Equivalent sets of solutions of the Klein-Gordon equation with a constant electric field

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Abstract

In connection with the problem of choosing the in- and out-states among the solutions of a wave equation with one-dimensional potential we study nonstationary and "stationary" families of complete sets. A nonstationary set consists of the solutions with the quantum number \( p_v = p^0 v - p_3 \). It can be obtained from the nonstationary set with quantum number \( p_3 \) by a boost along \( x_3 \)-axis (along the direction of the electric field) with velocity \(-v\). By changing the gauge the solutions in all sets can be brought to one and the same potential without changing quantum numbers. Then the transformations of solutions of one set (with quantum number \( p_v \)) to the solutions in another set (with quantum number \( p'_v \)) have the group properties.

The "stationary" solutions and sets possess the same properties as the nonstationary ones and are obtainable from stationary solutions with quantum number \( p^0 \) by the same boost.

It turns out that any set can be obtained from any other by gauge manipulations. So all sets are equivalent and the classification (i.e., ascribing the frequency sign and in-, out- indexes) in any set is determined by the classification in \( p_3 \)-set, where it is evident.

1 Introduction and statement of the problem

The choice of in- and out-solutions in \([1-2]\) (see also \([3]\)) disagrees with that in \([4-5]\). So the classification problem in different sets of solutions is on hand. We show that the choice of gauge of potential, describing the considered field, fixes the natural quantum number for this gauge. So each gauge define a complete set of solutions. Then we can go over to any other gauge without changing the quantum number. In this way we can relate solutions in different sets characterized by different quantum numbers. But changing gauge does not change neither the physical system nor the classification in the sets. These simple
considerations open the way to the solution of the stated problem, because the classification in the set with quantum number $p_3$ is evident.

Already in classical mechanics we can see in what sense fixing the gauge fixes the conserved number. The solutions of the classical equation of motion for a particle in a constant electric field have the form

$$\pi^0(t) = p_3 \sinh \varepsilon s + p^0 \cosh \varepsilon s, \quad \varepsilon = \frac{eE}{m},$$

$$\pi_3(t) = p_3 \cosh \varepsilon s + p^0 \sinh \varepsilon s,$$

$$eEt = p_3(\cosh \varepsilon s - 1) + p^0 \sinh \varepsilon s,$$

$$eEx_3 = p_3 \sinh \varepsilon s + p^0(\cosh \varepsilon s - 1).$$

We use the metric

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1).$$

The motion in the direction perpendicular to the field remains free and we are not interested in it here.

At first we take the vector-potential in the form

$$A_\mu = -\delta_\mu^3 Ex_v, \quad x_v = t - vx_3, \quad 0 \leq v < 0.$$  \hspace{1cm} (3)

It follows from (1) and (3) that

$$\pi^0 v - \pi_3 + e(vA_0 - A_3) = p^0 v - p_3 \equiv p_v = \text{Const}.$$  \hspace{1cm} (4)

(Vector-potential is taken on particle’s trajectory) The potential

$$A_3 = \frac{-Ex_v}{1 - v^2}, \quad A_0 = \frac{-vEx_v}{1 - v^2},$$

(5)

gives the same conserved number. This potential is obtainable from (3) at $v = 0$ by the above mentioned boost; the electric field remains unchanged.

Similarly, for the potential

$$A_0 = -Ex_s, \quad x_s = x_3 - st, \quad 0 \leq s < 1,$$

(6)

we find

$$\pi^0 - s\pi_3 + e(A_0 - sA_3) = p^0 - sp_3 \equiv p_s = \text{Const},$$

(7)

and the same for the potential

$$A_3 = \frac{-sEx_s}{1 - s^2}, \quad A_0 = \frac{-Ex_s}{1 - s^2},$$

(8)

obtainable from (6) at $s = 0$ by the boost along $x_3$ with velocity $-s$. We might denote $s$ by $v$ as before, but we prefer to have separate notation for a different situation.

