TOEPLITZ OPERATORS WITH QUASI-RADIAL QUASI-HOMOGENEOUS SYMBOLS AND BUNDLES OF LAGRANGIAN FRAMES

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Abstract. We prove that the quasi-homogenous symbols on the projective space \( \mathbb{P}^n(\mathbb{C}) \) yield commutative algebras of Toeplitz operators on all weighted Bergman spaces, thus extending to this compact case known results for the unit ball \( \mathbb{B}^n \). These algebras are Banach but not \( \mathcal{C}^\ast \). We prove the existence of a strong link between such symbols and algebras with the geometry of \( \mathbb{P}^n(\mathbb{C}) \). The latter is also proved for the corresponding symbols and algebras on \( \mathbb{B}^n \).

1. Introduction

The study of commutative algebras of Toeplitz operators has shown to be a very interesting subject. Some previous results in this topic serve as background to this work. First, it was shown the existence of symbols defining interesting commutative \( \mathcal{C}^\ast \)-algebras of Toeplitz operators on bounded symmetric domains (see [2], [7], [8] and [9]). Also, it was exhibited in [11] the existence of Banach algebras, which are not \( \mathcal{C}^\ast \), of commutative Toeplitz operators on the unit ball \( \mathbb{B}^n \). And, in [6] we constructed commutative \( \mathcal{C}^\ast \)-algebras of Toeplitz operators on complex projective spaces.

A remarkable fact is that the currently known commutative \( \mathcal{C}^\ast \)-algebras of Toeplitz operators on \( \mathbb{B}^n \) are naturally associated to Abelian subgroups of the group of biholomorphisms of \( \mathbb{B}^n \). In fact, their systematic description is best understood with the use of such groups of biholomorphisms (see [8] and [9]). Furthermore, this provided the guiding light to construct commutative \( \mathcal{C}^\ast \)-algebras of Toeplitz operators in the complex projective space \( \mathbb{P}^n(\mathbb{C}) \): the currently known \( \mathcal{C}^\ast \)-algebras for \( \mathbb{F}^n(\mathbb{C}) \) are naturally associated and described from the maximal tori of the group of isometric biholomorphisms of \( \mathbb{P}^n(\mathbb{C}) \) (see [6]).

The known Banach (not \( \mathcal{C}^\ast \)) algebras of commutative Toeplitz operators first introduced in [11] for \( \mathbb{B}^n \) are given by the so called quasi-homogeneous symbols. Such symbols are defined in terms of radial and spherical coordinates of components in \( \mathbb{B}^n \) (see Section 3 below). However, their introduction lacked the stronger connection with the geometry of the domain observed for the commutative \( \mathcal{C}^\ast \)-algebras of Toeplitz operators on \( \mathbb{B}^n \).

Given these lines of research, there are two natural problems to consider. First, to determine whether or not there are any interesting Banach algebras, that are not

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C∗, of commutative Toeplitz operators on \( \mathbb{P}^n(\mathbb{C}) \). And second, to find any possible special links between the geometry of \( \mathbb{B}^n \) and the Banach algebras, introduced in [11], of commutative Toeplitz operators. The goal of this work is to solve these problems.

On one hand, we define quasi-homogeneous symbols in the complex projective space \( \mathbb{P}^n(\mathbb{C}) \), and show that these provide Banach algebras of commutative Toeplitz operators on every weighted Bergman space of \( \mathbb{P}^n(\mathbb{C}) \). The results that exhibit the commuting Toeplitz operators are obtained in Section 4, where Theorem 4.5 is the main result. On the other hand, we also prove that the Banach algebras defined by quasi-homogeneous symbols turn out to have a strong connection with the geometry of the supporting space, for both \( \mathbb{B}^n \) and \( \mathbb{P}^n(\mathbb{C}) \); the main results in this case are presented in Section 5. In particular, we prove that the quasi-homogeneous symbols, for both \( \mathbb{B}^n \) and \( \mathbb{P}^n(\mathbb{C}) \), can be associated to an Abelian group of holomorphic isometries of the corresponding space (see Theorem 5.1). Such group is a subgroup of a maximal torus in the corresponding isometry group.

We further prove that the groups associated to quasi-homogeneous symbols afford pairs of foliations with distinguished Lagrangian and Riemannian geometry known as Lagrangian frames (see Section 5 below and [7], [8] and [9]). This recovers the behavior observed for the \( C^* \)-algebras of commutative Toeplitz operators constructed in [8], [9] and [6], for which such Lagrangian frames appear as well. Nevertheless, it is important to note a key difference between the \( C^* \) case and the Banach case. For the \( C^* \)-algebras of Toeplitz operators on \( \mathbb{B}^n \) and \( \mathbb{P}^n(\mathbb{C}) \), as constructed in [8], [9] and [6], the Lagrangian frames are obtained for the whole space, i.e. they come from Lagrangian submanifolds of the whole space, either \( \mathbb{B}^n \) or \( \mathbb{P}^n(\mathbb{C}) \). But for the Banach algebras defined by quasi-homogeneous symbols the Lagrangian frames are obtained on the fibers of a suitable fibration over either \( \mathbb{B}^n \) or \( \mathbb{P}^n(\mathbb{C}) \). The existence of such fibration and the Lagrangian frames on its fibers are obtained in Theorem 5.4. It is also proved in Theorem 5.6 that a full maximal torus of isometries continues to play an importante role, since the complement to the group that defines the fiberwise Lagrangian frames acts by automorphisms of such frames.

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2. Preliminaries on the Geometry and Analysis of \( \mathbb{P}^n(\mathbb{C}) \)

In this section we establish our notation concerning the \( n \)-dimensional complex projective space \( \mathbb{P}^n(\mathbb{C}) \). For further details we refer to the bibliography. There is a natural realization of \( \mathbb{C}^n \) as an open conull dense subset given by

\[
\mathbb{C}^n \to \mathbb{P}^n(\mathbb{C}), \quad z \mapsto [1, z],
\]

which defines a biholomorphism onto its image. Note that the points of \( \mathbb{P}^n(\mathbb{C}) \) are denoted by \([w]\), the complex line through \( w \in \mathbb{C}^{n+1} \setminus \{0\} \). We will refer to this embedding as the canonical embedding of \( \mathbb{C}^n \) into \( \mathbb{P}^n(\mathbb{C}) \).

Let us denote with \( \omega \) the canonical Kähler structure on \( \mathbb{P}^n(\mathbb{C}) \) that defines the Fubini-Study metric, whose volume is then given by \( \Omega = (\omega/2\pi)^n \). These induce
on $\mathbb{C}^n$ the following Kähler form and volume element, respectively
\[ \omega_0 = \frac{1}{2} \left( 1 + |z|^2 \right) \left( \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k - \sum_{k,l=1}^{n} \bar{z}_k z_l dz_k \wedge d\bar{z}_l \right), \]
\[ \Omega_0 = \frac{1}{\pi^n} \frac{dV(z)}{(1 + |z|^2)^{n+1}}, \]
where $dV(z)$ denotes the Lebesgue measure on $\mathbb{C}^n$.

