No purification for two copies of a noisy entangled state

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We consider whether two copies of a noisy entangled state can be transformed into a single copy of greater purity using local operations and classical communication. We show that it is never possible to achieve such a purification with certainty when the family of noisy states is twirlable (i.e., when there exists a local transformation that maps all states into the family, yet leaves the family itself invariant). This implies that two copies of a Werner state cannot be deterministically purified. Furthermore, due to the construction of the proof, it will hold not only in quantum theory, but in any generalised probabilistic theory. We use this to show that two copies of a noisy PR-box (a hypothetical device more non-local than is allowed by quantum theory) cannot be purified.

The ability to purify entanglement is a crucial feature of quantum theory, allowing imperfect states to be refined into those necessary to correctly implement quantum information protocols. Here we consider whether two copies of a noisy entangled state can be transformed into a single copy of greater purity, using local operations and classical communication (LOCC). For the purposes of this paper, we shall restrict our attention to purification protocols which work with certainty (i.e., without post-selection).

We will show that two-copy purification is impossible to achieve whenever the family of noisy entangled states is twirlable \[^1\]. That is, whenever there exists an LOCC transformation that maps all states into the family, yet leaves the family itself invariant. Twirling was first studied for Werner states \[^2\], where it can be implemented by applying an identical random unitary to both qubits. Our result therefore implies that two copies of a Werner state cannot be purified (although in \[^2\] it is shown that two copies can be purified using post-selection).

Aside from twirlability, the proof relies only on very weak assumptions that are not specifically quantum. Indeed it will apply in any reasonable theory admitting mixed states and entanglement, such as those defined in the general operational framework of \[^3\]. It is interesting that non-trivial results can be proved within such a general framework. Furthermore, identifying which features are important in the proof allows a deeper and simpler understanding of it, even if one is only concerned with quantum theory.

Recently however, much interest has focussed on PR-boxes \[^4, 5\], hypothetical devices which are more non-local than any quantum state, achieving the maximal possible violation of the CHSH inequality \[^6\]. Many information theoretic properties of PR-boxes have been studied. For example, it has been shown that they would allow any communication complexity problem to be solved with one bit of communication \[^7\]. Here, our proof implies that two noisy PR-boxes cannot be deterministically purified.

In our proof, we will use the same notation as the density matrix formalism of quantum theory. However, note that this notation is sufficient to describe a general probabilistic theory. In particular, we will denote a mixture of states \(s_i\) with probabilities \(p_i\) by \(s = \sum_i p_i s_i\), a bipartite state in which the first part is in state \(s_1\) and the second part is in state \(s_2\) by \(s_1 \otimes s_2\) (although we will not make use of the tensor product structure), and a transformation by \(s' = T[s]\).

The proof only requires three reasonable assumptions of our physical theory, that are common to quantum theory, and the generalised probabilistic framework of \[^3\].

1. **Existence of a separable state**: For the type of system considered, we assume that there exists at least one separable (i.e., un-entangled) state.

2. **Transformations act linearly on mixed states**: As probabilities may reflect a lack of knowledge about the state, rather than anything physical, we demand that transformations act linearly on mixed states. I.e., \(T[\sum_i p_i s_i] = \sum_i p_i T[s_i]\).

3. **Entanglement cannot be created by LOCC**: We require that a separable state cannot be transformed into an entangled one via LOCC. This follows from any reasonable definition of local operations and entanglement.

We now proceed to the proof of our main result, that two copies of a noisy entangled state from a twirlable family cannot be purified.

Consider a family of bipartite mixed states \(s(p) \in S\) of the form:

\[
s(p) = p s_0 + (1-p) s_1 \quad p \in [0, 1]
\]

where \(s_0\) is a desired entangled state, and \(s_1\) is the noise. As discussed above, we will assume that this family of states is twirlable.

