THE INITIAL MASS FUNCTION MODELED BY A LEFT TRUNCATED BETA DISTRIBUTION

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ABSTRACT

The initial mass function for stars is usually fitted by three straight lines, which means it has seven parameters. The presence of brown dwarfs (BDs) increases the number of straight lines to four and the number of parameters to nine. Another common fitting function is the lognormal distribution, which is characterized by two parameters. This paper is devoted to demonstrating the advantage of introducing a left truncated beta probability density function, which is characterized by four parameters. The constant of normalization, the mean, the mode, and the distribution function are calculated for the left truncated beta distribution. The normal beta distribution that results from convolving independent normally distributed and beta distributed components is also derived. The chi-square test and the Kolmogorov–Smirnov test are performed on a first sample of stars and BDs that belongs to the massive young cluster NGC 6611, and on a second sample that represents the masses of the stars of the cluster NGC 2362.

Key words: methods: statistical – stars: fundamental parameters – stars: luminosity function, mass function

1. INTRODUCTION

The mass distribution of stars was first fitted with a power law by Salpeter (1955). He suggested \( \xi(m) \propto m^{-\alpha} \), where \( \xi(m) \) represents the probability of having a mass between \( m \) and \( m+dm \). He found \( \alpha = 2.35 \) in the range \( 10 M_\odot > M \geq 1 M_\odot \). This value has changed little with time and a recent evaluation quotes 2.3; see Kroupa (2001). Subsequent research has begun to analyze the initial mass function (IMF) with three power laws (see Scalo 1986; Kroupa et al. 1993; Binney & Merrifield 1998), and four power laws (see Kroupa 2001). A first comment on this temporal evolution is that the name is not appropriate because the power function distribution \( \xi(m) \propto m^b \) is defined only for positive values of \( b \); see Evans et al. (2000). Second, the exact name for a probability density function (PDF) \( \xi(m) \propto m^{-(c+1)} \) with \( c > 0 \) is the Pareto distribution. Third, this progressive increase in the number of segments has limited the development of new or modified PDFs. The continuous distribution approach to the IMF has been modeled by the lognormal distribution in order to fit both the range of the stars and the brown dwarfs (BDs) regime; see Chabrier (2003). Recall that standard PDFs such as the lognormal, gamma, generalized gamma, and Weibull are usually defined in the interval \( 0 \leq x < \infty \). The fact that the number of stars with mass \( m < 0.07 M_\odot \) is nearly zero suggests a left truncated PDF. Our analysis has therefore focused on the beta distribution, which by definition has an upper bound. From the previous analysis, the following questions can be raised.

1. Is it possible to find the constant of normalization for a left truncated beta PDF?
2. Is it possible to derive an analytical expression for the mean, the mode, and the distribution function (DF) of a left truncated beta PDF?
3. Is a left truncated beta PDF an acceptable model for the IMF as well as a real sample of masses?

In order to answer the above questions, we first review some standard PDFs in Section 2. We subsequently introduce the various beta PDFs, the convolution of a beta PDF with a normal PDF, and a left truncated beta in Section 3. In order to determine which PDF performs best, the two main criteria that report the goodness of fit are found in Section 4. A comparison between various continuous PDFs and the left truncated beta is carried out in Sections 5.2 and 5.3 for two samples of stars.

2. DISTRIBUTIONS COMMONLY USED

This section reviews some standard PDFs, namely, the lognormal, gamma, generalized gamma, Pareto, truncated Pareto, and the recently developed Double Pareto-lognormal distribution.

2.1. Lognormal Distribution

If \( X \) is a random variable with values for \( x \) in the interval \([0, \infty]\), then the lognormal PDF, following Evans et al. (2000) or formula (14.2) in Johnson et al. (1994), is

\[
    f_{LN} = \frac{1}{x \sigma \sqrt{2\pi}} \exp\left(-\ln\left(x/m\right)^2/2\sigma^2\right),
\]

or

\[
    f_{LN} = \frac{1}{x \sigma \sqrt{2\pi}} \exp\left(-\left(ln\left(x - \mu_{LN}\right)/\sigma^2\right)^2\right),
\]

where \( m = \exp \mu_{LN} \) and \( \mu_{LN} = \log m \).

