A DG/CR discretization for the variational phase-field approach to fracture

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Abstract
Variational phase-field models of fracture are widely used to simulate nucleation and propagation of cracks in brittle materials. They are based on the approximation of the solutions of a free-discontinuity fracture energy by two smooth functions: a displacement and a damage field. Their numerical implementation is typically based on the discretization of both fields by nodal $P^1$ Lagrange finite elements. In this article, we propose a nonconforming approximation by discontinuous elements for the displacement and nonconforming elements, whose gradient is more isotropic, for the damage. The handling of the nonconformity is derived from that of heterogeneous diffusion problems. We illustrate the robustness and versatility of the proposed method through series of examples.

Keywords Phase-field · Fracture · Discontinuous Galerkin

1 Introduction

Variational phase-field models of fracture [1–4], are increasingly popular approaches to the numerical simulation of crack nucleation and propagation in brittle materials. Their essence is the approximation of Francfort and Marigo’s free discontinuity energy [5] by a functional depending on smooth variables: a continuous displacement field and a phase-field representing the cracks.

The strengths of these methods are now well-established, particularly concerning their ability to handle complex crack paths, reproduce crack nucleation in multiple settings [6,7], or extend to complex multi-physics problems [8–11]. As such, the vast majority of numerical implementations are based on functionals derived from that introduced in [12] for the Mumford-Shah image segmentation problem (see also [13]), regularized using Lagrange finite elements [2,14,15].

Their main strength, the representation of the crack geometry in terms of a smooth field allowing to easily handle complex geometries can become a liability in some situations. In the context of hydraulics, for example, fracturing accounting for crack aperture in the fluid flow simulations in the fracture system is particularly challenging [16]. This issue has been typically tackled in an ad-hoc manner [17–19], or through complex geometric reconstructions [20]. A rigorous treatment, based on non-local averages of the phase-field along lines intersecting fractures is available but computationally costly [11].

Combining the strength of phase-field models with an explicit representation of the crack geometry, using discontinuous finite elements, for instance could have significant advantages for complex coupled problems, but this idea has been left relatively unexplored, with the exception of [21] where a symmetric discontinuous Galerkin discretization for the displacement was introduced, while the phase-field variable remained discretized using nodal Lagrange elements and [22] where a fully hybridized discontinuous Galerkin discretization is chosen for both displacement and damage.

In this article, we propose a novel approach using a non-symmetric discontinuous Galerkin discretization for the displacement and a Crouzeix–Raviart discretization for the damage. In this approach, $P^1$ discontinuous elements are...
used to approximate the displacements whose discontinuities are then localized along element faces or edges, so that the cracks, their orientation or normal aperture can easily be obtained. The rationale for using \( P^1 \) Crouzeix–Raviart finite elements to approximate the damage is that the gradient of Crouzeix–Raviart functions is more isotropic than classical \( P^1 \) nodal finite elements, as shown in [23]. The choice of a non-symmetric version over a symmetric one is justified by its stability with respect to the value of the penalty parameter. The robustness of the method regarding crack propagation and crack nucleation is asserted on numerical tests. Because this discretization scheme allows displacement jumps along element faces, instead of along a long strip of elements, the proposed scheme leads to a better approximation of the surface energy, and reduces the need to adjust for “effective toughness” as discussed in [3] (Section 8.1.1).

In Sect. 2, we present the continuous model adopted to model crack propagation under quasi-static loading and then derive the evolution equations for the displacement and crack propagation under quasi-static loading and then toughness” as discussed in [3] (Section 8.1.1).

The robustness of the method regarding crack propagation and crack nucleation is asserted on numerical tests. Because this discretization scheme allows displacement jumps along element faces, instead of along a long strip of elements, the proposed scheme leads to a better approximation of the surface energy, and reduces the need to adjust for “effective toughness” as discussed in [3] (Section 8.1.1).

In Sect. 2, we present the continuous model adopted to model crack propagation under quasi-static loading and then derive the evolution equations for the displacement and the damage. In Sect. 3, the discretization is introduced and proved to have discrete solutions. The convergence towards the continuous model is proved in Appendix A. In Sect. 4, numerical tests show the reliability of the method in the computation of crack propagation and crack nucleation in both one and two dimensions. Finally, a comparison with experimental results obtained in [24] is performed. In Sect. 5, some conclusions are drawn and the potential for future work is explored.

## 2 Phase-field models of quasi-static brittle fracture

Consider a brittle material occupying a bounded domain \( \Omega \), a polyhedron in \( \mathbb{R}^d \) \((d = 2, 3)\) which can be perfectly fitted by simplicial meshes, with Hooke’s law \( \mathcal{C} = \lambda I \otimes I + 2\mu I \), where \( I \) is the unit tensor of order 2, \( I_d \) the unit tensor of order four, and \( \lambda \) and \( \mu \) are the Lamé coefficients. For any \( x \in \Omega \), the displacement is written \( u(x) \in \mathbb{R}^d \), the linearized strain is \( e(u) = \frac{1}{2} \left( \nabla u + \nabla u^T \right) \in \mathbb{R}^{d \times d} \) and the stress is \( \sigma(u) = \mathcal{C}e(u) \in \mathbb{R}^{d \times d} \). Let \( \partial \Omega = \partial \Omega_D \cup \partial \Omega_N \) be a partition of the boundary of \( \Omega \). We assume that \( \partial \Omega_D \) is a closed set and \( \partial \Omega_N \) is open in \( \partial \Omega \).

We consider a set of discrete time steps \( 0 = t_0 < \ldots < t_N = T \) and at each step \( t_i \), a Dirichlet boundary condition \( u_i = w_i \), with \( w_i \in \left( H^{1/2}(\partial \Omega) \right)^d \), is imposed on \( \partial \Omega_D \) while the remaining part of the boundary \( \partial \Omega_N \) remains stress free. Let \( \ell > 0 \) be a small regularization length, \( \alpha \in A := \{ \alpha \in H^1(\Omega); [0, 1] \}; \alpha = 0 \) on \( \partial \Omega_D \) the phase-field variable representing cracks, and

\[
U_i := \left\{ u \in \left( H^1(\Omega) \right)^d ; u = w_i \text{ on } \partial \Omega_D \right\}
\]

be the space of admissible displacement. We also define the associated homogeneous space \( U_0 \), with homogeneous Dirichlet boundary conditions on \( \partial \Omega_D \). To any \((u, \alpha) \in V_i := U_i \times A\), we associate the generalized phase-field energy [2,3,13]

