RECONSTRUCTION OF SINGULAR AND DEGENERATE INCLUSIONS IN CALDERÓN’S PROBLEM

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Abstract. We consider the reconstruction of the support of an unknown perturbation to a known conductivity coefficient in Calderón’s problem. In a previous result by the authors on monotonicity-based reconstruction, the perturbed coefficient is allowed to simultaneously take the values 0 and $\infty$ in some parts of the domain and values bounded away from 0 and $\infty$ elsewhere. We generalise this result by allowing the unknown coefficient to be the restriction of an $A_2$-Muckenhoupt weight in parts of the domain, thereby including singular and degenerate behaviour in the governing equation. In particular, the coefficient may tend to 0 and $\infty$ in a controlled manner, which goes beyond the standard setting of Calderón’s problem. Our main result constructively characterises the outer shape of the support of such a general perturbation, based on a local Neumann-to-Dirichlet map defined on an open subset of the domain boundary.

Keywords: Calderón’s problem, electrical impedance tomography, monotonicity method, inclusion detection, degenerate elliptic problem, singular elliptic problem.

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1. Introduction

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 2$, with connected complement and let $\Gamma \subseteq \partial \Omega$ be relatively open. For an appropriate nonnegative scalar-valued function $\gamma$, we consider the conductivity equation

$$- \nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega.$$  

(1.1)

The task of determining $\gamma$ from Cauchy data of the solutions to (1.1) is called Calderón’s inverse conductivity problem, which corresponds to the real-world imaging modality electrical impedance tomography (EIT); see [2, 3, 6, 25]. Instead of the general Calderón problem, this work concentrates on the simpler task of inclusion detection: Our aim is to reconstruct the outer shape of $\text{supp}(\gamma - \gamma_0)$ based on the local Neumann-to-Dirichlet (ND) map $\Lambda(\gamma)$ on $\Gamma$ for (1.1), with $\gamma_0$ being a known background conductivity coefficient that belongs to $L_+^\infty(\Omega) = \{ \varsigma \in L^\infty(\Omega; \mathbb{R}) \mid \text{ess inf } \varsigma > 0 \}$ and satisfies a weak unique continuation principle (UCP) in connection to the conductivity equation; cf., e.g., [4, Definition 3.3]. The outer shape $D$ of $\text{supp}(\gamma - \gamma_0)$ refers to the smallest closed set in $\Omega$ having connected complement and satisfying $\text{supp}(\gamma - \gamma_0) \subseteq D$.

In our analysis, the coefficient $\gamma$ may formally take values 0 (perfectly insulating) and $\infty$ (perfectly conducting) in certain parts of $\Omega$; we refer to such regions as extreme inclusions. If $\gamma$ is characterised by such extreme inclusions as well as finite positive and negative perturbations to $\gamma_0$, the monotonicity method has been proven to produce an exact reconstruction of the outer shape of $\text{supp}(\gamma - \gamma_0)$ under only mild geometric assumptions [4, Theorem 3.7]. However, this previous result requires that the restriction of $\gamma$ to the complement of the extreme inclusions belongs to $L_+^\infty$, thus only allowing (infinite) jump discontinuities at the boundaries of the extreme inclusions and no singular or degenerate behaviour elsewhere in $\Omega$.

The purpose of this paper is to provide a simple extension to the proof of [4, Theorem 3.7] so that it applies to a much larger class of unknown conductivity coefficients $\gamma$. To be more precise, we permit singular and degenerate behaviour by letting $\gamma$ be the restriction of an $A_2$-Muckenhoupt weight in parts of the domain, thereby including singular and degenerate behaviour in the governing equation. In particular, the coefficient may tend to 0 and $\infty$ in a controlled manner, which goes beyond the standard setting of Calderón’s problem. Our main result constructively characterises the outer shape of the support of such a general perturbation, based on a local Neumann-to-Dirichlet map defined on an open subset of the domain boundary.
weight in parts of \( \Omega \), which corresponds to a standard way of relaxing the assumptions on a coefficient in an elliptic partial differential equation [11, 20]. Unlike in [4, Theorem 3.7], we also allow \( \gamma \) to tend to \( \gamma_0 \) in a controlled manner rather than insisting on jump discontinuities at the boundary of \( \text{supp}(\gamma - \gamma_0) \); see [18, Theorem 4.7] for allowing such behaviour without extreme inclusions.