2.2. Gamma Distribution

If \( X \) is a random variable with values for \( x \) in the interval \([0, \infty]\), then the gamma PDF is

\[
    p(x; b, c) = \left(\frac{z}{b}\right)^{c-1} e^{-z/b} / b \Gamma(c),
\]

where \( \Gamma(z) \) is the gamma function

\[
    \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt;
\]

see formula (17.1) in Johnson et al. (1994).
2.3. Generalized Gamma Distribution

If X is a random variable with values for x in the interval [0, ∞], then the generalized gamma PDF, following Evans et al. (2000), is

\[ f(x; a, b, c) = c \frac{b^a e^{-bx^c}}{\Gamma(a/c)} x^{a-1} \exp(-bx^c); \]

(5)

see formula (17.116) in Johnson et al. (1994).

2.4. The Pareto and the Truncated Pareto Distributions

If X is a random variable with values for x in the interval [a, ∞], where a > 0, then the Pareto PDF is defined by

\[ f(x; a, b) = \frac{ca^x}{x^{(a+c)}}. \]

(6)

with c > 0; see formula (20.3) in Johnson et al. (1994). The traditional Salpeter slope is therefore -(c+1). An upper truncated Pareto random variable is defined in the interval [a, b] and the corresponding PDF, following Goldstein et al. (2004), Aban et al. (2006), Zaninetti & Ferraro (2008), and White et al. (2008), is

\[ f_T(x; a, b, c) = \frac{ca^x x^{-(a+c+1)}}{1 - \left(\frac{x}{b}\right)^{a+c+1}}. \]

(7)

Their means are

\[ E(x; a, c) = \frac{ac}{c-1} \]

(8)

and

\[ E(x; a, b, c)_T = \frac{ca\left(-1 + \left(\frac{x}{b}\right)^{a+1}\right)}{(c-1)\left(-1 + \left(\frac{x}{b}\right)^{a+1}\right)}. \]

(9)

2.5. The Double Pareto-lognormal Distribution

The double Pareto-lognormal distribution was recently derived (see formula (22) in Reed & Jorgensen 2004) and has been used to fit the actual sizes of cities; see Giesen et al. (2010). Its PDF is

\[ f(x; \alpha, \beta, \mu, \sigma) = \frac{1}{2\alpha} \alpha^{\alpha} \beta^{\beta} \left(\frac{\exp\left(\frac{1}{2\alpha} (\alpha + 2) \mu - 2 \ln(x)\right)}{\pi}\right) \]

\[ \times \text{erfc} \left(\frac{1}{2\alpha} \mu + \ln(x)\right) \]

\[ \times 2 \beta \left(\frac{\alpha \sigma^2 - 2 \mu + 2 \ln(x)}{\sigma}\right) \times \left(\frac{\beta \sigma^2 - \mu + \ln(x)}{\sigma}\right)^{\frac{1}{2}} \]

\[ \times (\alpha + \beta)^{-1}, \]

(10)

where \( \alpha \) and \( \beta \) are the Pareto coefficients for the upper and the lower tail, respectively, \( \mu \) and \( \sigma \) are the lognormal body parameters, and \( \text{erfc} \) is the complementary error function. The parameters can be found by minimizing the maximum distance, \( D \), of the Kolmogorov–Smirnov (K-S) test; see Section 4.

3. VARIOUS BETA DISTRIBUTIONS

This section reviews the beta PDF defined in [0, 1], the beta with a scale PDF defined in [0, 1], and the general beta defined in [a, b]. The left truncated beta PDF defined in [a, b], with a finite value of probability at \( x = a \), is explored. The convolution of a beta distribution with a normal distribution is also discussed.

