Research Article

On System of Nonlinear Sequential Hybrid Fractional Differential Equations

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In this study, the existence and uniqueness of the solution for a system consisting of sequential fractional differential equations that contain Caputo–Hadamard (CH) derivative are verified. To study the existence and uniqueness of these solutions, some of the most important results from the fixed point theorems in Banach space were used. A practical example is also given to support the theoretical side that was obtained.

1. Introduction

Many problems in various fields can be successfully formulated by fractional differential equations, such as theoretical physics, Biology, viscosity, electrochemistry, and other physical processes (see [1–7].) In the last decade, the fractional differential equation has attracted the attention of mathematicians, physicists, and engineers as well [8, 9].

A fractional differential equation is an equation that contains fractional derivatives and differentials of some mathematical functions and appears in the form of variables. The goal of solving these equations is to find these mathematical functions whose derivatives achieve these equations. Before starting to search for solutions to these equations, studying the conditions of existence and uniqueness is a major matter. To study these conditions, most researchers use the most important fixed point theorems in Banach space, such as Banach contraction principle and Leray Schauder’s theorem (see [10–24]).

In 2014, Zhang et al. [19] published a study investigating the existence results for

\[ \begin{aligned}
& \left( H^{D^{p}_{1}} + \lambda H^{D^{p-1}_{1}} \right) x(t) = f_{1}(t, x(t), y(t), H^{D^{r}_{1}} y(t)), \\
& \left( H^{D^{q}_{1}} + \lambda H^{D^{q-1}_{1}} \right) y(t) = f_{2}(t, x(t), H^{D^{r}_{1}} y(t), y(t)), \\
& x(1) = 0, \quad x(\epsilon) = H^{D^{\beta}_{1}} x(\xi), \\
& y(1) = 0, \quad y(\epsilon) = H^{D^{\beta}_{1}} y(\xi),
\end{aligned} \tag{1} \]

where \( H^{D^{p}_{1}}, p \in (1, 2) \) is the Hadamard fractional derivative, \( g \in C([1, \epsilon] \times \mathbb{R}, \mathbb{R} \setminus \{0\}) \).

In 2016, Algoudi et al. [25] published a study investigating the existence results for the following boundary value problem (sequential Hadamard type):

\[ \begin{aligned}
& \left( H^{D^{p}_{1}} + \lambda H^{D^{p-1}_{1}} \right) x(t) = f_{1}(t, x(t), y(t), H^{D^{r}_{1}} y(t)), \\
& \left( H^{D^{q}_{1}} + \lambda H^{D^{q-1}_{1}} \right) y(t) = f_{2}(t, x(t), H^{D^{r}_{1}} y(t), y(t)), \\
& x(1) = 0, \quad x(\epsilon) = H^{D^{\beta}_{1}} x(\xi), \\
& y(1) = 0, \quad y(\epsilon) = H^{D^{\beta}_{1}} y(\xi),
\end{aligned} \tag{2} \]
where $H^D_{\alpha^+} \in (1,2], r, \nu \in (0,1)$ is the Hadamard fractional derivative and $H^D_{\alpha^+}$ is the Hadamard fractional integral with order $\theta_1, \theta_2 > 0$, $f_1, f_2 \in C([1,e] \times \mathbb{R}^3, \mathbb{R})$, $\eta, \xi \in [1,e]$.

Some researchers went deeper into their research and verified the stability of the solutions to these equations (see [26, 27]). Furthermore, many specialists in the field have paid attention to hybrid fractional differential equations; the importance of fractional hybrid differential equations is that they have a different dynamic than ordinary differential equations and that the hybrid type describes the nonlinear relationship in the derivative of the hybrid function (see [28–31]).

Some focus on having solutions to a system of equations (see [32–36]).

