ON SIGNED ARC TOTAL DOMINATION IN DIGRAPHS

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Abstract. Let $D = (V, A)$ be a finite simple digraph and $N(uv) = \{u'v' \neq uv \mid u = u'$ or $v = v'\}$ be the open neighbourhood of $uv$ in $D$. A function $f : A \to \{-1, +1\}$ is said to be a signed arc total dominating function (SATDF) of $D$ if $\sum_{e' \in N(uv)} f(e') \geq 1$ holds for every arc $uv \in A$. The signed arc total domination number $\gamma'_st(D)$ is defined as $\gamma'_st(D) = \min\{\sum_{e \in A} f(e) \mid f$ is an SATDF of $D\}$. In this paper we initiate the study of the signed arc total domination in digraphs and present some lower bounds for this parameter.

Keywords: signed arc total dominating function, signed arc total domination number, domination in digraphs.

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1. INTRODUCTION

In this paper we continue the study of signed dominating functions in graphs and digraphs. Let $G$ be a simple graph with edge set $E(G)$ and let $N(e) = N_G(e)$ be the open neighborhood of the edge $e$. A signed edge total dominating function (SETDF) on a graph $G$ is defined in [6] as a function $f : E(G) \to \{-1, 1\}$ such that $\sum_{e' \in N_G(e)} f(e') \geq 1$ for every $e \in E(G)$. The weight of an SETDF $f$ on a graph $G$ is $\omega(f) = \sum_{e \in E(G)} f(e)$. The signed edge total domination number $\gamma'_st(G)$ of $G$ is the minimum weight of an SETDF on $G$. This concept has been studied by several authors (see, for example, [1, 5, 7]).

Let $D$ be a finite simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. A digraph without directed cycles of length 2 is an oriented graph. The order $n = n(D)$ and the size $m = m(D)$ of a digraph $D$ is the number of its vertices and arcs, respectively. We write $d^+_D(v)$ for the out-degree of a vertex $v$ and $d^-_D(v)$ for its in-degree. The minimum and maximum in-degrees and minimum and maximum out-degrees of $D$ are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If $uv$ is an arc of $D$, then we also write $u \rightarrow v$, and we say that $v$ is an out-neighbor of $u$ and $u$ is an in-neighbor of $v$. For each vertex $v \in V$, let $N^-_D(v)$ be the
in-neighbor set which consists of all vertices of $D$ from which arcs go into $v$ and $N_D^+(v)$ be the out-neighbor set which consists of all vertices of $D$ into which arcs go from $v$. The degree of a vertex $u$ in $D$ is defined by $d_D(u) = d_D^+(u) + d_D^-(u)$ and the minimum degree of $D$ is $\delta(D) = \min\{d_D(u) \mid u \in V\}$. If $d_D(v) = 1$, then we call $v$ a pendant vertex in $D$. If $X \subseteq V$, then $D[X]$ is the subdigraph induced by $X$. For every $uv \in A$, we define $d_D(uv) = d_D^+(u) + d_D^-(v) - 2$ to be the degree of the arc $uv$ in $D$. The minimum and maximum arc degrees of $D$ are denoted by $\delta' = \delta'(D)$ and $\Delta' = \Delta'(D)$, respectively. An arc of $D$ is said to be a pendant arc if it is incident with a pendant vertex in $D$. For $uv \in A$, define $N_D(uv) = N(uv) = \{u'v' \neq uv \mid u = u' \text{ or } v = v'\}$ as the open neighborhood of $uv$. An orientation of a graph $G$ is a digraph obtained from $G$ by replacing every edge of $G$ with a directed edge.

For a real-valued function $f : A(D) \rightarrow \mathbb{R}$, the weight of $f$ is $\omega(f) = \sum_{e \in A(D)} f(e)$, and for $S \subseteq A(D)$, we define $f(S) = \sum_{e \in A(D)} f(e)$, so $\omega(f) = f(A(D))$. Consult [4] for the notation and terminology which are not defined here.

Recently, Meng [2] defined a signed edge dominating function (SEDF) on a digraph $D$ as a function $f : A \rightarrow \{-1, 1\}$ such that $\sum_{e \in N[e]} f(e') \geq 1$ for every $e \in A$, where $N[e] = N(e) \cup \{e\}$. The signed edge domination number $\gamma_s'(D)$ of $D$ is the minimum weight of a signed edge dominating function on $D$. Following the ideas in [2] and [6], we initiate the study of signed arc total dominating functions in digraphs.

A function $f : A \rightarrow \{-1, +1\}$ is called a signed arc total dominating function (SATDF) on a digraph $D$, if $f(N(uv)) \geq 1$ for each arc $uv \in A$. The minimum of the values of $\omega(f) = f(A)$, taken over all SATDF $f$ of $D$, is called the signed arc total domination number of $D$ and denoted by $\gamma_{st}'(D)$. A $\gamma_{st}'(D)$-function is an SATDF on $D$ of weight $\gamma_{st}'(D)$. Obviously, $\gamma_{st}'(D)$ is defined only for digraphs $D$ with $\delta' \geq 1$. In this note we initiate the study of the signed arc total domination in digraphs and present some (sharp) bounds for this parameter.

A nonempty digraph $D$ with an SATDF $f$ on $D$, denoted by $(D, f)$, is called a signed arc total digraph. Let $(D, f)$ be a signed arc total digraph and let $u$ be an arbitrary vertex in $D$, then define

$A_+(u^+, f) = \{uv \in A \mid f(uv) = 1\}$, \hspace{1cm} $A^-(u^+, f) = \{uv \in A \mid f(uv) = -1\}$,

$A^+(u^-, f) = \{vu \in A \mid f(vu) = 1\}$, \hspace{1cm} $A^-(u^-, f) = \{vu \in A \mid f(vu) = -1\}$,

$A_-(f) = \{e \in A \mid f(e) = -1\}$, \hspace{1cm} $f(u^+) = |A^+(u^+, f)| - |A^-(u^+, f)|$,

$A_+(f) = \{e \in A \mid f(e) = 1\}$, \hspace{1cm} $f(u^-) = |A^+(u^-, f)| - |A^-(u^-, f)|$.

We make use of the following observations in this paper.

**Observation 1.1.** If $f$ is an SATDF on a digraph $D$ of size $m$, then

(a) $\omega(f) = |A_+(f)| - |A_-(f)|$,
(b) $m = |A_+(f)| + |A_-(f)|$,
(c) $\gamma_{st}'(D) \equiv m \pmod{2}$.

