The Sutured Thurston Norm

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Abstract

For sutured three-manifolds $M$, there is a sutured Thurston norm $x^s$ due to M. Scharlemann [10]. Here, we show how depth one foliations of $M$ can be useful tools for computing this norm. This uses the relation of these foliations with fibrations of $DM$ (the double of $M$ along the manifold $R \subset \partial M$ given by the sutured structure). We also prove and use the fact that a natural doubling map $D_* : H_2(M, \partial M) \to H_2(DM, \partial DM)$ is “norm doubling” with respect to the norms $x^s$ and $x$ on $H_2(M, \partial M)$ and $H_2(DM, \partial DM)$, respectively. All of this implies significant relations between the foliation cones of [5] and the sutured norm but, in general, these relations are difficult to pin down.

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1 Introduction

If $M$ is a compact 3-manifold, Thurston [11] defines a (semi)norm $x$ on the real vector space $H_2(M, \partial M)$ (coefficients $\mathbb{R}$ will be understood throughout), with unit ball polyhedral, and proves:

**Theorem 1.1** The fibrations of $M$ over the circle that are transverse to $\partial M$ correspond up to isotopy to the rays through lattice points in the open cones over certain top dimensional faces (called fibered faces) of the unit ball of the Thurston norm.

The cones over fibered faces of the Thurston ball will be called *fibration cones*. This is slightly misleading since the classes lying in the interior of fibration cones correspond to foliations without holonomy, “most” of which are dense-leaved.

Let $(M, \gamma)$ be a compact, connected, oriented, sutured 3–manifold [8]. Write $\partial M = \partial_r M \cup \partial_h M$.

This notation, introduced in earlier papers of ours and in [1], anticipates a foliation tangent to $\partial_r M$ and transverse to $\partial_h M$. Wherever these parts of $\partial M$ meet, $M$ has a convex corner. This notation relates to the standard sutured manifold notation as follows:

\[
\partial_h M = \gamma = A(\gamma) \cup T(\gamma), \\
\partial_r M = R(\gamma) = R_+ \cup R_-.
\]

Here, $A(\gamma)$ is a union of annuli and $T(\gamma)$ is a union of tori, while $R_\pm$ are, respectively, the outwardly and inwardly oriented portions of $R(\gamma)$. The choice of orientations is part of the sutured structure and each component of $R_-$ is separated from a component of $R_+$ by annular components of $\gamma$. Finally, each suture is a closed curve in the interior of a component of $A(\gamma)$, parallel to and oriented with the boundary curves of this annulus. The union of the sutures is denoted by $s$.

We will be interested in taut foliations of $M$, hence will require that $M$ be irreducible and, as a sutured manifold, *taut*. This latter requirement means that each component of $\partial_r M$ is norm-minimizing in $H_2(M, \partial_h M)$. In particular, if $\sigma \subset \partial_r M$ is an imbedded loop bounding a disk in $M$, it also bounds a disk in $\partial_r M$.

In [5], we proved the following analog of Thurston’s Theorem (Theorem 1.1) for depth one foliations.
Theorem 1.2  Let \((M, \gamma)\) be a compact, connected, oriented, irreducible, taut, sutured 3–manifold. There are finitely many closed, convex, polyhedral cones in \(H_2(M, \partial M)\), called foliation cones, having disjoint interiors and such that the taut, transversely oriented, depth one foliations of \((M, \gamma)\) that are transverse to \(\partial_0 M\) and have the components of \(\partial M\) as sole compact leaves correspond to the rays through integer lattice points of \(H_2(M, \partial M)\) in the interior of the foliation cones.

Remark  Set \(M_0 = M \setminus \partial_\tau M\) and remark that a depth one foliation as above restricts to a fibration of \(M_0\) over the circle. The classes in the interior of foliation cones that are not on rays through integer lattice points correspond to foliations “almost without holonomy” with each leaf in \(M_0\) dense in \(M\).

Remark  It is known [3] that the “foliated ray” \(\langle F\rangle\) corresponding to the depth one foliation \(F\) determines \(F\) up to a \(C^0\) isotopy that is smooth in \(M_0\).

Remark  In contrast to Thurston’s result, the cones in Theorem 1.2 are generally not defined by a norm. Indeed, they are not generally symmetric with respect to multiplication by \(-1\).

Remark  The proof of Theorem 1.2 in [5] had some serious gaps. The authors are preparing a revised version [6] of that paper that resolves these problems.

There is a seminorm \(x^s\) for sutured manifolds, called the sutured Thurston norm. This is due to M. Scharlemann [10] and, if \(s = \emptyset\), \(x^s\) reduces to the usual norm \(x\). In this note we develop ideas relating \(x^s\) to the depth one foliations classified by Theorem 1.2 and show how this theory can be used to compute the norm. This makes the computations of the norm, done in the examples at the end of [5], rigorous. In those examples, the foliation cones are unions of cones over some faces of the Thurston ball of \(x^s\), but this fails in Example 2 of the present paper. However, even in this example, \(x^s\) is closely enough related to the foliation cones that we are able to compute the Thurston norm.

2 Doubling

There are three basic topics to be treated here, namely: the doubling map in singular homology, the Thurston norm in sutured manifolds and their doubles, and the process of inducing fibrations in the double \(DM\) from certain depth one foliations on \(M\).
2.1 The Doubling Map

If $M$ is a smooth, connected, oriented, sutured manifold, we form the double $DM$ along $\partial_r M$ (assumed to be nonempty). This is defined in complete analogy with the usual definition of the double of a manifold along its full boundary. Thus $DM$ is an oriented manifold formed by taking a second copy of $M$, but with opposite orientation, and gluing the two together via the identity map on $\partial_r M$. We write

$$DM = M \cup (-M)/\sim.$$ 

There is a standard way to put a smooth, oriented structure on $DM$ so that $\partial DM$ is also smooth and so that the natural reflection map $\rho : DM \to DM$ is an orientation–reversing diffeomorphism. This map interchanges the corresponding points of $M$ and $-M$, hence has $\partial_r M$ as its set of fixed points.

Let $S \subset M$ be a smooth, properly imbedded, oriented surface. Reversing orientations gives $-S \subset -M$. The double $DS = S \cup (-S) \subset DM$ can be viewed as an oriented, properly imbedded submanifold of $DM$. There is a technical problem that, if $S \cap \partial_r M \neq \emptyset$, smoothness of $DS$ might fail along this set. To avoid this, one introduces a $\rho$–invariant Riemannian metric on $DM$. There is a $\rho$–invariant normal neighborhood $U$ of $\partial_r M$ in $DM$ and an isotopy of $S$ makes $S \cap U$ saturated by the normal fibers of $U \cap M$. Now $DS$ is a smooth, $\rho$–invariant subsurface of $DM$. Of course, if $S \cap \partial_r M = \emptyset$, $DS$ is the disjoint union of $S$ and $-S$. Note also that $\rho|DS$ is an orientation–reversing diffeomorphism of this surface.

A smooth triangulation of $S$ determines a smooth triangulation of $DS$, producing singular cycles mod the boundary in $M$ and $DM$ respectively. The corresponding classes $[S] \in H_2(M, \partial M)$ and $[DS] \in H_2(DM, \partial DM)$ are well defined, independently of the choice of triangulation. We will define a canonical “doubling” map

$$D_* : H_2(M, \partial M) \to H_2(DM, \partial DM)$$

such that $D_*[S] = [DS]$ and show that this map is “norm doubling”.

