Buckling Analyses of Edge-cracked Functionally Graded Graphene Reinforced Composite Piezoelectric Beam

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Abstract. This paper investigates the buckling behaviors of edge-cracked functionally graded graphene reinforced composite (FG-GRC) beam with piezoelectric actuators. The bending stiffness of the crack section is equivalent to the bending stiffness of massless rotational spring. The modified Halpin-Tsai model and rule of mixture are used to estimate the effective mechanical properties of the FG-GRC beam. The electro-mechanical governing equations of buckling behaviors for the edge-cracked FG-GRC beam are derived by Timoshenko beam theory, von Kármán strain displacement relationship and Ritz method. Along the thickness of the beam, both functionally graded (FG) and uniformly distributed graphene nanoplatelets (GPLs) are considered. The influences of GPL mass fraction, GPL distribution patterns, GPLs geometry, crack depth, crack location and thickness of piezoelectric layer on the buckling characters of the FG-GRC beam are studied in this paper.

1. Introduction
Due to the unique structure and excellent properties of graphene [1], graphene has received widespread attention in the field of materials. Many experimental and theoretical studies show that the mechanical properties of graphene reinforced materials have been significantly improved. For example, Rafiee et al. [2] measured and compared mechanical properties of graphene plates (GPLs), single wall carbon nanotubes (SWCNTs) and multi wall carbon nanotubes (MWCNTs) reinforced epoxy nanocomposites. They found that the young's modulus of the nanocomposites increased to 131% when 0.1% GPLs was added, and the reinforcing effect of graphene is better than other reinforcing materials. GPL reinforced composites is expected to be one of the most promising new materials.

Functionally graded materials (FGMs) with gradient properties and functions are widely used because they can significantly improve the stability and damage resistance of structures and adapt to complex environments [3,4]. In order to efficiently utilize the graphene reinforcements, Yang et al. [5] firstly introduced functionally graded materials into GPLs reinforced composites. After that, a large number of scholars have studied the properties of functionally graded graphene reinforced structure [6-8].

However, all of the above studies are based on the intact functionally graded graphene reinforced composites. In the process of manufacturing and employing, the structural damage, such as crack, caused by fatigue or impact can never be completely avoided. Cracks in engineering structures may significantly reduce the local stiffness and strength, and significantly affect the performance of the structure [9]. It is of great significance to understand the structural behavior of cracked beam structures to avoid structural failure and accidental performance, and to detect damage. Many scholars...
have studied the vibration and stability performance of functionally graded structures with an edge crack [10,11]. But only Song and his colleagues [12-14] researched the edge-cracked FG-GRC beam. In addition, the intelligent structures composed of piezoelectric layers on the upper and lower surfaces of the structure can control the vibration and improve the stability of the structure [15,16], which start to be popular in the field of intelligence.

This paper combines Timoshenko beam theory and Ritz method to investigate the buckling behaviour of edge-cracked FG-GRC beam with piezoelectric actuators. The bending stiffness of crack section is calculated by rotating massless spring model. Three different GPL distribution patterns are considered. The modified Halpin-Tsai model and mixing rules are used to calculate the effective Young's modulus and other mechanical properties of FG-GRC beams. The effects of GPL distribution, GPL mass fraction, crack depth, crack location and piezoelectric layer thickness on the vibration of FG-GRC beams are studied.

2. Theoretical formulation

As shown in figure 1, the multilayer FG-GRC beam with length $L$, width $b$ and thickness $h$ is contained an edge crack of depth $a$ at a distance $L_1$ from the left end. There are two piezoelectric actuator layers with the same thickness $h_p$ on the upper and lower surfaces of the FG-GRC beam. The total thickness of this system is $H$, i.e., $H=h+2h_p$. The multilayer FG-GRC beam consists of isotropic polymer matrix and GPLs of length $l_G$, width $w_G$ and thickness $h_G$. The thickness of each layer is assumed to be the same and expressed as $\Delta h$. The three different distribution patterns (U, FG-O, FG-X) of GPLs are considered along the thickness direction of the GRC beam, as shown in figure 3. For the U pattern, the GPLs is uniformly distributed in each layer of GRC beam and the content of GPLs in each layer is the same. For the FG-O and FG-X patterns, GPLs are symmetrically distributed, and GPL weight fractions increase/decrease linearly from the top and bottom to the middle respectively. Therefore, the GPL mass fraction of the middle layer is the largest in FG-O and the smallest in FG-X.