Now we consider the Klein-Gordon equation

$$\partial_\mu \partial^\mu \psi = [2ieA_\mu \partial_\mu + ie(\partial_\mu A^\mu) + e^2 A^2 + m^2] \psi.$$  \hspace{1cm} (9)
The vector-potential, describing the constant electric field, is taken at first in the general form
\[ A^\mu = a^\mu \varphi, \quad \varphi = k \cdot x, \quad k^\mu = (k^0, 0, 0, k_3), \quad a^\mu = (a^0, 0, 0, a_3). \] (10)

The solution for (9) is sought in the form
\[ \psi_p = C_p \exp \{ i [p \cdot x - \frac{p \cdot k}{k^2} \varphi + \frac{e a \cdot k}{2k^2} \varphi^2] \} v_p. \] (11)

Substituting this in (9) gives
\[ \left[ \frac{d^2}{d\varphi^2} + \left( \frac{p \cdot k}{k^2} \right)^2 + 2c_1 \varphi + c_2 \varphi^2 \right] v_p = 0, \] (12)
\[ c_1 = \frac{e_a \cdot p}{k^2} - \frac{p \cdot k e a \cdot k}{(k^2)^2}, \quad c_2 = \left( \frac{e_a \cdot k}{k^2} \right)^2 - \frac{e^2 a^2}{k^2} = \frac{e^2 E^2}{(k^2)^2}. \] (13)

Going over to the variable
\[ \tau = -\frac{i}{c_2} (\varphi + \frac{c_1}{c_2}) \] (14)
reduces (12) to the equation for the parabolic cylinder function [6]
\[ \left[ \frac{d^2}{d\tau^2} + \tau^2 + \tilde{\lambda} \right] v_p = 0, \quad \tilde{\lambda} = -\lambda \text{sign}k^2, \quad \lambda = \frac{m^2 + p_1^2 + p_2^2}{|eE|}. \] (15)

In contrast to [4, 7-8] we assume in this paper that the charge of a scalar particle \( e = -|e| \). Now we consider two separate families of potential (10).

2 Nonstationary solutions

We specify (10) as follows
\[ a_\mu = \delta_{\mu 3} a, \quad k^\mu = (\omega, 0, 0, v \omega), \quad A_3 = -Ex_v, \quad a \omega = E, \quad x_v = t - vx_3, \quad k^2 = \omega^2 (v^2 - 1). \] (16)

The conserved quantum number is \( p_v = p^0 v - p_3 \), cf (4). It follows from (13) and (14) that
\[ \tau_v = \frac{\pi_v}{\sqrt{|eE|(1 - v^2)}}, \quad \pi_v = p_v - eEx_v, \quad p_v = p^0 v - p_3. \] (17)

For brevity reasons we drop the dependence of the wave function on \( x_1, x_2 \) (i.e. we drop the factor \( \exp[ip_1 x_1 + ip_2 x_2] \)). Then for the phase in (11) we have
\[ p_3 x_3 - p^0 t - \frac{k \cdot p}{k^2} \varphi + \frac{e a \cdot k}{2k^2} \varphi^2 = \frac{p_v(t v - x_3)}{1 - v^2} - \frac{e E v x_v^2}{2(1 - v^2)}. \] (18)

In this Section we compare solutions bringing them to the potential
\[ \tilde{A}_\mu = -\delta_{\mu 3} Et = A_\mu + \frac{\partial \eta}{\partial x^\mu}. \] (19)
Taking into account $A_\mu$ in (16) we find
\[ \eta = -\frac{Evx^2_3}{2}, \quad \psi_{pv}(x|\tilde{A}) = e^{i\eta} \psi_{pv}(x|A). \] (20)

So instead of (18) we have
\[ \vartheta_v(x|\tilde{A}) \equiv \vartheta_v = \frac{p_v(tv - x_3)}{1 - v^2} - \frac{eEvx^2_v}{2(1 - v^2)} - \frac{eEvx^2_3}{2}. \] (21)

For uniformity we denote the wave function with quantum number $p_v$ as $\psi_{pv}$. Yet it should be remembered that for $v \to 0$ this function goes over into $\psi_{p3}$, not into $\psi_{-p3}$ as one might think looking at the definition of $p_v$ in (17).

Now we are in a position to write down and classify $\psi_{pv} \equiv \psi_{pv}(x|\tilde{A})$-solutions. We put the frequency sign (see [4]) before $\psi$-function in the lower position for in- solution and in the upper position for out- solution. The in- (out-)solution has only one wave of indicated frequency for $t \to -\infty (t \to \infty)$. So
\[ \pm \psi_{pv} = C_{pv} e^{i\vartheta_v} D_{\pm i\frac{\pi}{2}}(-e^{\mp \pi T_v}), \quad \pm \bar{\psi}_{pv} = C_{pv} e^{i\vartheta_v} D_{\mp i\frac{\pi}{2}}(e^{\pm i\pi T_v}), \]
\[ T_v = \sqrt{2}\tau_v = \sqrt{\frac{2}{|eE|(1 - v^2)}}(p_v - eEx_v), \quad C_{pv} = [2|eE|(1 - v^2)]^{-\frac{1}{4}} e^{-\frac{\pi}{12}}. \] (22)

$\pm \psi_{pv}$ are normalized as follows
\[ \int_{-\infty}^{\infty} dx_{3} \bar{\psi}^{*}_{pv} i \frac{d}{dt} \psi_{pv} = \pm 2\pi \delta(p'_v - p_v) \] (23)
and similarly for $\pm \bar{\psi}_{pv}$. The classification is called forth by the condition that for $v \to 0$ $\psi_{pv}$ goes over into $\psi_{p3}$ and the classification of the latter functions is substantiated in [4].