4. **Twirlability**: A family of states \(S\) is twirlable if there exists an LOCC transformation \(T\) which leaves all states in \(S\) invariant, and maps all allowed states into \(S\).
As twirling cannot transform a separable state into an entangled one (assumption A), and there exists at least one separable initial state (assumption C), S must also contain a separable state. We denote the maximal value of p for which s(p) is separable by p_s.

We say that deterministic two-copy entanglement purification of the family S is possible if there exists an LOCC transformation M such that M[s(p) ⊗ s(p)] = s(p'), for some p, p' ∈ [0, 1] satisfying p' > p > p_s.

Given any transformation M, we can always implement the twirled transformation M̃ which consists of carrying out M and then twirling (i.e. M = T · M). Furthermore, as twirling leaves all states in S invariant, if M achieves a purification then so will M̃. Without loss of generality, we therefore restrict our attention to twirled transformations.

Consider applying a particular twirled transformation M̃ to the states s_i ⊗ s_j, i, j ∈ [0, 1]. Due to twirlability (assumption B), each must be mapped to some state in S, and we denote the corresponding purities by q_ij ∈ [0, 1]:

\[ M̃[s_i ⊗ s_j] = s(q_{ij}) \] (2)

From assumption B, transformations must act linearly on probabilistic mixtures, hence

\[ s(p') = M̃[(ps_0 + (1-p)s_1) ⊗ (ps_0 + (1-p)s_1)] = p^2 M̃[s_0 ⊗ s_1] + (1-p) M̃[s_0 ⊗ s_1] + p(1-p) M̃[s_1 ⊗ s_1] + (1-p)^2 M̃[s_1 ⊗ s_1] = p^2 s(q_{00}) + p(1-p)s(q_{01}) + (1-p)^2 s(q_{11}) \] (3)

Writing

\[ p' = Q(p) = p^2 q_{00} + p(1-p) (q_{01} + q_{10}) + (1-p)^2 q_{11}, \] (4)

we now establish four properties of the function Q, that are necessary to achieve a deterministic two-copy entanglement purification:

**A Universal**: Because all input states are mapped into S under M, Q(p) ∈ [0, 1] for all p ∈ [0, 1].

**B Separability-preserving**: It must be the case that Q(p_s) ≤ p_s, otherwise one would be able to transform the separable state s(p_s) into an entangled state via LOCC (which would violate assumption B).

**C Useful**: In order to achieve a useful purification of the state, there must exist some p_e such that Q(p_e) > p_e > p_s.

**D Quadratic**: From equation (3): Q(p) is a quadratic (and hence continuous) function of p.

However, we now show that no function Q(p) satisfying these four requirements exists. From requirements A-C above, we obtain four relations between Q(p) and p, for increasing values of p:

\[ Q(0) ≥ 0, Q(p_s) ≤ p_s, Q(p_e) > p_e, \text{ and } Q(1) ≤ 1 \] (5)

As Q(p) is continuous, it is clear from these conditions that the function p' = Q(p) must intersect p' = p at 3 or more points in the interval [0, 1]. The only quadratic function to achieve this is Q(p) = p, but this is ruled out by the third relation, Q(p_e) > p_e. There are therefore no functions Q(p) obeying all four necessary conditions (see figure 1). Consequently, deterministic two-copy entanglement purification is impossible for twirled families of states.

**Werner states**: A important example of a twirlable family of entangled states are the Werner states in quantum theory. Here s_0 and s_1 are the two-qubit density matrices

\[ s_0 = |ψ^−⟩⟨ψ^−| = \frac{1}{2} (|01⟩ - |10⟩)(⟨01| - ⟨10|) \] (6)

\[ s_1 = \frac{1}{3} (I - |ψ^−⟩⟨ψ^−|) \] (7)

The Werner states have p_s = \frac{1}{2} (they are entangled for p > \frac{1}{2} and separable otherwise). The corresponding twirling operation consists of applying the same randomly chosen unitary to both qubits, i.e.

\[ s → \int (U ⊗ U^†) s(U ⊗ U^*) dU \] (8)

where the integral is taken according to the (unitarily-invariant) Haar measure.
As they form a twirlable family of states, the above proof implies that two copies of an entangled Werner state cannot be deterministically purified.

