Lecture I: Phase Transitions in the Early Universe

INTRODUCTION

If one calculates the equilibrium properties of the Standard Model at high temperatures, one finds several different phases. Below 100 MeV, the elementary particles behave more or less as at zero temperature. If we heat them up above this temperature, there is a transition to a deconfined phase in which quarks and gluons are not confined into hadrons. At 100 GeV, there is another transition at which the weak and electromagnetic interactions unify. Extensions of the Standard Model, such as Grand Unified Theories (GUTs), typically predict new phases at even higher temperatures.

This phase structure is important in cosmology because the temperature of the universe was initially very high. It seems likely that had to be higher than 100 GeV, because otherwise it is extremely difficult to explain the baryon asymmetry of the universe — the fact that there is so much matter in the universe, but practically no antimatter. As the universe expanded and cooled down to the current 2.7 K, it must have therefore undergone at least the deconfinement and electroweak phase transitions mentioned above. Typical inflationary models predict much higher temperatures, usually at least $10^{10}$ GeV, and it is therefore likely that there were also other phase transitions.

The cosmological consequences of these phase transitions depend sensitively on their properties. In many cases the consequences would be disastrous, and one can then rule out the theory that predicts the transition, or at least constrain its parameters. Examples of this are grand unified theories with strongly first-order phase transitions, as they would lead to a period of "old inflation", and to a very inhomogeneous universe, or theories that predict massive domain walls or too many magnetic monopoles. In other cases, the effects of the phase transition may be subtle, only barely observable, such as gravitational waves, or even completely unobservable. Or, it may be that the consequences are absolutely crucial for our own existence, as in the theory of electroweak baryogenesis, which uses the electroweak phase transition to explain the baryon asymmetry.

In this lecture, I will concentrate on those phase transitions that can be understood as a spontaneous breakdown of either a global or a local symmetry. This covers the electroweak and GUT transitions, and to a certain extent also the deconfinement transition. I will also point out similarities to phase transitions in condensed matter systems such as superfluids and superconductors. Unlike cosmological phase transitions, they can be studied experimentally and can therefore be used to test our theoretical understanding of phase transitions in quantum field theories.

DEFINITIONS

The properties of a thermodynamic system in thermal equilibrium are given by the partition function $Z$ (see Ref. [1]), which can be generally written as

$$Z = \text{Tr} \rho = \text{Tr} e^{-\beta \hat{H}},$$  \hspace{1cm} (1)

where $\hat{H}$ is the Hamiltonian of the system and $\beta = 1/kT$ is the inverse temperature.

In many cases, it is more convenient to use its logarithm, which defines the free energy $F = -T \ln Z$. This quantity has the property that its value is minimised in thermal equilibrium.

Neither the partition function nor the free energy are observable quantities. However, any observable can be expressed as the derivative of $F$ with respect to some external field. For instance, the expectation value of the...
FIG. 1: The phase diagram of water (greatly simplified). The lines are first-order phase transitions. The transition line between water and vapour ends at a second-order critical point.

operator $\hat{X}$ is defined as

$$\langle \hat{X} \rangle = Z^{-1} \text{Tr} \, \hat{X} \hat{\rho},$$

and if we define an external field $J_X$ by

$$Z(J_X) = \text{Tr} \, e^{-\beta \hat{H} + J_X \hat{X}},$$

we find

$$\langle \hat{X} \rangle = \frac{\partial}{\partial J_X} \ln Z = -\frac{1}{T} \frac{\partial F}{\partial J_X}.$$  \hspace{1cm} (4)

A phase transition is defined as a point at which the free energy (or, equivalently, the partition function) is non-analytic. In general, it follows that at least some observables are non-analytic at the transition point and act as order parameters.

The simplest type of phase transitions are first-order transitions, in which some expectation value, say $\langle \hat{X} \rangle$, is discontinuous. An everyday example of this is the transition between water and ice. Let us assume that the transition takes place at temperature $T_c$. Because of the discontinuity, the state of the system at $T_c$ depends on which side the critical temperature was approached from. Consequently, it is possible to have two phases coexisting at $T_c$, which is exactly what happens when we put ice into a glass of water. As the temperature is gradually changed through $T_c$, the discontinuous jump from one value of $\langle \hat{X} \rangle$ to the other does not necessarily take place instantaneously at $T_c$. For instance, water can remain liquid in a supercooled state at temperatures below 0°C for a relatively long time, if it is pure enough and completely unperturbed. This happens in a meteorological condition known as freezing rain, in which the supercooled water freezes as soon as it hits the ground, causing hazardous driving conditions. More generally, this phenomenon is known as metastability.

Second-order phase transitions are those in which every expectation value is continuous, but some expectation value $\langle \hat{X} \rangle$ has a discontinuous derivative. This is a limiting case of a first-order transition in which the discontinuity is taken to zero, and therefore it typically requires fine tuning of some parameter. On a phase diagram, second order transitions are typically end points of first order transition lines. One example of this is the critical point of water at which the first-order transition line between water and vapour ends. To reach this point, one has to tune both the temperature and the pressure to $T = 647.096$ K and $p = 22.064$ MPa. In contrast, reaching a first order transition point, one only has to tune the temperature provided that the pressure is below the critical value.

There is, however, an exception to this rule. There are cases in which some symmetry forces the system to be on the first-order transition line. This happens, for instance, in a ferromagnet. If the temperature is below the Curie temperature, the magnetisation $\vec{M}$ becomes non-zero even if there is no external magnetic field $\vec{H}$. Because the setting is rotation invariant, no direction of $\vec{M}$ is preferred. Therefore, the direction of $\vec{M}$, which depends on how the external field was taken to zero, breaks the rotation invariance. This is known as spontaneous symmetry breaking.

