Klyshko’s Advanced-Wave Picture in Stimulated Parametric Down-Conversion with a Spatially Structured Pump Beam

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(Dated: May 18, 2018)

The advanced-wave picture is “...an intuitive treatment of two-photon correlation with the help of the concept of an effective field acting upon one of the two detectors and formed by parametric conversion of the advanced wave emitted by the second detector ...” [A. V. Belinskii and D. N. Klyshko, JETP 78, 259 (1994)]. This quote from Belinskii and Klyshko nicely describes the concept of the advanced-wave picture; an intuitive tool for designing and predicting results from coincidence-based two-photon experiments. Up to now, the advanced-wave picture has been considered primarily for the case of an ideal plane-wave pump beam and only for design purposes. Here we study the advanced-wave picture for a paraxial pump beam. This suggests stimulated parametric down-conversion as a useful experimental tool for testing the experimental sets designed with the advanced-wave picture. We present experimental results demonstrating the strategy of designing the experiment with advanced-wave picture and testing with stimulated emission.

PACS numbers: 42.50.Dv, 03.67.Mn

I. INTRODUCTION

Optical correlations from spontaneous parametric down-conversion (SPDC) have been largely used to experimentally investigate fundamental aspects of quantum mechanics and to implement quantum information protocols. These correlations can be explored using several optical degrees of freedom, and are measured at the single-photon level by detecting the down-converted photon pairs with mode analyzers and coincidence electronics.

In a two-photon experiment, the coincidence-count distribution \( C(\phi_1, \phi_2) \) is obtained by projecting photons 1 and 2 in the optical modes described by (or labeled as) \( \phi_1 \) and \( \phi_2 \), respectively. The coincidence-count distribution is proportional to the joint probability \( P(\phi_1, \phi_2) \) for the detection of the photon pair in these optical modes. Among the optical degrees of freedom for which down-converted photons may exhibit correlations, transverse modes are an interesting subject of study [1], as in this case the two-photon correlations extend over a wide range of spatial modes \( \{\phi_1, \phi_2\} \).

Even though the structure of the two-photon spatial correlations \( P(\phi_1, \phi_2) \) may be intricate [2-4], D. N. Klyshko developed a simple method, introduced in 1988, for describing these correlations when considering a SPDC source excited by a plane-wave pump-photon [5]. Instead of considering the non-local joint detections that take place in a real-life two-photon experiment, Klyshko’s advanced-wave picture (AWP) is based on a prepare-and-measure scenario. In this scenario, the detection event at one of the detectors is replaced with (or thought of as) an emission event. Then, in place of detecting a photon (say photon 1) in the spatial mode \( \phi_1 \), in the AWP one is effectively preparing a photon with transverse spatial mode \( \phi_1 \). This prepare and measure scheme is assigned with a conditional probability \( P_{AWP}(\phi_2|\phi_1) \) of detecting the emitted photon in the spatial mode \( \phi_2 \) after some propagation, given that it was prepared in spatial mode \( \phi_1 \). Klyshko’s AWP is constructed in such a way that this single-photon prepare-and-measure probability equals the two-photon joint probability for the detection of photon 2 in spatial mode \( \phi_2 \) given that its correlated down-converted partner photon was detected in spatial mode \( \phi_1 \): \( P_{AWP}(\phi_2|\phi_1) = P_{SPDC}(\phi_2|\phi_1) \). Being based on the preparation and measurement of an advanced wave, the AWP can be cast in terms of a classical optics experiment in which the detected intensities of a classical optical field are proportional to the SPDC joint probabilities measured at the single-photon level. According to Belinskii and Klyshko [6], this advanced wave “...plays the role of an effective field, whose intensity determines the probability of detecting a photon at the point-event 2 given that the other photon has been detected at the point 1...”.

In 1994, Belinskii and Klyshko [6] theoretically analyzed two-photon image and diffraction effects, where the conditions for the formation of a two-photon image and diffraction were predicted using a classical optics setup based on the AWP. This was an interesting demonstration of the usefulness of the AWP for designing experiments and predicting results. Two-photon imaging and diffraction are optical effects observed in the spatial distribution of the joint detection of the down-converted...
A feature of the AWP is that this advanced wave emitted from detector 1’s location is analogous to the temporal reversion of the down-converted field 1, implying that the advanced wave is emitted towards the SPDC source. Also, the propagation undergone by the advanced wave is given by the propagation from detector 1 to the SPDC source, followed by the propagation from the SPDC source to detector 2, so that exactly the same optical elements (lenses, apertures, free-space, etc...) are taken into account. Then, the SPDC source acts as a mirror for the advanced wave, where the shape of the mirror is dictated by the pump beam size and curvature. In words of Belinskii and Klyshko [6], “...the advanced wave is effectively reflected by the wavefronts of the pump wave inside the crystal, i.e., the thin crystal serves as a mirror for the advanced wave...”. For a plane-wave pump beam considered by Klyshko [5, 6], the nonlinear crystal is simply replaced with a plane mirror that has infinite size in the transverse directions. In this case, the law of reflection on a specular surface, i.e., the angle of incidence equals the angle of reflection, accounts for the perfect transverse wave-vector correlations between photon pairs created from an SPDC pump photon with well defined transverse wave-vector. Pump beam phase curvatures lead to a curved, transversely infinite mirror, and have also been considered in the context of AWP [10]. More recently, an experimental demonstration of the equivalence between two-photon image and the AWP prepare and measure scenario was carried out using a camera-based coincidence system [13], and a classical optics “prepare and measure” experiment was used to predict novel orbital angular momentum correlations in SPDC [14].

