Nonequilibrium dynamics of spin-orbit coupled lattice bosons

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We study the nonequilibrium dynamics of two-component bosonic atoms in a one-dimensional optical lattice in the presence of spin-orbit coupling. In the Mott insulating regime, the two-component bosonic system at unity filling can be described by the quantum spin model. The atoms are initially prepared in their lower spin states. The system becomes out of equilibrium by suddenly introducing the spin-orbit coupling to the atoms. The system shows the relaxation and non-stationary dynamics, respectively, in the different interaction regimes. We find that the time average of magnetization is useful to characterize the many-body dynamics. The effect of even and odd numbers of sites is discussed. Our result sheds light on the nonequilibrium dynamics due to the interplay between spin-orbit coupling and atomic interactions.

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I. INTRODUCTION

Non-equilibrium dynamics of a closed many-body system is of fundamental importance in quantum physics and statistical mechanics. For example, the local observables appear to be in thermal states even if the entire many-body system is in a pure state. The eigenstate thermalization hypothesis was proposed to explain such phenomena for complex quantum systems. However, it cannot applied to integrable systems. In fact, a general mechanism of thermalization is still lacking.

Ultracold atoms offer experimental platforms to study many-body dynamics of closed systems. For instance, ultracold atoms can provide a relatively long accessible time to study the non-equilibrium dynamics, and the high level of controllability to adjust the interaction parameters for initiating the many-body dynamics. In addition, ultracold atoms have been exploited to simulate a lot of intriguing quantum phenomena such as quantum phase transition from a Mott insulating regime to superfluid. The advanced detection techniques enable one to individually address the single atom. Therefore, this can be used for probing the dynamics in the microscopic description. Recently, the relaxation dynamics of closed ultracold atomic systems have been observed.

In this paper, we consider a system of two-component bosonic atoms in a one-dimensional (1D) optical lattice. In the Mott insulating regime, the system, with unit filling, can be described by a quantum spin XXZ model. We consider all atoms to be initially prepared in their lower spin states. To study the non-equilibrium dynamics, the spin-orbit (SO) coupling is suddenly turned on. In fact, SO coupling, which produces the interaction between the particle’s spin and momentum, naturally exists in solid-state materials. It gives rise to a number of intriguing effects such as topological insulators and superconductors, etc. Spin-orbit coupling in atomic Bose-Einstein condensates can be produced by inducing two-photon Raman transition using a pair of lasers.

Alternatively, the SO coupling between the atoms in the lattice can be induced by periodically shaking the lattice potential. More recently, the techniques for adjusting SO coupling have been shown. The SO coupling gives rise to pair interactions between two neighbouring atoms and Dzyaloshinskii-Moriya (DM) interactions in the lattice.

We consider the magnetization as an observable to study the many-body dynamics. We find that the time average of magnetization is useful for characterizing the non-equilibrium dynamics. The system exhibits the relaxation and non-stationary dynamics in the different interaction regimes which depend on the SO coupling strength and the ratio of inter-component interaction to intra-component interaction. Indeed, the dynamical behaviours greatly depend on the overlap between the initial state and eigenstates. The interplay between the SO coupling and atomic interactions leads to the changes of this overlap, and results in the different dynamical behaviours.

Thermalization occurs when a subsystem evolves to a mixed state even if the system is a pure state. In the relaxation regime, the spins rapidly relax to the thermal states just after the SO coupling is turned on. The degree of quantum coherence of local spins can be measured by using the purity. We show that an atom in each site can be described by an almost completely mixed state.

The transition from the relaxation dynamics to non-stationary evolution occurs when the SO coupling strength is strong and the inter-component interaction strength becomes sufficiently larger than the intra-component interaction. In the non-stationary regime, we find that the distinct dynamics are displayed for the even and odd numbers of sites. For even-number cases, the effective two-level dynamics is shown. This forms a superposition of two distinct states during the time evolution. These superposition states are useful for quantum metrology. On the other hand, the spin system becomes ferromagnetic in the odd-number cases.
This paper is organized as follows: In Sec. II, we introduce the system. In Sec. III, we study the non-equilibrium dynamics of this system. We characterize the many-body dynamics by using the time average of magnetization. We discuss the relaxation dynamics of local spins and the non-stationary dynamics. The effect of even and odd number of sites on the dynamics is discussed. We provide a discussion and a conclusion.

