Research Article

Regularity of Semigroups of Transformations Whose Characters Form the Semigroup of a Δ-Structure

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1. Introduction and Preliminaries

For any semigroup $\mathcal{S}$, we call an element $a$ of $\mathcal{S}$ a regular element of $\mathcal{S}$ if there exists an element $b$ of $\mathcal{S}$ such that $aba = a$. A semigroup $\mathcal{S}$ is said to be regular if every element of $\mathcal{S}$ is regular. The semigroup of transformations on a nonempty set $X$, denoted by $T(X)$, is a well-known regular semigroup. In [1], Nenthein et al. studied the regularity of the following two sub-semigroups of $T(X)$:

$$T(X,Y) = \{a \in T(X): \text{Ran}(a) \subseteq Y\},$$

$$T(X) = \{a \in T(X): \text{a}(Y) \subseteq Y\},$$

where $Y$ is a fixed nonempty subset of $X$ and $\text{Ran}(a)$ denotes the range of $a$ for all $a \in T(X)$. The authors obtained that the two semigroups are regular if and only if $|Y| = 1$ or $Y = X$. The regularity of some other sub-semigroups of $T(X)$ has been studied by many people (see [2–6] for some recent works).

In this paper, by a partition of a nonempty set $X$, we mean a family $\mathcal{F} = \{Y_i: i \in I\}$ of nonempty subsets of $X$ such that $X = \bigcup_{i \in I} Y_i$ and $Y_i \cap Y_j = \emptyset$ for all $i, j \in I$ with $i \neq j$. For a given partition $\mathcal{F} = \{Y_i: i \in I\}$ of a nonempty set $X$, let

$$T_{\mathcal{F}}(X) = \{a \in T(X): \forall i \in I \exists j \in I, a(Y_i) \subseteq Y_j\}. $$

Note that $T_{\mathcal{F}}(X)$ is exactly the semigroup of all transformations on $X$ preserving the equivalence relation induced by partition $\mathcal{F}$.

Throughout this paper, let $X$ be a nonempty set, and let $\mathcal{F} = \{Y_i: i \in I\}$ be a partition of $X$, which are arbitrarily fixed. From the definition of a partition, Purisang and Rakbud [11] defined a function $\chi^{(a)}: I \rightarrow I$ associated with each $a \in T_{\mathcal{F}}(X)$ called the character of $a$, by

$$\forall i,j \in I, \chi^{(a)}i = j \Leftrightarrow a(Y_i) \subseteq Y_j.$$

They also defined and studied the regularity of the following sub-semigroup of $T(X)$:

$$T_{\mathcal{F}}^{(j)}(X) = \{a \in T(X): \forall i \in I \exists j \in I, a(Y_i) \subseteq Y_j\} = \{a \in T(X): \chi^{(a)}(I) \subseteq T(I, J)\},$$

where $J$ is an arbitrarily fixed nonempty subset of the index set $I$. We summarize some of their results as follows.

Lemma 1 (see [11], Lemma 2.3). For every $a, \beta \in T_{\mathcal{F}}^{(j)}(X)$, $\chi^{(a\beta)} = \chi^{(a)} \chi^{(\beta)}$. 


By making use of the notion of the character, the following two equivalence relations \( \chi \) and \( \overline{\chi} \) on \( T^I(Z)(X) \) were defined:

\[
\forall \alpha, \beta \in T^I(Z)(X), \quad (\alpha, \beta) \in \chi \iff \chi^{(\alpha)} = \chi^{(\beta)},
\]
\[
\forall \alpha, \beta \in T^I(Z)(X), \quad (\alpha, \beta) \in \overline{\chi} \iff \overline{\chi}^{(\alpha)} = \chi^{(\beta)}.
\]

(5)

Note that, by Lemma 1, they both are congruence relations. The authors studied the regularity of the quotient semigroups \( T^I(Z)(X)/\chi \) and \( T^I(Z)(X)/\overline{\chi} \). The following are what they obtained.

**Theorem 1** (see [11], Theorem 2.4). The following statements hold:

1. \( T^I(Z)(X)/\chi \cong T(I, I) \) by the isomorphism \([a] \mapsto \chi^{(a)}\);,
2. \( T^I(Z)(X)/\overline{\chi} \cong T(I, I) \) by the isomorphism \([a] \mapsto \chi^{(a)}\),

where for each \( \alpha \in T^I(Z)(X) \), \([a]\) and \([a]\) denote the equivalence classes of \( \alpha \) under the equivalence relations \( \chi \) and \( \overline{\chi} \), respectively.

**Corollary 1** (see [11], Corollary 2.5). The following statements hold.

1. The three statements below are all equivalent:
   a. The quotient semigroup \( T^I(Z)(X)/\chi \) is regular;
   b. The semigroup \( T(I, I) \) is regular;
   c. \( I = I \) or \(|I| = 1\).
2. The quotient semigroup \( T^I(Z)(X)/\chi \) is regular.
3. The quotient semigroup \( T^I(Z)(X)/\overline{\chi} \) is regular.

The regularity of the semigroup \( T^I(Z)(X) \) was obtained as follows.

**Theorem 2** (see [11], Theorem 2.6). The semigroup \( T^I(Z)(X) \) is regular if and only if \(|T^I(Z)(X)| = 1 \) or \( T^I(Z)(X) = T(X) \).

It is easy to see that for each \( \alpha \in T_Z(X) \), the equivalence class \([\alpha]\) of \( \alpha \) under the equivalence relation \( \chi \) is a subsemigroup of \( T_Z(X) \) if and only if \( \chi^{(a)} \) is an idempotent element of \( T(I) \). In [11], the authors studied the regularity of the semigroup \([a]\) when \( \alpha \) is an idempotent element of \( T(I) \). They also defined some further sub-semigroups of \( T_Z(X) \) by making use of the notion of the character as follows: let \( I_Z(X) \), \( S_Z(X) \), and \( B_Z(X) \) be the sets of all elements of \( T_Z(X) \) whose characters are injective, surjective, and bijective, respectively. Note that, by Lemma 1, the sets \( I_Z(X) \), \( S_Z(X) \), and \( B_Z(X) \) are sub-semigroups of \( T_Z(X) \). The regularity of each of these three semigroups was also studied.

It was observed by Rakbud [12] that the semigroups \( T^I(Z)(X) \), \([a]\) when \( \chi^{(a)} \) is idempotent, \( I_Z(X) \), \( S_Z(X) \), and \( B_Z(X) \) can simultaneously be generalized by making use of the notion of the character as follows: for every sub-semigroup \( \delta \) of \( T(I) \), let

\[
T^I(Z)(X) = \{ \alpha \in T_Z(X): \chi^{(a)} \in \delta \}.
\]

(6)

Note that, for each \( \gamma \in T(I) \), the function \( \alpha: X \to X \) defined on each \( Y_i \) by \( \alpha(Y_i) = \{ z \} \), where \( z_i \) is a fixed element of \( Y_i \) for all \( i \), is an element of \( T_Z(X) \) whose character is exactly \( \gamma \). Hence, \( T^I(Z)(X) \neq \emptyset \), and by Lemma 1, it is a sub-semigroup of \( T_Z(X) \).