3.1. Beta Distribution

If X is a random variable with values for x in the interval [0, 1], then the beta PDF is

\[ f(x; \alpha, \beta) = \frac{x^{\alpha-1} (1-x)^{\beta-1}}{B(\alpha, \beta)}, \]

(11)

with \( \alpha > 0 \) and \( \beta > 0 \); see Evans et al. (2000) or formula (25.2) in Johnson et al. (1995). Here, B is the beta function defined by

\[ B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} \]

(12)

Its mean is

\[ E(x; \alpha, \beta) = \frac{\alpha}{\alpha + \beta}, \]

(13)

and its variance is

\[ \sigma^2(x; \alpha, \beta) = \frac{\alpha \beta}{(1 + \alpha + \beta)(\alpha + \beta)^2}. \]

(14)

see formula (25.15a) in Johnson et al. (1995). The mode is at

\[ m(x; \alpha, \beta) = \frac{\alpha - 1}{\alpha - 2 + \beta}. \]

(15)

The method of matching moments gives the following parameter estimation:

\[ \tilde{\alpha} = \tilde{x} \left(1 - \frac{\tilde{x}}{s^2} - 1\right), \]

(16)

and

\[ \tilde{\beta} = (1 - \tilde{x}) \left(1 - \tilde{x} - \frac{s^2}{s^2} - 1\right), \]

(17)

where \( \tilde{x} \) and \( s^2 \) are the mean and the variance of the sample. The DF is

\[ DF(x; \alpha, \beta) = \frac{x^{\alpha-2} F_1(\alpha, -\beta + 1; \alpha + 1; x)}{\alpha \beta (\alpha, \beta)\alpha}, \]

(18)

where \( F_1(a, b; c; z) \) is the regularized hypergeometric function (Abramowitz & Stegun 1965; von Seggern 1992; Thompson 1997; Gradshteyn et al. 2007; Olver et al. 2010).

3.2. Beta Distribution with Scale

If X is a random variable with values for x in the interval [0, b], then the beta with scale PDF is

\[ f_s(x; b, \alpha, \beta) = \frac{(\frac{x}{b})^{\alpha-1} (1 - \frac{x}{b})^{\beta-1}}{B(\alpha, \beta) b}. \]

(19)

Its expected mean is

\[ E(x; b, \alpha, \beta) = \frac{\alpha b}{\alpha + \beta}, \]

(20)

and its variance is

\[ \sigma(x; b, \alpha, \beta)^2 = \frac{\beta b^2}{(1 + \alpha + \beta)(\alpha + \beta)^2}. \]

(21)

The mode is at

\[ m(x; b, \alpha, \beta) = \frac{b (\alpha - 1)}{\beta - 2 + \alpha}. \]

(22)
The DF is
\[ \text{DF}_b(x; b, \alpha, \beta) = x^{\alpha-1} \frac{\Gamma(a, \alpha + 1)}{\alpha + 1} \]
where the constant is
\[ K = \frac{\Gamma(a + 1)}{b^{\alpha+1} \Gamma(a + \alpha + 1)} \]
and
\[ H = \frac{\Gamma(a + 1)}{b^{\alpha+1} \Gamma(a + \alpha + 1)} \]

The constant of normalization can be obtained from the integral of the beta with scale PDF as represented by Equation (19). The integral 3.194.1 of Gradshteyn et al. (2007, p. 315),
\[ \int_0^1 x^{\alpha-1} (1 + \beta x)^{-1} \, dx = \frac{\alpha \ln(x)}{\alpha + 1} \]

The constant is new and therefore cannot be found in Johnson et al. (1995), at \( x = a \) is not zero but takes the finite value
\[ f_T(a; b, \alpha, \beta) = K a^{\alpha-1} (b-a)^{\beta-1} \]

Its expected mean is
\[ E(x; a, b, \alpha, \beta)_T = \frac{b(\alpha - 1)}{\alpha - 2 + \beta} \]
and in order to exist, one must have \( m_T > a \). The DF is
\[ \text{DF}_T(x; a, b, \alpha, \beta) = \Gamma(\alpha + \beta) x^{\alpha-1} \frac{\Gamma(a, \alpha + 1)}{\alpha + 1} \]
\[ \times H_D^{-1} \frac{a^\alpha \Gamma(a + \beta - 1)}{b^\alpha \Gamma(a + \beta)} \]

The survival function is
\[ S_T(x; a, b, \alpha, \beta) = 1 - \text{DF}_T(x; a, b, \alpha, \beta) \]

