On $L_1$-embeddability of unions of $L_1$-embeddable metric spaces and of twisted unions of hypercubes

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December 20, 2021

Abstract

We study properties of twisted unions of metric spaces introduced by Johnson, Lindenstrauss, and Schechtman, and by Naor and Rabani. In particular, we prove that under certain natural mild assumptions twisted unions of $L_1$-embeddable metric spaces also embed in $L_1$ with distortions bounded above by constants that do not depend on the metric spaces themselves, or on their size, but only on certain general parameters. This answers a question stated by Naor and by Naor and Rabani.

In the second part of the paper we give new simple examples of metric spaces such their every embedding into $L_p$, $1 \leq p < \infty$, has distortion at least 3, but which are a union of two subsets, each isometrically embeddable in $L_p$. This extends an analogous result of K. Makarychev and Y. Makarychev from Hilbert spaces to $L_p$-spaces, $1 \leq p < \infty$.

2020 Mathematics Subject Classification. Primary: 46B85; Secondary: 30L05, 46B20, 51F30, 68R12.

Keywords. Banach space, distortion of a bilipschitz embedding, stable metric space

1 Introduction

One of natural general questions about metric spaces is the following:

\textbf{Question 1.1.} Let a metric space $(X,d)$ be a union of its metric subspaces $A$ and $B$. Assume that $A$ and $B$ have a certain metric property $\mathcal{P}$. Does this imply that $X$ also has property $\mathcal{P}$, possibly in some weakened form?

This question can be viewed as a part of a general theme of “local-global” properties, when one wants to analyze whether spaces (or other mathematical objects) that have certain properties “locally”, i.e. on certain subspaces/subsets, also have related properties “globally”, i.e. on the whole space. The study of the “local-global” theme is prevalent in many (if not all) areas of mathematics, including functional analysis, and of theoretical computer science. Questions in the “local-global” theme usually assume that all subspaces/subsets of a specified size satisfy the investigated property. Question 1.1 is different since it assumes that property $\mathcal{P}$ is satisfied by only one pair of complementing subsets, at least one of which has to be at least half the size of $X$.

We are particularly interested in the embeddability properties of metric spaces. We are aware of three embeddability properties for which the answers to Question 1.1 are positive, interesting, and useful. We state them below after recalling the necessary definitions.

\textbf{Definition 1.2.} Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces. An injective map $F : X \to Y$ is called a \textit{bilipschitz embedding} if there exist constants $C_1, C_2 > 0$ so that for all $u, v \in X$

$$C_1d_X(u, v) \leq d_Y(F(u), F(v)) \leq C_2d_X(u, v).$$
The distortion of $F$ is defined as \( \text{Lip}(F) \cdot \text{Lip}(F^{-1}|_{F(X)}) \), where \( \text{Lip}() \) denotes the Lipschitz constant.

For \( p \in [1, \infty] \), the \( L_p \)-distortion \( c_p(X) \) is defined as the infimum of distortions of all bi-Lipschitz embeddings of \( X \) into any space \( L_p(\Omega, \Sigma, \mu) \).

**Definition 1.3.** A map \( f : (X, d_X) \to (Y, d_Y) \) between two metric spaces is called a coarse embedding if there exist non-decreasing functions \( \rho_1, \rho_2 : [0, \infty) \to [0, \infty) \) such that

\[
\forall u, v \in X \quad \rho_1(d_X(u, v)) \leq d_Y(f(u), f(v)) \leq \rho_2(d_X(u, v)).
\]

(Observe that this condition implies that \( \rho_2 \) has finite values, but that it does not imply that \( f \) is injective.)

**Definition 1.4.** A metric space \((Y, d_Y)\) is called ultrametric if for any \( u, v, w \in Y \)

\[
d_Y(u, w) \leq \max\{d_Y(u, v), d_Y(v, w)\}.
\]

**Theorem 1.5** (Dadarlat, Guentner [7 Corollary 4.5]). If a metric space \( X \) is a finite union of subsets each admitting a coarse embedding into a Hilbert space, then \( X \) also admits a coarse embedding into a Hilbert space.

**Theorem 1.6** (Mendel, Naor [20 Theorem 1.4]). Let a metric space \((X, d)\) be a union of its metric subspaces \( A \) and \( B \). Assume that \( A \) and \( B \) embed into, possibly different, ultrametric spaces with distortions \( D_A \) and \( D_B \), respectively. Then the metric space \( X \) embeds into an ultrametric space with distortion at most \((D_A + 2)(D_B + 2) - 2\).

**Theorem 1.7** (K. Makarychev, Y. Makarychev [18]). Suppose that a metric space \((X, d)\) is the union of two metric subspaces \( A \) and \( B \) that embed into \( \ell^a_2 \) and \( \ell^b_2 \) (where \( a \) and \( b \) may be finite or infinite) with distortions \( D_A \) and \( D_B \), respectively. Then \( X \) embeds into \( \ell^{a+b+1}_2 \) with distortion \( D \leq 7D_AD_B + 2(D_A + D_B) \).

If \( D_A = D_B = 1 \), then \( X \) embeds into \( \ell^{a+b+1}_2 \) with distortion at most 8.93.

**Remark 1.8.** We note that there is an extensive literature on the property of \( L_1 \)-embeddability within the “local-global” theme. For example, Arora, Lovász, Newman, Rabani, Rabinovich and Vempala [11] asked what is the least distortion with which one can embed the metric space \( X \) into \( L_1 \), given that every subset of \( X \) of cardinality \( k \) is embeddable into \( L_1 \) with distortion at most \( D \). An answer to this question was given by Charikar, K. Makarychev, and Y. Makarychev [5], who proved, among other results, that if even if a small fraction \( \alpha \) (say 1%) of all subsets of size \( k \) of a metric space \( X \), with \(|X| = n\), embeds into \( \ell_p \) with distortion at most \( D \), then the entire space \( X \) embeds into \( \ell_p \), \( 1 \leq p < \infty \), with distortion at most \( D \cdot O(\log(n/k) + \log \log(1/\alpha) + \log p) \). In particular, if \( k \) is proportional to \( n \), then one obtains a bounded distortion embedding of \( X \) into \( \ell_p \).

On the other hand, there exist absolute constants \( a, b, A, B > 0 \) such that for every \( N \in \mathbb{N} \) there exists an \( N \)-point metric space \( X \) such that every subset of \( X \) of size at most \( A(\exp(\log \log N)^a) \) embeds isometrically into \( L_1 \), but every embedding of \( X \) into \( L_1 \) requires distortion at least \( B(\log \log N)^b \), see [26] and the expositions in [14] Section 1.3 and [28] Section 4.4].

In connection with Theorem 1.7 it became natural to investigate the problem whether analogous results are valid for metric spaces embeddable into \( L_p \), when \( p \neq 2 \), explicitly stated e.g. in [20 Remark 4.2], [18 Question 5], [23 Open Problem 9.6], and [24 Remark 18].
Problem 1.9. Suppose \((X, d)\) is a metric space and \(X = A \cup B\), with \(c_p(A)\) and \(c_p(B)\) finite. Does this imply that \(c_p(X)\) is finite? Can \(c_p(X)\) be bounded from above only in terms of \(c_p(A)\) and \(c_p(B)\)?

It is easy to see that the answer is positive for \(p = \infty\). Theorem 1.7 states that the answer is positive for \(p = 2\). K. Makarychev and Y. Makarychev [18, Question 5] conjectured that the answer is negative for every \(p \in [1, \infty)\) except 2 and \(\infty\).

