Quantum breaks in a model for the evolution of neutrinos during their decoupling era in the big bang

R. F. Sawyer

1Department of Physics, University of California at Santa Barbara, Santa Barbara, California 93106

An active two-neutrino state’s coupling to an intermediate scalar meson, and thence to a two-antineutrino state, is one idea for catalyzing transitions from active to sterile neutrinos, in an implementation of the Dodelson-Widrow proposal for the production of sterile and somewhat massive neutrinos as dark matter candidates. We propose some mechanics that promises to use the very same model with the same mass and coupling parameters, in order to achieve similar results, but orders of magnitude faster; or alternatively, similar results but beginning with vastly reduced coupling constants. The model then would become much less constrained by consistency with laboratory and astrophysical data.

1. INTRODUCTION

Recently the reaction $\bar{\nu}_e + \nu_e \leftrightarrow \nu_x + \nu_x$, as mediated by an intermediate scalar meson, in conjunction with a standard bilinear coupling with a sterile neutrino $\nu_x$, has been used [1] in a promising implementation of the Dodelson-Widrow proposal [2] for the production of dark matter as composed of sterile neutrinos. Here $\nu_x$ stands for another active neutrino species, $\nu_\mu$ or $\nu_\tau$. The time frame is centered in the era of conventional neutrino decoupling.

Another recent work [3] has addressed the changes in standard early universe results in the same scalar coupling model, but leaving out the sterile neutrino altogether. These authors find some relatively minor changes but ones that might be checked in the current era of ever more precise data. The present paper bears on both of the above-cited works; more definitively on the second, more precise data. The present paper bears on both of the above-cited works; more definitively on the second, more precisely on the second, but more interestingly on the first.

In both exercises we have $\bar{\nu}_e + \nu_e \leftrightarrow \nu_x + \nu_x$, in which all participating neutrinos are left-handed, coupled through an intermediate scalar meson that has no other couplings except to active neutrinos. Here $\nu_x$ stands for some mixture of $\nu_\mu$ and $\nu_\tau$. In both of the works cited above, this reaction enters only through its cross-section in vacuum. This is of the order $G^2$ in some generic 4-Fermi coupling constant $G$, leading to a characteristic time $T_e$ for a macroscopic effect of $T_e \sim [G^2 E^2_\nu n_\nu]^{-1}$. Our replacement, based on coherent many-body amplitudes, is capable, under some circumstances, of achieving total flavor mixing in time $T_e \sim [G n_\nu]^{-1} \log \Lambda$, where $\log \Lambda$ could be as big as 30. The word “rate” is inapplicable here since the behavior consists of a long wait with only a tiny bit happening, followed by a rapid transition.

We shall consider only time intervals in which ordinary scattering in the medium is negligible and for which $T << m_\nu^{-1} E_\nu$, in order to preserve the coherence that drives the effect. And we consider only amplitudes in which, in any two-$\nu$ sub-amplitude, only reactions like $\nu(p) + \nu(q) \leftrightarrow \bar{\nu}(p) + \bar{\nu}(q)$ enter, with momentum “preservation”, not just momentum conservation. With genuinely massless particles (such as the gravitons and photons considered in ref. [3]) we then would have energy and momentum exactly conserved in every reaction.

Fig. 1 illustrates a miscellaneous 8-$\nu$ process, showing some conversion vertices (black-circles) amidst a swarm of eight neutrinos that are incoming from different directions. It is amazing at first glance that there is a coherent part to this graph that would survive averaging over angles when we are simulating an initial isotropic distribution and then summing over energy distributions. The reader can imagine how the combinatorial factors build up in a complete perturbative expansion with the numbers of particles that participate coherently increasing, order by order, as a power of total particle number $N$. In what follows we develop the applicable equations for the non-perturbative approach and solve them numerically. We do all calculations in a periodic box somewhat larger than $cT$.

Secs. 2-5 are devoted to the system without the coupling to steriles. Sec. 6 incorporates the mixing with them, but is less definitive, because of computational stresses connected to getting to the extremely small mixing angles that the literature demands and because of sensitive parameter dependence.
2. ANALYTIC TREATMENT OF MODELS.

By now several types of systems with features in common with the ones we shall discuss have been analyzed: a) Bose condensates of atoms in wells [5]-[7]; b) polarization exchanges in collisions of high intensity photon beams [8]; c) “Fast” neutrino flavor transformation at the supernova neutrino-sphere [9]-[23]; d) A cloud of very long wavelength gravitons at very high number density (in vacuum otherwise) transforming to photon pairs in a time proportional to \(G^{-1}\), not as \(G^{-2}\) [24].