In the majority of SPDC experiments, the pump beam is a well collimated zero-order Gaussian beam, often approximated in theory by a plane wave. Although this approximation reveals itself useful for understanding a huge variety of phenomena, a number of experiments have been performed that exploit the spatial structure of the pump beam. These include the control of two-photon interference [19, 20], manipulation of correlations in orbital angular momentum [21, 23], creation of novel optical vortex structures [24, 25], violation of Bell’s inequalities [26], increase of spatial entanglement [4, 27] and exploration of higher-order quantum correlations [28, 29]. These experiments fall outside the usual treatment of the AWP. In this paper, we consider a more general Parametric Down-Conversion (PDC) experiment in which the pump beam has arbitrary spatial structure. We show that the two-photon coincidence distribution is equivalent to propagation of an optical field through an optical element with profile equivalent to the pump profile. In this treatment, the crystal can be considered as an optically addressed spatial light modulator, where both phase and amplitude are controlled by the pump laser beam. In order to illustrate these concepts, we use Stimulated PDC (StimPDC), in which one of the down-converted fields is stimulated with an auxiliary laser.

StimPDC is a parametric amplifier without cavity and has been used to demonstrate image and coherence transfer [30, 32], phase conjugation [33] and conservation of the orbital angular momentum [44]. The auxiliary laser stimulating the emission in the signal beam acts as the advanced wave for the idler, which is reflected by the pump in the AWP.

Another issue related to the design and realization of experiments involving pairs of photons and transverse spatial effects concerns the measurement of the coincidence patterns. The first approach at hand is to scan the photon detectors through the detection planes using small pinholes or optical fibers and reconstruct the conditional spatial structure. This task can be very time consuming, and becomes prohibitive if one needs to follow some iterative procedure to align or optimize some parameter of the setup. The second possibility is to use intensified CCD cameras that allow the direct measurement of the whole two-photon coincidence [35, 36] pattern. However, even though this is more efficient than scanning, it also requires some time due to the weak flux of photons in the spontaneous parametric down-conversion in addition to other technical limitations of these devices. We propose and demonstrate here an alternative approach using stimulated emission for testing and aligning the experimental set-up.

In section [11], we briefly review the Fourier optics description of a paraxial field propagating through a linear optical system. In section [11] we describe the two-photon quantum state produced in spontaneous and stimulated down-conversion with an arbitrary pump beam, and discuss the AWP in this context. In section [V] we provide some examples of the utility of the AWP and stimulated emission approach in the description of two-photon experiments with structured pump beams. We provide concluding remarks in section [VI].
where

\( k \)  

The optical system can be described, in a very general way, by the following input/output relation [37]

\[
\phi(\mathbf{q}) = \frac{1}{2\pi} \int d\rho \, \mathcal{E}(\rho) e^{-i\mathbf{q}\cdot\mathbf{\rho}},
\]

where \( \mathbf{q} = (k_x, k_y) \) are the transverse wave vector coordinates and \( \rho = (x, y) \) are its conjugate position coordinates. \( \mathcal{H}(\mathbf{q}', \mathbf{q}) \) is the impulse-response function of the optical device in the transverse wave vector domain, which we henceforth refer to simply as transfer function. All integrals have limits \(-\infty\) to \(\infty\) unless otherwise noted.

Let us consider a few particular cases that are going to be useful further in this article. The first one is the free paraxial propagation over a distance \( z \):

\[
H_z(\mathbf{q}', \mathbf{q}) = \delta(\mathbf{q}' - \mathbf{q}) \exp \left[ -i \frac{q^2}{2k} z \right],
\]

where \( k = \sqrt{k_x^2 + k_y^2} \) is the wave number and \( k_z \) is the \( z \) component of the wave vector \( k \). The above transfer function, after integration in Eq. (1), simply multiplies the initial angular spectrum by a phase factor.