Problem 1.9 is particularly interesting in the case of \(p = 1\). In this case, in addition to the Makarychev-Makarychev conjecture of the negative answer to Problem 1.9, since 2015 in the literature there were conjectures that a construction known as a twisted union of hypercubes might be a possible method of constructing a family of counterexamples.

Problem 1.10 (Naor [23, Open problem 3.3], Naor, Rabani [24, Remark 18]). Must any embedding of a twisted union of hypercubes described in Examples 3.1 and 3.2 below into \(L_1\) incur a bi-Lipschitz distortion that tends to \(\infty\) as the size of the hypercube tends to \(\infty\)? That is, does a twisted union of hypercubes give a negative answer to Problem 1.9 in the case \(p = 1\)?

The idea of the construction of a twisted union of metric spaces can be traced back to [17] and has been used in [12] and [24] to provide examples that demonstrate that for \(\alpha \in (1/2, 1]\), the \(\alpha\)-extension constants from \(\ell_\infty\) to \(\ell_2\) are not bounded. Variants of this construction were also used in [16, 4].

The most general idea is explained in [24, Remark 19], and is as follows: Suppose that \((X, d_X)\) and \((Y, d_Y)\) are finite metric spaces with \(X\) and \(Y\) disjoint as sets. Given mappings \(\sigma : X \rightarrow Y\) and \(r : X \rightarrow (0, \infty)\), we define the weighted graph structure on \(X \cup Y\) by defining the following weighted edges: If \(x_1, x_2 \in X\) then \(x_1\) and \(x_2\) are joined by an edge of weight \(d_X(x_1, x_2)\); if \(y_1, y_2 \in Y\) then \(y_1\) and \(y_2\) are joined by an edge of weight \(d_Y(y_1, y_2)\). Also, for every \(x \in X\), the elements \(x\) and \(\sigma(x)\) are joined by an edge of weight \(r(x)\). We endow \(X \cup Y\) with the shortest-path metric induced by this weighted graph.

Naor and Rabani point out that all metric spaces that they construct in [24] to exhibit a maximal unbounded growth of certain extension constants can be described as subsets of this general construction, and they indicate that usefulness of this construction is probably yet to be fully explored.

One of the main goals of the present paper is to start an exploration of general twisted unions of metric spaces. Under certain natural restrictions on the metrics of the original spaces and the function \(r : M \rightarrow (0, \infty)\) we give detailed formulas for the induced metric of the twisted unions, see Section 3.

We then study the embedding properties of twisted unions. Somewhat to our surprise, we answer Problem 1.10 negatively. In fact we prove a number of results (see Theorems 4.1, 5.1, 5.2 and Corollaries 4.3, 5.3) establishing that all twisted unions of \(L_1\)-embeddable spaces that satisfy certain natural mild assumptions, also embed into \(L_1\) with distortions bounded by constants that depend only on certain general parameters and not on the metrics themselves or the size of the sets.

While we do not obtain the ultimate answer to Problem 1.9, we provide a fairly large new class of metric spaces for which the answer to Problem 1.9 is affirmative. While it is interesting to search for a counterexample to Problem 1.9 (which is widely believed to exist), identifying classes of spaces for which the answer to Problem 1.9 is affirmative may
have a greater potential for future applications. For this reason we investigate here what conditions on twisted unions assure their $L_1$-embeddability, see Sections 4 and 5.

In Section 6 we show that the lower bound on distortion of the embedding of a union of metric spaces found by K. Makarychev and Yu. Makarychev [18, Theorem 1.2 and Section 3] for the Hilbert space, is also valid for all $L_p$ with $1 \leq p < \infty$, and for many other Banach and metric spaces. Our proof uses the theory of stable metric spaces (see Definition 6.1) and our examples are infinite metric spaces. For spaces whose stability is known our proof is very simple (see Example 6.5).

## 2 Preliminary facts and notation

We use the standard terminology of the theories of Banach Spaces and Metric Embeddings, see [2, 25].

Suppose that $K \geq 1$, $M$ is a set, and $f : M \times M \rightarrow [0, \infty)$ is an arbitrary function that is not necessarily a metric on $M$, such that there exists a map $\Psi : M \rightarrow L_1$ such that for all $x, y \in M$, we have

$$f(x, y) \leq \|\Psi(x) - \Psi(y)\| \leq K f(x, y).$$

In this situation, even if the function $f$ is not a metric (we even allow the function $f$ to equal to 0 on an arbitrary subset of $M \times M$), with a slight abuse of notation, we will say that $(M, f)$ embeds in $L_1$ and write $c_1(M, f) \leq K < \infty$.

Please see Remark 4.2 for a brief discussion of functions $f$ that can satisfy this condition.

We will need the following two results of Mendel and Naor [21].

**Theorem 2.1.** For every $\lambda > 0$, $L_1$ with the truncated metric

$$\varrho(x, y) = \min\{\lambda, \|x - y\|_1\}$$

embeds into $L_1$ with distortion not exceeding $e/(e - 1)$.

This is a powerful result which, using the theory of concave functions of Brudnyi and Krugljak (see [3, Section 3.2] and [19, Remark 5.4]), implies:

**Corollary 2.2.** [21] There exists a universal constant $\Delta \leq (2\sqrt{2} + 3)e/(e - 1) < 10$, such that if $\omega : [0, \infty) \rightarrow [0, \infty)$ is any concave non-decreasing function with $\omega(0) = 0$ and $\omega(t) > 0$ for $t > 0$, then the metric space $(L_1, \omega(\|x - y\|_1))$ embeds into $L_1$ with distortion at most $\Delta$.

We will also use the following immediate corollary of Theorem 2.1

**Corollary 2.3.** Let $M$ be a set, $D \geq 1$, and $f : M \times M \rightarrow [0, \infty)$ be a function such that there exists a map $\phi : M \rightarrow L_1$ such that for all $x, y \in M$, we have

$$f(x, y) \leq \|\phi(x) - \phi(y)\| \leq D f(x, y).$$

Then, for any constant $\lambda > 0$, $c_1(M, \min\{f(x, y), \lambda\}) \leq eD/(e - 1)$.

**Proof.** By Theorem 2.1 there exists a map $T : L_1 \rightarrow L_1$ such that for all $u, v \in L_1$

$$\min\{\|u - v\|, \lambda\} \leq \|Tu - Tv\| \leq \frac{e}{e - 1} \min\{\|u - v\|, \lambda\}.$$
Thus for all \( x, y \in M \)
\[
\|T\phi(x) - T\phi(y)\| \leq \frac{e}{e-1} \min\{\|\phi(x) - \phi(y)\|, \lambda\} \leq \frac{e}{e-1} \min\{Df(x, y), \lambda\}
\]
and
\[
\|T\phi(x) - T\phi(y)\| \geq \min\{\|\phi(x) - \phi(y)\|, \lambda\} \geq \min\{f(x, y), \lambda\},
\]
which ends the proof.

If \( G \) and \( H \) are real valued quantities or functions, we use notation \( G \asymp H \) to mean that there exist \( 0 < \alpha \leq \beta < \infty \) such that \( \alpha G \leq H \leq \beta G \). The numbers \( \alpha \) and \( \beta \) can depend on the parameters mentioned in the statements of the results we are proving, but not on elements \( x, y \) of the considered metric space.

## 3 Twisted unions of hypercubes and of general metric spaces

We start by presenting the concrete examples of twisted unions of hypercubes from \cite{12, 23, 24}.