A first pair for \( \hat{a} \) and \( \hat{b} \) can be obtained from those of the beta distribution with a scale as given by Equations (25) and (26). A subsequent numerical loop around the previous values gives the pair that minimizes the \( \chi^2 \).
This integral has an analytical solution for the \( \alpha \) and \( \beta \) integers; for example, when \( \alpha = 1 \) and \( \beta = 1 \),

\[
NB(z; a, b, 1, 1, \sigma) = 1/2 \text{erf} \left( \frac{1}{2} \frac{\sqrt{2a}}{\sigma} - 1/2 \frac{\sqrt{2z}}{\sigma} \right) \\
\times (a - b)^{-1} - 1/2 \text{erf} \left( \frac{1}{2} \frac{\sqrt{2b}}{\sigma} - 1/2 \frac{\sqrt{2z}}{\sigma} \right) (a - b)^{-1},
\]

where \( \text{erf} \) is the error function.

4. GOODNESS-OF-FIT TESTS

The occasional reader may question which is the best fit for the distributions analyzed here. In order to answer this question, we first introduce \( \chi^2 \), which is computed according to the formula

\[
\chi^2 = \sum_{i=1}^{n} \frac{(T_i - O_i)^2}{T_i},
\]

where \( n \) is the number of bins, \( T_i \) is the theoretical value, and \( O_i \) is the experimental value represented by the frequencies. The theoretical frequency distribution is given by

\[
T_i = N \Delta x_i \cdot p(x),
\]

where \( N \) is the number of elements of the sample, \( \Delta x_i \) is the magnitude of the size interval, and \( p(x) \) is the PDF under examination. The size of the bins, \( \Delta x_i \), is equal for each bin in the case of linear histograms, but different for each bin when logarithmic histograms are considered.

A reduced merit function \( \chi^2_{\text{red}} \) is evaluated by

\[
\chi^2_{\text{red}} = \chi^2 / \text{NF},
\]

where \( \text{NF} = n - k \) is the number of degrees of freedom, \( n \) is the number of bins, and \( k \) is the number of parameters. The goodness of the fit can be expressed by the probability \( Q \) (see Equation (15.2.12) in Press et al., 1992), which involves the degrees of freedom and the \( \chi^2 \). According to Press et al. (1992), the fit “may be acceptable” if \( Q > 0.001 \). The Akaike information criterion (AIC) [see Akaike (1974)], is defined by

\[
\text{AIC} = 2k - 2 \ln(L),
\]

where \( L \) is the likelihood function and \( k \) is the number of free parameters in the model. We assume a Gaussian distribution for the errors, and the likelihood function can be derived from the \( \chi^2 \) statistic \( L \propto \exp(-\frac{\chi^2}{2}) \), where \( \chi^2 \) has been computed by Equation (47); see Liddle (2004) and Godlowski \\& Szydowski (2005). Now, the AIC becomes

\[
\text{AIC} = 2k + \chi^2.
\]

We also perform the K-S test (see Kolmogoroff 1941; Smirnov 1948; and Massey 1951), which does not require binning the data. The K-S test, as implemented by the FORTRAN subroutine KSONE in Press et al. (1992), finds the maximum distance, \( D \), between the theoretical and the astronomical DF as well the significance level \( P_{KS} \); see formulae 14.3.5 and 14.3.9 in Press et al. (1992). The values of \( P_{KS} \geq 0.1 \) assure us that the fit is acceptable.

5. ASTROPHYSICAL APPLICATIONS

This section reviews the galactic IMF as modeled by three and four power-law PDFs and fits the masses of the cluster NGC 2362 and the cluster NGC 6611 with the various PDFs considered here.

5.1. Galactic IMF

The IMF is usually modeled by two or three power laws of the type

\[
p_{\text{stars}}(m) \propto m^{-\alpha},
\]

with each zone being characterized by a different exponent \( \alpha_i \). In order to have a PDF normalized to unity, one must have

\[
\sum_{i=1}^{n} \int_{m_i}^{m_{i+1}} c_i m^{-\alpha_i} \, dm = 1,
\]

