Higher symmetries of symplectic Dirac operator

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Abstract

We construct in projective differential geometry of the real dimension 2 higher symmetry algebra of the symplectic Dirac operator \( D_s \) acting on symplectic spinors. The higher symmetry differential operators correspond to the solution space of a class of projectively invariant overdetermined operators of arbitrarily high order acting on symmetric tensors. The higher symmetry algebra structure corresponds to a completely prime primitive ideal having as its associated variety the minimal nilpotent orbit of \( \mathfrak{sl}(3, \mathbb{R}) \).

Key words: Symplectic Dirac operator, Higher symmetry algebra, Projective differential geometry, Minimal nilpotent orbit, \( \mathfrak{sl}(3, \mathbb{R}) \).

MSC classification: 53D05, 35Q41, 58D19, 17B08, 53A20.

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1 Introduction

It is always desirable to convert a purely algebraic construct into its geometrical realization with the hope to gain its better understanding, as well as a potential generalization of the former algebraic structure. The present article is an example of this phenomenon: the algebraic structure is the algebra of higher symmetries of the symplectic Dirac operator $\mathcal{D}_s$ (cf. the seminal work [13]) realized in projective differential geometry of the real dimension two. This algebra corresponds to a completely prime primitive ideal which has as its associated variety the minimal nilpotent orbit of the complexification of $\mathfrak{sl}(3, \mathbb{R})$, while the geometric realization pursued in our article relies on the use of certain class of projectively invariant systems of differential equations of arbitrarily high order together with the convenient calculus of symmetric powers of projective adjoint tractor bundle. Consequently, our geometric construction can be generalized to any smooth manifold equipped with a projective differential structure, thereby making contact with the appearance of linear and bilinear differential invariants for a given (curved) projective structure.

There is a well established notion of higher symmetry operators for a system of partial differential equations given by differential operators which preserve its solution space, see e.g. [17], [18], [19]. In the case when the symmetry algebra is derived from a semi-simple Lie algebra, there is its direct relationship to the fundamental structural properties of semi-simple Lie algebra including ideal structure in its universal enveloping algebra, its adjoint group orbit structure, etc. The geometric approach, allowing to construct these algebraic invariants locally on manifolds with a geometric structure, has been successfully completed in the case of Laplace operator and conformal differential structure, [7], and since then many other cases were treated (cf. the bibliography and extensive references therein.)

The present article is devoted to analogous questions in the case of symplectic Dirac operator $\mathcal{D}_s$ in the real dimension two. The first order differential symmetries of $\mathcal{D}_s$ were already identified with $\mathfrak{sl}(3, \mathbb{R})$, cf. [2]. As we shall observe, the algebra of higher symmetries leads to projective differential structure and the minimal nilpotent orbit of the complexification of $\mathfrak{sl}(3, \mathbb{R})$. We shall construct the vector space
of symmetries as the solution space of a class of projectively invariant overdetermined systems of arbitrarily high order (a substantial difference from the known cases with a uniform bound on the orders of higher symmetry differential operators) studied on a locally flat projective manifold. The most convenient geometrical language allowing a uniform treatment is based on tractor bundles for projective parabolic geometries, cf. [4]. Projectively invariant cup product allows to introduce associative algebra structure on higher symmetries of \( \mathcal{D}_a \), and leads to an identification of the higher symmetry algebra with an ideal in the universal enveloping algebra of the complexification of \( \mathfrak{sl}(3, \mathbb{R}) \) given by a quantization of the coordinate ring of its minimal nilpotent orbit.

2 Summary of the results

The present section is a brief and non-technical summary of the content of our article. Throughout the article we work over a smooth manifold \( M \) equipped with projective geometric structure. We shall focus particularly on the real dimension 2, where in addition we assume the structure group to be the double cover of \( SL(2, \mathbb{R}) \cong Sp(2, \mathbb{R}) \). (This is needed to introduce the symplectic spinors.) We shall use the Penrose abstract indices \( a, b, \ldots \) for tensor bundles, together with the Einstein summation convention understood. That is, \( \mathcal{E}_a = T^*M, \mathcal{E}^a = TM, \mathcal{E}^{ab} = \otimes^2 TM \) etc., and we shall use the same notation for the spaces of sections. Symmetric tensor products will be denoted by round brackets, e.g. \( S^2TM = \mathcal{E}^{(ab)} \). We shall use suitable projectively invariant calculus based on the notion of Cartan geometries, and we refer to [4] for details. We use the notation \( \mathbb{N} \) for natural numbers \( 1, 2, \ldots \), while \( \mathbb{N}_0 \) for \( 0, 1, 2, \ldots \). The abbreviation ”lot” denotes lower order terms of a given differential operator with a fixed order and symbol, and all considered operators are differential. We denote by \( tf \) the trace-free part of the corresponding bundle. For example, \( tf(\mathcal{E}^a_b) \) denotes the bundle of trace-free endomorphisms of \( TM \). Though we are interested mainly in \( M \) of real dimension 2, we shall discuss relevant projectively invariant overdetermined operators in any dimension.

We shall introduce the symplectic Dirac operator in the case of real even dimension \( 2n \geq 2 \). This means that we start with \((M, \omega, \nabla)\), a smooth manifold \( M \) equipped with symplectic 2-form \( \omega_{ab} \) and symplectic covariant derivative \( \nabla \) fulfilling \( \nabla_a \omega_{bc} = 0 \). The symplectic
2-form $\omega_{ab}$ allows the identification of $TM$ with $T^*M$. A double cover of the symplectic structure gives rise to the associated (the typical fiber being infinite dimensional) bundle of symplectic spinors $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$ induced from the Segal-Shale-Weil representation, [13]. As was already mentioned, by abuse of notation we denote by $\mathcal{S}$ also the space of sections of $\mathcal{S}$. The symplectic Clifford algebra can be realized via symplectic gamma-matrices $\gamma_a \in \mathcal{E}_a \otimes \text{End} \mathcal{S}$, satisfying

$$\gamma_a \gamma_b - \gamma_b \gamma_a = 2\omega_{ab}. \tag{2.1}$$

Here $\nu^a \gamma_a : \mathcal{S}_\pm \to \mathcal{S}_\mp$ for any vector field $\nu^a \in \mathcal{E}^a$. Then the symplectic Dirac operator $\slashed{D}_s$ (cf., [11, 5]) is defined as

$$\omega^{rs} \gamma_s \nabla_r : \mathcal{S}_\pm \to \mathcal{S}_\mp.$$

We refer to [14] for a thorough discussion of the notion of convenient analysis and differential operators acting on sections of infinite dimensional bundles over finite dimensional smooth manifolds.

Denoting the space of differential operators acting on $\mathcal{S}$ by $\text{Diff}(\mathcal{S})$, we term $O \in \text{Diff}(\mathcal{S})$ a higher symmetry or briefly a symmetry of $\slashed{D}_s$ provided there exists another differential operator $O' \in \text{Diff}(\mathcal{S})$ such that

$$\slashed{D}_s O = O' \slashed{D}_s. \tag{2.2}$$

Operators of the form $O = T \slashed{D}_s$ for some differential operator $T \in \text{Diff}(\mathcal{S})$ are called trivial symmetries. Since the composition of symmetries is also a symmetry, the vector space of symmetries is an algebra denoted $\mathcal{A}'$ together with the ideal $\mathcal{A}'' \subset \mathcal{A}'$ of trivial symmetries. The aim of our article is to understand the quotient-algebra $\mathcal{A} := \mathcal{A}'/\mathcal{A}''$ in the case of $\slashed{D}_s$. The filtration $\text{Diff}^k(\mathcal{S}) \subseteq \text{Diff}(\mathcal{S})$ by order of operators induces filtration $\mathcal{A}^k = \mathcal{A} \cap \text{Diff}^k(\mathcal{S})$ on the algebra $\mathcal{A}$. In dimension 2 and for the flat connection $\nabla$, a direct computation shows that the Lie algebra of first order symmetries of $\slashed{D}_s$ is isomorphic to $\mathfrak{sl}(3, \mathbb{R})$, [2]. Further, invariance of $\slashed{D}_s$ with respect to the projective geometrical structure was recognized in [15]. This conclusion does not apply in higher dimensions $2n \geq 4$ and not much is known in this direction.

Assume the real dimension equals to 2. Then we can extend the symplectic structure $(M, \omega, \nabla)$ to the projective class of connections $[\nabla]$. (Notice the choice $\hat{\nabla} \in [\nabla]$ uniquely determines a symplectic form $\hat{\omega}$ such that $\hat{\nabla}_a \hat{\omega}_{ab} = 0$.) Then the Lie algebra of first order symmetries of $\slashed{D}_s$ is indeed the algebra of infinitesimal projective symmetries
for \((M, [\nabla])\). Further, we shall need density bundles \(\mathcal{E}(w)\) for a weight \(w \in \mathbb{R}\) with standard notational convention in the projective geometry. That is, the bundle of volume forms is isomorphic to \(\mathcal{E}(-3)\). In analogy to [8], the bundle of suitably weighted projective frames (or rather its double cover) leads to the associated bundle of projective/symplectic spinors denoted by \(\mathcal{S}\) again (see section 3.3 for details). We put \(\mathcal{S}_\pm(w) := \mathcal{S}_\pm \otimes \mathcal{E}(w)\). Then we have a projectively invariant version of the symplectic Dirac operator,

\[
\mathcal{D}_s := \omega^{rs} \gamma^s \nabla_r : \mathcal{S}_\pm(-\frac{3}{4}) \to \mathcal{S}_\mp(-\frac{9}{4}),
\]

(2.3)
in the sense that \(\omega^{rs} \gamma^s \nabla_r\) does not depend on the choice of \(\nabla \in [\nabla]\) for this particular choice of weights.

Results in [15] and [2] presume the existence of a flat connection in the projective class \([\nabla]\). This corresponds to the notion of projectively flat structures, see section 3.2 for a precise definition. Indeed, characterizing symmetries of \(\mathcal{D}_s\) on curved projective manifolds is much more complicated problem, hence we assume \((M, [\nabla])\) is projectively flat for the rest of this section.

Principal symbols of operators in \(\text{Diff}^k(\mathcal{S}(w))\) are sections of \(S^kTM \otimes \text{End}(\mathcal{S})\), where \(\text{End}(\mathcal{S})\) is the bundle of symplectic Clifford algebras with the typical fiber isomorphic to \((3.1)\). Assuming \(\mathcal{O} \in \text{Diff}^k(\mathcal{S})\) is a symmetry of \(\mathcal{D}_s\), we first observe that, modulo \(\mathcal{D}_s\), the principal symbol of \(\mathcal{O}\) is a section of \(S^kTM = (S^kTM \otimes 1) \subseteq S^kTM \otimes \text{End}(\mathcal{S})\), see Corollary [12]. Moreover, given a section \(\sigma \in S^kTM\), we construct a canonical operator \(\mathcal{O}^\sigma \in \text{Diff}^k(\mathcal{S}(w))\) with the principal symbol \(\sigma\) for any \(w \in \mathbb{R}\), cf. [3.27]. Another ingredient are the differential constraints for principal symbols \(\sigma \in S^kTM\) of symmetries of \(\mathcal{D}_s\). They are provided by a family of overdetermined projectively invariant differential operators \(\Phi_k\) acting on \(S^kTM\) and introduced in section 3.2, cf. [3.15]. Now we are ready to characterize the algebra of symmetries \(\mathcal{A}\) of \(\mathcal{D}_s\) as a vector space:

**Theorem 1.** (i) Given \(\sigma \in S^kTM\), the operator \(\mathcal{O}^\sigma\) is a symmetry of \(\mathcal{D}_s\) if and only if \(\Phi_k(\sigma) = 0\).

