ON TOLERANCES REPRESENTABLE AS $R \circ R^-$

PAOLO LIPPARINI

Abstract. We give examples and counterexamples concerning varieties in which every tolerance is representable as $R \circ R^-$, for some reflexive and admissible relation $R$.

In [L] we introduced the following definitions.

Definition 1. A tolerance $\Theta$ of some algebra $A$ is representable if and only if there exists a compatible and reflexive relation $R$ on $A$ such that $\Theta = R \circ R^-$ (here, $R^-$ denotes the converse of $R$).

A tolerance $\Theta$ of some algebra $A$ is weakly representable if and only if there exists a set $K$ (possibly infinite) and there are compatible and reflexive relations $R_k$ ($k \in K$) on $A$ such that $\Theta = \bigcap_{k \in K} (R_k \circ R_k^-)$.

The definitions are motivated by the following Theorem from [L].

Theorem 2. For every variety $V$ and for every pair of terms $p, q$ (of the same arity) for the language $\{\circ, \cap\}$, if $p$ is regular, then the following are equivalent:

(i) $V$ satisfies the congruence identity $p(\alpha_1, \ldots, \alpha_n) \subseteq q(\alpha_1, \ldots, \alpha_n)$.

(ii) The tolerance identity $p(\Theta_1, \ldots, \Theta_n) \subseteq q(\Theta_1, \ldots, \Theta_n)$ holds for every algebra $A$ in $V$ and for all representable tolerances $\Theta_1, \ldots, \Theta_n$ of $A$.

(iii) The tolerance identity $p(\Theta_1, \ldots, \Theta_n) \subseteq q(\Theta_1, \ldots, \Theta_n)$ holds for every algebra $A$ in $V$ and for all weakly representable tolerances $\Theta_1, \ldots, \Theta_n$ of $A$.

(iv) $V$ satisfies the tolerance identity $p(\Theta_1 \circ \Theta_1, \ldots, \Theta_n \circ \Theta_n) \subseteq q(\Theta_1 \circ \Theta_1, \ldots, \Theta_n \circ \Theta_n)$.

We say that a term $p$ is regular if and only if in the labeled graph associated with $p$ no vertex is adjacent with two distinct edges labeled with the same name (see [C1, C2, CD, L] for details).

The aim of the present paper is to study the notion of a (weakly) representable tolerance in more detail.

We first show that all tolerances in algebras without operations are weakly representable.

Proposition 3. If $A$ is an algebra belonging to the variety of sets (that is, an algebra without operations) then every tolerance of $A$ is weakly representable.

2000 Mathematics Subject Classification. Primary 08A30, 08B05.

Key words and phrases. Representable, weakly representable tolerance, permutable variety.

The author has received support from MPI and GNSAGA. We acknowledge useful correspondence with G. Czédli.
Proof. Let $A$ be an algebra without operations. For every pair of distinct elements $a, b \in A$ let $\Theta_{ab}$ be the reflexive and symmetric relation such that $(x, y) \in \Theta$ if and only if $\{x, y\} \neq \{a, b\}$.

$\Theta_{ab}$ is representable: define $R$ by $x \in R y$ if and only if either $x = y = a$, or $x = y = b$, or $x \notin \{a, b\}$. $R$ is clearly reflexive, and is compatible since $A$ has no operations. It is easy to see that $\Theta_{ab} = R \circ R^\circ$.

If $\Theta$ is any tolerance of $A$ then $\Theta$ is weakly representable, since $\Theta = \bigcap_{(a,b) \notin \Theta} \Theta_{ab}$. \]

In contrast to Proposition 3 in algebras without operations there can be non representable tolerances. Such tolerances remain non representable if we add a certain kind of operations.

**Proposition 4.** (i) In the 5-element algebra without operations there is a non representable tolerance.

(ii) There exists a 7-element semilattice with a non representable tolerance.

(iii) There exists a 7-element algebra with a majority operation with a non representable tolerance (a majority operation is a ternary operation $f$ satisfying $x = f(x, x, y) = f(x, y, x) = f(y, x, x)$).

Proof. (i) Let $a, b_1, b_2, b_3, c$ denote the elements of the 5-element algebra without operations, and let $\Theta$ be the smallest reflexive and symmetric relation such that $a \Theta b_i$ and $b_i \Theta c$ for $i = 1, 2, 3$.

$\Theta$ is a tolerance, since the algebra has no operations, and it is easy to see that $\Theta$ is not representable. Indeed, if $R$ is reflexive and $\Theta = R \circ R^\circ$ then $R \subseteq \Theta$ and $R^\circ \subseteq \Theta$, hence either $a R b_1$ or $b_1 R a$. Suppose that $a R b_1$ (the case $b_1 R a$ is similar). If $c R b_1$ then $a R \circ R^\circ c$, that is, $\Theta c$, which is false, hence necessarily $b_1 R c$. Continuing in the same way we obtain both $b_2 R a$ and $b_3 R a$, which implies $b_2 R \circ R^\circ b_3$, hence $b_2 \Theta b_3$, contradiction.