Assuming the crack is perpendicular to the top surface of the FG-GRC beam and remains open. Therefore, the massless rotating spring model can be used to simulate the presence of crack, as shown in figure 2, and its bending stiffness can be derived from

$$K_r = \frac{1}{G},$$

(1)
where $G$ is the flexibility. Based on Brock’s approximation and introducing the normalized SIF $F(a)$ which was proposed by Song et al. [12], $G$ can be derived as

$$G = \frac{72\pi}{h^2} \left[ 1 - \nu(N_c) \right] \frac{c_{ij}(k)}{E(N_c)} \zeta F(\zeta)^2 d\zeta + \sum_{i=1}^{k-1} \frac{1 - \nu(k)}{E(k)} \frac{d_{ij}(k)}{h^2} \zeta F(\zeta)^2 d\zeta \right].$$  \hspace{1cm} (2)

where $\zeta = a/h$. $E(N_c)$, $\nu(N_c)$ and $E(k)$, $\nu(k)$ are Young’s modulus and Poisson’s ratio at the $N_c$-th and $k$-th layer of GRC beam, respectively. The tip of crack is in the $N_c$-th GRC layer.

Assuming that the total GPL mass fractions of the three different patterns are all $f_G$, the GPL volume fractions $V^{(k)}_G$ of the $k$-th ($k = 1, 2, \cdots, N$) layer of the FG-GRC beam with three different GPL distributions are respectively

$$U: \quad V^{(k)}_G = V^{(k)}_G; \quad FG-O: \quad V^{(k)}_G = 2V^{(k)}_G \left(1 - \frac{2k - N - 1}{N}\right); \quad FG-X: \quad V^{(k)}_G = 2V^{(k)}_G \left(\frac{2k - N - 1}{N}\right)$$

where

$$V^{(k)}_G = \frac{f_G}{f_G + (\rho_G/\rho_u)(1 - f_G)}$$

and $\rho_G$ and $\rho_u$ are respectively the mass density of the GPLs and polymer matrix.

According to the modified Halpin-Tsai model and the rule of mixture, the Young’s modulus $E^{(k)}_C$, Poisson’s ratio $\nu^{(k)}_C$ and mass density $\rho^{(k)}_C$ of the FG-GRC beam can be evaluated. Based on Timoshenko beam theory, the displacements $u(x, z, t)$ and $w(x, z, t)$ of an arbitrary point in the edge-cracked FG-GRC beam along x- and z- axes can be expressed by position coordinates $u(x, t)$, $w(x, t)$ and $\varphi(x, t)$ on the middle plane of the beam. The linear constitutive relationships for the $k$-th GRC layer and the piezoelectric actuator layers respectively take the forms of

$$\sigma_{zz}^{(k)} = Q_{11}^{(k)} \left(\frac{\partial u}{\partial z} + z \frac{\partial \varphi}{\partial x}\right), \quad \tau_{xz}^{(k)} = Q_{15}^{(k)} \left(\frac{\partial w}{\partial z} + \varphi\right);$$

$$\sigma_x^{(p)} = Q_{11}^{(p)} \left(\frac{\partial u}{\partial z} + z \frac{\partial \varphi}{\partial x}\right) + d_{zz}^{(p)} E_z, \quad \tau_{xz}^{(p)} = Q_{15}^{(p)} \left(\frac{\partial w}{\partial z} + \varphi\right).$$

where

$$Q_{11}^{(k)} = \frac{E^{(k)}_C}{1 - \nu^{(k)}_C}, \quad Q_{15}^{(k)} = \frac{E^{(k)}_C}{2(1 + \nu^{(k)}_C)}, \quad Q_{11}^{(p)} = \frac{E_p}{1 - \nu_p}, \quad Q_{15}^{(p)} = \frac{E_p}{2(1 + \nu_p)}.$$  \hspace{1cm} (6)