The second case is the amplitude-and-phase mask. It is described in the position space by a function \( T(\rho) \) relating the field \( \mathcal{E}_+ \) immediately after the mask to the field \( \mathcal{E}_- \) immediately before:

\[
\mathcal{E}_+(\rho) = T(\rho) \mathcal{E}_-(\rho).
\]

In transverse momentum space, the equation above is written in terms of the function \( t \), the Fourier transform of \( T \):

\[
\phi_+(\mathbf{q}) = \int d\mathbf{q}' \, t(\mathbf{q}' - \mathbf{q}) \phi_-(\mathbf{q}'),
\]

from which \( t(\mathbf{q}' - \mathbf{q}) \) is the mask’s transfer function.

Finally, we examine an optical system composed of three subsystems as shown in Fig. 2 where, in between two subsystems of arbitrary transfer functions \( A \) and \( B \), we find a mask as described by Eq. (1). The output angular spectrum of this optical system is related to its input by

\[
\phi_{\text{out}}(\mathbf{q}) = \int d\mathbf{q}' \, d\mathbf{q}'' \, d\mathbf{q}'' \, B(\mathbf{q}', \mathbf{q}'' \, t(\mathbf{q}'' - \mathbf{q}') \times A(\mathbf{q}'', \mathbf{q}'') \phi_{\text{in}}(\mathbf{q}'').
\]

The output amplitude in spatial coordinates is therefore given by:

\[
\mathcal{E}_{\text{out}}(\rho) = \frac{1}{2\pi} \int d\mathbf{q}' \, d\mathbf{q}'' \, d\mathbf{q}'' \, e^{i\mathbf{q}'\cdot\mathbf{\rho}} B(\mathbf{q}', \mathbf{q}'' \, t(\mathbf{q}'' - \mathbf{q}') \times A(\mathbf{q}'', \mathbf{q}'') \phi_{\text{in}}(\mathbf{q}''),
\]

where \( \phi_{\text{in}} \) can yet be written in terms of \( \mathcal{E}_{\text{in}} \), allowing us to express the output field in terms of the integral transform of the input field. The last equation will be of particular help to build the Advance Wave Picture for both spontaneous and stimulated PDC scenarios. Note that \( A \) and \( B \) are completely arbitrary paraxial transfer functions that may comprise free propagation and/or any spatial modulations possibly introduced by the optical subsystems.

### III. TWO-PHOTON QUANTUM STATE GENERATED BY SPDC

SPDC is a wave-mixing process involving three fields: pump, signal and idler. The pump is usually a laser beam, while signal and idler are weak fields, due to the low conversion efficiency of the spontaneous process. In the monochromatic, paraxial and thin-crystal approximations, the two-photon state produced by SPDC is well described by [11][12][18][35]

\[
|\psi\rangle = |\text{vac}\rangle + C \int dq_1 dq_2 \, \psi(q_1 + q_2) \, |1; q_1\rangle \, |1; q_2\rangle,
\]

where \( |\text{vac}\rangle \) represents a single-photon state in the mode with transverse momentum \( \mathbf{q} \) and \( \psi(q) \) is the normalized angular spectrum of the pump beam at the exit plane of the crystal. 1 and 2 are indices relative to signal and idler fields, respectively.

The two-photon detection probability is proportional to the fourth-order correlation function [39]

\[
P(\rho_1, \rho_2) \propto \langle \psi | E^\dagger_1(\rho_1) E^\dagger_2(\rho_2) E_1(\rho_1) E_2(\rho_2) | \psi \rangle,
\]

where \( E(\rho) \) is the detection operator for a photon detected at position \( \rho \). Assuming that the pump beam is sufficiently weak so that the production of multiple photon pairs is negligible, one can associate a wave function \( \Psi \) to the two photon state so that \( P(\rho_1, \rho_2) = |\Psi(\rho_1, \rho_2)|^2 \), where [40]

\[
\Psi(\rho_1, \rho_2) = \langle 0 | E_1(\rho_1) E_2(\rho_2) | \psi \rangle.
\]
FIG. 3: a) Spontaneous PDC using a pump laser with spatial structure. The down-converted photons travel through optical systems $H_1$ and $H_2$. Photon 1 is projected onto spatial mode $\phi$. b) In the similar scheme for Stimulated PDC, an auxiliary laser is sent along the signal direction, stimulating generation of signal photons in the laser mode.