At the level of singular chains, the map $\rho|M : M \to DM$ induces a linear map

$$\rho_\# : C_\#(M, \partial M) \to C_\#(DM, \partial_r M \cup \partial DM)$$

commuting with the singular boundary operator $\partial_\#$. Thus, we can define

$$D_\#(c) = c - \rho_\#(c), \quad \forall c \in C_\#(M, \partial M),$$

noting that this also commutes with $\partial_\#$. At this point, there is a small technical problem. The map $D_*$ induced by $D_\#$ takes its image in the space
$H_*(DM, \partial_r M \cup \partial DM)$, whereas we want to interpret it as a map into the space $H_*(DM, \partial DM)$. The crucial property to notice is that, if the singular chain $c$ is supported in $\partial_r M$, then $D_\#(c) = 0$.

Consider the open cover $\Phi = \{U, V\}$ of $DM$, where $U = \text{int} \ DM$ and $V$ is a normal neighborhood of $\partial DM$ with normal fibers along $\partial(\partial_r M)$ lying entirely within $\partial_r M$. Let $A = \partial_r M \cap V$ and note that $\partial DM$ is a deformation retract of $A \cup \partial DM$. By abuse of notation, we also let $\Phi$ denote the induced open cover on any subspace of $DM$ and we compute singular homology on $DM$ and any of its subspaces using the $\Phi$–small singular chain complex $C_\Phi^\#$. That is, each singular simplex in a chain $c \in C_\Phi^\#$ is supported either in $U$ or in $V$. It is standard that the $\Phi$–small homology $H_\Phi^\#$ is canonically equal to the ordinary singular homology $H_\#^\#$, the equality being induced by $C_\Phi^\# \subset C_\#^\#$.

If $c \in C_\#^\#(\partial M)$, then, since $D_\#$ annihilates all singular simplices in $\partial_r M$, $D_\#(c)$ is a chain on $A \cup \partial DM$. We obtain homomorphisms

$$D_\# : C_\#^\#(M) \to C_\#^\#(DM)$$

$$D_\# : C_\#^\#(\partial M) \to C_\#^\#(A \cup \partial DM),$$

of chain complexes, hence a chain homomorphism

$$D_\# : C_\#^\#(M, \partial M) \to C_\#^\#(DM, A \cup \partial DM).$$

This defines

$$D_* : H_*(M, \partial M) \to H_*(DM, A \cup \partial DM) = H_*(DM, \partial DM),$$

the desired doubling map.

**Remark** The above supposes that $\partial_r M$ meets $\partial_h M$. Otherwise, $\partial_h M = T(\gamma)$ and the proof that $D_* : H_*(M, \partial M) \to H_*(DM, \partial DM)$ is even easier, not requiring the use of $\Phi$-small homology.

**Lemma 2.1** If $S \subset M$ is a properly imbedded surface, then $D_*[S] = [DS]$.

**Proof** Indeed, if $c_S \in Z_2(M, \partial M)$ is a fundamental cycle for $S$ obtained by a smooth triangulation, it is an elementary consequence of the orientation-reversing property of $\rho : DS \to DS$ that $c_S - \rho_\#(c_S) \in Z_2(DM, \partial DM)$ is a fundamental cycle for $DS$.

Consider the inclusion map $i : M \hookrightarrow DM$ and the induced homomorphism

$$i^* : H^1(DM) \to H^1(M)$$

in real cohomology. Using Lefschetz duality, we view this as

$$i^* : H_2(DM, \partial DM) \to H_2(M, \partial M).$$
Lemma 2.2 The composition $i^* \circ D_\ast$ is equal to the identity on $H_2(M, \partial M)$. In particular, the doubling map is injective on $H_2(M, \partial M)$.

Proof It will be enough to prove this for elements $[S] \in H_2(M, \partial M)$, where $S$ is a properly imbedded, oriented surface in $M$. Indeed, these constitute the integer lattice in $H_2(M, \partial M)$. By Lemma 2.1, we must show that $i^*[DS] = [S]$. The Lefschetz dual of $[DS]$ is represented by a 1–form $\omega$ as follows. Fix a normal neighborhood $V$ of $DS$ in $DM$. This can be chosen so that $V \cap \partial_M$ is saturated by normal fibers, as is $V \cap \partial DM$. The closed form $\omega$ is supported in $V$ and has integral along each normal fiber equal to 1. Evidently, $V \cap M$ is a normal neighborhood of $S$ and $\omega$ restricts in $M$ to a representative of the Lefschetz dual of $[S]$.

Remark It is easy to give a geometric definition of

$$i^* : H_2(DM, \partial DM) \to H_2(M, \partial M)$$

on each element $[\Sigma]$ of the integer lattice. Represent this class by a properly imbedded surface $\Sigma \subset DM$ that is transverse to $\partial_M$ and note that $\Sigma_+ = \Sigma \cap M$ is a properly imbedded surface in $M$. Then $i^*[\Sigma] = [\Sigma_+]$.

2.2 The Thurston Norm

Roughly speaking, we define the Thurston norm in a sutured manifold by doubling along $\partial_\ast M$, computing the Thurston norm in the doubled manifold, and dividing by two. This is half the norm defined by Scharlemann in [10, Definition 7.4].

More precisely, let $S$ be properly imbedded as usual and connected. By a small isotopy, $\partial S$ can be assumed to be transverse to $\partial_\ast M$ and we compute $\chi_\ast(S)$ by doubling along $\partial_\ast M$, computing the usual $\chi_-$ of the doubled surface and dividing by two. (The superscript $s$ stands for “sutured"). One can give an intrinsic formula for this number as follows.

The components of $S \cap \partial_\ast M$ are circles and/or properly imbedded arcs in annular components of $\partial_\ast M$. These circles need not be essential and some of the arcs might also fail to be essential in the sense that they start and end on the same boundary component of an annular component in $\partial_\ast M$. We will see that these inessential arcs and circles can be eliminated, but for the moment
they are allowed. Let \( n(S) \) denote the number of arc components of \( S \cap \partial_{h}M \). Then the reader can verify that the formula for \( \chi_{-}^{s} \) is

\[
\chi_{-}^{s}(S) = \begin{cases} 
-\chi(S) + \frac{1}{2}n(S), & \text{if this number is positive,} \\
0, & \text{otherwise.}
\end{cases}
\]

As usual, if \( S \) is not connected, one defines \( \chi_{-}^{s}(S) \) as the sum of the values on each component. If \( z \) is an element of the integer lattice in \( H_{2}(M, \partial M) \), \( x^{s}(z) \) is defined to be the minimum value of \( \chi_{-}^{s}(S) \) taken over all surfaces \( S \in z \).

Continuing to follow Thurston’s lead, we extend \( x^{s} \) canonically to a pseudonorm on the vector space \( H_{2}(M, \partial M) \) and call this the sutured Thurston norm.

**Remarks** Instead of computing the sutured norm by doubling in \( \partial_{h}M \), one can equally well double in \( \partial_{r}M \). Again the components of \( S \cap \partial M \) are properly imbedded arcs and/or circles and the number of arc components is the same number \( n(S) \). One then notes that \( 2\chi_{-}^{s}(S) = \chi_{-}(DS) \), where \( \chi_{-}(DS) \) is defined as for the ordinary Thurston norm.