Let \( \delta \) be a sub-semigroup of \( T(I) \). Then, by considering the congruence relation \( \chi \) on \( T_Z(X) \) restricted to \( T^I(Z)(X) \), we have the quotient semigroup \( T^I(Z)(X)/\chi \). Obviously, \( T^I(Z)(X)/\chi = \{ [a]: a \in T^I(Z)(X) \} \) and \( T^I(Z)(X)/\chi \) is a sub-semigroup of \( T^I(Z)(X)/\chi \). Analogously to Theorem 1, the following result was established.

**Theorem 3** (see [12], Theorem 1.14). \( T^I(Z)(X)/\chi \cong \delta \) by the isomorphism defined by \([a] \mapsto \chi^{(a)}\).

Immediately from Theorem 3, the following corollary was obtained.

**Corollary 2** (see [12], Corollary 1.15). The quotient semigroup \( T^I(Z)(X)/\chi \) is regular if and only if the semigroup \( \delta \) is regular.

Besides the above results, in [12], the author also used the notion of the character to define the notion of a weakly regular transformation and study the regularity of a semigroup of weakly regular transformations in that sense. However, the regularity of \( T^I(Z)(X) \) has not been studied in general yet. This will be in our attention here when \( \delta \) has a certain property. We are mentioning that \( \delta \) is “the semigroup of a \( \Delta \)-structure” on \( I \).

We now refer to the definition of a \( \Delta \)-structure on a set and some other related ones from [13] by Magill and Subbiah. Let \( \mathcal{A} \) be a family of nonempty subsets of \( X \) such that \( X \in \mathcal{A} \). And, let \( \mathcal{M} = \{ \text{Hom}(A, B): A, B \in \mathcal{A} \} \), where \( \text{Hom}(A, B) \) is a nonempty set of functions from \( A \) into \( B \) for all \( A, B \in \mathcal{A} \), with the following properties:

1. \( \text{End}(X) = \text{Hom}(X, X) \) is a monoid;
2. \( \text{Ran}(a) \in \mathcal{A} \) for all \( a \in \text{End}(X) \);
3. For all \( a \in \mathcal{A} \), \( a \in \text{End}(X) \) and \( \beta \in \text{Hom}(\text{Ran}(a), B, \beta a \in \text{End}(X)) \);
4. For all \( a, \beta \in \text{End}(X) \) and \( A, B \in \mathcal{A} \) with \( \alpha(B) \subseteq A \) and \( \beta (A) \subseteq B \), if \( (\alpha \beta)_A = \text{id}_A \) and \( (\beta \alpha)_B = \text{id}_B \), then \( \beta|_A \in \text{Hom}(A, B) \) and \( a|_B \in \text{Hom}(B, A) \), where \( \text{id}_Z \) denotes the identity function on \( Z \) for any non-empty subset \( Z \).

The pair \( (\mathcal{A}, \mathcal{M}) \) is called a \( \Delta \)-structure on \( X \), and the monoid \( \text{End}(X) \) is called the semigroup of the \( \Delta \)-structure \( (\mathcal{A}, \mathcal{M}) \). A subset \( A \) of \( X \) is called a \( \Delta \)-retract of \( X \) if there is an idempotent element \( \rho \) of \( \text{End}(X) \) such that \( A = \text{Ran}(\rho) \). For any \( A, B \in \mathcal{A} \), an element \( \lambda \) of \( \text{Hom}(A, B) \) is called a \( \Delta \)-isomorphism (from \( A \) onto \( B \) if there is an \( \sigma \in \text{Hom}(B, A) \) such that \( \sigma \lambda = \text{id}_A \) and \( \lambda \sigma = \text{id}_B \). It is clear for any \( A, B \in \mathcal{A} \) that an element \( \lambda \) of \( \text{Hom}(A, B) \) is \( \Delta \)-isomorphism if and only if \( \lambda \) is bijective and \( \lambda^{-1} \in \text{Hom}(B, A) \).

In [13], the authors gave some characterizations of regular elements of the semigroup \( \text{End}(X) \) as follows.
Theorem 4 (see [13], Theorem 2.4). Let \( a \in \text{End}(X) \). Then, the following statements are equivalent:

1. \( a \) is regular;
2. \( \text{Ran}(a) \) is a \( \Delta \)-retract of \( X \), and there is a \( \Delta \)-retract \( A \) of \( X \) such that \( a|_A \) is a \( \Delta \)-isomorphism from \( A \) onto \( \text{Ran}(a) \);
3. \( \text{Ran}(a) \) is a \( \Delta \)-retract of \( X \), and there is \( A \in \mathcal{A} \) such that \( a|_A \) is a \( \Delta \)-isomorphism from \( A \) onto \( \text{Ran}(a) \).

It is clear that \( T(X) = \text{End}(X) \) when \( X \) is equipped with the \( \Delta \)-structure \((\mathcal{A}, \mathcal{M})\), where \( \mathcal{A} \) is the family of all nonempty subsets of \( X \), and \( \text{Hom}(A,B) \) is the set of all functions from \( A \) into \( B \) for all \( A, B \in \mathcal{A} \). In this setting, the \( \Delta \)-retracts of \( X \) are exactly the nonempty subsets of \( X \), and the \( \Delta \)-isomorphisms are exactly the bijective functions. More interesting semigroups of \( \Delta \)-structures on \( X \) were given in [13] as follows:

(i) The semigroup \( S(X) \) of all continuous maps on \( X \) is exactly \( \text{End}(X) \) when \( X \) is a topological space equipped with the \( \Delta \)-structure \((\mathcal{A}, \mathcal{M})\), where \( \mathcal{A} \) is the family of all nonempty subsets of \( X \) and \( \text{Hom}(A,B) \) is the set of all continuous maps from \( A \) into \( B \) for all \( A, B \in \mathcal{A} \). In this setting, the \( \Delta \)-retracts of \( X \) are exactly the \( \Delta \)-retracts of the topological space \( X \), and the \( \Delta \)-isomorphisms are exactly the homeomorphisms.

(ii) The semigroup \( L(X) \) of all linear transformations on \( X \) coincides with \( \text{End}(X) \) when \( X \) is a vector space equipped with the \( \Delta \)-structure \((\mathcal{A}, \mathcal{M})\), where \( \mathcal{A} \) is the family of all subspaces of \( X \) and \( \text{Hom}(A,B) \) is the set of all linear transformations from \( A \) into \( B \) for all \( A, B \in \mathcal{A} \). In this setting, the \( \Delta \)-retracts of \( X \) are exactly the subspaces of the vector space \( X \), and the \( \Delta \)-isomorphisms are exactly the isomorphisms.