Interestingly, it was shown by Bennett et al. [1] that two copies of an entangled Werner state can be purified using post-selection. This also implies that three (or more) copies of a Werner state can be deterministically purified. One simply applies the protocol given in [1] to the first two copies. If the post-selection succeeds, the resultant higher-purity state is output, and if the post-selection fails, the third copy is output (with the original purity). This achieves

\[ p' = \frac{1}{9}(-8p^3 + 14p^2 + 2p + 1), \]

which satisfies \( p' > p \) whenever \( 1 > p > p_s \). Note that having three copies of the state allows \( p' \) to be a cubic function of \( p \), which is able to satisfy requirements A-C given above.

**Noisy PR-boxes:** The ability to generate ‘non-local’ correlations (i.e. correlations that cannot be explained by any local hidden variable model [6, 8]), is one of the most surprising aspects of quantum theory. However, it is possible to consider hypothetical systems that yield even stronger non-local correlations than those attainable in quantum theory [4], yet which still cannot be used to signal.

The simplest devices of this type are known as PR-boxes. These are composed of two terminals, each of which takes a binary input and emits a binary output. Denoting the inputs by \( x \) and \( y \) and the corresponding outputs by \( a \) and \( b \), the behavior of the PR-box is characterised by the conditional probability distribution:

\[ P_{PR}(ab|xy) = \begin{cases} \frac{1}{2} : & a \oplus b = xy \pmod{2} \\ 0 : & \text{otherwise} \end{cases}, \]

where \( \oplus \) denotes addition modulo 2. With a PR-box, one can achieve the maximal possible violation of the CHSH inequality [3] (CHSH=4, compared to \( 2\sqrt{2} \) for quantum theory, or \( \leq 2 \) for local classical theories).

We can also consider the anti-PR-box described by

\[ P_{PR}(ab|xy) = \begin{cases} \frac{1}{2} : & a \oplus b \neq xy \pmod{2} \\ 0 : & \text{otherwise} \end{cases}, \]

Mixing PR-boxes (\( s_0 \)) and anti-PR-boxes (\( s_1 \)) we obtain a family of noisy PR-boxes \( s(p) \in S \) with probability distributions:

\[ P_{s(p)}(ab|xy) = pP_{PR}(ab|xy) + (1-p)P_{\overline{PR}}(ab|xy) \]

Note that the ‘maximally mixed state’ in which both outputs are random and uncorrelated, is \( s(1/2) \).

There exists a twirling operation of all devices (all probability distributions \( P(ab|xy) \)) into \( S \) [10]. To perform the twirling, the two parties generate three maximally random shared bits \( \alpha, \beta, \gamma \), and then perform local transformations on the inputs and outputs of their terminals as follows:

\[ x \rightarrow x \oplus \alpha \]
\[ y \rightarrow y \oplus \beta \]
\[ a \rightarrow a \oplus \beta x \oplus \alpha \beta \oplus \gamma \]
\[ b \rightarrow b \oplus \alpha y \oplus \gamma \]

These transformations can be achieved either by re-labeling, or by adding wires and gates to their local terminals. Note that these are all the local reversible transformations that leave the PR-box rule \( a \oplus b = xy \) invariant. It is straightforward to check that this achieves the desired twirling.

We can consider any two-terminal device to be ‘separable’ if the probability distribution for its inputs and outputs can be replicated by a local hidden variable model. I.e. if there exist probability distributions \( P_0(i) \), \( P_A(a|xi) \), \( P_B(b|yi) \) such that

\[ P(ab|xy) = \sum_i P_0(i)P_A(a|xi)P_B(b|yi) \]

We say a device is ‘entangled’ if it is not separable. The family of noisy PR-boxes defined above contains both entangled and separable states, with an entanglement threshold of \( p_s = 3/4 \).