From the relations between the parabolic cylinder functions it follows
\[ +\psi_{pv} = c_{1p}^+ \psi_{pv} + c_{2p}^+ \bar{\psi}_{pv}, \]
\[ -\psi_{pv} = c_{2p}^0 \psi_{pv} + c_{1p}^0 \bar{\psi}_{pv} \] (24)
The Bogoliubov coefficients $c_{1p}, c_{2p}$ depend only on $\lambda$, i.e. only on $p_{\perp}^2 = p_1^2 + p_2^2$ [4]:
\[ c_{1p} = \sqrt{2\pi} \Gamma^{-1}\left(1 - \frac{i\lambda}{2}\right) \exp[-\frac{\pi}{4}(\lambda - i)], \quad c_{2p} = \exp[-\frac{\pi}{4}(\lambda + i)], \quad |c_{1p}|^2 - |c_{2p}|^2 = 1. \] (25)

As shown in [7-8], $\psi_{pv}$ can be obtained from $\bar{\psi}_{p3}$ by a boost along $x_3-$axis with velocity $-v$ with subsequent regauging to the potential $\tilde{A}_\mu$ in (19).

Now we consider the limiting case $v \to 1$ and obtain the set of functions with quantum number $p^- = p^0 - p_3$ (a set in a system "with infinite momentum"). As seen from (17)
\[ \tau_v |_{v \to 1} \to \infty \text{sign} \pi_v, \quad \pi_v \to \pi^- = p^- - eEx^- \] (26)
and we need the asymptotic expansions of the parabolic cylinder functions. These expansions contain factors \( \exp[\pm i\tau_v^2/2] \). For example, we have

\[
D_{(1-i)\tau_v^{-\frac{3}{2}}} \left[ \tau_v \rightarrow -\infty \right] \rightarrow (2\tau_v^2)^{-\frac{1}{4}} \exp[\frac{\pi}{8}(\lambda+i)+\frac{\lambda}{4}\ln 2+\frac{i\tau_v^2}{2}+\frac{i\lambda}{2}\ln(-\tau_v)],
\]

\[
D_{(1-i)\tau_v^{-\frac{3}{2}}} \left[ \tau_v \rightarrow \infty \right] \rightarrow \sqrt{2\pi}\Gamma^{-1}(\frac{1-i\lambda}{2})(2\tau_v^2)^{-\frac{1}{4}} \exp[-\frac{\pi}{8}(\lambda-i)-\frac{\lambda}{4}\ln 2-\frac{i\tau_v^2}{2}-\frac{i\lambda}{2}\ln(\tau_v)] + (2\tau_v^2)^{-\frac{1}{4}} \exp[-\frac{3\pi}{8}(\lambda+i)+\frac{i\lambda}{4}\ln 2+\frac{i\tau_v^2}{2}+\frac{i\lambda}{2}\ln(\tau_v)].
\]

Therefore the asymptotic expressions of \( \psi \)-functions in (22) contain the factors \( \exp[i(\vartheta_v \pm \tau_v^2/2)] \). Simple calculations give

\[
\vartheta_v \pm \frac{\tau_v^2}{2} = \pm \frac{p_v^2}{2|eE|(1-v^2)} \pm \frac{|eE|x_v^2}{2(1+v)} \pm \frac{p_v x^\pm}{1+v} + \frac{|eE|x_v x^2}{2}, \quad x^\pm = t \pm x_3.
\]

If we form a wave packet using functions with phases (29), then only the lower sign case contributes in the limit \( v \rightarrow 1 \). The divergence of the constant term on the right-hand side of (29) is not dangerous and can be removed by the compensating term in the phase of the weight function of the packet. Hence in the limit \( v \rightarrow 1 \) terms with phase \( (\vartheta_v + \tau_v^2/2) \) may be dropped.