**Observation 1.2.** Let $e$ be an arc with degree at most 2 in $D$. If $f$ is an SATDF on $D$, then $f$ assigns 1 to each arc of $N(e)$.
For every arc $e \in A$, define

$$A_{\text{odd}} = \{ e \in A \mid d_{D}(e) \text{ is odd} \} \quad \text{and} \quad A_{\text{even}} = \{ e \in A \mid d_{D}(e) \text{ is even} \}.$$ 

Denote $m_{o} = |A_{\text{odd}}|$ and $m_{e} = |A_{\text{even}}|$.

**Observation 1.3.** Let $f$ be a signed arc total dominating function on $D$ and $e \in A$. If $e \in A_{\text{odd}}$, then $\sum_{e' \in N(e)} f(e') \geq 1$ and $\sum_{e' \in N(e)} f(e') \geq 2$, when $e \in A_{\text{even}}$.

A directed graph is called connected if replacing all of its arcs with undirected edges produces a connected (undirected) graph.

**Observation 1.4.** If $D_{1}, D_{2}, \ldots, D_{s}$ be the components of $D$, then

$$\gamma'_{st}(D) = \sum_{i=1}^{s} \gamma'_{st}(D_{i}). \quad (1.1)$$

**Theorem 1.5.** Let $D$ be a digraph of size $m$. Then $\gamma'_{st}(D) = m$ if and only if for each arc $e \in A(D)$ there is an arc $e' \in N(e)$ such that $d_{D}(e') \leq 2$.

**Proof.** One side is clear by Observation 1.2. Let $\gamma'_{st}(D) = m$. Assume, to the contrary, there exists an arc $e = uv \in A(D)$ such that for every $e' \in N(e)$, $d_{D}(e') \geq 3$. It is easy to verify that the function $f : A(D) \to \{-1, 1\}$ that assigns $-1$ to $uv$ and $+1$ to the remaining arcs, is an SATDF of $D$ of weight $m - 2$, and so $\gamma'_{st}(D) \leq m - 2$, a contradiction. This completes the proof. \qed

**Remark 1.6.** We remark that the signed edge total domination and signed arc total domination are not comparable. If $D_{1}$ is an orientation of $K_{1,4}$ such that $d_{D_{1}}^{+}(w) = d_{D_{1}}^{-}(w)$, where $w$ is the central vertex of $K_{1,4}$, then $\gamma'_{st}(D_{1}) = 4 > \gamma'_{st}(K_{1,4}) = 2$. If $D_{2}$ is an orientation of $K_{2,2}$ such that $\delta' \geq 1$, then $\gamma'_{st}(D_{2}) = \gamma'_{st}(K_{2,2}) = 4$. Let $U = \{u_{1}, u_{2}, u_{3}\}$ and $V = \{v_{1}, v_{2}, v_{3}\}$ be the partite sets of $K_{3,3}$ and let $D_{3}$ be an orientation of $K_{3,3}$ such that

$$A(D_{3}) = \{u_{i}v_{j}, u_{j}u_{2}, u_{3}v_{i}, u_{2}v_{1} \mid 1 \leq i \leq 3, \ 2 \leq j \leq 3\}.$$ 

Define $f$ on $A(D_{3})$ by $f(u_{1}v_{2}) = f(u_{3}v_{2}) = -1$ and $f(x) = 1$ otherwise. Obviously, $f$ is an SATDF on $D_{3}$ with weight 5. Thus $\gamma'_{st}(D_{3}) < \gamma'_{st}(K_{3,3}) = 7$.

2. **BOUNDS ON THE SIGNED ARC TOTAL DOMINATION NUMBER**

In this section, we present some lower bounds for the signed arc total domination number of a digraph $D$.

**Theorem 2.1.** For any digraph $D$ of size $m \geq 2$ and $\delta' \geq 1$,

$$\gamma'_{st}(D) \geq \max\{\delta' + 3 - m, \Delta' + 1 - m\}.$$ 

Furthermore, this bound is sharp.
Proof. Let $f$ be an SATDF on $D$ and let $uv \in A$. Then $f$ assigns 1 to at least $\lceil \frac{\delta' + 1}{2} \rceil$ arcs in $N(uv)$. Let $u'v' \in N(uv)$ such that $f(u'v') = 1$. Also $f$ assigns 1 to at least $\lceil \frac{\delta' + 1}{2} \rceil$ arcs in $N(u'v')$. Therefore

$$|A_-(f)| \leq m - \frac{\delta' + 1}{2} - 1,$$

which implies that

$$\gamma_{st}'(D) = |A_+(f)| - |A_-(f)| \geq \frac{\delta' + 1}{2} + 1 - \left( m - \frac{\delta' + 1}{2} - 1 \right) = \delta' + 3 - m,$$

as desired. Now let $uv \in A(D)$ be an arc with maximum arc degree in $D$, then

$$\frac{m + \gamma_{st}'(D)}{2} \geq |A_+(f)| \geq |A_+(f) \cap N(uv)| \geq \frac{\Delta' + 1}{2},$$

and this leads to $\gamma_{st}'(D) \geq \Delta' + 1 - m$. If $D$ is an orientation of $K_{1,2}$ with central vertex $v$ such that $d_D^+(v) = 2$, then obviously $\gamma_{st}'(D) = 2 = \delta' + 3 - m$. \qed

**Theorem 2.2.** Let $D$ be a digraph with order $n$ and size $m \geq 2$ with $\delta' \geq 1$. Then

$$\gamma_{st}'(D) \geq \frac{m - (\Delta^+ - \delta^+)(n - \delta^-)(\Delta^- - 1) - (\Delta^- - \delta^-)(n - \delta^+)(\Delta^+ - 1)}{\Delta^+ + \Delta^- - 2}.$$

**Proof.** Let $f$ be a $\gamma_{st}'(D)$-function. We have

$$\gamma_{st}'(D) = |A_+(f)| - |A_-(f)| = \sum_{u \in V} |A^+(u^+, f)| - \sum_{u \in V} |A^-(u^+, f)| = \sum_{u \in V} f(u^+). \quad (2.1)$$

Similarly, we have

$$\gamma_{st}'(D) = |A_+(f)| - |A_-(f)| = \sum_{u \in V} |A^+(u^-, f)| - \sum_{u \in V} |A^-(u^-, f)| = \sum_{u \in V} f(u^-). \quad (2.2)$$