Two-photon coincidence imaging has been considered by a number of authors \[12, 41, 42\]. Let us suppose that photons 1 and 2 propagate through optical systems described by the transfer functions $H_1$ and $H_2$, as illustrated in Fig. 3b. We also assume that photon 2 will be detected by a point detector. In this case the detection operator is

$$E_2(\rho_2) = \frac{1}{2\pi} \int dq_2 dq_2' e^{i q_2' \rho_2} H_2(q_2', q_2') a_2(q_2),$$

where the annihilation operator $a_2(q_2)$ annihilates a photon in the optical mode with transverse momentum component $q_2$.

Consider now that photon 1 is projected onto the spatial mode $\phi$. There are a number of possible strategies allowing this kind of projection. One nice example is the projection onto a Laguerre-Gaussian mode using a single mode optical fiber and a holographic mask \[21\]. In all cases, projection onto a spatial mode $\phi$ can be performed using an optical mode selector and a single mode fiber. The detection operator in this case is given by $E_1(\rho_1) \rightarrow E_{1\phi}$, where

$$E_{1\phi} = \int dq_1 dq_1' H_1(q_1, q_1') \phi^*(q_1') a_1(q_1),$$

and $\phi(q)$ is the mode’s angular spectrum. Using operators \[11\] and \[12\] in

$$\Psi_\phi(\rho_2) = \langle 0 | E_{1\phi} E_2(\rho_2) | \psi \rangle,$$

the two-photon wave function for a thin crystal becomes

$$\Psi_\phi(\rho_2) = \frac{1}{2\pi} \int dq_1 dq_2 dq_1' dq_2' v(q_1 + q_2) \int dq_1 dq_2 \int dq_1' dq_2' \int dq_1'' dq_2'' v(q_1' + q_2') \int dq_1 dq_2 \int dq_1' dq_2' \int dq_1'' dq_2'' \int dq_1''' dq_2''' v(q_1''' + q_2''') \int dq_1 dq_2 \int dq_1' dq_2' \int dq_1'' dq_2'' \int dq_1''' dq_2''' \int dq_1'''' dq_2'''' v(q_1'''' + q_2''''),$$

where $1$ and $2$ are indices for signal and idler, respectively, $v(q)$ is the angular spectrum of the stimulating field at $z = 0$ (at the crystal) and $|v_s(q)\rangle$ is the corresponding multimode coherent state in the continuous mode representation (p. 565 of Ref. \[39\]).

The idler intensity at a distance $z$ from the crystal and transverse coordinates $\rho_2$ is given by the second-order correlation function

$$I(\rho_2) = \langle E_2^{(-)}(\rho_2) E_2^{(+)}(\rho_2) \rangle,$$

where $E_2^{(+)}(\rho_2)$ is the field propagated through the optical system of transfer function $H_2$ between the crystal and the detection planes:

$$E_2^{(+)}(\rho_2) = \frac{1}{2\pi} \int dq_2 dq_2' H_2(q_2, q_2') a(q_2') e^{iq_2' \rho_2}.$$
As the electric field operator applied to vacuum yields 0, only the second term in Eq. (15) contributes to the intensity. Thus,

\[ \mathbf{E}_2^{(+)}(\mathbf{q}_j) |\psi\rangle \propto \int d\mathbf{q}' d\mathbf{q}_1 d\mathbf{q}_2 H_2(\mathbf{q}_2, \mathbf{q}') e^{i\mathbf{q}' \cdot \mathbf{p}_2} \times v_p(q_1 + q_2) a^\dagger(q_1) |v_s(q)\rangle |0\rangle \]  

(18)

and, therefore,

\[ \mathcal{I}(\mathbf{p}_2) \propto \int d\mathbf{q}' d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}' d\mathbf{q}_1 d\mathbf{q}_2 H_2(\mathbf{q}_2, \mathbf{q}') \times e^{-i(\mathbf{q}' - \mathbf{q}) \cdot \mathbf{p}_2} v_p^\ast(q_1' + q_2') v_p(q_1 + q_2) \times \langle v_s(q) | a(q_1') a^\dagger(q_1) |v_s(q)\rangle \]  

(19)

The commutation relation for the bosonic operators yields \( \langle v_s(q) | a(q_1') a^\dagger(q_1) |v_s(q)\rangle = \delta(q_1' - q_1) + v_s^\ast(q_1) v_s(q_1) \), splitting the integral of Eq. (19) in two parts:

\[ \mathcal{I}(\mathbf{p}_2) = \mathcal{I}_{\text{spont}}(\mathbf{p}_2) + \mathcal{I}_{\text{stim}}(\mathbf{p}_2) . \]  

(20)

The first part – the one involving \( \delta(q_1' - q_1) \) – describes the spontaneous emission, while the second one – which carries \( v_s^\ast(q_1) v_s(q_1) \) – represents the stimulated process. This nice and simple result was obtained in Ref. [31] for free propagation after the crystal and is generalized here for any given transfer function \( H_2 \). In order to prove this statement, we use the fact that the angular spectrum \( v \) is the Fourier transform of the amplitude profile \( \mathcal{V} \), which allows us to write, in particular:

\[ v(q_1 + q_2) = \int d\rho \; \mathcal{W}(\rho) e^{-i(q_1 + q_2) \cdot \rho} . \]  