We further remark that, by a \( \chi_{-}^{s} \)-reducing homology and/or isotopy, \( S \) can be assumed to meet each annular component of \( \partial_{h}M \) only in essential arcs, each crossing the suture once, or in essential circles, each parallel to the suture and disjoint from it. It can be assumed also that \( S \) meets each toral component only in essential circles, although this remark is not particularly consequential. At any rate, \( n(S) \) is now just the number of times that \( \partial S \) crosses the sutures and it is elementary that this number is even. Thus, \( \chi_{-}^{s}(S) \) is an integer, as is \( x^{s}[S] \).

**Example** A decomposing disk \( \Delta \) in the sense of Gabai [8] has \( \chi_{-}^{s}(\Delta) = 0 \) if it meets the sutures twice, \( \chi_{-}^{s}(\Delta) = 1 \) if it meets them four times, etc.

**Theorem 2.3** The map

\[
D_{s} : H_{2}(M, \partial M) \to H_{2}(DM, \partial DM)
\]

is norm–doubling, where the sutured Thurston norm is used on the first space and the usual Thurston norm is used on the second. Thus, if \( B \) is the Thurston ball of \( M \) and \( B^{*} \) that of \( DM \), then \( D_{s}(B/2) = B^{*} \cap D_{s}(H_{2}(M, \partial M)) \).

**Proof** It is enough to prove this on elements of the integer lattice. Let \( [S] \) be represented by a \( \chi_{-}^{s} \)-minimal surface \( S \). We have already noted that \( \chi_{-}(DS) = 2\chi_{-}^{s}(S) \), hence it will be enough to show that \( DS \) is a \( \chi_{-} \)-minimal representative of \( [DS] = D_{s}[S] \). If not, let \( \Sigma \in [DS] \) have \( \chi_{-}(\Sigma) < \chi_{-}(DS) \).
Isotope $\Sigma$ smoothly to be transverse to $\partial_\tau M$ and let $\Sigma_+ = \Sigma \cap M$ and $\Sigma_- = \Sigma \cap (-M)$. If no component of $\Sigma_\pm$ has positive Euler characteristic, one verifies the relation

$$\chi_-(\Sigma) = \chi_+(\Sigma_+) + \chi_-(\Sigma_-).$$

The only possible components with positive Euler characteristic are spheres or disks. In the first case, irreducibility of $M$ permits elimination of the offensive component. In the second, there will be no problem if the boundary of the disk $\Delta$ meets $\partial_\tau M$ in arcs. Otherwise, $\partial \Delta$ is a simple closed loop either in $\partial_0 M$ or $\partial_\tau M$. In the first case, $\Delta$ is also a component of $\Sigma$ in $DM$ and has zero Thurston norm. In $M$ it has zero sutured norm, so this case also causes no problem. In the remaining case, $\partial \Delta \subset \partial_\tau M$ and tautness of the sutured manifold structure, together with irreducibility, yields an isotopy of $\Sigma$ pulling the disk $\Delta$ through $\partial_\tau M$, hence eliminating it as a component of $\Sigma_\pm$. Thus $(\ast)$ can be assumed to hold. Interchanging the roles of $M$ and $-M$, if necessary, we can then assume that $\chi_+(\Sigma_+) < \chi_-(S)$. But

$$[S] = i^* [DS] = i^* [\Sigma] = [\Sigma_+]$$

contradicting $\chi_-$-minimality of $S$ in $[S]$. \hfill \square

### 2.3 Inducing Fibrations on $DM$

In this subsection, we assume that $M$, as a sutured manifold, is not a product $\partial_\tau M \times I$. This insures that $\partial_\tau M$ cannot be a fiber in a fibration of $DM$ over the circle. We sketch some facts that are treated in greater detail in [3], [4] and [5].

Let $\mathcal{F}$ be a smooth, depth one foliation of $M$, transverse to $\partial_0 M$ and having the components of $\partial_\tau M$ as sole compact leaves. A depth one leaf $L \subset M_0$ determines an element $\lambda(\mathcal{F}) \in H^1(M; \mathbb{Z})$ of the integer lattice in the real cohomology space $H^1(M)$ via the intersection product with loops in $M_0$. This class can also be represented by a closed, nonsingular 1-form $\omega$ on $M_0$ that “blows up nicely” at $\partial_\tau M$ (meaning that $\omega$ becomes unbounded near $\partial_\tau M$ in such a way that the 2-plane field $\ker \omega$ extends smoothly to a 2-plane field on $M$ tangent to $\partial_\tau M$). The form $\omega$ defines $\mathcal{F}|M_0$, hence also determines $\mathcal{F}$, and its cohomology class can be viewed as a class on $M$ via the homotopy equivalence $M_0 \hookrightarrow M$ (the natural inclusion map). For any positive constant $\alpha$, the form $\alpha \omega$ also defines $\mathcal{F}$, so we obtain a “foliated ray” $\langle \mathcal{F} \rangle \subset H^1(M)$ corresponding to $\mathcal{F}$. This ray, in turn, determines $\mathcal{F}$ up to an isotopy that is smooth in $M_0$ and continuous on $M$ [3, Theorem 1.1]. We often think of a foliated ray as
an isotopy class of foliations. These foliated rays are exactly the rays meeting integer lattice points in the interiors of the foliation cones of [5].

Remark Poincaré duality identifies $H^1(M) = H_2(M, \partial M)$.

The leaves of $F|M_0$ spiral in a well-understood way on each component $F$ of $\partial_r M$, giving rise to a nondivisible cohomology class

$$\nu : \pi_1(F) \to \mathbb{Z}$$

called the juncture of the spiral (cf. [3, §3]). The juncture on $F$ depends only on the class $\lambda(F)$ [3, Lemma 3.1]. It can be represented by a compact, properly imbedded, oriented, nonseparating 1-manifold $N \subset F$ which need not be connected [3, pp. 159–160] and each component is assigned an integer weight.

If there is a depth one foliation $\mathcal{G}$ such that $\lambda(\mathcal{G}) = -\lambda(F)$, we will denote $\mathcal{G}$ by $-F$ and call this the opposite foliation to $F$. Remark that this is not the foliation defined by the form $-\omega$, even up to isotopy, since this foliation would require that the outwardly oriented components of $\partial_r M$ become inwardly oriented and vice versa. These orientations are part of the given sutured structure on $M$ and may not be reversed. While, in many cases, $-F$ exists, examples show that it may not. Indeed, the three vertices in Figure 2 of Section 5 are not foliated classes, but they are the negatives of foliated classes. Of course, at the cohomology level, $[-\omega] = \lambda(-F)$. By the ideas in the proof of [3, Lemma 3.1], the juncture for $-F$ can be represented by $-N$, the manifold obtained by reversing the orientation of $N$. Intuitively, the foliations $F$ and $-F$ spin in “opposite directions” along $F$, appearing to be “mirror images” of one another in a small normal neighborhood of $F$ in $M$.

Suppose that $F$ admits an opposite foliation $-F$. We can produce a taut foliation $F \cup -F$ on $DM$ by using $F$ in $M$ and $-F$ in $-M$, the components of $\partial_r M$ being the sole compact leaves. Since the foliation is taut, each of the compact leaves is a properly imbedded, incompressible surface in $DM$.

