nu)_{ac} \partial_\mu \lambda_d = 0\) as it must. \(^{14}\)

3.3 Axial-Vector Supermultiplet \((U_\mu, \tilde{\lambda}_c)\)

For the axial vector supermultiplet one obtains from simply repeating the steps used in (3.9) - (3.12) the results below.

\[
\{D_a, D_b\} U_\mu = i 2 (\gamma^\nu)_{ab} \partial_\nu U_\mu - \partial_\mu \tilde{\lambda}^{(AVS)}_{ab} , \quad \tilde{\lambda}^{(AVS)}_{ab} = i 2 (\gamma^\nu)_{ab} U_\nu ,
\]

(3.13)

\[
\{D_a, D_b\} \tilde{\lambda}_c = i 2 (\gamma^\nu)_{ab} \partial_\nu \tilde{\lambda}_c - Z^{(AVS)}_{abc} ,
\]

(3.14)

\[
Z^{(AVS)}_{abc} = \left[ i \frac{1}{2} (\gamma^\nu)_{ab} (\gamma_\nu \gamma_\mu)_c^d - i \frac{1}{16} ([\gamma_\alpha, \gamma_\beta])_{ab} ([\gamma_\alpha, \gamma_\beta, \gamma^\mu])_c^d \right] \partial_\mu \tilde{\lambda}_d .
\]

(3.15)

\(^{14}\)There is a typo in [9], which disagrees in the last term of the last line of Eq. (3.11) by a minus sign
Notice the slight difference in the central charges for the vector multiplets and axial-vector multiplets, Eqs. (3.11) and (3.15) respectively. The only difference is the minus sign of the second term. This is because the axial-vector and vector multiplets are related by the following replacements

$$A_\mu \rightarrow U_\mu \quad , \quad \lambda_a \rightarrow -i(\gamma^5)_a \tilde{\lambda}_b \quad .$$

(3.16)

Since \(\gamma^5\) anti-commutes with all \(\gamma^\mu\), moving \(\gamma^5\) past two \(\gamma^\mu\) matrices in the leftmost term in the last line of Eq. (3.11) gives two minus signs and moving it past three \(\gamma^\mu\) matrices in the rightmost term gives three minus signs.

### 3.4 Matter-Gravitino Multiplet \((B_\mu, \psi_{\mu c})\)

The supermultiplet with propagating spins of one and three-halves has been called the “matter gravitino supermultiplet.” Applying the procedure used in the previous sections of this chapter yields the expressions in (3.17) - (3.24).

$$\{D_a, D_b\} B_\mu = i2(\gamma^\nu)_{ab} \partial_\nu B_\mu - \partial_\mu \lambda^{(MGM)}_{ab} \quad , \quad \lambda^{(MGM)}_{ab} = i2(\gamma^\nu)_{ab} B_\nu \quad ,$$

(3.17)

$$\{D_a, D_b\} \psi_{\mu c} = i2(\gamma^\nu)_{ab} \partial_\nu \psi_{\mu c} - \partial_\mu \varepsilon^{(MGM)}_{abc} - Z^{(MGM)}_{\mu abc} \quad ,$$

(3.18)

These include the gauge transformation

$$\varepsilon^{(MGM)}_{abc} = 2i(\gamma^\alpha)_{ab} \psi_{\alpha c} \quad ,$$

(3.19)

for the spin 3/2 field and in addition the non-closure terms,

$$Z^{(MGM)}_{\mu abc} = i(\gamma^\rho)_{ab} Z^{(MGM,1)}_{\mu c} + i[\gamma^\rho, \gamma^\sigma]_{ab} Z^{(MGM,2)}_{\mu \rho \sigma c} \quad ,$$

(3.20)

where

$$Z^{(MGM,1)}_{\mu c} = i \left[ (-\eta_{\mu \rho} + \frac{3}{4} \gamma^\rho \gamma_\rho \right] d(\gamma^\kappa)_{c} d - (\gamma_{[\mu}) c^{b} \delta_{\kappa]} b \right] E_{kb} \quad ,$$

(3.21)

$$Z^{(MGM,2)}_{\mu \rho \sigma c} = i \frac{1}{2} \left[ \eta_{\mu \rho} \delta_{\sigma} d + i \frac{1}{2} \epsilon_{\mu \rho \sigma \alpha} \gamma^\alpha \right] d \left( \delta_{d} b \delta_{\alpha} \kappa - \frac{1}{4} (\gamma_{\alpha} \gamma^\kappa)_{d} b \right) E_{kb} \quad ,$$

(3.22)

and \(E_{kb}\),

$$E_{kb} = \epsilon_{\kappa [\nu \alpha \beta} (\gamma^5 \gamma^\mu)_{b} d \partial_\alpha \psi_{\beta d} \quad ,$$

(3.23)

when set equal to zero, yields the Rarita-Schwinger equations of motion for the gravitino and with its trace being

$$R_a = ([\gamma^\alpha, \gamma^\beta]_a b \partial_\alpha \psi_{\beta b} = i(\gamma^\kappa)_a b E_{kb} \quad .$$

(3.24)

Notice the central charge terms vanish upon enforcement of the gravitino’s equations of motion, i.e. setting \(E_{kb} = R_a = 0\), as they must.
3.5 Supergravity Multiplet \((h_{\mu\nu}, \psi_{\mu c})\)

For the supergravity supermultiplet the process goes along the now familiar path and yields the results seen in (3.25) - (3.30).

