Loop quantum cosmology of Bianchi type IX models

Edward Wilson-Ewing*

Institute for Gravitation and the Cosmos, Physics Department,
The Pennsylvania State University, University Park, PA 16802, USA

The loop quantum cosmology “improved dynamics” of the Bianchi type IX model are studied. The action of the Hamiltonian constraint operator is obtained via techniques developed for the Bianchi type I and type II models, no new input is required. It is shown that the big bang and big crunch singularities are resolved by quantum gravity effects. We also present the effective equations which provide modifications to the classical equations of motion due to quantum geometry effects.

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I. INTRODUCTION

Loop quantum cosmology (LQC) [1, 2] is an approach to quantum cosmology following the ideas of loop quantum gravity (LQG) [3–5]. One of the major results of LQC in the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) models is that, while general relativity approximates the dynamics very well in the low (with respect to the Planck scale) curvature regime, the classical big bang singularity is avoided: when the matter energy density approaches the Planck energy density, deviations from general relativity become significant and a “quantum bounce” due to quantum gravity effects occurs when the matter energy density reaches a critical energy density of the order of the the Planck density [6–14].

More recently, it has been shown that the singularity is also resolved in the improved dynamics approach of loop quantum cosmology in the anisotropic Bianchi type I and type II cosmological models [15, 16] and in the hybrid loop-Fock quantization of the inhomogeneous Gowdy model [17]. The goal of this paper is to extend the LQC improved dynamics analysis of the Bianchi type I and type II models to the more complicated Bianchi type IX models.

At the classical level, the Bianchi IX model has a much richer phenomenology than Bianchi I and II models as it displays Mixmaster dynamics as the singularity is approached [18]. In essence, a space-time which exhibits Mixmaster dynamics is one which can be described for long periods of time (known as epochs) as a Bianchi I space-time characterized by three anisotropic expansion rates. Such a space-time will occasionally undergo a “Mixmaster bounce” from one epoch to another where the three expansion rates change in a specific manner. Bianchi I models approach the singularity in a rather straightforward way as they do not undergo any Mixmaster bounces while Bianchi II models may undergo a single Mixmaster bounce between two epochs as the singularity is approached (see [18] and references therein). The Bianchi IX model, on the other hand, undergoes many Mixmaster bounces and this behaviour is chaotic [18, 19]. Since much of this behaviour occurs when the curvature is of the Planck scale, quantum gravity effects cannot be neglected and the Mixmaster behaviour may be significantly modified when they are taken into account.

*Electronic address: wilsonewing@gravity.psu.edu
Bianchi IX models are also thought to play an important role near generic singularities in classical general relativity. The Belinskii, Khalatnikov, Lifshitz (BKL) conjecture suggests that as a generic space-like singularity is approached, time derivatives dominate over spatial derivatives whence physical fields at each point evolve independently from those at neighbouring points. Dynamics can therefore be approximated by the ODE’s used in homogeneous space-times, most generally a Bianchi IX solution with a massless scalar field [20, 21]. Since other matter fields do not contribute to the dynamics as the singularity is approached, we will only consider the case of a massless scalar field in this work. There has recently been a considerable amount of numerical work supporting this paradigm (see, e.g., [18]) and the conjecture has also been rewritten in terms of variables suitable to a loop quantization [22]. If the BKL conjecture is indeed correct, it follows that a good understanding of the quantum dynamics of the Bianchi IX model in the deep quantum regime could lead to major insights into the behaviour of generic space-times in regions where the curvature reaches the Planck scale.

Because of the Bianchi IX model’s importance, it has already been the subject of studies within the framework of loop quantum cosmology, both in a pre-$\mu_o$-type Hamiltonian framework [23, 24] and in a spin-foam-inspired dipole cosmology model (first introduced for the isotropic case in [12]) which also allows inhomogeneities [25]. However, it is important to study the improved $\bar{\mu}_i$-type dynamics of LQC since it has been shown that the predictions of the pre-$\mu_o$ approach are unphysical in the infrared limit. In particular, in isotropic cosmological models quantum gravity effects can become important at energy densities arbitrarily below the Planck scale in this scheme. To ensure that quantum gravity effects only become important at the Planck scale, one must instead use the improved dynamics approach in the Hamiltonian framework\(^1\). On the other hand, since the dipole cosmology model presented in [25] is inspired by spin foam models, that approach is complementary to ours and it will be interesting to compare the results of these two frameworks.

As pointed out above, chaotic behaviour appears as the singularity is approached in classical Bianchi IX space-times. It has been argued in the pre-$\mu_o$ LQC treatment of the model that this behaviour is avoided in LQC due to quantum gravity effects [24]. In essence, the argument is that quantum gravity effects become important before a significant number of Mixmaster bounces occur. Since the quantum gravity effects are repulsive, the space-time will exit the Planck regime having only undergone a small number of Mixmaster bounces and hence the dynamics are not chaotic. In this paper, we see that in some cases this occurs already in the effective theory which incorporates the quantum geometry effects into the dynamics. However, we cannot yet show that this is a generic result. We will go into more detail in Sec. IV.

The outline of the paper is as follows. In Sec. II, we will briefly review the necessary classical properties of the Bianchi type IX model in order to proceed with the quantization. In Sec. III we will study the quantum properties of the model, first recalling the kinematics which are the same as for the Bianchi type I and type II models studied in [15, 16]. We will then study the Hamiltonian constraint operator for the Bianchi IX model with a massless scalar field as the matter field. The Hamiltonian constraint operator gives an evolution equation where the scalar field acts as a relational time parameter. The dynamics of the model are obtained by using the same technology that was developed during the study of the

\(^1\) It is possible that once the curvature reaches the Planck scale a scheme other than $\bar{\mu}_i$ may be the correct one but we will only consider the $\bar{\mu}_i$ scheme here.
improved LQC dynamics of the Bianchi type I and type II models [15, 16]; *there is no need to introduce any new operators*. In Sec. IV we will derive effective equations which provide the first order quantum corrections to the classical equations of motion and in Sec. V we summarize our results and discuss open issues.

II. CLASSICAL THEORY

In Bianchi models [26–28], one restricts oneself to those phase space variables which admit a 3-dimensional group of symmetries which act simply and transitively. The symmetries allowed in the Bianchi IX group are the three spatial rotations on $S^3$. It follows that the three Killing (left invariant) vector fields $\xi^a_i$ satisfy

$$[\xi_i, \xi_j] = \frac{2}{r_o} \epsilon^k_{ij} \xi_k, \quad (2.1)$$

where the structure constants are given by the completely antisymmetric tensor $\epsilon_{ijk}$ times $2/r_o$ where $r_o$ is the radius of the 3-sphere with respect to the fiducial metric. $\epsilon_{ijk}$ is defined such that $\epsilon_{123} = 1$, note that the internal indices $i, j, k, \ldots$ can always be freely raised and lowered. There is also a canonical triad $\hat{e}^a_i$—the right invariant vector fields— which is Lie dragged by $\xi^a_i$. It is convenient to use $\hat{e}^a_i$ and its dual co-triad $\hat{\omega}^i_a$ as fiducial frames and co-frames. They satisfy:

$$[\hat{e}^i, \hat{e}^j] = -\frac{2}{r_o} \epsilon^k_{ij} \hat{e}_k, \quad d\hat{\omega}^k = \frac{1}{r_o} \epsilon^k_{ij} \hat{\omega}^i \wedge \hat{\omega}^j. \quad (2.2)$$

The form of the equations above indicates that $M$ admits global coordinates $\alpha \in [0, 2\pi), \beta \in [0, \pi)$ and $\gamma \in [0, 4\pi)$ such that for $r_o = 2$ the Bianchi IX co-triads have the form

$$\hat{\omega}^1_a = \sin \beta \sin \gamma (d\alpha)_a + \cos \gamma (d\beta)_a, \quad \hat{\omega}^2_a = -\sin \beta \cos \gamma (d\alpha)_a + \sin \gamma (d\beta)_a, \quad \hat{\omega}^3_a = \cos \beta (d\alpha)_a + (d\gamma). \quad (2.3)$$

The fiducial co-triads determine a fiducial 3-metric $\hat{q}_{ab} := \hat{\omega}^i_a \hat{\omega}^i_b$,

$$\hat{q}_{ab} dx^a dx^b = d\alpha^2 + d\beta^2 + d\gamma^2 + 2 \cos \beta d\alpha d\gamma. \quad (2.4)$$

It follows that $\sqrt{q} = \sin \beta$ and one can see that $\hat{q}_{ab}$ is the metric of a 3-sphere with a volume $V_o = 16\pi^2$, this agrees with $V_o = 2\pi^2 r_o^3$ for $r_o = 2$ as specified above. Finally, we introduce the length-scale $\ell_o = V_o^{1/3}$ for later convenience.