For an arbitrary $uv \in A$, $f(N(uv)) = f(u^+) + f(v^-) - 2f(uv) \geq 1$. Therefore,

$$m + 2\gamma_{st}'(D) \leq \sum_{uv \in A} (f(u^+) + f(v^-) - 2f(uv)) + 2 \sum_{uv \in A} f(uv)$$

$$= \sum_{uv \in A} (f(u^+) + f(v^-)) = \sum_{u \in V} f(u^+)d_D^+(u) + \sum_{v \in V} f(v^-)d_D^-(v).$$

Let

$$B_+^\perp = \{ u \in V \mid f(u^+) \geq 1 \}, \quad B_0^\perp = \{ u \in V \mid f(u^+) = 0 \}, \quad B_-^\perp = \{ u \in V \mid f(u^+) \leq -1 \},$$

$$B_+^- = \{ u \in V \mid f(u^-) \geq 1 \}, \quad B_0^- = \{ u \in V \mid f(u^-) = 0 \}, \quad B_-^- = \{ u \in V \mid f(u^-) \leq -1 \}. $$
Then by (2.1)–(2.3), we have
\[
m + 2\gamma'_{st}(D) \leq \sum_{u \in V} f(u^+)d^+_D(u) + \sum_{v \in V} f(v^-)d^-_D(v)
\]
\[
= \sum_{u \in B^+_+} f(u^+)d^+_D(u) + \sum_{u \in B^-_+} f(u^+)d^-_D(u)
\]
\[
+ \sum_{v \in B^-_-} f(v^-)d^-_D(v) + \sum_{v \in B^-_-} f(v^-)d^-_D(v)
\]
\[
\leq \Delta^+ \sum_{u \in B^+_+} f(u^+) + \delta^+ \sum_{u \in B^-_+} f(u^+)
\]
\[
+ \Delta^- \sum_{v \in B^-_-} f(v^-) + \delta^- \sum_{v \in B^-_-} f(v^-)
\]
\[
= \Delta^+ \sum_{u \in V} f(u^+) + (\delta^+ - \Delta^+) \sum_{u \in B^+_+} f(u^+)
\]
\[
+ \Delta^- \sum_{v \in V} f(v^-) + (\delta^- - \Delta^-) \sum_{v \in B^-_-} f(v^-)
\]
\[
= \Delta^+ \gamma'_{st}(D) + (\delta^+ - \Delta^+) \sum_{u \in B^+_+} f(u^+)
\]
\[
+ \Delta^- \gamma'_{st}(D) + (\delta^- - \Delta^-) \sum_{v \in B^-_-} f(v^-).
\]

Hence
\[
(\Delta^+ + \Delta^- - 2)\gamma'_{st}(D) \geq m + (\Delta^+ - \delta^+) \sum_{u \in B^+_+} f(u^+) + (\Delta^- - \delta^-) \sum_{v \in B^-_-} f(v^-).
\]

For each \(u \in B^+_+\) and \(v \in N^+(u)\), we have \(v \in B^+_- \cup B^-_0\). Since
\[
f(u^+) + f(v^-) - 2f(uv) \geq 1,
\]
it follows that
\[
\delta^+ \leq |N^+(u)| \leq |B^+_-| + |B^-_0| = n - |B^-_-|.
\]

Therefore
\[
|B^-_-| \leq n - \delta^+.
\]

(2.4)

Similarly, for each \(v \in B^-_-\) and \(u \in N^-(v)\), we have \(u \in B^+_+ \cup B^-_0\), which implies
\[
|B^+_+| \leq n - \delta^-.
\]

(2.5)
On the other hand, for each $u \in B^+$, there must be a vertex $v \in N^+(u)$ such that $f(uv) = -1$. Using this and the fact that $f(u^+) + f(v^-) - 2f(uv) \geq 1$, we get $f(u^+) + f(v^-) \geq -1$.

Since $f(v^-) \leq \Delta^+ - 2$, we have

$$f(u^+) \geq 1 - \Delta^-.$$  \hfill (2.6)

Similarly, for each $v \in B^-$, we have

$$f(v^-) \geq 1 - \Delta^+.$$  \hfill (2.7)

Applying (2.3)–(2.7), we obtain

$$(\Delta^+ + \Delta^- - 2)\gamma'_{st}(D) \geq m - (\Delta^+ - \delta^+)(n - \delta^-)(\Delta^- - 1)
- (\Delta^- - \delta^-)(n - \delta^+)(\Delta^+ - 1)$$

as desired.

A digraph $D$ is regular if $\Delta^+ = \delta^+ = \Delta^- = \delta^-$. As an application of Proposition 2.2, we obtain a lower bound on the signed arc total domination number for $r$-regular digraphs.

**Corollary 2.3.** If $D$ is an $r$-regular digraph of size $m$ with $r \geq 2$, then

$$\gamma'_{st}(D) \geq \left\lceil \frac{m}{2r - 2} \right\rceil.$$  

**Theorem 2.4.** For any digraph $D$ of order $n$ and size $m$,

$$\gamma'_{st}(D) \geq 2 \left\lceil \frac{m^2}{n(\Delta^+ + \Delta^- - 2)} - \frac{m_o}{2(\Delta^+ + \Delta^- - 2)} \right\rceil - m.$$

**Proof.** Let $f$ be a $\gamma'_{st}(D)$-function and let $e = uv$ be an arc in $D$. If $e$ is an arc of odd degree, then

$$|N(e) \cap A_+(f)| \geq \frac{1}{2}(d_D^+(u) + d_D^-(v) - 1)$$

and if $e$ is an arc of even degree, then

$$|N(e) \cap A_+(f)| \geq \frac{1}{2}(d_D^+(u) + d_D^-(v)).$$

Thus

$$\sum_{e \in A} |N(e) \cap A_+(f)| \geq \frac{1}{2} \sum_{uv \in A} (d_D^+(u) + d_D^-(v)) - \frac{1}{2}m_o$$

$$= \frac{1}{2} \left( \sum_{u \in V} (d_D^+(u))^2 + \sum_{v \in V} (d_D^-(v))^2 \right) - \frac{1}{2}m_o$$

$$\geq \frac{1}{2n} \left[ \left( \sum_{u \in V} d_D^+(u) \right)^2 + \left( \sum_{v \in V} d_D^-(v) \right)^2 \right] - \frac{1}{2}m_o = \frac{m^2}{n} - \frac{m_o}{2}.$$
On the other hand,

\[
(\Delta^+ + \Delta^- - 2)|A_+(f)| \geq \sum_{e \in A_+(f)} |N(e)|
\]

\[
= \sum_{e \in A_+(f)} (|N(e) \cap A_+(f)| + |N(e) \cap A_-(f)|)
\]

\[
= \sum_{e \in A_+(f)} |N(e) \cap A_+(f)| + \sum_{e \in A_+(f)} |N(e) \cap A_-(f)|
\]

\[
= \sum_{e \in A_+(f)} |N(e) \cap A_+(f)| + \sum_{e \in A_-(f)} |N(e) \cap A_+(f)|
\]

\[
= \sum_{e \in A_+(f)} |N(e) \cap A_+(f)| \geq \frac{m^2}{n} - \frac{m_o}{2}.
\]