\[
\{D_a, D_b\} h_{\mu\nu} = i 2 (\gamma^{\rho})_{ab} \partial_{\rho} h_{\mu\nu} - \partial (\mu)_{\rho ab} , \quad \xi_{\rho ab} = i (\gamma^{\rho})_{ab} h_{\nu\rho} , \tag{3.25}
\]

\[
\{D_a, D_b\} \psi_{\mu c} = i 2 (\gamma^{\rho})_{ab} \partial_{\rho} \psi_{\mu c} - \partial (\mu)_{\rho ab} \varepsilon^{(SG)}_{abc} - Z^{(SG)}_{\mu abc} , \tag{3.26}
\]

\[
\varepsilon^{(SG)}_{abc} = i \frac{1}{8} \left\{ 10 (\gamma^\alpha)_{ab} \delta^d_c - [\gamma^\alpha, \gamma^\beta]_{ab} (\gamma^\beta)_c^d + (\gamma^\beta)_{ab} ([\gamma^\beta, \gamma^\alpha])_c^d - (\gamma^5 [\gamma^\alpha, \gamma^\beta])_{ab} (\gamma^5 \gamma^\beta)_c^d \right\} \psi_{\alpha d} \tag{3.27}
\]

\[Z^{(SG)}_{\mu abc} = i (\gamma^{\rho})_{ab} Z^{(SG,1)}_{\mu pc} + i [\gamma^{\rho}, \gamma^{\sigma}]_{ab} Z^{(SG,2)}_{\mu pc} \tag{3.28}
\]

where

\[
Z^{(SG,1)}_{\mu pc} = \frac{1}{4} (\gamma_{\mu}, \gamma_{\rho})_c^d E_{pd} + \frac{i}{2} (\gamma_{\rho})_c^d E_{pd} , \tag{3.29}
\]

\[
Z^{(SG,2)}_{\mu pc} = i \frac{1}{2} \eta_{\mu \rho} E_{\sigma c} + i \frac{1}{32} (\gamma_{\rho}, \gamma_{\sigma})_c^d E_{pd} + \frac{1}{8} \varepsilon_{\mu \rho \sigma \kappa} (\gamma^5)_c^d E_{nd} \tag{3.30}
\]

with \(R_a\) and \(E_{\mu a}\) taking the same form as for the matter gravitino supermultiplet.

With these results, we have reviewed what is known about the structure of on-shell SUSY theories for fields of spins less than or equal to two.
4 A Geometry For Fermionic Non-closure Equation of Motion Terms

In this chapter, we will focus on the equations of motion terms that generate central charges in the SUSY algebra.

Let us start by gathering all of the results from the previous chapter in one place.

\[
Z^{(CS)}_{abc} = i(\gamma^\rho)_{ab}(\gamma_\rho \gamma^\mu)_c d \partial_\mu \psi_d \\
Z^{(VS)}_{abc} = i \left[ \frac{1}{2} (\gamma^\nu)_{ab}(\gamma_\nu \gamma^\mu)_c \right] d + \frac{1}{16} ([\gamma_\alpha, \gamma_\beta])_{ab} ([\gamma^\alpha, \gamma^\beta] \gamma^\mu)_c d \partial_\mu \lambda_d .
\]

\[
Z^{(AVS)}_{abc} = i \left[ \frac{1}{2} (\gamma^\nu)_{ab}(\gamma_\nu \gamma^\mu)_c \right] d - \frac{1}{16} ([\gamma_\alpha, \gamma_\beta])_{ab} ([\gamma^\alpha, \gamma^\beta] \gamma^\mu)_c d \partial_\mu \tilde{\lambda}_d .
\]

\[
Z^{(MGM)}_{\mu abc} = i(\gamma^\rho)_{ab} Z^{(MGM,1)}_{\mu \rho c} + i([\gamma^\rho, \gamma^\sigma])_{ab} Z^{(MGM,2)}_{\mu \rho \sigma c} ,
\]

where

\[
Z^{(MGM,1)}_{\mu \rho c} = i \left[ -\eta_{\mu \rho} + \frac{3}{4} \gamma_\mu \gamma_\rho \right] c (\gamma_\kappa)_d \gamma_5 b - (\gamma_\mu)_c (\gamma_\rho)_b \nabla_\kappa b \right] E_{\mu b} ,
\]

\[
Z^{(MGM,2)}_{\mu \rho \sigma c} = i \frac{1}{2} \left( \eta_{\mu \rho} \delta_\sigma^\alpha + i \frac{1}{2} \epsilon_{\mu \rho \sigma} \alpha \gamma_\beta \right) c (\delta_\kappa^b \gamma_\rho^\sigma - \frac{1}{4} (\gamma_\alpha \gamma_\kappa)_d b) E_{\mu b} ,
\]

with \( E_{\mu b} \)

\[
E_{\mu b} = \epsilon^{\nu \alpha \beta} (\gamma_5 \gamma^\nu)_b \partial_\alpha \psi_b ,
\]

when set equal to zero, yields the Rarita-Schwinger equations of motion for the gravitino and with its trace being

\[
R_a = ([\gamma^\alpha, \gamma^\beta])_b \partial_\alpha \psi_b = i(\gamma_\kappa)_a b E_{\mu b} .
\]

\[
\varepsilon^{(SG)}_{abc} = i \frac{1}{8} \left\{ 10(\gamma_\alpha)_{ab} \delta_\epsilon^c - \left[ \gamma_\alpha, \gamma_\beta \right]_{ab}(\gamma_\epsilon)_c d + (\gamma_\beta)_{ab}(\gamma_\epsilon)_c d \right\} \psi_{ad} ,
\]