**Example 3.1.** \cite{12, 23, 24} Let \( \mathbb{F}_2^n = \{0, 1\}^n \) be the Hamming cube embedded into \( \ell_1^n \) in a natural way and for \( x, y \in \mathbb{F}_2^n \) we use the notation \( \|x - y\| \) for the \( \ell_1 \)-norm. Let \( r \in (0, \infty) \) and \( \alpha \in (1/2, 1] \) be fixed constants.

Define the metric on \( \mathbb{F}_2^n \times \mathbb{F}_2 = \{0, 1\}^n \times \{0, 1\} \) as the shortest path metric when \( \mathbb{F}_2^n \times \mathbb{F}_2 \) is considered as a graph with the following edges and weights:

- for every \( x, y \in \mathbb{F}_2^n \) there is an edge with ends \( (x, 0) \) and \( (y, 0) \) of weight \( \frac{\|x - y\|}{2^\alpha r} \),
- for every \( x, y \in \mathbb{F}_2^n \) there is an edge with ends \( (x, 1) \) and \( (y, 1) \) of weight \( \frac{\|x - y\|}{2^\alpha r} \),
- for every \( x \in \mathbb{F}_2^n \) there is an edge with ends \( (x, 0) \) and \( (x, 1) \) of weight \( r \).

The exact formula for the distance between any two points of \( \mathbb{F}_2^n \times \mathbb{F}_2 \) in this metric is computed in \cite{24 Lemma 15}, cf. also (3.2) and Remark 3.5 below.

\[
d((x, a), (y, b)) = \begin{cases} 
\|x - y\| + \frac{1}{r} & \text{if } a = b = 0, \\
\min\{\frac{\|x - y\|}{r}, 2r + \|x - y\| + \frac{1}{r}\} & \text{if } a = b = 1, \\
r + \min\{\frac{\|x - y\|}{r(2^{\alpha} - 1)}, \|x - y\| + \frac{1}{r}\} & \text{if } x = y, a \neq b, \\
r + \min\{\frac{\|x - y\|}{r(2^{\alpha} - 1)}, \|x - y\| + \frac{1}{r}\} & \text{if } a \neq b.
\end{cases}
\]

In \cite{24} it was shown that Example 3.1 is a special case of the following more general example.

**Example 3.2.** \cite{24 Remark 18} Let \( \mathbb{F}_2^n \) be the Hamming cube, \( r \in (0, \infty) \) be a constant, and \( \omega_0 \) and \( \omega_1 \) be concave non-decreasing continuous functions on \([0, \infty)\) vanishing at 0 and such that for all \( t > 0 \), \( \omega_0(t) > 0 \) and \( \omega_1(t) > 0 \). Define the metric on \( \mathbb{F}_2^n \times \mathbb{F}_2 \) as the shortest path metric when \( \mathbb{F}_2^n \times \mathbb{F}_2 \) is considered as a graph with the following edges and weights:
• for every \( x, y \in \mathbb{F}_2^n \) there is an edge with ends \((x, 0)\) and \((y, 0)\) of weight \( \omega_0(\|x - y\|) \),
• for every \( x, y \in \mathbb{F}_2^n \) there is an edge with ends \((x, 1)\) and \((y, 1)\) of weight \( \omega_1(\|x - y\|) \),
• for every \( x \in \mathbb{F}_2^n \) there is an edge with ends \((x, 0)\) and \((x, 1)\) of weight \( r \).

Since the functions \( \omega_0 \) and \( \omega_1 \) are concave, it is easy to see that the above weights imply that the shortest path metric on \( \mathbb{F}_2^n \times \mathbb{F}_2 \) satisfies

\[
d((x, a), (y, b)) = \left\{ \begin{array}{ll}
\min\{\omega_0(\|x - y\|), 2r + \omega_1(\|x - y\|)\}, & \text{if } a = b = 0, \\
\min\{\omega_1(\|x - y\|), 2r + \omega_0(\|x - y\|)\}, & \text{if } a = b = 1, \\
r, & \text{if } x = y, a \neq b.
\end{array} \right.
\]

Moreover, by Lemma 3.3(c) below, if \( a \neq b \),

\[
d((x, a), (y, b)) = r + \min\{\omega_0(\|x - y\|), \omega_1(\|x - y\|)\}.
\]

Notice that if we define

\[
\varpi_0(t) = \min\{\omega_0(t), 2r + \omega_1(t)\}, \\
\varpi_1(t) = \min\{\omega_1(t), 2r + \omega_0(t)\},
\]

then the functions \( \varpi_0, \varpi_1 \) are concave, non-decreasing, continuous, vanishing only at \( t = 0 \), and for all \( t \)

\[
|\varpi_0(t) - \varpi_1(t)| \leq 2r.
\]

It is clear that we could have used functions \( \varpi_0(\|x - y\|), \varpi_1(\|x - y\|) \) as weights in place of \( \omega_0(\|x - y\|), \omega_1(\|x - y\|) \), respectively.

Naor and Rabani [24, Theorem 16] proved that the \( \alpha \)-snowflake (for the same \( \alpha \in (1/2, 1] \) that is used as a parameter in the definition of the space) of the metric space considered in Example 3.1 on the set \( \mathbb{F}_2^n \times \{0\} \) embeds isometrically into \( \ell_2 \), but that for certain choices of the value of \( r > 0 \), the minimum distortion of any embedding of the space \( \langle \mathbb{F}_2^n \times \mathbb{F}_2, d_\alpha \rangle \) into \( \ell_2 \) is at least of the order \( n^{\frac{2\alpha - 1}{4\alpha}} \), and thus the \( \alpha \)-extension constant grows to \( \infty \) when the size of the hypercube grows to \( \infty \): \( e_n^\alpha(\ell_\infty, \ell_2) \gtrsim (\log n)^{\frac{2\alpha - 1}{4\alpha}} \).

Naor and Rabani [24, Remark 18] also noted that, by a result from [21] (see Corollary 2.2), for all \( n \), the metric spaces \( \langle \mathbb{F}_2^n \times \{0\}, \varpi_0(\|x - y\|) \rangle \) and \( \langle \mathbb{F}_2^n \times \{1\}, \varpi_1(\|x - y\|) \rangle \), embed into \( L_1 \) with bounded distortions, and they asked whether any embedding of \( \langle \mathbb{F}_2^n \times \mathbb{F}_2, d \rangle \) must incur a bi-Lipschitz distortion that tends to \( \infty \) as \( n \to \infty \), and commented that this would be the first known example of a metric space that can be partitioned into two subsets, each of which well-embeds into \( \ell_1 \) yet the entire space does not, see also [23, Open Problem 3.3].

Following the ideas of Examples 3.1 and 3.2 and of [24, Remark 19] we state the definitions of a twisted union and a generalized twisted union of metric spaces.

**Definition 3.3** (Twisted Union and Generalized Twisted Union of Metric Spaces). Let \( M \) be a finite set which is endowed with two metrics, \( d_0 \) and \( d_1 \), and let \( r \) be a fixed positive number (respectively, let \( r \) be a function from \( M \) to \((0, \infty)\)) such that for all \( x, y \in M \):

\[
|d_0(x, y) - d_1(x, y)| \leq 2r,
\]

(3.3)
or, respectively,
\[ |d_0(x, y) - d_1(x, y)| \leq r(x) + r(y), \quad (3.4) \]
and \[ |r(x) - r(y)| \leq d_0(x, y) + d_1(x, y). \quad (3.5) \]

Define the metric \( d \) on \( M \times \mathbb{F}_2 = M \times \{0, 1\} \) as the shortest path metric when \( M \times \mathbb{F}_2 \) is considered as a graph with the following edges and weights:

- for every \( x, y \in M \) there is an edge with ends \((x, 0)\) and \((y, 0)\) of weight \( d_0(x, y) \),
- for every \( x, y \in M \) there is an edge with ends \((x, 1)\) and \((y, 1)\) of weight \( d_1(x, y) \),
- for every \( x \in M \) there is an edge with ends \((x, 0)\) and \((x, 1)\) of weight \( r \) (resp., \( r(x) \)).