We are now in a position to address the purification of noisy PR-boxes. Given two noisy PR-boxes \( s(p) \) with \( p > p_s \), can we produce a single PR-box with higher purity \( s(p') \) with \( p' > p \) using local operations (adding wires and gates to the local terminals) and classical communication?

An example strategy is shown in figure 2 and it is not obvious a priori whether such a purification is possible. Now, given the theory proved above, we know that it is not. Two copies of a noisy PR-box cannot be purified by local wirings and classical communication.

**Generalized probabilistic theories:** Both Werner states and PR-boxes can be considered within a general
probabilistic framework, in which a state is characterised by the joint probability distribution for some set of fiducial measurements on each subsystem, and the allowed states form a convex set.

In the case of qubits in quantum theory, these fiducial measurements can be taken to be measurements of the three Pauli operators $\sigma_x, \sigma_y, \sigma_z$. The convex set of probability distributions corresponding to allowed single qubit states will then be isomorphic to the Bloch sphere. States comprised of multiple qubits, such as the Werner states, can be completely characterised by giving the joint outcomes for every combination of Pauli measurements on the subsystems.

For the PR-box, each terminal can be thought of as a primitive subsystem with two binary-outcome fiducial measurements (represented by the two possible inputs). The probability distribution $P_{PR}(ab|xy)$ can then be understood as the joint probability of obtaining outcomes $a$ and $b$ when fiducial measurements $x$ and $y$ are performed on the subsystems. There are no quantum systems that can be completely characterised by two binary outcome measurements, and even if there were they could not generate the non-locality inherent in the PR-box, so $P_{PR}(ab|xy)$ lies outside the set of allowed quantum states. However, it can be embedded within a different theory, called ‘generalised non-signalling theory’ (or ‘box-world’), in which all non-signalling probability distributions for the fiducial measurements are allowed states.

Once the set of allowed states has been fixed, all allowed operations and non-fiducial measurements can be represented by linear maps acting on the fiducial measurement probabilities, and twirlability can be understood in terms of these transformations. As in (17), a state is considered separable if it can be represented by a convex combination of product states (factorisable probability distributions in which both subsystems have an allowed state), and entangled otherwise. It is easy to see that separable states remain separable under LOCC.

All generalised probabilistic theories represented within such a framework satisfy the three requirements given in the introduction. Hence in all such theories, two copies of a noisy entangled state cannot be transformed into a single copy of higher purity via LOCC. Due to the very minimal requirements of the proof, this result will hold not just in quantum theory, but in any theory that can be expressed within a generalised probabilistic framework. In quantum theory, this leads to the particular result that two copies of a Werner state cannot be purified, and in box-world that two copies of a noisy PR-box cannot be purified.

Interestingly, an analogous argument should apply to two-copy purification of other properties (in place of entanglement), given a class of allowed operations that cannot generate that property (instead of LOCC).

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**Conclusions:** We have shown that two copies of a noisy entangled state, taken from some twirlable family, cannot be transformed into a single copy of higher purity via LOCC. Due to the very minimal requirements of the proof, this result will hold not just in quantum theory, but in any theory that can be expressed within a generalised probabilistic framework. In quantum theory, this leads to the particular result that two copies of a Werner state cannot be purified, and in box-world that two copies of a noisy PR-box cannot be purified.

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12. For simplicity, we assume that the set of separable states is closed. If it is an open set, we take $p_s$ to be maximal boundary point of the set of $p$ for which $s(p)$ is separable, and the proof follows from the continuity of $Q(p)$.
13. It is also possible to achieve this twirling by choosing a random unitary from a finite set corresponding to a unitary 2-design.
14. We would be happy to restrict to non-signalling probability distributions, but in fact this is not necessary, and we can prove the stronger result given.
15. A probability distribution is non-signalling if the marginals for each party do not depend on the other parties’ measurement choices.