Carrollian hydrodynamics from symmetries

Laurent Freidel\(^1\) and Puttarak Jai-akson\(^{1,2,\ast}\)

\(^1\) Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada
\(^2\) Department of Physics and Astronomy, University of Waterloo, 200 University Avenue West, Waterloo, ON N2L 3G1, Canada

E-mail: puttarak.jaiakson@gmail.com

Received 17 November 2022
Accepted for publication 9 January 2023
Published 10 February 2023

Abstract
In this work, we revisit Carrollian hydrodynamics, a type of non-Lorentzian hydrodynamics which has recently gained increasing attentions due to its underlying connection with dynamics of spacetime near null boundaries, and we aim at exploring symmetries associated with conservation laws of Carrollian fluids. With an elaborate construction of Carroll geometries, we generalize the Randers–Papapetrou metric by incorporating the fluid velocity field and the sub-leading components of the metric into our considerations and we argue that these two additional fields are compulsory phase space variables in the derivation of Carrollian hydrodynamics from symmetries. We then present a new notion of symmetry, called the near-Carrollian diffeomorphism, and demonstrate that this symmetry consistently yields a complete set of Carrollian hydrodynamic equations. Furthermore, due to the presence of the new phase space fields, our results thus generalize those already presented in the previous literatures. Lastly, the Noether charges associated with the near-Carrollian diffeomorphism and their time evolutions are also discussed.

Keywords: Carrollian, hydrodynamics, Carroll geometry, symmetries

(Some figures may appear in colour only in the online journal)

1. Introduction
The fascinating discovery journey of Carrollian physics has begun purely out of the mathematical curiosity of Lévy-Leblond \(^1\) when he first proposed a new non-Lorentzian limit of flat spacetime and derived its resulting contracted isometry group. The novel Carrollian\(^3\) limit (also referred to as the ultra-relativistic limit and the ultra-local limit by different authors), lying at

\(^\ast\) Author to whom any correspondence should be addressed.

\(^3\) It was named after Lewis Carroll, the author of Through the Looking-Glass.
the opposite side to the familiar Galilean (non-relativistic) limit, along with associated geometries, symmetries, and rich physics unfolded in this limit have recently gained unprecedented attentions from many fields of theoretical physics, especially from the flat space holography community.

Given any relativistic theory, non-Lorentzian variants are regarded as limits of the original relativistic theory as the speed of light, \( c \), approaches extreme values. There are two types of non-Lorentzian limit—the Galilean limit and the Carrollian limit. The former corresponds to the limit \( c \to \infty \) while the latter corresponds to the opposite limit, \( c \to 0 \). Changing the speed of light affects spacetime structures, with a notable example being a structure of light cones. In the well-familiar Galilean case, light cones expand as \( c \to \infty \) so that a free particle traverses spacetime without a speed limit, and there exists a notion of absolute time. Light cones, however collapse in the Carrollian limit \( c \to 0 \), hence freezing a free particle’s motion and thereby completely inhibiting causal interactions between any spacetime events. It is in this sense that the Carrollian limit is sometimes called the ultra-local limit\(^5\). In addition, the trademark of Carrollian theories, contrary to the Galilean case, is the existence of absolute space. Spacetime symmetries are also contracted to the Galilei group and the Carroll group in their respective limits and their associated Lie algebras are derived from the Inönü–Wigner contraction\([2]\).

Although Lévy-Leblond deemed practical utilization of the Carrollian limit and the Carroll group problematic, interest in Carrollian physics has recently been rejuvenated and gained ever-increasing attention due to its wealth of interesting aspects and applications. Developments in this topic include the generalization of Carroll geometries beyond flat spacetime\([3,4]\). Non-trivial dynamics of systems of Carroll particles which occurs when turning on interactions and when particles are coupled to non-trivial background fields has been explored in\([5–8]\). Carrollian limit has also been studied in a wide range of relativistic theories:\([9–17]\) for strings and branes,\([18]\) for supergravity theories,\([19–21]\) for electrodynamics, and aspects of the Carrollian gravity have also been addressed in\([22–37]\). Furthermore, a recent resurgence of Carrollian physics was largely catalyzed by the deep connection between Carroll geometries and null boundaries. At asymptotic null infinities, the connection between the (conformal) Carroll group and the Bondi–van der Burg–Metzner–Sachs group\([38,39]\) plays a central role in understanding of holography of asymptotically flat spacetime\([40–44]\) and thereby motivates the studies on Carrollian field theories\([45–47]\). Carrollian physics has also appeared in the context of inflationary cosmology\([48]\).

In this work, we are interested in Carrollian fluids, which is another non-Lorentzian counterpart of relativistic fluids along with non-relativistic Galilean (or Navier–Stokes) fluids. Hydrodynamic equations governing the dynamics of Carrollian fluids are derived from the \( c \to 0 \) limit of the relativistic conservation laws of general relativistic fluids\([49]\). While seemingly irrelevant to real-world fluids, Carrollian hydrodynamics has been shown to have applications in the field of black holes and holography\([50–59]\). Akin to the Galilean case, the hydrodynamics equations of Carrollian fluids include the evolution equation of Carrollian energy density and

\(^4\) To be more rigorous, one would rather need to consider the dimensionless parameter \( \frac{v}{c} \) where a characteristic velocity \( v \) of a problem under consideration. The final results, however do not differ from naively using \( c \) as the varying parameter.

\(^5\) Clarification of terminology is in order here. In terms of a dimensionless parameter \( \frac{c}{v} \), the ultra-local limit corresponds to the case where \( \frac{c}{v} \to 0 \), meaning that the characteristic velocity of the problem tends to zero slower than \( c \), in turn freezing the dynamics. On the other hand, the ultra-relativistic limit corresponds to the limit \( \frac{c}{v} \to 1 \), inferring that \( v \) trends to \( c \) in this limit. Unfortunately, these two terminologies have been mixed up and used interchangeably in the literature.
the evolution equations of Carrollian momentum density (which are the Carrollian analog of the Navier–Stokes equations). One apparent difference between the two non-Lorentzian fluids lies in their respective continuity equations. In the Galilean case, there is a notion of a spin-0 quantity, the fluid mass density, which is conserved. Carrollian fluids instead exhibit a conserved spin-1 quantity, the Carrollian heat current. Carrollian hydrodynamics also has one more constraint equation.

Since there are conservation laws for Carrollian fluids, a natural question followed by the Noether theorem then arises—what symmetries are associated with these Carrollian conservation laws? This question has already been addressed in [51, 59] where it has been demonstrated that Carrollian symmetry corresponds to the energy density and momentum density evolutions of Carrollian hydrodynamics. Thus, these works only managed to derive a part of the Carrollian fluid dynamics and the continuity equation of the Carrollian heat current and the constraint equation needed to be additionally supplemented. Our objective is to complete and hence generalize their results and provide a complete derivation of Carrollian hydrodynamics from symmetries. The incompleteness in their derivations that we are trying to fix stems from the following:

(a) Phase space of Carrollian hydrodynamics presented in [51, 59] lacked two fluid momenta, namely the Carrollian energy density and the sub-leading Carrollian viscous stress tensor, which appears in the constraint equation. We will show that these two momenta are conjugate to the fluid velocity field and the sub-leading sphere metric.

(b) Carrollian symmetry is too restrictive. Because there are more equations of Carrollian hydrodynamics than those in the original relativistic hydrodynamics, more symmetries are required. The main result in this work is the enhanced symmetries, called the near-Carrollian symmetries, that yield all equations of Carrollian fluids.

The article is structured as follows. We start in section 2 with the introduction of Carroll structures, which serves as the most basic building block of Carroll geometries and Carrollian physics. We will discuss Carrollian hydrodynamics in section 3 starting from relativistic conservation laws and then carefully consider the Carrollian limit. This closely follows the idea first explored in [49] and we formalize it using the language of Carroll structures. Finally, in section 4, we present a new viewpoint on Carrollian hydrodynamics based on symmetries. We propose a new notion of symmetries, which we call near-Carrollian symmetries, that extends the usual Carroll symmetries. We then demonstrate that these symmetries are associated to the full set of Carrollian hydrodynamics and derive the corresponding Noether charges. Lastly, we conclude in section 5 and comment about the possible avenue of investigations.

2. Carroll structures

We dedicate this section to describe universal building blocks of Carroll geometries which underpin the research field of Carrollian physics: the so-called Carroll structures. In what follows, we consider a three-dimensional space $H$ endowed with a null metric $q$ whose kernel is generated by a nowhere vanishing vector field $\ell$, meaning that $q(\ell, \cdot) = 0$. The triplet $(H, \ell, q)$ provides a (weak) definition of Carroll structures $^6$ [3, 4, 38, 39]. Carroll structures are universal

$^6$ A strong definition of Carroll structure requires, in addition, an affine connection that parallel transports both the metric $q$ and the vector $\ell$ [3, 38, 39]. This connection however is not uniquely determined from the pair $(\ell, q)$ due to the non-degenerate nature of $q$. 

3
intrinsic structures of null surfaces both at finite distances \[60–62\]^7 and at asymptotic infinities \[63, 64\].

Carroll structures are naturally described in the language of fiber bundle [4]. This specifically means that the space \(H\) is a fiber bundle, \(p : H \to S\), with a one-dimensional fiber. The two-dimensional base space \(S\) can be chosen, for relevant physics at hand, to have a topology of a two-sphere (and will be dubbed the sphere in this article). We denote local coordinates on the sphere \(S\) by \(\{\sigma^A\}\) and denote by \(q_{AB}\, d\sigma^A \otimes d\sigma^B\) a metric on \(S\).

Stemming from the fiber bundle structure of the space \(H\), one defines the vertical subspace of the tangent space \(TH\), denoted by \(\text{vert}(H)\), to be a one-dimensional kernel of the differential of the projection map, \(dp : TH \to TS\),

\[
\text{vert}(H) := \ker(dp).
\]

A vertical vector field \(\ell \in \text{vert}(H)\) that belongs to the Carroll structure is a preferred representative of the equivalence class \([\ell]_\sim\) with the equivalence relation being the rescaling that preserve the direction of \(\ell\), that is \(\ell \sim \varepsilon \ell\), where \(\varepsilon\) is any arbitrary function on the space \(H\). In this sense, the Carrollian vector \(\ell\) also serves as a basis of the vertical subspace. Another element of the Carroll structure is a null Carrollian metric \(q\) whose one-dimensional kernel coincides with the vertical subspace, inferring that \(q(\ell, \cdot) = 0\). The null metric can be obtained by pulling back a metric on the sphere \(S\) by the projection map, that is

\[
q = p^*(q_{AB}\, d\sigma^A \otimes d\sigma^B) = q_{AB}e^A \otimes e^B,
\]

where we introduced the co-frame field \(e^A\) which is the pullback of the coordinate form \(d\sigma^A\) on the sphere \(S\) by the projection map,

\[
e^A := p^*(d\sigma^A), \quad \text{such that} \quad \iota_\ell e^A = 0.
\]

Note that the co-frame field, by definition, is a closed form on \(H\), \(de^A = 0\).