\[
\mathcal{E}_i(u, \alpha) := \frac{1}{2} \int_\Omega (a(\alpha) + \eta_\ell) \mathcal{C}e(u) \cdot e(u) \, dx + \int_\Omega \left( \frac{w(\alpha)}{\ell} + \ell |\nabla \alpha|^2 \right) \, dx,
\]

\[
= \mathcal{E}_i^{(0)}(u, \alpha) + \mathcal{E}_i^{(\ell)}(u, \alpha)
\]

where \( a \) and \( w \) are two non-negative \( C^1 \) monotonic functions such that \( a(0) = w(1) = 1 \) and \( a(1) = w(0) = 0 \). \( \eta_\ell \) is a regularization parameter, \( c_w := \int_0^1 \sqrt{w(s)} \, ds \) is a normalization parameter, and \( G_c \) is the critical elastic energy release rate, a material property. The first term in \( 1 \) approximates the elastic energy of the system while the second represents the fracture energy, i.e. in the limit of \( \ell \to 0 \) the aggregate crack length in two space dimensions and area in three dimensions, scaled by \( G_c \). For the \( AT_1 \) model subsequently used in this article, \( a(\alpha) = (\alpha - 1)^2 \) and \( w(\alpha) = \alpha \). Another classic choice is the \( AT_2 \) model for which \( a(\alpha) = (\alpha - 1)^2 \) and \( w(\alpha) = \alpha^2 \).

Following [3], at each time step \( t_i \) \((0 < i \leq N)\), we seek a pair \((u_i, \alpha_i)\) solution of

\[
(u_i, \alpha_i) = \arg \min_{(v, \beta) \in V_i} \mathcal{E}_i(v, \beta),
\]

with the convention that \( \alpha_0 = 0 \). Note that a simple truncation argument shows that the upper bound \( \alpha \leq 1 \) is naturally satisfied by the global minimizers of \( \mathcal{E}_i \) and does not have to be explicitly enforced. At each step \( t_i, i > 0 \), the Euler-Lagrange first order necessary conditions for optimality consist in searching for \( u_i \in U_i \) such that

\[
\int_\Omega (a(\alpha_i) + \eta_\ell) \mathcal{C}e(u_i) \cdot e(v) \, dx = 0 \quad \text{(3a)}
\]

for any \( v \in U_0 \) and \( \alpha_i \in A, \alpha_{i-1} \leq \alpha_i \) a.e. such that

\[
\int_\Omega \mathcal{C}e(u_i) \cdot e(u_i) \frac{1}{2} a'(\alpha_i)(\beta - \alpha_i) \, dx + \frac{G_c}{c_w} \int_\Omega \left( 2\ell \nabla \alpha_i : \nabla (\beta - \alpha_i) + \frac{w'(\alpha_i)}{\ell} (\beta - \alpha_i) \right) \, dx \geq 0.
\]

(3b) for any \( \beta \in K_i := \{ \beta \in H^1(\Omega); \beta \geq \alpha_i \text{ a.e., } \beta = 0 \text{ on } \partial \Omega_D \}. \)
It is now well-known that as $\ell \to 0$, the phase-field energy $E_\ell, \Gamma$ converges to Francfort and Marigo’s generalized Griffith energy

$$E(u, \Gamma) := \frac{1}{2} \int_{\Omega \setminus \Gamma} \mathcal{C}(u) \cdot \mathbf{e}(u) \, dx + G_c \mathcal{H}^{d-1}(\Gamma \cap \Omega), \quad (4)$$

so that the solutions of (2) converge to unilateral minimizers of $E$ subject to a crack growth hypothesis, and that the set $\{\alpha(x) > 0\}$ converges in some sense to the crack $\Gamma$ in (4) (see [3,25,26], for instance).

### 3 Space discretization

For a given mesh $T$, the diameter of the mesh is defined as $h := \max_c \varepsilon T \rho_c$, where $\rho_c$ is the diameter of the cell $c$. We consider a sequence of matching simplicial meshes $(T_h)_h$ indexed by $h$ with $h \to 0$. The mesh is supposed to be shape regular in the sense of [27], i.e., there exists a parameter $\rho > 0$, independent of $h$, such that, for all $c \in T_h$, $\rho_c c \leq r_c$, where $r_c$ is the radius of the largest ball inscribed in $c$. Let $P^1_d(T_h)$ be the set of broken polynomials of order one and dimension $d$ on the mesh $T_h$, and $P^1(T_h)$ a similar set for scalar polynomials, see [28]. The set of mesh facets of the mesh $T_h$ is written $\mathcal{F}_h$ and is partitioned into $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^b$, where $\mathcal{F}_h^i$ is the set on internal facets and $\mathcal{F}_h^b$ is the set of boundary facets, thus $\forall F \in \mathcal{F}_h^b, F \subset \partial \Omega$. The broken gradient $\nabla h$ is defined for a function $f$ as $\nabla h f = \sum_{c \in T_h} \nabla (f|_c)$, where $f|_c$ is the restriction of a function $f$ to a cell $c$. The discrete strain $e_h$ is then defined as the symmetric part of $\nabla h$. For an inner facet $F \in \mathcal{F}_h^i$, we define $c_{F,+}, c_{F,-} \in T_h$ to be the cells sharing the facet $F$. A normal vector to each inner facet $F$, written $n_F$, is directed from $c_{F,+}$ towards $c_{F,-}$. The average and jump of a function $u_h \in P^1_d(T_h)$ over the inner facet $F \in \mathcal{F}_h^i$ are respectively defined as $[u_h]_F := \frac{1}{2} (u_{c_{F,+}} + u_{c_{F,-}})$ and $[u_h]_F := u_{c_{F,+}} - u_{c_{F,-}}$, where $u_c$ is the restriction of $u_h$ to the cell $c \in T_h$.

#### 3.1 Definition of discrete spaces

The Dirichlet boundary conditions are imposed strongly both for the displacement and the damage. We define $\pi_h$ to be the $L^2$ projection onto $P^1_d(T_h)$ or $P^1(T_h)$, i.e., $\pi_h f_i$ is the solution of

$$\int_{\Omega} (\pi_h f_i - f_i) \cdot v_h = 0, \quad \forall v_h \in P^1_d(T_h).$$

Since $w_i \in (H^{1/2}(\partial \Omega))^d$, there exists $f_i \in (H^1(\Omega))^d$ such that $f_i|_{\partial \Omega_D} = w_i$ on $\partial \Omega_D$. Let $U_{i,h} := \{u_h \in P^1_d(T_h)| u_h = \pi_h f_i \text{ on } \partial \Omega_D\}$ and $A_h := \{\alpha_h \in P^1(T_h)| \forall F \in \mathcal{F}_h, [\alpha_h]_F = 0 \text{ and } \alpha_h = 0 \text{ on } \partial \Omega_D\}.$

As a consequence, the displacement $u_h$ is cellwise $P^1$ and discontinuous across facets and the damage $\alpha_h$, which is $P^1$ Crouzeix–Raviart, is continuous only at the barycentre of the inner facets [29]. Figure 1 sketches the location of the degrees of freedom (dofs) for the elements mentioned above.