For example, we start with \( c_1 = 1 \). \( c_2 \) will be determined by the following equation:

\[
c_1(0.5 - \epsilon)^{-\alpha_1} = c_2(0.5 + \epsilon)^{-\alpha_2},
\]

where \( \epsilon \) is a small number, e.g., \( \epsilon = 10^{-4} \). In the previous equation, we insert \( \alpha_1 = 1.3 \) and \( \alpha_2 = 2.3 \), and therefore \( c_2 = 0.503 \). The same procedure applied to \( c_3 \) gives \( c_3 = 0.506 \). The integral of \( p_{\text{stars}}(m) \) over the field of existence now gives 4.14, but, according to the requirement of normalization as given by Equation (53), it should be 1. Consequently, the three constants are now \( c_1 = 0.24 \), \( c_2 = 0.1205 \), and \( c_3 = 0.1206 \), which is the same as Equation (59) in Kroupa et al. (2012)

\[
p(m) = \begin{cases} 
0.24 m^{-1.3} & \text{if } 0.07 M_\odot < m \leq 0.5 M_\odot \\
0.12 m^{-2.3} & \text{if } 0.5 M_\odot < m \leq 1.0 M_\odot \\
0.12 m^{-2.7} & \text{if } 1.0 M_\odot < m \leq 10 M_\odot.
\end{cases}
\]

The mean of the galactic IMF is given by a numerical integration over the three zones

\[
\bar{m} = \sum_{i=1,3} \int_{m_i}^{m_{i+1}} c_i m^{-\alpha_i} \, dm = 0.389 M_\odot.
\]

The presence of the BDs means that we use four power laws instead of three power laws:

\[
p(m) = \begin{cases} 
2.194 m^{-0.3} & \text{if } 0.01 M_\odot < m \leq 0.07 M_\odot \\
0.153 m^{-1.3} & \text{if } 0.07 M_\odot < m \leq 0.5 M_\odot \\
0.076 m^{-2.3} & \text{if } 0.5 M_\odot < m \leq 1.0 M_\odot \\
0.076 m^{-2.7} & \text{if } 1.0 M_\odot < m \leq 10 M_\odot.
\end{cases}
\]

where, in order to have a continuous PDF, the BDs have the range \( 0.01 M_\odot < m \leq 0.07 M_\odot \) rather than \( 0.01 M_\odot < m \leq 0.15 M_\odot \); see Equation (59) in Kroupa et al. (2012). Now that we have covered the galactic four power laws, we introduce the generalized four power laws \( p_G(m; -\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4, m_1, m_2, m_3, m_4, m_5) \), which in the case of NGC 2362 are

\[
p_G(m; -0.01, -0.02, -1.1, -2.7, 0.01, 0.01, 0.07, 0.50, 1.0, 10) \hspace{1cm} \text{NGC 2362 case},
\]

and in the case of NGC 6611 are

\[
p_G(m; -0.01, -0.6, -2.4, -2.7, 0.01, 0.01, 0.07, 0.50, 1.0, 10) \hspace{1cm} \text{NGC 6611 case}.
\]
Table 1

| PDF                  | Parameters | AIC  | $\chi^2_{\text{red}}$ | $Q$     | $D$     | $P_{\text{KS}}$ |
|----------------------|------------|------|------------------------|---------|---------|-----------------|
| Lognormal            | $\sigma = 0.5$, $\mu_{\text{LN}} = -0.55$ | 37.64 | 1.86                   | 0.013   | 0.07305 | 0.10486         |
| Double Pareto-lognormal | $\sigma = 0.44$, $\mu_{\text{LN}} = -0.52$ | 40.42 | 2.02                   | 0.008   | 0.066103 | 0.17882         |
| General beta         | $a = 0.12$, $b = 1.47$ | 29.09 | 1.31                   | 0.17    | 0.059141 | 0.288813        |
| General beta +normal (NB) | $a = 0.12$, $b = 1.47$, $\alpha = 1.67$, $\beta = 2.77$, $\sigma = 0.001$ | 31.09 | 1.40                   | 0.13    | 0.06412  | 0.20612         |
| Left truncated beta  | $a = 0.12$, $b = 1.47$, $\alpha = 2.23$, $\beta = 3.09$ | 31.19 | 1.44                   | 0.1     | 0.06158  | 0.24572         |
| Four power laws      | Equation (58) | 77.608 | 4.89                   | 1.17 $\times 10^{-8}$ | 0.16941 | 2.60363 $\times 10^{-7}$ |

Note. The number of linear bins, $n$, is 20.