(ii) The operator \(\mathcal{O} \in \text{Diff}^k(\mathcal{S})\) is a symmetry of \(\mathcal{D}_s\) if and only if \(\mathcal{O}\) is, modulo trivial symmetries, of the form

\[
\mathcal{O} = \mathcal{O}^{\sigma[k]} + \mathcal{O}^{\sigma[k-1]} + \ldots + \mathcal{O}^{\sigma[0]} : \mathcal{S}(-\frac{3}{4}) \to \mathcal{S}(-\frac{9}{4})
\]

where \(\sigma[i] \in S^iTM\) and \(\Phi_i(\sigma[i]) = 0\) for all \(i = 0, \ldots, k\). This in particular means that \(\mathcal{O}\), modulo a trivial symmetry, preserves the decomposition \(\mathcal{S}(-\frac{3}{4}) = \mathcal{S}_+(-\frac{3}{4}) \oplus \mathcal{S}_-(-\frac{3}{4})\).
This theorem follows from Theorem 13 and the discussion before this theorem.

The zero order symmetries are given by multiplication with a constant zero order differential operators, i.e. $A^0 \cong \mathbb{R}$. Further, we write explicitly the first order symmetries (modulo constants): an operator $O \in A^1/A^0$ is a symmetry of $\mathcal{D}_s$ if and only if

\begin{align*}
O &= \sigma^r \nabla_r + \frac{1}{4}(\nabla_r \sigma^s)\gamma^r \gamma_s + \frac{1}{3}(\nabla_r v^r), \\
O' &= \sigma^r \nabla_r + \frac{1}{4}(\nabla_r \sigma^s)\gamma^r \gamma_s + \frac{5}{6}(\nabla_r v^r)
\end{align*}

modulo trivial symmetries. Here the symbol $\sigma^a$ of $O, O'$ satisfies projectively invariant overdetermined differential system $\Phi_1(\sigma) = 0$:

\begin{equation*}
\Phi_1(\sigma) = tf((\nabla_a \nabla_b)\sigma^c + \text{Ric}_{ab}\sigma^c),
\end{equation*}

with $\text{Ric}_{ab}$ the Ricci tensor of $\nabla$. That is, principal symbols of first order symmetries (modulo constants $A^0$) are exactly infinitesimal projective symmetries isomorphic to the Lie algebra $\mathfrak{sl}(3, \mathbb{R})$. Of course, this is expected since $\mathcal{D}_s$ is a projectively invariant differential operator, cf. [15].

The algebra structure on $A$ is the content of our second main result proved in Section 5.

**Theorem 2.** The algebra of symmetries $A$ of $\mathcal{D}_s$ is isomorphic to the quotient of the tensor algebra $\bigoplus_{k=0}^{\infty} \otimes^k(\mathfrak{sl}(3, \mathbb{R}))$ by a two sided ideal generated by quadratic relations

\begin{equation*}
I \otimes \bar{I} - I \boxtimes \bar{I} - \frac{1}{2}[I, \bar{I}] + \frac{1}{32}(I, \bar{I})_K
\end{equation*}

for $I, \bar{I} \in \mathfrak{sl}(3, \mathbb{R})$. Here $[ , ]$ and $\langle , \rangle_K$ are the Lie bracket and the Killing form on $\mathfrak{sl}(3, \mathbb{R})$, respectively, and $\boxtimes$ denotes the Cartan product.

Equivalently, $A$ is the quotient of the universal enveloping algebra $U(\mathfrak{sl}(3, \mathbb{R}))$ by a two sided ideal generated by quadratic relations

\begin{equation*}
I\bar{I} + \bar{I}I - 2I \boxtimes \bar{I} + \frac{1}{16}(I, \bar{I})_K.
\end{equation*}

The ideal defined on the last display corresponds to a completely prime primitive ideal having as its associated variety the minimal nilpotent orbit of the complexification of $\mathfrak{sl}(3, \mathbb{R})$, cf. [12] [21] [10] [9]. The present article can be regarded as a geometric construction of this exceptional ideal.
3 Symplectic spinors and projective geometry

3.1 Symplectic spinors

We present basic algebraic preliminaries related to the construction and realization of symplectic spinors, cf. [11, 5]. The metaplectic Lie group $Mp(2n, \mathbb{R})$ is the non-trivial double covering of the symplectic Lie group $Sp(2n, \mathbb{R})$ of automorphisms of the standard symplectic space $(\mathbb{R}^{2n}, \omega)$. Their Lie algebras are denoted by $mp(2n, \mathbb{R})$ and $sp(2n, \mathbb{R})$, respectively. The symplectic spinor representation for the metaplectic Lie algebra $mp(2n, \mathbb{R})$ is given by two simple metaplectic modules of the Segal-Shale-Weil representation, modeled on the vector space of polynomials on a Lagrangian subspace $\mathbb{R}^n$ of $(\mathbb{R}^{2n}, \omega)$.

The $\mathbb{C}$-algebra of endomorphisms of their direct sum is called symplectic Clifford algebra (or, the Weyl algebra) and is isomorphic to the quotient of the tensor algebra $T(\mathbb{R}^{2n})$ by a two sided ideal $I$:

$$ Cl_s(\mathbb{R}^{2n}, \omega) := T(\mathbb{R}^{2n})/I, $$

$$ I = \{ e_i e_j - e_j e_i = 2 \omega_{ij} | i,j = 1, \ldots, 2n, e_i, e_j \in \mathbb{R}^{2n} \}. $$

We denote by $e_1, \ldots, e_{2n}$ a basis of $\mathbb{R}^{2n}$ and $\omega_{ab} = \omega(e_a, e_b)$.

The inverse of the symplectic 2-form $\omega_{ab}$ is denoted $\omega^{ab}$, and we use the convention $\omega^{ja} \omega_{jb} = \delta^a_b$ with the summation over $j = 1, \ldots, 2n$ understood. The composition of the isomorphisms $\omega_{ab} : TM \to T^*M$, $v_a \mapsto \omega^{aj} v_j$, and $\omega^{ab} : T^*M \to TM$, $v^a \mapsto \omega_{ja} v^j$, is then the identity endomorphism of $TM$:

$$ v_a \mapsto \omega^{aj} v_j \mapsto \omega_{ka} \omega^{kj} v_j = v_a. $$

This is equivalent to $\omega^{ab} = -\delta^a_b$, $\omega^a_b = \delta^a_b$. As for the scalar product induced by the symplectic form, we have $v^j w_j = \omega_{jk} v_k \omega^{lj} w^l$ so that $v^j w_j = -v^j w_j$.

We denote basis elements $e_1, \ldots, e_{2n}$ of $\mathbb{R}^{2n}$ by $\gamma_1, \ldots, \gamma_{2n}$ when regarded as generators of $Cl_s(\mathbb{R}^{2n}, \omega)$, i.e., $\gamma_i \in T^*M \otimes Cl_s(\mathbb{R}^{2n}, \omega)$ for all $i = 1, \ldots, 2n$. Then

$$ \gamma_i \gamma_j - \gamma_j \gamma_i = 2 \omega_{ij}, \quad \gamma_i \gamma_j = \omega_{ij}, $$

where their traces are

$$ \gamma^k \gamma_k = -2n, \quad \gamma_k \gamma^k = 2n, $$

(3.3)
and the commutator with symmetrized product of $\gamma$'s is
\[
\gamma^a \gamma^{(a_1 \ldots a_j)} - \gamma^{(a_1 \ldots a_j)} \gamma^a = 2 j \omega^{a(a_1 \gamma^{a_2} \ldots \gamma^{a_j})}
\] (3.4)
for any $j \in \mathbb{N}$. As already indicated, in the present article we are mostly concerned with the case $n = 1$.

We have the inclusion $\mathfrak{sp}(2n, \mathbb{R}) \cong S^2 \mathbb{R}^{2n} \subseteq Cl_s(\mathbb{R}^{2n}, \omega)$ realized by quadratic elements in the generators $\mathbb{R}^{2n} \subseteq Cl_s(\mathbb{R}^{2n}, \omega)$. This promotes to the bundle level as follows: the section $F^{ab} \in \mathcal{E}^{(ab)}$ acts on $\varphi \in \mathcal{S}$ by
\[
\varphi \mapsto F(\varphi) = -\frac{1}{4} F^{ab} \gamma_a \gamma_b \varphi,
\]
where the coefficient $\frac{1}{4}$ ensures this indeed corresponds to the Lie algebra representation. If $n = 1$, this can be expressed as the action of endomorphisms $F^a_b \in \mathfrak{t}f(\mathcal{E}^{ab})$,
\[
\varphi \mapsto \frac{1}{4} F^a_b \gamma_a \gamma_b \varphi.
\] (3.5)

The symplectic Dirac operator $D_s$ acts on symplectic spinors induced from the half-integral Segal-Shale-Weil $SL(2, \mathbb{R})$-representation, \[13\], \[11\]. It was introduced in the seminal work \[13\] for the purpose of geometric quantization on any symplectic manifold with a metaplectic structure. In (2.3), the algebraic map $TM \otimes \mathcal{S}^\pm \rightarrow \mathcal{S}^\mp$ follows from the embedding of $TM$ into the bundle of symplectic Clifford algebras, and $\nabla$ denotes the induced symplectic covariant derivative on $\mathcal{S}^\pm$.

3.2 Projective geometry and tractor calculus

Projective structure on a smooth manifold $M$ of real dimension greater than equal to 2 is a class $[\nabla]$ of torsion-free volume preserving connections, which define the same family of unparametrized geodesics. A connection is projectively flat if and only if it is locally equivalent to a flat connection, which means there exists a local isomorphism with the flat model of $n$-dimensional homogeneous projective geometry on $\mathbb{R}^n$ equipped with the flat projective structure given by the absolute parallelism. The homogeneous model of projective geometry in the real dimension 2 is $\mathbb{R}P^2 \cong G/P$, where $G \simeq SL(3, \mathbb{R})$ and $P \subset G$ the parabolic subgroup stabilizing the line $[v] \in \mathbb{R}^3$ generated by a non-zero vector $v$ in the defining representation $\mathbb{R}^3$ of $G$. The construction of associated vector bundles induced from half integral modules of $P$ (e.g., the simple metaplectic submodules of the Segal-Shale-Weil representation for the metaplectic group) on
\( G/P \) requires the double (universal) cover \( \tilde{G} = \tilde{SL}(3, \mathbb{R}) \) and its parabolic subgroup \( \tilde{P} \). The Lie group \( \tilde{SL}(3, \mathbb{R}) \) acts transitively on \( S^2 \cong \mathbb{C}P^1 \), the double (universal) cover of \( \mathbb{R}P^2 \), with parabolic stabilizer \( \tilde{P} = (GL(1, \mathbb{R}) + \tilde{SL}(2, \mathbb{R})) \rtimes \mathbb{R}^2 \). The double (universal) cover \( \tilde{SL}(3, \mathbb{R})/\tilde{P} \cong S^2 \cong \mathbb{C}P^1 \) is a symplectic manifold, while \( \mathbb{R}P^2 \) is non-orientable and hence not symplectic.