\[
\varepsilon^{(SG)}_{\mu abc} = i(\gamma^\rho)_{ab} Z^{(SG,1)}_{\mu \rho c} + i([\gamma^\rho, \gamma^\sigma])_{ab} Z^{(SG,2)}_{\mu \rho \sigma c} ,
\]

where

\[
Z^{(SG,1)}_{\mu \rho c} = i \frac{1}{4} ([\gamma_\mu, \gamma_\rho]) c d R_d - i \frac{1}{2} (\gamma_\rho)_c d E_{\mu d} + i \frac{3}{4} (\gamma_\rho)_c d E_{\mu d} ,
\]

\[
Z^{(SG,2)}_{\mu \rho \sigma c} = i \frac{1}{4} \eta_{\mu \rho} E_{\sigma c} + i \frac{1}{32} ([\gamma_\rho, \gamma_\sigma]) c d E_{\mu d} + \frac{1}{8} \epsilon_{\mu \rho \sigma} (\gamma_5)_c d E_{\kappa d} .
\]
5 Review of 4D, $\mathcal{N}=1$ Off-Shell Holoraumy Tensors From Previous Work

In this chapter, we will present a review of off-shell holoraumy.

For the spinor fields of the minimal representation supermultiplets (2.1) - (2.3) we can introduce a “collective” notation whereby these spinors are denoted by the symbol $\Psi^{(\hat{R})}_a$ where the “representation index ($\hat{R}$)” distinguishes between the chiral/complex linear, vector, or axial-vector supermultiplets. Using this notation, the holoraumy calculations take the form,

$$[D_a, D_b] \Psi^{(\hat{R})}_c = -i2 [\hat{h}^{\mu}(\Pi)]_{abc} d \partial_\mu \Psi^{(\hat{R})}_d,$$  

(5.1)

where

$$[\hat{h}^{\mu}(\Pi)]_{abc}^d = \begin{bmatrix} p C_{ab} (\gamma^\mu)_c^d + q (\gamma^5)_ab (\gamma^5\gamma^\mu)_c^d \\
+ r (\gamma^5\gamma^\mu)_ab (\gamma^5)_c^d + \frac{1}{2} s (\gamma^5\gamma_\nu)_ab (\gamma^\nu\gamma^\mu)_c^d \end{bmatrix},$$

(5.2)

and the integers $p$, $q$, $r$, and $s$ take the values below\textsuperscript{15}.

| $\hat{R}$ | $p$ | $q$ | $r$ | $s$ |
|-----------|-----|-----|-----|-----|
| (CS)      | 0   | 0   | 0   | 1   |
| (VS)      | 1   | 1   | 1   | 0   |
| (AVS)     | -1  | -1  | 1   | 0   |

Table 1: Holoraumy Integers For 4D, $\mathcal{N}=\text{Chiral, Vector, and Axial-Vector Supermultiplets.}$

Finally, the quantity $\Pi$ is simply a notation for the set of integers $p$, $q$, $r$, and $s$

$$\Pi = (p, q, r, s).$$

(5.3)

While the anti-commutator algebra of the fermionic operator $D_a$ with itself takes a universal form being the sum of a translation operator, gauge transformation operator on bosonic fields and including terms involving the equations of motion for the fermionic fields, the commutator algebra of the fermionic operator $D_a$ yields results that are specific to each off-shell supermultiplet.

As we are focused upon the on-shell holoraumy in this work, we will re-write (5.2) to separate out the portions of those results that depend on the equations of motion for the spinor in each supermultiplet from those that do not depend on such terms. This is done by “splitting” according

$$[\hat{h}^{\mu}(\Pi)]_{abc}^d = [\hat{h}^{\mu,En-M}(\Pi)]_{abc}^d + [\hat{h}^{\mu,On-Sh}(\Pi)]_{abc}^d$$

$$= \begin{bmatrix} p C_{ab} (\gamma^\mu)_c^d + q (\gamma^5)_ab (\gamma^5\gamma^\mu)_c^d + s (\gamma^5\gamma_\nu)_ab (\gamma^\nu\gamma^\mu)_c^d \\
+ (r - s) (\gamma^5\gamma^\nu)_ab (\gamma^\nu)_c^d \end{bmatrix},$$

(5.4)

where all we have done is to make use of the identity

$$(\gamma^5[\gamma^\nu, \gamma^\mu])_c^d = (\gamma^5\gamma^\nu\gamma^\mu)_c^d - (\gamma^5\gamma^\mu\gamma^\nu)_c^d = -2i\eta^\mu\nu(\gamma^5)_c^d + 2(\gamma^5\gamma^\nu\gamma^\mu)_c^d,$$

(5.5)

which is valid for the $\gamma$-matrices.

\textsuperscript{15}Here we follow the definitions of $p$, $q$, $r$, and $s$ as in [1,2] as opposed to [3] which used the opposite definitions of $r$ and $s.
Accordingly, the first line of terms on the RHS of (5.4) are all proportional to the equations of motion for the fermions, while the single term on the second line is not. Owing to the Clifford algebra satisfied by the $\gamma^\mu$-matrices, Eqs. (5.2) and (5.4) are just two of an infinite number of ways to separate $\hat{h}^\mu(\Pi)$ in terms proportional to equations of motion and terms that are not. Our convention will be to simply commute the gamma matrix that contracts with the derivative (and/or gravitino for the $SG$ and $MGM$ multiplets we will investigate) to the right until it is no longer possible to produce terms proportional to equations of motion by doing so.