Based on this experience, we state requirements for our work:

A. Mean field

We require there to be a sensible mean-field theory of the phenomena. The standard formulation as given in detail by Raffelt and Sigl [4] is the background for everything we do. But it will be modified in what is to follow, in which the vector four-Fermi coupling is replaced by a scalar coupling. Then in \(\nu + \nu \leftrightarrow \bar{\nu} + \bar{\nu}\) the reactions proceed with all of the \(\nu, \bar{\nu}\)’s staying within the left-handed sector. The basic variables are \(\bar{\bar{\nu}}_j^c\nu_j\) and \(\bar{\nu}_k\nu^{*}_j\), quadratics in the neutrino fields, with \(j, k\) as flavor indices. The notation \(\nu^c_j\) stands for the charge-conjugate field.

B. Instability

Within the standard mean field context, if we begin with a pure unmixed flavor state, it is clear that nothing whatever would happen. But once the mass (i.e. \(\nu\) oscillation) terms begin to mix states, then a lot may happen in a very short time, at least in the much-studied vector-coupled case. Analytically we can distinguish such cases by linearizing the non-linear equations of evolution, through taking the diagonal matrices in flavor space as the annihilators for the same two momentum states of the neutrinos, over the time scale that we need to consider. Then, with fingers crossed, and some consistency checks, we apply it to systems with \(10^{30}\) \(\nu\)’s.

C. Quantum break

Taking the neutrino masses as the seed for the instability in all of the “fast transformation” literature for the standard vector coupled case obscured a fascinating fact, namely that the same fast processes for the unstable cases would have turned themselves on anyway, even in the absence of neutrino oscillations, and on nearly the same time scale, through a “quantum break”. That term was used, for example, in a classic article on Bose condensates of atoms in wells [5]. The question then addressed was: when, in the context of mean-field theory, we find a system that is in unstable equilibrium, how do we formulate the theory of the fluctuation that gives it a push to get it moving? Our test system of quantum fields spread out over a macroscopic region that contains \(10^{30}\) particles is far different from 100 atoms in a well, but the methods that can be applied are related.

In our new application of similar ideas to the scalar interaction \(\nu + \nu \leftrightarrow \bar{\nu} + \bar{\nu}\) the quantum break becomes an indispensable necessity, because we are not introducing a lepton-number breaking interaction that mixes \(\nu\) and \(\bar{\nu}\) on a single-particle basis, which otherwise could have provided a seed for the instability. We shall develop a modified form of mean-field theory that provides a solution that withstands testing in soluble systems of \(N < 1000\) neutrinos, over the time scale that we need to consider. Then, with fingers crossed, and some consistency checks, we apply it to systems with \(10^{30}\) \(\nu\)’s.

3. SOLUTIONS

We are working toward describing the development of systems containing \(e^\pm, \nu, \bar{\nu}\), and photons, and which embody perfect particle-antiparticle symmetry and isotropy. But we consider only time intervals that are short on the scale of free paths for scattering, and in which the neutrino-neutrino interactions, if unstable, can dominate the dynamics. And in calculating coherent effects we must trace what each quantum state of the multi-particle system does, before adding up to see how the statistical ensemble (which is where the isotropy, e.g., is encoded) evolves.

We begin by considering a beam of \(\nu_e\)’s in a narrow angular cluster impinging on another beam of \(\nu_e\)’s to make reactions, \(2\nu_e \leftrightarrow 2\bar{\nu}_e\), where \(\bar{\nu}_e\) is another active flavor. Let \(a^j\) be the annihilator for \(\nu_e\) with momentum \(p_j\) in one incident \(\nu_e\) beam and \(b^k\) the annihilator for a state \(q_k\) in the opposed \(\nu_e\) beam; with \(c^j, d^k\) being the annihilators for the same two momentum states of the \(\bar{\nu}_e\) system. Now we wish to take the piece of the four-Fermi interaction that is the local limit for the processes \(\nu_e + \nu_e \leftrightarrow \bar{\nu}_e + \bar{\nu}_e\) when the intermediate scalar meson mass is large compared to the neutrino energies (eq. 1.1 of [3]) and express the answer in terms of the operators...
We define the bilinears $\sigma_j^i = c_j^i a_j$ for $j = 1$ to $N$, and $\tau_k^\pm = d_k^\pm b_k$, for $k = 1$ to $N$, together with their Hermitian conjugates, $\sigma_j^\pm$, $\tau_k^\pm$. We shall also use $\sigma_j$, $\tau^k$ as sets of 3-vectors that separately have the commutation relations of independent Pauli spin matrices, e.g. $[\sigma_j^+, \sigma_k^\pm] = \delta^{jk} \sigma_j^3$ etc.