(iii) The semigroup \( \Gamma(X) \) of all closed maps on \( X \) is exactly \( \text{End}(X) \) when \( X \) is a \( T_1 \)-space equipped with the \( \Delta \)-structure \((\mathcal{A}, \mathcal{M})\), where \( \mathcal{A} \) is the family of all nonempty closed subsets of \( X \) and \( \text{Hom}(A,B) \) is the set of all closed maps from \( A \) into \( B \) for all \( A, B \in \mathcal{A} \). In this setting, the \( \Delta \)-retracts of \( X \) are exactly the nonempty closed subsets of the topological space \( X \), and the \( \Delta \)-isomorphisms are exactly the homeomorphisms.

The regularity of the semigroups \( S(X), L(X) \), and \( \Gamma(X) \) was also deduced via Theorem 4. In addition, we note here that the semigroup \( T_\mathcal{F}(X) \) of all transformations on \( X \) preserving an equivalence relation \( \mathcal{E} \) on \( X \) can be considered as the semigroup of all continuous maps on \( X \), where \( X \) is equipped with the topology having the family of all equivalence classes as a base. This was proved by Huisheng [8] (see Theorem 2.8).

The main aim of this paper is to study the regularity of the semigroup \( T_\mathcal{F}(X) \), introduced in [12], when \( \mathcal{F} \) is a \( \Delta \)-structure on the index set \( I \) of the partition \( \mathcal{F} \). We also define, in this situation, a sub-semigroup of \( T_\mathcal{F}(X) \) whose regularity coincides with that of the semigroup \( \Delta \).

2. The Semigroup and Its Regularity

For any nonempty sets \( Z \) and \( W \) and partitions \( \mathcal{F} = \{Z_i : i \in I\} \) and \( \mathcal{G} = \{W_j : j \in K\} \) of \( Z \) and \( W \), respectively, let \( T_{\mathcal{F},\mathcal{G}}(Z, W) \) be the set of all functions \( \lambda : Z \rightarrow W \) satisfying the condition that for all \( i \in I \), there is \( j \in K \) such that \( \lambda(Z_i) \in W_j \). And, for each \( \lambda \in T_{\mathcal{F},\mathcal{G}}(Z, W) \), let \( \chi_{\mathcal{F},\mathcal{G}} : H \rightarrow K \) be defined by

\[
\chi_{\mathcal{F},\mathcal{G}}(\lambda)(j) = i \iff \lambda(Z_i) \in W_j.
\]

It is easy to verify that the map \( \lambda \mapsto \chi_{\mathcal{F},\mathcal{G}}(\lambda) \) from the set \( T_{\mathcal{F},\mathcal{G}}(Z, W) \) into the set of all functions from \( H \) into \( K \) is surjective. If we have three nonempty sets \( Z, W, \) and \( U \) with partitions \( \mathcal{F} = \{Z_i : i \in I\}, \mathcal{G} = \{W_j : j \in K\}, \) and \( \mathcal{R} = \{U_i : i \in M\} \) of \( Z, W, \) and \( U \), respectively, we see for any \( \lambda \in T_{\mathcal{F},\mathcal{G}}(Z, W) \) and \( \rho \in T_{\mathcal{G},\mathcal{R}}(W, U) \) that \( \rho \lambda \in T_{\mathcal{F},\mathcal{R}}(Z, U) \) and \( \chi_{\mathcal{F},\mathcal{R}}(\rho \lambda) = \chi_{\mathcal{F},\mathcal{G}}(\lambda) \circ \chi_{\mathcal{G},\mathcal{R}}(\rho) \).

Let \( (J, \mathcal{C}) \) be a \( \Delta \)-structure on the index set \( I \) of the partition \( \mathcal{F} \), and let

\[
\text{End}_\mathcal{F}(X) = T\upharpoonright(\text{End}(I))(X) = \{ \alpha \in T_\mathcal{F}(X) : \chi(\alpha) \in \text{End}(I) \}.
\]

By the property (\( \Delta \)) of the \( \Delta \)-structure \( (J, \mathcal{C}) \) on \( I \), we have that \( \chi_{\mathcal{F},\mathcal{G}}(\chi_{\mathcal{D},\mathcal{F}}) = \text{id}_I \in \text{End}(I) \), which yields that \( \text{id}_I \in \text{End}_\mathcal{F}(X) \). Hence, \( \text{End}_\mathcal{F}(X) \) is a submonoid of \( T_\mathcal{F}(X) \).

Theorem 5. There is a \( \Delta \)-structure on \( X \) such that \( \text{End}(X) = \text{End}_\mathcal{F}(X) \).

Proof. Let

\[
\mathcal{A} = \{ \text{Ran}(\alpha) : \alpha \in \text{End}_\mathcal{F}(X) \}.
\]

Since \( \text{End}_\mathcal{F}(X) \) is a monoid, we have \( X \in \mathcal{A} \). For any \( \alpha, \beta \in \text{End}_\mathcal{F}(X) \), let

\[
\text{Hom} (\text{Ran}(\alpha), \text{Ran}(\beta)) = \left\{ \lambda \in T_\mathcal{F}(\alpha, \beta) : \chi_{\mathcal{F}}(\lambda) \in \text{Hom}(\text{Ran}(\alpha), \text{Ran}(\beta)) \right\}.
\]

Then, by the surjectivity of the map \( \lambda \mapsto \chi_{\mathcal{F}}(\lambda) \) from the set \( T_\mathcal{F}(\alpha, \beta) \) into the set of all functions from \( \text{Ran}(\chi_{\mathcal{A}}(\alpha)) \) into \( \text{Ran}(\chi_{\mathcal{A}}(\beta)) \), we have that \( \text{Hom} (\text{Ran}(\alpha), \text{Ran}(\beta)) \neq \emptyset \) for all \( \alpha, \beta \in \text{End}_\mathcal{F}(X) \). Note that for all \( \alpha, \beta, \lambda, \rho \in \text{End}_\mathcal{F}(X) \), if \( \text{Ran}(\alpha) = \text{Ran}(\lambda) \) and \( \text{Ran}(\beta) = \text{Ran}(\rho) \), then \( \text{Ran}(\chi_{\mathcal{A}}(\alpha)) = \text{Ran}(\chi_{\mathcal{A}}(\beta)) \).
Ran($\chi^{(1)})$ and Ran($\chi^{(\beta)}) = \text{Ran}(\chi^{(\psi)})$, which yields that Hom(Ran ($\alpha)$, Ran ($\beta)$) = Hom(Ran ($\lambda)$, Ran ($\rho)$). Let

$$\mathcal{M} = \{\text{Hom}(\text{Ran} (\alpha), \text{Ran} (\beta)) : \alpha, \beta \in \text{End}_\mathcal{F} (X)\}. \quad (13)$$