If we imagine changing the external field $\vec{H}$ gradually through zero, we see that $\vec{M}$ is discontinuous at $\vec{H} = 0$. In other words, there is a first-order transition at $\vec{H} = 0$. If we increase the temperature keeping $\vec{H}$ zero, we are therefore guaranteed to reach the second-order end point at the Curie temperature.
The above situation is very important in field theories, because they typically have a high degree of symmetry. As in the case of the ferromagnet, it often happens at low temperatures that the state of the system is not invariant under a symmetry of the theory and this typically leads to a second-order phase transition as the temperature is increased.

GLOBAL SYMMETRY BREAKING IN FIELD THEORIES

As a simple example of a field theory with spontaneous symmetry breaking, let us consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2,$$

where $\phi$ is a real scalar field. The Lagrangian is symmetric under $\phi \rightarrow -\phi$, which correspond to the group $\mathbb{Z}_2$.

The vacuum of the system is found by solving the Euler-Lagrange equations for a constant field,

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0,$$

which gives

$$\lambda (\phi^3 - v^2 \phi) = 0.$$  \hspace{1cm} (7)

In addition to $\phi = 0$, which is an unstable solution, we find two solutions, $\phi = \pm v$, which are the possible vacuum states of the system. Neither choice is invariant under the $\mathbb{Z}_2$ symmetry, and therefore the symmetry is spontaneously broken in the vacuum.

At a non-zero temperature, the value of $\phi$ fluctuates from point to point. If these thermal fluctuations are small, we can write the field as the sum of the homogeneous background contribution $\phi_0$ and the fluctuation $\delta \phi$,

$$\phi(\vec{x}) = \phi_0 + \delta \phi(\vec{x}).$$  \hspace{1cm} (8)

We assume that these fluctuations are Gaussian, which is equivalent to the one-loop approximation in perturbation theory, i.e., in the expansion in terms of Feynman diagrams. In many cases, this is a good approximation. The homogeneous mode $\phi_0$ is chosen in such a way that the mean value of the fluctuation in zero, $\langle \delta \phi \rangle = 0$. The variance, however, is non-zero, and has the value

$$\langle \delta \phi^2 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega} e^{ip\omega} + (\text{zero-temperature part}),$$  \hspace{1cm} (9)

where $\omega = (\vec{p}^2 + m^2)^{1/2}$. Here $m$ is the mass of the field in the vacuum around which the expansion is done, and is given by the second derivative of the potential,

$$m^2 = \frac{\partial^2 V}{\partial \phi^2}.$$  \hspace{1cm} (10)

The zero-temperature part is actually divergent, but we do not have to worry about that, because it is already absorbed in the definition of the renormalised zero-temperature couplings. For $T \gg m$, Eq. (11) can be expanded as

$$\langle \delta \phi^2 \rangle = \frac{T^2}{12} - \frac{mT}{4\pi},$$  \hspace{1cm} (11)

When the fluctuations $\delta \phi$ are taken into account, the homogeneous more $\phi_0$ feels an effective potential that is different from the “tree-level” potential in Eq. (5). This is easiest to see by averaging Eq. (7) over fluctuations $\delta \phi$,

$$0 = \lambda (\langle \phi^3 \rangle - v^2 \langle \phi \rangle) = \lambda (\phi^2_0 + 3 \langle \delta \phi^2 \rangle \phi_0 - v^2 \phi_0),$$  \hspace{1cm} (12)

where we have used the fact that the fluctuations are symmetric and odd moments must therefore vanish. If we now use the leading term from Eq. (11), $\langle \delta \phi^2 \rangle \approx T^2/12$, we find

$$\lambda \left( \phi^2_0 + \frac{T^2}{4} - v^2 \right) \phi_0 = 0.$$  \hspace{1cm} (13)

This shows that the two non-zero minima only exist below a critical temperature $T_c = 2v$, and above that, the symmetry is restored.

One can easily generalise this discussion to somewhat more complicated cases. If $\phi_i$ is a multicomponent real scalar field with $i = 1, \ldots, N$, we can consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{\lambda}{4} (\phi_i^4 - v_i^2)^2,$$

where summation over the indices $i$ is assumed. This Lagrangian is invariant under SO($N$) rotations of the field $\phi_i$. For two components, we can define a complex scalar field $\Phi = (\phi_1 + i \phi_2)$, and we have

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^* \partial^\mu \phi - \frac{\lambda}{4} (\phi^* \phi - v^2)^2.$$  \hspace{1cm} (15)

This Lagrangian is invariant under rotations of the complex phase angle of the scalar field $\phi$, which means that it has a U(1) symmetry. Because of this symmetry, we can choose that the vacuum expectation value is real, and expand $\phi_1 = \phi_0 + \delta \phi_1$ and $\phi_2 = \delta \phi_2$. The variance of both $\delta \phi_1$ and $\delta \phi_2$ are given by Eq. (14). The averaged Euler-Lagrange equation for the real part is

$$0 = \frac{\langle \partial \mathcal{L} \rangle}{\partial \phi_1} = \lambda \left( \langle \phi_1^3 \rangle + \langle \phi_2^3 \rangle \phi_1 - v^2 \langle \phi_1 \rangle \right) = \lambda (\phi^2_0 + 3 \langle \delta \phi^2_1 \rangle + \langle \delta \phi^2_2 \rangle - v^2) \phi_0.$$  \hspace{1cm} (16)