Further, we follow conventions for projective structures as in [1]. Recall the notation \( E(w) \) for density bundles, \( w \in \mathbb{C} \). The difference between two connections \( \nabla, \hat{\nabla} \in [\nabla] \) for a given projective structure \( [\nabla] \) is controlled by a one-form \( \Upsilon_a := \nabla_a \log(f), f \in \mathcal{E} \equiv \mathcal{C}^\infty(M) \).

Specifically, we have
\[
\hat{\nabla}_a \alpha = \nabla_a \alpha + w \Upsilon_a, \quad \alpha \in \mathcal{E}(w),
\]
\[
\hat{\nabla}_a V^b = \nabla_a V^b + \Upsilon_a V^b + \Upsilon_c V^c \delta^b_a, \quad V^b \in \mathcal{E}^b,
\]
\[
\hat{\nabla}_a \mu_b = \nabla_a \mu_b - \Upsilon_a \mu_b - \Upsilon_b \mu_a, \quad \mu_a \in \mathcal{E}_a.
\]

The curvature tensor \( R_{abcd} \) of \( \nabla \) is defined by \( (\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = R_{abcd} V^d \) and it decomposes as
\[
R_{abcd} = W_{abcd} + 2 \delta[a]_{P[b]c},
\]
where the Schouten tensor \( P_{ab} \) is symmetric. Note the Ricci tensor equals to \( \text{Ric}_{ab} = (n-1)P_{ab} \). Here \( W_{abcd} \) is projectively invariant (and irreducible) Weyl tensor, and the Cotton-York tensor \( Y_{abc} := 2\nabla_a P_{bc} \) is projectively invariant in dimension 2. The projective structure \( (M, [\nabla]) \) is locally flat if and only if \( W_{abcd} = 0 \) (in dimension \( 2n \geq 3 \)) or \( Y_{abc} = 0 \) (for \( n = 1 \)). There is a \( \nabla \)-parallel volume form \( \epsilon \in \Lambda^2 \mathcal{T}^\ast M \) and we have the projective volume form \( \tilde{\epsilon} \in \Lambda^{2n+1} \mathcal{T}^\ast M \) which is parallel for any \( \nabla \in [\nabla] \). Any choice \( \nabla \in [\nabla] \) in dimension 2 gives a symplectic structure \( \omega = \epsilon \) and we shall use the notation \( \omega \) in this dimension. Similarly, we have the weighted version \( \omega_{ab} = \epsilon_{ab} \in \mathcal{E}_{(ab)}(3) \) with its dual \( \omega^{ab} \in \mathcal{E}^{(ab)}(-3) \).

We shall write sections of the standard projective tractor bundle \( \mathcal{E}^A = \mathcal{E}^a(-1) \oplus \mathcal{E}(-1) \), resp. its dual \( \mathcal{E}_A = \mathcal{E}(1) \oplus \mathcal{E}_a(1) \) using the injectors \( Y^A, X^A, \) resp. \( Y_A, X_A \) as
\[
\begin{pmatrix} \sigma^a \\ \rho \end{pmatrix} = Y^A a \sigma^a + X^A \rho \in \mathcal{E}^A, \quad \text{resp.} \quad \begin{pmatrix} \nu \\ \mu_a \end{pmatrix} = Y_A \nu + X_A a \mu_a \in \mathcal{E}_A.
\]
(3.7)

These splittings of \( \mathcal{E}^A \) and \( \mathcal{E}_A \) are determined by choices of projective connections and we call them projective splittings. The change of the
splitting under the change of connection parametrized by \( \gamma_a \in \mathcal{E}_a \) is

\[
\left( \sigma^a \atop \rho \right) = \left( \sigma^a \atop \rho - \gamma_a \sigma^a \right), \quad \text{i.e.} \quad \hat{Y}^A_a = Y^A_a + X^A \gamma_a, \quad \hat{X}^A_a = X^A_a.
\]

\[
\left( \nu \atop \mu_a \right) = \left( \nu \atop \mu_a + \gamma_a \nu \right), \quad \text{i.e.} \quad \hat{Y}_A = Y_A - X_A a \gamma_a, \quad \hat{X}^A_a = X^A_a.
\]

That is, \( X^A \in \mathcal{E}^A(1), X^A_a \in \mathcal{E}^a(-1) \) are invariant and \( Y^A_a \in \mathcal{E}^A_a(1), Y_A \in \mathcal{E}^A(-1) \) depend on the choice of the projective scale. We assume the normalization of these in which \( Y_A X^B + X^A a Y^B_a = \delta_A^B \), i.e., \( Y_C X^C = 1 \) and \( X^a a Y^C_b = \delta^a_b \).

The normal covariant derivative is given by

\[
\nabla_c \left( \sigma^a \atop \rho \right) = \left( \nabla_c \sigma^a + \rho \delta^a_c \atop \nabla_c \rho - P^c_{cp} \sigma^p \right) \quad \text{and} \quad \nabla_c \left( \nu \atop \mu_a \right) = \left( \nabla_c \nu - \mu_c \atop \nabla_c \mu_a + P^c_{cp} \nu \right), \quad \text{i.e.} \nabla_c Y^A_a = -X^A a P^a_{ca}, \quad \nabla_c X^A_a = Y^A_a P^a_{ca}, \quad \nabla_c X^A_a = -Y^A_a \delta^a_c,
\]

and its curvature \( \Omega \) has the form

\[
\Omega_{ab}^E_F = Y^E_c X^F_d W_{ab} e^d - X^E_c X^F_d Y_{abf} \in \mathcal{E}_{[ab]} \otimes A_9.
\]

That is, \( A_9 : = \text{tf}(\mathcal{E}_{EF}) \) is the projective adjoint tractor bundle, hence the curvature action on \( \mathcal{E}_C \) is \( (\nabla_a \nabla_b - \nabla_b \nabla_a) F_C = -\Omega_{ab}^D C F_D \).

We shall be interested in symmetric tensor powers of \( A_9 \subseteq \mathcal{E}^A_B \). Injectors for the bundle \( \mathcal{E}^A_B \) defined as

\[
\forall a A B := Y^A a Y^B_a, \quad Z^b a A B := Y^A a X^b_B, \\
\forall a B := X^A a Y^B_a, \quad X^b a B := X^A a X^b_B,
\]

are acted upon by the covariant derivative,

\[
\nabla_c \forall a A B = -P^a_{ca} \forall a A B + P^a_{ca} Z^b a A B, \\
\nabla_c Z^b a A B = -P^a_{ca} X^b_K a B - \forall a A B \delta^b_c, \\
\nabla_c \forall a B = \forall a a B + P^a_{ac} \forall a A B, \\
\nabla_c X^b a B = Z^b a a B - \forall a A B \delta^b_c.
\]

As for the tensor product of injectors, we have

\[
Z^R^a B R Z^C^d R = Z^{a C B} C \delta^d a, \quad Z^R^b R^b a R = \delta^a s R^b a B, \\
X^R^a B R Y^C^d R = \forall a B C \delta^d b, \quad X^{R R} B^a a R = X^{a a B}, \\
X^R^a B R Y^C^d R = Z^d a B C, \quad Z^S a R Y^R a B = \delta a R a B, \quad \delta^a s R a B = \delta^a s R a B. \quad (3.10)
\]
and their contractions are

\[ Y_a^A B x^b_B A = \delta_a^b, \quad Z_a^b A B Z^c_B A = \delta_a^d \delta_c^b, \quad \mathcal{W}^A_B \mathcal{W}^B_A = 1, \quad (3.11) \]

respectively. By (3.7), we write sections of \( \mathcal{A}_3 \) as

\[ I^A_B := \left( \begin{array}{c} \mu^a_b \\ \varphi \end{array} \right) = v^a \mathcal{V}_c^A B + \mu^d_c \mathcal{Z}_d^c A B + \varphi \mathcal{W}_B^A + \rho_c \mathcal{X}_B^c A, \]

where the trace-free condition \( \mu^a_a + \varphi = 0 \), equivalent to \( I^A_A = 0 \), is implied by \( sl(3, \mathbb{R}) \)-irreducibility. The set of formulas (3.9) can be rewritten as

\[ \nabla_c \left( \begin{array}{c} \mu^a_b \\ \varphi \\ \rho_a \end{array} \right) = \nabla_c \left( v^a \mathcal{V}_c^A B + \mu^d_c \mathcal{Z}_d^c A B + \varphi \mathcal{W}_B^A + \rho_d \mathcal{X}_d^d A \right) \]

\[ = \begin{pmatrix} \nabla_c v^a - \mu^a_c + \varphi \delta^a_c & \nabla_c \varphi - v^a P_{ca} - \rho_c \\ \nabla_c \rho_a + \varphi P_{ac} - \mu^a_b P_{cb} \end{pmatrix}, \]

(3.12)

where \( \mu^a_a + \varphi = 0 \) is preserved by the tractor covariant derivative. If \( I^A_B \in \mathcal{A}_3 \) is covariantly constant,

\[ \nabla_c I^A_B = \nabla_c \left( \begin{array}{c} \mu^a_b \\ \varphi \\ \rho_a \end{array} \right) = 0, \quad (3.13) \]

then the top slot \( v^a \in \mathcal{E}_a \) satisfies the projectively invariant equation

\[ tf \left( \nabla_{(a} \nabla_{b)} v^c + P_{ab} v^c \right) = 0. \quad (3.14) \]

This follows from (3.12) after a short computation. Moreover, this equation is equivalent to \( \nabla_c I^A_B = 0 \) on projectively flat manifolds. If we use \( \omega_{ab} \) and consider \( v_a \in \mathcal{E}_a(3) \), the previous display converts into \( \nabla_{(a} \nabla_{b)} v_{(c} + P_{(ab} v_{c)} = 0 \) due to specific properties of projective geometry in the real dimension 2. The equation (3.14) is the special case of the projectively invariant first BGG operator on symmetric tensors,

\[ \Phi : \mathcal{E}^{(a_1...a_k)} \to tf(\mathcal{E}^{(a_0...c_k a_1...a_k)}) \equiv \mathcal{E}^{(c_0...c_k a_1...a_k)}(-3(k+1)), \quad (3.15) \]

\[ \sigma^{a_1...a_k} \mapsto tf(\nabla_{(c_0} \ldots \nabla_{c_k)} \sigma^{a_1...a_k}) + \text{lot}. \]

The following lemma is easily verified.
Lemma 3. The projection $\Pi : \mathcal{E}^A_B \to \mathcal{E}^a$ defined on sections by
\[
\begin{pmatrix}
\mu^a_b \\
\upsilon^a \\
\phi \\
\rho_a
\end{pmatrix} \mapsto v^a,
\]
has projectively invariant differential splitting $\Sigma : \mathcal{E}^a \to \mathcal{E}^A_B$ given by
\[
v^a \mapsto (\Sigma v^a)^A_B = Y^a_A B v^a + Z^a_b A_B (\langle \nabla_b v^a \rangle o + \frac{1}{6} \delta^a_b \nabla_c v^c) - \frac{1}{3} \gamma^A B \nabla_c v^c - X^a_A B (\frac{1}{2} \nabla_a \nabla_b + P_{ab}) v^b.
\]

Apart from the adjoint tractor bundle $A_d$, we shall need the bundles $\text{tf}(\mathcal{E}(A_1...A_k)(B_1...B_k))$. Analogously as above, we have the projection $\Pi$ and its differential splitting $\Sigma$ (by abuse of notation we use the same letters as in Lemma 3)
\[
\Pi : \text{tf}(\mathcal{E}(A_1...A_k)(B_1...B_k)) \to \mathcal{E}^{(a_1...a_k)},
\]
\[
\Sigma : \mathcal{E}^{(a_1...a_k)} \to \text{tf}(\mathcal{E}(A_1...A_k)(B_1...B_k)),
\]
i.e. $\Pi \circ \Sigma = \text{id}$. The existence of projectively invariant splitting $\Sigma$ follows from the BGG machinery. Moreover, $\Sigma$ is unique in the projectively flat case and, under certain normalization condition, also in the curved case. This is known as the BGG splitting operator and henceforth will be our choice of $\Sigma$. It follows from the BGG machinery that $\Sigma$ has the following essential property:

Proposition 4. Assume $(M, [\nabla])$ is a projectively flat structure and let $\sigma^{a_1...a_k} \in \mathcal{E}^{(a_1...a_k)}$. Then $\Phi(\sigma) = 0$ if and only if $\nabla \Sigma(\sigma) = 0$. That is, there is a bijective correspondence
\[
\{\sigma^{a_1...a_k} \in \mathcal{E}^{(a_1...a_k)} \mid \Phi(\sigma) = 0\} \xymatrix{\cong & \mathbb{Z}^k \mathfrak{sl}(3, \mathbb{R}).}
\]

We recall the notation $\mathbb{Z}^k \mathfrak{sl}(3, \mathbb{R})$ for the $k$-th Cartan power of $\mathfrak{sl}(3, \mathbb{R})$ in $\otimes^k \mathfrak{sl}(3, \mathbb{R}), k \in \mathbb{N}$. 