Since \(\gamma'_st(D) = 2|A^+(f)| - m\), we get

\[
\gamma'_st(D) \geq 2 \left[ \frac{m^2}{n(\Delta^+ + \Delta^- - 2)} - \frac{m_o}{2(\Delta^+ + \Delta^- - 2)} \right] - m. \tag{2.10}
\]

**Theorem 2.5.** Let \(D\) be a digraph of size \(m\). Then

\[
\gamma'_st(D) \geq (2 + \delta' - \Delta')m + 2m_e.
\]

**Proof.** Let \(f\) be a \(\gamma'_st(D)\)-function and \(\sum_{e \in A} d_D(e) = L\). By Observation 1.3, we have

\[
\sum_{e \in A} \sum_{e' \in N(e)} f(e') = \sum_{e \in A_{even}} \sum_{e' \in N(e)} f(e') + \sum_{e \in A_{odd}} \sum_{e' \in N(e)} f(e') \geq 2|A_{even}| + |A_{odd}| = m_e + m. \tag{2.8}
\]

On the other hand,

\[
\sum_{e \in A} \sum_{e' \in N(e)} f(e') = \sum_{e \in A} d_D(e) f(e) = \sum_{e \in A_+} d_D(e) f(e) + \sum_{e \in A_-} d_D(e) f(e)
\]

\[
= \sum_{e \in A_+} d_D(e) - \sum_{e \in A_-} d_D(e) = 2 \sum_{e \in A_+} d_D(e) - \sum_{e \in A} d_D(e)
\]

\[
\leq 2\Delta'|A_+(f)| - L. \tag{2.9}
\]

Similarly, we have

\[
\sum_{e \in A} \sum_{e' \in N(e)} f(e') = \sum_{e \in A} d_D(e) f(e) = \sum_{e \in A_+} d_D(e) f(e) + \sum_{e \in A_-} d_D(e) f(e)
\]

\[
= \sum_{e \in A_+} d_D(e) - \sum_{e \in A_-} d_D(e) = \sum_{e \in A} d_D(e) - 2 \sum_{e \in A_-} d_D(e)
\]

\[
\leq \sum_{e \in A} d_D(e) - 2|A_-(f)|\delta' = L - 2(m - |A_+(f)|)\delta'. \tag{2.10}
\]
By (2.8)–(2.10), we deduce the following inequalities:

\[ m + m_e + L \leq 2\Delta' |A_+(f)| \quad \text{and} \quad m + 2m\delta' + m_e - L \leq 2\delta' |A_+(f)|. \quad (2.11) \]

Summing the inequalities in (2.11), we have

\[ |A_+(f)| \geq \frac{(1 + \delta')m + m_e}{\delta' + \Delta'}, \]

and hence

\[ \gamma_{st}'(D) = 2|A_+(f)| - m \geq \frac{(2 + \delta' - \Delta')m + 2m_e}{\delta' + \Delta'}. \]

\[ \square \]

**Theorem 2.6.** Let \( D \) be a digraph of size \( m \) with the arc degree sequence \( d'_1 \geq d'_2 \geq \ldots \geq d'_m \). Then

\[ \gamma_{st}'(D) = 2|A_+(f)| - m \geq 2 \left[ \frac{m + m_e + L - 2L_t + 2td'_{t+1}}{2d'_{t+1}} \right] - m, \]

where \( t = \max\{ \left\lfloor \frac{m(1+2\delta') - L + m_e}{2\delta'} \right\rfloor, \left\lfloor \frac{m + L + m_e}{2\Delta'} \right\rfloor \} \), \( L_t = \sum_{i=1}^{t} d'_i \) and \( L = \sum_{e \in A} d_D(e) \).

**Proof.** Let \( f \) be a \( \gamma_{st}'(D) \)-function on \( D \). From (2.11), we have

\[ |A_+(f)| \geq \frac{m + L + m_e}{2\Delta'}, \quad |A_+(f)| \geq \frac{m(1 + 2\delta') - L + m_e}{2\delta'}. \]

So

\[ |A_+(f)| \geq t = \max \left\{ \left\lfloor \frac{m(1+2\delta') - L + m_e}{2\delta'} \right\rfloor, \left\lfloor \frac{m + L + m_e}{2\Delta'} \right\rfloor \right\}. \]

It follows from inequality (2.8) and the inequality chain (2.9) that

\[ m + m_e \leq 2 \sum_{e \in A_{+}(f)} d_D(e) - \sum_{e \in A} d_D(e) \]

\[ \leq 2 \left( \sum_{i=1}^{t} d'_i + (|A_+(f)| - t)d'_{t+1} \right) - L \]

\[ = 2(L_t + (|A_+(f)| - t)d'_{t+1}) - L. \]

Therefore

\[ |A_+(f)| \geq \left\lfloor \frac{m + m_e + L - 2L_t + 2td'_{t+1}}{2d'_{t+1}} \right\rfloor \]

and hence

\[ \gamma_{st}'(D) = 2|A_+(f)| - m \geq 2 \left[ \frac{m + m_e + L - 2L_t + 2td'_{t+1}}{2d'_{t+1}} \right] - m. \]

\[ \square \]
Theorem 2.7. For every simple connected digraph $D$ with $2 \leq \delta' \leq \Delta' \leq 6$, $\gamma'_{st}(D) \geq 6$.