$E_p$ and $\nu_p$ are respectively the Young’s modulus and Poisson’s ratio of the piezoelectric layer, and $d_{zz}$ is the piezoelectric strain constant. Since the thickness of the piezoelectric layer is very thin and the self-induced potential is far less than the applied actuator voltage, the relationship between the applied actuator voltage $V_a$ and the electric field strength $E_z$ in the piezoelectric actuator can be described as follow [16]

$$E_z = V_a/h_p.$$

Define the following components

$$\begin{align*}
(A_{11}, B_{11}, D_{11}) &= \sum_{k=1}^{N} \int_{-H/2}^{H/2} Q_{11}^{(k)} \left(1, z, z^2\right) dz + \int_{-H/2}^{H/2} Q_{11}^{(p)} \left(1, z, z^2\right) dz + \int_{-H/2}^{H/2} Q_{11}^{(p)} \left(1, z, z^2\right) dz, \\
A_{55} &= \sum_{k=1}^{N} \int_{-H/2}^{H/2} Q_{55}^{(k)} \left(1, z\right) dz + \int_{-H/2}^{H/2} Q_{55}^{(p)} \left(1, z\right) dz, \\
(N_p, M_p) &= \int_{-H/2}^{H/2} Q_{11}^{(p)} \left(1, z\right) dz + \int_{-H/2}^{H/2} Q_{11}^{(p)} \left(1, z\right) dz.
\end{align*}$$

(8)  \hspace{1cm} (9)  \hspace{1cm} (10)
where \( k_j = 5/6 \) is shear correction factor and the subscript “P” represents the electric load. Obviously, \( M_P \) equals to zero as the two piezoelectric layers are symmetric by the plane \( y = 0 \). The ends \((x = 0 \text{ and } x = L)\) of the beam are assumed as immovable, namely
\[
0 = \left[ u_{x=0}^+ - u_{x=L}^- \right] + \left[ u_{x=L}^- - u_{x=L}^- \right] = \int_0^L \frac{\partial u}{\partial x} \, dx + \int_0^L \frac{\partial u}{\partial x} \, dx
\]

The potential energy \( U \) of the edge-cracked FG-GRC beam with piezoelectric layers can be rewritten as
\[
U = \frac{1}{2} \int_0^L \left[ A_{11} \left( \frac{\partial u}{\partial x} \right)^2 + 2B_{11} \frac{\partial u}{\partial x} \frac{\partial \varphi}{\partial x} + D_{11} \left( \frac{\partial \varphi}{\partial x} \right)^2 + A_{55} \left( \varphi_1 + \frac{\partial w_1}{\partial x} \right)^2 \right] \, dx
\]
\[
+ \frac{1}{2} \int_0^L \left[ A_{11} \left( \frac{\partial u}{\partial x} \right)^2 + 2B_{11} \frac{\partial u}{\partial x} \frac{\partial \varphi_2}{\partial x} + D_{11} \left( \frac{\partial \varphi_2}{\partial x} \right)^2 + A_{55} \left( \varphi_2 + \frac{\partial w_2}{\partial x} \right)^2 \right] \, dx
\]
\[
+ \frac{1}{2} K_T [\varphi_2(L) - \varphi_1(L)]^2.
\]

where \( \varphi_i = 1, 2 \) respectively refer to the left sub-beam and right sub-beam divided by the crack.