II. SYSTEM

We consider the two-component bosonic atoms to be trapped in a one-dimensional optical lattice. We assume that this system to have open boundary conditions. The two-component Bose-Hubbard model can be used to describe the interactions of two-component bosons in an optical lattice. The Hamiltonian $H_{BH}$ can be written as,

$$H_{BH} = \sum_{\alpha, i} \left[ J_\alpha (\alpha)_{i+1} + H.c. \right] + \sum_{\alpha, i} \frac{U_{\alpha}}{2} n^\alpha_i (n^\alpha_i - 1) + U_{ab} n^a_i n^b_i ,$$

(1)

where $\alpha$ and $\alpha^\dagger$ are the annihilation and creation operators of an atom in the atomic spin state $|\alpha\rangle$, and $n^\alpha_i$ is the number operator at site $i$, and $\alpha = a, b$. The parameter $J_\alpha$ is the tunnel coupling, $U_{ab}$ and $U_{ab}$ are the intra- and inter-component interaction strengths of atoms, respectively. We assume that the tunnel coupling and atom-atom interaction strength of each component to be nearly equal, i.e., $J_a \approx J_b \approx J$ and $U_a \approx U_b \approx U$.

To study the nonequilibrium dynamics, we consider the SO coupling to be suddenly applied to the atoms. The Hamiltonian, which describes SO coupling, can be written as

$$H_{SO} = t_{so} \sum_i \left( a_i^\dagger b_{i+1} - a_i b_{i-1} + H.c. \right),$$

(2)

where $t_{so}$ is the strength of spin-orbit coupling.

We assume that the atom-atom interaction strengths $U$ and $U_{ab}$ are repulsive, and also they are much larger than the parameters such as $J$ and $t_{so}$. In this strongly interacting regime with unit filling, it is convenient to write the two-mode bosonic operators in terms of angular momentum operators, i.e., $S^+_i = a_i^\dagger b_i$, $S^-_i = b_i^\dagger a_i$ and $S^z_i = (b_i^\dagger b_i - a_i^\dagger a_i)/2$. Now the effective Hamiltonian is written as

$$H_{eff} = \sum_{i=1}^{N-1} \left\{ 2 \left[ \frac{t_{so}}{J} \right]^2 - 1 \right\} (2U_r - 1) S^z_i S^z_{i+1}$$

$$+ \left( \frac{t_{so}}{J} \right)^2 (S^+_i S^+_i + S^-_i S^-_{i+1}) - (S^+_i S^-_{i+1} + S^-_i S^+_{i+1})$$

$$- 4 \frac{U_{ab}}{U} \left( \frac{t_{so}}{J} \right) (S^z_i S^z_{i+1} - S^z_i S^z_{i+1}) \right\}$$

(3)

where $\lambda = 2J^2/U_{ab}$ and $U_r = U_{ab}/U$. The last terms in Eq. (3) are called the DM interactions $[24, 25]$. The system can be described by the XYZ spin model with the DM interactions $[24, 26]$.

III. MANY-BODY DYNAMICS

We investigate the quantum dynamics of the system by suddenly applying the SO coupling to the atoms. Initially, all atoms are prepared in their lower spin states, i.e.,

$$|\Psi(0)\rangle = |↓↓ . . . ↓↓\rangle.$$  

(4)

Then, the SO coupling is suddenly turned on. The system becomes out of equilibrium. We numerically simulate the dynamics of this spin chain by using exact diagonalization (see [36] and references therein).

![Contour plot of the average of magnetization $|M|$ versus parameters $t_{so}$ and $U_r$, where the total time taking for average is $\tau = 10\lambda$. The system's sizes $N = 11$ and $N = 12$ are shown in (a) and (b), respectively.](image)

We study the dynamics of magnetization which is given by

$$M = 2 \frac{N}{N} \sum_{i=1}^{N} \langle S^z_i \rangle.$$  

(5)

The magnetization $M$ is equal to $+1(-1)$ when all spins are in up(down) states. We take the average of magneti-
zation $|M|$ within a period $\tau = 10\lambda$, i.e.,
\[
\bar{M} = \frac{1}{\tau} \int_0^\tau |M(t)| dt.
\]  

(6)

By taking the absolute sign of the magnetization, we can ensure that $\bar{M}$ is positive.