Thus the discrete space to look for a solution is $V_{i,h} := U_{i,h} \times A_h$ and the associated test space is $V_0,h := U_{0,h} \times A_h$ where functions in $U_{0,h}$ verify homogeneous Dirichlet boundary conditions on $\partial \Omega_D$. We define the following discrete norms on these spaces to study the discrete problem. The jump semi-norm is defined as

$$|u_h|^2 := \sum_{F \in \mathcal{F}_h^b} \frac{1}{h_F} [[u_h]_F]_F^2.$$

The interior penalty norm is defined as

$$\|u_h\|^2_{ip} := \sum_{c \in T_h} \|e(u_c)\|_{L^2(c)}^2 + |u_h|^2.$$

#### 3.2 Naive discretization of displacements

Let $(u_h, \alpha_h) \in V_{i,h}$ and $v_h \in V_{0,h}$. Naively, one may want to discretize the bilinear form in (3a) as

$$\int_{\Omega} (a(\alpha_h) + \eta_h) \mathcal{C}(u_h) \cdot \nabla h v_h \, dx = 0, \quad \forall v_h \in V_{0,h}. \quad (5)$$

However, unlike functions in $U_{0,h}$, the functions in $U_{0,h}$ can jump between cells so that the solution $u_i$ of (3a) may not satisfy (5). To make sure that it is the case, a consistency term has to be added to the term of (5) to write a discrete formulation. To build this consistency term, one uses an integration by parts on each cell and gets

$$\int_{\Omega} (a(\alpha_h) + \eta_h) \mathcal{C}(u_h) \cdot \nabla h v_h \, dx$$

$$- \sum_{c \in \mathcal{F}_h} \int_{c} \text{div} ((a(\alpha_h) + \eta_h) \mathcal{C}(u_h)) \cdot v_h \, dx$$

$$+ \int_{\partial c} ((a(\alpha_h) + \eta_h) \mathcal{C}(u_h) v_h) \cdot n \, dS,$$

so that

$$\int_{\Omega} (a(\alpha_h) + \eta_h) \mathcal{C}(u_h) \cdot \nabla h v_h \, dx$$
Thus, the variational parameters. Thus, the last term in the right-hand side of (6) becomes

\[ - \sum_{c \in T_h} \int_{c} \text{div} \left( (a(\alpha_h) + \eta_c) \mathbb{C} e(u_h) \right) v_h \, dx \]

\[ + \sum_{F \in \mathcal{F}_h} \int_F \left[ (a(\alpha_h) + \eta_c) \mathbb{C} e(u_h) v_h \right] F \cdot n_F \, dS. \tag{6} \]

Testing for consistency by replacing \( u_h \) with \( u \), on any inner facet \( F \in \mathcal{F}_h \), the last term in the right-hand side of (6) becomes

\[ \int_F \left[ (a(\alpha_h) + \eta_c) \mathbb{C} e(u) v_h \right] F \cdot n_F \, dS \]

\[ = \int_F \left[ \left( (a(\alpha_h) + \eta_c) \mathbb{C} e(u) \right) v_h \right] F \]

\[ + \left( (a(\alpha_h) + \eta_c) \mathbb{C} e(u) \right) v_h F \cdot n_F \, dS. \tag{7} \]

The last term in the right-hand side of (7) is usual in a discontinuous Galerkin framework, see [30], for instance. However, the first term in the right-hand side does not vanish since the Crouzeix–Raviart finite element is not globally continuous on an inner facet \( F \in \mathcal{F}_h \) and \( a \) is not an affine function. Indeed, for an inner facet \( F \in \mathcal{F}_h \), one has \( \int_F [a(\alpha_h) F dS = 0 \) and thus one would have \( \int_F [f(\alpha_h) F dS = 0 \) for an affine function \( f \).

### 3.3 Discretization of displacement evolution

Instead, we acknowledge the existence of jumps of \( a(\alpha_h) \) and will instead consider the material studied as heterogeneous in the sense that \( C(a) := (a(a) + \eta_c) C \). A method to handle heterogeneous materials was introduced in [31] and analysed in [32]. However, it requires to have cellwise constant material parameters. Thus, the \( a(\alpha_h) \) in (5) is projected onto the set \( \mathbb{P}^0(T_h) \) of cellwise constant functions. We define \( \Pi_h \) as the \( L^2 \)-orthogonal projection from \( L^2(\Omega) \) onto \( \mathbb{P}^0(T_h) \) and thus for \( \phi \in L^2(\Omega) \), one has

\[ \Pi_h \phi := \sum_{c \in T_h} \left( \frac{1}{|c|} \int_c \phi dx \right) \chi_c, \]

where \( \chi_c \) is the indicator function of the cell \( c \in T_h \). We thus define \( a_h(\alpha_h) := \Pi_h \alpha(\alpha_h) \), where \( \alpha(\alpha_h) \in L^2(\Omega) \). Note that \( a_h(\alpha_h) \geq 0 \), by definition of \( a \). Following [32], we define the weighted average over an inner facet \( F \in \mathcal{F}_h \) as, for \( u_h \in U_{i,h} \),

\[ \{ \sigma_h(u_h) \cdot n_F \}_{w,F} := \frac{(a_h(\alpha_h) + \eta_c) \sigma_h(u_h) \cdot n_F)_F}{(a_h(\alpha_h) + \eta_c)_F}, \]

where \( \langle a_h \rangle_F := \frac{1}{2} (a_{c,F} + \alpha_{c,F}) \) is the average value of \( a_h \) over an inner facet \( F \in \mathcal{F}_h \). The corresponding consistency term is

\[ - \sum_{F \in \mathcal{F}_h} \int_F \left[ (a_h(\alpha_h) + \eta_c) \sigma_h(u_h) \cdot n_F \right] w_F \cdot [v_h] F dS, \]

where \( u_h \in U_{i,h} \) and \( v_h \in U_{0,h} \). It is used to have the strong consistency of the discrete bilinear form \( U_h \) defined below in Equation (9) in the sense that \( U_h(a_h; u, v_h) = 0 \), for all \( v_h \in U_{0,h} \). Following [31], we define the penalty term as

\[ \sum_{F \in \mathcal{F}_h} \frac{\zeta}{h_F} \gamma_F \int_F [u_h] F \cdot [v_h] F dS, \]

where \( \zeta > 0 \) is a penalty parameter and we define for an inner facet \( F \in \mathcal{F}_h \),

\[ \gamma_F := \frac{(a_h(\alpha_h) + \eta_c)a_h(\alpha_h) \sigma_h(u_h) + \eta_c}{a_h(\alpha_h) + \eta_c} > 0, \tag{8} \]

where \( \gamma_F \) is twice the harmonic average of \( a_h(\alpha_h) + \eta_c \) over \( F \).