The equation in (5.4) reveals that all the information associated with $p$, $q$, and $s$ separately is lost (as these occur in the $[\hat{h}^\mu_{EoM}(\Pi)]$ tensor) for the on-shell theories describing the minimal representations. We have shown by first calculating the off-shell fermionic holoraumy and then eliminating the portion of it that depends on the equations of motion for the fermions leads to

$$[D_a, D_b] \Psi_c^{(\hat{R})} = -i2(r - s)(\gamma^5\gamma^\mu)_{ab}(\gamma^5)_{cd}\partial_\mu \Psi_d^{(\hat{R})},$$  \hspace{1cm} (5.6)$$

So the only information retained relates to the combination $(r - s)$ which occurs in the $[\hat{h}^\mu_{On-Shell}(\Pi)]$ tensor. Thus, there is a severe loss of information upon enforcing the equations of motion. We return to this in the conclusions.
6 New Results of 4D, $\mathcal{N}=1$ On-Shell Holoraumy Tensors

The general method to calculate the fermionic holoraumy is an expansion in a basis of antisymmetric matrices

$$[D_a, D_b] = \frac{1}{2} C_{ab} C^{cd} D_c D_d + \frac{1}{2} (\gamma^5)_{ab} (\gamma^5)_{cd} D_c D_d - \frac{1}{2} (\gamma^5 \gamma^\kappa)_{ab} (\gamma^5 \gamma^\kappa)_{cd} D_c D_d$$

(6.1)
due to the identities

$$C_{ab} C_{ab} = 4, \quad C_{ab} (\gamma^5)_{ab} = 0, \quad C_{ab} (\gamma^5 \gamma^\kappa)_{ab} = 0, \quad (\gamma^5)_{ab} (\gamma^5)_{ab} = 4, \quad (\gamma^5 \gamma^\kappa)_{ab} (\gamma^5 \gamma^\kappa)_{ab} = 0, \quad (\gamma^5 \gamma^\rho)_{ab} (\gamma^5 \gamma^\kappa)_{ab} = -4\eta^{\rho\kappa}.$$  

(6.2)

So all that remains is to evaluate the terms quadratic in the D-operators on the RHS of (5.4) making use of the expressions in (2.1) - (2.5).

6.1 Chiral/Complex Linear Supermultiplet On-Shell $(A, B, \psi_c)$

We begin with the spin-0 bosons and derive

$$[D_a, D_b] A = -2(\gamma^5 \gamma^\mu)_{ab} \partial_\mu B,$$

(6.3)

$$[D_a, D_b] B = 2(\gamma^5 \gamma^\mu)_{ab} \partial_\mu A,$$

(6.4)

and find for the fermion

$$[D_a, D_b] \psi_c = i(\gamma^5 \gamma^\mu)_{ab} (\gamma^5)_{cd} \partial_\mu \psi_d - i\frac{1}{2}(\gamma^5 \gamma^\kappa)_{ab} (\gamma^5 [\gamma^\kappa, \gamma^\mu])_{cd} \partial_\mu \psi_d.$$  

(6.5)

These can be cast into the following forms

$$[D_a, D_b] \begin{pmatrix} A \\ B \end{pmatrix} = -i2 \mathcal{B}^\mu_{ab} \partial_\mu \begin{pmatrix} A \\ B \end{pmatrix},$$

(6.6)

for the bosonic fields and for the fermions as

$$[D_a, D_b] \psi_c = -i2 \left( \mathcal{F}^\mu_{ab} \right)_{cd} \partial_\mu \psi_d - Z_{abc},$$

(6.7)

where the on-shell bosonic and fermionic holoraumy tensors $\mathcal{B}^\mu_{ab}$, and $\left( \mathcal{F}^\mu_{ab} \right)_{cd}$, along with the EoM portion $Z_{abc}$ are explicitly written as

$$\mathcal{B}^\mu_{ab} = \sigma^2 \otimes (\gamma^5 \gamma^\mu)_{ab},$$

(6.8)

$$\left( \mathcal{F}^\mu_{ab} \right)_{cd} = -(\gamma^5 \gamma^\mu)_{ab} (\gamma^5)_{cd},$$

(6.9)

$$Z_{abc} = (\gamma^5 \gamma^\kappa)_{ab} (\gamma^5 \gamma^\kappa)_{cd} \mathcal{D}^\mu_{d},$$

(6.10)

where

$$\mathcal{D}^\mu_{d} = i(\gamma^\mu)_{de} \partial_\mu \psi_e.$$  

(6.11)
and when this is set to zero it yields the equation of motion for $\psi_e$, thus it is acting like a central charge for holoraumy.

These results also reveal something else...the presence of an electromagnetic type duality rotation. This is seen in the following manner.

In electromagnetism, if we start with the source-free Maxwell Equations, they are invariant under a transformation where we ‘rotate’ the electric field $\vec{E}$ and magnetic field $\vec{B}$ one into the other via

\begin{align}
\vec{E}' &= \vec{E}\cos \Theta - \vec{B}\sin \Theta , \\
\vec{B}' &= \vec{E}\sin \Theta + \vec{B}\cos \Theta .
\end{align}

(6.12)

Since $\vec{E}$ transforms as a vector under a parity transformation of field variables while $\vec{B}$ transforms as an axial-vector under the same transformation, the rotation in (6.12) mixes field variables that possess different parity properties. Now let us note that bosonic field variables ($A, B$) of the chiral supermultiplet are just such a pair, thus under the transformation of (6.12), the electric and magnetic field variable can be replaced by the of the derivatives of the spin-0 scalar and pseudoscalar field variables leading to

\begin{align}
(\partial_\mu A)' &= (\partial_\mu A)\cos \Theta - (\partial_\mu B)\sin \Theta , \\
(\partial_\mu B)' &= (\partial_\mu A)\sin \Theta + (\partial_\mu B)\cos \Theta .
\end{align}

(6.13)

Upon comparing the form of the RHS of (6.6) with (6.13), we see the angle variable has the value of $\Theta = \pi/2$.