Provided the Carroll structure on \(H\), it then becomes possible to have an intrinsic separation of the tangent space \(TH = \text{vert}(H) \oplus \text{hor}(H)\) into the aforementioned vertical subspace, \(\text{vert}(H)\), and its complement, the horizontal subspace denoted by \(\text{hor}(H)\). This splitting can be achieved by introducing a connection 1-form, \(k \in T^*H\), dual to the vertical vector \(\ell\),

\[
\iota_\ell k = 1.
\]

The 1-form \(k\) is known as the *Ehresmann connection* in the literature [4, 59, 61]. Its kernel, seen as a linear map \(k : TH \to \mathbb{R}\), thus defines the two-dimensional horizontal subspace. This equivalently means that

\[
\text{hor}(H) := \{X \in TH | \iota_X k = 0\}.
\]

In the following, we will denote a basis of the horizontal subspace by \(e_A \in \text{hor}(H)\) which, by definition, obeys the condition \(\iota_\ell e_A = 0\). Furthermore, without loss of generality, we can choose these horizontal basis vector fields to be ones that are dual to the co-frame field,

\[
\iota_\ell e_A^B = \delta_A^B.
\]

The frame fields \((\ell, e_A)\) and the dual co-frame fields \((k, e^A)\) therefore serve as a complete basis for the tangent space \(TH\) and the cotangent space \(T^*H\), respectively (see figure 1). In this basis, any vector field \(X \in TH\) and any 1-forms \(\omega \in T^*H\) can therefore be uniquely decomposed as follows:

---

[^7]: In [60], a complete universal structure of a null surface, viewed as a hypersurface embedded in an ambient spacetime, also includes an in-affinity function \(\kappa\) of the null vector \(\ell\). \(\kappa\) is defined as a time connection which transforms under rescaling \(\ell \to e^\kappa \ell\) as \(\kappa \to e^\kappa (\kappa + \ell(\kappa))\).
The space $H$ endowed with the Carroll structure. The general coordinates are $x^i = (u, y^A)$ where the surfaces at the cuts $u =$ constant are identified with the sphere $S$. The vertical vector $\ell$ and the horizontal vector $e_A$ span the tangent space $TH$.

$$X = (\iota_X k) \ell + (\iota_X e_A) e_A,$$

and

$$\omega = (\iota_x \omega) k + (\iota_{e_A} \omega) e^A.$$ (7)

Similarly, a differential of a function $F$ on the space $H$ can be expressed as

$$dF = \ell [F] k + e_A [F] e^A.$$ (8)

Lastly, having the intrinsic splitting of the tangent space $TH = \text{vert}(H) \oplus \text{hor}(H)$, one can naturally define the horizontal projector from the tangent space $TH$ to its horizontal components as

$$q^i_j := e^A_i e_A^j = \delta^i_j - k^i \ell,$$

and it satisfies the conditions $q^i_j k_j = 0$ and $\ell^i q^i_j = 0$. (9)

2.1. Acceleration, vorticity, and expansion

Next, we introduce two important objects that are naturally inherited from the Carroll structure and they will later appear when discussing Carrollian hydrodynamics [49, 51, 59]. These objects are the Carrollian acceleration, denoted by $\varphi_A$, and the Carrollian vorticity, denoted by $w_{AB}$. They are components of the curvature of the Ehresmann connection 1-form,

$$dk := - (\varphi_A k \wedge e^A + \frac{1}{2} w_{AB} e^A \wedge e^B).$$ (10)

Let us also recall that the co-frame $e^A$ is closed, i.e. $de^A = 0$. One can then show that the components $(\varphi_A, w_{AB})$ are also determined by the commutators of the basis vector fields. This correspondence can be established by invoking the identity $[x, L_Y] \omega = \iota_{[x,Y]} \omega$ for any vector.
fields $X, Y \in TH$ and any 1-form $\omega \in T^*H$. By making use of the Cartan formula, $\mathcal{L}_X = d_X + \iota_X d$, one can show that
\begin{equation}
\iota_{[Y,X]}d = \iota_{[X,Y]} + \mathcal{L}_Y(\iota_X) - \mathcal{L}_X(\iota_Y).
\end{equation}
Using this result and the property $de^A = 0$, we show that the commutators of the frame fields satisfy the conditions,
\begin{equation}
\iota_{[\ell, e_A]} e^B = 0, \quad \text{and} \quad \iota_{[\ell, e_A]} e^C = 0,
\end{equation}
suggesting that both commutators $[\ell, e_A]$ and $[e_A, e_B]$ lie in the vertical subspace. Similarly, using the definition (10), it then follows that,
\begin{equation}
\varphi_A = \iota_{[\ell, e_A]} k, \quad \text{and} \quad w_{AB} = \iota_{[e_A, e_B]} k.
\end{equation}
All these conditions therefore determines the commutation relations of the frame fields$^8$,
\begin{equation}
[e_A, e_B] = w_{AB} \ell, \quad \text{and} \quad [\ell, e_A] = \varphi_A \ell.
\end{equation}
We comment here that the Jacobi identity of the commutators determines the evolution of the Carrollian vorticity,
\begin{equation}
\ell[w_{AB}] = e_A[\varphi_B] - e_B[\varphi_A].
\end{equation}

It is important to appreciate that, as we have already derived, the commutator between horizontal basis vectors $[e_A, e_B]$ does not lie in the horizontal subspace hor$(H)$ when the Carrollian vorticity $w_{AB}$ does not vanish. Geometrically speaking, following from the Frobenius theorem, this means that the horizontal subspace hor$(H)$ is not integrable in general, meaning that it cannot be treated as a tangent space to a two-dimensional submanifold of the space $H$.

Given the metric $q_{AB}$ on the sphere $S$, we define the expansion tensor $\theta_{AB}$ as the change of the sphere metric along the vertical direction,
\begin{equation}
\theta_{AB} := \frac{1}{2} \mathcal{L}_\ell q_{AB} = \frac{1}{2} \ell[q_{AB}].
\end{equation}
The trace of the expansion tensor, called the expansion and denoted by $\theta$, computes the change of the area element of the sphere $S$ along the vector $\ell$,
\begin{equation}
\theta := q^{AB} \theta_{AB} = \ell[\ln \sqrt{q}].
\end{equation}

2.2. Horizontal covariant derivative

Another ingredient that is needed in order to write the Carrollian conservation laws is the notion of the horizontal covariant derivative. To this end, we introduce the Christoffel–Carroll symbols [49] defined in the same manner as the standard Christoffel symbols but using the two-sphere metric and the horizontal basis vectors,
\begin{equation}
(\partial)_{BC}^A := \frac{1}{2} q^{AD} (e_B[q_{DC}] + e_C[q_{BD}] - e_D[q_{BC}]).
\end{equation}
It is torsion-free, $(\partial)_{BC}^A = (\partial)_{CB}^A$ by definition. We then define the horizontal covariant derivative (or sometimes called the Levi-Civita–Carroll covariant derivative [49]) $\nabla_A$ which acts on a horizontal tensor $T = T^B e_A \otimes e^B$ as
\begin{equation}
\nabla_A T^B_C = e_A[T^B_C] + (\partial)_{BC}^A T^D - (\partial)_{DA}^B T^D_C - (\partial)_{CA}^D T^B_D,
\end{equation}

$^8$ Our definition of the Carrollian vorticity $w_{AB}$ differs from [49, 59] by a factor of 2.
and it can straightforwardly be generalized to a tensor of any degrees. By construction, the sphere metric $g_{AB}$ is compatible with this connection, that is $\mathcal{D}_C g_{AB} = 0$.

One useful formula will be that the horizontal divergence of a horizontal vector $X = X^A e_A$ is given by

$$\mathcal{D}_A X^A = \frac{1}{\sqrt{q}} e_A \left[ \sqrt{q} X^A \right].$$

More details on this covariant derivative are provided in appendix B.

2.3. Adapted coordinates for the Carroll structure

Up until this stage, we have always kept our presentation of the Carroll structure abstract and is thus completely independent of the choices of coordinates on the space $H$. We can pretty much continue this trend for the rest of this article. However, some physical pictures can be easily garnered when working explicitly with coordinates and, for practical purposes, some computations are conveniently carried out when expressing in coordinates. We will discuss the coordinate choices in this section.

Since the space $H$ is structured as the fiber bundle over the sphere $S$, we can, without loss of generality, choose a general coordinate system $x^i = (u, y^A)$ such that open sets of the cuts at $u = \text{constant}$, which denoted by $S_u$, are identified with open sets of the sphere $S$ through the projection map, $S_u \rightarrow S$, which maps the coordinates $y^A$ to the coordinates on the sphere $S$,

$$y^A \rightarrow \sigma^A = p^A(u, y^B).$$

In what follows, we will denote the Jacobian of the push-forward by $J: TS_u \rightarrow TS$, and it is explicitly given in coordinates by $J_\alpha^\beta = \partial_\alpha p^\beta$, where we have used the notation $\partial_\alpha := \frac{\partial}{\partial x^\alpha}$. In this general coordinate system, the Carroll structure is then characterized by a scale factor $\alpha$ and a velocity field $V^A$ such that

$$\ell = e^{-\alpha} D_u,$$

and

$$e^A = (dy^B - V^B du) J_B^A,$$

where we defined $D_u := (\partial_\alpha + V^A \partial_A)$. Following from the definition of the co-frame field $e^A := p^A(\sigma^{\alpha} d\sigma^A)$, the velocity field $V^A$ can be expressed in terms of the projection map as

$$V^A = -\partial_\beta p^\beta (J^{-1})^\beta_A,$$

such that $D_u p^A = 0$, (23)

where we introduced the matrix $J^{-1}$ to be the inverse of the Jacobian such that $J_\alpha^B (J^{-1})^B_C = (J^{-1})_A^C$. Let us also remark here that a change of the scale factor $\alpha$ preserves the Carroll structure while a variation of the velocity field $V^A$ changes the Carroll structure. It follows from the definition of the Jacobian that

$$\partial_\beta J_\alpha^A = \partial_C J_\beta^A.$$ (24)

In addition, the property $\partial_\beta e^A = 0$ imposes the following constraint on the Carrollian velocity and the Jacobian,

$$D_u J^A_B = -(\partial_B V^C) J^A_C,$$

and

$$D_u (J^{-1})^A_B = (J^{-1})^A_C \partial_C V^A.$$ (25)

The Ehresmann connection, obeying the condition $\iota_\ell k$, is characterized by the Carrollian connection density, $\beta_A$, and it can be parameterized as

$$k = e^\alpha (du - \beta_A e^A).$$ (26)

More rigorously, $p^A$ is a transition map, $p^A := (\sigma \circ p \circ x^{-1}(u, y))^A$, where $x: H \rightarrow \mathbb{R}^{d-1}$ and $\sigma: S \rightarrow \mathbb{R}^{d-2}$ provide, respectively, local coordinates on $H$ and $S$. 