Although there exist a few recent results on the unique solvability of the general Calderón problem with certain classes of singular and degenerate coefficients in two dimensions [1, 5, 23], we are not aware of any previous extensions of inclusion detection methods to such frameworks. Moreover, we pose no restrictions on the spatial dimension \( d \geq 2 \) and only require local measurements.

As a motivation for the use of \( \mathcal{A}_2 \)-coefficients, let us mention a couple of examples that are outside the standard setting for Calderón’s problem but can still be tackled within our setting. An \( \mathcal{A}_2 \)-weight is allowed to locally behave as \( \text{dist}(\cdot, \Sigma)^s \) for any \( s \in (-1, 1) \) and Lipschitz hypersurface \( \Sigma \); see, e.g., [9, Lemma 3.3]. In particular, such a hypersurface can be a subset of the boundary of an extreme inclusion, allowing a continuous decay to 0 or a continuous growth to \( \infty \) when approaching the inclusion boundary. Since even stronger singular or degenerate behaviour is possible for \( \mathcal{A}_2 \)-weights near a point, the coefficient \( \gamma \) can also behave as \( \text{dist}(\cdot, x_0)^s \) close to an arbitrary \( x_0 \in \Omega \) for any \( s \in (-d, d) \).

The main theoretical developments and related properties of monotonicity-based reconstruction in connection to Calderón’s problem and EIT can be found in [4, 10, 12, 16, 17, 18, 21, 22, 24] for the continuum model and in [13, 14, 16, 19] for related practical electrode models. For an extensive list of references to papers employing similar monotonicity-based arguments in other inverse coefficient problems, we refer to [16]. More general information on the theoretical aspects of Calderón’s problem is available in the review article [25] and the references therein.

The rest of this article is organised as follows. Section 2 introduces the main result on reconstruction of inclusions as Theorem 2.2; the required assumptions are summarised in Assumption 2.1. Section 3 further elaborates on Assumption 2.1. Section 4 rigorously defines the Neumann problem for (1.1) and the associated local ND map in the context of a Muckenhoupt coefficient and extreme inclusions. Finally, Section 5 concludes the paper by proving Theorem 2.2.

2. MAIN RESULT

The family of admissible inclusions is defined as

\[
\mathcal{A} = \{ C \subset \Omega \mid C \text{ is the closure of an open set,}
\hspace{1cm}
\text{has connected complement,}
\hspace{1cm}
\text{and has Lipschitz boundary } \partial C \}. 
\]

Let \( D \in \mathcal{A} \) be the set representing the inclusions in \( \Omega \), that is, \( D \) is (the outer shape of) the closure of the set where the investigated conductivity \( \gamma \) differs from the known background conductivity \( \gamma_0 \in L^\infty_{\text{loc}}(\Omega) \). More precisely, \( D \) is assumed to be composed of ‘negative’ and ‘positive’ parts as \( D = D^- \cup D^+ \),

\[
D^- = D_{\text{deg}} \cup D_0 \cup D_{F^-} \quad \text{and} \quad D^+ = D_{\text{sing}} \cup D_\infty \cup D_{F^+},
\]

where \( D_{\text{deg}}, D_{\text{sing}}, D_0, D_\infty, D_{F^-}, D_{F^+} \) are mutually disjoint measurable sets, each of which may be empty or have multiple components with the exact conditions on their geometry given in Assumption 2.1 below.