Based on what has been studied in the articles mentioned above, existence and uniqueness of the following nonlinear coupled differential equations are investigated. Unlike the previous studies, the main results of this article are different; that is, we generalize the problem mentioned in [36] by converting one single fractional differential equation into a system, using different fractional derivatives; here, we consider the problems in the context of sequential type.

$$
\begin{align*}
\left\{ 
&(H^D_{\alpha^+} + \lambda H^D_{\alpha^+}^{\nu-1}) \left( \frac{x(t)}{f(t, x(t), y(t))} \right) = \psi(t, x(t), y(t)), \\
&(H^D_{\alpha^+} + \mu H^D_{\alpha^+}^{\nu-1}) \left( \frac{y(t)}{g(t, x(t), y(t))} \right) = \varphi(t, x(t), y(t)), \\
x(a^+) = x(t(a^+)) = 0, \\
y(a^+) = y(t(a^+)) = 0,
\end{align*}
$$

(3)

where $H^D_{\alpha^+} \psi(t) = \delta^n H^D_{\alpha^+}^{n-1} \psi(t)$

$$
= \left( \frac{d}{dt} \right)^n \frac{1}{\Gamma(n-\theta)} \int_a^t \left( \ln \frac{t}{r} \right)^{n-\theta-1} \psi(r) \frac{dr}{r}, \quad n-1 < \theta < n, n = [\theta] + 1,
$$

(5)

where $\delta = t (d/dt)$, $[\theta]$ denotes the integer part of the real number $\theta$.

Let $C([a,T], \mathbb{R})$ denote the Banach space of all real valued continuous functions defined on $[a,T]$ and $C^r_\psi([a,T], \mathbb{R})$ denote the Banach space of all real valued functions $\varphi$ such that $\delta^n \varphi \in C([a,T], \mathbb{R})$ (see [38]).

In this work, we will follow the steps of researchers and specialists in the field. By organizing our results on existence of a solution to the problem as follows, Section 2 contains some fundamental results of fractional calculus and important result for establishing our main results. In Section 3, we introduce our main results. In Section 4, a practical example shows the applicability of our results is given. In Section 5, conclusion and future work are presented.

### 2. Preliminaries

In this section, we introduce some useful definitions, lemmas, and notations of fractional calculus.

**Definition 1 (see [36]).** The Hadamard fractional integral of order $\theta$ for a continuous function $\psi: [a, \infty) \rightarrow \mathbb{R}$ is defined as

$$
H^\theta \psi(t) = \frac{1}{\Gamma(\theta)} \int_a^t \left( \ln \frac{t}{x} \right)^{\theta-1} \psi(x) \frac{dx}{x}, \quad \theta > 0.
$$

(4)

**Definition 2 (see [36]).** The Hadamard fractional derivative of order $\theta > 0$ for a continuous function $\psi: [a, \infty) \rightarrow \mathbb{R}$ is defined as

$$
H^\theta \psi(t) = \delta^n (H^\theta \psi)(t)
$$

(5)

where $H^\theta \psi(t)$ is the Hadamard fractional derivative of order $1 < p, q < 2$, $a \leq t \leq T$, $f, g \in C([a,T] \times \mathbb{R}^2, \mathbb{R} \setminus \{0\})$, and $\lambda, \mu \in \mathbb{R}$.

**Lemma 1 (see [37]).** Let $z \in C^n_\psi([a,T], \mathbb{R})$, where $C^n_\psi([a,T] \rightarrow \mathbb{R}: \delta^n z \in C([a,T])$.

Then,

$$
H^\theta (H^\theta z)(t) = z(t) - \sum_{j=1}^n t^\theta j^\theta \frac{(\ln t)^{\theta-j}}{\theta-j}.
$$

(6)
Lemma 2. Given $x \in C^2_b([a, T], \mathbb{R})$, and $h_1 \in C([a, T], \mathbb{R})$, and

\[
\begin{cases}
(H^\alpha D^p_{x^a} + \lambda H^\alpha D^{p-1}_{x^a}) \left( \frac{x(t)}{f(t, x(t), y(t))} \right) = h_1(t), & 1 < p < 2, 0 < a \leq t \leq T, \\
x(a^+) = x'(a^+) = 0, & \lambda \in \mathbb{R}.
\end{cases}
\]  

(7)

Then, the solution of problem (7) is given by

\[
x(t) = f(t, x(t), y(t)) \times \left[ t^{-\mu} \int_a^t s^{\mu-1} I_{a^+}^{p-1} \varphi(s, x(s), y(s))ds \right].
\]  

(8)

Proof. Applying $H^\alpha I^p_{x^a}$ to (2), we get

\[
H^\alpha I^p_{x^a} h_1(t) = \left( \frac{x(t)}{f(t, x(t), y(t))} \right) + b_1 \left( \ln \frac{t}{a} \right)^{p-1} + b_2 \left( \ln \frac{t}{a} \right)^{p-2} + \lambda H^\alpha I^p_{x^a} \left( \frac{x(t)}{f(t, x(t), y(t))} \right) + c_1 \left( \ln \frac{t}{a} \right)^{p-1},
\]  