If \( r > 0 \) is a constant, the space \((M \times \mathbb{F}_2, d)\) is called the twisted union of metric spaces \((M, d_0)\) and \((M, d_1)\) with the joining parameter \( r \), or the \( r \)-twisted union, for short.

If \( r(\cdot) \) is a function, \((M \times \mathbb{F}_2, d)\) is called the generalized twisted union of \((M, d_0)\) and \((M, d_1)\) with the joining function \( r : M \to (0, \infty) \).

We note that, unlike the situation in Example 3.2 for a space \( M \) with arbitrary metrics \( d_0, d_1 \) that are not necessarily given as concave functions of another underlying metric, it is not clear how to derive an analogue of the formula (3.2), that is, a formula for the metric of the twisted union restricted to the sets \( M \times \{0\} \) and \( M \times \{1\} \). For this reason we impose conditions on relations between the metrics \( d_0, d_1 \), and the joining parameter \( r \) or the joining function \( r : M \to (0, \infty) \), respectively. The condition (3.3) (resp., conditions (3.4) and (3.5)) are necessary and sufficient for the weights of all edges to be equal to the weighted graph distance between their ends. Hence the metric \( d \) on the twisted union (resp., the generalized twisted union) \( M \times \mathbb{F}_2 \) satisfies

\[
d((x, a), (y, b)) = \begin{cases} 
  d_0(x, y) & \text{if } a = b = 0, \\
  d_1(x, y) & \text{if } a = b = 1, \\
  r & (\text{resp., } r(x)) \quad \text{if } x = y, a \neq b.
\end{cases} \quad (3.6)
\]

Computation of the formula for \( d((x, a), (y, b)) \) when \( x \neq y \) and \( a \neq b \) is more delicate and we did it only when certain additional conditions on \( d_0, d_1 \) and \( r \) (resp. \( r(\cdot) \)) are satisfied.

We prove, under two different natural assumptions that are independent of each other, that for all \( x, y \in M \),

\[
d((x, 0), (y, 1)) \asymp \min\{d_0(x, y), d_1(x, y)\} + r(x). \tag{3.7}
\]

By Lemma 3.4 below, the formula (3.7) is valid if the joining function \( r(\cdot) \) is Lipschitz with respect to both metrics \( d_0 \) and \( d_1 \), not just with respect to the sum of these metrics as required in (3.5), that is, if there exists a constant \( L > 0 \), such that for all \( x, y \in M \),

\[
|r(x) - r(y)| \leq L \min\{d_0(x, y), d_1(x, y)\}. \tag{3.8}
\]

In particular, (3.7) is satisfied when the joining function \( r(\cdot) \) is constant, since in this case (3.8) holds with \( L = 0 \).

By Lemma 3.6 below, the formula (3.7) is valid if there exists a constant \( C > 0 \) such that for all \( x, y \in M \),

\[
d_1(x, y) \leq C d_0(x, y). \tag{3.9}
\]
Condition (3.9) is satisfied, for example, in Example 3.1 since there, independent of the value of \( r > 0 \) or \( \alpha \in (1/2, 1] \), for all \( x, y \in M \), we have

\[
\varpi_1(x - y) = \min \left\{ \frac{\| x - y \|}{r^{2\alpha - 1}}, 2r + \| x - y \| \frac{1}{r^{2\alpha}} \right\} \leq 2\| x - y \| \frac{1}{r^{2\alpha}} = 2\varpi_0(\| x - y \|). \tag{3.10}
\]

It is clear that conditions (3.8) and (3.9) are independent of each other. We note that (3.8) implies that

\[
\text{Condition (3.9) is satisfied, for example, in Example 3.1, since there, independent of the value of } \alpha \in (0, 1] .
\]

On the other hand, (3.9) implies that \( \min \{d_0(x, y), d_1(x, y)\} \approx d_1(x, y) \). Thus, if (3.9) holds, then

\[
d((x, 0), (y, 1)) \approx \min \{d_0(x, y), d_1(x, y)\} + r(x) \approx d_1(x, y) + r(x) . \tag{3.11}
\]

However in (3.11) it is not possible to replace \( r(x) \) by either \( r(y) \), \( \max \{r(x), r(y)\} \), or \( \min \{r(x), r(y)\} \), since (3.9) does not imply that the function \( r(\cdot) \) is Lipschitz with respect to the metric \( d_1 \).

We now prove the lemmas mentioned above. The common assumption of Lemmas 3.4 and 3.8 is:

Let \( M \) be a metric space endowed with two metrics \( d_0 \) and \( d_1 \), and let \( r : M \to (0, \infty) \) be a function such that (3.1) and (3.5) are satisfied, and let \( (M \times F_2, d) \) be the generalized twisted union of \( (M, d_0) \) and \( (M, d_1) \) with the joining function \( r(\cdot) \).

**Lemma 3.4.** Suppose (\$) and that the joining function \( r(\cdot) \) is Lipschitz with respect to both metrics \( d_0 \) and \( d_1 \), that is, suppose that there exists a constant \( L > 0 \), such that for all \( x, y \in M \)

\[
|r(x) - r(y)| \leq L \min \{d_0(x, y), d_1(x, y)\} . \tag{3.8}
\]

Denote \( \min \{d_0(x, y), d_1(x, y)\} \) by \( h(x, y) \).

(a) Then for all \( x, y \in M \)

\[
\frac{1}{A} (h(x, y) + \max \{r(x), r(y)\}) \leq d((x, 0), (y, 1)) \leq h(x, y) + \max \{r(x), r(y)\} , \tag{3.12}
\]

where \( A = \max \{2L + 1, 3\} \).

(b) If the joining function is constant with \( r(x) = r > 0 \), for all \( x \in M \), then for all \( x, y \in M \)

\[
\frac{1}{3} (r + h(x, y)) \leq d((x, 0), (y, 1)) \leq r + h(x, y) . \tag{3.13}
\]

(c) If \( h(x, y) = \min \{d_0(x, y), d_1(x, y)\} \) is a metric on \( M \) and the joining function is constant with \( r(x) = r > 0 \), for all \( x \in M \), then for all \( x, y \in M \)

\[
d((x, 0), (y, 1)) = r + h(x, y) . \tag{3.14}
\]
Remark 3.5. We included (3.11) above, because we are particularly interested in spaces described in Example 3.2 when $M$ is a subset of $L_1$ and $d_i(x, y) = \varpi_i(\|x - y\|)$ for $i = 0, 1,$ where functions $\varpi_0, \varpi_1$ are concave, non-decreasing, continuous, and vanishing only at $t = 0,$ see Corollary 4.3. In this situation $\min\{d_0(x, y), d_1(x, y)\}$ is a metric on $M.$

We note that if $\min\{d_0(x, y), d_1(x, y)\}$ is a metric on $M,$ then one can obtain slightly better constants also in (3.12) above.

