Abstract

We present a study of the dynamics of a cup of coffee while walking with it in terms of a lumped model. The model considered is a planar (nonlinear) pendulum with parametric and direct excitation. The position of the pivot is changed with time, which leads to parametric excitation in the vertical direction; and parametric/direct excitation in the horizontal one. For the former case, the Method of Averaging is used to determine the regions of stability for two steady-state solutions. In the vertical/horizontal excitation, we determined the resonances of the system using the Method of Multiple Scales and show the results computed with exact numerical integration.

Introduction

Spilling coffee while walking is a common problem. The sloshing dynamics of the coffee, while walking, can be understood in terms of an oscillator with parametric excitation caused by the motion of the cup along the path. The main objective of this project was to study the slosh dynamics of a cup of coffee (while walking) in terms of a lumped mechanical model [8].

According to [3], the motion of a free-liquid-surface has three regimes (linear, weakly nonlinear, and strongly nonlinear). Depending on the regime that we want to study the system, we can use a different —lumped— mechanical model to understand its dynamics. A common way to model the sloshing of liquids in a cylindrical container is to consider it as a pendulum, and obtain the parameters (length, mass, dissipation coefficient) from the original system [3, 5, 8]. The model considered is a pendulum with a (parametric) excitation that comes from the human walking, this excitation is composed of two degrees of freedom: one in the plane (back-and-forth and lateral) and one vertical. Due
to the main frequencies in the human walking, the model will consist of a single pendulum\(^1\).

We present a study of the dynamics of a cup of coffee while walking with it in terms of a lumped model. The model considered is a planar (nonlinear) pendulum with parametric and direct excitation. The position of the pivot is changed with time, which leads to parametric excitation in the vertical direction; and parametric/direct excitation in the horizontal one. For the former case, the Method of Averaging is used to determine the regions of stability for two steady-state solutions. In the vertical/horizontal excitation, we determined the resonances of the system using the Method of Multiple Scales and show the results computed with exact numerical integration. An interesting outcome of an analytical solution is the capability of relating the parameters of the system to obtain some insight into the design process.\(^2\).

1 Modeling

A cup of coffee is, essentially, a cylindrical-like container filled with liquid — that can be assumed to behave like water. And this liquid will present some motion when a person walks with a cup of coffee at hand. The behavior of this system (for linear and weakly nonlinear regimes) can be interpreted in terms of the natural frequencies of the fluid. The natural frequencies of oscillation of a frictionless, vorticity-free, and incompressible liquid in a cylindrical container (a cup of coffee in this case) with a free liquid surface are given by [3]

\[
\omega_{nm}^2 = \frac{g\epsilon_{mn}}{R} \tanh \left( \frac{\epsilon_{mn} H}{R} \right) \left[ 1 + \frac{\sigma}{\rho g} \left( \frac{\epsilon_{mn}}{R} \right)^2 \right],
\]

where \(m = 0, 1, 2, \ldots\) and \(n = 1, 2, \ldots\).\(^3\) In this equation \(H\) is the height of the cup, \(R\) its radius, \(g\) the gravity, \(\rho\) fluid density, and \(\sigma\) is the surface tension. \(\epsilon_{mn}\) are the roots of the first derivative of the \(m\)-th order Bessel function — \(J'_m(\epsilon) = 0\).

One can match each of the modes of vibration of the liquid with a single degree of freedom oscillator, e.g., a mass-spring system or a pendulum. According to Ibrahim in [3]:

\(^1\)In [3], the author use several coupled pendulums to take into account different modes of vibration in the model.

\(^2\)Something that can be useful in the development of devices like the one described in this patent [11].

\(^3\)In our case the surface tension is negligible and the only relevant parameters are the geometrical ones.
A realistic representation of the liquid dynamics inside closed containers can be approximated by an equivalent mechanical system... For linear planar liquid motion, one can develop equivalent mechanical models in the form of a series of mass-spring dashpot systems or a set of simple pendulums. For nonlinear sloshing phenomena, other equivalent models such as spherical or compound pendulum may be developed to emulate rotational and chaotic sloshing.