We define \( \gamma \) as

\[
\gamma = \begin{cases} 
0 & \text{in } D_0, \\
\infty & \text{in } D_\infty, \\
\gamma_{\text{deg}} & \text{in } D_{\text{deg}}, \\
\gamma_{\text{sing}} & \text{in } D_{\text{sing}}, \\
\gamma_{F^-} & \text{in } D_{F^-}, \\
\gamma_{F^+} & \text{in } D_{F^+}, \\
\gamma_0 & \text{in } \Omega \setminus D. 
\end{cases}
\]
Before introducing the exact assumptions on the different types of inclusions and the associated conductivities, we summarise that $D_0$ and $D_\infty$ correspond to the subsets of $\Omega$ where $\gamma$ is characterised by extreme inclusions and $D_{\deg}$ and $D_{\sing}$ to the subsets where $\gamma$ is allowed to be an $A_2$-Muckenhoupt weight. In the subsets $D^\pm_F$ the coefficient $\gamma$ is bounded away from 0 and $\infty$. However, it should be noted that in the nonextreme parts, $\gamma$ is only assumed to deviate from $\gamma_0$ near $\partial D$.

Some implications and interpretations of the following assumptions are presented in Section 3.

**Assumption 2.1.** We assume the following about $D$ and $\gamma$.

(i) $\gamma \leq \gamma_0$ in $D^-$ and $\gamma \geq \gamma_0$ in $D^+$.  

(ii) $\gamma^-_F \in L^\infty_+(D^-_F)$ and $\gamma^+_F \in L^\infty_+(D^+_F)$.

(iii) The sets $D_{\deg}$ and $D_{\sing}$ are compactly contained in the interior of $D$, and $\gamma_{\deg}$ and $\gamma_{\sing}$ are restrictions of an $A_2$-Muckenhoupt weight.

(iv) The sets $D_0$, $D_\infty$, $D_0 \cup D_{\deg} \cup D_{\sing}$, and $D_\infty \cup D_{\deg} \cup D_{\sing}$ are closures of open sets with finitely many components and Lipschitz boundaries. Moreover, $D_0$ and $D_0 \cup D_{\deg} \cup D_{\sing}$ have connected complements.

(v) For every open neighbourhood $W$ of $x \in \partial D$, there exists a relatively open set $V \subset D$ that intersects $\partial D$, and $V \subset \widetilde{D} \cap W$ for one set $\widetilde{D} \in \{D_0, D_\infty, D^-_F, D^+_F\}$.

(a) If $\widetilde{D} = D^-_F$, there exists an open ball $B \subset V$ such that $\text{ess sup}_B (\gamma^-_F - \gamma_0) < 0$. 

(b) If $\widetilde{D} = D^+_F$, there exists an open ball $B \subset V$ such that $\text{ess inf}_B (\gamma^+_F - \gamma_0) > 0$.

Let $\Lambda(\gamma)$ be the local ND map on $\Gamma$ corresponding to the coefficient $\gamma$; see Section 4 for its precise definition, including an explanation on how extreme, singular, and degenerate inclusions are included in the associated elliptic Neumann boundary value problem. For $C \in \mathcal{A}$, we define $\Lambda_0(C)$ as the ND map with coefficient 0 in $C$ and $\gamma_0$ outside $C$. In the same manner, we define $\Lambda_\infty(C)$ to be the ND map with coefficient $\infty$ in $C$ and $\gamma_0$ outside $C$.

This prelude finally leads to the following general result on reconstructing inclusions from a local ND map.

**Theorem 2.2.** Let $\gamma$ and $D$ satisfy Assumption 2.1 and $\gamma_0$ satisfy the UCP. For $C \in \mathcal{A}$, it holds

$$D \subseteq C \quad \text{if and only if} \quad \Lambda_0(C) \geq \Lambda(\gamma) \geq \Lambda_\infty(C).$$

In particular, 

$$D = \cap \{ C \in \mathcal{A} \mid \Lambda_0(C) \geq \Lambda(\gamma) \geq \Lambda_\infty(C) \}. $$

It follows immediately from the proof of Theorem 2.2 that if $D^+$ is empty, then only the inequality $\Lambda_0(C) \geq \Lambda(\gamma)$ needs to be considered, and if $D^-$ is empty, then only $\Lambda(\gamma) \geq \Lambda_\infty(C)$ needs to be considered.