Proof. We fix $x, y \in M.$ The condition (3.4) implies that on a shortest path from $(x, 0)$ to $(y, 1)$ we may avoid moving from $M \times \{0\}$ to $M \times \{1\}$ more than once. Thus

$$d((x, 0), (y, 1)) = \inf_{z \in M} (d_0(x, z) + d_1(z, y) + r(z)).$$

If $h(x, y) = d_1(x, y),$ we pick $z = x,$ otherwise we pick $z = y,$ so that we get

$$d((x, 0), (y, 1)) \leq h(x, y) + \max\{r(x), r(y)\}.$$  

On the other hand, by (3.4), for all $u, v \in M$ we have $d_0(u, v) \leq d_1(u, v) + r(u) + r(v).$ Thus, for every $x, y, z \in M$ we have

$$h(x, y) \leq d_0(x, y) \leq d_0(x, z) + d_0(z, y) \leq d_0(x, z) + d_1(z, y) + r(z) + r(y)$$

$$\leq d_0(x, z) + d_1(z, y) + r(z) + \max\{r(x), r(y)\}.$$  

(3.15)

Hence, for every $x, y, z \in M$ and $T > 1$ we have

$$d_0(x, z) + d_1(z, y) + r(z) - \frac{1}{T}(h(x, y) + \max\{r(x), r(y)\})$$

$$\geq d_0(x, z) + d_1(z, y) + r(z) - \frac{1}{T}(d_0(x, z) + d_1(z, y) + r(z) + 2\max\{r(x), r(y)\})$$

$$\geq \left(1 - \frac{1}{T}\right)(d_0(x, z) + d_1(z, y)) + \left(1 - \frac{3}{T}\right)r(z) - \frac{2}{T}\max\{r(x), r(y)\}$$

$$\geq \left(1 - \frac{1}{T}\right)(h(x, z) + h(z, y)) + \left(1 - \frac{3}{T}\right)r(z) - \frac{2L}{T}\max\{h(x, z), h(z, y)\}$$

$$\geq \left(1 - \frac{2L + 1}{T}\right)\max\{h(x, z), h(z, y)\} + \left(1 - \frac{3}{T}\right)r(z).$$

If $T = \max\{2L + 1, 3\},$ the ultimate quantity is nonnegative, which proves (3.12).

Formula (3.13) immediately follows from (3.12), since when the function $r(\cdot)$ is constant then it is Lipschitz with the Lipschitz constant $L = 0.$

If $h(x, y) = \min\{d_0(x, y), d_1(x, y)\}$ is a metric on $M,$ then for all $z \in M$ we have

$$d_0(x, z) + d_1(z, y) + r \geq \min\{d_0(x, z), d_1(x, z)\} = \min\{d_0(x, y), d_1(x, y)\} + r$$

$$\geq \min\{d_0(x, y), d_1(x, y)\} + r,$$

which proves (3.14). $\square$

Lemma 3.6. Suppose (*). Then conditions (3.4), (3.5), and

$$\forall x, y \in M \quad d_1(x, y) \leq Cd_0(x, y).$$

(3.9)

imply that

$$\frac{1}{2C + 1}(d_1(x, y) + r(x)) \leq d((x, 0), (y, 1)) \leq d_1(x, y) + r(x).$$

(3.16)
Moreover, if \( r(x) = r > 0 \) for all \( x \in M \), then (3.4), (3.5), and (3.9) imply that
\[
\frac{1}{C+1} \left( d_1(x, y) + r \right) \leq d((x, 0), (y, 1)) \leq d_1(x, y) + r.
\] (3.17)

**Proof.** The condition (3.4) implies that on a shortest path from \((x, 0)\) to \((y, 1)\) we may avoid moving from \(M \times \{0\}\) to \(M \times \{1\}\) more than once. Thus
\[
d((x, 0), (y, 1)) = \inf_{z \in M} (d_0(x, z) + d_1(z, y) + r(z)).
\]

If we pick \( z = x \), we get
\[
d((x, 0), (y, 1)) \leq d_1(x, y) + r(x).
\]

On the other hand, for every \( z \in M \) and \( T > 1 \) we have
\[
d_0(x, z) + d_1(z, y) + r(z) - \frac{1}{T} \left( d_1(x, y) + r(x) \right)
\geq d_0(x, z) + d_1(z, y) - \frac{1}{T} \left( d_1(x, y) - \frac{1}{T} \left( r(x) - r(z) \right) \right)
\geq d_0(x, z) + d_1(z, y) - \frac{1}{T} \left( d_0(x, z) + d_1(x, z) \right)
\geq \left( 1 - \frac{1}{T} \right) d_0(x, z) + d_1(z, y) - \frac{1}{T} \left( d_1(x, y) - \frac{1}{T} d_1(x, z) \right)
\geq \left( 1 - \frac{1}{T} \right) d_0(x, z) - \frac{1}{T} d_1(x, z) - \frac{1}{T} d_1(x, z)
\geq \frac{1}{C} \left( 1 - \frac{1}{T} \right) d_1(x, z) - \frac{2}{T} d_1(x, z)
\geq \frac{T-1-2C}{CT} d_1(x, z).
\]
The ultimate quantity is nonnegative if \( T = 2C + 1 \). This proves (3.16). The proof of (3.17) is essentially the same if we notice that \( r(x) - r(z) = 0 \). \( \Box \)

## 4 On \( L_1 \)-embeddability of twisted unions

In this section we study \( L_1 \)-embeddability of twisted unions with uniform weight on all edges joining points from distinct original metric spaces. We start from a general result and, as an application, we obtain that all metric spaces from Examples 3.1 and 3.2 are \( L_1 \)-embeddable, see Corollary 4.3. Another corollary of Theorem 4.1 is presented in Section 5 see Corollary 5.3.

**Theorem 4.1.** Let \( r > 0 \) and \( M \) be a metric space endowed with two metrics \( d_0 \) and \( d_1 \), such that (3.3) is satisfied, and for \( i = 0, 1 \), the space \((M, d_i)\) embeds into \( L_1 \) with distortion at most \( D_i \). Suppose also that there exist a constant \( K \geq 1 \) and a map \( \psi : M \rightarrow L_1 \) such that for all \( x, y \in M \)
\[
\min\{d_0(x, y), d_1(x, y)\} \leq \|\psi(x) - \psi(y)\| \leq K \min\{d_0(x, y), d_1(x, y)\}.
\] (4.1)

Then the \( r \)-twisted union \((M \times \mathbb{F}_2, d)\) of \((M, d_0)\) and \((M, d_1)\) embeds into \( L_1 \) with distortion bounded by a constant \( D \) that depends only on \( D_0, D_1 \) and \( K \).
Remark 4.2. In general the minimum of two metrics does not need to be a metric. However if (4.1) is satisfied, then there exists a metric $\gamma$ on $M$, and a constant $\beta \in (0, 1]$ such that

$$\frac{1}{4}(\min\{d_0(x, y), d_1(x, y)\})^\beta \leq \gamma(x, y) \leq (\min\{d_0(x, y), d_1(x, y)\})^\beta.$$ 

This follows from a routine adjustment of a result of Kalton, Peck, and Roberts [13, Theorem 1.2], who studied properties of generalizations of $F$-norms that instead of the usual triangle inequality satisfy the ultimate inequality in

$$h(x, y) \leq K(h(x, z) + h(z, y)) \leq 2K \max \{h(x, z), h(y, z)\},$$

as in (4.2).