It follows from (22) and (27-28) that

\[
+\psi_{p_v}(x|\vec{A}) \bigg|_{v \rightarrow 1} \rightarrow \exp[-\frac{i\pi}{8} - \frac{i\lambda}{4}\ln \frac{2}{1-v^2} - \frac{ip_v^2}{2|eE|(1-v^2)}] + \psi_{p^-}(x|\vec{A}),
\]

and similarly for other functions in (22). In other words, for \( v \rightarrow 1 \) the \( \psi \)-functions in (22) go over into \( \psi_{p^-} \) up to an inessential phase factor.

The functions \( \psi_{p^-} \) for the potential (19) are defined as follows [5]

\[
+\psi_{p^-}(x|\vec{A}) = +\psi_{p^-} = (4|eE|)^{-\frac{1}{4}} \exp[-\frac{i}{2}p^- x^+ - i eE \left(\frac{x_3^2}{2} - \frac{(x^-)^2}{4}\right) + v^* \ln \frac{\pi^-}{\sqrt{|eE|}}],
\]

\[
-\psi_{p^-} = \theta(\pi^-)c_1 \psi_{p^-}, \quad +\psi_{p^-} = \theta(-\pi^-)c_1 \psi_{p^-}, \quad \pi^- = p^- - eEx^-,
\]

\[
\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}, \quad v^* = -\frac{i\lambda}{2} - \frac{1}{2}, \quad p^\pm = p^0 \pm p_3, \quad x^\pm = t \pm x_3.
\]

For \( \pi^- < 0 \) in (31) one must take \( \pi^- = -\pi^- \exp[-i\pi] \). The function \( -\psi_{p^-} \) is obtained from \( +\psi_{p^-} \) by changing sign of \( \pi^- \) under the logarithm sign:

\[
-\psi_{p^-}(x|\vec{A}) = (4|eE|)^{-\frac{1}{2}} \exp[-\frac{i}{2}p^- x^+ - i eE \left(\frac{x_3^2}{2} - \frac{(x^-)^2}{4}\right) + v^* \ln \frac{-\pi^-}{\sqrt{|eE|}}].
\]

For \( \pi^- > 0 \) in (33) one must take \( -\pi^- = \pi^- \exp[-i\pi] \). (For \( e = |e| \) see [7-8].)

The solutions \( \psi_{p^-} \) are connected with the solutions with quantum number \( p_3 \) by the integral transformations [5]

\[
\varphi_{p_3} = \int_{-\infty}^{\infty} dp^- N(p_3, p^-) \psi_{p^-}, \quad N(p_3, p^-) = (2\pi|eE|)^{-\frac{1}{4}} \exp\left\{ -\frac{1}{4}(p^-)^2 + 4p^- p_3 + 2p_3^2 \right\}. \quad (34)
\]
In these two expressions the charge $e$ may have any sign, but the expressions for $\psi_p^-$ and $\varphi_{p_3}$ depend on the sign of $e$, cf. [4, 7-8]. It is clear that (34) and similar expressions below are valid in any gauge. The functions $\varphi_{p_3}$ differ from $\psi_{p_3}$ only by the phase factor [5]

$$\varphi_{p_3} = \exp\left[\frac{-i\lambda}{4} \ln 2 + i\frac{3\pi}{8}\right] \psi_{p_3}. \quad (35)$$

Taking into account the unitarity conditions for $N(p_3, p^-)$

$$\int_{-\infty}^{\infty} dp^- N(p_3, p^-) N^*(p_3', p^-) = \delta(p_3 - p_3'),$$
$$\int_{-\infty}^{\infty} dp_3 N(p_3, p^-) N^*(p_3, p^-') = \delta(p^- - p^-'), \quad (36)$$

it is easy to express $\psi_{p^-}$ through $\varphi_{p_3}$

$$\psi_{p^-} = \int_{-\infty}^{\infty} dp_3 N^*(p_3, p^-) \varphi_{p_3}. \quad (37)$$

Using the same boost as in obtaining $\psi_{p_v}$ from $\psi_{p_3}$, we get from (34)

$$\varphi_{p_v} = \int_{-\infty}^{\infty} dp^- N(p_v, p^-) \psi_{p^-}, \quad (38)$$

$$N(p_v, p^-) = (2\pi|eE|(1 - v))^{-\frac{1}{2}} \exp\left\{-i\left[p^- (1 + v)\right]^2 - 4p^- p_v (1 + v) + 2p_v^2 \right\}. \quad (39)$$

This ”matrix” is also unitary. Hence the reversed relation is

$$\psi_{p^-} = \int_{-\infty}^{\infty} dp_v N^*(p_v, p^-) \varphi_{p_v}, \quad (40)$$