Then, from the above explanation, the family $\mathcal{M}$ is well defined. We will show that $(\mathcal{A}, \mathcal{M})$ is a $\Delta$-structure on $X$. It is easy to see that

$$\text{Hom}(X, X) = \text{Hom}(\text{Ran}(\text{id}_X), \text{Ran}(\text{id}_X))$$

$$= \{\lambda \in \text{End}_\mathcal{F} (X) : \chi^{(1)} \in \text{End}(I)\}$$

$$= \text{End}_\mathcal{F} (X). \quad (14)$$

Thus, the property ($\Delta1$) is satisfied. From the definition of the family $\mathcal{A}$, we immediately have that the property ($\Delta2$) holds. Next, we will show that the property ($\Delta3$) is satisfied. Let $\alpha, \beta \in \text{End}_\mathcal{F} (X)$, and let $\lambda \in \text{Hom}(\text{Ran} (\alpha), \text{Ran} (\beta))$. We want to show that $\lambda \alpha \in \text{End}_\mathcal{F} (X)$. Since $\lambda \in \text{Hom}(\text{Ran} (\alpha), \text{Ran} (\beta))$, we have $\lambda \alpha \in \text{Hom}(\text{Ran} (\chi^{(\alpha)}), \text{Ran} (\chi^{(\beta)}))$. Since $\alpha \in \text{End}_\mathcal{F} (X)$, we have that $\alpha \in \text{End}(I)$. Thus, by the property ($\Delta3$) of the $\Delta$-structure $(\mathcal{F}, \mathcal{C})$ on $I$, we get $\chi^{(\alpha)} \in \text{End}(I)$. It is clear that $\chi^{(\alpha)} \alpha \in \text{End}(I)$, and that $\chi^{(\alpha)} \alpha \in \chi^{(\alpha)}$. Hence, by the membership of $\chi^{(\alpha)}$ in $\text{End}(I)$, we obtain that $\alpha \in \text{End}_\mathcal{F} (X)$ as desired. Finally, we will show that the property ($\Delta4$) holds. Let $\alpha, \beta, \psi, \phi \in \text{End}_\mathcal{F} (X)$ be such that $\alpha(\text{Ran} (\phi)) \subseteq \text{Ran} (\psi)$. Since $\alpha(\text{Ran} (\phi)) \subseteq \text{Ran} (\psi)$, we get that $\alpha \phi (x) \subseteq \text{Ran} (\psi)$. Therefore, there is $z \in \mathcal{Z}$ such that $\phi (x) = \psi (z)$. Let $k \in I$ be such that $z \in Y_k$, and let $v = \chi^{(\psi)} (k)$. Then, $\psi (Y_k) \subseteq \mathcal{Y}$. Since $\alpha(\text{Ran} (\phi)) \subseteq \text{Ran} (\psi)$, we get that $\alpha \phi (x) \subseteq \text{Ran} (\psi)$. Therefore, there is $z \in \mathcal{Z}$ such that $\phi (x) = \psi (z)$. Let $k \in I$ be such that $z \in Y_k$, and let $v = \chi^{(\psi)} (k)$. Then, $\psi (Y_k) \subseteq \mathcal{Y}$. Since $\alpha(\text{Ran} (\phi)) \subseteq \text{Ran} (\psi)$, we get that $\alpha \phi (x) \subseteq \text{Ran} (\psi)$. Therefore, there is $z \in \mathcal{Z}$ such that $\phi (x) = \psi (z)$. Let $k \in I$ be such that $z \in Y_k$, and let $v = \chi^{(\psi)} (k)$. Then, $\psi (Y_k) \subseteq \mathcal{Y}$. Since $\alpha(\text{Ran} (\phi)) \subseteq \text{Ran} (\psi)$, we get that $\alpha \phi (x) \subseteq \text{Ran} (\psi)$. Therefore, there is $z \in \mathcal{Z}$ such that $\phi (x) = \psi (z)$. Let $k \in I$ be such that $z \in Y_k$, and let $v = \chi^{(\psi)} (k)$. Then, $\psi (Y_k) \subseteq \mathcal{Y}$. Since $\alpha(\text{Ran} (\phi)) \subseteq \text{Ran} (\psi)$, we get that $\alpha \phi (x) \subseteq \text{Ran} (\psi)$. Therefore, there is $z \in \mathcal{Z}$ such that $\phi (x) = \psi (z)$. Let $k \in I$ be such that $z \in Y_k$, and let $v = \chi^{(\psi)} (k)$. Then, $\psi (Y_k) \subseteq \mathcal{Y}$. Since $\alpha(\text{Ran} (\phi)) \subseteq \text{Ran} (\psi)$, we get that $\alpha \phi (x) \subseteq \text{Ran} (\psi)$. Therefore, there is $z \in \mathcal{Z}$ such that $\phi (x) = \psi (z)$. Let $k \in I$ be such that $z \in Y_k$, and let $v = \chi^{(\psi)} (k)$. Then, $\psi (Y_k) \subseteq \mathcal{Y}$. Since $\alpha(\text{Ran} (\phi)) \subseteq \text{Ran} (\psi)$, we get that $\alpha \phi (x) \subseteq \text{Ran} (\psi)$. Therefore, there is $z \in \mathcal{Z}$ such that $\phi (x) = \psi (z)$. Let $k \in I$ be such that $z \in Y_k$, and let $v = \chi^{(\psi)} (k)$. Then, $\psi (Y_k) \subseteq \mathcal{Y}$. Since $\alpha(\text{Ran} (\phi)) \subseteq \text{Ran} (\psi)$, we get that $\alpha \phi (x) \subseteq \text{Ran} (\psi)$. Therefore, there is $z \in \mathcal{Z}$ such that $\phi (x) = \psi (z)$. Let $k \in I$ be such that $z \in Y_k$, and let $v = \chi^{(\psi)} (k)$. Then, $\psi (Y_k) \subseteq \mathcal{Y}$. Since $\alpha(\text{Ran} (\phi)) \subseteq \text{Ran} (\psi)$, we get that $\alpha \phi (x) \subseteq \text{Ran} (\psi)$. Therefore, there is $z \in \mathcal{Z}$ such that $\phi (x) = \psi (z)$.

From now on, we will consider the $\Delta$-structure $(\mathcal{A}, \mathcal{M})$ on $X$ defined in the proof of Theorem 5, and the semigroup $\text{End}_\mathcal{F} (X)$ of this $\Delta$-structure will be in our attention. By Theorem 4, the regularity of elements of the semigroup $\text{End}_\mathcal{F} (X)$ can roughly be characterized. To get more precise characterizations, according to Theorem 4, the notions of a $\Delta$-retract and a $\Delta$-isomorphism in the $\Delta$-structure $(\mathcal{A}, \mathcal{M})$ on $X$ should particularly be studied. For that purpose, the following elementary theorem, stating some characterizations of idempotent elements in a transformation semigroup, is needed.

