Do Majorana zero modes emerge in the hybrid nanowire under a strong magnetic field?

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The hybrid nanowire consisting of semiconductor with proximity to superconductor is expected to serve as an experimental platform to display Majorana zero modes. By rederiving its effective Kitaev model with spins, we discover a novel topological phase diagram, which assigns a more precise constraint on the magnetic field strength for the emergence of Majorana zero modes. It then turns out the effective pairing strength dressed by the proximity effect exhibits a significant dependence on the magnetic field, and thus the topological phase region is refined as a closed triangle in the phase diagram with chemical potential vs. Zeeman energy (which is obviously different from the open hyperbolic region known before). This prediction is confirmed again by an exact calculation of quantum transport, where the zero bias peak of $2e^2/h$ in the differential conductance spectrum, as the necessary evidence for the Majorana zero modes, disappears when the magnetic field grows too strong. For illustrations with practical hybrid systems, in the InSb nanowire coupled to NbTiN, the accessible magnetic field range is around 0.1 – 1.5 T; when coupled to aluminum shell, the accessible magnetic field range should be smaller than 0.12 T. These predictions obviously clarify the current controversial issues about some experiments of Majorana zero modes with hybrid nanowire.

Introduction - The experiments for the Majorana zero modes (MZMs) have attracted extensive attentions in recent years [1–7], for their novel fractional statistics and potential applications in quantum computation [1, 8, 9]. In particular, the possible conductance signature for MZMs has been shown in hybrid semiconductor-superconductor (HSS) systems [10, 11], where a semiconductor nanowire with appreciable spin-orbit coupling is contacted with an s-wave superconductor (SC) providing the SC proximity effect [see Fig. 1]. Under a proper magnetic field, effectively a p-wave pairing could be induced in the hybrid nanowire, which gives birth to the MZMs localized at the two ends of the open wire. In transport experiments, the existence of MZMs would result in a zero bias peak (ZBP) in the differential conductance spectrum, and the height of the ZBP should be $2e^2/h$ in the idealistic case at the zero temperature [12, 13].

However, most of the current ZBP detections for MZMs do not reach the idealistic height of $2e^2/h$ but lower [10, 11, 14, 15]. As a result, such a ZBP signature alone cannot sufficiently confirm the existence of MZMS as the unique reason apart from other possible physical effects, such as Andreev bound states [7, 16–18], and the Kondo effect [19, 20]. To help narrow down the searching range in experiments, a more precise phase diagram for different parameter regimes is urgently needed to find the MZMs, for example, what is the valid extent for the magnetic field where the MZMs could exist.

In previous investigations, the SC proximity effect was assumed to be an s-wave pairing in the nanowire with the pairing strength, which is approximated as a constant [2, 21–25], and this is consistent with the result from the Bogoliubov-de-Gennes equation projected in the low energy band [5, 6, 26–30]. It follows from this over-approximation that the topological region bearing MZMs fills up the whole upper half of a hyperbolic curve in the $\mu$-$B$ diagram ($\mu$, $B$ are the chemical potential and Zeeman splitting of the nanowire respectively). Namely, these approximated results indicate MZMs always exist no matter how strong the magnetic field grows. In this letter, however, we find that indeed this conclusion is not proper, and MZMs emerge only when the magnetic field lies within a modest regime.

First, we utilize the Fröhlich-Nakajima (Schrieffer-Wolff) transformation to obtain an effective Hamiltonian for the nanowire, which encloses the SC proximity effect and then gives an effective Kitaev model in the HSS nanowire [31–33]. It turns out the effective pairing strength exhibits significant dependence on the strong magnetic field. As a result, the topological region bearing MZMs turns out to be a closed triangle in the $\mu$-$B$ diagram in comparison with the hyperbolic curve known before. Namely, when the magnetic field is too strong, the system enters the non-topological region, and thus MZMs disappear; when the magnetic field is weak, this triangle region just returns the previous hyperbolic curve. It is predicted from this phase diagram that the MZMs only emerge when the magnetic field strength lies within a modest regime. For the current experiments with InSb-NbTiN and InSb-Al, the valid...
magnetic field ranges are 0.1 – 1.5 T and 0.013 – 0.12 T respectively, the latter of which is out of the range used in some of the current experiments.

To confirm the above prediction based on the effective theory, we further make an exact calculation on the differential conductance in the transport measurement of the HSS nanowire by a quantum Langevin equation \[34\] – \[37\]. When the magnetic field increases from zero, a ZBP with \(2e^2/h\) appears at a certain field strength, keeps for a while, and then disappears at a certain higher strength, which is just consistent with the conclusion from the effective Hamiltonian. This result provides a more precise phase diagram which refines the searching region of MZMs in experiments.

**Effective Kitaev model in low energy scale** - In this HSS system, the semiconductor nanowire is described by the Hamiltonian \[3\]

\[
\hat{H}_w = \int dx \hat{\psi}^\dagger(x) \left[ -\frac{\partial^2}{2m_w} - \mu - i\alpha \gamma \partial_x + B \sigma^z \right] \hat{\psi}(x),
\]

\(\hat{\psi}(x) := [\hat{\psi}_\uparrow(x), \hat{\psi}_\downarrow(x)]^T\), and \(\sigma^{y,z}\) are the Pauli matrices. Here, \(\uparrow, \downarrow\) indicate the electron spins, \(\alpha\) the spin-orbit coupling strength, \(m_w\) the effective mass, \(\mu\) the chemical potential of the nanowire, and \(B\) the Zeeman splitting from the external magnetic field respectively.

The semiconductor nanowire is placed in contact with an s-wave SC, whose mode energies are zero, and they are just the MZMs. Hereafter the chemical potential \(\mu\) of the s-wave SC is set as \(\mu_{sc} \equiv k^2/2m_{sc} - \mu_{sc}\). Herewith the chemical potential of the s-wave SC is set as \(\mu_{sc} \equiv 0\). The whole nanowire is contacted with the surface of the nanowire by a quantum Langevin equation \[34\] – \[37\]. For a fixed chemical potential \(\mu\), the MZMs could emerge only if the magnetic field strength must properly lie in a modest range. When the magnetic field strength exceeds the range of the magnetic field determined by refined phase region, the effective Hamiltonian \[4\] of the hybrid nanowire does not support the existence of the MZMs\[1\].

For the weak field situation \((B \ll \Delta_s)\), the induced pairing strength can be approximated as a constant \(\Delta_k \approx \gamma_s [\text{see Eq. (5)}]\). Correspondingly, the above topological phase condition is reduced as \(B^2 - \mu^2 = \gamma_s^2/(1 - \gamma_s/\Delta_s)^2 \approx \gamma_s^2\), which just returns the hyperbolic curve sufficiently studied in previous literatures \[3, 38\]. Indeed the bottom part of the close triangle region and the hyperbolic curve fit well with each other (Fig. 2), which is consistent with the fact that the hyperbolic curve comes from an effective theory in the low energy regime.

To have a more clear understanding on above observations, we consider the above effective Kitaev Hamiltonian in a new representation [by taking the first three bracket terms of (4) into diagonalization],

\[
\hat{H}_{eff} = \int \frac{dk}{2\pi} \left\{ \tilde{\epsilon}_{w,k} \hat{\phi}^\dagger_{\uparrow}(k) \hat{\phi}_{\uparrow}(k) + \tilde{\epsilon}_{w,k} \hat{\phi}^\dagger_{\downarrow}(k) \hat{\phi}_{\downarrow}(k) \right\}
\]

\[
+ \frac{1}{2} \tilde{\Delta}_k^{(p)} \left[ \hat{\phi}_{\uparrow}^\dagger(k) \hat{\phi}_{\downarrow}(k) + \hat{\phi}_{\uparrow}(k) \hat{\phi}_{\downarrow}^\dagger(k) + \hat{H.c.} \right] + \tilde{\Delta}_k^{(s)} \left[ \hat{\phi}_{\uparrow}^\dagger(k) \hat{\phi}_{\downarrow}(k) + \hat{H.c.} \right].
\]

\[1\] With the increase of the magnetic field, the s-wave SC gap \(\Delta_s\) would also decrease. Here this effect is not considered, and \(\Delta_s\) is treated as a constant independent of the magnetic field. If this effect is considered, the topological region in Fig. 2 would be smaller.
Quantum transport - To confirm the above observations from the effective Hamiltonian (1) we make an exact calculation on the transport behavior of the hybrid nanowire. To this end, the above continuous Hamiltonian (1) is firstly discretized with respect to an $N$-site system as [25, 41]

$$
\hat{H}_w = \sum_{n,s} \left( \frac{J}{2} (\hat{d}_{n,s}^{\dagger} \hat{d}_{n+1,s} + \hat{d}_{n+1,s}^{\dagger} \hat{d}_{n,s}) - (\mu - J) \hat{d}_{n,s}^{\dagger} \hat{d}_{n,s} \right) + \sum_n \alpha \left( \hat{d}_{n,s}^{\dagger} \hat{d}_{n+1,s} - \hat{d}_{n+1,s}^{\dagger} \hat{d}_{n,s} + \text{H.c.} \right) + \sum_n B (\hat{d}_{n,s}^{\dagger} \hat{d}_{n,s} - \hat{d}_{n,s}^{\dagger} \hat{d}_{n,s}).
$$

Correspondingly, the tunneling term (3) between the nanowire and the $s$-wave SC becomes $\hat{H}_{w_{sc}} = -\sum_{n,k,s} \left( J_{n,k,s} \hat{d}_{n,s}^{\dagger} \hat{c}_{k,s} + \text{H.c.} \right)$, with the tunneling strength $|J_{n,k,s}| = |J_{n,k,s}| := \mathcal{J}_k$ for $m \neq n, s \neq s'$. We consider two electron leads are contacted with the two ends of the nanowire, which are described by $\hat{H}_{e-x} = \sum_{k,s} \hat{d}_{x,k}^{\dagger} \hat{b}_{s,k,s} + \text{H.c.}$, with $x = 1, N$ as the site number contacting with the electron lead. The tunneling interaction between the nanowire and lead-$x$ is $\hat{H}_{e-x} = -\sum_{k,s} (\mathcal{G}_{x,k} \hat{d}_{x,k}^{\dagger} \hat{b}_{s,k,s} + \text{H.c.})$.

Notice that, similar to the two electron leads, indeed the $s$-wave SC also can be regarded as the third fermionic bath interacting with the nanowire. Thus, it follows from the Heisenberg equations of $\hat{d}_{n,s}$ (nanowire), $\hat{c}_{k,s}$ ($s$-wave SC), and $\hat{b}_{k,s}$ (electron leads) that a quantum Langevin equation is derived to describe the transport dynamics (supplemental materials) as [34–37]

$$\partial_t \hat{d} = -i \mathbf{H}_w \cdot \hat{d} - \int_0^t d\tau \mathbf{D}(t - \tau) \cdot \hat{d}(\tau) + i \xi_{\text{ac}} + i \hat{\xi}_e. \quad (9)$$

Here, $\hat{d}(t) := (\hat{d}_1, \ldots, \hat{d}_N)^T$ is a $4N$-vector form with $N$ blocks $\hat{d}_n := (\hat{d}_{n\uparrow}, \hat{d}_{n\downarrow}, \hat{d}_{n\uparrow}^{\dagger}, \hat{d}_{n\downarrow}^{\dagger})^T$. Then the nanowire Hamiltonian (8) is rewritten as $\hat{H}_w = \frac{1}{2} \mathbf{H}_w \cdot \mathbf{d}$, with a $4N \times 4N$ matrix $\mathbf{H}_w$. The dissipation kernel $\mathbf{D}(t) = \mathbf{D}_w(t) + \mathbf{D}_{\text{ac}}(t)$ contains the contributions from both the two electron leads and the $s$-wave SC, and $\xi_{\text{ac}}(t)$ and $\hat{\xi}_e(t)$ are the corresponding random forces respectively.