3.3 Projective connection on symplectic spinors

From now on we assume $n = 1$ and use the symplectic form $\omega_{ab} = \epsilon_{ab} \in \mathcal{E}_{[ab]}(3)$. Then the symplectic spinor bundle can be realized and described in projective geometry similarly to the spinor bundle in conformal geometry, cf. \[5\]. It is convenient to induce bundles from symplectic frames of $\mathcal{E}_a(\mathbb{R})$ where the pairing $\langle \mu, \nu \rangle = \omega_{ab} \mu_a \nu_b$ is
projectively invariant for $\mu_a, \nu_a \in \mathcal{E}_a(\frac{3}{2})$. We have a weighted version of symplectic gamma matrices $\gamma_a$,

$$\gamma_a \gamma_a - \gamma_b \gamma_a = 2\omega_{ab}, \quad \gamma_a \in \mathcal{E}_a \otimes (\text{End} \mathcal{S})(\frac{3}{2}),$$

cf. (2.1), which generate the bundle of weighted Clifford algebras. The projective transformation (3.6) implies

$$\tilde{\nabla}_a \mu_b = \nabla_a \mu_b + \frac{1}{2} \gamma_a \mu_b - \gamma_b \mu_a = \nabla_a \mu_b - \Gamma_a^c b \mu_c, \quad \text{where} \quad \Gamma_{abc} = \omega_{a(b} \gamma_{c)}.$$

Considering $\Gamma_a^c b$ as a one form valued in the bundle of Lie algebras $\mathfrak{sl}(2, \mathbb{R})$ and using the action (3.5), we obtain

$$\tilde{\nabla}_a \varphi = \nabla_a \varphi + \frac{1}{4} \gamma_a \gamma^b \varphi = \nabla_a \varphi - \frac{1}{4} \gamma_a \gamma^b \varphi, \quad \varphi \in \mathcal{S}.$$

By (3.6), we obtain the weighted version

$$\tilde{\nabla}_a \varphi = \nabla_a \varphi + (w + \frac{1}{4}) \gamma_a \gamma^b \varphi = \nabla_a \varphi - \frac{1}{4} \gamma_a \gamma^b \varphi, \quad \varphi \in \mathcal{S}(w). \quad (3.19)$$

**Lemma 5.** The symplectic Dirac operator $\mathcal{D}_s = \gamma^a \nabla_a$ acting on $\mathcal{S}(w)$ is projectively invariant differential operator if and only if $w = -\frac{3}{4}$,

$$\mathcal{D}_s : \mathcal{S}(-\frac{3}{4}) \rightarrow \mathcal{S}(-\frac{9}{4}). \quad (3.20)$$

**Proof.** This results from the contraction of (3.19) by $\gamma^a$. \qed

Next we observe there is a spinor version of the Thomas $D$-operator $D_A : \mathcal{E}(w) \rightarrow \mathcal{E}_A(w - 1)$, [1]. Its analogue acting on symplectic spinors is given as follows:

**Theorem 6.** The tractor $D$-operator acting on $\mathcal{S}(w)$ and defined by

$$D_A : \mathcal{S}(w) \rightarrow \mathcal{S}(w - 1) \otimes \mathcal{E}_A,$$

$$D_A = (4w + 3)(w + \frac{1}{4}) \gamma_A + (4w + 3) X^b_A \nabla_b + \gamma_b X^b_A \mathcal{D}_s, \quad (3.21)$$

is a projectively invariant first order differential operator.

The projective invariance of $D_A$ is a straightforward consequence of the transformation property (3.19). In the case $w = -\frac{3}{4}$, $D_A$ reduces to the invariant bottom slot of the standard tractor bundle and hence $\mathcal{D}_s$ is projectively invariant operator for this value of $w$ in a full agreement with Lemma 5. Another interesting case corresponds to $w = -\frac{1}{4}$ for which $D_A$ reduces to the projectively invariant symplectic twistor differential operator acting on spinors,

$$\nabla_a + \frac{1}{2} \gamma_a \mathcal{D}_s : \mathcal{S}(-\frac{1}{4}) \rightarrow \mathcal{E}_a \otimes \mathcal{S}(-\frac{1}{4}).$$
Theorem 7. The tractor operator $\mathbb{D}^B$ acting on $\mathcal{S}(w)$ and defined by

$$
\mathbb{D}^B_A : \mathcal{S} \to \text{tf}(\mathcal{E}^B_A) \otimes \mathcal{S}(w),
$$

$$
\varphi \mapsto \mathbb{D}^B_A \varphi := \begin{pmatrix}
\frac{1}{4} \gamma^a \gamma_b \varphi + (-\frac{w}{3} + \frac{1}{4}) \delta^a b \varphi & \frac{2w}{3} \varphi
\end{pmatrix}
$$

or in terms of injectors by

$$
\mathbb{D}^B_A = Y^A_c X^b_B (\frac{1}{4} \gamma^a \gamma_b + (-\frac{w}{3} + \frac{1}{4}) \delta^a b) + \frac{2w}{3} X^A Y_B + X^A X^b_B \nabla_b,
$$

is a projectively invariant first order differential operator. We have the commutation relation of invariant differential operators

$$
\mathcal{D}_s \mathbb{D}^B_A = \mathbb{D}^B_A \mathcal{D}_s : \mathcal{S}(-\frac{3}{4}) \to \text{tf}(\mathcal{E}^B_A) \otimes \mathcal{S}(-\frac{3}{4})
$$

Proof. The first claim, related to projective invariance, is straightforward and follows from the transformation properties of injectors $\gamma^A, X^B, X^A, Y_B$ and $\nabla$. The second claim (3.24) follows from an explicit computation. Then

$$
\mathcal{D}_s \mathbb{D}^B_A = (-P_{ca} X^b_A - \gamma^A_B \delta^a b)(\frac{1}{4} \gamma^a \gamma_b + \frac{1}{2} \gamma^c \delta^a b)
$$

$$
- \frac{1}{2} \gamma^c (Y^a_X B + P_{ac} X^a_B) + \gamma^c (\zeta^b_B - \mathcal{W}^A_B \delta^a b)_b
$$

$$
+ \zeta^b_B (\frac{1}{4} \gamma^a \gamma_b + \frac{1}{2} \gamma^c \delta^a b)_c - \frac{1}{2} \mathcal{W}^A_B \gamma^c \gamma_b + X^b_A \gamma^c \nabla_c \nabla_b
$$

$$
= Y^A_B (\frac{1}{4} \gamma^c \gamma^a \gamma^b - \frac{1}{2} \gamma^a \gamma_b - \frac{1}{4} \gamma^c \gamma_b) + \mathcal{W}^A_B (\gamma^a \nabla_b + \frac{1}{4} \gamma^c \gamma^a \gamma_b + \frac{1}{2} \gamma^c \delta^a b)_c
$$

$$
+ \mathcal{W}^A_B (-\gamma^c \delta^a b \nabla_b - \frac{1}{2} \gamma^c \nabla_b) + \mathcal{X}^b_A (-\frac{1}{4} P_{ca} \gamma^c \gamma_b - \frac{1}{2} P_{ca} \gamma^c \delta^a b
$$

The identities (2.1) and (3.3) imply the coefficient of $\gamma^A_B$ is zero.

The operator of $\mathcal{W}^A_B$ quotients from the right through $\mathcal{D}_s$.

Let us now discuss the operator coefficient of $X^b_A$. The curvature of $\nabla$ acts on symplectic spinors as $\nabla_a \nabla_b - \nabla_b \nabla_a = -\frac{1}{4} R_{ab} e^c d \gamma^d$. In the real dimension 2 holds the curvature identity $R_{ab} e^c d = \delta^c b P_{bd} - \delta^d c P_{ad}$, hence

$$
\gamma^c R_{ab} e^f \gamma^d e^f = \gamma^c (\delta^c b P_{bf} - \delta^d c P_{cf}) \gamma^d
$$

$$
= -2 \gamma^f P_{bf} - \gamma^c b P_{cf} = -4 \gamma^f P_{bf} - \gamma^f \gamma^d \gamma^d P_{cf}.
$$

The substitution for $\gamma^c \nabla_c \nabla_b$ results in the contribution $X^b_A \nabla_b \mathcal{D}_s$. 