**Theorem 6.** Let $Z$ be a nonempty set and $a \in T(Z)$. Then, the following statements are equivalent:

1. $\alpha$ is idempotent;
2. $\alpha|_{\text{Ran} (\phi)} = \text{id}_{\text{Ran} (\phi)}$;
3. There is a partition $\{Z_j : j \in E\}$ of $Z$ and a subset $\{z_j : j \in E\} \subset Z$ such that $z_j \in Z_j$ for all $j \in E$ and $\alpha(Z_j) = \{z_j\}$ for all $j \in E$.

In this situation, the partition $\{Z_j : j \in E\}$ of $Z$ are unique determined by $a$.

For each nonempty subset $A$ of $X$, we see that there is a unique subset of $I$, denoted by $I_A$, such that the family $\{A \cap Y_i : i \in I_A\}$ is a partition of $A$. In particular, we have that $I_A$ is exactly $\text{Ran} (\chi^{(a)})$ if $A = \text{Ran} (\alpha)$ for some $\alpha \in T_\mathcal{F} (X)$.

**Proposition 1.** Let $\gamma \in T(I)$, and let $A \subseteq X$. If $\gamma$ is idempotent, and that $I_A = \text{Ran} (\gamma)$, then there is an idempotent element $\alpha$ of $T_\mathcal{F} (X)$ such that $\chi^{(a)} = \gamma$ and $A = \text{Ran} (\alpha)$.

**Proof.** Suppose that $\gamma$ is idempotent, and that $I_A = \text{Ran} (\gamma)$. Then, by Theorem 6, there is a partition $\{I_j : j \in E\}$ of $I$ and a subset $\{i_j : j \in E\}$ of $I$ such that $i_j \in I_j$, and $\gamma ([i_j]) = [i_j]$ for all $j \in E$ from this, we have that $I_A = \text{Ran} (\gamma) = \{i_j : j \in E\}$. For each $j \in E$, let $a_j \in T_\mathcal{F} (U_{i \in I_j} Y_i)$, where $U_{i \in I_j} Y_i = \{Y_j : i \in I_j\}$, be idempotent such that $\chi^{(a_j)} (i) = i$ for all $i \in I_j$ and $\text{Ran} (\alpha_j) \subseteq A \cap Y_i$. And, finally, let $a : X \longrightarrow X$ be defined by $\alpha|_{U_{i \in I_j} Y_i} = a_j$ for all $j \in E$. Since $\bigcup_{i \in I_j} Y_i = \{i_j : j \in E\}$ is a partition of $X$, it follows that $a$ is well defined. It is clear by the way
of defining $a$ that $a \in T_\varphi(X)$ with $\chi^{(a)} = y$. It is also clear that $a$ is idempotent with $\text{Ran}(a) = A$. □

From Proposition 1, the following corollary is easily obtained.

**Corollary 3.** Let $A \subseteq X$. Then, $A$ is a $\Delta$-retract of $X$ if and only if $J_A$ is a $\Delta$-retract of $I$.

**Proof.** Suppose that $A$ is a $\Delta$-retract of $X$. Then, there is an idempotent element $\alpha$ of $\text{End}_\varphi(X)$ such that $A = \text{Ran}(\alpha)$, which yields that $J_A = \text{Ran}(\chi^{(\alpha)})$. Since $\alpha$ is idempotent, we have by Lemma 1 that $\chi^{(\alpha)}$ is an idempotent element of $\text{End}(I)$. Hence, $J_A$ is a $\Delta$-retract of $I$. Conversely, suppose that $J_A$ is a $\Delta$-retract of $I$. Then, there is an idempotent element $\gamma$ of $\text{End}(I)$ such that $J_A = \text{Ran}(\gamma)$. Thus, by Proposition 1, the set $A$ is a $\Delta$-retract of $X$. □

**Proposition 2.** Let $\alpha, \beta \in \text{End}_\varphi(X)$, and let $\lambda \in \text{Hom}(\text{Ran}(\alpha), \text{Ran}(\beta))$. If $\alpha$ is a $\Delta$-isomorphism, then $\lambda^{(\alpha)}$ is a $\Delta$-isomorphism.

**Proof.** Let $\lambda \in \text{Hom}(\text{Ran}(\alpha), \text{Ran}(\beta))$. Then, $\lambda^{(\alpha)} \in \text{Hom}(\text{Ran}(\chi^{(\alpha)}), \text{Ran}(\chi^{(\beta)}))$. Suppose that $\lambda$ is a $\Delta$-isomorphism. Then, there exists $\psi \in \text{Hom}(\lambda(\beta), \text{Ran}(\alpha))$ such that $\lambda \psi = \text{id}_{\text{Ran}(\beta)}$ and $\lambda \psi = \text{id}_{\text{Ran}(\alpha)}$. By the membership of $\psi$ in $\text{Hom}(\text{Ran}(\beta), \text{Ran}(\alpha))$, we have $\lambda^{(\psi)}(\beta) = \lambda^{(\psi)}(\alpha)$ in $\text{Hom}(\chi^{(\alpha)}, \text{Ran}(\chi^{(\beta)}))$. Since $\lambda \psi = \text{id}_{\text{Ran}(\beta)}$, we get $\lambda^{(\alpha)}(\beta) = \lambda^{(\psi)}(\psi) = \lambda^{(\alpha)}(\psi)$. Similarly, since $\lambda \psi = \text{id}_{\text{Ran}(\alpha)}$, we get $\lambda^{(\beta)}(\beta) = \lambda^{(\alpha)}(\beta)$. Thus, $\lambda^{(\alpha)}$ is a $\Delta$-isomorphism. □

In the following theorem, we provide a characterization of the regularity of elements of $\text{End}_\varphi(X)$ in terms of the $\Delta$-retract and the $\Delta$-isomorphism of $I$.

**Theorem 7.** Let $\alpha \in \text{End}_\varphi(X)$. Then, $\alpha$ is regular if and only if $\text{Ran}(\chi^{(\alpha)})$ is a $\Delta$-retract of $I$, and there is $A \subseteq X$ such that each of the following statements holds true:

(i) $J_A$ is a $\Delta$-retract of $I$;

(ii) $\chi^{(\alpha)}|_{J_A}$ is a $\Delta$-isomorphism from $J_A$ onto $\text{Ran}(\chi^{(\alpha)})$;

(iii) $\alpha|_{\text{Ran}(\lambda)}$ is a bijective function from $\text{Ran}(\lambda) \cap Y_i$ onto $\text{Ran}(\alpha) \cap Y_i$ for all $i \in J_A$.