(9)

where $b_1, b_2, c_1 \in \mathbb{R}$. The condition $x(a^+) = 0$ implies that $b_2 = 0$. The first derivative of (4) is calculated as follows:

\[
\frac{d}{dt} \left( \frac{x(t)}{f(t, x(t), y(t))} \right) = \varphi(t, x(t), y(t)) + \left( \frac{x(t)}{f(t, x(t), y(t))} \right) \left[ t^{-\mu} \int_a^t s^{\mu-1} I_{a^+}^{p-1} \varphi(s, x(s), y(s))ds \right].
\]  

(10)

\[
\frac{d}{dt} \left( \frac{x(t)}{f(t, x(t), y(t))} \right) = \left[ t^{-\mu} \int_a^t s^{\mu-1} I_{a^+}^{p-1} \varphi(s, x(s), y(s))ds \right].
\]  

(11)

Note that $x(t) = 0$ implies $(x(t)/f(t, x(t), y(t)))' = 0$, and $b_1 = 0$. The integrating factor $\eta(t) = e^{\int_0^t b_1 s^{p-1} ds}$; then, multiplying $\eta(t)$ by (10), we get

\[
\frac{d}{dt} \left( \frac{x(t)}{f(t, x(t), y(t))} \right) = t^{\lambda-1} I_{a^+}^{p-1} h_1(t),
\]  

(12)

consequently,

\[
x(t) = f(t, x(t), y(t)) \times \left[ t^{-\lambda} \int_a^t s^{\lambda-1} I_{a^+}^{p-1} h_1(s)ds \right].
\]  

(13)

In a like manner of Lemma 2, one can easily find the solution $y(t)$ as

\[
y(t) = g(t, x(t), y(t)) \times \left[ t^{-\lambda} \int_a^t s^{\lambda-1} I_{a^+}^{p-1} h_1(s)ds \right].
\]  

(14)

\[\square\]

Remark 1. For $\lambda = 0$, the solution is still valid, as $H^\alpha I^p_{x^a} h_1(t) = H^\alpha I^p_{x^a} h_1(t) = \int_a^t (t^{p-1} h_1(s)/x)ds$ similar logic is applied for the case when $\mu = 0$. Here, these cases will not be taken for consideration in this study.

3. Main Results

In this section, we will present the main results to be obtained from this study.

The space $H = \{ (x(t), y(t)): (x, y) \in C^2_b \times C^3_b \}$ is a Banach space with the norm defined as $\| (x, y) \|_H = \| x \| + \| y \| \forall (x, y) \in H$. Based on Lemma 1, we define an operator $\mathcal{N}: H \rightarrow H$ as

\[
\mathcal{N} (x, y) (t) = \begin{pmatrix} \mathcal{N}_1 (x, y) (t) \\ \mathcal{N}_2 (x, y) (t) \end{pmatrix},
\]  

(15)
Proof. To ease our computations, we set

\[
\Lambda_1 = \sup_{a \in \mathbb{R}^+} \left\{ \int_a^t s^{-2} (\ln \frac{s}{r})^{-1} \frac{dr}{r} \right\} \leq \frac{(\ln T/a)^{p-1}1-(a/T)^p}{|\lambda|\Gamma(p)},
\]

\[
\Lambda_2 = \sup_{a \in \mathbb{R}^+} \left\{ \int_a^t s^{-1} \left( \int_a^t \ln \frac{s}{r} \right)^{-1} \frac{dr}{r} \right\} \leq \frac{(\ln T/a)^{p-1}1-(a/T)^p}{|\mu|\Gamma(q)}.
\]

We have

\[
|f(t, x, y)| \leq \lambda_f, \quad |g(t, x, y)| \leq \lambda_g \forall (t, x, y) \in [a, T] \times \mathbb{R}^2.
\]

(C1) Assume that both \( f, g \) are continuous and \( \exists \lambda_f, \lambda_g > 0 \) such that

\[
|\psi(t, x_1, y_1) - \psi(t, x_2, y_2)| \leq \nu_1|x_1 - x_2| + \nu_2|y_1 - y_2|,
\]

\[
|\varphi(t, x_1, y_1) - \varphi(t, x_2, y_2)| \leq \tau_1|x_1 - x_2| + \tau_2|y_1 - y_2|, \quad \forall t \in [a, T], \quad x_i, y_i \in \mathbb{R}, \quad (i = 1, 2).
\]