7
The choice of the Ehresmann connection also fixes the form of the horizontal basis vectors $e_A$ by the conditions, $\iota_{e_A} k = 0$ and also $\iota_{e_A} e^B = \delta^B_A$. In our parameterization, the horizontal basis is given by
\[
e_A = (J^{-1})A^B \partial_B + \beta_A D_u.
\] (27)

In this general coordinate system, we can evaluate the Carrollian commutators and in turn obtain the coordinate expression of the Carrollian acceleration $\varphi_A$ and the Carrollian vorticity $w_{AB}$ (see appendix A). They are given by
\[
\varphi_A = D_u \beta_A + e_A [\alpha],
\] (28)

\[
w_{AB} = e^\alpha (e_A [\beta_B] - e_B [\beta_A]).
\] (29)

In this article, we will always work with the general coordinates $x^i = (u, y^A)$ on the space $H$ as they are, by construction, independent of the Carroll structure. Let us, however, mention that we can also choose to work with the adapted coordinates $(u, \sigma^A)$ on $H$ which are such that the action of the projection is trivial, $p : (u, \sigma) \rightarrow \sigma$. With this choice, the coordinate $u$ is regarded as the fiber coordinate. By definition, the velocity field $V^A = 0$ vanishes in the adapted coordinates. These coordinates are therefore co-moving coordinates, which are such that
\[
\ell = e^{-\alpha} \partial_u, \quad \text{and} \quad e_A = d\sigma^A.
\] (30)

To connect with the previous parameterization, one can derive, given the coordinates $y^A(u, \sigma)$, the following relations
\[
V^A = \partial y^A / \partial u, \quad \text{and} \quad (J^{-1})^A_B = \partial y^B / \partial \sigma^A.
\] (31)

The Ehresman connection in the adapted coordinates therefore reads
\[
k = e^\alpha (du - \beta_A d\sigma^A).
\] (32)

The expressions for the Carrollian acceleration and the Carrollian vorticity simplifies in the co-moving coordinates becomes
\[
\varphi_A = \left( \frac{\partial}{\partial \sigma^A} + \beta_A \partial_u \right) \alpha + \partial_u \beta_A,
\] (33)

\[
w_{AB} = e^\alpha \left( \left( \frac{\partial}{\partial \sigma^A} + \beta_A \partial_u \right) \beta_B - \left( \frac{\partial}{\partial \sigma^B} + \beta_B \partial_u \right) \beta_A \right).
\] (34)

The co-moving coordinates have been widely adopted in the Carrollian literature (see for example [3, 4, 51]) as the apparent absence of the velocity field and the Jacobian factor heavily simplifies all computations. Also, this choice of coordinates works well when considering field variations that leave the Carroll structure unchanged. We will, however, be more general by considering the set of variations that can change the Carroll structure, and will therefore work with the general, field-independent, coordinates $x^i = (u, y^A)$.

Let us also comment that the vorticity is the curvature of the Witt connection
\[
w_{AB} = e^\alpha \left( \frac{\partial}{\partial \sigma^A} \beta_B - \frac{\partial}{\partial \sigma^B} \beta_A + [\beta_A, \beta_B]_W \right), \quad \text{where} \quad [a, b]_W := a \partial_u b - b \partial_u a.
\] (35)
The bracket $[\cdot,\cdot]_W$ is the Witt bracket$^{10}$. This means that the corresponding symmetry group is the group $\text{Diff}(\mathbb{R})$ of a space-dependent time reparameterizations. An element of this group is denoted $U$ and simply represented by a function $U: u \rightarrow U(u, \sigma)$. The demand that the vorticity vanishes $w_{AB} = 0$ means that $\beta_A \partial_k = - U^{-1} \circ \partial_k U$ is a flat $\text{Diff}(\mathbb{R})$ connections. This implies that the coefficient $\beta_A$ is given$^{11}$, in a comoving coordinate system, by

$$\beta_A = - \frac{\partial_k U}{\partial_k U}. \quad (37)$$

The slices $U = \text{constant}$ are then the Bondi slices. In an arbitrary coordinate system $\beta_A$ can be written simply as $\beta_A = c_A [u + U]$.  

2.4. Carrollian transformations

We conclude our geometrical setup on Carroll structures by discussing Carrollian diffeomorphism. In general, there are two types of diffeomorphism of the space $H$—one that preserves the fiber bundle structure and one that changes it. Here we will focus on the former case and we will discuss the latter case when considering hydrodynamics in the next section.

Transformations that preserve the fiber bundle structure of the space $H$, which has been particularly referred to as Carrollian transformations or Carrollian diffeomorphism in the literature, are such that

$$u \rightarrow u' (u, \sigma^A), \quad \text{and} \quad \sigma^A \rightarrow \sigma'^A (\sigma^B). \quad (38)$$

In this class of transformations, the co-frame field $e^A$ only changes by the diffeomorphism on the sphere $S$, inferring that the basis vector $\ell$ can only change by rescaling, $\delta^{\text{Car}} \ell \propto \ell$. In other words, the new Carrollian vector still belongs to the equivalence class $[\ell]_{\sim}$. This therefore means that the velocity field is unchanged under Carrollian transformations,

$$\delta^{\text{Car}} Y^A = 0. \quad (39)$$

We now compute how the components $(\alpha, \beta_A, q_{AB})$ of the Carroll structure vary under infinitesimal Carrollian diffeomorphism generated by a vector field

$$\xi = \tau \ell + Y^A e_A, \quad \text{where} \quad \ell [Y^A] = 0, \quad (40)$$

and $\tau$ is a generic function on the space $H$. It follows from

$$\delta \xi = L_\xi \ell = [\xi, \ell] = - (\ell [\tau] + Y^A \varphi_A) \ell, \quad (41)$$

and $\delta^{\text{Car}} \ell = - (\delta^{\text{Car}} \alpha) \ell$. So the transformation of the scale factor is

$$\delta^{\text{Car}} (\tau, Y) \alpha = \ell [\tau] + Y^A \varphi_A. \quad (42)$$

For the Carrollian connection $\beta_A$, we use that $\delta^{\text{Car}} (\tau, Y) k = L_\xi k$ to read off the transformation of $\beta_A$, which is

$^{10}$ This means that $\beta_A \partial_k$ is an element of the Witt algebra. Let us also comment that it is more common to work with the Laurent polynomial basis $L_n := - u^{m+1} \partial_k$ where now $\beta_A (u, \sigma) = \sum_{m \in \mathbb{Z}} \beta^{(m)}_A (\sigma)L_n$. In this basis, the Witt algebra is in the well-familiar form, $[L_m, L_n]_W = (n - m) L_{m+n}$.

$^{11}$ This follows from the fact that $[U^{-1} \phi (u, \sigma) := \phi (U(u, \sigma), \sigma)$ which gives

$$[\partial_\sigma U^{-1}] \phi = [\partial_\sigma U] [\partial_\sigma \phi] (U, \sigma), \quad \left[\hat{\beta}_A \circ U^{-1}\right] \phi (u, \sigma) = \beta_A \partial_k \phi (U, \sigma) = [\beta_A \partial_k U] (\partial_\phi (U, \sigma), \quad (36)$$

where we denoted $\hat{\beta}_A := \beta_A \partial_n$. 

In practice, it is the square of the speed of light, \(c^2\), that will enter the computations.

The second condition \(h(\ell, e_A) = 0\), in fact, can be relaxed by choosing \(h(\ell, e_A) = c^{-2}B_\ell\) for an arbitrary function \(B_\ell\). The choice of \(B_\ell\) is gauge as one can always absorb \(B_\ell\) into the definition of the horizontal basis \(e_\ell\), and correspondingly redefine the Ehresmann connection \(\kappa\) and the sphere metric \(q_{AB}\), by shifting the Carrollian connection \(\beta_A \rightarrow \beta_A + B_\ell\). This new basis \(e'_\ell = e_\ell + B_\ell D_\ell\) then satisfies the second condition \(h(\ell, e'_\ell) = 0\).
At first glance, doing this \( c^2 \)-expansion of the sphere metric may seem like we have introduced unnecessary complications to the problems. We will later demonstrate that this \( c^2 \)-expansion is necessary to derive the hydrodynamic conservation equations from symmetries.

(b) In our derivations, it is sufficient to expand the Lorentzian metric \( h \) to the order \( c^2 \). Therefore, we can assume that the components \( \alpha \) and \( \beta_A \) do not admit this \( c^2 \)-expansion.

Since we now have the \( c^2 \)-expansion of the sphere metric, some objects will also inherit this similar expansion. The obvious ones are the expansion tensor and its trace, which exhibit the following expansion

\[
\theta_{AB} = \dot{\theta}_{AB} + c^2 \ell [\lambda_{AB}] + \mathcal{O}(c^4), \quad \text{and} \quad \theta = \dot{\theta} + c^2 \ell [\lambda] + \mathcal{O}(c^4),
\]

where the zeroth-order terms are

\[
\dot{\theta}_{AB} = \frac{1}{2} (\dot{q}_{AB}), \quad \text{and} \quad \dot{\theta} = \dot{q}^{AB} \dot{q}_{AB} = \ell \left[ \ln \sqrt{\dot{q}} \right].
\]

Another object that will admits the \( c^2 \)-expansion is the Christoffel–Carroll symbols \( (\gamma_{\ell\ell})^{A B C} \), and we present its expansion in appendix B.

In order to do integration on the space \( H \), we need the volume form on the \( H \). We define the volume form as

\[
\epsilon_H := k \wedge \epsilon_S, \quad \epsilon_S := \sqrt{q} \left( \frac{\epsilon_{AB}}{2} d\sigma^A \wedge d\sigma^B \right),
\]

where \( \epsilon_{AB} \) is the standard Levi-Civita symbol (satisfying \( \epsilon_{AC} \epsilon^{CB} = \delta_A^B \)), \( \epsilon_S \) denotes the canonical volume form on the sphere \( S \), which satisfies the relation \( \iota_{\epsilon} \epsilon_H = p^*(\epsilon_S) \). As before, using that \( \sqrt{q} = \sqrt{q} \left( 1 + c^2 \lambda \right) + \mathcal{O}(c^4) \), we thus obtain the \( c^2 \)-expansion of the volume form,

\[
\dot{\epsilon}_H = (1 + c^2 \lambda) \dot{\epsilon}_H + \mathcal{O}(c^4), \quad \text{and} \quad \epsilon_S = \left( 1 + c^2 \lambda \right) \epsilon_S + \mathcal{O}(c^4),
\]

where \( \dot{\epsilon}_H \) and \( \dot{\epsilon}_S \) denote the zeroth-order of the volume form on \( H \) and on \( S \), respectively.

---

\[14\] We use \( \odot \) to denote the symmetric tensor product of tensors, i.e. \( A \odot B = \frac{1}{2} (A \otimes B + B \otimes A) \).
3.2. Covariant derivative

Before considering Carrollian hydrodynamics, let us now consider the Levi-Civita connection $\nabla$ compatible with the metric (46), that is $\nabla h = 0$. Let us compute the covariant derivative the basis vector fields, namely $\nabla \ell, \nabla \epsilon, \nabla e_A$, and $\nabla e_B$, as they will become handy tools when evaluating the hydrodynamic conservation equations. We start with the covariant derivative $\nabla \ell$, which we will present the computation in full detail here. Complete derivations of the others, which are done in a similar vein, are provided for the readers in appendix D. The term $\nabla \ell$, can be decomposed as

$$\nabla \ell = (k_i \nabla e^i) \ell + (q^{AB} e_B \nabla e^A) e_A.$$

Using the metric $h$ and the Leibniz rule, one can show that the vertical component vanishes\(^{15}\) as follows:

$$k_i \nabla \ell^i = - \frac{1}{c^2} h(\ell, \nabla \ell) = - \frac{1}{2c^2} \ell [h(\ell, \ell)] = 0,$$

as $h(\ell, \ell) = - c^2$ is constant. The horizontal components can be evaluated with the help of the commutation relations (14) as follows:

$$e_B \nabla \ell^i = h(e_B, \nabla \ell) = - h(\ell, \nabla e_B)$$

$$= - h(\ell, [\ell, e_B]) - \frac{1}{2} e_B [h(\ell, \ell)]$$

$$= c^2 \varphi_B.$$

Therefore, the covariant derivative of the vertical vector field along itself is given by

$$\nabla \ell = c^2 \varphi^A e_A + \mathcal{O}(c^4).$$

Observe that it vanishes in the Carrollian limit $c^2 \to 0$, dictating that the vector $\ell$ is the null generator of null geodesics on the space $H$.