In Fig. (a) we plot the time average of magnetization $\bar{M}$ versus the parameters $t_{so}$ and $U_r$. When $t_{so}/J$ ranges from 0.75 to 2, $\bar{M}$ is close to zero. The initial magnetization is -1. After taking the average, it becomes nearly zero. This means that the system reaches a steady state and the magnitude of magnetization is small in the long time.

As $t_{so}$ and $U_r$ increase, $\bar{M}$ gradually increases. The system is no longer stationary in the long time. This shows the different layers in the figure. When both parameters $t_{so}/J$ and $U_r$ become sufficiently large, $\bar{M}$ can reach nearly 0.5 and 1, respectively, for the even and odd numbers of sites. This shows that the even-odd effect in the dynamical behaviours.

It should be noted that the average magnetization is close to one in Fig. (a) if $t_{so}/J$ is less than 0.75. In fact, we cannot characterize the dynamics of this interaction range because the time evolution is too slow and they do not reach the steady state within the period $\tau = 10\lambda$.

According to Fig. (a) the many-body dynamics can be mainly classified into the two different types. They are relaxation dynamics and non-stationary evolution, respectively. We will discuss them in the following subsections.

A. Relaxation dynamics

Now we study the quantum dynamics of local spins in the parameter region in which the average of magnetization $\bar{M}$ is about zero in Fig. (a). In Figs. (a) and (b), we plot the dynamics of magnetization $\bar{M}$, for the odd and even numbers of sites, respectively. Initially, the magnetization $M$ is equal to -1. When the SO coupling is turned on, the magnetization swiftly increases in a short time. Then, it becomes saturated to around zero in a longer time. This shows that the spins nearly relax to their steady states. As the system size increases, the magnetization becomes more steady. Besides, the relaxation dynamics is no different between the odd- and even-number cases.

To study the thermalization of local spins, we examine their purities. The purity is a quantity which measures the degree of quantum coherence of a system. It is defined as
\[
\Lambda_{\text{pur}} = \text{Tr}(\rho^2),
\]  

(7)

where $\rho$ is the density matrix of a system. If the system is in a pure state, then the purity is equal to one. Otherwise, the purity is less than one.

In Fig. (a), we plot the purities of local spins versus time. The initial purity is equal to 1. Afterwards, the purities rapidly drop and the purity $\Lambda_{\text{pur}}$ of a local spin decreases to about 0.5 in a long time. The purity of a spin-half particle in a completely mixed state is 0.5. This means that the spins are nearly thermalized. Also, we study the purity $\Lambda_{\text{half}}^{\text{pur}}$ of a half of the spin chain in Fig. (b). The purity shows to be saturated to a value 0.018. It is very close to the totally mixed state which gives the purity $\Lambda_{\text{half}}^{\text{pur}} = 2^{N/2-N} \approx 0.015625$ in our case. This means that the half of a spin chain can be approximately described by a totally mixed state.

Then, we compare the relaxation dynamics with the different $U_r$-s. The average magnetization $\bar{M}$ increases when $U_r$ becomes larger. In Fig. (a), we plot the magnetization $\bar{M}$ as a function of time. When $U_r$ slightly increases, $\bar{M}$ gives a larger magnitude of fluctuations. This means that the average $|M|$ also increases. Then, we study the purity of a local spin in Fig. (b). The purity increases when $U_r$ increases. This means that this local spin has a higher degree of quantum coherence. This suggests that $\bar{M}$ is an useful quantity to characterize the relaxation dynamics in our case.

Indeed, the occurrence of thermalization can be understood by examining the overlap between the initial state and eigenvectors of the system. We consider the probability coefficients of the initial state and eigenvectors
\[
P_n = |\langle \Psi(0)|E_n \rangle|^2,
\]  

(8)

where $|\Psi(0)\rangle$ and $|E_n\rangle$ are the initial state and the $n$-th eigenvectors.
In Fig. 3(a), we plot the overlap probabilities $P_n$ versus $n$, where $t_{so}/J = U_r = 1$ and $n$ is an index of the $n$-th eigenstate. This corresponds to the previous case in Fig. 2. We can see that the initial state has a large overlap with the eigenstates in Fig. 3(a). The initial state overlaps with almost entire eigen-spectrum. In Fig. 3(b), we also compare the case in Fig. 3 where the magnetization shows stronger fluctuations in the dynamics. Obviously, the overlap between the initial state and the eigenstates is much smaller than that in Fig 3(a). Here the off-diagonal terms of the observables are suppressed if there is a large overlap between the initial states and the eigenstates of the system. Our result coincides with the mechanism of thermalization by Rigol et al. [27].