### 3. Some remarks on Assumption 2.1

Let us present a few remarks on Assumption 2.1:

- The condition (iii) ensures that at $\partial D$ the coefficient $\gamma$ has a jump to 0 or $\infty$, or a well-behaving finite transition characterised by $\gamma^-_F$ or $\gamma^+_F$. In particular, singular and degenerate behaviour of an $A_2$-weight is not permitted exactly at the boundary of $D$ but only in its interior.

- However, (iii) still allows $\gamma$ to exhibit singular and degenerate behaviour in the interior of $D$, e.g., by approaching 0 or $\infty$ with a limited rate near the extreme inclusions, certain hypersurfaces, curves, or points.

- The condition (iv) is related to technical assumptions for a convergence result in [4] for potentials and ND maps in connection with extreme inclusions.

- The condition (v) excludes certain pathological cases such as $\gamma - \gamma_0$ changing its sign arbitrarily often everywhere near an open part of $\partial D$. This type of oscillating behaviour is, however, allowed near points on $\partial D$ if these points are separated by at least a fixed positive distance. Indeed, the condition (v) ensures that one can access $D$ everywhere along $\partial D$ through an open set that only intersects with one of the inclusion types.
Moreover, (v) still allows $\gamma$ to approach $\gamma_0$ in a controlled manner near some parts of the inclusion boundary, while also allowing finite jump discontinuities as has previously been considered in connection to extreme inclusions in [4]. The former could, e.g., correspond to $\gamma$ equalling $\gamma_0$ on an open subset $\Sigma \subset \partial D$ but exhibiting a local strict increase or decrease inside $D$ with respect to the distance from $\Sigma$.

4. Forward Problem with a Muckenhoupt Coefficient

In this section we consider the Neumann problem for (1.1) when the coefficient is a restriction of an $A_2$-Muckenhoupt weight that is well-behaved near $\partial \Omega$ and allows perfectly insulating and perfectly conducting parts in $\Omega$. We denote the coefficient in the Neumann problem by $\sigma$ to distinguish it from the fixed coefficient $\gamma$ in (1.1). This provides the means to introduce the corresponding local ND map $\Lambda(\sigma)$, which in turn defines the forward map $\sigma \mapsto \Lambda(\sigma)$ associated to the considered inverse problem.

We refer to [11, 20] for an introduction to Muckenhoupt weights and weighted Sobolev spaces, although for weighted Poincaré inequalities we refer to results in [8] since those are shown for more general domains. A nonnegative function $w$ on $\mathbb{R}^d$ is called an $A_2$-Muckenhoupt weight provided that $w$ and $1/w$ are locally integrable and satisfy

$$\exists C > 0, \forall B \text{ open ball in } \mathbb{R}^d : \left( \int_B w \, dx \right) \left( \int_B \frac{1}{w} \, dx \right) \leq C.$$ 

A common equivalent definition integrates over cubes rather than balls. Let $C_0 \Subset \Omega$ be such that $\Omega = \Omega \setminus C_0$ is a Lipschitz domain; $C_0$ will play the role of perfectly insulating inclusions in the following. For an $A_2$-weight $w$, define the norms

$$\|v\|_{L^2(\tilde{\Omega}, w)} = \int_{\tilde{\Omega}} w|v|^2 \, dx,$$

$$\|v\|_{H^1(\tilde{\Omega}, w)} = \|\nabla v\|_{L^2(\tilde{\Omega}, w)}$$

where $\|\nabla v\|_{L^2(\tilde{\Omega}, w)}$ refers to the $L^2(\tilde{\Omega}, w)$-norm of $|\nabla v|$. The weighted spaces $L^2(\tilde{\Omega}, w)$ and $H^1(\tilde{\Omega}, w)$ are then defined as the completions of the spaces of $C^\infty(\tilde{\Omega})$-functions with finite norms with respect to (4.1) and (4.2). In particular, both $L^2(\tilde{\Omega}, w)$ and $H^1(\tilde{\Omega}, w)$ are Hilbert spaces.