If $F$ is one of these compact leaves, it inherits an orientation so that it is inwardly oriented with respect to $M$ or $-M$ and outwardly oriented with respect to the other. Thus the junctures in $F$ for the respective foliations can be taken to be physically the same submanifold of $F$, but with opposite orientations. It follows that the procedure in [4, pp. 379–381] applies, allowing us to erase these compact leaves by deleting their “spiral ramp” neighborhoods and fitting the resulting foliations together, matching convex corners of one to concave corners of the other and vice versa (cf. [4, Fig. 4]). Actually, our situation is a bit
more complicated than that envisioned in [4] because our juncture need not be connected, but essentially the same construction goes through. In this way we erase all leaves that are components of \( \partial_\tau M \). The resulting foliation of \( DM \), denoted by \( D\mathcal{F} \), has only compact leaves since the construction amputates the finitely many ends of all leaves and joins together their compact cores. Thus, \( D\mathcal{F} \) is a fibration of \( DM \) over the circle, the fibers being transverse to \( \partial DM \).

The reader should be warned that \( D\mathcal{F} \) is not uniquely determined by \( \mathcal{F} \) and \( -\mathcal{F} \). The topology of the fiber depends on the choices of spiral ramp neighborhoods of the components \( F \) of \( \partial_\tau M \). With a little care, this construction can be carried out so that the following is true.

**Lemma 2.4** If the depth one foliation \( \mathcal{F} \) admits an opposite foliation, then there are associated fibrations \( D\mathcal{F} \) of \( DM \) over the circle with fibers transverse to \( \partial DM \). Furthermore, there is a smooth, one–dimensional foliation \( \mathcal{L} \) of \( DM \), tangent to \( \partial DM \) and transverse both to \( \mathcal{F} \cup -\mathcal{F} \) and \( D\mathcal{F} \).

While each component \( F \) of \( \partial_\tau M \) fails to be a leaf of \( D\mathcal{F} \), it remains an incompressible surface in \( DM \) with a special relationship to \( D\mathcal{F} \).

**Lemma 2.5** The surface \( F \) is isotopic through properly imbedded surfaces in \( DM \) to a surface that has only positive saddle tangencies with \( D\mathcal{F} \).

**Proof** The tangent bundles \( \tau = \tau(\mathcal{F} \cup -\mathcal{F}) \) and \( \tau_0 = \tau(D\mathcal{F}) \) are both transverse to \( \mathcal{L} \) and transversely oriented so that both induce the same orientation along \( \mathcal{L} \). It follows that \( \tau \) and \( \tau_0 \) are homotopic as oriented 2–plane bundles, hence have the same (relative) Euler class \( e(\tau) = e(\tau_0) \in H^2(M, \partial M) \). Thus

\[
\int_F e(\tau_0) = \int_F e(\tau) = \chi(F).
\]

We can assume, via a small isotopy near \( \partial DM \), that each component of \( \partial F \) is either transverse to \( D\mathcal{F} \) or lies in a fiber of \( D\mathcal{F} \). The two possibilities correspond, respectively, to the cases in which the component of \( \partial F \) does or does not meet the juncture for \( \mathcal{F} \). Thus, Thurston’s general position result [11, Theorem 4] allows us to perform an isotopy of \( F \), putting it in a position so that all tangencies with \( D\mathcal{F} \) are saddles. (The possibility that \( F \) could be isotoped onto a fiber is eliminated by our assumption that \( M \) is not a product.) If some tangency is not positive (that is, the orientations of \( \tau(F) \) and \( \tau(D\mathcal{F}) \) at the tangency are opposite), it would follow that \( \int_F e(\tau_0) \neq \chi(F) \), a contradiction. \( \square \)
**Remark** Lemma 2.5 can also be proven more directly by a Morse theoretic argument.

**Proposition 2.6** If the depth one foliation $\mathcal{F}$ admits an opposite foliation and if $K \subset DM$ is a properly imbedded surface having only positive saddle tangencies with $D\mathcal{F}$, then $[K] \in H_2(DM, \partial DM)$ lies in the cone over a fibered face of the Thurston ball and $K$ is a norm minimizing representative of $[K]$.

**Proof** Let $C \subset H_2(DM, \partial DM)$ be the cone over a top dimensional face of the Thurston ball, the interior of which contains contains the “fibered ray” $\langle D\mathcal{F} \rangle$ associated to $D\mathcal{F}$ as in Theorem 1.1. Let $[D\mathcal{F}] \in \langle D\mathcal{F} \rangle \smallsetminus \{0\}$. Then, by a standard argument of Thurston [11], the fact that the tangencies are positive saddles implies that the convex combination $t[D\mathcal{F}]+(1-t)[K] \in \text{int } C$, $0 < t \leq 1$. Consequently, $[K] \in C$. The norm $x$ is linear in $C$, coinciding there with the linear functional $-e(\tau(D\mathcal{F})) : H_2(DM, \partial DM) \to \mathbb{R}$, and so

$$x([K]) = -e(\tau(D\mathcal{F}))(\tau([K])) = -\chi(K).$$

This latter equality is due to the fact that the tangencies are positive saddles [11] (see also [2, Lemma 10.1.13]).

**Corollary 2.7** If the depth one foliation $\mathcal{F}$ admits an opposite foliation and if $F \subset DM$ is as in Lemma 2.5, then $[F]$ lies in the cone over a lower dimensional face of a fibered face of the Thurston ball and $F$ is norm minimizing in $[F]$.

**Proof** Indeed, by Proposition 2.6 and Lemma 2.5, $F$ is norm minimizing in $[F]$ and that class lies in the cone over a fibered face. It cannot be in the interior of that cone since $F$ is not the fiber of a fibration of $DM$.

**Corollary 2.8** If the depth one foliation $\mathcal{F}$ admits an opposite foliation and if $S \subset M$ is a properly imbedded surface such that $DS$ is smooth and has only positive saddle tangencies with $D\mathcal{F}$, then $x^S[S] = -\frac{1}{2}\chi(DS)$ and $S \in [S]$ realizes this minimal sutured norm.

**Proof** Indeed, by Proposition 2.6, $DS$ is norm minimizing in $[DS]$. The assertion follows by Lemma 2.1 and Theorem 2.3.
3 Sutured Handlebodies

Lemma 3.1 There is a canonical decomposition
\[ H_2(DM, \partial DM) = H_2(M, \partial M) \oplus \ker i^*, \]
where \( H_2(M, \partial M) \) is imbedded as the image of \( D_* \).

Proof Since \( i^* \circ D_* \) is the identity on \( H_2(M, \partial M) \), this is immediate. \( \square \)

Lemma 3.2 \( \ker i^* \cong H_2(M, \partial\gamma M) \).

Proof By the long exact cohomology sequence of the pair \((DM, M)\)
\[ H^0(DM) \xrightarrow{i_*} H^0(M) \xrightarrow{\partial^*} H^1(DM, M) \rightarrow H^1(DM) \xrightarrow{i_*} H^1(M) \cdots, \]
and the fact that \( i^*: H^0(DM) \rightarrow H^0(M) \) is an isomorphism, it follows that \( \partial^*(H^0(M)) = 0 \). Thus, the kernel of \( i^*: H^1(DM) \rightarrow H^1(M) \) is isomorphic to \( H^1(-M, \partial(-M)) \). There is no harm in dropping the minus sign and employing Lefschetz duality to identify this space with \( H_2(M, \partial\gamma M) \). Here, the version of Lefschetz duality we are using is the seldom quoted one proven in [9, Theorem 3.43]. \( \square \)

Let \( M \) be a sutured handlebody of genus \( n \). We will let \( \gamma_i, 1 \leq i \leq m \), denote the sutures and also the homology class each suture represents in \( H_1(M) \). Let \( \{\gamma_i\}_{i=1}^m \) denote the basis of \( H_1(\partial\gamma M) \) represented by these sutures. Let \( X \subset M \) be a bouquet of circles \( \alpha_j \subset M, 1 \leq j \leq n \), that is a deformation retract of \( M \). Viewing \( \alpha_j \) as representing a homology class in \( H_1(M) \) as well as a curve, one obtains a basis \( \{\alpha_j\}_{j=1}^n \) of \( H_1(M) \).