Using Eq. (14), we find

$$\lambda \left( \phi^2_0 + \frac{T^2}{3} - v^2 \right) \phi_0 = 0,$$  \hspace{1cm} (17)
| $T_c$  | theory  | symmetry                          | order parameter |
|--------|---------|----------------------------------|-----------------|
| $10^{16}$ GeV | GUT     | SU(5)$\rightarrow$SU(3)$\times$SU(2)$\times$U(1) | $\Phi$, 24 real components |
| 100 GeV | electroweak | $\rightarrow$SU(3)$\times$U(1) | $\bar{\phi}$, 2 complex components |
| 100 MeV | QCD     | $"\rightarrow U(1)"$             | $\bar{q}q$ composite |

TABLE I: The phase transitions predicted by particle physics models.

showing that this time the critical temperature is $T_c = \sqrt{3}\nu$, but otherwise the conclusions are unchanged.

The restoration of broken symmetries at high temperatures is important for the early universe, because the universe was initially extremely hot and because the theories of particle physics contain several symmetries that are spontaneously broken at zero temperature. As shown in Table I these are associated with the hypothetical Grand Unified Theory (GUT), the electroweak theory and quantum chromodynamics (QCD).

The QCD deconfinement phase transition takes place at around 100 MeV, and is the only one of the three transitions that can be studied experimentally. The transition cannot be fully understood within perturbation theory, and it can be characterised in two different ways, either as a transition from a confining to a deconfined phase, or as spontaneous breakdown of the chiral symmetry. We will adopt the latter point of view.

If the up and down quarks were massless, the left and right handed quarks would decouple. One could therefore carry out independent rotations between the up and down quarks of the given handedness. In the QCD vacuum, this symmetry is spontaneously broken by the quark condensate $(\bar{q}q)$, which shows how the left and right handed quarks are aligned in the flavour space. Just like in a ferromagnet, one would expect the symmetry to be restored at high temperatures, and this seems to imply a second-order phase transition.

However, the quarks are not actually massless and therefore the chiral symmetry is not exact. The quark mass term plays the same role as the magnetic field $\vec{H}$ in the ferromagnet phase diagram in Fig. 2 and shifts the system a little bit away from the first-order transition line. Because there is no symmetry in the first place, there is no second order transition either. A first-order transition is still possible if the phase diagram is more complicated than that in Fig. 2.

The GUT and electroweak phase transitions, on the other hand, involve exact symmetries, but local gauge symmetries instead of global ones. Therefore, the above discussion does not apply them either, as we shall see in the next section.

GAUGE SYMMETRY BREAKING

The Lagrangians discussed above had global symmetries. One has to carry out the same rotation at every point in order for the Lagrangian to remain unchanged.

It is possible to modify the theory in such a way that it is invariant under position-dependent rotations as well. This is done by introducing a gauge field $A_\mu$, and the local symmetries are therefore also known as gauge symmetries. For the theory with a complex scalar in Eq. [15], the modified Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu \phi^* D^\mu \phi - \frac{\lambda}{4} (\phi^* \phi - v^2)^2, \quad (18)$$

where we have used the covariant derivative $D_\mu = \partial_\mu + ig A_\mu$ and the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This model is essentially a relativistic version of the Ginzburg-Landau theory of superconductivity. Our results will therefore apply to superconductors on a qualitative level. The quantitative results differ, though.

The gauge coupling constant $g$ is simply the electric charge of the field $\phi$. (For superconductors, $g = -2e$, because a Cooper pair consists of two electrons.) The Lagrangian is invariant under local (position-dependent) rotations of the phase angle

$$\phi \rightarrow e^{i\alpha} \phi, \quad A_\mu \rightarrow A_\mu - \frac{1}{g} \partial_\mu \alpha. \quad (19)$$

Again, we write $\phi = \phi_0 + \delta \phi$, where we now choose $\phi_0$ to be real and positive. The gauge field $A_\mu$ does not have a homogeneous mode, and therefore we can treat is as a pure fluctuation. Again, we average the Euler-Lagrange equation for the real part $\phi_1$ over fluctuations $\delta \phi_1$, $\delta \phi_2$ and $A_\mu$,

$$0 = \left\langle \frac{\partial \mathcal{L}}{\partial \dot{\phi}_1} \right\rangle = \lambda \left( \langle \dot{\phi}_1^2 \rangle + \langle \delta \phi_1^2 \rangle - v^2 \langle \phi_1 \rangle \right) + g^2 \langle A_\mu A^\mu \phi_1 \rangle \quad (21)$$

The gauge field contains three physical degrees of freedom, and therefore we have

$$\langle A_\mu A^\mu \rangle = 3\langle \delta \phi^2 \rangle = \frac{T^2}{4} - \frac{3m_\tau T}{4\pi} + (T = 0 \text{ part}). \quad \quad \quad (22)$$

Crucially, however, the mass $m_\tau$ of the gauge field depends on $\phi_0$ because of the Higgs mechanism, and is $m_\tau = g \phi_0$. Including this contribution, we find

$$\lambda \left[ \phi_0^2 - \frac{3}{4\pi} \frac{g^3}{\lambda} T \phi_0 + \frac{1}{3} \left( \frac{g^2}{4\lambda} \right) T^2 - v^2 \right] \phi_0 = 0. \quad (22a)$$
To interpret this results, we define the effective potential as the tree-level potential that would give this same Euler-Lagrange equation. That is obtained by simply integrating the left hand side of the equation,

\[ V_{\text{eff}}(\phi_0) = \frac{1}{2} \left[ \left( \frac{\lambda}{3} + \frac{g^2}{4} \right) T^2 - \lambda v^2 \right] \phi_0^2 - \frac{g^3 T}{4 \pi} \phi_0^3 + \frac{\lambda}{4} \phi_0^4. \]  