(21)

With that, a straight-forward calculation provides the expression for the first term on the right-hand side of Eq. (20):

\[ \mathcal{I}_{\text{spont}}(\mathbf{p}_2) \propto \int d\rho \; |\mathcal{W}_p(\rho)|^2 \times \left| \int d\mathbf{q}_2' d\mathbf{q}_2 H_2(\mathbf{q}_2, \mathbf{q}_2') e^{i\mathbf{q}_2' \cdot \mathbf{p}_2} \right|^2 , \]  

(22)

which, on the grounds that it does not depend on the profile of the auxiliary laser, gives the total intensity due to the spontaneous emission after propagation through the optical system of transfer function \( H_2 \).

It is worth noting that, in the case of paraxial free propagation from crystal to detector (Eq. 3), the second line in Eq. (22) reduces to 1. This implies that the detected intensity after free propagation contains no information on the spatial structure of the pump beam, since the squared modulus of the pump profile \( \mathcal{W}_p \) is integrated over the transverse spatial coordinates.

The second term on the right-hand side of Eq. (20) can be written as:

\[ \mathcal{I}_{\text{stim}}(\mathbf{p}_2) \propto \left| \int d\mathbf{q}_2' d\mathbf{q}_1 d\mathbf{q}_2 H_2(\mathbf{q}_2, \mathbf{q}_2') e^{i\mathbf{q}_2' \cdot \mathbf{p}_2} \times v_p(q_1 + q_2) v_s^\ast(q_1) \right|^2 . \]  

(23)

which gives the intensity due to StimPDC. We can see that it depends on \( v_s^\ast(\mathbf{p}) \), the angular spectrum of the auxiliary beam in the crystal plane. The contribution from the stimulated emission can be made much stronger than that from spontaneous emission if the auxiliary laser intensity is high enough. Typically, a few milliwatts are enough to produce a stimulated emission 100 times stronger than the spontaneous emission.

To see the relation between StimPDC and the AWP, we note that the last equation is isomorphic to Eq. (7). Indeed, if we assume that the angular spectrum of the auxiliary field is prepared by sending an initial field \( \phi^\ast(q_1') \) back through an optical system represented by the transfer function \( H_1 \), we can replace

\[ v_s^\ast(q_1) = \int d\mathbf{q}_1 H_1^B(q_1', q_1) \phi^\ast(q_1') \]  

(24)

in Eq. (23), where \( H_1^B \) is the transfer function of the optical system \( H_1 \) in the backwards direction (from the detector to the crystal). Then, we get

\[ \mathcal{I}_{\text{stim}}(\mathbf{p}_2) \propto \left| \int d\mathbf{q}_1' d\mathbf{q}_2' d\mathbf{q}_1 d\mathbf{q}_2 e^{i\mathbf{q}_2' \cdot \mathbf{p}_2} \phi^\ast(q_1') \times H_1^B(q_1', q_1) v_p(q_1 + q_2) H_2(q_2, q_2') \right|^2 , \]  

(25)

which is identical in form to Eq. (7). We conclude that the AWP, in this sense, also applies to StimPDC.

The auxiliary laser field \( v_s^\ast(q_1) \) can be properly prepared using a spatial light modulator (SLM) for instance, in order to help in the design of experiments with twin photons from SPDC. Moreover, it can be helpful for alignment, since one can align the set-up with the auxiliary laser and the stimulated idler beam, which is so intense that we can observe it with a common and inexpensive CCD camera, or even the naked eye, depending on the wavelength. Once the set-up is aligned, one can just turn off the auxiliary laser and perform coincidence counting experiments.

In the next section, we present a number of examples of how this can be used for novel optical experiments.

V. EXAMPLES AND EXPERIMENTS WITH STIMULATED EMISSION

We illustrate the usefulness of stimulated parametric down-conversion in the design of experiments with spontaneous emission. The experimental set-up is sketched in Fig. 1. A diode laser oscillating at 405 nm pumps a BBO nonlinear crystal. We work in a non-collinear phase-matching configuration with very small angle between signal and idler, \( 1^\circ \), and use 10 nm bandwidth interference filters centered at 780 nm (signal) and 840 nm (idler). Another diode laser (auxiliary laser) at 780 nm is aligned with the signal direction and stimulates
the emission in both signal and idler. Pump and auxiliary lasers can be spatially modulated on demand, which is represented on the picture by reflection on an SLM.