(C2) Suppose that \( \psi, \varphi \) are continuous and \( \forall t, \tau_i > 0, \quad (i = 1, 2) \) such that

\[
|f(t, x, y)| \leq \omega_0 + \omega_1|x| + \omega_2|y|,
\]

\[
|g(t, x, y)| \leq \theta_0 + \theta_1|x| + \theta_2|y|, \quad \forall t \in [a, T], \quad x, y \in \mathbb{R}, \quad (i = 1, 2).
\]

Theorem 1. Assume that (C1) and (C2) hold if \( |\lambda_f\Lambda_1 (\nu_1 + \nu_2) + \lambda_g\Lambda_2 (\tau_1 + \tau_2)| < 1 \). Then, problem (1) has a unique solution.

Proof. Consider \( \mathcal{N} \) defined by (9) and let \( \mathcal{B}_{\mathcal{N}} = \{ (x, y) \in \mathcal{N} \} \) be a closed ball in \( H \) with \( \| (x, y) \| \leq \gamma \) be a closed ball in \( H \) with \( \gamma \geq \lambda_f\Lambda_1 N_{\psi} + \lambda_g\Lambda_2 N_{\psi} /1 - (\lambda_f\Lambda_1 (\nu_1 + \nu_2) + \lambda_g\Lambda_2 (\tau_1 + \tau_2)) \), where \( N_{\psi} = \sup_{a \in \mathbb{R}^+} \| \psi(t, 0, 0) \|, \quad N_{\psi} = \sup_{a \in \mathbb{R}^+} \| \psi(t, 0, 0) \|). \)

Observe that \( \| \psi(t, x, y) \| = \| \psi(t, x, y) \| - \psi(t, 0, 0) + \psi(t, 0, 0) \leq \nu_1|\| x \| + \nu_2|\| y \| + N_{\psi} \leq (\nu_1 + \nu_2)\gamma + N_{\psi}. \)

First, we show that \( \mathcal{N}{\mathcal{B}}_{\mathcal{N}} \subset \mathcal{B}_{\mathcal{N}} \). For any \( (x, y) \in \mathcal{N}{\mathcal{B}}_{\mathcal{N}}, \) \( t \in [a, T], \) we have
\[ |N_1(x, y)(t)| = |f(t, x(t), y(t))| + t^{-1}\int_a^t s^{\lambda - 1} \int_a^s \psi(s, x(s), y(s))ds, \]
\[ = \lambda f(t) + t^{-1}\int_a^t s^{\lambda - 1} \int_a^s \psi(s, x(s), y(s))ds, \]
\[ \leq \lambda f(v_1 + v_2)y + N_\psi \times \sup_{\tilde{t} \leq t} \left\{ -\lambda \int_a^t s^{\lambda - 1} \int_a^s \left( \ln \frac{s}{r} \right)^{p-2} dr ds \right\}, \]
\[ \leq \lambda f \Lambda_1[(v_1 + v_2)y + N_\psi]. \]

In a same manner, we find that
\[ \|N_2(x, y)\| \leq \lambda y \Lambda_2[(r_1 + r_2)y + N_\psi]. \] (23)

From (14) and (15), we deduce that \( \|N(x, y)\| \leq y \) Next for \( (x_1, y_1), (x_2, y_2) \in H, \forall t \in [a, T], \) we have
\[ |N_1(x_1, y_1)(t) - N_1(x_2, y_2)(t)| \leq \lambda f \times \sup_{a \leq t \leq T} \left\{ t^{-1}\int_a^t s^{\lambda - 1} \int_a^s \psi(s, x_1(s), y_1(s)) - \psi(s, x_2(s), y_2(s))ds \right\}, \]
\[ \leq \lambda f(v_1\|x_1 - x_2\| + v_2\|y_1 - y_2\|) \times \sup_{a \leq t \leq T} \left\{ t^{-1}\int_a^t s^{\lambda - 1} \int_a^s \left( \ln \frac{s}{r} \right)^{p-2} dr ds \right\}, \]
\[ \leq \lambda f \Lambda_1(v_1\|x_1 - x_2\| + v_2\|y_1 - y_2\|), \]
\[ \leq \lambda f \Lambda_1(v_1 + v_2)(\|x_1 - x_2\| + \|y_1 - y_2\|). \]

