Static potential and a new generalized connection in three dimensions

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Abstract

For a recently proposed pure gauge theory in three dimensions, without a Chern-Simons term, we calculate the static interaction potential within the structure of the gauge-invariant variables formalism. The result coincides with that of the Maxwell-Chern-Simons theory in the short distance regime, which shows the confining nature of the potential.

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I. INTRODUCTION

Systems in (2+1) dimensions have been extensively discussed in the last few years [1–3]. As is well known, the interest in studying three-dimensional theories is mainly due to the possibility of realizing fractional statistics, where the physical excitations obeying it are called anyons. The three-dimensional Chern-Simons gauge theory is the key example, so that Wilczek’s charge-flux composite model of the anyon can be implemented [4]. In this context it may be recalled that when the Chern-Simons term is added to the usual Maxwell term, the gauge field becomes massive, leading to topologically massive gauge theories.

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Interestingly, the statistics of anyons changes with their mutual distance in the presence of the Maxwell term [5]. We further note that recently a novel way to describe anyons has been considered [6]. The crucial ingredient of this development is to introduce a generalized connection in (2+1) dimensions which permits to realize fractional statistics, such that the Chern-Simons term needs not be introduced. More precisely, it was shown that a Lagrangian which describes Maxwell theory coupled to the current via the generalized connection leads to fractional statistics by the same mechanism, as in the case of the Maxwell-Chern-Simons theory, of attaching a magnetic flux to the electrons.

On the other hand, we also recall that the ideas of screening and confinement play a central role in gauge theory. In this connection the interaction potential between static charges is an object of considerable interest, and its physical content can be understood when a correct separation of the physical degrees of freedom is made.

With this in view, we will apply in this work a formalism in terms of physical (gauge-invariant) quantities. This method was used previously for studying features of screening and confinement in two-dimensional quantum electrodynamics, generalized Maxwell-Chern-Simons gauge theory and for the Yang-Mills field [7]. This method is applied below to study the interaction energy in the recently proposed three-dimensional electrodynamics [6]. The procedure exploits the rich structure of the dressing around static fermions, where we refer to the cloud made out of the vector potentials around the fermions as dressing. The methodology presented here provides a physically-based alternative to the usual Wilson loop approach.

II. INTERACTION ENERGY

As already stated, our objective is to compute explicitly the interaction energy between static pointlike sources for this new electrodynamics. For this purpose we shall first carry out its Hamiltonian analysis. The gauge theory we are considering is defined by the following Lagrangian in three-dimensional space-time:
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 - A_\mu^\theta J^\mu = -\frac{1}{4} F_{\mu\nu}^2 + \frac{\theta}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\rho} J^\nu - A_\mu J^\mu.
\] (1)

Here \( A_\mu^\theta = A_\mu - \frac{\theta}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\nu\rho} \) is the generalized connection, \( J_\mu \) is the external current and \( \theta \) is a parameter with dimension \( M^{-1} \). The canonical momenta are \( \Pi^\mu = F^{\mu\nu} + \theta \varepsilon^{\mu\nu\rho} J_\rho \), which results in the usual primary constraint \( \Pi^0 = 0 \) and \( \Pi^i = F^{i\rho} + \theta \varepsilon^{ij} J_j \) \( (i, j = 1, 2) \). The canonical Hamiltonian is given by

\[
H_C = \int d^2 x \left\{ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} F_{ij} F^{ij} - A_0 \left( \partial_i \Pi^i - J^0 \right) - \frac{\theta}{2} \varepsilon_{ij} J^0 F^{ij} + A_i J^i \right\},
\] (2)

and it is straightforward to see that the preservation in time of the primary constraint leads to the secondary constraint

\[
\Omega_1 (x) \equiv \partial_i \Pi^i (x) - J^0 (x) = 0.
\] (3)

The above constraints are the first-class constraints of the theory since no more constraints are generated by the time preservation of the secondary constraint (3). The corresponding total (first class) Hamiltonian that generates the time evolution of the dynamical variables then reads

\[
H = H_C + \int d^2 x \left\{ c_0 (x) \Pi_0 (x) + c_1 (x) \Omega_1 (x) \right\},
\] (4)

where \( c_0 (x) \) and \( c_1 (x) \) are the Lagrange multiplier fields to implement the constraints. Since \( \Pi^0 = 0 \) for all time and \( \dot{A}_0 (x) = [A_0 (x), H] = c_0 (x) \), which is completely arbitrary, we eliminate \( A^0 \) and \( \Pi^0 \) because they add nothing to the description of the system. Thus the Hamiltonian takes the form

\[
H = \int d^2 x \left\{ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} F_{ij} F^{ij} + c'(x) \left( \partial_i \Pi^i - J^0 \right) - \frac{\theta}{2} \varepsilon_{ij} J^0 F^{ij} + A_i J^i \right\}
\] (5)

where \( c'(x) = c_1 (x) - A_0 (x) \).