By density and using [8, Theorem 3.3] with $\alpha = 1$, $p = q = 2$, and the weight function $\sqrt{w}$, we arrive at the weighted Poincaré inequality

$$\inf_{c \in \mathbb{C}} \|v - c\|_{L^2(\tilde{\Omega}, w)} \leq C\|\nabla v\|_{L^2(\tilde{\Omega}, w)}, \quad v \in H^1(\tilde{\Omega}, w).$$

(4.3)

In particular, the quotient space $H^1(\tilde{\Omega}, w)/\mathbb{C}$ can be equipped with the norm

$$\|v\|_{H^1(\tilde{\Omega}, w)/\mathbb{C}} = \|\nabla v\|_{L^2(\tilde{\Omega}, w)}, \quad v \in H^1(\tilde{\Omega}, w)/\mathbb{C},$$

which is equivalent to the standard quotient norm of $H^1(\tilde{\Omega}, w)/\mathbb{C}$ due to (4.3).

In order to define a local ND map, we need to ensure that the elements of the considered space $H^1(\Omega, w)$ have well-defined Dirichlet traces. To this end, suppose there exist $K \Subset \Omega$ and $c \in (0, 1)$ such that $\tilde{\Omega} \setminus K$ is a Lipschitz domain and $c \leq w \leq c^{-1}$ almost everywhere in $\tilde{\Omega} \setminus K$. This is sufficient for guaranteeing the existence of a bounded Dirichlet trace map from $H^1(\Omega, w)$ to $H^{1/2}(\partial \Omega)$, and thus also to $L^2(\Gamma)$: First, notice that if $v \in H^1(\tilde{\Omega}, w)$, then there is a sequence $(\phi_n)$ in $C^\infty(\tilde{\Omega})$ such that $\phi_n \to v$ in $H^1(\tilde{\Omega}, w)$. Consequently, it also holds that $\phi_n|_{\tilde{\Omega} \setminus K} \to v|_{\tilde{\Omega} \setminus K}$ in $H^1(\tilde{\Omega} \setminus K, w) = H^1(\tilde{\Omega} \setminus K)$. Hence, we may apply the standard trace theorem for $H^1(\tilde{\Omega} \setminus K)$ to deduce

$$\|v\|_{L^2(\Gamma)} \leq \|v|_{H^{1/2}(\partial \Omega)} \leq C\|v\|_{H^1(\tilde{\Omega} \setminus K)} \leq C\|v\|_{H^1(\tilde{\Omega}, w)}.$$  

Moreover,

$$\|v\|_{L^2(\Gamma)/\mathbb{C}} \leq C\|v\|_{H^1(\tilde{\Omega}, w)/\mathbb{C}}, \quad v \in H^1(\tilde{\Omega}, w)/\mathbb{C},$$

due to an obvious generalisation.
Finally, let $C_\infty \Subset \Omega \setminus C_0$ be the closure of an open set with Lipschitz boundary; in our analysis, $C_\infty$ corresponds to perfectly conducting inclusions. We define the considered conductivity coefficient as

$$
\sigma = \begin{cases} 
0 & \text{in } C_0, \\
\infty & \text{in } C_\infty, \\
w & \text{in } \Omega \setminus (C_0 \cup C_\infty)
\end{cases}
$$

and the corresponding closed subspace of $H^1(\bar{\Omega}, w)/\mathbb{C}$ via

$$
\mathcal{H}(\sigma) = \{ v \in H^1(\Omega \setminus C_0, w)/\mathbb{C} \mid \nabla v = 0 \text{ in } C_\infty \},
$$

where $C_\infty^\circ$ denotes the interior of $C_\infty$. We abbreviate this Hilbert space as $\mathcal{H}$ if there is no room for misinterpretation. The induced norm on $\mathcal{H}$ is

$$
\|v\|^2_\mathcal{H} = \int_{\Omega \setminus (C_0 \cup C_\infty)} \sigma|\nabla v|^2 \, dx, \quad v \in \mathcal{H}.
$$