Combining (34) and (40) we find

$$\varphi_{p_3} = \int_{-\infty}^{\infty} dp_v N(p_3, p_v) \varphi_{p_v}. \quad (41)$$

$$N(p_3, p_v) = \int_{-\infty}^{\infty} dp^- N(p_3, p^-) N^*(p_v, p^-) =$$

$$(2\pi|eE|v)^{-\frac{1}{2}} \exp\left\{i\frac{p_3^2 (1 - v^2) + 2p_v p_3 (1 - v^2) + p_v^2}{2veE(1 - v^2)} - i\frac{\pi}{4}\right\}. \quad (42)$$

This matrix is also unitary, so

$$\varphi_{p_v} = \int_{-\infty}^{\infty} dp_3 N^*(p_3, p_v) \varphi_{p_3}. \quad (43)$$

Now $\varphi_{p_3}$ satisfy the normalization condition [4]

$$\int_{-\infty}^{\infty} dx_3 \pm \varphi_{p_3}^* \frac{i}{\mu} \frac{d}{dt} \varphi_{p_3} = \pm 2\pi \delta(p_3' - p_3) \quad (44)$$
and similarly for $\pm \varphi_{p_3}$. Using (43) we have

$$\int_{-\infty}^{\infty} dx_3 \pm \varphi_{p_3}^* \frac{d}{dt} \pm \varphi_{p_3} = \int_{-\infty}^{\infty} dp_3 \int_{-\infty}^{\infty} dp_3' N(p_3, p_3') N^*(p_3, p_3) \int_{-\infty}^{\infty} dx_3 \pm \varphi_{p_3}^* \frac{d}{dt} \pm \varphi_{p_3}. \quad (45)$$

Taking into account (44) and the unitarity of $N(p_3, p_3)$ we get (23), see (35).

We note also that relation (38) can be checked by direct calculation. So inserting into it $+\psi_{p^-}$ from (31) and using eq. (3.462.3) in [9], we obtain

$$+\varphi_{p_3} = \left[2|eE|(1-v^2)\right]^{-\frac{1}{2}} \exp\left\{-\frac{\pi \lambda}{8} + i \frac{3 \pi}{8} + i \frac{\lambda}{4} \ln \frac{1+v}{2(1-v)} + i \theta_v \right\} D_v^*[\left(1+i\tau_v\right)] \quad (46)$$

and similarly for other function of this set. Here $\theta_v$ is the same as in (21). Comparison with (22) shows that $\varphi_{p_3}$ differs from $\psi_{p^-}$ only by a phase factor

$$\varphi_{p_3} = \exp\left[i \frac{3 \pi}{8} + i \frac{\lambda}{4} \ln \frac{1+v}{2(1-v)}\right] \psi_{p_3}. \quad (47)$$

As it should be the relation (41) goes over into (34) for $v \to 1$. Really, we can write $N(p_3, p_3)$ in (42) in the form

$$N(p_3, p_3) = \tilde{N}(p_3, p_3) \exp\left[-\frac{i \pi}{4} + \frac{i p_v^2}{2|eE|(1-v^2)}\right], \quad (48)$$

$$\tilde{N}(p_3, p_3) = (2\pi|eE|v)^{-\frac{1}{2}} \exp\left\{i \frac{p_3^2(1+v) + 2p_3 p_3(1+v)}{2v|eE|(1+v)} \right\}. \quad (48a)$$

It is easy to see that

$$\tilde{N}(p_3, p_3) \big|_{v \to 1} \to N(p_3, p^-), \quad (49)$$

see (34). Besides, from (47) and (30) we have

$$\varphi_{p_3} \big|_{v \to 1} \to \exp\left\{i \frac{\pi}{4} - \frac{i p_v^2}{2|eE|(1-v^2)}\right\} \psi_{p^-}. \quad (50)$$

So the phase factor on the right-hand side of (48) is cancelled by phase factor on the right-hand side of (50).

Combining now (43) and (41) we find

$$\varphi_{p_3} = \int_{-\infty}^{\infty} dp_v N(p_3, p_v) \varphi_{p_3},$$

$$N(p_3, p_v) = \int_{-\infty}^{\infty} dp_3 N^*(p_3, p_v) N(p_3, p_v) =$$

$$(-i(v'-v)2\pi|eE|)^{-\frac{1}{2}} \exp\left\{-i \frac{p_{v'}^2(1-v'v)}{2|eE|(1-v'^2)(v'-v)} - i \frac{p_{v'}^2(1-v'v)}{2|eE|(1-v'^2)(v'-v)} + \frac{i p_v p_v}{|eE|(v'-v)}\right\}. \quad (51)$$
Here
\[ \sqrt{-i(v' - v)} = \begin{cases} e^{-i\frac{\pi}{4}}\sqrt{v' - v}, & v' > v \\ e^{i\frac{\pi}{4}}\sqrt{v' - v}, & v' < v. \end{cases} \tag{52} \]