In diagonal Bianchi models, the physical triads $e^a_i$ are related to the fiducial ones by

$$\omega^i_a = a^i(\tau) \hat{\omega}^i_a \quad \text{and} \quad a^i(\tau) e^a_i = \hat{e}^a_i, \quad (2.5)$$

---

2 Here we are following the conventions used in [25]. A different, although equivalent, choice is used in [10, 23, 24] where the structure constants differ by an overall sign.

3 There is no sum if repeated indices are both covariant or contravariant. As usual, the Einstein summation convention holds if a covariant index is contracted with a contravariant index.
where the \( a_i \) are the three directional scale factors.

For later use, let us calculate the spin connection determined by physical triads \( e_i^a \). Since \( \Gamma^i_a \) is given by

\[
\Gamma^i_a = -\varepsilon^{ijk} e_j^b \left( \partial_a \omega_{bj}^k + \frac{1}{2} e^c_k \omega^l_a \partial_l \omega_{bj}^c \right),
\]

(2.6)

it follows that

\[
\Gamma^1_a = \varepsilon \frac{\ell_o}{r_o} \left( \frac{a_1^2}{a_2 a_3} - \frac{a_2}{a_3} - \frac{a_3}{a_2} \right) \dot{\omega}_a^1,
\]

(2.7)

where \( \varepsilon := \varepsilon_{123} \) is +1 for right-handed physical triads and -1 for left-handed physical triads. Note that while the \( \dot{\varepsilon}^{ijk} \) appearing in the Bianchi IX structure constants are not affected by the handedness of the physical triads, \( \varepsilon^{ijk} \) and \( \varepsilon \) on the other hand are affected by the handedness of \( e_i^a \). \( \Gamma^2_a \) and \( \Gamma^3_a \) can be obtained by permutations of Eq. (2.7).

As is usual in LQC, we will now use the fiducial triads and co-triads in order to introduce a convenient parametrization of the phase space variables \( E_i^a \) and \( A_i^a \). Because we have restricted ourselves to the diagonal model and these fields are symmetric under the Bianchi IX group, from each equivalence class of gauge related phase space variables we can choose a pair of the form

\[
E_i^a = \frac{p_i}{\ell_o^2} \sqrt{|q|} \varepsilon_i^a \quad \text{and} \quad A_i^a = \frac{c_i}{\ell_o} \dot{\omega}_i^a,
\]

(2.8)

where, as spelled out in footnote 3, there is no sum over \( i \). Note that the length \( \ell_o \) plays a similar role to that of the lengths of the fiducial cell in noncompact space-times in terms of the form of the basic variables \( (A, E) \). In this case the manifold is compact and there is no fiducial cell.

It is straightforward to relate the scale factors \( a_i \) to the \( p_i \):

\[
p_1 = \text{sgn}(a_1)|a_2 a_3|\ell_o^2, \quad p_2 = \text{sgn}(a_2)|a_1 a_3|\ell_o^2, \quad p_3 = \text{sgn}(a_3)|a_1 a_2|\ell_o^2,
\]

(2.9)

it follows that \( \sqrt{|q|} = \sqrt{|p_1 p_2 p_3| V_o^{-1}} \sqrt{q} \).

Thus, a point in the phase space is now coordinatized by six real numbers \( (p_i, c^i) \). One can use the symplectic structure in full general relativity to induce a symplectic structure on the six-dimensional phase space. The non-zero Poisson brackets are given by

\[
\{c_i, p_j\} = 8\pi G \gamma \delta_{ij},
\]

(2.10)

where \( \gamma \) is the Barbero-Immirzi parameter.

Our choice (2.8) of physical triads and connections has fixed the internal gauge as well as the diffeomorphism freedom. Furthermore, it is easy to explicitly verify that the Gauss and the diffeomorphism constraints are automatically satisfied due to Eq. (2.8). Thus we are left with the Hamiltonian constraint

\[
\mathcal{C}_H = \int_{\mathcal{M}} \left[ -\frac{N}{16\pi G^2 \sqrt{|q|}} \varepsilon^{ij}_k \left( F^{ab}_k - (1 + \gamma^2) \Omega^{ab}_k \right) + N \mathcal{H}_\text{matt} \right] d^3 x \approx 0,
\]

(2.11)

where \( F^{ab}_k \) and \( \Omega^{ab}_k \) are the curvature of \( A^i_a \) and \( \Gamma^i_a \) respectively, while \( \mathcal{H}_\text{matt} \) is the matter Hamiltonian density. The \( \approx 0 \) indicates that \( \mathcal{C}_H \) is a constraint and must vanish for physical solutions. Since we are most interested in the gravitational sector, our matter field will consist only of a massless scalar field \( T \) which will later serve as a relational time variable.
à la Liebniz. (Additional matter fields can be incorporated in a straightforward manner, modulo possible intricacies of essential self-adjointness.) Thus,

\[ \mathcal{H}_{\text{matt}} = \frac{1}{2} \frac{p_v^2}{\sqrt{|q|}} \]  

(2.12)

Since we want to use the massless scalar field as relational time, it is convenient to use a harmonic-time gauge, i.e., assume that the time coordinate \( \tau \) satisfies \( \Box \tau = 0 \). The corresponding lapse function is \( N = \sqrt{|p_1p_2p_3|} \). With this choice, the Hamiltonian constraint simplifies considerably.

In terms of \( p_i \), the first component of the spin connection is given by

\[ \Gamma^1_a = \frac{\varepsilon}{r_\circ} \left( \frac{p_2 p_3}{p^2_1} - \frac{p_2}{p_3} - \frac{p_3}{p_2} \right) \dot{\omega}^1_a, \]  

(2.13)

the other two spin connection components can be obtained via permutations. The curvature of \( \Gamma^i_a \) is in turn

\[ \Omega_{ab}^1 = 2 \partial_{[a} \Gamma^1_{b]} + \epsilon^{ijk} \Gamma^j_a \Gamma^k_b \]

\[ = \frac{2 \varepsilon}{r_\circ^2} \left( 3 \frac{p_2 p_3}{p^2_1} + 2 \frac{p^2_1}{p_2 p_3} - 2 \frac{p_2}{p_3} - 2 \frac{p_3}{p_2} - \frac{p^2_1 p_2}{p^3_3} - \frac{p^2_3 p_2}{p^3_3} \right) \dot{\omega}^2_{[a} \dot{\omega}^3_{b]}, \]  

(2.14)

the other components of \( \Omega_{ab}^k \) can again be obtained via permutations.

Finally, it is straightforward to calculate the curvature of \( A^i_a \). For example,

\[ F_{ab}^1 = 2 \partial_{[a} A^1_{b]} + \epsilon^{ijk} A^j_a A^k_b \]

\[ = 2 \left( \frac{2c_1}{r_\circ} + \frac{\varepsilon c_2 c_3}{r_\circ^2} \right) \dot{\omega}^2_{[a} \dot{\omega}^3_{b]}. \]  

(2.15)

Using these results, one finds that the Hamiltonian constraint (2.11) is given by

\[ \mathcal{C}_H = -\frac{1}{8 \pi G \gamma^2} \left( p_1 p_2 c_1 c_2 + p_2 p_3 c_2 c_3 + p_3 p_1 c_1 c_3 + \frac{2 \ell_\circ \varepsilon}{r_\circ} (p_1 p_2 c_3 + p_2 p_3 c_1 + p_3 p_1 c_2) \right. 

\[ + \frac{\ell_\circ^2}{r_\circ^2} (1 + \gamma^2) \left[ 2 p_1^2 + 2 p_2^2 + 2 p_3^2 - \left( \frac{p_1 p_2}{p_3} \right)^2 - \left( \frac{p_2 p_3}{p_1} \right)^2 - \left( \frac{p_3 p_1}{p_2} \right)^2 \right] \right) 

\[ + \frac{1}{2} p_T^2 \approx 0. \]  

(2.16)

Note that the constraint for the closed isotropic case is recovered for \( p_1 = p_2 = p_3 \) while the Bianchi I constraint is recovered in the limit \( r_\circ \to \infty \) or, equivalently, \( \ell_\circ \to 0 \). We will take advantage of this correspondence and set \( r_\circ = 2 \) for the remainder of the paper. The Bianchi I limit can be obtained by taking \( \ell_\circ \to 0 \).

