EQUIDISTRIBUTION OF DIGITS IN POWERS AND
DIOPHANTINE APPROXIMATIONS

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Abstract. Given integers $a, b > 1$ with $\log_b a$ irrational, we investigate
the connection between the conjectured asymptotic equidistribution of
digits in the base-$b$ expansion of $a^n$ and the (non-)Diophantine proper-
ties of the number $\theta_{a,b} = \log_b a$.

1. Introduction

Given two integers $a$ and $b$, both greater than 1, consider the expansions
of the powers $a^n$ in base $b$, say,

$$a^n = [d_{1,n}d_{2,n}\cdots d_{k_n,n}]_b = \sum_{j=1}^{k_n} d_{j,n}b^{k_n-j},$$

where $d_{j,n} \in \{0, 1, \ldots, b - 1\}$ for each $j$, and where $k_n = \lfloor n \log_b a \rfloor$ is the
number of digits in the expansion. What is the statistical behaviour of such
digits as $n \to \infty$? In [FT], we formulated a general conjecture implying that
such digits are asymptotically equidistributed. We reproduce the statement
here.

Conjecture 1. If $\log_b a$ is irrational, then for each digit $d = 0, 1, \ldots, b - 1$,
we have

$$\lim_{n \to \infty} \frac{1}{k_n} \# \{1 \leq j \leq k_n : d_{j,n} = d\} = \frac{1}{b}. \quad (1)$$

There is strong numerical evidence in favour of this conjecture, as well as
a fairly robust heuristic argument lending support to it, see [FT §5].

Our goal in this short note is to show that Conjecture 1, if true, has an
interesting consequence of a Diophantine nature. Namely, if we know that
the statement of the conjecture holds true for a given pair $(a, b)$ as above,
then we can say something non-trivial about how well (or how badly) the
irrational number $\theta_{a,b} = \log_b a$ is approximated by rationals. It is known

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that for every \( \epsilon > 0 \) there exists a positive constant \( C > 0 \) (depending on \( \theta_{a,b} \) and \( \epsilon \)) such that for all \( p/q \in \mathbb{Q} \) in irreducible form we have

\[
\left| \theta_{a,b} - \frac{p}{q} \right| \geq \frac{C}{qe^{q\epsilon}}.
\]

This can be derived as an application of the non-trivial theory of linear forms in logarithms developed primarily by A. Baker (see [Ba]).

The result we can prove assuming Conjecture 1 is not quite as strong. Still, it is sufficiently non-trivial as to lend further credibility to the conjecture, and it has the merit that its proof is more elementary by comparison. First of all we note that, since we are interested only in the approximation of the irrational number \( \theta_{a,b} \) by rationals, it suffices to consider the cases when \( 1 < a < b \). If \( a > b \), we consider instead \( \log_a b = (\log_b a)^{-1} \), and use the fact that the Diophantine properties of a non-zero number \( \theta \) are the same as those of \( 1/\theta \). Hence, from now on we assume that \( 1 < a < b \).

**Theorem 1.** If Conjecture 1 is true for the pair \((a, b)\), then for each \( \epsilon > \frac{\log a}{b} \) there exists a constant \( C > 0 \) such that

\[
\left| \theta_{a,b} - \frac{p}{q} \right| \geq \frac{C}{qe^{q\epsilon}}.
\]

for all rationals \( p/q \) in irreducible form. Moreover, the constant \( C \) can be replaced by 1 in the above inequality whenever \( q \) is sufficiently large.

We also prove below (see Theorem 2) a weaker version of the above theorem without resorting to the conjecture.

2. A PIECEWISE AFFINE CIRCLE MAP

The key to establishing a connection between the Diophantine approximation problem discussed above and Conjecture 1 is the piecewise affine map \( T_{a,b} : [b^{-1}, 1] \to [b^{-1}, 1] \) introduced in [FT], given by

\[
T_{a,b}(x) = \begin{cases} 
ax, & \text{if } \frac{1}{b} \leq x < \frac{1}{a}, \\
\frac{ax}{b}, & \text{if } \frac{1}{a} \leq x \leq 1.
\end{cases}
\]