(23)

The presence of the cubic term means that as long as the coefficient of the quadratic term is positive and small enough, there are two non-negative minima, one at \( \phi_0 = 0 \) and the other at a non-zero value. The two minima are degenerate at a certain critical value of \( T_c \),

\[ T_c = \sqrt{\frac{12 \lambda v^2}{4 \lambda + 3 g^2 - (3/2 \pi^2)(g^6/\lambda)}}. \]

(24)

This corresponds to a first-order transition line. Thus we see that due to thermal fluctuations, we have a first-order transition instead of a second-order one.\(^1\) The discontinuity in \( \phi_0 \) is

\[ \Delta \phi_0 = \frac{1}{2 \pi} \frac{g^3}{\lambda} T_c. \]

(25)

It is also straightforward to calculate the range of temperatures in which a second, metastable minimum exists,

\[ \frac{12 \lambda v^2}{4 \lambda + 3 g^2} < T^2 < \frac{12 \lambda v^2}{4 \lambda + 3 g^2 - (27/16 \pi^2)(g^6/\lambda)} \]

(26)

The critical temperature \( T_c \), together with the metastability range is shown in Fig. 3 for \( g = 0.5 \). Fig. 4 shows the shape of the effective potential at \( T_c \) and at the ends of the metastability range for \( g = 0.5, \lambda = 0.05 \).

\(^1\) Note that this is the reason why Type I superconductors have a first-order transition.

When deriving Eq. (23), we have assumed that the fluctuations are Gaussian. It is important to check whether this approximation, which is equivalent to a one-loop calculation in terms of Feynman diagrams, is consistent. At higher orders in the loop expansion, one finds that each loop gives typically a contribution of the order \( g^2 T/m_\gamma \). At \( T_c \), the photon mass in the broken phase is

\[ m_\gamma = g \Delta \phi_0 = \frac{1}{2 \pi} \frac{g^4}{\lambda} T_c, \]

(27)

and therefore each loop order gives a contribution proportional to \( \lambda/g^2 \). If \( \lambda \ll g^2 \), this is small and means that the contribution from higher loop diagrams can be ignored. One can therefore trust the calculation of the effective potential in Eq. (26). On the other hand, if \( \lambda \gtrsim g^2 \), the higher loop orders are not suppressed relative to the one-loop contribution, and the whole approximation breaks down.

In the electroweak theory the ratio \( \lambda/g^2 \) is essentially given by the ratio of the Higgs and W boson masses. The Higgs boson has not been found and its mass is therefore unknown, but there is an experimental limit that its mass has to be at least 115 GeV. This means that \( \lambda > g^2 \), and the perturbative calculation breaks down. This is also the case in Type II superconductors. In both cases, one needs non-perturbative methods to study the properties of the phase transition.

**NON-PERTURBATIVE RESULTS**

In the global theory, one can be certain that no matter what non-perturbative effects there may be near the transition, the transition must exist, because one can measure the value of \( \left\langle \phi \right\rangle = \phi_0 \). It vanishes in the symmetric phase and is non-zero in the broken phase, and a function that does that cannot be analytic. In other words, \( \left\langle \phi \right\rangle \) acts as an order parameter.

When one considers non-perturbative effects in the gauge theory, one makes the simple but rather striking
Lecture II: Defect Formation

INTRODUCTION

Topological defects are non-linear, time-independent solutions of field theories, which are made stable by the topology of the theory. They have been studied extensively in cosmology [11], condensed matter physics [12] and, at a more theoretical level, in the context of quantum or classical field theory.

In cosmology, magnetic monopoles formed at the GUT phase transition would have disastrous effects later on in the evolution of the universe, and part of the motivation behind the inflationary theory [12] was to wipe them out before that. Cosmic strings, on the other hand, are fully compatible with what is known about cosmology. If they were formed at some early stage, we may well be able to detect them with astronomical observations.

Topological defects also exist in many different condensed matter systems. The ones that are closest to defects of cosmological importance are vortices in superfluids and Abrikosov flux lines in superconductors.

It was pointed out by Kibble in 1976 [14] that topological defects are generally formed at symmetry breaking phase transitions and that we should therefore expect that they were formed in the early universe, too. However, as discussed in Ref. [15], the known cosmological phase transitions are not simple symmetry breaking transitions, but they involve a breakdown of a local gauge symmetry. Nevertheless, as we shall see, the same conclusion still holds: Topological defects are formed.

In recent years, defect formation has been studied extensively in condensed matter systems such as liquid crystals [16, 17, 18], superfluids [19, 20, 21, 22] and superconductors [23, 24, 25, 26]. The first two of these involve global symmetries, and indeed, they seem to confirm Kibble’s theory. Superconductors have a U(1) gauge symmetry and would therefore be closer to the cosmological phase transitions. The superconductor experiments carried out so far have not produced conclusive results, but hopefully this will change soon.
In this lecture, I will first explain how topological defects form in transitions associated with a breakdown of a global symmetry. This is known as the Kibble-Zurek mechanism. Then, I will discuss gauge symmetries and show that the mechanism is actually rather different in that case. For a more comprehensive review, see Refs. 22, 23.

