Euclid’s Number-Theoretical Work

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Abstract: When people mention the mathematical achievements of Euclid, his geometrical achievements always spring to mind. But, his Number-Theoretical achievements (See Books 7, 8 and 9 in his magnum opus Elements [1]) are rarely spoken. The object of this paper is to affirm the number-theoretical role of Euclid and the historical significance of Euclid’s algorithm. It is known that almost all elementary number-theoretical texts begin with Division algorithm. However, Euclid did not do like this. He began his number-theoretical work by introducing his algorithm. We were quite surprised when we began to read the Elements for the first time. Nevertheless, one can prove that Euclid’s algorithm is essentially equivalent with the Bezout’s equation and Division algorithm. Therefore, Euclid has preliminarily established Theory of Divisibility and the greatest common divisor. This is the foundation of Number Theory. After more than 2000 years, by creatively introducing the notion of congruence, Gauss published his Disquisitiones Arithmeticae in 1801 and developed Number Theory as a systematic science. Note also that Euclid’s algorithm implies Euclid’s first theorem (which is the heart of ‘the uniqueness part’ of the fundamental theorem of arithmetic) and Euclid’s second theorem (which states that there are infinitely many primes). Thus, in the nature of things, Euclid’s algorithm is the most important number-theoretical work of Euclid. For this reason, we further summarize briefly the influence of Euclid’s algorithm. Knuth said ‘we might call Euclid’s method the granddaddy of all algorithms’. Based on our discussion and analysis, it leads to the conclusion Euclid’s algorithm is the greatest number-theoretical achievement of the Euclidean age.

Keywords: Elements, Euclid’s number-theoretical work, Euclid’s algorithm, Division algorithm, Euclid’s second theorem, Euclid’s first theorem

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1 A brief introduction about Euclid’s number-theoretical work

Euclid, who was a Greek mathematician best known for his Elements which influenced the development of Western mathematics for more than 2000 years, is one of the greatest mathematicians of all time and popularly considered as the ‘Father of Geometry’, also known as Euclid of Alexandria, who lived probably around 300 BC, is the most famous mathematician of antiquity. For Euclid and the traditions about him, see [1, Introduction].

His magnum opus Elements which covers much of the geometry known to the ancient Greeks as well as some elementary number theory (See Elements: Books 7, 8 and 9) is probably the most successful textbook ever written and has appeared in over a thousand different editions from ancient to modern times. Heath [2, Introduction] called it ‘the greatest textbook of elementary mathematics that there was written twenty-two centuries ago. Nor does the reading of it require the ‘higher mathematics’. Any intelligent person with a fair recollection of school work in elementary geometry would find it (progressing as it does by gradual and nicely contrived steps) easy reading, and should feel a real thrill in following its development, always assuming that enjoyment of the book is not marred by any prospect of having to pass an examination in it! for everybody ought to read it who can, that is all educated persons except the very few who are constitutionally incapable of mathematics.’ Heath [1, Preface] pointed out ‘Euclid’s work will live long after all the text-books of the present day are superseded and forgotten. It is one of the noblest monuments of antiquity; no mathematician worthy of the name can afford not to know Euclid, the real Euclid as distinct from any revised or rewritten versions which will serve for schoolboys of engineers. And, to know Euclid, it is necessary to know his language, and so far as it can traced, the history of the ’elements’ which he collected in his immortal work.’ We might not see such a number-theoretical book any more, in which numbers are represented by line segments and so have a geometrical appearance and aesthetic feeling.

From his Elements, we know that Euclid’s main number-theoretical achievements should be reflected in the following Propositions.

Proposition 1 (Book 7): Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original
numbers will be prime to one another.

**Proposition 2** (Book 7): Given two numbers not prime to one another, to find their greatest common measure.

**Proposition 20** (Book 7): The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the lesser the lesser. Namely, if \( a \) and \( b \) are the smallest numbers such that \( a : b = c : d \), then \( a \) divides \( c \) and \( b \) divides \( d \).

**Proposition 30** (Book 7): If two numbers by multiplying one another make some number, and any prime number measures the product, it will also measure one of the original numbers. Namely, if \( p \) is prime, and \( p | ab \), then \( p | a \) or \( p | b \).

**Proposition 20** (Book 9): Prime numbers are more than any assigned multitude of prime numbers. Namely, there are infinitely many primes.

**Proposition 36** (Book 9): If as many as we please beginning from a unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect. Namely, if \( n = 2^p - 1 \), where \( p \) is prime such that \( 2^p - 1 \) is also prime, then, \( n \) is even perfect number.

Propositions 1 and 2 in Book 7 of *Elements* are exactly the famous Euclidean algorithm for computing the greatest common divisor of two positive integers. According to Knuth [3], 'we might call Euclid’s method the granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day'.

In their book *An Introduction to the Theory of Numbers*, Hardy and Wright [4] called Proposition 20 (Book 9) Euclid’s second theorem. Hardy like particularly Euclid’s proof of for the infinitude of primes. Hardy [5] called it is ‘as fresh and significant as when it was discovered—two thousand years have not written a wrinkle on it’. According to Hardy [5], ‘Euclid’s theorem which states that the number of primes is infinite is vital for the whole structure of arithmetic. The primes are the raw material out of which we have to build arithmetic, and Euclid’s theorem assures us that we have plenty of material for the task’. André Weil [6] also called ‘the proof for the existence of infinitely many primes represents undoubtedly a major advance, but there is no compelling reason either for attributing it to Euclid or for
dating them back to earlier times. What matters for our purposes is that the very broad diffusion of Euclid in later centuries, while driving out all earlier texts, made them widely available to mathematicians from then on'.

I think that anyone who likes Number Theory must like Euclid’s second theorem. In his book The book of prime number records, Paulo Ribenboim [7] cited nine and a half proofs of this theorem. For other beautiful proofs, see [41–55].

Hardy and Wright [4] called Proposition 30 (Book 7) Euclid’s first theorem which is the heart of ‘the uniqueness part’ of the fundamental theorem of arithmetic. Recently, David Pengelley and Fred Richman [8] published a readable paper entitled ‘Did Euclid need the Euclidean algorithm to prove unique factorization’. They called Proposition 30 (Book 7) Euclid’s Lemma and pointed out that Euclid’s Lemma can be derived from Porism of Proposition 2. But ‘it is not at all apparent that Euclid himself does this’. In their paper, David Pengelley and Fred Richman explored that how Euclid proved Proposition 30 using his algorithm. More precisely, they proved Proposition 30 using the porism which states that if a number divides two numbers, then it divides their greatest common divisor as follows: if \( p \) does not divide \( a \), then \( \gcd(pb, ab) = b \), so \( p \) divides \( b \). However, in their proof, they assumed such a clear result \( \gcd(pb, ab) = b \) which was not proven. Namely, in order to prove that the porism of Proposition 2 implies that Proposition 30 holds, one need prove \( \gcd(pb, ab) = b \) when \( p \) does not divide \( a \). After giving Propositions 1, 2 and the porism of Proposition 2, Euclid must want to prove that if \( \gcd(a, b) = 1 \) then \( \gcd(ac, bc) = c \), which implies \( \gcd(pb, ab) = b \) when \( p \) does not divide \( a \). In fact, Euclid proved in spite of his expression is not like this, that \( \gcd(ac, bc) = c \) if and only if \( \gcd(a, b) = 1 \), see [1]: Book 7, Propositions 17, 18, 19, 20, 21 (which states that numbers prime to one another are the least of those which have the same ratio with them) and 22 (which states that the least numbers of those which have the same ratio with them are prime to one another). So, Euclid’s algorithm implies Euclid first theorem because his proof of Proposition 30 refers exactly to Propositions 19 and 20.