This Langevin equation (9) of $\hat{d}(t)$ is exactly solvable in the Fourier space, namely,

$$\hat{d}(\omega) = \mathbf{G}(\omega) \cdot \left[ \hat{d}(t=0) + i \xi_{\text{ac}}(\omega) + i \hat{\xi}_e(\omega) \right], \quad (10)$$

where $[\mathbf{G}(\omega)]_{4N \times 4N}$ is the Green function of the nanowire,

$$\mathbf{G}(\omega) = i \left[ \mathbf{\Omega}_{w} + \mathbf{\Sigma}_{\text{sc}}(\omega) + i \mathbf{\Sigma}_{e}(\omega) \right]^{-1} \equiv i \left\{ \mathbf{\Omega}_{w} + \mathbf{\Sigma}_{\text{sc}}(\omega) + i \left[ \mathbf{\Gamma}_{\text{sc}}(\omega) + \mathbf{\Gamma}_{e}(\omega) \right] \right\}^{-1}, \quad (11)$$

with $\omega^+ \equiv \omega + i\epsilon$ ($\epsilon$ is an infinitesimal). Here $\mathbf{\Sigma}_{\text{sc}}(\omega) \equiv \mathbf{\Gamma}_{\text{sc}}(\omega)/2 + i \mathbf{\nu}_{\text{sc}}(\omega)$ is the Fourier image of the dissipation kernel $\mathbf{D}_{\text{sc}}(t)$ from the $s$-wave SC, where the “real part” $\mathbf{\Gamma}_{\text{sc}}(\omega)$ leads to the system dissipation, and the “imaginary part” $\mathbf{\nu}_{\text{sc}}(\omega)$ provides an effective interaction to the nanowire Hamiltonian.

Specifically, $\mathbf{D}_{\text{sc}}(\omega) := \text{diag} \{ D_1, \ldots, D_N \}$ is a block-diagonal matrix, with blocks $D_n(\omega) := \Gamma_{\text{sc}}(\omega)/2 + i \nu_{\text{sc}}(\omega)$,
where

$$\tilde{V}_s(\omega) := \frac{\Theta(\Delta_s - |\omega|)\gamma_s}{\sqrt{\Delta_s^2 - \omega^2}} \begin{bmatrix} \omega & -\Delta_s \\ -\Delta_s & \omega \end{bmatrix},$$

$$\tilde{\Gamma}_s(\omega) := \frac{2\Theta(|\omega| - \Delta_s)\gamma_s}{\sqrt{\omega^2 - \Delta_s^2}} |\omega|^{14 \times 4}. \quad (12)$$

Here, $\gamma_s := \pi \sum_k |j_k|^2 \delta(\omega - \epsilon_k^s) \rightarrow \pi |j_0(\omega)|^2 \rho_s(\omega)$ is introduced as the spectral density of the coupling with the $s$-wave SC, which is approximated as a constant coupling strength $\gamma_s \approx \gamma_s$ [this notion is just consistent with the one in the above continuous equation (5)].

The dissipation kernels of the two electron leads also give $D_v(\omega) \equiv \Gamma_v(\omega)/2 + i V_v(\omega)$, while $V_v(\omega) \approx 0$ in usual transport experiments, only with $\Gamma_v(\omega) := \Gamma_1 + \Gamma_N$ left providing dissipation to the system. Here $\Gamma_1 := \text{diag}\{\Gamma_1, 0, \ldots, 0\}$ and $\Gamma_N := \text{diag}\{0, \ldots, 0, \Gamma_N\}$ are the dissipation matrices from the two electron leads respectively, where $\Gamma_x := \gamma_x 1_{4 \times 4} (x = 1, N)$, and $\gamma_x$ indicates the coupling strength with lead-$x$, defined from the coupling spectral density $\gamma_x(\omega) := 2\pi \sum_k |g_{x,k}|^2 \delta(\omega - \epsilon_{x,k}) \approx \gamma_x$.

It is worth noting that, without deriving the effective Hamiltonian in priori with any approximations, the SC proximity effect is naturally presented as the off-diagonal elements of $V_x(\omega)$ in the dynamical propagator (11), which just indicates the onsite $s$-wave pairing for the nanowire [7, 27, 39, 41]. Moreover, a Heaviside function naturally appears in both $V_x(\omega)$ and $\Gamma_x(\omega)$, which indicates a complementary effect of the SC proximity: when the system energy scale lies within the $s$-wave gap $|\omega| < \Delta_s$, the $s$-wave SC just provides the effective pairing interaction without any dissipation; in contrast, outside the gap $|\omega| > \Delta_s$, the SC proximity does not give the effective pairing, but only brings in the dissipation effect similarly as the normal leads.

**Differential conductance** - To study the transport current, we consider the initial states of the three baths (the two normal leads, and the $s$-wave SC) are in the Fermi-Dirac distributions at the zero temperature. The chemical potentials of the $s$-wave SC and the electron lead-$N$ are set as $\mu_s = \mu_N = 0$, while the lead-1 is $\mu_1 = eV$ with $V$ as the bias voltage.

The electric current flowing from lead-1 to the nanowire is obtained from the changing rate of the total electron number in lead-1, i.e.,

$$I_1(t) := -e \partial_t \sum_{k, \sigma} \langle b_{1, k, \sigma} \rangle. \quad \text{After a long enough time relaxation } t \to \infty, \text{ a steady current is achieved.}$$

From the above Langevin equation, the differential conductance $\sigma \equiv dI_1/dV$ is obtained as (see supplemental materials) [35, 37]

$$\sigma = \frac{e^2}{h} \left\{ \text{tr}[G^\dagger T^+_1 G N]_{(eV)} + \text{tr}[G^\dagger T^+_1 G N]_{(-eV)} \right\}.
\quad (13)$$

Here the dissipation matrices $\Gamma^+_1, \Gamma^-_N$ are given by $\Gamma^+_1 := \text{diag}\{\Gamma^+_1, 0, \ldots, 0\}$, and $\Gamma^-_N := \text{diag}\{0, \ldots, 0, \Gamma^-_N\}$, with the $4 \times 4$ blocks $\Gamma^+_x := \gamma_x \text{diag}\{1, 1, 0, 0\}$, and $\Gamma^-_x := \gamma_x \text{diag}\{0, 0, 1, 1\}$.

The first two terms in Eq. (13) come from the electron exchanges among lead-1 to lead-$N$ and the $s$-wave SC respectively; the last two terms indicate the contribution from the Andreev reflection between lead-1 and the nanowire, which gives the ZBP of $2e^2/h$ as the necessary signature for the emergence of MZMs at the zero temperature [12, 13, 35, 37]. Up to now, no other approximations are made except the form of the coupling spectral density, thus the obtained result is exact enough for the situations that the coupling strength or the magnetic field is quite strong.

The numerical results for the differential conductance (13) under different physical conditions are illustrated in Fig. 3. It is shown that a ZBP with height $2e^2/h$ appears in the conductance spectrum when the magnetic field $B$ properly lies in a continuous regime of modest strength, and the ZBP of $2e^2/h$ does not represent Majorana zero modes when the magnetic field strength exceeds the range of the magnetic field determined by refined phase region [see Fig. 2]. Then the ZBP disappears when the magnetic field strength is too weak or too strong, thus, it is confirmed that the MZMs do not exist in these regimes. This observation is just consistent with the result obtained from the above effective Hamiltonian (Fig. 2).

The variation trend of ZBP signature with the magnetic field is fit with the observed result in experiment [10, 11, 42].
It is specially worth noting that under certain conditions here the ZBP could be even higher than $2e^2/h$ [see Fig. 3(e, f)]. In finite temperatures, such a ZBP would be as low as $2e^2/h$ and thus might be confused with the signature from MZMs. Therefore it should be emphasized that the ZBP of $2e^2/h$ is the necessary but not sufficient condition for the MZMs.

**Summary** - By examining the the low-energy effective model for the hybrid system with the semiconductor nanowire in proximity to the $s$-wave superconductor, we obtain a refined topological phase diagram where the Majorana zero modes (MZMs) could exist. The valid topological phase region bearing MZMs appears as a closed triangle in the $\mu$-$B$ phase diagram, in comparison with the open hyperbolic region known before. These predictions are also confirmed by the exact calculation about the quantum transport based on the quantum Langevin equation: in the transport spectrum, the zero bias peak with $2e^2/h$, as the necessary signature for MZMs, disappears when the magnetic field grows too strong. Therefore, to search MZMs in this hybrid nanowire system, we suggest that the magnetic field strength be properly set within a moderate range according to our novel phase diagram.

For the electron-doped InSb nanowire coupled to NbTiN ($\Delta_s \simeq 26$ K), the chemical potential is around $\mu \approx 0$–10 K, the spin-orbit energy is around $U \equiv 2m_0c^2 \approx 1$–3 K [4, 10, 11, 23, 40, 43–45], and the above results show that the proper range for the magnetic field is around $B \sim 0.1$–1.5 T where MZMs could exist. And for InSb nanowire coupled to aluminum shell ($\Delta_s \simeq 2$ K) [4, 40], the convincing range for the magnetic field is no greater than 0.12 T. However, some of the recent experiments claim the emergence of MZMs when the magnetic field strength exceeds much beyond the valid range that we predicted in this letter. It is believed that our current theoretical study eventually solves the corresponding controversies about MZM experiments.

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Appendix A: The low energy effective Hamiltonian of nanowire

In this section, we derive the effective Hamiltonian of the nanowire by using the Fröhlich-Nakajima transformation. For the hybrid semiconductor-superconductor nanowire system, the total Hamiltonian has three basic terms: $\hat{H} = \hat{H}_w + \hat{H}_{sc} + \hat{H}_{w-sc}$. The nanowire Hamiltonian in continuous model is

$$\hat{H}_w = \int dx \hat{\psi}^\dagger(x) \left[ -\frac{\partial^2}{2m_w} - \mu - i\alpha \sigma \gamma \sigma \partial_x + B \sigma^z \right] \hat{\psi}(x), \quad (A1)$$

where $\hat{\psi}(x) = [\hat{\psi}_\uparrow(x), \hat{\psi}_\downarrow(x)]^T$, and $\sigma^y, \sigma^z$ are the Pauli matrices. Here, $\uparrow, \downarrow$ indicate the electron spins, $\alpha$ the spin-orbit coupling strength, $m_w$ the effective mass, $\mu$ the chemical potential of the nanowire, and $B$ the Zeeman splitting from the external magnetic field respectively. The $s$-wave superconductor (SC) providing the SC proximity effect for the nanowire is described by

$$\hat{H}_{sc} = \int \frac{d^3k}{(2\pi)^3} \epsilon_{sc}^s(\mathbf{k})\hat{c}_\uparrow^\dagger(\mathbf{k})\hat{c}_\uparrow(\mathbf{k}) - \hat{c}_\downarrow(-\mathbf{k})\hat{c}_\downarrow^\dagger(-\mathbf{k}) + \Delta_s(\hat{c}_\uparrow^\dagger(\mathbf{k})\hat{c}_\downarrow(-\mathbf{k}) + H.c.) \quad (A2)$$

with the kinetic energy $\epsilon_{sc}^s$ and real pairing potential $\Delta_s$ of SC. The fermion operator $\hat{c}_{\uparrow, \downarrow}(\mathbf{k})$ follows the anti-commutation relation $\{\hat{c}_s(\mathbf{k}), \hat{c}_{s'}^\dagger(\mathbf{k'})\} = (2\pi)^3\delta_{ss'}\delta(\mathbf{k} - \mathbf{k'})$. The tunneling interaction between the nanowire and $s$-wave superconductor is

$$\hat{H}_{w-sc} = -J_s \sum_s \int dx \left[ \hat\psi^\dagger_s(x)\hat{c}_s(x, 0, 0) + \hat{c}_s^\dagger(x, 0, 0)\hat{\psi}_s(x) \right]. \quad (A3)$$

Here, $J_s$ describes the tunneling strength between the nanowire and the $s$-wave SC, and $\hat{c}_s(x)$ is Fourier image of $\hat{c}_s(\mathbf{k})$:

$$\hat{c}_s(\mathbf{x}) = \frac{1}{(2\pi)^3} \int \hat{c}_s(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3k \quad (A4)$$

Similarly, the above nanowire Hamiltonian (A1) and tunneling Hamiltonian (A3) in Fourier space become

$$\hat{H}_w = \int \frac{dk_x}{2\pi} \hat{\varphi}^\dagger(k_x) (\epsilon_{k_x} + \alpha k_x \sigma^y + B \sigma^z) \hat{\varphi}(k_x),$$

$$\hat{H}_{w-sc} = -J_s \sum_s \int \frac{d^3k}{(2\pi)^3} [\hat{\varphi}^\dagger_s(k_x)\hat{c}_s(k) + \hat{c}_s^\dagger(k)\varphi_s(k)]. \quad (A5)$$

Here, $\hat{\varphi}(k_x) = [\hat{\varphi}_\uparrow(k_x), \hat{\varphi}_\downarrow(k_x)]^T$ is obtained by Fourier transform of two-component operator $\hat{\psi}(x)$ in real space, $\epsilon_{k_x} = \frac{\hbar^2 k_x^2}{2m_w} - \mu$ is the electron kinetic energy of the nanowire and $k_x$ is the $x$ component of $\mathbf{k}$.