Let $H$ denote the dual bundle of the tautological line bundle of $\mathbb{P}^n(\mathbb{C})$. We recall that $H$ carries a canonical Hermitian metric $h$ obtained from the (flat) Hermitian metric of $\mathbb{C}^{n+1}$. Then, it is also well known that the curvature $\Theta$ of $(H, h)$ satisfies the identity
\[ \Theta = -i\omega, \]
which amounts to say that $(H, h)$ is a quantum line bundle over $\mathbb{P}^n(\mathbb{C})$.

We will denote with $\Gamma(\mathbb{P}^n(\mathbb{C}), H^m)$ and $\Gamma_{\text{hol}}(\mathbb{P}^n(\mathbb{C}), H^m)$ the smooth and holomorphic sections of $H^m$, respectively. Note that $H^m$ denotes the $m$-th tensorial power of $H$. Clearly, both of these spaces lie inside $L_2(\mathbb{P}^n(\mathbb{C}), H^m)$.

For every $m \in \mathbb{Z}_+$ and with respect to the canonical embedding of $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C})$, we define the weighted measure on $\mathbb{P}^n(\mathbb{C})$ with weight $m$ by
\[ d\nu_m(z) = \frac{(n + m)!}{m!} \frac{\Omega(z)}{(1 + |z|^2)^{n+1}} \frac{dV(z)}{dV(z)} = \frac{(n + m)!}{\pi^n m!} \frac{dV(z)}{(1 + |z|^2)^{n+m+1}}. \]
A simply computation shows that $d\nu_m$ is a probability measure for all $m \in \mathbb{Z}_+$. For simplicity, we will use the same symbol $d\nu_m$ to denote the weighted measures for both $\mathbb{P}^n(\mathbb{C})$ and $\mathbb{C}^n$. It is also straightforward to show that the canonical embedding of $\mathbb{C}^n$ into $\mathbb{P}^n(\mathbb{C})$ induces a canonical isometry
\[ \Phi : L_2(\mathbb{C}^n, \nu_m) \to L_2(\mathbb{P}^n(\mathbb{C}), H^m) \]
with respect to which we will identify these spaces in the rest of this work. Also, we will denote with $\langle \cdot, \cdot \rangle_m$ the inner product of this Hilbert spaces.

The weighted Bergman space on $\mathbb{P}^n(\mathbb{C})$ with weight $m \in \mathbb{Z}_+$ is defined by:
\[ \mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C})) = \{ \zeta \in L_2(\mathbb{P}^n(\mathbb{C}), H^m) : \zeta \text{ is holomorphic} \} = \Gamma_{\text{hol}}(\mathbb{P}^n(\mathbb{C}), H^m). \]

These Bergman spaces are finite-dimensional and are described by the following well known result.

**Proposition 2.1.** For every $m \in \mathbb{Z}_+$, the Bergman space $\mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))$ satisfies the following properties.

(i) $\mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))$ can be identified with the space $P^{(m)}(\mathbb{C}^{n+1})$ of homogeneous polynomials of degree $m$ over $\mathbb{C}^{n+1}$.

(ii) For $\Phi : L_2(\mathbb{C}^n, \nu_m) \to L_2(\mathbb{P}^n(\mathbb{C}), H^m)$ the canonical isometry described above, we have $\Phi(\mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))) = P_m(\mathbb{C}^n)$, the space of polynomials on $\mathbb{C}^n$ of degree at most $m$.

In what follows, we will use this realization of the Bergman spaces without further notice.
Recall the following notation for multi-indices $\alpha, \beta \in \mathbb{N}^n$ and $z \in \mathbb{C}^n$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$
$$\alpha! = \alpha_1! \cdots \alpha_n!, \ \alpha \in \mathbb{N}^n,$$
$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$
$$\delta_{\alpha, \beta} = \delta_{\alpha_1, \beta_1} \cdots \delta_{\alpha_n, \beta_n}.$$

The Bergman space $A^2_m(\mathbb{P}^n(\mathbb{C}))$ has a basis consisting of the polynomials $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ where $\alpha \in \mathbb{N}^n$ and $|\alpha| \leq m$. Hence we will consider the set

$$J_n(m) = \{ \alpha \in \mathbb{N}^n : |\alpha| \leq m \}.$$

More precisely, an easy computation shows that the set

$$(2.1) \quad \left\{ \left( \frac{m!}{\alpha!(m-|\alpha|)!} \right)^{\frac{1}{2}} z^\alpha : \alpha \in J_n(m) \right\}$$

is an orthonormal basis of $A^2_m(\mathbb{P}^n(\mathbb{C}))$.

For $\psi \in L_2(\mathbb{P}^n(\mathbb{C}), H^m)$, and considering the identification $\Phi$, we define the Bergman projection by

$$B_m(\psi)(z) = \frac{(n+m)!}{\pi^m m!} \int_{\mathbb{C}^n} \frac{\psi(w)K(z, w)dV(w)}{(1 + w_1 \overline{w}_1 + \cdots + w_n \overline{w}_n)^{n+m+1}}$$

where

$$K(z, w) = (1 + z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n)^m.$$

This operator satisfies the well known reproducing property.

**Proposition 2.2.** If $\psi \in L_2(\mathbb{P}^n(\mathbb{C}), H^m)$, then $B_m(\psi)$ belongs to the weighted Bergman space $A^2_m(\mathbb{P}^n(\mathbb{C}))$. Also, $B_m(\psi) = \psi$ if $\psi \in A^2_m(\mathbb{P}^n(\mathbb{C}))$.

Using this, we define the Toeplitz operator $T_a$ on $A^2_m(\mathbb{P}^n(\mathbb{C}))$ with bounded symbol $a \in L_\infty(\mathbb{P}^n(\mathbb{C}))$ by

$$T_a(\varphi) = B_m(a \varphi),$$

for every $\varphi \in A^2_m(\mathbb{P}^n(\mathbb{C}))$.

Let us denote by $S^n$ the unit sphere in $\mathbb{C}^n$. In this work, we will use the following identity on the $n$-sphere $S^n$

$$(2.2) \quad \int_{S^n} \xi^\alpha \overline{\xi}^\beta dS(\xi) = \delta_{\alpha, \beta} \frac{2\pi^n |\alpha|!}{(n-1 + |\alpha|)!}$$

where $dS$ be the corresponding surface measure on $S^n$ (see [12]).