In most elementary number-theoretical texts, Euclid’s first theorem is derived from the Bezout’s equation \( ax + by = \gcd(a, b) \). Of course, this is true (see also the following theorem 2). Note that the Bezout’s equation also can be derived from Euclid’s algorithm. It is a pity that Euclid himself does not obtain the equation \( ax + by = \gcd(a, b) \) by making use of his algorithm. If he had known about \( ax + by = \gcd(a, b) \)! He perhaps knew, he just didn’t
express explicitly.

According to Hardy and Wright [4], the fundamental theorem of arithmetic which says that every natural number is uniquely a product of primes ‘does not seem to have been stated explicitly before Gauss. It was, of course, familiar to earlier mathematicians; but Gauss was the first to develop arithmetic as a systematic science.’ They further remarked: ‘It might seem strange at first that Euclid, having gone so far, could not prove the fundamental theorem itself; but this view would rest on a misconception. Euclid had no formal calculus of multiplication and exponentiation, and it would have been most difficult for him even to state the theorem. He had not even a term for the product of more than three factors. The omission of the fundamental theorem is in no way casual or accidental; Euclid knew very well that the theory of number turned upon his algorithm, and drew from it all the return he could.’ In his Disquisitiones Arithmeticae, Gauss [9] proved definitely that ‘a composite number can be resolved into prime factors in only one way’. Maybe, Euclid did not consider this problem how to prove unique factorization or how to resolve a composite number into prime factors in only one way. But, he took the first step and proved the following Propositions 31 and 32 by the definition of composite number without using Euclid’s algorithm and its porism (see [1]: Book 7, the proofs of Propositions 31 and 32). Of course, he assumed that a finite composite number has only a finite number of prime factors.

**Proposition 31** (Book 7): Any composite number is measured by some prime number.

**Proposition 32** (Book 7): Every number is either prime or is measured by some prime number.

About the problem on Euclid and the ‘fundamental theorem of arithmetic’, we have not pursued it. Knorr W. [56] gave a reasonable discussion of the position of unique factorization in Euclid’s theory of numbers.

Proposition 36 (Book 9) is Euclid’s a great number-theoretical achievement because he gave a sufficient condition for even numbers to be perfect. In Weil’s view, it is the apex of Euclid’s number-theoretical work [10]. A perfect number is defined as a positive integer which is the sum of its proper positive divisors, that is, the sum of the positive divisors excluding the number itself. Euler proved further that the sufficient condition about even perfect numbers given by Euclid 2000 years ago is also necessary. Namely, $n \ldots$
is even perfect number if and only if $n = 2^{p-1}(2^p - 1)$, where $p$ is prime such that $2^p - 1$ (also called Mersenne primes) is also prime. Thus, Euclid’s work on perfect numbers is not perfect. According to Littlewood [11], ‘perfect number certainly never did any good, but then they never did any particular harm.’ Therefore, in this paper, we do not further talk about Euclid’s this work any more.

Finally, we should mention again Proposition 20 in Euclid’s Elements Book 7. B. L. van der Waerden [12] pointed out that Proposition 20 plays a central role in Euclid’s arithmetical books. C. M. Taisbak [13] also announced Proposition 20 is the core of Euclid’s arithmetical books. Proposition 20 can be derived from Euclid’s algorithm. Although ‘Central to Euclid’s development is the idea of four numbers being proportional: $a$ is to $b$ as $c$ is to $d’ [8], one can see again that the key of studying divisibility is essentially Euclid’s algorithm by David Pengelley and Fred Richman’s work. Generally speaking, Euclid’s algorithm is based on the following two results (Division algorithm and Theorem 1) in Elementary Number Theory. As we know, Division algorithm is the basis of Theory of Divisibility. Almost all number-theoretical texts begin with it. For example, see [14 19]. However, Euclid did not do like this. Euclid began his number-theoretical work by introducing his algorithm (See [1]: Book 7, Propositions 1 and 2). In this paper, we will prove that Euclid’s algorithm is essentially equivalent with Division algorithm. More precisely, for any positive integer $a$ and $b$, that there exist unique integers $q$ and $r$ such that $a = bq + r$ and $0 \leq r < b$ is equivalent with that there exist integers $x$ and $y$ such that $ax + by = \gcd(a, b)$. This implies that Division algorithm, Euclid’s algorithm and the Bezout’s equation are equivalent. For the details, see Section 2.

2 Proof that Euclid’s algorithm is equivalent with Division algorithm

Strictly speaking, Division algorithm is essentially a theorem, but Euclid’s algorithm is an algorithm, a theorem and an algorithm are not the same thing. Therefore, we should view Euclid’s algorithm as a theorem as the follows:

Euclid’s algorithm: For two distinct positive integers, replace continually the larger number by the difference of them until both are equal, then the answer is their greatest common divisor.
Division algorithm (called also Division with remainder): For any positive integer \( a \) and \( b \), that there exist unique integers \( q \) and \( r \) such that \( a = bq + r \) and \( 0 \leq r < b \).

Proof of Division algorithm: The uniqueness is clear. Therefore, it is sufficient that we only prove the existence. If \( a \mid b \), then \( q = \frac{a}{b} \) and \( r = 0 \). Otherwise, consider the set \( S = \{ a - bt : t = 0, \pm 1, \pm 2, \ldots, \infty \} \). It is manifest that there are positive integers in this set. Therefore, there must be the least positive integer in this set. Denote this number by \( c = a - bt_0 \). Obviously, \( c < b \) (otherwise \( c - b > 0 \) in the set \( S \), and \( c \) is not the least positive integer in this set). Set \( q = t_0 \) and \( r = c \). This completes the proof.

Another proof of Division algorithm: The uniqueness is clear. Therefore, it is sufficient that we only prove the existence. For any given positive integer \( b \), when \( a = 1 \), if \( b = 1 \), then let \( q = 1, r = 0 \), and the existence satisfies; if \( b > 1 \), then let \( q = 0, r = 1 \), and the existence satisfies again. Now, we assume that the existence satisfies when \( a = n \). We write \( n = bq_1 + r_1 \). Then, when \( a = n + 1 \), we have \( n + 1 = bq_1 + r_1 + 1 \). Note that \( 0 \leq r_1 < b \). If \( r_1 = b - 1 \), then let \( q = q_1 + 1, r = 0 \). If \( r_1 < b - 1 \), then let \( q = q_1, r = r_1 + 1 \), and the existence satisfies still. Therefore, by induction, Division algorithm is true.