For a current density $f$ belonging to the $\Gamma$-mean free space

$$
L^2_\infty(\Gamma) = \{ g \in L^2(\Gamma) \mid \langle g, 1 \rangle_{L^2(\Gamma)} = 0 \},
$$

we define the electric potential $u$, corresponding to the conductivity coefficient $\sigma$, as the unique solution in $\mathcal{H}$ to the variational problem

$$
\int_{\Omega \setminus (C_0 \cup C_\infty)} \sigma \nabla u \cdot \nabla v \, dx = \langle f, v \rangle_{L^2(\Gamma)}, \quad \forall v \in \mathcal{H},
$$

(4.4)

where there is no ambiguity in the right hand-side because of the mean free condition for $f$. The unique solvability of (4.4) is a straightforward consequence of the Lax–Milgram lemma. We will occasionally write $u = u_\sigma^f$ if the connection of $u$ to the specific Neumann boundary value $f$ and conductivity coefficient $\sigma$ needs to be emphasised.

Since the left hand-side of (4.4) defines a symmetric sesquilinear form on $\mathcal{H}$, $u$ is also the unique minimiser of the following functional (cf., e.g., [15, Remark 12.23] and [7, Theorem 1.1.2]):

$$
J_\sigma(u) = \int_{\Omega \setminus (C_0 \cup C_\infty)} \sigma|\nabla v|^2 \, dx - 2 \text{ Re} \langle f, v \rangle_{L^2(\Gamma)}, \quad v \in \mathcal{H}.
$$

(4.5)

The dependence on $\sigma$ in the notation $J_\sigma$ also encodes the extreme inclusions $C_0$ and $C_\infty$, as well as the domain $\mathcal{H}$ for the functional.

The corresponding local ND map $\Lambda(\sigma) \in \mathcal{L}(L^2_\infty(\Gamma))$, in the space of bounded linear operators on $L^2_\infty(\Gamma)$, is defined as $\Lambda(\sigma)f = u_\sigma^f|_\Gamma$. Here the notation is slightly abused by denoting with $u_\sigma^f$ the unique $\Gamma$-mean free element in the corresponding equivalence class of the quotient space $\mathcal{H}$, which makes the definition of $\Lambda(\sigma)$ concordant with the material in [4]. In particular, $\Lambda(\sigma)$ is compact, self-adjoint, and satisfies

$$
\langle \Lambda(\sigma)f, f \rangle_{L^2(\Gamma)} = \int_{\Omega \setminus (C_0 \cup C_\infty)} \sigma|\nabla u_\sigma^f|^2 \, dx, \quad f \in L^2_\infty(\Gamma),
$$

(4.6)

which are also standard properties of ND maps for conductivity coefficients in $L^\infty_-(\Omega)$.

**Remark 4.1.** We have decided to use (4.4) as the definition of the Neumann problem for the conductivity equation with $A_2$-Muckenhoupt weights and extreme inclusions. For the connection of (4.4) to the underlying partial differential equation and associated Neumann and Dirichlet boundary conditions on $\partial C_0$ and $\partial C_\infty$, respectively, we refer to [4], where the situation is analysed for standard $L^\infty_-$-coefficients. In particular, we consider the direct employment of the variational problem (4.4) as the natural generalisation to the case of $A_2$-weights.
5. Proof of Theorem 2.2

As we will see, the proof can essentially be reduced to showing that \( \Lambda(\gamma_L) \geq \Lambda(\gamma) \geq \Lambda(\gamma_U) \) for certain modified variants of \( \gamma \) satisfying \( \gamma_L \leq \gamma \leq \gamma_U \). Specifically, for \( D_{A2} = D_{\text{deg}} \cup D_{\text{sing}} \) we define

\[
\gamma_L = \begin{cases} 0 & \text{in } D_{A2}, \\ \gamma & \text{in } \Omega \setminus D_{A2}, \end{cases} \quad \gamma_U = \begin{cases} \infty & \text{in } D_{A2}, \\ \gamma & \text{in } \Omega \setminus D_{A2}. \end{cases}
\]

Notice that \( \gamma_L \) and \( \gamma_U \) satisfy the conditions required by \([4, \text{Theorem 3.7}]\) if one modifies its statement and proof to also allow \( \gamma \) to approach \( \gamma_0 \) in a controlled manner (cf. Assumption 2.1(v)). This slight generalisation of \([4, \text{Theorem 3.7}]\) is given below.