Proof. Let $f$ be a $\gamma'_{st}(D)$-function. Since $2 \leq \delta' \leq \Delta' \leq 6$, we have $|N_D(e) \cap A_+(f)| \geq 2$ and $|N_D(e) \cap A_-(f)| \leq 2$. Now it is clear that

$$2|A_-(f)| \leq \sum_{e \in A_-(f)} |N_D(e) \cap A_+(f)| = \sum_{e \in A_+(f)} |N_D(e) \cap A_-(f)| \leq 2|A_+(f)|.$$ 

Thus $|A_-(f)| \leq |A_+(f)|$ and hence, $\gamma'_{st}(D) = |A_+(f)| - |A_-(f)| \geq 0$. \hfill \Box

3. SIGNED ARC TOTAL DOMINATION IN ORIENTED GRAPHS

Let $G$ be the complete bipartite graph $K_{2,3}$ with bipartite sets $V = \{v_1, v_2\}$ and $U = \{u_1, u_2, u_3\}$. Let $D_1$ be an orientation of $G$ such that all arcs go from $V$ into $U$ and let $D_2$ be an orientation of $G$ such that $A(D_2) = \{(v_1, u_j), (u_j, v_2) \mid j = 1, 2, 3\}$. It is easy to see that $\gamma'_{st}(D_1) = 2$ and $\gamma'_{st}(D_2) = 6$. Therefore, two distinct orientations of a graph can have different signed total arc domination numbers. Motivated by this observation, we define lower orientable signed total arc domination number $\text{dom}'_{st}(G)$ and upper orientable signed total arc domination number $\text{Dom}'_{st}(G)$ of a graph $G$ as follows:

$$\text{dom}'_{st}(G) = \min \{ \gamma'_{st}(D) \mid D \text{ is an orientation of } G \text{ with } \delta' \geq 1 \},$$

and

$$\text{Dom}'_{st}(G) = \max \{ \gamma'_{st}(D) \mid D \text{ is an orientation of } G \text{ with } \delta' \geq 1 \}.$$

An immediate consequence of Proposition 1.5 now follows.

Corollary 3.1. For $n \geq 3$, $\text{dom}'_{st}(P_n) = n - 1$, $\text{dom}'_{st}(C_n) = n$.

Proposition 3.2. If $G = K_{1,m}$ is a star, then $\text{dom}'_{st}(K_{1,m}) = \begin{cases} 3 & m \text{ is odd,} \\ 2 & m \text{ is even.} \end{cases}$

Proof. Consider the graph $K_{1,m}$ with bipartite sets $\{v_1\}$ and $\{u_1, u_2, \ldots, u_m\}$. Let $D$ be an orientation of $K_{1,m}$ and let $f$ be a $\gamma'_{st}(D)$-function. If $d_D^+(v_1) = 0$ or $d_D^-(v_1) = 0$, then $|A_-(f)| = (m - 2)/2$ if $m$ is even and $|A_-(f)| = (m - 3)/2$ if $m$ is odd. Hence, $\gamma'_{st}(D) = 2$ if $m$ is even and $\gamma'_{st}(D) = 3$ if $m$ is odd. Suppose that $d_D^+(v_1)$ and $d_D^-(v_1) \geq 1$. If either $d_D^+(v_1) = 1$ or $d_D^-(v_1) = 1$, then there is an arc $e = v_1u_i$ with $d_D(e) = 0$, a contradiction. So $d_D^+(v_1)$ and $d_D^-(v_1) \geq 2$. Let, without loss of generality, that $u_1 \in N^+(v_1)$ and $u_2 \in N^-(v_1)$. If $m$ is odd, then either $f(N(v_1u_1)) \geq 2$ or $f(N(u_2v_1)) \geq 2$. Thus $\gamma'_{st}(D) \geq 3$. If $m$ is even, since $f(N(v_1u_1)) \geq 1$ and $f(N(u_2v_1)) \geq 1$, it follows that $\gamma'_{st}(D) \geq 2$. This completes the proof. \hfill \Box

Lemma 3.3. For $m \geq 2$, $\gamma'_{st}(K_{2,m}) = \begin{cases} 4 & \text{if } m \text{ is even,} \\ 2 & \text{if } m \text{ is odd.} \end{cases}$
Proof. Let $X = \{u_1, u_2\}$ and $Y = \{v_1, v_2, \ldots, v_m\}$ be the partite sets of $K_{2,m}$ and let $f$ be a $\gamma'_{st}(K_{2,m})$-function. We consider two cases.

Case 1. $m$ is odd.

Since

\[ f(N(u_1v_1)) = f(u_2v_1) + \sum_{i=2}^{m} f(u_1v_i) \geq 1 \]

and

\[ f(N(u_2v_1)) = f(u_1v_1) + \sum_{i=2}^{m} f(u_2v_i) \geq 1, \]

we have

\[ \omega(f) = \sum_{i=1}^{m} f(u_1v_i) + \sum_{i=1}^{m} f(u_2v_i) = f(N(u_1v_1)) + f(N(u_2v_1)) \geq 2. \]

Define $g : E(K_{2,m}) \to\{−1,1\}$ by $g(u_1v_1) = g(u_2v_1) = 1$ and $g(u_1v_i) = g(u_2v_i) = (−1)^i$ for $2 \leq i \leq m$. Obviously, $g$ is an SETDF of $K_{2,m}$ of weight 2 and so $\gamma'_{st}(K_{2,m}) \leq 2$. Therefore $\gamma'_{st}(K_{2,m}) = 2$.

Case 2. $m$ is even.

Define $g : E(K_{2,m}) \to\{−1,1\}$ by $g(u_1v_1) = g(u_2v_2) = 1$ for $i = 1, 2$ and $g(u_1v_i) = g(u_2v_i) = (−1)^i$ for $3 \leq i \leq m$. Obviously $g$ is an SETDF of $K_{2,m}$ of weight 4 and hence $\gamma'_{st}(K_{2,m}) \leq 4$. Now we show that $\gamma'_{st}(K_{2,m}) = 4$. Since $m$ is even, $f(N(u_1v_1)) \geq 2$ and $f(N(u_2v_1)) \geq 2$. Hence,

\[ \omega(f) = f(N(u_1v_1)) + f(N(u_2v_1)) \geq 4. \]

Therefore $\gamma'_{st}(K_{2,m}) = 4$. 

\[ \square \]

Proposition 3.4. For $m \geq 2$, $\text{dom}'_{st}(K_{2,m}) = \begin{cases} 
2 & \text{if } m \text{ is odd,} \\
4 & \text{if } m \text{ is even.} 
\end{cases} $

Proof. Let $U = \{u_1, u_2\}$ and $V = \{v_1, v_2, \ldots, v_m\}$ be the partite sets of $K_{2,m}$, $D$ be an orientation on $K_{2,m}$ and $f$ be a $\gamma'_{st}(D)$-function. If $d_D^+(v_i) = 2$ (or $d_D^-(v_i) = 2$) for each $1 \leq i \leq m$, then we are done by Lemma 3.3. Without loss of generality, suppose that $d_D^+(u_1) \geq d_D^-(u_1)$. We distinguish two cases.

Case 1. $d_D^+(v_i) = d_D^-(v_i) = 1$, for some $i$, say $i = 1$.