From these we construct an effective Hamiltonian that embodies the action of the coupling through the scalar meson intermediary as long as the mass of the scalar is large compared to the neutrino energies,

$$H_{\text{eff}} = \frac{2\pi G}{V} \sum_{j,k}^N \left[ \sigma_j^+ \tau_k^+ + \sigma_j^- \tau_k^- \right] (1 - \cos \theta_{j,k}).$$

The above form embodies terms from the interaction Lagrangian $-\mathcal{L}_I$ of eqn. 1.1 in [3], but simplified by taking all flavor-diagonal couplings of the scalar field to vanish, and coupling only to a single pair of neutrinos, $\nu_e, \nu_x$. This produces not only the terms in (1) but some additional additional terms that we believe do not affect the qualitative behavior in the large $N$ limit. The latter terms, in sum, can be shown not to contribute to results in the large $N$ limit, and we omit them here.

However, for the sake of being sure in this first look at this category of model, we change the framework slightly to rid ourselves of these terms, which are unwanted complications at their least. We take a complex scalar field, $\Phi$ to mediate the interaction, massive on the scale of the energies, with coupling,

$$\mathcal{L}_{\text{int}} = g_{i,j} \bar{\nu}_{iL} \nu_{jL} \Phi + g_{i,j} \bar{\nu}_{iL} \nu_{jL} \Phi^*$$

exactly as in ref. [3], eq. 1.1, and then specializing to the case $g_{1,2} = g_{2,1} = \sqrt{G}$, $g_{1,3} = g_{2,3} = 0$. Now (1) needs no supplementation. Note that in (2) $i$ and $j$ are flavor indices while in (1) they are momentum labels.

Because we are extracting only the part in which momentum flows as $\vec{p} + \vec{q} \rightarrow \vec{p} + \vec{q}$ for particles that are nearly massless, kinetic energy is conserved and the kinetic term in the Lagrangian is irrelevant. The answer above is independent of the absolute values of momentum that enter. But the directions (in whatever frame we choose for the calculation) only enter through the final factor in (1) where $\cos \theta_{j,k} = \vec{p}_j \cdot \vec{p}_k / (|p_j| |p_k|)$.

Our plan is first to solve the model numerically, but precisely, for the simplest case of head-on collision of two beams; this for the largest number of particles our modest computer resources can manage. Next we shall test the mean-field (MF) approaches (which are to come) that will be applied to cases with orders of magnitude more particle number. Finally we shall shift from two clashing beams to spherically symmetric initial distributions.

For the head-on case we replace the angular factor in (1) by 2. Then we sum over a set of momentum magnitudes indexed by $j, k$ to define collective operators $\sigma = N^{-1/2} \sum_j^N \sigma_j^i$ etc, and a Hamiltonian

$$H_{\text{eff}} = 4\pi n G (\sigma_+ \tau^+_3 + \sigma_- \tau^-_3),$$

where $n = N/V$ is the neutrino number density. The sets $\sigma_\pm$, $\tau_3$ still have the commutation rules of Pauli matrices; that is of angular momentum matrices times two. With the same number $N$ of $\nu_e$'s initially moving in the $+\hat{z}$ direction and of $\nu_x$'s in the $-\hat{z}$ directions, there is a single ladder of states connected by the interaction (1). Taking $k$ as the index for the number of $\nu_x$ pairs that have been created from our initial state with a prescribed set of momenta we have the matrix elements

$$\langle k + 1 | H_{\text{eff}} | k \rangle = g k (N - k + 1),$$

for $k = 1, ..., N + 1$, where $g = 4\pi G V^{-1}$. For small values of $N$ we can easily solve the Schrodinger equation based on (4), numerically, to find the time dependent wave function of the system. We define the retention amplitude,

$$\zeta(t) = N^{-1} \langle \Phi(t) | (1/2 + \sigma_3/2) \rangle \langle \Phi(t) \rangle,$$

In fig. 2 we plot results for $\zeta(s)$ for a series of three $N$'s spaced by powers of 4 from $N=64$ to 1024 as a function of scaled time $s = (gN)^{-1} t$. As $N$ is increased, the system's behavior will be to sit there for a longer time organizing itself, then making a sudden total turnover; that is as switch from particles of one flavor to anti-particles of the other. The actual turn-over time, in view of the scaling, is of the order of $(nG)^{-1}$, in contrast to a mean free time for scattering that is of order $(nG^2 E^2)^{-1}$, where $E$ is the energy scale of the cloud.

Those remarks ignore the steady march to the right as $N$ is increased in the curves shown in fig. 2; they are the hint of an additional log$|N|$ dependence that we shall verify in the mean field approach.