Consider the map \( W: H_1(\partial\gamma M) \rightarrow H_1(M) \)
induced by the inclusion \( \partial\gamma M \hookrightarrow M \).

Lemma 3.3 The vector space \( H_2(M, \partial\gamma M) \) is canonically imbedded in the vector space \( H_1(\partial\gamma M) \) as \( \ker W \).

Proof This follows from the long exact sequence
\[ \cdots \rightarrow 0 = H_2(M) \rightarrow H_2(M, \partial\gamma M) \xrightarrow{\partial} H_1(\partial\gamma M) \xrightarrow{W} H_1(M) \cdots. \] \( \square \)
Remark In the above long exact sequence, the map $W$ can be represented by the $n \times m$ matrix
\[
W = \begin{bmatrix}
w_{11} & \cdots & w_{1m} \\
\vdots & \ddots & \vdots \\
w_{n1} & \cdots & w_{nm}
\end{bmatrix}.
\]
Here, we coordinatize $H_1(\partial_\eta M)$ by the basis $\{\gamma_i\}_{i=1}^m$ and $H_1(M)$ by $\{\alpha_j\}_{j=1}^n$. The columns of $W$ are the vectors $\gamma_i$, $1 \leq i \leq m$. The column rank $r$ of this matrix is the rank of the linear map $W$ and the dimension of the kernel of $W$ is $d = m - r$.

**Theorem 3.4** $H_2(DM, \partial DM) \cong H_2(M, \partial M) \oplus \mathbb{R}^d$.

**Proof** Indeed,
\[
H_2(DM, \partial DM) \cong H_2(M, \partial M) \oplus \ker i^* \quad \text{(Lemma 3.1)}
\]
\[
\cong H_2(M, \partial M) \oplus H_2(M, \partial_\eta M) \quad \text{(Lemma 3.2)}
\]
\[
\cong H_2(M, \partial M) \oplus \ker W \quad \text{(Lemma 3.3)}
\]

Let $c$ be the number of components of $\partial_\tau M = R_+ \cup R_-$.

**Theorem 3.5** One has $d \geq c - 1$, with equality if and only if the linear map $W$ has rank $m - c + 1$ if and only if the identification in Lemma 3.1 is
\[
H_2(DM, \partial DM) = H_2(M, \partial M) \oplus \mathbb{R}^{c-1}.
\]
If $d = c - 1$, the factor $\mathbb{R}^{c-1}$ is generated by the classes represented by any $c - 1$ of the components of $R_+ \cup R_-$. 

**Proof** The first equivalence follows since the rank of $W$ equals $m - d$ while the second equivalence is immediate by Theorem 3.4. By Lemma 3.1, the factor $\mathbb{R}^{c-1}$ is identified in $H_2(DM, \partial DM)$ as $\ker i^*$ and it is clear that each component $N_i$ of $R_+ \cup R_-$ determines a homology class $\nu_i = [N_i] \in \ker i^*$. Thus, it will be sufficient to show that any $c - 1$ of these classes are linearly independent. This will also show that $d \geq c - 1$.

First note that the classes determined by the components of $R_+$ are linearly independent, as are those determined by the components of $R_-$. Indeed, there is a loop in $DM$ having intersection number 1 with any given component of $R_+$ and intersection number 0 with all others. The same argument works for the
components of $R_-$, proving that there is no nontrivial linear relation between the classes corresponding to the components of one of $R_\pm$. 

Next, choosing the indexing appropriately, let \( \{\nu_i = [N_i]\}_{i=1}^{c-1} \) be a choice of \( c - 1 \) of the classes and let \( \nu_c = [N_c] \) be the omitted one. For definiteness, suppose that \( N_c \) is a component of \( R_+ \). We consider a linear relation

\[
0 = \sum_{i=1}^{c-1} a_i \nu_i
\]

and show that each \( a_i \) is forced to be zero. For each component \( N_i \) of \( R_- \), there is an arc in \( M \) from \( N_c \) to \( N_i \) and this doubles to a loop in \( DM \) that has intersection number \( a_i \) with the right hand side of the above relation. Thus, \( a_i = 0 \) whenever \( N_i \) is a component of \( R_- \). The above relation, therefore, involves only terms corresponding to components of \( R_+ \). As already observed, there is no such nontrivial relation. An entirely similar argument works when \( N_c \) is a component of \( R_- \).

**Corollary 3.6** The linear map \( W \) has rank \( m - 1 \) if and only if the identification in Lemma 3.1 is

\[
H_2(DM, \partial DM) = H_2(M, \partial M) \oplus \mathbb{R}.
\]

In this case, the factor \( \mathbb{R} \) is generated by \( [R_+] = [R_-] \) and both \( R_+ \) and \( R_- \) are connected.

Let \( g \) be the genus of \( R_+ \cup R_- \).

**Theorem 3.7** \( m - c + 1 + g = n \).

**Proof** The disjoint union of \( R_+ \) and \( R_- \) has genus \( g \). The proof consists of sequentially pasting together adjoining components of the disjoint union of \( R_+ \) and \( R_- \) along a common suture. This operation either reduces the number of components by one or adds a handle. The totality of such pastings produces a surface homeomorphic to \( \partial M \), a connected surface of genus \( n \). Since there are \( c \) components, \( c - 1 \) of the pastings along sutures reduce the number of components and the remaining \( m - (c - 1) \) pastings add handles to give a total of \( m - c + 1 + g \) handles. The assertion follows. \( \square \)
4 Computing the Sutured Thurston Norm

Our goal in this section is to state and prove a proposition that can often be used to find top dimensional faces of the Thurston ball. It applies to all the examples at the end of [5] and Example 2 of Section 5. We let \([a_1, \ldots, a_n]\) denote the closed, convex hull of a set of points \([a_1, \ldots, a_n]\) in \(H_2(M, \partial M)\) or \(H_2(DM, \partial DM)\) and we let \(\langle a_1, \ldots, a_n \rangle\) be the cone with base \([a_1, \ldots, a_n]\) and cone point \(0\).

**Definition** A simple disk decomposition of \(M\) is a complete disk decomposition of \(M\) in which all the disks are disjoint proper disks in \(M\). That is we can assume all the disk are there at the beginning when we do the disk decomposition rather than having to do the disk decomposition sequentially.

The following lemmas are consequences of Gabai’s procedure of disk decomposition [7]. If \(D_i \subset M\) is a disk of a simple disk decomposition, we will denote the class \([D_i] \in H_2(M, \partial M)\) by \(e_i\).

**Lemma 4.1** If \(\{+D_1, \ldots, +D_n\}\) is a simple disk decomposition of \(M\) giving the depth one foliation \(\mathcal{F}\), then each \(D_i, 1 \leq i \leq n\), meets \(\mathcal{F}\) in positive saddles. Furthermore, \(\langle e_1, \ldots, e_n \rangle\) is a subcone of a foliation cone and \(\langle \mathcal{F} \rangle \setminus \{0\} \subset \text{int} \langle e_1, \ldots, e_n \rangle\).