Clearly, if $h(x, y) \defeq \min\{d_0(x, y), d_1(x, y)\}$, then (4.1) implies (4.2) and $h$ is separating ($h(x, y) = 0$ iff $x = y$) and symmetric ($h(x, y) = h(y, x)$ for all $x, y \in M$).

We note that such functions were studied already by Fréchet [8, 9], who called any symmetric, separating function $h : M \times M \to [0, \infty)$ satisfying a condition slightly weaker than (4.2) a voisinage. Chittenden [6] proved that any space with a voisinage is homeomorphic to a metric space. For modern theory of similar types of spaces see the monograph of Kalton, Peck, and Roberts [13].

Proof. Define for all $x, y \in M$ and $i = 0, 1$

$$h(x, y) = \min\{d_0(x, y), d_1(x, y)\} \quad \text{(4.3)}$$

$$\rho_i(x, y) = \min\{d_i(x, y), 2r\} \quad \text{(4.4)}$$

Then, for $i = 0, 1, x, y \in M$ we have

$$d_i(x, y) \leq \rho_i(x, y) + h(x, y) \leq 2d_i(x, y). \quad \text{(4.5)}$$

Indeed, by (4.3) and (4.4), the rightmost inequality is clear. If for some $x, y \in M$, either $\rho_i(x, y) = d_i(x, y)$ or $h(x, y) = d_i(x, y)$ then the leftmost inequality also holds. The remaining case is when $\rho_i(x, y) = 2r$ and $h(x, y) = d_{i+1}(x, y)$, where $i + 1 = 1$ if $i = 0$, and $i + 1 = 0$ if $i = 1$. In this case, by (4.3),

$$d_i(x, y) \leq 2r + d_{i+1}(x, y),$$

which proves (4.5).

Since $c_1(M, d_i) \leq D_i$, for $i = 0, 1$, by Corollary 2.3 there exist maps $\varphi_i : M \to L_1$ such that for all $x, y \in M$

$$\min\{d_i(x, y), 2r\} \leq \|\varphi_i(x) - \varphi_i(y)\| \leq \frac{eD_i}{e - 1} \min\{d_i(x, y), 2r\}.$$ 

Let $m_0 \in M$ be a fixed element of $M$. We define an embedding of $(M \times \mathbb{F}_2, d)$ into $L_1 \oplus L_1 \oplus L_1 \oplus \mathbb{R}$ by

$$G(x, 0) = (\varphi_0(x), \varphi_1(m_0), \psi(x), r)$$

$$G(x, 1) = (\varphi_0(m_0), \varphi_1(x), \psi(x), 0).$$

We have

$$\|G(x, 0) - G(y, 0)\| = \|\varphi_0(x) - \varphi_0(y)\| + \|\varphi_1(m_0) - \varphi_1(m_0)\| + \|\psi(x) - \psi(y)\| + |r - r|$$

$$\leq \rho_0(x, y) + h(x, y)$$

$$\leq d_0(x, y).$$
\[ \|G(x,1) - G(y,1)\| = \|\varphi_0(m_0) - \varphi_0(m_0)\| + \|\varphi_1(x) - \varphi_1(y)\| + \|\psi(x) - \psi(y)\| + |0| \leq \rho_1(x,y) + h(x,y) \leq d_1(x,y). \]

By Lemma 3.4(b) we have
\[ d((x,0), (y,1)) \leq r + h(x,y). \]

We need to compare this with
\[ \|G(x,0) - G(y,1)\| = \|\varphi_0(x) - \varphi_0(m_0)\| + \|\varphi_1(m_0) - \varphi_1(y)\| + \|\psi(x) - \psi(y)\| + |r - 0| \leq \rho_0(x,m_0) + \rho_1(m_0,y) + h(x,y) + r \leq r + h(x,y). \]

As an application of Theorem 4.1, we obtain the \( L_1 \)-embeddability of twisted unions that are described in Example 3.2 and includes, in particular, the twisted union described in Example 3.1. Thus Corollary 4.3 answers Problem 1.10 in the negative (see also Corollary 5.3 below, for different proofs that the twisted union from Example 3.1 is \( L_1 \)-embeddable).

**Corollary 4.3.** Let \( M \) be subset of \( L_1 \), \( r > 0 \) be a constant, \( \varpi_0 \) and \( \varpi_1 \) be concave non-decreasing continuous functions on \([0, \infty)\) vanishing only at 0, such that for all \( t > 0 \), \( |\varpi_0(t) - \varpi_1(t)| \leq 2r \), that is (3.3) is satisfied for the metrics \( d_i(x,y) = \varpi_i(||x - y||) \), \( i = 0,1 \).

Then the \( r \)-twisted union \((M \times \mathbb{F}_2, d)\) of \((M, \varpi_0(||x - y||))\) and \((M, \varpi_1(||x - y||))\) embeds into \( L_1 \) with distortion \( D \) bounded by an absolute constant \((D \leq 1 + 2.776\Delta < 26.6)\).

**Proof.** Define for all \( t \geq 0 \)
\[ \varpi_2(t) = \min\{\varpi_0(t), \varpi_1(t)\}. \]

Then \( \varpi_2 \) is a concave non-decreasing continuous function on \([0, \infty)\) vanishing only at 0. Thus, by Corollary 2.2 for \( i = 0, 1, 2 \), \( j = 0, 1 \), the spaces \((M, \varpi_i(||x - y||))\) and \((M, \rho_j)\) embed into \( L_1 \) with distortions bounded above by \( \Delta \leq (2\sqrt{2} + 3)e/(e - 1) \). Thus (4.1) is satisfied and the corollary follows by Theorem 4.1.

To obtain the estimate in the parenthesis we use (3.14) and the maps \( \varphi_j : M \to L_1 \) for \( j = 0,1 \), such that for all \( x,y \in M \)
\[ 0.694 \cdot \rho_j(x,y) \leq \|\varphi_j(x) - \varphi_j(y)\| \leq 0.694\Delta \rho_j(x,y). \]

The computation is easy but a little tedious. We leave it to an interested reader. \( \square \)

### 5 On \( L_1 \)-embeddability of generalized twisted unions

In this section we present two general results (Theorems 5.1 and 5.2) on \( L_1 \)-embeddability of generalized twisted unions which satisfy different natural restrictions, described in Lemmas 3.4 and 3.6 for which we obtained an equivalent formula for the twisted union distance between every pair of points of the union. As an application we obtain another proof, different from the one in Section 4, that the twisted union described in Example 3.1 embeds into \( L_1 \) with bounded distortion, see Corollary 5.3.
\textbf{Theorem 5.1.} Let $M$ be a metric space endowed with two metrics $d_0$ and $d_1$, and let $r : M \to (0, \infty)$ be a function such that (3.4) and (3.5) are satisfied.

Suppose that the function $r(\cdot)$ is Lipschitz with respect to both metrics $d_0$ and $d_1$, that is, suppose that there exists a constant $L > 0$, such that for all $x, y \in M$

$$|r(x) - r(y)| \leq L \min\{d_0(x,y), d_1(x,y)\}. \quad (5.1)$$

Denote

$$h(x,y) \triangleq \min\{d_0(x,y), d_1(x,y)\},$$

and, for $i = 0, 1$,

$$g_i(x,y) \triangleq \min\{d_i(x,y), r(x) + r(y)\}.$$

If there exist constants $C_0, C_1, C_2$ such that for $i = 0, 1$,

$$c_1(M, g_i) \leq C_i \quad \text{and} \quad c_1(M, h) \leq C_2,$$

then the generalized twisted union $(M \times \mathbb{F}_2, d)$ of $(M, d_0)$ and $(M, d_1)$ with the joining function $r(\cdot)$ embeds into $L_1$ with distortion bounded above by a constant which depends only on $C_0, C_1, C_2$, and $L$.