**Proof.** Suppose that $\chi^{(\alpha)}|_{J_A}$ is a $\Delta$-retract of $I$, and that there exists a subset $A$ of $X$ such that each of the following statements holds true:

(i) $J_A$ is a $\Delta$-retract of $I$;

(ii) $\chi^{(\alpha)}|_{J_A}$ is a $\Delta$-isomorphism from $J_A$ onto $\text{Ran}(\chi^{(\alpha)})$;

(iii) $\alpha|_{\text{Ran}(\lambda)}$ is a bijective function from $\text{Ran}(\alpha) \cap Y_i$ onto $\text{Ran}(\alpha) \cap Y_i$ for all $i \in J_A$.

Since $\text{Ran}(\chi^{(\alpha)})$ is a $\Delta$-retract of $I$, we have by Corollary 3 that $\text{Ran}(\alpha)$ is a $\Delta$-retract of $X$. And, since $J_A$ is a $\Delta$-retract of $I$, there is an idempotent element $\gamma$ of $\text{End}(I)$ such that $J_A = \text{Ran}(\gamma)$. Thus, by Proposition 1, there is an idempotent element $\beta$ of $T_\varphi(X)$ such that $\chi^{(\beta)} = y$ and $A = \text{Ran}(\beta)$. This yields that $A$ is a $\Delta$-retract of $X$. By condition (ii), we have that $\chi^{(\alpha)}|_{J_A}$ is a bijective function from $J_A$ onto $\text{Ran}(\chi^{(\alpha)})$. Thus, by condition (iii), we get that $\alpha|_{\text{Ran}(\lambda)}$, which is an element of $T_\varphi(\beta, A)$, is bijective. By condition (ii) once again, we have $\chi^{(\alpha)}|_{J_A} = \chi^{(\alpha)}|_{J_A} \in \text{Hom}(J_A, \text{Ran}(\chi^{(\alpha)}))$ and $\lambda^{(\alpha)}(\alpha)^{-1} = (\chi^{(\alpha)}|_{J_A})^{-1}$ in $\text{Hom}(\text{Ran}(\chi^{(\alpha)}), J_A)$. It follows that $\alpha|_{\text{Ran}(\lambda)}$ is a $\Delta$-isomorphism from $A$ onto $\text{Ran}(\alpha)$. Therefore, by Theorem 4, we obtain that $\alpha$ is regular. Conversely, suppose that $\alpha$ is regular. Then, by Theorem 4, we get that $\text{Ran}(\alpha)$ is a $\Delta$-retract of $X$, and that there is a $\Delta$-retract $A$ of $X$ such that $\alpha|_{\text{Ran}(\lambda)}$ is a $\Delta$-isomorphism from $A$ onto $\text{Ran}(\alpha)$. Since $\text{Ran}(\alpha)$ and $A$ are $\Delta$-retracts of $X$, we have by Corollary 3 that $\text{Ran}(\chi^{(\alpha)})$ and $J_A$ are $\Delta$-retracts of $I$, respectively. Since $\alpha|_{\text{Ran}(\lambda)}$ is a $\Delta$-isomorphism from $A$ onto $\text{Ran}(\alpha)$, we have that (iii) holds. And, by Proposition 2, we get that (ii) holds. □

If $\text{End}(I) = T(I)$, then $\text{End}_\varphi(X)$ becomes $T_\varphi(X)$. Hence, by Theorem 7, the following corollary is immediately obtained.

**Corollary 4.** Let $\alpha \in T_\varphi(X)$. Then, $\alpha$ is regular if and only if there is $A \subseteq X$ such that each of the following statements holds true:

(i) $\chi^{(\alpha)}|_{J_A}$ is a bijective function from $J_A$ onto $\text{Ran}(\chi^{(\alpha)})$;

(ii) $\alpha|_{\text{Ran}(\lambda)}$ is a bijective function from $A \cap Y_i$ onto $\text{Ran}(\alpha) \cap Y_i$ for all $i \in J_A$.

Note that by considering $T_\varphi(X)$ as the semigroup of all continuous maps on $X$, where $X$ is equipped with the topology having the family of all equivalence classes as a base, the regularity of $\alpha \in T_\varphi(X)$ can be deduced from Theorem 4 as well. This was provided by Huisheng [9]. The author obtained for any $\alpha \in T_\varphi(X)$ that $\alpha$ is regular if and only if for each $i \in I$, there is $\lambda \in I$ such that $Y_i \cap \text{Ran}(\alpha) \subseteq \alpha(Y_i)$. Here, we get another characterization of the regularity of elements of $T_\varphi(X)$ in terms of the character.

The following three corollaries are immediately obtained from Theorem 7 as well.

**Corollary 5.** Suppose that $I$ is a topological space, and let $\text{End}(I) = S(I)$. Let $\alpha \in \text{End}_\varphi(X)$. Then, $\alpha$ is regular if and only if there is $A \subseteq X$ such that each of the following statements holds true:

(i) $\chi^{(\alpha)}|_{J_A}$ is a homeomorphism from $J_A$ onto $\text{Ran}(\chi^{(\alpha)})$;

(ii) $\alpha|_{\text{Ran}(\lambda)}$ is a bijective function from $A \cap Y_i$ onto $\text{Ran}(\alpha) \cap Y_i$ for all $i \in J_A$.

**Corollary 6.** Suppose that $I$ is a $T_1$-space, and let $\text{End}(I) = \Gamma(I)$. Let $\alpha \in \text{End}_\varphi(X)$. Then, $\alpha$ is regular if and only if there is $A \subseteq X$ such that each of the following statements holds true:

(i) $\alpha|_{J_A}$ is a closed subset of $I$;

(ii) $\chi^{(\alpha)}|_{J_A}$ is a homeomorphism from $J_A$ onto $\text{Ran}(\chi^{(\alpha)})$;
(iii) $a|_{A \cap Y_i}$ is a bijective function from $A \cap Y_i$ onto $\text{Ran}(\alpha) \cap Y_i$ for all $i \in I_A$.

Corollary 7. Suppose that $I$ is a vector space, and let $\text{End}(I) = L(I)$. Let $\alpha \in \text{End}^o(X)$. Then, $\alpha$ is regular if and only if there is $A \subseteq X$ such that each of the following statements holds true:

(i) $I_A$ is a subspace of $I$;

(ii) $\chi(\alpha)|_{I_A}$ is an isomorphism from $I_A$ onto $\text{Ran}(\chi(\alpha))$;

(iii) $a|_{A \cap Y_i}$ is a bijective function from $A \cap Y_i$ onto $\text{Ran}(\alpha) \cap Y_i$ for all $i \in I_A$.

We end this section with a discussion on the regularity of the quotient semigroup $\text{End}^o(X)/\chi$, where $\chi$ is the congruence relation on $\text{End}^o(X)$ defined by

$$\forall \alpha, \beta \in \text{End}^o(X),$$

$$\alpha \chi \beta \iff \chi(\alpha) = \chi(\beta).$$

(15)

By virtue of Corollary 2, we immediately get that the semigroup $\text{End}^o(X)/\chi$ is regular if and only if the semigroup $\text{End}(I)$ is regular.