By the following Bogoliubov transformation

$$\eta^\uparrow(\mathbf{k}) := \cos \theta_{k_x} \hat{c}_\uparrow(\mathbf{k}) + \sin \theta_{k_x} \hat{c}_\downarrow^\dagger(-\mathbf{k}),$$

$$\eta^\downarrow(-\mathbf{k}) := -\sin \theta_{k_x} \hat{c}_\uparrow(\mathbf{k}) + \cos \theta_{k_x} \hat{c}_\downarrow^\dagger(-\mathbf{k}), \quad \text{(A6)}$$

with $\tan 2\theta_{k_x} = \Delta_s/\epsilon_{sc}^s$, the Hamiltonian of the $s$-wave SC can be diagonally reduced to

$$\hat{H}_{sc} = \int \frac{d^3k}{(2\pi)^3} \epsilon_{sc}^s(\mathbf{k}) [\eta^\uparrow(\mathbf{k})\eta^\dagger(\mathbf{k}) + \eta^\downarrow(-\mathbf{k})\eta^\dagger(-\mathbf{k})], \quad (A7)$$
with the excitation spectrum of the SC $E_\mathbf{k}^{w,sc} = \sqrt{\left|\epsilon_\mathbf{k}^{w,sc}\right|^2 + \Delta_\mathbf{k}^2}$. Then the tunneling Hamiltonian $\hat{H}_{w,sc}$ [Eq. (A3)] is rewritten by the quasi-particle operators $\hat{n}_s(\mathbf{k})$ as

$$\hat{H}_{w,sc} = -J_s \int \frac{d^3k}{(2\pi)^3} \left\{ \hat{n}_s(\mathbf{k}) \left[ -\cos \theta_\mathbf{k} \hat{\phi}_\mathbf{k}^\dagger(x_k) + \sin \theta_\mathbf{k} \hat{\phi}_\mathbf{k}(k_-) \right] + \hat{n}_s^\dagger(\mathbf{k}) \left[ \cos \theta_\mathbf{k} \hat{\phi}_\mathbf{k}(x_k) - \sin \theta_\mathbf{k} \hat{\phi}_\mathbf{k}^\dagger(k_-) \right] \right\}.$$  

(A8)

In order to obtain the effective theory for the nanowire, we utilize the Fröhlich-Nakajima transformation to eliminate the quasi-particle excitation in SC. For the total Hamiltonian $\hat{H} = [\hat{H}_w + \hat{H}_{w,sc}] + \hat{H}_{w,sc} := \hat{H}_0 + \hat{H}_1$, we apply a unitary transformation

$$\hat{H}_S = e^{-\hat{S}} \hat{H} e^{\hat{S}} = \hat{H}_0 + \frac{1}{2} \hat{H}_1 + \frac{1}{2} \hat{H}_0, \quad S + \ldots,$$

(A9)

and the anti-Hermitian operator $\hat{S}$ should be properly set to make sure $\hat{H}_1 + [\hat{H}_0, \hat{S}] \equiv 0$, so the effective Hamiltonian becomes $\hat{H}_{eff} = \hat{H}_0 + \frac{1}{2} \hat{H}_1$.

Specifically, here we adopt an ansatz that $\hat{S}$ has the following form

$$\hat{S} = \int \frac{d^3k}{(2\pi)^3} \left\{ \hat{n}_s(\mathbf{k}) \left[ A_k \hat{\phi}_\mathbf{k}^\dagger(x_k) + B_k \hat{\phi}_\mathbf{k}(k_-) + E_k \hat{\phi}_\mathbf{k}(x_k) + F_k \hat{\phi}_\mathbf{k}^\dagger(k_-) \right] + \hat{n}_s^\dagger(\mathbf{k}) \left[ A_k' \hat{\phi}_\mathbf{k}(x_k) + B_k' \hat{\phi}_\mathbf{k}^\dagger(k_-) + E_k' \hat{\phi}_\mathbf{k}(k_-) + F_k' \hat{\phi}_\mathbf{k}^\dagger(x_k) \right] + \hat{n}_s(\mathbf{k}) \left[ C_k \hat{\phi}_\mathbf{k}^\dagger(x_k) + D_k \hat{\phi}_\mathbf{k}(k_-) + H_k \hat{\phi}_\mathbf{k}(k_-) + L_k \hat{\phi}_\mathbf{k}^\dagger(x_k) \right] + \hat{n}_s^\dagger(\mathbf{k}) \left[ C_k' \hat{\phi}_\mathbf{k}(x_k) + D_k' \hat{\phi}_\mathbf{k}^\dagger(k_-) + H_k' \hat{\phi}_\mathbf{k}(k_-) + L_k' \hat{\phi}_\mathbf{k}^\dagger(x_k) \right] \right\}.$$  

(A10)

Then by using the above Eqs. (A5, A7, A8, A10), the condition $\hat{H}_1 + [\hat{H}_0, \hat{S}] \equiv 0$ gives

$$\int \frac{d^3k}{(2\pi)^3} \left\{ \left( -E_k^{w,sc} + \epsilon_{k_z,\downarrow} \right) E_k + i \alpha k_x A_k \right\} \hat{n}_s(\mathbf{k}) \hat{\phi}_\mathbf{k}^\dagger(x_k) - \left( -E_k^{w,sc} - \epsilon_{k_z,\uparrow} \right) A_k + i \alpha k_x E_k - J_s \cos \theta_\mathbf{k} \hat{n}_s(\mathbf{k}) \hat{\phi}_\mathbf{k}(k_-) + \left( -E_k^{w,sc} - \epsilon_{k_z,\downarrow} \right) E_k' + i \alpha k_x A_k' \hat{n}_s^\dagger(\mathbf{k}) \hat{\phi}_\mathbf{k}^\dagger(k_-) + \left( -E_k^{w,sc} + \epsilon_{k_z,\uparrow} \right) B_k + i \alpha k_x F_k + J_s \sin \theta_\mathbf{k} \hat{n}_s^\dagger(\mathbf{k}) \hat{\phi}_\mathbf{k}^\dagger(k_-) + \left( -E_k^{w,sc} + \epsilon_{k_z,\downarrow} \right) F_k + i \alpha k_x B_k \hat{n}_s(\mathbf{k}) \hat{\phi}_\mathbf{k}(k_-) + \left( -E_k^{w,sc} + \epsilon_{k_z,\downarrow} \right) B_k' - i \alpha k_x F_k' - J_s \sin \theta_\mathbf{k} \hat{n}_s(\mathbf{k}) \hat{\phi}_\mathbf{k}^\dagger(k_-) + \left( -E_k^{w,sc} + \epsilon_{k_z,\uparrow} \right) F_k' + i \alpha k_x B_k' \hat{n}_s^\dagger(\mathbf{k}) \hat{\phi}_\mathbf{k}^\dagger(k_-) + \left( -E_k^{w,sc} - \epsilon_{k_z,\downarrow} \right) C_k - i \alpha k_x H_k - J_s \cos \theta_\mathbf{k} \hat{n}_s^\dagger(\mathbf{k}) \hat{\phi}_\mathbf{k}(k_-) + \left( -E_k^{w,sc} - \epsilon_{k_z,\downarrow} \right) H_k - i \alpha k_x C_k \hat{n}_s(\mathbf{k}) \hat{\phi}_\mathbf{k}(k_-) + \left( -E_k^{w,sc} - \epsilon_{k_z,\downarrow} \right) C_k' + i \alpha k_x H_k' - J_s \sin \theta_\mathbf{k} \hat{n}_s^\dagger(\mathbf{k}) \hat{\phi}_\mathbf{k}^\dagger(k_-) + \left( -E_k^{w,sc} - \epsilon_{k_z,\uparrow} \right) H_k' - i \alpha k_x C_k' \hat{n}_s(\mathbf{k}) \hat{\phi}_\mathbf{k}(k_-) + \left( -E_k^{w,sc} - \epsilon_{k_z,\downarrow} \right) D_k - i \alpha k_x L_k - J_s \sin \theta_\mathbf{k} \hat{n}_s(\mathbf{k}) \hat{\phi}_\mathbf{k}(k_-) + \left( -E_k^{w,sc} + \epsilon_{k_z,\downarrow} \right) L_k - i \alpha k_x D_k \hat{n}_s^\dagger(\mathbf{k}) \hat{\phi}_\mathbf{k}^\dagger(k_-) + \left( -E_k^{w,sc} + \epsilon_{k_z,\uparrow} \right) D_k' + i \alpha k_x L_k' - J_s \sin \theta_\mathbf{k} \hat{n}_s^\dagger(\mathbf{k}) \hat{\phi}_\mathbf{k}^\dagger(k_-) + \left( -E_k^{w,sc} + \epsilon_{k_z,\downarrow} \right) L_k' - i \alpha k_x D_k' \hat{n}_s(\mathbf{k}) \hat{\phi}_\mathbf{k}(k_-) \right\} = 0,$$

with $\epsilon_{k_z,\uparrow(\downarrow)} \equiv \epsilon_{k_z} \pm B$. The coefficients of each term in the above integral should be zero, and then the undetermined coefficients in $S$ are solved as

\[
\begin{align*}
A_k &= A_k' = \frac{J_s \cos \theta_\mathbf{k} [E_k^{w,sc} - \epsilon_{k_z,\downarrow}]}{\Pi_-(\mathbf{k})}, & B_k &= B_k' = -\frac{J_s \sin \theta_\mathbf{k} [E_k^{w,sc} - \epsilon_{k_z,\uparrow}]}{\Pi_+(\mathbf{k})}, \\
C_k &= C_k' = \frac{J_s \cos \theta_\mathbf{k} [E_k^{w,sc} - \epsilon_{k_z,\downarrow}]}{\Pi_-(\mathbf{k})}, & D_k &= D_k' = \frac{J_s \sin \theta_\mathbf{k} [E_k^{w,sc} + \epsilon_{k_z,\uparrow}]}{\Pi_+(\mathbf{k})}, \\
E_k &= -E_k' = -H_k = H_k' = \frac{i \alpha k_x J_s \cos \theta_\mathbf{k}}{\Pi_-(\mathbf{k})}, & F_k &= -F_k' = L_k = L_k' = -\frac{i \alpha k_x J_s \sin \theta_\mathbf{k}}{\Pi_+(\mathbf{k})},
\end{align*}
\]

(A12)

with $\Pi_\pm(\mathbf{k}) = (E_k^{w,sc} \pm \epsilon_{k_z})^2 - (B^2 + \alpha^2 k_\mathbf{p}^2)$.