Since the sloshing dynamics of the coffee (while walking) can be understood in terms of an oscillator with parametric excitation caused by the motion of the cup along the path [8]. A first approach is to consider the system as a planar pendulum with parametric pumping. Taking into account the parameters that we use parametric excitation for this model can come from: the change of length, the motion of the point of gyration, the motion of the center of mass\(^4\), and the change of mass\(^5\).

The system can be considered as a pendulum with varying length, which is a system that has been studied before [1, 12], and with moving —both, vertically and horizontally— pivot. The parameters of the system are: \(r_0\) original length of the rod, \(r\) length of the rod, \(m\) mass of the bob, \(g\) gravitational acceleration, \(x_0\) is a function of time that describes the horizontal position of the pivot, and \(z_0\) is a function of time that describes the vertical position of the pivot. The present work restricts the excitation of the system to the motion of the pivot, which leads to direct and parametric excitations.\(^6\)

The Lagrangian of the system is given by [4]

\[
L = T - V = \frac{m}{2} [(\dot{x} - \dot{x}_0)^2 + (\dot{z} + \dot{z}_0)^2] + mg[z + z_0] \quad (1)
\]

taking

\[x = r(t) \sin \theta, \quad z = -r(t) \cos \theta\]

and using the Euler-Lagrange equation we get the differential equation, we get

\[
r^2 \ddot{\theta} + r[g + \ddot{z}_0] \sin \theta + r\dot{x}_0 \cos \theta + 2r \dot{r} \dot{\theta} = 0 \quad . \quad (2)
\]

In the general case \(x_0\), \(z_0\) and \(r\) are functions of time, and we can talk about having modulated stiffness, damping, and inertia. Where the \(r\) accompanies the second derivative of the angle \(\theta\) and the last term that appears due to the

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\(^4\)This is the case of a playground swing, where the center of mass oscillates around a given position. When the oscillation is symmetric it can be proved to be equivalent to the varying length pendulum [1].

\(^5\)Although this is less common.

\(^6\)Appendix A shows the parameters of the lumped model in terms of the original cup of coffee.
Considered system: A pendulum with the point of gyration moving in the plane. This motion cause direct and parametric excitation in the system.

If we expand Eq. (2) in a two terms Taylor series in $\theta$ around 0, we get

$$r^{2}\ddot{\theta} + r[g + \ddot{z}_0] \left\{ \theta - \frac{\theta^3}{6} \right\} + r\ddot{x}_0 \frac{\theta^2}{2} + 2\dot{r}\ddot{\theta} = -r\ddot{x}_0,$$

that is a Mathieu equation with quadratic an cubic nonlinearities and a direct excitation term.

## 2 Results

In [8] the authors already presented an analysis based on numerical and experimental results for the sloshing dynamics of the coffee. Here, we emphasize our effort in approximated solutions via perturbation methods, namely: the Method of Averaging and the Method of Multiple Scales [2].

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This term can be compared with a damping term due to its dependence on the first derivative of $\theta$. 
2.1 Vertical motion of the pivot: Method of Averaging

If we consider just the vertical motion of the pivot, the (nondimensional) differential equations turn to be

\[ u'' + \left[1 + \varepsilon \lambda \Omega^2 \cos \Omega \tau \right] \left(u - \frac{\varepsilon^2 u^3}{6}\right) = 0, \tag{4} \]

where

\[ \tau = \omega t \]
\[ \omega_0^2 = \frac{g}{r_0} \]
\[ \varepsilon \lambda = -\frac{\Delta z}{r_0} \]
\[ \Omega = \frac{\omega}{\omega_0}, \]

being \( \lambda \) the ratio between the effective length and amplitude of oscillation, and \( \varepsilon \) a bookkeeping parameter.