Remark 1: The proofs above need Peano axioms [14] which give the strict definition of the set of natural numbers and imply the induction and that if there are positive integers in a set, then, there must be a least positive integer in this set and also imply the existence and uniqueness the great common divisor.

Theorem 1: For any positive integer \( a \) and \( b \), that there exist unique integers \( q \) and \( r \) such that \( a = bq + r \) and \( 0 \leq r < b \). Moreover, if \( a = bq + r \), then \( \gcd(a, b) = \gcd(b, r) \).

Note that if \( b \) divides \( a \), then \( r = 0 \) in Division algorithm and \( \gcd(a, b) = b \). Hence, in order to accord with the notation of the greatest common divisor of two positive integers, we set \( \gcd(b, r) = b \) when \( r = 0 \).

Proof of Theorem 1: By Division algorithm, it is sufficient that we only prove \( \gcd(a, b) = \gcd(b, r) \). Let \( \gcd(a, b) = d \) and \( \gcd(b, r) = t \). Since \( r = a - bq \) and \( d \mid a, d \mid b \), hence \( d \mid r \). Now, \( d \mid b, d \mid r \), so \( d \mid \gcd(b, r) \) and \( d \mid t \). On the other hand, since \( a = bq + r \), and \( t \mid b, t \mid r \), hence \( t \mid a \) and \( t \mid d \). So, \( t = d \) and Theorem 1 is true.
Remark 2: In this proof, we use the definition of the greatest common divisor of positive integers \( x \) and \( y \), namely, the greatest common divisor \( \gcd(x, y) \) is not less than any other common divisor \( x \) and \( y \). Any two numbers have the greatest common divisor simply because the set of common divisors is finite. But, this does not imply that the greatest common divisor is divisible by any other common divisor. By Peano axioms, there exists a least positive integer \( d \) in the set \( S = \{ax + by : a \in \mathbb{Z}, b \in \mathbb{Z}\} \). By Division algorithm, we have \( x = dq + h \) and \( y = dp + r \), but \( r \in S, h \in S \), so \( r = h = 0 \) since \( d \) is least. Thus \( d \) is a common divisor \( x \) and \( y \). But \( d = ax + by \) implies that \( d \) is divisible by any other common divisor of \( x \) and \( y \). Specially, \( d \) is not less than any other common divisor of \( x \) and \( y \). By the definition of the greatest common divisor, \( d \) is the greatest common divisor. The uniqueness of the greatest common divisor is clear. It immediately shows that the greatest common divisor of \( x \) and \( y \) is divisible by any other common divisor of \( x \) and \( y \). This is exactly the porism of Proposition 2 in *Elements* Book 7.

Nevertheless, in his Elements Book 7, Euclid did not give the definition of the greatest common divisor. He only gave the definition of numbers prime to one another, in this case, the greatest common divisor of numbers is 1, see [1: Book 7, definition 12]. Then, he gave the sufficient condition of two numbers prime to one another, see [1: Book 7, Proposition 1]. I wonder why he did not give a necessary condition or the necessary and sufficient condition of two numbers prime to one another as Proposition 1. More precisely, Proposition 1 should be expressed as follows: two numbers will be prime to one another if and only if the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unite (i.e. 1) is left. After proving Proposition 1, Euclid considered how to find the greatest common divisor of two numbers, see [1: Book 7, Proposition 2]. He proved firstly that the output of his algorithm is a common divisor of two numbers, and then, he showed that any common divisor has to divide it, so has to be smaller. Thus, he gave naturally the porism of Proposition 2 which states that if a number is a common divisor of two numbers, then it divides their greatest common divisor.

By Division algorithm and Theorem 1, we see again that Euclid’s algorithm is true without using Euclid’s proof in Proposition 2. So, we can say that Division algorithm implies Euclid’s algorithm by giving the definition of the greatest common divisor. In fact, from his algorithm (Propositions 1 and 2), one can observe that Euclid assumed as two axioms. One is of that,
if \(a\) and \(b\) are both divisible by \(c\), so is \(a - bq\). Another is of that for any positive integer \(a\) and \(b\), there exist integers \(q\) and \(r\) such that \(a = bq + r\) and \(0 \leq r < b\). We do not know why Euclid began his number-theoretical work by introducing his algorithm without introducing any postulates or common notions (axioms). This is not his style for constructing Geometry system. He must have some reason for this. In my eyes, Euclid is not only a mathematician but also a philosopher. Perhaps, he felt that his algorithm is essentially equivalent with Division algorithm, or that he needed more complicated expression for some axioms like Peano axioms.

Now, we consider the set \(T = \{ax + by : x \in \mathbb{Z}, y \in \mathbb{Z}\}\) for any given positive integers \(a\) and \(b\). Note that there are positive integers in this set. Therefore, there must be a least positive integer in this set. Denote this number by \(d = ax_0 + by_0\). Of course, \(d\) is just the greatest common divisor of \(a\) and \(b\). Thus we get the following Bezout’s equation.

**Theorem 2:** For any given positive integers \(a\) and \(b\), there are integers \(x\) and \(y\) such that \(ax + by = \gcd(a, b)\).

By Euclid’s algorithm, it is also easy to find such integers \(x\) and \(y\) so that \(ax + by = \gcd(a, b)\). In fact, we have the well-known procedure involves repeated division, resulting in a sequence as following: \(a = bq_1 + r_1 (0 \leq r_1 < b)\), \(b = q_2r_1 + r_2 (0 \leq r_2 < r_1)\), \(r_1 = q_3r_2 + r_3 (0 \leq r_3 < r_2)\),... The process terminates when some remainder \(r_k = 0\), which must happen eventually. Then \(\gcd(a, b) = r_{k-1}\), and \(x\) and \(y\) are found as follows: Compute \(r_1 = a - q_1b\), substitute into \(b = q_2r_1 + r_2\) and obtain \(r_2 = -q_2a + (1 + q_1q_2)b\), and so on. Eventually, we must find integers \(x\) and \(y\) such that \(ax + by = \gcd(a, b)\).

So, Euclid’s algorithm implies Theorem 2.