Because the penalty coefficient (8) can become locally very small when \( a_F \rightarrow 0 \), we resort to a non-symmetric discontinuous Galerkin formulation [33, 34], which has the main advantage of being stable even with small penalty terms, as proved in Proposition 1 below. Let us then write the discretization of the bilinear form in (3a), for \( u_{i,h} \in U_{i,h} \) and \( v_h \in U_{0,h} \):

\[ U_h(a_h; u_h, v_h) := \int_{\Omega} \left( a_h(\alpha_h) + \eta_c \right) \mathbb{C} e(u_h) \cdot e_h(v_h) \, dx \]

\[ - \sum_{F \in \mathcal{F}_h} \int_F \left( (a_h(\alpha_h) + \eta_c) \sigma_h(u_h) \cdot n_F \right) w_F \cdot [v_h] F dS \]

\[ - \{ (a_h(\alpha_h) + \eta_c) \sigma_h(u_h) \cdot n_F \}_{w,F} \cdot [v_h] F dS + \sum_{F \in \mathcal{F}_h} \frac{\zeta}{h_F} \gamma_F \int_F [u_h] F \cdot [v_h] F dS, \tag{9} \]

As a consequence, the discretized Euler–Lagrange equation consists in searching for \( u_h \in U_{i,h} \) such that

\[ U_h(a_{i-1,h}; u_{i-1,h}, v_h) = 0, \quad \forall v_h \in U_{0,h}. \tag{10} \]

Equation (10) admits a unique solution as proved in Proposition 1.

### 3.4 Discretization of damage evolution

The second Euler–Lagrange equation is actually a variational inequality due to the irreversibility constraint \( a_{i-1,h} \leq a_h \), where \( a_{i-1,h} \) is the value of the damage variable at the previous time step \( t_{i-1} \). The associated bilinear form for any
\( \beta_h \in A_h \) is

\[
A_h (u_h; \alpha_h, \beta_h) = \int_\Omega \nabla u_h \cdot \nabla \alpha_h \beta_h \, dx
+ \frac{2G_c}{c_w} \int_\Omega \nabla \alpha_h \cdot \nabla \beta_h \, dx,
\]

and the linear form is

\[
f (u_h; \beta_h) := \int_\Omega \nabla u_h \cdot \nabla \beta_h \, dx
- \frac{G_c}{c_w} \int_\Omega \beta_h \, dx.
\]

Thus, we introduce the cone \( K_{i,h} := \{ \beta_h \in A_h | \alpha_i-h \leq \beta_h \} \) and the Euler–Lagrange equation becomes: search for \( \alpha_i-h \in K_{i,h} \),

\[
A_h (u_{i,h} ; \alpha_{i,h}, \beta_{i,h} - \alpha_{i,h}) \geq f (u_{i,h} ; \beta_{i,h} - \alpha_{i,h}), \quad \forall \beta_{i,h} \in K_{i,h}.
\]

Equation (11) admits a unique solution as proved in Proposition 2.

### 3.5 Well-posedness and convergence of the scheme

The regularization length \( \ell \) is kept constant in this proof. We only explore the effect of having the mesh size \( h \to 0 \).

**Proposition 1** Equation (10) admits a unique solution.

**Proof** As \( V_{i,h} \) is a finite-dimensional space, uniqueness and existence of a solution \( u_{i,h} \) to (10) are equivalent. Let us prove uniqueness by considering \( u_h \in U_{0,h} \) solution of (10).

We want to prove that \( u_h = 0 \in \Omega \). One has

\[
U_h (\alpha_h ; u_h, u_h) = 0,
\]

and thus

\[
\eta \ell \frac{2 \mu}{\xi} \left\| u_h \right\|^2_{H^1(\Omega)} + \frac{\xi \eta_h^2}{1 + \eta_h} \left| u_h \right|^2 \leq 0.
\]

Therefore, \( \left| u_h \right| \leq F^2 \), for all \( F \in F_{h}^j \), and \( u_h \) is continuous in \( \Omega \). We deduce that \( 0 = e_h (u_h) = e (u_h) \in \Omega \). Thus, \( u_h \) is a constant rigid body motion in \( \Omega \). However, taking into account the homogeneous Dirichlet boundary conditions in \( U_{0,h} \), \( u_h = 0 \) in \( \Omega \).

**Remark 1** (Non-variational discretization) Because the bilinear form that is the second term in the left-hand side of (9) is skew-symmetric, its curl in the second variable \( u_h \) is not zero. As a consequence, this term cannot be the gradient of a potential energy and thus a discrete equivalent to (1) cannot be provided.

**Proposition 2** Equation (11) admits a unique solution.

\[
\text{Proof} \quad \text{Equation (11) is the variational inequality related to the following constrained minimization problem:}
\]

\[
\min_{\alpha_{i-1,h} \leq \alpha_h \leq 1} E_{i,h} (u_h, \alpha_h).
\]

\( A_h \) is finite dimensional thus the existence of a solution to the unconstrained minimization problem is ensured by the fact that \( E_{i,h} \) is continuous in \( \alpha_h \) and proper \( (E_{i,h} (u_h, \alpha_h) \to +\infty \) when \( |\alpha_h| \to +\infty \)). The solution is unique because \( E_{i,h} \) is strictly convex in \( \alpha_h \). The irreversibility constraint is taken into account by adding to \( E_{i,h} \) the following indicator function which is proper convex lower semi-continuous

\[
X_{[0, \alpha_{i-1,h}]} = \begin{cases} 0 & \text{if } \alpha_{i-1,h} \leq \alpha_h \\ +\infty & \text{otherwise} \end{cases}
\]

Thus \( E_{i,h} (u_h, \cdot) + X_{[0, \alpha_{i-1,h}]} \) verifies the necessary hypotheses to ensure the existence and uniqueness of a solution to (11).