The covariant derivative of the vertical vector along the horizontal vectors can be computed using the same technique. One can show that (see appendix D) it is given by one could more simply write

$$\nabla \epsilon = \left( \frac{\partial}{\partial \epsilon} + c^2 \left( \frac{1}{2} w^a B^a + \ell [\lambda^B \ell] \right) \right) e_B + \mathcal{O}(c^4),$$

where $\lambda^B = q^{BC} \omega_{AC}$. The covariant derivative of the horizontal basis along the vertical basis, $\nabla \ell e_A$, is already determined from $\nabla \ell \ell$ and the commutator $[\ell, e_A]$. We are left with the remaining covariant derivative, $\nabla \epsilon e_B$. Its vertical component, $k_i \nabla \epsilon e_B^i$ can be inferred from $\nabla \epsilon \ell$. For the horizontal components, $e^C \nabla \epsilon e_B^C$, using that $q^{AB} = h(e_A, e_B)$ and the definition of the Christoffel–Carroll symbols (18), we can show that

$$\nabla \epsilon e_B = \left( \frac{1}{c^2} \theta_{AB}^B + \left( \frac{1}{2} w_{AB} + \ell [\lambda_{AB}] \right) \right) e_B + \mathcal{O}(c^4)$$

$$+ c^2 \left( \mathcal{D}_A \lambda^B + \mathcal{D}_B \lambda^A - \mathcal{D} e_C e_B \right).$$

With all these results, one can calculate the spacetime divergence of the basis vectors. Using the decomposition (9), we obtain

$$\nabla \ell^i = \delta^i \nabla \ell^i = (k_i \ell^i + e^B \epsilon e_B^i) \nabla \ell^i = \tilde{\ell} + c^2 \ell [\lambda],$$

\(^{15}\) This correspond to a choice of vanishing infinity $\kappa = \ell [\ln c] = 0.$
and in a similar manner,
\[ \nabla e_A^i = \delta^i_j \nabla_j e_A^j = (k_i^\ell + \epsilon^\ell, e_B^\ell) \nabla_j e_A^j = \varphi_A + c^2 \delta^i_A c^2 + \mathcal{O}(c^4). \quad (60) \]

It is important to remark that the three-dimensional metric compatible connection \( \nabla_j \) contains a component that diverges when taking the Carrollian limit \( c \to 0 \). This is to be expected since the inverse metric \( (46) \) diverges in this limit. This also suggests that, in practical, computations have to be carried out at finite value of \( c \) and the Carrollian limit needs to be taken at the last step.

### 3.3. Carrollian hydrodynamics

Armed with all these tools, we are ready to discuss the hydrodynamics of Carrollian fluid. Let us start from the general form of relativistic energy–momentum tensors,
\[ T^A = (\mathcal{E} + \mathcal{P}) \epsilon^A + \mathcal{E}^B + \frac{q^A}{c^2} + \frac{\tau^A}{c^2} + \mathcal{O}(c^4), \quad (61) \]

where we chose the vertical vector \( \ell \) to be the fluid velocity. The variables appeared in the fluid energy–momentum tensor consist of the fluid internal energy density \( \mathcal{E} \), the fluid pressure \( \mathcal{P} \), the heat current \( q^A \), and the viscous stress tensor \( \tau^A \), which is symmetric and traceless. The latter two quantities represent dissipative effects of the fluid and, by construction, they obey the orthogonality conditions with the fluid velocity, \( q_i^\ell = 0 \) and \( \tau^i_{\ell} = 0 \). This means that, in light of Carrollian geometry we have introduced, these dissipative tensors are horizontal tensors,
\[ q^i = q^A e_A^i, \quad \text{and} \quad \tau_{ij} = \tau^{AB} e_A^i e_B^j. \quad (62) \]

We are interested in the mixed indices version of the fluid energy–momentum tensor. Using the metric \( (46) \), it is given by
\[ T_i^j = - (\mathcal{E}^\ell + q^A e_A^i) k_i + \left( \frac{1}{c^2}q_{AB}q^B \ell^i + (q_{AC}^\ell + \mathcal{P}) \delta^i_A \right) e^j, \quad (63) \]

Furthermore, we choose the following \( c^2 \)-dependence \( \mathcal{E}^A = c^2 \mathcal{E} + c^2 \pi_A + \mathcal{O}(c^4) \) of the dissipative tensors,
\[ q^A = \mathcal{E}^A + c^2 (\pi^A - 2 \lambda_B^A \mathcal{J}^B), \quad \tau^{AB} = \frac{\Sigma^{AB}}{c^2} + \mathcal{J}^{AB}. \quad (64) \]

Note also that \( q_{AB}^B = \mathcal{J}^A + c^2 \pi_A + \mathcal{O}(c^4) \). Following from this parameterization, the fluid energy–momentum tensor can be expressed as the expansion in \( c^2 \) as
\[ T_i^j = \frac{1}{c^2} T_i^{(-1)j} + T_i^{(0)j} + \mathcal{O}(c^2), \quad (65) \]

where each term reads
\[ T_i^{(-1)j} = (\mathcal{E}^\ell + \Sigma^B e_B^i) e^j, \quad (66a) \]
\[ T_i^{(0)j} = (\mathcal{E}^\ell + \mathcal{J}^A e_A^i) k_i + (\pi^A e^i + (\mathcal{J}^B + \mathcal{P} \delta^B_A) e^j) e^j, \quad (66b) \]

and we defined for convenience the combination,
\[ \mathcal{J}^A_B = \mathcal{J}^A + 2 \lambda_{AC}^B \mathcal{J}^C. \quad (67) \]
The dynamics of the relativistic fluid is governed by the relativistic conservation laws, $\nabla_j T^i_j$. Let us first evaluate the vertical component of the conservation equations. With all the tools we derived previously, we show that

$$
eq -\nabla_j \left( \varepsilon \ell^i + q^i e_A \right) - \frac{1}{c^2} q^i \left( e_A \nabla_c \ell^i \right) = \frac{1}{c^2} \left( \varepsilon \ell^i + q^i \right) e_A \nabla_c \ell^i
- \left( \ell^i + \theta \right) e_A \nabla_c \ell^i - \left( \theta A + 2 \varphi_A \right) q^i - \tau^{AB} \theta_{AB}
= \frac{1}{c^2} C + E + O(c^2),
$$

where the coefficients of the $c^2$-expansion are

$$E = -\left( \ell + \theta \right) e_A \nabla_c \ell^i - \left( \theta A + 2 \varphi_A \right) J^A - \tau^{AB} \theta_{AB} = \Sigma^{AB} \ell [\lambda_{AB}],
$$

$$C = -\Sigma^{AB} \theta_{AB}.
$$

Imposing $\ell \nabla_i T^i_j = 0$ as one taking the limit $c \to 0$ demands $E = 0$ and $C = 0$. The first equation is the Carrollian energy evolution equation and second equation is the constraint equation. Note that the expression $E$ for the energy equation differs from the original work [49] due to the presence of the tensor $\lambda_{AB}$ and the fluid velocity $V^A$ contained implicitly in the Carrollian $\ell$. As we will discuss in the next section, these two additional variables are part of the phase space of Carrollian fluids and they are necessary when one wants to derive Carrollian conservation laws from symmetries. In this sense, our results generalizes those presented in [49].

In a similar manner to the vertical component, we compute the horizontal components of the conservation laws and consider the $c^2$-expansion. This is given by

$$e_A \nabla_i T^i_j = \nabla_j \left( e_A \ell^i \right) - \ell^i \nabla_j e_A
= \nabla_j \left( \frac{1}{c^2} q_{AB} q^B e_A \right) + \left( \varepsilon \ell^i + \frac{1}{c^2} q^i e_B \right) \nabla_i e_A
+ \left( q^B k_i - \left( q_{CD} \tau^{CD} + \varphi_B \delta^B_{CD} \right) e_i \right) \nabla_i e_A
= \frac{1}{c^2} \left( \ell + \theta \right) q_{AB} q^B e_A + \varepsilon \varphi_A - \delta_{AB} q^B + \left( \theta B + \varphi_B \right) \left( q_{AC} \tau^{CB} + \varphi_B \right)
= \frac{1}{c^2} J_i + \mathbb{P}_A + O(c^2),
$$

where the zeroth-order term is

$$\mathbb{P}_A = \left( \ell + \theta \right) \left[ \pi A + \varepsilon \varphi_A - \delta_{AB} q^B + \left( \theta B + \varphi_B \right) \left( \pi A + \varepsilon \varphi_A - \delta_{AB} q^B \right) \right],
$$

while the other term is

$$J_i = \left( \ell + \theta \right) \left[ \pi A + \theta B \right].
$$

Taking the Carrollian limit $c \to 0$ of the conservation laws, $e_A \nabla_i T^i_j = 0$, imposes the Carrollian momentum evolution, $\mathbb{P}_A = 0$ and the conservation of Carrollian current, $J_B = 0$. Again, our expression for $\mathbb{P}_A$ is the generalization of [49].

Let us comment here the case when the sub-leading components of the sphere metric vanishes, $\lambda_{AB} = 0$ simplifies the Carrollian evolution equations,

$$E = -\left( \ell + \theta \right) \left[ \varepsilon \right] - \varphi_B - \left( \theta A + 2 \varphi_A \right) J^A - \tau^{AB} \theta_{AB}.
$$

(74a)
\[ \mathcal{P}_A = (\ell + \dot{\theta})[\pi_A] + \delta\varphi_A - w_{AB} \mathcal{F}^B + (\dot{\mathcal{D}}_B + \varphi_B)(\mathcal{A}_A^B + \mathcal{D}_A^B), \]  
(74b)
\[ \mathcal{J}_A = (\ell + \dot{\theta})[\mathcal{J}_A] + (\dot{\mathcal{D}}_B + \varphi_B)\Sigma^B_A, \]  
(74c)
\[ \mathcal{C} = -\Sigma^{AB}\dot{\mathcal{D}}_{AB}. \]  
(74d)

These are the Carrollian fluid equations given in the literature [49, 51]. Note that the solutions of these equations are invariant under the shift \((\mathcal{D}, \pi_A, \mathcal{A}_A^B) \to (\mathcal{D}, \pi_A + a \mathcal{A}_A^B, \mathcal{A}_A^B + a\Sigma^B_A)\) where \(a\) is an arbitrary parameter and where \((\mathcal{P}, \mathcal{J}_A, \Sigma^B_A)\) is unchanged.

4. Hydrodynamics from symmetries

In this section, we tackle the Carrollian hydrodynamics from a different, but nonetheless equivalent, perspective. Our objective is to re-derive the equations that govern Carrollian hydrodynamics (69), (70), (72), and (73) from the symmetries of the space \(H\).