Next, we prove that Theorem 2 implies Division algorithm. By symmetry it is safe to assume that \(x \geq 0\). If \(x = 0\), then \(b|a\), so there exist integers \(q\) and \(r\) such that \(a = bq + r\) and \(0 \leq r < b\). If \(x = 1\), then \(a = -yb + \gcd(a, b)\). If \(\gcd(a, b) = b\), then \(b|a\). So we can assume that \(\gcd(a, b) < b\). Let \(q = -y, r = \gcd(a, b)\), then Division algorithm holds. Now let \(x > 1\). Since \(\gcd(a, b) = ax + by\), hence we can write \(a = b(-y - x + 1) + (x - 1)(b - a) + \gcd(a, b)\). If \(0 \leq (x - 1)(b - a) + \gcd(a, b) < b\), then let \(r = (x - 1)(b - a) + \gcd(a, b)\) and Division algorithm holds. If \((x - 1)(b - a) + \gcd(a, b) > b\), then \(b > a\) since \(x > 1\) and \(\gcd(a, b) \leq b\). Let \(q = 0\) and \(r = a\). Division algorithm holds. If \((x - 1)(b - a) + \gcd(a, b) = b\), then \(b|a\) since \(a = b(-y - x + 1) + (x - 1)(b - a) + \gcd(a, b)\). Division algorithm holds still. Therefore, it suffices to consider the case of \((x - 1)(b - a) + \gcd(a, b) < 0\) which implies that \(0 < b < a - \frac{x\gcd(a,b)}{x-1} < a\).
We write $(x - 1)(b - a) + \gcd(a, b) = -c$. Thus, $0 < c < b(-y - x + 1)$. Let $d = -y - x + 1$. Clearly, $d$ is a positive integer. Note that $x$ and $y$ are given since $ax + by = \gcd(a, b)$. Therefore $d$ is decided. So, we can consider a finite number of intervals $[0, b), [b, 2b), \ldots, [(d - 1)b, db)$. $c$ must be in some interval among these intervals. Let $ib \leq c < (i + 1)b$, where $1 \leq i < d$. Thus, $a = bd - c = b(d - i - 1) + (i + 1)b - c$. Let $q = d - i - 1$ and $r = (i + 1)b - c$. Therefore, there exist integers $q$ and $r$ such that $a = bq + r$ and $0 \leq r < b$. Of course, the uniqueness of $q$ and $r$ is obvious. Thus, Division algorithm is true. Therefore, Euclid’s algorithm is essentially equivalent with Division algorithm. Thus, we proved the following theorem 3.

**Theorem 3:** For any positive integer $a$ and $b$, that there exist unique integers $q$ and $r$ such that $a = bq + r$ and $0 \leq r < b$ is equivalent with that there exist integers $x$ and $y$ such that $ax + by = \gcd(a, b)$.

Theorem 3 shows that Division algorithm, Euclid's algorithm and the Bezout's equation are equivalent.

Note that Euclid’s algorithm implies also Euclid’s second theorem. Let’s go back to Euclid’s proof for the infinitude of prime numbers: Supposed that there are only finitely many primes, say $k$ of them, which denoted by $p_1, \ldots, p_k$. Consider the number $E = 1 + \prod_{i=1}^{i=k} p_i$. If $E$ is prime, it leads to the contradiction since $E \neq p_i$ for any $1 \leq i \leq k$. If $E$ is not prime, $E$ has a prime divisor $p$ by Proposition 31 (Book 7). But $p \neq p_i$ for any $1 \leq i \leq k$. Otherwise, $p$ divides $\prod_{i=1}^{i=k} p_i$. Since it also divides $1 + \prod_{i=1}^{i=k} p_i$, it will divide the difference or unity, which is impossible.

In his proof, we see that Euclid used Proposition 31 (Book 7). Of course, he also used a unexpressed axiom which states that if $A$ divides $B$, and $B$ divides $C$, then $A$ divides $C$. Thomas Little Heath had noted that Euclid used the aforementioned axioms. We would be quite surprised
if he did use these axioms because on one hand, Proposition 31 (Book 7) and Proposition 20 (Book 9) can be deduced early by definitions, on the other hand, we expect him to make use of his algorithm which is his first number-theoretical proposition in his *Elements*. In [69, Appendix], we try to supplement this work and give a simple proof.

For his other number-theoretical work, see [Appendix] in which we give a list of 22 definitions, 102 propositions and 3 porisms on Number Theory in Euclid’s *Elements* Books 7, 8 and 9 (Translated by Thomas Little Heath). We omit Euclid’s proofs and Heath’s commentaries for anyone who would like to learn Euclid’s Number Theory in his own way.

Based on the discussion and analysis above, we can say that we ourselves ‘require the Euclidean algorithm to prove unique factorization’ because it is the basis of Theory of Divisibility from Peano axioms. *Elements* established elementarily Theory of Divisibility and the greatest common divisor. This is the base of Number Theory (After more than 2000 years, by introducing creatively the notion of congruence, Gauss published his *Disquisitiones Arithmeticae* in 1801 and developed Number Theory as a systematic science.). In the nature of things, Euclid’s algorithm is the most important number-theoretical achievement of Euclid. In next section, we will further summarize briefly the influence of Euclid’s algorithm.

### 3 Euclid’s Algorithm and Our Conclusions

Historically, many mathematicians and computer scientists studied Euclid’s algorithm. For example, D. H. Lehmer, J. D. Dixon, L. K. Hua, Donald E. Knuth, Andrew C. Yao, H. Lenstra, A. K. Lenstra, H. Davenport, J. Barkley Rosser, P.M. Cohn, Heilbronn, Viggo Brun and so on. So many people like Euclid’s because it is not only simple and beautiful but also useful. Although more than 2000 years have passed, the study on Euclid’s algorithm still goes on and on. Heath [1, Preface to the second edition] said: ‘Euclid is far from being defunct or even dormant, and that, so long as mathematics is studied, mathematicians will find it necessary and worth while to come back again and again, for one purpose or another, to the twenty-two centuries-old book which, notwithstanding its imperfections, remains the greatest elementary textbook in mathematics that the world is privileged to possess.’

In 1968, Heilbronn [22] studied the average length of a class of finite continued fractions. This is an important result on Euclid’s algorithm. T.
Toukov [23] and J. W. Porter [24] improved Heilbronns estimate respectively. In 1975, using an idea of Heilbronn, Andrew C. Yao and Donald E. Knuth [25] studied the sum of the partial quotients $q_i$ in Euclid’s algorithm. They proved a well-known result which states that the sum $S$ of all the partial quotients of all the regular continued fractions for $\frac{a}{b}$ with $1 \leq b \leq a$ is $6\pi^2 - 2a(\log a)^2 + O(a \log a (\log \log a)^2)$. This implies that $S \ll a(\log a)^2$. In 1994, ZhiYong Zheng [26] improved the result of Andrew C. Yao and Donald E. Knuth. As an application, in 1996, J. B. Conrey, Eric Frensen, and Robert Klein [27] studied the mean values of Dedekind Sums. For a positive integer $k$ and an arbitrary integer $h$, the Dedekind sum $s(h, k)$ is defined by

$$s(h, k) = \sum_{a=1}^{\infty} \left( \left[ \frac{a}{k} \right] \left( \frac{ah}{k} \right) \right),$$

where $\left( \left[ x \right] \right) = x - \left[ x \right] - \frac{1}{2}$ if $x \neq 0$, otherwise $\left( \left[ x \right] \right) = 0$. The most famous property of the Dedekind sums is the reciprocity formula as follows