Similarly, we can find
\[ \|N_2(x_1, y_1) - N_2(x_2, y_2)\| \leq \lambda y \Lambda_2[(r_1 + r_2)(\|x_1 - x_2\| + \|y_1 - y_2\|)] \]. (25)

Combining (24) and (25) yields
\[ \|N_2(x_1, y_1) - N_2(x_2, y_2)\| \leq \lambda f \Lambda_1(v_1 + v_2) + \lambda y \Lambda_2[(r_1 + r_2)\times(\|x_1 - x_2\| + \|y_1 - y_2\|)]. \] (26)

**Theorem 2.** Assume (C1) and (C3) and (C4) hold if \( (\lambda _1 \omega _1 + \lambda _2 \delta _1) < 1 \) and \( (\lambda _1 \omega _2 + \lambda _2 \delta _2) < 1 \). Then, problem (1) has at least one solution.

**Proof.** We first prove that the operator \( N: H \longrightarrow H \) is completely continuous; obviously, the operator is continuous as a result that \( f, g, \psi, \) and \( \varphi \) are all assumed to be continuous.
By (C4), \( \forall (x, y) \in S \), we have

\[
\left| X_1(x, y)(t) \right| \leq \lambda_f \sup_{a \in [T]} \left\{ t^{-\lambda} \int_a^T \frac{1}{\alpha-a} \frac{1}{\beta-1} \frac{1}{\theta-2} \frac{1}{\theta-3} \frac{1}{\theta-4} \frac{1}{\theta-5} \psi(s, x(s), y(s)) \, ds \right\} \leq \lambda_f \Lambda_1 \sigma_1, \tag{27}
\]

\[
\left\| X_2(x, y) \right\| \leq \lambda_g \Lambda_2 \sigma_2. \tag{28}
\]

Combining inequalities (27) and (28) yields \( \left\| X_1(x, y) \right\| \leq \lambda_f \Lambda_1 \sigma_1 + \lambda_g \Lambda_2 \sigma_2 \); that is, the operator \( N \) is uniformly bounded.

Next, we prove equicontinuity for the operator \( N \); for this, we let \( t_1, t_2 \in [a, T] \), \( t_1 < t_2 \).

Then,

\[
\left| X_1(x, y)(t_2) - X_1(x, y)(t_1) \right| \leq \lambda_f \left( \sup_{a \in [T]} \left\{ t_2^{-\lambda} \int_a^{t_2} \frac{1}{\alpha-a} \frac{1}{\beta-1} \frac{1}{\theta-2} \frac{1}{\theta-3} \frac{1}{\theta-4} \frac{1}{\theta-5} \psi(s, x(s), y(s)) \, ds \right\} \right) \leq \lambda_f \Lambda_1 \sigma_1, \tag{29}
\]

\[
\left| X_2(x, y)(t_2) - X_2(x, y)(t_1) \right| \leq \lambda_g \left( \sup_{a \in [T]} \left\{ t_2^{-\mu} \int_a^{t_2} \frac{1}{\alpha-a} \frac{1}{\beta-1} \frac{1}{\theta-2} \frac{1}{\theta-3} \frac{1}{\theta-4} \frac{1}{\theta-5} \psi(s, x(s), y(s)) \, ds \right\} \right) \leq \lambda_g \Lambda_2 \sigma_2. \tag{30}
\]

Inequality (34) can be written as follows:

\[
\left\| (x, y) \right\| \leq \left( \lambda_f \Lambda_1 \omega_0 + \lambda_g \Lambda_2 \theta_0 \right) + \left( \lambda_f \Lambda_1 \omega_1 + \lambda_g \Lambda_2 \theta_1 \right) \left\| x \right\| + \left( \lambda_f \Lambda_1 \omega_2 + \lambda_g \Lambda_2 \theta_2 \right) \left\| y \right\|. \tag{35}
\]

where \( \Lambda_0 = \min \{ 1 - (\lambda_f \Lambda_1 + \lambda_g \Lambda_2 \theta_0), 1 - (\lambda_f \Lambda_1 \omega_2 + \lambda_g \Lambda_2 \theta_2) \} \). By (35), we conclude that \( \Omega \) is bounded. Hence, R.H.Ss are both independent on \( (x, y) \); in addition, R.H.Ss of both (29) and (30) approach to zero when \( t_1 \to t_2 \) and they imply that the operator \( N(x, y) \) is equicontinuous; consequently, the operator \( N(x, y) \) is completely continuous.