The following theorem proves the convergence of the scheme when \( h \to 0 \).

**Theorem 1** (Convergence) There exists a solution \( (u_h, \alpha_h) \in V_{i,h} \) of (10) and (11). Furthermore, the sequence \( (u_h, \alpha_h) \) converges when \( h \to 0 \) towards \( (u_i, \alpha_i) \in U_i \times K_i \) which is a solution of (3a) and (3b).

For concision, the proof is postponed to Appendix A.

### 4 Numerical experiments

In all the following numerical experiments, the penalty parameter \( \xi \) appearing in (9) is chosen as \( \xi := 2 \mu \), where \( \mu \) is the second Lamé coefficient and \( \eta \ell \) is taken to be such that \( \eta \ell = 10^{-6} \). The numerical implementation is based on the C++ and Python library FEniCS [35].

#### 4.1 Surfing boundary conditions

Owing to classical [36] and more modern [37] studies, a single crack propagating in an isotropic homogenous medium typically does so in mode-I. Formally, the jump of the displacement field is expected to be along a direction orthogonal to the tangent of the crack near its tip. Consider a polar coordinate system emanating at the crack tip along its tangent direction. In this setting, the asymptotic behavior of the displacement field is the plane-stress mode-I \( K \)-dominant field \( U_I \) defined by

\[
u_I (r, \theta) := \frac{K_I}{2 \mu} \left( \frac{r}{2\pi} (\kappa - \cos(\theta)) \left( \cos \left( \frac{\theta}{2} \right), \sin \left( \frac{\theta}{2} \right) \right) \right.
\]

\[
(12)
\]
Consider a rectangular domain \((0, W) \times (-H/2, H/2)\) occupied by a brittle material (see Fig. 2). The idea of the surfing boundary condition, introduced in [38] is to prescribe the boundary displacement associated to a translating crack \(\Gamma(t) = (0, Vt) \times \{0\}\) in mode-I:

\[
 w_I(x, y) = U_I(x - Vt, y),
\]

where \(U_I\) denotes the expression of \(u_I\) in cartesian coordinates. The expected solution for this problem consists of a phase-field representation of \(\Gamma(t)\) and associated equilibrium displacement.

In the classical (CG/CG) regularization, it is now well understood that the surface energy (hence the elastic energy release rate) is overestimated. This effect can be explained by the fact that phase-field “cracks” in the AT1 model correspond to a one-element wide strip of element where the damage field \(\alpha\) takes values close to 1 and the displacement jumps sided by transition zones where the damage field decays quadratically back to 0. The energetic contribution of the former zone is of order \(3G_c h^2/8\ell\) while that of the later is close to \(G_c\) (see Section 8.1.1 in [3]). In contrast, in the proposed discretization, the displacement is allowed to jump along edges of internal elements so that the damage field can localize along a set of edges. Subsequently, for a given \(h\) and \(\ell\), our scheme is expected to provide a better approximation of the surface energy.

Of course this could come at a cost since for a given mesh, both \(P^1\)-discontinuous and Crouzeix-Raviart elements lead to a larger number of degrees of freedom compared to standard \(P^1\)-Lagrange elements. In Fig. 3, we performed surfing simulations on a domain of width \(W = 5\) and height \(H = 1\) with an initial crack of length \(l_0 = 0.3\) using the classical and proposed discretization. In order to obtain a comparable number of degrees of freedom for each problem, we set the mesh size to \(h \approx 0.0048\) (leading to 2,069,307 degrees of freedom) for the CG/CG scheme and \(h \approx 0.011\) (2,028,499 degrees of freedom) for the proposed scheme. The non-dimensional material properties are the same as in Fig. 1 of [6], that is \(E = 1, \nu = 0.3\) and \(G_c = 1.5\) in plane stress, and the loading rate is \(V = 4\). In all cases, after a loading stage, a phase-field crack propagates at a constant rate. The normalized crack “velocity” \(\dot{\alpha} \frac{\varepsilon^{(e)}(u, \alpha)}{G_c V}\), computed by linear fitting of the surface energy as a function of a loading is shown in Table 1. We see that even for a comparable number of degrees of freedom and despite a larger element size, our scheme leads to a much better approximation of the surface energy.

Figure 3 shows the normalized elastic energy release rate \(G := -\frac{1}{G_c} \partial P/\partial \ell\), where \(P\) denotes the elastic energy and \(l\) the crack length, computed with the \(G-\theta\) method [39–41], as a function of the loading parameter for the classical and proposed schemes for the same values of the ratio \(\ell/h\). Again, we see that our discretization scheme outperforms the standard CG/CG scheme even for a comparable number of degrees of freedom.
4.2 Three-point bending test

We present numerical simulations of a V-notched three-point bending test, using the specific geometry and loading from [42], shown in Fig. 4.

The vertical displacement \( w_i(y) = (0, -t_i) \) is prescribed on a region of width 4.0mm centered on the upper edge of the domain. The lower-left corner of the sample is clamped and the vertical displacement of the lower-right corner has its vertical displacement blocked, while the rest of the boundary is left stress-free (\( \sigma \cdot n = 0 \)). The boundary value of the damage variable is set to 1 along both sides of the V-notch, following the guidelines of [6]. The material properties are \( \lambda = 12.0\text{kN mm}^{-2} \), \( \mu = 8.0\text{kN mm}^{-2} \) (corresponding to \( E = 45\text{kN mm}^{-2} \) and \( v = 0.25 \) in plane-stress conditions) and \( G_c = 5.4 \times 10^{-4}\text{kN mm}^{-1} \). The mesh is uniform and has a size \( h = 0.016\text{mm} \) and the regularization length is \( \ell = 2h \). The computation uses 2,348,592 dofs. Increments in the Dirichlet boundary conditions are chosen as \( \Delta u_D = 10^{-3}\text{mm} \). Figure 5 shows the crack pattern and the load–displacement curve. The crack path and the load-displacement curve are consistent with the results reported in [42], with a peak load of \( 4.65 \times 10^{-2}\text{kN} \) attained for a displacement load of 4.50mm.

4.3 Mixed-mode Compact-Tension test

Again, we repeat the variant of a compact-tension test initially presented in [42] and repeated in [22]. A Dirichlet boundary condition \( w_i(y) = (0, t_i) \) is imposed in the upper hole on the left of the sample whereas the lower hole on the left of the sample is clamped. The rest of the boundary of the domain is left stress-free \( \sigma \cdot n = 0 \), and again, the value of the damage parameter is prescribed to 1 along the initial crack faces. The geometry and loading are shown in Fig. 6.