Without loss of generality, suppose that $u_1v_1, v_1u_2 \in A(D)$. Since $f(N(u_1v_1)) \geq 1$, there is at least one arc $e' \in N(u_1v_1)$ such that $f(e') = 1$. Similarly, there is an arc $e'' \in N(v_1u_2)$ such that $f(e'') = 1$. Since

\[ |N(e') \cap \{u_2v_i \mid u_2v_i \in A(D)\}| \leq 1 \]

and

\[ |N(e'') \cap \{v_iu_1 \mid v_iu_1 \in A(D)\}| \leq 1, \]
we have
\[ \gamma'_st(D) \geq \sum_{u_2v_i \in A(D)} f(u_2v_i) + f(e') + f(N(e')) + f(e'') + f(N(e'')) \]
\[ + \sum_{v_iu_1 \in A(D)} f(v_iu_1) - 2 \]
\[ \geq 4 - 2 = 2. \]

Hence, if \( m \) is odd, then the statement is true. Assume that \( m \) is even. If either \( |N(e')| \) and \( |N(e'')| \) are even or
\[ |N(e') \cap \{u_2v_i \mid u_2v_i \in A(D)\}| = |N(e'') \cap \{v_iu_1 \mid v_iu_1 \in A(D)\}| = 0, \]
then by an argument similar to that described above we get \( \gamma'_st(D) \geq 4 \). We consider two subcases.

Subcase 1.1. \( |N(e')| \) is odd and \( |N(e') \cap \{u_2v_i \mid u_2v_i \in A(D)\}| = 1 \) (the case \( |N(e'')| \) is odd and \( |N(e') \cap \{v_iu_1 \mid v_iu_1 \in A(D)\}| = 1 \) is similar).

Then \( |N(u_1v_1)| \) is even. Let
\[ \{x\} = N(e') \cap \{u_2v_i \mid u_2v_i \in A(D)\}. \]

If \( f(x) = -1 \), then \( \sum_{u_1v_i \in A(D)} f(u_1v_i) \geq 3 \) and \( \sum_{u_2v_i \in A(D)} f(u_2v_i) \geq -1 \) and if \( f(x) = 1 \), then \( \sum_{u_1v_i \in A(D)} f(u_1v_i) \geq 1 \) and \( \sum_{u_2v_i \in A(D)} f(u_2v_i) \geq 1 \). Consequently, \( \sum_{u_1v_i \in A(D)} f(u_1v_i) + \sum_{u_2v_i \in A(D)} f(u_2v_i) \geq 2. \) Moreover, since \( f(N(e'')) \geq 1 \), we have \( \sum_{v_iu_2 \in A(D)} f(v_iu_2) \geq 1 \). If there is an arc \( y = v_iu_1 \) (note that since \( m \) and \( |N(u_1v_1)| \) are even, there is at least one arc \( v_iu_1 \) in \( A(D) \)) such that \( f(y) = 1 \), then \( \sum_{v_iu_1 \in A(D)} f(v_iu_1) \geq 1 \). Therefore
\[ \gamma'_st(D) = \sum_{u_1v_i \in A(D)} f(u_1v_i) + \sum_{u_2v_i \in A(D)} f(u_2v_i) \]
\[ + \sum_{v_iu_1 \in A(D)} f(v_iu_1) + \sum_{v_iu_2 \in A(D)} f(v_iu_2) \]
\[ \geq 4. \]

Suppose that \( f(v_iu_1) = -1 \) for each \( v_iu_1 \in A(D) \). Then \( d_D^-\{u_1\} = 1 \). Without loss of generality, suppose that \( \{v_m\} = N_D^-\{u_1\} \). Since \( \sum_{e \in N(v_mu_1)} f(e) \geq 1 \), we have \( f(v_mu_2) = 1 \) and since \( f(N(v_mu_2)) \geq 1 \), we have \( \sum_{v_iu_2 \in A(D)} f(v_iu_2) \geq 3. \) Therefore,
\[ \gamma'_st(D) = \sum_{u_1v_i \in A(D)} f(u_1v_i) + \sum_{u_2v_i \in A(D)} f(u_2v_i) + f(v_mu_1) + \sum_{v_iu_2 \in A(D)} f(v_iu_2) \geq 4. \]

Subcase 1.2. \( |N(e')| \) is odd and \( |N(e'') \cap \{v_iu_1 \mid v_iu_1 \in A(D)\}| = 1 \) (the case \( |N(e'')| \) is odd and \( |N(e') \cap \{u_2v_i \mid u_2v_i \in A(D)\}| = 1 \) is similar).
Let \( \{z\} = N(e'') \cap \{v_iu_1 \mid v_iu_1 \in A(D)\} \). If \( f(z) = -1 \), then \( \sum_{v_iu_2 \in A(D)} f(v_iu_2) \geq 3 \) and \( \sum_{v_iu_1 \in A(D)} f(v_iu_1) \geq -1 \) and if \( f(z) = 1 \), then \( \sum_{v_iu_2 \in A(D)} f(v_iu_2) \geq 1 \) and \( \sum_{v_iu_1 \in A(D)} f(v_iu_1) \geq 1 \). Hence, \( \sum_{v_iu_1 \in A(D)} f(v_iu_1) + \sum_{v_iu_2 \in A(D)} f(v_iu_2) \geq 2 \). If \( d_D^+(u_2) = 0 \), since \( f(N(e')) \geq 1 \), then \( \sum_{u_1v_i \in A(D)} f(u_1v_i) \geq 2 \) and if there is an arc \( y = u_2v_1 \) such that \( f(y) = 1 \), then \( \sum_{u_2v_i \in A(D)} f(u_2v_i) \geq 1 \). Therefore
\[
\gamma'_{st}(D) = \sum_{u_1v_i \in A(D)} f(u_1v_i) + \sum_{u_2v_i \in A(D)} f(u_2v_i) + \sum_{v_iu_1 \in A(D)} f(v_iu_1) + \sum_{v_iu_2 \in A(D)} f(v_iu_2) \geq 4.
\]

Suppose that \( f(u_2v_i) = -1 \) for each \( u_2v_i \in A(D) \). Then \( d_D^+(u_2) = 1 \). Without loss of generality, suppose that \( \{v_m\} = N_D^+(u_2) \). Then \( f(u_1v_m) = 1 \) and \( \sum_{u_1v_i \in A(D)} f(u_1v_i) \geq 3 \). Therefore,
\[
\gamma'_{st}(D) = \sum_{u_1v_i \in A(D)} f(u_1v_i) + f(u_2v_m) + \sum_{v_iu_1 \in A(D)} f(v_iu_1) + \sum_{v_iu_2 \in A(D)} f(v_iu_2) \geq 4.
\]