For a proof, see [6, Corollary 2.8].

**Lemma 4.2** If \(\{+D_1, \ldots, +D_n\}\) is a simple disk decomposition of \(M\) giving the foliation \(\mathcal{F}\), then \(\{+D_1, \ldots, +D_n\}\) is a simple disk decomposition of \(-M\) giving the foliation \(\mathcal{F}\). Each \(D_i \subset -M, 1 \leq i \leq n\), meets \(\mathcal{F}\) in positive saddles. Furthermore, \(\langle e_1, \ldots, e_n \rangle\) is a subcone of a foliation cone of \(-M\).

**Proof** Each \(D_i, 1 \leq i \leq n\), and \(\mathcal{F}\) have the opposite transverse orientation in \(-M\) as in \(M\), as does \(R(\gamma)\).

**Lemma 4.3** If \(\{-D_1, \ldots, -D_n\}\) is a simple disk decomposition of \(M\) giving the foliation \(\mathcal{F}\), then each \(-D_i\) meets \(\mathcal{F}\) in positive saddles and so the cone \(\langle -e_1, \ldots, -e_n \rangle = -\langle e_1, \ldots, e_n \rangle\) is a subcone of a foliation cone of both \(M\) and \(-M\).

**Proof** Apply Lemma 4.1 and 4.2
In the following, a boundary component of a properly imbedded surface $S$ is said to cross the sutures essentially if its intersections with annular components of $\partial_n M$ are essential arcs. Indeed, a small isotopy of $S$ removes any inessential intersections of $\partial S$ with sutures. When $S = D$ is a disk of a disk decomposition, the term “essentially” is redundant by Gabai’s definition of disk decomposition, but we will use it anyway for emphasis.

**Proposition 4.4** If $\{D_1, \ldots, D_n\}$ and $\{-D_1, \ldots, -D_n\}$ are simple disk decompositions of $M$, then there is a fibration $D^F$ of $DM$ over the circle such that the surfaces $D_i \cup -D_i$, $1 \leq i \leq n$, and $R_+$ have only positive saddle tangencies with the fibration. Further the Thurston norm of $D_*e_i = [D_i \cup -D_i] \in H_2(DM, \partial DM)$ is the number of times $\partial D_i$ essentially crosses the sutures minus 2 and the sutured Thurston norm of $e_i$ is half this number.

**Proof** The disk decomposition $\{D_1, \ldots, D_n\}$ (respectively $\{-D_1, \ldots, -D_n\}$) gives the subcone $\langle e_1, \ldots, e_n \rangle$ of a foliation cone of $M$ (respectively, it gives the subcone $\langle -e_1, \ldots, -e_n \rangle$ of a foliation cone of $-M$). If $\langle F \rangle \subset \text{int} \langle e_1, \ldots, e_n \rangle$, then $\langle -F \rangle \subset \text{int} \langle -e_1, \ldots, -e_n \rangle$. Then by Lemma 2.4, $F$ and $-F$ can be matched up across $\partial \tau M$ to give a fibration $D\mathcal{F}$. Further Lemmas 4.1 and 4.3 imply that $D_i \cup -D_i$, $1 \leq i \leq n$, has only positive saddle tangencies with $D\mathcal{F}$ while Lemma 2.5 implies that (after a small isotopy of $DM$ moving $D\mathcal{F}$ and all $D_i \cup -D_i$) $R_+$ has only positive saddle tangencies with $D\mathcal{F}$.

Let $b_i$ be the number of times $\partial D_i$ essentially crosses the sutures. Then the surface $D_i \cup -D_i$ is a punctured sphere with $b_i$ boundary components and thus $-\chi(D_i \cup -D_i) = b_i - 2$. Since this surface has only positive saddle tangencies with the fibration, Proposition 2.6 implies that $x(D_*e_i) = b_i - 2$ and Corollary 2.8 implies that $x^s(e_i) = x(D_*e_i)/2$. □

In the examples we are interested in, the matrix $W$ of Section 3 has rank $m-1$ so, by Corollary 3.6, $\partial_\tau M$ has one positive component $R_+$ and one negative component $R_-$ and $H_2(DM, \partial DM) = H_2(M, \partial M) \oplus \mathbb{R}$ where the $\mathbb{R}$ factor is generated by $R = [R_+] = [R_-]$, and, without loss, we can assume

$$D_*(H_2(M, \partial M)) = H_2(M, \partial M) \oplus \{0\} \subset H_2(DM, \partial DM).$$

In the following corollary, the integer $m$ and the matrix $W$ are as in Section 3.

**Corollary 4.5** All four of the cones

$$\langle D_*e_1, \ldots, D_*e_n, \pm \mathbb{R} \rangle$$

and

$$\langle D_*(-e_1), \ldots, D_*(-e_n), \pm \mathbb{R} \rangle$$



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are subcones of fibration cones (full-dimensional if \( \text{rank } W = m - 1 \)) and thus each lies in a cone over a fibered face of the Thurston ball of \( DM \). Also, the sutured Thurston norm is linear on both the cones \( \langle e_1, \ldots, e_n \rangle, \langle -e_1, \ldots, -e_n \rangle \subset H_2(M, \partial M) \) and both of these cones are full-dimensional subcones of foliation cones and are contained in cones over top dimensional faces of the Thurston ball of \( M \).

**Proof** Since \( \{D_1, \ldots, D_n\} \) and \( \{-D_1, \ldots, -D_n\} \) are simple disk decompositions of \( M \) and \( -M \) respectively, Proposition 4.4 gives a fibration \( D \mathcal{F} \) meeting the surfaces \( D_i \cup -D_i, 1 \leq i \leq n \), and \( R_+ \) in positive saddles. Thus, \( \langle D_e e_1, \ldots, D_e e_n, R \rangle \) is a subcone of a fibration cone (Proposition 2.6). If \( \text{rank } W = m - 1 \), this cone is full-dimensional by Corollary 3.6 and the fact that \( R \) is not in the image of \( D_e \). Similarly, since \( \{-D_1, \ldots, -D_n\} \) and \( \{D_1, \ldots, D_n\} \) are simple disk decompositions of \( M \) and \( -M \) respectively, one sees that the cone \( \langle D_e(-e_1), \ldots, D_e(-e_n), R \rangle \) is a subcone of a fibration cone (full-dimensional if \( \text{rank } W = m - 1 \)). One obtains the other two fibration cones because the Thurston ball and its fibered faces are symmetric under multiplication by \(-1\).

We prove the second part of the corollary for \( \langle e_1, \ldots, e_n \rangle \). The proof for the cone \(-\langle e_1, \ldots, e_n \rangle\) is identical. We must show that if \( p = u \cdot p_1 + v \cdot p_2 \), with \( p, p_1, p_2 \in \langle e_1, \ldots, e_n \rangle \) and \( u, v \in \mathbb{R} \) then \( x^s(p) = u \cdot x^s(p_1) + v \cdot x^s(p_2) \). Suppose on the contrary that \( x^s(p) \neq u \cdot x^s(p_1) + v \cdot x^s(p_2) \). Then, by Theorem 2.3, \( x(D_e p) \neq u \cdot x(D_e p_1) + v \cdot x(D_e p_2) \). This contradicts the linearity of the Thurston norm over faces of the Thurston ball of \( DM \).

Since the sutured Thurston norm is linear on \( \langle e_1, \ldots, e_n \rangle \), this is an (obviously full-dimensional) subcone of the cone over a fibered face of the Thurston ball. It is also a subcone of a foliation cone by Lemma 4.1.