In the ordinary mean-field approach the Heisenberg equations for the operators are easily constructed as,

$$\dot{\sigma}_3 = n g \tau_- \sigma_3, \quad \dot{\tau}_3 = n g \sigma_+ \tau_3, \quad \dot{\sigma}_3 = -\dot{\tau}_3 = 2n g (\sigma_+ \tau_3 - \sigma_- \tau_3).$$

The conventional mean field assumption (though the authors of ref. 4 do not phrase it quite in our terms) is that the operator equations (5), are valid for the $c$-number expectations of the various operator products in the equations, $\langle AB \rangle = \langle A \rangle \langle B \rangle$. But it is obvious that when we begin with pure flavor states, $\sigma_+(0) = 0$, $\tau_+(0) = 0$, $\sigma_3(0) = 0$, $\tau_3(0) = 0$, then nothing at all happens at the mean field level; the system stays exactly where it is. On the other hand if we look just at the two coupled equations for $\sigma_+(t), \tau_+(t)$ keeping $\tau_3 = \sigma_3 = N$ as fixed at the initial values, we see that a small perturbation, $\Delta \sigma_+$ will grow at a rate proportional to $\exp(nGt)$. The instability that is required for fast turnover is there. But the formalism has not supplied the break.
4. MODIFIED MEAN FIELD APPROXIMATION AND QUANTUM BREAK

The key to extending the MF approximation, as was done in the atomic physics problems of ref. [5] and also for the collisions of circularly polarized protons with each other enabled through the Heisenberg-Euler coupling, in ref. [10], is first to write equations of motion for some products of our basic operators. In our case we choose combinations like $σ_+ τ_+$. Now the right-hand-sides will contain higher order polynomials in the fields. We can then try judiciously to close the system through factorizations of the generic forms $⟨ABC⟩ → ⟨AB⟩⟨C⟩$.

The authors of ref. [3] invoke references to BBGKY hierarchy in their description of their method, which is akin to ours, although the present author might say “not even that”, in the belief that it is much less than a quantum field theoretic form of what N. N. Bogolyubov had in mind [23].

Beginning with the operators defined in the last section $σ$ for the p stream and $τ$ for the q stream, and before rescaling, we define $X = σ_+ τ_+, Y = σ_− τ_−$. We rename $σ_3 = τ_3 = Z$ (their equality being chosen in the initial condition, and then maintained throughout).

The Hamiltonian is now

$$H_{8,γ} = \frac{8πNG}{V}[X + X^†],$$

and the Heisenberg equations of motion are,

$$i\dot{X} = \frac{8πG}{V}(ZY - Z^2),$$

$$i\dot{Y} = \frac{16πG}{V}Z(X^† - X),$$

$$i\dot{Z} = \frac{16πG}{V}(X - X^†).$$

The $Z^2$ term in the first equation comes from a second commutation to get operators into the correct order; implicitly it carries an additional power of $\hbar$ and is the source of the “quantum break” to come. Our modified mean field method (MMF) is to replace each of the operators $X,Y,Z$ in (8) by its expectation value in the medium, thus implicitly assuming that, e.g., $⟨ZY⟩ = ⟨Z⟩⟨Y⟩$.

Next we do a rescaling in which each one of the single particle operators $a, b, c, d, a^†...$ is redefined by extracting a factor of $N^{1/2}$, so that $x = X/N^2, y = Y/N^2, z = Z/N$ and at the same time defining $n = N/V$, the number density, and the scaled time variable, $s$, according to $s = 8πGnt$, where $n$ = the number density $N/V$ of each beam.

The rescaled equations are,

$$\frac{dx}{ds} = zy - z^2/N, $$

$$\frac{dy}{ds} = 2z(x^† - x), $$

$$\frac{dz}{ds} = 2(x - x^†),$$

(9)

and the rescaled initial condition (all $ν_ε$’s at the beginning, say) is $z(s = 0) = 1$. The value $z(s) = -1$ signifies a complete transformation to $ν_ε$’s. The retention fractions $ζ(s) = (z(s) + 1)/2$ calculated from the solutions of (9) are plotted against scaled time, shown as the solid curves in fig. 2 for the same three values of $N$ used in that plot. We see an excellent fit at early times, in view of the fact that there are no free parameters, as the retention $ζ(t)$ decreases from unity to .75. After that there is qualitative agreement down to the turn-around. There is also a hint that the fit is improving for higher values of $N$.

We can also use the authoritative solutions of the preceding section to explore solutions of the ordinary mean field equations (13) of the last section, where we have to provide some kind of seeding in order to make anything happen at all. We find that initial values of order $σ_+(s = 0) = τ_+(s = 0) = ±N^{-1/2}i$ give a point of $ζ ≈ 0$ at about the same time as found from the modified MF solutions for the three plotted cases, though not fitting the shape as well for the earlier times.