### B. Even-odd effect

When both parameters $t_{so}/J$ and $U_r$ increase, the dynamics of the system becomes non-stationary. We find that the dynamical behaviours are totally different between the even and odd numbers of sites. For even-number cases, the system undergoes an effective two-level dynamics. In contrast, the system becomes ferromagnetic if the number of sites is odd.

#### 1. Even number case: Effective two-level dynamics

When $t_{so}/J$ and $U_r \gg 1$, the states $(|↑↑...↑↑⟩ ± |↓↓...↓↓⟩)/\sqrt{2}$ are two nearly degenerate eigenstates. In this regime, the entire many-body dynamics can be effectively described by these two degenerate states if the system starts with $|↓↓...↓↓⟩$. We plot the time evolution of the magnetization in Fig. 4(a). The magnetization shows periodic oscillations. The effective Rabi frequency decreases as the parameter $U_r$ increases and a larger magnitude can be attained. In fact, the superposition of the two degenerate ground states can be produced, i.e.,

$$|\Psi(t)⟩ \approx c_1 |↑↑...↑↑⟩ + c_2 |↓↓...↓↓⟩,$$

where $|c_1|^2 + |c_2|^2 = 1$.

The two degenerate states can be coupled via the high order virtual transitions. The effective Rabi frequency can be approximately obtained which can be derived from the high-order perturbation theory in Appendix A. Since this effective Rabi frequency inversely scales with the power $N$, the rate of evolution becomes very slow as the system’s size increases. This limits the creation of superposition of two spin states when the system goes large.

We compare the overlap between the initial state in Eq. (4) and the eigenstates in Fig. 5(c). The overlap is much smaller than the two previous cases in Figs. 4(a) and (b) which shows the relaxation dynamics. In fact,
FIG. 5. (Color online) Overlap probability $P_n$ versus the index $n$ of the $n$-th eigenstate, for $N = 12$. In (a), the parameters $t_{so} = J$ and $U_r = 1$ are used. The different parameters $U_r = 1$ and 2 are used in (b) and (c), respectively, but with the same SO coupling, $t_{so} = 8J$.

the dynamics cannot be thermalized in this degenerate system.

2. Odd number case: Ferromagnetic

In Fig. 6(b), we plot the magnetization versus time for odd-number cases. The spins tend to maintain in their ground states when $U_r$ increases. The spin system is ferromagnetic. Indeed, the initial state in Eq. (4) is an eigenstate if the parameters $t_{so}$ and $U_r$ go large. Therefore, the magnetization is about $-1$ for the large $t_{so}$ and $U_r$. The small fluctuations around -1 are shown due to the perturbation terms from the SO coupling.

The odd-number cases are totally different to the even-number cases. In even-number cases, the effective two-level dynamics occurs due to the virtual fluctuations of the perturbation terms $\sum_i S_i^+ S_{i+1}^- + H.c.$ However, these perturbation terms alter the spin state in pair only and therefore they cannot contribute the dynamics between the two degenerate states $| \uparrow \uparrow \cdots \uparrow \rangle$ and $| \downarrow \downarrow \cdots \downarrow \rangle$ for odd-number cases. Also, the DM terms cancel the contributions from the spin states and their reflection states. For example, the states $|001\rangle$ and its reflection states $|100\rangle$ will be cancelled in the perturbation series. Therefore, the DM terms cannot lead to the effective two-level dynamics in this case.

IV. DISCUSSION

We have investigated the non-equilibrium dynamics of a closed quantum system by suddenly applying the SO coupling. It is necessary to produce and tune the required SO couplings to the atoms. There are several ways to create SO coupling such as two-photon Raman transition [16, 17] and shaking lattice [18, 19]. Recently, the techniques for tuning SO coupling [20] have been demonstrated by using Raman coupling with laser fields. However, this method may produce unwanted heating due to spontaneous emissions of atoms.

Alternatively, the SO coupling can be exploited by shaking the lattice periodically. This “shaking” method has been used to successfully generate the artificial gauge potential to cold atoms in lattices [37]. More recently, the theoretical proposals for generation of SO coupling have been put forward [18, 19]. This method is able to create SO coupling without heating if the appropriate driving conditions are met [18]. The other scheme, which overcomes the problems of spontaneous emissions, has also been proposed [38].
In addition, it is required to adjust the inter- and intra-component interaction strengths for observing the different dynamics. This can be made by using Feshbach resonance [39]. The scattering length between the different components of atoms can be modified by applying the appropriate magnetic fields to the atoms [40].