\textbf{Proof.} First we observe, that for $i = 0, 1$, and all $x, y \in M$ we have

$$d_i(x,y) \leq g_i(x,y) + h(x,y) \leq 2d_i(x,y). \quad (5.2)$$

Indeed, by the definitions of $h$ and $g_i$, the rightmost inequality is clear. If for some $x, y \in M$, either $g_i(x,y) = d_i(x,y)$ or $h(x,y) = d_i(x,y)$ then the leftmost inequality also holds. The remaining case is when $g_i(x,y) = r(x) + r(y) < d_i(x,y)$ and $h(x,y) = d_{i+1}(x,y) < d_i(x,y)$, where $i + 1 = 1$ if $i = 0$, and $i + 1 = 0$ if $i = 1$. In this case, by (3.4),

$$d_i(x,y) \leq d_{i+1}(x,y) + r(x) + r(y),$$

which proves (5.2).

Denote by $\psi, \varphi_0, \varphi_1$, bi-Lipschitz embeddings into $L_1$ of $(M, h)$, $(M, g_0)$, and $(M, g_1)$, respectively.

Let $m_0 \in M$ be such that

$$r(m_0) = \min_{x \in M} r(x). \quad (5.3)$$

We define an embedding of $(M \times \mathbb{F}_2, d)$ into $L_1 \oplus_1 L_1 \oplus_1 L_1 \oplus_1 \mathbb{R}$ by

$$G(x, 0) = (\varphi_0(x), \varphi_1(m_0), \psi(x), r(x))$$

$$G(x, 1) = (\varphi_0(m_0), \varphi_1(x), \psi(x), 0).$$

We have

$$\|G(x,0) - G(y,0)\| = \|\varphi_0(x) - \varphi_0(y)\| + \|\varphi_1(m_0) - \varphi_1(m_0)\| + \|\psi(x) - \psi(y)\| + |r(x) - r(y)| \quad (5.4)$$

$$\overset{3.4}{\asymp} g_0(x,y) + 0 + h(x,y) \overset{5.2}{\asymp} d_0(x,y).$$

$$\|G(x,1) - G(y,1)\| = \|\varphi_0(m_0) - \varphi_0(m_0)\| + \|\varphi_1(x) - \varphi_1(y)\| + \|\psi(x) - \psi(y)\| + |0|$$

$$\overset{3.4}{\asymp} 0 + g_1(x,y) + h(x,y) + 0 \overset{3.4}{\asymp} d_1(x,y).$$
By Lemma 3.4(a) we have
\[ d((x,0),(y,1)) \asymp h(x,y) + \max\{r(x),r(y)\}. \]

We need to compare this with
\[
\|G(x,0) - G(y,1)\| = \|\varphi_0(x) - \varphi_0(m_0)\| + \|\varphi_1(m_0) - \varphi_1(y)\| + \|\psi(x) - \psi(y)\| + |r(x) - 0| \\
\asymp g_0(x,m_0) + g_1(m_0,y) + h(x,y) + r(x) \\
\asymp \min\{d_0(x,m_0),r(x) + r(m_0)\} + \min\{d_1(y,m_0),r(y) + r(m_0)\} \\
+ h(x,y) + r(x) \\
\asymp h(x,y) + \max\{r(x),r(y)\}. 
\]

\[ \square \]

**Theorem 5.2.** Let \( M \) be a metric space endowed with two metrics \( d_0 \) and \( d_1 \), and let \( r : M \to (0,\infty) \) be a function such that (3.1) and (3.5) are satisfied.

Suppose that there exists a function \( f : M \times M \to [0,\infty) \) and constants \( C_1, C_2, C_3, C_4 > 0 \) such that \( c_1(M,d_1) \leq C_1, c_1(M,\min\{f(x,y),r(x) + r(y)\}) \leq C_2, \) and for all \( x,y \in M \)
\[
\frac{1}{C_3}d_0(x,y) \leq d_1(x,y) + f(x,y) \leq C_4d_0(x,y). \tag{5.4}
\]

Then the generalized twisted union \((M \times \mathbb{F}_2, d)\) of \((M, d_0)\) and \((M, d_1)\) with the joining function \( r(\cdot) \) embeds into \( L_1 \) with distortion bounded above by a constant which depends only on \( C_1, C_2, C_3, \) and \( C_4 \).

**Proof.** Similarly as in the proof of (5.2), by a straightforward analysis of two cases, the conditions (3.1) and (3.4) imply that for all \( x,y \in M \)
\[
\frac{1}{C_3}d_0(x,y) \leq d_1(x,y) + \min\{f(x,y),r(x) + r(y)\} \leq C_4d_0(x,y). \tag{5.5}
\]

Let \( \varphi_1 \) and \( \psi \) be bi-Lipschitz embeddings into \( L_1 \) of \((M, d_1)\) and \((M, \min\{f(x,y),r(x) + r(y)\})\), respectively. Let \( m_0 \in M \) be such that
\[
r(m_0) = \min_{x \in M} r(x). \tag{5.6}
\]

We define the embedding of \((M \times \mathbb{F}_2, d)\) into \( L_1 \oplus L_1 \oplus \mathbb{R} \) by
\[
F(x,0) = (\varphi_1(x),\psi(x),r(x)) \\
F(x,1) = (\varphi_1(x),\psi(m),0). \tag{5.7}
\]

Then we have
\[
\|F(x,0) - F(y,0)\| = \|\varphi_1(x) - \varphi_1(y)\| + \|\psi(x) - \psi(y)\| + |r(x) - r(y)| \\
\asymp d_1(x,y) + \min\{f(x,y),r(x) + r(y)\} + |r(x) - r(y)| \\
\asymp d_0(x,y). 
\]

\[
\|F(x,1) - F(y,1)\| = \|\varphi_1(x) - \varphi_1(y)\| + \|\psi(m_0) - \psi(m_0)\| + |0| \asymp d_1(x,y).
\]
The condition (5.4) implies that for all \( x, y \in M \), \( d_1(x, y) \leq C_4d_0(x, y) \), and therefore, by Lemma 3.6 we have
\[
d((x, 0), (y, 1)) \leq d_1(x, y) + r(x).
\]

We need to compare this with
\[
\|F(x, 0) - F(y, 1)\| = \|\varphi_1(x) - \varphi_1(y)\| + \|\psi(x) - \psi(m_0)\| + |r(x)|
\leq d_1(x, y) + \min\{f(x, m_0), r(x) + r(m_0)\} + r(x)
\leq d_1(x, y) + r(x).
\]

Thus, since \( c_1(M, d_1) \leq D_1 \), we have
\[
\min\left\{1, 1/C\right\}d_1(x, y) \leq \min\{d_0(x, y), d_1(x, y)\} \leq d_1(x, y).
\]

Thus, since \( c_1(M, d_1) \leq D_1 \), we have that \( c_1(M, \min\{d_0(x, y), d_1(x, y)\}) \leq \infty \) and (4.1) is satisfied, so Corollary 5.3 follows from Theorem 4.1.

Moreover, when the function \( r(\cdot) \) is constant, then the functions \( g_i \), for \( i = 0, 1 \), defined in Theorem 5.1 are \( g_i(x, y) = \min\{d_i(x, y), 2r\} \), and thus, by Corollary 2.3 \( c_1(M, g_i) \leq eD_1/(e - 1) \). Hence Corollary 5.3 follows from Theorem 5.1.