3. **Toeplitz operators with quasi-homogeneous symbols**

Quasi-homogeneous symbols were introduced in [11] on the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. However, we will also consider quasi-homogeneous symbols for the projective space $\mathbb{P}^n(\mathbb{C})$ as functions defined on $\mathbb{C}^n$. The basic definitions of the quasi-radial and quasi-homogeneous symbols are very similar for both $\mathbb{B}^n$ and $\mathbb{C}^n$, so we will recall them here together. This will also be useful latter on for our common geometric treatment of quasi-homogeneous symbols on $\mathbb{B}^n$ and $\mathbb{P}^n(\mathbb{C})$. To simplify our notation, from now on $U^n$ will denote one of either $\mathbb{B}^n$ or $\mathbb{C}^n$, corresponding to whether we are dealing with $\mathbb{B}^n$ or $\mathbb{P}^n(\mathbb{C})$, respectively. For the latter case, we are considering $\mathbb{C}^n$ canonically embedded into $\mathbb{P}^n(\mathbb{C})$ as described in Section 2.
Let $k = (k_1, \ldots, k_l) \in \mathbb{Z}_+^l$ be a multi-index so that $|k| = n$. We will call such multi-index $k$ a partition of $n$. For the sake of definiteness, we will assume that $k_1 \leq \cdots \leq k_l$. This partition provides a decomposition of the coordinates $z \in \mathbb{U}^n$ as $z = (z(1), \ldots, z(l))$ where

$$z(j) = (z_{k_1+\cdots+k_{j-1}+1}, \ldots, z_{k_1+\cdots+k_l}),$$

for every $j = 1, \ldots, l$, and the empty sum is $0$ by convention. For $z \in \mathbb{U}^n$, we define $r_j = |z(j)|$ and

$$\xi(j) = \frac{z(j)}{r_j}$$

if $z(j) \neq 0$. Besides the quasi-radii $(r_1, \ldots, r_l)$, this provides a set of coordinates $(\xi(1), \ldots, \xi(l)) \in \mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_l}$.

For a partition $k$ of $n$, a $k$-quasi-radial symbol $a$ on $\mathbb{U}^n$ is a function $a : \mathbb{U}^n \to \mathbb{C}$ which depends only on the coordinates $(r_1, \ldots, r_l)$ introduced above. In other words, $a(z) = a(r_1, \ldots, r_l)$. The set of all $k$-quasi-radial functions is denoted by $\mathcal{R}_k$. The family of sets $\mathcal{R}_k$, while $k$ varies over the partitions of $n$, is a partially ordered by inclusion. The minimal element among these sets is the set $\mathcal{R}_{(1, \ldots, 1)}$ of separately radial functions.

Also, for $k$ a partition of $n$, a $k$-quasi-homogeneous symbol $\varphi$ on $\mathbb{U}^n$ is a function $\varphi : \mathbb{U}^n \to \mathbb{C}$ of the form

$$\varphi(z) = a(r_1, \ldots, r_l)\xi^p \xi^q,$$

where $a$ is a $k$-quasi-radial symbol and $p, q \in \mathbb{N}^n$ satisfy

$$p \cdot q = p_1 q_1 + \cdots + p_n q_n = 0.$$

Recall that every $k$ the family $\mathcal{R}_k$ is contained in $\mathcal{R}_{(1, \ldots, 1)}$. Hence, the Toeplitz operators $T_a$ for symbols $a \in \mathcal{R}_k$ can be simultaneously diagonalized with respect to the monomial basis in the corresponding Bergman space (see [8] and [6]). Furthermore, for the case $\mathbb{U}^n = \mathbb{C}^n$ the following result provides the multiplication operator so obtained. Note that this result is similar to Lemma 3.1 from [11].

**Lemma 3.1.** Consider the case $\mathbb{U}^n = \mathbb{C}^n$, and let $k$ be a partition of $n$. For any $k$-quasi-radial bounded measurable symbol $a(r_1, \ldots, r_l)$, we have

$$T_a z^\alpha = \gamma_{a,k,m}(\alpha) z^\alpha,$$

for every $\alpha \in J_n(m)$, where

$$\gamma_{a,k,m}(\alpha) = \gamma_{a,k,m}(|\alpha(1)|, \ldots, |\alpha(l)|)$$

(3.1)

$$= \frac{2^l(n+m)!}{(m-|\alpha|)! \prod_{j=1}^l (k_j - 1 + |\alpha(j)|)} \times \int_{\mathbb{R}^n_+} a(r_1, \ldots, r_l) (1 + r^2)^{-(n+m+1)} \prod_{j=1}^l r_j^{2|\alpha(j)|+2k_j-1} dr_j.$$

**Proof.** Let $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq m$. Then, we have

$$\langle T_a z^\alpha, z^\alpha \rangle_m = \langle a z^\alpha, z^\alpha \rangle_m$$

$$= \frac{(n+m)!}{\pi^m} \int_{\mathbb{C}^n} \frac{a(r_1, \ldots, r_l) z^\alpha dV(z)}{(1 + |z_1|^2 + \cdots + |z_n|^2)^{n+m+1}}.$$
Lemma 3.2. Consider the case $\mathbb{U} = C^n$, and let $a\xi^p\bar{\xi}^q = a(r_1, \ldots, r_l)\xi^p\bar{\xi}^q$ be a $k$-quasi-homogeneous symbol for $k$ a partition of $n$. Then, the Toeplitz operator $T_{a\xi^p\bar{\xi}^q}$ acts on monomials $z^\alpha$ with $\alpha \in \mathbb{Z}^n_+$ and $|\alpha| \leq m$ as follows

$$T_{a\xi^p\bar{\xi}^q}z^\alpha = \begin{cases} \hat{\gamma}_{a,k,p,q,m}(\alpha)z^{\alpha+p-q} & \text{for } \alpha + p - q \in J_n(m) \\ 0 & \text{for } \alpha + p - q \not\in J_n(m) \end{cases}$$

where

$$\hat{\gamma}_{a,k,p,q,m}(\alpha) = \frac{2^l(\alpha + p)!(n + m)!}{(\alpha + p - q)!((m - |\alpha + p - q|)! \prod_{j=1}^l(k_j - 1 + |\alpha_j + p(j)|))} \times \int_{\mathbb{R}^+} a(r_1, \ldots, r_l)(1 + r^2)^{-(n+m+1)} \prod_{j=1}^l r_j^{2|\alpha_j + p(j) - q_j| + 2k_j - 1} dr_j$$

Proof. Let $\alpha, \beta \in \mathbb{Z}^n_+$ satisfy $|\alpha|, |\beta| \leq m$. Then, we have

$$\langle T_{a\xi^p\bar{\xi}^q}z^\alpha, z^\beta \rangle_m = \langle a\xi^p\bar{\xi}^qz^\alpha, z^\beta \rangle_m = \frac{(n + m)!}{\pi^m!} \int_{\mathbb{C}^n} a(r_1, \ldots, r_l)\xi^p\bar{\xi}^q z^{n+2} dV(z).$$
Applying the change of the variables $z(j) = r_j \xi(j)$, where $r_j \in [0, \infty)$ and $\xi(j) \in S^{k_j}$, for $j = 1, \ldots, l$, this yields

\[
\langle a \xi_p \xi_q z^\alpha, z^\beta \rangle_m = \frac{(n + m)!}{\pi^{n+m} m!} \int_{\mathbb{R}^n_+} a(r_1, \ldots, r_l) (1 + r^2)^{-(n+m+1)}
\]

\[
\times \prod_{j=1}^l r_j^{|\alpha(j)| + |\beta(j)| + 2k_j - 1} dr_j
\]

\[
\times \prod_{j=1}^l \int_{S^{k_j}} \xi_{(j)}^{\alpha(j) + p(j)} \xi_{(j)}^{\beta(j) + q(j)} dS(\xi_{(j)})
\]

\[
= \delta_{\alpha + p, \beta + q} \frac{2^l (\alpha + p)! (n + m)!}{m! \prod_{j=1}^l (k_j - 1 + |\alpha(j) + p(j)|)}
\]

\[
\times \int_{\mathbb{R}^n_+} a(r_1, \ldots, r_l) (1 + r^2)^{-(n+m+1)}
\]

\[
\times \prod_{j=1}^l r_j^{2|\alpha(j) + p(j) - q(j)| + 2k_j - 1} dr_j
\]

