HOMOGENIZATION OF 2D CAHN-HILLIARD-NAVIER-STOKES SYSTEM

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Abstract. In the current work, we are performing the asymptotic analysis, beyond the periodic setting, of the Cahn-Hilliard-Navier-Stokes system. Under the general deterministic distribution assumption on the microstructures in the domain, we find the limit model equivalent to the heterogeneous one. To this end, we use the sigma-convergence concept which is suitable for the passage to the limit.

1. Introduction

There are numerous natural phenomena that have emphasized the flow of fluids with different scales of behavior: sap, river, concrete. Some contexts involving multiphasic flows in natural or artificial media are the depollution of soils [27], filtering [26], design of composite materials [28, 35] for chemical industry, blood flow or equally the flow of liquid-gases in the energetic cell [5]. Otherwise these flows are observed in media where the microscopic structure is extremely variable. It is therefore very important to understand the multiphasic flows in some media presenting a periodic structure or heterogeneity of size which is much smaller than the dimension of the domain, with different scale of space and time. Thus it is convenient to identify and analyze interfacial processes that happen at the microscopic scale, in order to describe their manifestation at the macroscopic scale.

The phase-field approach is a popular tool for the modeling and simulation of multiphase flow problems, see for instance [2, 20, 25] for an overview.

A typical model for the evolution of a mixture of two incompressible, immiscible and isothermal fluids occupying a domain $Q \subset \mathbb{R}^2$, on the time interval $(0,T)$ consists of a system of Cahn-Hilliard-Navier-Stokes equations

$$\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p + \kappa \phi \nabla \mu &= g \quad \text{in } Q_T = (0,T) \times Q \\
\text{div} u &= 0 \quad \text{in } Q_T \\
\frac{\partial \phi}{\partial t} + (u \cdot \nabla)\phi - \Delta \mu &= 0 \quad \text{in } Q_T \\
\mu &= -\lambda \Delta \phi + \alpha f(\phi) \quad \text{in } Q_T
\end{aligned}$$

\hspace{1cm} (1.1)
where $Q$ is a Lipschitz domain in $\mathbb{R}^2$ and $T$ a given positive real number, $u$, $p$ and $\phi$ are unknown velocity, pressure and the order parameter (which represents the relative concentration of one of the fluids) respectively. Here the constants $\nu > 0$ and $\kappa > 0$ correspond to the kinematic viscosity of the fluid and to the capillarity (stress) coefficient respectively, while $\lambda, \alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, $\lambda$ is related with the thickness of the interface separating the two fluids (see [21] for details) and it is reasonable to assume that $\lambda < \alpha$. The quantity $\mu$ is the variational derivative of the functional

$$\mathcal{F}(\phi) = \int_Q \left( \frac{\lambda}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) ds$$

where $F$ is a homogeneous free energy functional defined by

$$F(r) = \int_0^r f(\upsilon) d\upsilon \text{ for } r \in \mathbb{R}.$$ 

A common choice for $F$ is a quadratic double-well free energy functional

$$F(s) = \frac{1}{4}(s^2 - 1)^2.$$ 

The first two equations in (1.1) are the incompressible Navier-Stokes equations, where the nonlinear term $\phi \nabla \mu$ models the surface tension effects, cf. [23]. The two last equations in (1.1) are Cahn-Hilliard type equations with advection effect modeled by the term $(u \cdot \nabla)\phi$.

The study of almost periodic homogenization for a single phase flow has been done in [19]. For numerical homogenization approaches for single phase Navier-Stokes flow we mention [4, 6, 32]. The multiscale analysis for two-phase flow is less developed as shown by the very few works existing in the literature; see e.g. [22, 39, 3] where the homogenization result of sharp interface models for two-phase flows is performed.

In this work we are concerned with the deterministic homogenization of Cahn-Hilliard-Navier-Stokes system in a fixed bounded open two-dimensional domain. Here, the usual Laplace operator involved in the classical Navier-Stokes equations is replaced by an elliptic linear differential operator of order two, in divergence form and modeling fluids with variable oscillating viscosities. We refer to [34, 10] for the study of the Stokes flow with variable viscosity in thin domains. More precisely, the problem under study is stated in (2.1)-(2.4); the hypotheses on the variable oscillating viscosity are given in (A1). We first study problem (2.1)-(2.4) in the periodic setting. Indeed, up to our knowledge, there is no existing result in the literature. For the Stokes-Cahn-Hilliard equation with fixed small viscosity we refer the reader to [39] and [3] for the formal derivation and the mathematical derivation of the homogenized problem, respectively. Our main result for the periodic case is stated in Theorem 3.4. We notice that the homogenized problem (3.41) is of the same type as the initial one, with still variable but not anymore oscillating viscosity. Once the periodic case is completed, we are in position to extend this result to the more general deterministic homogenization setting, under hypothesis (4.6) for the viscosity. Our approach is based on the sigma-convergence concept; see e.g. [29, 37, 41]. The main result is stated in Theorem 4.3. The results presented in this paper are in particular valid when $Q$ is a finite cylinder. However, it would be very interesting to study the case when $Q$ is an infinite cylinder;
for the study of such kind of problems in infinite cylinders, we refer the reader e.g. to [12, 13, 14, 15, 16, 17].

The work is organized as follows. In Section 2, we state the ε-problem and prove the a priori estimates. Section 3 deals with the systematic study of the homogenization of (2.1)-(2.4) in the periodic setting. Finally in Section 4, we treat the homogenization problem for (2.1)-(2.4) in the more general setting.

2. Setting of the problem and uniform estimates

2.1. Statement of the problem. We start by introducing the functional setup. If $X$ is a real Hilbert space with inner product $(\cdot, \cdot)_X$, then we denote the induced norm by $|\cdot|_X$, while $X^*$ will indicate its dual. Moreover, we indicate by $\mathcal{X}$ the space $X \times X$ endowed with the product structure. Especially, by $\mathbb{H}$ and $\mathcal{V}$ we denote the Hilbert spaces defined as the closure in $L^2(Q) = L^2(Q)^2$ (resp. $H_0^1(Q) = H^1_0(Q)^2$) of the space $\{u \in C_0^\infty(Q) : \text{div}u = 0 \text{ in } Q\}$. The space $\mathbb{H}$ is endowed with the scalar product denoted by $(\cdot, \cdot)$ with the associated norm denoted by $|\cdot|$. The space $\mathcal{V}$ is equipped with the scalar product

$$(u, v) = \sum_{i=1}^{2} \left( \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)$$

whose associated norm is the norm of the gradient and is denoted by $\|\cdot\|$. Owing to the Poincaré’s inequality, the norm in $\mathcal{V}$ is equivalent to the $H^1$-norm. We also define the space $L^2_0(Q) = \{v \in L^2(Q) : \int_Q v dx = 0\}$. We refer the reader to [21, Section 2] for more details on these spaces.

This being so, our aim is to study the asymptotic behavior, as $\varepsilon \to 0$, of the solution of the system (2.1)-(2.4) below:

\[
\begin{align*}
\frac{\partial u_\varepsilon}{\partial t} - \text{div}(A_0^\varepsilon \nabla u_\varepsilon) + (u_\varepsilon \cdot \nabla)u_\varepsilon + \nabla p_\varepsilon - \kappa \mu_\varepsilon \nabla \phi_\varepsilon &= g \text{ in } Q_T, \\
\text{div}u_\varepsilon &= 0 \text{ in } Q_T \\
\frac{\partial \phi_\varepsilon}{\partial t} + u_\varepsilon \cdot \nabla \phi_\varepsilon - \Delta \mu_\varepsilon &= 0 \text{ in } Q_T, \\
\mu_\varepsilon &= -\lambda \Delta \phi_\varepsilon + \alpha f(\phi_\varepsilon) \text{ in } Q_T
\end{align*}
\]

where $Q$ is a fixed Lipschitz bounded domain in $\mathbb{R}^2$ and $T$ a given positive real number. Here $u_\varepsilon$, $p_\varepsilon$ and $\phi_\varepsilon$ are microscopic unknowns velocity, pressure and the order parameter respectively. In (2.1)-(2.4) $\nabla$ (resp. div) stands for the gradient (resp. divergence) operator in $Q$ and where the functions $A_0^\varepsilon$ and $g$ are constrained as follows:

(A1) Uniform ellipticity. The oscillating viscosity $A_0^\varepsilon$ is defined by $A_0^\varepsilon(t, x, t^\varepsilon, x^\varepsilon) = A_0(t, x, \frac{t^\varepsilon}{\varepsilon}, \frac{x^\varepsilon}{\varepsilon})$ for $(t, x) \in Q_T$ where $A_0 \in C(\overline{Q}_T; L^\infty(\mathbb{R}_r^3)^{2\times2})$ is a symmetric matrix satisfying

$$(\gamma |\xi|^2 \leq A_0 \xi \cdot \xi \leq \gamma^{-1} |\xi|^2) \text{ for all } \xi \in \mathbb{R}^2 \text{ and a.e. in } Q_T \times \mathbb{R}_r^3,$$

where $\gamma > 0$ a given constant independent of $x, t, y, \tau, \xi$. 
The function $g$ lies in $L^2(0,T;\mathbb{H}^{-1}(Q))$, and $f \in C^2(\mathbb{R})$ satisfies
\[ \liminf_{|r| \to \infty} f'(r) > 0 \quad \text{and} \quad |f''(r)| \leq c_f(1 + |r|^{k-1}) \quad \forall r \in \mathbb{R} \]  
where $c_f$ is some positive constant and $1 \leq k \leq \mathbb{R}$ is fixed.

It follows from \((2.5)\) that
\[ |f'(r)| \leq c_f(1 + |r|^k) \quad \text{and} \quad |f(r)| \leq c_f(1 + |r|^{k+1}) \quad \forall r \in \mathbb{R} \]  
(2.6)

The quantity $\mu_\varepsilon$ is the variational derivative of the functional
\[ \mathcal{F}(\phi_\varepsilon) = \int_Q \left( \frac{\lambda}{2} |\nabla \phi_\varepsilon|^2 + \alpha F(\phi_\varepsilon) \right) ds \]  
where
\[ F(r) = \int_0^r f(v)dv \quad \text{for} \quad r \in \mathbb{R}. \]  
(2.7)

Regarding the boundary conditions for this model, as in \cite{21}, we assume that the boundary conditions for $\phi_\varepsilon$ and $\mu_\varepsilon$ are the natural no-flux condition
\[ \frac{\partial \phi_\varepsilon}{\partial \nu} = \frac{\partial \mu_\varepsilon}{\partial \nu} = 0 \quad \text{on} \quad \partial Q \]  
(2.8)

where $\nu$ is the outward normal vector to $\partial Q$. These conditions ensure the mass conservation of the following quantity
\[ \langle \phi_\varepsilon(t) \rangle = \int_Q \phi_\varepsilon(t,x)dx \]  
(2.9)

where $\int_Q = \frac{1}{|Q|} \int_Q$ and $|Q|$ stands for the Lebesgue measure of $Q$. More precisely, we have
\[ \langle \phi_\varepsilon(t) \rangle = \langle \phi_\varepsilon(0) \rangle \quad \forall t > 0. \]

Concerning the boundary condition for $u_\varepsilon$, we assume the Dirichlet (no-slip) boundary condition
\[ u_\varepsilon = 0 \quad \text{on} \quad (0,T) \times \partial Q. \]  
(2.10)

The initial condition is given by
\[ (u_\varepsilon, \phi_\varepsilon)(0) = (u_0^*, \phi_0^*) \]  
(2.11)

where
\[ (A3) \quad u_0^* \in \mathbb{H}, \quad \phi_0^* \in H^1(Q). \]

Here above in \((A1)\) and henceforth, the numerical space $\mathbb{R}^3_{\tau,y}$ stands for the product space $\mathbb{R}_\tau \times \mathbb{R}^2_y$, where $\mathbb{R}^d$ denotes the space $\mathbb{R}^d$ of variable $\zeta$. We shall need the following bilinear operator $B_0$ (and its related trilinear form $b_0$)
\[ (B_0(u,v),w) = \int_Q [(u \cdot \nabla)v] \cdot wdx = b_0(u,v,w) \quad \forall u,v,w \in \mathbb{V}. \]
Remark 2.1. The operator defined above enjoys continuity properties which depend on the space dimension (cf., e.g., [36, Chap. 9]):

\[ b_0(\mathbf{u}, \mathbf{w}, \mathbf{v}) = -b_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{H}; \]

\[ |b_0(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq c|\mathbf{u}|^{1/2} |\nabla \mathbf{u}|^{1/2} |\nabla \mathbf{v}|^{1/2} |\mathbf{w}|^{1/2}. \tag{2.12} \]