Similarly, if, in the notation of Theorem 5.2, we define the function \( f \) to be equal to \( d_0 \). Then, by (5.8), the inequality (5.5) is satisfied with \( C_3 = 1 \) and \( C_4 = C + 1 \). Since \( c_1(M, d_0) \leq D_0 \), by Corollary 2.3 \( c_1(M, \min\{d_0(x, y), 2r\}) \leq eD_0/(e - 1) \). Thus, by Theorem 5.2, Corollary 5.3 is satisfied.

6 Lower bound on distortion

The goal of this section is to show that the lower bound on distortion of the union which was found in [18, Theorem 1.2 and Section 3] for Hilbert space is also valid for \( L_1 \) and many other Banach and metric spaces. Also, in some sense, our proof is simpler.

Definition 6.1. A metric space \((X, d)\) is called stable if for any two bounded sequences \( \{x_n\} \) and \( \{y_m\} \) in \( X \) and for any two free ultrafilters \( \mathcal{U} \) and \( \mathcal{V} \) on \( \mathbb{N} \)
\[
\lim_{n, \mathcal{U}} \lim_{m, \mathcal{V}} d(x_n, y_m) = \lim_{m, \mathcal{V}} \lim_{n, \mathcal{U}} d(x_n, y_m).
\]
This notion was introduced in the context of Banach spaces by Krivine and Maurey [15]. In the context of metric spaces this definition was introduced, in a slightly different, but equivalent form, in [10, p. 126]. (See [11] for an account on stable Banach spaces.)

To put our example into context we recall two simple well-known observations.

**Observation 6.2** ([15]). Hilbert space is stable.

**Proof.**

\[
\lim_{n \to \infty} \lim_{m \to \infty} \|x_n - y_m\|^2 = \lim_{n \to \infty} \|x_n\|^2 - 2 \lim_{n \to \infty} \lim_{m \to \infty} \langle x_n, y_m \rangle = \lim_{m \to \infty} \lim_{n \to \infty} \|x_n - y_m\|^2. \]

**Observation 6.3** ([27]). The space \(L_1(\mathbb{R})\) with the metric \(\|x - y\|_1^2\) is isometric to a subset of Hilbert space.

**Proof.** ([22]) We define a map \(T : L_1(\mathbb{R}) \to L_\infty(\mathbb{R} \times \mathbb{R})\) by:

\[
T(f)(t, s) \overset{\text{def}}{=} \begin{cases} 
1 & \text{if } 0 < s \leq f(t), \\
-1 & \text{if } f(t) < s < 0, \\
0 & \text{otherwise.}
\end{cases}
\]

For all \(f, g \in L_1(\mathbb{R})\) we have:

\[
|T(f)(t, s) - T(g)(t, s)| = \begin{cases} 
1 & \text{if } g(t) < s \leq f(t) \text{ or } f(t) < s \leq g(t), \\
0 & \text{otherwise.}
\end{cases}
\]

Therefore

\[
\|T(f) - T(g)\|_{L_2(\mathbb{R} \times \mathbb{R})}^2 = \int_{\mathbb{R}} \left( \int_{(g(t), f(t)) \text{ or } (f(t), g(t))} 1 \, ds \right) dt = \int_{\mathbb{R}} |f(t) - g(t)| dt
\]

\[
= \|f - g\|_{L_1(\mathbb{R})}. \]

**Corollary 6.4** ([15]). The space \(L_1\) is stable.

**Example 6.5.** Consider the disjoint union of two copies of \(\mathbb{N}\):

\[
\{1, 2, \ldots, \bar{n}, \ldots\} \cup \{\bar{1}, \bar{2}, \ldots, \bar{n}, \ldots\}.
\]

Endow this union with the following graph structure: Each \(i\) is adjacent to \(\bar{j}\) if and only if \(j \geq i\), and there are no other edges. Then

\[
\lim_{i \to \infty} d(j, \bar{i}) = 1 \quad \text{and} \quad \lim_{j \to \infty} d(j, \bar{i}) = 3.
\]

Observe that \(d(i, \bar{j}) = 2\) and \(d(\bar{i}, j) = 2\) for all \(i \neq j\).

Therefore both copies of \(\mathbb{N}\) are equilateral and thus embed isometrically into \(\ell_1\). On the other hand, since by Corollary 6.4 \(L_1\) is stable, the distortion of any embedding of the set constructed in Example 6.5 into \(L_1\) is at least 3.

Of course the same example can be used for any stable metric space containing an isometric copy of a countable equilateral set. Known theory [10, 11] implies that, for example, spaces \(L_p\) for \(1 \leq p < \infty\) satisfy this condition.

Finally we would like to mention that the distortion 3 in Example 6.5 cannot be increased using the same idea. Namely we prove:
Proposition 6.6. Let \( \{x_n\} \) and \( \{y_m\} \) be two bounded sequences in a metric space \( X \). Then
\[
\lim_{n, \mathcal{U}} \lim_{m, \mathcal{V}} d(x_n, y_m) \leq 3 \lim_{m, \mathcal{U}} \lim_{n, \mathcal{V}} d(x_n, y_m)
\]
for any free ultrafilters \( \mathcal{U} \) and \( \mathcal{V} \) on \( \mathbb{N} \).

Proof. Let \( d_{nm} = d(x_n, y_m) \). Passing to subsequences we may assume that the following limits exist: \( \lim_{n \to \infty} d_{nm} = S_m \), \( \lim_{m \to \infty} S_m = S \), \( \lim_{m \to \infty} d_{nm} = L_n \), \( \lim_{n \to \infty} L_n = L \), where \( S = \lim_{m, \mathcal{V}} \lim_{n, \mathcal{U}} d(x_n, y_m) \) and \( L = \lim_{n, \mathcal{V}} \lim_{m, \mathcal{U}} d(x_n, y_m) \).

We need to show \( S \leq 3L \).

Given \( \varepsilon > 0 \), let \( M \in \mathbb{N} \) be such that \( S_m > S - \varepsilon \) for all \( m \geq M \) and let \( N \in \mathbb{N} \) be such that \( L_n < L + \varepsilon \) for all \( n \geq N \).

Let \( m_N \in \mathbb{N} \) be such that \( m_N \geq M \) and \( d(x_N, y_{m_N}) < L + \varepsilon \).

Let \( n_M \in \mathbb{N} \) be such that \( n_M \geq N \) and \( d(x_{n_M}, y_{m_N}) > S - \varepsilon \).

Finally let \( f \in \mathbb{N} \) be such that \( d(x_N, y_f) < L + \varepsilon \) and \( d(x_{n_M}, y_f) < L + \varepsilon \).

Using the triangle inequality we get
\[
S - \varepsilon < d(x_{n_M}, y_{m_N}) \leq d(x_{n_M}, y_f) + d(y_f, x_N) + d(x_N, y_{m_N}) < 3(L + \varepsilon).
\]

Since \( \varepsilon > 0 \) is arbitrary, we get \( S \leq 3L \). \( \square \)

Acknowledgements. We thank Alexandros Eskenazis for useful information related to the problems studied in this paper and Gilles Godefroy for his interest in this research. The first named author gratefully acknowledges the support by the National Science Foundation grants NSF DMS-1700176 and DMS-1953773, and the hospitality of the Department of Mathematics of Miami University where this research started.

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