We can study this nonlinear differential equation using the Method of Averaging. In Cartesian coordinates, we propose a solution of the form

\[ u(\tau) = X(\tau) \cos \left(\frac{\Omega}{2} \tau\right) + Y(\tau) \sin \left(\frac{\Omega}{2} \tau\right) \]
\[ u'(\tau) = -\frac{\Omega}{2} X(\tau) \sin \left(\frac{\Omega}{2} \tau\right) + \frac{\Omega}{2} Y(\tau) \cos \left(\frac{\Omega}{2} \tau\right) \tag{5} \]
\[ u''(\tau) = -\frac{\Omega^2}{4} X(\tau) \cos \left(\frac{\Omega}{2} \tau\right) - \frac{\Omega^2}{4} Y(\tau) \sin \left(\frac{\Omega}{2} \tau\right) \tag{6} \]

that is a constrained coordinate transformation. From the restriction in the first derivative, the substitution in the differential equation and averaging over one period we obtain two (autonomous) differential equations for \( X' \) and \( Y' \) and introducing a detuning parameter (\( \varepsilon \sigma = \Omega - 2 \)), namely

\[ X' = -\frac{\pi \varepsilon Y}{12(\varepsilon \sigma + 2)^2} [2\lambda \varepsilon^2 Y^2 - 3\varepsilon Y^2 - 3\varepsilon X^2 - 6\varepsilon \sigma^2 - 24 \sigma - 12 \lambda] \tag{7} \]
\[ Y' = -\frac{\pi \varepsilon X}{12(\varepsilon \sigma + 2)^2} [3\varepsilon Y^2 + 2\lambda \varepsilon^2 X^2 + 3\varepsilon X^2 + 6 \varepsilon \sigma^2 + 24 \sigma - 12 \lambda], \tag{8} \]

we can solve this differential equation for the transient response, or equate \( (X', Y') = (0, 0) \) and solve for \( X \) and \( Y \) to get the steady-state solution. The solutions for the steady-state case are

\[ X = Y = 0 \]
or

\[ X^2 = -\frac{3}{\lambda \varepsilon^3} (\varepsilon^2 \sigma^2 + 4 \varepsilon \sigma - 2 \lambda \varepsilon + 6) \]
\[ Y^2 = \frac{3}{\lambda \varepsilon^3} (\varepsilon^2 \sigma^2 + 4 \varepsilon \sigma + 2 \lambda \varepsilon + 6) . \]

We can compute the Jacobian to obtain

\[
J(X,Y) = \begin{bmatrix}
-\frac{\pi \varepsilon X Y}{2(\varepsilon \sigma + 2)^2} & \frac{\pi \varepsilon (2 \lambda \varepsilon^2 Y^2 - 3 \varepsilon Y^2 - \varepsilon X^2 - 2 \varepsilon \sigma^2 - 8 \sigma - 4 \lambda)}{4(\varepsilon \sigma + 2)^2} \\
\frac{\pi \varepsilon Y^2 + 2 \lambda \varepsilon^2 X^2 + 3 \varepsilon X^2 + 2 \varepsilon \sigma^2 + 8 \sigma - 4 \lambda}{4(\varepsilon \sigma + 2)^2} & \frac{\pi \varepsilon^2 X Y}{2(\varepsilon \sigma + 2)^2}
\end{bmatrix}
\]

around \((0,0)\) gives

\[
J(0,0) = \frac{\pi \varepsilon}{2(\varepsilon \sigma + 2)^2} \begin{bmatrix}
0 & \frac{\pi \varepsilon^2 + 4 \sigma + 2 \lambda}{\varepsilon \sigma^2 + 4 \sigma + 2 \lambda} \\
\varepsilon \sigma^2 + 4 \sigma + 2 \lambda & 0
\end{bmatrix},
\]

the stability of this fixed point is given by the determinant (since the trace is equal to zero)

\[
\Delta = \frac{\pi^2 \varepsilon^2}{4(\varepsilon + 2)^4} (\varepsilon \sigma^2 + 4 \sigma - 2 \lambda)(\varepsilon \sigma^2 + 4 \sigma + 2 \lambda) ,
\]

this gives the critical values

\[
\lambda = -\frac{1}{2}(\varepsilon \sigma_1^2 + 4 \sigma_1), \quad \lambda = -\frac{1}{2}(\varepsilon \sigma_2^2 + 4 \sigma_2) .
\]

The stability of the system is depicted in figure 2.