One can now derive the time evolution of any classical observable \( \mathcal{O} \) by taking its Poisson bracket with \( \mathcal{C}_H \):

\[ \dot{\mathcal{O}} = \{ \mathcal{O}, \mathcal{C}_H \}, \]  

(2.17)

where the ‘dot’ stands for derivative with respect to the harmonic time \( \tau \). This gives

\[ \dot{p}_1 = \frac{p_1}{\gamma} \left( p_2 c_2 + p_3 c_3 + \ell_\circ \varepsilon \frac{p_2 p_3}{p_1} \right), \]  

(2.18)
\[
\dot{c}_1 = -\frac{1}{\gamma} \left( p_2 c_1 c_2 + p_3 c_1 c_3 + \ell_0 \varepsilon (p_2 c_3 + p_3 c_2) + \ell_0^2 (1 + \gamma^2) \left( p_1 + \frac{p_2^2 p_3^2}{2p_1^2} - \frac{p_1 p_2^2}{2p_3^2} - \frac{p_1 p_3^2}{2p_2^2} \right) \right). \tag{2.19}
\]

As usual, the other equations of motion can be obtained by permutations. Any initial data satisfying the Hamiltonian constraint can be evolved by these equations of motion. It is particularly interesting to study the Hubble rates \( H_i \) which are given by

\[
H_i = \frac{1}{a_i} \frac{da_i}{dt}, \tag{2.20}
\]

where \( t \) is the proper time and is related to the harmonic time \( \tau \) (which is the time coordinate used until now) by

\[
\frac{d}{dt} = \frac{1}{\sqrt{|p_1 p_2 p_3|}} \frac{d}{d\tau}. \tag{2.21}
\]

It follows that the Hubble rates are related to the \((c_i, p_i)\) by, e.g.,

\[
c_1 p_1 = \gamma \sqrt{|p_1 p_2 p_3|} H_1 + \frac{\ell_0}{2} \left( \frac{p_2 p_3}{p_1} - \frac{p_1 p_2}{p_3} - \frac{p_1 p_3}{p_2} \right). \tag{2.22}
\]

The mean Hubble rate \( H \) of the mean scale factor \( a = (a_1 a_2 a_3)^{1/3} \) is given by

\[
H = \frac{1}{a} \frac{da}{dt} = \frac{1}{3} (H_1 + H_2 + H_3), \tag{2.23}
\]

and the Friedmann equation is

\[
H^2 = \frac{8\pi G}{3} \rho + \frac{1}{6} \sigma^2 - \frac{\ell_0^2}{12} V(p), \tag{2.24}
\]

where the energy density of the scalar field is \( \rho = p_1^2/2|p_1 p_2 p_3| \), the shear term is given by

\[
\sigma^2 = \frac{1}{3} [(H_1 - H_2)^2 + (H_2 - H_3)^2 + (H_3 - H_1)^2], \tag{2.25}
\]

and the potential is

\[
V(p) = \frac{1}{p_1 p_2 p_3} \left[ 2(p_1^2 + p_2^2 + p_3^2) - \left( \frac{p_2 p_3}{p_1} \right)^2 - \left( \frac{p_3 p_1}{p_2} \right)^2 - \left( \frac{p_1 p_2}{p_3} \right)^2 \right]. \tag{2.26}
\]

Clearly, these dynamics are quite complex already at the classical level and, as mentioned in the introduction, become chaotic as a singularity is approached. The one exception is the case when the matter field is a massless scalar field which is precisely what is considered here. In this case, as the singularity is approached, the Friedmann equation is asymptotically velocity term dominated (AVTD) which means that the potential can be safely neglected [18]. Thus, as the singularity is approached, the dynamics are the same as those of the Bianchi I space-time with a massless scalar field. This behaviour will be important for the study of the effective equations later. However, the quantum Hamiltonian constraint operator derived in the following section will hold everywhere and it will be relatively straightforward to extend it for other types of matter fields which classically allow the full Mixmaster dynamics.
Finally, before moving on to the quantum theory, let us consider the parity transformation \( \Pi_k \) which flips the \( k \)th physical triad vector \( e^a_k \). (Keep in mind that this transformation does not act on any of the fiducial quantities which carry the label \( o \).) These correspond to residual discrete gauge transformations. Under this map, we have: \( q_{ab} \rightarrow q_{ab} \), \( \epsilon_{abc} \rightarrow \epsilon_{abc} \) but \( \epsilon_{ijk} \rightarrow -\epsilon_{ijk} \), \( \varepsilon \rightarrow -\varepsilon \). The canonical variables \( c_i, p_i \) transform as proper internal vectors and co-vectors. For example,

\[
\Pi_1(c_1, c_2, c_3) \rightarrow (-c_1, c_2, c_3) \quad \text{and} \quad \Pi_1(p_1, p_2, p_3) \rightarrow (-p_1, p_2, p_3). \tag{2.27}
\]

Consequently, both the symplectic structure and the Hamiltonian constraint are left invariant under any of the parity maps \( \Pi_k \).

The Hamiltonian description given in this section will serve as the starting point for the loop quantization in the next section.

III. QUANTUM THEORY

This section is divided into three parts. In the first, we discuss the kinematics of the model and in the second we introduce the Hamiltonian constraint operator and describe its action on physical states. Finally, in the third subsection we show that the dynamics of a wave function sharply peaked around an isotropic geometry are well approximated by the LQC dynamics of the closed FRW model.

A. LQC Kinematics

The kinematics for the LQC of Bianchi IX models is identical to that of the Bianchi II models [16], but we will briefly present the kinematics here as well for the sake of completeness.

The elementary functions on the classical phase space that have unambiguous analogs in the quantum theory are the momenta \( p_i \) and holonomies \( h^{(\mu)}_k \) of the gravitational connection \( A^\mu_a \) along the integral curves of \( e^a_k \) of length \( \mu \ell_o \) with respect to the fiducial metric \( \hat{q}_{ab} \). These holonomies are given by

\[
h^{(\mu)}_k(c_1, c_2, c_3) = \exp (\mu c_k \tau_k) = \cos \frac{\mu c_k}{2} \mathbb{I} + 2 \sin \frac{\mu c_k}{2} \tau_k, \tag{3.1}
\]

where the \( \tau_k \) are \(-i/2\) times the Pauli matrices. This family of holonomies is completely determined by the almost periodic functions \( \exp(i\mu c_k) \) of the connection. These almost periodic functions will be the elementary configuration variables which will be promoted unambiguously to operators in the quantum theory.

It is simplest to use the \( p \)-representation to specify the gravitational sector \( \mathcal{H}^{grav}_{kin} \) of the kinematic Hilbert space. The basis is orthonormal in the sense that

\[
\langle p_1, p_2, p_3 | p_1', p_2', p_3' \rangle = \delta_{p_1p_1'} \delta_{p_2p_2'} \delta_{p_3p_3'}, \tag{3.2}
\]

where the right side features Kronecker delta symbols rather than Dirac delta distributions. Kinematical states consist of countable linear combinations

\[
|\Psi\rangle = \sum_{P_1, P_2, P_3} \Psi(P_1, P_2, P_3) |P_1, P_2, P_3\rangle \tag{3.3}
\]
of these basis states for which the norm

\[ ||\Psi||^2 = \sum_{p_1,p_2,p_3} |\Psi(p_1,p_2,p_3)|^2 \]  

(3.4)
is finite.