**DISCRETE GLOBAL SYMMETRIES**

Consider first a real scalar field in a potential with a $\mathbb{Z}_2$ symmetry,

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4.$$  \hfill (29)

If $m^2$ is positive, the only minimum is at $\phi = 0$, and the state of the system is symmetries under $\mathbb{Z}_2$. However, for $m^2 < 0$, there are two degenerate minima at $\phi = \pm \sqrt{-m^2/\lambda}$.

If the field is initially exactly zero, it remains zero even if $m^2$ is negative. However, in real systems there are always both quantum and thermal fluctuations. Although the arguments we will use in this section are probably also valid for pure quantum fluctuations, we assume that the temperature is non-zero and the dominant fluctuations are thermal. Therefore, we will not consider any quantum effects.

If either the value of $\phi$ or its velocity is initially non-zero, it determines which minimum the system ends up in. If the initial condition is inhomogeneous, i.e., position-dependent, different parts of the system may end up in different vacua. On the whole, each minimum is equally probable, and therefore the system ends up in a configuration of positive and negative regions, which both cover half of the whole system. The boundaries between these domains are topological defects known as domain walls. The system is in vacuum on both sides of the wall, but because of continuity, the field must leave the vacuum inside the wall to interpolate between the positive and negative values. This means that the wall has positive energy per unit area.

The most obvious question about the domain configuration is whether it has a characteristic length scale. In general, the initial fluctuations have a finite correlation length $\xi$, which is defined by

$$\langle \phi(\vec{x})\phi(\vec{y}) \rangle - \langle \phi(\vec{x}) \rangle \langle \phi(\vec{y}) \rangle \sim e^{-|\vec{x}-\vec{y}|/\xi}.$$  \hfill (30)

This means that if the points $\vec{x}$ and $\vec{y}$ are further away from each other than $\xi$, the initial conditions at these points are independent. If we then assume that the domains form in a short enough time that no signal can travel from $\vec{x}$ to $\vec{y}$ before they have formed, there is a 50% chance that the system will end up in different vacua at these two points, in which case there would have to be (at least) one domain wall between them. Thus, one expects the typical distance between the walls to be roughly $\xi$.

Let us now assume that $m^2$ is initially positive but decreases with time. Initially, the system is in thermal equilibrium in the symmetric phase, but when $m^2$ reaches a critical value, a symmetry breaking phase transition takes place. Because the leading effect of a non-zero temperature is to change the effective mass term [15], we can equally well assume that $m^2$ is constant, but $T$ decreases due to, say, the expansion of the universe or a contact with a colder heat bath.

It is a general property of second-order phase transitions that the correlation length diverges at the transition point. This divergence can be characterised by a critical exponent $\nu$, which is universal in the sense that it only depends on the universality class and not on any microscopic details,

$$\xi(T) = \xi_0 \left( \frac{T_c - T}{T_c} \right)^{-\nu}.$$  \hfill (31)

Typical values of $\nu$ are around 2/3.

If we could cool the system adiabatically so that it stays in equilibrium, the correlation length $\xi$ would be infinite at the transition temperature $T_c$. Consequently, the whole system could end up in the same vacuum and there would be no domain walls.

However, this is not possible in practice, because of critical slowing down [23]. The relaxation time $\tau$, i.e., the time it takes for a small perturbation to equilibrate, also diverges exponentially, with exponent $\mu$,

$$\tau(T) = \tau_0 \left( \frac{T_c - T}{T_c} \right)^{-\mu}.$$  \hfill (32)

This means that if we are cooling the system with a constant rate characterised by the “quench time” $\tau_Q$,

$$T(t) = \left( 1 - \frac{t}{\tau_Q} \right) T_c,$$  \hfill (33)

the equilibrium correlation length and relaxation time diverge as

$$\xi(t) = \xi_0 \left( \frac{|t|}{\tau_Q} \right)^{-\nu}, \quad \tau(t) = \tau_0 \left( \frac{|t|}{\tau_Q} \right)^{-\mu}.$$  \hfill (34)

Once $\tau(t) > |t|$, any deviation from equilibrium will not have time to disappear before the transition. The fluctuations in the system at that time freeze out and survive all the way to the transition. In particular this means that the correlation length cannot grow significantly after that time, because it is simply a property of these fluctuations.

This means that the actual correlation length at the time of the transition is

$$\xi(\hat{t}) = \xi(t),$$  \hfill (35)
where $\hat{t}$ is the freeze-out time determined by the equation $\tau(\hat{t}) = |\hat{t}|$. This is straightforward to solve and gives

$$\hat{t} = - (\tau_0 \tau_Q^\mu)^{1/(1+\mu)},$$

whereby the frozen-out correlation length is

$$\hat{\xi} = \xi_0 \left( \frac{\tau_0}{\tau_Q} \right)^{\mu/(1+\mu)} \propto \tau_Q^{\nu/(1+\mu)}.$$

As we saw before, this is the typical distance between domain walls formed in the transition.

**CONTINUOUS GLOBAL SYMMETRIES**

It is straightforward to generalise these arguments to systems with continuous symmetries. In the case of a complex scalar field with a $U(1)$ symmetry, the set of possible vacua, i.e., the vacuum manifold, is a circle. The vacua are characterised by the phase angle $\phi = ve^{i\varphi}$. After the transition, the angle $\varphi$ is uncorrelated at distances longer than $\hat{\xi}$.