The material properties are \( \lambda = 1.94\text{kN mm}^{-2} \), \( \mu = 2.45\text{kN mm}^{-2} \) and \( G_c = 2.28 \times 10^{-3}\text{kN mm}^{-2} \). The mesh is uniform and has a size \( h = 0.39\text{mm} \) and the regularization length is \( \ell = 0.78\text{mm} \). The computation uses 2,348,592 dofs. Increments in the Dirichlet boundary conditions are chosen as \( \Delta u_D = 10^{-3}\text{mm} \). The offset location of the hole leads to a mixed loading mode ahead of the initial crack tip, producing a curved crack path connecting to the hole.

The loading upon which the crack nucleates from the hole and reaches for the free edge is strongly \( \ell \)-dependent, as expected from the analysis of [6]. This effect is studied below and we focus therefore on the crack path between the initial crack and the offset hole. Figure 7(left,center) compares the crack path obtained using the DG/CR scheme and that obtained using a standard CG/CG discretization with the open-source code mef90/vDef [43] with a mesh size \( h = 0.15\text{mm} \) and regularization length \( \ell = 0.6\text{mm} \). Both
paths are essentially identical and also match that of [42]. The load–displacement curves for the DC/CR and CG/CG calculations, shown in Fig. 7(right) are also in strong agreement. Conclusive comparison with a carefully validated code, give credentials to the proposed scheme.

4.4 One-dimensional nucleation test

The link between regularization length and nucleation stress has become an important feature of phase-field models of fracture, when predicting crack nucleation [6]. The main ingredient in establishing this link is the study of a one-dimensional bar under uniaxial tension as studied in [44] (continuous setting) and [45] (discrete setting). In these articles, a purely elastic evolution is observed for small-enough loadings. A critical load upon which the elastic configuration becomes unstable, leading to the nucleation of a single fully-developed crack (a stable critical point of the phase-field energy) can be computed in closed-form. In [45], the authors have developed a method to study the stability of an evolution with a classical CG/CG discretization.

Because of the loss of the variational structure of our discrete problem, it is not clear that the outcome of the stability analysis on the continuous problem provides any insight on the nucleation properties of the proposed model. We performed a series of numerical simulations to establish that it is the case.

The computations are performed under a plane stress assumption. The domain is the rectangle \((0, 1) \times (-0.05, 0.05)\). The material parameters are \(E = 100, \nu = 0\) and \(G_c = 1\). A Dirichlet boundary condition \(w_i = t_i\) is applied on the normal component of the displacement on the left and right edges of the beam. Homogeneous Neumann boundary conditions are enforced on the upper and lower boundaries. The internal length is chosen as \(\frac{\ell}{h} = 5\). The values of \(\ell\) are given in Fig. 9. Let \(t_f\) be the time corresponding to nucleation for a space-continuous evolution, one has

\[
t_f := \frac{L}{E \sqrt{\frac{3G_c}{8E\ell}}},
\]

which corresponds to a stress

\[
\sigma_f := \sqrt{\frac{3G_c}{8E\ell}}.
\]

The simulation is performed over the time-interval \([0, T]\) where \(T = t_f \times 1.01\). For \(t = t_f \times 0.99\), the evolution is purely elastic for all computations. Nucleation is detected at \(t = t_f \times 1.01\) as shown in Fig. 8.

Figure 9 shows the critical time computed compared with the analytical solution.
4.5 Two-dimensional nucleation test

The behaviour of a sample under uniaxial tension has been generalized to the case of a uniformly loaded two-dimensional sample in [46, Appendix A]. The domain $\Omega$ is the unit disk centered at the origin. Dirichlet boundary conditions are imposed on the entire boundary as $w_i(x) = t_i \bar{E} \cdot x$, where $\bar{E} \in \mathbb{R}^{2 \times 2}$ is such that

$$\bar{E} = \frac{1}{E} \begin{pmatrix} \cos(\theta) - \nu \sin(\theta) & 0 \\ 0 & \sin(\theta) - \nu \cos(\theta) \end{pmatrix},$$

where $\theta$ is the angle in polar coordinates in $\Omega$. Therefore, one has

$$\sigma_i = t_i \begin{pmatrix} \cos(\theta) & 0 \\ 0 & \sin(\theta) \end{pmatrix}.$$

Following [46], nucleation occurs at

$$t_f := \frac{3G_c E}{8\ell(1 - \nu \sin(2\theta))}.$$

As in Sect. 4.4, the nucleation time is computed through a stability analysis involving the continuous energy of the system. Thus, a similar analysis cannot be performed any longer with the proposed non-variational method.

The material parameters are $E = 1$, $\nu = 0.3$ and $G_c = 1.5$. The length of the phase-field is chosen as $\frac{\ell}{h} = 5$ and three meshes of sizes $h = 0.04$, $h = 0.02$ and $h = 0.01$ are used for the test. We want to verify that the computed elastic domain corresponds to the theoretical one. Figure 10 shows the principal stresses (the eigenvalues of $\sigma_i$) depending on the sampled values of $\theta$ with respect to the analytical solution given above.

5 Conclusion

In this article, a mixed $\mathbb{P}^1$ discontinuous Galerkin and $\mathbb{P}^1$ Crouzeix–Raviart non-symmetric approximation of phase-field models for brittle fracture has been proposed in Sect. 3. The non-symmetry brings improved stability compared to a symmetric method. However, the methods lose its variational property in the sense that the discrete first order stability conditions are not discrete Euler–Lagrange equations associated to the minimization of a discrete energy. The discretization is proved to converge towards the continuous phase-field model presented in Sect. 2. In Sect. 4, numerical evidence of the capabilities of the method regarding both crack nucleation and crack propagation are given. Additional investigations into the approximation of second order stability conditions for the non-symmetric method could prove valuable.

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Code availability The source code and data files for all examples are available at https://github.com/marazzaf/DG_CR.git.
Therefore,
\[ C_1 \|u_{i,h}\|_{L^p}^2 \leq U_h(\alpha_{i,h}; u_{i,h}, u_{i,h}) = U_h(\alpha_{i,h}; u_{i,h}, f_1) \leq C_2 \|u_{i,h}\|_{L^p}, \]

where \( C_1 \) and \( C_2 \) are non-negative constants.