Next, recall that on the classical phase space the three reflections \( \Pi_i : \epsilon_i^a \rightarrow -\epsilon_i^a \) are large gauge transformations under which physics does not change since both the metric and the extrinsic curvature are left invariant. These large gauge transformations have a natural induced action, denoted by \( \hat{\Pi}_i \), on the space of wave functions \( \Psi(p_1,p_2,p_3) \). For example,

\[ \hat{\Pi}_1 \Psi(p_1,p_2,p_3) = \Psi(-p_1,p_2,p_3). \]  

(3.5)

Since \( \hat{\Pi}_i^2 \) is the identity, for each \( i \) the group of these large gauge transformations is simply \( \mathbb{Z}_2 \). As in Yang-Mills theory, physical states belong to its irreducible representation. For definiteness, as in the isotropic and the Bianchi type I and type II models, we will work with the symmetric representation. It then follows that \( \mathcal{H}_{\text{kin}}^{\text{grav}} \) is spanned by wave functions \( \Psi(p_1,p_2,p_3) \) which satisfy

\[ \Psi(p_1,p_2,p_3) = \Psi(|p_1|,|p_2|,|p_3|) \]  

(3.6)

and have a finite norm.

The action of the elementary operators on \( \mathcal{H}_{\text{kin}}^{\text{grav}} \) is as follows: the momenta act by multiplication whereas the almost periodic functions in \( c_i \) shift the \( i \)th argument. For example,

\[ [\hat{p}_1 \Psi](p_1,p_2,p_3) = p_1 \Psi(p_1,p_2,p_3) \quad \text{and} \quad \left[ \exp(i\mu c_1) \Psi \right](p_1,p_2,p_3) = \Psi(p_1-8\pi\gamma G\hbar \mu,p_2,p_3). \]  

(3.7)
The expressions for \( \hat{p}_2, \hat{p}_3 \) and \( \exp(i\mu c_2), \exp(i\mu c_3) \) are analogous. Finally, we must define the operator \( \hat{\epsilon} \) since \( \epsilon \) features in the expression of the Hamiltonian constraint. Following [16], we define

\[ \hat{\epsilon} |p_1,p_2,p_3\rangle := \begin{cases} 
|p_1,p_2,p_3\rangle & \text{if } p_1 p_2 p_3 \geq 0, \\
-|p_1,p_2,p_3\rangle & \text{if } p_1 p_2 p_3 < 0.
\end{cases} \]  

(3.8)

Finally, the full kinematical Hilbert space \( \mathcal{H}_{\text{kin}} \) will be the tensor product \( \mathcal{H}_{\text{kin}} = \mathcal{H}_{\text{kin}}^{\text{grav}} \otimes \mathcal{H}_{\text{kin}}^{\text{matt}} \), where \( \mathcal{H}_{\text{kin}}^{\text{matt}} = L^2(\mathbb{R},dT) \) is the matter kinematical Hilbert space for the homogeneous scalar field. On \( \mathcal{H}_{\text{kin}}^{\text{matt}} \), \( \hat{T} \) will act by multiplication and \( \hat{p}_T := -i\hbar d_T \) will act by differentiation.

\[ \text{B. The Quantum Hamiltonian Constraint} \]

To define the quantum Hamiltonian constraint, we must express the Hamiltonian constraint in terms of almost periodic functions of the connection which can be directly promoted to operators. For isotropic and/or spatially flat space-times, this can be done by expressing the field strength \( F_{ab}^k \) in terms of holonomies and this is what is done for the \( \bar{\mu}_i \) approach in LQC in [8, 10, 11, 15]. However, this is not possible for space-times which are both anisotropic and spatially curved such as the Bianchi type II and type IX models. In this case we need to extend the strategy: the connection itself —rather than the field
strength— has to be expressed in terms of holonomies. This task was carried out in [16]. The connection operator is given by

\[
\hat{c}_k = \frac{\sin(\bar{\mu}_k c_k)}{\bar{\mu}_k},
\]

(3.9)

where

\[
\bar{\mu}_1 = \sqrt{\frac{|p_1| \Delta \ell_{\text{Pl}}^2}{|p_2 p_3|}}, \quad \bar{\mu}_2 = \sqrt{\frac{|p_2| \Delta \ell_{\text{Pl}}^2}{|p_1 p_3|}}, \quad \bar{\mu}_3 = \sqrt{\frac{|p_3| \Delta \ell_{\text{Pl}}^2}{|p_1 p_2|}},
\]

(3.10)

and \( \Delta \ell_{\text{Pl}}^2 = 4\sqrt{3}\pi \gamma \ell_{\text{Pl}}^2 \) is the ‘area gap’. Note that the choice for this operator is motivated by LQG: it is obtained in [16] by expressing the connection in terms of holonomies, a procedure commonly used in LQG, and then ensuring that this approach is equivalent to what is done for simpler cosmological models. Although the precise value of the area gap may change as the relation between LQG and LQC is better understood, the form of \( \bar{\mu}_i \) in terms of the \( p_i \) is necessary in order to obtain the correct infrared, low curvature behaviour.

Using the connection operator, it is possible to promote the classical Hamiltonian constraint in Eq. (2.16) to an operator. Ignoring factor ordering ambiguities and inverse triad operators for the moment, \( \hat{C}_H \) is given by

\[
\hat{C}_H = -\frac{1}{8\pi G \gamma^2 \Delta \ell_{\text{Pl}}^2} \left[ p_1 p_2 |p_3| \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_2 c_2 + |p_1| p_2 p_3 \sin \bar{\mu}_2 c_2 \sin \bar{\mu}_3 c_3 
+ p_1 |p_2| p_3 \sin \bar{\mu}_3 c_3 \sin \bar{\mu}_1 c_1 \right] - \frac{\ell_0 \bar{\epsilon}}{8\pi G \gamma^2 \sqrt{\Delta \ell_{\text{Pl}}^2}} \left[ p_1 p_2 \sqrt{\frac{|p_1|}{|p_3|}} \sin \bar{\mu}_3 c_3 
+ p_2 p_3 \sqrt{\frac{|p_2 p_3|}{|p_1|}} \sin \bar{\mu}_1 c_1 + p_3 p_1 \sqrt{\frac{|p_3 p_1|}{|p_2|}} \sin \bar{\mu}_2 c_2 \right] - \frac{\ell_0^2 (1 + \gamma^2)}{32\pi G \gamma^2} \left[ 2 \left( p_1^2 + p_2^2 + p_3^2 \right) 
- \left( \frac{p_1 p_2}{p_3} \right)^2 - \left( \frac{p_2 p_3}{p_1} \right)^2 - \left( \frac{p_3 p_1}{p_2} \right)^2 \right] + \frac{1}{2} \bar{p}_T^2,
\]

(3.11)

where for simplicity of notation here and in what follows we have dropped the hats on the \( p_i \) and \( \sin \bar{\mu}_i c_i \) operators.

To obtain the action of the \( \sin \bar{\mu}_i c_i \) operators (or, equivalently, the \( \exp(i\bar{\mu}_i c_i) \) operators) we will use the same strategy as in [15]. As shown there, it is simplest to introduce the dimensionless variables

\[
\lambda_i = \frac{\text{sgn}(p_i) \sqrt{|p_i|}}{(4\pi \gamma \sqrt{\Delta \ell_{\text{Pl}}^2})^{1/3}}.
\]

(3.12)

Then the kets \( |\lambda_1, \lambda_2, \lambda_3\rangle \) constitute an orthonormal basis in which the operators \( p_k \) are diagonal

\[
p_k |\lambda_1, \lambda_2, \lambda_3\rangle = \text{sgn}(\lambda_k) (4\pi \gamma \sqrt{\Delta \ell_{\text{Pl}}^2})^{2/3} \lambda_k^2 |\lambda_1, \lambda_2, \lambda_3\rangle,
\]

(3.13)

and quantum states are represented by functions \( \Psi(\lambda_1, \lambda_2, \lambda_3) \). Then the operator \( \exp(i\bar{\mu}_i c_i) \) acts by shifting the wavefunction,

\[
\left[ \exp(i\bar{\mu}_i c_i) \Psi \right](\lambda_1, \lambda_2, \lambda_3) = \Psi(\lambda_1 - \frac{1}{|\lambda_2 \lambda_3|}, \lambda_2, \lambda_3)
= \Psi\left(\frac{v - 2\text{sgn}(\lambda_2 \lambda_3)}{v} : \lambda_1, \lambda_2, \lambda_3\right),
\]

(3.14)
where we have introduced the variable \( v = 2\lambda_1\lambda_2\lambda_3 \) which is proportional to the volume \( V \) of the space-time:

\[
\dot{V} \Psi(\lambda_1, \lambda_2, \lambda_3) = [2\pi \gamma \sqrt{\Delta} |v| \ell^3_{P_3}] \Psi(\lambda_1, \lambda_2, \lambda_3),
\]

(3.15)

The action of the operators \( e^{i\mu_2 c_2} \) and \( e^{i\mu_3 c_3} \) is analogous.