Definition 2.1. Let \( \mathbf{u}_0^\varepsilon \in \mathbb{H}, \phi_0^\varepsilon \in L^2(Q), \) with \( F(\phi_0^\varepsilon) \in L^1(Q) \) and \( 0 < T < +\infty \) be given. Then the triplet \( (\mathbf{u}_\varepsilon, \phi_\varepsilon, \mu_\varepsilon)_{\varepsilon > 0} \) is a weak solution to the problem (2.1)-(2.4) if

\[ \mathbf{u}_\varepsilon \in L^\infty(0; T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \quad \mathbf{\partial}_t \mathbf{u}_\varepsilon \in L^2(0, T; \mathbb{V}^*) \]

\[ \phi_\varepsilon \in L^\infty(0, T; H^1(Q)) \cap L^\infty(0, T; L^4(Q)) \quad \mathbf{\partial}_t \phi_\varepsilon \in L^2(0, T; (H^1(Q))^*) \]

\[ \mu_\varepsilon \in L^2(0, T; H^1(Q)) \]

and for all \( \psi \in L^2(0, T; \mathbb{V}) \) and \( \varphi, \chi \in L^2(0, T; H^1(Q)) \),

\[ \int_0^T \left( \frac{\partial \mathbf{u}_\varepsilon}{\partial t}, \psi \right) dt + \int_{Q_T} A_0^\varepsilon \nabla \mathbf{u}_\varepsilon \cdot \nabla \psi dx \]

\[ + \int_{Q_T} (\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon \psi dx dt - \kappa \int_{Q_T} \mu_\varepsilon \nabla \phi_\varepsilon \psi dx dt = \int_{Q_T} g \psi dx dt, \tag{2.13} \]

\[ \int_0^T \left( \frac{\partial \phi_\varepsilon}{\partial t}, \varphi \right) dt - \int_{Q_T} \mathbf{u}_\varepsilon \phi_\varepsilon \cdot \nabla \varphi dx dt + \int_{Q_T} \nabla \mu_\varepsilon \cdot \nabla \varphi dx dt = 0, \tag{2.14} \]

\[ \int_{Q_T} \mu_\varepsilon \chi dx dt = \lambda \int_{Q_T} \nabla \phi_\varepsilon \cdot \nabla \chi dx dt + \alpha \int_{Q_T} f(\phi_\varepsilon) \chi dx dt. \tag{2.15} \]

Furthermore, with each weak solution \( (\mathbf{u}_\varepsilon, \phi_\varepsilon, \mu_\varepsilon) \), we associate a pressure \( p_\varepsilon \in L^\infty(0, T; L^2_0(Q)) \) which satisfies (2.1) in the distributional sense.

The existence of a weak solution in the sense of Definition 2.1 has been extensively studied by many authors see e.g. [1, 8, 20]. Theorem 2.1 below can be proved as its homologue in [8, Theorem 1] (see also [1, 18, 20]).

Theorem 2.1. Under assumptions (A1)-(A3), there exists (for each fixed \( \varepsilon > 0 \)) a unique weak solution \( (\mathbf{u}_\varepsilon, \phi_\varepsilon, \mu_\varepsilon) \), to the problem (2.1)-(2.4) in the sense of Definition 2.1. Furthermore, there exists a unique \( p_\varepsilon \in L^\infty(0, T; L^2_0(Q)) \) such that (2.1) is satisfied in the distributional sense.

Proof. Using assumptions (A1)-(A3), the method used in [1, 8] provides us with the existence of a unique weak solution \( (\mathbf{u}_\varepsilon, \phi_\varepsilon, \mu_\varepsilon) \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \times L^\infty(0, T; H^1(Q)) \cap L^\infty(0, T; L^4(Q)) \times L^2(0, T; H^1(Q)) \). Indeed, although we have variable viscosity in our assumptions, the proof follows the same way of reasoning like the one in [8, Theorem in Subsection 3.2] by relying on the ellipticity of the operator \(-\text{div}(A_0^\varepsilon \nabla)\). For the existence
of the pressure, since \( g \in L^2(0, T; H^{-1}(Q)) \) the necessary condition of \([40, \text{Section 4}]\) for the existence of the pressure is satisfied. Coming back to \((2.1)\) let denoted by:

\[
\text{h}_\varepsilon = g - \frac{\partial u_\varepsilon}{\partial t} + \text{div}(A_0^\varepsilon \nabla u_\varepsilon) - (u_\varepsilon \cdot \nabla)u_\varepsilon + \kappa \mu_\varepsilon \nabla \phi_\varepsilon,
\]

and \( \langle \text{h}_\varepsilon, v \rangle = 0 \) for every \( v \in C_0^\infty(Q)^2 \) with \( \text{div} v = 0 \) where \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( \mathcal{D}'(Q)^2 \) and \( \mathcal{D}(Q)^2 \). Arguing as in the proof of \([40, \text{Proposition 5}]\) we are led to \( \text{h}_\varepsilon \in L^2(0, T; H^{-1}(Q)^2) \) so that there exists a unique \( p_\varepsilon \in L^2(0, T; L^2(Q)) \) such that \( \nabla p_\varepsilon = \text{h}_\varepsilon \) and \( \int_Q p_\varepsilon \, dx = 0 \).

### 2.2. A priori estimates

We now derive some a priori estimates to show that the sequences \( (u_\varepsilon, \phi_\varepsilon, \mu_\varepsilon, p_\varepsilon)_\varepsilon \) are bounded independently of \( \varepsilon \) in suitable function spaces.

**Lemma 2.1.** Suppose that \( (u_\varepsilon, \phi_\varepsilon, \mu_\varepsilon) \) is a smooth solution of \((2.1)-(2.4)\). Then the following dissipative energy equality holds:

\[
\frac{d}{dt} \left[ \frac{1}{2\kappa} \int_Q |u_\varepsilon(t, x)|^2 \, dx + \frac{\lambda}{2} \int_Q |\nabla \phi_\varepsilon(t, x)|^2 \, dx + \alpha \int_Q F(\phi_\varepsilon(t, x)) \, dx \right] + \frac{1}{\kappa} (\text{A}_0^\varepsilon \nabla u_\varepsilon(t), \nabla u_\varepsilon(t)) - \frac{1}{\kappa} (u_\varepsilon(t), g(t)) + \int_Q |\nabla \mu_\varepsilon(t, x)|^2 \, dx = 0
\]

where \( u_\varepsilon(t) = u_\varepsilon(t, \cdot) \) and \( g(t) = g(t, \cdot) \).

**Proof.** By taking the scalar product in \( \mathcal{H} \) of equation \((2.1)\) with \( u_\varepsilon \), and using \((2.10)\), we obtain

\[
\frac{1}{2\kappa} \frac{d}{dt} \int_Q |u_\varepsilon(t, x)|^2 \, dx - \int_Q \mu_\varepsilon(t, x) \nabla \phi_\varepsilon(t) \cdot u_\varepsilon(t, x) \, dx + \frac{1}{\kappa} (\text{A}_0^\varepsilon \nabla u_\varepsilon(t), \nabla u_\varepsilon(t)) - \frac{1}{\kappa} (u_\varepsilon(t), g(t)) = 0 \quad (2.17)
\]

Next, taking the scalar product in \( L^2(Q) \) of equation \((2.3)\) with \( \mu_\varepsilon \), we get

\[
\frac{d}{dt} \left[ \int_Q \frac{\lambda}{2} |\nabla \phi_\varepsilon(t, x)|^2 \, dx + \alpha \int_Q F(\phi_\varepsilon(t, x)) \, dx \right] + \int_Q |\nabla \mu_\varepsilon(t, x)|^2 \, dx + \int_Q \mu_\varepsilon(t, x) \nabla \phi_\varepsilon(t, x) \cdot u_\varepsilon(t, x) \, dx = 0 \quad (2.18)
\]

Summing up equations \((2.17)\) and \((2.18)\) gives the result. \(\square\)

It is also worth mentioning that \((2.16)\) is a consequence of the orthogonality properties of the products below, which will also be employed in the sequel, namely,

\[
(B_0(u, v), v) = 0 \quad \forall u, v \in \mathcal{V} \quad (2.19)
\]

**Lemma 2.2.** Under the assumptions \((A1)-(A3)\), the weak solution \((u_\varepsilon, \phi_\varepsilon, \mu_\varepsilon)\) of \((2.1)-(2.4)\) in the sense of Definition \(2.1\) satisfies the following estimate:

\[
\frac{1}{2\kappa} \|u_\varepsilon(t)\|^2_{L^2(Q)^2} + \frac{\gamma}{\kappa} \|u_\varepsilon\|^2_{L^2[0, T; \mathcal{V}]} + \frac{\lambda}{2} \|\nabla \phi_\varepsilon\|^2_{L^2[0, T; L^2(Q)^2]} + \|\nabla \mu_\varepsilon\|^2_{L^2[0, T; L^2(Q)^2]} \leq C + C \int_0^t \|u_\varepsilon(s)\|^2_{L^2(Q)^2} \, ds. \quad (2.20)
\]
We then have the estimates
\[
\|\phi_\varepsilon\|_{L^\infty(0,T;H^1(Q))} \leq C, \quad (2.21)
\]
\[
\|u_\varepsilon\|_{L^\infty(0,T;H^1) \cap L^2(0,T;V)} \leq C, \quad (2.22)
\]
\[
\|\partial_t u_\varepsilon\|_{L^2(0,T;V')} \leq C, \quad (2.23)
\]
\[
\|\mu_\varepsilon\|_{L^2(0,T;H^1(Q))} \leq C, \quad (2.24)
\]
\[
\|\partial_t \phi_\varepsilon\|_{L^2(0,T;H^1(Q)')} \leq C, \quad (2.25)
\]
\[
\|f(\phi_\varepsilon)\|_{L^2(Q_T)} \leq C \quad (2.26)
\]
where the positive constant $C$ does not depend on $\varepsilon$.