The intensity of the emission in the idler field is strongly enhanced in comparison to the case of only spontaneous emission, generating a beam with macroscopic intensity whose transverse profile is detected by a CCD camera, along with a relatively small background coming from spontaneous emission. In practice, we can monitor the stimulated emission profile in real time. The auxiliary laser transverse profile is also monitored with a CCD camera.

A. Phase conjugation effects

The physical phenomenon behind the AWP is phase conjugation. For time-dependent waves, phase conjugation is equivalent to temporal reversion. Therefore, the advanced wave (as in the AWP) is exactly the time reversal of the signal beam. In stimulated down-conversion, the phase conjugation is evident in Eq. (23). It is the conjugate of the auxiliary beam’s angular spectrum that, together with the pump, determines the properties of the idler beam. For instance, if the pump beam has a flat transverse field distribution, or in other words, if its angular spectrum is strongly concentrated around $q = 0$, the idler beam will propagate forward as if it were the reflection of the advanced wave (signal propagating backwards).

Experiment. In reference [33], the authors observed effects of phase conjugation by analyzing the symmetry of images transferred from the pump and auxiliary beams to the idler. Here, we illustrate the phase conjugation effects by observing the focusing of the idler beam as we make the auxiliary laser diverge. The scheme is sketched in Fig. 5. The pump is collimated, so that its wave front and amplitude distribution are practically flat. The auxiliary laser is sent to the SLM, where a divergent lens of variable focal length is implemented, and its profile is monitored with a CCD camera. We start with very long focal length, so that the auxiliary beam stays collimated. In this case, the idler beam has its largest spot size. As we force the auxiliary laser to diverge (by bringing the focal length from $-\infty$ toward 0), its spot size in the camera increases. Consequently, the idler spot starts decreasing. That is to say, because the idler beam reproduces the auxiliary’s advanced wave reflected by a flat mirror, its spot size varies in the opposite way as compared to the auxiliary. If the auxiliary field is converging, the idler is diverging, and vice-versa. We have observed this behavior, and the results are shown in Fig. 6.

B. Phase-modulated pump - Fractional Fourier Transform

The Fractional Fourier Transform (FRFT) has been studied in the context of coincidence imaging by several authors [44–50]. In these experiments, the down-converted photons pass through optical systems that are used to implement the FRFT. In our first example, we show that the FRFT can be implemented in the AWP by controlling the pump beam alone. The FRFT is a generalization of the usual Fourier transform, and it is parameterized by the angle $\alpha$. Its kernel is given by

$$F_{\alpha}(q, q') \propto \exp \left\{ -\frac{i}{2} \left[ \cot \alpha (q^2 + q'^2) + 2 \frac{q' \cdot q}{\sin \alpha} \right] \right\},$$

which transforms a function $\Psi(q)$ into a function $\Phi(q')$. Here $q$ and $q'$ are dimensionless variables, which can be represented by different coordinate axes in a bosonic phase space. For $\alpha = \pi/2$, one recovers the usual Fourier transform.
which is equivalent to the FRFT kernel (26). We write

\[ F_\pi(q, q') \propto \delta(q + q'), \]

(28)

where \( q \) and \( q' \) live in the same space, which can be understood in the context of optics as an imaging system.

Let us now return to the wavefunction (14), and consider that the pump beam is a Gaussian beam \([37]\) with angular spectrum given by:

\[ \psi(q_1 + q_2, Z) = f(q_1 + q_2) e^{i\frac{Z(q_1^2 + q_2^2)}{2K}} e^{i\varphi(Z)}, \]

(29)

where \( f \) is the Gaussian envelope defined at the waist position, and \( \varphi(z) \) is the Gouy phase. \( Z \) is the distance from the pump beam waist to the crystal. As we are in the thin crystal approximation, we will neglect its thickness. Let us also consider the special case where the down-converted photons propagate freely the distance \( z \) between the crystal and the corresponding detection planes. For the present purpose it is convenient to assume equal distances \( z \) for signal and idler. The free propagation transfer functions are for signal and idler are given by Eq. (3). Then, the detection amplitude is

\[ \Psi(\rho_2) = e^{i\varphi(Z)} \int dq_1 dq_2 f(q_1 + q_2) \phi^*(q_1) e^{-iq_2\rho_2} \]

\[ \exp \left\{ i \left[ \frac{Z(q_1 + q_2)^2}{2K} - \frac{z(q_1^2 + q_2^2)}{2k} \right] \right\}. \]

(30)