For the other fixed point the determinant is

\[
\Delta = \frac{\pi^2 (\varepsilon^2 \sigma^2 + 4 \varepsilon \sigma - 2 \lambda \varepsilon + 6) (\varepsilon^2 \sigma^2 + 4 \varepsilon \sigma + 2 \lambda \varepsilon + 6)}{(\varepsilon \sigma + 2)^4},
\]

this gives the critical values

\[
\lambda = -\frac{1}{2 \varepsilon}(\varepsilon \sigma_1^2 + 4 \varepsilon \sigma_1 + 6), \quad \lambda = -\frac{1}{2 \varepsilon}(\varepsilon \sigma_2^2 + 4 \varepsilon \sigma_2 + 6) .
\]

This curves present an extremum at \(\sigma = -2\varepsilon\). The stability of the system is depicted in figure 3.

For parametric pumping, the optimum effect is achieved by pumping at twice the natural frequency of the pendulum. This is constant for the linear
**Figure 2:** Stability of the system around the point (0,0). Regions in blue refer to stable behavior, while regions in red to unstable behavior.

**Figure 3:** Stability of the system around the second fixed point. Regions in blue refer to stable behavior, while regions in red to unstable behavior.
pendulum, due to isochrony condition. When the pendulum increases the amplitude the period increase as

\[ T = 4 \sqrt{\frac{r_0}{2g}} \int_{0}^{\theta_0} \frac{1}{\sqrt{\cos \theta - \cos \theta_0}} d\theta, \]

being \( \theta_0 \) the initial amplitude, and the angular frequency diminishes.\(^8\) This gives us an explanation about the wider stability for the second (non-trivial) solution. Around \((0,0)\) the linear analysis of stability resembles the linear case, while in the second fixed point we are not around this point and the amplitude of the oscillation would affect the (parametric) resonances of the system.

### 2.2 Vertical and horizontal motion of the pivot: Method of Multiple Scales

If we keep both, the vertical and horizontal motion of the pivot we get parametric and direct excitation in the differential equation. After normalization reads

\[ u'' + [1 + \varepsilon \lambda_1 \Omega^2 \cos \Omega \tau] \left\{ u - \frac{\varepsilon^2 u^3}{6} \right\} - \varepsilon^2 \lambda_2 \Omega^2 \cos \Omega \tau \frac{u^2}{2} = -\lambda_2 \Omega^2 \cos \Omega \tau \]

with

\[ \tau = \omega t \omega_0 \]
\[ \omega_0^2 = \frac{g}{r_0} \]
\[ \varepsilon \lambda_1 = -\frac{\Delta z}{r_0} \]
\[ \Omega = \frac{\omega}{\omega_0} \]
\[ \varepsilon \lambda_1 = -\frac{\Delta z}{r_0}, \]

when, one more time \( \lambda_1 \) and \( \lambda_2 \) refers to the ratios between the amplitude of the parametric excitations and the length of the pendulum.

In this case, it is interesting to find the resonances that the system can achieve, that will differ from the single resonance from the linear case. Using the method of multiple scales we propose a solution

\[ u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \]

and

\[ t = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \cdots. \]

\(^8\)Hence, as the amplitude increases the optimal frequency of parametric excitation decreases.
Replacing, and grouping by powers of $\varepsilon$ we get

\[ \varepsilon^0 : D_0^2 u_0 + u_0 = -\lambda_2 \Omega^2 \cos \Omega T_0 \]
\[ \varepsilon^1 : D_0^2 u_1 + u_1 = -2D_0 D_1 u_0 - u_0 \lambda_1 \cos \Omega T_0 + \frac{1}{2} \lambda_2 \Omega^2 u_0^2 \cos \Omega T_0 \]
\[ \varepsilon^2 : D_0^2 u_2 + u_2 = -\lambda_1 u_1 \cos \Omega T_0 - 2D_0 D_1 u_1 - D_1^2 u_0 + \frac{u_0^3}{6} + \lambda_2 u_0 u_1 \Omega^2 \cos \Omega T_0 \]

Solving sequentially the set of equations, and enforcing secular terms to be zero we get as resonances

\[ \Omega \in \left\{ \frac{1}{2}, 1, 2, 3 \right\} \]

The appearance of three of the resonances was obtained via numerical integration and are presented in Figure 4. Figure 5 show solutions for a range of frequencies around the direct and primary parametric resonances, due to nonlinearities the higher values are not present exactly at $\Omega = 1, 2$.