Proof. If $(u_\varepsilon, \phi_\varepsilon, \mu_\varepsilon)$ is a weak smooth solution of (2.1)-(2.4), it verifies the dissipative equality (2.16) so that, using the assumption (A3), we obtain (after integrating (2.16) over $(0, t)$) and using the following inequality
\[
(u_\varepsilon(t), g(t)) \leq (4\delta)^{-1} \|g(t)\|_{H^{-1}(Q)}^2 + \frac{\delta}{2} \|u_\varepsilon(t)\|^2,
\]
the following inequality
\[
\frac{1}{2\kappa} |u_\varepsilon(t)|^2 + \frac{\lambda}{2} |\nabla \phi_\varepsilon(t)|^2 + \alpha \int_Q F(\phi_\varepsilon(t)) dx + \int_0^t \frac{\gamma}{\kappa} |\nabla u_\varepsilon(s)|^2 + |\nabla \mu_\varepsilon(s)|^2 ds
\]
\[
\leq \frac{1}{\kappa}(4\delta)^{-1} \int_0^t \|g(s)\|_{H^{-1}(Q)}^2 ds + \delta \int_0^t |u_\varepsilon(s)|^2 ds + \frac{1}{2\kappa} \|u_\varepsilon(0)\|^2 + \frac{\lambda}{2} \|\nabla \phi_\varepsilon(0)\|^2 + \int_Q F(\phi_\varepsilon(0)) dx.
\]
It follows that
\[
\frac{1}{2\kappa} |u_\varepsilon(t)|^2 + \frac{\lambda}{2} |\nabla \phi_\varepsilon(t)|^2 + \alpha \int_Q F(\phi_\varepsilon(t)) dx + \frac{\gamma}{\kappa} \|u_\varepsilon\|_{L^2(0,T;V)} + \|\nabla \mu_\varepsilon\|_{L^2(0,T;L^2(Q))}
\]
\[
\leq C_0(\|u_0^\varepsilon\|_H, \|\phi_0^\varepsilon\|_{H^1(Q)}, \alpha, \delta, \kappa, \lambda, \|g\|_{L^2(0,T;H^{-1}(Q))}^2) + \delta \int_0^t |u_\varepsilon(s)|^2 ds.
\]
(2.27)
So by (2.27) we get,
\[
|u_\varepsilon(t)|^2 \leq C + C \int_0^t |u_\varepsilon(s)|^2 ds
\]
and, using Gronwall’s inequality,
\[
\|u_\varepsilon(t)\|_{L^2(Q)} \leq C \text{ for all } 0 \leq t \leq T, \varepsilon > 0.
\]
Whence
\[
\|u_\varepsilon\|_{L^\infty(0,T;H)} \leq C. \quad (2.28)
\]
Moreover, using (2.27) and (2.28) it holds that
\[
\|u_\varepsilon\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \leq C, \quad (2.29)
\]
\[
\|\nabla \phi_\varepsilon\|_{L^\infty(0,T;L^2(Q))} \leq C, \quad (2.30)
\]
\[
\|\nabla \mu_\varepsilon\|_{L^2(0,T;L^2(Q))} \leq C, \quad (2.31)
\]
From the mass conservation \( \int_Q \phi_\varepsilon(t) \, dx = \int_Q \phi_\varepsilon(0) \, dx = \int_Q \phi_0 \, dx \) (where \( f_Q = |Q|^{-1} \int_Q \)) we have that \( \left| \int_Q \phi_\varepsilon(t) \, dx \right| \leq C \). Next, using Poincaré-Wirtinger inequality, we obtain
\[
\left( \int_Q |\phi_\varepsilon(t)|^2 \, dx \right)^{\frac{1}{2}} \leq C \| \nabla \phi_\varepsilon(t) \|_{L^2(Q)} + C \left| \int_Q \phi_\varepsilon(t) \, dx \right| \leq C.
\]
This leads to
\[
\| \phi_\varepsilon \|_{L^\infty(0,T;H^1(Q))} \leq C. \tag{2.32}
\]
From (2.13) we obtain for all \( \psi \in V \),
\[
\left| \frac{\partial u_\varepsilon}{\partial t}, \psi \right| \leq C \| \nabla u_\varepsilon \|_{L^2(Q)} \| \nabla \psi \|_{L^2(Q)} + \| u_\varepsilon \|_{L^4(Q)} \| \nabla u_\varepsilon \|_{L^2(Q)} \| \psi \|_{L^4(Q)^2} + \kappa \| \mu_\varepsilon \|_{\dot{H}^1_0(Q)} \| \nabla \phi_\varepsilon \|_{L^2(Q)} \| \psi \|_{L^4(Q)} + \| g \|_{V'} \| \psi \|_{L^2(Q)} \\
\leq C \| \nabla u_\varepsilon \|_{L^2(Q)} \| \psi \|_{H^1_0(Q)} + \| u_\varepsilon \|_{L^4(Q)} \| \nabla u_\varepsilon \|_{L^2(Q)} \| \psi \|_{H^1_0(Q)} + \kappa \| \mu_\varepsilon \|_{H^1(Q)} \| \nabla \phi_\varepsilon \|_{L^2(Q)} \| \psi \|_{H^1_0(Q)} + \| g \|_{V'} \| \psi \|_{L^2(Q)},
\]
where above we have used the continuous embedding \( H^1(Q) \hookrightarrow L^4(Q) \). Thus,
\[
\sup_{\| \psi \|_V \leq 1} \left| \frac{\partial u_\varepsilon}{\partial t}, \psi \right| \leq C \| \nabla u_\varepsilon \|_{L^2(Q)} + \| u_\varepsilon \|_{H^1_0(Q)} \| \nabla u_\varepsilon \|_{L^2(Q)} + \kappa \| \mu_\varepsilon \|_{H^1(Q)} \| \nabla \phi_\varepsilon \|_{L^2(Q)} + \| g \|_{V'}.
\]
We integrate the square of \( \sup_{\| \psi \|_V \leq 1} \left| \frac{\partial u_\varepsilon}{\partial t}, \psi \right| \) on \((0,T)\) and get by (2.29)-(2.31) the bound
\[
\left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^2(0,T;V')} \leq C.
\]
Using Cauchy-Schwarz’s inequality we get from (2.14) that
\[
\left| \frac{\partial \phi_\varepsilon}{\partial t}, \varphi \right| \leq C \| u_\varepsilon(t) \|_{L^4(Q)} \| \phi_\varepsilon(t) \|_{L^4(Q)} \| \nabla \varphi \|_{L^2(Q)} + \| \nabla \mu_\varepsilon(t) \|_{L^2(Q)} \| \nabla \varphi \|_{L^2(Q)} \\
\leq C \| u_\varepsilon(t) \|_{L^4(Q)} \| \phi_\varepsilon(t) \|_{L^4(Q)} \| \nabla \varphi \|_{H^1(Q)} + \| \nabla \mu_\varepsilon(t) \|_{L^2(Q)} \| \varphi \|_{H^1(Q)} \\
\leq C \| u_\varepsilon(t) \|_{H^1(Q)} \| \phi_\varepsilon(t) \|_{H^1(Q)} \| \nabla \varphi \|_{H^1(Q)} + \| \nabla \mu_\varepsilon(t) \|_{L^2(Q)} \| \varphi \|_{H^1(Q)}.
\]
Thus,
\[
\sup_{\| \varphi \|_{H^1(Q)} \leq 1} \left| \frac{\partial \phi_\varepsilon}{\partial t}, \varphi \right| \leq C \| u_\varepsilon(t) \|_{H^1(Q)} \| \phi_\varepsilon(t) \|_{H^1(Q)} + \| \nabla \mu_\varepsilon(t) \|_{L^2(Q)}.
\]
We integrate the square of \( \sup_{\| \varphi \|_{H^1(Q)} \leq 1} \left| \frac{\partial \phi_\varepsilon}{\partial t}, \varphi \right| \) on \((0,T)\) and obtain by (2.29)-(2.31) the estimate
\[
\left\| \frac{\partial \phi_\varepsilon}{\partial t} \right\|_{L^2(0,T;H^1(Q))} \leq C. \tag{2.33}
\]
The next step is to deduce an estimate for the sequence $\mu_\varepsilon$ in $L^2(0,T;H^1(Q))$. To this aim we first observe that $(-\Delta \phi_\varepsilon, 1) = 0$ and using (2.6) we have
\[
\left| \int_Q \mu_\varepsilon dx \right| = |(\mu_\varepsilon, 1)| = |(\alpha f(\phi_\varepsilon), 1)| \leq \alpha \int_Q |f(\phi_\varepsilon)| dx \leq C \int_Q (1 + |\phi_\varepsilon|^{k+1}) dx \tag{2.34}
\]
where in (2.34) we have used the inequality $f(r) \leq C(1 + |r|^{k+1})$. Now, since $k + 1 \geq 2$, we deduce from the Sobolev embedding $H^1(Q) \hookrightarrow L^{k+1}(Q)$ that there is a positive constant $C$ such that
\[
\left( \int_Q |\phi_\varepsilon|^{k+1} dx \right)^{\frac{1}{k+1}} \leq C \|\phi_\varepsilon\|_{H^1(Q)}.
\]
Whence in view of (2.34),
\[
\left| \int_Q \mu_\varepsilon dx \right| \leq C. \tag{2.35}
\]
Applying Poincaré-Wirtinger’s inequality we obtain
\[
\int_Q \left| \mu_\varepsilon - \int_Q \mu_\varepsilon \right|^2 dx \leq C \int_Q |\nabla \mu_\varepsilon|^2 dx
\]
where $C$ is independent of $\varepsilon$. It follows that
\[
\int_Q |\mu_\varepsilon|^2 dx \leq C \left[ \int_Q |\nabla \mu_\varepsilon|^2 dx + \left| \int_Q \mu_\varepsilon dx \right|^2 \right],
\]
from which, using (2.31) and (2.35), we obtain
\[
\|\mu_\varepsilon\|_{L^2(Q)} \leq C. \tag{2.36}
\]
Integrating (2.36) over $(0,T)$ and using (2.31) we obtain
\[
\|\mu_\varepsilon\|_{L^2(0,T;H^1(Q))} \leq C.
\]
This completes the proof. \hfill \square

We recall the following properties of the Bogovskii operator which will be used to estimate the pressure $p_\varepsilon$.

**Lemma 2.3** ([33 Lemma 3.17, p. 169]). Let $1 < p < \infty$. There exists a linear operator $\mathfrak{B} : L^p_0(Q) \to W^{1,p}_0(Q)^N$ with the following properties
\begin{enumerate}[(i)]
    
    \item $\text{div}\mathfrak{B}(f) = f$ a.e in $Q$ for any $f \in L^p_0(Q)$,
    \item $\|\mathfrak{B}(f)\|_{W^{1,p}_0(Q)^N} \leq c(p,Q) \|f\|_{L^p(Q)}$,
    \item If $f = \text{div}g$ with $g \in L^r(Q)^N$ and $g \cdot \nu = 0$ on $\partial Q$ for some $1 < r < \infty$, where $\nu$ is the outward unit vector normal to $\partial Q$, then $\|\mathfrak{B}(f)\|_{L^r(Q)^N} \leq c(r,Q) \|g\|_{L^r(Q)^N}$,
    \item If $f \in C_0^\infty(Q) \cap L^p_0(Q)$, then $\mathfrak{B}(f) \in C_0^\infty(Q)^N$.
\end{enumerate}

**Lemma 2.4.** Assume that Lemma 2.2 and hypotheses (A1)-(A3) are satisfied, then $p_\varepsilon \in L^2(0,T;L^2_0(Q))$ and the following estimate holds
\[
\sup_{\varepsilon > 0} \|p_\varepsilon\|_{L^2(0,T;L^2_0(Q))} \leq C
\]
for some positive constant $C$ independent of $\varepsilon$.

**Proof.** First, we know that $p_\varepsilon \in \mathcal{D}'(Q_T)$. Let $h \in C_0^\infty(Q_T) \cap L^2(0,T;L^2_0(Q))$ (recall that the dual of $L^2_0(Q)$ is $L^2_0(Q)$). By [parts (i) and (iv) of] Lemma 2.3, let $S \in C^\infty_0(Q)$ be such that $\text{div} S = h$. Then

$$\langle \nabla p_\varepsilon, S \rangle = - \langle p_\varepsilon, \text{div} S \rangle = - \langle p_\varepsilon, h \rangle$$

i.e.

$$\langle p_\varepsilon, h \rangle = - \langle \nabla p_\varepsilon, S \rangle = - \langle Z_\varepsilon, S \rangle.$$ 

In view of (2.1) where

$$\partial_t u_\varepsilon - \text{div}(A_0^\varepsilon \nabla u_\varepsilon) + (u_\varepsilon \cdot \nabla)u_\varepsilon + \nabla p_\varepsilon - \kappa \mu_\varepsilon \nabla \phi_\varepsilon = g$$

$$Z_\varepsilon = g - \partial_t u_\varepsilon + \text{div}(A_0^\varepsilon \nabla u_\varepsilon) - (u_\varepsilon \cdot \nabla)u_\varepsilon + \kappa \mu_\varepsilon \nabla \phi_\varepsilon.$$

But by Lemma 2.2

$$|\langle p_\varepsilon, h \rangle| \leq C(\|S\|_{L^2(0,T;H^1_0(Q))} + \|S\|_{L^2(Q_T)^2})$$

$$\leq C \|S\|_{L^2(0,T;H^1_0(Q))}$$

$$\leq C \|h\|_{L^2(Q_T)^2}$$

where the last inequality is valid owing to [part (ii) of] Lemma 2.3. By the above inequality, $p_\varepsilon \in L^2(Q_T)$ i.e., $p_\varepsilon \in L^2(0,T;L^2_0(Q))$ with $\|p_\varepsilon\|_{L^2(Q_T)} \leq C$ for a constant $C > 0$ independent of $\varepsilon$. □

Thus modulo extracting of subsequence (keeping the same notation) we have:

$$u_\varepsilon \rightharpoonup u_0 \text{ in } L^\infty(0,T;\mathbb{H})\text{-weak*} \quad (2.37)$$

$$u_\varepsilon \rightarrow u_0 \text{ in } L^2(0,T;\mathbb{V})\text{-weak} \quad (2.38)$$

$$u_\varepsilon \rightarrow u \text{ in } L^2(0,T;\mathbb{H})\text{-strong} \quad (2.39)$$

$$\phi_\varepsilon \rightharpoonup \phi_0 \text{ in } L^\infty(0,T;H^1(Q))\text{-weak*} \quad (2.40)$$

$$\phi_\varepsilon \rightarrow \phi_0 \text{ in } L^2(0,T;L^2(Q))\text{-strong} \quad (2.41)$$

$$p_\varepsilon \rightarrow p_0 \text{ in } L^2(Q_T)\text{-weak.} \quad (2.42)$$

### 3. Homogenization results: the periodic setting

We assume once for all that $N = 2$. We set $\mathcal{Y} = (0,1)^N$ ($\mathcal{Y}$ considered as a subset of $\mathbb{R}^N_y$, the numerical space $\mathbb{R}^N$ of variables $y = (y_1,\ldots,y_N)$ and $\mathcal{T} = (0,1)$ ($\mathcal{T}$ considered as a subset of $\mathbb{R}_\tau$). Our purpose is to study the homogenization problem associated to (2.1)-(2.4) under periodicity hypothesis on $A_0(t,x,\ldots)$, i.e., under the assumption that

$$A_0(t,x,\ldots) \text{ is } Z\text{-periodic, where } Z = \mathcal{T} \times \mathcal{Y},$$

(3.1)

that is, $A_0(t,x,\tau + \ell,y + k) = A_0(t,x,\tau,y)$ for a.e. $(t,x,\tau,y) \in Q_T \times \mathbb{R} \times \mathbb{R}^2$ and all $(\ell,k) \in Z \times Z^2$. 

...
3.1. Preliminaries. Let us first recall that a function \( u \in L^1_{\text{loc}}(\mathbb{R}^{1+N}) \) is said to be \( \mathbb{Z} \)-periodic if for each \((l, k) \in \mathbb{Z}^{1+N} \) (\( \mathbb{Z} \) denotes the integers), we have \( u(\tau + l, y + k) = u(\tau, y) \) almost everywhere (a.e.) in \((\tau, y) \in \mathbb{R}^{1+N} \). The space of all \( \mathbb{Z} \)-periodic continuous complex functions on \( \mathbb{R}^{1+N} \) is denoted by \( \mathcal{C}_{\text{per}}(\mathbb{T} \times Y) \), and that of all \( \mathbb{Z} \)-periodic functions in \( L^p_{\text{loc}}(\mathbb{R}^{1+N}) \) (\( 1 \leq p \leq \infty \)) is denoted by \( L^p_{\text{per}}(\mathbb{T} \times Y) \). In the sequel, we need the space \( H^1_\#(Y) \) of \( Y \)-periodic functions \( u \in H^1_{\text{loc}}(\mathbb{R}^N_Y) \) such that \( \int_Y u(y)dy = 0 \). Equipped with the gradient norm,

\[
\|u\|_{H^1_\#(Y)} = \left( \int_Y |\nabla_y u(y)|^2dy \right)^{1/2} \quad (u \in H^1_\#(Y)),
\]

\( H^1_\#(Y) \) is a Hilbert space. Throughout the rest of the work, the letter \( E \) will denote any ordinary sequence \((\varepsilon_n)_{n \in \mathbb{N}} \) with \( 0 < \varepsilon_n \leq 1 \) and \( \varepsilon_n \to 0 \) as \( n \to \infty \).