It is convenient to change variables, so that the detection amplitude can be written in terms of dimensionless variables \( \tilde{q}_1 = sq_1 \) and \( \tilde{q}_2 = sq_2 \):

\[ \Psi(\rho_2) = e^{i\varphi(Z)} \frac{1}{s^2} \int dq_1 dq_2 f \left( \frac{q_1 + q_2}{s} \right) \phi^* \left( \frac{q_1}{s} \right) e^{-s^2q_2\rho_2} \]

\[ \times \exp \left\{ -\frac{i}{2s^2} \left[ \frac{Z}{k} (q_1 \cdot \hat{q}_2) - \left( \frac{z}{s} + \frac{Z}{K} \right) (q_1^2 + q_2^2) \right] \right\}. \]

(31)

where we considered that \( z_1 = z_2 = z \). If we choose

\[ \frac{Z}{Ks^2} = \frac{1}{\sin \alpha} \quad \text{and} \quad \left( \frac{z}{k} + \frac{Z}{K} \right) \frac{1}{s^2} = \cot \alpha, \]

(32)

it is possible to show that the last phase term in Eq. (31) becomes

\[ \exp \left\{ -\frac{i}{2} \left[ \cot \alpha(q_1^2 + q_2^2) + \frac{2(q_1 \cdot \hat{q}_2)}{\sin \alpha} \right] \right\}, \]

(33)

which is equivalent to the FRFT kernel (26). We write the detection amplitude explicitly:

\[ \Psi(\rho_2) = e^{i\varphi(Z)} \frac{1}{s^2} \int dq_1 dq_2 f \left( \frac{q_1 + q_2}{s} \right) \phi^* \left( \frac{q_1}{s} \right) e^{-s^2q_2\rho_2} \]

\[ \times \phi^* \left( \frac{q_1}{s} \right) \exp \left\{ -\frac{i}{2} \left( \cot \alpha(q_1^2 + q_2^2) + \frac{2(q_1 \cdot \hat{q}_2)}{\sin \alpha} \right) \right\}. \]

Let us further assume that the pump beam envelope function \( f \) is much larger than the function \( \phi^* \). Then, after integration in \( q_2 \), we have:

\[ \Psi(\rho) = e^{i\varphi(Z)} \]

\[ \times \int dq \phi^* \left( \frac{q}{s} \right) \exp \left\{ -\frac{i}{2} \left[ (q^2 + \rho^2) \tan \alpha + 2q \cdot \rho \cos \alpha \right] \right\}, \]

(35)

which is the FRFT of the function \( \phi^*(q) \), written in terms of the conjugate variable \( \rho \).

A number of previous results and possible applications can be obtained as special cases of Eqs. (31) and (36). For example, the imaging experiment performed by Pittman et al. \([30]\) is obtained from Eq. (36) by choosing \( \alpha = \pi \). In this case \( \Psi(\rho) \) gives the Fourier transform of \( \phi^*(q) \). Recalling that \( \phi^*(q) \) represents a certain transverse mode preparation of the AWP light source, its Fourier transform is actually the image of some aperture function (which implements the mode filtering) placed in front of the detector. We note that the a similar effect was used to optimize the pair collection efficiency in SPDC \([51]\).

In the more general case, proper control of the phase curvature of the pump beam allows for lensless implementation of the FRFT, which has applications in imaging and signal processing \([52]\). We note that a lensless implementation of the FRFT was performed using partially coherent light and coincidence detection in Ref. \([43]\), though phase curvature of the beam was not relevant in that case.

**Experiment.** We now illustrate the effect of phase modulation of the pump transverse profile in the stimulated emission. Consider that the function \( \phi(q) \) is given by

\[ \phi(q) = \int d\rho \varphi(\rho) \exp(-i q \cdot \rho), \]

(36)

where \( \varphi(\rho) = \begin{cases} 1 & \text{if } \frac{d-\delta}{2} < |\rho| < \frac{d+\delta}{2} \\ 0 & \text{elsewhere} \end{cases} \)

(37)

which characterizes the image of a double slit, centered at \( x = 0 \), with slit width \( \delta \) and separation \( d \). For this kind of AWP source and performing the FRFT, we expect to see transverse distributions varying between the usual Young double slit interference pattern when the FRFT parameter \( \alpha = \pi/2 \) and the image of the double slit when \( \alpha = 0 \). The FRFT parameter \( \alpha \) is varied by changing
distance aperture, which is imaged onto a plane situated at where the pump beam propagates through some diffraction in the StimPDC case. Let us consider the situation counting rate in the SPDC case, or to the idler intensity in an SLM (Fig. 7).

The curvature of the pump beam wave front. We have done this with a variable focal-length lens simulated by an SLM (Fig. 7).

The function \( \phi \) is realized by sending the auxiliary laser through a double slit, which is imaged to the detection plane by a lens located between the double slit and the crystal. The results are shown in Fig. 8, which compares, for different values of \( \alpha \) implemented, the observed intensity of the idler beam with the simulated FRFT of the imaged auxiliary laser.