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The operator coefficient of $Z^a_{\alpha B}$ can be rewritten as $\frac{1}{2}\gamma^a \nabla_b - \frac{1}{2}\gamma_b \nabla^a + \frac{1}{2}\gamma^b \gamma_a \mathcal{D}_s + \frac{1}{2}\delta^a_b \mathcal{D}_s$, hence in the real dimension 2 we get
\[ \frac{1}{2}\gamma^a \nabla_b - \frac{1}{2}\gamma_b \nabla^a = \frac{1}{2}\delta^a_b \gamma^c \nabla_c = \frac{1}{2}\delta^a_b \mathcal{D}_s, \]
because any 2-form is proportional to $\omega_{ab}$ and the application of trace results in the constant $\frac{1}{2}$ on the right hand side. A further simplification gives $Z^b_{\alpha B}(\delta^a_b + \frac{1}{2}\gamma^a \gamma_b) \mathcal{D}_s$. All the three non-trivial contributions quotient from the right by $\mathcal{D}_s$, and it is elementary to see that the overall expression equals to the composition $\mathbb{D}^B_A \mathcal{D}_s$ as claimed in (3.24). The proof is complete. \hfill \Box

It is an immediate consequence of (3.22) that $\mathbb{D}$ preserves irreducible components $S_{\pm}(w)$. We shall need the explicit formulæ for $\mathbb{D}^B_A$ acting on $\varphi \in \mathcal{S}(-\frac{3}{4})$ and $\psi \in \mathcal{S}(-\frac{9}{4})$:
\[
\mathbb{D}^B_A \varphi = Y^A_{\alpha B} X^b_B \left( \frac{1}{2} \gamma^a \gamma_b + \frac{1}{2} \delta^a_b \right) \varphi - \frac{1}{2} X^A Y_B \varphi + X^A X^B \nabla_b \varphi, \quad (3.25)
\]
\[
\mathbb{D}^B_A \psi = Y^A_{\alpha B} X^b_B \left( \frac{1}{2} \gamma^a \gamma_b + \delta^a_b \right) \psi - \frac{3}{2} X^A Y_B \psi + X^A X^B \nabla_b \psi. \quad (3.26)
\]
Combining the BGG splitting $\Sigma$ in (3.18) with the operator $\mathbb{D}$, we obtain a differential operator $\mathcal{O}^\sigma$ on symplectic spinors with the prescribed symbol $\sigma$. Specifically, we have the map
\[
\mathcal{Q}^k : \mathcal{E}^{(a_1 \ldots a_k)} \to \text{Diff}^k(\mathcal{S}(w)),
\]
\[
\sigma^{a_1 \ldots a_k} \mapsto (\Sigma(\sigma))^{A_1 \ldots A_k} B_{B_1 \ldots B_k} \mathbb{D}^{B_1}_{A_1} \ldots \mathbb{D}^{B_k}_{A_k}, \quad (3.27)
\]
where $\text{Diff}^k(\mathcal{S}(w))$ denotes the space of differential operators acting on $\mathcal{S}(w)$ of order $\leq k$. It follows from the properties of $\Sigma$ and $\mathbb{D}$ that $\mathcal{Q}(\sigma)$ has indeed the symbol $\sigma$, i.e., for $\sigma^{a_1 \ldots a_k} \in \mathcal{E}^{(a_1 \ldots a_k)}$ we have
\[
\mathcal{Q}^k(\sigma) = \sigma^{a_1 \ldots a_k} \nabla_{a_1} \ldots \nabla_{a_k} + \text{lot} : \mathcal{S}(w) \to \mathcal{S}(w), \quad (3.28)
\]
for any $w \in \mathbb{R}$. Let us note that it follows from invariance of $\Sigma$ and $\mathbb{D}$ that $\mathcal{Q}^k$ yields the projectively invariant bilinear operator $(\sigma, \varphi) \mapsto (\mathcal{O}^k(\sigma))(\varphi)$ for $\varphi \in \mathcal{S}(w)$.

Recall that $\mathcal{D}_s$ is projectively invariant for a specific weight computed in Lemma 3. To construct higher symmetries of $\mathcal{D}_s$, we shall need operators $\mathcal{Q}^k(\sigma)$ for these specific weights. We put
\[
\mathcal{O}^\sigma := \mathcal{Q}^k(\sigma) : \mathcal{S}(-\frac{3}{4}) \to \mathcal{S}(-\frac{9}{4}),
\]
\[
\overline{\mathcal{O}}^\sigma := \mathcal{Q}^k(\sigma) : \mathcal{S}(-\frac{9}{4}) \to \mathcal{S}(-\frac{3}{4}), \quad (3.29)
\]
where $\mathcal{O}^\sigma$ preserves $\mathcal{S}_{\pm}(-\frac{3}{4})$ and similarly for $\overline{\mathcal{O}}^\sigma$. The crucial information relating $\mathcal{O}^\sigma$ to $\Phi(\sigma)$ is provided by the following lemma:
Lemma 8. Let $\sigma$ and $\Sigma(\sigma)$ be as above. Then

\[
\left[\nabla_b(\Sigma(\sigma))\right]^{A_1\ldots A_k}_{B_1\ldots B_k}D^{B_1}_{A_1} \ldots D^{B_k}_{A_k} = \\
= (\frac{1}{4})^k (\Phi_b)(\sigma)c_{1\ldots k}^{a_1\ldots a_k} \gamma_c \ldots \gamma_c \gamma_a \ldots \gamma_a : S(w) \rightarrow E_b \otimes S(w).
\]

Proof. First, we need to use the properties of $\nabla_b(\Sigma(\sigma))$. It follows from the BGG machinery, cf. [3], that

\[
\left[\nabla_b(\Sigma(\sigma))\right]^{A_1\ldots A_k}_{B_1\ldots B_k} = Z_{a_1} c_{A_1} B_1 \ldots Z_{a_k} c_{A_k} B_k (\Phi_b)(\sigma) b_{c_1\ldots c_k}^{a_1\ldots a_k} \]

\[
+ \sum_{i=1}^{k} X \ldots X Z \ldots Z \mu_i,
\]

where we suppressed all abstract indices in the second summand. Here $\mu_i$ is a section of suitable bundle which we do need to know explicitly.

Secondly, it follows from (3.22) for the composition

\[
D^{B_1}_{A_1} \ldots D^{B_k}_{A_k} = Z_{a_1} c_{A_1} B_1 \ldots Z_{a_k} c_{A_k} B_k (\Phi_b)(\sigma) b_{c_1\ldots c_k}^{a_1\ldots a_k} \]

\[
+ \text{remaining terms containing } X, Y, Z \text{ and } W.
\]

Now combining $Z \ldots Z$-terms of both displays yields the right hand side as stated in lemma. Since $(\Phi_b)(\sigma) b_{c_1\ldots c_k}^{a_1\ldots a_k}$ is totally trace free, the ”trace terms” in the previous display cannot contribute. In addition we exploit the notion of homogeneity given by $h(Y) := 1$, $h(Z) := 0$ and $h(X) := -1$, extended to concatenations of $Y, Z$ and $X$ in an obvious way. Now since $X \ldots X Z \ldots Z$-terms of $\nabla_b(\Sigma(\sigma))$ with at least one $X$ have homogeneity $\leq -1$ and all “remaining terms” in the previous display have homogeneity $\leq 0$, lemma follows.

Now we substitute $w = -\frac{3}{4}$ and use (3.24) to conclude:

Theorem 9. Assume $(M, [\nabla])$ is a projectively flat structure, and $\sigma^{a_1\ldots a_k} \in E^{a_1\ldots a_k}$ for $k \in \mathbb{N}$. Then the operator $\mathcal{D}_s \mathcal{O}^\sigma - \overline{\mathcal{O}} \mathcal{D}_s : S(-\frac{3}{4}) \rightarrow S(-\frac{9}{4})$ satisfies

\[
\mathcal{D}_s \mathcal{O}^\sigma - \overline{\mathcal{O}} \mathcal{D}_s = (\frac{1}{4})^k (\Phi_b)(\sigma) b_{c_1\ldots c_k}^{a_1\ldots a_k} \gamma_b \gamma_{c_1} \ldots \gamma_c \gamma_{a_1} \ldots \gamma_{a_k}.
\]

In particular, $\mathcal{O}^\sigma$ is a symmetry of $\mathcal{D}_s$ if and only if $\Phi_b(\sigma) = 0$.

Notice that once we define $\mathcal{O}^\sigma$ and $\overline{\mathcal{O}}^\sigma$ as the left multiplication by $\sigma$ and $\Phi_b(\sigma)$ as the differential $d\sigma$, the theorem holds for $k = 0$ as well.
4 Symbols and construction of higher symmetries of $\mathcal{D}_s$

Let us consider a $k$-th order differential operator $O$ on weighted symplectic spinors. Its principal symbol is $\sigma^{a_1 \ldots a_k} \in \mathcal{E}^{(a_1 \ldots a_k)} \otimes \text{End}(S_{\pm})$, and

$$\text{End}(S_{\pm}) \cong \bigoplus_{i \in \mathbb{N}_0} \mathcal{E}^{(b_1 \ldots b_{2i})}(-3i)$$

(4.1)

since the typical fiber of $\text{End}(S)$ is isomorphic to $\mathbb{C}l_s(\mathbb{R}^2, \omega\omega\omega)$ up to a weight, cf. (3.1). The decomposition (4.1) implies that $\sigma^{a_1 \ldots a_k}$ can be written as

$$\sigma^{a_1 \ldots a_k} = \sum_{i \in \mathbb{N}_0} \sigma_i^{a_1 \ldots a_k b_1 \ldots b_{2i}}, \quad \sigma_i^{a_1 \ldots a_k b_1 \ldots b_{2i}} \in \mathcal{E}^{(a_1 \ldots a_k)(b_1 \ldots b_{2i})}(-3i),$$

where all $\sigma_i^{a_1 \ldots a_k b_1 \ldots b_{2i}}$ up to finitely many vanish. Thus we have for any $w \in \mathbb{R}$

$$O : S_{\pm}(w) \rightarrow S_{\pm}(w),$$

$$\varphi \mapsto \sum_{i \in \mathbb{N}_0} \sigma_i^{a_1 \ldots a_k b_1 \ldots b_{2i}} \gamma_{b_1} \ldots \gamma_{b_{2i}} \nabla_{a_1} \ldots \nabla_{a_k} \varphi + \text{lot} \quad (4.2)$$

where we have explicitly mentioned the leading term of $O$. The subspaces $\mathcal{E}^{(a_1 \ldots a_k)(b_1 \ldots b_{2i})}$ are in general not irreducible, namely, we have a distinguished Cartan component $\mathcal{E}^{(a_1 \ldots a_k)(b_1 \ldots b_{2i})} \subset \mathcal{E}^{(a_1 \ldots a_k)(b_1 \ldots b_{2i})}$. Moreover, one easily observes that once $\sigma_i^{a_1 \ldots a_k b_1 \ldots b_{2i}}$ lives in the complement to the Cartan component it gives rise to the operator

$$\sigma_i^{a_1 \ldots a_k b_1 \ldots b_{2i}} \gamma_{b_1} \ldots \gamma_{b_{2i}} \nabla_{a_1} \ldots \nabla_{a_k}$$

which factors through the symplectic Dirac operator $\mathcal{D}_s = \mathcal{D}_s(\omega\omega\omega, \nabla_{a_1} \ldots \nabla_{a_k})$ for the weight $w = -\frac{3}{4}$. Henceforth we specialize to this weight and conclude

**Lemma 10.** The operator $O$ from (4.2) has, modulo $\mathcal{D}_s$, the principal symbol

$$\sigma^{a_1 \ldots a_k} = \sum_{i \in \mathbb{N}_0} \sigma_i^{a_1 \ldots a_k b_1 \ldots b_{2i}}, \quad \sigma_i^{a_1 \ldots a_k b_1 \ldots b_{2i}} \in \mathcal{E}^{(a_1 \ldots a_k b_1 \ldots b_{2i})}(-3i).$$

We shall turn our attention to properties of the principal symbol for $O$ in order to become a symmetry of $\mathcal{D}_s$, i.e., $\mathcal{D}_s O = O' \mathcal{D}_s$ for some operator $O'$. The following lemma concerns algebraic properties of such symbols.
Proposition 11. Assume the principal symbol of $\mathcal{O}$ is non-trivial in the sense that $\sigma_{a_1 \ldots a_k b_1 \ldots b_{2i}} \neq 0$ for some $i \geq 1$. Then, modulo $\mathcal{D}_s$, the composition $\mathcal{D}_s \mathcal{O}$ is a differential operator of order $k + 1$ and its principal symbol is non-vanishing.