To finish, we establish the bounded set given by \( \Omega = \{ (x, y) \in H : (x, y) = \beta N(x, y), \beta \in [0, 1] \} \), and then \( \forall t \in [0, 1] \), with \( (x, y) = \beta N(x, y) \), we obtain

\[
x(t) = \beta N_1(x, y)(t), \tag{31}
\]

\[
y(t) = \beta N_2(x, y)(t). \tag{32}
\]

By (C3), we get

\[
\left\| x \right\| \leq \lambda_f \Lambda_1 \left( \omega_0 + \omega_1 \left\| x \right\| + \omega_2 \left\| y \right\| \right), \tag{33}
\]

\[
\left\| y \right\| \leq \lambda_g \Lambda_2 \left( \theta_0 + \theta_1 \left\| x \right\| + \theta_2 \left\| y \right\| \right). \tag{34}
\]

Consequently, we have

\[
\left\| x \right\| + \left\| y \right\| \leq \left( \lambda_f \Lambda_1 \omega_0 + \lambda_g \Lambda_2 \theta_0 \right) + \left( \lambda_f \Lambda_1 \omega_1 + \lambda_g \Lambda_2 \theta_1 \right) \left\| x \right\| + \left( \lambda_f \Lambda_1 \omega_2 + \lambda_g \Lambda_2 \theta_2 \right) \left\| y \right\|. \tag{35}
\]

Leray–Schauder alternative applies; that is, problem (1) has at least one solution. This completes the proof. \( \square \)

4. Example

Consider the following initial value problem:
\[
\left( H D_{a^+}^{\alpha} + H D_{a^+}^{\beta/4} \right) \left( \frac{x(t)}{1/3|\sin x(t)|+2} \right) = \ln t + \frac{1}{11\sqrt[3]{t^3 + 15}} \frac{|x|}{1 + |x|} + \frac{1}{44} \tan^{-1} y, \quad 0 < t \leq e, \\
\left( H D_{a^+}^{\alpha} + \mu H D_{a^+}^{\beta/4} \right) \left( \frac{y(t)}{1/3|\sin y(t)|+1} \right) = e^{-t} \cos t + \frac{1}{20} \left( \tan^{-1} y + \tan^{-1} x \right), \\
x(1) = x'(1) = 0, \\
y(1) = y'(1) = 0.
\]

Here, \( 1 = \lambda = \mu, \quad p = 7/4, q = 5/4, \)
\( f(t, x, y) = 1/3(|\sin x| + 2), \quad g(t, x, y) = 1/3(|\cos y| + 1), \)
\( \psi(t, x(t), y(t)) = 3 \ln t + \frac{1}{11\sqrt[3]{t^3 + 15}} \frac{|x|}{1 + |x|} + \frac{1}{44} \tan^{-1} y, \)
\( \varphi(t, x(t), y(t)) = e^{-t} \cos t + \frac{1}{20} \left( \tan^{-1} y + \tan^{-1} x \right). \)

Observe that
\[
\left| \psi(t, x_1, y_1) - \psi(t, x_2, y_2) \right| \leq \frac{1}{44} \left| x_2 - x_1 \right| + \frac{1}{44} \left| y_2 - y_1 \right|, \\
\left| \varphi(t, x_1, y_1) - \varphi(t, x_2, y_2) \right| \leq \frac{1}{20} \left| x_2 - x_1 \right| + \frac{1}{20} \left| y_2 - y_1 \right|, \\
\left[ \lambda_j \Lambda_1 (v_1 + v_2) + \lambda_m \Lambda_2 (r_1 + r_2) \right] \leq 0.1363483 < 1.
\]

Thus, problem (36) satisfies all the conditions of Theorem 1; accordingly, we conclude that the B.V.P has a unique solution on \([1,e]. \)

5. Conclusion and Future Work

In this article, the existence and uniqueness theory of solutions for sequential fractional differential system involving Hadamard fractional derivatives of order \( 1 < p, \quad q < 2 \) with initial conditions were investigated. For the future work, the researcher may generalize our system by taking an \( n \times 1 \) system of sequential fractional differential equations and may apply another type of fractional derivatives such as Psi-Hilfer and Psi-Caputo fractional derivatives.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors’ Contributions

M.A and KA contributed to each part of this work equally and read and approved the final version of the manuscript.