In the modified mean-field approach we can extend the calculations to much higher values of $N$, provided the beams have very small angular dispersion. Plots are shown in fig. 3.

5. MULTI-BEAM SOLUTIONS, ISOTROPY

In the application to the early universe we seek the behavior for completely isotropic initial distributions and in view of the $(1 - \cos θ)$ factor in the effective interaction, we might expect, at the least, some diminution of the effect. In addressing this situation we take a forest of angles, for $N_i$ different incident beams, each with $N_a$
neutrinos distributed uniformly in elements of solid angle \(d \cos \theta d \phi\). This translates into an effective interaction,

\[
H_{\text{eff}} = \frac{4 \pi G}{V} \sum_{j,k}^N \left[ \sigma^j_+ \sigma^k_- + \sigma^j_- \sigma^k_+ \right] \lambda_{j,k},
\]

(10)

where the \(\lambda_{j,k} = (1 - \cos \theta_{j,k})\) for the rays \(j\) and \(k\).

The equations for the operators at the MF level involve the variables \(\sigma^j_+, \tau^j_+\) and \(\sigma^j_- = \tau^j_3\), and pose no particular problem in solution, but require seedings of the form \(\sigma^j_+(0) = \pm 1/(N_a N_b)^{-1}\). In the special case in which half of the beams are in one direction and the other half are in the opposite direction we can roughly confirm this at the MF level (with very small \(N_a\’s\) and \(N_b\’s\)). But the MF equations, even for \(N_b = 25\) which we use in our angular simulation in which each cell \(N_b\) uses the same fraction of the total solid angle, are too numerous for our computer.

In comparing the head-on beams scenario with the isotropic scenario, we shall calculate both in the MF method, with the same seeding algorithm, so that that the comparison is unbiased. There is in fact no discernible difference in the computational results for the two cases in a 25 angle simulation, and we are reassured that our effects will persist in an isotropic system.

FIG. 3:
The same as fig.2, except only the MMF solutions, for a series of higher values of \(N\).

As an application of the structures that we have been developing, we add an ordinary \(\nu_e - \nu_s\) mixing term to the system. We revert to the head-on clashing “monoegetic” beam model for the active neutrino clouds, in this first pass at a new problem. The single particle annihilation operators, as in sec.1, were \(a, c\) for the first beam and \(b, d\), leading to the quadratics \(\sigma\) and \(\tau\), expressed as \(\sigma_\pm, \sigma_3\). To add mixings of, say, \(\nu_s\) with a sterile \(\nu_S\) we will need introduce the annihilators \(\tau, s\) for a sterile \(\nu_s\) in the up or down beam respectively and four new operators,

\[
\begin{align*}
\sigma^j_+ &= c^j_1 a, & \tau^j_+ &= d^j_1 b, \\
\sigma^j_- &= a^j_1 c, & \tau^j_3 &= b^j_1 d - d^j_1 d, \\
\zeta^j_+ &= a^j_1 s_j, & \zeta^j_3 &= a^j_1 a_j - s^j_3 s_j, \\
\rho^j_+ &= c^j_3 s_j, & \rho^j_3 &= c^j_3 c_j - s^j_3 s_j, \\
\eta^j_1 &= s^j_3 c_j.
\end{align*}
\]

(11)

The Hamiltonian is given by \[8\] with an added active-sterile mixing term,

\[
H_{\text{eff}} = 2 \pi G \sum_{j,k}^N \left[ \sigma^j_+ \tau^k_+ + \sigma^j_- \tau^k_- \right] (1 - \cos \theta_{j,k}) + m_s \sum_{j}^N \left\{ \sin \theta_{s,e} (\zeta^j_+ + \zeta^j_3) + \eta^j_1 \right\},
\]

(12)

where \(\theta_{s,e}\) is the mixing angle for \(\nu_e\) and \(\nu_s\) mixing. For simplicity, we shall take no mixing coupling between \(\nu_e\) and \(\nu_s\). Following the previous path, we find Heisenberg equations for the collective variables that can be defined in the case of the two head-on clashing beams, where \((1 - \cos \theta_{j,k}) = 2\) for all \(j, k\). The eight coupled equations are,

\[
\begin{align*}
\dot{\sigma}_+ &= g \tau_+ \sigma_3 - g_s \rho_+ + i \sigma_, \\
\dot{\sigma}_3 &= 2g \sigma_+ \tau_+ + g_s (\zeta_+ - \zeta_-), \\
\dot{\tau}_+ &= 2g (\sigma_+ \tau_+ - \sigma_- \tau_-) + g_s (\eta_+ - \eta_-), \\
\dot{\rho}_+ &= -g \sigma_+ + g_s \sigma_3 + m_s \rho_+, \\
\dot{\eta}_+ &= -g \sigma_3 + m_s \zeta_+ + m_s \zeta_3, \\
\dot{\nu}_3 &= g (\sigma_+ \tau_+ - \sigma_- \tau_-) + 2g_s (\zeta_+ - \zeta_-) + m_s \rho_+, \\
\dot{\nu}_1 &= g_s (\zeta_1 - \zeta),
\end{align*}
\]