Proof. We shall consider the composition $\mathcal{D}_s \mathcal{O}$ which is an operator of order at most $k + 1$. In fact, we need just the leading term in this composition. Applying $\omega^s \gamma \nabla_s$ to $\mathcal{O}$ as given in (4.2) yields, modulo $\mathcal{D}_s$,

$$\mathcal{D}_s \mathcal{O} = 4 \sum_{i \in \mathbb{N}_0} i \sigma_{a_1 \ldots a_k b_1 \ldots b_{2i}} \omega^s \omega_{b_1} \gamma_{b_2} \ldots \gamma_{b_{2i}} \nabla_s \nabla_{a_1} \ldots \nabla_{a_k} + \text{lot}$$

using Lemma 10 and (3.4).

Next, consider the principal symbol $\bar{\sigma}$ of the operator $\mathcal{D}_s \mathcal{O}$ for some $\mathcal{O}$. Specifically, we shall need the component of $\bar{\sigma}$ with $2i - 1$ of $\gamma$'s which we denote by $\bar{\sigma}_i$. This yields the component in the leading term of $\mathcal{D}_s \mathcal{O}$ of the form

$$\bar{\sigma}_i^{a_1 \ldots a_k b_1 \ldots b_{2i}} \gamma_{b_2} \ldots \gamma_{b_{2i}} \nabla_{b_1} \nabla_{a_1} \ldots \nabla_{a_k} \quad \text{with}$$

$$\bar{\sigma}_i^{a_1 \ldots a_k b_1 \ldots b_{2i}} \in \mathcal{E}^{(a_1 \ldots a_k b_1 \ldots b_{2i})} \left( -\frac{3(2i - 1)}{2} \right).$$

We highlight the fact that $\bar{\sigma}_i^{a_1 \ldots a_k b_1 \ldots b_{2i}}$ lives in $\mathcal{E}^{(a_1 \ldots a_k b_1 \ldots b_{2i})}$, whereas $\bar{\sigma}_i^{a_1 \ldots a_k b_1 \ldots b_{2i}}$ lives in its complement. Consequently, the differential operator $\mathcal{D}_s \mathcal{O} - \mathcal{D}_s \mathcal{O}$ cannot have order $\leq k$, hence the proof follows.

Corollary 12. If $\mathcal{O}$ is a symmetry of $\mathcal{D}_s$ then, modulo $\mathcal{D}_s$, the principal symbol of $\mathcal{O}$ is

$$\sigma^{a_1 \ldots a_k} = \sigma_0^{a_1 \ldots a_k} \in \mathcal{E}^{(a_1 \ldots a_k)}.$$

That is, $\sigma^{a_1 \ldots a_k b_1 \ldots b_{2i}} = 0$ for all $i \geq 1$.

Given $\sigma^{a_1 \ldots a_k}$, we can use the operator $\mathcal{O}^\sigma$ from (4.2). Assuming $\mathcal{O}$ is a symmetry of $\mathcal{D}_s$, both operators $\mathcal{O}$ and $\mathcal{O}^\sigma$ have the same principal symbol, cf. (4.28). Therefore, we have $\mathcal{O} = \mathcal{O}^\sigma + \mathcal{O}'$ with $\mathcal{O}'$ of order $k - 1$. Denote the symbol of $\mathcal{O}'$ by $(\sigma')^{a_1 \ldots a_{k-1}} \in \mathcal{E}^{(a_1 \ldots a_{k-1})} \otimes \text{End}(S)$. This generally decomposes according to the number of $\gamma$'s. But since $\mathcal{O}$ is a symmetry of $\mathcal{D}_s$ and $\mathcal{D}_s \mathcal{O}^\sigma$ has zero order by Theorem 9, it
follows from Proposition 11 applied to $\mathcal{O}'$ that in fact $(\sigma')^{a_1...a_{k-1}} \in \mathcal{E}(a_1...a_{k-1})$. Thus we have

$$\mathcal{O} = \mathcal{O}' + \mathcal{O}''$$

with $\mathcal{O}''$ of order $k - 2$, and by inductive procedure (with respect to the strictly decreasing order on the algebra of differential operators) we conclude that after finitely many steps every symmetry $\mathcal{O}$ of $\mathcal{D}_s$ reduces, modulo $\mathcal{D}_s$, into the form

$$\mathcal{O} = \mathcal{O}^{[k]} + \mathcal{O}^{[k-1]} + \ldots + \mathcal{O}^{[0]} : S(-\frac{3}{2}) \to S(-\frac{3}{2}), \quad \text{where}$$

$$\sigma^{[i]}_{[i]} \in \mathcal{E}(\alpha_1...\alpha_i), \ i = 0, \ldots, k.$$  \hfill (4.3)

Here we have introduced a new notation: comparing the last two displays, we have $\sigma_{[k]} = \sigma$, $\sigma_{[k-1]} = \sigma'$, etc.

The final statement on the characterization of symbols of higher symmetries of $\mathcal{D}_s$ is a direct consequence of Theorem 9 and Corollary (4.3):

**Theorem 13.** The operator $\mathcal{O}$ is a symmetry of $\mathcal{D}_s$ if and only if $\Phi(\sigma_{[i]}) = 0$ for all $i = 0, \ldots, k$, cf. (3.15).

This result, together with Proposition 11 and Theorem 9, completes the description of $\mathcal{A}_k$ as a vector space,

$$\mathcal{A}^k = \bigoplus_{i=0}^{k} \mathbb{R}^k \mathfrak{sl}(3, \mathbb{R}).$$

The dimension of the right hand side can be easily computed from $\dim \mathbb{R}^k \mathfrak{sl}(3, \mathbb{R}) = (k + 1)^3$, cf. [16].

In order to construct symmetry differential operators for $\mathcal{D}_s$ explicitly, we start with parallel sections $I^A_B, \bar{I}^A_B \in \mathcal{E}^A_B$ of the adjoint tractor bundle $\mathcal{A}_0$ for $\mathfrak{sl}(3, \mathbb{R})$, cf. (3.13). Let us first observe that the composition $I^B_A \bar{I}^D_C \mathcal{D}[A_B \mathcal{D} C_D]$ decomposes into irreducibles for $\mathfrak{sl}(3, \mathbb{R})$ according to

$$[I, \bar{I}]^A_B = -(I^A_R \bar{I}^R_B - \bar{I}^A_R I^R_B),$$

$$\left(I \odot \bar{I}\right)^A_B = (I^A_R \bar{I}^R_B + \bar{I}^A_R I^R_B)_o,$$

$$\langle I, \bar{I} \rangle = I^R_S \bar{I}^S_R.$$ \hfill (4.4)
Note the latter pairing is related to the Killing form $\langle \cdot, \cdot \rangle_K$ on $\mathfrak{sl}(3, \mathbb{R})$ by $\langle \cdot, \cdot \rangle = \frac{1}{6} \langle \cdot, \cdot \rangle_K$. We denote the $\mathfrak{sl}(3, \mathbb{R})$-irreducible Cartan component in the tensor product $I^A_B \otimes \bar{I}^C_D$ by $(I \otimes \bar{I})^{(A \otimes B \otimes C \otimes D)}$. We have

$$(I \otimes \bar{I})^A_B = (I^A_R I^R_B + \bar{I}^A_R I^R_B) = [I, \bar{I}]^A_B + 2 \bar{I}^A_R I^R_B,$$  

where the leading term of the adjoint tractor expansion equals

$$\bar{I}^A_R I^R_B = Y_r^A B (v^s (\nabla_s \bar{v}^r))_o + \bar{v}^s (\nabla_s v^r) - \frac{2}{3} v^s (\nabla_s \bar{v}^r) - \frac{3}{2} \bar{v}^s (\nabla_s v^r)$$

+ "lower adjoint tractor slots"

and implies equality in the leading adjoint slot expansion

$$(I \otimes \bar{I})^A_B = \bar{Y}_r^A B (v^s (\nabla_s \bar{v}^r)) + \bar{v}^s (\nabla_s v^r) - \frac{2}{3} v^s (\nabla_s \bar{v}^r) - \frac{3}{2} \bar{v}^s (\nabla_s v^r)$$

+ "lower adjoint tractor slots".

**Lemma 14.** The first order differential operator $S^v = I^A_B D^B_A$, where $v$ is a solution of (3.14) and so is the projective component of the parallel tractor $I^A_B$, equals to

$$S^v := I^A_B B^B_D = v^r \nabla_r + \frac{1}{2} (\nabla_r v^s) \gamma^r \gamma_s + \frac{1}{3} (\nabla_r v^s).$$

It is a first order symmetry operator of $D_s$ with symbol $v^r \nabla_r$. The equivalence relation (2.2) is for $S^v$ completed by the operator

$$\bar{S}^v := I^A_B D^B_A = v^r \nabla_r + \frac{1}{2} (\nabla_r v^s) \gamma^r \gamma_s + \frac{5}{6} (\nabla_r v^s).$$

The Lie algebra structure on first order symmetry operators given by parallel tractors $I^A_B, \bar{I}^A_B \in \mathcal{E}_B^A$ is $[I, \bar{I}]^A_B \mathbb{D}^B_A$ with the leading term $[v, \bar{v}]^a \nabla_a$. Here $[v, \bar{v}]^a$ is the Lie bracket of vector fields $v$ and $\bar{v}$.

**Proof.** The first part of the claim follows from Theorem 13 and the explicit formulas for $S^v$ and $\bar{S}^v$ from (3.17), (3.25), (3.26). Analogously, we have

$$[I, \bar{I}]^A_B = -Y_a^R B^A (Z_r^s B^R (\nabla_s \bar{v}^r))_o + \frac{1}{6} \delta^s_r (\nabla_t \bar{v}^t) - \frac{1}{3} \bar{Y}_a^R B^A (\nabla_t v^t)$$

$$- Y_a^R B^A (Z^s_r B^R (\nabla_s v^r))_o + \frac{1}{6} \delta^s_r (\nabla_t v^t) - \frac{1}{3} \bar{Y}_a^R B^A (\nabla_t v^t)$$

$$+ Y_a^R B^A (Z^s_r B^R (\nabla_s \bar{v}^r))_o + \frac{1}{6} \delta^s_r (\nabla_t \bar{v}^t) - \frac{1}{3} \bar{Y}_a^R B^A (\nabla_t \bar{v}^t)$$

+ "lower adjoint tractor slots"

$$= Y_a^R B^A (v^s (\nabla_s \bar{v}^r)) - \bar{v}^s (\nabla_s v^r) + "lower adjoint tractor slots"$$

$$= Y_a^R B^A [v, \bar{v}]^a + "lower adjoint tractor slots",$$

which proves the second half of the claim. 

20
5 Algebra of higher symmetries of $\mathcal{D}_s$

In the present section we analyze the algebra structure on the vector space of higher symmetries of $\mathcal{D}_s$. Following Lemma 14, the first order symmetries generate the tensor algebra of higher symmetry differential operators and hence we have to produce $\mathfrak{sl}(3,\mathbb{R})$-invariant decomposition of the composition of operators $\mathcal{D}^B_A$. As we shall argue in the end of this section, it is sufficient to compute it just for $\mathcal{D}^B_A\mathcal{D}^D_C$.

Throughout this section we assume $w = -\frac{3}{4}$.