$$s(h, k) + s(k, h) = \frac{h^2 + k^2 + 1}{12hk} - \frac{1}{4}. $$

Dedekind Sums are closely related to the transformation theory of the Dedekind eta-function which is the infinite product $\eta(\tau) = q^{24} \prod_{n=1}^{\infty} (1 - q^n)$ and satisfies the transformation law $\eta(-\frac{1}{\tau}) = \sqrt{-i\tau} \eta(\tau)$ with $\tau \in H = \{ \tau \in \mathbb{C} : \text{Im} (\tau) > 0 \}$, where $q = e^{2\pi i \tau}, q_{24} = e^{\frac{1}{12} \pi i \tau}$. Thus, the Dedekind eta-function is a modular form of weight $\frac{1}{2}$ for the modular group $SL(2, \mathbb{Z}) = \{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \}$. Each element of the modular group $SL(2, \mathbb{Z})$ is also viewed as an automorphism (invertible self-map) of the Riemann sphere $\mathbb{C} \cup \{ \infty \}$, the fractional linear transformation $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) (\tau) = \frac{a\tau + b}{c\tau + d}, \tau \in \mathbb{C} \cup \{ \infty \}$. The Dedekind eta-function played a prominent role in Number Theory and in other areas of mathematics. From these results above, one can see again the importance of Euclid’s algorithm.

Note that Euclid’s algorithm is essentially a dynamical system. Namely, the Euclidean algorithm is the map defined by

$$f : (x, y) \in \mathbb{R}^2 \rightarrow \{ (x - y, y), x \geq y \\
(x, y - x), x < y \},$$

where $\mathbb{R}^+ = \{ x \in \mathbb{R} : x \geq 0 \}$. Clearly, when $x$ and $y$ are natural numbers, there must be a positive integer $k$ such that $f^k(x, y) = (0, d)$ or $(d, 0)$, where $d$ is the greatest common divisor of $x$ and $y$. Equivalently, the map $f$ is given by

$$\left( \begin{array}{c} x \\ y \end{array} \right) \rightarrow \left\{ \begin{array}{ll} \left( \begin{array}{c} 1 \ -1 \\ 0 \ 1 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right), x \geq y \\
\left( \begin{array}{c} 1 \ 0 \\ -1 \ 1 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right), x < y \end{array} \right\}. $$
Note that \((1, -1)^{-1}\) and \((1, 0)^{-1}\) generate the modular group \(SL(2, \mathbb{Z})\).

By a result of Hedlund [28], we know that the linear action of \(SL(2, \mathbb{Z})\) on \(\mathbb{R}^2\) is ergodic. By a result of A. Nogueira [29], we see also that the map \(f\) is ergodic relative to the Lebesgue measure on \(\mathbb{R}^2\). In 2007, Dani, S. G. and Nogueira, Arnaldo [30] showed that ‘the Euclidean algorithm \(f\) turns out to be an example of a dissipative transformation for which this is not the case via a natural extension of \(f\) constructed by using the action of \(SL(2, \mathbb{Z})\) on a subset of \(SL(2, \mathbb{R})\).’

Euclid’s algorithm gives the greatest common divisor \(a\) and \(b\) in a finite number of steps and its total running time is approximately \(O(\log ab)\) word operations [31]. Lamé’s theorem states that Euclid’s algorithm runs in no more than \(5k\) steps, where \(k\) is the number of (decimal) digits of \(\max\{a, b\}\) [17]. In 2003, Vallée, B. [32] published a paper entitled ‘Dynamical analysis of a class of Euclidean algorithms’ and developed a general technique for analyzing the average-case behavior of the Euclidean-type algorithms. ‘This is a deep and important paper which merits careful study, and will likely have a significant impact on future directions in algorithm analysis.’ Jeffrey O. Shallit reviewed, ‘The method involves viewing these algorithms as a dynamical system, where each step is a linear fractional transformation of the previous one. ... Then a generating function (Dirichlet series) is used to describe the cost of the algorithms, and Tauberian theorems are used to extract the coefficients.’ In 2006, Vallée, B. [33] further proposed a detailed and precise dynamical and probabilistic analysis of the more natural variants of Euclid’s algorithm. The paper ‘presented a clear, clever, and unified overview of the methodology (dynamical analysis) and of the tools.’ Valérie Berthé reviewed, ‘... One of the strengths of this approach comes from the fact that it combines sophisticated tools taken from, on the one hand, analytic combinatorics and functional analysis (moment generating functions, Dirichlet series, quasi-power theorems and Tauberian theorems), and on the other hand, from dynamical systems and ergodic theory (including Markovian dynamical systems, induction, transfer operators and related concepts such as zeta series, pressure and entropy)... In particular, Section 9 provides a detailed overview of the literature on the subject as well as a list of open problems, particularly, in the higher-dimensional case (including the lattice reduction problem or classical multidimensional continued fraction algorithms such as the Jacobi-Perron algorithm).’ From these results above, one would see again the vitality of Euclid’s algorithm. ‘Although more than 2000 years have passed, the Euclidean algorithm has not yet been completely
analyzed’—Vallée, B.

In Book 7, Proposition 3 of his Elements [1], Euclid considered how to compute the great common divisors of three positive integers \(a, b\) and \(c\). His method is simple and natural. Namely, firstly, compute \(\gcd(a, b) = d\), secondly, compute \(\gcd(d, c) = e\), then \(\gcd(a, b, c) = e\). It is easy to prove that this method is true, and this method can be readily generalized to the case for computing the greatest common divisor of several positive integers. Thus, using Euclid’s algorithm, one can solve the following two problems:

**Problem 1:** Let \(a_1, ..., a_m\) and \(b\) be any positive integers. Find an algorithm to determine whether \(b\) can be represented by \(a_1, ..., a_m\). Or equivalently, find an algorithm to determine whether \(b\) belongs to the ideal generated by \(a_1, ..., a_m\).

**Problem 2:** Let \(a_1, ..., a_m\) be any positive integers. Find an algorithm to determine whether there is an integer \(a_i\) among \(a_1, ..., a_m\) such that \(a_i\) is relatively prime with all of the others.

Generalizing Problem 1 to the case over the unique factorization domain \(F[x_1, ..., x_n]\), where \(F\) is a field. Let \(f_1, ..., f_m\) and \(g\) be polynomials in \(F[x_1, ..., x_n]\). Find an algorithm to determine whether \(g\) belongs to the ideal generated by \(f_1, ..., f_m\). As we know, this interesting generalization leads to the invention of Gröbner bases of polynomial ideals. Using S-polynomial, also combining with the multivariate division algorithm, in 1965, Buchberger [34] gave an algorithm for finding a basis \(g_1, ..., g_k\) of the ideal \(I = \langle f_1, ..., f_m \rangle\) such that the leading term of any polynomial in \(I\) is divisible by the leading term of some polynomial in \(G = \{g_1, ..., g_k\}\). Such a basis is called Gröbner bases by Buchberger. An analogous concept was developed independently by Heisuke Hironaka in 1964 [35, 36], who named it standard bases. As we know, Gröbner bases of polynomial ideals in modern Computational Algebraic Geometry are very important. They also are rather useful to Symbolic Computation and Cryptography and so on. The concept and algorithms of Gröbner bases have been further generalized to modules over a polynomial ring and to non-commutative (skew) polynomial rings such as Weyl algebras.