(13)

where we take \(g = 4 \pi n_b G\), \(g_s = m_s \sin \theta_{s,e}\).

We follow the basic mean field approach, in which the initial value of each one of the variables is replaced by
its expectation value. At the present time we are not able to follow the modified mean-field approach because of computer limitations and a much larger equation set. Earlier we discussed the alternative of seeding the initial values, \( \sigma_{\pm}(t = 0) = \pm i/N \), and similarly for \( \tau_{\pm}(t = 0) \), which should give approximate results. In our example, for a pure \( \nu_e \) gas, we have \( \sigma_3(t = 0) = 1 \), \( \tau_3(t = 0) = 1 \) after the rescalings used earlier.

Fig. 4 shows a specific example of a solution to the system for an initial state that is pure \( \nu_e \) in both directions, with sterile mixing angle \( \sin^2\theta_{e,\nu} = 10^{-5} \). It shows active-sterile transitions stimulated by the \( \nu_e \), \( \nu_x \) mixing that was the focus of the body of this paper. Note that it is at the first point of the dip in the \( \nu_e \) retention \( \zeta(t) \) (the “break”) that the sterile mixing ties into the instability. It then grows steadily until reaching the next break.

\[
\begin{align*}
\text{FIG. 4:} & \quad \text{Solid curve: retention amplitude, } \zeta(t), \text{ of an initial state with clashing } \nu_e \text{ beams, adding mixing with a sterile neutrino (with } \sin^2(\theta_{e,\nu}) = 10^{-5} \text{.) Dashed curve: } 5\eta(t) \\
& \quad \text{where } \eta \text{ is the average amplitude for } \nu_e \text{ in one of the states } j. \quad \text{Dotted curve at bottom: the extension of the retention amplitude (solid curve) when coupling to sterile flavor is turned off. Time, } t, \text{ is in arbitrary units.}
\end{align*}
\]

The conventional description of sterile neutrino production at early times, e.g., in ref.3 involves the slow development of transformed amplitude in the form of an oscillation with amplitude \( \sin^2(\theta_{e,\nu}) = 10^{-5} \). It's transformations into the sterile mode were approaching 10% in the same underlying theory.

In the above simulation, as far as we plotted it, amplitude mixings into the sterile mode were approaching 10%, and number conversions therefore of order 1%. The computational program, when extended in time, shows the number growing continuously, while the back and forth oscillations between \( \nu_e \)'s and \( \nu_x \)'s continue at an even faster rate. But it would be naive to believe any mean field approach over a domain in which so many drastic changes take place. It is far more likely, in the author’s opinion, that a large conversion can happen through the agency of repeated smaller events. If we look at a thermal neutrino number density of say, \( 10^6[\text{MeV}]^3 \) in the decoupling region and the choice from [3], \( G = 10^{-5}[\text{MeV}]^{-2} \) for the scalar coupling constant of the two \( \nu_e \) to the two \( \nu_x \), we could get a 1% conversion to sterile in \( 10^{-6} \) cm.

What was required in order to make this at all possible is first to have the fast transformation mechanics from the instability at the mean-field level. The principles of this part appear to be well established in several applications, though no doubt unfamiliar to cosmologists. But the ability of the fast flavor oscillation in the active-\( \nu \) sector to entrain the otherwise very slowly growing sterile fraction and then coherently usher it to macroscopic significance on the fast time scale is a novel possibility.

We can now speculate that this approach could provide a very fast way of populating the universe with dark matter by producing mass = 1 KeV sterile neutrinos during the conventional neutrino decoupling era, but we must go very carefully in view of the fact that a 5 MeV beam maintains its phase coherence (in the context that we discussed earlier) for only about 10 meters.

These speculations are now getting ahead of our abilities to systematically check ideas with our computationally based methods. For example, if we think about beginning in a particle-antiparticle symmetric medium (zero neutrino chemical potential) and with the number of \( \nu_e \)'s equal to the number of \( \nu_x \)'s, the first thing we might say is: “Now nothing can happen. The \( \nu_e + \nu_x \leftrightarrow \bar{\nu}_e + \bar{\nu}_x \) mechanism does not act to change this state at all, since at the same time we had an equal number of \( \bar{\nu}_x \)’s transformed back to \( \nu_e \)’s.”