We are now ready to write the Hamiltonian constraint explicitly in the \( \lambda_i \)-representation, again ignoring factor-ordering issues for the time being:

\[
\dot{\mathcal{C}}_H = \dot{\mathcal{C}}_1 + \dot{\mathcal{C}}_2 + \dot{\mathcal{C}}_3 + \dot{\mathcal{C}}_4 + \frac{1}{2} \hat{p}^2_7,
\]

(3.16)

where

\[
\dot{\mathcal{C}}_1 = -\frac{1}{2} \hbar \ell^2_{P_1} v^2 \left[ \text{sgn}(\lambda_1 \lambda_2) \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_2 c_2 + \text{sgn}(\lambda_2 \lambda_3) \sin \bar{\mu}_2 c_2 \sin \bar{\mu}_3 c_3 
+ \text{sgn}(\lambda_3 \lambda_1) \sin \bar{\mu}_3 c_3 \sin \bar{\mu}_1 c_1 \right];
\]

(3.17)

\[
\dot{\mathcal{C}}_2 = -2\pi \sqrt{\Delta} \hbar \ell^3_{P_1} \ell^2_{P_2} \left[ (\lambda_1 \lambda_2)^3 \frac{1}{\sqrt{|P_3|}} \sin \bar{\mu}_3 c_3 + (\lambda_2 \lambda_3)^3 \frac{1}{\sqrt{|P_1|}} \sin \bar{\mu}_1 c_1 
+ (\lambda_3 \lambda_1)^3 \frac{1}{\sqrt{|P_2|}} \sin \bar{\mu}_2 c_2 \right];
\]

(3.18)

\[
\dot{\mathcal{C}}_3 = -\frac{(4\pi \gamma \sqrt{\Delta})^{1/3} \sqrt{\Delta} \hbar \ell^2_{P_1} \ell^3_{P_2}}{4\gamma} \left[ \lambda_1^4 + \lambda_2^4 + \lambda_3^4 \right];
\]

(3.19)

\[
\dot{\mathcal{C}}_4 = \frac{1}{2} (16\pi^2 \gamma^2 \Delta)^{1/3} \pi \Delta \hbar \ell^2_{P_1} \ell^2_{P_2} \left[ (\lambda_1 \lambda_2)^4 \frac{1}{p_3^4} + (\lambda_2 \lambda_3)^4 \frac{1}{p_1^4} + (\lambda_3 \lambda_1)^4 \frac{1}{p_2^4} \right].
\]

(3.20)

It will be straightforward to deal with \( \dot{\mathcal{C}}_H \) since the terms in \( \dot{\mathcal{C}}_1 \) are the exact terms that appear in the Bianchi I model and have already been studied in [15] while the terms in \( \dot{\mathcal{C}}_2 \) and \( \dot{\mathcal{C}}_4 \) are of the same form as some of the terms in the Bianchi II model [16]. Finally, the only new terms —those in \( \dot{\mathcal{C}}_3 \) — act by multiplication and will not cause any difficulty.

All of the terms will be factor-ordered in a symmetric manner. For example, the first term in \( \dot{\mathcal{C}}_1 \) will be factor-ordered as

\[
-\frac{1}{16} \pi \hbar \ell^2_{P_1} \sqrt{|v|} \left[ (\sin \bar{\mu}_1 c_1 \text{sgn} \lambda_1 + \text{sgn} \lambda_1 \sin \bar{\mu}_1 c_1) |v| (\sin \bar{\mu}_2 c_2 \text{sgn} \lambda_2 + \text{sgn} \lambda_2 \sin \bar{\mu}_2 c_2) 
+ (\sin \bar{\mu}_2 c_2 \text{sgn} \lambda_2 + \text{sgn} \lambda_2 \sin \bar{\mu}_2 c_2) |v| (\sin \bar{\mu}_1 c_1 \text{sgn} \lambda_1 + \text{sgn} \lambda_1 \sin \bar{\mu}_1 c_1) \right] \sqrt{|v|},
\]

(3.21)

while the first term in \( \dot{\mathcal{C}}_2 \) will be

\[
-\pi \sqrt{\Delta} \hbar \ell^3_{P_1} \ell(\lambda_1 \lambda_2)^3 \frac{1}{|P_3|^{1/4}} \left[ \hat{\epsilon} \sin \bar{\mu}_3 c_3 + \sin \bar{\mu}_3 c_3 \hat{\epsilon} \right] \frac{1}{|p_3|^{1/4}}.
\]

(3.22)

Since all of the components in each term in \( \dot{\mathcal{C}}_3 \) and \( \dot{\mathcal{C}}_4 \) commute, there are no factor-ordering choices to be made for these terms.

The factor ordering given in Eq. (3.21) was first introduced in [17] for the study of the Gowdy model. It is a particularly nice choice as it causes the octants to decouple from each
other, one can then focus on the dynamics of a single octant and then derive the behaviour of the other octants via the parity properties of the wave function.

The only operators that remain to be defined are the inverse volume operators. Using a variation on the Thiemann inverse triad identities [29], one obtains the operator [16]

$$|p_1|^{-1/4} |\lambda_1, \lambda_2, \lambda_3\rangle = \frac{\sqrt{2} \text{sgn}(\lambda_1) \sqrt{|\lambda_2\lambda_3|}}{(4\pi\gamma\sqrt{\Delta^{3}})^{1/6}} \left( \sqrt{|v + \text{sgn}(\lambda_2\lambda_3)|} - \sqrt{|v - \text{sgn}(\lambda_2\lambda_3)|} \right) |\lambda_1, \lambda_2, \lambda_3\rangle. \tag{3.23}$$

This operator is diagonal in the eigenbasis $|\lambda_1, \lambda_2, \lambda_3\rangle$ and, on eigenkets with large volume, the eigenvalue is indeed well approximated by $|p_1|^{-1/4}$, whence on semi-classical states it behaves as the inverse of $|\rho|^{1/4}$, just as one would hope. Nonetheless, there are interesting nontrivialities in the Planck regime, the most important one being that the inverse triad operator annihilates states $|\lambda_1, \lambda_2, \lambda_3\rangle$ where $v = 2\lambda_1\lambda_2\lambda_3 = 0$.

Finally, the other inverse triad operator which is necessary for the study of Bianchi IX models can be defined by

$$\hat{p}_i^2 := \left(|p_1|^{-1/4}\right)^8. \tag{3.24}$$

Note that both of these operators were already introduced for the study of the Bianchi II model in [16].

As in the Bianchi I model, the action simplifies if we replace $(\lambda_i, \lambda_j, \lambda_k)$ by $(\lambda_i, \lambda_j, v)^4$. Because of the high symmetry of the Bianchi IX model, it does not matter which of the $\lambda_i$ is replaced; we will choose to replace $\lambda_3$ by $v$ here. This change of variables would be nontrivial if, as in the Wheeler-DeWitt theory, we had used the Lesbegue measure in the gravitational sector. However, it is quite tame here because the norms are defined using a discrete measure. The inner product on $\mathcal{H}_{\text{kin}}^{\text{grav}}$ is now given by

$$\langle \Psi_1 | \Psi_2 \rangle_{\text{kin}} = \sum_{\lambda_1, \lambda_2, \lambda_3, v} \Psi_1(\lambda_1, \lambda_2, v) \Psi_2(\lambda_1, \lambda_2, v) \tag{3.25}$$

and states are symmetric under the action of $\hat{\Pi}_k$. In the Appendix of [16], it is shown that under the action of the $\hat{\Pi}_i$, the operators $\sin \hat{\mu}_i c_i$ have the same transformation properties as $c_i$ under the reflections $\Pi_i$ in the classical theory. As a consequence, $\hat{C}_H$ is also reflection symmetric. Therefore, its action is well defined on $\mathcal{H}_{\text{kin}}^{\text{grav}}$: $\hat{C}_H$ is a densely defined, symmetric operator on this Hilbert space. In the isotropic and Bianchi I cases, its analog has been shown to be essentially self-adjoint [30, 31]. In what follows we will assume that (3.16) is essentially self-adjoint on $\mathcal{H}_{\text{kin}}^{\text{grav}}$ and work with its self-adjoint extension.