Upon identification of \( x_0 = b^{-1} \) with 1 via the translation \( x \mapsto x + 1 - x_0 \), the interval \([b^{-1}, 1]\) becomes an affine copy of the circle \( S^1 = \mathbb{R}/\mathbb{Z} \), and \( T_{a,b} \) becomes a piecewise affine circle homeomorphism. The rotation number of \( T_{a,b} \) is \( \theta_{a,b} = \log_b a \), and the forward orbit of \( x_0 \) under \( T_{a,b} \) is easily seen to

\[
x_n = T_{a,b}^n(x_0) = \frac{a^n}{b^{k_n}} = \sum_{i=1}^{k_n} \frac{d_{i,n}}{b^i} = [0.d_{1,n}d_{2,n} \cdots d_{k_n,n}]_b.
\]

\[\text{[Footnote]}\]

There are also fast algorithms for computing the continued fraction development of \( \theta_{a,b} \), the first of which was developed by D. Shanks; see [JM] and references therein.
We proved in [FT] that $T_{a,b}$ is conjugate to the rotation by $\theta_{a,b}$ via a bi-Lipschitz homeomorphism (the Lipschitz constant depends on $b$, but not on $a$). In particular, the orbit $(x_n)$ is dense in the interval $[b^{-1},1]$. Every time $x_n$ comes very close to $x_0 = [0.100\ldots]_b$ (through a sequence of so-called closest returns) this implies that $d_{1,n} = 1$ and $d_{2,n} = d_{3,n} = \cdots = d_{\ell_{\alpha},n} = 0$ for some $1 < \ell_{\alpha} < k_n$. If $\theta_{a,b}$ is very fastly approximated by rationals, then it turns out that $\ell_{\alpha}$ becomes a significant fraction of $k_n$, larger than $1/b$, and this spoils the conjecture. This is the rough idea, and it will be made precise below.

We recall some elementary facts from the theory of continued fractions. If $\theta \in \mathbb{R}$ is an irrational number, then there exists a sequence of rationals $p_n/q_n$ in irreducible form, called the convergents to $\theta$, which alternate around $\theta$ and satisfy

$$
\frac{1}{q_n(q_n + q_{n+1})} < \left| \frac{\theta - p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.
$$

This sequence of convergents is obtained from the continued fraction development of $\theta$ by truncation; see [La, ch.1] for what is needed here (and much more). It is well-known that the sequence of denominators $q_n$ of the convergents is precisely the sequence of closest return times in the orbit of any point in the circle under the rotation $x \mapsto x + \theta( \mod 1)$ (see for instance [He]).

Whether Conjecture [1] is true or not, the mere fact that the map $T_{a,b}$ has a dense orbit of rationals whose denominators are powers of $b$ already implies a Diophantine approximation result. More precisely, we have the following weak version of (2).

**Theorem 2.** There exists a constant $C = C(b) > 0$ such that

$$
\left| \theta_{a,b} - \frac{p}{q} \right| \geq \frac{C}{q e(\log a) q}.
$$

for all rationals $p/q$ in irreducible form.

**Proof.** It suffices to prove (5) for the sequence of convergents $p_n/q_n$. Since $T_{a,b}$ is conjugate to the rotation $R : t \mapsto t + \theta_{a,b} \mod 1$ by a Lipschitz homeomorphism, we have

$$
|q_n \theta_{a,b} - p_n| = |R^{q_n}(t) - t| \geq K \left| T_{a,b}^{q_n}(x_0) - x_0 \right| = K |x_{q_n} - x_0|.
$$

Here, $t$ denotes any point on the unit circle $\mathbb{R}/\mathbb{Z}$ and $K$ is the Lipschitz constant of the conjugating homeomorphism. As we proved in [FT], $K$ depends only on $b$. But $x_{q_n}$, when written in base $b$, has $k_n$ digits after the decimal period, and the rightmost digit in this expansion is non-zero. Therefore $|x_{q_n} - x_0| \geq b^{-k_n}$. Since $k_n = q_n \log_b a < 1 + q_n \log_b a$, it follows that

$$
|x_{q_n} - x_0| > \frac{1}{b^{1 + q_n \log_b a}} = \frac{b^{-1}}{e(\log a) q_n}.
$$
Putting (8) back into (7) and dividing the resulting inequality by $q_n$, we deduce that
\[ \left| \theta_{a,b} - \frac{p_n}{q_n} \right| \geq \frac{Kb^{-1}}{q_n e^{(\log a)q_n}}, \]
and this is exactly what we wanted. \(\square\)