To understand why this reasoning does not apply to our calculation, consider the momentum states that enter a calculation. For clarity, let us start at \( t=0 \) in a system in which we have equal numbers, \( 2(N_1 + N_2) \) of left-moving (L) momenta and \( 2(N_1 + N_2) \) right-moving momenta (R), with respect to some x-axis and detailed momentum sets obtained by picking from the Fermi distribution with zero chemical potential. Then we randomly assign \( 2N_1 \) of the momentum states either to the flavor \( \nu_e \) and \( 2N_2 \) to the flavor \( \nu_x \). Then there are four groups, in our initial state |ABCD|, respectively given by,

A. \( \nu_e[L], q_1, q_2, \ldots q_{N_1} \)  \quad B. \( \nu_e[R], p_1, p_2, \ldots p_{N_1} \)
C. \( \bar{\nu}_x[L], r_1, r_2, \ldots r_{N_2} \)  \quad D. \( \bar{\nu}_x[R], s_1, s_2, \ldots s_{N_2} \)

and we keep in mind that there is an equally big set of states that were initially just the anti particles of the above (but with different momenta draws.) The latter do not affect the evolution of the former, however. Now we mainly have the reactions \( R+L \rightarrow R+L \), rather than \( R+R \rightarrow R+R \) or \( L+L \rightarrow L+L \) even when we started with spherical symmetry (with a 3 to 1 advantage), because of the angular dependence of the interaction. And in the
simplified argument below we shall assume that we have only R+L interactions.

The initial momentum distributions in each set are derived from Fermi distributions with zero chemical potential (for particle-antiparticle symmetry). Then, independently of the temperature (as long as it is relativistic), we have a probability \((-\pi^2/9 + \zeta[3])/\zeta[3]1\) that the momentum of a randomly drawn state, say from group A matches any of the momenta in group C, or drawn from B that matches D. These sets are nearly disjoint, for a typical throw of the dice, and the small minority of overlapping states can be written off as duds.

Thus the calculation of progress of the system as it develops from an initial |ABC⟩ wave function in the 4N dimensional space is the same as that which can be calculated from separate calculations in 2N dimensional spaces with initial respective wave functions |AB⟩ and |CD⟩, then taking the direct product. Therefore let’s now consider the development of the flavor density matrix for a particular momentum state |p1⟩ in an |AB⟩ calculation. As we have seen, this sub-system oscillates back and forth between flavor e and flavor x, as will the particular momentum mode |p1⟩, even if we began in a flavor equilibrium situation for the whole system (the N1=N2 case).

In the absence of a coupling to a sterile state, the above remarks are inconsequential. In the complete system when there is flavor equilibrium the other part of the wave function, |CD⟩, produces the counter-behavior that preserves flavor balance. There is usually no reason to care about the single momentum state, |p1⟩. But when there is a coupling of a sterile mode, say to νe, we then need to follow the wave-function of that particular momentum state as it develops a sterile component. And the flavor assignment for that particular momentum state will be executing the very nonlinear oscillations that we have become accustomed to.

7. DISCUSSION

Everyone understands that in an interaction of a photon with a cloud of neutral particles, the index-of-refraction correction is proportional to the forward scattering amplitude, and that there will be situations in which it makes important and observable consequences over distances orders of magnitude less than (cross-section \times number density)^-1.

In the generalization to clouds of massless interacting particles that interact with each other, and the particles have some set of flavors (e,x,\bar{e},\bar{x} in the present neutrino case) we have a multiplicity of two body reaction channels in which flavors change, or which momentum goes with which flavor changes, while the set of momenta of the whole swarm stays the same. Then one finds an instability leading to exponential growth of perturbations, and a time-scale for total mixing of order \(G^{-1}\), where G is the four-Fermi coupling.

Of course, actual neutrinos have mass, and the above will be limited to distances D less than \((m_\nu^2/|p)|^{-1}\), in order to maintain coherence. Followers of the neutrino physics calculations in the type-2 supernovae calculations will remember a “fast transformation” possibility for that system as well, where the coupling was through Z meson exchanges. But that demanded rather specific attributes of the initial flavor and angular distributions. In contrast, in the scalar coupling model used in ref \[3\] and in the present paper, the instabilities are ubiquitous. We expect them to dominate, as long as the scalar coupling constant is greater than \(G_F\).