Proposition 5.1. Let \( \gamma \) and \( D \) satisfy Assumption 2.1, but with \( D_{\text{deg}} = D_{\text{sing}} = 0 \), and let \( \gamma_0 \) satisfy the UCP. For \( C \in A \), it holds

\[
D \subseteq C \quad \text{if and only if} \quad \Lambda_0(C) \geq \Lambda(\gamma) \geq \Lambda_\infty(C).
\]

Proof. The first part of the proof for \([4, \text{Theorem 3.7}]\) remains identical in such a setting, whereas its second part must be slightly modified: when choosing the ball \( B \) where potentials are localised in parts (a) and (b) of the proof, Assumption 2.1(v) guarantees that \( B \) can be chosen such that \( \gamma \) is bounded away from \( \gamma_0 \) in \( B \), without requiring a jump discontinuity at the inclusion boundary as in the original assumptions for \([4, \text{Theorem 3.7}]\).

In what follows we denote \( D_{\text{ext}} = D_0 \cup D_{\infty} \). As the coefficients \( \gamma_L \) and \( \gamma_U \) purposefully do not have singular or degenerate parts, the corresponding potentials \( u_{\gamma}^L \) and \( u_{\gamma}^U \) are minimisers of the appropriate energy functionals defined by (4.5), over the unweighted spaces \( \mathcal{H}(\gamma_L) \) and \( \mathcal{H}(\gamma_U) \), respectively. Moreover, \( u_{\gamma}^L \in \mathcal{H}(\gamma) \) obviously satisfies \( u_{\gamma}^L|_{\Omega \setminus (D_0 \cup D_{A2})} \in \mathcal{H}(\gamma_L) \) and, on the other hand, \( u_{\gamma}^U \in \mathcal{H}(\gamma) \) since it has vanishing gradient in the region where \( \gamma \) may exhibit \( A_2 \) behaviour. These observations are employed below when manipulating the energy functionals associated to different conductivity coefficients.

We start by proving \( \Lambda(\gamma_L) \geq \Lambda(\gamma) \). Using (4.4) and (4.6), we get

\[
-\langle \Lambda(\gamma)f, f \rangle_{L^2(\Gamma)} = \int_{\Omega \setminus D_{\text{ext}}} \gamma L |\nabla u_{\gamma}^L|^2 \, dx - 2\langle f, u_{\gamma}^L | r \rangle_{L^2(\Gamma)}
\]

\[
\geq \int_{\Omega \setminus (D_{\text{ext}} \cup D_{A2})} \gamma_L |\nabla u_{\gamma}^L|^2 \, dx - 2\langle f, u_{\gamma}^L | r \rangle_{L^2(\Gamma)}
\]

\[
\geq \int_{\Omega \setminus (D_{\text{ext}} \cup D_{A2})} \gamma_L |\nabla u_{\gamma}^U|^2 \, dx - 2\langle f, u_{\gamma}^U | r \rangle_{L^2(\Gamma)}
\]

\[
= -\langle \Lambda(\gamma)f, f \rangle_{L^2(\Gamma)},
\]

where the \( L^2(\Gamma) \) inner products have real values by (4.4) and (4.6). The first inequality follows from the nonnegativity of the integrand and the fact that \( \gamma \) equals \( \gamma_L \) outside \( D_{A2} \), whereas the second one is a consequence of \( u_{\gamma}^L \) being the minimiser of \( J_{\gamma} \) in \( \mathcal{H}(\gamma_L) \).