**Lemma 15.** The composition of two operators $\mathcal{D}^B_A$, see (3.25), equals to

\[
\mathcal{D}^B_A\mathcal{D}^D_C =
\begin{align*}
&= Z_b^aB^dZ_c^dC \left( \frac{1}{16} \sum_{\gamma \gamma \gamma} \gamma_b \gamma_a \gamma^d \gamma_c + \frac{1}{8} \left( \gamma_b \gamma_a \delta^d_c + \gamma^d \gamma_c \delta^d_a \right) \right) \\
&- \frac{1}{2} (Z_b^aB^dA^\gamma C + Z_c^dA^\gamma B^\gamma) \left( \frac{1}{8} \gamma_b \gamma_a + \frac{1}{2} \delta^d_a \right) + \frac{1}{2} \mathcal{W}^B_A \mathcal{W}^D_C \\
&- X_b^bA^\gamma C \left( \frac{1}{4} \sum_{\gamma \gamma \gamma} \gamma^d \gamma_b + \delta^d_b \right) + X_b^bA^\gamma A^\gamma \gamma C \left( \frac{1}{4} \gamma^d \gamma_c \nabla_b + \frac{1}{2} \delta^d_c \nabla_b + \delta^b \nabla c \right) \\
&+ Z_b^aB^dC \left( \frac{1}{4} \sum_{\gamma \gamma \gamma} \gamma_b \gamma_a + \frac{1}{2} \delta^d_b \right) \nabla_d - \frac{3}{2} X_b^bA^\gamma D^\gamma C \nabla_b - \frac{1}{2} \mathcal{W}^B_A \mathcal{X}^dD^\gamma C \nabla_d \\
&+ \mathcal{X}^bA^\gamma C \left( \nabla_b \nabla_d - \frac{1}{4} P_{bd} \gamma^d \gamma_d - P_{bd} \right). 
\end{align*}
\]

**Proof.** The proof is a straightforward computation based on (3.25), (3.9) and (3.10). \qed

The composition $\mathcal{D}^B_A\mathcal{D}^D_C$ can be invariantly decomposed according to the $\mathfrak{sl}(3,\mathbb{R})$-module structure on the second tensor power of its adjoint representation. We shall first examine the skew-symmetric part of this tensor product, which induces the Lie algebra structure on the linear span of symmetry operators $\mathcal{D}^A_B$. In fact, we shall show that the skew-symmetric component in their composition contains just the adjoint representation.

**Lemma 16.** The skew-symmetric component of (5.1) is given by

\[
\mathcal{D}^B_A\mathcal{D}^D_C - \mathcal{D}^D_C\mathcal{D}^B_A =
\begin{align*}
&= \frac{1}{8} Z_b^aB^dZ_c^dC \left( \gamma_b \gamma_a \delta^d_c - \gamma^d \gamma_a \delta^d_c + \gamma^d \gamma_c \omega^b - \gamma^d \gamma_c \omega^a \right) \\
&- (X_b^bA^\gamma D^\gamma C - X_d^dB^\gamma A^\gamma B^\gamma) \left( \frac{1}{4} \sum_{\gamma \gamma \gamma} \gamma^d \gamma_b + \delta^d_b \right) \\
&+ (X_b^bA^\gamma A^\gamma \gamma C - Z_d^dB^\gamma A^\gamma A^\gamma B^\gamma) \delta^d_c \nabla_c \\
&- (X_b^bA^\gamma D^\gamma C - \mathcal{W}^B_A A^\gamma D^\gamma C) \nabla_b. 
\end{align*}
\]
Proof. This is a straightforward consequence of Lemma 15. We notice that the contribution

\[ X^b A X^d C \left( - \frac{1}{4} R_{bd}^{r s} \gamma_r \gamma^s - \frac{1}{2} P_{r[b} \gamma^r \gamma_{d]} \right) \]  

(5.3)

is trivial due to the projective curvature tensor identity in real dimension 2, \( R_{bd}^{r s} \gamma_r \gamma^s = 2 P_{s[d} \gamma_{b]} \gamma^s \), and an elementary identity in the symplectic Clifford algebra.

Lemma 17. The commutator in the composition of \( D^B A \) and \( D^D C \),

\[ [D, D]_A^B = (\text{tr}(D \wedge D))_A^B = D^B R_D R_A - D^R A D^B R, \]  

(5.4)

is related to (5.2) via

\[ D^B A D^D C - D^D C D^B A = \frac{2}{3} \delta^{[A}_{B} (D[D, D]|_C)^{B]} - \frac{2}{3} \delta^{[B}_{C} (D[D, D]|_A)^{D)}. \]  

(5.5)

Proof. We recall the formula for the tractor volume form \( \epsilon^{[ABC]} := X^a [A X^b Y^c C] \omega^{bc} \), which satisfies the following properties:

\[ \epsilon^{EAC} X^a A X^c C = \frac{1}{3} X^E \omega^{ac}, \]

\[ \epsilon^{EAC} X^b [A Y_C] = -\frac{2}{3} X^A Y_e ^E Y_e C \omega^{ec} X^b [A Y_C] = \frac{1}{3} Y_e ^E \omega^{eb}. \]  

(5.6)

The contraction of

\[ 2D^{[B}_{A} D^D C] = \frac{1}{2} Y^B \gamma^D d X^a A X^c C (\gamma^{b (d]} \omega_{ac} + 2 \delta^{[b}_{[a} \gamma^{d]} \gamma_{c]} ) \]

\[ - X^{(B Y^D)} d X^b [A Y_C] (\gamma^d b + \delta^d b) \]

\[ + 2 X^{(B Y^D)} d X^b [A X^c C] \delta^d [b \nabla c] \]

\[ - 2 X^{(B Y^D)} d X^b [A Y_C] \nabla_b \]  

(5.7)

by \( \epsilon^{EAC} \) yields (notice that the last two terms on the last display cancel out after contraction by \( \epsilon \) due to (5.6))

\[ \epsilon^{EAC} D^{[B}_{A} D^D C] = \frac{1}{3} X^E Y^B Y^D \gamma^{(b} \gamma^{d]} - \frac{1}{3} Y^E X^B Y^D \gamma^{(d} \gamma^{b]} = 0. \]  

(5.8)

This result implies immediately that the only irreducible component in the skew-symmetric part of the composition is the Lie bracket, which completes the proof.

Lemma 18. The skew-symmetric component in the composition of two operators \( D^D C \) contains only the trace part, and can consequently be simplified as

\[ [D, D]_B^C = D^B R_D R_C - D^R C D^B R = 3 D^B C. \]  

(5.9)
Proof. We apply the trace $\delta^A_B$ to (5.2), and use the identities (3.9), (3.10) and (3.11):

\[
[D, D]_{BC} = \frac{1}{8} \mathbb{Z}_{ab}^{cB} C (2 \gamma^b \gamma_c + 2 \delta^b_c + \gamma_c \gamma^b + \gamma^b \gamma_c)
- \delta^b_d \mathbb{W}_{BC} (\frac{1}{4} \gamma^d \gamma_b + \delta^d_b) + \mathbb{Z}_{ab}^{bB} C (\frac{1}{4} \gamma^d \gamma_b + \delta^d_b)
+ \mathbb{X}_{cB}^B \delta^b_d \delta^d_c \nabla_c + \mathbb{X}_{cB}^B \nabla_b.
\] (5.10)

By elementary manipulations in the symplectic Clifford algebra, the last display equals to

\[
3(\mathbb{Z}_{b}^{cB} C (\frac{1}{4} \gamma^b \gamma_c + \frac{1}{2} \delta^b_c) - \frac{1}{2} \mathbb{W}_{BC} + \mathbb{X}_{bB}^B \nabla_b),
\] (5.11)

which proves by (3.25) the statement of the lemma.

Now we pass to the analysis of the symmetric part

\[
(D \circ D)^{A}_{D C} = (D^{B} A D^{D}_{C} + D^{D}_{C} D^{B} A)
\] (5.12)

of the composition $D^{A}_{B} D^{C}_{D}$. The following notation turns out to be convenient for our purposes:

\[
(D^2)^{A}_{B} = D^{A}_{R} D^{R}_{B} + D^{R}_{B} D^{A}_{R}, \quad (D, D) = D^{S}_{R} D^{R}_{S}, \quad D^2_o = tf(D^2).
\] (5.13)

**Lemma 19.** The composition $D^{R}_{B} D^{A}_{R}$ of $D^{R}_{B}$ is given by

\[
D^{R}_{B} D^{A}_{R} = \frac{1}{8} \mathbb{Z}_{ad}^{aA} B (-3 \gamma^d \gamma_a - 10 \delta^d_a) + \mathbb{X}_{bA}^B \delta^d_b (\gamma^b \gamma_a + \frac{1}{2} \delta^b a) \nabla_d
- \frac{3}{4} \mathbb{X}_{bA}^A B \nabla_c + \frac{1}{4} \mathbb{W}_{AB}.
\] (5.14)

Consequently, we have

\[
(D^2)^{A}_{B} = -\delta^A_B + \frac{1}{2} \mathbb{X}_{bA}^B \gamma_b \mathcal{D},
\]

\[
(D^2_o)^{A}_{B} = \frac{1}{2} \mathbb{X}_{bA}^B \gamma_b \mathcal{D}.
\] (5.15)

**Proof.** The first equality (5.14) is the trace component of Lemma (15). By (5.9), we have

\[
(D^2)^{A}_{B} = D^{A}_{R} D^{R}_{B} + D^{R}_{B} D^{A}_{R} = (D^{A}_{R} D^{R}_{B} - D^{R}_{B} D^{A}_{R}) + 2 D^{R}_{B} D^{A}_{R}
= 3 D^{A}_{B} + 2 D^{R}_{B} D^{A}_{R},
\] (5.16)

and the substitution from an elementary formula

\[
D^{R}_{B} D^{A}_{R} = -\frac{3}{2} D^{A}_{B} - \frac{1}{2} \delta^A_B + \frac{1}{4} \mathbb{X}_{bA}^A B \gamma_b \mathcal{D}
\] (5.17)

implies (5.15). We used the identity decomposition $\delta^A_B = \mathbb{Z}_{a}^{aA} B + \mathbb{W}_{aA}^A$ in order to separate the trace and the trace free components. The proof is complete.
Lemma 20. The symmetric component in the composition of two operators $D^A_B$ equals to

$$\frac{1}{2}(D^A_B D^C_D + D^C_D D^A_B) = D^{(A (B D^C)_D)} + D^{[A [B D^C]_D]},$$

(5.18)

where

$$D^{(A (B D^C)_D)} = \frac{1}{2}(D^{(A (B D^C)_D)} + \frac{2}{5} \delta^{(A (B D^2)_C)_D})$$

$$+ \frac{1}{12} \delta^{(A (B \delta^C)_D)_D} (D, D),$$

$$D^{[A [B D^C]_D]} = -2\delta^{[A [B (D^2)_C]_D]} - \frac{1}{6} \delta^{[A [B \delta^C]_D] (D, D)}.$$  

(5.19)

Moreover, $(D, D) = -\frac{3}{2}$.

Proof. As for the term $D^{(A (B D^C)_D)}$, we can write

$$D^{(A (B D^C)_D)} = \frac{1}{2}(D^{(A (B D^C)_D)} + \frac{2}{5} \delta^{(A (B D^2)_C)_D})$$

$$+ \frac{1}{12} \delta^{(A (B \delta^C)_D)_D} (D, D)$$

(5.20)

for some $\tilde{A}, \tilde{B} \in \mathbb{C}$. The double trace $\delta^C_D, \delta^A_D$ of the first equality in (5.19) implies $\tilde{B} = \frac{1}{12}$, while its first trace $\delta^C_D$ gives after the substitution for $\tilde{B}$ the value $\tilde{A} = \frac{2}{5}$. The second equality in (5.19) for $D^{[A [B D^C]_D]}$ is proved analogously.