Observe that this expression is non zero if and only if $\beta = \alpha + p - q$, which a priori belongs to $J_n(m)$. We conclude the result from the orthonormality of the basis defined in \ref{2.4}. \hfill \Box

4. Commutativity results for quasi-homogeneous symbols on $\mathbb{P}^n(\mathbb{C})$

In the rest of this section we will restrict ourselves to the case $\mathbb{U}^n = C^n$. The results in this section show that the commuting identities proved in \cite{11} for the unit ball $\mathbb{B}^n$ have corresponding ones for the complex projective space $\mathbb{P}^n(\mathbb{C})$.

**Theorem 4.1.** Let $k \in \mathbb{Z}_+^l$ be a partition of $n$ and $p, q \in \mathbb{N}^l$ a pair of orthogonal multi-indices. If $a_1$ and $a_2$ are non identically zero $k$-quasi-radial bounded symbols, then the Toeplitz operators $T_{a_1}$ and $T_{a_2 \xi_j}$ commute on each weighted Bergman space $A^2_m(\mathbb{P}^n(\mathbb{C}))$ if and only if $|p(j)| = |q(j)|$ for each $j = 1, \ldots, l$.

**Proof.** Let $\alpha \in J_n(m)$ be given. First note that if $\alpha + p - q \notin J_n(m)$, then the Lemmas \ref{3.1} and \ref{3.2} imply that both $T_{a_1} T_{a_2 \xi_j} z^\alpha$ and $T_{a_2 \xi_j} T_{a_1} z^\alpha$ vanish. Hence, we can assume that $\alpha + p - q \in J_n(m)$. Applying again Lemmas \ref{3.1} and \ref{3.2} we
From which we conclude that the special property of Toeplitz operators with quasi-homogeneous symbols.

We can combine the equations (3.2) to yield the following identity.

\[ T_{a_1}T_{a_2}\xi^\alpha z^\alpha = \frac{2^m(\alpha + p)!(n + m)!}{(\alpha + p - q)!(m - |\alpha + p - q|)!\prod_{j=1}^l (k_j - 1 + |\alpha_j + p_j|)} \times \int_{\mathbb{R}_+^n} a_2(r_1, \ldots, r_l)(1 + r^2)^{(n + m + 1)} \prod_{j=1}^l r_j^{2|\alpha_j| + p_j + q_j - 1} dr_j \times z^{\alpha + p - q}. \]

As a consequence of the previous computations, we also obtain the following very special property of Toeplitz operators with quasi-homogeneous symbols.

And similarly, we have

\[ T_{a_2}\xi^\alpha T_{a_1}z^\alpha = \frac{2^l(n + m)!}{(m - |\alpha|)!\prod_{j=1}^l (k_j - 1 + |\alpha_j|)} \times \int_{\mathbb{R}_+^n} a_1(r_1, \ldots, r_l)(1 + r^2)^{(n + m + 1)} \prod_{j=1}^l r_j^{2|\alpha_j| + 2k_j - 1} dr_j \times z^{\alpha + p - q}. \]

From which we conclude that \( T_{a_1}T_{a_2}\xi^\alpha z^\alpha = T_{a_2}\xi^\alpha T_{a_1}z^\alpha \) if and only if \( |p_j| = |q_j| \) where \( j = 1, \ldots, l \). \( \square \)

If we assume that \( |p_j| = |q_j| \) for all \( j = 1, \ldots, l \), then the equations (3.1) and (3.2) combine together to yield the following identity.

\[ \gamma_{a,k,p,q,m}(\alpha) = \frac{2^l(\alpha + p)!(n + m)!}{(\alpha + p - q)!(m - |\alpha + p - q|)!\prod_{j=1}^l (k_j - 1 + |\alpha_j + p_j|)} \times \int_{\mathbb{R}_+^n} a(r_1, \ldots, r_l)(1 + r^2)^{(n + m + 1)} \prod_{j=1}^l r_j^{2|\alpha_j| + 2k_j - 1} dr_j = \frac{(\alpha + p)!\prod_{j=1}^l (k_j - 1 + |\alpha_j|)}{(\alpha + p - q)!\prod_{j=1}^l (k_j - 1 + |\alpha_j + p_j|)} \gamma_{a,k,m}(\alpha) \]

\[ = \prod_{j=1}^l \left( \frac{(\alpha_j + p_j)!|k_j - 1 + |\alpha_j|)}{(\alpha_j + p_j - q_j)!|k_j - 1 + |\alpha_j + p_j|)} \right) \gamma_{a,k,m}(\alpha) \]

As a consequence of the previous computations, we also obtain the following very special property of Toeplitz operators with quasi-homogeneous symbols.
Corollary 4.2. Let \( k \in \mathbb{Z}_+^n \) be a partition of \( n \) and \( p, q \in \mathbb{N}^n \) a pair of orthogonal multi-indices such that \( |p(j)| = |q(j)| \) for all \( j = 1, \ldots, l \). Then for each non identically zero \( k \)-quasi-radial function \( a \), we have \( T_a \tilde{T}_{\xi^p \bar{\xi}^q} = \tilde{T}_{\xi^p \bar{\xi}^q} T_a = T_{a\tilde{\xi}^p \bar{\xi}^q} \).

Consider \( k = (k_1, \ldots, k_l) \) and a pair of multi-indices \( p, q \) such that \( p \perp q \) and \( |p(j)| = |q(j)| \) for \( j = 1, \ldots, l \), we define

\[
\tilde{p}(j) = (0, \ldots, 0, p(j), 0, \ldots, 0), \quad \tilde{q}(j) = (0, \ldots, 0, q(j), 0, \ldots, 0)
\]

where the only possibly non zero part is placed in the \( j \)-th position. In particular, we have \( p = \tilde{p}(1) + \ldots + \tilde{p}(l) \) and \( q = \tilde{q}(1) + \ldots + \tilde{q}(l) \). Now let \( T_j = T_{\tilde{q}(j) \bar{\xi}^p(j)} \) for every \( j = 1, \ldots, l \). As a consequence of the previous computations we obtain the following result.

**Corollary 4.3.** The Toeplitz operators \( T_j = T_{\tilde{q}(j) \bar{\xi}^p(j)} \), for \( j = 1, \ldots l \) mutually commute and

\[
\prod_{j=1}^m T_j = T_{\bar{\xi}^p \xi^q}
\]

Next, we obtain a necessary and sufficient condition for two given quasi-homogeneous symbols to determine Toeplitz operators that commute with each other.