We can now study the action of $\hat{C}_H$ on a wavefunction. For a complete derivation of the action of each term in the constraint, see [15, 16].

It is straightforward to write down the full Hamiltonian constraint on $\mathcal{H}_{\text{kin}}^{\text{grav}}$:

$$-\hbar^2 \partial_T^2 \Psi(\lambda_2, \lambda_3, v; T) = \Theta \Psi(\lambda_2, \lambda_3, v; T), \quad \text{where} \quad \Theta = -2\hat{C}_{\text{grav}}. \tag{3.26}$$

---

4 This cannot be done for states where $\lambda_1\lambda_2\lambda_3 = 0$ but since these states decouple under the action of $\hat{C}_H$, we can restrict our attention solely to states where $\lambda_1\lambda_2\lambda_3 \neq 0$.

5 Note that although $\hat{P}_i\hat{\epsilon}\hat{\Pi}_i = -\hat{\epsilon}$ (recall that classically $\epsilon \rightarrow -\epsilon$ under a parity transformation) only when $v \neq 0$, in the $v = 0$ case the wavefunction is annihilated by the gravitational part of the Hamiltonian constraint $\hat{C}_{\text{grav}}$ and therefore $\hat{P}_i\hat{C}_{\text{grav}}\hat{\Pi}_i|\Psi_{\text{sing}}\rangle = 0 = \hat{C}_{\text{grav}}|\Psi_{\text{sing}}\rangle$ where $|\Psi_{\text{sing}}\rangle$ is a state that only has support on $v = 0$. It is then straightforward to show that $\hat{P}_i\hat{C}_H\hat{\Pi}_i|\Psi\rangle = \hat{C}_H|\Psi\rangle$ for all wavefunctions.
As in the isotropic case [32], one can obtain the physical Hilbert space $\mathcal{H}_{\text{phy}}$ by a group averaging procedure and the final result is completely analogous. Elements of $\mathcal{H}_{\text{phy}}$ consist of ‘positive frequency’ solutions to (3.26), i.e., solutions to

$$-i\hbar\partial_T\Psi(\lambda_1, \lambda_2, v; T) = \sqrt{|\Theta|}\Psi(\lambda_1, \lambda_2, v; T),$$

which are symmetric under the three reflection maps $\hat{\Pi}_i$:

$$\Psi(\lambda_1, \lambda_2, v; T) = \Psi(|\lambda_1|, |\lambda_2|, |v|; T).$$

The scalar product is simply given by

$$\langle \Psi_1 | \Psi_2 \rangle_{\text{phys}} = \langle \Psi_1(\lambda_1, \lambda_2, v; T_o) | \Psi_2(\lambda_1, \lambda_2, v; T_o) \rangle_{\text{kin}}$$

$$= \sum_{\lambda_1, \lambda_2, v} \bar{\Psi}_1(\lambda_1, \lambda_2, v; T_o) \Psi_2(\lambda_1, \lambda_2, v; T_o),$$

where $T_o$ is any “instant” of internal time $T$.

Since elements of $\mathcal{H}_{\text{kin}}$ are invariant under the three parity maps $\hat{\Pi}_k$ and the Hamiltonian constraint satisfies $\hat{\Pi}_k \hat{C}_{\text{grav}} \hat{\Pi}_k = \hat{C}_{\text{grav}}$, knowledge of the restriction of the image $\hat{C}_{\text{grav}} \Psi$ of $\Psi$ to the positive octant suffices to determine $\hat{C}_{\text{grav}} \Psi$ completely. Therefore, in the remainder of this section we will restrict the argument of $\hat{C}_H \Psi$ to the positive octant. The full action is simply given by

$$\langle \hat{C}_{\text{grav}} \Psi(\lambda_1, \lambda_2, v) = \langle \hat{C}_{\text{grav}} \Psi(\lambda_1, \lambda_2, |v|).$$

Since all states with $v = 0$ are annihilated by $\hat{C}_{\text{grav}}$, their evolution is trivial:

$$\partial^2_T \Psi(\lambda_1, \lambda_2, v = 0; T) = 0.$$}

Such states correspond to classical geometries which are singular and therefore we will call these states ‘singular’, even though they are well defined in the quantum theory. Non-singular states on the other hand are physically much more interesting. On them, the explicit form of the full constraint is given by:

$$\partial^2_T \Psi(\lambda_1, \lambda_2; T) = \pi G \left[ \frac{\sqrt{v}}{8} (v + 2) \sqrt{v + 4} \Psi_0^+(\lambda_1, \lambda_2; T) - (v + 2) \sqrt{v} \Psi_0^+(\lambda_1, \lambda_2; T) 
- \theta_{v-2}(v - 2) \sqrt{v} \Psi_1^+(\lambda_1, \lambda_2; T) + \theta_{v-4}(v - 2) \sqrt{|v - 4|} \Psi_1^+(\lambda_1, \lambda_2; T)
- \frac{2i\ell_o \sqrt{\Delta}}{(4\pi \gamma \sqrt{\Delta})^{1/3}} \left( \sqrt{v + 1} - \sqrt{|v - 1|} \right) \left( \Phi^+ - \theta_{v-2} \Phi^- \right) (\lambda_1, \lambda_2; T)
+ \frac{8\Delta \ell_o^2 (1 + \gamma^2)}{(4\pi \gamma \sqrt{\Delta})^{2/3}} \left( \sqrt{v + 1} - \sqrt{|v - 1|} \right)^2 \left( (\lambda_1 \lambda_2)^8 + (\lambda_2 \lambda_3)^8 + (\lambda_3 \lambda_1)^8 \right)
- \frac{1}{8} \left( \lambda_1^4 + \lambda_2^4 + \lambda_3^4 \right) \Psi(\lambda_1, \lambda_2; T) \right],$$

where $\theta_x$ is the step function

$$\theta_x = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$
Note that the step function kills any terms that would allow the positive octant to interact with any of the other ones, this is a direct consequence of the factor ordering choices made earlier.

The $\Psi_{0,4}^{\pm}$ are defined as follows:

$$
\Psi_{4}^{\pm}(\lambda_1, \lambda_2, v; T) = \Psi \left( \frac{v + 4}{v + 2} \lambda_1, \frac{v + 2}{v} \lambda_2, v \pm 4; T \right) + \Psi \left( \frac{v + 4}{v + 2} \lambda_1, \lambda_2, v \pm 4; T \right)
$$

$$
+ \Psi \left( \frac{v + 2}{v} \lambda_1, \frac{v + 4}{v + 2} \lambda_2, v \pm 4; T \right) + \Psi \left( \frac{v + 2}{v} \lambda_1, \lambda_2, v \pm 4; T \right)
$$

$$
+ \Psi \left( \lambda_1, \frac{v + 2}{v} \lambda_2, v \pm 4; T \right) + \Psi \left( \lambda_1, \frac{v + 4}{v + 2} \lambda_2, v \pm 4; T \right),
$$

(3.34)

and

$$
\Psi_{0}^{\pm}(\lambda_1, \lambda_2, v; T) = \Psi \left( \frac{v + 2}{v} \lambda_1, \frac{v}{v + 2} \lambda_2, v; T \right) + \Psi \left( \frac{v + 2}{v} \lambda_1, \lambda_2, v; T \right)
$$

$$
+ \Psi \left( \frac{v}{v + 2} \lambda_1, \frac{v + 2}{v} \lambda_2, v; T \right) + \Psi \left( \frac{v}{v + 2} \lambda_1, \lambda_2, v; T \right)
$$

$$
+ \Psi \left( \lambda_1, \frac{v}{v + 2} \lambda_2, v; T \right) + \Psi \left( \lambda_1, \frac{v + 2}{v} \lambda_2, v; T \right),
$$

(3.35)

while $\Phi^{\pm}$ are given by

$$
\Phi^{\pm}(\lambda_1, \lambda_2, v; T) = (\sqrt{|v + 2 + 1|} - \sqrt{|v + 2 - 1|}) \times \left[ (\lambda_2 \lambda_3)^{4} \Psi \left( \frac{v + 2}{v} \lambda_1, \lambda_2, v \pm 2; T \right) \right]
$$

$$
+ (\lambda_3 \lambda_1)^{4} \Psi \left( \lambda_1, \frac{v + 2}{v} \lambda_2, v \pm 2; T \right) + (\lambda_1 \lambda_2)^{4} \Psi \left( \lambda_1, \lambda_2, v \pm 2; T \right).
$$

(3.36)

As expected, the quantum dynamics of the Bianchi IX model reduces to that of the Bianchi I model discussed in [15] in the limit $\ell_o \to 0$ in Eq. (3.32).