Thus \( \|u_{i,h}\|_{L^p} \) is bounded from above. We can apply Kolmogorov compactness criterion [28, p. 194]. Thus, there exists \( v_i \in U_i \) such that, up to a subsequence, \( u_{i,h} \rightharpoonup v_i \) strongly in \( (L^2(\Omega))^d \) and \( \nabla_h u_{i,h} \rightharpoonup \nabla v_i \) weakly in \( (L^2(\Omega))^d \times \Omega^d \).

Now let us get a bound on the damage. Testing (11) with \( \alpha_{i-1,h} \), one has
\[ A_h(u_{i,h}; \alpha_{i,h}, \alpha_{i-1,h}) \leq f(u_{i,h}; \alpha_{i,h} - \alpha_{i-1,h}). \]

Thus, using a Cauchy–Schwarz inequality and the fact that \( \alpha_h \leq 1 \) and \( \alpha_{i-1,h} \leq 1 \), one has
\[ \frac{2Ge}{c_w} \int_{\Omega} \|\nabla_h \alpha_{i,h}\|^2 dx \leq A_h(u_{i,h}; \alpha_{i,h}, \alpha_{i,h}) \leq A_h(u_{i,h}; \alpha_{i,h}, \alpha_{i-1,h}) + f(u_{i,h}; \alpha_{i,h} - \alpha_{i-1,h}) \leq 2 \int_{\Omega} \nabla_h \alpha_{i,h} \cdot e_h(u_{i,h}) dx + \frac{2Ge}{c_w} \|\nabla_h \alpha_{i-1,h}\|_{L^2(\Omega)} \|\nabla_h \alpha_{i-1,h}\|_{L^2(\Omega)} \leq C \|u_{i,h}\|_{L^p} + C' \|\nabla_h \alpha_{i,h}\|_{L^2(\Omega)} \]

where \( C > 0 \) and \( C' > 0 \) are generic non-negative constants.

The second order polynomial in the variable \( \|\nabla_h \alpha_{i,h}\|_{L^2(\Omega)} \) is negative between its two real roots and thus \( \|\nabla_h \alpha_{i,h}\|_{L^2(\Omega)} \) is bounded from above. Using the compactness of the Crouzeix–Raviart FE [47, p. 297], there exists \( \beta_h \in A \) such that, up to a subsequence, \( \alpha_{i,h} \rightharpoonup \beta_i \) strongly in \( L^2(\Omega) \) and \( \nabla_h \alpha_{i,h} \rightharpoonup \nabla \beta_i \) weakly in \( (L^2(\Omega))^d \).

**Proposition 3** (Existence of solution to the discretized problem) There exists \((u_{i,h}, \alpha_{i,h}) \in V_{i,h}\) solving (10) and (11) simultaneously.

**Proof** Let \( T : (v_i, \beta_i) \mapsto (u_{i,h}, \alpha_{i,h}) \), where \( u_{i,h} \) is the solution of \( U_h(\beta_h; u_{i,h}, \bullet, 0) = 0 \) over \( U_{i,h} \) and \( \alpha_{i,h} \) is the solution of \( A_h(v_i; \alpha_{i,h}, \bullet - \alpha_{i,h}) \geq f(v_i; \bullet - \alpha_{i,h}) \) over \( K_{i,h} \). Assuming \( v_i \) and \( \beta_h \) verify the bounds proved in the proof of Lemma 1, then \( u_{i,h} \) and \( \alpha_{i,h} \) verify these same bounds. Thus \( T \) is a mapping of a nonempty compact convex subset of \( V_{i,h} \) into itself. As, \( U_h(\beta_h) \) and \( A_h(v_i) \) are continuous bilinear forms, \( T \) is a continuous map. As a consequence, using Brouwer fixed point theorem [48, p. 179], there exists a fixed point \((u_{i,h}, \alpha_{i,h}) \) solving (10) and (11) simultaneously.

**Lemma 2** \((v_i, \beta_i) \) is a solution of (3a).

**Proof** Let \( \varphi \in (C^\infty_c(\Omega))^d \) be a function with compact support in \( \Omega \). Testing (10) with \( \pi_h \varphi \), one has
\[ \int_{\Omega} (a_h(\alpha_{i,h}) + \eta_i) \nabla_h u_{i,h} \cdot e_h(\pi_h \varphi) dx \]
\[ - \sum_{F \in F_h} \int_{F} n \cdot ((a_h(\alpha_{i,h}) + \eta_i) \sigma_h(u_{i,h})) F \cdot [\pi_h \varphi]_F \]
\[ - ((a_h(\alpha_{i,h}) + \eta_i) \sigma_h(\pi_h \varphi)) F \cdot [u_{i,h}]_F dS \]
\[ + \sum_{F \in F_h} \xi F h \int_{F} [u_{i,h}]_F \cdot [\pi_h \varphi]_F dS = 0. \]

The last two terms in the left-hand side vanish when \( h \to 0 \) because \( \varphi, v_i \in (H^1(\Omega))^d \). Regarding the first term in the left-hand side, one has
\[ \int_{\Omega} (a_h(\alpha_{i,h}) + \eta_i) \nabla_h u_{i,h} \cdot e_h(\pi_h \varphi) dx \]
\[ = \int_{\Omega} (a(\beta_i) + \eta_i) \nabla_h u_{i,h} \cdot e_h(\pi_h \varphi) dx \]
\[ + \int_{\Omega} (a_h(\alpha_{i,h}) - \Pi_h a(\beta_i)) \nabla_h u_{i,h} \cdot e_h(\pi_h \varphi) dx \]
\[ + \int_{\Omega} (\Pi_h a(\beta_i) - a(\beta_i)) \nabla_h u_{i,h} \cdot e_h(\pi_h \varphi) dx \]
\[ = (I) + (II) + (III) \]

Passing to the limit in (I), one obtains the expected term
\[ \int_{\Omega} (a(\beta_i) + \eta_i) \nabla u_{i,h} \cdot e(\varphi) dx. \]

Let us now prove that (II) and (III) vanish as \( h \to 0 \). Using a Cauchy–Schwarz inequality, one has
\[ (II) \leq \left( \int_{\Omega} (\nabla_h u_{i,h} \cdot e_h(\pi_h \varphi))^2 dx \right)^{1/2} \]
\[ \|\Pi_h a(\alpha_{i,h}) - \Pi_h a(\beta_i)\|_{L^2(\Omega)} \]
\[ \leq C \|\varphi\|_{W^{1,\infty}(\Omega)} \|u_{i,h}\|_{L^p} \|\Pi_h a(\alpha_{i,h}) - \Pi_h a(\beta_i)\|_{L^2(\Omega)}. \]