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References

[1] J. Wang, W. Wei, and M. Fečkan, "Nonlocal Cauchy problems for fractional evolution equations involving Volterra-Fredholm type integral operators," *Miskolc Mathematical Notes*, vol. 13, no. 1, pp. 127–147, 2012.
[2] J. Wang, Y. Zhou, and M. Fečkan, "On the nonlocal Cauchy problem for semilinear fractional order evolution equations," *Open Mathematics*, vol. 12, no. 6, pp. 911–922, 2014.
[3] M. Awadalla, Y. Y. N. Yameni, and K. A. Asbeh, "Psi-Caputo logistic population growth model," *Journal of Mathematics*, vol. 2021, Article ID 863428, 2021.
[4] M. Awadalla, Y. Y. Y. Noupoue, and K. Abuabeh, "Population growth modeling via Rayleigh-caputo fractional derivative," *Journal of Statistics Applications & Probability*, vol. 10, no. 1, pp. 11–16, 2021.
[5] M. Awadalla, Y. Y. Y. Nennick, and K. Asbeh, "Modeling the dependence of barometric pressure with altitude using caputo and caputo-fabrizio fractional derivatives," *Journal of Mathematics*, vol. 2020, Article ID 2417681, 2020.
[6] M. K. A. Kaabar, M. Shabibi, J. Alzabut et al., "Investigation of the fractional strongly singular thermostat model via fixed point techniques," *Mathematics*, vol. 9, no. 18, pp. 2298, 2021.
[7] M. M. Matar, M. I. Abbas, J. Alzabut, M. K. A. Kaabar, S. Etemad, and S. Rezapour, “Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives,” *Advances in Difference Equations*, vol. 2021, no. 1, pp. 68–18, 2021.
[8] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, UK, 1999.
[9] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, "Theory and Applications of Fractional Differential Equations," *North-Holland Mathematics Studies*, Elsevier Science B.V, Amsterdam, Netherlands, 2006.
[10] C. Bai, "Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative," *Journal of Mathematical Analysis and Applications*, vol. 384, no. 2, pp. 211–231, 2011.
B. Ahmad and J. J. Nieto, “Sequential fractional differential equations with three-point boundary conditions,” *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3046–3052, 2012.

B. Ahmad and J. J. Nieto, “Boundary value problems for a class of sequential integro-differential equations of fractional order,” *Journal of Function Spaces and Applications*, vol. 2013, Article ID 149659, 8 pages, 2013.

M. H. Aqlan, A. Alsaeidi, B. Ahmad, and J. J. Nieto, “Existence theory for sequential fractional differential equations with anti-periodic type boundary conditions,” *Open Mathematics*, vol. 14, no. 1, pp. 723–735, 2016.

M. Klimek, “Sequential fractional differential equations with Hadamard derivative,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 12, pp. 4689–4697, 2011.

H. Ye and R. Huang, “On the nonlinear fractional differential equations with Caputo sequential fractional derivative,” *Advances in Mathematical Physics*, vol. 2015, Article ID 174156, 2015.

A. Alsaeidi, S. Sivasundaram, and B. Ahmad, “On the generalization of second order nonlinear anti-periodic boundary value problems,” *Nonlinear Studies*, vol. 16, pp. 415–420, 2009.

B. Ahmad and J. J. Nieto, “Anti-periodic fractional boundary value problems,” *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1150–1156, 2011.

B. Ahmad, J. Losada, and J. J. Nieto, “On antiperiodic nonlinear three-point boundary value problems for nonlinear fractional differential equations,” *Discrete Dynamics in Nature and Society*, vol. 2015, Article ID 973783, 2015.

L. Zhang, B. Ahmed, and G. Wang, “Existence and approximation of positive solutions for nonlinear fractional integro-differential boundary value problems on an unbounded domain,” *Applied and Computational Mathematics V.*, vol. 15, no. 2, pp. 149–158, 2016.

Y. Huang, Z. Liu, and R. Wang, “Quasilinearization for higher order impulsive fractional differential equa-tions,” *Applied and Computational Mathematics V*. vol. 15, no. 2, pp. 159–171, 2016.