\( N(p_{v'}, p_v) \) is hermitian and have group property
\[ N(p_{v''}, p_v) = \int_{-\infty}^{\infty} dp_{v'} N(p_{v'}, p_{v'}) N(p_{v'}, p_v). \tag{53} \]

If we insert into the right-hand side of (51) \( + \varphi_{p_v} = + \varphi_{p_v} (x|\tilde{A}) \) from (46) and use eq.(2.11.4.7) in [10] we get the left-hand side of (51).

In this Section we have compared the solutions of Klein-Gordon equation with vector-potential (19). Utilizing transformations similar to (19, 20) we can go over to any vector-potential of the considered field.

### 3 Stationary solutions

We name ”stationary” the solutions with quantum number \( p_s = p^0 - sp_3 \). For \( s = 0 \) these solutions are stationary in the usual sense. Others are obtainable from these by boosts. All the consideration in this Section is quite analogous to the one in the previous Section.

In the potential (10) we put
\[ a_\mu = \delta_\mu 0 a, \quad k^\mu = (s\omega, 0, 0, \omega), \quad k^2 = \omega^2(1 - s^2), \quad A^0 = -A_0, \quad a\omega = E, \quad x_s = x_3 - st. \tag{54} \]

From (14) and (13) we have
\[ \tau_s = \frac{\pi_s}{\sqrt{|eE|(1 - s^2)}}, \quad \pi_s = p_s - |eE|x_s, \quad p_s = p^0 - sp_3. \tag{55} \]

For the phase in (11) we get
\[ p_3x_3 - p^0t - \frac{k \cdot p}{k^2} \varphi + \frac{e a \cdot k}{2k^2} \varphi^2 = -p_s(t - sx_3) + \frac{eEsx^2_s}{2(1 - s^2)}. \tag{56} \]

In this Section we bring the solutions with different \( s \) to the potential
\[ A_\mu = \delta_\mu 0 Ex_3, \quad A^0 = -A_0. \tag{57} \]

In this gauge phase (56) acquires an additional term:
\[ \psi_s(x|A) \equiv \psi_s = -p_s(t - sx_3) + \frac{eEsx^2_s}{2(1 - s^2)} + \frac{eEst^2}{2} = -p_s(t - sx_3) - \frac{|eE|s}{2(1 - s^2)}(x^2_3 - 2stx_3 + t^2). \tag{58} \]

The classification in the family of sets \( \psi_{p_s} \) is dictated by the requirement that for \( s = 0 \) we must obtain the classification in the set \( \psi_{p^0} \). (For some more details on classification in the latter set see [5]). So
\[ \pm \psi_{p_s} = C_{ps} e^{i\psi_s} D_{\mp i\frac{\pi}{4} - \frac{1}{2}}(\pm e^{\pm i\frac{\pi}{4}} Z_s), \quad \pm \psi_{p_s} = C_{ps} e^{i\psi_s} D_{\mp i\frac{\pi}{4} - \frac{1}{2}}(\pm e^{\mp i\frac{\pi}{4}} Z_s), \]
\[ Z_s = -\sqrt{2} \tau_s = -\sqrt{\frac{2}{|eE|(1 - s^2)}}(p_s - |eE|x_s), \quad C_{ps} = |eE|(1 - s^2)^{-\frac{1}{2}}e^{\frac{3\pi}{8}}. \] (59)

These functions satisfy the same relations (24) with the same \( c_1, c_2 \), see (25).

Taking into account that the asymptotic expansions for the parabolic cylinder functions contain \( \exp[\pm i\tau_s^2/2] \)-factors, we write down the expression analogues to (29)

\[ \vartheta_s \pm \tau_s^2/2 = \pm \frac{p_s^2}{2|eE|(1 - s^2)} \pm \frac{|eE|x_s^2}{2(1 \pm s)} - \frac{p_s x_s}{1 \pm s} - \frac{|eE|st^2}{2}. \] (60)