Problem 2 leads to the invention of W sequences [58] which play an interesting role in the study of primes and enable us to give new weakened forms of many classical problems which are open in Number Theory. For any integer \(n > 1\), the sequence of integers \(0 < a_1 < a_2 < ... < a_n\) is called
a W sequence, if there exists \( r \) with \( 1 \leq r \leq n \) such that \( a_r \) and each of the rest numbers are coprime.

W sequences in the case of consecutive positive integers relates to Grimm’s conjecture. Clearly, if there is a prime in the sequence \( m+1, \ldots, m+n \), then this sequence is a W sequence. Therefore, in order to determine whether a consecutive positive integer sequence is a W sequence, it is enough to consider the case of consecutive composite numbers. It leads to the further study of consecutive composite numbers. In 1969, C.A.Grimm [57] made an important conjecture that if \( m+1, \ldots, m+n \) are consecutive composite numbers, then there exist \( n \) distinct prime numbers \( p_1, \ldots, p_n \) such that \( m+i \) is divisible by \( p_i \) for \( 1 \leq i \leq n \). This implies that for all sufficiently large integer \( n \), there is a prime between \( n^2 \) and \((n+1)^2\). It is nice that for \( m \geq 1 \), that there exists a prime in the interval \((m^2, (m+1)^2)\) is equivalent with that \( m^2 + 1, \ldots, m^2 + 2m \) is a W sequence. In [59], we further refine the function \( g(m) \) on Grimm’s conjecture and obtain several interesting results. For example, we refine a result of Erdös and Selfridge without using Hall’s theorem.

Denote the largest integer \( n \) in \( m+1, \ldots, m+n \) by \( h(m) \) such that no one of \( m+1, \ldots, m+n \) is relatively prime with all of the others. Cramér’s conjecture [70] and Pillai’s result [71] imply \( 17 \leq h(m) \ll (\log m)^2 \). From these, one can see that the W sequences in the case of consecutive positive integers tie up the distribution of primes in short interval. Unfortunately, it is not easy to prove that a sequence is a W sequence.

In the non-consecutive case, W sequences enable us to get a new weakened form of Goldbach’s conjecture and reveal the internal relationship between Goldbach’s conjecture and the least prime in an arithmetic progression. We find that Goldbach Conjecture ties up Kanold Hypothesis and Chowla Hypothesis, for details, see [60]. For another example, for positive integers \( a \) and \( b \) with \( \gcd(a, b) = 1 \) and \( 1 \leq a < b \), if there is a prime in \( a, a+b, \ldots, a+(b-1)b \), then this sequence is a W sequence. Thus, in order to determine whether the sequence \( a, a+b, \ldots, a+(b-1)b \) is a W sequence, it is enough to consider the case that \( a, a+b, \ldots, a+(b-1)b \) are all composite numbers. It leads to the generalization of Grimm’s conjecture. In [61], we generalized a theorem about the binomial coefficient and got a slightly stronger result than Langevin’s [62]. This leads to possible generalizations of Grimm’s conjecture [68].

Generalize W sequences to the case over the unique factorization do-
main $F[x_1, \ldots, x_n]$, where $F$ is a field. It leads how to find an efficient algorithm for computing the greatest common divisor of any two polynomials in $F[x_1, \ldots, x_n]$.

By the aforementioned discussion, one see that Euclid’s algorithm implies his second theorem. We believe that one of substantive characteristics of the set of all integers is that it contains infinitely many prime numbers. It is known that $f(x) = x$ on $\mathbb{Z}$ is the simplest polynomial which represents infinitely many primes. By Dirichlet’s famous theorem, for any positive integer $l, k$ with $(l, k) = 1$, $f(x) = l + kx$ is a simpler polynomial which also represents infinitely many primes. More generally, it is possible to establish a generic model for the problem that several multivariable integral polynomials represent simultaneously primes. More concretely, let’s consider the map $F: \mathbb{Z}^n \to \mathbb{Z}^m$ for all integral points $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, $F(x) = (f_1(x), \ldots, f_m(x))$ for distinct polynomials $f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_n]$. How to determine whether $f_1(x), \ldots, f_m(x)$ represent simultaneously primes? In [67,72], we considered this problem and obtained some interesting results. We strongly wish that in the higher-dimension case, we have a similar Prime Number Theorem. Thus, it is also possible to generalize the problem of the least prime number in an arithmetic progression and an analogy of Chinese Remainder Theorem, moreover, give an analogy of Goldbach’s conjecture, and so on.

It is well-known that we can solve either linear Diophantine equations or a system of simultaneous linear congruences, and find also modular inversions, and expand continued fractions, testing primality, generating primes, factoring large integers and so on by using Euclid’s algorithm and extended algorithm. Moreover, due to the fact that Euclid’s algorithm can be not only readily generalized to polynomials in one variable over a field but also generalized multivariate polynomials over any unique factorization domain, Euclid’s algorithm also plays an important role in symbolic computation and cryptography even in science and engineering. Without Euclid’s algorithm there would be no the prosperity of computation nowadays, we are afraid. It is very nice that Viggo Brun [20, 21] studied the relations between Euclid’s algorithm and music theory. By coincidence, Euclid himself also reveled in music [1].

Looking back into Ancient Greek Number Theory history, it is not difficult to confirm that Euclid’s algorithm indeed is the greatest number-theoretical achievements of the age. Some scholars believe that Euclid’s algorithm probably was not Euclid’s own invention [19]. Many scholars
conjecture was actually Euclid’s rendition of an algorithm due to Eudoxus (c.375B.C.) [3]. Nevertheless, it first appeared in Euclid’s Elements, and more importantly, it is the first nontrivial algorithm that has survived to this day. Therefore, I think that this is why people would like to call it Euclid’s algorithm. Perhaps, it is suitable to call it Ancient Greek Algorithm.

Anyway, also closely relating to many famous algorithms such as Guass’ elimination, Buchberger’s algorithm [34], Schoof’s algorithm [37, 38], Cornacchia’s algorithm [39], LLL Algorithm [40], modern factorization algorithms (Continued Fraction Factoring Algorithm [63], the Elliptic Curve Factoring Algorithm [64], the Multiple Polynomial Quadratic Sieve [65] and the Number Field Sieve [66]) and so on, the Euclidean algorithm (together with the discovery of irrationals in Pythagoras’ School) is the greatest achievement of Ancient Greek Number Theory. Let’s cite Edna St. Vincent Millay’ sonnet Euclid Alone Has Looked on Beauty Bare to close this paper.