C. Amplitude-modulated pump

It is pedagogical to analyze the cases where the structure of the pump beam is transferred to the coincidence counting rate in the SPDC case, or to the idler intensity in the StimPDC case. Let us consider the situation where the pump beam propagates through some diffraction aperture, which is imaged onto a plane situated at distance \( z \) after the crystal. Let us assume that the distances from the detection planes and the crystal are also \( z \). It was shown in Ref. [12] that the coincidence counting rate is given by:

\[
C(p_1, p_2 = 0) \propto |W_p(p_1/2, z)|^2, \tag{38}
\]

where \( W_p \) is the pump beam field profile at a distance \( z \) from the crystal. We consider the particular case where \( p_1 = 0 \) is fixed. From the AWP perspective, we have the signal detector acting as point source located at \( p_1 = 0 \) and at a distance \( z \) from the crystal. In this ideal case we ignore the effects due to the finite size of the detector aperture. It emits an advanced wave, which reflects in a structured mirror that has the angular spectrum of the pump and it acquires essentially the same angular spectrum. Therefore, in the same way as the pump, after propagating a distance \( z \), the image given by \( W \) is formed. The only difference is that due to the different wavelength, the coincidence image is two times larger. This can be seen from the factor 2 in the argument of \( W \).

The equivalent scheme for stimulated down-conversion is obtained by replacing the signal point detector with the auxiliary laser focused on a plane at a distance \( z \) from the crystal. In the plane of the crystal, the auxiliary laser will have the same angular spectrum as the advanced wave coming from a point source, neglecting the effects due to the finite size of the laser in the focal plane. Approximating the angular spectrum of the auxiliary laser in the crystal by a plane wave, with \( \nu_s(q) = \delta(q) \), Eq. (23) becomes:

\[
I_{stim}(p_2) \propto \left| \int dq_2 \, v_p(q_2) \exp \left[ i \left( q_2 \cdot p_2 - \frac{q_2^2 z}{2k_2^2} \right) \right] \right|^2 = |W_p(p_2, z)|^2. \tag{39}
\]

Comparing Eqs. (38) and (39), we can see that in both cases the angular spectrum of the pump beam is transferred and, after propagation, the same image as the pump is measured in the idler side. The only difference is the scaling factor of 2 that appears only in the argument of \( W_p \) in Eq. (38).

Experiment. We present here a slightly different case, where an obstacle (a thin horizontal and/or a vertical wire) is placed in front of the pump and imaged onto the crystal plane with a 4f imaging system. In this way, when the pump reaches the crystal, its amplitude shows the exact shape of the obstacle and we have a purely amplitude-modulated pump.

The auxiliary beam is collimated and sent to the crystal. The idler’s intensity then results from the advanced wave propagating back from the detector to the crystal, “reflecting” in the amplitude-modulated pump and propagating towards the detection plane. In our specific experiment, the detection plane is about 30 cm behind the crystal, meaning that the advanced wave goes through some diffraction prior to detection.

Fig. 9 illustrates the experimental setup and Fig. 10 shows the result of the stimulated beam intensity, which displays approximately the same shape of the obstacle in front of the pump.
VI. CONCLUSION

Klyshko’s advanced-wave picture is an extremely useful tool for understanding and designing two-photon coincidence experiments. Here we studied the advanced-wave picture considering a spatially structured pump beam in the context of both spontaneous and stimulated PDC. We show that, when the pump beam angular spectrum is properly prepared, it works in analogy to a spatial light modulator, rather than as a simple mirror as described in the original version of the advanced-wave picture. This allows for a number of interesting applications in quantum imaging and the preparation of spatially entangled photons. Though the AWP has found widespread use in analyzing two-photon coincidence experiments (SPDC), we believe that this is the first time it has been applied to StimPDC. Linking the advanced-wave picture, typically used in analyzing correlations in SPDC, to StimPDC, which can be observed using a simple CCD camera or even the naked eye, suggests that StimPDC can be used to help design, build and align SPDC experiments. As a first example, we discussed how the fractional Fourier transform can be performed in a quantum imaging scenario using the phase curvature of the pump laser as the lens, and presented an experimental implementation using StimPDC. We also show how the advance wave picture applied to both the spontaneous and stimulated cases describe the transfer of the angular spectrum from the pump to the down-converted fields. We believe that this study can be helpful in the manipulation of the spatial correlations of twin photons from parametric down-conversion in several applications, as well in design and alignment of coincidence experiments.

Acknowledgments

The authors acknowledge financial support from the Brazilian funding agencies CNPq, CAPES, FAPERJ, FAPEAL and the National Institute for Science and Technology - Quantum Information.

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