(ii) Consider the semilattice $S$ with 6 minimal elements $a, b_1, b_2, b_3, b_4, c$ and with a largest element $1$. Let $\Theta$ be the smallest reflexive and symmetric relation such that $1$ is $\Theta$-related to all elements of $S$, and such that $a \Theta b_i$ and $b_i \Theta c$ for $i = 1, 2, 3, 4$.

It is easy to check that $\Theta$ is a tolerance. Suppose by contradiction that $\Theta$ is representable as $R \circ R^\circ$. If $x, y$ are minimal elements of $S$ and both $x R 1$ and $y R 1$, then $x R \circ R^\circ y$, hence $x \Theta y$. Thus $|\{x \in S| x \text{ is minimal and } x R 1\}| \leq 2$, since in $S$ there do not exist 3 pairwise $\Theta$-connected minimal elements.

We can now repeat the arguments in (i) restricting ourselves to minimal elements $x$ such that not $x R 1$.

(iii) Consider the lattice $\langle L, +, \cdot \rangle$ with 6 atoms $a, b_1, b_2, b_3, b_4, c$ and with a largest element $1$ and a smallest element $0$. If $f$ is the ternary operation defined by $f(x, y, z) = (x + y)(x + z)(y + z)$ then $\langle L \setminus \{0\}, f \rangle$ is an algebra, since $L \setminus \{0\}$ is closed under $f$. We have that $f$ is a majority operation, and the same tolerance as in (ii) is not representable. \]
Even if we have showed that a majority term does not necessarily imply representability of tolerances, we can show that lattices have representable tolerances.

**Proposition 5.** Suppose that the algebra \( A \) has binary terms \( \lor \) and \( \land \) such that \( \lor \) defines a join-semilattice operation, the identities \( a \land (a \lor b) = a \), \( (a \lor b) \land b = b \) are satisfied for all elements \( a, b \in A \), and the semilattice order induced by \( \lor \) is a compatible relation on \( A \). Then all tolerances of \( A \) are representable.

In particular, all tolerances in a lattice are representable.

**Proof.** If \( \Theta \) is a tolerance of \( A \), let \( R = \Theta \cap \leq \). \( R \) is compatible since both \( \Theta \) and \( \leq \) are compatible.

If \( a \Theta b \) then \( a = a \lor a \Theta a \lor b \), and \( a \leq a \lor b \), thus \( a R a \lor b \). Similarly, \( b R a \lor b \), that is, \( a \lor b R^{-} b \), thus \( \Theta \subseteq R \circ R^{-} \).

Conversely, if \( (a, b) \in R \circ R^{-} \), say \( a Rc R^{-} b \), then \( a \leq c \), hence \( c = a \lor c \), and \( a = a \land (a \lor c) = a \land c \); similarly, \( c \land b = b \), hence \( a = a \land c \Theta c \land b = b \), since both \( R \subseteq \Theta \) and \( R^{-} \subseteq \Theta \). Thus \( a \Theta b \). We have proved \( R \circ R^{-} \subseteq \Theta \). \( \square \)

We now proceed to show that if \( A \) has a tolerance \( \Theta \) which is not a congruence, then we can add operations to \( A \) in such a way that, in the expanded algebra, \( \Theta \) is not even weakly representable. As a consequence, a Mal’cev condition \( M \) implies that every tolerance is representable if and only if \( M \) implies congruence permutability (Corollary \( C \)).

**Proposition 6.** Let \( A \) be any algebra, and let \( \Theta \) be a tolerance of \( A \). Then there is an expansion \( A^{+} \) of \( A \) by unary operations such that \( \Theta \) is a tolerance of \( A^{+} \), and any non trivial reflexive compatible relation of \( A^{+} \) contains \( \Theta \).

**Proof.** Let \( A^{+} \) be obtained from \( A \) by adding, for every \( a, b \in A \) such that \( a \Theta b \), and for every function \( f : A \to \{a, b\} \), a new unary operation which represents the function. Since \( a \Theta b \), \( \Theta \) is a tolerance of \( A^{+} \).

If \( R \) is a non trivial reflexive compatible relation of \( A^{+} \), there exist \( c \neq d \in A \) such that \( c R d \). However, for every \( a \Theta b \) there is a function such that \( f(c) = a \) and \( f(d) = b \), thus \( a = f(c) R f(d) = b \), since \( R \) is compatible. This proves that \( R \subseteq \Theta \). \( \square \)

**Corollary 7.** If \( A \) is an algebra and \( \Theta \) is a tolerance of \( A \) which is not a congruence, then there is an expansion \( A^{+} \) of \( A \) by unary operations such that \( \Theta \) is a tolerance of \( A^{+} \) and \( \Theta \) is not representable in \( A^{+} \). Actually, \( \Theta \) is not even weakly representable in \( A^{+} \).