3. A Subsemigroup of $\text{End}^o(X)$ and Its Regularity

In this section, we define a sub-semigroup of $\text{End}^o(X)$ and study the regularity of that semigroup. Let $\kappa$ be a cardinal number, and let

$$M_{\kappa}^o(X) = \{ \alpha \in T_{\kappa}(X) : \forall i \in \text{Ran}(\chi(\alpha)), |Y_i \cap \text{Ran}(\alpha)| \leq \kappa \}.$$ (16)

Lemma 2. Let $\sigma, \rho \in M_{\kappa}^o(X)$ and $\gamma \in T_{\kappa}(\sigma, \rho)$. Then, $\gamma \sigma \rho \in M_{\kappa}^o(X)$.

Proof. Let $\gamma = \gamma \sigma \rho$. We now want to show that $\gamma \in M_{\kappa}^o(X)$. It is clear that $\gamma \in T_{\kappa}(X)$. Next, $i \in \text{Ran}(\chi(\gamma))$. Since $\chi(\rho) = \chi(\gamma \sigma \rho)$, we have $\text{Ran}(\chi(\rho)) = \chi(\rho)$, $\text{Ran}(\chi(\gamma) \cap \text{Ran}(\rho)) \subseteq \text{Ran}(\chi(\gamma)) \subseteq \text{Ran}(\chi(\gamma) \cap \text{Ran}(\rho))$. Thus, $i \in \text{Ran}(\chi(\rho))$, and there is $j \in \text{Ran}(\chi(\gamma))$ such that $\chi(\rho)(j) = i$, which yields that $\gamma(Y_i \cap \text{Ran}(\rho)) \subseteq Y_i \cap \text{Ran}(\rho)$. Since $j \in \text{Ran}(\chi(\gamma))$, there is $k \in I$ such that $\chi(\gamma)(k) = j$, which implies that $\sigma(Y_k) \subseteq Y_j$. Thus, $\gamma(Y_i \cap \text{Ran}(\rho)) \subseteq Y_i \cap \text{Ran}(\rho)$. From this, we obtain that $Y_i \cap \text{Ran}(\rho)$. Since $i \in \text{Ran}(\chi(\gamma))$, we have that $Y_i \cap \text{Ran}(\rho) \leq \kappa$. And, since $\text{Ran}(\rho) \subseteq \text{Ran}(\gamma) \cap \text{Ran}(\rho)$, it follows that $0 < |Y_i \cap \text{Ran}(\rho)| \leq |Y_i \cap \text{Ran}(\rho)| \leq \kappa$. Thus, $\gamma \rho \in M_{\kappa}^o(X)$. □

Corollary 8. The set $M_{\kappa}^o(X)$ is a sub-semigroup of $T_{\kappa}(X)$.

Proof. Let $\alpha, \beta \in M_{\kappa}^o(X)$. Then, $\text{al}_{\text{Ran}}(\beta) \in T_{\kappa}(\beta, \alpha)$. Thus, by Lemma 2, we get that $a \beta = \text{al}_{\text{Ran}}(\beta) \in M_{\kappa}^o(X)$. □

Let

$$E_{\kappa}^o(X) = \text{End}^o(X) \cap M_{\kappa}^o(X).$$ (17)

Note that for each $\gamma \in T(I)$, there is $\alpha \in M_{\kappa}^o(X)$ such that $\alpha(\gamma) = \gamma$. Such a function $\alpha$ can easily be defined as follows: for each $j \in \text{Ran}(\gamma)$, let $z_j$ be a fixed element of $Y_j$ and let $\alpha: X \rightarrow X$ be defined by $\alpha(\bigcup_{i \in Y_i} Y_i) = \{ z_j \}$ for all $j \in \text{Ran}(\gamma)$. Thus, by the note, we have that $E_{\kappa}^o(X) \neq \emptyset$. Furthermore, it is a sub-semigroup of $\text{End}^o(X)$.

Theorem 8. Let $\alpha \in E_{\kappa}^o(X)$. Then, the following statements are equivalent:

(1) $\alpha$ is a regular element of $E_{\kappa}^o(X);

(2) $\alpha$ is a regular element of $\text{End}^o(X);

(3) $\alpha(\gamma)$ is a regular element of $\text{End}(I)$. Proof. $(3) \Rightarrow (2)$. Suppose that $\chi(\gamma)$ is a regular element of $\text{End}(I)$. Then, by the regularity of $\chi(\gamma)$ in $\text{End}(I)$, we get by Theorem 4 that $\chi(\gamma)$ is a $\Delta$-retract of $I$, and that there is an idempotent $\gamma$ of $\text{End}(I)$ such that $\chi(\gamma)$ is a $\Delta$-isomorphism from $\text{Ran}(\gamma)$ onto $\text{Ran}(\chi(\gamma))$. Since $\gamma$ is an idempotent element of $\text{End}(I)$, we have by Theorem 6 that there is a partition $\{ I_j : j \in E \}$ of $I$ and a subset $\{ j_i : j_i \in E \}$ of $I$ with $i \in I_j$ for all $j_i \in E$ such that $\gamma(I_j) = \{ j_i \}$ for all $j_i \in E$. So, $\gamma(\text{Ran}(\gamma)) = \{ j_i : j_i \in E \}$. Since $\gamma(\text{Ran}(\gamma))$ is an $\Delta$-isomorphism from $\text{Ran}(\gamma)$ onto $\text{Ran}(\chi(\gamma))$, we have that $\chi(\gamma)$ $\in \text{Hom}(\text{Ran}(\gamma), \chi(\gamma))$ is bijective, and that $\chi(\gamma)**(\text{Ran}(\gamma))^{-1}$ $\in \text{Hom}(\text{Ran}(\gamma), \chi(\gamma))$. So, $\chi(\gamma) = \chi(\gamma)**(\gamma)^{-1}$ $\in \text{Hom}(\text{Ran}(\gamma), \chi(\gamma))$. Therefore, by Theorem 7, we obtain that $\alpha$ is regular.

$(2) \Rightarrow (1)$. Suppose that $\alpha$ is a regular element of $\text{End}(I)$. Then, by Theorem 4 and Proposition 1, there exists an idempotent element $\sigma$ of $\text{End}(X)$ such that $\chi(\sigma) = \chi(\gamma)$ and $\text{Ran}(\sigma) = \text{Ran}(\chi(\gamma))$. Since $\alpha \in E_{\kappa}^o(X)$, it follows that $\sigma \in E_{\kappa}^o(X)$. By Theorem 7, there exists an idempotent element $\rho$ of $\text{End}^o(X)$ such that

(a) $\chi(\sigma)$ $\in \text{Ran}(\chi(\gamma))$ is a $\Delta$-isomorphism from $\text{Ran}(\chi(\gamma))$ onto $\text{Ran}(\chi(\gamma))$;

(b) $\alpha_{\text{Ran}^{\rho}(\gamma)}$ is a bijective function from $\text{Ran}(\rho) \cap Y_i$ onto $\text{Ran}(\alpha) \cap Y_i$ for all $i \in \text{Ran}(\chi(\gamma))$.