When the magnetic field is not too strong and the electron tunneling strength between the nanowire and SC is weak, i.e. $\|\Pi_\pm(\mathbf{k})\| \gg (J_s k_\mathbf{p})^2$, the effective Hamiltonian of the nanowire is further obtained as
\[ \hat{H}_{\text{eff}} = \hat{H}_0 + \frac{J_s}{2} \{ \hat{H}_1, \hat{S} \}, \] which is calculated as follows [by using \{\hat{n}_s(k), \hat{n}^\dagger_s(k')\} + = (2\pi)^3 \delta_{ss'} \delta(k - k')]

\[
\frac{1}{2} \{ \hat{H}_1, \hat{S} \} = \frac{J_s}{2} \int \frac{d^3 k}{(2\pi)^3} \left\{ \left[ - \cos \theta_k \hat{\varphi}_1^\dagger(k_x) + \sin \theta_k \hat{\varphi}_1^\dagger(-k_x) \right] \left[ A_k \hat{\varphi}_1(k_x) + B_k \hat{\varphi}_1^\dagger(k_x) + E_k \hat{\varphi}_1^\dagger(-k_x) + F_k \hat{\varphi}_1(-k_x) \right] + \left[ \cos \theta_k \hat{\varphi}_1(k_x) - \sin \theta_k \hat{\varphi}_1(-k_x) \right] \left[ A_k \hat{\varphi}_1(k_x) + B_k \hat{\varphi}_1^\dagger(-k_x) + E_k \hat{\varphi}_1^\dagger(k_x) + F_k \hat{\varphi}_1(-k_x) \right] + \left[ - \cos \theta_k \hat{\varphi}_1(k_x) - \sin \theta_k \hat{\varphi}_1(-k_x) \right] \left[ C_k \hat{\varphi}_1(k_x) + D_k \hat{\varphi}_1^\dagger(k_x) + H_k \hat{\varphi}_1^\dagger(-k_x) + L_k \hat{\varphi}_1(-k_x) \right] \right\} + \frac{1}{2} J_s \int \frac{d^3 k}{(2\pi)^3} \left\{ \left[ - 2 A_k \cos \theta_k + 2 D_k \sin \theta_k \right] \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1(k_x) \right\} + \left[ - 2 B_k \sin \theta_k - 2 C_k \cos \theta_k \right] \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1(-k_x) + \left[ 2 \cos \theta_k E_k - 2 \sin \theta_k F_k \right] \left[ \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1(k_x) - \hat{\varphi}_1^\dagger(-k_x) \hat{\varphi}_1(k_x) - \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1(-k_x) - \hat{\varphi}_1^\dagger(-k_x) \hat{\varphi}_1(-k_x) \right] \right\} \}

\text{(A13)}

Finally, by substituting the coefficients obtained in Eq. (A12) here, the effective Hamiltonian is obtained

\[
\hat{H}_{\text{eff}} = \int \frac{dk}{2\pi} \left\{ \tilde{\varepsilon}_{k_x} \left[ \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1(k_x) + \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1(k_x) \right] + i \tilde{\alpha}_{k_x} \left[ \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1(k_x) - \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1(-k_x) \right] + \tilde{\Delta}_{k_x} \left[ \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1^\dagger(-k_x) - \hat{\varphi}_1^\dagger(-k_x) \hat{\varphi}_1(k_x) - \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1^\dagger(-k_x) - \hat{\varphi}_1^\dagger(-k_x) \hat{\varphi}_1^\dagger(-k_x) \right] \right\} \text{H.c.}

\text{(A14)}

The electron tunneling between the nanowire and the s-wave SC modifies the kinetic energy \( \tilde{\varepsilon}_{k_x} \), Zeeman splitting \( \tilde{B}_{k_x} \) and spin-orbit coupling \( \tilde{\alpha}_{k_x} \), and these parameters are

\[
\tilde{B}_{k_x} = [1 - J_s^2] \int \frac{dk_y dk_z}{(2\pi)^2} \frac{\cos^2 \theta_k}{\Pi_{-}(k)} \left[ \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1^\dagger(k_x) \right] + \frac{\sin^2 \theta_k}{\Pi_{+}(k)} \left, B \right, \]

\[
\tilde{\alpha}_{k_x} = [1 - J_s^2] \int \frac{dk_y dk_z}{(2\pi)^2} \frac{\cos^2 \theta_k}{\Pi_{-}(k)} \left[ \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1^\dagger(k_x) \right] \alpha \right, \]

\[
\tilde{\varepsilon}_{k_x} = \varepsilon_{k_x} - J_s^2 \int \frac{dk_y dk_z}{(2\pi)^2} \frac{\cos \theta_k E^e_{k_z}(k)}{\Pi_{-}(k)} - \frac{\sin \theta_k E^e_{k_z}(k)}{\Pi_{+}(k)} \right, \]

\[
\tilde{\Delta}_{k_x} = J_s^2 \int \frac{dk_y dk_z}{(2\pi)^2} \frac{\sin \theta_k E^e_{k_z}(k)}{\Pi_{-}(k)} - \frac{\sin \theta_k E^e_{k_z}(k)}{\Pi_{+}(k)} \right, \]

\[
\tilde{\Lambda}_{k_x} = J_s \int \frac{dk_y dk_z}{(2\pi)^2} \frac{\cos \theta_k E^e_{k_z}(k)}{\Pi_{-}(k)} - \frac{\cos \theta_k E^e_{k_z}(k)}{\Pi_{+}(k)} \right, \]

where \( E^e_{k_z}(k) \equiv E^e_{k_z} \pm \varepsilon_{k_x} \), and \( \Pi_{\pm}(k) \), \( \sin \theta_k \) and \( \cos \theta_k \) have been given in Eqs. (A6, A12) respectively.

To further calculate the integrals in the above result, we consider the Zeeman splitting \( B \) and spin-orbital coupling \( \alpha \) of the nanowire are much smaller than the s-wave SC gap, thus \( |E^e_{k_z} - \sqrt{B^2 + \alpha^2 k_z^2}| \gg |\varepsilon_{k_x}| \). Then in the above integrals we omit the second and higher-order terms of \( |\alpha k_x|/|E^e_{k_z} + \sqrt{B^2 + \alpha^2 k_z^2}| \) and \( |\varepsilon_{k_x}|/|E^e_{k_z} - \sqrt{B^2 + \alpha^2 k_z^2}| \), and the above parameters in Eq. (A15) are simplified as

\[
\tilde{\Delta}_{k_x} = \Upsilon_s(1 - \frac{\alpha^2 k_z^2 + B^2}{\Delta_s^2})^{-\frac{1}{4}}, \quad \tilde{\Lambda}_{k_x} = 0
\]

\[
\frac{\tilde{B}_{k_x}}{B} = \frac{\tilde{\alpha}_{k_x}}{\alpha} = \frac{\varepsilon_{k_x}}{\varepsilon_{k_x}} = \frac{\mu_{k_x}}{\mu} = 1 - \frac{\tilde{\Delta}_{k_x}}{\Delta_s}
\]

Here, \( \Upsilon_s = J_s^2 \rho_s \) describes the coupling strength between the s-wave SC and the nanowire, where \( \rho_s \) is two-dimensional superconducting density of states. Now we have obtained the effective Hamiltonian of the nanowire in the main text, i.e.,

\[
\hat{H}_{\text{eff}} = \int \frac{dk_x}{2\pi} \left\{ \tilde{\varepsilon}_{k_x} \left[ \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1(k_x) + \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1(k_x) \right] + i \tilde{\alpha}_{k_x} \left[ \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1(k_x) - \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1(-k_x) \right] + \tilde{\Delta}_{k_x} \left[ \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1^\dagger(-k_x) - \hat{\varphi}_1^\dagger(-k_x) \hat{\varphi}_1(k_x) + \hat{\varphi}_1^\dagger(k_x) \hat{\varphi}_1^\dagger(-k_x) - \hat{\varphi}_1^\dagger(-k_x) \hat{\varphi}_1^\dagger(-k_x) \right] \right\} \text{H.c.}
\]
Clearly, $\tilde{\Delta}_{k_x}$ is the pairing potential induced by the SC proximity effect, and all these parameters $\tilde{\epsilon}_{k_x}$, $\tilde{\alpha}_{k_x}$, $\tilde{\Delta}_{k_x}$ exhibit significant dependence on the magnetic field. In addition, when the SC gap is so large that $|E_m^c| \gg B$, $Y_s$, in the low energy regime ($k_x \approx 0$), the induced pairing strength can be approximated as a constant $\tilde{\Delta} \approx Y_s$, and the kinetic energy, spin-orbital coupling and Zeeman splitting of the nanowire are corrected by the constant factor $1 - Y_s/\Delta_a$ according to (A16).

Then the quasi-particle energy spectrum determined by the effective Hamiltonian in Eq. (A17) is

$$E_{\pm}(k_x) = \sqrt{\Delta_{k_x}^2 + \frac{\epsilon_{k_x}^2 + \epsilon_{k_x}^2}{2}} \pm (\epsilon_{k_x} + \epsilon_{k_x}^-) \sqrt{[\Delta_{k_x}^2] + \epsilon_{k_x}^2},$$

(A18)

with $\epsilon_{k_x} = \tilde{\epsilon}_{k_x} \pm \sqrt{B_{k_x}^2 + \tilde{\alpha}_{k_x}^2 k_x^2}$ and $\Delta_{k_x} = B\Delta_{k_x}/\sqrt{B^2 + \alpha^2 k_x^2}$. The energy level crossing point of the quasi-particle energy spectrum is $E_{\pm}(k_x = 0) = 0$, which gives the critical condition of topological phase: $\tilde{B}_{k_x = 0} = \sqrt{\tilde{\mu}_{k_x = 0}^2 + \Delta_{k_x = 0}^2}$.

When the corrected magnetic field satisfies

$$\tilde{B}_{k_x = 0}(B, Y_s) > \sqrt{\tilde{\mu}_{k_x = 0}(B, Y_s) + \Delta_{k_x = 0}(B, Y_s)},$$

(A19)

the energy gap reopens, and the topological phase with Majorana zero modes emerges in the region. However, when the magnetic field gets stronger, the corrected magnetic field decreases significantly according to Eq. (A16), which makes the corrected magnetic field become $\tilde{B}_{k_x = 0} < \sqrt{\tilde{\mu}_{k_x = 0} + \Delta_{k_x = 0}}$. Then the topological phase may disappear. Therefore, the topological phase only emerge at proper magnetic field strength.

**Appendix B: Majorana zero modes determined by the effective Hamiltonian of the nanowire**

Here, based on the effective Hamiltonian of the nanowire, we give the region of the existence of Majorana zero modes in the $\mu - B$ diagram, which is consistent with the topological phase region given by Eq. (A19). In the continuous limit, the effective Hamiltonian (A17) of the nanowire in the low energy regime ($k_x \approx 0$) becomes

$$\hat{H}_{\text{eff}} = \int dx \hat{\psi}^\dagger(x)(1 - Z(B))[ - \frac{\partial^2}{2m_w} - \mu - i\sigma^y \partial_x + B\sigma^z] \hat{\psi}(x) + Z(B)T_s[\hat{\psi}_x^\dagger(x)\hat{\psi}_x(x) + \hat{\psi}_L(x)\hat{\psi}_L(x)],$$

(B1)

where the corrected factor is defined as $Z(B) \equiv (1 - \frac{B^2}{\Delta_a^2})^{-\frac{1}{2}}$. In the Nambu representation, the above Hamiltonian becomes

$$\hat{H}_{\text{eff}} = \frac{1}{2} \int dx \hat{\Phi}^\dagger(x) \cdot \mathbf{H}_x \cdot \hat{\Phi}(x),$$

(B2)

with $\Phi(x) = [\hat{\psi}(x), \hat{\psi}_x^\dagger(x)]^T$ as the 4-component operator and

$$\mathbf{H}_x = \begin{bmatrix} [1 - Z(B)][ - \frac{\partial^2}{2m_w} - \mu - i\sigma^y \partial_x + B\sigma^z] & i\sigma^y Z(B) \mathbf{T}_s \cdot \mathbf{I} \\ -i\sigma^y Z(B) \mathbf{T}_s \cdot \mathbf{I} & [1 - Z(B)][ - \frac{\partial^2}{2m_w} + \mu + i\sigma^y \partial_x - B\sigma^z] \end{bmatrix}.$$ (B3)

Considering the Bogoliubov-de-Gennes (BdG) equation

$$\mathbf{H}_x \Psi_E(x) = E \Psi_E(x),$$

(B4)

the corresponding wave function is $\Psi_E(x) = [u_{\uparrow, E}(x), u_{\downarrow, E}(x), v_{\uparrow, E}(x), v_{\downarrow, E}(x)]^T$. Then the effective Hamiltonian (B1) is diagonalized as $\hat{H}_{\text{eff}} = \frac{1}{2} \sum_E E \hat{\gamma}_E \hat{\gamma}_E^\dagger$, where

$$\hat{\gamma}_E = \int dx \sum_s [u_{s, E}^*(x)\hat{\psi}_s(x) + v_{s, E}^*(x)\hat{\psi}_s^\dagger(x)].$$

(B5)

For the Majorana fermion, the antiparticle is itself, which means the quasi-particle operator is self-Hermitian $\hat{\gamma}_E = \hat{\gamma}_E^\dagger$, namely,

$$u_{s, E}(x) = v_{s, E}^*(x).$$

(B6)
If the wave function \(u\) with the 2-component wave function \(\xi\) is non-degenerate, we obtain relation of \(u_{s,E}(x)\) and \(v_{s,E}(x)\) by Eqs. (B5, B8) again, i.e.,

\[
u_{s,E}(x) = v^*_{s,-E}(x).
\]

Thus, only the zero mode quasi-particle could be self-Hermitian, i.e., the Majorana zero-mode.