## 5 Examples

In many case we can figure out the Thurston ball of knot or link complements cut apart along the Seifert surface using the methods of Sect ion 4. The methods of Example 1 can be used to make rigorous the computations of the Thurston norm in [5, §7].

**Example 1** Let \( M \) be the complement of the pretzel link \((2, 2, 2)\) cut apart along its Seifert surface as in [5, §7, Example 1] (see Figure 1). One can do disk decompositions using disks \( \{D_i, -D_j\} \) as long as \( i \neq j \in \{0, 1, 2\} \). These
Figure 1: (a) A Seifert surface for $(2, 2, 2)$  (b) The sutured manifold $M$ obtained from $(2, 2, 2)$

Figure 2: Thurston ball and foliation cones for $(2, 2, 2)$

disk decompositions are extremely easy to do using Gabai’s graphical algorithm in [7, Theorem 6.1]. Since each of the $\partial D_i$’s essentially crosses the sutures 4 times, it follows from Proposition 4.4 that $x(D_\ast e_i) = 2$ and $x^s(e_i) = 1$. By Corollary 4.5, it follows that the Thurston ball is the dotted hexagon $B$ of Figure 2.

The Markov process argument of [5, §7, Example 1 or Example 2] shows that $\langle e_1, e_2 \rangle$, $\langle e_2, e_0 \rangle$ and $\langle e_0, e_1 \rangle$ are the foliation cones.

Suitably labelling the sutures, we have that $\gamma_1 = -\alpha_1 + \alpha_2$, $\gamma_2 = \alpha_1 + \alpha_2$, and $\gamma_3 = \alpha_1 - \alpha_2$ in $H_1(M)$ (notation as in §3). The matrix

$$ W = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} $$

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Proposition 5.1 The Thurston ball of $DM$ is the double cone (suspension) over $D_{s}(B/2)$ with cone points $\pm R$.

**Remark** Similarly, if $M$ is any of the sutured manifolds in [5, §7], the Thurston ball of $DM$ is the double cone over $D_{s}(B/2)$ with cone points $\pm R/x(R)$.

**Example 2** Regard Figure 3 as drawn on $S^2$, the boundary of a solid ball $B$. Paste $D_1$ to $D_1$ so that $A$ (respectively $B$) on one copy of $D_1$ is matched to $A$ (respectively $B$) on the other copy of $D_1$ and the sutures match up, paste $D_2$ to $D_2$ so that $C$ (respectively $D$) on one copy of $D_2$ is matched to $C$ (respectively $D$) on the other copy of $D_2$ and the sutures match up, and paste $D_3$ to $D_3$ so that $E$ (respectively $F$) on one copy of $D_3$ is matched
to $E$ (respectively $F$) on the other copy of $D_3$ and the sutures match up. Then Figure 3 represents a sutured handlebody $M$ of genus 3 with sutures $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. Clearly, $H_2(M, \partial M) = \mathbb{R}^3$.

The arrows on the disks in Figure 3 define the positive orientation of the disks. Let $\alpha$ be a simple closed curve in Figure 3 going once around $D_1$, $D_2$, and $D_3$ in the negative sense and essentially crossing the sutures $\gamma_2$ twice and $\gamma_3$ and $\gamma_4$ once each. Then $\alpha$ bounds an oriented disk in the solid ball $B$ which we will denote $D_0$. In $H_2(M, \partial M)$, $e_0 + e_1 + e_2 + e_3 = 0$.

5.1 The Thurston Ball

Consider the compact, convex polyhedron depicted in Figure 4. One easily checks that the vertices of the quadrilateral faces really are coplanar. Two of these faces will present special problems in the following analysis.

**Definition** The two quadrilateral faces $Q^\pm = \pm[e_2, e_3, -e_0, -e_1]$ will be called the exceptional faces.

**Lemma 5.2** Each of the vertices in Figure 4 is represented by an oriented properly imbedded disk in $M$, the boundary of which essentially crosses the sutures four times.
Proof This is clear for $e_1, e_2, e_3$ and has already been observed for $e_0$. For $e_1 + e_2$, draw a closed, positively oriented curve on $\partial B$ meeting the suture $\gamma_1$ once, $\gamma_2$ twice, and $\gamma_3$ once. This bounds the desired disk in $M$. One argues similarly for $e_1 + e_3$, obtaining a disk with boundary meeting $\gamma_1$ once, $\gamma_2$ twice, and $\gamma_4$ once. The negatives of these classes are represented by the respective oppositely oriented disks.

Lemma 5.3 The vertices in Figure 4 all have sutured Thurston norm one and the sutured norm is identically equal to 1 on each of the nonexceptional faces.

Proof If $\{p_1, p_2, p_3\}$ are any three vertices of a nonexceptional face with corresponding representative disks $\{\Delta_1, \Delta_2, \Delta_3\}$, then these disks and their negatives give simple disk decompositions and each of the disks has boundary that essentially crosses the sutures 4 times. Verifying these disk decompositions by Gabai’s algorithm is routine but tedious. The lemma then follows by Proposition 4.4 and Corollary 4.5.

Let $\pm D_i$ denote the disk representing $\pm e_i$, $0 \leq i \leq 3$. Then there are simple disk decompositions $\{-D_1, D_2, D_3\}$ and $\{-D_0, D_2, D_3\}$ and simple disk decompositions $\{D_0, D_1, -D_2\}$ and $\{D_0, D_1, -D_3\}$. There can be no pairs of simple disk decompositions $\{\Delta_1, \Delta_2, \Delta_3\}$ and $\{-\Delta_1, -\Delta_2, -\Delta_3\}$ that can be used in Corollary 4.5 to show that $Q^\pm$ are faces. Instead we will show that $x^8(e_2 + e_3) = 2$, which proves, by convexity of the sutured Thurston ball, that $Q^+$ is a face. Of course, the norm of $-e_2 - e_3$ is also 2 and $Q^-$ is a face.

In the following, $\partial(D_2 \cup D_3)$ and the sutures $\gamma_i$ are viewed as 1-cycles on $\partial M$.

Lemma 5.4 The intersection numbers of $\partial(D_2 \cup D_3)$ with the sutures is given by: $\gamma_1 \cdot \partial(D_2 \cup D_3) = -2$, $\gamma_2 \cdot \partial(D_2 \cup D_3) = 4$, $\gamma_3 \cdot \partial(D_2 \cup D_3) = -1$, $\gamma_4 \cdot \partial(D_2 \cup D_3) = -1$.

Proof Let $n$ be an exterior normal to $\partial M$ and use a right hand rule to define the intersection number $\gamma_i \cdot D_j$, i.e. $\gamma_i \cdot D_j = \pm 1$ depending on whether $(\gamma_i, D_j, n)$ is a right or left handed system $1 \leq i, j \leq 3$. One can compute the intersection numbers:

$\gamma_1 \cdot \partial D_2 = -1$ $\gamma_2 \cdot \partial D_2 = +2$ $\gamma_3 \cdot \partial D_2 = -1$ $\gamma_4 \cdot \partial D_2 = 0$

$\gamma_1 \cdot \partial D_3 = -1$ $\gamma_2 \cdot \partial D_3 = +2$ $\gamma_3 \cdot \partial D_3 = 0$ $\gamma_4 \cdot \partial D_3 = -1$

The lemma follows. 