The potential energy \( V \) due to the external axial load \( N_x \) of the edge-cracked FG-GRC beam with piezoelectric layers is
\[
V = \frac{1}{2} \int_0^L N_x \left( \frac{\partial w}{\partial x} \right)^2 \, dx + \frac{1}{2} \int_0^L N_x \left( \frac{\partial w}{\partial x} \right)^2 \, dx.
\]

Introducing the following dimensionless quantities
\[
X = \frac{x}{L}, \quad (U, W) = \left( \frac{u, w}{H} \right), \quad \psi = \varphi, \quad \eta = \frac{L}{H}, \quad X_0 = \frac{L_0}{L},
\]

\[
(a_{11}, a_{55}, b_{11}, d_{11}) = \left( \frac{A_{11}}{A_{11}}, \frac{A_{55}}{A_{11}}, \frac{B_{11}}{A_{11}H}, \frac{D_{11}}{A_{11}H^2} \right), \quad P = \frac{N_x}{A_{11}},
\]

\[
\tilde{N}_x = \frac{N_x}{A_{11}}, \quad K_T = \frac{K_T}{A_{11}H^2/L}, \quad V = \frac{V}{A_{11}H^2/L}, \quad U = \frac{U}{A_{11}H^2/L}.
\]

Therefore, the energy function for the edge-cracked FG-GRC host beam with piezoelectric layers can be written as
\[
\Pi = U^* - V^*.
\]

### 3. Solution Method

In order to analyze the buckling behavior of the edge-cracked FG-GRC beam with piezoelectric actuator layer, the Ritz method [10] is employed to derive the governing eigenvalue equations of buckling for the edge-cracked FG-GRC beam with piezoelectric layers. The standard Ritz procedure [10] is employed to minimize the total energy with respect to unknown coefficients
\[
\frac{\partial \Pi}{\partial A_j} = 0, \quad \frac{\partial \Pi}{\partial B_j} = 0, \quad \frac{\partial \Pi}{\partial C_j} = 0.
\]

The eigenvalue equations of the dimensionless critical buckling load \( P_{cr} \) for the edge-cracked FG-GRC beam with piezoelectric layers are obtained
\[
(K_L - P_{cr} K_c) \mathbf{d} = 0.
\]

where \( \mathbf{d} = \left\{ \mathbf{A}_j \right\}_j, \left\{ \mathbf{B}_j \right\}_j, \left\{ \mathbf{C}_j \right\}_j \right., j = 1 \cdots n \). The linear stiffness matrix \( K_L \) and coefficient matrix \( K_c \) are all \( 6n \times 6n \) symmetric matrices.
4. Results and discussion
In the following calculations, the FG-GRC beam is constitutive of the epoxy and GPLs, and the piezoelectric layers are made by the Polyvinylidene fluoride (PVDF). The material parameters are as follows

Epoxy: \( E_M = 3.0 \text{ GPa} \), \( \rho_M = 1200 \text{ kg/m}^3 \), \( \nu_M = 0.34 \).

GPL: \( E_G = 1010 \text{ GPa} \), \( \rho_G = 1060 \text{ kg/m}^3 \), \( \nu_G = 0.186 \).

PVDF: \( E_P = 3.0 \text{ GPa} \), \( \rho_P = 1800 \text{ kg/m}^3 \), \( \nu_P = 0.29 \), \( d_{31} = 23 \times 10^{-12} \text{ m/V} \).

Unless otherwise stated, the edge-cracked FG-GRC beam with piezoelectric layers (\( a/h = 0.3 \) and \( L_1/L = 0.5 \)) subjected to the applied actuator voltage \( V_0 = 400 \text{ V} \) has slenderness ratio \( L/H = 10 \). The thicknesses of the FG-GRC beam and the piezoelectric layer are respectively \( h = 0.1 \text{ m} \) and \( h_p = 0.01 \text{ m} \). The length, width and thickness of the GPL are respectively \( L_G = 2.5 \mu\text{m} \), \( w_G = 1.5 \mu\text{m} \) and \( h_G = 1.5 \mu\text{m} \), and the GPL weight fraction \( f_G = 0.5\% \). In order to validate the correctness of the present analysis, the dimensionless critical buckling loads \( P_{cr} \) of intact C-C FG-GRC beam are compared with Yang et al.’s \([17]\) results.