For \(E = 0\), considering that the Majorana fermion requires \(u_{s,E}(x) = v^*_{s,E}(x)\), the BdG equation (B4) is reduced to

\[
[1 - Z(B)](-\frac{\partial^2}{\partial x^2} - \mu - i\alpha\sigma_y\partial_x + B\sigma^z)u(x) + i\sigma^y Z(B)\Upsilon_s u^*(x) = 0,
\]

with the 2-component wave function \(u(x) \equiv [u^\uparrow(x), u^\downarrow(x)]^T\). Here we consider the length of the nanowire is \(L\), and \(x \in [0, L]\). Then \(u(x)\) can be decomposed into the real and imaginary parts \(u(x) = u^{(r)}(x) + iu^{(i)}(x)\), and we assume they have the following forms

\[
u^{(r)}(x) = e^{-\xi_r x}[u^\uparrow(x), u^\downarrow(x)]^T,
\]

\[
u^{(i)}(x) = e^{-\xi_i x}[u^\uparrow(x), u^\downarrow(x)]^T,
\]

where \(\xi_r\) and \(\xi_i\) are real numbers. Taking the ansatz (B12) into the reduced BdG equation (B11), we get the equations of the real and imaginary parts respectively

\[
\begin{bmatrix}
-\frac{\xi_r^2}{2m_w} - \mu + B & (-\alpha\xi_r + \frac{Z(B)}{1 - Z(B)}\Upsilon_s) \\
(\alpha\xi_r - \frac{Z(B)}{1 - Z(B)}\Upsilon_s) & -\frac{\xi_r^2}{2m_w} - \mu - B
\end{bmatrix}
u^{(r)}(x) = 0,
\]

\[
\begin{bmatrix}
-\frac{\xi_i^2}{2m_w} - \mu + B & (-\alpha\xi_i + \frac{Z(B)}{1 - Z(B)}\Upsilon_s) \\
(\alpha\xi_i + \frac{Z(B)}{1 - Z(B)}\Upsilon_s) & -\frac{\xi_i^2}{2m_w} - \mu - B
\end{bmatrix}
u^{(i)}(x) = 0.
\]

If the above two equations have solutions, the determinants of the coefficient matrices in Eq. (B13) are zero

\[
0 = \left[ -\frac{1}{2m_w} \xi_r^2 - \mu \right]^2 + (\alpha\xi_r - \frac{Z(B)}{1 - Z(B)}\Upsilon_s)^2 - B^2 \equiv f(\xi_r),
\]

\[
0 = \left[ -\frac{1}{2m_w} \xi_i^2 - \mu \right]^2 + (\alpha\xi_i + \frac{Z(B)}{1 - Z(B)}\Upsilon_s)^2 - B^2 \equiv g(\xi_i).
\]

Here, the parameters \(\mu, m_w, \alpha\) and \(B\) are all positive, and the corrected factor satisfies \(0 < Z(B) < 1\), and thus \(Z(B)/(1 - Z(B)) > 0\).

(a) If the equation \(f(\xi) = 0\) has a positive root \(\xi_r > 0\), the real part \(u^{(r)}(x)\) has a solution localized around the end \(x = 0\). The existence condition for \(\xi_r\) can be given by examining the monotonicity \(f(\xi)\). Notice that the quartic function \(f(\xi)\) satisfies

\[
f''(\xi) = \frac{1}{m}\left[ 3m\xi^2 + 2\mu \right] + 2\alpha^2 > 0,
\]

thus \(f'(\xi)\) is monotonically increasing. Thus, in the interval \(\xi \in [0, \infty),\) the minimum of \(f'(\xi)\) appears at \(\xi = 0\), which is

\[
f'(0) = -2\alpha Z(B)\Upsilon_s/(1 - Z(B)) < 0.
\]

Therefore, there must exist a certain \(\xi_0 > 0\) satisfying \(f'(\xi_0) = 0\), that means, when
\text{Appendix C: Hamiltonian description for the nanowire system}

Now we study the transport behavior of the nanowire system. We consider the nanowire is contacted with two electron leads at the two ends, and then derive a quantum Langevin equation to describe the dynamics of the nanowire. First, the Hamiltonian (A1) of the nanowire is discretized as \( N \) sites, that is

\[
\hat{H}_w = \sum_{n,s} -J \left( \hat{d}_{n,s}^\dagger \hat{d}_{n+1,s} + \hat{d}_{n+1,s}^\dagger \hat{d}_{n,s} \right) - (\mu - J) \hat{d}_{n,s}^\dagger \hat{d}_{n,s}
\]

\[
+ \sum_n \frac{\alpha}{2} (\hat{d}_{n,\uparrow}^\dagger \hat{d}_{n,\uparrow} - \hat{d}_{n,\downarrow}^\dagger \hat{d}_{n,\downarrow} + \text{H.c.}) + \sum_n B (\hat{d}_{n,\uparrow}^\dagger \hat{d}_{n,\downarrow} - \hat{d}_{n,\downarrow}^\dagger \hat{d}_{n,\uparrow}). \tag{C1}
\]

Here, \( s = \uparrow, \downarrow \) indexes the electron spin, \( \alpha \) indicates the spin-orbit coupling strength, \( J \) the hopping amplitude, \( \mu \) the chemical potential of the nanowire, and \( B \) the Zeeman splitting from the external magnetic field.

All the \( N \) sites are contacted with an \( s \)-wave SC independently, and two electron leads are contacted with site-1 and site-\( N \). The \( s \)-wave SC and the two leads are treated as the fermion baths of the nanowire, and they are described by

\[
\hat{H}_{sc} = \sum_k \epsilon_{k \uparrow}^s (\hat{c}_{k \uparrow}^\dagger \hat{c}_{k \uparrow} - \hat{c}_{-k \downarrow}^\dagger \hat{c}_{-k \downarrow}) + \Delta_s (\hat{c}_{k \uparrow}^\dagger \hat{c}_{-k \downarrow}^\dagger + \text{H.c.}),
\]

\[
\hat{H}_{c-x} = \sum_{k,s} \varepsilon_{x,k} \hat{b}_{x,k,s}^\dagger \hat{b}_{x,k,s}, \quad (x = 1, N). \tag{C2}
\]

Both the two leads and the \( s \)-wave SC are coupled with the nanowire through the tunneling interaction [see also Eq. (A3)],

\[
\hat{H}_{w-sc} = - \sum_{n,k,s} \left( J_{n,k} \hat{d}_{n,s}^\dagger \hat{c}_{k,s} + \text{H.c.} \right), \quad \hat{H}_{w-x} = - \sum_{k,s} \left( \epsilon_{x,k} \hat{b}_{x,k,s}^\dagger \hat{b}_{x,k,s} + \text{H.c.} \right). \tag{C3}
\]

Denoting \( [\hat{d}(t)]_{1 \times 4N} := (\hat{d}_1, \ldots, \hat{d}_N)^T \) with blocks \( [\hat{d}_n]_{1 \times 4} := (\hat{d}_n^\uparrow, \hat{d}_n^\downarrow, \hat{d}_n^\dagger, \hat{d}_n^\dagger)^T \), \( \hat{c}_k(t) := (\hat{c}_{k \uparrow}, \hat{c}_{-k \downarrow}, \hat{c}_{k \uparrow}^\dagger, \hat{c}_{-k \downarrow}^\dagger)^T \), and \( \hat{b}_{x,k}(t) := (\hat{b}_{x,k}^\dagger, \hat{b}_{x,k}^\dagger, \hat{b}_{x,k}^\dagger, \hat{b}_{x,k}^\dagger)^T \), the above Hamiltonians (C1, C2) can be rewritten as

\[
\hat{H}_w = \frac{1}{2} \hat{d}^\dagger \cdot \hat{H}_{w'} \cdot \hat{d}, \quad \hat{H}_{sc} = \frac{1}{2} \sum_k \left[ \begin{array}{c} \epsilon_{k \uparrow}^s \\ \epsilon_{k \downarrow}^s \\ -\Delta_s \\ -\epsilon_{k \uparrow}^{sc} \\ \Delta_s \\ -\epsilon_{k \downarrow}^{sc} \\ -\epsilon_{k \downarrow}^{sc} \end{array} \right] \cdot \hat{H}_{sc}^{k \downarrow} \cdot \hat{c}_k, \quad \hat{H}_{c-x} = \frac{1}{2} \sum_k \hat{b}_{x,k}^\dagger \cdot \hat{H}_{c-x}^{k \downarrow} \cdot \hat{b}_{x,k}, \quad \hat{H}_w = \frac{1}{2} \hat{d}^\dagger \cdot \hat{H}_{w'} \cdot \hat{d}.
\]

\[
\hat{H}_{sc}^{k \downarrow} := \left[ \begin{array}{cccc} \epsilon_{k \uparrow}^s & \epsilon_{k \downarrow}^s + \Delta_s & \epsilon_{k \downarrow}^{sc} & -\epsilon_{k \uparrow}^{sc} \\ \epsilon_{k \downarrow}^s - \Delta_s & -\epsilon_{k \uparrow}^{sc} & \epsilon_{k \downarrow}^s & -\epsilon_{k \uparrow}^{sc} \\ -\epsilon_{k \uparrow}^{sc} & -\epsilon_{k \downarrow}^{sc} & -\epsilon_{k \downarrow}^{sc} & \epsilon_{k \uparrow}^{sc} \\ \epsilon_{k \uparrow}^{sc} & -\epsilon_{k \downarrow}^{sc} & \epsilon_{k \uparrow}^{sc} & -\epsilon_{k \downarrow}^{sc} \end{array} \right], \quad \hat{H}_{c-x}^{k \downarrow} := \left[ \begin{array}{c} \varepsilon_{x,k} \\ \varepsilon_{x,k} \\ -\varepsilon_{x,k} \\ -\varepsilon_{x,k} \end{array} \right]. \tag{C4}
\]
The system dynamics can be obtained by the Heisenberg equation. Here, we consider all the dynamical observables are corrected as $\hat{o}(t) = \hat{o}(0) e^{-\epsilon t} := \hat{o}(0) e^{\epsilon t}$ with $\epsilon \to 0$, namely, the dynamical evolution starts from $t = 0$. The infinitesimal $\epsilon$ guarantees the convergence of the evolution, and would naturally lead to the causality in the dynamical propagator. Then the equations of motions becomes \[\partial_t [\hat{o}(t) \Theta^{(t)}] = \delta(t) \hat{o}(0) - i \Theta^{(t)} [\hat{o}, \hat{H}],\] with $\hat{o}(0)$ as the initial state. Then the dynamics for the nanowire $\hat{a}_{n,s}$, s-wave SC $\hat{c}_{k,s}$, and electron leads $\hat{b}_{x,k,s}$ are given as

\[
\begin{align*}
\partial_t \hat{d}_{n,s} &= \delta(t) \hat{d}_{n,s}(0) - i [\hat{d}_{n,s}, \hat{H}_w] + i \sum_k J_{n,k} \hat{c}_{k,s} + i \sum_{x=1}^N \sum_k g_{x,k} \hat{b}_{x,k,s}, \\
\partial_t \hat{c}_{k,s} &= \delta(t) \hat{c}_{k,s}(0) - i [\hat{c}_{k,s}, \hat{H}_{sc}] + i \sum_n J_{n,k} \hat{d}_{n,s}, \\
\partial_t \hat{b}_{x,k,s} &= \delta(t) \hat{b}_{x,k,s}(0) - i \varepsilon_{x,k} \hat{b}_{x,k,s} + i g_{x,k} \hat{d}_{x,s}. \tag{C5}
\end{align*}
\]