From (a) and (b), we have that $\alpha_{\text{Ran}^{\rho}(\gamma)}^{-1} \in \text{Hom}(\text{Ran}(\sigma), \text{Ran}(\rho))$ and $\rho \in E_{\kappa}^o(X)$, respectively. Let $\beta = \alpha_{\text{Ran}^{\rho}(\gamma)}^{-1} \alpha$. Then, by the property (A3), we have that $\beta \in \text{End}^o(X)$ and, by Lemma 2, we immediately obtain that $\beta \in M_{\kappa}^o(X)$. Thus, $\beta \in E_{\kappa}^o(X)$. Finally, we will show that $\beta \alpha = \alpha$. Let $x \in X$. Then, $\alpha(x) \in \text{Ran}(\sigma)$. Hence, there is $z \in X$ such that $\alpha(x) = \sigma(z)$. And, since $\sigma$ is...
idempotent, it follows that \( a \beta a(x) = \sigma(a(x)) = \sigma(\sigma(z)) = \sigma(z) = a(x) \). Therefore, \( a \beta a = a \), and hence \( a \) is a regular element of \( E^I_X \).

\[(1) \Rightarrow (3) \]. It follows directly from Lemma 1.

**Corollary 9.** The semigroup \( E^I_X \) is regular if and only if \( \End(I) \) is regular.

4. Conclusions

The semigroup \( T^{(\sigma)}(X) \), where \( \sigma \) is a sub-semigroup of \( T(I) \), was first defined by Rakbud [12] in 2018 via the notion of the character introduced by Purisang and Rakbud [11] in 2016. Here, we focus on studying the regularity of the semigroup \( T^{(\sigma)}(X) \) when \( \sigma \) is the semigroup of a \( \Delta \)-structure on \( I \), which is written as \( \sigma = \End(I) \). In our study, we obtain that \( T^{(\sigma)}(X) \), which is denoted by \( \End_{\Gamma}(X) \), is the semigroup of a \( \Delta \)-structure on \( X \). From this, the regularity of elements of \( \End_{\Gamma}(X) \) can generally be explained via Theorem 4 established by Magill and Subbiah [13] in 1974. We also obtain a characterization of regular elements of \( \End_{\Gamma}(X) \) in terms of the \( \Delta \)-structure on \( I \) (see Theorem 7). From this result, we deduce the regularity of \( \End_{\Gamma}(X) \) when \( \End(I) \) is one of the following semigroups: the transformation semigroup \( T(I) \), the semigroup \( S(I) \) of continuous maps on \( I \) when \( I \) is a topological space, the semigroup \( \Gamma(I) \) of closed maps on \( I \) when \( I \) is a \( T_1 \)-space, and the semigroup \( L(I) \) of linear transformations on \( I \) when \( I \) is a vector space (see Corollaries 4–7). Apart from the regularity of \( \End_{\Gamma}(X) \), we provide a sub-semigroup of \( \End_{\Gamma}(X) \), namely, the semigroup \( E^{I,1}_{\Gamma}(X) \), whose regularity coincides with that of \( \End(I) \). In [13], Magill and Subbiah also generally gave some characterizations of Green’s relations for regular elements of the semigroup of a \( \Delta \)-structure. Since our semigroup \( \End_{\Gamma}(X) \) is the semigroup of a \( \Delta \)-structure on \( X \), some rough characterizations of Green’s relations for regular elements of \( \End_{\Gamma}(X) \) can immediately be deduced from the results of Magill Jr. and Subbiah.

We end this paper with some interesting questions:

1. Can Green’s relations for regular elements of \( \End_{\Gamma}(X) \) be characterized more deeply in terms of the \( \Delta \)-structure on \( I \)?

2. Can other notions such as the ideal, the rank, the left regularity, and the right regularity in the semigroup \( \End_{\Gamma}(X) \) be explained in terms of those in the semigroup \( \End(I) \)?

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] S. Nenthein, P. Youngpkong, and Y. Kenprasit, “Regular elements of some transformation semigroups,” Puma, vol. 16, no. 3, pp. 307–314, 2005.
[2] R. Chinram and W. Yonthanthum, “Regularity of the semigroups of transformations with a fixed point set,” Thai Journal of Mathematics, vol. 18, no. 3, pp. 1261–1268, 2020.
[3] C. Pookpienlert, P. Honyam, and J. Sanwong, “Regularity of a Semigroup of transformations with restricted range that preserves an equivalence relation and a cross-section,” Thai Journal of Mathematics, vol. 18, no. 2, pp. 819–830, 2020.
[4] K. Sangkhahan and J. Sanwong, “Regularity and Green’s relations on semigroups of transformations with restricted range that preserve an equivalence,” Semigroup Forum, vol. 100, no. 2, pp. 568–584, 2020.
[5] N. Sawatraksa, C. Namnak, and K. Sangkhahan, “Green’s relations and natural partial order on the regular sub-semigroup of transformations preserving an equivalence relation and fixed a cross-section,” Thai Journal of Mathematics, vol. 17, no. 2, pp. 431–444, 2019.
[6] R. Srithus, R. Chinram, and C. Khongthai, “Regularity in the semigroup of transformations preserving a zig-zag order,” Bulletin of the Malaysian Mathematical Sciences Society, vol. 43, no. 2, pp. 1761–1773, 2020.
[7] J. Araújo and J. Konieczny, “Semigroups of transformations preserving an equivalence relation and a cross-section,” Communications in Algebra, vol. 32, no. 5, pp. 1917–1935, 2004.
[8] P. Huisheng, “Equivalences, \( \alpha \)-semigroups and \( \alpha \)-congruences,” Semigroup Forum, vol. 49, no. 1, pp. 49–58, 1994.
[9] P. Huisheng, “Regularity and Green’s relations for \( \alpha \)-semigroups of transformations that preserve an equivalence,” Communications in Algebra, vol. 33, no. 1, pp. 109–118, 2005.
[10] P. Huisheng and Z. Dingyu, “Greens relations on semigroups of transformations preserving order and an equivalence,” Semigroup Forum, vol. 71, no. 2, pp. 241–251, 2005.
[11] P. Purisang and J. Rakbud, “Regularity of transformation semigroups defined by a partition,” Communications of the Korean Mathematical Society, vol. 31, no. 2, pp. 217–227, 2016.
[12] J. Rakbud, “Regularity of a particular subsemigroup of the semigroup of transformations preserving an equivalence,” Kyungpook Mathematical Journal, vol. 58, no. 4, pp. 627–635, 2018.
[13] K. D. Magill and S. Subbiah, “Green’s relations for regular elements of semigroups of endomorphisms,” Canadian Journal of Mathematics, vol. 26, no. 6, pp. 1484–1497, 1974.