This electromagnetic duality rotation must also effect the $\gamma$-matrices since $\gamma^\mu$ and $i\gamma^5\gamma^\mu$ must satisfy a condition similar in form to the equations in (6.12) and (6.13).

\begin{align}
(\gamma^\mu)' &= \gamma^\mu \cos \Theta - i\gamma^5\gamma^\mu \sin \Theta , \\
(i\gamma^5\gamma^\mu)' &= \gamma^\mu \sin \Theta + i\gamma^5\gamma^\mu \cos \Theta .
\end{align}

(6.14)

The $2 \times 2$ matrix that appears in (6.12) - (6.14) for convenience be written as

\begin{equation}
\mathcal{P}(\Theta) = \begin{bmatrix}
\cos \Theta & -\sin \Theta \\
\sin \Theta & \cos \Theta
\end{bmatrix}
\end{equation}

(6.15)

As a consistency check, we need to consider the on-shell fermionic holoraumy. i. e. the first term on the RHS of (6.7). The presence of the factor of $\gamma^5\gamma^\mu$ in (6.7) show the usual $\gamma^\mu$ associated with the SUSY algebra has been subjected to electromagnetic duality rotation where $\Theta = \pi/2$. However, the complete form of $(\mathcal{F}^{\mu(CS)}_{ab})_{cd}$ shows the electromagnetic duality rotation is realized as an operator

\begin{equation}
(\psi_a)' = \cos \Theta \psi_a - i\sin \Theta (\gamma^5)_a b \psi_b .
\end{equation}

(6.16)

acting on the on-shell fermion as well with the same value of $\Theta$.

In fact, in the remainder of this section, we find these conditions are satisfied uniformly on all of the supermultiplets under explicit investigation.
6.2 Vector Supermultiplet \((A_\mu, \lambda_c)\)

The on-shell holoraumy calculations for the vector supermultiplet yield

\[
[D_a, D_b] A_\mu = -i2 \left( \mathcal{R}^{(VS)}_{\mu\alpha\beta} \right)_{ab} \partial^\alpha A^\beta, \quad (6.17)
\]

\[
[D_a, D_b] \lambda_c = -i2 \left( \mathcal{F}^{(VS)}_{\mu} \right)_{cd} \partial_\mu \lambda_d - \mathcal{L}^{(VS)}_{abc}, \quad (6.18)
\]

with the expressions for the bosonic and fermionic holoraumy tensors given by

\[
\left( \mathcal{R}^{(VS)}_{\mu\alpha\beta} \right)_{ab} = -i \epsilon_{\mu\alpha\beta\nu} (\gamma^5 \gamma^\nu)_{ab}, \quad (6.19)
\]

\[
\left( \mathcal{F}^{(VS)}_{\mu} \right)_{cd} = (\gamma^5 \gamma^\mu)_{ab} (\gamma^5)_{cd}, \quad (6.20)
\]

along with the fermionic equation of motion term

\[
\mathcal{L}^{(VS)}_{abc} = \left\{ \frac{3}{2} C_{ab} \delta_c^d + \frac{3}{2} (\gamma^5)_{ab} (\gamma^5)_c^d - \frac{1}{2} (\gamma^5 \gamma^\kappa)_{ab} (\gamma^5 \gamma^\kappa)_c^d \right\} D^{(VS)}_d, \quad (6.21)
\]

with

\[
D^{(VS)}_d = i (\gamma^\mu)_d^e \partial_\mu \lambda_e. \quad (6.22)
\]

The term \(\mathcal{L}^{(VS)}_{abc}\) vanishes upon enforcing the equations of motion \(D^{(VS)}_d = 0\).

To check for the presence of an electromagnetic duality rotation as found for the case of the on-shell chiral/complex linear supermultiplet has two parts, i.e., an examination of the bosonic holoraumy and an examination of the fermionic holoraumy.

For the gauge field, we see combine the result in (6.17) with the one in (6.19) to write

\[
[D_a, D_b] A_\mu = -\epsilon_{\mu\alpha\beta\nu} (\gamma^5 \gamma^\nu)_{ab} F^{\alpha\beta}(A) = -2(\gamma^5 \gamma^\nu)_{ab} \tilde{F}_{\mu\nu}(A), \quad (6.23)
\]

where in the middle step the Maxwell Field Strength \(F^{\alpha\beta}(A)\) is introduced and in the final step the answer is expressed in terms of the dual Maxwell Field Strength \(\tilde{F}_{\mu\nu}(A) = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}(A)\). Of course, (6.23) is simply the manifestly relativistic formulation of (6.12).

Upon comparing this result with the one in (3.9), it clear that an electromagnetic duality rotation is present. It is realized precisely as in (6.12) though the same angle \(\Theta = \pi/2\).

The explicit introduction of the dual electromagnetic field strength also allows the discussion of the transformation under the action of the electromagnetic duality rotation according to

\[
F_{\alpha\beta}(A)' = F_{\alpha\beta}(A) \cos \Theta - \tilde{F}_{\alpha\beta}(A) \sin \Theta, \quad (6.24)
\]

Before leaving this point, it is also important to keep in mind that the electromagnetic duality rotation defined by the formula (6.24) can be applied to any field that possesses a field strength that is a two-form. This includes spin-3/2 fields as well as spin-2 fields where the first two (in our conventions) indices on the Riemann curvature tensor are treated as the indices on the Maxwell field strength.