Figure 4 illustrates the effects of GPL distribution patterns (U, FG-O and FG-X) and different GPL mass fraction (\( f_G = 0.1\% \), \( 0.3\% \), \( 0.5\% \)) on the critical buckling load ratio \( (P_{cr}/P_{cr0})/P_{cr0} \) of the edge-cracked FG-GRC piezoelectric. Here, \( P_{cr0} \) represents the critical buckling load of the edge-cracked pure epoxy piezoelectric beam. As expected, the critical buckling load ratios \( (P_{cr}/P_{cr0})/P_{cr0} \) extremely increases when a small amount of GPL mass fraction \( f_G \) is added. And among the three GPL distribution patterns in Figure 2, the FG-X produces the highest buckling load. That indicates that GPLs can effectively improve the stiffness of the pure epoxy piezoelectric and distributing more GPL at the top and bottom of the beam is the most effective way to improve the beam stiffness.

![Figure 4. The effect of the GPL distribution patterns and mass fraction on the critical buckling load.](image)

Figure 5 presents the effect of the width-to-thickness ratio \( w_G/h_G \) for the GPLs on the critical buckling loads of the edge-cracked X-GRC piezoelectric beam when \( l_G/h_G = 3 \). Results show that the critical buckling loads increases when GPL width-to-thickness ratios raise. This is because the larger the width thickness ratio, the bigger the surface area of GPLs, which leads to the increase of structural stiffness.

![Figure 5. The effect of the GPLs geometry on the critical buckling load.](image)

Figure 6 gives the effects of crack depth and crack location on the critical buckling load \( P_{cr} \) of the edge-cracked X-GRC beam with piezoelectric layers. Compared with intact FG-GRC beam with piezoelectric layers, the existence of edge-crack reduces the critical buckling load, and the increase of crack depth will further reduce the critical buckling load. The \( P_{cr} \) versus \( L_1/L \) curves is symmetric due
to the symmetric C-C boundary and the critical buckling loads are most sensitive to the crack when it is located at the section $L_1/L = 0.5$ of the C-C beam.

![Figure 6. The effect of the crack depth and crack location on the critical buckling load.](image)

![Figure 7. The effect of the thickness of piezoelectric layers on the critical buckling load.](image)

In addition, figure 7 gives the critical buckling loads of the edge-cracked FG-GRC beam with different thickness of piezoelectric layers. $h_p/h=0$ represents the edge-cracked FG-GRC beam without piezoelectric layers. It is shown that the critical buckling load $P_{cr}$ grows with the increasing thickness of the piezoelectric layer, mainly due to the growth of thickness of the piezoelectric layer can improve the system stiffness.

5. **Conclusion**

The buckling character of the cracked FG-GRC beam with piezoelectric layers is investigated on the basis of the Timoshenko beam theory, von Kármán strain displacement relationship and Ritz method. The bending stiffness of the cracked section is equivalent to the bending stiffness of a massless rotational spring model. The mechanical parameters of the FG-GRC beam are worked out by the modified Halpin-Tsai model and the rule of mixture. Three different GPL distribution patterns are considered to numerically analyze the effects of graphene weight fraction, GPL distribution patterns, crack depth, crack location and thickness of piezoelectric layer on the buckling behavior. Numerical results show that (1) The addition of a small amount of graphene can enhance the rigidity of the matrix, and distributing more graphene with a large surface area on the top and bottom of the beam is the most effective method to enhance the buckling load; (2) Compared with intact FG-GRC beam with piezoelectric layers, the existence of cracks accelerates the buckling process, and the degree of acceleration is related to the crack location; (3) The presence of the piezoelectric layer increases the rigidity of the structure and slows down the buckling process.

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