We focus on the second term in the right-hand side.
\[ \|\Pi_h a(\alpha_{i,h}) - \Pi_h a(\beta_i)\|_{L^2(\Omega)} \leq \|a(\alpha_{i,h}) - a(\beta_i)\|_{L^2(\Omega)}, \]

since \( \Pi_h \) is a projection in \( L^2(\Omega) \). Using the strong convergence \( \alpha_{i,h} \rightharpoonup \beta_i \) in \( L^2(\Omega) \) and the fact that \( a \) is continuous gives the desired result. Regarding (III), using a Cauchy–Schwarz inequality, one has
\[ (III) \leq \left( \int_{\Omega} (\nabla_h u_{i,h} \cdot e_h(\pi_h \varphi))^2 dx \right)^{1/2} \|a(\beta_i)\|_{L^2(\Omega)}, \]
\[ - \Pi_h a(\beta_i) \|_{L^2(\Omega)} \leq C \|\varphi\|_{W^{1,\infty}(\Omega)} \|u_{i,h}\|_{L^p} \|a(\beta_i)\|_{L^2(\Omega)} \]
\[ \leq C \|\varphi\|_{W^{1,\infty}(\Omega)} \|u_{i,h}\|_{L^p} \|a(\beta_i) - \Pi_h a(\beta_i)\|_{L^2(\Omega)}. \]
where $C > 0$ is a generic non-negative constant. Using a classical local approximation result (see [29, Proposition 1.135] for instance), one has:

$$||a(\beta_i) - \Omega_h a(\beta_i)||_{L^2(\Omega)} \leq C h ||\nabla(a(\beta_i))||_{L^2(\Omega)},$$

where $\nabla(a(\beta_i)) = a'(\beta_i)\nabla\beta_i \in (L^2(\Omega))^d$ because $\beta_i \in L^\infty(\Omega) \cap H^1(\Omega)$ and $a$ is $C^1$ and thus (III) vanishes as $h \to 0$.

**Lemma 3** $e_h(u_{i,h}) \to e(v_i)$ strongly in $(L^2(\Omega))^{d \times d}$, where $v_i$ is a solution of (3a).

**Proof** We consider again $f_i \in (H^1(\Omega))^d$ such that $f_i = w_i$ on $\partial \Omega_D$. We are going to test (10) with $\tilde{v}_h = u_{i,h} - \pi_h f_i$ so that $\tilde{v}_h \in U_{0,h}$. One thus has

$$\int_\Omega (a_h(\alpha_{i,h}) + \eta \ell) C e_h(u_{i,h}) \cdot \nabla \tilde{v}_h d\chi = \sum_{F \in F_h^e} n \cdot \left((a_h(\alpha_{i,h}) + \eta \ell) \sigma_h(u_{i,h})\right)_F \cdot [\tilde{v}_h]_F - \{a_h(\alpha_{i,h}) + \eta \ell \sigma_h(\tilde{v}_h)\}_F \cdot [u_{i,h}]_F dS - \sum_{F \in F_h^e} \xi g_F \int_\Omega [u_{i,h}]_F \cdot [\tilde{v}_h]_F dS.$$

Using the strong convergence in $u_{i,h}$ and $\alpha_{i,h}$ in the right-hand side gives 0 but it also ensures that the quadratic term in $e_h(u_{i,h})$ in the left-hand side has a limit when $h \to 0$. Thus $e_h(u_{i,h}) \to e(v_i)$ strongly in $(L^2(\Omega))^{d \times d}$, when $h \to 0$. \qed

**Lemma 4** $(v_i, \beta_i)$ is a solution of (3b).

**Proof** Let $\varphi \in C^\infty(\Omega), \varphi \geq 0$. We are going to test (11) with $\beta_i = \pi_h + \alpha_{i,h}$. One thus has

$$A_h(u_{i,h}; \alpha_{i,h}, \pi_h \varphi) \geq f(u_{i,h}; \pi_h \varphi_h).$$

Owing to the weak convergence of $\nabla h \alpha_{i,h}$, one has

$$\int_\Omega e(v_i) : \mathbb{C} : e(v_i)(\beta_i - 1) \varphi d\chi + \frac{G_c}{c_w} \int_\Omega (2\ell \nabla \beta_i \cdot \nabla \varphi + \frac{\varphi}{\ell}) d\chi \geq 0. \quad (14)$$

Therefore, for any $\beta \in K_i$,

$$\int_\Omega e(v_i) : \mathbb{C} : e(v_i)(\beta_i - 1)(\beta - \beta_i) d\chi + \frac{G_c}{c_w} \int_\Omega (2\ell \nabla \beta_i \cdot (\beta - \beta_i) + \frac{\beta - \beta_i}{\ell}) d\chi \geq 0. \quad (15)$$

**Lemma 5** $\nabla h \alpha_{i,h} \to \nabla \beta_i$ strongly in $(L^2(\Omega))^d$, where $\beta_i$ is the solution of (3b).

**Proof** Because of the weak convergence of $\nabla h \alpha_{i,h}$ towards $\nabla \beta_i$, one has

$$||\nabla \beta_i||_{L^2(\Omega)} \leq \liminf_{h \to 0} ||\nabla h \alpha_{i,h}||_{L^2(\Omega)}.$$

Let us test (11) with $\beta_i = \pi_h \beta_i$. One has

$$f(u_{i,h}; \beta_i - \alpha_{i,h}) + A_h(u_{i,h}; \alpha_{i,h}, \pi_h \beta_i) \leq A_h(u_{i,h}; \alpha_{i,h}, \beta_i).$$

Because of the weak convergence of $\nabla h \alpha_{i,h}$,

$$0 + \limsup_{h \to 0} A_h(u_{i,h}; \alpha_{i,h}, \alpha_{i,h}) \leq \int_\Omega C e(v_i) \cdot e(v_i) \beta_i^2 d\chi + \frac{2G_c \ell}{c_w} ||\nabla \beta_i||^2_{L^2(\Omega)},$$

Due to the strong convergence of $\alpha_{i,h}$ towards $\beta_i$, one finally gets

$$\limsup_{h \to 0} \frac{2G_c \ell}{c_w} ||\nabla h \alpha_{i,h}||^2_{L^2(\Omega)} \leq \frac{2G_c \ell}{c_w} ||\nabla \beta_i||^2_{L^2(\Omega)}.$$ (19)

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