Figure 4: Time response for two sets of different parameters of the equation; in both cases $\theta(0) = \pi/4$ and $\dot{\theta}(0) = 0$.

2.3 Numerical Results

We computed the numerical solution using odeint from Scipy.\(^9\)

\(^9\)odeint solve a system of ordinary differential equations using lsoda from the Fortran library odepack. http://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.odeint.html
Figure 5: Time response around the primary and direct resonances. The parameters used were: $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $\epsilon_1 = 2$, $\theta(0) = \pi/30$, and $\dot{\theta}(0) = 0$.

3 Conclusions

A model consisting of a planar pendulum was used to model the sloshing dynamics of coffee. Rotating sloshing was neglected since this will need two degrees of freedom (a spherical pendulum model).

The vertically excited pendulum presents a parametric pumping with an effective stiffness that varies with time. This is similar to the Mathieu equation with Duffing-like nonlinearity. When excited both, vertically and horizontally the system presents parametric and direct excitation. Once again, the equation reads similar to Mathieu equation with quadratic and cubic nonlinearities, which leads to multiple resonances.

If we consider the aspect ratio of cups constant we can have as design parameter the radius of the cup. Currently, the frequency ranges for the cups and regular walking are overlapped, but we can change this fact by tuning the size of the cup. An increase in the size will lead to a decrease in the natural frequency and will cease the overlapping. From a practical point of view, this is not feasible, since one doesn’t want to have a huge cup of coffee.

Small values of $\lambda$ will translate in longer walking with coffee without spilling, which can be achieved with higher sizes of cup or smaller amplitudes of oscillation—smoother walking. The latter can be obtained focusing in not spilling while walking, but this can be considered as a controlled system. This is one of the facts discussed in [8].

A single harmonic excitation was considered due to its main importance [8], but the real walking is a complex signal with a broadband frequency content.

\[\text{See Appendix A.}\]
and several resonances will appear in that case.

Appendix A: Physical Parameters

Figure 6 shows a schematic of the cup of coffee. According to [8], the parameters for common cups are

\[
R \in [2.5, 6.7] \text{ cm} \\
H \in [5.7, 8.9] \text{ cm} \\
h \in [5.10, 10] \text{ cm} \\
\alpha \in [8^\circ, 16^\circ]
\]
(e) $\Omega = 2.82$.  
(f) $\Omega = 2.99$.  
(g) $\Omega = 3.54$.  
(h) $\Omega = 4.00$.  

Figure 5: Phase portrait for different excitation frequencies. In all the cases $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $\varepsilon = 0.2$, $\theta(0) = 0.1$ and $\dot{\theta}(0) = 0$. The red square is the initial point in the phase space, while the blue one is the finishing one after $\tau = 500$.  

This combination of parameters gives a range of frequencies for the cup  

$$f_{\text{cup}} \in [2.6, \ 4.3] \text{ Hz },$$  

and the frequency of walking is [8]  

$$f_{\text{step}} \in [1, \ 2.5] \text{ Hz }.$$  

According to [3] the lumped parameters for a single degree of freedom
Figure 6: Geometrical description of the cup of coffee.

The system are

\[ m = \frac{c_1 m_0}{1 - c_1} \]

\[ m_0 = \rho \pi (H - h) R^2 \]

\[ c_1 = \frac{R}{2.2} \tanh \left( 1.84 \frac{H}{R} \right) \]

\[ r_0 = \frac{R}{1.84} \tanh \left( 1.84 \frac{H}{R} \right), \]

for simple calculation we can take \( H/R \approx 1.5 \), what yields

\[ r_0 \approx 0.54R. \]

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