**Theorem 4.4.** Let \( k \in \mathbb{Z}_+^n \) be a partition of \( n \) and \( p, q \in \mathbb{N}^n \) a pair of orthogonal multi-indices. Consider \( a(r_1, \ldots, r_l) \xi^p \bar{\xi}^q \) and \( b(r_1, \ldots, r_l) \xi^u \bar{\xi}^v \) two \( k \)-quasi-homogeneous functions on \( \mathbb{P}^n(\mathbb{C}) \) where \( a(r_1, \ldots, r_l) \) and \( b(r_1, \ldots, r_l) \) are \( k \)-quasi-radial measurable and bounded symbols. Assume that \( |p(j)| = |q(j)| \) and \( |u(j)| = |v(j)| \) for all \( j = 1, \ldots, l \). Then, the Toeplitz operators \( T_a \tilde{T}_{\xi^p \bar{\xi}^q} \) and \( T_b \tilde{T}_{\xi^u \bar{\xi}^v} \) commute on each weighted Bergman space \( \mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C})) \) if and only if for each \( j = 1, \ldots, l \) one of the following conditions hold

(i) \( p_j = q_j = 0 \)
(ii) \( u_j = v_j = 0 \)
(iii) \( p_j = u_j = 0 \)
(iv) \( q_j = v_j = 0 \)

**Proof.** First, we observe that for our hypotheses, the quantities \( T_a \tilde{T}_{\xi^p \bar{\xi}^q} z^\alpha \) and \( T_a \tilde{T}_{\xi^p \bar{\xi}^q} T_b \tilde{T}_{\xi^u \bar{\xi}^v} z^\alpha \) are always simultaneously zero or non zero. Hence, we compute \( T_a \tilde{T}_{\xi^p \bar{\xi}^q} z^\alpha \) and \( T_b \tilde{T}_{\xi^u \bar{\xi}^v} z^\alpha \), for \( \alpha \in J_n(m) \), assuming that both are non zero.

By (4.11), we have the following expression

\[
T_{a \tilde{T}_{\xi^p \bar{\xi}^q}} T_{b \tilde{T}_{\xi^u \bar{\xi}^v}} z^\alpha = \frac{2^l(\alpha + p + q + u + v)!(n + m)!}{(\alpha + p + q + u + v)!(n + m)!} (\alpha + p + q + u + v)!(n + m)! \prod_{j=1}^l r_j^{2|\alpha| + 2k_j - 1} dr_j
\]

where \( a(r_1, \ldots, r_l) \) and \( b(r_1, \ldots, r_l) \) are \( k \)-quasi-homogeneous functions on \( \mathbb{P}^n(\mathbb{C}) \) with \( |p(j)| = |q(j)| \) and \( |u(j)| = |v(j)| \) for all \( j = 1, \ldots, l \).
Similarly, we also have
\[
T_{a_{ξ^r}ξ^s}T_{b_{ξ^r}ξ^s}z^α = 2^l(α + u)!/(n + m)! \left( \frac{2r(α + u - v)!/(m - |α|)!}{(α + p - q + u - v)!/(m - |α|)!} \right)
\times \int_{\mathbb{R}^+_k} b(r_1, \ldots, r_l)(1 + r^2)^{(n + m + 1)} \prod_{j=1}^l r_j^{2|α(j)| + 2k_j - 1} dr_j
\times \frac{2l(α + u - v + p)!/(n + m)!}{(α + p - q + u - v)!/(m - |α|)!} \prod_{j=1}^l (k_j - 1 + |α(j) + p(j)|)
\times \int_{\mathbb{R}^+} a(r_1, \ldots, r_l)(1 + r^2)^{(n + m + 1)} \prod_{j=1}^l r_j^{2|α(j)| + 2k_j - 1} dr_j
\times 2^{a+\nu+u-v}.
\]
Therefore, we conclude that \( T_{a_{ξ^r}ξ^s}T_{b_{ξ^r}ξ^s}z^α = T_{a_{ξ^r}ξ^s}T_{b_{ξ^s}ξ^r}z^α \) if and only if
\[
(α_j + u_j - v_j + p_j)!/(α_j + u_j)! = (α_j + p_j - q_j + u_j)!/(α_j + p_j) \]
for \( j = 1, \ldots, l \). And this equality holds if and only if for each \( j = 1, \ldots, l \) one of the following conditions is fulfilled
\[(i) \quad p_j = q_j = 0, \]
\[(ii) \quad u_j = v_j = 0, \]
\[(iii) \quad p_j = u_j = 0, \]
\[(iv) \quad q_j = v_j = 0. \]

Finally, we present one of our main results: the construction of a Banach algebra of Toeplitz operators on \( \mathbb{P}^n(\mathbb{C}) \). Our construction is parallel to the one presented in [11].

As above, continue to consider \( k \in \mathbb{Z}^l_+ \) a partition of \( n \), and now let \( h \in \mathbb{Z}^l_+ \) be such that \( 1 \leq h_j \leq k_j - 1 \), for all \( j = 1, \ldots, l \). Let \( ϕ \) be a \( k \)-quasi-homogeneous symbol on \( \mathbb{C}^n \) of the form
\[
ϕ(z) = a(r_1, \ldots, r_l)ξ^pξ^q,
\]
where \( p, q \in \mathbb{N}^n \) satisfy \( p \cdot q = 0 \). Consider the decompositions of \( p \) and \( q \) given as follows
\[
p = (p_1, \ldots, p_l), \quad q = (q_1, \ldots, q_l),
\]
\[
p_j = (p_{j,1}, \ldots, p_{j,k_j}), \quad q_j = (q_{j,1}, \ldots, q_{j,k_j}),
\]
for \( j = 1, \ldots, l \). With respect to this decomposition, we will now assume that
\[
p_{j,r} = 0, \quad q_{j,s} = 0,
\]
for \( r > h_j \) and \( s \leq h_j \), where \( j = 1, \ldots, l \). As before, we also assume that \( |p_j| = |q_j| \) for all \( j \), which now becomes
\[
p_{j,1} + \cdots + p_{j,h_j} = q_{j,h_j+1} + \cdots + q_{j,k_j}.
\]
Let us denote with \( R_k(h) \) the space of all symbols obtained through this construction. As a consequence of Theorem 4.3.
Theorem 4.5 (Banach algebra of symbols with commuting operators). For \( k \) and \( h \) as above, the Banach algebra of Toeplitz operators generated by the symbols in \( \mathcal{R}_k(h) \) is commutative on each weighted Bergman space on \( \mathbb{F}^n(\mathbb{C}) \).

5. BUNDLES OF LAGRANGIAN FRAMES AND QUASI-HOMOGENEOUS SYMBOLS

In the rest of this work, \( \mathcal{R}_k(h) \) will denote the space of symbols described above for \( U^n \); recall that the latter is either \( \mathbb{C}^n \) or \( \mathbb{B}^n \). Whenever necessary, we will specify which of these two choices is under consideration.

Recall that the \( n \)-torus \( T^n \) acts by isometric biholomorphisms on \( U^n \) by the expression

\[
t \cdot z = (t_1 z_1, \ldots, t_n z_n)
\]

where \( t \in T^n \) and \( z \in U^n \). The following result associates to the space of symbols \( \mathcal{R}_k(h) \) a subgroup of the Abelian group of isometries corresponding to the \( T^n \)-action.