In accordance with the Dirac method, we impose one gauge constraint such that the full set of constraints becomes second class. A convenient choice is found to be

\[
\Omega_2 (x) \equiv \int_{C_{\lambda x}} dz^\nu A_\nu (z) = \int_0^1 d\lambda x^i A_i (\lambda x) = 0
\] (6)
where $\lambda$ ($0 \leq \lambda \leq 1$) is the parameter describing the spacelike straight path between the reference point $\xi^k$ and $x^k$, on a fixed time slice. For simplicity we have assumed the reference point $\xi^k = 0$. The choice (6) leads to the Poincaré gauge [7]. Through this procedure, we arrive at the following set of Dirac brackets for the canonical variables

$$\{A_i(x), A^j(y)\}^* = 0, \quad (7)$$

$$\{\pi_i(x), \pi^j(y)\}^* = 0, \quad (8)$$

$$\left\{A_i(x), \pi^j(y)\right\}^* = g^j_i \delta^{(2)}(x - y) - \partial^x_i \int_0^1 d\lambda x^j \delta^{(2)}(\lambda x - y). \quad (9)$$

In order to illustrate the discussion, we now write the Dirac brackets in terms of the magnetic ($B = \varepsilon_{ij} \partial^i A^j$) and electric ($E^i = \Pi^i - \theta \varepsilon^{ij} J_j$) fields as

$$\{E_i(x), E_j(y)\}^* = 0, \quad (10)$$

$$\{B(x), B(y)\}^* = 0, \quad (11)$$

$$\{E_i(x), B(y)\}^* = -\varepsilon_{ij} \partial^j \delta^{(2)}(x - y). \quad (12)$$

It is important to realize that, unlike the Maxwell-Chern-Simons theory, in the present model, the brackets (7) and (10) are commutative. One can now easily derive the equations of motion for the electric and magnetic fields. We find

$$\dot{E}_i(x) = -\varepsilon_{ij} \partial^j B(x) + \theta \varepsilon_{ij} \partial^j J^0(x) + J_i(x) + \int d^2 y J^k(y) \partial^k \int_0^1 d\lambda y_i \delta^{(2)}(\lambda x - y), \quad (13)$$

$$\dot{B}(x) = -\varepsilon_{ij} \partial_i E_j(x). \quad (14)$$

In the same way, we write the Gauss law as:

$$\partial_i E^i_L + \theta \varepsilon^{ij} \partial_i J_j - J^0 = 0, \quad (15)$$
where $E^i_L$ refers to the longitudinal part of $E^i$. This implies that for a static charge located at $x^i = 0$, and $J^i = 0$, the static electromagnetic fields are given by

\begin{equation}
B (x) = \theta J^0,
\end{equation}

\begin{equation}
E_i (x) = - \frac{\partial_i J^0}{\nabla^2},
\end{equation}

where $\nabla^2$ is the two-dimensional Laplacian. For $J^0(t, x) = \epsilon \delta^2(x)$, expressions (16) and (17) become

\begin{equation}
B (x) = \epsilon \theta \delta^{(2)}(x),
\end{equation}

\begin{equation}
E_i (x) = - \frac{\epsilon}{2\pi} \frac{x^i}{r^2},
\end{equation}

with $r \equiv |x|$. One immediately sees that the associated magnetic field has its support only at the position of the charge, and a long range electric field is also generated. As a consequence, the total magnetic flux associated to the magnetic field is

\begin{equation}
\Phi_B = \int_V d^2 x B (x) = e \theta.
\end{equation}

This tells us that the charged particle actually behaves like a magnetic flux point. Accordingly, the mechanism of attaching a magnetic flux to the charges has been implemented in a particularly simple way, as was claimed in Ref. [6].