For the gaugino field, we may compare the result for the fermionic holoraumy in (6.9) for the spinor in the chiral/complex linear supermultiplet with the one in (6.20) for the vector supermultiplet. There is a sign difference seen upon doing so. This is a feature to be discussed in the conclusion section.
6.3 Axial-Vector Supermultiplet

The calculations for the axial-vector supermultiplet simply reproduce exactly the ones carried out upon the vector supermultiplet. So in the discussion below we simply report the results for the sake of explicitness, but without further comment.

\[ [D_a, D_b] U_\mu = - i 2 \left( \mathcal{B}_{\mu}^{AVS} \right)_{ab} \partial^a U^b \]  
(6.25)

\[ [D_a, D_b] \tilde{\lambda}_c = - i 2 \left( \mathcal{F}_{\mu}^{AVS} \right) c\mu \partial_{\mu} \tilde{\lambda}_d - \mathcal{Z}_{abc}^{AVS} \]  
(6.26)

where the bosonic and fermionic holoraumy tensors are

\[ \left( \mathcal{B}_{\mu\alpha\beta}^{AVS} \right)_{ab} = - i \epsilon_{\mu\alpha\beta\nu} (\gamma^5 \gamma^\nu)_{ab} \]  
(6.27)

\[ \left( \mathcal{F}_{\mu\alpha\beta}^{AVS} \right)_{c\mu} = (\gamma^5 \gamma^\mu)_{ab} (\gamma^5)_{c} d \]  
(6.28)

\[ \mathcal{Z}_{abc}^{AVS} = \left\{ - \frac{3}{2} C_{ab} \delta_c d - \frac{3}{2} (\gamma^5)_{ab} (\gamma^5)_{c} d - \frac{1}{2} (\gamma^5 \gamma^\nu)_{ab} (\gamma^5 \gamma^\nu)_{c} d \right\} D_d^{AVS} \]  
(6.29)

where

\[ D_d^{AVS} = i (\gamma^\mu)_{e} d \partial_{\mu} \tilde{\lambda}_e \]  
(6.30)

is when set to zero the equation of motion for \( \tilde{\lambda}_e \). Notice the term \( \mathcal{Z}_{abc}^{AVS} \) vanishes upon enforcing the equations of motion \( D_d^{AVS} = 0 \).

As there has not been any published work looking at the off-shell holoraumy of the matter-gravitino supermultiplets, we will not in the following to be making comparisons between on-shell and off-shell holoraumies. However, as there are spin-1 fields in the vector, axial vector, and matter gravitino supermultiplets, we will make comments on this at the end of the next subsection.

6.4 Matter-Gravitino Multiplets \( (B_\mu, \psi_{\mu\nu}) \)

\[ [D_a, D_b] B_\mu = - i 2 \left( \mathcal{B}_{\mu}^{MG} \right)_{ab} \partial^a B^b \]  
(6.31)

\[ [D_a, D_b] \psi_{\mu\nu} = - i 2 \left( \mathcal{F}_{\mu\alpha\beta}^{MG} \right)_{abc} d \partial^{a} \psi_{b}^{\mu} - \mathcal{Z}_{\mu\nu}^{MG} \]  
(6.32)

where the bosonic and fermionic holoraumy tensors are

\[ \left( \mathcal{B}_{\mu\alpha\beta}^{MG} \right)_{ab} = i \epsilon_{\mu\alpha\beta\nu} (\gamma^5 \gamma^\nu)_{ab} \]  
(6.33)

\[ \left( \mathcal{F}_{\mu\alpha\beta}^{MG} \right)_{abc} d = \frac{1}{2} \eta_{\mu\nu} (\gamma^5)_{ab} (\gamma^5)_{c} d - i \frac{1}{2} \epsilon_{\mu\alpha\beta\nu} (\gamma^5 \gamma^\nu)_{ab} \delta_c d \]  
(6.34)

\[ \mathcal{Z}_{\mu\nu}^{MG} = i \frac{1}{3} \left\{ C_{ab} (\gamma_{\mu})_{c} d - (\gamma^5)_{ab} (\gamma^5 \gamma_{\mu})_{c} d - (\gamma^5 \gamma_{\rho})_{ab} (\gamma^5 \gamma_{\rho} \gamma_{\mu})_{c} d \right\} R_d \]  
(6.35)
where $E_{\mu d}$ and $R_a$ are as in Eqs. (3.23) and (3.24), when set to zero the equation of motion for the gravitino and its trace, respectively. We see then that $2^{(MGM)}_{\mu abc}$ vanishes upon enforcing the equations of motion for the gravitino.

Looking at the result in (6.19) in comparison to the one in (6.33), it can be seen that

$$\left( B^{(VS)}_{\mu \alpha \beta} \right)_{ab} = - \left( B^{(MGM)}_{\mu \alpha \beta} \right)_{ab}$$

(6.36)

which is one more of the points to be covered in the concluding discussion.