N. I. Mahmudov, M. Awadalla, and K. Abusessa, “Nonlinear sequential fractional differential equations with nonlocal boundary conditions,” *Advances in Difference Equations*, vol. 2017, no. 1, pp. 319–415, 2017.

Y. Zhou, C. Zhang, and H. Wang, “Boundary value methods for Caputo fractional differential equations,” *Journal of Computational Mathematics*, vol. 39, no. 1, pp. 108–129, 2021.

B. Ahmad, Y. Alruwaily, A. Alsaeidi, and S. K. Ntouyas, “Sequential Fractional Differential Equations with Nonlocal Integro-Multipoint Boundary Conditions,” *The Bulletin of the Malaysian Mathematical Society Series*, vol. 41, no. 4, 2016.

Z. Bouazza, S. Etemad, M. S. Souid, S. Rezapour, F. Martinez, and M. K. A. Kaabar, “A study on the solutions of a multiterm FBVP of variable order,” *Journal of Function Spaces*, vol. 2021, Article ID 9939147, 2021.

S. Alijoudi, B. Ahmad, J. J. Nieto, and A. Alsaeidi, “A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions,” *Chaos, Solitons & Fractals*, vol. 91, pp. 39–46, 2016.

R. Agarwal, D. O’Regan, and S. Hristova, “Stability of Caputo fractional differential equations by Lyapunov functions,” *Applications of Mathematics*, vol. 60, no. 6, pp. 653–676, 2015.

X. Wang, D. Luo, and Q. Zhu, “Ulam-Hyers stability of caputo type fuzzy fractional differential equations with time-delays,” *Chaos, Solitons & Fractals*, vol. 156, Article ID 111822, 2022.

Y. Zhao, Z. Sun, Z. Han, and Q. Li, “Theory of fractional hybrid differential equations,” *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1312–1324, 2011.

S. Sun, Y. Zhao, Z. Han, and Y. Li, “The existence of solutions for boundary value problem of fractional hybrid differential,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 12, pp. 4961–4967, 2012.

B. Ahmad and S. K. Ntouyas, “An Existence Theorem for Fractional Hybrid Differential Inclusions of Hadamard Type with Dirichlet Boundary Conditions,” *Abstract and Applied Analysis*, vol. 2014, Article ID 705809, 7 pages, 2014.

B. C. Dhage and S. K. Ntouyas, “Existence results for boundary value problems for fractional hybrid differential inclusions,” *Topological Methods in Nonlinear Analysis*, vol. 44, no. 1, pp. 229–238, 2014.

B. Ahmad, F. A. Amjad, K. N. Sotiris, and A. Bashir, “Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions,” *Applied Mathematics and Computation*, vol. 266, pp. 615–622, 2015.

X. Su, “Boundary value problem for a coupled system of nonlinear fractional differential equations,” *Applied Mathematics Letters*, vol. 22, no. 1, pp. 64–69, 2009.

H. Baghani, J. Alzabut, J. Farokhi-Ostad, and J. J. Nieto, “Existence and uniqueness of solutions for a coupled system of sequential fractional differential equations with initial conditions,” *Journal of Pseudo-Differential Operators and Applications*, vol. 11, no. 4, pp. 1731–1741, 2020.

A. Wongcharoen, S. K. Ntouyas, P. Wongsantisuk, and J. Tariboon, “Existence results for a nonlinear coupled system of sequential fractional differential equations involving-hilfer fractional derivaties,” *Advances in Mathematical Physics*, vol. 2021, Article ID 5554619, 9 pages, 2021.

M. M. Matar, J. Alzabut, and J. M. Jonnalagadda, “A coupled system of nonlinear Caputo-Hadamard Langevin equations associated with nonperiodic boundary conditions,” *Mathematical Methods in the Applied Sciences*, vol. 44, no. 3, pp. 2650–2670, 2021.

M. M. Mohammed, “Solution of sequential Hadamard fractional differential equations by variation of parameter technique,” *Abstract and Applied Analysis*, vol. 2018, Article ID 960553, 7 pages, 2018.

N. I. Mahmudov and M. Awadalla, “On sequential fractional differential equations with nonlocal integral boundary conditions,” *Journal of Computational Analysis and Applications*, vol. 26, no. 4, 628 pages, 2019.