After achieving the quantization we may now proceed to calculate the interaction energy between pointlike sources in the model under consideration. To do this, we will compute the expectation value of the energy operator $H$ in a physical state $|\Omega\rangle$. We also recall that the physical states $|\Omega\rangle$ are gauge-invariant [10]. In that case we consider the stringy gauge-invariant $|\Psi(y) \Psi(y')\rangle$ state,

\begin{equation}
|\Omega\rangle \equiv |\Psi(y) \Psi(y')\rangle = \overline{\psi(y)} \exp \left(-i\epsilon \int_y^{y'} dz^i A_i (z)\right) \psi(y') |0\rangle,
\end{equation}

5
where $|0\rangle$ is the physical vacuum state and the integral is to be over the linear spacelike path starting at $y$ and ending at $y^\prime$, on a fixed time slice. Note that the strings between fermions have been introduced to have a gauge-invariant state $|\Omega\rangle$, in other terms, this means that the fermions are now dressed by a cloud of gauge fields. We can write the expectation value of $H$ as

$$
\langle H \rangle_\Omega = \langle \Omega | \int d^2x \left( \frac{1}{2} E_i^2 + \frac{1}{2} B^2 - \theta B J^0 + J^i A_i \right) | \Omega \rangle.
$$

(22)

The preceding Hamiltonian structure thus leads to the following result

$$
\langle H \rangle_\Omega = \langle H \rangle_0 + \frac{e^2}{2} \int d^2x \left( \int dy \delta(y - z) \right) \left( x - z \right)^2,
$$

(23)

where $\langle H \rangle_0 = \langle 0 | H | 0 \rangle$. Following our earlier procedure [8], we see that the second term on the right-hand side of Eq. (23) is clearly dependent on the distance, and the potential for two opposite located at $y$ and $y^\prime$ takes the form

$$
V = \frac{e^2}{\pi} \ln |y - y^\prime|.
$$

(24)

Now we recall that the same calculation for the Maxwell-Chern-Simons theory [8,9] gives

$$
V = -\frac{e^2}{\pi} K_0 (\mu |y - y^\prime|),
$$

(25)

where $K_0$ is a modified Bessel function. We immediately see that the result (24) agrees with the behavior of the Maxwell-Chern-Simons theory in the limit of short separation. Eq.(24) displays the confining nature of the potential (the potential grows to infinity when the mutual separation grows), but the Maxwell-Chern-Simons theory result (25) does not. In summary, the above analysis reveals that, although both theories lead to fractional statistics by the same mechanism of attaching a magnetic flux to the charges, the physical content is quite different. However, the observation in the present work that the new electrodynamics is confining is new.

It is worth noting here that there is an alternative but equivalent way of obtaining the result (24). To do this we consider
\[ V \equiv e \left( A_0 (y) - A_0 (y') \right), \]  

(26)

where the physical scalar potential is given by

\[ A_0 (t, x) = \int_0^1 d\lambda x^i E_i (t, \lambda x). \]  

(27)

Two remarks are pertinent at this point. First, Eq.(27) follows from the vector gauge-invariant field [7]

\[ A_\mu (x) \equiv A_\mu (x) + \partial_\mu \left( - \int_\xi^x dz A_\mu (z) \right), \]  

(28)

where, as in Eq.(6), the line integral appearing in the above expression is along a spacelike path from the point \( \xi \) to \( x \), on a fixed time slice. Second, it should be noted that the gauge-invariant variables (28) commute with the sole first class constraint (Gauss’ law), corroborating the fact that these fields are physical variables [10].

Having made these observations and from Eq. (17), we can write immediately the following expression for the physical scalar potential

\[ A_0 (t, x) = \int_0^1 d\lambda x^i E_i (t, \lambda x) = \int_0^1 d\lambda x^i \partial_i \left( \frac{-J^0 (\lambda x)}{\nabla^2 x^i} \right), \]  

(29)

where \( J^0 \) is the external current. The static current describing two opposite charges \( e \) and \(-e\) located at \( y \) and \( y' \) is then given by \( J^0 (t, x) = e \left\{ \delta (t^2) (x - y) - \delta (t^2) (x - y') \right\}. \) Substituting this back into Eq. (29) we obtain

\[ V = \frac{e^2}{\pi} \ln |y - y'|. \]  

(30)

It is clear from this discussion that a correct identification of physical degrees of freedom is a key feature for understanding the physics hidden in gauge theories. According to this viewpoint, once that identification is made, the computation of the potential is achieved by means of Gauss law [11]. As a final comment, we point out that the methodology advocated in this paper provides yet another support to the dressed fields picture to compute the static potential in gauge theories.
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