### 6.5 Supergravity Multiplets ($h_{\mu \nu}, \psi_{\mu c}$)

Finally, we come to the case of the on-shell supergravity supermultiplets whose results are reported in this subsection.

$$[D_{\alpha}, D_{\beta}] h_{\mu \nu} = - i 2 \left( B^{(SG)}_{\mu \nu \rho \sigma \delta} \right)_{ab} \partial^\rho h^{\beta \delta}$$

(6.37)

$$[D_{\alpha}, D_{\beta}] \psi_{\mu c} = - i 2 \left( F^{(SG)}_{\mu \alpha \beta} \right)_{abc} d \partial^\alpha \psi^\beta - 2 \varphi^{(SG)}_{\mu abc} - \partial^\nu \gamma_{\mu abc}$$

(6.38)

where the bosonic and fermionic holoraumy tensors are

$$\left( B^{(SG)}_{\mu \nu \rho \sigma \delta} \right)_{ab} = - i \frac{1}{2} \eta_{\beta (\mu \rho \nu \sigma \delta \epsilon) \rho \sigma \kappa (\gamma^5 \gamma^\kappa)_{ab}}$$

(6.39)

$$\left( F^{(SG)}_{\mu \alpha \beta} \right)_{abc} = \frac{d}{2} \left\{ - C_{ab} (\gamma^\mu)_{c} d - (\gamma^5)_{ab} (\gamma^5 \gamma^\mu)_{c} d + \frac{1}{2} (\gamma^5 \gamma^\sigma)_{ab} (\gamma^5 \gamma_{\mu \sigma})_{c} d \
- \frac{1}{4} (\gamma^5 \gamma^\mu)_{ab} (\gamma^5)_{c} d \right\} R_d$$

(6.40)

$$\varphi^{(SG)}_{\mu abc} = \frac{1}{2} \left\{ 5 C_{ab} (\gamma^\beta)_{c} d + (\gamma^5)_{ab} (\gamma^5 \gamma^\beta)_{c} d + (\gamma^5 \gamma^\beta)_{ab} (\gamma^5)_{c} d \
- \frac{1}{6} (\gamma^5 \gamma_{\beta})_{ab} (\gamma^5 [\gamma^\kappa, \gamma^\beta]_{c} d \right\} \psi_{\beta d}$$

(6.41)

Notice all terms in $\left( B^{(SG)}_{\mu \nu \rho \sigma \delta} \right)_{abc} d$ are anti-symmetric in $\alpha$ and $\beta$ as in the case for MGM. Also, we see that $F^{(SG)}_{\mu \alpha \beta} d$ vanishes upon enforcing the equations of motion for the gravitino and its trace, Eqs. (3.23) and (3.24) set to zero, respectively. The term $\varphi^{(SG)}_{\mu abc} d$ acts like a gauge transformation that can not be included in $\left( B^{(SG)}_{\mu \nu \rho \sigma \delta} \right)_{abc} d$ while maintaining the $\alpha$ and $\beta$ antisymmetry. Comparing the SG fermionic holoraumy to that of MGM, Eq. (6.32), we point out that the former has a gauge-like term $\zeta^{(SG)}_{abc}$ that remains upon enforcing the equations of motion and the latter does not.
7 Conclusion

Perhaps, one of the fascinating observations uncovered by this study in the fact that the commutator, not the anti-commutator of two supercharges in the theories under investigation show a previously unnoticed uniformity.

Putting all of this together, we arrive at a conclusion that to our knowledge has not appeared in the physics literature. In a field independent manner, for an on-shell supermultiplet in four dimensions upon dropping dependences proportional to equations of motion and gauge transformations, the equation

$$[D_a, D_b] = 2 Q_{EMDC} \mathcal{P}(\Theta = \pi/2) \otimes (\gamma^\mu)_{ab} P_\mu,$$  \hspace{1cm} (7.1)

with

(a.) $\mathcal{P}(\Theta = \pi/2)$ being an electromagnetic duality rotation through an angle of $\pi/2$ acting on all quantities that follow it,

(b.) $P_\mu$ being the generator of spacetime translations is satisfied, and

(c.) $Q_{EMDC}$ denoting an electromagnetic-duality charge,

appears to be ubiquitously valid.

As several points in the discussions of the holobraumy tensors, we alluded to the fact that the signs that show up in some of the calculation might seem inconsistent with the action of the operator $\mathcal{P}(\Theta = \pi/2)$ realized on the fields of the various multiplets. The simplest way to resolve this is to conclude that when action on field variable, the operator $\mathcal{P}(\Theta = \pi/2)$ is also dependent on “an electromagnetic-duality charge” and different spinors in the distinct supermultiplets carry different values of this charge.

Let us give an expanded discussion of this “electromagnetic-duality charge” that can be denoted by $Q_{EMDC}$.

The calculations reviewed in chapter four are off-shell calculation where sets of auxiliary fields are included ab initio. We used the form of the off-shell holobraumy to separate it into on part proportional to terms involving equations of motion (the first line shown in (5.4) and a single term not proportional to the equations of motion (i. e. the fine line of (5.4). Thus a single term for the on-shell holobraumy, shown in (5.6), remained. The implication of (5.6) is

$$Q_{EMDC} = (r - s).$$ \hspace{1cm} (7.2)

With this understanding in place, all the signs are consistent and all the calculations take the form of the equation in (7.1).

We conjecture this is valid for all on-shell 4D, $\mathcal{N} = 1$ supermultiplets.

Of course, we always have the usual condition

$$\{D_a, D_b\} = 2 (\gamma^\mu)_{ab} P_\mu,$$ \hspace{1cm} (7.3)

and upon adding (7.1) and (7.3) we obtain the very powerful statement (without commutators nor anti-commutators of the supercovariant derivatives)

$$D_a D_b = (\gamma^\mu)_{ab} P_\mu + Q_{EMDC} \mathcal{P}(\Theta = \pi/2) \otimes (\gamma^\mu)_{ab} P_\mu,$$ \hspace{1cm} (7.4)
whose implications need further study.