5.2. IMF of NGC 2362

A photometric survey of NGC 2362 allows us to deduce the mass of 271 stars in the range $1.47 M_\odot > M \geq 0.11 M_\odot$; see Irwin et al. (2008) and the data in J/MNRAS/384/675 at the Centre de Données astronomiques de Strasbourg (CDS). Table 1 shows the values of $\chi^2_{\text{red}}$, the AIC, and the probability $Q$ for the astrophysical fits and the results of the K-S test.

Figure 1 shows the fit for the left truncated beta distribution of NGC 2362 and Figure 2 visually compares the four types of fits for NGC 2362.

5.3. IMF of NGC 6611

The massive young cluster NGC 6611 has been carefully analyzed from the point view of the IMF in the range $1.5 M_\odot > M \geq 0.02 M_\odot$. This means that the BD range is also covered; see Oliveira et al. (2009) for more details for data in J/MNRAS/392/1034 at the CDS. Figure 3 shows the fit with the left truncated beta distribution of NGC 6611 and Figure 4 shows a visual comparison of four types of fits for NGC 6611. Table 2 shows the values of $\chi^2_{\text{red}}$, the AIC, and the probability $Q$ of the

Figure 2. Histogram (step-diagram) of mass distribution as given by NGC 2362 cluster data (272 stars) with a superposition of the left truncated beta distribution (full line), the lognormal (dashed), the double Pareto lognormal (dotted), and the four power laws (dot-dash-dot-dash). The vertical and horizontal axes have logarithmic scales.

Figure 3. Logarithmic histogram of the mass distribution as given by NGC 6611 cluster data (207 stars + BDs) with a superposition of the left truncated beta distribution when the number of bins, $n$, are 12, $a = 0.019$, $b = 1.46$, $\alpha = 0.55$, and $\beta = 1.6$. The vertical and horizontal axes have logarithmic scales.
Table 2

Numerical Values of $\chi^2_{\text{red}}$, AIC, Probability $Q$, $D$, the Maximum Distance between Theoretical and Observed DF, and $P_{\text{KS}}$, Significance Level, in the K-S Test for the Mass Distribution of NGC 6611 Cluster Data (207 Stars + BDs)

| PDF                  | Parameters | AIC    | $\chi^2_{\text{red}}$ | $Q$    | $D$    | $P_{\text{KS}}$ |
|----------------------|------------|--------|------------------------|--------|--------|-----------------|
| Lognormal            | $\sigma = 1.029$, $\mu_{\text{LN}} = -1.258$ | 71.24  | 3.73                   | $1.3 \times 10^{-7}$ | 0.09366 | 0.04959         |
| Double Pareto-lognormal | $\sigma = 0.979$, $\mu_{\text{LN}} = -1.208$ | 70.3   | 3.89                   | $2.13 \times 10^{-7}$ | 0.07995 | 0.13523         |
| General beta         | $a = 0.019$, $b = 1.46$ | 39.29  | 1.956                  | 0.0123 | 0.11456 | 0.007924        |
| General beta +normal (NB) | $a = 0.019$, $b = 1.46$, $\sigma = 0.001$ | 41.3   | 2.08                   | 0.008  | 0.09476 | 0.04545         |
| Left truncated beta  | $a = 0.019$, $b = 1.46$ | 42.09  | 2.13                   | 0.005  | 0.06839 | 0.27781         |
| Four power laws      | Equation (59) | 81.39  | 5.18                   | $2.41 \times 10^{-9}$ | 0.12514 | $2.7239 \times 10^{-3}$ |

Note. The number of linear bins, $n$, is 20.

Figure 4. Histogram (step-diagram) of the mass distribution as given by NGC 6611 cluster data (207 stars + BDs) with a superposition of the left truncated beta distribution (full line), the lognormal (dashed), the double Pareto lognormal (dotted), and the four power laws (dot-dash-dot-dash). The vertical and horizontal axes have logarithmic scales.

Figure 5. Residuals of the fits to NGC 6611 cluster data when 12 logarithmic bins are considered. The empty stars represent the left truncated beta PDF and the filled triangles the lognormal PDF.

astrophysical fits and the results of the K-S test. Figure 5 shows the residuals and $\chi^2$ as a function of the middle value of the logarithmic bin considered, both for the left truncated beta and for the lognormal.