For reasons mentioned after eq.(29) only the case with the upper sign is needed. Now it is easy to verify that

\[ +\psi_{ps}(x|A)|_{s \to 1} \to \exp[-i\frac{3\pi}{8} - i\lambda \ln \frac{2}{1 - s^2} + \frac{i p_s^2}{2|eE|(1 - s^2)}] \psi_{p^+}(x|A), \] (61)

\[ +\psi_p(x|A) = e^{ieEtx_3} \psi_{p^-}(x|A), \] (62)

and similarly for other \( \psi_{ps} \). In other words, for \( s \to 1 \) \( \psi_{ps} \) go over into \( \psi_{p^-} \) up to an inessential phase factor. Now we note that instead potential (57) we can use (19). Then we may say that the set \( \psi_{p^-} \) constitute a bridge between \( \psi_{p^+} \) and \( \psi_{ps} \)-families. All sets in this joint (super)family are indistinguishable experimentally. Instead of classical solution (1) we have the superfamily of sets \( (\psi_{p^+}, \psi_{ps}) \) for the Klein-Gordon equation.

As in previous Section, we consider \( \varphi_{ps} \) along with \( \psi_{ps} \) (cf. (38-39) and (47)):

\[ \varphi_{ps} = \int_{-\infty}^{\infty} dp^+ S(p_s, p^-) \psi_{p^-}, \quad \psi_{p^-} = \int_{-\infty}^{\infty} dp_s S^*(p_s, p^-) \varphi_{ps}. \] (63)

\[ S(p_s, p^-) = (2\pi|eE|(1 - s^2)^{-\frac{1}{2}}} \exp\left\{-i\left[p^-(1 + s)\right]^2 - 4p^- p_s(1 + s) + 2p_s^2\right\}/4|eE|(1 - s^2), \] (63a)

\[ \varphi_{ps} = \exp[i\frac{\pi}{8} + i\frac{\lambda}{4} \ln \frac{1 + s}{2(1 - s)}] \psi_{ps}. \] (64)

For \( s = 0 \) we get from (63)

\[ \varphi_p = \int_{-\infty}^{\infty} dp S(p^0, p^-) \psi_{p^-}, \] (65)

and due to unitarity of \( S(p^0, p^-) \)

\[ \psi_{p^-} = \int_{-\infty}^{\infty} dp^0 S^*(p^0, p^-) \varphi_{p^0}. \] (66)

We note that (65) was given in [5] and (63) can be obtained from (65) by going over to the boosted system.

Combining the first relation in (63) and (66) we find

\[ \varphi_{ps} = \int_{-\infty}^{\infty} dp^0 S^*(p^0, p_s) \varphi_{p^0}, \] (67)
From (64) and (61) we have

\[ S^*(p^0, p_s) = \int_{-\infty}^{\infty} dp^- S(p_s, p^-) S^*(p^0, p^-) = (2\pi|eE|s) - \frac{i}{4} \exp\left\{ i \frac{p^0_s (1 - s^2)}{2s|eE|(1 - s^2)} - \frac{i\pi}{4} \right\}. \] (68)

From (64) and (61) we have

\[ \varphi_{p_s} \mid_{s \to 1} \rightarrow \exp\left\{ - \frac{i\pi}{4} + \frac{ip^2_s}{2|eE|(1 - s^2)} \right\} \psi_{p^-}, \] (69)

and (68) can be written as

\[ S^*(p^0, p_s) = \tilde{S}^*(p^0, p_s) \exp\left\{ - \frac{i\pi}{4} + \frac{ip^2_s}{2|eE|(1 - s^2)} \right\}, \]

\[ \tilde{S}^*(p^0, p_s) = (2\pi|eE|s) - \frac{i}{4} \exp\left\{ i \frac{p^2_s (1 + s)}{2s|eE|(1 + s)} \right\}. \] (70)

Now

\[ \tilde{S}^*(p^0, p_s) \mid_{s \to 1} \rightarrow S^*(p^0, p^-) = (2\pi|eE|) - \frac{i}{4} \exp\left\{ \frac{i(p^-)^2}{4|eE|} - \frac{ip^0 p^-}{|eE|} + \frac{ip^2_s}{2|eE|} \right\}. \]

From (69-71) it follows that for \( s \to 1 \) (67) goes over to (66).

As shown in [5], \( \psi_{p^0} \) (i.e. functions in (59) for \( s = 0 \)) are orthonormalized. The same is true for \( \psi_{p_s} \) in (52) due to unitarity of \( S(p^0, p_s) \), cf text near eqs. (44-45).