Eq. (3.32) also immediately implies that the steps in $v$ are uniform: the argument of the wave function only involves $v - 4, v - 2, v, v + 2$ and $v + 4$. Thus, there is a superselection in $v$. For each $\epsilon \in [0, 2)$, we can introduce a lattice $L_\epsilon$ of points $v = 2n + \epsilon$. Then the quantum evolution—and the action of the Dirac observables $\hat{p}_r$ and $\hat{V}|_T$ commonly used in LQC—preserves the subspaces $H_{\text{phy}}$ consisting of states with support in $v$ on $L_\epsilon$. The most interesting lattice is the one corresponding to $\epsilon = 0$ since it includes the classically singular points $v = 0$.

The form of the action of the Hamiltonian constraint operator also shows that the classical singularity is resolved. Using the scalar field $T$ as time, we find that if one starts with a wavefunction which only has support on singular states, that wavefunction does not evolve in $T$ and therefore will always only have support on singular states.

On the other hand, a state which does not have any support on the singular subspace will never have support on it. Restricting our argument to the positive octant for the sake of simplicity (it can easily be generalized to the other octants), it is easy to see that to go from $\lambda_1, \lambda_2, v > 0$ to $v = 0$, one must either have $v = 2$ and then $\Phi^-$ will give a term with $v = 0$ or have $v = 4$ and then $\Psi^-$ will give a term with $v = 0$. However, the prefactors in
front of $\Phi^-$ vanish for $v = 2$ just as the prefactors in front of $\Psi_4^-$ vanish for $v = 4$. Because of this, it is impossible for a wavefunction with no support on singular states to ever gain support on a singular state.

This shows that singular states decouple from nonsingular states under the relational $T$ dynamics given by Eqs. (3.31) and (3.32). In other words, if one starts with a nonsingular state at some ‘time’ $T_0$, it will remain nonsingular throughout its evolution. It is in this (rather strong) sense that the singularity is resolved.

IV. EFFECTIVE EQUATIONS

In the isotropic models, effective equations have been introduced via two different approaches—the embedding [33, 34] and the moment expansion [35] methods—in order to study the first order quantum-corrected equations of motion. In the isotropic case the effective equations following from the embedding approach provide an excellent approximation to the full quantum evolution of states which are Gaussians at late times, even in the $\Lambda \neq 0$ as well as $k=\pm 1$ cases where the models are not exactly soluble. However, the truncation method is more systematic and also more general in the sense that it is applicable to a wide variety of states. Nonetheless, in this section we will use the first method (although we will ignore the effect of fluctuations in this work) in order to gain qualitative insights into modifications of the equations of motion due to quantum geometry effects.

To obtain the effective equations we can restrict our attention to the positive octant of the classical phase space (where $\varepsilon = 1$) without loss of generality. Then the quantum corrected Hamiltonian constraint is given by the classical analogue of (3.11):

$$\frac{p_T^2}{2} + C_{\text{grav}}^{\text{eff}} = 0,$$

where

$$C_{\text{grav}}^{\text{eff}} = -\frac{p_1 p_2 p_3}{8 \pi G \gamma^2 \Delta \ell_{\text{Pl}}^2} \left[ \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_2 c_2 \sin \bar{\mu}_3 c_3 + \sin \bar{\mu}_3 c_3 \sin \bar{\mu}_1 c_1 \right]$$

$$- \frac{\ell_o}{8 \pi G \gamma^2 \sqrt{\Delta \ell_{\text{Pl}}}} \left[ \left( \frac{p_1 p_2}{\sqrt{p_3}} \right)^{3/2} \sin \bar{\mu}_3 c_3 + \left( \frac{p_2 p_3}{\sqrt{p_1}} \right)^{3/2} \sin \bar{\mu}_1 c_1 + \left( \frac{p_3 p_1}{\sqrt{p_2}} \right)^{3/2} \sin \bar{\mu}_2 c_2 \right]$$

$$- \frac{\ell_o^2}{32 \pi G \gamma^2} (1 + \gamma^2) \left[ 2(p_1^2 + p_2^2 + p_3^2) - \left( \frac{p_1 p_2}{p_3} \right)^2 - \left( \frac{p_2 p_3}{p_1} \right)^2 - \left( \frac{p_3 p_1}{p_2} \right)^2 \right].$$

Using the expressions (3.10) of $\bar{\mu}_k$, it is easy to verify that far away from the classical singularity—more precisely in the regime in which the Hubbles rates $H_i$ are well below the Planck scale—the effective Hamiltonian constraint (4.1) is well-approximated by the classical one given in Eq. (2.16).

---

6 Recall that every $\ell_o$ which appears in the constraint is divided by $r_o$ which has been set to 2. As $\ell_o/r_o$ is dimensionless, we must ignore $\ell_o$ when counting units.
The effective dynamics are obtained by taking Poisson brackets with the effective Hamiltonian constraint. This gives

\[ \dot{p}_1 = \gamma^{-1} \left( \frac{p_1^2}{\mu_1} (\sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_3 c_3) + \ell_o p_2 p_3 \right) \cos \bar{\mu}_1 c_1; \]  

(4.3)

\[ \dot{c}_1 = -\frac{1}{\gamma} \left[ \frac{p_2 p_3}{\Delta L^2 p_1} \left( \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_3 c_3 + \sin \bar{\mu}_2 c_2 \sin \bar{\mu}_3 c_3 \right) 
+ \frac{\bar{\mu}_1 c_1}{2} \cos \bar{\mu}_1 c_1 (\sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_3 c_3) - \frac{\bar{\mu}_2 c_2}{2} \cos \bar{\mu}_2 c_2 (\sin \bar{\mu}_1 c_1 + \sin \bar{\mu}_3 c_3) 
- \frac{\bar{\mu}_3 c_3}{2} \cos \bar{\mu}_3 c_3 (\sin \bar{\mu}_1 c_1 + \sin \bar{\mu}_2 c_2) \right] + \ell_o \left( \frac{3}{2} \frac{p_1 p_2}{p_3} \sin \bar{\mu}_3 c_3 + \frac{p_1 p_3}{p_2} \sin \bar{\mu}_2 c_2 
- \frac{p_2 p_3}{3 p_1} \sin \bar{\mu}_1 c_1 \right) + \frac{1}{2} \frac{p_2 p_3}{p_1} \cos \bar{\mu}_1 c_1 - \frac{1}{2} \frac{p_2 p_3}{p_1} \cos \bar{\mu}_3 c_3 - \frac{1}{2} \frac{p_3 c_2}{p_2} \cos \bar{\mu}_2 c_2 \right) 
+ \ell_o^2 \left( 4 p_1 - 2 p_1 \left( \frac{p_2^2}{p_3^2} + \frac{p_3^2}{p_2^2} \right) + 2 \frac{p_2 p_3}{p_1^2} \right). \]  

(4.4)

The equations for \( \dot{p}_2, \dot{p}_3, \dot{c}_2 \) and \( \dot{c}_3 \) are the same modulo the appropriate permutations. Note that it is easy to extend this for other matter fields and also to the vacuum case simply by appropriately modifying the matter part of the effective Hamiltonian constraint.

In the embedding approach these effective equations provide quantum geometry corrections to the classical equations of motion Eqs. (2.18) and (2.19) due to the area gap. However, careful numerical work comparing the full quantum dynamics to the effective dynamics is necessary to determine whether the effective equations are accurate beyond first order in \( \hbar \).

Now, it is well known that classical Bianchi IX space-times with a massless scalar field as a matter source behave in an asymptotically velocity term dominated (AVTD) manner\(^7\), that is to say that the potential term is negligible (see [18] and references therein). For certain regions of phase space, this will occur before quantum gravity effects become important and we will assume that in this case only quantum gravity corrections to the velocity terms are relevant.