Consider now three points A, B and C separated by more than $\hat{\xi}$ (see Fig. 5). Each one has a random value of $\varphi$, but field $\phi$ is still continuous. It will therefore have to somehow interpolate between the three points, and if the points are not separated by much more than $\hat{\xi}$, it is most likely that the field follows the shortest path on the vacuum manifold, because that costs least energy. This assumption is known as the geodesic rule. However, there is a finite probability that if the geodesic rule is satisfied between A and B, between B and C and between C and A, the phase angle changes by $2\pi$ as one follows it from A to B to C to A. In that case, continuity tell us that the field has to leave the vacuum manifold and vanish at some point within the circle formed by these points. That point is a vortex.

As with domain walls, the typical distance between vortices after the transition is $\xi$, and their number density is therefore

$$n \approx \xi^{-2} \propto \tau_Q^{-2\nu/(1+\mu)}.$$  \hspace{1cm} (38)

It is straightforward to generalise these arguments to more complicated symmetry groups. In particular, one finds that the number density of magnetic monopoles is $n \approx \xi^{-3}$.

In principle, Eq. (38) makes a prediction that can be tested in condensed matter experiments. However, the arguments that led to it neglected factors or order one in many places, and therefore even if one knew exactly the values of the parameters $\xi_0$ and $\tau_0$, the prediction could still easily be off by a factor of ten. A more robust prediction is the dependence on $\tau_Q$, but that would require carrying out experiments with a wide range of different cooling rates, and in many cases it is not possible to vary the cooling rate at all.

An even more robust test of the Kibble-Zurek mechanism is given by the spatial distribution of vortices. Let us define $N_W(R)$ as the winding number around a circle of radius $R$,

$$N_W(R) = \frac{1}{2\pi} \oint_{|\vec{r}|=R} d\vec{r} \cdot \vec{\nabla} \varphi.$$  \hspace{1cm} (39)

On average, this course vanishes, but its variance $\langle N_W(R)^2 \rangle$ is typically non-zero. After a phase transition in a global theory, the circle passes through around $2\pi R/\hat{\xi}$ domains that are uncorrelated with each other. Every time, the curve moves from one domain to another, the phase angle changes by some random amount $\Delta \varphi$. Labelling the domains by $i$, assuming the geodesic rule, and ignoring all factors of $2\pi$, the total winding number is then

$$N_W(R) \approx \sum_{i=1}^{R/\hat{\xi}} \Delta \varphi_i.$$  \hspace{1cm} (40)

Because the changes $\Delta \varphi_i$ are uncorrelated, the total variance is

$$\langle N_W(R)^2 \rangle \approx \sum_{i,j} \langle \Delta \varphi_i \Delta \varphi_j \rangle = \sum_i \langle \Delta \varphi_i^2 \rangle = \langle \Delta \varphi^2 \rangle R/\hat{\xi} \approx R/\hat{\xi}.$$  \hspace{1cm} (41)

More generally, one can define a scaling exponent $\nu_D$ by

$$\langle N_W(R)^2 \rangle \propto R^{4\nu_D}.$$  \hspace{1cm} (42)

The above result then implies that the Kibble-Zurek mechanism predicts $\nu_D = 1/4$. This can be compared
with a fully random distribution, in which the sign $s_a$ of each vortex $a \in \{1, \ldots, N_{\text{vort}}\}$ inside the curve is independent of all the others. In that case, the winding number is

$$N_W(R) = \sum_{a=1}^{N_{\text{vort}}} s_a, \quad (43)$$

and consequently

$$\langle N_W(R)^2 \rangle \approx \sum_{a,b}(s_a s_b) = \sum_a (s_a^2) = N_{\text{vort}} \propto R^2. \quad (44)$$

Thus, a random distribution of vortices would have $\nu_D = 1/2$.

**GAUGE SYMMETRIES**

In a gauge field theory, one cannot apply the Kibble-Zurek arguments as such, because the phase angle $\varphi$ is not a gauge invariant quantity. Furthermore, while a global system tries to minimise phase gradients $\nabla \varphi$ in order to minimise energy, the condition for minimum energy in a gauge theory is $\nabla \varphi + e \vec{A} = 0$. If the gauge field $\vec{A}$ is non-zero, the phase $\varphi$ does not even attempt to become uniform.

In the special case in which the magnetic field is zero everywhere, one can choose a gauge in which $\vec{A} = 0$. Then, the gradients of $\varphi$ cost energy, and one can expect the same arguments to hold as in a global theory.

However, if one considers a non-zero temperature, as we have been doing, one cannot make this assumption, because it is not consistent with thermal fluctuations. In fact, these fluctuations are nothing but thermal radiation, and in the symmetric phase, they can be approximated by the blackbody spectrum. To the extent that we can consider classical dynamics, we can therefore say that the initial conditions for the magnetic field are given by the Rayleigh-Jeans spectrum. A convenient way to express that is by defining the function $G(k)$,

$$\langle B_i(\vec{k}) B_j(\vec{\ell}) \rangle = G(k) \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) (2\pi)^3 \delta(\vec{k} + \vec{\ell}). \quad (45)$$

The Rayleigh-Jeans distribution corresponds to $G(k) = T$.

In the low-temperature phase, the long-wavelength part of the spectrum is suppressed by the Higgs effect,

$$G(k) = \frac{T k^2}{k^2 + m_\gamma^2}, \quad (46)$$

where $m_\gamma$ is the photon mass. These same considerations apply to superconductors, as well, and in that case $m_\gamma$ is the inverse penetration depth.

