Nonlinear relaxation field in charged systems under high electric fields

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The influence of an external electric field on the current in charged systems is investigated. The results from the classical hierarchy of density matrices are compared with the results from the quantum kinetic theory. The kinetic theory yields a systematic treatment of the nonlinear current beyond linear response. To this end the dynamically screened and field-dependent Lenard-Balescu equation is integrated analytically and the nonlinear relaxation field is calculated. The classical linear response result known as Debye-Onsager relaxation effect is only obtained if asymmetric screening is assumed. Considering the kinetic equation of one species the other species have to be screened dynamically while the screening with the same species itself has to be performed statically. Different other approximations are discussed and compared.

I. INTRODUCTION

High field transport has become a topic of current interest in various fields of physics. In semiconductors the nonlinear transport effects are accessible due to femto-second laser pulses and shrink devices [1]. In plasma physics these field effects can be studied within such short pulse periods [2]. One observable of interest is the current or the electrical conductivity which gives access to properties of dense nonideal plasmas [3]. In high energy physics the transport in strong electric fields is of interest due to pair creation [4]. In order to describe these field effects one can start conveniently from kinetic theory. Within this approach the crucial question is to derive appropriate kinetic equations which include field effects beyond linear response.

At low strength of the external electric field one expects the linear response regime to be valid. Then the contribution of field effects to the conductivity can be condensed into the Debye-Onsager relaxation effect [5–11] which was first derived within the theory of electrolytes [12–16]. Debye has given a limiting law of electrical conductivity [12] which stated that the external electric field $E$ on a single charge $Z = 1$ is diminished in an electrolyte solution by the amount

$$\delta E = E \left(1 - \frac{\kappa e^2}{6T} \right)$$

where $e$ is the elementary charge, $E$ the electric field strength, $T$ is the temperature of the plasma and $\kappa$ is the inverse screening radius of the screening cloud. This law is interpreted as a deceleration force which is caused by the deformed screening cloud surrounding the charge. Later it has been shown by Onsager [13] that this result has to be corrected to

$$\delta E = E \left(1 - \frac{\kappa e^2}{3(2 + \sqrt{2})T} \right)$$

if the dynamics of ions ($Z = 1$) is considered. While the linear response theory seems to reproduce this Onsager result [8,9,11], the kinetic theory seems to support more the Debye result [10,17,11].

The correct treatment is a matter of ongoing debate. In this paper we will give the result beyond linear response for the statically and dynamically screened approximation. Here different approximations of kinetic theory will be discussed and the one which leads to the closest form to the hydrodynamical approximation (Onsager result) is presented.

The kinetic approach describes the time evolution of the one particle distribution function within an external field $E$ as

$$\frac{\partial}{\partial t} f - eZ E \frac{\partial}{\partial k} f = I[f, E]$$

where the field dependent collision integral $I[f, E]$ has to be provided by different approximations. Integrating this kinetic equation over the momentum $k$ one obtains the balance of the current. For simplicity we assume that the distribution function can be parameterized by a displaced local equilibrium one with a field and time dependent momentum $f(k, t) = f_0(k - p(E, t))$ which is related to the current $J$ as

$$J(E) = nZe \frac{p(E)}{m}$$

if the charge is $Ze$, the density $n$ and the mass $m$. The balance equation for the field and time
dependent local momentum \( p(E,t) \) follows from (3) as
\[
\frac{\partial}{\partial t} p - e Z n (1 + \frac{\delta E(E)}{E}) E = R(E) e Z n J
\]
(5)
where the relaxation field \( \delta E(E) \) as well as the free conductivity \( R(E) \) follows from the field dependent collision integral. The total conductivity \( \sigma = \sigma J \) is then given by
\[
\sigma(E) = \frac{R(E)}{1 + \frac{\delta E(E)}{E}}
\]
(6)
The free conductivity \( R \) is the subject of intense investigations in the literature \[3\]. It is known that the Coulomb divergence for small wave vectors is omitted if screening is included and the divergence at large wave vectors is omitted by the De Broglie wavelength i.e. by the quantum effects. We will not consider the discussion of the free conductivity \( R \) here but concentrate on the relaxation field \( \delta E \). The free conductivity can be obtained by the same considerations as will be outlined here. We want to point out that the relaxation field will turn out to be free of long wave divergences in the classical limit in contrast to the free conductivity \( R \).

First we recall the hydrodynamical approach starting from the classical Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy which results into an analytical formula for the classical relaxation effect already reported \[3\]. This result is then compared with the quantum kinetic approach. We give a short rederivation of the field dependent kinetic equations in dynamically screened approximation from the Green’s function technique in Sec. III. Two approximations, the static screening as well as dynamically screened treatment are presented. In the fourth section we will derive the field dependent current analytically. We present both the statically as well as dynamically screened treatment as analytical results. The classical expressions for the statically screened result \[7\] is compared with the classical result from hydrodynamical approximation. The dynamical result is then derived analytically too and it will be shown that only for asymmetric screening the hydrodynamical result can be approached. In Sec. V we shortly discuss the physical limitation of field strengths for the local equilibrium assumption and the gradient approximation. Sec. VI summarizes and the appendix gives the calculation of some involved integrals appearing during the integration of the Lenard-Balescu equation.

II. APPROACH BY CLASSICAL BBGKY-HIERARCHY

The starting point for the classical considerations is the BBGKY hierarchy \[19,20\] which reads for the one - particle distribution function \( F_a \)
\[
\frac{\partial F_a}{\partial t} + \mathbf{v} \frac{\partial F_a}{\partial \mathbf{r}} + \frac{e_a}{m_a} \mathbf{E} \cdot \frac{\partial F_a}{\partial \mathbf{v}} - S_a F_a
\]
\[
= \sum_b \frac{n_b e_a e_b}{m_a} \frac{\partial}{\partial \mathbf{v}} \int d\mathbf{r}' d\mathbf{v}' F_{ab}(\mathbf{r}, \mathbf{r}', \mathbf{v}, \mathbf{v}') \frac{1}{|\mathbf{r} - \mathbf{r}'|}
\]
(7)
and the two - particle distribution function \( F_{ab} \)
\[
\frac{\partial F_{ab}}{\partial t} + \mathbf{v} \frac{\partial F_{ab}}{\partial \mathbf{r}} + \mathbf{v} \frac{\partial F_{ab}}{\partial \mathbf{r}'} + \frac{e_a}{m_a} \mathbf{E} \cdot \frac{\partial F_{ab}}{\partial \mathbf{v}}
\]
\[
+ \frac{e_b}{m_b} \frac{\partial F_{ab}}{\partial \mathbf{v}'} - S_a F_{ab} - S_b F_{ab}
\]
\[
= e_a e_b \frac{\partial}{\partial \mathbf{r}} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \left( \frac{1}{m_a} \frac{\partial F_{ab}}{\partial \mathbf{v}} - \frac{1}{m_b} \frac{\partial F_{ab}}{\partial \mathbf{v}'} \right)
\]
\[
+ \sum_c n_c e_c \int d\mathbf{r}'' d\mathbf{v}'' \left( \frac{e_a}{m_a} \frac{\partial}{\partial \mathbf{r}} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot \frac{\partial F_{abc}}{\partial \mathbf{v}''} \right)
\]
\[
+ \frac{e_b}{m_b} \frac{\partial}{\partial \mathbf{r}'} \frac{1}{|\mathbf{r}' - \mathbf{r}'|} \cdot \frac{\partial F_{abc}}{\partial \mathbf{v}''}
\]
(8)
with the external field \( \mathbf{E} \). \( S_a \) describes a collision integral with some background which we will specify later. This hierarchy is truncated approximating that \[20\]
\[
F_{ab} = F_a F_b + g_{ab}
\]
\[
F_{abc} = F_a F_b F_c + F_a g_{bc} + F_b g_{ac} + F_c g_{ab}
\]
(9)
where \( g_{ab}(\mathbf{r}_a, \mathbf{r}_b, \mathbf{v}_a, \mathbf{v}_b) \) is the two-particle correlation function.

Within the local equilibrium approximation we suppose a stationary (for example a local Maxwellian) distribution for the velocities in the one and two-particle distribution functions
\[
f_a(\mathbf{r}, \mathbf{v}, t) = n_a(\mathbf{r}, t) \left( \frac{m_a}{2\pi T} \right)^{3/2} \exp \left[ -\frac{m_a(\mathbf{v} - \mathbf{u}_a)^2}{2T} \right]
\]
\[
g_{ab}(\mathbf{r}, \mathbf{r}', \mathbf{v}, \mathbf{v}', t) = F_{ab} - F_a F_b
\]
\[
= \hbar_{ab}(\mathbf{r}, \mathbf{r}', t) \left( \frac{m_a m_b}{4\pi^2 T^2} \right)^{3/2}
\]
\[
\times \exp \left[ -\frac{m_a(\mathbf{v} - \mathbf{w}_{ab})^2}{2T} - \frac{m_b(\mathbf{v}' - \mathbf{w}_{ab})^2}{2T} \right]
\]
(10)
Here we have introduced the local one-particle density and the local average velocity
\[
n_a(\mathbf{r}, t) = \int d\mathbf{v} F_a(\mathbf{r}, \mathbf{v}, t)
\]
\[
\mathbf{u}_a = \frac{1}{n_a} \int d\mathbf{v} \mathbf{v} F_a(\mathbf{r}, \mathbf{v}, t)
\]
(11)
as well as the pair correlation function and the average pair velocity

\[ h_{ab}(\mathbf{r}, \mathbf{r}', t) = \int d\mathbf{v} d\mathbf{v}' g_{ab}(\mathbf{r}, \mathbf{r}', \mathbf{v}, \mathbf{v}', t), \]

\[ w_{ab}(\mathbf{r}, \mathbf{r}', t) = \frac{1}{h_{ab}} \int d\mathbf{v} d\mathbf{v}' v g_{ab}(\mathbf{r}, \mathbf{r}', \mathbf{v}, \mathbf{v}', t). \]

(12)

Further on, we suppose that the particles interact with some background (e.g., neutrals or electrolyte solvent) by the collision integrals \( S_a \) with the following properties

\[ \int d\mathbf{v} S_a f_a = 0 \]

\[ \int d\mathbf{v} v S_a f_a = \frac{1}{b_0 m_a} \rho_a u_a, \]

\[ \int d\mathbf{v} v S_a g_{ab}(\mathbf{r}, \mathbf{r}', \mathbf{v}, \mathbf{v}', t) = \frac{1}{b_0 m_a} h_{ab} w_{ab} \]

(13)

where \( b_0 \) is the mobility of particle of type \( a \). This friction with a background serves here to couple the two-particle equations and will be considered infinitesimal small in the end. However, as we will demonstrate this yields to a symmetry breaking in the system which leads basically to different results than neglecting this friction.

Fourier transform of the resulting two equations (9) into momentum space and assuming a homogeneous density \( n(\mathbf{r}) = n \) we arrive at the coupled equation system

\[ - \frac{e_a}{T} \bar{E} (\mathbf{v}_a - \mathbf{u}_a) f_a = \frac{\mathbf{u}_a}{b_0 m_a} + \sum b \frac{4 \pi n_b e_c e_b}{T} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{ik.(\mathbf{v}_a - \mathbf{w}_{ab})}{k^2} f_a (\mathbf{v}_a - \mathbf{w}_{ab} + \mathbf{u}_a) h_{ab}(\mathbf{k}) \]

(14)

and

\[ i \frac{d\mathbf{k}}{(2\pi)^3} \frac{4 \pi e_c e_b}{k^2} \bar{k}(\mathbf{v}_a - \mathbf{u}_a - \mathbf{v}_b + \mathbf{u}_b) f_a f_b \]

\[ -i \sum b \frac{4 \pi n_c}{T k^2} e_c k.(\mathbf{v}_a - \mathbf{u}_a - \mathbf{v}_b + \mathbf{u}_b) g_{cb}(\mathbf{k} - \mathbf{k}) \]

\[ - e_c \mathbf{k}.(\mathbf{v}_b - \mathbf{u}_b) f_b g_{ac}(\mathbf{k}) + S_a g_{ab} + S_b g_{ab} \]

(15)

By multiplying the above equation system by \( 1, \mathbf{v}_a, \mathbf{v}_b \) and integrating over the velocities we obtain the Onsager equation

\[ b_a \left[ T h_{ab}(\mathbf{k})(1 + \frac{e_a}{e} \Phi_a(-\mathbf{k})) + e_a \Phi_b(-\mathbf{k}) \right] \]

\[ = -b_b \left[ T h_{ab}(\mathbf{k})(1 - \frac{e_b}{e} \Phi_a(-\mathbf{k})) + e_b \Phi_a(\mathbf{k}) \right] \]

(17)

with

\[ k^2 \Phi_a(\mathbf{k}) = 4\pi e_a + \sum c n_c e_c h_{ac}(\mathbf{k}) \]

\[ k^2 \Phi_a(-\mathbf{k}) = 4\pi e_a + \sum c n_c e_c h_{ca}(\mathbf{k}) \]

(18)

for the two-particle correlation function \( h_{ab} \). Here we use

\[ a = \frac{e k \bar{E}}{k^2 T}. \]

(19)

Let us already remark here that the friction with a background described by the mobilities \( b \) couple the two sides of the equation (17). If we had not considered this friction, \( S_i = 0 \), we would have obtained that the left and the right hand side of (17) vanish separately. This will lead essentially to a different result even for infinite small friction. There is no continuous transition between these two extreme cases pointing to a symmetry breaking in the two treatments. Let us first discuss the case with background friction.

A. With background friction

The system (17) for electrons, \( e_c = e \), and ions, \( e_i = -Ze \), with charge \( Z \) reads expanded

\[ T h_{ee} = -\frac{\Phi_i(-\mathbf{k}) + \Phi_i(\mathbf{k})}{2} \]

(20)

This we can solve together with (18). First we calculate the effective field strength at the position of the electron in linear response the Onsager result

\[ \frac{\delta E}{E} = -i \int_0^{\infty} \frac{d\mathbf{k}}{(2\pi)^2} \int_0^{1} k^3 d\mathbf{k} \int d(\cos \theta) \cos \theta \Phi_e(\mathbf{k}) \]

\[ = \frac{\mathbf{e} \mathbf{k} e^2}{3T} \frac{Zq}{\sqrt{q^2 + 1}} \]

(21)
For single charged ions $Z = 1$ the influence of the mobilities drop out and we recover the result (2).

The numerical values of this result will be discussed in chapter (IV C).

B. Without background

Now we reconsider the steps from (13) to (17) without friction with the background. We obtain that both sides of (17) vanish separately

$$Th_{ab}(k)(1 + i e_a a + e_a \Phi_b(-k) = 0$$

$$h_{ab}(k)(1 - i e_a a + e_b \Phi_a(k) = 0. \tag{26}$$

Both equations have identical solutions $h_{ab}$ which can be easily verified using the symmetry $h_{ab}(k) = h_{ba}(-k)$. Together with (13) we can solve for $\Phi$ and the relaxation field is obtained instead of (24)

$$\delta E = -\frac{e^2 \kappa e \sqrt{1 + Z}}{6T} (Z + 1) E F_N \left( \frac{\epsilon E}{T \kappa e} \right) \tag{27}$$

which takes for $Z = 1$

$$\frac{\delta E}{E} = -\frac{e^2 \kappa e}{6T} \left\{ \frac{2 + o(E)}{3 \epsilon T + o(1/E)^2} \right\} \tag{28}$$

with

$$F_N(\alpha) = \frac{3}{(1 + Z) \alpha^2} \left[ \sqrt{4 + (1 + Z) \alpha^2} \right.$$

$$+ \frac{4}{\sqrt{1 + Z} \alpha} \log \frac{2}{\sqrt{1 + Z \alpha + \sqrt{4 + (1 + Z) \alpha^2}}} \right]. \tag{29}$$

We see that the linear response result for $Z = 1$ is twice the Debye result (4). For equal charged system $Z = -1$ which would coincide with a one component plasma no relaxation effect appears as one would expect. In other words in a perfectly symmetric mathematical two - component plasma there is another relaxation effect than in a system which distinguishes the components by a different treatment of friction. The Onsager result (28) does not vanish for the limit of one-component plasma $Z = -1$. This is due to the different treatment of ions and electrons there which assumes explicitly a two component plasma. Therefore the limit $Z = -1$ does not work there.

This result is quite astonishing. One would expect that the limiting procedure which transforms the system (18) into (24) would also lead to a
smooth transitions of the end results. However this is not the case. While the separate limit of $\beta_{r,i} \rightarrow \infty$ of \textbf{(15)} leads to \textbf{(26)} there is no possibility to transform the result \textbf{(21)} into the linear response result of \textbf{(28)}. This underlines that due to even infinitesimal small friction assumed in obtaining \textbf{(21)} there occurs a symmetry breaking in the sense that the electrons and ions are not anymore symmetrically treated.

This lesson we have to keep in mind when we now advance and investigate the systematic treatment by quantum kinetic theory. There we will find also complete different results when we use asymmetric screening compared to symmetric screening. Of course, we will not assume any phenomenological friction since the kinetic theory provides for a systematic description of all occurring processes. Here we want only to point out that the above symmetry breaking is the main reason for the confusion in literature. Following the linear response formalism an asymmetric treatment of two - particle correlation functions is used in \textbf{(21)} there occurs a symmetry breaking in the sense that the electrons are statically screened \textbf{(11)}. This seemingly innocent usage leads there to an occasional agreement for $Z = 1$ with the Onsager result \textbf{(2)}.

Another advantage of the kinetic theory we want to point out here. The classical local equilibrium or hydrodynamical approximation does not lead to a mass dependence of the relaxation effect. This will be provided by the kinetic theory.

### III. QUANTUM KINETIC THEORY

We will formulate the kinetic theory within gauge invariant functions not missing field effects. The most promising theoretical tool is the Green function technique \textbf{(17)} [22,23]. The resulting equations show some typical deviations from the ordinary Boltzmann equation: (i) A collision broadening which consists in a smearing out of the elementary energy conservation of scattering. This is necessary to ensure global energy conservation \textbf{(24)}.

(ii) The intra-collisional field effect, which gives additional retardation effects in the momentum of the distribution functions. This comes mainly from the gauge invariance.

One of the most important questions is the range of applicability of these kinetic equations. Up to which field strengths are such modifications important and appropriate described within one-particle equations? In \textbf{(25)} this question has been investigated for semiconductor transport. It was found that for high external fields the intra-collisional field effect becomes negligible. This range is given by a characteristic time scale of field effects $\tau_F^H = m\hbar/(eE \cdot q)$ which has to be compared with the inverse collision frequency. This criterion is a pure quantum one. It remains the question whether there are also criteria in the classical limit. For a plasma system we will discuss in Sec. \textbf{V} that there is indeed a critical value of the field strength which can be given by classical considerations.

#### A. Definitions

In order to describe correlations in highly nonequilibrium situations, we define various correlation functions by different products of creation and annihilation operators

$$G^R(1,2) = < \Psi(1)\Psi^\dagger(2) >$$
$$G^< (1,2) = <\Psi^\dagger(2)\Psi(1)>.$$ \textbf{(30)}

Here $<>$ is the average value with the unknown statistical nonequilibrium operator $\rho$ and $1$ denotes the cumulative variables $(r_1,s_1,t_1,...)$ of space, spin, time etc. The equation of motion for the correlation functions are given in the form of the Kadanoff-Baym equation \textbf{(24)} [22,23]

$$-i(G_0^{-1}G^< - G^< G_0^{-1}) = i(G^R \Sigma^< - \Sigma^< G^A)$$
$$-i(\Sigma^R G^< - G^< \Sigma^A)\] \text{(31)}$$

where the retarded and advanced functions are introduced as $A^R(1,2) = -i\Theta(t_1-t_2)[A^> \pm A^<]$ and $A^A(1,2) = i\Theta(t_2-t_1)[A^> \pm A^<]$. Here operator notation is employed where products are understood as integrations over intermediate variables (time and space) and the upper/lower sign stands for Fermions/Bosons respectively. The Hartree-Fock drift term reads

$$G_0^{-1}(1;1') = \left(i\hbar\frac{\partial}{\partial t_1} + \frac{\hbar^2}{2m}\nabla_1^2 - \Sigma_{HF}(1')\right)\delta(1-1')$$ \textbf{(32)}

with the Hartree-Fock self energy

$$\Sigma_{HF}(1,1') = \left(\mp\delta(r_1-r_1')\int d^3 r_2 V(r_1-r_2)G^< (r_2;r_1') r_1', t_1') + V(r_1-r_1')G^< (r_1,r_1') t_1'\right)\delta(t_1-t_1')\] \text{(33)}$$

where $G(r_2,t_1,r_2,t_1) = n(r_2,t_1)$ is the density.

#### B. Gauge invariance

In order to get an unambiguous way of constructing approximations we have to formulate our
theory in gauge invariant way. This can be done following a procedure known from field theory \[25\]. This method has been applied to high field problems in \[23\]. With the help of the Fourier transform of an arbitrary function \(G(x,X)\) over the relative coordinates \(x = (r_2 - r_1, t_2 - t_1) = (r, \tau)\) with the center of mass coordinates \(X = ((r_2 + r_1)/2, (t_2 + t_1)/2) = (R, t)\) one can introduce a generalized Fourier-transform of the difference coordinates \(x\)

\[
\tilde{G}(k, X) = \int dx G(x, X) \times \exp \left\{ \frac{i}{\hbar} \int d\lambda r_{\mu} \left[ k^\mu + \frac{e}{c} A^\mu \left( X + \lambda X\right) \right] \right\} . \tag{34}
\]

For constant electric fields, which will be of interest in the following, one obtains a generalized Fourier-transform

\[
\tilde{G}(k, X) = \int dx e^{i \frac{e}{c} \cdot (k - eE t)} G(x, X),
\]

where the \(\chi\) function was chosen in such a way that the scalar potential is zero \(A^\mu = (0, -eE t)\). Therefore, we have the following rule in formulating the kinetic theory gauge-invariantly

1. Fourier transformation of the 4-dimensional difference-variable \(x\) to canonical momentum \(p\).
2. Shifting the momentum to kinematic momentum according to \(p = k - eE t\).
3. The gauge invariant functions \(\tilde{G}\) are given by

\[
G(p, t) = G(k - eE t, t) = \tilde{G}(k, t)
= \tilde{G}(p + eE t, t). \tag{35}
\]

We shall make use of these rules in the following sections. In \[33\] this procedure has been generalized for two-particle Greens functions and leads to the field dependent Bethe-Salpeter equation.

**C. Equation for Wigner distribution**

In the relative and center of mass coordinates the time diagonal part of \(\{31\}\) reads \[12\]

\[
\begin{align*}
\frac{\partial}{\partial t} f(p, t) = & \left. \frac{\partial}{\partial t} \right|_{t=t_0} \\
& \int dt \left\{ G^>(p, t - \frac{\tau}{2}, -\tau), \Sigma^>(p, t - \frac{\tau}{2}, \tau) \right\} \\
& - \left\{ G^<(p, t - \frac{\tau}{2}, -\tau), \Sigma^>(p, t - \frac{\tau}{2}, \tau) \right\}
\end{align*}
\]

Here \(f(p, t) = G^<(p, R, t, \tau = 0)\) denotes the Wigner distribution function and we suppress the center of mass coordinates. \(\{,\}\) is the anti-commutator understood that the \(\tau\) variable at the first place comes with a minus sign respectively. This equation is exact in time, but according to the assumed slowly varying space dependence we have used gradient expansion for space variables and dropped all R-dependence for simplicity. This criterion is discussed in the last section \[26\]. With the help of the gauge invariant formulation of Green’s function \([31, 33]\), we can write the kinetic equation \[30\] finally in the following gauge-invariant form

\[
\frac{\partial}{\partial t} f(k, t) + eE \nabla_k f(k, t) = \int_0^{t-t_0} d\tau \left\{ \left[ G^>(k - \frac{eE}{2} \tau, \tau, t - \frac{\tau}{2}), \Sigma^>(k - \frac{eE}{2} \tau, -\tau, t - \frac{\tau}{2}) \right] + \\
- \left[ G^<(k - \frac{eE}{2} \tau, \tau, t - \frac{\tau}{2}), \Sigma^>(k - \frac{eE}{2} \tau, -\tau, t - \frac{\tau}{2}) \right] \right\} . \tag{37}
\]

This kinetic equation is exact in time convolutions. This is necessary because gradient expansions in time are connected with linearization in electric fields and consequently fail \[23\]. The gradient approximation in space has been applied assuming slow varying processes in space. This corresponds to the limit of a weakly coupled plasma, which we employed already in Section \[11\]. Please remind that due to Coulomb gauge we do not have space inhomogeneity by the electric field.

**D. Spectral function**

The spectral properties of the system are described by the Dyson equation for the retarded Green function. A free particle in a uniform electric field, where the field is represented by a vector potential \(E(t) = -\frac{1}{c} A(t)\) leads to the following equation

\[
\left[ i\hbar \frac{\partial}{\partial t} - e(p - \frac{e}{c} A(t)) \right] G_0^R(p, t\prime) = \delta(t - t\prime). \tag{38}
\]

This equation is easily integrated \[33, 34\]

\[
G_0^R(p, t\prime) = -i\Theta(t - t\prime) \exp \left[ \frac{ie}{\hbar} \int_{t_0}^{t} du \epsilon(p - \frac{e}{c} A(u)) \right] . \tag{39}
\]
For free particles and parabolic dispersions, the gauge invariant spectral function [33,34] follows

\[ A_0(k, \omega) = \frac{2}{\epsilon_E} \text{Ai}\left(\frac{k^2/2m - \hbar\omega}{\epsilon_E}\right) \]

where \( \text{Ai}(x) \) is the Airy function [35] and \( \epsilon_E = (\hbar^2 c^2 E^2/8m)^{1/3} \). It is instructive to verify that (40) satisfies the frequency sum rule \( \int d\omega A_0(\omega) = 2\pi \). The interaction-free but field-dependent retarded Green’s function \( G^R_0 \) can be obtained from the interaction-free and field-free Green’s function by a simple Airy transformation [36]. This is an expression of the fact that the solutions of the Schrödinger equation with constant electric field are Airy-functions. The retarded functions can therefore be diagonalized within those eigen-solutions [33,29]. It can be shown that (40) remains valid even within a quasiparticle picture [36], where we have to replace simply the free dispersion \( k^2/2m \) by the quasiparticle energy \( \epsilon_k \).

E. The Problem of the ansatz

In order to close the kinetic equation (36), it is necessary to know the relation between \( G^> \) and \( G^< \). This problem is known as an ansatz and must be constructed consistently with the required approximation of self-energy. Assuming the conventional KB ansatz [26] we have a relation between the two time Green functions and the distribution function

\[ G^< (k, \omega, r, t) = A(k, \omega, r, t) f(k, r, t) \]

\[ G^> (k, \omega, r, t) = A(k, \omega, r, t) (1 \mp f(k, r, t)) \]  \hspace{1cm} (41)

This is quite good as long as the quasi-particle picture holds and no memory effects play any role. As we shall see, the formulation of kinetic equations with high fields is basically connected with a careful formulation of retardation times. Therefore, the simple ansatz, called KB ansatz fails.

Another obscure discrepancy is the fact that with the old ansatz, one has some minor differences in the resulting collision integrals compared with the results from the density operator technique. With the old ansatz, one gets just one half of all retardation times in the various time arguments [11,33]. This annoying discrepancy remained obscure until the work of Lipavsky, et al. [38] where an expression is given for the \( G^< \) function in terms of expansion after various times. We can write in Wigner coordinates

\[ G^< (p, t, \tau) = f(p, t - \frac{|\tau|}{2}) A(p, \tau, t). \quad (42) \]

This generalized-Kadanoff-Baym (GKB) - ansatz is an exact relation as long as the selfenergy is taken in Hartree-Fock approximation. Together with the requirement of gauge invariance of Sec. 11, and using the quasiparticle spectral function [40] with quasiparticle energies \( \epsilon_k \) instead of \( k^2/2m \), the GKB ansatz finally reads

\[ G^< (k, \tau, r, t) = \exp \left\{ -\frac{i}{\hbar} \left( \epsilon_k \tau + \frac{e^2 E^2}{24m} \tau^3 \right) \right\} \times f(k - \frac{eE|\tau|}{2}, r, t - \frac{|\tau|}{2}). \quad (43) \]

In order to get more physical insight into this ansatz one transforms into the frequency representation

\[ G^< (k, \omega, r, t) = 2 \int_0^\infty d\tau f(k - \frac{eE\tau}{2}, t - \frac{\tau}{2}) \cos \left( \omega \tau - \epsilon(k, r, t) \frac{\tau}{2} - \frac{e^2 E^2}{24m} \tau^3 \right) \]  \hspace{1cm} (44)

Neglecting the retardation in \( f \) one recovers the ordinary ansatz (11) with the spectral function (10). The generalized ansatz takes into account history by an additional memory. This ansatz is superior to the Kadanoff-Baym ansatz in the case of high external fields in several respects [39]: (i) it has the correct spectral properties, (ii) it is gauge invariant, (iii) it preserves causality, (iv) the quantum kinetic equations derived with Eq. (42) coincide with those obtained with the density matrix technique [11,13], and (v) it reproduces the Debye-Onsager relaxation effect [10].

Other choices of ansatz can be appropriate for other physical situations. For a more detailed discussion see [42].

F. Kinetic equation in dynamically screened approximation

For Coulomb interaction it is unavoidable to consider screening if one does not want to obtain long range or short wave vector divergences. To obtain an explicit form for the kinetic equation we have to determine the selfenergy \( \Sigma^>^< \). The dynamically screened approximation is given by expressing the self energy by a sum of all ring diagrams. The resulting kinetic equation is the quantum Lenard - Balescu equation, which has been derived for high fields in [36]. We give this approximation in exact time convolutions. The selfenergy is given in terms of the dynamical potential \( V \)
\[
\Sigma^<_a(k, t, t') = \int \frac{dq}{(2\pi\hbar)^3} V^<_a(q, t, t') G^<_a(k - q, t, t')
\]
(45)

where the dynamical potential is expressed within Coulomb potentials \( V_{ab}(q) \)
\[
V^<_a(q, t, t') = \sum_{dc} V_{ad}(q) L^<_c(q, t, t') V_{ca}(q)
\]
(46)

via the density-density fluctuation
\[
L^<_c(q, t, t') = \delta_{ab} \int df dt \times (\mathcal{E}^r)^{-1}(q, t, \tilde{t}) L^{<\alpha}(q, \tilde{t}, \tilde{t}) (\mathcal{E}^a)^{-1}(q, \tilde{t}, t').
\]
(47)

Here \( L \) is the free density fluctuation
\[
L^<_c(q, t, t') = \int \frac{dp}{(2\pi\hbar)^3} G^<_c(p, t, t') G^>_a(p - q, t', t)
\]
(48)

and \( \mathcal{E}^r/a \) the retarded/advanced dielectric function
\[
\mathcal{E}^r/a(q, t, t') = \delta(t + t') \pm i\Theta[\pm(t - t')] \sum_b V_{bc}(q)
\]
\[
\times (L^>(q, t, t') - L^<_c(q, t, t')).
\]
(49)

One easily convince oneself that this set of equations (45)[49] is gauge invariant.

We can directly introduce this set of equations into the equation for the Wigner function (52) and obtain after some algebra for the in-scattering part of the collision integral
\[
I^{in}_a(k, t) = 2 \sum_q \int \frac{dq}{(2\pi\hbar)^3} V^<_a(q) 0 \int d\omega \frac{2\pi}{2\pi}
\]
\[
\times \cos \left( \epsilon^a_{k'-q} - \epsilon^a_k - \omega \right) \tau + \frac{e_aE\tau^2}{2m_a} \right]
\]
\[
\times f_a(k - q, \omega - e_aE\tau, t - \tau) (1 - f_a(k, \omega, t - \tau))
\]
\[
\times L^<_b(q, \omega, t - \frac{1}{2}\tau) \left| \mathcal{E}(q, \omega, t - \frac{1}{2}\tau) \right|^2
\]
(50)

with the free density fluctuation (48)
\[
L^<_b(q, \omega, t) = -2 \int \frac{dp}{(2\pi\hbar)^3} \int_0^\infty d\tau
\]
\[
\times \cos \left( \omega - e_b^p + e_b^p + \frac{e_b^p E\tau^2}{2m_b} \right)
\]
\[
\times f_b(p + q, t - \frac{1}{2}\tau) (1 - f_b(p, t - \frac{1}{2}\tau)).
\]
(51)

The out-scattering term \( I^{out}_a \) is given by \( f \leftrightarrow 1 - f \). Here we used the ansatz (43) and have employed the approximation \( t \pm \frac{1}{2}\tau \approx t \) in the density fluctuation (47) which corresponds to a gradient approximation in times for the density fluctuations. Since the center of mass time dependence is carried only by the distribution functions in (47), this approximation is exact in the quasistationary case which we investigate in the next section. All internal time integrations remain exact. Of course, for time dependent phenomena we have to question this approximation.

Eq. (44) represents the field dependent Lenard-Balescu kinetic equation (58) which was here slightly rewritten and which form will turn out to be very convenient for the later analytical integration. Other standard approximations like the T-matrix (50) approximation resulting into a field dependent Bethe-Salpeter equation can be given.

1. Kinetic equation in statically screened approximation

Using the static approximation for the dielectric function \( \mathcal{E}(q, 0, t) \) in (51), the kinetic equation for statically screened Coulomb potentials in high electric fields appears [10][31][36]
\[
\frac{\partial}{\partial t} f_a + eE \frac{\partial}{\partial k_a} f_a = \sum_b I_{ab}
\]
\[
I_{ab} = \frac{2(2s_b + 1)}{\hbar^2} \int \frac{dk'_a d\bar{k}_b dk'_b}{(2\pi\hbar)^6} \delta(k_a + k_b - k'_a - k'_b)
\]
\[
\times \{ f_{a'} f_{b'} (1 - f_a)(1 - f_{b'}) - f_a f_b (1 - f_{a'})(1 - f_{b'}) \}
\]
\[
\times V^2_b(k_a - k'_a) \int_0^\infty d\tau \cos \left( \epsilon_a + \epsilon_b - \epsilon_{a'} - \epsilon_{b'} \frac{\tau}{\hbar} \right)
\]
\[
- \frac{E\tau^2}{2\hbar} \left( e_a k_a + e_b k_b - e_{a'} k_{a'} - e_{b'} k_{b'} \right)
\]
(52)

with \( f_b = f_b(k_b - e_bE\tau, T - \tau) \). The potential is the static Debye one
\[
V_s(p) = \frac{4\pi e_a e_b \hbar^2}{p^2 + \hbar^2 k^2}
\]
(53)

and the static screening length \( \kappa \) is given by
\[
\kappa^2 = \sum_c \frac{4\pi e_c^2 n_c}{T_c}
\]
(54)
in the equilibrium and nondegenerated limit. Here \( T_c \) is the temperature of specie \( c \), charge \( e_c \), spin \( s_c \) and mass \( m_c \) respectively.

If we had used the conventional Kadanoff and Baym ansatz (41) we would have obtained a factor
1/2 in different retardations [31]. This would lead to no relaxation effect at all [14]. Furthermore it is assumed, that no charge or mass transfer will occur during the collision. Otherwise one would obtain an additional term in the cos-function proportional to $\tau^3$.

Two modifications of the usual Boltzmann collision integral can be deduced from (52): (i) A broadening of the $\delta$-distribution function of the energy conservation and an additional retardation in the center-of-mass times of the distribution functions. This is known as collisional broadening and is a result of the deformation integral can be deduced from (52): ($\delta E$)

\[
\frac{\partial}{\partial t} < k_n > - n_a e_q E = \sum_b < k_n \Gamma_{bf}^b > .
\]

Here we search for the relaxation field (3) which will be represented as renormalization of the external field $E$ similar to the Debye-Onsager Relaxation field in the theory of electrolyte transport [14] [16]. This effect is a result of the deformation of the two-particle correlation function by an applied electric field.

To proceed we assume some important restrictions on the distribution functions. First, we assume a nondegenerate situation, such that the Pauli blocking effects can be neglected. Second, to calculate the current for a quasistationary plasma we choose Maxwellian distributions analog to (11)

\[
f_i(p) = \frac{n_i}{2\nu_i + 1} \lambda_i^3 \exp \left\{-\frac{p^2}{2\nu_i T_i}\right\}
\]

with the thermal wave length $\lambda_i^2 = 2\pi \hbar^2/(m_i T_i)$, the spin $s_i$ and the partial temperature $T_i$ for species $i$ which can be quite different e.g. in a two-component system.

A. Statically screened result

Before we present the result for the dynamically screened approximation we want to give the static result. The momentum conservation in (52) can be carried out and we get for the relaxation field

\[
n_a e_q \frac{\delta E}{E} = - \sum_b (2s_a + 1)(2s_b + 1)
\]

\[
\times \frac{2}{\hbar^3} \int \frac{dq dq Q dq}{(2\pi \hbar)^9} V^2(q) \int d\tau (k + e_q E \tau)
\]

\[
\times \cos \left[\left(\frac{q^2}{2m_a \hbar} + \frac{q}{m_\pi \hbar} + \frac{q Q}{m_b \hbar}\right) \tau - \frac{e_b}{2\hbar} \left(\frac{e_b}{m_b} - \frac{e_a}{m_a}\right) \tau^2\right]
\]

\[
\times \left\{f_a(k)f_b\left(Q + \frac{q}{2}\right) - f_a(k + q)f_b\left(Q - \frac{q}{2}\right)\right\}
\]

(57)

where we have shifted the retardation into the distribution functions. The second part of the distribution functions can be transformed into the first one by putting $k + q \rightarrow k$ and $q \rightarrow -q$ with the result

\[
n_a e_q \frac{\delta E}{E} = \sum_b \frac{2s_a s_b}{(2\pi \hbar)^9} \int dq dq Q dq f_b(Q) f_a(k)
\]

\[
\times \int dq V^2(q) \int d\tau \cos \left[\left(\frac{q^2}{2m_a \hbar} - \frac{q}{m_\pi \hbar} + \frac{q Q}{m_b \hbar}\right) \tau
\]

\[
+ \frac{e_b}{2\hbar} \left(\frac{e_b}{m_b} - \frac{e_a}{m_a}\right) \tau^2\right]\]

(58)

with the reduced mass $\mu^{-1} = 1/m_a + 1/m_b$. The angular integrations can be carried out trivially and we get

\[
n_a e_q \frac{\delta E}{E} = \frac{E}{E} \sum_b I_1
\]

\[
I_1 = \frac{1}{\hbar^2 \pi \sigma^6} \int dq V^2(q) \int d\tau \sin\left(\frac{Q}{2\hbar} \left(\frac{e_b}{m_b} - \frac{e_a}{m_a}\right) \tau^2\right)
\]

\[
\times \sin\left(\frac{q^2}{2\hbar} \tau\right)I_2[a] I_2[b]
\]

(59)

with $\sin(x) = (x \cos x - \sin x)/x^2$. The two integrals over the distribution functions $I_2$ can be done with the result

\[
I_2[a] = \frac{\hbar m_a (2s_a + 1)}{q \tau} \int dk k f_a(k) \sin\left(\frac{k \tau}{m_a \hbar}\right)
\]

\[
= 2\hbar^2 n_a \pi^2 \frac{1}{m_a} \frac{-\frac{q^2}{2\nu_i T_i}}{\hbar^3}
\]

(60)

and correspondingly $I_2[b]$. We now introduce the new variables
\begin{align*}
q &= 2y \sqrt{\frac{\mu T_{ab}}{\hbar}} \\
t &= \frac{2T_{ab}^2}{\hbar} \\
T_{ab} &= \frac{1}{2} \left( \frac{m_b}{m_a + m_b} T_a + \frac{m_a}{m_a + m_b} T_b \right) \\
e &= \frac{\hbar \sqrt{\mu E}}{4T_{ab}^{3/2}} \left( \frac{e_b}{m_b} - \frac{e_a}{m_a} \right) \\
(61)
\end{align*}

and obtain

\begin{align*}
I_1 &= \frac{8n_n n_b e_q^2}{\pi^2 \hbar^4} T_{ab} \int_0^\infty dy y^2 V^2(2y \sqrt{\mu T_{ab}}) \\
&\quad \int_0^\infty dt js(yt^2) \sin(yt^2)e^{-yt^2}.
\end{align*}

(62)

Using the screened Debye potential (53) we finally obtain

\begin{align*}
I_1 &= \frac{8n_n n_b e_q^2}{\pi^2 \hbar^4} T_{ab} I_3 \\
I_3 &= \int_0^\infty dz \frac{z^3}{(z^2 + 1)^2} \int_0^\infty dt js(zt^2) \sin(zt^2) e^{-zt^2}.
\end{align*}

(63)

Therein we used \( y = z\zeta \) and \( l = t\zeta \) with the quantum parameter

\[ \zeta^2 = \frac{\hbar^2 \kappa^2}{4\mu T_{ab}} \]

(64)

and the classical field parameter

\[ x = \frac{e}{\zeta} = \frac{E}{2T_{ab}\kappa} \left( \frac{m_a}{m_a + m_b} e_b - \frac{m_b}{m_a + m_b} e_a \right). \]

(65)

With this form (63) we have given an extremely useful representation because the field effects, contained in \( x \), are separated from the quantum effects, which are contained in \( \zeta \). The integral in (63) can be performed analytically in the classical limit \( \zeta \to 0 \). For the more general quantum case with arbitrary \( \zeta \) the linear and cubic field effect can be given analytically and are discussed in (17).

We will not discuss them here.

Performing the classical limit \( \zeta \to 0 \) one obtains from (63) that (18, 17)

\begin{align*}
I_{3c} &= -\frac{\pi x}{24} F(|x|) \\
F(x) &= -\frac{3}{x^2} \left[ 3 - x + \frac{1}{1 + x} - \frac{4}{x} \ln(1 + x) \right].
\end{align*}

(66)

Introducing the classical result (63) into (3) we find from (52) and (53) the following relaxation field

\[ \frac{\partial}{\partial t} |k_a| = -n_a e_a E \left( 1 + \frac{\delta E_a}{E} \right) = n_a e_a J R(E) \]

(67)

with

\[ \frac{\delta E_a}{E} = -\frac{e_a \pi}{6\kappa} \sum_b 4\nu_b e_b^2 \frac{e_a}{m_a} - \frac{e_a}{m_a} \frac{T_a}{T_b + T_m} \]

\[ F(|x|) \]

(68)

and \( x \) from (63). We see that for a plasma consisting of particles with equal charge to mass ratios, no relaxation field appears. The link to the known Debye- Onsager relaxation effect can be found if we assume that we have a plasma consisting of electrons \( (m_e = e = e) \) and ions with charge \( e_i = eZ \) and temperatures \( T_e = T_i = T \). Then (68) reduces to

\[ \frac{\delta E_a}{E} = -\frac{e e_c^2}{6T} \left( \frac{1}{Z(1 + Z)(1 + \frac{m_e}{m_i})} \right) \]

\[ F(|x|) \]

\[ \frac{1}{Z} + o(E) \]

\[ \frac{1}{2} + o(1/E) \]

\[ \frac{3T}{2\pi E} + o(1/E^2) \]

(69)

This formula together with the general form (68) is the main result of this chapter. It gives the classical relaxation effect for statically screened approximation up to any field strength and represents a result beyond linear response. We see that in the case of single charged heavy ions the Debye result (10) is underestimated by a factor of two.

**B. Dynamically screened result**

The calculation of the current with the collision integral for dynamically screened potentials (4) can be performed analytically as well. For the quasistationary condition we can calculate the frequency integral in (47) analytically using the identity (43) for the classical limit \( o(h) \)

\[ \int \frac{d\omega}{2\pi} \frac{H(\omega)}{\omega} \text{Im}E^{-1}(q, \omega) = \frac{H(0)}{2} \text{Re} \left( 1 - \frac{1}{E(q, 0)} \right) \]

(70)

where we set \( H(\omega) = \omega / \text{Im}E \) and which relation is proven in appendix A. We will employ only classical screening. The quantum result for screening is more involved and not yet analytically integrable.

Observing that for the dielectric function (4) together with (58) holds
\[
\lim_{\omega \to 0} \frac{\omega}{\Im \mathcal{E}(q, \omega)} = \frac{q^3}{\sqrt{\pi} T^3} \left( \sum_b \frac{\kappa_b^2}{v_b} \right)^{-1}
\]  

(71)

with the partial screening length \( \kappa_b^2 = 4\pi\epsilon_b^2 n_b/T_b \) and the partial thermal velocity \( v_b^2 = 2T_b/m_b \), we obtain for the current (64) after similar integrations as in chapter IV_A instead of (63) and the linear relaxation field (68) takes the form

\[
\mathcal{I}^{\text{dyn}}_1 = \frac{8\kappa^2 e^2 n_e n_b \sqrt{m_a m_b}}{\sqrt{\pi} \mu T_a T_b} \mathcal{I}^{\text{dyn}}_3
\]

\[
\mathcal{I}^{\text{dyn}}_3 = \int_0^\infty dz z^2 \int_1^{-1} dx x \int_0^\infty dl e^{-z^2(l^2+i^2)}
\]

\[
\times \frac{1}{\kappa} \cos[M_a \zeta l_2 + B z l^2 x] \cos[M_a \zeta l_1 z^2 - A z l^2 x].
\]  

(72)

Here we used the same dimensionless variables as in chapter IV_A and the quantum parameter (64).

Further we abbreviated \( A = \frac{e \rho_E}{\kappa T_a}, \ B = \frac{e \rho_E}{\kappa T_b}, \ M_a = \sqrt{\frac{2\mu T_a}{m_a T_a}}, \ M_b = \sqrt{\frac{2\mu T_b}{m_b T_b}} \).

We like to remark that we neglect any field dependence on the screening \( \mathcal{E} \) itself here. As presented in [38] a field dependent screening function can be derived. However, this field dependence gives rise to a field dependence starting quadratically and will be not considered in this work.

The classical limit of (72) can be performed again by \( \zeta \to 0 \). We obtain

\[
\mathcal{I}^{\text{dyn}}_3 = \frac{1}{2} A M_a I[|A|, |B|] - (a \leftrightarrow b)
\]  

(73)

with the remaining 3-dimensional integral

\[
I[A, B] = \int_0^\infty dz z^3 \int_1^{-1} dx x^2 A^2 x^2 + z^2
\]

\[
\int_0^\infty dl e^{-z^2 l^2} \cos(B z l^2). 
\]  

(74)

1. Linear response

The linear response can be read off directly from (73) and is given by \( I[0, 0] \) of (74). We obtain

\[
\mathcal{I}^{\text{dyn}}_3 = \frac{\pi^{3/2}}{12} (M_a A - M_b B)
\]  

(75)

and the linear relaxation field (68) takes the form

\[
\frac{\delta \mathcal{E}^{\text{dyn}}}{E} = \frac{4e \pi \kappa}{3} \sum_c \frac{\epsilon_c^2}{\sqrt{m_c T_c}} \sum_b n_b e_b^2 \sqrt{\frac{m_a m_b}{T_a T_b}}
\]

\[
\times \left( \frac{e_a}{T_a^{3/2} \sqrt{m_a}} I[A, B] - \frac{e_b}{T_b^{3/2} \sqrt{m_b}} I[B, A] \right). 
\]  

(76)

The difference to (64) becomes more evident if we consider again only electrons and ions with equal temperature

\[
\frac{\delta \mathcal{E}^{\text{dyn}}}{E} = -\frac{\kappa e^2}{6e_0 T} 2Z I[A, B] + \sqrt{\frac{m_a}{m_b}} Z I[B, A]. 
\]  

(77)

The differences to (64) are obvious in the different mass dependence. This result overestimates the Debye result by a factor of two.

2. Complete classical result

Now we are able to present a complete field dependence beyond linear response. The integral (74) can be done analytically, which is sketched in appendix B. The result reads

\[
I[A, B] = \frac{\pi^{3/2}}{6} T[A, B]
\]

\[
I[A, B] = \frac{3}{2A^2} \left[ 4A(1-\sqrt{1+B}) + A^2 + \log(1-A^2) \right]
\]

\[
\times 2 \left( \text{ArcTanh}(\frac{1}{\sqrt{1+B}}) - \text{ArcTanh}(\frac{1}{\sqrt{1+A}}) \right)
\]

\[
- \text{ArcTanh}(\frac{1}{\sqrt{1+A}}) + \text{ArcTanh}(\frac{1}{\sqrt{1+B}}) \right]. 
\]  

(78)

We obtain for (73)

\[
\frac{\delta \mathcal{E}^{\text{dyn}}}{E} = \frac{4e \pi \kappa}{3} \sum_c \frac{\epsilon_c^2}{\sqrt{m_c T_c}} \sum_b n_b e_b^2 \sqrt{\frac{m_a m_b}{T_a T_b}}
\]

\[
\times \left( \frac{e_a}{T_a^{3/2} \sqrt{m_a}} I[A, B] - \frac{e_b}{T_b^{3/2} \sqrt{m_b}} I[B, A] \right). 
\]  

(79)

Expanding (78) in powers of \( E \) we recover (74). Once more we choose the case of electrons and ions with equal temperature and obtain

\[
\frac{\delta \mathcal{E}^{\text{dyn}}}{E} = -\frac{\kappa e^2}{6e_0 T} 2Z I[A, B] + \sqrt{\frac{m_a}{m_b}} Z I[B, A]. 
\]  

(80)
For single charged ions and big mass differences we can further simplify to

\[
\frac{\delta E^{\text{dyn}}}{E} = -\frac{\kappa^2 e^2}{6T} \mathcal{F}[\kappa T E]\n\]

\[
= -\frac{e^2 \kappa_e}{6T} \begin{cases} 
2 + o(E) \\
\frac{3 \sqrt{\kappa T \omega x^2}}{\sqrt{\kappa T}} + o(1/E)^2 
\end{cases}
\]

\[
\mathcal{F}[x] = \frac{3}{x^3} \left( 2 \left( -2x + 3 \left( -1 + \sqrt{1 + x} \right) \right) \right)
\]

\[
+ \frac{\sqrt{2}}{2} \left( -\text{ArcTanh}\left( \frac{1}{\sqrt{2}} \right) + \text{ArcTanh}\left( \frac{\sqrt{1 + x}}{\sqrt{2}} \right) \right)
\]

\[
+ \frac{x^2 + \log(1 - x^2)}{\sqrt{2}} \right) = 2 + o(E).
\]

This result will be compare with the statically screened result \((\mathbf{85})\) and the hydrodynamical result \((\mathbf{28})\) in section IV C. Here we remark already that the Debye result is twice overestimated here.

C. Thermally averaged dynamically screened result

We will now give an approximative treatment of the dynamical screening used in \((\mathbf{83})\). This approximation consists into the replacement of the dynamical screening in the collision integral \((\mathbf{50})\) approximation consists into the replacement of the dynamical screening used in \((\mathbf{8})\). This approximation leads to the Onsager result \((\mathbf{2})\) for small field strength \(2 - \sqrt{2}\). The statically screened result \((\mathbf{58})\) or \((\mathbf{69})\) leads to half the Debye result \((\mathbf{1})\). The thermally averaged approximation of the dynamical screening \((\mathbf{82})\) leads to the Debye result while the full dynamically screened approximation \((\mathbf{81})\) leads to twice the Debye result. Also the hydrodynamical result \((\mathbf{28})\) without background leads to twice the Debye result.

D. Asymmetric dynamical screened result

We want now to proceed and ask under which assumptions the Onsager result \((\mathbf{2})\) might be reproduced. Following the results we saw from the hierarchy we have consequently to treat the electrons (specie a) and ions (all other species) asymmetrically. This we will perform in the same spirit as Onsager in that the ions have to be treated dynamically. This we will perform in the same spirit as Onsager in that the ions have to be treated dynamically (as before) but the electrons are screened statically.

This means we consider as the potential not the bare Coulomb one but a statically screened Debye potential for specie a. The ions (all other species) will then form the dynamical screening. In comparison with the chapter before we can perform all steps analogously except two modifications, eq. \((\mathbf{85})\) and \((\mathbf{88})\). First we observe that instead of \((\mathbf{71})\) we have now

\[
\lim_{\omega \to 0} \frac{\omega}{\text{Im} \mathcal{E}(q, \omega)} = q v_b (q^2 + \hbar^2 k_b^2) \frac{\sqrt{\pi} \hbar \kappa_b^2}{\sqrt{\pi} \hbar \kappa_b^2}
\]

(83)

which leads to a replacement of the sum.
\[
\sum_c \frac{\kappa_c^2}{v_c} \to \frac{\kappa_i^2}{v_i}
\]

in the for-factor of (72) and (79). This leads in the limit of big mass differences to a for-factor in (77) and (80) respectively

\[
\text{modification I : } \frac{Z}{1+Z}
\]

The second modification is that in (72) one has to replace

\[
\frac{z^2}{1+z^2} \to \frac{z^2}{1+z^2} \frac{z^2}{q+z^2} = \frac{q z^2}{q-1 q+z^2} - \frac{1}{q-1 1+z^2}
\]

with

\[
q = \frac{\kappa_e^2}{\kappa_i^2}.
\]

This shows that in the end results (74), Eq. (80) has to be changed

\[
\text{modification II : } \mathcal{I}[A, B] \to \frac{\sqrt{q}}{q-1} \mathcal{I}\left[\frac{A}{\sqrt{q}}, \frac{B}{\sqrt{q}}\right] - \frac{1}{q-1} \mathcal{I}[A, B].
\]

Particularly we obtain for the linear response result (77) where for electron-ion plasma

\[
\delta E_{\text{asy}} = \frac{\kappa_e^2}{2T} \frac{Z q}{\sqrt{q} + 1} + o(E)
\]

which agrees with (21) if we consider that the mobilities are very different \(b_i/b_e \to 0\) in (22).

The same result we obtain from the thermally averaged result (80) with there appears no such function as (83) and therefore the modification I of (85) does not apply but solely the modification II of (88). We therefore obtain (89) but

\[
F_{\text{asy}}(x) = 2F^\text{dyn}(x) - \sqrt{2} F^\text{dyn}(\sqrt{2} x)
\]

with \(F^\text{dyn}\) of (82). The linear response leads then exactly to the same result as from the dynamical screening (89), i.e. the Onsager result with the same charge dependence.

The fact that we reproduce the classical Onsager result with the same charge dependence can be considered as very satisfactory. The more since we have seen how many different considerations are possible. Please note that the special case \(Z = 1\) could lead occasionally to a seemingly agreement between different treatments. We think that the charge dependence incriminates different treatments.

![FIG. 2. The different asymmetric screened approximations for the relaxation effect versus field strength.](image)

The hydrodynamical result (24) is compared with the thermally averaged asymmetric screened result of (90) as well as the asymmetric screened one of (88).

In figure 2, we see that the asymmetrical screened result (80) with (88) approaches the hydrodynamical or Onsager result (2) rather well for small fields while it is too low at high fields. On the other hand the thermally averaged symmetrical screened result (89) agrees with the hydrodynamical approximation (24) in the low and high field limit. Why the hydrodynamical result cannot be reproduced completely within the kinetic theory remains still a puzzle. Probably the remaining difference is due to the neglect of the field effect on the screening itself (89).

V. RANGE OF APPLICABILITY

During the derivation of the quantum kinetic equations there has been assumed the gradient approximations which restricted the spatial gradients of the system. Here we want to discuss up to which field strength this assumption is justified.

The electric field is limited to values \(x < < 1\) for \(x\) from (65). This can be deduced from the expression for the dynamical screened result (81). The expression has a removable singularity at \(x = 1\).
Therefore we see a smooth curve. Nevertheless this is the field strength where something is happening. For equal masses and temperatures of plasma components this condition translates into

\[ E < \frac{\kappa T}{e}. \]  

(91)

We interpret the occurrence of such singular point that no thermal distributions are pertained in the system. Then we have to take into account non-thermal field dependent distributions which have been employed to study nonlinear conductivity.

The condition (91) allows for different physical interpretations. Within the picture of the screening cloud we can rewrite (91) into

\[ eE < \frac{v_{\text{th}}^2}{r_D}. \]  

(92)

This means that a particle moving on the radius of the screening cloud \( r_D = 1/\kappa \) with thermal velocity \( v_{\text{th}}^2 = T/m \) should not be pulled away by the acting field force. We can discuss this limit also via the energy density which can be reached in a system by the applied field. We can reformulate once more the condition (91) to find equivalently

\[ \frac{E^2}{4\pi} < nT. \]  

(93)

This means that we have essentially non-thermal effects to be expected if the energy density of the field becomes comparable with the thermal energy density.

The validity criterion (91) can now be used to check the weak space inhomogeneity which has been assumed during our calculation. Quasi-equilibrium in charged systems with external fields can only be assumed if the field current is accompanied by an equivalent diffusion current

\[ j_{\text{field}} = e\mu En = -j_{\text{diff}} = eD \frac{dn}{dx}. \]  

(94)

using the Einstein condition \( \mu = eD/T \) one gets

\[ eE = T \frac{1}{n} \frac{dn}{dx}. \]  

(95)

Combining this elementary consideration with our condition (91) we obtain a limitation for space gradients

\[ \frac{dn}{d(\kappa x)} < n \]  

(96)

where our treatment of field effects and local equilibrium is applicable.

VI. SUMMARY

The nonlinear relaxation field of a charged system under the influence of high electric fields is investigated. The local equilibrium or hydrodynamical approach starting from the classical BBGKY hierarchy is compared with the results from the quantum kinetic equations.

We come to the same conclusion considering the hydrodynamical approximation or the kinetic theory that a perfectly symmetric two component plasma will lead to a different relaxation effect than the case where we consider the moving charge asymmetrically from the screening surrounding. In the hydrodynamic approach this has been achieved by friction with a background, in the kinetic approach we have realized it due to asymmetric screening. Within this asymmetric treatment the limit to a one component plasma which would be to set the ion charge to \( Z = -1 \) leads to a non-vanishing finite quantity. Oppositely in the perfectly symmetrical treatment this limit vanishes in that the relaxation field vanishes as it should. The perfectly symmetrical treatment of species in the system leads to twice the Debye result different from the Onsager result in linear response.

Different approximations of the kinetic approaches are compared and discussed. We found that the symmetrical treatment of species as well as the asymmetrical treatment leads to the same corresponding results as the hydrodynamical approach for linear response. But for higher field strengths there appear minor differences which are probably due to the neglect of the field dependent screening itself. The thermally averaged approximation of screening has the advantage to agree for low and high fields with the hydrodynamical of local equilibrium approach.

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APPENDIX A: INTEGRALS OVER DIELECTRIC FUNCTIONS

Here we proof a very useful relation, which has been given in [44]. Therefore we consider the fol-
following integral including the dielectric function

\[ I = \int \frac{d\omega}{2\pi} \frac{H(\omega)}{\omega} \text{Im} e^{-\omega} (\omega) \]

\[ = \int \frac{d\omega}{4\pi i} \left( \frac{1}{\omega + i\eta} + \frac{1}{\omega - i\eta} \right) H(\omega)(f^- - f^+) \quad (A1) \]

where \( f^+ = 1 - 1/e \) and \( f^- = (f^+)^* \). In the following we will assume that the function \( H(\omega) \) is analytical and vanishes with \( \sim \omega^{-2} \) for large \( \omega \). Since \( f^\pm(\omega) \) has no poles in the lower/upper half plane we have the identity

\[ \int \frac{d\omega}{2\pi i} H(\omega) \frac{f^\pm(\omega)}{(\omega \pm i\eta)} = \mp f^\pm(0) H(0) \quad (A2) \]

and all other combinations of \( f^\pm \) with the denominator vanish. If we would use the quantum dielectric function \( \epsilon \) we would have to add the residue of the poles at the Matsubara frequencies. Because we calculate only with the classical dielectric function we can use \( (A2) \). With the help of the relation \( (A3) \) we compute easily for

\[ I = \frac{1}{2} H(0) \text{Re} \left( 1 - \frac{1}{\epsilon(0)} \right) \quad (A3) \]

which proves relation \( (70) \).

**APPENDIX B: AN INTEGRAL**

Here we calculate the integral \( (74) \)

\[ I[a, b] = \int_0^\infty dz \frac{z^3}{z^2 + 1} \int_{-1}^1 dx \frac{x^2}{a^2 x^2 + z^2} \]

\[ \times \int_0^\infty dt e^{-z^2 t^2} \cos(b z t^2). \quad (B1) \]

The variable substitutions \( l \rightarrow p \) by \( p = \sqrt{x} \), \( x \rightarrow z \) by \( z = x^2 \) and \( p \rightarrow e \) by \( p \sqrt{x} = l \) leads to

\[ I[a, b] = \int_0^\infty dx \int_0^\infty dy \frac{y^{5/2}}{y^2 x^2 + 1} \frac{x^3}{a^2 + y^2} \int_0^\infty dt e^{-y^2 t^2} \cos(b e t^2) \]

\[ = \int_0^\infty dy \frac{y^{1/2}}{a^2 + y^2} \left( 1 - \frac{\log(1 + y^2)}{y^2} \right) \int_0^\infty dt e^{-y^2 t^2} \cos(b e t^2) \quad (B2) \]

where the trivial \( x \)-integration has been carried out. The variable substitution \( e \rightarrow q \) by \( \sqrt{q} = q \) and \( y \rightarrow z \) by \( y = 1/z \) leads to

\[ I[a, b] = \frac{1}{a^2} \int_0^\infty dq e^{-q^2} \int_0^\infty dy \cos(b q^2) \frac{1 - z^2 \log(1 + \frac{1}{y^2})}{z^2 + \frac{1}{a^2}}. \quad (B3) \]

Now we proceed and use an integral calculated in the next subsection \( 3 \)

\[ \int_{-\infty}^\infty dy e^{i c y} \frac{1 - y^2 \log(1 + \frac{1}{y^2})}{y^2 + \frac{1}{a^2} - x^2} = 2\pi \int_0^1 dx x^2 \frac{e^{-c x}}{x^2} \]

\[ + a \pi e^{-c/a} \left( 1 + \frac{\log(1 - a^2)}{a^2} \right) \quad (B4) \]

to obtain for \( (B3) \)

\[ I[a, b] = \frac{\pi^{3/2}}{4a} \frac{1 + \log(1 - a^2)}{\sqrt{b/a}} \]

\[ + \frac{\pi^{3/2}}{2a^2} \int_0^1 dx \frac{x^2}{(\frac{1}{a^2} - x^2) \sqrt{1 + bx}}. \quad (B5) \]

The last integrals is trivial and we end up with \( (78) \).

1. **Another Integral**

Our task remains now to solve the integral

\[ I = \int_{-\infty}^\infty dy e^{i c y} \frac{1 - y^2 \log(1 + \frac{1}{y^2})}{y^2 + \frac{1}{a^2}}. \quad (B6) \]

Because the complex function \( \log(1 + 1/y^2) \) has a cut from \((0, i)\) we perform the integration along the path as depicted in figure \( 3 \) and write

\[ \int_{-r}^{-R} + \int_{-R}^{-r} + C_R + C_r + \int_{i l}^{i l} \int_{-l}^{l} \]

\[ = 2\pi i \text{Res} \left[ \frac{1 - y^2 \log(1 + \frac{1}{y^2})}{y^2 + \frac{1}{a^2}}, i/a \right] \]

\[ = \pi a e^{-c/a} \left( 1 + \frac{\log(1 - a^2)}{a^2} \right). \quad (B7) \]

It is now easy to proof that in the limit \( r \rightarrow 0 \) and \( R \rightarrow \infty \) the integration parts \( C \) vanish. Since the first two parts of \( (B7) \) represent just the desired integral \( I \) we have to calculate

\[ \int_{i l}^{i l} + \int_{i l}^{r} = \int_{i l}^{r} \int_{i l}^{r} dy e^{i c y} \frac{1 - y^2 \log(1 + \frac{1}{y^2})}{y^2 + \frac{1}{a^2}} \]

\[ \int_{i l}^{r} \int_{i l}^{r} dy e^{i c y} \frac{1 - y^2 \log(1 + \frac{1}{y^2})}{y^2 + \frac{1}{a^2}} \]
\[ 
\begin{align*}
+ \int_{-r}^{i-r} dy e^{i y} \left( 1 - y^2 \log(1 + \frac{1}{y^2}) + 2\pi i \right) + y^2 + \frac{1}{y^2} \\
= -2\pi i \int_{0}^{1} dx \frac{x^2 e^{-cx}}{\sigma^2 - x^2}.
\end{align*}
\]

(B8)

Using (B8) and (B7) we obtain just (B4).

FIG. 3. The complex integration path for the integral (B6).

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