We return to a question posed in the introduction, “What would the effects on the outgoing neutrino spectra and flavor-balance be, given our treatment of exactly the \(\nu + \bar{\nu} \leftrightarrow \bar{\nu} + \bar{\nu}\) process as described in \[3\], but with no account of the back reaction on other physics of the era.” The answer is: complete equilibration of flavors and energies in the entire range \(10^{-10} < G^2 < 3 \times 10^{-6}\text{MeV}^{-2}\). For values moderately less than \(10^{-10}\) it could still do the job, depending on how it coexists with the purely refractive part of the standard model interactions. That is to say: the transformations would still be very fast compared to the scattering rate.

Therefore the issues raised in sections 1-5, should be further explored with respect to all of the conclusions of ref. \[3,11\] and other work on the effects of \(\nu + \nu \leftrightarrow \bar{\nu} + \bar{\nu}\) in the decoupling era. The results of the sec. 6 discussion are still somewhat provisional, as to possible relevance to promising models of sterile neutrinos as dark matter. This is due both to our computational shortcomings, and to the several novelties in our approach, which need to be better evaluated.

\[\text{[1]}\] A. de Gouvea, M. Sen, W. Tangarife, Y. Zhang, Phys. Rev. Lett. 124 081802, arXiv:1910.04901
\[\text{[2]}\] S. Dodelson and L. M. Widrow, Phys. Rev. Lett. 72, 17 (1994), arXiv:hep-ph/9303287
\[\text{[3]}\] E. Grohs, George M. Fuller, M. Sen, arXiv:2002.08557
\[\text{[4]}\] G. Raffelt and G. Sigl, Nucl. Phys. B406, 423 (1993)
\[\text{[5]}\] A. Vardi, J. R. Anglin, Phys. Rev. Lett. 86, 568 (2001), arXiv:physics/0007054
\[\text{[6]}\] F. Canetti, C. Presilla., Phys. Rev. Lett. 89, 040403 (2002), arXiv:quant-ph/0201147
\[\text{[7]}\] J. Keeling, Phys. Rev. A 79, 053825 (2009), arXiv:cond-mat/0901.4245
\[\text{[8]}\] R. F. Sawyer, Phys. Rev. Letters 93, 133601 (2004), arXiv:hep-ph/0404217
\[\text{[9]}\] R. F. Sawyer, Phys. Rev. D79, 105003 (2005), arXiv:hep-ph/0503013
\[\text{[10]}\] R. F. Sawyer, Phys Rev A89, 052321 (2014), arXiv:1402.5170
\[\text{[11]}\] R. F. Sawyer, Phys. Rev. Lett. 116, 081101 (2016), arXiv:1509.03323
[12] S. Chakraborty, R. S. Hansen, I. Izaguirre, G. Raffelt, Nucl. Phys. B908, 366 (2016), arXiv:1602.02766
[13] I. Izaguirre, G. Raffelt, I. Tamborra, Phys. Rev. Lett. 118, 021101 (2017), arXiv:1610.01612
[14] B. Dasgupta, A. Mirizzi, M. Sen, arXiv:1609.00528
[15] A. Dighe and M. Sen, Phys. Rev. D97, 043011 (2018), arXiv:1709.06858
[16] B. Dasgupta and M. Sen, Phys. Rev. D97, 023017 (2018), arXiv:1709.0867
[17] S. Abbar and H. Duan, Phys. Rev. D98, 043014 (2018), arXiv:1712.07013
[18] S. Airen, F. Capozzi, S. Chakraborty, B. Dasgupta, G. Raffelt, and T. Stirner, JCAP 1812, no. 12 019 (2018), arXiv: 1809.09137
[19] S. Abbar and M. C. Volpe, Phys. Lett. B790, 545 (2019), arXiv:1809.09137
[20] C. Yi, L. Ma, J. D. Martin, and H. Duan, Phys. Rev. D99, 063005 (2019), arXiv: 1809.09137
[21] F. Capozzi, G. Raffelt, and T. Stirner, JCAP 2019 (Sep. 2019) 80
[22] S. Shalgar and I. Tamborra, Astrophys. J. 883, 80 (2019), arXiv: 1904.07236
[23] R. Glas, H.-T. Janka, F. Capozzi, Manibrata Sen, B. Dasgupta, A. Mirizzi, G. Sigl, Phys. Rev. D101, 063001 (2020), arXiv:1912.00274
[24] R. F. Sawyer, Phys. Rev. Lett. 124, 101301 (2020), arXiv:1910.08835
[25] N. N. Bogolyubov, J. Phys. (USSR) 11, 23 (1947), reprinted in D. Pines, The Many-Body Problem, (W. A. Benjamin, New York, 1961).