These dynamical equations can be solved in the Fourier space. We adopt the following Fourier transform,

\[
\begin{align*}
\hat{d}(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{d}(\omega)e^{-i\omega t}, \quad \hat{d}(\omega) &= \int_{-\infty}^{\infty} dt \hat{d}(t)e^{i\omega t}, \\
\hat{d}^\dagger(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} [\hat{d}(\omega)]^\dagger e^{i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{d}^\dagger(-\omega)e^{-i\omega t}. \tag{C6}
\end{align*}
\]

Under this convention, the Fourier images for the vectors $\hat{d}_n(t), \hat{c}_k(t), \hat{b}_{x,k}(t)$ read

\[
\begin{align*}
\hat{d}_n(\omega) &= (\hat{d}_{n+}(\omega), \hat{d}_{n-}(\omega), \hat{d}_{n+}^0(-\omega), \hat{d}_{n-}^0(-\omega))^T, \\
\hat{c}_k(\omega) &= (\hat{c}_{k+}(\omega), \hat{c}_{k-}(\omega), \hat{c}_{k+}^0(-\omega), \hat{c}_{k-}^0(-\omega))^T, \\
\hat{b}_{x,k}(\omega) &= (\hat{b}_{x,k+}(\omega), \hat{b}_{x,k-}(\omega), \hat{b}_{x,k+}^0(-\omega), \hat{b}_{x,k-}^0(-\omega))^T. \tag{C7}
\end{align*}
\]

Then the Fourier images of the dynamical equations (C5) become ($\omega^+ := \omega + i\epsilon$)

\[
\begin{align*}
-i\omega^+ \hat{d}(\omega) &= \hat{d}(0) - i\hat{H}_w \cdot \hat{d}(\omega) + i \sum_k [T_k]_{4N \times 4} \cdot \hat{c}_k(\omega) + i \sum_{x=1}^N \sum_k [Y_{x,k}]_{4N \times 4} \cdot \hat{b}_{x,k}(\omega), \\
-i\omega^+ \hat{c}_k(\omega) &= \hat{c}_k(0) - i\hat{H}_{sc}(k) \cdot \hat{c}_k(\omega) + i [T_k]_{4N \times 4} \cdot \hat{d}(\omega), \\
-i\omega^+ \hat{b}_{x,k}(\omega) &= \hat{b}_{x,k}(0) - i\hat{H}_{e,x}(k) \cdot \hat{b}_{x,k}(\omega) + i [Y_{x,k}]_{4N \times 4} \cdot \hat{d}(\omega). \tag{C8}
\end{align*}
\]

Here $T_k$ and $Y_{x,k}$ are the the $4N \times 4$ tunneling matrices indicating the coupling with the s-wave SC and lead-$x$ respectively, and they are defined by

\[
T_k := [T_{1,k}, T_{2,k}, \ldots, T_{N,k}]^T, \quad Y_{1,k} := [Y_{1,k}, 0, \ldots, 0]^T, \quad Y_{N,k} := [0, \ldots, 0, Y_{N,k}]^T, \tag{C9}
\]

with $4 \times 4$ blocks $T_{n,k} := \text{diag} \{J_{n,k}, J_{n,k}^*, -J_{n,k}^*, -J_{n,k}^*\}$ and $Y_{x,k} := \text{diag} \{g_{x,k}, g_{x,k}^*, -g_{x,k}^*, -g_{x,k}^*\}$.

The dynamics for the s-wave SC and the two electron leads can be formally obtained with the help of Green functions

\[
\begin{align*}
\hat{c}_k(\omega) &= \Theta_{sc}(\omega, k) = \hat{c}_k(0) + i\hat{T}_k^\dagger \cdot \hat{d}(\omega), \\
\hat{b}_{x,k}(\omega) &= \Theta_{e,x}(\omega, k) = \hat{b}_{x,k}(0) + i\hat{Y}_{x,k}^\dagger \cdot \hat{d}(\omega), \\
\Theta_{sc}(\omega, k) &= i[\omega^+ - \hat{H}_{sc}^0]_{4N \times 4}^{-1}, \\
\Theta_{e,x}(\omega, k) &= i[\omega^+ - \hat{H}_{e,x}^0]_{4N \times 4}^{-1}. \tag{C10}
\end{align*}
\]

The Green function of lead-$x$ is $\Theta_{e,x}(\omega) = i \text{diag} \{\omega^+ - \varepsilon_{x,k}^-, \omega^+ - \varepsilon_{x,k}^-, \omega^+ + \varepsilon_{x,k}^-, \omega^+ + \varepsilon_{x,k}^-\}^{-1}$, while $\Theta_{sc}(\omega)$ for the s-wave SC requires calculating the inverse of the Hamiltonian matrix $\hat{H}_{sc}^0$. Taking $\hat{c}_k(\omega), \hat{b}_{x,k}(\omega)$ back into the equation of $\hat{d}(\omega)$, the dynamics of the nanowire becomes

\[
-i\omega^+ \hat{d}(\omega) = \hat{d}(0) - i\hat{H}_w \cdot \hat{d}(\omega) - [\hat{D}_{sc}(\omega) + \hat{D}_e(\omega)] \cdot \hat{d}(\omega) + i \hat{\xi}_{sc}(\omega) + i \hat{\xi}_e(\omega). \tag{C11}
\]

where $\hat{\xi}_{sc}(\omega), \hat{\xi}_e(\omega)$ are the random forces, $\hat{D}_{sc}(\omega), \hat{D}_e(\omega)$ are the dissipation kernels, and they are given by

\[
\begin{align*}
\hat{\xi}_{sc}(\omega) := \sum_k T_k \cdot \Theta_{sc}(\omega, k) \cdot \hat{c}_k(\omega), \\
\hat{\xi}_e(\omega) := \hat{\xi}_1 + \hat{\xi}_N := \sum_{x=1}^N \sum_k Y_{x,k} \cdot \Theta_{e,x}(\omega, k) \cdot \hat{b}_{x,k}(0), \\
\hat{D}_{sc}(\omega) := \sum_k T_k \cdot \Theta_{sc}(\omega, k) \cdot \hat{T}_k^\dagger, \\
\hat{D}_e(\omega) := \hat{D}_{e,1} + \hat{D}_{e,N} := \sum_{x=1}^N \sum_k Y_{x,k} \cdot \Theta_{e,x}(\omega, k) \cdot \hat{Y}_{x,k}^\dagger. \tag{C12}
\end{align*}
\]
In the time domain, Eq. (C11) gives the quantum Langevin equation in the main text

\[ \partial_t \hat{d} = \hat{d}(0)\delta(t) - iH_w \cdot \hat{d}(t) - \int_0^t d\tau \, D(t - \tau) \cdot \hat{d}(\tau) + i\xi_{se}(t) + i\xi_e(t), \]  

(C13)

where \( D(t) := D_{se}(t) + D_s(t) \), and \( \hat{d}(0)\delta(t) \) brings in the initial condition (omitted in the main text). Then the nanowire dynamics can be obtained from the Langevin equation (C11), i.e.,

\[ \hat{d}(\omega) = G(\omega) \cdot \left[ \hat{d}(0) + i\xi_{se}(\omega) + i\xi_e(\omega) \right], \]

\[ G(\omega) := i[\omega^+ - H_w + i\hat{D}_{se}(\omega) + i\hat{D}_e(\omega)]^{-1}, \]  

(C14)

where \( [G(\omega)]_{4N \times 4N} \) is the Green function for the nanowire.

Appendix D: The dissipation kernel and effective interaction

The dissipation kernels in Eq. (C12) provide both dissipation effect and effective interaction to the nanowire system. For the two electron leads, with the help of the Green functions (C10) and tunneling matrices (C9), the dissipation kernel \( \hat{D}_e(\omega) = [\hat{D}_{e,1} + \hat{D}_{e,N}] \) is given by

\[ \hat{D}_{e,1}(\omega) = \sum_k Y_{1,k} \cdot G_{e,1}(\omega, k) \cdot Y_{1,k}^\dagger = \text{diag}\{\hat{D}_{e,1}(\omega), 0, \ldots, 0\}, \quad \hat{D}_{e,N} = \text{diag}\{0, \ldots, 0, \hat{D}_{e,N}\}, \]

\[ \hat{D}_{e,x}(\omega) := \sum_k Y_{x,k} \cdot G_{e,x}(\omega, k) \cdot Y_{x,k}^\dagger = \sum_k |g_{x,k}|^2 \text{diag}\left\{ \frac{i}{\omega^+ - \varepsilon_{x,k}}, \frac{i}{\omega^+ - \varepsilon_{x,k}}, \frac{i}{\omega^+ + \varepsilon_{x,k}}, \frac{i}{\omega^+ + \varepsilon_{x,k}} \right\}. \]  

(D1)

The above summation over the electron modes \( k \) can be turned into an integral by introducing \( \Upsilon_x(\omega) \) as the spectral density of the coupling strength between the nanowire and lead-x, i.e.,

\[ \Upsilon_x(\omega) := 2\pi \sum_k |g_{x,k}|^2 \delta(\omega - \varepsilon_{x,k}) \]

\[ \sum_k \frac{|g_{x,k}|^2}{\omega^+ \pm \varepsilon_{x,k}} \rightarrow i \int_{-\infty}^{+\infty} \frac{d\varepsilon}{2\pi} \frac{\Upsilon_x(\varepsilon)}{\omega^+ \pm i\varepsilon} = i\mathcal{P} \int_{-\infty}^{+\infty} \frac{d\varepsilon}{2\pi} \frac{\Upsilon_x(\varepsilon)}{\omega^+ \pm \varepsilon} + \frac{1}{2} \Upsilon_x(\pm\omega). \]  

(D2)

The first term of principle integral provides an energy correction to the system Hamiltonian \( H_w \) in the Green function (C14), and the second term provides the dissipation effect. In transport measurements, the coupling spectral density is usually approximated as a constant \( \Upsilon_x(\omega) \equiv \Upsilon_x \), which is also known as the wide band limit. Thus, the above energy correction gives zero, only left \( \hat{D}_{e,x}(\omega) \approx \frac{1}{2} \Upsilon_x \text{diag}\{1, 1, 1, 1\} := \frac{1}{2} \Upsilon_x \Gamma_x \). As a result, the dissipation kernel of the two electron leads only provides the dissipation effect, i.e., \( \hat{D}_e(\omega) \approx \frac{1}{2}\Upsilon_e(\omega) := \frac{1}{2}(\Gamma_1 + \Gamma_N) \), where \( \Gamma_1 := \text{diag}\{0, \ldots, 0, \Gamma_1\} \), \( \Gamma_N := \text{diag}\{0, \ldots, 0, \Gamma_N\} \) are \( 4N \times 4N \) dissipation matrices.