Following Ref. [15], one can calculate $m_\gamma$ as a function of temperature, and to leading order one finds,

$$m_\gamma^2 = e^2 \phi_0^2 = \frac{e^2}{4} (T_c - T^2) \quad (47)$$

Assuming again linear cooling, one finds

$$m_\gamma^2 = \frac{e^2 T_c^2}{2} \tau_Q. \quad (48)$$

We will now assume that as the system is cooled through the transition, the scalar field remains close to equilibrium. This is not always true, and if it is not, then one would expect something like the Kibble mechanism. However, here we are interested in the opposite limit in which the gauge field falls out of equilibrium first. We also assume that the mean field result Eq. (48) is approximately valid.

According to Eq. (49), the amplitude $G(k)$ of a given mode decreases slowly at first, but when $m_\gamma \approx k$, it starts to drop sharply toward zero. Using Eq. (48), we see that this happens at time

$$t \approx t_k = \frac{2 k^2}{e^2 T_c^2 \tau_Q}. \quad (49)$$

After this time, the equilibrium value of $G(k)$ decreases as

$$G_{eq}(k) \approx \frac{2 k^2}{e^2 T_c} \tau_Q, \quad (50)$$

and we find

$$\frac{d\ln G_{eq}(k)}{dt} \approx -\frac{1}{t}. \quad (51)$$

This rate is faster at early times, and the fastest decay takes place at $t \approx t_k$. Thus,

$$\left. \frac{d\ln G_{eq}(k)}{dt} \right|_{\text{max}} \approx -\frac{1}{t_k} = -\frac{e^2 T_c^2}{2k^2 \tau_Q}. \quad (52)$$

This is the rate at which $G(k)$ has to be able to decay for the mode to stay in equilibrium. We can see that long wavelengths (low $k$) have to decay faster.

On the other hand, one would expect on physical grounds that the dynamics of longer wavelengths is slower and that they would therefore actually decay slower than short wavelengths. It is therefore inevitable that some very long-wavelength modes are unable to reach the necessary rate in Eq. (52) and fall out of equilibrium instead.

We can be more specific if we assume that the dynamics is governed by the conductivity $\sigma$ and Ohm’s law is valid,

$$\vec{j} = \sigma \vec{E}. \quad (53)$$

Maxwell’s equations then imply

$$\vec{B} = -\nabla \times \vec{\nabla} \times \vec{B} + \vec{\nabla} \times \vec{j} = -\nabla \times \vec{\nabla} \times \vec{B} - \sigma \dot{\vec{B}}. \quad (54)$$
In Fourier space (and ignoring vector indices), this becomes simply
\[ \dot{B} + \sigma B + k^2 B = 0, \tag{55} \]
which is nothing but a damped harmonic oscillator. If \( k < \sigma/2 \), it has an exponential solution
\[ B \propto \exp \left( -\frac{k^2}{\sigma} t \right). \tag{56} \]
For \( G(k) \), this means that any deviation from the equilibrium value will decay no faster than at this rate,
\[ \frac{d \ln G(k)}{dt} \gtrsim -\frac{2k^2}{\sigma}. \tag{57} \]
Thus, we find that the critical value \( k_c \), defined in such a way that modes with \( k < k_c \) fall out of equilibrium, is given by
\[ \frac{2k^2}{\sigma} = \frac{e^2 T^2}{2k^2 T_Q} \Rightarrow k_c \approx \left( \frac{2e^2 T^2 \sigma}{4\pi T} \right)^{1/4}. \tag{58} \]
One can then roughly say that as \( m_\gamma \to \infty \), modes with \( k > k_c \) decay to zero, whereas those with \( k < k_c \) retain their original amplitude. Thus, we would have the spectrum
\[ G(k) = \begin{cases} T, & k < k_c, \\ 0, & k > k_c. \end{cases} \tag{59} \]
What this means is that the magnetic field has not disappeared completely. However, because the field cannot penetrate the broken, superconducting, phase, the magnetic field must be confined into flux tubes, i.e., vortices. Thus, we have seen that in the gauge theory, vortices are formed by the non-equilibrium dynamics of the gauge field. This phenomenon is known as flux trapping \cite{27}, because these vortices originate in the thermal fluctuations of the magnetic field, which do not have time to decay and are instead trapped in vortices.

In general, the winding number \( N_W \) and the magnetic flux \( \Phi = \int d^2 \vec{S} \cdot \vec{B} \) are related by the flux quantum \( \Phi_0 = 2\pi/e \), and for the case of a circle of radius \( R \) in Eq. \( \text{(59)} \), we have
\[ N_W(R) = \Phi(R)/\Phi_0 \approx e\Phi(R). \tag{60} \]
On average, this is zero, but as in the global case, the variance is non-zero and can be calculated from \( G(k) \),
\[ \langle N_W(R)^2 \rangle \approx e^2 \langle \Phi(R)^2 \rangle 
= e^2 \int_0^R d^2 x d^2 y (B_z(x)B_z(y)) 
= e^2 \int_0^R d^2 x d^2 y \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} G(k) \tag{61} \]
Using Eq. \( \text{(59)} \), this is
\[ \langle N_W(R)^2 \rangle \approx \begin{cases} e^2 T_c R, & \text{if } R > 1/k_c, \\ e^2 T_c k_c^2 R^4, & \text{if } R < 1/k_c. \end{cases} \tag{62} \]
The scaling exponent \( \nu_D \) is therefore
\[ \nu_D = \begin{cases} 1, & \text{at short distances,} \\ 1/4, & \text{at long distances.} \end{cases} \tag{63} \]
Unfortunately, the long-distance value 1/4 is equal to what the Kibble-Zurek mechanism gives, and cannot therefore be used to distinguish between the two theories. Because it depends on the dimensionality, the situation is different in two-dimensional models that have been studied numerically \cite{31, 32}. Note, however, that a two-dimensional film in three-dimensional space, the typical setup of actual experiments, behaves like a three-dimensional system in this sense \cite{33}.