**Case 2.** \( d_D^+(v_i) = 2 \) and \( d_D^-(v_j) = 2 \), for some \( i, j \).

Without loss of generality, suppose that \( d_D^+(v_i) = 2 \) for \( 1 \leq i \leq t \) and \( d_D^-(v_j) = 2 \) for \( t + 1 \leq j \leq m \). Then by Lemma 3.3,
\[
\gamma'_{st}(D) = \gamma'_{st}(K_{2,t}) + \gamma'_{st}(K_{2,m-t}) \geq 2 + 2 = 4.
\]

This completes the proof. \( \square \)

**Theorem 3.5.** For any integer \( t \), there is a graph \( G \) with dom\(_{st}'(G) = -t \).

**Proof.** For a given positive integer \( r \geq 4 \), let \( T \) be a graph that obtained from a star \( K_1,r \) by subdividing all of its edges once and let \( G \) be the graph obtained from \( t+1 \) copies of \( T \) with central vertices \( v_1, v_2, \ldots, v_{t+1} \) by adding the edges \( v_1v_2, v_2v_3, \ldots, v_{t+1} \) (see Figure 1).

![Fig. 1. A digraph with \( \gamma_{st}'(D) = -4 \)](image)

Let \( \{v_j, v_{i,j}, u_{i,j} \mid 1 \leq i \leq t \} \) be the vertex set of \( j \)th copy of \( T \), where \( N(v_{i,j}) = \{v_j, u_{i,j}\} \) and \( u_{i,j} \) are leaves for each \( i \). Let \( D \) be an arbitrary orientation of \( G \) and let \( f \) be a \( \gamma_{st}'(D) \)-function. Clearly, either \( d_D^+(v_{i,j}) = 2 \) or \( d_D^-(v_{i,j}) = 2 \) for each \( i,j \) because \( \delta' \geq 1 \). In both cases, \( f \) assigns +1 to each non-pendant arc of each copy of \( T \).
Since the least possible weight for \( f \) will be achieved if \( f(e) = -1 \) for each other arcs, we have \( \omega(f) \geq (t+1)r - (t+1)r - t = -t \). In order to show that \( \text{dom}_s^t(G) \leq -t \), let \( D \) be an orientation of \( G \) such that

\[
A(D) = \{(v_j, v_{j+1}), (v_j, v_{i,j}), (u_{i,j}, v_{i,j}) : 1 \leq i \leq r, 1 \leq j \leq t\},
\]
as illustrated in Figure 1 for \( t = 4 \). Define \( f : A(D) \to \{-1, 1\} \) by \( f(v_j, v_{i,j}) = +1 \) and \( f(v_j, v_{j+1}) = f(u_{i,j}, v_{i,j}) = -1 \) for \( 1 \leq i \leq r \) and \( 1 \leq j \leq t \). Obviously, \( f \) is an SATDF on \( D \) of weight \(-t\). Therefore, \( \text{dom}_s^t(G) = -t \).

**Theorem 3.6.** If \( T \) is a tree of order \( n \geq 3 \), then

\[
\text{dom}_s^t(T) \geq \frac{7-n}{3}.
\]

Furthermore, this bound is sharp.

**Proof.** The proof is by induction on \( n \). The statement holds for all trees of order \( n = 3, 4, 5 \). Assume \( T \) is a tree of order \( n \geq 6 \) and that the statement holds for all trees with smaller orders. Let \( D \) be an arbitrary orientation of \( T \) with \( \delta' \geq 1 \) and let \( f \) be a \( \gamma'_s(D) \)-function. We consider two cases.

**Case 1.** There is a non-pendant arc, say \( e = uv \in A(D) \), for which \( f(e) = -1 \). Let \( D_1 \) and \( D_2 \) be the components of \( D - e \) with \( u \in D_1 \) and \( v \in D_2 \). Obviously, the order of \( D_1 \) and \( D_2 \) are greater than 3 and \( \gamma'_s(D) = f(A(D_1)) - 1 + f(A(D_2)) \). For \( i = 1, 2 \), the function \( f \), restricted to \( D_i \), is an SATDF of \( D_i \), and so \( \gamma'_s(D_i) \leq f(A(D_i)) \). By the inductive hypothesis,

\[
\gamma'_s(D_i) \geq \frac{7 - |A(D_i)|}{3}.
\]

Thus

\[
\gamma'_s(D) \geq -1 + \frac{7 - |A(D_1)|}{3} + \frac{7 - |A(D_2)|}{3} = \frac{11 - n}{3} > \frac{7 - n}{3}.
\]

**Case 2.** The only arcs \( e \) for which \( f(e) = -1 \) are pendant arcs. Then \( f(v^+) \geq 0 \) for each \( v \in V(D) \) with \( d^+_D(v) \geq 2 \) and \( f(v^-) \geq 0 \) for each \( v \in V(D) \) with \( d^-_D(v) \geq 2 \). Let

\[
P^+_D = \{v \in V(D) \mid d^+_D(v) \geq 2 \text{ and } f(v^+) = 0\} \text{ and } P^-_D = \{v \in V(D) \mid d^-_D(v) \geq 2 \text{ and } f(v^-) = 0\}.
\]

First, let \( P^+_D = P^-_D = \emptyset \). Then \( f \) is an SEDF of \( D \). Hence, \( \gamma'_s(D) \geq |V(D)| - |A(D)| \) (see [2]). Since \( n \geq 6 \) and \( |V(D)| = |A(D)| + 1 \), it follows that

\[
\gamma'_s(D) = f(A(D)) \geq \gamma'_s(D) \geq 1 > \frac{7-n}{3}.
\]

Without loss of generality, suppose that \( P^+_D \neq \emptyset \). Let \( P^+_D = \{u_1, u_2, \ldots, u_k\} \). Obviously, there is no +1 pendant arc out from \( u_i \) for each \( i \). Let

\[
M^+_D(u_i) = \{u \in N^+_D(u_i) \mid d^-_D(u) \geq 2\}.
\]
Let first $|M_+^D(v_i)| \geq 2$ for some $i$. Without loss of generality we may assume $|M_+^D(u_1)| \geq 2$ and $v_1, v_2 \in M_+^D(u_1)$. Let $D_1$ and $D_2$ be the connected components of $D - u_1v_1$ for which $v_1 \in V(D_1)$. Let $D_1'$ be obtained from $D_1$ by adding a new pendant arc $w_1v_1$ and let $D_2'$ be obtained from $D_2$ by deleting one of the $-1$ pendant arcs out from $u_1$. Now define $g : A(D_1') \rightarrow \{-1, +1\}$ by $g(w_1v_1) = +1$ and $g(e) = f(e)$ if $e \in A(D_1)$. Obviously, $g$ is an SATDF of $D_1'$ and $f|_{D_2'}$ is an SATDF of $D_2'$. By the inductive hypothesis,

$$
\gamma'_{st}(D_1') \geq \frac{7 - |A(D_1')|}{3}.
$$

Thus

$$
\gamma'_{st}(D) = f(A(D)) = g(A(D_1')) + f|_{D_2'}(A(D_2')) - 1
\geq -1 + \frac{7 - |A(D_1')|}{3} + \frac{7 - |A(D_2')|}{3} > \frac{7 - n}{3}.
$$