**Euclid alone has looked on Beauty bare**

Euclid alone has looked on Beauty bare.  
Let all who prate of Beauty hold their peace,  
And lay them prone upon the earth and cease  
To ponder on themselves, the while they stare  
At nothing, intricately drawn nowhere  
In shapes of shifting lineage; let geese  
Gabble and hiss, but heroes seek release  
From dusty bondage into luminous air.  
O blinding hour, O holy, terrible day,  
When first the shaft into his vision shone  
Of light anatomized! Euclid alone  
Has looked on Beauty bare. Fortunate they  
Who, though once only and then but far away,  
Have heard her massive sandal set on stone.

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Appendix: Euclid’s *Elements* Books 7, 8 and 9

Below is the list of 22 definitions, 102 propositions and 3 porisms on Elementary Number Theory in Euclid’s *Elements* Books 7, 8 and 9 (Translated by Thomas Little Heath).

**Book 7 Definitions**
1. A unit is that by virtue of which each of the things that exist is called one.
2. A number is a multitude composed of units.
3. A number is a part of a number, the lesser of the greater, when it measures the greater.
4. but parts when it does not measure it.
5. The greater number is a multiple of the lesser when it is measured by the lesser.
6. An even number is that which is divisible into two equal parts.
7. An odd number is that which is not divisible into two equal parts or that which differs by a unit from an even number.
8. An even-times-even number is that which is measured by an even number according to an even number.
9. An even-times-odd number is that which is measured by an even number according to an odd number.
10. An odd-times-odd number is that which is measured by an odd number according to an odd number.
11. A prime number is that which is measured by a unit alone.
12. Numbers prime to one another are those which are measured by a unit alone as a common measure.
13. A composite number is that which is measured by some number.
14. Numbers composite to one another are those which are measured by some number as a common measure.
15. A number is said to multiply a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.
16. And, when two numbers having multiplied one another make some number, the number so produced is called plane, and its sides are the numbers which have multiplied one another.
17. And, when three numbers having multiplied one another make some number, the number so produced is solid, and its sides are the numbers which have multiplied one another.
18. A square number is equal multiplied by equal, or a number which is
contained by two equal numbers.
19. And a cube is equal multiplied by equal and again by equal, or a
number which is contained by three equal numbers.
20. Numbers are proportional when the first is the same multiple, or the
same part, or the same parts, of the second that the third is of the fourth.
21. Similar plane and solid numbers are those which have their sides
proportional.
22. A perfect number is that which is equal to its own parts.

**Book 7 Propositions**

Proposition 1: Two unequal numbers being set out, and the less being
continually subtracted in turn from the greater, if the number which is left
never measures the one before it until a unit is left, the original numbers
will be prime to one another.

Proposition 2: Given two numbers not prime to one another, to find
their greatest common measure.

Porism: If a number measures two numbers then it will also measure
their greatest common measure.

Proposition 3: Given three numbers not prime to one another, to find
their greatest common measure.

Proposition 4: Any number is either a part or parts of any number, the
lesser of the greater.

Proposition 5: If a number be a part of a number, and another be the
same part of another, then the sum will also be the same part of the sum
that the one is of the one.

Proposition 6: If a number be parts of a number, and another be the
same parts of another, then the sum will also be the same parts of the sum
that the one is of the one.

Proposition 7: If a number be that part of a number, which a num-
ber subtracted is of a number subtracted, the remainder will also be the same
part of the remainder that the whole is of the whole.

Proposition 8: If a number be the same parts of a number that a num-
ber subtracted is of a number subtracted, the remainder will also be the same
parts of the remainder that the whole is of the whole.

Proposition 9: If a number be a part of a number, and another be the
same part of another, alternately also, whatever part or parts the first is of
the third, the same part, or the same parts, will the second also be of the
fourth.

Proposition 10: If a number is parts of a number, and another be the
same parts of another, alternately also, whatever parts or part the first is of the third, the same parts, or the same part will the second also be of the fourth.

Proposition 11: If, as the whole is to whole, so is a number subtracted to a number subtracted, the remainder will also be to the remainder as whole to whole.

Proposition 12: If there be as many numbers as we please in proportion, then, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents.

Proposition 13: If four numbers be proportional, they will also be proportional alternately.

Proposition 14: If there be as many numbers as we please, and others equal to them in multitude, which taken two and two are in the same ratio, they will also be in the same ratio ex aequali.

Proposition 15: If a unit measures any number, and another number measures any other number the same number of times, then, alternately also, the unit will measure the third number the same number of times that the second measures the fourth.

Proposition 16: If two numbers by multiplying one another make certain numbers, the numbers so produced will be equal to one another.

Proposition 17: If a number by multiplying two numbers make certain numbers, the numbers so produced will have the same ratio as the number multiplied.

Proposition 18: If two numbers by multiplying any number make certain numbers, the numbers so produced will have the same ratio as the multipliers.

Proposition 19: If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third; and, if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional.

Proposition 20: The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the lesser the lesser.

Proposition 21: Numbers prime to one another are the least of those which have the same ratio with them.

Proposition 22: The least numbers of those which have the same ratio with them are prime to one another.

Proposition 23: If two numbers be prime to one another, the number which measures the one of them will be prime to the remaining number.
Proposition 24: If two numbers be prime to any number, their produced
also will be prime to the same.

Proposition 25: If two numbers be prime to one another, the product of
one of them into itself will be prime to the remaining one.

Proposition 26: If two numbers be prime to two numbers, both to each,
their products also will be prime to one another.

Proposition 27: If two numbers be prime to one another, and each by
multiplying itself make a certain number, the products will be prime to
one another; and if the original numbers by multiplying the products make
certain numbers, the latter will also be prime to one another [and this is
always the case with the extremes].

Proposition 28: If two numbers be prime to one another, the sum will
also be prime to each of them; and if the sum of two numbers be primes to
any one of them, the original numbers will also be prime to one another.

Proposition 29: Any prime number is prime to any number which it does
not measure.

Proposition 30: If two numbers by multiplying one another make some
number, and any prime number measures the product, it will also measure
one of the original numbers.

Proposition 31: Any composite number is measured by some prime num-
ber.

Proposition 32: Every number is either prime or is measured by some
prime number.

Proposition 33: Given as many numbers as we please, to find the least
of those which have the same ratio with them.

Proposition 34: Given two numbers, to find the least number which
they measure.

Proposition 35: If two numbers measure any number, the least number
measured by them will also measure the same number.

Proposition 36: Given three numbers, to find the least number which
they measure.

Proposition 37: If a number be measured by any number, the number
which is measured will have a part called by the same name as the measuring
number.

Proposition 38: If a number has any part whatever, it will be measured
by a number called the same name as the part.

Proposition 39: To find the number which is the least that will have
given parts.

**Book 8 Propositions**
Proposition 1: If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the numbers are the least of those which have the same ratio with them.

Proposition 2: To find the numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio.

Porism: If three numbers in continued proportion be the least of those which have the same ratio with them, the extremes of them are square, and, if four numbers, cubes.