4.1. The action for Carrollian fluid

Since the metric \(h\) is defined on the space \(H\), we can consider the action of the fluid whose variation yields the fluid energy–momentum tensor. We will consider the fluid action that is finite when taking the Carrollian limit \(c \to 0\). The variation of the fluid action we will use takes the form

\[ \delta S_{\text{fluid}} = -\int_H \left( \mathcal{E}\delta\alpha - e^\alpha \delta\mathcal{A}_A + e^{-\alpha} \tilde{\pi}_A \delta V^A - \frac{1}{2} \left( \delta\mathcal{A}^C + \mathcal{D}\delta\mathcal{A}^B \right) \delta\tilde{q}_A^B - \Sigma^{AB}\delta\lambda_{AB} \right) \epsilon_H. \]  
(75)

We defined the momentum conjugated to the velocity field \(V^A\) and the leading-order sphere metric \(q_{AB}\) to be

\[ \tilde{\pi}_A := \pi_A + \lambda \mathcal{A}_A \]  
(76)
\[ \tilde{\mathcal{A}}^{AB} := \mathcal{A}^{AB} + \lambda \Sigma^{AB}. \]  
(77)

We also absorbed the Jacobian factors and the velocity field variation into the definition of the variation \(\delta\) as follows,

\[ \delta\alpha := \delta\alpha + \beta_A \delta V^A, \]  
(78)
\[ \delta\beta_A := (J^{-1})_A^C \delta (J_C^B \beta_B) - (\beta \cdot \delta V)\beta_A, \]  
(79)
\[ \delta\tilde{q}_{AB} := (J^{-1})_A^C (J^{-1})_B^D \delta (J_C^E J_D^F \tilde{q}_{EF}) - 2\tilde{q}_{AC}\beta_B \delta V^C, \]  
(80)
\[ \delta\lambda_{AB} := (J^{-1})_A^C (J^{-1})_B^D \delta (J_C^E J_D^F \lambda_{EF}) - 2\lambda_{AC}\beta_B \delta V^C, \]  
(81)

and that we define

\[ \delta V^A := (\delta V^B) J_B^A. \]  
(82)

The action (75) is simply derived from the fluid energy-momentum tensor \(T^i_j\) and the metric variation \(\delta h_{ij}\). To see this, let us consider an action \(S[h_{ij}]\) and its metric variation yields the energy–momentum tensor,

\[ \delta S = \int_H \left( \frac{1}{2} T^i_j \delta h_{ij} \right) \epsilon_H. \]  
(83)
Since the fluid energy–momentum tensor (65) has a part that diverges when taking the Carrollian limit \( c \to 0 \), the variation \( \delta S \) also diverges in this limit. To obtain the finite action (75), we subtract the divergent part from \( \delta S \) then take the Carrollian limit, that is

\[
\delta S_{\text{fluid}} := \lim_{c \to 0} \left( \frac{\delta S}{c^2} \right). 
\]  
(84)

We note that the divergent part is given by

\[
\delta S_{(-1)} := \lim_{c \to 0} \left( c^2 \delta S \right) = \int_H \left( \frac{1}{2} T^{(0)}_{\cdots(1)} \delta h_{(0)(1)} \right) \epsilon_H, 
\]  
(85)

where we used that the metric variation is regular as \( c \to 0 \) and schematically expands as \( \delta h_{ij} = \delta h_{(0)ij} + c^2 \delta h_{(1)ij} + O(c^4) \). The fluid action (75) is thus

\[
\delta S_{\text{fluid}} = \int_H \left( \frac{1}{2} T^{(0)}_{\cdots(1)} \delta h_{(0)(1)} + T^{(-1)}_{\cdots(1)} \delta h_{(1)(1)} + \lambda T^{(-1)}_{\cdots(1)} \delta h_{(0)(1)} \right) \epsilon_H. 
\]  
(86)

4.2. Near-Carrollian diffeomorphism

To derive the Carrollian hydrodynamic equations from the variation of the action (75) under certain symmetries, we first need to specify those symmetries and derive the symmetry transformations for the metric components, \((\alpha, \beta_A, \lambda_A, \tilde{q}_{AB}, \lambda_{AB})\). The seemingly obvious choice one could consider is the Carrollian diffeomorphism. However, Carrollian diffeomorphism is not sufficient to derive the complete set of hydrodynamic equations (69), (70), (72), and (73), as already shown in [51]. The reasons for this limitation are as follows:

(a) Carrollian diffeomorphism fixes the variation of the velocity field, \( \delta^{\text{Carroll}} V^A = 0 \), hence turning off a phase space degree of freedom conjugated to the velocity, that is the fluid momentum density.

(b) There are only two symmetry parameters \((\tau, Y^A)\) for the Carrollian diffeomorphism, while there are four hydrodynamic equations. The symmetries labelled by the parameter \( \tau \) and \( Y^A \) correspond, respectively, to the energy equation (69) and the momentum equation (72). To obtain the remaining two equations, the current conservation (73) and the constraint (70), we need two more symmetry parameters.

We therefore need to detach our consideration from the Carrollian diffeomorphism and consider a general diffeomorphism on the space \( H \). The general diffeomorphism on \( H \) is labelled by vector fields of the form,

\[
\xi = f^\ell + X^A e_A, 
\]  
(87)

where \( f \) and \( X^A \) are arbitrary functions on \( H \). This general diffeomorphism will definitely change the Carroll structure. In the same fashion as our prior discussions, let us expand the transformation parameters \((f, X^A)\) in the small parameter \( c^2 \) as

\[
f = \tau + c^2 \psi + O(c^4), \quad \text{and} \quad X^A = Y^A + c^2 Z^A + O(c^4), 
\]  
(88)

where now the parameter \((\tau, \psi, Y^A, Z^A)\) are functions on \( H \). This way, we have already secured four parameters we need for four equations of Carrollian fluid. It is of extreme importance to point out that expanding the diffeomorphism around \( c^2 = 0 \) can be regarded as the analog to the diffeomorphism of spacetime geometry in the close vicinity of a black hole horizon, the
near-horizon diffeomorphism, with \( c^2 \) plays the same role as the distance away from the black hole horizon. We will refer to this diffeomorphism as the near-Carrollian diffeomorphism\(^\text{16}\).

As stated previously, we need to find how the metric components vary under the near-Carrollian diffeomorphism. To carry out this task, we employ the technology of the anomaly operator \( \Delta_\xi \) which compares the spacetime transformation of the field to its field space transformation. The metric \( h \) is covariant under the near-horizon diffeomorphism, meaning that its anomaly \( \Delta_\xi h := \delta_\xi h - L_\xi h \) vanishes. The anomaly of the metric \( h \) decomposes as

\[
\Delta_\xi h = -2c^2(\iota_\xi k \circ k + \Delta_\xi q) - 2c^2(\iota_\xi q(\iota, e_A) - c^2(\iota_\xi \Delta_\xi k)) k \circ e^A + \Delta_\xi q(e_A, e_B) e^A \circ e^B.
\]

Demanding covariance, \( \Delta_\xi h = 0 \), imposes the following conditions,

\[
\iota_\xi \Delta_\xi k = 0, \quad \Delta_\xi q(\iota, e_A) = 2c^2(\iota_\xi \Delta_\xi k), \quad \text{and} \quad \Delta_\xi q(e_A, e_B) = 0.
\]

The problem then boils down to the computation of the anomaly of the Ehresmann connection \( k \) and the anomaly of the null Carrollian metric \( q \) (we defer the derivations to the appendix E).

Solving the above conditions for different powers of \( c^2 \) gives us the transformation of the metric components under the near-Carrollian diffeomorphism,

\[
\begin{align*}
\delta_\xi \alpha &= \delta^{\text{CM}}_{\iota_\xi Y}\alpha, \\
\iota_\xi \beta_A &= e^\alpha \delta^{\text{CM}}_{\iota_\xi Y}\beta_A + \dot{q}_A \ell^C Z^B, \\
\delta_\xi q_{AB} &= \delta^{\text{CM}}_{\iota_\xi Y} q_{AB}, \\
\delta_\xi \lambda_{AB} &= \frac{1}{2} \delta^{\text{CM}}_{\iota_\xi Y} \dot{q}_{AB} + \tau \ell[\lambda_{AB}] + Y^C \check{D}_C \lambda_{AB} + 2\lambda C_{(A} \check{D}_B Y^C, \\
\delta_\xi V^A &= -D_\xi Y^A.
\end{align*}
\]

where we recalled the functional form of the Carrollian transformations\(^\text{17}\) (42)–(44), and the transformation of the velocity field is given by.

4.3. Hydrodynamics from near-Carrollian diffeomorphism

The Carrollian hydrodynamic equations (69), (70), (72), and (73) can be recovered by demanding invariance, up to boundary terms, of the fluid action (75) under the near-Carrollian transformations, \( \delta_\xi S_{\text{fluid}} = 0 \). Using the near-Carrollian transformations (91) and (92) and the Stokes theorem (108), one can show that

\[
\delta_\xi S_{\text{fluid}} = - \int_H \left( \tau \mathbb{E} + \mathbb{F} + Y^A \mathbb{P}_A + Z^A \mathbb{J}_A \right) \dot{\epsilon}_H + \Delta_\xi \mathcal{Q}_\xi
\]

where we defined the combinations of the transformation parameters, \( \check{\psi} := \psi + \lambda \tau \) and \( Z^A := Z^A + \lambda Y^A \). The boundary term \( \Delta_\xi \mathcal{Q}_\xi \) is the difference of Noether charges corresponding to the near-Carrollian diffeomorphism at the two ends of \( H \). We clearly see that imposing \( \delta_\xi S_{\text{fluid}} = 0 \) up to the boundary term yields the fluid equations.

The Noether charges of these transformations have three components associated with different sectors of the near-Carrollian symmetries,

\(^{16}\) Expansion in \( c^2 \) has been dubbed pre-ultra-local expansion in [31].

\(^{17}\) Although now there is no constraint on \( Y^A \), unlike the Carrollian transformations where \( \ell[Y^A] = 0 \).
where each components are given by

\begin{align}
Q_{\tau} &= -\int_S \left( \epsilon + c^\alpha J^A \beta_A \right) \hat{\epsilon}_S, \\
Q_Y &= \int_S \left( \pi_A + c^\alpha \left( A^A + \mathcal{P}^B \delta^B_A \right) \beta_B \right) \hat{\epsilon}_S, \\
Q_Z &= \int_S \left( J_A + c^\alpha \Sigma^B_A \beta_B \right) \hat{\epsilon}_S,
\end{align}

5. Conclusion

In this work we have extended the analysis of the $c \to 0$ limit of relativistic fluids towards a Carrollian fluids. Starting from Carroll structures, we studied Carrollian fluids and presented two methods to derive the corresponding hydrodynamic conservation laws. In the first and conventional method, we started from energy–momentum tensors of general relativistic fluids, then properly consider the Carrollian limit ($c \to 0$) of the standard relativistic conservation laws.

---

18 One can more generally express the charges at the spheres $S_f = \{u = f(\sigma)\}$ as the same integrals with $\beta_A$ replaced by $\beta_A - e_A[f]$. 

---
laws. Our derivations could be viewed as the generalization of [49] due to the fact that we now have in our construction the fluid velocity $V^A$ and the sub-leading components of sphere metric $\lambda_{AB}$. These two quantities are indeed important parts of the phase space of Carrollian hydrodynamics. The second route, which was the highlight of this article, was to viewed Carrollian hydrodynamics as the consequence of symmetries. We argued that Carrollian diffeomorphism is not sufficient to derive the full set of Carrollian fluid equations (which has already been studied in [51, 59]) and that we need to go beyond Carrollian diffeomorphism. To this end, we introduced the notion of near-Carrollian symmetries (88) and finally showed that it leads to the complete set of Carrollian hydrodynamic equations.

Many directions however remain to be explored. Let us list some of them below.

(a) **Realization on stretched horizons and null boundaries:** The membrane paradigm [66–68] has established the correspondence between black hole physics and dynamics of fluids living on timelike surfaces, called stretched horizons or membranes, located near black hole horizons (which are null surfaces). As Carroll structures are universal structures of null boundaries, be they at finite distances [60–62] or infinities [63, 64], one would therefore expect the membrane fluids to be Carrollian fluids. This statement has just been realized recently in [54] (see also [50]), where it has been shown that the Einstein equations on black hole horizons can be displayed as Carrollian hydrodynamic equations and that the near-horizon diffeomorphism [69, 70] is Carrollian diffeomorphism. The analog of the Brown–York energy–momentum tensor of null boundaries and its conservation laws have also been studied in [61].