It then follows that this behaviour is identical to that of the Bianchi I model and therefore the effective Friedmann equation for the Planck regime to first order in \( \hbar \) is given by [36]

\[ H^2 = \frac{8 \pi G}{3 \rho} \left( 1 - \frac{\rho}{\rho_c} \right) + \frac{\Sigma^2}{6} - \frac{\Sigma}{2 \rho_c} - \frac{(\Sigma^2)^2}{32 \pi G \rho_c} + O(\ell_p^4), \]  

(4.5)

where \( \rho_c = 3/8 \pi \gamma^2 \Delta G \ell_p^2 \approx 0.41 \rho_0 \) (recall that \( \Delta = 4 \sqrt{3} \pi \gamma \) and \( \gamma \approx 0.2375 \) due to black hole entropy calculations [37]). The expression for \( \Sigma^2 \) is given by

\[ \Sigma^2 = \frac{1}{3 \gamma^2 p^3} \left[ (p_1 c_1 - p_2 c_2)^2 + (p_2 c_2 - p_3 c_3)^2 + (p_3 c_3 - p_1 c_1)^2 \right], \]  

(4.6)

and one can show that \( p^3 \Sigma^2 \) is a constant in the AVTD limit [36].

\(^7\) This is true so long as the constant of motion \( p_T^3 \) is large enough so that the three scale factors are all decreasing as the singularity is approached.
It is clear that there is a bounce \((H^2 = 0)\) when the matter energy density reaches

\[
\rho_{\text{bounce}} = \frac{1}{2} \left[ \rho_c - \frac{3\Sigma^2}{16\pi G} + \sqrt{\left( \rho_c - \frac{3\Sigma^2}{16\pi G} \right) \left( \rho_c - \frac{\Sigma^2}{16\pi G} \right)} \right],
\]  

(4.7)

at which point the energy density and curvature will both decrease and leave the Planck regime and the classical dynamics will once again become a good approximation. It follows that the matter energy density is always bounded above by the critical energy density \(\rho_c = 0.41\rho_{Pl}\). This is only an upper bound as the matter density at the bounce depends quite strongly on \(\Sigma^2\) which is a measure of the strength of the gravitational waves: the stronger the gravitational waves are, the lower \(\rho_{\text{bounce}}\) will be.

The scenario described above relies on the AVTD behaviour of the Bianchi IX cosmology with a massless scalar field occurring before quantum gravity effects become important. In this case, the true Friedmann equation can then be well approximated by Eq. (2.24) in the classical regime and by Eq. (4.5) in the AVTD limit. However, this scenario will not be valid for all regions of phase space, in particular where the scalar field momentum \(p_T\) is small enough for the chaotic Mixmaster behaviour to appear.

It has been suggested that, by bounding the strength of the potential terms due to inverse triad effects, quantum gravity effects could play an important role in Bianchi IX dynamics and that the chaotic Mixmaster behaviour would be avoided as a result of this for all types of matter fields [24]. In the effective equations presented above, we have ignored the effect of inverse volume corrections (which for the inverse volume operator used in this paper are only important for \(v < 4\)) and have only considered the effect of holonomy corrections. If the chaotic behaviour is to be generically avoided in this effective theory, it will be because the repulsive quantum gravity effects will ensure that the Bianchi IX space-time will not remain in high curvature regions for long enough for there to occur a sufficient number of Mixmaster bounces for chaos to appear.

For now, this remains a conjecture and one would have to study the Bianchi IX effective equations of motion more carefully, using both analytic and numerical methods, in order to determine whether the bounce is generic and also to see if chaotic behaviour is avoided or not in the effective theory for small \(p_T^2\).

V. DISCUSSION

In this paper we have studied the improved LQC dynamics for Bianchi IX cosmologies where the matter content is a massless scalar field which is used as a relational time parameter. We have shown that the singularities in the classical theory are resolved by quantum gravity effects in the usual manner in LQC as the singular states decouple from the regular ones under the relational dynamics given by the Hamiltonian constraint operator.

It is important to point out that all of the tools necessary for the task of deriving the LQC dynamics for Bianchi IX models were already available. First, the form of \(\tilde{\mu}_i\) was introduced in the study of Bianchi I models [15], as were the variables \(\lambda_i\) which greatly simplify the form of the action of the Hamiltonian constraint operator. The other two necessary ingredients to the results for this work are the connection operator and the inverse triad operators, both of which were introduced for the study of Bianchi II models in [16]. In addition, even the factor-ordering choices necessary in the Hamiltonian constraint operator had been made in [15–17]. Because of this, it is reasonable to expect that no additional machinery should
be necessary in order to complete the study of the loop quantum cosmology of the other Bianchi models of type A.

Finally, in addition to obtaining a well-defined LQC Hamiltonian constraint operator for Bianchi IX space-times and studying some of its properties, we also derived some effective equations which provide modifications to the classical equations of motion due to the area gap which is a manifestation of quantum geometry in LQG. Although all of the results presented in this paper were derived for the particular case of a massless scalar field as the matter field, it will be easy to extend the results presented here for other types of matter fields (as well as the vacuum case) for both the quantum and effective theories.

Of course, it is not enough to know the form of the equations of motion given in Eq. (3.32) in order to understand the full dynamics of the loop quantum cosmology of Bianchi IX models. Numerical studies will be particularly useful and help us understand how the quantum state of a Bianchi IX cosmology evolves with time. Most interesting would be a study of states which are sharply peaked around a semi-classical state at late times and to then evolve them back in time to see what happens as the curvature increases. Based upon previous experience with isotropic models, one might expect to see one or several bounces as the curvature reaches the Planck scale but careful numerical studies are needed to check this.

If the BKL conjecture is correct a good understanding of the quantum dynamics of Bianchi IX cosmologies will lead to a better understanding of the behaviour of generic space-times as their curvature reaches the Planck scale. If Bianchi IX models are sufficiently rich in order to understand the approach to such regions, it would appear that no singularities would form since an initially nonsingular Bianchi IX wave function must remain nonsingular as shown in Sec. III. It is therefore possible that a careful study of the BKL conjecture at the level of the quantum dynamics could provide a no-singularity theorem, a first step in this direction is provided by [22].

A simpler avenue to study quantum gravity effects in Bianchi IX models would be to study the effective dynamics presented in Sec. IV. In isotropic models it turns out that the effective dynamics are surprisingly accurate even in the deep Planck regime: the effective equations accurately predict the quantum trajectory throughout the quantum bounce for sharply peaked wave functions. Because of this, it would be interesting to study the dynamics given by the effective equations for Bianchi IX space-times. However, it is essential to see where the effective equations break down, if they do at all. This can be done by including higher order corrections to the effective equations via the moment expansion method and also by comparing the predictions of the effective equations to full numerical solutions of the Hamiltonian constraint operator.

An analysis of the effective equations of motion in the case where the asymptotically velocity term dominated behaviour begins before quantum gravity effects become important shows that there is a bounce when the curvature reaches the Planck scale and that the matter energy density is bounded above by the critical energy density \( \rho_c \approx 0.41 \rho_{Pl} \). This result relies on the AVTD behaviour and is not generic. Therefore, one must also examine other areas in the phase space in order to fully understand the predictions of the effective theory, particularly near Planck scales. As the Mixmaster behaviour appears for small \( p_T^2 \) during the approach to the singularity in the classical theory, the effective equations can provide a better understanding of how quantum gravity effects may modify the Mixmaster behaviour as well. In particular it is possible that, as for simpler isotropic models and in AVTD case here, these quantum gravity effects will be repulsive and cause a quantum
bounce. This would limit the amount of time that the Mixmaster behaviour occurs and the chaos which arises in the classical theory might be avoided due to the short time span of the Mixmaster dynamics. However, this remains a conjecture and much more work, both analytic and numerical, is needed in order to resolve this question.

Finally, it has been pointed out that the dipole cosmology model can be used in order to study the Bianchi IX model [25]. Although that paper studies the Euclidean theory, it would nonetheless be interesting to compare the model presented in [25] with the one developed in this paper. In particular, [25] suggests two possible approaches in order to obtain the Hamiltonian constraint operator for their model. Comparing the quantum dynamics resulting from these two possibilities to those derived in this paper could help determine which of the two approaches is the correct one and hence give some insight into the dipole cosmology models and also spin foam models in general. It is also possible to further probe the relation between the canonical and the covariant approaches to LQG via LQC by extending the Feynman path integral construction given in [38] for the flat FRW model to the Bianchi IX model; this extension would be nontrivial due to the additional degrees of freedom present in Bianchi IX space-times, but it could also improve our understanding of the connection between the canonical and covariant approaches to LQG as well as the relation between full LQG and the symmetry-reduced models of LQC.

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