Now let $M_+^D(u_i) = \{v_i\}$ for each $1 \leq i \leq k$. Since $f(N(u_i, v_i)) \geq 1$, we have $f(v_i^-) \geq 3$ for each $i$. Let $D'$ be obtained from $D$ by deleting all pendant vertices and the vertices of $P_D^\pm$. We distinguish three subcases.

**Subcase 2.1.** $d_D^+(v_1) \geq 1$, $e = vv_1 \in A(D')$ and $f(v^+) = 1$ in $D$.

By the construction of $D'$ we have $d_D^+(v) \geq 3$. Since $f(v^+) = 1$ and all arcs in $D'$ are $+1$ arcs, there exists a pendant arc $e'$ out from $v$ in $D$, say $e' = vz$. Let $D_1$ and $D_2$ be the connected components of $D - e$ containing $v_1$ and $v$, respectively. Let $D_1'$ be obtained from $D_1$ by adding a new pendant arc $v'v_1$ at $v_1$ and $D_2' = D_2 - z$. It is easy to see that the order of $D_1'$ and $D_2'$ are greater than 3. Define $g : A(D_1') \rightarrow \{-1, +1\}$ by $g(v'v_1) = 1$ and $g(e) = f(e)$ if $e \in A(D_1)$. Obviously, $g$ and $f|_{D_2'}$ are SATDFs of $D_1'$ and $D_2'$, respectively. By the inductive hypothesis,

$$
\gamma'_{st}(D_1') \geq \frac{7 - |V(D_1')|}{3}.
$$

Thus

$$
\gamma'_{st}(D) = f(A(D)) = g(A(D_1')) + f|_{D_2'}(A(D_2')) - 1
\geq -1 + \frac{7 - |V(D_1')|}{3} + \frac{7 - |V(D_2')|}{3} > \frac{7 - n}{3}.
$$

**Subcase 2.2.** $d_D^-(v_1) \geq 1$, $e = vv_1 \in A(D')$ and $f(v^+) \geq 2$ in $D$.

Let $D_1$ and $D_2$ be the connected components of $D - e$. Let $D_1'$ and $D_2'$ be obtained from $D_1$ and $D_2$ by adding new pendant arcs $v'v_1$ and $vv''$, respectively. Define $g_1 : A(D_1') \rightarrow \{-1, +1\}$ by $g_1(v'v_1) = 1$ and $g(e) = f(e)$ if $e \in A(D_1)$, and $g_2 : A(D_2') \rightarrow \{-1, +1\}$ by $g_2(vv'') = 1$ and $g(e) = f(e)$ if $e \in A(D_2)$. Obviously, $g_i$ is an SATDF of $D_i'$ for $i = 1, 2$. In addition, we have $|V(D_1')| + |V(D_2')| = n + 2$. By the inductive hypothesis,

$$
\gamma'_{st}(D) = f(A(T)) = g_1(A(D_1')) + g_2(A(D_2')) - 1 > \frac{7 - n}{3}.
$$
Subcase 2.3. $d_D^-(v_1) = 0$.
This implies that $u_i v_1 \in A(D)$ for each $1 \leq i \leq k$. If there exist two pendant arcs at $v_1$, say $e' = x v_1$, $e'' = y v_1$, such that $f(e') = -1$ and $f(e'') = 1$, then using the inductive hypothesis on $D - \{x, y\}$ we have

$$\gamma'_{st}(D) \geq \frac{7 - (n - 2)}{3} > \frac{7 - n}{3}.$$ 

Let $r$ be the number of pendant in-neighbors of $v_1$. By assumption $k - r = f(v_1^-) \geq 3$. Furthermore, since $f(u_i^+) = 0$, there exists a pendant arc $u_i w_i$ for each $i$. Therefore, $n \geq 2k + r + 1$ and hence, $r \leq \frac{n - 7}{3}$. If $D_1$ is the subdigraph induced by $(\cup_{i=1}^{k} N_D^+(u_i)) \cup N_D^-(v_1)$, then $\omega(f|_{D_1}) = -r$. Now let $D_2$ be the digraph obtained from $D$ by deleting all arcs of $D_1$ and all the isolated vertices. If $|V(D_2)| = 0$, then $D = D_1$ and we are done. Let $|V(D_2)| \neq 0$. Since $D$ is an oriented tree, it is easy to verify that $D_2$ has $t$ components, where $t = |V(D_1) \cap V(D_2)|$. Since the order of each component of $D_2$ is greater than 2, by the induction hypothesis and Observation 1.4, we have

$$\gamma'_{st}(D_2) \geq \frac{7t - |V(D_2)|}{3}.$$ 

Therefore

$$\gamma'_{st}(D) \geq \gamma'_{st}(D_1) + \gamma'_{st}(D_2) \geq \frac{7 - |V(D_1)|}{3} + \frac{7t - |V(D_2)|}{3} \geq \frac{7(t + 1) - (n + t)}{3} > \frac{7 - n}{3}.$$ 

In order to show the sharpness of the lower bound, let $D$ be a digraph with vertex set

$$V(D) = \{w, u_i, v_i, w_j \mid 1 \leq i \leq k, k \geq 3 \text{ and } 1 \leq j \leq k - 3\},$$

and arc set

$$A(D) = \{w w_j, u_i w_i, v_i u_i \mid 1 \leq i \leq k \text{ and } 1 \leq j \leq k - 3\}$$

(see Figure 2).

Fig. 2. Digraph $D$ with $k = 7$
Define $f : A(D) \to \{-1, 1\}$ by $f(ww_j) = f(v_iu_i) = -1$ and $f(wu_i) = 1$ for each $1 \leq i \leq k$ and $1 \leq j \leq k - 3$. Clearly, $f$ is an SATDF of $D$ with weight $\frac{7-n}{3}$. This completes the proof.

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