**Proof.** Let \( A^{+} \) be an expansion of \( A \) as given by Proposition \( B \). \( \Theta \) is a tolerance of \( A^{+} \) by Proposition \( B \); moreover, \( \Theta \) is non trivial, since the trivial tolerance is a congruence. Suppose by contradiction that \( \Theta = R \circ R^{-} \) for some reflexive and admissible relation \( R \) on \( A^{+} \), hence \( R \) and \( R^{-} \) are non trivial, thus \( R \varsupseteq \Theta \) and \( R^{-} \supseteq \Theta \), by Proposition \( B \). Then \( \Theta = R \circ R^{-} \varsupseteq \Theta \circ \Theta \), and this implies that \( \Theta \) is a congruence of \( A^{+} \), hence a congruence
of $A$, contradiction. The proof that $\Theta$ is not weakly representable in $A^+$ is similar. □

The following result is probably known, but we give a proof, since we know no reference for it.

**Proposition 8.** (a) If $A$ is an algebra, and every tolerance of $A$ is a congruence, then all congruences of $A$ permute.

(b) A variety $V$ is congruence permutable if and only if every tolerance of every algebra in $V$ is a congruence.

**Proof.** (a) If $\alpha, \beta$ are congruences of $A$, let $\alpha \cup \beta$ denote the smallest tolerance containing $\alpha$ and $\beta$, which is the smallest admissible relation containing $\alpha \cup \beta$. Notice that $\alpha \cup \beta \subseteq \beta \circ \alpha$.

By assumption, $\alpha \cup \beta$ is a congruence. Then $\alpha \circ \beta \subseteq \alpha \cup \beta \circ \alpha \cup \beta = \alpha \cup \beta \subseteq \beta \circ \alpha$.

(b) is immediate from (a) and the well known result that in permutable varieties every reflexive and admissible relation is a congruence (see [HM], [S, Proposition 143]). □

Trivially, every congruence $\alpha$ is representable, since $\alpha = \alpha \circ \alpha$. By Proposition 8(b), congruence permuitability, for varieties, implies that every tolerance is representable. The next result shows that if a Mal’cev condition $M$ implies that every tolerance is representable, then $M$ implies congruence permuitability.

**Corollary 9.** Let $M$ be either a Mal’cev condition, or a weak Mal’cev condition, or a strong Mal’cev condition. The following are equivalent:

(i) $M$ implies congruence permuitability.

(ii) $M$ implies that every tolerance is representable.

(iii) $M$ implies that every tolerance is weakly representable.

**Proof.** (i) $\Rightarrow$ (ii). Suppose that (i) holds. If $V$ satisfies $M$, then, by Proposition 8(b), every tolerance in every algebra in $V$ is a congruence, hence is representable. Thus, (ii) holds.

(ii) $\Rightarrow$ (iii) is trivial.

We shall prove (iii) $\Rightarrow$ (i) by contradiction.

Suppose that (i) fails. Then there exists some variety $V$ which satisfies $M$ but which is not congruence permutable. By Proposition 8(b), there is an algebra $A \in V$ with a tolerance $\Theta$ which is not a congruence. By Corollary 7, $A$ can be expanded to an algebra $A^+$ in which $\Theta$ is a tolerance which is not weakly representable. By well known properties of Mal’cev conditions, the variety generated by $A^+$ still satisfies $M$, and this contradicts (iii). □

**Corollary 10.** (i) The class of varieties $V$ such that every tolerance in every algebra in $V$ is representable cannot be characterized by a weak Mal’cev condition.

(ii) The class of varieties $V$ such that every tolerance in every algebra in $V$ is weakly representable cannot be characterized by a weak Mal’cev condition.
Proof. If any of those classes could be characterized by some weak Mal’cev condition \( \mathcal{M} \), then, by Corollary \( \mathcal{M} \) would imply permutability. This is a contradiction, since Propositions \( \mathcal{M} \) and \( \mathcal{M} \) provide examples of non-permutable varieties in which every tolerance is (weakly) representable. \( \square \)

References

[C1] G. Czédli, On properties of rings that can be characterized by infinite lattice identities, Studia Sci. Math. Hungar. 16 (1981), 45–60.

[C2] G. Czédli, A characterization for congruence semi-distributivity, Proc. Conf. Universal Algebra and Lattice Theory, Puebla (Mexico, 1982), Springer-Verlag Lecture Notes in Math. 1004, 104–110.

[CD] G. Czédli and Alan Day, Horn sentences with (W) and weak Mal’cev conditions, Algebra Universalis 19, (1984), 217–230.

[HM] J. Hagemann and A. Mitschke, On \( n \)-permutable congruences, Algebra Universalis 3 (1973), 8-12.

[L] P. Lipparini, From congruence identities to tolerance identities, manuscript.

[S] J. D. H. Smith, Mal’cev varieties, Springer-Verlag, Berlin, 1976, Lecture Notes in Mathematics, Vol. 554.

Dipartimento di Matematica, Viale della Ricerca Scientifica, II Università di Roma (Tor Vergata), ROME ITALY

E-mail address: lipparin@axp.mat.uniroma2.it

URL: http://www.mat.uniroma2.it/~lipparin