Similarly, the dissipation kernel from the s-wave SC [Eq. (C12)] is treated in the same way, which gives

\[ \hat{D}_{sc}(\omega) = \sum_k T_k \cdot G_{sc}(\omega, k) \cdot T_k^\dagger \approx \text{diag}\{\hat{D}_{1}(\omega), \ldots, \hat{D}_{N}(\omega)\}, \]

\[ \hat{D}_{n}(\omega) := \sum_k T_{n,k} \cdot \frac{i}{\omega^+ - H_{sc}(k)} \cdot T_{n,k}^\dagger \equiv \hat{D}_{n}(\omega). \]  

(D3)

Here, the off-diagonal blocks in \( \hat{D}_{sc}(\omega) \) are omitted for the local tunneling approximation, which means the tunneling processes from different sites of the nanowire to the s-wave SC do not have any interferences or correlations with each other. Besides, since the coupling strengths between different sites and the s-wave SC have the same amplitude, \( |J_{m,k}| = |J_{n,k}| := J_k \) for \( m \neq n \), all the \( N \) diagonal blocks in \( \hat{D}_{sc}(\omega) \) are equal to each other, and they are given by

\[ \hat{D}_{s}(\omega) = \sum_k T_k \cdot \frac{i}{\omega^+ - H_{sc}(k)} \cdot T_k^\dagger = \sum_k \frac{i|J_k|^2}{(\omega^+)^2 - (\varepsilon_{sc})^2 - \Delta_s^2} \begin{bmatrix} \omega^+ + \varepsilon_{sc} & \omega^+ - \varepsilon_{sc} & \Delta_s & -\Delta_s \\ \Delta_s & \omega^+ + \varepsilon_{sc} & \omega^+ - \varepsilon_{sc} & -\Delta_s \\ -\Delta_s & \omega^+ - \varepsilon_{sc} & \omega^+ + \varepsilon_{sc} & \Delta_s \\ \omega^+ + \varepsilon_{sc} & \omega^+ - \varepsilon_{sc} & -\Delta_s & \Delta_s \end{bmatrix}, \]  

(D4)

\[ \frac{1}{(\omega^+)^2 - E^2} = \frac{1}{2E} \left[ \frac{1}{\omega^+ - E} - \frac{1}{\omega^+ + E} \right] = \frac{1}{2E} \left[ \mathcal{P} \frac{1}{\omega - E} - i\pi \delta(\omega - E) - \mathcal{P} \frac{1}{\omega + E} + i\pi \delta(\omega + E) \right], \]
where $\Gamma_k := |J_k| \text{ diag } \{1, 1, -1, -1\}$. Similarly like Eq. (D2), the summation over $k$ can be turned into an integral by introducing $\Upsilon_s(\omega)$ as the the spectral density of the coupling strength between the nanowire and the s-wave SC, i.e.,

$$\Upsilon_s(\omega) := \pi \sum_k |J_k|^2 \delta(\omega - \epsilon_k^{sc}) \sim \pi |J_s(\omega)|^2 \rho_s(\omega). \quad (D5)$$

Here, $\rho_s(\omega)$ is the density of state from the s-wave SC, and approximately $\Upsilon_s(\omega) \simeq \Upsilon_s$ is a constant. Then the dissipation kernel $\tilde{D}_s(\omega)$ from the s-wave SC is obtained as

$$\tilde{D}_s(\omega) = \frac{1}{2} \tilde{\Gamma}_s(\omega) + i \tilde{V}_s(\omega),$$

$$\tilde{V}_s(\omega) = -\frac{\Theta(\Delta_s - |\omega|)\Upsilon_s}{\sqrt{\Delta_s^2 - \omega^2}} \begin{bmatrix} \omega & -\Delta_s \\ \Delta_s & \omega \\ -\Delta_s & \omega \end{bmatrix}, \quad \tilde{\Gamma}_s(\omega) = \frac{2\Theta(|\omega| - \Delta)\Upsilon_s}{\sqrt{\Delta^2 - \omega^2}} |\omega| \mathbf{1}_{4 \times 4}, \quad (D6)$$

where $\tilde{\Gamma}_s(\omega)$ indicates the dissipation effect, while $\tilde{V}_s(\omega)$ can be regarded as an effective interaction in the Green function (C14). Correspondingly, the full dissipation kernel (D3) can be written as $\tilde{D}_{sc}(\omega) := \frac{1}{2} \tilde{\Gamma}_{sc}(\omega) + i \tilde{V}_{sc}(\omega)$, with $\tilde{V}_{sc}(\omega) := \text{diag}(\tilde{V}_s, \ldots, \tilde{V}_s)$ and $\tilde{\Gamma}_{sc}(\omega) := \text{diag}(\tilde{\Gamma}_s, \ldots, \tilde{\Gamma}_s)$.

It is worth noticing that the Heaviside function appears in both $\tilde{\Gamma}_s(\omega)$ and $\tilde{V}_s(\omega)$. That means, when the system energy lies within the SC gap $|\omega| < \Delta_s$, the s-wave SC only provides the effective pairing (the SC proximity) without any dissipation effect; for the high energy modes outside the SC gap $|\omega| > \Delta_s$, the s-wave SC only gives the dissipation effect but does not induce the SC proximity.

**Appendix E: Steady state current**

Here we consider the electric current flowing from lead-1 to the nanowire, and the differential conductance measurement. Generally, the electric current can be calculated by the changing rate of the electron number in lead-1, that is,

$$\langle \dot{I}_1(t) \rangle = -e \sum_{k,s} \partial_t \langle \hat{b}^\dagger_{1,k,s} \hat{b}_{1,k,s} \rangle = \frac{ie}{\hbar} \sum_{k,s} \left[ g_{1,k} \langle \hat{d}^\dagger_{1,s}(t) \hat{b}_{1,k,s}(t) \rangle - g_{1,k} \langle \hat{b}^\dagger_{1,k,s}(t) \hat{d}_{1,s}(t) \rangle \right] := I_1(t) + \text{c.c.} \quad (E1)$$

In particular, we focus on the steady state current after a long enough time relaxation $t \rightarrow \infty$. This can be obtained from the Fourier image $\tilde{I}_1(\omega)$ based on the final value theorem, which gives $I_1(t \rightarrow +\infty) = \lim_{\omega \rightarrow 0} [ -i\omega \tilde{I}_1(\omega) ]$. With the help of the tunneling matrix $\mathbf{Y}_{1,k}$ and a projector operator $\mathbf{P}^+$ defined below, $\tilde{I}_1(\omega)$ can be rewritten as the following matrix form

$$\tilde{I}_1(\omega) = \frac{ie}{\hbar} \sum_k \int \frac{d\nu}{2\pi} \langle [\hat{d}(\nu)]^\dagger \cdot \mathbf{P}^+ \cdot \mathbf{Y}_{1,k} \cdot \hat{b}_{1,k,\nu + \omega} \rangle$$

$$= \frac{ie}{\hbar} \int d\nu \left[ -i \chi_{sc}^{\dagger}(\nu) - i \chi_{e}^{\dagger}(\nu) \right] \cdot \mathbf{G}^\dagger(\nu) \cdot \mathbf{P}^+ \cdot \left[ \xi_1(\nu + \omega) + i \Delta \mathbf{D}_{sc}(\nu + \omega) \cdot \hat{d}(\nu + \omega) \right]$$

$$= \frac{e}{\hbar} \int d\nu \left\{ \left[ \chi_{sc}^{\dagger} + \xi_1^{\dagger} + \xi_N^{\dagger} \right] \cdot \mathbf{G}^\dagger(\nu) \cdot \mathbf{P}^+ \cdot \left[ \xi_1 - \Delta \mathbf{D}_{sc} \cdot \mathbf{G} \cdot \left[ \chi_{sc} + \xi_1 + \xi_N \right] \right] \right\}_{(\nu + \omega)}. \quad (E2)$$

Here the projector $\mathbf{P}^+$ is defined as $\mathbf{P}^+ := \text{diag}\{ \mathbf{P}^+_1, \mathbf{P}^+_2, \ldots, \mathbf{P}^+_N \}$, with blocks $\mathbf{P}^+_1 := \text{diag}\{ 1, 1, 0, 0 \}$. The dynamics of $\hat{b}_{1,k,\nu + \omega}$, $[\hat{d}(\nu)]^\dagger$ has been given in Eqs. (C10, C14), and the initial states of the nanowire like $\hat{d}(0)$ do not appear here since their contributions would decay to zero in the steady state $t \rightarrow \infty$.

The quantum expectations in Eq. (E2) are calculated from the random forces $\xi_{sc}, \xi_e$ based on the initial states of lead-1, $N$ and the s-wave SC, and they can be expressed by the correlation matrices for these three fermionic baths, $[C_{1}(\omega, \omega)]_{mn} := \langle \xi_1^{\dagger}(\omega) \xi_1(\omega) \rangle_m \cdot [C_s(\omega, \omega)]_{mn} := \langle \xi_{sc}^{\dagger}(\omega) \xi_{sc}(\omega) \rangle_m$, which further give

$$\tilde{I}_1(\omega) = \frac{e}{\hbar} \int d\nu \text{tr} \left[ C_{1}(\nu, \nu + \omega) \cdot \mathbf{G}^\dagger(\nu) \cdot \mathbf{P}^+ \right] - \sum_{y=1,N} \text{tr} \left[ C_y(\nu, \nu + \nu) \cdot \mathbf{G}^\dagger(\nu) \cdot \mathbf{P}^+ \cdot \mathbf{D}_{sc}(\nu + \omega) \cdot \mathbf{G}(\nu + \omega) \right]$$

$$- \text{tr} \left[ C_s(\nu, \nu + \nu) \cdot \mathbf{G}^\dagger(\nu) \cdot \mathbf{P}^+ \cdot \mathbf{D}_{sc}(\nu + \omega) \cdot \mathbf{G}(\nu + \omega) \right]. \quad (E3)$$
According to the final value theorem, with the help of the correlation matrices $C_{x,n}(\tilde{\omega}, \omega)$ calculated in Sec. F, in the steady state $t \to \infty$, the electric current is obtained as

$$
\langle \hat{J}_1 \rangle_\infty = \frac{e}{h} \int d\nu \text{tr} \left[ G^{\dag}_1 \Gamma^+_1 G^+_N(\nu) \left( f_1(\nu) - f_N(\nu) \right) + \text{tr} \left[ G^{\dag}_1 \Gamma^+_1 G^{-}_N(\nu) \right] \left( f_1(\nu) - \bar{f}_N(\nu) \right) + \text{tr} \left[ G^{\dag}_1 \Gamma^{-}_1 G^+_N(\nu) \right] \left( f_1(\nu) - \bar{f}_N(\nu) \right) + \text{tr} \left[ G^{\dag}_1 \Gamma^{-}_1 G^{-}_N(\nu) \right] \left( f_1(\nu) - \bar{f}_N(\nu) \right) - \text{tr} \left[ G^{\dag}_1 \Gamma^+_1 K^+_s(\nu) \right] f_s(\nu) - \text{tr} \left[ G^{\dag}_1 \Gamma^{-}_1 K^{-}_s(\nu) \right] \bar{f}_s(\nu) \right].
$$

(E4)

Here the dot symbols for the matrix product “$\cdot$” are omitted for short. $f_x(\nu)$, $f_s(\nu)$ are the Fermi distributions of lead-x and the s-wave SC, and $f_{x,n}(\omega) := 1 - f_{x,n}(\omega)$. The dissipation matrices $\Gamma_x(\nu)$, $\Gamma_{x,n}$ have been given in Eqs. (D1, D2, D6). $\Gamma^\pm_x$ and $K^\pm_s(\nu)$ are dissipation matrices of lead-x and the s-wave SC, which are given in Eqs. (F4, F7). The following relation is needed when deriving the above result,

$$
\hat{G} + \hat{G}^\dag = \hat{G}^\dag \cdot \left[ i(\omega - H - i\bar{D}) - i(\omega - H + i\bar{D}) \right] \cdot \hat{G} = \hat{G}^\dag \cdot (\Gamma_1 + \Gamma_N + \tilde{\Gamma}_s) \cdot \hat{G}.
$$

(E5)

For a transport measurement, we set the chemical potentials as $\mu_N = \mu_{se} = 0$, $\mu_1 = eV$, with $V$ as the voltage bias. At the zero temperature, the chemical potential of lead-1 is $f_1(\nu) = \Theta(eV - \nu)$, thus the above electric current (E4) gives the differential conductance $\sigma = dI_1/dV$ as

$$
\sigma = \frac{e^2}{h} \left\{ \text{tr} \left[ G^{\dag}_1 \Gamma^+_1 G^+_N(\nu) \right] (eV) + \text{tr} \left[ G^{\dag}_1 \Gamma^+_1 G^+_s(\nu) \right] (eV) + \text{tr} \left[ G^{\dag}_1 \Gamma^+_1 G^+_1(\nu) \right] (eV) + \text{tr} \left[ G^{\dag}_1 \Gamma^+_1 G^-_1(\nu) \right] (-eV) \right\}.
$$

(E6)

The first two terms indicate the contributions from the electron exchange from lead-1 to lead-N and the s-wave SC. The last two terms come from the Andreev reflection between lead-1 and the nanowire. In the above derivations, except the local tunneling approximation and the constant coupling spectrum, no other approximations are made. Thus, in principle this result also applies for the situations when the magnetic field or coupling strength is strong.