We have learned in this work that Carroll structures can be endowed on any surfaces, regardless of whether they are null or timelike, inferring the possibility to assign the Carrollian hydrodynamic picture to timelike surfaces, say for example, stretched horizons. In fact, it is to be expected that stretched horizons encode some underlying informations of the null boundaries, in the same spirit as the near-Carrollian analysis (the value of $c^2$ deviates from zero) presented in this work. To make our argument more elaborate, further investigations are required and some aspects of it will be provided in our upcoming work [71].

(b) **Sub-subleading and higher order corrections:** In our analysis, only the sub-leading (order $c^{-2}$) terms were considered. One can indeed extend our construction by including sub-subleading (order $c^{-4}$) and higher order terms in the metric (46), which will introduce new variables to the phase space of Carrollian fluids and in turn activates new Carrollian fluid momenta conjugate to these higher-order variables. These momenta are corresponding to the $c^{-4}, c^{-6}, c^{-8}, \ldots$ corrections of the dissipation tensors (64) which we have truncated them at order $c^{-2}$ here. As a consequence, the near-Carrollian diffeomorphism (88) will be enhanced with the inclusion of higher-order corrections associated with new equations governing the dynamics of these new momenta and also new Noether charges.

Let us also mention that this picture has already been realized in the context of asymptotic null infinities [72–74] which exhibit the infinite tower of charges and their corresponding conservation equations. It would then be of interest to study the higher-order dynamics of the Carrollian hydrodynamics and bridge the findings with the results at infinities.

---

19 Usually in the literature, the Carrollian metric $q$ is treated as an induced metric on hypersurfaces and the null-ness property of $q$ then dictates the hypersurfaces to be null. This is not necessary as we can endowed, for example, the Lorentzian metric (46) on any type of hypersurfaces while incorporating all elements of the Carroll structure into the geometry of the surfaces.
(c) **Thermodynamics of Carrollian fluids:** Having established the Carrollian hydrodynamics, one natural question therefore emerges—what are thermodynamical properties of Carrollian fluids? Admittedly, although this question may not garner much interest in the field of fluid mechanics due to the sole fact that everyday life’s fluids are Galilean in nature, we believe that answering this question will provide useful insights to the realm of black hole physics. One possible direction to explore in the future is the notion of **thermodynamical horizons**, the type of surfaces that obey all laws of thermodynamics, and also the universal notion of equilibrium in any surface.

(d) **Galilean hydrodynamics from symmetries:** As the speed of light \( c \) now plays a role of the varying parameter when taking non-Lorentzian limits, it then suggests that similar analysis could be carried out for the Galilean case \((c \to \infty)\ limit\), therefore giving the derivation of Galilean hydrodynamics, e.g. the continuity equation and the Navier–Stokes equations, from symmetries. In this case, the underlying structure is the Newton–Cartan structure \([3]\) (see also \([75, 76]\)) instead of the Carroll structure.

**Data availability statement**

No new data were created or analysed in this study.

**Acknowledgments**

We would like to thank Céline Zwickel for helpful discussions and insights. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Colleges and Universities. The work of LF is funded by the Natural Sciences and Engineering Research Council of Canada (NSERC) and also in part by the Alfred P Sloan Foundation, Grant FG-2020-13768. This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No. 841923. P J’s research is supported by the DPST Grant from the government of Thailand, and Perimeter Institute for Theoretical Physics.

**Appendix A. Coordinate expressions for \( \varphi_A \) and \( w_{AB} \)**

Expressions for the Carrollian acceleration \( \varphi_A \) and the Carrollian vorticity in coordinates are straightforwardly computed from the Carrollian commutators. Let us start with the acceleration, we evaluate

\[
\varphi_A = [\ell, e_A] = [e^{-\alpha} D_u, e_A] = e_A[\alpha] \ell + e^{- \alpha} [D_u, (J^{-1})^B_A \partial_B + \beta_A D_u] = (D_u \beta_A + e_A[\alpha]) \ell + e^{- \alpha} (D_u (J^{-1})^B_A - (J^{-1})^C_A \partial_C V^B) \partial_B. \tag{98}
\]

The last term vanishes due to the condition \((25)\). We therefore obtain the expression

\[
\varphi_A = D_u \beta_A + e_A[\alpha]. \tag{99}
\]
Similarly, the Carrollian vorticity can be evaluated as follows,

\[ w_{AB}^\ell = \left[ e_A, e_B \right] \]

\[ = [(J^{-1})_A^C \partial_C + \beta_A D_u, (J^{-1})_B^D \partial_D + \beta_B D_u] \]

\[ = [(J^{-1})_A^C \partial_C, (J^{-1})_B^D \partial_D] + [(J^{-1})_A^C \partial_C, \beta_B D_u] + [\beta_A D_u, (J^{-1})_B^D \partial_D] \]

\[ + [\beta_A D_u, \beta_B D_u] \]

\[ = e^\alpha (e_A [\beta_B] - e_B [\beta_A]) \ell + (e_A [J_B^C] - \beta_A (J^{-1})_B^D \partial_D Y^C - (A \leftrightarrow B)) \partial_C. \] (100)

The last term, again, computes to zero by means of (25). The Carrollian vorticity is then given by

\[ w_{AB} = e^\alpha (e_A [\beta_B] - e_B [\beta_A]). \] (101)

One can alternatively check by computing the curvature of \( k = e^\alpha (du - \beta_A e^A) \), which is

\[ dk = d\alpha \wedge k - e^\alpha d\beta_A \wedge e^A \]

\[ = -(D_u \beta_A + e_A [\alpha]) k \wedge e^A - \frac{1}{2} e^\alpha (e_A [\beta_B] - e_B [\beta_A]) e^A \wedge e^B \]

\[ = -\varphi_A k \wedge e^A - \frac{1}{2} w_{AB} e^A \wedge e^B. \] (102)

Appendix B. Horizontal covariant derivative

One property of the horizontal covariant derivative \( \mathcal{D}_A \) is that we can define the analog of the Riemann tensor with this connection and it is called the Riemann–Carroll tensor, \( (\mathcal{R})_{BCD} \). Its components are determined from the commutator,

\[ [\mathcal{D}_C, \mathcal{D}_D] X^A = (\mathcal{R})_{BCD} X^B + w_{CDB} X^A, \] (103)

where the vertical derivative term \( \ell[X^A] \) appeared due to the non-integrability of the horizontal subspace. We can then define corresponding the Ricci–Carroll tensor, \( (\mathcal{R}_{AB}) := (\mathcal{R})_{CAB} q^{CD}, \)
and the Ricci–Carroll scalar, \( (\mathcal{R}) := (\mathcal{R})_{AB} q^{AB}. \) Let us also note that the Ricci–Carroll tensor is not symmetric, \( (\mathcal{R})_{AB} \neq (\mathcal{R})_{BA} \), in general.

Since we are dealing with the expansion in \( c^2 \) of the sphere metric, \( q_{AB} = \hat{q}_{AB} + 2c^2 \lambda_{AB} \), it then becomes essential to define the similar expansion for the connection \( (\mathcal{\Gamma})_{BC} \). With this in mind, let us define the following connection,

\[ (\mathcal{\Gamma})_{BC} := \frac{1}{2} q^{AD} (e_B [\hat{q}_{DC}] + e_C [\hat{q}_{BD}] - e_D [\hat{q}_{BC}]), \] (104)

and the new horizontal covariant derivative \( \mathcal{D}_A \) compatible with the zeroth-order of the sphere metric \( \hat{q}_{AB} \), that is \( \mathcal{D}_A \hat{q}_{BC} = 0 \). This operator \( \mathcal{D}_A \) acts on a horizontal tensor the same way as \( \mathcal{D}_A \) but with the new connection \( (\mathcal{\Gamma})_{BC} \) instead of \( (\mathcal{\Gamma})_{BC} \). One can therefore show that \( (\mathcal{\Gamma})_{BC} \) admits the following expansion in \( c^2 \),

\[ (\mathcal{\Gamma})_{BC} = (\mathcal{\Gamma})_{BC} + c^2 \left( \mathcal{D}_A \lambda_B^C + \mathcal{D}_B \lambda_A^C - \mathcal{D}_C \lambda_{AB} \right) + \mathcal{O}(c^4). \] (105)

Appendix C. Integration by parts

One can verify the following relations

\[ d(\varphi_S) = (\ell[f] + \theta f) e_H, \quad \text{and} \quad d(i_H e_H) = (\mathcal{D}_A X^A + \varphi_A X^A) e_H, \] (106)
for a function $f$ on $H$ and for a horizontal vector $X = X^A e_A \in \text{hor}(H)$. The second equation can be proven as follows:

\[
\mathbf{d}(\iota_X \epsilon_H) = \mathcal{L}_X \epsilon_H = (X \lrcorner \sqrt{q} + \iota_X \mathcal{L}_k + \iota_{e_A} \mathcal{L} e^A) \epsilon_H = (e_A [X^A] + X^A e_A \lrcorner \sqrt{q} + \varphi_A X^A) \epsilon_H = (\mathcal{D}_A X^A + \varphi_A X^A) \epsilon_H,
\]

(107)

where we recalled the expression of the volume form $\epsilon_H = \frac{1}{2} \epsilon_{AB} \sqrt{q} k \wedge e^A \wedge e^B$ and the curvature of the Ehresmann connection (10).

One can imagine the space $H$ to have a boundary $\partial H$ situated at a constant value of the coordinate $u$. This boundary, in our construction, is identified under the projection map with the sphere $S$, meaning that $\partial H = S_u$. In this setup, the Stokes theorem is written as

\[
\hat{H} \left( \iota_{[f]} + \theta f \right) \hat{\epsilon}_H = \int_{S_u} \epsilon^A X^A \beta_A \hat{\epsilon}_S.
\]

(108a)

\[
\hat{H} \left( \mathcal{D}_A X^A + \varphi_A X^A \right) \hat{\epsilon}_H = \int_{S_u} \epsilon^A X^A (\beta_A - e_A [f]) \hat{\epsilon}_S.
\]

(108b)

Alternatively, one can choose a more general cut, say $u = f(\sigma)$. In this case, we have instead

\[
\hat{H} \left( \mathcal{D}_A X^A + \varphi_A X^A \right) \hat{\epsilon}_H = \int_{S_u} \epsilon^A X^A (\beta_A - e_A [f]) \hat{\epsilon}_S.
\]

(109)

Let us observe that choosing $\beta_A = e_A [f]$ removes the boundary contribution. This is equivalent to the case of vanishing vorticity, $w_{AB} = 0$.

### Appendix D. Covariant derivatives

In the main text, we already presented the derivation of the covariant derivative $\nabla_\ell \ell$. Here we complete the detailed derivations of the remaining covariant derivatives, which are $\nabla_{e_A} \ell$, $\nabla_\ell e_A$, and $\nabla_{e_A} e_B$.