In three dimensions, the best way to distinguish between the theories is the short-distance exponent \( \nu \approx 1 \). It means that the typical value of \( N_W(R) \) grows as \( R^2 \), i.e., it is proportional to the area of the circle. This is only possible if all the vortices inside the circle have the same sign, or in other words, if there are clusters of equal-sign vortices (see Fig. \( \text{6} \)). Eq. \( \text{(62)} \) tells us that the radius of these clusters is \( \approx 1/k_c \), and the number of vortices in each is \( \approx \sqrt{e^2 T_c/k_c} \). Consequently, the number density per unit cross sectional area is
\[ n \approx \sqrt{e^2 T_c/k_c} \approx \sqrt{e^2 T_c k_c^2} \approx \sqrt{e^2 T} \left( \frac{e^2 T^2 \sigma}{\gamma Q} \right)^{3/8}. \tag{64} \]
Here we have only discussed Abelian gauge field theories, but similar arguments seem to apply to non-Abelian theories as well. In that case, one can also study the formation of magnetic monopoles, which may shed more light on the monopole problem in cosmology.

CONCLUSIONS

We have seen that if a symmetry breaking phase transition takes place in a finite time, topological defects are formed. In a global theory, they are produced by the Kibble-Zurek mechanism, which leads to a characteristic negative correlation between vortices that can be used to identify the mechanism. In a gauge theory, defects are also produced by thermal fluctuations of the magnetic field that are too slow to decay. The clearest prediction of this flux trapping scenario is the formation of clusters of equal-sign vortices.

These theories are currently being tested in condensed matter experiments, in particular with superconductors. Although the calculations done in these paper refer to relativistic field theories, the qualitative conclusions would be the same in condensed matter systems.

I would like to thank the organisers for a very successful workshop, and the ESF COSLAB programme and Churchill College for financial support.

[1] J. I. Kapusta, *Finite-temperature field theory*, (Cambridge University Press, Cambridge, 1989).
[2] A. Linde, *Particle Physics and Inflationary Cosmology*, (Harwood Academic Publishers, Chur, 1990).
[3] D. Bailin and A. Love, *Introduction to Gauge Field Theory*, (Institute of Physics Publishing, Bristol, 1993).
[4] E. W. Kolb and M. S. Turner, *The Early Universe*, (Westview Press, Boulder, 1994).
[5] M. E. Shaposhnikov, Cont. Phys. 39 (1998) 177.
[6] S. Elitzur, Phys. Rev. D 12 (1975) 3978.
[7] J. Smit, *Introduction to Quantum Fields on a Lattice*, (Cambridge University Press, Cambridge, 2002).
[8] K. Kajantie, M. Karjalainen, M. Laine and J. Peisa, Phys. Rev. B 57 (1998) 3011.
[9] K. Kajantie, M. Laine, K. Rummukainen and M. E. Shaposhnikov, Phys. Rev. Lett. 77 (1996) 2887.
[10] A. M. Polyakov, *Gauge Fields and Strings*, (Harwood Academic Publishers, Chur, 1987).
[11] Vilenkin and E.P.S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, 1994).
[12] G. E. Volovik, *Universe in a Helium Droplet*, (Oxford University Press, Oxford, 2003).
[13] M. J. Bowick, L. Chandar, E. A. Schiff and A. M. Srivastava, Science 263 (2001) 894.
[14] S. Digal, R. Ray and A. M. Srivastava, Phys. Rev. Lett. 83 (1999) 5030.
[15] P. C. Hendry et al., Nature 368 (1995) 315.
[16] M. E. Dodd et al., Phys. Rev. Lett. 81 (1998) 3703.
[17] V. M. Ruutu et al., Nature 383 (1996) 334.
[18] C. Bäumer et al., Nature 383 (1996) 332.
[19] R. Monaco, J. Mygind, and R. J. Rivers, Phys. Rev. Lett. 89 (2002) 080603; Phys. Rev. B 67 (2003) 104506.
[20] R. Carmi and E. Polturak, Phys. Rev. B 60 (1999) 7595.
[21] J. R. Kirtley, C. C. Tsuei and F. Tafuri, Phys. Rev. Lett. 90 (2003) 257001.
[22] A. Maniv, E. Polturak and G. Koren, Phys. Rev. Lett. 91 (2003) 197001.
[23] A. Rajantie, Int. J. Mod. Phys. A 17 (2002) 1.
[24] A. Rajantie, Contemp. Phys. 44 (2003) 485.
[25] W. H. Zurek, Nature 317 (1985) 505.
[26] S. Digal, R. Ray and A. M. Srivastava, Phys. Rev. Lett. 83 (1999) 5030.
[27] M. Hindmarsh and A. Rajantie, Phys. Rev. Lett. 85 (2000) 4660.
[28] G. J. Stephens, L. M. Bettencourt and W. H. Zurek, Phys. Rev. Lett. 88 (2002) 137004.
[29] T. W. B. Kibble and A. Rajantie, Phys. Rev. B 68 (2003) 174512.
[30] A. Rajantie, Phys. Rev. D 68 (2003) 021301.