Theorem 5.1 (Torus associated to \( \mathcal{R}_k(h) \)). Let us consider the space of symbols \( \mathcal{R}_k(h) \) on \( U^n \) where \( k \in \mathbb{Z}_+ \) is a partition of \( n \), and consider the subgroup of \( T^n \) given by

\[
T_k = \prod_{j=1}^l \mathbb{T}1_{[k_j]}
\]

where \( 1_{[k_j]} \) denotes the element of \( \mathbb{C}^{k_j} \) all of whose coordinates are 1. Then, for the \( T^n \)-action by isometries on \( U^n \), \( T_k \) is precisely the subgroup of \( T^n \) that leaves invariant the elements of \( \mathcal{R}_k(h) \). In other words, for \( t \in T^n \) we have

\[
t \in T_k \iff \varphi(tz) = \varphi(z) \text{ for every } \varphi \in \mathcal{R}_k(h), z \in U^n.
\]

Proof. Associated to the partition \( k = (k_1, \ldots, k_l) \), we recall the decomposition \( z = (z_{(1)}, \ldots, z_{(l)}) \) introduced before and note that we have a corresponding decomposition \( t = (t_1, \ldots, t_{(l)}) \) for every element \( t \in T^n \). With this notation at hand, we observe that for \( t \in T^n \) the following two conditions are equivalent

- \( t \in T_k \),
- there exist \( s \in T^l \) such that

\[
t_{(j)} = s_j 1_{[k_j]}
\]

for all \( j = 1, \ldots, l \).

First, let us consider \( \varphi \in \mathcal{R}_k(h) \) and \( t \in T_k \) as given in equation (5.1). If \( \varphi(z) = a(r)\xi^p \zeta^q \), then

\[
\varphi(tz) = a(r)(t\xi)^p (\overline{t}\xi)^q = a(r)t^p t^q \xi^p \zeta^q = \varphi(z) \prod_{j=1}^l t_{(j)}^{p_{(j)}} \zeta_{(j)}^{q_{(j)}} = \varphi(z) \prod_{j=1}^l s_j^{p_{(j)}} |s_j|^{q_{(j)}} = \varphi(z),
\]

where we have used that \( |p_{(j)}| = |q_{(j)}| \) for all \( j = 1, \ldots, l \), since \( \varphi \in \mathcal{R}_k(h) \); note that we have also used that the quasi-radial symbols \( a(r) \) is \( T^n \)-invariant.

Conversely, let us assume that \( t \in T^n \) satisfies \( \varphi(tz) = \varphi(z) \), for every \( \varphi \in \mathcal{R}_k(h) \) and \( z \in U^n \). We will pick a particular choice of \( p, q \in \mathbb{N}^n \). Given \( 1 \leq j_0 \leq l \) choose
$r, s$ such that $1 \leq r \leq h_j < s \leq k_j$ and define

$$(p_{(j)})_i = \begin{cases} 
1 & \text{if } j = j_0 \text{ and } i = r, \\
0 & \text{otherwise},
\end{cases}$$

$$(q_{(j)})_i = \begin{cases} 
1 & \text{if } j = j_0 \text{ and } i = s, \\
0 & \text{otherwise}.
\end{cases}$$

Then, it is easy to check that the symbol $\varphi(z) = \xi^r\xi^s$ belongs to $R_k(h)$. For this symbol, we have

$$\varphi(tz) = (t_{(j_0)})_r(t_{(j_0)})_s\varphi(z),$$

and so the invariance of $\varphi$ under $t$ implies that

$$(t_{(j_0)})_r = (t_{(j_0)})_s$$

for all $r, s$ satisfying $1 \leq r \leq h_j < s \leq k_j$ for our arbitrarily given $1 \leq j_0 \leq l$. This shows that $t$ satisfies the conditions given by equation (5.1) and thus completes the proof. □

The set of symbols $R_k(h)$ is determined by the partition $k \in \mathbb{Z}_l$ of $n$ and $h \in \mathbb{Z}_l$ (satisfying the properties described before). We will in turn associate to $k$ a fibration whose fibers carry distinguished symplectic geometry and that can be naturally associated to the set of symbols $R_k(h)$.

Let us start by defining the map

$$\pi_k : \mathbb{P}^n(\mathbb{C}) \to \prod_{j=1}^l \mathbb{P}^{k_j-1}(\mathbb{C})$$

$$[z_0, z] \mapsto \left(\left[\frac{1}{z_{(1)}}\right], \ldots, \left[\frac{1}{z_{(l)}}\right]\right),$$

where $z_0 \in \mathbb{C}$ and $z \in \mathbb{C}^n$. Then, $\pi_k$ is a rational map defined in the complement of the algebraic variety

$$P_k = \{[z_0, z] \in \mathbb{P}^n(\mathbb{C}) : z_0 \in \mathbb{C} \setminus \{0\}, z_{(j)} \in \mathbb{C}^{k_j} \setminus \{0\}, j = 1, \ldots, l\}.$$ 

In particular, it is well defined and holomorphic in the subset

$$V_k = \prod_{j=1}^l (\mathbb{C}^{k_j} \setminus \{0\}) \subset \mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C}),$$

where the last inclusion is given by the canonical embedding $z \mapsto [1, z]$ introduced before. In fact, with this realization of $V_k$ as subset of $\mathbb{P}^n(\mathbb{C})$ we have

$$\pi_k(z) = ([z_{(1)}], \ldots, [z_{(l)}]),$$

for every $z \in V_k$. We also note that the set $V_k$ is the natural domain of definition of the symbols that belong to $R_k(h)$.

We now consider the complexification of the torus $T_k$ introduced above. More precisely, we denote

$$A_k = \prod_{j=1}^l \mathbb{C}^*1_{[k_j]},$$

where, as in the statement of Theorem 5.1, $1_{[k_j]}$ denotes the element of $\mathbb{C}^{k_j}$ all of whose coordinates are 1. Note that $A_k$ is a subgroup of $\mathbb{C}^n$ isomorphic to $\mathbb{C}^l$ by
an obvious isomorphism with the same expression as the one described above for $T_k$ and $T^l$.

Recall, that the group $\mathbb{C}^{\ast n}$ acts on $\mathbb{P}^n(\mathbb{C})$ biholomorphically through the assignment
\[ (a, [z_0, z]) \mapsto [z_0, a_1 z_1, \ldots, a_n z_n], \]
for $a \in \mathbb{C}^{\ast n}$, $z_0 \in \mathbb{C}$ and $z \in \mathbb{C}^n$. Furthermore, this action restricts to the natural componentwise action of $\mathbb{C}^{\ast n}$ on $\mathbb{C}^n$. In particular, the restriction of this action to $A_k$ is given on $V_k$ as follows.

\[ A_k \times V_k \to V_k \]
\[ ((a_1 [k_i], \ldots, a_l [k_i]), z) \mapsto (a_1 z_1, \ldots, a_l z_l), \]

(5.3)

Also, note that the restriction of such action to $T_k \subset A_k$ yields the action that leaves invariant the symbols of $R_k(h)$. With this respect, the following result relates the set of symbols $R_k(h)$ to the rational map $\pi_k$, while providing geometric information on the latter. We note that our notion of principal fiber bundle is as defined in \cite{4}.