**Derivation of $\nabla_{e_A} \ell$:** For completeness, let us quote the result derived in the main text,

\[
\nabla_{e_A} \ell = e^2 \varphi^A e_A + O(e^4).
\]

(110)

**Derivation of $\nabla_{e_A} e_B$:** We begin by writing the vector $\nabla_{e_A} \ell$ in the Carrollian basis $(\ell, e_A)$,

\[
\nabla_{e_A} \ell = (k_i \nabla_{e_A} \ell^i) \ell + (e^B \nabla_{e_A} \ell^B) e_B.
\]

(111)

then consider each component separately. The vertical component is identically zero as one can easily see from

\[
k_i \nabla_{e_A} \ell^i = -\frac{1}{c^2} h(\ell, \nabla_{e_A} \ell) = -\frac{1}{2c^2} e_A [h(\ell, \ell)] = 0.
\]

(112)

The horizontal components are computed using repeatedly the Leibniz rule and the commutators (14),
\[ e^B \nabla_{e^i} e^\ell = q^{BC} h(e^C, \nabla_{e^i} e^\ell) \]
\[ = \frac{1}{2} q^{BC} (h(e^C, \nabla_{e^i} e^\ell) + h(e^C, \nabla_{e^i} e^\ell)) \]
\[ = \frac{1}{2} q^{BC} (-h(\nabla_{e^i} e^C, e^\ell) + h(e^C, \nabla_{e^i} e^\ell)) \]
\[ = \frac{1}{2} q^{BC} (-h([e^i, e^C], e^\ell) - h(\nabla_{e^i} e^A, e^\ell) + h(e^C, \nabla_{e^i} e^\ell)) \]
\[ = \frac{1}{2} q^{BC} (e^2 w_{AC} + h(e^A, \nabla_{e^i} e^\ell) + h(e^C, \nabla_{e^i} e^\ell)) \]
\[ = \frac{1}{2} q^{BC} (e^2 w_{AC} + h(e^A, \nabla_{e^i} e^\ell) + h(e^C, \nabla_{e^i} e^\ell)) \]
\[ = \frac{1}{2} q^{BC} (e^2 w_{AC} + 2 \theta_{AC}), \tag{113} \]

where we recalled \( 2 \theta_{AB} = \ell [q_{AB}] \). Expanding the metric \( q_{AB} \) in \( c^2 \), we therefore obtain

\[ \nabla_{e^i} e^\ell = \left( \delta^B_A + c^2 \left( \frac{1}{2} w^B_A + q^{BC} \ell [\lambda_{AC}] - 2 \lambda^{BC} \theta_{AC} \right) \right) e_B + O(c^4). \tag{114} \]

**Derivation of \( \nabla_{e^i} e_A \):** this term decomposes as

\[ \nabla_{e^i} e_A = (k_i \nabla_{e^i} e_A) \ell + (e^B_i \nabla_{e^i} e_A^B) e_B. \tag{115} \]

Its components are already determined by the components of \( \nabla_{e^i} \ell \) and \( \nabla_{e^i} e^\ell \). For the vertical component, we have

\[ k_i \nabla_{e^i} e_A^i = -\frac{1}{c^2} h(\ell^i, \nabla_{e^i} e_A) = \frac{1}{c^2} h(\nabla_{e^i} \ell, e_A) = \varphi_A, \tag{116} \]

and for the horizontal components, we have

\[ e^B_i \nabla_{e^i} e_A^i = q^{BC} h(e^C, \nabla_{e^i} e_A) \]
\[ = q^{BC} (h(e^C, \nabla_{e^i} e^\ell) + h(e^C, [\ell, e_A])) \]
\[ = \delta^B_A + c^2 \left( \frac{1}{2} w^B_A + q^{BC} \ell [\lambda_{AC}] - 2 \lambda^{BC} \theta_{AC} \right) + O(c^4). \tag{117} \]

Together, they give

\[ \nabla_{e^i} e_A = \varphi_A e^\ell + \left( \delta^B_A + c^2 \left( \frac{1}{2} w^B_A + q^{BC} \ell [\lambda_{AC}] - 2 \lambda^{BC} \theta_{AC} \right) \right) e_B + O(c^4). \tag{118} \]

**Derivation of \( \nabla_{e^i} e_B \):** for this covariant derivative, we write its decomposition in the Cartanian basis as

\[ \nabla_{e^i} e_B = (k_i \nabla_{e^i} e_B^i) \ell + (e^C_i \nabla_{e^i} e_B^C) e_C, \tag{119} \]

where the vertical component is

\[ k_i \nabla_{e^i} e_B^i = -\frac{1}{c^2} h(\ell, \nabla_{e^i} e_B) = \frac{1}{c^2} h(\nabla_{e^i} \ell, e_B) = \frac{1}{c^2} \delta_{AB} + \left( \frac{1}{2} w_{AB} + \ell [\lambda_{AB}] \right). \tag{120} \]
The horizontal components, \( e^C \nabla e_a e^b = q^{CD} h(e_D, \nabla e_a e^b) \), can be evaluated using the following trick. First, we use that the covariant derivative is metric compatible, which following from this the obvious identity, \( e_A[q_{DB}] = h(e_D, \nabla e_A e^b) + h(e_B, [e_A, e_D]) \). It then become a straightforward computation to show that

\[
e^A[q_{DB}] + e^B[q_{AD}] - e^D[q_{AB}] = 2 h(e_D, \nabla e_A e^b) + h(e_B, [e_A, e_D]) + h(e_D, [e_A, e_b]). \tag{121}
\]

Using the commutator \([e_A, e^B] = w_{AB} \ell\) and that \( h(e_A, \ell) = 0 \), we arrive at the expression for the horizontal components,

\[
e^C \nabla e_a e^b = \frac{1}{2} q^{CD} (e_A[q_{DB}] + e^B[q_{AD}] - e^D[q_{AB}] = (\ell^C)^A_{AB}. \tag{122}
\]

We finally obtain the covariant derivative \( \nabla e_a e^b \) expanded in \( c^2 \) as

\[
\nabla e_a e^b = \left( \frac{1}{c^2} \theta_{AB} + \left( \frac{1}{2} w_{AB} + \ell [\lambda_{AB}] \right) \right) \ell + (\ell^C)^A_{AB} e^C
+ c^2 \left( D_A \lambda_B e^C + D_B \lambda_A e^C - D^C \lambda_{AB} \right) e_C. \tag{123}
\]

### Appendix E. Anomaly computations

To evaluate the anomaly of the Ehresmann connection, \( \Delta \xi = \delta \xi - L \xi \), one first computes its variation under the near-Carrollian diffeomorphism. Using the fact that the coordinates \( x^i = (u, y^A) \) are field-independent and thus \( \delta d x^i = 0 \), we can straightforwardly write the variation of the Ehresmann connection as

\[
\delta \xi = \xi \alpha - e^a \xi \beta_A e^A. \tag{124}
\]

Next, we need to compute the Lie derivative of the Ehresmann connection. Using the Cartan formula and recalling the curvature of the Ehresmann connection \( (10) \), one can proof that

\[
L \xi = d(\xi) + \xi d k = d f + (X \cdot \phi) k + (-f \phi_A + w_{AB} X^B) e^A
+ (\ell [f] + X \cdot \phi) k + ((e_A - \phi_A) f + w_{AB} X^B) e^A. \tag{125}
\]

Expanding the transformation parameter \( f = \tau + c^2 \psi \) and \( X^A = Y^A + c^2 Z^A \), the anomaly of the Ehresmann connection \( \Delta \xi k \) decomposes as

\[
\Delta \xi k = (\iota_\xi \Delta \xi) k + (\iota_{e_A} \Delta \xi) e^A, \tag{126}
\]

where the components are

\[
\iota_\xi \Delta \xi = \xi \alpha - e^a \xi \beta_A + O(c^2), \nl_{e_A} \Delta \xi = -e^a (\xi \beta_A - e^a \beta_A) + O(c^2). \tag{127}
\]

Next, we compute the anomaly of the null Carrollian metric, \( q = q_{AB} e^A \circ e^B \). We begin by considering its variation under the near-Carrollian diffeomorphism and show that

\[
\delta q = -2 e^{-\alpha} (q_{AB} \xi k) k \circ e^A + \xi q_{AB} e^A \circ e^B. \tag{128}
\]
Using the Cartan formula and the fact that \( \text{d}e^A = 0 \), the Lie derivative of the null Carrollian metric thus given by

\[
\mathcal{L}_\xi q = \xi[q_{AB}]e^A \circ e^B + 2q_{AB}(\mathcal{L}_\xi e^A) \circ e^B \\
= 2q_{AB}E^k e^B + (\xi[q_{AB}] + q_{CD}e^C)[X^e] e^A \circ e^B.
\]  

(129)

The anomaly of the null Carrollian metric is

\[
\Delta_\xi q = \delta_\xi q = 2 \mathcal{L}_\xi q(\xi) e^A \circ e^B + \Delta_\xi q(e_A, e_B) e^A \circ e^B,
\]

(130)

where its components, given in the \( c^2 \)-expansion, are given by

\[
\Delta_\xi q(\xi) e^A = -\epsilon^{\alpha} (q_{AB} + 2\epsilon^2 \lambda_{AB}) (\delta \mathcal{C} V^B + D_a Y^B) - c^2 \hat{\ell} [Z^B] e^A,
\]

(131)

and

\[
\Delta_\xi q(e_A, e_B) = \left( \hat{\delta}_{AB} - \delta^C_{[Y, \ell]} \hat{q}_{AB} \right) \\
+ 2\epsilon^2 \left( \hat{\delta}_k \lambda_{AB} - \frac{1}{2} \delta^C_{[Y, \ell]} \hat{q}_{AB} - \tau \ell [\lambda_{AB}] - Y^C D_C \lambda_{AB} - 2 \lambda_C (\hat{D}_B Y^C) \right).
\]

(132)

### Appendix F. Carrollian fluid action

The detail of the derivation of the Carrollian fluid action (75) is provided here. Recalling that the variation of the action is given by

\[
\delta S_{\text{fluid}} = \int \frac{1}{2} \left( T^{(0)ij} \delta h_{ij} + T^{(-1)ij} \delta h_{ij} + \lambda T^{(-1)ij} \delta h_{ij} + \right) \epsilon_H,
\]

(133)

the derivation thus boils down the components of the relativistic energy-momentum tensor and the metric on \( H \) in the \( c^2 \)-expansion.

For the energy–momentum tensor (61), we have that

\[
T^{(-1)ij} = \mathcal{C} \ell^i \ell^j + \mathcal{A} (e_A^i \ell^j + e_A^j \ell^i) + 3 \lambda_{AB} e_A^i e_B^j,
\]

(134a)

\[
T^{(0)ij} = \left( \mathcal{A}^{AB} + \mathcal{C} \hat{q}_{AB} \right) e^i e^j + (\pi^A - 2 \lambda^A_{[B} \mathcal{F}^{B]})(e_A^i \ell^j + e_A^j \ell^i).
\]

(134b)

Using the results from appendix E, we can show that the components of the expansion of the metric variation are

\[
\delta h_{00ij} = -\epsilon^{\alpha} (\hat{q}_{AB} \delta V^B) (k_i e_A^j + k_j e_A^i) + (\hat{\delta}_{AB}) e_A^i e_B^j
\]

(135a)

\[
\delta h_{01ij} = -\epsilon^{\alpha} (\delta \mathcal{A}) k_i k_j + (\epsilon^C \epsilon^D \delta A - 2\epsilon^2 \lambda_{AB} \delta V^B)(k_i e_A^j + k_j e_A^i) + 2(\delta \lambda_{AB}) e_A^i e_B^j.
\]

(135b)

Straightforward computations of (133) yields the variation of the Carrollian fluid action (75) presented in the main text.