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Lemma 5.5 If $D$ is a properly embedded disk in $M$ and $\partial D$ crosses the sutures essentially at most twice, then $D$ is boundary compressible. If $S$ is a properly embedded, connected surface in $M$ which is not a boundary compressible disk and whose boundary crosses the sutures essentially (and does so cross some sutures), then $\chi_-(DS) \geq 2$.

Proof Suppose $D$ is a properly embedded disk with $\partial D$ meeting the sutures at most twice. Put $D$ into general position with respect to $D_1$, $D_2$, and $D_3$. The points of intersection of $D$ with $D_1$, $D_2$, and $D_3$ will consist of circles and arcs. Assume the ends of the arcs do not lie on sutures.

By an innermost circle on $D$ argument, we can get rid of all circles of intersection. Similarly, by an innermost arc argument on $D$ we can get rid of all arcs of intersection without increasing the number of intersections of $\partial D$ with the sutures. In fact, choose an arc of intersection $\alpha$ in $D$ having endpoints $x$ and $y$ such that there exists an arc $\beta \subset \partial D$ having endpoints $x$ and $y$ with $\alpha \cup \beta$ bounding a disk $D' \subset D$ such that int $D'$ meets none of the arcs in the innermost arc argument. Since there are at least two such $\alpha$ and $\beta$ and since $\partial D$ meets the sutures at most twice, we can assume $\alpha$ and $\beta$ chosen so that $\beta$ meets the sutures at most once. The arc $\alpha$ will be a properly embedded arc in $D_{i_0}$, some $1 \leq i_0 \leq 3$. Thus, there is an arc $\delta \subset \partial D_{i_0}$ with endpoints $x$ and $y$, such that $\alpha \cup \delta$ bounds a disk $D'' \subset D_{i_0}$. Since $\partial D_{i_0}$ meets the sutures four times and there are two possible choices of $\delta$, we can assume $\delta$ meets the sutures at most twice. Thus $\delta \cup \beta$ is a simple closed curve in $\partial M$ meeting the sutures at most three times, therefore never or twice. Therefore $\delta \cup \beta$ bounds a disk $D''' \subset \partial M$ ($D'''$ lies on the sphere represented in Figure 3 and $D'''$ contains none of $\pm D_j$, $1 \leq j \leq 3$) and a suture meets $\delta$ if and only if it meets $\beta$. Since $M$ is irreducible, the sphere $D' \cup D'' \cup D'''$ bounds a ball that can be used to give an isotopy of $D$ removing the arc of intersection $\alpha$. Indeed, $D'$ can be moved onto $D''$, keeping $\alpha$ fixed, and then an arbitrarily small isotopy pulls this image of $D'$ free of $D_{i_0}$. Since a suture meets $\delta$ if and only if it meets $\beta$, the isotopy does not change the number of intersections of $\partial D$ with the sutures. After finitely many isotopies, we may assume that $D$ does not meet $D_i$, $1 \leq i \leq 3$ and that $\partial D$ meets the sutures at most twice. Cut $M$ apart along $D_1$, $D_2$, and $D_3$ to give the solid ball $\mathcal{B}$ with boundary $S^2$ (see Figure 3). Clearly, $D$ is boundary compressible in the solid ball $\mathcal{B}$ and so in $M$.

Thus if $S$ has boundary meeting the sutures and $S$ is not a boundary compressible disk with $\partial S$ meeting the sutures twice, then either $S$ is a disk with
∂S meeting the sutures 4 or more times, or S has genus \( g \geq 1 \), or S has at least 2 boundary components and S has genus \( g = 0 \). In the first case \( \chi_-(DS) \geq 4 - 2 = 2 \) and, in the second case, \( \chi_-(DS) \geq 2 + 4g - 2 = 4g > 2 \). The third case falls into two subcases. If only one boundary component meets \( \partial \tau M \), then DS has genus 0 and at least four boundary components, in which case \( \chi_-(DS) \geq 4 + 0 - 2 = 2 \). If at least two boundary components of S meet \( \partial \tau M \), then DS has genus at least 1 and at least two boundary components, hence \( \chi_-(DS) \geq 2 + 2 - 2 = 2 \).

**Lemma 5.6** \( x^s(e_2 + e_3) = 2 \) and so \( x^s \equiv 1 \) on each of the exceptional faces \( Q^\pm \).

**Proof** The double of \( S = D_2 \cup D_3 \) consists of two four times punctured spheres with Euler characteristic \( 2 \cdot (2 - 4) = -4 \). Dividing by two we see that

\[
x^s(e_2 + e_3) \leq \chi_-(S) \leq | -2 | = 2.
\]

Let \( S \) be a surface representing \( [D_2 \cup D_3] \) in \( H_2(M, \partial M) \). Thus \( \chi_-(S) = \frac{1}{2} \chi_-(DS) \). By Lemma 5.4,

\[
\gamma_1 \cdot \partial S = -2, \gamma_2 \cdot \partial S = 4, \gamma_3 \cdot \partial S = -1, \gamma_4 \cdot \partial S = -1.
\]

Therefore, \( \partial S \) must meet the sutures at least eight times. If \( S \) has only one component \( S_1 \) whose boundary meets the sutures, then

\[
\chi^s(S) \geq \chi^s(S_1) \geq \frac{1}{2} \chi_-(DS_1) \geq \frac{1}{2} (8 + 4g - 2) \geq 3,
\]

where \( g \) is the genus of \( S_1 \). Otherwise \( S \) has at least two components, \( S_1 \) and \( S_2 \), whose boundaries meet the sutures. Thus, by Lemma 5.5,

\[
\chi^s(S) \geq \frac{1}{2} \chi_-(DS_1) + \frac{1}{2} \chi_-(DS_2) \geq 2.
\]

In any event, \( x^s(e_2 + e_3) \geq 2 \) and equality holds.

For the last assertion, the fact that \( x^s = 1 \) on \( \pm (e_2 + e_3) / 2 \) and on each vertex of \( Q^\pm \), together with convexity of the unit ball, implies that \( x^s|Q^\pm \equiv 1 \).

**Theorem 5.7** The polyhedron \( B \) in Figure 4 is the unit ball of \( x^s \).

Indeed, by Lemma 5.3 and Lemma 5.6, \( x^s \equiv 1 \) on each of the faces.
5.2 The Foliation Cones

Bases of the foliation cones are given in Figure 5 and can be found by doing the four simple disk decompositions using the disks \( \{D_1, D_2, D_3\} \), \( \{D_0, D_2, D_3\} \), \( \{D_1, D_0, D_3\} \), and \( \{D_1, D_2, D_0\} \). Thus every lattice point in the four open cones of Figure 5 correspond to depth one foliations. The foliation cones obtained this way are seen to be maximal by the Markov processes argument of [5, §7].

Remark The face \( Q^+ \) (respectively \( Q^- \)) meets the interior of both \( \langle e_1, e_2, e_3 \rangle \) and \( \langle e_0, e_2, e_3 \rangle \) (respectively \( \langle e_0, e_1, e_3 \rangle \) and \( \langle e_0, e_1, e_2 \rangle \)). Thus none of the foliation cones can be the union of cones over faces of the Thurston ball.

Remark In this example it is not true that the Thurston ball of \( DM \) is the double cone (suspension) of \( B/2 \). The dimension of \( H_2(DM, \partial DM) \) is 4 and \( x(\pm R/2) = 1 \) but the two exceptional faces, coned with \( \pm R/2 \) do not give faces of the unit ball. The cones over the other faces are faces of the unit ball.

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