**Theorem 5.2** (Principal bundle associated to $R_k(h)$). Let $k \in \mathbb{Z}_+$ be a partition of $n$. Then, the rational map $\pi_k$ satisfies the following property.

- The $A_k$-action on $V_k$ turns the map $\pi_k|V_k : V_k \to \prod_{j=1}^l \mathbb{P}^{k_j - 1}(\mathbb{C})$ into a complex principal fiber bundle with structure group $A_k$.

In particular, $\pi_k|V_k : V_k \to \prod_{j=1}^l \mathbb{P}^{k_j - 1}(\mathbb{C})$ is a holomorphic submersion. Also, for every $p \in \prod_{j=1}^l \mathbb{P}^{k_j - 1}(\mathbb{C})$, the fiber $\pi_k^{-1}(p)$ is a holomorphic $l$-dimensional submanifold of $V_k \subset \mathbb{C}^n$.

**Proof.** Using the expressions for $\pi_k$ given in \cite{5, 2}, we note that the $\pi_k|V_k$ is the product of the canonical quotient maps
\[ \mathbb{C}^{k_j} \setminus \{0\} \to \mathbb{P}^{k_j - 1}(\mathbb{C}), \quad j = 1, \ldots, l, \]
which are well known to be principal fiber bundles with structure group $\mathbb{C}^\ast$. Then, the expression for the $A_k$-action given by \cite{5, 3} implies that $\pi_k$ is indeed a principal fiber bundle with structure group $A_k$. The other claims follow from elementary properties of fiber bundles and submersions. \hfill \Box

We recall the definition of Lagrangian frame introduced in \cite{8, 9, 7}. We refer to these references for further details on the notions involved.

**Definition 5.3.** On a Kähler manifold $N$, a Lagrangian frame is a pair $(O, \mathcal{F})$ of smooth foliations that satisfy the following properties.

- Both foliations are Lagrangian. In other words, the leaves of both foliations are Lagrangian submanifolds of $N$.
- If $L_1$ and $L_2$ are leaves of $O$ and $\mathcal{F}$, respectively, then $T_x L_1 \perp T_x L_2$ at every $x \in L_1 \cap L_2$.
- The foliation $O$ is Riemannian. I.e. the Riemannian metric of $N$ is invariant by the leaf holonomy of $O$.
- The foliation $\mathcal{F}$ is totally geodesic. I.e. its leaves are totally geodesic submanifolds of $N$.

We will refer to $O$ and $\mathcal{F}$ as the Riemannian and totally geodesic foliations, respectively, of the Lagrangian frame.
The next result shows that the fibers of the submersion \( \pi_k \) carry Lagrangian frames naturally associated to the symbols \( \mathcal{R}_k(h) \). Note that the result holds for both the projective geometry and hyperbolic geometry of \( \mathbb{P}^n(\mathbb{C}) \) and \( \mathbb{B}^n \), respectively, even though their Kähler structures are different. We recall that, with our previous convention, \( \mathbb{U}^n \) denotes either \( \mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C}) \) or \( \mathbb{B}^n \). And we also emphasize that \( \mathbb{U}^n \) carries either the complex projective or complex hyperbolic Kähler metric according to the choice of either case.

**Theorem 5.4** (Lagrangian frames associated to \( \mathcal{R}_k(h) \)). For a partition \( k \in \mathbb{Z}_+^l \) of \( n \), let \( \mathcal{R}_k(h) \) be the set of quasi-radial quasi-homogeneous symbols on \( \mathbb{U}^n \). Then, every fiber of the holomorphic submersion \( \pi_k|_{V_k \cap \mathbb{U}^n} : V_k \cap \mathbb{U}^n \to \prod_{j=1}^l \mathbb{P}^{k_j-1}(\mathbb{C}) \) carries a Lagrangian frame for the Kähler structure inherited from \( \mathbb{U}^n \) so that the symbols belonging to \( \mathcal{R}_k(h) \) are constant on the leaves of the Riemannian foliation of the frame. More precisely, the following conditions hold for every fiber \( F_p = \pi_k^{-1}(p) \cap V_k \cap \mathbb{U}^n \), where \( p \in \pi_k(V_k \cap \mathbb{U}^n) \).

(i) The action of \( T_k \) restricted to \( F_p \) defines a Riemannian foliation \( \mathcal{O}_p \) on whose leaves every symbol that belongs to \( \mathcal{R}_k(h) \) is constant.

(ii) The vector bundle \( T\mathcal{O}_p^\perp \) defined as the orthogonal complement of \( T\mathcal{O}_p \) inside of \( TF_p \) is integrable to a totally geodesic foliation \( J\mathcal{O}_p \).

(iii) The pair \( (\mathcal{O}_p, J\mathcal{O}_p) \) is a Lagrangian frame of the complex manifold \( F_p \) for the Kähler structure inherited from \( \mathbb{U}^n \).

**Proof.** We fix a fiber \( F_p \) as described in the statement. We will also denote \( \hat{F}_p = \pi_k^{-1}(p) \), so that \( F_p \subset \hat{F}_p \) with proper inclusion precisely in the case \( \mathbb{U}^n = \mathbb{B}^n \). Note that both \( F_p \) and \( \hat{F}_p \) are complex \( l \)-dimensional submanifolds of \( \mathbb{C}^n \).

By Theorem 5.2, the map \( \pi_k \) is \( A_k \)-invariant, and since \( \mathbb{U}^n \) is \( T_k \)-invariant, it follows that \( F_p \) is \( T_k \)-invariant. Also, the action of \( A_k \) on \( \hat{F}_p \) is free and so \( T_k \) acts freely on \( F_p \) thus defining a foliation \( \mathcal{O}_p \) with real \( l \)-dimensional leaves. We recall that the \( T_k \)-action is a restriction of the natural \( \mathbb{T}^n \)-action on \( \mathbb{C}^n \). Since the latter is isometric on \( \mathbb{U}^n \), then the \( T_k \)-action is isometric as well. This last property implies that the foliation \( \mathcal{O}_p \) is Riemannian (see, for example, [7]). Moreover, by Theorem 5.1 the symbols that belong to \( \mathcal{R}_k(h) \) are \( T_k \)-invariant and so constant on the leaves of \( \mathcal{O}_p \). This proves (1).