Appendix F: Correlation matrices of the baths

Here we calculate the correlation matrices in the above current (E3). From the random force (C12), Green function (C10), and the tunneling matrix (C9), the $4N \times 4N$ correlation matrix $[C_{x}(\tilde{\omega}, \omega)]_{mn} := \langle [\xi^\dag_x(\tilde{\omega})]_n [\xi_x(\omega)]_m \rangle$ for lead-x is

$$
C_1(\tilde{\omega}, \omega) = \text{diag} \{ C_1(\tilde{\omega}, \omega), 0, \ldots, 0 \}, \quad C_N(\tilde{\omega}, \omega) = \text{diag} \{ 0, \ldots, 0, C_N(\tilde{\omega}, \omega) \},
$$

$$
[C_x(\tilde{\omega}, \omega)]_{4 \times 4} = \sum_k \text{diag} \left\{ \frac{|g_{x,k}|^2 f_x(\epsilon_k)}{(\omega - \epsilon_k)(\omega + \epsilon_k)}, \frac{|g_{x,k}|^2 f_x(\epsilon_k)}{(\omega - \epsilon_k)(\omega + \epsilon_k)}, \frac{|g_{x,k}|^2 f_x(\epsilon_k)}{(\omega - \epsilon_k)(\omega + \epsilon_k)}, \frac{|g_{x,k}|^2 f_x(\epsilon_k)}{(\omega - \epsilon_k)(\omega + \epsilon_k)} \right\}
$$

$$
= \text{diag} \left\{ \frac{i\Upsilon_x(\omega)f_x(\omega)}{(\omega - \epsilon_k)(\omega + \epsilon_k)}, \frac{i\Upsilon_x(\omega)f_x(\omega)}{(\omega - \epsilon_k)(\omega + \epsilon_k)}, \frac{i\Upsilon_x(\omega)f_x(\omega)}{(\omega - \epsilon_k)(\omega + \epsilon_k)}, \frac{i\Upsilon_x(\omega)f_x(\omega)}{(\omega - \epsilon_k)(\omega + \epsilon_k)} \right\}.
$$

(F1)

Here $f_x(\omega)$ is the Fermi distribution from the initial equilibrium state of lead-x, i.e.,

$$
f_x(\omega) = \langle \hat{b}^\dag_{x,k}(0) \hat{b}_{x,k}(0) \rangle = \frac{1}{e^{(\omega - \mu_x)/T_x} + 1} \frac{T_x \rightarrow 0}{\Theta(\mu_x - \omega)}, \quad f_x(\omega) := 1 - f_x(\omega).
$$

(F2)

The above summations of $k$ are turned into integrals by using the coupling spectral density $\Upsilon_x(\omega) \equiv \Upsilon_x$, i.e.,

$$
\sum_k \frac{|g_{x,k}|^2 f_x(\epsilon_k)}{(\omega - \epsilon_k)(\omega + \epsilon_k)} \rightarrow \int \frac{d\nu}{2\pi} \frac{\Upsilon_x(\nu)f_x(\nu)}{(\nu - \omega + i\epsilon)(\nu - \omega - i\epsilon)} = \frac{i\Upsilon_x(\omega)f_x(\omega)}{(\omega - \omega + 2i\epsilon)}
$$

$$
\sum_k \frac{|g_{x,k}|^2 f_x(\epsilon_k)}{(\omega + \epsilon_k)(\omega + \epsilon_k)} \rightarrow \int \frac{d\nu}{2\pi} \frac{\Upsilon_x(\nu)f_x(\nu)}{(\nu + \omega + i\epsilon)(\nu + \omega - i\epsilon)} = \frac{i\Upsilon_x(\omega)f_x(\omega)}{(\omega + \omega + 2i\epsilon)}.
$$

(F3)

Therefore, when calculating the steady state current from the final value theorem, the correlation matrix in Eq. (E3) gives

$$
\lim_{\omega \rightarrow 0} \left[ \langle -i\omega \rangle C_x(\nu, \omega + \nu) \right] = \Upsilon_x \text{diag} \{ f_x(\nu), f_x(\nu), f_x(-\nu), f_x(-\nu) \} := f_x(\nu) \Gamma^+_x + f_x(-\nu) \Gamma^-_x,
$$

$$
\lim_{\omega \rightarrow 0} \left[ \langle -i\omega \rangle C_x(\nu, \omega + \nu) \right] = f_x(\nu) \Gamma^+_x + f_x(-\nu) \Gamma^-_x,
$$

(F4)
where $\Gamma^+_s := \Upsilon_s \text{diag}\{1, 1, 0, 0\}$ and $\Gamma^-_s := \Upsilon_s \text{diag}\{0, 0, 1, 1\}$ are upper and lower parts of the dissipation matrix $\Gamma_s \equiv \Gamma^+_s + \Gamma^-_s$ respectively, and correspondingly $\Gamma^+_1 := \text{diag}\{\bar{\Gamma}^+_s, 0, \ldots, 0\}$, $\Gamma^-_1 := \text{diag}\{0, \ldots, 0, \bar{\Gamma}^-_s\}$.

On the other hand, to calculate the correlation matrix for the $s$-wave SC, we need to use the Bogoliubov eigen modes, which determine the initial Fermi distribution of the $s$-wave SC. The $s$-wave SC Hamiltonian is diagonalized as $H_{sc} = \frac{1}{2} \sum_k \hat{c}^\dagger_k \cdot \mathbf{H}_{sc}^k \cdot \hat{c}_k \equiv \frac{1}{2} \sum_k \hat{\eta}_k^\dagger \cdot \mathbf{E}_{sc}^k \cdot \hat{\eta}_k$, with $\hat{\eta}_k := U_k \cdot \hat{c}_k$ and

$$E_{sc}^k := \begin{bmatrix} E_{sc}^{+k} & E_{sc}^{-k} \\ -E_{sc}^{-k} & -E_{sc}^{+k} \end{bmatrix} = U_k \cdot \mathbf{H}_{sc}^k \cdot U_k^\dagger,$$

$$U_k = \begin{bmatrix} \cos \theta_k & \cos \theta_k - \sin \theta_k \\ \sin \theta_k & \cos \theta_k \end{bmatrix},$$

where $E_{sc}^{\pm k} = [(\epsilon_{sc}^k)^2 + \Delta_s^2]^{1/2}$, and $\tan 2\theta_k \equiv \Delta_s / \epsilon_{sc}^k$ [see also Eq. (A6)].

From the random force (C12), the correlation matrix of the $s$-wave SC gives $[C_s(\bar{\omega}, \omega)]_{mn} \equiv \langle [\xi_{sc}^+(\bar{\omega})]_n [\xi_{sc}^+(\omega)]_m \rangle \simeq \text{diag}\{C_s(\bar{\omega}, \omega), \ldots, C_s(\bar{\omega}, \omega)\}$, which is block-diagonal. Similarly as the treatment to the dissipation kernel (D3), the off-diagonal blocks are omitted for the local tunneling approximation. And the diagonal blocks $C_s(\bar{\omega}, \omega)$ are calculated as

$$[C_s(\bar{\omega}, \omega)]_{mn} = \sum_{k,ij} \langle \hat{c}_k^\dagger (0) \rangle \langle \mathbf{G}_{sc}^k (\bar{\omega}) \mathbf{T}_k \rangle_{in} \cdot \langle \mathbf{T}_k \mathbf{G}_{sc}^k (\omega) \rangle_{mj} \langle \hat{c}_k (0) \rangle_{j}$$

$$= \sum_k T_k \cdot \mathbf{G}_{sc}^k (\omega) \cdot \left[ 1 - \langle \hat{c}_k (0) \hat{c}_k^\dagger (0) \rangle \right] \cdot \mathbf{T}_k^\dagger = \sum_k T_k \cdot \mathbf{G}_{sc}^k (\omega) \cdot \left[ U_k^\dagger \cdot \left[ 1 - \langle \hat{\eta}_k^\dagger (0) \hat{\eta}_k (0) \rangle \right] \cdot U_k \cdot \mathbf{G}_{sc}^k (\bar{\omega}) \cdot \mathbf{T}_k^\dagger \right.$$

where $f_s(E_{sc}^k)$ is the Fermi distribution for the Bogoliubov eigen modes in the initial state. The summation over fermion modes $k$ can be turned into an integral with the help of the coupling spectral density $\Upsilon_s (\omega) \equiv \Upsilon_s$. Further, when calculating the steady state current from the final value theorem, the correlation matrix $C_s(\bar{\omega}, \omega)$ gives

$$\lim_{\omega \to 0} \left[ (-iw) C_s(\nu, \omega + \nu) \right] = \frac{2 \Upsilon_s}{\sqrt{\nu^2 - \Delta_s^2}} \left[ \Theta(\nu - \Delta_s) f_s(\nu) - \Theta(-\Delta_s - \nu) \tilde{f}_s(-\nu) \right] \begin{pmatrix} \nu & -\Delta_s \\ \Delta_s & \nu \end{pmatrix}$$

$$:= f_s(\nu) \hat{K}_s^+(\nu) + \tilde{f}_s(-\nu) \hat{K}_s^-(\nu),$$

$$\lim_{\omega \to 0} \left[ (-iw) C_s(\nu, \omega + \nu) \right] := f_s(\nu) \hat{K}_s^+(\nu) + \tilde{f}_s(-\nu) \hat{K}_s^-(\nu),$$

where $\hat{K}_s^\pm(\nu) := \text{diag}\{K_s^\pm(\nu), K_s^\pm(\nu), \ldots, K_s^\pm(\nu)\}$. To obtain this result, the following derivation is adopted [$\mathcal{F}(x)$ is an arbitrary function]:

$$\lim_{\omega \to \omega} \left[ -i(\omega - \bar{\omega}) \int_{-\infty}^{\infty} \frac{dx}{2\pi} \frac{\mathcal{F}(\sqrt{\omega^2 + \Delta_s^2})}{\omega^+ \pm \sqrt{\omega^2 + \Delta_s^2}} \right]$$

$$= \lim_{\omega \to \omega} \int_{-\infty}^{\infty} \frac{dx}{2\pi} i(\omega^+ - \omega^-) \left[ \frac{1}{\omega^+ \pm \sqrt{\omega^2 + \Delta_s^2}} - \frac{1}{\omega^- \pm \sqrt{\omega^2 + \Delta_s^2}} \right] \mathcal{F}(\sqrt{\omega^2 + \Delta_s^2})$$

$$= \int_{-\infty}^{\infty} d\omega \mathcal{F}(\sqrt{\omega^2 + \Delta_s^2}) \delta(\omega \pm \sqrt{\omega^2 + \Delta_s^2}) = 2 \int_{0}^{\infty} dE \frac{E \mathcal{F}(E) \delta(\omega \pm E)}{\sqrt{E^2 - \Delta_s^2}} = \pm \frac{\Theta(\mp \omega - \Delta_s)}{\sqrt{\omega^2 - \Delta_s^2}} \cdot 2\omega \mathcal{F}(\mp \omega).$$

(F8)