V. CONCLUSION

In summary, we have studied the many-body dynamics of two-component bosonic atoms in a 1D optical lattice by suddenly introducing the SO coupling. In Mott-insulating regime, the system can be described by a quantum spin system. We study the dynamics of magnetization of the system. We find that the time average of magnetization is useful for characterizing the non-equilibrium dynamics. The system shows the relaxation and non-stationary dynamics in the different interaction regimes. In the relaxation regime, the magnetization becomes nearly stationary in a long time and the local spins become nearly thermalized. When the SO coupling is strong and the inter-component interaction strength is sufficiently larger than the intra-component strength, the system becomes non-stationary. The totally different dynamical behaviours are shown for the even and odd numbers of sites in the non-stationary regime.

Appendix A: Derivation of effective Rabi frequency

We study the effective Rabi frequency between the two degenerate states from the perturbation theory. We consider the perturbation terms with pair excitations that contribute the virtual excitations:

\[ H_0 = \lambda \sum_{i=1}^{N-1} \left\{ 2 \left[ \left( \frac{\lambda}{J} \right)^2 - 1 \right] \left( \frac{U_{ab}}{U} - 1 \right) S_i^z S_{i+1}^z \right\}, \]

\[ V = \lambda \left( \frac{\lambda}{J} \right)^2 (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+), \]  

(A1)

where \( H_0 \) is treated as an unperturbed Hamiltonian and \( V \) as a perturbation. We can obtain the leading terms of the \( N/2 \)-th order eigenenergy:

\[ E_{n_1, n_2}^{(N/2)} = \sum_{j=1}^{\tilde{N}_c} \left( \frac{\lambda}{J} \right)^2 \left( \frac{\lambda}{J} \right)^2 \left( \frac{U_{ab}}{U} - 1 \right) \prod_{i=1}^{N/2-1} V_{k_i+k_{i+1}} V_{k_{i+1}} \]

\[ + \text{other terms,} \]  

(A2)

where \( V_{k_d}^l = j \langle l | V | k_d \rangle \), \( E_{k_d}^{l} = E_{k_d}^{(0)} - E_{k_d}^{(0)} \) and \( D \) denotes the degenerate subspace for \( n_1 = | \uparrow \uparrow \ldots \uparrow \rangle \) and \( n_2 = | \downarrow \downarrow \ldots \downarrow \rangle \), and \( \tilde{N}_c \) is the number of possible terms that connect the two degenerate states via virtual fluctuations. The number \( \tilde{N}_c \) can be obtained numerically by counting all possibilities to connect the two states. The leading term of \( E_{n_1, n_2}^{(N/2)} \) can be written as

\[ E_{n_1, n_2}^{(N/2)} \approx \lambda \left( \frac{\lambda}{J} \right)^2 2^{1-N/2} (2U_r - 1) 1^{1-N/2} \tilde{N}_c. \]  

(A3)

TABLE I. This table shows the effective Rabi frequencies from the numerical results and the perturbation theory for the different system’s sizes, and the percent error \( \eta \).

| \( N \) | \( \Omega_R/\lambda \) | \( \tilde{\Omega}_R/\lambda \) | \( \eta \) |
|---|---|---|---|
| 6 | 8.9760 | 8.1633 | 9.1% |
| 8 | 2.7013 | 2.3324 | 13.7% |
| 10 | 0.83642 | 0.66639 | 20.3% |
| 12 | 0.25964 | 0.19040 | 26.7% |

We compare the effective Rabi frequency between the numerics and the approximation from the perturbation theory in Table I for the different \( N \)-s. The percent error \( \eta \) shows the error between the exact numerical value and the approximation. It is defined as

\[ \eta = \frac{\| \Omega_R - \tilde{\Omega}_R \|}{\| \Omega_R \|} \times 100\%. \]  

(A4)

Here we denote \( \tilde{\Omega}_R = E_{n_1, n_2}^{(N/2)} \) as the approximation from the perturbation theory. This approximation is fairly good when \( N \) is small. However, as \( N \) increases, the error grows. In fact, we have taken account of the leading term from the \( N/2 \)-th order perturbation only. When \( N \) increases, the calculation should involve the higher order perturbation terms.

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