**ORCID iD**

Puttarak Jai-akson  https://orcid.org/0000-0001-7213-0339
References

[1] Lévy-Leblond J-M 1965 Une nouvelle limite non-relativiste du groupe de Poincaré Annales de l'I.H.P. Physique théorique 3 1–12
[2] Inonu E and Wigner E P 1953 On the contraction of groups and their representations Proc. Natl Acad. Sci. USA 39 510–24
[3] Duval C, Gibbons G W, Horvathy P A and Zhang P M 2014 Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time Class. Quantum Grav. 31 085016
[4] Ciambelli L, Leigh R G, Marteau C and Petropoulos P M 2019 Carroll structures, null geometry and conformal isometries Phys. Rev. D 100 046010
[5] Bergshoeff E, Gomis J and Longhi G 2014 Dynamics of Carroll particles Class. Quantum Grav. 31 205009
[6] Marsot L 2022 Planar Carrollean dynamics and the Carroll quantum equation J. Geom. Phys. 179 104574
[7] Bidussi L, Hartong J, Have E, Musaeus J and Prohazka S 2022 Fractional, dipole symmetries and curved spacetime SciPost Phys. 12 205
[8] Marsot L, Zhang P-M and Horvathy P To move or not to move: anomalous spin-Hall effect on the black hole horizon (arXiv:2207.06302)
[9] Gibbons G, Hashimoto K and Yi P 2002 Tachyon condensates, Carrollian contraction of Lorentz group and fundamental strings J. High Energy Phys. JHEP09(2002)061
[10] Bagchi A, Chakraborty S and Parekh P 2016 Tensionless strings from worldsheet symmetries J. High Energy Phys. JHEP01(2016)158
[11] Bagchi A, Chakraborty S and Parekh P 2016 Tensionless superstrings: view from the worldsheet J. High Energy Phys. JHEP10(2016)113
[12] Bagchi A, Banerjee A, Chakraborty S and Parekh P 2018 Inhomogeneous tensionless superstrings J. High Energy Phys. JHEP02(2018)065
[13] Bagchi A, Banerjee A, Chakraborty S and Parekh P 2020 Exotic origins of tensionless superstrings Phys. Lett. B 801 135139
[14] Bagchi A 2013 Tensionless strings and Galilean conformal algebra J. High Energy Phys. JHEP05(2013)141
[15] Bagchi A, Banerjee A, Chakraborty S and Chatterjee R 2022 A Rindler road to Carrollian worldsheets J. High Energy Phys. JHEP04(2022)082
[16] Bergshoeff E, Izquierdo J M and Romano L 2020 Carroll versus Galilei from a brane perspective J. High Energy Phys. JHEP10(2020)066
[17] Roychowdhury D 2019 Carroll membranes J. High Energy Phys. JHEP10(2019)258
[18] Ravera L and Zorba U Carrollian and non-relativistic Jackiw-Teitelboim supergravity (arXiv:2204.09643)
[19] Basu R and Chowdhury U N 2018 Dynamical structure of Carrollian electrodynamics J. High Energy Phys. JHEP04(2018)111
[20] Bagchi A, Basu R, Mehra A and Nandi P 2020 Field theories on null manifolds J. High Energy Phys. JHEP02(2020)141
[21] Banerjee K, Basu R, Mehra A, Mohan A and Sharma A 2021 Interacting conformal Carrollian theories: cues from electrodynamics Phys. Rev. D 103 105001
[22] Hartong J 2015 Gauging the Carroll algebra and ultra-relativistic gravity J. High Energy Phys. JHEP08(2015)069
[23] Bergshoeff E, Gomis J, Rollier B, Rosseel J and ter Veldhuis T 2017 Carroll versus Galilei gravity J. High Energy Phys. JHEP03(2017)165
[24] Duval C, Gibbons G W, Horvathy P A and Zhang P M 2017 Carroll symmetry of plane gravitational waves Class. Quantum Grav. 34 175003
[25] Morand K 2020 Embedding Galilean and Carrollian geometries I. gravitational waves J. Math. Phys. 61 082502
[26] Bergshoeff E, Izquierdo J M, Ortín T and Romano L 2019 Lie algebra expansions and actions for non-relativistic gravity J. High Energy Phys. JHEP08(2019)048
[27] Gomis J, Kleinschmidt A, Palmkvist J and Salgado-Rebolledo P 2020 Newton-Hooke/Carrollian expansions of (A)dS and Chern-Simons gravity J. High Energy Phys. JHEP02(2020)009
[28] Ballesteros A, Gubitosi G and Herranz F J 2020 Lorentzian Snyder spacetimes and their Galilei and Carroll limits from projective geometry Class. Quantum Grav. 37 195021
Gomis J, Hidalgo D and Salgado-Rebolledo P 2021 Non-relativistic and Carrollian limits of Jackiw-Teitelboim gravity J. High Energy Phys. JHEP05(2021)162

Grumiller D, Hartong J, Prohazka S and Salzer J 2021 Limits of JT gravity J. High Energy Phys. JHEP02(2021)134

Hansen D, Obers N A, Oling G and Sgaard B T Carroll expansion of general relativity (arXiv:2112.12684)

Concha P, Peñafiel D, Ravera L and Rodríguez E 2021 Three-dimensional Maxwellian Carroll gravity theory and the cosmological constant Phys. Lett. B 823 136735

Guerrieri A and Sobreiro R F 2021 Carroll limit of four-dimensional gravity theories in the first order formalism Class. Quantum Grav. 38 245003

Pérez A 2021 Asymptotic symmetries in Carrollian theories of gravity J. High Energy Phys. JHEP12(2021)173

Sengupta S Hamiltonian forms of ‘Carroll’ gravity (arXiv:2208.02983)

Pérez A Asymptotic symmetries in Carrollian theories of gravity with a negative cosmological constant (arXiv:2202.08768)

Camposoleoni A, Henneaux M, Pekar S, Pérez A and Salgado-Rebolledo P Magnetic Carrollian gravity from the Carroll algebra (arXiv:2207.14167)

Duval C, Gibbons G W and Horváthy P A 2014 Conformal Carroll groups and BMS symmetry Class. Quantum Grav. 31 092001

Duval C, Gibbons G W and Horváthy P A 2014 Conformal Carroll groups J. Phys. A 47 335204

Bagchi A, Banerjee S, Basu R and Dutta S 2022 Scattering amplitudes: celestial and Carrollian Phys. Rev. Lett. 128 241601

Camposoleoni A, Ciambelli L, Delfante A, Marteau C, Petropoulos P M and Ruzziconi R Holographic Lorentz and Carroll frames (arXiv:2208.07575)

Bagchi A, Grumiller D and Nandi P 2022 Carrollian superconformal theories and super BMS J. High Energy Phys. JHEP05(2022)044

Donnay L, Fiorucci A, Herfray Y and Ruzziconi R 2022 Carrollian perspective on celestial holography Phys. Rev. Lett. 129 071602

Donnay L, Pasterski S and Puhm A 2022 Goldilocks modes and the three scattering bases J. High Energy Phys. JHEP06(2022)124

Bagchi A, Mehra A and Nandi P 2019 Field theories with conformal Carrollian symmetry J. High Energy Phys. JHEP05(2019)108

Gupta N and Suryanarayana N V 2021 Constructing Carrollian CFTs J. High Energy Phys. JHEP03(2021)194

Bagchi A, Banerjee A, Dutta S, Kolekar K S and Sharma P Carroll covariant scalar fields in two dimensions (arXiv:2203.13197)

de Boer J, Hartong J, Obers N A, Sybesma W and Vandoren S 2022 Carroll symmetry, dark energy and inflation Front. Phys. 10 810405

Ciambelli L, Marteau C, Petkou A C, Petropoulos P M and Siampos K 2018 Covariant Galilean versus Carrollian hydrodynamics from relativistic fluids Class. Quantum Grav. 35 165001

Ciambelli L and Marteau C 2019 Carrollian conservation laws and Ricci-flat gravity Class. Quantum Grav. 36 085004

Ciambelli L, Marteau C, Petkou A C, Petropoulos P M and Siampos K 2018 Flat holography and Carrollian fluids J. High Energy Phys. JHEP07(2018)165

Camposoleoni A, Ciambelli L, Marteau C, Petropoulos P M and Siampos K 2019 Two-dimensional fluids and their holographic duals Nucl. Phys. B 946 114692

Donnay L and Marteau C 2019 Carrollian physics at the black hole horizon Class. Quantum Grav. 36 165002

Ciambelli L, Marteau C, Petropoulos P M and Ruzziconi R 2020 Gauges in three-dimensional gravity and holographic fluids J. High Energy Phys. JHEP11(2020)092

Ciambelli L, Marteau C, Petropoulos P M and Ruzziconi R 2020 Fefferman-Graham and Bondi Gauges in the Fluid/Gravity Correspondence PoS CORFU2019 154

Bagchi A, Chakrabartty S, Grumiller D, Radhakrishnan B, Riegler M and Sinha A 2021 Non-Lorentzian chaos and cosmological holography Phys. Rev. D 104 L101901

Bagchi A, Dutta S, Kolekar K S and Sharma P 2021 BMS field theories and Weyl anomaly J. High Energy Phys. JHEP07(2021)101
[59] Petkou A C, Petropoulos P M, Betancour D R and Siampos K Relativistic fluids, hydrodynamic frames and their Galilean versus Carrollian avatars (arXiv:2205.09142)

[60] Chandrasekaran V, Flanagan E E and Prabhu K 2018 Symmetries and charges of general relativity at null boundaries J. High Energy Phys. JHEP11(2018)125

[61] Chandrasekaran V, Flanagan E E, Shehzad I and Speranza A J 2022 Brown-York charges at null boundaries J. High Energy Phys. JHEP01(2022)029

[62] Ashtekar A, Khera N, Kolanowski M and Lewandowski J 2022 Non-expanding horizons: multipoles and the symmetry group J. High Energy Phys. JHEP01(2022)028

[63] Ashtekar A Geometry and physics of null infinity (arXiv:1409.1800)

[64] Ashtekar A, Campiglia M and Laddha A 2018 Null infinity, the BMS group and infrared issues Gen. Rel. Grav. 50 140–63

[65] Ciambelli L 2019 Paving the fluid road to flat holography PhD Thesis Ecole Polytechnique, CPHT

[66] Damour T 1978 Black hole Eddy currents Phys. Rev. D 18 3598–604

[67] K S Thorne, R H Price and D A Macdonald (eds) 1986 Black Holes: The Membrane Paradigm (New Haven, CT: Yale University Press) p 367

[68] Price R H and Thorne K S 1986 Membrane viewpoint on black holes: properties and evolution of the stretched horizon Phys. Rev. D 33 915–41

[69] Donnay L, Giribet G, González H A and Pino M 2016 Supertranslations and superrotations at the black hole horizon Phys. Rev. Lett. 116 091101

[70] Donnay L, Giribet G, González H A and Pino M 2016 Extended symmetries at the black hole horizon J. High Energy Phys. JHEP09(2016)100

[71] Freidel L and Jai-akson P Carrollian hydrodynamics and symplectic structure on stretched horizons (arXiv:2211.06415)

[72] Freidel L and Pranzetti D 2022 Gravity from symmetry: duality and impulsive waves J. High Energy Phys. JHEP04(2022)125

[73] Freidel L, Pranzetti D and Raclariu A-M 2022 Sub-subleading soft graviton theorem from asymptotic Einstein’s equations J. High Energy Phys. JHEP05(2022)186

[74] Freidel L, Pranzetti D and Raclariu A-M Higher spin dynamics in gravity and $w_{1+\infty}$ celestial symmetries (arXiv:2112.15573)

[75] Duval C, Burdet G, Kunzle H P and Perrin M 1985 Bargmann structures and Newton-Cartan theory Phys. Rev. D 31 1841–53

[76] Duval C, Gibbons G W and Horváth P 1991 Celestial mechanics, conformal structures and gravitational waves Phys. Rev. D 43 3907–22