The reversed relation for (67) has the form

\[ \varphi_{p^0} = \int_{-\infty}^{\infty} dp_s S(p^0, p_s) \varphi_{p_s}. \] (72)

Combining (67) and (72) we find

\[ \varphi_{p_{s'}} = \int_{-\infty}^{\infty} dp_s S(p_{s'}, p_s) \varphi_{p_s}, \]

\[ S(p_{s'}, p_s) = \int_{-\infty}^{\infty} dp^0 S^*(p^0, p_{s'}) S(p^0, p_s) = \]

\[ (i(s' - s)2\pi|eE|) - \frac{i}{4} \exp\left\{ i \frac{p^2_s (1 - ss')}{2|eE|(1 - s^2)(s' - s)} + i \frac{p^2_s (1 - ss')}{|eE|(1 - s^2)(s' - s)} - \frac{ip^2_s}{2|eE|} \right\}. \] (73)

Here

\[ \sqrt{i(s' - s)} = \begin{cases} e^{i\pi/4} \sqrt{s' - s}, & s' > s \\ e^{-i\pi/4} \sqrt{s' - s}, & s' < s. \end{cases} \] (74)

\( S(p_{s'}, p_s) \) is hermitian and have group property

\[ S(p_{s''}, p_s) = \int_{-\infty}^{\infty} dp_{s'} S(p_{s''}, p_{s'}) S(p_{s'}, p_s). \] (75)
Now combining the first relation in (63) and (40) we obtain

\[ \varphi_{p_s} = \int_{-\infty}^{\infty} dp_v I(p_s, p_v) \varphi_{p_v}, \quad (76) \]

\[ I(p_s, p_v) = \int_{-\infty}^{\infty} dp^- S(p_s, p^-) N^*(p_v, p^-) = \]

\[ (\pi |eE|f_+)^{-\frac{1}{2}} \exp \left\{ -\frac{if_-}{2|eE|f_+} \left[ \frac{p_s^2}{1 - s^2} - \frac{p_v^2}{1 - v^2} \right] + \frac{2ip_sp_v}{|eE|f_+} - \frac{i\pi}{4} \right\}, \quad f_\pm = (1+s)(1-v)\pm(1-s)(1+v). \quad (77) \]

Similarly, from (38) and (63) we find

\[ \varphi_{p_v} = \int_{-\infty}^{\infty} dp_s I^*(p_s, p_v) \varphi_{p_s}, \quad (78) \]

For \( s = v = 0 \) we get from (76)

\[ \varphi_{p^0} = \int_{-\infty}^{\infty} dp_3 I(p^0, -p_3) \varphi_{p_3}, \quad (79) \]

in agreement with eq. (117) in [5].

If we use in the right-hand side of (76) the expression for \( ^+\varphi_{p_v}(x|\mathcal{A}) = e^{-i|eE|tx_3} \varphi_{p_v} \), (where \( ^+\varphi_{p_v} \) is defined in (47), (22), and (21)) and utilize formula (2.11.4(7)) in [5], we get \( ^+\varphi_{p_s}(x|\mathcal{A}) \) defined in (64), (59), and (58).

4 Conclusion

Any set in the collection of sets(\( \psi_{p_v}, \psi_{p_s} \)) can be obtained from any other by gauge manipulations. The classification in any set is dictated by the classification in the set \( \psi_{p_3} \), where it is beyond doubt.

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References

1. A.Hansen, F.Ravndal, Physica Scripta, 23, 1036 (1981).

2. W.Greiner, B.Müller, J.Rafelski, Quantum Electrodynamics of Strong Field, Springer-Verlag (1985).
3. A.Calogeracos, N.Dombey, Contemp. Phys. 40, 313 (1999).

4. A.I.Nikishov, Tr. Fiz. Inst. Akad. Nauk SSSR 111, 152 (1979); J. Sov. Laser Res. 6, 619 (1985).

5. A.I.Nikishov, hep-th/0111137.

6. Higher Transcendental Functions, Vol. 2 (Bateman Manuscript Project), Ed. by A.Erdélyi (McGraw-Hill, New York, 1953; Nauka, Moscow, 1980; Pergamon, Oxford, 1982).

7. N.B.Narozhny and A.I.Nikishov, Teor. Mat. Fiz. 26, 16 (1976).

8. N.B.Narozhny and A.I.Nikishov, Tr. Fiz. Inst. Akad. Nauk SSSR 168, 175 (1985); in Issues in Intensive-Field Quantum Electrodynamics, Ed. by V.L.Ginzburg (Nova Science, Commack, 1987).

9. I.S.Gradstein, I.M.Ryzhik, Tables of Integrals, Sums, Series, and Products, Moscow, 1962.

10. A.P.Prudnikov, Yu. A.Brychkov, and O.I.Marichev, Integrals and Series. Special Functions, Moscow, Nauka, 1983.