It is known that the \( \mathbb{T}^n \)-action on \( \mathbb{U}^n \) defines a Lagrangian foliation: the \( \mathbb{T}^n \)-orbits are Lagrangian. This has been verified in [7] and [6] for \( \mathbb{U}^n = \mathbb{B}^n \) and \( \mathbb{U}^n = \mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C}) \), respectively. Since \( T_k \) is a subgroup of \( \mathbb{T}^n \), we conclude that \( \omega(T_z\mathcal{O}_p, T_z\mathcal{O}_h) = 0 \) for every \( z \in F_p \) and for the Kähler form \( \omega \) of \( \mathbb{U}^n \). Also, we observe that the real dimension of the leaves of \( \mathcal{O}_p \) and the complex dimension of the fiber \( F_p \) have the common value \( l \). This implies that \( \mathcal{O}_p \) is a Lagrangian foliation of \( F_p \). Since the orthogonal complement of a Riemannian foliation is totally geodesic (see, for example, [7]), to prove (2) and (3) it suffices to show that \( T\mathcal{O}_p^\perp = iT\mathcal{O}_p \) is integrable.

To prove the integrability of \( iT\mathcal{O}_p \) let us consider the vector field on \( \mathbb{U}^n \) given by the expression

\[
(X_j)_z = (i\delta_{j1}z(1), \ldots, i\delta_{jl}z(l)), \quad j = 1, \ldots, l,
\]

where \( \delta_{jr} \) denotes Kronecker’s delta for \( j, r = 1, \ldots, l \). Then, \( X_j \) is the vector field on \( \mathbb{U}^n \) induced by the action of a one-parameter subgroup of \( T_k \). More precisely,
for the action
\[ \psi_j : \mathbb{R} \times \mathbb{U}^n \rightarrow \mathbb{U}^n \]
\[ (\theta, z) \mapsto (e^{i\delta_{j1}\theta}z(1), \ldots, e^{i\delta_{jl}\theta}z(l)) \]
we have
\[ (X_j)_z = \frac{d}{d\theta} \bigg|_{\theta=0} \psi_j(\theta, z). \]
Note that either a direct computation or the fact that \( T_k \) is Abelian implies that
\[ [X_{j1}, X_{j2}] = 0, \]
for all \( j_1, j_2 = 1, \ldots, l \). Also, we observe that \((X_1)_z, \ldots, (X_l)_z\) defines a basis for \( T_zO_p \) for every \( z \in F_p \).

A similar construction provides an explicit expression for the local flow associated to the vector field \( JX_{j1} \), for \( j = 1, \ldots, l \). Here and in what follows, \( J \) denotes the complex structure of \( \mathbb{U}^n \). We now consider the assignment
\[ \beta_j : (r, z) \mapsto (r^{\delta_{j1}}z(1), \ldots, r^{\delta_{jl}}z(l)), \quad j = 1, \ldots, l, \]
for which we now have
\[ \frac{d}{dr} \bigg|_{r=1} \beta_j(r, z) = (\delta_{j1}z(1), \ldots, \delta_{jl}z(l)) = (JX_j)_z. \]
We note that the local flow defined by \( \beta_j \) is holomorphic.

We observe that \((JX_1)_z, \ldots, (JX_l)_z\) is a basis for \( iT_zO_p \) for every \( z \in F_p \); this is a consequence of the corresponding property for \( T_zO_p \) stated above. Next, we observe that, for \( j_1, j_2 = 1, \ldots, l \), we have
\[ [JX_{j1}, JX_{j2}] = J[X_{j1}, JX_{j2}] \]
\[ = J^2[X_{j1}, X_{j2}] \]
\[ = 0. \]
Here we used in the first and second identities the fact that \( JX_{j2} \) and \( X_{j1} \), respectively, define Lie derivatives that commute with \( J \); the latter is a consequence of the fact that both fields have holomorphic local flows (see [5]). For the third identity, we used the equations (5.4). Thus, we have proved that the bundle \( iTO_p \) has a set of sections that generate the fibers and commute pairwise. Hence, the integrability of \( iTO_p \) follows from Frobenius’ Theorem.

Finally, we prove that a suitable complement of \( A_k \) in \( \mathbb{C}^n \) acts by symmetries of the bundle obtained in Theorem 5.2.

For a partition \( k \in \mathbb{Z}_+^l \) of \( n \), let us consider the following subgroups. For every \( j = 1, \ldots, l \) we denote
\[ B_j = \{ z \in (\mathbb{C}^*)^{k_j} : z_1 \cdots z_{k_j} = 1 \} \]
and consider the group given by
\[ B_k = \prod_{j=1}^l B_j \subset \mathbb{C}^n. \]

**Lemma 5.5.** For any partition \( k \in \mathbb{Z}_+^l \) of \( n \) the map
\[ A_k \times B_k \rightarrow \mathbb{C}^n \]
\[ (a, b) \mapsto ab \]
is an isomorphism of Lie groups.

Proof. For a given integer \( m \in \mathbb{Z}_+ \) consider the group given by
\[
B = \{ z \in \mathbb{C}^m : z_1, \ldots, z_m = 1 \}.
\]
It is immediate that the product map
\[
\mathbb{C}^* \times B \to \mathbb{C}^m
\]
\[
(z, b) \mapsto (zb_1, \ldots, zb_m),
\]
is an isomorphism of Lie groups. The statement of the Lemma follows from this fact and the definitions of \( A_k \) and \( B_k \).

The group \( B_k \) has a natural action on \( \mathbb{P}^n(\mathbb{C}) \) as a subgroup of \( \mathbb{C}^* \). Also, for every \( j = 1, \ldots, l \) the group \( B_j \), as a subgroup of \( \mathbb{C}^* \), has a natural action on \( \mathbb{P}^{k_j-1}(\mathbb{C}) \). By taking the product of these actions, we obtain an action of \( B_k \) on \( \prod_{j=1}^l \mathbb{P}^{k_j-1}(\mathbb{C}) \). Furthermore, it is easy to verify that for these actions we have
\[
\pi_k(b \cdot z) = b\pi_k(z),
\]
for every \( b \in B_k \) and \( z \in V_k \). This implies the first part of the following result. The second part brings into play the subgroup of \( \mathbb{T}^n \) that can be considered the “complement” of \( T_k \). We refer to [\[4\]] for the definition of the automorphism of a principal bundle.

**Theorem 5.6.** The \( B_k \)-actions on \( V_k \) and \( \prod_{j=1}^l \mathbb{P}^{k_j-1}(\mathbb{C}) \) yield principal bundle automorphisms \( \pi_k|_{V_k} : V_k \to \prod_{j=1}^l \mathbb{P}^{k_j-1}(\mathbb{C}) \). In particular, the action of \( \mathbb{T}^n \cap B_k \) is an isometric action on \( V_k \cap \mathbb{U}^n \) that permutes the fibers of \( \pi_k|_{V_k \cap \mathbb{U}^n} \) and their Lagrangian frames defined in Theorem [\[5.3\]].

Proof. As noted above, only the second part requires justification. First note that \( \mathbb{T}^n \) acts isometrically and so \( \mathbb{T}^n \cap B_k \) acts isometrically as well. Also, the Lagrangian frames of the fibers are given by \( T_k \)-orbits and their orthogonal complements. But \( \mathbb{T}^n \cap B_k \) preserves the former since the \( B_k \)-action commutes with the \( T_k \)-action. Finally, \( \mathbb{T}^n \cap B_k \) preserves the orthogonal complement of the \( T_k \)-orbits because the action of \( \mathbb{T}^n \cap B_k \) is isometric.

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