Next we prove \( \Lambda(\gamma) \geq \Lambda(\gamma_U) \):

\[
-\langle \Lambda(\gamma_U)f, f \rangle_{L^2(\Gamma)} = \int_{\Omega \setminus (D_{\text{ext}} \cup D_{A2})} \gamma_U |\nabla u_{\gamma}^U|^2 \, dx - 2\langle f, u_{\gamma}^U | r \rangle_{L^2(\Gamma)}
\]

\[
= \int_{\Omega \setminus D_{\text{ext}}} \gamma |\nabla u_{\gamma}^U|^2 \, dx - 2\langle f, u_{\gamma}^U | r \rangle_{L^2(\Gamma)}
\]

\[
\geq \int_{\Omega \setminus D_{\text{ext}}} \gamma |\nabla u_{\gamma}^L|^2 \, dx - 2\langle f, u_{\gamma}^L | r \rangle_{L^2(\Gamma)}
\]

\[
= -\langle \Lambda(\gamma)f, f \rangle_{L^2(\Gamma)},
\]

where the second equality follows from \( |\nabla u_{\gamma}^U| \) vanishing in \( D_{A2} \), while the inequality is a consequence of \( u_{\gamma}^U \) being the minimiser of \( J_{\gamma} \) in \( \mathcal{H}(\gamma) \).
To prove the actual theorem, assume first $D \subseteq C$. Due to (5.1), (5.2), and Proposition 5.1 applied to $\gamma_L$ and $\gamma_U$, \begin{equation}
abla_0(C) \geq \Lambda(\gamma_L) \geq \Lambda(\gamma) \geq \Lambda(\gamma_U) \geq \Lambda_{\infty}(C), \tag{5.3}
abla\end{equation}
which proves the direction “$\Rightarrow$” of the assertion.

To prove the opposite direction “$\Leftarrow$”, assume that $D \not\subseteq C$, i.e., there is a part of $D \setminus C$ that can be connected to $\Gamma$ via a relatively open connected set $U \subset \Omega \setminus C$. By virtue of Assumption 2.1(v), we may assume that $U$ only intersects one of the inclusion types $D_0, D_{\infty}, D_0^\gamma$, or $D_0^\Lambda$. Proposition 5.1 does not directly reveal which one of its two operator inequalities that fails when applied to $\gamma_L$ or $\gamma_U$ in case $D \not\subseteq C$ (cf. (5.3)). However, according to [4, Proof of Theorem 3.7], this depends on the inclusion type that $U$ intersects, and we choose our approach to dealing with $\gamma$ accordingly: If (Case A) the intersected inclusion type belongs to $D^-$, we will investigate $\Lambda_0(C) - \Lambda(\gamma)$, and, on the other hand, if (Case B) the inclusion type belongs to $D^+$, we will investigate $\Lambda(\gamma) - \Lambda_{\infty}(C)$.

We start by considering Case A, meaning that the only part of $D$ that $U$ is chosen to intersect is either $D_0$ or $D_0^\gamma$. According to (5.2), \[ \Lambda_0(C) - \Lambda(\gamma_U) \geq \Lambda_0(C) - \Lambda(\gamma). \]
However, by repeating either part (b) (if $U$ intersects $D_0^\gamma$) or part (d) (if $U$ intersects $D_0$) in [4, Proof of Theorem 3.7], we deduce $\Lambda_0(C) \not\geq \Lambda(\gamma_U)$, and therefore also $\Lambda_0(C) \not\geq \Lambda(\gamma)$.

In Case B, the only part of $D$ that $U$ is chosen to intersect is either $D_{\infty}$ or $D_0^\gamma$. According to (5.1), \[ \Lambda(\gamma_L) - \Lambda_{\infty}(C) \geq \Lambda(\gamma) - \Lambda_{\infty}(C). \]
However, by repeating either part (a) (if $U$ intersects $D_0^\gamma$) or part (c) (if $U$ intersects $D_{\infty}$) in [4, Proof of Theorem 3.7], we deduce $\Lambda(\gamma_L) \not\geq \Lambda_{\infty}(C)$, and therefore also $\Lambda(\gamma) \not\geq \Lambda_{\infty}(C)$. \hfill $\Box$

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