We will close by carrying out a simple test of this construction in (7.1) as this equation makes a prediction that is simple to verify. In chapter two, there were two minimal 4D, \( \mathcal{N} = 1 \) supermultiplets that were not included...the tensor supermultiplet and the axial tensor supermultiplet. The reason for neglecting them is that the form of their supersymmetry variations of the fields \( (\varphi, B_{\mu\nu}, \chi_a) \)

\[
\begin{align*}
D_a \varphi &= \chi_a, \\
D_a B_{\mu\nu} &= -\frac{1}{4}([\gamma_\mu, \gamma_\nu])_a^b \chi_b, \\
D_a \chi_b &= i (\gamma^\mu)_{ab} \partial_\mu \varphi - (\gamma^5 \gamma^\mu)_{ab} \epsilon_{\mu\rho\sigma\tau} \partial_\rho B_{\sigma\tau}, \\
\end{align*}
\]

are seen to be the same whether on-shell or off-shell. On the basis of (7.1) and without a single calculation one can predict the on-shell bosonic holoraumy must involve an electromagnetic duality rotation being realized as

\[
(\partial_\mu \varphi') = (\partial_\mu \varphi) \cos \Theta - \epsilon_{\mu\lambda\nu}(\partial^\kappa B^{\lambda\nu}) \sin \Theta, \\
(\partial_\mu \varphi') = (\partial_\mu \varphi) \sin \Theta + \epsilon_{\mu\lambda\nu}(\partial^\kappa B^{\lambda\nu}) \cos \Theta .
\]

The precise results of the calculation are

\[
\begin{align*}
[D_a, D_b] \varphi &= -2(\gamma^5 \gamma^\mu)_{ab} \epsilon_{\mu\rho\sigma\tau} \partial_\rho B_{\sigma\tau} , \\
[D_a, D_b] B_{\mu\nu} &= \epsilon_{\mu\alpha\beta}(\gamma^5 \gamma^\alpha)_{ab} \partial^\beta \varphi + (\gamma^5 \gamma_{[\mu})_{ab} \epsilon_{\nu]\rho\sigma\tau} \partial^\rho B^{\sigma\tau} , \\
[D_a, D_b] \chi_c &= -2i \left( \mathcal{F}^{(TS)}_{ab}\right)_c^d \partial_\mu \chi_d - \mathcal{F}^{(TS)}_{abc} \\
\end{align*}
\]

where

\[
\begin{align*}
\left( \mathcal{F}^{(TS)}_{ab}\right)_c^d &= - (\gamma^5 \gamma^\mu)_{ab}(\gamma^5)_c^d , \\
\mathcal{F}^{(TS)}_{abc} &= 2 \left\{ - C_{ab} \delta_c^d + (\gamma^5)_{ab}(\gamma^5)_c^d \right\} \mathcal{D}^{(TS)}_d \\
\mathcal{D}^{(TS)}_d &= i(\gamma^\mu)_d^c \partial_\mu \chi_c
\end{align*}
\]

Neglecting the second term in the holoraumy of \( B_{\mu\nu} \) and comparing Eqs. (7.7) and (7.8) to (7.6) demonstrates that \( \Theta = \pi/2 \) as above. Thus, in the end, the appearance of the electromagnetic duality rotation within the holoraumy in on-shell 4D, \( \mathcal{N} = 1 \) is the final result of parity conservation.

One final observation about the presence of these electromagnetic duality rotations is the role they play in describing the dynamics of 4D, \( \mathcal{N} = 1 \) supersymmetrical systems. The evidence from our investigation here is that these electromagnetic duality rotations are ubiquitously present in all 4D, \( \mathcal{N} = 1 \) supersymmetrical theories.

There are three parts to understanding why this must be so.

(a.) As the example of the tensor supermultiplet shows, the off-shell holoraumy tensors can always be decomposed into one portion that includes the duality rotations and then other terms. Looking closely at the terms in (7.7) and (7.8) on the bosonic fields in the supermultiplet, while the leading term in (7.9) shows the effect of the electromagnetic duality rotations on the fermionic term.
(b.) This general behavior can be seen in all off-shell holoraumy calculations in 4D, $\mathcal{N} = 1$ supermultiplets [2].

(c.) The final portion to seeing the role the duality rotations play emerges from understanding how superspace Lagrangians describe dynamical systems. Every off-shell manifestly 4D, $\mathcal{N} = 1$ supersymmetrical component level Lagrangian $\mathcal{L}_{\text{comp}}$ can be written in the forms

$$\mathcal{L}_{\text{comp}} = [D^a, D^b][D_a, D_b]\mathcal{L}_{SF}$$

(7.13)

where $\mathcal{L}_{SF}$ is a superfield Lagrangian expressed in terms of unconstrained prepotentials. The operator terms $[D^a, D^b]$ acting on a monomial involving a single prepotential superfield will generate a holoraumy containing an electromagnetic duality rotation. In particular, the terms in the purely quadratic portion of any supersymmetrical Lagrangian is determined by the holoraumy of the fields in the supermultiplet. Thus, propagators explicitly depend upon the electromagnetic duality rotations.

The equation in (7.1) is a pristine example of the power of the “SUSY holography” concept [10] realized between the spaces of one dimensional adinkra representations of supersymmetry [11] in quantum mechanical systems and spaces of four dimensional representations of supersymmetry in field theories. In the very first introduction of the concept of 1d, $N = 4$ holoraumy [4], it was pointed out that the commutator of supercharges in such systems leads to the appearance of an operator of the form of a product of a 1d translation times and 1d R-symmetry rotation. The results expressed in (7.1) show that the commutator of supercharges in these systems leads to the appearance of an operator of the form of a product of a translation times an electromagnetic duality rotation. Thus the 1d R-symmetry rotation is the “shadow” of the 4D electromagnetic duality rotation.

“Our knowledge can only be finite, while our ignorance must necessarily be infinite.”

- Karl Popper

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