3. A special class of numbers

Now suppose we are given a function $\sigma : \mathbb{N} \to \mathbb{R}^+$, which we call the gauge, with the property that $\sigma(q) \to 0$ as $q \to \infty$. We define a special set of (rotation) numbers $A_\sigma$ for any given gauge $\sigma$ as follows:
\[ A_\sigma = \left\{ \theta \in [0,1] \setminus \mathbb{Q} : \limsup_q \frac{\log q_{n+1}}{q} > 1 \right\}. \]
Note that if $\theta \in A_\sigma$, then there are infinitely many rational solutions $p/q$ (in irreducible form) to the inequality
\[ \left| \theta - \frac{p}{q} \right| < \frac{1}{qe^{1-\sigma(q)}}. \]
Conversely, if $\theta \in [0,1] \setminus \mathbb{Q}$ is such that the above inequality has infinitely many solutions, then $\theta \in A_\sigma$. This follows from the first inequality in (5) combined with a classic theorem due to Legendre, which says that if $\theta$ is irrational and $p,q$ are relatively prime integers such that $|\theta - p/q| < 1/2q^2$, then $p/q$ must be a convergent to $\theta$.

The set $A_\sigma$ is very small from the point of view of measure, but rather large from the topological viewpoint. This will hardly seem surprising to those acquainted with the theory of Diophantine approximations. The proof below is inspired by [Ox, ch. 2].

**Lemma 1.** The set $A_\sigma$ has Hausdorff dimension equal to zero, but it is a residual set in $[0,1]$ in the sense of Baire.

**Proof.** For each $q \geq 1$ and each $0 \leq p \leq q$, consider the open interval
\[ \Delta_{p,q} = \left( \frac{p}{q} - \frac{1}{qe^{1-\sigma(q)}}, \frac{p}{q} + \frac{1}{qe^{1-\sigma(q)}} \right). \]
From the characterization of $A_\sigma$ given prior to the statement of the lemma, we see that
\[ A_\sigma = \mathbb{Q}^c \cap \bigcap_{n=1}^{\infty} A_n, \quad \text{where} \quad A_n = \bigcup_{q=n}^{\infty} \bigcup_{p=0}^{q} \Delta_{p,q}. \]
Each $A_n$ is clearly open. Since $A_n$ contains all irreducible fractions $p/q$ in $[0,1]$ with $q \geq n$, it is also dense in $[0,1]$. Hence $A_\sigma$, being the intersection of a residual set with a countable intersection of open dense sets, is itself residual in $[0,1]$.

Let us now estimate the Hausdorff $s$-measure $H_s(A_n)$ for any given $s > 0$. We do this only for sufficiently large $n$. More precisely, let $q_0$ be such that...
\( \sigma(q) < \frac{1}{2} \) for all \( q \geq q_0 \), and let \( n > q_0 \). Since \( A_n \) is already a countable union of intervals, using such intervals as covering we deduce that

\[
H_s(A_n) \leq \sum_{q=n}^{\infty} \sum_{p=0}^{q} |\Delta_{p,q}|^s = 2^s \sum_{q=n}^{\infty} \frac{q + 1}{q^s e^{q^{1-\sigma(q)}}} < 2^{s+1} \sum_{q=n}^{\infty} q e^{-s\sqrt{q}} < C_s n^{\frac{3}{2} - s} e^{-s\sqrt{n}},
\]

for some \( C_s > 0 \). Hence \( H_s(A_n) \to 0 \) as \( n \to \infty \), and therefore \( H_s(A_{\sigma}) = \inf H_s(A_n) = 0 \), for all \( s > 0 \). This shows that the Hausdorff dimension of \( A_{\sigma} \) is zero, as asserted. \( \square \)

**Remark 1.** In fact, it is not difficult to see that each element of \( A_{\sigma} \) is a Liouville number. Thus, the fact that \( A_{\sigma} \) has Hausdorff dimension zero is a consequence of the well-known fact that the set of all Liouville numbers has zero Hausdorff dimension. See the classic [Ox; ch. 2].

**Theorem 3.** Let \( 1 < a < b \) as before, and suppose we are given a gauge \( \sigma \) satisfying the further condition

\[
\liminf_{q \to \infty} q^{-\sigma(q)} > \frac{\log a}{b}.
\]

If \( \theta_{a,b} = \log_b a \in A_{\sigma} \), then Conjecture 7 fails for the pair \((a, b)\).