6. CONCLUSIONS

Motivations. In the last 50 years, the IMF has been modeled progressively by one power law, two power laws, three power laws, and four power laws. The three power-law distribution has seven parameters and the four power law has nine, and they both have a finite range of existence. A second widely used fitting function is the lognormal, which is characterized by two parameters and is defined in the interval $[0, \infty]$. In this paper, we have described a left truncated beta PDF that has (1) a lower and an upper bound, and (2) a finite value of probability on the lower bound rather than zero, (3) two parameters, $\alpha$ and $\beta$, which fix the shape of the distribution, and (4) an analytical expression for the average value. Two physical meanings are distinguished: (1) the upper limit of the left truncated beta is connected with the maximum stellar mass, which is $\approx 60 M_\odot$, and (2) the lower limit is connected with an unknown physical mechanism that limits the mass distribution. Further on, we recall that the lognormal PDF has the important disadvantage of missing the well-accepted Salpeter-type high-mass power law. The high-mass behavior of the various PDFs analyzed here is reported in Figure 6 for NGC 6611, where the Pareto and truncated Pareto PDFs are evaluated for $M \geq 0.43 M_\odot$, which...
means a Salpeter slope of $-2.3$. From the previous figure, the discrepancy between the lognormal and double Pareto-lognormal at high masses is evident.

**Goodness-of-fit tests.** The statistical tests performed here are split into two: (1) the first test requires binning the data in order to evaluate $\chi^2$, and the indicators are $\chi^2_{\text{red}}$, the AIC, and the probability $Q$; (2) the K-S test does not require binning the data, and the two indicators are $D$ and $P_{\text{KS}}$. These two tests, when applied to NGC 2362 and NGC 6611, indicate that the beta family (general and left truncated) performs better than the lognormal distribution, and when the binning of the data is computed (see tables example) the K-S test for the mass distribution of NGC 6611 indicates a confidence level of $27\%$ for the left truncated beta and $5\%$ for the lognormal. New confidence levels can be found with the Anderson-Darling test, which is a modification of the K-S test; see Stephens (1974) and the discussion at https://asaip psu.edu/Articles/beware-the-kolmogorov-smirnov-test. Currently, tables of critical values for the Anderson-Darling test are available for the lognormal PDF, but the critical values for other PDFs explored here are not yet available; see http://www.itl.nist.gov/div898/handbook.

**Convolution.** The random sum (convolution) of a general beta and a normal random variable, as represented by Equation (45), when applied to NGC 6611, introduces an additional parameter, $\sigma$, which increases $\chi^2_{\text{red}}$ and the AIC compared to that of the general beta, but decreases $D$ and $P_{\text{KS}}$ in the K-S test; see Table 2.

The mode. Careful attention should be paid to the falloff of the IMF toward the BDs. The left truncated beta PDF (see PDF (34)) once the numbers of the open cluster NGC 6611 are inserted (see Figure 3) decreases after the maximum at $m \approx 0.019 M_\odot$. This fact can be explained by the following Taylor expansion:

$$f_T(x; 0.019, 1.46, 0.715, 2.185) = 2.65 - 21.64(x - 0.039) + 336.35(x - 0.039)^2 + O((x - 0.039)^3).$$

The previous decreasing function converts itself into an increasing function when the integration is performed

$$\int_{0.019}^{x} f_T(x; 0.019, 1.46, 0.715, 2.185) = 3.50x - 10.82x^2 + 112.11(x - 0.039)^3,$$

and we recall that the evaluation of the frequencies corresponds to an integration.

**Lognormal family.** The recently formulated double Pareto-lognormal distribution draws attention to a possible alternative to the lognormal. Our tests show that the double Pareto-lognormal lowers the value of the maximum distance, $D$, of the K-S test; see Tables 1 and 2. Inconveniently, at the moment there are no analytical evaluations of the four parameters that characterize the double Pareto-lognormal.

The astronomical sample. The new PDFs presented here can be tested on an astronomical sample representative of the IMF. Currently, not all of the various catalogs available on CDS report the mass of the column. As an example, the promising IMF of IC 348 (see Figure 11 in Alves de Oliveira et al. 2012) is not available on CDS.

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