Proposition 3: If as many numbers as we please in continued proportion be the least of those which have the same ratio with them, the extremes of them are prime to one another.

Proposition 4: Given as many numbers as we please in least numbers, to find numbers in continued proportion which are the least in the given ratios.

Proposition 5: Plane numbers have to one another the ratio compounded of the ratios of their sides.

Proposition 6: If there be many numbers as we please in continued proportion, and the first does not measure the second, then neither will any other measure any other.

Proposition 7: If there be as many numbers as we please in continued proportion, and the first measures the last, it will measure the second also.

Proposition 8: If between two numbers there fall numbers in continued proportion with them, then, however many numbers fall between them in continued proportion, so many will also fall in continued proportion between the numbers which have the same ratio with the original numbers.

Proposition 9: If two numbers be prime to one another, and numbers fall between them in continued proportion, then, however many numbers fall between them in continued proportion, so many will also fall between each of them and a unit in continued proportion.

Proposition 10: If numbers fall between each of two numbers and a unit in continued proportion, however many numbers fall between each of them and a unit in continued proportion, so many also will fall between the numbers themselves in continued proportion.

Proposition 11: Between two square numbers there is one mean proportional number, and the square has to the square the ratio duplicate of that which the side has to the side.

Proposition 12: Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side.

Proposition 13: If there be as many numbers as we please in continued proportion, and each by multiplying itself makes some number, the product
will be proportional; and, if the original numbers by multiplying the product make certain numbers, the latter will also be proportional.

Proposition 14: If a square measures a square, the side will also measure the side, and, if the side measures the side, the square will also measure the square.

Proposition 15: If a cube number measures a cube number, the side will also measure the side; and, if the side measures the side, the cube will also measure the cube.

Proposition 16: If a square number does not measure a square number, neither will the side measure the side; and, if the side does not measure the side, neither will the square measure the square.

Proposition 17: If a cube number does not measure a cube number, neither will the side measure the side; and, if the side does not measure the side, neither will the cube measure the cube.

Proposition 18: Between two similar plane numbers there is one mean proportional number; and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side.

Proposition 19: Between two similar solid numbers there fall two mean proportional numbers; and the solid number has to the similar solid number the ratio triplicate of that which the corresponding side has to the corresponding side.

Proposition 20: If one mean proportional number falls between two numbers, the numbers will be similar plane numbers.

Proposition 21: If two mean proportional numbers fall between two numbers, the numbers will be similar solid numbers.

Proposition 22: If three numbers be in continued proportion, and the first be square, the third will also be square.

Proposition 23: If four numbers be in continued proportion, and the first is cube, the fourth will also be cube.

Proposition 24: If two numbers have to one another the ratio which a square number has to a square number, and the first be square, the second will also be square.

Proposition 25: If two numbers have to one another the ratio which a cube number has to a cube number, and the first be cube, the second will also be cube.

Proposition 26: Similar plane numbers have to one another the ratio which a square number has to a square number.

Proposition 27: Similar solid numbers have to one another the ratio which a cube number has to a cube number.
Book 9 Propositions

Proposition 1: If two similar plane numbers by multiplying one another make some number, the product will be square.

Proposition 2: If two numbers by multiplying one another make a square number, they are similar plane numbers.

Proposition 3: If a cube number by multiplying itself, the product will be cube.

Proposition 4: If a cube number by multiplying a cube number makes some number, the product will be cube.

Proposition 5: If a cube number by multiplying any number makes a cube number, the multiplied number will also be cube.

Proposition 6: If a number by multiplying itself make a cube number, it will itself also be cube.

Proposition 7: If a composite number by multiplying any number makes some number, the product will be solid.

Proposition 8: If as many numbers as we please beginning from a unit be in continued proportion, the third from the unit will be square, as will also those which successively leave out one; and the fourth will be cube, as will also those which leave out two; and the seventh will be at once cube and square, as will also those which leave out five.

Proposition 9: If as many numbers as we please beginning from a unit be in continued proportion, and the number after the unit be square, all the rest will also be square. And, if the number after the unit be cube, all the rest will also be cube.

Proposition 10: If as many numbers as we please beginning from a unit be in continued proportion, and the number after the unit be not square, neither will any other be square except the third from the unit and all those which leave out one. And, if the number after the unit be not cube, neither will any other be cube except the fourth from the unit and all those which leave out two.

Proposition 11: If as many numbers as we please beginning from a unit be in continued proportion, the less measures the greater according to some one of the numbers which have place among the proportional numbers.

Corollary: And it is manifest that whatever place the measuring number has, reckoned from the unit, the same place also has the number according to which it measures, reckoned from the number measured, in the direction of the number before it.

Proposition 12: If as many numbers as we please beginning from a unit be in continued proportion, by however many prime numbers the last is measured, the next to the unit will also be measured by the same.
Proposition 13: If as many numbers as we please beginning from a unit be in continued proportion, and the number after the unit be prime, the greatest will not be measured by any except those which have a place among the proportional numbers.

Proposition 14: If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those the originally measuring it.

Proposition 15: If three numbers in continued proportion be the least of those which have the same ratio with them, any two whatever added together will be prime to the remaining number.

Proposition 16: If two numbers be prime to one another, the second will not be to any other number as the first is to the second.

Proposition 17: If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the last will not be to any other number as the first is to the second.

Proposition 18: Given two numbers, to investigate whether it is possible to find a third proportional to them.

Proposition 19: Given three numbers, to investigate when it is possible to find a fourth proportional to them.

Proposition 20: Prime numbers are more than any assigned multitude of prime numbers.

Proposition 21: If as many even numbers as we please be added together, the whole is even.

Proposition 22: If as many odd numbers as we please be added together, and their multitude be even, the whole will be even.

Proposition 23: If as many odd numbers as we please be added together, and their multitude be odd, the whole will also be odd.

Proposition 24: If from an even number an even number be subtracted, the remainder will be even.

Proposition 25: If from an even number an odd number be subtracted, the remainder will be odd.

Proposition 26: If from an odd number an odd number be subtracted, the remainder will be even.

Proposition 27: If from an odd number an even number be subtracted, the remainder will be odd.

Proposition 28: If an odd number by multiplying an even number make some number, the product will be even.

Proposition 29: If an odd number by multiplying an odd number make some number, the product will be odd.
Proposition 30: If an odd number measures an even number, it will also measure the half of it.

Proposition 31: If an odd number be prime to any number, it will also be prime to the double of it.

Proposition 32: Each of the numbers which are continually doubled beginning from a dyad is an even-times even only.

Proposition 33: If a number has its half odd, it is an even-times odd only.

Proposition 34: If a number neither be one of those which are continually doubled from a dyad, nor have its half odd, it is both an even-times even and even-times odd.

Proposition 35: If as many numbers as we please be in continued proportion, and there be subtracted from the second and the last numbers equal to the first, then, as the excess of the second is to the first, so will the excess of the last be to all those before it.

Proposition 36: If as many as we pleas beginning from a unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect.