Study of the superconducting to normal transition

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I. INTRODUCTION

Shortly after the discovery of superconductivity, it was realized that applying a growing magnetic field $H$ turns superconducting electrons into normal ones at a critical value $H = H_c$. Besides, this process has been characterized as a reversible first order transition, i.e. decreasing $H$ from $H_c$ down to 0 brings normal electrons back to the superconducting state. However, this experimental procedure suffers from several drawbacks, when scrutinized from a thermodynamical standpoint:

- because all experiments have been made so far1,10 at fixed temperature $T$, the heat exchanged during the transition remains unknown. Likewise, since nobody bothered to measure the work performed by $H$, the binding energy of the superconducting phase with respect to the normal one $E_b$ could not be assessed with help of the first law of thermodynamics. Meanwhile the formula $E_b = \mu_0 H_c^2/2$ was assumed11, with $\mu_0$ being the magnetic permeability of vacuum, and became eventually ubiquitous in textbooks1–4. Unfortunately, a numerical application in case of Al turns out to underestimate12 by ten orders of magnitude the $E_b$ value, deduced from the BCS theory13.

- due to the Meissner effect12 and the finite ac conductivity3 in the superconducting state, the current density is spatially inhomogeneous14,15 and there is no one-to-one correspondence between the external magnetic field and the current distribution inside the sample, so that qualitative information only can be achieved from $H$ mediated experiments16,17. At last, letting high $T_c$ compounds go normal requires a huge, often unpractical magnetic field12,18,19.

Consequently, despite countless published $H_c$ data1–4,17,18, there is still no theory of the superconducting to normal transition, apart from the phenomenological15 approach, based on the grossly wrong assumption $E_b = \mu_0 H_c^2/2$. Thus the purpose of the present article is to design one, valid for both high and low $T_c$ superconductors, as well. Accordingly, since it has been argued recently20 that feeding a growing current into a superconductor drives continuously the superconducting phase to the normal one, this article will focus on a theoretical account of the current-driven, superconducting to normal transition. Such an experimental procedure enables one to dodge all of the shortcomings mentioned above, in particular because reliable data for the critical current are available in all superconductors, including those for which $H_c$ is so large that it cannot be reached experimentally. Furthermore it allows for a quantitative treatment, unlike the $H$ mediated procedure. At last, since the current, carried by the superconducting electrons, plays a paramount role hereafter, it is worth mentioning an original view19, which establishes the common significance of the persistent current20 and Josephson effect21.

The purpose of this work is then twofold:

- this transition will be studied with help of Newton’s law and thermodynamics;
- the resulting findings will be taken advantage of to shed light into the transport properties of high $T_c$ compounds in the controversial22–24 $T > T_c$ range and the magneto-elastic behaviour, observed in elementary superconductors24,25 for $T \leq T_c$.

The outline is as follows: the electrodynamical and thermodynamical properties of the superconducting to normal transition are worked out in sections 2, 3, respectively; a new derivation of the BCS variational procedure is given in section 4, which enables us to define two critical temperatures and to reckon the $T$ dependent critical current; this analysis is further applied to investigate the transport properties of high-$T_c$ compounds for $T > T_c$, in
and flown through, along the
of conduction electrons. Electro-
volume
Helmholz free energy of independent electrons per unit
section 5; magneto-elastic data
process, addressed in sections 2
set at the bottom of the conduction band; the tiny differ-
the reader’s convenience; the crossing point
the adiabatic process, discussed in section 5
Consider then a superconducting material of cylindri-
II. ELECTRODYNAMICAL DISCUSSION
As done previously, our analysis will proceed
the normal and superconducting electrons, in concentration \( c_n, c_s \), respectively. The normal electrons behave like a Fermi
gas, characterised by \( T \) and the Fermi energy \( E_F \). The
Helmholz free energy of independent electrons per unit volume \( F_n \) and \( E_F \) are related by \( E_F = \frac{\partial F_n}{\partial c_s} \). By contrast, the superconducting electrons are organised as a many bound electron state of eigenenergy per unit volume \( E_s(c_s) \), such that its chemical potential reads
\( \mu = \frac{\partial E_s}{\partial c_s} \). Gibbs and Duhem’s laws entails that the thermal equilibrium is characterised by
\[
E_F(T, c_n(T)) = \mu(c_s(T)) , \tag{1}
\]
with \( c_0 = c_n(T) + c_s(T) \) and \( c_0 \) being the concentration of conduction electrons.
Consider then a superconducting material of cylindrical shape, characterized by its symmetry axis \( z \) and radius \( r_0 \) in a cylindrical frame with coordinates \( (r, \theta, z) \) and flown through, along the \( z \) direction, by a time dependent current \( I(t) = \pi r_0^2 j(t) \), with \( j(t) \) being a uniform current density. The analysis of an isothermal, current-
F
and dashed lines, respectively; the origin \( E_F = \mu = 0 \) is
section 5; magneto-elastic data are discussed in section
the rest of section 1 below deals with a detailed, quantitative account of the isother-
process, outlined above and illustrated in Fig. 1.
Due to \( \gamma = \frac{\partial \mu}{\partial T} \neq 0 \), Newton’s law reads for the normal and superconducting current densities \( j_n(t), \)
\( j_s(t) \) (\( \Rightarrow j = j_n + j_s \); note that \( j_s \) is also referred to as a collective mode current)
\[
\tau_n \frac{dj_n}{dt} = \sigma_n E - j_n , \quad \tau_s \frac{dj_s}{dt} = \sigma_s (E - E_{s\rightarrow n}) - j_s . \tag{2}
\]
\( E \) and \( \tau_n, \tau_s \) are, respectively, the applied electric field and the decay times of \( j_n, j_s \), due to friction with the lattice, responsible for Ohm’s law, whereas \( \sigma_n = \frac{e^2 \nu}{8 \pi m}, \sigma_s = \frac{e^2 \nu}{8 \pi m} \) stand for the normal and superconducting conductivities \( \tau_n, \tau_s \) \( \Rightarrow \sigma_n << \sigma_s \) and \( m, e \) refer to the effective mass and charge of an electron. Moreover \( \tau_s \) being finite has been demonstrated elsewhere and shown furthermore to be consistent with observation of persistent currents at \( E = 0 \). The effective field \( E_{s\rightarrow n} \) is defined with respect to \( f_s\rightarrow n = c_s E_{s\rightarrow n} \), the interelectron force, which turns supercon-
ducting electrons into normal ones. The resulting \( f_s\rightarrow n \) is mediated by the interelectron coupling, also responsible for the binding energy of the superconducting electrons, i.e. \( E_b > 0 \). Actually \( E_{s\rightarrow n} \) was neglected in previous works \( \Rightarrow js \approx \sigma_s E \). But, as it will appear below that \( \left| \frac{\partial E_s}{\partial c_s} \right| << 1 \), such an approximation was fully vindicated.
During the elementary time-duration \( \delta t \), superconducting electrons in concentration \( \delta c_s \), moving at the mass center velocity \( v_s \) \( \Rightarrow vs = \frac{c_s}{c_s \epsilon} \), are driven normal at vanishing velocity by \( f_s\rightarrow n \), which corresponds to a momentum variation per unit volume of \( \dot{\rho} = -m \delta c_s v_s \). Thence \( f_s\rightarrow n \) is inferred from Newton’s law to read

\[
f_s\rightarrow n = \frac{\delta \rho}{\delta t} = -\frac{mc_s}{c_s \epsilon} j_s \Rightarrow E_{s\rightarrow n} = \frac{\dot{c}_s}{c_s \epsilon} j_s \tag{3}
\]
with \( \dot{c}_s = \frac{dc_s}{dt} \). Then combining Eqs. (2,3), while recalling that the inertial force \( \propto \frac{dc_s}{dt} \) is negligible, yields

\[
E = \frac{\dot{c}_s}{c_s \epsilon} = \frac{\dot{c}_s}{c_s} + E_{s\rightarrow n} = \frac{\dot{c}_s}{c_s \epsilon} = \frac{\dot{c}_s}{c_s \epsilon} \Rightarrow \sigma_s^* = \frac{\sigma_n}{1 - \tau_s} \tag{4}
\]
elsewhere, will be developed below with \( j(t) = \gamma t, \gamma > 0 \). Accordingly, the initial state of the whole electron system is defined as \( j(0) = 0, c_n = c_n(T), c_s = c_s(T) \) (see A in Fig. 1). As \( j(t) \) increases at constant \( T \), the electron system shifts away from the equilibrium position in \( A \) : the Fermi gas, represented by \( P_n \) in Fig. 1 moves, along the solid line, towards \( B \), corresponding to the normal state \( c_n = c_0 \), while the superconducting electrons, represented by \( P_s \), go, along the dashed line, towards the point characterized by \( c_s = 0 \) \( \Rightarrow c_s = \frac{\epsilon_s}{\epsilon} \) (\( \epsilon_s \) refers to the Cooper pair energy). As this process will be shown to be reversible, the pair \( P_n, P_s \) will shift back along the solid and dashed lines and will eventually merge into A, if \( j \) is brought back down to 0. The rest of section 1 below deals with a detailed, quantitative account of the isothermal process, outlined above and illustrated in Fig. 1.
Eq. (1) conveys the same meaning as Ohm’s law, written for \( j_n, j_s \) flowing parallel to each other, except for the effective conductivity \( \sigma_s^* \) showing up instead of \( \sigma_s \).

The elementary work \( \delta W \), needed for one superconducting electron, moving with velocity \( v_s \), to go normal with vanishing velocity, is reckoned to be equal to
\[
\delta W = \frac{mv_s^2}{2} = \frac{2}{2} \left( \frac{\sigma_s}{c_s} \right)^2,
\]
thanks to the kinetic energy theorem. On the other hand, for an isothermal process, \( \delta W \) is also equal to the difference of free energy \( \delta W \) between the superconducting and normal states, which leads thence to the identity
\[
\delta W = \frac{\partial F_n}{\partial c_n} - \frac{\partial F_s}{\partial c_s} = E_F(T, c_n) - \mu(c_s).
\]
Consequently, \( j_s \) reads finally
\[
j_s(c_s) = c_se\sqrt{\frac{2}{m} (E_F(T, c_0 - c_s) - \mu(c_s))} \quad (5)
\]

Note that, unlike the normal current \( j_n = \sigma_n E \), \( j_s \) is independent from the external field \( E \) and depends only on the concentration of bound electrons \( c_s \).

Eq. (1) can now be recast as an ordinary differential equation of first order for the unknown \( c_s(j) \)
\[
\gamma \frac{d\log c_s}{dj} = \frac{c_s}{c_n} \frac{(\beta c_s(T) - c_s(j) - 1)}{\tau_n} - \frac{1}{\tau_n},
\]
with \( c_n = c_0 - c_s \) and \( j_s \) given by Eq. (5).

For \( j \) increasing from 0, \( c_s \) decreases from \( c_s(T) \), while \( E_F - \mu \) increases from 0, proportionally to the length of the arrow linking \( P_0, P_s \) in Fig. 1. In addition, since \( j_s \) will eventually vanish for \( c_s \to 0 \), as inferred from Eq. (5), \( j_s \) is bound to rise from \( j_s = 0 \) at \( A \) up to a maximum \( j_s(c_m) \) at \( c_m(T) < c_s(T) \) defined by \( \frac{dc}{dc} (c_m) = 0 \). In order to solve Eq. (6), \( E_F - \mu \) will be replaced by its Taylor’s expansion at first order with respect to \( c_s - c_s(T) \)
\[
\frac{\sigma_s^*}{m} (E_F - \mu) = \beta (c_s(T) - c_s) \Rightarrow c_m = \frac{2}{3} c_s(T) \quad (7)
\]
with \( \beta = \frac{\sigma_s^*}{m} \left( \frac{\partial E_F}{\partial c_n}(c_n(T)) + \frac{\partial m}{\partial c}(c_n(T)) \right) \). Thus Eq. (6) has been integrated with \( c_0 = 10^{28}/m^3, c_s(T) = 1.1 c_0, \tau_s = 10^{-9} s, \tau_n = 10^{-4} s, \beta = 10^{-65} A^2/T^2 \) and initial condition \( c_s(j = 0) = c_s(T) \). The resulting data \( c_s(j), \sigma_s(j) \) (\( \sigma_s = \sigma_s + \sigma_s^* \) refers to the effective conductivity) have been plotted in Figs. 1 and 2 corresponding to \( j \leq j_m \) or \( j > j_m \), respectively, with \( j_m \) defined by \( j_m = j_s(c_m) \).

For \( j \leq j_m \), there is \( \tau_n \frac{\partial c_s}{\partial c_s} << 1 \), so that Eq. (4) yields
\[
\sigma_s^* \approx \sigma_s \quad \text{and Eq. (5) reduces to}
\]
\[
\frac{j_s(c_s)}{j} = 1 + \frac{\sigma_n}{\sigma_s} = 1 + \frac{\tau_n}{\tau_s} \left( \frac{c_0}{c_s} - 1 \right) \quad (8)
\]
Likewise, Eq. (5) implies \( j_s \approx \sigma_s E \Rightarrow \left| \frac{E_F}{\mu} \right| << 1 \), which confirms the validity of a previous assumption. As \( \gamma \) does not show up in Eq. (5), there is a one-to-one correspondence between \( j \) and \( c_s \), as seen in Fig. 2. Moreover,

\[ \tau_n << \tau_s \] entails that \( j_s \approx j \), so that the \( c_s(j), \sigma_s(j) \) plots cannot be distinguished from each other. Note that \( \frac{dc}{dj}(j = 0) = 0 \), while \( \frac{dc}{dj} \) becomes very large for \( j \to j_m \).

However, when \( j \) keeps growing beyond \( j_m, j_s \approx j \) is no longer valid because of \( j_s \leq j_m < j \). Consequently, as seen in Fig. 1, \( c_s(j), \sigma_s(j) \), obtained by integrating Eq. (6) for \( j > j_m \), falls steeply from \( c_s(j_m) = c_m \) down to 0, and \( \sigma_s \) sinks by the ratio \( \frac{\sigma_c(T)}{c_n(T)} = 10^3 \) from \( \sigma_s(T) \) down to \( \sigma_s(T_c) = \frac{\sigma_s(T)}{m} T_c \), typical of the normal metal. Meanwhile \( j \) undergoes a tiny increase from \( j_m \) up to \( J_M \), with \( j_m \) being weakly dependent, i.e., \( j_m/j_m - 1 \approx 10^{-7}, 10^{-8} \) for \( \gamma = 2 \times 10^9, 2 \times 10^7 A/(m^2 \times s) \), respectively (see Fig. 3). Finally, due to \( j_M \approx j_m = j_s(c_m) \), \( j_M(T < T_c) \) reads
\[
J_M = c_m(T) \left[ \sqrt{\frac{2}{m} (E_F(T, c_0 - c_m(T)) - \mu(c_m(T)))} \right] .
\]

Integrating Eq. (6) from \( j = j_m \) down to \( j = 0 \) with the initial condition \( c_s = c_s(j_m) \approx 0 \), while keeping \( \gamma \) unaltered, will produce the same solution \( c_s(j) \), as displayed in Figs. 2 and 3. This shows that the superconducting-normal transition is reversible and there is a one-to-one correspondence between \( j \) and \( c_s \), provided that \( \gamma \) keeps the same value for \( j \) increasing from 0 up to \( j_M \) or decreasing from \( j_m \) down to 0, as well. This property holds actually for any \( j(t) \), such that \( j(t) = j(t_p - t), \forall t \in [0, t_p / 2] \), with \( t_p \) taken such that \( j(t_p / 2) = J_M \).

Due to \( j_s \approx j \) for \( j < j_m \), measuring \( \sigma_s(j) \) and the \( j \) dependent London length, which gives access to \( c_s \), would enable one to chart \( E_F(T, c_n) - \mu(c_s) \) with help of Eq. (5). Given the highest observed \( J_M \) values, Eq. (5) provides the estimate \( E_F(T, c_n) - \mu(c_s) << 10^{-5} eV \). It is noticeable that the conductivity, decreasing by several orders of magnitude for \( j \to J_M \), as seen above, and for \( j \to J_M \), as discussed elsewhere [12], is to be ascribed, in
including all of the lattice and electron degrees of freedom. ΔE, ΔS will be calculated by working out the detailed thermal balance over the following trajectory: the sample is first taken at T < Tc and heated up to Tc with j = 0. Hence, the associated ΔE1, ΔS1 read

\[ \Delta E_1 = \int_{T_1}^{T_c} \left( C_p(u) + C_s(u) \right) du , \]
\[ \Delta S_1 = \int_{T_1}^{T_c} \left( C_p(u) + C_s(u) \right) \frac{du}{u} , \]

with C_p(T), C_s(T) standing for the respective contributions to the specific heat of the phonons (Debye) and of the conduction electrons in the superconducting state; then let the sample be cooled down back to T1 while being flown through by a current density j ≥ jM(T), so that the sample remains normal down to T. The associated ΔE2, ΔS2 read then

\[ \Delta E_2 = \int_{T_1}^{T} \left( C_p(u) + C_n(u) \right) du , \]
\[ \Delta S_2 = \int_{T_1}^{T} \left( C_p(u) + C_n(u) \right) \frac{du}{u} , \]

with C_n(T) standing for the T linear, specific heat of a Fermi gas, which is known to be independent from j, like C_p(T). At last, the searched expressions read

\[ E_b(T) = \Delta E_1 + \Delta E_2 = \int_{T_1}^{T_c} \left( C_p(u) - C_n(u) \right) du , \]
\[ W_{s-n}(T) = \Delta E_1 + T \Delta S_1 + \Delta E_2 + T \Delta S_2 \]
\[ = \int_{T_1}^{T_c} \left( C_p(u) - C_n(u) \right) \left( 1 - \frac{u}{\mu} \right) du , \]

with E_b(T) being the binding energy of the superconducting phase with respect to the normal one at T. Noteworthy is that the superconducting phase being stable (⇔ E_b(T) > 0) requires C_s(T) > C_n(T) in Eq. (12), which is confirmed experimentally, i.e., C_s(Tc) ≈ 3C_n(Tc).

W_{s-n} can actually be measured directly by feeding a growing current I(t) = πr^2γt into the superconducting sample, from t = 0 until t = tp with I(t) = πr^2jM(T), so that the sample goes normal at \( t_p \) (this is referred to as the Sihbee effect). Then I(t) is reduced, like I(t) = \( πr^2γ(t_p - t) \), from I(\( \frac{tp}{2} \)) down to I(tp) = 0. The work \( W(tp) \), performed by the electric field E from t = 0 until \( t = t_p \), reads then

\[ W(tp) = W_1 + W_2 \]
\[ W_1 = \int_{0}^{t_p} \frac{U(t)I(t)dt}{t_p} , \quad W_2 = \int_{0}^{t_p} U(t)I(t)dt \]

with U = El and l being the measured voltage drop across the sample and its length, respectively. Moreover, owing to Eq. (11), \( W_1, W_2 \) can be recast as

\[ \frac{W_{1}}{\pi r^2\gamma} = \int_{0}^{tp} \left( \frac{\beta^2}{\sigma_n} + \frac{\gamma^2}{\sigma_s} + j_s E_{n\rightarrow s} \right) dt , \]
\[ \frac{W_{2}}{\pi r^2\gamma} = \int_{0}^{tp} \left( \frac{\beta^2}{\sigma_n} + \frac{\gamma^2}{\sigma_s} + j_s E_{n\rightarrow s} \right) dt . \]

Likewise, recalling that \( j_n, \sigma_n, j_s, \sigma_s \) have been shown above to depend on j only, if \( \gamma \) is kept fixed, and furthermore

\[ W_{s-n} = \int_{0}^{tp} j_s E_{n\rightarrow s} dt , \quad W_{n\rightarrow s} = \int_{0}^{tp} j_s E_{n\rightarrow s} dt \]
\[ W_{s-n} = -W_{n\rightarrow s} \]
enables us to recast $W_1, W_2$ as
\[
\frac{W_1}{2\pi r_0^2} = Q_1 + W_{s \rightarrow n}, \quad \frac{W_2}{2\pi r_0^2} = Q_1 - W_{s \rightarrow n},
\]
with $Q_1$ expressing the Joule heat, released\(^{16}\) through process I per unit volume for $t \in \{0, t_p/2\}$. Finally it ensues from Eq. (15)
\[
W_1 - W_2 = \int^T_{T_0} (C_s(u) - C_n(u)) (1 - T/u) \, du.
\]
The validity of Eq. (16) should be checked experimentally first in a superconducting material, for which accurate data are available for $C_s, C_n$ and thence $T_c$ is low\(^{29}\) enough for $C_n > C_\phi, C_n > C_\phi$. Accordingly, Al ($T_c = 1.19 K$) might be a good candidate. In case of a successful test, Eq. (16) might then provide with a rather unique access to $C_s$ in high $T_c$ materials, for which the direct measurement of $C_s$ proves unreliable\(^{29}\) due to $C_s << C_\phi$. Note that $C_n$ can always be measured at low $T$ by feeding into the sample a current density $j > j_M(T)$, whereby the sample goes normal even at $T < T_c$, because $C_n$ is $j$ dependent, unlike $C_s$, and extrapolated further to higher $T$, by taking advantage of its $T$ linear behaviour\(^{29}\).

Although the superconducting to normal transition and ice melting into water are both first order processes, they differ in two respects:

- the role of the latent heat, typical of all usual first order transitions (melting or vaporisation), is played here by the latent work $W_{s \rightarrow n}$, because the superconducting-normal transition is controlled by current rather than by temperature;
- ice and water are separated by a clear-cut interface, whereas the mixture of superconducting and normal electrons is homogeneous. Consequently, the chemical potentials of ice and water remain uniquely defined all over the melting process, while the chemical potentials of ice and water remain

\[
\text{IV. CRITICAL CURRENT}
\]

As it will appear below that $\partial \mu/\partial c_s$ remains finite for $c_s \to 0$, the inequality (17) is bound not to hold any more for $c_s < c_c$ with the critical concentration $c_c$ defined by $r(c_c) = 1$.

To proceed further, $c_s$ must be assessed, which requires to reckon $\partial \mu/\partial c_s$. The only practical tool for this purpose is the BCS scheme\(^{13}\), but for some reason to become clear below, we shall refrain from using it, and rather develop our own procedure.

Thus let us consider a three-dimensional crystal containing $N$ sites and $2n$ itinerant electrons with $N > > 1, n > 1 \implies c_s = 2n/N$. These electrons of spin $\sigma = \pm 1/2$ populate a single band, accomodating at most two electrons per site $\implies n \leq N$. The independent electron motion is described, in reciprocal space, by the Hamiltonian $H_d$
\[
H_d = \sum_{k, \sigma} \varepsilon(k) c^\dagger_{k, \sigma} c_{k, \sigma},
\]
for which $\varepsilon(k)$, $k$ are the one-electron, spin-independent energy $\implies \varepsilon(k) = \varepsilon(-k)$ and a vector of the Brillouin zone, respectively, and the sum over $k$ is to be carried out over the whole Brillouin zone. Then $c^\dagger_{k, \sigma} c_{k, \sigma}$ are one-electron creation and annihilation operators on the Bloch state $|k, \sigma\rangle$
\[
|k, \sigma\rangle = c^\dagger_{k, \sigma} |0\rangle, \quad |0\rangle = c_{k, \sigma} |k, \sigma\rangle,
\]
with $|0\rangle$ being the no electron state. They enable us to introduce the two-electron creation and annihilation operators\(^{3, 13, 28}\)
\[
b^\dagger_{k, k'} c_{k', \sigma} c_{k, \sigma} = c^\dagger_{k, \sigma} c^\dagger_{k', -\sigma} c_{k', -\sigma} c_{k, \sigma},
\]
which operate on hard-core bosons and thence do not fulfill the boson commutation rules. The interacting electron motion is governed by a truncated Hubbard Hamiltonian $H_K$, used previously\(^{3, 13, 30}\),
\[
H_K = \sum_k \varepsilon(K, k) b^\dagger_{K, k} b_{K, k} + \frac{U}{N} \sum_{k, k'} b^\dagger_{K, k} b_{K, k'} b^\dagger_{K, k'} b_{K, k'},
\]
with $\varepsilon(K, k) = \varepsilon(k) + \varepsilon(K - k)$ and $U$ being the Hubbard coupling constant. Unlike previous authors\(^{3, 13, 28}\), we shall consider both cases $U > 0, U < 0$.

The eigenstate of the Schrödinger equation, pertaining to a single bound pair, $(H_K - \varepsilon_c(K)) |\varphi_c\rangle = 0$, is known as the Cooper pair\(^{28}\) state $|\varphi_c\rangle = \sum_k c_{k, \sigma} c^\dagger_{K, -\sigma} |K, K\rangle (0)$, with the eigenenergy $\varepsilon_c(K)$ being the solution of
\[
\frac{1}{U} = \frac{1}{N} \sum_k \varepsilon_c(K) - \varepsilon_c(K, k) = \int^{t_K}_{-t_K} \frac{\rho_K(\varepsilon)}{\varepsilon_c(K) - \varepsilon} \, d\varepsilon.
\]
(20)

$\pm t_K$ are the upper and lower bounds of the two-electron band, i.e. the maximum and minimum of $\varepsilon(K, k)$ over $k$, whereas $\rho_K(\varepsilon)$ is the corresponding two-electron density of states. For the sake of illustration, we shall solve Eq. (20) for $\rho_K(\varepsilon)$ as
\[
\rho_K(\varepsilon) = \frac{\pi}{4 t_K} \left( 1 - \frac{\varepsilon^2}{t_K^2} \right),
\]
where $t, a$ are the one-electron bandwidth and the lattice parameter, respectively. The dispersion curves $\varepsilon_c(K)$ are
\[
(\varepsilon_c(K))^2 = 4 |\varphi|^2 (1 - \frac{\varepsilon^2}{4 a^2}),
\]
for a weakly coupled pair state $\varphi = \sum_k c_{k, \sigma} c^\dagger_{K, -\sigma} |K, K\rangle (0)$.
given in Fig.4 for $U > 0$ only, because it can be deduced from Eq. (20) and $\rho_K(\varepsilon) = \rho_K(-\varepsilon)$ that $\varepsilon_c(K, U) = -\varepsilon_c(K, -U)$. A remarkable feature is that there is a Cooper pair, even for $t_K < 0$, i.e. the upper bound of the two-electron band, for $U$ decreasing toward $t_K/2$, so that there is no Cooper pair solution for $U < t_K/2$ (accordingly, the dashed curve is no longer defined in Fig.4 for $\Delta < .13$), in marked contrast with the opposite conclusion reached elsewhere, that there is a Cooper pair, even for $U > 0$. This discrepancy results from the three-dimensional Van Hove singularities, showing up at both two-electron band edges $\rho_K(\varepsilon \to \pm t_K) \propto \sqrt{t_K - |\varepsilon|}$, unlike the two-electron density of states, used previously, which displayed no such singularity.

$H_K$ operates within the Hilbert space $S_K$. A typical vector of its basis reads $\varphi = \prod_{i=1\ldots m} b_{K,k_i}^\dagger b_{K,k_i}^0$ with $m$ being any integer. We shall look for a variational approximation $\varphi_v$ of the single21 bound eigenstate of $H_K$ in the subset $\{ \varphi_v \} \subset S_K$, characterized by

$$\langle \varphi_v \rvert b_{K,k}^\dagger \lvert \varphi_v \rangle = (n_k(1-n_k))^\alpha, \quad \forall \varphi_v \in \{ \varphi_v \} \ .$$

The real parameter $\alpha$ will be assigned shortly and $n_k = \langle \varphi_v \rvert b_{K,k}^\dagger b_{K,k} \lvert \varphi_v \rangle$. The pair number operator $b_{K,k}^\dagger b_{K,k}$ has two eigenvalues 0, 1, associated with $\lvert 0 \rangle$ and $b_{K,k}^\dagger b_{K,k} \lvert 0 \rangle$, respectively, so that $0 \leq n_k \leq 1$ and $\sum_k n_k = N$. The energy of $\varphi_v$ per site reads

$$E = \frac{\langle \varphi_v \rvert H_K \lvert \varphi_v \rangle}{N} = \sum_k \varepsilon(K,k) n_k^\alpha + U \Delta^2$$

Hence minimising $E$ ($\Rightarrow dE = 0$), under the constraint of $n$ kept constant ($\Rightarrow dn = 0$), yields

$$\varepsilon(K,k) + 2\alpha U \Delta \frac{1 - 2n_k}{(n_k(1-n_k))^\alpha} = \lambda \ ,$$

with $\lambda = \frac{\partial E}{\partial n_k}, \forall k$ being a Lagrange multiplier, which implies that $\lambda = \frac{\partial E}{\partial n} = 2\mu(c_s = \frac{2N}{N})$. The $\alpha$ value will be assigned now by checking consistency with the Cooper pair properties in the limit $n \to 0 \Rightarrow n_k \to 0 \Rightarrow n_k \propto (2\mu - \varepsilon(K,k))^{1-\alpha}$. Comparing with $n_k = \langle \varphi_v \rvert b_{K,k}^\dagger b_{K,k} \lvert \varphi_v \rangle \propto (\varepsilon_c(K) - \varepsilon_c(K,k))^{-2}$, inferred from Eq. (20), yields finally $\alpha = 1/2$ and $\varepsilon_c(K) = 2\mu(c_s = 0)$, a conclusion which had already been reached by an independent rationale. Hence, our variational procedure can be summarised, with help of notations introduced elsewhere, as follows

$$\tan \theta_k = \frac{2U \Delta}{2n_k - \varepsilon_c(K,k)}, \quad n_k = \sin^2 \theta_k$$

$$\Delta = \sum_k \frac{\sin 2\theta_k}{2N} = \int_{-t_K}^{t_K} \frac{\rho_K(\varepsilon)d\varepsilon}{2} \ ,$$

$$c_s = 2 \sum_k \frac{\sin^2 \theta_k}{N} = 2 \int_{-t_K}^{t_K} \sin^2 \theta_k \rho_K(\varepsilon)d\varepsilon \ , \quad (21)$$

with $0 \leq \theta_k \leq \frac{\pi}{4}$. The formulae in Eqs. (21) are found to be identical to those of BCS. As an illustrative example, Eqs. (21) have been solved for $\mu(c_s), \Delta(c_s)$ and $r(c_s)$ defined by Eq. (17), with $U > 0$ and $c_s < 1$ electron per site, only, because it can be deduced from Eqs. (21) and $\rho_K(\varepsilon) = \rho_K(-\varepsilon)$ that $\mu(c_s, U) = -\mu(c_s, -U) = -\mu(2-c_s, U)$. The results, presented in Fig.5, exhibit $\Delta$ almost independent from $c_s$ and $\Delta(c_s \to 0) \propto \sqrt{c_s}$. An important inequality, holding for $U > 0$ and $U < 0$ as well, can be deduced from $\mu(c_s, U) = -\mu(c_s, -U)$ and Fig.5

$$U \frac{\partial \mu}{\partial c_s} < 0 \ . \quad (22)$$

Note that for $K = \frac{\pi}{a}$, the two-electron band is dispersionless because of $t_{K=\frac{\pi}{a}} = 0$. Then applying Eqs. (21) to the $K = \frac{\pi}{a}$ case gives $\Delta = \frac{\sin 2\theta_k}{2}$ and finally $\mu = \frac{U}{2}(1-c_s)$, the validity of which can be checked independently, because the $n$-pair, bound eigenstate of $H_K=\frac{\pi}{a}$ is known30 to read $\varphi = \prod_{i=1\ldots d} b_{n_i,k_i}^\dagger b_{n_i,k_i}^0$ with $d = \binom{N}{n}$, $\varphi_i = \prod_{i=1\ldots n} b_{n_i,k_i}^\dagger b_{n_i,k_i}^0$ and the sum with respect to $i$ runs over all of $n$ pair combinations $\left\{ b_{n_i,k_i}^\dagger \right\}$, chosen among $N$ of available pairs. As each $\varphi_i$ contributes $\frac{n}{N^N}(1 - \frac{n}{N}) U$ to $E$, it can thence be inferred $E = U \frac{n}{N^N}(1 - \frac{n}{N}) \Rightarrow \mu = \frac{\partial E}{\partial n} = \frac{U}{2}(1-c_s)$, which is seen to be identical to the above result, deduced from Eqs. (21). At last, there is $U \frac{\partial \mu}{\partial c_s} = -U^2/2 < 0$ in accordance with inequality (22).

Combining Eq. (20) with Taylor’s expansions of $\sin 2\theta_k$, $\sin \theta_k$, worked out from Eqs. (21), up to $\Delta^2$ for $c_s \to 0 \Rightarrow \Delta \to 0$, leads to

$$\frac{1}{U} = \int_{-t_K}^{t_K} \frac{\rho_K(\varepsilon)}{\varepsilon_c(K) - \varepsilon_c(K,k)} d\varepsilon$$

$$= \int_{-t_K}^{t_K} \left( 1 - 2 \left( \frac{\Delta}{2n_k} \right)^2 \right) \frac{\rho_K(\varepsilon)}{\varepsilon_c(K) - \varepsilon_c(K,k)} d\varepsilon$$

$$(\lambda) \Delta^2 = \frac{1}{2} \int_{-t_K}^{t_K} \frac{\rho_K(\varepsilon)}{\varepsilon_c(K) - \varepsilon_c(K,k)} d\varepsilon \ .$$

Subtracting the integrals equal to $1/U$ from each other, while taking advantage of $\mu(c_s \to 0) \Rightarrow \varepsilon_c(K)/2$, gives in
FIG. 5. plots of μ(ε), Δ(ε), r(ε) reckoned for K = 0, 2π/2a; the solid and dotted lines and the triangles, which pertain to μ, r, Δ, respectively, have been calculated with U/t = 1, whereas the dashed line and the ×, + symbols, which refer to μ, r, Δ, respectively, have been calculated with U/t = .6; the r data have been calculated with mv² = t/3; c_s = 1 corresponds to one electron per site turn

\[(U\Delta)^2 = \left(\frac{\varepsilon_c(K)}{2} - \mu\right)\int_{t_K}^{t_K} \frac{\rho_K(\varepsilon)}{(\varepsilon_c(K) - \varepsilon)^2} d\varepsilon.\]

Equating both expressions of \((U\Delta)^2\) yields finally

\[\frac{\partial \mu}{\partial c_s}(K, c_s = 0) = -\frac{1}{2} \left(\frac{\int_{t_K}^{t_K} \frac{\rho_K(\varepsilon)}{(\varepsilon_c(K) - \varepsilon)^2} d\varepsilon}{\int_{t_K}^{t_K} \frac{\rho_K(\varepsilon)}{(\varepsilon_c(K) - \varepsilon)^2} d\varepsilon}\right)^2.

Note that \(\frac{\partial \mu}{\partial c_s}(K, c_s = 0) \rightarrow \infty\) for \(U \rightarrow t_K/2\) and

\[U \frac{\partial \mu}{\partial c_s}(K, c_s = 0) < 0 \] in accordance with inequality (22).

With help of Fig.5, c_c, defined by r(ε) = 1 in Eq. (17), can now be assigned, for K = 0, to the values .7, .54, and for K = 2π/2a to the values .85, .6, associated with U/t = .6, 1, respectively. Noteworthy is that \(\varphi_v\) will sustain persistent currents or not, according to whether c_s > c_c or c_s < c_c, although \(\varphi_v\) undergoes no qualitative change for c_s = c_c. Accordingly, it still obeys Eqs. (21) for c_s > c_c and c_s < c_c, as well. Therefore, \(\varphi_v(c_s < c_c)\) will be referred to below, as the many-bound-electron, non-superconducting (MBENS) state. Moreover, applying Eq. (5) for c_s = c_c, while taking advantage of the Sommerfeld integral (26) and Eq. (11), yields the critical current density as

\[j_c(T) = c_c e^{\frac{\sqrt{2}}{m} (E_F(T, c_0 - c_c) - \mu(c_c))},\]

\[= \pi \kappa_B c_c e^{\frac{\rho(\varepsilon_F)}{\sqrt{2}\pi^2 m^3} (T_s^2 - T^2)} \] ,

with \(E_F = E_F(T, c_0 - c_c)\). \(\kappa_B, \rho(\varepsilon)\) designate Boltzmann's constant and the one-electron density of states and \(\rho(\varepsilon) = \frac{\partial \rho(\varepsilon)}{\partial \varepsilon}\), while \(T_s < T_c\) is defined by \(E_F(T_c, c_0 - c_c) = \mu(c_c) = c_c(T_c) \) = c_c. The calculated behavior \(j_c(T) \propto \sqrt{T_s^2 - T^2}\), resulting from Eq. (23), is found to agree with observations (24, 26). Consequently, the MBENS state and the superconducting one can be observed for \(T_s < T < T_c\) and \(T < T_s\), respectively. In a low Tc metal such as Sn, c_c has been shown (26) to grow steeply from c_c(T_c) = 0 up to c_c(T_c = .04 K) ≈ c_c(0) = T_s < T_s < .04 K, so that \(T_c, T_s\) are unlikely to be resolved experimentally from each other. However \(T_c - T_s\) will be argued in the next section to be quite sizeable in high Tc materials.

\[\frac{\partial \mu}{\partial c_s} < 0\] has been shown (16) to be a prerequisite for persistent currents. Hence, the inequality (22) entails \(U > 0\). Besides, an additional setback of the assumption (23, 28) \(U < 0 \Rightarrow \frac{\partial \mu}{\partial c_s} > 0\) is to preclude any thermal equilibrium for \(T \leq T_c\). Here is a proof by contradiction. Let us assume that the BCS state is indeed in equilibrium at \(T_c\), which implies \(E_F(T_c, c_0) = \mu(c_c) = 0\), because of \(c_c(T_s) = 0, c_s(T_s) = c_c\), in accordance with Eq. (11). When T decreases from \(T_c\) down to 0, charge conservation \(c_0 = c_0(T_s) + c_s(T_s)\) entails

\[E_F(T) = E_F(T_c) - \frac{c_s(T_c)}{\rho(E_F(T_c))} \mu(T_c) = \mu(T_c) + \frac{\partial \mu}{\partial c_s}(c_s(T)),\]

for which we have used (29, 30) \(\rho(E_F) = 1/\rho(E_F)\), while neglecting \(\frac{\partial E_F}{\partial T} \approx k_B T/E_F < 1\). Thus, \(\frac{\partial \mu}{\partial c_s} > 0\) implies \(E_F(T_c) < \mu(T_c) < \mu(T_c)\), so that the equilibrium condition \(\mu(c_s(T)) = E_F(T, c_0 = c_s(T))\) in Eq. (11) cannot be fulfilled for any \(T < T_c\). Q.E.D.

V. HIGH-\(T_c\) COMPOUNDS

Overdoped high \(T_c\) compounds are known (5, 10, 17, 18, 22, 23) to undergo, at \(T_c\), a crossover from a superconducting state of type II, observed for \(T < T_c\), to an ill-understood state, which sustains no persistent current, but the conduction properties of which differ yet markedly from those of usual metals up to \(T >> T_c\).
• contrary to the conductivity expected to be low, given the high doping rate \( > .15 \), it is observed to be large;

• the Hall coefficient is found to be \( T \) dependent, which hints at a \( T \) dependent carrier concentration, unlike what is observed in usual metals and alloys, behaving like a Fermi gas \( ^{20} \) with \( T \) independent concentration.

Assuming \( T_c = T_s \), both above mentioned features might be consistent with an electron system, comprising a Fermi gas \( ^{1} \) and a MBENS state in respective concentration \( c_n(T > T_s) \), \( c_s(T) \) and fulfilling Eq. \(( \text{I} ) \) with \( c_0 = c_n(T) + c_s(T) \). As a matter of fact, \( \tau_s >> \tau_n \) entails that the large conductivity is settled by the MBENS electrons only and the Hall coefficient, dominated by \( j_s \), is \( T \) dependent as is \( c_s(T) \). The main virtue of such an assumption is that it lends itself to an experimental check, as shown below.

Consider a thermally isolated sample, flown through by \( I(t) = \pi \sigma^2 j(t) \) with \( j(t) = \gamma t \), and taken at \( t = 0 \) in the thermal equilibrium state, represented by \( A \) in Fig. \( \text{I} \) i.e. \( T(t = 0) = T_s, I(t = 0) = 0, c_n(t = 0) = c_n(T_s), c_s(t = 0) = -c_n(T_s) \). While \( I(t) \) keeps growing, the bound electrons, pictured by \( Q_s \), in Fig. \( \text{I} \) are turned into independent ones, depicted by \( Q_n \), as explained in section 2. The experiment ends up at \( t = t_f \), when \( Q_n \), after traveling all along the dotted line, merges with \( C \), referring to the normal state and thence characterized by \( C(t_f) = T_f, c_n(t_f) = c_0, c_s(t_f) = 0 \). Thus applying the first law of thermodynamics to this adiabatic process yields

\[
\int_{T_s}^{T_f} C(T) \, dT = Q_1 + Q_2 \]

\[
Q_1 = \int_0^{t_f} \frac{\dot{U}(t)}{l} \, j(t) \, dt = \int_0^{t_f} \left( \frac{\dot{U}(t)}{\sigma_n} + \frac{\dot{U}(t)}{\sigma_j} \right) \, dt ,
\]

where \( Q_1 > 0, Q_2 < 0 \) stand for the Joule heat released in section I and II, respectively, and \( C_n(T > T_s) << C_0(T), C_s(T > T_s) << C_0(T) \). Besides, \( W_{s-n} \) is neglected. Derivating Eq. \(( \text{I} ) \) with respect to \( t \) gives finally

\[
C_0(T) \dot{T} = \frac{U(t)}{l} j(t) + \frac{j^2}{\sigma_j} = \frac{j_n^2}{\sigma_n} + j_s^2 \left( \frac{1}{\sigma_n} + \frac{1}{\sigma_j} \right) , \tag{25}
\]

with \( \dot{T} = \frac{dT}{dt} \). Because of \( \frac{1}{\sigma_n} + \frac{1}{\sigma_j} > 0 \), due to the very definition of \( T_s \), we predict \( \dot{T} > 0 \) \( \Rightarrow T_f > T_s \), with \( \sigma_j < 0 \) \( \Rightarrow C_0(T) \dot{T} < U(t) j(t) \). Despite \( \dot{T} > 0 \) like in a usual metal, the latter inequality would rather read \( C_0(T) \dot{T} = U(t) j(t) \), if the same experiment were carried out in a normal conductor. Conversely, would the experiment be performed at \( T < T_s \) \( \Rightarrow T_f < T_s \), with \( \sigma_j < 0 \), we should observe \( \dot{T} < 0 \), as remarked by De Gennes too \( ^{20} \). Besides, the sign of \( \dot{T} \) is independent of that of \( I \), because the Joule effect is irreversible. At last, due to the high doping rate, the local electron concentration is likely to display spatial fluctuations, which should eventually result into a sample, comprising both superconducting and MBENS domains. This case could be brought to experimental evidence by observing different values of \( \dot{T} \) in Eq. \(( \text{I} ) \), according to whether a \( dc \) \( (\Rightarrow \dot{T}_{dc}) \) or \( ac \) \( (\Rightarrow \dot{T}_{ac}) \) current is fed into the sample, because superconducting domains will contribute to the Joule effect only for \( ac \) current, whereas MBENS ones will do in both cases. Thus we predict \( \dot{T}_{ac} < \dot{T}_{dc} \).

VI. MAGNETOElasticity

Magnetoelastic effects were reported \( ^{24,25} \) long ago, in superconducting metals, at \( T \leq T_c \) and atmospheric pressure: when the magnetic field \( H \) starts growing from 0, the sample first expands by a tiny amount \( (\approx 10^{-7}) \) and then shrinks abruptly for \( H \) reaching some critical value \( H_c(T) \), at which the sample goes normal. Actually, because the superconducting electrons are known \( ^{1,2} \) to be in a macroscopic singlet spin state, \( H \) has no direct sway on them, but merely induces an eddy current according to Faraday’s law \( ^{24} \). This current, responsible for the Meissner effect, turns superconducting electrons into normal ones, as discussed in section 2, but only within a thin film of thickness \( \lambda_M \), located at the outer edge of the sample \( ^{26} \). Meanwhile, the partial pressure, stemming from the electrons, is altered, as will be shown now.

The free energy, associated with a sample of volume \( V \), containing \( n \) conduction electrons \( (\Rightarrow \frac{1}{\gamma} = c_0 = c_n + c_s) \), reads \( V F(T, c_0) \) with \( F(T, c_0) = E_n(T, c_n) + E_s(c_s) \) being the electronic free energy per unit volume. The partial pressure \( p_e \), exerted by the electrons, reads \( ^{27} \)

\[
p_e(H \neq 0) = -\frac{\partial\left(\frac{1}{\gamma} F\right)}{\partial H} = c_n E_n(T, c_n) + c_s E_s(c_s) - E_s , \tag{26}
\]

with \( c_n > c_n(T), F_n = \int_0^{c_n} E_F(T, u) \, du, c_s = c_0 - c_n, E_s = \int_0^{c_s} \mu(u) \, du \).

Eq. \(( \text{I} ) \) implies

\[
\frac{\partial p_e}{\partial c_n} = c_n \frac{\partial E_F}{\partial c_n} - c_s \frac{\partial E_F}{\partial c_s} - c_s \frac{\partial \mu}{\partial c_s} .
\]

Besides, \( \frac{\partial E_F}{\partial c_n} = \frac{1}{\rho(E_F)} \frac{\partial \rho(E_F)}{\partial c_n} < 0 \) entails \( \frac{\partial \rho}{\partial c_n} < 0 \), since \( c_n \) grows at the expense of \( c_s \) for increasing \( H \); the inequality \( \frac{\partial H}{\partial c_n} > 0 \) is always valid, which implies at last \( \frac{\partial p_e}{\partial H} > 0 \), in agreement with the observed \( H \) induced expansion \( ^{24,25} \).

For \( H = H_c(T) \), the sample goes normal, so that \( H \) penetrates suddenly into bulk matter and polares the whole set of normal electrons in concentration \( c_0 \). The associated paramagnetic energy per unit volume reads \( ^{28} \)

\[
\mathcal{E}_H = -\frac{(\mu_B H)^2}{2} \rho(E_F(T, c_0)) \text{ with } \mu_B \text{ being the Bohr magneton. Because Pauli's susceptibility is } T \text{ independent} \tag{26},
\]

\( \mathcal{E}_H \) is also equal to the magnetic contribution to the free energy, so that the partial pressure \( p_H \), associated with \( H \), reads

\[
p_H = c_0 \frac{\partial \mathcal{E}_H}{\partial c_0} = \mathcal{E}_H - \frac{(\mu_B H)^2}{2} \left( \rho(E_F) - c_0 \frac{\partial \rho(E_F)}{\partial E_F} \right) .
\]
with $E_F = E_F(T, c_0)$. As the sample was reported to shrink at $H_c(T)$, this implies $\rho_H < 0$, which can be realized only if $E_F(T, c_0)$ lies close to a Van Hove singularity at $\varepsilon_{VH} \Rightarrow \rho'(E_F) \propto (E_F - \varepsilon_{VH})^{-1/2} >> 1$.

This kind of $H$ driven experiment provides merely qualitative information, because of several drawbacks, related to the Meissner effect, as recalled in section 1. Consequently, the critical field $H_c(T)$ is ill-defined. To buttress this conclusion, we shall calculate $H(r)$ induced by the homogeneous current density $j_c$, parallel to the $z$ axis. $H(r)$ is normal to the unit vectors along the $r$ and $z$ coordinates and there is $H = r j_c/2$, thanks to the Ampère-Maxwell equation. Hence, $H$ is seen to vary from $H(r = 0) = 0$ up to $H(r_0) = r_0 j_c/2$, so that $H_c$ cannot be defined in a unique way, unlike $j_c(T)$. Likewise superconductors of type II make this proof more cogent, inasmuch as the whole superconducting sample is known to turn continuously normal over a broad range of critical values $H_c \in [H_{c1}, H_{c2}]$ with $H_{c1} << H_{c2}$.

**VII. CONCLUSION**

A unified picture, accounting for low and high $T_c$ superconductivity as well, has been developed. The physical significance of two different critical temperatures $T_s, T_c$, with $T_s < T_c$, characterizing the electrodynamical behavior of superconducting materials, has been analyzed. Whereas no persistent current can be observed for $T > T_s$, $T_c$ is the upper bound of the MBENS state ($\equiv c_s(T \geq T_c) = 0$) and is also identical to the usual critical temperature. The expression of the maximum persistent current $j_s(T)$ has been worked out and found to agree with observation. Unlike the normal current $j_n$, the bound electron current $j_s$ does not depend on the applied electric field, but rather on $c_s$. The many-body wave-function, describing the motion of bound electrons, is identical for both superconducting ($c_s > c_c$) and MBENS ($c_s < c_c$) states, and accurately approximated by the BCS variational scheme. Conversely, the critical field $H_c$ has been shown to lack a unique definition. Whereas $T_s, T_c$ are unlikely to be resolved from each other in conventional superconductors due to the steep variation of $c_s(T \rightarrow T_c)$, $T_c/T_s$ may be $> 10$ in high $T_c$ compounds. Moreover, their peculiar conduction properties in the contentious range $T \in [T_s, T_c]$ have been ascribed to a MBENS state and an experiment, taking full advantage of the interplay between the usual and anomalous Joule effects, has been outlined to check the validity of this assumption. The merit of a current driven experiment over a $H$ driven one has been emphasized. At last, it has been shown that a repulsive \((U > 0)\) Hubbard coupling is a prerequisite for superconductivity at thermal equilibrium, in accordance with the Coulomb force and Eq.\(^*\).

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