**Proof.** For ease of notation, we write \( \theta = \theta_{a,b} \) in this proof. Since \( \theta \in A_{\sigma} \), there exists a sequence \( n_i \to \infty \) such that

\[
|q_{n_i} \theta - p_{n_i}| < \frac{1}{q_{n_i+1}} < \frac{c_0}{e^{q_{n_i}^{1-\sigma(q_{n_i})}}},
\]

for some constant \( c_0 > 0 \). Here we have used (4). Note that either \( \{q_{n_i} \theta\} = |q_{n_i} \theta - p_{n_i}| \) or \( \{q_{n_i} \theta\} = 1 - |q_{n_i} \theta - p_{n_i}| \), depending on whether \( q_{n_i} \theta - p_{n_i} \) is positive or negative, respectively. One of these two cases must happen infinitely often along the sequence \( (n_i) \); we assume the former (if the latter, a similar argument applies, *mutatis mutandis*).

We now look at the orbit \((x_n)\) of \( x_0 = 1/b \). Note that, since \( k_{n_i} \) is \( \left[ n \log_b a \right] = n \theta - \{n \theta\} + 1 \) for each \( n \), we have \( x_n = b^{(n \theta) - 1} \), and therefore

\[
|x_n - \frac{1}{b}| = \frac{1}{b} |b^{(n \theta)} - 1| < \frac{c_1}{b} \{n \theta\},
\]

for some constant \( c_1 > 0 \) (we have used that \( x \mapsto b^x \) is Lipschitz in the interval \([0, 1]\)). Therefore, replacing \( n \) by \( q_{n_i} \) in (12) and using (11), we get

\[
|x_{q_{n_i}} - \frac{1}{b}| < \frac{c_2}{b e^{q_{n_i}^{1-\sigma(q_{n_i})}}},
\]
where $c_2 = c_0 c_1 > 0$. Writing $\gamma = 1/\log b$, letting $\beta$ be such that $b^\beta = c_2^{-1}$, and defining $\ell(q_{n_i}) = 1 + \beta + \gamma q_{n_i}^{1-\sigma(q_{n_i})}$, we have proved that

$$|x_{q_{n_i}} - \frac{1}{b}| < \frac{1}{b^\ell(q_{n_i})},$$

for a sequence $n_i \to \infty$. But this means that the first $\lfloor \ell(q_{n_i}) \rfloor$ digits in the base-$b$ expansion of $x_{q_{n_i}}$ agree with those of $x_0 = 1/b = [0.100\ldots]_b$. In particular, the number of zeros in the base-$b$ expansion of the power $a^{p_{n_i}}$ is at least $\lfloor \ell(q_{n_i}) \rfloor - 1$. Hence we have

$$\limsup_{n \to \infty} \frac{1}{k_n} \# \{1 \leq j \leq k_n : d_{j,n} = 0\} \geq \limsup_{i \to \infty} \frac{\ell(q_{n_i}) - 1}{k_{q_{n_i}}}. \quad (13)$$

But now, using (10), we see that

$$\limsup_{i \to \infty} \frac{\ell(q_{n_i}) - 1}{k_{q_{n_i}}} = \limsup_{i \to \infty} \frac{\beta + \gamma q_{n_i}^{1-\sigma(q_{n_i})}}{q_{n_i}^\beta} \geq \frac{\gamma \log a}{\theta b} = \frac{1}{b}. \quad (14)$$

Combining (13) with (14) we arrive at

$$\limsup_{n \to \infty} \frac{1}{k_n} \# \{1 \leq j \leq k_n : d_{j,n} = 0\} > \frac{1}{b}.$$

This contradicts (1), and so the theorem is proved. \qed

4. PROOF OF THEOREM 1

At last, we are in a position to prove the main result of this paper. Let $\sigma(q) = \log(\epsilon^{-1})/\log q$ and note that this gauge satisfies (10). Hence Theorem 3 implies that $\theta_{a,b} \notin A_\sigma$. This tells us that the inequality (9) with $\theta = \theta_{a,b}$ has only finitely many rational solutions, This proves both inequality (3) and the last assertion in the statement of Theorem 1. \qed

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