**Definition 3.1.** 1) A sequence \((u_\varepsilon)_{\varepsilon > 0} \subset L^p(Q_T)\) (\( 1 \leq p < \infty \)) is said to weakly two-scale converge in \( L^p(Q_T) \) to a limit \( u_0 \in L^p(Q_T; L^p_{\text{per}}(\mathbb{T} \times Y)) \) if as \( \varepsilon \to 0 \),

\[
\int_{Q_T} u_\varepsilon(t, x)\varphi(t, x)dxdt \to \int_{Q_T} u_0(t, x, \tau, y)\varphi(t, x, \tau, y)d\tau dy dt dx \quad (3.2)
\]

for all \( \varphi \in L^p_{\text{per}}(Q_T; \mathcal{C}_{\text{per}}(\mathbb{T} \times Y)) \) \((\frac{1}{p} = 1 - \frac{1}{p})\), where \( \psi^\varepsilon(t, x) = \psi(t, x, t/\varepsilon, x/\varepsilon) \) \(((t, x) \in Q_T)\). We denote this by \( u_\varepsilon \rightharpoonup u_0 \) in \( L^p(Q_T) \)-weak 2s.

2) The sequence \((u_\varepsilon)_{\varepsilon > 0} \subset L^p(Q_T)\) is said to strongly two-scale converge in \( L^p(Q_T) \) to some \( u_0 \in L^p(Q_T; L^p_{\text{per}}(\mathbb{T} \times Y)) \) if it is weakly two-scale convergent and further \( \|u_\varepsilon\|_{L^p(Q_T)} \to \|u_0\|_{L^p(Q_T \times T \times Y)} \). We denote this by \( u_\varepsilon \to u_0 \) in \( L^p(Q_T) \)-strong 2s.

We recall below some fundamental results that constitute the corner stone of the two-scale convergence concept; see e.g. [30] for the justification.

**Theorem 3.1.** Assume that \( 1 < p < \infty \) and further \( E \) is a fundamental sequence. Let a sequence \((u_\varepsilon)_{\varepsilon \in E} \) be bounded in \( L^p(Q_T) \). Then a subsequence \( E' \) can be extracted from \( E \) and there exist \( u \in L^p(Q_T \times T \times Y) \) such that \((u_\varepsilon)_{\varepsilon \in E'} \) two-scale converges to \( u \).

**Theorem 3.2.** Let \( E \) be a fundamental sequence. Suppose a sequence \((u_\varepsilon)_{\varepsilon \in E} \) is bounded in \( L^2(0, T; H^1(Q)) \). Then a subsequence \( E' \) can be extracted from \( E \) such that, as \( E' \ni \varepsilon \to 0 \),

\[
u_\varepsilon \to u_0 \text{ in } L^2(Q_T)-\text{weak 2s}
\]

\[
\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \quad \text{in } L^2(Q_T)\text{-weak 2s} \quad (1 \leq j \leq N),
\]

where \( u_0 \in L^2(0, T; L^2(T; H^1(Q))) \) with \( \tilde{u}_0 = \int_T u_0 d\tau \) and \( u_1 \in L^2(Q_T; L^2(T; H^1_\#(Y))) \). If further \((\frac{\partial u}{\partial y})_{\varepsilon \in E} \) is bounded in \( L^2(0, T; (H^1(Q))^*) \) then \( u_0 \) is independent of \( \tau \), that is, \( u_0 \in L^2(0, T; H^1(Q)) \).

**Theorem 3.3.** Let \( 1 < p, q < \infty \) and \( r \geq 1 \) be such that \( 1/r = 1/p + 1/q \leq 1 \). Assume \((u_\varepsilon)_{\varepsilon \in E} \subset L^q(Q_T)\) is weakly two-scale convergent in \( L^q(Q_T) \) to some \( u_0 \in L^q(Q_T; L^p_{\text{per}}(T \times Y)) \) and \((v_\varepsilon)_{\varepsilon \in E} \subset L^p(Q_T)\) is strongly two-scale convergent in \( L^p(Q_T) \) to some \( v_0 \in L^p(Q_T; L^p_{\text{per}}(T \times Y)) \).
Then the sequence \((u_\varepsilon v_\varepsilon)_{\varepsilon \in E}\) is weakly two-scale convergent in \(L^r(Q_T)\) to \(u_0v_0 \in L^r(Q_T; L^r(\mathcal{T} \times \mathcal{Y}))\).

### 3.2. Homogenization results

Here we assume that the coefficients of the problem (2.21)-(2.24) are periodic, that is, the matrix \(A_0(t, x, \cdot, \cdot)\) has periodic entries.

Set
\[
\mathcal{V} = \left\{ u \in L^2(0, T; \mathcal{V}) : u' = \frac{\partial u}{\partial t} \in L^2(0, T; \mathcal{V}^*) \right\}
\]
\[
\mathcal{W} = \left\{ \phi \in L^2(0, T; H^1(\mathcal{Q})) : \phi' \in L^2(0, T; (H^1(\mathcal{Q}))^*) \right\}.
\]

The spaces \(\mathcal{V}\) and \(\mathcal{W}\) are Hilbert spaces with obvious norms. Moreover, the embeddings \(\mathcal{V} \hookrightarrow L^2(0, T; \mathcal{H})\) and \(\mathcal{W} \hookrightarrow L^2(0, T; L^2(Q))\) are compact.

Now, in view of Lemma 2.2, the sequences \((u_\varepsilon)_{\varepsilon > 0}\) and \((\phi_\varepsilon)_{\varepsilon > 0}\) are bounded in \(\mathcal{V}\) and \(\mathcal{W}\) respectively. Hence, given a fundamental sequence \(E\), there exist a subsequence \(E'\) of \(E\) a couple \((u_0, \phi_0)\) in \(\mathcal{V} \times \mathcal{W}\) such that, as \(E' \ni \varepsilon \to 0\),

\[
\begin{align*}
&u_\varepsilon \to u_0 \text{ in } \mathcal{V}-\text{weak} \quad (3.4) \\
&\phi_\varepsilon \to \phi_0 \text{ in } \mathcal{W}-\text{weak} \quad (3.5) \\
&\phi_\varepsilon \to \phi_0 \text{ in } L^2(\mathcal{Q})-\text{strong}. \quad (3.6)
\end{align*}
\]

Also there exist a subsequence of \(E'\) still denoted by \(E'\) and a function \(\tilde{\mu}_0 \in L(0, T; L^2(\mathcal{T}; H^1(\mathcal{Q})))\) such that

\[
\mu_\varepsilon \to \tilde{\mu}_0 \text{ in } L^2(\mathcal{Q}_T)-\text{weak 2s} \quad (3.8)
\]

and

\[
\mu_\varepsilon \to \mu_0 \text{ in } L^2(0, T; H^1(\mathcal{Q}))-\text{weak} \text{ with } \mu_0 = \int_{\mathcal{T}} \tilde{\mu}_0 d\tau. \quad (3.9)
\]

Taking once again into account the estimates (2.21), (2.22) and (2.24) and appealing to Theorem 3.2, we derive by a diagonal process, the existence of a subsequence of \(E'\) not relabeled and of functions \(u_1 \in L^2(Q_T; L^2_{\text{per}}(\mathcal{T}; H^1_\#(\mathcal{Y})))\), \(\phi_1, \mu_1 \in L^2(Q_T; L^2_{\text{per}}(\mathcal{T}; H^1_\#(\mathcal{Y})))\) and \(\xi \in L^2(Q_T; L^2_{\text{per}}(\mathcal{T} \times \mathcal{Y}))\) such that, as \(E' \ni \varepsilon \to 0\), for \(i = 1, 2\), we have

\[
\begin{align*}
\frac{\partial u_\varepsilon}{\partial x_i} &\to \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \text{ in } L^2(Q_T)^{2\text{-weak 2s}} \quad (3.10) \\
\frac{\partial \phi_\varepsilon}{\partial x_i} &\to \frac{\partial \phi_0}{\partial x_i} + \frac{\partial \phi_1}{\partial y_i} \text{ in } L^2(Q_T)^{\text{weak 2s}} \quad (3.11) \\
\frac{\partial \mu_\varepsilon}{\partial x_i} &\to \frac{\partial \tilde{\mu}_0}{\partial x_i} + \frac{\partial \mu_1}{\partial y_i} \text{ in } L^2(Q_T)^{\text{weak 2s}} \quad (3.12) \\
\end{align*}
\]

\[
f(\phi_\varepsilon) \to \xi \text{ in } L^2(Q_T)^{\text{weak 2s}}. \quad (3.13)
\]

Arguing as above, we deduce from Lemma 2.4 the existence of \(p_\varepsilon \in L^2(Q_T; L^2_{\text{per}}(\mathcal{T} \times \mathcal{Y}))\) such that, as \(E' \ni \varepsilon \to 0\),

\[
p_\varepsilon \to p \text{ in } L^2(Q_T)^{\text{weak 2s}}. \quad (3.14)
\]

Next, since \(\text{div } u_\varepsilon = 0\), we can easily check that \(\text{div}_y u_1 = 0\), so that

\[
u_1 \in V' = \left\{ w \in L^2(Q_T; L^2_{\text{per}}(\mathcal{T}; H^1_\#(\mathcal{Y}))) : \text{div}_y w = 0 \right\}.
\]
Proposition 3.1. Let \( u = (u_0, u_1) \in \mathbb{F}_0^1 \) and \( \Phi = (\phi_0, \phi_1), \mu = (\mu_0, \mu_1) \in \mathbb{F}_0^2 \). For an element \( \Psi = (\Psi_0, \Psi_1) \in \mathbb{F}_0^1 \times \mathbb{F}_0^2 \), we obtain
\[
\left\{ \begin{array}{l}
- \int_{Q_T} u_0 \frac{\partial \Psi_0}{\partial t} \, dxdt + \int_{Q_T \times \mathbb{T} \times \mathcal{Y}} A_0 \mathbb{D}u \cdot \mathbb{D}\Psi dxdt dy d\tau \\
+ \int_{Q_T} (u_0 \cdot \nabla) u_0 \Psi_0 dxdt - \int_{Q_T \times \mathbb{T} \times \mathcal{Y}} p(\text{div} \, \Psi_0 + \text{div}_y \Psi_1) dxdt dy d\tau \\
- \kappa \int_{Q_T} \mu_0 \nabla \phi_0 \cdot \Psi_0 dxdt = \int_{Q_T} g \Psi_0 dxdt;
\end{array} \right.
\]
for all \( \Psi = (\Psi_0, \Psi_1) \in \mathbb{C}_0^\infty (Q_T)^2 \times (E)^2 \), \( \varphi = (\varphi_0, \varphi_1) \in \mathbb{C}^\infty_0 (Q_T) \times \mathcal{E} \) and \( \chi = (\chi_0, \chi_1) \in \mathbb{C}^\infty_0 (Q_T) \times \mathbb{C}_0^\infty (\mathcal{T}) \times \mathcal{E} \).

Proof. Let \( \Psi = (\Psi_0, \Psi_1), \varphi = (\varphi_0, \varphi_1) \) and \( \chi = (\chi_0, \chi_1) \) be as above, and define
\[
\Psi_\varepsilon = \Psi_0 + \varepsilon \Psi_1^{\varepsilon}, \quad \varphi_\varepsilon = \varphi_0 + \varepsilon \varphi_1^{\varepsilon}, \quad \chi_\varepsilon = \chi_0 + \varepsilon \chi_1^{\varepsilon}
\]
with \( \Psi_1^{\varepsilon}(t, x) = B \left( t, \frac{t}{\varepsilon}, \frac{x}{\varepsilon} \right) \) for \( (t, x) \in Q_T \).
\[
\int_{Q_T} \mu \nabla \phi \cdot \nabla \chi dx dt = \lambda \int_{Q_T} \nabla \psi \cdot \nabla \chi dx dt + \alpha \int_{Q_T} f(\phi) \chi dx dt. \quad (3.20)
\]

Using the identities
\[
\frac{\partial \Psi}{\partial t} = \frac{\partial \Psi_0}{\partial t} + \left( \frac{\partial \Psi_1}{\partial \tau} \right)^\varepsilon + \varepsilon \left( \frac{\partial \Psi_1}{\partial \tau} \right)^\varepsilon \quad \text{and a similar equality for } \frac{\partial \varphi}{\partial t},
\]

and similar equalities for \( \nabla \varphi \) and \( \nabla \chi \), we infer that, as \( \varepsilon \to 0 \),
\[
\frac{\partial \Psi}{\partial t} \to \frac{\partial \Psi_0}{\partial t} \quad \text{in } L^2(0, T; H^{-1}(Q)^2)-\text{weak} \quad (3.21)
\]
\[
\frac{\partial \varphi}{\partial t} \to \frac{\partial \varphi_0}{\partial t} \quad \text{in } L^2(0, T; H^{-1}(Q))-\text{weak} \quad (3.22)
\]
\[
\nabla \Psi \to \nabla \Psi_0 + (\nabla_y \Psi_1)^\varepsilon + \varepsilon (\nabla_y \Psi_1)^\varepsilon \quad \text{in } L^2(Q_T)^{2 \times 2}-\text{strong 2s} \quad (3.23)
\]
\[
\nabla \varphi \to \nabla \varphi_0 + (\nabla_y \varphi_1)^\varepsilon + \varepsilon (\nabla_y \varphi_1)^\varepsilon \quad \text{in } L^2(Q_T)^{2 \times 2}-\text{strong 2s} \quad (3.24)
\]
\[
\nabla \chi \to \nabla \chi_0 + (\nabla_y \chi_1)^\varepsilon + \varepsilon (\nabla_y \chi_1)^\varepsilon \quad \text{in } L^2(Q_T)^{2 \times 2}-\text{strong 2s} \quad (3.25)
\]
\[
\Psi \to \Psi_0 \quad \text{in } L^2(Q_T)^{2 \times 2}-\text{strong} \quad (3.26)
\]
\[
\varphi \to \varphi_0 \quad \text{in } L^2(Q_T)-\text{strong} \quad (3.27)
\]
\[
\chi \to \chi_0 \quad \text{in } L^2(Q_T)-\text{strong 2s}. \quad (3.28)
\]

Let us consider each of the equations (3.18)-(3.20) separately. Considering (3.18), only the term \( \int_{Q_T} \mu \nabla \phi \cdot \Psi dx dt \) needs careful treatment. To that end, we have
\[
\int_{Q_T} \mu \nabla \phi \cdot \Psi dx dt = - \int_{Q_T} (\mu \nabla \phi \cdot \Psi + \phi \nabla \mu) dxdtdydt
\]
\[
= -(I_1 + I_2).
\]

In order to compute the limit of \( I_1 \), we first notice that, since the sequence \( (\mu_\varepsilon) \) is bounded in \( L^2(0, T; H^1(Q)) \), we have that \( \mu_0 \) is independent of \( y \), i.e. \( \mu_0 \in L^2(0, T; H^1(Q)) \) and we have (3.3). It therefore comes from (3.3) that \( \mu_\varepsilon \phi \to \tilde{\mu} \phi_0 \) in \( L^1(Q_T)-\text{weak 2s} \). Using the equality \( \text{div} \Psi = \text{div} \Psi_0 + (\text{div}_y \Psi_1)^\varepsilon + \varepsilon (\text{div}_y \Psi_1)^\varepsilon \) and viewing \( \text{div} \Psi \) as a test function, we get at once
\[
I_1 \to \int_{Q_T \times \times \times} \phi \tilde{\mu}_0 (\text{div} \Psi_0 + \text{div}_y \Psi_1) dx dt
d
\]
\[
= \int_{Q_T} \phi_0 (\int \tilde{\mu}_0 d\tau) \text{div} \Psi_0 dx dt = \int_{Q_T} \phi_0 \mu_0 \text{div} \Psi_0 dx dt
\]
since \( \phi_0 \tilde{\mu}_0 \) in independent of \( y \) and \( \int_y \text{div}_y \Psi_1 dy = 0 \). As for \( I_2 \), we have
\[
I_2 \to \int_{Q_T \times \times \times} \phi_0 \Psi_0 \cdot \mu dx dt dy d\tau = \int_{Q_T} \phi_0 \Psi_0 \cdot \nabla (\int \tilde{\mu}_0 d\tau) dx dt
\]
\[
= \int_{Q_T} \phi_0 \Psi_0 \cdot \nabla \mu_0 dx dt.
\]
It follows that
\[
\int_{Q_T} \mu_\varepsilon \nabla \phi_\varepsilon \cdot \Psi_\varepsilon dxdt \to - \int_{Q_T} (\phi_0 \mu_0 \text{div} \Psi_0 + \phi_0 \Psi_0 \cdot \nabla \mu_0) dxdt
\]
\[
= \int_{Q_T} \mu_0 \nabla \phi_0 \cdot \Psi_0 dxdt.
\]

The limit passage in (3.19) is straightforward using for the second integral, the strong convergences of \((u_\varepsilon)\) and \((\phi_\varepsilon)\).

As for (3.20), since it involves the sequence \((\mu_\varepsilon)\) which is bounded in \(L^2(0,T;H^1(Q))\) with no knowledge about the boundedness of the sequence \((\partial \mu_\varepsilon/\partial t)\), we need, as in the case of (3.18), to explain how the limit equation (3.17) is derived. First we have from (3.8) and (3.28) that
\[
\int_{Q_T} \mu_\varepsilon \chi_\varepsilon dxdt \to \int_{Q_T \times T} \tilde{\mu}_0 \chi_0 dxdtd\tau.
\]

Next, using (3.11) and (3.25), we obtain
\[
\int_{Q_T} \nabla \phi_\varepsilon \cdot \nabla \chi_\varepsilon dxdt \to \int_{Q_T \times T \times Y} \nabla \Phi \cdot D\chi dxdtdy\tau.
\]

Concerning \(\int_{Q_T} f(\phi_\varepsilon) \chi_\varepsilon dxdt\), we use the continuity of \(f\) associated to (3.7) to pass to the limit. Indeed, from (3.7), we have, up to a subsequence of \((\phi_\varepsilon)_{\varepsilon \in E'}\) not relabeled, that \(\phi_\varepsilon \to \phi_0\) a.e. in \(Q_T\), so that, owing to the continuity of \(f\), \(f(\phi_\varepsilon) \to f(\phi_0)\) a.e. in \(Q_T\). It therefore follows (using (2.26)) that \(f(\phi_\varepsilon) \to f(\phi_0)\) in \(L^2(Q_T)\)-weak as \(E' \ni \varepsilon \to 0\).

It therefore emerges from (3.13) that
\[
\int_{Q_T} f(\phi_\varepsilon) \chi_\varepsilon dxdt \to \int_{Q_T \times T \times Y} \xi \chi_0 dx dtdy\tau \text{ and } \int_{T \times Y} \xi dyd\tau = f(\phi_0).
\]

This concludes the proof. □

3.3. Homogenized problem. Let \(p_0 = \int_{T \times Y} p dyd\tau\). We intend here to derive the equivalent problem whose the quadruplet \((u_0, \phi_0, \mu_0, p_0)\) is solution to. Let us recall that \(u_0 \in V\), \(\phi_0 \in \mathcal{W}\), \(\mu_0 \in L^2(0,T;H^1(Q))\) and \(p_0 \in L^2(0,T;L^2_0(Q))\) since from the equality \(\int_Q p \, dx = 0\) we deduce that \(\int_Q p_0 \, dx = 0\) as \(p_\varepsilon \to p_0\) in \(L^2(Q_T)\)-weak. To that end, we first uncouple the equations (3.15)-(3.17) in order to obtain the following equivalent system: (3.15) is equivalent to

\[
\begin{cases}
- \int_{Q_T} u_0 \frac{\partial \Psi_0}{\partial t} dxdt + \int_{Q_T \times T \times Y} A_0 D\mu \cdot \nabla \Psi_0 dx dtdyd\tau \\
+ \int_{Q_T} (u_0 \cdot \nabla) u_0 \Psi_0 dxdt - \int_{Q_T} p_0 \text{div} \Psi_0 dxdt \\
- \kappa \int_{Q_T} \mu_0 \nabla \phi_0 \cdot \Psi_0 dxdt = \int_{Q_T} g \Psi_0 dxdt
\end{cases}
\]  
(3.29)
\[
\int_{Q_T \times T \times Y} A_0 \mathbf{u} \cdot \nabla_y \Psi_1 \, dx \, dt \, dy \, d\tau - \int_{Q_T \times T \times Y} p \text{div}_y \Psi_1 \, dx \, dt \, dy \, d\tau = 0. \tag{3.30}
\]

Eq (3.16) is equivalent to (3.31)-(3.32) below

\[
- \int_{Q_T} \phi_0 \frac{\partial \varphi_0}{\partial t} \, dx \, dt + \int_{Q_T} \phi_0 \mathbf{u}_0 \cdot \nabla \varphi_0 \, dx \, dt + \int_{Q_T \times T \times Y} \mathbb{D} \mu \cdot \nabla \varphi_0 \, dx \, dt \, dy \, d\tau = 0 \tag{3.31}
\]

\[
\int_{Q_T \times T \times Y} D \mu \cdot \nabla \varphi_1 \, dx \, dt \, dy \, d\tau = 0 \tag{3.32}
\]

while (3.17) is equivalent to (3.33)-(3.34):

\[
\int_{Q_T} \tilde{\mu}_0 \chi_0 \, dx \, dt \, d\tau = \lambda \int_{Q_T \times T \times Y} \nabla \Phi \cdot \nabla \chi_0 \, dx \, dt \, dy \, d\tau + \alpha \int_{Q_T \times T \times Y} \xi \chi_0 \, dx \, dt \, dy \, d\tau \tag{3.33}
\]

\[
\int_{Q_T \times T \times Y} \nabla_y \chi_0 \, dx \, dt \, dy \, d\tau = 0 \quad \forall \chi_0 \in C^\infty_0(Q_T) \otimes C^\infty_{per}(T), \tag{3.34}
\]

Let us start with (3.17). In (3.34), we take \( \chi_1 \) under the form \( \chi_1(t,x,\tau,y) = \chi_1^0(t,x)\psi(y)\theta(\tau) \) with \( \chi_1^0 \in C^\infty_0(Q_T), \psi \in C^\infty_{per}(Y) \) and \( \theta \in C^\infty_{per}(T) \) to get

\[
\int_Y (\nabla \phi_0 + \nabla_y \phi_0) \cdot \nabla \psi \, dy = 0 \quad \forall \psi \in C^\infty_{per}(Y),
\]

which equation is easily seen to possess a unique solution \( \phi_1 \equiv 0 \). With this in mind, going back to (3.33) and choosing there \( \chi_0(t,x,\tau) = \chi_0^0(t,x)\psi(y)\theta(\tau) \) with \( \chi_0^0 \in C^\infty_0(Q_T) \) and \( \theta \in C^\infty_{per}(T) \), we obtain, after simplification by \( \theta \) and next integrating over \( T \),

\[
\int_{Q_T} \left( \int_T \tilde{\mu}_0 \, d\tau \right) \chi_0 \, dx \, dt \lambda \int_{Q_T} \nabla \phi_0 \cdot \nabla \chi_0 \, dx \, dt + \alpha \int_{Q_T \times T \times Y} \xi \chi_0 \, dx \, dt \, dy \, d\tau \chi_0 \, dx \, dt,
\]

i.e.

\[
\left\{ \begin{array}{c}
\int_{Q_T} \mu_0 \chi_0 \, dx \, dt = \lambda \int_{Q_T} \nabla \phi_0 \cdot \nabla \chi_0 \, dx \, dt + \alpha \int_{Q_T} f(\phi) \chi_0 \, dx \, dt \\
\text{for all } \chi_0 \in C^\infty_0(Q_T).
\end{array} \right.
\]

This yields the first homogenized equation, viz.

\[
\mu_0 = -\lambda \Delta \phi_0 + \alpha f(\phi_0) \text{ in } Q_T. \tag{3.35}
\]

The next step is to consider the equation (3.13). Here we first deal with (3.32) to see that, as for (3.31), it can be shown that \( \mu_1(t,x,\tau,y) = 0 \) for a.e. \((t,x,\tau,y)\), so that, moving to (3.31), we obtain the variational form of (where we take into account the fact that \( \text{div} \mathbf{u}_0 = 0 \))

\[
\frac{\partial \phi_0}{\partial t} + \mathbf{u}_0 \cdot \nabla \phi_0 - \Delta \mu_0 = 0 \text{ in } Q_T. \tag{3.36}
\]

Now, the most involved problem is (3.15). Therein we first consider the corrector equation (3.30) in which, with the choice of the test function under the form \( \Psi_1(t,x,\tau,y) = \)
\[ \Psi_0^0(t, x)\psi(y)\theta(\tau) \] with \( \Psi_0^0 \in C^\infty_0(Q_T), \psi \in C^\infty_{\text{per}}(\mathcal{Y})^2 \) and \( \theta \in C^\infty_{\text{per}}(\mathcal{T}) \), yields the variational form of the following Stokes type equation: for a.e. \((t, x, \tau, \cdot)\), \( u_1(t, x, \tau, \cdot) \) solves the equation

\[
\begin{cases}
- \text{div}_y(A_0(t, x, \tau, \cdot)(\nabla u_0(t, x, \tau, \cdot) + \nabla u_1(t, x, \tau, \cdot)) + \nabla y p(t, x, \tau, \cdot) = 0 \text{ in } \mathcal{Y} \\
\text{div}_y u_1(t, x, \tau, \cdot) = 0 \text{ in } \mathcal{Y} \\
u_1(t, x, \tau, \cdot) \text{ is } \mathcal{Y}\text{-periodic and } \int_\mathcal{Y} u_1(t, x, \tau, y) dy = 0.
\end{cases}
\tag{3.37}
\]

So, for \( r \in \mathbb{R}^{2 \times 2} \), let \( (\eta(r) = \eta_{t,x,\tau}(r), \pi(r) = \pi_{t,x,\tau}(r)) \) be the solution of the Stokes equation

\[
\begin{cases}
- \text{div}_y(A_0(t, x, \tau, \cdot)(r + \nabla \eta(r)) + \nabla y \pi(r) = 0 \text{ in } \mathcal{Y} \\
\text{div}_y \eta(r) = 0 \text{ in } \mathcal{Y} \\
\eta(r) \in H^1_\#(\mathcal{Y})^2, \pi(r) \in L^2_{\text{per}}(\mathcal{Y})/\mathbb{R}.
\end{cases}
\tag{3.38}
\]

Then as classically known, equation \(3.38\) possesses a unique solution. Choosing in \(3.15\) \( r = \nabla u_0(t, x) \) and the uniqueness of the solution to \(3.38\) leads to \( u_1 = \eta(\nabla u_0) \) and \( p = \pi(\nabla u_0) \) where \( \eta(\nabla u_0) \) stands for the function \((t, x, \tau) \mapsto \eta_{t,x,\tau}(\nabla u_0(t, x))\), which belongs to \( L^2(Q_T \times \mathcal{T}; H^1_\#(\mathcal{Y})^2) \). Clearly, if \( \eta_{j}^\ell \) is the solution of \(3.38\) corresponding to \( r = r_j^\ell = (\delta_{ij}\delta_{k\ell})_{1 \leq i, k \leq 2} \) (that is all the entries of \( r \) are zero except the entry occupying the \( j \)th row and the \( \ell \)th column which is equal to 1), then

\[
u_1 = \sum_{j, \ell=1}^{2} \frac{\partial u_{0j}^\ell}{\partial x_j} \eta_{j}^\ell \] where \( u_0 = (u_0^\ell)_{1 \leq \ell \leq 2} \).
\tag{3.39}
\]

Let us recall that \( \eta_{j}^\ell \) depends implicitly on \( t, x, \tau \) since \( A_0 \) does. Going back to the variational form of \(3.29\) and inserting there the value of \( u_1 \) obtained in \(3.39\), we are led to the following equation

\[
\frac{\partial u_0}{\partial t} - \text{div}(A_0 \nabla u_0) + (u_0 \cdot \nabla) u_0 + \nabla p_0 - \kappa \mu_0 \nabla \phi_0 = g \text{ in } Q_T
\tag{3.40}
\]

where

\[
\tilde{A}_0(t, x) = (\tilde{a}_{ij}^\ell(t, x))_{1 \leq i, j, k, \ell \leq 2}, \quad \tilde{a}_{ij}^\ell(t, x) = a_{\text{per}}(\eta_{j}^\ell + P_{j}^\ell, \eta_{i}^k + P_{i}^k)
\]

with \( P_{j}^\ell = y_j e^\ell \) (\( e^\ell \) the \( \ell \)th vector of the canonical basis of \( \mathbb{R}^2 \)) and

\[
a_{\text{per}}(u, v) = \sum_{i,j,k=1}^{2} \int_{T \times \mathcal{Y}} a_{ij} \frac{\partial u_k}{\partial y_j} \frac{\partial v_k}{\partial y_i} dy d\tau \text{ where } A_0 = (a_{ij})_{1 \leq i, j \leq 2}.
\]
Finally, let us put together the equations (3.40), (3.36), (3.35) associated to the boundary and initial conditions:

\[
\begin{align*}
\frac{\partial \mathbf{u}_0}{\partial t} - \text{div}(\hat{A}_0 \nabla \mathbf{u}_0) + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \nabla p_0 - \kappa \mu_0 \nabla \phi_0 &= g \text{ in } QT \\
\text{div } \mathbf{u}_0 &= 0 \text{ in } QT \\
\frac{\partial \phi_0}{\partial t} + \mathbf{u}_0 \cdot \nabla \phi_0 - \Delta \mu_0 &= 0 \text{ in } QT \\
\mu_0 &= -\lambda \Delta \phi_0 + \alpha f(\phi_0) \text{ in } QT \\
(\mathbf{u}_0, \phi_0)(0) &= (\mathbf{u}_0^0, \phi_0^0) \text{ in } Q \\
\frac{\partial \phi_0}{\partial \nu} &= \frac{\partial \mu_0}{\partial \nu} \text{ on } (0, T) \times \partial Q \\
\mathbf{u}_0 &= 0 \text{ on } (0, T) \times \partial Q.
\end{align*}
\]

Problem (3.41) is the homogenized problem. Contrasting with (3.15)-(3.17), it involves only the macroscopic limit \((\mathbf{u}_0, \phi_0, \mu_0, p_0)\) of the sequence \((\mathbf{u}_\varepsilon, \phi_\varepsilon, \mu_\varepsilon, p_\varepsilon)\) of solutions to (2.1)-(2.4). Thus it describes the macroscopic behavior of the above-mentioned sequence as the size of the heterogeneities goes to zero. It can be easily shown that the matrix \(\hat{A}_0\) of homogenized coefficients is uniformly elliptic, so that under the conditions (A2)-(A3), the problem (3.41) possesses a unique solution \((\mathbf{u}_0, \phi_0, p_0)\) with \(\mathbf{u}_0 \in L^2(0, T; V), \phi_0, \mu_0 \in L^2(0, T; H^1(Q))\) and \(p_0 \in L^2(0, T; L^2_0(Q))\). Since the solution to (3.41) is unique, we infer that the whole sequence \((\mathbf{u}_\varepsilon, \phi_\varepsilon, \mu_\varepsilon, p_\varepsilon)\) converges in the suitable spaces towards \((\mathbf{u}_0, \phi_0, \mu_0, p_0)\) as stated in the following result, which is the first main theorem of this work.

**Theorem 3.4.** Assume that (A1)-(A3) and (3.1) hold. For any \(\varepsilon > 0\) let \((\mathbf{u}_\varepsilon, \phi_\varepsilon, \mu_\varepsilon, p_\varepsilon)\) be the unique solution of problem (2.1)-(2.4). Then the sequence \((\mathbf{u}_\varepsilon, \phi_\varepsilon, \mu_\varepsilon, p_\varepsilon)_{\varepsilon \geq 0}\) converges strongly in \(L^2(Q_T)^2 \times L^2(Q_T)\) (with respect to the first two components \((\mathbf{u}_\varepsilon, \phi_\varepsilon)\)) and weakly in \(L^2(Q_T) \times L^2(0, T; H^1(Q))\) (with respect to \((\mu_\varepsilon, p_\varepsilon)\)) to the solution of problem (3.41).

**Proof.** The proof is a consequence of the previous steps. \(\square\)

4. Homogenization results: the general deterministic framework

Our purpose here is to extend the results of the preceding section to more general setting beyond the periodic framework. The basic notation and hypotheses (except the periodicity assumption) stated before are still valid.

In this section we recall some basic facts about the algebras with mean value [43] and the concept of sigma-convergence [29] (see also [31, 37]). Using the semigroup theory we reify the presentation of some essential results related to the previous concepts. We refer the reader to [41] for the details regarding most of the results of this section.

4.1. Algebras with mean value and sigma-convergence. Let \(A\) be an algebra with mean value (algebra wmv, in short) on \(\mathbb{R}^d\), that is, a closed subalgebra of the algebra of bounded uniformly continuous real-valued functions on \(\mathbb{R}^d\), \(\text{BUC}(\mathbb{R}^d)\), which contains the constants, is translation invariant and is such that any of its elements possesses a mean value in the following sense: for every \(u \in A\), the sequence \((u^\varepsilon)_{\varepsilon \geq 0}\) (where \(u^\varepsilon(x) = u(x/\varepsilon)\) for \(x \in \mathbb{R}^d\)) weakly*-converges in \(L^\infty(\mathbb{R}^d)\) to some real number \(M(u)\) (called the mean value of \(u\)) as \(\varepsilon \to 0\).
The mean value expresses as
\[ M(u) = \lim_{R \to +\infty} \int_{B_R} u(y) dy \] (4.1)
where \( B_R \) stands for the bounded open ball in \( \mathbb{R}^d \) with radius \( R \) and \( |B_R| \) denotes its Lebesgue measure, with \( \int_{B_R} = \frac{1}{|B_R|} \int_{B_R} \).

To an algebra wmv \( A \) we associate its regular subalgebras \( A^m = \{ \psi \in C^m(\mathbb{R}^d) : D_y^\alpha \psi \in A \ \forall \alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d \text{ with } |\alpha| \leq m \} \) \( (m \geq 0) \) an integer with \( A^0 = A \), and \( D_y^\alpha \psi = \frac{\partial^{\alpha |\alpha|}}{\partial y_{|\alpha|}}(\psi) \).

Under the norm \( \|\cdot\|_m = \sup_{|\alpha| \leq m} \|D_y^\alpha \psi\|_{\infty} \), \( A^m \) is a Banach space. We also define the space \( A^\infty = \{ \psi \in C^\infty(\mathbb{R}^d) : D_y^\alpha \psi \in A \ \forall \alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d \} \), a Fréchet space when endowed with the locally convex topology defined by the family of norms \( \|\cdot\|_m \). The space \( A^\infty \) is dense in any \( A^m \) (integer \( m \geq 0 \)).

The notion of a product algebra wmv will be very useful in this study. Before we can define it, let first and foremost deal with the concept of a vector-valued algebra wmv.

Let \( F \) be a Banach space. We denote by \( \text{BUC}(\mathbb{R}^d; F) \) the Banach space of bounded uniformly continuous functions \( u : \mathbb{R}^d \to F \), endowed with the norm \( \|u\|_{\infty} = \sup_{y \in \mathbb{R}^d} \|u(y)\|_F \).

We may now define the product algebra wmv.

**Definition 4.1.** Let \( A_y \) and \( A_\tau \) be two algebras wmv on \( \mathbb{R}^d \) and \( \mathbb{R} \), respectively. The vector-valued algebra wmv \( A := A_y(\mathbb{R}^d; A_\tau) = A_\tau(A_y; A_y) \) is an algebra wmv on \( \mathbb{R}^{d+1} \) denoted by \( A_y \otimes A_\tau \). The algebra \( A_y \otimes A_\tau \) is called the product algebra wmv of \( A_y \) and \( A_\tau \).

Now, let \( f \in A(\mathbb{R}^d; F) \) (integer \( d \geq 1 \)). Then, defining \( \|f\|_F \) by \( \|f\|_F(y) = \|f(y)\|_F \) \((y \in \mathbb{R}^d)\), we have that \( \|f\|_F \in \mathcal{A} \). Similarly we can define (for \( 0 < p < \infty \)) the function \( \|f\|_F^p \) and \( \|f\|_F^p \in \mathcal{A} \). This allows us to define the Besicovitch seminorm on \( A(\mathbb{R}^d; F) \) as follows: for \( 1 \leq p < \infty \), we define the Marcinkiewicz-type space \( \mathcal{M}^p(\mathbb{R}^d; F) \) to be the vector space of functions \( u \in L^p_{\text{loc}}(\mathbb{R}^d; F) \) such that
\[ \|u\|_{p, F} = \left( \lim_{R \to \infty} \sup_{R} \int_{B_R} \|u(y)\|_F^p \, dy \right)^{\frac{1}{p}} < \infty \]
where \( B_R \) is the open ball in \( \mathbb{R}^d \) centered at the origin and of radius \( R \). Under the seminorm \( \|\cdot\|_{p, F} \), \( \mathcal{M}^p(\mathbb{R}^d; F) \) is a complete seminormed space with the property that \( A(\mathbb{R}^d; F) \subset \mathcal{M}^p(\mathbb{R}^d; F) \) since \( \|u\|_{p, F} < \infty \) for any \( u \in A(\mathbb{R}^d; F) \). We therefore define the generalized
Besicovitch space $B^p_A(\mathbb{R}^d; F)$ as the closure of $A(\mathbb{R}^d; F)$ in $2\mathbb{R}^p(\mathbb{R}^d; F)$. The following hold true:

(i) The space $B^p_A(\mathbb{R}^d; F) = B^p_A(\mathbb{R}^d; F)/\mathcal{N}$ (where $\mathcal{N} = \{ u \in B^p_A(\mathbb{R}^d; F) : \| u \|_{p,F} = 0 \}$) is a Banach space under the norm $\| u + \mathcal{N} \|_{p,F} = \| u \|_{p,F}$ for $u \in B^p_A(\mathbb{R}^d; F)$.

(ii) The mean value $M : A(\mathbb{R}^d; F) \to F$ extends by continuity to a continuous linear mapping (still denoted by $M$) on $B^p_A(\mathbb{R}^d; F)$ satisfying

$$L(M(u)) = M(L(u)) \text{ for all } L \in F' \text{ and } u \in B^p_A(\mathbb{R}^d; F).$$

Moreover, for $u \in B^p_A(\mathbb{R}^d; F)$ we have

$$\| u \|_{p,F} = [M(\| u \|_{F})]^{1/p} = \left( \lim_{R \to \infty} \int_{B_R} \| u(y) \|_{F} \, dy \right)^{1/p},$$

for $u \in \mathcal{N}$ one has $M(u) = 0$.

It is to be noted that $B^2_A(\mathbb{R}^d; H)$ (when $F = H$ is a Hilbert space) is a Hilbert space with inner product

$$(u, v)_2 = M [(u, v)_H] \text{ for } u, v \in B^2_A(\mathbb{R}^d; H) \quad (4.2)$$

where $(\cdot, \cdot)_H$ stands for the inner product in $H$ and $(u, v)_H$ the function $y \mapsto (u(y), v(y))_H$ from $\mathbb{R}^d$ to $\mathbb{R}$, which belongs to $B^1_A(\mathbb{R}^d)$.

Of interest in the sequel is the special case $F = \mathbb{R}$ for which $B^p_A(\mathbb{R}^d) := B^p_A(\mathbb{R}^d; \mathbb{R})$ and $B^p_\mathcal{A}(\mathbb{R}^d) := B^p_A(\mathbb{R}^d; \mathbb{R})$. The Besicovitch seminorm in $B^p_A(\mathbb{R}^d)$ is merely denoted by $\| \cdot \|_p$, and we have $B^1_A(\mathbb{R}^d) \subset B^p_A(\mathbb{R}^d)$ for $1 \leq p \leq q < \infty$. From this last property one may naturally define the space $B^\infty_A(\mathbb{R}^d)$ as follows:

$$B^\infty_A(\mathbb{R}^d) = \left\{ f \in \bigcap_{1 \leq p < \infty} B^p_A(\mathbb{R}^d) : \sup_{1 \leq p < \infty} \| f \|_p < \infty \right\}.$$ 

We endow $B^\infty_A(\mathbb{R}^d)$ with the seminorm $[f]_\infty = \sup_{1 \leq p < \infty} \| f \|_p$, which makes it a complete seminormed space.

In this regard, we consider the space $B^{1,p}_A(\mathbb{R}^d) = \{ u \in B^p_A(\mathbb{R}^d) : \nabla_y u \in (B^p_A(\mathbb{R}^d))^d \}$ endowed with the seminorm

$$\| u \|_{1,p} = \left( \| u \|_p + \| \nabla_y u \|_p \right)^{1/p},$$

which is a complete seminormed space. The Banach counterpart of the previous spaces are defined as follows. We set $B^p_A(\mathbb{R}^d) = B^p_A(\mathbb{R}^d)/\mathcal{N}$ where $\mathcal{N} = \{ u \in B^p_A(\mathbb{R}^d) : \| u \|_p = 0 \}$. We define $B^{1,p}_A(\mathbb{R}^d)$ mutatis mutandis: replace $B^p_A(\mathbb{R}^d)$ by $B^p_\mathcal{A}(\mathbb{R}^d)$ and $\partial/\partial y_i$ by $\overline{\partial}/\partial y_i$, where $\overline{\partial}/\partial y_i$ is defined by

$$\overline{\partial}/\partial y_i (u + \mathcal{N}) := \frac{\partial u}{\partial y_i} + \mathcal{N} \text{ for } u \in B^{1,p}_A(\mathbb{R}^d). \quad (4.3)$$

It is important to note that $\overline{\partial}/\partial y_i$ is also defined as the infinitesimal generator in the $i$th direction coordinate of the strongly continuous group $\mathcal{T}(y) : B^{p}_A(\mathbb{R}^d) \to B^p_A(\mathbb{R}^d); \mathcal{T}(y)(u +$
\( \mathcal{N} = u(\cdot + y) + \mathcal{N} \). Let us denote by \( \varrho : B_A^{1,p}(\mathbb{R}^d) \to B_A^{1,p}(\mathbb{R}^d) = B_A^{1,p}(\mathbb{R}^d)/\mathcal{N} \), \( \varrho(u) = u + \mathcal{N} \), the canonical surjection. We remark that if \( u \in B_A^{1,p}(\mathbb{R}^d) \) then \( \varrho(u) \in B_A^{1,p}(\mathbb{R}^d) \) with further

\[
\frac{\partial \varrho(u)}{\partial y_i} = \varrho \left( \frac{\partial u}{\partial y_i} \right)
\]
as seen above in \([4.3]\).

We assume in the sequel that the algebra \( A \) is ergodic, that is, any \( u \in B_A^{1,p}(\mathbb{R}^d) \) that is invariant under \( (T(y))_{y \in \mathbb{R}^d} \) is a constant in \( B_A^{1,p}(\mathbb{R}^d) \), i.e., if \( T(y)u = u \) for every \( y \in \mathbb{R}^d \), then \( \|u - c\|_p = 0 \), \( c \) a constant. As in \([7]\) we observe that if the algebra with mean value \( A \) is ergodic, then \( u \in B_A^{1}(\mathbb{R}^d) \) is invariant if and only if \( \partial u/\partial y_i = 0 \) for all \( 1 \leq i \leq d \). We denote by \( B_A^{1,p}(\mathbb{R}^d) \) the space of invariant functions in \( B_A^{1,p}(\mathbb{R}^d) \). Let us also recall the following property \([41, 42]\).

(iii) The mean value \( M \) viewed as defined on \( A \), extends by continuity to a positive continuous linear form (still denoted by \( M \)) on \( B_A^{1,p}(\mathbb{R}^d) \). For each \( u \in B_A^{1,p}(\mathbb{R}^d) \) and all \( a \in \mathbb{R}^d \), we have \( M(u(\cdot + a)) = M(u) \), and \( \|u\|_p = [M(|u|^p)]^{1/p} \).

To the space \( B_A^{1,p}(\mathbb{R}^d) \) we also attach the following corrector space

\[
B_A^{1,p}(\mathbb{R}^d) = \{ u \in W_{1,p}^{1,p}(\mathbb{R}^d) : \nabla u \in B_A^{1,p}(\mathbb{R}^d) \text{ and } M(\nabla u) = 0 \}.
\]

We also define the space \( \nabla B_A^{1,p}(\mathbb{R}^d) = \{ \nabla u : u \in B_A^{1,p}(\mathbb{R}^d) \} \). Identifying an element of \( \nabla B_A^{1,p}(\mathbb{R}^d) \) with its class in \( (B_A^{1,p}(\mathbb{R}^d))^d \), \( \nabla B_A^{1,p}(\mathbb{R}^d) \) will be considered as a subspace of \( (B_A^{1,p}(\mathbb{R}^d))^d \). Moreover we identify two elements of \( B_A^{1,p}(\mathbb{R}^d) \) by their gradients: \( u = v \) in \( B_A^{1,p}(\mathbb{R}^d) \) iff \( \nabla (u - v) = 0 \), i.e. \( \|\nabla(u - v)\|_p = 0 \). We may therefore equip \( B_A^{1,p}(\mathbb{R}^d) \) with the gradient norm \( \|u\|_{1,p} = \|\nabla u\|_p \). This defines a Banach space \([11\] Theorem 3.12\) (actually \( \nabla B_A^{1,p}(\mathbb{R}^d) \) is closed in \( (B_A^{1,p}(\mathbb{R}^d))^d \), so that \( B_A^{1,p}(\mathbb{R}^d) \) is a Banach space) containing \( B_A^{1,p}(\mathbb{R}^d) \) as a subspace.

For \( u \in B_A^{1}(\mathbb{R}^d) \) (resp. \( v = (v_1, ..., v_d) \in (B_A^{p}(\mathbb{R}^d))^d \)), we define the gradient operator \( \nabla_y \) and the divergence operator \( \nabla_y \cdot \) by

\[
\nabla_y u := \left( \frac{\partial u}{\partial y_1}, ..., \frac{\partial u}{\partial y_d} \right) \quad \text{and} \quad \nabla_y \cdot v \equiv \nabla_y v := \sum_{i=1}^{d} \frac{\partial v_i}{\partial y_i}.
\]

Then the divergence operator sends continuously and linearly \( (B_A^{p}(\mathbb{R}^d))^d \) into \( (B_A^{1,p}(\mathbb{R}^d))^d \)' and satisfies

\[
\langle \nabla_y u, v \rangle = -\langle u, \nabla_y v \rangle \quad \text{for} \quad v \in B_A^{1,p}(\mathbb{R}^d) \text{ and } u = (u_i) \in (B_A^{p}(\mathbb{R}^d))^d,
\]

where \( \langle u, \nabla_y v \rangle := M(u \cdot \nabla_y v) \).

This being so, our next aim is to define the sigma-convergence concept. To this end, let \( A_y \) (resp. \( A_r \)) be an algebra wmv on \( \mathbb{R}^d \) (resp. \( \mathbb{R} \)) and let \( A = A_r \circ A_y \) be their product which is an algebra wmv on \( \mathbb{R} \times \mathbb{R}^d \). We will denote by the same letter \( M \), the mean value on \( \mathbb{R}^d \) and on \( \mathbb{R}^{d+1} \) as well. Finally, \( E, Q, T \) and \( Q_T \) are as in the previous sections.
Theorem 4.1. Let \((u_\varepsilon)_{\varepsilon>0}\) be a bounded sequence in \(L^2(Q_T)\). Then there exist a subsequence \(E'\) from \(E\) and a function \(u\) in \(L^2(Q_T; B^2_{A}(\mathbb{R}^{d+1}))\) such that the sequence \((u_\varepsilon)_{\varepsilon\in E'}\) weakly \(\Sigma\)-converges in \(L^2(Q_T)\) to \(u\).

Theorem 4.2. Let \((u_\varepsilon)_{\varepsilon\in E}\) be a bounded sequence in \(L^2(0,T; H^1(Q))\). There exist a subsequence \(E'\) from \(E\) and a couple \(u = (u_0, u_1)\) with \(u_0 \in L^2(0,T; B^2_{A_r}(\mathbb{R}_r; H^1(Q); I^2_{A_y}(\mathbb{R}_y)))\) and \(u_1 \in L^2(Q_T; B^2_{A_r}(\mathbb{R}_r; B^2_{A_y}(\mathbb{R}_y)))\) such that, as \(E' \ni \varepsilon \to 0\),

\[ u_\varepsilon \rightharpoonup u_0 \text{ in } L^2(Q_T) \text{-weak } \Sigma \]

and

\[ \frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^2(Q_T) \text{-weak } \Sigma \quad (1 \leq j \leq d). \]

If moreover the sequence \((\partial u_\varepsilon/\partial t)_{\varepsilon>0}\) is bounded in \(L^2(0,T; (H^1(Q))^*)\) then \(u_0 \in L^2(0,T; H^1(Q); I^2_{A_y}(\mathbb{R}_y)))\).

Remark 4.1. Assume that the algebra \(A_y\) is ergodic. Then \(I^2_{A_y}(\mathbb{R}_y)\) reduces to constants functions, so that in Theorem 4.2 the function \(u_0\) lies either in \(L^2(0,T; B^2_{A_r}(\mathbb{R}_r; H^1(Q)))\) or in \(L^2(0,T; H^1(Q))\) (if the algebra \(A_y\) is ergodic).

4.2. Homogenization results: passage to the limit. We assume that the algebras \(A_y\) and \(A_r\) are ergodic. The notations are those of the preceding sections, we remark that property (14.5) in Definition 4.2 still holds true for \(f \in C(Q_T; B^{p',\infty}_{A}(\mathbb{R}^{N+1}_y))\) where \(B^{p',\infty}_{A}(\mathbb{R}^{N+1}_y) = B^{p',\infty}_{A}(\mathbb{R}^{N+1}_y) \cap L^\infty(\mathbb{R}^{N+1}_y)\) and as usual \(p' = p/(p-1)\).
With this in mind, the use of the sigma-convergence method to solve the homogenization problem for (2.1)-(2.4) will be possible provided that the following assumption on the coefficient $A_0$ of (2.1) holds true.

$$A_0(t,x,\cdot,\cdot) \in [B^2_A(\mathbb{R}^{2+1})]^{2\times 2} \text{ for a.e. } (t,x) \in Q_T$$

(4.6)

where $A = A_\Gamma \cup A_y$. Let $(u_\varepsilon, \phi_\varepsilon, \mu_\varepsilon, p_\varepsilon)_{\varepsilon > 0}$ be the sequence determined in Section 2. Proceeding as in Section 3, and using the compactness results Theorems 4.1-4.2 and Remark 4.1, we derive the existence of functions $u$ and $\tilde{\mu}$ as shown for the periodic case.

Proceeding as in Section 3, and using the compactness results Theorems 4.1-4.2 and Remark 4.1, we derive the existence of functions $u_0 \in V, \phi_0 \in W$ (see (3.3) for the definitions of $V$ and $W$), $\tilde{\mu}_0 \in L^2(0,T;B^2_A(\mathbb{R};H^1(Q)))$, $u_1 \in L^2(Q_T;B^2_A(\mathbb{R};B^{1/2}_{#A_y}(\mathbb{R}^2)^2))$, $\phi_1, \mu_1 \in L^2(Q_T;B^2_{A_\Gamma}(\mathbb{R};B^{1/2}_{#A_y}(\mathbb{R}^2)^2))$, $p, \xi \in L^2(Q_T;B^2_A(\mathbb{R}^{2+1}))$, and a subsequence $E' \subseteq E$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ in } V\text{-weak}$$

(4.7)

$$u_\varepsilon \to u_0 \text{ in } L^2(0,T;\mathbb{H})\text{-strong}$$

(4.8)

$$\phi_\varepsilon \to \phi_0 \text{ in } W\text{-weak}$$

(4.9)

$$\phi_\varepsilon \to \phi_0 \text{ in } L^2(Q_T)\text{-strong}$$

(4.10)

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial x_i} \text{ in } L^2(Q_T)^2\text{-weak } \Sigma$$

(4.11)

$$\frac{\partial \phi_\varepsilon}{\partial x_i} \to \frac{\partial \phi_0}{\partial x_i} + \frac{\partial \phi_1}{\partial x_i} \text{ in } L^2(Q_T)\text{-weak } \Sigma$$

(4.12)

$$\frac{\partial \mu_\varepsilon}{\partial x_i} \to \frac{\partial \tilde{\mu}_0}{\partial x_i} + \frac{\partial \mu_1}{\partial x_i} \text{ in } L^2(Q_T)\text{-weak } \Sigma$$

(4.13)

$$p_\varepsilon \to p \text{ in } L^2(Q_T)\text{-weak } \Sigma$$

(4.14)

$$f(\phi_\varepsilon) \to \xi \text{ in } L^2(Q_T)\text{-weak } \Sigma.$$  

(4.15)

Since $\text{div } u_\varepsilon = 0$, it holds that $\text{div}_y u_1 = 0$. As in the preceding section, we set $\Phi = (\phi_0, \phi_1)$, $u = (u_0, u_1)$, $\mu = (\mu_0, \mu_1)$, and we use the same definition for $\mathbb{D}u$: $\mathbb{D}u = \nabla u_0 + \nabla_y u_1$. We also set $\mu_0 = M_\tau(\tilde{\mu}_0)$ and observe that

$$M(\xi) = f(\phi_0)$$

(4.16)

as shown for the periodic case.

We proceed as in the proof of Proposition 3.1 to get that $u = (u_0, u_1)$, $\Phi = (\phi_0, \phi_1)$, $\mu = (\tilde{\mu}_0, \mu_1)$, $\xi$ and $p$ determined above by (4.17)-(4.19), solve the system (4.17)-(4.19) below

\[
\begin{cases}
- \int_{Q_T} u_0 \frac{\partial \Psi_0}{\partial t} \, dx \, dt + \int_{Q_T} M(A_0 \mathbb{D}u \cdot \mathbb{D} \Psi) \, dx \, dt \\
+ \int_{Q_T} (u_0 \cdot \nabla) u_0 \Psi_0 \, dx \, dt - \int_{Q_T} M(p \text{div } \Psi_0 + \text{div}_y \Psi_0) \, dx \, dt \\
- \kappa \int_{Q_T} \mu_0 \nabla \phi_0 \cdot \Psi_0 \, dx \, dt = \int_{Q_T} g \Psi_0 \, dx \, dt
\end{cases}
\]

\[= - \int_{Q_T} \phi_0 \frac{\partial \phi_0}{\partial t} \, dx \, dt + \int_{Q_T} \phi_0 u_0 \cdot \nabla \phi_0 \, dx \, dt + \int_{Q_T} M(\mathbb{D} \mu \cdot \mathbb{D} \varphi) \, dx \, dt = 0;
\]
\[
\int_{Q_T} M_r(\tilde{\mu}_0) \chi_0 \, dx \, dt = \lambda \int_{Q_T} M(\nabla \Phi \cdot \nabla \chi_0) \, dx \, dt + \alpha \int_{Q_T} M(\xi \chi_0) \, dx \, dt
\]

for all \( \Psi = (\Psi_0, \Psi_1) \in C_0^\infty(Q_T)^2 \times (E)^2 \), \( \varphi = (\varphi_0, \varphi_1) \in C_0^\infty(Q_T) \times E \) and \( \chi = (\chi_0, \chi_1) \in (C_0^\infty(Q_T) \otimes A_r^\infty) \times E \) where \( E = C_0^\infty(Q_T) \otimes (A_r^\infty \otimes A_y^\infty) \).

Concerning the homogenized problem, we set \( p_0 = M(p) \) and we proceed as in Subsection 3.3 to uncouple (4.17), (4.18) and (4.19). Starting with (4.19), we choose there \( \chi_1(t, x, \tau, y) = \chi_0^0(t, x) \psi(y) \theta(\tau) \) with \( \chi_0^0 \in C_0^\infty(Q_T) \), \( \psi \in A_y^\infty \) and \( \theta \in A_r^\infty \), and we obtain
\[
M((\nabla \phi_0 + \nabla_y \phi_1) \cdot \nabla_y \psi) = 0 \quad \forall \psi \in A_y^\infty.
\]

For \( r \in \mathbb{R}^2 \), we consider the equation
\[
M((r + \nabla_y \pi(r)) \cdot \nabla_y \psi) = 0 \quad \forall \psi \in A_y^\infty,
\]
which is the variational form of \(-\text{div}_y(r + \nabla_y \pi(r)) = 0 \) in \( \mathbb{R}^2 \), i.e.
\[
-\Delta_y \pi(r) = 0 \quad \text{in} \ \mathbb{R}^2, \quad \pi(r) \in B_{1,2}^{1,2}(\mathbb{R}^2).
\]

Appealing to [24] Theorem 1.2], (4.20) possesses constant solutions \( \pi(r) \) since \( \nabla_y \pi(r) = 0 \).

This yields at once
\[
\int_{Q_T} \mu_0 \chi_0 \, dx \, dt = \lambda \int_{Q_T} \nabla \phi_0 \cdot \nabla \chi_0 \, dx \, dt + \alpha \int_{Q_T} f(\phi_0) \chi_0 \, dx \, dt
\]
for all \( \chi_0 \in C_0^\infty(Q_T) \), where \( \mu_0 = M_r(\tilde{\mu}_0) \) and \( M(\xi) = f(\phi) \),

which is the variational form of
\[
\mu_0 = -\lambda \Delta \phi_0 + \alpha f(\phi_0) \quad \text{in} \quad Q_T.
\]

We also uncouple (4.18) and proceed as above in (4.19) to obtain
\[
\frac{\partial \phi_0}{\partial t} + u_0 \cdot \nabla \phi_0 - \Delta \phi_0 = 0 \quad \text{in} \quad Q_T.
\]

Finally, considering (4.17), we fix \( r \in \mathbb{R}^{2 \times 2} \) and consider the equation
\[
\left\{
\begin{array}{l}
\text{Find } \eta(r) \in B_{1,2}^{1,2}(\mathbb{R}^2)^2 \text{ and } \pi(r) \in B_{A_y}^2(\mathbb{R}^2) \\
-\text{div}_y(A_0(t, x, \tau, \cdot)(r + \nabla_y \eta(r))) + \nabla_y \pi(r) = 0 \quad \text{in} \quad \mathbb{R}^2
\end{array}
\right.
\]

Then using standard method, we prove that there exists a unique \( \eta(r) \in B_{\text{div}}^{1,2}(\mathbb{R}^2) \) = \( \{ u \in B_{1,2}^{1,2}(\mathbb{R}^2)^2 : \text{div}_y u = 0 \} \) solution of (4.23) in the following sense
\[
M((A_0(t, x, \tau, \cdot)(r + \nabla_y \eta(r)) \cdot \nabla \psi) = 0 \quad \text{for all } \psi \in B_{\text{div}}^{1,2}(\mathbb{R}^2).
\]

Moreover, using [9] Theorem 2.1, we infer the existence of a function \( \pi(r) \in B_{A_y}^2(\mathbb{R}^2) \) with \( \pi(r) + N \) unique modulo \( I_{A_y}^2(\mathbb{R}^2) \) (where \( N = \{ u \in B_{A_y}^2(\mathbb{R}^2) : \| u \|_2 = 0 \} \)), such that
\[
-\text{div}_y(A_0(t, x, \tau, \cdot)(r + \nabla_y \eta(r))) + \nabla_y \pi(r) = 0 \quad \text{in} \quad \mathbb{R}^2.
\]

Choosing in (4.23) \( r = \nabla u_0(t, x) \) and the uniqueness of the solution to (4.23) leads to \( u_1 = \eta(\nabla u_0) \) and \( p = \pi(\nabla u_0) \) where \( \eta(\nabla u_0) \) stands for the function \( (t, x, \tau) \mapsto \eta_{t,x,\tau}(\nabla u_0(t, x)) \),
which belongs to $L^2(Q_T; B^2_{\lambda,4}(\mathbb{R}^3; B^1_{\text{div}}(\mathbb{R}^2)))$. Clearly, if $\eta_j^\ell$ is the solution of (4.23) corresponding to $r = r_j^\ell = (\delta_{ij}\delta_{k\ell})_{1 \leq i,k \leq 2}$ (that is all the entries of $r$ are zero except the entry occupying the $j$th row and the $\ell$th column which is equal to 1), then

$$\mathbf{u}_1 = \sum_{j,\ell=1}^{2} \frac{\partial \mathbf{u}_0}{\partial x_j} \eta_j^\ell \text{ where } \mathbf{u}_0 = (u_0^\ell)_{1 \leq \ell \leq 2}. \quad (4.24)$$

We recall again that $\eta_j^\ell$ depends on $t, x, \tau$ as it is the case for $A_0$. In the variational form of (4.17), we insert the value of $\mathbf{u}_1$ obtained in (4.24) to get the equation

$$\frac{\partial \mathbf{u}_0}{\partial t} - \text{div}(\hat{A}_0 \nabla \mathbf{u}_0) + (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 + \nabla p_0 - \kappa \mu_0 \nabla \phi_0 = g \text{ in } Q_T \quad (4.25)$$

where $\hat{A}_0(t, x) = (\hat{a}_{ij}^{\ell \ell}(t, x))_{1 \leq i,j,k,\ell \leq 2}$, $\hat{a}_{ij}^{\ell \ell}(t, x) = a_{\text{hom}}(\eta_i^\ell + P_j^\ell, \eta_i^\ell + P_j^\ell)$ with $P_j^\ell = y_j^\ell e^\ell$ ($e^\ell$ the $\ell$th vector of the canonical basis of $\mathbb{R}^2$) and

$$a_{\text{hom}}(\mathbf{u}, \mathbf{v}) = \sum_{i,j,k=1}^{2} M \left( a_{ij} \frac{\partial u^k}{\partial y_j} \frac{\partial v^k}{\partial y_i} \right) \text{ where } A_0 = (a_{ij})_{1 \leq i,j \leq 2}. \quad (4.26)$$

4.3 Homogenized problem. Putting together the equations (4.21), (4.22), (4.25) together with boundary and initial conditions, we obtain the following system

$$\begin{aligned}
\frac{\partial \mathbf{u}_0}{\partial t} - \text{div}(\hat{A}_0 \nabla \mathbf{u}_0) + (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 + \nabla p_0 - \kappa \mu_0 \nabla \phi_0 &= g \text{ in } Q_T \\
\text{div } \mathbf{u}_0 &= 0 \text{ in } Q_T \\
\frac{\partial \phi_0}{\partial t} + \mathbf{u}_0 \cdot \nabla \phi_0 - \Delta \phi_0 &= 0 \text{ in } Q_T \\
\mu_0 &= -\lambda \Delta \phi_0 + \alpha f(\phi_0) \text{ in } Q_T \\
(\mathbf{u}_0, \phi_0)(0) &= (\mathbf{u}_0^*, \phi_0^*) \text{ in } Q \\
\frac{\partial \phi_0}{\partial \nu} &= \frac{\partial \mu_0}{\partial \nu} \text{ on } (0, T) \times \partial Q \\
\mathbf{u}_0 &= 0 \text{ on } (0, T) \times \partial Q.
\end{aligned} \quad (4.26)$$

The following is the main homogenized result of this section. Its proof follows the same lines as the one of Theorem 3.4.

**Theorem 4.3.** Assume that (A1)-(A3) and (4.6) hold. For each $\varepsilon > 0$ let $(\mathbf{u}_\varepsilon, \phi_\varepsilon, \mu_\varepsilon, p_\varepsilon)$ be the unique solution of (2.1)-(2.4). Then the sequence $(\mathbf{u}_\varepsilon, \phi_\varepsilon, \mu_\varepsilon, p_\varepsilon)_{\varepsilon > 0}$ converges strongly in $L^2(Q_T)^2 \times L^2(Q_T)$ (with respect to the first two components $(\mathbf{u}_\varepsilon, \phi_\varepsilon)$) and weakly in $L^2(Q_T) \times L^2(0, T; H^1(Q))$ (with respect to $(\mu_\varepsilon, p_\varepsilon)$) to the solution of problem (4.26).

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