The last claim follows by taking the trace component of the first equality in (5.15), or equivalently (5.17). The proof is complete. □

Lemma 21. Let $I^A_B, \bar{I}^A_B \in E^A_B$ be parallel sections of the projective adjoint tractor bundle. Then $I^A_B D^B_A, \bar{I}^A_B D^B_A$ are first order symmetry operators of $D_s$, and there is $\mathfrak{sl}(3, \mathbb{R})$-invariant decomposition

$$I^A_B D^B_A I^C_D D^D_C = (I \otimes I)^{(A (B C)_D)_D} D^B_A D^C_D + \frac{6}{5} (I \otimes \bar{I})^A_B D^B_D C^B_A$$

$$+ \frac{1}{6} (I, \bar{I})^A_B D [D, D] B^B_A + \frac{1}{5} (I, \bar{I}) (D, D).$$

(5.21)

Proof. By (3.24) and the assumption that $I, \bar{I}$ are parallel tractors, $I^A_B D^B_A$ and $\bar{I}^A_B D^B_A$ are first order symmetry operators of $D_s$.

Secondly, we have by Lemma 17 and Lemma 20

$$I^A_B \bar{I}^C_D D^B_A D^D_C = \frac{1}{2} I^A_B \bar{I}^C_D (D^B_A D^D_C + D^D_C D^B_A)$$

$$+ \frac{1}{2} I^A_B \bar{I}^C_D (D^B_A D^D_C - D^D_C D^B_A)$$

$$= (I \otimes \bar{I})^A_B D^B_A D^D_C$$

$$+ I^A_B \bar{I}^C_D \left( \frac{2}{5} \delta^{(A (B D^2)_D)} - \delta^{(A (B D^2)_D)} D^B_A \bar{D}^D_C \right)$$

$$+ I^A_B \bar{I}^C_D \left( \frac{2}{5} \delta^{(A \delta^D)_D} (D, D) - \frac{1}{6} \delta^{(A \delta^D)_D} (D, D) \right)$$

$$+ \frac{1}{3} I^A_B \bar{I}^C_D \left( \delta^{(A [B D, D]_C)_D} D^B_A D^D_C + \delta^{(A [D, D]_C)_D} D^B_A D^D_C \right).$$
and the substitution of identities (4.4) yields the claim. The proof is complete.

**Theorem 22.** Let $I^A_B, \bar{I}^A_B \in E^A_B$ be parallel sections of the projective adjoint tractor bundle $A_\delta$ corresponding to the first order symmetry operators $S^g = I^A_B D^B_A$ and $S^\bar{g} = \bar{I}^A_B D^B_A$ of $\bar{\mathcal{D}}_s$. Then their composition equals

$$S^g \circ S^\bar{g} = I^A_B D^B_A \bar{I}^C D^D_C$$

$$= (I \otimes \bar{I})^{(A \otimes C)} (D_B D^B_A) \bar{I}^C D^D_C + \frac{1}{2} [I, \bar{I}]^A_B D^B_A - \frac{3}{16} \langle I, \bar{I} \rangle \mod \bar{\mathcal{D}}_s,$$  

hence the symmetry algebra of $\bar{\mathcal{D}}_s$ is isomorphic to the quotient of the tensor algebra $\bigoplus \otimes^k (\mathfrak{sl}(3, \mathbb{R}))$ by a two sided ideal generated by quadratic relations

$$I \otimes \bar{I} - I \otimes \bar{I} - \frac{1}{2} [I, \bar{I}] + \frac{3}{16} \langle I, \bar{I} \rangle.$$  

(5.22)

Equivalently, the symmetry algebra of $\bar{\mathcal{D}}_s$ is the quotient of the universal enveloping algebra $U(\mathfrak{sl}(3, \mathbb{R}))$ by a two sided ideal generated by quadratic relations

$$I \bar{I} + \bar{I} I - 2I \otimes \bar{I} + \frac{3}{2} \langle I, \bar{I} \rangle, \quad I, \bar{I} \in \mathfrak{sl}(3, \mathbb{R}).$$  

(5.23)

**Proof.** The proof goes along the same lines as in e.g., [17], so we shall give just a brief account of its exposition.

The identification of $g = \mathfrak{sl}(3, \mathbb{R})$ with differential symmetries is given by the mapping $I^A_B \mapsto \bar{I}^A_B D^B_A$, where $I^A_B$ is a parallel section of the projective adjoint tractor bundle $A_\delta$. This extends to

$$g \otimes \ldots \otimes g \mapsto (I^A_B D^B_A) \ldots (\bar{I}^C D^D_C)$$

with $I^A_B \otimes \ldots \otimes \bar{I}^D_C \in g \otimes \ldots \otimes g$, and hence to the full tensor algebra $\bigoplus \otimes^k g$ by linearity.

The first step in the proof is to express the composition $I^A_B D^B_A \bar{I}^C D^D_C$ for $I^A_B, \bar{I}^A_B \in g$ in terms of canonical symmetries. This was already done in Lemma [21].

To finish the proof, we have the following observations. The mapping (5.25) determines an associative algebra morphism $\bigoplus \otimes^k g \rightarrow \bar{\mathcal{A}}$ with $\mathcal{A}$ the algebra of symmetries, cf. the paragraph beyond (2.22), which is surjective as a consequence of the fact that the canonical
symmetries $I^4_B$ arise in the range of (5.25) (cf., Lemma 21). We want to find all relations, that is to identify the two-sided ideal of this algebra morphism. As we already identified the generators of the ideal (5.23), it remains to show that this ideal is large enough to have $A$ as the resulting quotient.

Since we know $A$ as a vector space, cf. Section 4, it is sufficient to consider the associated graded algebra (the symbol algebra of $A$.) The corresponding graded ideal contains $I \otimes I - I \mathcal{S} I$ for $I, I \in g$, hence contains the skew-symmetric component of the tensor product $g \wedge g$. Therefore, we can pass to symmetric tensors $\odot g$ in the tensor algebra and write $I$ for the ideal defined as the image of (5.23). Now we claim that as for the associated graded $A = \bigoplus_k A^k$, where the $A^k$ are defined as the $g$-submodules satisfying $A^k = \{ F(A_i(B_1 \cdots A_k)B_k) \text{ with all traces zero} \} \subset \odot^k g$.

To finish the proof, we need to show the vector space decomposition $\odot^k g = A^k \oplus I^k$ for $I^k := I \cap \odot^k g$. This is based on the following observation:

\[
(g \otimes A^{k-1}) \cap (A^{k-1} \otimes g) = A^k,
\]

which is elementary to check directly for $\mathfrak{sl}(3, \mathbb{R})$ (and at the same time holds for $\mathfrak{sl}(n, \mathbb{R})$ in general): the inclusion $\supseteq$ is obvious, and to prove the inclusion $\subseteq$ we consider $F(A_1B_1 \cdots A_kB_k)$ living in the intersection on the left hand side of the display. Then

\[
F(A_1B_1 \cdots A_i \cdots A_j \cdots A_kB_k) = F(A_1B_1 \cdots A_j \cdots A_i \cdots A_kB_k)
\]

for any $1 \leq i < j \leq k$. A similar conclusion holds for the symmetry in the collection of lower indices as well as for the trace-freeness.

The final step relies on the following standard fact in the representation theory of simple Lie algebras. There is a projection $\odot^k g \rightarrow A^k$ such that the induced projections $P^k : \odot^k g \rightarrow g \otimes A^{k-1}$ and $\tilde{P}^k : \odot^k g \rightarrow A^{k-1} \otimes g$ have their kernels in $g \otimes I^{k-1}$ and $I^{k-1} \otimes g$, respectively. In particular, it is contained in $I^k$ for both cases. By standard dimensional considerations in linear algebra,

\[
\odot^k g = (\text{Im}(P^k) \cap \text{Im}(\tilde{P}^k)) \oplus (\text{Ker}(P^k) + \text{Ker}(\tilde{P}^k))
\]

for all $k \geq 3$, so the claim above follows. This completes the proof of theorem.

It is well known (cf., [20], Section 4) that the representation of $\mathfrak{sl}(3, \mathbb{R})$ on Ker($\mathcal{D}_s$) is an unitary irreducible representation equivalent to the exceptional representation associated with the minimal
nilpotent orbit of $\mathfrak{sl}(3, \mathbb{R})$, cf. [6, 12, 21]. This result is based on the analysis of K-types in the underlying Harish-Chandra module of $\text{Ker}(\mathcal{H})$. We note that in the case of simple Lie algebras $A_n, n \in \mathbb{N}$, the Joseph ideal in $U(\mathfrak{g})$ is not uniquely defined and there is a one parameter family of completely prime primitive ideals having as its associated variety the minimal nilpotent orbit.

6 Comments and open questions

Let us conclude by observing that higher symmetries of $\mathcal{H}$ for dimensions $2n > 2$ are not induced from a semi-simple Lie algebra of symmetries. In particular, it is straightforward to see that for $(\mathbb{R}^{2n}, \omega)$ and the flat symplectic connection $\nabla$, a general first order symmetry differential operator is of the form

$$O = v^a \nabla_a + \sum_{j=0}^{\infty} \gamma^{a_1} \ldots \gamma^{a_{2j}} w^{j}_{a_1 \ldots a_{2j}}, \quad w^{j}_{a_1 \ldots a_{2j}} \in E_{(a_1 \ldots a_{2j})}$$

(with $w^{j}_{a_1 \ldots a_{2j}} = 0$ for almost all $j \in \mathbb{N}_0$) and the symbol $v^a$ fulfilling the differential system

$$\nabla_{(a} \nabla_{b} v_{c)} = 0 \quad \text{and} \quad \nabla^{[a} v_{b]} = 0,$$

where the subscript 0 indicates the “trace-free” part with respect to $\omega$.

Prolonging this system, we obtain the bundle $\mathcal{T} := E_a \oplus E_{(ab)} \oplus E$ with the connection

$$\nabla_c \left( \begin{array}{c} v_a \\ w_{ab} \\ \varphi \end{array} \right) = \left( \begin{array}{c} \nabla_c v_a - w_{ca} - \varphi \omega_{ca} \\ \nabla_c w_{ab} \\ \nabla_c \varphi \end{array} \right)$$

for $(v_a, w_{ab}, \varphi) \in E_a \oplus E_{(ab)} \oplus E$. In particular, the solution space of the system (6.1), i.e., the Lie algebra of first order symmetries of $\mathcal{H}$, is isomorphic to the space of covariantly constant sections of $\mathcal{T}$. This isomorphism is given by the projection to the top slot in one direction and by the differential splitting

$$v_a \mapsto \left( \begin{array}{c} \nabla_{(a} v_{b)} \\ v_a \\ \frac{1}{\pi} \omega^{kl} \nabla_k v_l \end{array} \right),$$

in the opposite direction. From this one can easily see that the solution space is a Lie algebra given by the semidirect product of $\mathfrak{sp}(n) \oplus \mathbb{R}$ with its representation on $\mathbb{R}^{2n}$.  

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So far we discussed higher symmetries for projectively flat manifolds in the real dimension 2 and indicated an analogous problem for flat affine symplectic manifolds in higher dimensions. The curved setting is far more complicated. However, we expect that at least the case of second order symmetries is this task manageable in the sense that one can find symmetry operators explicitly on the assumption of certain curvature conditions.

We shall analyze these questions in more detail elsewhere.

Acknowledgments

P. Somberg and J. Šilhan acknowledge the financial support from the grant GA CR P201/12/G028.

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